Sudan University of Science and Technology
College of Graduate Studies

Twisting Non-Commutative L^p Spaces with Triples and Derivations of Semifinite von Neumann Algebras

فضاءات-L^p غير التبديلية الألتوائية مع الثلاثيات و مشتقات جبريات فون نيومان شبه المنتهية

A Thesis Submitted in Fulfillment of the Requirements for the Degree of Ph.D in Mathematics

By:

Sahar Abrahim Bakrey

Supervisor:

Prof. Dr. Shawgy Hussein AbdAlla

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Dedication

To my Family

Acknowledgements

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Abstract

We introduces commutator estimates for interpolation scales with holomorphic structure and give a new and contractive spectral triples over the space of connections and study for crossed products. We show the noncommutative solenoids and their projective modules and deal with Gromov-Hausdorff propinquity and find the spectral triples for noncommutative solenoidal spaces from self-coverings. A description of certain homological properties with twisting of Schatten classes and non-commutative L^p-spaces are obtained. The derivations of τ-measurable operators and on symmetric quasi-Banach ideals of compact operators with continuous derivations in algebras of locally measurable operators are inner are studied. The structure of derivations and on various algebras of measurable operators for type I and with values in ideals of semifinite von Neumann algebras are constructed.

الخلاصة

قمنا بادخال تقديرات المبدل لأجل مقاييس الأستكمال مع البناء التحليلي واعطينا الثلاثي الطيفي الجديد والأنكماش فوق فضاء الاتصالات والضرب الاتجاهي. تم توضيح الملفات اللولبية غير التبديلية ودراسة مقاسات اسقاطاتها وتم التعامل مع تقارب قروموف– هاوسدورف وأوجدنا الثلاثيات الطيفية لأجل فضاءات الملف اللولبي غير التبديلي من الغطاءات–الذاتية. تم الحصول على وصف خصائص متماثلة معينة مع إلتواء عائلات شاتن وفضاءات–L^p غير التبديلية. تمت دراسة الاشتقاقات لمؤثرات المقيسية–τ وعلى مثاليات شبه–باناخ المتماثلة لمؤثرات التراص مع الاشتقاقات المستمرة في جبريات مؤثرات المقيسية الموضعية والتى هي داخلة. تم أنشاء بناء الاشتقاقات وعلى جبريات متنوعة لمؤثرات مقيسة لأجل النوع I ومع قيماً في مثاليات جبريات فون نيومان شبه المنتهية.

Introduction

It is well known that all derivations on a C^*-algebra are continuous and that all derivations on a von Neumann algebra are inner. We some quantitative estimates for nonlinear commutators under the complex interpolation methods and more general interpolation scales with holomorphic structures.

A new construction of a semifinite spectral triple on an algebra of holonomy loops is presented. For p be prime. A noncommutative p-solenoid is the C^*-algebra of $\mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right]$ twisted by a multiplier of that group, where $\mathbb{Z}\left[\frac{1}{p}\right]$ is the additive subgroup of the field $\mathbb{Q}$ of rational numbers whose denominators are powers of p. We survey our classification of these C^*-algebras up to *-isomorphism in terms of the multipliers on $\mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right]$, using techniques from noncommutative topology. Our work relies in part on writing these C∗- algebras as direct limits of rotation algebras, i.e. twisted group C^*-algebras of the group $\mathbb{Z}^2$, thereby providing a mean for computing the K-theory of the noncommutative solenoids, as well as the range of the trace on the K_0 groups.

Given a von Neumann algebra M denote by $S(M)$ and $LS(M)$ respectively the algebras of all measurable and locally measurable operators affiliated with M. For a faithful normal semi-finite trace τ on M let $S(M, \tau)$ be the algebra of all τ-measurable operators from $S(M)$. We give a complete description of all derivations on the above algebras of operators in the case of type I von Neumann algebra M. We show that any derivation of the $*$-algebra $LS(M)$ of all locally measurable operators affiliated with a properly infinite von Neumann algebra M is continuous with respect to the local measure topology $t(M)$.

An extension of Z by Y is a short exact sequence of quasi-Banach modules and homomorphisms $0 \to Y \to X \to Z \to 0$. When properly organized all these extensions constitute a linear space denoted by $\mathrm{Ext}_B(Z, Y)$, where B is the underlying (Banach) algebra. We compute the spaces of extensions for the Schatten classes when they are regarded in its natural (left) module structure over $B = B(H)$, the algebra of all operators on the ground Hilbert space. We make the first steps into the study of extensions ("twisted sums") of noncommutative L^p-spaces regarded as Banach modules over the underlying von Neumann algebra M. Our approach combines Kalton's description of extensions by centralizers (these are certain maps which are, in general, neither linear nor bounded) with a general principle, due to Rochberg and Weiss, saying that whenever one finds a given Banach space Y as an intermediate space in a (complex) interpolation scale, one automatically gets a self-extension $0 \to Y \to X \to Y \to 0$.

Connes showed that spectral triples encode (noncommutative) metric information. Further, Connes and Moscovici in their metric bundle construction showed that, as with the Takesaki duality theorem, forming a crossed product spectral triple can substantially simplify the structure. Bellissard, Marcolli and Reihani (among other things) studied in depth metric notions for spectral triples and crossed product spectral triples for Z-actions, with applications in number theory and coding theory. In the work of Connes and Moscovici, crossed products involving groups of diffeomorphisms and even of etale groupoids are required. We show that noncommutative solenoids are limits, in the sense of the Gromov-Hausdorff propinquity, of quantum tori. Examples of noncommutative self-coverings are described, and spectral triples on the base space are extended to spectral triples on the inductive family of coverings, in such a way that the covering projections are locally isometric. Such triples are shown to converge, in a suitable sense, to a semifinite spectral triple on the direct limit of the tower of coverings, which we call noncommutative solenoidal space. Some of the self-coverings described here are given by the inclusion of the fixed point algebra in a C^*-algebra acted upon by a finite abelian group.

Let I, J be symmetric quasi-Banach ideals of compact operators on an infinite-dimensional complex Hilbert space H, let $J : I$ be the space of multipliers from I to J. Obviously, ideals I and J are quasi-Banach algebras and it is clear that ideal J is a bimodule for I. We show that every derivation acting on the $*$-algebra $LS(M)$ of all locally measurable operators affiliated with a von Neumann algebra M is necessarily inner provided that it is continuous with respect to the local measure topology. For M be a semifinite von Neumann algebra with a faithful semifinite normal trace τ and let A be an arbitrary C^*-subalgebra of M. Assume that E is a fully symmetric function space on $(0, \infty)$ having Fatou property and order continuous norm and $E(M, \tau)$ is the corresponding symmetric operator space. We prove that every derivation $\delta : A \to E(M, \tau) := E(M, \tau) \cap M$ is inner, strengthening earlier results by Kaftal and Weiss [294].

The Contents

Chapter 1
Derivations and Commutator Estimates

We provide some extensions of the results to the algebra of τ-measurable operators affiliated with a von Neumann algebra equipped with a faithful semifinite normal trace τ. We also investigate the spectral behaviour of bounded linear operators under some kinds of interpolation methods.

Section (1.1): τ-Measurable Operators

For A be an algebra. A derivation on A is a linear map $D: A \to A$ such that $D(xy) = xD(y) + D(x)y$ for all $x, y \in A$. A derivation on A is said to be inner if there exists an $a \in A$ such that $D(x) = ax - xa$ for every $x \in A$. The study of derivations on C^*-algebras is well developed ([29], [13] and [30]). It is known that all derivations on a C^*-algebra are continuous ([29], Lemma 4.1.3) and that all derivations on a von Neumann algebra are inner ([29], Theorem 4.1.6).

The study of non-commutative integration, initiated by Segal in [31], brought about some interesting classes of $*$-algebras consisting of unbounded operators affiliated with a von Neumann algebra. Of special importance here is the $*$-algebra of measurable operators affiliated with a von Neumann algebra. Several other notions of measurability have arisen since Segal published. Of special importance to us is the $*$-algebra $\widetilde{M}$ of τ-measurable operators affiliated with a (semifinite) von Neumann algebra M equipped with a faithful semifinite normal trace τ. Equipped with the topology of convergence in measure, $\widetilde{M}$ is a complete metrizable topological $*$-algebra.

All of this naturally leads to questions about derivations on $\widetilde{M}$. We ask whether derivations on $\widetilde{M}$ are continuous and inner. These questions have recently been answered in the negative in [7] and [8], namely that if M is commutative and the projection lattice of M is not atomic, then $\widetilde{M}$ admits at least one discontinuous (hence non-inner) derivation.
We show that if the projection lattice of M is atomic, then all derivations on $\widetilde{M}$ are continuous. Conversely, we show that if M is a finite type I von Neumann algebra and all derivations on $\widetilde{M}$ are continuous, then the projection lattice of M is atomic.

We also consider the problem as to when all continuous derivations on $\widetilde{M}$ are inner. This problem is still open and Sh. A. Ayupov et al. have made some progress on this problem in [2]. We show that if the range of a derivation on $\widetilde{M}$ is contained inside a non-commutative L_p-space, then it is inner.

Throughout, M denotes a (semifinite) von Neumann algebra equipped with a faithful semifinite normal trace τ on a Hilbert space H.

Throughout, $\mathcal{D}(x)$ denotes the domain of an unbounded operator x. The notation $\mathcal{U}(M)$ denotes the set of closed densely defined operators affiliated with M. If M is finite, then $\mathcal{U}(M)$ is a $*$-algebra ([12]). The $*$-algebra of measurable operators affiliated with a von Neumann algebra M, as defined in [31], will be denoted by $S(M)$. If M is commutative, then $M \cong L_\infty(X, \Sigma, \mu)$ and $S(M) \cong L_0(X, \Sigma, \mu)$ for some localizable measure space (X, Σ, μ) ([31]).

The projection lattice of a von Neumann algebra M is denoted by M_p. We denote the centre of M by $Z(M)$.

Definition (1.1.1)[1]: ([19], Definition 2.1) Let E be a subspace of H. We say that E is τ-dense in H if for every $\delta > 0$, there exists a projection $p \in M$ such that $pH \subset E$ and $\tau(1 - p) < \delta$.

An unbounded operator x is said to be τ-measurable if $x \in \mathcal{U}(M)$ and $D(x)$ is τ-dense in H. The set of τ-measurable operators affiliated with M will be denoted by $\widetilde{M}$.

Proposition (1.1.2)[1]: ([19], p. 271; [31], Corollary 4.1) If $\tau(1) < \infty$, then $\widetilde{M} = \mathcal{U}(M)$.

For any finite von Neumann algebra M, one has $S(M) = \mathcal{U}(M)$.

Let $\epsilon, \delta > 0$ and let $\widetilde{M}(\epsilon, \delta)$ denote the set $\{x \in \widetilde{M}$: there is a $p \in M_p$ such that $pH \subset D(x), \|xp\| \leq \epsilon, \tau(1-p) < \delta\}$.

The sets $\{\widetilde{M}(\epsilon, \delta): \epsilon, \delta > 0\}$ form a system of basic neighbourhoods of zero for a topology γ_{cm} on $\widetilde{M}$, called the topology of convergence in measure on $\widetilde{M}$ ([19], p. 272). Furthermore, $\widetilde{M}$ is a $*$-algebra under strong sum, strong product, scalar multiplication and ordinary adjunction (the strong sum (resp. strong product) of unbounded operators x and y is defined to be $\overline{x+y}$ (resp. $\overline{xy}$), where x denotes the closure of the operator x). When equipped with the topology of convergence in measure, $\widetilde{M}$ is a complete metrizable topological $*$-algebra ([19], p. 272). Lastly, M is dense in $\widetilde{M}$ with respect to the topology of convergence in measure ([19], p. 272).

If $x, y \in \widetilde{M}$, we will write $x + y$ and xy to mean the strong sum and strong product respectively of x and y.

Example (1.1.3)[1]: ([36]) Let M be a commutative von Neumann algebra. Then $M \cong L_\infty(X, \Sigma, \mu)$ for some localizable measure space (X, Σ, μ), and M is equipped with the trace defined by $\tau(f) = (\int_X f\, d\mu$ for all (almost everywhere) positive $f \in L_\infty(X, \Sigma, \mu)$. The topologies of convergence in measure on $L_\infty(X, \Sigma, \mu)$ and $\bar{L}_\infty(X, \Sigma, \mu)$ are the restrictions of the topology of convergence in measure on $L_0(X, \Sigma, \mu)$ to $L_\infty(X, \Sigma, \mu)$ and $\bar{L}_\infty(X, \Sigma, \mu)$ respectively. By taking completions, we find that $\widetilde{M}$ is topologically $*$-isomorphic to $\bar{L}_\infty(X, \Sigma, \mu)$.

Theorem (1.1.4)[1]: ([32], Theorem 2.3(i)) Let $p \in M_p$ and $\tau_p = \tau|_{(pMp)^+}$. Then τ_p is a faithful semifinite normal trace on pMp. Furthermore, $\widetilde{pMp} = p\widetilde{M}p$, where $\widetilde{pMp}$ is the algebra of τ_p-measurable operators affiliated with pMp.

Proposition (1.1.5)[1]: ([32], Examples 2.2(3)) The following statements are equivalent.

(i) $\widetilde{M} = M$.

(ii) $\inf\{\tau(p) : 0 \neq p \in M_p\} > 0$.

(iii) The topology of convergence in measure coincides with the norm topology.

Let $x \in \widetilde{M}$. The generalized singular function ([19], Definition 2.1) of x, denoted by $\mu_t(x)$, is defined by

$$\mu_t(x) = \inf\{\|xp\|: p \in M_p, pH \subset D(x), and\ \tau(1-p) \leq t\}, t > 0.$$

Proposition (1.1.6)[1]: ([19], Lemma 2.5) Let $x, y, z \in \widetilde{M}$. The following statements hold for every $t > 0$.

(i) $\mu_t(xyz) \leq \|x\|\mu_t(y)\|z\|$, where the cases $\|x\| = \infty$ and $\|z\| = \infty$ are allowed.

(ii) $\mu_t(x^*) = \mu_t(x)$.

Theorem (1.1.7)[1]: ([37], Theorem 2.1) Let I be a γ_{cm}-closed two-sided ideal of $\widetilde{M}$. Then $I \cap M$ is a norm closed two-sided ideal of M and $I = \overline{I \cap M}^{\gamma_{cm}}$.

An atom of M is a nonzero projection in M having no nonzero proper subprojections in M. We say that M_p is atomic if for every nonzero $p \in M_p$, there exists an atom $q \in M_p$ such that $q \leq p$.

For $0 < p < \infty$, we define $L_p(M, \tau)$ to be the set of all $x \in \widetilde{M}$ such that

$$\|x\|_p = \left(\int_0^\infty \mu_t(x)^p \, dt\right)^{\frac{1}{p}} < \infty$$

(a special case of [16], Definition 4.1). In addition, we put $L_\infty(M,\tau) = M$ and denote by $\|\cdot\|_\infty$ the usual operator norm defined on M. It is well known that $L_p(M,\tau)$ is a Banach space under $\|\cdot\|_p$ whenever $1 \le p \le \infty$ ([16], Theorem 4.5). Also, whenever $1 \le p \le \infty$, $L_\infty(M,\tau)$ is a Banach M-module ([38], Proposition 2.5). If $1 < p < \infty$, then L_p is reflexive (this is a special case of [17], Corollary 5.16). The proof of the following lemma follows directly from Proposition (1.1.6).

Lemma (1.1.8)[1]: Let $1 \le p < \infty$ and let $u \in M$ be a unitary operator. Then $\|u^* x u\|_p = \|x\|_p$ for every $x \in L_p(M,\tau)$.

There are other definitions of non-commutative L_p-spaces which we will not discuss, see [19], p. 271, and [38]. These definitions are equivalent to the one given above.

The separating space $S(D)$ of a derivation D on $\widetilde{M}$ is defined to be the set $\{y \in \widetilde{M}$: there is a sequence (x_n) in $\widetilde{M}$ such that $x_n \to 0$ and $D(x_n) \to y\}$ with respect to the topology of convergence in measure on $\widetilde{M}$. Since $\widetilde{M}$ is a complete metrizable topological $*$-algebra in the topology of convergence in measure, the closed graph theorem tells us that D is $\gamma_{cm} - \gamma_{cm}$ continuous if and only if $S(D) = \{0\}$.

Every derivation on a C^*-algebra is continuous and that every derivation on a von Neumann algebra is inner. The following result demonstrates that these results do not carry over to algebras of measurable operators.

Theorem (1.1.9)[1]: (A.F. Ber, V.I. Chilin and F.A. Sukochev). ([7], Theorem 3, and [8], Theorem 3.4) Let M be a commutative von Neumann algebra. Then $S(M)$ admits a nonzero derivation if and only if M_p is not atomic.

Every commutative von Neumann algebra is finite, the following corollary follows from Proposition (1.1.2) and Theorem (1.1.9).

Corollary (1.1.10)[1]: Let M be a commutative von Neumann algebra with a faithful finite normal trace. Then $\widetilde{M}$ admits a nonzero derivation if and only if M_p is not atomic.

Note that if M is commutative, then a derivation on $\widetilde{M}$ is $\gamma_{cm} - \gamma_{cm}$ continuous if and only if it is zero ([2], p. 11).

Using Theorem (1.1.9) and the fact that every finite type I von Neumann algebra is a direct sum of finite matrix algebras over commutative von Neumann algebras ([29], Theorems 2.3.2 and 2.3.3), one can prove the following result.

Theorem (1.1.11)[1]: ([9], Theorem 5) If M is a finite type I von Neumann algebra and every derivation $D: S(M) \to S(M)$ is inner, then M_p is atomic.

The following corollary is an immediate consequence of Proposition (1.1.2) and Theorem (1.1.11).

Corollary (1.1.12)[1]: Let M be a type I von Neumann algebra with a faithful finite normal trace τ. If every derivation D: $\widetilde{M} \to \widetilde{M}$ is inner, then M_p is atomic.

Our next result was motivated by Corollary (1.1.10) and the remark immediately thereafter.

Theorem (1.1.13)[1] (see[35]): If M_p is atomic, then all derivations on $\widetilde{M}$ are $\gamma_{cm} - \gamma_{cm}$ continuous.

Proof. Let D be a derivation on $\widetilde{M}$ and q an atomic projection in M. Let $y \in S(D)$. Then there exists a sequence x_n in $\widetilde{M}$ with $x_n \to 0$ (γ_{cm}) and $D(x_n) \to y$ (γ_{cm}). Therefore

$$D(qx_nq) = qD(x_nq) + D(q)x_nq = qx_nD(q) + qD(x_n)q + D(q)x_nq \to qyq \ (\gamma_{cm}).$$

By Theorem (1.1.4), it follows that $q\,\widetilde{M}q=\widetilde{qMq}$. Since q is an atomic projection, $qMq=\mathbb{C}_q$.
We show that $qMq=\widetilde{qMq}$. Since q is an atomic projection, it is the only nonzero projection in qMq. Hence

$$\inf\{\tau(p): p \text{ a nonzero projection in } qMq\}=\tau(q)>0,$$

since $q\neq 0$ and the trace τ is faithful. Therefore, by Proposition (1.1.5), $qMq=\widetilde{qMq}$. Thus $q\widetilde{M}q=\widetilde{qMq}=qMq=\mathbb{C}q$. Since $x_n\to 0\ (\gamma_{cm}), qx_nq\to 0\ (\gamma_{cm})$.

Since $q\widetilde{M}q$ is finite-dimensional, $D|_{q\widetilde{M}q}$ is $\gamma_{cm}-\gamma_{cm}$ continuous, implying that $D(qx_nq)\to 0\ (\gamma_{cm})$. Recall that $D(qx_nq)\to\ qyq\ (\gamma_{cm})$. Therefore $qyq=0$. This is true for every atomic projection q in M and for every $y\in S(D)$.

We show next that $S(D)$ has no atomic projections of M. Suppose that $S(D)$ has at least one atomic projection q_0 of M. Then, from what we have proved above, $q_0q_0q_0=0$, i.e., $q_0=0$. This is a contradiction since $q_0=0$. Therefore $S(D)$ has no atomic projections of M.

For a contradiction, let us assume that $S(D)\neq\{0\}$. By Theorem (1.1.7), and the fact that $S(D)$ is a γ_{cm}-closed two-sided ideal of $\widetilde{M}$, it follows that $S(D)\cap M$ is a norm closed two-sided ideal of M. Furthermore, $S(D)\cap M\neq\{0\}$ since, by Theorem (1.1.7),

$$\overline{S(D)\cap M}^{\gamma_{cm}}=S(D)\neq\{0\}.$$

Therefore $S(D)\cap M$ has at least one nonzero projection p. Since M_p is atomic, there exists an atomic projection q_0 of M such that $0<q_0\leq p$. So $q_0=q_0p\in S(D)\cap M$ because $q_0\in M, p$ is in $S(D)$, and $S(D)\cap M$ is a two-sided ideal of M. This is a contradiction since $S(D)$ has no atomic projections of M. Hence $S(D)=\{0\}$, and so D is $\gamma_{cm}-\gamma_{cm}$ continuous.

In light of Corollary (1.1.10) and the remark thereafter, it would be interesting to know if the converse of Theorem (1.1.13) holds. We now solve this problem in the affirmative for finite type I von Neumann algebras, thereby extending Corollary (1.1.12). For this, we need the following four lemmas. The proof of [9], Theorem 5 (i.e., Theorem 3.3), relies strongly on [12], which itself depends on the fact that $S(M)$ is a regular algebra whenever M is finite.

In general, $\widetilde{M}$ is not regular even if M is commutative: A finite von Neumann algebra M can be enlarged to a regular algebra $\mathcal{U}(M)$, the algebra of closed densely defined operators affiliated with M. By [11], p. 211, $\mathcal{U}(M)$ is also the smallest regular algebra to contain M, in the sense that the only regular subalgebra of $\mathcal{U}(M)$ to contain M is $\mathcal{U}(M)$ itself. Thus, if M is commutative and $\widetilde{M}\neq\mathcal{U}(M)$, then $\widetilde{M}$ is not regular.

Lemma (1.1.14)[1]: Let p be a central projection of M and D a derivation on $\widetilde{pMp}$. Then D can be extended to a derivation $\overline{D}$ on $\widetilde{M}$.

Proof. Let $\overline{D}(x)=D(pxp)$ for every $x\in\widetilde{M}$. Since p is a central projection, $D(p)=0$. Using this, it is easily verified that $\overline{D}$ is a derivation on $\widetilde{M}$ extending D.

For $p\in M_p$, we denote by $c(p)$ the least central projection majorizing p, and we call $c(p)$ the central support of p. If $p,q\in M_p$, then $qxp=0$ for every $x\in M$ if and only if $c(p)c(q)=0$ ([33], Corollary V.1.7). This is needed in the proof of the following known result, which we give for completeness.

Lemma (1.1.15)[1]: ([21], Theorem 2.1) If q is an atom in a von Neumann algebra M, then the central support $c(q)$ of q is an atom in $Z(M)$.

Proof. Let p be a central projection of M such that $0<p\leq c(q)$. Then $qp\in M_p$ and $qp\leq q$. Since q is an atom, it follows that $qp=0$ or $qp=q$.

We show that $qp \neq 0$, so suppose that $qp = 0$. Then $qxp = qpx = 0$ for every $x \in M$, and so $c(q)c(p) = 0$, i.e., $pc(q) = c(q)p = 0$. Since $p \leq c(q)$, it follows that $p = 0$. This is a contradiction since $0 \neq p$. Thus $qp \neq 0$.
Therefore $qp = q$, implying that $q = qp \leq p$. Hence $c(q) \leq p$, since p is a central projection majorizing q, and thus $c(q) = p$, implying that $c(q)$ is an atom in $Z(M)$.
Lemma (1.1.16)[1]: Let $(p\alpha)$ be a family of central orthogonal projections in M such that $\sum_\alpha p\alpha = 1$. If $(p\alpha M p\alpha)_p$ is atomic for every α, then M_p is atomic.
Proof. Suppose $(p\alpha M p\alpha)_p$ is atomic for every α. Let $0 < p \in M_p$. Since $p\alpha$ is a central projection for every α, it follows that $pp_\alpha \in M_p$ for every α. By hypothesis, $\sum_\alpha p\alpha = 1$, and so $\sum_\alpha pp_\alpha = p$. Since $p \neq 0$, we have that $pp\alpha \neq 0$ for some α. It is easily verified that $p \geq pp_\alpha$ and that $pp_\alpha \leq p_\alpha$. Therefore, pp_α is a projection in $p_\alpha M p_\alpha$ and, by hypothesis, there exists an atom q_α in $p_\alpha M p_\alpha$ such that $p \geq pp_\alpha \geq q_\alpha$. Clearly, q_α is also an atom in M. This completes the proof.

For the next lemma, we recall the notion of a direct sum of von Neumann algebras ([15], pp. 20-21). For each α, let A_α be a von Neumann algebra on the Hilbert space H_α, and let $\|\cdot\|_\alpha$ denote the norm on A_α. If $x_\alpha \in A_\alpha$ for each α, and $sup_\alpha \|x_\alpha\|_\alpha < \infty$, define a bounded linear operator x on the direct sum H of the family of Hilbert spaces (H_α) by $x(\xi) = (x_\alpha(\xi_\alpha))$, where $\xi = (\xi_\alpha) \in H$. The direct sum A of the von Neumann algebras A_α is defined to be the set of all such operators x, and we write $A = \bigoplus_\alpha A_\alpha$. It can be verified that A is a von Neumann algebra on H with coordinate-wise operations, and norm $x \longmapsto sup_\alpha \|x_\alpha\|_\alpha$.
Lemma (1.1.17)[1]: Let $M = A \overline{\otimes} \mathcal{B}(H)$, where A is a commutative von Neumann algebra and H a finite-dimensional Hilbert space (so M is a type I von Neumann algebra). If A_p is atomic, so is M_p.
Proof. Since A is commutative and A_p is atomic, $A \cong L_\infty(X, \Sigma, \mu)$ for some localizable atomic measure space (X, Σ, μ). Therefore we can find a disjoint family of atoms $(A_\lambda : \lambda \in \Lambda)$ satisfying $X = \cup A_\lambda$. Recall that measurable functions on atoms are constant. Hence $A \cong l_\infty(\Lambda)$, where $l_\infty(\Lambda)$ is the space of all bounded nets in $\mathbb{C}$ indexed by Λ. Hence, by [15], p. 29,

$$M = A \overline{\otimes} B(H) \cong l_\infty(\Lambda) \overline{\otimes} B(H) = \bigoplus_{\alpha\in\Lambda} (\mathbb{C}_\alpha \overline{\otimes} B(H)) \cong \bigoplus_{\alpha\in\Lambda} B(H_\alpha),$$

where $H_\alpha = H$ and $\mathbb{C}_\alpha = \mathbb{C}$ for every $\alpha \in \Lambda$. Every $(B(H_\alpha))_p$ is atomic, and one can find a family of central orthogonal projections (p_α) in M such that $\sum_\alpha p_\alpha = 1$ and $p_\alpha M p_\alpha \cong B(H_\alpha)$, namely $p_\alpha = 1_\alpha$, where 1_α denotes the identity operator on H_α, for every $\alpha \in \Lambda$. By Lemma (1.1.16), M_p is atomic.

If M is a finite von Neumann algebra, then M can be imbedded into the maximal ring of right quotients Q_M of M ([12]). Furthermore, $\mathcal{U}(M) \cong Q_M$([12]). Therefore, if A and B are $*$-isomorphic finite von Neumann algebras, then $\mathcal{U}(A) \cong \mathcal{U}(B)$. In particular, if A and B are equipped with faithful finite normal traces, then $\bar{A} = \mathcal{U}(A) \cong \mathcal{U}(B) = \bar{B}$. This is needed in the proof of our next proposition.
If A is a von Neumann algebra with a faithful finite normal trace, then $\widetilde{M_n(A)} = M_n(\bar{A})$ for every $n \in N$ ([12], Lemma 2). Let $(a_{i,j}^m)$ be a sequence in $M_n(\bar{A})$ and $(a_{i,j}) \in M_n(\bar{A})$. Then $(a_{i,j}^m) \to (a_{i,j})$ in measure as $m \to \infty$ if and only if $a_{i,j}^m \to a_{i,j}$ as $m \to \infty$ for every i, j (this is an immediate consequence of [18], Lemma 2.1).
The proof of the following proposition is a slight modification of the proof [9], Theorem 5, and we give the proof for completeness. It is a special case of Theorem (1.1.20).

Proposition (1.1.18)[1]: Suppose that M is a type I von Neumann algebra with a faithful finite normal trace. If all derivations on $\widetilde{M}$ are $\gamma_{cm} - \gamma_{cm}$ continuous, then M_p is atomic.

Proof. Since M is a finite type I von Neumann algebra, there exists a sequence of central projections (p_n) such that $M = \bigoplus_{n=1}^{\infty} p_n M p_n$, and $p_n M p_n \cong M_{k_n}(A_n)$, where A_n is a commutative von Neumann algebra for every n ([29], Theorems 2.3.2 and 2.3.3), and the k_n are integers. Assume that M_p is not atomic. Then, by Lemmas (1.1.16) and (1.1.17), it follows that $(A_r)_p$ is not atomic for some $r \in \mathbb{N}$. By Corollary (1.1.10), there exists a derivation δ on $\widetilde{A_r}$ which is not $\gamma_{cm} - \gamma_{cm}$ continuous.

We show that there exists a derivation D_r on $\widetilde{M_{k_r}(A_r)} = M_{k_r}(\widetilde{A_\iota})$ which is not $\gamma_{cm} - \gamma_{cm}$ continuous. Define D_r to be the linear map defined by $D((a_{i,j})) = (\delta(a_{i,j}))$ for every $(a_{i,j}) \in M_{k_r}(\widetilde{A_r})$. It is easily verified that D_r is a derivation. Since δ is not $\gamma_{cm} - \gamma_{cm}$ continuous, there exists a sequence (a_m) in $\widetilde{A_r}$ such that $a_m \to 0$ in measure and $\delta(a_m)$ does not converge to zero in measure. Let $(a_{i,j}^m)$ be the sequence in $M_{k_r}(A_r)$ defined as $a_{i,j}^m = a_m$ for every m and for all i, j. By the preceding remarks, $(a_{i,j}^m) \to (b_{i,j})$ as $m \to \infty$, where $b_{i,j} = 0$ for all i, j. Also, by the preceding remarks, $D_r((a_{i,j}^m)) = (\delta(a_{i,j}^m))$ does not converge to $D_r((b_{i,j})) = (\delta(b_{i,j}))$ in measure. Therefore, D_r is not $\gamma_{cm} - \gamma_{cm}$ continuous. Since $p_r M p_r \cong M_{k_r}(A_r)$ and the traces of $p_r M p_r$ and $M_{k_r}(A_r)$ are finite, it follows that the algebra $M_{k_r}(\widetilde{A_r}) = \widetilde{M_{k_r}(A_r)}$ is isomorphic to $\widetilde{p_r M p_r}$, and this isomorphism is also $\gamma_{cm} - \gamma_{cm}$ bicontinuous (this follows from [25], Proposition 4.7). Therefore, the derivation D_r can be identified with a derivation on $\widetilde{p_r M p_r}$ which is not $\gamma_{cm} - \gamma_{cm}$ continuous. By Lemma (1.1.14), D_r can be extended to a derivation D on $\widetilde{M}$. It follows that D is not $\gamma_{cm} - \gamma_{cm}$ continuous. This contradicts the hypothesis, implying that M_p is atomic.

The following proposition is the first part of the proof of Theorem (1.1.20).

Proposition (1.1.19)[1]: Let M be a finite type I von Neumann algebra. If all derivations on $\widetilde{M}$ are $\gamma_{cm} - \gamma_{cm}$ continuous, then $(Z(M))_p$ is atomic.

Proof. Suppose that M is a finite type I von Neumann algebra such that all derivations on $\widetilde{M}$ are $\gamma_{cm} - \gamma_{cm}$ continuous. Since M is a finite von Neumann algebra, $\tau|_{Z(M)}$ is semifinite ([15], Proposition 10, p. 12).

Let q be nonzero central projection such that $\tau(q) < \infty$. Let D be a derivation on $\widetilde{qMq}$. By Lemma (1.1.14), D can be extended to a derivation $\overline{D}$ on $\widetilde{M}$. By hypothesis, $\overline{D}$ is $\gamma_{cm} - \gamma_{cm}$ continuous, and thus, D is $\gamma_{cm} - \gamma_{cm}$ continuous. Since qMq is a type I von Neumann algebra (on the Hilbert space qH) with finite trace, it follows from Proposition (1.1.19) that $(qMq)_p$ is atomic. Therefore there exists an atom q_0 in qMq such that $q \geq q_0$. It is clear that q_0 is also an atom in M.

Let $p \in Z(M)_p$. Using again the fact that $\tau|_{Z(M)}$ is semifinite, it follows that there exists a nonzero $p_1 \in Z(M)_p$ such that $\tau(p_1) < \infty$ and $p \geq p_1$. By the previous paragraph, there is an atom p_2 in M such that $p_1 \geq p_2$. Hence every central projection majorizes an atom of M. By Lemma (1.1.15), $c(p_2)$ is an atom in $Z(M)$. Since $p \in Z(M)_p$, it follows that $p \geq c(p_2)$. Thus $Z(M)_p$ is atomic.

Theorem (1.1.20)[1]: Let M be a finite type I von Neumann algebra. If all derivations on $\widetilde{M}$ are $\gamma_{cm} - \gamma_{cm}$ continuous, then M_p is atomic.

Proof. Since M is a finite type I von Neumann algebra, we can write $M \cong \bigoplus_{n=1}^{\infty} M_n$, where every$M_n$ is a type I_n algebra such that $M_n \cong Z(M_n) \overline{\otimes} B(H_n)$, where H_n is a finite-

dimensional Hilbert space for each n ([29], Theorems 2.3.2 and 2.3.3). To be more precise, one can find a sequence of central projections (p_n) in M such that, for every n, $M_n \cong p_n M p_n$ and $\sum_{n=1}^{\infty} p_n = 1$. By Theorem (1.1.4), $\widetilde{M_n} = p_n \widetilde{M} p_n$ for every n.
Consider now a fixed n and let D_n be a derivation on $\widetilde{M_n}$. It follows from Lemma (1.1.14) that D_n can be extended to a derivation $\overline{D_n}$ on $\widetilde{M}$. By hypothesis, $\overline{D_n}$ is $\gamma_{cm} - \gamma_{cm}$ continuous, and so the same holds for D_n. Hence all derivations on $\widetilde{M_n}$ are $\gamma_{cm} - \gamma_{cm}$ continuous. This holds for every n. By Proposition (1.1.19), $Z(M_n)_p$ is atomic for every n. It is an immediate consequence of Lemma (1.1.20), and the fact that $M_n \cong Z(M_n) \overline{\otimes} B(H)$ for every n, that every $(M_n)_p$ is atomic. Finally, by Lemma (1.1.16), M_p is atomic.
The following example demonstrates that the converse of Theorem (1.1.13) does in general not hold.
Example (1.1.21)[1]: Let $A = L_\infty([0,1]), H$ an infinite-dimensional Hilbert space, and $M = A \overline{\otimes} B(H)$. Then M is a type I_∞ von Neumann algebra.
Now

$$Z(M) = Z(A \overline{\otimes} B(H)) = Z(A) \overline{\otimes} Z(B(H)) = A \overline{\otimes} \mathbb{C}1 \cong A = L_\infty([0,1]).$$

Therefore $Z(M)$ has no atoms. Therefore, by Lemma (1.1.15), M has no atoms, implying that M_p is not atomic. Recently, $F.A.$ Sukochev, $A.F.$ Ber and $B.$ de Pagter proved that all derivations on $\widetilde{M}$ are $\gamma_{cm} - \gamma_{cm}$ continuous ([10]). This result has also been proved in [4].
We have already seen that derivations on $\widetilde{M}$ need not be continuous and not inner. Therefore, the following problem presents itself.
Lemma (1.1.22)[1]: ([6]) If D is a $\gamma_{cm} - \gamma_{cm}$ continuous derivation on $\widetilde{M}$ such that $D(M) \subset M$, then there exists $a \in M$ such that $D(x) = ax - xa$ for all $x \in \widetilde{M}$, i.e., D is inner.
Proof. Let D be a derivation as in the hypothesis. Then, since $D(M) \subset M$, there exists $a \in M$ such that $D(x) = ax - xa$ for every $x \in M$ ([29], Theorem 4.1.6).

Since D is continuous and M is dense in $\widetilde{M}$, it follows that $D(x) = ax - xa$ for every $x \in \widetilde{M}$, i.e., D is inner.

Derivations on $\widetilde{M}$ have recently also been studied in [3] and [4].
Proposition (1.1.23)[1]: Let $D: \widetilde{M} \to \widetilde{M}$ be a derivation. If $D|_M$ is $\gamma_{cm} - \gamma_{cm}$ continuous, then D is $\gamma_{cm} - \gamma_{cm}$ continuous.
Proof. Every derivation on a $*$-algebra A can be written as a linear combination of two $*$-derivations ([13], p. 229). Therefore, we may assume without loss of generality that D is a $*$-derivation. It suffices to show that $S(D) = \{0\}$. Let $y \in S(D)$. Then there is a sequence (x_n) in $\widetilde{M}$ such that $x_n \to 0\ (\gamma_{cm})$ and $D(x_n) \to y\ (\gamma_{cm})$. Since $x_n(1 + x_n^* x_n)^{-1}$ is affiliated with M for every n, it follows that $x_n(1 + x_n^* x_n)^{-1} \in M$ for every n. Since inversion is continuous on $\widetilde{M}$ in the measure topology ([34]) and $1 + x_n^* x_n \to 1(\gamma_{cm})$, it is immediate that $x_n(1 + x_n^* x_n)^{-1} \to 0\ (\gamma_{cm})$. Observe that

$$D(x_n(1 + x_n^* x_n)^{-1}) = x_n D((1 + x_n^* x_n)^{-1}) + D(x_n)(1 + x_n^* x_n)^{-1},$$

and

$$\begin{aligned} 0 = D(1) &= D((1 + x_n^* x_n)(1 + x_n^* x_n)^{-1}) \\ &= (1 + x_n^* x_n) D((1 + x_n^* x_n)^{-1}) + D(1 + x_n^* x_n)(1 + x_n^* x_n)^{-1}. \end{aligned}$$

Therefore

$$(1 + x_n^* x_n) D((1 + x_n^* x_n)^{-1}) = -D(1 + x_n^* x_n)(1 + x_n^* x_n)^{-1},$$

implying that

$$\begin{aligned} D((1 + x_n^* x_n)^{-1}) &= -(1 + x_n^* x_n)^{-1} D(1 + x_n^* x_n)(1 + x_n^* x_n)^{-1} \\ &= -(1 + x_n^* x_n)^{-1} D(x_n^* x_n)(1 + x_n^* x_n)^{-1} \end{aligned}$$

$$= -(1 + x_n^* x_n)^{-1}(x_n^* D(x_n) + D(x_n)^* x_n)(1 + x_n^* x_n)^{-1}$$
$$\to -1^{-1}(0.y + y * .0)1^{-1}(\gamma_{cm})$$
$$= 0$$

by continuity of inversion. Hence $D(x_n(1 + x_n^* x_n)^{-1}) \to y\ (\gamma_{cm})$. Let $y_n = x_n(1 + x_n^* x_n)^{-1}$ for every n. Then (y_n) is a sequence in M such that $y_n \to 0\ (\gamma_{cm})$ and $D(y_n) \to y\ (\gamma_{cm})$. By hypothesis, $y = 0$, implying that $S(D) = \{0\}$.

In what follows, we will need the following result.

Theorem (1.1.24)[1]: (Ryll-Nardzewski fixed point theorem). ([26], p. 444) Suppose that X is a locally convex Hausdorff space and $\emptyset \neq K \subset X$ a weakly compact convex subset of X. Let $\mathcal{F}: K \to K$ be a non-contracting semigroup of weakly continuous affine maps (here, non-contracting means that for every $x, y \in K$ with $x \neq y$, there exists a seminorm p such that $inf_{\phi \in \mathcal{F}}\, p(\phi(x) - \phi(y)) > 0$). Then there exists $x \in K$ such that x is a fixed point of $\mathcal{F}$.

The proof of the next result is similar to that of [22], Lemma 4.1, and [20], Theorem 1. Recall from the notion of the non-commutative L_p-spaces $L_p(M, \tau), 0 < p < \infty$, and that every $L_p(M, \tau), p \geq 1$, is a Banach M-module with the norm $\|\cdot\|_p$ as defined.

Theorem (1.1.25)[1]: Let D be a derivation on $\widetilde{M}$ such that $D(\widetilde{M}) \subset L_p(M, \tau)$ for some $1 < p < \infty$. Then there exists an $a \in L_p(M, \tau)$ such that $D(x) = ax - xa$ for every $x \in \widetilde{M}$.

Proof. We denote the unitary group of M by M_u. Let $K = \{u^* D(u): u \in M_u\}$ and $\mathcal{L}$ be the $\sigma(L_p, (L_p))$-closed convex hull of K (here $L_p = L_p(M, \tau)$). Let $D_b = D|_M$. Then D_b is a derivation on M into L_p. By [27], Theorem 2, D_b is norm-norm continuous.

Therefore $\mathcal{L}$ is a bounded subset of (the Banach space) L_p: $\|u^* D(u)\|_p \leq \|u^*\| \|D(u)\|_p = \|D(u)\|_p$. Now M_u is a norm bounded subset of M. Since D_b is (norm-norm) continuous, $\{D(u): u \in M_u\}$ is a norm bounded subset of L_p. So $sup_{u \in M_u} \|D(u)\|_p < \infty$. Hence $sup_{u \in M_u} \|u^* D(u)\|_p < \infty$, meaning that K, and thus $\mathcal{L}$, is a norm bounded subset of L_p.

Since L_p is reflexive, it follows from the Banach-Alaoglu theorem that $\mathcal{L}$ is $\sigma(L_p, (L_p)^*)$-compact. For each $u \in M_u$, define the affine map $A_u(x) = u^* x u + u^* D(u)$ for all $x \in L_p$. Since L_p is a Banach M-module, $A_u(L_p) \subset L_p$ for every $u \in M_u$. Since $\|u^* x u\|_p = \|x\|_p$ for every $u \in M_u$ and every $x \in L_p$ (Lemma 1.1.7), it follows easily that $A_u: L_p \to L_p$ is norm continuous for every $u \in M_u$. A Standard result tells us that A_u is $\sigma(L_p, (L_p)^*)$-continuous for every $u \in M_u$.

Let $u, v \in M_u$. Then an easy computation shows that $A_v(u^* D(u)) = (uv)^* D(uv)$. Now $uv \in M_u$. Thus $A_v(K) \subset K$. Therefore, since A_v is $\sigma(L_p, (L_p)^*)$-continuous, $A_v(\mathcal{L}) \subset \mathcal{L}$. Since $A_u A_v(x) = A_{uv}(x)$ for every $x \in L_p$, it follows that $\{A_u: u \in M_u\}$ is a semigroup. Now let $x, y \in \mathcal{L}$ with $x \neq y$. Then, by Lemma (1.1.8),

$$\|A_u(x) - A_u(y)\|_p = \|u^* x u + u^* D(u) - u^* y u - u^* D(u)\|_p = \|u^* x u - u^* y u\|_p$$
$$= \|u^*(x - y)u\|_p = \|x - y\|_p$$

for all $u \in M_u$ and for all $1 < p < \infty$. Therefore

$$inf_{u \in M_u} \|A_u(x) - A_u(y)\|_p > 0$$

for every $1 < p < \infty$. Hence $\{A_u: u \in M_u\}$ is non-contracting. By the Ryll-Nardzewski fixed point theorem, there exists $a_0 \in \mathcal{L}$ such that $A_u(a_0) = a_0$ for every $u \in M_u$. Therefore $u^* a_0 u + u^* D(u) = a_0$ for every $u \in M_u$. Let $a = -a_0$.

It follows that $D(u) = au - ua$ for every $u \in M_u$.

Let $x \in M$. Then it is well known that $x = \sum_{i=1}^{4} \lambda_i u_i$, where $\lambda_i \in \mathbb{C}$ and $u_i \in M_u$ for all $1 \leq i \leq 4$. Once again, an easy calculation shows that $D(x) = ax - xa$. This holds for every $x \in M$. Thus $D|_M$ is $\gamma_{cm} - \gamma_{cm}$ continuous.

By Proposition (1.1.23), D is $\gamma_{cm} - \gamma_{cm}$ continuous. Therefore, since M is dense in $\widetilde{M}$ with respect to the measure topology, it follows that $D(x) = ax - xa$ for every $x \in \widetilde{M}$, implying that D is inner.

In the proof of [29], Theorem 4.1.6, a similar argument was used for the case where M is a countably decomposable finite von Neumann algebra: Sakai introduced the maps $T_u(x) = uxu^* + D(u)u^*$ for all $x \in M$ and for all $u \in M_u$. He showed, by using Zorn's Lemma, instead of the Ryll-Nardzewski fixed point theorem, that the maps T_u have a fixed point.

Section (1.2): Interpolation Scales with Holomorphic Structure

In 1983, Rochberg and Weiss [54] studied the behaviour of the commutators for bounded linear operators and some derivation operators, which are usually unbounded and nonlinear, under the complex interpolation methods. The similar results for then real interpolation methods were obtained by Jawerth et al. [50] in 1985. Recently, Cwikel et al. constructed a general interpolation method with holomorphic structure in [47]. This new setting includes the classical real and complex methods, and even the so called ± methods given by Peetre and Gustavsson as special cases. The main idea behind this approach is to give a unified treatment on several different interpolation methods currently in use and the corresponding commutator estimates.

For a Banach couple $\bar{X} = (X_0, X_1)$, let $[\bar{X}]_\theta$ be the complex interpolation space with parameter $\theta \in (0,1)$ in the sense of Calderón. If $T: \sum \bar{X} \to \sum \bar{X}$ is a linear operator whose restrictions to both X_0 and X_1 are bounded, then many properties of T can be inherited to the interpolation space $[\bar{X}]_\theta$ and can vary with θ. The main topic is to investigate how the interpolation norms of commutators depend explicitly on θ. We present such norm estimates for commutators and some applications on operator inequalities. We formulate the similar quantitative commutator estimates for the above-mentioned general interpolation methods with holomorphic structure by using the mixed reiteration. We devoted to some related results concerning the spectral behaviour of bounded linear operators under interpolation.

The notations $\subseteq$ and $=$ between Banach spaces stand for continuous inclusion and isomorphic equivalence, respectively. For complex Banach spaces X and Y, we denote by $\mathcal{B}(X, Y)$ the Banach space of all bounded linear operators from X to Y. For two Banach couples $\bar{X}$ and $\bar{Y}$, we denote by $\mathcal{B}(\bar{X}, \bar{Y})$ the Banach space of all bounded linear operators T from $\bar{X}$ to $\bar{Y}$ with $\|T\|_j = \|T\|_{X_j, X_j} (j = 0,1)$. The space $\mathcal{B}(\bar{X}, \bar{Y})$ is equipped with the norm $\|T\|_{\bar{X},\bar{Y}} = \|T_0\|_0 \vee \|T_1\|_1$. We simply write $\mathcal{B}(X) = \mathcal{B}(X, X)$ and $\mathcal{B}(\bar{X}) = \mathcal{B}(\bar{X}, \bar{X})$. See [44] for the further information concerning interpolation theory.

For a Banach couple $\bar{X} = (X_0, X_1)$ on the strip $\boldsymbol{S} = \{z \in \boldsymbol{C} | 0 \leq \operatorname{Re} z \leq 1\}$, let $\mathcal{A}^b(\boldsymbol{S}, \bar{X})$ be the Banach space of all continuous functions $f: \boldsymbol{S} \to \sum \bar{X}$, where f is analytic in the interior of $\boldsymbol{S}$ and, for $j = 0,1$, the function $t \mapsto f(j + it)$ is boundedly continuous from $\boldsymbol{R}$ to X_j. This space is equipped with the norm

$$\|f\|_\infty = \max_{j=0,1} \left\{ \sup_{t \in \boldsymbol{R}} \|f(j + it)\|_j \right\}$$

for $f \in \mathcal{A}^b(\boldsymbol{S}, \bar{X})$. Given $\theta \in (0,1)$, the complex interpolation space $[\bar{X}]_\theta$ is defined by

$$[\bar{X}]_\theta = \{x \in \textstyle\sum \bar{X} | x = f(\theta),\ f \in \mathcal{A}^b(\boldsymbol{S}, \bar{X})\}$$

with the norm $\|x\|_\theta = \inf\{\|f\|_\theta | x = f(\theta)\}$. In what follows, we will use different variants of the complex interpolation in different situations.

Let $C_{\text{opt}} > 1$ be a fixed constant. For each $x \in [\bar{X}]_\theta$, select an analytic function $f_x \in \mathcal{A}^b(\boldsymbol{S}, \bar{X})$ with $x = f_x(\theta)$ and

$$f_x(\theta) \leq C_{\text{opt}}\|x\|_\theta.$$

Define the derivation operator Ω on $[\bar{X}]_\theta$ by

$$\Omega(x) = \Omega_{\bar{X}}(x) = f_x^{'}(\theta).$$

It was shown in [54] that, if $T \in \mathcal{B}(\bar{X}, \bar{Y})$ for Banach couples $\bar{X}$ and $\bar{Y}$, then the commutato

$$[T, \Omega] = T\Omega_{\bar{X}} - \Omega_{\bar{Y}}T$$

is a bounded nonlinear operator from $[\bar{X}]_\theta$ to $[\bar{Y}]_\theta$. We present now a norm estimate for this commutator depending more explicitly on the numerical index θ. For $t_0, t_1 > 0$, let

$$\rho_{\theta,1}(t_0, t_1) = t_0^{1-\theta} t_1^\theta \left(\sqrt{1 + \left(\frac{\sin \pi\theta}{\pi} \log \left| \frac{t_1}{t_0} \right| \right)^2} + \frac{\sin \pi\theta}{\pi} \log \left| \frac{t_1}{t_0} \right| \right).$$

Proposition (1.2.1)[39]: With the preceding assumptions, we have

$$\frac{\sin \pi\theta}{\pi} \|T\Omega_{\bar{X}}(x) - \Omega_{\bar{Y}}(Tx)\|_\theta \leq C_{\text{opt}} \rho_{\theta,1}(\|T\|_\theta, \|T\|_1)\|x\|_\theta$$

for all $x \in [\bar{X}]_\theta$.

Proof. We consider a variant of the space $[\bar{X}]_\theta$ on the unit disk

$$\boldsymbol{D} = \{z \in \boldsymbol{C} | |z| \leq 1\}.$$

Let

$$I_0^\theta = \big[-\pi(1-\theta), \pi(1-\theta)\big), \ \ I_1^\theta = \big[\pi(1-\theta), \pi(1+\theta)\big).$$

We denote by $\mathcal{A}_\theta(\boldsymbol{D}, \bar{X})$ the Banach space of all functions $f: \boldsymbol{D} \to \sum \bar{X}$, where f is analytic in the interior of $\boldsymbol{D}$ and the function $t \mapsto f(e^{it})$ is continuous from I_j^θ to $X_j (j = 0,1)$, with the norm

$$\|f\|_{\mathcal{A}_\theta} = \max_{j=0,1} \sup \left\{ \left\| f(e^{it}) \right\|_j \middle| t \in I_j^\theta \right\}.$$

Let now

$$[\bar{X}]_{D\theta} = \{x \in \textstyle\sum \bar{X} | x = f(0) \ \text{ for some } \ f \in \mathcal{A}_\theta(\boldsymbol{D}, \bar{X})\}$$

with the norm $\|x\|_{D\theta} = \inf\{\|f\|_{\mathcal{A}_\theta} | x = f(0)\}$. According to [48],

$$[\bar{X}]_{D\theta} = [\bar{X}]_\theta$$

isometrically via the conformal mapping $m_\theta: \boldsymbol{S} \to \boldsymbol{D}$, for which

$$m_\theta(z) = \frac{\exp(i\pi(z-\theta)) - 1}{\exp(-i\pi\theta) - \exp(i\pi z)} \ \ for \ \ z \in \boldsymbol{S}.$$

For each $x \in [\bar{X}]_{D\theta}$, we may choose $f_x \in \mathcal{A}_\theta(\boldsymbol{D}, \bar{X})$ with $| x = f_x(0)$ and

$$\|f_x\|_{\mathcal{A}_\theta} \leq C_{\text{opt}}\|x\|_\theta.$$

Thus,

$$\Omega_{\bar{X}}(x) = (f_x \circ m_\theta)^{'}(\theta) = f_x^{'}(0) m_\theta^{'}(\theta) = -\frac{\pi}{2\sin \pi\theta} f_x^{'}(0),$$

and $Tf_x(0) - f_{T_x}(0) = x - x = 0$. Let $\|T\|_j (j = 0,1)$. By [48], $\exists \phi \in H^\infty(\boldsymbol{D})$ such that

$$\phi(0) = 1, \qquad \phi^{'}(0) = \frac{\sin \pi\theta}{\pi} \log(M_1/M_0),$$

and

$$\|\phi\|_\infty = \sqrt{1+\left(\frac{\sin\pi\theta}{\pi}\log(M_1/M_0)\right)^2} + \frac{\sin\pi\theta}{\pi}|\log(M_1/M_0)|.$$

We choose $\psi \in H^\infty(\boldsymbol{D})$, for which

$$\psi(0) = 1, \qquad \psi'(0) = \frac{\sin\pi\theta}{\pi}\log(M_1/M_0),$$

and

$$\left|\exp\left(-\psi(e^{it})\right)\right| = (M_1/M_0)^{j-\theta} \ \text{ for } \ t \in I_j^\theta (j = 0,1).$$

Let $h_x = \phi\exp(-\psi)f_x$, then $h_x \in \mathcal{A}^b(\boldsymbol{S}, \bar{X})$ with

$$h_x(0) = f_x(0) = x \ \text{ and } \ h_x'(0) = f_x'(0) = -\frac{2\sin\pi\theta}{\pi}\Omega_{\bar{X}}(x),$$

Moreover,

$$\left\|h_x(e^{it})\right\|_j \le (M_1/M_0)^{j-\theta}\|\phi\|_\infty\left\|f_x(e^{it})\right\|_j$$

for $j = 0,1$ and for $t \in I_j^\theta$. We define now

$$g(z) = \frac{Th_x(z) - f_{T_x}(z)}{z} \ \ for \ \ z \in \boldsymbol{D}\backslash\{0\},$$

ana

$$g(0) = Th_x'(0) - f_{T_x}'(0) = -\frac{2\sin\pi\theta}{\pi}\left(T\Omega_{\bar{X}}(x) - \Omega_{\bar{Y}}(Tx)\right).$$

Then $g \in \mathcal{A}_\theta(\boldsymbol{D}, \bar{Y})$ such that

$$\left\|g(e^{it})\right\|_j \le \left\|Th_x(e^{it})\right\|_j + \left\|f_{T_x}(e^{it})\right\|_j \le \left(M_0^{1-\theta}M_1^\theta\|\phi\|_\infty\|f_x\|_\infty + \left\|f_{T_x}\right\|_\infty\right)$$

for $t \in I_j^\theta (j = 0,1)$. This, together with (1), (3), (4) and [44], implies that

$$\frac{\sin\pi\theta}{\pi}\|T\Omega_{\bar{X}}(x) - \Omega_{\bar{Y}}(Tx)\|_\theta \le \frac{1}{2}\|g\|_{\mathcal{A}_\theta} \le C_{\text{opt}}\cdot\rho_{\theta,1}(\|T\|_0, \|T\|_1)\|x\|_\theta,$$

which completes the proof.

We consider now the completely bounded linear maps on operator spaces. A (concrete) operator space E on a Hilbert space H is a subspace of $\mathcal{B}(H)$. For an arbitrary positive integer n, let $M_n(E)$ be the Banach space of all $n \times n$ matrices with entries in E with the norm introduced by the natural inclusion

$$M_n(E) \subseteq M_n(\mathcal{B}(H)) = \mathcal{B}(H^n).$$

The operator space E has a matrical structure with these norms on $M_n(E)$. Given a linear operator between two operator spaces $T: E \to F$, we define $T_n: M_n(E) \to M_n(F)$ by

$$T_n\left((x_{ij})\right) = \left(T(x_{ij})\right)$$

for all $(x_{ij})_{i,j=1}^n \in M_n(E)$. Themap T is called completely bounded ($c.b.$ for short) if

$$\|T\|_{cb} = \sup\{\|T_n\| | n = 1,2,...\} < \infty.$$

The space of all $c.b.$ maps from E to F is denoted by $\mathcal{CB}(E,F)$, which is also an operator space with matrix norms defined by

$$M_n(\mathcal{CB}(E,F)) = \mathcal{CB}(E, M_n(F))$$

for all $n \ge 1$. Let $\bar{E} = (E_0, E_1)$ be a Banach couple of operator spaces. We equip the complex interpolation space $[\bar{E}]_\theta$ with the matrical structure by defining

$$M_n([\bar{E}]_\theta) = [M_n(E_0), M_n(E_1)]_\theta$$

for each $n \ge 1$. See [53] for details. We have

Proposition (1.2.2)[39]: For Banach couples of $\bar{E} = (E_0, E_1)$ and $\bar{F} = (F_0, F_1)$ of operator spaces, let Ω be the derivation operator, and let $T: \sum \bar{E} \to \sum \bar{F}$ be $a c.b.$ map such that $T: E_j \to F_j$ is $c.b.$ with the $c.b.$ norm $\|T\|_{cb_j} (j = 0,1)$. Then

$$\frac{\sin \pi\theta}{\pi} \|T\Omega_{\bar{X}}(x) - \Omega_{\bar{Y}}(Tx)\|_{M_n([\bar{F}]_\theta)} \le C_{\mathrm{opt}} \rho_{\theta,1}\left(\|T\|_{cb_0}, \|T\|_{cb_1}\right) \|x\|_{M_n([\bar{E}]_\theta)}$$

for all $n \ge 1$ and all $x \in M_n([\bar{E}]_\theta)$.

The next result concerns with the commutator estimates for bounded bilinear operators under the complex interpolation. Let $\bar{X}, \bar{Y}$ and $\bar{X}$ be Banach couples, and let T be a bounded bilinear operator from $\bar{X} \times \bar{Y}$ to $\bar{Z}$ with

$$\|T\|_j = \|T\|_{X_j, Y_j} = \sup\left\{\frac{\|T(x,y)\|_{Z_j}}{\|x\|_{X_j}\|y\|_{Y_j}} \middle| x \neq 0 \text{ in } X_j, y \neq 0 \text{ in } Y_j\right\} < \infty$$

for $j = 0,1$. We improve the estimate given in [43] as follows.

Proposition (1.2.3)[39]: Let Ω be the derivation operator. We have

$$\frac{\sin \pi\theta}{\pi} \left\|T(\Omega_{\bar{X}}(x), y) + T\left(x, \Omega_{\bar{Y}}(y)\right) - \Omega_{\bar{Z}} T(x,y)\right\|_\theta \le C_{\mathrm{opt}} \rho_{\theta,1}(\|T\|_0, \|T\|_1) \|x\|_\theta \|y\|_\theta$$

for all $x \in [\bar{X}]_\theta$ and $y \in [\bar{Y}]_\theta$.

Proof. For any $x \in [\bar{X}]_{D\theta}$ and $y \in [\bar{Y}]_{D\theta}$, we choose $f_x \in \mathcal{A}_\theta(\boldsymbol{D}, \bar{X})$ with

$$x = f_x(0) \text{ and } \|f_x\|_{\mathcal{A}_\theta} \le C_{\mathrm{opt}} \|x\|_\theta,$$

and choose $f_y \in \mathcal{A}_\theta(\boldsymbol{D}, \bar{Y})$ with

$$y = f_y(0) \text{ and } \left\|f_y\right\|_{\mathcal{A}_\theta} \le C_{\mathrm{opt}} \|y\|_\theta.$$

Let $h_x \in \mathcal{A}_\theta(\boldsymbol{D}, \bar{X})$ and $h_y \in \mathcal{A}_\theta(\boldsymbol{D}, \bar{Y})$ be as in the proof of Proposition (1.2.1). We define now

$$g(z) = \frac{T\left(h_x(z), h_y(z)\right) - f_{T(x,y)}(z)}{z} \quad \text{for } z \in \boldsymbol{D} \backslash \{0\},$$

and

$$g(0) = T\left(h_x'(0), h_y(0)\right) + T\left(h_x(0), h_y'(0)\right) - f_{T(x,y)}'(0)$$
$$= -\frac{2\sin\pi\theta}{\pi}\left(T(\Omega_{\bar{X}}x, y) + T(x, \Omega_{\bar{Y}} y) - \Omega_{\bar{Z}} T(x,y)\right).$$

Observe that

$$\|T(x,y)\|_\theta \le \|T\|_0^{1-\theta} \|T\|_1^\theta \|x\|_\theta \|y\|_\theta$$

by [44]. We can obtain

$$\frac{\sin \pi\theta}{\pi} \left\|T(\Omega_{\bar{X}}(x), y) + T\left(x, \Omega_{\bar{Y}}(y)\right) - \Omega_{\bar{Z}} T(x,y)\right\|_\theta \le \frac{1}{2} \|g\|_{\mathcal{A}_\theta}$$
$$\le C_{\mathrm{opt}} \rho_{\theta,1}(\|T\|_0, \|T\|_1) \|x\|_\theta \|y\|_\theta$$

in a similar way for the proof of Proposition (1.2.1).

As applications of Propositions (1.2.1) and (1.2.3), we can show the following operator inequalities.

Proposition (1.2.4)[39]: Let H be a Hilbert space, and let $A, B \in \mathcal{B}(H)$ be invertible positive operators. If $0 < \theta < 1$, then the inequality

$$\frac{\sin \pi\theta}{\pi} \|T(\log A) - (\log B)T\| \le \rho_{\theta,1}\left(\left\|B^{-\theta} T A^\theta\right\|, \left\|B^{1-\theta} T A^{\theta-1}\right\|\right)$$

holds for all $T \in \mathcal{B}(H)$.

Proof. For the positive operator A, let H_A be the Hilbert space H equipped with the norm $\|x\|_A = \|Ax\|$ for all $x \in H$, and let $\overline{H}_A = (H, H_A)$.Moreover, for $0 < \theta < 1$, we define H_{A^θ} to be the Hilbert space H equipped with the norm $\|x\|_\theta = \|A^\theta x\|$ for all $x \in H$. Thus,
$$\|\overline{H}_A\|_\theta = H_{A^\theta}$$
with equal norms. For $x \in H_{A^\theta}$ and $z \in \boldsymbol{S}$, let $f_x(z) = A^{\theta - z}x$. Then $f_x \in \mathcal{A}^b(\boldsymbol{S}, \overline{H}_A)$ with $x = f_x(\theta)$. Moreover,
$$\|f_x\|_\infty \coloneqq \sup\{\|f_x(it)\| \vee \|f_x(1+it)\|_{H_A} | t \in \boldsymbol{R}\} = \|A^\theta x\| = \|x\|_\theta.$$
Thus, we may choose
$$\Omega_{\overline{H}_A}(x) = f_x^{'}(\theta) = -(\log A)x.$$
By Proposition (1.2.1), we have
$$\frac{\sin \pi\theta}{\pi} \|T(\log A) - (\log B)T\|_{H_{A^\theta}, H_{B^\theta}} \le \rho_{\theta,1}(\|T\|, \|T\|_{H_A, H_B})$$
for all $T \in \mathcal{B}(H)$. This implies the required inequality by replacing T with $B^{-\theta}TA^\theta$.

Let $\mathcal{M}$ be a semifinite von Neumann algebra acting on a Hilbert space H with a normal faithful semifinite trace τ and the identity 1. The densely-defined closed linear operator x on H is said to be affiliated with $\mathcal{M}$ if $xu = ux$ for all unitary operators u commuting with $\mathcal{M}$. The operator x is said to be τ-measurable if, for each $\epsilon > 0$, there is a project e in $\mathcal{M}$ for which $e(H)$ is included in the domain of x and $\tau(1 - e) < \epsilon$. We denote by $\widetilde{\mathcal{M}}$ the space of all τ-measurable operators affliated with $\mathcal{M}$, which is a complete Hausdorff topological $*$-algebra equipped with the measure topology [52]. For $x \in \widetilde{\mathcal{M}}$ and $t > 0$, the corresponding singular number is defined by
$$\mu_t(x) = \inf\{\|xe\|_{\mathcal{M}} | e \text{ is a projection e in } \mathcal{M} \text{ with } \tau(1-e) \le t\}.$$
It is known that, if $x \in \widetilde{\mathcal{M}}$ is positive, then
$$\tau(x) = \int_0^\infty \mu_t(x)dt.$$
For $1 < p < \infty$, let $L^p(\mathcal{M}) = L^p(\mathcal{M}, \tau)$ be the non-commutative L^p-space consisting of all $x \in \widetilde{\mathcal{M}}$ for which
$$\|x\|_p = \tau(|x|^p)^{1/p} = \left(\int_0^\infty \mu_t(|x|^p)dt\right)^{1/p}.$$
In addition, $L^\infty(\mathcal{M}) = \mathcal{M}$ equipped with the usual operator norm. We formulate now the non-commutative analogue of Hölder’s inequality, which is a generalization of [43].

Proposition (1.2.5)[39]: Let $1 < p, p^{'} < \infty$ with $1/p + 1/p^{'} = 1$. If $x \in L^p(\mathcal{M})$ and $y \in L^{p^{'}}(\mathcal{M})$, then
$$\frac{\sin(\pi/p^{'})}{\pi}\tau(|pxy\log(|x|/\|x\|_p) - p^{'}xy\log(|y|/\|y\|_{p^{'}})|) \le \|x\|_p\|y\|_{p^{'}}.$$
Proof. Let $\theta = 1/p^{'}$. Then
$$[L^1(\mathcal{M}), L^\infty(\mathcal{M})]_\theta = L^p(\mathcal{M}) \text{ and } [L^\infty(\mathcal{M}), L^1(\mathcal{M})]_\theta = L^{p^{'}}(\mathcal{M})$$
isometric ally. We set $\bar{X} = (L^1(\mathcal{M}), L^\infty(\mathcal{M}))$ and $\bar{Y} = (L^\infty(\mathcal{M}), L^1(\mathcal{M}))$. For $x \in L^p(\mathcal{M})$, if we choose
$$f_x(x) = x(|x|/\|x\|_p)^{p(\theta - z)},$$
then $f_x \in \mathcal{A}^b(\boldsymbol{S}, \bar{X})$ with $x = f_x(\theta)$ and $\|f_x\|_\infty = \|x\|_p$. Now we have
$$\Omega_{\bar{X}}(x) = f_x^{'}(\theta) = -px\log(|x|/\|x\|_p).$$

Similarly, if $y \in L^{p'}(\mathcal{M})$, then

$$\Omega_{\bar{Y}}(y) = -p' y \log\left(|y|/\|y\|_{p'}\right).$$

We define now a bounded bilinear operator $T: \bar{X} \times \bar{Y} \to \left(L^1(\mathcal{M}), L^1(\mathcal{M})\right)$ by

$$T(x, y) = xy.$$

Then $\|T\|_0 = \|T\|_1 = 1$. By applying Proposition (1.2.3) on this operator, we obtain

$$\frac{\sin \pi\theta}{\pi} \tau\left(\left|pxy \log\left(|x|/\|x\|_p\right) - p' xy \log\left(|y|/\|y\|_{p'}\right)\right|\right) \le \|x\|_p \|y\|_{p'}$$

for all $x \in L^p(\mathcal{M})$ and $y \in L^{p'}(\mathcal{M})$.

As introduced in [47], we consider **Ban**, the class of all complex Banach spaces, and the exact pseudolattice Φ: **Ban** $\to$ **Ban** in the sense that, for each $X \in$ **Ban**, $\Phi(X)$ consists of X-valued sequences $(x_v)_{v \in \boldsymbol{Z}}$ satisfying

$$\|x_k\|_X \le \|(x_v)_v\|_{\Phi(X)} \text{ when } k \in \boldsymbol{Z};$$

for each closed subspace Y of $X, \Phi(Y)$ is a closed subspace of $\Phi(X)$; and, for all $X, Y \in$ **Ban**, for all T ∈ B(X, Y), and for all $(x_v)_v \in \Phi(X)$, the estimate

$$\|(Tx_v)_v\|_{\Phi(Y)} \le \|T\|_{X.Y} \|(x_v)_v\|_{\Phi(X)}$$

holds. For a given pair of exact pseudolattices $\bar{\Phi} = \{\Phi_0, \Phi_1\}$ and for each Banach couple $\bar{X} = (X_0, X_1)$, we denote by $\mathcal{J}(\bar{\Phi}, \bar{X})$ the space of all $\Delta\bar{X}$-valued sequences $(x_v)_{v \in \boldsymbol{Z}}$, for which $\left(e^{jv} x_v\right)_v \in \Phi_j\left(X_j\right)(j = 0,1)$, with the norm

$$\|(x_v)_v\|_{\mathcal{J}} = \max_{j=0,1} \left\|\left(e^{jv} x_v\right)_v\right\|_{\Phi_j(X_j)}.$$

We assume that $\bar{\Phi} = \{\Phi_0, \Phi_1\}$ is a fixed pair of exact pseudolattices which is Laurent compatible and admits differentiation in the sense of [47].

For a fixed $s \in \boldsymbol{A} = \{z \in \boldsymbol{C} | 1 < |z| < e\}$, we define the following space

$$\bar{X}_{\bar{\Phi},s} = \left\{ x = \sum_v s^v x_v \,\middle|\, (x_v)_v \in \mathcal{J}(\bar{\Phi}, \bar{X}) \right\}$$

as in [47] with the natural quotient norm

$$\|x\|_{\bar{\Phi},s} = \inf\left\{ \|(x_v)_v\|_{\mathcal{J}} \,\middle|\, x = \sum_v s^v x_v \right\}.$$

According to [47], $\bar{X}_{\bar{\Phi},s}$ is an interpolation space for the Banach couple X such that

$$\|T\|_{\bar{X}_{\bar{\Phi},s}, \bar{Y}_{\bar{\Phi},s}} \le \|T\|_0 \vee \|T\|_1$$

for all Banach couples $\bar{X}, \bar{Y}$, and for all $T \in \mathcal{B}(\bar{X}, \bar{Y})$. Furthermore, if we identify each sequence $(x_v)_v \in \mathcal{J}(\bar{\Phi}, \bar{X})$ with the function $f(z) = \sum_v z^v x_v$ for $z \in \boldsymbol{A}$, then $\mathcal{J}(\bar{\Phi}, \bar{X})$ can be considered as the space consisting of all analytic functions $f: A \to \sum\bar{X}$, for which $\left(e^{jv} x_v\right)_v \in \Phi_j\left(X_j\right)(j = 0,1)$, with the norm given in (6). Consequently, the interpolation space $\bar{X}_{\bar{\Phi},s}$ consists of all $x \in \sum\bar{X}$ such that $x = f(s)$ for some analytic function f on $\boldsymbol{A}$ given above.

According to [47], we can introduce an equivalent version of the space $\bar{X}_{\bar{\Phi},s}$. For $s \in \boldsymbol{A}$ fixed, let $\mathcal{K}_s(\bar{\Phi}, \bar{X})$ be the space of all pairs of $\sum\bar{X}$-valued sequences $x' = \left(\left(x_{0,v}\right)_v, \left(x_{1,v}\right)_v\right)$ such that $\left(e^{jv} x_{j,v}\right)_v \in \Phi_j\left(X_j\right)(j = 0,1)$, and $\kappa(\bar{x}) = x_{0,0} + x_{1,0} = s^v\left(x_{0,v} + x_{1,v}\right)$ for each $v \in \boldsymbol{Z}$. This space is normed by

$$\|\bar{x}\|_{\mathcal{K}_s} = \left\|(x_{0,v})_v\right\|_{\Phi_0(X_0)} + \left\|(e^v x_{1,v})_v\right\|_{\Phi_1(X_1)}.$$

Let now $\bar{X}_{\bar{\Phi},s;\mathcal{K}}$ be the space of all $x \in \sum\bar{X}$, which can be represented by $x = \kappa(\bar{x})$ for some $\bar{x} \in \mathcal{K}_s(\bar{\Phi}, \bar{X})$, with the seminorm

$$\|x\|_{\bar{\Phi},s;\mathcal{K}} = \inf\{\|\bar{x}\|_{\mathcal{K}_s} | \bar{x} \in \mathcal{K}_s(\bar{\Phi}, \bar{X}), x = \kappa(\bar{x})\}.$$

Let S denote the left shift operator on two-sided (vector valued) sequences given in [47] by

$$S((x_v)_{v\in\boldsymbol{Z}}) = (x_{v+1})_{v\in\boldsymbol{Z}}.$$

Then S^{-1} is the right shift operator given by $S^{-1}((x_v)_v) = (x_{v-1})_v$. If S is isometric on $\Phi_j(X_j)(j = 0,1)$, then S is also isometric on $\bar{X}_{\bar{\Phi},s}$. In this case,

$$\|T\|_{\bar{X}_{\bar{\Phi},s}, \bar{Y}_{\bar{\Phi},s}} \le e\|T\|_0^{1-\theta}\|T\|_1^{\theta}$$

for all $T \in \mathcal{B}(\bar{X}, \bar{Y})$, where $\theta = \log|s|$, by [47]. Moreover, the following equivalence

$$\bar{X}_{\bar{\Phi},s} = \bar{X}_{\bar{\Phi},s;\mathcal{K}}$$

holds true by [47].

For suitable choices of pseudolattice pairs, we can recover the classical interpolation methods. Let X be a Banach space.We denote by $l^p(X)$ the space of all X-valued sequences $(x_v)_{v\in\boldsymbol{Z}}$ for $1 \le p < \infty$, for which

$$\|(x_v)_v\|_{l^p(X)} = \left(\sum_v \|x_v\|_X^p\right)^{1/p};$$

denote by $FC(X)$ the space of all X-valued sequences $(x_v)_{v\in\boldsymbol{N}}$ such that, for some continuous function $f: \boldsymbol{T} = [0,2\pi] \to X$ and for all $v \in \boldsymbol{N}$,

$$x_v = \frac{1}{2\pi}\int_0^{2\pi} e^{-ivt} f(t)dt$$

with the norm

$$\|(x_v)_v\|_{FC(X)} = \sup_{t\in\boldsymbol{T}}\|f(t)\|_X,$$

and denote by $UC(X)$ (resp. $WUC(X)$) the space of all X-valued sequences $(x_v)_v$ such that the series $\sum_v x_v$ is unconditionally (resp. weakly unconditionally) convergent in X with the norm

$$\|(x_v)_v\|_{UC(X)} \sup\left\{\left\|\sum_{v\in\boldsymbol{F}} \epsilon_v x_v\right\| | \boldsymbol{F} \text{ is a finite subset of } \boldsymbol{Z}, \epsilon_v = \pm 1\right\}$$

(resp. $\|(x_v)_v\|_{WUC(X)} = \sup\{\|\sum_v \epsilon_v x_v\| | \epsilon_v = \pm 1\}$). For the pseudolattices $\Phi = l^p, FC, UC$ and WUC, the left shift opertor S given in (7) and the rotation operators $R_t, t \in \boldsymbol{R}$, on two-sided (vector valued) sequences given by

$$R_t((x_v)_{v\in\boldsymbol{Z}}) = \left(e^{itv}x_v\right)_{v\in\boldsymbol{Z}}$$

are isometric on $\Phi(X)$ for all Banach spaces X. Let now $\overline{l^p} = (l^p, l^p), \overline{FC} = (FC, FC), \overline{UC} = (UC, UC)$ and $\overline{WUC} = (WUC, WUC)$ be the corresponding pairs of pseudolattices. For those pairs,wemay assume that $s = e^{\theta}$ for some $\theta \in (0,1)$. It is known that, for all Banach couples $\bar{X}$, the spaces $\bar{X}_{\overline{l^p},s}, \bar{X}_{\overline{FC},s}, \bar{X}_{\overline{UC},s}$ and $\bar{X}_{\overline{WUC},s}$ are equivalent to the real interpolation space $[\bar{X}]_{\theta,p}$, the complex interpolation space $[\bar{X}]_{\theta}$, the Peetre±interpolation space $\langle\bar{X}\rangle_{\theta}$, and its Gustavsson–Peetre variant $\langle\bar{X}, \theta\rangle$, respectively.

We formulate now the following mixed reiteration.

Proposition (1.2.6)[39]: Assume that the left shift operator S and the rotation operators R_t are isometric on $\Phi_j(X_j)(j=0,1)$ for all $t \in \boldsymbol{R}$. Let $s_0, s_1, s, \tau \in (1, e)$ and $\alpha \in (0,1)$ such that $s = s_0^{1-\alpha} s_1^{\alpha}$ and $\tau = e^{\alpha}$. Then

$$\bar{X}_{\bar{\Phi},s} = \left(\bar{X}_{\bar{\Phi},s_0}, \bar{X}_{\bar{\Phi},s_1}\right)_{\overline{FC},\tau}.$$

Proof. Let

$$X = \left(\bar{X}_{\bar{\Phi},s_0}, \bar{X}_{\bar{\Phi},s_1}\right)_{\overline{FC},\tau} \text{ and } Y = \left(\bar{X}_{\bar{\Phi},s_0;\mathcal{K}}, \bar{X}_{\bar{\Phi},s_1;\mathcal{K}}\right)_{\overline{FC},\tau}.$$

It is enough to show that $\bar{X}_{\bar{\Phi},s} \subseteq X$ and $Y \subseteq \bar{X}_{\bar{\Phi},s;\mathcal{K}}$.

First assume that $x \in \bar{X}_{\bar{\Phi},s}$ and $\epsilon > 0$ such that $x = \sum_v s^v x_v$ for some $(x_v)_v \in \mathcal{J}(\bar{\Phi}, \bar{X})$ with $\|(x_v)_v\|_{\mathcal{J}} \le \|x\|_{\bar{\Phi},s} + \epsilon$. Set $\theta_j = \log s_j (j=0,1)$, and let

$$f(z) = \sum_v s_0^v z^{v(\theta_1 - \theta_0)} x_v \quad \text{for } z \in \boldsymbol{A}.$$

Then f is an analytic function on $\boldsymbol{A}$ with $x = f(\tau)$. Moreover, for each $t \in \boldsymbol{T}$ and for $j = 0,1$,

$$f\left(e^{j+it}\right) = \sum_v e^{(j+it)v(\theta_1-\theta_0)} s_0^v x_v = f(z) = \sum_v e^{itv(\theta_1-\theta_0)} s_j^v x_v.$$

Let $f_j(t) = f\left(e^{j+it}\right)(j=0,1)$, and $y_v(t) = e^{itv(\theta_1-\theta_0)} x_v$ for $v \in \boldsymbol{Z}$. Then

$$f_j(t) = \sum_v s_j^v y_v(t) \in \bar{X}_{\bar{\Phi},s_j} \quad \text{for } t \in \boldsymbol{T},$$

and $f_j : \boldsymbol{T} \to \bar{X}_{\bar{\Phi},s_j}$ is continuous $(j = 0,1)$. It follows that

$$\|x\|_X \le \max_{j=0,1} \sup_{t\in\boldsymbol{T}} \left\|f_j(t)\right\|_{\bar{\Phi},s_j} \le \sup_{t\in\boldsymbol{T}} \|(y_v(t))_v\|_{\mathcal{J}} = \sup_{t\in\boldsymbol{T}} \left\|R_{t(\theta_1-\theta_0)}((x_v))\right\|_{\mathcal{J}} \le \|(x_v)_v\|_{\mathcal{J}} \le \|x\|_{\bar{\Phi},s} + \epsilon,$$

and hence $x \in X$ with $\|x\|_X \le \|x\|_{\bar{\Phi},s}$ by letting $\epsilon \to 0$.

Conversely, let $Y_j \subseteq \bar{X}_{\bar{\Phi},s_j;\mathcal{K}}(j=0,1)$ and $\bar{Y} = (Y_0, Y_1)$. For $x \in Y$ and $\epsilon > 0$, there exists an analytic function $g: \boldsymbol{A} \to \sum \bar{Y}$ such that $x = g(\tau), t \mapsto g(e^{j+it})$ is a continuous function from $\boldsymbol{T}$ to $Y_j(j=0,1)$, and

$$\sup_{t\in\boldsymbol{T}} \left\|g(e^{j+it})\right\|_{Y_j} \le \|x\|_Y + \epsilon \ \ (j=0,1).$$

Morover, there exists a Lebesgue measurable function. $\bar{x}_j = \left(x_{j,0,v}(t), x_{j,1,v}(t)\right)(j=0,1)$ from $\boldsymbol{T}$ to $\mathcal{K}_s(\bar{\Phi}, \bar{X})$ such that, for each $t \in \boldsymbol{T}$,

$$g\left(e^{j+it}\right) = \kappa\left(\bar{x}_j(t)\right) = s_j^v\left(x_{j,0,v}(t) + x_{j,1,v}(t)\right), \qquad v \in \boldsymbol{Z},$$

with $\left\|\bar{x}_j(t)\right\|_{\mathcal{K}_{s_j}} \le \left\|g\left(e^{j+it}\right)\right\|_{Y_j} + \epsilon(j=0,1)$. For each $v \in \boldsymbol{Z}$ and $v \in \boldsymbol{A}$, let

$$h_v(z) = (s_1/s_0)^{(\alpha - \log z)v} g(z) = e^{(\alpha - \log z)(\theta_1-\theta_0)v} g(z).$$

Then $x = h_v(\tau)$. By the Cauchy integral formula, we obtain

$$x = \frac{1}{2\pi}\int_0^{2\pi} h_v\left(e^{it}\right)\frac{e^{it}dt}{\tau - e^{it}} + \frac{1}{2\pi}\int_0^{2\pi} h_v\left(e^{1+it}\right)\frac{e^{it}dt}{e^{1+it} - \tau}$$

$$= s^v\left(\frac{1}{2\pi}\int_0^{2\pi} e^{ivt(\theta_0-\theta_1)}\left(x_{0,0,v}(t) + x_{0,1,v}(t)\right)\frac{e^{it}dt}{\tau - e^{it}} + \frac{1}{2\pi}\int_0^{2\pi} e^{ivt(\theta_0-\theta_1)}\left(x_{1,0,v}(t) + x_{1,1,v}(t)\right)\frac{e^{it}dt}{e^{1+it} - \tau}\right).$$

For $j=0,1$, let

$$y_{j,v}=\frac{1}{2\pi}\int_0^{2\pi} e^{ivt(\theta_0-\theta_1)}x_{0,j,v}(t)\frac{e^{it}dt}{\tau-e^{it}}+\frac{1}{2\pi}\int_0^{2\pi} e^{ivt(\theta_0-\theta_1)}x_{1,j,v}(t)\frac{e^{it}dt}{e^{1+it}-\tau},$$

and let $\bar{y}=\left(\left(y_{0,v}\right)_v,\left(y_{1,v}\right)_v\right)$. Then $\bar{y}\in\mathcal{K}(\bar{\Phi},\bar{X})$ with $x=\kappa(\bar{y})$ and

$$\begin{aligned}\|\bar{y}\|_{\mathcal{K}_s}&=\left\|\left(y_{0,v}\right)_v\right\|_{\Phi_0(X_0)}+\left\|\left(e^v y_{1,v}\right)_v\right\|_{\Phi_1(X_1)}\\&\le\frac{1}{2\pi}\int_0^{2\pi}\|\bar{x}_0(t)\|_{\mathcal{K}_{s_0}}\frac{dt}{|\tau-e^{it}|}+\frac{1}{2\pi}\int_0^{2\pi}\|\bar{x}_0(t)\|_{\mathcal{K}_{s_1}}\frac{dt}{|e^{1+it}-\tau|}\\&\le c(\|x\|_Y+2\epsilon),\end{aligned}$$

where $c=(\tau-1)^{-1}\vee(e-\tau)^{-1}$. Therefore, $x\in\bar{X}_{\bar{\Phi},s;\mathcal{K}}$ with

$$\|x\|_{\bar{\Phi},s;\mathcal{K}}\le c\|x\|_Y$$

by letting $\epsilon\to 0$.

Now we turn our attention to the derivation operator $\Omega_{\bar{X}}$ associated with the interpolation space $\bar{X}_{\bar{\Phi},s}$ as given in [47]. Let us fix a constant $C_{\text{opt}}>1$, a Laurent compatible pair of exact pseudolattices $\bar{\Phi}$, and a point $s\in\boldsymbol{A}$. For each Banach couple $\bar{X}$ and each $x\in\bar{X}_{\bar{\Phi},s}$, let $\bar{\Omega}_{\bar{X}}(x)$ denote the set of all elements $x'\in\sum\bar{X}$ of the form $x'=\sum_v vs^{v-1}x_v$ for all choices of the sequences $(x_v)_v$ in $\mathcal{J}(\bar{\Phi},\bar{X})$ for which $x=\sum_v s^{v-1}x_v$ and

$$\|(x_v)_v\|_{\mathcal{J}(\bar{\Phi},\bar{X})}\le C_{\text{opt}}\|x\|_{\bar{X}_{\bar{\Phi},s}}.$$

We choose some element $\Omega_{\bar{X}}(x)\in\widetilde{\Omega}$. Observe that the operator $\Omega_{\bar{X}}$ is equivalent to that for the complex interpolation space $[\bar{X}]_{\overline{FC},e^{\theta}}$. According to [47], for Banach couples $\bar{X}$ and $\bar{Y}$, for $T\in\mathcal{B}(\bar{X},\bar{Y})$, and for $x\in\bar{X}_{\bar{\Phi},s}$, we have

$$T\Omega_{\bar{X}}(x)-\Omega_{\bar{Y}}(Tx)\in\bar{Y}_{\bar{\Phi},s}$$

satisfying

$$\|T\Omega_{\bar{X}}(x)-\Omega_{\bar{Y}}(Tx)\|_{\bar{Y}_{\bar{\Phi},s}}\le\tilde{C}\|T\|_{\bar{X},\bar{Y}}\|x\|_{\bar{X}_{\bar{\Phi},s}}$$

for some positive constant $\tilde{C}$ not depending on x and T. In case that the left shift operator S is isometric on $\Phi_j\left(\bar{X}_j\right)(j=0,1)$, it is possible to give an alternative version of the estimate in (11) depending even more explicitly on $\theta=\log|s|$. As mentioned in [47], the factor $\|T\|_{\bar{X},\bar{Y}}$ can be replaced by

$$\|T\|_0^{1-\theta}\|T\|_1^{\theta}\left(1+e(\|T\|_1/\|T\|_0)^{\theta}\right)^{1-\theta}\left(1+e(\|T\|_0/\|T\|_1)^{1-\theta}\right)^{\theta}.$$

We have the following result for commutator estimates.

Proposition (1.2.7)[39]: Assume that the left shift operator S and the rotation operators R_t are isometric on $\Phi_j\left(\bar{X}_j\right)(j=0,1)$ for all $t\in\boldsymbol{R}$. Then

$$\|T\Omega_{\bar{X}}(x)-\Omega_{\bar{Y}}(Tx)\|_{\bar{Y}_{\bar{\Phi},s}}\le\tilde{C}\rho_{\theta,1}(\|T\|_0,\|T\|_1)\|x\|_{\bar{X}_{\bar{\Phi},s}}$$

for some positive constant $\tilde{C}$ not depending on x and T, where $\theta=\log|s|$.

Proof. Without loss of generality,wemay assume that $s\in(1,e)$ with $\theta=\log s\le 1/2$, and choose now $\theta_0,\theta_1\in(0,1)$ with

$$\theta_0<\theta<1/2<1-\theta_0=\theta_1.$$

Let $s_j=e_j^{\theta}(j=0,1)$. Then $s=s_0^{1-\alpha}s_1^{\alpha}$ for some $\alpha\in(0,1)$ Thus,

$$\alpha=\frac{\theta-\theta_0}{1-2\theta_0}\le\theta\le\frac{1}{2}$$

and hence

$$0 \le (\theta_1 - \theta_0)\sin \pi\alpha \le \sin \pi\alpha \le \sin \pi\theta.$$

Let $\bar{V} = (V_0, V_1)$ with $V_j = \bar{X}_{\Phi,s_j}$ and $\bar{W} = (W_0, W_1)$ with $W_j = \bar{Y}_{\Phi,s_j} (j = 0,1)$. According to Proposition (1.2.6),

$$X = \bar{X}_{\Phi,s} = \bar{V}_{\overline{FC}.\tau} = V \text{ and } Y = \bar{Y}_{\Phi,s} = \bar{W}_{\overline{FC}.\tau} = W,$$

where $\tau = e^{\alpha}$. For any $x \in \bar{X}_{\bar{\Phi},s}$, we choose $(x_v)_v \in \mathcal{J}(\bar{\Phi}, \bar{X})$ for which $x = \sum_v s^v x_v$, $\|(x_v)_v\|_{\mathcal{J}(\bar{\Phi},\bar{X})} \le \|x\|_{\bar{X}_{\bar{\Phi},s}}$, and

$$\Omega_{\bar{X}}(x) = \sum_v s^v x_v.$$

Let

$$f(z) = \sum_v s_0^v z^{v(\theta_1-\theta_0)} x_v \quad \text{for } z \in \boldsymbol{A}$$

as in the proof of Proposition (1.2.6). Then f is an analytic function on $\boldsymbol{A}$ with

$$x = f(\tau) \text{ and } \|f(j+it)\|_{Y_j} \le \|(x_v)_v\|_{\mathcal{J}(\bar{\Phi},\bar{X})}\ (j = 0,1)$$

for all $t \in \boldsymbol{R}$. Thus,

$$\Omega_{\bar{V}}(x) = f'(\tau) = (\theta_1 - \theta_0)\sum_v v s_0^v \tau^{v(\theta_1-\theta_0)} x_v = (\theta_1 - \theta_0)\Omega_{\bar{X}}(x),$$

and similarly $\Omega_{\bar{W}}(Tx) = (\theta_1 - \theta_0)\Omega_{\bar{Y}}(Tx)$. Combining these equalities with Proposition (1.2.6), we have

$$\|T\Omega_{\bar{X}}(x) - \Omega_{\bar{Y}}(Tx)\|_Y \le C_1\|T\Omega_{\bar{V}}(x) - \Omega_{\bar{W}}(Tx)\|_W \le C_2\rho_{\alpha,1}\left(\|T\|_{V_0,W_0}, \|T\|_{V_1,W_1}\right)\|x\|_W,$$

where C_1 and C_1 are positive constants not depending on x and T. Therefore,

$$\|T\Omega_{\bar{X}}(x) - \Omega_{\bar{Y}}(Tx)\|_Y \le \tilde{C}\rho_{0,1}(\|T\|_0, \|T\|_1)\|x\|_X$$

for some positive constant $\tilde{C}$ not depending on x and T.

These results can be easily extended to the analytic interpolation methods by using the mixed reiteration formulated. We present them here without proof. In Propositions (1.2.8)–(1.2.10), we always assume that the left shift operator S and the rotation operators R_t are isometric on $\Phi_j(X_j) (j = 0,1)$ for all $t \in \boldsymbol{R}$. We first study the stability of compactness and extend [46] to the analytic interpolation.

Proposition (1.2.8)[39]: Let $T \in \mathcal{B}(\bar{X}, \bar{Y})$. If $T: \bar{X}_{\bar{\Phi},s} \to \bar{Y}_{\bar{\Phi},s}$ is compact for some $s \in (1, e)$, then $T: \bar{X}_{\bar{\Phi},u} \to \bar{Y}_{\bar{\Phi},u}$ is compact for each $u \in (1, e)$.

The next result is about the local uniqueness-of-resolvent property which is a generalization of [41].

Proposition (1.2.9)[39]: Let $T \in \mathcal{B}(\bar{X})$ and let $s \in (1, e)$. If T is invertible on $\bar{X}_{\bar{\Phi},s}$ with the inverse T_s^{-1}, then there exists $\epsilon > 0$ such that, for all $u \in (1, e)$ with $|u - s| \in \epsilon, T$ is also invertible on $\bar{X}_{\bar{\Phi},u}$ with the inverse T_u^{-1} satisfying

$$T_u^{-1}x = T_s^{-1}x$$

for $x \in \Delta\bar{X}$.

For a Banach space X, and for $T \in \mathcal{B}(X)$, we denote by $S_p(T, X), r(T, X)$ and $r_e(T, X)$ the spectrum, the spectral radius and the essential spectral radius of T on X, respectively. By [40], we have

$$r\left(T, \bar{X}_{\bar{\Phi},s}\right) \le r(T, X_0)^{1-\theta} r(T, X_1)^{\theta},$$

and, $\Delta\bar{X}$ is dense in $\bar{X}_{\bar{\Phi},s}$,

$$r_e\left(T, \bar{X}_{\bar{\Phi},s}\right) \le r_e(T, X_0)^{1-\theta} r_e(T, X_1)^{\theta}.$$

We adapt the proof of [55] to our situation and obtain the continuity of spectral radius and essential spectral radius for a single operator.

Proposition (1.2.10)[39]: If $T \in \mathcal{B}(\bar{X})$, then the mappings

$$s \mapsto r\left(T, \bar{X}_{\bar{\Phi},s}\right) \text{ and } s \mapsto r_e\left(T, \bar{X}_{\bar{\Phi},s}\right)$$

are continuous for $s \in (1, e)$.

For $s \in \boldsymbol{A}$, for $n = 1,2, \dots$, and for an exact pseudolattice pair $\bar{\Phi}$, we may define spaces $\bar{X}_{\bar{\Phi},s}^{(\pm n)}$ associated with the nth derivative at $s \in \boldsymbol{A}$ in the following way:

$$\bar{X}_{\bar{\Phi},s}^{(n)} = \left\{x \in \sum \bar{X} \middle| x = f^{(n)}(s), f \in \mathcal{J}(\bar{\Phi}, \bar{X})\right\}$$

with the norm $\|x\|_{\bar{\Phi},s(n)} = \inf\left\{\|f\|_{\mathcal{J}} \middle| x = f^{(n)}(s)\right\}$, and

$$\bar{X}_{\bar{\Phi},s}^{(-n)} = \left\{x \in \sum \bar{X} \middle| x = f^{(n)}(s), f \in \mathcal{J}(\bar{\Phi}, \bar{X}), f^{(k)}(s) = 0 \text{ for } 1 \le k \le n\right\}$$

with the norm

$$\|x\|_{\bar{\Phi},s(-n)} = \inf\left\{\|f\|_{\mathcal{J}} | x = f(s) \text{ and } f^{(k)}(s) = 0 \text{ for } 1 \le k \le n\right\}.$$

Spaces $\bar{X}_{\bar{\Phi},s}^{(n)}$ and $\bar{X}_{\bar{\Phi},s}^{(-1)}$ are defined by [47]. These spaces arise naturally when the commutators of the derivation mappings $\Omega_{\bar{X}}$ given are concerned at least for $n = 1$. In fact, by [47], $\bar{X}^{(-1)}$ and $\bar{X}^{(1)}$ are domain and range spaces of $\Omega_{\bar{X}}$, respectively. For the sake of completeness, we write

$$\bar{X}_{\bar{\Phi},s}^{(0)} = \bar{X}_{\bar{\Phi},s}.$$

It is easy to show that $\bar{X}_{\bar{\Phi},s}^{(\pm n)}$ are interpolation spaces for the Banach couple $\bar{X}$. In particular, for $\bar{\Phi} = \overline{FC}$ and $s \in (1, e)$ with $\theta = \log s$, we have

$$\bar{X}_{\bar{\Phi},s}^{(\pm n)} = [\bar{X}]_{\theta(\pm n)},$$

which are systematically studied in [48]. The following result is an extension of [45], [48] and [49].

Proposition (1.2.11)[39]: Assume that the left shift operator S and the rotation operators R_t are isometric on $\Phi_j\left(X_j\right)(j = 0,1)$ for all $t \in \boldsymbol{R}$.

(i) Let $s_1, s_0, s, \tau \in (1, e)$ and $\alpha \in (0,1)$ such that $s = s_0^{1-\alpha} s_1^{\alpha}$ and $\tau = e^{\alpha}$. Then

$$\bar{X}_{\bar{\Phi},s}^{(\pm 1)} = \left(\bar{X}_{\bar{\Phi},s_0}^{(\pm 1)}, \bar{X}_{\bar{\Phi},s_1}^{(\pm 1)}\right)_{\overline{FC},\tau}.$$

(ii) Let $T \in \mathcal{B}(\bar{X})$ and let $s \in (1, e)$. Then

$$S_p\left(T, \bar{X}_{\bar{\Phi},s}^{(\pm 1)}\right) \subseteq S_p\left(T, \bar{X}_{\bar{\Phi},s}\right).$$

Proof. As in the proof of Proposition (1.2.7), we set $\bar{Y} = \left(\bar{X}_{\bar{\Phi},s_0}, \bar{X}_{\bar{\Phi},s_0}\right)$. Then $\bar{Y}_{\overline{FC},\tau} = \bar{X}_{\bar{\Phi},s}$ and

$$\Omega_{\bar{Y}} = (\theta_1 - \theta_0)\Omega_{\bar{X}}.$$

Consequently, $\Omega_{\bar{X}}$ and $\Omega_{\bar{Y}}$ have the same domain and range spaces, which implies that

$$\bar{X}_{\bar{\Phi},s}^{(\pm 1)} = \left(\bar{X}_{\bar{\Phi},s_0}^{(\pm 1)}, \bar{X}_{\bar{\Phi},s_1}^{(\pm 1)}\right)_{\overline{FC},\tau}.$$

The inclusion

$$S_p\left(T, \bar{X}_{\bar{\Phi},s}^{(\pm 1)}\right) \subseteq S_p\left(T, \bar{X}_{\bar{\Phi},s}\right)$$

follows by this reiteration result and Proposition (1.2.9) in a similar way for the proof of [49].

Let us recall the concept of distance between closed subspaces of a given Banach space X. For two closed subspaces V_0 and V_1 of X, we define the distance between V_0 and V_1 by

$$d(V_0,V_1)=\max_{j=0,1}\sup\left\{d\left(v_j,V_{1-j}\right)\middle|v_j\in V_j,\left\|v_j\right\|_X\le 1\right\},$$

where $d\left(v_j,V_{1-j}\right)$ is the distance between the point v_j and the set $V_{1-j}(j=0,1)$. We also denote

$$d(X/V_0\,,X/V_1)=d(V_0,V_1)$$

as the distance between the reduced quotient spaces. If $\vec{T}=(T_1,T_2,\dots,T_m)$ is an m-tuple of commuting operators in B(X), then $\vec{T}$ induces m-tuples $\vec{T}_j$ of commuting operators in $\mathcal{B}\left(X/V_j\right)(j=0,1)$. According to [47] and [51], there exists a constant c depending on s such that

$$d\left(\bar{X}_{\bar{\Phi},s},\bar{X}_{\bar{\Phi},u}\right)\le c|s-u|,$$

for all $u\in\boldsymbol{A}$ and for all Banach couples $\bar{X}$. This estimate can be extended to spaces $\bar{X}^{(\pm n)}_{\bar{\Phi},s}$.

Lemma (1.2.12)[39]: If $s,u\in(1,e)$, then

$$d\left(\bar{X}^{(\pm n)}_{\bar{\Phi},s},\bar{X}^{(\pm n)}_{\bar{\Phi},u}\right)\le c|s-u|.$$

where

$$c=\frac{1}{d(s,\partial\boldsymbol{A})}\sum_{k=0}^{n-1}\left(1+\frac{1}{d(s,\partial\boldsymbol{A})}\right)^k.$$

Proof. Fix an arbitrary $f\in\mathcal{J}(\bar{\Phi},\bar{X})$ with $f(s)=0$ and $\|f\|_{\mathcal{J}}\le 1$. We define $g\in\mathcal{J}(\bar{\Phi},\bar{X})$ by $g(z)=0$ and

$$g(z)=\left(\frac{z-u}{z-s}\right)^n f(z)\ for\ z\ne s,$$

Then $g(u)=g^{'}(u)=\cdots=g^{(n)}(u)=0$. Observe that, for $z\ne s$,

$$g(z)-f(z)=\left(\left(\frac{z-u}{z-s}\right)^n-1\right)f(z)=\frac{s-u}{z-s}\sum_{k=0}^{n-1}\left(1+\frac{s-u}{z-s}\right)^k f(z).$$

which implies that

$$\|g-f\|_{\mathcal{J}}\le\frac{|s-u|}{d(s,\partial\boldsymbol{A})}\sum_{k=0}^{n-1}\left(1+\frac{1}{d(s,\partial\boldsymbol{A})}\right)^k.$$

This shows that

$$d\left(\bar{X}^{(n)}_{\bar{\Phi},s},\bar{X}^{(n)}_{\bar{\Phi},u}\right)\le c|s-u|.$$

The inequality

$$d\left(\bar{X}^{(-n)}_{\bar{\Phi},s},\bar{X}^{(-n)}_{\bar{\Phi},u}\right)\le c|s-u|$$

can be obtained by assuming $f^{'}(s)=\cdots=f^{(n)}(s)=0$ additionally.

See [56], [57] for the spectral theory for several commuting operators developed by Taylor in 1970. For an m-tuple of commuting operators $\vec{T}=(T_1,T_2,\dots,T_m)$ in $\mathcal{B}(X)$, let $H^k\left(\vec{T},X\right)$ be the corresponding homology modules. If $\vec{T}$ is a semi-Fredholm m-tuple, then the index of $\vec{T}$ is defined by

$$\operatorname{ind}\left(\vec{T},X\right)=\sum_{k=1}^{m+1}(-1)^k\dim\left(\vec{T},X\right).$$

If $\operatorname{ind}(\vec{T}, X) \neq \pm\infty$, then $K(\vec{T}, X)$ is a Fredholm complex, and hence $\vec{T}$ is a Fredholm m-tuple. Let $S_p(\vec{T}, X)$ and $S_{p_e}(\vec{T}, X)$ be the Taylor spectrum and the Taylor essential spectrum of $\vec{T}$ on X, respectively. By combining [40] and [42] with Lemma (1.2.12), we obtain

Proposition (1.2.13)[39]: Let $\vec{T} = (T_1, T_2, \dots, T_m)$ be an m-tuple of commuting operators in $\mathcal{B}(\bar{X})$, and let $s, u \in (1, e)$.

(i) If $\vec{T}$ is semi-Fredholm on $\bar{X}^{(\pm n)}_{\bar{\Phi},s}$, then there exists $\epsilon > 0$ such that, whenever $|u - s| < \epsilon, \vec{T}$ is also semi-Fredholm on $\bar{X}^{(\pm n)}_{\bar{\Phi},u}$ satisfying

$$\dim H^k\left(\vec{T}, \bar{X}^{(\pm n)}_{\bar{\Phi},u}\right) \leq \dim H^k\left(\vec{T}, \bar{X}^{(\pm n)}_{\bar{\Phi},s}\right) \quad for \;\; 1 \leq k \leq m,$$

and

$$\operatorname{ind}\left(\vec{T}, \bar{X}^{(\pm n)}_{\bar{\Phi},u}\right) = \operatorname{ind}\left(\vec{T}, \bar{X}^{(\pm n)}_{\bar{\Phi},s}\right).$$

In particular, if $\vec{T}$ is exact on $\bar{X}^{(\pm n)}_{\bar{\Phi},s}$, then $\vec{T}$ is also exact on $\bar{X}^{(\pm n)}_{\bar{\Phi},u}$.

(ii) The mappings $\theta \mapsto S_p\left(T, \bar{X}^{(\pm 1)}_{\bar{\Phi},e^\theta}\right)$ and $\theta \mapsto S_{p_e}\left(T, \bar{X}^{(\pm 1)}_{\bar{\Phi},e^\theta}\right)$ from the interval $(0,1)$ to the set of all compact subsets of $\boldsymbol{C}^n$ are upper semicontinuous.

Chapter 2
New Spectral Triple and Noncommutative Solenoids

We show that the construction is canonically associated to quantum gravity and is an alternative version of the spectral triple presented in [41]. We establish a necessary and sufficient condition for the simplicity of the noncommutative solenoids. Then, using the computation of the trace on K_0, we discuss two different ways of constructing projective modules over the noncommutative solenoids.

Section (2.1): A Space of Connections

[61], [62] commenced a programme of combining Connes noncommutative geometry with quantum gravity. This programme is motivated by the formulation of the Standard Model coupled to gravity in terms of noncommutative geometry, see [66]. A spectral triple, and the classical action is obtained via a spectral action principle natural to noncommutative geometry. The fact that the classical Standard Model is so readily translated into the language of noncommutative geometry raises the question wether there exist a corresponding translation of the quantization procedure of QFT. Since noncommutative geometry is essentially gravitational such a translation would presumable involve quantum gravity.

We successfully constructed a semifinite spectral triple over a space of connections [59], [60]. The spectral triple involves an algebra of holonomy loops and the interaction between the Dirac type operator and the algebra reproduces the Poisson structure of General Relativity when formulated in Ashtekar variables [64]. The associated Hilbert space corresponds, up to a discrete symmetry group, to the Hilbert space of diffeomorphism invariant states known from Loop Quantum Gravity [67].

We construct a new semifinite spectral triple which differs from the triple constructed in [59], [60] through the form of the Dirac type operator. The operator presented is significantly simpler and thus possible more suitable for actual spectral computations. The construction of the operator is based on a reparameterization of the space of connections, such that the structure maps are deleting copies of the structure group. Hence the spectral triple can be constructed by writing a Dirac operator on each copy of the structure group. Whereas the reparameterized Dirac operator is simpler than the one in [61], [62], its interaction with the algebra of loops becomes more complicated.

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We recall from [60] how we constructed the completion of spaces of connections. The construction is a variant of the Ashtekar-Lewandowski construction, see [63]. The setup is a manifold M an a trivial G-principal fiber bundle over M, where G is a compact connected Lie group. Denote by $\mathcal{A}$ the space of smooth G-connections. We start with a system $\mathcal{S}$ of graphs on M. The system has to be dense and directed according to the definitions 2.1.6 and 2.1.7 in [60]. The specific examples we have in mind are the following two:

Example (2.1.1)[58]: Let $\mathcal{T}$ be a triangulation of M. We let Γ_0 be the graph consisting of all the edges in this triangulation. Strictly speaking this is not a graph if the manifold is not compact, but in this case we can consider Γ_0 as a system of graphs instead. Let $\mathcal{T}_n$ be the triangulation obtained by barycentric subdividing each of the simplices in $\mathcal{T}$ n times. The graph Γ_n is the graph consisting of the edges of $\mathcal{T}_n$. In this way we get a directed and dense system $\mathcal{S}_\Delta = \{\Gamma_n\}$ of graphs.

Example (2.1.2)[58]: Let Γ_0 be a finite, d-dimensional lattice and let Γ_1 be the lattice obtained by subdividing each cell in Γ_0 into 2^d cells, see figure 1. Correspondingly, let Γ_i be the lattice obtained by repeating i such subdivisions of Γ_0. In this way we get a directed and dense system $\mathcal{S}_\square = \{\Gamma_n\}$ of graphs.

We will for simplicity assume that the system $\mathcal{S}$ we are dealing with is of

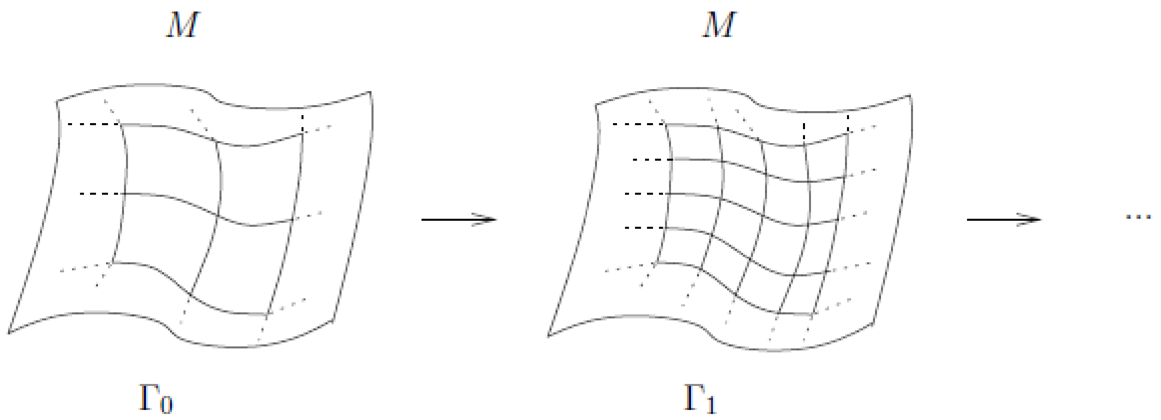


Figure (1)[58]: Repeated subdivisions of a lattice

the form $\mathcal{S} = \{S_n\}_n \in \mathbb{N}$, where S_n are finite graphs and $S_n \subset S_{n+1}$. Also we assume the edges to be oriented and we assume the embeddings $S_n \subset S_{n+1}$ to preserve the orientation. This is clearly the case in Example (2.1.1) and (2.1.2).

We define

$$\mathcal{A}_n = G^{e(S_n)},$$

where $e(S_n)$ denotes the number of edges in S_n. In other words we have just associated to each edge a copy of G. We think of $\mathcal{A}_n$ as $\mathcal{A}$ restricted to S_n; namely for each connection we associate to each edge in S_n the holonomy of the connection along the edge, which is just an element of G.

There are natural maps

$$P_{n,n+1}: \mathcal{A}_{n+1} \to \mathcal{A}_n$$

defined in the following way: If an edge $e_i \in S_n$ is the composition $e_{i_1} e_{i_2} \dots e_{i_k}$, where $e_{i_1}, e_{i_2}, \dots, e_{i_k} \in S_{n+1}$ then $\left(g_{i_1}, \dots, g_{i_k}\right)$ gets mapped to $g_{i_1}, \dots, g_{i_k}$ in the i'component of $\mathcal{A}_n$. If $e_l \in S_n$ is not the subdivision of any edges in S_n the map $P_{n,n+1}$ just forgets the i's component in $\mathcal{A}_{n+1}$. See Figure (2).

Given these maps we can define

$$\bar{\mathcal{A}}^{\mathcal{S}} = \lim_{\leftarrow} \mathcal{A}_n.$$

Since $\mathcal{A}_n$ has a natural compact Hausdorff topology, and the maps $P_{n,n+1}$ are continuous, $\bar{\mathcal{A}}^{\mathcal{S}}$ has a natural compact Hausdorff topology.

A smooth connection ∇ gives rise to an element in $\bar{\mathcal{A}}^{\mathcal{S}}$ by

$$\nabla \to \left(H\,ol(e_1, \nabla), \dots, H\,ol\left(e_{e(S_n)}, \nabla\right)\right) \in \mathcal{A}_n,$$

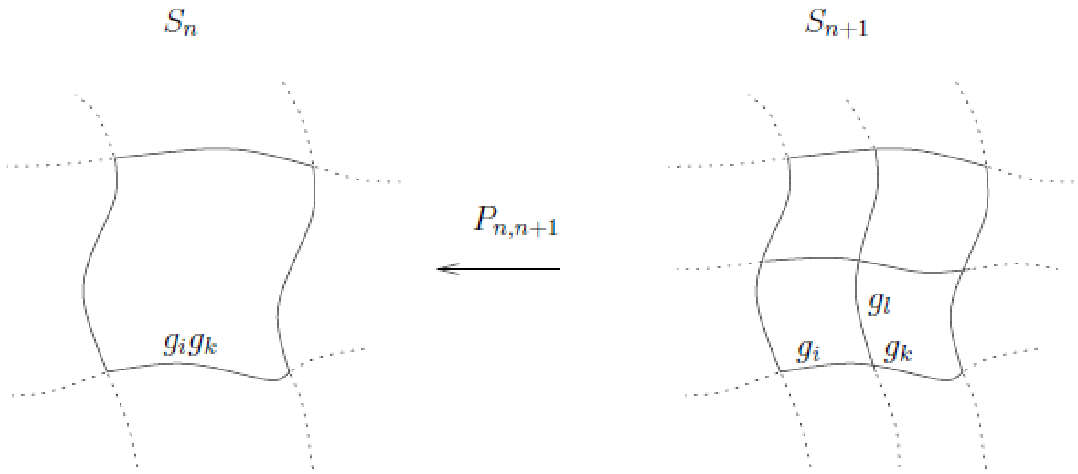


Figure (2)[58]: The structure map of one subdivision

where $H\,ol(e_i,\nabla)$ denotes the holonomy of ∇ along e_i.
We therefore get a map from $\mathcal{A}$ to $\bar{\mathcal{A}}^{\mathcal{S}}$. This map a dense embedding, see [60].

We assume that edges from S_n get subdivided into to two in S_{n+1}. This is clearly the case in Example (2.1.1) and (2.1.2). Therefore for a single edge, the projective system looks like

$$G \leftarrow G^2 \leftarrow G^4 \leftarrow \cdots G^{2n} \leftarrow G^{2^{n+1}} \leftarrow \cdots$$

with structure maps

$$P_{n,n+1}(g_1,\dots,g_{2^{n+1}}) = (g_1g_2,\dots,g_{2^{n+1}-1}g_{2^{n+1}}).$$

We will from now on focus on the case of a single edge, since the general case is basically just more notation.
Like in [60] we define the coordinate transformation

$$\Theta_n\colon \mathcal{A}_n = G^{2^n} \to G^{2^n}$$

by

$$\Theta_n(g_1,\dots,g_{2^n}) = (g_1g_2,\dots,g_{2^n},g_2g_3\dots g_{2^n},\dots,g_{2^n-1}g_{2^n},g_{2^n}).$$

It is easy to see that Θ_n preserves the Haar measure on G^{2^n}. The inverse of Θ_n is given by

$$\Theta_n^{-1}(g_1,\dots,g_{2^n}) = \left(g_1g_2^{-1},g_2g_3^{-1},\dots g_{2^n-1}g_{2^n}^{-1},g_{2^n}\right).$$

The important feature of the coordinate change is the following:

$$\Theta_n\left(P_{n,n+1}(\Theta_n^{-1})\right)(g_1,\dots,g_{2^{n+1}}) = (g_1,g_3,\dots,g_{2^{n+1}-1}).$$

We will from now on use Θ to identify $\mathcal{A}$ with a projective system of the form

$$G \leftarrow G^2 \leftarrow G^4 \leftarrow \cdots G^{2n} \leftarrow G^{2^{n+1}} \leftarrow \cdots$$

with structure maps

$$P_{n,n+1}(g_1,\dots,g_{2^{n+1}}) = (g_1g_3,\dots,g_{2^{n+1}-1}).$$

Hence the structure maps has been simplified significantly.

This way of writting the projective system can be seen in the following way:
The edge is divided into 2^n smaller edges. The coordinate g_1 corresponds to holonomy along the entire edge. The coordinate g_2 corresponds to the holonomy along the entire edge minus the first of the 2^n edges. The coordinate g_3 corresponds to the holonomy along the entire edge minus the first two of the 2^n edges and so on and so fort. See Figure (3).

We now choose a left and right invariant metric $\langle\cdot,\cdot\rangle$ on G. We will consider a metric on T^*G. We will equip $T^*\mathcal{A}_n = T^*G^{2^n}$ with the product metric and denote it by $\langle\cdot,\cdot\rangle_n$. Note that

$$\langle P_{n,n+1}^*(v), P_{n,n+1}^*(u)\rangle,$$

and hence the family of metrics $\langle\cdot,\cdot\rangle_n$ descends to a metric on $T^*\bar{\mathcal{A}}^{\mathcal{S}} = \lim_n T^*\mathcal{A}_n$, which we will also denote by $\langle\cdot,\cdot\rangle$.

Denote by $L^2(\mathcal{A}_n, Cl(T^*\mathcal{A}_n))$ the Hilbert space $L^2\left(G^{2^n}, Cl\left(T^*G^{2^n}\right)\right)$, where $Cl\left(T^*G^{2^n}\right)$ is the Clifford bundle with respect $\langle\cdot,\cdot\rangle_n$, and G^{2^n} is equipped with

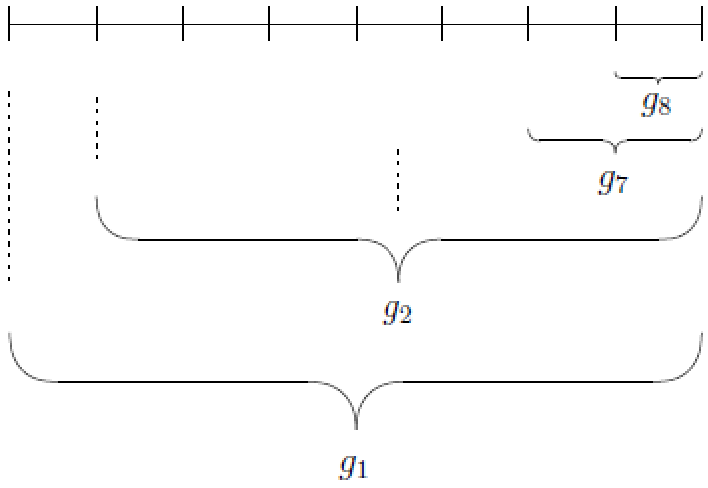


Figure (3)[58]: The new parameterization

the Haar mass. Because of (1), and because the Haar mass of G^{2^n} is one, the map $P^*_{n,n+1}$ defines a Hilbert space embedding of

$$P^*_{n,n+1}: L^2(\mathcal{A}_n, Cl(T^*\mathcal{A}_n)) \to L^2(\mathcal{A}_{n+1}, Cl(T^*\mathcal{A}_{n+1})).$$

We can thus define

$$L^2\left(\bar{\mathcal{A}}^{\mathcal{S}}, Cl(T^*\bar{\mathcal{A}}^{\mathcal{S}})\right) = \lim_{\rightarrow} L^2(\mathcal{A}_n, Cl(T^*\mathcal{A}_n)).$$

We want to construct a spectral triple related to $\bar{\mathcal{A}}^{\mathcal{S}}$. Let v be a vertex in $\mathcal{S}$ and assume that G is represented as matrices. A loop L in $\mathcal{S}$ with base point v define a matrix valued function h_L over $\bar{\mathcal{A}}^{\mathcal{S}}$ via

$$h_L(\nabla) = H\,ol(L,\nabla), \qquad \nabla \in \bar{\mathcal{A}}^{\mathcal{S}}.$$

Definition (2.1.3). The algebra B_v of holonomy loops based in v is the $*$-algebra generated by the h_L's, where L is running through all the loops in $\mathcal{S}$ based in v.

Since the representation of G is unitary h_L are bounded functions and therefore defines bounded operators on $L^2(\bar{\mathcal{A}}^{\mathcal{S}}, M_N)$, where M_N are the $N \times M$ matrices in which G is represented. In particular $\mathcal{B}_v$ can be completed to a C^*-algebra.

We want to construct a spectral triple for $\mathcal{B}_v$. Since $\mathcal{B}_v$ is an algebra of functions over $\bar{\mathcal{A}}^{\mathcal{S}}$, we will do this by constructing a Dirac type operator on $\bar{\mathcal{A}}^{\mathcal{S}}$. To be more precise the operator will act on $L^2\left(\bar{\mathcal{A}}^{\mathcal{S}}, Cl(T^*\bar{\mathcal{A}}^{\mathcal{S}})\right)$.

Let g be the Lie algebra of G. We choose an orthonormal basis $\{e_i\}$ for g with respect to $\langle\cdot,\cdot\rangle$. We also denote by $\{e_i\}$ the corresponding left translated vectorfields. On G define the bare Dirac type operator by

$$D_b(\xi) = \sum_i e_i \cdot d_{e_i}\xi\,, \qquad \xi \in L^2(G, Cl(TG)),$$

where d_{e_i} means deriving with respect to e_i in the trivialization given by $\{e_i\}$, and $\cdot$ means Clifford multiplication.

On G^{2^n} we define the operator $D_{n,j}$ acting on $L^2\left(G^{2^n}, Cl(TG^{2^n})\right)$ simply as D_b acting on the j'es copy of G. Since $\langle\cdot,\cdot\rangle_n$ identifies TG^{2^n} with $T^*G^{2^n}$, we can consider $D_{n,j}$ as an operator acting on $L^2\left(\mathcal{A}_n, Cl(T^*\mathcal{A}_n)\right)$.

Note that given a Dirac type operator D acting on $L^2\left(\mathcal{A}_{n-1}, Cl(T^*\mathcal{A}_{n-1})\right)$ we can define an operator $E_n(D)$ acting on $L^2\left(\mathcal{A}_n, Cl(T^*\mathcal{A}_n)\right)$ simply by letting it act on the odd variables of $\mathcal{A}_n = G^{2^n}$.

Definition (2.1.4). Let $\left\{a^{j,k}\right\}_{j\in\mathbb{N}_0, 1\le k\le 2^{j-1}}$ (with the odd convention that $2^{-1} = 1$) be a sequence of non zero real numbers. The n'th Dirac type operator operator is defined inductively via $D_0 = D_b$ and

$$D_n = E_n(D_{n-1}) + \sum_k a^{n,k} D_{n,2k}.$$

By construction it is clear that

$$P^*_{n,n+1}\left(D_n(\xi)\right) = D_{n+1}\left(P^*_{n,n+1}(\xi)\right).$$

Proposition (2.1.5). The family of operators $\{D_n\}$ descends to a densely defined essentially self adjoint operator D on $L^2\left(\bar{\mathcal{A}}^{\mathcal{S}}, Cl(T^*\bar{\mathcal{A}}^{\mathcal{S}})\right)$.

Proof. By (2) it follows that $\{D_n\}$ descends to a densely defined operator D on $L^2\left(\bar{\mathcal{A}}^{\mathcal{S}}, Cl(T^*\bar{\mathcal{A}}^{\mathcal{S}})\right)$. The operators D_n are formally self adjoint elliptic differential operators on compact manifolds, and hence orthonormal diagonalizable. Because of (2) we can find a orthonormal basis for $L^2\left(\bar{\mathcal{A}}^{\mathcal{S}}, Cl(T^*\bar{\mathcal{A}}^{\mathcal{S}})\right)$ diagonalizing D with real eigenvalues. In particular D is essentially self adjoint.

Proposition (2.1.6). The commutator $[h_L, D]$ is bounded for all $h_L \in \mathcal{B}_v$.

Proof. A given loop L belongs to $\mathcal{S}_n$ for some n. Therefore the action of h_L on $L^2\left(\bar{\mathcal{A}}^{\mathcal{S}}, M_N \otimes Cl(T^*\bar{\mathcal{A}}^{\mathcal{S}})\right)$ depends only, by construction of the coordinate change, of the coppies of G arising at the n'level. Therefore $[h_L, D] = [h_L, D_n]$. On the other hand $[h_L, D_n]$ is an order zero operator on a compact manifold, and hence bounded.

We will assume that G has the property that the kernel of the bare Dirac type operator is $Cl(g)$, where $Cl(g)$ is understood as $CL(TG)$ generated by left invariant vectorfields. For $U(1)$ this is trivial, and the computation in the appendix of [60] shows that this is also the case for $SU(2)$, which is the example of most interest. We do not know if all compact Lie groups possesses this property.

One of the crucial demands of being a unital spectral triple is that the Dirac operator should have compact resolvent. This is however clearly not the case for D, since it has infinite dimensional kernel. We will however see that we have a semifinite spectral triple. For a semifinite spectral triple one replaces the compact resolvent condition with the condition that

$$\frac{1}{D^2 + 1}$$

is compact with respect to a certain trace, i.e. the trace should be thought of as integrating out the infinite degeneracy in the spectrum of D.

The following definition first appeared in [65].

Definition (2.1.7). Let $\mathcal{N}$ be a semifinite von Neumann algebra with a semifinite trace τ. Let $\mathcal{K}_\tau$ be the τ-compact operators. A semifinite spectral triple $(\mathcal{B}, \mathcal{H}, D)$ is a $*$-subalgebra

$\mathcal{B}$ of $\mathcal{N}$, a representation of $\mathcal{N}$ on the Hilbert space $\mathcal{H}$ and an unbounded densely defined self adjoint operator D on $\mathcal{H}$ affiliated with $\mathcal{N}$ satisfying
(i) $b(\lambda - D)^{-1} \in \mathcal{K}_\tau$ for all $b \in \mathcal{B}$ and $\lambda \notin \mathbb{R}$..
(ii) $[b, D]$ is densely defined and extends to a bounded operator.

We will now prove that
$$\left(\mathcal{B}_v, L^2\left(\bar{\mathcal{A}}^{\mathcal{S}}, M_N \otimes Cl(T^*\bar{\mathcal{A}}^{\mathcal{S}})\right), D\right),$$
is a semifinite spectral triple. We therefore need to specify a semifinite von Neumann algebra $\mathcal{N}$ with a semifinite trace τ.
We can use $\{e_i\}$ to trivialize T^*G. Doing this in each copy of G we can also trivialize $T^*\mathcal{A}_n$. Hence we can factorize
$$L^2\left(\bar{\mathcal{A}}^{\mathcal{S}}, M_N \otimes Cl(T^*\bar{\mathcal{A}}^{\mathcal{S}})\right) = L^2(\bar{\mathcal{A}}^{\mathcal{S}}) \otimes M_N \otimes Cl(T^*_{id}\bar{\mathcal{A}}^{\mathcal{S}}),$$
where
$$Cl(T^*_{id}\bar{\mathcal{A}}^{\mathcal{S}}) = \lim_{n} Cl(T^*_{id}\mathcal{A}_n).$$
Since the problem arises from the infinite dimensionality of $Cl(T^*_{id}\bar{\mathcal{A}}^{\mathcal{S}})$ we will take the algebra
$$\mathcal{N} = \mathcal{B}\left(L^2(\bar{\mathcal{A}}^{\mathcal{S}})\right) \otimes M_N \otimes C,$$
where C is the following von Neumann algebra acting on $Cl(T^*_{id}\bar{\mathcal{A}}^{\mathcal{S}})$:
• We write
$$T^*_{id}\mathcal{A}_{n+1} = T^*_{id}\mathcal{A}_n \otimes V_{n,n+1},$$
and
$$Cl(T^*_{id}\mathcal{A}_{n+1}) = Cl(T^*_{id}\mathcal{A}_n) \hat{\otimes} Cl\left(V_{n,n+1}\right),$$
then, with abuse of notation,
$$P^*_{n,n+1}: Cl(T^*_{id}\mathcal{A}_n) \to Cl(T^*_{id}\mathcal{A}_{n+1})$$
is given by
$$P^*_{n,n+1}(v) = v \otimes 1_{Cl(V_{n,n+1})}.$$
Define C as the weak closure of the C^*-algebra
$$B = \lim_{\rightarrow} Cl(T^*_{id}\mathcal{A}_n)$$
with respect to the representation on $Cl(T^*_{id}\bar{\mathcal{A}}^{\mathcal{S}})$.
We denote by P^*_n the natural map from $Cl(T^*_{id}\mathcal{A}_n)$ to B.
Note that B is a UHF-algebra. Since the dimension of the Clifford algebra is a power of 2 when $n \geq 1, B$, is the CAR-algebra and has a normalized trace. This trace can be described in the following way: $Cl(T^*_{id}\mathcal{A}_n)$ is a matrix algebra, and hence has a normalized trace τ_n. By definition of the normalized trace we have
$$\tau_{n+1} \circ P^*_{n,n+1} = \tau_n.$$
Thus $\{\tau_n\}$ descends to a trace τ on B. In particular $\tau(1) = 1$. This remedies the defect that $Cl(T^*_{id}\bar{\mathcal{A}}^{\mathcal{S}})$ is infinite dimensional.

Note that the action of B on $Cl(T^*_{id}\bar{\mathcal{A}}^{\mathcal{S}})$ is just the GNS-representation of B with respect to the normalized trace on B. Therefore C is the hyperfinite II_1 factor, and τ extends to a finite trace on C.

Tensoring with the ordinary operator trace tr on $\mathcal{B}(L^2(\bar{\mathcal{A}}^{\mathcal{S}}) \otimes M_N)$ we obtain a semifinite trace Tr on $\mathcal{N}$.
Theorem (2.1.8). The triple
$$\left(\mathcal{B}_v, L^2\left(\bar{\mathcal{A}}^{\mathcal{S}}, M_N \otimes Cl(T^*\bar{\mathcal{A}}^{\mathcal{S}})\right), D\right)$$

is a semifinite spectral triple with respect to $(\mathcal{N}, Tr)$ when $a^{j,k} \to \infty$.
Proof. Clearly $\mathcal{B}_v \subset \mathcal{N}$. Also by proposition 4.0.4, the commutators

$$[h_L, D]$$

are bounded. We therefore only need to check that D is affiliated with $\mathcal{N}$ and that D has Tr-compact resolvent.

Let $P_{n,\lambda}$ be the spectral projection of D_n corresponding to the eigenvalue λ. To this projection we associate a projection $P_{n,\lambda}^{\infty}$ in $\mathcal{N}$ in the following way:
The embedding of $L^2(\mathcal{A}_n) \to L^2(\bar{\mathcal{A}}^{\mathcal{S}})$ induces an embedding

$$I_n: \mathcal{B}(L^2(\mathcal{A}_n) \otimes M_N) \to \mathcal{B}(L^2(\bar{\mathcal{A}}^{\mathcal{S}}) \otimes M_N).$$

Define $P_{n,\lambda}^{\infty} = (I_n \otimes P_n^*)(P_{n,\lambda})$, where $P_n^*: Cl(T_{id}^* \mathcal{A}_n) \to B$ is the natural map.
Suppose ξ is an eigenvector for D_n with eigenvalue λ. Since

$$D_{n+1}(v) = 0, \qquad v \in Cl(V_{n,n+1}).$$

we see that $P_{n,n+1}(\xi) \otimes v$ is an eigenvector for D_{n+1}. This shows that $P_{n,\lambda}^{\infty}$ is a subprojection of P_λ, the spectral projection of D corresponding to the eigenvalue λ.
Since $P_{n,\lambda}^{\infty} \nearrow P_\lambda$ weakly, $P_\lambda \in \mathcal{N}$, and hence D is affiliated to $\mathcal{N}$.
By the assumption on the bare Dirac type operator and since $a^{j,k} \to \infty$ the only new eigenvectors with eigenvalues in a given bounded set introduced by going from D_n to D_{n+1} will from a certain step be of the form $P_{n,n+1}^*(\xi) \otimes v$, where ξ is an eigenvector of D_n with eigenvalue in the bounded set and $v \in Cl(V_{n,n+1})$. Thus in every bounded set of $\mathbb{R}$ there are only finitely many eigenvalues of D and the associated spectral projections are finite with respect to Tr.
The present construction of the spectral triple is based on the reparameterization of $\bar{\mathcal{A}}^{\mathcal{S}}$ into a projective system of the form

$$G \leftarrow G^2 \leftarrow G^4 \leftarrow \cdots G^{2n} \leftarrow G^{2^{n+1}} \leftarrow \cdots$$

with structure maps

$$P_{n,n+1}(g_1, \dots, g_{2^{n+1}}) = (g_1 g_3, \dots, g_{2^{n+1}-1}).$$

The Dirac operator we have constructed is just a weighted sum of Dirac operators on each of the copies of G.

The reparameterization we have chosen is unique. The reparameterization relies on a choice of labeling of the new degrees of freedom which are generated by going from step n to step $n+1$. Another choice of labeling is indicated in figure 4. The labeling can in general be done in many different ways, where each choice of labeling gives rise to different spectral triples. At the end one would expect some symmetry condition singling out the spectral triple which might be relevant in physics.

The spectral triple constructed by means of the reparameterization differs from the one constructed in [60]. The spectral analysis of the one constructed in [60] appears to be more complicated than the reparameterized ones. However the original Dirac type operator appears to be more natural since it is more symmetrical. This is related to the interaction between the Dirac type operators with the loop algebra. In fact, the interaction of

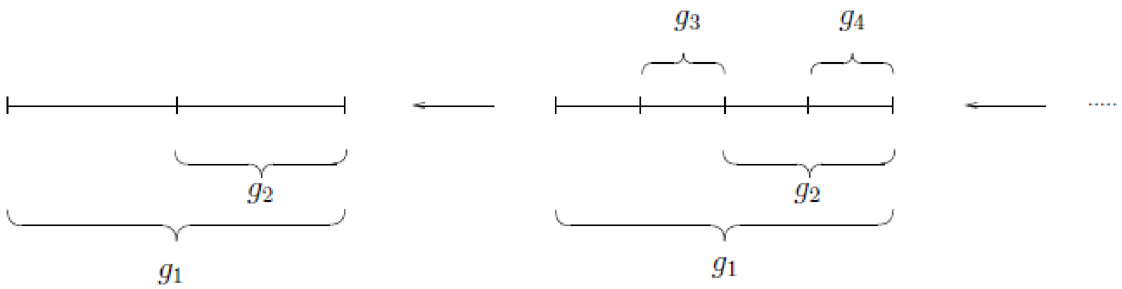


Figure (4)[58]: A different reparameterization

the algebra with the reparameterized Dirac type operator seems to be less natural due to an asymmetry which arises through the reparameterization. For example a loop L running through the first half of an edge, see figure 5, has in the reparameterization the action of the form

$$(L\xi)(\dots, g_1, g_2, \dots) = \cdots g_2^{-1} g_1 \dots \xi(\dots, g_1, g_2, \dots),$$

whereas a loop running through the second half of the edge has an action of the form

$$(L\xi)(\dots, g_1, g_2, \dots) = \cdots g_1 \dots \xi(\dots, g_1, g_2, \dots).$$

Therefore the construction has a build in asymmetry. It remains to be clarified whether any of these different Dirac type operators are singled out by some arguments of symmetry related to physical principles.

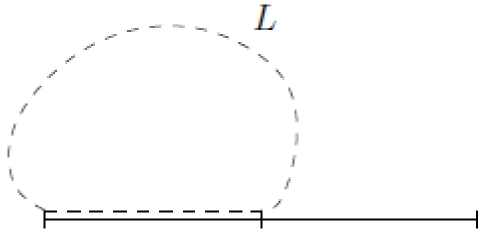


Figure (5)[58]: Loop running through first half of an edge

Section (2.2): The Projective Modules

Twisted group algebras and transformation group C^*-algebras have been studied since the early 1960's [76] and provide a rich source of examples and problems in C^*-algebra theory. Much progress has been made in studying such C^*-algebras when the groups involved are finitely generated (or compactly generated, in the case of Lie groups). Even when $G = \mathbb{Z}^n$, these C^*-algebras give a rich class of examples which have driven much development in C^*-algebra theory, including the foundation of noncommutative geometry by Connes [70], the extensive study of the geometry of quantum tori by Rieffel [82], [84], [85], [86], the expansion of the classification problem from AF to AT algebras by G. Elliott and D. Evans [72], and many more (L. Baggett and A. Kleppner [69], and S. Echterhoff and J. Rosenberg [71]).

We present the work on twisted group C^*-algebras of the Cartesian square of the discrete group $\mathbb{Z}\left[\frac{1}{p}\right]$ of p-adic rationals, i.e. the additive subgroup of $\mathbb{Q}$ whose elements have denominators given by powers of a fixed $p \in \mathbb{N}, p \geq 1$. The Pontryagin duals of these groups are the p-solenoid, thereby motivating our terminology in calling these C^*-algebras noncommutative solenoids. We review our computation of the K-groups of these C^*-algebras, derived in their full technicality in [78], and which in and of itself involves an intriguing problem in the theory of Abelian group extensions. We were also able to compute the range of the trace on the K_0-groups, and use this knowledge to classify these C^*-algebras up to $*$-isomorphism, in [78], and these facts are summarized in a brief survey of [78].

We concern with the open problem of classifying noncommutative solenoids up to Morita equivalence. We demonstrate a method of constructing an equivalence bimodule between two noncommutative solenoids using methods due to M. Rieffel [84], and will note how this method has relationships to the theory of wavelet frames.

We provide a survey of the main results proven in [78] concerning the computation of the K-theory of noncommutative solenoids and its application to their classification up to $*$-isomorphism. An interesting connection between the K-theory of noncommutative solenoids and the p-adic integers is unearthed, and in particular, we prove that the range of the K_0 functor on the class of all noncommutative solenoids is fully described by all Abelian extensions of the group of p-adic rationals by $\mathbb{Z}$. These interesting matters are the subject, whereas we start with the basic objects of our study.

We shall fix, an arbitrary $p \in \mathbb{N}$ with $p > 1$. We have the following groups:

Definition (2.2.1)[68]: Let $p \in \mathbb{N}, p > 1$. The group $\mathbb{Z}\left[\frac{1}{p}\right]$ of p-adic rationals is the inductive limit of the sequence of groups:

$$\mathbb{Z} \xrightarrow{z \mapsto pz} \mathbb{Z} \xrightarrow{z \mapsto pz} \mathbb{Z} \xrightarrow{z \mapsto pz} \mathbb{Z} \xrightarrow{z \mapsto pz} \cdots$$

which is explicitly given as the group:

$$\mathbb{Z}\left[\frac{1}{p}\right] = \left\{\frac{z}{p^k} \in \mathbb{Q} : z \in \mathbb{Z}, k \in \mathbb{N}\right\}$$

endowed with the discrete topology.

From the description of $\mathbb{Z}\left[\frac{1}{p}\right]$ as an injective limit, we obtain the following result by functoriality of the Pontryagin duality. We denote by $\mathbb{T}$ the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ in the field $\mathbb{C}$ of complex numbers.

Proposition (2.2.2)[68]: Let $p \in \mathbb{N}, p > 1$. The Pontryagin dual of the group $\mathbb{Z}\left[\frac{1}{p}\right]$ is the p-solenoid group, given by:

$$\wp_p = \left\{(z_n)_{n\in\mathbb{N}} \in \mathbb{T}^{\mathbb{N}} : \forall_n \in \mathbb{N}\ z_{n+1}^p = z_n\right\},$$

endowed with the induced topology from the injection $\wp_p \hookrightarrow \mathbb{T}^{\mathbb{N}}$. The dual pairing between $\mathbb{Q}_N$ and $\wp_N$ is given by:

$$\langle \frac{q}{p^k}, (z_n)_{n\in\mathbb{N}} \rangle = z_k^q,$$

where $\frac{q}{p^k} \in \mathbb{Z}\left[\frac{1}{p}\right]$ and $(z_n)_{n\in\mathbb{N}} \in \wp_p$.

We study in [78] the following C^*-algebras.

Definition (2.2.3)[68]: A noncommutative solenoid is a C^*-algebra of the form

$$C^*\left(\mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right], \sigma\right),$$

where p is a natural number greater or equal to 2 and σ is a multiplier of the group $\mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right]$.

The first matter to attend in the study of these C^*-algebras is to describe all the multipliers of the group $\mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right]$ up to equivalence, where our multipliers are $\mathbb{T}$-valued unless otherwise specified, with $\mathbb{T}$ the unit circle in $\mathbb{C}$. Note that the group $\mathbb{Z}\left[\frac{1}{p}\right]$ has no nontrivial multiplier, so our noncommutative solenoids are the natural object to consider.

Using [76], we compute in [78] the group $H^2\left(\mathbb{Z}\left[\frac{1}{p}\right]\times\mathbb{Z}\left[\frac{1}{p}\right],\mathbb{T}\right)$ of $\mathbb{T}$-valued multipliers of $\mathbb{Z}\left[\frac{1}{p}\right]\times\mathbb{Z}\left[\frac{1}{p}\right]$ up to equivalence, as follows:

Theorem (2.2.4)[68]: [78] Let $p\in\mathbb{N}, p>1$. Let:

$$\Xi_p=\{(\alpha_n):\alpha_0\in[0,1)\wedge(\forall_n\in\mathbb{N}\ \exists k\in\{0,\dots,N-1\}\ p\alpha_{n+1}=\alpha_n+k)\}$$

which is a group for the pointwise addition modulo one. There exists a group isomorphism $\rho: H^2\left(\mathbb{Z}\left[\frac{1}{p}\right]\times\mathbb{Z}\left[\frac{1}{p}\right],\mathbb{T}\right)\to\Xi_p$ such that if $\sigma: H^2\left(\mathbb{Z}\left[\frac{1}{p}\right]\times\mathbb{Z}\left[\frac{1}{p}\right],\mathbb{T}\right)$ and $\alpha=\rho(\sigma)$, and if f is a multiplier of class σ, then f is cohomologous to:

$$\Psi_\alpha:\left(\left(\frac{q_1}{p^{k_1}},\frac{q_2}{p^{k_2}}\right),\left(\frac{q_3}{p^{k_3}},\frac{q_4}{p^{k_4}}\right)\right)\mapsto\exp\left(2i\pi\alpha_{(k_1+k_4)q_1q_4}\right).$$

For any $p\in\mathbb{N},p>1$, the groups Ξ_p and $\wp_p$ are obviously isomorphic as topological groups; yet it is easier to perform our computations in the additive group Ξ_p in what follows. Thus, as a topological group, $H^2\left(\mathbb{Z}\left[\frac{1}{p}\right]\times\mathbb{Z}\left[\frac{1}{p}\right],\mathbb{T}\right)$ is isomorphic to $\wp_p$. We observe that a corollary of Theorem (2.2.4) is that Ψ_α and Ψ_β are cohomologous if and only if $\alpha=\beta\in\Xi_p$. The proof of Theorem (2.2.4) involves the standard calculations for cohomology classes of multipliers on discrete Abelian groups, due to A. Kleppner, generalizing results of Backhouse and Bradley.

With this understanding of the multipliers of $\mathbb{Z}\left[\frac{1}{p}\right]\times\mathbb{Z}\left[\frac{1}{p}\right]$, we thus propose to classify the noncommutative solenoids $C^*\left(\mathbb{Z}\left[\frac{1}{p}\right]\times\mathbb{Z}\left[\frac{1}{p}\right],\sigma\right)$. We start by recalling [88] that for any multiplier σ of a discrete group Γ, the C^*-algebra $C^*(\Gamma,\sigma)$ is the C^*-completion of the involutive Banach algebra $(\ell^1(\Gamma),*_\sigma,\cdot^*)$, where the twisted convolution $*_\sigma$ is given for any $f_1,f_2\in\ell^1(\Gamma)$ by

$$f_1*_\sigma f_2:\gamma\in\Gamma\mapsto\sum_{\gamma_1\in\Gamma}f_1(\gamma_1)f_2(\gamma-\gamma_1)\sigma(\gamma_1,\gamma-\gamma_1),$$

while the adjoint operation is given by:

$$f_1^*:\gamma\in\Gamma\mapsto\overline{\sigma(\gamma,-\gamma)f_1(-\gamma)}.$$

The C^*-algebra $C^*(\Gamma,\sigma)$ is then shown to be the universal C^*-algebra generated by a family $(W_\gamma)_{\gamma\in\Gamma}$ of unitaries such that $W_\gamma W_\delta=\sigma(\gamma,\delta)W_{\gamma\delta}$ for any $\gamma,\delta\in\Gamma$ [88]. We shall henceforth refer to these generating unitaries as the canonical unitaries of $C^*(\Gamma,\sigma)$.

One checks easily that if σ and η are two cohomologous multipliers of the discrete group Γ, then $C^*(\Gamma,\sigma)$ and $C^*(\Gamma,\eta)$ are $*$-isomorphic [88]. Thus, by Theorem (2.2.4), we shall restrict our attention to multipliers of $\mathbb{Z}\left[\frac{1}{p}\right]\times\mathbb{Z}\left[\frac{1}{p}\right]$ of the form Ψ_α with $\alpha\in\Xi_p$. With this in mind, we introduce the following notation:

Notation (2.2.5)[68]: For any $p\in\mathbb{N},p>1$ and for any $\alpha\in\Xi_p$, the C^*-algebra $C^*\left(\mathbb{Z}\left[\frac{1}{p}\right]\times\mathbb{Z}\left[\frac{1}{p}\right],\Psi_\alpha\right)$, with Ψ_α defined in Theorem (2.2.4), is denoted by $\mathring{A}_\alpha^{\wp}$.

Noncommutative solenoids, defined in Definition (2.2.3) as twisted group algebras of $\mathbb{Z}\left[\frac{1}{p}\right]\times\mathbb{Z}\left[\frac{1}{p}\right]$, also have a presentation as transformation group C^*-algebras, in a manner similar to the situation with rotation C^*-algebras:

Proposition (2.2.6)[68]: [78] Let $p \in \mathbb{N}, p > 1$ and $\alpha \in \Xi_p$. Let θ^α be the action of $\mathbb{Z}\left[\frac{1}{p}\right]$ on $\wp_p$ defined for all $\frac{q}{p^k} \in \mathbb{Z}\left[\frac{1}{p}\right]$ and for all $(z_n)_{n\in\mathbb{N}} \in \wp_p$ by:

$$\theta^\alpha_{\frac{q}{p^k}}((z_n)_{n\in\mathbb{N}}) = \left(e^{(2i\pi\alpha_{(k+n)}q)} z_n\right)_{n\in\mathbb{N}}.$$

The C^*-crossed-product $C(\wp_p) \rtimes_{\theta^\alpha} \mathbb{Z}\left[\frac{1}{p}\right]$ is $*$-isomorphic to $\mathring{A}^{\wp}_\alpha$.

Whichever way one decides to study them, there are longstanding methods in place to determine whether or not these C^* algebras are simple. For now, we concentrate on methods from the theory of twisted group C^*-algebras.

In [78], we thus characterize when the symmetrizer group of the multipliers of $\mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right]$ given by Theorem (2.2.4) is non-trivial:

Theorem (2.2.7)[68]: [78] Let $p \in \mathbb{N}, p > 1$. Let $\alpha \in \Xi_p$. Denote by Ψ_α the multiplier defined in Theorem (2.2.4). The following are equivalent:

(i) the symmetrizer group S_{Ψ_α} is non-trivial,

(ii) the sequence α has finite range, i.e. the set $\{\alpha_j : j \in \mathbb{N}\}$ is finite,

(iii) there exists $k \in \mathbb{N}$ such that $(p^k - 1)\alpha_0 \in \mathbb{Z}$,

(iv) the sequence α is periodic,

(v) there exists a positive integer $b \in \mathbb{N}$ such that:

$$S_{\Psi_\alpha} = b\mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right] = \left\{(br_1, br_2), (r_1, r_2) \in \mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right]\right\}.$$

Theorem (2.2.8), when applied to noncommutative solenoids via Theorem (2.2.7), allows us to conclude:

Theorem (2.2.8)[68]: [78] Let $p \in \mathbb{N}, p > 1$ and $\alpha \in \Xi_p$. Then the following are equivalent:

(i) the noncommutative solenoid $\mathring{A}^{\wp}_\alpha$ is simple,

(ii) the set $\{\alpha_j : j \in \mathbb{N}\}$ is infinite,

(iii) for every $k \in \mathbb{N}$, we have $(p^k - 1)\alpha_0 \notin \mathbb{Z}$.

In particular, if $\alpha \in \Xi_p$ is chosen with at least one irrational entry, then by definition of Ξ_p, all entries of α are irrational, and by Theorem (2.2.8), the noncommutative solenoid $\mathring{A}^{\wp}_\alpha$ is simple. We observe that, even if $\alpha \in \Xi_p$ only has rational entries, the noncommutative solenoid may yet be simple — as long as α has infinite range. We called this situation the aperiodic rational case in [78].

Example (2.2.9)[68]: (Aperiodic rational case). Let $p = 7$, and consider $\alpha \in \Xi_7$ given by

$$\alpha = \left(\frac{2}{7}, \frac{2}{49}, \frac{2}{343}, \frac{2}{2401}, \cdots\right) = \left(\frac{2}{7^n}\right)_{n\in\mathbb{Z}}.$$

Note that $\alpha_j \in \mathbb{Q}$ for all $j \in \mathbb{N}$, yet Theorem (2.2.8) tells us that the noncommutative solenoid $\mathring{A}^{\wp}_\alpha$ is simple!

The following is an example where the symmetrizer subgroup is non-trivial, so that the corresponding C^*-algebra is not simple.

Example (2.2.10)[68]: (Periodic rational case). Let $p = 5$, and consider $\alpha \in \Xi_5$ given by

$$\alpha = \left(\frac{1}{62}, \frac{25}{62}, \frac{5}{62}, \frac{1}{62}, \cdots\right).$$

Theorem (2.2.7) shows that the symmetrizer group of the multiplier Ψ_α of $\left(\mathbb{Z}\left[\frac{1}{5}\right]\right)^2$ given by Theorem (2.2.4) is:

$$S_\alpha = \left\{\left(\frac{62j_1}{5^k}, \frac{62j_2}{5^k}\right) \in \mathbb{Q}: j_1, j_2 \in \mathbb{Z}, k \in \mathbb{N}\right\}.$$

Hence the noncommutative solenoid $\mathring{A}_\alpha^{\wp}$ is not simple by Theorem (2.2.8).

We conclude with the following result about the existence of traces on noncommutative solenoids, which follows from [75], since the Pontryagin dual $\wp_p \times \wp_p$ of $\mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right]$ acts ergodically on $\mathring{A}_\alpha^{\wp}$ for any $\alpha \in \Xi_p$ via the dual action:

Theorem (2.2.11)[68]: [78] Let $p \in \mathbb{N}, p > 1$ and $\alpha \in \Xi_p$. There exists at least one tracial state on the noncommutative solenoid $\mathring{A}_\alpha^{\wp}$. Moreover, this tracial state is unique if, and only if α is not periodic.

Moreover, since noncommutative solenoids carry an ergodic action of the compact groups $\wp_p$, if one chooses any continuous length function on $\wp_p$, then one may employ the results found in [86] to equip noncommutative solenoids with quantum compact metric spaces structures and, use [87] and [77] to obtain various results on continuity for the quantum Gromov-Hausdorff distance of the family of noncommutative solenoids as the multiplier and the length functions are left to vary. We shall focus our attention on the noncommutative topology of our noncommutative solenoids, rather than their metric properties.

In [78], we provide a full description of noncommutative solenoids as bundles of matrix algebras over the space $\wp_p^2$, while in contrast, in [78], we note that for α with at least (and thus all) irrational entry, the noncommutative solenoid $\mathring{A}_\alpha^{\wp}$ is an inductive limit of circle algebras (i.e. AT), with eal rank zero. Both these results follow from writing noncommutative solenoids as inductive limits of quantum tori, which is the starting point.

Noncommutative solenoids are classified by their K-theory; more precisely by their K_0 groups and the range of the traces on K_0. The main content in [78] is the computation of the K-theory of noncommutative solenoids and its application to their classification up to $*$-isomorphism.

The starting point of this computation is the identification of noncommutative solenoids as inductive limits of sequences of noncommutative tori. A noncommutative torus is a twisted group C^*-algebra for $\mathbb{Z}^d$, with $d \in \mathbb{N}, d > 1$ [82]. In particular, for $d = 2$, we have the following description of noncommutative tori. Any multiplier of $\mathbb{Z}^2$ is cohomologous to one of the form:

$$\sigma_\theta: \left(\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) \longmapsto \exp(2i\pi\theta_{z_1 y_2}).$$

for some $\theta \in [0,1)$. Consequently, for a given $\theta \in [0,1)$, the C^*-algebra $C^*(\mathbb{Z}^2, \sigma_\theta)$ is the universal C^*-algebra generated by two unitaries U, V such that:

$$UV = e^{2i\pi\theta} VU.$$

Notation (2.2.12)[68]: The noncommutative torus $C^*(\mathbb{Z}^2, \sigma_\theta)$, for $\theta \in [0,1)$, is denoted by A_θ. Moreover, the two canonical generators of A_θ (i.e. the unitaries corresponding to $(1,0), (0,1) \in \mathbb{Z}^2$), are denoted by U_θ and V_θ, so that $U_\theta V_\theta = e^{2i\pi\theta} V_\theta U_\theta$.

For any $\theta \in [0,1)$, the noncommutative torus A_θ is $*$-isomorphic to the crossedproduct C^*-algebra for the action of $\mathbb{Z}$ on the circle $\mathbb{T}$ generated by the rotation of angle $2i\pi\theta$, and thus A_θ is also known as the rotation algebra for the rotation of angle θ — a name by which it was originally known.

The following question naturally arises: since A_θ is a twisted $\mathbb{Z}^2$ algebra, and $\mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right]$ can be realized as a direct limit group built from embeddings of $\mathbb{Z}^2$ into itself, is it possible to build our noncommutative solenoids $\mathring{A}_\alpha^\wp$ as a direct limits of rotation algebras? The answer is positive, and this observation provides much structural information regarding noncommutative solenoids.

Theorem (2.2.13)[68]: [78] Let $p \in \mathbb{N}, p > 1$ and $\alpha \in \Xi_p$. For all $n \in \mathbb{N}$, let φ_n be the unique $*$-morphism from $A_{\alpha_{2n}}$ into $A_{\alpha_{2n+2}}$ given by:

$$\begin{cases} U_{\alpha_{2n}} \longmapsto U_{\alpha_{2n+2}}^p \\ V_{\alpha_{2n}} \longmapsto V_{\alpha_{2n}}^p \end{cases}$$

Then:

$$A_{\alpha_0} \overset{\alpha_0}{\to} A_{\alpha_2} \overset{\alpha_1}{\to} A_{\alpha_4} \overset{\alpha_2}{\to} \cdots$$

converges to the noncommutative solenoid $\mathring{A}_\alpha^\wp$. Moreover, if $\left(W_{r_1,r_2}\right)_{(r_1,r_2)\in\mathbb{Z}\left[\frac{1}{p}\right]\times\mathbb{Z}\left[\frac{1}{p}\right]}$ is the family of canonical unitary generators of $\mathring{A}_\alpha^\wp$, then, for all $n \in \mathbb{N}$, the rotation algebra $A_{\alpha_{2n}}$ embeds in $\mathring{A}_\alpha^\wp$ via the unique extension of the map:

$$\begin{cases} U_{\alpha_{2n}} \longmapsto W_{\left(\frac{1}{p^n},0\right)} \\ V_{\alpha_{2n}} \longmapsto W_{\left(0,\frac{1}{p^n}\right)} \end{cases}$$

to a $*$-morphism, given by the universal property of rotation algebras; one checks that this embeddings, indeed, commute with the maps φ_n.

Our choice of terminology for noncommutative solenoids is inspired, in part, by Theorem (2.2.13), and the well established terminology of noncommutative torus for rotation algebras. As we shall now see, our study of noncommutative solenoids is firmly set within the framework of noncommutative topology.

The main result from [78] under survey and the previous one is the computation of the K-theory of noncommutative solenoid and its application to their classification. An interesting connection between the work on noncommutative solenoid and classifications of Abelian extensions of $\mathbb{Z}\left[\frac{1}{p}\right]$ by $\mathbb{Z}$, which in turn are classified by means of the group of p-adic integers, emerges as a consequence of our computation. We shall present this result now, starting with some reminders about the p-adic integers and Abelian extensions of $\mathbb{Z}\left[\frac{1}{p}\right]$, see [78] for the involved proof leading to it.

We define the group of p-adic integer simply as the set of sequences valued in $\{0, \dots, p-1\}$ with the appropriate operation, but our choice of definition will make our exposition clearer. We note that we have a natural embedding of $\mathbb{Z}$ as
a subgroup of $\mathbb{Z}_p$ by sending $z \in \mathbb{Z}$ to the sequence $(z \mod p^k)_{K\in\mathbb{N}}$. We shall henceforth identify $\mathbb{Z}$ with its image in $\mathbb{Z}_p$ when no confusion may arise.

We can associate, to any p-adic integer, a Schur multiplier of $\mathbb{Z}\left[\frac{1}{p}\right]$, i.e. a map $\xi_j: \mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right] \to \mathbb{Z}$ which satisfies the (additive) 2-cocycle identity, in the
following manner:

Theorem (2.2.14)[68]: [78] Let $p \in \mathbb{N}, p > 1$ and let $J = (J_k)_{k\in\mathbb{N}} \in \mathbb{Z}_p$. Define the map $\xi_J : \mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right] \to \mathbb{Z}$ by setting, for any $\frac{q_1}{p^{k_1}}, \frac{q_2}{p^{k_2}} \in \mathbb{Z}\left[\frac{1}{p}\right]$:

$$\xi_J\left(\frac{q_1}{p^{k_1}}, \frac{q_2}{p^{k_2}}\right) = \begin{cases} -\frac{q_1}{p^{k_1}}\left(J_{k_2} - J_{k_1}\right) \text{ if } k_2 > k_1, \\ -\frac{q_2}{p^{k_2}}\left(J_{k_1} - J_{k_2}\right) \text{ if } k_1 > k_2, \\ \frac{q}{p^r}\left(J_{k_2} - J_{k_1}\right) \text{ if } k_1 = k_2, \text{ with } \frac{q}{p^r} = \frac{q_1}{p^{k_1}} + \frac{q_2}{p^{k_2}}, \end{cases}$$

where all fractions are written in their reduced form, i.e. such that the exponent of p at the denominator is minimal (this form is unique). Then:

(a) ξ_J is a Schur multiplier of $\mathbb{Z}\left[\frac{1}{p}\right]$ [78].

(b) For any $J, K \in \mathbb{Z}_p$, the Schur multipliers ξ_J and ξ_K are cohomologous if, and only if $J - K \in \mathbb{Z}$ [78].

(c) Any Schur multiplier of $\mathbb{Z}\left[\frac{1}{p}\right]$ is cohomologous to ξ_J for some $J \in \mathbb{Z}_p$ [78].

In particular, Ext $\left(\mathbb{Z}\left[\frac{1}{p}\right], \mathbb{Z}\right)$ is isomorphic to $\mathbb{Z}_p/\mathbb{Z}$.

Schur multipliers provide us with a mean to describe and classify Abelian extensions of $\mathbb{Z}\left[\frac{1}{p}\right]$ by $\mathbb{Z}_p$. The interest in Theorem (2.2.14) lies in the remarkable observation that the K_0 groups of noncommutative solenoids are exactly given by these extensions:

Theorem (2.2.15)[68]: [78] Let $p \in \mathbb{N}, p > 1$ and let $\alpha = (\alpha_k)_{k\in\mathbb{N}} \in \Xi_p$. For any $k \in \mathbb{N}$, define $J_k = p^k \alpha_k - \alpha_0$, and note that by construction, $J \in \mathbb{Z}_p$. Let ξ_J be the Schur multiplier of $\mathbb{Z}\left[\frac{1}{p}\right]$ defined in Theorem (2.2.14), and let Q_J be the group with underlying set $\mathbb{Z} \times \mathbb{Z}\left[\frac{1}{p}\right]$ and operation:

$$(z_1, r_1) \boxplus (z_2, r_2) = \left(z_1 + z_2 + \xi_J(r_1, r_2), r_1 + r_2\right)$$

for all $(z_1, r_1), (z_2, r_2) \in \mathbb{Z} \times \mathbb{Z}\left[\frac{1}{p}\right]$. By construction, Q_J is an Abelian extension of $\mathbb{Z}\left[\frac{1}{p}\right]$ by $\mathbb{Z}$ given by the Schur multiplier ξ_J.

Then:

$$K_0\left(\text{Å}_\alpha^{\wp}\right) = Q_J$$

and, moreover, all tracial states of $\text{Å}_\alpha^{\wp}$ lift to a single trace τ on $K_0\left(\text{Å}_\alpha^{\wp}\right)$, characterized by:

$$\tau : (1,0) \mapsto 1 \text{ and } \left(0, \frac{1}{p^k}\right) \mapsto \alpha_k.$$

Furthermore, we have:

$$K_1\left(\text{Å}_\alpha^{\wp}\right) = \mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right].$$

We observe, in particular, that given any Abelian extension of $\mathbb{Z}\left[\frac{1}{p}\right]$ by $\mathbb{Z}$, one can find, by Theorem (2.2.14), a Schur multiplier of $\mathbb{Z}\left[\frac{1}{p}\right]$ of the form ξ_J for some $J \in \mathbb{Z}_p$, and, up to an arbitrary choice of $\alpha_0 \in [0,1)$, one may form the sequence $\alpha = \left(\frac{\alpha_0 + J_k}{p^k}\right)_{k\in\mathbb{N}}$, and check that $\alpha \in \Xi_p$; thus all possible Abelian extensions, and only Abelian extensions of $\mathbb{Z}\left[\frac{1}{p}\right]$ by $\mathbb{Z}$ are given as K_0 groups of noncommutative solenoids. With this observation, the K_0 groups of noncommutative solenoids are uniquely described by a p-adic integer modulo an integer, and the information contained in the pair $\left(K_0\left(\mathring{A}_\alpha^\wp\right), \tau\right)$ of the K_0 group of a noncommutative solenoid and its trace, is contained in the pair (J, α_0) with $J \in \mathbb{Z}_p/\mathbb{Z}$ as defined in Theorem (2.2.15).

Theorem (2.2.16)[68]: [78] Let p, q be two prime numbers and let $\alpha \in \Xi_p$ and $\beta \in \Xi_q$. Then the following are equivalent:

(i) The noncommutative solenoids $\mathring{A}_\alpha^\wp$ and $\mathring{A}_\beta^\wp$ are $*$-isomorphic,

(ii) $p = q$ and a truncated subsequence of α is a truncated subsequence of β or $(1 - \beta_k)_{k\in\mathbb{N}}$.

Theorem (2.2.16) is given in greater generality in [78], where p, q are not assumed prime; the second assertion of the Theorem must however be phrased in a more convoluted manner: essentially, p and q must have the same set of prime factors, and there is an embedding of Ξ_p and Ξ_q in a larger group Ξ, whose elements are still sequences in [0, 1), such that the images of α and β for these embeddings are sub-sequences of a single element of Ξ.

We conclude with an element of the computation of the K_0 groups in Theorem (2.2.15). Given $\gamma = \left(0, \frac{1}{p^k}\right) \in K_0\left(\mathring{A}_\alpha^\wp\right)$, if α_0 is irrational, then there exists a Rieffel-Powers projection in $A_{\alpha_{2k}}$ whose image in $\mathring{A}_\alpha^\wp$ for the embedding given by Theorem (2.2.13) has K_0 class the element γ, whose trace is thus naturally given by Theorem (2.2.15). Much work is needed, however, to identify the range of K_0 as the set of all Abelian extensions of $\mathbb{Z}\left[\frac{1}{p}\right]$ by $\mathbb{Z}$, and parametrize these, in turn, by $\mathbb{Z}_p/\mathbb{Z}$, as we have shown.

We now turn to the question of the structure of the category of modules over noncommutative solenoids. We show how to apply some constructions of equivalence bimodules to the case of noncommutative solenoids as a first step toward solving the still open problem of Morita equivalence for noncommutative solenoids.

Projective modules for rotation algebras and higher dimensional noncommutative tori were studied by M. Rieffel ([84]). F. Luef has extended this work to build modules with a dense subspace of functions coming from modulation spaces (e.g., Feichtinger's algebra) with nice properties ([79], [80]). One approach to building projective modules over noncommutative solenoids is to build the projective modules from the "inside out".

We first make some straightforward observations in this direction. We recall that, by Notation (2.2.5), for any $p \in \mathbb{N}, p > 1$, and for any $\alpha \in \Xi_p$, where Ξ_p is defined in Theorem (2.2.4), the C^*-algebra $C^*\left(\mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right], \Psi_\alpha\right)$, where the multiplier Ψ_α was defined in Theorem (2.2.4), is denoted by $\mathring{A}_\alpha^\wp$. We will work with p a prime number. Last, we also recall that by Notation (2.2.12), the rotation algebra for the rotation of angle $\theta \in [0,1)$ is denoted by A_θ, while its canonical unitary generators are denoted by U_θ and V_θ, so that $U_\theta V_\theta = e^{2i\pi\theta} V_\theta U_\theta$.

Theorem (2.2.15) describes the K_0 groups of noncommutative solenoids, and, among other conclusions, state that there always exists a unique trace on the K_0 of any noncommutative solenoid, lifted from any tracial state on the C^*-algebra itself. We state:

Proposition (2.2.17)[68]: Let p be a prime number, and fix $\alpha \in \Xi_p$, with $\alpha_0 \notin \mathbb{Q}$. Let $\gamma = z + q\alpha_N$ for some $z, q \in \mathbb{Z}$ and $N \in \mathbb{N}$, with $\gamma > 0$. Then there is a left projective module over $\mathring{A}_\alpha^\wp$ whose K_0 class has trace γ, or equivalently, whose K_0 class is given by $\left(z, \frac{q}{p^k}\right) \in \mathbb{Z} \times \mathbb{Z}\left[\frac{1}{p}\right]$.

Proof. γ is the image of some class in $K_0(\mathring{A}_\alpha^\wp)$ for the trace on this group. Now, since $\alpha_{N+1} = p\alpha_N + j$ for some $j \in \mathbb{Z}$ by definition of Ξ_α, we may as well assume N is even. As $K_0(\mathring{A}_\alpha^\wp)$ is the inductive limit of $K_0\left(A_{\alpha_N}\right)_{k\in 2\mathbb{N}}$ by Theorem (2.2.13), γ is the trace of an element of $K_0(A_{\alpha_N})$), where A_{α_N} is identified as a subalgebra of $\mathring{A}_\alpha^\wp$ (again using Theorem (2.2.13). By [82], there is a projection P_γ in A_{α_N} whose K_0 class has trace γ, and it is then easy to check that the left projective module $P\mathring{A}_\alpha^\wp$ over $\mathring{A}_\alpha^\wp$ fulfills our proposition.

So, for example, with the notations of the proof of Proposition (2.2.17), if P_γ is a projection in $A_{\alpha_n} \subset \mathring{A}_\alpha^\wp$ with trace $\gamma \in (0,1)$, one can construct the equivalence bimodule

$$\mathring{A}_\alpha^\wp - \mathring{A}_\alpha^\wp P_\gamma - P_\gamma \mathring{A}_\alpha^\wp P_\gamma.$$

From this realization, not much about the structure of $P_\gamma \mathring{A}_\alpha^\wp P_\gamma$ can be seen, although it is possible to write this C^*-algebra as a direct limit of rotation algebras. Let us now discuss this matter.

Suppose we have two directed sequences of C^*-algebras:

$$A_0 \overset{\varphi_0}{\to} A_1 \overset{\varphi_1}{\to} A_2 \overset{\varphi_2}{\to} \dots$$

and

$$B_0 \overset{\psi_0}{\to} B_1 \overset{\psi_1}{\to} B_2 \overset{\psi_2}{\to} \dots$$

Suppose further that for each $n \in \mathbb{N}$ there is an equivalence bimodule X_n between A_n and B_n

$$A_n - X_n - B_n,$$

and that the $(X_n)_{n\in\mathbb{N}}$ form a directed system, in the following sense: there exists a direct system of module monomorphisms

$$X_0 \overset{i_0}{\to} X_1 \overset{i_1}{\to} X_2 \overset{i_2}{\to} \dots$$

satisfying, for all $f, g \in X_n$ and $b \in B_n$:

$$\langle i_n(f), i_n(g)\rangle_{B_{n+1}} = \psi_n\left(\langle f, g\rangle_{B_n}\right)$$

and

$$i_n(f) = i_n(f) \cdot \psi_n(b),$$

with analogous but symmetric equalities holding for the X_n viewed as left-A_n modules.

Now let $\mathcal{A}$ be the direct limit of $(A_n)_{n\in\mathbb{N}}$, $\mathcal{B}$ be the direct limit of $(B_n)_{n\in\mathbb{N}}$ and χ be the direct limit of $(X_n)_{n\in\mathbb{N}}$ (completed in the natural C^*-module norm). Then χ is an $\mathcal{A} - \mathcal{B}$ bimodule. If one further assumes that the algebra of adjointable operators on χ viewed as a $\mathcal{A} - \mathcal{B}$ bimodule, $\mathcal{L}(\chi)$, can be obtained via an appropriate limiting process from the sequence of adjointable operators $\{\mathcal{L}(X_n)\}_{n=1}^\infty$ (where each X_n is a $A_n - B_n$ bimodule), then in addition one has that χ is a strong Morita equivalence bimodule between $\mathcal{A}$ and $\mathcal{B}$.

So suppose that $\gamma \in (0,1)$ is as in the statement of Proposition (2.2.17), for some $\alpha \in \Xi_p$ not equal to zero, and suppose that we know that there is a positive integer N and a projection P_γ in A_{α_N} whose K_0 class has trace γ. Again, without loss of generality, we assume that N is even. Then setting

$$A_n = A_{\alpha_{N+2n}}, X_n = A_{\alpha_{N+2n}} P_\gamma, \text{ and } B_n = P_\gamma A_{\alpha_{N+2n}} P_\gamma,$$

all of the conditions in the above paragraphs hold a priori, since $\mathring{A}_\alpha^\wp$ is a direct limit of the $A_{\alpha_{N+2n}}$, so that certainly $\mathcal{B} = P_\gamma \mathring{A}_\alpha^\wp P_\gamma$ is a direct limit of the $P_\gamma A_{\alpha_{N+2n}} P_\gamma$, and $\chi = \mathring{A}_\alpha^\wp P_\gamma$ can be expressed as a direct limit of the $X_n = A_{\alpha_{N+2n}} P_\gamma$, again by construction, with the desired conditions on the adjointable operators satisfied by construction.

We discuss very simple examples, to show how the directed system of bimodules is constructed.

Example (2.2.18)[68]: Fix an irrational $\alpha_0 \in [0,1)$, let $p = 2$, and consider $\alpha \in \Xi_2$ given by

$$\alpha = \left(\alpha_0, \alpha_1 = \frac{\alpha_0}{2}, \alpha_2 = \frac{\alpha_0}{4}, \dots, \alpha_n = \frac{\alpha_n}{2^n}, \dots, \right),$$

Consider $A_{\alpha_0} \in A_{\alpha_0} \subset A_{\alpha_1}$ a projection of trace $\alpha_0 = 2\alpha_1$. The bimodule

$$A_{\alpha_0} - A_{\alpha_0} \cdot P_{\alpha_0} - P_{\alpha_0} A_{\alpha_0} P_{\alpha_0}$$

is equivalent to Rieffel's bimodule

$$A_{\alpha_0} - \overline{C_c(\mathbb{R})} - A_{\frac{1}{\alpha_0}} = B_0.$$

Let $\beta_0 = \frac{1}{\alpha_0}$. Rieffel's theory, specifically Theorem 1.1 of [83], again shows there is a bimodule

$$A_{\alpha_2} - A_{\alpha_2} \cdot P_{\alpha_0} - P_{\alpha_0} A_{\alpha_2} P_{\alpha_0}$$

is the same as

$$A_{\alpha_2} - A_{\alpha_2} \cdot P_{4\alpha_2} - P_{4\alpha_2} A_{\alpha_2} P_{4\alpha_4}$$

which is equivalent to Rieffel's bimodule

$$A_{\alpha_2} - \overline{C_c(\mathbb{R} \times F_4)} - C(\mathbb{T} \times F_4) \rtimes_{\tau_1} \mathbb{Z} = B_1,$$

where $F_4 = \mathbb{Z}/4\mathbb{Z}$, and the action of $\mathbb{Z}$ on $\mathbb{T} \times F_4$ is given by multiples of $\left(\frac{\beta_2}{4}, [1]_{F_4}\right)$, for $\beta_2 = \frac{1}{\alpha_2}$, i.e. multiples of $\left(\frac{1}{\alpha_2}, [1]_{F_4}]\right)$, i.e. multiples of $\left(\beta_0, [1]_{F_4}\right)$.

At the n^{th} stage, using Theorem 1.1 of [83] again, we see that

$$A_{\alpha_{2n}} - A_{\alpha_{2n}} \cdot P_{\alpha_0} - P_{\alpha_0} A_{\alpha_{2n}} P_{\alpha_0}$$

is the same as

$$A_{\alpha_{2n}} - A_{\alpha_{2n}} \cdot P_{2^n \alpha_n} - P_{2^n \alpha_n} A_{\alpha_n} P_{2^n \alpha_n}$$

which is equivalent to

$$A_{\alpha_n} - \overline{C_c(\mathbb{R} \times F_{4^n})} - C(\mathbb{T} \times F_{4^n}) \rtimes_{\tau_n} \mathbb{Z} = B_n,$$

where the action of $\mathbb{Z}$ on $\mathbb{T} \times F_{4^n}$ is given by multiples of $\left(\frac{\beta_{2n}}{4^n}, [1]_{F_{4^n}}\right)$, for $\beta_{2n} = \frac{1}{\alpha_{2n}} = \frac{4^n}{\alpha_0}$, i.e. multiples of $\left(\frac{1}{\alpha_2}, [1]_{F_{4^n}}\right)$, i.e. multiples of $\left(\beta_0, [1]_{F_{4^n}}\right)$, for $F_{4^n} = \mathbb{Z}/4^n\mathbb{Z}$.

From calculating the embeddings, we see that for $\alpha = \left(\alpha_0, \frac{\alpha_0}{2}, \dots, \frac{\alpha}{2^n}, \dots\right) \in \Xi_2$, we have that

$$\mathcal{A}_\alpha^{\mathcal{S}}$$

is strongly Morita equivalent to a direct limit $\mathcal{B}$ of the B_n. The structure of $\mathcal{B}$ is not clear in this description, although each B_n is seen to be a variant of a rotation algebra. As expected, one calculates

$$tr\left(K_0(\mathcal{A}_\alpha^{\mathcal{S}})\right) = \alpha_0 \cdot tr\big(K_0(\mathcal{B})\big).$$

Under certain conditions, one can construct equivalence bimodules for $\mathring{A}_\alpha^{\wp}$ ($\alpha \in \Xi_p, p$ prime) by using a construction of M. Rieffel [84]. The idea is to first embed $\Gamma = \mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right]$ as a co-compact 'lattice' in a larger group M, and the quotient group M/Γ will be exactly the solenoid $\wp_p$.

We start with a brief description of the field of p-adic numbers, with p prime. Algebraically, the field $\mathbb{Q}_p$ is the field of fraction of the ring of p-adic integers $\mathbb{Z}_p$ — we introduce $\mathbb{Z}_p$ as a group, though there is a natural multiplication on $\mathbb{Z}_p$ turning it into a ring. A more analytic approach is to consider $\mathbb{Q}_p$ as the completion of the field Q for the p-adic metric d_p, defined by $d_p(r, r') = |r - r'|_p$ for any $r, r' \in \mathbb{Q}$, where $|\cdot|_p$ is the p-adic norm defined by:

$$|r| = \begin{cases} p^{-n} & \text{if } r \neq 0 \text{ and where } r = p^n \dfrac{a}{b} \text{ with } a, b \text{ are both relatively prime with } p, \\ 0 & \text{if } r = 0. \end{cases}$$

If we endow $\mathbb{Q}$ with the metric d_p, then series of the form:

$$\sum_{j=k}^{\infty} a_j p^j$$

will converge, for any $k \in \mathbb{Z}$ and $a_j \in \{0, \dots, p-1\}$ for all $j = k, \dots$. This is the p-adic expansion of a p-adic number. One may easily check that addition and multiplication on $\mathbb{Q}$ are uniformly continuous for d_p and thus extend uniquely to $\mathbb{Q}_p$ to give it the structure of a field. Moreover, one may check that the group $\mathbb{Z}_p$ of padic integer defined in $\mathbb{Q}_p$ as the group of p-adic numbers of the form $\sum_{j=0}^{\infty} a_j p^j$ with $a_j \in \{0, \dots, p-1\}$ for all $j \in \mathbb{N}$. Now, with this embedding, one could also check that $\mathbb{Z}_p$ is indeed a subring of $\mathbb{Q}_p$ whose field of fractions is $\mathbb{Q}_p$ (i.e. $\mathbb{Q}_p$ is the smallest field containing $\mathbb{Z}_p$ as a subring) and thus, both constructions described. Last, the quotient of the (additive) group $\mathbb{Q}_p$ by its subgroup $\mathbb{Z}_p$ is the Prüfer p-group $\mathbb{Z}(p^\infty) = \{z \in \mathbb{T} : \exists n \in \mathbb{N} \quad z^{(p^n)} = 1\}$.

Since $\mathbb{Q}_p$ is a metric completion of $\mathbb{Q}$ and $\mathbb{Z}\left[\frac{1}{p}\right]$ is a subgroup of $\mathbb{Q}$, we shall identify, $\mathbb{Z}\left[\frac{1}{p}\right]$ as a subgroup of $\mathbb{Q}_p$ with no further mention. We now define a few group homomorphisms to construct a short exact sequence at the core of our construction.

Let $\omega : \mathbb{R} \to \wp_p$ be the standard "winding line" defined for any $t \in \mathbb{R}$ by:

$$\omega(t) = \left(e^{2\pi i t}, e^{2\pi i \frac{t}{p}}, e^{2\pi i \frac{t}{p^2}}, \dots, e^{2\pi i \frac{t}{p^n}}, \dots\right).$$

Let $\gamma \in \mathbb{Q}_p$ and write $\gamma = \sum_{j=k}^{\infty} a_j p^j$ for a (unique) family $(a_j)_{j=k,\dots}$ of elements in $\{0, \dots, p-1\}$. We define the sequence $\zeta(\gamma)$ by setting for all $j \in \mathbb{N}$:

$$\zeta_j(\gamma) = e^{2\pi i \left(\sum_{m=k}^{j} \frac{a_m}{p^{j-k}}\right)}$$

with the convention that $\sum_j^k \dots$ is zero if $k < j$.

We thus may define the map

$$\Pi : \begin{cases} \mathbb{Q}_p \times \mathbb{R} \longrightarrow \wp_p \\ \gamma \longmapsto \Pi(\gamma, t) = \zeta_j(\gamma) \cdot \omega(t). \end{cases}$$

If we set

$$\iota : \begin{cases} \mathbb{Z}\left[\dfrac{1}{p}\right] \to \mathbb{Q}_p \times \mathbb{R} \\ r \longmapsto \iota(r) = (r, -r), \end{cases}$$

then one checks that the following is an exact sequence:

$$1 \longrightarrow \mathbb{Z}\left[\frac{1}{p}\right] \xrightarrow{\iota} \mathbb{Q}_p \times \mathbb{R} \xrightarrow{\Pi} \wp_p \to 1$$

It follows that there is an exact sequence

$$1 \longrightarrow \mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right] \to [\mathbb{Q}_p \times \mathbb{R}] \times [\mathbb{Q}_p \times \mathbb{R}] \to \wp_p \times \wp_p \to 1.$$

Indeed, we will show later that it is possible to perturb the embeddings of the different terms in $\mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right]$ by elements of $\mathbb{Q}_p \backslash \{0\}$ and $\mathbb{R}\backslash\{0\}$ to obtain a family of different embeddings of $\mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right]$ into $[\mathbb{Q}_p \times \mathbb{R}]^2$.

We now observe that $M = \mathbb{Q}_p \times \mathbb{R}$ is self-dual.

Notation (2.2.19)[68]: The Pontryagin dual of a locally compact group G is denoted by $\hat{G}$. The dual pairing between a group and its dual is denoted by $\langle \cdot,\cdot \rangle : G \times \hat{G} \to \mathbb{T}$.

Let us show that $M \cong \widehat{M}$. To every $x \in \mathbb{Q}_p$, we can associate the character

$$\chi_x : q \in \mathbb{Q}_p \mapsto e^{2i\pi i\{x \cdot q\}}$$

where $\{x \cdot q\}_p$ is the fractional part of the product $x \cdot q$ in $\mathbb{Q}_p$, i.e. it is the sum of the terms involving the negative powers of p in the p-adic expansion of $x \cdot q$. The map $x \in \mathbb{Q}_p \mapsto \chi_x \in \widehat{Q_p}$ is an isomorphism of topological group. Similarly, every character of $\mathbb{R}$ is of the form $\chi_x : t \in \mathbb{R} \mapsto e^{2i\pi rt}$ for some $r \in \mathbb{R}$. Therefore every character of M is given by

$$\chi_{(x,r)} : (q,t) \in \mathbb{Q}_p \times \mathbb{R} \longmapsto \chi_x(q)\chi_r(t)$$

for some $(x,r) \in \mathbb{Q}_p \times \mathbb{R}$ (see [74]) for further details on characters of specific locally compact abelian groups). It is possible to check that the map $(x,r) \mapsto \chi_{(x,r)}$ is a group isomorphism between M and $\widehat{M}$, so that $M = \mathbb{Q}_p \times \mathbb{R}$ is indeed self-dual.

We write $\Gamma = \mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right]$ where p is some prime number, and now let $M = [\mathbb{Q}_p \times \mathbb{R}]$. We have shown that M is self-dual, since both $\mathbb{Q}_p$ and $\mathbb{R}$ are self-dual. Now suppose there is an embedding $\iota : \Gamma \to M \times \widehat{M}$. Let the image $\iota(\Gamma)$ be denoted by D. In the case we are considering, D is a discrete co-compact subgroup of $M \times \widehat{M}$. Following the method of M. Rieffel [84], the Heisenberg multiplier $\eta : (M \times \widehat{M}) \times (M \times \widehat{M}) \to \mathbb{T}$ is defined by:

$$\eta((m,s),(n,t)) = \langle m,t \rangle, (m,s),(n,t) \in M \times \widehat{M}.$$

(We note we use the Greek letter 'η' rather than Rieffel's 'β', because we have used 'β' elsewhere. Following Rieffel, the symmetrized version of η is denoted by the letter ρ, and is the multiplier defined by:

$$\rho((m,s),(n,t)) = \eta((m,s),(n,t))\overline{\eta((n,t),(m,s))}, (m,s),(n,t) \in M \times \widehat{M}.$$

M. Rieffel [84] has shown that $C_C(M)$ can be given the structure of a left $C^*(D,\eta)$ module, as follows. One first constructs an η-representation of $M \times \widehat{M}$ on $L^2(M)$, defined as π, where

$$\pi_{(m,s)}(f)(n) = \langle n,s \rangle f(n+m), (m,s) \in M \times \widehat{M}, n \in M.$$

When the representation π is restricted to D, we still have a projective η-representation of D, on $L^2(M)$, and its integrated form gives $C_C(M)$ the structure of a left $C^*(D,\eta)$ module, i.e. for $\Phi \in C_C(D,\eta), f \in C_C(M)$,

$$\pi(\Phi) \cdot f(n) = \sum_{(d,\chi)\in D} \Phi((d,\chi))\pi_{(d,\chi)}(f)(n) = \sum_{(d,\chi)\in D} \Phi((d,\chi))\langle n,\chi \rangle f(n+d).$$

There is also a $C_C(D,\eta)$ valued inner product defined on $C_C(M)$ given by:

$$\langle f,g \rangle_{C_C(D,\eta)} = \int_M f(n)\overline{\pi_{(d,\chi)}(g)(n)}dn = \int_M f(n)\overline{\langle n,\chi \rangle g(n+d)}dn.$$

Moreover, Rieffel has shown that setting

$$D^{\perp} = \{(n,t) \in M \times \widehat{M} : \forall (m,s) \in D \ \ \rho((m,s),(n,t)) = 1\},$$
$C_C(M)$ has the structure of a right $C^*(D^{\perp}, \bar{\eta})$ module. Here the right module structure is given for all $f \in C_C(M), \Omega \in C_C(D^{\perp})$ and $n \in M$ by:
$$f \cdot \Omega(n) = \sum_{(c,\xi) \in D^{\perp}} \pi^*_{(c,\xi)}(f)(n)\Omega(c,\xi),$$
and the $C_C(D^{\perp}, \bar{\eta})$-valued inner product is given by
$$\langle f, g\rangle_{C_C(D^{\perp},\bar{\eta})}(c,\xi) = \int_M \overline{f(n)}\pi_{(c,\xi)}(g)(n)dn = \int_M \overline{f(n)}\langle n,\xi\rangle g(n+c)dn,$$
where $f, g \in C_C(M), \Omega \in C_C(D^{\perp}, \bar{\eta})$, and $(c,\xi) \in D^{\perp}$.

Moreover, Rieffel shows in [84] that $C^*(D,\eta)$ and $C^*(D^{\perp},\bar{\eta})$ are strongly Morita equivalent, with the equivalence bimodule being the completion of $C_C(M)$ in the norm defined by the above inner products.

In order to construct explicit bimodules, we first define the multiplier η more precisely, and then discuss different embeddings of $\mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right]$ into $M \times \widehat{M}$.

In the case examined here, the Heisenberg multiplier $\eta: [\mathbb{Q}_p \times \mathbb{R}]^2 \times [\mathbb{Q}_p \times \mathbb{R}]^2 \to \mathbb{T}$ is given by:

Definition (2.2.20)[68]: The Heisenberg multiplier $\eta: [\mathbb{Q}_p \times \mathbb{R}]^2 \times [\mathbb{Q}_p \times \mathbb{R}]^2 \to \mathbb{T}$ is defined by
$$\eta[((q_1,r_1),(q_2,r_2)),((q_3,r_3),(q_4,r_4))] = e^{2\pi i r_1 r_4} e^{2\pi i \{q_1 q_4\}_p},$$
where $\{q_1q_4\}_p$ is the fractional part of the product $q_1 \cdot q_4$, i.e. the sum of the terms involving the negative powers of p in the p-adic expansion of q_1q_4.

The following embeddings of $\mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right]$ in $[\mathbb{Q}_p \times \mathbb{R}]^2$ will prove interesting:

Definition (2.2.21)[68]: For $\theta \in \mathbb{R}, \theta \neq 0$, we define $\iota_\theta: \mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right] \to [\mathbb{Q}_p \times \mathbb{R}]^2$ by
$$\iota_\theta(r_1,r_2) = [(r_1, \theta \cdot r_1),(r_2,r_2)].$$
We examine the structure of the multiplier η more precisely and then discuss different embeddings of $\mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right]$ into $[\mathbb{Q}_p \times \mathbb{R}]^2$ and their influence on the different equivalence bimodules they allow us to construct.

We start by observing that for $r_1, r_2, r_3, r_4 \in \mathbb{Z}\left[\frac{1}{p}\right]$:
$$\eta(\iota_\theta(r_1,r_2),\iota_\theta(r_3,r_4)) = e^{2\pi i\{r_1r_4\}_p} e^{2\pi i r_1 r_4} = e^{2\pi i r_1 r_4} e^{2\pi i \theta r_1 r_4} = e^{2\pi i(\theta+1) r_1 r_4}.$$
(Here we used the fact that for $r_i, r_j \in \mathbb{Z}\left(\frac{1}{p}\right), \{r_ir_j\}_p \equiv r_ir_j$ modulo $\mathbb{Z}$.)

One checks that setting $D_\theta = \iota_\theta\left(\mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right]\right)$, the C^*-algebra $C^*(D_\theta,\eta)$ is exactly $*$-isomorphic to the noncommutative solenoid $\mathring{A}^{\wp}_{\alpha}$ for
$$\alpha = \left(\theta+1, \frac{\theta+1}{p}, \cdots, \frac{\theta+1}{p^n}\right) = \left(\frac{\theta+1}{p^n}\right)_{n\in\mathbb{N}}.$$
For this particular embedding of $\mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right]$ as the discrete subgroup D inside $M \times \widehat{M}$, we calculate that
$$D_\theta^{\perp} = \left\{\left(r_1, -\frac{r_1}{\theta}\right), (r_1, -r_2) : r_1, r_2 \in \mathbb{Z}\left[\frac{1}{p}\right]\right\}.$$
Moreover,

$$\bar{\eta}\left(\left[\left(r_1,-\frac{r_1}{\theta}\right),(r_2,-r_2)\right],\left[\left(r_3,-\frac{r_3}{\theta}\right),(r_4,-r_4)\right]\right)=e^{-2\pi i\left(\frac{1}{\theta}+1\right)r_1 r_4}.$$

It is evident that $C^*(D_\theta^\perp,\eta)$ is also a non-commutative solenoid $\mathring{A}_\beta^\wp$ where $\beta = \left(1-\frac{\theta+1}{p^{n\theta}}\right)_{n\in\mathbb{N}}$.

Note that for

$$\alpha=\left(\theta+1,\frac{\theta+1}{p},\cdots,\frac{\theta+1}{p^n},\cdots\right),$$

and

$$\beta=\left(1-\frac{\theta+1}{p^{n\theta}}\right)_{n\in\mathbb{N}},$$

we have

$$\theta\cdot\tau\left(K_0\left(\mathring{A}_\alpha^\wp\right)\right)=\tau\left(K_0\left(\mathring{A}_\beta^\wp\right)\right)$$

with the notations of Theorem (2.2.15). Thus in this case we do see the desired relationship mentioned: the range of the trace on the K_0 groups of the two C^*-algebras are related via multiplication by a positive constant.

We can now generalize our construction above as follows.

Definition (2.2.22)[68]: For any $x\in\mathbb{Q}_p\backslash\{0\}$, and any $\theta\in\mathbb{R}\backslash\{0\}$, there is an embedding

$$\iota_{x,\theta}:\mathbb{Z}\left[\frac{1}{p}\right]\times\mathbb{Z}\left[\frac{1}{p}\right]\to\left[\mathbb{Q}_p\times\mathbb{R}\right]^2$$

defined for all $r_1,r_2\in\mathbb{Z}\left(\frac{1}{p}\right)$ by

$$\iota_{x,\theta}(r_1,r_2)=[(x\cdot r_1,\theta\cdot r_1),(r_2,r_2)].$$

Then, we shall prove that for all $\alpha\in\Xi_p$ there exists $x\in\mathbb{Q}_p\backslash\{0\}$ and $\theta\in\mathbb{R}\backslash\{0\}$ such that, by setting

$$D_{x,\theta}=\iota_{x,\theta}\left(\mathbb{Z}\left[\frac{1}{p}\right]\times\mathbb{Z}\left[\frac{1}{p}\right]\right)$$

the twisted group C^*-algebra $C^*(D,\eta)$ is $*$-isomorphic to $\mathring{A}_\alpha^\wp$.

As a first step, we prove:

Lemma (2.2.23)[68]: Let p be prime, and let $M=\mathbb{Q}_p\times\mathbb{R}$. Let $(x,\theta)\in\left[\mathbb{Q}_p\backslash\{0\}\right]\times[\mathbb{R}\backslash\{0\}]$, and define $\iota_{x,\theta}:\mathbb{Z}\left[\frac{1}{p}\right]\times\mathbb{Z}\left[\frac{1}{p}\right]\to\left[\mathbb{Q}_p\times\mathbb{R}\right]^2\cong M\times\bar{M}$ by:

$$\iota_{x,\theta}(r_1,r_2)=[(x\cdot r_1,\theta\cdot r_1),(r_2,r_2)]\ \ for\ all\ \ r_1,r_2\in\mathbb{Z}\left(\frac{1}{p}\right).$$

Let η denote the Heisenberg cocycle defined on $\left[M\times\widehat{M}\right]^2$ and let

$$D=\iota_{x,\theta}\left(\mathbb{Z}\left[\frac{1}{p}\right]\times\mathbb{Z}\left[\frac{1}{p}\right]\right).$$

Then

$$D_{x,\theta}^\perp=\left\{\left[(t_1,-t_1),\left(x^{-1}t_2,-\frac{t_2}{\theta}\right)\right]:t_1,t_2\in\mathbb{Z}\left[\frac{1}{p}\right]\right\}.$$

Proof. By definition,

$$D_{x,\theta}^{\perp} = \left\{[(q_1,s_1),(q_2,s_2)]: \forall r_1,r_2 \in \mathbb{Z}\left[\frac{1}{p}\right] \ \rho\left(\left[\iota_{x,\theta}(r_1,r_2)\right],[(q_1,s_1),(q_2,s_2)]\right)=1\right\}$$
$$= \left\{[(q_1,s_1),(q_2,s_2)]: \forall r_1,r_2 \in \mathbb{Z}\left[\frac{1}{p}\right] \ \rho([(x\cdot r_1,\theta\cdot r_1),(r_2,r_2)],[(q_1,s_1),(q_2,s_2)])=1\right\}$$
$$= \left\{[(q_1,s_1),(q_2,s_2)]: \forall r_1,r_2 \in \mathbb{Z}\left[\frac{1}{p}\right] \ e^{2\pi i\theta r_1 s_2} e^{2\pi i\{x\cdot r_1 q_2\}_p}\overline{e^{2\pi i s_1 r_2} e^{2\pi i\{q_1 r_2\}_p}} = 1\right\}.$$

Now if $r_2 = 0$, and $r_1 = p^n$, for any $n \in \mathbb{Z}$, this implies

$$\forall n \in \mathbb{Z}\ e^{2\pi i\theta p^n s_2} e^{2\pi i\{x\cdot p^n q_2\}_p} = 1,$$

so that if we choose $s_2 = -\frac{t_2}{\theta}$ for some $t_2 \in \mathbb{Z}\left[\frac{1}{p}\right] \subseteq \mathbb{R}$, we need $q_2 = x^{-1}t_2$.
Likewise, if we take $r_1 = 1$, and $r_2 = p^n$, for any $n \in \mathbb{Z}$, we need (q_1,s_1) such that

$$\forall n \in \mathbb{Z}\ e^{2\pi i s_1 p^n} e^{2\pi i\{q_1 p^n\}_p} = 1.$$

Again fixing $q_1 = t_1 \in \mathbb{Z}\left[\frac{1}{p}\right]$, this forces $s_1 = -t_1$. Thus

$$D_{x,\theta}^{\perp} = \left\{\left[(t_1,-t_1),\left(x^{-1}t_2,-\frac{t_2}{\theta}\right)\right]: t_1,t_2 \in \mathbb{Z}\left[\frac{1}{p}\right]\right\},$$

as we desired to show.
One thus sees that the two C^*-algebras $C^*(D_{x,\theta},\eta)$ and $C^*(D_{x,\theta}^{\perp},\bar{\eta})$ are strongly Morita equivalent (but not isomorphic, in general), and also the proof of this lemma shows that $C^*(D_{x,\theta}^{\perp},\bar{\eta})$ is a noncommutative solenoid.

We can use Lemma (2.2.23) to prove the following Theorem:

Theorem (2.2.24)[68]: Let p be prime, and let $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \Xi_p$, with $\alpha_0 \in (0,1)$. Then there exists $(x,\theta) \in [\mathbb{Q}_p\backslash\{0\}] \times [\mathbb{R}\backslash\{0\}]$ with $C^*(D_{x,\theta},\eta)$ isomorphic to the noncommutative solenoid $\mathring{A}_{\alpha}^{\wp}$, where $D_{x,\theta} = \iota_{x,\theta}\left(\mathbb{Z}\left[\frac{1}{p}\right]\times\mathbb{Z}\left[\frac{1}{p}\right]\right)$. Moreover, the method of Rieffel produces an equivalence bimodule between $\mathring{A}_{\alpha}^{\wp}$ and another unital C^*-algebra $\mathcal{B}$, and B is itself isomorphic to a noncommutative solenoid.

Proof. By definition of Ξ_p, for all $j \in \mathbb{N}$ there exists $b_j \in \{0,\dots,p-1\}$ such that $p\alpha_{j+1} = \alpha_j + b_j$. We construct an element of the p-adic integers, $x = \sum_{j=0}^{\infty} b_j p^j \in \mathbb{Z}_p \subset \mathbb{Q}_p$. Let $\theta = \alpha_0$, and now consider for this specific x and this specific θ the C^*-algebra $C^*(D_{x,\theta},\eta)$. By Definition (2.2.22), $\iota_{x,\theta}(r_1,r_2) = [(x\cdot r_1,\theta\cdot r_1),(r_2,r_2)]$, for $r_1,r_2 \in \mathbb{Z}\left[\frac{1}{p}\right]$. Then

$$\eta\left(\iota_{x,\theta}(r_1,r_2),\iota_{x,\theta}(r_3,r_4)\right) = \eta([(x\cdot r_1,\theta\cdot r_1),(r_2,r_2)],[(x\cdot r_3,\theta\cdot r_3),(r_4,r_4)])$$
$$= e^{2\pi i r_1 r_4} e^{2\pi i\{x r_1 r_4\}_p}, r_1,r_2,r_3,r_4 \in \mathbb{Z}\left[\frac{1}{p}\right],$$

and, setting $r_i = \frac{j_i}{p^{k_i}}, 1 \le i \le 4$, and setting $\theta = \alpha_0$, we obtain

$$\eta\left(\iota_{x,\alpha_0}\left(\frac{j_1}{p^{k_1}},\frac{j_2}{p^{k_2}}\right),\iota_{x,\alpha_0}\left(\frac{j_3}{p^{k_3}},\frac{j_4}{p^{k_4}}\right)\right) = e^{2\pi i r_1 r_4} e^{2\pi i\left\{x\frac{j_1 j_4}{p^{k_1+k_4}}\right\}_p}$$

for all

$$\frac{j_1}{p^{k_1}},\frac{j_2}{p^{k_2}},\frac{j_3}{p^{k_3}},\frac{j_4}{p^{k_4}} \in \mathbb{Z}\left[\frac{1}{p}\right].$$

We now note that the relation $p\alpha_{j+1} = \alpha_j + b_j, b_j \in \{0, \dots, p-1\}$ allows us to prove inductively that

$$\forall n \geq 1 \ \ \alpha_n = \frac{\alpha_0 + \sum_{j=0}^{n-1} b_j p^j}{p^n}.$$

By Theorem (2.2.4), the multiplier Ψ_α on $\mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right]$ is defined by:

$$\Psi_\alpha\left(\left(\frac{j_1}{p^{k_1}}, \frac{j_2}{p^{k_2}}\right), \left(\frac{j_3}{p^{k_3}}, \frac{j_4}{p^{k_4}}\right)\right) = e^{2\pi i\left(\alpha_{(k_1+k_4)} j_1 j_4\right)}$$

$$= e^{2\pi i \frac{\alpha_0 j_1 j_4}{p^{k_1+k_4}}} e^{2\pi i\left(\sum_{j=0}^{k_1+k_4-1} b_j p^j j_1 j_4\right)/p^{k_1+k_2}}.$$

A p-adic calculation now shows that for $\frac{j_1}{p^{k_1}}$ and $\frac{j_4}{p^{k_4}} \in \mathbb{Z}\left[\frac{1}{p}\right]$ and $x = \sum_{j=0}^{\infty} b_j p^j \in \mathbb{Z}_p$, we have $\left\{x \frac{j_1 j_4}{p^{k_1+k_4}}\right\}_p = \left(\sum_{j=0}^{k_1+k_4-1} b_j p^j j_1 j_4\right) \cdot \frac{j_1 j_4}{p^{k_1+k_2}}$ modulo $\mathbb{Z}$, so that

$$e^{2\pi i\left\{x \frac{j_1 j_4}{p^{k_1+k_4}}\right\}_p} = e^{2\pi i\left(\sum_{j=0}^{k_1+k_4-1} b_j p^j j_1 j_4\right)/p^{k_1+k_2}}.$$

We thus obtain

$$\eta\left(\iota_{x,\theta}(r_1, r_2), \iota_{x,\theta}(r_3, r_4)\right) = \Psi_\alpha\left((r_1, r_2), (r_3, r_4)\right)$$

for all $r_1, r_2, r_3, r_4 \in \mathbb{Z}\left[\frac{1}{p}\right]$,as desired.

To prove the final statement of the Theorem, we use Lemma (2.2.23). We have shown $\mathring{A}_\alpha^\wp$ is isomorphic to $C^*\left(D_{x,\theta}, \eta\right)$, and the discussion prior to the statement of Lemma (2.2.23) shows that $C^*\left(D_{x,\theta}, \eta\right)$ is stronglyMorita equivalent to $C^*\left(D_{x,\theta}^\perp, \bar{\eta}\right) = \mathcal{B}$. But the proof of Lemma (2.2.23) gives that $D_{x,\theta}^\perp$ is isomorphic to $\mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right]$, so that $C^*\left(D_{x,\theta}^\perp, \bar{\eta}\right) = \mathcal{B}$ is a noncommutative solenoid, as we desired to show.

Chapter 3
Structure and Continuity of Derivations

We show that if M is of type I_∞ then every derivation on $LS(M)$ (resp. $S(M)$ and $S(M,\tau)$) is inner. Building an extension of a derivation $\delta : M \to LS(M)$ up to a derivation from $LS(M)$ into $LS(M)$, it is further established that any derivation from M into $LS(M)$ is $t(M)$-continuous.

Section (3.1): Various Algebras of Measurable Operators for Type I von Neumann Algebras

Derivations on unbounded operator algebras, in particular on various algebras of measurable operators affiliated with von Neumann algebras, appear to be a very attractive special case of the general theory of unbounded derivations on operator algebras. The present continues the series of [89], [2] devoted to the study and a description of derivations on the algebra $LS(M)$ of locally measurable operators with respect to a von Neumann algebra M and on various subalgebras of $LS(M)$.

For A be an algebra over the complex number. A linear operator $D: A \to A$ is called a derivation if it satisfies the identity $D(xy) = D(x)y + xD(y)$ for all $x, y \in A$ (Leibniz rule). Each element $a \in A$ defines a derivation D_a on A given as $D_a(x) = ax - xa, x \in A$. Such derivations D_a are said to be inner derivations. If the element a implementing the derivation D_a on A, belongs to a larger algebra B, containing A (as a proper ideal as usual) then D_a is called a spatial derivation.

In the particular case where A is commutative, inner derivations are identically zero, i.e. trivial. One of the main problems in the theory of derivations is automatic innerness or spatialness of derivations and the existence of noninner derivations (in particular, nontrivial derivations on commutative algebras).

On this way A.F. Ber, F.A. Sukochev, V.I. Chilin [90] obtained necessary and sufficient conditions for the existence of nontrivial derivations on commutative regular algebras. In particular they have proved that the algebra $L^0(0,1)$ of all (classes of equivalence of) complex measurable functions on the interval (0,1) admits nontrivial derivations. Independently A.G. Kusraev [99] by means of Boolean-valued analysis has established necessary and sufficient conditions for the existence of nontrivial derivations and automorphisms on universally complete complex f-algebras. In particular he has also proved the existence of nontrivial derivations and automorphisms on $L^0(0,1)$. It is clear that these derivations are discontinuous in the measure topology, and therefore they are neither inner nor spatial. It seems that the existence of such pathological example of derivations deeply depends on the commutativity of the underlying von Neumann algebra M. In this connection the study of the above problems in the noncommutative case [89], [2], by considering derivations on the algebra $LS(M)$ of all locally measurable operators with respect to a semi-finite von Neumann algebra M and on various subalgebras of $LS(M)$. Recently another approach to similar problems in the framework of type I $AW*$-algebras has been outlined in [93].

The main result of [2] states that if M is a type I von Neumann algebra, then every derivation D on LS(M) which is identically zero on the center Z of the von Neumann algebra M (i.e. which is Z-linear) is automatically inner, i.e. $D(x) = ax - xa$ for an appropriate $a \in LS(M)$. In [2] we also gave a construction of noninner derivations D_δ on the algebra $LS(M)$ for type I_{fin} von Neumann algebra M with nonatomic center Z, where δ is a nontrivial derivation on the algebra $LS(Z)$ (i.e. on the center of $LS(M)$) which is isomorphic with the algebra $L^0(\Omega, \sum, \mu)$ of all measurable functions on a nonatomic measure space $(\Omega, \sum, \mu)$.

The main idea of the mentioned construction is the following.

Let A be a commutative algebra and let $M_n(A)$ be the algebra of $n \times n$ matrices over A. If $e_{i,j}, i,j = \overline{1,n}$, are the matrix units in $M_n(A)$, then each element $x \in M_n(A)$ has the form

$$x = \sum_{i,j=1}^{n} \lambda_{ij} e_{ij}, \qquad \lambda_{i,j} \in A, \qquad i,j = \overline{1,n}.$$

Let $\delta: A \to A$ be a derivation. Setting

$$D_\delta = \left(\sum_{i,j=1}^{n} \lambda_{ij} e_{ij} \right) = \sum_{i,j=1}^{n} \delta(\lambda_{ij}) e_{ij}$$

we obtain a well-defined linear operator D_δ on the algebra $M_n(A)$. Moreover D_δ is a derivation on the algebra $M_n(A)$ and its restriction onto the center of the algebra $M_n(A)$ coincides with the given δ.

In [89] we have proved spatialness of all derivations on the noncommutative Arens algebra $L^\omega(M, \tau)$ associated with an arbitrary von Neumann algebra M and a faithful normal semi-finite trace τ. Moreover if the trace τ is finite then every derivation on $L^\omega(M, \tau)$ is inner.

We give a complete description of all derivations on the algebra $LS(M)$ of all locally measurable operators affiliated with a type I von Neumann algebra M, and also on its subalgebras $S(M)$—of measurable operators and $S(M, \tau)$ of τ-measurable operators, where τ is a faithful normal semi-finite trace on M. We prove that the above mentioned construction of derivations D_δ from [2] gives the general form of pathological derivations on these algebras and these exist only in the type I_{fin} case, while for type I_∞ von Neumann algebras M all derivations on $LS(M), S(M)$ and $S(M, \tau)$ are inner. We prove that an arbitrary derivation D on each of these algebras can be uniquely decomposed into the sum $D = D_a + D_\delta$ where the derivation D_a is inner (for $LS(M), S(M)$ and $S(M, \tau)$) while the derivation D_δ is constructed in the above mentioned manner from a nontrivial derivation δ on the center of the corresponding algebra.

We give necessary definition and preliminaries from the theory of measurable operators, Hilbert–Kaplansky modules and also prove some key results concerning the structure of the algebra of locally measurable operators affiliated with a type I von Neumann algebra.

We describe derivations on the algebra $LS(M)$ of all locally measurable operators for a type I von Neumann algebra M.

We devoted to derivation respectively on the algebra $S(M)$ of all measurable operators and on the algebra $S(M, \tau)$ of all τ-measurable operators with respect to M, where M is a type I von Neumann algebra and τ is a faithful normal semi-finite trace on M.

Finally, contains an application of the above results to the description of the first cohomology group for the considered algebras.

Let H be a complex Hilbert space and let $B(H)$ be the algebra of all bounded linear operators on H. Consider a von Neumann algebra M in $B(H)$ with the operator norm $\|\cdot\|_M$. Denote by $P(M)$ the lattice of projections in M.

A linear subspace $\mathcal{D}$ in H is said to be affiliated with M (denoted as $\mathcal{D}\eta M$), if $u(\mathcal{D}) \subset \mathcal{D}$ for every unitary u from the commutant

$$M' = \{y \in B(H) : xy = yx, \forall x \in M\}$$
of the von Neumann algebra M.

A linear operator x on H with the domain $\mathcal{D}(x)$ is said to be affiliated with M (denoted as $x\eta M$) if $\mathcal{D}(x)\eta M$ and $u\big(u(\xi)\big) = x(u(\xi))$ for all $\xi \in \mathcal{D}(x)$.

A linear subspace $\mathcal{D}$ in H is said to be strongly dense in H with respect to the von Neumann algebra M, if
(i) $\mathcal{D}\eta M$;
(ii) there exists a sequence of projections $\{p_n\}_{n=1}^{\infty}$ in $P(M)$ such that $p_n \uparrow \mathbf{1}, p_n(H) \subset \mathcal{D}$ and $p_n^{\perp} = \mathbf{1} - p_n$ is finite in M for all $n \in \mathbb{N}$, where $\mathbf{1}$ is the identity in M.

A closed linear operator x acting in the Hilbert space H is said to be measurable with respect to the von Neumann algebra M, if $x\eta M$ and $\mathcal{D}(x)$ is strongly dense in H. Denote by $S(M)$ the set of all measurable operators with respect to M.

A closed linear operator x in H is said to be locally measurable with respect to the von Neumann algebra M, if $x\eta M$ and there exists a sequence $\{z_n\}_{n=1}^{\infty}$ of central projections in M such that $z_n \uparrow \mathbf{1}$ and $z_n x \in S(M)$ for all $n \in \mathbb{N}$.

It is well known [100] that the set $LS(M)$ of all locally measurable operators with respect to M is a unital $*$-algebra when equipped with the algebraic operations of strong addition and multiplication and taking the adjoint of an operator.

Let τ be a faithful normal semi-finite trace on M. We recall that a closed linear operator x is said to be τ-measurable with respect to the von Neumann algebra M, if $x\eta M$ and $\mathcal{D}(x)$ is τ-dense in H, i.e. $\mathcal{D}(x)\eta M$ and given $\varepsilon > 0$ there exists a projection $p \in M$ such that $p(H) \subset \mathcal{D}(x)$ and $\tau(p^{\perp})\varepsilon$. The set $S(M,\tau)$ of all τ-measurable operators with respect to M is a solid $*$-subalgebra in $S(M)$ (see [101]).

Consider the topology $t\tau$ of convergence in measure or measure topology on $S(M,\tau)$, which is defined by the following neighborhoods of zero:
$$V(\varepsilon,\delta) = \{x \in S(M,\tau) : \exists e \in P(M), \tau(e^{\perp}) \le \delta, xe \in M, \ \|xe\|_M \le \varepsilon\},$$
where ε, δ are positive numbers.

It is well known [101] that $S(M,\tau)$ equipped with the measure topology is a complete metrizable topological $*$-algebra.

Note that if the trace τ is a finite then
$$S(M,\tau) = S(M) = LS(M).$$
The following result describes one of the most important properties of the algebra $LS(M)$ (see [100], [102]).

Proposition (3.1.1)[3]: Suppose that the von Neumann algebra M is the C^*-product of the von Neumann algebras $M_i, i \in I$, where I is an arbitrary set of indices, i.e.
$$M = \bigoplus_{i\in I} M_i = \left\{\{x_i\}_{i\in I} : x_i \in M_i, i \in I, \sup_{i\in I}\|x_i\|_{M_i} < \infty\right\}$$
with coordinate-wise algebraic operations and involution and with the C^*-norm $\|\{x_i\}_{i\in I}\|_M = \sup_{i\in I}\|x_i\|_{M_i}$. Then the algebra $LS(M)$ is $*$-isomorphic to the algebra $\prod_{i\in I} LS(M_i)$ (with the coordinate-wise operations and involution), i.e.
$$LS(M) \cong \prod_{i\in I} LS(M_i)$$
($\cong$ denoting $*$-isomorphism of algebras). In particular, if M is a finite, then
$$S(M) \cong \prod_{i\in I} S(M_i).$$

It should be noted that similar isomorphisms are not valid in general for the algebras $S(M), S(M,\tau)$ (see [100]).

Proposition (3.1.1) implies that given any family $\{z_i\}_{i\in I}$ of mutually orthogonal central projections in M with $\bigvee_{i\in I} z_i = \mathbf{1}$ and a family of elements $\{x_i\}_{i\in I}$ in $LS(M)$ there exists a unique element $x \in LS(M)$ such that $z_i x = z_i x_i$ for all $i \in I$. This element is denoted by $x = \sum_{i\in I} z_i x_i$.

We shall prove several crucial results concerning the properties of algebras of measurable operators for type I von Neumann algebras. In particular we present an alternative and shorter proof of the statement that the algebra of locally measurable operators in this case is isomorphic to the algebra of bounded operators acting on a Hilbert–Kaplansky module (cf. [2]).

It is well known [31] that every commutative von Neumann algebra M is $*$-isomorphic to the algebra $L^\infty(\Omega) = L^\infty(\Omega, \sum, \mu)$ of all (classes of equivalence of) complex essentially bounded measurable functions on a measure space $(\Omega, \sum, \mu)$ and in this case $LS(M) = S(M) \cong L^0(\Omega)$, where $L^0(\Omega) = L^0(\Omega, \sum, \mu)$ the algebra of all (classes of equivalence of) complex measurable functions on $(\Omega, \sum, \mu)$.

Further we shall need the description of the centers of the algebras $S(M)$ and $S(M,\tau)$ for type I von Neumann algebras.

It should be noted, that if M is a finite von Neumann algebra with a faithful normal semi-finite trace τ, then the restriction τ_Z of the trace τ onto the center Z of M is also semi-finite.

Indeed by [33] M admits the canonical center valued trace $T: M \to Z$. It is known that $T(x) = \overline{co}\{uxu^*, u \in U\} \cap Z$, where $\overline{co}\{uxu^*, u \in U\}$ denotes the norm closure in M of the convex hull of the set $\{uxu^*, u \in U\}$ and U is the set of all unitaries from M. Therefore given any finite trace ρ (since it is norm-continuous and linear on M) one has $\rho(Tx) = \rho(x)$ for all $x \in M$. Given a normal semi-finite trace τ on M there exists a monotone increasing net $\{e_\alpha\}$ of projection in M with $\tau(e_\alpha) < \infty$ and $e_\alpha \uparrow \mathbf{1}$. The trace $\rho_\alpha(x) = \tau(e_\alpha x), x \geq 0, x \in M$, is finite for any α and therefore for all $x \in M, x \geq 0$, we have $\tau(Tx) = \lim_\alpha \tau(e_\alpha Tx) = \lim_\alpha \tau(e_\alpha x) = \tau(x)$. Now given any projection $z \in Z$ there exists a non-zero projection $p \in M$ such that $p \leq z$ and $\tau(p) < \infty$. Consider the element $T(p) \in Z$. From properties of T it follows that $T(p)$ is a non-zero positive element in Z with $\tau(T(p))\tau(p) < \infty$ and $T(p) \leq T(z) = z$. By the spectral theorem there exists a nonzero projection z_0 in Z such that $z_0 \leq \lambda T(p)$ for an appropriate positive number λ. Therefore $\tau(z_0) \leq \tau\big(\lambda T(p)\big) = \lambda\tau(p) < \infty$ and $z_0 \lambda z$, i.e. $z_0 \leq z$, and thus the restriction of τ onto Z is also semi-finite.

Proposition (3.1.2)[3]: If M is finite von Neumann algebra of type I with the center Z and a faithful normal semi-finite trace τ, then $Z\big(S(M)\big) = S(Z)$ and $Z\big(S(M,\tau)\big) = S(Z,\tau_Z)$, where τ_Z is the restriction of the trace τ on Z.

Proof. Given $x \in S(Z)$ take a sequence of orthogonal projections $\{z_n\}$ in Z such that $z_n x \in Z$ for all n. Since M is finite, one has that $LS(M) = S(M)$ and therefore $x = \sum_n z_n x \in LS(M) = S(M)$, i.e. $x \in Z(S(M))$.

Conversely, let $x \in Z\big(S(M)\big), x \geq 0$ and let $x = \int_0^\infty \lambda d\lambda$ be it spectral resolution. Put $z_1 = e_1$ and $z_k = e_k - e_{k-1}, k \geq 2$. Then $\{z_k\}$ is a family of mutually orthogonal central projections with $\bigvee_k z_k = 1$. It is clear that $z_k x \in Z$ for all k. Therefore $x = \sum_n z_n x \in S(Z)$, and thus $Z\big(S(M)\big) = S(Z)$. In a similar way we obtain that $Z\big(S(M,\tau)\big) = S(Z,\tau_Z)$. The proof is complete.

Recall that M is a type I_∞ if M is of type I and does not have non-zero finite central projections.

Proposition (3.1.3)[3]: Let M be a type I_∞ von Neumann algebra with the center Z. Then the centers of the algebras $S(M)$ and $S(M,\tau)$ coincide with Z.

Proof. Suppose that $z \in S(M), z \geq 0$, is a central element and let $z = \int_0^\infty \lambda de_\lambda$ be its spectral resolution. Then $e_\lambda \in Z$ for all $\lambda > 0$. Assume that $e_n^\perp \neq 0$ for all $n \in \mathbb{N}$. Since M is of type I_∞, Z does not contain non-zero finite projections. Thus $e_n^\perp$ is infinite for all $n \in \mathbb{N}$, which contradicts the condition $z \in S(M)$. Therefore there exists $n_0 \in \mathbb{N}$ such that $e_n^\perp = 0$ for all $n \geq n_0$, i.e. $z \leq n_0\mathbf{1}$. This means that $z \in Z$, i.e. $Z(S(M)) = Z$. Similarly $Z(S(M,\tau)) = Z$. The proof is complete.

Let M be a von Neumann algebra of type $I_n (n \in \mathbb{N})$ with the center Z. Then M is $*$-isomorphic to the algebra $M_n(Z)$ of $n \times n$ matrices over Z (see [29]).

In this case the algebras $S(M,\tau)$ and $S(M)$ can be described in the following way.

Proposition (3.1.4)[3]: Let M be a von Neumann algebra of type $I_n, n \in \mathbb{N}$, with a faithful normal semi-finite trace τ and let $Z(S(M,\tau))$ denote the center of the algebra $S(M,\tau)$. Then $S(M,\tau) \cong M_n(Z(S(M,\tau)))$.

Proof. Let $\{e_{ij}: i,j \in \overline{1,n}\}$ be matrix units in $M_n(Z)$. Consider the $*$-subalgebra in $S(M,\tau)$ generated by the set

$$Z\big(S(M,\tau)\big) \cup \{e_{ij}: i,j \in \overline{1,n}\}.$$

This $*$-subalgebra consists of elements of the form

$$\sum_{i,j=1}^{n} \lambda_{ij} e_{ij}, \qquad \lambda_{i,j} \in Z\big(S(M,\tau)\big), \qquad i,j = \overline{1,n}$$

and it is $*$-isomorphic with $M_n(Z(S(M,\tau))) \subseteq S(M,\tau)$. Since M is $t\tau$-dense in $S(M,\tau)$, it is sufficient to show that the subalgebra $M_n(Z(S(M,\tau)))$ is closed in $S(M,\tau)$ with respect to the topology t_τ. The center $Z(S(M,\tau))$ is t_τ-closed in $S(M,\tau)$ and therefore the subalgebra

$$e_{11} Z\big(S(M,\tau)\big) e_{11} = Z\big(S(M,\tau)\big) e_{11},$$

is also t_τ-closed in $S(M,\tau)$.

Consider a sequence $x_m = \sum_{i,j=1}^{n} \lambda_{ij}^{(m)} e_{ij}$ in $M_n(Z(S(M,\tau)))$ such that $x_m \to x \in S(M,\tau)$ in the topology t_τ. Fixing $k,l \in \overline{1,n}$ we have that $e_{1k} x_m e_{l1} \to e_{1k} x e_{l1}$. Since $e_{1k} x_m e_{l1} = \lambda_{kl}^{(m)} e_{11}$ one has $\lambda_{kl}^{(m)} e_{11} \to e_{1k} x e_{l1}$. The t_τ-closedness of $Z\big(S(M,\tau)\big) e_{11}$ in $S(M,\tau)$ implies that

$$\lambda_{kl}^{(m)} e_{11} \to \lambda_{kl} e_{11}$$

for an appropriate $\lambda_{kl} \in Z\big(S(M,\tau)\big)$. Multiplying (2) by ek1 from the left side and by e_{l1} from the right-hand side we obtain that $\lambda_{kl}^{(m)} e_{kl} \to \lambda_{kl} e_{kl}$. Therefore $x_m \to \sum_{i,j=1}^{n} \lambda_{ij} e_{i,j}$ and thus $x = \sum_{i,j=1}^{n} \lambda_{ij} e_{i,j}$. This implies that $S(M,\tau) \cong M_n(Z(S(M,\tau)))$. The proof is complete.

Proposition (3.1.5)[3]: Let M be a von Neumann algebra of type $I_n, n \in \mathbb{N}$, and let $Z(S(M))$ denote the center of $S(M)$. Then $S(M) \cong M_n(Z(S(M,\tau)))$.

Proof. Let τ be a faithful normal semi-finite trace on M and consider a family $\{z_\alpha\}$ of mutually orthogonal central projections in M with $\bigvee_\alpha z_\alpha = \mathbf{1}$ and such that $\tau_\alpha = \tau|_{z_\alpha M}$ is finite for every α (such family exists because M is of type $I_n, n < \infty$). Then

$$M = \bigoplus_\alpha z_\alpha M.$$

Since each trace τ_α is finite one has

$$S(z_\alpha M) = S(z_\alpha M, \tau_\alpha) = M_n(Z(S(z_\alpha M, \tau_\alpha))) = M_n(Z(S(z_\alpha M))),$$

i.e.

$$S(z_\alpha M) = M_n(Z(S(z_\alpha M))).$$

This and Proposition (3.1.1) imply that

$$S(M) \cong \prod_\alpha S(z_\alpha M) = \prod_\alpha M_n\left(Z\big(S(z_\alpha M)\big)\right) = M_n\left(\prod_\alpha \left(z_\alpha Z\big(S(M)\big)\right)\right)$$
$$= M_n\left(Z\big(S(M)\big)\right),$$

i.e.

$$S(M) \cong M_n(Z(S(M))).$$

The proof is complete.

The last proposition enables us to obtain the following important property of the algebra $LS(M)$ in the case of type I von Neumann algebra M.

Proposition (3.1.6)[3]: Let M be a type I von Neumann algebra. Then for any element $x \in LS(M)$ there exists a countable family of mutually orthogonal central projections $\{z_k\}_{k\in\mathbb{N}}$ in M such that $\bigvee_k z_k = \mathbf{1}$ and $z_k x \in M$ for all k.

Proof. Case 1. The algebra M has type $I_n, n \in \mathbb{N}$. In this case $LS(M) = S(M)$ and Proposition (3.1.5) implies that $S(M) \cong M_n(Z(S(M)))$. Consider $x = \sum_{i,j=1}^n \lambda_{ij} e_{i,j} \in M_n(Z(S(M)))$. Put $c = \bigvee_{i,j=1}^n |\lambda_{ij}|$. Then $c \in Z(S(M))$ and if $c = \int_0^\infty \lambda d\lambda$ is its spectral resolution, put $z_1 = e_1$ and $z_k = e_k - e_{k-1}, k \geq 2$. Then $\{z_k\}$ is the family of mutually orthogonal central projections with $\bigvee_k z_k = \mathbf{1}$ and by definition $z_k c \in Z$ for all k. Therefore $z_k|\lambda_{ij}| \in Z$ for every $k \in \mathbb{N},\ 1 \leq i,j \leq n$. Thus $z_k x \in M$ for all k.

Case 2. M is a finite von Neumann algebra of type I. Then there exists a family $\{q_n\}_{n\in F}, F \subseteq \mathbb{N}$, of central projections from M with $\sup_{n\in F} q_n = \mathbf{1}$ such that the algebra M is $*$-isomorphic with the C^*-product of von Neumann algebras $q_n M$ of type I_n respectively, $n \in F$, i.e.

$$M \cong \bigoplus_{n\in F} q_n M.$$

By Proposition (3.1.1) we have that

$$S(M) \cong \prod_{n\in F} S(q_n M).$$

Take $x = \{x_n\}_{n\in F} \in \prod_{n\in F} S(q_n M)$. The case 1 implies that for every $n \in F$ there exists a family $\{x_{n,m}\}$ of mutually orthogonal central projections with $\bigvee_n z_{n,m} = q_n$ and $z_{n,m} z_n \in q_n M$ for all $m \in \mathbb{N}$.

In this case we have the countable family $\{z_{n,k}\}_{(n,k)\in F\times\mathbb{N}}$ of mutually orthogonal central projections with $\bigvee_{(n,k)\in F\times\mathbb{N}} z_{n,k} = \mathbf{1}$ and $z_{n,k} x \in M$ for all $(n,k) \in F \times \mathbb{N}$.

Case 3. M is an arbitrary von Neumann algebra of type I and $x \in S(M)$. Without loss of generality we may assume that $x \geq 0$.

Let $x = \int_0^\infty \lambda d\lambda$ be the spectral resolution of x. Since $x \in S(M)$ by the definition there exists $\lambda_0 > 0$ such that $e_{\lambda_0}^\perp$ is a finite projection. Thus $e_{\lambda_0}^\perp M e_{\lambda_0}^\perp$ is a finite von Neumann algebra of type I and $x e_{\lambda_0}^\perp \in S(e_{\lambda_0}^\perp M e_{\lambda_0}^\perp)$. From the case 2 we have that there exists a family of mutually orthogonal projections $\{z'_m\}_{m\in\mathbb{N}}$ in $e_{\lambda_0}^\perp M e_{\lambda_0}^\perp$ such that $\bigvee_{m\geq 1} z'_m = e_{\lambda_0}^\perp$ and $z'_m x e_{\lambda_0}^\perp \in e_{\lambda_0}^\perp M e_{\lambda_0}^\perp$ for all $m \in \mathbb{N}$. Each central projection z' in $e_{\lambda_0}^\perp M e_{\lambda_0}^\perp$ has the form $z' = e_{\lambda_0}^\perp z e_{\lambda_0}^\perp$ for an appropriate central projection $z \in M$. Moreover passing if necessary to $z(e_{\lambda_0}^\perp)z$ one may chose z with $z \leq z(e_{\lambda_0}^\perp)$, where $z(e_{\lambda_0}^\perp)$ is the central cover of the projection $e_{\lambda_0}^\perp$ in M. Let

$z_m' = e_{\lambda_0}^{\perp} z_m e_{\lambda_0}^{\perp}, m \in \mathbb{N}$. Mutually orthogonality of the family $\{z_m'\}$ then implies the similar property of the corresponding $\{z_m\}$. Denote $z_0 = z(e_{\lambda_0}^{\perp})^{\perp}$. Then $\bigvee_{m\geq 0} z_m = \mathbf{1}$ and

$$z_0 x = z_0 x e_{\lambda_0} + z_0 x e_{\lambda_0}^{\perp} = z_0 x e_{\lambda_0} \in M,$$
$$z_m x = z_m x e_{\lambda_0} + z_m x e_{\lambda_0}^{\perp} = z_m x e_{\lambda_0} + z_m' x e_{\lambda_0}^{\perp} \in M$$

for all $m \in \mathbb{N}$, i.e. $z_m x \in M$ for all $m \geq 0$.

Case 4. The general case, i.e. M is a type I von Neumann algebra and $x \in LS(M)$. By the definition there exists a sequence $\{f_n\}_{m\in\mathbb{N}}$ of mutually orthogonal central projection with $\bigvee_n f_n = \mathbf{1}$ and $f_n x \in S(M)$ for all $n \in \mathbb{N}$. Then the case 3 implies that for each $n \in \mathbb{N}$ there exists a sequence $\{z_{n,m}\}$ of mutually orthogonal central projections with $\bigvee_m z_{n,m} = f_n$ and $z_{n,m} x_n \in f_n M$ for all $m \in \mathbb{N}$.

Now we have that $\{z_{n,k}\}_{(n,k)\in\mathbb{N}\times\mathbb{N}}$ is a countable family of mutually orthogonal central projections with $\bigvee_{(n,k)\in\mathbb{N}\times\mathbb{N}} z_{n,k} = \mathbf{1}$ and $z_{n,k} x \in M$ for all $(n,k) \in \mathbb{N}\times\mathbb{N}$. The proof is complete.

Now let us recall some notions and results from the theory of Hilbert–Kaplansky modules.

Let $(\Omega, \sum, \mu)$ be a measure space and let H be a Hilbert space. A map $s\colon \Omega \to H$ is said to be simple, if $s(\omega) = \sum_{k=1}^{n} \chi_{A_k}(\omega) c_k$, where $A_k \in \sum, A_i \cap A_j = \emptyset, i \neq j, c_k \in H$. A map $u\colon \Omega \to H$ is said to be measurable, if for any $A \in \sum$ with $\mu(A) < \infty$ there is a sequence (s_n) of simple maps such that $\|s_n(\omega) - u(\omega)\| \to 0$ almost everywhere on A.

Let $\mathcal{L}(\Omega, H)$ be the set of all measurable maps from Ω into H, and let $L^0(\Omega, H)$ denote the space of all equivalence classes with respect to the equality almost everywhere. Denote by $\hat{u}$ the equivalence class from $L^0(\Omega, H)$ which contains the measurable map $u \in \mathcal{L}(\Omega, H)$. Further we shall identify the element $u \in \mathcal{L}(\Omega, H)$ and the class $\hat{u}$. Note that the function $\omega \to \|u(\omega)\|$ is measurable for any $u \in \mathcal{L}(\Omega, H)$. The equivalence class containing the function $\|u(\omega)\|$ is denoted by $\|\hat{u}\|$. For $\hat{u}, \hat{v} \in L^0(\Omega, H), \lambda \in L^0(\Omega)$ put $\hat{u} + \hat{v} = \widehat{u(\omega) + v(\omega)}, \lambda\hat{u} = \widehat{\lambda(\omega) u(\omega)}$. Equipped with the $L^0(\Omega)$-valued inner product

$$\langle x, y\rangle = \langle x(\omega), y(\omega)\rangle_H,$$

where $\langle\cdot,\cdot\rangle_H$ in the inner product in $H, L^0(\Omega, H)$ becomes a Hilbert–Kaplansky module over $L^0(\Omega)$. The space

$$L^\infty(\Omega, H) = \{x \in L^0(\Omega, H) \colon \langle x, x\rangle \in L^\infty(\Omega)\}$$

is a Hilbert–Kaplansky module over $L^\infty(\Omega)$.

It should be noted that $L^\infty(\Omega, H)$ is a Banach space with respect to the norm $\|x\|_\infty = \left\|(x, x)^{\frac{1}{2}}\right\|_{L^\infty(\Omega)}$.

Let us show that if $\dim H = \alpha$ then the Hilbert–Kaplansky module $L^\infty(\Omega, H)$ is α-homogeneous.

Indeed, let $\{\varphi_i\}_{i\in J}$ be an orthonormal basis in H with the cardinality α, and consider the equivalence class $\hat{\varphi}_i$ from $L^\infty(\Omega, H)$ containing the constant vector-function

$$\omega \in \Omega \to \varphi_i.$$

From the definition of the inner-product it is clear that

$$\langle\hat{\varphi}_i, \hat{\varphi}_j\rangle = \delta_{ij}\mathbf{1},$$

where δ_{ij} is the Kroenecker symbol, $\mathbf{1}$ is the identity from $L^\infty(\Omega)$.

Let us show that if $y \in L^\infty(\Omega, H)$ and $\langle\hat{\varphi}_i, y\rangle = 0$ for all $i \in J$ then $y = 0$. Put

$$Sp\{\hat{\varphi}_i\} = \left\{\sum_{k=1}^{n} \lambda_k \hat{\varphi}_{i_k} : \lambda_k \in L^\infty(\Omega), i_k \in J, k = \overline{1,n}, n \in \mathbb{N}\right\}.$$

Since the set of elements of the form $\sum_{k=1}^{n} t_k \varphi_{i_k}$, where $t_k \in \mathbb{C}, i_k \in J, k = \overline{1,n}, n \in \mathbb{N}$, is norm dense in H, we have that $\inf\{\|\psi - y\| : \psi \in Sp\{\hat{\varphi}_i\}\} = 0$, for each fixed $y \in L^\infty(\Omega, H)$. The set $Sp\{\hat{\varphi}_i\}$ is an $L^\infty(\Omega)$-submodule in $L^\infty(\Omega, H)$, and by [92] there exists a sequence $\{\psi_k\}$ in $Sp\{\hat{\varphi}_i\}$ such that $\|\psi_k - y\| \downarrow 0$, i.e. the set $Sp\{\hat{\varphi}_i\}$ is (bo)-dence in $L^\infty(\Omega, H)$.

Now let $y \in L^\infty(\Omega, H)$ be such an element that $\langle \hat{\varphi}_i, y\rangle = 0$ for all $i \in J$. Then $\langle \xi, y\rangle = 0$ for all $\xi \in Sp\{\hat{\varphi}_i\}$. Since the set $Sp\{\hat{\varphi}_i\}$ is (bo)-dense in $L^\infty(\Omega, H)$, we have that $\langle \xi, y\rangle = 0$ for all $\xi \in L^\infty(\Omega, H)$. In particular $\langle y, y\rangle = 0$, i.e. $y = 0$.

Therefore $\{\hat{\varphi}_i\}_{i\in J}$ is an orthogonal basis in $L^\infty(\Omega, H)$ with the cardinality α, i.e. $L^\infty(\Omega, H)$ is α-homogeneous, where $\alpha = \dim H$.

Denote by $B\big(L^0(\Omega, H)\big)$ the algebra of all $L^0(\Omega)$-bounded $L^0(\Omega)$-linear operators on $L^0(\Omega, H)$ and by $B\big(L^\infty(\Omega, H)\big)$—the algebra of all $L^\infty(\Omega)$-bounded $L^\infty(\Omega)$-linear operators on $L^\infty(\Omega, H)$.

In [91] it was proved that $B\big(L^0(\Omega, H)\big)$ is a C^*-algebra over $L^0(\Omega)$.

Put

$$B(L^0(\Omega, H)_b) = \left\{x \in B\big(L^0(\Omega, H)\big) : \|x\| \in L^\infty(\Omega)\right\}.$$

Note that the correspondence

$$x \mapsto x|_{L^\infty(\Omega,H)}$$

gives a $*$-isomorphism between the $*$-algebras $B(L^0(\Omega, H)_b)$ and $B\big(L^\infty(\Omega, H)\big)$. We further shall identify $B(L^0(\Omega, H)_b)$ with $B\big(L^\infty(\Omega, H)\big)$, i.e. the operator x from $B(L^0(\Omega, H)_b)$ is identified with its restriction $x|_{L^\infty(\Omega,H)}$.

Since $L^\infty(\Omega, H)$ is a Hilbert–Kaplansky module over $L^\infty(\Omega)$, [95] implies that $B\big(L^\infty(\Omega, H)\big)$ is an AW^*-algebra of type I and its center is $*$-isomorphic with $L^\infty(\Omega)$. If $\alpha = \dim H$, then $L^\infty(\Omega, H)$ is α-homogeneous and by [95] the algebra $B\big(L^\infty(\Omega, H)\big)$ has the type I_α. The center $Z\left(B\big(L^\infty(\Omega, H)\big)\right)$ of this AW^*-algebra coincides with the von Neumann algebra $L^\infty(\Omega)$ and thus by [96] $B\big(L^\infty(\Omega, H)\big)$ is also a von Neumann algebra. Thus for $\dim H = \alpha$ we have that $B\big(L^\infty(\Omega, H)\big)$ is a von Neumann algebra of type I_α.

Now let M be a homogeneous von Neumann algebra of type I_α with the center $L^\infty(\Omega)$. Since two von Neumann algebras of the same type I_α with isomorphic center are mutually $*$-isomorphic, it follows that the algebra M is $*$-isomorphic to the algebra $B\big(L^\infty(\Omega, H)\big)$, where $\dim H = \alpha$.

It is well known [33] that given any type I von Neumann algebra M, there exists a (cardinalindexed) system of central orthogonal projections $(q_\alpha)_{\alpha\in J} \subset \mathcal{P}(M)$ with $\sum_{\alpha\in J} q_\alpha = \mathbf{1}$ such that $q_\alpha M$ is a homogeneous von Neumann algebra of type I_α, i.e. $q_\alpha M \cong B\big(L^\infty(\Omega_\alpha, H_\alpha)\big)$ with $\dim H = \alpha$, and the algebra M is $*$-isomorphic to the C^*-product of the algebras M_α, i.e.

$$M \cong \bigoplus_{\alpha\in J} M_\alpha$$

Note that if $L^\infty(\Omega)$ is the center of M then $q_\alpha L^\infty(\Omega) \cong L^\infty(\Omega_\alpha)$ for an appropriate $\Omega_\alpha, \alpha \in J$. Therefore

$$L^\infty(\Omega) \cong \bigoplus_{\alpha\in J} L^\infty(\Omega_\alpha)$$

The product

$$\bigoplus_{\alpha\in J} L^{\infty}(\Omega_\alpha, H_\alpha)$$

equipped with coordinate-wise algebraic operations and inner product becomes a Hilbert–Kaplansky module over $L^{\infty}(\Omega_\alpha)$. The product

$$\bigoplus_{\alpha\in J} B\big(L^{\infty}(\Omega_\alpha, H_\alpha)\big)$$

equipped with coordinate-wise algebraic operations and involution becomes a $*$-algebra and moreover

$$\bigoplus_{\alpha\in J} B\big(L^{\infty}(\Omega_\alpha, H_\alpha)\big) \cong B\left(\bigoplus_{\alpha\in J} L^{\infty}(\Omega_\alpha, H_\alpha)\right).$$

Indeed, take $x \in B\big(\bigoplus_{\alpha\in J} L^{\infty}(\Omega_\alpha, H_\alpha)\big)$. For each α define the operator xα on $L^{\infty}(\Omega_\alpha, H_\alpha)$ by

$$x_\alpha(\varphi_\alpha) = q_\alpha x(\varphi_\alpha), \varphi_\alpha \in L^{\infty}(\Omega_\alpha, H_\alpha).$$

Then

$$\{x_\alpha\} \in \bigoplus_{\alpha\in J} B\big(L^{\infty}(\Omega_\alpha, H_\alpha)\big)$$

and the correspondence

$$x \mapsto \{x_\alpha\}$$

gives a $*$-homomorphism from $B\big(\bigoplus_{\alpha\in J} L^{\infty}(\Omega_\alpha, H_\alpha)\big)$ into $\bigoplus_{\alpha\in J} B\big(L^{\infty}(\Omega_\alpha, H_\alpha)\big)$.

Conversely, consider

$$\{x_\alpha\} \in \bigoplus_{\alpha\in J} B\big(L^{\infty}(\Omega_\alpha, H_\alpha)\big).$$

Define the operator x on $\bigoplus_{\alpha\in J} L^{\infty}(\Omega_\alpha, H_\alpha)$ by

$$x(\{\varphi_\alpha\}) = \{x_\alpha(\varphi_\alpha)\}, \qquad \{\varphi_\alpha\} \in \bigoplus_{\alpha\in J} L^{\infty}(\Omega_\alpha, H_\alpha).$$

Then $x \in B\big(\bigoplus_{\alpha\in J} L^{\infty}(\Omega_\alpha, H_\alpha)\big)$ and therefore

$$M \cong \bigoplus_{\alpha\in J} B\big(L^{\infty}(\Omega_\alpha, H_\alpha)\big) \cong B\left(\bigoplus_{\alpha\in J} L^{\infty}(\Omega_\alpha, H_\alpha)\right).$$

The direct product

$$\prod_{\alpha\in J} L^{0}(\Omega_\alpha, H_\alpha)$$

equipped with the coordinate-wise algebraic operations and inner product forms a Hilbert–Kaplansky module over $L^{0}(\Omega) \cong \prod_{\alpha\in J} L^{0}(\Omega_\alpha)$.

The proof of the following proposition in [2] has a small gap, therefore here we shall give an alternative proof for this result.

Proposition (3.1.7)[3]: If von Neumann algebra M is $*$-isomorphic with $B\big(\bigoplus_{\alpha\in J} L^{\infty}(\Omega_\alpha, H_\alpha)\big)$ then the algebra $LS(M)$ is $*$-isomorphic with $B\big(\prod_{\alpha\in J} L^{0}(\Omega_\alpha, H_\alpha)\big)$.

Proof. Let Φ be a $*$-isomorphism between M and $B\big(\bigoplus_{\alpha\in J} L^{\infty}(\Omega_\alpha, H_\alpha)\big)$. Take $x \in B\big(\prod_{\alpha\in J} L^{0}(\Omega_\alpha, H_\alpha)\big)$ and let $\|x\|$ be its $L^{0}(\Omega)$-valued norm. Consider a family of mutually orthogonal projections $\{z_n\}_{n\in\mathbb{N}}$ in $L^{\infty}(\Omega)$ with $\bigvee z_n = \mathbf{1}$ such that $z_n\|x\| \in L^{\infty}(\Omega)$ for all $n \in \mathbb{N}$. Then $z_n x \in M$ for all $n \in \mathbb{N}$ and $\sum_n z_n \Phi(z_n x) \in LS(M)$. Put

$$\psi: x \to \sum_n z_n \Phi(z_n x).$$

It is clear that ψ is a well-defined $*$-homomorphism from $B\big(\prod_{\alpha\in J} L^{0}(\Omega_\alpha, H_\alpha)\big)$ into $LS(M)$. Since given any element $x \in LS(M)$ there exists a sequence of mutually orthogonal central projections $\{z_n\}$ in M such that $z_n x \in M$ for all $n \in \mathbb{N}$ (Proposition 3.1.6) and $x = \sum_n z_n x$, this implies that ψ is a $*$-isomorphism between $LS(M)$ and $B\left(\prod_{\alpha\in J} L^0(\Omega_\alpha, H_\alpha)\right)$. The proof is complete.

It is known [91] that $B\left(\prod_{\alpha\in J} L^0(\Omega_\alpha, H_\alpha)\right)$ is a C^*-algebra over $L^0(\Omega)$ and therefore there exists a map $\|\cdot\|: LS(M) \to L^0(\Omega)$ such that for all $x, y \in LS(M), \lambda \in L^0(\Omega)$ one has

$$\|x\| \geq 0, \qquad \|x\| = 0 \iff x = 0;$$
$$\|\lambda x\| = |\lambda|\|x\|;$$
$$\|x + y\| \leq \|x\| + \|y\|;$$
$$\|xy\| \leq \|x\|\|y\|;$$
$$\|xx^*\| = \|x\|^2.$$

This map $\|\cdot\|: LS(M) \to L^0(\Omega)$ is called the center-valued norm on $LS(M)$.

We shall give a complete description of derivations on the algebra $LS(M)$ of all locally measurable operators affiliated with a type I von Neumann algebra M. It is clear that if a derivation D on $LS(M)$ is inner then it is Z-linear, i.e. $D(\lambda x) = \lambda D(x)$ for all $\lambda \in Z, x \in LS(M)$, where Z is the center of the von Neumann algebra M. The following main result of [2] asserts that the converse is also true.

Theorem (3.1.8)[3]: Let M be a type I von Neumann algebra with the center Z. Then every Z-linear derivation D on the algebra $LS(M)$ is inner.

Proof. (See [2].)

We are now in position to consider arbitrary (non-Z-linear, in general) derivations on $LS(M)$. The following simple but important remark is crucial in our further considerations.

Lemma (3.1.9)[3]: Let M be a homogenous von Neumann algebra of type $I_n, n \in \mathbb{N}$. Every derivation D on the algebra $LS(M)$ can be uniquely represented as a sum

$$D = D_a + D_\delta,$$

where D_a is an inner derivation implemented by an element $a \in LS(M)$ while D_δ is the derivation of the form (1) generated by a derivation δ on the center of $LS(M)$ identified with $S(Z)$.

Proof. Let D be an arbitrary derivation on the algebra $LS(M) \cong M_n(S(Z))$. Consider its restriction δ onto the center $S(Z)$ of this algebra, and let D_δ be the derivation on the algebra $M_n(S(Z))$ constructed as in (1). Put $D_1 = D - D_\delta$. Given any $\lambda \in S(Z)$ we have

$$D_1(\lambda) = D(\lambda) - D_\delta(\lambda) = D(\lambda) - D(\lambda) = 0,$$

i.e. D_1 is identically zero on $S(Z)$. Therefore D_1 is Z-linear and by Theorem (3.1.8) we obtain that D_1 is inner derivation and thus $D_1 = D_a$ for an appropriate $a \in M_n(S(Z))$. Therefore $D = D_a + D_\delta$.

Suppose that

$$D = D_{a_1} + D_{\delta_1} = D_{a_2} + D_{\delta_2}.$$

Then $D_{a_1} - D_{a_2} = D_{\delta_2} - D_{\delta_1}$. Since $D_{a_1} - D_{a_2}$ is identically zero on the center of the algebra $M_n(S(Z))$ this implies that $D_{\delta_2} - D_{\delta_1}$ is also identically zero on the center of $M_n(S(Z))$. This means that $\delta_1 = \delta_2$, and therefore $D_{a_1} = D_{a_2}$, i.e. the decomposition of D is unique. The proof is complete.

Now let M be an arbitrary finite von Neumann algebra of type I with the center Z. There exists a family $\{z_n\}_{n\in F}, F \subseteq \mathbb{N}$, of central projections from M with $\sup_{n\in F} z_n = \mathbf{1}$ such that the algebra M is $*$-isomorphic with the C^*-product of von Neumann algebras $z_n M$ of type I_n respectively, $n \in F$, i.e.

$$M \cong \bigoplus_{n\in F} z_n M.$$

By Proposition(3.1.1) we have that

$$LS(M) \cong \prod_{n \in F} LS(z_n M).$$

Suppose that D is a derivation on $LS(M)$, and δ is its restriction onto its center $S(Z)$. Since δ maps each $z_n S(Z) \cong Z(LS(z_n M))$ into itself, δ generates a derivation δ_n on $z_n S(Z)$ for each $n \in F$.

Let D_{δ_n} be the derivation on the matrix algebra $M_n(z_n Z(LS(M))) \cong LS(z_n M)$ defined as in (1). Put

$$D_\delta(\{x_n\}_{n\in F}) = \{D_{\delta_n}(x_n)\}, \qquad \{x_n\}_{n\in F} \in LS(M).$$

Then the map D_δ is a derivation on $LS(M)$.

Now Lemma (3.1.9) implies the following result:

Lemma (3.1.10)[3]: Let M be a finite von Neumann algebra of type I. Each derivation D on the algebra $LS(M)$ can be uniquely represented in the form

$$D = D_a + D_\delta,$$

where D_a is an inner derivation implemented by an element $a \in LS(M)$, and D_δ is a derivation given as (3).

In order to consider the case of type I_∞ von Neumann algebra we need some auxiliary results concerning derivations on the algebra $L^0(\Omega) = L(\Omega, \sum, \mu)$.

Recall that a net $\{\lambda_\alpha\}$ in $L^0(\Omega)(o)$-converges to $\lambda \in L^0(\Omega)$ if there exists a net $\{\xi_\alpha\}$ monotone decreasing to zero such that $|\lambda_\alpha - \lambda| \le \xi_\alpha$ for all α.

Denote by ∇ the complete Boolean algebra of all idempotents from $L^0(\Omega)$, i.e. $\nabla = \{\tilde{\chi}_A : A \in \sum\}$, where $\tilde{\chi}_A$ is the element from $L^0(\Omega)$ which contains the characteristic function of the set A. A partition of the unit in ∇ is a family (π_α) of orthogonal idempotents from ∇ such that $\bigvee_\alpha \pi_\alpha = \mathbf{1}$.

Lemma (3.1.11)[3]: Any derivation δ on the algebra $L^0(\Omega)$ commutes with the mixing operation on $L^0(\Omega)$, i.e.

$$\delta\left(\sum_\alpha \pi_\alpha \lambda_\alpha\right) = \sum_\alpha \pi_\alpha \delta(\lambda_\alpha)$$

for an arbitrary family $(\lambda_\alpha) \subset L^0(\Omega)$ and any partition $\{\pi_\alpha\}$ of the unit in ∇.

Proof. Consider a family $\{\lambda_\alpha\}$ in $L^0(\Omega)$ and a partition of the unit $\{\pi_\alpha\}$ in $\nabla \subset L^0(\Omega)$. Since $\delta(\pi) = 0$ for any idempotent $\pi \in \nabla$, we have $\delta(\pi_\alpha) = 0$ for all α and thus $\delta(\pi_\alpha \lambda) = \pi_\alpha \delta(\lambda)$ for any $\lambda \in L^0(\Omega)$. Therefore for each π_{α_0} from the given partition of the unit we have

$$\pi_{\alpha_0}\delta\left(\sum_\alpha \pi_\alpha \lambda_\alpha\right) = \delta\left(\pi_{\alpha_0}\sum_\alpha \pi_\alpha \lambda_\alpha\right) = \delta\left(\pi_{\alpha_0}\lambda_{\alpha_0}\right) = \pi_{\alpha_0}\delta\left(\lambda_{\alpha_0}\right).$$

By taking the sum over all α_0 we obtain

$$\delta\left(\sum_\alpha \pi_\alpha \lambda_\alpha\right) = \sum_\alpha \pi_\alpha \delta(\lambda_\alpha).$$

The proof is complete.

Recall [97] that a subset $K \subset L^0(\Omega)$ is called cyclic, if $\sum_{\alpha\in J} \pi_\alpha u_\alpha \in K$ for each family $(u_\alpha)_{\alpha\in J} \subset K$ and for any partition of the unit $(\pi_\alpha)_{\alpha\in J}$ in ∇.

We need the following technical result.

Lemma (3.1.12)[3]: Let A be a cyclic subset in $L^0(\Omega)$. If the set πA is unbounded above for each non-zero $\pi \in \nabla$, then given any $n \in \mathbb{N}$ there exists $\lambda_n \in A$ such that $\lambda_n \ge n\mathbf{1}$.

Proof. For fixed $n \in \mathbb{N}$ and an arbitrary $\lambda \in A$ denote
$$\pi_\lambda = \bigvee\{q \in \nabla: q\lambda \geq qn\}.$$
Then
$$\pi_\lambda \lambda \geq \pi_\lambda n$$
and
$$\pi_\lambda^\perp \lambda \leq \pi_\lambda^\perp n,$$
Put
$$\pi_0 = \bigvee\{\pi_\lambda: \lambda \in A\}.$$
Since
$$\pi_0^\perp = \bigwedge\{\pi_\lambda^\perp: \lambda \in A\}$$
from (5) we obtain
$$\pi_0^\perp \lambda \leq \pi_0^\perp n$$
for all $\lambda \in A$, i.e. $\pi_0^\perp A$ is bounded above. By the assumption of lemma $\pi_0^\perp = 0$, i.e.
$$\pi_0 = \bigvee\{\pi_\lambda: \lambda \in A\} = \mathbf{1}.$$
By [103] there exists a partition of unit $\{\pi_i\}$ in ∇ such that for any π_i there exists $\lambda_i \in A$ with $\pi_i \leq \pi_{\lambda_i}$. Put $\lambda_n = \sum_i \pi_i \lambda_i$. Since A is a cyclic we have $\lambda_n \in A$. From (4) one has $\pi_{\lambda_i}\lambda_i \geq \pi_{\lambda_i} n$ for all i. Thus $\pi_i \lambda_i \geq \pi_i n$ for all i, therefore $\lambda_n \geq n\mathbf{1}$. The proof is complete.

Given an arbitrary derivation δ on L^0 the element
$$z_\delta = \inf\{\pi \in \nabla: \pi\delta = \delta\}$$
is called the support of the derivation δ.

Lemma (3.1.13)[3]: Given any nontrivial derivation $\delta: L^0(\Omega) \to L^0(\Omega)$ there exist a sequence $\{\lambda_n\}_{n=1}^\infty$ in $L^\infty(\Omega)$ with $|\lambda_n| \leq \mathbf{1}, n \in \mathbb{N}$ such that
$$|\delta(\lambda_n)| \geq n z_\delta$$
for all $n \in \mathbb{N}$.

Proof. Considering if necessary the algebra $z_\delta L^0(\Omega)$ instead $L^0(\Omega)$ and the derivation $z_\delta \delta$ instead δ, we may assume that $z_\delta = \mathbf{1}$.

Put $A = \{\delta(\lambda): \lambda \in L^0(\Omega), |\lambda| \leq \mathbf{1}\}$ and let us prove that for any non-zero $\pi \in \nabla$ the set πA is unbounded from above. Suppose that the set $\pi\{\delta(\lambda): \lambda \in L^0(\Omega), |\lambda| \leq \mathbf{1}\}$ is order bounded in $L^0(\Omega)$ for some $\pi \in \nabla, \pi \neq 0$. Then $\pi\delta$ maps any uniformly convergent sequence in $L^\infty(\Omega)$ to an (o)-convergent sequence in $L^0(\Omega)$. The algebra $L^\infty(\Omega)$ coincides with the uniform closure of the linear span of idempotents from ∇. Since $\pi\delta$ is identically zero on ∇ it follows that $\pi\delta \equiv 0$ on $L^\infty(\Omega)$. Since δ commutes with the mixing operation and every element $\lambda \in L^0(\Omega)$ can be represented as $\lambda = \sum_\alpha \pi_\alpha \lambda_\alpha$, where $\{\lambda_\alpha\} \subset L^\infty(\Omega)$, and $\{\pi_\alpha\}$ is a partition of unit in ∇, we have $\delta(\lambda) = \delta(\sum_\alpha \pi_\alpha \lambda_\alpha) = \sum_\alpha \pi_\alpha \delta(\lambda_\alpha) = 0$, i.e. $\pi\delta \equiv 0$ on $L^0(\Omega)$. This is contradicts with $z_\delta = \mathbf{1}$. This contradiction shows that the set $\pi\{\delta(\lambda): \lambda \in L^0(\Omega), |\lambda| \leq \mathbf{1}\}$ is not order bounded in $L^0(\Omega)$ for all $\pi \in \nabla, \pi \neq 0$. Further, since δ commutes with the mixing operations and the set $\{\lambda: \lambda \in L^0, |\lambda| \leq \mathbf{1}\}$ is cyclic, the set $\{\delta(\lambda): \lambda \in L^0(\Omega), |\lambda| \leq \mathbf{1}\}$ is also cyclic. By Lemma (3.1.11) there exist a sequence $\{\lambda_n\}_{n=1}^\infty$ in $L^\infty(\Omega)$ with $|\lambda_n| \leq \mathbf{1}$ such that $|\lambda_n| \geq n\mathbf{1}, n \in \mathbb{N}$. The proof is complete.

Now we are in position to consider derivations on the algebra of locally measurable operators for type I_∞ von Neumann algebras.

Theorem (3.1.14)[3]: If M is a type I_∞ von Neumann algebra, then any derivation on the algebra $LS(M)$ is inner.

Proof. Since M is of type I_∞ there exists a sequence of mutually orthogonal and mutually equivalent abelian projections $\{p_n\}_{n=1}^{\infty}$ in M with the central cover **1** (i.e. faithful projections).

For any bounded sequence $\Lambda = \{\lambda_k\}$ in Z define an operator x_Λ by

$$x_\Lambda = \sum_{k=1}^{\infty} \lambda_k p_k.$$

Then

$$x_\Lambda p_n = p_n x_\Lambda = \lambda_n p_n.$$

Let D be a derivation on $LS(M)$, and let δ be its restriction onto the center of $LS(M)$, identified with $L^0(\Omega)$.

Take any $\lambda \in L^0(\Omega)$ and $n \in \mathbb{N}$. From the identity

$$D(\lambda p_n) = D(\lambda)p_n + \lambda D(p_n)$$

multiplying it by p_n from both sides we obtain

$$p_n D(\lambda p_n) p_n = p_n D(\lambda) p_n + \lambda p_n D(p_n) p_n.$$

Since p_n is a projection, one has that $p_n D(p_n) p_n = 0$, and since $D(\lambda) = \delta(\lambda) \in L^0(\Omega)$, we have

$$p_n D(\lambda p_n) p_n = \delta(\lambda) p_n.$$

Now from the identity

$$D(x_\Lambda p_n) = D(x_\Lambda)p_n + x_\Lambda D(p_n),$$

in view of (6) one has similarly

$$p_n D(\lambda_n p_n) p_n = p_n D(x_\Lambda) p_n + \lambda_n p_n D(p_n) p_n,$$

i.e.

$$p_n D(\lambda_n p_n) p_n = p_n D(x_\Lambda) p_n.$$

Eqs. (7) and (8) imply

$$p_n D(x_\Lambda) p_n = \delta(\lambda_n) p_n.$$

Further for the center-valued norm $\|\cdot\|$ on $LS(M)$ we have:

$$\|p_n D(x_\Lambda) p_n\| \le \|p_n\| \|D(x_\Lambda)\| \|p_n\| = \|D(x_\Lambda)\|$$

and

$$\|\delta(\lambda_n) p_n\| = |\delta(\lambda_n)|.$$

Therefore

$$\|D(x_\Lambda)\| \ge |\delta(\lambda_n)|$$

for any bounded sequence $\Lambda = \{\lambda_n\}$ in Z.

If we suppose that $\delta \ne 0$ then $\pi = z_\delta \ne 0$. By Lemma (3.1.12) there exists a bounded sequence $\Lambda = \{\lambda_n\}$ in Z such that

$$|\delta(\lambda_n)| \ge n\pi$$

for any $n \in \mathbb{N}$. Thus

$$\|D(x_\Lambda)\| \ge n\pi$$

for all $n \in \mathbb{N}$, i.e. $\pi = 0$—that is a contradiction. Therefore $\delta \equiv 0$, i.e. D is identically zero on the center of $LS(M)$, and therefore it is Z-linear. By Theorem (3.1.8) D is inner. The proof is complete.

We shall now consider derivations on the algebra $LS(M)$ of locally measurable operators with respect to an arbitrary type I von Neumann algebra M.

Let M be a type I von Neumann algebra. There exists a central projection $z_0 \in M$ such that

(a) $z_0 M$ is a finite von Neumann algebra;

(b) $z_0^{\perp} M$ is a von Neumann algebra of type I_∞.

Consider a derivation D on $LS(M)$ and let δ be its restriction onto its center $Z(S)$. By Theorem (3.1.14) $z_0^{\perp}D$ is inner and thus we have $z_0^{\perp}\delta \equiv 0$, i.e. $\delta = z_0\delta$.

Let D_δ be the derivation on $z_0 LS(M)$ defined as in (3) and consider its extension D_δ on $LS(M) = z_0 LS(M) \oplus z_0^{\perp} LS(M)$ which is defined as

$$D_\delta(x_1 + x_2) := D_\delta(x_1), \qquad x_1 \in z_0 LS(M), \qquad x_2 \in z_0^{\perp} LS(M).$$

The following theorem gives the general form of derivations on the algebra $LS(M)$.

Theorem (3.1.15)[3]: Let M be a type I von Neumann algebra. Each derivation D on $LS(M)$ can be uniquely represented in the form

$$D = D_a + D_\delta$$

where D_a is an inner derivation implemented by an element $a \in LS(M)$, and D_δ is a derivation of the form (9), generated by a derivation δ on the center of $LS(M)$.

Proof. It immediately follows from Lemma (3.1.10) and Theorem (3.1.14).

We describe derivations on the algebra $S(M)$ of measurable operators affiliated with a type I von Neumann algebra M.

Let M be a type I von Neumann algebra and let $\mathcal{A}$ be an arbitrary subalgebra of $LS(M)$ containing M. Consider a derivation $D: \mathcal{A} \to LS(M)$ and let us show that D can be extended to a derivation $\widetilde{D}$ on the whole $LS(M)$.

Since M is a type I, for an arbitrary element $x \in LS(M)$ there exists a sequence $\{z_n\}$ of mutually orthogonal central projections with $\bigvee_{n\in\mathbb{N}} z_n = \mathbf{1}$ and $z_n x \in M$ for all $n \in \mathbb{N}$. Set

$$\widetilde{D}(x) = \sum_{n\geq 1} z_n D(z_n x).$$

Since every derivation $D: \mathcal{A} \to LS(M)$ is identically zero on central projections of M, the equality (10) gives a well-defined derivation $\widetilde{D}: LS(M) \to LS(M)$ which coincides with D on $\mathcal{A}$.

In particular, if D is Z-linear on $\mathcal{A}$, then $\widetilde{D}$ is also Z-linear and by Theorem (3.1.8) the derivation $\widetilde{D}$ is inner on $LS(M)$ and therefore D is a spatial derivation on $\mathcal{A}$, i.e. there exists an element $a \in LS(M)$ such that

$$D(x) = ax - xa$$

for all $x \in \mathcal{A}$.

Therefore we obtain the following

Theorem (3.1.16)[3]: Let M be a type I von Neumann algebra with the center Z, and let $\mathcal{A}$ be an arbitrary subalgebra in $LS(M)$ containing M. Then any Z-linear derivation $D: \mathcal{A} \to LS(M)$ is spatial and implemented by an element of $LS(M)$.

Corollary (3.1.17)[3]: Let M be a type I von Neumann algebra with the center Z and let D be a Z-linear derivation on $S(M)$ or $S(M, \tau)$. Then D is spatial and implemented by an element of $LS(M)$.

We are now in position to improve the last result by showing that in fact such derivations on $S(M)$ and $S(M, \tau)$ are inner.

Let us start by the consideration of the type I_∞ case.

Let M be a type I_∞ von Neumann algebra with the center Z identified with the algebra $L^\infty(\Omega)$ and let ∇ be the Boolean algebra of projection from $L^\infty(\Omega)$.

Denote by $St(\nabla)$ the set of all elements $\lambda \in L^0(\Omega)$ of the form $\lambda = \sum_\alpha \pi_\alpha t_\alpha$, where $\{\pi_\alpha\}$ is a partition of the unit in ∇, and $\{t_\alpha\} \subset \mathbb{R}$ (so-called step-functions).

Suppose that $a \in LS(M), a = a^*$ and consider the spectral family $\{e_\lambda\}_{\lambda\in\mathbb{R}}$ of the operator a. For $\lambda \in St(\nabla), \lambda = \sum_\alpha \pi_\alpha t_\alpha$ put $e_\lambda = \sum_\alpha \pi_\alpha e_{t_\alpha}$.

Denote by $P_\infty(M)$ the family of all faithful projections p from M such that pMp is of type I_∞.

Set

$$\Lambda_- = \{\lambda \in St(\nabla): e_\lambda \in P_\infty(M)\}$$

and

$$\Lambda_+ = \{\lambda \in St(\nabla): e_\lambda^\perp \in P_\infty(M)\}.$$

Lemma (3.1.18)[3]:
(a) $\Lambda_- \neq \emptyset$ and $\Lambda_+ \neq \emptyset$;
(b) the set Λ_+ (resp. Λ_-) is bounded from above (resp. from below);
(c) if $\lambda_+ = \sup \Lambda_+$ (resp. $\lambda_- = \inf \Lambda_-$) then $\lambda \in \Lambda_+$ (resp. $\lambda \in \Lambda_-$) for all $\lambda \in St(\nabla)$ with $\lambda + \varepsilon\mathbf{1} \leq \lambda_+$ (resp. $\lambda - \varepsilon\mathbf{1} \geq \lambda_-$) for some $\varepsilon > 0$;
(d) if $\lambda_+ \in L^\infty(\Omega)$ and $\lambda_- \in L^\infty(\Omega)$, then $a \in S(M)$.

Proof. (a) Take a sequence of projections $\{z_n\}$ from ∇ such that $z_n a \in M$ for all $n \in \mathbb{N}$. Then for $t_n < \|z_n a\|_M$ we have $z_n e_{t_n} = 0$ or $z_n e_{t_n}^\perp = z_n$ for all $n \in \mathbb{N}$. Therefore for $\lambda = \sum z_n t_n$ one
has $e_\lambda^\perp = \sum z_n e_{t_n}^\perp = \sum z_n = \mathbf{1}$, i.e. $\lambda \in \Lambda_+$ and hence $\Lambda_+ \neq \emptyset$. Similarly $\Lambda_- \neq \emptyset$.
(b) Suppose that the element $\lambda = \sum \pi_\alpha \lambda_\alpha \in St(\nabla)$, satisfies the condition $\pi_0 \lambda \geq \pi_0\|a\| + \varepsilon\pi_0$ for an appropriate non-zero $\pi_0 \in \nabla$, where $\|\cdot\|$ is the center-valued norm on $LS(M)$. Without loss of generality we may assume that $\pi_0 = \pi_\alpha$ for some α, i.e. $\pi_\alpha t_\alpha \geq \pi_\alpha\|a\| + \varepsilon\pi_\alpha$. Then $t_\alpha \geq \|\pi_\alpha a\|_M + \varepsilon$ and therefore $\pi_\alpha e_{t_\alpha} = \pi_\alpha \mathbf{1}$, i.e. $\pi_\alpha e_{t_\alpha}^\perp = 0$. Since $\pi_\alpha e_{t_\alpha}^\perp = 0$, we have $z(e_{t_\alpha}^\perp) = \mathbf{1}$ and so $\lambda \notin \Lambda_+$. Therefore Λ_+ is bounded from above by the element $\|a\|$. Similarly the set Λ_- is bounded from below by the element $-\|a\|$.
(c) Put

$$\lambda_+ = \sup \Lambda_+$$

and

$$\lambda_- = \sup \Lambda_-.$$

Take an element $\lambda \in St(\nabla)$ such that $\lambda + \varepsilon\mathbf{1} \leq \lambda_+$, where $\varepsilon > 0$. Suppose that $e_\lambda^\perp \notin P_\infty(M)$. Then $\pi_0 e_\lambda^\perp M e_\lambda^\perp$ is a finite von Neumann algebra for some non-zero $\pi_0 \in \nabla$. Without loss of generality we may assume that $\pi_0 = \pi_\alpha$ for some α, i.e. $\pi_\alpha e_{t_\alpha}^\perp$ is a finite projection. Then $\pi_\alpha e_t^\perp$ is finite for all $t > t_\alpha$. This means that $\pi_\alpha \lambda_+ \leq \pi_\alpha t_\alpha$.

On the other hand multiplying by π_α the unequality $\lambda + \varepsilon\mathbf{1} \leq \lambda_+$ we obtain that $\pi_\alpha t_\alpha + \pi_\alpha\varepsilon \leq \pi_\alpha\lambda_+$. Therefore $\pi_\alpha \varepsilon \leq 0$. This contradiction implies that $\lambda \in \Lambda_+$ for all $\lambda \in St(\nabla)$ with $\lambda + \varepsilon\mathbf{1} \leq \lambda_+$.
(d) Let $\lambda_+, \lambda_- \in L^\infty(\Omega)$. Take a number $n \in \mathbb{N}$ such that $\lambda_+ \leq n\mathbf{1}$ and $\lambda_- \geq -n\mathbf{1}$. Take a largest element $\pi \in \nabla$ such that $\pi e_{n+1}^\perp$ is a finite projection and $\pi^\perp e_{n+1}^\perp$ is an infinite projection. For $\lambda' \in \Lambda_+$ put $\lambda'' = \pi\lambda' + \pi^\perp(n+1)$. Then $\lambda'' \in \Lambda_+$ and therefore $\lambda'' \leq \lambda_+$. Hence $\pi^\perp\lambda'' \leq \pi^\perp\lambda_+$, i.e $\pi^\perp(n+1) \leq \pi^\perp\lambda_+$. That contradicts the inequality $\lambda_n \leq n\mathbf{1}$. Therefore $\pi = \mathbf{1}$, i.e. $e_{n+1}^\perp$ is a finite projection. Similarly $e_{-(n+1)}^\perp$ is a finite projection. Therefore $a \in S(M)$. The proof is complete.

Lemma (3.1.19)[3]: If M is a type I_∞ von Neumann algebra then every derivation $D: M \to S(M)$ has the form

$$D(x) = ax - xa, \qquad x \in M,$$

for an appropriate $a \in S(M)$.

Proof. By D maps the center Z of M into the center of $S(M)$ which coincides with Z by Proposition (3.1.3), i.e. we obtain a derivation D on commutative von Neumann algebra Z.

Therefore $D|_Z = 0$. Thus $D(\lambda x) = D(\lambda)x + \lambda D(x) = \lambda D(x)$ for all $\lambda \in Z$, i.e. D is Z-linear.

By Theorem (3.1.16) there exists an element $a \in LS(M)$ such that $D(x) = ax - xa$ for all $x \in M$.

Let us prove that one can choose the element a from $S(M)$.

For $x \in M$ we have

$$(a + a^*)x - x(a + a^*) = (ax - xa) - (ax^* - x^*a)^* = D(x) - D(x^*)^* \in S(M)$$

and

$$(a - a^*)x - x(a - a^*) = D(x) + D(x^*)^* \in S(M).$$

This means that the elements $a + a^*$ and $a - a^*$ implement derivations from M into $S(M)$. Since $a = \frac{a+a^*}{2} + i\frac{a-a^*}{2i}$, it is sufficient to consider the case where a is a self-adjoint element.

Consider the elements $\lambda_+, \lambda_- \in L^0$ defined in Lemma (3.1.18) (c) and let us prove that $\lambda_+, \lambda_- \in L^\infty(\Omega)$. Lemma (3.1.18) (c) implies that there exists an element $\lambda_1 \in \Lambda_-$ such that $-\frac{1}{4} \le \lambda_- - \lambda_1 \le -\frac{1}{8}$. Since $D(x) = (a - \lambda_1)x - x(a - \lambda_1)$, replacing a by $a - \lambda_1$, we may assume that $-\frac{1}{4} \le \lambda_- \le -\frac{1}{8}$. Then $e_\varepsilon \in P_\infty(M)$ for all $\varepsilon < -\frac{1}{8}$ and e_ε is a finite for all $\varepsilon < -\frac{1}{4}$. In particular $\left(e_{-\frac{1}{16}} - e_{-\frac{1}{2}}\right) M \left(e_{-\frac{1}{16}} - e_{-\frac{1}{2}}\right)$ is of type I_∞, and moreover $\lambda_+ \ge -\frac{1}{2}$.

Suppose that $\lambda_+ \notin L^\infty(\Omega)$. Since $\lambda_+ \ge -\frac{1}{2}$, we have that λ_+ is unbounded from above and thus passing if necessary to the subalgebra zM, where z is a non-zero central projection in M with $z\lambda_+ \ge z$, we may assume without loss of generality that $\lambda_+ \ge \mathbf{1}$.

First let us consider the particular case where M is of type $I_{\aleph_0}$, where $\aleph_0$ is the countable cardinal number. Take an element $\lambda_0 \in St(\nabla)$ such that $\lambda_+ - \frac{1}{2} \le \lambda_0 \le \lambda_+ - \frac{1}{4}$. By Lemma (3.1.18) (c) we have $e^\perp_{\lambda_0} \in P_\infty(M)$. Since algebras $e^\perp_{\lambda_0} M e^\perp_{\lambda_0}$ and $\left(e_{-\frac{1}{16}} - e_{-\frac{1}{2}}\right) M \left(e_{-\frac{1}{16}} - e_{-\frac{1}{2}}\right)$ are algebras of type $I_{\aleph_0}$, then there exists a sequences of pairwise equivalent and pairwise orthogonal abelian projections $\{f_k\}_{k\in\mathbb{N}}$ and $\{g_k\}_{k\in\mathbb{N}}$ such that $\bigvee f_k = e^\perp_{\lambda_0}, \bigvee g_k = e_{-\frac{1}{16}} - e_{-\frac{1}{2}}$. Since $z(e^\perp_{\lambda_0}) = z\left(e_{-\frac{1}{16}} - e_{-\frac{1}{2}}\right) = \mathbf{1}$, then $z(f_k) = z(g_k) = \mathbf{1}$ for all k, and therefore $f_k \sim g_k$ for all k. Thus the projections $p_1 = e^\perp_{\lambda_0}$ and $p_2 = e_{-\frac{1}{16}} - e_{-\frac{1}{2}}$ are equivalent. From $\lambda_0 e^\perp_{\lambda_0} \le a e^\perp_{\lambda_0}$ it follows that $\lambda_0 p_1 \le p_1 a p_1$. Since $p_1 M p_1$ is of type $I_{\aleph_0}$, the center of the algebra $S(p_1 M p_1)$ coincides with the center of the algebra $p_1 M p_1$ (Proposition 3.1.3). Due to the fact that $\lambda_0 \notin L^\infty(\Omega)$ and $z(p_1) = \mathbf{1}$, we see that $\lambda_0 p_1$ is an unbounded linear operator from $LS(p_1 M p_1) \backslash S(p_1 M p_1)$. Therefore $a p_1 = p_1 a p_1 \notin S(p_1 M p_1)$.

Let u be a partial isometry in M such that $uu^* = p_1, u^*u = p_2$. Put $p = p_1 + p_2$. Consider the derivation D_1 from pMp into $pS(M)p = S(pMp)$ defined as

$$D_1(x) = pD(x)p, \qquad x \in pMp.$$

This derivation is implemented by the element $ap = pap$, i.e.

$$D_1(x) = apx - xap, \qquad x \in pMp.$$

Since $p_2 = e_{-\frac{1}{16}} - e_{-\frac{1}{2}}$ then $-\frac{1}{2} e_{-\frac{1}{2}} \le \left(e_{-\frac{1}{16}} - e_{-\frac{1}{2}}\right) a \left(e_{-\frac{1}{16}} - e_{-\frac{1}{2}}\right) \le -\frac{1}{16} e_{-\frac{1}{16}}$. Therefore $ap_2 = pMp$, the element $b = ap_1 = ap - ap_2$ implements a derivation D_2 from pMp into $S(pMp)$.

Since $D_2(u+u^*) = b(u+u^*) - (u+u^*)b$, it follows that $b(u+u^*) - (u+u^*)b \in S(M)$. From $up_1 = p_1u^* = 0$ it follows that $bu - u^*b \in S(M)$. Multiplying this by u from the left side we obtain $ubu - uu^*b \in S(M)$. From , $ub = 0, uu^* = p_1$, it follows that $p_1b \in S(M)$, i.e. $ap_1 \in S(M)$. This contradicts the above relation $ap_1 \notin S(M)$. The contradiction shows that $\lambda_+ \in L^\infty(\Omega)$. Now Lemma (3.1.18) (d) implies that $a \in S(M)$.

Let us consider the case of general type I_∞ von Neumann algebra M. Take an element $\lambda_0 \in St(\nabla)$ such that $\lambda_+ - \frac{1}{2} \le \lambda_0 \le \lambda_+ - \frac{1}{4}$. Lemma (3.1.18)(c) implies that $e_{\lambda_0}^\perp \in P_\infty(M)$. Consider projections p_1 and p_2 with the central cover **1** such that $p_1 \le e_{\lambda_0}^\perp, p_2 \le e_{\frac{1}{4}}$ and such that p_iMp_i are of type $I_{\aleph_0}, i = 1,2$. Put $p = p_1 + p_2$. Consider the derivation D_p from pMp into $pS(M)p = S(pMp)$ defined as

$$D_p(x) = pD(x)p, \qquad x \in pMp.$$

Since pMp is of type $I_{\aleph_0}$ the above case implies that $pap \in S(M)$ and therefore $p_1ap_1 \in S(M)$. On the other hand $\lambda_0p_1 \le p_1ap_1$ and $\lambda_0p_1 \notin S(M)$. From this contradiction it follows that $\lambda_+ \in L^\infty(\Omega)$. By Lemma (3.1.18) (d) we obtain that $a \in S(M)$. The proof is complete.

We obtain

Lemma (3.1.20)[3]: Let M be a type I von Neumann algebra with the center Z. Then every Z-linear derivation D on the algebra $S(M)$ is inner. In particular, if M is a type I_∞ then every derivation on $S(M)$ is inner.

Now let M be an arbitrary type I von Neumann algebra and let z_0 be the central projection in M such that z_0M is a finite von Neumann algebra and $z_0^\perp M$ is a von Neumann algebra of type I_∞. Consider a derivation D on $S(M)$ and let δ be its restriction onto its center $Z(S)$. By Lemma (3.1.20) the derivation $z_0^\perp D$ is inner and thus we have $z_0^\perp \delta \equiv 0$, i.e. $\delta = z_0\delta$.

Since z_0M is a finite type I von Neumann algebra, we have that $z_0LS(M) = z_0S(M)$. Let D_δ be the derivation on $z_0S(M) = z_0LS(M)$ defined as in (3).

Finally Lemmas (3.1.10) and (3.1.20) imply the following.

Theorem (3.1.21)[3]: Let M be a type I von Neumann algebra. Then every derivation D on the algebra $S(M)$ can be uniquely represented in the form

$$D = D_a + D_\delta,$$

where D_a is inner and implemented by an element $a \in S(M)$ and D_δ is the derivation of the form (3) generated by a derivation δ on the center of $S(M)$.

We present a general form of derivations on the algebra $S(M,\tau)$ of τ-measurable operators affiliated with a type I von Neumann algebra M and a faithful normal semi-finite trace τ.

Theorem (3.1.22)[3]: Let M be a type I von Neumann algebra with the center Z and a faithful normal semi-finite trace τ. Then every Z-linear derivation D on the algebra $S(M,\tau)$ is inner. In particular, if M is a type I_∞ then every derivation on $S(M,\tau)$ is inner.

Proof. By Theorem (3.1.16) $D(x) = ax - xa$ for some $a \in LS(M)$ and all $x \in S(M,\tau)$. Let us show that the element a can be chosen from the algebra $S(M,\tau)$. As in Lemma (3.1.18) we may assume that $a = a^*$.

Case 1. M is a homogeneous type $I_n, n \in \mathbb{N}$ von Neumann algebra. Then $LS(M) = S(M) \cong M_n(L^0(\Omega))$. By [98] a $*$-isomorphism between $S(M)$ and $M_n(L^0(\Omega))$ can be a chosen such that the element a can be represented as $a = \sum_{i=1}^n \lambda_i e_{i,i}$, where $\lambda_i = \overline{\lambda_i} \in L^0(\Omega), i = \overline{1,n}, \lambda_1 \ge \cdots \ge \lambda_n$.

Put $u = \sum_{j=1}^n e_{j,n-j+1}$. Then

$$D_a(u) = au - ua = \sum_{i=1}^{n} (\lambda_i - \lambda_{n-i+1}) e_{j,n-j+1}$$

and

$$D_a(u)^* = \sum_{i=1}^{n} (\lambda_i - \lambda_{n-i+1}) e_{n-j+1,i}.$$

Therefore $D_a(u)^* D_a(u) = \sum_{i=1}^{n} (\lambda_i - \lambda_{n-i+1})^2 e_{i,i}$, and thus $|D_a(u)| = \sum_{i=1}^{n} |\lambda_i - \lambda_{n-i+1}| e_{i,i}$, Since $\lambda_1 \geq \cdots \geq \lambda_n$, we have

$$|\lambda_i - \lambda_{n-i+1}| \geq \left|\lambda_i - \lambda_{\left[\frac{n+1}{2}\right]}\right|$$

for all $i = \overline{1,n}$.

Denote $b = \sum_{i=1}^{n} \left|\lambda_i - \lambda_{\left[\frac{n+1}{2}\right]}\right| e_{i,i}$. From (11) we obtain that $|D_a(u)| \geq b$, and thus $b \in S(M,\tau)$.

Put $v = \sum_{i=1}^{\left[\frac{n+1}{2}\right]} e_{i,i} - \sum_{j=\left[\frac{n+1}{2}\right]}^{n} e_{j,j}$. Then $vb = a - \lambda_{\left[\frac{n+1}{2}\right]} \mathbf{1}$ and $vb \in S(M,\tau)$. Therefore $a - \lambda_{\left[\frac{n+1}{2}\right]} \mathbf{1} \in S(M,\tau)$ and this element also implements the derivation D_a.

Case 2. Let M be a finite type I von Neumann algebra. Then

$$LS(M) = S(M) \cong M_n\big(L^0(\Omega)\big),$$

where $F \subseteq \mathbb{N}$. Therefore $a = \{a_n\}$, where $a_n = \sum_{i=1}^{n} \lambda_i^{(n)} e_{i,i}^{(n)}, \lambda_1^{(n)} \geq \cdots, \geq \lambda_n^{(n)}, \lambda_i^{(n)} \in L^0(\Omega)$ and $e_{i,j}^{(n)}$ are the matrix units in $M_n\big(L^0(\Omega)\big), i,j = \overline{1,n}, n \in F$.

For each $n \in F$ consider the following elements in $M_n\big(L^0(\Omega)\big)$

$$b_n = \sum_{i=1}^{n} \left|\lambda_i^{(n)} - \lambda_{\left[\frac{n+1}{2}\right]}^{(n)}\right| e_{i,i}^{(n)}$$

and

$$v_n = \sum_{i=1}^{\left[\frac{n+1}{2}\right]} e_{i,i}^{(n)} - \sum_{j=\left[\frac{n+1}{2}\right]}^{n} e_{j,j}^{(n)}.$$

Set $b = \{b_n\}_{n\in F}$ and $v = \{v_n\}_{n \in F}$. Consider the element

$$\lambda = \left\{\lambda_{\left[\frac{n+1}{2}\right]}\right\}_{n\in F} \in L^0(\Omega) \cong \prod_{n\in F} L^0(\Omega_n).$$

Similar to the case 1 we obtain that $a - \lambda\mathbf{1} = vb \in S(M,\tau)$.

Case 3. M is a type I_∞ von Neumann algebra. Since $S(M,\tau) \subseteq S(M)$ by Lemma (3.1.18) there exists an element $a \in S(M)$ such that $D(x) = ax - xa$ for all $x \in M$. Let us show that a can be picked from the algebra $S(M,\tau)$. Since $a \in S(M,\tau)$, there exists $\lambda \in \mathbb{R}, \lambda > 0$ such that $f = e_{-\lambda} \vee e_\lambda^\perp$ is a finite projection.

Suppose that $z_0 \in Z$ is a central projection such that $z_0 gMg$ is a finite von Neumann algebra, where $g = e_\lambda^\perp \wedge e_\lambda = e_\lambda - e_{-\lambda}$. $z_0\mathbf{1} = z_0 f + z_0 g$ is a finite projection and thus $z_0 = 0$. Therefore gMg is a type I_∞ von Neumann algebra, in particular $z(g) = \mathbf{1}$. There exists a central projection z in M such that $zf \preccurlyeq zg$ and $z^\perp f \succcurlyeq z^\perp g$. Since gMg is a type I_∞ von Neumann algebra, we have that $z^\perp g = 0$. From $z(g) = \mathbf{1}$ one has $z^\perp = 0$ and therefore $f \preccurlyeq g$. This means that there exists $q \leq g$ such that $q \sim f$. Let u be a partial isometry in M such that $uu^* = q, u^*u = f$. Similar to Lemma (3.1.19) we obtain that $uafu - uu^*af \in S(M,\tau)$ and $af = a(e_{-\lambda} \vee e_\lambda^\perp) \in S(M,\tau)$. Therefore $a \in S(M,\tau)$. The proof is complete.

Let N be a commutative von Neumann algebra, then $N \cong L^\infty(\Omega)$ for an appropriate measure space $(\Omega, \sum, \mu)$. It has been proved in [90], [99] that the algebra $LS(N) = S(N) \cong L^0(\Omega)$ admits nontrivial derivations if and only if the measure space $(\Omega, \sum, \mu)$ is not atomic.

Let τ be a faithful normal semi-finite trace on the commutative von Neumann algebra N and suppose that the Boolean algebra $P(N)$ of projections is not atomic. This means that there exists a projection $z \in N$ with $\tau(z) < \infty$ such that the Boolean algebra of projection in zN is continuous (i.e. has no atom). Since $zS(N,\tau) = zS(N) = S(zN)$, the algebra $zS(N,\tau)$ admits a nontrivial derivation δ. Putting

$$\delta_0(x) = \delta(zx), \qquad x \in S(N,\tau),$$

we obtain a nontrivial derivation δ_0 on the algebra $S(N,\tau)$. Therefore, we have that if a commutative von Neumann algebra N has a nonatomic Boolean algebra of projections then the algebra $S(N,\tau)$ admits a non-zero derivation.

Lemma (3.1.23)[3]: If N is a commutative von Neumann algebra with a faithful normal semi-finite trace τ and δ is a derivation on $S(N,\tau)$ then $\tau(z_\delta) < \infty$, where z_δ is the support of the derivation δ.

Proof. Suppose the opposite, i.e. $\tau(z_\delta) = \infty$. Then there exists a sequence of mutually orthogonal projections $z_n \in N, n = 1,2,\ldots$, with $z_n \le z_\delta, 1 \le \tau(z_n) < \infty$. For $z = \sup_n z_n$ we have $\tau(z) = \infty$. Since $\tau(z_n) < \infty$ for all $n = 1,2,\ldots$, it follows that $z_n S(N,\tau) = z_n S(N) = S(z_n N)$. Define a derivation $\delta_n : S(z_n N) \to S(z_n N)$ by

$$\delta_n(x) = z_n \delta(x), \qquad x \in S(z_n N).$$

Since $z_{\delta_n} = z_n$, Lemma (3.1.11) implies that for each $n \in \mathbb{N}$ there exists an element $\lambda_n \in z_n N$ such that $|\lambda_n| \le z_n$ and $|\delta_n(\lambda_n)| \ge n z_n$.

Put $\lambda = \sum_{n\ge1} \lambda_n$. Then $|\lambda| \le \sum_{n\ge1} z_n \le \mathbf{1}$ and therefore $\lambda \in S(N,\tau)$. On the other hand

$$|\delta(\lambda)| = \left|\delta\left(\sum_{n\ge1} \lambda_n\right)\right| = \left|\delta\left(\sum_{n\ge1} z_n\lambda_n\right)\right| = \left|\sum_{n\ge1} z_n\delta(\lambda_n)\right| = \sum_{n\ge1} |\delta_n(\lambda_n)| \ge \sum_{n\ge1} n z_n,$$

i.e. $|\delta(\lambda)| \ge \sum_{n\ge1} n z_n$. But $\tau(z_n) \ge 1$ for all $n \in \mathbb{N}$, i.e. $\sum_{n\ge1} n z_n \notin S(N,\tau)$. Therefore $\delta(\lambda) \notin S(N,\tau)$. The contradiction shows that $\tau(z_\delta) < \infty$. The proof is complete.

Let M be a homogeneous von Neumann algebra of type $I_n, n \in \mathbb{N}$, with the center Z and a faithful normal semi-finite trace τ. Then the algebra M is $*$-isomorphic with the algebra $M_n(Z)$ of all $n \times n$- matrices over Z, and the algebra $S(M,\tau)$ is $*$-isomorphic with the algebra $M_n(S(Z,\tau_Z))$ of all $n \times n$ matrices over $S(Z,\tau_Z)$, where τ_Z is the restriction of the trace τ onto the center Z.

Now let M be an arbitrary finite von Neumann algebra of type I with the center Z and let $\{z_n\}_{n\in F}, F \subseteq \mathbb{N}$, be a family of central projections from M with $\sup_{n\in F} z_n = \mathbf{1}$ such that the algebra M is $*$-isomorphic with the C^*-product of von Neumann algebras $z_n M$ of type In respectively, $n \in F$, i.e.

$$M \cong \bigoplus_{n\in F} z_n M.$$

In this case we have that

$$S(M,\tau) \subseteq \prod_{n\in F} S(z_n M, \tau_n),$$

where τ_n is the restriction of the trace τ onto $z_n M, n \in F$.

Suppose that D is a derivation on $S(M,\tau)$, and let δ be its restriction onto the center $S(Z,\tau_Z)$. Since δ maps each $z_n S(Z,\tau_Z) \cong Z\big(S(z_n M,\tau_n)\big)$ into itself, δ generates a derivation δ_n on $z_n S(Z,\tau_Z)$ for each $n \in F$.

Let D_{δ_n} be the derivation on the matrix algebra $M_n\left(z_n Z\big(S(M,\tau)\big)\right) \cong S(z_n M,\tau_n)$ defined as in (1). Put

$$D_\delta(\{x_n\}_{n\in F}) = \{D_{\delta_n}(x_n)\}, \qquad \{x_n\}_{n\in F} \in S(M,\tau).$$

By Lemma (3.1.23) $\tau(z_\delta) < \infty$, thus

$$z_\delta S(M,\tau) = z_\delta S(M) \cong z_\delta \prod_{n\in F} S(z_n M) = z_\delta \prod_{n\in F} S(z_n M,\tau_n),$$

and therefore $\{D_{\delta_n}(x_n)\} \in z_\delta S(M,\tau)$ for all $\{x_n\}_{n\in F} \in S(M,\tau)$. Hence we obtain that the map D_δ is a derivation on $S(M,\tau)$.

Similar to Lemma (3.1.10) one can prove the following.

Lemma (3.1.24)[3]: Let M be a finite von Neumann algebra of type I with a faithful normal semi-finite trace τ. Each derivation D on the algebra $S(M,\tau)$ can be uniquely represented in the form

$$D = D_a + D_\delta,$$

where D_a is an inner derivation implemented by an element $a \in S(M,\tau)$, and D_δ is a derivation given as (10).

Finally Theorem (3.1.22) and Lemma (3.1.24) imply the following.

Theorem (3.1.25)[3]: Let M be a type I von Neumann algebra with a faithful normal semi-finite trace τ. Then every derivation D on the algebra $S(M,\tau)$ can be uniquely represented in the form

$$D = D_a + D_\delta,$$

where D_a is inner and implemented by an element $a \in S(M,\tau)$ and D_δ is the derivation of the form (12) generated by a derivation δ on the center of $S(M,\tau)$.

If we consider the measure topology t_τ on the algebra $S(M,\tau)$ then it is clear that every non-zero derivation of the form D_δ is discontinuous in t_τ. Therefore the above Theorem (3.1.25) implies

Corollary (3.1.26)[3]: Let M be a type I von Neumann algebra with a faithful normal semi-finite trace τ. A derivation D on the algebra $S(M,\tau)$ is inner if and only if it is continuous in the measure topology.

Let A be an algebra. Denote by $Der(A)$ the space of all derivations (in fact it is a Lie algebra with respect to the commutator), and denote by $\ln Der(A)$ the subspace of all inner derivations on A (it is a Lie ideal in $Der(A)$).

The factor-space $H^1(A) = Der(A) \setminus \ln Der(A)$ is called the first (Hochschild) cohomology group of the algebra A (see [14]). It is clear that $H^1(A)$ measures how much the space of all derivations on A differs from the space on inner derivations.

The following result shows that the first cohomology groups of the algebras $LS(M), S(M)$ and $S(M,\tau)$ are completely determined by the corresponding cohomology groups of their centers (cf. [90]).

Theorem (3.1.27)[3]: Let M be a type I von Neumann algebra with the center Z and a faithful normal semi-finite trace τ. Suppose that z_0 is a central projection such that $z_0 M$ is a finite von Neumann algebra, and $z_0^\perp M$ is of type I_∞. Then

(a) $H^1\big(LS(M)\big) = H^1\big(S(M)\big) \cong H^1\big(S(z_0 Z)\big)$;

(b) $H^1\big(S(M,\tau)\big) \cong H^1\big(S(z_0 Z,\tau_0)\big)$, where τ_0 is the restriction of τ onto $z_0 Z$.

Proof. It immediately follows from Theorems (3.1.15), (3.1.21) and (3.1.25).

Corollary (3.1.28)[3]: Let M be a type I von Neumann algebra with the center Z and a faithful normal semi-finite trace τ. Consider the topological algebra $S(M,\tau)$ equipped with the measure topology. Then $H_c^1(S(M,\tau)) = \{0\}$.

Section (3.2): Algebras of Locally Measurable Operators

The theory of derivations of various classes of Banach $*$-algebras (e.g. C^*, AW^* and W^*-algebras) is very well developed (see, for example, [13], [29], [30]). It is well known that every derivation of a C^*-algebra is norm continuous and every derivation of a AW^*-algebra (in particular, of a W^*-algebra) is inner [115], [29]. The development of the theory of noncommutative integration, initiated by I. Segal's [31] prompted the introduction of numerous non-trivial $*$-algebras of unbounded operators, which, in a certain sense, are close to AW^* and W^*-algebras. The main interest here is represented by the $*$-algebra $LS(M)$ (respectively, $S(M)$) of all locally measurable (respectively, measurable) operators, affiliated with a W^*-algebra (or with a AW^*-algebra) M and also by the $*$-algebra $S(M,\tau)$ of all τ-measurable operators from $S(M)$, where τ is a faithful normal semifinite trace on M [52], [117]. The importance of the algebra $LS(M)$ for the theory of unbounded derivations on von Neumann algebras may be seen from the following classical example. Consider the algebra $M = L_\infty(0,\infty)$ equipped with the semifinite trace given by Lebesgue integration and consider (a partially defined) derivation $\delta = d/dt$ on M. A simple argument shows that the algebra $LS(M)$, which in this case coincides with the space of all measurable complex functions on $(0,\infty)$ is the only natural receptacle of δ. Similar examples can be produced in much more sophisticated circumstances and clearly indicate that the algebra $LS(M)$ is the most suitable object for studying unbounded derivations on a given von Neumann algebra M. However, the study of derivations in the setting of $LS(M)$ has been greatly impeded by the fact that it is not a Banach algebra (it is not even a Frechet algebra or locally convex algebra when endowed with its natural topology). An additional difficulty (especially, in comparison with rather well studied algebras $S(M,\delta)$) is represented by the lack of developed analytical techniques in $LS(M)$. In [2], [105], [106], [107], [108] meaningful attempts have been made to study the structure of derivations on such algebras. Of particular interest is the problem of identifying the class of von Neumann algebras, for which any derivation of the $*$-algebra $LS(M)$ is inner. In the setting of commutative W^*-algebras (respectively, commutative AW^*-algebras) this problem is fully resolved in [107] (respectively, in [112]). In the setting of von Neumann algebras of type I, a thorough treatment of this problem may be found in [2] and [106]. In [2], [107] contain examples of non-inner derivations of the $*$-algebra $LS(M)$, which are not continuous with respect to the topology $t(M)$ of local convergence in measure on $LS(M)$. The latter topology is the only topology considered on algebras $LS(M)$ to date, it may be also viewed as a noncommutative generalization of the classical topology of convergence in measure on the sets of finite measure in the case when M is given by the algebra $L^\infty(\Omega,\Sigma,\mu)$, where (Ω,Σ,μ) is a σ-finite measure space (in this case the algebra $LS(M)$ coincides with the algebra of all measurable complex functions on). It is shown in [2] that in the special case when M is a properly infinite von Neumann algebra of type I, every derivation of $LS(M)$ is continuous with respect to the local measure topology $t(M)$. Moreover, all such derivations are inner. Using a completely different technique, a similar result was also obtained in [106] under the additional assumption that the predual space M_* to M is separable. It is of interest to observe

that an analogue of this result (that is the continuity of an arbitrary derivation of $(LS(M), t(M))$) also holds for any von Neumann algebra M of type III [105]. In [105] the following problem is formulated. Let M be a von Neumann algebra of type II and let τ be a faithful, normal, semifinite trace on M. Is any derivation of a $*$-algebra $S(M, \tau)$ equipped with the (classical) measure topology generated by the trace necessarily continuous? In [108] this problem is solved affirmatively for a properly infinite algebra M. In view of the example we mentioned above, a natural problem (analogous to Problem 3 from [105]) is whether any derivation in a $*$-algebra $LS(M)$ is necessarily continuous with respect to the topology $t(M)$, where M is a properly infinite von Neumann algebra of type II. The main results provide an affirmative solution to this problem. In fact, we establish a much stronger result that any derivation $\delta: A \longrightarrow LS(M)$, where A is any subalgebra in $LS(M)$ containing the algebra M, is necessarily continuous with respect to the topology $t(M)$. The proof proceeds in two stages. Firstly, we establish the $t(M)$-continuity of any derivation $\delta: LS(M) \longrightarrow LS(M)$ for a properly infinite von Neumann algebra M. A special construction of extension of a derivation δ: $M \longrightarrow LS(M)$ up to a derivation defined on the whole algebra $LS(M)$ is given (here M is actually an arbitrary von Neumann algebra). We also hope that our approach to unbounded derivations on M as well as techniques developed for dealing with locally measurable operators and the topology of local convergence in measure are of interest in their own right and may be used elsewhere.
See von Neumann algebra theory [29], [33] and theory of locally measurable operators from [113], [117], [118].

For H be a Hilbert space, let $B(H)$ be the $*$-algebra of all bounded linear operators on H, and let 1 be the identity operator on H. Given a von Neumann algebra M acting on H, denote by $Z(M)$ the center of M and by $P(M) = \{p \in M: p = p^2 = p^*\}$ the lattice of all projections in M. Let $P_{fin}(M)$ be the set of all finite projections in M. Denote by τ_{so} the strong operator topology on $B(H)$, that is the locally convex topology generated by the family of seminorms $p_\xi(x) = \|x\xi\|_H, \xi \in H$, where $\|\cdot\|_H$ is the Hilbert norm on H.

A linear operator $x: \mathfrak{D}(x) \to H$, where the domain $\mathfrak{D}(x)$ of x is a linear subspace of H, is said to be affiliated with M if $yx \subseteq xy$ for all y from the commutant M' of algebra M.

A densely-defined closed linear operator x (possibly unbounded) affiliated with M is said to be measurable with respect to M if there exists a sequence $\{p_n\}_{n=1}^{\infty} \subset P(M)$ such that $p_n \uparrow 1, p_n(H) \subset \mathfrak{D}(x)$ and $p_n^{\perp} = 1 - p_n \in P_{fin}(M)$ for every $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers. Let us denote by $S(M)$ the set of all measurable operators.
Let $x, y \in S(M)$. It is well known that $x + y, xy$ and x^* are densely-defined and preclosed operators. The closures $\overline{x+y}$ (strong sum), $\overline{xy}$ (strong product) and x^* are also measurable, and equipped with this operations (see [31]) $S(M)$ is a unital $*$-algebra over the field $\mathbb{C}$ of complex numbers. It is clear that M is a $*$-subalgebra of $S(M)$.
A densely-defined linear operator x affiliated with M is called locally measurable with respect to M if there is a sequence $\{z_n\}_{n=1}^{\infty}$ of central projections in M such that $z_n \uparrow 1$ and $z_n x \in S(M)$ for all $n \in \mathbb{N}$.
The set $LS(M)$ of all locally measurable operators (with respect to M) is a unital $*$-algebra over the field $\mathbb{C}$ with respect to the same algebraic operations as in $S(M)$ [118] and $S(M)$ is a $*$-subalgebra of $LS(M)$. If M is finite, or if $dim(Z(M)) < \infty$, the algebras $S(M)$ and $LS(M)$ coincide [113]. If von Neumann algebra M is of type III and $dim(Z(M)) = \infty$, then $S(M) = M$ and $LS(M) \neq M$[113].

For every subset $E \subset LS(M)$, the sets of all self-adjoint (resp., positive) operators in E will be denoted by E_h (resp. E_+). The partial order in $LS(M)$ is defined by its cone $LS_+(M)$ and is denoted by $\leq$.

We shall need the following important property of the $*$-algebra $LS(M)$. Let $\{z_i\}_{i\in I}$ be a family of pairwise orthogonal non-zero central projections from M with $sup_{i\in I}\, z_i = 1$, where I is an arbitrary set of indices (in this case, the family $\{z_i\}_{i\in I}$ is called a central decomposition of the unity 1). Consider the $*$-algebra $\prod_{i\in I} LS(z_i M)$ with the coordinate-wise operations and involution and set

$$\phi: LS(M) \to \prod_{i\in I} LS(z_i M), \phi(x) := \{z_i x\}_{i\in I}.$$

Proposition (3.2.1)[104]: [113],[102]. The mapping ϕ is a $*$-isomorphism from $LS(M)$ onto $\prod_{i\in I} LS(z_i M)$.

Observe that the analogue of Proposition (3.2.1) for the $*$-algebra $S(M)$ does not hold in general [113].

Proposition (3.2.1) implies that given any central decomposition $\{z_i\}_{i\in I}$ of the unity and any family of elements $\{x_i\}_{i\in I}$ in $LS(M)$, there exists a unique element $x \in LS(M)$ such that $z_i x = z_i x_i$ for all $i \in I$. This element is denoted by $x = \sum_{i\in I} z_i x_i$.

It is shown in [114] that if M is of type I or III, then for any $x \in LS(M)$ there exists a countable central decomposition of unity $\{z_n\}_{n=1}^{\infty}$, such that $x = \sum_{n=1}^{\infty} z_n x$ and $z_n x \in M$ for all $n \in \mathbb{N}$.

Let x be a closed operator with dense domain $\mathfrak{D}(x)$ in H, let $x = u|x|$ be the polar decomposition of the operator x, where $|x| = (x^* x)^{\frac{1}{2}}$ and u is a partial isometry in $B(H)$ such that $u^* u$ is the right support $r(x)$ of x. It is known that $x \in LS(M)$ (respectively, $x \in S(M)$) if and only if $|x| \in LS(M)$ (respectively, $|x| \in S(M)$) and $u \in M$ [113]. If x is a self-adjoint operator affiliated with M, then the spectral family of projections $\{E_\lambda(x)\}_{\lambda\in R}$ for x belongs to M [113]. A locally measurable operator x is measurable if and only if $E_\lambda^{\perp}(|x|) \in P_{fin}(M)$ for some $\lambda > 0$ [113].

Let us now recall the definition of the local measure topology. First let M be a commutative von Neumann algebra. Then M is $*$-isomorphic to the $*$-algebra $L^\infty(\Omega, \Sigma, \mu)$ of all essentially bounded measurable complex-valued functions defined on a measure space (Ω, Σ, μ) with the measure μ satisfying the direct sum property (we identify functions that are equal almost everywhere) (see e.g. [33]). The direct sum property of a measure μ means that the Boolean algebra of all projections of the $*$-algebra $L^\infty(\Omega, \Sigma, \mu)$ is order complete, and for any non-zero $p \in P(M)$ there exists a non-zero projection $q \leq p$ such that $\mu(q) < \infty$. The direct sum property of a measure μ is equivalent to the fact that the functional $\tau(f) := \int_\Omega f\, d\mu$ is a semi-finite normal faithful trace on the algebra $L^\infty(\Omega, \sigma, \mu)$.

Consider the $*$-algebra $LS(M) = S(M) = L^0(\Omega, \Sigma, \mu)$ of all measurable almost everywhere finite complex-valued functions defined on (Ω, Σ, μ) (functions that are equal almost everywhere are identified).

On $L^0(\Omega, \Sigma, \mu)$, define the local measure topology $t(L^\infty(\Omega))$, that is, the Hausdorff vector topology, whose base of neighborhoods of zero is given by $W(B, \varepsilon, \delta) := \{f \in L^0(\Omega, \Sigma, \mu) :$ there exists a set $E \in \Sigma$ such that

$$E \subseteq B, \mu(B \backslash E) \leq \delta, f\chi_E \in L^\infty(\Omega, \Sigma, \mu), \|f\chi_E\|_{L^\infty(\Omega,\Sigma,\mu)} \leq \varepsilon\},$$

where $\varepsilon, \delta > 0, B \in \Sigma, \mu(B) < \infty$, and

$$\chi(\omega) = \begin{cases} 1, & \omega \in E, \\ 0, & \omega \notin E. \end{cases}$$

Convergence of a net $\{f_\alpha\}$ to f in the topology $t(L^\infty(\Omega))$, denoted by $f_\alpha \xrightarrow{t(L^\infty(\Omega))} f$, means that $f_\alpha \chi_B \longrightarrow f\chi_B$ in measure μ for any $B \in \Sigma$ with $\mu(B) < \infty$. Note, that the topology $t(L^\infty(\Omega))$ does not change if the measure μ is replaced with an equivalent measure [118].

Now let M be an arbitrary von Neumann algebra and let φ be a $*$-isomorphism from $Z(M)$ onto the $*$-algebra $L^\infty(\Omega, \Sigma, \mu)$, where μ is a measure satisfying the direct sum property. Denote by $L^+(\Omega, \Sigma, m)$ the set of all measurable real-valued functions defined on (Ω, Σ, μ) and taking values in the extended half-line $[0, \infty]$ (functions that are equal almost everywhere are identified). It was shown in [31] that there exists a mapping

$$D: P(M) \to L^+(\Omega, \Sigma, \mu)$$

that possesses the following properties:

(D1) $D(p) \in L_+^0(\Omega, \Sigma, \mu) \Leftrightarrow p \in P_{fin}(M)$;

(D2) $D(p \vee q) = D(p) + D(q)$ if $pq = 0$;

(D3) $D(u^*u) = D(uu^*)$ for any partial isometry $u \in M$;

(D4) $D(zp) = \varphi(z)D(p)$ for any $z \in P(Z(M))$ and $p \in P(M)$;

(D5) if $p_\alpha, p \in P(M), \alpha \in A$ and $p_\alpha \uparrow p$, then $D(p) = sup_{\alpha \in A} D(p_\alpha)$.

A mapping $D: P(M) \to L^+(\Omega, \Sigma, \mu)$ that satisfies properties (D1)—(D5) is called a dimension function on $P(M)$.

A dimension function D also has the following properties [31]:

(D6) if $p_n \in P(M), n \in \mathbb{N}$, then $D(sup_{n \geq 1} p_n) \leq \sum_{n=1}^\infty D(p_n)$, in addition, when $p_n p_m = 0, n \neq m$, the equality holds;

(D7) if $p_n \in P_{fin}(M), n \in \mathbb{N}, p_n \downarrow 0$, then $D(p_n) \to 0$ almost everywhere.

For arbitrary scalars $\varepsilon, \delta > 0$ and a set $B \in \Sigma, \mu(B) < \infty$, we set

$V(B, \varepsilon, \delta) := \{x \in LS(M)$: there exist $p \in P(M), z \in P(Z(M))$,

such that $xp \in M, \|xp\|_M \leq \varepsilon, \varphi(z^\perp) \in W(B, \varepsilon, \delta), D(zp^\perp) \leq \varepsilon\varphi(z)\}$, where $\|\cdot\|_M$ is the C^*-norm on M.

It was shown in [118] that the system of sets

$$\{x + V(B, \varepsilon, \delta): x \in LS(M), \varepsilon, \delta > 0, B \in \Sigma, \mu(B) < \infty\}$$

defines a Hausdorff vector topology $t(M)$ on $LS(M)$ such that the sets $\{x + V(B, \varepsilon, \delta)\}, \varepsilon, \delta > 0, B \in \Sigma, \mu(B) < \infty$ form a neighborhood base of an operator $x \in LS(M)$. It is known that $(LS(M), t(M))$ is a complete topological $*$-algebra, and the topology $t(M)$ does not depend on a choice of dimension function D and on the choice of $*$-isomorphism φ (see e.g. [113], [118]).

The topology $t(M)$ on $LS(M)$ is called the local measure topology (or the topology of convergence locally in measure). Note, that in case when $M = B(H)$ the equality $LS(M) = M$ holds [113] and the topology $t(M)$ coincides with the uniform topology, generated by the C^*-norm $\|\cdot\|_{B(H)}$.

We will need the following criterion for convergence of nets from $LS(M)$ with respect to this topology.

Proposition (3.2.2)[104]: ([113]). (i). A net $\{p_\alpha\}_{\alpha \in A} \subset P(M)$ converges to zero with respect to the topology $t(M)$ if and only if there is a net $\{z_\alpha\}_{\alpha \in A} \subset P(Z(M))$ such that $z_\alpha p_\alpha \in P_{fin}(M)$ for all $\alpha \in A, \varphi(z_\alpha^\perp) \xrightarrow{t(L^\infty(\Omega))} 0$, and $D(z_\alpha p_\alpha) \xrightarrow{t(L^\infty(\Omega))} 0$, where $t(L^\infty(\Omega))$ is the local measure topology on $L^0(\Omega, \Sigma, \mu)$, and φ is a $*$-isomorphism of $Z(M)$ onto $L^\infty(\Omega, \Sigma, \mu)$.

(ii). A net $\{x_\alpha\}_{\alpha\in A} \subset LS(M)$ converges to zero with respect to the topology $t(M)$ if and only if $E_\lambda^\perp(|x_\alpha|) \xrightarrow{t(M)} 0$ for every $\lambda > 0$, where $\{E_\lambda(|x_\alpha|)\}$ is the spectral projection family for the operator $|x_\alpha|$.

Proposition (3.2.3)[104]: If $x_\alpha \in LS(M), 0 \neq z \in P(Z(M))$, then

$$zx_\alpha \xrightarrow{t(M)} 0 \Leftrightarrow zx_\alpha \xrightarrow{t(zM)} 0.$$

Proof. Fix a $*$-isomorphism $\varphi: Z(M) \to L^\infty(\Omega, \Sigma, \mu)$ and $0 \neq z \in P(Z(M))$. Let $E \in \Sigma$ be such that $\varphi(z) = \chi_E$. Define the mapping

$$\psi: Z(zM) = zZ(M) \to L^\infty(E, \Sigma_E, \mu|_E)$$

by setting

$$\psi(za) = \varphi(za)|_E, for\ a \in Z(M).$$

Here, $\Sigma_E := \{A \cap E : A \in \Sigma\}$ and $\mu|_E$ is the restriction of μ to Σ_E.

It is clear that is a $*$-isomorphism. Now define $D_z: P(zM) \to L_+(E, \Sigma_E, \mu|_E)$ by setting $D_z(q) = D(q)|_E$ for $q \in P(zM)$. It is straightforward that D_z is a dimension function on $P(zM)$.

Let $\{q_\alpha\}_{\alpha\in A} \subset P(zM)$. We claim

$$q_\alpha \xrightarrow{t(M)} 0 \Leftrightarrow q_\alpha \xrightarrow{t(zM)} 0.$$

To see the claim, assume that the first convergence holds and observe that by Proposition (3.2.2)(i), there exists a net $\{z_\alpha\}_{\alpha\in A} \subset P(Z(M))$ such that $z_\alpha q_\alpha \in P_{fin}(M)$ for any $\alpha \in A, \varphi(z_\alpha^\perp) \xrightarrow{t(L^\infty(\Omega))} 0$, and $D(z_\alpha q_\alpha) \xrightarrow{t(L^\infty(\Omega))} 0$. The projection $r_\alpha = zz_\alpha$ belongs to the center $Z(zM)$ of the von Neumann algebra zM, and $r_\alpha q_\alpha = z_\alpha q_\alpha$ is a finite projection in zM for each $\alpha \in A$. Also

$$\psi(z - r_\alpha) = \psi(z(1 - z_\alpha)) = \varphi(zz_\alpha^\perp)|_E = \varphi(z)\varphi(z_\alpha^\perp)|_E \xrightarrow{t(L^\infty(E))} 0,$$

where $t(L^\infty(E))$ is the local measure topology on $L^0(E, \Sigma_E, \mu|_E)$, and

$$D_z(r_\alpha q_\alpha) = D_z(z_\alpha q_\alpha) = D(z_\alpha q_\alpha)|_E \xrightarrow{t(L^\infty(E))} 0.$$

Hence, by Proposition (3.2.2) (i) we get that $q \xrightarrow{t(zM)} 0$.

We will show now that the convergence $q_\alpha \xrightarrow{t(zM)} 0$ for $\{q_\alpha\}_{\alpha\in A} \subset P(zM)$ implies the convergence $q_\alpha \xrightarrow{t(M)} 0$.

Let $\{r_\alpha\}_{\alpha\in A}$ be a net in $P(Z(zM))$ such that $r_\alpha q_\alpha \in P_{fin}(zM)$ for every $\alpha \in A$,

$$\psi(z - r_\alpha) \xrightarrow{t(L^\infty(E))} 0$$

and

$$D_z(r_\alpha q_\alpha) \xrightarrow{t(L^\infty(E))} 0.$$

Put $z_\alpha = z^\perp + r_\alpha$. Then $z_\alpha \in P(Z(M))$ and $z_\alpha q_\alpha = r_\alpha q_\alpha \in P_{fin}(M)$.

Since $z_\alpha^\perp = z(1 - r_\alpha)$, we have $\varphi(z_\alpha^\perp) = \chi_E\varphi(z_\alpha^\perp)$ and

$$\varphi(z_\alpha^\perp)|_E = \chi_E\varphi(z(1 - r_\alpha))|_E = \chi_E\psi(z - r_\alpha) \xrightarrow{t(L^\infty(E))} 0.$$

Also

$$D(z_\alpha q_\alpha) = D(zr_\alpha q_\alpha) = \chi_E D(r_\alpha q_\alpha),$$

and so $D(z_\alpha q_\alpha) \xrightarrow{t(L^\infty(\Omega))} 0$, since $D(r_\alpha q_\alpha)|_E = D_z(r_\alpha q_\alpha) \xrightarrow{t(L^\infty(E))} 0$. Again appealing to Proposition (3.2.2)(i), we conclude that $q_\alpha \xrightarrow{t(M)} 0$.

Now let $\{x_\alpha\} \subset LS(zM)$ and $x_\alpha \xrightarrow{t(M)} 0$. By Proposition (3.2.2) (ii), we have that $E_\lambda^\perp(|x_\alpha|) \xrightarrow{t(M)} 0$ for any $\lambda > 0$, where $\{E_\lambda(|x_\alpha|)\}$ is the spectral family for $|x_\alpha|$. Denote by $\{E_\lambda^z(|x_\alpha|)\}$ the family of spectral projections for $|x_\alpha|$ in $LS(zM), \lambda > 0$. It is clear that $E_\lambda(|x_\alpha|) = z^\perp + E_\lambda^z(|x_\alpha|)$ and $E_\lambda^\perp(|x_\alpha|) = z - E_\lambda^z(|x_\alpha|)$ for all $\lambda > 0$. It follows from above that $z - E_\lambda^z(|x_\alpha|) \xrightarrow{t(zM)} 0$ for all $\lambda > 0$. Hence, by Proposition (3.2.2)(ii), it follows that $x_\alpha \xrightarrow{t(zM)} 0$.

The proof of the implication $x_\alpha \xrightarrow{t(zM)} 0 \Rightarrow x_\alpha \xrightarrow{t(M)} 0$ is similar and therefore omitted.

The lattice $P(M)$ is said to have a countable type, if every family of non-zero pairwise orthogonal projections in $P(M)$ is, at most, countable. A von Neumann algebra is said to be σ-finite, if the lattice $P(M)$ has a countable type. It is shown in [31] that a finite von Neumann algebra M is σ-finite, provided that the lattice $P(Z(M))$ of central projections has a countable type.

If M is a commutative von Neumann algebra and $P(M)$ has a countable type, then M is $*$-isomorphic to a $*$-algebra $L^\infty(\Omega, \Sigma, \mu)$ with $\mu(\Omega) < \infty$. In this case, the topology $t(L^\infty(\Omega))$ is metrizable and has a base of neighborhoods of 0 consisting of the sets $W(, 1/n, 1/n), n \in \mathbb{N}$. In addition, $f_n \xrightarrow{t(L^\infty(\Omega))} 0 \Leftrightarrow f_n \longrightarrow 0$ in measure μ, where $f_n, f \in L^0(\Omega, \Sigma, \mu) = LS(M)$.

Let M be a commutative von Neumann algebra such that $P(M)$ does not have a countable type. Denote by φ a $*$-isomorphism from Mon $L^\infty(\Omega, \Sigma, \mu)$, where μ is a measure with the direct sum property.

Due to the latter property, there exists a family $\{z_i\}_{i \in I}$ of non-zero pairwise orthogonal projections from $P(M)$, such that $sup_{i \in I}\, z_i = 1$ and $\mu(\varphi(z_i)) < \infty$ for all $i \in I$, in particular, $P(z_i Z(M))$ has a countable type. Select $A_i \in \Sigma$ so that $\varphi(z_i) = \chi_{A_i}$ and set

$$\Sigma_{A_i} = \{A \cap A_i : A \in \Sigma\}, \mu_i(A \cap A_i) = \mu(A \cap A_i), \qquad i \in I.$$

Let $t(L^\infty(A_i))$ be the local measure topology on $L^0(A_i, \Sigma_{A_i}, \mu_i)$. Since $\mu_i(A_i) < \infty$, we see that the topology $t(L^\infty(A_i))$ coincides with the topology of convergence in measure μ_i in $L^0(A_i, \Sigma_{A_i}, \mu_i)$.

Proposition (3.2.4)[104]: For a net $\{f_\alpha\}_{\alpha \in A}$ and f from $L^0(\Omega, \Sigma, \mu)$ the following conditions are equivalent:

(i). $f_\alpha \xrightarrow{t(L^\infty(\Omega))} f$;

(ii). $f_\alpha \chi_{A_i} \xrightarrow{t(L^\infty(A_i))} f_\alpha A_i$ for all $i \in I$.

Proof. The implication (i) ⇒ (ii) follows from the definitions of topologies $t(L^\infty(\Omega))$ and $t(L^\infty(A_i))$.

(ii) ⇒ (i). It is sufficient to consider the case when $f = 0$.

Consider the set Γ of all finite subsets γ from I and order it with respect to inclusion. Consider an increasing net $\chi_{D_\gamma} \uparrow \chi_\Omega$ in $L_h^0(\Omega, \Sigma, \mu)$, where $D_\gamma = \cup_{i \in \gamma} A_i, \gamma \in \Gamma$. Take an arbitrary neighborhood of zero U (in the topology $(L^\infty(\Omega))$) and select $W(B, \varepsilon, \delta)$ in such a way that $W(B, \varepsilon, \delta) + W(B, \varepsilon, \delta) \subset U$. Since $\mu(B \cap D_\lambda) \uparrow \mu(B) < \infty$, we can locate such $\gamma_0 \in \Gamma$ that $\mu(B \setminus D_{\gamma_0}) \leq \delta$. Hence, $f_\alpha \chi_\Omega \setminus D_{\gamma_0} \in W(B, \varepsilon, \delta)$ for all $\alpha \in A$.

Since $f_\alpha \chi_{A_i} \xrightarrow{t(L^\infty(A_i))} 0$ for all $i \in 0$ and 0 is a finite set, it follows $f_\alpha \chi_{D_{\gamma_0}} = \sum_{i \in \gamma_0} f_\alpha \chi_{A_i} \xrightarrow{t(L^\infty(\Omega))} 0$.

Thus, there exists such $\alpha_0 \in A$ that $f_\alpha \chi_{D_{\gamma_0}} \in W(B,\varepsilon,\delta)$ for all $\alpha \geq \alpha_0$.

In particular,

$$f_\alpha = f_\alpha \chi_{D_{\gamma_0}} + f_\alpha \chi_{\Omega} \backslash D_{\gamma_0} \in W(B,\varepsilon,\delta) + W(B,\varepsilon,\delta) \subset U, \forall \alpha \geq \alpha_0,$$

which implies the convergence $f_\alpha \xrightarrow{t(L^\infty(\Omega))} 0$.

Let us now establish a variant of Proposition (3.2.4) for an arbitrary von Neumann algebra M.

Let φ be a $*$-isomorphism from $Z(M)$ onto $L^\infty(\Omega,\Sigma,\mu)$ and let $\{z_i\}_{i\in I}$ be a central decomposition of the unity. As before, we denote Γ the directed set of all finite subsets from I. For every $\gamma \in \Gamma$ we set $z^{(\gamma)} = \sum_{i\in\gamma} z_i$.

Since $\varphi(z_i) = \chi_{A_i}$ for some $A_i \in \Sigma$, we see that $\varphi(z^{(\gamma)}) = \chi_{D_\gamma}$, where $D_\gamma = \bigcup_{i\in\gamma} A_i$, and, in addition, $z^{(\gamma)} \uparrow 1$ which implies $z^{(\gamma)} \xrightarrow{t(M)} 1$ (see Proposition (3.2.2) (i) for $p_\alpha = z_\alpha^\perp$). As it the proof of Proposition (3.2.4) for a given $V(B,\varepsilon,\delta)$ we choose $\gamma_0 \in \Gamma$ such that $x(1 - z^{(\gamma_0)}) \in V(B,\varepsilon,\delta)$ for every $x \in LS(M)$. If $x_\alpha \in LS(M)$ and $x_\alpha z_i \xrightarrow{t(z_iM)} 0$ for all $i \in I$, then by Proposition (3.2.3), we have $x_\alpha z_i \xrightarrow{t(M)} 0$ for all $i \in I$, and so $x_\alpha z^{(\gamma_0)} = \sum_{i\in\gamma_0} x_\alpha z_i \xrightarrow{t(M)} 0$. Hence, there exists such $\alpha_0 \in A$ that $x_\alpha z^{(\gamma_0)} \in V(B,\varepsilon,\delta)$ for all $\alpha \geq \alpha_0$. This means that

$$x_\alpha = x_\alpha z^{(\gamma_0)} + x_\alpha(1 - z^{(\gamma_0)}) \in V(B,\varepsilon,\delta) + V(B,\varepsilon,\delta) \subset V(B,2\varepsilon,2\delta).$$

The argument above justifies the following result.

Proposition (3.2.5)[104]: Let M be an arbitrary von Neumann algebra, $x_\alpha, x \in LS(M), 0 \neq z_i \in P(Z(M)), z_i z_j = 0$ when $i \neq j, sup_{i\in I} z_i = 1$. The following conditions are equivalent:

(i). $x_\alpha \xrightarrow{t(M)} x$;

(ii). $z_i x_\alpha \xrightarrow{t(z_iM)} z_i x$ for any $i \in I$.

Corollary (3.2.6)[104]: Let M and let $\{z_i\}_{i\in I}$ satisfy the same assumptions of Proposition (3.2.5), and let $T: LS(M) \to LS(M)$ be a linear operator such that $T(z_i x) = z_i T(x)$ for all $x \in LS(M), i \in I$. The following conditions are equivalent:

(i). The mapping $T: (LS(M), t(M)) \to (LS(M), t(M))$ is continuous;

(ii). The mapping $Tz_i: (LS(z_iM), t(z_iM)) \to (LS(z_iM), t(z_iM))$ is continuous for every $i \in I$.

Let M be an arbitrary von Neumann algebra, let A be a subalgebra in $LS(M)$. A linear mapping $\delta: A \to LS(M)$ is called a derivation on A with values in $LS(M)$, if $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in A$.

Each element $a \in A$ defines a derivation $\delta_a(x) := ax - xa$ on A with values in A. Derivations $\delta_a, a \in A$ are said to be inner derivations on A. Since the operation of multiplication is continuous with respect to the topology $t(M)$, it immediately follows that any inner derivation of A is continuous with respect to the topology $t(M)$.

We list a few properties of derivations on A which we shall need below.

Lemma (3.2.7)[104]: If $P(Z(M)) \subset A, \delta$ is a derivation on A and $z \in P(Z(M))$, then $\delta(z) = 0$ and $\delta(zx) = z\delta(x)$ for all $x \in A$.

Proof. We have that $\delta(z) = \delta(z^2) = \delta(z)z + z\delta(z) = 2z\delta(z)$. Hence, $z\delta(z) = z(2z\delta(z)) = 2z\delta(z)$, that is $z\delta(z) = 0$. Therefore, we have $\delta(z) = 0$. In particular, $\delta(zx) = \delta(z)x + z\delta(x) = z\delta(x)$.

Let A be an $*$-subalgebra in $LS(M)$, let δ be a derivation on A with values in $LS(M)$. Let us define a mapping

$$\delta^*: A \to LS(M),$$

by setting $\delta^*(x) = (\delta(x^*))^*, x \in A$. A direct verification shows that δ^* is also a derivation on A.
A derivation δ on A is said to be self-adjoint, if $\delta = \delta^*$. Every derivation δ on A can be represented in the form $\delta = Re(\delta) + iIm(\delta)$, where $Re(\delta) = (\delta + \delta^*)/2, Im(\delta) = (\delta - \delta^*)/2i$ are self-adjoint derivations on A.
Since $(LS(M), t(M))$ is a topological $*$-algebra, the following result holds.
Lemma (3.2.8)[104]: A derivation $\delta: A \to LS(M)$ is continuous with respect to the topology $t(M)$ if and only if the self-adjoint derivations $Re(\delta)$ and $Im(\delta)$ are continuous with respect to that topology.
As we already stated , in the special case, when M is a properly infinite von Neumann algebra of type I or von Neumann algebra of type III, any derivation of the algebra $LS(M)$ is continuous with respect to the topology $t(M)$ [105]. The next theorem extends this result to an arbitrary properly infinite von Neumann algebra.
Theorem (3.2.9)[104]: If M properly infinite von Neumann algebras, then any derivation $\delta: LS(M) \to LS(M)$ is continuous with respect to the topology $t(M)$ of local convergence in measure.
Proof. By Lemma (3.2.8), we may assume that $\delta^* = \delta$. Since $Z(M)$ is a commutative von Neumann algebra, there exists a system $\{z_i\}, i \in I$ of non-zero pairwise orthogonal projections from $Z(M)$ such that $sup_{i\in I} z_i = 1$ and the Boolean algebra $P(z_i Z(M))$ has a countable type for all $i \in I$. By Lemma (3.2.7) we have that $\delta(z_i x) = z_i \delta(x)$ for all $x \in LS(M), i \in I$. Therefore, by Corollary (3.2.6), it is sufficient to prove that each derivation δ_{z_i} is $t(z_i M)$-continuous, $i \in I$. Thus, we may assume without loss of generality that the Boolean algebra $P(Z(M))$ has a countable type.

In this case the topology $t(M)$ is metrizable, and the sets $V(\Omega, 1/n, 1/n), n \in \mathbb{N}$ form a countable base of neighborhoods of 0; in particular, $(LS(M), t(M))$ is an F-space. Therefore it is sufficient to show that the graph of the linear operator δ is a closed set.

Arguing by a contradiction, let us assume that the graph of δ is not closed. This means that there exists a sequence $\{x_n\} \subset LS(M)$, such that $x_n \xrightarrow{t(M)} 0$ and $\delta(x_n) \xrightarrow{t(M)} x \neq 0$. Recalling that $(LS(M), t(M))$ is a topological $*$-algebra and that $\delta = \delta^*$, we may assume that $x = x^*, x_n = x_n^*$ for all $n \in \mathbb{N}$. In this case, $x = x_+ - x_-$, where $x_+, x_- \in LS_+(M)$ are respectively the positive and negative parts of x. Without loss of generality, we shall also assume that $x_+ \neq 0$, otherwise, instead of the sequence $\{x_n\}$ we consider the sequence $\{-x_n\}$. Let us select scalars $0 < \lambda_1 < \lambda_2$ so that the projection

$$p := E_{\lambda_2}(x) - E_{\lambda_1}(x)$$

does not vanish. We have that $0 < \lambda_1 p \leq pxp = px \leq \lambda_2 p$ and $\|px\|_M \leq \lambda_2$. Replacing, if necessary, x_n on x_n/λ_1, we may assume that

$$pxp \geq p. \tag{1}$$

By the assumption, M is a properly infinite von Neumann algebra and therefore, there exist pairwise orthogonal projections $\left\{p_m^{(1)}\right\}_{m=1}^{\infty} \subset P(M)$, such that $sup_{m\geq 1}\, p_m^{(1)} = 1$ and $p_m^{(1)} \sim 1$ for all $m \in \mathbb{N}$, in particular, $p \preccurlyeq p_m^{(1)}$. Here, the notation $p \sim q$ denotes the equivalence of projections $p, q \in P(M)$, and the notation $p \preccurlyeq q$ means that there exists a projection $e \leq q$ such that $p \sim e$. In course of the proof of our main result we shall frequently use the following well-known fact: if $p \sim q$ and $z \in P(Z(M))$ then $pz \sim qz$.

For every $m \in \mathbb{N}$ we select a projection $p_m \leq p_m^{(1)}$, for which $p_m \sim p$ and denote by v_m a partial isometry from M such that $v_m^* v_m = p, v_m v_m^* = p_m$. Clearly, we have $p_m p_k = 0$ whenever $m \neq k$ and the projection

$$p_0 := sup_{m\geq 1} p_m \qquad (2)$$

is infinite as the supremum of pairwise orthogonal and equivalent projections. Taking into account that

$$p_m = v_m p {v^*}_m \overset{(1)}{\leq} v_m pxp v_m^* = v_m x v_m^* \in p_m M p_m,$$

and

$$\|v_m x v_m^*\|_M = \|v_m pxp v_m^*\|_M \leq \|pxp\|_M \leq \lambda_2,$$

we see that the series $\sum_{m=1}^{\infty} v_m x v_m^*$ converges with respect to the topology τ_{so} to some operator $y \in M$ satisfying

$$\|y\|_M = sup_{m\geq 1}\|v_m x v_m^*\|_M \leq \|pxp\|_M, \text{ and } y \geq p_0. \qquad (3)$$

In what follows, we shall assume that the central support $c(p_0)$ of the projection p_0 is equal to 1 (otherwise, we replace the algebra M with the algebra $c(p_0)M$).

Let φ be a $*$-isomorphism from $Z(M)$ onto $L^{\infty}(\Omega, \Sigma, \mu)$. By the assumption, the Boolean algebra $P(Z(M))$ has a countable type, and so we may assume that $\mu(\Omega) = \int_{\Omega} 1_{L^{\infty}(\Omega)} d\mu = 1$, where $1_{L^{\infty}(\Omega)}$ is the identity of the $*$-algebra $L^{\infty}(\Omega, \Sigma, \mu)$. In this case, the countable base of neighborhoods of 0 in the topology $t(M)$ is formed by the sets $V(\Omega, 1/n, 1/n), n \in \mathbb{N}$.

Recalling that we have $x_n \xrightarrow{t(M)} 0$ and $\delta(x_n) \xrightarrow{t(M)} x$, we obtain

$$v_m x_n v_m^* \xrightarrow{t(M)} 0, \delta(v_m) x_n v_m^* \xrightarrow{t(M)} 0, v_m \delta(x_n) v_m^* \xrightarrow{t(M)} v_m x v_m^*$$

when $n \to \infty$ for every fixed $m \in \mathbb{N}$.

Fix $k \in \mathbb{N}$, and using the convergence $v_m x_n v_m^* \xrightarrow{t(M)} 0$ for $n \to 0$, for each $m \in \mathbb{N}$ select an index $n_1(m,k)$ and projections $q_{m,n}^{(1)} \in P(M), z_{m,n}^{(1)} \in P(Z(M))$, such that

$$\left\| v_m x_n v_m^* q_{m,n}^{(1)} \right\|_M \leq 2^{-m}(k+1)^{-1};$$

$$\int_{\Omega} \varphi(1 - z_{m,n}^{(1)}) d\mu \leq 3^{-1} 2^{-m-k-1}$$

and

$$D(z_{m,n}^{(1)}(1 - q_{m,n}^{(1)})) \leq 3^{-1} 2^{-m-k-1} \varphi(z_{m,n}^{(1)})$$

for all $n \geq n_1(m,k)$.

Similarly, using the convergence $\delta(v_m) x_n v_m^* \xrightarrow{t(M)} 0$ (respectively, $v_m \delta(x_n) v_m^* \xrightarrow{t(M)} v_m x v_m^*$) for $n \to \infty$, for each $m \in \mathbb{N}$ select indexes $n_2(m,k)$ and $n_3(m,k)$ and projections $q_{m,n}^{(2)}, q_{m,n}^{(3)} \in P(M), z_{m,n}^{(2)}, z_{m,n}^{(3)} \in P(Z(M))$, such that

$$\left\| \delta(v) x_n v_m^* q_{m,n}^{(2)} \right\|_M \leq (3(k+1)2^m)^{-1}$$

(respectively, $\left\| (v_m \delta(x_n) v_m^* - v_m x v_m^*) q_{m,n}^{(3)} \right\|_M \leq (3(k+1)2^m)^{-1}$);

$$\int_{\Omega} \varphi(1 - z_{m,n}^{(i)}) d\mu \leq 3^{-1} 2^{-m-k-1}$$

and $D(1 - q_{m,n}^{(i)}) \leq 3^{-1} 2^{-m-k-1} \varphi(z_{m,n}^{(i)}), i = 2, 3$, for all $n \geq n_2(m,k)$ (respectively, $n \geq n_3(m,k)$).

Set $n(m,k) = max_{i=1,2,3}\, n_i(m,k), z_m = inf_{i=1,2,3}\, z^{(i)}_{m,n(m,k)}, q_m = inf_{i=1,2,3} q^{(i)}_{m,n(m,k)}$. Due to the selection of projections $q_m \in P(M), z_m \in P(Z(M))$ and indexes $n(m,k)$, we have that for each $m \in \mathbb{N}$ inequalities hold

$(A1)$ $\|v_m x_{n(m,k)} v_m^* q_m\|_M \leq 2^{-m}(k+1)^{-1}$;

$(A2)$ $\|\delta(v_m) x_{n(m,k)} v_m^* q_m\|_M \leq (3(k+1)2^m)^{-1}$;

$(A3)$ $\|q_m(v_m \delta(x_{n(m,k)}) v_m^* - v_m x v_m^*)\|_M \leq (3(k+1)2^m)^{-1}$;

$(A4)$ $D(z_m(1-q_m)) \overset{(D6)}{\leq} D(z_m(1-q^{(i)}_{m,n(m,k)})) \leq 2^{-m-k-1}\varphi(z_m)$;

$(A5)$ $1 - \int_\Omega \varphi(z_m) d\mu = \int_\Omega \varphi(1-z_m) d\mu \leq$

$$\leq \sum_{i=1}^{3} \int_\Omega \varphi(1 - z^{(i)}_{m,n(m,k)}) d\mu \leq 2^{-m-k-1}.$$

Fix $m_1, m_2 \in \mathbb{N}$ with $m_1 < m_2$ and set

$$q_{m_1,m_2} := inf_{m_1<m\leq m_2} q_m, z_{m_1,m_2} := inf_{m_1<m\leq m_2} z_m.$$

Since $(1 - z_{m_1,m_2}) = sup_{m_1<m\leq m_2}(1 - z_m)$ and $(1 - q_{m_1,m_2}) = sup_{m_1<m\leq m_2}(1 - q_m)$, it follows that $\varphi(1 - z_{m_1,m_2}) = sup_{m_1<m\leq m_2}\varphi(1 - z_m)$ and $\varphi(1 - q_{m_1,m_2}) = sup_{m_1<m\leq m_2}\varphi(1 - q_m)$, and therefore

$$1 - \int_\Omega \varphi(z_{m_1,m_2}) d\mu = \varphi(1 - z_{m_1,m_2}) d\mu \leq \varphi(1 - z_m) d\mu \overset{(A5)}{\leq} 2^{-m_1-k-1}; \quad (4)$$

$$(z_{m_1,m_2}(1 - q_{m_1,m_2})) \overset{(D6)}{\leq} \sum_{m=m_1+1}^{m_2} D\left(z_{m_1,m_2}(1 - q_m)\right) \overset{(A4)}{\leq} 2^{-m_1-k-1}\varphi(z_{m_1,m_2}) \quad (5)$$

and

$$\left\|\sum_{m=m_1+1}^{m_2} (v_m x_{n(m,k)} v_m^*) q_{m_1,m_2}\right\|_M$$

$$\leq \sum_{m=m_1+1}^{m_2} \|(v_m x_{n(m,k)} v_m^*) q_m\|_M \overset{(A1)}{\leq} 2^{m_1}(k+1)^{-1}. \quad (6)$$

Inequalities (4)-(6) mean that the sequence

$$S_{l,k} = \sum_{m=1}^{l} v_m x_{n(m,k)} v_m^*, l \geq 1$$

is a Cauchy sequence in the F-space $(LS(M, t(M)))$ for each fixed $k \in \mathbb{N}$. Consequently, there exists $y_k \in LS(M)$ such that $S_{l,k} \xrightarrow{t(M)} y_k$ for $l \to \infty$, i.e. the series

$$y_k = \sum_{m=1}^{\infty} v_m x_{n(m,k)} v_m^* \quad (7)$$

converges in $LS(M)$ with respect to the topology $t(M)$. Since the involution is continuous in topology $t(M)$ and $S^*_{l,k} = S_{l,k}$, we conclude $y_k = y^*_k$.

Setting

$$r_m := p_m \wedge q_m, m \in \mathbb{N}, \quad (8)$$

and using the relation $z_m(p_m - p_m \wedge q_m) \sim z_m(p_m \vee q_m - q_m)$ (see e.g. [33]) we have

$$D(z_m(p_m - r_m)) = D(z_m(p_m - p_m \wedge q_m)) \overset{(D3)}{=} D(z_m(p_m \vee q_m - q_m))$$
$$\leq D(z_m(1 - q_m)) \overset{(A4)}{\leq} 2^{-m-k-1}\varphi(z_m). \qquad (9)$$

Setting

$$q_0^{(k)} := sup_{m\geq 1} r_m, z_0^{(k)} := inf_{m\geq 1} z_m, \qquad (10)$$

we have (see (2), (3) and (8))

$$y \geq p_0 \geq q_0^{(k)}, k \in \mathbb{N}. \qquad (11)$$

From (4) it follows that

$$1 - \int_\Omega \varphi(z_0^{(k)})d\mu \ = \int_\Omega \varphi(1 - z_0^{(k)})d\mu \ \leq 2^{-k-1}. \qquad (12)$$

Since $p_m p_j = 0, m \neq j$, and $r_m \leq p_m$ (see (8)) we obtain $p_0 - q_0^{(k)} = sup_{m\geq 1}(p_m - r_m)$ and hence, by (9),

$$D(z_0^{(k)}(p_0 - q_0^{(k)})) \overset{(D6)}{=} \sum_{m=1}^{\infty} D(z_0^{(k)}(p_m - r_m)) \overset{(9)}{\leq} 2^{-k-1}\varphi(z_0^{(k)}). \qquad (13)$$

Due to (8), we have $p_m q_0^{(k)} = r_m q_0^{(k)} = r_m = r_m q_m$ for all $m \in \mathbb{N}$.
Hence,

$$v_m x_{n(m,k)} v_m^* q_0^{(k)} = v_m x_{n(m,k)} v_m^* p_m q_0^{(k)} = v_m x_{n(m,k)} v_m^* r_m$$

and

$$\left\| y_k q_0^{(k)} \right\|_M = \left\| \left(\sum_{m=1}^{\infty} v_m x_{n(m,k)} v_m^* \right) q_0^{(k)} \right\|_M =$$
$$= \left\| \sum_{m=1}^{\infty} v_m x_{n(m,k)} v_m^* \, q_0^{(k)} \right\|_M \leq sup_{m\geq 1} \left\| v_m x_{n(m,k)} v_m^* r_m \right\|_M$$
$$\leq sup_{m\geq 1} \left\| v_m x_{n(m,k)} v_m^* q_m \right\|_M \overset{(A1)}{\leq} (k+1)^{-1}. \qquad (14)$$

Using the properties of the derivation δ and equalities $p_n v_n = v_n, v_n^* = v_n^* \, p_n$ and (8), (10), we have

$$q_0^{(k)} \delta\big(v_m x_{n(m,k)} v_m^*\big) q_0^{(k)} = q_0^{(k)}((\delta(v_m x_{n(m.k)} v_m^*) - v_m x v_m^*) + v_m x v_m^*) q_0^{(k)}$$
$$= (q_0^{(k)} \delta(v_m) x_{n(m,k)} v_m^* q_0^{(k)} + q_0^{(k)} v_m x_{n(m,k)} \delta(v_m^*) q_0^{(k)})$$
$$+ q_0^{(k)}(v_m \delta(x_{n(m,k)}) v_m^* - v_m x v_m^*) q_0^{(k)} + q_0^{(k)}(v_m x v_m^*) q_0^{(k)}$$
$$= q_0^{(k)} \delta(v_m) x_{n(m,k)} v_m^* q_m r_m + r_m q_m v_m x_{n(m,k)} \delta(v_m^*) q_0^{(k)}$$
$$+ r_m q_m (v_m \delta(x_{n(m,k)}) v_m^* - v_m x v_m^*) q_m r_m + q_0^{(k)}(v_m x v_m^*) q_0^{(k)}.$$

Consider the following formal series suggested by the preceding

$$\sum_{m=1}^{\infty} q_0^{(k)} \delta(v_m) x_{n(m,k)} v_m^* q_m r_m \, ; \qquad (15)$$

$$\sum_{m=1}^{\infty} r_m q_m v_m x_{n(m,k)} \delta(v_m^*) q_0^{(k)} \ ; \qquad (16)$$

$$\sum_{m=1}^{\infty} r_m q_m (v_m \delta(x_{n(m,k)}) v_m^* - v_m x v_m^*) q_m r_m \, ; \qquad (17)$$

$$\sum_{m=1}^{\infty} q_0^{(k)}\left(v_m x_{n(m,k)} v_m^*\right) q_0^{(k)} . \quad (18)$$

By the condition $(A2)$ the first series (15) and the second series (16) converge with respect to the norm $\|.\|_M$ to some elements $a, b \in M$ respectively and $\|a\|_M \leq (3(k+1))^{-1}$ and $\|b\|_M \leq (3(k+1))^{-1}$.

Similarly, by the condition $(A3)$, the third series (17) also converges with respect to the norm $\|.\|_M$ to some element $c \in M$, satisfying $\|c\|_M \leq (3(k+1))^{-1}$. Finally, since $y = \sum_{m=1}^{\infty} v_m x v_m^*$ (the convergence of the latter series is taken in the τ_{so} topology), we see that the fourth series (18) converges with respect to the topology τ_{so} to some element $q_0^{(k)} y q_0^{(k)}$. Hence, the series

$$\sum_{m=1}^{\infty} q_0^{(k)} \delta(v_m x_{n(m,k)} v_m^*) q_0^{(k)} \quad (19)$$

converges with respect to the topology τ_{so} to some element $a_k \in M$, and, in addition, we have

$$\left\| a_k - q_0^{(k)} y q_0^{(k)} \right\|_M \leq (k+1)^{-1}. \quad (20)$$

We shall show that

$$a_k = q_0^{(k)} \delta(y_k) q_0^{(k)}, \quad (21)$$

where $y_k = v_m x_{n(m,k)} v_m^*$ (the convergence of the latter series is taken in the $t(M)$-topology (see (7)). Using (10) for any $m_1, m_2 \in \mathbb{N}$ we have

$$r_{m_1} q_0^{(k)} \delta(y_k) q_0^{(k)} r_{m_2} = \delta(r_{m_1} q_0^{(k)} y_k) q_0^{(k)} r_{m_2} - \delta(r_{m_1} q_0^{(k)}) y_k q_0^{(k)} r_{m_2}$$
$$= \delta(r_{m_1} v_{m_1} x_{n(m_1,k)} v_{m_1}^*) r_{m_2} - \delta(r_{m_1}) v_{m_2} x_{n(m_2,k)} v_{m_2}^* r_{m_2}.$$

Since the series $\sum_{m=1}^{\infty} q_0^{(k)} \delta(v_m x_{n(m,k)} v_m^*) q_0^{(k)}$ converges with respect to the topology τ_{so} (see (19)), it follows that the series

$$\sum_{m=1}^{\infty} r_{m_1} (q_0^{(k)} \delta(v_m x_{n(m,k)} v_m^*) q_0^{(k)}) r_{m_2}$$

also converges with respect to this topology ([116]), in addition, the following equalities hold

$$r_{m_1} a_k r_{m_2} = r_{m_1} (q_0^{(k)} \delta(v_m x_{n(m,k)} v_m^*) q_0^{(k)}) r_{m_2}$$
$$\stackrel{(10)}{=} \sum_{m=1}^{\infty} r_{m_1} \delta(v_m x_{n(m,k)} v_m^*) r_{m_2}$$
$$= \sum_{m=1}^{\infty} (\delta(r_{m_1} v_m x_{n(m,k)} v_m^*) r_{m_2} - \delta(r_{m_1}) v_m x_{n(m,k)} v_m^* r_{m_2})$$
$$\stackrel{(8)}{=} \delta(r_{m_1} v_{m_1} x_{n(m_1,k)} v_{m_1}^*) r_{m_2} - \delta(r_{m_1}) v_{m_2} x_{n(m_2,k)} v_{m_2}^* r_{m_2},$$

which guarantees

$$r_{m_1} q_0^{(k)} \delta(y_k) q_0^{(k)} r_{m_2} = r_{m_1} a_k r_{m_2}. \quad (22)$$

Since

$$r_{m_1} (\delta(y_k) - a_k) r_m \stackrel{(22)}{=} 0,$$

we see that for the right support $r(r_{m_1}(\delta(y_k)-a_k))$ of the operator $r_{m_1}(\delta(y_k) - a_k)$ satisfies the inequality

$$r(r_{m_1}(\delta(y_k)-a_k)) \leq 1 - r_m, m \in \mathbb{N},$$

and therefore

$$r(r_{m_1}(\delta(y_k)-a_k)) \leq inf_{m\geq 1}(1-r_m) \overset{(10)}{=} 1-q_0^{(k)}.$$

Consequently, $r_{m_1}(\delta(yk)-a_k)q_0^{(k)} = 0$ for all $m_1 \in \mathbb{N}$.

Similarly, using the left support of the operator $(\delta(y_k)-a_k)q_0^{(k)}$, we claim that $q_0^{(k)}(\delta(y_k)-a_k)q_0^{(k)} = 0$.

Since $q_0^{(k)}a_k q_0^{(k)} = a_k$, the equality (21) holds.

Thus, the inequality (20) can be restated as follows

$$\left\| q_0^{(k)}(\delta(y_k)-y)q_0^{(k)} \right\|_M \leq (k+1)^{-1}. \quad (23)$$

It follows from the inequalities (3) and (23), that

$$\left\| (k+1)q_0^{(k)}y_k \right\|_M = \left\| (k+1)y_k q_0^{(k)} \right\|_M \leq 1 \quad (24)$$

and

$$\left\| q_0^{(k)}\delta((k+1)y_k)q_0^{(k)} - (k+1)q_0^{(k)}yq_0^{(k)} \right\|_M \leq 1. \quad (25)$$

Due to (25), and taking into account (11), we obtain

$$(k+1)q_0^{(k)} - q_0^{(k)}\delta((k+1)y_k)q_0^{(k)} \leq (k+1)q_0^{(k)}yq_0^{(k)} - q_0^{(k)}\delta((k+1)y_k)q_0^{(k)} \leq q_0^{(k)},$$

that is

$$kq_0^{(k)} \leq q_0^{(k)}\delta((k+1)y_k)q_0^{(k)}. \quad (26)$$

Let us now consider the projections

$$q_0 := inf_{k\geq 1}q_0^{(k)}, z_0 := inf_{k\geq 1}z_0^{(k)}. \quad (27)$$

Using (11), (27) we have that $p_0 - q_0 = sup_{k\geq 1}(p_0 - q_0^{(k)})$. Therefore, combining (13) and (27), we obtain

$$D(z_0(p_0-q_0)) = D(sup_{k\geq 1}(z_0(p_0-q_0^{(k)}))) \overset{(D6)}{\leq} \sum_{k=1}^{\infty} D(z_0(p_0-q_0^{(k)})) \overset{(13)}{\leq} \varphi(z_0), \quad (28)$$

that is the projection $z_0(p_0-q_0)$ is finite (see (D1)). Moreover, due to inequalities (24) (respectively, (26)), we have

$$\|(k+1)q_0 y_k\|_M = \|(k+1)y_k q_0\|_M \leq 1, k \in \mathbb{N} \quad (29)$$

(respectively,

$$kq_0 \leq q_0\delta((k+1)y_k)q_0, k \in \mathbb{N}.) \quad (30)$$

Since φ is a $*$-isomorphism from $Z(M)$ onto $L^\infty(\Omega, \Sigma, \mu)$, by (12), we have that

$$\int_\Omega \varphi(1-z_0)d\mu = \int_\Omega sup_{k\geq 1}\varphi(1-z_0^{(k)})d\mu \leq \sum_{k=1}^{\infty}\int_\Omega \varphi(1-z_0^{(k)})d\mu \overset{(12)}{\leq} 2^{-1},$$

in particular, $z_0 \neq 0$. Since $1 = c(p_0)$ and $c(p_0 z_0) = c(p_0)z_0 = z_0 \neq 0$, we have $z_0 p_0 \neq 0$, and therefore there exists such $n \in \mathbb{N}$ that $z_0 p_n \neq 0$ (see (2)). Since $z_0 p_n \sim z_0 p_m$, we have $z_0 p_m \neq 0$ for all $m \in \mathbb{N}$. Hence, $z_0 p_0$ is an infinite projection. Since the projection $z_0(p_0-q_0)$ is finite (see (28)), we see that the projection $z_0 q_0$ must be infinite. By [23], there exists a central projection

$$0 \neq e_0 \in P(Z(M)), e_0 \leq z_0,$$
such that $e_0 q_0$ is properly infinite, in particular, there exist pairwise orthogonal projections
$$e_n \leq e_0 q_0, e_n \sim e_0 q_0 \tag{31}$$
for all $n \in \mathbb{N}$ (see, for example, [29]). In addition,
$$\int_\Omega \varphi(c(q_0)e_0)\, d\mu \neq 0. \tag{32}$$
For every $n \in \mathbb{N}$ the operator
$$b_n := \delta(e_n)e_n$$
is locally measurable, and therefore there exists such a sequence $\{z_m^{(n)}\} \subset P(Z(M))$ that $z_m^{(n)} \uparrow 1$ when $m \to \infty$ and $z_m^{(n)} b_n \in S(M)$ for all $m \in \mathbb{N}$. Since $\varphi(z_m^{(n)}) \uparrow \varphi(1) = 1_{L^\infty}(\Omega)$ it follows that $\int_\Omega \varphi(z_m^{(n)})d\mu \uparrow \mu(1_{L^\infty}(\Omega)) = 1$ when $m \to \infty$, and therefore, by (32), for every $n \in \mathbb{N}$ there exists such a projection $z^{(n)} \in P(Z(M))$, that $z^{(n)} b_n \in S(M)$ and
$$1 - 2^{-n-1} \int_\Omega \varphi(c(q_0)e_0)d\mu < \int_\Omega \varphi(z^{(n)})d\mu. \tag{33}$$
Consider the central projection
$$g_0 := inf_{n \geq 1} z^{(n)}.$$
Since $z^{(n)} b_n \in S(M), g_0 = g_0 z^{(n)}$ we have that $g_0 b_n \in S(M)$ for all $n \in \mathbb{N}$. Due to (33) we have
$$1 - \int_\Omega \varphi(g_0)d\mu = \int_\Omega \varphi(1 - g_0)d\mu = \int_\Omega sup\, \varphi(1 - z^{(n)})d\mu \leq$$
$$\sum_{n=1}^{\infty} \int_\Omega \varphi(1 - z^{(n)})d\mu = \sum_{n=1}^{\infty} (1 - \int_\Omega \varphi(z^{(n)})d\mu) \leq 2^{-1} \int_\Omega \varphi(c(q_0)e_0)d\mu\,.$$
Consequently, $1 - 2^{-1} \int_\Omega \varphi(c(q_0)e_0)d\mu \leq \int_\Omega \varphi(g_0)d\mu$, and therefore
$$1 + 2^{-1} \int_\Omega \varphi(c(q_0)e_0)d\mu \leq \int_\Omega \varphi(g_0)d\mu + 2^{-1} \int_\Omega \varphi(c(q_0)e_0)d\mu. \tag{34}$$
From (32) and inequality (34), it follows that $2^{-1} \int_\Omega \varphi(g_0 c(q_0)e_0)d\mu > 0$, i.e. $g_0 c(q_0)e_0 \neq 0$ and so $g_0 e_0 q_0 \neq 0$. Since $e_0 q_0$ is a properly infinite projection it follows that $g_0 e_0 q_0$ is a properly infinite projection. From the relationship $g_0 e_n \overset{(31)}{\sim} g_0 e_0 q_0$, we see that the projection $g_0 e_n$ is also properly infinite for all $n \in \mathbb{N}$. Since
$$c(g_0 e_n) = g_0 c(e_n) \overset{(31)}{\leq} q_0 c(q_0 e_0) = g_0 c(q_0)e_0,$$
it follows that ze_n is also properly infinite projection for every $0 \neq z \in P(Z(M))$ with $z \leq g_0 c(q_0)e_0$. Indeed, if $z' \in P(Z(M))$ and $z'ze_n \neq 0$, then $0 \neq z'ze_n = (z'zc(q_0)e_0)g_0 e_n$, and therefore, since the projection $g_0 e_n$ is properly infinite, we have $(z'zc(q_0)e_0)g_0 e_n \notin P_{fin}(M)$. Consequently, the projection ze_n is also properly infinite.
Passing, if necessary to the algebra $g_0 c(q_0)e_0 M$, we may assume that $g_0 c(q_0)e_0 = 1$. In this case, we also may assume that $b_n \in S(M), e_n \sim q_0, c(e_n) = 1$ and ze_n is a properly infinite projection for every non-zero
$z \in P(Z(M))$.
The assumption $b_n \in S(M)$ means that for every fixed $n \in \mathbb{N}$ there exists such a sequence $\{p_m^{(n)}\}_{m=1}^{\infty} \subset P_{fin}(M)$, that $p_m^{(n)} \downarrow 0$ when $m \to \infty$ and $b_n(1 - p_m^{(n)}) \in M$ for all $m \in \mathbb{N}$. Since $D(p_m^{(n)}) \in L^0(\Omega, \Sigma, \mu)$ and $D(p_m^{(n)}) \downarrow 0$ (see (D7)), it follows that $\{D(p_m^{(n)})\}_{n=1}^{\infty}$ converges in measure μ to zero. Consequently, we may select a central projection f_n and a

finite projection $s_n = p_{m_n}^{(n)} \in P_{fin}(M)$ as to guarantee $D(f_n s_n) < 2^{-n}\varphi(f_n), 1 - 2^{-n-1} < \int \varphi(f_n) d\mu$ and

$$f_n b_n (1 - s_n) \in M \tag{35}$$

for all $n \in \mathbb{N}$.
Setting

$$f := inf_{n\geq 1} f_n, s := sup_{n\geq 1} s_n,$$

we have that

$$1/2 < \int \varphi(f) d\mu, \qquad D(fs) \overset{(D6)}{\leq} \sum_{n=1}^{\infty} D(fs_n) \leq \varphi(f).$$

This means that $f \neq 0$ and $fs \in P_{fin}(M)$ (see (D1)). In addition, since $f \leq f_n, (1 - s) \leq (1 - s_n)$ from (35) it follows that $fb_n(1 - s) \in M$ for all $n \in \mathbb{N}$.
Consider the projections $t = f(1 - s)$ and $g_n = f(e_n \wedge (1 - s)), n \in \mathbb{N}$.
Clearly (see (31)),

$$g_n \leq fe_n \leq q_0, b_n g_n \in M, g_n \leq t \tag{36}$$

for all $n \in \mathbb{N}$, and also

$$fe_n - g_n = f(e_n - e_n \wedge (1 - s)) \sim f(e_n \vee (1 - s) - (1 - s)) \leq fs,$$

that is $fe_n - g_n \in P_{fin}(M)$. Hence, for every non-zero central projection $z \leq f$, we have that the projection $ze_n - zg_n$ is finite. Since the projection ze_n is infinite, the projection zg_n is also infinite, i.e.

$$zg_n \notin P_{fin}(M) \tag{37}$$

for any $0 \neq z \in P(Z(M))$ and $n \in \mathbb{N}$.
Since $b_n t = fb_n(1 - s) \in M$, we see that there exists such an increasing sequence $\{l_n\} \subset \mathbb{N}$ that $l_n > n + 2\|b_n t\|_M$ for all $n \in \mathbb{N}$.
Appealing to the inequalities (29), (36) and taking into account the equality $b_n = \delta(e_n)e_n$, we deduce

$$\begin{aligned}\left\|g_n(l_n + 1)y_{l_n}\delta(e_n)e_n g_n\right\|_M &\leq \left\|g_n(l_n + 1)y_{l_n}\right\|_M \|\delta(e_n)e_n g_n\|_M \\ &\leq \left\|q_0(l_n + 1)y_{l_n}\right\|_M \|\delta(e_n)e_n t\|_M < (l_n - n)/2.\end{aligned}$$

Hence,

$$\left\|g_n e_n \delta(e_n)(l_n + 1)y_{l_n} g_n + g_n(l_n + 1)y_{l_n}\delta(e_n)e_n g_n\right\|_M \leq l_n - n. \tag{38}$$

For every $x = x^* \in M$ the inequalities $-\|x\|_M 1 \leq x \leq \|x\|_M 1$ holds, in particular, $-g_n\|x\|_M \leq q_n x q_n \leq g_n\|x\|_M$. Hence, inequality (38) implies that

$$g_n e_n \delta(e_n)(l_n + 1)y_{l_n} g_n + g_n(l_n + 1)y_{l_n}\delta(e_n)e_n g_n \geq (n - l_n)g_n. \tag{39}$$

Since $e_n e_m = 0$ whenever $n \neq m$, we see (due to inequalities (29) and (36)) that the series $\sum_{n=1}^{\infty} e_n(l_n + 1)y_{l_n}e_n$ converges with respect to the topology τ_{so} to a self-adjoint operator $h_0 \in M$, satisfying

$$\|h_0\|_M \leq sup_{n\geq 1}\left\|e_n(l_n + 1)y_{l_n}e_n\right\|_M \leq 1.$$

Again appealing to the inequalities (30), (36) and (39), we infer that

$$\begin{aligned} ng_n &= l_n g_n + (n - l_n)g_n \\ &\leq g_n(l_n + 1)\delta(y_{l_n})g_n + g_n e_n \delta(e_n)(l_n + 1)y_{l_n} g_n + g_n(l_n + 1)y_{l_n}\delta(e_n)e_n g_n \\ &= (l_n + 1)(g_n\delta(y_{l_n})g_n + g_n e_n\delta(e_n)y_{l_n} g_n + g_n y_{l_n}\delta(e_n)e_n g_n) \\ &= (l_n + 1)g_n\delta(e_n y_{l_n} e_n)g_n \\ &= \delta(g_n e_n(l_n + 1)y_{l_n} e_n)g_n - \delta(g_n)e_n(l_n + 1)y_{l_n} e_n g_n \\ &= \delta(g_n h_0)g_n - \delta(g_n)h_0 g_n = g_n\delta(h_0)g_n.\end{aligned}$$

Thus,

$$ng_n \leq g_n \delta(h_0) g_0 \quad (40)$$

for every $n \in \mathbb{N}$.

Set $g_n^{(0)} = g_n \wedge E_{n-1}(\delta(h_0)), n \in \mathbb{N}$, where $\{E_\lambda(\delta(h_0))\}$ is the spectral family of projections for self-adjoint operator $\delta(h_0)$. For every $n \in \mathbb{N}$ we have

$$\begin{aligned} ng_n^{(0)} &= ng_n^{(0)} g_n g_n^{(0)} \overset{(40)}{\leq} g_n^{(0)}(g_n \delta(h_0) g_n) g_n^{(0)} \\ &= g_n^{(0)} \delta(h_0) g_n^{(0)} = g_n^{(0)} E_{n-1}(\delta(h_0)) \delta(h_0) g_n^{(0)} \\ &\leq g_n^{(0)} (n-1) E_{n-1}(\delta(h_0)) g_n^{(0)} = (n-1) g_n^{(0)}. \end{aligned}$$

Hence, $g_n \wedge E_{n-1}(\delta(h_0)) = g_n^{(0)} = 0$ which implies

$g_n = g_n - g_n \wedge E_{n-1}(\delta(h_0)) \sim g_n \vee E_{n-1}(\delta(h_0)) - E_{n-1}(\delta(h_0)) \leq 1 - E_{n-1}(\delta(h_0))$, i.e. $g_n \preccurlyeq 1 - E_{n-1}(\delta(h_0))$.

Then $g_n \overset{(36)}{\leq} f g_n \preccurlyeq f(1 - E_{n-1}(\delta(h_0)))$, and therefore

$$D(g_n) \overset{(D3)}{\leq} D(f(1 - E_{n-1}(\delta(h_0)))) \quad (41)$$

for all $n \in \mathbb{N}$.

Since $|f\delta(h_0)| \in LS(M)$, we see that there exists such a non-zero central projection $f_0 \leq f$, that $|f_0 \delta(h_0)| \in S_h(M)$. Hence, we may find such $\lambda_0 > 0$, that $(f_0 - E_\lambda(|f_0 \delta(h_0)|)) \in P_{fin}(M)$ for all $\lambda \geq \lambda_0$ ([113]), that is $D(f_0(1 - E_\lambda(|f_0 \delta(h_0)|))) \in L_+^0(\Omega, \Sigma, \mu)$ when $\lambda > \lambda_0$.

Since $f_0(1 - E_\lambda(|f_0 \delta(h_0)|)) = f_0(1 - E_\lambda(|\delta(h_0)|))$, we infer from (41) that

$$D(f_0 g_n) \in L_+^0(\Omega, \Sigma, \mu)$$

for all $n \geq \lambda_0 + 1$ which contradicts with the property (D1) in the definition of the dimension function D, since $f_0 g_n$ is an infinite projection (see (37)).

Hence, our assumption that the derivation δ fails to be continuous in $(LS(M), t(M))$ has led to a contradiction.

Observe that in the special case of properly infinite von Neumann algebras of type I or III, Theorem (3.2.9) gives a new proof of the results concerning the continuity of a derivation of $(LS(M), t(M))$ established earlier in [2], [105], [106].

The construction of extension of any derivation, acting on a von Neumann algebra M with values in $LS(M)$, up to a derivation from $LS(M)$ into $LS(M)$ is given. Using this extension and Theorem 3.3 it is established that in case the of a properly infinite von Neumann algebra M, any derivation $\delta: A \to LS(M)$ from a subalgebra A satisfying $M \subset A \subset LS(M)$ is continuous with respect to the local measure topology.

Let M be an arbitrary von Neumann algebra and let $\{z_n\}_{n=1}^\infty$ be a sequence of central projections from M, such that $z_n \uparrow 1$. A sequence $\{x_n\}_{n=1}^\infty$ is called consistent with the sequence $\{z_n\}_{n=1}^\infty$, if for any $n, m \in \mathbb{N}$ the equality $x_m z_n = x_n z_n$ holds for $n < m$.

Proposition (3.2.10)[104]: Let $\{x_n\}_{n=1}^\infty \subset LS(M)$ (respectively, $\{y_n\}_{n=1}^\infty \subset LS(M)$) be a sequence consistent with the sequence $\{z_n\}_{n=1}^\infty \subset P(Z(M))$ (respectively, with the sequence $\{z_n'\}_{n=1}^\infty \subset P(Z(M))), z_n \uparrow 1 (z_n' \uparrow 1)$.

Then

(i). There exists a unique $x \in LS(M)$, such that $x z_n = x_n z_n$ for all $n \in \mathbb{N}$, in addition, $x_n \xrightarrow{t(M)} x$;

(ii). If $x_n z_n z_m' = y_m z_n z_m'$ for all $n, m \in \mathbb{N}$, then $(x_n z_n - y_n z_n') \xrightarrow{t(M)} 0$ for $n \to \infty$.

Proof. (i). Consider a neighborhood $V(B,\varepsilon,\delta)$ of zero in topology $t(M)$, where $\varepsilon,\delta > 0, B \in \Sigma, \mu(B) < \infty$ (see the definition of topology $t(M)$). Since $z_n^{\perp} = (1 - z_n) \downarrow 0$, it follows that $\varphi(z_n^{\perp}) \in W(B,\varepsilon,\delta)$ for $n \geq n(B,\varepsilon,\delta)$. Taking $x \in LS(M), q_n = z_n$, we have $(xz_n^{\perp})q_n = 0, D(z_n^{\perp}\, q_n) = 0$, i.e. $xz_n^{\perp} \in V(B,\varepsilon,\delta)$ for all $x \in LS(M), n \geq n(B,\varepsilon,\delta)$. For $m > n$, we have

$$x_m z_m - x_n z_n = x_m z_m - x_m z_n = x_m(z_m - z_n) = x_m z_m z_n^{\perp} \in V(B,\varepsilon,\delta)$$

for all $n \geq n(B,\varepsilon,\delta)$. It means that $\{x_n z_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(LS(M), t(M))$. Consequently, there exists $x \in LS(M)$ such that $x_n z_n \xrightarrow{t(M)} x$.

Since $x_n z_n^{\perp} \in V(B,\varepsilon,\delta)$ for all $n \geq n(B,\varepsilon,\delta)$, it follows that $x_n z_n^{\perp} \xrightarrow{t(M)} 0$, and therefore $x_n = x_n z_n + x_n z_n^{\perp} \xrightarrow{t(M)} x$. Fixing $k \in \mathbb{N}$, for $n > k$ we have $x_k z_k = x_n z_k \xrightarrow{t(M)} x z_k$ for $n \to \infty$, i.e. $x z_k = x_k z_k$ for all $k \in \mathbb{N}$.

If $a \in LS(M)$ and $a z_n = x_n z_n = x z_n$ for all $n \in \mathbb{N}$, then $0 = (a - x) z_n \xrightarrow{t(M)} (a - x)$, i.e. $a = x$.

(ii). If $x_m z_m z_n^{'\perp} \xrightarrow{t(M)} 0$ for $n \to \infty, y_n z_n^{'}\, z_m^{\perp} \xrightarrow{t(M)} 0$ for $m \to \infty$, and $x_n z_n - x_m z_m \xrightarrow{t(M)} 0$ for $n, m \to \infty$, then

$$\begin{aligned} x_n z_n - y_n z_n^{'} &= x_n z_n - x_m z_m + x_m z_m z_n^{'} + x_m z_m z_n^{'\perp} - y_n z_n^{'} \\ &= (x_n z_n - x_m z_m) + y_n z_m z_n^{'} + x_m z_m z_n^{'\perp} - y_n z_n^{'} = \\ &= (x_n z_n - x_m z_m) - y_n z_n^{'} z_m^{\perp} + x_m z_m z_n^{'\perp} \xrightarrow{t(M)} 0 \end{aligned}$$

for $n, m \to \infty$.

Now, we consider a derivation δ from $S(M)$ into $LS(M)$ and construct an extension $\tilde{\delta}$ from $LS(M)$ into $LS(M)$. Recall that for an arbitrary operator $x \in LS(M)$ there exists a sequence $\{z_n\}_{n=1}^{\infty} \subset P(Z(M))$ such that $z_n \uparrow 1$ and $x z_n \in S(M)$ for all $n \in \mathbb{N}$.

Since $\delta(x z_n) z_m = \delta(x z_n z_m)$ (see Lemma (3.2.7)), the sequence $\{\delta(x z_n)\}_{n=1}^{\infty}$ is consistent with the sequence $\{z_n\}_{n=1}^{\infty}$. By Proposition (3.2.10)(i), there exists a unique $y(x) \in LS(M)$ such that $\delta(x z_n) \xrightarrow{t(M)} y(x)$ (notation: $y(x) = t(M) - \lim_{n\to\infty} \delta(x z_n)$). Set $\tilde{\delta}(x) = y(x)$. According to Proposition (3.2.10) (ii), the definition of operator $\tilde{\delta}(x)$ does not depend on a choice of a sequence $\{z_n\}_{n=1}^{\infty} \subset P(Z(M))$, for which $z_n \uparrow 1$ and $x z_n \in S(M), n \in \mathbb{N}$. If $x \in S(M)$, then, taking $z_n = 1, n \in \mathbb{N}$, we obtain $\tilde{\delta}(x) = \delta(x)$.

Proposition (3.2.11)[104]: The mapping $\tilde{\delta}$ is a unique derivation from $LS(M)$ into $LS(M)$ such that $\tilde{\delta}(x) = \delta(x)$ for all $x \in S(M)$.

Proof. Let $x, y \in LS(M)$, and let $z_n, p_n \in P(Z(M))$ be such that $z_n \uparrow 1, p_n \uparrow 1, x z_n, y p_n \in S(M), n \in \mathbb{N}$. Observing that

$$z_n p_n \in P(Z(M)), (z_n p_n) \uparrow 1, x z_n p_n, y z_n p_n, (x + y) z_n p_n \in S(M), n \in \mathbb{N},$$

we have

$$\begin{aligned} \tilde{\delta}(x + y) &= t(M) - \lim_{n\to\infty} \delta((x + y) z_n p_n) \\ &= \left(t(M) - \lim_{n\to\infty} \delta(x z_n p_n)\right) + \left(t(M) - \lim_{n\to\infty} \delta(y z_n p_n)\right) \\ &= \tilde{\delta}(x) + \tilde{\delta}(y). \end{aligned}$$

Similarly, $\tilde{\delta}(\lambda x) = \lambda \tilde{\delta}(x), \lambda \in \mathbb{C}$. Further, using convergences

$$x z_n \xrightarrow{t(M)} x, y p_n \xrightarrow{t(M)} y, \delta(x z_n) \xrightarrow{t(M)} \tilde{\delta}(x), \delta(y p_n) \xrightarrow{t(M)} \tilde{\delta}(y)$$

and the inclusion $x y z_n p_n \in S(M), n \in \mathbb{N}$, we have

$$\tilde{\delta}(xy) = t(M) - \lim_{n\to\infty} \delta(x y z_n p_n) = t(M) - \lim_{n\to\infty} \delta((x z_n)(y p_n))$$

$$= t(M) - \lim_{n\to\infty}(\delta(xz_n)yp_n + xz_n\delta(y_np_n)) = \tilde{\delta}(x)y + x\tilde{\delta}(y).$$

Consequently, $\tilde{\delta}: LS(M) \to LS(M)$ is a derivation, in addition, $\tilde{\delta}(x) = \delta(x)$ for all $x \in S(M)$.

Assume that $\delta_1: LS(M) \to LS(M)$ is also a derivation for which $\delta_1(x) = \delta(x)$ for all $x \in S(M)$. Let us show that $\tilde{\delta} = \delta_1$.

If $x \in LS(M), z_n \uparrow 1, xz_n \in S(M), n \in \mathbb{N}$, then, by Lemma (3.2.7) and Proposition (3.2.10) (i), we obtain

$$\tilde{\delta}(x) = t(M) - \lim_{n\to\infty}\delta(xz_n) = t(M) - \lim_{n\to\infty}\delta_1(xz_n)$$
$$= t(M) - \lim_{n\to\infty}\delta_1(x)\,z_n = \delta_1(x).$$

Now, we give the construction of extension of a derivation $\delta: M \to LS(M)$ up to a derivation $\hat{\delta}: S(M) \to LS(M)$. For each $x \in LS(M)$ set $s(x) := l(x) \vee r(x)$, where $l(x)$ is the left and $r(x)$ is the right support of x. If $x = u|x|$ is a polar decomposition of $x \in LS(N)$, then $u \in M$ [113] and, due to equalities $l(x) = uu^*, r(x) = u^*u$, we have $l(x) \sim r(x)$. We need the following lemma.

Lemma (3.2.12)[104]: If D is a dimension function of a von Neumann algebra M, then for any derivation δ from M into $LS(M)$ the following inequality

$$D(s(\delta(x))) \leq 3D(s(x))$$

holds for all $x \in M$.

Proof. For $x \in M$ we have

$$l(\delta(x)s(x)) \sim r(\delta(x)s(x)) \leq s(x),$$
$$r(x\delta(s(x))) \sim l(x\delta(s(x))) = l(s(x)x\delta(s(x))) \leq s(x),$$

i.e.

$$l(\delta(x)s(x)) \preccurlyeq s(x)$$

and

$$r(x\delta(s(x))) \preccurlyeq s(x),$$

that implies the inequalities (see (D2), (D3))

$$D(l(\delta(x)s(x))) \leq D(s(x)), D(r(x\delta(s(x)))) \leq D(s(x)).$$

Since

$$\delta(x) = \delta(xs(x)) = \delta(x)s(x) + x\delta(s(x)),$$

we have

$$s(\delta(x)) = s(\delta(x)s(x) + x\delta(s(x))) \leq s(x) \vee l(\delta(x)s(x)) \vee r(x\delta(s(x))).$$

Due to (D6), we have

$$D(s(\delta(x))) \leq D(s(x)) + D(l(\delta(x)s(x))) + D(r(x\delta(s(x)))) \leq 3D(s(x)).$$

As in the definition of the topology $t(M)$, denote by φ a $*$-isomorphism from $Z(M)$ onto the $*$-algebra $L^\infty(\Omega, \Sigma, \mu)$, where μ is a measure satisfying the direct sum property. By Proposition (3.2.2)(i), the convergence of the sequence of projections $p_n \xrightarrow{t(M)} 0$ is equivalent to existence of a sequence $\{z_n\} \subset P(Z(M))$ such that $z_np_n \in P_{fin}(M)$ for all $n, \varphi(z_n^\perp) \xrightarrow{t(L^\infty(\Omega))} 0$ and $D(z_np_n) \xrightarrow{t(L^\infty(\Omega))} 0$.

Lemma (3.2.13)[104]: If $\{x_n\}_{n=1}^\infty \subset LS(M), s(x_n) \in P_{fin}(M), D(s(x_n)) \xrightarrow{t(L^\infty(\Omega))} 0$, then $x_n \xrightarrow{t(M)} 0$.

Proof. Taking $z_n = 1$ for all $n \in \mathbb{N}$, we have

$$z_ns(x_n) \in P_{fin}(M), \varphi(z_n^\perp) = 0, n \in \mathbb{N},$$

and

$$D(z_n s(x_n)) = D(s(x_n)) \xrightarrow{t(L^\infty(\Omega))} 0.$$

Consequently, $s(x_n) \xrightarrow{t(M)} 0$ (see Proposition 2.2(i)).

Since $E_\lambda^\perp(|x_n|) \le s(x_n)$ for all $\lambda > 0, n \in \mathbb{N}$, it follows $E_\lambda^\perp(|x_\alpha|) \xrightarrow{t(M)} 0$, and therefore $x_n \xrightarrow{t(M)} 0$.

If $p_n \in P_{fin}(M)$ and $p_n \downarrow 0$, then $D(p_n) \in L_+^0(\Omega, \Sigma, \mu)$ (see (D1)) and $D(p_n) \downarrow 0$ (see (D2) and D(7)), in particular, $D(p_n) \xrightarrow{t(L^\infty(\Omega))} 0$. Hence, Lemma (3.2.13) implies the following

Corollary (3.2.14)[104]: If $\{p_n\}_{n=1}^\infty \subset P_{fin}(M), p_n \downarrow 0$, then $p_n \xrightarrow{t(M)} 0$.

Lemma(3.2.15)[104]: Let $x \in S(M), p_n, q_n \in P(M), p_n \uparrow 1, q_n \uparrow 1, xp_n, xq_n \in M, p_n^\perp, q_n^\perp \in P_{fin}(M), n \in \mathbb{N}$. If $\delta: M \to LS(M)$ is a derivation, then there exists $\hat{\delta}(x) \in LS(M)$, such that

$$t(M) - \lim_{n\to\infty} \delta(xp_n) = \hat{\delta}(x) = t(M) - \lim_{n\to\infty} \delta(xq_n).$$

Proof. For $n < m$ we have

$$l(x(p_m - p_n)) \sim r(x(p_m - p_n)) \le p_m - p_n,$$

and therefore, applying Lemma (3.2.12) and properties (D2), (D3), we obtain

$$D(s(\delta(xp_m - xp_n))) = D(s(\delta(x(p_m - p_n)))) \le 3D(s(x(p_m - p_n)))$$
$$\le 3D(l(x(p_m - p_n)) \vee (p_m - p_n)) \le 6D(p_m - p_n) \le 6D(p_n^\perp).$$

Since $D(p_n^\perp) \in L_+^0(\Omega, \Sigma, \mu)$ (see (D1)) and $D(p_n^\perp) \downarrow 0$ (see (D7)) it follows that $D(p_n^\perp) \xrightarrow{t(L^\infty(\Omega))} 0$ (see (D7)). Hence,

$$D(s(\delta(xp_m) - \delta(xp_n))) \xrightarrow{t(L^\infty(\Omega))} 0$$

for $n, m \to \infty$. By Lemma (3.2.13), we have that $(\delta(xp_m) - \delta(xp_n)) \xrightarrow{t(M)} 0$ for $n, m \to \infty$, i.e. $\{\delta(xp_n)\}_{n=1}^\infty$ is a Cauchy sequence in $(LS(M), t(M))$.

Consequently, there exists $\hat{\delta}(x) \in LS(M)$, such that

$$t(M) - \lim_{n\to\infty} \delta(xp_n) = \hat{\delta}(x).$$

Let us show that $t(M) - \lim_{n\to\infty} \delta(xq_n) = \hat{\delta}(x)$.

For each $n \in \mathbb{N}$ we have

$$(p_n - q_n)((p_n - p_n \wedge q_n) \vee (q_n - p_n \wedge q_n)) =$$
$$= ((p_n - p_n \wedge q_n) - (q_n - p_n \wedge q_n))((p_n - p_n \wedge q_n) \vee (q_n - p_n \wedge q_n))$$
$$= (p_n - p_n \wedge q_n) - (q_n - p_n \wedge q_n) = p_n - q_n.$$

Hence,

$$r(p_n - q_n) \le ((p_n - p_n \wedge q_n) \vee (q_n - p_n \wedge q_n)).$$

Since

$$r(x(p_n - q_n)) \le r(p_n - q_n)$$

and

$$l(x(p_n - q_n)) \sim r(x(p_n - q_n)),$$

it follows

$$D(s(x(p_n - q_n))) = D(l(x(p_n - q_n)) \vee r(x(p_n - q_n)))$$
$$\overset{(D6)}{\le} D(l(x(p_n - q_n))) + D(r(x(p_n - q_n))) = 2D(r(x(p_n - q_n)))$$
$$\overset{(D6)}{\le} 2D(p_n - p_n \wedge q_n) + 2D(q_n - p_n \wedge q_n) \le 4D(1 - p_n \wedge q_n)$$
$$= 4D(p_n^\perp \vee q_n^\perp) \le 4(D(p_n^\perp) + D(q_n^\perp)).$$

Since (see Lemma (3.2.12))

$$D(s(\delta(xp_n)-\delta(xq_n)))=D(s(\delta(x(p_n-q_n)))) \leq 3D(s(x(p_n-q_n))),$$

we have

$$D(s(\delta(xp_n)-\delta(xq_n)))\leq 12(D(p_n^\perp)+D(q_n^\perp))\downarrow 0.$$

By Lemma (3.2.13), we obtain

$$t(M)-\lim_{n\to\infty}\delta(xq_n)=t(M)-\lim_{n\to\infty}\delta(xp_n)=\hat{\delta}(x).$$

Now, equipped with Lemma (3.2.15), we may extend any derivation $\delta: M\to LS(M)$ up to a derivation $\hat{\delta}$ from $S(M)$ into $LS(M)$.

For each $x\in S(M)$ there exists a sequence $\{p_n\}\in P(M)$, such that $p_n\uparrow 1, p_n^\perp\in P_{fin}(M), xp_n\in M$ for all $n\in\mathbb{N}$. By Lemma (3.2.12), there exists $\hat{\delta}(x)\in LS(M)$, such that $t(M)-\lim_{n\to\infty}\delta(xp_n)=\hat{\delta}(x)$. In addition, the definition of $\hat{\delta}(x)$ does not depend on a choice of sequence $\{p_n\}_{n\geq 1}$ satisfying the above mentioned property, in particular, $\hat{\delta}(x)=\delta(x)$ for all $x\in M$ (in this case, $p_n=1, n\in\mathbb{N}$).

Proposition (3.2.16)[104]: The mapping $\hat{\delta}$ is a unique derivation from $S(M)$ into $LS(M)$, such that $\hat{\delta}(x)=\delta(x)$ for all $x\in M$.

Proof. For $x,y\in S(M)$ select $p_n,q_n\in P(M), n\in\mathbb{N}$, such that

$$p_n\uparrow 1, q_n\uparrow 1, p_n^\perp, q_n^\perp\in P_{fin}(M), xp_n, yq_n\in M$$

for all $n\in\mathbb{N}$. The sequence of projections $e_n=p_n\wedge q_n$ is increasing, and, in addition,

$$xe_n=xp_ne_n\in M, ye_n=yq_ne_n\in M,$$
$$e_n^\perp=p_n^\perp\vee q_n^\perp\in P_{fin}(M), D(e_n^\perp)\leq D(p_n^\perp)+D(q_n^\perp)\downarrow 0.$$

The last estimate implies the convergence $e_n^\perp\downarrow 0$ (see (D7)), or $e_n\uparrow 1$.

By Lemma (3.2.15), we have

$$\hat{\delta}(x+y)=t(M)-\lim_{n\to\infty}\delta((x+y)e_n)=$$
$$=\left(t(M)-\lim_{n\to\infty}\delta(xe_n)\right)+\left(t(M)-\lim_{n\to\infty}\delta(ye_n)\right)=\hat{\delta}(x)+\hat{\delta}(y).$$

Similarly, $\hat{\delta}(\lambda x)=\lambda\hat{\delta}(x)$ for all $\lambda\in\mathbb{C}$.

Let us show that $\hat{\delta}(xy)=\hat{\delta}(x)y+x\hat{\delta}(y), x,y\in S(M)$.

Due to polar decomposition $y=u|y|, u^*u=r(y)$, we have $y_n=yE_n(|y|)\in M$ for all $n\in\mathbb{N}$. Set

$$g_n=1-r(E_n^\perp(|x|)y_n), s_n=g_n\wedge E_n(|y|).$$

Since

$$g_n^\perp=r(E_n^\perp(|x|)y_n)\sim l(E_n^\perp(|x|)y_n)\leq E_n^\perp(|x|),$$

we obtain

$$g_n^\perp\preccurlyeq E_n^\perp(|x|).$$

Since $x\in S(M)$, there exists $n_0\in\mathbb{N}$ such that $E_n^\perp(|x|)\in P_{fin}(M)$ for all $n\geq n_0$, and therefore $g_n^\perp\in P_{fin}(M)$ for all $n\geq n_0$. The equality

$$y_ng_n=E_n(|x|)y_ng_n+E_n^\perp(|x|)y_ng_n=E_n(|x|)y_ng_n$$

implies that

$$E_{n+1}^\perp(|x|)y_{n+1}s_n=E_{n+1}^\perp(|x|)E_n^\perp(|x|y_{n+1}E_n(|y|))s_n=$$
$$=E_{n+1}^\perp+1(|x|)(E_n^\perp(|x|)y_nE_n(|y|))s_n=$$
$$=E_{n+1}^\perp(|x|)(E_n^\perp(|x|)y_ns_n)=E_{n+1}^\perp(|x|)(E_n^\perp(|x|)y_ng_n)s_n=0,$$

in particular,

$$s_n\leq 1-r(E_{n+1}^\perp(|x|)y_{n+1})=g_{n+1}$$

for all $n \in \mathbb{N}$. From here and from the inequalities $s_n \le E_n(|y|) \le E_{n+1}(|y|)$ it follows that $s_n \le s_{n+1}$.
Since $y \in S(M)$, we have $E_n^{\perp}(|y|) \in P_{fin}(M)$ for $n \ge n_1$ for some $n_1 \ge n_0$. Hence,
$$s_n^{\perp} = g_n^{\perp} \vee E_n^{\perp}(|y|) \in P_{fin}(M)$$
for $n \ge n_1$ and
$$D(s_n^{\perp}) \le D(g_n^{\perp}) + D(E_n^{\perp}(|y|)) \le (D(E_n^{\perp}(|x|)) + D(E_n^{\perp}(|y|))) \downarrow 0,$$
i.e. $s_n^{\perp} \downarrow 0$ or $s_n \uparrow 1$.
Using Corollary (4.2.14), Lemma (3.2.15), the inclusions $xE_n(|x|) \in M, yE_n(|y|) \in M$ and equalities
$$xys_n = xyE_n(|y_n|)s_n = xy_ns_n = xy_ng_ns_n = \\ = xE_n(|x|)y_nq_ns_n = xE_n(|x|)yE_n(|y|)s_n,$$
we obtain
$$\hat{\delta}(xy) = t(M) - \lim_{n\to\infty} \delta(xys_n) = t(M) - \lim_{n\to\infty} \delta(xE_n(|x|)yE_n(|y|s_n)) = \\ = t(M) - \lim_{n\to\infty}(\delta(xE_n(|x|))ys_n + xE_n(|x|)\delta(ys_n)) = \\ = \left(t(M) - \lim_{n\to\infty} \delta(xE_n(|x|))\right) \cdot \left(t(M) - \lim_{n\to\infty} ys_n\right) + \\ + \left(t(M) - \lim_{n\to\infty} xE_n(|x|)\right) \cdot \left(t(M) - \lim_{n\to\infty} \delta(ys_n)\right) = \hat{\delta}(x)y + x\hat{\delta}(y).$$
Consequently, $\hat{\delta}: S(M) \to LS(M)$ is a derivation, such that $\hat{\delta}(x) = \delta(x)$ for all $x \in M$.
Let $\delta_1: S(M) \to LS(M)$ also be a derivation, for which $\delta_1(x) = \delta(x)$ for all $x \in M$. If $x \in S(M)$, then $E_n(|x|) \uparrow 1, xE_n(|x|) \in M, n \in \mathbb{N}, E_n^{\perp}(|x|) \in P_{fin}(M)$ for all $n \ge n_3$ for some $n_3 \in \mathbb{N}$.
Hence, $E_n(|x|) \xrightarrow{t(M)} 1$ (see Corollary 3.2.14). Since $(LS(M), t(M))$ is a topological algebra, it follows that
$$\delta_1(x) = t(M) - \lim_{n\to\infty} \delta_1(x)E_n(|x|) = \\ = \left(t(M) - \lim_{n\to\infty} \delta_1(xE_n(|x|))\right) - \left(t(M) - \lim_{n\to\infty} x\delta_1(E_n(|x_n|))\right) = \\ = \left(t(M) - \mathit{lim}_{n\to\infty} \delta(xE_n(|x|))\right) - \left(t(M) - \mathit{lim}_{n\to\infty} x\delta(E_n(|x_n|))\right) = \\ = \hat{\delta}(x) - x(t(M) - \mathit{lim}_{n\to\infty} \delta(E_n(|x_n|))).$$
Since $\delta(1) = 0, s(x) = s(-x)$ for $x \in LS(M)$, it follows via Lemma (3.2.12), that
$$D(s(\delta(E_n(|x|)))) = D(s(\delta(-E_n(|x|)))) = \\ = D(s(\delta(1 - E_n(|x|)))) \le 3D(E_n^{\perp}(|x|)) \downarrow 0.$$
By Lemma (3.2.13), we obtain $\delta(E_n(|x|)) \xrightarrow{t(M)} 0$, that implies the equality $\delta_1(x) = \hat{\delta}(x)$.

Propositions (3.2.11) and (3.2.16) imply the following theorem.

Theorem (3.2.17)[104]: Let A be a subalgebra of $LS(M), M \subset A$ and let $\delta: A \to LS(M)$ be a derivation. Then there exists a unique derivation $\delta_A: LS(M) \to LS(M)$ such that $\delta_A(x) = \delta(x)$ for all $x \in A$.

Proof. Since $M \subset A$, the restriction δ_0 of the derivation δ on M is a well-defined derivation from M into $LS(M)$. Hence, by Propositions (3.2.11) and (3.2.16), the mapping $\delta_A = \tilde{\hat{\delta}}$ is a unique derivation from $LS(M)$ into $LS(M)$ such that $\delta_A(x) = \delta_0(x)$ for all $x \in M$. Let us show that $\delta_A(a) = \delta(a)$ for every $a \in A$. If $a \in A$, then there exists a sequence $\{z_n\}_{n=1}^{\infty} \subset P(Z(M))$, such that $z_n \uparrow 1$ and $az_n \in S(M), n \in \mathbb{N}$. Since $z_n \xrightarrow{t(M)} 1$ (see Proposition (3.2.10)(i)), we have, by Lemma (3.2.7),

$$\delta_A(a) = t(M) - \lim_{n\to\infty} \delta_A(a)\, z_n = t(M) - \lim_{n\to\infty} \delta_A(az_n),$$

and, similarly, $\delta(a) = t(M) - \lim_{n\to\infty} \delta(az_n)$.

Using the equality $\delta_A(x) = \delta_0(x) = \delta(x)$ for each $x \in M$, and following the proof of uniqueness of the derivation $\hat{\delta}$ from Proposition (3.2.16), we obtain $\delta_A(az_n) = \delta(az_n)$ for all $n \in \mathbb{N}$, that implies the equality $\delta_A(a) = \delta(a)$.

The following corollary immediately follows from Theorems (3.2.9) and (3.2.17).

Corollary (3.2.18)[104]: Let M be a properly infinite von Neumann algebra, A is a subalgebra in $LS(M)$ and $M \subset A$. Then any derivation $\delta: A \to LS(M)$ is continuous with respect to the local measure topology $t(M)$.

In particular, Corollary (3.2.18) implies that for a properly infinite von Neumann algebra M any derivation $\delta: S(M) \to S(M)$ is $t(M)$-continuous. Note, that in case, when M is of type I_∞, any derivation of $S(M)$ is inner [105], and therefore is automatically continuous with respect to the topology $t(M)$.

Let M be a semifinite von Neumann algebra acting on Hilbert space H, τ be a faithful normal semifinite trace on M. An operator $x \in S(M)$ with domain $D(x)$ is called τ-measurable if for any $\varepsilon > 0$ there exists a projection $p \in P(M)$ such that $p(H) \subset \mathfrak{D}(x)$ and $\tau(p^\perp) < \infty$.

The set $S(M,\tau)$ of all τ-measurable operators is a $*$-subalgebra of $S(M)$ such that $M \subset S(M,\tau)$. If the trace τ is finite, then $S(M,\tau) = S(M)$. The algebra $S(M,\tau)$ is a noncommutative version of the algebra of all measurable complex functions f defined on (Ω, Σ, μ), for which $\mu(\{|f| > \lambda\}) \to 0$ for $\lambda \to \infty$. For each $x \in S(M,\tau)$ it is possible to define the generalized singular value function

$$\mu_t(x) = \inf\{\lambda > 0: \tau(E_\lambda^\perp(|x|) < t\} = \inf\{\|x(1-e)\|_M: e \in P(M), \tau(e) < t\},$$

which allows to define and study a noncommutative version of rearrangement invariant function spaces. For the theory of the latter spaces, See [16], [111].

Let t_τ be the measure topology [52] on $S(M,\tau)$, whose base of neighborhoods of zero is given by $U(\varepsilon,\delta) = \{x \in S(M,\tau): \exists p \in P(M), \tau(p^\perp) \le \delta, xp \in M, \|xp\|_M \le \varepsilon\}, \varepsilon > 0, \delta > 0$.

The pair $(S(M,\tau), t_\tau)$ is a complete metrizable topological $*$-algebra.

Here, the topology t_τ majorizes the topology $t(M)$ on $S(M,\tau)$ and, if τ is a finite trace, the topologies t_τ and $t(M)$ coincide [113].

However, if $\tau(1) = \infty$, then on $(S(M,\tau), t_\tau)$ topologies t_τ and $t(M)$ do not coincide in general [109]. For example, when $M = L^\infty(\Omega,\Sigma,\mu), \tau(f) = \int_\Omega f d\mu, f \in L_+^\infty(\Omega)$, where μ is a σ-finite measure, $\mu(\Omega) = \infty$, the topology t_τ in $S(L^\infty(\Omega),\tau)$ coincide with the topology of convergence in measure μ, and the topology $t(L^\infty(\Omega))$ is the topology of convergence locally in measure μ, in particular, if $A_n \in \Sigma, \mu(A_n) = \infty, n \in \mathbb{N}$ and $\chi_{A_n} \downarrow 0$, then $\chi_{A_n} \xrightarrow{t(L^\infty(\Omega))} 0$, whereas $\chi_{A_n} \overset{t_\tau}{\nrightarrow} 0$. See the detailed comparison of topologies t_τ and $t(M)$ in [109].

It is proved in [108] that in a properly infinite von Neumann algebra M any derivation $\delta: S(M,\tau) \to S(M,\tau)$ is continuous with respect to the topology t_τ. Corollary (3.2.18) implies that, in this case, derivation $\delta : S(M,\tau) \to S(M,\tau)$ is continuous with respect to the topology $t(M)$ too.

Now, we give an application of Theorem (3.2.17) to derivations defined on absolutely solid $*$-subalgebras of the algebra $LS(M)$.

Recall [106], that a $*$-subalgebra A of $LS(M)$ is called absolutely solid if conditions $x \in LS(M), y \in A, |x| \le |y|$ imply that $x \in A$. In [106] it is proved that, if δ is a derivation on absolutely solid $*$-subalgebra $A \supset M$ and $\delta(x) = [w, x]$ for all $x \in A$ and some $w \in LS(M)$, then there exists $w_1 \in A$, such that $\delta(x) = [w_1, x]$ for all $x \in A$, i.e. the derivation δ is inner on the $*$-subalgebra A. This observation and Theorem (3.2.17) yield our final result

Corollary (3.2.19)[104]: Suppose that all derivations on the algebra $LS(M)$ are inner and let $A \supset M$ be an absolutely solid $*$-subalgebra of $LS(M)$. Then all derivations on A are inner. In particular, any derivation on the algebras $S(M)$ and $S(M, \tau)$ are inner.

The result of Corollary (3.2.19) extends and generalizes [105] and [106]. Note, that the conditions of Corollary (3.2.19) hold, in particular, for properly infinite von Neumann algebras, which do not have direct summand of type II_∞ [2], [105], [106].

Corollary (3.2.20)[301]: If $x_\alpha \in LS(M), 0 \neq x + 2\epsilon \in P(Z(M))$, then
$$(x + 2\epsilon)x_\alpha \xrightarrow{t(M)} 0 \Leftrightarrow (x + 2\epsilon)x_\alpha \xrightarrow{t((x+2\epsilon)M)} 0.$$
Proof. Fix a $*$-isomorphism $\varphi: Z(M) \to L^\infty(\Omega, \Sigma, \mu)$ and $0 \neq x + 2\epsilon \in P(Z(M))$. Let $E \in \Sigma$ be such that $\varphi(x + 2\epsilon) = \chi_E$. Define the mapping
$$\psi: Z((x + 2\epsilon)M) = (x + 2\epsilon)Z(M) \to L^\infty(E, \Sigma_E, \mu|_E)$$
by setting
$$\psi((x + 2\epsilon)a) = \varphi((x + 2\epsilon)a)|_E, for\ a \in Z(M).$$
Here, $\Sigma_E := \{A \cap E: A \in \Sigma\}$ and $\mu|_E$ is the restriction of μ to Σ_E.

It is clear that ψ is a $*$-isomorphism. Now define $D_{x+2\epsilon}: P((x + 2\epsilon)M) \to L_+(E, \Sigma_E, \mu|_E)$ by setting $D_{x+2\epsilon}(q) = D(q)|_E$ for $q \in P((x + 2\epsilon)M)$. It is straightforward that $D_{x+2\epsilon}$ is a dimension function on $P((x + 2\epsilon)M)$.

Let $\{q_\alpha\}_{\alpha \in A} \subset P((x + 2\epsilon)M)$. We claim
$$q_\alpha \xrightarrow{t(M)} 0 \Leftrightarrow q_\alpha \xrightarrow{t((x+2\epsilon)M)} 0.$$
To see the claim, assume that the first convergence holds and observe that by Proposition (3.2.2)(i), there exists a net $\{x_\alpha + 2\epsilon\}_{\alpha \in A} \subset P(Z(M))$ such that $(x_\alpha + 2\epsilon)q_\alpha \in P_{fin}(M)$ for any $\alpha \in A, \varphi((x_\alpha + 2\epsilon)^\perp) \xrightarrow{t(L^\infty(\Omega))} 0$, and $D((x_\alpha + 2\epsilon)q_\alpha) \xrightarrow{t(L^\infty(\Omega))} 0$. The projection $r_\alpha = (x + 2\epsilon)(x_\alpha + 2\epsilon)$ belongs to the center $Z((x + 2\epsilon)M)$ of the von Neumann algebra $(x + 2\epsilon)M$, and $r_\alpha q_\alpha = (x_\alpha + 2\epsilon)q_\alpha$ is a finite projection in $(x + 2\epsilon)M$ for each $\alpha \in A$. Also
$$\psi(x + 2\epsilon - r_\alpha) = \psi((x + 2\epsilon)(1 - (x_\alpha + 2\epsilon))) = \varphi((x + 2\epsilon)(x_\alpha + 2\epsilon)^\perp)|_E$$
$$= \varphi(x + 2\epsilon)\varphi((x_\alpha + 2\epsilon)^\perp)|_E \xrightarrow{t(L^\infty(E))} 0,$$
where $t(L^\infty(E))$ is the local measure topology on $L^0(E, \Sigma_E, \mu|_E)$, and
$$D_{x+2\epsilon}(r_\alpha q_\alpha) = D_{x+2\epsilon}((x_\alpha + 2\epsilon)q_\alpha) = D((x_\alpha + 2\epsilon)q_\alpha)|_E \xrightarrow{t(L^\infty(E))} 0.$$
Hence, by Proposition (3.2.2) (i) we get that $q \xrightarrow{t((x+2\epsilon)M)} 0$.

We will show now that the convergence $q_\alpha \xrightarrow{t((x+2\epsilon)M)} 0$ for $\{q_\alpha\}_{\alpha \in A} \subset P((x + 2\epsilon)M)$ implies the convergence $q_\alpha \xrightarrow{t(M)} 0$.

Let $\{r_\alpha\}_{\alpha \in A}$ be a net in $P(Z((x + 2\epsilon)M))$ such that $r_\alpha q_\alpha \in P_{fin}((x + 2\epsilon)M)$ for every $\alpha \in A$,
$$\psi(x + 2\epsilon - r_\alpha) \xrightarrow{t(L^\infty(E))} 0$$
and
$$D_{x+2\epsilon}(r_\alpha q_\alpha) \xrightarrow{t(L^\infty(E))} 0.$$

Put $(x_\alpha + 2\epsilon) = (x + 2\epsilon)^\perp + r_\alpha$. Then $(x_\alpha + 2\epsilon) \in P(Z(M))$ and $(x_\alpha + 2\epsilon)q_\alpha = r_\alpha q_\alpha \in P_{fin}(M)$.
Since $(x_\alpha + 2\epsilon)^\perp = (x + 2\epsilon)(1 - r_\alpha)$, we have $\varphi((x_\alpha + 2\epsilon)^\perp) = \chi_E \varphi((x_\alpha + 2\epsilon)^\perp)$ and

$$\varphi((x_\alpha + 2\epsilon)^\perp)|_E = \chi_E \varphi((x + 2\epsilon)(1 - r_\alpha))|_E = \chi_E \psi(x + 2\epsilon - r_\alpha) \xrightarrow{t(L^\infty(E))} 0.$$

Also

$$D((x_\alpha + 2\epsilon)q_\alpha) = D((x + 2\epsilon)r_\alpha q_\alpha) = \chi_E D(r_\alpha q_\alpha),$$

and so $D((x_\alpha + 2\epsilon)q_\alpha) \xrightarrow{t(L^\infty(\Omega))} 0$, since $D(r_\alpha q_\alpha)|_E = D_{x+2\epsilon}(r_\alpha q_\alpha) \xrightarrow{t(L^\infty(E))} 0$. Again appealing to Proposition (3.2.2)(i), we conclude that $q_\alpha \xrightarrow{t(M)} 0$.

Now let $\{x_\alpha\} \subset LS((x + 2\epsilon)M)$ and $x_\alpha \xrightarrow{t(M)} 0$. By Proposition (3.2.2) (ii), we have that $E_\lambda^\perp(|x_\alpha|) \xrightarrow{t(M)} 0$ for any $\lambda > 0$, where $\{E_\lambda(|x_\alpha|)\}$ is the spectral family for $|x_\alpha|$. Denote by $\{E_\lambda^{x+2\epsilon}(|x_\alpha|)\}$ the family of spectral projections for $|x_\alpha|$ in $LS((x + 2\epsilon)M), \lambda > 0$. It is clear that $E_\lambda(|x_\alpha|) = (x + 2\epsilon)^\perp + E_\lambda^{x+2\epsilon}(|x_\alpha|)$ and $E_\lambda^\perp(|x_\alpha|) = x + 2\epsilon - E_\lambda^{x+2\epsilon}(|x_\alpha|)$ for all $\lambda > 0$. It follows from above that $x + 2\epsilon - E_\lambda^{x+2\epsilon}(|x_\alpha|) \xrightarrow{t((x+2\epsilon)M)} 0$ for all $\lambda > 0$. Hence, by Proposition (3.2.2)(ii), it follows that $x_\alpha \xrightarrow{t((x+2\epsilon)M)} 0$.

The proof of the implication $x_\alpha \xrightarrow{t((x+2\epsilon)M)} 0 \Rightarrow x_\alpha \xrightarrow{t(M)} 0$ is similar and therefore omitted.

Corollary (3.2.21)[301]: (See [104]) For a net $\{f_\alpha\}_{\alpha \in A}$ and f from $L^0(\Omega, \Sigma, \mu)$ the following conditions are equivalent:

(i). $f_\alpha \xrightarrow{t(L^\infty(\Omega))} f$;

(ii). $f_\alpha \chi_{A_i} \xrightarrow{t(L^\infty(A_i))} f_\alpha A_i$ for all $i \in I$.

Proof. The implication (i) ⇒ (ii) follows from the definitions of topologies $t(L^\infty(\Omega))$ and $t(L^\infty(A_i))$.

(ii) ⇒ (i). It is sufficient to consider the case when $f = 0$.

Consider the set Γ of all finite subsets γ from I and order it with respect to inclusion. Consider an increasing net $\chi_{D_\gamma} \uparrow \chi_\Omega$ in $L_h^0(\Omega, \Sigma, \mu)$, where $D_\gamma = \bigcup_{i \in \gamma} A_i, \gamma \in \Gamma$. Take an arbitrary neighborhood of zero U (in the topology $(L^\infty(\Omega))$) and select $W(B, \varepsilon, \delta)$ in such a way that $W(B, \varepsilon, \delta) + W(B, \varepsilon, \delta) \subset U$. Since $\mu(B \cap D_\lambda) \uparrow \mu(B) < \infty$, we can locate such $\gamma_0 \in \Gamma$ that $\mu(B \setminus D_{\gamma_0}) \le \delta$. Hence, $f_\alpha \chi_\Omega \backslash D_{\gamma_0} \in W(B, \varepsilon, \delta)$ for all $\alpha \in A$.

Since $f_\alpha \chi_{A_i} \xrightarrow{t(L^\infty(A_i))} 0$ for all $i \in 0$ and 0 is a finite set, it follows $f_\alpha \chi_{D_{\gamma_0}} = \sum_{i \in \gamma_0} f_\alpha \chi_{A_i} \xrightarrow{t(L^\infty(\Omega))} 0$.

Thus, there exists such $\alpha_0 \in A$ that $f_\alpha \chi_{D_{\gamma_0}} \in W(B, \varepsilon, \delta)$ for all $\alpha \ge \alpha_0$.

In particular,

$$f_\alpha = f_\alpha \chi_{D_{\gamma_0}} + f_\alpha \chi_\Omega \backslash D_{\gamma_0} \in W(B, \varepsilon, \delta) + W(B, \varepsilon, \delta) \subset U, \forall \alpha \ge \alpha_0,$$

which implies the convergence $f_\alpha \xrightarrow{t(L^\infty(\Omega))} 0$.

Corollary (3.2.22)[301]: If $P(Z(M)) \subset A, \delta$ is a derivation on A and $(x + 2\epsilon) \in P(Z(M))$, then $\delta(x + 2\epsilon) = 0$ and $\delta((x + 2\epsilon)x) = (x + 2\epsilon)\delta(x)$ for all $x \in A$.

Proof. We have that $\delta(x + 2\epsilon) = \delta((x + 2\epsilon)^2) = \delta(x + 2\epsilon)(x + 2\epsilon) + (x + 2\epsilon)\delta(x + 2\epsilon) = 2(x + 2\epsilon)\delta(x + 2\epsilon)$. Hence, $(x + 2\epsilon)\delta(x + 2\epsilon) = (x + 2\epsilon)(2(x + 2\epsilon)\delta(x +$

$2\epsilon)) = 2(x + 2\epsilon)\delta(x + 2\epsilon)$, that is $(x + 2\epsilon)\delta(x + 2\epsilon) = 0$. Therefore, we have $\delta(x + 2\epsilon) = 0$. In particular, $\delta((x + 2\epsilon)x) = \delta(x + 2\epsilon)x + (x + 2\epsilon)\delta(x) = (x + 2\epsilon)\delta(x)$.

Corollary (3.2.23)[301]: If M properly infinite von Neumann algebras, then any derivation $\delta: LS(M) \to LS(M)$ is continuous with respect to the topology $t(M)$ of local convergence in measure.

Proof. By Lemma (3.2.8), we may assume that $\delta^* = \delta$. Since $Z(M)$ is a commutative von Neumann algebra, there exists a system $\{x_i + 2\epsilon\}, i \in I$ of non-zero pairwise orthogonal projections from $Z(M)$ such that $\sup_{i\in I} x_i + 2\epsilon = 1$ and the Boolean algebra $P((x_i + 2\epsilon)Z(M))$ has a countable type for all $i \in I$. By Corollary (3.2.22) we have that $\delta((x_i + 2\epsilon)x) = (x_i + 2\epsilon)\delta(x)$ for all $x \in LS(M), i \in I$. Therefore, by Corollary 2.8, it is sufficient to prove that each derivation $\delta_{x_i+2\epsilon}$ is $t((x_i + 2\epsilon)M)$-continuous, $i \in I$. Thus, we may assume without loss of generality that the Boolean algebra $P(Z(M))$ has a countable type.

In this case the topology $t(M)$ is metrizable, and the sets $V(\Omega, 1/n, 1/n), n \in \mathbb{N}$ form a countable base of neighborhoods of 0; in particular, $(LS(M), t(M))$ is an F-space. Therefore it is sufficient to show that the graph of the linear operator δ is a closed set.

Arguing by a contradiction, let us assume that the graph of δ is not closed. This means that there exists a sequence $\{x_n\} \subset LS(M)$, such that $x_n \xrightarrow{t(M)} 0$ and $\delta(x_n) \xrightarrow{t(M)} x \neq 0$. Recalling that $(LS(M), t(M))$ is a topological $*$-algebra and that $\delta = \delta^*$, we may assume that $x = x^*, x_n = x_n^*$ for all $n \in \mathbb{N}$. In this case, $x = x_+ - x_-$, where $x_+, x_- \in LS_+(M)$ are respectively the positive and negative parts of x. Without loss of generality, we shall also assume that $x_+ \neq 0$, otherwise, instead of the sequence $\{x_n\}$ we consider the sequence $\{-x_n\}$. Let us select scalars $0 < \lambda_1 < \lambda_2$ so that the projection

$$q + \epsilon := E_{\lambda_2}(x) - E_{\lambda_1}(x)$$

does not vanish. We have that $0 < \lambda_1(q + \epsilon) \leq (q + \epsilon)x(q + \epsilon) = (q + \epsilon)x \leq \lambda_2(q + \epsilon)$ and $\|(q + \epsilon)x\|_M \leq \lambda_2$. Replacing, if necessary, x_n on x_n/λ_1, we may assume that

$$(q + \epsilon)x(q + \epsilon) \geq (q + \epsilon). \tag{42}$$

By the assumption, M is a properly infinite von Neumann algebra and therefore, there exist pairwise orthogonal projections $\left\{(q_m + \epsilon)^{(1)}\right\}_{m=1}^{\infty} \subset P(M)$, such that $\sup_{m\geq 1} (q_m + \epsilon)^{(1)} = 1$ and $(q_m + \epsilon)^{(1)} \sim 1$ for all $m \in \mathbb{N}$, in particular, $q + \epsilon \preccurlyeq (q_m + \epsilon)^{(1)}$. Here, the notation $q + \epsilon \sim q$ denotes the equivalence of projections $q + \epsilon, q \in P(M)$, and the notation $\epsilon \preccurlyeq 0$ means that there exists a projection $e \leq q$ such that $q + \epsilon \sim e$. In course of the proof of our main result we shall frequently use the following well-known fact: if $\epsilon \sim 0$ and $(x + 2\epsilon) \in P(Z(M))$ then $(q + \epsilon)(x + 2\epsilon) \sim q(x + 2\epsilon)$.

For every $m \in \mathbb{N}$ we select a projection $q_m + \epsilon \leq (q_m + \epsilon)^{(1)}$, for which $q_m \sim q$ and denote by v_m a partial isometry from M such that $v_m^* v_m = q + \epsilon, v_m v_m^* = q_m + \epsilon$. Clearly, we have $(q_m + \epsilon)(q_k + \epsilon) = 0$ whenever $m \neq k$ and the projection

$$q_0 + \epsilon := \sup_{m\geq 1} q_m + \epsilon \tag{43}$$

is infinite as the supremum of pairwise orthogonal and equivalent projections. Taking into account that

$$q_m + \epsilon = v_m(q + \epsilon)^{v^*}{}_{m} \overset{(1)}{\leq} v_m(q + \epsilon)x(q + \epsilon)v_m^* = v_m x v_m^* \in (q_m + \epsilon)M(q_m + \epsilon),$$

and

$$\|v_m x v_m^*\|_M = \|v_m(q + \epsilon)x(q + \epsilon)v_m^*\|_M \leq \|(q + \epsilon)x(q + \epsilon)\|_M \leq \lambda_2,$$

we see that the series $\sum_{m=1}^{\infty} v_m x v_m^*$ converges with respect to the topology τ_{so} to some operator $x + \epsilon \in M$ satisfying

$$\|x+\epsilon\|_M = sup_{m\geq 1}\|v_m x v_m^*\|_M \leq \|(q+\epsilon)x(q+\epsilon)\|_M, \text{ and } x \geq q_0. \tag{44}$$

In what follows, we shall assume that the central support $c(q_0+\epsilon)$ of the projection $q_0+\epsilon$ is equal to 1 (otherwise, we replace the algebra M with the algebra $c(q_0+\epsilon)M$).

Let φ be a $*$-isomorphism from $Z(M)$ onto $L^\infty(\Omega,\Sigma,\mu)$. By the assumption, the Boolean algebra $P(Z(M))$ has a countable type, and so we may assume that $\mu(\Omega) = \int_\Omega 1_{L^\infty(\Omega)} d\mu = 1$, where $1_{L^\infty(\Omega)}$ is the identity of the $*$-algebra $L^\infty(\Omega,\Sigma,\mu)$. In this case, the countable base of neighborhoods of 0 in the topology $t(M)$ is formed by the sets $V(\Omega,1/n,1/n), n\in\mathbb{N}$.

Recalling that we have $x_n \xrightarrow{t(M)} 0$ and $\delta(x_n) \xrightarrow{t(M)} x$, we obtain

$$v_m x_n v_m^* \xrightarrow{t(M)} 0, \delta(v_m)x_n v_m^* \xrightarrow{t(M)} 0, v_m\delta(x_n)v_m^* \xrightarrow{t(M)} v_m x v_m^*$$

when $n\to\infty$ for every fixed $m\in\mathbb{N}$.

Fix $k\in\mathbb{N}$, and using the convergence $v_m x_n v_m^* \xrightarrow{t(M)} 0$ for $n\to 0$, for each $m\in\mathbb{N}$ select an index $n_1(m,k)$ and projections $q_{m,n}^{(1)}\in P(M), \left(x_{m,n}+2\epsilon\right)^{(1)}\in P(Z(M))$, such that

$$\left\|v_m x_n v_m^* q_{m,n}^{(1)}\right\|_M \leq 2^{-m}(k+1)^{-1};$$

$$\int_\Omega \varphi(1-\left(x_{m,n}+2\epsilon\right)^{(1)})d\mu \leq 3^{-1}2^{-m-k-1}$$

and

$$D(\left(x_{m,n}+2\epsilon\right)^{(1)}(1-q_{m,n}^{(1)})) \leq 3^{-1}2^{-m-k-1}\varphi\left(\left(x_{m,n}+2\epsilon\right)^{(1)}\right)$$

for all $n\geq n_1(m,k)$.

Similarly, using the convergence $\delta(v_m)x_n v_m^* \xrightarrow{t(M)} 0$ (respectively, $v_m\delta(x_n)v_m^* \xrightarrow{t(M)} v_m x v_m^*$) for $n\to\infty$, for each $m\in\mathbb{N}$ select indexes $n_2(m,k)$ and $n_3(m,k)$ and projections $q_{m,n}^{(2)}, q_{m,n}^{(3)}\in P(M), \left(x_{m,n}+2\epsilon\right)^{(2)}, \left(x_{m,n}+2\epsilon\right)^{(3)}\in P(Z(M))$, such that

$$\left\|\delta(v_m)x_n v_m^* q_{m,n}^{(2)}\right\|_M \leq (3(k+1)2^m)^{-1}$$

(respectively, $\left\|(v_m\delta(x_n)v_m^* - v_m x v_m^*)q_{m,n}^{(3)}\right\|_M \leq (3(k+1)2^m)^{-1}$);

$$\int_\Omega \varphi(1-\left(x_{m,n}+2\epsilon\right)^{(i)})d\mu \leq 3^{-1}2^{-m-k-1}$$

and $D(1-q_{m,n}^{(i)}) \leq 3^{-1}2^{-m-k-1}\varphi(\left(x_{m,n}+2\epsilon\right)^{(i)}), i=2,3,$ for all $n\geq n_2(m,k)$ (respectively, $n\geq n_3(m,k)$).

Set $n(m,k) = max_{i=1,2,3}\, n_i(m,k), x_m+2\epsilon = inf_{i=1,2,3}\left(x_{m,n}+2\epsilon\right)^{(i)}, q_m = inf_{i=1,2,3} q_{m,n(m,k)}^{(i)}$. Due to the selection of projections $q_m\in P(M), x_m+2\epsilon\in P(Z(M))$ and indexes $n(m,k)$, we have that for each $m\in\mathbb{N}$ inequalities hold

$(A1)$ $\left\|v_m x_{n(m,k)} v_m^* q_m\right\|_M \leq 2^{-m}(k+1)^{-1}$;

$(A2)$ $\left\|\delta(v_m)x_{n(m,k)} v_m^* q_m\right\|_M \leq (3(k+1)2^m)^{-1}$;

$(A3)$ $\left\|q_m(v_m\delta(x_{n(m,k)})v_m^* - v_m x v_m^*)\right\|_M \leq (3(k+1)2^m)^{-1}$;

$(A4)$ $D((x_m+2\epsilon)(1-q_m)) \overset{(D6)}{\leq} D((x_m+2\epsilon)(1-q_{m,n(m,k)}^{(i)})) \leq 2^{-m-k-1}\varphi(x_m+2\epsilon)$;

$(A5)$ $\quad 1-\int_\Omega \varphi(x_m+2\epsilon)d\mu=\int_\Omega \varphi(1-(x_m+2\epsilon))d\mu \le \sum_{i=1}^{3}\int_\Omega \varphi(1-\left(x_{m,n(m,k)}+2\epsilon\right)^{(i)})d\mu \le 2^{-m-k-1}$.

Fix $m_1, m_2 \in \mathbb{N}$ with $m_1 < m_2$ and set

$$q_{m_1,m_2} := inf_{m_1<m\le m_2} q_m, x_{m_1,m_2}+2\epsilon := inf_{m_1<m\le m_2}(x_m+2\epsilon).$$

Since $(1-(x_{m_1,m_2}+2\epsilon)) = sup_{m_1<m\le m_2}(1-(x_m+2\epsilon))$ and $(1-q_{m_1,m_2}) = sup_{m_1<m\le m_2}(1-q_m)$, it follows that $\varphi(1-(x_{m_1,m_2}+2\epsilon)) = sup_{m_1<m\le m_2}\varphi(1-(x_m+2\epsilon))$ and $\varphi\left(1-q_{m_1,m_2}\right) = sup_{m_1<m\le m_2}\varphi(1-q_m)$, and therefore

$$1-\int_\Omega \varphi\left(x_{m_1,m_2}+2\epsilon\right)d\mu = \varphi\left(1-\left(x_{m_1,m_2}+2\epsilon\right)\right)d\mu$$

$$\le \varphi\left(1-(x_m+2\epsilon)\right)d\mu \overset{(A5)}{\le} 2^{-m_1-k-1}; \tag{45}$$

$$\left(\left(x_{m_1,m_2}+2\epsilon\right)\left(1-q_{m_1,m_2}\right)\right) \overset{(D6)}{\le} \sum_{m=m_1+1}^{m_2} D\left((x_{m_1,m_2}\right.$$

$$\left.+2\epsilon)(1-q_m)\right) \overset{(A4)}{\le} 2^{-m_1-k-1}\varphi(x_{m_1,m_2}+2\epsilon) \tag{46}$$

and

$$\left\|\sum_{m=m_1+1}^{m_2}(v_m x_{n(m,k)} v_m^*)q_{m_1,m_2}\right\|_M$$

$$\le \sum_{m=m_1+1}^{m_2}\left\|\left(v_m x_{n(m,k)} v_m^*\right)q_m\right\|_M \overset{(A1)}{\le} 2^{m_1}(k+1)^{-1}. \tag{47}$$

Inequalities (45)-(47) mean that the sequence

$$S_{l,k}=\sum_{m=1}^{l} v_m x_{n(m,k)} v_m^*, l\ge 1$$

is a Cauchy sequence in the F-space $(LS(M,t(M)))$ for each fixed $k\in\mathbb{N}$. Consequently, there exists $x_k+\epsilon \in LS(M)$ such that $S_{l,k}\xrightarrow{t(M)} x_k+\epsilon$ for $l\to\infty$, i.e. the series

$$x_k+\epsilon=\sum_{m=1}^{\infty} v_m x_{n(m,k)} v_m^* \tag{48}$$

converges in $LS(M)$ with respect to the topology $t(M)$. Since the involution is continuous in topology $t(M)$ and $S_{l,k}^*=S_{l,k}$, we conclude $x_k+\epsilon=(x_k+\epsilon)^*$.

Setting

$$r_m := (q_m+\epsilon)\wedge q_m, m\in\mathbb{N}, \tag{49}$$

and using the relation $(x_m+2\epsilon)((q_m+\epsilon)-(q_m+\epsilon)\wedge q_m) \sim (x_m+2\epsilon)((q_m+\epsilon)\vee q_m-q_m)$ (see e.g. [33]) we have

$$D((x_m+2\epsilon)((q_m+\epsilon)-r_m))$$

$$= D((x_m+2\epsilon)((q_m+\epsilon)-(q_m+\epsilon)\wedge q_m)) \overset{(D3)}{=} D((x_m+2\epsilon)((q_m+\epsilon)\vee q_m-q_m))$$

$$\le D((x_m+2\epsilon)(1-q_m)) \overset{(A4)}{\le} 2^{-m-k-1}\varphi(x_m+2\epsilon). \tag{50}$$

Setting

$$q_0^{(k)} := sup_{m\geq 1} r_m, (x_0+2\epsilon)^{(k)} := inf_{m\geq 1}(x_m+2\epsilon), \tag{51}$$

we have (see (43), (44) and (49))

$$x+\epsilon \geq q_0+\epsilon \geq q_0^{(k)}, k\in\mathbb{N}. \tag{52}$$

From (45) it follows that

$$1-\int_\Omega \varphi((x_0+2\epsilon)^{(k)})d\mu \ = \int_\Omega \varphi(1-(x_0+2\epsilon)^{(k)})d\mu \ \leq 2^{-k-1}. \tag{53}$$

Since $(q_m+\epsilon)(q_j+\epsilon)=0, m\neq j$, and $r_m\leq (q_m+\epsilon)$ (see (49)) we obtain $(q_0+\epsilon)-q_0^{(k)} = sup_{m\geq 1}((q_m+\epsilon)-r_m)$ and hence, by (50),

$$D((x_0+2\epsilon)^{(k)}(q_0+\epsilon - q_0^{(k)})) \overset{(D6)}{=} \sum_{m=1}^{\infty} D((x_0+2\epsilon)^{(k)}(q_m+\epsilon - r_m)) \overset{(50)}{\leq} 2^{-k-1}\varphi((x_0+2\epsilon)^{(k)}). \tag{54}$$

Due to (49), we have $(q_m+\epsilon)q_0^{(k)} = r_m q_0^{(k)} = r_m = r_m q_m$ for all $m\in\mathbb{N}$.

Hence,

$$v_m x_{n(m,k)} v_m^* q_0^{(k)} = v_m x_{n(m,k)} v_m^*(q_m+\epsilon)q_0^{(k)} = v_m x_{n(m,k)} v_m^* r_m$$

and

$$\left\|(x_k+\epsilon)q_0^{(k)}\right\|_M = \left\|\left(\sum_{m=1}^{\infty} v_m x_{n(m,k)} v_m^*\right) q_0^{(k)}\right\|_M = \left\|\sum_{m=1}^{\infty} v_m x_{n(m,k)} v_m^*\, q_0^{(k)}\right\|_M$$
$$\leq sup_{m\geq 1}\left\|v_m x_{n(m,k)} v_m^* r_m\right\|_M$$
$$\leq sup_{m\geq 1}\left\|v_m x_{n(m,k)} v_m^* q_m\right\|_M \overset{(A1)}{\leq} (k+1)^{-1}. \tag{55}$$

Using the properties of the derivation δ and equalities $(q_n+\epsilon)v_n = v_n, v_n^* = v_n^*\,(q_n+\epsilon)$ and (49), (51), we have

$$q_0^{(k)}\delta\big(v_m x_{n(m,k)} v_m^*\big)q_0^{(k)} = q_0^{(k)}((\delta(v_m x_{n(m.k)} v_m^*) - v_m x v_m^*) + v_m x v_m^*)q_0^{(k)}$$
$$= (q_0^{(k)}\delta(v_m)x_{n(m,k)}v_m^* q_0^{(k)} + q_0^{(k)} v_m x_{n(m,k)}\delta(v_m^*)q_0^{(k)})$$
$$+ q_0^{(k)}(v_m\delta(x_{n(m,k)})v_m^* - v_m x v_m^*)q_0^{(k)} + q_0^{(k)}(v_m x v_m^*)q_0^{(k)}$$
$$= q_0^{(k)}\delta(v_m)x_{n(m,k)}v_m^* q_m r_m + r_m q_m v_m x_{n(m,k)}\delta(v_m^*)q_0^{(k)}$$
$$+ r_m q_m(v_m\delta(x_{n(m,k)})v_m^* - v_m x v_m^*)q_m r_m + q_0^{(k)}(v_m x v_m^*)q_0^{(k)}.$$

Consider the following formal series suggested by the preceding

$$\sum_{m=1}^{\infty} q_0^{(k)}\delta(v_m)x_{n(m,k)}v_m^* q_m r_m\,; \tag{56}$$

$$\sum_{m=1}^{\infty} r_m q_m v_m x_{n(m,k)}\delta(v_m^*)q_0^{(k)}\ ; \tag{57}$$

$$\sum_{m=1}^{\infty} r_m q_m(v_m\delta(x_{n(m,k)})v_m^* - v_m x v_m^*)q_m r_m\,; \tag{58}$$

$$\sum_{m=1}^{\infty} q_0^{(k)}\big(v_m x_{n(m,k)} v_m^*\big)q_0^{(k)}\ . \tag{59}$$

By the condition $(A2)$ the first series (56) and the second series (57) converge with respect to the norm $\|.\|_M$ to some elements $a, b \in M$ respectively and $\|a\|_M \leq (3(k+1))^{-1}$ and $\|b\|_M \leq (3(k+1))^{-1}$.

Similarly, by the condition $(A3)$, the third series (58) also converges with respect to the norm $\|.\|_M$ to some element $c \in M$, satisfying $\|c\|_M \leq (3(k+1))^{-1}$. Finally, since $x + \epsilon = \sum_{m=1}^{\infty} v_m x v_m^*$ (the convergence of the latter series is taken in the τ_{so} topology), we see that the fourth series (59) converges with respect to the topology τ_{so} to some element $q_0^{(k)}(x+\epsilon)q_0^{(k)}$. Hence, the series

$$\sum_{m=1}^{\infty} q_0^{(k)} \delta(v_m x_{n(m,k)} v_m^*) q_0^{(k)} \tag{60}$$

converges with respect to the topology τ_{so} to some element $a_k \in M$, and, in addition, we have

$$\left\| a_k - q_0^{(k)}(x+\epsilon) q_0^{(k)} \right\|_M \leq (k+1)^{-1}. \tag{61}$$

We shall show that

$$a_k = q_0^{(k)} \delta(x_k + \epsilon) q_0^{(k)}, \tag{62}$$

where $x_k + \epsilon = v_m x_{n(m,k)} v_m^*$ (the convergence of the latter series is taken in the $t(M)$-topology (see (48)). Using (51) for any $m_1, m_2 \in \mathbb{N}$ we have

$$\begin{aligned} r_{m_1} q_0^{(k)} \delta(x_k+\epsilon) q_0^{(k)} r_{m_2} &= \delta(r_{m_1} q_0^{(k)} (x_k+\epsilon)) q_0^{(k)} r_{m_2} - \delta(r_{m_1} q_0^{(k)})(x_k+\epsilon) q_0^{(k)} r_{m_2} \\ &= \delta(r_{m_1} v_{m_1} x_{n(m_1,k)} v_{m_1}^*) r_{m_2} - \delta(r_{m_1}) v_{m_2} x_{n(m_2,k)} v_{m_2}^* r_{m_2}. \end{aligned}$$

Since the series $\sum_{m=1}^{\infty} q_0^{(k)} \delta(v_m x_{n(m,k)} v_m^*) q_0^{(k)}$ converges with respect to the topology τ_{so} (see 19), it follows that the series

$$\sum_{m=1}^{\infty} r_{m_1} (q_0^{(k)} \delta(v_m x_{n(m,k)} v_m^*) q_0^{(k)}) r_{m_2}$$

also converges with respect to this topology ([116]), in addition, the following equalities hold

$$\begin{aligned} r_{m_1} a_k r_{m_2} = r_{m_1} (q_0^{(k)} \delta(v_m x_{n(m,k)} v_m^*) q_0^{(k)}) r_{m_2} &\overset{(51)}{=} \sum_{m=1}^{\infty} r_{m_1} \delta(v_m x_{n(m,k)} v_m^*) r_{m_2} \\ &= \sum_{m=1}^{\infty} (\delta(r_{m_1} v_m x_{n(m,k)} v_m^*) r_{m_2} - \delta(r_{m_1}) v_m x_{n(m,k)} v_m^* r_{m_2}) \\ &\overset{(49)}{=} \delta(r_{m_1} v_{m_1} x_{n(m_1,k)} v_{m_1}^*) r_{m_2} - \delta(r_{m_1}) v_{m_2} x_{n(m_2,k)} v_{m_2}^* r_{m_2}, \end{aligned}$$

which guarantees

$$r_{m_1} q_0^{(k)} \delta(x_k+\epsilon) q_0^{(k)} r_{m_2} = r_{m_1} a_k r_{m_2}. \tag{63}$$

Since

$$r_{m_1}(\delta(x_k+\epsilon) - a_k) r_m \overset{(63)}{=} 0,$$

we see that for the right support $r(r_{m_1}(\delta(x_k+\epsilon)-a_k))$ of the operator $r_{m_1}(\delta(x_k+\epsilon) - a_k)$ satisfies the inequality

$$r(r_{m_1}(\delta(x_k+\epsilon)-a_k)) \leq 1 - r_m, m \in \mathbb{N},$$

and therefore

$$r(r_{m_1}(\delta(x_k+\epsilon)-a_k)) \leq inf_{m\geq 1}(1-r_m) \overset{(51)}{=} 1 - q_0^{(k)}.$$

Consequently, $r_{m_1}(\delta(x_k+\epsilon)-a_k)q_0^{(k)}=0$ for all $m_1\in\mathbb{N}$.

Similarly, using the left support of the operator $(\delta(x_k+\epsilon)-a_k)q_0^{(k)}$, we claim that $q_0^{(k)}(\delta(x_k+\epsilon)-a_k)q_0^{(k)}=0$.

Since $q_0^{(k)}a_kq_0^{(k)}=a_k$, the equality (62) holds.

Thus, the inequality (61) can be restated as follows

$$\left\|q_0^{(k)}\big(\delta(x_k+\epsilon)-(x+\epsilon)\big)q_0^{(k)}\right\|_M\le(k+1)^{-1}. \tag{64}$$

It follows from the inequalities (44) and (64), that

$$\left\|(k+1)q_0^{(k)}(x_k+\epsilon)\right\|_M=\left\|(k+1)(x_k+\epsilon)q_0^{(k)}\right\|_M\le 1 \tag{65}$$

and

$$\left\|q_0^{(k)}\delta((k+1)(x_k+\epsilon))q_0^{(k)}-(k+1)q_0^{(k)}(x+\epsilon)q_0^{(k)}\right\|_M\le 1. \tag{66}$$

Due to (66), and taking into account (52), we obtain

$$(k+1)q_0^{(k)}-q_0^{(k)}\delta((k+1)(x_k+\epsilon))q_0^{(k)}$$
$$\le(k+1)q_0^{(k)}(x+\epsilon)q_0^{(k)}-q_0^{(k)}\delta((k+1)(x_k+\epsilon))q_0^{(k)}\le q_0^{(k)},$$

that is

$$kq_0^{(k)}\le q_0^{(k)}\delta((k+1)(x_k+\epsilon))q_0^{(k)}. \tag{67}$$

Let us now consider the projections

$$q_0:=inf_{k\ge1}q_0^{(k)},(x_0+2\epsilon):=inf_{k\ge1}(x_0+2\epsilon)^{(k)}. \tag{68}$$

Using (52), (68) we have that $\epsilon=sup_{k\ge1}(q_0+\epsilon-q_0^{(k)})$. Therefore, combining (54) and (68), we obtain

$$D\big((x_0+2\epsilon)(\epsilon)\big)$$
$$=D\left(sup_{k\ge1}\left((x_0+2\epsilon)\left(q_0+\epsilon-q_0^{(k)}\right)\right)\right)\overset{(D6)}{\le}\sum_{k=1}^{\infty}D\left((x_0\right.$$
$$\left.+2\epsilon)\left(q_m+\epsilon-q_0^{(k)}\right)\right)\overset{(54)}{\le}\varphi(x_0+2\epsilon), \tag{69}$$

that is the projection $(x_0+2\epsilon)(\epsilon)$ is finite (see (D1)). Moreover, due to inequalities (65) (respectively, (67)), we have

$$\|(k+1)q_0(x_k+\epsilon)\|_M=\|(k+1)(x_k+\epsilon)q_0\|_M\le 1,k\in\mathbb{N} \tag{70}$$

(respectively,

$$kq_0\le q_0\delta((k+1)(x_k+\epsilon))q_0,k\in\mathbb{N}.) \tag{71}$$

Since φ is a $*$-isomorphism from $Z(M)$ onto $L^\infty(\Omega,\Sigma,\mu)$, by (53), we have that

$$\int_\Omega\varphi(1-(x_0+2\epsilon))d\mu=\int_\Omega sup_{k\ge1}\varphi(1-\ (x_0+2\epsilon)^{(k)})d\mu$$
$$\le\sum_{k=1}^{\infty}\int_\Omega\varphi(1-(x_0+2\epsilon)^{(k)})d\mu\overset{(53)}{\le}2^{-1},$$

in particular, $x_0+2\epsilon\neq0$. Since $1=c(q_0+\epsilon)$ and $c((q_0+\epsilon)(x_0+2\epsilon))=c(q_0+\epsilon)(x_0+2\epsilon)=x_0+2\epsilon\neq0$, we have $(x_0+2\epsilon)(q_0+\epsilon)\neq0$, and therefore there exists such $n\in\mathbb{N}$ that $(x_0+2\epsilon)(q_n+\epsilon)\neq0$ (see (43)). Since $(x_0+2\epsilon)(q_n+\epsilon)\sim(x_0+2\epsilon)(q_m+\epsilon)$, we have $(x_0+2\epsilon)(q_m+\epsilon)\neq0$ for all $m\in\mathbb{N}$. Hence, $(x_0+2\epsilon)(q_0+\epsilon)$ is an infinite projection. Since the projection $(x_0+2\epsilon)(\epsilon)$ is finite (see (69)), we see that the projection $(x_0+2\epsilon)q_0$ must be infinite. By [23], there exists a central projection

$$0\neq e_0\in P(Z(M)),e_0\le x_0+2\epsilon,$$

such that e_0q_0 is properly infinite, in particular, there exist pairwise orthogonal projections

$$e_n \leq e_0q_0, e_n \sim e_0q_0 \tag{72}$$

for all $n \in \mathbb{N}$ (see, for example, [29]). In addition,

$$\int_\Omega \varphi(c(q_0)e_0)\, d\mu \neq 0. \tag{73}$$

For every $n \in \mathbb{N}$ the operator

$$b_n := \delta(e_n)e_n$$

is locally measurable, and therefore there exists such a sequence $\{(x_m + 2\epsilon)^{(n)}\} \subset P(Z(M))$ that $(x_m + 2\epsilon)^{(n)} \uparrow 1$ when $m \to \infty$ and $(x_m + 2\epsilon)^{(n)}b_n \in S(M)$ for all $m \in \mathbb{N}$. Since $\varphi((x_m + 2\epsilon)^{(n)}) \uparrow \varphi(1) = 1_{L^\infty}(\Omega)$ it follows that $\int_\Omega \varphi((x_m + 2\epsilon)^{(n)})d\mu \uparrow \mu(1_{L^\infty}(\Omega)) = 1$ when $m \to \infty$, and therefore, by (73), for every $n \in \mathbb{N}$ there exists such a projection $(x + 2\epsilon)^{(n)} \in P(Z(M))$, that $(x + 2\epsilon)^{(n)}b_n \in S(M)$ and

$$1 - 2^{-n-1}\int_\Omega \varphi(c(q_0)e_0)d\mu < \int_\Omega \varphi((x + 2\epsilon)^{(n)})d\mu. \tag{74}$$

Consider the central projection

$$g_0 := inf_{n\geq 1}(x + 2\epsilon)^{(n)}.$$

Since $(x + 2\epsilon)^{(n)}b_n \in S(M), g_0 = g_0(x + 2\epsilon)^{(n)}$ we have that $g_0b_n \in S(M)$ for all $n \in \mathbb{N}$. Due to (74) we have

$$\begin{aligned} 1 - \int_\Omega \varphi(g_0)d\mu &= \int_\Omega \varphi(1 - g_0)d\mu = \int_\Omega sup\, \varphi(1 - (x + 2\epsilon)^{(n)})d\mu \\ &\leq \sum_{n=1}^{\infty}\int_\Omega \varphi(1 - (x + 2\epsilon)^{(n)})d\mu = \sum_{n=1}^{\infty}(1 - \int_\Omega \varphi((x + 2\epsilon)^{(n)})d\mu) \\ &\leq 2^{-1}\int_\Omega \varphi(c(q_0)e_0)d\mu. \end{aligned}$$

Consequently, $1 - 2^{-1}\int_\Omega \varphi(c(q_0)e_0)d\mu \leq \int_\Omega \varphi(g_0)d\mu$, and therefore

$$1 + 2^{-1}\int_\Omega \varphi(c(q_0)e_0)d\mu \leq \int_\Omega \varphi(g_0)d\mu + 2^{-1}\int_\Omega \varphi(c(q_0)e_0)d\mu. \tag{75}$$

From (73) and inequality (75), it follows that $2^{-1}\int_\Omega \varphi(g_0c(q_0)e_0)d\mu > 0$, i.e. $g_0c(q_0)e_0 \neq 0$ and so $g_0e_0q_0 \neq 0$. Since e_0q_0 is a properly infinite projection it follows that $g_0e_0q_0$ is a properly infinite projection. From the relationship $g_0e_n \overset{(72)}{\sim} g_0e_0q_0$, we see that the projection g_0e_n is also properly infinite for all $n \in \mathbb{N}$. Since

$$c(g_0e_n) = g_0c(e_n) \overset{(72)}{\leq} q_0c(q_0e_0) = g_0c(q_0)e_0,$$

it follows that $(x + 2\epsilon)e_n$ is also properly infinite projection for every $0 \neq (x + 2\epsilon) \in P(Z(M))$ with $x + 2\epsilon \leq g_0c(q_0)e_0$. Indeed, if $(x + 2\epsilon)' \in P(Z(M))$ and $(x + 2\epsilon)'(x + 2\epsilon)e_n \neq 0$, then $0 \neq (x + 2\epsilon)'(x + 2\epsilon)e_n = ((x + 2\epsilon)'(x + 2\epsilon)c(q_0)e_0)g_0e_n$, and therefore, since the projection g_0e_n is properly infinite, we have $((x + 2\epsilon)'(x + 2\epsilon)c(q_0)e_0)g_0e_n \notin P_{fin}(M)$. Consequently, the projection $(x + 2\epsilon)e_n$ is also properly infinite.

Passing, if necessary to the algebra $g_0c(q_0)e_0M$, we may assume that $g_0c(q_0)e_0 = 1$. In this case, we also may assume that $b_n \in S(M), e_n \sim q_0, c(e_n) = 1$ and $(x + 2\epsilon)e_n$ is a properly infinite projection for every non-zero $(x + 2\epsilon) \in P(Z(M))$.

The assumption $b_n \in S(M)$ means that for every fixed $n \in \mathbb{N}$ there exists such a sequence $\{(q_m+\epsilon)^{(n)}\}_{m=1}^{\infty} \subset P_{fin}(M)$, that $(q_m+\epsilon)^{(n)} \downarrow 0$ when $m \to \infty$ and $b_n(1-(q_m+\epsilon)^{(n)}) \in M$ for all $m \in \mathbb{N}$. Since $D((q_m+\epsilon)^{(n)}) \in L^0(\Omega,\Sigma,\mu)$ and $D((q_m+\epsilon)^{(n)}) \downarrow 0$ (see (D7)), it follows that $\{D((q_m+\epsilon)^{(n)})\}_{n=1}^{\infty}$ converges in measure μ to zero. Consequently, we may select a central projection f_n and a finite projection $s_n = \left(q_{m_n}+\epsilon\right)^{(n)} \in P_{fin}(M)$ as to guarantee $D(f_n s_n) < 2^{-n}\varphi(f_n), 1-2^{-n-1} < \int \varphi(f_n)d\mu$ and

$$f_n b_n(1-s_n) \in M \tag{76}$$

for all $n \in \mathbb{N}$.
Setting

$$f := inf_{n\geq 1} f_n, s := sup_{n\geq 1} s_n,$$

we have that

$$1/2 < \int \varphi(f)d\mu, \qquad D(fs) \overset{(D6)}{\leq} \sum_{n=1}^{\infty} D(fs_n) \leq \varphi(f).$$

This means that $f \neq 0$ and $fs \in P_{fin}(M)$ (see (D1)). In addition, since $f \leq f_n, (1-s) \leq (1-s_n)$ from (76) it follows that $fb_n(1-s) \in M$ for all $n \in \mathbb{N}$.
Consider the projections $t = f(1-s)$ and $g_n = f(e_n \wedge (1-s)), n \in \mathbb{N}$.
Clearly (see (72)),

$$g_n \leq fe_n \leq q_0, b_n g_n \in M, g_n \leq t \tag{77}$$

for all $n \in \mathbb{N}$, and also

$$fe_n - g_n = f(e_n - e_n \wedge (1-s)) \sim f(e_n \vee (1-s) - (1-s)) \leq fs,$$

that is $fe_n - g_n \in P_{fin}(M)$. Hence, for every non-zero central projection $x+2\epsilon \leq f$, we have that the projection $(x+2\epsilon)e_n - (x+2\epsilon)g_n$ is finite. Since the projection $(x+2\epsilon)e_n$ is infinite, the projection $(x+2\epsilon)g_n$ is also infinite, i.e.

$$(x+2\epsilon)g_n \notin P_{fin}(M) \tag{78}$$

for any $0 \neq (x+2\epsilon) \in P(Z(M))$ and $n \in \mathbb{N}$.

Since $b_n t = fb_n(1-s) \in M$, we see that there exists such an increasing sequence $\{l_n\} \subset \mathbb{N}$ that $l_n > n + 2\|b_n t\|_M$ for all $n \in \mathbb{N}$.

Appealing to the inequalities (70), (77) and taking into account the equality $b_n = \delta(e_n)e_n$, we deduce

$$\left\|g_n(l_n+1)(x_{l_n}+\epsilon)\delta(e_n)e_n g_n\right\|_M \leq \left\|g_n(l_n+1)(x_{l_n}+\epsilon)\right\|_M \|\delta(e_n)e_n g_n\|_M$$
$$\leq \left\|q_0(l_n+1)(x_{l_n}+\epsilon)\right\|_M \|\delta(e_n)e_n t\|_M < (l_n-n)/2.$$

Hence,

$$\left\|g_n e_n \delta(e_n)(l_n+1)(x_{l_n}+\epsilon)g_n + g_n(l_n+1)(x_{l_n}+\epsilon)\delta(e_n)e_n g_n\right\|_M \leq l_n - n. \tag{79}$$

For every $x = x^* \in M$ the inequalities $-\|x\|_M 1 \leq x \leq \|x\|_M 1$ holds, in particular, $-g_n\|x\|_M \leq q_n x q_n \leq g_n\|x\|_M$. Hence, inequality (79) implies that

$$g_n e_n \delta(e_n)(l_n+1)(x_{l_n}+\epsilon)g_n + g_n(l_n+1)(x_{l_n}+\epsilon)\delta(e_n)e_n g_n \geq (n-l_n)g_n. \tag{80}$$

Since $e_n e_m = 0$ whenever $n \neq m$, we see (due to inequalities (70) and (77)) that the series $\sum_{n=1}^{\infty} e_n(l_n+1)(x_{l_n}+\epsilon)e_n$ converges with respect to the topology τ_{so} to a self-adjoint operator $h_0 \in M$, satisfying

$$\|h_0\|_M \leq sup_{n\geq 1}\left\|e_n(l_n+1)(x_{l_n}+\epsilon)e_n\right\|_M \leq 1.$$

Again appealing to the inequalities (71), (77) and (80), we infer that

$$\begin{aligned} ng_n &= l_n g_n + (n-l_n)g_n \\ &\le g_n(l_n+1)\delta(x_{l_n}+\epsilon)g_n + g_n e_n\delta(e_n)(l_n+1)(x_{l_n}+\epsilon)g_n + g_n(l_n \\ &\quad +1)(x_{l_n}+\epsilon)\delta(e_n)e_n g_n \\ &= (l_n+1)(g_n\delta(x_{l_n}+\epsilon)g_n + g_n e_n\delta(e_n)(x_{l_n}+\epsilon)g_n + g_n(x_{l_n} \\ &\quad +\epsilon)\delta(e_n)e_n g_n) = (l_n+1)g_n\delta(e_n(x_{l_n}+\epsilon)e_n)g_n \\ &= \delta(g_n e_n(l_n+1)(x_{l_n}+\epsilon)e_n)g_n - \delta(g_n)e_n(l_n+1)(x_{l_n}+\epsilon)e_n g_n \\ &= \delta(g_n h_0)g_n - \delta(g_n)h_0 g_n = g_n\delta(h_0)g_n. \end{aligned}$$

Thus,

$$ng_n \le g_n\delta(h_0)g_0 \tag{81}$$

for every $n \in \mathbb{N}$.

Set $g_n^{(0)} = g_n \wedge E_{n-1}(\delta(h_0)), n \in \mathbb{N}$, where $\{E_\lambda(\delta(h_0))\}$ is the spectral family of projections for self-adjoint operator $\delta(h_0)$. For every $n \in \mathbb{N}$ we have

$$\begin{aligned} ng_n^{(0)} &= ng_n^{(0)}g_n g_n^{(0)} \overset{(81)}{\le} g_n^{(0)}(g_n\delta(h_0)g_n)g_n^{(0)} = g_n^{(0)}\delta(h_0)g_n^{(0)} \\ &= g_n^{(0)}E_{n-1}(\delta(h_0))\delta(h_0)g_n^{(0)} \le g_n^{(0)}(n-1)E_{n-1}(\delta(h_0))g_n^{(0)} \\ &= (n-1)g_n^{(0)}. \end{aligned}$$

Hence, $g_n \wedge E_{n-1}(\delta(h_0)) = g_n^{(0)} = 0$ which implies

$g_n = g_n - g_n \wedge E_{n-1}(\delta(h_0)) \sim g_n \vee E_{n-1}(\delta(h_0)) - E_{n-1}(\delta(h_0)) \le 1 - E_{n-1}(\delta(h_0))$, i.e. $g_n \preccurlyeq 1 - E_{n-1}(\delta(h_0))$.

Then $g_n \overset{(77)}{\le} fg_n \preccurlyeq f(1 - E_{n-1}(\delta(h_0)))$, and therefore

$$D(g_n) \overset{(D3)}{\le} D(f(1 - E_{n-1}(\delta(h_0)))) \tag{82}$$

for all $n \in \mathbb{N}$.

Since $|f\delta(h_0)| \in LS(M)$, we see that there exists such a non-zero central projection $f_0 \le f$, that $|f_0\delta(h_0)| \in S_h(M)$. Hence, we may find such $\lambda_0 > 0$, that $(f_0 - E_\lambda(|f_0\delta(h_0)|)) \in P_{fin}(M)$ for all $\lambda \ge \lambda_0$ ([113]), that is $D(f_0(1 - E_\lambda(|f_0\delta(h_0)|))) \in L_+^0(\Omega, \Sigma, \mu)$ when $\lambda > \lambda_0$.

Since $f_0(1 - E_\lambda(|f_0\delta(h_0)|)) = f_0(1 - E_\lambda(|\delta(h_0)|))$, we infer from (82) that

$$D(f_0 g_n) \in L_+^0(\Omega, \Sigma, \mu)$$

for all $n \ge \lambda_0 + 1$ which contradicts with the property (D1) in the definition of the dimension function D, since $f_0 g_n$ is an infinite projection (see (78)).

Hence, our assumption that the derivation δ fails to be continuous in $(LS(M), t(M))$ has led to a contradiction.

Corollary (3.2.24)[301]: (See [104]) Let $\{x_n\}_{n=1}^\infty \subset LS(M)$ (respectively, $\{x_n + \epsilon\}_{n=1}^\infty \subset LS(M)$) be a sequence consistent with the sequence $\{x_n + 2\epsilon\}_{n=1}^\infty \subset P(Z(M))$ (respectively, with the sequence $\{(x_n + 2\epsilon)'\}_{n=1}^\infty \subset P(Z(M))), (x_n + 2\epsilon) \uparrow 1((x_n + 2\epsilon)' \uparrow 1)$.

Then

(i). There exists a unique $x \in LS(M)$, such that $x(x_n + 2\epsilon) = x_n(x_n + 2\epsilon)$ for all $n \in \mathbb{N}$, in addition, $x_n \xrightarrow{t(M)} x$;

(ii). If $x_n(x_n + 2\epsilon)(x_m + 2\epsilon)' = (x_m + \epsilon)(x_n + 2\epsilon)(x_m + 2\epsilon)'$ for all $n, m \in \mathbb{N}$, then $(x_n(x_n + 2\epsilon) - (x_n + \epsilon)(x_n + 2\epsilon)') \xrightarrow{t(M)} 0$ for $n \to \infty$.

Proof. (i). Consider a neighborhood $V(B,\varepsilon,\delta)$ of zero in topology $t(M)$, where $\varepsilon,\delta > 0, B \in \Sigma, \mu(B) < \infty$ (see the definition of topology $t(M)$). Since $(x_n + 2\epsilon)^{\perp} = (1-(x_n + 2\epsilon)) \downarrow 0$, it follows that $\varphi((x_n + 2\epsilon)^{\perp}) \in W(B,\varepsilon,\delta)$ for $n \geq n(B,\varepsilon,\delta)$. Taking $x \in LS(M), q_n = (x_n + 2\epsilon)$, we have $(x(x_n + 2\epsilon)^{\perp})q_n = 0, D((x_n + 2\epsilon)^{\perp} q_n) = 0$, i.e. $x(x_n + 2\epsilon)^{\perp} \in V(B,\varepsilon,\delta)$ for all $x \in LS(M), n \geq n(B,\varepsilon,\delta)$. For $m > n$, we have

$$x_m(x_m + 2\epsilon) - x_n(x_n + 2\epsilon) = x_m(x_m + 2\epsilon) - x_m(x_n + 2\epsilon) = x_m(x_m - x_n)$$
$$= x_m(x_m + 2\epsilon)(x_n + 2\epsilon)^{\perp} \in V(B,\varepsilon,\delta)$$

for all $n \geq n(B,\varepsilon,\delta)$. It means that $\{x_n(x_n + 2\epsilon)\}_{n=1}^{\infty}$ is a Cauchy sequence in $(LS(M), t(M))$. Consequently, there exists $x \in LS(M)$ such that $x_n(x_n + 2\epsilon) \xrightarrow{t(M)} x$. Since $x_n(x_n + 2\epsilon)^{\perp} \in V(B,\varepsilon,\delta)$ for all $n \geq n(B,\varepsilon,\delta)$, it follows that $x_n(x_n + 2\epsilon)^{\perp} \xrightarrow{t(M)} 0$, and therefore $x_n = x_n(x_n + 2\epsilon) + x_n(x_n + 2\epsilon)^{\perp} \xrightarrow{t(M)} x$. Fixing $k \in \mathbb{N}$, for $n > k$ we have $x_k(x_k + 2\epsilon) = x_n(x_k + 2\epsilon) \xrightarrow{t(M)} x(x_k + 2\epsilon)$ for $n \to \infty$, i.e. $x(x_k + 2\epsilon) = x_k(x_k + 2\epsilon)$ for all $k \in \mathbb{N}$.

If $a \in LS(M)$ and $a(x_n + 2\epsilon) = x_n(x_n + 2\epsilon) = x(x_n + 2\epsilon)$ for all $n \in \mathbb{N}$, then $0 = (a - x)(x_n + 2\epsilon) \xrightarrow{t(M)} (a - x)$, i.e. $a = x$.

(ii). If $x_m(x_m + 2\epsilon)(x_n + 2\epsilon)^{'\perp} \xrightarrow{t(M)} 0$ for $n \to \infty, (x_n + \epsilon)(x_n + 2\epsilon)^{'}(x_m + 2\epsilon)^{\perp} \xrightarrow{t(M)} 0$ for $m \to \infty$, and $x_n(x_n + 2\epsilon) - x_m(x_m + 2\epsilon) \xrightarrow{t(M)} 0$ for $n, m \to \infty$, then

$$\begin{aligned} x_n(x_n + 2\epsilon) &- (x_n + \epsilon)(x_n + 2\epsilon)^{'} \\ &= x_n(x_n + 2\epsilon) - x_m(x_m + 2\epsilon) + x_m(x_m + 2\epsilon)(x_n + 2\epsilon)^{'} + x_m(x_m \\ &\quad + 2\epsilon)(x_n + 2\epsilon)^{'\perp} - (x_n + \epsilon)(x_n + 2\epsilon)^{'} \\ &= (x_n(x_n + 2\epsilon) - x_m(x_m + 2\epsilon)) + (x_n + \epsilon)(x_m + 2\epsilon)(x_n + 2\epsilon)^{'} + x_m(x_m \\ &\quad + 2\epsilon)(x_n + 2\epsilon)^{'\perp} - (x_n + \epsilon)(x_n + 2\epsilon)^{'} \\ &= (x_n(x_n + 2\epsilon) - x_m(x_m + 2\epsilon)) - (x_n + \epsilon)(x_n + 2\epsilon)^{'}(x_m + 2\epsilon)^{\perp} \\ &\quad + x_m(x_m + 2\epsilon)(x_n + 2\epsilon)^{'\perp} \xrightarrow{t(M)} 0 \end{aligned}$$

for $n, m \to \infty$.

Corollary (3.2.25)[301]: (See [104]) The mapping $\tilde{\delta}$ is a unique derivation from $LS(M)$ into $LS(M)$ such that $\tilde{\delta}(x) = \delta(x)$ for all $x \in S(M)$.

Proof. Let $x, x + \epsilon \in LS(M)$, and let $(x_n + 2\epsilon), (q_n + \epsilon) \in P(Z(M))$ be such that $(x_n + 2\epsilon) \uparrow 1, (q_n + \epsilon) \uparrow 1, x(x_n + 2\epsilon), (x + \epsilon)(q_n + \epsilon) \in S(M), n \in \mathbb{N}$. Observing that

$$\begin{aligned} &(x_n + 2\epsilon)(q_n + \epsilon) \in P(Z(M)), ((x_n + 2\epsilon)(q_n + \epsilon)) \\ &\quad \uparrow 1, x(x_n + 2\epsilon)(q_n + \epsilon), (x + \epsilon)(x_n + 2\epsilon)(q_n + \epsilon), (2x + \epsilon)(x_n + 2\epsilon)(q_n \\ &\quad + \epsilon) \in S(M), n \in \mathbb{N}, \end{aligned}$$

we have

$$\begin{aligned} \tilde{\delta}(2x + \epsilon) &= t(M) - \lim_{n\to\infty} \delta((2x + \epsilon)(x_n + 2\epsilon)(q_n + \epsilon)) \\ &= \left(t(M) - \lim_{n\to\infty} \delta(x(x_n + 2\epsilon)(q_n + \epsilon))\right) \\ &\quad + \left(t(M) - \lim_{n\to\infty} \delta((x + \epsilon)(x_n + 2\epsilon)(q_n + \epsilon))\right) = \tilde{\delta}(x) + \tilde{\delta}(x + \epsilon). \end{aligned}$$

Similarly, $\tilde{\delta}(\lambda x) = \lambda \tilde{\delta}(x), \lambda \in \mathbb{C}$. Further, using convergences

$$x(x_n + 2\epsilon) \xrightarrow{t(M)} x, (x + \epsilon)(q_n + \epsilon) \xrightarrow{t(M)} x + \epsilon, \qquad \delta(x(x_n + 2\epsilon)) \xrightarrow{t(M)} \tilde{\delta}(x),$$
$$\delta((x + \epsilon)(q_n + \epsilon)) \xrightarrow{t(M)} \tilde{\delta}(x + \epsilon)$$

and the inclusion $x(x + \epsilon)(x_n + 2\epsilon)(q_n + \epsilon) \in S(M), n \in \mathbb{N}$, we have

$$\tilde{\delta}(x(x+\epsilon)) = t(M) - \lim_{n\to\infty} \delta(x(x+\epsilon)(x_n+2\epsilon)(q_n+\epsilon))$$
$$= t(M) - \lim_{n\to\infty} \delta\Big((x(x_n+2\epsilon))\big((x+\epsilon)(q_n+\epsilon)\big)\Big)$$
$$= t(M)$$
$$- \lim_{n\to\infty}\Big(\delta(x(x_n+2\epsilon))(x+\epsilon)(q_n+\epsilon) + x(x_n+2\epsilon)\delta\big((x_n+\epsilon)(q_n+\epsilon)\big)\Big)$$
$$= \tilde{\delta}(x)(x+\epsilon) + x\tilde{\delta}(x+\epsilon).$$

Consequently, $\tilde{\delta}: LS(M) \to LS(M)$ is a derivation, in addition, $\tilde{\delta}(x) = \delta(x)$ for all $x \in S(M)$.

Assume that $\delta_1: LS(M) \to LS(M)$ is also a derivation for which $\delta_1(x) = \delta(x)$ for all $x \in S(M)$. Let us show that $\tilde{\delta} = \delta_1$.

If $x \in LS(M), (x_n + 2\epsilon) \uparrow 1, x(x_n + 2\epsilon) \in S(M), n \in \mathbb{N}$, then, by Corollary (3.2.22) and Corollary (3.2.24) (i), we obtain

$$\tilde{\delta}(x) = t(M) - \lim_{n\to\infty} \delta(x(x_n+2\epsilon)) = t(M) - \lim_{n\to\infty} \delta_1(x(x_n+2\epsilon))$$
$$= t(M) - \lim_{n\to\infty} \delta_1(x)\,(x_n+2\epsilon) = \delta_1(x).$$

Corollary (3.2.26)[301]: If D is a dimension function of a von Neumann algebra M, then for any derivation δ from M into $LS(M)$ the following inequality

$$D(s(\delta(x))) \le 3D(s(x))$$

holds for all $x \in M$.

Proof. For $x \in M$ we have

$$l(\delta(x)s(x)) \sim r(\delta(x)s(x)) \le s(x),$$
$$r(x\delta(s(x))) \sim l(x\delta(s(x))) = l(s(x)x\delta(s(x))) \le s(x),$$

i.e.

$$l(\delta(x)s(x)) \preccurlyeq s(x)$$

and

$$r(x\delta(s(x))) \preccurlyeq s(x),$$

that implies the inequalities (see (D2), (D3))

$$D(l(\delta(x)s(x))) \le D(s(x)), D(r(x\delta(s(x)))) \le D(s(x)).$$

Since

$$\delta(x) = \delta(xs(x)) = \delta(x)s(x) + x\delta(s(x)),$$

we have

$$s(\delta(x)) = s(\delta(x)s(x) + x\delta(s(x))) \le s(x) \vee l(\delta(x)s(x)) \vee r(x\delta(s(x))).$$

Due to (D6), we have

$$D(s(\delta(x))) \le D(s(x)) + D(l(\delta(x)s(x))) + D(r(x\delta(s(x)))) \le 3D(s(x)).$$

Corollary (3.2.27)[301]: (See [104]) If $\{x_n\}_{n=1}^{\infty} \subset LS(M), s(x_n) \in P_{fin}(M), D(s(x_n)) \xrightarrow{t(L^\infty(\Omega))} 0$, then $x_n \xrightarrow{t(M)} 0$.

Proof. Taking $x_n + 2\epsilon = 1$ for all $n \in \mathbb{N}$, we have

$$(x_n+2\epsilon)s(x_n) \in P_{fin}(M), \varphi(x_n+2\epsilon)^\perp = 0, n \in \mathbb{N},$$

and

$$D((x_n+2\epsilon)s(x_n)) = D(s(x_n)) \xrightarrow{t(L^\infty(\Omega))} 0.$$

Consequently, $s(x_n) \xrightarrow{t(M)} 0$ (see Proposition (3.2.2)(i)).

Since $E_\lambda^\perp(|x_n|) \le s(x_n)$ for all $\lambda > 0, n \in \mathbb{N}$, it follows $E_\lambda^\perp(|x_\alpha|) \xrightarrow{t(M)} 0$, and therefore $x_n \xrightarrow{t(M)} 0$.

Corollary (3.2.28)[301]: (See [104]) Let $x \in S(M), (q_n+\epsilon), q_n \in P(M), (q_n+\epsilon) \uparrow 1, q_n \uparrow 1, x(q_n+\epsilon), xq_n \in M, (q_n+\epsilon)^{\perp}, q_n^{\perp} \in P_{fin}(M), n \in \mathbb{N}$. If $\delta: M \to LS(M)$ is a derivation, then there exists $\hat{\delta}(x) \in LS(M)$, such that

$$t(M) - \lim_{n\to\infty} \delta(x(q_n+\epsilon)) = \hat{\delta}(x) = t(M) - \lim_{n\to\infty} \delta(xq_n).$$

Proof. For $n < m$ we have

$$l\big(x(q_m-q_n)\big) \sim r\big(x(q_m-q_n)\big) \le q_m - q_n,$$

and therefore, applying Corollary (3.2.26) and properties (D2), (D3), we obtain

$$D(s(\delta(xq_m - xq_n))) = D(s(\delta(x(q_m-q_n)))) \le 3D(s(x(q_m-q_n)))$$
$$\le 3D(l(x(q_m-q_n)) \vee (q_m-q_n)) \le 6D(q_m-q_n) \le 6D((q_n+\epsilon)^{\perp}).$$

Since $D((q_n+\epsilon)^{\perp}) \in L_+^0(\Omega,\Sigma,\mu)$ (see (D1)) and $D((q_n+\epsilon)^{\perp}) \downarrow 0$ (see (D7)) it follows that $D((q_n+\epsilon)^{\perp}) \xrightarrow{t(L^\infty(\Omega))} 0$ (see (D7)). Hence,

$$D(s(\delta(x(q_m+\epsilon)) - \delta(x(q_n+\epsilon)))) \xrightarrow{t(L^\infty(\Omega))} 0$$

for $n, m \to \infty$. By Corollary (3.2.27), we have that $(\delta(x(q_m+\epsilon)) - \delta(x(q_n+\epsilon))) \xrightarrow{t(M)} 0$ for $n, m \to \infty$, i.e. $\{\delta(x(q_n+\epsilon))\}_{n=1}^{\infty}$ is a Cauchy sequence in $(LS(M), t(M))$.

Consequently, there exists $\hat{\delta}(x) \in LS(M)$, such that

$$t(M) - \lim_{n\to\infty} \delta(x(q_n+\epsilon)) = \hat{\delta}(x).$$

Let us show that $t(M) - \lim_{n\to\infty} \delta(xq_n) = \hat{\delta}(x)$.

For each $n \in \mathbb{N}$ we have

$$\epsilon(((q_n+\epsilon) - ((q_n+\epsilon)) \wedge q_n) \vee (q_n - (q_n+\epsilon) \wedge q_n))$$
$$= (((q_n+\epsilon) - (q_n+\epsilon) \wedge q_n) - (q_n - (q_n+\epsilon) \wedge q_n))(((q_n+\epsilon) - (q_n+\epsilon) \wedge q_n) \vee (q_n - (q_n+\epsilon) \wedge q_n))$$
$$= ((q_n+\epsilon) - (q_n+\epsilon) \wedge q_n) - (q_n - (q_n+\epsilon) \wedge q_n) = \epsilon.$$

Hence,

$$r\epsilon \le (((q_n+\epsilon) - (q_n+\epsilon) \wedge q_n) \vee (q_n - (q_n+\epsilon) \wedge q_n)).$$

Since

$$r(x((q_n+\epsilon) - q_n)) \le r\epsilon$$

and

$$l(x(\epsilon)) \sim r(x(\epsilon)),$$

it follows

$$D(s(x(\epsilon))) = D(l(x(\epsilon)) \vee r(x(\epsilon))) \overset{(D6)}{\le} D(l(x(\epsilon)) + D(r(x(\epsilon))) = 2D(r(x(\epsilon)))$$
$$\overset{(D6)}{\le} 2D((q_n+\epsilon) - (q_n+\epsilon) \wedge q_n) + 2D(q_n - (q_n+\epsilon) \wedge q_n)$$
$$\le 4D(1 - (q_n+\epsilon) \wedge q_n)$$
$$= 4D((q_n+\epsilon)^{\perp} \vee q_n^{\perp}) \le 4(D((q_n+\epsilon)^{\perp}) + D(q_n^{\perp})).$$

Since (see Corollary (3.2.26))

$$D(s(\delta(x(q_n+\epsilon)) - \delta(xq_n))) = D(s(\delta(x(\epsilon)))) \le 3D(s(x(\epsilon))),$$

we have

$$D(s(\delta(x(q_n+\epsilon)) - \delta(xq_n))) \le 12(D((q_n+\epsilon)^{\perp}) + D(q_n^{\perp})) \downarrow 0.$$

By Corollary (3.2.27), we obtain

$$t(M) - \lim_{n\to\infty} \delta(xq_n) = t(M) - \lim_{n\to\infty} \delta(x(q_n+\epsilon)) = \hat{\delta}(x).$$

Corollary (3.2.29)[301]: (See [104]) The mapping $\hat{\delta}$ is a unique derivation from $S(M)$ into $LS(M)$, such that $\hat{\delta}(x) = \delta(x)$ for all $x \in M$.

Proof. For $x, x+\epsilon \in S(M)$ select $(q_n+\epsilon), q_n \in P(M), n \in \mathbb{N}$, such that
$$(q_n+\epsilon) \uparrow 1, q_n \uparrow 1, (q_n+\epsilon)^{\perp}, q_n^{\perp} \in P_{fin}(M), x(q_n+\epsilon), (x+\epsilon)q_n \in M$$
for all $n \in \mathbb{N}$. The sequence of projections $e_n = (q_n+\epsilon) \wedge q_n$ is increasing, and, in addition,
$$xe_n = x(q_n+\epsilon)e_n \in M, \qquad (x+\epsilon)e_n = (x+\epsilon)q_ne_n \in M,$$
$$e_n^{\perp} = (q_n+\epsilon)^{\perp} \vee q_n^{\perp} \in P_{fin}(M), D(e_n^{\perp}) \le D((q_n+\epsilon)^{\perp}) + D(q_n^{\perp}) \downarrow 0.$$
The last estimate implies the convergence $e_n^{\perp} \downarrow 0$ (see (D7)), or $e_n \uparrow 1$.
By Corollary (3.2.28), we have
$$\begin{aligned}\hat{\delta}(2x+\epsilon) &= t(M) - \lim_{n\to\infty} \delta((2x+\epsilon)e_n)\\ &= \left(t(M) - \lim_{n\to\infty}\delta(xe_n)\right) + \left(t(M) - \lim_{n\to\infty}\delta((x+\epsilon)e_n)\right)\\ &= \hat{\delta}(x) + \hat{\delta}(x+\epsilon).\end{aligned}$$
Similarly, $\hat{\delta}(\lambda x) = \lambda\hat{\delta}(x)$ for all $\lambda \in \mathbb{C}$.
Let us show that $\hat{\delta}(x(x+\epsilon)) = \hat{\delta}(x)(x+\epsilon) + x\hat{\delta}(x+\epsilon), x, x+\epsilon \in S(M)$.
Due to polar decomposition $x+\epsilon = u|x+\epsilon|, u^*u = r(x+\epsilon)$, we have $x_n+\epsilon = (x+\epsilon)E_n(|x+\epsilon|) \in M$ for all $n \in \mathbb{N}$. Set
$$g_n = 1 - r(E_n^{\perp}(|x|)(x_n+\epsilon)), s_n = g_n \wedge E_n(|x+\epsilon|).$$
Since
$$g_n^{\perp} = r(E_n^{\perp}(|x|)(x_n+\epsilon)) \sim l(E_n^{\perp}(|x|)(x_n+\epsilon)) \le E_n^{\perp}(|x|),$$
we obtain
$$g_n^{\perp} \preccurlyeq E_n^{\perp}(|x|).$$
Since $x \in S(M)$, there exists $n_0 \in \mathbb{N}$ such that $E_n^{\perp}(|x|) \in P_{fin}(M)$ for all $n \ge n_0$, and therefore $g_n^{\perp} \in P_{fin}(M)$ for all $n \ge n_0$. The equality
$$(x_n+\epsilon)g_n = E_n(|x|)(x_n+\epsilon)g_n + E_n^{\perp}(|x|)(x_n+\epsilon)g_n = E_n(|x|)(x_n+\epsilon)g_n$$
implies that
$$\begin{aligned}E_{n+1}^{\perp}(|x|)(x_{n+1}+\epsilon)s_n &= E_{n+1}^{\perp}(|x|)E_n^{\perp}(|x|(x_{n+1}+\epsilon)E_n(|x+\epsilon|))s_n\\ &= E_{n+1}^{\perp} + 1(|x|)(E_n^{\perp}(|x|)(x_n+\epsilon)E_n(|x+\epsilon|))s_n\\ &= E_{n+1}^{\perp}(|x|)(E_n^{\perp}(|x|)(x_n+\epsilon)s_n) = E_{n+1}^{\perp}(|x|)(E_n^{\perp}(|x|)(x_n+\epsilon)g_n)s_n = 0,\end{aligned}$$
in particular,
$$s_n \le 1 - r(E_{n+1}^{\perp}(|x|)(x_{n+1}+\epsilon)) = g_{n+1}$$
for all $n \in \mathbb{N}$. From here and from the inequalities $s_n \le E_n(|x+\epsilon|) \le E_{n+1}(|x+\epsilon|)$ it follows that $s_n \le s_{n+1}$.

Since $x+\epsilon \in S(M)$, we have $E_n^{\perp}(|x+\epsilon|) \in P_{fin}(M)$ for $n \ge n_1$ for some $n_1 \ge n_0$. Hence,
$$s_n^{\perp} = g_n^{\perp} \vee E_n^{\perp}(|x+\epsilon|) \in P_{fin}(M)$$
for $n \ge n_1$ and
$$D(s_n^{\perp}) \le D(g_n^{\perp}) + D(E_n^{\perp}(|x+\epsilon|)) \le (D(E_n^{\perp}(|x|)) + D(E_n^{\perp}(|x+\epsilon|))) \downarrow 0,$$
i.e. $s_n^{\perp} \downarrow 0$ or $s_n \uparrow 1$.

Using Corollary (3.2.14), Corollary (3.2.28), the inclusions $xE_n(|x|) \in M, (x+\epsilon)E_n(|x+\epsilon|) \in M$ and equalities
$$\begin{aligned}x(x+\epsilon)s_n &= x(x+\epsilon)E_n(|x_n+\epsilon|)s_n = x(x_n+\epsilon)s_n = x(x_n+\epsilon)g_ns_n\\ &= xE_n(|x|)(x_n+\epsilon)q_ns_n = xE_n(|x|)(x+\epsilon)E_n(|x+\epsilon|)s_n,\end{aligned}$$
we obtain

$$\hat{\delta}(x(x+\epsilon)) = t(M) - \lim_{n\to\infty} \delta(x(x+\epsilon)s_n)$$
$$= t(M) - \lim_{n\to\infty} \delta\big(xE_n(|x|)(x+\epsilon)E_n(|x+\epsilon|s_n)\big)$$
$$= t(M) - \lim_{n\to\infty} \Big(\delta\big(xE_n(|x|)\big)(x+\epsilon)s_n + xE_n(|x|)\delta\big((x+\epsilon)s_n\big)\Big)$$
$$= \Big(t(M) - \lim_{n\to\infty} \delta\big(xE_n(|x|)\big)\Big) \cdot \Big(t(M) - \lim_{n\to\infty} (x+\epsilon)s_n\Big)$$
$$+ \Big(t(M) - \lim_{n\to\infty} xE_n(|x|)\Big) \cdot \Big(t(M) - \lim_{n\to\infty} \delta\big((x+\epsilon)s_n\big)\Big)$$
$$= \hat{\delta}(x)(x+\epsilon) + x\hat{\delta}(x+\epsilon).$$

Consequently, $\hat{\delta}: S(M) \to LS(M)$ is a derivation, such that $\hat{\delta}(x) = \delta(x)$ for all $x \in M$. Let $\delta_1: S(M) \to LS(M)$ also be a derivation, for which $\delta_1(x) = \delta(x)$ for all $x \in M$. If $x \in S(M)$, then $E_n(|x|) \uparrow 1, xE_n(|x|) \in M, n \in \mathbb{N}, E_n^{\perp}(|x|) \in P_{fin}(M)$ for all $n \geq n_3$ for some $n_3 \in \mathbb{N}$.

Hence, $E_n(|x|) \xrightarrow{t(M)} 1$ (see Corollary (3.2.14)). Since $(LS(M), t(M))$ is a topological algebra, it follows that

$$\delta_1(x) = t(M) - \lim_{n\to\infty} \delta_1(x)E_n(|x|)$$
$$= \Big(t(M) - \lim_{n\to\infty} \delta_1(xE_n(|x|))\Big) - \Big(t(M) - \lim_{n\to\infty} x\delta_1(E_n(|x_n|))\Big)$$
$$= \Big(t(M) - \lim_{n\to\infty} \delta(xE_n(|x|))\Big) - \Big(t(M) - \lim_{n\to\infty} x\delta(E_n(|x_n|))\Big)$$
$$= \hat{\delta}(x) - x(t(M) - \lim_{n\to\infty} \delta(E_n(|x_n|))).$$

Since $\delta(1) = 0, s(x) = s(-x)$ for $x \in LS(M)$, it follows via Corollary (3.2.26), that $D(s(\delta(E_n(|x|)))) = D(s(\delta(-E_n(|x|)))) = D(s(\delta(1 - E_n(|x|)))) \leq 3D(E_n^{\perp}(|x|)) \downarrow 0.$

By Corollary (3.2.26), we obtain $\delta(E_n(|x|)) \xrightarrow{t(M)} 0$, that implies the equality $\delta_1(x) = \hat{\delta}(x)$.

Corollary (3.2.30)[301]: Let A be a subalgebra of $LS(M), M \subset A$ and let $\delta: A \to LS(M)$ be a derivation. Then there exists a unique derivation $\delta_A: LS(M) \to LS(M)$ such that $\delta_A(x) = \delta(x)$ for all $x \in A$.

Proof. Since $M \subset A$, the restriction δ_0 of the derivation δ on M is a well-defined derivation from M into $LS(M)$. Hence, by Propositions (3.2.25) and (3.2.29), the mapping $\delta_A = \hat{\tilde{\delta}}$ is a unique derivation from $LS(M)$ into $LS(M)$ such that $\delta_A(x) = \delta_0(x)$ for all $x \in M$. Let us show that $\delta_A(a) = \delta(a)$ for every $a \in A$. If $a \in A$, then there exists a sequence $\{x_n + 2\epsilon\}_{n=1}^{\infty} \subset P(Z(M))$, such that $(x_n + 2\epsilon) \uparrow 1$ and $a(x_n + 2\epsilon) \in S(M), n \in \mathbb{N}$. Since $x_n + 2\epsilon \xrightarrow{t(M)} 1$ (see Corollary (3.2.24)(i)), we have, by Corollary (3.2.22),

$$\delta_A(a) = t(M) - \lim_{n\to\infty} \delta_A(a)\,(x_n + 2\epsilon) = t(M) - \lim_{n\to\infty} \delta_A(a(x_n + 2\epsilon)),$$

and, similarly, $\delta(a) = t(M) - \lim_{n\to\infty} \delta(a(x_n + 2\epsilon))$.

Using the equality $\delta_A(x) = \delta_0(x) = \delta(x)$ for each $x \in M$, and following the proof of uniqueness of the derivation $\hat{\delta}$ from Corollary (3.2.29), we obtain $\delta_A(a(x_n + 2\epsilon)) = \delta(a(x_n + 2\epsilon))$ for all $n \in \mathbb{N}$, that implies the equality $\delta_A(a) = \delta(a)$.

Chapter 4
Twisting and Schatten Classes

We show that the results can be summarized as follows: $\mathrm{Ext}_B(S^p, S^q) = \begin{cases} 0 & \text{if } 0 < q < p \leq \infty \text{ or } p = q = \infty, \\ \mathrm{Ext}_C(S^1, C) & \text{if } q = p \text{ is finite,} \\ \mathrm{Ext}_C(H) & \text{if } 0 < p < q \leq \infty \end{cases}$. In the first case, every extension $0 \to S^q \to X \to S^p \to 0$ splits and so $X = S^q \oplus S^p$. In the second case, every self-extension of S p arises (and gives rise) to a minimal extension of S^1 in the quasi-Banach category, that is, a short exact sequence $0 \to C \to M \to S^1 \to 0$. In the third case, each extension corresponds to a "twisted Hilbert space", that is, a short exact sequence $0 \to H \to T \to H \to 0$. We show that Kalton twisting of Schatten classes is strictly singular as $B(H)$-modules. We also identify the dual construction. For semifinite algebras, considering $L^p = L^p(M, \tau)$ as an interpolation space between M and its predual M_* one arrives at a certain self-extension of L^p that is a kind of noncommutative Kalton-Peck space and carries a natural bimodule structure. Some interesting properties of these spaces are presented. For general algebras, including those of type III, the interpolation mechanism produces two (rather than one) extensions of one sided modules, one of left-modules and the other of right-modules. Whether or not one may find (nontrivial) self-extensions of bimodules in all cases is left open.

Section (4.1): Certain Homological Properties

For Z and Y be quasi-Banach modules over a fixed Banach algebra A. An extension (of Z by Y) is a short exact sequences of (quasi-) Banach modules and homomorphisms

$$0 \longrightarrow Y \longrightarrow X \longrightarrow Z \longrightarrow 0.$$

Less technically we may think of X as a module containing Y as a closed submodule in such a way that X/Y is (isomorphic to) Z. The extension is said to be trivial (or to split) if Y is complemented in X through a homomorphism.

When properly classified and organized the extensions of Z by Y constitute a linear space denoted by $Ext_A(Z, Y)$.

While the homomorphisms between a given couple of modules display the most basic links between them, extensions reect much more subtle connections, often in a encrypted or disguised form.

We study extensions between Schatten classes when these are regarded as modules over $B = B(\mathcal{H})$, the algebra of all (linear, bounded) operators on the underlying Hilbert space $\mathcal{H}$.

Thus we are concerned with short exact sequences of (say left) B-modules

$$0 \longrightarrow S^q \longrightarrow X \longrightarrow S^p \longrightarrow 0. \quad (1)$$

Our main results can be summarized as follows, according to the relative position between p and q:

$$Ext_B(S^p, S^q) = \begin{cases} 0 & if\ 0 < q < p \leq \infty\ or\ p = q = \infty, \\ Ext_{\mathbb{C}}(S^1; \mathbb{C}) & if\ q = p\ is\ finite, \\ Ext_{\mathbb{C}}(\mathcal{H}) & if\ 0 < p < q \leq \infty. \end{cases}$$

(S^∞ is the ideal of compact operators, with the operator norm.) In the first case every extension is trivial and we have $X = S^p \oplus S^q$. In the second case we see that $Ext_B(S^p)$ does not depend on $p \in (0, \infty)$ and, in fact, each self-extension of S^p corresponds to a minimal extension of S^1, that is, an exact sequence of quasi-Banach spaces and operators

$$0 \longrightarrow \mathbb{C} \longrightarrow M \longrightarrow S^1 \longrightarrow 0.$$

Notice that such an extension is nontrivial precisely when M is not locally convex, despite the fact that both S^1 and $\mathbb{C}$ are. In the third case, each extension of S^p by S^q gives rise to (and arises from) a "twisted Hilbert space", that is, a short exact sequence of Banach spaces and operators

$$0 \longrightarrow \mathcal{H} \longrightarrow T \longrightarrow \mathcal{H} \longrightarrow 0$$

which arises as its "spatial part". By the well-known projection property of Hilbert spaces such an extension is (non-) trivial if and only if T is (not) isomorphic to a Hilbert space.

It is remarkable that the results are so cleanly connected with the early "three space" problems. See [140], [125], [120], [137] or [138] for basic information on the topic.

The study of the modular structure of noncommutative L^p spaces built over a general von Neumann algebra M goes back to their inception. However, the computation of the spaces of homomorphisms, which plays a role, see [132].

Not much is known about the corresponding spaces of extensions $Ext_M(L^p, L^q)$ for general M. By following ideas of Kalton [136] it is proved in [124] that $Ext_M(L^p) \neq 0$ for every (infinite-dimensional) M and other related results.

The approach also originates in Kalton's work. Indeed, the idea of representing extensions by centralizers is already in [134]. Even if the connection between centralizers and extensions is deliberately neglected in both [135] and [136], the should be considered as the first serious studies on self-extensions of the Schatten classes within the category of quasi-Banach bimodules over B.

The commutative situation is settled in [122] with quite different techniques. Considering the usual Lebesgue spaces $L^p = L^p(\mu)$ for an arbitrary measure μ as L^∞-modules with "pointwise" multiplication we have $Ext_{L^\infty}(L^q, L^p) = 0$ when $p \neq q$ and $Ext_{L^\infty}(L^p) = Ext_{L^\infty}(L^1)$ for every $p \in (0,1)$. The preceding identity had been proved by Kalton in [134] for $p \in (1, \infty)$.

Some consider a more restrictive notion of extension by requiring the splitting in the (quasi-) Banach category. This leads to the study of the amenability of the underlying algebra, a major theme in homology of Banach algebras [129]. Although we will not pursue this point here, the results imply that if (1) splits as an extension of quasi-Banach spaces, then so it does as an extension of quasi-Banach modules over B, a result which is easy to prove when $q \geq 1$.

We consider modules on the left unless otherwise stated. Let A be a Banach algebra that for all purposes will be a C^*-algebra. A quasi-normed module over A is a quasi-normed space X together with a jointly continuous outer multiplication $A \times X \to X$ satisfying the traditional algebraic requirements. If the underlying space is complete (that is, a quasi-Banach space) we speak of a quasi-Banach module. Given quasi-normed modules X and Y, a homomorphism $u: X \to Y$ is an operator such that $u(ax) = au(x)$ for all $a \in A$ and $x \in X$. Operators and homomorphisms are assumed to be continuous unless otherwise stated. If no continuity is assumed, we speak of linear maps and morphisms. We use $Hom_A(X, Y)$ for the space of homomorphisms and $\mathcal{M}_A(X, Y)$ for the morphisms. If there is no possible confusion about the underlying algebra A, we omit the subscript.

Quasi-normed right modules and bimodules and their homomorphisms are defined in the obvious way.

In general, $Hom_A(X, Y)$ carries no module structure. However, if X is a bimodule instead of a mere left module, then $Hom_A(X, Y)$ can be given a structure of left module letting $(ah)(x) = h(xa)$, where $h \in Hom_A(X, Y), x \in X, a \in A$. Similarly, if Y is a bimodule, then the multiplication $ha(x) = h(x)a$ makes $Hom_A(X, Y)$ into a right module.

These structures are functorial in the obvious sense.

An extension of Z by Y is a short exact sequence of quasi-Banach modules and homomorphisms

$$0 \longrightarrow Y \xrightarrow{\iota} X \xrightarrow{\pi} Z \longrightarrow 0. \quad (2)$$

The open mapping theorem guarantees that ι embeds Y as a closed submodule of X in such a way that the corresponding quotient is isomorphic to Z. Two extensions $0 \longrightarrow Y \longrightarrow X_i \longrightarrow Z \longrightarrow 0$ $(i = 1,2)$ are said to be equivalent if there exists a homomorphism u making commutative the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X_1 & \longrightarrow & Z & \longrightarrow & 0 \\ & & \| & & \downarrow u & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & X_2 & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

By the five-lemma [130], and the open mapping theorem, u must be an isomorphism. We say that (2) splits if it is equivalent to the trivial sequence $0 \longrightarrow Y \longrightarrow Y \oplus Z \longrightarrow Z \longrightarrow 0$. This just means that Y is a complemented submodule of X, that is, there is a homomorphism $X \longrightarrow Y$ which is a left inverse for the inclusion $\iota: Y \longrightarrow X$; equivalently, there is a homomorphism $Z \longrightarrow X$ which is a right inverse for the quotient $\pi : X \longrightarrow Z$.

Given quasi-Banach modules Y and Z, we denote by $Ext_A(Z, Y)$ the set of all possible module extensions (2) modulo equivalence. When $Y = Z$ we just write $Ext_A(Z)$. By using pull-back and pushout constructions, it can be proved (see [123] for the details in the F-space setting) that $Ext_A(Z, Y)$ carries a "natural" linear structure (without topology) in such a way that trivial extensions correspond to 0. (The usual approach using injective or projective representations completely fails dealing with quasi-Banach modules since there are neither injective nor projective objects.) Thus, $Ext_A(Z, Y) = 0$ means "every extension $0 \to Y \to X \to Z \to 0$ splits".

Taking A as the ground field one recovers extensions in the quasi-Banach space setting.

Definition (4.1.1)[119]: (Kalton). Let Z and Y be quasi-normed modules over the Banach algebra A and let $\tilde{Y}$ be another module containing Y in the purely algebraic sense. A centralizer from Z to Y with ambient space $\tilde{Y}$ is a homogeneous mapping $\Omega: Z \longrightarrow \tilde{Y}$ having the following properties.

(a) It is quasi-linear, that is, there is a constant Q so that if $f, g \in Z$, then $\Omega(f + g) - \Omega(f) - \Omega(g) \in Y$ and $\|\Omega(f + g) - \Omega(f) - \Omega(g)\|_Y \le Q(\|f\|_Z + \|g\|_z)$.

(b) There is a constant C so that if $a \in A$ and $f \in Z$, then $\Omega(af) - a\Omega(f) \in Y$ and $\|\Omega(af) - a\Omega(f)\|_Y \le C\|a\|_A\|f\|_Z$.

We denote by $Q[\Omega]$ the least constant for which (a) holds and by $C[\Omega]$ the least constant for which (b) holds. We refer to the number $\Delta[\Omega] = \max\{Q[\Omega], C[\Omega]\}$ as the centralizer constant of Ω.

We now indicate the connection between centralizers and extensions. Let Z and Y be quasi-Banach modules. Suppose $\Omega : Z_0 \to \tilde{Y}$ is a centralizer from Z_0 to Y, where Z_0 is a dense submodule of Z.Then

$$Y \oplus_\Omega Z_0 = \{(g, f) \in \tilde{Y} \times Z_0: g - \Omega f \in Y\}$$

is a linear subspace of $\tilde{Y} \times Z_0$ and the functional $\|(g, f)\|_\Omega = \|g - \Omega f\|_y + \|f\|_Z$ is a quasi-norm on it. Moreover, the map $\iota : Y \to Y \oplus_\Omega Z_0$ sending g to $(g, 0)$ preserves the quasi-norm, while the map $\pi: Y \oplus_\Omega Z_0 \longrightarrow Z_0$ given as $\pi(g, f) = f$ is open, so that we have a short exact sequence of quasi-normed spaces and relatively open operators

$$0 \longrightarrow Y \xrightarrow{\iota} Y \oplus_{\Omega} Z_0 \xrightarrow{\pi} Z_0 \longrightarrow 0 \qquad (3)$$

Actually only quasi-linearity (a) is necessary here. The estimate in (b) implies that the multiplication a $(g,f) = (ag, af)$ makes $Y \oplus_{\Omega} Z_0$ into a quasi-normed module over A in such a way that the arrows in (3) become homomorphisms. Indeed,

$$\|a(g,f)\|_{\Omega} = \|ag - \Omega(af)\|_Y + \|af\|_Z = \|ag - \Omega af + af - \Omega(af)\|_Y + \|af\|_Z$$
$$\leq M\|a\|_A \|(g,f)\|_{\Omega}.$$

Let X_{Ω} be the completion of $Y \oplus_{\Omega} Z_0$. This is a quasi-Banach module and there is a unique homomorphism $X_{\Omega} \to Z$ extending the quotient in (3) we still denote π. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & Y & \longrightarrow & Y \oplus_{\Omega} Z_0 & \longrightarrow & Z_0 & \longrightarrow 0 \\ & \| & & \downarrow & & \downarrow & \\ 0 \longrightarrow & y & \longrightarrow & X_{\Omega} & \xrightarrow{\pi} & Z & \longrightarrow 0 \end{array} \qquad (4)$$

in which the vertical arrows are inclusions and the horizontal rows are exact. We will always refer to the lower row in this diagram as the extension (of Z by Y) induced by Ω.

It is easily seen that two centralizers Ω and Φ (acting between the same sets, say Z_0 and $\tilde{Y}$) induce equivalent extensions if and only if there is a morphism $h: Z_0 \to \tilde{Y}$ such that $\|\Omega(f) - \Phi(f) - h(f)\|_Y \leq K\|f\|_Z$. We write $\Omega \sim \Phi$ in this case and $\Omega \approx \Phi$ if the preceding inequality holds for $h = 0$. In particular Ω induces a trivial extension if and only if $\|\Omega(f) - h(f)\|_Y \leq K\|f\|_Z$ for some morphism $h: Z_0 \to \tilde{Y}$ (that is, $dist(\Omega, h) < \infty$). In this case we say that Ω is a trivial centralizer.

We now move to the concrete modules we shall deal with. For $p \in (0, \infty)$, let ℓ^p denote quasi-Banach space of (complex) sequences (t_n) for which the quasi-norm $|(t_n)|_p = (\sum_n |t_n|^p)^{1/p}$ is finite.

Let f be a compact operator on the Hilbert space $\mathcal{H}$. The singular numbers of f are the sequence of eigenvalues of $|f| = (f^* f)^{1/2}$ arranged in decreasing order and counting multiplicity. The Schatten class S^p consists of those operators on $\mathcal{H}$ whose singular numbers $(s_n(f))$ are in ℓ^p. It is a quasi-Banach space under the quasi-norm $\|f\|_p = |(s_n(f))|_p$. Each $f \in S^p$ has an expansion $f = \sum_n s_n x_n \otimes y_n$, where (s_n) are its singular numbers and (x_n) and (y_n) are orthonormal sequences in $\mathcal{H}$. This is called an Schmidt representation of $f. S^p$ is a quasi-Banach bimodule over B in the obvious way: given $f \in S^p$ and $a, b \in B$ one has $afb \in S^p$ and $\|afb\|_p \leq \|a\|_B \|f\|_p \|b\|_B$. The submodule of finite rank operators is denoted by S_0^p . The structure of homomorphisms between Schatten classes is fairly simple. Indeed, one has

$$Hom_B(S^p, S^q) = \begin{cases} S^r & if\ 0 < q < p < \infty, where\ p^{-1} + r^{-1} = q^{-1}; \\ B & if\ p \leq q. \end{cases} \qquad (5)$$

This should be understood as follows: each operator g in the left-hand side defines a homomorphism $\gamma: S^p \to S^q$ by multiplication on the right $\gamma(f) = fg$. Moreover, the norm of g in in the corresponding space equals $\|\gamma : S^p \to S^q\|$ and every homomorphism arises in this way. All this can be seen in Simon [147].

It will be convenient at some places to consider right module structures. We indicate this just by putting the (algebra) subscript on the right. Thus, for instance, $Hom(Z,Y)_A$ is the space of homomorphisms of right modules from Z to Y, which are assumed to be (quasi-normed) right modules over A. The meaning of $\mathcal{M}(Z, Y)_A, Ext(Z, Y)_A$ or “right centralizer” should be clear.

The right module structure of Schatten classes is "isomorphic" to the left one throughout the involution: $fa = (a^*f^*)^*$. Thus, for instance, if $u: S^p \to B$ is a morphism of left (respectively, right) modules, then we obtain a morphism of right (respectively, left) modules thus: $f \longmapsto (u(f^*))^*$. The same formula can be used to exchange left and right homomorphisms, centralizers, and the like. We will use this fact without further mention.

Lemma (4.1.2)[119]: (a) $\mathfrak{F}$ is a projective B-module in the pure algebraic sense: if X is any algebraic B-module and $\pi: X \to \mathfrak{F}$ is a surjective morphism, then there is another morphism $s: \mathfrak{F} \to X$ such that $\pi \circ s = I_{\mathfrak{F}}$.

(b) $\mathcal{M}(\mathfrak{F}, B)_B = L(\mathcal{H})$ in the sense that for every morphism of right modules $\alpha : \mathfrak{F} \to B$ there is a unique linear endomorphism ℓ of $\mathcal{H}$ such that $\alpha(f) = (\ell \circ f^*)^*$ for every $f \in \mathfrak{F}$.

(c) Similarly, $\mathcal{M}_B(\mathfrak{F}, B) = L(\mathcal{H})$ in the sense that for every morphism of left modules $\alpha: \mathfrak{F} \to B$ there isq a unique linear endomorphism ℓ of $\mathcal{H}$ such that $\alpha(f) = (\ell \circ f^*)^*$ for every $f \in \mathfrak{F}$.

(d) Let $\ell : \mathfrak{F} \to \mathbb{C}$ be a linear map such that for each fixed $y \in \mathcal{H}$ one has $\ell(x \otimes y) \to 0$ as $x \to 0$ in $\mathcal{H}$. Then there is a linear endomorphism L of $\mathcal{H}$ such that $\ell(f) = tr(L \circ f)$ for all $x, y \in \mathcal{H}$.

Proof. (a) Of course, B is a projective B-module. $\mathcal{H}$ is a B-module under the obvious action $(a, h) \longmapsto a(h)$. Fix any norm one $\eta \in \mathcal{H}$. Then the map $\eta \otimes -: \mathcal{H} \to B$ given by $h \longmapsto \eta \otimes h$ is an injective (homo)morphism. The evaluation map$\delta_\eta: B \to \mathcal{H}$ given by $\delta_\eta(u) = u(\eta)$ provides a left inverse (homo)morphism for $\eta \otimes -$. Being a direct factor in $B, \mathcal{H}$ is projective too.

On the other hand, $\mathfrak{F} = \mathcal{H}' \otimes \mathcal{H}$ (as bimodules). If I is a Hamel basis for $\mathcal{H}'$, we have $\mathcal{H}' = \bigoplus_I \mathbb{C}$ as linear spaces. Combining, we have

$$\mathfrak{F} = \mathcal{H}' \otimes \mathcal{H} \simeq (\textstyle\bigoplus_I \mathbb{C}) \otimes \mathcal{H} = \bigoplus_I (\mathbb{C} \otimes \mathcal{H}) = \bigoplus_I \mathcal{H},$$

as (left) modules, and a direct sum of projective modules is again projective.

(b) is very easy. Take $x, y \in \mathcal{H}$, with $\|x\| = 1$. Then $\alpha(x \otimes y) = \alpha((x \otimes y)(x \otimes x)) = (\alpha(x \otimes y))(x \otimes x)$. Hence there is $z = z(x, y) \in \mathcal{H}$ such that $\alpha(x \otimes y) = x \otimes z$. It is easily seen that z does not depend on the first variable while it depends linearly on the second one. Thus the rule $\ell(y) = z$ is an endomorphism of $\mathcal{H}$. Quite clearly one has $\alpha(f) = \ell \circ f$ when f has rank one and the same is true for every $f \in \mathfrak{F}$.

(c) is just the left version of (b).

(d) Fix $y \in \mathcal{H}$. The hypothesis implies that $x \longmapsto \ell(x \otimes y)$ is a continuous, conjugate-linear functional on $\mathcal{H}$ and by Riesz representation theorem there is $z \in \mathcal{H}$ such that $\ell(x \otimes y) = \langle z | x \rangle$.

Putting $z = L(y)$ we obtain a transformation of $\mathcal{H}$ which is easily seen to be linear. And since $\ell(x \otimes y) = \langle L(y) | x \rangle = tr(x \otimes L(y)) = tr(L \circ (x \otimes y))$ we are done.

Corollary (4.1.3)[119]: Up to equivalence, every extension of S^p by an arbitrary quasi-Banach module Y comes from a centralizer $\Omega: S_0^p \to Y$.

Proof. Let $0 \longrightarrow Y \longrightarrow X \xrightarrow{\pi} S^p \longrightarrow 0$ be an extension of quasi-Banach modules over B. With no serious loss of generality we may assume $Y = ker\, \pi$. Putting $X_0 = \pi^{-1}(S_0^p)$ we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X_0 & \longrightarrow & S_0^p & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Y & \longrightarrow & X & \xrightarrow{\pi} & S^p & \longrightarrow & 0 \end{array}$$

where the vertical arrows are plain inclusions. We shall show there is a centralizer $\Omega: S_0^p \to Y$ and an isomorphism of quasi-normed normed modules u making commutative the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & Y \oplus_\Omega S_0^p & \longrightarrow & S_0^p & \longrightarrow & 0 \\ & & \| & & u \downarrow & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & X_0 & \xrightarrow{\pi} & S_0^p & \longrightarrow & 0 \end{array}$$

This obviously implies that u extends to an isomorphism between X_Ω and X fitting in the corresponding diagram.

The identification of Ω is as follows. Let $b: S^p \longrightarrow X$ be a homogeneous bounded of the quotient map $\pi: X \to S^p$, that is, a map satisfying $\pi \circ b = I_{S^p}$ and $\|b(f)\|_M \le M\|f\|_p$ for some M independent on $f \in S^p$. Such exists because π is open. Notice, moreover, that $b(f) \in X_0$ if $f \in S_0^p$.

Now we use Lemma (4.1.2)(a) to get a morphism $s: S_0^p \to X_0$ such that $\pi \circ s = I_{S_0^p}$ and we set $\Omega(f) = b(f) - s(f)$ for $\in S_0^p$. Clearly, $\pi\big(\Omega(f)\big) = \pi\big(b(f)\big) - \pi(s(f)) = 0$ and so Ω takes values in Y.

That Ω is a centralizer is nearly trivial: given $f, g \in S_0^p$ and $a \in B$ one has

$$\|\Omega(f+g) - \Omega f - \Omega g\|_Y = \|b(f+g) - b(f) - b(g)\|_X \le M\big(\|f\|_p + \|g\|_p\big),$$
$$\|\Omega(af) - a\Omega \mathrm{f}\|_Y = \|b(af) - ab(f)\|_X \le M\|a\|_B\|f\|_p.$$

We define $u: Y \oplus_\Omega S_0^p \to X_0$ by $u(y,f) = y + s(f)$. This is a homomorphism in view of the bound

$$\|u(y,f)\|_X = \|y + s(f)\|_X \le M(\|y - b(f) + s(f)\|_X + \|b(f)\|_X)$$
$$\le M(\|y - \Omega f\|_X + \|f\|_p) \le M\|(y,f)\|_\Omega.$$

The inverse of u is given by $v(x) = (x - s\big(\pi(x)\big), \pi(x))$ for $x \in X_0$. It is continuous since

$$\|v(x)\|_\Omega = \big\|x - s\big(\pi(x)\big) - \Omega(\pi(x))\big\|_Y + \|\pi(x)\|_p = \|x - b(\pi(x))\|_Y + \|\pi(x)\|_p$$
$$\le M\|x\|_X.$$

This completes the proof.

We prove that $Ext_B(S^p, S^q) = 0$ when $0 < p < q < \infty$. We will imagine, what we prove is that every centralizer $S_0^p \to S^q$ is trivial. First we need to break a given centralizer into "small pieces" without losing the relevant information it encodes.

Let $\Phi: S_0^p \to S^q$ be a centralizer and e a finite rank projection. Then we can define a centralizer $\Phi_e: S^p \to S^q$ by the formula $\Phi_e(f) = \Phi(fe)$. Of course, Φ_e is trivial. Indeed taking $g = \Phi(e)$ we have

$$\|\Phi_e(f) - fg\|_q = \|\Phi(fe) - f\Phi(e)\|_q \le C[\Phi]\|f\|_B\|e\|_p \le C[\Phi]\, rk(e)^{1/p}\|f\|_p,$$

where $rk(e)$ is the dimension of the image of e.

Lemma (4.1.4)[119]: Let $\Phi: S_0^p \to S^q$ be a centralizer, with q finite. Then

$$dist\Big(\Phi, \mathcal{M}_B\big(S_0^p, S^q\big)\Big) = sup_e dist\big(\Phi_e, \mathcal{M}_B(S^p, S^q)\big),$$

where e runs over all finite rank projections in B.

Proof. That $dist(\Phi, \mathcal{M}_B(S_0^p, S^q)) \ge dist(\Phi_e, \mathcal{M}_B(S^p, S^q))$ for every e is obvious. Let us prove the other inequality. Let D be a constant such that for every e there is a morphism ϕ_e so that

$$\|\Phi_e f - \phi_e(f)\|_q \le D\|f\|_p \qquad (f \in S^p).$$

Let $\mathcal{U}$ be an ultrafilter refining the Frechet filter on the set of finite rank projections in B. We define a mapping $\phi : S_0^p \to S^q$ by the formula

$$\phi(f) = \lim_{\mathcal{U}} \phi_e(fe) \tag{6}$$

where the limit is taken in the WOT. The definition makes sense because for each $f \in S_0^p$ one has $fe = e$ for sufficiently large e. For these projections we have $\|\Phi(f) - \phi_e(f)\|_q \leq D\|f\|_p$ and thus the net $(\phi_e(fe))_e$ is (essentially) bounded in S^q and so in B. As bounded subsets of B are relatively compact in the WOT we see that (6) defines a map from S_0^p to B. But $\|\cdot\|_q$is lower semicontinuous with respect to the restriction of the WOT to S^q (see [126]) and so

$$\|\Phi(f) - \phi(f)\|_q \leq \lim_{\mathcal{U}} inf\, \|\Phi(f) - \phi_e(f)\|_q \leq D\|f\|_p \qquad \left(f \in S_0^p\right).$$

In particular $\phi(f)$ belongs to S^q. Finally that ϕ is a morphism follows from the fact that, for fixed $a \in B$, the map $b \longmapsto ab$ is WOT-continuous on bounded sets of B.

The sought-after result reads as follows.

The proof uses a simple ultraproduct technique, but requires some noncommutative gadgetry.

Let X be a quasi-Banach space, I an index set and $\mathcal{U}$ a countably incomplete ultrafilter on I. Let $\ell^\infty(I, X)$ be the space of bounded families of X indexed by I (furnished with the sup quasi-norm) and let $N_{\mathcal{U}}$ be the (closed) subspace of those $x \in \ell^\infty(I, X)$ such that $\|x_i\|_X \to 0$ along $\mathcal{U}$. The ultrapower of X with respect to $\mathcal{U}$ is the quotient space $\ell^\infty(I, X)/N_{\mathcal{U}}$ with the quotient quasi-norm. The class of the family (x_i) in $X_{\mathcal{U}}$ is denoted by $[(x_i)]$. Notice that if the quasi-norm of X is continuous one can compute the quasi-norm in $X_{\mathcal{U}}$ by the formula $\|[(x_i)]\| = \lim_{\mathcal{U}} \|x_i\|_X$. Clearly, if A is a Banach algebra, then so is $A_{\mathcal{U}}$ when equipped with the coordinatewise product $[(a_i)][(b_i)] = [(a_i b_i)]$. If besides X is a quasi-Banach module over A, then the multiplication $[(a_i)][(x_i)] = [(a_i x_i)]$ makes $X_{\mathcal{U}}$ into a quasi-Banach module over $A_{\mathcal{U}}$.

What we need to prove Theorem (4.1.12) is the following.

Lemma (4.1.5)[119]: Let $p, q, r \in (0, \infty)$ satisfy $q^{-1} = p^{-1} + r^{-1}$. If $\gamma: S_{\mathcal{U}}^p \to S_{\mathcal{U}}^q$ is a homomorphism of (left) modules over $B_{\mathcal{U}}$, then there is bounded family (g_i) in S^r such that $\gamma[(f_i)] = [(f_i g_i)]$ whenever (f_i) is bounded in S^p.

Proof. This can be obtained as a combination of results by Raynaud, and Junge and Sherman. Let us explain how.

(i) There is a general construction, due to Haagerup, that associates to a given von Neumann algebra M the so-called (Haagerup, non-commutative) L^p spaces $L^p(M)$ for $0 < p \leq \infty$. These spaces consist of certain (densely defined, closable, but in general discontinuous) operators acting on a common suitable Hilbert space which is related to M in a highly nontrivial way and M itself can be identified with $L^\infty(M)$, as von Neumann algebras. As it happens this provides the following generalization of Holder inequality: suppose $p, q, r \in (0, \infty]$ are such that $q^{-1} = p^{-1} + r^{-1}$; if $f \in L^p(M)$ and $g \in L^r(M)$, then $fg \in L^q(M)$ and $\|fg\|_q \leq \|f\|_p \|g\|_r$, where the subscript indicates the quasi-norm of the corresponding Haagerup space. Letting $p = \infty$ or $r = \infty$ one gets the module structures over $L^\infty(M)$. See [128], [142], [144].

(ii) After that it is clear that that every $g \in L^r(M)$ gives rise to a homomorphism (of left $L^\infty(M)$-modules) $\gamma : L^p(M) \to L^q(M)$ by multiplication: $\gamma(f) = fg$. Moreover, $\|\gamma: L^p(M) \to L^q(M)\| = \|g\|_r$. Junge and Sherman proved in [132] that all such homomorphisms arise in this way, which is crucial for us.

(iii) The Haagerup spaces do not form any "scale". Indeed, by the very definition, one has $L^q(M) \cap L^q(M) = 0$ unless $p = q$. In particular, $L^p(B)$ (the Haagerup L^p space

corresponding to the choice $M = B$) cannot be the same as 'our' S^p. Nevertheless there is a system of isometric bimodule isomorphisms $\iota_p : S^p \to L^p(B)$ which are compatible with the product maps in the sense that $\iota_q(fg) = \iota_p(f)\iota_r(g)$ whenever $f \in S^p$ and $g \in S^q$ with $q^{-1} = p^{-1} + r^{-1}$.
The obvious consequence of this is that a map $u: S^p \to S^q$ is a homomorphisms of B-modules if and only if $\iota_q \circ u \circ \iota_p^{-1}: L^p(B) \to L^q(B)$ is a homomorphism of $L^\infty(B)$-modules. Therefore replacing Schatten classes by Haagerup spaces and B by $L^\infty(B)$ does not alter the Lemma.
(iv) Raynaud proved in [143] that given a von Neumann algebra M and a countably incomplete ultrafilter $\mathcal{U}$ one can represent the ultrapowers of the whole family of Haagerup spaces $L^p(M)$ (for finite p) as the Haagerup spaces associated to some von Neumann algebra independent on p. Precisely: there is a von Neumann algebra N containing $L^\infty(M)_\mathcal{U}$ and a system of surjective isometries $\kappa_p: L^p(M)_\mathcal{U} \to L^p(N)$ for $0 < p < \infty$ compatible with the product maps in the following sense: $p, q, r \in (0, \infty)$ are such that $q^{-1} = p^{-1} + r^{-1}$ and (f_i) and (g_i) are bounded families in $L^p(M)$ and $L^r(M)$, respectively, then

$$\big(\kappa_p[(f_i)]\big)(\kappa_r[(g_i)]) = \kappa_q[(f_i g_i)],$$

where the product in the left-hand side refers to spaces over N and those in the right-hand side to M.
(v) Therefore we can regard $L^p(M)_\mathcal{U}$ as a module over N and every homomorphism of N-modules $\gamma: L^p(M)_\mathcal{U} \to L^q(M)_\mathcal{U}$ can be represented as $\gamma[(f_i)] = [(f_i g_i)]$, where (g_i) is a bounded family in $L^r(M)$.
(vi) The proof of the Lemma will be complete if we show that every homomorphism of $L^\infty(M)_\mathcal{U}$-modules $\gamma: L^p(M)_\mathcal{U} \to L^q(M)_\mathcal{U}$ is automatically a homomorphism of N-modules. And this is so because on one hand $L^\infty(M)_\mathcal{U}$ is dense in N in the strong operator topology induced by the (module) action on $L^2(N)$ and, on the other hand, the restriction to bounded subsets of N of the strong operator topology induced by the action on $L^p(N)$ does not depend on $0 < p < \infty$ (see [132]).
Theorem (4.1.6)[119]: Given $0 < q < p < \infty$, there is a constant $K = K(p, q)$ so that, for every centralizer $\Omega: S_0^p \to S^q$ there is a morphism $\omega: S_0^p \to S^q$ satisfying $\|\Omega(f) - \omega(f)\|_q \le K\Delta[\Omega]\|f\|_p$ for every $f \in S_0^p$.
Proof. Assuming the contrary there is a sequence of centralizers $\Omega_n: S_0^p \to S^q$ with $\Delta[\Omega_n] \le 1$ and $dist(\Omega_n, \mathcal{M}_B(S_0^p, S^q)) \to \infty$. In view of Lemma (4.1.4) we may and do assume that for each n there is a finite rank projection $e_n \in B$ such that $\Omega_n(f) = \Omega_n(fe_n)$ for all $f \in S_0^p$. Thus we may assume Ω_n defined on the whole of S^p and also that $dist(\Omega_n, \mathcal{M}_B(S^p, S^q))$ is finite for every n.
For each n we take a morphism $\phi_n: S^p \to S^q$ such that

$$\delta_n = dist(\Omega_n, \phi_n) \le dist\big(\Omega_n, \mathcal{M}_B(S^p, S^q)\big) + 1/n.$$

Of course,$\delta_n \to 1$ as $n \to \infty$. Put

$$v_n = \frac{\Omega_n - \phi_n}{\delta_n},$$

so that v_n is a homogeneous mapping from S^p to S^p with $\|v_n: S^p \to S^q\| \le 1$ and $\Delta[v_n] \le \delta_n^{-1}\Delta[\Omega] \to 0$as $n \to \infty$.
Let $\mathcal{U}$ be a free ultrafilter on the integers and consider the corresponding ultrapowers $S_\mathcal{U}^p$ and $S_\mathcal{U}^q$.
We can use the (probably nonlinear) maps v_n to define $v: S_\mathcal{U}^p \to S_\mathcal{U}^q$ by

$$v[(f_n)] = [(v_n(f_n))].$$

Let us check that v is well defined. First, suppose $[(f_n)] = 0$, that is, $\|f_n\|_p \to 0$ along $\mathcal{U}$. As $\|v_n(f_n)\|_q \leq \|f_n\|_p$ we have $[(v_n(f_n))] = 0$. Suppose now $[(f_n)] = [(g_n)]$. We must prove that $[(v_n(f_n))] = [(v_n(g_n))]$. But

$$\lim_{\mathcal{U}} \|v_n(f_n) - v_n(g_n)\|_q = \lim_{\mathcal{U}} \|v_n(f_n) - v_n(g_n) - v_n(f_n - g_n)\|_q$$
$$\leq \lim_{\mathcal{U}} Q[v_n](\|g_n\|_p + \|f_n - g_n\|_p) = 0$$

and the definition of v makes sense. Now it is nearly obvious that v is a continuous homomorphism of $B_{\mathcal{U}}$-modules. By Lemma (4.1.5) there is a bounded sequence (u_n) in S^r representing v in the sense that $v[(f_n)] = [(f_n u_n)]$ whenever (f_n) is a bounded sequence in S^p, where $r^{-1} + p^{-1} = q^{-1}$. This implies that $dist(v_n, u_n) \to 0$ along $\mathcal{U}$. In particular, for every $\varepsilon > 0$, the set $S = \{n \in \mathbb{N}: 0 < dist(\delta_n^{-1}(\Omega_n - \phi_n), u_n) < \varepsilon\}$ belongs to $\mathcal{U}$ and it contains infinitely many indices n. For these n we get

$$dist(\Omega_n, \phi_n + \delta_n u_n) < \varepsilon\delta_n < 2\varepsilon dist\big(\Omega_n, \mathcal{M}_B(S^p, S^q)\big),$$

in striking contradiction with our choice of ϕ_n.

Corollary (4.1.7)[119]: $Ext_B(S^p, S^q) = 0$ for $0 < q < p \leq \infty$.

Proof. Corollary (4.1.3) and Theorem (4.1.6). For $p = \infty$ use Lemma (4.1.4).

Once we know that Ext vanishes at certain couples, it is easy to use the functor Hom to compare different spaces of extensions. Let us begin with the covariant case. Suppose we are given an extension of modules

$$0 \to Y \xrightarrow{\iota} X \xrightarrow{\pi} Z \to 0. \tag{7}$$

If E is another module we can apply $Hom(E, -)$ to get an exact sequence (of linear spaces)

$$0 \longrightarrow Hom(E, Y) \xrightarrow{\iota_\circ} Hom(E, X) \xrightarrow{\pi_\circ} Hom(E, Z) \xrightarrow{\alpha} Ext(E, Y) \longrightarrow \cdots \tag{8}$$

Notice that $\iota_\circ$ is just the functorial image of ι, and similarly with $\pi_\circ$. The connecting map α sends a given homomorphism ϕ into the (class of the) lower extension in the pull-back diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\iota} & X & \xrightarrow{\pi} & Z & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow \phi & & \\ 0 & \longrightarrow & Y & \longrightarrow & PB & \longrightarrow & E & \longrightarrow & 0 \end{array}$$

Thus, if $Ext(E, Y)$ vanishes, then (8) represents an extension of $Hom(E, Z)$ by $Hom(E, Y)$. If, besides, E is a bimodule, then (8) is an extension of (left) modules. All this can be seen in [123].

In a similar vein, if we apply $Hom(-, E)$ to (7) we obtain

$$0 \longrightarrow Hom(Z, E) \xrightarrow{\pi^\circ} Hom(X, E) \xrightarrow{\iota^\circ} Hom(Y, E) \xrightarrow{\beta} Ext(Z, E) \longrightarrow \cdots \tag{9}$$

Here, β sends a given homomorphism $\phi: Y \to E$ into the (class of the) lower row of the push-out diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\iota} & X & \longrightarrow & Z & \longrightarrow & 0 \\ & & \phi \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & E & \longrightarrow & PO & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

Hence, if $Ext(Z, E)$ vanishes, then (9) is an extension of $Hom(Y, E)$ by $Hom(Z, E)$ which lives in the category of right modules provided E is a bimodule.

Theorem (4.1.8)[119]: Let $0 < r < p_1 \leq p_2 < \infty$ be fixed. Then $Hom_B(-, S^r)$ defines an isomorphism from $Ext_B(S^{p_1}, S^{p_2})$ onto $Ext(S^{q_2}, S^{q_1})_B$, where $q_i^{-1} + p_i^{-1} = r^{-1}$ for $i = 1,2$. Similarly, $Hom(-, S^r)_B$ is an isomorphism from $Ext(S^{p_1}, S^{p_2})_B$ onto $Ext_B(S^{q_2}, S^{q_1})$.

Proof. Suppose we are given an extension of left modules $0 \longrightarrow S^{p_2} \longrightarrow X \longrightarrow S^{p_1} \longrightarrow 0$. Applying $Hom_B(-,S^r)$ we get

$$0 \longrightarrow Hom_B(S^{p_1},S^r) \longrightarrow Hom_B(X,S^r) \longrightarrow Hom_B(S^{p_2},S^r) \longrightarrow Ext_B(S^{p_1},S^r) \longrightarrow \cdots \quad (10)$$

But $Ext_B(S^{p_1},S^r) = 0$, so the above diagram is in fact an extension of $Hom(S^{p_2},S^r) = S^{q_2}$ by $Hom_B(S^{p_1},S^r) = S^{q_1}$ in the category of right modules over B. It is pretty obvious that this procedure preserves equivalences and so it defines a mapping

$$Hom(-,S^r)_B: Ext_B(S^{p_1},S^{p_2}) \longrightarrow Ext(S^{q_2},S^{q_1})_B.$$

To see that it is indeed an isomorphism, consider now $Hom(-,S^r)_B$ as a map from $Ext(S^{q_2},S^{q_1})_B$ to $Ext_B(S^{p_1},S^{p_2})$ −take into account that $r < q_2$ − and let us check that the two maps are inverse to each other. Indeed, if we apply $Hom(-,S^r)_B$ to (10), we obtain another extension

$$0 \to Hom(Hom_B(S^{p_2},S^r),S^r)_B \to Hom(Hom_B(X,S^r),S^r)_B \to Hom(Hom_B(S^{p_1}S^r),S^r)_B \to 0.$$

But after the identification $S^{p_i} = Hom(Hom_B(S^{p_i},S^r),S^r)_B$ this extension is equivalent to the starting one since the diagram

$$\begin{array}{ccccc} S^{p_2} & \longrightarrow & X & \longrightarrow & S^{p_1} \\ \| & & \downarrow \delta & & \| \\ Hom(Hom(S^{p_2},S^r),S^r)_B & \longrightarrow & Hom(Hom(X,S^r),S^r)_B & \longrightarrow & Hom(Hom(S^{p_1},S^r),S^r)_B \end{array}$$

is commutative − the middle arrow is the obvious evaluation homomorphism given by $\delta(x)(\phi) = \phi(x)$.

Theorem (4.1.9)[119]: Let $0 < p_1 \le p_2 < s < \infty$ be fixed. Then $Hom(S^s,-): Ext(S^{p_1},S^{p_2}) \longrightarrow Ext(S^{q_1},S^{q_2})$ is an isomorphism, where $s^{-1} + q_i^{-1} = p_i^{-1}$ for $i = 1,2$.

Proof. (Absence of subscript indicates left module structure.) This can be proved in several ways.

Perhaps the simplest one is checking that if $r^{-1} + s^{-1} = t^{-1}$, then one has $Hom(-,S^r) \circ Hom(S^s,-) = Hom(-,S^t)$ at $Ext(S^{p_1},S^{p_2})$, that is, the composition

$$Ext(S^{p_1},S^{p_2}) \xrightarrow{Hom(S^s,-)} Ext(S^{q_1},S^{q_2}) \xrightarrow{Hom(-,S^r)} Ext(S^{\ell_2},S^{\ell_1})_B$$

agrees with $Hom(-,S^t): Ext(S^{p_1},S^{p_2}) \longrightarrow Ext(S^{\ell_2},S^{\ell_1})_B$, where $\ell_i^{-1} + p_i^{-1} = t^{-1}$ for $i = 1, 2$.

Incidentally this will show that $Hom(S^s,-) = Hom(-,S^r)_B \circ Hom(-\,,S^t)$ since $Hom(-,S^r)_B$ is the inverse of $Hom_B(-,S^r)$. See the proof of Theorem (4.1.8).

Recall that $S^r = Hom(S^s,S^t)$, so that the composition $Hom(-,S^r) \circ Hom(S^s,-)$ agrees with $Hom(Hom(S^s,-),Hom(S^s,S^t))$.

To each quasi-Banach left B-module M we attach the homomorphism of right modules

$$-_\circ: Hom(M,S^t) \longrightarrow Hom(Hom(S^s,M),Hom(S^s,S^t))$$

sending a given homomorphism $u: M \longrightarrow S^t$ into the transformation $u_\circ : Hom(S^s,M) \longrightarrow Hom(S^s,S^t)$ defined by $u_\circ(v) = u \circ v$.

This is in fact a natural transformation from $Hom(-,S^t)$ to $Hom(Hom(S^s,-),Hom(S^s,S^t))$ meaning that for every homomorphism of (left) modules $\alpha: M \to N$ the following diagram is commutative

$$\begin{array}{ccc} Hom(M,S^t) & \xrightarrow{-_\circ} & Hom(Hom(S^s,M),Hom(S^s,S^t)) \\ \alpha^\circ \uparrow & & \uparrow (\alpha_\circ)^\circ \\ Hom(N,S^t) & \xrightarrow{-_\circ} & Hom(Hom(S^s,N),Hom(S^s,S^t)) \end{array}$$

The point is that the preceding natural transformation behaves as a natural equivalence at S^{p_1} and S^{p_2} and so it induces an isomorphism at $Ext(S^{p_1}, S^{p_2})$. Indeed, if we are given an extension

$$0 \longrightarrow S^{p_2} \xrightarrow{\iota} X \xrightarrow{\pi} S^{p_1} \longrightarrow 0$$

and we apply $Hom(-, S^t)$ on one hand and $Hom(Hom(S^s, -), Hom(S^s, S^t)) = Hom(Hom(S^s, -), S^r)$ on the other we have the commutative diagram

$$\begin{array}{ccccc} Hom(S^{p_1}, S^t) & \xrightarrow{\iota^\circ} & Hom(X, S^t) & \xrightarrow{\pi^\circ} & Hom(S^{p_2}, S^t) \\ -_\circ\downarrow & & -_\circ\downarrow & & -_\circ\downarrow \\ Hom(Hom(S^s, S^{p_1}), S^r) & \xrightarrow{(\iota_\circ)^\circ} & Hom(Hom(S^s, X), S^r) & \xrightarrow{(\pi_\circ)^\circ} & Hom(Hom(S^s, S^{p_2}), S^r) \end{array}$$

where the rows are extensions. Now, the left and right vertical arrows are isomorphisms (they are the identity after identifying both $Hom(S^{p_i}, S^t)$ and $Hom(Hom(S^s, S^{p_i}), S^r)$ with S^{ℓ_i} for $i = 1,2$) and so is the middle one.

We take a look at the actions of $Hom(-, S^r)$ and $Hom(S^s, -)$ on centralizers. To take advantage of the extra simplification provided by Lemma (4.1.2) (b) we shall work with right centralizers.

Fix numbers $0 < r < p_1 \leq p_2 \leq \infty$ and let $\Omega: S_0^{p_1} \longrightarrow S^{p_2}$ be a right centralizer − note we are allowing $p_2 = \infty$ here. Given $g \in Hom(S^{p_2}, S^r)_B = S^{q_2}$ (isometric isomorphism of bimodules), we consider the mapping $f \in S_0^{p_1} \longmapsto g(\Omega f) \in S^r$. Clearly, this is a right centralizer with constant at most $\|g\|_{q_2}\Delta[\Omega]$ and since $r < p_1$ there is a morphism $\phi_g \in \mathcal{M}\left(S_0^{p_1}, S^r\right)_B = \mathcal{M}(\mathfrak{F}, B)_B$ such that $\left\|\Omega(f)g + \phi_g(f)\right\|_r \leq M\Delta[\Omega]\|f\|_{p_1}\|g\|_{q_2}$. By Lemma (4.1.2) there is $\ell \in L(\mathcal{H})$ that implements ϕ_g in the sense that $\phi_g(f) = \ell \circ f$ and we can define a mapping $\Phi: S^{q_2} \rightarrow L(\mathcal{H})$ just taking $\Phi(g) = \ell$. Of course this can be done homogeneously and we have the estimate

$$\|g(\Omega f) + (\Phi g)f\|_r \leq M\Delta[\Omega]\|g\|_{q_2}\|f\|_{p_1}\left(f \in S_0^{p_1}, g \in S^{q_2}\right). \qquad (11)$$

Obviously, S^{q_1} is a left submodule of $L(\mathcal{H})$. That Φ is a left centralizer from S^{q_2} to S^{q_1} now follows from (11), taking into account that for $\ell \in L(\mathcal{H})$ one has $\|\ell\|_{q_1} = \|\ell_\circ: S^{p_1} \longrightarrow S^r\| = \left\|\ell_\circ: S_p^{p_1} \longrightarrow S^r\right\|$, where $\ell_\circ(f) = \ell \circ f$.

Let X_Ω denote the completion of $S^{p_2} \bigoplus_\Omega S_0^{p_1}$. It is possible to identify $Hom(X_\Omega, S^r)_B$ and $S^{q_1} \bigoplus_\Phi S^{q_2}$ as follows: for $(h, g) \in S^{q_1} \bigoplus_\Phi S^{q_2}$ (hence $h - \Phi g \in S^{q_1}$) and $(f', f) \in S^{p_2} \bigoplus_\Omega S_0^{p_1}$, we put

$$(h, g)\left(f', f\right) = hf + gf'.$$

One then has

$$\begin{aligned} \|hf + gf'\|_r &= \|hf - (\Phi g)f + (\Phi g)f + g\Omega f - g\Omega f + gf'\|_r \\ &\leq M\left(\|h - \Phi g\|_{q_1}\|f\|_{p_1} + \|g\|_{q_2}\|f\|_{p_1} + \|g\|_{q_2}\left\|f' - \Omega f\right\|_{p_2}\right) \\ &\leq M\left(\|(h, g)\|_\Phi\left\|\left(f', f\right)\right\|_\Omega\right), \end{aligned}$$

and since $S^{p_2} \bigoplus_\Omega S_0^{p_1}$ is a dense submodule of X_Ω we see that $S^{q_1} \bigoplus_\Phi S^{q_2}$ embeds in $Hom(X_\Omega, S^r)_B = Hom(S^{p_2} \bigoplus_\Omega S_0^{p_1}, S^r)_B$. That embedding is onto (and open) in view of the commutativity of the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S^{q_1} & \longrightarrow & S^{q_1} \bigoplus_\Phi S^{q_2} & \longrightarrow & S^{q_2} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & Hom(S^{p_1}, S^r)_B & \longrightarrow & Hom(X_\Omega, S^r)_B & \longrightarrow & Hom(S^{p_2}, S^r)_B & \longrightarrow & 0 \end{array}$$

whose rows are exact.

We next turn our attention to the covariant case. We fix $0 < p_1 \leq p_2 < s$ and for $i = 1,2$, we put $q_i^{-1} + s^{-1} = p_i^{-1}$ so that $Hom(S^s, S^{p_i})_B = S^{q_i}$ in the obvious way. As before, we consider a right centralizer $\Omega: S_0^{p_1} \to S^{p_2}$.

Given $g \in S^{q_1}$ we consider the map $\in S_0^s \longmapsto \Omega(gf) \in S^{p_2}$. This is again a right centralizer, with constant at most $\|g\|_{q_1} C[\Omega]$. As $s > p_2$, there is a linear map ψ on $\mathcal{H}$ such that $\|\Omega(gf) - \psi_\circ(f)\|_{p_2} \leq M\Delta[\Omega]\|g\|_{q_1}\|f\|_s$. Taking $\Psi(g) = \psi$ homogeneously we get a mapping $\Psi: S^{q_1} \to L(\mathcal{H})$ such that

$$\|\Omega(gf) - (\Psi g)f\|_{p_2} \leq M\Delta[\Omega]\|g\|_{q^2}\|f\|_s, (g \in S^{q_2}, f \in S_0^s). \qquad (12)$$

Let us verify that Ψ is a right centralizer from S^{q_1} to S^{q_2} . Take $g, g' \in S^{q_1}$ and $a \in B$ and recall that for $\ell \in L(\mathcal{H})$ one has $\|\ell\|_{q_2} = \|\ell_\circ : S_0^s \to S^{p_2}\|$. For $f \in S_0^s$ we have on account of (12):

$$\begin{aligned}
&\left\|(\Psi(g + g') - \Psi g - \Psi g')f\right\|_{p_2} \\
&\qquad \leq \left\|(\Omega(gf + g'f) - \Omega(gf) - (g'f)\right\|_{p_2} + M(\|g + g'\|_{q_2} + \|g\|_{q_2} \\
&\qquad + \|g'\|_{q_2})\|f\|_s \\
&\qquad \leq Q[\Omega](\|gf\|_{p_1} + \|g'f\|_{p_1}) + M(\|g\|_{q_2} + \|g'\|_{q_2})\|f\|_s \\
&\qquad \leq M(\|g\|_{q_2} + \|g'\|_{q_2})\|f\|_s
\end{aligned}$$

The estimate $\|\Psi(ga) - (\Psi g)a\|_{q_2} \leq MC[\Omega]\|g\|_{q_2}\|a\|_B$ is even easier and we leave it to the reader. As before, $S^{q_1} \oplus_\Psi S^{q_2}$ is isomorphic to $Hom(S^s, X_\Omega)_B$ (as quasi-Banach right modules), where X_Ω is the completion of $S^{p_2} \oplus_\Omega S_0^{p_1}$. Indeed, take $(h, g) \in S^{q_1} \oplus_\Psi S^{q_2}$. Given $f \in S_0^s$ we define

$$(h, g)(f) = (hf, gf).$$

The definition is correct because f has finite rank and thus hf is bounded even if h is not. Moreover (hf, gf) falls in $S^{p_2} \oplus_\Omega S_0^{p_1}$ and we have

$$\begin{aligned}
\|(hf, gf)\|_\Omega &= \|hf - \Omega(gf)\|_{p_2} + \|gf\|_{p_1} \\
&\leq \|hf - \Omega(gf) + (\Psi g)f - (\Psi \& g)f\|_{p_2} + \|gf\|_{p_1} \\
&\leq M(\|h - \Psi g\|_{q_2} + \|g\|_{q_1})\|f\|_s.
\end{aligned}$$

This shows that $S^{q_1} \oplus_\Psi S^{q_2}$ embeds into $Hom(S^s, X_\Omega)_B = Hom(S_0^s, X_\Omega)_B$. Finally, the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S^{q_2} & \longrightarrow & S^{q_2} \oplus_\Psi S^{q_1} & \longrightarrow & S^{q_1} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \| & & \\
0 & \longrightarrow & Hom(S^s, S^{p_2})_B & \longrightarrow & Hom(S^s, X_\Omega)_B & \longrightarrow & Hom(S^s, S^{p_1})_B & \longrightarrow & 0
\end{array}$$

provides the required isomorphism.

We focus on a different kind of transformation.

Proposition (4.1.10)[119]: Let $\Psi: S_0^{q_1} \to S^{q_2}$ be a right-centralizer, with $0 < q_1 \leq q_2 \leq \infty$. Take $0 < s < 1$ and let p_i be given by the identity $p_i^{-1} = q_i^{-1} + s^{-1}$ for $i = 1,2$. We define a mapping $\Psi^{(s)}: S_0^{p_1} \to S^{p_2}$ by

$$\Psi^{(s)}(h) = \Psi(u|h|^{p_1/q_1})|h|^{p_1/s},$$

where $u|h|$ is the polar decomposition of h. Then $\Psi^{(s)}$ is a right-centralizer.

Moreover, $\Psi^{(s)}$ is bounded if and only if Ψ is bounded.

Proof. Actually one can take $\Psi^{(s)}(h) = (\Psi f)g$ provided $h = fg$, with $M\|h\|_{p_1} \leq \|f\|_{q_1}\|g\|_s$.

We will show that $\Psi^{(s)}$ can be obtained as $(Hom(\Psi, S^\tau)_B,\ S^t)$ for suitable r and t. We pick any $r < q_1$ and then we take t so that $t^{-1} = s^{-1} + r^{-1}$. Applying $Hom(-, S^r)_B$ to Ψ we get a map$\Phi : S^{\ell_2} \to L(\mathcal{H})$ which is a left-centralizer from S^{ℓ_2} to S^{ℓ_1}, where $\ell_i^{-1} + q_i^{-1} = r^{-1}$ and satisfying an estimate

$$\|g(\Psi f) + (\Phi g)f\|_r \leq M\|g\|_{\ell_2}\|f\|_{q_1}, \left(g \in S^{\ell_2}, f \in S_0^{q_1}\right). \quad (13)$$

Now we apply $Hom_B(-, S^t)$ to Φ as follows (notice that $S^{p_i} = Hom_B(S^{\ell_1}, S^t)$ for $= 1,2$; in particular $t < \ell_2$). For each $h \in S^{p_1}$ we consider the map $g \in S^{\ell_2} \mapsto (\Phi g)h \in L(\mathcal{H})$. This is a left-centralizer from S^{ℓ_2} to S^t having constant proportional to $\|h\|_{p_1}$. Therefore there is $\Lambda(h) \in \mathcal{M}_B(S^{\ell_2}, L(\mathcal{H}))$ such that

$$\|g(\Lambda h)(g) + (\Phi g)h\|_t \leq M\|h\|_{p_1}\|g\|_{\ell_2}, \quad \left(h \in S^{p_1}, g \in S^{\ell_2}\right). \quad (14)$$

Even if we know no representation for arbitrary morphisms in $\mathcal{M}_B(S^{\ell_2}, L(\mathcal{H}))$ we claim that we may take $\Lambda(h)(g) = g(\Omega f)k$ provided $h = fk$ is the factorization appearing in the statement of the theorem. Indeed, by (13),

$$\|g(\Omega f)k + (\Phi g)h\|_t \leq M\|g\|_{\ell_2}\|f\|_{q_1}\|k\|_s \leq 2M\|g\|_{\ell_2}\|h\|_{p_1}$$

and we are done. The last statement obviously follows from the estimate (11).

As we mentioned, $Ext_B(S^p)$ is essentially independent on $p \in (0,1)$. Of course this follows from Theorem (4.1.9): indeed, if $p < q < 1$, then $Hom_B(S^s, -): Ext_B(S^p) \to Ext_B(S^q)$ is an isomorphism provided s is given by $p^{-1} = q^{-1} + s^1$. Let us record here the (right) centralizer version of this fact.

Corollary (4.1.11)[119]: Let the numbers $p, q, s \in (0, \infty)$ satisfy $p^{-1} = q^{-1} + s^{-1}$. Given a right centralizer $\Psi: S_0^q \to S^q$, we define $\Psi^{(s)}: S_0^p \to S^p$ by $\Psi^{(s)}(f) = (u|f|^{p/q})|f|^{p/s}$, where $u|f|$ is the polar decomposition of f. Then $\Psi^{(s)}$ is a right centralizer and every right centralizer on S_0^p is at finite distance from one obtained in this way.

Proof. Everything but the last part is a particular case of the preceding Proposition. Please note that $q < \infty$ is required here, while $q_2 = \infty$ was allowed in Proposition (4.1.10).
Let Ω be a right centralizer on S_0^p and let Ψ be any centralizer obtained by applying $Hom(S^s, -)_B$ to Ω. According to (12) we have $\|\Omega(gf) - (\Psi g)f\|_p \leq M\|g\|_q\|f\|_s$ for $g \in S^{q_2}$ and $f \in S_0^s$, from where is follows that $\Omega \approx \Psi^{(s)}$.

We describe the extensions of S^p by S^q, with $0 < p < q \leq \infty$, by means of the so-called twisted Hilbert spaces. These are self-extensions of $\mathcal{H}$ in the category of (quasi-) Banach spaces, that is, short exact sequences of (quasi-) Banach spaces and operators

$$0 \longrightarrow \mathcal{H} \longrightarrow T \longrightarrow \mathcal{H} \longrightarrow 0. \quad (15)$$

As a matter of fact, the middle space T must be (isomorphic to) a Banach space [133]. Moreover, T is itself isomorphic to a Hilbert space if and only if (15) splits. The existence of nontrivial twisted Hilbert spaces was first established by Enflo, Lindenstrauss, and Pisier [127]. Later on Kalton and Peck [139] constructed fairly concrete examples, among them the nowadays famous Kalton-Peck space Z_2.

As it is well-known, twisted Hilbert spaces are in correspondence with quasi-linear maps on $\mathcal{H}$, that is, homogeneous maps $\phi: \mathcal{H} \to \mathcal{H}$ satisfying an estimate of the form

$$\|\phi(x + y) - \phi(x) - \phi(y)\|_{\mathcal{H}} \leq Q(\|x\|_{\mathcal{H}} + \|x\|_{\mathcal{H}}) \qquad (x, y \in \mathcal{H}).$$

(We can replace the target space by a larger ambient space, or consider ϕ defined only on some dense subspace, or both. However, as linear spaces are free modules over the ground field, this is unnecessary to elaborate the theory.) See [120], [125], [137], [138].

Theorem (4.1.12)[119]: Let ϕ be a quasi-linear map on $\mathcal{H}$. We define a map $\tilde{\phi}$ on $\mathfrak{F}$ as follows. For each $f \in \mathfrak{F}$ we choose a Schmidt expansion $f = \sum_n s_n x_n \otimes y_n$ (homogeneously) and we put

$$\tilde{\phi}(f) = \sum_n s_n x_n \otimes (y_n). \tag{16}$$

Then $\tilde{\phi}: S_0^p \to S^q$ defines a right-centralizer whenever $0 < p < q \leq \infty$. Moreover, if $\Phi: S_0^p \to S^q$ is a right-centralizer, then $\Phi \approx \tilde{\phi}$ for some quasi-linear ϕ, where $\tilde{\phi}$ has the form given by (16).

Proof. "Homogeneously" means that if $f = \sum_n s_n x_n \otimes y_n$ is the Schmidt expansion attached to f and $\lambda \in \mathbb{C}$ is not zero, then the expansion for λf is $\sum_n |\lambda| s_n x_n \otimes |\lambda|^{-1} \lambda y_n$. This makes $\tilde{\phi}$ homogeneous.

Let us beging by checking the first part when $q = \infty$ so that $S^q = K$, the ideal of compact operators on $\mathcal{H}$. To this end, recall that an operator $u: X \to Y$ acting between (quasi-) Banach spaces is said to be p-nuclear $(0 < p < 1)$ if it admits a representation as

$$u = t_n x_n' \otimes y_n, \qquad (x_n' \in X', y_n \in Y) \tag{17}$$

with $\|x_n'\| = \|y_n\| = 1$ and (t_n) in ℓ^p. The class of all p-nuclear operators $X \to Y$ is denoted by $\mathfrak{N}^p(X, Y)$. The p-nuclear norm of u is then defined as the infimum of the (quasi-) norm in ℓ^p of the sequences (t_n) that can arise in (17). Notice that $S^p = \mathfrak{N}^p(\mathcal{H})$, with equal (quasi-) norms.

Now, let

$$0 \longrightarrow Y \overset{\iota}{\longrightarrow} X \overset{\pi}{\longrightarrow} Z \longrightarrow 0 \tag{18}$$

be an extension of quasi-Banach spaces and U another quasi-Banach space. Without loss of generality we assume $Y = ker\pi$. If we fix $0 < p < \infty$ and we apply $\mathfrak{N}^p(U, -)$ to the quotient map $\pi: X \to Z$ we obtain the operator $\pi_\circ: \mathfrak{N}^p(U, X) \longrightarrow \mathfrak{N}^p(U, Z)$ which is easily seen to be open.

Observe that $ker\,\pi_\circ$ consists of certain Y-valued compact operators. Moreover, if $u \in ker\,\pi_\circ$, then

$$\|u: U \longrightarrow Y\| = \|u: U \longrightarrow X\| \leq \|u\|_{\mathfrak{N}^p(U,X)},$$

so that the embedding $ker\,\pi_\circ \to K(U, Y)$ is continuous and we may form the push-out diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & ker\pi_\circ & \longrightarrow & \mathfrak{N}^p(U,X) & \overset{\pi_\circ}{\longrightarrow} & \mathfrak{N}^p(U,Z) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & K(U,Y) & \longrightarrow & PO & \longrightarrow & \mathfrak{N}^p(U,Y) & \longrightarrow & 0 \end{array} \tag{19}$$

We recall that if we are given an arbitrary push-out diagram of quasi-Banach spaces and operators

$$\begin{array}{ccccccccc} 0 & \longrightarrow & ker\varpi & \longrightarrow & A & \overset{\varpi}{\longrightarrow} & C & \longrightarrow & 0 \\ & & s\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & D & \longrightarrow & PO & \longrightarrow & C & \longrightarrow & 0 \end{array} \tag{20}$$

then, a quasi-linear map associated to the lower row can be constructed as follows: if $b: C \longrightarrow A$ is a (homogeneous) bounded selection for the quotient $\varpi: A \to C$ and $\ell: C \to A$ a linear (surely unbounded) selection, then the difference $b - \ell: C \to ker\,\varpi$ is associated to the upper extension in (20) and so $\sigma = s \circ (b - \ell): C \to D$ is the desired quasi-linear map. See [123] for the missing details.

This applies to the diagram (19) as follows. Suppose $X = Y \oplus_\phi Z$ arises from the quasi-linear map $\phi: Z \to Y$ and that $\phi = \beta - \lambda$, with $\beta: Z \to X$ homogeneous and bounded and $\lambda: Z \to X$ linear. Then, if $u \in \mathfrak{N}^p(U, Z)$ has finite rank and we choose (homogeneously) an expansion $u=\sum_n u_n^{'}\otimes z_n$ with finitely many summands and $\|u\|_p \geq (1+\varepsilon)(\sum\|u_n^{'}\|^p\|z_n\|^p)^{1/p}$ we may define

$$B(u)=\sum_n u_u^{'}\otimes\beta(z_n) \quad and \quad \Lambda(u)=\sum_n u_n^{'}\otimes\lambda(z_n).$$

Then B is homogeneous and bounded, $\Lambda = I_{U'}\otimes\lambda$ is linear and, therefore, we can define a quasi-linear map $\tilde{\phi}: \mathfrak{N}^p(U,Z)\to K(U,Y)$ taking

$$\tilde{\phi}(u)=B(u)-\Lambda(u)=\sum_n u_n^{'}\otimes(z_n), \tag{21}$$

at least when u has finite rank. Notice, moreover, that if $u=\sum_m v_m^{'}\otimes\zeta_m$ is another representation with $\|u\|_p\geq(1+\varepsilon)(\sum\|v_m^{'}\|^p\|\zeta_m\|^p)^{1/p}$, then

$$\left\|\tilde{\phi}(u)-\sum_m v_m^{'}\otimes\phi(\zeta_m)\right\|_{\mathfrak{N}^p(U,X)} = \left\|\sum_n u_n^{'}\otimes\beta(z_n)-\sum_m v_m^{'}\otimes\beta(\zeta_m)\right\|_{\mathfrak{N}^p(U,X)}$$
$$\leq 2(1+\varepsilon)2^{\frac{1}{p}-1}\|\beta\|\|u\|_{\mathfrak{N}^p(U,Z)}$$

(with the factor $2^{\frac{1}{p}-1}$ deleted if $p\geq 1$). Hence $\tilde{\phi}$ is essentially independent on the chosen representation of u.

Going back to the Schatten classes, consider the case where the starting extension (18) is the self-extension induced by ϕ, and we take $U=\mathcal{H}$ so that (19) becomes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & ker\,\pi_\circ & \longrightarrow & \mathfrak{N}^p(\mathcal{H},X) & \xrightarrow{\pi_\circ} & S^p & \longrightarrow & 0\\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & K & \longrightarrow & PO & \longrightarrow & S^p & \longrightarrow & 0 \end{array} \tag{22}$$

The preceding diagram lives in the category of quasi-Banach right modules over B (the multiplication in $\mathfrak{N}^p(\mathcal{H},X)$ given by composition on the right) and, according to (21) the map $\phi: S_0^p\to K$ given by $\tilde{\phi}(u)=\sum t_n x_n\otimes(y_n)$ is a right-centralizer inducing its lower row. Here, $u=\sum t_n x_n\otimes y_n$ is the Schmidt-expansion appearing in the statement of the Theorem and $X=\mathcal{H}\oplus_\phi\mathcal{H}$.

Notice that $\tilde{\phi}$ is essentially independent on the prescribed representations since any other choice yields a centralizer $S_0^p\to K$ at finite distance from $\tilde{\phi}$.

Next we prove that the map $\tilde{\phi}$ is still a right-centralizer when regarded as a map from S_0^p to S^q.

To this end we consider p and q fixed and take r so that $p^{-1}=q^{-1}+r^{-1}$. We already know that $\tilde{\phi}: S_0^r\to K$ is a centralizer. We introduce a second choice of the Schmidt expansions on S^r as follows.

Given a normalized $h\in S^r$ set $f=|h|^{r/p}$, so that if u is the phase of h, then $h=uf^{p/r}$, with f normalized in S^p. Now, if $uf=\sum_n s_n x_n\otimes y_n$ is the prescribed representation, we have

$$h=\sum_n s_n^{p/r}x_n\otimes y_n$$

and we can define a map $\Gamma: S_0^r\to K$ by the formula $\Gamma(h)=\sum_n s_n^{p/r}x_n\otimes\phi(y_n)$. This is in fact a centralizer and we even know that $\Gamma\approx\tilde{\phi}$.

Let us activate Proposition (4.1.10) with $s=q$ and $\Psi=\Gamma$ to conclude that if $u|f|$ is the polar decomposition of $f\in S_0^p$, then the formula

$$\Gamma^{(q)}(f) = \Phi(u|f|^{p/r})|f|^{p/q}$$
defines a centralizer from S_0^p to S^q. But $\Gamma^{(q)}$agrees with our old friend $\tilde{\phi}$. Indeed, if $f = \sum_n s_n x_n \otimes y_n$ is the prescribed representation of f, then $\Gamma(u|f|^{p/r}) = \sum_n s_n^{p/q} x_n \otimes \phi(y_n)$ and since $|f|^{p/q} = \sum_n s_n^{p/q} x_n \otimes x_n$ and $p(r^{-1} + q^{-1}) = 1$ we have
$$\Gamma^{(q)}(f) = \left(\sum_n s_n^{\frac{p}{r}} x_n \otimes \phi(y_n)\right)\left(\sum_n s_n^{\frac{p}{q}} x_n \otimes x_n\right) = \sum_n s_n x_n \otimes \phi(y_n) = \tilde{\phi}(f).$$
This completes the proof of the first part.

We finally prove the 'moreover' part. Let $\Phi: S_0^p \to S^q$ be a right-centralizer for which we may assume (and do) that $\Phi(fe) = \Phi(f)e$ for every $f \in S_0^p$ and every projection $e \in B$. Here, $p, q \in (0, \infty]$ are arbitrary; in particular we are not assuming $p < q$. Fixing a norm one $\eta \in \mathcal{H}$, we see that $\Phi(\eta \otimes y) = \eta \otimes \phi$ for some $\phi \in H$ depending on y (and η). Taking $\phi = \phi_\eta(y)$ we obtain a self-map on $\mathcal{H}$ which is easily seen to be quasi-linear. Let ζ be another normalized vector in $\mathcal{H}$ and define $\phi_\zeta$by the identity $\Phi(\zeta \otimes y) = \zeta \otimes \phi_\zeta(y)$. Let $u \in B$ be an isometry of $\mathcal{H}$ sending ζ to η, so that $(\eta \otimes y)u = u^*(\eta) \otimes y = \zeta \otimes y$. One has
$$\|\phi_\zeta(y) - \phi_\eta(y)\| = \|\eta \otimes (\phi_\zeta(y) - \phi_\eta(y))\|_q = \|\Phi((\eta \otimes y)u) - (\Phi(\eta \otimes y)u)\|_q$$
$$\le C[\Phi]\|\eta\|\|y\|\|u\|_B \le M\|y\|.$$
Therefore,$\phi_\eta \approx \phi_\zeta$ and so there is a quasi-linear map ϕ on $\mathcal{H}$ that could be properly called the spatial part of Φ and satisfies
$$\|\Phi(x \otimes y) - x \otimes \phi(y)\|_q \le M\|x\|\|y\| \quad (x, y \in \mathcal{H}). \tag{23}$$
We want to see that when $0 < p < q \le \infty$ one has $\Phi \approx \tilde{\phi}$ as long as (23) holds true. Clearly, we may and so assume that $\Phi(x \otimes y) = x \otimes \phi(y)$ for all $x, y \in \mathcal{H}$ and we must prove that if $f = \sum_n t_n x_n \otimes y_n$ is a Schmidt representation, then
$$\left\|\Phi(f) - \sum_n t_n x_n \otimes \phi(y_n)\right\|_q \le M\|f\|_p \tag{24}$$
for some constant M depending only on Φ, p and q.

Assume first $p < 1$. Then $(x_n \otimes y_n)$ is (isometrically) equivalent to the unit basis of ℓ^p and since S^q is an m-Banach space for $m = min(1, q)$ the bound (24) follows from an inequality due to Kalton [133] − indeed one may take
$$M = \left(\sum_{k=1}^{\infty} \left(\frac{2}{k}\right)^{m/p}\right)^{1/p} Q[\Phi].$$
Now, if $p \ge 1$ we can use Proposition (4.1.10) to lower Φ to a centralizer defined on $S^{1/2}$, say. So, take s such that $p^{-1} + s^{-1} = 2$ and let q' be given by $1/q' = p^{-1} + s^{-1}$. We know from Proposition (4.1.10) that the map $\Phi^{(s)}: S_0^{1/2} \to S^{q'}$ defined by
$$\Phi^{(s)}(h) = \Phi(u|h|^{\frac{1}{2p}})|h|^{\frac{1}{2s}}$$
is a right-centralizer. Here, $u|h|$ is the polar decomposition of h. Notice that for $\|x\| = \|y\| = 1$, the polar decomposition of $x \otimes y$ is $(x \otimes y)(x \otimes x)$ and since $(x \otimes x)^t = x \otimes x$ for all $t > 0$ we have
$$\Phi^{(s)}(x \otimes y) = (\Phi(x \otimes y))(x \otimes x) = (x \otimes \phi(y))(x \otimes x) = x \otimes \phi(y)$$

and $\Phi^{(s)} \approx \tilde{\phi}$ on $S^{1/2}$. On the other hand we know that $\tilde{\phi}^{(s)} \approx \tilde{\phi}$ on $S_0^{1/2}$ and thus, $\Phi^{(s)} - \tilde{\phi}^{(s)} \approx (\Phi - \tilde{\phi})^{(s)}$ is bounded as a map from $S_0^{1/2}$ to $S^{q'}$. The 'moreover' part of Proposition (4.1.10) now yields $\Phi \approx \tilde{\phi}$ on S_0^p.

Recall that a (complex) quasi-Banach space Z is said to be a K-space if every minimal extension (of quasi-Banach spaces) $0 \to \mathbb{C} \to X \to Z \to 0$ splits. Equivalently, if for every dense subspace Z_0 of Z and every quasi-linear map $\varphi: Z_0 \to \mathbb{C}$ there is a linear map $\ell: Z_0 \to \mathbb{C}$ such that $\operatorname{dist}(\varphi, \ell) < \infty$. The main examples of K-spaces were supplied by Kalton and coworkers: it turns out that ℓ^p (or L^p) is a K-space if and only if $p \in (0, \infty]$ is different from 1. See [145], [133], [146], [141]. In contrast to the commutative situation, one has:

Theorem (4.1.13)[119]: S^p is a K-space for no $p \in (0,1)$.

Proof. Letϕ be quasi-linear on $\mathcal{H}$and $\tilde{\phi} : S_0^p \to S^1$ the right centralizer given by Theorem (4.1.12).

Composing with $tr: S^1 \to \mathbb{C}$ we get a quasi-linear function $\varphi: S_0^p \to \mathbb{C}$ such that

$$\varphi(x \otimes y) = tr(\tilde{\phi}(x \otimes y)) = tr(x \otimes \phi(y)) = \langle \phi(y) | x \rangle.$$

Suppose there is a linear$\ell: S_0^p \to \mathbb{C}$ at fiinite distance from φ. As $\varphi(x \otimes y) \to 0$ for fixed y when $x \to 0$ in $\mathcal{H}$ we see that $\ell(x \otimes y) \to 0$ for fixed y when $x \to 0$ in $\mathcal{H}$ and by Lemma (4.1.2)(d) there is a
linear map L on $\mathcal{H}$ such that $\ell(x \otimes y) = \langle L(y) | x \rangle$. This obviously implies $dist(\phi, L) < \infty$. Starting with a non-trivial ϕ we get a non-trivial, minimal extension of S^p.

Of course S^1 is not a K-space as it contains a complemented subspace isomorphic to ℓ^1, while S^p is a K-space for $p \in (1, \infty)$, as all B-convex spaces are.

We finally add a result which partially answers a question raised by Kalton and Montgomery-Smith at the end of the survey [138].

Proposition (4.1.14)[119]: Let $\Phi: S_0^2 \to L(\mathcal{H})$ be a left centralizer from S_0^2 to S^2. Then the function $\varphi: S_0^1 \to \mathbb{C}$ given by

$$\varphi(f) = tr\left(u|f|^{1/2}\Phi(|f|^{1/2})\right) \tag{25}$$

is quasi-linear, where $u|f|$ is the polar decomposition of f. Every quasi-linear (complex) function on S_0^1 is at finite distance from one arising in this way.

Proof. Let us see the first part assuming that Φ takes values in S^2. An specialization ($q_1 = q_2 = s = 2$) of the obvious left version of Proposition (4.1.10) shows that the map$\Phi^{(2)}: S_0^2 \to S^1$ defined by $\Phi^{(2)}(f) = u|f|^{1/2}\Phi(|f|^{1/2})$ is a centralizer, hence a quasi-linear map. Since the trace is bounded and linear on S^1, the composition $\varphi = tr \circ \Phi^{(2)}$ is quasi-linear, too.

In any case, we know from Corollary (4.1.3) that there is a centralizer $\Psi: S_0^2 \to S^2$ that induces an extension equivalent to that induced by Φ. Hence there exist a morphism $\alpha: S_0^2 \to L(\mathcal{H})$ and a bounded homogeneous map $b: S_0^2 \to S^2$ such that $\Phi = \Psi + \alpha + b$. We have

$$\varphi(f) = tr\left(u|f|^{1/2}(|f|^{1/2})\right) + tr\left(u|f|^{1/2}\alpha(|f|^{1/2})\right) + tr\left(u|f|^{1/2}b(|f|^{1/2})\right).$$

We have just proved that the first summand in the right-hand side of the preceding equality is a quasi-linear function of f. The second one is linear since $u|f|^{1/2}\alpha(|f|^{1/2}) = \alpha(u|f|^{1/2}|f|^{1/2}) = \alpha(f)$. The third one is clearly bounded. Thus φ is itself quasi-linear.

As for the second one, let $\phi: S_0^1 \to \mathbb{C}$ be quasi-linear. Consider the map $S_0^2 \times S_0^2 \to \mathbb{C}$ sending (f, g) into $\phi(fg)$. For fixed $g \in S_0^2$, the function $f \longmapsto \phi(fg)$ is quasi-linear on S_0^2 , with constant at most $\|g\|_2 Q[\phi]$. But S^2 is a K-space and so there is a linear map $\ell_g: S_0^2 \to \mathbb{C}$ (depending on g) such that

$$\left|\phi(fg) - \ell_g(f)\right| \le K\|g\|_2 Q[\phi]\|f\|_2 \tag{26}$$
where $K \le 37$ is the "K-space constant" of S^2.
Next we want to see that $\ell(f) = tr(L \circ f) = tr(fL)$ for some $L \in L(\mathcal{H})$. According to Lemma (4.1.2)(d) it suffices to check that for each fixed $y \in \mathcal{H}$ one has $\ell(x \otimes y) \to 0$ as $x \to 0$ in $\mathcal{H}$. As (26) must hold, it suffices to verify that for fixed $g \in S_0^2$ and $y \in \mathcal{H}$ one has

$$\phi\big((x \otimes y)g\big) \to 0 \quad (x \to 0). \tag{27}$$

Write $g = \sum_{n=1}^{m} t_n x_n \otimes y_n$. Then

$$(x \otimes y)g = g^*(x) \otimes y = t_n \langle x|y_n\rangle x_n \otimes y.$$

As ϕ is quasi-linear we have the estimate (see [133])

$$\left|\phi\big((x \otimes y)g\big) - \sum_{n=1}^{m} t_n \langle x|y_n\rangle\, \phi(x_n \otimes y)\right| \le \sum_{n=1}^{m} |n t_n \langle x|y_n\rangle| \|x_n\| \|y\|$$

and (27) follows.
To sum up, there is homogeneous map $\Phi: S_0^2 \to L(\mathcal{H})$ such that

$$|\phi(fg) - tr(f\Phi(g))| \le M\|f\|_2\|g\|_2 \quad (f, g \in S_0^2). \tag{28}$$

Clearly, $\phi \approx \varphi$, where φ is given by (25). It only remains to check that Φ is a centralizer. Take $g, f \in S_0^2$, $a \in B$. We have:

$$\left|\phi\big(f(ag)\big) - tr(f\Phi(ag))\right| \le M\|f\|_2\|ag\|_2$$
$$\left|\phi\big((fa)g\big) - tr(fa\Phi(g))\right| \le M\|fa\|_2\|g\|_2,$$

so

$$\|\Phi(ag) - a\Phi(g)\|_2 = sup_{\|f\|_2 \le 1}|tr(f(\Phi(ag) - a\Phi(g))| \le M\|a\|_B\|g\|_2$$

and we are done.

A bicentralizer is just a left centralizer which is also a right centralizer. This amounts to modifying Definition (4.1.1) by requiring Z, Y and $\tilde{Y}$ to be bimodules and replacing the estimate in (b) by

$$\|\Omega(afb) - a\Omega(f)b\|_Y \le C_2\|a\|_A\|f\|_Z\|b\|_A \qquad (a, b \in A, f \in Z).$$

Bicentralizers on Schatten classes are the subject of [135] and [136]. It can be proved that every extension of quasi-Banach B-bimodules $0 \to S^q \to X \to S^p \to 0$ arises from a bicentralizer $\Omega: S_0^p \to S^q$ although we will refrain from entering into the details here. We draw some consequences of the results proved so far.

Theorem (4.1.15)[119]: Let $\Omega: S_0^p \to S^q$ be a bicentralizer, with $p \ne q$. Then there is $t \in \mathbb{C}$ such that $\|\Omega(f) - tf\|_q \le D\|f\|_p$ for some constant D independent on $f \in S_0^p$.

Proof. Case $q < p$. If $\Omega: S^p \to S^q$ is a bicentralizer, with q finite for which we may assume it preserves both left and right supports, then given a finite rank projection $e \in B$ we have that Ω maps eS^pe to eS^qe, as a bicentralizer over eBe. Proceeding as in Lemma (4.1.4) we see that the distance from Ω to the space of bimodule morphisms $S_0^p \to S^q$ equals $sup_e \delta_e$, where δ_e is the distance from $\Phi: eS^pe \to eS^pe$ to the corresponding space of bimodule homomorphisms over the corner algebra eBe (they all given by multiplication by some constant) and e runs over all finite rank projections in B. After that one should consider the obvious version of Lemma (4.1.5) for bimodules using ultraproducts (instead of ultrapowers) of the families $(eS^pe)_e, (eS^qe)_e$ and the corresponding ultraproduct algebra $(eBe)_{\mathcal{U}}$. The remainder of the proof of Theorem (4.1.6) goes undisturbed to get the desired conclusion.

In case $q > p$, as Ω is a right-centralizer, we know from Theorem (4.1.12) that there is a quasi-linear map ϕ on $\mathcal{H}$ such that $\|\Omega(x\otimes y) - x\otimes\phi(y)\|_q \le M\|x\|\|y\|$ for some M independent on $x, y \in \mathcal{H}$. But Ω is also a left centralizer and so $\left\|\Omega\big(a(x\otimes y)\big) - a\Omega(x\otimes y)\right\|_q \le M\|x\|\|y\|$, which yields

$$\|x\otimes\phi(ay) - x\otimes a\phi(y)\|_q = \|x\|\|\phi(ay) - a\phi(y)\| \\ \le M\|a\|_B\|x\|\|y\| \ (a \in B, x, y \in \mathcal{H}).$$

As $\{ay : \|a\|_B \le 1\}$ is the ball of radius $\|y\|$ in $\mathcal{H}$ we see that ϕ is bounded and so is Ω.

As for "self-bicentralizers" on S^p, we have the following extension of a result by Kalton.

Theorem (4.1.16)[119]: Let $\phi: \ell_0^p \to \ell^p$ be a symmetric centralizer over ℓ^∞, with $p \in (0,1)$. Define a self map on S_0^p as follows. Given $f \in S_0^p$ choose a Schmidt expansion $f = \sum_n s_n x_n \otimes y_n$. Let $(t_n) = \phi((s_n))$ and put $\Phi f = \sum_n t_n x_n \otimes y_n$. Then $\Phi: S_0^p \to S^p$ is a bicentralizer. Moreover, every bicentralizer is at finite distance from one obtained in this way.

Proof. Symmetric means that there is a constant M such that $|\phi(f\circ\sigma) - \phi(f)\circ\sigma|_p \le M|f|_p$ for every $f \in \ell_0^p$ whenever σ is a bijection of $\mathbb{N}$.

The proof is based on the following four facts:

(i) The statement holds for $p > 1$ as proved by Kalton in [136].

(ii) Corollary (4.1.11) is true replacing right centralizer by bicentralizer everywhere.

(iii) The commutative version of Corollary (4.1.11) holds: let $p, q, s \in (0, \infty)$ satisfy $p^{-1} = q^{-1} + s^{-1}$ and let $\omega: \ell_0^p \to \ell^q$ be a centralizer over ℓ^∞, where ℓ_0^q stands for the finitely supported sequences in ℓ^q. Define $\omega^{(s)} : \ell_0^p \to \ell^p$ taking$\omega^{(s)}(f) = \omega(u|f|^{p/q})f^{p/s}$, where u is the signum of f. Then $\omega^{(s)}$ is a centralizer and every centralizer on ℓ_0^p is at finite distance from one obtained in this way.

(iv) Referring to the preceding statement, $\omega^{(s)}$ is symmetric if and only if ω is.

Now, let ϕ be a symmetric ℓ^∞-centralizer on ℓ_0^p, where $p \le 1$. By (iii) and (iv), there is a symmetric centralizer ω on ℓ_0^2 such that $\phi \approx \omega^{(s)}$, where $p^{-1} = 2^{-1} + s^{-1}$ and we may assume $\phi = \omega^{(s)}$. Applying (i) to ω we can extend it to a B-bicentralizer $\Omega : S_0^2 \to S^2$ just taking

$$\Omega(f) = \sum_n t_n x_n \otimes y_n\,,$$

where $s_n x_n \otimes y_n$ is the prescribed Schmidt expansion of f and $\omega((s_n)) = (t_n)$. Finally, applying Corollary (4.1.11) to Ω with $q = 2$ one realizes that $\Phi = \Omega^{(s)}$ from where it follows that Φ is a bicentralizer.

The "moreover" part follows from the case $p = 2$ and the "moreover" part of Corollary (4.1.11).

Section (4.2): Twisting Schatten Classes

Nigel J. Kalton proved in [136] that it is possible to twist the Schatten classes, meaning that there exists a $B(H)$-module, namely Θ_p, containing a non-complemented copy of S_p-the corresponding Schatten class-such that the quotient is again S_p. Although not explicitly stated in this way, [136] contains this fact and much more. For example, the same theorem contains the statement that every bicentralizer on the Schatten classes arises as a derivation, which is a very deep result. The proof of all these facts requires heavy machinery, of course. A natural question about Θ_p is to identify its dual. The answer is implicit in the works of Kalton by juxtaposition of [136] and some results in [134]. So, one may conclude that $\Theta_p^* = \Theta_q$ for conjugated p, q although, as far as we know, it has never been explicitly stated. However, the necessary proofs to conclude it are not easy to follow. To find the precise form of the duality, one needs to go inside the proof of [139] and combine it again with [136]. We provide a direct computation of the dual so one can skip hard proofs and see explicitly how duality is working. A second comment related to this construction is whether Θ_p is an extremal twisting. Precisely, is the natural quotient map $\Theta_p \to S_p$ strictly singular? Recall that an operator is said to be strictly singular if it is never an isomorphism when restricted to an infinite dimensional subspace. This is equivalent to saying that the corresponding bicentralizer is never trivial when restricted to an infinite dimensional subspace. We show that Θ_p is extremal in the category of $B(H)$-modules. That is, the quotient $map\ \Theta_p \to S_p$ is not an isomorphism when restricted to an infinite dimensional $B(H)$-submodule, say V , if and only if

$$\max\{rk(T): T \in V\} = +\infty,$$

where, by $rk\ (T)$, we denote the rank of T. As far as we know, this result is new. Thus, the aim is to simplify and clarify a couple of points‑duality and singularity‑for Kalton twisting of Schatten classes. Let us sketch briefly the main definitions necessary.

Definition (4.2.1)[149]: Let Z and Y be quasi-normed modules over the Banach algebra A, and let $\tilde{Y}$ be another module containing Y in the purely algebraic sense. A bicentralizer from Z to Y with ambient space $\tilde{Y}$ is a homogeneous mapping $\Omega: Z \to \tilde{Y}$ having the following properties.

(a) It is quasi-linear, that is, there is a constant Q so that if $f, g \in Z$, then $\Omega(f+g) - \Omega(f) - \Omega(g) \in Y$ and $\| \Omega(f+g) - \Omega(f) - \Omega(g) \|_Y \leq Q(\| f \|_Z + \| g \|_Z)$.

(b) There is a constant C so that if $a, b \in A$ and $f \in Z$. Then $\Omega(afb) - a\Omega(f)b \in Y$ and

$$\| \Omega(afb) - a\Omega(f)b \|_Y \leq C \| a \|_A \| f \|_Z \| b \|_A.$$

We now indicate the connection between bicentralizers and extensions. Let Z and Y be quasi-Banach modules and $\Omega: Z \to \tilde{Y}$ is a bicentralizer from Z to . Then

$$Y \oplus_\Omega Z = \{(g, f) \in \tilde{Y} \times Z: g - \Omega f \in Y\}$$

is a linear subspace of $\tilde{Y} \times Z$, and the functional $\| (g, f) \|_\Omega = \| g - \Omega f \|_Y + \| f \|_Z$ is a quasi-norm on it. Moreover, the map $i: Y \to Y \oplus_\Omega Z$ sending g to $(g, 0)$ preserves the quasi-norm, while the map $p: Y \oplus_\Omega Z \to Z$ given as $p(g, f) = f$ is open, so that we have a short exact sequence of quasi-normed spaces:

$$0 \longrightarrow Y \xrightarrow{i} Y \oplus_\Omega Z \xrightarrow{p} Z \longrightarrow 0$$

with relatively open maps. This already implies that $Y \oplus_\Omega Z$ is complete, i.e., a quasi-Banach space. Actually, only quasi-linearity (a) is necessary here. The estimate in (b) implies that the multiplication $a(g, f)b = (agb, afb)$ makes $Y \oplus_\Omega Z$ into a quasi-Banach bimodule over A in such a way that the arrows in the exact sequence become homomorphisms. We say that Ω induces a trivial extension if and only if $\| \Omega(f) - h(f) \|_Y \leq K \| f \|_Z$ for some morphism $h: Z \to \tilde{Y}$. In this case, we say that Ω is a trivial bicentralizer. In particular, if h is a morphism of A-modules, we say that Ω is a A-trivial bicentralizer. In our setting, both notions agree. The following lemma is known to specialists in twisted sums:

Lemma (4.2.2)[149]: If a bicentralizer Ω from S_p to S_p with $1 < p < \infty$(and ambient space $B(H)$) is trivial, then it is $B(H)$-trivial.

Details of the proof can be found in [119].

Kalton twisting of Schatten classes is done by constructing a nontrivial bicentralizer. The precise one, called Kalton bicentralizer and denoted by $\Omega_p: S_p \to B(H)$, is defined as follows: given an operator $T \in S_p$ with spectral form $T = \Sigma a_i(T) f_i \otimes e_i$,

$$\Omega_p(T) := \sum a_i(T)\ \log\left(\frac{\| T \|_{S_p}}{a_i(T)}\right) f_i \otimes e_i.$$

We are ready to prove the singularity of the Kalton bicentralizer:

Proposition (4.2.3)[149]: Let V be a $B(H)$-submodule of S_p for $1 < p < \infty$.
The following conditions are equivalent:
(i) The restriction of Ω_p to V is not trivial.
(ii) $max\{rk\ (T): T \in V\} = +\infty$.

Proof. We prove (ii) implies (i): Let $T \in V$ be a norm one operator with spectral representation $T = \sum_{i=1}^{N} a_i(T) f_i \otimes e_i$. It is possible to find for every i an operator $P_i \in B(H)$ that $P_i(T) = f_i \otimes e_i$. Assume that (i) does not hold and pick, by Lemma (4.2.2), a morphism of $B(H)$-modules $\Lambda: V \to S_p$ such that $\| \Omega_p - \Lambda \| \le K$. Since $\Omega_p(f_i \otimes e_i) = 0$, then $\| \Lambda(f_i \otimes e_i) \| \le K$ and, since Λ is a morphism of $B(H)$-modules, $\Lambda(f_i \otimes e_i) = \varphi_i \otimes e_i$.

Claim A.

$$\mathbb{E}\left\|\Lambda\left(\sum_{i=1}^{N} r_i f_i \otimes e_i\right)\right\|_{S_p} = \mathbb{E}\left\|\sum_{i=1}^{N} r_i \varphi_i \otimes e_i\right\|_{S_p} \le CKN^{1/p}$$

holds for $1 \le p < \infty$ where C is a universal constant depending at most on p.
Once the claim is proved we will find that, on the other hand,

$$\mathbb{E}\left\|\Omega_p\left(\sum_{i=1}^{N} r_i f_i \otimes e_i\right)\right\|_{S_p} \overset{(a)}{=} \mathbb{E}\left\|logN^{1/p}\sum_{i=1}^{N} r_i f_i \otimes e_i\right\|_{S_p}$$

$$\overset{(b)}{=} N^{1/p}\ logN^{1/p}$$

making $\| \Omega_p - \Lambda \| < +\infty$ impossible for all $T \in V$ if (ii) holds. To check the last equalities (a) and (b) displayed above, notice that for any $t \in [0, 1]$ the operator $T_t = \sum_{i=1}^{N} r_i(t) f_i \otimes e_i$ is a diagonal operator. Since $r_i(t) = \pm 1$, it follows that $\{\pm 1 f_i\}$ is still an orthonormal basis. Thus, $a_i(T_t) = 1$, and consequently, $\| T_t \|_p = N^{1/p}$. And now, one just needs to apply the definition of Ω_p to every T_t to obtain

$$\Omega_p(T_t) = \sum a_i(T_t)\ \log\left(\frac{\| T_t \|}{a_i(T_t)}\right) r_i(t) f_i \otimes e_i$$
$$= \sum logN^{1/p} r_i(t) f_i \otimes e_i\,.$$

Finally, one just needs to integrate to obtain equality (a). Equality (b) is immediate by the definition of norm in S_p. We are ready to prove Claim A.

Proof. Consider $\varphi_i = \sum_j a_{ij} e_j$, and thus $\sum_{i=1}^{N} \varphi_i \otimes e_i = \sum_{ij} a_{ij} e_j \otimes e_i$. We know that

$$\left(\sum_j a_{ij}^2\right)^{1/2} = \left\|\sum_j a_{ij} e_j \otimes e_i\right\| = \|\Lambda(f_i \otimes e_i)\| \le K.$$

Then, to prove Claim A, it is enough to show that

$$\mathbb{E}\left\|\sum_{ij} a_{ij} r_i e_j \otimes e_i\right\|_{S_p} \leq C\left(\sum_{i=1}^{N}\left\|\sum_j a_{ij} e_j \otimes e_i\right\|_{S_p}^p\right)^{1/p}$$

for $1 \leq p < \infty$ and some universal constant C depending at most on p.
For $1 \leq p \leq 2$, the result follows by noting that the corresponding S_p has type p, so we just have to deal with the case $2 \leq p < \infty$. We need a tool slightly better than the type, namely, the noncommutative version of the Khintchine inequality for Schatten classes. More precisely, in [153], it was proved that, for $2 \leq p < \infty$, the following holds:

(∗∗) $$\mathbb{E}\left\|\sum_{i=1}^N r_i A_i\right\|_{S_p} \approx \left\|\left(\sum_{i=1}^N A_i^* A_i\right)^{1/2}\right\|_{S_p} + \left\|\left(\sum_{n=1}^N A_i A_i^*\right)^{1/2}\right\|_{S_p}.$$

In our case, set $A_i := \sum_j a_{ij} e_j \otimes e_i$. It is clear, since A_i is a row matrix, that $A_i^* A_i = A_i A_i^* = \sum_j a_{ij}^2 e_i \otimes e_i$ and

$$\left(\sum_j a_{ij}^2\right)^{1/2} = \| A_i \|_{S_p} \leq K.$$

Then the right side of (∗∗) turns into

$$2\left\|\left(\sum_{i=1}^N A_i^* Ai\right)^{1/2}\right\|_{S_p} = 2\left\|\left(\sum_{i=1}^N \left(\sum_j a_{ij}^2\right) e_i \otimes e_i\right)^{1/2}\right\|_{S_p}$$

$$= 2\left\|\sum_{i=1}^N \left(\sum_j a_{ij}^2\right)^{1/2} e_i \otimes e_i\right\|_{S_p}$$

$$= 2\left\|\sum_{i=1}^N \| A_i \| e_i \otimes e_i\right\|_{S_p}$$

$$= 2\left(\sum_{i=1}^N \| A_i \|^p\right)^{1/p} \leq 2KN^{1/p},$$

and Claim A is proved. To prove (i) implies (ii); assume (ii) does not hold and pick $T \in V$ of finite rank. We claim now that the least constant $c(T)$ making

$$\| \Omega_p(T) \|_p \leq c(T) \| T \|_p$$

is exactly $c(T) = log(rk\ (T)^{1/p})$, and thus, since (ii) does not hold,

$$sup_{T \in V} c(T) < \infty.$$

This last result means that $\Omega_{p|V}$ is trivial, and the proof is done. So we just need to prove our claim.

This claim can be found in [150] under a more general form. Let us reproduce the argument for the sake of completeness. First, observe that Ω_p is a homogeneous map in the sense

$$\Omega_p(\lambda T) = \lambda \Omega_p(T)$$

with $\lambda \in \mathbb{K}$ and $T \in S_p$. To prove it, notice that, by writing $T \in S_p$ in spectral form and using a similar argument as in the previous proof for the orthonormal basis, one has: $a_i(\lambda T) =$

$|\lambda|a_i(T)$. Now by putting the definition of $\Omega_p(\lambda T)$ and comparing it with $\lambda\Omega_p(T)$, it follows that Ω_p is homogeneous. Thus, since the expression $\| \Omega_p(T) \|_p \le c(T) \| T \|_p$ is homogeneous, one may assume that $T = a_i(T) f_i \otimes e_i$ is norm one, i.e., $\sum_{i=1}^N a_i(T)^p = 1$. It only remains to prove the following.

Claim B.

$$\sup\left\{\left(\sum_{i=1}^{N} |a_i|^p \, (-log\,|a_i|)^p\right)^{1/p} : \sum_{i=1}^{N} |a_i|^p = 1\right\} = logN^{1/p},$$

where $1 < p < \infty$.

Proof. To compute the supremum, we use Lagrange' s multiplier theorem. Thus, we write

$$\Lambda(a_i, \lambda) = \sum_{i=1}^{N} |a_i|^p \, (-log\,|a_i|)^p + \lambda\left(1 - \sum_{i=1}^{N} |a_i|^p\right).$$

There is no loss of generality to assume that $0 < a_i < 1$ for all i. From $d\Lambda/da_i = 0$, we get $(-log\,a_i)^p - (-log\,a_i)^{p-1} = \lambda$. It is routine to check that the function $\psi(a) := (-log\,a)^p - (-log\,a)^{p-1}$ is injective, and thus $a_i = a_{i'}$ for $i, i' \in \{1, \dots, N\}$. Consequently, it must be $a_i = N^{-1/p}$ for $i = 1, \dots, N$ and Claim B is proved.

This result is, in a sense, optimal. It was proven by Kalton and Peck that the map Ω_p from ℓ_p to ℓ_p (with ambient space ℓ_∞) is singular for $1 < p < \infty$, which means that its restriction to any infinite dimensional subspace is not trivial.

We make the analogue proof of [139]. The crucial step in the proof is the following trivial inequality.

Lemma (4.2.4)[149]: The following expression holds:

$$\left|ts\left(log\frac{|t|}{|s|^{p-1}}\right)\right| \le \frac{p-1}{e}(|t|^q + |s|^p),$$

for $1 = 1/p + 1/q$ and $1 < p, q < \infty$.

This lemma corresponds to the case $n = 1$ of [139].

Theorem (4.2.5)[149]: There exists an isomorphism φ making the following diagram commute:

$$0 \longrightarrow S_q \xrightarrow{j} \Theta_q \xrightarrow{q} S_q \longrightarrow 0 \equiv \frac{1}{1-p}\Omega_q,$$

$$\downarrow tr \qquad \downarrow \varphi \qquad \downarrow tr$$

$$0 \longrightarrow (S_p)^* \xrightarrow{j^*} (\Theta_p)^* \xrightarrow{q^*} (S_p)^* \longrightarrow 0 \equiv (\Omega_p)^*,$$

where tr denotes trace duality and $1 = 1/p + 1/q$ for $1 < p, q < \infty$.

Proof. Let us denote by $\mathfrak{F}$ the space of finite rank operators acting between a Hilbert space and by $\mathfrak{F}_r$ when endowed with the corresponding S_r norm. Given $T \in \mathfrak{F}$ with spectral decomposition $T = \sum a_i(T) e_i \otimes f_i$, let us write by simplicity:

$$\Omega_p(T) = \sum a_i(T)\; log\left(\frac{\| T \|_{S_p}}{a_i(T)}\right) e_i \otimes f_i,$$

$$\Omega_q(T) = \frac{1}{1-p}\sum a_i(T)\; log\left(\frac{\| T \|_{S_p}}{a_i(T)}\right) e_i \otimes f_i.$$

We define the map $\varphi: \mathfrak{F}_q \oplus_{\Omega_q(T)} \mathfrak{F}_q \to \Theta_p^*$ by the formula

$$(\varphi(S, T))\,(V, W) = tr\,(SW) + tr\,(TV).$$

Let us rewrite the expression above as:

$$(*) \qquad tr(SW) + tr\,(TV\,) = tr\,(T(V - \Omega_p(W))) + tr\,(T\Omega_p(W)) + tr\,(\Omega_q(T)W) + tr\,((S - \Omega_q(T))W).$$

Setting $T = \sum a_i(T) e_i \otimes f_i$ and $W = \sum a_j(W) u_j \otimes v_j$, we easily get

$$T\Omega_p(W) = \sum a_j(W) a_i(T)\ log\left(\frac{\| W \|_{S_p}}{a_j(W)}\right)(e_i|v_j) u_j \otimes f_i,$$

$$\Omega_q(T)W = \frac{1}{1-p} \sum a_j(W) a_i(T)\ log\left(\frac{\| T \|_{S_p}}{a_i(T)}\right)(e_i|v_j) u_j \otimes f_i,$$

$$tr\,(T\Omega_p(W)) = \sum a_j(W) a_i(T)\ log\left(\frac{\| W \|_{S_p}}{a_j(W)}\right)(e_i|v_j)(f_i|u_j),$$

$$tr(\Omega_q(T)W) = \frac{1}{1-p} \sum a_j(W) a_i(T)\ log\left(\frac{\| T \|_{S_p}}{a_i(T)}\right)(e_i|v_j)(f_i|u_j).$$

We need to prove that expression $(*)$ is bounded. To this end, we write

$$|tr\,(SW) + tr\,(TV\,)| \le |tr\,(T(V - \Omega_p(W)))| + |tr\,(T\Omega_p(W)) + tr\,(\Omega_q(T)W)| + |tr\,((S - \Omega_q(T))W)|.$$

The quantities $|tr(T(V - \Omega_p(W)))|$ and $|tr((S - \Omega_q(T))W)|$ can be easily bounded. Assume for a moment that

$$|tr(T\Omega_p(W)) + tr\,(\Omega_q(T)W)| \le \| W \| \| T \|. \tag{29}$$

Then observe

$$\begin{aligned} |tr(SW) + tr\,(TV\,)| &\le |tr(T(V - \Omega_p(W)))| + |tr\,(T\Omega_p(W)) + tr\,(\Omega_q(T)W)| + |tr((S - \Omega_q(T))W)| \\ &\le \| (V - \Omega_p(W) \|_{S_p} \| T \|_{S_p} + \| W \|_{S_p} \| T \|_{S_p} + \| W \|_{S_q} \| (S - \Omega_q(T) \|_{S_p} \\ &\le \| (S,T) \|_{\Theta_q} \| (V,W) \|_{\Theta_p}. \end{aligned}$$

Therefore, the expression $(*)$ defines a bounded operator. So all the rest is to convince us that the bound (29) holds for arbitrary $T, W \in \mathfrak{F}$. To this end, we may bound the left side of expression (29) by:

$$\begin{aligned} (**) &= \left| \sum a_j(W) a_i(T) (e_i|v_j)(f_i|u_j)\ log\left(\frac{\| W \|_{S_p}}{a_j(W)}\right) + \frac{1}{1-p} \sum a_j(W) a_i(T)\,(e_i|v_j)(f_i|u_j)\ log\left(\frac{\| T \|_{S_p}}{a_i(T)}\right) \right| \\ &= \left| \sum a_j(W) a_i(T)\,(e_i|v_j)(f_i|u_j)\ log\left(\frac{\|W\|_{S_p}}{a_j(W)} \left(\frac{a_i(T)}{\|T\|_{S_p}}\right)^{1/p-1}\right) \right| \\ &= \frac{1}{p-1} \left| \sum a_j(W) a_i(T)\,(e_i|v_j)(f_i|u_j) \times log\left(\left(\frac{\| W \|_{S_p}}{a_j(W)}\right)^{p-1} \frac{a_i(T)}{\| T \|_{S_p}} \right) \right| \\ &\le \frac{1}{p-1} \sum_{i,j} \left| a_j(W) a_i(T) (e_i|v_j)(f_i|u_j) \times log\left(\left(\frac{\| W \|_{S_p}}{a_j(W)}\right)^{p-1} \frac{a_i(T)}{\| T \|_{S_p}} \right) \right| \\ &\le \frac{1}{p-1} \left(\sum_{i,j} |c_{i,j}\,| |(e_i|v_j)|^2 \right)^{1/2} \left(\sum_{i,j} |c_{i,j}\,| |(f_i|u_j)|^2 \right)^{1/2}, \end{aligned}$$

where

$$c_{i,j} := a_j(W)a_i(T)\, log\left(\left(\frac{\| W \|_{S_p}}{a_j(W)}\right)^{p-1} \frac{a_i(T)}{\| T \|_{S_p}}\right).$$

Now recall that, by Lemma (4.2.4),

$$c_{i,j} \leq \frac{p-1}{e} \| W \| \| T \| \left(\frac{a_i(T)^q}{\| T \|^q} + \frac{a_j(W)^p}{\| W \|^p}\right).$$

Thus, we find that the last expression $(**)$ is bounded by

$$\frac{\| W \| \| T \| \frac{p-1}{e}}{p-1}\left(\sum_{i,j}\left(\frac{a_i(T)^q}{\| T \|^q} + \frac{a_j(W)^p}{\| W \|^p}\right)|(e_i|v_j)|^2\right)^{1/2}$$

$$\left(\sum_{i,j}\left(\frac{a_i(T)^q}{\| T \|^q} + \frac{a_j(W)^p}{\| W \|^p}\right)|(f_i|u_j)|^2\right)^{1/2}.$$

To finish, let us observe the following upper bounds for the last expression:

$$\frac{\| W \| \| T \|}{e}\left(\sum_{i,j}\frac{a_i(T)^q}{\| T \|^q}|(e_i|v_j)|^2 + \sum_{i,j}\frac{a_j(W)^p}{\| W \|^p}|(e_i|v_j)|^2\right)^{1/2}$$

$$\left(\sum_{i,j}\frac{a_i(T)^q}{\| T \|^q}|(f_i|u_j)|^2 + \sum_{i,j}\frac{a_j(W)^p}{\| W \|^p}|(f_i|u_j)|^2\right)^{1/2}$$

$$\leq \frac{\| W \| \| T \|}{e}\left(\sum_{i}\frac{a_i(T)^q}{\| T \|^q}\sum_j|(e_i|v_j)|^2 + \sum_{j}\frac{a_j(W)^p}{\| W \|^p}\sum_i|(e_i|v_j)|^2\right)^{1/2}$$

$$\left(\sum_{i}\frac{a_i(T)^q}{\| T \|^q}\sum_j|(f_i|u_j)|^2 + \sum_{j}\frac{a_j(W)^p}{\| W \|^p}\sum_i|(f_i|u_j)|^2\right)^{1/2}$$

$$\leq \frac{\| W \| \| T \|}{e}\left(\sum_{i}\frac{a_i(T)^q}{\| T \|^q} + \sum_{j}\frac{a_j(W)^p}{\| W \|^p}\right) = \frac{2}{e}\| W \| \| T \|.$$

This last result means that φ is bounded and can be extended to a bounded map. Clearly, φ makes the diagram of Theorem (4.2.5) commute so, by the 3-lemma, [125], it is an isomorphism.

Corollary (4.2.6)[149]: Θ_p^* is isomorphic to Θ_q for $1/p + 1/q = 1$.

Section (4.3): Non-Commutative L^p Spaces

We make the first steps into the study of extensions of noncommutative L^p-spaces. An extension (of Z by Y) is a short exact sequence of Banach spaces and (linear, continuous) operators

$$0 \longrightarrow Y \longrightarrow X \longrightarrow Z \longrightarrow 0. \tag{30}$$

This essentially means that X contains Y as a closed subspace so that the corresponding quotient is (isomorphic to) Z.

We believe that the convenient setting in studying extensions of L^p-spaces is not that of Banach spaces, but that of Banach modules over the underlying von Neumann algebra M. Accordingly, one should require the arrows in (30) to be homomorphisms.

It is remarkable and perhaps a little ironic that, while the study of the module structure of general L^p-spaces goes back to its inception, the only where one can find some relevant information about extensions, namely [151] and [136], deliberately neglected this point.

We deal with the tracial (semifinite) case. It is shown that whenever one has a reasonably "symmetric" self-extension of the commutative L^p (the usual Lebesgue space of p-integrable functions on the line) one can get a similar self-extension

$$0 \longrightarrow L^p(M,\tau) \longrightarrow X \longrightarrow L^p(M,\tau) \longrightarrow 0$$

of bimodules over any semifinite von Neumann algebra M, equipped with a trace τ.
Our approach combines Kalton's description of extensions by centralizers (these are certain maps which are, in general, neither linear nor bounded) with a general principle, due to Rochberg and Weiss that we can express by saying that whenever one finds a given Banach space Y as an intermediate space in a (complex) interpolation scale, one automatically gets a self-extension $0 \to Y \to X \to Y \to 0$.
Thus for instance, considering $L^p(M,\tau)$ as an interpolation space between M and its predual M_* one arrives at a certain self-extension of $L^p(M,\tau)$ that we regard as a kind of noncommutative Kalton-Peck space. Some interesting properties of these spaces are presented.

We consider L^p-spaces over general (but σ-finite) algebras, including those of type III. In this case the interpolation trick still works but produces two (rather than one) extensions of one sided modules, one of left-modules and the other of right-modules. Whether or not one can find (nontrivial) self-extensions of bimodules in all cases is left open.

For A be a Banach algebra. A quasi-Banach (left) moduFlore over A is a quasi-Banach space X together with a jointly continuous outer multiplication $A \times X \to X$ satisfying the traditional algebraic requirements.
An extension of Z by Y is a short exact sequence of quasi-Banach modules and homomorphisms

$$0 \longrightarrow Y \xrightarrow{i} X \xrightarrow{\pi} Z \longrightarrow 0. \tag{31}$$

The open mapping theorem guarantees that $\imath$ embeds Y as a closed submodule of X in such a way that the corresponding quotient is isomorphic to Z. Two extensions $0 \to Y \to X_i \to Z \to 0$ $(i = 1, 2)$ are said to be equivalent if there exists a homomorphism u making commutative the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X_1 & \longrightarrow & Z & \longrightarrow & 0 \\ & & \| & & \downarrow u & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & X_2 & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

By the five-lemma [130], and the open mapping theorem, u must be an isomorphism. We say that (31) splits if it is equivalent to the trivial sequence $0 \to Y \to Y \oplus Z \to Z \to 0$. This just means that Y is a complemented submodule of X, that is, there is a homomorphism $X \to Y$ which is a left inverse for the inclusion $Y \to X$; equivalently, there is a homomorphism $Z \to X$ which is a right inverse for the quotient $X \to Z$.
Operators and homomorphisms are assumed to be continuous. Otherwise we speak of linear maps and "morphisms".
Taking $A = \mathbb{C}$ one recovers extensions in the Banach space setting.
Every extension of (quasi-) Banach modules is also an extension of (quasi-) Banach spaces. Clearly, if an extension of modules is trivial, then so is the underlying extension of (quasi-) Banach spaces. Simple examples show that the converse is not true in general. A Banach

algebra A is amenable if every extension of Banach modules (31) in which Y is a dual module splits as long as it splits an extension of Banach spaces. This is not the original definition but an equivalent condition. The original definition reads as follows: A is amenable if every continuous derivation from A into a dual bimodule is inner. Here "derivation" means "operator satisfying Leibniz's rule" and has nothing to do with the derivations appearing.

Every Banach space is a quasi-Banach space and it is possible that the middle space X in (31) is only a quasi-Banach space even if both Z and Y are Banach spaces (see [152]). This will never occur, among other things because X will invariably be a quotient of certain Banach space of holomorphic functions. Anyway, Kalton proved in [133] that if Z has nontrivial type $p > 1$ and Y is a Banach space, then X must be locally convex and so isomorphic to a Banach space. In particular, any quasi-norm giving the topology of X must be equivalent to a norm, and hence to the convex envelope norm. If Z is super-reflexive the proof is quite simple; see [121].

We introduce the main tool in the study of extensions.

Definition (4.3.1)[154]: Let Z and Y be quasi-normed modules over the Banach algebra A and let $\tilde{Y}$ be another module containing Y in the purely algebraic sense. A centralizer from Z to Y with ambient space $\tilde{Y}$ is a $\mathbb{C}$-homogeneous mapping $\Omega: Z \to \tilde{Y}$ having the following properties.

(a) It is quasi-linear, that is, there is a constant Q so that if $f, g \in Z$, then $\Omega(f + g) - (f) - (g) \in Y$ and $\|\Omega(f+g) - \Omega(f) - \Omega(g)\|_Y \leq Q(\|f\|_z + \|g\|_z)$.

(b) There is a constant C so that if $a \in A$ and $f \in Z$, then $\Omega(af) - a\Omega(f) \in Y$ and $\|\Omega(af) - a\Omega(f)\|_Y \leq C\|a\|_A\|f\|_z$.

We denote by $Q[\Omega]$ the least constant for which (a) holds and by $C[\Omega]$ the least constant for which (b) holds.

We now indicate the connection between centralizers and extensions. Let Z and Y be quasi-Banach modules and $\Omega: Z \to \tilde{Y}$ is a centralizer from Z to Y. Then

$$Y \oplus_\Omega Z = \{(g, f) \in \tilde{Y} \times Z : g - \Omega f \in Y\}$$

is a linear subspace of $\tilde{Y} \times Z$ and $\|(g,f)\|_\Omega = \|g - \Omega f\|_Y + \|f\|_Z$ is a quasi-norm on it (here is the only point where the assumption about the homogeneity of Ω is used). Moreover, the map $\iota: Y \to Y \oplus_\Omega Z$ sending g to $(g, 0)$ preserves the quasi-norm, while the map $\pi: Y \oplus_\Omega Z \to Z$ given as $\pi(g,f) = f$ is open, so that we have a short exact sequence of quasi-normed spaces

$$0 \longrightarrow Y \xrightarrow{\iota} Y \oplus_\Omega Z \xrightarrow{\pi} Z \longrightarrow 0 \tag{32}$$

with relatively open maps. This already implies that $Y \oplus_\Omega Z$ is complete, i.e., a quasi-Banach space. Actually only the quasi-linearity of Ω is necessary here. The estimate in (b) implies that the multiplication $a(g,f) = (ag, af)$ makes $Y \oplus_\Omega Z$ into a quasi-Banach module over A in such a way that the arrows in (32) become homomorphisms. Indeed,

$$\|a(g,f)\|_\Omega = \|ag - \Omega(af)\|_Y + \|af\|_Z = \|ag - a\Omega f + a\Omega f - \Omega(af)\|_Y + \|af\|_Z$$
$$\leq M\|a\|_A\|(g,f)\|_\Omega.$$

We will always refer to Diagram 3 as the extension (of Z by Y) induced by Ω.

It is easily seen that two centralizers Ω and Φ (acting between the same sets, say Z and $\tilde{Y}$) induce equivalent extensions if and only if there is a morphism $h: Z \to \tilde{Y}$ such that $\|\Omega(f) - \Phi(f) - h(f)\|_Y \leq K\|f\|_Z$. If the preceding inequality holds for $h = 0$ we say that Ω and Φ are equivalent and we write $\Omega \approx \Phi$. In particular Ω induces a trivial extension if

and only if $\|\Omega(f)-h(f)\|_Y \le K\|f\|_Z$ for some morphism $h: Z \to \tilde{Y}$. In this case we say that Ω is a trivial centralizer.
The corresponding definitions for right modules and bimodules are obvious. Thus, for instance, we define bicentralizers from Z to Y (which are now assumed to be Banach bimodules over the Banach algebra A) by requiring $\tilde{Y}$ to be also a bimodule and replacing the estimate in Definition (4.3.1)(b) by

$$\|\Omega(azb)-a\Omega(z)b\|_Y \le C\|a\|_A\|f\|_Z\|b\|_A \qquad (a,b\in A, z\in Z).$$

We insist that we are interested in the case of Banach spaces here, so one can assume Z and Y to be Banach spaces. However, the Ribe function $\|\cdot\|_\Omega$ will be only a quasi-norm on $Y\oplus_\Omega Z$, even if it is equivalent to a true norm. See the paragraph closing and [125].

The push-out construction appears naturally when one considers two operators defined on the same space. Given operators $\alpha: Y\to A$ and $\beta: Y \to B$, the associated push-out diagram is

$$\begin{array}{ccc} Y & \xrightarrow{\alpha} & A \\ \beta\downarrow & & \downarrow\beta' \\ B & \xrightarrow{\alpha'} & PO \end{array} \tag{33}$$

Here, the push-out space $PO=PO(\alpha,\beta)$ is the quotient of the direct sum $A\oplus B$ (with the sum norm, say) by S, the closure of the subspace $\{(\alpha y,-\beta y): y\in Y\}$. The map α' is given by the inclusion of B into $A\oplus B$ followed by the natural quotient map $A\oplus B \to (A\oplus B)/S$, so that $\alpha'(b)=(0,b)+S$ and, analogously, $\beta'(a)=(a,0)+S$.
The diagram (iv) is commutative: $\beta'\alpha=\alpha'\beta$. Moreover, it is "minimal" in the sense of having the following universal property: if $\beta'': A\to C$ and $\alpha'': B\to C$ are operators such that $\beta''\alpha=\alpha''\beta$, then there is a unique operator $\gamma: PO\to C$ such that $\alpha''=\gamma\alpha'$ and $\beta''=\gamma\beta'$. Clearly, $\gamma((a,b)+S)=\beta''(a)+\alpha''(b)$ and one has $\|\gamma\|\le \max\{\|\alpha''\|,\|\beta''\|\}$.
Suppose we are given an extension (31) and an operator $t: Y\to B$. Consider the push-out of the couple (ι,t) and draw the corresponding arrows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\iota} & X & \longrightarrow & Z & \longrightarrow & 0 \\ & & t\downarrow & & \downarrow t' & & & & \\ & & B & \xrightarrow{\iota'} & PO & & & & \end{array}$$

Clearly, ι' is an isomorphic embedding. Now, the operator $\pi: X\to Z$ and the null operator $n: B\to Z$ satisfy the identity $\pi\iota = nt = 0$, and the universal property of push-outs gives a unique operator $\varpi: PO\to Z$ making the following diagram commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\iota} & X & \xrightarrow{\pi} & Z & \longrightarrow & 0 \\ & & t\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & B & \xrightarrow{\iota} & PO & \xrightarrow{\varpi} & Z & \longrightarrow & 0 \end{array} \tag{34}$$

Or else, just take $\varpi((x,b)+S)=\pi(x)$, check commutativity, and discard everything but the definition of PO. Elementary considerations show that the lower sequence in the preceding diagram is exact. That sequence will we referred to as the push-out sequence. The universal property of push-out diagrams yields:
Lemma (4.3.2)[154]: With the above notations, the push-out sequence splits if and only if t extends to X, that is, there is an operator $T: X\to B$ such that $T\iota=t$.

These lines explain the main connection between interpolation and twisted sums we use throughout. See [54], [158], [138], [136], [156]. Let (X_0,X_1) be a compatible couple of complex Banach spaces. This means that both X_0 and X_1 are embedded into a third topological vector space W and so it makes sense to consider its sum $\Sigma = X_0 + X_1 = \{w \in W: w = x_0 + x_1\}$ which we furnish with the norm $\|w\|_\Sigma = \inf\{\|x_0\|_0 + \|x_1\| : w = x_0 + x_1\}$ as well as the intersection $\Delta = X_0 \cap X_1$ with the norm $\|x\|_\Delta = \max\{\|x\|_0, \|x\|\}$. We attach a certain space of analytic functions to (X_0, X_1) as follows.

Let $\mathbb{S}$ denote the closed strip $\mathbb{S} = \{z \in \mathbb{C}: 0 \le \Re z \le 1\}$, and $\mathbb{S}^\circ$ its interior. We denote by $\mathcal{G} = \mathcal{G}(X_0, X_1)$ the space of functions $g: \mathbb{S} \to \Sigma$ satisfying the following conditions:

(i) g is $\|\cdot\|_\Sigma$-bounded ;

(ii) g is $\|\cdot\|_\Sigma$-continuous on $\mathbb{S}$ and $\|\cdot\|_\Sigma$-analytic on $\mathbb{S}^\circ$;

(iii) $g(it) \in X_0, g(it+1) \in X_1$ for each $t \in \mathbb{R}$;

(iv) the map $t \longmapsto g(it)$ is $\|\cdot\|_0$-bounded and $\|\cdot\|_0$-continuous on $\mathbb{R}$;

(v) the map $t \longmapsto g(it+1)$ is $\|\cdot\|_1$-bounded and $\|\cdot\|_1$-continuous on $\mathbb{R}$.

Then $\mathcal{G}$ is a Banach space under the norm $\|g\|_\mathcal{G} = \sup\{\|g(j+it)\|_j : j = 0,1; t \in \mathbb{R}\}$.

For $\theta \in [0,1]$, define the interpolation space $X_\theta = [X_0, X_1] = \{x \in \Sigma : x = g(\theta)$ for some $g \in \mathcal{G}\}$ with the norm $\|x\|_\theta = \inf\{\|g\|_\mathcal{G} : x = g(\theta)\}$. We remark that $[X_0, X_1]_\theta$ is the quotient of $\mathcal{G}$ by $ker\ \delta_\theta$, the closed subspace of functions vanishing at θ, and so it is a Banach space.

The basic result is the following.

Lemma (4.3.3)[154]: With the above notations, the derivative $\delta'_\theta: \mathcal{G} \to \Sigma$ is bounded from $ker\ \delta_\theta$ onto X_θ for $0 < \theta < 1$.

Proof. For a fixed $\theta \in]0,1[$, let φ be a conformal map of $\mathbb{S}^\circ$ onto the open unit disc sending θ to 0, for instance that given by

$$\varphi(z) = \frac{exp(i\pi z) - exp(i\pi\theta)}{exp(i\pi z) - exp(-i\pi\theta)} \qquad \text{for } z \in \mathbb{S}. \tag{35}$$

If $g \in \mathcal{G}$ vanishes at θ, then one has $g = \varphi h$, with $h \in \mathcal{G}$ and $\|h\|_\mathcal{G} = \|g\|_\mathcal{G}$. Therefore, $g'(\theta) = \varphi(\theta)h(\theta)$, so $g'(\theta) \in X_\theta$ and

$$\|g'(\theta)\|_{X_\theta} = |\varphi'(\theta)|\|h(\theta)\|_{X_\theta} \le |\phi'(\theta)|\|h\|_\mathcal{G} = |\phi'(\theta)|\|g\|_\mathcal{G}.$$

Hence $\left\|\delta'_\theta: ker\ \delta_\theta \to X_\theta\right\| \le |\varphi'(\theta)|$. Notice that $|\varphi'(\theta)| = \pi/(2\, sin(\pi\theta))$ when φ is given by (35).

Let us see that δ'_θ maps $ker\ \delta_\theta$ onto X_θ. Take $x \in X_\theta$, with $\|x\|_\theta = 1$ and choose $g \in \mathcal{G}$ so that $g(\theta) = x$, with $\|g\|_\mathcal{G} \le 1 + \epsilon$. Then $h = \varphi g$ belongs to $ker\ \delta_\theta$ and $h'(\theta) = \varphi'(\theta)x$.

In this way, for each $\theta \in]0,1[$ we have a push-out diagram

$$\begin{array}{ccccc} ker\ \delta_\theta & \longrightarrow & \mathcal{G} & \xrightarrow{\delta_\theta} & X_\theta \\ \delta'_\theta \downarrow & & \downarrow & & \| \\ X_\theta & \longrightarrow & PO & \longrightarrow & X_\theta \end{array} \tag{36}$$

whose lower row is a self extension of X_θ. The derivation associated with the preceding diagram is the map $\Omega: X_\theta \to \Sigma$ obtained as follows: given $x \in X_\theta$ we choose $g = g_x \in \mathcal{G}$ (homogeneously) such that $x = g(\theta)$ and $\|g\|_\mathcal{G} \le (1+\epsilon)\|x\|_{X_\theta}$ for small $\epsilon > 0$ and we set $\Omega(x) = g'(\theta) \in \Sigma$. (Note that (x) lies in X_θ at least for $x \in \Delta = X_0 \cap X_1$.) Homogeneously means that if g is the function attached to x and λ is a complex number, then the function attached to λx is λg – this makes $\Omega : X_\theta \to \Sigma$ homogeneous.

Needless to say, the map Ω depends on the choice of g. However, if $\widetilde{\Omega}(x)$ is obtained as the derivative (at θ) of another $\tilde{g} \in \mathcal{G}$ such that $\tilde{g}(\theta) = x$ and $\|\tilde{g}\|_\mathcal{G} \le M\|x\|$, then $\tilde{g} - g$ vanishes at θ, so

$$\left\|\widetilde{\Omega}(x) - \Omega(x)\right\|_{X_\theta} = \left\|\delta'_\theta(\tilde{g} - g)\right\|_{X_\theta} \le \left\|\delta'_\theta: ker\ \delta_\theta \to X_\theta\right\|(M+1+\epsilon)\|x\|_{X_\theta},$$

and thus $\widetilde{\Omega} \approx \Omega$.

Lemma (4.3.4)[154]: The just defined map Ω is quasi-linear on X_θ. The extension induced by is (equivalent to) the push-out sequence in (36).

Proof. That Ω is quasi-linear is straightforward from Lemma (4.3.3).

As for the second part, look at the basic Diagram 7. Consider the map $(\delta_\theta^{'}, \delta_\theta)\, \mathcal{G} \to X_\theta \bigoplus_\Omega X_\theta$ given by $(\delta_\theta^{'}, \delta_\theta)(f) = (f'(\theta), f(\theta))$. Notice that $(f'(\theta), f(\theta))$ belongs to $X_\theta \bigoplus_\Omega X_\theta$ for every $f \in \mathcal{G}$. Indeed, letting $x = f(\theta)$ we have $f'(\theta) - \Omega(f(\theta)) = \delta_\theta^{'}(f - g_x) \in X_\theta$. Moreover,

$$\|(f'(\theta), f(\theta))\|_\Omega = \left\|\delta_\theta^{'}(f - g_x)\right\|_\theta + \|f(\theta)\|_\theta \le M\|f\|_{\mathcal{G}}.$$

There is an obvious map $\imath: X_\theta \to X_\theta \bigoplus_\Omega X_\theta$ sending x to $(x, 0)$. If $f \in ker\, \delta_\theta$ one has

$$(\delta_\theta^{'}, \delta_\theta)(f) = (f'(\theta), 0) = \imath\delta_\theta^{'}(f)$$

and the universal property of the push-out construction yields an operator u making commutative the following diagram

$$\begin{array}{ccccc} ker\,\delta_\theta & \longrightarrow & \mathcal{G} & \xrightarrow{\delta_\theta} & X_\theta \\ \delta_\theta^{'} \downarrow & & \downarrow & & \| \\ X_\theta & \longrightarrow & PO & \longrightarrow & X_\theta \\ \| & & \downarrow & & \| \\ X_\theta & \longrightarrow & X_\theta \bigoplus_\Omega X_\theta & \longrightarrow & X_\theta \end{array}$$

This completes the proof.

The preceding argument is closely related to the observation, due to Rochberg and Weiss [54], that $X_\theta \bigoplus_\Omega X_\theta = \mathcal{G}/(ker\,\delta_\theta \cap ker\,\delta_\theta^{'}) = \{(f'(\theta), f(\theta)) : f \in \mathcal{G}\}$, where the third space carries the obvious (infimum) norm.

An important feature of the derivation process is that if we start with a couple (X_0, X_1) of Banach modules over an algebra A (this terminology should be self-explanatory by now), then the diagram (36) lives in the category of Banach modules and Ω is a centralizer over A.

We introduce the spaces of measurable functions we shall use along. Our default measure space is the half line $\mathbb{R}^+ = (0, \infty)$. We write $\mathfrak{B}$ for the algebra of Borel sets of $\mathbb{R}^+$ and we denote by λ the Lebesgue measure on $\mathfrak{B}$.

Let L^0 be the space of all (Borel) measurable functions $f: \mathbb{R}^+ \to \mathbb{C}$ equipped with the topology of convergence in measure on sets of finite measure. Here we apply the usual convention of identifying functions agreeing almost everywhere. According to Lindenstrauss and Tzafriri [163], a Kothe space on $\mathbb{R}^+$ is a linear subspace X of L^0 consisting of locally integrable functions, equipped with a monotone norm (if $f \in X$ and $|g| \le |f|$ almost everywhere, then $g \in X$ and $\|g\|_X \le \|f\|_X$) rendering it complete and containing the characteristic function of each Borel set of finite measure. A symmetric space is a Kothe space X satisfying:

(a) If $|f|$ and $|g|$ have the same distribution and $f \in X$, then $g \in X$ and $\|g\|_X = \|f\|_X$.

(b) The Fatou property: if (f_n) is an increasing sequence of nonnegative functions of X converging almost everywhere to f and $sup_n \|f_n\|_X < \infty$, then $f \in X$ and $\|f\|_X = lim_n \|f_n\|_X$.

If u is a measure-preserving automorphism of $\mathbb{R}^+$, then the mapping $f \mapsto u^\circ(f) = f \circ u$ defines an isometry on every symmetric space. We have included the Fatou property in the definition to avoid any difficulty when dealing with spaces of operators. The present definition guarantees that our symmetric spaces are both "fully symmetric" (in the sense of [159]) and "rearrangement invariant" in the sense of [163] and [160]; anyway see cite [111]

for a discussion and related results. If X is a symmetric space, then $L^\infty \cap L^1 \subset X \subset L^\infty + L^1$ and the inclusion are continuous; see [160] for a proof.
It is clear from the definition that every Kothe space X is an L^∞-module under "pointwise" multiplication which turns out to be a submodule of L^0. Let $\Phi: X \to L^0$ be an L^∞-centralizer on X. Then Φ is said to be:
· Real if it takes real functions to real functions.
· Symmetric if (X is symmetric and) there is a constant S so that, whenever u is a measure-preserving automorphism of $\mathbb{R}^+$ one has $\|\Phi(u^\circ f) - u^\circ(\Phi f)\|_X \leq S\|f\|_X$.
· Lazy if, whenever $\mathcal{A}$ is a σ-subalgebra of $\mathfrak{B}$ and $f \in X$ is $\mathcal{A}$ -measurable, $\Phi(f)$ is $\mathcal{A}$ -measurable.
Observe that Φ is lazy if and only if, for every $f \in X$, the function $\Phi(f)$ is measurable with respect to the σ-algebra generated by f, namely $\mathcal{A}(f) = \{f^{-1}(A): A \text{ is a Borel subset of } \mathbb{C}\}$. Also, if Φ is lazy and $f = \sum_{k=1}^\infty t_k 1_{A_k}$ is a function taking only countably many values (σ-simple from now on), one has $\Phi(f) = \sum_{k=1}^\infty s_k 1_{A_k}$ for certain sequence of scalars (s_k).
Important examples of centralizers are given as follows (see [134]). Let $\varphi : \mathbb{R}^2 \to \mathbb{C}$ be a Lipschitz function. Then the map $L^p \to L^0$ given by

$$f \longmapsto f\varphi\left(log\frac{|f|}{\|f\|_p}, log\, r_f\right). \tag{37}$$

is a (symmetric) centralizer on L^p which is real when φ is real-valued. Here r_f is the so called rank-function of $f \in L^0$ defined by

$$r_f(t) = \lambda\{s \in \mathbb{R}^+: |f(s)| > |f(t)| \text{ or } s \leq t \text{ and } |f(s)| = |f(t)|\},$$

which arises in real interpolation (cf. [50]).
For what is concerned, the crucial result on L^∞-centralizers is the following.
Theorem (4.3.5)[154]: (Kalton [136], Theorem 7.6). There is a (finite) constant K so that whenever $1 < p \leq 2$ and X is a p-convex and q-concave Kothe function space with $p^{-1} + q^{-1} = 1$ and Φ is a real centralizer on X with $C[\Phi] < 200/q$ then there is a pair of Kothe function spaces (X_0, X_1) so that $X = [X_0, X_1]_{1/2}$ (with equivalent norms) and if $\Omega: X \to L^0$ is the corresponding derivation, then $\|\Phi(f) - \Omega(f)\| \leq K\|f\|$ for $f \in X$. In particular $\Phi \approx \Omega$.
If Φ is symmetric, then X_0 and X_1 can be taken to be symmetric.
We see some useful consequences.
Lemma (4.3.6)[154]: Let $p \in (1, \infty)$.
(a) Every centralizer on L^p is equivalent to a linear combination of two derivations.
(b) Every symmetric centralizer on L^p is equivalent to a lazy centralizer.
(c) Every symmetric centralizer on L^p takes values in $L^1 + L^\infty$.
Proof. (a) It is obvious from Theorem (4.3.5) that if Φ is a real centralizer on L^p then $c\Phi$ is equivalent to a derivation for $c > 0$ sufficiently small, hence Φ is equivalent to a constant multiple of a derivation. If Φ is any (symmetric) centralizer on L^p, then letting $\Phi_1(f) = \Re\Phi(\Re(f)) + i\Re\Phi(\Im(f))$ and $\Phi_2(f) = \Im\Phi(\Re(f)) - i\Im\Phi(\Im(f))$ one has $\Phi \approx \Phi_1 + i\Phi_2$ with Φ_1 and Φ_2 real (symmetric) centralizers and the result follows.
(b) Let Φ be a symmetric centralizer on L^p, where $1 < p < \infty$. We shall prove that Φ "almost commutes" with every conditional expectation operator in the following sense: there is a constant C such that for every σ-algebra $\mathcal{A} \subset \mathfrak{B}$ and every $f \in L^p$, one has

$$\left\|E^{\mathcal{A}}\Phi f - \Phi(E^{\mathcal{A}}(f))\right\|_{L^p} \leq C\|f\|_{L^p}, \tag{38}$$

where $E^{\mathcal{A}}$ is the conditional expectation operator; see [163] for the definition. After that the result follows just considering the mapping $f \longmapsto E^{\mathcal{A}(f)}(\Phi f)$ which gives a (necessarily symmetric) lazy centralizer equivalent to _. By (a) we may assume that Φ is a derivation, so that there are a couple of symmetric spaces X_0, X_1 so that $[X_0, X_1]_{\frac{1}{2}} = L^p$ with equivalent norms and $\Phi(f) = G_f'\left(\frac{1}{2}\right)$, where $G_f \in \mathcal{G}(X_0, X_1)$ is such that $G_f\left(\frac{1}{2}\right) = f$ and $\|G_f\|_{\mathcal{G}(X_0,X_1)} \leq M\|f\|_{L^p}$ for some constant M independent on f. Since $L^\infty + L^1$ contains both X_0 and X_1, it also contains its sum, so $\Phi(f) \in L^\infty + L^1$ and $E^{\mathcal{A}}\Phi f$ is correctly defined. On the other hand, if $\mathcal{A} \subset \mathfrak{B}$ is a σ-algebra, $E^{\mathcal{A}}$ is a contractive projection on every symmetric space (see [163]), hence if $g = \mathcal{G}(X_0, X_1)$, then $E^{\mathcal{A}} \circ g$ also belongs to $\mathcal{G}(X_0, X_1)$ and $\|E^{\mathcal{A}} \circ g\| \leq \|g\|$.
Now, if $f \in L^p$ and $\mathcal{A} \subset \mathfrak{B}$ is a σ-algebra, letting $h = E^{\mathcal{A}}(f)$ we consider the functions G_f and G_h. Then $E^{\mathcal{A}} \circ G_f - G_h$ vanishes at $z = \frac{1}{2}$, so

$$\left\|E^{\mathcal{A}}\Phi f - \Phi(E^{\mathcal{A}}(f))\right\|_{L^p} = \left\|\delta_{1/2}' E^{\mathcal{A}} \circ G_f - G_h\right\| \leq M\left\|\delta_{1/2}'\right\|\|f\|_{L^p}.$$

This proves (b) and (c) for derivations and the general case follows from (a).

We depend on the Spectral theorem that we now recall, mainly to fix notations. See [116] for a complete exposition. Let $\mathcal{H}$ be a Hilbert space. A closed and densely defined operator $x: D(x) \to \mathcal{H}$ is self-adjoint when $D(x) = D(x^*)$ and $x^* = x$.
For every self-adjoint x there exists a unique "spectral measure" $e^x: \mathfrak{B}(\mathbb{R}) \to B(\mathcal{H})$ (this means that $e^x(B)$ is an orthogonal projection for each Borel B and that $e^x(\cdot)$ is σ-additive with respect to the strong operator topology of $B(\mathcal{H})$ such that

$$x = \int_{\mathbb{R}} \lambda de^x(\lambda).$$

If x is a closed, densely defined operator, then x^*x is self-adjoint (and, actually, positive). The modulus of x is then defined as

$$|x| = (x^*x)^{1/2} = \int_{\mathbb{R}^+} \lambda^{1/2} de^{(x^*x)}(\lambda).$$

One has the "polar decomposition" $x = u|x|$, where u is a partial isometry which is ofted called the phase of x.
Let M be a semifinite von Neumann algebra with a faithful, normal, semifinite (fns) trace , acting on $\mathcal{H}$. A closed densely defined operator on $\mathcal{H}$ is affiliated with M if its spectral projections (that is, the projections $e^{(x^*x)}(B)$ for $B \in \mathfrak{B}(\mathbb{R})$) belong to M. A closed, densely defined operator x affiliated with M is called τ -measurable if, for any $\epsilon > 0$, there exists a projection $e \in M$ such that $e\mathcal{H} \subset D(x)$ and $\tau(1 - e) \leq \epsilon$. We denote the set of all τ -measurable operators affiliated with a von Neumann algebra M by $\widetilde{M}$. The so called measure topology on $\widetilde{M}$ is the least linear topology containing the sets
$\{x \in \widetilde{M}$: there exists a projection $e \in M$ such that $\tau\ (1 - e) < \varepsilon, xe \in M$ and $\|xe\| < \varepsilon\}$, with $\varepsilon > 0$. Endowed with measure topology, strong sum, strong product and adjoint operation as involution, $\widetilde{M}$ becomes a topological *-algebra (see [52], [157] for basic information). The trace τ has a natural extension to $\widetilde{M}_+$.
We define $L^p(M, \tau\ )$ as the space of all τ -measurable operators x such that $\tau\ (|x|^p) < \infty$, with norm $\|x\|_p = (\tau\ (|x|^p))^{1/p}$.

More general spaces of operators can be introduced as follows [159], [164], [111]. Let x be a measurable operator, so that $\tau(e^{|x|}(\lambda,\infty))$ is finite for some $\lambda > 0$. The generalized singular value function of x is the function $\mu(x): \mathbb{R}^+ \to [0,\infty]$ given by

$$\mu(x)(t) = \inf\{\lambda > 0 : \tau(e^{|x|}(\lambda,\infty)) \le t\}.$$

Now, if X is a symmetric function space, the corresponding "symmetric operator space" is

$$X(M,\tau) = \{x \in \widetilde{M}: \mu(x) \in X\}, \qquad with\ \|x\| = \|\mu(x)\|_X.$$

An important feature of these spaces is that they are bimodules over M with the obvious outer multiplications.

In order to state the main result, let us consider a self-adjoint $y \in \widetilde{M}$ and let M_y be the (von Neumann) subalgebra of M generated by the spectral projections of y. By general representation results one can construct a *-homomorphism $\xi: M_y \to L^\infty$ preserving the trace, that is, such that $\tau(a) = \int_0^\infty \xi(a)d\lambda$ for every nonnegative $a \in M_y$. A simple proof of this fact appears in [165] (note that "our" ξ is the inverse of the map that Pisier and Xu call S). A different proof for finite (von Neumann) algebras can be seen in [166] (the argument works for semifinite algebras as well). For a more general result, see [159].

If $\widetilde{M}_y$ denotes the closure of M_y in $\widetilde{M}$, then ξ extends to a continuous *-homomorphism $\widetilde{M}_y \to L^0$ that we denote again by ξ. Clearly, $\xi(M_y) = L^\infty(\mathbb{R}^+, \mathcal{A}, \lambda)$, where $\mathcal{A}$ is a σ-subalgebra of $\mathfrak{B}$. It follows that for every $\mathcal{A}$ -measurable $f \in L^1$ there is $z \in \widetilde{M}_y$ (actually in $L^1(M,\tau))$ such that $f = \xi(z)$ and so $\xi^{-1}(f)$ is correctly defined if $f \in L^1 + L^\infty$ is $\mathcal{A}$ -measurable. Besides, ξ preserves every "symmetric" norm in the following sense: if X is a symmetric function space on $\mathbb{R}^+$ and $f \in X$ is $\mathcal{A}$ -measurable, then there is $x \in \widetilde{M}_y$ such that $\xi(x) = f$ and $\|x\|_{X(M,\tau)} = \|f\|_X$. This is obvious since $\mu(x)$ and f have the same distribution.

The following result and its proof are modeled on [136]:

Theorem (4.3.7)[154]: Let Φ be a lazy, symmetric L^∞-centralizer on L^p, where $1 < p < \infty$. Given a semifinite von Neumann algebra (M,τ) we define a mapping $\Phi_\tau: L^p(M,\tau) \to \widetilde{M}$ as follows: For each $x \in L^p(M,\tau)$ we choose a trace preserving *-homomorphism $\xi: \widetilde{M}_{|x|} \to L^0$ (depending only on $M_{|x|}$) as before and we set

$$\Phi_\tau(x) = u \cdot \xi^{-1}(\Phi(\xi(|x|))), \tag{39}$$

where $x = u|x|$ is the polar decomposition. Φ_τ is an M-bicentralizer on $L^p(M,\tau)$ and all mappings defined in this way are equivalent, independently of the choice of ξ.

Proof. First of all observe that the definition of Φ_τ makes sense since $\Phi(\xi(|x|))$ belongs to $L^\infty + L^1$ and it is measurable with respect to the σ-algebra generated by $\xi(|x|)$ and so $\xi^{-1}\Phi(\xi(|x|))$ is well defined. Also, note that since $M_{|\lambda x|} = M_{|x|}$ for each nonzero $\lambda \in \mathbb{C}$ and ξ depends only on its domain algebra the resulting map Φ_τ is homogeneous.

Let us prove that Φ_τ is a bicentralizer assuming that Φ is a derivation. Precisely, we are assuming there is a couple of symmetric Kothe spaces on $\mathbb{R}^+$ such that $L^p = [X_0, X_1]^{1/2}$, with equivalent norms in such a way that, for each $f \in L^p$ one has $\Phi(f) = g'(\frac{1}{2})$, where $g \in \mathcal{G}(X_0, X_1)$ satisfies $g(\frac{1}{2}) = f$ and $\|g\|_{\mathcal{G}} \le K\|f\|_p$.

Set $X = [X_0, X_1]_{1/2}$, with the natural norm. This is a symmetric Kothe space on $\mathbb{R}^+$. The key point is that the formula

$$[X_0(M,\tau), X_1(M,\tau)]_{\frac{1}{2}} = X(M,\tau) \tag{40}$$

holds for all semifinite algebras (M,τ) –see [159] and [165].

Of course $X(M,\tau) = L^p(M,\tau)$, up to equivalence of norms and we may consider the corresponding derivation on $L^p(M,\tau)$. That is, given $x \in L^p(M,\tau)$ we choose $G_x \in \mathcal{G}(X_0(M,\tau), X_1(M,\tau))$ such that $G_x(\frac{1}{2}) = x$ and $\|G_x\|_{\mathcal{G}} \le (1+\epsilon)\|x\|_{X(M,\tau)} \le K\|x\|_p$ and then we put

$$\Omega(x) = \delta'_{1/2} G_x \in \widetilde{M}.$$

The fact that such an Ω turns out to be an M-bicentralizer on $L^p(M,\tau)$ should be obvious by now, but let us record the proof for future reference. Take $x \in X(M,\tau)$ and $a, b \in M$ (that we regard as constant functions on S). We have $G_{axb} - aG_x b \in ker\ \delta_{1/2}$ by the very definition. Moreover,

$$\|G_{axb} - aG_x b\|_{\mathcal{G}} \le \|G_{axb}\|_{\mathcal{G}} + \|aG_x b\|_{\mathcal{G}} \le 2(1+\epsilon)\|a\|\|x\|_{X(M,\tau)}\|b\|,$$

so

$$\|\Omega(axb) - a\Omega(x)b\|_{X(M,\tau)} = \left\|\delta'_{1/2}(G_{axb} - aG_x b)\right\|_{X(M,\tau)}$$
$$\le \left\|\delta'_{1/2}\ \ ker\ \delta_{1/2} \to X(M,\tau)\right\| 2(1+\epsilon)\|a\|\|x\|_{X(M,\tau)}\|b\|$$
$$\le (1+\epsilon)\pi\|a\|\|x\|_{X(M,\tau)}\|b\|.$$

Thus, to complete the proof that the formula (10) defines a bicentralizer on $L^p(M,\tau)$, it suffices to see that one can choose the functions G_x in such a way that $G'_x(\frac{1}{2}) = \Phi_\tau$.

So, pick a normalized $x \in L^p(M,\tau)$ and put $f = \xi(|x|)$. Then f is normalized in L^p and we have $\Phi(f) = \delta'_{1/2} g$ where $g \in \mathcal{G}(X_0, X_1)$ is the corresponding extremal–recall that we are assuming that Φ is itself a derivation.

We claim that the mapping $G: \mathbb{S} \to \widetilde{M}$ given by $G(z) = u \cdot \xi^{-1} E^{\mathcal{A}}(g(z))$ is allowable for x. We have

$$G(\frac{1}{2}) = u \cdot \xi^{-1} E^{\mathcal{A}}(g(\frac{1}{2})) = u \cdot \xi^{-1} E(f) = u\xi^{-1} f = u|x| = x.$$

That G belongs to $\mathcal{G}(X_0(M,\tau), X_1(M,\tau))$ is obvious since $E^{\mathcal{A}}$ is contractive on X_0 and X_1 (hence on $X_0 + X_1$) and ξ preserves all symmetric norms: actually the norm of G in $\mathcal{G}(X_0(M,\tau), X_1(M,\tau))$ cannot exceed that of g in $\mathcal{G}(X_0, X_1)$.

Finally, applying the chain rule and taking into account that Φ is lazy,

$$G'\left(\frac{1}{2}\right) = u \cdot \xi^{-1} E^{\mathcal{A}}\left(g'\left(\frac{1}{2}\right)\right) = u \cdot \xi^{-1} E^{\mathcal{A}}(\Phi(f)) = u\xi^{-1}\Phi(f) = u\xi^{-1}\Phi(\xi(|x|))$$
$$= \Phi_\tau(x).$$

And so Φ_τ is a bicentralizer.

To complete the proof (still under the assumption that Φ is a derivation) we must prove that Φ_τ is essentially independent of the family of *-homomorphisms ξ. Indeed, if $\xi_1: \widetilde{M}_{|x|} \to L^0$ is another trace-preserving *-homomorphism and $\mathcal{A}_1 \subset \mathfrak{B}$ is the corresponding σ-algebra, letting $f_1 = \xi_1(|x|)$ and taking any allowable $g_1 \in \mathcal{G}(X_0, X_1)$ so that $g_1(\frac{1}{2}) = f_1$ with $g_1 \le (1+\varepsilon)\|f_1\|_X = \|x\|_{X(M,\tau)}$ we have that if

$$G_1(z) = u \cdot \xi_1^{-1}\, E^{\mathcal{A}_1}(g_1(z)) \qquad (z \in \mathbb{S}),$$

then G_1 belongs to $\mathcal{G}(X_0(M,\tau), X_1(M,\tau))$, is allowable for x and $G'_1(\frac{1}{2}) = u \cdot \xi_1^{-1}\Phi(\xi_1(|x|))$ and since $\delta'_{1/2}$ is bounded from $ker\ \delta_{1/2}$ to $X(M,\tau)$ (see Lemma (4.3.3)) we have that $G'_1(\frac{1}{2}) - G'(\frac{1}{2})$ falls in $X(M,\tau)$ and

$$\|u\xi_1^{-1}\Phi(\xi_1(|x|)) - u\xi^{-1}\Phi(\xi(|x|))\|_{X(M,\tau)} = \left\|\delta'_{1/2}(G_1 - G)\right\| \le M\|x\|_{X(M,\tau)}.$$

This completes the proof when Φ is a derivation–or a linear combination of derivations.

To finish, observe that if Ψ and Φ are two equivalent lazy centralizers on L^p, then the maps Ψ_τ and Φ_τ are equivalent on $L^p(M,\tau)$–at least if the "prescribed" family of *-homomorphisms $x \longmapsto \xi$ is fixed. Indeed, if $x \in L^p(M,\tau)$, then

$$\|\Psi_\tau(x) - \Phi_\tau(x)\|_{L^p(M,\tau)} = \|u\xi^{-1}(\Psi(\xi(|x|) - \Phi(\xi(|x|))\| \le \|\Psi - \Phi\|\|(\xi(|x|))\| \\ \le M\|x\|.$$

Now, the result follows from Lemma (4.3.6).

The action of Φ_τ on σ-elementary operators if quite simple. Here, a σ-elementary operator is one of the form $x = \sum_{k=1}^{\infty} \lambda_k e_k$, with e_k disjoint projections and $\lambda_k \in \mathbb{C}$. Indeed, for such an x we have $|x| = \sum_{k=1}^{\infty} |\lambda_k| e_k$ and $u = \sum_{k=1}^{\infty} u_k e_k$, where u_k is the signum of λ_k. Hence, if $\xi\colon \widetilde{M}_{|x|} \to L^0$ is any trace-preserving *-homomorphism, then $f = \xi(|x|) = \sum_{k=1}^{\infty} |\lambda_k| 1_{A_k}$, where (A_k) is a sequence of disjoint Borel sets of $\mathbb{R}^+$ and $\xi(e_k) = 1_{A_k}$ for every $k \in \mathbb{N}$. Now, as Φ is lazy, we have $\Phi(f) = \sum_{k=1}^{\infty} s_k 1_{A_k}$ for some sequence (s_k) and

$$\Phi_\tau(x) = u\xi^{-1}\left(\sum_{k=1}^{\infty} s_k 1_{A_k}\right) = \sum_{k=1}^{\infty} u_k s_k e_k\,.$$

The following result applies to many centralizers appearing in nature. In particular, it applies to the centralizers given by (37) when φ depends only on the first variable, by just taking $\phi(t) = t\varphi(log\ t)$ for $t \in \mathbb{R}^+$.

Corollary (4.3.8)[154]: Let Φ be a centralizer on L^p, where $1 < p < \infty$. Suppose there is a Borel function $\varphi\colon \mathbb{R}^+ \to \mathbb{C}$ such that $\Phi(f) = \phi \circ f$ for every $f \ge 0$ normalized in L^p. Then, for every semifinite von Neumann algebra (M,τ), the map $x \longmapsto \|x\|_p u\phi(|x|/\|x\|_p)$ is an M-bicentralizer on $L^p(M,\tau)$.

Proof. It is obvious that Φ is both symmetric and lazy. In view of Theorem (4.3.7) it suffices to check that$\Phi_\tau(x) = \phi(x)$ for x positive and normalized in $L^p(M,\tau)$. Let ξ be the prescribed trace-preserving *-homomorphism. Since $\Phi_\tau(x) = \xi^{-1}(\xi(x)) = \xi^{-1}(\phi \circ (\xi(x))$ the proof will be complete if we show that $\phi \circ (\xi(x) = \xi(\varphi(x))$, where $\phi(x)$ is given by the functional calculus:

$$\phi(x) = \int_0^\infty \phi(\lambda) de^x(\lambda), \text{ where } x = \int_0^\infty \lambda de^x(\lambda).$$

Let us consider L^∞ as a von Neumann algebra with trace λ (to be true, the trace of $a \in L^\infty$ is $\int_{\mathbb{R}^+} a d\lambda$) acting by multiplication on L^2 and let $\widetilde{L}^\infty$ be the space of λ-measurable operators (affiliated with L^∞). Set $f = \xi(x)$ which we may now regard also as a self-adjoint operator in $\widetilde{L}^\infty$. Then, if

$$f = \int_0^\infty \lambda de^f(\lambda)$$

is the spectral representation it is obvious that $\xi e^x = e^f$ in the sense that for every $B \in \mathfrak{B}$ one has $\xi(B) = e^f(B)$. Moreover, $e^f(B)$ can be identified with $1_B \circ f = 1_{f^{-1}(B)}$ and so

$$\xi(\phi(x)) = \xi\left(\int_0^\infty \phi(\lambda) de^x(\lambda)\right) = \int_0^\infty \phi(\lambda) d\xi e^x(\lambda) = \int_0^\infty \phi(\lambda) de^f(\lambda) = \phi \circ f,$$

and we are done.

We discuss the simplest case of self-extensions, namely that one obtains out from the identity $[M, L^1(M,\tau)]_\theta = L^p(M,\tau)$ at $\theta = 1/p$. In order to simplify the computation of extremals we introduce a larger space of holomorphic functions as follows. We consider both M and $L^1(M,\tau)$ as subspaces of $\widetilde{M}$ and we set $\Delta = M \cap L^1(M,\tau)$ and $\Sigma = M + L^1(M,\tau)$. Let $\mathcal{H} = \mathcal{H}(M,\tau)$ be the space of functions $h\colon \mathbb{S} \to \Sigma$ satisfying the following conditions:

(i) h is $\|\cdot\|_\Sigma$-bounded.
(ii) For each $x \in \Delta$ the function $z \longmapsto \tau(xh(z))$ is continuous on $\mathbb{S}$ and analytic on $\mathbb{S}^\circ$.
(iii) $h(it) \in M, h(it+1) \in L^1(\tau)$ for each $t \in \mathbb{R}$;
(iv) the map $t \longmapsto h(it)$ is $\|\cdot\|_\infty$-bounded and $\sigma(M, L^1(\tau))$-continuous on $\mathbb{R}$;
(v) the map $t \longmapsto h(it+1)$ is $\|\cdot\|_1$-bounded and $\|\cdot\|_1$-continuous on $\mathbb{R}$.
We equip $\mathcal{H}$ with the norm $\|h\|_{\mathcal{H}} = \sup\{\|h(it)\|_M, \|h(it+1)\|_1 : t \in \mathbb{R}\}$. Note that the elements of $\mathcal{H}$ are in fact $\|\cdot\|_\Sigma$-analytic on $\mathbb{S}^\circ$.
Letting $\theta = 1/p \in (0,1)$ we have that δ_θ maps $\mathcal{H}$ onto $L^p(\tau)$ (without increasing the norm) and replacing $\mathcal{G}$ by $\mathcal{H}$ everywhere in the proof of Lemma (4.3.3) we see that the restriction of δ_θ' to $ker\ \delta_\theta$ is a bounded operator onto $L^p(\tau)$ and we can form the push-out diagram

$$\begin{array}{ccccc} ker\ \delta_\theta & \longrightarrow & \mathcal{H} & \xrightarrow{\delta_\theta} & L^p(\tau) \\ \delta_\theta' \downarrow & & \downarrow & & \| \\ L^p(\tau) & \longrightarrow & PO & \longrightarrow & L^p(\tau) \end{array} \tag{41}$$

Please note that the above diagram lives in the category of bimodules over M. Also, as $\mathcal{H}$ contains the Calderon space $\mathcal{G}$ it is really easy to see that this new push-out extension is in fact the same one gets by using $\mathcal{G}$.
Let us compute the extremals associated to the quotient $\delta_\theta: \mathcal{H} \to L^p(\tau)$. Suppose $f \in L^p(\tau)$ is a positive operator with $\|f\|_p = 1$. It is easily seen that the function $h(z) = f^{pz}$ belongs to $\mathcal{H}$ (although it is not in $\mathcal{G}$ in general) and also that $\|h\|_{\mathcal{H}} = 1$. Of course, $h'(\theta) = pf\ log\ f$ and thus, the derivation associated to Diagram 12 is given by

$$\Omega_p(f) = pf\ log(|f|/\|f\|_p) \qquad (f \in L^p(\tau)). \tag{42}$$

Let us denote the corresponding twisted sum $L^p(\tau) \oplus_{\Omega_p} L^p(\tau)$ by $Z_p(\tau)$. Our immediate aim is to prove the following.
Theorem (4.3.9)[154]: $Z_p(M,\tau)$ is a nontrivial self extension of $L^p(M,\tau)$ as long as M is infinite dimensional and $1 < p < \infty$.
Proof. Needless to say $Z_p(M,\tau)$ is a bimodule extension over M. We shall prove that it doesn't split even as an extension of Banach spaces. As M is infinite dimensional there is a sequence (e_i) of mutually orthogonal projections having finite trace. Let A be the von Neumann subalgebra of M spanned by these projections. Notice that we may consider $\tilde{A}$ as a *-subalgebra of $\widetilde{M}$ and $L^p(A,\tau)$ as a subspace of $L^p(M,\tau)$.
Clearly, p maps $L^p(A,\tau)$ to $\tilde{A}$ as an A-centralizer and we have a commutative diagram of inclusions

$$\begin{array}{ccccc} L^p(A,\tau) & \longrightarrow & L^p(A,\tau) \oplus_{\Omega_p} L^p(A,\tau) & \longrightarrow & L^p(A,\tau) \\ \downarrow & & \downarrow & & \downarrow \\ L^p(M,\tau) & \longrightarrow & Z_p(M,\tau) & \longrightarrow & L^p(M,\tau) \end{array}$$

On the other hand, the "conditional expectation" given by

$$E_A(f) = \sum_i \frac{\tau(fe_i)}{\tau(e_i)} e_i$$

is a contractive projection on $L^p(M,\tau)$ whose range is $L^p(A,\tau)$. The immediate consequence of all this is that if the lower extension of the preceding diagram splits, then so does the upper one.
Let us check that this is not the case. As A is amenable (it is isometrically *-isomorphic to the algebra ℓ^∞) and $L^p(A,\tau)$ is a dual bimodule (it is isometrically isomorphic to ℓ^p, which

is reflexive) we have that the upper row in the above diagram splits as an extension of Banach spaces if and only if it splits as an extension of Banach A-modules. And this happens if and only if there is a morphism $\phi: L^p(A,\tau) \to \tilde{A}$ approximating Ω_p in the sense that

$$\left\|\Omega_p(f) - \phi(f)\right\|_p \leq \delta\|f\|_p \tag{43}$$

for some constant δ and every $f \in L^p(A,\tau)$. It is clear that every morphism $\phi: L^p(A,\tau) \to \tilde{A}$ has the form $\phi(\sum_i(t_i e_i)) = \sum_i \phi_i t_i e_i$ for some sequence of complex numbers (ϕ_i). Taking $f = e_i$ in (43) we see that $|\phi_i + log\,\tau\,(e_i)| \leq \delta$. It follows that if (43) holds for some $\phi = (\phi_i)$ then it must hold for $\phi_i = -log\,\tau\,(e_i)$, possibly doubling the value of δ. Fix $n \in \mathbb{N}$ and take $f = t_i e_i$ normalized in $L^p(\tau)$ in such a way that the nonzero summands in the norm of f agree:

$$f = \sum_{i=1}^{n} (n\tau(e_i))^{-1/p}\, e_i.$$

For this f and $\phi_i = -log\,\tau\,(e_i)$ we have $p(f) - \phi(f) = -log(n)f$, so $\left\|\Omega_p(f) - \phi(f)\right\|_p = log(n)$, which makes impossible the estimate in (43).

We extend Kalton-Peck duality results in [139] to all semifinite algebras by showing that for every trace τ the dual space of $Z_p(M,\tau)$ is isomorphic to $Z_q(M,\tau)$, where p and q are conjugate exponents, that is, $p^{-1} + q^{-1} = 1$. (see [149] for the particular case of Schatten classes). In order to achieve a sharp adjustment of the parameters, let us agree that, given $p \in (1, \infty)$ and a Lipschitz function $\varphi: \mathbb{R} \to \mathbb{C}$, the associated Kalton-Peck centralizer $\Phi_p : L^p(M,\tau) \to \widetilde{M}$ is defined by $\Phi_p(f) = f\varphi(p\,log(|f|/\|f\|_p)$ and the corresponding Kalton-Peck space is $Z_p^{\varphi}(M,\tau) = L^p(\tau) \oplus_{\Phi_p} L^p(\tau)$. This is coherent with (42), where φ is the identity on $\mathbb{R}$.

Theorem (4.3.10)[154]: Let p and q be conjugate exponents, φ a Lipschitz function, and τ be a trace. Then $Z_q^{\varphi}(\tau)$ is isomorphic to the conjugate of $Z_p^{\varphi}(\tau)$ under the pairing

$$\langle (x,y),(v,w)\rangle = \tau(xw - yv) \qquad ((x,y) \in Z_q^{\varphi}(\tau), (v,w) \in Z_p^{\varphi}(\tau)) \tag{44}$$

Proof. The proof depends on the following elementary inequality: given $s, t \in \mathbb{C}$ one has

$$\left|ts\left(log\frac{|t|^q}{|s|^p}\right)\right| \leq \frac{p}{e}(|t|^q + |s|^p)\,. \tag{45}$$

This is (a rewording of) the case $n = 1$ of [139] that Kalton and Peck use in the proof of [139]. Let us see that the pairing is continuous. To this end write

$$xw - yv = \left(x - \Phi_q(y)\right)w \ + \Phi_q(y)w - y(v - \Phi_p(w)) - y\Phi_p(w).$$

As $\left\|(x - \Phi_p(y))w\right\|_1 \leq \left\|x - \Phi_q\right\|_q \|w\|_p$ and, similarly, $\left\|y(v - \Phi_p(w))\right\|_1 \leq \|y\|_q \left\|v - \Phi_p(w)\right\|_p$ it suffices to obtain an estimate of the form

$$|\tau(\Phi_q(y)w - y\Phi_p(w))| \leq M\|y\|_q\|w\|_p. \tag{46}$$

First, let us assume y and w are σ-elementary operators with $\|y\|_q = \|w\|_p = 1$ and representations $y = \sum t_i y_i$ and $w = \sum s_j w_j$ converging in $L^q(\tau)$ and $L^p(\tau)$, respectively. We may assume with no loss of generality that $\sum_i y_i = \sum_j w_j = 1_M$ (summation in the $\sigma(M, M_*)$ topology). We have

$$\Phi_q(y)w - y\Phi_p(w)$$
$$= \left(\sum_i t_i\varphi(q\log|t_i|)y_i\right)\left(\sum_j s_jw_j\right) - \left(\sum_i t_iy_i\right)\left(\sum_j s_j\varphi(plog\,|s_j|)w_j\right)$$
$$= \sum_{i,j} t_is_j\,(\varphi(log\,|t_i|^q) - \varphi(log\,|s_j|^p))y_iw_j.$$

Applying (45) and taking into account that the product of any two projections has positive trace we can estimate the left-hand of (46) as follows:

$$|\tau\,(\Phi_q(y)w - y\Phi_pw)| \leq \sum_{i,j} |t_i||s_j||\varphi(log\,|t_i|^q) - \varphi(log\,|s_j|^p)|\tau(y_iw_j)$$
$$\leq L_\varphi \sum_{i,j} |t_i||s_j| \left|log\frac{|t_i|^q}{|s_j|^p}\right| \tau(y_iw_j)$$
$$\leq L_\varphi \sum_{i,j} \frac{p}{e}(|t_i|^q + |s_j|^p)\,\tau(y_iw_j)$$
$$= \frac{p}{e}L_\varphi \left(\sum_i |t_i|^q\left(\sum_j \tau(y_iw_j)\right) + \sum_j |s_j|^p\left(\sum_j \tau(y_iw_j)\right)\right)$$
$$= \frac{p}{e}L_\varphi\left(\sum_i |t_i|^q\tau(y_i) + \sum_j |s_j|^p\tau\,(w_j)\right)$$
$$= \frac{2pL_\varphi}{e},$$

where L_φ denotes the Lipschitz constant of φ. Assuming for instance that $1 < p \leq 2$, by homogeneity

$$|\tau(\Phi_q(y)w - y\Phi_p(w))| \leq 2L_\varphi\|y\|_q\|w\|_p, \tag{47}$$

whenever y and w are σ-elementary operators. Now, suppose y and w are self-adjoint. It is easy to find a sequences of σ-elementary operators (y_n) and (w_n) such that the numerical sequences

$$\|y_n - y\|_q, \qquad \left\|\Phi_qy_n - \Phi_qy\right\|_q, \qquad \|w_n - w\|_p, \left\|\Phi_pw_n - \Phi_pw\right\|_p$$

are all convergent to zero. This implies that

$$\left\|(\Phi_q(y)w - y\Phi_p(w)) - (\Phi_q(y_n)w_n - y_n\Phi_p(w_n))\right\|_1 \to 0$$

and so (47) holds when y and w are self-adjoint. Next, if $y \in L^p(M,\tau\,)$ and $w \in L^p(M,\tau\,)$ is self-adjoint, we can write $y = y_1 + iy_2$, with each y_i self-adjoint and since Φ_q is quasi-linear

one has $\left\|\Phi_q(y) - \Phi_q(y_1) - i\Phi_q(y_2)\right\|_q \leq Q[\Phi_q](\|y_1\|_q + \|y_2\|_q) \leq 2Q[\Phi_q]\|y\|_q$ and

$$|\tau(\Phi_q(y)w - y\Phi_qw)|$$
$$= |\tau((\Phi_qy - \Phi_qy_1 - i\Phi_qy_2)w + (\Phi_qy_1 + i\Phi_qy_2)w - (y_1 + iy_2)\Phi_p(w))|$$
$$\leq 2Q[\Phi_q]\|y\|_q\|w\|_p + 2L_\varphi(\|y_1\|_q + \|y_2\|_q)\|w\|_p$$
$$\leq (2Q[\Phi_q] + 4L_\varphi)\|y\|_q\|w\|_p.$$

Finally, writing $w = w_1 + iw_2$ with each w_i self-adjoint and using the quasilinearity of Φ_p one arrives to (46), where M depends on p, q and L_φ, but not on f or g.

Therefore, going back to (44) we have

$$|\tau(xw-yv)| = |\tau((x-\Phi_q(y))w+\Phi_q(y)w-y(v-\Phi_p(w))-y\Phi_p(w))|$$
$$\le \left\|(v-\Phi_p(w)\right\|_p\|y\|_q + M\|w\|_p\|y\|_q + \|w\|_p\left\|(x-\Phi_q(y)\right\|_q$$
$$\le (M+1)\|(x,y)\|_{\Phi_q}\|(v,w)\|_{\Phi_p}.$$

The remainder of the proof is quite easy: we have just seen that the map $u: Z_q^{\varphi}(\tau) \to (Z_p^{\varphi}(\tau))^*$ given by $(u(x,y))(v,w) = \tau\,(xw-yv)$ is bounded. On the other hand, the following diagram is commutative:

$$\begin{array}{ccccc} L^q(\tau) & \longrightarrow & Z_q^{\varphi}(\tau) & \longrightarrow & L^q(\tau) \\ \| & & \downarrow u & & \downarrow -1 \\ L^q(\tau)^* & \longrightarrow & \left(Z_q^{\varphi}(\tau)\right)^* & \longrightarrow & L^q(\tau)^* \end{array}$$

Here, the lower row is the adjoint (in the Banach space sense) of the extension induced by Φ_p. It follows that u is one-to-one, onto, and open.

Theorem (4.3.9) cannot be extended to arbitrary centralizers. Actually, the following example shows that the behavior of Φ_τ may depend strongly on the trace τ.

Example (4.3.11)[154]: For each $\pm$ and $p \in (1,\infty)$, consider the centralizer on $L^p(\mathbb{R}^+)$ given by $\Phi^{\pm}(f) = f(\iota^{\pm}(log(|f|/\|f\|_p)))$, where $\iota^+(t) = \max\{0,t\}$ and $\iota^-(t) = \min\{0,t\}$ Then, with the notation of Theorem (4.3.7):

(a) $\Phi^{\pm}$ is nontrivial on $L^p(\mathbb{R}^+)$.

(b) If τ is bounded away from zero on the projections of M then Φ_τ^+ is trivial on $L^p(M,\tau)$, while Φ_τ^- is nontrivial as long as M is infinite-dimensional.

(c) If $\tau(1_M) < \infty$ then Φ_τ^- is trivial on $L^p(M,\tau)$, while Φ_τ^+ is nontrivial as long as M is infinite-dimensional.

Proof. Let $\Psi: L^p \to L^0$ be any centralizer. Let (A_i) be a sequence of disjoint measurable sets, with finite and positive measure and let $\mathcal{A}$ be the least σ-algebra of Borel sets containing every A_i. Then, if Ψ maps $L^p(\mathbb{R}^+,\mathcal{A},\lambda)$ to $L^0(\mathbb{R}^+,\mathcal{A},\lambda)$, in particular if Ψ is lazy, then it defines an $L^\infty(\mathbb{R}^+,\mathcal{A},\lambda)$ centralizer on $L^p(\mathbb{R}^+,\mathcal{A},\lambda)$. Moreover, if Ψ is trivial on $L^p(\mathbb{R}^+,\mathcal{A},\lambda)$ (as a quasi-linear map), then it is also trivial as an $L^\infty(\mathbb{R}^+,\mathcal{A},\lambda)$ centralizer.

(a) To check that Φ^+ is nontrivial on $L^p(\mathbb{R}^+)$ just take a sequence (A_i) with $|A_i| = 2^{-i}$. To check that Φ^- is nontrivial, take A_i with $|A_i| = 1$ for all $i \in \mathbb{N}$.

(b) We may assume $\tau(e) \ge 1$ for every projection $e \in M$. Pick a positive, σ-elementary f normalized in $L^p(\tau)$ so that $f = \sum_{n=1}^{\infty} f_i e_i$, with $f_i \ge 0$ and e_i disjoint projections. Obviously $f_i \le 1$ for every i and so $\Phi^+(f) = 0$. It follows that Φ^+ is bounded on $L^p(\tau)$. As $\Phi_\tau^+ + \Phi_\tau^- = \Omega_p$ and Φ^+ is trivial we see that Φ^- must be nontrivial since $\Omega_p$is nontrivial unless M is finite-dimensional.

(c) We may assume $\tau(1_M) = 1$. Take a positive, normalized $f \in L^p(M,\tau)$ and write $f = \int_0^\infty \lambda de(\lambda)$ to be its spectral resolution. Set $g = \int_0^1 \lambda de(\lambda)$ and $h = \int_{1^+}^\infty \lambda de(\lambda)$. One has

$$\|\Phi_\tau^-(f) - \Phi_\tau^-(g) - \Phi_\tau^-(h)\|_p \le Q[\Phi_\tau^-](\|g\|_p + \|h\|_p) \le 2Q[\Phi_\tau^-].$$

Obviously, $\Phi_\tau^-(h) = 0$, while $g \in M$, with $\|g\|_\infty \le 1$. Hence

$$\|\Phi_\tau^-(g) - g\Phi_\tau^-(1_M)\|_p \le C[\Phi_\tau^-]\|g\|_\infty\|1\|_p \le C[\Phi_\tau^-].$$

But $\Phi_\tau^-(1_M) = 0$ and so $\|\Phi_\tau^-(f)\|_p \le 2Q[\Phi_\tau^-] + C[\Phi_\tau^-]$.

The centralizers Φ_τ^- appearing in Theorem (4.3.7) have the property that, if $x \in L^p(M,\tau)$ is self-adjoint, then x and $\Phi_\tau(x)$ commute. This is not by accident. Indeed, suppose that $\Psi: L^p(M,\tau) \to \widetilde{M}$ is any bicentralizer and that x is selfadjoint. Let A be a maximal abelian self-adjoint subalgebra containing the spectral projections of x, so that $ax = xa$ for every $a \in A$. Then

$$\|a\Psi x - (\Psi x)a\|_p = \|a\Psi x - \Psi(ax) + \Psi(xa) - (\Psi x)a\|_p \leq 2C[\Psi]\|a\|\|x\|,$$

and $\|\Psi(x) - u(\Psi x)u^*\| \leq M\|x\|$ for every unitary $u \in A$. Averaging the difference $\Psi(x) - u(\Psi x)u^*$ over the unitary group of A one obtains an element $B(x) \in L^p(M, \tau)$ such that $\|B(x)\|_p \leq M\|x\|_p$ and such that $\Psi(x) - B(x)$ commutes with A. Thus, if we define $\widetilde{\Psi}(x) = \Psi(x) - B(x)$ we get a centralizer with the additional property that x is self-adjoint, then $\widetilde{\Psi}(x)$ commutes with (the spectral projections of) x.

One may wonder what is the role of the symmetry of the starting centralizer Φ in Theorem (4.3.7). In general one cannot expect to get bicentralizers out from arbitrary centralizers, as shown by Kalton in [136]. And this is so because, if M is large enough, the bimodule structure of $L^p(M, \tau)$ already encodes the "symmetric" structure of its "commutative" subspaces. Actually even the definition of $X(M, \tau)$ requires the symmetry of the function space X. To explain this, let us consider the following situation. Let $\mathcal{H} = \ell^2$ be the standard Hilbert space of 2-summable sequences $f: \mathbb{N} \to \mathbb{C}$ and consider the algebra $B(\mathcal{H})$ of all bounded operators on $\mathcal{H}$, with the usual trace. Then the corresponding L^p spaces are just the Schatten classes S^p.

Each bounded sequence $b \in \ell^\infty$ induces a multiplication operator $M_b(f) = b \cdot f$, which is "diagonal" with respect to the unit basis of $\mathcal{H}$.

It is clear from the preceding remark that if Ψ is any bicentralizer on S_p, then one may assume that $\Psi(x)$ is "diagonal" whenever x is so. Since diagonal operators in S_p correspond with multiplication operators by a sequence in ℓ^p we see that Ψ gives rise to a mapping ψ (actually an ℓ^∞-centralizer) on ℓ^p defined by $\Psi(M_f) = M_{\psi}(f)$.

Let us see that ψ must be symmetric. Indeed, let u be a permutation of $\mathbb{N}$ and consider the isometry of $\mathcal{H}$ given by $U(h) = h \circ u$. Then $U^* = U^{-1}$ is given by $h \longmapsto h \circ u^{-1}$. Note that if $b \in \ell^\infty$, then $UM_bU^* = M_{b \circ u}$ since for $h \in \mathcal{H}$

$$UM_bU^*(h) = U(M_b(h \circ u^{-1})) = U(b \cdot (h \circ u^{-1})) = (b \circ u) \cdot h = M_{b \circ u}(h).$$

Thus, if $f \in \ell^p$, and taking $b = \psi(f)$, we have

$$\|\psi(f \circ u) - (\psi(f)) \circ u\|_{\ell^p} = \left\|M_{\psi(f \circ u)} - M(\psi(f)) \circ u\right\|_{Sp}$$

$$= \left\|\Psi(M_{f \circ u}) - U\Phi(M_f)U^*\right\|_{S_p}$$

$$= \left\|U\Psi(M_f)U^* - U\Phi(M_f)U^*\right\|_{S_p} \leq C[\Psi]\left\|M_f\right\|_{Sp} = C[\Psi]\|f\|_{\ell^p}$$

and ψ is symmetric.

We face the problem of twisting arbitrary L^p spaces, including those built over type III von Neumann algebras. There are several constructions of these L^p spaces, none of them elementary. All provide bimodule structures on the resulting spaces that turn out to be equivalent at the end.

It is natural to ask for (nontrivial) self-extensions of $L^p(M)$ in the category of Banach bimodules over M. Unfortunately we have been unable to construct such objects; nevertheless we can still use the interpolation trick to obtain self extensions as (one-sided) modules. In this regard the most suited representation of L^p spaces is one due to Kosaki.

For the sake of clarity, we can restrict here to σ-finite algebras so that we can take functionals from M_*. So, let M be a von Neumann algebra and $\phi \in M_*$ a faithful positive functional. (We don't normalize ϕ because the restriction of a state to a direct summand is not a state; see Lemma (4.3.12)(b) below.) We "include" M into M_* just taking $a \in M \longmapsto$

$a\phi \in M_*$ thus starting the interpolation procedure with $\Sigma = M_*$ as "ambient" space and $\Delta = M\phi$, to which the norm and $\sigma(M, M_*)$ topology are transferred without further mention. Then, the Kosaki (left) version of the space $L^p(M)$ is defined as

$$L^p(\phi) = L^p(M, \phi) = [M\phi, M_*]_\theta, \qquad (\theta = 1/p).$$

We emphasize we are referring to Kosaki's construction [162], [144], [165] and not to that of Terp [167], [168]. Recall that M_* is an M-bimodule with product given by

$$\langle a\psi b, x\rangle = \langle \psi, bxa\rangle \qquad (\psi \in M_*;\ a, b, x \in M).$$

The inclusion $\cdot\,\phi: M \to M_*$ is, however, only a left-homomorphism: $(ba)\cdot\phi = b(a\phi)$. Asking for a two-sided homomorphism means that one should also have

$$(ab)\cdot\phi = ab\phi = a\phi b = (a\phi)\cdot b.$$

In particular (take $a = 1$) $b\phi = \phi b$ for all $b \in M$, which happens if and only if ϕ is a trace.

Let $\mathcal{G} = \mathcal{G}(M, \phi)$ denote the Calderon space associated to the couple $(M\phi, M_*)$ and put $\mathcal{G}_0 = \mathcal{G}(M, \phi)_0 = \{g \in \mathcal{G}: g(\theta) = 0\}$, where $\theta = 1/p$ is fixed. These are left M-modules in the obvious way and so are the quotients $L^p(M, \phi) = \mathcal{G}/\mathcal{G}_0$. Plug and play to get the push-out diagram

$$\begin{array}{ccccc} \mathcal{G}_0 = ker\ \delta_\theta & \longrightarrow & \mathcal{G} & \xrightarrow{\delta_\theta} & L^p(M,\phi) \\ \delta_\theta^{'} \downarrow & & \downarrow & & \| \\ L^p(M,\phi) & \longrightarrow & PO & \longrightarrow & L^p(M,\phi) \end{array} \qquad (48)$$

(where $\theta = 1/p$) and observe that every arrow here is a homomorphism of left M-modules. Let us denote by $Z_p(M, \phi)$ or $Z_p(\phi)$ the push-out space in the preceding diagram. This is coherent with the notation used in the tracial case.

We have mentioned that there is also a right action of M on $L^p(M, \phi)$ which is compatible with the given left action and makes $L^p(M, \phi)$ into a bimodule. All known descriptions of that action are quite heavy and depend on Tomita-Takesaki theory. That action is in general incompatible with the arrows in the preceding diagram.

Now, we are confronted with the problem of deciding whether the lower extension in Diagram 19 is trivial or not. The pattern followed in the proof of Theorem (4.3.9)cannot be used now because we have only a left multiplication in $Z_p(M, \phi)$.

Suppose we are given two von Neumann algebras M and N with distinguished faithful normal states ϕ and ψ. If $u_\infty: M \to N$ and $u_1: M_* \to N_*$ are operators making the square

$$\begin{array}{ccc} M & \xrightarrow{u_\infty} & N \\ \cdot\,\phi \downarrow & & \downarrow \cdot\,\psi \\ M_* & \xrightarrow{u_1} & N_* \end{array}$$

commutative, then interpolation yields operators $u_p: L^p(M, \phi) \to L^p(N, \psi)$ for each $p \in (1, \infty)$.

Lemma (4.3.12)[154]: Let M be a von Neumann algebra with a faithful positive normal functional ϕ. Let N be a subalgebra of M equipped with the restriction of ϕ. Suppose either (a) N is a von Neumann subalgebra of M and there is a normal conditional expectation $\varepsilon: M \to N$ leaving ϕ invariant; or

(b) N is a von Neumann algebra, and a direct summand in M.

Then, for each $p \in [1, \infty]$, there are homomorphisms of N-modules $\iota_p : L^p(N, \phi|_N) \to L^p(M, \phi)$ and $\varepsilon_p: L^p(M, \tau) \to L^p(N, \phi|_N)$ such that $\varepsilon_p \circ \iota_p$ is the identity on $L^p(N, \phi|_N)$.

Proof. (a) We have assembled the hypotheses in order to guarantee the commutativity of the diagram

$$\begin{array}{ccccc} N & \xrightarrow{\iota} & M & \xrightarrow{\varepsilon} & N \\ \cdot\,\phi|_N \downarrow & & \cdot\,\phi \downarrow & & \downarrow \cdot\,\phi|_N \\ N_* & \xrightarrow{\varepsilon_*} & M_* & \xrightarrow{\iota_*} & N* \end{array} \tag{49}$$

Here, $\iota: N \to M$ the inclusion map and the subscript indicates preadjoint (in the Banach space sense), in particular ι_* is plain restriction.

Indeed, for $a \in N$, one has $\varepsilon_*(a\phi) = a\varepsilon_*(\phi) = a\phi$, so the left square commutes. As for the right one, taking $a \in N, b \in M$ we have

$$\langle \varepsilon(b)\phi, a\rangle = \langle \phi, a\varepsilon(b)\rangle = \langle \phi, \varepsilon(ab)\rangle = \langle \phi, ab\rangle = \langle b\phi, a\rangle.$$

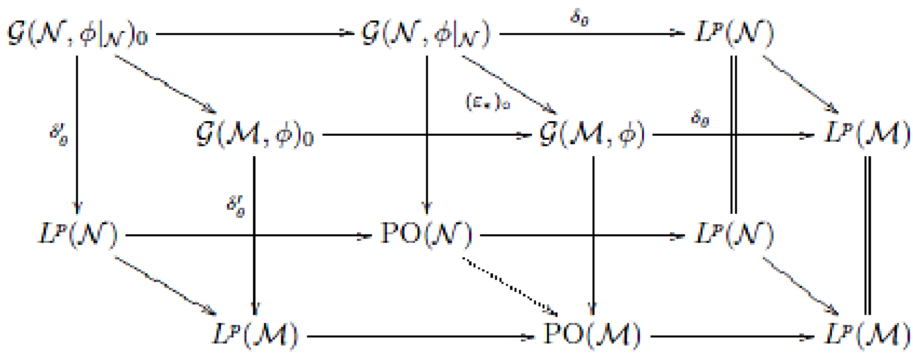
Notice, moreover, that $\varepsilon \circ \iota$ is the identity on N, while $\iota_* \circ \varepsilon_*$ is the identity on N_*. Therefore, interpolating (ι, ε_*) we get operators $\iota_p: L^p(N, \phi|_N \to L^p(M, \phi)$ for $1 \le p \le \infty$, while (ε, ι_*) gives operators $\varepsilon_p: L^p(M, \phi) \to L^p(N, \phi|_N)$. And since $\varepsilon_p \circ \iota_p$ is the identity on $L^p(N, \phi|_N)$ we are done.

(b) In this case we can use the same diagram, just replacing ε by the projection $P: M \to N$ given by $P(a) = eae$, where e is the unit of N. Then $P_*: N_* \to M_*$ is given by $\langle P_*(\psi), b\rangle = \langle \psi, ebe\rangle$.

The following step is the result we are looking for.

Lemma (4.3.13)[154]: With the same hypotheses as in Lemma (4.3.12), $Z_p(N, \phi|_N)$ is a complemented subspace of $Z_p(M, \phi)$ for every $1 < p < \infty$.

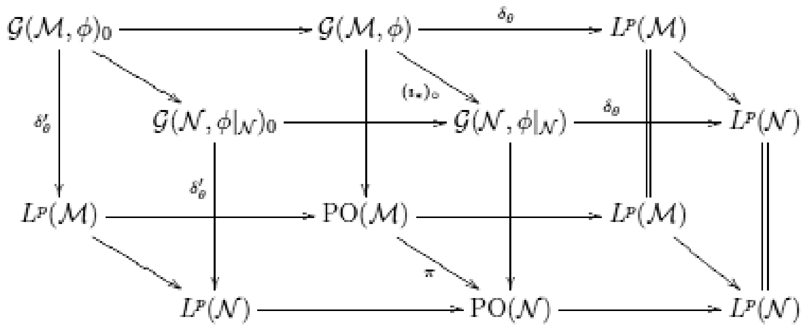
Proof. We write the proof assuming (a). The other case requires only minor modifications that are left to the reader. Let us begin with the embedding of $PO(N) = Z_p(N, \phi|_N)$ into $PO(M) = Z_p(M, \phi)$. Consider the diagram



Here, $(\varepsilon_*)_\circ$ sends a given function $f: \mathbb{S} \to N_*$ to the composition $\varepsilon_* \circ f: \mathbb{S} \to N_* \to M_*$ and the mappings from $L^p(N)$ to $L^p(M)$ are all given by ι_p. It is not hard to check that this is a commutative diagram. Therefore, we can insert an operator $\kappa: PO(N) \to PO(M)$ making the resulting diagram commutative because of the universal property of the push-out square

$$\begin{array}{ccc} \mathcal{G}(N, \phi|_N)_0 & \longrightarrow & \mathcal{G}(N, \phi|_N) \\ \delta_\theta' \downarrow & & \downarrow \\ L^p(N) & \longrightarrow & PO(N) \end{array}$$

A similar argument shows the existence of an operator $\pi: PO(M) \to PO(N)$ making commutative the diagram



The arrows from $L^p(M)$ to $L^p(N)$ are now given by ε_p. Putting together the two preceding diagrams it is easily seen that $\pi \circ \kappa$ is the identity on $PO(N)$.

Here is the main result about the twisting of Kosaki's L^p. As we shall see later $Z_p(M,\phi)$ doesn't depend on ϕ and so the conclusion of the following theorem holds for any ϕ.

Theorem (4.3.14)[154]: Let M be an infinite dimensional von Neumann algebra. There is a faithful normal state ϕ for which the lower extension of the push-out diagram

$$\begin{array}{ccccc} ker\ \delta_\theta & \longrightarrow & \mathcal{G}(M,\phi) & \xrightarrow{\delta_\theta} & L^p(M,\phi) \\ \delta'_\theta \downarrow & & \downarrow & & \| \\ L^p(M,\phi) & \longrightarrow & PO & \longrightarrow & L^p(M,\phi) \end{array}$$

is nontrivial.

Proof. The idea of the proof is to choose ϕ in such a way that its "centralizer subalgebra"

$$M^\phi = \{a \in M : a\phi = \phi a\}$$

is infinite dimensional. After that we proceed as follows. By [155], there is a normal conditional expectation $\varepsilon: M \to M^\phi$ leaving ϕ invariant: $\phi = \phi|_{M^\phi} \circ \varepsilon$. Actually ε is unique, by [155].

Apply now Lemma (4.3.13) to embed $PO(M^\phi, \phi)$ as a complemented subspace (in fact as a "complemented subextension") of $PO(M,\phi)$ and please note that the restriction of ϕ to M^ϕ is a (finite) trace by the very definition of M^ϕ.

The nonsplitting of $PO(M^\phi,\phi)$ is nothing but a particular case of Theorem (4.3.9) as for a finite trace τ one has $M \subset L^1(\tau)$ and, after identifying $L^1(\tau)$ with M_*, the inclusion agrees with Kosaki's left method.

In order to find out the required ϕ, let us decompose $M = N \oplus \mathcal{L}$, with N semifinite and $\mathcal{L}$ without direct summands of type I (This can be done in several ways: for instance, taking N as the semifinite part and $\mathcal{L}$ as the type III part of M, or taking N as the discrete part and $\mathcal{L}$ as the continuous part, see [155].)

By Lemma (4.3.13) we have an isomorphism $PO(M,\phi) = PO(N,\phi|_N) \oplus PO(\mathcal{L},\phi|_\mathcal{L})$ and we can consider the two cases separately.

(a) First, assume M has no direct summand of type I (so that it is either type II or III). Then, if ψ is any faithful normal state on M, there is a faithful normal state ϕ (in the closure of the orbit of ψ under the inner automorphisms of M) whose centralizer subalgebra M^ϕ is of type II_1 ([161]) and we are done.

(b) Now, suppose M semifinite and let us see that any ϕ works. Let τ be a (fns) trace on M and let us identify $M_*$with $L^1(\tau)$ so that we may consider ϕ as a τ -measurable operator on the ground Hilbert space. If ϕ is elementary, let us write it as $\phi = \sum_{i=1}^n t_i e_i$, where the e_i are mutually orthogonal projections in M. Letting $M_i = e_i M e_i$ we see that $\oplus_i M_i$ is an infinite dimensional subalgebra of M^ϕ, which is enough. Otherwise ϕ has infinite spectrum and its spectral projections already generate an infinite dimensional subalgebra of M^ϕ.

We now give a description of the dual of $Z_p(M,\phi)$ for general M. To this end we consider the right embedding of M into M_* given by $a \mapsto \phi a$ which is a homomorphism of right modules and the new couple $(\phi M, M_*)$. The former couple using the left embedding is denoted by $(M\phi, M_*)$. The right version of Kosaki $L^p$is

$$L^p(M,\phi)^r = [\phi M, M_*]_{1/p} = [M_*, \phi M]_{1-1/p}.$$

Let us define $Z_p(M,\phi)^r$ as the push-out space (actually right module on M) in the ubiquitous diagram

$$\begin{array}{ccccc} ker\,\delta_\theta & \longrightarrow & \mathcal{G}(M,\phi)^r & \xrightarrow{\delta_\theta} & L^p(M,\phi)^r \\ \delta_\theta' \downarrow & & \downarrow & & \| \\ L^p(M,\phi)^r & \longrightarrow & PO & \longrightarrow & L^p(M,\phi)^r \end{array} \tag{50}$$

where $\theta = 1/p$ and $\mathcal{G}(M,\phi)^r$ is the Calderon space associated to the couple $(\phi M, M_*)$.
We want to see that if $p, q \in (1, \infty)$ are conjugate exponents, then the conjugate of $Z_p(M,\phi)^\ell$ (our former $Z_p(M,\phi)$) is well isomorphic to $Z_q(M,\phi)^r$.
Consider the couples $(M\phi, M_*)$ and $(M_*, \phi M)$ (not $(\phi M, M_*)$!). Then

$$\Delta^\ell = \Delta(M\phi, M_*) = M\phi \qquad and \qquad \Delta^r = \Delta(M_*, \phi M) = \phi M.$$

Both $M\phi$ and ϕM are dense in M_* since ϕ is faithful. Define a bilinear form $\beta: \Delta^\ell \times \Delta^r \to \mathbb{C}$ by $\beta(a\phi, \phi b) = \phi(ba)$. The key point in that

$$\beta(a\phi, \phi b) = \langle a\phi, b\rangle = \langle a, \phi b\rangle,$$

where the brackets refer to the dual pairing between M_* and M. (Notice, moreover, that β is balanced in the sense that $\beta(cf, g) = \beta(f, gc)$ for $f \in \Delta^\ell, g \in \Delta^r$ and $c \in M$.)
Then β is bounded both at $\theta = 0$ and $\theta = 1$. Indeed, for $\theta = 0$ one has

$$|\phi(ba)| \le \|a\|_M \|\phi b\|_{M_*}.$$

Similarly, when $\theta = 1$,

$$|\phi(ba)| \le \|a\phi\|_{M_*} \|b\|_M.$$

By bilinear interpolation [44] β extends to a bounded bilinear form on $L^p(M,\phi)^\ell \times L^q(M,\phi)^r = [M\phi, M_*]_\theta^\ell \times [M_*, \phi M]_\theta^r$ which provides the dual pairing between $L^p(M,\phi)^\ell$ and $L^q(M,\phi)^r$ (see [162] or [144]). Let us call β to that extension.
For $1 < p < \infty$, let $\Omega_p^\ell: L^p(M,\phi) \to M_*$ be the derivation associated to the identity $[M\phi, M_*]_{1/p}^\ell = L^p(M,\phi)$ and $\Omega_p^r: L^p(M,\phi)^r \to M_*$ that associated to $[\phi M, M_*]_{1/p}^\ell = L^p(M,\phi)^r$. Note that if $\theta = 1/p$, then the derivation associated to $[M_*, \phi M]_\theta^r = L^q(M,\phi)^r$ is just $-\Omega_q^r$. Proposition 1.3 in [39] yields

$$|\beta(\Omega_p^\ell(f), g) - \beta(f, \Omega_q^r(g))| \le \frac{\pi}{sin(\pi\theta)} \|f\|_p \|g\|_q$$

at least when $f \in M\phi$ and $g \in \phi M$. The following result is implicit in [54].
Theorem (4.3.15)[154]: Given conjugate exponents $p, q \in (1, \infty)$, the dual of

$$Z_p(M,\phi)^\ell = L^p(M,\phi)^\ell \oplus_{\Omega_p^\ell} L^p(M,\phi)^\ell$$

is isomorphic to

$$-Z_q(M,\phi)^r = L^q(M,\phi)^r \oplus_{-\Omega_q^r} L^q(M,\phi)^r.$$

More precisely, there is an isomorphism of right Banach modules over M making commutative the following diagram

$$\begin{array}{ccccc} (L^p(M,\phi)^\ell)^* & \xrightarrow{\pi^*} & (Z_p(M,\phi)^\ell)^* & \xrightarrow{\iota^*} & (L^p(M,\phi)^\ell)^* \\ \| & & \uparrow u & & \| \\ L^q(M,\phi)^r & \longrightarrow & L^q(M,\phi)^r \oplus_{-\Omega_q^r} L^q(M,\phi)^r & \longrightarrow & L^q(M,\phi)^r \end{array} \tag{51}$$

Proof. Put

$$(u(g',g))(f',f) = \beta(f,g') + \beta(f',g) \qquad (g \in \Delta^r, f \in \Delta^\ell). \quad (52)$$

We have

$$\begin{aligned} &|\beta(f,g') + \beta(f',g)| \\ &= |\beta(f,g' + \Omega_q^r(g)) - \beta(f,\Omega_q^r(g)) + \beta(f' - \Omega_p^\ell(f),g) + \beta(\Omega_p^\ell(f),g)| \\ &\le \|f\|_p \left\|g' + \Omega_q^r(g)\right\|_q + \left\|f' - \Omega_p^\ell(f)\right\|_p \|g\|_q + \frac{\pi}{sin(\pi/p)} \|f\|_p \|g\|_q \\ &\le \frac{\pi}{sin(\pi/p)} \|(g',g)\|_{-\Omega_q^r} \|(f',f)\|_{\Omega_p^\ell}. \end{aligned}$$

As Δ^ℓ is dense in $L^p(M,\phi)^\ell$, we see that $L^p(M,\phi)^\ell \oplus_{\Omega_p^\ell} \Delta^\ell$ is dense in $Z_p(M,\phi)^\ell$ and so (52) shows that $u(g',g)$ acts, as a bounded linear functional on $Z_p(M,\phi)^\ell$, with

$$\left\|u(g',g): Z_p(M,\phi)^\ell \to \mathbb{C}\right\| \le M \left\|(g',g)\right\|_{-\Omega_q^r},$$

at least when g is in Δ^r. This defines an operator making the following diagram commute:

$$\begin{array}{ccccc} (L^p(M,\phi)^\ell)^* & \xrightarrow{\pi^*} & (Z_p(M,\phi)^\ell)^* & \xrightarrow{\iota^*} & (L^p(M,\phi)^\ell)^* \\ \| & & \uparrow u & & \| \\ L^q(M,\phi)^r & \longrightarrow & L^q(M,\phi)^r \oplus_{-\Omega_q^r} \Delta^r & \longrightarrow & \Delta^r \end{array} \quad (53)$$

and where Δ^r is treated as a submodule of $L^q(M,\phi)^r$. By density u extends to an operator that we still call u fitting in (51). The five-lemma and the open mapping theorem guarantee that u is a linear homeomorphism. It remains to check it is also a homomorphism of right M-modules. But for $g \in \Delta^r$ and $f \in \Delta^\ell$ one has

$$\begin{aligned} u((g',g)a)(f',f) &= (u(g'a,ga))(f',f) = \beta(f,g'a) + \beta(f',ga) \\ &= \beta(af,g') + \beta(af',g) = u(g',g)(af',af) = (u(g',g)a)(f',f). \end{aligned}$$

This completes the proof.

We prove the extension $L^p(M,\phi) \to Z_p(M,\phi) \to L^p(M,\phi)$ is essentially independent on the reference state ϕ in the following precise sense.

Proposition (4.3.16)[154]: Let ϕ_0 and ϕ_1 be faithful normal states on M and $p \in (1, \infty)$. Then there is a commutative diagram

$$\begin{array}{ccccc} L^p(M,\phi_0) & \longrightarrow & Z_p(M,\phi_0) & \longrightarrow & L^p(M,\phi_0) \\ \alpha \downarrow & & \downarrow & & \downarrow \alpha \\ L^p(M,\phi_1) & \longrightarrow & Z_p(M,\phi_1) & \longrightarrow & L^p(M,\phi_1) \end{array}$$

in which the vertical arrows are isomorphisms of left M-modules.

Proof. The proof is based on an idea explained and discarded by Kosaki in [162]. We remark that our proof provides a very natural isometry between L^p spaces based on two different states.

It will be convenient to consider two more spaces of analytic functions. The first one is the obvious adaptation of the space $\mathcal{H}$ appearing to the nontracial setting. So, given a faithful state $\phi \in M_*$, we consider the couple $(M\phi, M_*)$, and the space $\mathcal{H} = \mathcal{H}(M,\phi)$ of bounded functions $H: \mathbb{S} \to M_*$ such that:

(i) H is continuous on $\mathbb{S}$ and analytic on $\mathbb{S}^\circ$ with respect to $\sigma(M_*, M)$.

(ii) $H(it) \in M\phi$ for every $t \in \mathbb{R}$. The function $t \in \mathbb{R} \longmapsto H(it) \in M\phi$ is M-bounded and $\sigma(M, M_*)$-continuous.

(iii) The function $t \in \phi \longmapsto H(1+it) \in M_*$ is continuous in the norm of M_*.

As one may expect we furnish $\mathcal{H}$ with the norm $\|H\|_{\mathcal{H}} = sup_t(\|H(it)\|_M, \|H(1+it)\|_{M_*})$. Of course, $\mathcal{H}$ is larger than $\mathcal{G}$. The second space we shall denote by $\mathcal{F} = \mathcal{F}(M,\phi)$ is the space of those $f \in \mathcal{G}(M,\phi)$ satisfying the additional condition that $f(it) \to 0$ in $M = M\phi$ as $|t| \to \infty$ and $f(1+it) \to 0$ in M_* as $|t| \to \infty$. Moreover the complex method of interpolation, applied to the couple (M, M_*), leads to the same scale using $\mathcal{F}, \mathcal{G}$ or $\mathcal{H}$:

$$[M, M_*]_\theta^{\mathcal{F}} = [M, M_*]_\theta^{\mathcal{G}} = [M, M_*]_\theta^{\mathcal{H}} = L^p(M,\phi) \quad (\theta = 1/p)$$

with identical norms. This is very easy to check, once we know that $L^p(M,\phi)$ is reflexive and agrees with the dual of the right space $L^q(M,\phi)^r$, where q is the conjugate exponent of p. As Lemma (4.3.3) is true (with the same proof) replacing $\mathcal{G}$ by $\mathcal{F}$ or by $\mathcal{H}$ we see that the lower extension in Diagram 19 does not vary after replacing $\mathcal{G}$ by $\mathcal{F}$ or by $\mathcal{H}$.

We shall use the following notations:

$$\mathcal{F}_0(M,\phi) = \{F \in \mathcal{F}(M,\phi) : F(\theta) = 0\},$$
$$\mathcal{F}_1(M,\phi) = \{F \in \mathcal{F}(M,\phi) : F(\theta) = F'(\theta) = 0\}$$

and similarly for $\mathcal{G}$ and $\mathcal{H}$. As we mentioned after Lemma (4.3.4) one has isomorphisms

$$Z_p(M,\phi) = \frac{\mathcal{F}(M,\phi)}{\mathcal{F}_1(M,\phi)} = \frac{\mathcal{G}(M,\phi)}{\mathcal{G}_1(M,\phi)} = \frac{\mathcal{H}(M,\phi)}{\mathcal{H}_1(M,\phi)}.$$

It is important to realize how these quotient spaces arise as self-extensions of $L^p = L^p(M,\phi)$. We describe the details for the smaller space $\mathcal{F}$; replacing it by $\mathcal{G}$ or $\mathcal{H}$ makes no difference. Recall that we have $\mathcal{F}_1 \subset \mathcal{F}_0 \subset \mathcal{F}$ and therefore an exact sequence

$$0 \longrightarrow \mathcal{F}_0/\mathcal{F}_1 \xrightarrow{\mathcal{I}} \mathcal{F}/\mathcal{F}_1 \xrightarrow{\varpi} \mathcal{F}/\mathcal{F}_0 \longrightarrow 0$$

where $\mathcal{I}$ and ϖ are the obvious maps. This becomes a self-extension of L^p after identifying $\mathcal{F}/\mathcal{F}_0$ with L^p through the (factorization) of the evaluation map $\delta_\theta : \mathcal{F} \to L^p$ at $\theta = 1/p$, while the identification of $\mathcal{F}_0/\mathcal{F}_1$ with L^p is provided by the (factorization) of the derivative $\delta_\theta' : \mathcal{F}_0 \to L^p$ (at $\theta = 1/p$) which is an isomorphism of left modules over M.

We conclude these prolegomena with the following observation. Let $\mathcal{E}(M,\phi)$ denote the subspace of those $F \in \mathcal{F}(M,\phi)$ having the form $F(z) = f(z)\phi$, where $f : \mathbb{S} \to M$ is continuous and analytic on the interior. It turns out that $\mathcal{E}(M,\phi)$ is dense in $\mathcal{F}(M,\phi)$. Indeed, the set of functions having the form $F(z) = f(z)\phi$, with

$$f(z) = exp(\lambda z^2) \sum_{i=1}^{n} \exp(\lambda_i z)\, a_i \quad (\lambda, \lambda_i \in \mathbb{R}, a_i \in M)$$

is already a dense subspace of $\mathcal{F}(M,\phi)$. See [44].

Let $\varphi : \mathbb{S} \to \mathbb{D}$ be the function given by (35). Replacing $\mathcal{G}$ by $\mathcal{F}$ everywhere in the proof of Lemma (4.3.3) we see that $\mathcal{F}_0 = \varphi\mathcal{F}$ in the sense that the multiplication operator $f \longmapsto \varphi f$ is an isomorphism between $\mathcal{F}$ and $\mathcal{F}_0$. Similarly, $f \longmapsto \varphi^2 f$ is an isomorphism between $\mathcal{F}$ and $\mathcal{F}_1$. It follows that $\mathcal{E} \cap \mathcal{F}_0$ and $\mathcal{E} \cap \mathcal{F}_1$ are dense in $\mathcal{F}_0$ and in $\mathcal{F}_1$, respectively.

Now we need a bit of (relative) modular theory see [162] or [144]. We fix two faithful states $\phi_0, \phi_1 \in M_*$ and we consider the Connes-Radon-Nikodym cocycle of ϕ_0 relative to ϕ_1:

$$(D\phi_0; D\phi_1)_t = \Delta_{\phi_0\varphi_1}^{it} \Delta_{\phi_0}^{-it} \; (t \in R).$$

As it happens, $t \longmapsto (D\phi_0; D\phi_1)_t$ is a strongly continuous path of unitaries in M and so

$$t \longmapsto (D\phi_0; D\phi_1)_t \phi_1 \tag{54}$$

defines a continuous function from $\mathbb{R}$ to M_*. Now the point is that (54) extends to a function from the horizontal strip $-i\mathbb{S} = \{z \in \mathbb{C} : -1 \leq \Im(z) \leq 0\}$ to M_* we may denote by $\overline{(D\phi_0; D\phi_1)_{(\cdot)}\phi_1}$ having the following properties:

(a) For each $x \in M$, the function $z \longmapsto \langle \overline{(D\phi_0; D\phi_1)_{(z)}\phi_1}, x \rangle$ is continuous on $-i\mathbb{S}$ and analytic on $i\mathbb{S}^\circ$.

(b) $\overline{(D\phi_0;D\phi_1)_{(-\imath+t)}\phi_1} = \phi_0(D\phi_0;D\phi_1)_t$ for every real t.
We are going to define an isometric embedding of modules $I\colon \mathcal{F}(M,\phi_0) \to \mathcal{H}(M,\phi_1)$.
First, for $F \in \mathcal{E}(M,\phi_0)$, we put

$$(IF)(z) = f(z)\overline{(D\phi_0;D\phi_1)_{(-\imath z)}\phi_1}\ (F(z) = f(z)\phi_0, z \in \mathbb{S}). \tag{55}$$

We observe that for such an F one has $\|F\|_{\mathcal{F}} = \max\{\|f(it)\|_M, \|f(1+it)\phi_0\|_{M_*} : t \in \mathbb{R}\}$.
Let us check that $IF \in \mathcal{H}(M,\phi_1)$. That IF satisfies (30) is obvious from (a). Regarding the values of IF on the boundary of $\mathbb{S}$ we have for real t:

$$(IF)(it) = f(it)(D\phi_0;D\phi_1)_t\phi_1 \tag{56}$$

which certainly falls in $M\phi_1$ since $(D\phi_0;D\phi_1)_t$ is unitary and, besides, $\|f(it)(D\phi_0;D\phi_1)_t\|_M = \|f(it)\|_M$. Moreover, the function $t \in \mathbb{R} \mapsto f(it)(D\phi_0;D\phi_1)_t \in M$ is $\sigma(M,M_*)$ continuous since $t \in \mathbb{R} \longmapsto f(it) \in M$ is continuous for the norm and $t \in \mathbb{R} \longmapsto (D\phi_0;D\phi_1)_t \in M$ is strongly (hence $\sigma(M,M_*)$) continuous. So (31) holds as well.
On the other hand,

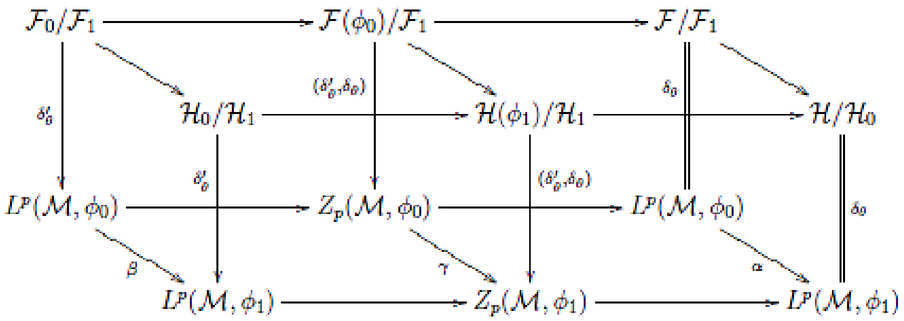
$$(IF)(1+it) = f(1+it)\overline{(D\phi_0;D\phi_1)_{(-\imath+t)}\phi_1} = f(1+it)\phi_0(D\phi_0;D\phi_1)_t,$$

so $\|(IF)(1+it)\|_{M_*} = \|f(1+it)\phi_0(D\phi_0;D\phi_1)_t\|_{M_*} = \|f(1+it)\phi_0\|_{M_*}$ and $(IF)(1+it)$ is continuous in t for the norm topology of M_*. Finally, that IF is M_* bounded on the whole $\mathbb{S}$ now follows by interpolation, using (56). Hence IF belongs to $\mathcal{H}(M,\phi_1)$ and, moreover, the norm of IF in $\mathcal{H}(M,\phi_1)$ and the norm of F in $\mathcal{F}(M,\phi_0)$ coincide.
By density, I extends to an isometric homomorphism of left M-modules from $\mathcal{F}(M,\phi_0)$ into $\mathcal{H}(M,\phi_1)$ that we call again I.

Now we observe that I maps $\mathcal{F}_0(M,\phi_0)$ into $\mathcal{H}_0(M,\phi_1)$. Indeed, it is obvious from (55) that IF vanishes at θ if $F \in \mathcal{E}$ vanishes at θ and for arbitrary F the result follows by a density argument, taking into account that $\mathcal{H}_0(M,\phi_1)$ is closed in $\mathcal{H}(M,\phi_1)$. In particular, I induces a contractive homomorphism from $L^p(M,\phi_0)$ to $L^p(M,\phi_1)$.
Similarly, I maps $\mathcal{F}_1(M,\phi_0)$ into $\mathcal{H}_1(M,\phi_1)$. Indeed, for $F \in \mathcal{E}(M,\phi_0)$ one has

$$(IF)'(z) = f'(z)\overline{(D\phi_0;D\phi_1)_{(-\imath z)}\phi_1} + f(z)\frac{d}{dz}\overline{(D\phi_0;D\phi_1)_{(-\imath z)}\phi_1}(F(z) = f(z)\phi_0).$$

Thus, if $F(\theta) = F'(\theta) = 0$ then $f(\theta) = f'(\theta) = 0$ and therefore $(IF)(\theta) = (IF)'(\theta) = 0$ and we proceed as before for general F.
Therefore we have a commutative diagram



The arrows in the preceding Diagram can be described as follows. First, all arrows in the upper face going from left to right are the obvious ones. All arrows in the upper face going from spaces based on ϕ_0 to spaces based on ϕ_1 are induced by I.

All vertical arrows are given by (factorization of) evaluations at $\theta = 1/p$, as indicated in the diagram. They are all isomorphisms of left modules over M. Thus, for instance

$(\delta_\theta', \delta_\theta): \mathcal{H}(\phi_1)/\mathcal{H}_1 \to Z_p(M, \phi_1)$ takes (the class of) $H \in \mathcal{H}(M, \phi_1)$ into the pair $(H'(\theta), H(\theta)) \in Z_p(M, \phi_1)$ and so on.

The arrows lying in the bottom face are mere "shadows" of the corresponding arrows in the top face. It is really easy to see that all arrows in the bottom face going from left to right act as expected. Let us identify the arrows of the bottom face going from objects based on ϕ_0 to objects based on ϕ_1. We begin with α. Suppose $x \in L^p(\phi_0)$ has the form $x = a\phi_0$, with $a \in M$. Let $\varepsilon: \mathbb{S} \to \mathbb{C}$ be an analytic function such that $\varepsilon(\theta) = 1$ and $\varepsilon(\infty) = 0$. Letting $F(z) = \varepsilon(z)a\phi_0$ we have $(IF)(z) = \varepsilon(z)\overline{a(D\phi_0; D\phi_1)_{(-\imath z)}\phi_1}$ and so $\alpha(a\phi_0) = (IF)(\theta) = \overline{a(D\phi_0; D\phi_1)_{(-\imath\theta)}\phi_1}$.

In order to identify β we take again $x = a\phi_0$ and we "jump" to $\mathcal{F}_0$ taking $F(z) = (\varphi'(\theta))^{-1}\varepsilon(z)\varphi(z)a\phi_0$, where φ is the function defined by (35). Note that

$$\varphi'(\theta) = \frac{\pi}{2}\frac{\exp(i\pi\theta)}{sin(\pi\theta)} \neq 0.$$

One has

$$F'(\theta) = \frac{(\varepsilon\varphi)'(\theta)}{\varphi'(\theta)}a\phi_0 = a\phi_0.$$

Hence $\beta(a\phi_0) = (IF)'(\theta)$, where $(IF)(z) = (\varphi'(\theta))^{-1}\varepsilon(z)\varphi(z)\overline{a(D\phi_0; D\phi_1)_{(-\imath z)}\phi_1}$ and so by Leibniz's rule

$$\beta(a\phi_0) = (IF)'(\theta) = \overline{a(D\phi_0; D\phi_1)_{(-\imath\theta)}\phi_1} = \alpha(a\phi_0).$$

Finally, the same argument shows that

$$\gamma(b\phi_0, a\phi_0) = (\overline{b(D\phi_0; D\phi_1)_{(-\imath\theta)}\phi_1}, \overline{a(D\phi_0; D\phi_1)_{(-\imath\theta)}\phi_1}.$$

To complete the proof we have to prove that α is an isomorphism – that γ is an isomorphism then follows from the five-lemma. This is not automatic because $I: \mathcal{F}(\phi_0) \to \mathcal{H}(\phi_1)$ is not surjective.

Anyway, reversing the roles of ϕ_0 and ϕ_1 we know that there is a homomorphism of left M modules $\omega: L^p(M, \phi_1) \to L^p(M, \phi_0)$ such that

$$\omega(x) = a\left(\overline{(D\phi_0; D\phi_1)_{(-\imath\theta)}\phi_0}\right) \qquad (x = a\phi_1 \in L^p(\phi_1)).$$

We will prove that α and ω are inverse of each other.

To this end, let us say that $a \in M$ is "analytic" if the map

$$t \in \mathbb{R} \longmapsto a(D\phi_0; D\phi_1)_t \in M$$

extends to an entire function we shall denote by $\overline{a(D\phi_0; D\phi_1)_{(\cdot)}}$. This is a "left" version of the usual definition; see, e.g., [162]. Let A denote the set of "analytic" operators in M. It is not hard to see that the $\mathcal{A}$ is σ-weak $(= \sigma(M, M_*))$ dense in M and so the set $\mathcal{A}\phi_0 = \{a\phi_0 : a \in \mathcal{A}\}$ is dense in $L^p(\phi_0)$. Thus the proof will be complete if we show that $\omega(\alpha(a\phi_0)) = a\phi_0$ for $a \in \mathcal{A}$. But for such an a we have

$$\alpha(a\phi_0) = (\overline{a(D\phi_0; D\phi_1)_{(-\imath\theta)}}) \cdot \phi_1 = a \cdot (\overline{(D\phi_0; D\phi_1)_{(-\imath\theta)}\phi_1})$$

by the uniqueness of analytic continuation. Therefore, as $\overline{a(D\phi_0; D\phi_1)_{(-\imath\theta)}}$ belongs to M,

$$\omega(\alpha(a\phi_0)) = \omega\left((\overline{a(D\phi_0; D\phi_1)_{(-\imath\theta)}}) \cdot \phi_1\right)$$
$$= (\overline{a(D\phi_0; D\phi_1)_{(-\imath\theta)}}) \cdot (\overline{(D\phi_1; D\phi_0)_{(-\imath\theta)}\phi_0}) = a\phi_0,$$

again by the uniqueness of analytic continuation, taking into account that $(D\phi_0; D\phi_1)_t = ((D\phi_1; D\phi_0)_t)^* = ((D\phi_1; D\phi_0)_t)^{-1}$ for $t \in \mathbb{R}$.

Chapter 5
Contractive Spectral and Noncommutative Solenoids with Spectral Triples

We develops part of the Bellissard-Marcolli-Reihani theory for a general discrete group action, and in particular, introduces coaction spectral triples and their associated metric notions. The isometric condition is replaced by the contractive condition. We show that noncommutative solenoids can be approximated by finite dimensional quantum compact metric spaces, and that they form a continuous family of quantum compact metric spaces over the space of multipliers of the solenoid, properly metrized. In all the examples treated here, the noncommutative solenoidal spaces have the same metric dimension and volume as on the base space, but are not quantum compact metric spaces, namely the pseudo-metric induced by the spectral triple does not produce the $weak^*$ topology on the state space.

Section (5.1): Triples for Crossed Products

Throughout, $X = (A, \mathcal{H}, D)$ will be a spectral triple in the sense of Connes ([172], [173]). This can be defined as follows. First, A is a C^*-algebra, which we will always assume to be unital, equipped with a faithful (non-degenerate) representation π on a Hilbert space $\mathcal{H}$, and second, D is a (usually unbounded) self-adjoint operator on $\mathcal{H}$ with compact resolvent. Third, we require that the set $C^1(X)$ of a's in A for which $\pi(a)Dom(D) \subset Dom(D)1$ and $\|[D, \pi(a)]\| < \infty$ is dense in A. (The operator $[D, \pi(a)]$ is at this stage, of course, only defined on $Dom\ D$, but since the latter is dense in $\mathcal{H}$ and $[D, \pi(a)]$ is bounded, it extends by continuity to an element of $B(\mathcal{H})$ with the same norm, and so can be regarded as actually belonging to $B(\mathcal{H})$.) We will sometimes regard A as a subalgebra of $B(\mathcal{H})$ and omit reference to the π.

In his development of noncommutative geometry, Connes showed that spectral triples not only give a context for K-homology and cyclic cohomology . Particularly notable was his observation (e.g. [173]) that for a compact spin manifold M, one can recover, among other things, the (geodesic) distance d on M from the canonical spectral triple $(C(M), \mathcal{H}, D)$ where ([190],[198]) $\mathcal{H}$ is the Hilbert space of L^2-spinors on M and D is the (self-adjoint) Dirac operator of M. This recovery is achieved by considering the space of Lipschitz functions $\mathcal{A}$ on M. Indeed, each $a \in \mathcal{A}$ can be regarded as a multiplication operator on $\mathcal{H}$, and the commutator $[D, a]$ is densely defined and extends to a bounded linear operator on $\mathcal{H}$. The distance function d on M is then determined for $p, q \in M$ by:

$$d(p,q) = sup\{|a(p) - a(q)|: |||[D,a]||| \leq 1\}. \quad (1)$$

In particular, the right-hand side of (1) determines a metric for the topology of M. We can, think of points of M as states on the C^*-algebra $C(M)$, and Connes pointed out that, more generally, if we replace $a(p) - a(q)$ by $\phi(a) - \psi(a)$ above, we can extend the metric d to a metric (also denoted d) on the state space $S(C(M))$ (i.e. the set of probability measures on M) of $C(M)$. Further, the metric topology of d on the state space is just the weak*-topology. This approach is motivation for replacing the special spectral triple $(C(M), \mathcal{H}, D)$ by an arbitrary spectral triple $X = (A, \mathcal{H}, D)$, and this gives a pseudo-metric d_X, or simply d, on $S(A)$. Following [171], we will refer to d as the Connes pseudo-metric. So for $\phi, \psi \in S(A)$,

$$d(\phi, \psi) = sup\{|\phi(a) - \psi(a)|: a \in C^1(X), \|[D,a]\| \leq 1\}. \quad (2)$$

Two natural questions arise. (See the discussion in [171].) First, when is d actually a metric on $S(A)$? Referring to (2), we see that obstacles to this are (1) the degeneracy of the representation π of A on $\mathcal{H}$, and (2) there are non-trivial a's (i.e. a's that are not multiples of the identity) in the metric commutant $(\{a \in C^1(X): [D, a] = 0\})$ of D. In fact ([192],

[86], [194]) non-degeneracy for π and triviality of the metric commutant are necessary and sufficient conditions for d to be a metric. The second question was raised and studied by Rieffel: given that d is a metric on $S(A)$, when does its metric topology coincide with the weak*-topology? The answer in the unital case ([192], [86], [195], [191]) is that the two topologies coincide if and only if the image of the Lipschitz ball has compact closure in $A/\mathbb{C}1$. The corresponding result for the non-unital case was given by Latremoliere ([189]). The main inspiration for the present is the recent work on spectral triples for group actions on the C^*-algebra A by Bellissard, Marcolli and Reihani ([171]), in particular in the case when the group is $\mathbb{Z}$ (so that only a single automorphism of A is involved). For an ordinary metric space, there are a number of geometric notions associated with an action of a group on the space by homeomorphisms. These include, in particular, the familiar notions of quasi-isometric, equicontinuous and isometric. It is shown in [171] that there are corresponding notions in the noncommutative case, i.e. for spectral triples.

These noncommutative versions are used in a central theme of the investigations of [171], viz. given a spectral triple $X = (A, \mathcal{H}, D)$ where A supports an action α of $\mathbb{Z}$ by automorphisms, how to define a dual spectral triple Y on the (reduced) crossed product C^*-algebra $A \rtimes_{\alpha,r} \mathbb{Z}$. ([171], write $Y = X \rtimes_{\alpha} \mathbb{Z}$ and call it the regular representation of the metric dynamical system (X, α).) Motivation for such a study is that taking an appropriate dual action can greatly simply the study of the original spectral triple. A remarkable example of how an appropriate crossed product can simplify the study of the original is in the von Neumann algebra category, where the Takesaki duality theorem (e.g. [33]) says (among other things) that taking the crossed product of a von Neumann algebra for the action of the modular automorphism group (corresponding to a faithful normal state) transforms a type III factor into a type II_∞ von Neumann algebra. (The C^*-algebra version of this is given by the Imai-Takai duality theorem ([182], [188], [179]) which in its general form, uses the dual coaction - in particular, G does not have to be abelian.) A philosophically similar, but geometrical, situation arose in the work of Connes and Moscovici ([174]) in the context of diffeomorphism invariant geometry. There, one needs to consider the crossed product $C_0(W) \rtimes \Gamma$ where W is a compact Riemannian manifold and Γ a subgroup of $Diff(W)$. In general, the action preserves no structure at all, in particular, no Riemannian metric is invariant under the action. However, if we replace W by the metric bundle $\mathcal{W}$ over W, whose fiber over $w \in W$ is the space of Euclidean metrics on the tangent space $T_x W$ then there is an invariant metric on $\mathcal{W}$ invariant under the natural action of Γ, and the shift from W to $\mathcal{W}$ corresponds to the shift from the type III situation to one of type II as above. (See [171] for a detailed description of the construction of the metric bundle.)

Among a number of results [171], show the following (for a $\mathbb{Z}$-action α on A). Given that X is equicontinuous, there exists a natural “dual” spectral triple Y for the reduced crossed product $A \rtimes_{\alpha,r} \mathbb{Z}$, where

$$Y = (A \rtimes_{\alpha,r} \mathbb{Z}, K \otimes \mathbb{C}^2, \widehat{D}).$$

Here, K is the space of sequences $\ell^2(\mathbb{Z}, \mathcal{H}) = \mathcal{H} \otimes \ell^2(\mathbb{Z})$, and $\widehat{D}$ is given by a diagonal operator whose entry over $n \in \mathbb{Z}$ is the 2×2 matrix with zero diagonal entries and off-diagonal entries $D \mp \imath n$. One considers the dual action of $\widehat{\mathbb{Z}} = \mathbb{T}$ on $A \rtimes_{\alpha,r} \mathbb{Z}$. Further results in [171] are:

(i) Y is isometric;

(ii) if X is such that the metric commutant is trivial and the image of the Lipschitz ball has compact closure in $A/\mathbb{C}1$, then the Connes metrics induced on the state space of A by both X, Y are equivalent (and give the weak* topology of A);
(iii) if X is not equicontinuous but is quasi-isometric, it can effectively be replaced by a spectral triple that is equicontinuous (using a "metric bundle" construction inspired by that of Connes-Moscovici above).

A number of interesting examples illustrating the theory is given.

These results involve, actions by the group $\mathbb{Z}$. However, it is desirable to extend them to actions by general discrete groups. We saw this above in the discussion of the metric bundle, where the group acting could be any subgroup of $Diff(W)$. More generally, in further work of Connes and Moscovici ([175]), allowing for local rather than just global diffeomorphisms, one needs to consider the case where the transformation group is replaced by an etale groupoid. We will prove the general version of (i) for a discrete group acting on A. While there is, of course, much more to be done to extend to this general context the other results in [171], even in the case of (i) alone, there are, as we shall see, questions that first have to be resolved.

We now define the two geometrical notions that we will require for an action α of a discrete group on a spectral triple $X = (A, \mathcal{H}, D)$. First, we say that X is pointwise bounded if the set

$$C_b^1(G, X) = \{a \in C^1(X): \alpha_g(a) \in C^1(X)\ for\ all\ g \in G\ and\ sup_{g \in G} \left\|[D, \alpha_g(a)]\right\| < \infty\}$$

is dense in A (or equivalently dense in $C^1(X)$). This is weaker than the "equicontinuous" condition used in [171]: there, X is equicontinuous if $C_b^1(G, X) = C^1(X)$. The motivation for the terminology "pointwise bounded" is that for each appropriate "point" $a \in A$, the maps $g \to [D, \alpha_g(a)]$ are uniformly bounded, so that the set of "functions" $a \to [D, \alpha_g(a)] (g \in G)$ is pointwise bounded. We can think of this condition as corresponding to the "pointwise bounded" condition in the classical Arzelela-Ascoli theorem (cf. the use of equicontinuity in the noncommutative Arzela-Ascoli theorem, [171]). Pointwise boundedness is a natural condition to require. (Indeed, the density of $C^1(X)$ in A in the spectral triple definition is just pointwise bound-edness for the trivial group action.) It is surely a weaker condition than equicontinuity, but unfortunately I do not have an example where pointwise boundedness holds but equicontinuity does not. As in [171], X will be called isometric if $C^1(X) = C_b^1(G, X)$ and

$$\left\|[D, \alpha_g(a)]\right\| = \|[D, a]\|$$

for all $a \in C^1(X), g \in G$.

One problem that arises when trying to prove a version of (i) for a general discrete group acting on A is how to define $\widehat{D}$. What should we put in place of the $\mp n$? However, n can be recognized as coming from the usual word metric on the group $\mathbb{Z}$, and so for a general finitely generated, infinite group G, we need to replace n by $c(g)$ where c is the word metric on G associated with a symmetric generating subset of G. In fact, such a word metric is naturally associated ([172]) with a spectral triple $(C_r^*(G), \ell^2(G), M_c)$, where M_c is the multiplication operator by c on $\ell^2(G), C_r^*(G)$ is the reduced C^*-algebra of G, and $\widehat{D}$ gives the unbounded Fredholm operator determining the Kasparov product of K-homology classes in the unbounded Fredholm picture ([170], [186], [173]).
A second problem is that in the $\mathbb{Z}$ case, the group was abelian, and we had the dual group $\mathbb{T}$ available to act on the crossed product. This is no longer the case for general G. Instead, as in the Imai-Takai duality theorem, we have to consider the dual coaction

$$\hat{\alpha}: A \rtimes_{\alpha,r} G \to (A \rtimes_{\alpha,r} G) \otimes C_r^*(G)$$

where for $F \in C_c(G,A)$,

$$\hat{\alpha}(F) = \tilde{\pi}(F(s))\tilde{\lambda}_s \otimes \lambda_s. \quad (3)$$

Here, $\hat{\alpha}$ is continuous for the $A \rtimes_{\alpha,r} G$ norm restricted to $C_c(G,A)$, and so extends by continuity to the whole of $\rtimes_{\alpha,r} G$. Since, in the situation, G will be discrete, the integrals involved are just summations, but we will stay with the familiar integral notations. (This may also prove useful if the result can be extended to general G.) Further, $(\tilde{\pi}, \tilde{\lambda})$ is the covariant representation giving the regular representation of $\rtimes_{\alpha,r} G$ as realized on $\mathcal{H} \otimes \ell^2(G) = \ell^2(G,\mathcal{H})$, and λ is the left regular representation of G on $\ell^2(G)$. As we will see, the dual spectral triple associated with $X = (A, \mathcal{H}, D)$ will be the triple $Y = (A \rtimes_{\alpha,r} G, \mathcal{H} \otimes \ell^2(G) \otimes \mathbb{C}^2, \widehat{D})$. In the abelian case, the dual coaction reduces to the familiar action of the dual group on the crossed product.

We need geometric definitions of metric notions (such as "isometry") for coactions just as we had for actions. It is not immediately clear what they should be. However, it is reasonable to think that if we dualize the coaction in some sense, then we should have something like an action (though not necessarily of a group). Let $P_r(G)$ be the state space of $C_r^*(G)$. This is a subsemigroup of $(C_r^*(G))^*$, which is an ideal in the Fourier-Stieltjes algebra $B(G)$ of G. (The multiplication is just pointwise multiplication on G when we regard the elements ϕ of $P_r(G)$ as functions on G by setting $\phi(s) = \phi(\lambda_s)$.) It is this semigroup that we want acting in place of the group G. For a general C^*-algebra B with a coaction $\delta: B \to B \otimes C_r^*(G)$, the action β of $P_r(G)$ on B is given by slicing by $\phi \in P_r(G)$: so for $b \in B, \beta_\phi(b) = S_\phi(\delta(b))$.

When G is abelian, so that we are dealing with the dual action in place of the dual coaction, the dual action on the crossed product is just β restricted to the characters of G, the set of extreme points of $P_r(G)$. The isometric condition makes good sense in this case since under the dual action, each character acts by multiplication as a unitary on $\ell^2(G)$. However, when we extend this action to convex combinations in $P_r(G)$ of these characters, this is no longer the case. Instead we need to replace the isometric condition by the contractive one. In the case of a spectral triple $Y = (B, K, D')$ - and we have in mind primarily the dual spectral triple - for which a coaction on B is given, the idea is that Y is contractive if the set

$$\{b \in C^1(Y): \|[\widehat{D}, \beta_\phi(b)]\| \leq \|[\widehat{D}, b]\| < \infty \; for\ all\ \phi \in P_r(G)\}$$

is dense in B (or equivalently dense in $C^1(Y)$).

The main result (Theorem (5.1.2)) states that if X is a pointwise bounded spectral triple, then the dual spectral triple Y is contractive for the dual coaction.

As is well-known, some care has to be exercised with unbounded operators on a Hilbert space because of their partially defined domains. In particular, it gives a proof that the operator $\widehat{D}$ used in the dual spectral triple really is a self-adjoint operator with compact resolvent. We have tried to incorporate into a simple account of the material, as self-contained as possible, that we need from the theory of reduced crossed products and coactions. In particular, substantial simplifications result because, we only deal with the reduced case, the group G is discrete and the C^*-algebras involved unital. (For a short, informative exposition (with proofs) of the general theory for full and reduced crossed products and coactions, Appendix A of the memoir [179] by Echterhoff, Kaliszewski, Quigg and Raeburn is recommended.)

I am grateful to Kamran Reihani for helpful conversations.

For G be a discrete group. A length function on G is a function $c: G \to \mathbb{R}$ satisfying: for every $s \in G$,

$$sup_{t\in G}|c(t) - c(st)| < \infty \quad (4)$$

and $|c(t)| \to \infty$ as $t \to \infty$. In particular, it is assumed that G is countably infinite.
The most important example of a length function is that of the word length function on G (e.g. [180], [172]). Suppose that G is infinite and finitely generated, and let S be a finite, symmetric set of generators for G. For $t \in G$, let $c(t)$ be the word norm associated with S, i.e. $c(t)$ is smallest integer n such that t can be written as a product of n elements of S. Then (as is easy to check) c is indeed a length function.
We now establish notation for reduced crossed products for a locally compact group G. (In our case, of course, G is discrete.) Let A be a unitary C^*-algebra and (A, G, α) be a dynamical system; so $\alpha: G \to Aut\ A$ is a homomorphism which is pointwise norm continuous. Then (e.g. [193]) $C_c(G, A)$ is a convolution normed algebra under the L^1-norm, and with product and involution given by:

$$f * g(t) = \int f(s)\alpha_s(g(s^{-1}t))d\lambda(s) \quad f^*(t) = \Delta(t)^{-1}\alpha_t(f(t^{-1})^*).$$

The completion $L^1(G, A)$ of $C_c(G, A)$ is then a Banach algebra, and the full crossed product $A \rtimes_\alpha G$ is defined to be the enveloping C^*-algebra of $L^1(G, A)$. The (non-degenerate) representations of $A \rtimes_\alpha G$ are determined by the covariant representations (π, u) on a Hilbert space K of (A, G, α), i.e. a pair for which π is a representation of A and u a unitary representation of G on the same Hilbert space K and for which $\pi(\alpha_t(a)) = u_t\pi(a)u_t^*$ for all $a \in A, t \in G$. Such a covariant representation determines the corresponding representation $\pi \times u$ of $A \rtimes_\alpha G$ by defining

$$\pi \times u(F) = \int \pi(F(s))u_s\, d\lambda(s). \quad (5)$$

In the present day study of crossed products and coactions, it is, for categorical reasons, usually desirable to work in the full setting because of the good universal properties. (See [179] for a discussion of the pros and cons of using the full or reduced theories.) However, since we are concerned with spectral triples and such a triple involves an explicit Hilbert space, we will work with the reduced crossed product (as was the case in the early work on the subject, e.g. [182], [188]).
The reduced crossed product of G and A will denoted by $A \rtimes_{\alpha,r} G$. It is a homomorphic image of the full crossed product and can be constructed as follows. Let $\pi: A \to B(\mathcal{H})$ be a faithful, non-degenerate representation of A on a Hilbert space $\mathcal{H}$. Then (e.g. [193]) there are a representation $\tilde{\pi}$ of A on $L^2(G, \mathcal{H})$ and a homomorphism $\tilde{\lambda} : G \to U(B(L^2(G, \mathcal{H})))$ defined by:

$$\tilde{\pi}(a)\xi(t) = \pi\left(\alpha_t^{-1}(a)\right)\xi(t), \;\; \tilde{\lambda}_s\xi(t) = \xi(s^{-1}t)$$

for $\xi \in L^2(G, \mathcal{H})$. Let λ be the left regular representation of G on $L^2(G)$: $\lambda_s f(t) = f(s^{-1}t)$.) The pair $(\tilde{\pi}, \tilde{\lambda})$ is a covariant representation of (A, G, α) and hence determines a representation $\hat{\pi} = \tilde{\pi} \times \tilde{\lambda}$ of $A \rtimes_\alpha G$. From (5), for $F \in C_c(G, A), \xi \in L^2(G, \mathcal{H})$,

$$\hat{\pi}(F)\xi(t) = \int \tilde{\pi}(F(s))(\widetilde{\lambda_s}\xi)(t) = \pi(\alpha_{t-1}(F(s)))\xi(s^{-1}t). \quad (6)$$

The image of this representation is the reduced crossed product $A \rtimes_{\alpha,r} G$, realized spatially on $L^2(G, \mathcal{H})$. As is customary, for notational simplicity, we sometimes identify $F \in C_c(G, A)$ with its image $\hat{\pi}(F)$.

If G is abelian, then (e.g. [188], [179]) there is an action of the dual group $\hat{G}$ on $A \rtimes_{\alpha,r} G$ called the dual action. This action is defined by: for $\gamma \in \hat{G}$ and $F \in C_c(G,A)$, we take $\hat{\alpha}_\gamma F(s) = \gamma(s)F(s)$. The map $\hat{\alpha}_\gamma$ extends by continuity to give an automorphism on $A \rtimes_{\alpha,r} G$, and $(A \rtimes_{\alpha,r} G, \hat{G}, \hat{\alpha})$ is a C^*-dynamical system. (Often, authors define the dual action using the complex conjugate of $\gamma(s)$, i.e. $\hat{\alpha}F(s) = \overline{\gamma(s)}F(s)$, and (cf. [199]) either choice is fine, depending on how we identify elements in the dual group with actual functions on the group. However, in the study of coactions, it is more convenient to use the $\hat{\alpha}F(s) = \gamma(s)F(s)$ version for the dual action.)

In the non-abelian case, the dual group is no longer relevant for duality purposes, and instead one replaces the dual action in the abelian case by the dual coaction. The definition of coaction which we now give is for the case where G is discrete and the C^*-algebra B unital, the general case being more involved (in particular, requiring the use of multiplier algebras). So let B be a unital C^*-algebra, and id_B be the identity map on B and id_G the identity map on $C_r^*(G)$. Let $\delta_G: C_r^*(G) \to C_r^*(G) \otimes C_r^*(G)$ be the homomorphism determined by: $\delta_G(\lambda_s) = \lambda_s \otimes \lambda_s$. (This extends continuously to $C_r^*(G)$ since (e.g. [177] or [179]) $\lambda \otimes \lambda$ is weakly contained in λ.) A (reduced) coaction for B (with respect to G) is a unital injective homomorphism $\delta: B \to B \otimes C_r^*(G)$ that satisfies the coaction identity:

$$(\delta \otimes id_G) \circ \delta = (id_B \otimes \delta_G) \circ \delta.$$

Of course, if (as will be in our case) B is a C^*-subalgebra of $B(K)$ (K a Hilbert space) then $B \otimes C_r^*(G) \subset B(\ell^2(G,K))$ so that δ will also be an injective homomorphism into $B(\ell^2(G,K))$. (Coactions are also required to be non-degenerate (e.g.[188]) - this condition is always satisfied by the dual coaction, the only coaction with which we will be concerned, and so we will not define non-degeneracy here.)

Of particular importance is the dual coaction $\hat{\alpha}$ for $A \rtimes_{\alpha,r} G$, defined in (3). It is easily checked that $\hat{\alpha}$ satisfies the coaction identity. If G is abelian, then the dual action and the dual coaction are effectively the same, the relation between them being given by: $(1 \otimes 1 \otimes \sigma_\chi)(\hat{\alpha}(F)) = \alpha_\chi(F)$ where σ_χ is the state on $C_r^*(G) \cong C_0(\hat{G})$ associated with point evaluation at χ: $\sigma_\chi(\lambda_s) = \chi(s)$. (We will return to this more generally, and for this, as we will see, it is helpful to use slice maps (below).)

First, let $P_r(G)$ be the state space of $C_r^*(G)$. Since $C_r^*(G)$ is unital, $P_r(G)$ is a weak* compact, convex subset of $B_r(G) = C_r^*(G)^*$. (A brief discussion of $B_r(G)$ is given on [188].) The canonical embedding of $B_r(G)$ into $B(G) = C^*(G)^*$ (itself coming from the canonical homomorphism from $C^*(G)$ onto $C_r^*(G)$) identifies the Banach space $B_r(G)$ with a subspace of $B(G)$, the Fourier-Stieltjes algebra of G, and $P_r(G)$ with a weak*-compact convex subset of the state space $P(G)$ of $C^*(G)$. Now regard $B(G)$ as a space of functions on G, where, for $\phi \in B(G), \phi(s) = \phi(\lambda_s^u)$, where $s \to \lambda_s^u$ is the canonical homomorphism from G into the unitary group of $C^*(G)$.Then $B_r(G)$ is a (normed closed) ideal in $B(G)$ and $P_r(G)$ is a subsemigroup of $P(G)$. As a function on $G, \phi \in B_r(G)$ is given by: $\phi(s) = \phi(\lambda_s)$, and since $\delta_G(s) = \lambda_s \otimes \lambda_s$, the product on $B_r(G)$ can be defined by: $\phi\psi = (\phi \otimes \psi) \circ \delta_G$. In order to associate an action of $B_r(G)$ - and hence of $P_r(G)$ - on B for a coaction with respect to G, we use slice maps (e.g. [187], [179]). (As commented in [179], slicing in tensor products is one of the basic tools in the theory of coactions.)

If A_1, A_2 are C^*-algebras realized on Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, let $A_1 \odot A_2$ be the span of simple tensors $a_1 \otimes a_2$ in $B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ ($a_i \in A_i$). The closure of $A_1 \odot A_2$ is the spatial tensor product $A_1 \otimes A_2$ of A_1 and A_2. If $\phi \in A_2^*$, then the slice map $S_\phi: A_1 \otimes A_2 \to A_1$ is a well-defined bounded linear map of norm $\|\phi\|$ and is determined by its value on $A_1 \odot A_2$:

$$S_\phi(a_1 \odot a_2) = a_1\phi(a_2).$$

Next for $c \in A_1 \otimes A_2$, the map $\phi \to S_\phi(c)$ is weak*-norm continuous on bounded subsets of A_2^*. To show this, let $\phi_n \to \phi$ weak* in a bounded subset of A_2^*. Trivially, $S_{\phi_n}(a \otimes b) = \phi_n(b)a \to \phi(b)a = \phi(a \otimes b)$ in norm for every simple tensor $a \otimes b$. Hence this result is also true for elements in the span C of such tensors in $A_1 \otimes A_2$. A "uniform convergence" type argument, using the density of C in $A_1 \otimes A_2$ and the norm boundedness of $\{S_{\phi_n}\}$, then gives the result.

Given a coaction $\delta: B \to B \otimes C_r^*(G)$, the (left) action β of $B_r(G)$ on B is defined by: $\beta_\phi(b) = \phi.b$ where

$$\phi.b = S_\phi(\delta(b)). \tag{7}$$

To check that this is an action, we have to show (among other things) that $\phi.(\psi.b) = (\phi\psi)$.b, i.e. $S_\phi(\delta(S_\psi(\delta(b)))) = S_{\phi\psi}(\delta(b))$. This amounts to showing that for $b \in B$,

$$S_\phi(\delta(S_\psi(\delta(b)))) = S_{\phi\otimes\psi}((1 \otimes \delta_G)(\delta(b))) = S_{\phi\otimes\psi}((\delta \otimes 1)(\delta(b)))$$

(using the coaction identity). It is simple to prove this by approximating $\delta(b)$ by a finite sum of simple tensors $\sum b_i \otimes s_i$ ($b_i \in B, s_i \in G$) then similarly, each $\delta(b_i)$ by a finite sum $\sum b_{ij} \otimes s_{ij}$. The remaining verifications that B is a left Banach $B_r(G)$-module are easy.

We saw above that every coaction on a C^*-algebra B gives rise to an action $\phi \to \beta_\phi$ of the semigroup $P = P_r(G)$ on B. In the case which concerns us, viz. where δ is the dual coaction for an action α of G on a C^*-algebra A, the action β of $B_r(G)$ on $B = A \rtimes_{\alpha,r} G \subset B(\mathcal{H} \otimes \ell^2(G))$ is easy to calculate, and fits in well with the familiar dual action for the commutative case.

Indeed (cf. [188]) for $F \in C_c(G, A)$,

$$\beta_\phi F = S_\phi(\int \tilde{\pi}(F(s))\tilde{\lambda}_s \otimes \lambda_s) = \int \tilde{\pi}(\phi(s)F(s))\tilde{\lambda}_s \tag{8}$$

so that $\beta_\phi F$ is just pointwise multiplication by ϕ on $C_c(G, A)$, exactly the same as what happens in the abelian case with the characters of G. In that case, $C_r^*(G) = C_0(\hat{G})$, so that $B_r(G) = M(\hat{G})$, and P is just the set of probability measures on $\hat{G}$. The extreme points of P are just the characters of G, and restricting the action of P to these gives the dual action of $\hat{G}$ on $A \rtimes_{\alpha,r} G$. Of course, when G is not abelian, the extreme points of P are the pure states on $C_r^*(G)$ which are usually not characters. There is no advantage in restricting the action of P to the pure states, and by doing that, we also lose the semigroup structure of P. For these reasons, for general G, we use the action of P on $A \rtimes_{\alpha,r} G$.

Now let c be a length function on G, and M_c be the multiplication operator by c on $\ell^2(G)$: so $(M_c\xi)(t) = c(t)\xi(t)$ defined for the subspace D of elements $\xi \in \ell^2(G)$ for which $\sum_{t\in G} c(t)^2\,|\xi(t)|^2 < \infty$. Then ([23]) M_c is an unbounded self-adjoint operator on $\ell^2(G)$ with domain D. Further, $C_c(G) \subset \ell^2(G)$ is a core for M_c. Also, since each $c(t)$ is an integer and $|c(t)| \to \infty$, the operator $(M_c - \iota)^{-1}$ is compact, so that M_c has compact resolvent. Let $Z = (C_r^*(G), \ell^2(G), M_c)$. For each $s \in G$ let $m_s = sup_{t\in G}|c(t) - c(s^{-1}t)| < \infty$ (by (2.1)) . Let B be the space of functions $f \in \ell^1(G)$ for which $mf \in \ell^1(G)$ where $(mf)(s) = m_s f(s)$, and let π_r be the left regular representation of $C_r^*(G)$. Then $C_c(G) \subset B \hookrightarrow C_r^*(G)$, so that B is a dense subspace of $C_r^*(G)$. Then for $f \in B$ and $\xi \in D$,

$$|[M_c, \pi_r(f)]\xi(t)| = \left|\sum_{s\in G}[c(t) - c(s^{-1}t)]f(s)\xi(s^{-1}t)\right| \le \sum\nolimits_{s\in G} m_s|f(s)|\,|\xi|\,(s^{-1}t)$$
$$= (|mf| * |\xi|)(t).$$

So

$$\|[M_c, \pi_r(f)]\xi\|_2 \leq \|mf\|_1 \|\xi\|_2 \tag{9}$$

from which it follows that $f \in C^1(Z)$ and that Z is a spectral triple.

So we now have two spectral triples $X = (A, \mathcal{H}, D)$ and $Z = (C_r^*(G), \ell^2(G), M_c)$. In particular, both D, M_c are self-adjoint operators with compact resolvent, and so by Proposition (5.1.4) with $K = \mathcal{H} \otimes \ell^2(G)$, the operator $\widehat{D}$ on $\mathcal{H} \otimes \ell^2(G) \otimes \mathbb{C}^2$, where

$$\widehat{D} = \begin{bmatrix} 0 & \widehat{D}_- \\ \widehat{D}_+ & 0 \end{bmatrix} \tag{10}$$

is self-adjoint with compact resolvent, where $\widehat{D}_\mp = D \otimes 1 \mp \iota \otimes M_c$. Again from Proposition (5.1.4), the domain of $Dom\, \widehat{D}_\mp$ is $\hat{V}$ and $Dom\, \widehat{D} = \hat{V}^2 = \hat{V} \oplus \hat{V}$ where $\hat{V}$.

We will show that if X is pointwise bounded (for the G-action), then the triple $(A \rtimes_{\alpha,r} G, \mathcal{H} \otimes \ell^2(G) \otimes \mathbb{C}^2, \widehat{D})$ is in fact a spectral triple, which we will call the dual spectral triple for X. Further, Y will be shown to be contractive for the dual coaction.

Let $X = (A, \mathcal{H}, D)$ be a spectral triple. It is obvious from the definition of $C^1(X)$ and the Leibniz formula that $C^1(X)$ is a subalgebra of A. (In fact ([171]) $C^1(X)$ is a Banach *-algebra invariant under the holomorphic functional calculus where the Banach algebra norm is given by: $\|a\|_1 = \|a\| + \|[D, a]\|$.)

Let G be a locally compact group and α as above be an action of G on A (i.e. a strongly continuous homomorphism α of G into the *-automorphism group $Aut(A)$ of A). [171], define three noncommutative geometric properties with respect to the group action on A. The names given are those used in the "commutative" case of a group action on a locally compact metric space. Let $C^1(G, X)$ be the set of $a \in A$ such that $\alpha_t(a) \in C^1(X)$ for all $t \in G$ and the map $t \to [D, \alpha_t(a)]$ is norm continuous. Note that taking $t = e$ in this definition gives that $C^1(G, X) \subset C^1(X)$. Now define $C_b^1\,(G, X)$ to be the set of a's in $C^1(G, X)$ such that $sup_t\, \|[D, \alpha_t(a)]\| < \infty$. Note that if $a \in C_b^1(G, X)$ then so also is every $\alpha_t(a)$. Then X is called quasi-isometric if $C^1(G, X) = C^1(X)$. The spectral triple X is called equicontinuous if $C_b^1(G, X) = C^1(X)$. Last, it is called isometric if it is quasi-isometric and for all $a \in C^1(X)$ and all $t \in G$, we have $\|[D, \alpha_t(a)]\| = \|[D, a]\|$. (In particular, X is equicontinuous if it is isometric.) The condition that we will be concerned is similar to that of equicontinuity but not quite so strong. We will call X pointwise bounded if $C_b^1\,(G, X)$ is dense in $C^1(X)$ (and hence by the spectral triple requirement, dense in A.)

We now turn to the corresponding definitions for a coaction $\delta: B \to B \otimes C_r^*(G)$ of a unital C^*-algebra B instead of an action of G on A. So let $Y = (B, K, D')$ be a spectral triple. Then (7) associated with δ is the semigroup action $\phi \to \beta_\phi$ of $P = P_r(G)$ on B. Then, similar to the definitions for an action, we define $C^1(P, Y)$ to be the set of b's in $C^1(Y)$ such that for all $\phi \in P, \beta_\phi(b) \in C^1(Y)$ and the map $\phi \to [D', \beta_\phi(b)]$ is weak*-norm continuous. Next $C_b^1(P, Y)$ is defined to be the set of b's in $C^1(P, Y)$ such that $sup_{\phi \in P} \left\|[D, \beta_\phi(b)]\right\| < \infty$. As in the group action case, we say that Y is quasi-isometric if $C^1(P, Y) = C^1(Y)$. The spectral triple Y is called equicontinuous if $C_b^1(P, Y) = C^1(Y)$. We replace the isometric condition of the action case by the contractive condition: Y is called contractive if

$$C_{contr}^1(P, Y) = \{b \in B: \left\|[D', \beta_\phi(b)]\right\| \leq \left\|[\widehat{D}, b]\right\| < \infty\, for\ all\ \phi \in P_r(G)\}$$

is dense in B. (For justification of this definition (and as we will see later), for abelian discrete G, the isometric condition for the dual action is equivalent to the contractive condition for the dual coaction.) Last, the spectral triple Y is called pointwise bounded if $C_b^1(P, Y)$ is dense in $C^1(Y)$ (and hence dense in B).

We will only have occasion to use pointwise boundedness for group actions and the contractive condition for coactions. As in the previous, X will be the spectral triple $(A, \mathcal{H}, D)$ and Y the triple $(A \rtimes_{\alpha,r} G, \ell^2(G, \mathcal{H}) \otimes \mathbb{C}^2, \widehat{D})$.

Proposition (5.1.1)[169]: Suppose that X is pointwise bounded. Then Y is a spectral triple.

Proof. We only need to show that $C^1(Y)$ is dense in B, since Y satisfies all the other requirements for a spectral triple. Since X is pointwise bounded, $C_b^1(G, X)$ is dense in A. Now let C be the space of functions $F: G \to C_b^1(G, X)$ that vanish outside a finite subset of G. It is obvious that C is dense in $\ell^1(G, A)$ and hence its image, also denoted C, is dense in $A \rtimes_{\alpha,r} G$. It is sufficient, then, to show that $C \subset C^1(Y)$. Let $\mathcal{E} = Dom\ D \odot C_c(G) \subset \mathcal{H} \otimes \ell^2(G)$. Note that since G is discrete, $\mathcal{E} = C_c(G, Dom\ D)$ and that $\mathcal{E}$ is invariant under both $D \otimes 1$ and $1 \otimes M_c$. Since $Dom\ D$ and $C_c(G)$ are respectively cores for D, M_c, it follows by Proposition (5.1.4) that $\mathcal{E}^2$ is a core for $\widehat{D}$. Next we claim that for $F \in C$, we have

$$\hat{\pi}(F)\mathcal{E} \subset \mathcal{E}. \tag{11}$$

To see this, let $\xi \in \mathcal{E}$. Then $\hat{\pi}(F)\xi(t) = \int \pi(\alpha_{t^{-1}}(F(s))\tilde{\lambda}_s \xi(t)$. Now for each $s, \tilde{\lambda}_s \xi \in \mathcal{E}$ and since each $F(s) \in C_b^1(G, X)$, so also does every $\alpha_{t^{-1}}(F(s))$, in particular, it belongs to $C^1(X)$ and so preserves the domain of D. So the map F_s given by $t \to \pi(\alpha_{t^{-1}}(F(s))\tilde{\lambda}_s \xi(t)$ sends G into $Dom\ D$. Further, F_s has finite support since ξ has and so $F_s \in \mathcal{E}$. Since F vanishes off a finite subset of $G, \hat{\pi}(F)\xi$ is a finite sum of F_s's and so $\hat{\pi}(F)$ maps $\mathcal{E}$ into $\mathcal{E}$ giving (11). It also follows that the commutators $[D \otimes 1, F], [1 \otimes M_c]$ are operators on $\mathcal{E}$, and we now calculate them. (Recall that, when convenient, we identify F with $\hat{\pi}(F)$.)

First we claim that for $\xi \in \mathcal{E}, t \in G$,

$$([D \otimes 1, F]\xi)(t) = \int [D, \pi(\alpha_{t^{-1}}(F(s))]\xi(s^{-1}t). \tag{12}$$

For, recalling that $\hat{\pi}(F)$ is a finite sum of F_s's,

$$\begin{aligned}[D \otimes 1, F]\xi(t) &= ((D \otimes 1)\hat{\pi}(F)\xi)(t) - (\hat{\pi}(F)(D \otimes 1)\xi)(t)\\ &= (D \otimes 1)\int \pi(\alpha_{t^{-1}}(F(s))\tilde{\lambda}_s\xi(t) - \int \pi(\alpha_{t^{-1}}(F(s)))\tilde{\lambda}_s((D \otimes 1)\xi)(t)\\ &= \int D(\pi(\alpha_{t^{-1}}(F(s))))\xi(s^{-1}(t) - \int \pi(\alpha_{t^{-1}}(F(s))D(\xi(s^{-1}t))\\ &= \int [D, \pi(\alpha_{t^{-1}}(F(s))]\xi(s^{-1}t).\end{aligned}$$

Next, we show that

$$[1 \otimes M_c, F]\xi(t) = \int \pi(\alpha_{t^{-1}}(F(s)))[c(t) - c(s^{-1}t)]\xi(s^{-1}t). \tag{13}$$

For

$$\begin{aligned}[1 \otimes M_c, \hat{\pi}(F)]\xi(t) &= c(t)\int \pi(\alpha_{t^{-1}}(F(s)))\tilde{\lambda}_s\xi(t) - \int \pi(\alpha_{t^{-1}}(F(s)))\tilde{\lambda}_s(c\xi)(t)\\ &= \int \pi(\alpha_{t^{-1}}(F(s)))[c(t) - c(s^{-1}t)]\xi(s^{-1}t).\end{aligned}$$

We now want to show that each of the commutators in (12), (13) is a bounded map on $\mathcal{E}$. For the first of these, suppose that $a \in C_b^1(G, X)$, $f \in C_c(G)$ and take $F = a \otimes f \in C$. Then

$$\|[D \otimes 1, F]\| \leq (sup_t\|[D, \alpha_{t^{-1}}(a)]\|)\|f\|_1. \tag{14}$$

For let $M = sup_t\ \|[D, \alpha_{t^{-1}}(a)]\|$ and $\xi, \eta \in \mathcal{E}$. Then $M < \infty$ since $a \in C_b^1(G, X)$, and by (12),

$$|\langle[D \otimes 1, F]\xi, \eta\rangle| = \left|\iint \langle[D, \pi(\alpha_{t^{-1}}(f(s)a))]\xi(s^{-1}t), \eta(t)\rangle\, ds\, dt\right|$$

$$\leq \iint |\langle[D,\pi(\alpha_{t^{-1}}(a))]\xi(s^{-1}t),\eta(t)\rangle||f(s)|\,ds\,dt$$
$$\leq \iint |f(s)|M\|\xi(s^{-1}t)\|\|\eta(t)\|ds\,dt$$
$$\leq \int M|f(s)|\,(\int \|\,\xi(s^{-1}t)\|^2\,dt)^{1/2}(\int \|\eta(t)\|^2 dt)^{1/2}ds \leq M\|f\|_1\|\xi\|\|\eta\|$$
giving (14).

Since every $F \in C$ is a linear combination of terms of the form $a \otimes f$, it follows that for general $F \in C, [D \otimes 1, F]$ is a bounded operator on $\mathcal{E}$. The boundedness of the second commutator on $\mathcal{E}$ follows similarly using (13) since for each $s, sup_t|c(t) - c(s^{-1}t)| < \infty$ by (4) and $f(s) \neq 0$ for only finitely many s. Precisely,
$$\|[1 \otimes M_c, a \otimes f]\| \leq \|a\|\|f\|_1 \sup\{c(t) - c(s^{-1}t\colon f(s) \neq 0, t \in G\}.$$
So for $F \in C$, the commutators
$$[D \otimes 1 \pm \imath 1 \otimes M_c, F] = [D \otimes 1, F] \pm \imath[1 \otimes M_c, F]$$
are bounded operators on $\mathcal{E}$. Let $F' = F \oplus F$, a diagonal operator on $\ell^2(G,\mathcal{H}) \otimes \mathbb{C}^2$. Then the commutator $[\widehat{D}\,, F']$ has zero diagonal and off-diagonal entries $[D \otimes 1\, \pm \imath 1 \otimes c, F]$, and is as well a bounded operator on $\mathcal{E} \oplus \mathcal{E}$. Now apply (11) and Proposition (5.1.3) to conclude that $F' \in C^1(Y\,)$, and that Y is a spectral triple. The next theorem is the main result. (Note that if G is discrete and finitely generated, then we can take c to be any word length function on G.)

Theorem (5.1.2)[169]: Let A be a unital C^*-algebra, $X = (A, \mathcal{H}, D)$ a spectral triple, G a discrete countably infinite group with length function c and (A, G, α) be a C^*-dynamical system. Suppose that X is pointwise bounded for G. Then the dual spectral triple $Y = (A \rtimes_{\alpha,r}\, G, \ell^2(G,\mathcal{H}) \otimes \mathbb{C}^2, \widehat{D})$ is contractive for the dual coaction $\delta\colon A \rtimes_{\alpha,r} G \to (A \rtimes_{\alpha,r}\, G) \otimes C_r^*(G)$.

Proof. By Proposition (5.1.1), Y is a spectral triple. It remains to show that Y is contractive. It is sufficient to show that $C = C_c(G, C_b^1(G,X)$ (which we used in the previous proof) is a subspace of $C^1_{contr}(P, Y\,)$. First, C is P-invariant.

This is trivial, since if $F \in C$, i.e. the map $F\colon G \to C_b^1(G,X)$ has finite support, so also does $\beta_\phi F$ (since $\beta_\phi F(s)\; = \phi(s)F(s)$). It remains to show that
$$\left\|[D, \beta_\phi(F)]\right\| \leq \|[D,F]\|$$
and that the map $\phi \to [D, \beta_\phi(F)]$ is weak*-norm continuous.

To this end, define ([182], [188]) the unitary W on $\ell^2(G\, \times G, \mathcal{H})$ by:
$$W\zeta(s,t) = \zeta(s, s^{-1}t).$$
Then $W^*\zeta(s,t) = \zeta(s,st)$ and trivially, W is unitary. We shall also use W for the case $\mathcal{H} = \mathbb{C}$. For $t \in G, \zeta \in \ell^2(G \times G, \mathcal{H}), \zeta_t \in \ell^2(G,\mathcal{H})$ is given by: $\zeta_t(s) =\; \zeta(s,t)$.

Let δ be the dual coaction on B. Then for $F \in C_c(G, A)$,
$$W(F \otimes 1)W^*\zeta(v,t) = (F \otimes 1)W^*\zeta(v, v^{-1}t) = F((W^*\zeta)_{v^{-1}t})(v)$$
$$= \int \pi(\alpha_{v^{-1}}(F(s))(W^*\zeta)_{v^{-1}t}(s^{-1}v)\,ds$$
$$= \int \pi(\alpha_{v^{-1}}(F(s))(W^*\zeta)(s^{-1}v, v^{-1}t)\,ds$$
$$= \int \pi(\alpha_{v^{-1}}(F(s))\zeta(s^{-1}v, s^{-1}t)\,ds = \delta(F)$$
using the formula (6). It follows by continuity that for $w \in A\, \rtimes_{\alpha,r} G$,
$$W(w \otimes 1)W^* = \delta(w). \qquad (15)$$

So we can extend δ to a homomorphism, also denoted $\delta: B(\ell^2(G,\mathcal{H})) \to B(\ell^2(G\times G,\mathcal{H}))$, by defining

$$\delta(T) = W(T\otimes 1)W^*. \tag{16}$$

We want to extend it to certain unbounded operators associated with $\widehat{D}$, specifically, the unbounded operators $D\otimes 1$ and $1\otimes M_c$ on $B(\ell^2(G,\mathcal{H}))$. To this end, let $Z=\mathcal{E}\odot C_c(G)$. Then Z is a dense subspace of $\ell^2(G\times G,\mathcal{H})$ that is invariant under both W, W^*. Also, Z is invariant for $\hat{\pi}(F)\otimes 1, D\otimes 1\otimes 1$ and $1\otimes M_c\otimes 1$ because of the corresponding properties for $\hat{\pi}(F), D\otimes 1, 1\otimes M_c$ for $\mathcal{E}$.

We now claim that on Z and conjugating with W as in (15) to define δ on $D\otimes 1, 1\otimes M_c$,

$$\delta(D\otimes 1) = D\otimes 1\otimes 1, \delta(1\otimes M_c) = 1\otimes M_c\otimes 1. \quad (17)$$

These follow since, for a simple tensor $\zeta = h\otimes\xi$, where $h\in Dom\ D$ and $\xi\in C_c(G\times G)\in Z$,

$$(W(D\otimes 1\otimes 1)W^*\zeta)(s,t) = ((D\otimes 1\otimes 1)W^*\eta)(s,s^{-1}t)$$
$$= D(h)(W^*\eta)(s,s^{-1}t) = ((D\otimes 1\otimes 1)\zeta)(s,t),$$

and

$$(W(1\otimes M_c\otimes 1)W^*\zeta)(s,t) = ((1\otimes M_c\otimes 1)W^*\zeta)(s,s^{-1}t)$$
$$= hc(s)(W^*F)(s,s^{-1}t) = ((1\otimes M_c\otimes 1)\zeta)(s,t).$$

We will use use the notation $\widehat{D}_\pm$ for $D\otimes 1\pm\iota 1\otimes M_c$. To prove the contractive property for Y, we recall that for $F\in C$, the operator matrices $[\widehat{D},(\beta_\phi\oplus\beta_\phi)(F\oplus F)]$ are off-diagonal, and considering their entries, it is sufficient to prove that

$$\|[\widehat{D}_\pm,\beta_\phi F]\| \le \|[\widehat{D}_\pm,F]\|. \tag{18}$$

and establish the continuity of the maps $\phi\to[\widehat{D}_\pm,\beta_\phi F]$.

From (17) and (16),

$$\delta([\widehat{D}_\pm,F]) = [\delta(D\otimes 1)\pm\iota\delta(1\otimes M_c),\delta(F)]$$
$$= [(D\otimes 1\pm\iota 1\otimes M_c)\otimes 1, \int\hat{\pi}(F(s))\,\hat{\lambda}_s\otimes\lambda_s] = \int([\widehat{D}_\pm,\hat{\pi}(F(s))\,\hat{\lambda}_s)]\otimes\lambda_s\,.$$

Of particular significance, this gives that $\delta([\widehat{D}_\pm,F])$ belongs to $B(\mathcal{H}\otimes\ell^2(G))\otimes C_r^*(G)$ and we can then use slice maps. Precisely, if $\phi\in P$, then

$$S_\phi(\delta([\widehat{D}_\pm,F])) = S_\phi(\int([\widehat{D}_\pm,\hat{\pi}(F(s))\,\widehat{\lambda_s})]\ \otimes\lambda_s)$$
$$= \int([\widehat{D}_\pm,\hat{\pi}(F(s))\,\widehat{\lambda_s})]\phi(s) = \int([\widehat{D}_\pm,\hat{\pi}(\phi(s)F(s))\,\widehat{\lambda_s})] = [\widehat{D}_\pm,\beta_\phi F].$$

The continuity of the maps $\phi\to[\widehat{D}_\pm,\beta_\phi F]$ now follows from the corresponding continuity property for slice maps. (18) also follows using $\|S_\phi\|\le\|\phi\|=1$ and the fact that δ is a homomorphism (and so norm decreasing).

We now discuss how the theorem above simplifies when G is abelian. The case where $G=\mathbb{Z}$ was examined in detail in [171], which relates equicontinuity for X to the isometric condition for Y. Suppose that X is pointwise bounded. Then we know that Y is contractive. Let $\chi\in\hat{G}\subset P$. Then for $F\in C$,

$$\|[\widehat{D}_\pm,F]\| = \|[\widehat{D}_\pm,\beta_{\chi^{-1}}\beta_\phi F]\| \le \|[\widehat{D}_\pm,\beta_\chi F]\| \le \|[\widehat{D}_\pm,F]\|$$

so that Y is isometric, at least with respect to $F\in C$. However, because G is abelian, there is a nice formula for $\hat{\pi}(\beta_\chi F)$. As is easily proved (and well-known)

$$\hat{\pi}(\beta_\chi F) = (1\otimes M_\chi)\hat{\pi}(F)(1\otimes M_\chi)^{-1}$$

where M_χ is the unitary on $\ell^2(G)$ given by: $f\ \to\ \chi f$ (pointwise multiplication). It is left to the reader to check that for all $b\in C^1(Y), [\widehat{D}_\pm,\beta_\chi(b)] = (1\otimes\ M_\chi)[\widehat{D}_\pm,b](1\otimes M_\chi)^{-1}$

and that we get the isometry condition for all $b \in C^1(Y)$. This generalizes part of [171], extending from $\mathbb{Z}$ to general abelian G and using the weaker pointwise boundedness condition in place of equicontinuity. Incidentally, going in the other direction, the isometry condition for $\hat{G}$ gives the contractive property for P. Indeed, con-tractivity for $\phi \in co\ \hat{G} \subset P$ follows trivially, and by weak*-norm continuity, the contractive inequality follows for all $\phi \in P\ (= \overline{co\ \hat{G}})$.

Lastly, from the above, in the abelian case, a stronger version of Theorem (5.1.2) holds, in which the contractive condition holds for all $b \in C^1(Y)$ and not just for $b \in C$ as in Theorem (5.1.2). We do not know if this stronger version also holds for the non-abelian case.

We briefly recall some basic information about unbounded operators on a Hilbert space (e.g. [23], [33], [185], [197], [178].)

Let D be an unbounded linear operator on a Hilbert space $\mathcal{H}$ with domain $Dom\, D$. The operator D is called closed if its graph $\mathcal{G}(D)$ is closed in $\mathcal{H} \times \mathcal{H}$. It is called preclosed if the closure of $\mathcal{G}(D)$ is itself the graph of a linear operator $\overline{D}$. In particular, in that case, $\overline{D}$ is a closed operator, the minimal closed operator that restricts to D. If D is closed, a subspace $\mathcal{E}$ of $Dom\, D$ is called a core for D if the graph of D restricted to $\mathcal{E}$ is dense in $\mathcal{G}(D)$. (In particular, $\mathcal{E}$ is dense in $Dom\, D$.) We will require the following simple and (no doubt) well known result; we give the proof.

Proposition (5.1.3)[169]: Let D be a closed operator on the Hilbert space $\mathcal{H}$. Let $\mathcal{E}$ be a core for D and $D_{\mathcal{E}}$ be the restriction of D to $\mathcal{E}$. Suppose that $T \in B(\mathcal{H})$ is such that $T\mathcal{E} \subset \mathcal{E}$ and the commutator operator $[D_{\mathcal{E}}, T]$ on $\mathcal{E}$ is bounded. Let $[D, T]$ be the continuous extension of $[D_{\mathcal{E}}, T]$ to $Dom\, D$. Then $T(Dom\, D) \subset Dom\, D$ and the commutator $[D, T]$ on $Dom\, D$ is bounded.

Proof. Let $\xi \in Dom\, D$. Since $\mathcal{E}$ is a core for D, there exists a sequence $\{\xi_n\}$ in $\mathcal{E}$ such that $\xi_n \to \xi, D(\xi_n) \to D\xi$. Then $T\xi_n \in \mathcal{E}, T\xi_n \to T\xi$ and $D(T\xi_n) = TD\xi_n + [D,T]\xi_n \to TD\xi + [D,T]\xi$. So $T\xi \in Dom\, D$.

Now let D have dense domain. Its adjoint D^* has as its domain the set of $\eta \in \mathcal{H}$ for which there is a ζ (which will be unique) such that for all $\xi \in Dom\, D, \langle D\xi, \eta\rangle = \langle \xi, \zeta\rangle$, and for such an $\eta, D^*\eta$ is defined to be ζ.

The unbounded operator D^* is always closed, and D is called self-adjoint if $D = D^*$. In particular, such a D is closed. If D is self-adjoint, then (e.g. [197], [23]) $(D \pm \imath 1)$ is a one-to-one map from $Dom\, D$ onto $\mathcal{H}$, and its inverse $(D \pm \imath I)^{-1}$ is bounded.

We will have to consider tensor products of unbounded operators. Let $D_1, \ldots, D_n$ be densely defined closed operators on Hilbert spaces $H_1, \ldots, H_n$.

Then the tensor product $D_1 \odot \cdots \odot D_n$ is defined in the obvious way on the algebraic tensor product $Dom\, D_1 \odot \cdots \odot Dom\, D_n$. This operator is pre-closed, and its closure is denoted by $D_1 \otimes \cdots \otimes D_n$. The algebraic tensor product of cores for the D_i is a core for D ([33]). If the D_i's are self-adjoint then ([33]) $D_1 \otimes \cdots \otimes D_n$ is also self-adjoint.

We next describe some of the basic properties of a self-adjoint unbounded operator D on a Hilbert space $\mathcal{H}$ with compact resolvent ([185]). Having a compact resolvent means that for some $\zeta \in \mathbb{C}$, the map $(D - \zeta)\colon Dom(D) \to \mathcal{H}$ is one to one and onto, and the resolvent $R(\zeta) = (D - \zeta)^{-1} : \mathcal{H} \to Dom\, D \subset \mathcal{H}$ is a compact linear operator. Then ([185]) since D is closed (as it is self-adjoint), the compact resolvent property ensures the remarkable facts that the entire spectrum of D consists of isolated eigenvalues $\{\lambda_k\}$ with finite-dimensional eigenspaces E_k, and for every complex number λ which is not an eigenvalue of $D, R(\lambda)$ is compact. Further ([185]) all the λ_k's are real, and ([185]) for ζ not in the spectrum of D, the eigenvalues of $R(\zeta)$ are of the form $(\lambda_k - \zeta)^{-1}$ and have the same set of mutually orthogonal eigenspaces E_k and eigenprojections P_k as D. Further, $R(\zeta) = \sum_k (\lambda_k - \zeta)^{-1} P_k$ in the norm topology. In particular, since by compactness, $(\lambda_k - \zeta)^{-1} \to 0$, we have $|\lambda_k| \to \infty$. By the spectral theorem for self-adjoint compact operators, $\sum_k P_k = 1$ in the strong operator topology. From these facts we can determine $Dom\, D$. In fact, $Dom\, D$ is the subspace of all vectors ξ of the form $\sum_k \xi_k$ where $\xi_k = P_k \xi \in E_k$ and $\sum_k \|\xi_k\|^2 < \infty, \sum_k \lambda_k^2 \|\xi_k\|^2 < \infty$, and for such an $\xi, D(\sum_k \xi_k) = \sum_k \lambda_k \xi_k$. So we can write $D = \sum_k \lambda_k P_k$ on $Dom\, D$, convergence being in the strong operator topology. Conversely given real λ_k with $|\lambda_k| \to \infty$ and a family E_k of mutually orthogonal finite dimensional subspaces of $\mathcal{H}$ with associated orthogonal projections P_k and $\sum_k P_k = 1$, then $D = \sum_k \lambda_k P_k$ defines a self-adjoint operator on $\mathcal{H}$ with compact resolvent. To show this, it is obvious that D is densely defined, and it is simple to check from the definition that if $\eta \in Dom\, D^*$, then $\eta \in Dom\, D$ and D is self-adjoint. Using the facts that $|\lambda_k| \to \infty$ and that the P_k form a complete orthonormal set of projections, one shows that $(D - \iota)^{-1}$ is a compact normal operator. So D is self-adjoint with compact resolvent as claimed.

The following proposition gives the information that we will need about the operator $\widehat{D}$. (We note, by the way, that in the general Hilbert C^*-module context, Kaad and Lesch ([183], [184]) give general conditions that ensure self-adjointness and regularity for a class of two-by-two matrix operators that include $\widehat{D}$ below.)

Proposition (5.1.4)[169]: Let D_1, D_2 be self-adjoint unbounded operators with compact resolvents on the Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2, K = \mathcal{H}_1 \otimes \mathcal{H}_2$ and (as above) $D_1 \otimes 1, 1 \otimes D_2$, be the closures of the operators $D_1 \odot 1, 1 \odot D_2$. Define operators $D', \widehat{D}$ on $K \otimes \mathbb{C}^2 = K^2$ by:

$$(A.1) \qquad D' = \begin{bmatrix} 0 & D_1 \odot 1 - \iota 1 \odot D_2 \\ D_1 \odot 1 + \iota 1 \odot D_2 & 0 \end{bmatrix}$$

and

$$(A.2) \qquad \widehat{D} = \begin{bmatrix} 0 & D_1 \otimes 1 - \iota 1 \otimes D_2 \\ D_1 \otimes 1 + \iota 1 \otimes D_2 & 0 \end{bmatrix}.$$

Then $\widehat{D}$ is a self-adjoint unbounded operator on K^2 and is the closure of D'.
Further, if $\mathcal{E}_1, \mathcal{E}_2$ are cores for D_1, D_2, then $\mathcal{E}^2$, where $\mathcal{E} = \mathcal{E}_1 \odot \mathcal{E}_2$, is a core for $\widehat{D}$.

Proof. By the preceding, the operators $D_1 \otimes 1$ and $1 \otimes D_2$ are self-adjoint and $V = Dom\, D_1 \odot Dom\, D_2$ is a core for both. Since

$$V = (Dom\, D_1 \odot \mathcal{H}_2) \cap (\mathcal{H}_1 \odot Dom\, D_2) = Dom(D_1 \odot 1) \cap Dom(1 \odot D_2),$$

it follows that $DomD' = V^2 = V \oplus V$. We now adapt the approach of [171].
In the above notation, we can write $D_1 = \sum_k \lambda_k P_k, D_2 = \sum_r \mu_r Q_r$, where the eigenspaces for D_1, D_2 associated with λ_k, μ_r are E_k, F_r. Of course, these are also the ranges of the projections P_k, Q_r. Let $E_{k,r} = E_k \otimes F_r$. Then $E_{k,r}^2$ is an eigenspace for the operator D', and the restriction $D_{k,r}^{'}$ of D' to $E_{k,r}^2$ is the 2×2 matrix $\begin{pmatrix} 0 & (\lambda_{k-\iota\mu_r})I \\ (\lambda_{k+\iota\mu_r})I & 0 \end{pmatrix}$ where I is the identity operator on $E_{k,r}$. An elementary calculation shows that the eigenvalues of $D_{k,r}^{'}$ are $\pm\sqrt{\lambda_k^2 + \mu_r^2}$. Let λ be any one of these eigenvalues, and suppose that $\lambda \neq 0$. Then the eigenspace for λ is

$$E_{k,r}^{\lambda} = \{(\xi, \eta)' \in E_{k,r}^2 : \lambda\xi = (\lambda_k - \iota\mu_r)\eta\}.$$

Since $D_{k,r}^{'}$ is self-adjoint, $E_{k,r}^2 = E_{k,r}^{\lambda} \oplus E_{k,r}^{-\lambda}$ (orthogonal direct sum). Let $P_{\lambda,kr} : K^2 \to E_{k,r}^{\lambda}$ be the orthogonal projection. So $(P_k \otimes Q_r) \otimes 1 = P_{\lambda,k,r} \oplus P_{-\lambda,k,r}$. If $\lambda = 0$, then $D_{k,r}^{'} =$

0, and trivially $E_{k,r}^{\lambda} = E_{k,r}^{2}$ and $(P_k \otimes Q_r) \otimes 1 = P_{\lambda,k,r}$. Then $\{P_{\lambda,k,r}\}$ $(\lambda^2 = \lambda_k^2 + \mu_r^2)$ is a complete orthonormal family of projections on $K^2 = \oplus_{\lambda,k,r} (E_{k,r}^{\lambda})^2$ (since $\{(P_k \otimes Q_r) \otimes 1\}$ is) and $|\lambda| \to \infty$ as $k^2 + r^2 \to \infty$.

Let L be the self-adjoint operator with compact resolvent associated above with the λ's and $P_{\lambda,k,r}$: so $Dom\, L$ is the space of $[\xi,\eta]' = \{[\xi_{\lambda,k,r}, \eta_{\lambda,k,r}]'\}$ in K^2 for which $\sum \lambda[\xi_{\lambda,k,r}, \eta_{\lambda,k,r}]' \in K^2$, and for such an $[\xi,\eta]'$, $L[\xi,\eta]' = \sum \lambda[\xi_{\lambda,k,r}, \eta_{\lambda,k,r}]'$. Let W be the space of $\xi \in Dom\, L$ for which $\xi_{\lambda,k,r} = 0$ except for a finite number of triples (λ, k, r). (So W is just the linear span of $\cup_{k,r} E_{k,r}^2$ in K^2.) It is left to the reader to check that $L, D', \widehat{D}$ coincide on W, and that W is dense in K^2, and is a core for $L, \widehat{D}$ and D'. So the closure of D' is L. It remains to show that $\widehat{D} = L$.

To this end, we first determine the domain of $\widehat{D}$. First, a core for $D_1 \odot 1$ is the space of all linear combinations of elements of the form $\xi_{k,r} \in E_{k,r}$ over k, r. Since by definition, $D_1 \otimes 1$ is the closure of $D_1 \odot 1$, its domain is the space of elements $\xi \in K$ such that $\sum \lambda_k \xi_{k,r} \in K$ and $(D_1 \otimes 1)(\xi) = \sum \lambda_k \xi_{k,r}$. Similarly, the domain of $1 \otimes D_2$ is the space of elements $\eta \in K$ such that $\sum \mu_r \eta_{k,r} \in K$ and $(1 \otimes D_2)(\eta) = \sum \mu_r \eta_{k,r}$. Hence the domain of $D_1 \otimes 1 \mp \imath 1 \otimes D_2$ is the space

$$\widehat{V} = \{\xi \in K\colon\ both\ \lambda_k \xi_{k,r}, \mu_r \xi_{k,r} \in K\}. \tag{19}$$

Obviously, if $\xi \in K$, then $\xi \in \widehat{V}$ if and only if $(\lambda_k \mp \imath\mu_r)\xi_{k,r} \in K$, since that amounts to saying that $\sum(\lambda_k^2 + \mu_r^2) \|\xi_{k,r}\|^2 < \infty$. The domain of $\widehat{D}$ is then $\widehat{V}^2$, and this is the same as $Dom\, L$. Indeed, $\sum_{k,r}(\lambda_k^2 + \mu_r^2) \|[\xi_{k,r}, \eta_{k,r}]'\|^2 = \sum_{k,r}(\lambda_k^2 + \mu_r^2) (\|\xi_{k,r}\|^2 + \|\eta_{k,r}\|^2) = \sum \lambda^2 \|[\xi_{\lambda,k,r}, \eta_{\lambda,k,r}]'\|^2$. Since both $\widehat{D}, L$ coincide on every $E_{k,r}^2$, they are the same on their domain $\widehat{V}^2$.

Now let $\mathcal{E}_i$ be cores for D_i and $\mathcal{E} = \mathcal{E}_1 \odot \mathcal{E}_2$. Then trivially, $\mathcal{E}^2 \subset Dom\, \widehat{D}$. Since $\widehat{V}^2$ is the domain of $\widehat{D}$, we just have to show that each pair $(\zeta, D'\zeta)$, where $\zeta = [\xi_k \otimes \eta_r, \xi_k' \otimes \eta_r']'$ with $\xi_k, \xi_k' \in E_k, \eta_r, \eta_r' \in F_r$, is in the closure of the graph of $\widehat{D}$ restricted to $\mathcal{E}^2$. To prove this, we need only show that $(\xi_k \otimes \eta_r, D_1\xi_k \otimes \eta_r \mp \imath\xi_k \otimes D_2\eta_r)$ is in the closure of the graph of $D_1 \odot 1 \mp \imath 1 \odot D_2$ restricted to $\mathcal{E}$. This follows since there are sequences $\{v_n\}, \{w_n\}$ in $\mathcal{E}_1, \mathcal{E}_2$ such that $(v_n, D_1 v_n) \to (\xi_k, D_1\xi_k) = (\xi_k, \lambda_k\xi_k)$ and $(w_n, D_2 w_n) \to (\eta_r, D_2\eta_r) = (\eta_r, \mu_r\eta_r)$.

Section (5.2): The Gromov-Hausdorff Propinquity

The quantum Gromov-Hausdorff propinquity, introduced by [213], [210], is a distance on quantum compact metric spaces which extends the topology of the Gromov-Hausdorff distance [205], [204]. Quantum metric spaces are generalization of Lipschitz algebras [220] first discussed by Connes [172] and formalized by Rieffel [86]. The propinquity strengthens Rieffel's quantum Gromov-Hausdorff distance [87] to be well-adapted to the C^*-algebraic framework, in particular by making $*$-isomorphism a necessary condition for distance zero [211]. The propinquity thus allows us to address questions from mathematical physics, such as the problem of finite dimensional approximations of quantum space times [202], [215], [203], [219], [216]. Matricial approximations of physical theory motivates our project, which requires, at this early stage, the study of many different examples of quantum spaces.

Recently, the quantum tori form a continuous family for the propinquity, and admit finite dimensional approximations via so-called fuzzy tori [207]. Together with the work on AF algebras done in [201], explores the connection between our geometric approach to limits of C^*-algebras and the now well studied approach via inductive limits, which itself played a role is quantum statistical mechanics [2]. We thus bring noncommutative solenoids, studied by [78], [68], [214], and which are inductive limits of quantum tori, into the realm of noncommutative metric geometry. Our techniques apply to more general inductive limits on which projective limits of compact metrizable groups act ergodically. Noncommutative solenoids are interesting examples since they also are C^*-crossed products, whose metric structures are still a challenge to understand. Irrational noncommutative solenoids [78] are non-type I C^*-algebras, and many are even simple, thus they are examples of quantum spaces which are far from commutative.

We show that noncommutative solenoids are limits, for the quantum Gromov-Hausdorff propinquity, of quantum tori. As corollaries, we then show that the map from the solenoid group to the family of noncommutative solenoids is continuous for the quantum propinquity, and that noncommutative solenoids are limits of fuzzy tori, namely C^*-crossed products of finite cyclic groups acting on themselves by translation. As noncommutative solenoids have nontrivial K_1 group [78], they are not AF algebras, so our proof that they are limits of finite dimensional C^*-algebras illustrates the difference and potential usefulness of our metric geometric approach. Moreover, noncommutative solenoids' connection with wavelet theory [214] means that our result is a first step in what could be a metric approach to wavelet theory, by means of finite dimensional approximations. Last, metric approximations may prove a useful tool in the study of modules over non-commutative solenoids, initiated in [68], [214], as recent research in noncommutative metric geometry is concerned in part with the category of modules over quantum metric spaces [218]
Noncommutative solenoids, introduced in [78] and studied further in [68], [214], are the twisted group C^*-algebras of the Cartesian square of the subgroups of Q consisting of the p-adic rationals for some $p \in \mathbb{N}\setminus\{0,1\}$. We begin with the classification of the multipliers of these groups.

Theorem-Definition (5.2.1)[200]: ([78]). Let $p \in \mathbb{N}\setminus\{0,1\}$. The inductive limit of:

$$\mathbb{Z} \xrightarrow{k\mapsto pk} \mathbb{Z} \xrightarrow{k\mapsto pk} \mathbb{Z} \xrightarrow{k\mapsto pk} \cdots$$

is the group of p-adic rational numbers:

$$\mathbb{Z}\left[\frac{1}{p}\right] = \left\{\frac{q}{p^k} : q \in \mathbb{Z}, k \in \mathbb{N}\right\}.$$

The Pontryagin dual of $\mathbb{Z}\left[\frac{1}{p}\right]$ is the solenoid group:

$$S_p = \varprojlim \mathbb{T} \xleftarrow{z\mapsto z^p} \mathbb{T} \xleftarrow{z\mapsto z^p} \mathbb{T} \xleftarrow{z\mapsto z^p} \cdots = \left\{(z_n)_{n\in\mathbb{N}} \in \mathbb{T}^{\mathbb{N}} : \forall n \in \mathbb{N}\ z_{n+1}^p = z_n\right\},$$

where the dual pairing is given, for all $q \in \mathbb{Z},\ k \in \mathbb{N}$, and $(z_n)_{n\in\mathbb{N}} \in S_p$, by $\langle \frac{q}{p^k}, (z_n)_{n\in\mathbb{N}}\rangle = z_k^q$.

For any $\theta = (\theta_n)_{n\in\mathbb{N}} \in S_p$, and for all $q_1, q_2, q_3, q_4 \in \mathbb{Z}$ and $k_1, k_2, k_3, k_4 \in \mathbb{N}$, we define:

$$\Psi_\theta : \left(\left(\frac{q_1}{p^{k_1}}, \frac{q_2}{p^{k_2}}\right), \left(\frac{q_3}{p^{k_3}}, \frac{q^4}{p^{k_4}}\right)\right) = \theta_{k_1+k_4}^{q_1 q_4}.$$

For any multiplier f of $\mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right]$, there exists a unique $\theta \in S_p$ such that f is cohomologous to Ψ_θ.

Thus, formally, noncommutative solenoids are defined by:

Definition (5.2.2)[200]: A noncommutative solenoid $\mathfrak{S}_\theta$, for some $\theta \in S_p$, is the twisted group C^*-algebra $C^*\left(\mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right], \Psi_\theta\right)$.

We compute the K-theory of noncommutative solenoids in [78] in terms of the multipliers of $\mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right]$, identified with elements on the solenoid via Theorem- Definition (5.2.1); we then classify noncommutative solenoids up to their multiplier.

As the compact group S_p^2 acts on $\mathfrak{S}_\theta$ for any $\theta \in S_p$ via the dual action, any continuous length function on S_p^2 induces a quantum metric structure on $\mathfrak{S}_\theta$, as de-scribed in [86]. A quantum metric structure is given by a noncommutative analogue of the Lipschitz seminorm as follows:

Notation (5.2.3)[200]: If $\mathfrak{A}$ is a C^*-algebra with unit, then the norm on $\mathfrak{A}$ is denoted by $\|\cdot\|_{\mathfrak{A}}$, while the unit of $\mathfrak{A}$ is denoted by $1_{\mathfrak{A}}$. The state space of $\mathfrak{A}$ is denoted by $S(\mathfrak{A})$, and the subspace of self-adjoint elements in A is denoted by $\mathfrak{sa}(\mathfrak{A})$.

Definition (5.2.4)[200]: ([86], [195], [213]). A pair $(\mathfrak{A}, L)$ is a Leibniz quantum compact metric space when $\mathfrak{A}$ is a unital C^*-algebra and L is a seminorm defined on some dense Jordan-Lie subalgebra $dom(L)$ of the space of self-adjoint elements $\mathfrak{sa}(\mathfrak{A})$ of $\mathfrak{A}$, called a Lip-norm, such that:

(i) $\{a \in \mathfrak{sa}(\mathfrak{A}): L(a) = 0\} = \mathbb{R}1_{\mathfrak{A}}$,

(ii) $\max\{L\left(\frac{ab+ba}{2}\right), L\left(\frac{ab-ba}{2i}\right) \leq \|a\|_{\mathfrak{A}}L(b) + \|b\|_{\mathfrak{A}}L(a)$,

(iii) the Monge-Kantorovich metric mk_L dual to L on $S(\mathfrak{A})$ by setting, for all $\varphi, \psi \in S(\mathfrak{A})$ by $mk_L(\varphi, \psi) = sup\, \{|\varphi(a) - \psi(a)|: a \in dom(L), L(a) \leq 1\}$ induces the weak* topology on $S(\mathfrak{A})$,

(iv) L is lower semi-continuous with respect to $\|\cdot\|_{\mathfrak{A}}$.

Classical examples of Lip-norms are given by the Lipschitz seminorms on the C^*- algebras of $\mathbb{C}$-valued continuous functions on compact metric spaces. An important source of noncommutative example is given by:

Theorem-Definition (5.2.5)[200]: ([86]). Let α be a strongly continuous action by $*$-automorphisms of a compact group G on a unital C^*-algebra $\mathfrak{A}$ and let ℓ be a continuous length function on G. For all $a \in \mathfrak{sa}\,(\mathfrak{A})$, we define:

$$L_{\alpha,\ell}(a) = \sup\left\{\frac{\|a - \alpha^g(a)\|_{\mathfrak{A}}}{\ell(g)} : g \in \mathfrak{G}, g\ is\ not\ the\ unit\ of\ G\right\}.$$

Then $L_{\alpha,\ell}$ is a Lip-norm on $\mathfrak{A}$ if and only if α is ergodic, i.e. $\{a \in \mathfrak{A} : \forall g \in G\ \alpha^g(a) = a\} = \mathbb{C}1_{\mathfrak{A}}$. We note that $L_{\alpha,\ell}$ is always lower semi-continuous.

Theorem (5.2.5) is thus, in particular, applicable to any dual action on the twisted group C^*-algebra of some discrete Abelian group, such as noncommutative solenoids or quantum tori.

We continue the study of the geometry of classes of quantum compact metric spaces under noncommutative analogues of the Gromov-Hausdorff distance, with the perspective that such a new geometric approach to the study of C^*-algebras may prove useful in mathematical physics and C^*-algebra theory. Our focus is a noncommutative analogue of the Gromov-Hausdorff distance devised by the first author [213] as an answer to many early challenges in this program, and whose construction begins with a particular mean to relate two Leibniz quantum compact metric spaces via an object akin to a correspondence.

Definition (5.2.6)[200]: A bridge from a unital C^*-algebra $\mathfrak{A}$ to a unital C^*-algebra $\mathfrak{B}$ is a quadruple $(\mathfrak{D}, \omega, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ where:

(i) $\mathfrak{D}$ is a unital C^*-algebra,

(ii) the element ω, called the pivot of the bridge, satisfies $\omega \in \mathfrak{D}$ and $S_1(\mathfrak{D}|\omega) \neq \emptyset$, where:

$$S_1(\mathfrak{D}|\omega) = \{\varphi \in S(\mathfrak{D}) : \varphi((1-\omega^*\omega)) = \varphi((1-\omega\omega^*)) = 0\}$$

is called the 1-level set of ω,

(iii) $\pi_{\mathfrak{A}}: \mathfrak{A} \hookrightarrow \mathfrak{D}$ and $\pi_{\mathfrak{B}}: \mathfrak{B} \hookrightarrow \mathfrak{D}$ are unital $*$-monomorphisms.

There always exists a bridge between any two arbitrary Leibniz quantum compact metric spaces [213]. The quantum propinquity is computed from a numerical quantity called the length of a bridge. We will denote the Hausdorff (pseudo)distance associated with a (pseudo)metric d by $Haus_d$[206].

First introduced in [213], the length of a bridge is computed from two numbers, the height and the reach of a bridge. The height of a bridge assesses the error we make by replacing the state spaces of the Leibniz quantum compact metric spaces with the image of the 1-level set of the pivot of the bridge, using the ambient Monge-Kantorovich metric.

Definition (5.2.7)[200]: Let $(\mathfrak{A}, L_{\mathfrak{A}})$ and $(\mathfrak{B}, L_{\mathfrak{B}})$ be two Leibniz quantum compact metric spaces. The height $\varsigma(\gamma|L_{\mathfrak{A}}, L_{\mathfrak{B}})$ of a bridge $\gamma = (\mathfrak{D}, \omega, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ from $\mathfrak{A}$ to $\mathfrak{B}$, and with respect to $L_{\mathfrak{A}}$ and $L_{\mathfrak{B}}$, is given by:

$$\max\left\{Haus_{mk_{L_{\mathfrak{A}}}}(S(\mathfrak{A}), \{\varphi \circ \pi_{\mathfrak{A}}: \varphi \in S_1(\mathfrak{D}|\omega)\}), Haus_{mk_{L_{\mathfrak{B}}}}(S(\mathfrak{B}), \{\varphi \in \pi_{\mathfrak{B}}: \varphi \in S_1(\mathfrak{D}|\omega)\})\right\}.$$

The second quantity measures how far apart the images of the balls for the Lip-norms are in $\mathfrak{A} \oplus \mathfrak{B}$; to do so, they use a seminorm on $\mathfrak{A} \oplus \mathfrak{B}$ built using the bridge:

Definition (5.2.8)[200]: ([213]). Let $\mathfrak{A}$ and $\mathfrak{B}$ be two unital C^*-algebras. The bridge seminorm $bn_\lambda\,(\cdot)$ of a bridge $\gamma = (\mathfrak{D}, \omega, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ from $\mathfrak{A}$ to $\mathfrak{B}$ is the seminorm defined on $\mathfrak{A} \oplus \mathfrak{B}$ by $bn_\lambda\,(a, b) = \|\pi_{\mathfrak{A}}(a)\omega - \omega\pi_{\mathfrak{B}}(b)\|_{\mathfrak{D}}$ for all $(a, b) \in \mathfrak{A} \oplus \mathfrak{B}$.

We implicitly identify $\mathfrak{A}$ with $\mathfrak{A} \oplus \{0\}$ and $\mathfrak{B}$ with $\{0\} \oplus \mathfrak{B}$ in $\mathfrak{A} \oplus \mathfrak{B}$ in the next definition, for any two spaces $\mathfrak{A}$ and $\mathfrak{B}$.

Definition (5.2.9)[200]: ([213]). Let $(\mathfrak{A}, L_{\mathfrak{A}})$ and $(\mathfrak{B}, L_{\mathfrak{B}})$ be two Leibniz quantum compact metric spaces. The reach $\varrho\,(\gamma|L_{\mathfrak{A}}, L_{\mathfrak{B}})$ of a bridge $\gamma = (\mathfrak{D}, \omega, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ from $\mathfrak{A}$ to $\mathfrak{B}$, and with respect to $L_{\mathfrak{A}}$ and $L_{\mathfrak{B}}$, is given by:

$$Haus_{bn_\lambda(\cdot)}\,(\{a \in \mathfrak{sa}\,(\mathfrak{A}) : L_{\mathfrak{A}}(a) \leq 1\}, \{b \in \mathfrak{sa}\,(\mathfrak{B}): L_{\mathfrak{B}}(b) \leq 1\}).$$

We thus choose a natural synthetic quantity to summarize the information given by the height and the reach of a bridge:

Definition (5.2.10)[200]: ([213]). Let $(\mathfrak{A}, L_{\mathfrak{A}})$ and $(\mathfrak{B}, L_{\mathfrak{B}})$ be two Leibniz quantum compact metric spaces. The length $\lambda\,(\gamma|L_{\mathfrak{A}}, L_{\mathfrak{B}})$ of a bridge $\gamma = (\mathfrak{D}, \omega, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ from $\mathfrak{A}$ to $\mathfrak{B}$, and with respect to $L_{\mathfrak{A}}$ and $L_{\mathfrak{B}}$, is given by $max\,\{\varsigma(\gamma|L_{\mathfrak{A}}, L_{\mathfrak{B}}), \varrho(\gamma|L_{\mathfrak{A}}, L_{\mathfrak{B}})\}$.

The quantum Gromov-Hausdorff propinquity is constructed from bridges, though the construction requires some care. We refer to [213] for the construction.

Theorem-Definition (5.2.11)[200]: ([213]). Let $\mathcal{L}$ be the class of all Leibniz quantum compact metric spaces. There exists a class function Λ from $\mathcal{L} \times \mathcal{L}$ to $[0, \infty) \subseteq \mathbb{R}$ such that:

(i) for any $(\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}}) \in \mathcal{L}$ we have:

$0 \leq \Lambda\big((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})\big) \leq\ max\,\{diam(S(\mathfrak{A}), mk_{L_{\mathfrak{A}}}), diam(S(\mathfrak{B}), mk_{L_{\mathfrak{B}}})\}$,

(ii) for any $(\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}}) \in \mathcal{L}$ we have:

$$\Lambda((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})) = \Lambda((\mathfrak{B}, L_{\mathfrak{B}}), (\mathfrak{A}, L_{\mathfrak{A}})),$$

(iii) for any $(\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}}), (\mathfrak{C}, L_{\mathfrak{C}}) \in \mathcal{L}$ we have:

$$\Lambda((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{C}, L_{\mathfrak{C}})) \leq \Lambda((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})) + \Lambda((\mathfrak{B}, L_{\mathfrak{B}}), (\mathfrak{C}, L_{\mathfrak{C}})),$$

(iv) for all $(\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}}) \in \mathcal{L}$ and for any bridge γ from $\mathfrak{A}$ to $\mathfrak{B}$, we have

$$\Lambda((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})) \leq \lambda\,(\gamma|L_{\mathfrak{A}}, L_{\mathfrak{B}}),$$

(v) for any $(\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}}) \in \mathcal{L}$, we have $\Lambda((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})) = 0$ if and only if $(\mathfrak{A}, L_{\mathfrak{A}})$ and $(\mathfrak{B}, L_{\mathfrak{B}})$ are isometrically isomorphic, i.e. if and only if there exists a $*$-isomorphism $\pi: \mathfrak{A} \to \mathfrak{B}$ with $L_{\mathfrak{B}} \circ \pi = L_{\mathfrak{A}}$, or equivalently there exists a $*$-isomorphism $\pi: \mathfrak{A} \to \mathfrak{B}$ whose dual map π^* is an isometry from $(S(\mathfrak{B}), mk_{L_{\mathfrak{B}}})$ into $(S(\mathfrak{A}), mk_{L_{\mathfrak{A}}})$,
(vi) if Ξ is a class function from $\mathcal{L} \times \mathcal{L}$ to $[0, \infty)$ which satisfies Properties (ii), (iii) and (iv) above, then $\Xi((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})) \leq \Lambda((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}}))$ for all $(\mathfrak{A}, L_{\mathfrak{A}})$ and $(\mathfrak{B}, L_{\mathfrak{B}})$ in $\mathcal{L}$,
(vii) the topology induced by Λ on the class of classical metric spaces agrees with the topology induced by the Gromov-Hausdorff distance.
The study of finite dimensional approximations of quantum compact metric spaces for the quantum propinquity is an important topic in noncommutative metric geometry, with results about the quantum tori [77], [207], spheres [217], [218], and AF algebras [201]. It is in general technically very difficult to construct natural approximations, while their existence is only known under certain certain quantum topological properties (pseudo-diagonality) [212]. Moreover, quantum tori have been an important test case for our theory, with work on the continuity of the family of quantum tori [207], and perturbations of metrics for curved quantum tori [209]. See [211] for a survey of the theory of quantum compact metric spaces and the Gromov-Hausdorff propinquity.
All our results are valid for the dual Gromov-Hausdorff propinquity [210], [208] and therefore for Rieffel's quantum Gromov-Hausdorff distance [87].

The first step in obtaining our results about noncommutative solenoids consists in constructing a natural metric on the countable product $\prod_{n \in \mathbb{N}} G_n$ of a sequence $(G_n)_{n \in \mathbb{N}}$ of compact metrizable groups. Our metric is inspired by a standard construction of metrics on the Cantor set, and is motivated by the desire to have the sequence of subgroups $(\prod_{n>N} G_n)_{N \in \mathbb{N}}$ converge to the trivial group for the induced Hausdorff distance. This latter property will be the key to our computation of estimates on the propinquity later on. Our metrics are constructed from length functions. We recall that ℓ is a length function on a group G with unit e when:
(i) for any $x \in G$, the length $\ell(x)$ is 0 if and only if $x = e$,
(ii) $\ell(x) = \ell(x^{-1})$ for all $x \in G$,
(iii) $\ell(xy) \leq \ell(x) + \ell(y)$ for all $x, y \in G$.
Hypothesis (5.2.12)[200]: Let $(G_n)_{n \in \mathbb{N}}$ be a sequence of compact metrizable groups, and for each $n \in \mathbb{N}$ let ℓ_n be a continuous length function on G_n. Let$M \geq diam(G_0, \ell_0)$.
Let:

$$\mathbb{G} = \prod_{n \in \mathbb{N}} G_n = \{(g_n)_{n \in \mathbb{N}} : \forall n \in \mathbb{N} \quad g_n \in G_n\},$$

endowed with the product topology. With the pointwise operations, $\mathbb{G}$ is a compact group. We denote the unit of $\mathbb{G}$ by 1 and, by abuse of notation, we also denote the unit of G_n by 1 for all $n \in \mathbb{N}$.
Definition (5.2.13)[200]: Let Hypothesis (5.2.12) be given. We define the length function ℓ_∞ on $\mathbb{G}$ by setting, for any $g = (g_n)_{n \in \mathbb{N}}$ in $\mathbb{G}$:

$$\ell_\infty(g) = \inf\{\varepsilon > 0 : \forall n \in \mathbb{N}\, n < \frac{M}{\varepsilon} \Longrightarrow \ell_n(g_n) \leq \varepsilon\}.$$

The basic properties of the metric are given by:
Proposition (5.2.14)[200]: Assume Hypothesis (5.2.12). The length function ℓ_∞ on $\mathbb{G} = \prod_{n \in \mathbb{N}} G_n$ from Definition (5.2.13) is continuous for the product topology on the compact

group $\mathbb{G}$, and thus metrizes this topology. Moreover, if for all $N \in \mathbb{N}$, we set $\mathbb{G}^{(N)} = \{(g_n)_{n\in\mathbb{N}} \in \mathbb{G} : \forall j \in \{0,\dots,N\} \quad g_j = 1\}$, then $\mathbb{G}^{(N)}$ is a closed subgroup of $\mathbb{G}$ and:

$$diam\left(\mathbb{G}^{(N)}, \ell_\infty\right) \le \frac{M}{N+1}, \tag{20}$$

and thus in particular, if $1 \in \mathbb{G}$ is the unit of $\mathbb{G}$:

$$\lim_{N\to\infty} Haus_{\ell_\infty}\left(\mathbb{G}^{(N)}, \{1\}\right) = 0. \tag{21}$$

Proof. We easily note that $diam(\mathbb{G}, \ell_\infty) \le diam(G_0, \ell_0)$. Indeed, if $g = (g_n)_{n\in\mathbb{G}} \in \mathbb{G}$ then for $n = 0 < 1 = \frac{M}{diam(G_0,\ell_0)}$ we have $\ell_0(g_0) \le diam(G_0, \ell_0)$. So by definition, $\ell_\infty(g) \le diam(G_0, \ell_0)$.

Now, let $N \in \mathbb{N}$. We observe that if $g = (g_n)_{n\in\mathbb{N}} \in \mathbb{G}^{(N)}$, then for all $n \le N < \frac{M}{\frac{M}{N+1}}$ we have $\ell_n(g_n) = 0 \le \frac{M}{N+1}$. Thus, $\ell_\infty(z) \le \frac{M}{N+1}$.

This proves both Expressions (20) and (21).

Assume now that $(g^m)_{m\in\mathbb{N}}$ converges in $\mathbb{G}$ to some g, i.e. converges pointwise.

Let $\varepsilon > 0$. Let $N = \lfloor\frac{M}{\varepsilon}\rfloor$. For each $j \in \{0,\dots,N\}$, there exists $K_j \in \mathbb{N}$ such that for all $m \ge K_j$, we have $\ell_j(g_j^m g_j^{-1}) \le \varepsilon$, by pointwise convergence. Let $K = \max\{K_j : j \in \{0,\dots,N\}\}$. Then by construction, for all $m \ge K$, we have, for all $n < \frac{M}{\varepsilon}$, that $\ell_n(g_n^m\, g_j^{-1}) \le \varepsilon$, so $\ell_\infty(g^m g^{-1}) \le \varepsilon$. Thus ℓ_∞ is continuous and induces a weaker topology on $\mathbb{G}$ than the topology of pointwise convergence.

Assume now that $\ell_\infty((g_n)_{n\in\mathbb{N}}) = 0$. Fix $k \in \mathbb{N}$. Let $N > k$. Then $\ell_\infty((g_n)_{n\in\mathbb{N}}) \le \frac{M}{N+1}$. Thus by definition, $\ell_k(g_k) \le \frac{M}{N+1}$ for all $N > k$. Thus $\ell_k(g_k) = 0$ for all $k \in \mathbb{N}$ and thus g_k is the unit of G_k for all $k \in \mathbb{N}$.

Thus the topology induced by ℓ_∞ is Hausdorff, and thus, as the product topology on $\mathfrak{G}^N$ is compact by Tychonoff theorem, ℓ_∞ induces the product topology on $\mathfrak{G}^N$. This could also be easily verified directly.

We shall apply Definition (5.2.13) and Proposition (5.2.14) to projective limits, and thus we record the following corollary. We note that all our projective sequences of groups involve only epimorphisms.

Corollary (5.2.15)[200]: Let $G_0 \xleftarrow{\rho_0} G_1 \xleftarrow{\rho_1} G_2 \xleftarrow{\rho_2} \cdots = (G_n, \rho_n)_{n\in\mathbb{N}}$ be a projective sequence of compact metrizable groups, and let ℓ_n be a continuous length function on G_n for all $n \in \mathbb{N}$. Let $M \ge diam(G_0, \ell_0)$. Let:

$$G = \varprojlim (G_n, \rho_n)_{n\in\mathbb{N}} = \left\{(g_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} G_n : \forall n \in \mathbb{N} \quad g_n = \rho_n(g_{n+1})\right\}.$$

The restriction to G of the length function ℓ_∞ on $\prod_{n\in\mathbb{N}} G_n$ from Definition (5.2.13) metrizes the projective topology on G; moreover if $G_N = G \cap \mathbb{G}^{(N)}$ for all $N \in \mathbb{N}$, then $Haus_{\ell_\infty}(G_N, \{1\}) \le \frac{M}{N+1}$ with $1 \in G$ the unit of G.

Proof. This is all straightforward as G is a closed subgroup of $\mathbb{G}$.

We begin our study of quantum metrics on inductive limits with the observation that the proof of [211] includes the following fact, which will be of great use to us in view of Corollary (5.2.15):

Lemma (5.2.16)[200]: Let G be a compact metrizable group, $H \subseteq G$ be a normal closed subgroup, ℓ a continuous length function on G and $\mathfrak{A}$ *a* unital C^*-algebra endowed with a

strongly continuous ergodic action α of G. Let $K = G/H$ and let ℓ_K be the continuous length function $\ell_K: k \in K \longmapsto \inf\{\ell(g): g \in kH\}$ where kH, for any $k \in K$, is the coset associated with k.
Let $\mathfrak{A}_K = \{a \in \mathfrak{A} : \forall g \in H \alpha^g(a) = a\}$ be the fixed point C^*-subalgebra of $\mathfrak{A}$ for the action α of K on $\mathfrak{A}$. Note that α induces an ergodic, strongly continuous action β of K on $\mathfrak{A}_K$. Using Theorem (1.5), Let L be the Lip-norm on $\mathfrak{A}$ given by the action α of G and the length function ℓ, and let L_K be the Lip-norm on $\mathfrak{A}_K$ given by the action β of K and the length function ℓ_K. Then:

$$\Lambda((\mathfrak{A}, L), (\mathfrak{A}_K, L_K)) \leq diam(H, \ell).$$

Proof. We first note that since H is closed, ℓ_K is easily checked to be a length function on K. Moreover, if $\pi: G \twoheadrightarrow K$ is the canonical surjection, then the trivial inequality $\ell_K(\pi(g)) \leq \ell(g)$ for all $g \in G$ proves that ℓ_K is continuous on K since $g \in G \longmapsto \ell_K(\pi(g))$ is 1-Lipschitz, by characterization of continuity for the final topology on K.
Let μ be the Haar probability measure on H. For all $a \in \mathfrak{A}$, we define:

$$\mathbb{E}(a) = \int_H \alpha^g(a)\, d\mu(g)\,.$$

A standard argument shows that $\mathbb{E}$ is a unital conditional expectation on $\mathfrak{A}$ with range $\mathfrak{A}_K$. In particular, $\mathbb{E}$ maps $\mathfrak{sa}(\mathfrak{A})$ onto $\mathfrak{sa}(\mathfrak{A}_K)$.
Moreover, we note that since H is normal, we have $gH = Hg$ for all $g \in G$, and thus:

$$L_K\big(\mathbb{E}(a)\big) = \sup\left\{\frac{\left\|\alpha^g\left(\int_H \alpha^h(a)d\mu(h)\right) - \int_H \alpha^h(a)d\mu(h)\right\|_{\mathfrak{A}}}{\ell(g)} : g \in G \setminus \{1\}\right\}$$

$$= sup\left\{\frac{\left\|\int_{gH} \alpha^h(a)d\mu(h) - \int_H \alpha^h(a)d\mu(h)\right\|_{\mathfrak{A}}}{\ell(g)} : g \in G \setminus \{1\}\right\}$$

$$= \ sup\left\{\frac{\left\|\int_{Hg} \alpha^h(a)d\mu(h) - \int_H \alpha^h(a)d\mu(h)\right\|_{\mathfrak{A}}}{\ell(g)} : g \in G \ \ \{1\}\right\}$$

$$\leq sup\left\{\frac{\int_H \left\|\alpha^{hg}(a) - \alpha^h(a)d\mu(h)\right\|_{\mathfrak{A}}}{\ell(g)} : g \in G \setminus \{1\}\right\}$$

$$= \ sup\left\{\frac{\|a - \alpha^g(a)\|_{\mathfrak{A}}}{\ell(g)} : g \in G \setminus \{1\}\right\} = \ L(a).$$

Hence $\mathbb{E}$ is a weak contraction from $(\mathfrak{A}, L)$ onto $(\mathfrak{A}_K, L_K)$.
Let now id be the identity operator on $\mathfrak{A}$ and $\vartheta: \mathfrak{A}_K \hookrightarrow \mathfrak{A}$ be the canonical inclusion map. We thus define a bridge $\gamma = (\mathfrak{A}, 1_{\mathfrak{A}}, \vartheta, id)$ from $\mathfrak{A}_K$ to $\mathfrak{A}$, whose height is null since its pivot is $1_{\mathfrak{A}}$. We are thus left to compute the reach of γ.
To begin with, if $a \in \mathfrak{sa}\,(\mathfrak{A}_K)$ with $L_K(a) \leq 1$, then an immediate computation proves that $L(a) = L_K(a) \leq 1$ and thus $\|a1_{\mathfrak{A}} - 1_{\mathfrak{A}}a\|_{\mathfrak{A}} = 0$.
Now let $a \in \mathfrak{sa}(\mathfrak{A})$ with $L(a) \leq 1$. Then $L_K(\mathbb{E}(a)) \ \leq \ 1$, and we have:
$\|a - \mathbb{E}(a)\|_{\mathfrak{A}} = \left\|\int_H \alpha^h(a) - a\, d\mu(h)\right\|_{\mathfrak{A}}$ since μ probability measure,

$$\leq \int_H \left\|\alpha^h(a) - a\right\|_{\mathfrak{A}} d\mu(h)$$

$$\leq \int_H \ell(h)L(a)\, d\mu(h)$$

$$\leq \int_H diam(H,\ell)\, d\mu(h) \;=\; diam(H,\ell).$$

Thus, the reach, and hence the length of γ is no more than $diam(H,\ell)$, which, by Theorem-Definition (5.2.11), concludes our proof for our lemma.
We prove one of the main results.

Theorem (5.2.17)[200]: Let $G_0 \overset{\rho_0}{\longleftarrow} G_1 \overset{\rho_1}{\longleftarrow} G_2 \overset{\rho_2}{\longleftarrow} \cdots = (G_n,\rho_n)_{n\in\mathbb{N}}$ be a projective sequence of compact metrizable groups, and for each $n \in \mathbb{N}$, let ℓ_n be a continuous length function on G_n. Let $\mathfrak{A}$ be a unital C^*-algebra endowed with a strongly continuous action α of $G = \varprojlim(G_n,\rho_n)_{n\in\mathbb{N}}$. Let$\varrho_n: G \twoheadrightarrow G_n$ be the canonical surjection for all $n \in \mathbb{N}$.
We endow G with the continuous length function ℓ_∞ from Definition (5.2.13) for some $M \geq diam(G_0,\ell_0)$.
For all $N \in \mathbb{N}$, let:

$$G^{(N)} = ker\varrho_N = \{(g_n)_{n\in\mathbb{N}} \in G: \forall n \in \{0,\ldots,N-1\}\;\; g_n = 1\} \trianglelefteq G$$

and let $\mathfrak{A}_N$ be the fixed point C^*-subalgebra of α restricted to $G^{(N)}$. We denote by α_n the action of G_n induced by α on $\mathfrak{A}_n$ for all $n \in \mathbb{N}$.
Moreover, for all $n \in \mathbb{N}$ and $g \in G_n$ we set:

$$\ell_\infty^n(g) = \inf\{\ell_\infty(h): \varrho_n(h) = g\}\,.$$

If, for some $n \in \mathbb{N}$, the action of G_n induced by α on $\mathfrak{A}_n$ is ergodic, then:
(i) α is ergodic on $\mathfrak{A}$ and α_n is ergodic on $\mathfrak{A}_n$ for all $n \in \mathbb{N}$
(ii) If L is the Lip-norm induced by α and ℓ_∞ on $\mathfrak{A}$ and L_n is the Lip-norm induced by α_n and ℓ_∞^n on $\mathfrak{A}_n$ using Theorem (5.2.5), then for all $n \in \mathbb{N}$:

$$\Lambda((\mathfrak{A},L),(\mathfrak{A}_n,L_n)) \leq \frac{M}{n+1}$$

and thus: $\lim_{n\to\infty} \Lambda((\mathfrak{A},L),(\mathfrak{A}_n,L_n)) = 0$.

Proof. For any given $n \in \mathbb{N}$, the group G_n is isomorphic to $G/G^{(n)}$ and we are in the setting of Lemma (5.2.16) — in particular, ℓ_∞^n is a continuous length function on G_n and α_n is a well-defined action.
We note that by construction, for all $n \in \mathbb{N}$:

$$\{a \in \mathfrak{A}_n: \forall g \in G_n\;\; \alpha_n^g(a) = a\} = \{a \in \mathfrak{A}_n: \forall g \in G\; \alpha^g(a) = a\}\,. \tag{22}$$

Let us now assume that the action α_n is ergodic for some $n \in \mathbb{N}$. Let $a \in \mathfrak{A}$ such that for all $g \in G$ we have $\alpha^g(a) = a$. Then $a \in \mathfrak{A}_n$ in particular, since a is invariant by the action of α restricted to $G^{(n)}$. Moreover, a is invariant by the action α_n by Expression (22) and thus $a \in \mathfrak{A}1_{\mathfrak{A}}$. Thus α is ergodic. This, in turn, proves that for all $n \in \mathbb{N}$, the action α_n is ergodic by Expression (22).
Thus, L and L_n are now well-defined. By Lemma (5.2.16) and Corollary (5.2.15), we obtain:

$$\Lambda((\mathfrak{A},L),(\mathfrak{A}_n,L_n)) \leq \frac{M}{n+1}\,.$$

This concludes our proof.
Theorem (5.2.17) involves an ergodic action of a projective limit of compact groups on a unital C^*-algebra and one may wonder when such actions exist. The following theorem proves that one may obtain such actions on inductive limits, under reasonable compatibility conditions. Thus the next theorem provides us with a mean to construct Leibniz Lip-norms on inductive limits of certain Leibniz quantum compact metric spaces.

Theorem (5.2.18)[200]: Let $G_0 \overset{\rho_0}{\longleftarrow} G_1 \overset{\rho_1}{\longleftarrow} G_2 \overset{\rho_2}{\longleftarrow} \cdots = (G_n,\rho_n)_{n\in\mathbb{N}}$ be a projective sequence of compact groups. Let:

$$G = \left\{ (g_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} G_n : \forall n \in \mathbb{N}\ \rho_n(g_{n+1}) = g_n \right\},$$

noting that $G = \varprojlim(G_n, \rho_n)_{n\in\mathbb{N}}$.

Let $\mathfrak{A}_0 \xrightarrow{\varphi_0} \mathfrak{A}_1 \xrightarrow{\varphi_1} \mathfrak{A}_2 \xrightarrow{\varphi_2} \cdots = (\mathfrak{A}_n, \varphi_n)_{n\in\mathbb{N}}$ be an inductive sequence of unital C^*-algebras where, for all $n \in \mathbb{N}$, we assume:

(i) φ_n is a $*$-monomorphism,

(ii) there exists an ergodic action α n of G_n on $\mathfrak{A}_n$,

(iii) for all $g = (g_n)_{n\in\mathbb{N}} \in G$ we have:

$$\varphi_n \circ \alpha_n^{g_n} = \alpha_{n+1}^{g_{n+1}} \circ \varphi_n. \tag{23}$$

We denote by $\mathfrak{A}$ the inductive limit of $(\mathfrak{A}_n, \varphi_n)_{n\in\mathbb{N}}$.

Then there exists an ergodic strongly continuous action α of $G = \varprojlim(G_n, n)_{n\in\mathbb{N}}$ on $\mathfrak{A}$.

Proof. For all $(a_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathfrak{A}_n$, we set $\|(a_n)\|_\infty = \lim sup_{n\to\infty} \|a_n\|_{\mathfrak{A}_n}$, which defined a C^*-seminorm on $\prod_{n\in\mathbb{N}} \mathfrak{A}_n$. The quotient of $\prod_{n\in\mathbb{N}} \mathfrak{A}_n$ by $\{a \in \prod_{n\in\mathbb{N}} \mathfrak{A}_n : \|a\|_\infty = 0\}$, endowed with the quotient seminorm of $\|\cdot\|_\infty$, which we still denote by $\|\cdot\|_\infty$, is a C^*-algebra, which we denote by $\lim sup_{n\to\infty} \mathfrak{A}_n$. Let π be the canonical surjection from $\prod_{n\in\mathbb{N}} \mathfrak{A}_n$ onto $\lim sup_{n\to\infty} \mathfrak{A}_n$.

Up to a $*$-isomorphism, $\mathfrak{A} = \varinjlim(\mathfrak{A}_n, \varphi_n)$ is the completion of the image by π of the set:

$$\mathfrak{A}_\infty = \{(a_n)_{n\in\mathbb{N}} : \exists N \in \mathbb{N}\ \forall n > N \quad a_n = \varphi_{n-1} \circ \ldots \circ\ \varphi_N(a_N)\},$$

in $\lim\ sup_{n\to\infty} \mathfrak{A}_n$.

We begin with a useful observation. Let $a = (a_n)_{n\in\mathbb{N}}$ and $b = (b_n)_{n\in\mathbb{N}}$ in $\mathfrak{A}_n$ with $\|a - b\|_\infty = 0$. Let $N \in \mathbb{N}$ such that, for all $n > \mathbb{N}$, we have $a_{n+1} = \varphi_n(a_n)$ and $b_{n+1} = \varphi_n(b_n)$: note that by definition, such a number N exists. If $\|a_N - b_N\|_{\mathfrak{A}} > \varepsilon$ for some $\varepsilon > 0$, then since φ_n is a $*$-monomorphism for all $n \in \mathbb{N}$, it is an isometry, and thus $\lim\ sup_{n\to\infty}\|a_n - b_n\|_{\mathfrak{A}_n} \geq \varepsilon$, which is a contradiction. Hence, for all $n \geq N$ we have $\|a_n - b_n\|_{\mathfrak{A}_n} = 0$. Informally, if two sequences in $\mathfrak{A}_\infty$ describe the same element of $\mathfrak{A}$, then their predictable tails are in fact equal.

We now define the action of G on $\mathfrak{A}$. For $g = (g_n)_{n\in\mathbb{N}} \in G$ and $(a_n)_{n\in\mathbb{N}} \in \mathfrak{A}_\infty$, we set $\alpha^g((a_n)_{n\in\mathbb{N}}) = (\alpha^{g_n}(a_n))_{n\in\mathbb{N}}$, which is a $*$-morphism of norm 1. Condition (23) ensures that α^g maps $\mathfrak{A}_\infty$ to itself. It induces an action of G on $\pi(\mathfrak{A})$ by norm 1 $*$-automorphisms in the obvious manner, and thus extends to $\mathfrak{A}$ by continuity (we use the same notation for this extension). It is easy to check that α is an action of G on $\mathfrak{A}$.

Let $a \in \pi(\mathfrak{A}_\infty)$ such that $\alpha^g(a) = a$ for all $g \in G$. Let $(a_n)_{n\in\mathbb{N}} \in \mathfrak{A}_\infty$ with $\pi((a_n)_{n\in\mathbb{N}}) = a$. Let $N \in \mathbb{N}$ such that for all $n \geq N$, we have $a_{n+1} = \varphi_n(a_n)$.

By definition of the action α, we have for all $g = (g_n)_{n\in\mathbb{N}} \in G$ that $\alpha^g(a) = (\alpha_n^{g_n}(a_n))_{n\in\mathbb{N}}$, and we note that:

$$\alpha_{n+1}^{g_{n+1}}(a_{n+1}) = \alpha_{n+1}^{g_{n+1}}(\varphi_n(a_n)) = \varphi_n(\alpha_n^{g_n}(a)),$$

by Condition (23). Thus by our earlier observation, we conclude that $\alpha_N^{g_N}(a_N) = a_N$ for all $g \in G$. Thus, as ρ_N is surjective, and α_N is ergodic, we conclude that $a_N = \lambda 1_{\mathfrak{A}_N}$. Thus for all $n \geq N$ we have $a_n = \varphi_n \circ \cdots \circ \varphi_N(\lambda 1_{\mathfrak{A}_N})$. Consequently, $a \in \mathbb{C}1_{\mathfrak{A}}$ by definition.

Now, let μ be the Haar probabilitymeasure on G and define $\mathbb{E}(a) = \int_G \alpha^g(a)\, d\mu(g)$ for all $a \in \mathfrak{A}$. It is straightforward to check that $\mathbb{E}(a)$ is invariant by α for all $a \in \mathfrak{A}$.

Let $a \in \mathfrak{A}$ such that $\alpha^g(a) = a$ for all $g \in G$. Thus $\mathbb{E}(a) = a$. Let $\varepsilon > 0$. There exists $a_\varepsilon \in \mathfrak{A}_\infty$ such that $\|a - a_\varepsilon\|_{\mathfrak{A}} \leq \frac{\varepsilon}{2}$. Now:

$$\|\mathbb{E}(a)-\mathbb{E}(a_\varepsilon)\|_{\mathfrak{A}}=\|\mathbb{E}(a-a_\varepsilon)\|_{\mathfrak{A}}\le\|a-a_\varepsilon\|_{\mathfrak{A}}\ \le\frac{\varepsilon}{2}.$$
and yet $\mathbb{E}(a_\varepsilon)\in\mathbb{C}1_{\mathfrak{A}}$ since G is ergodic on $\mathfrak{A}_\infty$. Thus, as $\varepsilon>0$ is arbitrary, $\mathbb{E}(a)$ lies in the closure of $\mathbb{C}1_{\mathfrak{A}}$, i.e. in $\mathbb{C}1_{\mathfrak{A}}$, and thus α is ergodic.
Finally, again let $a\in\mathfrak{A}$ and $\varepsilon>0$, and let $a\varepsilon\in\pi(\mathfrak{A}_\infty)$ such that $\|a-a_\varepsilon\|_{\mathfrak{A}}\le\frac{\varepsilon}{3}$.
Let $(a_n)_{n\in\mathbb{N}}\in\mathfrak{A}_\infty$ such that $\pi((a_n)_{n\in\mathbb{N}})=a_\varepsilon$. There exists $N\in\mathbb{N}$ such that $\varphi_n(a_n)=a_{n+1}$ for all $n\ge N$. Since α_N is strongly continuous, there exists a neighborhood V of $1\in G_N$ such that $\left\|\alpha_N^g(a_N)-a_N\right\|_{\mathfrak{A}_N}<\frac{\varepsilon}{3}$ for all $\in V$. Let $W=\rho_N^{-1}(V)$ which is an open neighborhood of $1\in G$. Then since φ_n is an isometry for all $n\in\mathbb{N}$, we have for all $g=(g_n)_{\in\mathbb{N}}\in W$:
$$\left\|\alpha_n^{g_n}(a_n)-a_n\right\|_{\mathfrak{A}_n}=\left\|\alpha_N^{g_N}(a_N)-a_N\right\|_{\mathfrak{A}_N}\le\frac{\varepsilon}{3}.$$
Thus for all $g\in W$ we have:
$$\|a-\alpha^g(a)\|_{\mathfrak{A}}\ \le\|a-a_\varepsilon\|_{\mathfrak{A}}+\|a_\varepsilon-\alpha^g(a_\varepsilon)\|_{\mathfrak{A}}+\|\alpha^g(a_\varepsilon-a)\|\le\varepsilon.$$
Thus α is strongly continuous.
Thus, Theorem (5.2.18) can provide ergodic, strongly continuous actions on certain inductive limits, which then fit Theorem (5.2.17) and provide us with convergence of certain Leibniz quantum compact metric spaces to inductive limit C^*-algebras:
Corollary (5.2.19)[200]: We assume the same assumptions as Theorem (5.2.18). Moreover, for each $n\in\mathbb{N}$, let ℓ_n be a continuous length function on G_n. Let ℓ_∞ and, for all $n\in\mathbb{N}$, let ℓ_∞^n be given as in Theorem (5.2.17), for some $M\ge\ diam(G_0,\ell_0)$.
We denote by $\mathfrak{A}$ the inductive limit of $(\mathfrak{A}_n,\varphi_n)_{n\in\mathbb{N}}$.
Let α be the action of G on $\mathfrak{A}$ constructed in Theorem (5.2.17). For all $n\in\mathbb{N}$, let $\mathfrak{B}_n$ is the fixed point C^*-subalgebra of the restriction of α to $ker\ \rho_n$, let L_n be the Lip-norm defined from the restriction of α to G_n on $\mathfrak{B}_n$ using the length function ℓ_∞^n. If L is the Lip-norm on $\mathfrak{A}$ induced by α and ℓ_∞ via Theorem (5.2.5) then:
$$\lim_{n\to\infty}\Lambda((\mathfrak{A},L),(\mathfrak{B}_n,L_n))=0.$$
Proof. Apply Theorem (5.2.16) to Theorem (5.2.18).

We apply the work to the noncommutative solenoids. We begin by setting our framework. We begin with some notation.
Notation (5.2.20)[200]: For any $\theta\in S_p$, the noncommutative solenoid $\mathfrak{S}_\theta$ is, by Definition (5.2.2), the universal C^*-algebra generated by unitaries $W_{x,y}$ with $x,y\in\mathbb{Z}\left[\frac{1}{p}\right]\times\mathbb{Z}\left[\frac{1}{p}\right]$, subject to the relations: $W_{x,y}W_{x',y'}=\Psi_\theta((x,y),(x',y'))W_{x+x',y+y'}$.
By functoriality of the twisted group C^*-algebra construction, we note that non-commutative solenoids are inductive limits of quantum tori. All the quantum tori are rotation C^*-algebras, and we shall employ a slightly unusual notation, which will make our presentation clearer:
Notation (5.2.21)[200]: The rotation C^*-algebra $\mathfrak{A}_\theta$, for $\theta\in\mathbb{T}$, is the C^*-algebra generated by two unitaries U_θ and V_θ which is universal for the relation $VU=\theta UV$.
Theorem (5.2.22)[200]: ([78]). Let $p\in\mathbb{N}\setminus\{0\}$ and $\theta\in S_p$. For each $n\in\mathbb{N}$, we define the map $\Theta_n:\mathfrak{A}_{\theta_{2n}}\to\mathfrak{A}_{\theta_{2n+2}}$ as the unique $*$-monomorphism such that:
$$\Theta_n(U_{\theta_{2n}})=U_{\theta_{2n+2}}^p\quad\text{and }\Theta_n(V_{\theta_{2n}})=V_{\theta_{2n+2}}^p.$$
Then:
$$\mathfrak{S}_\theta=\varinjlim(\mathfrak{A}_{\theta_{2n}},\Theta_n)_{n\in\mathbb{N}}.$$
Moreover, the canonical injection ρ_n from $\mathfrak{A}_{\theta_{2n}}$ into $\mathfrak{S}_\theta$ is given by extending the map:

$$U_{\theta_{2n}} \longmapsto W_{\frac{1}{p^n},0} \text{ and } V_{\theta_{2n}} \longmapsto W_{0,\frac{1}{p^n}}.$$

Theorem (5.2.23)[200]: Let $\theta \in S_p$ and ℓ a continuous length function on $\mathbb{T}^2$. We let ℓ_∞ be the length function of Definition (5.2.13) on S_p^2 for $M = diam(\mathbb{T}^2, \ell)$. For all $n \in \mathbb{N}$ and all $z \in \mathbb{T}^2$, let:

$$\ell_\infty^n(z) = \inf\{\ell_\infty(\omega) : \omega \in S_p^2, \omega = (z^{p^n}, z^{p^{n-1}}, \dots, z, \dots)\}.$$

Then ℓ_∞^n is a continuous length function on $\mathbb{T}^2$. Let L_n be the Lip-norm on the quantum torus $\mathfrak{A}_{\theta_{2n}}$ defined by ℓ_∞^n, the dual action of $\mathbb{T}^2$ on $\mathfrak{A}_{\theta_{2n}}$, and Theorem- Definition (5.2.5). Let L be the Lip-norm on $\mathfrak{S}_\theta$ defined by the dual action α of S_p^2 and the length ℓ_∞ via Theorem-Definition (5.2.5).

We then have, for all $n \in \mathbb{N}$:

$$\Lambda^*((\mathfrak{A}_\theta, L), (\mathfrak{A}_{\theta_{2n}}, L_n)) \leq \frac{diam(\mathbb{T}^2, \ell)}{n+1}.$$

In particular:

$$\lim_{n\to\infty} \Lambda^*((\mathfrak{S}_\theta, L), (\mathfrak{A}_{\theta_{2n}}, L_n)) = 0.$$

Proof. Let $N \in \mathbb{N}$ and let $S_{p,N} = \{(z_n)_{n\in\mathbb{N}} : \forall n \leq N \; z_n = 1\}$. If $\mathbb{G} = S_p^2$ then $S_{p,N}^2 = \mathbb{G}^{(N)}$ using the notation of Theorem (5.2.17).

The quotient $S_p/S_{p,N}$ is given by:

$$\{(z_n)_{0\leq n\leq N} \in \mathbb{T}^{N+1} : \forall n \in \{0, \dots, N\} \; z_{n+1}^p = z_n\}.$$

The map $z \in \mathbb{T} \longmapsto (z^{p^N}, z^{p^{N-1}}, \dots, z)$ is an isomorphism from $\mathbb{T}$ onto $S_p/S_{p,N}$.

Moreover, the dual of $S_p/S_{p,N}$ is isomorphic to the subgroup:

$$Z_N = \left\{\frac{q}{p^k} : k \in \{0, \dots, N\}\right\}$$

of $\mathbb{Z}\left[\frac{1}{p}\right]$; this subgroup is trivially isomorphic to $\mathbb{Z}$ via the map $z \in \mathbb{Z} \longmapsto \frac{z}{p^N}$. In fact, this isomorphism is also (up to changing the codomain to make it a monomor-phism) the canonical injection of the N^{th} copy of $\mathbb{Z}$ to $\mathbb{Z}\left[\frac{1}{p}\right]$, with range $Z_N \lhd \mathbb{Z}\left[\frac{1}{p}\right]$, when writing $\mathbb{Z}\left[\frac{1}{p}\right]$ as the inductive limit of $\mathbb{Z} \xrightarrow{k\mapsto pk} \mathbb{Z} \xrightarrow{k\mapsto pk} \cdots$.

By Theorem (5.2.17), it is thus sufficient, to conclude, that we identify the fixed point C^*-subalgebra of $\mathfrak{S}_\theta$ for the subgroup $S_{p,N}^2$.

Let μ be the Haar probability measure on S_p^2. As in the proof of Lemma (5.2.16), We define the conditional expectation $\mathbb{E}_N$ of $\mathfrak{S}_\theta$ by setting for all $a \in \mathfrak{S}_\theta$:

$$\mathbb{E}_N(a) = \int_{S_{p,N}^2} \alpha^g(a) \, d\mu(g).$$

Let $(z, y) \in S_p^2$, and $q_1, q_2 \in \mathbb{Z}, k_1, k_2 \in \mathbb{N}$. By Theorem-Definition (5.2.1) and by definition of the dual action α of S_p^2 on $\mathfrak{S}_\theta$, we compute:

$$\alpha^{z,y}\left(W_{\frac{q_1}{p^{k_1}},\frac{q_2}{p^{k_2}}}\right) = z_{k_1}^{q_1} y_{k_2}^{q_2} W_{\frac{q_1}{p^{k_1}},\frac{q_2}{p^{k_2}}}.$$

Thus, if $(z, y) \in S_{p,N}^2$ then $\alpha^{z,y}\left(W_{\frac{q_1}{p^{k_1}},\frac{q_2}{p^{k_2}}}\right) = W_{\frac{q_1}{p^{k_1}},\frac{q_2}{p^{k_2}}}$ for all $\frac{q_1}{p^{k_1}}, \frac{q_2}{p^{k_2}} \in Z_N$.

On the other hand, $\mathbb{E}_N\left(W_{\frac{q_1}{p^{k_1}},\frac{q_2}{p^{k_2}}}\right) = 0$ for all $\frac{q_1}{p^{k_1}}, \frac{q_2}{p^{k_2}} \notin Z_N$.

Thus the range of $\mathbb{E}_N$, which is the fixed point C^*-subalgebra for $S^2_{p,N}$, is the C^*-subalgebra of $\mathfrak{S}_\theta$ generated by:

$$\left\{W_{\frac{1}{p^{k_1}},\frac{1}{p^{k_2}}}: \frac{q_1}{p^{k_1}}, \frac{q_2}{p^{k_2}} \in Z_N\right\}.$$

Now, by definition:

$$W_{\frac{1}{p^{k_1}},\frac{1}{p^{k_2}}} = \overline{\Psi_\theta\left(\left(\frac{q_1}{p^{k_1}},0\right),\left(0,\frac{q_2}{p^{k_2}}\right)\right)}\left(W_{\frac{1}{p^N},0}\right)^{q_1k_1p}\left(W_{0,\frac{1}{p^N}}\right)^{q_2k_2p}.$$

Thus, the range of $\mathbb{E}_N$ is the C^*-subalgebra of $\mathfrak{S}_\theta$ generated by $W_{\frac{1}{p^N},0}, W_{0,\frac{1}{p^N}}$.

By Theorem (5.2.22), the range of $\mathbb{E}_N$ is the image of $\mathfrak{A}_{\theta_{2n}}$ in $\mathfrak{S}_\theta$ via the canonical injection ρ_N defined in Theorem (5.2.22). Now, note that ρ_N is an isometry from L_N to the Lip-norm $L_{S^2_{p,N}}$ defined by Theorem (5.2.5), the restriction of the dual action α to $S^2_{p,N}$, acting on $\mathbb{E}_{N(\mathfrak{S}_\theta)}$ (as in Lemma (5.2.16)). Thus:

$$\Lambda((\mathbb{E}_{N(\mathfrak{S}_\theta)}, L_{S^2_{p,N}}), (\mathfrak{A}_{\theta_{2n}}, L_N)) = 0.$$

By Theorem (5.2.17), we thus conclude:

$$\begin{aligned}\Lambda((\mathfrak{S}_\theta, L), (\mathfrak{A}_{\theta_{2N}}, L_N)) &= \Lambda((\mathfrak{S}_\theta, L), (\mathbb{E}_N(\mathfrak{S}_\theta), L_{S^2_{p,N}})) \\ &\le diam\left(\mathfrak{S}_{p,N^2}, \ell_\infty\right) \\ &\le \frac{diam(\mathbb{T}^2,\ell)}{N+1} \text{ by Corollary (5.2.15).}\end{aligned}$$

This completes our proof.

We note that since convergence for the quantum propinquity implies convergence in the sense of the Gromov-Hausdorff distance for classical metric spaces, we have proven that $(\mathbb{T}^2, \ell^n_\infty)_{n\in\mathbb{N}}$ converges to S^2_p in the Gromov-Hausdorff distance, using the notations of Theorem (5.2.23).

We begin with the immediate observation that, since quantum tori are limits of fuzzy tori for the quantum propinquity, so are the noncommutative solenoids.

Corollary (5.2.24)[200]: Let $p \in \mathbb{N}\backslash\{0\}$ and $\theta \in S_p$. Fix a continuous length function ℓ on $\mathbb{T}^2$ and let ℓ_∞ be the induced length function on S^2_p given in Definition (5.2.13).

There exists a sequence $(\omega_n)_{n\in\mathbb{N}} \in \mathbb{T}^N$ and a sequence $(k_n)_{n\in\mathbb{N}}$ in $\mathbb{N}^N$ with $\lim_{n\to\infty} k_n = \infty$, $\lim_{n\to\infty} |\theta_{2n} - \omega_n| = 0$, and $\omega_n^{k_n} = 1$ for all $n \in \mathbb{N}$, such that:

$$\lim_{n\to\infty} \Lambda\left((C^*(\mathbb{Z}^2_{k_n}, \sigma_n), L_n), (\mathfrak{S}_\theta, L)\right) = 0$$

where $\mathbb{Z}_k = Z/k\mathbb{Z}$, Ln and L are the Lip-norms given by Theorem (5.2.5) for the dual actions, respectively, of the groups of k_n roots of unit and the solenoid group S_p, and:

$$\sigma_n: ((z_1,z_2),(y_1,y_2)) \in \mathbb{Z}^2_{k_n} \times \mathbb{Z}^2_{k_n} \longmapsto \exp(2i\pi\omega_n(z_1y_2 - z_2y_1)).$$

Proof. This follows from a standard diagonal argument using Theorem (5.2.23) and [207].

Quantum tori form a continuous family for the quantum propinquity, and together with Theorem (5.2.23), we thus can prove:

Theorem (5.2.25)[200]: Let ℓ be a continuous length function on $\mathbb{T}^2$. For each $\theta \in S_p$, let L_θ be the Lip-norm defined by Theorem (5.2.5) for the dual action of S^2_p on $\mathfrak{S}_\theta$ and the continuous length function ℓ_∞ of Definition (5.2.13).

The function $\theta \in S_p \longmapsto (\mathfrak{S}_\theta, L_\theta)$ is continuous from S_p to the class of Leibniz quantum compact metric spaces endowed with the quantum Gromov-Hausdorff propinquity.

Proof. Fix some continuous length function m on $\mathbb{T}$. This length function need not be related to ℓ. Its purpose is simply to provide us with a metric λ_m for the topology of S_p.

Let $\varepsilon > 0$. Let $N \in \mathbb{N}$ be chosen so that $\frac{diam(\mathbb{T}^2,\ell)}{N+1} \leq \frac{\varepsilon}{3}$. By Theorem (5.2.23), for all $\theta \in S_p$, we have:

$$\Lambda^*((\mathfrak{S}_\theta, L), (\mathfrak{A}_{\theta_{2N}}, L)) \leq \frac{\varepsilon}{3}.$$

By [207], there exists $\delta > 0$ such that, for all $\omega, \eta \in [0, 1)$ with $m(\omega\eta^{-1}) \leq \delta$, we have $\Lambda^*((\mathfrak{S}_\omega, L), (\mathfrak{A}_\eta, L)) \leq \frac{\varepsilon}{3}$.

Let $\varsigma = \min\left\{\delta, \frac{diam(\mathbb{T},m)}{N+1}\right\}$. Let $\theta, \xi \in S_p$ with $\lambda_m(\theta, \xi) \leq \varsigma$. By definition of λ_m, we have $m(\theta_{2n}\xi_{2n}^{-1}) \leq \delta$. Consequently:

$$\begin{aligned}\Lambda((\mathfrak{S}_\theta, L_\theta), (\mathfrak{S}_\theta, L_\xi)) &\\ &\leq \Lambda((\mathfrak{S}_\theta, L_\theta), (\mathfrak{A}_{\theta_{2N}}, L)) + \Lambda((\mathfrak{A}_{\theta_{2N}}, L), (\mathfrak{A}_{\xi 2N}, L)) \\ &\quad + \Lambda((\mathfrak{A}_{\xi 2N}, L), (\mathfrak{S}_\xi, L_\xi)) \leq \varepsilon,\end{aligned}$$

which concludes our theorem.

Section (5.3): Noncommutative Solenoidal Spaces from Self-Coverings

Given a noncommutative self-covering consisting of a C^*-algebra with a unital injective endomorphism $(\mathcal{A}, \alpha)$, we study the possibility of extending a spectral triple on $\mathcal{A}$to a spectral triple on the inductive limit C^*-algebra, where, as in [238], the inductive family associated with the endomorphism α is

$$\mathcal{A}_0 \overset{\alpha}{\rightarrow} \mathcal{A}_1 \overset{\alpha}{\rightarrow} \mathcal{A}_2 \overset{\alpha}{\rightarrow} \mathcal{A}_3 \ldots, \tag{24}$$

all the $\mathcal{A}_n$ being isomorphic to $\mathcal{A}$. The algebra $\mathcal{A}_n$ may be considered as the n-th covering of the algebra $\mathcal{A}_0$ w.r.t. the endomorphism α. As a remarkable byproduct, all the spectral triples we construct on the inductive limit C^*-algebra are semifinite spectral triples.

Let us recall that the first notion of type II noncommutative geometry appeared in [237], where semifinite Fredholm modules were introduced, a notion then generalized in [230], see also [231], with that of semifinite unbounded Fredholm module. The latter is essentially the same definition as that of von Neumann spectral triples of [225], where some previous constructions [224], [234], [248], [243] were reinterpreted as examples of such concept. In the same period, [245] considered semifinite spectral triples for graph algebras and posed the problem of exhibiting more examples of the kind, which was done in [246] using k-graph algebras and in [58] inspired by quantum gravity. Further examples have been considered in [229], [250]. In the cases we analyze, it is possible to construct natural spectral triples on the C^*-algebras $\mathcal{A}_n$ of the inductive family, which converge, in a suitable sense, to a triple on the inductive limit, and the latter triple is indeed semifinite.

The leading idea is that of producing geometries on each of the noncom-mutative coverings $\mathcal{A}_n$ which are locally isomorphic to the geometry on the original noncommutative space $\mathcal{A}$. This means in particular that the covering projections should be local isometries or, in algebraic terms, that the noncommutative metrics given by the Lip-norms associated with the Dirac operators via $L_n(a) = \|[D_n, a]\|$ (cf. [235], [195]) should be compatible with the inductive maps, i.e. $L_{n+1}(\alpha(a)) = L_n(a), a \in \mathcal{A}_n$. In one case, this property will be weakened to the existence of a finite limit for the sequences $L_{n+p}(\alpha^p(a)), a \in \mathcal{A}_n$.

The above request produces two related effects. On the one hand, the noncom-mutative coverings are metrically larger and larger, so that their radii diverge to infinity, and the inductive limit is topologically compact (the C^*-algebra has a unit) but not totally bounded (the metric on the state space does not induce the weak*-topology). On the other hand, the spectrum of the Dirac operator becomes more and more dense in the real line, so that the resolvent of the limiting Dirac operator cannot be compact, being indeed τ-compact w.r.t. a suitable trace, and thus producing a semifinite spectral triple on the inductive limit.

Pursuing this idea means also that we see the elements of the inductive family in a more geometric way, namely as distinct (though isomorphic) algebras of "functions" on noncommutative coverings, and the inductive maps as embeddings of a sub-algebra into an algebra of "less periodic" functions. In this sense the inductive limit is a noncommutative version of the solenoidal spaces in [244], see also the noncommutative solenoids in [68], [200].

We devoted to the study of noncommutative regular (self-) coverings with finite abelian group, namely in particular of a C^*-algebra $\mathcal{A}_1$ acted upon by a finite abelian group Γ whose fixed point algebra $\mathcal{A}_0$ is isomorphic to $\mathcal{A}_1$. A further property, which we call regularity, requires that the eigenspaces of $\mathcal{A}_1$ w.r.t. the action of Γ contain invertible elements. This turns out to imply that $\mathcal{A}_1$ can be seen as a subalgebra of a matrix algebra on $\mathcal{A}_0$, and in this way the algebras $\mathcal{A}_n$ forming the inductive family described above are naturally embedded into $\mathcal{A}_0 \otimes M_r(\mathbb{C})^{\otimes n}, r = |\Gamma|$. The resulting embedding of the inductive limit into $\mathcal{A}_0 \otimes UHF(r^\infty)$ provides the semifinite environment for the spectral triple on the inductive limit. The regularity assumption also implies that the "can" map is an isomorphism, namely the regularity property of the covering according to [226].

We study coverings of the torus and generalizations to noncommutative tori and to crossed products with $\mathbb{Z}^n$, based on a non-degenerate integer-valued matrix B. This implies in particular that the regularity property holds and the invertible elements in the eigenspaces of the action Γ can be chosen in terms of the exact sequence

$$0 \to \mathbb{Z}^p \to (B^T)^{-1}\mathbb{Z}^p \to (B^T)^{-1}\mathbb{Z}^p/\mathbb{Z}^p \to 0, \qquad (25)$$

cf. equation (25). The choice of a particular plays no role in the definition of the Dirac operator and of the Lip-norm on the n-th covering quantum spaces, however such enters the formulas for the identification of the covering algebras as algebras of matrices on the base algebra. It turns out that choosing "minimal" and requiring the matrix B to be purely expanding guarantees a suitable convergence of the Dirac operators on the n-th covering to a Dirac operator on the inductive limit.

As mentioned above, the first example of regular covering is described, and is indeed a classical covering, namely the self-covering of the p-torus $\mathbb{R}^p/\mathbb{Z}^p$ given by a non-degenerate matrix $B \in M_p(\mathbb{Z})$. We assume $detB \neq \pm 1$ to avoid the automorphism case. The covering map is the projection $\mathbb{R}^p/B\mathbb{Z}^p \to \mathbb{R}^p/\mathbb{Z}^p$, the group of deck transformations being $\Gamma = \mathbb{Z}^p/B\mathbb{Z}^p$. The corresponding embedding for the algebras is $C(\mathbb{R}^p/\mathbb{Z}^p) \hookrightarrow C(\mathbb{R}^p/B\mathbb{Z}^p)$, the group $\mathbb{Z}^p/B\mathbb{Z}^p$ acts on the larger algebra having the smaller as fixed point algebra. As mentioned before, the algebras $\mathcal{A}_n$, consisting of continuous functions on the n-th covering, can be represented as matrices on $\mathcal{A}_0$, namely embed into $\mathcal{A}_0 \otimes M_r(C)^{\otimes n}$. Endowing the n-th covering with the pullback of the metric on the base space makes the covering projections locally isometric. The corresponding Dirac operator on $\mathcal{A}_n$ is formally identical to that on $\mathcal{A}_0$; when $\mathcal{A}_n$ is described as a sub-algebra of $\mathcal{A}_0 \otimes M_r(C)^{\otimes n}$, the Dirac operator D_n is affiliated to $B(\mathcal{H}) \otimes M_r(C)^{\otimes n}$, $\mathcal{H}$ being the Hilbert space of the spectral triple on $\mathcal{A}_0$.

When $n \to \infty, A_\infty \subset \mathcal{A}_0 \otimes UHF(r^\infty) \subset \mathcal{B}(\mathcal{H}) \otimes \mathcal{R}$, where $\mathcal{R}$ is the unique injective type II_1 factor obtained as the weak closure of the UHF algebra in the GNS representation of the unital trace. Moreover, with a suitable choice in (25) and under the assumption that the matrix B is purely expanding, the sequence D_n converges to an operator D_∞ affiliated with $\mathcal{B}(\mathcal{H}) \otimes \mathcal{R}$. The triple $(\mathcal{A}_\infty, \mathcal{B}(\mathcal{H}) \otimes \mathcal{R}, D_\infty)$ turns out to be a semifinite spectral triple and the metric dimension, given by the abscissa of convergence d_∞ of the zeta function $\tau((1 + D_\infty^2)^{s/2})$, and the noncommutative volume, given by the residue in d_∞ of the zeta function, coincide with the corresponding quantities for the base torus. We contain the extension of the results for the torus to the case of rational noncommutative 2-tori $A_\vartheta, \vartheta = p/q$. In this case we get a self-covering if $detB \equiv 1 \; mod \; q$. With this proviso, for B purely expanding, we get a semifinite spectral triple on the inductive limit, and the metric dimension and the noncommutative volume are the same as those of the base torus.

The third example of noncommutative regular self-coverings with finite abelian group is treated, where we consider the covering $\mathcal{Z} \rtimes_\rho \mathbb{Z}^p \hookrightarrow \mathcal{Z} \rtimes_\rho (B^T)^{-1}\mathbb{Z}^p, B \in M_p(\mathbb{Z})$ with $detB \notin \{0, \pm 1\}$ as above. The action of $\mathbb{Z}^p/B\mathbb{Z}^p$ by translation on $C(\mathbb{R}^p/B\mathbb{Z}^p)$ induces an action on $\mathcal{Z} \rtimes_\rho (B^T)^{-1}\mathbb{Z}^p$ whose fixed point algebra is $\mathcal{Z} \rtimes_\rho \mathbb{Z}^p$. We get a self-covering if the actions $g \in \mathbb{Z}^p \to \rho_g \in Aut(\mathcal{Z})$ and $g \in \mathbb{Z}^p \to \rho_{(B^T)^{-1}g} \in Aut(\mathcal{Z})$ are conjugate by an automorphism β of $\mathcal{Z}$. Under this assumption the C^*-algebras $\mathcal{A}_n$ are all isomorphic and may be described as $\mathcal{Z} \rtimes (B^T)^{-n}\mathbb{Z}^p$, with the action defined in terms of the action of $\mathbb{Z}^p$ and the automorphism β.

The study of the extension of a spectral triple on a C^*-algebra to a triple on crossed products was initiated in [171] and then pursued in [241], [169]. Such extension requires a choice of a (proper, translation bounded, matrix-valued) length function on the group. We assume $\mathcal{Z}$ is endowed with a spectral triple, and use the mentioned results to extend such triple to the algebras $\mathcal{A}_n$, with a suitable choice of a length function on $(B^T)^{-n}\mathbb{Z}^p$. However a further assumption is required in order to prove that the covering projection is locally isometric.

The same results as in the previous hold for the spectral triple on the inductive limit C^*-algebra.

The subsequent contains the analysis of an example of covering which is not given by an action of a finite abelian group. It consists of a UHF algebra with the shift endomorphism, the spectral triple being the one described in [233]. In this case the C^*-algebras $\mathcal{A}_n$ are naturally given by tensoring $\mathcal{A}_0$ with $\mathcal{M}(\mathbb{C})^{\otimes n}$, and we choose the Dirac operators to have the same form as that on $\mathcal{A}_0$. However in this way local isometricity is not exactly satisfied, while, as mentioned above, the sequence $L_{n+p}(\alpha^p(a))$ converges when $n \to \infty$, for a in a suitable dense sub-algebra of $\mathcal{A}_n$. We obtain a semifinite spectral triple on the inductive limit C^*-algebra with the same metric dimension and noncommutative volume of the base space.

We deal with the metric properties induced by the limit spectral triple on the inductive limit C^*-algebra. We show that, in all the examples considered, the radii of the algebras $\mathcal{A}_n$ diverge, giving rise to a non totally bounded noncommutative space. As explained before, this is a consequence of the fact that the covering projections are locally isometric.

We compare our results with those in [200], where the direct limits of noncommutative tori, called noncommutative solenoids, are seen as twisted group C^*-algebras acted upon by the solenoid group. It is shown there that the inductive sequence of noncommutative tori converges in the quantum Gromov-Hausdorff metric (and in the Gromov-Hausdorff propinquity) to the noncommutative solenoid, when the latter is endowed with a Lip-norm given by a suitable choice of the length function on the solenoid

group (cf. [86]). It turns out that our seminorm on the inductive limit is also induced, *a la* Rieffel, by a length function on the solenoid group, but our function is infinite on some elements, thus giving rise to a non totally bounded space. It would be interesting to know if also our sequence of quantum compact metric spaces tends to the direct limit w.r.t. some kind of (pointed) quantum Gromov-Hausdorff convergence.
We mention in conclusion that a motivation for the study of spectral triples for direct limits is the attempt to extend the constructions in [171], [241], [169] to the case of a crossed product by a single endomorphism, that is, to the case of self-covering. The corresponding results are contained in [222]. An example of self-covering with ramification and the study of the corresponding crossed product is contained in [223].

We describe a spectral decomposition of an algebra in terms of an action of a finite abelian group. For more details and a general theory the interested reader is referred to [193]. For $\mathcal{B}$ be a C^*-algebra and Γ be a finite abelian group which acts on $\mathcal{B}$ (we denote the action by γ). Let

$$\mathcal{B}_k := \{ b \in \mathcal{B} \ s.t. \gamma_g(b) = \langle k, g \rangle b \ \ \forall g \in \Gamma \}, \qquad k \in \hat{\Gamma}.$$

Proposition (5.3.1)[221]: With the above notation,
(i) $\mathcal{B}_h \mathcal{B}_k \subset \mathcal{B}_{hk}$; in particular each $\mathcal{B}_k$ is an $\mathcal{A}$-bimodule, where $\mathcal{A}$ is the fixed point subalgebra,
(ii) if $b_k \in \mathcal{B}_k$ is invertible, then $b_k^{-1}, b_k^* \in \mathcal{B}_{k^{-1}}$,
(iii) each $b \in \mathcal{B}$ may be written as $\sum_{k\in\hat{\Gamma}} b_k$ with $b_k \in \mathcal{B}_k$.
Before proving this proposition we recall that by the Schur orthogonality relations, [247], given Γ a finite abelian group, $\hat{\Gamma}$ its dual,

$$\sum_{k\in\hat{\Gamma}} \langle k, g \rangle = \delta_{g,e} \cdot \ |\Gamma| \quad \forall g \in \Gamma. \tag{26}$$

Proof. The first two properties follow by definition. Let us set

$$b_k \equiv E_k(b) \stackrel{\text{def}}{=} \frac{1}{|\Gamma|} \sum_{g\in\Gamma} \langle k^{-1}, g \rangle \, \gamma_g(b).$$

Then, by (26),

$$\sum_{k\in\hat{\Gamma}} b_k = \frac{1}{|\Gamma|} \sum_k \sum_g \langle k^{-1}, g \rangle \, \gamma_g(b) = \frac{1}{|\Gamma|} |\Gamma| \delta_{g,e} \gamma_g(b) = b.$$

Finally, b_k belongs to $\mathcal{B}_k$ since, for any $g \in \Gamma$,

$$\gamma_g(b_k) = \frac{1}{|\Gamma|} \sum_{h\in\Gamma} \langle k^{-1}, h \rangle \, \gamma_g \gamma_h(b) = \langle k^{-1}, g^{-1} \rangle \frac{1}{|\Gamma|} \sum_{h\in\Gamma} \langle k^{-1}, h \rangle \, \gamma_h(b) = \langle k, g \rangle b_k.$$

Definition (5.3.2)[221]: A finite (noncommutative) covering with abelian group is an inclusion of (unital) C^*-algebras $\mathcal{A} \subset \mathcal{B}$ together with an action of a finite abelian group Γ on $\mathcal{B}$ such that $\mathcal{A} = \mathcal{B}^\Gamma$. We will say that $\mathcal{B}$ is a covering of $\mathcal{A}$ with deck transformations given by the group Γ.
Let us denote by $M_{\hat{\Gamma}}(\mathcal{B})$ the algebra of matrices, whose entries belong to $\mathcal{B}$ and are indexed by elements of $\hat{\Gamma}$. Then, to any $b \in \mathcal{B}$, we can associate the matrix $\widetilde{M}(b) \in M_{\hat{\Gamma}}(\mathcal{B})$ with the following entries

$$\widetilde{M}(b)_{hk} = b_{h-k}, \qquad h, k \in \hat{\Gamma}.$$

By the definition of b_k the following formula easily follows

$$\widetilde{M}(b)\widetilde{M}(b') \ = \widetilde{M}(bb'). \tag{27}$$

The following definition is motivated by Theorem (5.3.8) below.

Definition (5.3.3)[221]: We say that the finite covering $\mathcal{A} \subset \mathcal{B}$ w.r.t. Γ is regular if each $\mathcal{B}_k$ has an element which is unitary in $\mathcal{B}$, namely we may choose a map $\sigma: \hat{\Gamma} \to \mathcal{B}$ such that $\sigma(k) \in U(\mathcal{B}) \cap \mathcal{B}_k$, with $\sigma(e) = I$.

Theorem (5.3.4)[221]: Under the regularity hypothesis, the algebra $\mathcal{B}$ is isomorphic to a subalgebra of matrices with coefficients in $\mathcal{A}$, i.e. we have an embedding

$$\mathcal{B} \hookrightarrow \mathcal{A} \otimes M_{\hat{\Gamma}}(\mathbb{C}). \quad (28)$$

Proof. It is easy to show that $M(b^*)_{jk} = (M(b)_{kj})^*, \forall b \in \mathcal{B}, j, k \in \hat{\Gamma}$. That the product is preserved, namely

$$M(bb')_{hk} = \sum_j M(b)_{hj} M(b')_{jk}$$

follows easily from (28).

We are mainly interested in self-coverings, namely when there exists an isomorphism $\phi: \mathcal{B} \to \mathcal{A}$ or, equivalently, $\mathcal{A}$ is the image of $\mathcal{B}$ under a unital endomorphism $\alpha = j \circ \phi$, where j is the embedding of $\mathcal{A}$ in $\mathcal{B}$.

Theorem (5.3.5)[221]: Given a (noncommutative) regular self-covering with abelian group Γ, we may construct an inductive family $\mathcal{A}_i$ associated with the endomorphism α as in [238]. Then, setting $r = |\hat{\Gamma}| = |\Gamma|$, we have the following embedding:

$$\lim_{\rightarrow} \mathcal{A}_i \hookrightarrow \mathcal{A} \otimes UHF(r^\infty).$$

Proof. By applying Theorem (5.3.4) j times, we get an embedding of $\mathcal{A}_j$ into $\mathcal{A} \otimes M_r^{\otimes j}$. The result immediately follows.

The following example shows that the regularity property in Definition (5.3.3) is not always satisfied.

Example (5.3.6)[221]: Let $\mathcal{B} = M_3(\mathbb{C}), \Gamma = \mathbb{Z}_2 = \{0, 1\}$. We have the following action γ on $\mathcal{B}$: $\gamma_0 = id, \gamma_1 = ad(J)$, where

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Therefore

$$\mathcal{B}_0 = \mathcal{A} = \mathcal{B}^\Gamma = \left\{ x \in \mathcal{B}: x = \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & d & e \end{pmatrix} \right\}, \qquad \mathcal{B}_1 = \left\{ x \in \mathcal{B}: x = \begin{pmatrix} 0 & a & b \\ c & 0 & 0 \\ d & 0 & 0 \end{pmatrix} \right\}.$$

Hence $\mathcal{B}_1$ has no invertible elements.

Proposition (5.3.7)[221]: Consider a (noncommutative) regular self-covering $\mathcal{A} \subset \mathcal{B}$ with abelian group Γ.

(i) A representation π of $\mathcal{A}$ on a Hilbert space H produces a representation $\tilde{\pi}$ of $\mathcal{B}$ on $H \otimes \mathbb{C}^r, r = |\hat{\Gamma}|$, given by $\tilde{\pi}(b) := [\pi(M(b)_{hk})]_{h,k \in \hat{\Gamma}} \in M_{\hat{\Gamma}}(\mathcal{B}(H)) = \mathcal{B}(H \otimes \mathbb{C}^r), \forall b \in \mathcal{B}$.

(ii) If the representation of $\mathcal{A}$ is induced by a state φ via the GNS mechanism, the corresponding representation of $\mathcal{B}$ on $H \otimes \mathbb{C}^r$ is a GNS representation induced by the state $\tilde{\varphi}$, where $\tilde{\varphi}(b) = \varphi \circ E_\Gamma$, and E_Γ is the conditional expectation from $\mathcal{B}$ to $\mathcal{A}$. Moreover, the map

$$\begin{matrix} \mathcal{B} \to \mathcal{A} \otimes \mathbb{C}^r \\ b \mapsto (a_j)_{j \in \hat{\Gamma}} \end{matrix}, a_j = \sigma(j)^{-1} b_j \quad (29)$$

extends to an isomorphism of the Hilbert spaces $L^2(\mathcal{B}, \tilde{\varphi})$ and $L^2(\mathcal{A}, \varphi) \otimes \mathbb{C}^r$.

Proof. (i) It is a simple computation.

(ii) Denoting by $\xi_\varphi$the GNS vector in H, we set $\tilde{\xi}_\varphi$ to be the vector ξ_φ in He and 0 in the other summands. It is cyclic, because

$$\tilde{\pi}(b)\tilde{\xi}_{\varphi} = \bigoplus_{k\in\hat{\Gamma}} \sigma(k)^{-1} b_k \xi_{\varphi}.$$

Since ξ_{φ} is cyclic for $\mathcal{A}, \{\sigma(k)^{-1} b_k \xi_{\varphi}: b_k \in \mathcal{B}_k\}$ is dense in H. It induces the state $\tilde{\varphi}$, since

$$\left(\tilde{\xi}_{\varphi}, \tilde{\pi}(b)\tilde{\xi}_{\varphi}\right) = \left(\xi_{\varphi}, b_e \xi_{\varphi}\right) = \varphi\left(\frac{1}{|\Gamma|}\sum_{g\in\Gamma} \gamma_g(b)\right) = \varphi \circ E_{\Gamma}(b).$$

The isomorphism in (29) follows by the GNS theorem.

We discuss the relation between our definition of (noncommutative) finite regular covering and the classical notion of regular covering. As a byproduct of an analysis on actions of compact quantum groups, it was proved in [226] that a finite covering is regular iff the "can" map is an isomorphism. More precisely, if X and Y are compact Hausdorff spaces and $\pi: X \to Y$ is a covering map with finite group of deck transformations Γ, X is a regular covering of Y if and only if the canonical map

$$can: C(X) \otimes_{C(Y)} C(X) \to C(X) \otimes C(\Gamma)$$
$$f_1 \otimes f_2 \to (f_1 \otimes 1)\delta(f_2),$$

is an isomorphism of C^*-algebras, where $\delta f = \gamma_{g^{-1}}(f) \otimes \chi_g, \gamma_g: \Gamma \to Aut(C(X))$ and χ_g denote the action induced by Γ and the characteristic function on elements of Γ, respectively. The map can for classical coverings makes perfect sense in our case too

$$can: \mathcal{B} \otimes_{\mathcal{A}} \mathcal{B} \to \mathcal{B} \otimes C(\Gamma),$$

where $\text{can}(x \otimes y) = (x \otimes 1)\delta(y)$ and $\delta(y) = \sum_{g\in\Gamma} \gamma_{g^{-1}}(y) \otimes \chi_g$. In our framework, however, the canonical map is no longer a morphism of C^*-algebras, it is a morphism of $(\mathcal{B} - \mathcal{A})$-bimodules. In fact, this map clearly commutes with the left action of $\mathcal{B}$. Moreover, the right action $\mathcal{A}$ commutes with can since $\delta|_{\mathcal{A}} = id$. The following theorem shows that, under the regularity property of Definition (5.3.3), the can map is an isomorphism, that is, the regularity property according to [226].

Theorem (5.3.8)[221]: Under the above hypotheses, the map can : $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{B} \to \mathcal{B} \otimes C(\Gamma)$ is an isomorphism of $(\mathcal{B} - \mathcal{A})$-bimodules.

Proof. The group $\Gamma \times \Gamma$ clearly acts on $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{B}$, the eigenspaces being $(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{B})_{j,k} = \{\sigma(j)a \otimes \sigma(k): a \in \mathcal{A}\}, (j,k) \in \hat{\Gamma} \times \hat{\Gamma}$. Therefore the elements of $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{B}$ can be written as

$$z = \sum_{j,k\in\hat{\Gamma}} \sigma(j)a_{j,k} \otimes \sigma(k) \qquad a_{j,k} \in \mathcal{A}.$$

Suppose that $\text{can}(z) = 0$. We want to prove that $z = 0$. Using the fact that $\mathcal{B}$ is the direct sum of its eigenspaces we get

$$can(z) = \sum_{g\in\Gamma}\sum_{j,k\in\hat{\Gamma}} \langle g^{-1}, k\rangle \sigma(j)a_{j,k}\sigma(k) \otimes \chi_g = 0$$
$$\Rightarrow \sum_{j,k\in\hat{\Gamma}} \langle g^{-1}, k\rangle \sigma(j)a_{j,k}\sigma(k) = 0 \quad \forall g \in \Gamma,$$

where $a_{j,k} \in \mathcal{A}$. Now we show that any $a_{j,k}$ is zero. In fact, multiplying by $\langle g, \ell\rangle$ and summing over $g \in \Gamma$, we get

$$0 = \sum_{g\in\Gamma}\langle g, \ell\rangle \sum_{j,k\in\hat{\Gamma}} \langle g^{-1}, k\rangle\sigma(j)a_{j,k}\sigma(k) = \sum_{j,k\in\hat{\Gamma}} \langle g, \ell k^{-1}\rangle \sigma(j)a_{j,k}\sigma(k)$$
$$= |\Gamma| \sum_{j\in\hat{\Gamma}} \sigma(j)a_{j,k}\sigma(k),$$

which implies that $a_{j,k} = 0$ for all $j, k \in \hat{\Gamma}$, so that $z = 0$.
Consider $\sum_{g\in\Gamma} b(g) \otimes \chi_g$, we want to show that it can be obtained as can(z) for some $z \in \mathcal{B} \otimes_{\mathcal{A}} \mathcal{B}$. By the above computations, it suffices to solve the following equation, for any $\ell \in \hat{\Gamma}$,

$$\sum_{g\in\Gamma} \langle g, \ell\rangle b(g) = \sum_{g\in\Gamma} \langle g, \ell\rangle \sum_{j,k\in\hat{\Gamma}} \langle g^{-1}, k\rangle \sigma(j) a_{j,k} \sigma(k)\,,$$

which, using (26), may be rewritten as

$$\sum_{g\in\Gamma} \langle g, \ell\rangle b(g)\sigma(k)^{-1} = |\Gamma| \sum_{j\in\hat{\Gamma}} \sigma(j) a_{j,k}\,.$$

Since each $b(g)$ is given, the cofficients $a_{j,k}$ can be uniquely determined using again the decomposition of $\mathcal{B}$ in its eigenspaces.

We consider the p-torus $\mathbb{T}^p = \mathbb{R}^p/\mathbb{Z}^p$ endowed with the usual metric, inherited from $\mathbb{R}^p$. On this Riemannian manifold we have the Levi-Civita connection $\nabla^{LC} = d$ and we can define the Dirac operator acting on the Hilbert space $\mathbb{C}^{2^{[p/2]}} \otimes L^2(\mathbb{T}^p, dm)$

$$D = -i \sum_{a=1}^{p} \varepsilon^a \otimes \partial^a,$$

where $\varepsilon^a = (\varepsilon^a)^* \in M_{2^{[p/2]}}(\mathbb{C})$ furnish a representation of the Clifford algebra for the p-torus (see [190] for more details on Dirac operators). Therefore, we have the following spectral triple

$$(C^1(\mathbb{T}^p), \mathbb{C}^{2^{[p/2]}} \otimes L^2(\mathbb{T}^p, dm), D = -i \sum_{a=1}^{p} \varepsilon^a \otimes \partial^a).$$

Consider an integer-valued matrix $B \in M_p(\mathbb{Z})$ with $|det(B)| =: r > 1$. This defines a covering of $\mathbb{T}^p$ as follows. Let us set $\mathbb{T}_1 = \mathbb{R}^p/B\mathbb{Z}^p$ seen as a covering space of $\mathbb{T}_0 := \mathbb{T}^p$. Clearly $\mathbb{Z}^p$ acts on $\mathbb{T}_1$ by translations, the subgroup $B\mathbb{Z}^p$ acting trivially by definition, namely we have an action of $\mathbb{Z}_B := \mathbb{Z}/B\mathbb{Z}^p$ on $\mathbb{T}_1$, which is simply the group of deck trasformations for the covering. We denote this action by γ. We are now in the situation described, with $\mathcal{A} = C(\mathbb{T}_0)$ the fixed point algebra of $\mathcal{B} = C(\mathbb{T}_1)$ under the action of $\mathbb{Z}_B$. These algebras can be endowed with the following states, respectively

$$\tau_0(f) = \int_{\mathbb{T}_0} f dm\,, \qquad f \in \mathcal{A},$$

$$\tau_1(f) = \frac{1}{|det(B)|} \int_{\mathbb{T}_1} f dm\,, f \in \mathcal{B},$$

where dm is Haar measure.

Proposition (5.3.9)[221]: The GNS representation $\pi_1: \mathcal{B} \to B(L^2(\mathcal{B}, \tau_1)) = B(L^2(\mathbb{T}_1, dm))$ is unitarily equivalent to the representation $\tilde{\pi}_0$ obtained by $\pi_0: \mathcal{A} \to B(L^2(\mathcal{A}, \tau_0)) = B(L^2(\mathbb{T}_0, dm))$ according to Proposition (5.3.7).

Proof. By the GNS theorem it is enough to check that $\tau_1 = \tau_0 \circ E$, where E denotes the conditional expectation from $\mathcal{B}$ to $\mathcal{A}$. This follows from the following observation on the associated measures: they are both probability measures that are traslation invariant, by the results on Haar measures the claim follows.

In order to apply the results, we need to choose unitaries in the eigenspaces $\mathcal{B}_k, k \in c\widehat{\mathbb{Z}_B}$, namely a map $\sigma: g \in c\widehat{\mathbb{Z}_B} \to \mathcal{U}(\mathcal{B}) \cap \mathcal{B}_g$.

With $\mathbb{T}_0 = \mathbb{R}^p/\mathbb{Z}^p, \mathbb{T}_1 = \mathbb{R}^p/B\mathbb{Z}^p, \mathbb{Z}_B = \mathbb{Z}^p/B\mathbb{Z}^p$ as above, set $A = (B^T)^{-1}, \langle x, y\rangle = exp\left(2\pi i \sum_{a=1}^p x^a y^a\right), x, y \in \mathbb{R}^p$.

Lemma (5.3.10)[221]: With the above notation

(i) The cardinality $|\mathbb{Z}_B|$ of $\mathbb{Z}_B$ is equal to r,

(ii) the following duality relations hold: $\widehat{\mathbb{T}}_0 = (\mathbb{R}^p/\mathbb{Z}^p)\widehat{} = \mathbb{Z}^p, \widehat{\mathbb{T}}_1 = (\mathbb{R}^p/B\mathbb{Z}^p)\widehat{} = A\mathbb{Z}^p, \widehat{\mathbb{Z}_B} = (\mathbb{Z}^p/B\mathbb{Z}^p)\widehat{} = A\mathbb{Z}^p/\mathbb{Z}^p$.

In particular, the duality $\langle z, g\rangle, g \in \mathbb{T}_1, z \in A\mathbb{Z}^p$ induces the duality $\langle k, g\rangle_o$, $g \in \mathbb{Z}_B, k \in \widehat{\mathbb{Z}_B}$, namely if $g \in \mathbb{Z}_B \subset \mathbb{T}_1, \langle z, g\rangle = \langle \dot{z}, g\rangle_o$, where $\dot{z}$ denotes the class of z in $\widehat{\mathbb{Z}_B}$. For this reason we drop the subscript o in the following.

Proof. The proofs of the claims are all elementary. We only make some comments on the first one. It is well known that each finite abelian group is the direct sum of cyclic groups and that the order of these groups can be obtained with the following procedure. Let $D = SBT$ the Smith normal form of B, where $S, T \in GL(p, \mathbb{Z})$ and $D = diag(d_1, \cdots d_p) > 0$. Therefore, we have that $\mathbb{Z}_B = \mathbb{Z}^p/B\mathbb{Z}^p \cong \mathbb{Z}^p/D\mathbb{Z}^p$. As B is invertible, so is D and all the diagonal elements are non-zero. Thus, $\mathbb{Z}_B = \mathbb{Z}_{d_1} \oplus \ldots \oplus \mathbb{Z}_{d_p}$ and $|\mathbb{Z}_B| = d_1 \cdot \ldots \cdot d_p = det(D) = \pm det(B)$.

Let us consider the short exact sequence of groups

$$0 \longrightarrow \mathbb{Z}^p \longrightarrow A\mathbb{Z}^p \longrightarrow \widehat{\mathbb{Z}_B} \longrightarrow 0. \tag{30}$$

Such central extension $A\mathbb{Z}^p$ of $\widehat{\mathbb{Z}_B}$ via $\mathbb{Z}^p$ can be described either with a section $s: \widehat{\mathbb{Z}_B} \to A\mathbb{Z}^p$ or via a $\mathbb{Z}^p$-valued 2-cocycle $\omega(k, k') = s(k) + s(k') - s(k + k')$, see e.g. [228]. We choose the unique such that, for any $k \in \widehat{\mathbb{Z}_B}, s(k) \in [0, 1)^p$.

Remark (5.3.11)[221]: The mentioned choice of the section s will play a role only later. For the moment, we only note that it implies $s(0) = 0$, hence $\omega(k, 0) = 0 = \omega(0, k)$.

The covering we are studying is indeed regular according to Definition (5.3.3), since we may construct the map σ as follows:

$$\sigma(k)(t) := \overline{\langle s(k), t\rangle}, \qquad k \in \widehat{\mathbb{Z}_B}, t \in \mathbb{T}_1. \tag{31}$$

Given the integer-valued matrix $B \in M_p(\mathbb{Z})$ as above, if $\mathbb{T}^p$ is identified with $\mathbb{R}^p/\mathbb{Z}^p$, then there is an associated self-covering $\pi: t \in \mathbb{T}^p \longmapsto Bt \in \mathbb{T}^p$. We denote by α the induced endomorphism of $C(\mathbb{T}^p)$, i.e. $\alpha(f)(t) = f(Bt)$. Then we consider the inductive limit $\mathcal{A}_\infty = \underset{\longrightarrow}{\lim} \mathcal{A}_n$ described in (24), where $\mathcal{A}_n = \mathcal{A}$ for any n.

In the next pages it will be convenient to consider the following isomorphic inductive family: $\mathcal{A}_n$ consists of continuous $B^n\mathbb{Z}^p$-periodic functions on $\mathbb{R}^p$, and the embedding is the inclusion. In this way $\mathcal{A}_\infty$ may be identified with a generalized solenoid C^*-algebra (cf. [244], [68]).

Since $\mathbb{T}_n = \mathbb{R}^p/B^n\mathbb{Z}^p$ is a covering space of $\mathbb{T}_0 := \mathbb{T}^p$, the formula of the Dirac operator on $\mathbb{T}_n$ doesn't change. Therefore, we will consider the following spectral triple

$$(C^1(\mathbb{T}_n), \mathbb{C}^{2^{[p/2]}} \otimes L^2(\mathbb{T}_n, \frac{1}{r^n} dm), D = -i \sum_{a=1}^{p} \varepsilon^a \otimes \partial^a).$$

We describe the spectral triple on $\mathbb{T}_n$ in terms of the spectral triple on $\mathbb{T}_0$. Consider the short exact sequences of groups

$$0 \longrightarrow B^n\mathbb{Z}^p \longrightarrow B^{n-1}\mathbb{Z}^p \longrightarrow \mathbb{Z}_B \longrightarrow 0, \tag{32}$$

$$0 \longrightarrow A^{n-1}\mathbb{Z}^p \longrightarrow A^n\mathbb{Z}^p \longrightarrow \widehat{\mathbb{Z}_B} \longrightarrow 0, \tag{33}$$

where $\mathbb{Z}_B$ is now identified with the finite group in (32), hence is a subgroup of $\mathbb{T}_n$. The central extension $A^n\mathbb{Z}^p$ of $\widehat{\mathbb{Z}_B}$ via $A^{n-1}\mathbb{Z}^p$ can be described either with a section $s_n: \widehat{\mathbb{Z}_B} \to$

$A^n\mathbb{Z}^p$ or via a $A^{n-1}\mathbb{Z}^p$-valued 2-cocycle $\omega_n(k,k') = s_n(k) + s_n(k') - s_n(k+k')$, see e.g. [228]. We choose the unique such that, for any $k \in \widehat{\mathbb{Z}_B}, s_n(k) \in A^{n-1}[0,1)^p$, and observe that this is the same as choosing $s_n(k) = A^{n-1}s_1(k)$. In the same way, the second extension $B^{n-1}\mathbb{Z}^p$ of $\mathbb{Z}_B$ via $B^n\mathbb{Z}^p$ can be described either with a section $\hat{s}_n: \mathbb{Z}_B \to B^{n-1}\mathbb{Z}^p$ or via a $B^n\mathbb{Z}^p$-valued 2-cocycle $\widehat{\omega}_n(k,k') = \hat{s}_n(k) + \hat{s}_n(k') - \hat{s}_n(k+k')$. We choose the unique such that, for any $k \in \mathbb{Z}_B, \hat{s}_n(k) \in B^n[0,1)^p$. The following result holds

Proposition (5.3.12)[221]: Any function ξ on $\mathbb{T}_i$ can be decomposed as $\xi = \sum_{k\in\widehat{Z_B}} \xi_k$, where

$$\xi_k(t) \equiv E_k(\xi)(t) = \frac{1}{r}\sum_{g\in\mathbb{Z}_B} \langle -k, g\rangle \xi(t-g), \qquad t \in \mathbb{T}_i = \mathbb{R}^p/B^i\mathbb{Z}^p.$$

Moreover, this correspondence gives rise to unitary operators

$$v_i: L^2(\mathbb{T}_i, dm/r^i) \to \sum_{k\in\widehat{Z_B}}^{\oplus} L^2(\mathbb{T}_{i-1}, dm/r^{i-1}) = L^2(\mathbb{T}_{i-1}, dm/r^{i-1}) \otimes \mathbb{C}^r$$

$$\xi \ \mapsto \ \sum_{k\in\widehat{Z_B}}^{\oplus} \sigma(k)^{-1}\xi_k \ .$$

The multiplication operator by the element $f \in \mathcal{A}_i$ is mapped to the matrix $M_r(f)$ acting on $L^2(\mathbb{T}_{i-1}, dm/r^{i-1}) \otimes \mathbb{C}^r$ given by

$$M_r(f)_{j,k}(t) = \langle s(j) - s(k), t\rangle f_{j-k}(t), \qquad j,k \in \widehat{\mathbb{Z}_B}.$$

In particular, when f is $B^{i-1}\mathbb{Z}$-periodic, namely it is a function on $\mathbb{T}_{i-1}$, then $M_r(f)_{j,k}(t) = f(t)\delta_{j,k}$, i.e. a function f on $\mathbb{T}_{i-1}$ embeds into $\mathcal{B}(L^2(\mathbb{T}_{i-1}, dm/r^{i-1})) \otimes\ M_r(\mathbb{C}) as\ f \otimes I$.

Proof. The statement follows from the analysis of Proposition (5.3.7), in particular here $b_k = \xi_k, M(b) = M_r(f)$.

Theorem (5.3.13)[221]: The Dirac operator D_n acting on $\mathbb{C}^{2^{[p/2]}} \otimes L^2(\mathbb{T}_n, \frac{1}{r^n}\,dm)$ gives rise to an operator, which we denote by $\widehat{D}_n$, when the Hilbert space is identified with the Hilbert space $\mathbb{C}^{2^{[p/2]}} \otimes L^2(\mathbb{T}_0, dm) \otimes (\mathbb{C}^r)^{\otimes n}$ as above. The Dirac operator $\widehat{D}_n$ has the following form:

$$\widehat{D}_n = V_nD_nV_n^* = D_0 \otimes I - 2\pi\sum_{a=1}^{p} \varepsilon^a \otimes I \otimes \left(\sum_{h=1}^{n} I^{\otimes h-1} \otimes diag(s_h(\cdot)^a) \otimes I^{\otimes n-h}\right),$$

where $diag(s_h(\cdot)^a)_{j,k} = \delta_{j,k}s_j(k)^a$ for $j,k \in \widehat{\mathbb{Z}_B}$, the unitary operator V_n: $\mathbb{C}^{2^{[p/2]}} \otimes L^2(\mathbb{T}_2, \frac{1}{r^n}dm) \to \mathbb{C}^{2^{[p/2]}} \otimes L^2(\mathbb{T}_0, dm) \otimes (\mathbb{C}^r)^{\otimes n}$ is defined as $V_n := I \otimes [(v_n \otimes \otimes_{j=1}^{n-1} I) \circ (v_n \otimes \otimes_{j=1}^{n-2}\ I) \circ \cdots \circ v_n]$. Moreover, we have the following spectral triple

$$(\mathcal{L}_n := C^1(\mathbb{T}_n), \mathbb{C}^{2^{[p/2]}} \otimes L^2(\mathbb{T}_0, dm) \otimes (\mathbb{C}^r)^{\otimes n}, \widehat{D}_n).$$

Proof. First of all we prove the formula for $n = 1$. We give a formula for D_1 acting on $\mathbb{C}^{2^{[p/2]}} \otimes L^2(\mathbb{T}_1, \frac{1}{r}\ dm) \cong \mathbb{C}^{2^{[p/2]}} \otimes L^2(\mathbb{T}_0, dm) \otimes \mathbb{C}^r$. Let us denote by $\{\eta_k\}_{k\in\widehat{Z_B}}$ a r-tuple of vectors in $\mathbb{C}^{2^{[p/2]}} \otimes L^2(\mathbb{T}_0, dm)$, so that $\xi := \sum_{k\in\widehat{\mathbb{Z}_B}} \sigma(k)\eta_k$ is an element in $\mathbb{C}^{2^{[p/2]}} \otimes L^2(\mathbb{T}_1, \frac{1}{r}\ dm)$, and $E_k(\xi) = \sigma(k)\eta_k, k \in \widehat{\mathbb{Z}_B}$. Then, for any $t \in \mathbb{T}_1$, we get

$$\widehat{D}_1\left(\sum_{k\in\widehat{Z_B}}^{\oplus}\eta_k(t)\right)=V_1D_1V_1^*\left(\sum_{k\in\widehat{Z_B}}^{\oplus}\eta_k(t)\right)$$

$$=\sum\nolimits_{j\in\widehat{Z_B}}^{\oplus}\frac{1}{r}\langle s(j),t\rangle\sum\nolimits_{g\in\widehat{Z_B}}\langle -j,g\rangle D\left(\sum\nolimits_{k\in\widehat{Z_B}}\langle s(k),-t+g\rangle\eta_k(t-g)\right)$$

$$=\sum\nolimits_{j\in\widehat{Z_B}}^{\oplus}\sum_{k\in\widehat{Z_B}}\frac{1}{r}\langle s(j),t\rangle\sum\nolimits_{g\in\widehat{Z_B}}\langle k-j,g\rangle D\left(\langle s(k),-t\rangle\eta_k(t)\right)$$

$$=\sum\nolimits_{k\in\widehat{\mathbb{Z}_B}}^{\oplus}\langle s(k),t\rangle D(\langle s(k),-t\rangle\eta_k(t))$$

$$=-i\sum_{a=1}^{p}\sum\nolimits_{k\in\widehat{\mathbb{Z}_B}}^{\oplus}\langle s(k),t\rangle\varepsilon^a\partial^a(\langle s(k),-t\rangle\eta_k(t))$$

$$=-i\sum_{a=1}^{p}\sum\nolimits_{k\in\widehat{\mathbb{Z}_B}}^{\oplus}\varepsilon^a(-2\pi i s(k)^a\eta_k(t)+\partial^a\eta_k(t))$$

$$=\sum_{a=1}^{p}\left(-2\pi\varepsilon^a\otimes I\otimes diag(s(k)^a)_{k\in\widehat{\mathbb{Z}_B}}-i\varepsilon^a\otimes\partial^a\otimes I\right)\sum\nolimits_{k\in\widehat{\mathbb{Z}_B}}^{\oplus}\eta_k(t).$$

The formula for $n>1$ can be obtained by iterating the above procedure.

We construct a spectral triple for the inductive limit $\overset{\lim}{\longrightarrow}\mathcal{A}_n$. We begin with some preliminary results. A matrix $B\in M_p(\mathbb{Z})$ is called purely expanding if, for all vectors $v\neq 0$, we have that $\|B^nv\|$ goes to infinity.

Proposition (5.3.14)[221]: Assume $detB\neq 0, A=(B^T)^{-1}$. Then the following are equivalent:

(i) B is purely expanding,

(ii) $\|A^n\|\to 0$,

(iii) the spectral radius $spr(A)<1$,

(iv) $\sum_{n\geq 0}\|A^n\|<\infty$.

Proof. (i) ⇔ (ii) Consider a vector $w=B^nv/\|B^nv\|$, then, from the identity

$$\|B^nw\|=\frac{\|v\|}{\|B^nv\|},$$

we deduce that (i) is equivalent to $\|B^{-n}v\|\to 0$, for all $v\neq 0$. The latter is equivalent to (ii) by the identity $(A^nv,u)=(v,B^{-n}u)$, for any vectors u,v. (ii) ⇒ (iii) We argue by contradiction. Let $\lambda\in sp(A)$ have modulus $|\lambda|\geq 1$, and consider an associated eigenvector $v\neq 0$. Then, we have that $\|A^nv\|=|\lambda|^n\|v\|\nrightarrow 0$.

(iii) ⇒ (iv) Let $A=C^{-1}(D+N)C$ be the Jordan decomposition of A, where D is the diagonal part, and N the nilpotent one. Then

$$\|(D\;+\;N)^n\|=\left\|\sum_{j=0}^{p-1}\binom{n}{j}D^{n-j}N^j\right\|\leq\sum_{j=0}^{p-1}\binom{n}{j}\|D^{n-j}\|$$

$$=\sum_{j=0}^{p-1}\binom{n}{j}spr(A)^{n-j}\leq spr(A)^n\left(\sum_{j=0}^{p-1}n^j spr(A)^{-j}\right)\qquad(34)$$

$$= spr(A)^n \frac{(n/spr(A))^p - 1}{n/spr(A) - 1} < \frac{n^p}{n-1} spr(A)^{n-p},$$
where we used $N^p = 0, \|N^j\| \le 1, \|D\| = spr(A)$, so that the series $\sum_{n\ge0}\|A^n\|$ converges.
(iv) ⇒ (ii) is obvious.

Theorem (5.3.15)[221]: Assume now that B is purely expanding and consider the C^*-algebras $\mathcal{A}_n = C(\mathbb{R}^p/B^n\mathbb{Z}^p)$, which embed into $M\ \ (\mathbb{C}) \otimes \mathcal{B}(L^2(\mathbb{T}_0, dm)) \otimes M_{r^n}(C)$, and the Dirac operators $\widehat{D}_n \mathrel{\widehat{\in}} \mathcal{B}(\mathcal{H}_0) \otimes UHF(r^\infty)$, where $\mathcal{H}_0 := \mathbb{C}^{2^{[p/2]}} \otimes L^2(\mathbb{T}_0, dm)$. As a consequence, $\mathcal{A}_\infty$ embeds in the injective limit
$$\overset{\lim}{\longrightarrow} \mathcal{B}(\mathcal{H}_0) \otimes M_{r^n}(\mathbb{C}) = \mathcal{B}(\mathcal{H}_0) \otimes UHF(r^\infty)$$
hence in $\mathcal{B}(\mathcal{H}_0) \otimes \mathcal{R}$, where $\mathcal{R}$ is the injective type II_1 factor. Moreover, the operator $\widehat{D}_\infty$ has the following form:
$$\widehat{D}_\infty = D_0 \otimes I - 2\pi \sum_{a=1}^{p} \varepsilon^a \otimes I \otimes \left(\sum_{h=1}^{\infty} I^{\otimes h-1} \otimes diag(s_h(\cdot)^a) \right).$$
In particular, $\widehat{D}_\infty$ is affiliated to $\mathcal{B}(\mathcal{H}_0) \otimes \mathcal{R} = \mathcal{M}$ and has the form $D_0 \otimes I + C$, with $C = C^* \in \mathcal{B}(\mathcal{H}_0) \otimes UHF(r^\infty) \subset \mathcal{B}(\mathcal{H}_0) \otimes \mathcal{R} = \mathcal{M}$.

Proof. The formula and the fact that $\widehat{D}_\infty$ is affiliated to $\mathcal{M}$ follow from what has already been proved and the following argument. We posed $s_n(k) \in A^{n-1}[0,1)^p$, therefore
$$\max_{k\in\widehat{\mathbb{Z}_B}} \|s_n(k)\| \le sup_{x\in[0,1)^p} \|A^{n-1}x\| \le \|A^{n-1}\|\sqrt{p}.$$
As a consequence, for any $a \in \{1, \ldots, p\}$,
$$\left\|diag(s_n(k)_a)_{k\in\widehat{\mathbb{Z}_B}}\right\| = \max_{k\in\widehat{\mathbb{Z}_B}} |s_n(k)_a| \le \max_{k\in\widehat{\mathbb{Z}_B}} \|s_n(k)\| \le \|A^{n-1}\|\sqrt{p}.$$
Recalling that $\widehat{D}_\infty = D_0 \otimes I + C,$ with $C = 2\pi \sum_{a=1}^{p} \varepsilon^a \otimes I \otimes \left(\sum_{h=1}^{\infty} I^{\otimes h-1} \otimes diag(s_h(k)^a)\right)$, we get, by Proposition (5.3.14) and the estimate above, that C is bounded and belongs to $M_{2^{[p/2]}}(\mathbb{C}) \otimes \mathbb{C} \otimes UHF(r^\infty)$, while $D_0 \mathrel{\widehat{\in}} \mathcal{B}(\mathcal{H}_0)$.

Theorem (5.3.16)[221]: Let $\{(\mathcal{A}_n, \varphi_n) : n \in \mathbb{N} \cup \{0\}\}$ be an inductive system, with $\mathcal{A}_n \cong \mathcal{A}_0$, and $\varphi_n : \mathcal{A}_n \hookrightarrow \mathcal{A}_{n+1}$ is the inclusion, for all $n \in \mathbb{N}$. Suppose that, for any $n \in N \cup \{0\}$, there exists a spectral triple $(\mathcal{L}_n, \mathcal{H}_n, \widehat{D}_n)$ on $\mathcal{A}_n$, with
$\mathcal{H}_n = \mathcal{H}_0 \otimes (\mathbb{C}^r)^{\otimes n}, \widehat{D}_n = D_0 \otimes I + C_n, C_n \in \mathcal{B}(\mathcal{H}_0) \otimes M_r(\mathbb{C})^{\otimes n} \subset \mathcal{B}(\mathcal{H}_0) \otimes$
$UHF(r^\infty)$ is a self-adjoint sequence converging to $C \in \mathcal{B}(\mathcal{H}_0) \otimes UHF(r^\infty)$, and $\widehat{D}_\infty = D_0 \otimes I + C$. Let d be the abscissa of convergence of ζ_{D_0} and suppose that $res_{s=d}(\tau(D_0^2 + 1)^{-s/2})$ exists and is finite. Let $\mathcal{L}_\infty := \cup_{n=0}^{\infty} \mathcal{L}_n$. Then $(\mathcal{L}_\infty, \mathcal{B}(\mathcal{H}_0) \otimes \mathcal{R}, \mathcal{H}_0 \otimes L^2(\mathcal{R}, \tau), \widehat{D}_\infty)$ is a finitely summable, semifinite, spectral triple, with the same Hausdorff dimension of $(\mathcal{L}_0, \mathcal{H}_0, D_0)$. Moreover, the volume of this noncommutative manifold coincides with the volume of $(\mathcal{L}_0, \mathcal{H}_0, D_0)$, namely the Dixmier trace τ_ω of $(\widehat{D}_\infty^2 + 1)^{-d/2}$ coincides with that of $(D_0^2 + 1)^{-d/2}$ (hence does not depend on ω) and may be written as:
$$\tau_\omega((\widehat{D}_\infty^2 + 1)^{-d/2}) = \lim_{t\to\infty} \frac{1}{log\ t} \int_0^t \left(\mu_{(D_0^2+1)^{-1/2}}(s)\right)^d ds\,.$$
Proof. As for the commutator condition, we observe that for each $f \in \mathcal{L}_n$ we have that $[\widehat{D}_\infty, f]$ is bounded since $[\widehat{D}_n, f]$ is bounded.

We now show that $\widehat{D}_\infty$ has τ-compact resolvent, where τ is the unique f.n.s.
trace on $\mathcal{B}(\mathcal{H}_0) \otimes \mathcal{R}$. Indeed, on a finite factor, any bounded operator has τ-finite rank, hence is τ-compact. Therefore, since D_0 has compact resolvent in $\mathcal{B}(\mathcal{H}_0)$, $D_0 \otimes I$ has τ-

compact resolvent in $\mathcal{B}(\mathcal{H}_0)\otimes\mathcal{R}$. We have $(D_0\otimes I+C+i)^{-1}=[I+(D_0\otimes I+i)^{-1}C]^{-1}(D_0\otimes I+i)^{-1}=(D_0\otimes I+i)^{-1}[I+C(D_0\otimes I+i)^{-1}]^{-1}$, where $I+C(D_0\otimes I+i)^{-1}$ and $I+(D_0\otimes I+i)^{-1}C$ have trivial kernel and cokernel. Indeed $Ran(I+(D_0\otimes I+i)^{-1}C)^{\perp}=ker(I+C(D_0\otimes I-i)^{-1})$, and $(I+C(D_0\otimes I\pm i)^{-1})x=0$ means $(C+D_0\otimes I)y=\mp iy$ with $y=(D_0\otimes I\pm i)^{-1}x$, which is impossible since $C+D_0\otimes I$ is self-adjoint. Moreover, $ker(I+(D_0\otimes I+i)^{-1}C)$ is trivial.
In fact, $(I+(D_0\otimes I\pm i)^{-1}C)x=0$ implies that $(D_0\otimes I+C)x=\mp ix$ which is impossible because $D_0\otimes I+C$ is self adjoint. Therefore $I+C(D_0\otimes I+i)^{-1}$ has bounded inverse, hence $D_0\otimes I+C$ has τ-compact resolvent.
Since D_0 has spectral dimension d, $res_{s=d}(\tau(D_0^2+1)^{-s/2})$ exists and is finite. Then, applying Proposition A.4, in the appendix, we get $res_{s=d}(\tau(D_0^2+1))=res_{s=d}(\tau(D_\infty^0+1)^{-s/2})$. The result follows by [232], Thm 4.11.

Corollary (5.3.17)[221]: Let $(\mathcal{L}_n,\mathcal{H}_n,\widehat{D}_n)$ be the spectral triple on $\mathbb{T}_n$ constructed in Theorem (5.3.15), and let us set $\mathcal{L}_\infty:=\cup_{n=0}^{\infty}\mathcal{L}_n,\mathcal{M}_\infty:=\mathcal{B}(\mathcal{H}_0)\otimes\mathcal{R},\mathcal{H}_\infty:=\mathcal{H}_0\otimes L^2(\mathcal{R},\tau)$. Then $(\mathcal{L}_\infty,\mathcal{M}_\infty,\mathcal{H}_\infty,\widehat{D}_\infty)$ is a finitely summable, semifinite, spectral triple, with Hausdorff dimension p. Moreover, the Dixmier trace τ_ω of $(\widehat{D}_0^2+1)^{-p/2}$ coincides with that of $(D_0^2+1)^{-p/2}$ (hence does not depend on ω)
and may be written as:

$$\tau_\omega((\widehat{D}_\infty^2+1)^{-p/2})=\lim_{t\to\infty}\frac{1}{log\ t}\int_0^t\left(\mu_{(D_0^2+1)^{-1/2}}(s)\right)^p ds.$$

Proof. By construction, $\mathcal{L}_\infty$ is a dense $*$-subalgebra of the C^*-algebra $\mathcal{A}_\infty\subset\mathcal{M}_\infty$. The thesis follows from Theorem (5.3.16) and the above results.

Let A_ϑ be the noncommutative torus generated by U,V with $UV=e^{2\pi i\vartheta}VU,\vartheta\in[0,1)$. Given a matrix $B\in M_2(\mathbb{Z}),detB\neq 0,B=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$, we may consider the C^*-subalgebra A_ϑ^B generated by the elements

$$U_1=U^aV^b,\quad V_1=U^cV^d. \tag{40}$$

We may set $W(n):=U^{n_1}V^{n_2}$ with $n\in\mathbb{Z}^2$. By using the commutation relation between U and V, it is easy to see that

$$W(m)W(n)=e^{-2\pi i\theta m_2n_1},$$
$$W(n)^k=e^{-\pi i\theta k(k-1)n_1n_2}W(kn),\quad\forall k\in\mathbb{Z}. \tag{41}$$

Lemma (5.3.18)[221]:
(i) $A_\vartheta^B=A_\vartheta\Leftrightarrow r=|detB|=1$.
(ii) $A_\vartheta^B\cong A_{\vartheta'}$, where $\vartheta'=r\vartheta$.
(iii) $A_\vartheta^B\cong A_\vartheta$ iff $r\equiv_q\pm 1$.

Proof. (i)($\Leftarrow$) By using equation (41) it can be shown that the generators of $(A_\vartheta^B)^{B^{-1}}$ are

$$U_2=e^{\pi i\vartheta bd(1-a+c)\ detB}U,$$
$$V_2=e^{\pi i\vartheta ac(1+b-d)\ detB}V.$$

Hence $A_\vartheta=(A_\vartheta^B)^{B^{-1}}\subset A_\vartheta^B\subset A_\vartheta$, namely these algebras coincide.
(ii) We compute the commutation relations for U_1 and V_1, getting $U_1V_1=e^{2\pi i\ detB\vartheta}V_1U_1$. Since $A_{detB\vartheta}\cong A_{r\vartheta}$, the statement follows.
(iii) We have $A_\vartheta\cong A_{\vartheta'}\Leftrightarrow\vartheta\pm\vartheta'\in\mathbb{Z}\Leftrightarrow(r\pm 1)\vartheta\in\mathbb{Z}$. This means in particular that $\vartheta=p/q$, for some relatively prime $p,q\in\mathbb{N}$, and $r\equiv_q\pm 1$.
(i)($\Rightarrow$) Finally, we observe that $A_\vartheta=A_\vartheta^B\Rightarrow A_\vartheta\cong A_{\vartheta'}$. We show that A_ϑ^B is a proper subalgebra of A_ϑ when $r\neq\pm 1$, thus completing the proof of (i).

On the one hand, the previous Lemma shows that, setting $\vartheta_n = r^{-n}\vartheta$, the algebras A_{ϑ_n} form an inductive family, where $A_{\vartheta_{k-1}}$ can be identified with the subalgebra $A^B_{\vartheta_k}$of A_{ϑ_k}. The inductive limit is a noncommutative solenoid according to [68], [200].
On the other hand, since we are mainly concerned with selfcoverings, we will, in the following, consider only the rational case $\vartheta = p/q$, with $r \equiv_q \pm 1$. Possibly replacing B with $-B$, this is the same as assuming $detB \equiv_q 1$.

We give a description of the rational rotation algebra making small modifications to the description of $A_\theta, \theta = p/q \in \mathbb{Q}$, seen in [227]. Consider the following matrices

$$(U_0)_{hk} = \delta_{h,k}e^{2\pi i(k-1)\theta}, \qquad (V_0)_{hk} = \delta_{h+1,k} + \delta_{h,q}\delta_{k,1} \in M_q(\mathbb{C})$$

and

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in M_2(\mathbb{C}).$$

We have that $U_0V_0 = e^{2\pi i\theta}V_0U_0$. Let $n = (n_1, n_2) \in \mathbb{Z}^2$ and set $W_0(n) \stackrel{\text{def}}{=} U_0^{n_1}\, V_0^{n_2}, \tilde{\gamma}_n(f)(t) := ad(W_0(Jn))[f(t-n)] = V_0^{n_1}U_0^{-n_2}f(t-n)U_0^{n_2}V_0^{-n_1}$. Since formula (41) holds whenever two operators satisfy the commutation relation $UV = e^{2\pi i\theta}VU$, the following formula holds

$$W_0(n)^k = e^{-\pi i\theta k(k-1)n_1n_2}W_0(kn) \qquad \forall k \in \mathbb{Z}. \tag{42}$$

We have the following description of A_θ (cf. [227])

$$A_\theta = \{f \in C(\mathbb{R}^2, M_q(\mathbb{C})) : f = \tilde{\gamma}_n(f), n \in \mathbb{Z}^2\}.$$

This algebra comes with a natural trace

$$\tau(f) := \frac{1}{q}\int_{\mathbb{T}_0} tr(f(t))dt\,,$$

where we are considering the Haar measure on $\mathbb{T}_0$ and $tr(A) = \sum_i a_{ii}$. We observe that the function $tr(f(t))$ is $\mathbb{Z}^2$-periodic. The generators of the algebra are

$$U(t_1, t_2) = e^{2\pi i\theta t_1}U_0,$$
$$V(t_1, t_2) = e^{2\pi i\theta t_2}V_0.$$

They satisfy the following commutation relation

$$U(t)^\alpha V(t)^\beta = e^{2\pi i\theta\alpha\beta}V(t)^\beta U(t)^\alpha, \qquad \alpha, \beta \in \mathbb{Z}.$$

We set $W(n,t) = U(t)^{n_1}V(t)^{n_1}\,, \forall t \in \mathbb{R}^n, n \in \mathbb{Z}^2$, and note that

$$W(m,t)W(n,t) = e^{2i\pi\theta(m,J_n)}W(n,t)W(m,t),$$
$$U(t) = W((1,0),t),$$
$$V(t) = W((0,1),t).$$

We observe that $\tilde{\gamma}_n(f)(t) = ad(W(Jn,t))[f(t-n)], \forall t \in \mathbb{R}^2, n \in \mathbb{Z}^2$.

Define

$$\mathcal{L}_\theta := \left\{\sum_{r,s} a_{rs}U^rV^s : (a_{rs}) \in S(\mathbb{Z}^2)\right\},$$

where $S(\mathbb{Z}^2)$ is the set of rapidly decreasing sequences. It is clear that the derivations ∂_1 and ∂_2, defined as follows on the generators, extend to $\mathcal{L}_\theta$

$$\partial_1(U^hV^k) = 2\pi ihU^hV^k$$
$$\partial_2(U^hV^k) = 2\pi ikU^hV^k.$$

Moreover, the above derivations extend to densely defined derivations both on A_θ and $L^2(A_\theta, \tau)$.
We still denote these extensions with the same symbols. We may consider the following spectral triple (see [236], or section 12.3 in [240]):

$$(\mathcal{L}_\theta, \mathbb{C}^2 \otimes L^2(A_\theta,\tau), D = -i(\varepsilon^1 \otimes \partial_1 + \varepsilon^2 \otimes \partial_2)),$$
where $\varepsilon^1, \varepsilon^2$ denote the Pauli matrices. In order to fix the notation we recall that the Pauli matrices are self-adjoint, in particular they satisfy the condition $(\varepsilon^k)^2 = I, k = 1, 2$.

Let $A \doteq A_\theta$ be a rational rotation algebra, $\vartheta = p/q, B \in M_2(\mathbb{Z})$ be a matrix such that $detB \equiv_q 1, r := |detB| > 1$, and set $C_B = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$ the cofactor matrix of B, and $A = (B^T)^{-1}$. Then a self-covering of $\mathcal{A}$ may be constructed in analogy with the construction for the classical torus. Consider the C^*-algebra
$$\mathcal{B} := \{f \in C(\mathbb{R}^2, M_q(\mathbb{C})) : f = \tilde{\gamma}_{B_n}(f), n \in \mathbb{Z}^2\}.$$
This algebra is generated by the elements
$$\begin{aligned} U_{\mathcal{B}}(t) &= e^{\pi i \vartheta bd(1-a+c)} e^{2\pi i \theta\langle Ae_1, t\rangle} W_0(C_B e_1), \\ V_{\mathcal{B}}(t) &= e^{\pi i \vartheta ac(1+b-d)} e^{2\pi i \theta\langle Ae_2, t\rangle} W_0(C_B e_2), \end{aligned} \quad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{43}$$
and can be endowed with a natural trace
$$\tau_1(f) := \frac{1}{q|detB|} \int_{\mathbb{T}_1} tr(f(t))dt, \qquad f \in \mathcal{B}.$$
The action $\tilde{\gamma}$ of $\mathbb{Z}^2$ on $\mathcal{B}$, being trivial when restricted to $B\mathbb{Z}^2$, induces an action of $\mathbb{Z}_B$.

Proposition (5.3.19)[221]: The GNS representation $\pi_1: \mathcal{B} \to B(L^2(\mathcal{B}, \tau_1))$ is unitarily equivalent to the representation obtained by $\pi_0: \mathcal{A} \to B(L^2(\mathcal{A}, \tau))$ according to Proposition (5.3.7).

Proof. It is enough to prove that $\tau_1 = \tau_0 \circ E$, where E is the conditional expectation from $\mathcal{B}$ to $\mathcal{A}$. We have that
$$\tau_0[E(f)] = \frac{1}{q} \int_{\mathbb{T}_0} tr[E(f)(t)] = \frac{1}{qr} \int_{\mathbb{T}_0} \sum_{n \in \mathbb{Z}_B} tr[\gamma_n(f)(t)] = \frac{1}{qr} \int_{\mathbb{T}_0} \sum_{n \in \mathbb{Z}_B} tr[(f)(t-n)]$$
$$= \frac{1}{qr} \int_{\mathbb{T}_1} tr[f(t)] = \tau_1(f).$$

Given the integer-valued matrix $B \in M_2(\mathbb{Z})$ as above, there is an associated endomorphism $\alpha: A_\theta \to A_\theta$ defined by $\alpha(f)(t) = f(Bt)$. Then, we consider the inductive limit $\mathcal{A}_\infty = \overset{\lim}{\longrightarrow} \mathcal{A}_n$ described in (0.1), where $\mathcal{A}_n = \mathcal{A}$ for any n.

It will be convenient to consider the following isomorphic inductive family: $\mathcal{A}_n$ consists of continuous $B^k\mathbb{Z}^2$-invariant matrix-valued functions on $\mathbb{R}^n$, i.e
$$\mathcal{A}_k := \{f \in C(\mathbb{R}^2, M_q(\mathbb{C})) : f = \tilde{\gamma}_{B^k n}(f), n \in \mathbb{Z}^2\},$$
with trace
$$\tau_k(f) = \frac{1}{q|detB^k|} \int_{\mathbb{T}_k} tr(f(t))dt,$$
and the embedding is unital inclusion $\alpha_{k+1,k}: \mathcal{A}_k \hookrightarrow \mathcal{A}_{k+1}$. In particular, $\mathcal{A}_0 = \mathcal{A}$, and $\mathcal{A}_1 = \mathcal{B}$. This means that $\mathcal{A}_\infty$ may considered as a generalized solenoid C^*-algebra (cf. [244], [68]).

On the n-th noncommutative covering $\mathcal{A}_n$, the formula of the Dirac operator doesn't change and we can consider the following spectral triple
$$(L_\theta^{(n)}, \mathbb{C}^2 \otimes L^2(\mathcal{A}_n, \tau), D = -i(\varepsilon^1 \otimes \partial_1 + \varepsilon^2 \otimes \partial_2)).$$

We describe the spectral triple on $\mathcal{A}_n$ in terms of the spectral triple on $\mathcal{A}_0 = A_\theta$.

We will consider the two central extensions (32) and (33) (case $p = 2$) with the associated $s_n: \widehat{\mathbb{Z}_B} \to A^n\mathbb{Z}^2$ and $\hat{s}_n: \mathbb{Z}_B \to B^{n-1}\mathbb{Z}^2$ defined earlier.

The following result holds:

Theorem (5.3.20)[221]: Any b in $\mathcal{A}_i$ can be decomposed as $b = \sum_{k\in\widehat{\mathbb{Z}_B}} b_k$, where

$$b_k(t) = \frac{1}{r}\sum_{g\in\mathbb{Z}_B} \langle -k, g\rangle \gamma_g(b(t)) \in (\mathcal{A}_i)_k. \tag{44}$$

Let u_g be the unitary operator on $L^2(\mathcal{A}_i, \tau_i)$ implementing the automorphism γ_g. Then, any $\xi \in L^2(\mathcal{A}_i, \tau_i)$ can be decomposed as $\xi = \sum_{k\in\widehat{\mathbb{Z}_B}} \xi_k$, where

$$\xi_k(t) = \frac{1}{r}\sum_{g\in\mathbb{Z}_B} \langle -k, g\rangle u_g\,(\xi(t)). \tag{45}$$

Moreover, this correspondence gives rise to unitary operators $v_i: L^2(\mathcal{A}_i, \tau_i) \to L^2(\mathcal{A}_{i-1}, \tau_{i-1}) \otimes \mathbb{C}^r$ defined by $v_i(\xi) = \{\sigma(k)^{-1}\xi_k\}_{k\in\widehat{\mathbb{Z}_B}}$. The multiplication operator by an element f on $\mathcal{A}_i$ is mapped to the matrix $M_r(f)$ acting on $L^2(\mathcal{A}_{i-1}, \tau_{i-1}) \otimes \mathbb{C}^r$ given by

$$M_r(f)_{h,k}(t) = \langle s(k) - s(h), -t\rangle f_{h-k}(t), \qquad t \in \mathbb{R}^2, h, k \in \widehat{\mathbb{Z}_B}.$$

Theorem (5.3.21)[221]: Set $\mathcal{H}_0 := \mathbb{C}^2 \otimes L^2(\mathcal{A}_0, \tau_0)$. Then the Dirac operator D_n acting on $\mathbb{C}^2 \otimes L^2(\mathcal{A}_n, \tau_n)$ gives rise to the operator $\widehat{D}_n$ when the Hilbert space is identified with $\mathcal{H}_0 \otimes (\mathbb{C}^r)^{\otimes n}$ as above. Moreover, the Dirac operator $\widehat{D}_n$ has the following form:

$$\begin{aligned}\widehat{D}_n &:= V_n D_n V_n^* \\ &= D_0 \otimes I - 2\pi \sum_{a=1}^{2} \varepsilon^a \otimes I \otimes \left(\sum_{j=1}^{n} I^{\otimes j-1} \otimes diag(s_j(k)^a)_{k\in\widehat{\mathbb{Z}_B}} \otimes I^{\otimes n-j}\right),\end{aligned}$$

where $V_n: \mathbb{C}^2 \otimes L^2(\mathcal{A}_n, \tau_n) \to \mathcal{H}_0 \otimes (\mathbb{C}^r)^{\otimes n}$ is defined as $V_n := I \otimes [(v_1 \otimes_{j=1}^{n-1} I) \circ (v_2 \otimes_{j=1}^{n-2} I) \circ \cdots \circ v_n]$.

Proof. We prove the formula for $n = 1$, the case $n > 1$ can be obtained by iterating the procedure. Let us denote by $\{\eta_k\}_{k\in\widehat{\mathbb{Z}_B}}$ an element in $\mathbb{C}^2 \otimes L^2(\mathcal{A}_0, \tau_0)$.

$$\begin{aligned}
V_1 D_1 V_1^* \left(\sum\nolimits^{\oplus}_{k\in\widehat{\mathbb{Z}_B}} \eta_k(t)\right) &= V_1 D_1 \left(\sum_{k\in\widehat{\mathbb{Z}_B}} \langle s(k), -t\rangle \eta_k(t)\right) \\
&= \sum\nolimits^{\oplus}_{j\in\widehat{\mathbb{Z}_B}} \langle s(j), t\rangle \frac{1}{r}\sum_{g\in\mathbb{Z}_B} \langle -j, g\rangle u_g \left(\sum_{k\in\widehat{\mathbb{Z}_B}} D(\langle s(k), -t\rangle \eta_k(t))\right) \\
&\overset{(a)}{=} \sum\nolimits^{\oplus}_{j\in\widehat{\mathbb{Z}_B}} \sum_{k\in\widehat{\mathbb{Z}_B}} \langle s(j), t\rangle \frac{1}{r}\sum_{g\in\mathbb{Z}_B} \langle -j, g\rangle D(\langle s(k), -t+g\rangle \eta_k(t)) \\
&= \sum\nolimits^{\oplus}_{j\in\widehat{\mathbb{Z}_B}} \sum_{k\in\widehat{\mathbb{Z}_B}} \langle s(j), t\rangle \frac{1}{r}\sum_{g\in\mathbb{Z}_B} \langle k-j, g\rangle D(\langle s(k), -t\rangle \eta_k(t)) \\
&= \sum\nolimits^{\oplus}_{k\in\widehat{\mathbb{Z}_B}} \langle s(k), t\rangle\, D(\langle s(k), -t\rangle \eta_k(t)) \\
&= -i \sum\nolimits^{\oplus}_{k\in\widehat{\mathbb{Z}_B}} \langle s(k), t\rangle \sum_{a=1}^{2} \langle s(k), -t\rangle\, \varepsilon^a(-2\pi i s(k)^a \eta_k(t) + \partial^a \eta_k(t)) \\
&= \left(-i\sum_{a=1}^{2} \varepsilon^a \otimes \partial^a \otimes\ I - 2\pi \sum_{a=1}^{2} \varepsilon^a \otimes I \otimes diag(s(k)^a)_{k\in\widehat{\mathbb{Z}_B}}\right) \sum\nolimits^{\oplus}_{k\in\widehat{\mathbb{Z}_B}} \eta_k(t),
\end{aligned}$$

where in (a) we used the facts that $u_g \circ D = D \circ u_g$, and $u_g \equiv id$ on $\mathbb{C}^2 \otimes L^2(\mathcal{A}_0, \tau_0)$.

Proposition (5.3.22)[221]: The C^*-algebra $\mathcal{A}_n$ embeds into $\mathcal{B}(\mathcal{H}_0) \otimes \mathcal{M}_{r^n}(\mathbb{C})$. As a consequence, $\mathcal{A}_\infty$ embeds into the injective limit

$$\underset{\longrightarrow}{\lim} \mathcal{B}(\mathcal{H}_0) \otimes \mathcal{M}_{r^n}(\mathbb{C}) = \mathcal{B}(\mathcal{H}_0) \otimes UHF(r^\infty)$$

hence in $\mathcal{B}(\mathcal{H}_0) \otimes \mathcal{R}$, where $\mathcal{R}$ is the injective type II_1 factor.

Theorem (5.3.23)[221]: Assume that B is purely expanding and that $det(B) \equiv_q 1$. Let us set $\mathcal{L}_\infty = \cup_n \mathcal{L}_\theta^{(n)}, \mathcal{M} = \mathcal{B}(\mathcal{H}_0) \otimes \mathcal{R}$, and define

$$\widehat{D}_\infty := D_0 \otimes I - 2\pi \sum_{a=1}^{2} \varepsilon^a \otimes I \otimes \left(\sum_{j=1}^{\infty} I^{\otimes j-1} \otimes diag(s_j(k)^a)_{k \in \widehat{\mathbb{Z}_B}} \right).$$

Then $(\mathcal{L}, \mathcal{M}, \mathcal{H}_0 \otimes L^2(\mathcal{R}, \tau), \widehat{D}_\infty)$ is a finitely summable, semifinite, spectral triple, with Hausdorff dimension 2. Moreover, the Dixmier trace τ_ω of $(\widehat{D}_\infty^2 + 1)^{-1}$ coincides with that of $(D_0^2 + 1)^{-1}$ (hence does not depend on the generalized limit ω) and may be written as:

$$\tau_\omega((\widehat{D}_\infty^2 + 1)^{-1}) = \lim_{t \to \infty} \frac{1}{log\ t} \int_0^t \left(\mu_{(D_0^2+1)^{-1/2}}(s) \right)^2 ds.$$

Proof. The formula for $\widehat{D}_\infty$ follows from what has already been proved. We want to prove that $\widehat{D}_\infty$ is of the form $D_0 \otimes I + C$, with $C = -2\pi \sum_{a=1}^{2} \varepsilon^a \otimes I \otimes$

$$\left(\sum_{j=1}^{\infty} I^{\otimes j-1} \otimes diag(s_j(k)^a) \right) \in \mathcal{B}(\mathcal{H}_0) \otimes \mathcal{R} \ and\ \widehat{D}_\infty \,\widehat{\in}\, \mathcal{B}(\mathcal{H}_0) \otimes \mathcal{R}.$$

By construction, $\mathcal{L}_\infty$ is a dense $*$-subalgebra of the C^*-algebra $\mathcal{A}_\infty \subset \mathcal{M}$. We now prove that $\widehat{D}_\infty$ is affiliated to $\mathcal{M}$. We posed $s_n(k) \in A^{n-1}[0,1)^2$, therefore

$$\max_{k \in \widehat{\mathbb{Z}}_B} \|s_n(k)\| \leq sup_{x \in [0,1)^2} \|A^{n-1}x\| \leq \|A^{n-1}\|\sqrt{2}.$$

As a consequence, for $a = 1, 2, j \in \mathbb{N}$,

$$\left\| diag(s_j(k)^a) \right\| = \max_{k \in \widehat{\mathbb{Z}}_B} |s_j(k)^a| \leq \max_{k \in \widehat{\mathbb{Z}}_B} \left\| s_j(k) \right\| \leq \left\| A^{j-1} \right\| \sqrt{2}.$$

By Proposition (5.3.14) and the estimate above, we get that C is bounded and belongs to $M_2(\mathbb{C}) \otimes \mathbb{C} \otimes UHF(r^\infty)$, while $D_0 \otimes I \,\widehat{\in}\, \mathcal{B}(\mathcal{H}_0) \otimes \mathbb{C}$.

The thesis follows from Theorem (5.3.16) and what we have seen above.

The algebra and the noncommutative covering

Let $B \in M_p(\mathbb{Z})$, with $r = |det(B)| > 1$, and set $A = (B^T)^{-1}$. Consider a finitely summable spectral triple $(\mathcal{L}_\mathcal{Z}, \mathcal{H}, D)$ on the C^*-algebra $\mathcal{Z}$ and assume the following:

(a) there is an action $\rho: G_1 = A\mathbb{Z}^p \to Aut(\mathcal{Z})$;

(b) $sup_{g \in G_1} \left\| [D, \rho_g(a)] \right\| < \infty$, for any $a \in \mathcal{L}_\mathcal{Z}$.

Assuming, for simplicity, that $\mathcal{Z} \subset \mathcal{B}(\mathcal{H})$, recall that the crossed product $\mathcal{A}_{G_1} = \mathcal{Z} \rtimes_\rho G_1$ is the C^*-subalgebra of $\mathcal{B}(\mathcal{H} \otimes \ell^2(G_1))$ generated by $\pi_{G_1}(\mathcal{Z})$ and $U_h, h \in G_1$, where

$$(\pi_{G_1}(z)\xi)(g) := \rho_g^{-1}(z)\xi(g),$$

$$(U_h\xi)(g) := \xi(g-h), z \in \mathcal{Z}, g, h \in G_1, \xi \in \ell^2(G_1; \mathcal{H}) \cong \mathcal{H} \otimes \ell^2(G_1).$$

Set $G_0 = \mathbb{Z}^p \subset G_1$. The embedding $\mathcal{Z} \rtimes_\rho G_0 \subset \mathcal{Z} \rtimes_\rho G_1$ is a finite covering with respect to the action $\gamma: \mathbb{Z}_B \to Aut(\mathcal{Z} \rtimes_\rho G_1)$ defined as

$$\gamma_j \left(\sum_{g \in G_1} a_g U_g \right) = \sum_{g \in G_1} \langle \hat{s}(j), g \rangle a_g U_g, \qquad j \in \mathbb{Z}_B,$$

where $\hat{s}: \mathbb{Z}_B \to \mathbb{Z}^p$ of the short exact sequence

$$0 \to B\mathbb{Z}^p \to \mathbb{Z}^p \to \mathbb{Z}_B \to 0.$$
In fact, the fixed point algebra of this action is $\mathcal{A}_{G_0} := \mathcal{Z} \rtimes_p G_0$.

Define the map $\ell: \mathbb{Z}^p \to M_{2^{[p/2]}}(\mathbb{C})$ as $\ell(m) := \sum_{\mu=1}^{p} m_\mu \varepsilon_{\mu+1}^{(p+1)}$, where $\{\varepsilon_i^{(p+1)}\}_{i=1}^{p+1}$ denote the generators of the Clifford algebra $\mathbb{C}l(\mathbb{R}^{p+1})$, and $m \in \mathbb{Z}^p$.

Theorem (5.3.24)[221]: The following triple is a spectral triple for the crossed product $\mathcal{A}_{G_0} = \mathcal{Z} \rtimes_\rho G_0$
$$(\mathcal{L}_0 = C_c(\mathbb{Z}^p, \mathcal{Z}), \mathcal{H}_0 = \mathcal{H} \otimes \mathbb{C}^{2^{[p/2]}} \otimes \ell^2(\mathbb{Z}^p), D_0 = D \otimes \varepsilon_1^{(p+1)} \otimes I + I \otimes M_\ell).$$
where $C_c(\mathbb{Z}^p, \mathcal{Z}) := \{\sum_{g \in \mathbb{Z}^p} \pi_{G_1}(z_g) U_g : z_g \in \mathcal{L}_\mathcal{Z}, z_g \neq 0$ for finitely many $g \in \mathbb{Z}^p\}$, and M_ℓ is the operator of multiplication by the generalized length function ℓ(cf. [240]). If the Hausdorff dimension $d(\mathcal{L}_\mathcal{Z}, \mathcal{H}, D) = d$, then $d(\mathcal{L}_0, \mathcal{H}_0, D_0) = d + p$.

Proof. The triple in the statement is indeed an iterated spectral triple in the sense of [241], sec. 2.4. Equivalently, $\ell(g)$ is a proper translation bounded matrixvalued function (cf. [241]). For the sake of completeness we sketch the proof of the statement. For the bounded commutator property it is enough to show that the commutators with $\pi_{G_0}(z), z \in \mathcal{L}_z$, and with $U_h, h \in G_0$ are bounded. The norm of the first is bounded by $sup_{g \in G_0} \|[D, \rho_g(a)]\|$, which is finite for any $a \in \mathcal{L}_z$, the norm of the second is bounded by $\|\ell(h)\|$. We then explicitly compute the eigenvalues of D_0^2: they are given by $\lambda^2 + \|g\|_2^2$, with λ belong to the sequence of eigenvalues of D and $g \in \mathbb{Z}^p$. The compact resolvent property follows. The formula for the dimension can be obtained as in [241].

In a similar way we define the following spectral triple for the crossed product $\mathcal{A}_{G_1} = \mathcal{Z} \rtimes_\rho G_1$
$$(\mathcal{L}_1 = C_c(G_1, \mathcal{Z}), \mathcal{H}_{G_1} = \mathcal{H} \otimes \mathbb{C}^{2^{[p/2]}} \otimes \ell^2(G_1), D_1 = D \otimes \varepsilon_1^{(p+1)} \otimes I + I \otimes M_{\ell_1}).$$
where $\ell_1: G_1 = A\mathbb{Z}^p \to M_{2^{[p/2]}}(\mathbb{C})$ is defined as $\ell_1(g) := g_\mu \varepsilon_{\mu+1}^{(p+1)}, g \in G_1$.

In order to show that the covering is regular according to Definition (5.3.3), we need to define a map σ which takes values in the spectral subspaces of γ. Consider the $s: \widehat{\mathbb{Z}_B} \to A\mathbb{Z}^p$ defined for the short exact sequence (30). Define $\sigma: \widehat{\mathbb{Z}_B} \to \mathcal{U}(\mathcal{Z} \rtimes_\rho A\mathbb{Z}^p)$ as
$$\sigma(k) = U_s(k). \tag{46}$$
We observe that $U_{s(k)} \in (\mathcal{Z} \rtimes_\rho A\mathbb{Z}^p)_k, k \in \widehat{\mathbb{Z}_B}$.

We first consider the crossed-product C^*-algebras $\mathcal{A}_{G_0}$ and $\mathcal{A}_{G_1}$ as acting on the Hilbert spaces $\mathcal{H} \otimes \mathbb{C}^{2^{[p/2]}} \otimes \ell^2(G_0)$ and $\mathcal{H} \otimes \mathbb{C}^{2^{[p/2]}} \otimes \ell^2(G_1)$. A short exact sequence of groups can be described either via a section $s: \widehat{\mathbb{Z}_B} \to G_1$ or a 2-cocycle $\omega: \widehat{\mathbb{Z}_B} \times \widehat{\mathbb{Z}_B} \to G_0, \omega(j,k) = s(j) + s(k) - s(j+k)$, where $G_1/G_0 = \widehat{\mathbb{Z}_B}$. Since G_1 is a central extension of $\widehat{\mathbb{Z}_B}$ by G_0, the group G_1 may be identified with $(G_0, \widehat{\mathbb{Z}_B})$, with $g \in G_1$ identified with $(g - s \circ p(g), p(g)), p(g)$ denoting the projection of g to $\widehat{\mathbb{Z}_B}$. The multiplication rule is given by $(a,b) \cdot (a', b') = (a + a' - \omega(b, b'), b + b')$, [228]. The above choice of the section s implies that in particular $s(0) = 0$, hence $\omega(0,g) = \omega(g,0) = 0$.

Consider the unitary operator
$$V: \xi \in \ell^2\left(G_1; \mathcal{H} \otimes \mathbb{C}^{2^{\left[\frac{p}{2}\right]}}\right) \to V\xi \in \ell^2\left(G_0 \times \frac{G_1}{G_0}; \mathcal{H} \otimes \mathbb{C}^{2^{\left[\frac{p}{2}\right]}}\right)(V\xi)(m,j)$$
$$:= \xi(m + s(j)), \qquad m \in G_0 j \in G_1/G_0\,. \tag{47}$$

Proposition (5.3.25)[221]: The representation $\pi_{G_1}: \mathcal{Z} \rtimes_\rho G_1 \to \ell^2(G_1; \mathcal{H} \otimes \mathbb{C}^{2^{[p/2]}})$ is unitarily equivalent, through, to the representation obtained by $\pi_{G_0}: \mathcal{Z} \rtimes_\rho G_0 \to \ell^2(G_0; \mathcal{H} \otimes \mathbb{C}^{2^{[p/2]}})$ according to Proposition (5.3.7).

Proof. Since $\mathcal{A}_{G_1}$is generated by $\pi_{G_1}(z), z \in \mathcal{Z}$, and $U_h, h \in G_1$, it is enough to prove the statement for the generators. Observe that, for any $z \in \mathcal{Z}, m, n \in G_0, j, k \in G_1/G_0, \eta \in \ell^2(G_0 \times G_1/G_0; \mathcal{H} \otimes \mathbb{C}^{2^{[p/2]}})$, we have

$$(V\,\pi_{G_1}(z)V^*\eta)(n,k) = (\pi_{G_1}(z)V^*\eta)(n+s(k)) = (\rho^{-1}_{n+s(k)}(z)V^*\eta)(n+s(k))$$
$$= \rho^{-1}_{n+s(k)}(z)\eta(n,k),$$
$$(VU_{m+s(j)}V^*\eta)(n,k) = (U_{m+s(j)}V^*\eta)(n+s(k) = (V^*\eta)(n-m+s(k)-s(j))$$
$$= \eta(n-m-\omega(j,k-j),k-j)\,.$$

In order to obtain the representation of these operators in $MG_1/G_0(\mathcal{B}(\ell^2(G_0;\mathcal{H} \otimes \mathbb{C}^{2^{[p/2]}}))$, choose any $\varphi, \psi \in \ell^2(G_0; \mathcal{H} \otimes \mathbb{C}^{2^{[p/2]}})$, and denote by $\{e_j\}_{j\in G_1/G_0}$ the canonical basis of $\ell^2(G_1/G_0)$, so that, for any $j,k \in G_1/G_0$, we get

$$\langle \varphi, (V\pi_{G_1}(z)V^*)_{jk}\psi\rangle = \langle \varphi \otimes e_j, V\pi_{G_1}(z)V^*(\psi \otimes e_k)\rangle$$
$$= \sum_{i\in G_1/G_0}\sum_{n\in G_0} e_j(i)e_k(i)\langle \varphi(n), \rho^{-1}_{n+s(i)}(z)\xi(n)\rangle$$
$$= \delta_{jk}\sum_{n\in G_0}\langle \varphi(n), (\pi_{G_0}(\rho^{-1}_{s(j)}(z))\xi)(n)\rangle,$$

which implies that $(V\pi_{G_1}(z)V^*)_{jk} = \delta_{jk}\pi_{G_0}(\rho^{-1}_{s(j)}(z))$; analogously, for $m \in G_0$ $\ell \in G_1/G_0$,

$$\langle \varphi, (VU_{m+s(\ell)}V^*)_{jk}\psi\rangle = \langle \varphi \otimes e_j, VU_{m+s(\ell)}V^*(\psi \otimes e_k)\rangle$$
$$= \sum_{i\in G_1/G_0}\sum_{n\in G_0} e_j(i)e_k(i-\ell)\langle \varphi(n), \psi(n-m-\omega(\ell,i-\ell)\rangle$$
$$= \delta_{k,j-\ell}\sum_{n\in G_0}\langle \varphi(n), \psi(n-m-\omega(\ell,j-\ell)\rangle,$$

which implies that $(VU_{m+s(\ell)}V^*)_{jk} = \delta_{k,j-\ell}U_{m+\omega(\ell,k)}$. On the other hand,

$$M(\pi_{G_1}(z))_{jk} = U^*_{s(j)}E_{j-k}(\pi_{G_1}(z))U_{s(k)} = \delta_{jk}U^*_{s(j)}\pi_{G_1}(z)U_{s(k)},$$

so that

$$\langle \varphi, M(\pi_{G_1}(z))_{jk}\psi\rangle = \delta_{jk}\langle \varphi \otimes e_j, U^*_{s(j)}\pi_{G_1}(z)U_{s(k)}(\psi \otimes e_k)\rangle$$
$$= \delta_{j,k}\langle \varphi \otimes e_j, \pi_{G_1}(\rho_{-s(j)}(z))(\psi \otimes e_k)\rangle$$
$$= \delta_{jk}\sum_{i\in G_1/G_0}\sum_{n\in G_0} e_j(i)e_k(i)\langle \varphi(n), \rho_n^{-1}(\rho^{-1}_{s(j)}(z))\psi(n)\rangle$$
$$= \delta_{j,k}\sum_{n\in G_0}\langle \varphi(n), (\pi_{G_0}(\rho^{-1}_{s(j)}(z))\psi)(n)\rangle,$$

which implies that $M(\pi_{G_1}(z))_{jk} = \delta_{jk}\pi_{G_1}(\rho^{-1}_{s(j)}(z))$. Finally,

$$M(U_{m+s(\ell)})_{jk} = U^*_{s(j)}E_{j-k}(U_{m+s(\ell)})U_{s(k)} = \frac{1}{r}\sum_{g\in\mathbb{Z}_B}\langle k-\;j,g\rangle\,U^*_{s(j)}\gamma_g(U_{m+s(\ell)})U_{s(k)}$$
$$= \frac{1}{r}\sum_{g\in\mathbb{Z}_B}\langle k-\;j,g\rangle\,\langle \hat{s}(g), m+s(\ell)\rangle U^*_{s(j)}U_{m+s(\ell)}U_{s(k)}$$
$$= \frac{1}{r}\sum_{g\in\mathbb{Z}_B}\langle k-j+\ell,g\rangle\,U_{m+s(\ell)+s(k)-s(j)} = \delta_{k,j-\ell}U_{m+\omega(\ell,j-\ell)},$$

which ends the proof.

Corollary (5.3.26)[221]: The following diagram commutes:

$$\begin{array}{ccc} \mathcal{A}_{G_0} & \longrightarrow & \mathcal{A}_{G_1} \\ \downarrow & \circlearrowright & \downarrow \\ \mathcal{B}(\mathcal{H}\otimes\mathbb{C}^{2^{[p/2]}}\otimes\ell^2(G_0)) & \longrightarrow & \mathcal{B}(\mathcal{H}\otimes\mathbb{C}^{2^{[p/2]}}\otimes\ell^2(G_0))\otimes M_r(\mathbb{C}) \end{array} \quad (48)$$

where vertical arrows are the representations, the elements of $\mathcal{A}_{G_1}$ being identified with matrices as in the previous Proposition, and the horizontal arrows are given by the monomorphisms $a \to M_a, (M_a)_{j,k} = \delta_{j,k} U^*_{s(j)} a U_{s(j)}$, both for $a \in \mathcal{A}_{G_0}$ and for $a \in \mathcal{B}(\mathcal{H} \otimes \ell^2(G_0))$.

So far we have defined a finite noncommutative covering. In order to obtain a self-covering, $\mathcal{B}$ has to be isomorphic to $\mathcal{A}$, and we have to make further assumptions. Suppose that there exists an automorphism $\beta \in Aut(\mathcal{Z})$ such that

$$\beta \circ \rho_{A_g} \circ \beta^{-1} = \rho_g, \quad g \in \mathbb{Z}^p; \quad (49)$$

The following result tells us that the above algebras yield a noncommutative self-covering.

Proposition (5.3.27)[221]: ([199]) Under the above hypotheses, the sub-algebra $\mathcal{A}_{G_0} = \mathcal{Z} \rtimes G_0 \subset \mathcal{A}_{G_1}$ is isomorphic to $\mathcal{A}_{G_1}$, the isomorphism being given by

$$\alpha: \sum_{g\in\mathbb{Z}^p} a_g U_{Ag} \in \mathcal{A}_{G_1} \longmapsto \sum_{g\in\mathbb{Z}^p} \beta(a_g) U_g \in \mathcal{A}_{G_0}.$$

The map α may also be seen as an endomorphism of $\mathcal{A}_{G_1}$.

As above, given an integer-valued matrix $B \in M_p(\mathbb{Z})$ we may define an endomorphism $\alpha: \mathcal{A}_{G_1} \to \mathcal{A}_{G_1}$. Then, we may describe the inductive limit $\mathcal{A}_\infty \overset{\lim}{=\!\!\longrightarrow} \mathcal{A}_n$ where $\mathcal{A}_n = \mathcal{A}_{G_n}, G_n = A^n\mathbb{Z}^p$, and the embedding is the inclusion. Endow G_n with the length function $\ell_n: G_n \to M_{2^{[p/2]}}(\mathbb{C})$ defined as $\ell_n(g) := \sum_{\mu=1}^p g_\mu \varepsilon_{\mu+1}^{(p+1)}, g = (g_1,\dots,g_p) \in G_n$ (ℓ_n is indeed a proper translation bounded matrix-valued function, [241]). Let us observe that $G_n \subset G_{n+1}$ and that $|G_n/G_{n-1}| = |detB| =: r$.

Let us define the action $\rho^{(n)}$ of G_n on $\mathcal{Z}$ as follows:

$$\rho^{(n)}_{A^n g} = \beta^{-n} \circ \rho_g \circ \beta^n, \qquad g \in G_0.$$

Lemma (5.3.28)[221]: For any $m < n, g \in G_m$, we have that $\rho^{(n)}_g = \rho^{(m)}_g$, namely the family $\{\rho^{(n)}\}_{n\in\mathbb{N}}$ defines an action ρ of $\cup_n G_n$.

Proof. From equation (49), we have

$$\rho^{(m+1)}_g = \beta^{-(m+1)} \circ \rho_{A^{-m-1}g} \circ \beta^{m+1} = \beta^{-m} \circ \rho_{A^{-m}g} \circ \beta^m = \rho^{(m)}_g, \qquad g \in G_m.$$

The thesis follows.

Suppose that

$$sup_{g\in G_n} \left\| [D, \rho^{(n)}_g(a)] \right\| < \infty,$$

for any $a \in \mathcal{L}_n := C_c(G_n, \mathcal{Z})$. Then, the algebra $\mathcal{A}_{G_n}$ has a natural spectral triple

$$(\mathcal{L}_n, \mathcal{H} \otimes \mathbb{C}^{2^{\lceil\frac{p}{2}\rceil}}$$

Theorem (5.3.29)[221]: Set $\mathcal{H}_0 := \mathcal{H} \otimes \mathbb{C}^{2^{[p/2]}} \otimes \ell^2(G_0)$. Then the Dirac operator D_n acting on $\mathcal{H} \otimes \mathbb{C}^{2^{[p/2]}} \otimes \ell^2(G_n)$ gives rise to the operator $\widehat{D}_n$ when the Hilbert space is identified with $\mathcal{H}_0 \otimes \otimes_{i=1}^n \ell^2(G_i/G_{i-1})$ as above, where $G_i/G_{i-1} \cong \widehat{\mathbb{Z}_B}$. The Dirac operator $\widehat{D}_n$ has the following form:

$$\widehat{D}_n := V_n D_n V_n^* = D_0 \otimes I^{\otimes n} + C_n,$$

with $C_n \in \mathcal{B}(\mathcal{H}_0) \otimes M_r(\mathbb{C})^{\otimes n}$ defined, for $\eta \in \ell^2(G_0 \times G_1/G_0 \times \ldots \times G_n/G_{n-1}; \mathcal{H} \otimes \mathbb{C}^{2^{[p/2]}})$, as

$$(C_n\eta)(m, j_1, \ldots, j_n) := \sum_{h=1}^{n} (I \otimes \ell_h(s_h(j_h)))\,(\eta(m, j_1, \ldots, j_n)),$$

and $V_n: \mathcal{H} \otimes \mathbb{C}^{2^{[p/2]}} \otimes \ell^2(G_n) \to \mathcal{H}_0 \otimes \otimes_{j=1}^{n} \ell^2(G_j/G_{j-1})$ given by $V_n := (v_1 \otimes \otimes_{j=1}^{n-1} I) \circ (v_2 \otimes \otimes_{j=1}^{n-2} I) \circ \cdots \circ v_n$.

Proof. For simplicity, we prove the case $n = 1$, the case $n > 1$ can be proved by iterating the procedure. For any $\eta \in \mathcal{H} \otimes \mathbb{C}^{2^{[p/2]}} \otimes \ell^2(G_0) \otimes \ell^2(G_1/G_0) \cong \ell^2(G_0 \times (G_1/G_0); \mathcal{H} \otimes \mathbb{C}^{2^{[p/2]}})$, we get, for $m \in G_0, j \in G_1/G_0$,

$$\begin{aligned}(V_1D_1V_1^*\eta)(m,j) &= (D_1V_1^*\eta)(m + s(j)) \\ &= (D \otimes \varepsilon_1^{(p+1)})(V_1^*\eta)(m + s(j)) + (I \otimes \ell_1(m + s(j)))(V_1^*\eta)(m + s(j)) \\ &= (D \otimes \varepsilon_1^{(p+1)})(\eta(m,j)) + (I \otimes \ell_1(m + s(j)))(\eta(m,j)) \\ &= (D \otimes \varepsilon_1^{(p+1)} + I \otimes \ell_1(m))(\eta(m,j)) + (I \otimes \ell_1(s(j)))(\eta(m,j)) \\ &= (D_0\eta)(m,j) + (C_1\eta)(m,j),\end{aligned}$$

where $(C_1\eta)(m,j) := (I \otimes \ell_1(s(j)))(\eta(m,j))$ belongs to $I \otimes \mathcal{B}(\mathbb{C}^{2^{[p/2]}} \otimes \ell^2(G_0 \times G_1/G_0))$. We stress that $(C_1\eta)(m,j)$ dos not depend on m because ℓ_1 is a linear map.

For any $n \in \mathbb{N}_0$ and $x \in \mathcal{L}_n$, set $L_{D_n}(x) := \|[D_n, x]\|$. An immediate consequence of the previous result is that, under a suitable assumption, these seminorms are compatible.

Corollary (5.3.30)[221]: Suppose that

$$\left\|[D_0, Ad(U_g)(x)]\right\| = \|[D_0, x]\| \qquad \forall x \in \cup_n \mathcal{L}_n, \forall g \in \cup_n G_n.$$

Then for any positive integer m, we have that

$$L_{D_{m+1}}(x) = L_{D_m}(x) \quad \forall x \in \mathcal{L}_m.$$

Proof. We give the proof for $m = 0$. As the elements in $\mathcal{A}_1$ may be seen as matrices with entries in $\mathcal{A}_0$ acting on $\ell^2(G_1/G_0; \mathcal{H}_0)$. $\mathcal{A}_0$ itself is then embedded in $\mathcal{A}_1$ as diagonal matrices, the matrix $M(x)$ associated with $x \in \mathcal{A}_0$ being $M(x)_{kk} = (\sigma(k)^* x \sigma(k))_{kk} = (U_{-s(k)} x U_{s(k)})_{kk} = (\rho_{-s(k)}(x))_{kk}$, where the action ρ has been naturally extended to $\mathcal{A}_0$. $D_0 \otimes I$ may as well be identified with the diagonal matrix $(D_0 \otimes I)_{kk} = D_0$, therefore their commutator is the diagonal matrix $([D_0, \rho_{-s(k)}(x)])_{kk}$. As for the commutator with the second term of $\widehat{D}_1$, let us describe the Hilbert space as $\ell^2(G_1/G_0; (\mathcal{H} \otimes \ell^2(G_0)) \otimes \mathbb{C}^{2^{[p/2]}})$. Then both x and C_1 act as diagonal matrices, whose entries jj are $\rho_{-s(j)}(x) \otimes I$ for the first operator and $I \otimes \ell_1(s(j))$ for the second, showing that the corresponding commutator vanishes. The thesis now follows by the assumption.

We describe the Dirac operator on $\mathcal{A}_\infty$.

Theorem (5.3.31)[221]: Assume B is purely expanding, set $\mathcal{H}_0 := \mathcal{H} \otimes \mathbb{C}^{2^{[p/2]}} \otimes \ell^2(G_0)$, $\mathcal{L} = \cup_n \mathcal{L}_n$, $\mathcal{M} = \mathcal{B}(\mathcal{H}_0) \otimes \mathcal{R}$, and define the Dirac operator $\widehat{D}_\infty$ as follows:

$$\widehat{D}_\infty := D_0 \otimes I_{UHF} + C,$$

where $C = \lim C_n$, $C_n = C_n^* \in \mathcal{B}(\mathcal{H}_0) \otimes UHF(r^\infty)$. Then $(\mathcal{L}, \mathcal{M}, \mathcal{H}_0 \otimes L^2(\mathcal{R}, \tau), \widehat{D}_\infty)$ is a finitely summable, semifinite, spectral triple, with the same Hausdorff dimension of $(\mathcal{L}_0, \mathcal{H}_0, D_0)$ (which we denote by d). Moreover, the Dixmier trace $\tau_\omega$of $(\widehat{D}_\infty^2 + 1)^{-d/2}$ coincides with that of $(D_0^2 + 1)^{-d/2}$ (hence does not depend on the generalized limit ω) and may be written as:

$$\tau_\omega((\widehat{D}_\infty^2+1)^{-d/2}) = \lim_{t\to\infty}\frac{1}{\log t}\int_0^t \left(\mu_{(D_0^2+1)^{-1/2}}(s)\right)^d ds\,.$$

Proof. The Dirac operator $\widehat{D}_\infty$ is of the form $D_0 \otimes I + C$. First of all, we prove that $\widehat{D}_\infty \,\widehat{\in}\, \mathcal{B}(\mathcal{H}_0) \otimes \mathcal{R}$ by showing that $C \in \mathcal{B}(\mathcal{H}_0) \otimes \mathcal{R}$. This claim and the formula follow from what has already been proved and the following argument.

Since we posed $s_n(k) \in A^{n-1}[0,1)^p$, by using the properties of the Clifford algebra and the linearity of ℓ_n, we get

$$\left\|\ell_n\left(\sum_{h=1}^n (s_h(j_h))\right)\right\| = \left\|\sum_{h=1}^n \ell_n(s_h(j_h))\right\| = \left\|\sum_{h=1}^n s_h(j_h)\right\|$$
$$\le \sum_{h=1}^n \|s_h(j_h))\| \le \sqrt{p}\sum_{h=1}^n \|A^{h-1}\|\,,$$

so that

$$\|C_n\| = \left\|\ell_n\left(\sum_{h=1}^n (s_h(j_h))\right)\right\| \le \sqrt{p}\sum_{h=1}^n \|A^{h-1}\|\,.$$

As $\widehat{D}_\infty = D_0 \otimes I + C$, we get, by Proposition (5.3.14) and the estimate above, that C is bounded and belongs to $\mathcal{B}(\mathcal{H}_0) \otimes UHF(r^\infty)$, while $D_0 \,\widehat{\in}\, \mathcal{B}(\mathcal{H}_0)$.

Moreover, by construction, $\mathcal{L}$ is a dense $*$-subalgebra of the C^*-algebra $\mathcal{A}_\infty \subset \mathcal{M}$.

The thesis follows from Theorem (5.3.16) and the above results.

We want to consider the C^*-algebra $UHF(r^\infty)$. This algebra is defined as the inductive limit of the following sequence of finite dimensional matrix algebras:

$$M_0 = M_r(\mathbb{C})$$
$$M_n = M_{n-1} \otimes M_r(\mathbb{C}) \qquad n \ge 1,$$

with maps $\phi_{ij}: M_j \to M_i$ given by $\phi_{ij}(a_i) = a_i \otimes 1$. We denote by $\mathcal{A}$ the $UHF(r^\infty)$ C^*-algebra and set $M_{-1} = \mathbb{C}1_{\mathcal{A}}$ in the inductive limit defining the above algebra. The C^*-algebra $\mathcal{A}$ has a unique normalized trace that we denote by τ.

Now we follow [233]. Consider the projection $P_n: L^2(\mathcal{A},\tau) \to L^2(M_n, T_r)$, where $T_r: M_r(\mathbb{C}) \to M_r(\mathbb{C})$ is the normalized trace, and define

$$Q_n = P_n - P_{n-1}, \qquad n \ge 0,$$
$$E(x) = \tau(x)1_{\mathcal{A}}\,.$$

Lemma (5.3.32)[221]: The projection $Q_n: L^2(\mathcal{A},\tau) \to L^2(M_n,\tau) \ominus L^2(M_{n-1}, Tr)$ $(n \ge 0)$ is given by

$$Q_n(x_0 \otimes \cdots \otimes x_n \otimes \cdots) = x_0 \otimes \cdots \otimes x_{n-1} \otimes [x_n - Tr(x_n)1_{M_d(\mathbb{C})}]\tau(x_{n+1} \otimes \cdots),$$

where $Tr: M_r(\mathbb{C}) \to \mathbb{C}$ is the normalized trace.

Proof. The proof follows from direct computations.

For any $s > 1$, Christensen and Ivan ([233]) defined the following spectral triple for the algebra $UHF(r^\infty) \stackrel{\text{def}}{=} \mathcal{A}$

$$(\mathcal{L}, L^2(\mathcal{A},\tau), D_0 = \sum_{n\ge 0} r^{ns} Q_n)$$

where $\mathcal{L}$ is the algebra consisting of the elements of $\mathcal{A}$ with bounded commutator with D_0. It was proved that for any such value of the parameter s, this spectral triple induces a metric which defines a topology equivalent to the weak*-topology on the state space ([233]).

Introduce the endomorphism of $\mathcal{A}$ given by the right shift, $\alpha(x) = 1 \otimes x$. Then, according to [238], we may consider the inductive limit $\mathcal{A}_\infty = \xrightarrow{\lim} \mathcal{A}_n$ with $\mathcal{A}_n = \mathcal{A}$ as described in (i). We have the following isomorphic inductive family: $\mathcal{A}_i$ is defined as

$$\mathcal{A}_0 = \mathcal{A},$$
$$\mathcal{A}_n = M_r(\mathbb{C})^{\otimes n} \otimes \mathcal{A}_0,$$
$$\mathcal{A}_\infty = \xrightarrow{\lim} \mathcal{A}_i$$

and the embedding is the inclusion.

We want to stress that this case cannot be described within the framework considered. In fact, it would be necessary to exhibit a finite abelian group that acts trivially on $1_{M_r(\mathbb{C})} \otimes \otimes_{i=1}^{\infty} M_r(\mathbb{C})$ and that has no fixed elements in $M_r(\mathbb{C}) \otimes 1_{\otimes_{t=1}^{\infty} M_r(\mathbb{C})}$. However, since all the automorphisms of $M_r(\mathbb{C})$ are inner, there cannot be any such group.

Each algebra $\mathcal{A}_p$ has a natural Dirac operator (the one considered earlier)

$$(\mathcal{L}^p, H = L^2(\mathcal{A}_p, \tau), D_p = \sum_{n \geq -p} r^{ns} Q_n),$$

where $\mathcal{L}_p$ is the algebra formed of the elements of $\mathcal{A}_p$ with bounded commutator.

We are going to describe the Dirac operator on the first covering.

Lemma (5.3.33)[221]: Let $\xi_1 \otimes \xi_\infty \in L^2(\mathcal{A}, \tau), \xi_1 \in M_r(\mathbb{C})$, we have that

$$Q_n(\xi_1 \otimes \xi_\infty) = \begin{cases} (1 \otimes Q_{n-1})(\xi_1 \otimes \xi_\infty) \; if \; n > 0 \\ (F \otimes Q_{n-1})(\xi_1 \otimes \xi_\infty) \; if \; n = 0, \end{cases}$$

where $F: M_r(\mathbb{C}) \to M_r(\mathbb{C})^\circ$ is defined by $F(x) = x - tr(x)$, and $M_r(\mathbb{C})^\circ$ are the matrices with trace 0.

Proposition (5.3.34)[221]: The following relation holds:

$$D_1 = r^{-s} F \otimes E + I \otimes D_0.$$

Proof. Let $e_{ij} \otimes x \in \mathcal{D}(D_1) \subset L^2(\mathcal{A}_1, \tau)$. We have that

$$D_1(e_{ij} \otimes x) = \sum_{n \geq -1} r^{ns} Q_n(e_{ij} \otimes x) =$$
$$= r^{-s} F e_{ij} \otimes Ex + \sum_{n \geq 0} r^{ns} e_{ij} \otimes (Q_n x) =$$
$$= [r^{-s} F \otimes E + I \otimes D_0](e_{ij} \otimes x).$$

The thesis follows by linearity.

The spectral triple on $\mathcal{A}_n$ and the inductive limit spectral triple.

We will consider the Dirac operators on $\mathcal{A}_n$ and $\mathcal{A}_\infty$**.**

Theorem (5.3.35)[221]: The Dirac operator D_n has the following form

$$D_n = I^{\otimes n} \otimes D_0 + \sum_{k=1}^{n} r^{-sk} I_r^{n-k} \otimes F \otimes E. \tag{50}$$

Proof. Let $x \in \mathcal{D}(D_n) \subset L^2(\mathcal{A}_n, \tau)$. We have that

$$D_n x = \sum_{k \geq 0} r^{(k-n)s} Q_k x = \sum_{k \geq 0} r^{(k-n)s} (I^{\otimes k} \otimes F \otimes E)x$$
$$= \sum_{k=0}^{n-1} r^{(k-n)s} (I^{\otimes k} \otimes F \otimes E)x + \sum_{k \geq n} r^{(k-n)s} (I^{\otimes k} \otimes F \otimes E)x$$

$$= \sum_{h=1}^{n} r^{-sh} \, (I^{\otimes k} \otimes F \otimes E)x + (I^{\otimes n} \otimes D_0)x.$$

Corollary (5.3.36)[221]: The Dirac operator D_∞ has the following form

$$D_\infty = I_{-\infty,-1} \otimes D_0 + \sum_{k=1}^{\infty} r^{-sh} \, I_{-\infty,-k-1} \otimes F \otimes E, \tag{51}$$

where $I_{-\infty,k}$ is the identity on the factors with indices in $[-\infty, k]$.

Theorem (5.3.37)[221]: Set $\mathcal{L} = \cup_n \mathcal{L}_n, \mathcal{M} = \mathcal{R} \otimes \mathcal{B}(L^2(\mathcal{A}_9, \tau))$. Then the triple $(\mathcal{L}, \mathcal{M}, L^2(\mathcal{R}, \tau) \otimes L^2(\mathcal{A}_9, \tau), D_\infty)$ is a finitely summable, semifinite, spectral triple, with Hausdorff dimension $\frac{2}{s}$. Moreover, the Dixmier trace τ_ω of $(D_\infty^2 + 1)^{-1/s}$ coincides with that of $(D_0^2 + \ 1)^{-1/s}$ (hence does not depend on ω) and may be written as:

$$\tau_\omega((D_\infty^2 + 1)^{-1/s}) = \lim_{t\to\infty} \frac{1}{\log t} \int_0^1 \left(\mu_{(D_0^2+1)^{-1/2}}(s)\right)^{\frac{2}{s}} ds \, .$$

Proof. By construction, $\mathcal{L}$ is a dense $*$-subalgebra of the C^*-algebra $\mathcal{A}_\infty \subset \mathcal{M}$. Since $D_\infty = I_{-\infty,-1} \otimes D_0 + C$, where $C \in \mathcal{R} \otimes I, D_\infty$ is affiliated to $\mathcal{M}$.

The thesis follows from Theorem (5.3.21) and what we have seen above.

First of all, we recall some definitions. Let $(\mathcal{L}, H, D)$ be a spectral triple over a unital C^*-algebra $\mathcal{A}$. Then we can define the following pseudometric on the state space

$$\rho_D(\phi, \psi) = sup\{|\phi(x) - \psi(x)| : x \in \mathcal{A}, L_D(x) \le 1\}, \qquad \phi, \psi \in S(\mathcal{A}),$$

where $L_D(x)$ is the seminorm $\|[D, x]\|$.

We have the following result proved by Rieffel.

Theorem (5.3.38)[221]: ([86]) The pseudo-metric ρ_D induces a topology equivalent to the weak*-topology if and only if the ball

$$B_{L_D} := \{x \in \mathcal{A} : L_D(x) \le 1\}.$$

is totally bounded in the quotient space $\mathcal{A}/\mathbb{C}1$

If the above condition is satisfied, the seminorm L_D is said a Lip-norm on $\mathcal{A}$.

In our examples we determined a semifinite spectral triple on $\mathcal{A}_\infty$. Our aim is to prove that the seminorm $L_{\widetilde{D}_\infty}$, restricted to $\mathcal{A}_n$, is a Lip-norm equivalent to L_{D_n}, for any n, while it is not a Lip-norm on the whole inductive limit A_∞. Therefore, the pair $(\mathcal{A}_\infty, L_{\widehat{D}_\infty})$ is not a quantum compact metric space, whilst $\mathcal{A}_\infty$ is topologically compact (i.e. it is a unital C^*-algebra).

Theorem (5.3.39)[221]: Consider the Dirac operators $\widehat{D}_\infty$ determined. Then the sequence of the normic radii of the balls $B_{L_{D_n}}$ diverges. In particular, the seminorm $L_{\widehat{D}_\infty}$ on the inductive limit is not Lipschitz.

Proof. Our aim is to show that $B_{L_{\widehat{D}_\infty}}$ is unbounded. Actually, we will exhibit a sequence in $B_{L_{\widehat{D}_\infty}}$ with constant seminorm and diverging quotient norm, which means that it is an unbounded set in $\overset{\lim}{\longrightarrow} \mathcal{A}_k/\mathbb{C}$.

In the first place we consider the cases of the commutative and noncommutative torus. The noncommutative rational torus has centre isomorphic to the algebra of continuous functions on the torus. Thus, it is enough to exhibit a sequence only in the case of the torus.

Consider the following sequence

$$x_k = e^{2\pi i}(A^k e_1, t)$$

where $A := (B^T)^{-1}$.

Each $x_k \in C(\mathbb{T}_k) \subset lim_i C(\mathbb{T}_i)$. We have that

$$[D_k, x_k] = \sum_a \varepsilon^a [\partial_a, x_k] \leq \sum_a [\partial_a, x_k]$$
$$= \sum_a \partial_a(x_k) = \sum_a 2\pi i (A^k e_1, e_a) x_k$$
$$\leq 2p\pi \|A^k e_1\| \leq 2p\pi \|A^k\| \to 0$$

where we used Proposition (5.3.14).

Consider the sequence $y_k := x_k / \|[D_k, x_k]\|$. This sequence has constant seminorm $L_{\widehat{D}_\infty} \equiv L_{D_k}$. Since each element x_k has spectrum $\mathbb{T}$, then the quotient norm of x_k is equal to $\|x_k\|$ and thus the sequence $\{y_k\}$ is unbounded.

We now consider the case of the crossed products. With the same notations as above, consider the following sequence

$$x_k = U_{A^k e_1}$$

Each $x_k \in \mathcal{A}_k \subset \overset{\lim}{\longrightarrow} \mathcal{A}_i$. We have that

$$\|[D_k, x_k]\| = \|[M_{\ell_k}, U_{A^k e_1}]\| \leq sup_g |\ell_k(g) - \ell_k(g - A^k e_1)| \leq \|A^k e_1\| \leq \|A^k\| \to 0.$$

Since $sp(x_k) = \mathbb{T}$, again the sequence $y_k := x_k / \|[D_k, x_k]\|$ has constant seminorm $L_{\widehat{D}_\infty} \equiv L_{D_k}$ and increasing quotient norm.

Finally we take care of the UHF-algebra. Consider any matrix $b \in (M_r(\mathbb{C}) \setminus \mathbb{C}I) \subset UHF(r^\infty)$. We define the following sequence

$$x_n = I_{[-\infty, -n-1]} \otimes b \otimes I_{[-n+1, +\infty]},$$

where with the above symbol we mean that the matrix b is in the position $-n$ inside an infinite bilateral product where each factor is labelled by an integer.

A quick computation shows that

$$[Q_k, x_n] = \begin{cases} 0 & if\ k > -n \\ id_{-\infty, k-1} \otimes (bTr(\cdot) - Tr(b\,\cdot)) \otimes \tau & if\ k = -n \\ id_{-\infty, k-1} \otimes F \otimes \left(\otimes_{i=k+1}^{-n-1} Tr(\cdot)\right) \otimes (Tr(b\,\cdot) - bTr(\cdot)) \otimes \tau & if\ k < -n. \end{cases}$$

This means that $[D_\infty, x_n] = \sum_{k \leq -n} r^{ks} [Q_k, x_n]$.

We observe that each x_n has non-zero seminorm. In fact,

$$\|[D_\infty, x_n]\| = sup_{\|\xi\|=1} \|[D_\infty, x_n]\xi\|$$
$$\geq \|[D_\infty, x_n] x_n^*\|$$
$$= r^{-ns} \|Tr(bb^*) - bTr(b^*)\| > 0$$

where in the last line we used that $[Q_k, x_n] x_n^* = 0$ for all $k \neq -n$. Moreover, we have that

$$\|[D_\infty, x_n]\| \leq 2\|b\| \left(\sum_{k \leq -n} r^{ks} \right) = 2\|b\| \frac{r^{s-ns}}{1 - r^s}$$

which tends to zero as n goes to infinity.

The sequence $y_k := x_k / \|[D_\infty, x_k]\|$ has bounded seminorm $L_{\widehat{D}_\infty}$ and increasing quotient norm.

We end this proof with an explanation of the second part of the statement of this Theorem. First of all, we observe that if the sequence of the normic radii of the balls $B_{L_{D_n}}$ diverges, then $B_{L_{\widehat{D}_\infty}}$ contains an unbounded subset with unbounded quotient norm. Therefore, since a compact subset is bounded, the ball $B_{L_{\widehat{D}_\infty}}$ cannot be compact.

In ([200]) Latremoliere and Packer studied the metric structure of noncommutative solenoids, namely of the inductive limits of quantum tori. In particular, they considered noncommutative tori as quantum compact metric spaces and proved that their inductive limits, seen as quantum compact metric spaces, are also limits in the sense of Gromov-Hausdorff propinquity (hence quantum Gromov-Hausdorff) of the inductive families. In our setting the inductive limit of the quantum tori is no longer a quantum compact metric space. The different result is a consequence of the different metric structure considered. Latremoliere and Packer described the inductive limit as a twisted group C^*-algebra on which there is an ergodic action of $G_\infty := \overset{\lim}{\longleftarrow} \mathbb{T}$, and according to Rieffel ([86]) a continuous length function on G_∞ gives rise to a Lip-seminorm. In our setting the seminorm may also be described in the same way, however the corresponding length function is unbounded, thus not continuous. We give an explicit description of this situation in a particular example.

Example (5.3.40)[221]: Consider the two-dimensional rational rotation algebra A_θ, with $\theta = 1/3$. With the former notation, set

$$B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

and define the morphism $\alpha: A_\theta \to A_\theta$ by $\alpha(U) = U^2, \alpha(V) = V^2$. Now we may consider the inductive limit $\overset{\lim}{\longrightarrow} \mathcal{B}_n$ where $\mathcal{B}_n = A_\theta$ (see (0.1)). We observe that this case also fits in the setting of Latremoliere and Packer (see [200]). Then, there exists a length function that induces the seminorm $L_{\widehat{D}_\infty}$.

Proof. Consider the standard length function on the circle $l(e^{2\pi i t}) := |t|$ for $t \in (-1/2, 1/2]$. There is an induced length function on $\mathbb{T}^2$, namely $\ell_0(z_1, z_2) := max\{l(z_1), l(z_2)\}$. We define the following length function $\ell(g) := sup_n\, 2^n \ell_0(g_n)$ on the direct product $\prod \mathbb{T}^2$, thus by restriction also on the projective limit $G_\infty := \overset{\lim}{\longleftarrow} \mathbb{T}$ (with respect to the projection $\pi \equiv \alpha^*: \mathbb{T}^2 \to \mathbb{T}^2, \pi((z_1, z_2)) = (2z_1, 2z_2))$. For any $\phi \in \mathbb{R}^2$ we define the following action on A_θ: $\tilde{\rho}_\phi(f)(t) := f(t + 3\phi)$. Since $\theta = 1/3, \tilde{\rho}$ is the identity on A_θ when $\phi \in \mathbb{Z}^2$, hence there is an induced action of $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ on A_θ. We denote this action with ρ. There is a naturally induced action ρ^∞ of the group $\prod_{i=0}^{\infty} \mathbb{T}^2$ on $\prod_{i=0}^{\infty} A_\theta$ given by $\rho_g^\infty(f_0, f_1, \ldots) := (\rho_{g0}(f_0), \rho_{g1}(f_1), \ldots)$ for any $g \in \prod_{i=0}^{\infty} \mathbb{T}^2$ and any $(f_0, f_1, \ldots) \in \prod_{i=0}^{\infty} A_\theta$. We now check that the restriction of this action to G_∞ gives rise to an action on $\overset{\lim}{\longrightarrow} A_\theta$. It is enough to prove the claim on the algebraic inductive limit alg$-\overset{\lim}{\longrightarrow} A_\theta$. Let $(f_0, f_1, \ldots) \in alg \overset{\lim}{\longrightarrow} A_\theta$. By definition there exists $n \in \mathbb{N}$ such that $f_{n+i} = \alpha^i(f_n)$ for all $i \in \mathbb{N}$. For any $g \in G_\infty$, we have that

$$\rho_{g_{n+i}}(f_{n+i}) = \rho_{g_{n+i}}(\alpha^i(f_n)) = \alpha^i(\rho_{\pi^n(g_{n+i})}(f_n)) = \alpha^i(\rho_{g_n}(f_n)).$$

For any $g \in G_\infty$ and any $X \in \overset{\lim}{\longrightarrow} A_\theta$ we define the following seminorm

$$L_{\rho^\infty, \ell}(X) := sup_{g \in G_\infty} \frac{\left\|\rho_g^\infty(X) - X\right\|}{\ell(g)}.$$

Any element $f_n \in \mathcal{B}_n$ embeds into $\overset{\lim}{\longrightarrow} A_\theta$ as $X = (\underbrace{0, \ldots, 0}_{n}, f_n, \alpha(f_n), \alpha^2(f_n) \ldots)$.

We have that

$$L_{\rho^\infty, \ell}(X) = sup_{g \in G_\infty} \frac{\left\|\rho_g^\infty(X) - X\right\|}{\ell(g)}$$

$$= sup_{g\in G_\infty} \limsup_i \frac{\|\alpha^i(f_n)(z+3g_{n+i})-\alpha^i(f)(z)\|}{\ell(g)}$$
$$= sup_{g\in G_\infty} \frac{f_n(z+3g_n)-f_n(z)}{\ell_0(g_n)}\frac{\ell_0(g_n)}{\ell(g)}$$
$$= \left(sup_{g_n}\frac{\|f_n(z+3g_n)-f_n(z)\|}{\ell_0(g_n)}\right)\left(sup_{g\in G_\infty}\frac{\ell_0(g_n)}{\ell(g)}\right)$$
$$= \frac{L_0(f)}{2^n},$$

where the last two equalities hold because, for any $g_n \in \mathbb{T}^2$, we may find a sequence $g = \{g_i\}$ such that $\ell(g) = 2^n\ell_0(g_n)$ (if $g_n = e^{2\pi it}$ for $t \in (-1/2, 1/2]$ consider $g_{n+k} = e^{2\pi it/2^k}$) and L_0 is the Lipschitz seminorm $sup_{h\in\mathbb{T}^2}\frac{\|f(z+h)-f(z)\|}{\ell_0(h)}$, which is equivalent to L_{D_0} (see [86]). Denote by $\varphi_n: \mathcal{B}_n = A_\theta \to \mathcal{A}_n$ the natural isomorphism given by $\varphi_n(W(m,t)) := e^{2\pi i\theta(2^{-n}m,t)}W_0(2^n m)$ (cf. (43)) and consider the following seminorm on $\mathcal{B}_n$: $L_n(x) := L_D(\varphi_n(x)) = \|[D_n, \varphi_n(x)]\|$. Since the seminorm L_D is expressed in terms of the norm of some linear combinations of the two derivatives, one has that $L_n(x) = 2^{-n}L_0(x)$. Therefore, the former computation leads to $L_{\rho\,\infty,\ell} = L_n$, when restricted to $\mathcal{B}_n$.

Let $(\mathcal{M},\tau)$ be a von Neumann algebra with a f.n.s. trace, $T \,\widehat{\in}\, \mathcal{M}$ a self-adjoint operator. We use the notation $e_T(\Omega)$ for the spectral projection of T relative to the measurable set $\Omega \subset \mathbb{R}$, and $\lambda_T(t) := \tau(e_{|T|}[t, +\infty)), \mu_T(t) := \inf\{s: \lambda_T(s) \le t\}$, for a τ-compact operator T.

Lemma (5.3.41)[221]: Let $(\mathcal{M},\tau)$ be a von Neumann algebra with a f.n.s. trace, $T \,\widehat{\in}\, \mathcal{M}$ a self-adjoint operator, such that $\Lambda_T(s) := \tau(e_T(-s,s)) < \infty$ for any $s > 0$. Then

(i) $\Lambda_T(s) = \sup\{\tau(e): \|Te\| < s, e \in Proj(\mathcal{M})\}, s > 0$,

(ii) if $C \in \mathcal{M}_{sa}$, and $c := \|C\|$, then $\tau(e_T + C(-s,s)) < \infty$ for any $s \ge 0$, and $\Lambda_{T+C}(s) \le \Lambda_T(s+c)$,

(iii) if $e_T(\{0\}) = 0, T^{-1}$ is τ-compact and $\Lambda_T(s) = \lambda_{|T|^{-1}}(s^{-1}), s > 0$.

Proof. (i) Indeed,
$$a := \tau(e_T(-s,s)) = \sup\{\tau(e_T(-\sigma,\sigma)): 0 \le \sigma < s\} \le \sup\{\tau(e): \|Te\| < s\}.$$
Assume, by contradiction, there is $e \in Proj(\mathcal{M})$ such that $\tau(e) > a$ and $\|Te\| < s$. For $\xi \in e\mathcal{H} \cap e_{|T|}[s, \infty)\mathcal{H}, \|\xi\| = 1$, we have $(\xi, T^*T\xi) < s^2$ and $(\xi, T^*T\xi) \le s^2$, namely $e \wedge e_{|T|}[s, \infty) = \{0\}$. As a consequence,
$$e_{|T|}[s, \infty) = e_{|T|}[s, \infty) - e \wedge e_{|T|}[s, \infty) \sim e \vee e_{|T|}[s, \infty) - e \le I - e$$
where $\sim$ stands for Murray - von Neumann equivalence. Passing to the orthogonal complements we get $a = \tau(e_T(-s,s)) \ge \tau(e) > a$, which is absurd.

(ii) Set $\Omega_{T,s} = \{e \in Proj(\mathcal{M}): \|Te\| < s\}$; since $\|Te\| \le \|(T + C)e\| + c$, we have that $\Omega_{T+C,s} \subseteq \Omega_{T,s+c}$, . The thesis follows from (i).

(iii) A straightforward computation shows that $e_{|T|^{-1}}(s, +\infty) = e_T(-1/s, 1/s)$. Therefore T^{-1} is τ-compact [239] and the equality follows.

Lemma (5.3.42)[221]: Let $(\mathcal{M},\tau)$ be a von Neumann algebra with a f.n.s. trace, $T \,\widehat{\in}\, \mathcal{M}$ a positive self-adjoint operator T, with τ-compact resolvent, $d, t > 0$. Then, the following are equivalent

(i) exists $res_{s=d}\, \tau(T^{-s}e_T[t, +\infty)) = \alpha \in \mathbb{R}$,

(ii) exists $res_{s=d}\, \tau((T^2+1)^{-s/2}) = \alpha \in \mathbb{R}$.

Proof. Let us first observe that

$$\tau(T^{-s}e_T[t,+\infty)) = \int_t^{\infty} \lambda^{-s} d\tau(e_T(0,\lambda)), \tag{52}$$

$$\tau\left((T^2+1)^{-\frac{s}{2}}\right) = \int_0^{\infty} (\lambda^2+1)^{-\frac{s}{2}} d\tau\big(e_T(0,\lambda)\big), \tag{53}$$

and

$$(t^2+1)^{-s/2} \le (\lambda^2+1)^{-\frac{s}{2}} \le 1, \qquad \forall \lambda \in [0,t],$$

$$t^s(1+t^2)^{-s/2}\lambda^{-s} \le (\lambda^2+1)^{-\frac{s}{2}} \le \lambda^{-s}, \qquad \forall \lambda \in [t,+\infty),$$

therefore the finiteness of any of the two residues in the statement implies the finiteness of the two integrals (52), (53) above for any $s > d$. Then,

$$\begin{aligned} &|\tau(T^{-s}e_T[t,+\infty)) - \tau((T^2+1)^{-s/2})| \\ &\qquad = \left| \int_t^{\infty} \lambda^{-s} d\tau(e_T(0,\lambda)) - \int_0^{\infty} (\lambda^2+1)^{-s/2} d\tau(e_T(0,\lambda)) \right| \\ &\qquad \le \int_0^t (\lambda^2+1)^{-\frac{s}{2}} d\tau\big(e_T(0,\lambda)\big) + \frac{s}{2}\int_t^{\infty} \lambda^{-s-2} d\tau(e_T(0,\lambda)), \end{aligned}$$

where the inequality follows by

$$\lambda^{-s} - (\lambda^2+1)^{-s/2} = \lambda^{-s}[1-(1+\frac{1}{\lambda^2})^{-s/2}] \le \frac{s}{2}\lambda^{-s-2},$$

which, in turn, follows by

$$g(x) = 1-(1+x)^{-s/2} \le sup_{\xi\in[0,x]} g'(\xi)x = \frac{s}{2}x, for\ x \ge 0$$

Finally, taking the limit for $s \to d^+$, we get

$$\lim_{s\to d^+} |\tau(T^{-s}e_T[t,+\infty)) - \tau((T^2+1)^{-s/2})| \le \tau(e_T(0,t)) + \frac{d}{2}\int_t^{\infty} \lambda^{-(d+2)} d\tau(e_T(0,\lambda)) < \infty,$$

where the last integral is (52) with $s = d+2$, hence is finite, and we have proven the thesis.

Lemma (5.3.43)[221]: Let $(\mathcal{M},\tau)$ be a von Neumann algebra with a f.s.n. trace, T a self-adjoint operator affiliated with M with bounded compact inverse, $C \in \mathcal{M}_{sa}$ such that $T + C$ has bounded inverse. Then, the following are equivalent

(i) exists $res_{s=d}\, \tau(|T\,|^{-s}) = \alpha \in \mathbb{R}$,

(ii) exists $res_{s=d}\tau(|T+C|^{-s}) = \alpha \in \mathbb{R}$.

Proof. It is enough to prove that $(i) \Longrightarrow (ii)$. Set $c := \|C\|$. From Lemma A.1, we get $\Lambda_{T+C}(s) \le \Lambda_T(s+c)$ for every $s > 0$, hence $\lambda_{|T+C|^{-1}}(s) \le \lambda_{|T|^{-1}}\left(\frac{s}{1+cs}\right)$.

Then, for $0 < \vartheta < 1$,

$$\begin{aligned} \mu_{|T+C|^{-1}}(t) &= \inf\{s \ge 0 : \lambda_{|T+C|^{-1}}(s) \le t\} \\ &\le \inf\{s \ge 0 : \lambda_{|T|^{-1}}\left(\frac{s}{1+cs}\right) \le t\} \\ &= \inf\{\frac{h}{1-ch} \ge 0 : \lambda_{|T|^{-1}}(h) \le t\} \\ &= \inf\{\frac{h}{1-ch} : 0 \le h < c^{-1}, \lambda_{|T|-1}(h) \le t\} \\ &\le \inf\{\frac{h}{1-ch} : 0 \le h \le \vartheta c^{-1}, \lambda_{|T|^{-1}}(h) \le t\} \\ &\le (1-\vartheta)^{-1}\inf\{h : 0 \le h \le \vartheta c^{-1}, \lambda_{|T|^{-1}}(h) \le t\} \end{aligned}$$

$$= \begin{cases} (1-\vartheta)^{-1} \inf\{h \geq 0 : \lambda_{|T|^{-1}}(h) \leq t\}, & if\ \lambda_{|T|^{-1}}(c^{-1}\vartheta) \leq t, \\ +\infty, & otherwise, \end{cases}$$
$$= \begin{cases} (1-\vartheta)^{-1}\mu_{|T|^{-1}}(t), & if\ \lambda_{|T|^{-1}}(c^{-1}\vartheta) \leq t, \\ +\infty, & otherwise. \end{cases}$$
As a consequence,
$$\tau(|T+C|^{-s}) = \int_0^\infty \mu_{|T+C|^{-1}}(t)^s dt$$
$$\leq \int_0^{\lambda_{|T|^{-1}}(c^{-1}\vartheta)} \mu_{|T+C|^{-1}}(t)^s dt + \int_{\lambda_{|T|^{-1}}(c^{-1}\vartheta)}^{+\infty} (1-\vartheta)^{-s}\mu_{|T|^{-1}}(t)^s dt$$
$$= \int_0^{\lambda_{|T|^{-1}}(c^{-1}\vartheta)} (\mu_{|T+C|^{-1}}(t)^s - (1-\vartheta)^{-s}\mu_{|T|^{-1}}(t)^s)dt + (1-\vartheta)^{-1}\tau(|T|^{-s})$$
$$\leq (\|(T+C)^{-1}\|^s + (1-\vartheta)^{-s}\|T^{-1}\|^s)\lambda_{T^{-1}}(c^{-1}\vartheta) + (1-\vartheta)^{-s}\tau(|T|^{-s}) < \infty.$$
Passing to the residues, we get $\lim sup_{s\to d^+}(s-d)\tau(|T+C|^{-s}) \leq (1-\vartheta)^{-d} res_{s=d}\tau(|T|^{-s})$, hence, by the arbitrariness of ϑ, $\lim sup_{s\to d^+}(s-p)\tau(|T+C|^{-s}) \leq res_{s=d}\tau(|T|^{-s})$.
Exchanging T with $T+C$ we get $res_{s=d}\tau(|T|^{-s}) \leq \lim sup_{s\to d^+}(s-p)\tau(|T+C|^{-s})$, hence the thesis.
Proposition (5.3.44)[221]: Let $(\mathcal{M},\tau)$ be a von Neumann algebra with a f.n.s. trace, T a self-adjoint operator affiliated with $\mathcal{M}$ with compact resolvent, $C \in \mathcal{M}_{sa}$.
Then, the following are equivalent
(i) exists $res_{s=d}\tau((T^2+1)^{-s/2}) = \alpha \in \mathbb{R}$,
(ii) exists $res_{s=d}\tau((T+C)^2+1)^{-s/2}) = \alpha \in \mathbb{R}$.
In particular, the abscissas of convergence coincide.
Proof. By Lemma (5.3.42), the thesis may be rewritten as
$$\exists\ res_{s=d}\tau\left(|T|^{-s}e_{|T|}[t,+\infty)\right) = \alpha \in \mathbb{R} \iff \exists\ res_{s=d}\tau\left(|T+C|^{-s}e_{|T+C|}[t,+\infty)\right) = \alpha.$$
Since the operator
$$C' := (T+C)e_{|T+C|}[t,+\infty) - Te_{|T|}[t,+\infty)$$
$$= (T+C)e_{|T+C|}[0,+\infty) - Te_{|T|}[0,+\infty) - (T+C)e_{|T+C|}[0,t) + Te_{|T|}[0,t)$$
$$= C - (T+C)e_{|T+C|}[0,t) + Te_{|T|}[0,t)$$
is bounded and self-adjoint, we may apply Lemma (5.3.43) to the operators $(T+C)e_{|T+C|}[t,+\infty)$ and $Te_{|T|}[t,+\infty)$, proving the Proposition.

Chapter 6
Continuous Derivations and Derivations with Values

We study the set of all derivations from I into J. We show that any such derivation is automatically continuous and there exists an operator $a \in J : I$ such that $\delta(\cdot) = [a,\cdot]$, moreover $\|a + \alpha\mathbb{1}\|_{B(H)} \leq \|\delta\|_{I\to J} \leq 2C\|a\|_{J:I}$ for some complex number α, where C is the modulus of concavity of the quasi-norm $\|\cdot\|_J$ and $\mathbb{1}$ is the identity operator on H. In the special case, when $I = J = K(H)$ is a symmetric Banach ideal of compact operators on H our result yields the classical fact that any derivation δ on $K(H)$ may be written as $\delta(\cdot) = [a,\cdot]$, where a is some bounded operator on H and $\|a\|_{B(H)} \leq \|\delta\|_{I\to I} \leq 2\|a\|_{B(H)}$. We show that every derivation on $LS(M)$ is inner provided that M is a properly infinite von Neumann algebra. Furthermore, any derivation on an arbitrary von Neumann algebra M with values in a Banach M-bimodule of locally measurable operators is inner. In the case when M is a semifinite non-finite factor, we show that our assumptions on $E(0,\infty)$ are sharp.

Section (6.1): Symmetric Quasi-Banach Ideals of Compact Operators

For I , J be ideals of compact operators on an infinite-dimensional complex Hilbert space H. Obviously, J is an I-module and we can consider the set $\mathrm{Der}(I,J)$ of all derivations $\delta : I \to J$. Consider two closely related questions (here, $\mathcal{B}(\mathrm{H})$ is the set of all bounded linear operators on H):

Question (6.1.1)[251]: Let $I \in \mathrm{Der}(I,J)$. Does there exist a bounded operator $a \in \mathcal{B}(\mathrm{H})$ such that $\delta(x) = [a,x]$ for every $x \in I$?

Question (6.1.2)[251]: What is the set $\mathrm{D}(I,J) = \{a \in \mathcal{B}(\mathrm{H}): [a,x] \in J, \forall x \in I\}$?

The second question was completely answered by Hoffman in [252], who also coined the term J-essential commutant of I for the set $D(I,J)$. We completely answer the first question in the setting when the ideals I,J are symmetric quasi-Banach. In this setting, it is also natural to ask.

Question (6.1.3)[251]: Let $\delta \in Der(I,J)$. Is it continuous?

Of course, if $\delta \in Der(I,J)$ is such that $\delta(x) = [a,x]$ for some $a \in \mathcal{B}(H)$ (that is when δ is implemented by the operator a), then δ is a continuous mapping from $(I, \|\cdot\|_I)$ to $(J, \|\cdot\|_J)$, that is a positive answer to Question (6.1.1) implies also a positive answer to Question (6.1.3). However, we are establishing a positive answer to Question (6.1.1) via firstly answering Question (6.1.3) in positive. Both these results (Theorems (6.1.12) and (6.1.13)). We also provide a detailed discussion of the J-essential commutant of I.

It is also instructive to outline a connection between Questions (6.1.1) and (6.1.3) with some classical results. It is well known [29] that every derivation on a C^*-algebra is norm continuous. In fact, this also easily follows from the following well-known fact [29] that every derivation on a C^*-algebra $M \subset \mathcal{B}(H)$ is given by a reduction of an inner derivation on a von Neumann algebra $\bar{M}^{wo}$ (the weak closure of M in the C^*-algebra $\mathcal{B}(H)$). The latter result [29], in the setting when M is a C^*-algebra $K(H)$ of all compact operators on H states that for every derivation δ on M there exists an operator $a \in \mathcal{B}(H)$ such that $\delta(x) = [a,x]$ for every $x \in K(H)$, in addition, $\| a \|_{\mathcal{B}(H)} \leq \| \delta \|_{M\to M}$. The ideal $K(H)$ equipped with the uniform norm is an element from the class of so-called symmetric Banach operator ideals in $\mathcal{B}(H)$ and evidently this example also suggests the statements of Questions (6.1.1) and (6.1.3). In the case of Schatten ideals $C_p(H) = \{x \in K(H) : \| x \|_p = tr(|x|^p)^{\frac{1}{p}} < \infty\}$,

where $|x| = (x^*x)^{\frac{1}{2}}, 1 \leq p < \infty$, somewhat similar problems concerning derivations from $C_p(H)$ into $C_r(H)$ were also considered in the work by Kissin and Shulman [253]. In particular, it is shown in [253] that every closed $*$-derivation δ from $C_p(H)$ into $C_r(H)$ is implemented by a symmetric operator S, in addition the domain $D(\delta)$ of δ is dense $*$-subalgebra in $C_p(H)$. In our case, we have $D(\delta) = C_p$ and it follows from our results that the derivation δ is necessarily continuous and implemented by an operator $a \in \mathcal{B}(H)$.
It is also worth to mention that Hoffman's results in [252] were an extension of earlier results by Calkin [254] who considered the case when $I = \mathcal{B}(H)$. Recently, Calkin's and Hoffman's results were extended to the setting of general von Neumann algebras in [5,6] and, in the special setting when $I = J$, Questions (6.1.1) and (6.1.3) were also discussed in [257]. However, our methods are quite different from all the approaches applied in [252], [253], [254], [255], [256].
As a corollary of solving Questions (6.1.1) and (6.1.3), in Theorem (6.1.17) we present a description of all derivations δ acting from a symmetric quasi-Banach ideal I into a symmetric quasi-Banach ideal J. Indeed, every such derivation δ is an inner derivation $\delta(\cdot) = \delta a(\cdot) = [a,\cdot]$, where a is some operator from J-dual space $J : I$ of I. Recall that $D(I,J) = J : I + \mathbb{C}1$[252], where 1 is the identity operator in $\mathcal{B}(H)$. Theorem (6.1.17) gives a complete answer to Question (6.1.2). In particular, using the equality $C_r: C_p = C_q, 0 < r < p < \infty, \frac{1}{q} = \frac{1}{r} - \frac{1}{p}$, we recover Hoffman's result that any derivation $\delta: C_p \to C_r$ has a form $\delta = \delta_a$ for some $a \in C_q$. If $0 < p \leq r < \infty$, then $D(C_p, C_r) = \mathcal{B}(H)$.
When I, J are arbitrary symmetric quasi-Banach ideals of compact operators and $I \subseteq J$, then $J: I = \mathcal{B}(H)$, and, in this case, a linear operator $\delta: I \to J$ is a derivation if and only if $\delta = \delta_a$ for some $a \in \mathcal{B}(H)$. However, if $I \nsubseteq J$, then to obtain a complete description of J-essential commutant of I we need a procedure of finding $J: I$.
We use the classical Calkin's correspondence between two-sided ideals I of compact operators and rearrangement invariant solid sequence subspaces E_I of the space c_0 of null sequences. The meaning of this correspondence is the following. Take a compact operator $x \in I$ and consider a sequence of eigenvalues $\{\lambda_n(x)\}_{n=1}^{\infty} \in c_0$. For each sequence $\xi = \{\xi_n\} \in c_0$, let $\xi^* = \{\xi_n^*\}_{n=1}^{\infty}$ denote a decreasing rearrangement of the sequence $|\xi| = \{|\xi_n|\}_{n=1}^{\infty}$. The set
$E_I := \{\{\xi_n\}_{n=1}^{\infty} \in c_0 : \{\xi_n^*\}_{n=1}^{*} = \{\lambda_n^*(|x|)\}_{n=1}^{\infty}$ for some $x \in I\}$,
is a solid linear subspace in the Banach lattice c_0. In addition, the space E_I is rearrangement invariant, that is if $\eta \in c_0, \xi \in E_I, \eta * = \xi^*$, then $\eta \in E_I$. Conversely, if E is a rearrangement invariant solid sequence subspace in c_0, then

$$C_E = \{x \in K(H) : \{\lambda_n(|x|)\}_{n=1}^{\infty} \in E\}$$

is a two-sided ideal of compact operators from $\mathcal{B}(H)$.
For the proof of the following theorem see Calkin's original, [254], and to Simon's book, [258].
Theorem (6.1.4)[251]: The correspondence $I \leftrightarrow E_I$ is a bijection between rearrangement invariant solid spaces in c_0 and two-sided ideals of compact operators.

In [111] this correspondence has been extended to symmetric quasi-Banach (Banach) ideals and p-convex symmetric quasi-Banach (Banach) sequence spaces. We use the notation $\|\cdot\|_{\mathcal{B}(H)}$ and $\|\cdot\|_{\infty}$ to denote the uniform norm on $\mathcal{B}(H)$ and on l_{∞} respectively.

Recall, that a two-sided ideal I of compact operators from $B(H)$ is said to be symmetric quasi-Banach (Banach) ideal if it is equipped with a complete quasi-norm (respectively, norm) $\|\cdot\|_I$ such that

$$\| axb \|_I \leq \| a \|_{\mathcal{B}(H)} \| x \|_I \| b \|_{\mathcal{B}(H)}, x \in I, a, b \in \mathcal{B}(H).$$

A symmetric quasi-Banach (Banach) sequence space $E \subset c_0$ is a rearrangement invariant solid sequence space equipped with a complete quasi-norm (respectively, norm) $\|\cdot\|_E$ such that $\| \eta \|_E \leq \| \xi \|_E$ for every $\xi \in E$ and $\eta \in c_0$ such that $\eta^* \leq \xi^*$.

It is clear that if $(I, \|\cdot\|_I)$ is a symmetric quasi-Banach ideal of compact operators, $x \in I$ and $y \in K(H)$ is such that $\{\lambda_n^* (|y|)\}_{n=1}^{\infty} \leq \{\lambda_n^*(|x|)\}_{n=1}^{\infty}$, then $y \in I$ and $\| y \|_I \leq \| x \|_I$. In Theorem (6.1.22) we show that if E_I is a rearrangement invariant solid space in c_0 corresponding to symmetric quasi-Banach ideal I, then setting $\| \xi \|_{E_I} := \| x \|_I$ (where $x \in I$ is such that $\xi^* = \{\lambda_n^* (|x|)\}_{n=1}^{\infty}$) we obtain that $(E_I, \|\cdot\|_{E_I})$ is a symmetric quasi-Banach sequence space. The converse implication is much harder [111].

Theorem (6.1.5)[251]: If $(E, \|\cdot\|_E)$ is a symmetric Banach (respectively, p-convex symmetric quasi-Banach) sequence space in c_0, then C_E equipped with the norm

$$\| x \|_{C_E} := \| \{\lambda_n^*(|x|)\}_{n=1}^{\infty} \|_E$$

is a symmetric Banach (respectively, p-convex quasi-Banach) ideal of compact operators from $\mathcal{B}(H)$.

In [259] it was shown that for $J = C_1$ is the trace class and an arbitrary two-sided ideal I with $C_1 \subset I \subset K(H)$ the C_1-dual space (also sometimes called the Köthe dual) $I^{\times} := C_1 : I$ of I is precisely an ideal corresponding to symmetric sequence space $l_1 : E_I$, where $l_1 : E_I$ is l_1-dual space of E_I. If I is a symmetric Banach ideal of compact operators, then C_1-dual space $I^{\times}$ is symmetric Banach ideal of compact operator and norms on $C_1 : I$ and $C_{l_1 : E_I}$ are equal [17]. We extend these results to arbitrary symmetric quasi-Banach ideals I, J of compact operators with $I \nsubseteq J$, that allows to describe completely all derivations from one symmetric quasi-Banach ideal to another. In addition, we use the technique of J-dual spaces in order to obtain the estimation $\| \delta_a \|_{I \to J} \leq 2 \| a \|_{J:I}$ for an arbitrary derivation $\delta = \delta_a : I \to J, a \in J : I$.

Let H be an infinite-dimensional Hilbert space over the field $\mathbb{C}$ of complex numbers and $\mathcal{B}(H)$ be the C^*-algebra of all bounded linear operators on H. Set

$$\mathcal{B}_h(H) = \{x \in \mathcal{B}(H): x^* = x\},$$
$$\mathcal{B}_+(H) = \{x \in \mathcal{B}_h(H): \forall \varphi \in H\ (x(\varphi), \varphi) \geq 0\},$$
$$\mathcal{P}(H) = \{p \in \mathcal{B}(H): p = p^2 = p^*\}.$$

It is well known [23] that $B_+(H)$ is a proper cone in $\mathcal{B}_h(H)$ and with the partial order given by $x \leq y \Leftrightarrow y - x \in \mathcal{B}_+(H)$ the set $\mathcal{B}_h(H)$ is a partially ordered vector space over the field $\mathbb{R}$ of real numbers, satisfying $y^*xy \geq 0$ for all $y \in \mathcal{B}(H), x \in \mathcal{B}_+(H)$. Note, that $-\| x \|_{\mathcal{B}(H)} 1 \leq x \leq \| x \|_{\mathcal{B}(H)} 1$ for all $x \in \mathcal{B}_h(H)$, where 1 is the identity operator on H. It is known (see e.g. [23]) that every operator x in $\mathcal{B}_h(H)$ can be uniquely written as follows: $x = x_+ - x_-$, where $x_+, x_- \in \mathcal{B}_+(H)$ and $x_+ x_- = 0$. In addition, every operator $x \in \mathcal{B}(H)$ can be represented as $x = u|x|$ (the polar decomposition of the operator x), where $|x| = (x^*x)^{\frac{1}{2}}$ and u is a partial isometry in $\mathcal{B}(H)$ such that u^*u is the right support of x [116].

We need the following.

Proposition (6.1.6)[251]: ([113]). If $x, y \in \mathcal{B}_+(H), x \leq y$, then there exists an operator $a \in \mathcal{B}(H)$ such that $\| a \|_{\mathcal{B}(H)} \leq 1$ and $x = a^*ya$.

Let $K(H)$ be a two-sided ideal in $\mathcal{B}(H)$ of all compact operators and $x \in K(H)$. The eigenvalues $\{\lambda_n(|x|)\}_{n=1}^{\infty}$ of the operator $|x|$ arranged in decreasing order and repeated according to algebraic multiplicity are called singular values of the operator x, i.e. $s_n(x) = \lambda_n(|x|), n \in \mathbb{N}$, where $\lambda_1(|x|) \geq \lambda_2(|x|) \geq \cdots$ and $\mathbb{N}$ is the set of all natural numbers. We need the following properties of singular values.

Proposition (6.1.7)[251]: ([260]).

(a) $s_n(x) = s_n(x^*), s_n(\alpha x) = |\alpha| s_n(x)$ for all $x \in K(H), \alpha \in \mathbb{C}$;

(b) $s_n(xb) \leq s_n(x) \parallel b \parallel_{\mathcal{B}(H)}, s_n(bx) \leq s_n(x) \parallel b \parallel_{\mathcal{B}(H)}$ for all $x \in K(H), b \in \mathcal{B}(H)$.

Let $\mathcal{F}(H)$ be a two-sided ideal in $\mathcal{B}(H)$ of all operators with finite range and let I be an arbitrary proper two-sided ideal in $\mathcal{B}(H)$. Then I is a $*$-ideal [23] and the following inclusion holds: $\mathcal{F}(H) \subseteq I$ [23], in particular, I contains all finite-dimensional projections from $\mathcal{P}(H)$. If H is a separable Hilbert space, then the inclusion $I \subseteq K(H)$ also holds [254]. If, however, H is not separable, then for proper two-sided ideals in $\mathcal{B}(H)$ we have the following proposition.

Proposition (6.1.8)[251]: ([259]).

(i) $D = \{x \in \mathcal{B}(H): x(H)$ is separable$\}$ is a proper two-sided ideal in $\mathcal{B}(H)$, in addition $K(H) \subset \mathcal{D}$;

(ii) If I is an ideal in $\mathcal{B}(H)$, then either $I \subseteq K(H)$ or $\mathcal{D} \subseteq I$.

Let X be a linear space over the field $\mathbb{C}$. A function $\|\cdot\|$ from X to $\mathbb{R}$ is a quasi-norm, if for all $x, y \in X, \alpha \in \mathbb{C}$ the following properties hold:

(a) $\| x \| \geq 0, \| x \| = 0 \Leftrightarrow x = 0$;

(b) $\| \alpha x \| = |\alpha| \| x \|$;

(c) $\| x + y \| \leq C(\| x \| + \| y \|), C \geq 1$.

The couple $(X, \|\cdot\|)$ is called a quasi-normed space and the least of all constants C satisfying the inequality (c) above is called the modulus of concavity of the quasi-norm $\|\cdot\|$.

It is known (see [137]) that for each quasi-norm $\|\cdot\|$ on X there exists an equivalent p-additive quasinorm $|||\cdot|||$, that is a quasi-norm $|||\cdot|||$ on X satisfying the following property of p-additivity: $|||x + y|||^p \leq |||x|||^p + |||y|||^p$, where p is such that $C = 2^{\frac{1}{p}-1}$, in particular, $0 < p \leq 1$ since $C \geq 1$. In this case, the function $d: X^2 \to \mathbb{R}$ defined by $d(x, y) := |||x - y|||^p, x, y \in X$ is an invariant metric on X, and in the topology τ_d, generated by the metric d, the linear space X is a topological vector space. If (X, d) is a complete metric space, then $(X, \|\cdot\|)$ is called a quasi-Banach space and the quasi-norm $\|\cdot\|$ is a complete quasi-norm; in this case, (X, τ_d) is an F-space.

Proposition (6.1.9)[251]: Let $(X, \|\cdot\|)$ be a quasi-Banach space with the modulus of concavity C, let $|||\cdot|||$ be a p-additive quasi-norm equivalent to the quasi-norm $\|\cdot\|, C = 2^{\frac{1}{p}-1}$. If $x_n \in X, n \geq 1$ and $\sum_{n=1}^{\infty} |||x_n|||^p < \infty$, then the series $\sum_{n=1}^{\infty} x_n$ converges in $(X, \|\cdot\|)$, i.e. there exists $x \in X$ such that $\left\|x - \sum_{n=1}^{k} x_n\right\| \to 0$ for $k \to \infty$.

Proof. For partial sums $S_k = \sum_{n=1}^{k} x_n$ we have

$d(S_{k+l}, S_k) = |||S_{k+l} - S_k|||^p = |||\sum_{n=l+1}^{k+l} x_n|||^p \leq \sum_{n=l+1}^{k+l} \|x_n\|^p \to 0$ for $k, l \to \infty$,

i.e. $\{S_k\}_{k=1}^{\infty}$ is a Cauchy sequence in (X, d). Since the metric space (X, d) is complete, there exists $x \in X$ such that $d(S_k, x) = |||S_k - x|||^p \to 0$ for $k \to \infty$. Since quasi-norms $\|\cdot\|$ and $|||\cdot|||$ are equivalent we have that $\| S_k - x \| \to 0$ for $k \to \infty$.

Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be quasi-normed spaces and let $\mathcal{B}(X, Y)$ be the linear space of all bounded linear mappings $T: X \to Y$. For each $T \in \mathcal{B}(X, Y)$ set $\| T \|_{\mathcal{B}(X,Y)} = sup\{\| Tx \|_Y$:

$\| x \| \leq 1\}$. As in the case of normed spaces, the set $\mathcal{B}(X,Y)$ coincides with the set of all continuous linear mappings from X into Y, moreover, the function $\|\cdot\|_{\mathcal{B}(X,Y)}: \mathcal{B}(X,Y) \to \mathbb{R}$ is a quasi-norm on $\mathcal{B}(X,Y)$ whose modulus of concavity, does not exceed the modulus of concavity of the quasi-norm $\|\cdot\|_Y$ [137]. Furthermore, $\| Tx \|_Y \leq \| T \|_{\mathcal{B}(X,Y)} \| x \|_X$ for all $T \in \mathcal{B}(X,Y)$ and $x \in X$.

Proposition (6.1.10)[251]: If $(Y, \|\cdot\|_Y)$ is a quasi-Banach space, then $(\mathcal{B}(X,Y), \|\cdot\|_{\mathcal{B}(X,Y)}$ is a quasi-Banach space too.

Proof. Since $\|\cdot\|_Y$ is a quasi-norm on Y, there exists a p-additive quasi-norm $|||\cdot|||_Y$ equivalent to $\|\cdot\|_Y$, i.e. $\alpha_1 |||y|||_Y \leq \| y \|_Y \leq \beta_1 |||y|||_Y$ for all $y \in Y$ and some constants $\alpha_1, \beta_1 > 0$. Similarly, there exists a q-additive quasi-norm $|||\cdot|||_{\mathcal{B}(X,Y)}$ equivalent to the quasi-norm $\|\cdot\|_{\mathcal{B}(X,Y)}$, i.e. $\alpha_2 |||T|||_{\mathcal{B}(X,Y)} \leq \| T \|_{\mathcal{B}(X,Y)} \leq \beta_2 |||T|||_{\mathcal{B}(X,Y)}$ for all $T \in \mathcal{B}(X,Y)$ and some $\alpha_2, \beta_2 > 0, 0 < p, q \leq 1$.

Let $\{T_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $(\mathcal{B}(X,Y), d)$, where $d(T, S) = |||T - S|||_{\mathcal{B}(X,Y)}^q, T, S \in \mathcal{B}(X,Y)$. Fix $\varepsilon > 0$ and select a positive integer $n(\varepsilon)$ such that $|||T_n - T_m|||_{\mathcal{B}(X,Y)}^q < \varepsilon^q$ for all $n, m > n(\varepsilon)$. For every $x \in X$ we have

$$|||T_n x - T_m x|||_Y^p \leq \frac{1}{\alpha_1^p} \| T_n x - T_m x \|_Y^p \leq \frac{1}{\alpha_1^p} \| T_n - T_m \|_{\mathcal{B}(X,Y)}^p \| x \|_X^p$$

$$\leq \left(\frac{\beta_2}{\alpha_1}\right)^p \left|||T_n - T_m|||\right|_{\mathcal{B}(X,Y)}^p \| x \|_X^p < \left(\frac{\beta_2}{\alpha_1}\right)^p \| x \|_X^p \varepsilon^p \; for \; n, m \geq n(\varepsilon).$$

Thus, $\{T_n x\}_{n=1}^{\infty}$ is a Cauchy sequence in (Y, d_Y), where $d_Y(x,y) = |||x - y|||_Y^p$. Since the metric space (Y, d_Y) is complete, there exists $T(x) \in Y$ such that $|||T_n(x) - T(x)|||_Y^p \to 0$ for $n \to \infty$. The verification that $T \in \mathcal{B}(X,Y)$ and $|||T_n - T|||_{\mathcal{B}(X,Y)}^q \to 0$ for $n \to \infty$ is routine and is therefore omitted.

Let I be a nonzero two-sided ideal in $\mathcal{B}(H)$.

A quasi-norm $\|\cdot\|_I: I \to \mathbb{R}$ is called symmetric quasi-norm if

(a) $\| axb \|_I \leq \| a \|_{\mathcal{B}(H)} \| x \|_I \| b \|_{\mathcal{B}(H)}$ for all $x \in I, a, b \in \mathcal{B}(H)$;

(b) $\| p \|_I = 1$ for any one-dimensional projection $p \in I$.

Proposition (6.1.11)[251]: (Compare [260]). Let $\|\cdot\|_I$ be a symmetric quasi-norm on a two-sided ideal I. Then

(a) $\| x \|_I = \| x^* \|_I = |x|_I$ for all $x \in I$;

(b) If $x \in I \subset K(H), y \in K(H), s_n(y) \leq s_n(x), n = 1, 2, \ldots,$ then $y \in I$ and $\| y \|_I \leq \| x \|_I$;

(c) If $I \subset K(H)$, then $\| x \|_{\mathcal{B}(H)} \leq \| x \|_I$ for all $x \in I$.

Proof. (a) Let $x = u|x|$ be the polar decomposition of the operator x. Then $\| x \|_I = \| u|x| \|_I \leq |x|_I$. Since $u^* x = |x|$, the Inequality $|x|_I \leq \| x \|_I$ holds and so $|x|_I = \| x \|_I$. Using the equalities $x^* = |x| u^*, x^* u = |x|$ in the same manner, we obtain that $|x|_I = \| x^* \|_I$.

(b) Since x, y are compact operators and $s_n(y) \leq s_n(x)$ we have $s_n(y) = \alpha_n s_n(x)$, where $0 \leq \alpha_n \leq 1, n \in \mathbb{N}$. By the Hilbert–Schmidt theorem, there exists an orthogonal system of eigenvectors $\{\varphi_n\}_{n=1}^{\infty}$ for the operator $|y|$ such that

$|y|(\varphi) = \sum_{n=1}^{\infty} s_n(y) c_n \varphi_n$, where $c_n = (\varphi, \varphi_n), \varphi \in H$. Since $s_n(y) = \alpha_n s_n(x)$, it follows that $card\{\varphi_n\} \leq card\{\psi_n\}$, where $\{\psi_n\}_{n=1}^{\infty}$ is an orthogonal system of eigenvectors for the operator $|x|$. Thus, there exists a unitary operator $u \in \mathcal{B}(H)$ such that $u(\psi_n) = \varphi_n$, in addition, $u|x|u^{-1} \geq |y|$.

By Proposition (6.1.6), there exists an operator $a \in \mathcal{B}(H)$ with $\| a \|_{\mathcal{B}(H)} \leq 1$ such that $|y| = a^* u|x|u^{-1} a$. Consequently, $|y| \in I$ and $|y|_I \leq |x|_I$, thus $y \in I$ and $\| y \|_I \leq \| x \|_I$.
(c) Let $y(\cdot) = s_1(x)(\cdot, \varphi)\varphi$, where φ is an arbitrary vector in H with $\| \varphi \|_H = 1$. Whereas $s_n(y) \leq s_n(x)$, we have $\| x \|_{\mathcal{B}(H)} = s_1(x) = \| y \|_{\mathcal{B}(H)} = \| y \|_I \leq \| x \|_I$ (see (b)).
A two-sided ideal I of compact operators from $\mathcal{B}(H)$ is called a symmetric quasi-Banach (respectively, Banach) ideal, if I is equipped with a complete symmetric quasi-norm (respectively, norm).
Let I, J be two-sided ideals of compact operators from $\mathcal{B}(H)$. A linear mapping $\delta: I \to J$ is called a derivation, if $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in I$. If, in addition, $\delta(x^*) = (\delta(x))^*$ for all $x \in I$, then δ is called a $*$-derivation. Denote by
$Der(I, J)$ the linear space of all derivations from I into J.
For each derivation $\delta: I \to J$ define the mappings $\delta_{Re}(x) := \frac{\delta(x)+\delta(x^*)^*}{2}$ and $\delta_{Im}(x) := \frac{\delta(x)-\delta(x^*)^*}{2i}$, $x \in I$. It is easy to see that δ_{Re} and δ_{Im} are $*$-derivations from I into J, moreover $\delta = \delta_{Re} + i\delta_{Im}$.
If $a \in \mathcal{B}(H)$, then the mapping $\delta_a: \mathcal{B}(H) \to \mathcal{B}(H)$ given by $\delta_a(x) := [a, x] = ax - xa, x \in \mathcal{B}(H)$, is a derivation.
Derivations of this type are called inner. When I is a two-sided ideal in $\mathcal{B}(H)$, then $\delta_a(I) \subset I$ for all $a \in \mathcal{B}(H)$. If J is also a two-sided ideal in $\mathcal{B}(H)$ and $a \in J$, then $\delta_a(I) \subset I \cap J$.

The following theorem gives a positive answer to Question (6.1.3).

Theorem (6.1.12)[251]: Let I, J be symmetric quasi-Banach ideals of compact operators from $\mathcal{B}(H)$ and δ is a derivation from I into J. Then δ is a continuous mapping from I into J, i.e. $\delta \in \mathcal{B}(I, J)$.

Proof. Without loss of generality we may assume that δ is a $*$-derivation. The spaces $(I, \|\cdot\|_I), (J, \|\cdot\|_J)$ are F-spaces, and therefore it is sufficient to prove that the graph of δ is closed. Suppose a contrary, that is there exists a sequence $\{x_n\}_{n=1}^\infty \subset I$ such that $\|\cdot\|_I - \lim_{n\to\infty} x_n = 0$ and $\|\cdot\|_J - \lim_{n\to\infty} \delta(x_n) = x \neq 0$.
Since $x_n = Rex_n + iImx_n$ for all $n \in \mathbb{N}$, where $Rex_n = \frac{x_n + x_n^*}{2}$, $Imx_n = \frac{x_n - x_n^*}{2}$, and $\| x_n \|_I \to 0, \| x_n^* \|_I = \| x_n \|_I \to 0$, we have

$$\| Rex_n \|_I = \left\| \frac{x_n + x_n^*}{2} \right\|_I \leq \frac{C(\| x_n \|_I + \| x_n^* \|_I)}{2} \to 0$$

and

$$\| Imx_n \|_I = \left\| \frac{x_n - x_n^*}{2} \right\|_I \leq \frac{C(\| x_n \|_I + \| x_n^* \|_I)}{2} \to 0,$$

where C is the modulus of concavity of the quasi-norm $\|\cdot\|_I$. Consequently, we may assume that $x_n^* = x_n$ for all $n \in \mathbb{N}$. In this case, from the relationships

$$x \stackrel{\|\cdot\|_J}{\longleftarrow} \delta(x_n) = \delta(x_n^*) = \delta(x_n)^* \stackrel{\|\cdot\|_J}{\longrightarrow} x^*,$$

we obtain $x = x^*$.
Writing $x = x_+ - x_-$, where $x_+, x_- \geq 0$ and $x_+ x_- = 0$, we may assume that $x_+ \neq 0$, otherwise we consider the sequence $\{-x_n\}_{n=1}^\infty$. Since x_+ is a nonzero positive compact operator, $\lambda = \| x_+ \|_{\mathcal{B}(H)}$ is an eigenvalue of x_+ corresponding to a finite-dimensional eigensubspace. Let q be a projection onto this subspace.
Fix an arbitrary non-zero vector $\varphi \in q(H)$ and consider the projection p onto the one-dimensional subspace spanned by φ. Combining the inequality $p \leq q$ with the equality

$qx_+q = \lambda q$, we obtain $pxp = pqxqp = \lambda pqp = \lambda p$. Replacing, if necessary, the sequence $\{x_n\}_{n=1}^{\infty}$ with the sequence $\{\frac{x_n}{\lambda}\}_{n=1}^{\infty}$, we may assume

$$pxp = p. \tag{1}$$

Since p is one-dimensional, it follows that $pap = \alpha p, \alpha \in C$ for any operator $a \in \mathcal{B}(H)$, in particular, $px_np = \alpha_n p$, therefore $|\alpha_n| = \| px_np \|_I \to 0$ for $n \to \infty$. Writing

$$\| \delta(p)x_np \|_J \leq \| \delta(p) \|_J \| x_np \|_{\mathcal{B}(H)} \leq \| \delta(p) \|_J \| x_n \|_{\mathcal{B}(H)} \leq \| \delta(p) \|_J \| x_n \|_I,$$

we infer $\| \delta(p)x_np \|_J \to 0$ and $\| px_n\delta(p) \|_J = \| (\delta(p)x_np)^* \|_J \to 0$.

Since $pxp \overset{(1)}{=} p \in J$, we have

$$\| \delta(px_np) - pxp \|_J = \| \delta(p)x_np + p\delta(x_n)p + px_n\delta(p) - pxp \|_J$$
$$\leq C_1 \| \delta(p)x_np + px_n\delta(p) \|_J + C_1 \| p\delta(x_n)p - pxp \|_J$$
$$\leq C_1^2 \| \delta(p)x_np \|_J + C_1^2 \| px_n\delta(p) \|_J + C_1 \| p\delta(x_n)p - pxp \|_J \to 0,$$

where C_1 is the modulus of concavity of the quasi-norm $\|\cdot\|_J$, i.e. $\delta(px_np) \overset{\|\cdot\|_J}{\longrightarrow} pxp$. Hence

$$p \overset{(1)}{=} pxp = \|\cdot\|_J - \lim_{n\to\infty} \delta(px_np) = \|\cdot\| - \lim_{n\to\infty} \delta(\alpha_n p) = \|\cdot\|_J - \lim_{n\to\infty} \alpha_n\delta(p) = 0,$$

which is a contradiction, since $p \neq 0$.

Consequently, δ is a continuous mapping from $(I, \|\cdot\|_I)$ into $(J, \|\cdot\|_J)$.

Note, that in [257] a version of Theorem (6.1.12) is obtained for the case of an arbitrary symmetric Banach ideal $I = J$ of τ -compact operators in a von Neumann algebra M equipped with a semi-finite normal faithful trace τ .

The following theorem gives a positive answer to Question (6.1.1).

Theorem (6.1.13)[251]: If I, J are symmetric quasi-Banach ideals of compact operators from $\mathcal{B}(H)$, then for every derivation $\delta: I \to J$ there exists an operator $a \in \mathcal{B}(H)$ such that $\delta(\cdot) = \delta a(\cdot) = [a,\cdot]$, in addition, $\| a \|_{\mathcal{B}(H)} \leq \| \delta \|_{\mathcal{B}(I,J)}$.

Proof. Fix an arbitrary vector $\varphi_0 \in H$ with $\| \varphi_0 \|_H = 1$ and consider projection $p_0(\cdot) := (\cdot, \varphi_0)\varphi_0$ onto one-dimensional subspace spanned by φ_0. Obviously, $p_0 \in I \cap J$.

Let $x \in I, x(\varphi_0) = 0$ and $\varphi \in H$. Since

$$xp_0(\varphi) = x(p_0(\varphi)) = x((\varphi, \varphi_0)\varphi_0) = (\varphi, \varphi_0)x(\varphi_0) = 0,$$

it follows that $xp_0 = 0$, and so $\delta(xp_0)(\varphi_0) = 0$. Consequently, the linear operator $a(z(\varphi_0)) = \delta(zp_0)(\varphi_0)$ is correctly defined on the linear subspace $L := \{z(\varphi_0) : z \in I\} \subset H. If\ \varphi \in H, z(\cdot) = (\cdot, \varphi_0)\varphi$, then $z \in I$ and $z(\varphi_0) = \varphi$, which implies $L = H$. For arbitrary $z \in \mathcal{B}(H), \varphi \in H$, we have

$$|zp_0|^2(\varphi) = (p_0z^*zp_0)(\varphi) = (p_0z^*z)((\varphi, \varphi_0)\varphi_0) = (\varphi, \varphi_0)p_0(z^*z(\varphi_0))$$
$$= (z\varphi_0, z\varphi_0)(\varphi, \varphi)\varphi_0 = (z\varphi_0, z\varphi_0)p_0(\varphi) = \| z(\varphi_0) \|_H^2 \, p_0(\varphi),$$

in particular, $\| zp_0 \|_{\mathcal{B}(H)} = \| |zp_0| \|_{\mathcal{B}(H)} = \| \| z(\varphi_0) \|_H \, p_0 \|_{\mathcal{B}(H)} = \| z(\varphi_0) \|_H$. Applying this observation together with Theorem (6.1.12) guaranteeing $\| \delta(x) \|_J \leq \| \delta \|_{\mathcal{B}(I,J)} \| x \|_I$ for all $x \in I$, we have

$$\| a(x(\varphi_0)) \|_H = \| \delta(xp_0)(\varphi_0) \|_H = \| \delta(xp_0)p_0 \|_{\mathcal{B}(H)} \leq \| \delta(xp_0) \|_{\mathcal{B}(H)} \| p_0 \|_{\mathcal{B}(H)}$$
$$\leq \| \delta(xp_0) \|_J \leq \| \delta \|_{\mathcal{B}(I,J)} \| xp_0 \|_I$$
$$\leq \| \delta \|_{\mathcal{B}(I,J)} \| p_0 \|_I \| xp_0 \|_{\mathcal{B}(H)} = \| \delta \|_{\mathcal{B}(I,J)} \| x(\varphi_0) \|_H.$$

This shows that a is a bounded operator on H and $\| a \|_{\mathcal{B}(H)} \leq \| \delta \|_{\mathcal{B}(I,J)}$.

Finally, for all $x, z \in I$ we have

$$[a, x](z(\varphi_0)) = ax(z(\varphi_0) - xa(z(\varphi_0)) = a(xz(\varphi_0)) - xa(z(\varphi_0))$$
$$= \delta(xzp_0)(\varphi_0) - x\delta(zp_0)(\varphi_0) = \delta(x)zp_0(\varphi_0) = \delta(x)z(\varphi_0)$$

and since $L = H$, it follows $\delta(\cdot) = [a,\cdot] = \delta a(\cdot)$.

Let I,J be arbitrary two-sided ideals in $\mathcal{B}(H)$. The set

$$D(I,J)=\{a\in\mathcal{B}(H): ax-xa\in J, \forall x\ \in I\}$$

is called the J-essential commutant of I, and the set

$$J:I=\{a\in B(H): ax\in J, \forall x\in I\}$$

is called the J-dual space of I. It is clear that $J:I$ is a two-sided ideal in $\mathcal{B}(H)$. Hence $J:I$ is a $*$-ideal, and therefore $xa\in J$ for all $x\in I, a\in J:I$. If $I\nsubseteq J$, then $1\notin J:I, i.e. J:I\neq\mathcal{B}(H)$, and so $J:I$ is a proper ideal in $\mathcal{B}(H)$. However, in case when $I\subseteq J$ we have $J:I=\mathcal{B}(H)$, in particular, $C_r:C_p=\mathcal{B}(H)$ for all $0<p\leq r$, where $C_p=\{x\in K(H): \|x\|_p=\left(tr\left(|x|_p\right)\right)^{\frac{1}{p}}<\infty\}$ is the Schatten ideal of compact operators from $\mathcal{B}(H), 0<p<\infty, tr$ is the standard trace on $\mathcal{B}_+(H)$.

Proposition (6.1.14)[251]: If I,J are proper two-sided ideals of compact operators in $\mathcal{B}(H)$ and $I\nsubseteq J$, then $J:I\subset K(H)$.

Proof. Since $I\nsubseteq J, J:I$ is a proper two-sided ideal in $\mathcal{B}(H)$. If H is a separable Hilbert space, then $J:I\subset K(H)$ [254]. Suppose that H is not separable and $J:I\nsubseteq K(H)$. By Proposition (6.1.8), the proper two-sided ideal $D=\{x\in\mathcal{B}(H): x(H)$ is separable $\}\subset J:I$. Since $I\nsubseteq J$ there exists a positive compact operator $a\in I\setminus J$. Since $a\in D$, we have that $L:=a(H)$ is separable. Let $p\in P(H)$ be the orthogonal projection onto L. Since $a\notin J$, it follows that L is infinite-dimensional subspace. Indeed, if it were not the case, then a would be a finite rank operator and automatically belonging to $a\in J$. Therefore $p\in D\setminus K(H)\subset J:I$, in addition, $0\neq a=pap\in(pIp)\setminus(pJp), i.e. pIp\nsubseteq pJp$. Since L is a separable Hilbert space, we have $(pJp):(pIp)\subset K(L)$.

Let $y\in pIp$, i.e. $y=py'p$ for some $y'\in I$. Since $p\in D\subset J:I$ we have $py'\in J$, hence, $p(py')p\in pJp$. Consequently, $p\in(pJp):(pIp)$, i.e. p is a compact operator in L, which is a contradiction. Thus, $J:I\subset K(H)$.

For arbitrary two-sided ideals I,J in $\mathcal{B}(H)$ we denote by $d(I,J)$ the set of all derivations δ from $\mathcal{B}(H)$ into $\mathcal{B}(H)$ such that $\delta(I)\subset J$. To characterize the set $d(I,J)$ we need the following theorem.

Theorem (6.1.15)[251]: ([252]). $D(I,J)=J:I+\mathbb{C}1$.

It should be noted that Theorem (6.1.14) holds for arbitrary von Neumann algebras, i.e. for any two-sided ideals I,J in von Neumann algebra M we have $D(I,J)=J:I+Z(M)$, where $Z(M)$ is the center of M [255].

Proposition (6.1.16)[251]: $d(I,J)=\{\delta a: a\in D(I,J)\}=\{\delta a: a\in J:I\}$.

Proof. Let $\delta\in d(I,J)$. Since δ is a derivation from $\mathcal{B}(H)$ into $\mathcal{B}(H)$ there exists an operator $a\in\mathcal{B}(H)$ such that $\delta=\delta_a$. If $x\in I$, then $[a,x]=\delta(x)\in J$, i.e. $a\in D(I,J)$. Using Theorem (6.1.15), we have that $a=b+\alpha 1$, where $b\in J:I, \alpha\in\mathbb{C}$, and therefore $\delta=\delta_a=\delta_b$.

Further, let $\delta_a(\cdot)=[a,\cdot]$ be the inner derivation on $\mathcal{B}(H)$ generated by an operator $a\in J:I$. For all $x\in I$ we have $\delta_a(x)=[a,x]=ax-xa\in J$. Consequently, $\delta_a\in d(I,J)$.

Now, let I,J be arbitrary symmetric quasi-Banach ideals of compact operators from $\mathcal{B}(H)$. According to Theorem (6.1.13), for each derivation $\delta\in Der(I,J)$ there exists an operator $a\in\mathcal{B}(H)$ such that $\delta(x)=\delta_a(x)=[a,x]$ for all $x\in I$. Since $\delta(I)\subset J$ we have $[a,x]\in J$ for all $x\in I$, i.e. $a\in D(I,J)$. Hence, $\delta_a\in d(I,J)$ see Proposition (6.1.16). On the other hand, if $a\in J:I$, then $\delta_a\in d(I,J)$ (see Proposition (6.1.16)), in particular, $\delta_a(I)\subset J$.

Hence, in view of Proposition (6.1.16) and Theorem (6.1.13), the following theorem holds.

Theorem (6.1.17)[251]: For arbitrary symmetric quasi-Banach ideals I, J of compact operators in $\mathcal{B}(H)$ each derivation $\delta: I \rightarrow J$ has a form $\delta = \delta_a$ for some $a \in J: I$, in addition $\| a + \alpha 1 \|_{\mathcal{B}(H)} \leq \| \delta_a \|_{\mathcal{B}(I,J)}$ for some $\alpha \in \mathbb{C}$. Conversely, if $a \in J: I$ then the restriction of the derivation δ_a on I is a derivation from I into J.

If $0 < r < p < \infty$, then we have $C_r: C_p = C_q$, where $\frac{1}{q} = \frac{1}{r} - \frac{1}{p}$ [252]. Therefore, the following corollary follows immediately from Theorem (6.1.17).

Corollary (6.1.18)[251]: If $0 < p \leq r < \infty$, then the mapping $\delta: C_p \rightarrow C_r$ is a derivation if and only if $\delta = \delta_a$ for some $a \in \mathcal{B}(H)$. If $0 < r < p < \infty$, then the mapping $\delta: C_p \rightarrow C_r$ is a derivation if and only if $\delta = \delta_a$ for some $a \in C_q$, where $\frac{1}{q} = \frac{1}{r} - \frac{1}{p}$.

We show that any symmetric quasi-Banach ideal $(I, \|\cdot\|_I)$ of compact operators from $\mathcal{B}(H)$ has a form of $I = C_{E_I}$ with the quasi-norm $\|\cdot\|_I = \|\cdot\|_{C_{E_I}}$ for a special symmetric quasi-Banach sequence space $(E_I, \|\cdot\|_{E_I})$ in c_0 constructed by I with the help of Calkin correspondence. The equality $J: I = C_{E_J:E_I}$ established provides a full description of all derivations $\delta \in Der(I,J)$ in terms of E_J-dual space $E_J: E_I$ of E_I of symmetric quasi-Banach sequence spaces E_I and E_J in c_0.

A quasi-Banach lattice E is a vector lattice with a complete quasi-norm $\|\cdot\|_E$, such that $\| a \|_E \leq \| b \|_E$ whenever $a, b \in E$ and $|a| \leq |b|$. In this case, $\||a|\|_E = \| a \|_E$ for all $a \in E$ and the lattice operations $a \vee b$ and $a \wedge b$ are continuous in the topology τ_d, generated by the metric $d(a,b) = |||a - b|||_E^p$, where $||| \cdot |||_E$ is a p-additive quasi-norm equivalent to the quasi-norm $\|\cdot\|_E$.

Consequently, the set $E_+ = \{a \in E: a \geq 0\}$ is closed in (E, τ_d). Thus, for any increasing sequence $\{a_k\}_{k=1}^{\infty} \subset E$ converging in the topology τ_d to some $a \in E$, we have $a = sup_{k\geq 1} a_k$ [261].

A sequence $\{a_n\}_{n=1}^{\infty}$ from a vector lattice E is said to be (r)-convergent to $a \in E$ (notation: $a_n \xrightarrow{(r)} a$) with the regulator $b \in E_+$, if and only if there exists a sequence of positive numbers $\varepsilon_n \downarrow 0$ such that $|a_n - a| \leq \varepsilon_n b$ for all $n \in \mathbb{N}$ (see [262].)

Observe, that in any quasi-Banach lattice $(E, \|\cdot\|_E)$ it follows from $a_n \xrightarrow{(r)} a,\ a_n, a \in E$ that $\| a_n - a \|_E \rightarrow 0$.

The following proposition is a quasi-Banach version of the well-known criterion of sequential convergence in Banach lattices.

Proposition (6.1.19)[251]: (Compare [262]). Let $(E, \|\cdot\|_E)$ be a quasi-Banach lattice, $a, a_n \in E$. The following conditions are equivalent:

(i) $\| a_n - a \|_E \rightarrow 0$ for $n \rightarrow \infty$;

(ii) for any subsequence a_{n_k} there exists a subsequence anks such that $a_{n_{k_s}} \xrightarrow{(r)} a$.

Proof. Without loss of generality we may assume that $a = 0$.

(i) ⇒ (ii) For an equivalent p-additive quasi-norm $||| \cdot |||_E$ we have $||||a_n||||_E \rightarrow 0$ for $n \rightarrow \infty$. Hence, we may choose an increasing sequence of positive integers $n_1 < n_2 < \cdots < n_k < \cdots$ such that $||||a_{n_k}||||^p \leq \frac{1}{k^3}$. The estimate

$$\sum_{k=1}^{\infty} |||k^{\frac{1}{p}} |a_{n_k}||||^p = \sum_{k=1}^{\infty} k||||a_{n_k}||||^p \leq \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty,$$

shows that the series $\sum_{k=1}^{\infty} k^{\frac{1}{p}}|a_{n_k}|$ converges in $(E, \|\cdot\|_E)$ to some $b \in E_+$ (see Proposition (6.1.20)) and therefore there exists $b = sup_{n\geq 1} \sum_{k=1}^{n} k^{\frac{1}{p}}|a_{n_k}|$ such that we also have $k^{\frac{1}{p}}|a_{n_k}| \leq b$ for all $k \in \mathbb{N}$. In particular, $|a_{n_k}| \leq k^{-\frac{1}{p}}b$, which immediately implies $a_{n_k} \xrightarrow{(r)} 0$. The same reasoning may be repeated for any subsequence $\{a_{n_k}\}_{k=1}^{\infty}$.

The proof of the implication (ii) ⇒ (i) is the verbatim repetition of the analogous result for Banach lattices [262].

Let m be the Lebesgue measure on the semi-axis $(0, \infty)$, let $L_1(0, \infty)$ be the Banach space of all integrable functions on $(0, \infty)$ with the norm $\| f \|_1 := \int_0^{\infty} |f| dm$ and let $L_\infty(0, \infty)$ be the Banach space of all essentially bounded measurable functions on $(0, \infty)$ with the norm $\| f \|_\infty := esssup\{|f(t)| : 0 < t < \infty\}$. For each $f \in L_1(0, \infty) + L_\infty(0, \infty)$ we define the decreasing rearrangement f^* of f by setting

$$f^*(t) := \inf\{s > 0 : m(\{|f| > s\}) \leq t\}, t > 0.$$

The function $f^*(t)$ is equimeasurable with $|f|$, in particular, $f^* \in L_1(0, \infty) + L_\infty(0, \infty)$ and $f^*(t)$ is non-increasing and right-continuous.

We need the following properties of decreasing rearrangements (see [263]).

Proposition (6.1.20)[251]: Let $f, g \in L_1(0, \infty) + L_\infty(0, \infty)$. We have

(i) if $|f| \leq |g|$, then $f^* \leq g^*$;

(ii) $(\alpha f)^* = |\alpha| f^*$ for all $\alpha \in \mathbb{R}$;

(iii) if $f \in L_\infty(0, \infty)$, then $(fg)^* \leq \| f \|_\infty g^*$;

(iv) $(f + g)^*(t + s) \leq f^*(t) + g^*(s)$;

(v) if $fg \in L_1(0, \infty) + L_\infty(0, \infty)$, then $(fg)^*(t + s) \leq f^*(t)g^*(s)$.

Let l_∞ be the Banach lattice of all bounded real-valued sequences $\xi := \{\xi_n\}_{n=1}^{\infty}$ equipped with the norm $\| \xi \|_\infty = sup_{n\geq 1} |\xi_n|$. For each $\xi = \{\xi_n\}_{n=1}^{\infty} \in l_\infty$ the function $f_\xi(t) := \sum_{n=1}^{\infty} \xi_n \chi_{[n-1,n)}(t), t > 0$ is contained in $L_\infty(0, \infty)$. For the decreasing rearrangement f_ξ^*, we obviously have $f_\xi^*(t) = \sum_{n=1}^{\infty} \xi_n^* \chi_{[n-1,n)}(t), t > 0$, where $\xi^* := \{\xi_n^*\}_{n=1}^{\infty}$ is a decreasing sequence of nonnegative numbers with $|\xi_1^*| = sup_{n\geq 1}|\xi_n|$, which, in case when $\xi \in c_0$, coincides with the decreasing rearrangement of the sequence $\{|\xi_n|\}_{n=1}^{\infty}$. By Proposition (6.1.20)(i), (ii) we have $\xi^* \leq \eta^*$ for $\xi, \eta \in l_\infty$ with $|\xi| \leq |\eta|$, and $(\alpha\xi)^* = |\alpha|\xi^*, \alpha \in \mathbb{R}$.

A linear subspace $\{0\} \neq E \subset l_\infty$ is said to be solid rearrangement-invariant, if for every $\eta \in E$ and every $\xi \in l_\infty$ the assumption $\xi^* \leq \eta^*$ implies that $\xi \in E$. Every solid rearrangement-invariant space E contains the space c_{00} of all finitely supported sequences from c_0. If E contains an element $\{\xi_n\}_{n=1}^{\infty} \notin c_0$, then $E = l_\infty$. Thus, for any solid rearrangement-invariant space $E \neq l_\infty$ the embeddings $c_{00} \subset E \subset c_0$ hold.

A solid rearrangement-invariant space E equipped with a complete quasi-norm (norm) $\|\cdot\|_E$ is called symmetric quasi-Banach (Banach) sequence space, if

(a) $\| \xi \|_E \leq \| \eta \|_E$, provided $\xi^* \leq \eta^*, \xi, \eta \in E$;

(b) $\| \{1, 0, 0, \dots\} \|_E = 1$.

The inequality $\| a\xi \|_E \leq \| a \|_\infty \| \xi \|_E$ for all $a \in l_\infty, \xi \in E$ immediately follows from Proposition(6.1.20)(iii). In particular, if $E = l_\infty$, then the norm $\|\cdot\|_E$ is equivalent to $\|\cdot\|_\infty$; for example, this is the case for any Lorentz space $(l_\psi, \|\cdot\|_\psi)$, where $\psi: [0, \infty) \to \mathbb{R}$ is an arbitrary nonnegative increasing concave function with the properties $\psi(0) = 0, \psi(+0) \neq 0, \lim_{t\to\infty} \psi(t) < \infty$(see details in [263]).

The spaces $(c_0, \|\cdot\|_\infty), (l_p, \|\cdot\|_p), 1 \le p < \infty$(respectively, $(l_p, \|\cdot\|_p)$ for $0 < p < 1$), where

$$l_p = \left\{ \{\xi_n\}_{n=1}^\infty \in c_0 : \| \{\xi_n\} \|_p = \left(\sum_{n=1}^\infty |\xi_n|^p \right)^{\frac{1}{p}} < \infty \right\}$$

are examples of the classical symmetric Banach (respectively, quasi-Banach) sequence spaces in c_0.

Let $(E, \|\cdot\|_E)$ be a symmetric quasi-Banach sequence space. For every $\xi = \{\xi_n\}_{n=1}^\infty \in E, m \in \mathbb{N}$, we set

$$\sigma_m(\xi) = (\underbrace{\xi_1, \dots, \xi_1}_{m\ times}, \underbrace{\xi_2, \dots, \xi_2}_{m\ times}, \dots),$$
$$\eta^{(1)} = (\xi_1, \underbrace{0, \dots, 0}_{m-1\ times}, \xi_2, \underbrace{0, \dots, 0}_{m-1\ times}, \dots),$$
$$\eta^{(2)} = (0, \xi_1, \underbrace{0, \dots, 0}_{m-2\ times}, 0, \xi_2, \underbrace{0, \dots, 0}_{m-2\ times}, \dots),$$
$$\cdots,$$
$$\eta^{(m)} = (\underbrace{0, \dots, 0}_{m-1\ times}, \xi_1, \underbrace{0, \dots, 0}_{m-1\ times}, \xi_2, \dots).$$

Since $(\eta^{(1)})^* = (\eta^{(2)})^* = \cdots = (\eta^{(m)})^* = \xi^* \in E$, it follows $\eta^{(1)}, \dots, \eta^{(m)} \in E$. Consequently, $\sigma_m(\xi) = \eta^{(1)} + \eta^{(2)} + \cdots + \eta^{(m)} \in E$, i.e. σ_m is a linear operator from E into E. In addition, we have

$$\| \sigma_m(\xi) \|_E = \| \eta^{(1)} + \eta^{(2)} + \cdots + \eta^{(m)} \|_E \le C(\| \eta^{(1)} \|_E + \| \eta^{(2)} + \eta^{(3)} + \cdots + \eta^{(m)} \|_E)$$
$$\le C(\| \eta^{(1)} \|_E + C(\| \eta^{(2)} \|_E + \| \eta^{(3)} + \cdots + \eta^{(m)} \|_E))$$
$$\le (C + C^2 + \cdots + C^{m-1} + C^{m-1}) \| \xi \|_E,$$

where C is the modulus of concavity of the quasi-norm $\|\cdot\|_E$, in particular $\| \sigma_m \|_{\mathcal{B}(E,E)} \le C + C^2 + \cdots + C^{m-2} + 2C^{m-1}$ for all $m \in \mathbb{N}$.

Proposition (6.1.21)[251]: The inequalities

$$(\xi + \eta)^* \le \sigma_2(\xi^* + \eta^*), (\xi\eta)^* \le \sigma_2(\xi^*\eta^*)$$

hold for all $\xi = \{\xi_n\}_{n=1}^\infty, \eta = \{\eta_n\}_{n=1}^\infty \in l_\infty$.

Proof. Since $f_{\xi+\eta}(t) = \sum_{n=1}^\infty (\xi_n + \eta_n)\chi_{[n-1,n)}(t) = f_\xi(t) + f_\eta(t), t > 0$, we have by Proposition (6.1.20) (iv) that

$$\sum_{n=1}^\infty (\xi_n + \eta_n)^* \chi_{[n-1,n)}(2t) = f_{\xi+\eta}^*(2t) = (f_\xi + f_\eta)^*(2t)$$
$$\le f_\xi^*(t) + f_\eta^*(t) = \sum_{n=1}^\infty (\xi_n^* + \eta_n^*)\chi_{[n-1,n)}(t) = (\sigma_2(\xi^* + \eta^*))_n \chi_{[n-1,n)}(2t)$$

for all $t > 0$, where $\{(\sigma_2(\xi^* + \eta^*))_n\}_{n=1}^\infty = \sigma_2(\xi^* + \eta^*)$. In other words, $(\xi + \eta)^* \le \sigma_2(\xi^* + \eta^*)$. The proof of the inequality $(\xi\eta)^* \le \sigma_2(\xi^*\eta^*)$ is very similar (one needs to use Proposition (6.1.20)(v)) and is therefore omitted.

For a symmetric quasi-Banach sequence space $(E, \|\cdot\|_E)$, we set

$$C_E := \{x \in K(H) : \{s_n(x)\}_{n=1}^\infty \in E\}, \qquad \| x \|_{C_E} := \| s_n(x) \|_E, x \in C_E.$$

If $E = l_p$ (respectively, $E = c_0$) then $C_{l_p} = C_p, \|\cdot\|_{C_{l_p}} = \|\cdot\|_{C_p}, 0 < p < \infty$ (respectively, $C_{c_0} = K(H), \|\cdot\|_{C_{c_0}} = \|\cdot\|_{\mathcal{B}(H)}$).

A quasi-Banach vector sublattice $(E, \|\cdot\|_E)$ in l_∞ is said to be p-convex, $0 < p < \infty$, if there is a constant M, so that

$$\left\|\left(\sum_{i=1}^{n}|x_i|^p\right)^{\frac{1}{p}}\right\|_E \leq M\left(\sum_{i=1}^{n}\| x_i \|_E^p\right)^{\frac{1}{p}} \qquad (2)$$

for every finite collection $\{x_i\}_{i=1}^{n} \subset E, n \in \mathbb{N}$.

If the estimate (2) holds for elements from a symmetric quasi-Banach ideal $(I, \|\cdot\|_I)$ of compact operators from $\mathcal{B}(H)$, then the ideal $(I, \|\cdot\|_I)$ is said to be p-convex. As already stated in Theorem (6.1.5), for every symmetric Banach (respectively, symmetric p-convex quasi-Banach, $0 < p < \infty$) sequence space E in c_0 the couple $(C_E, \|\cdot\|_{C_E})$ is a symmetric Banach (respectively, p-convex symmetric quasi-Banach) ideal of compact operators in $\mathcal{B}(H)$.

Thus, for every symmetric Banach (p-convex quasi-Banach) sequence space $(E, \|\cdot\|_E)$ the corresponding symmetric Banach (p-convex quasi-Banach) ideal $(C_E, \|\cdot\|_{C_E})$ of compact operators from $\mathcal{B}(H)$ is naturally constructed. This extends the classical Calkin correspondence [254].

Conversely, if $(I, \|\cdot\|_I)$ is a symmetric quasi-Banach ideal $(I, \|\cdot\|_I)$ of compact operators from $\mathcal{B}(H)$, then it is of the form C_{E_I} with $\|\cdot\|_I = \|\cdot\|_{C_{E_I}}$ for the corresponding symmetric quasi-Banach sequence space $(E_I, \| \xi \|_{E_I})$. The definition of the latter space is given below. Denote by E_I the set of all $\xi \in c_0$, for which there exists some $x \in I$, such that $\xi^* = \{s_n(x)\}_{n=1}^{\infty}$. For $\xi \in E_I$ with $\xi^* = \{s_n(x)\}_{n=1}^{\infty}, x \in I$ set $\| \xi \|_{E_I} = \| x \|_I$.

Fix an orthonormal set $\{e_n\}_{n=1}^{\infty}$ in H and for every $\xi = \{\xi_n\}_{n=1}^{\infty} \in c_0$ consider the diagonal operator $x_\xi \in K(H)$ defined as follows

$$x_\xi(\varphi) = \sum_{n=1}^{\infty} \xi_n c_n(\varphi) e_n\,,$$

where $c_n(\varphi) = (\varphi, e_n), \varphi \in H$. If $\xi \in E_I$, then $\xi^* = \{s_n(x)\}_{n=1}^{\infty}$ for some $x \in I$, and due to equalities $\{s_n(x_{\xi^*})\}_{n=1}^{\infty} = \{\xi_n^*\} = \{s_n(x)\}_{n=1}^{\infty}$ we have $x_{\xi^*} \in I$ and $\| x_{\xi^*} \|_I = \| x \|_I = \| \xi \|_{E_I}$ (see Proposition (6.1.11)(b)). Moreover, since $\{s_n(x_\xi)\}_{n=1}^{\infty} = \{s_n(x_{\xi^*})\}_{n=1}^{\infty}$ and $x_{\xi^*} \in I$, it follows that $x_\xi \in I$ and $\| \xi \|_{E_I} = \| x_\xi \|_I$. Thus, a sequence $\xi \in c_0$ is contained in E_I, if and only if operators x_ξ and x_{ξ^*} are in I, in addition, $\| \xi \|_{E_I} = \| x_{\xi^*} \|_I = \| x_\xi \|_I$. In particular, if $\eta \in c_0, \xi \in E_I, \eta^* \leq \xi^*$, then $\eta \in E_I$ and $\| \eta \|_{E_I} \leq \| \xi \|_{E_I}$.

Theorem (6.1.22)[251]: For any symmetric quasi-Banach ideal I of compact operators from $\mathcal{B}(H)$ the couple $(E_I, \|\cdot\|_{E_I})$ is a symmetric quasi-Banach sequence space in c_0 with the modulus of concavity which does not exceed the modulus of concavity of the quasinorm $\|\cdot\|_I$, in addition, $C_{E_I} = I$ and $\|\cdot\|_{C_{E_I}} = \|\cdot\|_I$.

Proof. If $\xi, \eta \in E_I$, then $x_\xi, x_\eta \in I$, hence $x_\xi + x_\eta \in I$. Since

$$(x_\xi + x_\eta)(\varphi) = \sum_{n=1}^{\infty} \xi_n c_n(\varphi) e_n + \sum_{n=1}^{\infty} \eta_n c_n(\varphi) e_n = \sum_{n=1}^{\infty} (\xi_n + \eta_n) c_n(\varphi) e_n = x_{\xi+\eta}(\varphi),$$
$$\varphi \in H,$$

we have $x_{\xi+\eta} \in I$. Consequently, $\xi + \eta \in E_I$, moreover,

$$\| \xi + \eta \|_{E_I} = \| x_{\xi+\eta} \|_I = \| x_\xi + x_\eta \|_I \leq C(\| x_\xi \|_I + \| x_\eta \|_I) = C(\| \xi \|_{E_I} + \| \eta \|_{E_I}),$$

where C is the modulus of concavity of the quasi-norm $\|\cdot\|_I$.

Now, let $\xi \in E_I, \alpha \in \mathbb{R}$. Since

$$x_{\alpha\xi}(\varphi) = \sum_{n=1}^{\infty} \alpha\xi_n c_n(\varphi) e_n = \alpha x_\xi(\varphi), \qquad \varphi \in H,$$

we have $\alpha\xi \in E_I$ and $\| \alpha\xi \|_{E_I} = \| x_{\alpha\xi} \|_I = \| \alpha x_\xi \|_I = |\alpha| \, \| x_\xi \| = |\alpha| \, \| \xi \|_{E_I}$.
It is easy to see that $\| \xi \|_{E_I} \geq 0$ and $\| \xi \|_{E_I} = 0 \Leftrightarrow \xi = 0$.
Hence, E_I is a solid rearrangement-invariant subspace in c_0 and $\|\cdot\|_{E_I}$ is a quasi-norm on E_I.
Let us show that $(E_I, \|\cdot\|_{E_I})$ is a quasi-Banach space. Let $|||\cdot|||_I$ (respectively, $|||\cdot|||_{E_I}$) be a p-additive (respectively, q-additive) quasi-norm equivalent to the quasi-norm $\|\cdot\|_I$ (respectively, $\|\cdot\|_{E_I}$), $0 < p, q \leq 1$.
Let $\xi^{(k)} = \{\xi_n^{(k)}\}_{n=1}^{\infty} \in E_I$ and $|||\xi^{(k)} - \xi^{(m)}||| \to 0$ for $k, m \to \infty$. Then $\| x_{\xi(k)} - x_{\xi\,(m)} \|_I \to 0$ and $|||x_{\xi\,(k)} - x_{\xi\,(m)}|||_I^p \to 0$ for $k, m \to \infty$, i.e. $x_{\xi\,(k)}$ is a Cauchy sequence in (I, d_I), where $d_I(x, y) = |||x - y|||_I^p$. Since (I, d_I) is a complete metric space, there exists an operator $x \in I$ such that $|||x_{\xi(k)} - x|||_I^p \to 0$ for $k \to \infty$. If pn is the one-dimensional projection onto subspace spanned by e_n, then

$$\xi^{(k)} p_n = p_n x_{\xi_n^{(k)}} p_n \xrightarrow{\|\cdot\|_I} p_n x p_n := \lambda_n p_n,$$
$$0 = p_n x \xi_n^{(k)} p_m \to p_n x p_m, \qquad n \neq m.$$

Hence, x is also a diagonal operator, i.e. $= x_\xi$, where $\xi = \{\lambda_n\}_{n=1}^{\infty}$. Since $x \in I$ we have $\xi \in E_I$, moreover, $\| \xi^{(k)} - \xi \|_{E_I} = \| x_{\xi(k)} - x_\xi \|_\xi \to 0$ for $k \to \infty$.
Consequently, $(E_I, \|\cdot\|_{E_I})$ is a symmetric quasi-Banach sequence space in c_0.
Now, let us show that $C_{E_I} = I$ and $\| x \|_{C_{E_I}} = \| x \|_I$ for all $x \in I$. Let $\in C_{E_I}$, i.e. $\{s_n(x)\}_{n=1}^{\infty} \in E_I$. Hence, there exists an operator $y \in I$, such that $s_n(x) = s_n(y), n \in \mathbb{N}$. Consequently, $x \in I$, moreover, $\| x \|_I = \| \{s_n(x)\}_{n=1}^{\infty} \|_{E_I} = \| x \|_{C_{E_I}}$. Conversely, if $x \in I$, then $\{s_n(x)\}_{n=1}^{\infty} \in E_I$ and therefore $x \in C_{E_I}$.
The definition of symmetric Banach (p-convex quasi-Banach) ideal $(C_E, \|\cdot\|_{C_E})$ of compact operators from $\mathcal{B}(H)$ jointly with Theorem (6.1.22) implies the following corollary:
Corollary (6.1.23)[251]: Let $(E, \|\cdot\|_E)$ be a symmetric Banach (p-convex quasi-Banach) sequence space from c_0. Then $E_{C_E} = E$ and $\|\cdot\|_{E_{C_E}} = \|\cdot\|_E$.
Proof. If $\xi \in E$, then $x_{\xi^*} \in C_E$, and due to the equality $\{s_n(x_{\xi^*})\}_{n=1}^{\infty} = \xi^*$, we have $\xi \in E_{C_E}$ and $\| \xi \|_{E_{C_E}} = \| x_{\xi^*} \|_{C_E} = \| \xi^* \|_E = \| \xi \|_E$. The converse inclusion $E_{C_E} \subset E$ may be proven similarly.
Let G, F be solid rearrangement-invariant spaces in c_0. It is easy to see that G and F are ideals in the algebra l_∞, in particular, it follows from the assumptions $|\xi| \leq |\eta|, \xi \in l_\infty, \eta \in G$ that $\xi \in G$, i.e. G and F are solid linear subspaces in l_∞. We define F -dual space $F{:}\,G$ of G by setting

$$F{:}\,G = \{\xi \in l_\infty : \xi\eta \in F, \forall \eta \in G\}.$$

It is clear that $F{:}\,G$ is an ideal in l_∞ containing c_{00}. If $\subset F$, then $F{:}\,G = l_\infty$, in particular, $l_\infty{:}\,G = l_\infty$ for any solid rearrangement-invariant space G. However, if $G \nsubseteq F$, then $F{:}\,G \neq l_\infty$.
Proposition (6.1.24)[251]: If $F{:}\,G \neq l_\infty$, then $F{:}\,G \subset c_0$.
Proof. Suppose that there exists $\xi = \{\xi_n\}_{n=1}^{\infty} \in (F{:}\,G), \xi \notin c_0$. Let $\alpha_n = sign\xi_n, n \in \mathbb{N}, \eta = \{\eta_n\}_{n=1}^{\infty} \in G$. Obviously, $\{\alpha_n\eta_n\}_{n=1}^{\infty} \in G$ and hence, $|\xi|\eta = \{\xi_n\alpha_n\eta_n\}_{n=1}^{\infty} \in F$ for all $\eta \in G$, that is $|\xi| \in (F{:}\,G)$, and, in addition, $|\xi\,| \notin c_0$. This implies that there exists a

subsequence $0 \neq |\xi_{n_k}| \to \alpha > 0$ for $k \to \infty$. Consider a sequence $\zeta = \{\zeta_k\}_{k=1}^{\infty}$ from $l_\infty \setminus c_0$ such that $\zeta_k = |\xi_{n_k}|$ and show that $\zeta \in F\colon G$.
For every $\eta = \{\eta_n\}_{n=1}^{\infty} \in G$ define the sequence $a_\eta = \{a_n\}_{n=1}^{\infty}$ such that $a_{n_k} = \eta_k$ and $a_n = 0$, if $n \neq n_k, k \in \mathbb{N}$. Since $a_\eta^* = \eta^*$, we have $a_\eta \in G$, and therefore $\zeta\eta = \{|\xi_{n_k}|\eta_k\}_{k=1}^{\infty} = \{|\xi_n|a_n\}_{n=1}^{\infty} = |\xi\,|a_\eta \in F$ for all $\eta \in G$. Consequently, $\zeta = \{\zeta_n\}_{n=1}^{\infty} \in F\colon G$, moreover, $\zeta_n \geq \beta$ for some $\beta > 0$ and all $n \in \mathbb{N}$. Since $F\colon G$ is an ideal in l_∞, it follows that $F\colon G$ is a solid linear subspace in l_∞, containing the sequence $\{\zeta_n\}_{n=1}^{\infty}$ with $\zeta_n \geq \beta > 0, n \in \mathbb{N}$, that implies $F\colon G = l_\infty$.
Proposition (6.1.25)[251]: If $F\colon G \neq l_\infty$, then $F\colon G = \{\xi \in c_0\colon \xi^*\eta^* \in F, \forall \eta \in G\}$.
Proof. By Proposition (6.1.24), we have that $F\colon G \subset c_0$. Let $\xi = \{\xi_n\}_{n=1}^{\infty} \in c_0$ and $\xi^*\eta^* \in F$ for all $\eta \in G$. Due to Proposition (6.1.21), we have $(\xi\eta)^* \leq \sigma_2(\xi^*\eta^*) \in F$, i.e. $(\xi\eta)^* \in F$. Since F is a symmetric sequence space, it follows that $\xi\eta \in F$ for all $\eta \in G$, i.e. $\xi \in F\colon G$. Conversely, suppose that $\xi = \{\xi_n\}_{n=1}^{\infty} \in F\colon G$. Let $\alpha_n = sign\xi_n, \eta = \{\eta_n\}_{n=1}^{\infty} \in G$. Then $\{\alpha_n\eta_n\}_{n=1}^{\infty} \in G$, and therefore $|\xi|\eta = \{\xi_n\alpha_n\eta_n\}_{n=1}^{\infty} \in F$ for all $\eta \in G$, i.e. $|\xi| \in F\colon G \subset c_0$. Since $|\xi\,| = \{|\xi_n|\}_{n=1}^{\infty} \in c_0$, there exists a bijection of the set $\mathbb{N}$ of natural numbers, such that $\xi^* = |\xi_{\pi(n)}|$. For linear bijective mapping $U_\pi\colon l_\infty \to l_\infty$ defined by $U_\pi(\{\eta_n\}_{n=1}^{\infty}) = \{\eta_{\pi(n)}\}_{n=1}^{\infty}$ we have $U_\pi(\eta\zeta) = U_\pi(\eta)U_\pi(\zeta), \left(U_\pi(\zeta)\right)^* = \zeta^*, (U_\pi^{-1}(\zeta))^* = \zeta^*$ for all $\zeta \in l_\infty$, in particular, $U_\pi(E) = E$ for any solid rearrangement-invariant space $E \subset l_\infty$. Consequently, for all $\eta \in G$ we have $\xi^*\eta^* = U_\pi(|\xi\,|)U_\pi(U_\pi^{-1}(\eta^*)) = U_\pi(|\xi\,|U_\pi^{-1}(\eta^*)) \in F$.
Propositions (6.1.24) and (6.1.25) imply the following corollary.
Corollary (6.1.26)[251]: $F\colon G$ is a solid rearrangement-invariant space, moreover, if $F\colon G \neq l_\infty$, then $c_{00} \subset F\colon G \subset c_0$.
Proof. The definition of $F\colon G$ immediately implies that $F\colon G$ is an ideal in l_∞ and $c_{00} \subset F\colon G$. If $F\colon G \neq l_\infty$, then, due to Proposition (6.1.24), we have $F\colon G \subset c_0$.

In the case when $F\colon G \neq l_\infty$, we have for any $\xi \in c_0, \eta \in F\colon G, \xi^* \leq \eta^*, \zeta \in G$ that $\xi^*\zeta^* \leq \eta^*\zeta^* \in F$ (see Proposition (6.1.25)). Consequently, $\xi^*\zeta^* \in F$ for any $\zeta \in G$, which implies the inclusion $\xi \in F\colon G$.
We need some complementary properties of singular values of compact operators. For every operator $x \in \mathcal{B}(H)$ define the decreasing rearrangement $\mu(x,t)$ of x by setting

$$\mu(x,t) = \inf\{s > 0\colon tr(|x| > s) \leq t\}, \qquad t > 0$$

(see e.g. [19]). If $x \in K(H)$, then

$$\mu(x,t) = \sum_{n=1}^{\infty} s_n(x)\chi_{[n-1,n)}(t) = f^*_{\{s_n(x)\}_{n=1}^{\infty}}(t).$$

In [19] it is established that for every $x, y \in \mathcal{B}(H)$ the inequalities

$$\mu(x+y, t+s) \leq \mu(x,t) + \mu(y,s),$$
$$\mu(xy, t+s) \leq \mu(x,t)\mu(y,s)$$

hold, in particular, if $x, y \in K(H)$, then

$$\{s_n(x+y)\}_{n=1}^{\infty} \leq \sigma_2(\{s_n(x) + s_n(y)\}_{n=1}^{\infty}), \tag{3}$$
$$\{s_n(xy)\}_{n=1}^{\infty} \leq \sigma_2(\{s_n(x)s_n(y)\}_{n=1}^{\infty}). \tag{4}$$

Let I, J be symmetric quasi-Banach ideals of compact operators from $\mathcal{B}(H)$ and $I \nsubseteq J$. In this case, $J\colon I \subset K(H)$ see Proposition (6.1.14) and $E_I \nsubseteq E_J$ (see Theorem (6.1.22)), therefore $E_J\colon E_I \subset c_0$ see Proposition (6.1.24). The following proposition establishes that the set of operators belonging to the J-dual space $J\colon I$ of I coincides with the set

$$C_{E_J:E_I} = \{x \in K(H): \{s_n(x)\}_{n=1}^{\infty} \in E_J: E_I\}.$$

Proposition (6.1.27)[251]: $J: I = C_{E_J:E_I}$.

Proof. Let $a \in J: I$. We claim that $a \in C_{E_J:E_I}$, i.e. $\xi = \{s_n(a)\}_{n=1}^{\infty} \in E_J: E_I$. For any sequence $\eta \in E_I$ consider operators x_ξ and x_{η^*}. Since $x_\xi \in J: I, x_{\eta^*} \in I$, we have $x_\xi x_{\eta^*} \in J$. On the other hand, $x_\xi x_{\eta^*}(\varphi) = \|\cdot\|_H - \lim_{n\to\infty} \left(\sum_{k=1}^{n} s_n(a) c_k \left(x_{\eta^*}(\varphi)\right) e_k\right) = s_n(a)\eta_n^* c_n(\varphi) e_n = x_{\xi\eta^*}(\varphi)$ for all $\varphi \in H$. Thus $x_{\xi\eta^*} \in J, i.e. \xi\eta^* \in E_J$. Consequently, $\{s_n(a)\}_{n=1}^{\infty} \in E_J: E_I$ see Proposition (6.1.25) yielding our claim .

Conversely, let $\in C_{E_J:E_I}$, i.e. $\{s_n(a)\}_{n=1}^{\infty} \in E_J: E_I$. Due to (4), for all $x \in I$ we have $\{s_n(ax)\}_{n=1}^{\infty} \le \sigma_2(\{s_n(a)s_n(x)\}_{n=1}^{\infty})$. Since $\{s_n(a)s_n(x)\}_{n=1}^{\infty} \in E_J$, it follows that $\sigma_2(\{s_n(a)s_n(x)\}_{n=1}^{\infty}) \in E_J$, and therefore $\{s_n(ax)\}_{n=1}^{\infty} \in E_J$, i.e. $ax \in J$. Consequently, $a \in J: I$.

Let I, J be symmetric quasi-Banach ideals of compact operators from $\mathcal{B}(H), I \nsubseteq J$ and $J: I$ be the J-dual space of I. For any $a \in J: I$ define a linear mapping $Ta: I \to J$ by setting $T_a(x) = ax, x \in I$.

Proposition (6.1.28)[251]: T_a is a continuous linear mapping from I into J for every $a \in J: I$.

Proof. Let $a \in J: I, \xi = \{s_n(a)\}_{n=1}^{\infty}, x_k \in I$ and $\| x_k \|_I \to 0$ for $k \to \infty$. Then $\xi^{(k)} = \{s_n(x_k)\}_{n=1}^{\infty} \in E_I$ and $\| \xi^{(k)} \|_{E_I} \to 0$. By Proposition (6.1.19), for every subsequence $\{\xi^{(k_l)}\}_{l=1}^{\infty}$ there exists a subsequence $\{\xi^{(k_{l_s})}\}_{s=1}^{\infty}$ such that $\xi^{(k_{l_s})} \xrightarrow{(r)} 0$ for $s \to \infty$, i.e. there exist $0 \le \eta \in E_I$ and a sequence $\{\varepsilon_s\}_{s=1}^{\infty}$ of positive numbers decreasing to zero such that $|\xi^{(k_{l_s})}| \le \varepsilon_s \eta$. Since $a \in J: I$, we have $\xi \in E_J : E_I$ (see Proposition (6.1.27)), and therefore $\zeta = \xi\eta \in E_J$, in addition, $\zeta \ge 0$. Since $|\xi\xi^{(k_{l_s})}| \le \varepsilon_s \zeta$, it follows that $\xi\, \xi^{(k_{l_s})} \xrightarrow{(r)} 0$. By Proposition (6.1.19), we have $\| \xi\xi^{(k)} \|_{E_J} \to 0$. Consequently,

$$\| ax_k \|_J = \| \{s_n(ax_k)\} \|_{E_J} \le \| \sigma_2(\xi\, \xi^{(k)}) \|_{E_J} \le 2C \| \xi\, \xi^{(k)} \|_{E_J} \to 0 \text{ for } k \to \infty.$$

By Proposition (6.1.28), T_a is a bounded linear operator from I into J, therefore $\| T_a \|_{\mathcal{B}(I,J)} = \sup\{\| T_a(x) \|_J: \| x \|_I \le 1\} = sup\{\| ax \|_J: \| x \|_I \le 1\} < \infty$, i.e. for all $a \in J: I$ the quantity

$$\| a \|_{J:I} := \sup\{\| ax \|_J : x \in I, \| x \|_I \le 1\}$$

is well-defined.

Theorem (6.1.29)[251]: Let I, J be symmetric quasi-Banach ideals of compact operators in $\mathcal{B}(H)$ such that $I \nsubseteq J$. Then $(J: I, \|\cdot\|_{J:I})$ is a symmetric quasi-Banach ideal of compact operators whose modulus of concavity does not exceed the modulus of concavity of the quasi-norm $\|\cdot\|_J$, in addition, $\| ax \|_J \le \| a \|_{J:I} \| x \|_I$ for all $a \in J: I, x \in I$.

Proof. Since $\| \cdot\|_{\mathcal{B}(I,J)}$ is a quasi-norm with the modulus of concavity which does not exceed the modulus of concavity of the quasi-norm $\|\cdot\|_J$, we see that $\|\cdot\|_{J:I}$ is a quasi-norm on $J: I$ with the modulus of concavity which does not exceed the modulus of concavity of the quasi-norm $\|\cdot\|_J$.

If $y \in \mathcal{B}(H), a \in J: I$, then

$$\| ya \|_{J:I} = \sup\{\| (ya)x \|_J: x \in I, \| x \|_I \le 1\}$$
$$\le \sup\{\| y \|_{\mathcal{B}(H)} \| ax \|_J: x \in I, \| x \|_I \le 1\} = \| y \|_{\mathcal{B}(H)} \| a \|_{J:I}.$$

Since $yx \in I$ for all $x \in I$ and $\| yx \|_I \leq \| y \|_{\mathcal{B}(H)} \| x \|_I$ then for $y \neq 0$ and $\| x \|_I \leq 1$ we have $\left\| \frac{yx}{\|y\|_{\mathcal{B}(H)}} \right\|_I \leq 1$. Hence,

$$\| ay \|_{J:I} = \sup\{\| a(yx) \|_J : x \in I, \| x \|_I \leq 1\}$$
$$= \| y \|_{\mathcal{B}(H)} \sup\left\{ \left\| a\left(\frac{yx}{\| y \|_{\mathcal{B}(H)}}\right) \right\|_J : x \in I, \| x \|_I \leq 1 \right\}$$
$$\leq \| y \|_{\mathcal{B}(H)} \sup\{\| ax \|_J : x \in I, \| x \|_I \leq 1\} = \| y \|_{\mathcal{B}(H)} \| a \|_{I:J}.$$

If p is a one-dimensional projection from $\mathcal{B}(H)$, then $p \in I, \| p \|_I = 1$, and so

$$\| p \|_{J:I} = \sup\{\| px \|_J : x \in I, \| x \|_I \leq 1\} \geq \| p \|_J = 1.$$

On the other hand, for $x \in I$ with $\| x \|_I \leq 1$ we have $\| x \|_{\mathcal{B}(H)} \leq 1$ (see Proposition (6.1.11)(c)), and therefore

$$\| px \|_J = \| p(px) \|_J \leq \| px \|_{\mathcal{B}(H)} \| p \|_J \leq 1.$$

Consequently, $\| p \|_{J:I} = 1$.

Thus, $\|\cdot\|_{J:I}$ is a symmetric quasi-norm on the two-sided ideal $J:I$. The inequality $\| ax \|_J \leq \| a \|_{J:I} \| x \|_I$ immediately follows from the definition of $\|\cdot\|_{J:I}$.

Let us show that $(J:I, \|\cdot\|_{J:I})$ is a quasi-Banach space.

Denote by $|||\cdot|||_J$ (respectively $|||\cdot|||_{J:I}$) a p-additive (respectively, q-additive) quasi-norm on J (respectively, on $J:I$) which is equivalent to the quasi-norm $\|\cdot\|_J$ (respectively, $\|\cdot\|_{J:I}$), where $0 < p, q \leq 1$. In particular, we have $\alpha_1 |||x|||_J \leq \| x \|_J \leq \beta_1 |||x|||_J$ and $\alpha_2 |||a|||_{J:I} \leq \| a \|_{J:I} \leq \beta_2 |||a|||_{J:I}$ for all $x \in J, a \in J:I$ and some constants $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$.

Let $d_J(x,y) = |||x-y|||_J^p, d_{J:I}(a,b) = |||a-b|||_{J:I}^q$ be metrics on J and $J:I$ respectively.

Let $\{a_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $(J:I, d_{J:I})$, i.e. $|||a_n - a_m|||_{J:I}^q \leq \varepsilon^q$ for all $n, m > n(\varepsilon), \varepsilon > 0$, thus

$$|||a_n x - a_m x|||_J \leq \frac{1}{\alpha_1} \| a_n x - a_m x \|_J \leq \frac{1}{\alpha_1} \| a_n - a_m \|_{J:I} \| x \|_I$$
$$\leq \frac{\beta_2}{\alpha_1} \left|||a_n - a_n||\right|_{J:I} \| x \|_I \leq \frac{\beta_2}{\alpha_1} \varepsilon \| x \|_I \tag{5}$$

for all $x \in I, n, m > n(\varepsilon)$. Consequently, the sequence $\{a_n x\}_{n=1}^{\infty}$ is a Cauchy sequence in $(J, d_J), x \in I$. Since the metric space (J, d_J) is complete, there exists an operator $z(x) \in J$ such that $|||a_n x - z(x)|||_J^p \to 0$ for $n \to \infty$. Since

$$\| a_n x - z(x) \|_{\mathcal{B}(H)} \leq \| a_n x - z(x) \|_J \leq \beta_1 |||a_n x - z(x)|||_J,$$

it follows that $\| a_n x - z(x) \|_{\mathcal{B}(H)} \to 0$.

Since

$$\| a_n - a_m \|_{\mathcal{B}(H)} \leq \| a_n - a_m \|_{J:I} \leq \beta_2 |||a_n - a_m|||_{J:I} \to 0$$

for $n, m \to \infty$, there exists $a \in \mathcal{B}(H)$ such that $\| a_n - a \|_{\mathcal{B}(H)} \to 0$ for $n \to \infty$. For an arbitrary $x \in I$, we have

$\| a_n x - ax \|_{\mathcal{B}(H)} \leq \| a_n - a \|_{\mathcal{B}(H)} \| x \|_I \to 0$ for $n \to \infty$.

Thus, $ax = z(x)$ for all $x \in I$. Since $z(x) \in J$ for all $x \in I$, it follows that $a \in J:I$, moreover, due to (5), $\| a_n x - ax \|_J \leq \frac{\beta_1 \beta_2}{\alpha_1} \varepsilon \| x \|_I$ for $n \geq n(\varepsilon)$ and for all $x \in I$. Consequently,

$$|||a_n - a|||_{J:I} \leq \frac{1}{\alpha_2} \| a_n - a \|_{J:I} = \frac{1}{\alpha_2} sup\{\| a_n x - ax \|_J : x \in I, \| x \|_J \leq 1\} \leq \frac{\beta_1 \beta_2}{\alpha_1 \alpha_2} \varepsilon$$

for $n \geq n(\varepsilon)$, i.e. $|||a_n - a|||_{J:I} \to 0$. Thus, the metric space $(J: I, d_{J:I})$ is complete, i.e. $(J: I, \|\cdot\|_{J:I})$ is a quasi-Banach space.

Remark (6.1.30)[251]: Since the quasi-norms $\|\cdot\|_J$ and $\|\cdot\|_{J:I}$ are symmetric, for all $a \in J: I$ the relations

$$\| a \|_{J:I} = \| a^* \|_{J:I} = \sup\{\| a^* x \|_J : x \in I, \| x \|_I \leq 1\}$$
$$= \sup\{\| x^* a \|_J : x \in I, \| x \|_I \leq 1\} = \sup\{\| xa \|_J : x \in I, \| x \|_I \leq 1\}$$

hold, i.e. for any $a \in J: I$ we have

$$\| a \|_{J:I} = \sup\{\| xa \|_I : x \in I, \| x \|_I \leq 1\}. \quad (6)$$

When $I \subseteq J$ we have $J: I = \mathcal{B}(H)$ and for any $a \in J: I$ the mapping $T_a(x) = ax$ is a bounded linear operator from I into J. As in the proof of Theorem (6.1.29) we may establish that $\| a \|_{J:I} = \sup\{\| ax \|_J : x \in I, \| x \|_I \leq 1\}$ is a complete symmetric quasi-norm on $J: I$. In addition, in case $I = J$ we have

$$\| a \|_{I:I} = \sup\{\| ax \|_I : x \in I, \| x \|_I \leq 1\}$$
$$\leq \sup\left\{\| a \|_{\mathcal{B}(H)} \| x \|_I : x \in I, \| x \|_I \leq 1\right\} \leq \| a \|_{\mathcal{B}(H)},$$

i.e.

$$\| a \|_{I:I} \leq \| a \|_{\mathcal{B}(H)} \text{ for all } a \in I: I. \quad (7)$$

Thus, the norm $\|\cdot\|_{\mathcal{B}(H)}$ and the quasi-norm $\|\cdot\|_{I:I}$ are equivalent.

Now, let G and F be arbitrary symmetric quasi-Banach sequence spaces in l_∞. For every $\xi \in F: G$ set

$$\| \xi \|_{F:G} = \sup\{\| \xi\eta \|_F : \eta \in G, \| \eta \|_G \leq 1\}.$$

The following theorem is a ‘‘commutative’’ version of Theorem (6.1.29).

Theorem (6.1.31)[251]: If $G \not\subseteq F$, then $(F: G, \|\cdot\|_{F:G})$ is a symmetric quasi-Banach sequence space in c_0 with the modulus of concavity, which does not exceed the modulus of concavity of the quasi-norm $\|\cdot\|_F$, in addition, $\| \xi\eta \|_F \leq \| \xi \|_{F:G} \| \eta \|_G$ for all $\xi \in F: G, \eta \in G$.

Proof. Since $G \not\subseteq F$, it follows that $F \neq l_\infty, F: G \neq l_\infty$, and therefore, according to Corollary(6.1.26), $F: G$ is a solid rearrangement invariant space and $F: G \subset c_0$.

As in the proof of Theorem (6.1.29) it is established that $\|\cdot\|_{F:G}$ is a complete quasi-norm on $F: G$ with the modulus of concavity which does not exceed the modulus of concavity of the quasi-norm $\|\cdot\|_F$.

If $\xi, \eta \in F: G$ and $\xi^* \leq \eta^*$, then $\xi^* = a\eta^*$ for some $a \in l_\infty$ with $\| a \|_\infty \leq 1$. Hence,

$$\| \xi^* \|_{F:G} = \| a\eta^* \|_{F:G} = \sup\{\| a\eta^*\zeta \|_F : \zeta \in G, \| \zeta \|_G \leq 1\}$$
$$\leq \| a \|_\infty \sup\{\| \eta^*\zeta \|_F : \zeta \in G, \| \zeta \|_G \leq 1\} \leq \| \eta^* \|_{F:G}.$$

Let us show that $\| \xi \|_{F:G} = \| \xi^* \|_{F:G}$ for all $\xi = \{\xi_n\}_{n=1}^\infty \in F: G$. Since $\xi \in c_0$ there exists a bijection $\pi: \mathbb{N} \to \mathbb{N}$ such that $U_\pi(\xi) := \{\xi_{\pi(n)}\}_{n=1}^\infty = \{\xi_n^*\}_{n=1}^\infty = \xi^*$. It is clear that the mapping $U_\pi: l_\infty \to l_\infty$ defined by the equality $U_\pi(\eta) = U_\pi(\{\eta_n\}_{n=1}^\infty) = \{\eta_{\pi(n)}\}_{n=1}^\infty, \eta = \{\eta_n\}_{n=1}^\infty \in l_\infty$, is a linear bijective mapping, such that $U_\pi(\eta\zeta) = U_\pi(\eta)U_\pi(\zeta), \eta, \zeta \in l_\infty$. In addition, $U_\pi(G) = G, U_\pi(F) = F$, and $\| U_\pi(\eta) \|_G = \| \eta \|_G, \| U_\pi(\zeta) \|_F = \| \zeta \|_F$ for all $\eta \in G, \zeta \in F$.

Since $U_\pi(\xi) = \xi^*$, we have

$$\| \xi^* \|_{F:G} = \sup\{\| U_\pi(\xi)\eta \|_F : \eta \in G, \| \eta \|_G \leq 1\} = \sup\{\| U_\pi(\xi U_\pi^{-1}(\eta)) \|_F : \eta \in G, \| \eta \|_G \leq 1\}$$
$$= \sup\{\| \xi U_\pi^{-1}(\eta) \|_F : \eta \in G, \| \eta \|_G \leq 1\} = \sup\{\| \xi\zeta \|_F : U_\pi(\zeta) \in G, \| U_\pi(\zeta) \|_G \leq 1\}$$
$$= \sup\{\| \xi\zeta \|_F : \zeta \in G, \| \zeta \|_G \leq 1\} = \| \xi \|_{F:G}.$$

Thus, from $\xi, \eta \in F: G, \xi^* \leq \eta^*$ it follows that

$$\| \xi \|_{F:G} = \| \xi^* \|_{F:G} \leq \| \eta^* \|_{F:G} = \| \eta \|_{F:G}.$$

The equality $\| \xi \|_{F:G} = 1$ is established similarly to the equality $\| p \|_{J:I} = 1$, where p is a one-dimensional projection from $\mathcal{B}(H)$ (see the proof of Theorem (6.1.29)).

Consequently, $(F:G, \|\cdot\|_{F:G})$ is a symmetric quasi-Banach sequence space in c_0. The inequality $\| \xi\eta \|_F \leq \| \xi \|_{F:G} \| \eta \|_G$ immediately follows from the definition of $\|\cdot\|_{F:G}$.

Let I, J be symmetric quasi-Banach ideals of compact operators from $\mathcal{B}(H), I \nsubseteq J$. By Proposition (6.1.27), $: I = C_{E_J:E_I}$, i.e. $C_{E_J:E_I}$ is a two-sided ideal of compact operators from $\mathcal{B}(H)$. For every $a \in C_{E_J:E_I}$ we set

$$\| a \|_{C_{E_J:E_I}} := \| \{s_n(a)\} \|_{E_J:E_I} .$$

Proposition (6.1.32)[251]: $\|\cdot\|_{C_{E_J:E_I}}$ is a symmetric quasi-norm on $C_{E_J:E_I}$.

Proof. Obviously, $\| a \|_{C_{E_J:E_I}} \geq 0$ for all $a \in C_{E_J:E_I}$ and $\| a \|_{C_{E_J:E_I}} = 0 \Leftrightarrow a = 0$. If $a, b \in C_{E_J:E_I}$, $\lambda \in \mathbb{C}$, then

$$\| \lambda a \|_{C_{E_J:E_I}} = \| \{s_n(\lambda a)\}_{n=1}^{\infty} \|_{E_J:E_I} = |\lambda| \| a \|_{E_J:E_I}$$

and

$$\begin{aligned} \| a + b \|_{C_{E_J:E_I}} = \| \{s_n(a + b)\} \|_{E_J:E_I} &\overset{(3)}{\leq} \| \sigma_2(\{s_n(a) + s_n(b)\}) \|_{E_J:E_I} \\ &\leq 2C \| \{s_n(a)\} + \{s_n(b)\} \|_{E_J:E_I} \\ &\leq 2C^2(\| \{s_n(a)\} \|_{E_J:E_I} + \| \{s_n(b)\} \|_{E_J:E_I}) \\ &= 2C^2(\| a \|_{C_{E_J:E_I}} + \| b \|_{C_{E_J:E_I}}). \end{aligned}$$

Hence, $\|\cdot\|_{C_{E_J:E_I}}$ is a quasi-norm on $C_{E_J:E_I}$ and the modulus of concavity of $\|\cdot\|_{C_{E_J:E_I}}$ does not exceed $2C^2$, where C is the modulus of concavity of the quasi-norm $\|\cdot\|_{E_J}$.

Since $s_n(xay) \leq \| x \|_{\mathcal{B}(H)} \| y \|_{\mathcal{B}(H)} s_n(a)$ for all $a \in K(H), x, y \in \mathcal{B}(H), n \in \mathbb{N}$ (see Proposition (6.1.7)), it follows

$$\| xay \|_{C_{E_J:E_I}} = \| \{s_n(xay)\} \|_{E_J:E_I} \leq \| x \|_{\mathcal{B}(H)} \| y \|_{\mathcal{B}(H)} \| a \|_{C_{E_J:E_I}}.$$

It is clear that $\| p \|_{C_{E_J:E_I}} = 1$ for every one-dimensional projection p.

Thus, $\|\cdot\|_{C_{E_J:E_I}}$ is a symmetric quasi-norm on $C_{E_J:E_I}$.

Theorem (6.1.33)[251]: Let I, J be symmetric quasi-Banach ideals of compact operators from $\mathcal{B}(H)$ and $I \nsubseteq J$. Then

(i) $E_{J:I} = E_J : E_I$ and $\|\cdot\|_{E_J:E_I} \leq \|\cdot\|_{E_{J:I}} \leq 2C \|\cdot\|_{E_J:E_I}$, where C is the modulus of concavity of the quasi-norm $\|\cdot\|_J$;

(ii) $J : I = C_{E_J:E_I}$ and $\|\cdot\|_{C_{E_J:E_I}} \leq \|\cdot\|_{J:I} \leq 2C \|\cdot\|_{C_{E_J:E_I}}$, where C is the modulus of concavity of the quasi-norm $\|\cdot\|_{E_J}$.

Proof. If $\xi = \xi^* \in E_{J:I}$, then $x_\xi \in J : I$ see Theorem (6.1.22). Hence, for every $\eta = \eta^* \in E_I$ we have $x\eta \in I$ and $x_{\xi\eta} = x_\xi \, x_\eta \in J$, i.e. $\xi\eta \in E_J$. Therefore, due to Proposition (6.1.25), $\xi \in E_J : E_I$, in addition,

$$\begin{aligned} \| \xi \|_{E_{J:I}} = \| x_\xi \|_{J:I} &= \sup\{\| x_\xi y \|_J : y \in I, \| y \|_J \leq 1\} \\ &\geq \sup\{\| x_\xi \, x_\eta \|_J : \eta \in E_I, \| \eta \|_{E_I} \leq 1\} \\ &= sup\{\| x_\xi \eta \|_J : \eta \in E_I, \| \eta \|_{E_I} \leq 1\} \\ &= \sup\{\| \xi\eta \|_{E_I} : \eta \in E_I, \| \eta \|_{E_I} \leq 1\} = \| \xi \|_{E_J:E_I}. \end{aligned}$$

Conversely, if $\xi = \xi^* \in E_J: E_I$, then $x_\xi \in C_{E_J:E_I} = J: I$ (see Proposition (6.1.27)), and so $\xi \in E_{J:I}$. Moreover,

$$\| \xi \|_{E_{J:I}} = \| x_\xi \|_{J:I} = \sup\{\| x_\xi\, y \|_J: y \in I, \| y \|_I \leq 1\}$$
$$= \sup\{\| x_{\{s_n(x_\xi\, y)\}} \|_J: y \in I, \| y \|_I \leq 1\}$$
$$\overset{(4)}{\leq} \sup\{\| x_{\sigma_2(\{\xi\, s_n(y)\})} \|_J: y \in I, \| y \|_I \leq 1\}$$
$$\leq 2C\, sup\{\| \xi\, \{s_n(y)\} \|_{E_J} : y \in I, \| y \|_I \leq 1\}$$
$$\leq 2C \sup\left\{\| \xi\eta \|_{E_J} : \eta \in E_I, \| \eta \|_{E_I} \leq 1\right\} = 2C \| \xi \|_{E_J:E_I} .$$

Thus, $E_{J:I} = E_J: E_I$ and $\| \xi \|_{E_J:E_I} \leq \| \xi \|_{E_{J:I}} \leq 2C \| \xi \|_{E_J:E_I}$ for all $\xi \in E_{J:I}$.
(ii) The equality $J: I = C_{E_J:E_I}$ is proven in Proposition (6.1.27). For an arbitrary $a \in J: I$ we have

$$\| a \|_{C_{E_J:E_I}} = \| \{s_n(a)\} \|_{E_J:E_I} = \sup\left\{\| \{s_n(a)\}\eta \|_{E_J} : \eta \in E_I, \| \eta \|_{E_I} \leq 1\right\}$$
$$= \sup\left\{\| x_{\{s_n(a)\}} x_\eta \|_J: x_\eta \in I, \| x_\eta \|_I \leq 1\right\}$$
$$\leq \sup\left\{\| x_{\{s_n(a)\}} y \|_J: y \in I, \| y \|_I \leq 1\right\} = \| x_{\{s_n(a)\}} \|_{J:I} = \| a \|_{J:I}.$$

On the other hand,

$$\| a \|_{J:I} = \sup\left\{\| ay \|_J: y \in I, \| y \|_I \leq 1\right\} = \sup\left\{\| \{s_n(ay)\}_{n=1}^{\infty} \|_{E_J} : y \in I, \| y \|_I \leq 1\right\}$$
$$\overset{(4)}{\leq} \sup\{\| \sigma_2(\{s_n(a)s_n(y)\}_{n=1}^{\infty}) \|_{E_J} : y \in I, \| y \|_I \leq 1\}$$
$$= 2C \sup\{\| \{s_n(a)s_n(y)\} \|_{E_J} : y \in I, \| y \|_I \leq 1\}$$
$$= 2C \| \{s_n(a)\} \|_{E_J:E_I} = 2C \| a \|_{C_{E_J:E_I}}.$$

Since $(J : I, \|\cdot\|_{J:I})$ is a quasi-Banach space (see Theorem (6.1.29)) and quasi-norms $\|\cdot\|_{J:I}$ and $\|\cdot\|_{C_{E_J:E_I}}$ are equivalent (see Theorem (6.1.33)(ii)), we have the following corollary.

Corollary (6.1.34)[251]: For any symmetric quasi-Banach ideals I, J of compact operators from $\mathcal{B}(H), I \nsubseteq J$, the couple $(C_{E_J:E_I}, \|\cdot\|_{C_{E_J:E_I}})$ is a symmetric quasi-Banach ideal of compact operators from $\mathcal{B}(H)$.

The following theorem gives the full description of the set $Der(I, J)$.

Theorem (6.1.35)[251]: (i) Let I and J be symmetric quasi-Banach ideals of compact operators from $\mathcal{B}(H), I \nsubseteq J$. Then any derivation δ from I into J has a form $\delta = \delta_a$ for some $a \in C_{E_J:E_I}$ and $\| a + \alpha 1 \|_{\mathcal{B}(H)} \leq \| \delta_a \|_{\mathcal{B}(I,J)}$ for some $\alpha \in C$. Conversely, if $\in C_{E_J:E_I}$, then the restriction of δ_a on I is a derivation from I into J. In addition, $\| \delta_a \|_{\mathcal{B}(I,J)} \leq 2C \| a \|_{J:I}$, where C is the modulus of concavity of the quasi-norm $\|\cdot\|_J$;
(ii) Let G and F be symmetric Banach (respectively, F is a p-convex, G is a q-convex quasi-Banach with $0 < p, q < \infty$) sequence spaces in c_0 and $G \nsubseteq F$. Then any derivation $\delta: C_G \to C_F$ has a form $\delta = \delta_a$ for some $a \in C_{F:G}$ and $\| a + \alpha 1 \|_{\mathcal{B}(H)} \leq \| \delta_a \|_{\mathcal{B}(C_G,C_F)}$ for some $\alpha \in C$. Conversely, if $a \in C_{G:F}$, then the restriction of δ_a on C_G is a derivation from C_G into C_F. In addition, $\| \delta_a \|_{\mathcal{B}(C_G,C_F)} \leq 2C \| a \|_{C_F:C_G}$, where C is the modulus of concavity of the quasi-norm $\|\cdot\|_{C_F}$.

Proof. (i) By Theorem (6.1.17), any derivation $\delta : I \to J$ has a form $\delta = \delta_a$ for some $a \in J: I$, in addition $\| a + \alpha 1 \|_{\mathcal{B}(H)} \leq \| \delta_a \|_{\mathcal{B}(I,J)}$ for some $\alpha \in C$. Since $J: I = C_{E_J:E_I}$ (see Theorem (6.1.33)), we have $a \in C_{E_J:E_I}$.
Conversely, if $\in C_{E_J:E_I}$, then $a \in J: I$, and, according to Theorem (6.1.17), $\delta_a(I) \subset J$.

Moreover,

$$\begin{aligned}\| \delta_a \|_{\mathcal{B}(I,J)} &= \sup\{\| \delta_a(x) \|_J : x \in I, \| x \|_I \le 1\} \\ &= \sup\{\| ax - xa \|_J : x \in I, \| x \|_I \le 1\} \\ &\le \sup\{C(\| ax \|_J + \| xa \|_J) : x \in I, \| x \|_I \le 1\} \\ &\overset{(6)}{=} 2C \sup\{\| ax \|_J : x \in I, \| x \|_I \le 1\} = 2C \| a \|_{J:I}. \qquad (8)\end{aligned}$$

Item (ii) follows from (i) and Theorems (6.1.5) and (6.1.33). The inequality $\| \delta_a \|_{\mathcal{B}(C_G,C_F)} \le 2C \| a \|_{C_G:C_F}$ is proven in the same manner.

We illustrate Theorem (6.1.35) with an example drawn from the theory of Lorentz and Marcinkiewicz sequence spaces. Let $\omega = \{\omega_n\}_{n=1}^{\infty}$ be a decreasing weight sequence of positive numbers. Letting $W(j) = \sum_{n=1}^{j} w_n, j \in \mathbb{N}$, we shall assume that $W(\infty) = \sum_{n=1}^{\infty} w_n = \infty$.

The Lorentz sequence space $l_\omega^p, 1 \le p < \infty$, consists of all sequences $\xi = \{\xi_n\}_{n=1}^{\infty} \in c_0$ such that

$$\| \xi \|_{l_\omega^p} = \left(\sum_{n=1}^{\infty} (\xi_n^*)^p w_n\right)^{\frac{1}{p}} < \infty.$$

The Lorentz (Marcinkiewicz) sequence space $m_W^p, 1 \le p < \infty$, is the space of all sequences $\xi = \{\xi_n\}_{n=1}^{\infty} \in c_0$ satisfying

$$\| \xi \|_{m_W^p} = sup_{k\ge 1} \left(\frac{\sum_{n=1}^{k} (\xi_n^*)^p}{W_k}\right)^{\frac{1}{p}} < \infty.$$

It is well known (see e.g. [264] and [265]) that $(l_\omega^p, \|\cdot\|_{l_\omega^p})$ and $(m_W^p, \|\cdot\|_{m_W^p})$ are symmetric Banach sequence spaces in c_0.

Hence, $(C_{l_\omega^p}, \|\cdot\|_{C_{l_\omega^p}})$ and $(C_{m_W^p}, \|\cdot\|_{C_{m_W^p}})$ are symmetric Banach ideals of compact operators Theorem (6.1.5). Since $l_1 : l_\omega = m_W^1$ (see e.g. [264]) it follows that $l_p : l_\omega^p = m_W^p$ for every $1 \le p < \infty$ [265]. By Theorem (6.1.33), $C_p : C_{l_\omega^p} = C_{m_W^p}$ and $\| a \|_{C_p:C_{l_\omega^p}} \le 2 \| a \|_{C_{m_W^p}}$ for all $a \in C_p : C_{l_\omega^p}$. From Theorem (6.1.35) (ii), we obtain the following example significantly extending similar results from [252].

Corollary (6.1.36)[251]: A linear mapping $\delta: C_{l_\omega^p} \to C_p, 1 \le p < \infty$is a derivation if and only if $\delta = \delta_a$ for some $a \in C_{m_W^p}$, in addition, $\| \delta \|_{\mathcal{B}(C_{l_\omega^p},C_p)} \le 2 \| a \|_{C_p:C_{l_\omega^p}} \le 4 \| a \|_{C_{m_W^p}}$.

In conclusion, note that, by Theorem (6.1.13), (8), any derivation δ from a symmetric quasi-Banach ideal I into a symmetric quasi-Banach ideal J, such that $I \subseteq J$, has a form $\delta = \delta_a$ for some $a \in \mathcal{B}(H)$ and, in addition, $\| a \|_{\mathcal{B}(H)} \le \| \delta_a \|_{\mathcal{B}(I,J)} \le 2C \| a \|_{J:I}$, where C is the modulus of concavity of the quasi-norm $\|\cdot\|_J$. Moreover, for the case when $I = J$ we have $\| a \|_{\mathcal{B}(H)} \le \| \delta_a \|_{\mathcal{B}(I,J)} \le 2C \| a \|_{\mathcal{B}(H)}$, where C is the modulus of concavity of the quasi-norm $\|\cdot\|_I$ (see (7)). This complements results from [257].

Section (6.2): Algebras of Locally Measurable Operators are Inner

We first recall an important result due to Ringrose [27] that any derivation acting on a C^*-algebra M with values in a Banach M-bimodule is automatically norm-continuous, which extends the classical result of Sakai [29] that every derivation of a C^*-algebra is norm-continuous. In the special case when M is an AW^*-algebra (in particular, a W^*-algebra), these results are strongly linked with another classical fact that every derivation on M is inner [115], [29]. The objects of study are analogues of these classical results in numerous topological $*$-algebras of unbounded operators, which by their algebraic and order-topological structure are still somewhat close to C^*, W^* and AW^*-algebras, but which are neither Banach, nor even locally convex. The algebras which we study are of prime importance for the theory of non-commutative integration, initiated by Segal's [31].

Let M be a W^*-algebra. There are three fundamentally important $*$-algebras of 'measurable' operators affiliated with M, which may be thought of as far reaching generalizations of the algebra of all measurable functions on an arbitrary measure space (the latter situation can be made precise when M is commutative). The first two algebras, $LS(M)$ and $S(M)$ of all locally measurable and, respectively, of all measurable operators are defined for every M, the third algebra, $S(M,\tau)$, of all τ-measurable operators from $S(M)$ is defined for a semifinite M equipped with a faithful normal semifinite trace τ (see [52], [117], [31], [118]). All M-bimodules of interest (such as non-commutative L_p-spaces, or more generally non-commutative symmetric spaces associated with M) in non-commutative integration theory and/or in (semifinite version of) noncommutative geometry are solid subspaces in these algebras [278], [33]. We study derivations on such algebras and in such bimodules. The main technical impediment in this study is the fact that all these three $*$-algebras are not even locally convex algebras [279] and this fact renders all rich techniques developed for such study in Banach algebras (see, for example, [14]).

We detailed discussion of our main results, we would like to recall a slightly larger picture, indicating other areas of study of algebras of unbounded operators to which our results are immediately applicable or relevant. In particular, there is a direct connection between our algebras of measurable operators and extended W^*-algebras (briefly, EW^*-algebras) introduced in [276] and generalized B^*-algebras (briefly, GB^*-algebras) (introduced in [268] for locally convex case and further generalized in [275]). The bounded part $A(B_0)$ of every GB^*-algebra A is a C^*-algebra [275] and the algebra A is a topological bimodule over $A(B_0)$. In the case when A is an EW^*-algebra, its bounded part is a W^*-algebra. It is natural to ask whether results similar to Ringrose and Sakai-Kadison theorems hold for GB^* and EW^*-algebras. Some particular results in this direction may be found in [89], [267], [271], [22], [277].

Due to the characterization of EW^*-algebras given in [273], we know that every such algebra A with the bounded part $A_b = M$ is in fact a solid $*$-subalgebra in the $*$-algebra $LS(M)$, in particular, the algebra $LS(M)$ is the largest EW^*-algebra in the class of all EW^*-algebras with the bounded part coinciding with M. In addition, every EW^*-algebra A with the bounded part $A_b = M$ is a topological $*$-algebra of unbounded operators with respect to the nonlocally convex topology $t(M)$, the so-called local measure topology on $LS(M)$. The preceding comment brings in focus the main theme and its connection with our article [270], where we proved that each derivation $\delta: M \to A$ extends up to a derivation from $LS(M)$ into $LS(M)$. In this respect, the problem of $t(M)$-continuity of a given derivation from a von Neumann algebra M with values in EW^*-algebra A with the bounded part $A_b = M$ and related question whether δ is inner can be reduced to the similar questions stated for extension of δ up to a derivation on $LS(M)$.

In the setting of commutative W^*-algebras these problem are fully resolved in [107]. In the setting of von Neumann algebras of type I, a thorough treatment of this problem may be found in [2], [106]. In [2], [107] contain examples of non-inner derivations on the $*$-algebra $LS(M)$, which are not continuous with respect to the topology $t(M)$ of convergence

locally in measure on $LS(M)$. On the other hand, it is shown in [2] that in the special case when M is a properly infinite von Neumann algebra of type I, every derivation of $LS(M)$ is continuous with respect to the local measure topology $t(M)$. Using a completely different technique, a similar result was also obtained in [106] under the additional assumption that the predual space M_* to M is separable. It is of interest to observe that an analogue of this result (that is the continuity of an arbitrary derivation on $(LS(M), t(M))$) also holds for any von Neumann algebra M of type III [105]. In [105], the following problem is formulated (Problem 3): Let M be a von Neumann algebra of type II and let τ be a faithful normal semifinite trace on M. Is any derivation on the $*$-algebra $S(M,\tau)$ equipped with the measure topology t_τ necessarily continuous? In [108], this problem is solved affirmatively for properly infinite algebras M. In view of the example we mentioned above, a natural problem (similar to Problem 3 from [105]) is whether any derivation on the $*$-algebra $LS(M)$ is necessarily continuous with respect to the topology $t(M)$, where M is a properly infinite von Neumann algebra of type II. A positive solution of this problem may be found in [270], where it is established that any derivation $\delta: A \to LS(M)$ is $t(M)$-continuous, where M is a properly infinite von Neumann algebra, A is arbitrary EW^*-subalgebra in $LS(M)$ with $A_b = M$. This result naturally suggests that δ is actually inner.
For von Neumann algebras of type I and III this problem is solved in [2], [105]. It is proved that for every $t(M)$-continuous derivation δ acting on the algebra $LS(M)$ there exists $a \in LS(M)$, such that $\delta(x) = ax - xa = [a, x]$ for all $x \in LS(M)$, that is the derivation δ is inner. As a corollary, we obtain complete resolution of similar questions for EW^*-algebras.

The proof proceeds in several stages. We introduce and study the properties of so-called λ-systems for a self-adjoint derivation $\delta: LS(M) \to LS(M)$. After that we supply the proof of the main result (Theorem (6.2.19)) showing that every $t(M)$-continuous derivation $\delta: LS(M) \to LS(M)$ is necessarily inner. In particular, in view of the result of [270], this implies that for a properly infinite von Neumann algebra M every derivation on $LS(M)$ is inner. We give quick applications of Theorem (6.2.19) for $t(M)$-continuous derivations acting on an EW^*-algebra A with the bounded part $A_b = M$. We consider the class of Banach M-bimodules of locally measurable operators $\mathcal{E} \subset LS(M)$. This class contains all non-commutative symmetric spaces. We contain our second main result, Theorem (6.2.32), showing that every derivation δ from a von Neumann algebra M with values in a Banach M-bimodule $\mathcal{E}$ is inner, that is, it has the form $\delta(x) = [d, x] = \delta_d(x)$for all $x \in M$ and some $d \in \mathcal{E}$. In particular, δ is a continuous derivation from $(M, \|\cdot\|_M)$ into $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$. In addition, the operator $d \in \mathcal{E}$ may be chosen so that $\|d\|_{\mathcal{E}} \leq 2\|\delta\|_{M\to\mathcal{E}}$.
We use terminology and notation from the von Neumann algebra theory [23], [29], [33] and the theory of locally measurable operators from [113], [117], [118].

Let H be a Hilbert space, let $B(H)$ be the $*$-algebra of all bounded linear operators on H, and let $\mathbf{1}$ be the identity operator on H. Given a von Neumann algebra M acting on H, denote by $Z(M)$ the centre of M and by $P(M) = \{p \in M : p = p^2 = p^*\}$ the lattice of all projections in M.
Recall that two projections $e, f \in P(M)$ are called equivalent (notation: $e \sim f$) if there exists a partial isometry $u \in M$ such that $u^*u = e$ and $uu^* = f$. For projections $e, f \in P(M)$ notation $e \preccurlyeq f$ means that there exists a projection $q \in P(M)$ such that $e \sim q \leq f$. A projection $0 \neq p \in P(M)$ is called finite if the conditions $q \leq p, q \sim p$ imply that $q = p$. Denote by $P_{fin}(M)$ the set of all finite projections in M.

A linear operator $x: \mathfrak{D}(x) \to H$, where the domain $\mathfrak{D}(x)$ of x is a linear subspace of H, is said to be affiliated with M if $yx \subseteq xy$ for all y from the commutant M' of the algebra M.
A densely defined closed linear operator x (possibly unbounded) affiliated with M is said to be measurable with respect to M if there exists a sequence $\{p_n\}_{n=1}^{\infty} \subset P(M)$ such that $p_n \uparrow \mathbf{1}, p_n(H) \subset \mathfrak{D}(x)$ and $p_n^{\perp} = \mathbf{1} - p_n \in P_{fin}(M)$ for every $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers. Let us denote by $S(M)$ the set of all measurable operators.
Let $x, y \in S(M)$. It is well known that $x + y$ and xy are densely defined and preclosed operators. Moreover, the closures $\overline{x+y}$ (strong sum), $\overline{xy}$ (strong product) and x^* are also measurable, and equipped with these operations (see [31]) $S(M)$ is a unital $*$-algebra over the field $\mathbb{C}$ of complex numbers. It is clear that M is a $*$-subalgebra of $S(M)$.
A densely defined linear operator x affiliated with M is called locally measurable with respect to M if there is a sequence $\{z_n\}_{n=1}^{\infty}$ of central projections in M such that $z_n \uparrow \mathbf{1}, z_n(H) \subset \mathfrak{D}(x)$ and $xz_n \in S(M)$ for all $n \in \mathbb{N}$.
The set $LS(M)$ of all locally measurable operators (with respect to M) is a unital $*$-algebra over the field $\mathbb{C}$ with respect to the same algebraic operations as in $S(M)$ [118] and $S(M)$ is a $*$-subalgebra of $LS(M)$. For every operator $x \in LS(M)$ the left support $l(x)$ and the right support $r(x)$ are always equivalent [113], in addition, $r(|x|) = r(x) = l(x^*)$, where $|x| = (x^*x)^{1/2}$.

If M is finite, or if $dim(Z(M)) < \infty$, the algebras $S(M)$ and $LS(M)$ coincide [113]. If a von Neumann algebra M is of type III and $\dim(Z(M)) = \infty$, then $S(M) = M$ and $LS(M) \neq M$ [113].
For every $x \in S(Z(M))$, there exists a sequence $\{z_n\}_{n=1}^{\infty} \subset P(Z(M))$ such that $z_n \uparrow \mathbf{1}$ and $xz_n \in M$ for all $n \in \mathbb{N}$. This means that $x \in LS(M)$. Hence, $S(Z(M))$ is a $*$-subalgebra of $LS(M)$ and $S(Z(M))$ coincides with the centre of the $*$-algebra $LS(M)$.
For every subset $E \subset LS(M)$, the sets of all self-adjoint (respectively, positive) operators in E is denoted by E_h (respectively, E_+). The partial order in $LS(M)$ is defined by its cone $LS_+(M)$ and is denoted by $\leq$.
Let $\{z_i\}_{i\in I}$ be a family of pairwise orthogonal non-zero central projections from M with $\bigvee_{i\in I} z_i = \mathbf{1}$, where I is an arbitrary set of indexes (in this case, the family $\{z_i\}_{i\in I}$ is called a central decomposition of the unity $\mathbf{1}$). Consider the $*$-algebra $\prod_{i\in I} LS(z_i M)$ with the coordinate-wise operations and involution and for every $x \in LS(M)$ set

$$\phi(x) := \{z_i x\}_{i\in I} .$$

In [102], it is proved that the mapping ϕ is a $*$-isomorphism from the $*$-algebra $LS(M)$ onto $\prod_{i\in I} LS(z_i M)$. From here immediately follows:
Proposition (6.2.1)[266]: Given any central decomposition $\{z_i\}_{i\in I}$ of the unity $\mathbf{1}$ and any family of elements $\{x_i\}_{i\in I}$ in $LS(M)$ there exists a unique element $x \in LS(M)$ such that $z_i x = z_i x_i$ for all $i \in I$.
Let x be a closed operator with the dense domain $\mathfrak{D}(x)$ in H, let $x = u|x|$ be the polar decomposition of the operator x, where u is a partial isometry in $B(H)$ such that u^*u is the right support $r(x)$ of x. It is known that $x \in LS(M)$ (respectively, $x \in S(M)$) if and only if $|x| \in LS(M)$ (respectively, $|x| \in S(M)$) and $u \in M$ [113].
Let $\varphi_\lambda(t) = \chi_{(-\infty,\lambda]}(t)$ be the real-valued function on $(-\infty, +\infty)$ for which $\varphi_\lambda(t) = 1$, if $t \leq \lambda$ and $\varphi_\lambda(t) = 0$, if $t > \lambda$. For every self-adjoint operator x affiliated with M the spectral family of projections $E_\lambda(x) = \varphi_\varphi(x), \lambda \in \mathbb{R}$, of x belongs to M [113]. A locally measurable operator x is measurable if and only if $E_\lambda^{\perp}(|x|) \in P_{fin}(M)$ for some $\lambda > 0$ [113].

We recall the definition of the local measure topology. Firstly, let M be a commutative von Neumann algebra. Then M is $*$-isomorphic to the $*$-algebra $L^{\infty}(\Omega,\Sigma,\mu)$ of all essentially bounded measurable complex-valued functions defined on a measure space (Ω,Σ,μ) with the measure μ satisfying the direct sum property (we identify functions that are equal almost everywhere) (see [33]). The direct sum property of the measure μ means that the Boolean algebra of all projections of the $*$-algebra $L^{\infty}(\Omega,\Sigma,\mu)$ is order complete, and for any non-zero $p \in P(M)$ there exists a non-zero projection $q \leq p$ such that $\mu(q) < \infty$. The direct sum property of the measure μ is equivalent to the fact that the functional $\tau(f) := \int_{\Omega} f\, d\mu$ is a semi-finite normal faithful trace on the algebra $L^{\infty}(\Omega,\Sigma,\mu)$.
Consider the $*$-algebra $LS(M) = S(M) = L^{0}(\Omega,\Sigma,\mu)$ of all measurable almost everywhere finite complex-valued functions defined on (Ω,Σ,μ) (functions that are equal almost everywhere are identified). Define on $L^{0}(\Omega,\Sigma,\mu)$ the local measure topology $t(L^{\infty}(\Omega))$, that is, the Hausdorff vector topology, whose base of neighbourhoods of zero is given by

$$W(B,\varepsilon,\delta) := \{f \in L^{0}(\Omega,\Sigma,\mu): there\ exists\ a\ set\ E \in \Sigma\ such\ that$$
$$E \subseteq B, \mu(B \setminus E) \leq \delta, f\chi_E \in L^{\infty}(\Omega,\Sigma,\mu), \|f\chi_E\|_{L^{\infty}(\Omega,\Sigma,\mu)} \leq \varepsilon\},$$

where $\varepsilon,\delta > 0, B \in \Sigma, \mu(B) < \infty, \chi(\omega) = 1, \omega \in E$ and $\chi(\omega) = 0$, when $\omega \notin E$.
Convergence of a net $\{f_\alpha\}$ to f in the topology $t(L^{\infty}(\Omega))$, denoted by $f_\alpha \xrightarrow{t(L^{\infty}(\Omega))} f$, means that $f_\alpha\chi_B \to f\chi_B$ in measure μ for every $B \in \Sigma$ with $\mu(B) < \infty$. Note, that the topology $t(L^{\infty}(\Omega))$ does not change if the measure μ is replaced with an equivalent measure [118].
Now let M be an arbitrary von Neumann algebra and let φ be a $*$-isomorphism from $Z(M)$ onto the $*$-algebra $L^{\infty}(\Omega,\Sigma,\mu)$, where μ is a measure satisfying the direct sum property. Denote by $L^{+}(\Omega,\Sigma,m)$ the set of all measurable real-valued functions defined on (Ω,Σ,μ) and taking values in the extended half-line $[0,\infty]$ (functions that are equal almost everywhere are identified). It was shown in [31] that there exists a mapping

$$D: P(M) \longrightarrow L^{+}(\Omega,\Sigma,\mu)$$

that possesses the following properties.
(D1) A projection $p \in M$ is finite if and only if $D(p) \in L^{0}_{+}(\Omega,\Sigma,\mu)$.
(D2) If $pq = 0$, then $D(p \vee q) = D(p) + D(q)$.
(D3) If $u \in M$ is a partial isometry, then $D(u^*u) = D(uu^*)$.
(D4) If $z \in P(Z(M))$ and $p \in P(M)$, then $D(zp) = \varphi(z)D(p)$.
(D5) If $p_\alpha, p \in P(M), \alpha \in A$ and $p_\alpha \uparrow p$, then $D(p) = \bigvee_{\alpha \in A} D(p_\alpha)$.
A mapping $D: P(M) \to L^{+}(\Omega,\Sigma,\mu)$ satisfying properties $(D1)$—$(D5)$ is called dimension function on $P(M)$.
The dimension function D also has the following properties [31].
$(D6)$ If $p_n \in P(M)$, $n \in \mathbb{N}$, then $D(\bigvee_{n\geq 1} p_n) \leq \sum_{n=1}^{\infty} D(p_n)$, in addition, when $p_n p_m = 0, n \neq m$, the equality holds.
(D7) If $p_n \in P_{fin}(M), n \in \mathbb{N}, p_n \downarrow 0$, then $D(p_n) \to 0$ almost everywhere.
For arbitrary scalars $\varepsilon,\delta > 0$ and a set $B \in \Sigma, \mu(B) < \infty$, we set

$$V(B,\varepsilon,\delta) := \{x \in LS(M) : \text{there exist } p \in P(M), z \in P(Z(M)), \text{such that } xp \in M,$$
$$\|xp\|_M \leq \varepsilon, \varphi(z^{\perp}) \in W(B,\varepsilon,\delta), D(zp^{\perp}) \leq \varepsilon\varphi(z)\}, \qquad (9)$$

where $\|\cdot\|_M$ is the C^*-norm on M.
It was shown in [118] that the system of sets

$$\{x + V(B,\varepsilon,\delta): x \in LS(M), \varepsilon,\delta > 0, B \in \Sigma, \mu(B) < \infty\}$$

defines a Hausdorff vector topology $t(M)$ on $LS(M)$ such that the sets $\{x + V(B,\varepsilon,\delta)\}, \varepsilon,\delta > 0, B \in \Sigma, \mu(B) < \infty$ form a neighbourhood base of an operator $x \in LS(M)$. It is known that $(LS(M), t(M))$ is a complete topological $*$-algebra, and the topology $t(M)$ does not depend on a choice of dimension function D and on a choice of $*$-isomorphism φ (see [113], [118]).

The topology $t(M)$ on $LS(M)$ is called the local measure topology (or the topology of convergence locally in measure). Note, that in case when $M = B(H)$ the equality $LS(M) = M$ holds [113] and the topology $t(M)$ coincides with the uniform topology, generated by the C^*-norm $\|\cdot\|_{B(H)}$.

We will need the following criterion for convergence of nets with respect to this topology.

Proposition (6.2.2)[266]: [113]. (i) A net $\{p_\alpha\}_{\alpha\in A} \subset P(M)$ converges to zero with respect to the topology $t(M)$ if and only if there is a net $\{z_\alpha\}_{\alpha\in A} \subset P(rZ(M))$ such that $z_\alpha p_\alpha \in P_{fin}(M)$ for all $\alpha \in A, \phi(z_\alpha^\perp) \xrightarrow{t(L^\infty(\Omega))} 0$, and $D(z_\alpha p_\alpha) \xrightarrow{t(L^\infty(\Omega))} 0$, where $t(L^\infty(\Omega))$ is the local measure topology on $L^0(\Omega, \Sigma, \mu)$ and φ is a $*$-isomorphism of $Z(M)$ onto $L^\infty(\Omega, \Sigma, \mu)$.

(ii) A net $\{x_\alpha\}_{\alpha\in A} \subset LS(M)$ converges to zero with respect to the topology $t(M)$ if and only if $E_\lambda^\perp(|x_\alpha|) \xrightarrow{t(M)} 0$ for every $\lambda > 0$, where $E_\lambda^\perp(|x_\alpha|)$ is the spectral family for the operator $|x_\alpha|$.

Since the involution is continuous in the topology $t(M)$, the set $LS_h(M)$ is closed in $(LS(M), t(M))$. The cone $LS_+(M)$ of positive elements is also closed in $(LS(M), t(M))$ [118].

Using Proposition (6.2.2), the following is established:

Proposition (6.2.3)[266]: [270]. If $x_\alpha \in LS(M), 0 \neq z \in P(Z(M))$, then

$$zx_\alpha \xrightarrow{t(M)} 0 \Leftrightarrow zx_\alpha \xrightarrow{t(zM)} 0.$$

Moreover, from Proposition (6.2.2) , it immediately follows that:

Corollary (6.2.4)[266]: If $\{z_\alpha\}_{\alpha\in A} \subset P(Z(M))$ and $z_\alpha \downarrow 0$ then $z_\alpha \xrightarrow{t(M)} 0$.

Let us mention the following important property of the topology $t(M)$.

Proposition (6.2.5)[266]: The von Neumann algebra M is everywhere dense in $(LS(M), t(M))$.

Proof. If $x \in LS(M)$, then there exists a sequence $\{z_n\}_{n=1}^\infty \subset P(Z(M))$ such that $z_n \uparrow \mathbf{1}$ and $xz_n \in S(M)$ for all $n \in \mathbb{N}$. By Corollary (6.2.4), $z_n \xrightarrow{t(M)} \mathbf{1}$, and therefore $xz_n \xrightarrow{t(M)} x$. Consequently, the algebra $S(M)$ is everywhere dense in $(LS(M), t(M))$.

Now let $x \in S(M)$. Then there exists a sequence $\{p_n\}_{n=1}^\infty \subset P(M)$ such that $p_n \uparrow \mathbf{1}, p_n^\perp \in P_{fin}(M)$ and $xp_n \in M$ for any $n \in \mathbb{N}$. According to (D7), we have that $D(p_n^\perp) \xrightarrow{t(L^\infty(\Omega))} 0$, therefore, Proposition (6.2.2)(i) implies the convergence $p_n \xrightarrow{t(M)} \mathbf{1}$ (we set $z_n = \mathbf{1}$). Then $xp_n \xrightarrow{t(M)} x$. It means that the algebra M is everywhere dense in the algebra $S(M)$ with respect to the topology $t(M)$. Thus, the von Neumann algebra M is everywhere dense in $(LS(M), t(M))$.

The lattice $P(M)$ is said to have a countable type if every family of non-zero pairwise orthogonal projections in $P(M)$ is, at most, countable. A von Neumann algebra is said to be σ-finite if the lattice $P(M)$ has a countable type. It is shown in [31] that a finite von Neumann algebra M is σ-finite, provided that the lattice $P(Z(M))$ of central projections has a countable type.

If M is a commutative von Neumann algebra and $P(M)$ has a countable type, then M is $*$-isomorphic to the $*$-algebra $L^\infty(\Omega, \Sigma, \mu)$ with $\mu(\Omega) < \infty$. In this case, the topology $t(L^\infty(\Omega))$ is metrizable and has a base of neighbourhoods of zero consisting of the sets

$W(\Omega, 1/n, 1/n), n \in N$. In addition, $f_n \xrightarrow{t(L^\infty(\Omega))} 0 \Leftrightarrow f_n \to 0$ in measure μ, where $f_n, f \in L^0(\Omega, \Sigma, \mu) = LS(M)$.

We need another basis of neighbourhoods of zero in topology $t(M)$ in the case when the algebra $Z(M)$ is σ-finite. If φ is a $*$-isomorphism from $Z(M)$ onto $L^\infty(\Omega, \Sigma, \mu), \mu(\Omega) < \infty$, then $\tau(x) = \int_\Omega \varphi(x)\, d\mu$ is a faithful normal finite trace on $Z(M)$. For arbitrary positive scalars ε, β and γ, we set

$$V(\varepsilon, \beta, \gamma) := \{x \in LS(M) : \text{there exist } p \in P(M), z \in P(Z(M)), \text{ such that } xp \in M, \|xp\|_M\ \varepsilon, \tau\ (z^\perp)\ \beta, D(zp^\perp) \le \gamma\varphi(z)\}. \quad (10)$$

Proposition (6.2.6)[266]: If the centre $Z(M)$ of the von Neumann algebra M is a σ-finite algebra, then the system of sets given by (10) forms a basis of neighbourhoods of zero in the topology $t(M)$.

Proof. Let $V(\Omega, \varepsilon, \delta)$ be a neighbourhood of zero of the form (10). If $x \in V(\varepsilon, \delta, \varepsilon)$, then there exist $p \in P(M), z \in P(Z(M))$, such that $xp \in M, \|xp\|_M \le \varepsilon, \int_\Omega \varphi(z^\perp) d\mu \le \delta$ and $D(zp^\perp) \le \varepsilon\varphi(z)$. The inequality $\int_\Omega \varphi(z^\perp) d\mu \le \delta$ means that $\varphi(z^\perp) \in W(\Omega, \varepsilon, \delta)$. Hence, $x \in V(\Omega, \varepsilon, \delta)$, that implies the inclusion $V(\varepsilon, \delta, \varepsilon) \subset V(\Omega, \varepsilon, \delta)$.

If $x \in V(\Omega, \varepsilon, \delta)$, then there exist $p \in P(M), z \in P(Z(M))$, such that $\|xp\|_M \le \varepsilon, \phi(z^\perp) \in W(\Omega, \varepsilon, \delta)$ and $D(zp^\perp) \le \varepsilon\varphi(z)$. The inclusion $\varphi(z^\perp) \in W(\Omega, \varepsilon, \delta)$ means that there exists $E \in \Sigma$, such that $\mu(\Omega \setminus E) \le \delta$ and $0 \le \varphi(z^\perp)\chi_E \le \varepsilon$. If $0 < \varepsilon < 1$, then $\phi(z^\perp)\chi_E = 0$, that is $\varphi(z^\perp) \le \chi_\Omega \setminus E$, and therefore $\tau(z^\perp) \le \delta$, hence $x \in V(\varepsilon, \delta, \varepsilon)$.

Let M be an arbitrary von Neumann algebra, let A be a subalgebra in $LS(M)$. A linear mapping $\delta: A \to LS(M)$ is called a derivation on A with values in $LS(M)$, if $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in A$. Each element $a \in A$ defines a derivation $\delta_a(x) := [a, x] = ax - xa$ on A with values in A. The derivations $\delta_a, a \in A$, are said to be inner derivations on A.

We list a few properties of derivations on A which we shall need below.

Lemma (6.2.7)[266]: If $P(Z(M)) \subset A, \delta$ is a derivation on A and $z \in P(Z(M))$, then $\delta(z) = 0$ and $\delta(zx) = z\delta(x)$ for all $x \in A$.

Proof. We have that $\delta(z) = \delta(z^2) = \delta(z)z + z\delta(z) = 2z\delta(z)$. Hence, $z\delta(z) = z(2z\delta(z)) = 2z\delta(z)$, that is $z\delta(z) = 0$. Therefore, we have $\delta(z) = 0$. In particular, $\delta(zx) = \delta(z)x + z\delta(x) = z\delta(x)$.

Lemma (6.2.7) immediately implies the following:

Corollary (6.2.8)[266]: If $z \in P(Z(M)) \subset A$ and δ is a derivation on A, then $\delta(zA) \subset zA$ and the restriction $\delta(z)$ of the derivation δ to zA is a derivation on zA, in addition, if δ is $t(M)$-continuous, then $\delta(z)$ is $t(zM)$-continuous.

Proof. The inclusion $\delta(zA) \subset zA$ holds. Moreover, the linear mapping $\delta^{(z)}: zA \to zA$ has the following property:

$$\delta^{(z)}((zx)(zy)) = \delta(zx)zy + zx\delta(zy) = \delta^{(z)}(zx)zy + zx\delta^{(z)}(zy)$$

for all $x, y \in A$.

If $x_\alpha, x \in zA, x_\alpha \xrightarrow{t(zM)} x$, then $x_\alpha \xrightarrow{t(M)} x$ Proposition (6.2.8), and therefore $\delta^{(z)}(x_\alpha) = z\delta(x_\alpha) \xrightarrow{t(M)} z\delta(x) = \delta^{(z)}(x)$, that implies the convergence $\delta^{(z)}(x_\alpha) \xrightarrow{t(zM)} \delta(z)(x)$ Proposition (6.2.3).

Let A be a subalgebra in $LS(M), 0 \ne e \in P(M) \cap A$, let δ be a derivation on A and let $\delta^{(e)}$ be a linear mapping from eAe into eAe defined by the equality $\delta^{(e)}(x) = e\delta(x)e, x \in$

$eAe \subset A$. If $e = z \in P(Z(M))$, then $\delta^{(e)}$ coincides with the restriction $\delta^{(z)}$ of the derivation δ to $zA = zAz$.

Lemma (6.2.9)[266]: $\delta^{(e)}$ is a derivation on eAe.

Proof. If $x, y \in eAe$, then $x, y \in A$ and

$$\delta^{(e)}(xy) = e(\delta(x)y)e + e(x\delta(y))e$$
$$= (e\delta(x)e)(eye) + (exe)(e\delta(y)e) = \delta^{(e)}(x)y + x\delta^{(e)}(y).$$

Let A be a $*$-subalgebra in $LS(M)$, let δ be a derivation on A with values in $LS(M)$. Let us define a mapping

$$\delta^*: A \to LS(M),$$

by setting $\delta^*(x) = (\delta(x^*))^*, x \in A$. A direct verification shows that δ^* is also a derivation on A.

A derivation δ on A is said to be *self*-adjoint, if $\delta = \delta^*$. Every derivation δ on A can be represented in the form $\delta = Re(\delta) + i\, Im(\delta)$, where $Re(\delta) = (\delta + \delta^*)/2, Im(\delta) = (\delta - \delta^*)/2i$ are self-adjoint derivations on A.

Since $(LS(M), t(M))$ is a topological $*$-algebra, the following result holds.

Lemma (6.2.10)[266]: If A is a $*$-subalgebra in $LS(M)$, then a derivation $\delta: A \to LS(M)$ is continuous with respect to the topology $t(M)$ if and only if the self-adjoint derivations $Re(\delta)$ and $Im(\delta)$ are continuous with respect to this topology.

The following lemma shows that a derivation δ on a $*$-subalgebra of $LS(M)$ is inner if and only if the derivations $Re(\delta)$ and $Im(\delta)$ are inner.

Lemma (6.2.11)[266]: Let A be a $*$-subalgebra in $LS(M)$. A derivation $\delta: A \to A$ is inner derivation on A if and only if $Re(\delta)$ and $Im(\delta)$ are inner derivations.

The proof of the lemma immediately follows from the decomposition of every element $a \in A$ in the form $a = Re(a) + i\, Im(a)$, where $Re(a) = (a + a^*)/2, Im(a) = (a - a^*)/2i$.

Lemma (6.2.12)[266]: Let A and δ be the same as in Corollary (6.2.8) and let $\{z_i\}_{i\in I}$ be a central decomposition of the unity $\mathbf{1}$. If $\delta^{(z_i)} = \delta_{d_i}, di \in z_iA$ is an inner derivation on z_iA for every $i \in I$, then there exists an operator $d \in LS(M)$, such that $\delta(x) = [d, x]$ for all $x \in A$ and $z_id = d_i$ for every $i \in I$.

Proof. Since $\{z_i\}_{i\in I}$ is a central decomposition of the unity $\mathbf{1}, d_i \in z_iA \subset z_iLS(M)$, by Proposition (6.2.1) there exists $d \in LS(M)$, such that $z_id = d_i$ for every $i \in I$. Using Lemma (6.2.7) and the equalities $\delta^{(z_i)}(x) = [d_i, x], x \in z_iA, i \in I$ we have that for all $y \in A, i \in I$ the equalities

$$z_i\delta(y) = \delta^{(z_i)}(z_iy) = [d_i, z_iy] = [z_id, z_iy] = z_i[d, y]$$

hold. Since $\bigvee_{i\in I} z_i = \mathbf{1}$ it follows that $\delta(y) = [d, y]$.

We need the following technical lemma.

Lemma (6.2.13)[266]: Let δ be a derivation on a subalgebra A of $LS(M)$ and $P(M) \subset A$. If $p, q \in P(M)$ and $p\delta(q)p \geq \lambda p$ for some $\lambda > 0$, then

$$r(qp)\delta(l(qp))r(qp) \geq \lambda r(qp).$$

Proof. Set $e = l(qp)$ and $f = r(qp)$. It is clear that $eq = e$ and $pf = f$. In addition, $e = r((qp)^*) = r(pq)$ and $f = l((qp)^*) = l(pq)$.

Since

$$ef = (eq)(pf) = e(qp)f = l(qp)qpr(qp) = qp = (qp)f = q(pf) = qf,$$

we have

$$fe = (ef)^* = (qf)^* = fq$$

and

$$f\delta(e)f = f(f\delta(e))f = f\delta(fe)f - f(\delta(f)e)f$$

$$= f\delta(fq)f - f\delta(f)qf = f\delta(f)qf + f(f\delta(q))f - f\delta(f)qf$$
$$= f\delta(q)f = f(p\delta(q)p)f \geq \lambda fpf = \lambda f.$$

For every $x \in LS(M)$, we set $s(x) = l(x) \vee r(x)$, where $l(x)$ and $r(x)$ are the left and right supports of x, respectively.

Let $M_{h,1} = \{x \in M_h : \|x\|_M \leq 1\}$. We are now prepared to introduce the main technical tool.

Definition (6.2.14)[266]: Fix a positive number λ and a self-adjoint derivation $\delta: LS(M) \to LS(M)$. The set of pairs $S = \{(p_j, x_j) \in P(M) \times M_{h,1} : p_j \neq 0, j \in J\}$ is called a λ-system (for the derivation δ), if

(i) $(p_j \vee s(x_j))(p_i \vee s(x_i)) = 0$ and $(p_j \vee s(x_j))s(\delta(p_i)) = 0$ for $j \neq i, j, i \in J$;

(ii) $s(x_j) \sim p_j$ for all $j \in J$;

(iii) $p_j\delta(x_j)p_j \geq \lambda p_j$ for all $j \in J$.

The projection $\bigvee_{j\in J}(p_j \vee s(x_j) \vee s(\delta(p_j)) \vee s(\delta(p_j \vee s(x_j))))$ is called the support of the λ-system S and is denoted by $s(S)$. If λ-system S is empty, then we set $s(S) = 0$.

A λ-system is said to be maximal if it is not contained in any larger λ-system. The role which is played by the notions introduced above in our study of derivations becomes clearer from the following result. Recall that the (reduced) derivation $\delta(g)$ is introduced in Lemma (6.2.9).

Theorem (6.2.15)[266]: Let $S = \{(p_j, x_j)\}_{j\in J}$ be the maximal λ-system for a self-adjoint derivation $\delta: LS(M) \to LS(M), g = s(S)^{\perp}$ and $\delta^{(g)}(x) = g\delta(x)g, x \in gLS(M)g$. Then

$$\delta^{(g)}(gMg) \subset gMg. \tag{11}$$

Proof. Let us first prove that

$$\delta^{(g)}(q) \subset gMg \text{ and } \left\|\delta^{(g)}(q)\right\|_M \leq \lambda \tag{12}$$

for any projection $q \in P(gMg)$. Since $\delta^* = \delta$, it follows that $\delta(q) \in LS_h(M)$ and therefore $\delta^{(g)}(q) \in LS_h(gMg)$. Let $\delta^{(g)}(q) \neq 0$ and let p be the spectral projection for $\delta^{(g)}(q)$ corresponding to the interval $(\lambda, +\infty)$. It is clear that $p \leq s(\delta^{(g)}(q)) \leq g$.

Suppose that $p \neq 0$, then

$$0 \neq \lambda p \leq p\delta^{(g)}(q)p = p\delta(q)p. \tag{13}$$

Since

$$0 \neq p\delta(q)p = \delta(pq)p - \delta(p)qp = \delta(pq)p - \delta(p)(pq)^*,$$

it follows that $qp = (pq)^* \neq 0$. Consequently,

$$e = l(qp) \neq 0 \quad \text{and} \quad f = r(qp) \neq 0,$$

in addition, $e \sim f$. Since $g = s(S)^{\perp}$, from the inequalities $f \leq p \leq g$ and $e \leq q \leq g$ it follows that

$$(f \vee e)\left(p_j \vee s(x_j)\right) = 0, \ (f \vee e)s(\delta(p_j)) = 0$$

and

$$(p_j \vee s(x_j))\delta(f) = \delta((p_j \vee s(x_j))f) - \delta(p_j \vee s(x_j))f = 0$$

for all $j \in J$. Moreover, according to (13) and Lemma (6.2.13) we have that $f\delta(e)f \leq \lambda f$. Thus, the system $S \cup \{(f, e)\}$ is a λ-system, that contradicts to the maximality of the λ-system S. Consequently, $p = 0$, which implies the inequality $\delta^{(g)}(q) \leq \lambda\mathbf{1}$. Similarly, for projection $(g - q) \leq g$, we obtain that

$$g(\delta(g - q))g = \delta^{(g)}(g - q) \leq \lambda\mathbf{1}.$$

By Lemma (6.2.9), $g\delta(g)g = 0$, and therefore $-g\delta(q)g \leq \lambda\mathbf{1}$. Thus,

$$-\lambda\mathbf{1} \leq g\delta(q)g \leq \lambda\mathbf{1},$$
that is $\delta^{(g)}(q) \in gMg$ and $\left\|\delta^{(g)}(q)\right\|_M \leq \lambda$.
Now, suppose that the inclusion (11) false, that is there exists an element $x \in M_{h,1} \cap (gMg)$, such that $\delta^{(g)}(x) \in LS_h(M) \setminus gMg$. It means that the spectral projection $r = E_{-3\lambda}(\delta(g)(x))$ (or $r = E_{-3\lambda}(\delta(g)(x)))$ for $\delta^{(g)}(x)$ corresponding to the interval $(3\lambda, +\infty)$ (respectively, $(-\infty, -3\lambda)$), is not equal to zero. Replacing, if necessary, x by $-x$, we may assume that $r = E_{3\lambda}^{\perp}(\delta^{(g)}(x)) \neq 0$. It is clear that $r \leq s(\delta^{(g)}(x)) \leq g$ and
$$0 < 3\lambda r \leq r\delta^{(g)}(x)r = r\delta(x)r. \tag{14}$$
By (12), we have that $\left\|\delta^{(g)}(r)\right\|_M \leq \lambda$, and therefore the inclusion $x \in M_{h,1} \cap gMg$ and the equality
$$r\delta(r)xr + rx\delta(r)r = rg\delta(r)gxr + rxg\delta(r)gr$$
imply that
$$\|r\delta(r)xr + rx\delta(r)r\|_M \leq 2\lambda.$$
Consequently,
$$-2\lambda r \leq r\delta(r)xr + rx\delta(r)r \leq 2\lambda r. \tag{15}$$
Using (14) and (15) for $y = rxr$, we obtain that
$$r\delta(y)r = r\delta(rxr)r = r\delta(r)xr + r\delta(x)r + rx\delta(r)r \geq \lambda r > 0, \tag{16}$$
in particular, $y \neq 0$ and $q = s(y) \neq 0$. Let us show that the collection $S \cup \{(q, y)\}$ forms a λ-system. Since $q \leq r \leq g$, from (16) it follows that $q\delta(y)q \geq \lambda q$, in addition
$$(q \vee s(y))(p_j \vee s(x_j)) = 0 = q(p_j \vee s(x_j)), q\delta(p_j) = 0$$
and
$$(p_j \vee s(x_j))\delta(q) = \delta((p_j \vee s(x_j))q) - \delta(p_j \vee s(x_j))q = 0$$
for all $j \in J$. It means that the set $S \cup \{(q, y)\}$ is a λ-system, that contradicts to the maximality of the λ-system S. From obtained contradiction follows the validity of inclusion (11).

The main result is given in Theorem (6.2.18). We need two simple technical lemmas for its proof.

Lemma (6.2.16)[266]: If $\{x_j\}_{j\in J} \subset M_{h,1}, x_i x_j = 0, i \neq j, i, j \in J$, then there exists a unique element $x \in M_{h,1}$, denoted by $\sum_{j\in J} x_j$, such that $xs(x_j) = x_j$ for all $j \in J$ and $\bigvee_{j\in J} s(x_j) = s(x)$.

Proof. Denote by A the commutative von Neumann subalgebra of M, containing the family $\{x_j\}_{j\in J}$. Since A_h is an order complete vector lattice, $\{x_j\}_{j\in J}$ is the family of pairwise disjoint element of A_h and $|x_j| \leq \mathbf{1} \in A$ for all $j \in J$, it follows that there exists a unique element x of A_h such that $|x| \leq \mathbf{1}, xs(x_j) = x_j$ and $s(x) = \bigvee_{j\in J} s(x_j)$.

Let y be another element of $M_{h,1}$, such that $ys(x_j) = x_j$ for all $j \in J$ and $\bigvee_{j\in J} s(x_j) = s(y)$. Then $(x - y)s(x_j) = x_j - x_j = 0$ for any $j \in J$. Therefore, $s(x) = \bigvee_{j\in J} s(x_j) \leq (r(x - y))^{\perp}$ and then
$$x - y = xs(x) - ys(y) = xs(x) - ys(x) = (x - y)s(x) = 0.$$
Lemma (6.2.17)[266]: Let $x \in LS_h(M), p, q \in P(M), \rho, \lambda \in \mathbb{R}, \rho < \lambda$,
$$pxp \leq \rho p \tag{17}$$
and
$$qxq \geq \lambda q. \tag{18}$$
Then $p \preccurlyeq q^{\perp}$ and $q \preccurlyeq p^{\perp}$.

Proof. Set $r = p \wedge q$. Multiplying both parts on both sides of inequalities (17) and (18) by r, we obtain that

$$\lambda r \leq rxr \leq \rho r,$$

that is possible if $r = 0$ only. Therefore, $p = p - p \wedge q \sim p \vee q - q \leq q^{\perp}$, that is $p \preccurlyeq q^{\perp}$. Similarly, $q \preccurlyeq p^{\perp}$.

Now, we are prepared to estimate the dimension function of the support of a λ-system from the above.

Theorem (6.2.18)[266]: Let $S = \{(p_j, x_j)\}_{j\in J}$ be a λ-system for a self-adjoint derivation δ: $LS(M) \to LS(M)$, let D be a dimension function on $P(M)$. Then

$$D(s(S)) \leq 8D\left(E_{\rho}^{\perp}\left(\delta\left(\sum_{j\in J} x_j\right)\right)\right) \text{ for any } \rho < \lambda. \tag{19}$$

Proof. Set $x = \sum_{j\in J} x_j$ (see Lemma (6.2.16)) and $p = \bigvee_{j\in J} p_j$. The following inequality is our main technical tool. It may be thought of as 'glueing' of the inequalities given in Definition (6.2.14)(iii). We claim that

$$p\delta(x)p \ \geq \ \lambda p. \tag{20}$$

To prove the claim, we note firstly that $(p_j \vee s(x_j))(p_i \vee s(x_i)) = 0$ and $(p_j \vee s(x_j))s(\delta(p_i)) = 0$ for $i \neq j$ imply that $x_i p_j = x_i s(x_i) p_j = 0$ and $x_i\delta(p_j) = x_i s(x_i)s(\delta(p_j))\delta(p_j) = 0$ for $i \neq j$. Therefore,

$$\delta(x_i)p_j = \delta(x_i p_j) - x_i\delta(p_j) = 0,$$

that implies the equality

$$s\big(\delta(x_i)\big)p_j = 0 \; for \; i = j.$$

From here and from the equality $p = \bigvee_{j\in J} p_j$ it follows that

$$s(\delta(x_i))p = s(\delta(x_i))p_i.$$

Thus,

$$\delta(x_i)p = \delta(x_i)s(\delta(x_i))p = \delta(x_i)p_i. \tag{21}$$

By Lemma (6.2.16), we have that

$$p_i x = p_i s(x) x = \big(p_i \textstyle\bigvee_{j\in J} s(x_j)\big)x = p_i s(x_i) x = p_i x_i \tag{22}$$

and

$$x_i p = x_i\big(s(x_i) \textstyle\bigvee_{j\in J} p_j\big) = x_i(s(x_i)p_i) = x_i p_i. \tag{23}$$

Similarly,

$$\delta(p_i)xp = \delta(p_i)\left(s(\delta(p_i)) \bigvee_{j\in J} s(x_j)\right)xp$$
$$= \delta(p_i)s(x_i)xp = \delta(p_i)x_i p = \delta(p_i)x_i p_i. \tag{24}$$

By (21)-(23), we obtain

$$\delta(p_i x)p = \delta(p_i x_i)p = \delta(p_i)x_i p + p_i\delta(x_i)p$$
$$= \delta(p_i)x_i p_i + p_i\delta(x_i)p_i = \delta(p_i x_i)p_i,$$

that by (24) implies the equalities

$$p_i(p\delta(x)p) = p_i\delta(x)p = \delta(p_i x)p - \delta(p_i)xp$$
$$= \delta(p_i x_i)p_i - \delta(p_i)x_i p_i = p_i\delta(x_i)p_i.$$

Hence,

$$p_i(p\delta(x)p) = p_i\delta(x_i)p_i, \tag{25}$$

in particular, the projection p_i commutes with the operator $p\delta(x)p$. Set $y = p\delta(x)p - \lambda p$ and by $y- = (|y| - y)/2$ denote the negative part of the operator y. Since $p_i y = y p_i$ (see (25)) and $p_i\delta(x_i)p_i \geq \lambda p_i$ (see Definition (6.2.14)(iii)), it follows that

$$y - p_i = p_i y - = (p_i(p\delta(x)p - \lambda p)) - \overset{(25)}{=} (p_i\delta(x_i)p_i - \lambda p_i) - = 0 \tag{26}$$

for all $i \in J$. From equalities (26) and $p = \bigvee_{j\in J} p_j$ by [113], it follows that

$$(pyp) - = \ p(p\delta(x)p - \lambda p) - p = py - p = 0. \tag{27}$$

Therefore,

$$pyp = (pyp)_+ - (pyp) - \overset{(27)}{=} (pyp)_+ \geq 0,$$

which implies inequality (20).

Having established our claim, the rest is an easy application of the properties of the dimension function D.

Fix a real number $\rho < \lambda$ and set $q = E_\rho(\delta(x))$. By Lemma (6.2.17), we obtain

$$p \preccurlyeq q^\perp. \tag{28}$$

For every fixed $j \in J$ we have that

$$\delta(p_j) = \delta(p_j^2) = \delta(p_j)p_j + p_j\delta(p_j) = \delta(p_j)p_j + (\delta(p_j)p_j)^*$$

and therefore

$$s(\delta(p_j)) \leq l(\delta(p_j)p_j) \vee p_j,$$

that implies

$$D(s(\delta(p_j))) \leq D(l(\delta(p_j)p_j)) + D(p_j). \tag{29}$$

Since $l(\delta(p_j)p_j) \sim r(\delta(p_j)p_j) \leq p_j$, by (29) we have

$$D(s(\delta(p_j))) \leq 2D(p_j) \tag{30}$$

for all $j \in J$. Similarly,

$$D(s(\delta(p_j \vee s(x_j)))) \leq 2D(p_j \vee s(x_j)),$$

and in view of the equivalence $p_j \sim s(x_j)$ (see the definition of λ-system), we obtain

$$D(s(\delta(p_j \vee s(x_j)))) \leq 4D(p_j). \tag{31}$$

Denote by A the directed set of all finite subsets of J ordered by inclusion and for every $\alpha \in A$ set

$$e_\alpha := \bigvee_{j\in\alpha} (p_j \vee s(x_j) \vee s(\delta(p_j)) \vee s(\delta(p_j \vee s(x_j)))).$$

From properties $(D2)$ and $(D3)$ of the dimension function D and from inequalities (28), (30) and (31) we have that

$$D(e_\alpha) \leq \sum_{j\in\alpha} D(p_j \vee s(x_j) \vee s(\delta(p_j)) \vee s(\delta(p_j \vee s(x_j))))$$

$$\leq \sum_{j\in\alpha} (D(p_j) + D(s(x_j)) + D(s(\delta(p_j))) + D(s(\delta(p_j \vee s(x_j)))))$$

$$\leq 8 \sum_{j\in\alpha} D(p_j) = 8D\left(\sum_{j\in\alpha} p_j\right) \leq 8D(p) \leq 8D(q^\perp).$$

Since $e_\alpha \uparrow s(S)$ the last inequality and property $(D6)$ of the dimension function D imply that

$$D(s(S)) = D\left(\bigvee_{\alpha\in A} e_\alpha\right) = \bigvee_{\alpha\in A} D(e_\alpha) \leq 8D(q^\perp).$$

Let M be an arbitrary von Neumann algebra and let $\delta_a(x) = [a, x]$ be an inner derivation on $LS(M), a \in LS(M)$. Since $(LS(M), t(M))$ is a topological $*$-algebra, every derivation δ_a is continuous with respect to the topology $t(M)$.
The main result establishes the converse implication.
Theorem (6.2.19)[266]: Every derivation on the $*$-algebra $LS(M)$ continuous with respect to the topology $t(M)$ is necessarily inner.
Proof. Let δ be an arbitrary derivation on the $*$-algebra $LS(M)$ and let δ be continuous with respect to the topology $t(M)$. By Lemmas (6.2.10) and (6.2.11), we may assume that δ is a self-adjoint derivation.
Choose a central decomposition $\{z_i\}_{i\in I}$ of the unity $\mathbf{1}$, such that every Boolean algebra $z_i P(Z(M))$ has a countable type, $i \in I$. By Corollary (6.2.8) the restriction $\delta^{(z_i)}$ of the derivation δ to $z_i LS(M) = LS(z_i M)$ is a $t(z_i M)$-continuous derivation on $LS(z_i M)$. If every derivation $\delta^{(z_i)}, i \in I$ is inner, then, by Lemma (6.2.12), the derivation δ is inner too. Thus, in the proof of Theorem (6.2.19) we may assume that the centre $Z(M)$ of the von Neumann algebra M is σ-finite algebra. In this case, there exists a faithful normal finite trace $\tau(x) = \int \varphi(x)\, d\mu$ on $Z(M)$ and the vector topology $t(M)$ has the basis of neighbourhoods of zero consists of the sets $V(\varepsilon, \beta, \gamma)$ given by (10) (see Proposition (6.2.6)). Since the derivation δ is $t(M)$-continuous, for arbitrary $\varepsilon, \beta, \gamma > 0$ there exist $\varepsilon_1, \beta_1, \gamma_1 > 0$, such that $\delta(V(\varepsilon_1, \beta_1, \gamma_1)) \subset V(\varepsilon, \beta, \gamma)$. It is clear that

$$M_1 := \{x \in M : \|x\|_M \le 1\} \subset V(1, \beta_1, \gamma_1) = \varepsilon_1^{-1} V(\varepsilon_1, \beta_1, \gamma_1),$$

and therefore

$$\delta(M_1) \subset \varepsilon_1^{-1} V(\varepsilon, \beta, \gamma) = V\,(\varepsilon/\varepsilon_1, \beta, \gamma).$$

Hence, for the $t(M)$-continuous self-adjoint derivation $\delta : LS(M) \to LS(M)$ and for arbitrary positive numbers β and γ there exists a number $\Delta(\beta, \gamma)$, such that

$$\delta(M_1) \subset V(\Delta(\beta, \gamma), \beta, \gamma). \tag{32}$$

Let D, φ, τ be the same as in the definition of the set $V(\varepsilon, \beta, \gamma)$ from (10). Fix an arbitrary $2\Delta(\beta, \gamma)$-system $S = \{(p_j, x_j)\}_{j\in J}$ for the derivation δ and show that there exists a central projection $z \in P(Z(M))$, such that

$$\tau(z^\perp) \le \beta \text{ and } D(zs(S)) \le 8\gamma\varphi(z). \tag{33}$$

If S is empty, then $s(S) = 0$ and, in this case, relations (33) hold for $z = \mathbf{1}$. Now, let $S = \{(p_j, x_j)\}_{j\in J}$ be non-empty $2\Delta(\beta, \gamma)$-system. By Lemma (6.2.16), there exists $x = x_j \in M_{h,1}$. From (32) it follows that $\delta(x) \in V(\Delta(\beta, \gamma), \beta, \gamma)$ for all $\beta, \gamma > 0$. Therefore, there exist projections $z \in P(Z(M))$ and $q \in P(M)$, such that

$$\tau(z^\perp) \le \beta, \delta(x)q \in M, \|\delta(x)q\|_M \le \Delta(\beta, \gamma) \text{ and } D(zq^\perp) \le \gamma\varphi(z). \tag{34}$$

Since $x = x^*$ and $\delta = \delta^*$, it follows that $\delta(x) = (\delta(x))^*$ and, by (34), we have

$$-\Delta(\beta, \gamma)q \le q\delta(x)q \le \Delta(\beta, \gamma)q. \tag{35}$$

Set $\rho = \frac{3}{2} \cdot \Delta(\beta, \gamma)$. Using inequalities (35) and

$$\rho E_\rho^\perp(\delta(x)) \le E_\rho^\perp(\delta(x))\delta(x)E_\rho^\perp(\delta(x)),$$

we obtain that $E_\rho^\perp(\delta(x)) \le q^\perp$ (Lemma (6.2.17)). Consequently, $zE_\rho^\perp(\delta(x)) \le zq^\perp$ and, by (19) and (34), we have that

$$D(zs(S)) \overset{(D4)}{=} \varphi(z)D(s(S)) \overset{(19)}{=} 8\varphi(z)D(E_\rho^\perp(\delta(x)))$$
$$\overset{(D4)}{=} 8D(zE_\rho^\perp(\delta(x))) \overset{(D2),(D3)}{\le} 8D(zq^\perp) \overset{(34)}{\le} 8\gamma\varphi(z),$$

that is (33) holds.
For every $n \in \mathbb{N}$ choose the maximal (possible, empty) $2\Delta(2^{-n}, 2^{-n})$-system S_n for the derivation δ. Set $q_n' = s(S_n)^\perp$. By Theorem (6.2.15), we have that

$$\delta(q_n^{'})(q_n^{'}Mq_n^{'}) \subset q_n^{'}Mq_n^{'} \tag{36}$$

for all $n \in \mathbb{N}$. Moreover, in view of (33), there exists a projection $z_n^{'} \in P(Z(M))$, such that

$$\tau(z_n^{'\perp}) \leq 2^{-n} \text{ and } D(z_n^{'}q_n^{'\perp}) \leq 2^{-n+3}\varphi(z_n^{'}). \tag{37}$$

We set $q_n := \bigwedge_{k=n+1}^{\infty} q_k^{'}$ and $z_n := \bigwedge_{k=n+1}^{\infty} z_k^{'}$ and consider the derivation $\delta^{(q_n)}$ on $q_n LS(M) q_n$. We shall show that

$$\delta^{(q_n)}(q_nMq_n) \subset q_nMq_n.$$

Clearly, the sequences $\{q_n\}$ and $\{z_n\}$ are increasing and, in addition,

$$\tau(z_n^{\perp}) \leq \tau\left(\bigvee_{k\geq n+1} z_n^{'\perp}\right) \leq \sum_{k\geq n+1} \tau(z_n^{'\perp}) \overset{(37)}{=} \sum_{k\geq n+1} 2^{-k} = 2^{-n} \tag{38}$$

and

$$\begin{aligned} D(z_nq_n^{\perp}) &= \varphi(z_n)D\left(\bigvee_{k\geq n+1} z_nq_k^{'\perp}\right) \\ &\leq \varphi(z_n)D\left(\bigvee_{k\geq n+1} z_k^{'}q_k^{'\perp}\right) \overset{(D6)}{\leq} \varphi(z_n)\sum_{k\geq n+1} D(z_k^{'}q_k^{'\perp}) \\ &\overset{(37)}{\leq} \varphi(z_n)\sum_{k\geq n+1} 2^{-k+3}\phi(z_k^{'}) = \sum_{k\geq n+1} 2^{-k+3}\varphi(z_nz_k^{'}) \\ &= \sum_{k\geq n+1} 2^{-k+3}\varphi(z_n) = 2^{-n+3}\varphi(z_n). \end{aligned} \tag{39}$$

If $x \in q_nMq_n$, then $x \in q_{n+1}^{'}Mq_{n+1}^{'}$ and therefore, by (36),

$$\begin{aligned} \delta^{(q_n)}(x) &= q_n\delta(x)q_n = q_nq_{n+1}^{'}\delta(x)q_{n+1}^{'}q_n \\ &= q_n\delta^{(q_{n+1}^{'})}(x)q_n \in q_nq_{n+1}^{'}Mq_{n+1}^{'}q_n = q_nMq_n. \end{aligned}$$

Hence, the restriction $\delta^{(q_n)}|_{q_nMq_n}$ of the derivation $\delta^{(q_n)}$ to q_nMq_n is a derivation on the von Neumann algebra q_nMq_n. By Sakai Theorem [29], there exists an element $c_n \in q_nMq_n$, such that $\delta(q_n)(x) = [c_n, x]$ for all $x \in q_nMq_n$.

Now, we replace the sequence $\{c_n\}$ with a sequence $\{d_n\}$, which is somewhat similar to a sequence of 'martingale differences'. More precisely, we shall construct a sequence $\{d_n\}$ of M, such that

$$q_nd_mq_n = d_n \text{ for all } n \leq m,$$
$$\delta^{(q_n)}(x) = [d_n, x] \text{ for all } x \in q_nMq_n. \tag{40}$$

Set $d_1 = c_1$ and assume that elements $d_1, \ldots, d_n$ are already constructed. Since $\delta^{(q_n)}(q_nxq_n) = q_n\delta^{(q_{n+1})}(q_nxq_n)q_n$, it follows that

$$[d_n, q_nxq_n] = q_n[c_{n+1}, q_nxq_n]q_n = [q_nc_{n+1}q_n, q_nxq_n]$$

for any $x \in M$. Consequently, the element $d_n - q_nc_{n+1}q_n$ is contained in the centre of algebra q_nMq_n. By [274] there exists an element z of the centre of algebra $q_{n+1}Mq_{n+1}$, such that $d_n - q_nc_{n+1}q_n = zq_n$. Set $d_{n+1} = c_{n+1} + z$. It is clear that

$$\delta^{(q_{n+1})}|_{q_{n+1}Mq_{n+1}}(x) = [c_{n+1}, x] = [d_{n+1}, x] \tag{41}$$

for all $x \in q_{n+1}Mq_{n+1}$, in addition,

$$d_{n+1} \in q_{n+1}Mq_{n+1} \; and \; q_nd_{n+1}q_n = q_nc_{n+1}q_n + zq_n = d_n$$

for every $n \in \mathbb{N}$. Moreover, for $k \in \mathbb{N}, k < n+1$ the equalities

$$q_kd_{n+1}q_k = q_kq_nd_{n+1}q_nq_k = q_kd_nq_k = \cdots = q_kd_{k+1}q_k = d_k \tag{42}$$

hold.

Thus we have constructed the sequence $\{d_n\}$ of elements of M which has property (40).

By [272], the topology $t(M)$ induces on $q_nLS(M)q_n = LS(q_nMq_n)$ the topology $t(q_nMq_n)$, and therefore the derivation $\delta(q_n)$ is continuous on $(LS(q_nMq_n), t(q_nMq_n))$. By Proposition (6.2.5), we have that $q_nMq_nt(q_nMq_n) = LS(q_nMq_n)$. Consequently, the equality $\delta^{(q_n)}(x) = [d_n, x]$ holds for all $x \in LS(q_nMq_n)$.
Our next objective is to show that the sequence $\{d_n\}$ is a Cauchy sequence in $(LS(M), t(M))$. If $n, m \in \mathbb{N}, n < m$, then

$$d_m - d_n \overset{(40)}{=} q_md_mq_m - q_nd_mq_n = (q_m - q_n)d_mq_m + q_nd_m(q_m - q_n).$$

Since

$$r((q_m - q_n)d_mq_m) \sim l((q_m - q_n)d_mq_m) \leq q_n^\perp,$$

it follows that

$$D(z_nr(d_m - d_n)) \leq D(z_nr((q_m - q_n)d_mq_m) \vee z_nq_n^\perp)$$
$$\leq 2D(z_nq_n^\perp) \overset{(39)}{\leq} 2^{-n+4}\varphi(z_n).$$

From here, by taking $p = r(d_m - d_n)^\perp$ in view of (10) and (38), we obtain

$$d_m - d_n \in V(0, 2^{-n}, 2^{-n+4}) \subset V(1/n, 2^{-n}, 2^{-n+4}).$$

It means that $\{d_n\}$ is a Cauchy sequence in $(LS(M), t(M))$, and therefore, since the space $(LS(M), t(M))$ is complete there exists $d \in LS(M)$, such that $d_n \xrightarrow{t(M)} d$.
Finally, let us show that $\delta(x) = [d, x]$ for all $x \in LS(M)$. By (38) and (39) we have that $q_n^\perp \in V(0, 2^{-n}, 2^{-n+3})$ for all $n \in \mathbb{N}$, and therefore $q_n^\perp \xrightarrow{t(M)} 0$. Consequently, $q_n \xrightarrow{t(M)} \mathbf{1}$ and for every $x \in LS(M)$ we have that $q_nxq_n \xrightarrow{t(M)} x$. We just need to use the $t(M)$-continuity of the derivation δ, which implies the following:

$$\delta(x) = t(M) - \lim_{n\to\infty}(q_n\delta(q_nxq_n)q_n) = t(M) - lim_{n\to\infty} \delta^{q_n}(q_nxq_n)$$
$$= t(M) - \lim_{n\to\infty}[d_n, q_nxq_n] = [t(M) - \lim_{n\to\infty} d_n, t(M) - \lim_{n\to\infty} q_nxq_n] = [d, x].$$

A combination of Theorem (6.2.19) with results from [270] yields the full description of all derivations on the $*$-algebra $LS(M)$ in case when M is a properly infinite von Neumann algebra. This result significantly strengthens earlier results from [2], [106].
Corollary (6.2.20)[266]: If M is a properly infinite von Neumann algebra, then every derivation on the $*$-algebra $LS(M)$ is inner.
Proof. By [270] every derivation $\delta: LS(M) \to LS(M)$ is $t(M)$-continuous. Consequently, by Theorem (6.2.19), there exists $d \in LS(M)$, such that $\delta(x) = [d, x]$ for all $x \in LS(M)$.

We give applications of Theorem (6.2.19) and Corollary (6.2.20) to the description of continuous derivations on EW^*-algebras. The class of EW^*-algebras (extended W^*-algebras) was introduced in [276] for the purpose of description of $*$-algebras of unbounded closed operators, which are 'similar' to W^*-algebras by their algebraic and order properties. Let A be a set of closed, densely defined operators on the Hilbert space H which is a $*$-algebra under strong sum, strong product, scalar multiplication and the usual adjoint of operators. The set A is said to be EW^*-algebra [276] if the following conditions hold:
(i) $(\mathbf{1} + x^*x)^{-1} \in A$ for every $x \in A$;
(ii) the subalgebra A_b of bounded operators in A is a W^*-algebra.
The meaningful connection between EW^*-algebras A and solid subalgebras of $LS(Ab)$ is given in [273]. Recall [106], that a $*$-subalgebra A of $LS(M)$ is called solid if conditions $x \in LS(M), y \in A, |x| \leq |y|$ imply that $x \in A$. It is clear that every solid $*$-subalgebra A in $LS(M)$ with $M \subset A$ is an EW^*-algebra and $A_b = M$. The converse implication is given in [273], where it is established that every EW^*-algebra A with the bounded part $A_b = M$ is a

solid $*$-subalgebra in the $*$-algebra $LS(M)$, that is $LS(M)$ is the greatest EW^*-algebra of all EW^*-algebras with the bounded part coinciding with M.
In the case when the bounded part A_b of an EW^*-algebra A is a properly infinite W^*-algebra we have that any derivation $\delta: A \to LS(A_b)$ is continuous with respect to the local measure topology $t(A_b)$ [270].
Now, let A_b be an arbitrary W^*-algebra and let $\delta: A \to A$ be a $t(A_b)$-continuous derivation. Since $A_b \subset A, A_b$ is everywhere dense in $(LS(A_b), t(A_b))$ (Proposition (6.2.5)) and $(LS(A_b), t(A_b))$ is a topological $*$-algebra, there exists a unique $t(A_b)$-continuous derivation $\hat{\delta}: LS(A_b) \to LS(A_b)$ such that $\hat{\delta}(x) = \delta(x)$ for all $x \in A$. By Theorem (6.2.19), the derivation $\hat{\delta}$ is inner. In [106] it is proved that, if δ is a derivation on a solid $*$-subalgebra $A \supset M$ and $\delta(x) = [w, x]$ for all $x \in A$ and some $w \in LS(M)$, then there exists an element $w_1 \in A$, such that $\delta(x) = [w_1, x]$ for all $x \in A$, that is the derivation δ on the $*$-subalgebra A is inner.
Thus the following result holds.
Theorem (6.2.21)[266]: (i) Every $t(A_b)$-continuous derivation on a EW^*-algebra A is inner.
(ii) If the bounded part A_b of an EW^*-algebra A is a properly infinite W^*-algebra, then every derivation on A is inner.

We specialize the result given in Theorem (6.2.21) to the algebra of τ-measurable operators.

Let M be a semifinite von Neumann algebra acting on the Hilbert space H and let τ be a faithful normal semifinite trace on M. An operator $x \in S(M)$ with the domain $\mathfrak{D}(x)$ is called τ-measurable if for any $\varepsilon > 0$ there exists a projection $p \in P(M)$ such that $p(H) \subset \mathfrak{D}(x)$ and $\tau(p^\perp) < \varepsilon$.
The set $S(M, \tau)$ of all τ-measurable operators is a solid $*$-subalgebra of $LS(M)$ such that $M \subset S(M, \tau) \subset S(M)$. If the trace τ is finite, then $S(M, \tau) = S(M)$. The algebra $S(M, \tau)$ is a non-commutative version of the algebra of all measurable complex functions f defined on (Ω, Σ, μ), for which $\mu(\{|f| > \lambda\}) \to 0$ as $\lambda \to \infty$.
Let t_τ be the measure topology [52] on $S(M, \tau)$ whose base of neighbourhoods of zero is given by

$$U(\varepsilon, \delta) = \{x \in S(M, \tau) : \text{there exists a projection } p \in P(M), \text{ such that } \tau(p^\perp) \le \delta, xp \in M, \|xp\|_M \le \varepsilon\}, \varepsilon > 0, \delta > 0.$$

The pair $(S(M, \tau), t_\tau)$ is a complete metrizable topological $*$-algebra. Here, the topology t_τ majorizes the topology $t(M)$ on $S(M, \tau)$ and, if the trace τ is finite, the topologies t_τ and $t(M)$ coincide [113]. Denote by $t(M, \tau)$ the topology on $S(M, \tau)$ induced by the topology $t(M)$. It is not true in general that, if the topologies t_τ and $t(M, \tau)$ are the same, then the von Neumann algebra M is finite. Indeed, if $M = B(H), dim(H) = \infty, \tau = tr$ is the canonical trace on $B(H)$, then $LS(M) = S(M) = S(M, \tau) = M$, and the two topologies t_τ and $t(M)$ coincide with the uniform topology on $B(H)$.
At the same time, if M is a finite von Neumann algebra with a faithful normal semifinite trace τ and $t_\tau = t(M, \tau)$, then necessarily $\tau(\mathbf{1}) < \infty$ [272].
It is shown in [269] that every t_τ -continuous derivation on $S(M, \tau)$ is inner. In addition, if M is a properly infinite von Neumann algebra, then every derivation on $S(M, \tau)$ is t_τ–continuous [108]. Thus, in view of Theorem (6.2.21) we obtain the following strengthening of earlier results from [2], [106].

Corollary (6.2.22)[266]: Let M be a semifinite von Neumann algebra, let τ be a faithful normal semifinite trace on M, let δ be a derivation on $S(M,\tau)$. Then the following conditions are equivalent:
(i) δ is $t(M)$-continuous;
(ii) δ is t_τ -continuous;
(iii) δ is inner.
In addition, if M is a properly infinite von Neumann algebra then every derivation on $S(M,\tau)$ is inner.

We give one more application of Theorem (6.2.19) to derivations with values in Banach M-bimodules of locally measurable operators.

Let M be a von Neumann algebra. A linear subspace $\mathcal{E}$ of $LS(M)$, is called an M-bimodule of locally measurable operators if $uxv \in \mathcal{E}$ whenever $x \in \mathcal{E}$ and $u, v \in M$. If $\mathcal{E}$ is an M-bimodule of locally measurable operators, $x \in \mathcal{E}$ and $x = v|x|$ is the polar decomposition of the operator x, then $|x| = v^*x \in E$ and $x^* = |x|v^* \in \mathcal{E}$. In addition, we have

$$\text{if } |a| \le |b|, b \in \mathcal{E}, a \in LS(M) \text{ then } a \in \mathcal{E}. \quad (43)$$

Property (43) of an M-bimodule of locally measurable operators follows from the following proposition:

Proposition (6.2.23)[266]: Let M be a von Neumann algebra acting in a Hilbert space H, and let $a, b \in LS(M), 0 \le a \le b$. Then $a^{1/2} = cb^{1/2}$ for some $c \in s(b)Ms(b), \|c\|_M \le 1$, in particular, $a = cbc^*$. In addition, if $c_1 \in M$ and $a^{1/2} = c_1 b^{1/2}$, then $s(b) \cdot c_1 \cdot s(b) = c$.

Proof. Let us first show that $s(a) \le s(b)$. Since

$$0 \le (\mathbf{1} - s(b))a(\mathbf{1} - s(b)) \le (\mathbf{1} - s(b))b(\mathbf{1} - s(b)) = 0,$$

if follows that $(\mathbf{1} - s(b))a^{1/2} = 0$, which implies the equality $(\mathbf{1} - s(b))a = 0$, that is, $s(b)a = a = a^* = a^*s(b) = as(b)$. Consequently, $s(a) \le s(b)$.

Thus, passing if necessary to the reduced algebra $s(b)Ms(b)$, we may assume that $s(b) = \mathbf{1}$. For every $n \in \mathbb{N}$ denote by p_n the spectral projection for the operator b corresponding to the interval $[1/n, n]$. Since $p_n \uparrow s(b) = \mathbf{1}$ it follows that the linear subspace $H_0 = \bigcup_{n=1}^{\infty} p_n H$ is dense in H and $H_0 \subset \mathfrak{D}(b) \cap \mathfrak{D}(b^{1/2})$. Furthermore, according to the inequalities $0 \le p_n a p_n \le p_n b p_n \le n p_n$ we have that $a^{1/2}p_n \in M$ and $\left\|a^{1/2}p_n\right\|_M \le \sqrt{n}$ for all $n \in \mathbb{N}$. In particular, $H_0 \subset \mathfrak{D}(a^{1/2})$.

Since $b^{1/2}p_n \le n^{1/2}p_n$ and $b^{1/2}(p_n H) = p_n b^{1/2}(p_n H) \subset p_n H$ for all $n \in \mathbb{N}$ we have $b^{1/2}(H_0) \subset H_0$. Consequently, it is possible to define a linear mapping $d: b^{1/2}(H_0) \to H$ by setting $d(b^{1/2}\xi) = a^{1/2}\xi, \xi \in H_0$. The definition of the operator d is correct since the equality $b^{1/2}\xi = 0$ and the inequality

$$\left\|a^{1/2}\xi\right\|_H^2 = (a^{1/2}\xi, a^{1/2}\xi) = (a\xi, \xi) \le (b\xi, \xi) = \left\|b^{1/2}\xi\right\|_H^2$$

imply that $a^{1/2}\xi = 0$.

In addition, for every $\xi \in H_0$, we have

$$\left\|d(b^{1/2}\xi)\right\|_H^2 = \left\|a^{1/2}\xi\right\|_H^2 \le \left\|b^{1/2}\xi\right\|_H^2,$$

that is d is a continuous linear operator on $b^{1/2}(H_0)$ and $\|d\|_{b^{1/2}(H_0) \to H} \le 1$.

Since $n^{-1}p_n \le bp_n \le np_n$, by Proposition [113], we have $n^{-1/2}p_n \le b^{1/2}p_n \le n^{1/2}p_n$.

Therefore, the restriction of the operator $b^{1/2}$ to $p_n(H_0)$ has inverse bounded operator b_n, in addition $n^{-1/2}p_n \le b_n p_n \le n^{1/2}p_n$. Hence, $b^{1/2}(p_n H) = p_n H$, that implies the equality $b^{1/2}(H_0) = H_0$.

Thus, the operator d uniquely extends to the Hilbert space H up to a bounded linear operator c, moreover, $\|c\|_{B(H)} \leq 1$ and $cb^{1/2}\xi = a^{1/2}\xi$ for all $\xi \in H_0$.
If u is a unitary operator from the commutant M', then $u(p_nH) = p_nH$ for all $n \in \mathbb{N}$ and therefore $u(H_0) = H_0$. If $\eta \in H_0$, then $\eta = b^{1/2}\xi$ for some $\xi \in H_0$ and
$$u^{-1}cu\eta = u^{-1}cub^{1/2}\xi = u^{-1}cb^{1/2}u\xi$$
$$= u^{-1}a^{1/2}u\xi = u^{-1}ua^{1/2}\xi = a^{1/2}\xi = cb^{1/2}\xi = c\eta.$$
Consequently, $u^{-1}cu = c$, that implies the inclusion $c \in M$.
Since $p_ncb^{1/2}p_n = p_na^{1/2}p_n$ for all $n \in \mathbb{N}$ and $p_n \uparrow \mathbf{1}$, by Proposition [113] we have $cb^{1/2} = a^{1/2}$.
If $c_1 \in M$ and $c_1b^{1/2} = a^{1/2}$, then the operators c_1 and c coincide on the everywhere dense subspace H_0 and therefore $c_1 = c$.
If $s(b) = \mathbf{1}, c_1 \in M$ and $c_1b^{1/2} = a^{1/2}$, then, using equalities
$$a^{1/2}s(b) = s(b)a^{1/2} = a^{1/2}$$
and
$$b^{1/2}s(b) = s(b)b^{1/2} = b^{1/2},$$
we obtain $(s(b)c_1s(b))b^{1/2} = a^{1/2}$. Uniqueness of the operator c in the reduced algebra $s(b)Ms(b)$ implies that $s(b) \cdot c_1 \cdot s(b) = c$.
Let $\mathcal{E}$ be an M-bimodule of locally measurable operators. A linear mapping $\delta: M \to \mathcal{E}$ is called a derivation, if $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in M$. A derivation $\delta: M \to \mathcal{E}$ is called inner, if there exists an element $d \in \mathcal{E}$, such that $\delta(x) = [d, x] = dx - xd$ for all $x \in M$.
We recall the following result.
Theorem (6.2.24)[266]: [255]. Let M be a von Neumann algebra and $a \in LS_h(M)$. Then there exist a self-adjoint operator c in the centre of the $*$-algebra $LS(M)$ and a family $\{u_\varepsilon\}_{\varepsilon>0}$ of unitary operators from M such that
$$|[a, u_\varepsilon]| \geq (1-\varepsilon)|a - c|. \qquad (44)$$
The following theorem shows that every derivation $\delta: M \to \mathcal{E}$ is inner, provided that M is a properly infinite von Neumann algebra. In this special case, we are in a position to significantly strengthen Ringrose Theorem [27] by omitting the assumption that $\mathcal{E}$ is a Banach M-bimodule.
Theorem (6.2.25)[266]: Let M be a properly infinite von Neumann algebra and let $\mathcal{E}$ be an M bimodule of locally measurable operators. Then any derivation $\delta: M \to \mathcal{E}$ is inner.
Proof. By [270], there exists a derivation $\delta: LS(M) \to LS(M)$, such that $\delta(x) = \delta(x)$ for all $x \in M$. By Corollary (6.2.20), there exists an element $a \in LS(M)$, such that $\delta(x) = [a, x]$ for all $x \in LS(M)$. It is clear that $[a, M] = \delta(M) = \delta(M) \subset \mathcal{E}$.
Let $a_1 = Re(a), a_2 = Im(a)$. Since $[a^*, x] = -[a, x^*]^* \in \mathcal{E}$ for any $x \in M$, it follows that $[a_1, x] = [a + a^*, x]/2 \in \mathcal{E}$ and $[a_2, x] = [a - a^*, x]/2i \in \mathcal{E}$ for all $x \in M$.
Taking $\varepsilon = \frac{1}{2}$ in (44), we obtain that there exist $c_1, c_2 \in Z_h(LS(M))$ and unitary operators $u_1, u_2 \in M$ such that
$$2|[a_i, u_i]| \geq |a_i - c_i|, \qquad i = 1, 2.$$
Since $[a_i, u_i] \in \mathcal{E}$ and $\mathcal{E}$ is an M-bimodule we have that $d_i := a_i - c_i \in \mathcal{E}, i = 1,2$ (see (43)). Therefore, $d = d_1 + id_2 \in \mathcal{E}$. Since c_1 and c_2 are central elements from $LS(M)$ it follows that $\delta(x) = [a, x] = [d, x]$ for all $x \in M$.
Let M be a von Neumann algebra. If an M-bimodule of locally measurable operators $\mathcal{E}$ is equipped with a norm $\|\cdot\|_{\mathcal{E}}$, satisfying

$$\|uxv\|_{\mathcal{E}} \le \|u\|_M \|v\|_M \|x\|_{\mathcal{E}}, x \in \mathcal{E}, u, v \in M, \tag{45}$$

then $\mathcal{E}$ is called a normed M-bimodule of locally measurable operators. If, in addition, the pair $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ is a Banach space, then $\mathcal{E}$ is called a Banach M-bimodule of locally measurable operators.

It is easy to see that the norm $\|\cdot\|_{\mathcal{E}}$ on a normed M-bimodule of locally measurable operators $\mathcal{E}$ satisfies the following properties:

$$\||a|\|_{\mathcal{E}} = \|a^*\|_{\mathcal{E}} = \|a\|_{\mathcal{E}} \text{ for any } a \in \mathcal{E}; \tag{46}$$

$$\|a\|_{\mathcal{E}} \le \|b\|_{\mathcal{E}} \text{ for any } a, b \in \mathcal{E}, 0 \le a \le b. \tag{47}$$

$$\text{If } q \in \mathcal{E} \cap P(M), p \in P(M), p \preccurlyeq q, \text{ then } p \in \mathcal{E} \text{ and } \|p\|_{\mathcal{E}} \le \|q\|_{\mathcal{E}}, \text{ in particular, if } p \sim q, \text{ then } \|p\|_{\mathcal{E}} = \|q\|_{\mathcal{E}}. \tag{48}$$

Proposition (6.2.26)[266]: If $\{p_k\}_{k=1}^n \subset P(M) \cap \mathcal{E}, n \in \mathbb{N}$, then

$$\bigvee_{k=1}^{n} p_n \in \mathcal{E} \text{ and } \left\|\bigvee\nolimits_{k=1}^{n} p_n\right\|_{\mathcal{E}} \le \sum\nolimits_{k=1}^{n} \|p_k\|_{\mathcal{E}}. \tag{49}$$

Proof. If $p, q \in P(M) \cap \mathcal{E}$, then $p \vee q - p \sim q - p \wedge q \le q$, and therefore, $p \vee q - p \in \mathcal{E}$ and $\|p \vee q - p\|_{\mathcal{E}} \le \|q\|_{\mathcal{E}}$ (see (48)). Hence, $p \vee q = (p \vee q - p) + p \in \mathcal{E}$ and

$$\|p \vee q\|_{\mathcal{E}} - \|p\|_{\mathcal{E}} \le \|p \vee q - p\|_{\mathcal{E}} \le \|q\|_{\mathcal{E}}.$$

For an arbitrary finite set $\{p_k\}_{k=1}^n \subset P(M) \cap \mathcal{E}$ the assertion is proved via mathematical induction.

We devoted to another strengthening of Ringrose Theorem. In Theorem (6.2.32), we show that any derivation on a von Neumann algebra M with values in a Banach M-bimodule is inner.

In Lemmas (6.2.28)-(6.2.31), we assume that τ is a faithful normal finite trace on a finite von Neumann algebra M and $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ is a Banach M-bimodule of locally measurable operators in $LS(M)$. In this case, $LS(M) = S(M) = S(M, \tau), t(M) = t_\tau$ and $(LS(M), t(M))$ is an F-space. Later we need the following:

Proposition (6.2.27)[266]: [113]. Let M be a von Neumann algebra with a faithful normal finite trace τ and $\{p_n\}_{n=1}^\infty \subset P(M)$. Then

$$p_n \xrightarrow{t(M)} 0 \Leftrightarrow \tau(p_n) \longrightarrow 0.$$

Lemma (6.2.28)[266]: If $\{p_n\}_{n=1}^\infty \subset P(M) \cap \mathcal{E}$ and the series $\sum_{n=1}^\infty \|p_n\|_{\mathcal{E}}$ converges, then $p = \bigvee_{n=1}^\infty p_n \in \mathcal{E}$ and $\|p\|_{\mathcal{E}} \le \sum_{n=1}^\infty \|p_n\|_{\mathcal{E}}$.

Proof. Set $q_n = \bigvee_{k=1}^n p_k$. By (49), we have $q_n \in \mathcal{E}$ and $\|q_n\|_{\mathcal{E}} \le \sum_{k=1}^n \|p_k\|_{\mathcal{E}}$.

Let $n, m \in \mathbb{N}, n < m$. By (48) and (49), we have that

$$\|q_m - q_n\|_{\mathcal{E}} = \left\|q_n \vee \bigvee_{k=n+1}^{m} p_k - q_k\right\|_{\mathcal{E}}$$

$$= \left\|\bigvee_{k=n+1}^{m} p_k - q_n \wedge \bigvee_{k=n+1}^{m} p_k\right\|_{\mathcal{E}} \le \left\|\bigvee_{k=n+1}^{m} p_k\right\|_{\mathcal{E}} \le \sum_{k=n+1}^{m} \|p_k\|_{\mathcal{E}}.$$

Consequently, $\{q_n\}$ is a Cauchy sequence in $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$, and therefore there exists $q \in \mathcal{E}$, such that $\|q_n - q\|_{\mathcal{E}} \to 0$, in addition $\|q\|_{\mathcal{E}} \le \sum_{k=n+1}^m \|p_n\|_{\mathcal{E}}$.

Since

$$\|qp - q_n\|_{\mathcal{E}} = \|qp - q_n p\|_{\mathcal{E}} \le \|p\|_M \|q - q_n\|_{\mathcal{E}},$$

it follows that $qp = q = q^* = pq$. Hence, $s(p - q) \le p$. Fix $n_0 \in \mathbb{N}$, then for $n > n_0$, we have

$$\|q_{n_0} q - q_{n_0}\|_{\mathcal{E}} = \|q_{n_0} q - q_{n_0} q_n\|_{\mathcal{E}}$$

$$\leq \|q_{n_0}\|_M \|q-q_n\|_\mathcal{E} \leq \|q-q_n\|_\mathcal{E}.$$

Passing to the limit for $n \to \infty$, we obtain $q_{n_0}q = q_{n_0}$. Therefore, $q_n(p-q)q_n = 0$ for all $n \in \mathbb{N}$. The inequality $s(p-q) \leq p$ and the convergence $q_n \uparrow p$ by [113] imply that $q = p$.

The following lemma shows that a Banach bimodule $(\mathcal{E}, \|\cdot\|_\mathcal{E})$ of locally measurable operator is continuously embedded into $(LS(M), t(M))$.

Lemma (6.2.29)[266]: If $\{a_n\}_{n=1}^\infty \subset \mathcal{E}$ and $\|a_n\|_\mathcal{E} \to 0$, then $a_n \xrightarrow{t(M)} 0$.

Proof. Let us show firstly that every convergent to zero in the norm $\|\cdot\|_\mathcal{E}$ sequence of projections from $\mathcal{E}$ has a subsequence convergent to zero in the topology $t(M)$. Consider a sequence $\{p_n\}_{n=1}^\infty \in P(M) \cap \mathcal{E}$, such that $\|p_n\|_\mathcal{E} \to 0$. Choose a subsequence $\{p_{n_k}\}_{k=1}^\infty$so that $\|p_{n_k}\|_\mathcal{E} \leq 2^{-k}$. By Lemma (6.2.28), for the sequence of projections $q_k = \bigvee_{l \leq k+1} p_{n_l}$, we have $q_k \in \mathcal{E}$ and $\|q_k\|_\mathcal{E} \leq 2^{-k}$. If $q = \bigwedge_{k\geq 1} q_k$, then $q \in \mathcal{E}$ and $\|q\|_\mathcal{E} \leq \|q_k\|_\mathcal{E} \leq 2^{-k}$ for all $k \in \mathbb{N}$, that implies $q = 0$. Consequently, $q_k \downarrow 0$, and therefore $\tau(q_k) \downarrow 0$.

Since $p_{n_{k+1}} \leq q_k$ for all $k \in \mathbb{N}$ we have $\tau(p_{n_k}) \to 0$. By Proposition (6.2.27), we infer the convergence $p_{n_k} \xrightarrow{t(M)} 0$.

Now, let us show that every sequence $\{p_n\}_{n=1}^\infty \in P(M) \cap \mathcal{E}$ convergent to zero in the norm$\|\cdot\|_\mathcal{E}$ automatically converges to zero in the topology $t(M)$. Otherwise, there exists a subsequence $\{p_{n_k}\}_{k=1}^\infty$ and a $t(M)$-neighborhood U of zero, which does not contain $\{p_{n_k}\}_{k=1}^\infty$.

From the above, there exists a subsequence $\{p_{n_{k_s}}\}_{s=1}^\infty$ converging to zero in the topology $t(M)$, that contradicts to the relation $p_{n_{k_s}} \notin U$.

Now, let $\{a_n\}_{n=1}^\infty \subset \mathcal{E}$ and $\|a_n\|_\mathcal{E} \to 0$. For every $\lambda > 0$ the inequality $\lambda E_\lambda^\perp(|a_n|) \leq |a_n| E_\lambda^\perp(|a_n|) \leq |a_n|$ imply that

$$\|E_\lambda^\perp(|a_n|)\|_\mathcal{E} \overset{(47)}{\leq} \lambda^{-1}\||a_n|\|_\mathcal{E} \overset{(46)}{=} \lambda^{-1}\|a_n\|_\mathcal{E} \longrightarrow 0.$$

By the preceding argument, we have that $E_\lambda^\perp(|a_n|) \longrightarrow 0$. Finally, by Proposition (6.2.2)(ii), we obtain $a_n \xrightarrow{t\,(M)} 0$.

Lemma (6.2.30)[266]: If $\{a_n\}_{n=1}^\infty \subset LS(M)$ and $a_n \xrightarrow{t(M)} 0$, then there exists a sequence $\{a_{n_k}\}_{k=1}^\infty$ such that $a_{n_k} = b_k + c_k$, where $b_k \in M, c_k \in LS(M), k \in \mathbb{N}, \|b_k\|_M \to 0$ and $s(|c_k|) \xrightarrow{t(M)} 0$ for $s \to \infty$.

Proof. Since $(LS(M), t(M))$ is an F-space there exists a countable basis $\{U_k\}_{k=1}^\infty$ of neighbourhoods of zero of the topology $t(M)$.

By Proposition (6.2.2)(ii), we have $E_\lambda^\perp(|a_n|) \xrightarrow{t(M)} 0$ for every $\lambda > 0$. Therefore, there exists a sequence $(a_{n_k})_{k\geq 1}$ such that $E_{1/k}^\perp(|a_{n_k}|) \in U_k$ for all $k \in \mathbb{N}$. Set $b_k = a_{n_k}E_{1/k}(|a_{n_k}|)$ and $c_k = a_{n_k}E_{1/k}^\perp(|a_{n_k}|)$. It is clear that $b_k \in M$ and $\|b_k\|_M \leq 1/k$. Since

$$|c_k| = (c_k^* c_k)^{1/k} = (E_{1/k}^\perp(|a_{n_k}|)|a_{n_k}|^2 E_{1/k}^\perp(|a_{n_k}|))^{1/2}$$
$$= E_{1/k}^\perp(|a_{n_k}|)|a_{n_k}|E_{1/k}^\perp(|a_{n_k}|) = |a_{n_k}|E_{1/k}^\perp(|a_{n_k}|),$$

it follows that

$$s(|c_k|) \leq E_{1/k}^\perp(|a_{n_k}|) \in U_k.$$

Since $\{U_k\}_{k=1}^\infty$ is a basis of neighbourhoods of zero of the topology $t(M)$, we have $E_{1/k}^\perp(|a_{n_k}|) \xrightarrow{t(M)} 0$, which, in its turn, guarantees the convergence $\tau(E_{1/k}^\perp(|a_{n_k}|)) \to 0$

Proposition (6.2.27). From the inequality $\tau(s(|c_k|)) \le \tau\,(E_{1/k}^{\perp}(|a_{n_k}|))$ and Proposition (6.2.27), we obtain $s(|c_k|) \xrightarrow{t(M)} 0$.

Lemma (6.2.31)[266]: Let M be a von Neumann algebra with faithful normal finite trace τ and let $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ be a Banach M-bimodule. Every derivation $\delta: LS(M) \to LS(M)$ with $\delta(M) \subset \mathcal{E}$ is $t(M)$-continuous.

Proof. Since $(LS(M), t(M))$ is an F-space it is sufficient to show that the graph of the linear operator δ is closed.

Suppose that the graph of the operator δ is not closed. Then there exists a sequence $\{a_n\}_{n=1}^{\infty} \subset LS(M)$ and $0 \neq b \in LS(M)$ such that $a_n \xrightarrow{t(M)} 0$ and $\delta(a_n) \xrightarrow{t(M)} b$.

By Lemma (6.2.30) and passing, if necessary, to a subsequence, we may assume that $a_n = b_n + c_n$, where $b_n \in M, c_n \in LS(M), n \in \mathbb{N}, \|b_n\|_M \to 0$ and $s(|c_n|) \xrightarrow{t(M)} 0$ for $n \to \infty$.

Since the restriction $\delta|_M$ of the derivation δ to the von Neumann algebra M is a derivation from M into the Banach M-bimodule $\mathcal{E}$, by Ringrose Theorem [27], we have $\|\delta(b_n)\|_{\mathcal{E}} \to 0$.

Lemma (6.2.29) now implies that $\delta(b_n) \xrightarrow{t(M)} 0$.

From the equalities

$$\delta(c_n) = \delta(c_n s(|c_n|)) = \delta(c_n)s(|c_n|) + c_n\delta(s(|c_n|))$$

we have that

$$s(\delta(c_n)) \le l(\delta(c_n)s(|c_n|)) \vee r(\delta(c_n)s(|c_n|)) \vee l(c_n\delta(s(|c_n|))) \vee r(c_n\delta(s(|c_n|))).$$

Since

$$l(c_n) \sim r(c_n) = s(|c_n|), l(\delta(c_n)s(|c_n|)) \sim r(\delta(c_n)s(|c_n|)) \le s(|c_n|),$$
$$r(c_n\delta(s(|c_n|))) \sim l(c_n\delta(s(|c_n|))) \le l(c_n) \preccurlyeq s(|c_n|),$$

it follows that

$$\tau(s(\delta(c_n))) \le 4\tau(s(|c_n|)).$$

By Proposition (6.2.28), $\tau(s(|c_n|)) \to 0$, and therefore $\tau(s(\delta(c_n))) \to 0$ and $\tau(s(|\delta(c_n)|)) \to 0$, that implies the convergence $\tau(E_{\lambda}^{\perp}\,(|\delta(c_n)|)) \to 0$ for every $\lambda > 0$. Hence by Propositions (6.2.2)(ii) and (6.2.27), we obtain $\delta(c_n) \xrightarrow{t(M)} 0$.

Thus, $\delta(a_n) = \delta(b_n) + \delta(c_n) \xrightarrow{t(M)} 0$. The latter convergence contradicts to the inequality $b \neq 0$. Consequently, δ is $t(M)$-continuous.

Now, we present the main result.

Theorem (6.2.32)[266]: Let M be an arbitrary von Neumann algebra and let $\mathcal{E}$ be a Banach M-bimodule of locally measurable operators. Then any derivation $\delta: M \to \mathcal{E}$ is inner. In addition, there exists $d \in \mathcal{E}$ such that $\delta(x) = [d, x]$ for all $x \in M$ and $\|d\|_{\mathcal{E}} \le 2\|\delta\|_{M\to\mathcal{E}}$. If $\delta^* = \delta$ or $\delta^* = -\delta$ then d may be chosen so that $\|d\|_{\mathcal{E}} \le \|\delta\|_{M\to\mathcal{E}}$.

Proof. By [270], there exists a derivation $\bar{\delta}: LS(M) \to LS(M)$ such that $\bar{\delta}(x) = \delta(x)$ for all $x \in M$.

Choose a central decomposition $\{z_\infty, z_i\}_{j\in J}$ of the unity $\mathbf{1}$ such that Mz_∞ is a properly infinite von Neumann algebra and on every von Neumann algebra Mz_j there exists a faithful normal finite trace. By [270], the derivation $\bar{\delta}^{(z_\infty)} := \bar{\delta}|_{LS(Mz_\infty)}: LS(Mz_\infty) \to LS(Mz_\infty)$ is $t(Mz_\infty)$-continuous. Lemma (6.2.31) implies that every derivation $\bar{\delta}^{(z_j)} := \bar{\delta}|_{LS(Mz_j)}: LS(Mz_j) \to LS(Mz_j)$ is also $t(Mz_j)$-continuous for all $j \in J$. Therefore, by [270], the derivation $\bar{\delta}$ is $t(M)$-continuous. By Theorem (6.2.19) the derivation δ is inner.

Repeating the proof of Theorem (6.2.25), we obtain that there exists an element $d \in \mathcal{E}$ such that $\delta(x) = [d, x]$ for all $x \in M$.
Now, suppose that $\delta^* = \delta$. In this case, $[d + d^*, x] = [d, x] - [d, x^*]^* = \delta(x) - (\delta(x^*))^* = \delta(x) - \delta^*(x) = 0$ for any $x \in M$. Consequently, the operator $Re(d) = (d + d^*)/2$ commutes with every elements from M, hence, by Proposition (6.2.5), $Re(d)$ is a central element in the algebra $LS(M)$. Therefore, we may assume that $\delta(x) = [d, x], x \in M$, where $d = ia, a \in \mathcal{E}_h$.
By Theorem (6.2.24), there exist $c = c^*$ from the centre of the algebra $LS(M)$ and a family $\{u_\varepsilon\}_{\varepsilon>0}$ of unitary operators from M such that

$$|[a, u_\varepsilon]| \leq (1 - \varepsilon)|a - c|.$$

For $b = ia - ic$ and $\varepsilon = \frac{1}{2}$, we have

$$|b| = |a - c| \leq 2|[a, u_{1/2}]| = 2|[-id, u_{1/2}]| = 2|[d, u_{1/2}]| \in \mathcal{E}.$$

Consequently, $b \in \mathcal{E}$ (see (43)), moreover,

$$\delta(x) = [d, x] = [ia, x] = [b, x]$$

for all $x \in M$. Since

$$(1 - \varepsilon)|b| = (1 - \varepsilon)|a - c| \overset{(44)}{\leq} |[a, u_\varepsilon]| = |[d, u_\varepsilon]| = |\delta(u_\varepsilon)|,$$

it follows that

$$(1 - \varepsilon)\|b\|_{\mathcal{E}} \overset{(47)}{\leq} \|\delta(u_\varepsilon)\|_{\mathcal{E}} \leq \|\delta\|_{M\to\mathcal{E}}$$

for all $\varepsilon > 0$, that implies the inequality $\|b\|_{\mathcal{E}} \leq \|\delta\|_{M\to\mathcal{E}}$.
If $\delta^* = -\delta$, then taking $Im(d)$ instead of $Re(d)$ and repeating the preceding argument, we obtain that $\delta(x) = [b, x]$, where $b \in \mathcal{E}$ and $\|b\|_{\mathcal{E}} \leq \|\delta\|_{M\to\mathcal{E}}$.
Now, suppose that $\delta \neq \delta^*$ and $\delta \neq -\delta^*$. Equality (46) implies that

$$\|\delta^*\|_{M\to\mathcal{E}} = \sup\{\|\delta(x^*)^*\|_{\mathcal{E}} : \|x\|_M \leq 1\}$$
$$= \sup\{\|\delta(x)\|_{\mathcal{E}} : \|x\|_M \leq 1\} = \|\delta\|_{M\to\mathcal{E}}.$$

Consequently,

$$\|Re(\delta)\|_{M\to\mathcal{E}} = 2^{-1}\|\delta + \delta^*\|_{M\to\mathcal{E}} \leq \|\delta\|_{M\to\mathcal{E}}.$$

Similarly, $\|Im(\delta)\|_{M\to\mathcal{E}} \leq \|\delta\|_{M\to\mathcal{E}}$. Since $(Re(\delta))^* = Re(\delta), (Im(\delta))^* = Im(\delta)$, there exist $d_1, d_2 \in E$, such that $Re(\delta)(x) = [d_1, x], Im(\delta)(x) = [d_2, x]$ for all $x \in M$ and $\|d_i\|_{\mathcal{E}} \leq \|\delta\|_{M\to\mathcal{E}}, i = 1, 2$. Taking $d = d_1 + id_2$, we have that $d \in \mathcal{E}, \delta(x) = (Re(\delta) + i \cdot Im(\delta))(x) = [d_1, x] + i[d_2, x] = [d, x]$ for all $x \in M$, in addition $\|d\|_{\mathcal{E}} \leq 2\|\delta\|_{M\to\mathcal{E}}$.
Theorem (6.2.32) strengthens [257], where Banach M-bimodule was assumed to be separable or reflexive.
One of the important classes of Banach M-bimodules of locally measurable operators is given by non-commutative symmetric spaces.
Let M be a semifinite von Neumann algebra and τ be a faithful normal semifinite trace on M. Let $S(M, \tau)$ be the $*$-algebra of all τ-measurable operators affiliated with M.
For each $x \in S(M, \tau)$, it is possible to define the generalized singular value function $t \mapsto \mu(t, x), t > 0$ given by

$$\mu(t, x) := \inf\{\lambda > 0 : \tau(E_\lambda^\perp(|x|)) \leq t\}$$
$$= \inf\{\|x(\mathbf{1} - e)\|_M : e \in P(M), \tau(e) \leq t\}.$$

The latter notion is the main technical tool in the theory of non-commutative symmetric spaces [16], [111], [278].

Let E be a linear subspace in $S(M, \tau)$ equipped with a Banach norm $\|\cdot\|_{\mathcal{E}}$ satisfying the following condition:

if $x \in S(M, \tau), y \in \mathcal{E}$ and $\mu(x) \leq \mu(y)$ then $x \in \mathcal{E}$ and $\|x\|_{\mathcal{E}} \leq \|y\|_{\mathcal{E}}$.

In this case, the pair $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ is called a symmetric space of measurable operators. Every symmetric spaces of measurable operators is a Banach M-bimodule [16], and therefore Theorem (6.2.32) implies the following:

Corollary (6.2.33)[266]: Let $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ be a symmetric spaces of measurable operators, affiliated with a semifinite von Neumann algebra M and with a faithful semifinite normal trace τ. Then any derivation $\delta: M \to \mathcal{E}$ is continuous and there exists $d \in \mathcal{E}$ such that $\delta(x) = [d, x]$ for all $x \in M$ and $\|d\|_{\mathcal{E}} \leq 2\|\delta\|_{M\to\mathcal{E}}$.

Corollary (6.2.34)[301]: The von Neumann algebra M_m is everywhere dense in $(LS(M_m), t(M_m))$.

Proof. If $x^m \in LS(M_m)$, then there exists a sequence $\{z_n^m\}_{m,n=1}^{\infty} \subset P(Z(M_m))$ such that $z_n^m \uparrow 1$ and $x^m z_n^m \in S(M_m)$ for all $n \in \mathbb{N}$. By Corollary (6.2.4), $z_n^m \xrightarrow{t(M_m)} 1$, and therefore $x^m z_n^m \xrightarrow{t(M_m)} x^m$. Consequently, the algebra $S(M_m)$ is everywhere dense in $(LS(M_m), t(M_m))$.

Now let $x^m \in S(M_m)$. Then there exists a sequence $\{p_n\}_{n=1}^{\infty} \subset P(M_m)$ such that $p_n \uparrow 1, p_n^{\perp} \in P_{fin}(M_m)$ and $x^m p_n \in M_m$ for any $n \in \mathbb{N}$. According to (D7), we have that $D(p_n^{\perp}) \xrightarrow{t(L^{\infty}(\Omega))} 0$, therefore, Proposition (6.2.2)(i) implies the convergence $p_n \xrightarrow{t(M_m)} 1$ (we set $z_n^m = 1$). Then $x^m p_n \xrightarrow{t(M_m)} x^m$. It means that the algebra M_m is everywhere dense in the algebra $S(M_m)$ with respect to the topology $t(M_m)$. Thus, the von Neumann algebra M_m is everywhere dense in $(LS(M_m), t(M_m))$.

Corollary (6.2.35)[301]: If the centre $Z(M_m)$ of the von Neumann algebra M_m is a σ-finite algebra, then the system of sets given by (10) forms a basis of neighbourhoods of zero in the topology $t(M_m)$.

Proof. Let $V(\Omega, \varepsilon, \delta)$ be a neighbourhood of zero of the form (10). If $x^m \in V(\varepsilon, \delta, \varepsilon)$, then there exist $p \in P(M_m), z^m \in P(Z(M_m))$, such that $x^m p \in M_m, \|x^m p\|_{M_m} \leq \varepsilon, \int_{\Omega} \varphi(z^{m\perp}) d\mu \leq \delta$ and $D(z^m p^{\perp}) \leq \varepsilon\varphi(z^m)$. The inequality $\int_{\Omega} \varphi(z^{m\perp}) d\mu \leq \delta$ means that $\varphi(z^{m\perp}) \in W(\Omega, \varepsilon, \delta)$. Hence, $x^m \in V(\Omega, \varepsilon, \delta)$, that implies the inclusion $V(\varepsilon, \delta, \varepsilon) \subset V(\Omega, \varepsilon, \delta)$.

If $x^m \in V(\Omega, \varepsilon, \delta)$, then there exist $p \in P(M_m), z^m \in P(Z(M_m))$, such that $\|x^m p\|_{M_m} \leq \varepsilon, \phi(z^{m\perp}) \in W(\Omega, \varepsilon, \delta)$ and $D(z^m p^{\perp}) \leq \varepsilon\varphi(z^m)$. The inclusion $\varphi(z^{m\perp}) \in W(\Omega, \varepsilon, \delta)$ means that there exists
$E \in \Sigma$, such that $\mu(\Omega \setminus E) \leq \delta$ and $0 \leq \varphi(z^{m\perp})\chi_E \leq \varepsilon$. If $0 < \varepsilon < 1$, then $\phi(z^{m\perp})\chi_E = 0$, that is $\varphi(z^{m\perp}) \leq \chi_{\Omega} \backslash E$, and therefore $\tau(z^{m\perp}) \leq \delta$, hence $x^m \in V(\varepsilon, \delta, \varepsilon)$.

Corollary (6.2.36)[301]: If $P(Z(M_m)) \subset A, \delta$ is a derivation on A and $z^m \in P(Z(M_m))$, then $\delta(z^m) = 0$ and $\delta(z^m x^m) = z^m \delta(x^m)$ for all $x^m \in A$.

Proof. We have that $\delta(z^m) = \delta(z^{2m}) = \delta(z^m)z^m + z^m\delta(z^m) = 2z^m\delta(z^m)$. Hence, $z^m\delta(z^m) = z^m(2z^m\delta(z^m)) = 2z^m\delta(z^m)$, that is $z^m\delta(z^m) = 0$. Therefore, we have $\delta(z^m) = 0$. In particular, $\delta(z^m x^m) = \delta(z^m)x^m + z^m\delta(x^m) = z^m\delta(x^m)$.

Corollary (6.2.37)[301]: If $z^m \in P(Z(M_m)) \subset A$ and δ is a derivation on A, then $\delta(z^m A) \subset z^m A$ and the restriction $\delta(z^m)$ of the derivation δ to $z^m A$ is a derivation on $z^m A$, in addition, if δ is $t(M_m)$-continuous, then $\delta(z^m)$ is $t(z^m M_m)$-continuous.

Proof. The inclusion $\delta(z^m A) \subset z^m A$ holds. Moreover, the linear mapping $\delta^{(z^m)}: z^m A \to z^m A$ has the following property:

$$\delta^{(z^m)}((z^m x^m)(z^m y^m)) = \delta(z^m x^m) z^m y^m + z^m x^m \delta(z^m y^m)$$
$$= \delta^{(z^m)}(z^m x^m) z^m y^m + z^m x^m \delta^{(z^m)}(z^m y^m)$$
for all $x^m, y^m \in A$.

If $x_\alpha^m, x^m \in z^m A, x_\alpha^m \xrightarrow{t(z^m M_m)} x^m$, then $x_\alpha^m \xrightarrow{t(M_m)} x^m$ Proposition (6.2.8), and therefore $\delta^{(z^m)}(x_\alpha^m) = z^m \delta(x_\alpha^m) \xrightarrow{t(M_m)} z^m \delta(x^m) = \delta^{(z^m)}(x^m)$, that implies the convergence $\delta^{(z^m)}(x_\alpha^m) \xrightarrow{t(z^m M_m)} \delta(z^m)(x^m)$ Proposition (6.2.3).

Corollary (6.2.38)[301]: $\delta^{(e)}$ is a derivation on eAe.

Proof. If $x^m, y^m \in eAe$, then $x^m, y^m \in A$ and
$$\delta^{(e)}(x^m y^m) = e(\delta(x^m) y^m) e + e(x^m \delta(y^m)) e$$
$$= (e\delta(x^m)e)(ey^m e) + (ex^m e)(e\delta(y^m)e) = \delta^{(e)}(x^m) y^m + x^m \delta^{(e)}(y^m).$$

Corollary (6.2.39)[301]: Let A and δ be the same as in Corollary (6.2.37) and let $\{z_i^m\}_{m,i\in I}$ be a central decomposition of the unity $\mathbf{1}$. If $\delta^{(z_i^m)} = \delta_{(d_m)_i}, (d_m)_i \in z_i^m A$ is an inner derivation on $z_i^m A$ for every $i \in I$, then there exists an operator $d_m \in LS(M_m)$, such that $\delta(x^m) = [d_m, x^m]$ for all $x^m \in A$ and $z_i^m d_m = (d_m)_i$ for every $i \in I$.

Proof. Since $\{z_i^m\}_{m,i\in I}$ is a central decomposition of the unity $\mathbf{1}, (d_m)_i \in z_i^m A \subset z_i^m LS(M_m)$, by Proposition (6.2.1) there exists $d_m \in LS(M_m)$, such that $z_i^m d_m = (d_m)_i$ for every $i \in I$. Using Corollary (6.2.36) and the equalities $\delta^{(z_i^m)}(x^m) = [(d_m)_i, x^m], x^m \in z_i^m A, m, i \in I$ we have that for all $y^m \in A,$ the equalities
$$z_i^m \delta(y^m) = \delta^{(z_i^m)}(z_i^m y^m) = [(d_m)_i, z_i^m y^m] = [z_i^m d_m, z_i^m y^m] = z_i^m [d_m, y^m]$$
hold. Since $\bigvee_{i\in I} z_i^m = \mathbf{1}$ it follows that $\delta(y^m) = [d_m, y^m]$.

Corollary (6.2.40)[301]: Let δ be a derivation on a subalgebra A of $LS(M_m)$ and $P(M_m) \subset A$. If $p, q \in P(M_m)$ and $p\delta(q)p \geq \lambda_m p$ for some $\lambda_m > 0$, then
$$r(qp)\delta(l(qp))r(qp) \geq \lambda_m r(qp).$$

Proof. Set $e = l(qp)$ and $f = r(qp)$. It is clear that $eq = e$ and $pf = f$. In addition, $e = r((qp)^*) = r(pq)$ and $f = l((qp)^*) = l(pq)$.

Since
$$ef = (eq)(pf) = e(qp)f = l(qp)qpr(qp) = qp = (qp)f = q(pf) = qf,$$
we have
$$fe = (ef)^* = (qf)^* = fq$$
and
$$f\delta(e)f = f(f\delta(e))f = f\delta(fe)f - f(\delta(f)e)f = f\delta(fq)f - f\delta(f)qf$$
$$= f\delta(f)qf + f(f\delta(q))f - f\delta(f)qf = f\delta(q)f = f(p\delta(q)p)f \geq \lambda_m fpf$$
$$= \lambda_m f.$$

Corollary (6.2.41)[301]: Let $S = \{(p_j, x_j^m)\}_{j\in J}$ be the maximal λ_m-system for a self-adjoint derivation $\delta: LS(M_m) \to LS(M_m), g = s(S)^\perp$ and $\delta^{(g)}(x^m) = g\delta(x^m)g, x^m \in gLS(M_m)g$. Then
$$\delta^{(g)}(gM_m g) \subset gM_m g. \tag{50}$$

Proof. Let us first prove that
$$\delta^{(g)}(q) \subset gM_m g \text{ and } \left\|\delta^{(g)}(q)\right\|_{M_m} \leq \lambda_m \tag{51}$$
for any projection $q \in P(gM_m g)$. Since $\delta^* = \delta$, it follows that $\delta(q) \in LS_{h_m}(M_m)$ and therefore $\delta^{(g)}(q) \in LS_{h_m}(gM_m g)$. Let $\delta^{(g)}(q) \neq 0$ and let p be the spectral projection for $\delta^{(g)}(q)$ corresponding to the interval $(\lambda_m, +\infty)$. It is clear that $p \leq s(\delta^{(g)}(q)) \leq g$. Suppose that $p \neq 0$, then

$$0 \neq \lambda_m p \leq p\delta^{(g)}(q)p = p\delta(q)p. \quad (52)$$

Since

$$0 \neq p\delta(q)p = \delta(pq)p - \delta(p)qp = \delta(pq)p - \delta(p)(pq)^*,$$

it follows that $qp = (pq)^* \neq 0$. Consequently,

$$e = l(qp) \neq 0 \quad \text{and} \quad f = r(qp) \neq 0,$$

in addition, $e \sim f$. Since $g = s(S)^{\perp}$, from the inequalities $f \leq p \leq g$ and $e \leq q \leq g$ it follows that

$$(f \vee e)\left(p_j \vee s\left(x_j^m\right)\right) = 0, \ (f \vee e)s(\delta(p_j)) = 0$$

and

$$(p_j \vee s(x_j^m))\delta(f) = \delta((p_j \vee s(x_j^m))f) - \delta(p_j \vee s(x_j^m))f = 0$$

for all $j \in J$. Moreover, according to (52) and Corollary (6.2.40) we have that $f\delta(e)f \leq \lambda_m f$. Thus, the system $S \cup \{(f, e)\}$ is a λ_m-system, that contradicts to the maximality of the λ_m-system S. Consequently, $p = 0$, which implies the inequality $\delta^{(g)}(q) \leq \lambda_m \mathbf{1}$. Similarly, for projection $(g - q) \leq g$, we obtain that

$$g(\delta(g - q))g = \delta^{(g)}(g - q) \leq \lambda_m \mathbf{1}.$$

By Corollary (6.2.38), $g\delta(g)g = 0$, and therefore $-g\delta(q)g \leq \lambda_m \mathbf{1}$. Thus,

$$-\lambda_m \mathbf{1} \leq g\delta(q)g \leq \lambda_m \mathbf{1},$$

that is $\delta^{(g)}(q) \in gM_m g$ and $\left\|\delta^{(g)}(q)\right\|_{M_m} \leq \lambda_m$.

Now, suppose that the inclusion (50) false, that is there exists an element $x^m \in (M_m)_{h_m,1} \cap (gM_m g)$, such that $\delta^{(g)}(x^m) \in LS_{h_m}(M_m) \setminus gM_m g$. It means that the spectral projection $r = E_{-3\lambda_m}(\delta(g)(x^m))$ (or $r = E_{-3\lambda_m}(\delta(g)(x^m)))$ for $\delta^{(g)}(x^m)$ corresponding to the interval $(3\lambda_m, +\infty)$ (respectively, $(-\infty, -3\lambda_m)$), is not equal to zero. Replacing, if necessary, x^m by $-x^m$, we may assume that $r = E^{\perp}_{3\lambda_m}(\delta^{(g)}(x^m)) \neq 0$. It is clear that $r \leq s(\delta^{(g)}(x^m)) \leq g$ and

$$0 < 3\lambda_m r \leq r\delta^{(g)}(x^m)r = r\delta(x^m)r. \quad (53)$$

By (51), we have that $\left\|\delta^{(g)}(r)\right\|_{M_m} \leq \lambda_m$, and therefore the inclusion $x^m \in (M_m)_{h_m,1} \cap gM_m g$ and the equality

$$r\delta(r)x^m r + rx^m\delta(r)r = rg\delta(r)gx^m r + rx^m g\delta(r)gr$$

imply that

$$\|r\delta(r)x^m r + rx^m\delta(r)r\|_{M_m} \leq 2\lambda_m.$$

Consequently,

$$-2\lambda_m r \leq r\delta(r)x^m r + rx^m\delta(r)r \leq 2\lambda_m r. \quad (54)$$

Using (53) and (54) for $y^m = rx^m r$, we obtain that

$$r\delta(y^m)r = r\delta(rx^m r)r = r\delta(r)x^m r + r\delta(x^m)r + rx^m\delta(r)r \geq \lambda_m r > 0, \quad (55)$$

in particular, $y^m \neq 0$ and $q = s(y^m) \neq 0$. Let us show that the collection $S \cup \{(q, y^m)\}$ forms a λ_m-system. Since $q \leq r \leq g$, from (55) it follows that $q\delta(y^m)q \geq \lambda_m q$, in addition

$$(q \vee s(y^m))(p_j \vee s(x_j^m)) = 0 = q(p_j \vee s(x_j^m)), q\delta(p_j) = 0$$

and

$$(p_j \vee s(x_j^m))\delta(q) = \delta((p_j \vee s(x_j^m))q) - \delta(p_j \vee s(x_j^m))q = 0$$

for all $j \in J$. It means that the set $S \cup \{(q, y^m)\}$ is a λ_m-system, that contradicts to the maximality of the λ_m-system S. From obtained contradiction follows the validity of inclusion (50).

Corollary (6.2.42)[301]: If $\{x_j^m\}_{m,j\in J} \subset (M_m)_{h_m,1}, x_i^m x_j^m = 0, i \neq j, i,j \in J$, then there exists a unique element $x^m \in (M_m)_{h_m,1}$, denoted by $\sum_{j\in J} x_j^m$, such that $x^m s(x_j^m) = x_j^m$ for all $j \in J$ and $\bigvee_{j\in J} s(x_j^m) = s(x^m)$.

Proof. Denote by A the commutative von Neumann subalgebra of M_m, containing the family $\{x_j^m\}_{m,j\in J}$. Since A_{h_m} is an order complete vector lattice, $\{x_j^m\}_{m,j\in J}$ is the family of pairwise disjoint element of A_{h_m} and $|x_j^m| \leq \mathbf{1} \in A$ for all $j \in J$, it follows that there exists a unique element x^m of A_{h_m} such that $|x^m| \leq \mathbf{1}, x^m s(x_j^m) = x_j^m$ and $s(x^m) = \bigvee_{j\in J} s(x_j^m)$.

Let y^m be another element of $(M_m)_{h_m,1}$, such that $y^m s(x_j^m) = x_j^m$ for all $j \in J$ and $\bigvee_{j\in J} s(x_j^m) = s(y^m)$. Then $(x^m - y^m)s(x_j^m) = x_j^m - x_j^m = 0$ for any $j \in J$. Therefore, $s(x^m) = \bigvee_{j\in J} s(x_j^m) \leq (r(x^m - y^m))^{\perp}$ and then

$$x^m - y^m = x^m s(x^m) - y^m s(y^m) = x^m s(x^m) - y^m s(x^m) = (x^m - y^m)s(x^m) = 0.$$

Corollary (6.2.43)[301]: Let $x^m \in LS_{h_m}(M_m), p, q \in P(M_m), \rho_m, \lambda_m \in \mathbb{R}, \rho_m < \lambda_m$,

$$p x^m p \leq \rho_m p \tag{56}$$

and

$$q x^m q \geq \lambda_m q. \tag{57}$$

Then $p \preccurlyeq q^{\perp}$ and $q \preccurlyeq p^{\perp}$.

Proof. Set $r = p \wedge q$. Multiplying both parts on both sides of inequalities (56) and (57) by r, we obtain that

$$\lambda_m r \leq r x^m r \leq \rho_m r,$$

that is possible if $r = 0$ only. Therefore, $p = p - p \wedge q \sim p \vee q - q \leq q^{\perp}$, that is $p \preccurlyeq q^{\perp}$. Similarly, $q \preccurlyeq p^{\perp}$.

Corollary (6.2.44)[301]: Let $S = \{(p_j, x_j^m)\}_{m,j\in J}$ be a λ_m-system for a self-adjoint derivation δ:
$LS(M_m) \to LS(M_m)$, let D be a dimension function on $P(M_m)$. Then

$$D(s(S)) \leq 8D\left(E_{\rho_m}^{\perp}\left(\delta\left(\sum_{j\in J} x_j^m\right)\right)\right) \text{ for any } \rho_m < \lambda_m. \tag{58}$$

Proof. Set $x^m = \sum_{j\in J} x_j^m$ see Corollary (6.2.42) and $p = \bigvee_{j\in J} p_j$. The following inequality is our main technical tool. It may be thought of as 'glueing' of the inequalities given in Definition (6.2.14)(iii). We claim that

$$p\delta(x^m)p \;\geq\; \lambda_m p. \tag{59}$$

To prove the claim, we note firstly that $(p_j \vee s(x_j^m))(p_i \vee s(x_i^m)) = 0$ and $(p_j \vee s(x_j^m))s(\delta(p_i)) = 0$ for $i \neq j$ imply that $x_i^m p_j = x_i^m s(x_i^m) p_j = 0$ and $x_i^m \delta(p_j) = x_i^m s(x_i^m) s(\delta(p_j))\delta(p_j) = 0$ for $i \neq j$. Therefore,

$$\delta(x_i^m)p_j = \delta(x_i^m p_j) - x_i^m \delta(p_j) = 0,$$

that implies the equality

$$s(\delta(x_i^m))p_j = 0 \; for \; i = j.$$

From here and from the equality $p = \bigvee_{j\in J} p_j$ it follows that

$$s(\delta(x_i^m))p = s(\delta(x_i^m))p_i.$$

Thus,

$$\delta(x_i^m)p = \delta(x_i^m)s(\delta(x_i^m))p = \delta(x_i^m)p_i. \tag{60}$$

By Corollary (6.2.42), we have that

$$p_i x^m = p_i s(x^m) x^m = \left(p_i \bigvee_{j\in J} s(x_j^m)\right) x^m = p_i s(x_i^m) x^m = p_i x_i^m \qquad (61)$$

and

$$x_i^m p = x_i^m \left(s(x_i^m) \bigvee_{j\in J} p_j\right) = x_i^m (s(x_i^m) p_i) = x_i^m p_i. \qquad (62)$$

Similarly,

$$\delta(p_i) x^m p = \delta(p_i) \left(s\big(\delta(p_i)\big) \bigvee_{j\in J} s\big(x_j^m\big) \right) x^m p = \delta(p_i) s(x_i^m) x^m p = \delta(p_i) x_i^m p$$
$$= \delta(p_i) x_i^m p_i. \qquad (63)$$

By (60)-(62), we obtain

$$\delta(p_i x^m) p = \delta(p_i x_i^m) p = \delta(p_i) x_i^m p + p_i \delta(x_i^m) p$$
$$= \delta(p_i) x_i^m p_i + p_i \delta(x_i^m) p_i = \delta(p_i x_i^m) p_i,$$

that by (63) implies the equalities

$$p_i(p\delta(x^m)p) = p_i \delta(x^m) p = \delta(p_i x^m) p - \delta(p_i) x^m p = \delta(p_i x_i^m) p_i - \delta(p_i) x_i^m p_i$$
$$= p_i \delta(x_i^m) p_i.$$

Hence,

$$p_i(p\delta(x^m)p) = p_i \delta(x_i^m) p_i, \qquad (64)$$

in particular, the projection p_i commutes with the operator $p\delta(x^m)p$. Set $y^m = p\delta(x^m)p - \lambda_m p$ and by $y^m - = (|y^m| - y^m)/2$ denote the negative part of the operator y^m. Since $p_i y^m = y^m p_i$ (see (64)) and $p_i \delta(x_i^m) p_i \geq \lambda_m p_i$ (see Definition (6.2.13)(iii)), it follows that

$$y^m - p_i = p_i y^m - = (p_i(p\delta(x^m)p - \lambda_m p)) - \overset{(64)}{=} (p_i \delta(x_i^m) p_i - \lambda_m p_i) - = 0 \qquad (65)$$

for all $i \in J$. From equalities (65) and $p = \bigvee_{j\in J} p_j$ by [113], it follows that

$$(py^m p) - = p(p\delta(x^m)p - \lambda_m p) - p = py^m - p = 0. \qquad (66)$$

Therefore,

$$py^m p = (py^m p)_+ - (py^m p) - \overset{(66)}{=} (py^m p)_+ \geq 0,$$

which implies inequality (59).

Having established our claim, the rest is an easy application of the properties of the dimension function D.

Fix a real number $\rho_m < \lambda_m$ and set $q = E_{\rho_m}(\delta(x^m))$. By Corollary (6.2.43), we obtain

$$p \preccurlyeq q^{\perp}. \qquad (67)$$

For every fixed $j \in J$ we have that

$$\delta(p_j) = \delta(p_j^2) = \delta(p_j) p_j + p_j \delta(p_j) = \delta(p_j) p_j + (\delta(p_j) p_j)^*$$

and therefore

$$s(\delta(p_j)) \leq l(\delta(p_j) p_j) \vee p_j,$$

that implies

$$D(s(\delta(p_j))) \leq D(l(\delta(p_j) p_j)) + D(p_j). \qquad (68)$$

Since $l(\delta(p_j) p_j) \sim r(\delta(p_j) p_j) \leq p_j$, by (68) we have

$$D(s(\delta(p_j))) \leq 2D(p_j) \qquad (69)$$

for all $j \in J$. Similarly,

$$D(s(\delta(p_j \vee s(x_j^m)))) \leq 2D(p_j \vee s(x_j^m)),$$

and in view of the equivalence $p_j \sim s(x_j^m)$ (see the definition of λ_m-system), we obtain

$$D(s(\delta(p_j \vee s(x_j^m)))) \leq 4D(p_j). \qquad (70)$$

Denote by A the directed set of all finite subsets of J ordered by inclusion and for every $\alpha \in A$ set

$$e_\alpha := \bigvee_{j\in\alpha} (p_j \vee s(x_j^m) \vee s(\delta(p_j)) \vee s(\delta(p_j \vee s(x_j^m)))).$$

From properties $(D2)$ and $(D3)$ of the dimension function D and from inequalities (67), (69) and (70) we have that

$$D(e_\alpha) \le \sum_{j\in\alpha} D(p_j \vee s(x_j^m) \vee s(\delta(p_j)) \vee s(\delta(p_j \vee s(x_j^m))))$$

$$\le \sum_{j\in\alpha} (D(p_j) + D(s(x_j^m)) + D(s(\delta(p_j))) + D(s(\delta(p_j \vee s(x_j^m)))))$$

$$\le 8 \sum_{j\in\alpha} D(p_j) = 8D\left(\sum_{j\in\alpha} p_j\right) \le 8D(p) \le 8D(q^\perp).$$

Since $e_\alpha \uparrow s(S)$ the last inequality and property $(D6)$ of the dimension function D imply that

$$D(s(S)) = D\left(\bigvee_{\alpha\in A} e_\alpha\right) = \bigvee_{\alpha\in A} D(e_\alpha) \le 8D(q^\perp).$$

Corollary (6.2.45)[301]: Every derivation on the $*$-algebra $LS(M_m)$ continuous with respect to the topology $t(M_m)$ is necessarily inner.

Proof. Let δ be an arbitrary derivation on the $*$-algebra $LS(M_m)$ and let δ be continuous with respect to the topology $t(M_m)$. By Lemmas 3.4 and 3.5, we may assume that δ is a self-adjoint derivation.

Choose a central decomposition $\{z_i^m\}_{m,i\in I}$ of the unity $\mathbf{1}$, such that every Boolean algebra $z_i^m P(Z(M_m))$ has a countable type, $i \in I$. By Corollary (6.2.37) the restriction $\delta^{(z_i^m)}$ of the derivation δ to $z_i^m LS(M_m) = LS(z_i^m M_m)$ is a $t(z_i^m M_m)$-continuous derivation on $LS(z_i^m M_m)$. If every derivation $\delta^{(z_i^m)}, i \in I$ is inner, then, by Corollary (6.2.39), the derivation δ is inner too. Thus, in the proof of Corollary (6.2.45) we may assume that the centre $Z(M_m)$ of the von Neumann algebra M_m is σ-finite algebra. In this case, there exists a faithful normal finite trace $\tau(x^m) = \int \varphi(x^m)\, d\mu$ on $Z(M_m)$ and the vector topology $t(M_m)$ has the basis of neighbourhoods of zero consists of the sets $V(\varepsilon, \beta, \gamma)$ given by (10) (see Corollary (6.2.35)). Since the derivation δ is $t(M_m)$-continuous, for arbitrary $\varepsilon, \beta, \gamma > 0$ there exist $\varepsilon_1, \beta_1, \gamma_1 > 0$, such that $\delta(V(\varepsilon_1, \beta_1, \gamma_1)) \subset V(\varepsilon, \beta, \gamma)$. It is clear that

$$(M_m)_1 := \{x^m \in M_m : \|x^m\|_{M_m} \le 1\} \subset V(1, \beta_1, \gamma_1) = \varepsilon_1^{-1} V(\varepsilon_1, \beta_1, \gamma_1),$$

and therefore

$$\delta((M_m)_1) \subset \varepsilon_1^{-1} V(\varepsilon, \beta, \gamma) = V\,(\varepsilon/\varepsilon_1, \beta, \gamma).$$

Hence, for the $t(M_m)$-continuous self-adjoint derivation $\delta: LS(M_m) \to LS(M_m)$ and for arbitrary positive numbers β and γ there exists a number $\Delta(\beta, \gamma)$, such that

$$\delta((M_m)_1) \subset V(\Delta(\beta, \gamma), \beta, \gamma). \tag{71}$$

Let D, φ, τ be the same as in the definition of the set $V(\varepsilon, \beta, \gamma)$ from (10). Fix an arbitrary $2\Delta(\beta, \gamma)$-system $S = \{(p_j, x_j^m)\}_{j\in J}$ for the derivation δ and show that there exists a central projection $z^m \in P(Z(M_m))$, such that

$$\tau(z^{m\perp}) \le \beta \text{ and } D(z^m s(S)) \le 8\gamma\varphi(z^m). \tag{72}$$

If S is empty, then $s(S)=0$ and, in this case, relations (72) hold for $z^m=\mathbf{1}$. Now, let $S=\{(p_j,x_j^m)\}_{m,j\in J}$ be non-empty $2\Delta(\beta,\gamma)$-system. By Corollary (6.2.42), there exists $x^m=x_j^m\in(M_m)_{h_m,1}$. From (71) it follows that $\delta(x^m)\in V(\Delta(\beta,\gamma),\beta,\gamma)$ for all $\beta,\gamma>0$. Therefore, there exist projections $z^m\in P(Z(M_m))$ and $q\in P(M_m)$, such that

$$\tau(z^{m\perp})\leq\beta,\delta(x^m)q\in M_m,\|\delta(x^m)q\|_{M_m}\leq\Delta(\beta,\gamma)\text{ and }D(z^mq^\perp)\leq\gamma\varphi(z^m).\quad(73)$$

Since $x^m=x^{m*}$ and $\delta=\delta^*$, it follows that $\delta(x^m)=(\delta(x^m))^*$ and, by (73), we have

$$-\Delta(\beta,\gamma)q\leq q\delta(x^m)q\leq\Delta(\beta,\gamma)q.\quad(74)$$

Set $\rho_m=\frac{3}{2}\cdot\Delta(\beta,\gamma)$. Using inequalities (74) and

$$\rho_mE_{\rho_m}^\perp(\delta(x^m))\leq E_{\rho_m}^\perp(\delta(x^m))\delta(x^m)E_{\rho_m}^\perp(\delta(x^m)),$$

we obtain that $E_{\rho_m}^\perp(\delta(x^m))\leq q^\perp$. Consequently, $z^mE_{\rho_m}^\perp(\delta(x^m))\leq z^mq^\perp$ and, by (58) and (73), we have that

$$D(z^ms(S))\overset{(D4)}{=}=\varphi(z^m)D(s(S))\overset{(58)}{=}8\varphi(z^m)D(E_{\rho_m}^\perp(\delta(x^m)))$$
$$\overset{(D4)}{=}8D(z^mE_{\rho_m}^\perp(\delta(x^m)))\overset{(D2),(D3)}{\leq}8D(z^mq^\perp)\overset{(73)}{\leq}8\gamma\varphi(z^m),$$

that is (72) holds.

For every $n\in\mathbb{N}$ choose the maximal (possible, empty) $2\Delta(2^{-n},2^{-n})$-system S_n for the derivation δ. Set $q_n^{'}=s(S_n)^\perp$. By Corollary (6.2.41), we have that

$$\delta(q_n^{'})(q_n^{'}M_mq_n^{'})\subset q_n^{'}M_mq_n^{'}\quad(75)$$

for all $n\in\mathbb{N}$. Moreover, in view of (72), there exists a projection $z_n^{m'}\in P(Z(M_m))$, such that

$$\tau(z_n^{m'\perp})\leq2^{-n}\text{ and }D(z_n^{m'}q_n^{'\perp})\leq2^{-n+3}\varphi(z_n^{m'}).\quad(76)$$

We set $q_n:=\bigwedge_{k=n+1}^\infty q_k^{'}$ and $z_n^m:=\bigwedge_{k=n+1}^\infty z_k^{m'}$ and consider the derivation $\delta^{(q_n)}$ on $q_nLS(M_m)q_n$. We shall show that

$$\delta^{(q_n)}(q_nM_mq_n)\subset q_nM_mq_n.$$

Clearly, the sequences $\{q_n\}$ and $\{z_n^m\}$ are increasing and, in addition,

$$\tau(z^{m\perp}_{n})\leq\tau\left(\bigvee_{k\geq n+1}z_n^{m'\perp}\right)\leq\sum_{k\geq n+1}\tau(z_n^{m'\perp})\overset{(76)}{=}\sum_{k\geq n+1}2^{-k}=2^{-n}\quad(77)$$

and

$$D(z_n^mq_n^\perp)=\varphi(z_n^m)D\left(\bigvee_{k\geq n+1}z_n^mq_k^{'\perp}\right)$$
$$\leq\varphi(z_n^m)D\left(\bigvee_{k\geq n+1}z_k^{m'}q_k^{'\perp}\right)\overset{(D6)}{\leq}\varphi(z_n^m)\sum_{k\geq n+1}D(z_k^{m'}q_k^{'\perp})$$
$$\overset{(76)}{\leq}\varphi(z_n^m)\sum_{k\geq n+1}2^{-k+3}\phi(z_k^{m'})=\sum_{k\geq n+1}2^{-k+3}\varphi(z_n^mz_k^{m'})\quad(78)$$
$$=\sum_{k\geq n+1}2^{-k+3}\varphi(z_n^m)=2^{-n+3}\varphi(z_n^m).$$

If $x^m\in q_nM_mq_n$, then $x^m\in q_{n+1}^{'}M_mq_{n+1}^{'}$ and therefore, by (75),

$$\delta^{(q_n)}(x^m)=q_n\delta(x^m)q_n=q_nq_{n+1}^{'}\delta(x^m)q_{n+1}^{'}q_n=q_n\delta^{(q_{n+1}^{'})}(x^m)q_n$$
$$\in q_nq_{n+1}^{'}M_mq_{n+1}^{'}q_n=q_nM_mq_n.$$

Hence, the restriction $\delta^{(q_n)}|_{q_nM_mq_n}$ of the derivation $\delta^{(q_n)}$ to $q_nM_mq_n$ is a derivation on the von Neumann algebra $q_nM_mq_n$. By Sakai Theorem [29], there exists an element $(c_m)_n\in q_nM_mq_n$, such that $\delta(q_n)(x^m)=[(c_m)_n,x^m]$ for all $x^m\in q_nM_mq_n$.

Now, we replace the sequence $\{(c_m)_n\}$ with a sequence $\{(d_m)_n\}$, which is somewhat similar to a sequence of 'martingale differences'. More precisely, we shall construct a sequence $\{(d_m)_n\}$ of M_m, such that

$$q_n(d_m)_m q_n = (d_m)_n \text{ for all } n \leq m,$$
$$\delta^{(q_n)}(x^m) = [(d_m)_n, x^m] \text{ for all } x^m \in q_n M_m q_n. \quad (79)$$

Set $(d_m)_1 = (c_m)_1$ and assume that elements $(d_m)_1, \dots, (d_m)_n$ are already constructed. Since $\delta^{(q_n)}(q_n x^m q_n) = q_n \delta^{(q_{n+1})}(q_n x^m q_n) q_n$, it follows that

$$[(d_m)_n, q_n x^m q_n] = q_n[(c_m)_{n+1}, q_n x^m q_n] q_n = [q_n (c_m)_{n+1} q_n, q_n x^m q_n]$$

for any $x^m \in M_m$. Consequently, the element $(d_m)_n - q_n(c_m)_{n+1}q_n$ is contained in the centre of algebra $q_n M_m q_n$. By [274] there exists an element z^m of the centre of algebra $q_{n+1}M_m q_{n+1}$, such that $(d_m)_n - q_n(c_m)_{n+1}q_n = z^m q_n$. Set $(d_m)_{n+1} = (c_m)_{n+1} + z^m$. It is clear that

$$\delta^{(q_{n+1})}|_{q_{n+1}M_m q_{n+1}}(x^m) = [(c_m)_{n+1}, x^m] = [(d_m)_{n+1}, x^m] \quad (80)$$

for all $x^m \in q_{n+1}M_m q_{n+1}$, in addition,

$$(d_m)_{n+1} \in q_{n+1}M_m q_{n+1} \text{ and } q_n(d_m)_{n+1}q_n = q_n(c_m)_{n+1}q_n + z^m q_n = (d_m)_n$$

for every $n \in \mathbb{N}$. Moreover, for $k \in \mathbb{N}, k < n+1$ the equalities

$$q_k(d_m)_{n+1}q_k = q_k q_n (d_m)_{n+1} q_n q_k = q_k (d_m)_n q_k = \cdots = q_k (d_m)_{k+1} q_k$$
$$= (d_m)_k \quad (81)$$

hold.

Thus we have constructed the sequence $\{(d_m)_n\}$ of elements of M_m which has property (79).

By [272], the topology $t(M_m)$ induces on $q_n LS(M_m) q_n = LS(q_n M_m q_n)$ the topology $t(q_n M_m q_n)$, and therefore the derivation $\delta(q_n)$ is continuous on $(LS(q_n M_m q_n), t(q_n M_m q_n))$. By Corollary (6.2.34), we have that $q_n M_m q_n t(q_n M_m q_n) = LS(q_n M_m q_n)$. Consequently, the equality $\delta^{(q_n)}(x^m) = [(d_m)_n, x^m]$ holds for all $x^m \in LS(q_n M_m q_n)$.

Our next objective is to show that the sequence $\{(d_m)_n\}$ is a Cauchy sequence in $(LS(M_m), t(M_m))$. If $n, m \in \mathbb{N}, n < m$, then

$$(d_m)_m - (d_m)_n \overset{(4.9)}{=} q_m(d_m)_m q_m - q_n (d_m)_m q_n$$
$$= (q_m - q_n)(d_m)_m q_m + q_n (d_m)_m (q_m - q_n).$$

Since

$$r((q_m - q_n)(d_m)_m q_m) \sim l((q_m - q_n)(d_m)_m q_m) \leq q_n^{\perp},$$

it follows that

$$D(z_n^m r((d_m)_m - (d_m)_n)) \leq D(z_n^m r((q_m - q_n)(d_m)_m q_m) \vee z_n^m q_n^{\perp})$$
$$\leq 2D(z_n^m q_n^{\perp}) \overset{(78)}{\leq} 2^{-n+4}\varphi(z_n^m).$$

From here, by taking $p = r((d_m)_m - (d_m)_n)^{\perp}$ in view of (10) and (77), we obtain

$$(d_m)_m - (d_m)_n \in V(0, 2^{-n}, 2^{-n+4}) \subset V(1/n, 2^{-n}, 2^{-n+4}).$$

It means that $\{(d_m)_n\}$ is a Cauchy sequence in $(LS(M_m), t(M_m))$, and therefore, since the space $(LS(M_m), t(M_m))$ is complete there exists $d_m \in LS(M_m)$, such that $(d_m)_n \xrightarrow{t(M_m)} d_m$. Finally, let us show that $\delta(x^m) = [d_m, x^m]$ for all $x^m \in LS(M_m)$. By (77) and (78) we have that $q_n^{\perp} \in V(0, 2^{-n}, 2^{-n+3})$ for all $n \in \mathbb{N}$, and therefore $q_n^{\perp} \xrightarrow{t(M_m)} 0$. Consequently, $q_n \xrightarrow{t(M_m)} \mathbf{1}$ and for every $x^m \in LS(M_m)$ we have that $q_n x^m q_n \xrightarrow{t(M_m)} x^m$. We just need to use the $t(M_m)$-continuity of the derivation δ, which implies the following:

$$\delta(x^m) = t(M_m) - \lim_{n\to\infty}(q_n \delta(q_n x^m q_n) q_n) = t(M_m) - \lim_{n\to\infty} \delta^{q_n}(q_n x^m q_n)$$

$$= t(M_m) - \lim_{n\to\infty}[(d_m)_n, q_n x^m q_n] = [t(M_m) - \lim_{n\to\infty}(d_m)_n, t(M_m) - \lim_{n\to\infty} q_n x^m q_n] = [d_m, x^m].$$

Corollary (6.2.46)[301]: If M_m is a properly infinite von Neumann algebra, then every derivation on the $*$-algebra $LS(M_m)$ is inner.

Proof. By [270] every derivation $\delta: LS(M_m) \to LS(M_m)$ is $t(M_m)$-continuous. Consequently, by Corollary (6.2.45), there exists $d_m \in LS(M_m)$, such that $\delta(x^m) = [d_m, x^m]$ for all $x^m \in LS(M_m)$.

Corollary (6.2.47)[301]: Let M_m be a von Neumann algebra acting in a Hilbert space H, and let $a_m, b_m \in LS(M_m), 0 \le a_m \le b_m$. Then $a_m^{\frac{1}{2}} = c_m b_m^{\frac{1}{2}}$ for some $c_m \in s(b_m)M_m s(b_m), \|c_m\|_{M_m} \le 1$, in particular, $a_m = c_m b_m c_m^*$. In addition, if $(c_m)_1 \in M_m$ and $a_m^{\frac{1}{2}} = (c_m)_1 b_m^{\frac{1}{2}}$, then $s(b_m) \cdot (c_m)_1 \cdot s(b_m) = c_m$.

Proof. Let us first show that $s(a_m) \le s(b_m)$. Since

$$0 \le (\mathbf{1} - s(b_m))a_m(\mathbf{1} - s(b_m)) \le (\mathbf{1} - s(b_m))b_m(\mathbf{1} - s(b_m)) = 0,$$

if follows that $\left(\mathbf{1} - s(b_m)\right)a_m^{\frac{1}{2}} = 0$, which implies the equality $(\mathbf{1} - s(b_m))a_m = 0$, that is, $s(b_m)a_m = a_m = a_m^* = a_m^* s(b_m) = a_m s(b_m)$. Consequently, $s(a_m) \le s(b_m)$.

Thus, passing if necessary to the reduced algebra $s(b_m)M_m s(b_m)$, we may assume that $s(b_m) = \mathbf{1}$. For every $n \in \mathbb{N}$ denote by p_n the spectral projection for the operator b_m corresponding to the interval $[1/n, n]$. Since $p_n \uparrow s(b_m) = \mathbf{1}$ it follows that the linear subspace $H_0 = \bigcup_{n=1}^{\infty} p_n H$ is dense in H and $H_0 \subset \mathfrak{D}(b_m) \cap \mathfrak{D}(b_m^{\frac{1}{2}})$. Furthermore, according to the inequalities $0 \le p_n a_m p_n \le p_n b_m p_n \le n p_n$ we have that $a_m^{\frac{1}{2}} p_n \in M_m$ and $\left\|a_m^{\frac{1}{2}} p_n\right\|_{M_m} \le \sqrt{n}$ for all $n \in \mathbb{N}$. In particular, $H_0 \subset \mathfrak{D}(a_m^{\frac{1}{2}})$.

Since $b_m^{\frac{1}{2}} p_n \le n^{1/2} p_n$ and $b_m^{\frac{1}{2}}(p_n H) = p_n b_m^{1/2}(p_n H) \subset p_n H$ for all $n \in \mathbb{N}$ we have $b_m^{\frac{1}{2}}(H_0) \subset H_0$. Consequently, it is possible to define a linear mapping $d_m: b_m^{\frac{1}{2}}(H_0) \to H$ by setting $d_m(b_m^{\frac{1}{2}}\xi) = a_m^{\frac{1}{2}}\xi, \xi \in H_0$. The definition of the operator d is correct since the equality $b_m^{\frac{1}{2}}\xi = 0$ and the inequality

$$\left\|a_m^{\frac{1}{2}}\xi\right\|_H^2 = (a_m^{\frac{1}{2}}\xi, a_m^{\frac{1}{2}}\xi) = (a_m\xi, \xi) \le (b_m\xi, \xi) = \left\|b_m^{\frac{1}{2}}\xi\right\|_H^2$$

imply that $a_m^{\frac{1}{2}}\xi = 0$.

In addition, for every $\xi \in H_0$, we have

$$\left\|d_m(b_m^{\frac{1}{2}}\xi)\right\|_H^2 = \left\|a_m^{\frac{1}{2}}\xi\right\|_H^2 \le \left\|b_m^{\frac{1}{2}}\xi\right\|_H^2,$$

that is d_m is a continuous linear operator on $b_m^{\frac{1}{2}}(H_0)$ and $\|d_m\|_{b_m^{\frac{1}{2}}(H_0)\to H} \le 1$.

Since $n^{-1}p_n \le b_m p_n \le n p_n$, by Proposition [113], we have $n^{-1/2}p_n \le b_m^{\frac{1}{2}} p_n \le n^{1/2} p_n$.

Therefore, the restriction of the operator $b_m^{\frac{1}{2}}$ to $p_n(H_0)$ has inverse bounded operator $(b_m)_n$, in addition $n^{-1/2}p_n \le (b_m)_n p_n \le n^{1/2}p_n$. Hence, $b_m^{\frac{1}{2}}(p_nH) = p_nH$, that implies the equality $b_m^{\frac{1}{2}}(H_0) = H_0$.
Thus, the operator d_m uniquely extends to the Hilbert space H up to a bounded linear operator c_m, moreover, $\|c_m\|_{B(H)} \le 1$ and $c_m b_m^{\frac{1}{2}}\xi = a_m^{\frac{1}{2}}\xi$ for all $\xi \in H_0$.
If u is a unitary operator from the commutant M_m', then $u(p_nH) = p_nH$ for all $n \in \mathbb{N}$ and therefore $u(H_0) = H_0$. If $\eta \in H_0$, then $\eta = b_m^{\frac{1}{2}}\xi$ for some $\xi \in H_0$ and

$$u^{-1}c_m u\eta = u^{-1}c_m u b_m^{\frac{1}{2}}\xi = u^{-1}c_m b_m^{\frac{1}{2}}u\xi$$
$$= u^{-1}a_m^{\frac{1}{2}}u\xi = u^{-1}u a_m^{\frac{1}{2}}\xi = a_m^{\frac{1}{2}}\xi = c_m b_m^{\frac{1}{2}}\xi = c_m\eta.$$

Consequently, $u^{-1}c_m u = c_m$, that implies the inclusion $c_m \in M_m$.
Since $p_n c_m b_m^{\frac{1}{2}} p_n = p_n a_m^{\frac{1}{2}} p_n$ for all $n \in \mathbb{N}$ and $p_n \uparrow \mathbf{1}$, by Proposition [113] we have $c_m b_m^{\frac{1}{2}} = a_m^{\frac{1}{2}}$.
If $(c_m)_1 \in M_m$ and $(c_m)_1 b_m^{\frac{1}{2}} = a_m^{\frac{1}{2}}$, then the operators $(c_m)_1$ and c_m coincide on the everywhere dense subspace H_0 and therefore $(c_m)_1 = c_m$.
If $s(b_m) = \mathbf{1}$, $(c_m)_1 \in M_m$ and $(c_m)_1 b_m^{\frac{1}{2}} = a_m^{\frac{1}{2}}$, then, using equalities

$$a_m^{\frac{1}{2}} s(b_m) = s(b_m) a_m^{\frac{1}{2}} = a_{\mathrm{m}}^{\frac{1}{2}}$$

and

$$b_m^{\frac{1}{2}} s(b_m) = s(b_m) b_m^{\frac{1}{2}} = b_m^{\frac{1}{2}},$$

we obtain $(s(b_m)(c_m)_1 s(b_m)) b_m^{\frac{1}{2}} = a_m^{\frac{1}{2}}$. Uniqueness of the operator c_m in the reduced algebra $s(b_m)M_m s(b_m)$ implies that $s(b_m)\cdot(c_m)_1 \cdot s(b_m) = c_m$.
Corollary (6.2.48)[301]: Let M_m be a properly infinite von Neumann algebra and let $\mathcal{E}$ be an M_m bimodule of locally measurable operators. Then any derivation $\delta: M_m \to \mathcal{E}$ is inner.
Proof. By [270], there exists a derivation $\delta: LS(M_m) \to LS(M_m)$, such that $\delta(x^m) = \delta(x^m)$ for all $x^m \in M_m$. By Corollary (6.2.46), there exists an element $a_m \in LS(M_m)$, such that $\delta(x^m) = [a_m, x^m]$ for all $x^m \in LS(M_m)$. It is clear that $[a_m, M_m] = \delta(M_m) = \delta(M_m) \subset \mathcal{E}$.

Let $(a_m)_1 = Re(a_m), (a_m)_2 = Im(a_m)$. Since $[a_m^*, x^m] = -[a_m, x^{m*}]^* \in \mathcal{E}$ for any $x^m \in M_m$, it follows that $[(a_m)_1, x^m] = [a_m + a_m^*, x^m]/2 \in \mathcal{E}$ and $[(a_m)_2, x^m] = [a_m - a_m^*, x^m]/2i \in \mathcal{E}$ for all $x^m \in M_m$.
Taking $\varepsilon = \frac{1}{2}$ in (44), we obtain that there exist $(c_m)_1, (c_m)_2 \in Z_{h_m}(LS(M_m))$ and unitary operators $u_1, u_2 \in M_m$ such that

$$2|[(a_m)_i, u_i]| \ge |(a_m)_i - (c_m)_i|, \qquad i = 1, 2.$$

Since $[(a_m)_i, u_i] \in \mathcal{E}$ and $\mathcal{E}$ is an M_m-bimodule we have that $(d_m)_i := (a_m)_i - (c_m)_i \in \mathcal{E}, i = 1,2$ (see (43)). Therefore, $d_m = (d_m)_1 + i(d_m)_2 \in \mathcal{E}$. Since $(c_m)_1$ and $(c_m)_2$ are central elements from $LS(M_m)$ it follows that $\delta(x^m) = [a_m, x^m] = [d_m, x^m]$ for all $x^m \in M_m$.

Corollary (6.2.49)[301]: If $\{p_k\}_{k=1}^n \subset P(M_m) \cap \mathcal{E}, n \in \mathbb{N}$, then

$$\bigvee_{k=1}^{n} p_n \in \mathcal{E} \text{ and } \left\|\bigvee_{k=1}^{n} p_n\right\|_{\mathcal{E}} \le \sum_{k=1}^{n} \|p_k\|_{\mathcal{E}} . \qquad (82)$$

Proof. If $p, q \in P(M_m) \cap \mathcal{E}$, then $p \vee q - p \sim q - p \wedge q \le q$, and therefore, $p \vee q - p \in \mathcal{E}$ and $\|p \vee q - p\|_{\mathcal{E}} \le \|q\|_{\mathcal{E}}$ (see (48)). Hence, $p \vee q = (p \vee q - p) + p \in \mathcal{E}$ and

$$\|p \vee q\|_{\mathcal{E}} - \|p\|_{\mathcal{E}} \le \|p \vee q - p\|_{\mathcal{E}} \le \|q\|_{\mathcal{E}} .$$

For an arbitrary finite set $\{p_k\}_{k=1}^n \subset P(M_m) \cap \mathcal{E}$ the assertion is proved via mathematical induction.

Corollary (6.2.50)[301]: If $\{p_n\}_{n=1}^\infty \subset P(M_m) \cap \mathcal{E}$ and the series $\sum_{n=1}^\infty \|p_n\|_{\mathcal{E}}$ converges, then p = $\bigvee_{n=1}^\infty p_n \in \mathcal{E}$ and $\|p\|_{\mathcal{E}} \le \sum_{n=1}^\infty \|p_n\|_{\mathcal{E}}$.

Proof. Set $q_n = \bigvee_{k=1}^n p_k$. By (82), we have $q_n \in \mathcal{E}$ and $\|q_n\|_{\mathcal{E}} \le \sum_{k=1}^n \|p_k\|_{\mathcal{E}}$.
Let $n, m \in \mathbb{N}, n < m$. By (48) and (82), we have that

$$\|q_m - q_n\|_{\mathcal{E}} = \left\|q_n \vee \bigvee_{k=n+1}^{m} p_k - q_k\right\|_{\mathcal{E}}$$

$$= \left\|\bigvee_{k=n+1}^{m} p_k - q_n \wedge \bigvee_{k=n+1}^{m} p_k\right\|_{\mathcal{E}} \le \left\|\bigvee_{k=n+1}^{m} p_k\right\|_{\mathcal{E}} \le \sum_{k=n+1}^{m} \|p_k\|_{\mathcal{E}} .$$

Consequently, $\{q_n\}$ is a Cauchy sequence in $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$, and therefore there exists $q \in \mathcal{E}$, such that $\|q_n - q\|_{\mathcal{E}} \to 0$, in addition $\|q\|_{\mathcal{E}} \le \sum_{k=n+1}^m \|p_n\|_{\mathcal{E}}$.
Since

$$\|qp - q_n\|_{\mathcal{E}} = \|qp - q_n p\|_{\mathcal{E}} \le \|p\|_{M_m} \|q - q_n\|_{\mathcal{E}} ,$$

it follows that $qp = q = q^* = pq$. Hence, $s(p - q) \le p$. Fix $n_0 \in \mathbb{N}$, then for $n > n_0$, we have

$$\left\|q_{n_0} q - q_{n_0}\right\|_{\mathcal{E}} = \left\|q_{n_0} q - q_{n_0} q_n\right\|_{\mathcal{E}}$$
$$\le \left\|q_{n_0}\right\|_{M_m} \|q - q_n\|_{\mathcal{E}} \le \|q - q_n\|_{\mathcal{E}} .$$

Passing to the limit for $n \to \infty$, we obtain $q_{n_0} q = q_{n_0}$. Therefore, $q_n(p - q)q_n = 0$ for all $n \in \mathbb{N}$. The inequality $s(p - q) \le p$ and the convergence $q_n \uparrow p$ by [113] imply that $q = p$.

Corollary (6.2.51)[301]: If $\{(a_m)_n\}_{m,n=1}^\infty \subset \mathcal{E}$ and $\|(a_m)_n\|_{\mathcal{E}} \to 0$, then $(a_m)_n \xrightarrow{t(M_m)} 0$.

Proof. Let us show firstly that every convergent to zero in the norm $\|\cdot\|_{\mathcal{E}}$ sequence of projections from $\mathcal{E}$ has a subsequence convergent to zero in the topology $t(M_m)$. Consider a sequence $\{p_n\}_{n=1}^\infty \in P(M_m) \cap \mathcal{E}$, such that $\|p_n\|_{\mathcal{E}} \to 0$. Choose a subsequence $\{p_{n_k}\}_{k=1}^\infty$so that $\left\|p_{n_k}\right\|_{\mathcal{E}} \le 2^{-k}$. By Corollary (6.2.50), for the sequence of projections $q_k = \bigvee_{l \le k+1} p_{n_l}$, we have $q_k \in \mathcal{E}$ and $\|q_k\|_{\mathcal{E}} \le 2^{-k}$. If $q = \bigwedge_{k \ge 1} q_k$, then $q \in \mathcal{E}$ and $\|q\|_{\mathcal{E}} \le \|q_k\|_{\mathcal{E}} \le 2^{-k}$ for all $k \in \mathbb{N}$, that implies $q = 0$. Consequently, $q_k \downarrow 0$, and therefore $\tau(q_k) \downarrow 0$.
Since $p_{n_{k+1}} \le q_k$ for all $k \in \mathbb{N}$ we have $\tau\,(p_{n_k}) \to 0$. By Proposition (6.2.27), we infer the convergence $p_{n_k} \xrightarrow{t(M_m)} 0$.
Now, let us show that every sequence $\{p_n\}_{n=1}^\infty \in P(M_m) \cap \mathcal{E}$ convergent to zero in the norm$\|\cdot\|_{\mathcal{E}}$ automatically converges to zero in the topology $t(M_m)$. Otherwise, there exists a subsequence $\{p_{n_k}\}_{k=1}^\infty$ and a $t(M_m)$-neighborhood U of zero, which does not contain $\{p_{n_k}\}_{k=1}^\infty$.

From the above, there exists a subsequence $\{p_{n_{k_s}}\}_{s=1}^{\infty}$ converging to zero in the topology $t(M_m)$, that contradicts to the relation $p_{n_{k_s}} \notin U$.

Now, let $\{(a_m)_n\}_{m,n=1}^{\infty} \subset \mathcal{E}$ and $\|(a_m)_n\|_{\mathcal{E}} \to 0$. For every $\lambda_m > 0$ the inequality $\lambda_m E_{\lambda_m}^{\perp}(|(a_m)_n|) \leq |(a_m)_n| E_{\lambda_m}^{\perp}(|(a_m)_n|) \leq |(a_m)_n|$ imply that

$$\left\|E_{\lambda_m}^{\perp}(|(a_m)_n|)\right\|_{\mathcal{E}} \overset{(47)}{\leq} \lambda_m^{-1} \||(a_m)_n|\|_{\mathcal{E}} \overset{(46)}{=} \lambda_m^{-1} \|(a_m)_n\|_{\mathcal{E}} \longrightarrow 0.$$

By the preceding argument, we have that $E_{\lambda_m}^{\perp}(|(a_m)_n|) \longrightarrow 0$. Finally, by Proposition (6.2.2)(ii), we obtain $(a_m)_n \xrightarrow{t\,(M_m)} 0$.

Corollary (6.2.52)[301]: If $\{(a_m)_n\}_{m,n=1}^{\infty} \subset LS(M_m)$ and $(a_m)_n \xrightarrow{t(M_m)} 0$, then there exists a sequence $\{(a_m)_{n_k}\}_{m,k=1}^{\infty}$ such that $(a_m)_{n_k} = (b_m)_k + (c_m)_k$, where $(b_m)_k \in M_m, (c_m)_k \in LS(M_m), k \in \mathbb{N}, \|(b_m)_k\|_{M_m} \to 0$ and $s(|(c_m)_k|) \xrightarrow{t(M_m)} 0$ for $s \to \infty$.

Proof. Since $(LS(M_m), t(M_m))$ is an F-space there exists a countable basis $\{U_k\}_{k=1}^{\infty}$ of neighbourhoods of zero of the topology $t(M_m)$.

By Proposition (6.2.2)(ii), we have $E_{\lambda_m}^{\perp}(|(a_m)_n|) \xrightarrow{t(M_m)} 0$ for every $\lambda_m > 0$. Therefore, there exists a sequence $((a_m)_{n_k})_{m,k \geq 1}$ such that $E_{1/k}^{\perp}(|(a_m)_{n_k}|) \in U_k$ for all $k \in \mathbb{N}$. Set $(b_m)_k = (a_m)_{n_k} E_{1/k}(|(a_m)_{n_k}|)$ and $(c_m)_k = (a_m)_{n_k} E_{1/k}^{\perp}(|(a_m)_{n_k}|)$. It is clear that $(b_m)_k \in M_m$ and $\|(b_m)_k\|_{M_m} \leq 1/k$. Since

$$|(c_m)_k| = ((c_m)_k^*(c_m)_k)^{1/k} = (E_{1/k}^{\perp}(|(a_m)_{n_k}|)\left|(a_m)_{n_k}\right|^2 E_{1/k}^{\perp}(\left|(a_m)_{n_k}\right|))^{1/2}$$
$$= E_{1/k}^{\perp}(|(a_m)_{n_k}|)\left|(a_m)_{n_k}\right| E_{1/k}^{\perp}(|(a_m)_{n_k}|) = \left|(a_m)_{n_k}\right| E_{1/k}^{\perp}(|(a_m)_{n_k}|),$$

it follows that

$$s(|(c_m)_k|) \leq E_{1/k}^{\perp}(|(a_m)_{n_k}|) \in U_k.$$

Since $\{U_k\}_{k=1}^{\infty}$ is a basis of neighbourhoods of zero of the topology $t(M_m)$, we have $E_{1/k}^{\perp}(|(a_m)_{n_k}|) \xrightarrow{t(M_m)} 0$, which, in its turn, guarantees the convergence $\tau(E_{1/k}^{\perp}(|(a_m)_{n_k}|)) \to 0$ Proposition (6.2.27). From the inequality $\tau(s(|(c_m)_k|)) \leq \tau\,(E_{1/k}^{\perp}(|(a_m)_{n_k}|))$ and Proposition (6.2.27), we obtain $s(|(c_m)_k|) \xrightarrow{t(M_m)} 0$.

Corollary (6.2.53)[301]: Let M_m be a von Neumann algebra with faithful normal finite trace τ and let $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ be a Banach M_m-bimodule. Every derivation $\delta: LS(M_m) \to LS(M_m)$ with $\delta(M_m) \subset \mathcal{E}$ is $t(M_m)$-continuous.

Proof. Since $(LS(M_m), t(M_m))$ is an F-space it is sufficient to show that the graph of the linear operator δ is closed.

Suppose that the graph of the operator δ is not closed. Then there exists a sequence $\{(a_m)_n\}_{m,n=1}^{\infty} \subset LS(M_m)$ and $0 \neq b_m \in LS(M_m)$ such that $(a_m)_n \xrightarrow{t(M_m)} 0$ and $\delta((a_m)_n) \xrightarrow{t(M_m)} b_m$.

By Corollary (6.2.52) and passing, if necessary, to a subsequence, we may assume that $(a_m)_n = (b_m)_n + (c_m)_n$, where $(b_m)_n \in M_m, (c_m)_n \in LS(M_m), n \in \mathbb{N}, \|(b_m)_n\|_{M_m} \to 0$ and $s(|(c_m)_n|) \xrightarrow{t(M_m)} 0$ for $n \to \infty$.

Since the restriction $\delta|_{M_m}$ of the derivation δ to the von Neumann algebra M_m is a derivation from M_m into the Banach M_m bimodule $\mathcal{E}$, by Ringrose Theorem [27], we have $\|\delta((b_m)_n)\|_{\mathcal{E}} \to 0$.

Corollary (6.2.51) now implies that $\delta((b_m)_n) \xrightarrow{t(M_m)} 0$.
From the equalities

$$\delta((c_m)_n) = \delta((c_m)_n s(|(c_m)_n|)) = \delta((c_m)_n) s(|(c_m)_n|) + (c_m)_n \delta(s(|(c_m)_n|))$$

we have that

$$s(\delta((c_m)_n)) \leq l(\delta((c_m)_n) s(|(c_m)_n|)) \vee r(\delta((c_m)_n) s(|(c_m)_n|)) \vee l((c_m)_n \delta(s(|(c_m)_n|))) \vee r((c_m)_n \delta(s(|(c_m)_n|))).$$

Since

$$l((c_m)_n) \sim r((c_m)_n) = s(|(c_m)_n|), l(\delta((c_m)_n) s(|(c_m)_n|)) \sim r(\delta((c_m)_n) s(|(c_m)_n|)) \leq s(|(c_m)_n|),$$
$$r((c_m)_n \delta(s(|(c_m)_n|))) \sim l((c_m)_n \delta(s(|(c_m)_n|))) \leq l((c_m)_n) \preccurlyeq s(|(c_m)_n|),$$

it follows that

$$\tau(s(\delta((c_m)_n))) \leq 4\tau(s(|(c_m)_n|)).$$

By Proposition (6.2.28), $\tau(s(|(c_m)_n|)) \to 0$, and therefore $\tau(s(\delta((c_m)_n))) \to 0$ and $\tau(s(|\delta((c_m)_n)|)) \to 0$, that implies the convergence $\tau(E^{\perp}_{\lambda_m}(|\delta((c_m)_n)|)) \to 0$ for every $\lambda_m > 0$. Hence by Propositions (6.2.2)(ii) and (6.2.27), we obtain $\delta((c_m)_n) \xrightarrow{t(M_m)} 0$.

Thus, $\delta((a_m)_n) = \delta((b_m)_n) + \delta((c_m)_n) \xrightarrow{t(M_m)} 0$. The latter convergence contradicts to the inequality $b_m \neq 0$. Consequently, δ is $t(M_m)$-continuous.

Corollary (6.2.54)[301]: Let M_m be an arbitrary von Neumann algebra and let $\mathcal{E}$ be a Banach M_m-bimodule of locally measurable operators. Then any derivation $\delta: M_m \to \mathcal{E}$ is inner. In addition, there exists $d_m \in \mathcal{E}$ such that $\delta(x^m) = [d_m, x^m]$ for all $x^m \in M_m$ and $\|d_m\|_{\mathcal{E}} \leq 2\|\delta\|_{M_m \to \mathcal{E}}$. If $\delta^* = \delta$ or $\delta^* = -\delta$ then d_m may be chosen so that $\|d_m\|_{\mathcal{E}} \leq \|\delta\|_{M_m \to \mathcal{E}}$.

Proof. By [270], there exists a derivation $\bar{\delta}: LS(M_m) \to LS(M_m)$ such that $\bar{\delta}(x^m) = \delta(x^m)$ for all $x^m \in M_m$.

Choose a central decomposition $\{z^m_\infty, z^m_i\}_{m,j \in J}$ of the unity **1** such that $M_m z^m_\infty$ is a properly infinite von Neumann algebra and on every von Neumann algebra $M_m z^m_j$ there exists a faithful normal finite trace. By [270], the derivation $\bar{\delta}^{(z^m_\infty)} := \bar{\delta}|_{LS(M_m z^m_\infty)}: LS(M_m z^m_\infty) \to LS(M_m z^m_\infty)$ is $t(M_m z^m_\infty)$-continuous. Corollary (6.2.53) implies that every derivation $\bar{\delta}^{(z^m_j)} := \bar{\delta}|_{LS(M_m z^m_j)}: LS(M_m z^m_j) \to LS(M_m z^m_j)$ is also $t(M_m z^m_j)$-continuous for all $j \in J$. Therefore, by [270], the derivation $\bar{\delta}$ is $t(M_m)$-continuous. By Corollary (6.2.45) the derivation δ is inner. Repeating the proof of Corollary (6.2.48), we obtain that there exists an element $d_m \in \mathcal{E}$ such that $\delta(x^m) = [d_m, x^m]$ for all $x^m \in M_m$.

Now, suppose that $\delta^* = \delta$. In this case, $[d_m + d_m^*, x^m] = [d_m, x^m] - [d_m, x^{m*}]^* = \delta(x^m) - \left(\delta(x^{m*})\right)^* = \delta(x^m) - \delta^*(x^m) = 0$ for any $x^m \in M_m$. Consequently, the operator $Re(d_m) = (d_m + d_m^*)/2$ commutes with every elements from M_m, hence, by Corollary (6.2.34), $Re(d_m)$ is a central element in the algebra $LS(M_m)$. Therefore, we may assume that $\delta(x^m) = [d_m, x^m], x^m \in M_m$, where $d_m = ia_m,\ a_m \in \mathcal{E}_{h_m}$.

By Theorem (6.2.24), there exist $c_m = c_m^*$ from the centre of the algebra $LS(M_m)$ and a family $\{u_\varepsilon\}_{\varepsilon > 0}$ of unitary operators from M_m such that

$$|[a_m, u_\varepsilon]| \leq (1 - \varepsilon)|a_m - c_m|.$$

For $b_m = ia_m - ic_m$ and $\varepsilon = \frac{1}{2}$, we have

$$|b_m| = |a_m - c_m| \le 2|[a_m, u_{1/2}]| = 2|[-id_m, u_{1/2}]| = 2|[d_m, u_{1/2}]| \in \mathcal{E}.$$
Consequently, $b_m \in \mathcal{E}$ (see (43)), moreover,
$$\delta(x^m) = [d_m, x^m] = [ia_m, x^m] = [b_m, x^m]$$
for all $x^m \in M_m$. Since
$$(1-\varepsilon)|b_m| = (1-\varepsilon)|a_m - c_m| \overset{(44)}{\le} |[a_m, u_\varepsilon]| = |[d_m, u_\varepsilon]| = |\delta(u_\varepsilon)|,$$
it follows that
$$(1-\varepsilon)\|b_m\|_{\mathcal{E}} \overset{(47)}{\le} \|\delta(u_\varepsilon)\|_{\mathcal{E}} \le \|\delta\|_{M_m\to\mathcal{E}}$$
for all $\varepsilon > 0$, that implies the inequality $\|b_m\|_{\mathcal{E}} \le \|\delta\|_{M_m\to\mathcal{E}}$.
If $\delta^* = -\delta$, then taking $Im(d_m)$ instead of $Re(d_m)$ and repeating the preceding argument, we obtain that $\delta(x^m) = [b_m, x^m]$, where $b_m \in \mathcal{E}$ and $\|b_m\|_{\mathcal{E}} \le \|\delta\|_{M_m\to\mathcal{E}}$.
Now, suppose that $\delta \ne \delta^*$ and $\delta \ne -\delta^*$. Equality (46) implies that
$$\|\delta^*\|_{M_m\to\mathcal{E}} = \sup\{\|\delta(x^{m*})^*\|_{\mathcal{E}} : \|x^m\|_{M_m} \le 1\}$$
$$= \sup\{\|\delta(x^m)\|_{\mathcal{E}} : \|x^m\|_{M_m} \le 1\} = \|\delta\|_{M_m\to\mathcal{E}}.$$
Consequently,
$$\|Re(\delta)\|_{M_m\to\mathcal{E}} = 2^{-1}\|\delta + \delta^*\|_{M_m\to\mathcal{E}} \le \|\delta\|_{M_m\to\mathcal{E}}.$$
Similarly, $\|Im(\delta)\|_{M_m\to\mathcal{E}} \le \|\delta\|_{M_m\to\mathcal{E}}$. Since $(Re(\delta))^* = Re(\delta), (Im(\delta))^* = Im(\delta)$, there exist $(d_m)_1, (d_m)_2 \in E$, such that $Re(\delta)(x^m) = [(d_m)_1, x^m], Im(\delta)(x^m) = [(d_m)_2, x^m]$ for all $x^m \in M_m$ and $\|(d_m)_i\|_{\mathcal{E}} \le \|\delta\|_{M_m\to\mathcal{E}}, i = 1, 2$. Taking $d_m = (d_m)_1 + i(d_m)_2$, we have that $d_m \in \mathcal{E}, \delta(x^m) = (Re(\delta) + i \cdot Im(\delta))(x^m) = [(d_m)_1, x^m] + i[(d_m)_2, x^m] = [d_m, x^m]$ for all $x^m \in M_m$, in addition $\|d_m\|_{\mathcal{E}} \le 2\|\delta\|_{M_m\to\mathcal{E}}$.

Section (6.3): Ideals of Semifinite von Neumann Algebras

For N be a von Neumann algebra and let J be an N-bimodule. A derivation $\delta: N \to J$ is a linear mapping satisfying $\delta(XY) = \delta(X)Y + X\delta(Y), X, Y \in N$. In particular, if $K \in J$, then $\delta_K(X) := KX - XK$ is a derivation. Such derivations implemented by elements in J are called inner. One of the classical problems in operator algebra theory is the question whether every derivation from N into J is automatically inner. The classical result in this direction is that in the special case, when the N-bimodule J coincides withalgebra N, every derivation $\delta: N \to N$ is necessarily inner [29], [30]. At the same time, when one considers more general N-bimodules, there are examples of non-inner derivations from N into some ideals of a von Neumann algebra M with $N \subset M$ (see [297]).

Johnson and Parrott[293]considered the special case, when the larger algebra M coincides with the algebra $B(H)$ of all bounded linear operators on a Hilbert space H and the bimodule J is the ideal $K(H)$of all compact operators on H. The authors proved that if N is an abelian von Neumman subalgebra of M, then every derivation δ from $N$to $K(H)$ is automatically inner. As an easy consequence, they were also able to treat the case when N has no certain type II_1factors as direct summands. The remaining case, when N is a von Neumann algebra of type $II_1$was later resolved by Popa in[296].

Motivated by [293], derivations relative to semifinite von Neumann algebras have been widely investigated (see [270],[3], [297], [294] and [285]). Accompanied with the rapid development of the theory of symmetrically normed spaces (see [111], [278], [17], [298]), interesting results concerning derivations with values in symmetrically (quasi-)normed spaces are also established in[282], [251], [257], [256], [255].
In [294], Kaftal and Weiss studied the derivation problem in the setting when M is an arbitrary semifinite von Neumann algebra with a semifinite faithful normal trace τ and $N$is a unital abelian or properly infinite von Neumann subalgebra of M. Under this hypothesis,

it is proved in [294] that every derivation $\delta: N \to \mathcal{L}_p(M,\tau) := \mathcal{L}_p(M,\tau) \cap M, 1 \le p < \infty$, is inner, where $L_p(M,\tau)$ is the noncommutative L_p-space relative to τ (see [142]). However, the question whether every derivation from an arbitrary von Neumann subalgebra $N$of M into $\mathcal{L}_p(M,\tau), 1 \le p < \infty$, is inner was left unresolved. Our main objective is to answer this question in the affirmative. Furthermore, rather than just studying derivations with values in $\mathcal{L}_p(M,\tau), 1 \le p < \infty$, we prove that every derivation from an arbitrary C^*-subalgebra $A$of M into ideal $E(M,\tau) \cap M$ is inner, whenever $E(M,\tau)$ is the symmetric operator space corresponding to a fully symmetric function space $E(0,\infty)$on $(0,\infty)$having Fatou property and order continuous norm, extending earlier results by Kaftal and Weiss [294]. In Theorem (6.3.8), we demonstrate the sharpness of our assumptions on $E(0,\infty)$.

We denote symmetric space (of possible unbounded operators) on $M$by $(E(M,\tau), \|\cdot\|_E)$, while the corresponding ideal in M by $\mathcal{E}(M,\tau) = E(M,\tau) \cap M$. The latter ideal is equipped with the norm $\|\cdot\|_E$, however no assumption on completeness of $\mathcal{E}(M,\tau)$ with respect to $\|\cdot\|_E$is imposed.

We recall main notions of the theory of noncommutative integration and introduce some properties of generalized singular values.

In what follows, H is a Hilbert space and $B(H)$ is the $*$-algebra of all bounded linear operators on H equipped with uniform norm $\|\cdot\|_\infty$, and $\mathbf{1}$is the identity operator on H. Let M be a von Neumann algebra on H. For details on von Neumann algebra theory, see [274], [23], [52] or [33]. General facts concerning measurable operators may be found in [52], [31](see also [299] and [289]).

A linear operator $X: \mathfrak{D}(X) \to H$, where the domain $\mathfrak{D}(X)$ of X is a linear subspace of H, is said to be affiliated with M if $YX \subseteq XY$for all $Y \in M'$, where M' is the commutant of M. A linear operator $X: \mathfrak{D}(X) \to H$ is termed measurable with respect to M if X is closed, densely defined, affiliated with M and there exists a sequence $\{P_n\}_{n=1}^{\infty}$in the logic of all projections of $M, P(M)$, such that $P_n \uparrow \mathbf{1}, P_n(H) \subseteq \mathfrak{D}(X)$and $\mathbf{1} - P_n$ is a finite projection (with respect to M) for all n. It should be noted that the condition $P_n(H) \subseteq \mathfrak{D}(X)$ implies that $XP_n \in M$. The collection of all measurable operators with respect to M is denoted by $S(M)$, which is a unital $*$-algebra with respect to strong sums and products (denoted simply by $X + Y$ and XY for all $X, Y \in S(M)$).

For X be a self-adjoint operator affiliated with M. We denote its spectral measure by $\{e^X\}$. It is well known that if X is a closed operator affiliated with M with the polar decomposition $X = U|X|$, then $U \in M$ and $E \in M$ for all projections $E \in \{e^{|X|}\}$. Moreover, $X \in S(M)$ if and only if X is closed, densely defined, affiliated with M and $e^{|X|}(\lambda,\infty)$ is a finite projection for some $\lambda > 0$. It follows immediately that in the case when M is a von Neumann algebra of type III or a type I factor, we have $S(M) = M$. For type II von Neumann algebras, this is no longer true.

We would not consider type III von Neumann algebra. From now on, let M be an arbitrary semifinite von Neumann algebra equipped with a faithful normal semifinite trace τ. An operator $X \in S(M)$is called τ-measurable if there exists a sequence $\{P_n\}_{n=1}^{\infty}$in $P(M)$ such that $P_n \uparrow \mathbf{1}, P_n(H) \subseteq \mathfrak{D}(X)$ and $\tau(\mathbf{1} - P_n) < \infty$ for all n. The collection $S(M,\tau)$ of all τ-measurable operators is a unital $*$-subalgebra of $S(M)$ denoted by $S(M,\tau)$. It is well known that a linear operator X belongs to $S(M,\tau)$ if and only if $X \in S(M)$ and there exists $\lambda > 0$ such that $\tau(e^{|X|}(\lambda,\infty)) < \infty$. Alternatively, an unbounded operator X affiliated with M is τ-measurable (see [239]) if and only if

$$\tau\left(e^{|X|}(n,\infty)\right) \to 0, \qquad n \to \infty.$$

We also recall the definition of the measure topology t_τ on the algebra $S(M,\tau)$. For every $\varepsilon,\delta > 0$, we define the set
$V(\varepsilon,\delta) = \{X \in S(M,\tau) : \exists P \in P(M)$ such that $\|X(\mathbf{1} - P)\| \le \varepsilon, \tau(P) \le \delta\}$.

The linear topology generated by the sets $V(\varepsilon,\delta), \varepsilon,\delta > 0$, is called the measure topology t_τ on $S(M,\tau)$ [289], [239], [52]. It is well known that the algebra $S(M,\tau)$ equipped with the measure topology is a complete metrizable topological algebra [52](see also [113]). A sequence $\{X_n\}_{n=1}^{\infty} \subset S(M,\tau)$ converges to zero with respect to measure topology t_τ if and only if $\tau\left(e^{|X_n|}(\varepsilon,\infty)\right) \to 0$as $n \to \infty$ for all $\varepsilon > 0$ [289].
Another important vector topology on $S(M,\tau)$ is the local measure topology. For convenience we denote by $P_f(M)$the collection of all τ-finite projections in M, that is the set of all $E \in P(M)$ satisfying $\tau(E) < \infty$. A neighborhoodbase for this topology is given by the sets $V(\varepsilon,\delta; E), \varepsilon,\delta > 0, E \in P_f(M)$, where

$$V(\varepsilon,\delta;E) = \{X \in S(M,\tau) : EXE \in V(\varepsilon,\delta)\}.$$

Obviously, local measure topology is weaker than measure topology[288]. We note here, that the local measure topology used differs from the local measure topology defined in e.g. [270], [266].
Definition (6.3.1)[280]: Let a semifinite von Neumann algebra M be equipped with a faithful normal semi-finite trace τ and let $X \in S(M,\tau)$. The generalized singular value function $\mu(X): t \to \mu(t; X)$ of the operator X is defined by setting

$$\mu(s;X) = \inf\{\|XP\| : P = P^* \in M \text{ is a projection}, \tau(\mathbf{1} - P) \le s\}.$$

An equivalent definition in terms of the distribution function of the operator X is the following. For every self-adjoint operator $X \in S(M,\tau)$, setting

$$d_X(t) = \tau\left(e^X(t,\infty)\right), \qquad t > 0,$$

we have (see e.g. [239])

$$\mu(t;X) = \inf\{s \ge 0 : d_{|X|}(s) \le t\}. \tag{83}$$

Consider the algebra $M = L^\infty(0,\infty)$ of all Lebesgue measurable essentially bounded functions on $(0,\infty)$. The algebra M can be seen as an abelian von Neumann algebra acting via multiplication on the Hilbert space $H = L^2(0,\infty)$, with the trace given by integration with respect to Lebesgue measure m. It is easy to see that the algebra of all τ-measurable operators affiliated with M can be identified with the subalgebra $S(0,\infty)$ of the algebra of Lebesgue measurable functions $L_0(0,\infty)$ which consists of all functions x such that $m(\{|x| > s\})$ is finite for some $s > 0$. It should also be pointed out that the generalized singular value function $\mu(x)$ is precisely the decreasing rearrangement $\mu(x)$ of the function x (see e.g. [263]) defined by

$$\mu(t; x) = \inf\{s \ge 0 : m(\{|x| \ge s\}) \le t\}.$$

If $M = B(H)$ (respectively, l_∞) and τ is the standard trace Tr (respectively, the counting measure on $\mathbb{N}$), then it is not difficult to see that $S(M) = S(M,\tau) = M$. In this case, for $X \in S(M,\tau)$ we have

$$\mu(n;X) = \mu(t;X), \qquad t \in [n, n+1), \qquad n \ge 0.$$

The sequence $\{\mu(n; X)\}_{n\ge 0}$ is just the sequence of singular values of the operator X.
Definition (6.3.2)[280]: A linear subspace E of $S(M,\tau)$ equipped with a complete norm $\|\cdot\|_E$, is called symmetric space (of τ-measurable operators) if $X \in S(M,\tau), Y \in E$ and $\mu(X) \le \mu(Y)$ imply that $X \in E$ and $\|X\|_E \le \|Y\|_E$.

It is well-known that any symmetric space E is a normed M-bimodule, that is $AXB \in E$ for any $X \in E, A, B \in M$ and $\|AXB\|_E \leq \|A\|_\infty \|B\|_\infty \|X\|_E$.
If $X, Y \in S(M,\tau)$, then $X$is said to be submajorized by Y, denoted by $X \prec\prec Y$, if

$$\int_0^t \mu(s;X)ds \leq \int_0^t \mu(s;\ Y)ds$$

for all $t \geq 0$. A linear subspace $E$of $S(M,\tau)$ equipped with a complete norm $\|\cdot\|_E$, is called fully symmetric space(of τ-measurable operators) if $X \in S(M,\tau), Y \in E$ and $X \prec\prec Y$ imply that $X \in E$ and $\|X\|_E \leq \|Y\|_E$.

A symmetric space $E \subset S(M,\tau)$ is said to have the Fatou property if for every upwards directed net $\{X_\beta\}$ in E^+, satisfying $sup_\beta \|X_\beta\|_E < \infty$, there exists an element $X \in E^+$ such that $X_\beta \uparrow X$ in E and $\|X\|_E = sup_\beta \|X_\beta\|_E$. Examples such as Schatten–von Neumann operator ideals, Lorentz operator ideals, Orlicz operator ideals, etc. all have symmetric norms which have Fatou property.

If $E \subset S(M,\tau)$ is a symmetric space, then the norm $\|\cdot\|_E$is called order continuous if $\|X_\alpha\|_E \to 0$ whenever $\{X_\alpha\}$ is downwards directed net in E^+ satisfying $X_\alpha \downarrow 0$.
The so-called Köthe dual is identified with an important part of the dual space. If $E \subset S(M,\tau)$ is a symmetric space, then the Köthe dual $E^\times$ of E is defined by setting

$$E^\times = \{X \in S(M,\tau): sup_{\|Y\|_E \leq 1, Y \in E}\ \tau(|XY\ |) < \infty\}.$$

A wide class of symmetric operator spaces associated with the von Neumman algebra M can be constructed from concrete symmetric function spaces studied extensively in e.g. [263]. Let $(E(0,\infty),\ \|\cdot\|_{E(0,\infty)})$ be a symmetric function space on the semi-axis $(0,\infty)$. Then the pair

$$E(M,\tau) = \{X \in S(M,\tau): \mu(X) \in E(0,\infty)\}, \|X\|_{E(M,\tau)} := \|\mu(X)\|_{E(0,\infty)}$$

is a symmetric space on M [111](see also [278]). For convenience, we denote $\|\cdot\|_{E(M,\tau)}$ by $\|\cdot\|_E$. Many properties of symmetric spaces, such as reflexivity, Fatou property, order continuity of the norm as well as Köthe duality carry over from the commutative symmetric function space $E(0,\infty)$ to its noncommutative counterpart $E(M,\tau)$ (see e.g. [288]).

We introduce two classical examples of symmetric spaces.

The symmetric space of τ-compact operators is the symmetric space associated to the algebra of functions from $S(0,\infty)$ vanishing at infinity (see [278]), that is,

$$S_0(M,\tau) = \{A \in S(M,\tau): \mu(\infty, A) = 0\}.$$

The ideal $S_0(M,\tau) := S_0(M,\tau) \cap M$ of all τ-compact bounded operatorscan be described as the closure in the norm $\|\cdot\|_\infty$of the linear span of all τ-finite projections.
The noncommutative L_p-space $L_p(M,\tau), p \geq 1$, is the symmetric space corresponding to the classical L_p-space of functions $L_p(0,\infty)$, that is

$$L_p(M,\tau) = \{X \in S(M,\tau) : \mu(X) \in L_p(0,\infty)\}.$$

This space can be also described as the space of all τ-measurable operators X such that $\tau(|X|^p) < \infty$. It is well-known [288] that for all $1 \leq p < \infty$, the symmetric space $L_p(M,\tau)$ is fully symmetric, has Fatou property and order continuous norm. In addition, for $1 < p < \infty$the space $L_p(M,\tau)$ is reflexive [142].

Let M be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace τ, and let A be a C^*-subalgebra of M.

Before we proceed to the study of derivations with values in symmetric ideals in M, we consider derivations with values in reflexive symmetric operator spaces (of possibly

unbounded operators) affiliated with M. These typesof questions were considered in e.g. [266], [292], [257], [1].
The approach is based on Ryll-Nardzewski's fixed point theorem, which was suggested in [293](see [291]due to Hoover).
The following result (see [292], [257], [1]for some similar results) enables us to extend [294] to the case when A is an arbitrary C^*-subalgebra. We note that, for every reflexive symmetric function space $E(0,\infty)$, the corresponding operator space $E(M,\tau)$ is also reflexive (see [17], see also [288]). We always use the term "weakly" to refer the weak topology of the Banach space $(E(M,\tau),\ \|\cdot\|_E)$.

Theorem (6.3.3)[280]: Let A be a unital C^*-subalgebra of M and let $E(M,\tau)$ be a reflexive symmetric space. Then, for every derivation $\delta: A \to E(M,\tau)$, there exists a $T \in E(M,\tau)$ such that $\delta = \delta_T$ and $\|T\|_E \le \|\delta\|_{A\to E}$. Moreover, $T \in \overline{co\{\delta(U)U^*: U \in \mathcal{U}(A)\}}^{\|\cdot\|_E}$, where $co(S)$ denotes the convex hull of a set S.

Proof. By Ringrose's theorem [27], the derivation $\delta: (A, \|\cdot\|_\infty) \to (E(M,\tau), \|\cdot\|_E)$ is bounded. Let us define the sets $K_{00} := \{\delta(U)U^*: U \in \mathcal{U}(A)\} \subset E(M,\tau)$ and $K_0 := co(K_{00})$. It is clear that K_{00}, and therefore K_0, lie in the ball of radius $\|\delta\|_{A\to E}$ in $E(M,\tau)$. We set $K := \overline{K_0}^{\|\cdot\|_E}$. Then, K is weakly closed in $E(M,\tau)$. Since $E(M,\tau)$ is reflexive, K is a convex weakly compact subset of $E(M,\tau)$, contained in the ball of radius $\|\delta\|_{A\to E}$.
For every $U \in \mathcal{U}(A)$, we have $\delta(U) \in E(M,\tau)$, and therefore we can define the mapping $\alpha_U: E(M,\tau) \longrightarrow E(M,\tau)$, by setting

$$\alpha_U(X) := UXU^* + \delta(U)U^*.$$

For every $U, V \in \mathcal{U}(A)$, we have

$$\begin{aligned}\alpha_U(\alpha_V(X)) &= UVXV^*U^* + U\delta(V)V^*U^* + \delta(U)U^* \\ &= (UV)X(UV)^* + U\delta(V)V^*U^* + \delta(U)VV^*U^* \\ &= (UV)X(UV)^* + \delta(UV)(UV)^* = \alpha_{UV}(X).\end{aligned}$$

In addition, the equality $\delta(\mathbf{1}) = \delta(\mathbf{1^2}) = 2\delta(\mathbf{1})$ implies that $\delta(\mathbf{1}) = 0$, and therefore $\alpha_1(X) = X, X \in E(M,\tau)$. Thus, α is an action of the group $\mathcal{U}(A)$ on $E(M,\tau)$.
We claim that the set K is invariant with respect to α. Since $\delta(U)U^* = \alpha_U(0)$, it follows that K_{00} is an orbit of 0 with respect to α, and therefore, is an invariant subset with respect to α. In addition, for any positive scalars s and t with $s + t = 1$, we have

$$\begin{aligned}\alpha_U(sX + tY) &= sUXU^* + tUYU^* + (s+t)\delta(U)U^* \\ & s\alpha_U(X) + t\alpha_U(Y), \qquad X, Y \in E(M,\tau).\end{aligned}$$

Hence, for every $U \in \mathcal{U}(A)$ the mapping $\alpha_U$is affine, which implies that $K_0 = co(K_{00})$ is also an invariant subset with respect to α. Now, the equality $\alpha_U(X) - \alpha_U(Y) = U(X - Y)U^*, X, Y \in E(M,\tau)$ implies that every $\alpha_U, U \in \mathcal{U}(A)$, is a distance preserving isometry on $E(M,\tau)$. Hence, K is an invariant subset with respect to α.

Furthermore, the fact that α_U is an isometry implies that the family $\{\alpha_U: U \in \mathcal{U}(A)\}$ is a noncontracting family of affine mappings. Clearly, $\alpha_U$is weakly continuous for every $U \in \mathcal{U}(A)$. Thus, the set K and the family $\{\alpha_U: U \in \mathcal{U}(A)\}$ satisfy the assumptions of Ryll-Nardzewski's fixed point theorem [284]. Hence, there exists a point $T \in K$ fixed with respect to α, that is, we have $T = \alpha_U(T) = UTU^* + \delta(U)U^*$ for every $U \in \mathcal{U}(A)$ and $T \in K$ (and therefore $\|T\|_E \le \|\delta\|_{A\to E}$). Hence, $TU = UT + \delta(U)$ for every $U \in \mathcal{U}(A)$. Thus, $\delta(U) = [T, U]$ for every $U \in \mathcal{U}(A)$. Since every element $X \in A$ is a linear combination of four elements from $\mathcal{U}(A)$, we obtain that $\delta = \delta_T$on A.

Since $L_p(M,\tau)$ is reflexive when $p > 1$, the above theorem together with [281] implies the following result, which is also an easy corollary of [281].

Corollary (6.3.4)[280]: Let Abe a unital C^*-subalgebra of M and $\delta: A \to L_p(M,\tau), p \geq 1$, be a derivation. Then, there exists an element $T \in L_p(M,\tau)$ such that $\delta = \delta_T$ and $\|T\|_{L_p} \leq \|\delta\|_{A\to L_p}$.

Proposition (6.3.5)[280]: Let Abe a C^*-subalgebra of M and let $E(M,\tau)$ be a reflexive symmetric space affiliated with M. Then, for every derivation $\delta: A \to E(M,\tau)$, there exists an element $T \in \mathcal{E}(M,\tau)$ such that $\delta = \delta_T$with $\|T\|_E \leq \|\delta\|_{A\to E}$ and $\|T\|_E \leq \|\delta\|_{A\to M}$.

Proof. We firstly consider the case when A is unital.

Since $E(M,\tau)$ is reflexive, Theorem (6.3.3) implies that there exists a $T \in E(M,\tau)$ such that $\delta = \delta_T$ and $\|T\|_E \leq \|\delta\|_{A\to E}$. Therefore, it remains to show that $T \in M$ and $\|T\|_\infty \leq \|\delta\|_{A\to M}$.

By Ringrose's theorem [27], we have that $\delta: (A, \|\cdot\|_\infty) \to (M, \|\cdot\|_\infty)$ is a bounded mapping. Hence, $K_0 := co\{\delta(U)U^*: U \in \mathcal{U}(A)\}$ lies in the ball of radius $\|\delta\|_{A\to M}$ in M. By Theorem (6.3.3), we have $T \in \overline{K_0}^{\|\cdot\|_E}$. Let $\{X_0\} \subset K_0$ be such that $\|T - X_n\|_E \longrightarrow 0$. By [288], we have $X_n \longrightarrow T$in local measure topology. Since M has Fatou property (see [288]), it follows from [287] that the closed ball in $(M,\ \|\cdot\|_M)$with radius $\|\delta\|_{A\to M}$ is closed with respect to the local measure topology. Noticing that $\|X_n\|_M \leq \|\delta\|_{A\to M}$ and $X_n \longrightarrow T$in local measure topology, we conclude that $T \in M$with $\|T\|_\infty \leq \delta_{A\to M}$.

Next, if A does not contain the identity of M, we let $A_1 := A + \mathbb{C}\mathbf{1}$ and define $\delta_1: A_1 \to \mathcal{E}(M,\tau)$ by $\delta_1(a + \alpha\mathbf{1}) = \delta(a), a \in A$ and $\alpha \in \mathbb{C}$. It is clear that $\delta_1$is a derivation from a unital C^*-subalgebra A_1into $\mathcal{E}(M,\tau)$. Hence, the assertion follows from the one proved above.

In the main result, Theorem(6.3.6) below, we extend the result of Proposition(6.3.5) to a wider class of symmetric ideals in M. To this end we recall the notation of p-convexification of a symmetric space of τ-measurable operators.

Let $E(0,\infty)$ be a symmetric function space on $(0,\infty)$ and let $(E(M,\tau),\ \|\cdot\|_E)$ be the corresponding noncommutative operator space. Following the notation introduced in [300], for $1 < p < \infty$, we set

$$E(M,\tau)^{(p)} = \{X \in S(M,\tau) : |X|^p \in E\}, \quad \|X\|_{E^{(p)}} = \||X|^p\|_E^{1/p}.$$

It is well-known (see e.g. [286]) that $E^{(p)}(M,\tau) = E(M,\tau)^{(p)}$, where $E^{(p)}(M,\tau)$ is the symmetric space corresponding to the p-convexification $E^{(p)}(0,\infty)$ of the symmetric function space $E(0,\infty)$.

The next theorem is the main result, which substantially extends [294]. The prototype of the proof of the following theorem for the case of Schatten ideals $L_p(H)$ when H is separable can be found in[291]. This theorem generalizes [294] in two directions. Firstly, instead of imposing additional condition on the von Neumann subalgebra A, we can consider the case, when A is an arbitrary C^*-algebra. Secondly, we have extended significantly the class of symmetric ideals associated with M for which the result is applicable.

Theorem (6.3.6)[280]: Let Abe a C^*-subalgebra of M and let E be a fully symmetric function space on $(0,\infty)$ having Fatou property and order continuous norm. Then every derivation $\delta: A \to \mathcal{E}(M,\tau)$ is inner, that is there exists an element $T \in \mathcal{E}(M,\tau)$ such that $\delta = \delta_T$ with $\|T\|_\infty \leq \|\delta\|_{A\to M}$ and $\|T\|_E \leq \|\delta\|_{A\to M}$.

Proof. Without loss of generality, we may assume that $\|\delta\|_{A\to M} \leq 1$.

Since $\mathcal{E}(M,\tau) \subset M$, it follows that $|X|^q$ is a bounded operator for every $X \in \mathcal{E}(M,\tau)$ and $q \geq 0$. Therefore, for $p \geq p' \geq 1$and every $X \in E^{(p')}(M,\tau)$ we have that

$$|X|^p = |X|^{p'} \cdot |X|^{p-p'} \in \mathcal{E}(M,\tau),$$

that is $X \in \mathcal{E}^{(p)}(M,\tau)$. Thus,

$$\mathcal{E}^{(p')}(M,\tau) \subset \mathcal{E}^{(p)}(M,\tau), \quad p \geq p' \geq 1. \tag{84}$$

In particular, from inclusion (84) we have that $\mathcal{E}(M,\tau) \subset \mathcal{E}^{(p)}(M,\tau)$ for every $p > 1$. Hence the derivation δ can be considered as a derivation defined on A with values in the symmetric ideal $\mathcal{E}^{(p)}(M,\tau)$. By [159], every $\mathcal{E}^{(p)}(M,\tau)$is reflexive and therefore, it follows from Proposition(6.3.5) that there exists a $T_p \in \mathcal{E}^{(p)}(M,\tau)$ such that $\delta = \delta_{T_p}$on $A$with $\|T_p\|_\infty \leq \|\delta\|_{A\to M}$ and $\|T_p\|_{E^{(p)}} \leq \|\delta\|_{A\to E^{(p)}}$.

We note that for $p, p' > 1$ with $p \geq p'$, inclusions $T_{p'} \in \mathcal{E}^{(p')}(M,\tau)$ and (84) imply that $T_{p'} \in \mathcal{E}^{(p)}(M,\tau)$. Moreover, since $\|T_{p'}\|_\infty \leq \|\delta\|_{A\to M} \leq 1$, we have that

$$\begin{aligned}\|T_{p'}\|_{E^{(p)}}^{p} &= \||T_{p'}|^p\|_E \leq \||T_{p'}|^{p'}\|_E \cdot \||T_{p'}|^{p-p'}\|_\infty \\ &\leq \|T_{p'}\|_{E^{(p')}}^{p'} \leq \|\delta\|_{A\to E^{(p')}}^{p'} = sup_{\,X\in A,\|X\|_\infty=1} \||\delta(X)|^{p'}\|_E \\ &\leq sup_{X\in A,\|X\|_\infty=1}\|\delta(X)\|_E \cdot \|\delta(X)\|_\infty^{p'-1} \\ &\leq sup_{X\in A,\|X\|_\infty=1}\|\delta(X)\|_E \cdot \|\delta\|_{A\to M}^{p'-1} \leq \|\delta\|_{A\to E}.\end{aligned} \tag{85}$$

We define $M:=\left\{T_{1+\frac{1}{m}}\right\}_{m\in\mathbb{N}}$. Since $1+\frac{1}{m} \leq 2, m \in \mathbb{N}$, inclusion (84)implies that $M \subset \mathcal{E}^{(2)}(M,\tau)$. Now, let us construct inductively a subsequence $\{T_{n,m}\}_m$of M for every $n \geq 1$ such that

(i) for every fixed $n \geq 1$, $T_{n,m} \subset E^{(1+\frac{1}{n})}(M,\tau), m \in \mathbb{N}$ and $T_{n,m} \to S_n \in E^{\left(1+\frac{1}{n}\right)}(M,\tau)$ as $m \to \infty$ in the weak topology of $E^{(1+\frac{1}{n})}(M,\tau)$ with $\|S_n\|_{E^{(1+\frac{1}{n})}}^{1+\frac{1}{n}} \leq \|\delta\|_{A\to E}$.

(ii) $\{T_{n+1,m}\}_m \subset \{T_{n,m}\}_m$for every $n \geq 1$.

Let $M_{1,0} := M \subset E^{(2)}(M,\tau)$ and $M_1 := \overline{co(M_{1,0})}^{\|\cdot\|_{E^{(2)}}}$. Since M_1 is a convex bounded norm-closed subset of the reflexive space $E^{(2)}(M,\tau)$, it follows that $M_1$is a convex weakly compact subset of $E^{(2)}(M,\tau)$. Hence, by the Eberlein–Smulian Theorem [290], there is a subsequence $\{T_{1,m}\}$ of $M_{1,0}$ converging to an element $S_1 \in M_1 \subset E^{(2)}(M,\tau)$ in the weak topology of $E^{(2)}(M,\tau)$. Since $S_1 \in M_1$ and $M_1$is the $\|\cdot\|_{E^{(2)}}$-norm closure of $co(M_{1,0})$, inequality (85)implies that $\|S_1\|_{E^{(2)}}^2 \leq \|\delta\|_{A\to E}$.

Assume that the construction up to $n-1, n \geq 2$, is completed. We let $M_{n,0} = \{T_{n-1,m}\}_m \cap \{T_{1+\frac{1}{m}} : m \geq n, m \in \mathbb{N}\} \subset E^{\left(1+\frac{1}{n}\right)}(M,\tau)$. Note, that this intersection is non-empty (and infinite) as the elements of $\{T_{n-1,m}\}_m$ are chosen from the sequence $\left\{T_{1+\frac{1}{m}}\right\}_{m\in\mathbb{N}}$. We set $M_n := \overline{co(M_{n,0})}^{\|\cdot\|_{E^{(1+\frac{1}{n})}}}$. Then, M_n is a convex weakly compact subset of $E^{(1+\frac{1}{n})}(M,\tau)$. Then, by the Eberlein–Smulian Theorem, there is a subsequence $\{T_{n,m}\}_m$ of $M_{n,0}$ converging to an element $S_n \in M_n \subset E^{(1+\frac{1}{n})}(M,\tau)$ in the weak topology of $E^{(1+\frac{1}{n})}(M,\tau)$, in particular, $\|S_n\|_{E^{(1+\frac{1}{n})}}^{1+\frac{1}{n}} \leq \|\delta\|_{A\to E}$, which completes the induction.

Now, we show that every $S_n$belongs to M. For every $n \geq 1$, there is a sequence $\{X_{n,m}\} \subset co(M_{n,0})$ such that $\|X_{n,m} - S_n\|_{E^{(1+\frac{1}{n})}} \to 0$ as $m \to \infty$. Hence, by [288], we have

$X_{n,m} \to S_n$ as $m \to \infty$ in local measure topology. It follows from [287] that the closed ball of radius $\|\delta\|_{A\to M}$ of $(M, \|\cdot\|_\infty)$ is closed with respect to the local measure topology. Since $co(M_{n,0})$ lies in the closed ball of radius $\|\delta\|_{A\to M}$of $(M, \|\cdot\|_\infty)$, it follows that $S_n \in M$ and $\|S_n\|_\infty \le \|\delta\|_{A\to M}$.

We claim that all of the S_n are the same. Since S_n and $S_{n+1}$are τ-compact operators, the operator $S_n - S_{n+1}$ is also τ-compact. Let $S_n - S_{n+1} = U|S_n - S_{n+1}|$ be the polar decomposition. Then, for any $\varepsilon > 0$, $e^{|S_n - S_{n+1}|}(\varepsilon, \infty)$ is a τ-finite projection. Hence, by [288] we have that $e^{|S_n - S_{n+1}|}(\varepsilon, \infty) \in E^{(1+\frac{1}{n})}(M, \tau)^\times$ for every $n \in \mathbb{N}$. Since the Köthe dual space $E^{(1+\frac{1}{n})}(M, \tau)^\times$ can be identified with a subspace of the Banach dual, conditions (i) and (ii) imply that

$$\tau\left(S_n e^{|S_n - S_{n+1}|}(\varepsilon, \infty)U^*\right) \overset{(i)}{=} \lim_{m\to\infty} \tau\left(T_{n,m} e^{|S_n - S_{n+1}|}(\varepsilon, \infty)U^*\right)$$
$$\overset{(ii)}{=} \lim_{m\to\infty} \tau\left(T_{n+1,m} e^{|S_n - S_{n+1}|}(\varepsilon, \infty)U^*\right)$$
$$\overset{(i)}{=} \tau\left(S_{n+1} e^{|S_n - S_{n+1}|}(\varepsilon, \infty)U^*\right)$$

and therefore

$$\tau\left(|S_n - S_{n+1}| e^{|S_n - S_{n+1}|}(\varepsilon, \infty)\right) = \tau\left(U^*(S_n - S_{n+1}) e^{|S_n - S_{n+1}|}(\varepsilon, \infty)\right) = 0$$

for any $\varepsilon > 0$, which implies that $S_n = S_{n+1}$. In what follows we denote S_n by T. In particular, we have $\|T\|_\infty \le \|\delta\|_{A\to M}$ and $\|T\|^p_{E^{(p)}} \le \|\delta\|_{A\to E}$ for every $p \in (1, 2]$.

Next, we claim that $\delta = \delta_T$. Consider $\mathcal{E}(M, \tau)$ as a subspace of $E^{(2)}(M, \tau)$. For every $X \in A$, $\delta_{T_p}(X) = \delta(X)$ for every $p > 1$. By condition (i) above, we have $T_{1,m} \to T$ in the weak topology of $E^{(2)}(M, \tau)$. Thus, for every $f \in (E^{(2)}(M, \tau))^*$ and $X \in A$, we have $f(T_{1,m}X) \to f(TX)$ and $f(XT_{1,m}) \to f(XT)$ as $m \to \infty$, which implies that $f(\delta_{T_{1,m}}(X)) \to f(\delta_T(X))$ as $m \to \infty$. That is, $\delta_{T_{1,m}}(X) \to \delta_T(X)$ in the weak topology of $E^{(2)}(M, \tau)$ as $m \to \infty$. On the other hand, every $\delta_{T_{1,m}}(X), m \in \mathbb{N}$, is equal to $\delta(X)$, and therefore, we conclude that $\delta(X) = \delta_{T_{1,m}}(X) = \delta_T(X)$ for every m, and therefore $\delta = \delta_T$ on A.

By the construction of T, we have that $T \in \cap_{p>1} E^{(p)}(M, \tau)$ and $\||T|^p\|_E = \|T\|^p_{E^{(p)}} \le \|\delta\|_{A\to E}$ for every $p \in (1, 2]$.Since $\|T\|_\infty \le \|\delta\|_{A\to M} \le 1$, we have $|T|^p \uparrow |T|$as $p \downarrow 1$. Since $(E(M, \tau), \|\cdot\|_E)$ has Fatou property, we have $T \in E(M, \tau)$ with $\|T\|_E \le \|\delta\|_{A\to E}$, which completes our proof.

It is well-known, that the space $L_p(0, \infty)$is fully symmetric and has Fatou property and order continuous norm. Therefore, as an immediate corollary of Theorem (6.3.6), we obtain the following result extending the earlier results by Kaftal and Weiss [294], which were proved there under restrictive conditions that A is either an abelian or properly infinite von Neumann subalgebra, to the case when A is an arbitrary C^*-subalgebra of M.

Corollary (6.3.7)[280]: Let Abe a C^*-subalgebra of M and let $\delta: A \to \mathcal{L}_p(M, \tau),\ p \ge 1$, be a derivation. Then, there exists an element $T \in \mathcal{L}_p(M, \tau)$ such that $\delta = \delta_T$and $\|T\|_p \le \|\delta\|_{A\to\mathcal{L}_p}$.

Theorem (6.3.8)[280]: Let M be a semifinite non-finite factor. If $E_0(M, \tau) \ne E(M, \tau)$ and $E(M, \tau) \cap M \ne M$, then we can always find a non-inner derivation from a C^*-subalgebra of M into $E_0(M, \tau) \cap M$.

Proof. Since $E(M, \tau) \cap M \ne M$, every operator in $E(M, \tau) \cap M$ is τ-compact. Since $E_0(M, \tau) \ne E(M, \tau)$, there exists $T \in E(M, \tau) \setminus E_0(M, \tau)$ with $T \in M$. We claim that δ_T is a non-inner derivation from some C^*-subalgebra of M into $E_0(M,\tau)\cap M$. Consider δ_T acting on $S_0(M,\tau)$. For every $X\in S_0(M,\tau), e^{|X|}(\varepsilon,\infty)$ is τ-finite for every $\varepsilon>0$. Thus,

$$\left\|TX - TXe^{|X|}(\varepsilon,\infty)\right\|_E \le \left\|TXe^{|X|}(0,\varepsilon]\right\|_E \le \varepsilon\|T\|_E$$

implies that $TX\in E_0(M,\tau)$. Similarly, $XT\in E_0(M,\tau)$ and therefore $\delta_T(S_0(M,\tau))\subset E_0(M,\tau)$. Moreover, $T\in M$ and $S_0(M,\tau)\subset M$ imply that $\delta_T(S_0(M,\tau))\subset E_0(M,\tau)\cap M$. Finally, if there exists an operator $K\in E_0(M,\tau)\cap M$ such that $\delta_T=\delta_K$, then $T-K\in S_0(M,\tau)'$. For every $B\in S_0(M,\tau)$ and $A\in S_0(M,\tau)'$, we have $BA = BA$. Then, noticing that M is the weak operator closure of $S_0(M,\tau)$ (see e.g. [278]), we have $BA = AB$ for every $B\in M$ and $A\in S_0(M,\tau)'$and therefore $S_0(M,\tau)'\subset M'$. Since $M'\subset S_0(M,\tau)'$, we have $S_0(M,\tau)'=M'$. However, M is a factor and therefore $T-K\in\mathbb{C}\mathbf{1}$. Noticing that $T$and $K$are τ-compact, we have $T=K$, which is a contradiction.

List of Symbols

Symbol		Page
L^p:	Lebesgue space	1
L_∞:	essential Lebesgue space	1
cm:	Convergence in measure	2
inf:	infimum	2
H_α:	Hilbert space	5
sup:	supremum	5
$\oplus$:	orthogonal sum	5
$\otimes$:	tensor product	5
max:	maximum	10
$c.b.$:	completely bounded	11
L^1:	Lebesgue space on the real line	13
ℓ^p:	The space of sequences	15
UC:	un conditionally convergent	15
WUC:	Weakly un conditionally convergent	15
ind:	index	20
dim:	dimension	20
DFG:	German Research Foundation	22
Hol:	Holonmy	23
L^2:	Hilbert space	25
cl:	Clifford bunde	25
tr:	trace	27
mod:	modulo	34
Fin:	finite	45
co:	convex	48
Der:	derivations	51
In:	Inner	64
Re:	Real	64
Im:	Imaginary	72
Ext_c:	Extension	72
S^p:	Suatter class	103
L^q:	Dual of Lebesgue space	103
Hom:	Homomorphism	104
dist:	distante	104
ker:	kernel	106
WOT:	Weak operator Topology	107
ℓ^∞:	The essential Banach space of sequences	109
min:	minimum	109
L^0:	The space of all measurable functions	118
Dom:	domain	132
ℓ^2:	Hilbert space of sequences	151

$\odot$:	Span of tensor	152
Aut:	Automorphism	157
$contr$:	Contractive	158
sa:	Self-adjoint	166
$Haus$:	Hausderff	167
$diam$:	diameter	167
Lip:	Lipchitz	171
can:	canonical	181
spr:	Spectral radius	185
Res:	Residue	186
Ran:	Range	187
$Proj$:	projection	201
$card$:	cardinality	208
ess:	essential	213
l_w^p:	Lorertz sequence space	223

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