Sudan University of Science and Technology

College of Graduate Studies

Classification of Lie Algebras and its Applications in Physics

تصنيف جبر لي و تطبيقاته في الفيزياء

Thesis submitted in fulfillment for the requirement of the Ph. D degree in Mathematics

By

Anwar Adam Ahmed Mohammed

Supervisor:

Prof. Mohammed Ali Basher

2021

Acknowledgments

I would like to express my gratitude to my supervisor Professor Mohammed Ali Basher for his valuable help and fruthful suggestions. Thanks are extended to Sudan University of Science and Technology, graduate College, College of science, the Department of mathematics. Iam also deeply indepted to my family

Dedication

To all those who devote their work to Allah, to my parents, my brothers, sisters, my wife, my children, my all family, friends, colleagues, my students, To all researchers and scientists, to all those who helped me and all those who asked Allah for me.

Abstract

In this work we studied Lie groups and Lie algebras. We considered several examples of Lie groups and their associated Lie algebras. We treated the representation theory of Lie groups and used this theory in the problem of classification of some Lie algebras such as semi –simple and solvable Lie algebras. Other Lie algebras have also been characterized; we also applied Lie algebras to problems in physics

المستخلص

في هذا البحث درسنا زمر لي و جبر لي تناولنا عدة امثله لزمر لي جبر لي المتعلقة بتلك لزمر عالجنا نظرية التمثيل لزمر لي واستخدمنا هذه النظريه في مسالة تصنيف بعض جبر لي كجبر شبه بسيط وكذلك قابليةجبر الحل كما تم تصنيف جبر لي اخر.قمناكذلك بتطبيق جبر لي علي بعض المسائل في الفيزياء

Contents

Dedication--I

Acknowledgments---II

Abstract--III

Abstract (Arabic) --IV

Contents---V

Introduction---VIII

Chapter one

Lie Groups

(1.1) Definition---1
(1.2) Examples of Lie Groups--1
(1.3) Lie Groups, Subgroups, and Cosets--1
(1.4) Action of Lie Groups on Manifolds and Representations---------------7
(1.5) Orbits and Homogeneous Spaces --8
(1.6) Left, Right, and Adjoin Action--11
(1.7) Classical Groups---12

Chapter two

Lie Algebras

(2.1) Definition---20

(2.2) Some Motivation for Lie Algebras---------------------------------------22

(2.3) Some Low-dimensional Lie Algebras-------------------------------------24

(2.4) The Lie Algebra of Vector Fields on Manifolds---------------------------24

(2.5) The Lie Algebra of Matrix Groups---29

(2.6) The Free Lie Algebra---30

(2.7) Linear Lie Algebra---33

(2.8) Representation of Lie Algebra--34
(2.9) Lie Algebra of GL (n, R) --35
(2.10) Definition---36
(2.11) the Exponential Map---37
(2.12) Examples of Lie Algebras--38
(2.13) Theorem--40

Chapter Three

Classification of Lie Algebras

(3.1) Definition--41
(3.2) The Systemic Classification of Semi-simple Lie Algebras---------------45
(3.3) Carton Sub Algebra--45
(3.4) Nilpotent Lie Algebra---51
(3.5) Solvable Lie Algebra--53
(3.6) Semi-simple Lie Algebra--58

Chapter four

Symmetries and Lie Algebra

(4.1) Introduction---60
(4.2) Importance of Symmetries--61
(4.3) Local one Parameter Point Transformations---------------------------------62
(4.4) Local one Parameter Point Transformation Groups-----------------------63
(4.5) Generate Point Symmetries---65
(4.6) Lie Group of the Heat Equation--71

(4.7) Theorem--75

(4.8) Two Dimensional Heat Equation--77

(4.9) Determining Equation of Two Dimensional Heat Equation-------------79

(4.10) Invariant Solution --83

Chapter five

Applications of Lie Algebra

(5.1) Introduction--88

(5.2) Representation Theory--89

(5.3) Lie Group---92

(5.4) Lie Algebra---100

(5.5) Physical Application---107

(5.6) Solvable Lie Algebra Application--111

Introduction

The problem before us has origin in the theory of finite continuous groups of transformations. The usual method of classification depends on the existence or non-existence of invariant subgroups of different types so that we know (thanks to the work of killing, Cartan and Weyl) all about simple and semi simple groups on the one hand and integrable or solvable groups on the other. If the group does not belong to either of these classes we know very little about the different types except in the case of groups of comparatively small order. In this research we propose to give an outline of a method of classification which does not depend so much on the order of the group as it does on its genus. Unfortunately the method does not apply or only partially applied in good many cases however it may be possible to further refine it so as to exclude at least some of the exceptional cases. Another remark the first part of what follows applied to finite linear algebras in general and we have already applied it some simple cases of linear associative algebras Lie groups and their Lie algebras are essential tools in the study of several mathematical fields. These include partial differential equations.

Homogenous spaces, symmetric spaces and differential geometry in general we first give a brief introduction to differentiable manifolds and then define Lie groups that allow us to study Lie algebra. A Lie group is a space **e**ndowed with two structures

Chapter One

Lie Groups

1.1. Definition

A Lie group G is a differential manifold such that the composition of elements of G As group and the inverse operation are differential this means that: For any $gin\ G\ g^{-1}$ is differentiable operation. For g_1 and $g_2\ in\ G$ the composition $g_1 \text{o} g_2$ is differentiable operation.

1.2. Examples of Lie Groups

The basic example of Lie group is of course the general linear group $GL_n(R)$ of invertible $n \times n$ real matrices, this is an open subset of the vector space of all $n \times n$ matrices, and gets its manifold structure accordingly (so that the entries of the matrix are coordinates on $GL_nR)$. That the multiplication map $GL_nR \times GL_nR \to GL_nR$ is differentiable is clear, that the inverse map $GL_nR \to GL_nR$ is follows from Crammers' formula for the inverse. Occasionally GL_nR will come to us as the group of automorphism of an n-dimensional real vector space, when we want to think of GL_nR in this way (e. g without choosing a basis for V and there by identifying G with the group of matrices).

We will write it as $GL(V)\ or\ Aut(V)$. Representation of a Lie group, of course, is amorphism from $G\ to\ GL(V)$

1.3. Lie Groups, Subgroups, and Coset

1.3.1. Definition

A Lie group is a set G with two structures: G is a group and G is a (smooth, real) manifold. These structures agree in the following sense: multiplication and inversion are smooth maps.

A morphism of Lie groups is a smooth map which also preserves the group operation:

$$f(gh) = f(g)f(h), f(1) = 1$$

In a similar way, one defines complex Lie groups. However, unless specified otherwise, 'Lie group' means a real group.

1.3.2 .Remark

The word smooth in the definition above can be understood in different ways:

C^1, C^∞ Analytic it turns out that all of them are equivalent: every C^0 Lie group has a unique analytic structure. This a highly non-trivial result and we are not going to prove it

1.3.3. Example

The following are examples of Lie groups

(1) $\mathbb{R}^n$, with the group operation given by addition
(2) $\mathbb{R}^*, \times\ \mathbb{R}_+, \times$
(3) $S^1 = \{z \in \mathbb{C}: |z| = 1\}, \times$
(4) $GL(n, \mathbb{R}) \subset \mathbb{R}^{n^2}$. Many of the groups we will consider will be subgroups of $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$
(5) $SU(2) = \{A \in GL(2, \mathbb{C}) | A\ \bar{A}^t = 1, detA = 1\}$. Indeed, one can easily see that

$$SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C}\ |\alpha|^2 + |\beta|^2 = 1 \right\}$$

Writing$\alpha = x_1 + ix_2\beta = x_3 + ix_4x_i \in \mathbb{R}$ we see that $SU(2)$ is diffeomorphism to $S^3 = \{x^2{}_1 + \cdots + x^2{}_4\} \subset \mathbb{R}^4$
(6) In fact, all usual groups of linear algebra, such as $GL(n, \mathbb{R}), SL(n, \mathbb{R}), O(n, \mathbb{R}), U(n), SO(n, \mathbb{R}), SU(n), Sp(2n, \mathbb{R})$ are Lie groups. This will be proved later

Note that the definition of a Lie group does not require that G be connected. Thus, any finite group is a 0-dimensional Lie group. Since the theory of finite groups is complicated enough, it makes sense to separate the finite (or, more generally, discrete) part. It can be done as follows

1.3.4. Theorem

Let G be a Lie group. Denote by G^0 the connected component of unity. Then G^0 is a normal subgroup of G and is a Lie group it self the quotient group $G \backslash G^0$ is discrete.

Proof

We need to show that G^0 is closed under the operations of multiplication and inversion since the image of a connected topological space under a Continuous map is connected the inversion map i must take G^0 to one component of G, that which contains $i(1) = 1, namely\ G^0$. In a similar way one shows that G^0 is closed under multiplication.

To check that this is a normal subgroup, we must show that if $g \in G\ and\ h \in G^0$, then $ghg^{-1} \in G^0$. Conjugation by g is continuous and thus will take G^0 to some connected component of G, since it fixes 1, this component is G^0

The fact that the quotient is discrete is obvious

This theorem mostly reduces the study of arbitrary Lie groups to study of finite groups and connected Lie groups. In fact, one can go further and reduce the study of connected Lie groups to connected simply –connected Lie groups

1.3.5. Theorem

If G is a connected Lie group then its universal cover $\tilde{G}$ has a canonical Structure of a Lie group such that the covering map $P: \tilde{G} \to G$ is a morphism of Lie groups, and $\ker p = \pi_1(G)$ as a group. Moreover, in this case $kerp$ is a discrete central subgroup in $\tilde{G}$

Proof

The proof follows from the following general result of topology: if M, N are connected manifolds (or, more generally, nice enough topological spaces), then any continuous map $f: M \to N$ can be lifted to a map $\tilde{f}: \widetilde{M} \to \widetilde{N}$. Moreover, if we choose $m \in M, n \in N$ such that $f(m) = n$ and choose lifting $\widetilde{m} \in \widetilde{M}, \tilde{n} \in \widetilde{N}$ such that $p(\widetilde{m}) = m, p(\tilde{n}) = n,$ then there is a unique lifting $\tilde{f}\ of\ f$ such that $\tilde{f}(\widetilde{m}) = \tilde{n}$

Now Let us choose some element $\tilde{1} \in \tilde{G}$ such that $p(\tilde{1}) = 1 \in G$.Then, by The above theorem, there is a unique map $\tilde{\imath}: \tilde{G} \to \tilde{G}$ which lifts the inversion map $i: G \to G$ and satisfies $\tilde{\imath}(\tilde{1}) = \tilde{1}$. In a similar way one constructs the multiplication map $\tilde{G} \times \tilde{G} \to \tilde{G}$.

1.3.6. Definition

A Lie subgroup H of a Lie group G is a subgroup which is also a sub manifold

1.3.7. Remark

In this definition, the word sub manifold should be understood as imbedded sub manifold. In particular, this means that H is locally closed but not necessarily closed, as we will show below, it will automatically be closed.

1.3.8. Theorem

(1) Any Lie subgroup is closed in G

(2) Any closed subgroup of a Lie group is a Lie subgroup

Proof

The proof of the first part is given in definition 1.1. The second part is much harder and will not be proved here. The proof uses the technique of Lie algebras and can be found

1.3.9. Corollary

(1) If G is a connected Lie group and U is a neighborhood of 1, then U generates$G$.

(2) Let $f: G_1 \to G_2$ be morphism of Lie groups with G_2 connected and $f_*: T_1G_1 \to T_1G_2$ is surjective. Then f is surjective

Proof

(1) Let H be the subgroup generated by$U$. Then H is open in$G$: for any element $h \in H, the\ set\ h.U$ is a neighborhood of $h\ in\ G$. since it is an open subset of a manifold, it is a sub manifold, so H is a Lie subgroup. Therefore, by theorem 1.1.8 it is closed , and is nonempty, so $H = G$

(2) Given the assumption, the inverse function theorem says that f is surjective onto some neighborhood$U$ of $1 \in G_2$.Since an image of group morphism is a subgroup, and U generates G_2, f is surjective.

As in the theory of discrete groups, given a subgroup $H \subset G,$ we can define the notion of cosets and define the coset space G / H as the set of equivalence classes. The following theorem shows that the coset space is actually a manifold.

(1) Let G be a Lie group of dimension $n\ and\ H \subset G$ a Lie subgroup of dimension k. then the coset space G / H has a natural structure of a manifold of dimension $n - k$ such that the canonical map $p: G \to G / H$ is a fiber

bundle, with fiber diffeomorphism to$H$. The tangent space at $\bar{1} = p(1)$ is given by$T_{\bar{1}}(G/H) = T_1G\ /\ T_1H$.

(2) If H is a normal Lie subgroup then G/H has a canonical structure of a Lie group

Proof

Denote by $p: G \to G\ /\ H$ the canonical map. Let $g \in G$ and $\bar{g} = p(g) \in G/H$. Then the set $g.H$ is a sub manifold in Gas it is an image of H under diffeomorphism$x \to gx$. Choose a sub manifold $M \subset G$ such that $g \in M\ and\ M$ is transversal to the manifold $gH, i.e. T_gG = T_g(gH))\oplus T_gM$

(This implies that dim$M = dimG - dimH$). Let $U \subset M$ be a sufficiently small neighborhood of g in$M$. Then the set $UH = \{uh/u \in U, h \in H\}$is open in G(which easily follows from inverse function theorem applied to the map$U \times H \to G$).Consider $\overline{U} = p(U)$; since $p^{-1}(\overline{U}) = UH$ is open, $\overline{U}$ is an open neighborhood of $\bar{g}$ in GH and the map $U \to \overline{U}$ is homeomorphism. This gives a local chart for $G\ /\ H$ and at the same time shows that $G \to G\ /\ H$ is a fiber bundle with fiber H. we leave it to the reader to show that transition functions between such charts are smooth and that the smooth structure does not depend on the choice of g, M.

This argument also shows that the kernel of the projection $p_*: T_gG \to T_{\bar{g}}(G\ /\ H)$ is equal to$T_g(gH)$. In particular, for $g = 1$ this gives an isomorphism $T_{\bar{1}}(G/H) = T_1G\ /\ T_1H$

1.3.10. Corollary

(1) If H is connected, then the set of connected components $\pi_0(G) = \pi_0(G/H)$. In particular, if $H, G\ /\ H$ are connected, then so is G

(2) If G, H are connected, then there is an exact sequence of groups

$$\pi_2(G/H) \to \pi_1(H) \to \pi_1(G) \to \pi_1(G/H) \to \{1\}$$

This corollary follows from more general long exact sequence of homotopy groups associated with any fiber bundle. We will later use it to compute fundamental groups of classical groups such as $GL(n)$.

Finally, there is an analog of the standard homomorphism theorem for Lie groups

1.3.11. Theorem

Let $f: G_1 \to G_2$ be a morphism of Lie groups. Then $H = kerf$ is normal Lie subgroup in G_1 and f gives rise to an injective morphism$G_1 / H \to G_2$, which is an immersion of manifolds. If Imf is closed, then it is a Lie subgroup in$G_2 and\ f$gives an isomorphism of Lie groups$G_1 / H \simeq Imf$. The proof of this theorem will be given later (see Corollary 3.27)

Corollary 3.27

.Let $f: G_1 \to G_2$ be a morphism of Lie groups, and

$f_*: g_1 \to g_2$ the corresponding morphism of Lie algebras Then $kerf$ is a Lie subgroup with Lie algebra ker f_*, and the map $G_1/kerf \to G_2\ is$ immersion if and only if f is closed, then we have an isomorphism

$$f \simeq G_1 / kerf$$

Proof . Consider the action of G_1 on G_2 given by $\rho(g).h = f(g)h, g \in G_1, h \in G_2$. Then the stabilizer of $1 \in G_2$ is exactly $kerf$, so by the previous theorem, it is a Lie group with Lie algebra $kerf_*$

Note that it shows in particular that an image of f is a subgroup in G_2which is an immersed sub manifold, however, it may not be a Lie subgroup as the example below shows. Such more general kinds of subgroups are called immersed subgroups

1.3.12. Example

Let$G_1 = \mathbb{R}\, , G = T^2 = \mathbb{R}^2 / \mathbb{Z}^2$. Define the map $f: G_1 \to G_2\ by\ f(t) = (t\ mod\ \mathbb{Z}, \alpha t\ \ mod\ \mathbb{Z}\ )$, where α is some fixed irrational number. Then it is well-known that the image of this map is everywhere dense in T^2(it is sometimes called irrational winding on the torus).

1.4. Action of Lie Groups on Manifolds and Representations

The primary reason why Lie groups are so frequently used is that they usually appear as groups of symmetry of various geometric objects. We will show several examples.

1.4.1. Definition

An action of a Lie group G and a manifold M is an assignment to each $g \in G$ a diffeomorphism $\rho(g) \in Diff M$ such that $\rho(1) = id, \rho(gh) = \rho(g)\rho(h)$ and such that the map $G \times M \to M: (g,m) \to \rho(g).m$ Is a smooth map

1.4.2. Example

(1) The group$GL(n, \mathbb{R})$ (and thus, any it's Lie subgroup) acts on $\mathbb{R}^n$
(2) The group $O(n, \mathbb{R})$ acts on the sphere$S^{n-1} \subset \mathbb{R}^n$. The group $U(n)$ acts on the sphere$S^{2n-1} \subset \mathbb{C}^n$.

Closely related with the notion of group acting on a manifold is the notion of representation

1.4.3. Definition

A representation of a Lie group G is a vector space V together with group morphism $\rho: G \to End(V). If\ V$ is finite-dimensional, we also require that the map $G \times V \to V: (g, v) \to \rho(g).v$ be a smooth map, so that ρ is a morphism of Lie groups.

Morphism between two representations V, W is a linear map
$f: V \to W$ Which commutes with the action of?

$$G: f\rho v(g) = \rho w(g) f.$$

In other words, we assign to every $g \in G$ a linear map
$\rho(g): V \to V\ so\ that\ \rho(g)\rho(h) = \rho(gh)$ We will frequently use the shorter notation $g.m, g.v$ instead of $\rho(g).m$ in the cases when there is no ambiguity about the representation being used

1.4.4. Remark

Note that we frequently consider representation on a complex vector space V even for a real Lie group$G$.

Any action of the group G on manifold M gives rise to several representations of G on various vector spaces associated with M:

(1) Representation of G on the (infinite-dimensional) space of function $C^\infty(M)$ defined by

$$(\rho(g)f)(m) = f(g^{-1}.m)$$

$$(note\ that\ we\ need\ g^{-1} rather\ than\ g\ to\ satisfy\ \rho(g)\rho(h) = \rho(gh))$$

(2) Representation of G on the (infinite-dimensional) space of vector fields $Vect(M)$ defined by

$$(\rho(g).v)(m) = g_*\big(v(g^{-1}.m)\big).$$

In a similar way, we define the action of G on the spaces of differential forms and other types of tensor fields on M

(3) Assume that $m \in M$ is a stationary point: $g.m = m$ for any $g \in G$. Then we have a canonical action of G on the tangent space $T_m M$ given by $\rho(g) = g_*: T_m M \to T_m M,$ and similarly for the spaces $T^*{}_m M, \Lambda^k T^*{}_m M.$

1.5. Orbits and Homogeneous Spaces

Let G act on a manifold M. Then for every point $m \in M$ we define its orbit by $O_m = G_m = \{g.m \ \setminus g \in G\}$.

1.5.1. Lemma

Let M be a manifold with an action of G. Choose a point $m \in M$ and Let $H = stab_G(m) = \{g \in G \setminus g.m = m\}$. Then H is a Lie subgroup in , and $g \to g.m$ is an injective immersion $G/H \to M$ whose image coincides with the orbit O_m.

Proof

The fact that the orbit is in bijection with G/H is obvious. For the proof of the fact that H is a closed subgroup, we could just refer to theorem 1.8. However, this would not help proving that $G/Stab(m) \to M$ is an immersion both of these statements are easiest proved using the technique of Lie algebras.

1.5.2. Corollary

The orbit O_m is an immersed sub manifold in M with tangent space $T_m O_m = T_1 G / T_1 H$. If O_m is closed, then $g \to g.m$ is a diffeomorphism $G/Stab(m) \to O_m$

An important special case is when the action of G is transitive, i.e. when there is only one orbit.

1.5.3. Definition

A G-homogeneous space is a manifold with a transitive action of G As an immediate corollary of corollary 1.3.2, we see that each homogeneous space is diffeomorphism to a coset space G/H. Combining it with theorem 1.10 we get the following results

1.5.4. Corollary

Let M be a G-homogeneous space and choose $m \in M$. Then the map $G \to M: g \to gm$ is a fiber bundle over M with fiber $H = Stab_G m$.

1.5.5. Example

(1) Consider the action of $SO(n, \mathbb{R})$ on the sphere $S^{n-1} \subset \mathbb{R}^n$. Then it is a homogeneous space, so we have a fiber bundle

$$SO(n-1, \mathbb{R}) \to SO(n, \mathbb{R}) \to S^{n-1}$$

(2) Consider the action of $SU(n)$ on the sphere $S^{2n-1} \subset \mathbb{C}^n$. Then it is a homogeneous space, so we have a fiber bundle

$$SU(n-1) \to SU(n) \to S^{2n-1}$$

In fact, action of G can be used to define smooth structure on a set. Indeed, if M is a set (no smooth structure yet) with a transitive action of a Lie group G, then M is in bijection with G/H, $H = Stab_G(m)$ and thus, by theorem 1.10. M has a canonical structure of a manifold of dimension equal to $dimG - dimH$

1.5.6. Example

Define a flag in $\mathbb{R}^n$ to be a sequence of subspaces

$$\{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{R}^n \ , dimV_i = i$$

Let $B_n(\mathbb{R})$ be the set of all flags in $\mathbb{R}^n$. It turns out that $B_n(\mathbb{R})$ has a canonical structure of a smooth manifold which is called the flag manifold (or

sometimes flag variety). The easiest way to define it is to note that we have an obvious action of the group$GL(n,\mathbb{R})$ on $B_n(\mathbb{R})$. This action is transitive:
By a change of basis, any flag can be identified with the standard flag

$$V^{st} = (\{0\} \subset e_1 \subset (e_1, e_2) \subset \cdots \subset (e_1, \dots, e_{n-1}) \subset \mathbb{R}^n)$$

Where $(e_1, \dots, e_k)$ stands for the subspace spanned by $e_1, \dots, e_k$ Thus $B_n(\mathbb{R})$ can be identified with the coset $\frac{GL(n,\mathbb{R})}{B(n,\mathbb{R})}$, $whereB(n,\mathbb{R}) = StabV^{st}$ is the group of all invertible upper-triangular matrices. Therefore, B_n is a manifold of dimension equal to $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$
Finally, we should say a few words about taking the quotient by the action of a group. In many cases when we have an action of a group G on a manifold M one would like to consider the quotient space, i.e. the set of all G-orbits. This set is commonly denoted by M/G .It has a canonical quotient topology. However, this space can be very singular, even if G is a Lie group, for example, it can be non-Hausdroff. For example, if $G = GL(n,\mathbb{C})$ acting on the set of all $n \times n$ matrices by conjugation, then the set of orbits described by Jordan canonical form however, it is well-known that by a small perturbation, any matrix can be made diagonalizable. Thus, if X, Y are matrices with the same eigenvalues but different Jordan form, then any neighborhood of the orbit of X contains points from orbit of Y
There are several ways of dealing with this problem. One of them is to impose additional requirements on the action, for example assuming that the action is proper. In this case it can be shown thatM/G is indeed a Hausdroff topological space, and under some additional conditions, it is actually a manifold.

1.6. Left, Right, and Adjoin Action

Important examples of group action are the following actions of G on itself:
Left action: $L_g: G \to G$ is defined by $L_g(h) = gh$
Right action: $R_g: G \to G$ is defined by $R_g(h) = hg^{-1}$
Adjoin action:$Ad_g: G \to G$ is defined by $Ad_g(h) = ghg^{-1}$
One easily sees that left and right actions are transitive; in fact, each of them is simply transitive.
It is also easy to see that the left and right actions commute and that$Ad_g = L_g R_g$.

As mentioned above, each of these actions also defines the action of G on the spaces of functions, vector fields, forms, etc. on G. For simplicity, for a tangent vector $v \in T_m G$, we with frequently write just $gv \in T_{gm} G$ instead of technically more accurate but cumbersome notation $(Lg)_* v$. Similarly, we will write vg for $(R_{g^{-1}})_* v$.

Since the ad joint action preserves the identity element $1 \in G$, it also defines an action of G on the (finite-dimensional) space $T_1 G$. Slightly abusing the notation, we will denote this action also by

$Ad_g \colon T_1 G \to T_1 G$.

1.6.1. Definition

A vector field $v \in Vect(G)$ is left-invariant if $g.v = v$ for every $g \in G$, and right-invariant if $v.g = v$ for every $g \in G$. A vector field is called bi-invariant if it is both left and right-invariant.

In a similar way one defines left, right, and bi-invariant differential forms and other tensors.

1.6.2. Theorem

The map $v \longmapsto v(1)$ (where 1 is the identity element of the group) defines an isomorphism of the vector space of left-invariant vector fields on G with the vector space $T_1 G$, and similarly for right-invariant vector spaces.

Proof

It suffices to prove that every $x \in T_1 G$ can be uniquely extended to a left-invariant vector field on G. Let us define the extension by $v(g) = gx \in T_g G$. Then one easily sees that so defined vector field is left-invariant, and $v(1) = x$. This proves existence of extension, uniqueness is obvious.

Describing bi-invariant vector fields on G is more complicated: any $x \in T_1 G$ can be uniquely extended to a left-invariant vector field and to a right-invariant vector field, but these extensions may differ

1.6.3. Theorem

The map $v \longmapsto v(1)$ defines an isomorphism of the vector space of bi-invariant vector fields on G with the vector space of invariants of adjoint action:

$$(T_1 G)^{AdG} = \{x \in T_1 G \setminus Adg(x) = x \; for \; all \; g \in G\}$$

The proof of this result is left to the reader. Note also that a similar result holds for other types of tensor fields: convector fields, differential forms, etc.

1.7. Classical groups

In this section, we discuss the so-called classical groups, or various subgroups of the general linear group which are frequently used in linear algebra. Traditionally, the name "classical group" is applied to the following groups:

- $GL(n, \mathbb{K})$ (Here and below, $\mathbb{K}$ is either$\mathbb{R}$, which gives a real Lie group, or$\mathbb{C}$, which gives a complex Lie group)
- $SL(n, \mathbb{K})$
- $O(n, \mathbb{K})$
- $SO(n, \mathbb{K})$ and more general groups $SO(p, q; \mathbb{R})$
- $U(n)$
- $SU(n)$
- $Sp(2n, \mathbb{K}) = \{A: \mathbb{K}^{2n} \to \mathbb{K}^{2n} \parallel \omega(Ax, Ay) = \omega(x, y)\}$. Here $\omega(x, y)$ is the skew-symmetric bilinear form $\sum_{i=1}^{n} x_i y_{i+n} - y_i x_{i+n}$ (which, up to a change of basis, is the unique non-degenerate skew-symmetric bilinear form on $\mathbb{K}^{2n}$). Equivalently, one can write $\omega(x, y) = (Jx, y), where$ $(,)$ is the standard symmetric bilinear form on $\mathbb{K}^n$ and
- $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$

1.7.1. Remark

There is some ambiguity with the notation for symplectic group: the group we denoted $Sp(2n, \mathbb{K})$. would be written as$Sp(n, \mathbb{K})$. Also, it should be noted that there is a closely related compact group of quaternionic unitary transformations

This group, which is usually denoted simply$Sp(n)$, is a "compact form" of the group $Sp(2n, \mathbb{C})$ in the sense we will describe

To avoid confusion, we have not included this group in the list of classical groups.

We have already shown that $GL(n)$ and $SU(2)$ are Lie groups. we will show that each of these groups is a Lie group and will find their dimensions.

Straightforward approach, based on implicit function theorem, is hopeless: for example, $SO(n, \mathbb{K})$ is defined by n^2equations in$\mathbb{K}^{n^2}$, and Finding the rank of this system is not an easy task.

We could just refer to the theorem about closed subgroups; this would prove that each of them is a Lie group, but would give us no other information-not even the dimension of G. Thus, we will need another approach

Our approach is based on the use of exponential map. Recall that for matrices, the exponential map is defined by

$$\exp(x) = \sum_{0}^{\infty} \frac{x^k}{k!}$$

It is well-known that this power series converges and defines an analytic map $gl(n, \mathbb{K}) \to gl(n, \mathbb{K})$, where $gl(n)$ is the set of all $n \times n$ matrices. In a similar way, we define the logarithmic map by

$\log(1 + x) = \sum_{1}^{\infty} \frac{(-1)^{k+1} x^k}{k}$.

So defined log is an analytic map defined in a neighborhood of $1 \in gl(n, \mathbb{K})$. The following theorem summarizes properties of exponential and logarithmic maps. Most of the properties are the same as for numbers; however, there are also some differences due to the fact that multiplication of matrices is not commutative. All of the statements of this theorem apply equally well in real and complex cases

1.7.2. Theorem

2. $\log(\exp(x)) = x; \exp(\log(X)) = X$ Whenever they are defined.
3. $\exp(x) = 1 + x + \cdots$ This means $\exp(0) = 1$ *and* $d \exp(0) = id$.
4. $If xy = yx \ then \exp(x + y) = \exp(x) \exp(y) . If XY = YX \ then \log(XY) = \log(X) + \log(Y)$ In some neighborhood of the identity. In particular, for any $x \in gl(n, \mathbb{K}), \exp(x) \exp(-x) = 1, so \ expx \in GL(n, \mathbb{K})$.
5. For $fixed \ x \in gl(n, \mathbb{K})$, consider the map $\mathbb{K} \to GL(n, \mathbb{K}): t \to \exp(tx)$. $then \exp\big((t + s)x\big) = \exp(tx)\exp(sx)$. In other words, this map is a morphism of Lie groups.

6. The exponential map agrees with change of basis and transposition $\exp(AxA^{-1}) = Aexp(x)A^{-1}, \exp(x^t)\,(\exp(x))^t$

Full proof of this theorem will not be given here; instead, we just give a sketch. First two statements are just equalities of formal power series in one variable; thus, it suffices to check that they hold for $x \in \mathbb{R}$. similarly, the third one is an identity of formal power series in two commuting variables, so it again follows from well-known equality for $x, y \in \mathbb{R}$. The fourth follows from the third, and the fifth follows from $(AxA^{-1})^n = Ax^nA^{-1}$ and $(A^t)^n = (A^n)^t$

Note that group morphism $\mathbb{R} \to G$ is frequently called one-parameter subgroups in G. This is not a quite accurate name, as the image may not be a Lie subgroup

However, the name is so widely used that it is too late to change it. Thus, we can reformulate part (4) of the theorem by saying that exp (tx) is one-parameter subgroup in $GL(n, \mathbb{K})$.

How does it help us to study various matrix groups? The key idea is that the logarithmic map identifies some neighborhood of the identity in $GL(n, \mathbb{K})$ with some neighborhood of 0 in a vector space. It turns out that it also does the same for all of the classical groups

1.7.3. Theorem

For each classical group $G \subset GL(n, \mathbb{K})$, there exists a vector space $g \subset gl(n, \mathbb{K})$ such that for some neighborhood $U\ of 1\ in GL(n, \mathbb{K})$ and some neighborhood $u\ of\ 0\ in\ gl(n, \mathbb{K})$ the following maps are mutually inverse

$$(U \cap G) \stackrel{log}{\longrightarrow} (u \cap g) \stackrel{exp}{\longrightarrow} (U \cap G)$$

Before proving this theorem, note that it immediately implies the following important corollary.

1.7.4. Corollary

Each classical group is a Lie group, with tangent space at identity $T_1G = g$ and $dimG = dimg$

Let us prove this corollary first because it is very easy. Indeed, Theorem1.5.3 shows that near $1, G$is identified with an open set in a vector space. So it is immediate that near $1, G$ is smooth. If $g \in G then g. U$ is a neighborhood of $g in GL(n, \mathbb{K}), and (g. U) \cap G = g.(U \cap G)$ is a neighborhood of $g in G$; thus, G is smooth near g.

For the second part, consider the differential of the exponential map$exp_*: T_0 g \to T_1 G$. Since g is a vector space, $T_0 g = g$, and since $\exp(x) = 1 + x + \cdots$, the derivative is the identity; thus, $T_0 g = g = T_1 G$

Proof of theorem 1.7.3

The proof is case by case; it cannot be any other way, as "classical groups" are defined by a list rather than by some general definition

$GL(n, \mathbb{K})$: Immediate from theorem 1.5.2 in this case, $g = gl(n, \mathbb{K})$ is the space of all matrices.$SL(n, \mathbb{K}): Suppose X \in SL(n, \mathbb{K})$ Is close enough to identity then$X = \exp(x)$ for some$x \in gl(n, \mathbb{K})$ The condition that $X \in SL(n, \mathbb{K})$ is equivalent to $det X = 1, or det exp(x) = 1$. But it is well-known that $det exp(x) = \exp(tr(x))$ (which is easy to see by finding a basis in which x is upper-triangular), so $\exp(x) \in SL(n, \mathbb{K})$ if and only if $tr(x) = 0$. Thus, in this case the statement also holds, with $g = \{x \in gl(n, \mathbb{K}) \parallel tr x = 0\}$.

$O(n, \mathbb{K}), SO(n, \mathbb{K})$: The group O_n is defined by$XX^t = I$. Then X, X^t commute. Writing $X = \exp(x), X^t = \exp(x^t)$ (since exponential map a grees with transposition), we see that x, x^t also commute, and thus $\exp(x) \in O(n)$ implies $\exp(x)\exp(x^t) = \exp(x + x^t) = 1$, so $x + x^t = 0; conversely, if x + x^t = 0, then x, x^t$ commute, so we can reverse the argument to get$\exp(x) \in O(n, \mathbb{K})$. Thus, in this case the theorem also holds, with $g = \{x \parallel x + x^t = 0\}$ the space of skew-symmetric matrix

What about SO $(n, \mathbb{K})$ In this case, we should add to the condition $XX^t = 1$ (which gives$x + x^t = 0$) also the condition$det X = 1$, which gives $tr(x) = 0$. However, this last condition is unnecessary, because$x + x^t = 0$ Implies that all diagonal entries of x are zero.

So both O $(n, \mathbb{K})$ and SO $(n, \mathbb{K})$ correspond to the same space of matrices$g = \{x \parallel x + x^t = 0\}$. This might seem confusing until one realizes that SO$(n,\mathbb{K})$ is exactly the connected component of identity in O$(n,\mathbb{K})$; thus, neighborhood of 1 in O$(n,\mathbb{K})$ coincides with the neighborhood of 1in SO$(n,\mathbb{K})$.

$U(n), SU(n)$: Similar argument shows that $expx \in U(n) \Leftrightarrow x + x^* = 0$ (where$x^* = x^{-t}$) and $expx \in SU(n) \Leftrightarrow x + x^* = 0, tr(x) = 0$.Note that in this case, $x + x^*$ does not imply that x has zeroes on the diagonal: it only implies that the diagonal entries are purely imaginary. Thus, $trx = 0$ does not follow automatically from$x + x^* = 0$, so in this case the tangent spaces for $U(n), SU(n)$ Are different.

$SP(2n,\mathbb{K})$: Similar argument shows that $\exp(x) \in SP(n,\mathbb{K}) \Leftrightarrow x + Jx^tJ^{-1} = 0$; thus, in this case the theorem also holds.

The vector space $g = T_1G$ is called the Lie algebra of the corresponding group G (this will be justified later, when we actually define an algebra operation on it). Traditionally the Lie algebra is denoted by lower case gothic letters: for example, the Lie algebra of group $SU(n)$ is denoted by $su(n)$.

The following table summarizes results of the theorem 1.

In addition, it also contains information about topological structure of classical Lie groups. Proofs of them can be found in exercises

G	$GL(n,\mathbb{R}$	$SL(n,\mathbb{R}$	O $(n,\mathbb{R})$	SO $(n,\mathbb{R})$	$U((n$	$SU(n)$	SP $(2n,\mathbb{R})$
g	$gl(n,\mathbb{R}$	$trx = 0$	$x + x^t = 0$	$x + x^t = 0$	$x + x^* = 0$	$x + x^* = 0, tr = 0$	$x + Jx^tJ^-$
$dimG$	n^2	$n^2 - 1$	$\frac{n(n-}{2}$	$\frac{n(n-}{2}$	n^2	$n^2 - 1$	$n(2n + 1)$
$\pi_0(G$	$\mathbb{Z}_2$	$\{1\}$	$\mathbb{Z}_2$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
$\pi_1(G$	$\mathbb{Z}_2(n \geq 3)$	$\mathbb{Z}_2(n \geq 3)$	$\mathbb{Z}_2(n \geq 3)$	$\mathbb{Z}_2(n \geq 3)$	$\mathbb{Z}$	$\{1\}$	$\mathbb{Z}$

For complex classical groups, the Lie algebra and dimension are given by the same formula as for real groups. However, the topology of complex Lie groups is different and is given in the table below.

G	$GL(n,\mathbb{C})$	$SL(n,\mathbb{C})$	$\mathrm{O}(n,\mathbb{C})$	$\mathrm{SO}(n,\mathbb{C})$
$\pi_0(G)$	$\{1\}$	$\{1\}$	$\mathbb{Z}_2$	$\{1\}$
$\pi_1(G)$	$\mathbb{Z}$	$\{1\}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Note that some of the classical groups are not simply-connected. As was shown in theorem 1.1.5, in this case the universal cover has a canonical structure of Lie group. Of special importance is the universal cover of $\mathrm{SO}(n,\mathbb{R})$ which is called the spin group and is denoted $\mathrm{Spin}(n)$; since $\pi_1(\mathrm{SO}(n,\mathbb{R}))= \mathbb{Z}_2$, this is a twofold cover

Chapter Two

Lie Algebras

2.1. Definition

A Lie algebra g is a vector space over a field F on which a product[,], called the bracket, is defined with the properties

(i) $X, Y \in g$ imply $[X, Y] \in g$

(ii) $[X, \alpha Y + \beta Z] = \alpha [X, Y] + \beta [X, Z]$

For $\alpha, \beta \in F \; and \; X, Y, Z, \in g$

(iii) $[X, Y] = -[Y, X]$

(iv) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

Properties (iii) and (iv) are known as skew –symmetry and the Jacobi identity, respectively .A Lie algebra is equivalently defined as the tangent space at the identify of a Lie group, which is much easier to visualize than.

2.1.2. Definition

The Lie bracket may be defined in various ways; as long as it fulfills the conditions of definition (2.1).A linear Lie algebra has matrix elements, where the Lie bracket is defined as the commulator.

$[X, Y] = XY - YX$. Linear Lie algebras are frequently used in quantum mechanics. Another common example of Lie algebra is the vector space R^3 with the Lie bracket defined as the cross product

$$[X, Y] = X \times Y$$

The basis of a Lie algebra from the infinitesimal generators of its associated Lie group, which are extremely useful in physical applications of finding differential symmetries. Let's go ahead and find the infinitesimal generators of a Lie group, we begin by finding the tangent vectors at any element in the Lie group.

For the Lie group action

$(x\ ,y) \quad \rightarrow \quad (\bar{x},\bar{y})$ The tangent vector to $(\bar{x}\ ,\bar{y})\ is\ (\varphi\ (x,y)\ ,\tau(x\ ,y))$ where

$(\frac{d\bar{x}}{d\varphi}) = \varphi(\bar{x},\bar{y})\ \ ,\ (\frac{d\bar{y}}{d\varphi}) = \tau(\bar{x},\bar{y})$

The tangent vector at a point (x,y)

$$\big(\varphi(x,\tilde{y}),\tau(x\ ,y)\big) = (\frac{d\bar{x}}{d\varphi},\frac{d\bar{y}}{d\varphi}\ \ \varphi = 0)$$

So, if we Taylor expand, our Lie group action becomes, to first order in Ɛ

$$\bar{x} = x + \varphi\varepsilon(x,y) + O\ (\varepsilon^2)$$

$$\bar{y} = y + \tau\varepsilon(x,y) + O\ (\varepsilon^2)$$

And our infinitesimal generators X is

$X\ = \varphi(x,y)\ d\ x\ +\ \tau(x,y)dy$ .For this to make more sense, let's do a simple example inspired

2.1.3. Theorem

For every Lie algebra $(V,[,])$ there exists a Lie group G with g isomorphic to subalgebra of $g^1(n,R)$ for some n

Combining this proposition (1.4.2) (b) we get

2.1.4. Example

Let V be a Fs-vector space let g^1(v) be the vector space of linear maps→ V define [- , -] on g^1(v) by

$$[x,y]\ =\ x\ o\ y - y\ o\ x$$

Where o is composition of maps? This Lie algebra is known as general linear algebra sometime it is convenient to fix a basis and work with matrices rather than linear maps – if we do this we get

2.1.5. Example

Let ${g^1}_n(f)$ be the vector space of all $n \times n$ matrices with entries in f-define the Lie bracket by

$$[X,Y] = XY - YX$$

Where XY is the product of the matrices $x$and$y$. As a vector space $g^i{}_n(f)$ have as basis the matrix units $e_{ij}$for $1 \leq i\,, j\ \leq\ n$ when calculating with this basis the formula $[e_{ij}, e_{kl}]\ =\ \delta_{jk}e_{il} - \delta_{il}e_{kj}$ is often useful

2.2. Some Motivation for Lie Algebras

Lie algebras were discovered by sophism Lie (1842 - 1899) while he was attempting to classify certain smooth sub group of general linear groups. The groups he considered are now called Lie groups. He found that by taking the tangent space at identity element of such a group one obtained Lie algebra.

Question about the group could be reduced to questions about the Lie algebra in which form they usually proved more tractable.

2.2.1. Example

Let

$$SL_2(R)\ =\ \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc = 1\right\}$$

We give a 2×2matrix$X$, ask when is $1\ +\ \mathcal{E}X \in SL_2$

If we neglect terms in $\mathcal{E}^2$ we get the statement $1\ +\ \mathcal{E}\,X \in SL_2\ (R)\ \leftrightarrow$ $t\,r\,x\ =\ o$; this define the Lie algebra $SL_2(R)$.

We refer to this kind of argument as an argument by naive calculus

The main disadvantage of the naive approach is that it doesn't explain how the Lie bracket on $Sl_2(R)$ comes from the group multiplication in $SL_2(R)$.

For a short explanation of this accessible to those who know a small amount about tangent spaces to manifolds.

2.2.2. Example

Let $GL_n(n\,, R)$ be the group of invertible $n \times n$ matrices with entries in R.

Let S be an element of $GL_n(R)$ and let

$$G_s(R)\ =\ \{X\ \in GL_n(R){:}\ X^t\,S\,X\ = S\}$$

WhereX^t is the transpose of the matrix$x$. Then$G_s(R)$ is a group .The associated Lie algebra is

$$g^i{}_s(R) = \{X \in g^i{}_n(R): X^tS + S\,X = 0\}$$

2.2.3. Example

Certain subspace of$g^1{}_n(f)$ turn out to be Lie algebra in their own right

(i) Let $SL_n(R)$ be the vector subspace of $g^1{}_n(f)$ consisting of all matrices with trace 0. This is known as the special linear algebra.

(ii) Let $b_n\,(f)$ be a vector subspace of $g^1{}_n(f)$ consisting of all upper triangular matrices.

(iii) Let $h_n(f)$ be the vector subspace of $b_n\,(f)$ consisting oa all strictly upper triangular matrices.

2.2.4. Definition

The center of a Lie algebra L is Z$(L) = \{X \in L: [X\,,Y] = 0$, for all $Y \in L\}\,if\,[X\,,Y] = 0$ we say that X and Y commute so the centre consists of those elements which commute with every element of L , so $L = Z\,(L)$ if and only if L is abelian .

2.2.5. Definition

An ideal is particular sub algebra of L . But sub algebra need not be an ideal for instance if M = $b_2\,(C)$ and L = $g^1{}_2(c)$ then M is a sub algebra of L but not an ideal. Whenever one has a collection of objects (here Lie algebras) one should expect to define maps between them the interesting maps are those that are structure preserving.

2.2.6. Definition

Let L and M be algebras. A linear map $\varphi: L \to M$ is a Lie algebra homomorphism if $\varphi([x\ ,y]\ = [\ \varphi(x)\ ,\varphi(y)]$ for all $x, y \in L$ objective Lie algebra homomorphism is an isomorphism

2.3. Some low-Dimensional Lie Algebras

To get a little more practice at working with Lie algebras we attempt to classify Lie algebras of small dimension. Any 1 dimensional Lie algebra is abelian, so up to isomorphism, there is just one 1- dimensional Lie algebra over any given field.

2.3.1. Theorem

If L is a 2 dimensional non- abelian Lie algebra than L has a basis X, Y such that $[X, Y]\ =\ X$ thus up to isomorphism there are exactly two
2- dimensional Lie algebras over any given field.

2.3.2. Theorem

Suppose that L is a 3- dimensional Lie algebra such that L^1 is 1-dimensional and L^1 is not contained in Z(L). Then L has a basis,x, y, z such that z is centre and $[x, y]\ =\ x$

2.3.3. Lemma

Suppose that L is a3- dimensional Lie algebra such that L^1 is 2-dimensional. Then L^1 is algebra If $X\ \in\ L \setminus L^1$ then ad acts on L^1 as an invertible linear transformation.

2.4. The Lie Algebra of Vector Fields on Manifold

2.4.1. Definition

A vector space L over R is a (real) Lie algebra if in addition to its vector space structure it possesses a product that is a map $L \times L \to L$

Taking the pair (x, y) to the element $[x, y]$ of L, which has the following properties:

(i) It is bilinear over

$$[a_1x_1 , y + a_2x_2, y] = a_1[x_1, y] + a_2[x_2, y]$$

$$[x, a_1y_1 + a_2y_2] = a_1[x, y_1] + a_2[x, y_2]$$

(ii) It is skew commutative $[x, y] = -[y, x]$

(iii) It satisfies the Jacobi identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

2.4.2. Example

A vector space V^3 of dimension 3 over R with the usual vector product of vector calculus is Lie algebra.

2.4.3. Example

Let $M_n(R)$ denote the algebra of n × n matrices over R with $X , Y \in X(M)$, in general the operator $f \to x_p(YF)$ defined on $C^\infty(p) \to F$being a C^∞ function on a neighborhood of P does not define C^∞, vector field however ,oddly enough $XY - YX$ dose define a vector field $Z \in X(M)$ according to the prescription .

$$Z_pf = (XY - YX)_pf = X_p(Yf) – Y_p(Xf)$$

For if $F, g \in C^\infty(p)$ then Xf and Yf are C^∞on a neighborhood of P and this prescription determines a linear map of $C^\infty(p) \to R$.

Therefore if the Leibniz rule holds for $Z_p$an element of $T_p(M)$ at each $\in M$. Consider $F, g \in C^\infty(p)$ then f $, g \in C^\infty(U)$, for some open set U containing P using the nation $(xf)_p$ for x_p the value of xf at P we have relations .

$$(XY - YX)_p\,(fg) = \ X_p\,(Yfg) - \ Y_p\,(Xfg)$$

$$= \ X_p(Yfg + gYf) - \ Y_p(Xfg + gXf)$$

$$= \big(X_p\,f\big)(Yg)_p + \ f(p)X_p\,(Yg) + \ (X_p g(Yf)_p + g(p)X_p(Yf)$$

$$- \ \big(Y_p f\big)(Xg)_p - \ f(p)Y_p(Xg - \big(Y_p g\big)(Xf)_p - g(p)(Y_p Xf)$$

So that

$$Z_p(fg) = \ (XY - YX)_p(fg) = f(p)(YX - YX)_p g + g(p)(XY - YX)_p f$$

$$= f(p)Z_p g + g(p)Z_p f$$

Finally if F is C^∞ on any open set ⊂M then so is $(XY - YX)f$ and there fore Z is a C^∞ vector field on M as claimed.

We may define the product on M using this fact namely define the product of X and Y by

$$[X\,,Y] \ = \ XY - YX$$

2.4.4. Theorem

M with the product [X, Y] is Lie algebra

Proof

If α, β ∈ R and X_1, X_2, Y are C^∞vector fields then it is straight to verify that $[\alpha X_1 + \ \beta\, X_2\,, Y]f \ = \ \alpha\,[X_1, Y]f \ + \ \beta[X_2, Y]f$.

Thus [X, Y] is linear in first variable since the skew commutatively [X , Y] = -[Y , X] is immediate from the definition we see that linearity in the first variable implies linearity in the second there for [X , Y] is bilinear and skew commutative .

There remains the Jacobi identity which follows immediately if we evaluate

$$[X\,,[Y\,,Z]] + \ [Y\,,[Z\,,X]] + [Z\,,[X\,,Y]]$$

Applied to a C^∞- function F – using the definition, we obtain

$$[X\,,[Y\,,Z]]F \ = \ X([Y\,,Z])F - [Y\,,Z](XF)$$

$$= X(Y(ZF)) - X(Z(YF)) - Y(Z(XF))$$

Permuting cyclically and adding establishes the identity. now we discuss the Lie algebra analytically and geometrically we have two Lie algebra associated with the tangent space at the identity $T_e G$ G with the bracket induce by ad and the left invariant vector fields L(G) with the Lie bracket . in this illustration we will demonstrate that they are isomorphic as vector spaces.

Define a map $V: T_e G \rightarrow X(G)$ by

$V_\xi(g) = T_e L_g(\xi)$, for all $\xi \in T_e$ G and $g \in G$.

Because tangent maps are linear, so is V. for all $\xi \in T_e G$ and $g, h \in G$ we have

$$(T_n L_g)\ (V_\xi(h)) = (T_n L_g)\ (T_e L_n(\xi\ )) =$$

$$T_e\left(L_g\ o\ L_n\right)(\xi) = T_e L_{gh}(\xi) = V_\xi(gh) = \left(V_\xi o\ L_g\right)(h)$$

Therefore V_ξ is left invariant so V really is a map

$T_e\ G \rightarrow L(G)$ it is inverse (immediately) give by the map

$$L(G)T_e \rightarrow G,\quad X \rightarrow X(e) \in T_e\ G$$

Now we discuss T_(e) G ≅ L(G) as Lie algebras .

To show that $T_e G$ and $L(G)$, are isomorphic as Lie algebras as well vector field , we must show that n the map

V= $T_e\ G \rightarrow$ L(G)

$V_{ad(\xi)_\eta} = [V_\xi, V_\eta]$

Define a$d_g = L_g: L_g M \rightarrow T_g M$, for all $\eta \in T_e G$

Since the Lie bracket of vector fields can be described easily in terms of flows it might be help full to know what the flows of these vector fields look like.

2.4.5. Lemma

Let$\xi \in T_e G$, and $V_\xi \in G$.Then the flows of V_ξ through g is the curve $L: R \rightarrow G$ Given by $C(t) = L_g o \exp(t\xi)$

Ext (t)

Proof

Note that $C(o) = L_g o\, exp(o) = L_g(e) = g$.

Let t∈ R then

$$c(o) = \frac{d}{d_{s\ s=t}} = \frac{d}{d_{s\ s=0}}\ c(s + t)$$

$$= \frac{d}{d_{s\ s=0}} L_g\ o\ exp((s+t)\xi) = \frac{d}{d_{s\ s=0}} L_g\ o\ exp((t+s)\xi)$$

$$= \frac{d}{d_{s\ s=0}} L_g(exp(t\xi))\, exp(s\xi) = \frac{d}{d_{s\ s=0}} L_g\ o\ exp(s\xi)$$

$$= \frac{d}{d_{s\ s=0}} L_g\ exp(t\xi)(exp(s\xi))$$

$$= (T_e L_{gexp(t\xi)(\xi)} = V_\xi(g - exp(t\xi))) = V_\xi(c(t))$$

2.4.6. Theorem

Let η t ∈ $T_e\ G$ then $V_{ad(\xi)\eta} = [V_\xi, V_\eta]$

Proof

Recall that the flow of $V_\xi$at time t ∈R is the map G → G given by $R_{\exp(t\xi)}$. Let g ∈ G, and then using the definition of ψ , Ad , and the linearity of tangent maps O we calculate

$$[V_\xi, V_\eta](g) = \frac{d}{d_{s\ t=0}}\ ((R_{\exp(t\xi)}))(g)$$

$$= \frac{d}{d_{s\ t=0}}\ \mathrm{T}\, R_{\exp(t\xi)}\ o\ V_{eta} R^{-1}{}_{\exp(t\xi)}(g)$$

$$= \frac{d}{d_{s\ t=0}}\ \mathrm{T}\, R_{\exp(t\xi)}\ o\ V_{eta}(gexp(-t\xi))$$

$$= \frac{d}{d_{s\ t=0}} \quad \mathrm{T}\, R_{\exp(t\xi)} o \mathrm{T} L_{g\ \exp(-t\xi)}(\eta)$$

$$= \frac{d}{d_{s\ t=0}} \quad \mathrm{T}\, (R_{\exp(t\xi)} o L \emptyset_{\exp(t\xi)})$$

$$= (TL_g) \frac{d}{ds}\ Ad(expt\xi)\eta$$

$$= (TL_g)[\xi, \eta] = V_{[\xi,\eta]}(g)$$

2.5. The Lie Algebra of Matrix Groups

Let us consider (n , k) , as a coordinates we choose the centries of the matrices so that matrix g is parameterize by g = g_j^i .

In particular the identity is e = δ_j^i then the left translation as multiplication acts is

$$L_h g = hg = \ h_k^j g_j^k$$

It is differential is

$$(dL_n)_{jk}^{ij} = \ \frac{\partial (hg)_j^i}{\partial g_l^k} = h_k^i \delta^i{}_j$$

The left t invariant vector fields can be obtained from the tangent vector at the identity denoted such a vector by

V = $V_j^i \frac{\partial}{\partial g_l^i}$g=e

The field X_v corresponding to V is given by acting on V with differential

$X_{v\mathrm{h}} = dl_h v = (dl_n)_{jk}^{il} v_l^k \frac{\partial}{\partial h_l^i} = h_k^i f_j^i v_l^k \frac{\partial}{\partial h_l^i} = (hv)_l^k \frac{\partial}{\partial h_l^i}$

The component of X at the point is just V interpret end, as a matrix product .This give us a very important formula for the Lie bracket.

Let x_v and x_w be two vector fields obtained from tangent vectors V and was above. The Lie bracket is new vector field which at point h is given by

$[X_v, X_w]$h= $\left((X_v\backslash h)^i_j \frac{\partial}{\partial h^i_l} \ (X_w\backslash h)\right)^k_l - (X_w\backslash h)^i_j \frac{\partial}{\partial h^i_l} ((X_v\backslash h))^k_l \frac{\partial}{\partial h^k_l} =$

$(h^i_m v^m_j \frac{\partial}{\partial h^i_l} h^k_n w^n_l - h^i_m w^m_j \frac{\partial}{\partial h^i_l} h^k_n v^h_j) \frac{\partial}{\partial h^k_l} = h^k_m (v^m_j w^i_j - w^m_j v^j_i) \frac{\partial}{\partial h^k_l} =$

$h[v, w] \ \frac{\partial}{\partial h}$

2.5.1. Remark

In the last line above the square brackets indicate is not the Lie bracket of vector fields , but the matrix commentator , (it means that we can identify the Lie algebra of GL(n , c) with the components ${V^i}_j$ of tangent vector and use the usual matrix commentator as the product which is huge simplification).

2.6. The Free Lie Algebra

In A_x let be two – sided ideal generated by all elements of the form a , a ∈ A_x and (a b) c + (b c) a , a , b , c ∈ A_x we set $L_x = A_x\backslash$ I and call L_x the free Lie algebra on X any map from X to a Lie algebra L extends to a unique algebra homomorphism from L_x to L .

We claim that the ideal I defining L_x is graded .This means that if

A=$\sum a_n$is decomposition of an element of I into it is homogenous component, then each of the a_n also belong to I. To prove this, let JCI denote the set of all A=$\sum a_n$ with the property that all the homogenous component a_n belong to I, clearly J is a two sided ideal we must show that JCI . For this it is enough to prove the corresponding fact for the generating elements clearly if

$$A = \sum a_p \ \ , b = \sum b_q \ , c = \sum c_r$$

Then $(ab)c + (bc)a + (ca)b =$

$$\sum_{p(q)r}((a_p b_q)c_r + (b_q c_r)a_p + (cra_p)b_q)$$

But also if $X = \sum x_m$ then

$$x^2 = \sum x_n^2 + \sum(x_m x_n + x_n x_m)$$

= $x_m x_n + x_n x_m = (x_m x_n)^2 - x_m^2 - x_n^2 \in I$

So JCI

The fact I is graded means that L_x in her it's the structure of a graded algebra.

In the following we will discuss the free associative algebra ASS (X) let V_x be the vector space of all finite formal linear combinations of elements of X define

$$ASS(x) = T(V_x)$$

The tensor algebra of V_x.Any map of x into an associative algebra A extends to a unique linear map from $V_x$to A and hence to a unique algebra homomorphism from Ass_x to A so Ass_x is the free associative algebra on x

We have the map $x \rightarrow L_x$ and$\epsilon: L_x \rightarrow U(L_x)$ and hence their composition maps x →As$s_x$thus give rise to a lie algebra homomorphism

$$L_x \rightarrow Ass_x$$

Which determines an associative algebra homomorphism?

ϕ : U (L_x) → Ass_x

Both composition $\phi o \psi$ and$\psi o \phi$ are the identity on X and hence , by uniqueness , the identity everywhere .we obtain the important result that U(L_x) and Ass_x are canonically isomorphism

U (L_x) ≅ Ass_x

Now the Poincare – Birkhoff- Witt theorem guarantees that the map

ξ: $L_x \rightarrow$ u(L_x) is injective ,so under the above isomorphism the map

$L_x \to$ Ass_xin injective on the other hand by construction the map

$X \to$ V_x Induces a subjective Lie sub algebra homomorphism from L_x into the Lie sub algebra of Ass_xgenerated by$X$.

So we see that the under the isomorphism

$L_x \subset$ U (L_x) is mapped isomorphism ally on to the Lie sub algebra of $\mathrm{A}ss_x$ generated by, X Now the map

$L_x \to \mathrm{Ass}_x$, $\mathrm{Ass}_x \to x$

Extends to a unique algebra homomorphism

Δ: Ass_x $\to$ Ass_x , x $\to$ Ass_x

Under the identification this is none other than the map

$\Delta: U(L_x) \to U(L_x) \times U(L_x)$ And hence we conclude that L_x is the set of primitive elements of

$$h_k^i \delta_j^i v_l^k \frac{\partial}{\partial h_j^i} = (hv)_l^k \frac{\partial}{\partial h_j^i}$$

Ass_x

L_x= {w∈ $\frac{\mathrm{Ass}_x}{\Delta\,(w)} = w \otimes I + I \otimes w$ }

Under the identification let us know discuss the algebra proof of CBH and explicit formulas we recall our constructs above x denotes a set $L_x$the free Lie algebra on x and Ass_x the free associative algebra on x so that Ass_x may be identified with the universal enveloping algebra of L_x since Ass_x may be identified

With non-commutative polynomials indexed by x we may consider is completion f_x the algebra of formal power series indexed by x – since the free Lie algebra L is graded we may also consider it is completion which we shall denoted by L_x finally let m denote the ideal in f_x generated by$x$. The maps

Ext: m $\to$ 1 + m

Are well defining by their formal power series and mutual inverse
Are well defining by their formal power series and mutual inverse
(There is no convergence issue since every things is within the real m of formal power series)

Furthermore $expo$ is abjection of the set of α∈m satisfying

$\Delta \alpha = \alpha \otimes 1 + 1 \otimes \alpha$ to the set of all $\beta \in 1 + m$ satisfying

$\Delta \beta : \beta \otimes \beta$

Now we will discuss abstract version of CBH and it is algebraic proof in particular, since the set $\{ \beta \in 1 + m \backslash \Delta\beta = \beta \otimes \beta \}$ forms a group we conclude that for any A , B ∈ L_x there exists ac ∈ $L_x$such that

Expc = (exp A) (exp B)

This is abstract version of the Campbell – Baker – Hausdroff formula – ponds basically on two algebra facts that the universal enveloping algebra of the Lie algebra is the free associative algebra, and that the set of primitive elements in the universal enveloping algebra is precisely the original Lie algebra.

2.7. Linear Lie Algebra

In algebra a linear Lie algebra is subalgebra of the Lie algebra $g^1(v)$ consisting of endomorphism of a vector space V. in other words a linear Lie algebra is the image of Lie algebra representation.

Any Lie algebra is a linear Lie algebra in the sense that there is g^1always faithful representation of g (in fact on a finite dimensional vector space by Ado's theorem if g is itself finite dimensional).

Let V be a finite – dimensional vector space over a field of characteristic zero and g subalgebra of $g^1(v)$ then v is semi- simple as a module over g if and only if (i) it is a direct sum of the center and a semi-simple ideal and (ii) the elements of the center are diagonalizable cover some extension field.

2.8. Representation of Lie Algebra

We will study representation of the simplest possible Lie algebras SL(2, c) Recall that this Lie algebra has a basis e, f , h with commutation relations

[e, f] = h, [h, e] = 2e [h, f] = -2 f

As we proved easier, this Lie algebra is simple the main idea of the study of representation of SL (2, c) is to start by diagonal zing the operator h.

2.8.1. Definition

Let V be a representation of SL (2, c). A vector V $\in V$ is called vector of Weight λ, λ ∈ C if it is an eigenvector for h with eigenvalue

Hv = λ v

We denoted by v[λ] c v the subspace of vector of weight λ the following lemma plays a key role in the study of representations of SL (2, c)

2.8.2. Lemma

a v[λ] c v [λ + 2]

f v[λ] c v [λ - 2]

Proof

Let v ∈V [λ]. Then

Hev = [h, e] v + e h v = 2 ev+λev = ($\lambda + 2$) e v

So ev ∈[λ + 2] . The proof for F is similar

2.8.3. Theorem

Every finite – dimensional representation V of SL (2, C) can be written in the form

$$V = \otimes_{\lambda} V [\lambda]$$

Where V[λ] is defined in definition, this decomposition is called weight decomposition of V.

Proof

Since every representation of $sl(2\,, C)$ is completely reducible, it suffices to prove this for irreducible, v, so assume that V is irreducible let

$V = \sum_{\lambda} v[\lambda]$ Be the sub space

Spanned by eigen vector of h – by well – known result of linear algebra, Eigen vector with different eigen value are linearly independent, so V^1=⊗v[λ]by lemma V^1 is stable under action of set f and h.Thus V^1 is a sub representation .since we assume that V is irreducible and $V^1 \neq 0$ (h has at least one Eigen vector) we see that $V^1 = V$. Our main goal will be classification of irreducible representation of$SL(2\,, C)$. Let λ be a weight of V(L ,e , v[λ] ≠ 0) which is maximal in the following sense R e λ ≥ R e $\lambda^1$for every weight $\lambda^1$of V such a weight will be called highest weight of $(V'')''$and vectors $V \in V[\lambda]$-highest weight vectors it is obvious that every finite – dimensional representation has at least one – zero highest weight vector.

2.8.4. Lemma

Let λ ∈ C define M to be the infinite – dimensional vector space with basis V°, V`. …

Irreducible representation $V_n$can also be described more explicitly, as symmetric powers of the usual two- dimensional presentation.

2.9.Lie Algebra of GL (n, R)

Consider the Lie group GL (n, R) we have $T_1 GL(n, R) = M_n(R)$the set of all $n \times n$ real matrices for any T∈ $M_n(R)$, the Lie bracket is the commeutator that is [A, B] = AB – BA

To prove this we compute, X_A the left invariant vector field associated with the matrix A ∈ T GL(n ,R) now on $M_n(R)$, we have global coordinate maps given by

$X_{ij}(A)$=A_{ij} , the ijth entry of the matrices B(n , R)

($x_A(x_{ij})$)(g)= $x_A(I)(x_{ij}oL_g)$ also if h ∈ GL(n , R)

Then $(X_{ij}oL_g)(h) = X_{ij}(gh)$

$$\sum_k g_{ik}h_{kj} = \sum_k g_{ik}X_{kj}\,(h)$$

Which implies that?

$$X_{ij}oL_g \sum_k g_{ik}X_{kj}$$

Now if $f \in C^{\infty}\big(GL(n, R)\big), X_A(I)f = \frac{d}{dt}$t(I + t$A$),. t=0

So that $X_A(I)X_{ij} = \frac{d}{dt}X_{ij}(1 + tA) = A_{ij}$

Putting these remarks together we see that

$$X_A\big(X_{ij}oL_g\big) = \sum_k g_{ik}A_{kj} = \sum_k X_{kj}(g)A_{kj}$$

We are now in a position to calculate the Lie bracket of the left invariant vector fields associated with element of M_n(R).

$$([X_A, X_B](I))_{ij} = [X_A, X_B](I)X_{ij} = X_AX_B\big(X_{ij}\big) - X_BX_A\big(X_{ij}\big)$$

$$X_A\left(\sum B_{kj}\,X_{ik}\right) - X_B\left(\sum A_{kj}\,X_{ik}\right)\left(\sum B_{kj}\,X_{il}B_{ik} - A_{kj}\,x_{il}B_{ik}\right)(I)$$

$$= \sum_{,kl} A_{kj}\delta_{il}A_{ik} - A_{kj}\delta_{il}B_{ik} = \sum_k A_{ik}B_{kj} - \sum_k B_{ik}A_{kj}$$

$$= (\mathrm{AB} - \boldsymbol{BA})_{\boldsymbol{ij}}$$

So [A, B] = AB –BA

2.10. Definitions

Lie algebra g is a vector space together with a skew-symmetric bilinear map

$$[,] : g \times g \to g$$

Satisfying the Jacobi identity

We should take a moment out here to make one important point. Why, you might ask, do we define the bracket operation in terms of the relatively difficult operations Ad and ad, instead of just defining $[X,Y]$ to be the commutator $X.Y - Y.X$ is answer is that the composition $X.Y$ of elements of Lie algebras is not well defined? Specifically , any time we embed a Lie group G in a general linear group GL(V) , we get a corresponding embedding of its Lie algebra g in the space $End(V)$, and can talk about the composition $X.Y \in End(V)$ of elements of g in this context , but it must be borne in mind that this composition $X.Y$ will depend on the embedding of g , and for that matter need not even be an element of g .Only the commutator $X.Y - Y.X$ is always an element of g , independent of the representation.

The terminology sometimes heightens the confusion : for example , when we speak of embedding a Lie algebra in the algebra$End(V)$ of endomorphism of V , the word algebra may mean two very different things

In general, when we want to refer to the endomorphism of a vector space V $(resp. R^n)$ as a Lie algebra.

2.11. The Exponential Map

The essential ingredient in studying the relationship between a Lie group G and its Lie algebra g is the exponential map. This may be defined in very straight forward fashion, using the notion of one –parameter subgroups, which we study next.

Suppose that $X \in g$ is any element, viewed simply as a tangent vector to G at the identity. For any element$g \in G$, denoted by $m_g: G \to G$ the map of manifolds given by multiplication on the left by g.

Then we can define a vector field v_x on all of G simply by setting

$$v_x(g) = \left(m_g\right)_*(X)$$

This vector field is clearly invariant under left translation (i.e., it is carried into itself under the diffeomorphism$m_g \; fort \; all \; g$), and it is not hard to see that this gives an identification of g with the space of the all left invariant vector fields on G .Under these identification, the bracket operation on the Lie algebra g corresponds to Lie bracket of vector fields, indeed, this may be adopted as the definition of the Lie algebra associated to a Lie group.

Given any vector field v on a manifold M and a point$p \in M$, a basic theorem form differential equations allows us to integrate the vector field.

This given differentiable map $\varphi: I \to M$, defined on some open interval I containing 0, with $\varphi(0) = p$,whose tangent vector at any point is the vector assigned to that point by v i.e., such that

$$\varphi'(t) = v(\varphi(t))$$

For all t in I . The map φ is uniquely characterized by these properties.

2.12. Examples of Lie Algebras

We start with the Lie algebras associated to each of the groups .each of these groups is given as a subgroup of $GL(V) = GL_n R$, so their Lie algebras will be subspaces of

$$End(V) = gI_n R$$

Consider first the special linear group $SL_n R$.If $\{A_t\}$ is an arc in $SL_n R$ with

$$A_0 = I$$

And tangent vector

$$A'_0 = X \;\; at \; t = 0,$$

Then by definition we have for any basis $e_1, \dots, e_n$ of $V = \mathbb{R}^n$,

$$A_t(e_1) \wedge \dots \wedge A_t(e_n) \equiv e_1 \wedge \dots \wedge e_n$$

Taking the derivative and evaluating at t=0 we have by the product rule

$0 = \frac{d}{dt}\left(A_t(e_1) \wedge \ldots \wedge A_t(e_n)\right) =$ `

$\sum e_1 \wedge \ldots \wedge X(e_t) \wedge \ldots \wedge e_n = Trace(X).(e_1 \wedge \ldots \wedge e_n)_{-}\ t = 0$

The tangent vectors to $SL_n(\mathbb{R})$are thus all endomorphism of trace 0, comparing dimensions we can see that the Lie algebra $\mathfrak{s}I_n(\mathbb{R})$ is exactly the vector space of traceless $n \times n$ matrices.

The orthogonal and simplistic cases are somewhat simpler for example, the orthogonal group $O_n(\mathbb{R})$ is defined to be the auto orphism$A$ of an n-dimensional vector space V preserving a quadratic form φ , so that if $\{A_t\}$ is an arc in$O_n(\mathbb{R})$ with $A_0 = I$ and $A'_0 = X$ we have for every pair of vectors $v, w \in V$

$$\varphi(A_t(v), A_t(w)) \equiv \varphi(v, w)$$

Taking derivatives, we see that

$$\varphi(X(v), w) + \varphi\left(v, X(w)\right) = 0$$

For all $v, w \in V$, this is exactly the condition that describes the orthogonal Lie Algebra $o_n(\mathbb{R})$. In coordinates, if the quadratic formφ is given on $V = \mathbb{R}^n$ as

$$\varphi(v, w) = v^t. M. w$$

For some symmetric $n \times n$matrix$M$, then as we have seen the condition on $A \in GL_n\mathbb{R}$ to be in $O_n\mathbb{R}$ is that

$$A^t. M. A = M$$

Differentiating the condition on a $n \times n$ matrix X to be in the Lie algebra $\mathfrak{s}o_n(\mathbb{R})$ of the orthogonal group is that

$$X^t. M + M. X = 0$$

Nilpotent [i.e.,$p(X)_s = 0$], whereas under the representation

$$p_3 : t \to \begin{pmatrix} t & t \\ 0 & 0 \end{pmatrix}$$

Not only are the images $p(X)$ neither diagonalizable nor nilpotent, the diagonalizable and nilpotent parts of $p(X)$ are not even in the image $p(g)$ of the representation.

If we assume the Lie algebra g is semi-simple, however, the situation is radically different. Specifically, we have.

2.14. Theorem

(Preservation of Jordan Decomposition)

Let g be a semi simple Lie algebra. For any element$x \in g$, there exist X_s and $X_n \in g$ such that for any representation $p: g \to g^i(V$

Chapter three

Classification of Lie Algebras

There are three important classes of Lie algebras with very different behavior solvable Lie algebras, semi-simple Lie algebras and nilpotent Lie algebras.

3.1.1. Definition

Lie algebras g is solvable if there exists a sequence of Lie subalgebra.

$O \sqsubset \ldots \sqsubset g_2 \sqsubset g_1 \sqsubset g_0 = g$

Such that for all I, $g_{i+1}$is an ideal in $g_i$and that quotient Lie algebra $g_{i+1} \backslash g_i$ is abelian

The standard examples to keep in mind are the Lie algebras of upper triangular matrices other examples are we will see include the Heisenberg Lie algebra and Boral subalgebra.

3.1.2. Definition

Lie algebra g is semi- simple if it contains no non-zero solvable ideals

3.1.3. Example

SL (n), the Lie algebra of trace-matrices, as well as Lie algebras compact simple Lie group (Su (n) so (n),)

Lie algebra can be decomposed as direct sum:

$$g = rad(g) \times g^1$$

Where $rad(g)$ is a (unique) maximal solvable ideal (the "radical" of g) and g^1 is semi-simple

A more general class are that one of ten wants to consider are Lie algebra where red (g) is rather trivial just the center of$g$.

3.1.5. Definition

Lie algebra is reductive if red$(g) = Z(g)$ any reductive Lie algebra is a sum.

$$g = Z(g) \times g^1$$

Where $Z(g)$is Abelian sub algebra (the center of g note this is not the same as $Z(g) \complement U(g)$ and g^1 is semi-simple.

3.1.6. Example

Reductive Lie algebras included ${g^1}_n$ and U(n) the classification of complex semi-simple Lie algebras g relies upon the existence in this case of a carton sub algebra h which is a maximal total (commutative with semi-simple elements) . Subalgebra it is dimension is the rank of the algebra (and the corresponding Lie group) all possible choices of h are related by conjugation. Note that this fails over other fields $g_{i+1}$is an ideal in $g_i$and the quotient.

Lie algebra $g_{i+1} \backslash g_i$ is abeliane, g, R, the ad joint action of h is used to decompose g as: $g = h_{at\Delta} \times g_\alpha$

When the root space $g_\alpha$is the Eigen space of the h action with eigenvalue αh^* and Δ is the set of root α.

For any Lie algebra one can define an invariant bilinear form, the killing form.

3.1.7. Definition

The killing form for Lie algebra g is defined by

{X, Y} = tr (ad(x) ad(y))

Where ad it's the adjoin representation for a complex simple Lie algebra this provides a non-degenerate bilinear form. Restricting to the Cartan subalgebra h this remains non-degenerate so can be used to define an isomorphism $h = h^*$ and thus a non-degenerated bilinear form on h^*. This is positive definition of non-zero elements of h_R the sub space of real linear combinations of the roots inΔ . For each $\alpha \in \Delta$one gets a reflection map S_α on h^* that takes $\alpha \rightarrow -\alpha$ presving the killing from this generate a finite group of permutation of the roots, the Weyl group W (g, h) the set of roots can be decomposed as:

$\Delta = \Delta^+ + \Delta^-$, such that:

$n^+ = \otimes_{\propto \in \Delta +} + g_\alpha and n^- = \otimes_{\alpha \epsilon \Delta -} g_\alpha$

Are (nilpotent) Lie sub algebras with $\alpha \in \Delta^-$ if $-\alpha \in \Delta^+$ taking?

b=$h \otimes n^+$

Gives an important sub algebra of g called a Boral sub algebra (for which n^+ is the nil potent radical) b is maximal solvable sub algebra, of g the space of Boral sub algebras of g can be identified with G\B. the Lie group corresponding to b. this is a complex projective variety and will play a crucial role in the study of representation theory of G via geometry its often known as the "flag variety" since it parameterizes flags on C^n in the case .

g =SL (n, c), a generalization is the hot ion of paretic "sub algebra". This Lie algebra p such that b $\subset$ p $\subset g$ with corresponding Lie group p. G\P it's also a complex projective variety.

One ends up with the following lists classifies complex simple Lie algebras (the sub scripts n give the rank)

$A_{n,}$ n= 1,2, 3… ,SL (n+1, C)

B_{n} , n= 2,3,4…,SO (2n +1, C)

C_{n} , n= 2,3,4… ,SP (2n +1, C)

P_{n} ,n= 4,5,6…,SO (2n ,C)

G_2 ,F_4 , E_6,E_7 ,E_8 corresponding exceptional Lie algebra.

In this research we will continually use. A example and consider its representations in ideal, so you should become familiar with how things work in that case.

We will also cover in some detail later the B and D case, but from the perspective of the spin representation.

When we discuss the highest weight theory of finite dimensional representations well review the story of the Weyl group and how it acts on h^*Lie algebra over other fields.

The classification of semi-simple Lie algebras over R is quite a bit more complicated. For each semi-simple Lie algebra$g$ there wills multiple non-isomorphic "real form" these are real Lie algebras g_α such that:

$$g = g_R \otimes C = g_R \otimes i g_R$$

3.1.8. Example

Real forms of SL (2, c) are SL (2,R) real form of SO (n, c) include Lie algebras of orthogonal groups for quadratic forms of different signatures SO (p ,q ,R) for (p+q =n) . It turns out that there will always be one "compact" real from which corresponding to a compact Lie group. We will always using this specific real from until later, when we will deal with this just one example of different real, with a non-compact Lie groups SL (2, R).

It turns out that one can find not just real forms for a complex semi-simple Lie algebra, but a Z-form, using a basis for Lie algebra due to chivalry, in which all the defining relation of the Lie algebra is have Z coefficients . The group of adjoins transformations of the Lie algebra is then an algebraic group defined over Z. This means that one can use it to define a group over any commutative ring giving for each complex Lie algebra wide range of different kinds of groups to study for example $g^1(2,c)$ the adjoin group of P GL (2, C). and one can construct and study groups like PG L (2, F_q) where F_q is finite field.

3.2. The Systemic Classification of Semi-simple Lie Algebras

The simple Lie algebras have been completely classified be Cartan. They come into four infinite classes and five exceptional Lie algebras the four infinite classes and five are" classical algebras" associated with classical groups.

Classical notation	**Rank**	**Carton's notation**
SU(n+1, C)	n≥1	A_n
SO(2n+1, C)	n≥1	B_n
SP(2n, C)	n≥1	C_n
SO(2n, C)	n≥1	D_n

Here Su (n+1, C) denote the complexification of Lie algebra of Su (n+1), i-e, it consists of complex linear combinations of the traceless hermetical $n + 1 \times n + 1$matrices. SO (2n, C) and so (2n+1, C) are defined analogously. The Lie algebra SP (2n) is the complexification of Lie algebra of the Lie group SP (2n).

They are also five exceptional Lie algebra denote G_2,F_4,E_6,E_7,E8 which have dimension 14, 52, 78, 133, and 248 respectively.

The rank of the algebra is the dimension of maximal commuting sub algebra.

3.3. Carton Subalgebra

3.3.1. Definition

Let L be a complex semi-simple Lie algebra and H a complex subspace such that:

(i) If $h_1, h_2 \in H then [h_1, h_2] = 0$

(ii) For all $V \in L if [v, h] = 0 \; for all \; h \in H, then V \in H.$

(iii) For all $h \in H, theopreatorad(h)isdiagonalizable$.

The condition (i) and (ii) imply that it is maximal commuting subalgebra of L. its straight forward to construct sub algebra satisfying (i) and (ii), by induction, but its non-trivial to satisfy (iii). It can be shown (but not here) that if L is a complex semi-simple Lie algebra then L has a Cart an subalgebra. Cart an sub algebra are not unique, it can be shown that if H_1 and H_2 are two Cartan subalgebra of matrix subalgebra L(G) then there exists some $g \in G$ $such\ that\ H_1 = g_1 H g_2{}^{-1}$. Hence the dimension of all Carton sub algebras in equal, the dimension of Cart an sub algebra it's called the rank r of L. the diagonalizability in condition (iii) together with (i) is sufficient to ensure that if $\{h_{1,\ldots,}h_r\}$ is a basis for H then ad (h_1) ,…,ad(h_r) can be simultaneously diagonalzed.

3.3.2. Definition

Suppose that L is a complexities semi-simple Lie algebra of rank n, and H as Carton sub algebra let $\{h_1,\ldots,h_n\}$ be a basis for H. then as the ad (h_i) can be simultaneously diagonal zed it follows that L can be decomposed as:

$$L = H \otimes \sum L_{\underline{\propto}}$$

Where the $\underline{\alpha} \neq 0$. Are vectors in R^n with $L_{\underline{\alpha}} = \{V \in L: [h_i, V] = \alpha_i V\}$. The vectors $\underline{\alpha} \neq 0$. Are called roots and $L_{\underline{\alpha}} \neq 0$. Is called root space. At thought is not a root we will set L_0= H for convenience.

3.3.3. Lemma

(i) L_α, L_β, if $\underline{\alpha} + \underline{\beta} \neq 0$.

(ii) The restriction of killing form K to H is non-degenerate.

(iii) Suppose$\underline{\alpha}$,$\underline{\beta}$, are roots. If $\underline{\alpha} + \underline{\beta}$, is not root then:

$[L_{\underline{\alpha}}, L_{\underline{\beta}}]$ C$L_{\underline{\alpha+\beta}}$ if $\underline{\alpha}$+$\underline{\beta}$ is not root then $[L_{\underline{\alpha}}, L_{\underline{\beta}}] = 0$.

(iv) If $\underline{\alpha}$ is a root then SO is, -$\underline{\alpha}$.

Proof:

(i) Suppose$X_{\underline{\alpha}} \in L_{\underline{\alpha}}$ and $X_{\underline{\beta}} \in L_{\underline{\beta}}$

$\alpha_i + \beta_i$) k ($X_{\underline{\alpha}}$,$X_{\underline{\beta}}$) = k (h_i ,X_α) +k ($X_{\underline{\alpha}}$, [h_i ,X_β])(

But k ([h_i ,α_i] ,$X_{\underline{\beta}}$)+k($X_{\underline{\alpha}}$, [h_i ,X_β]) $= 0$ by associatively of k. Hence ($\alpha_i + \beta_i$) k ($X_{\underline{\alpha}}$,$X_{\underline{\beta}}$) $= 0$ so if $\underline{\alpha} + \underline{\beta} \neq 0$ then

K ($X_{\underline{\alpha}}$,$X_{\underline{\beta}}$) $= 0$

For all $X_{\underline{\alpha}} \in L_{\underline{\alpha}}$ and $X_{\underline{\beta}} \in L_{\underline{\beta}}$

(ii) Suppose that K restricted to H is degenerate then there exists some $V \in H$ such that k (V,L)=0 for all $h \in H$. And if $\underline{\alpha}$ is a root then by reasoning used in (i) it follows that K (V, $X_{\underline{\alpha}}$)= 0, for all

$X_{\underline{\alpha}} \in L_{\underline{\alpha}}$. so it follows that K(V, L) =0, for all $L \in L$, contradiction with the fact that K is non-degenerated on L. so K restricted to H is non-generate. Hence the equation $\alpha_i = K(h_i, u_\alpha)$ be solved for unique$u_{\underline{\alpha}}$.

(iii) Suppose that$X_{\underline{\alpha}} \in L_{\underline{\alpha}}$ and $X_{\underline{\beta}} \in L_{\underline{\beta}}$then from the Jacobi identity

$$[h_i \left[X_{\underline{\alpha}} , X_{\underline{\beta}}\right] = [\left[h_i , X_{\underline{\beta}}\right] + X_{\underline{\beta}} + \left[X_{\underline{\beta}}, h_i, X_{\underline{\beta}}\right]\right] =$$

$$\underline{\alpha}_i \left[X_{\underline{\alpha}}, X_{\underline{\beta}}\right] + \beta_i \left[X_{\underline{\alpha}}, X_{\underline{\beta}}\right] = (\alpha_i + \beta_i) \left[X_{\underline{\alpha}}, X_{\underline{\beta}}\right]$$

Hence if $\alpha + \beta$is a root then this implies that $[X_{\underline{\alpha}}, X_{\underline{\beta}}] \in L_{\alpha+\beta}$

If how ever $\underline{\alpha} + \underline{\beta}$ is not a root then one must have $[X_{\underline{\alpha}}, X_{\underline{\beta}}] = 0$

4\ suppose -$\underline{\alpha}$ is not a root .suppose$X_{\underline{\alpha}} \in L_{\underline{\alpha}}$. Then if $\underline{\beta}$ is any root $\underline{\alpha} + \underline{\beta} \neq 0$ then by (1) if $X_{\underline{\beta}} \in L_{\underline{\beta}}$ then K ($X_{\underline{\alpha}}, X_{\underline{\beta}}$) $= 0$

Similarly also by reasoning in (i) K ($X_{\underline{\alpha}}$,h) =0 for all$h \in H$. Thus then implies that$X_{\underline{\alpha}} = 0$, so $L_{\underline{\alpha}} = 0$ a contradiction.

3.3.4. Corollary

If $\underline{\alpha}$ is a root then $[L_{\underline{\alpha}}, L_{\underline{\alpha}}] \in H$

Proof:

From the reasoning used to prove (iii) in the above lemma if

$X_{\underline{\alpha}} \in L_{\underline{\alpha}}$ And $X_{\underline{-\alpha}} \in L_{\underline{-\alpha}}$, then $\left[h_i, \left[X_{\underline{\alpha}}, X_{\underline{-\alpha}}\right]\right] = 0$

Which implies $[X_{\underline{\alpha}}, X_{\underline{-\alpha}}] \in H$.

3.3.5. Lemma

If$\underline{\alpha}$ is the root then there exists a unique $Y_{\underline{\alpha}} \in H$ such that

$$\alpha_i = K\left(h_i, Y_{\underline{\alpha}}\right)$$

Proof:

As the restriction of K to H is non-degenerate, the equation $\alpha_i = K\ (h_i, Y_{\underline{\alpha}})$ can be solved uniquely for $Y_{\underline{\alpha}} \in H$.

Note that as $\underline{\alpha} \neq 0$, $Y_{\underline{\alpha}} \neq 0$.

3.3.6. Corollary

Suppose that $\underline{\alpha}$ is a root .if $X \in L_{\underline{\alpha}}, Y \in L_{\underline{-\alpha}}$ then [X, Y]= K(X , Y)$Y_{\underline{\alpha}}$.

Proof:

From the above if follows that [X, Y]$\in H$. If $h = u^i h_i \in H$ then $K([X,Y],h) = K(X[Y,h])$

$$K\left(X, u^i \alpha_i Y\right) = u^i \alpha_i K(X,Y) = K\left(Y_{\underline{\alpha}}, h\right) K(X,Y)$$

Hence $K([X,Y] - K(X,Y)Y_{\underline{\alpha}}, h) = 0, for all h \in H$. But K restricted to H is non-degenerate, so $[X,Y] = K(X,Y)Y_{\underline{\alpha}}$ as required.

3.3.7. Lemma

Suppose that $\underline{\alpha}$ is a root. There exits some $X_{\underline{\alpha}} \in L_{\underline{\alpha}}$ and $X_{\underline{-\alpha}} \in L_{\underline{-\alpha}}$ such that $Y_{\underline{\alpha}} = [X_{\underline{\alpha}}, X_{\underline{-\alpha}}] andK = (Y_{\underline{\alpha}}, Y_{\underline{-\alpha}}) \neq 0$.

Proof:

Pick some

$X_{\underline{-\alpha}} \in L_{\underline{-\alpha}} withX_{-\alpha} \neq 0 \; supposethatK = (X_{\underline{-\alpha}}, X_{\underline{\alpha}}) = 0$. For all

$X_{\underline{\alpha}} \in L_{\underline{\alpha}}$. Then $L_{\alpha} \perp L_{\beta}$ for all roots β and$L_{-\alpha} \perp H$. Hence $L_{-\alpha} \perp L$. But K^{α} is non-degenerate on L so this implies $L_{\underline{-\alpha}} = 0$, a contradiction. So there must exists some $X_{\alpha} \in L_{\underline{\alpha}} withk(X_{\underline{-\alpha}}, X_{\underline{\alpha}}) \neq 0$ by reasoning we can $k(X_{\underline{\alpha}}, X_{\underline{\alpha}}) = 1$.

Then by the corollary above one finds $Y_{\underline{\alpha}} = [X_{\underline{\alpha}}, X_{\underline{-\alpha}}]$ next suppose that $\underline{\beta}$ is a root. Consider $W = \otimes_{j\epsilon z} L_{\beta+\underline{\alpha}}$. By (iii) of lemma it follows that W is an invariant sup space of ad $X_{\underline{+\alpha}}$ and W is also an invariant sub space of ad $Y_{\underline{\alpha}}$ then:

$$T_{rw}(adY_{\underline{\alpha}}) = T_{rw}(ad[X_{\underline{\alpha}}, X_{\underline{-\alpha}}]) = T_{rw}\left[adX_{\underline{\underline{\alpha}}}, adX_{\underline{-\alpha}}\right] = 0$$

Where here the trace T_{rw} denotes the trace restricted to the sub space W. As

$$Y_{\underline{\alpha}} \in HsetY_{\underline{\alpha}} = Y_{\underline{\alpha}}^{i} h_i.$$

Then note that:

$$T_{rw}(adY_{\underline{\alpha}}) = \otimes_{J\epsilon z} Y_{\underline{\alpha}}^{i}(\beta_i + j\alpha_i) \dim L_{\underline{\beta}+J\underline{\alpha}}$$

we therefore obtain the equality:

$$Y_{\underline{\alpha}}^{i}\beta_i \dim w + \; Y_{\underline{\alpha}}^{i} X_i \sum jdimL_{\underline{\beta}+J\underline{\alpha}} = 0$$

Note that:

$$K(Y_{\underline{\alpha}}, Y_{\underline{-\alpha}}) = K(Y_{\underline{\alpha}}[X_{\underline{\alpha}}, X_{\underline{-\alpha}}])$$

$=K\left[Y_{\underline{\alpha}}, X_{\underline{\alpha}}\right], X_{-\alpha} = K\left(Y_{\underline{\alpha}}^{i}, \alpha_i, X_{\underline{\alpha}} X_{\underline{-\alpha}}\right)$

$$= Y_{\underline{\alpha}}^{i} \alpha_i K\left(X_{\underline{\alpha}}, X_{\underline{-\alpha}}\right)$$

$$= Y_{\underline{\alpha}}^{i} \alpha_i$$

Hence $Y_{\underline{\alpha}}^{i} \beta_i \dim W + K\left(Y_{\underline{\alpha}}, Y_{\underline{-\alpha}}\right) \sum j dim L_{\underline{\beta}+j\underline{\alpha}} = 0, if K\left(Y_{\underline{\alpha}}, Y_{\underline{-\alpha}}\right) = 0$ $then Y_{\underline{\alpha}}^{i} \beta_i = 0,$ for all roots β

this implies that $[Y_{\underline{\alpha}}, X_{\beta}] = 0, for all L_{\underline{\alpha}} \in L_{\underline{\beta}}$. hence $[Y_{\underline{\alpha}}, L] = 0,$ $for all L \in L$. It follows that ad $Y_{\alpha} = 0$ $so K\left(Y_{\underline{\alpha}}, L\right) = T_r(ad Y_{\alpha} ad L) = 0$ $for all L \in L$ where the trace is how taken over L. As K is non-degenerate this implies that $y_{\underline{\alpha}} = 0$, a contradiction. Hence $k(y_{\alpha}, y_{\alpha}) \neq 0$.

3.3.8. Proposition

Suppose $\underline{\alpha}, \underline{\beta}$ are roots, consider the vectors $\beta_{-} + n\alpha_{-}$ for $n \in Z$this sequence consist of string of roots for p ≤ n ≤ q for some p\q ∈ Z with p ≤ o ≤ q.

Moreover $\frac{2k(y_{\beta}, y_{\alpha})}{k(y_{\alpha}, y_{\beta})}$ = -(p + q)

Proof:

Consider the space$V = \otimes_{n \in z} L_{\underline{\beta}+n\alpha}$ where the sum is taken over those n. such that $\underline{\beta} + n\alpha$is a root then V is an invariant under the adjoin action of the su(2) generators $h_{\alpha}, J_{\mp\alpha}$

As each $L_{\beta+n\alpha}$ is one dimensional it follows that the representation is irreducible the elements of V consist of elements of the form:

$$(ad J_{\alpha})^{m1} J_{\beta} and\ (ad J_{-\alpha})^{m2}\ for m_1, m_2 \in N$$

Hence there exists integers p, q with p ≤ o ≤ q such that $\underline{\beta} + n\alpha$ is a root if and any if p ≤ n≤ q. Note that if $x \in L_{\beta+n\alpha}$ then

$[h_{\alpha}, x] = \frac{1}{k(y_{\alpha}, y_{\alpha})}$ $[y_{\alpha}, x]$

$=\frac{1}{k(y_\alpha,y_\alpha)}\left(y^i{}_\alpha\right)[h_i,x]$

$$=\frac{1}{k(y_\alpha,y_\alpha)}\left(y^i{}_\alpha\right)(\beta i+n\alpha i)x=\frac{(n+k(y_\alpha,y_\alpha))x}{k(y_\alpha,y_\alpha)}$$

But the largest and smallest of the possible Eigen values is ±r where 2r ∈ N, $i-e$

$$q+\frac{k(y_\beta,y_\alpha)}{k(y_\alpha,y_\alpha)}=r\ ,p\ +\frac{k(y_\beta,y_\alpha)x}{k(y_\alpha,y_\alpha)}=-r$$

And hence $\frac{2k(y_\beta,y_\alpha)}{k(y_\alpha,y_\beta)}=(p+q)$ as required

3.4. Nilpotent Lie Algebra

3.4.1. Definition

Lie algebra g is said to be nilpotent if it admits filtration.

$g=a_0\supset a_1\supset\cdots\supset a_r=0$ (3-4-1)

By ideals such that:

$[g,g_i]\subset a_{i+1}\ \ for\ 0\le i\le r-1$

Such that filtration is called a nilpotent series the condition (3-4-1) to be a nilpotent series is that $a_i\backslash a_{i+1}$be in the center of $g\backslash a_{i+n}$ for $0\le i\le r-1$.

Thus nilpotent Lie algebras by successive central extensions

$$0\longrightarrow a_1\backslash a_2\longrightarrow g\backslash a_2\longrightarrow g\backslash a_1\longrightarrow 0$$

$$0\longrightarrow a_2\backslash a_3\longrightarrow g\backslash a_3\longrightarrow g\backslash a_2\longrightarrow 0$$

In other words the nilpotent Lie algebras from the smallest class containing the commutative Lie algebras and closed under contrail extension. The lower central series of g is:

$$g\supset g^1\supset\cdots\supset g^{i+1}\supset\cdots$$

$$with\ g^1=[g,g]\ ,g^2=[g,g^1],\ldots,g^{i+1}=\left[g,g^i\right]\ldots$$

3.4.2. Proposition

A Lie algebra g is nilpotent if and only if it is lower central series terminates with zero.

Proof:

If the lower central series

Terminates with zero, then it is a nilpotent series conversely, if

$g \supset a_1 \supset a_2 \supset \cdots \supset a_r = 0$ is nilpotent series, then $a \supset g^1$ because $g \backslash a_1$ is commutative $a_2 \supset [g, a_1] \supset [g, g^1] = g^2$ and so on, until we arrive at $0 = a_r \supset g^r$Let V be a vector space of dimension and let $f = V = V_0 \supset V_1 \supset \cdots \supset V_n = 0 \dim V = n - 1$, be a maximal flag in let n(f) be the Lie sub algebra of ${g^1}_v$consisting the element $x \; such \; that \; x(V_i) \subset V_{i+1}$ for all i the lower central series of n(f) has:

$$n(f)^j = \left\{x \in {g^1}_v \;\; x(v_i) \subset v_{\,i+1+j}\right\}$$

For J = 1,…,n. in particular n(f) is nilpotent for example:

$$n_3 = \left\{\begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}\right\} \supset \left\{\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right\} \supset \{0\}$$

Is nilpotent series for n_3.

An extension of nilpotent algebra is solvable but necessary nilpotent. For example n_3 is nilpotent and $b_3 \backslash n_3$is commutative, butb_3 is not nilpotent when $n \geq 3$.

3.4.3. Proposition

(i) Sub algebras and quotient algebras of nilpotent Lie algebras are nilpotent.

(ii) A Lie algebra g is nilpotent if g\a is nilpotent for some ideal a contained in Z(g).

(iii) A nonzero nilpotent Lie algebra has non zero center.

Proof:

(i) The intersection of nilpotent series for g with a Lie algebra h is nilpotent series with h, and the image of a nilpotent series for g in a quotient algebra g$g$is a nilpotent series for$g$.

(ii) For any ideal $a \subset Z(g)$ the inverse image of a nilpotent for g\a becomes a nilpotent series for g when extended by 0.

(iii) If g is nilpotent then the last nonzero term a, in, a nilpotent series for g is contained in Z (g).

3.4.4. Proposition

Let h be a proper Lie algebra of a nilpotent Lie algebra g then $h \neq h_g(h)$

Proof:

We use induction on the dimension of$g$. Because g is nilpotent and non-zero it is center Z (g) is non-zero. $if(z(g) \not\subset h, then\, n_g(H) \neq h$ Because Z (g) normalizes h. $if\ Z(g) \subset h,$ then we can apply induction to the Lie sub algebra h\Z (g) of g\Z (g s).

3.5. Solvable Lie Algebra

3.5.1. Definition

A Lie algebra g is said to be solvable if it admits a filtration by ideal such that $[a_i, a_i] \sqsubset a_{i+1}\, for\ 0 \leq i \leq r-1.$ such a filtration is called a solvable series the condition (3.1) to be a solvable series is that the quotient $a_i \backslash a_{i+1}$ commutative for $0 \leq i \leq r-1$. thus the solvable Lie algebras are exactly those that can be obtained from commutative Lie algebras by successive extensions.

$$0 \to a_1 \backslash a_2 \to g \backslash a_2 \to g \backslash a_1 \to 0$$

$$0 \to a_2 \backslash a_3 \to g \backslash a_3 \to g \backslash a_2 \to 0$$

In other words the solvable Lie algebras from the smallest class containing the commutative Lie algebras and closed under extensions the classic ideal $[g,g]$ is called the derived algebra of and is denoted D_g clearly D_g is contained in every ideal a such that $g\backslash g$ is commutative, and so $g\backslash D_g$ is the largest commutative quotient of g. write D^2 for the second derived , algebra $D\left(D_g\right), D_g^3$ for the third derived algebra $D(D_g^2)$, and so on these are classific and the derived series of g is the sequence.

$$g \sqsupset D_g \sqsupset D_g^2 \sqsupset \cdots$$

We sometimes write g^1 for D_g and $g^{(n)}$ for D_g^n

3.5.2. Proposition

Lie algebra $g$is solvable if and only if its derived series terminates with zero.

Proof:

If the derived series terminates with zero then it is a solvable series conversely if $g \sqsupset a_1 \sqsupset a_2 \sqsupset \cdots a_r = 0$ is a solvable series then $a_1 \sqsupset g^1$ because $g\backslash a_1$ is commutative. $a_2 \sqsupset a^1 \sqsupset g^n$, Because$a_1\backslash a_2$ is commutative and so on until $0 = a_r \sqsupset g^{(n)}$.

Let v be a vector space of dimension and let:

$$F\colon\ V = V_0 \sqsupset V_1 \sqsupset \cdots \sqsupset V_n = 0, \dim V_i = n-1$$

Be a maximal flag in. Let b (F) be the Lie sub algebra of g_v^1 consisting of the elements x such that $x(v_i) \subset v_i$ for all i. then

D (b (F)) = n (F) and so b (F) is solvable for example:

$$b_3 = \left\{\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}\right\} \sqsupset \left\{\begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}\right\} \sqsupset \{0\}$$

In nilpotent series for b_3.

3.5.3. Proposition

Let k^1 be a field containing k. a Lie algebra g over k is a solvable if and only if $gk^1 = k \times_k g$ is solvable.

Proof:

Obviously, for any sub algebras h and h[1] of g $[h, h^1]_{k1} = [\begin{matrix} h & h^1 \\ k & k^1 \end{matrix}]$ and so under extension of the base field derived series of g maps to that of g_{k^1}

Note we say that an ideal is solvable if it is solvable as Lie algebra.

3.5.4. Proposition

(i) Sub algebras and quotient algebra of solvable Lie algebra are solvable.

(ii) A Lie algebra $g$it's solvable if it contains an ideal n such that both n and g\n are solvable.

(iii) Let n be an ideal in Lie algebra$g$, and let h be sub algebra in$g$. if n and h is solvable then $n + h$ is solvable.

Proof:

(i) The intersection of solvable series for g with Lie sub algebra is a solvable series for h and the image of a solvable series for g in quotient algebra $g$is solvable series for$g$.

(ii) Because g\n is solvable $g^m \subset n$ for some m. now $g^{(m+n)} \subset n^n$ which is zero of some n.

(iii) This follows from (ii) because $h + n\backslash n \simeq h\backslash h \cap n$ which is solvable by (i).

3.5.5. Corollary

Every Lie algebra contains a largest solvable ideal.

Proof:

Let n be a maximal solvable ideal. If h is also a solvable ideal then $h + n$ is solvable by (iii) and so equals, therefore $h \sqsubset n$ the radial r = r(g) of G is the largest solved ideals in. The radical of g is classific ideal.

3.5.6. Lemma

For any $n \times n$ matrices $A = (a_{ij})$ and where

$$Tr(A) = \sum_{ij} a_{ij}b_{ij} = Tr(AB)$$

Hence $Tr_v(xoy) = Tr_v(yox)$ for any endomorphism x, y of a vector space v, and so

$$Tr_v([x, y]oz) = Tr(xoyoz) - Tr(yoxoz) = Tr(xoyoz) - Tr(xozoy)$$
$$= Tr(xo)[y, z]$$

3.5.7. Theorem

Let g be sub algebra of g_v^1, where V is a finite-dimensional vector space a field k of characteristic zero then g is solved if $Tr_v(xoy) = 0$ for all $x, y\ over\ g$

We first observe that if k^1 is a field containing k , then the theorem is true for $g \subset g_v^1$if and only if it is true for $g_k \subset g_k^1$ (because g is solvable if and only if $g_k$is solvable from proposition).

Therefore we may assume that the field k is finitely generated over 0, hence embeddable in c and then that k=c

We shall show that the condition implies that each $x \in [g, g]$ define nilpotent endomorphism of V the Engle is theorem will show that $[g, g]$ is nilpotent in particular solvable and it follows that g is solvable because

$$g^{(n)} = (D_g)^{(n-1)}$$

Let $x \in [g, g]$, and choose a basis of V for which the matrix of x is diagonal say dig($a, \dots, a_n$), and the matrix of x_n is strictly upper triangular . we have to show that x =0 and for this it suffices to show that

$$\bar{a}_1 a_1 + \cdots + \overline{a}_n a_n = 0$$

Where $\bar{a}$ is the complex conjugate of a .Note that $Tr_v(x^{-}{}_s o x) = \bar{a}_1 a_1 + \cdots + \bar{a}_n a_n$

Because $\bar{x}$ has matrix dia$g$ $(\bar{a}_1, \dots \dots \dots, \bar{a}_n)$. By assumption x is a sum of commentator [y ,z] and so it suffices to show that

$Tr_v(\bar{x}_s o[y, z]) = 0\ , all\ x, y, z \in g$ Form the trivial identity (3.2) we see that it suffices to show that

$$Tr_v([\bar{x}_s, y]oz) = 0\ , all\ \ Y, Z \in g$$

This will follow from the hypothesis once we have shown that $[\bar{x}, y] \in g$. According to

$\bar{x}_s = c_1 x + c_1 x^2 + \cdots \dots \dots . . + c_r x^r$, for some $c_i \in g$

And so $[\bar{x}_s, y] \complement g$ Because $[x, y] \complement g$

3.5.8. Corollary

Let V be a finite-dimensional vector space over a field of characteristic zero and let g be sub algebra of $g^1{}_v$ .if g is solvable then $Tr_v(xoy) = 0$ for all $, y \in [g, g]$. Conversely if $Tr_v(xoy) = 0\ , for\ \ all\ x, y \in [g, g]$then g is solvable

If g is solvable then $Tr_v(xoy) = 0\ , for\ x \in g\ and\ y \in [g, g]$ For the converse note that the condition implies that $[g, g]$ is solvable by .But this implies that $g$is solvable, because

$$g^{(n)} = (D_g)^{(n-1)}$$

3.6. Semi -simple Lie Algebras

As is clear from the above many of the spaces of the representation theory of finite groups that were essential to our approach are no longer valid in the context of general Lie algebras and Lie groups. Most obvious of these is complete reducibility which we have seen fails for Lie groups, another is the vector space be non-diagonalizable, the action of some element of Lie algebra may be diagonalizable under one representation and not under another.

That is the bad news .The good news is that if we just restrict ourselves to semi –simple Lie algebras, everything is once more as well behaved as possible. For one thing we have complete reducibility again:

3.6.1. Theorem

(Complete Reducibility) Let V be a representation of the semi-simple Lie algebra $g\ and\ W \subset V$ a subspace invariant under the action of g .

Then there exists a subspace$W' \subset V$ complementary to W and invariant under$g$. The proof of this basic result will be deferred to Appendix C. The other question the diagonalizability of elements a Lie algebra under a representation ,requires a little more discussion. Recall first the statement of Jordan decomposition: any endomorphism X of a complex vector space V can be uniquely written in the form

$$X = X_s + X_n$$

Where X_s is diagonalizable x_n is nilpotent, and the two commute. Moreover,$X_s\ and\ X_n$ may be expressed as polynomials in X

Now suppose that g is an arbitrary Lie algebra, $X \in g$ any element, and $p: g \to gI_nC$ any representation. We have seen that the image $p(X)$ behaves with respect to the Jordan decomposition. The answer is that in general, absolutely nothing need be true. For example, just taking $g = C$, we see that under the representation

$$p_1 : t \to (t)$$

Every element is diagonalizable i.e.,$p(X)_s = p(X)$, under the representation

$$p_2 : t \to \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$$

Every element is $p(X)_s = p(X_s) \; and \; p(X)_n = p(X_n)$

In other words ,if we think of p as injective and g accordingly as a Lie sub algebra of $gI(V)$, the diagonalizable and nilpotent parts of any element X of g are again in g and are independent of the particular representation p. The proofs we will give of the last two theorems both involve introducing objects that are not essential for the rest of this reference and we therefore relegate them to Appendix C. It is worth remarking however that another approach was used classically by Hermann Weyl, this is the famous unitary t that not only can the action of elements of Lie group or algebra on a vector trick, which we will describe briefly

Chapter Four

Symmetries and Lie Algebras

4.1. Introduction

In order to understand symmetries of differential equations, it is help full to consider symmetries of simpler objects roughly speaking, symmetry of a geometrical – object is a transformation whose action leaves the object apparently unchanged. For instance, consider the result of rotating an equilateral triangle antic lock wise about its centre. After rotation of $2\pi/3$, the triangle looks the same as it did before the rotation, so this transformation is symmetry rotation of $4\pi/3$ and 2π are also symmetries of the equilateral triangle. In fact, rotating by 2π is equivalent to doing nothing, because each points its mapped to itself. The transformation mapping each point to itself is asymmetry of any geometrical object it is called the trivial symmetry.

On the other hand, consider a circle; any rotation by angle ξ may be represented in Cartesian coordinates as the mapping

$$f_{\xi}(x, y) \;\; (x \cos \xi - ysin\xi\,, xsin\xi + ycos\xi)$$

In this case ξis continuous parameter, similar the set of reflection of circle can be represented by the mapping.

$$F_p: (x, y) \to (-x, y)$$

Followed by arotation f_{ξ} there fore in the case of circle we are examining asymmetric group which is not discrete, these kinds of symmetries are known as Lie symmetries, and they form a Lie group Which will be explored in the next section symmetries are commonly used to classify geometrical objects. There are certain constraints of on symmtries of geometrical objects. Each, symmetry has aunique inverse, which is itself is symmetry. The combined action of the symmetry and its inverse up on the objects leaves of coordinates q(t) and $q^{'}$ (t) are soluation of the same set of equations. This explains the view of symmetry transformation as mapping between different soluations of the equation of motion motivates an alternative view of symmetry transformation, btween different time evolutions of the system, describe in the some coordinate frame. This is called an active transformation. It is transformation between defferent physical stations, rather than atransformation between two different descriptions of the same situation.

4.2. Importance of Symmetries

Symmetries are for many reasons a highly important subject to study in physics. Symmetries of the fundamental laws of nature tells as something basic about the nature that in many cases can be viewed as even more basic than the laws themselves. A well –known example is the space time symmetries of the special theory of relativity. The Lorentz transformation were first detected as symmetries of Max wells equations, but Einstein realized that they are more fundamental than being symmetries of equation. After him the relativistic symmetries space-time have been the guiding principles for formulation of all fundamental laws of nature.

In the similar may, when extending the laws of the nature in to new realms, in particular in elementary particle physics, the study of observes symmetries have often been used as a tool in the construction of new theoretical models.

When systems become complex and detailed description becomes difficult, the symmetries may still shine through the complexities as a simplifying principle. In the study of the condensed matter physics the identification of important symmetries is often used as a guiding principle to obtain a correct description of the observed phenomena.

In the present case where we focus on the description of mechanical systems we have notice the possible use of symmetries as generates new solution s from old solutions of the equations of the motion. There is another important effect of symmetries that we will focus on, the connection between symmetries and constant of motion. Loosely speaking, to any (continuous) symmetry there is associated a conserved quantity. The study of constants of the motion is important for the following reasons. These constants may tell us something important about time evolution of the system even if we are not able to solve the full problem given by equations of motion. The presence of constants of motion may simplify the problems since they effectively reduce the number of variables of the system.

In symmetry, at ran for motion is symmetry if it satisfies the following:

1. The transformation preserves the structure.
2. The transformation is a diffeomorphism
3. The transformation maps are object to itself

4.3. Local One Parameter Point Transformations

To begin local one –parameter point transformation, we consider the following equation

$\bar{x} = G(x, \varepsilon)$ **(4-1)**

Be a family of one –parameter ε ∈R invertible transformation, of points $x = (x^1, \dots, x^n) \in R^n$ Into point$\bar{x} = (x^{-1}, \dots, x^{-n}) \in R^n$

These are known as one- parameter transformations, and subject to the conditions.

$\bar{x}|_{\varepsilon=0} = x$ This is $G(x, \varepsilon)|_{\varepsilon=0} = x$ $(\mathbf{4-2})$

Equation (4-1) using Taylor expansion in the some , neighborhood$\varepsilon = 0$, we get

$\bar{x} = x + \varepsilon_i \left(\frac{\partial G}{\partial \varepsilon_i}|_{\varepsilon=0}\right) + \frac{\varepsilon_i^2}{2}\left(\frac{\partial^2 G}{\partial \varepsilon_i^2}|_{\varepsilon=0}\right) + 0(\mathcal{E}^2)$ **(4-3)**

Putting

$$\xi(x) = \frac{\partial G}{\partial \varepsilon_i}|_{\varepsilon=0} \quad \textbf{(4 . 4)}$$

Reduces the expansion to

$$\bar{x} = x + \varepsilon\xi(x) + O(\epsilon^2) \quad \textbf{(4. 5)}$$

The expression

$$\bar{x} = x + \varepsilon\xi(x) \quad \textbf{(4. 6)}$$

Is called a local one –parameter point transformation and the component of $\varepsilon\xi(x)$ are the infinitesimals (4.1).

4.4. Local One Parameter Point Transformation Groups

The set G of transformation

$$\bar{x}_{\varepsilon i} = x + \varepsilon_i \left(\frac{\partial G}{\partial \varepsilon_i}|_{\varepsilon_i=0}\right) + \frac{\varepsilon_i^2}{2}\left(\frac{\partial^2 G}{\partial \varepsilon_i^2}|_{\varepsilon_i=0}\right) + \cdots, i = 1,2,3 \quad \textbf{(4.7)}$$

Becomes a group only when truncated at $O(\varepsilon^2)$

4.4.1. The Group Generator

The local one – parameter point transformation in equation (4.6) we can rewritten in the form

$$\bar{x} = x + \varepsilon\xi(x) \,.\nabla_x$$

So $\bar{x} = (1 + \varepsilon\xi(x).\nabla_x)$, an operator **(4. 8)**

$$G = \xi(x).\nabla \quad \textbf{(4. 9)}$$

These implies that

$$\bar{x} = \big(1 + \varepsilon\xi(x)\big)x \quad \textbf{(4. 10)}$$

An operator (4.9) has the expanded form

$$G = \sum_{K=1}^{N} \xi K \frac{\partial}{\partial x^k} \quad \textbf{(4. 11)}$$

4.4.2. Prolongation Formulas

The prolongation is happens when the function $F(x,y)$ dose not only depend on point $x$alone, but also on the derivatives. When this case is happened then we have use the prolonged form of operation G.

When

N = 2 with $x^1 = x$ and $x^2 = y$ reduces (4.11)

To

$$G = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} \qquad \mathbf{(4.12)}$$

In the deterring the prolongations, it is convenient to use the operator of total differentiation

$$D = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \cdots \qquad \mathbf{(4.13)}$$

Where $y' = \frac{dy}{dx}$, $y'' = \frac{d^2y}{dx^2}$

The derivative of the transformation point is

$\tilde{y}' = \frac{dy'}{dx'}$ Since

$$\bar{x} = x + \varepsilon\xi and \bar{y} + \varepsilon\eta \qquad \mathbf{(4.14)}$$

Then

$$\overline{Y}' = \frac{dy + \varepsilon\eta}{dx + \varepsilon d\xi} \qquad \mathbf{(4.15)}$$

So that

We can introduce the operator D:

$$\overline{Y}' = \frac{y' + \varepsilon D(\eta)}{1 + \varepsilon D(\xi)} \qquad \mathbf{(4.16)}$$

From the (4.16) implies that

$$\overline{Y}' = y' + \varepsilon(D(\eta) - y'D(C)) \qquad \mathbf{(4.17)}$$

Or $\bar{Y}' = y' + \varepsilon\xi'$ $\quad(\mathbf{4.18})$

With

$$\xi' = D(\eta) - y'D(\xi)$$

It expands into

$$\xi' = \eta_x + (\eta_y - \xi_x)y' - y'^2\xi_y \quad (\mathbf{4.19})$$

The first prolongation of G is

$$G''' = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \zeta'\frac{\partial}{\partial y'} \quad (\mathbf{4.20})$$

For the second prolongation, we have

$$\bar{Y}'' = \frac{y' + \varepsilon(D(\xi')}{1 + \varepsilon D(\xi)} \approx y'' + \varepsilon\zeta^2 \quad (\mathbf{4.21})$$

With $\zeta^2 = D(\zeta') - y''D(\xi)$this expands into

$\zeta^2 = \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - y'^3\xi_{yy} + (\eta_y - 2\xi_x - 3y'\xi_y)y''$ (4.22)

The second prolongation of G is

$$G^{|2|} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \zeta'\frac{\partial}{\partial y'} + \zeta^2\frac{\partial}{\partial y''} \quad (\mathbf{4.23})$$

Most applications movie up to second order, derivatives for this reasonable we pause here third order.

4.5. Generate Point Symmetries

4.5.1. One Dependent and Two Independent Variables

Considering the Equation:

$$u_t = u_{xx} \quad (\mathbf{4.24})$$

In order to generating point symmetries for equation (4.24) we first consider a change of variables from t, x and u to t^*, x^* and u^*involving an infinitesimal parameterε. A Taylors expansion in $\varepsilon = 0$ is

$$\bar{t} = t + \varepsilon T(t, x, u)$$

$$\bar{x} = x + \varepsilon\xi(t, x, u),$$

$$\bar{u} = u + \varepsilon\zeta(t, x, u) \quad \mathbf{(4.25)}$$

Differentiating (4.25) with respect to ε

$$\frac{\partial \bar{t}}{\partial \varepsilon}|_{\varepsilon=0} = T(t, x, u),$$

$$\frac{\partial \bar{x}}{\partial \varepsilon}|_{\varepsilon=0} = \xi(t, x, u),$$

$$\frac{\partial \bar{u}}{\partial \varepsilon}|_{\varepsilon=0} = \zeta(t, x, u) \quad \mathbf{(4.26)}$$

The tangent vector field (4.26) is associated with an operator

$$G = T\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \zeta\frac{\partial}{\partial u} \quad \mathbf{(4.27)}$$

This operator is called a symmetries generator this leads to the invariance condition

$$G^{|2|} = [F(t, x, u_t, u_x, u_{tx}, u_{tt}, u_{xx})]|_{[F(t,x,u_t,u_x,u_{tx},u_{tt},u_{xx})=0]} \quad \mathbf{(4.28)}$$

Where

$G^{|2|}$ is the second prolongation of G. It is obtained from:

$$G^{|2|} = G + \zeta_t^{'}\frac{\partial}{\partial u_t} + \zeta_x^{'}\frac{\partial}{\partial u_x} + \zeta_{tt}^{2}\frac{\partial}{\partial u_{tt}} + \zeta_{tx}^{2}\frac{\partial}{\partial u_{tx}} + +\zeta_{xx}^{'}\frac{\partial}{\partial u_{xx}} \quad \mathbf{(4.29)}$$

Where

$$\zeta_t^{'} = \frac{\partial g}{\partial t} + u\frac{\partial f}{\partial t} + \left[F - \frac{\partial T}{\partial t}\right]u_t - \frac{\partial \xi}{\partial x}u_x$$

$$\zeta_x^{'} = \frac{\partial g}{\partial x} + u\frac{\partial f}{\partial x} + \left[F - \frac{\partial \xi}{\partial x}\right]u_x - \frac{\partial T}{\partial t}u_t$$

$$\zeta_{tt}^{2} = \frac{\partial^2 g}{\partial t^2} + u\frac{\partial^2 F}{\partial t^2} + \left[2\frac{\partial F}{\partial t} - \frac{\partial^2 F}{\partial t^2}\right]u_t - \frac{\partial^2 \xi}{\partial t^2}u_x + \left[F - 2\frac{\partial T}{\partial t}\right]u_{tt} - 2\frac{\partial \xi}{\partial t}u_{tx}$$

$$\zeta_{xx}^{2} = \frac{\partial^2 g}{\partial x^2} + u\frac{\partial^2 F}{\partial x^2} + \left[2\frac{\partial F}{\partial x} - \frac{\partial^2 \xi}{\partial x^2}\right]u_x - \frac{\partial^2 T}{\partial x^2}u_t + \left[F - 2\frac{\partial T}{\partial x}\right]u_{xx-}2\frac{\partial T}{\partial x}u_{tx}$$

And

$$\zeta^2{}_{tx} = \frac{\partial^2 g}{\partial t \partial x} + u\frac{\partial^2 F}{\partial t \partial x} + \left[2\frac{\partial F}{\partial x} - \frac{\partial^2 T}{\partial t \partial x}\right]u_t + \left[2\frac{\partial F}{\partial t} - \frac{\partial^2 \xi}{\partial t \partial x}\right]u_x$$
$$- \left[F - \frac{\partial T}{\partial t} - \frac{\partial^2 \xi}{\partial x}\right]u_{tx} - \frac{\partial T}{\partial t}u_{tt} - \frac{\partial \xi}{\partial t}u_{xx}$$

4.5.2. One Dependent and Three Independent Variables

In order to generate point symmetries for equation $u_t = u_{xx} + u_{yy}$, we first consider change of variables from $t, x, y$and u to t^*, x^*, y^* and u^* involve an infinitesimal parameter ε.use Taylor's series in ε near $\varepsilon = 0$ yields.

$$\bar{t} = t + \varepsilon T(t, x, y, u),$$
$$\bar{x} = x + \varepsilon \xi(t, x, y, u),$$
$$\bar{y} = y + \varepsilon \varphi(t, x, y, u),$$
$$\bar{u} = u + \varepsilon \zeta(t, x, y, u) \qquad (\mathbf{4.30})$$

$$\frac{\partial \bar{t}}{\partial \varepsilon}|_{\varepsilon=0} = T(t, x, y, u),$$
$$\frac{\partial \bar{x}}{\partial \varepsilon}|_{\varepsilon=0} = \xi(t, x, y, u),$$
$$\frac{\partial \bar{y}}{\partial \varepsilon}|_{\varepsilon=0} = \varphi(t, x, y, u),$$
$$\frac{\partial \bar{u}}{\partial \varepsilon}|_{\varepsilon=0} = \zeta(t, x, y, u) \qquad (\mathbf{4.31})$$

The tangent vector field (4.31) is associated with an operator:

$$G = T\frac{\partial}{\partial x} + \xi\frac{\partial}{\partial x} + \varphi\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial u} \qquad (\mathbf{4.32})$$

Called asymmetry generator this in turn leads to the invariant condition

$$G^{|2|} = \left[F(t, x, u_t, u_y, u_{ty}, u_{tt}, u_{yy})\right]|_{[F(t,x,u_t,u_x,u_{ty},u_{tt},u_{yy})=0]}$$
$$= 0 \qquad (\mathbf{4.33})$$

Where $G^{|2|}$is the second prolongation of G. it is obtained from the formulas:

$$\zeta_t^{'} = \frac{\partial g}{\partial t} + u\frac{\partial f}{\partial t} + \left[F - \frac{\partial T}{\partial t}\right]u_t - \frac{\partial \xi}{\partial x}u_x - u_y\frac{\partial \varphi}{\partial t}$$

$$\zeta_x' = \frac{\partial g}{\partial x} + u\frac{\partial f}{\partial x} + \left[F - \frac{\partial \xi}{\partial x}\right]u_x - \frac{\partial T}{\partial t}u_t - u_y\frac{\partial \varphi}{\partial t}$$

$$\zeta_y' = \frac{\partial g}{\partial t} + u\frac{\partial f}{\partial y} + \left[F - \frac{\partial \varphi}{\partial y}\right]u_y - \frac{\partial \varphi}{\partial t}u_t - u_x\frac{\partial T}{\partial y}$$

$$\zeta_{tt}^2 = \frac{\partial^2 g}{\partial t^2} + u\frac{\partial^2 F}{\partial t^2} + \left[2\frac{\partial F}{\partial t} - \frac{\partial^2 F}{\partial t^2}\right]u_t - \frac{\partial^2 \xi}{\partial t^2}u_x - \frac{\partial^2 \xi}{\partial t^2}u_y + \left[F - 2\frac{\partial T}{\partial t}\right]u_{tt}$$

$$- 2\frac{\partial \xi}{\partial t}u_{tx} - 2\frac{\partial \varphi}{\partial t}u_{yt}$$

$$\zeta_{xx}^2 = \frac{\partial^2 g}{\partial x^2} + u\frac{\partial^2 F}{\partial x^2} + \left[2\frac{\partial F}{\partial x} - \frac{\partial^2 \xi}{\partial x^2}\right]u_x\frac{\partial^2 \varphi}{\partial x^2}u_y - \frac{\partial^2 T}{\partial x^2}u_t + \left[F - 2\frac{\partial \xi}{\partial x}\right]u_{xx}$$

$$- 2\frac{\partial \varphi}{\partial x}u_{xy} - 2\frac{\partial T}{\partial x}u_{tx}$$

$$\zeta_{tx}^2 = \frac{\partial^2 g}{\partial t\partial x} + u\frac{\partial^2 F}{\partial t\partial x} + \left[\frac{\partial F}{\partial x} - \frac{\partial^2 F}{\partial t\partial x}\right]u_t + \left[2\frac{\partial F}{\partial t} - \frac{\partial^2 \xi}{\partial t\partial x}\right]u_x - u_y\frac{\partial^2 \varphi}{\partial t\partial x}$$

$$- \left[2F - 2\frac{\partial T}{\partial t} - \frac{\partial T}{\partial x}\right]u_{tx} - \left[\frac{\partial \xi}{\partial t} - \frac{\partial \xi}{\partial x}\right]u_{xx} - \left[2\frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial x}\right]u_{xy}$$

$$\zeta_{ty}^2 = \frac{\partial^2 g}{\partial t\partial y} + u\frac{\partial^2 F}{\partial t\partial y} + \left[2\frac{\partial F}{\partial y} - \frac{\partial^2 \varphi}{\partial t\partial y}\right]u_y - \frac{\partial^2 \xi}{\partial t\partial y}u_x - \frac{\partial^2 T}{\partial y^2}u_t$$

$$- \left[2\frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial y}\right]u_{yy} - 2\left[\frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial y}\right]u_{xy} + \left[2F - 2\frac{\partial T}{\partial t} - \frac{\partial T}{\partial y}\right]u_{yt}$$

$$\zeta_{yy}^2 = \frac{\partial^2 g}{\partial y^2} + u\frac{\partial^2 F}{\partial y^2} + \left[2\frac{\partial F}{\partial y} - \frac{\partial^2 \varphi}{\partial y^2}\right]u_y - \frac{\partial^2 \xi}{\partial y^2}u_x - \frac{\partial^2 T}{\partial y^2}u_t$$

$$+ \left[F - 2\frac{\partial \varphi}{\partial y}\right]u_{yy} - 2\left[\frac{\partial \xi}{\partial y}\right]u_{xy} + \left[2F - 2\frac{\partial T}{\partial y}\right]u_{yt} \qquad (\mathbf{4.34})$$

4.5.3. One Dependent and N Independent Variables

The local one – parameter point transformation

$$x = x_i(x,u,\varepsilon) = x_i + \varepsilon\xi(x,u) + O(\varepsilon^2),$$

$$u = u_i(x,u,\varepsilon) = u_i + \varepsilon\eta(x,u) + O(\varepsilon^2), i = 1, \dots., n \qquad (\mathbf{4.35})$$

Acting on (x,u) space has generator

$$X = \xi_i(x,u)\frac{\partial}{\partial xi} + \eta(x,u)\frac{\partial}{\partial u} \qquad (4.36)$$

The Kth extended infinitesimals are given by

$$\xi(x,u), \eta(x,u), \eta^{(1)}\big(x,u,\partial u, \dots, \eta(x,u,\partial u, \dots, \partial u)\big) \qquad (\mathbf{4.37})$$

And the corresponding Kth extended generator is

$$x^{(k)} = x_i\frac{\partial}{\partial xi} + \eta\frac{\partial}{\partial u}\zeta_i^1\frac{\partial}{\partial ui} + \cdots + \zeta_i^k\frac{\partial}{\partial ui} \ , l = 1, \dots. n \qquad (\mathbf{4.38})$$

4.5.4. Dependent and N Independent Variables

We consider the case of n independent variables $x = (x^1, \dots, x^n)$ and m dependent variables $u(x) = (u^1(x), \dots, u^n(x))$ with partial derivatives denoted by

$u_i^\mu = \frac{\partial u^\mu}{\partial xi}$ The notation

$$\partial^p u = \partial^1 u = u_1^1(x), \dots, u_n^1(x) \dots. u_1^m(x) \dots u_n^m(x)$$

Denotes the set of all first order partial derivatives

$$\partial_u^p = \{u_{i,\dots ip}^p|_{\mu=1} = i_1 \dots i_p = 1 \dots n\}$$

$$\frac{\partial^p u_{(x)}^\mu}{\partial x_{1\dots}^i \partial x^{ip}}|_{\mu=1} = i_1 \dots i_p = 1 \dots n$$

Denotes the set all partial derivatives of order P point transformations of the form

$$\bar{x} = f(x,u),$$

$$\bar{u} = g(x,u) \qquad (\mathbf{4.39})$$

Acting on the $n+m$ dimensional space (x,y) has as its p^{th}extended transformation

$$(\bar{x})^i = f^i(x,u),$$

$$(\bar{u})^\mu = g^\mu(x,u),$$

$$\left(u_i^{-\mu}\right) = h_i^\mu(x,u,\partial u),$$

$$\left(u_{i1\ldots ip}^{-\mu}\right) = h_{i,\ldots i_p}^\mu(x,u,\partial u,\ldots\partial^p u) \qquad (\mathbf{4.40})$$

With $i = i_1,\ldots,i_p, \mu = 1\,,\ldots,m = \frac{\partial u^{-\mu}}{\partial(x)^i}$

The trans for med components of the first order derivatives are determined by

$$\begin{bmatrix} (\bar{u})_1^\mu \\ (\bar{u})_2^\mu \\ \cdot \\ \cdot \\ \cdot \\ (\bar{u})_n^\mu \end{bmatrix} = \begin{bmatrix} h_1^\mu \\ h_2^\mu \\ \cdot \\ \cdot \\ \cdot \\ h_n^\mu \end{bmatrix} = A^{-1} \begin{bmatrix} D_1 g^\mu \\ D_2 g^\mu \\ \cdot \\ \cdot \\ \cdot \\ D_n g^\mu \end{bmatrix}$$

Where A^{-1} is the inverse of the matrix

$$A = \begin{bmatrix} D^1 f^1 & \ldots & D_1 f^n \\ \vdots & \ddots & \vdots \\ D_n f^1 & \ldots & D_n f^n \end{bmatrix}$$

In term of the total derivatives operator

$$D_i = \frac{\partial}{\partial x^i} + u_i^\mu \frac{\partial}{\partial u^\mu} + u_{iii}^\mu \frac{\partial}{\partial u_{i1}^\mu} + \cdots i = 1,\ldots,n$$

The transformed components of the higher –order derivatives are determined by

$$\begin{bmatrix}(\bar{u})^{\mu}_{i,\dots ip1}\\(\bar{u})^{\mu}_{i,\dots ip2}\\.\\.\\.\\(\bar{u})^{\mu}_{i,\dots ipn}\end{bmatrix}=\begin{bmatrix}h^{\mu}_{i,\dots ip1}\\h^{\mu}_{i,\dots ip2}\\.\\.\\.\\h^{\mu}_{i,\dots ipn}\end{bmatrix}=A^{-1}\begin{bmatrix}D_1h^{\mu}_{i,\dots ip1}\\D_2h^{\mu}_{i,\dots ip2}\\.\\.\\.\\D_nh^{\mu}_{i,\dots ipn}\end{bmatrix}$$

The situation where the point transformation (4.39) is a one –parameter group of transformation give by

$$x^i=f^i(x,u,\varepsilon)=x^i+\varepsilon\xi^i(x,u)+O(\varepsilon^2),i=1,\dots,n$$

$$u^{\mu}=g^{\mu}(x,u,\varepsilon)=u^{\mu}+\varepsilon\eta^{\mu}(x,u)+O(\varepsilon^2),\mu=1,\dots,m \qquad (\mathbf{4.41})$$

Will have the corresponding generator given by

$$X=\xi^i(x,u)\frac{\partial}{\partial i}+\eta^{\mu}(x,u)\frac{\partial}{\partial u^{\mu}} \qquad (\mathbf{4.42})$$

4.6. Lie Group of the Heat Equation

In this section we consider the symmetry analysis and Lie symmetries of one –dimensional and two dimensional heat equation, also we find the invariant solutions of certain symmetry generator of heat equations

4.6.1.One-Dimensional Heat Equation

Consider the heat equation given by

$$u_{xx}-u_t=0 \qquad (\mathbf{4.43})$$

Let x and t two independent variables, and u a dependent variable The total derivatives are given by

$$D_x=\frac{\partial}{\partial x}+u_x\frac{\partial}{\partial u}+u_{xx}\frac{\partial}{\partial u_x}+u_{xt}\frac{\partial}{\partial u_t}+\cdots$$

$$D_t=\frac{\partial}{\partial t}+u_t\frac{\partial}{\partial u}+u_{tt}\frac{\partial}{\partial u_t}+\cdots$$

The infinitesimal generator is given by

$$X = T(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} \qquad \mathbf{(4.44)}$$

The second prolongation of X is define by

$$X^{|2|} = x + \zeta_x^1\frac{\partial}{\partial u_x} + \zeta_t^1\frac{\partial}{\partial u_t} + \zeta_{xx}^2\frac{\partial}{\partial u_{xt}} + \zeta_{xt}^2\frac{\partial}{\partial u_{xt}} + \zeta_{tt}^2\frac{\partial}{\partial u_{tt}} \qquad \mathbf{(4.45)}$$

The coefficients $\zeta_x^1, \zeta_t^1, \zeta_{xx}^2, \zeta_{tt}^2, \zeta_{tx}^2$ are define by

$$\zeta_x^1 = D_x(\eta) - u_x D_x(\xi) - u_t D_t(T) = g_x + f_x u + u_x(F - \xi_x) - u_t T_x$$

$$\zeta_t^1 = D_t(\eta) - u_x D_t(\xi) - u_t D_t(T) = g_t + f_t u + u_t(F - T_t) - u_x \xi_t$$

$$\zeta_{xx}^2 = D_x(\zeta_x^1) - u_{xx} D_{xt}(\xi) - u_t D_x(T)$$
$$= g_{xx} + u\, f_{xx} + u_x(2F_x - \xi_{xx}) - u_t T_{xx} + u_{xx}(F - 2\xi_x)$$
$$- 2u_{xt} T_x$$

$$\zeta_{xt}^2 = D_t(\zeta_x^1) - u_{xx} D_t(\xi) - u_{xt} D_x(T)$$
$$= g_{xt} + f_{xt} u + u_x(2F_t - \xi_{xt}) - u_t(F_x - T_{xt}) - \xi_t u_{xx}$$
$$+ (f - \xi_x - T_t)u_{tx} - T_x u_{tt}$$

$$\zeta_{tt}^2 = D_t(\zeta_t^1) - u_{xt} D_t(\xi) - u_{tt} D_t(T)$$
$$= g_{tt} + f_{tt} u + u_t(\xi_{tt}) - u_t(2F_t - T_{tt}) - 2u_{xt}\xi_t + u_{tt}(f - 2T_t)$$

4.6.2. Symmetries of One Dimensional Heat Equation

The deterring equation is obtained from invariance condition

$$\left(T(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} + \zeta_x^1\frac{\partial}{\partial u_x} + \zeta_t^1\frac{\partial}{\partial u_t} + \zeta_{xx}^2\frac{\partial}{\partial u_{xt}} + \zeta_x^1\frac{\partial}{\partial u_{xt}} + \zeta_{tt}^1\frac{\partial}{\partial u_{tt}}\right)(u_t - u_{tt})\,\Big|_{u_t = u_{tt} = 0})$$

Where

$(\zeta_t^1 - \zeta_{tt}^2)\,\Big|_{u_t = u_{tt} = 0} \rightarrow (4.46)$

After substituting ζ_t^1 , ζ_{xx}^2 and $u_t = u_{xx}$ in equation **(4.46)**

We get $g_t + F_t u(F - T_t) - u_x \xi_t - [g_{xx} + uF_{xx} + u_x(2F_x - \xi_{xx}) - u_t T_{xx} + u_t(F - 2\xi_x) - 2u_{xx}T_t] = 0 \;\; \rightarrow$ (4.47)

Separate coefficients in (4.47) having the following monomials

$$C = g_t - g_{xx} = 0 \qquad \mathbf{(4.48)}$$

$$U = f - f_{xx} = 0 \qquad \mathbf{(4.49)}$$

$$u_t = T_t - T_{xx} - 2\xi_x = 0 \qquad \mathbf{(4.50)}$$

$$u_x = \xi_t - \xi_{xx} - 2f_x = 0 \qquad \mathbf{(4.51)}$$

$$u_{xt} = T_x = 0 \qquad \mathbf{(4.52)}$$

Integrating equation (4.53) with respect to x we get

$$T = \alpha(t) \qquad \mathbf{(4.53)}$$

Substituting T in to (4.50) and integrating with respect to x we obtain

$$\xi_t = \frac{1}{2}\alpha_{tt}x + b(t) \qquad \mathbf{(4.54)}$$

Differentiating (4.54) with respect to t we get

$$\xi_t = \frac{1}{2}\alpha_{tt}x + b_t \qquad \mathbf{(4.55)}$$

Substituting ξ_t in (4.51) and integrating with respect to x yields

$$F = \frac{1}{8}\alpha_{tt}x^2 - \frac{1}{2}b_t x + c(x) \qquad \mathbf{(4.56)}$$

Substituting f in (4.49) we obtain

$$-\frac{1}{8}\alpha_{ttt}x^2 - \frac{1}{2}b_{tt}x + c_t + \frac{1}{4}\alpha_{tt} = 0 \qquad \mathbf{(4.57)}$$

Splitting equation (4.57) with respect to the power x we get

$$X^2 = \alpha_{ttt} = 0$$

$$X^1 = b_{tt} = 0$$

$$X^0 = c_t + \frac{1}{4}\alpha_{tt} = 0$$

And integrating three equation with respect to respectively yields

$$\alpha(t) = \frac{A}{2}t^2 + A_2 t + A_3 \qquad (\mathbf{4.58})$$

$$b(t) = A_4 t + A_5 \qquad (\mathbf{4.59})$$

$$c(t) = -\frac{1}{4}A_1 t + A_6 \qquad (\mathbf{4.60})$$

The infinitesimals

$$T = \propto_1 t^2 + 2 \propto_2 t + \propto_3 \qquad (\mathbf{4.61})$$

$$\xi = \propto_1 tx^2 + 2 \propto_2 x + \propto_4 t + \propto_5 \qquad (\mathbf{4.62})$$

$$f = -\frac{1}{4} \propto_1 x^2 - \frac{1}{2} \propto_4 x - \propto_1 t + \propto_6 \qquad (\mathbf{4.63})$$

4.6.3. Symmetries

The equation

$$T\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial t} + (fu + g)\frac{\partial}{\partial t} \qquad (\mathbf{4.64})$$

The corresponding symmetries are given by

$$X_1 = \frac{\partial}{\partial t} \qquad (\mathbf{4.65})$$

$$X_2 = \frac{\partial}{\partial t} \qquad (\mathbf{4.66})$$

$$X_3 = x\frac{\partial}{\partial t} + 2t\frac{\partial}{\partial t} \qquad (\mathbf{4.67})$$

$$X_4 = xt\frac{\partial}{\partial t} + t^2\frac{\partial}{\partial t} - (\frac{1}{4}x^2 + \frac{1}{2}u)u\frac{\partial}{\partial u} \qquad (\mathbf{4.68})$$

$$X_5 = t\frac{\partial}{\partial t} - \frac{1}{2}xt\frac{\partial}{\partial u} \qquad (\mathbf{4.69})$$

$$X_6 = u\frac{\partial}{\partial u} \qquad (\mathbf{4.70})$$

$$X_\infty = g\frac{\partial}{\partial u} \qquad \textbf{(4.71)}$$

4.6.4. Invariant Solutions

Useful tools of the symmetries groups that conserve the set of all solutions in the differential equations admitting these groups .That is the symmetries transformation simply permute those integrals curves among themselves. Such integral curves are termed invariant solutions.

4.7. Theorem

A function F(x , y) is called an invariant of the group G if and only if solving the following first – order linear differential equations.

$$XF = \xi(x,y)\frac{\partial F}{\partial x} + \eta(x,y)\frac{\partial F}{\partial y} \qquad \textbf{(4.72)}$$

$$X(t,x,u)\frac{\partial u}{\partial x} + T(t,x,u)\frac{\partial u}{\partial t} = \eta(t,x,u) \qquad \textbf{(4.73)}$$

Is the general partial differential equation of invariant surface, with the following characteristic equation?

$$\frac{dx}{x(t,x,u)} = \frac{dt}{T(t,x,u)} = \frac{du}{\eta(t,x,u)} \qquad \textbf{(4.74)}$$

For the symmetry equation (s_5) characteristic equation is

$$\frac{xdx}{2t} + \frac{du}{u} = 0 \qquad \textbf{(4.75)}$$

Integrating (4.68) we obtain

$$u = \beta(t)e^{\frac{-x^2}{4t}} \qquad \textbf{(4.76)}$$

Differentiating (4.76) with respect to t and twice with respect to x respectively we obtain

$$u_t = \beta'(t)e^{\frac{-x^2}{4t}} + \beta(t)e^{\frac{-x^2}{4t}}\frac{x^2}{4t^2} \qquad \textbf{(4.77)}$$

$$u_{xx} = \frac{x^2}{4t^2} e^{\frac{-x^2}{4t}} \quad \beta(t) - \frac{1}{2} t e^{\frac{-x^2}{4t}} \beta(t) \qquad \mathbf{(4.78)}$$

Substituting u_t and u_{xx} in (4.43) yields

$$\beta'(t) + \frac{1}{2}\beta(t) = 0 \qquad \mathbf{(4.79)}$$

Hence the solution is

$$u = \frac{k}{t^{1/2}} e^{\frac{-x^2}{4t}} \qquad \mathbf{(4.80)}$$

Finding the invariant solution the symmetries equation (s_4) is used since it contains (x, t, u) then the invariance condition becomes

$$xtu_x + t^2 u_t = \left(\frac{1}{4}x^2 + \frac{1}{2}t\right)u \qquad \mathbf{(4.81)}$$

The corresponding characteristic equation are given by

$$\frac{dx}{xt} = \frac{dt}{t^2} = \frac{du}{\left(\frac{1}{4}x^2 + \frac{1}{2}t\right)u} \qquad \mathbf{(4.82)}$$

By separation of variables and integration the solution of the characteristic equation yields two invariant of

x_4 , $\zeta \ = \ \frac{x}{t}$ and $v = \sqrt{te^{\frac{2x}{4x}}}$u

Then the solution of the invariant surface equation (4.81) is given by the invariant form

$$\sqrt{te^{\frac{x}{4t}}} = \phi\left(\frac{x}{t}\right)$$

$$u = \phi(\mathrm{x}, \mathrm{t}) = \frac{1}{\sqrt{\mathrm{t}}} e^{\frac{2x}{4t}} \phi(\zeta) \qquad \mathbf{(4.83)}$$

Finding the solution:

$$u_t = \frac{x^2 u}{4t^2} -$$

$$\frac{u}{2t} - \phi\left(\frac{x}{t}\right)\left(\frac{xt}{t^2}\right) \quad (4.84)$$

$$u_{xx} = \ddot{\emptyset}\left(\frac{x}{t}\right)\left(\frac{u}{t^2}\right) - \dot{\emptyset}\left(\frac{x}{t}\right)\left(\frac{xu}{t^2}\right) + \left(\frac{x^2}{4t^2} - \frac{1}{2t}\right)u \qquad (\mathbf{4.85})$$

Substituting u_t and u_{xx} in (4.83) we obtain

$$\phi^{\cdot\cdot}\left(\frac{x}{t}\right)\frac{u}{t^2} = 0 \qquad (\mathbf{4.86})$$

Hence $\phi^{\cdot\cdot} = 0$

Solving the differential equation obtain

$$U = \frac{1}{\sqrt{t}}[C_1 + C_2]e^{\frac{-x^2}{4t}} \qquad (\mathbf{4.87})$$

4.8. Two Dimensional Heat Equations

Consider two dimensional head equation

$$u_t - u_{tt} - u_{xx} = 0 \qquad (\mathbf{4.88})$$

The dependent variable is u and independent variable are t , x and y .

The infinitesimal generator is given by

$$X = T(t,x,y,u)\frac{\partial}{\partial t} + \xi(t,x,y,u)\frac{\partial}{\partial x} + \varphi s(t,x,y,u)\frac{\partial}{\partial y} + \zeta(t,x,y,u)\frac{\partial}{\partial u} \qquad (\mathbf{4.89})$$

The second prolongation of X is

$$e^{|2|} = x + \left(\zeta_t^1\frac{\partial}{\partial u_t} + \zeta_x^1\frac{\partial}{\partial u_x} + \zeta_{tt}^2\frac{\partial}{\partial u_{tt}} + \zeta_y^1\frac{\partial}{\partial u_y} + \zeta_{tx}^2\frac{\partial}{\partial u_{tx}} + \zeta_{ty}^2\frac{\partial}{\partial u_{ty}} + \zeta_{xx}^2\frac{\partial}{\partial u_{xx}} + \zeta_{xy}^2\frac{\partial}{\partial u_{xy}} + \zeta_{yy}^2\frac{\partial}{\partial u_{yy}}\right)$$

With invariance condition

$$X^{|2|}(u_t + u_{xx} + u_{yy})|_{(u_t - u_{xx} - u_{yy})=0} = 0 \qquad (\mathbf{4.90})$$

That yields

$$\zeta_t^1 - \zeta_{xx}^2 - \zeta_{yy}^2|_{(u_t - u_{xx} - u_{yy})=0} = 0 \quad (4.91)$$

Where ζ_t^1 , ζ_{xx}^2 and ζ_{yy}^2 are substituting in (4.91)

$$\zeta_t^1 = \frac{\partial g}{\partial t} + u\frac{\partial f}{\partial t} + \left[f - \frac{\partial T}{\partial t}\right]u_t - \frac{\partial \xi}{\partial x}u_x - u_y\frac{\partial \varphi}{\partial t}$$

$$\zeta_x^1 = \frac{\partial g}{\partial t} + u\frac{\partial f}{\partial x} + \left[f - \frac{\partial \xi}{\partial x}\right]u_x - \frac{\partial T}{\partial t} - u_y\frac{\partial \varphi}{\partial t}$$

$$\zeta_y^1 = \frac{\partial g}{\partial t} + u\frac{\partial f}{\partial y} + \left[f - \frac{\partial \varphi}{\partial y}\right]u_y - \frac{\partial \varphi}{\partial t}u_t - u_x\frac{\partial \xi}{\partial y} - u_t\frac{\partial T}{\partial y}$$

$$\zeta_{tt}^2 = \frac{\partial^2 g}{\partial t^2} + u\frac{\partial^2 f}{\partial t^2} + \left[2\ \frac{\partial F}{\partial t} - \frac{\partial^2 T}{\partial t^2}\right]u_t - \frac{\partial^2 \xi}{\partial t^2}u_x - \frac{\partial^2 \xi}{\partial t^2}u_y + \left[t - 2\frac{\partial T}{\partial t}\right]u_{tt}$$

$$- 2\frac{\partial \xi}{\partial t}u_{tx} - 2\frac{\partial \varphi}{\partial t}u_{yt}$$

$$\zeta_{tx}^2 = \frac{\partial^2 g}{\partial t\partial x} + u\frac{\partial^2 f}{\partial t\partial x} + \left[\frac{\partial F}{\partial t} - \frac{\partial^2 T}{\partial t\partial x}\right]u_t + \left[\ 2\frac{\partial F}{\partial t} - \frac{\partial^2 \xi}{\partial t\partial x}\right]u_x - u_y\frac{\partial^2 \varphi}{\partial t\partial x}$$

$$- \left[2f - 2\frac{\partial T}{\partial t} - \frac{\partial T}{\partial x}\right]u_{tx} - \left[\frac{\partial \xi}{\partial t} - \frac{\partial \xi}{\partial x}\right]u_{xx} - 2\left[\frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial x}\right]u_{xy}$$

$$\zeta_{ty}^2 = \frac{\partial^2 g}{\partial t\partial y} + u\frac{\partial^2 f}{\partial t\partial y} + \left[\frac{2\partial F}{\partial y} - \frac{\partial^2 \varphi}{\partial t\partial y}\right]u_y - \frac{\partial^2 \xi}{\partial t\partial y}u_x - \frac{\partial^2 T}{\partial y^2}u_t$$

$$- \left[2\ \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial y}\right]u_{yy} - \left[2\ \frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial y}\right]u_{xy} + \left[2f - 2\frac{\partial T}{\partial t} - \frac{\partial T}{\partial y}\right]u_{yt}$$

$$\zeta_{yy}^2 = \frac{\partial^2 g}{\partial y^2} + u\frac{\partial^2 f}{\partial y^2} + \left[2\ \frac{\partial F}{\partial y} - \frac{\partial^2 \varphi}{\partial y^2}\right]u_y - \frac{\partial^2 \xi}{\partial y^2}u_x - \frac{\partial^2 \xi}{\partial y^2}u_t$$

$$+ \left[t - 2\frac{\partial \varphi}{\partial y}\right]u_{yy} - 2\left[\frac{\partial \xi}{\partial y}u_{xy}\right] + \left[2F - 2\frac{\partial T}{\partial y}\right]u_{yt} \quad (4.92)$$

4.9. Determining Equation of Two Dimensional Heat Equation

Considering equation

$$\zeta_t^1 = \zeta_{xx}^2 - \zeta_{yy}^2 = 0 \quad (\mathbf{4.93})$$

Implies that

$$\frac{\partial g}{\partial t} - u\frac{\partial f}{\partial t} + \left[f - \frac{\partial T}{\partial t}\right]u_t - \frac{\partial \xi}{\partial x}u_x - u_y\frac{\partial \varphi}{\partial t}$$

$$= \frac{\partial^2 g}{\partial x^2} + u\frac{\partial^2 f}{\partial x^2} + \left[2\frac{\partial F}{\partial x} - \frac{\partial^2 \xi}{\partial x^2}\right]u_x - \frac{\partial^2 \varphi}{\partial x^2}u_y - \frac{\partial^2 T}{\partial x^2}u_t$$

$$+ \left[f - 2\frac{\partial \xi}{\partial x}\right]u_{xx} - 2\frac{\partial \varphi}{\partial x}u_{xy} - 2\frac{\partial T}{\partial x}u_{tx} + \frac{\partial^2 g}{\partial y^2} + u\frac{\partial^2 f}{\partial y^2}$$

$$+ \left[2\frac{\partial F}{\partial y} - \frac{\partial^2 \varphi}{\partial y^2}\right]u_y - \frac{\partial^2 \xi}{\partial y^2}u_x - \frac{\partial^2 T}{\partial y^2}u_t + \left[F - 2\frac{\partial \varphi}{\partial y}\right]u_{yy}$$

$$- 2\left[\frac{\partial \xi}{\partial y}\right]u_{xy} + \left[2f - 2\frac{\partial T}{\partial y}\right]u_{yt} \quad (\mathbf{4.94})$$

Compare coefficient of constant and u yields

$$C = g_t - g_{xx} - g_{yy}$$

$$= 0 \quad (\mathbf{4.95})$$

$$u = f_t - f_{xx} - f_{yy} = 0 \quad (\mathbf{4.96})$$

$$u_t = T_t - T_{xx} - T_{yy} = 0 \quad (\mathbf{4.97})$$

$$u_x = \xi_t - \xi_{xx} - \xi_{yy} = 0 \quad (\mathbf{4.98})$$

$$u_y = \varphi_t - \varphi_{xx} - \varphi_{yy} = 0 \quad (\mathbf{4.99})$$

$$u_{xx} = \xi_x - \varphi_y = 0 \quad (\mathbf{4.100})$$

$$u_{xy} = \varphi_x - \xi_y = 0 \quad (4.101)$$

$$u_{xt} = T_x = 0 \quad (4.102)$$

$$u_{yt} = T_y = 0 \quad (4.103)$$

Integrating (4.102) and (4.103) with respect to x and y respectively getting

$$T = \propto (t) \quad (4.104)$$

Substituting T in (4.97) and integrate with respect to y we obtain

$$\varphi = \frac{1}{2} \propto_t y + b(t, x) \qquad (4.105)$$

Differentiating G twice with respect to x and y

$$\varphi_{xx} = b_{xx}(t, x)\varphi_{yy} = 0 \qquad (4.106)$$

Differentiating φ with respect to y we get

$$\varphi_t = \frac{1}{2} \propto_{tt} y + b_t(t, x) = 0 \qquad (4.107)$$

Differentiating φs with respect to y and substituting in (4.100) and integrate with respect to x obtain

$$\xi = \frac{1}{2} \propto_t x + c(t, y) \qquad (4.108)$$

Differentiating ξ with respect to t and twice with respectively we obtain the following equations

$$\xi_t = \frac{1}{2} \propto_{tt} x + c_t(t, y) \qquad (4.109)$$

$$\xi_{xx} = 0 \, , \xi_{yy} = c_{yy}(t, y) \qquad (4.110)$$

Differentiating (4.110) with respect to x and y respectively we obtain the following equations

$$\varphi_{xx} = 0 \quad b_{xx}, \xi_{yy} = 0 = c_{yy} \qquad (4.111)$$

Integrating (4.111) with respect to x and y respectively we obtained

$$b = A_1 x + A_2 \qquad (4.112)$$

$$c = A_3 y + A_4 \qquad (4.113)$$

Substituting ξ_t , ξ_{xx} , ξ_{yy} in (4.98) and $\varphi_t, \varphi_{xx}, \varphi_{yy}$ in (4.99) and integrating with respect to x and y respectively we obtains

$$f = -\frac{1}{8} \propto_{tt} x^2 - \frac{1}{8} \propto_{tt} y^2 - \frac{1}{2} c_t(t,y)x - \frac{1}{2} b_t(t,y)y + d(t,x) + e(t,y) \quad (4.114)$$

Differentiating f with respect to t and twice with respect to x and y respectively we obtain

$$f_t = -\frac{1}{8} \propto_{ttt} x^2 - \frac{1}{8} \propto_{ttt} y^2 - \frac{1}{2} c_{tt}(t,y)x - \frac{1}{2} b_{tt}(t,x)y + d_t(t,x) + e_t(t,y) \quad (4.115)$$

$$f_{xx} = -\frac{1}{4} \propto_{tt} + d_{xx}(t,x), f_{yy} = -\frac{1}{4} \alpha_{tt} + c_{yy}(t,y) \quad (4.116)$$

Substituting f_t , f_{xx} in (4.96) yields

$$-\frac{1}{8} \propto_{tt} x^2 - \frac{1}{8} \propto_{tt} y^2 - \frac{1}{2} c_{tt}(t,y)x - \frac{1}{2} b_{tt}(t,x)y + d_t(t,x) + e_t(t,y) + \frac{1}{2} \propto_{tt} - c_{yy}(t,y) - d_{xx}(t,x) = 0 \quad (4.117)$$

Splitting (4.117) we obtains minimal equations

$:\propto_{tt}(t) = 0$ (4.118)

$:d_{xx}(t,x) = 0$ (4.119)

$:e_{yy}(t,y) = 0$ (4.120)

$:d_t(t,x) = 0$ (4.121)

$: e_t(t,y) = 0$ (4.122)

Integrating (4.121) , (4.122) and (4.120)

$$\alpha(t) = \frac{1}{2} A_5 t^2 + A_6 t + A_7 \quad (4.123)$$

$$d(x,t) = A_8 x + A_9 \quad (4.124)$$

$$e(x,t) = A_{10} y + A_{11} \quad (4.125)$$

The infinitesimals:

$$T = \beta_1 t^2 + 2\beta_2 t + \beta_3 \tag{4.126}$$

$$\xi = \beta_1 tx + 2\beta_2 x + \beta_4 y + \beta_5 \tag{4.127}$$

$$\varphi = \beta_1 ty + 2\beta_2 y + \beta_4 x + \beta_6 \tag{4.128}$$

$$f = -\frac{1}{4}\beta_1 (x^2 + y^2) + \beta_7 x + \beta_8 y + \beta_9 \tag{4.129}$$

Symmetries:

$$x_1 = t\frac{\partial}{\partial t} \tag{4.130}$$

$$x_2 = \frac{\partial}{\partial x} \tag{4.131}$$

$$x_3 = \frac{\partial}{\partial y} \tag{4.132}$$

$$x_4 = u\frac{\partial}{\partial u} \tag{4.133}$$

$$x_5 = xu\frac{\partial}{\partial u} \tag{4.134}$$

$$x_6 = yu\frac{\partial}{\partial u} \tag{4.135}$$

$$x_7 = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \tag{4.136}$$

$$x_8 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \tag{4.137}$$

$$x_9 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + ty\frac{\partial}{\partial y} - \frac{1}{4}(x^2 + y^2)u\frac{\partial}{\partial u} \tag{4.138}$$

$$\vdots$$

$$x_\infty = g\frac{\partial}{\partial u} \tag{4.139}$$

4.10. Invariant Solution

For $x_3 = \frac{\partial}{\partial y}$, the characteristic condition is giving by

$$x_3 I = \frac{\partial 1}{\partial y} \tag{4.140}$$

The characteristic equation is

$$\frac{dy}{1} = \frac{dx}{0} = \frac{du}{0} = \frac{dt}{0} \quad \rightarrow (4.141)$$

This implies that d$y = 0$, thus the invariant solution $u = \emptyset(t)$ or $= \emptyset(t)$, we substitute this in the original equation $u_t = u_{xx} + u_{yy}$ to get $\emptyset'(t) = 0$ thus $\emptyset(t) = k$

Similarly for $u = \emptyset(x)$ we have $u_{xx} = \ddot{\emptyset}(t) = 0$ which implies $\emptyset(x) = c_1 x + c_2 x$ then the invariant solution because

$$u = c_1 x + c_2 \tag{4.142}$$

For $x_2 = \frac{\partial}{\partial x}$

Similarly we obtain the invariant solution

$$u = A_1 y + A_2 \tag{4.143}$$

For

$$x_7 = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \tag{4.144}$$

The invariant condition is

$$x_7 I = y\frac{\partial I}{\partial x} + x\frac{\partial T}{\partial y} \tag{4.144}$$

The characteristic equation is given by

$$\frac{dx}{y} = \frac{dy}{x} = \frac{du}{0} = \frac{dt}{0} \tag{4.145}$$

Then $c = y^2 - x^2$

The first invariant $\psi_1 = y^2 - x^2$ and the second invariant $\psi_1 = \emptyset(t)$ the invariant this solution is $u = \emptyset(t)\ (y^2 - x^2)$ substituting this solution in the original equation obtain

$$\emptyset'(t)\ (y^2 - x^2) = 0, \emptyset'(t) = 0\ , \emptyset(t) = c \qquad (4.146)$$

Thus invariant solution

$$u = c(y^2 - x^2) \qquad (4.147)$$

For $x_9 = t^2 \frac{\partial}{\partial t} + tx \frac{\partial}{\partial x} + ty \frac{\partial}{\partial y} - \frac{1}{4}(x^2 + y^2)u\ \frac{\partial I}{\partial u}$ (4.148)

The invariant condition is given by

$$x_9 I = t^2 \frac{\partial I}{\partial t} + tx \frac{\partial I}{\partial x} + ty \frac{\partial I}{\partial y} - \frac{1}{4}(x^2 + y^2)u\ \frac{\partial I}{\partial u} \qquad (4.149)$$

The characteristic equation is giver by equation

$$\frac{dx}{t^2} = \frac{dy}{tx} = \frac{du}{dy} = \frac{du}{\frac{1}{4}(x^2 + y^2)u} \qquad (4.150)$$

From the equation (4.150)

$$\frac{dt}{t^2} = \frac{dx}{tx} \rightarrow (4.151)$$

Integrating the equation (4.151) leads to $\frac{x}{t} = \emptyset_1$ (4.152)

From the (4.150) we have

$$\frac{dt}{t^2} + \frac{du}{\frac{1}{4}(x^2 + y^2)u} \qquad (4.153)$$

Integrating (4.153) obtain

$$u = F\left(\frac{x}{t}\right) e^{\left(\frac{x^2+y^2}{4t}\right)} \qquad (4.154)$$

Differentiating (4.154) with respect to t and twice with respect to x and y respectively we obtain

$$u_t = -F\frac{x}{t^2}e^{\left(\frac{-x^2+y^2}{4t}\right)} - F\frac{x}{t}e^{\left(\frac{x^2+y^2}{4t}\right)}\left(\frac{x^2+y^2}{4t}\right) \qquad (4.155)$$

$$u_{xx} = F''\frac{x}{t^2}e^{\left(\frac{-x^2+y^2}{4t}\right)} + F'\frac{x}{t^2}e^{\left(\frac{x^2+y^2}{4t}\right)} + F\frac{1}{2t}e^{\left(\frac{x^2+y^2}{4t}\right)} \qquad (4.156)$$

$$u_{yy} = F''e^{\left(\frac{-x^2+y^2}{4t}\right)}\left(\frac{1}{2t}\right) + Fe^{\left(\frac{x^2+y^2}{4t}\right)}\left(\frac{y^2}{4t^2}\right) \qquad (4.157)$$

Substituting u_t , u_{xx} and u_{yy} in (4.88) yields

$$F''\frac{1}{t^2} + F'\frac{2x}{t^2} + F\left(\frac{1}{t} + \frac{x^2}{4t^2} + e^{\left(\frac{x^2+y^2}{4t}\right)}\right)$$

$$= 0 \qquad s \qquad (4.158)$$

The second order differential equation (4.158) reduces tos

$$F'' \propto +F'\beta + F\lambda = 0 \qquad (4.159)$$

From characteristic equation (4.150)

$$\frac{dt}{tx} = \frac{dy}{ty} \qquad (4.160)$$

Integrating (4.160) obtain

$$\frac{x}{y} = \varphi_2 \qquad (4.161)$$

From (4.150)

$$\frac{dy}{ty} + \frac{du}{\frac{1}{4}(x^2+y^2)u} = 0 \qquad (4.162)$$

Integrating (4.162) obtain

$$u = R(y)e^{\frac{(-2x^2 lny + y^2)}{8t}} \qquad (4.163)$$

Differentiating (4.163) with respect to x and twice with respect to x and y respectively yields

$$u_t = R(\frac{x}{y})e^{\frac{-(2x^2 lny+y^2)}{8t}}\frac{(2x^2 lny + y^2)}{8t^2}$$

$$u_{xx} = R''\frac{1}{y^2}e^{\frac{-(2x^2 lny+y^2)}{8t}} - R'\left(\frac{1}{y}e^{\frac{-(2x^2 lny+y^2)}{8t}}\left(\frac{xlny}{ty}\right)\right)$$

$$- R\left(e^{\frac{-(2x^2 lny+y^2)}{8t}}\frac{lny}{2t} + e^{\frac{-(2x^2 lny+y^2)}{8t}}\frac{x^2 lny^2}{4t^2}\right)$$

$$u_{yy} = R''\left(\frac{x^2}{y^4}e^{\frac{-(2x^2 lny+y^2)}{8t}}\right)$$

$$+ R'\left(\frac{2x}{y^3}e^{\frac{-(2x^2 lny+y^2)}{8t}} + \frac{x}{y^2}e^{\frac{-(2x^2 lny+y^2)}{8t}}\frac{(x^2+y^2)}{2t}\right)$$

$$+ R\left(e^{\frac{-(2x^2 lny+y^2)}{8t}}\frac{(x^2+y^2)}{4ty}\right) + e^{\frac{-(2x^2 lny+y^2)}{8t}}\left(\frac{1}{4t}\right)\frac{(x^2+y^2)}{y^2}$$

Substituting u_t , u_{xx} and u_{yy} in (4.88) we obtain

$$R''\left(\frac{1}{y^2} - \frac{x^2}{y^2}\right) - R'\left(\frac{xlny}{ty^2} - \frac{2x}{y^3} - \frac{x(x^2+y^2)}{2ty^2}\right)$$

$$- R\left(\frac{2x^2 lny + y^2}{8t} + \frac{lny}{4t^2} + \frac{x^2 lny}{4t^2} - \frac{(x^2+y^2)}{4ty} - \frac{x+y^2}{4ty^2}\right)$$

$$= 0 \qquad (4.164)$$

This reduce to

$$R''S - R'K - RY = 0 \qquad (4.165)$$

From [14] for two – dimensional Lie algebra spanned by x , y has invariant $r = \sqrt{x^2+y^2}$ and $v = u^{-kt}$ looking for invariant solution in the form $V = \phi(r)$ when $u = \phi(r)e^{kt}$substituting in (4.88) and multiplying by r the result equation because

$$r\emptyset'' + \emptyset' - Kr\emptyset = 0 \qquad (4.166)$$

Letting K < 0 , then setting K = - $\propto^2$ and$r' =\propto r$ the equation because Bessel function n $J_0(r')$ of order zero

$$r\emptyset'' + \emptyset' + r'\emptyset = 0 \qquad (4.167)$$

Where

$\emptyset = J_0(r')$ and the invariant solution is given by

$$u = J_0(\alpha r)e^{-\alpha^2 t}$$

Also we can obtain the solution of (4.159) and (4.165) similar to (4.167).

Chapter Five

Applications of Lie Algebra

5.1. Introduction

This chapter introduces basic concepts from representation theory, Lie group, Lie algebra, and topology and their applications in physics, particularly, in particle physic. The main focus will be on matrix groups, especially the special unitary groups and the special orthogonal groups. They play crucial roles in particle physics in modeling the symmetries of the subatomic particles of the many physical applications the chapter will introduce two concept phenomena known

Here, we define basic concept that will be used later on. Note that the scalar field of the vector space will be the complex number, C, unless mentioned otherwise.

5.1.1. Definitions

Give a vector space V over field C a norm on V is a function

$$P = V \rightarrow (R)$$

With the following properties for all $a \in C$ and all $x, y \in V$

i. $P(ax) = |a|P(x),$

ii. $P(x+y) = P(x) + P(y)$

iii. $P(x) = 0 \rightarrow x = 0$

The particular norm, which we will use in this chapter for a vector $x \in C^n$ will be denoted by $||x||$ and defined as following:

$$\left||x|\right| = \sqrt{\sum_{i=1}^{n} |x_i|^2}$$

Whereas the norm of nxn complex matrix, A, will be defined as following:

$$||A|| = \frac{sup||Ax||}{x \in c^n/\{0\}||x||}$$

It is not hard to check that these two norms satisfy the three properties mentioned above.

5.1.2. Definitions

Let V be a vector space will basic $\{vi\}_{i=1}^{n}$ then the symmetric product sym^2v is defined as

$sym^2v = V \otimes V/(vi \otimes vj - vj \otimes vi)$ With $1 \leq i,j \leq n$ the alternating product, A^2V is defined as

$A^2V = V \otimes V/(vi \otimes vjv - vj \otimes vi)$ With $1 \leq i,j \leq n$

5.1.3. Definition

A group is a set G together with an operation, $\therefore G \times G \rightarrow G$ satisfying four requirements known as the group axioms:

i. $\forall\, a,b, \in G\,, a\,.\,b \in G$

ii. $\forall\, a,b,c \in G\,, a.(b.c) = (a.b).c$

iii. $\exists\, e \in G$ such that $a.e = e.a = a \forall a \in G$

iv. $for\ each\ a \in G\,, \exists\, a^{-1} \in G$ such that $a.a^{-1} = a^{-1}.a = e$

5.2. Representation theory

Here, we define representation and its associated concepts and state some examples. Before starting, it could be helpful to first understand why the concept of representation is important. A representation can be thought of as an action of group on a vector space. Such actions can arise naturally in mathematics and physics, and it is important to study and understand them. For example, consider a differential equation in their –dimensional apace with rotational symmetry. If the equation has a rotational symmetry, then the solution space will be invariant under rotations. Thus, the solution space will constitute of are presentation of the rotation group so (3). Hence, knowing what all the representations of so (3) are, it is easy to narrow down what the solution space of the equation will be. In fact, one of the main applications of representation is exploiting the system's symmetry. In system with symmetry, the set of symmetries form a group, and the representation of this symmetry group allows you to use that symmetry to simplify the given system. We will see more of these applications, particularly in physics, in the final section.

5.2.1. Definition

A representation of a group G on a vector space V is defined as a homomorphism $\emptyset : G \to GL(v)$. To each $g \in G$, the representation map assigns a linear map $P_g : V \to V$. A though V is actually the representation space, one may, for short, refer to V as the representation of G.

5.2.2. Definition

A sub representation of a representation V is a vector subspace w of v, which is invariant under G. This means Im (w) = w under the action of each $g \in G$.

5.2.3. Definition

A representation V is called irreducible if there is no proper nonzero invariant subspace w of v. that is has no sub representation, except itself and the trivial space.

Now that the basic concepts are defined, we can look at some examples of representation.

5.2.4. Example

(Trivial representation) every element $g \in G$ gets mapped to the identify mapping between the vector space V and itself. Hence, all elements of G act as the identity on all$v \in V$.

5.2.5. Example

(Standard Representation) If we let G be S_n , the symmetric group on n elements, then G is obviously represented by vector space $V \cong C^n$ with n basis vector. Now, there is one dimensional sub representation W of V spanned by sum of the basis vector. The standard representation is n-1 irreducible representation V/W the equation space.

5.2.6. Example

(Dual Representation) Let $\emptyset$ be a representation of G on V for V, we can define its dual space V^* =Hom (V, C). We define

$\emptyset^*: G \to GL(V^*)$ by

$\emptyset^*(g) = \emptyset^t(g^{-1})$ For all $g \in G$. $\emptyset^*$ Is the dual representation . It is easy to check that this is actually a representation = let $g, h \in G$

And p_g, p_n be their associated mapping in$GL(V)$. Then,

$$P_{gh}^* = P_{(gh)^{-1}}^t = \left(P_{h^{-1}}\ o\ P_{g^{-1}}\right)^t = P_{g^{-1}}^t o\ P_{h^{-1}}^t = P_g^* o\ P_h^*$$

As we wanted

5.2.7. Example

If V, W is representation of G, the direct sum $V \otimes W$ and the tensor product $V \otimes W$ are also representation, the letter via

$$g(V \otimes W) = gv \otimes gw$$

For a representation V, the n^{th} tensor power $V^{\otimes n}$ is again a representation of G by this rule, and exterior powerV^n (v) and symmetric power sy $m^n(v)$ are sub representation of it.

5.3. Lie Group

Here, we introduce concept of Lie group, which plays crucial role in physics, particularly in studies of particle physics. We make a slight detour to introduce an application in physics and the necessary concept in topology.

5.3.1. Definition

A real Lie group is a group that is also a finite-dimensional real smooth manifold, in which the group operations of multiplication and inversion are smooth maps smoothness of the group multiplication.

$$\mu: G \times G \to G, \mu(x, y) = xy$$

Means that μ is a smooth mapping of the product $G \times G$ into G.

These two requirement can be combined to the single requirement that the mapping.

$$(x, y) \to x^{-1}y$$

Be a smooth mapping of $G \times G$ into G.

As defined above Lie group embodies three different forms of mathematic structure. Firstly, it has the group structure. Secondly, the elements of this group also form a “topological space“so that is may be described as being a special case of a “topological group” .finally, the elements also constitute an “analytic manifold”. Consequently, a Lie group may be defined in several different (but equivalent) ways, depending on degree of emphasis on its various a sepects in particular it can be defined as a topological group with certain analytic properties or, alternatively, as an analytic manifold with group properties. But, formulating in these ways would require many set of other definitions (such as manifold, smooth mapping, and etc), which may not be every important in understanding the applications of lie groups in physic.

In fact, we are mainly interested in a particular type of Lie group for problems in physic, the matrix Lie group, for which a straight forward definition can be give using the general linear group, $GL(n, c)$

5.3.2. Definition

The general linear group over the real numbers, denoted by $GL(n, R)$ is the group of all $n \times n$ invertible matrices with real number entries.

We can similarly define it over the complex numbers C denoted by $GL(n, c)$.

5.3.3. Definition

A matrix Lie group is any sub group H of $GL(n, c)$with the following properties. If A_n is any sequence of matrices in H and A_n converges to some matrix A, then either$A \in H$, or A is not invertible. A mounts to saying that H is a closed subset of $GL(n, c)$. Thus one can think matrix Lie group as simply a closed sub group of $GL(n, c)$.

5.3.4. Definition

Counter examples. An example of a sub group of $GL(n, c)$ in not closed is the set of all $n \times n$ invertible matrices whose entries are real and rational number. One can easily have a sequence of invertible matrices with rational number entries converging to an invertible matrix with some irrational number entries.

5.3.5. Example

(The general linear groups, $GL(n, c)$ and$GL(n, R)$) the general linear groups are themselves matrix Lie groups. Of course, $GL(n, c)$is a sub group of itself. Also, if A is a sequence of matrices in $GL(n, c)$ and A converges, to A, then by definition of (n, c) , either A is in $GL(n, c)$ or A is not invertible. Moreover, $GL(n, R)$ is a subgroup of $GL(n, c)$ and converges to A, then the entries of A are, of course, real. Thus, either A is not invertible, or A $\in$ $GL(n, c)$.

5.3.6. Example

The special linear groups $SL(n, c)$ and $SL(n, R)$ the special linear groups is group of $n \times n$ invertible matrices having determent1. Since determent is a continuous function, if a sequence A_n in SL (n, c) converges to A, then A also has a determinant 1 and$A \in SL(n, c)$.

5.3.7. Example

The orthogonal and special orthogonal group O(n) and $so(n)$

A $n \times n$ matrix A is orthogonal if the column vectors that makeup A are orthogonal, that is, if

$$\sum_{L=i}^{n} A_{lj}\, A_{ik} = \delta_{jk}$$

Equivalently, A is orthogonal if it preserves inner product, namely if

$$\langle x,y \rangle = \langle Ax,Ay \rangle$$

For all $x, y \in R^n$. Another equivalent definition is that A is orthogonal if $A^t A = I$, $i.e$ if $A^t = A^{-1}$. Since $detA^t = \det A$ if A is orthogonal, then $= \pm 1$. Hence, orthogonal matrix must be invertible. Furthermore, if A is an orthogonal matrix, then

$$\langle A^{-1}x, A^{-1}y \rangle = \langle A(A^{-1}x, A(A^{-1}y)) \rangle = \langle x,y \rangle$$

Thus, the inverse is also orthogonal. Also, the product of two orthogonal matrices is orthogonal. Therefore, the set of $n \times n$ orthogonal matrices forms a group, called orthogonal group and it is a subgroup of $GL(n, c)$. The limit of a sequence of orthogonal matrices is orthogonal since the relation $A^t A = I$ is preserved under limits. Thus O(n) is a matrix Lie group. Similar to $SL(n, c)$. The special orthogonal group, denoted by So(n), is defined as subgroup of O(n) whose matrices have determinant 1. Again, this is a matrix Lie group.

5.3.8. Remark

Geometrically speaking, the elements of O (n) are either rotation, or combinations of rotations and reflections, while the element of So (n) are just the rotations. Due to this geometric nature, the special orthogonal group appears frequently in physic problem dealing with rotation symmetry. An example of this would be a problem dealing with hydrogen atom potentials, which has a spherical symmetry.

5.3.9. Example

(The unitary and special unitary groups, U (n) and SU(n)) an $n \times n$ complex matrix A is unitary if the column vector of A are orthogonal that is

$$\sum_{L=i}^{n} \bar{A}_{lj} A_{ik} = \delta_{jk}$$

Similar to an orthogonal matrix, a unitary matrix has to nother equivalent definitions. A matrix A is unitary

i. If it preserves an inner product.

ii. If $A^*A = 1$ $i,e.$ if $A^* = A^{-1}$(where A^* is adjoin of A) since $\det A^* = det\ A| \ detA| = 1$ for all

Unitary matrices A. This shows unitary matrices are invertible. The same argument as for the orthogonal group can used to show that the set of unitary matrices from a group, called unitary group $U(n)$.

This is clearly a subgroup of $GL(n, c)$ and since limit of unitary matrices is unitary, $U(n)$ is a matrix Lie group. The special unitary group $SU(n)$. it is easy to see that this is also a matrix Lie group.

Here, we take a short detour to cover some needed topics in topology to help understand an application of special unitary group, $SU(2)$ and special orthogonal group, $SO(3)$ in physics.

5.3.10. Definition

A covering space of a space x is space $\tilde{x}$ together with a map $P: \tilde{x} \rightarrow x$, satisfying the following condition: there exists an open cover $\{U_\infty\}$ of X such that, for each $x, P^{-1}(U_\alpha)$ is a disjoint union of open sets in $\tilde{x}$ likewise, covering group can be defined similarly on topological groups, in particular matrix Lie groups.

5.3.11. Definition

A path in topological space X is a continuous function f from unit interval $I = [0,1]$ to X

$$f = I \rightarrow x$$

A space X is said to be path –connected if a

ny two points on X can be joined by a path. The stronger notion, the simple-connected space X, is if:

i. X is path-connected

ii. And every path between two points can be continuously transformed, staying within space, into any other such path while preserving two end points.

5.3.12. Definition

A covering space is a universal covering space if it is simple-connected. The name universal cover comes from the property that the universal cover (of the space X) covers any connected cover (of the space X), i.e. if the mapping $P: \bar{x} \rightarrow x$ Is the universal cover of the space X and the mapping $\emptyset = \tilde{x}$in any cover in the space X where $\tilde{x}$ is connected, then there exists a cover map $\psi: \bar{x} \rightarrow \tilde{x}$ such that$\emptyset \; o \; \psi = p$.

5.3.13. Proposition

The matrix Lie group $SU(2)$ can be identifiedwith the manifold S^3

Proof:

Consider $U = \begin{pmatrix} \alpha & M \\ \beta & V \end{pmatrix} \in SU(2)$

For $\alpha, \beta, M, V \in C$. This is $SU(2)$ if

$U * U = I$ And $det\ U = 1$

Using the inverse matrix formula with $det\ U = 1$

$$U^{-1} = \begin{pmatrix} V & -M \\ -\beta & \alpha \end{pmatrix}$$

Since $U^{-1} = U^{*}$, we have

$$\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$$

Where $U = \alpha\bar{\alpha} + \beta\bar{\beta} = 1$

This is a generic form of element of$SU(2)$. Now, set $\alpha = y_0 + iy_3,$

$\beta = -y_2 + iy_1$ For $y_0, y_1, y_2, y_3 \in R$

Then, it is straight forward to see that

$$U = y_0 I + \sum_{n=1}^{3} y_n\, \sigma_n$$

Where σ_n are Pa uli matrices defined as following?

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Now, the previous condition $\alpha\bar{\alpha} + \beta\bar{\beta} = 1$ is equivalent to $y_0^2 + y_1^2 + y_2^2 + y_3^2 = 1$ $i.e.$ $y = (y_0, y_1, y_2, y_3) \in S^3$. This establishes a smooth, invertible map between $SU(2)$andS^3.

5.3.14. Remark

The Pauli matrices, $\{\sigma_n\}$ which are familiar from quantum mechanics, can be taken as generators for Lie algebra of $SU(2)$, often with an extra factor of I, in physics$\{i\sigma_n\}$.

5.3.15. Remark

The point of establishing the correspondence between $SU(2)$ and S^3 is to provide an explicit way of seeing that $SU(2)$is simply-connected, since unit S^n (with$n \geq 2$) is simply-connected, in particular, S^3.

5.3.16. Proposition

The special unitary group $SU(2)$is a double-cover of the special orthogonal group $SU(3)$. There is 2-1 correspondence $\emptyset = SU(2) \rightarrow SO(3)$ and $\emptyset$ is a group homomorphism additionally it is a universal cover.

<u>Proof</u>:

Suppose $U \in SU(2)$. Define a 3 x 3 matrix $\emptyset(U)$ via

$$\emptyset(U)_{mn} = \frac{1}{3} trace\ (\sigma_m U \sigma_n U^*),$$

Where σ_n are Pauli matrices defined earlier. By writing

$U = y_0 I + \sum_{n=1}^{3} y_n\, \sigma_n\ for\ y_0, y_n \in R$, satisfying

$y_0^2 + \sum_{p=1}^{3} y_p y_p = 1$, it is straight forward from here to show that

$$\emptyset(U)_{mn} = \left(y_0^2 - \sum_{p=1}^{3} (y_p y_p)\, \delta_{nm} + 2_{\in m,n,q} \sum_{q=1}^{3} y_0 y_p + 2\, y_m y_n \right)$$

It is clear that if $y_p = 0$ for $p = 1,2,3$ so that

$U = \underline{+}\, I, than\ \emptyset(U) = I\ , so\ \emptyset(U) \in So(3)$. More generally, suppose that $y_p \neq 0$ than we can set $y_0 = \sin(\propto)$,

$y_p = \cos(\propto)\ zp\ for\ 0 < \propto < 2\pi\ , \propto \neq \pi$. Than the constraint

$$y_0^2 + \sum_{p=1}^{3} y_p y_p = 1\ implies\ that\ \sum_{p=1}^{3} Z_p Z_p = 1\ , i.e$$

$\vec{Z} := (Z_1, Z_2, Z_3)$ Is a unit vector in R^3. Then $\emptyset(U)_{mn}$ can be rewritten as following

$$\emptyset(U)_{mn} = \cos(2 \propto)\, \delta_{mn} + \sum_{q=1}^{3} \sin(2 \propto) \in_{m,n,q} Z_q + (1 - \cos(2 \propto))\, Z_m Z_n$$

It is then apparent that

$$\emptyset(U)_{mn} Z_n = Z_m$$

And if $\vec{X}$ is orthogonal to $\vec{Z}$ then

$$\emptyset(U)_{mn} X_n = \cos(2 \propto)\, X_m + \sum_{q=1}^{3} \sin(2 \propto)_{\in_{m,n,q}} X_n Z_q$$

The transformation $\emptyset$ is therefore corresponds to a rotation by $2\propto$

In the plane unit normal vector $\vec{Z}$

It is clean that any non-trivial rotation in $So(3)$ can be written in this way. However, the correspondence in not 1-1, but 2-1 .To see this explicitly, it requires bit of echoes algebra, so we will skip this part of the proof. The end picture will be that $\emptyset(U)$ from $So(3)$ will correspond to both U and –U from$SU(2)$. Now, check the group homomorphism, $i.e, \emptyset(U_1) = \emptyset(U_1)\emptyset(U_2)forU_1, U_2 \in SU(2)$. Let us write U_1, U_2 by following: (Note that we are now using the Einstein notation, where there is implicitly a summation over rerated indices).

$U_1 = y_0 I + i y_n \sigma_n$, $U_2 = w_0 I + i w_n \sigma_n$

For $y_0, y_p, w_0, w_p \in R$ satisfying $y_0^2 + y_p y_p = w_0^2 + w_p w_p = 1$then

$$U_1 U_2 = u_0\, I + i u_n \sigma_n$$

Where

$u_0 = y_0 w_0 - y_p w_p$ And $u_m = y_0 w_m + y_m w_0 - \in_{mpq} y_p w_q$

Satisfy $u_0^2 + u_p u_p = 1$, it then suffices to evaluate

Directly $\emptyset(U_1U_2)_{mn} = \left(u_0^2 - u_pu_p\right)\delta_{mn} + 2 \in_{mpq} u_0u_q + 2u_mu_n$

And compare this with

$$\emptyset(U_1)_{mp}\emptyset(U_2)_{pn}$$
$$= \left[(y_0^2 - y_1y_2)\delta_{mp} + 2 \in_{mpq} y_0y_q + 2y_my_p\right]\left[(w_0^2 - w_rw_r)\delta_{pn}\right.$$
$$\left. +\in_{pmr} w_0w_k + 2w_pw_n\right]$$

From here, it is again a simple but tedious matter of expanding out two expressions in terms of y and w and checking $\emptyset(U_1U_2)_{mn} = \emptyset(U_1)_{mp}\emptyset(U_2)_{pn}$ as required. The final statement about the universal over follows straight forward from simple connectedness of $SU(2)$.And the definition of universal cover

5.3.17. Remark

To quote from W. S. Massey, algebraic topology an induction, "A simple – connected space admits no non-trivial converging equivalently speaking, a universal cover is unique up to a homomorphism. In linear group (or matrix Lie group), in particular, the only cover of simply-connected group is an isomorphism, thus, the universal covers are isomorphic. This fact, in addition, to the relationship between $SU(2)$and $SO(3)$ has a well-known, physical outcome, there are particle with only integer or half-integer spins (boons or fermions), i.e. there is no $\frac{1}{3}$ or $\frac{1}{N}$ $(N \neq 2)$ spin particles.

5.4. Lie Algebra

Now, we move on to Lie algebras. The concept of Lie algebra can be motivated as the tangent space of the associated lie group at the identity, using Lie group's smooth manifold structure. Install, we can also define Lie algebra, using the matrix exponential, which is much more straightforward first, we define the matrix exponential.

5.4.1. Definition

Let X be any $n \times n$ complex matrix. We define the exponential of X , $e^x$by the usual power series

$$e^x = \sum_{m=0}^{\infty} \frac{X^m}{m!}$$

5.4.2. Proposition

For any $n \times n$complex matrix X, the series above converges.

The matrix exponential, e^x is a continuous function of X

proof

Recall that the norm of matrix A is defined as

$$|| X^m || \leq ||X||^m$$

$$\sum_{m=0}^{\infty} \left\| \frac{X^m}{m!} \right\| \leq \sum_{m=0}^{\infty} \frac{\|x\|^m}{m!} = e^{\|x\|} < \infty$$

Thus, the exponential series converges absolutely. But, given absolute convergence, we can take partial sums of the series to form a Cauchy sequence. Now, using the fact every Cauchy sequence converges in complete space, we now have that the exponential series converges. As for continuity, we just have to note that X^m is continuous of X.

5.4.3. Proposition

Let X ,Y be arbitrary $n \times n$ complex matrices. Then following are true

i. $e^o = I$

ii. e^x is invertible and $(e^x)^{-1} = e^{-x}$

iii. $e^{(\alpha+\beta)x} = e^{\alpha x} e^{\beta x}$ for all complex number α ,β.

iv. $if\ XY = YX$, then$e^{x+y} = e^x e^y = e^y e^x$.

v. $if\ c\ is\ an\ n \times n$ Invertible complex matrix, then$e^{c+c^{-1}} = Ce^x C^{-1}$.

vi. $|| e^x || \leq || x ||$.

Proof:

Point (i) is obvious from the fact that $x^o = I$ for all matrices X and $O^m = 0$ for all $m > 1$. Point (ii) and (iii) are special cases of (iv). To verify (iv) we simply multiply power series out term by term

$$e^x e^y = \left(1 + \frac{x}{1!} + \cdots\right)\left(1 + Y + \frac{Y^2}{2!} + \cdots\right)$$

By collecting the term where the power of X and power of Y add up to m, we get

$$e^x e^y = \sum_{m=0}^{\infty} \sum\nolimits_{k=0}^{m} \frac{x^k}{k!} \frac{Y^{m-k}}{(m-k)!} = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k=0}^{m} \frac{m!}{k!\,(m-k)!} X^k Y^{m-k}$$

Since we are give XY=YX, we have.

$$(X+Y)^m = \sum_{k=0}^{m} \frac{m!}{k!\,(m-k)!} X^k Y^{m-k}$$

Substituting this back in to what we had earlier, we get

$$e^x e^y = \sum_{m=0}^{\infty} \frac{1}{m!} (X+Y)^m = e^{X+Y}$$

For (v) we simply note that$(C \times C^{-1})^m = CX^m C^{-1}$. The proof of (vi) was already made from the proof of proposition (3.5.16).

5.4.4. Definition

Let G be a matrix Lie group. Then the Lie algebra of G denoted g, is the set of all matrices X such that e^{tx} is in G for all real number t, together with a bracket operation [. , .] : $g \times g \to g$,called the Lie bracket, with the following properties:

i. [. , .] is anti-symmetric, i.e. for all $X, Y, \in g$ $[X, Y] = -[Y, X]$

ii. [. , .] is bilinear, i.e. for all $a, b \in C$and $X, Y, Z \in g$ $[ax + by, z] = a[x, z] + b[Y, Z][x, aY + bZ] = a[X, Y] + b[X, Z]$

iii. [. , .] satisfies the Jacobi identity, i.e. foe all $X, Y, Z \in g$ $\big[X, [Y, Z]\big] + \big[Y, [Z, X]\big] + \big[Z, [X, Y]\big] = 0$

In fact, for a matrix Lie group G, the Lie bracket associated with its Lie algebra g is simply given by commentator of matrices, i.e. for all $a, b \in c$ and $X, Y, Z \in g$, $[X, Y]$ is defined by:

$$[X, Y] = XY - YX$$

And it is easy to check that three properties mentioned above are satisfied.

5.4.5. Remark

Physic is are accustomed to considering the map $X \to e^{ix}$instead of usual $X \to e^{x}$. Thus, in physics lie algebra of G is often defined as set od matrices X such that $e^{itx} \in G$ for all real t. in physics lie algebra is frequently referred to as the space of infinitesimal group elements, which actually connects the concept of the Lie algebra back to its original definition as the tangent space.

5.4.6. Example

(The general linear group) let X be any $n \times n$ complex matrix, then by proposition 5.20, e^{tx} is invertible thus, the Lie algebra of $GL(n, c)$ is the space of all $n \times n$ complex matrix. This Lie algebra is donated by $gI(n, c)$. If X is Any $n \times n\ real\ matrix$ then e^{tx}will be invertible and real. On the other hand if e^{tx} is real for all t, then $X = \frac{d}{dt}|_{t=0} = e^{tx}$ will also be real Thus the Lie algebra of $GL(n, R)$ is the space of all $n \times n$ real matrices$gI(n, R)$.

5.4.7. Proposition

Let X be $n \times n$ complex matrix. Then

$$det(e^{x}) = e^{trace(x)}$$

Proof:

We can divide up the proof into three cases:

X is diagonalizable, nilpotent, or arbitrary. The reason we can do this is because any matrix X can be written in the form X=S+N with diagonalizable, nilpotent, and S/V =NS. this following form the Jordan canonical form. Since S and N commute.

$$e^{s+n} = e^{s}e^{n}$$

And we can, then use the results for diagonalizable and nilpotent matrices to compute for the arbitrary matrices.

Case 1: suppose X is diagonalizable. Then, there exists an invertible matrix C such that

$$X = C\begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}C^{-1}$$

Then

$$e^{x} = C\begin{pmatrix} e^{\lambda_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n} \end{pmatrix}C^{-1}$$

Using the proposition 3.20 thus trace$(x)\ = \sum \lambda i$, and $\det(e^{x}) = \pi e^{\lambda i} = e^{\xi \lambda i}$

Case 2:

Suppose X is nilpotent. If x is nilpotent cannot have any non-zero eigenvalues. Thus all the roofs of characteristic polynomial must be zero. Hence the Jordan canonical form of x will be strictly upper-triangular. X can be written as:

$$X = C\begin{pmatrix} 0 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}C^{-1}$$

Hence, e^{x} will be upper-triangular with it’s on the diagonals:

$$e^{x} = C\begin{pmatrix} 1 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}C^{-1}$$

Thus, if X is nilpotent, trace (x) =0 and$dete^{x} = 1$.

Case 3:

Let X is arbitrary. Then $e^{x} = e^{s}e^{n}$ as described above

$$dete^{x} = dete^{S}dete^{N} = e^{trace(S)}e^{trace(N)} = e^{trace(x)}$$

As we wanted to show

5.4.8. Example

(The special linear group) we have$\det(e^{x}) = e^{trace(x)}$. Thus if trace X =0, then $\det(e^{tx}) = 1$foe all real number t. on the other hand, if X is any $n \times n$ matrix such that $\det(e^{tx}) = 1$ for all number t, then $e^{(t)(tracex)} = 1$ for all t, this means that (t)(trace x) is an integer multiple of $2\pi i$ for all t, which is only possible if trace x =0. Thus the Lie algebra of $SL(N,C)$ is the space of all $n \times n$ real matrices with trace zero denoted $SL(N,R)$.

5.4.9. Example

(The unitary group) recall that matrix U is unitary if and only if

$U^{*} = U^{-1}$. Thus, e^{tx} is unitary if and only if

$$(e^{tx})^{*} = (e^{tx})^{-1} = e^{-tx}$$

But by taking adjoins term-by-term, we see that

$(e^{tx})^{*} = e^{tx*}$, and so the above

$e^{tx*} = e^{-tx}$.

Clearly, a sufficient condition is $X^{*} = -X$. On the other hand, if above hold for all real t, then by differentiating at t=0, we see that $X^{*} = -X$ is necessary condition. Thus the Lie algebra of U(n) is the space of all $n \times n$ complex matrices X such that $X^{*} = -X$ and trace x=0, denoted su(n).

5.4.10. Proposition

Let G be a matrix Lie group, and X and element of its Lie algebra. Then e^{x} is an element of identity component of G. the identity component of a topological group G is the connected component G of G that contains the identity element of the group.

Proof:

By definition of Lie algebra, e^{tx} lies in G for all real t. but as t varies from 0 to 1, e^{tx} is a continuous path connecting identity to e^{tx}.

5.4.11. Example

(The orthogonal group) the identity component of $O(n)$ is just $So(n)$. by the proposition (3.27) the exponential of a matrix in Lie algebra is automatically in the identity component.

So, the Lie algebra of $O(n)$ is the same as the Lie algebra of $O(n)$. Now, an $n \times n$ real matrix X is orthogonal if and only if $X^{tr} = X^{-1}$. (Note, here, we used X^{tr} instead of X^{t} to not cause confusion with the t for the exponential) So, given as $n \times n$ realmatrix X, e^{tx} is orthogonal if and only if

$$(e^{tx})^{tr} = (e^{tx})^{-1}$$

Or $e^{tx} = e^{-tx}$

Clearly, a sufficient condition for above to hold is that

$X^{tr} = -X$.

Meanwhile, if above equality holds for all t, then by differentiating at t=0, we must have $X^{tr} = -X$. Thus the Lie algebra of O(n), as well as So(n), is the space of all $n \times n$ real matrices X with

$X^{tr} = -X$, denoted So(n). Note that the condition

$$X^{tr} = -X$$

Forces the diagonal entries of X to be zero, and so explicitly the trace of X are zero.

5.5. Physical Application

Here, we look at how the concept introduces earlier have been applied in the field of physics, particularly, particle physics

5.5.1. Is spin and SU (2).

The simplest case an application in physics can be found in a Lie algebra generated from the bilinear products of creation and annihilation operators where there are only two quantum states this is often referred to as the " old fashioned " . Isospin as it was originally conceived for systems of neutrons and protons before the discovery of mesons and strange particles. The concept of isospin was first introduced by Heisenberg in 1932 to explain the symmetries of newly discovered neutrons. Although the proton has appositive charge, and neutron is neutral, they are almost identical in other respects such as their masses. Hence, the term 'nucleon' was coined: treating two particles as two different states of the same particle, the nucleon in fact , the strength of strong interaction. The force which is responsible forming the nucleon of an atom-between any pair of nucleons is independent of whether they are interacting as protons and nucleons. More precisely, the isospin symmetry is given by the invariance of Hamiltonian of the strong interactions under the action of Lie group, SU (2). The neutrons and protons are assigned to the doublets with spin $\frac{1}{2} -$ representation of SU (2) .

Let us take a more detailed look in the mathematical for mulation.

Let a^t and a^t be operators for the creation of proton and neutron, respectively, and let a_p and a_n be the corresponding annihilation operators. Now, construct the lie algebra of all possible bilinear products of these operators which do not change the number of partials (strong interaction invariance). There are four possible bilinear products

$$a_p^t a_n^t \, , a_n^t a_p^t \, , a_p^t a_p , a_p^t a_p$$

The first operator turns a neutron into a proton, while the second operator turns a neutron into a proton. Let us denote the first two operators by T_+ and T_-. Recall, this whole symmetry is based on the idea that the proton and the neutron are simple two different states of the same particle : we can treat the proton as havening spin-up and the neutron as having spin-down, i.e, associating them with doublets $\left(\frac{1}{0}\right)$ and $\left(\frac{0}{1}\right)$, respectively. Which that in mind, the notation would seen more natural: T_+ being the raising operator, while T- being the lowering operator. Now, the last two operators simply annihilate A proton or neutron and then ereate them back. These are just the number operators which count the number of protons and neutron. Together, they are total number operators, which commute which the all the other operators, as they do not change the total number. It is therefore convenient to divide the set of four operators into a set of three operators plus the total number, or baryon number, operator, which commutes with other:

$$B = a_p^t a_p + a_n^t a_n$$

$$T_+ = a_p^t a_n$$

$$T_- = a_n^t a_p$$

$$T_0 = \frac{1}{2}\left(a_p^t a_p - a_n^t a_n\right) = Q - \frac{1}{2}B$$

Where

Q is just the total change (since proton has a positive change where as neutron has no change).

Now, the set of three operators, T_+ , T_- , T_0 satisfy following commutation relation, which is exactly like that of angular moment a:

$[T_0 T_+] = T_+$

$[T_0 T_-] = -T_-$

$$[T_+ T_-] = 2T_0$$

This has led to the designation isospin for these operators and to the description of rotations in the fictitious isospin space.

Let us now consider which Lie group is associated with these isospin operators.

By analogy with angular momentum operator, we allow these operators to generate infinitesimal transformation such as

$$\psi^1 = \{1 + i \in ([T_+ + T_-)\}\psi$$

We use the linear combination $[T_+ + T_-$ because these are not individually ttermition (or self-ad joint).

Note that such a transformation changes a proton or neutron into something which is a linear combination of the proton and the neutron state. These transformations are thus transformation in a two-dimensional proton neutron Hilbert space. The transformation are unitary thus the lie group of unitary transformation in a two-dimensional space is generated by the set of four operators, however, the unitary transformation generated by the operator B are of a trivial nature they are multiplication of any state by a phase factor. Since the three isospin operators form a lie group by themselves, the associated group is the subgroup of full unitary group in two-dimensional, the space unitary group, SU(2).

5.5.2. Eight fold way and SU (2)

As noted before, the above isospin SU (2) symmetry is old fashioned in that it does not consider mesons and the "strangeness" a property in particles expressed as quantum number, for describing a decay of particles in strong and electromagnetic in tractions, which occur in a short amount of time. Thus was first introduced by Murray Gell-Mann and Kazuhiko ivijishima to explain that certain particles such as the Kasson or certain hyperons were created easily in particle collisions, yet decayed much more slowly than expected for their large masses. To account for the newly added property (quantum number), the Lie group SU(3) was chosen over SU(2) to construct a theory which organizes baryons and mesons into octets(thus, the term English fold way), where the acutest are the representation of the Lie group.

Why this question needed to be asked. I don't think we introduce Lie groups and algebras property to our students. They are missing from most if not all of the basic courses. Except for the orthogonal and possibly the unitary group, they are not mentioned much in differential geometry course. They are too often introduce to students in a separate Lie group and algebra course, where everything is discussed too abstractly and too isolated from other subject for my taste. Here is a very fundamental way to ereate interesting Riemannian manifolds: let G be a semi-simple lie group let K be its maximal compact subgroup, let be a discrete subgroup of G, and from G/K this quotient is called the symmetric space attached to G the Riemannian structure come from an invariant metric ton G, and so G acts as isometrics on G / K by left transformation. If you consider the case G= $SL_2(R)$you get $SL_2(R)$ / $SO(2)$ which is naturally identified with the complex upper half. Plane (on which $SL_2(R)$acts via Moby's transformation, note that the point i is stabilized precise by $SO(2)$, which is also the hyperbolic, other groups give higher dimensional hyperbolic space ($e.gSL_2(R)$) gives hyperbolic space

5.6. Solvable Lie Algebra Application

I am starting to study lie algebra and when I reached the notion of solvable lie algebra, I tried to find concrete applications (in physics for example) and I couldn't find one. For example, solvable group are very important for the insolvability of quantic equation (and by the way, it's the only application I know of them). In the same manner, can we fine application for solvable lie algebras? There are actually lots of applications of solvable Lie algebras, especially in the field of enterable systems where the solvability of Hamilton's equations of motion is frequently related to the integrals of motion of the system.

5.6.1. Examples

For any Hamiltonian system on R^{2n} with its standard simplistic structure if there are n integrals of motion F_i , which are functionally independent and they from a solvable lie algebra (under the poison bracket). Plus some technical can be integrated on the level-set of the integrals by quadrature. This is a classic result. See for example perelomov's book, P-34, 35 theorem 2 , the sub sequent example and the references theorem.

Noether is Theorem

Consider the variation of the shape of the field without changing the space-time coordinates which defined as

$$\delta\emptyset_i(x) \equiv {\emptyset_i}^{\Delta}(x) - \emptyset_i(x) \qquad (5.1)$$

We can also define another type of variation which is closely related, a local variation. It is defined as the difference between the fields evaluated in the same space-time point but in two different coordinates systems:

$\bar{\delta}\emptyset_i(x) \equiv \emptyset_i{}^{\Delta}(x^{\Delta}) - \emptyset_i(x).$ (5.2)

Now consider a continuous space time translation which define as following

$x^{\mu} \to x^{\Delta\mu} = x^{\mu} + \Delta x^{\mu}$ (5.3)

Which be proper orthochronous Lorentz transformation or space-time Transformation at first order in Δx, $\bar{\delta}\emptyset_i(x)$

Write as:

$\bar{\delta}\emptyset_i(x) \equiv \emptyset_i{}^{\Delta}(x^{\Delta}) - \emptyset_i(x) = \emptyset_i{}^{\Delta}(x + \Delta x) - \emptyset_i(x) \approx \emptyset_i{}^{\Delta}(x) + \left(\partial_{\mu}\emptyset_i{}^{\Delta}(x)\right)\Delta x^{\mu} - \emptyset_i(x) \approx \emptyset_i{}^{\Delta}(x) + \left(\partial_{\mu}\emptyset_i(x)\right)\Delta x^{\mu} - \emptyset_i(x) = \delta\emptyset_i(x) + (\partial_{\mu}\emptyset_i(x))\Delta x^{\mu}$ (5.4)

Therefore, we have found the following relation between $\delta\emptyset_i(x)$ and $\bar{\delta}\emptyset_i(x)$ for an infinitesimal transformation of the type (5.3)

$\bar{\delta}\emptyset_i(x) = \delta\emptyset_i(x) + (\partial_{\mu}\emptyset_i(x))\Delta x^{\mu}$ (5.5)

If $\emptyset_i{}^{\Delta}(x^{\Delta}) = \emptyset_i(x)$ (which is in general the case for scalar field, it is also the case for spin or fields under space-time translations) then

$\delta\emptyset_i(x) = -(\partial_{\mu}\emptyset_i(x))\Delta x^{\mu}$ (5.6)

Thus, in this case, an equivalent way of making a transformation of the type (5.3) which acts on the coordinates is by making an opposite transformation on the field:

$\emptyset_i(x) \to \emptyset^{\Delta}{}_i(x) = \emptyset_i(x - \Delta x)$ (5.7)

References

[1] A. Baker. Matrix groups - Springer Under graduate Mathematic series - Springer -Verlag London Ltd –London, 2002, an introduction to Lie group theory

[2] R. W .Cater Lie algebras of finite and a fine type, volume 96 of Cambridge studies in advanced Mathematics Cambridge University Press Cambridge 2005

[3] A.C.desilva Lecture on simplistic geometry .http:www.math .ist .ut1 pt/ cannas Book isg pdf, 2006

[4] K .Erdmann and M .J, wild on introduction to Lie algebras Springer Under graduate Mathematics series-Springer Verlag-London, Ltd London, 2006

[5] W.Fulton and J, Harris Representation theory volume 129 of graduate Texts in Mathematics Springer-Verlag New York 1991. A first course Readings in Mathematics

[6] V. Guillemin and S Sternberg Symplectic techniques in physics Cambridge University Press-Cambridge second edition, 1990

[7] B. C .Hall, Lie groups, Lie algebras, and representations volume 222 of graduate Texts in Mathematics Springer –Verlag New York, 2003 An elementary introduction

[8] J. E Humphreys, Introduction to Lie algebras and Representation theory, Springer –Verlag-New York, 1972 Graduate Texts in Mathematics , VOL 90

[9] A. Kirillov, Jar An introduction to Lie groups and Lie algebras volume 113 of Cambridge studies in Advanced Mathematics Cambridge University Press Cambridge, 2008

[10] D , Mond . Lecture notes for MA455 manifolds http: //home pages war.[11] Corn well, John Group theory in physics. Academic Press.1997.

[12] Lip kin, Harry J. Lie Groups for pedestrians' .Dover publications InC.1966.

[13] Fulton, William and Joe Harris Representation theory spring, 2004.

[14] M.A .Basher: Lecture notes AlNeelainUniversity,2015

[15] Stattinger, D.H. and O.L. Weaver . Lie Groups and algebras with Application to physics, Geometry , and Mechanics .

- springer-Verla1986

[16] introduction to Lie Groups and Lie Algebras Alexander Kirillov Jr. Department of Mathematics , Suny At Stony Brook ,NY 11794,USA

E-mail address: Kirillov@math.sunyb.edu

URL:http://www.math.sunysb.edu/kiril

[17] A. Baker. Matrix groups - Springer Under graduate Mathematic series - Springer –Verlag London Ltd –London, 2002, an introduction to Lie group theory

[18] R. W .Cater Lie algebras of finite and a fine type, volume 96 of Cambridge studies in advanced Mathematics Cambridge University Press Cambridge 2005

[19] A.C.desilva Lecture on simplistic geometry .http:www.math .ist .ut1 pt/ cannas Book isg pdf, 2006

[20] K .Erdmann and M .J, wild on introduction to Lie algebras Springer Under graduate Mathematics series-Springer Verlag-London, Ltd London, 2006

[21] W.Fulton and J, Harris Representation theory volume 129 of graduate Texts in Mathematics Springer-Verlag New York 1991. A first course Readings in Mathematics

[22] V. Guillemin and S Sternberg Symplectic techniques in physics Cambridge University Press-Cambridge second edition, 1990

[23] B. C .Hall, Lie groups, Lie algebras, and representations volume 222 of graduate Texts in Mathematics Springer –Verlag New York, 2003 An elementary introduction

[24] J. E Humphreys, Introduction to Lie algebras and Representation theory, Springer –Verlag-New York, 1972 Graduate Texts in Mathematics , VOL 90

[25] A. Kirillov, Jar An introduction to Lie groups and Lie algebras volume 113 of Cambridge studies in Advanced Mathematics Cambridge University Press Cambridge, 2008

[26] D , Mond . Lecture notes for MA455 manifolds http: //home pages war w.

[27] V. Arnold, Mathematical methods of classical mechanics,

[28] N. Bourbaki, Lie groups and Lie algebras. Springer, Berlin, 1998, Lie groups Lie algebras, Springer, Berlin, 2002,

[29] D. Bump, Lie groups, Graduate Texts in Mathematics, vol. 225, Springer, New York, 2004

[30] T. Brocker, T.tomDieck, Representations of compact Lie groups, Graduate Texts in mathematics, vol.98, Springer, New York, 1985

[31] R. Carter, G. Segal, I. Macdonald, Lectures on Lie groups and Lie algebras, London Mathematical Society student Texts, 32. Cambridge University Press, Cambridge, 1995

[32] J. Dixmier, Enveloping algebras, Amer. Math. Soc, Providence RI 1996. [This is a reprint of 1977 translation of 1974 French original]

[33] B .L .Feigin, D. B. Fuchs, Co homologies of Lie groups and Lie algebras, in Lie groups and Lie algebras II Encyclopedia math. Sci. vol. 21; Springer. Berlin, 2000

[34] V.V. Gorbatsevich, A.L. Onischik E.B. Venberg, Structure of Lie groups and Lie algebras, in of Lie groups and Lie algebras Ш Encyclopadia math. Sci. vol. 41, Springer. Berlin, 1994

[35] B. Hall, Lie Groups, Lie algebras, and Representations. And elementary introduction, Graduate Texts in mathematics, Vol.222, Springer, 2003. Corr. 2nd printing, 2004

[36] J.J Duistermaat, J. A. C. Kolk, Lie groups, Springer-Verlag, Berlin, 2000.

[37] N. Jacobson, Lie algebras, Republication of the 1962 original. Dover Publication, Inc. , New York , 1979.

[38] J.Humphreys, Introduction of Lie algebras and Representation theory

[39] J. Humphreys Reflection Groups and Coxeter Groups, Cambridge Univ. Press, Cambridge 1990

[40] A. A .Kirillov lecture on the orbit method, Graduate Studies in mathematics, 64 Amer. Math. Soc., Providence, RI, 2004

[41] A. Knapp, Lie groups beyond an introduction, Second edition. Progress in Mathematics, 140.Birkhauser Boston, Inc. , Boston,MA,2002

[42] Landau, Lifshits.

[43] D. Mumford, J .Fogarty, F. Kirwan, Geometric invariant theory, third edition, Springer-Verlag, Berlin, 1994.

[44] A. L. Onischik, E.B. Vinberg Foundation of Lie Theory, in Lie Groups and Lie algebras II Enc Math Sci. vol. 20