كلية الدراسات العليا

Sudan University of Science and Technology
College of Graduate Studies

جامعة السودان للعلوم والتكنولوجيا

Compact and Weighted Composition Operators on μ-Bergman and Maeda-Ogasawara Spaces with Rigidity on Hardy Spaces

التراص ومؤثرات التركيب المرجحة على فضاءات بيرجمان $-\mu$ ومايدا- اوقاساوارا مع الصلابة على فضاءات هاردي

A Thesis Submitted in Fulfillment for the Degree of Ph.D

in Mathematics

By:

Bushara Eisa Hamad AbdAlla

Supervisor:

Prof. Dr. Shawgy Hussein AbdAlla

April-2020

Dedication

To My Family.

Acknowledgements

I would like to express my deep gratitude to my supervisor Prof. Dr. Shawgy Hussein AbdAlla for many helpful suggestions and good following.

Abstract

It is shown that composition operators on the Bloch space in polydiscs and on μ-Bloch type spaces are dealt with. The weighted composition operators between μ- Bloch spaces on the unit ball, of $C_0(X)$ and on Maeda-Ogasawara spaces are considered. The compact and weakly compact operators on BMOA, on Bergman and μ-Bergman spaces in the unit ball are studied. In addition the isometries between function spaces, on Banach-Stone theorem, atomic decomposition of μ-Bergman spaces in unitary space and strict singularity of Volterra-type integral operator on Hardy space are characterized. We show the linear isometries spaces of Lipschitz and vector-valued Lipschitz functions. The new properties, approximation numbers and rigidity of composition operators are discussed.

الخلاصة

تم التعامل مع إيضاح مؤثرات التركيب على فضاء بلوش في الأقراص البولي وعلى الفضاءات نوع بلوش-μ. قمنا بأعتبار مؤثرات التركيب المرجحة بين فضاءات بلوش-μ على كرة الوحدة و $C_0(X)$ وعلى فضاءات مايدا-اوقاساوارا. تمت دراسة مؤثرات التركيب الضعيفة والمتراصة على BMOA وعلى فضاءات بيرجمان وبيرجمان-μ في كرة الوحدة. أضافة تم تشخيص الأيزوميتريس بين فضاءات الدالة وعلى مبرهنة باناخ - ستون والتفكيك الذري لفضاءات بيرجمان-μ في الفضاء الواحدي والشذوذية التامة لمؤثر التكامل نوع-فولتيرا على فضاء هاردي. أوضحنا فضاءات الأيزوميتريس الخطية لدوال ليبشيتز ودوال ليبشيتز قيمة-المتجة. قمنا بمناقشة الخصائص الجديدة وأعداد التقريب والصلابة لمؤثرات التركيب.

Introduction

For Ω be a bounded Bergman domain in $\mathbb{C}^n$ (A domain $\Omega \subset \mathbb{C}^n$ is called boanded Bergman domain if it is bounded and there exists a constant C depending only on Ω, such that $H_{\psi(z)}(J\psi(z)u, J\psi(z)u) \leq CH_z(u,u)$, for each $z \in \Omega, u \in \mathbb{C}^n$ and holomorphic serlf-map ψ of Ω, where $H_z(u,u)$ denote the Bergman metric of Ω and $J\psi$ the Jacobian of ψ) and ϕ a holomorphic self-map of Ω. Necessary and sufficient conditions are given for the weighted composition operator $T_{\psi,\phi}$ to be bounded or compact from the space β_μ to β_ν (or $\beta_{\mu,0}$ to $\beta_{\nu,0}$) on the unit ball of $\mathbb{C}^n$.

Surjective isometries between some classical function spaces are investigated. We give a simple technical scheme which verifies whether any such isometry is given by a homeomorphism between corresponding Hausdorff compact spaces. We investigated for the isometries of $C(T)$and for the holoorphic maps which are isometries for the Caratheodory-Kobayashi differential metric of $B(T)$.

We give examples of results on composition operators connected with lens maps. The first two concern the approximation numbers of those operators acting on the usual Hardy space H^2. We show that the approximation numbers of a compact composition operator on the Hardy space H^2 or on the weighted Bergman spaces $\mathcal{B}_\alpha$ of the unit disk can tend to 0 arbitrarily slowly, but that they never tend quickly to 0. Any analytic map ϕ of the unit disc $\mathbb{D}$ into itself induces a composition operator C_ϕ on BMOA, mapping $f \to f \circ \phi$, where BMOA is the Banach space of analytic functions $f : \mathbb{D} \to \mathbb{C}$ whose boundary values have bounded mean oscillation on the unit circle. We show that C_ϕ is weakly compact on BMOA precisely when it is compact on BMOA, thus solving a question initially posed by Tjani and by Bourdon, Cima and Matheson in the special case of VMOA.

Let μ be a normal function on $[0,1)$. The atomic decomposition of the μ-Bergman space in the unit ball B is given for all $p > 0$. Let $p > 0$ and μ be a normal function on $[0,1), \nu(r) = (1-r^2)^{1+\frac{n}{p}}\mu(r)$ for $r \in [0,1)$. The bounded or compact weighted composition operator $T_{\phi,\psi}$ from the μ-Bergman space $A^p(\mu)$ to the normal weight Bloch type space β_ν in the unit ball is characterized. The briefly sufficient and necessary condition that the composition operator C_ϕ is compact from A^p (μ) to β_ν is given.

We state a Lipschitz version of a known Holsztyński's theorem on linear isometries of $C(X)$-spaces. Let $\mathrm{Lip}(X)$ be the Banach space of all scalar-valued Lipschitz functions f on a compact metric space X endowed with the norm $\|f\| = \max\{\|f\|_\infty, L(f)\}$, where $L(f)$ is the Lipschitz constant of f. Every Archimedean Riesz space can be embedded as an order dense subspace of some $C^\infty(X)$, the Riesz space of all extended continuous functions on a Stonean space X, called its Maeda–Ogasawara space. Furthermore, it is a fact that every Riesz homomorphism between spaces of ordinary continuous functions on compact Hausdorff spaces is a weighted composition operator.

We show that the Volterra-type integral operator $T_g f(z) = \int_0^z f(\zeta)\, g(\zeta) d\zeta, z \in D$, defined on the Hardy spaces H^p fixes an isomorphic copy of p if it is not compact. In particular, the strict singularity of T_g coincides with its compactness on spaces H^p. Let φ be an analytic map taking the unit disk $\mathbb{D}$ into itself. We establish that the class of composition operators $f \to C_\varphi(f) = f \circ \varphi$ exhibits a rather strong rigidity of non-compact behaviour on the Hardy space H^p, for $1 \leq p < \infty$ and $p \neq 2$. Our main result is the following trichotomy, which states that exactly one of the following alternatives holds: (i) C_φ is a compact operator $H^p \to H^p$, (ii) C_φ fixes a (linearly isomorphic) copy of ℓ^p in H^p, but C_φ does not fix any copies of ℓ^2 in H^p, (iii) C_φ fixes a copy of ℓ^2 in H^p. Moreover, in case (iii) the operator C_φ actually fixes a copy of $L^p(0,1)$ in H^p provided $p > 1$.

The Contents

Chapter 1
Compact and Weighted Composition Operators

We show that the composition operator C_ϕ induced by ϕ is always bounded on the Bloch space $\beta(\Omega)$. For $\Omega = U^n$ the unit polydisc of $\mathbb{C}^n$, we give a necessary and sufficient condition for C_ϕ to be compact $\beta(U^n)$. Under a mild condition we show that a composition operator C_ϕ is compact on the Bergman space A^p_α of the open unit ball in $\mathbb{C}^n$ if and only if $\frac{1-|z|}{1-|\phi(z)|} \to 0$ as $|z| \to 1^-$.

Section (1.1): Bloch Space in Polydiscs

Let Ω be a bounded domain in $\mathbb{C}^n$.The class of all holomorphic functions with domain Ω will be denoted by $H(\Omega)$. If ϕ holomorphic maps Ω into itself, the composition operator C_ϕ induced by ϕ is defined by

$$(C_\phi f)(z) = f(\phi(z)),$$

for z in Ω and $f \in H(\Omega)$.

Let $K(z,z)$ be the Bergman kernel function of Ω, the Bergman metric $H_z(u,u)$ in Ω is defined by

$$H_z(u,u) = \frac{1}{2}\frac{\partial^2 \log K(z,z)}{\partial z_l \partial \bar{z}_k} u_l \bar{u}_k, \tag{1}$$

where $z \in \Omega$ and $u = (u_1, \ldots, u_n) \in \mathbb{C}^n$.

Following Timoney [3], we say that $f \in H(\Omega)$ is in the Bloch space $\beta(\Omega)$, if

$$\|f\|_{\beta(\Omega)} = \sup_{z\in\Omega} Q_f(z) < \infty, \tag{2}$$

where

$$Q_f(z) = \sup\left\{\frac{|\nabla f(z)u|}{H_z^{(1/2)}(u,u)} : u \in \mathbb{C}^n - \{0\}\right\},$$

and $\nabla f(z) = \left((\partial f(z)/\partial z_1), \ldots, (\partial f(z)/\partial z_n)\right), \nabla f(z)u = \sum_{l=1}^n \frac{\partial f(z)}{\partial z_1} u_l$.

A domain Ω is called bounded Bergman domain if Ω is bounded and there exists a constant C depending only on Ω, such that

$$H_{\phi(z)}(J\phi(z)u, J\phi(z)u) \le C H_z(u,u), \tag{3}$$

for each $z \in \Omega, u \in \mathbb{C}^n$ and holomorphic self-map of Ω, where $H_z(u,u)$ denotes the Bergman metric of Ω, $J\phi(z)u = \left((\partial\phi_l(z)/\partial z_k)\right)_{1\le l,k\le n}$ denotes the Jacobian matrix of ϕ and $J\phi(z)u$ denotes a vector, which lth component is $(J\phi(z)u)_l = \sum_{k=1}^n (\partial\phi_l(z)/\partial z_k)u_k$, $l = 1,2,\ldots,n$.

The goal of the study composition operators will be to answer the question when C_ϕ will be a bounded or compact operator on Bloch space $\beta(\Omega)$.

Madigan and Matheson [I] studied the problem on the Bloch space $\beta(\Omega)$ on the unit disk U. They proved that C_ϕ, is always bounded on $\beta(U)$, they also gave the sufficient and necessary conditions that C_ϕ is compact on $\beta(U)$.

We show that C_ϕ is always bounded on $\beta(\Omega)$, where Ω is a bounded Bergman domain in $\mathbb{C}^n$. For $\Omega = U^n$ the unit polydisc of $\mathbb{C}^n$,we give a sufficient and necessary condition that the composition operator C_ϕ, is compact on $\beta(U^n)$ Some new methods and techniques have been used because of the difference between one complex variable and several complex variables.

Theorem (1.1.1)[1]: Let Ω be a bounded Bergman domain in $\mathbb{C}^n$ and ϕ a holomorphic self-map of Ω. Then C_ϕ is bounded on $\beta(\Omega)$.

Theorem (1.1.2)[1]: Let U^n be the unit polydisc of $\mathbb{C}^n$.If ϕ holomorphic maps U^n into itself, then C_ϕ is compact on $\beta(U^n)$ if and only if for every $\varepsilon > 0$, there exists a $\delta > 0$, such that

$$\frac{H_{\phi(z)}(J\phi(z)u, J\phi(z)u)}{H_z(u,u)} < \varepsilon, (4)$$

For all $u \in \mathbb{C}^n\{0\}$ whenever $dist(\phi(z), \partial U^n) < \delta$.

When $n = 1$, the Bergman metric of the unit disk U is $H_z(u,u) = (|u^2|/(1-|z|^2)^2), z \in U, u \in \mathbb{C}$. Hence

$$\frac{H_{\phi(z)}(\phi'(z)u, \phi'(z)u)}{H_z(u,u)} = \left\{\frac{1-|z|^2}{1-|\phi(z)|^2}\right\}^2 |\phi'(z)|^2$$

where ϕ is a holomorphic self-map from U to U Thus. by Theorem (1.1.2), we can also obtain Theorem (1.1.2) in [2] .

In what follows, Ω always denotes a bounded Bergman domain in $\mathbb{C}^n$, and U^n the unit polydisc in $\mathbb{C}^n$. ϕ holomorphic maps Ω or U^n into itself and C is a positive constant not necessary the same at each occurence.

In order to prove Theorem (1.1.1) and Theorem (1.1.2), we need the following Lemmas.

By the Bergman distance in $\beta(\Omega)$ and Montel's Theorem, according to the definition of compact operators, it is easy to prove the following Lemma which is a characterization of compactness of composition operators C_ϕ in terms of sequential convergence, we omit the details.

Lemma (1.1.3)[1]: C_ϕ is compact on $\beta(\Omega)$ if and only if for any bounded sequence $\{f_k\}$ in ,$\beta(\Omega)$ which converges to zero uniformly on compact subsets of Ω, we have $\|f_k \circ \phi\|_{\beta(\Omega)} \to 0, as\ \ k \to \infty$.

Lemma (1.1.4)[1]: let $F(z) = (1 - z/1 - \lambda z)$. If $0 < \lambda < 1, |z| \le 1$ then

$$|F(z)| < 2.$$

Proof. Since $0 < \lambda < 1, |1 - \lambda z| \ge 1 - \lambda|z| \ge 1 - \lambda > 0$, it follows that

$$|F(z)| = \left|\frac{1-z}{1-\lambda z}\right| = \left|\frac{(1-\lambda z)-(1-\lambda)z}{1-\lambda z}\right|$$
$$= \left|1 - z\frac{1-\lambda}{1-\lambda z}\right| \le 1 + (1-\lambda)\frac{1}{1-|\lambda z|} < 2.$$

The desired inequality follows.

Lemma (1.1.5)[1]: Let $G(z) = \sqrt{1-z} + \sqrt{1-\lambda z}$. If $0 < \lambda < 1, |z| < 1$, then

$$|G(z)| \ge \sqrt{2(1-|z|)}.$$

Proof. We write $z = x + iy, 1 - z = d_1 e^{i\theta_1}, 1 - \lambda z = d_2 e^{i\theta_2}$, where $d_1 = |1 - z|, \theta_1 = \arg(1 - z), d_2 = |1 - \lambda z|, \theta_2 = \arg(1 - \lambda z)$.

It is clear that $\theta_1 = \arctan(y/1 - x), \theta_2 = \arctan(y/1 - \lambda x), 1 - x > 0, 1 - \lambda x > 0$, so

$$-\pi/2 \le \theta_1 \le \pi/2, \qquad -\pi/2 \le \theta_2 \le \pi/2$$

furthermore

$$-\pi/2 \le \frac{\theta_1 - \theta_2}{2} \le \pi/2.$$

$$G(z) = \sqrt{1-z} + \sqrt{1-\lambda z} = \sqrt{d_1} e^{i(\theta_1/2)} + \sqrt{d_2} e^{i(\theta_2/2)}$$
$$= \left(\sqrt{d_1}\cos\frac{\theta_2}{2} + \sqrt{d_2}\cos\frac{\theta_2}{2}\right) + i\left(\sqrt{d_1}\sin\frac{\theta_1}{2} + \sqrt{d_2}\sin\frac{\theta_2}{2}\right)$$
$$|G(z)| = \sqrt{d_1 + d_2 + 2\sqrt{d_1}\sqrt{d_2}\cos\frac{\theta_1 - \theta_2}{2}} \geq \sqrt{d_1 + d_2} = \sqrt{|1-z| + |1-\lambda z|}$$
$$\geq \sqrt{2(1-|z|)}.$$
The proof of this Lemma is completed.

Using the chain rule, we get
$$\nabla(f \circ \phi)(z) = \nabla(f)\big(\phi(z)\big) J_\phi(z)$$
if $u \in \mathbb{C}^n - \{0\}$ and $J\phi(z)u = 0$, from the above equality, it follows that $\nabla(f \circ \phi)(z)u = 0$.

If $u \in \mathbb{C}^n - \{0\}$ and $J\phi(z)u \neq 0$,
$$\frac{\nabla(f \circ \phi)(z)u}{H_z^{(1/2)}(u,u)} = \frac{\nabla(f)\big(\phi(z)\big)J\phi(z)u}{H_{\phi(z)}^{(1/2)}(J\phi(z)u, J\phi(z)u)} \times \frac{H_{\phi(z)}^{(1/2)}(J\phi(z)u, J\phi(z)u)}{H_z^{(1/2)}(u,u)}.$$
Hence
$$Q_{f\circ\phi}(z) = \sup\left\{\frac{|\nabla(f \circ \phi)(z)u|}{H_z^{(1/2)}(u,u)} : u \in \mathbb{C}^n - \{0\}\right\}$$
$$= \sup\left\{\frac{|\nabla(f \circ \phi)(z)u|}{H_z^{(1/2)}(u,u)} : u \in \mathbb{C}^n - \{0\}, J\phi(z)u \neq 0\right\}$$
$$\leq Q_f\big(\phi(z)\big)\sup\left\{\frac{H_{\phi(z)}^{(1/2)}(J\phi(z)u, J\phi(z)u)}{H_z^{(1/2)}(u,u)} : u \in \mathbb{C}^n - \{0\}\right\}. \quad (5)$$
$$Q_{f\circ\phi}(z) \leq C Q_f\big(\phi(z)\big)$$
so
$$\left\|C_\phi(f)\right\|_{\beta(\Omega)} = \|f \circ \phi\|_{\beta(\Omega)} = \sup_{z\in\Omega} Q_{f\circ\phi}(z) \leq C \sup_{z\in\Omega} Q_f\big(\phi(z)\big) \leq C\|f\|_{\beta(\Omega)}.$$
It means that C_ϕ is bounded on $\beta(\Omega)$. Theorem (1.1.1) is proved.

Lemma (1.1.6)[1]: If $\phi : \Omega \to \Omega$ is a holomorphic self-map, where Ω is a bounded Bergman domain in $\mathbb{C}^n$.Then C_ϕ is compact on $\beta(\Omega)$ if for every $\varepsilon > 0$, there exists a $\delta > 0$, such that
$$\frac{H_{\phi(z)}(J\phi(z)u, J\phi(z)u)}{H_z(u,u)} < \varepsilon,$$
For all $u \in \mathbb{C}^n - \{0\}$ whenever $dist(\phi(z), \partial\Omega) < \delta$.

Proof. By Lemma (1.1.3), it is enough to show that if $\{f_k\}$ is a bounded sequence in $\beta(\Omega)$ which converges to zero uniformly on compact subsets of Ω, then $\|f_k \circ \phi\|_{\beta(\Omega)} \to 0$ as $k \to \infty$.

Let $M = \|f_k\|_{\beta(\Omega)}$, for given $\varepsilon > 0$, there exists a $\delta > 0$, such that
$$\frac{H_{\phi(z)}(J\phi(z)u, J\phi(z)u)}{H_z(u,u)} < \left(\frac{\varepsilon}{M}\right)^2, \quad (6)$$
For all $u \in \mathbb{C}^n - \{0\}$ whenever $dist(\phi(z), \partial\Omega) < \delta$.

$f_k \in \beta(\Omega)$, so (5) gives

$$Q_{f_k\circ\phi}(z) \le Q_{f_k}\big(\phi(z)\big) \sup\left\{\left[\frac{H_{\phi(z)}(J\phi(z)u, J\phi(z)u)}{H_z(u,u)}\right]^{(1/2)} : u \in \mathbb{C}^n - \{0\}\right\}. \quad (7)$$

Combining (6) and (7), we have

$$Q_{f_k\circ\phi}(z) < \varepsilon, \quad (8)$$

for $\operatorname{dist}(\phi(z), \partial\Omega) < \delta$.

On the other hand, it is easy to see that

$$\inf\left\{H_w^{(1/2)}(u,u) : |u| = 1, dist(w, \partial\Omega) \ge \delta\right\} = m > 0.$$

So

$$\frac{|\nabla(f_k)(w)u|}{H_w^{(1/2)}(u,u)} = \frac{|\nabla(f_k)(w)||u|}{H_w^{(1/2)}(u,u)} = \frac{|\nabla(f_k)(w)|}{H_w^{(1/2)}\left(\left(\frac{u}{|u|}\right),\left(\frac{u}{|u|}\right)\right)} \le \frac{|\nabla(f_k)(w)|}{m} \quad (9)$$

if $dist(w, \partial\Omega) \ge \delta$. Now the hypothesis that $\{f_k\}$ converges to zero uniformly on compact subsets of Ω implies $Q_{f_k}(w) \to 0$ uniformly for $dist(w, \partial\Omega) \ge \delta$ as $k \to \infty$. So by (3) and (9), (7) gives that for large enough k,

$$Q_{f_k\circ\phi}(z) \le C Q_{f_k}\big(\phi(z)\big) < \varepsilon, \quad (10)$$

if $dist(\phi(z), \partial\Omega) \ge \delta$.

It follows from (8) and (10) that $\|f_k \circ \phi\|_{\beta(\Omega)} < \varepsilon$ for large enough k.

The compactness of C_ϕ on $\beta(\Omega)$ follows by Lemma (1.1.3).

We will prove Theorem (1.1.2) in the following.

It is obvious that the sufficiency of condition (4) has been proved by Lemma (1.1.6), so we only need to prove the condition (4) is necessary.

Suppose C_ϕ is compact on $\beta(U^n)$ and the condition (4) fails, then there exists a sequence $\{z'\}$ in U^n with $\phi(z') \to \partial U^n$, as $j \to \infty, u^j \in \mathbb{C}^n - \{0\}$, such that

$$\frac{H_{\phi(z^j)}\big(J\phi(z^j)u^j, J\phi(z^j)u^j\big)}{H_{z^j}(u^j,u^j)} \ge \varepsilon_0 \quad (11)$$

for all $j = 1, 2, \ldots$.

Using the condition (1l), we will construct a sequence of functions $\{f_j\}$ satisfying the following three conditions:

(i) $\{f_j\}$ is a bounded sequence in $\beta(U^n)$;

(ii) $\{f_j\}$ tends to zero uniformly on coiiipaci subsets of U^n,

(iii) $\|C_\phi f_j\|_{\beta(U^n)} \nrightarrow 0$,as $j \to \infty$.

This contradicts the compactness of C_ϕ by Lemma (1.1.3).

To construct the sequence of $\{f_j\}$, we first assume that

$$\phi(z^j) = r_j e_1, j = 1,2, \ldots,$$

where $e_1 = (1,0,\ldots,0)$, the 1th coordinate is 1 and the others are 0.

It is clear that $0 < r_j < 1$. From $\phi(z^j) \to \partial U^n$, we know $r_j \to 1$, as $j \to \infty$. Denote $J(\phi(z')) = w^j$.

It is well known that the Bergman metric of U^n

$$H_z(u,u) = \sum_{k=1}^{n} \frac{|u_k|^2}{(1-|z_k|^2)^2},$$

where $z \in U^n$ and $u = (u_1, \dots, u_n) \in \mathbb{C}^n$. So

$$H_{\phi(z^j)}(w^j, w^j) = \frac{\left|w_1^j\right|^2}{\left(1 - r_j^2\right)^2} + \sum_{k=2}^{n} \left|w_k^j\right|^2, \tag{12}$$

We construct the functions according to two different cases:
Case 1. If tor some j,

$$\sum_{k=2}^{n} \left|w_k^j\right|^2 \leq \frac{\left|w_1^j\right|^2}{\left(1 - r_j^2\right)^2}, \tag{13}$$

then set

$$f_j(z) = \log\left(1 - e^{-a(1-r_j)} z_1\right) - \log(1 - z_1) \tag{14}$$

Where a is any positive number.
Case 2. If for some j,

$$\sum_{k=2}^{n} \left|w_k^j\right|^2 > \frac{\left|w_1^j\right|^2}{\left(1 - r_j^2\right)^2}, \tag{15}$$

then set

$$f_j(z) = \left(\sum_{k=2}^{n} e^{-i\theta_k^j} z_k\right) \sqrt{1 - z_1} \left(\frac{1}{\sqrt{1 - e^{-a(1-r_j)} z_1}} - \frac{1}{\sqrt{1 - z_1}}\right), \tag{16}$$

where a is any positive number, and $\theta_k^j = \arg w_k^j$, $k = 2,3, \dots, n$. If $w_k^j = 0$ for some k replace the corresponding term $e^{-i\theta_k^j} z_k$ by 0.

First we prove that the functions defined by (14) satisfy the conditions (i), (ii) and (iii).

By (14), we know

$$\frac{\partial f_j}{\partial z_k} = 0 (2 \leq k \leq n), \quad \frac{\partial f_j}{\partial z_1} = \frac{1}{1 - z_1} - \frac{e^{-a(1-r_j)}}{1 - e^{-a(1-r_j)} z_1}$$

$$\left|\nabla f_j(z) u\right| = \left|\frac{\partial f_j}{\partial z_1} u_1 + \frac{\partial f_j}{\partial z_2} u_2 + \cdots + \frac{\partial f_j}{\partial z_n} u_n\right| = \left|\left(\frac{1}{1 - z_1} - \frac{e^{-a(1-r_j)}}{1 - e^{-a(1-r_j)} z_1}\right) u_1\right|,$$

$$\begin{aligned}
\left|\nabla f_j(z) u\right| &= \frac{\left|(1/1 - z_1) - \left(e^{-a(1-r_j)}/1 - e^{-a(1-r_j)} z_1\right)\right| |u_1|}{\left(\sum_{l=1}^{n} (|u_l|^2/(1 - |z_l|^2)^2)\right)^{1/2}} \\
&\leq \frac{\left|(1/1 - z_1) - \left(e^{-a(1-r_j)}/1 - e^{-a(1-r_j)} z_1\right)\right| |u_1|}{(|u_1|/1 - |z_1|)^2} \\
&= (1 - |z_1|^2) \left|\frac{1}{1 - z_1} - \frac{e^{-a(1-r_j)}}{1 - e^{-a(1-r_j)} z_1}\right| \\
&\leq (1 - |z_1|^2) \left(\frac{1}{1 - |z_1|} + \frac{1}{1 - |z_1|}\right) = 2(1 + |z_1|) \leq 4
\end{aligned}$$

so

$$\left\|f_j\right\|_{\beta(U^n)} = \sup_{z \in U^n} Q_{f_j}(z) = \sup_{z \in U_n} \sup \left\{\frac{\left|\nabla f_j(z) u\right|}{H_z^{(1/2)}(u, u)}, u \in \mathbb{C}^n - \{0\}\right\} \leq 4,$$

If means that $f_j \in \beta(U^n)$ and $\{f_j\}$ is bounded on $\beta(U^n)$.

Let E be a compact subset of U^n, it is clear that there exists a $\rho(0 < \rho < 1)$ such that

$$|z_1| \le \rho, \tag{17}$$

for every $z = (z_1, \ldots, z_n) \in E$.

$$f_j(z) = \log\left(1 - e^{-a(1-r_j)}z_1\right) - \log(1 - z_1) = \log\frac{1 - e^{-a(1-r_j)}z_1}{1 - z_1}.$$

Since

$$\begin{aligned}\left|\frac{1 - e^{-a(1-r_j)}z_1}{1 - z_1} - 1\right| &= \left|\frac{1 - e^{-a(1-r_j)}z_1 - 1 + z_1}{1 - z_1}\right| \\ &= \left|\frac{z_1}{1 - z_1}\right|\left|1 - e^{-a(1-r_j)}\right| \\ &\le \frac{1}{1-\rho}\left(1 - e^{-a(1-r_j)}\right) \to 0,\end{aligned}$$

as $j \to \infty$. So $\left(1 - e^{-a(1-r_j)}z_1/1 - z_1\right)$ converges to 1 uniformly on compact subset E, that is, $f_j(z) = \log\left(1 - e^{-a(1-r_j)}z_1/1 - z_1\right)$ converges to zero uniformly on compact subsets of U^n.

We now prove that $\|C_\phi f_j\|_{\beta(U^n)} \nrightarrow 0$. In fact, by (1 l), (12) and (13), we get

$$\|C_\phi f_j\|_{\beta(U^n)} = \|f_j \circ \phi\|_{\beta(U^n)} \ge Q_{f_j \circ \phi}(z^j)$$

$$\ge \frac{\left|\nabla(f_j \circ \phi)(z^j)u^j\right|}{H_{z^j}^{(1/2)}(u, u)} = \frac{\left|\nabla(f_j)(\phi(z^j))J\phi(z^j)u^j\right|}{H_{z^j}^{(1/2)}(u, u)}$$

$$= \frac{\left|\nabla(f_j)(\phi(z^j))J\phi(z^j)u^j\right|}{H_{\phi(z^j)}^{1/2}(J\phi(z^j)u^j, J\phi(z^j)u^j)}\left\{\frac{H_{\phi(z^j)}\left(J\phi(z^j)u^j, J\phi(z^j)u^j\right)}{H_{z^j}(u^j, u^j)}\right\}^{(1/2)}$$

$$\ge \sqrt{\varepsilon_0}\frac{\left|\nabla(f_j)(r_je_1)w^j\right|}{H_{r_je_1}^{(1/2)}(w^j, w^j)} = \sqrt{\varepsilon_0}\frac{\left|(\partial f_j/\partial z_1)(r_je_1)w_1^j\right|}{\left((|w_1^j|^2/(1-r_j^2)^2) + \sum_{k=2}^n |w_k^j|^2\right)^{(1/2)}}$$

$$\ge \sqrt{\frac{\varepsilon_0}{2}}\frac{\left|(\partial f_j/\partial z_1)(r_je_1)w_1^j\right|}{\left(|w_1^j|/1 - r_j^2\right)}$$

$$= \sqrt{\frac{\varepsilon_0}{2}}\left(1 - r_j^2\right)\left|\frac{1}{1 - r_j} - \frac{e^{-a(1-r_j)}}{1 - e^{-a(1-r_j)}r_j}\right|$$

$$\ge \sqrt{\frac{\varepsilon_0}{2}}\left(1 - \frac{(1 - r_j)e^{-a(1-r_j)}}{1 - e^{-a(1-r_j)}r_j}\right),$$

From the fact

$$\lim_{j\to\infty}\left[1 - \frac{(1 - r_j)e^{-a(1-r_j)}}{1 - e^{-a(1-r_j)}r_j}\right] = \frac{a}{a+1} \ne 0,$$

we know $\|C_\phi f_j\|_{\beta(U^n)} \nrightarrow 0$, as $j \to \infty$.

Now we prove that the functions defined by (16) also satisfy the conditions (i), (ii) and (iii).

In fact,

$$\frac{\partial f_j}{\partial z_1} = \frac{1}{2}\left(\sum_{k=2}^{n} e^{-i\theta_k^j} z_k\right)\left[\sqrt{1-z_1}\left(\frac{e^{-a(1-r_j)}}{\left(1-e^{-a(1-r_j)}z_1\right)^{(3/2)}} - \frac{1}{(1-z_1)^{(3/2)}}\right)\right.$$

$$\left. - \frac{1}{\sqrt{1-z_1}}\left(\frac{1}{\sqrt{1-e^{-a(1-r_j)}z_1}} - \frac{1}{\sqrt{1-z_1}}\right)\right]$$

$$\frac{\partial f_j}{\partial z_k} = e^{-i\theta_k^j}\sqrt{1-z_1}\left(\frac{1}{\sqrt{1-e^{-a(1-r_j)}z_1}} - \frac{1}{\sqrt{1-z_1}}\right),$$

$$\frac{\left|\nabla f_j(z)u\right|}{H_z^{(1/2)}(u,u)} = \frac{\left|\sum_{k=2}^{n}\left(\partial f_j/\partial z_k\right)u_k + \left(\partial f_j/\partial z_1\right)u_1\right|}{\left(\sum_{l=1}^{n}(|u_l|^2/(1-|z_l|^2)^2)\right)^{(1/2)}}$$

$$\leq \frac{\sum_{k=2}^{n}\left|\left(\partial f_j/\partial z_k\right)\right||u_k|}{\left(\sum_{l=1}^{n}(|u_l|^2/(1-|z_l|^2)^2)\right)^{(1/2)}} + \frac{\left|\left(\partial f_j/\partial z_1\right)\right||u_1|}{\left(\sum_{l=1}^{n}(|u_l|^2/(1-|z_l|^2)^2)\right)^{(1/2)}}$$

$$\leq \sum_{k=2}^{n}(1-|z_k|^2)\left|\frac{\partial f_j}{\partial z_k}\right| + (1-|z_1|^2)\left|\frac{\partial f_j}{\partial z_1}\right|$$

$$\leq \sum_{k=2}^{n}\left|\sqrt{\frac{1-z_1}{1-e^{-a(1-r_j)}z_1}} - 1\right| + \frac{n-1}{2}(1-|z_1|^2)$$

$$\times\left|\left(\frac{e^{-a(1-r_j)}\sqrt{1-z_1}}{\left(1-e^{-a(1-r_j)}z_1\right)^{(3/2)}} - \frac{1}{1-z_1}\right) - \frac{1}{1-z_1}\left(\sqrt{\frac{1-z_1}{1-e^{-a(1-r_j)}z_1}} - 1\right)\right|$$

$$\leq (n-1)\left(\sqrt{\frac{1-z_1}{1-e^{-a(1-r_j)}z_1}} + 1\right) + (n-1)(1-|z_1|)$$

$$\times\left[\left(\frac{1}{1-|z_1|}\sqrt{\frac{1-z_1}{1-e^{-a(1-r_j)}z_1}} + \frac{1}{1-|z_1|}\right) + \frac{1}{1-|z_1|}\left(\sqrt{\left|\frac{1-z_1}{1-e^{-a(1-r_j)}z_1}\right|} + 1\right)\right]$$

$$= (n-1)\left(\sqrt{\left|\frac{1-z_1}{1-e^{-a(1-r_j)}z_1}\right|} + 1\right)$$

$$+2(n-1)\left(\sqrt{\left|\frac{1-z_1}{1-e^{-a(1-r_j)}z_1}\right|} + 1\right)$$

$$= 3(n-1)\left(\sqrt{\left|\frac{1-z_1}{1-e^{-a(1-r_j)}z_1}\right|} + 1\right)$$

If we choose $\lambda = e^{-a(1-r_j)}$, then $0 < \lambda < 1$, it follows from Lemma (1.1.4) that

$$\left|\frac{1-z_1}{1-e^{-a(1-r_j)}z_1}\right| < 2,$$

$$\frac{|\nabla f_j(z)u|}{H_z^{(1/2)}(u,u)} < 3(n-1)(\sqrt{2}+1),$$

it means that $f_j \in \beta(U^n)$ and $\{f_j\}$is bounded on $\beta(U^n)$.

For the compact subset E of U^n. By (16)

$$f_j(z) \quad = \left(\sum_{k=2}^{n} e^{-i\theta_k^j} z_k\right) \frac{\sqrt{1-z_1} - \sqrt{1-e^{-a(1-r_j)}z_1}}{\sqrt{1-e^{-a(1-r_j)}z_1}}$$

$$= \left(\sum_{k=2}^{n} e^{-i\theta_1^j} z_1\right) \frac{\left(e^{-a(1-r_j)}-1\right)z_1}{\sqrt{1-e^{-a(1-r_j)}z_1}\left(\sqrt{1-z_1}+\sqrt{1-e^{-a(1-r_j)}z_1}\right)}.$$

Since $|z_1| < 1, 0 < \lambda = e^{-a(1-r_j)} < 1, \sqrt{1-e^{-a(1-r_j)}z_1} \geq \sqrt{1-|z_1|}$, and from Lemma (1.1.5), it follow that

$$\left|\sqrt{1-z_1} + \sqrt{1-e^{-a(1-r_j)}z_1}\right| \geq \sqrt{2(1-z_1)}$$

By (17), $1-|z_1| \geq 1-\rho$, thus

$$|f_j(z)| \leq (n-1)\frac{1-e^{-a(1-r_j)}}{\sqrt{1-|z_1|}\sqrt{2(1-|z_1|)}} \leq \frac{n-1}{\sqrt{2}(1-\rho)}\left(1-e^{-a(1-r_j)}\right),$$

it is clear that $\lim_{j\to\infty}\left(1-e^{-a(1-r_j)}\right) = 0$, thus $\{f_j\}$ converges to zero uniformly on compact subset E of U^n.

We now prove that$\|C_\phi f_j\|_{\beta(U^n)} \nrightarrow 0$. By (1 1) , (12) and (15), we get

$$\|C_\phi f_j\|_{\beta(U^n)} = \|f_j \circ \phi\|_{\beta(U^n)} \geq Q_{f_j \circ \phi}(z^j)$$

$$\geq \frac{|\nabla(f_j \circ \phi)(z^j)u^j|}{H_{z^j}^{(1/2)}(u,u)} = \frac{|\nabla(f_j)(\phi(z^j))J\phi(z^j)u^j|}{H_{z^j}^{(1/2)}(u,u)}$$

$$= \frac{|\nabla(f_j)(\phi(z^j))J\phi(z^j)u^j|}{H_{\phi(z^j)}^{1/2}(J\phi(z^j)u^j, J\phi(z^j)u^j)}\left\{\frac{H_{\phi(z^j)}\left(J\phi(z^j)u^j, J\phi(z^j)u^j\right)}{H_{z^j}(u^j,u^j)}\right\}^{(1/2)}$$

$$\geq \sqrt{\varepsilon_0}\frac{|\nabla(f_j)(r_je_1)w^j|}{H_{r_je_1}^{(1/2)}(w^j,w^j)} = \sqrt{\varepsilon_0}\frac{\left|\sum_{k=2}^{n}(\partial f_j/\partial z_k)(r_je_1)w_k^j\right|}{\left((|w_1^j|^2/(1-r_j^2)^2) + \sum_{k=2}^{n}|w_k^j|^2\right)^{1/2}}$$

$$\geq \sqrt{\frac{\varepsilon_0}{2}}\frac{\left|\sum_{k=2}^{n}(\partial f_j/\partial z_k)(r_je_1)w_k^j\right|}{\left(\sum_{k=2}^{n}|w_k^j|^2\right)^{1/2}}$$

$$= \sqrt{\frac{\varepsilon_0}{2}} \frac{\sqrt{1-r_j^2}\left|\left(1/\left(\sqrt{1-e^{-a(1-r_j)}r_j}\right)-\left(1/\left(\sqrt{1-r_j}\right)\right)\right)\right|\sum_{k=2}^{n}\left|w_k^j\right|}{\left(\sum_{k=2}^{n}\left|w_k^j\right|^2\right)^{1/2}}$$

$$= \sqrt{\frac{\varepsilon_0}{2}}\left|\sqrt{\frac{1-r_j}{1-e^{-a(1-r_j)}r_j}}-1\right| = \sqrt{\frac{\varepsilon_0}{2}}\left(1-\sqrt{\frac{1-r_j}{1-e^{-a(1-r_j)}r_j}}\right)$$

From the fact

$$\lim_{j\to\infty}\frac{1-r_j}{1-e^{-a(1-r_j)}r_j} = \lim_{r\to 1}\frac{1-r}{1-e^{-a(1-r)}r} = \frac{1}{a+1},$$

we know $\left\|C_\phi f_j\right\|_{\beta(U^n)} \nrightarrow 0$, as $j\to\infty$.

In general situation, set $(z^j) = \left(t_j^1, t_j^2, \dots, t_j^n\right) = \sum_{i=1}^{n} t_j^l e_l$, where $e_l = (0,\dots,1,\dots,0)$, the lth coordinate is 1 and the others are 0. Since $\phi\left(t^i\right)\to\partial U^n$, for some k, there exists a subsequence in $\left\{t_j^k\right\}$ (we write still $\{t_j^k\}$) such that $\left|t_j^k\right|\to 1$ as $j\to\infty$, without loss of generality, we may assume $k=1$. Let $r_j = \left|t_j^1\right|, \theta = \arg r_j\, r_j^l = e^{-i\theta}t_j^l, 2\le l\le n$. Then $0<\left|r_j^l\right| = \left|t_j^l\right| < 1$,$0<r_j<1$ and $r_j\to 1$.

Let $\psi^j(z) = (\psi_1^j(z),\dots,\psi_n^j(z))$, where $\psi_l^j(z) = e^{-i\theta}\left(z_l - t_j^l/1-t_j^{-l}z_l\right)(2\le l\le n)$ and $\psi_1^j(z) = e^{-i\theta}z_1$, then

$$\psi^j(\phi(z^j)) = r_j e_1.$$

Set $g_j = f_j\circ\psi^j$, then

$$\begin{aligned}\left|\nabla(g_j)(\phi(z^j))w^j\right| &= \left|\nabla(f_j\circ\psi^j)(\phi(z^j))w^j\right| \\ &= \left|\nabla(f_j)(\psi^j\circ\phi(z^j))J\psi^j(\phi(z^j))w^j\right| \\ &= \left|\nabla(f_j)(r_je_1)J\psi^j(\phi(z^j))w^j\right|. \end{aligned} \tag{18}$$

It is well known that for $\psi^j\in Aut(U^n)$,

$$H_{\psi^j(z)}\left(J\psi^j(z)u, J\psi^j(z)u\right) = H_z(u,u), \tag{19}$$

For each $z\in\Omega$. So

$$\begin{aligned}H_{\phi^j(z^j)}\left(w^j,w^j\right) &= H_{\psi^j\circ\phi(z^j)}\left(J\psi^j(\phi(z^j))w^j, J\psi^j(\phi(z^j))w^j\right) \\ &= H_{r_je^k}\left(J\psi^j(\phi(z^j))w^j, J\psi^j(\phi(z^j))w^j\right). \end{aligned} \tag{20}$$

It follows from (18) and (20) that

$$\begin{aligned}\frac{\left|\nabla(g_j)(\phi(z^j))w^j\right|}{H_{\phi(z^j)}^{(1/2)}(w^j,w^j)} &= \frac{\left|\nabla(f_j)\left(r_je_1\right)J\psi^j(\phi(z^j))w^j\right|}{H_{r_je_1}^{(1/2)}(J\psi^j(\phi(z^j))w^j, J\psi^j(\phi(z^j))w^j)} \\ &= \frac{\left|\nabla(f_j)\left(r_je_1\right)W^j\right|}{H_{r_je_1}^{(1/2)}(W^j,W^j)}, \end{aligned} \tag{21}$$

Where $W^j = J\psi^j(\phi(z^j))$.

$$\begin{aligned}\left\|C_\phi g_j\right\|_{\beta(U^n)} &= \left\|g_j\circ\phi\right\|_{\beta(U^n)} \ge Q_{g_j\circ\phi}(z^j) \\ &\ge \frac{\left|\nabla(g_j\circ\phi)(z^j)u^j\right|}{H_{z^j}^{1/2}(u^j,u^j)} = \frac{\left|\nabla(g_j)(\phi(z^j))J\phi(z^j)u^j\right|}{H_{z^j}^{1/2}(u^j,u^j)}\end{aligned}$$

$$= \frac{\left|\nabla(g_j)\left(\phi(z^j)\right)J\phi(z^j)u^j\right|}{H^{1/2}_{\phi(z^j)}(J\phi(z^j)u^j,J\phi(z^j)u^j)}\left\{\frac{H_{\phi(z^j)}\left(J\phi(z^j)u^j,J\phi(z^j)u^j\right)}{H_{z^j}(u^j,u^j)}\right\}^{\frac{1}{2}}$$

$$\geq \sqrt{\varepsilon_0}\frac{\left|\nabla(g_j)(\phi(z^j))w^j\right|}{H^{1/2}_{\phi(z^j)}(w^j,w^j)}$$

$$= \sqrt{\varepsilon_0}\frac{\left|\nabla(f_j)\left(r_je_1\right)\left(J\psi^j(\phi(z^j))w^j\right)\right|}{H^{1/2}_{r_je_1}(J\psi^j(\phi(z^j))w^j,J\psi^j(\phi(z^j))w^j)}$$

$$= \sqrt{\varepsilon_0}\frac{\left|\nabla(f_j)\left(r_je_1\right)W^j\right|}{H^{1/2}_{r_je_1}(W^j,W^j)}. \tag{22}$$

If for some j,$\sqrt{\sum_{k=2}^{n}\left|W_k^j\right|^2} \leq \left(\left|W_1^j\right|/1-r_j^2\right)$, then choose the functions $\{f_j\}$ defined by (14). If for some j, $\sqrt{\sum_{k=2}^{n}\left|W_k^j\right|^2} > \left(\left|W_1^j\right|/1-r_j^2\right)$, then choose the functions $\{f_j\}$ defined by (16).

Now we prove $\{g_j = f_j \circ \psi^j\}$ satisfies condition (i),(ii) and (iii). In fact, by (19), we have

$$\frac{\left|\nabla(g_j)(z)u\right|}{H_z^{1/2}(u,u)} = \frac{\left|\nabla(f_j)(\psi^j(z))J\psi^j(z)u\right|}{H^{1/2}_{\psi^j(z)}(J\psi^j(z)u,J\psi^j(z)u)} \leq \left|f_j\right|_{\beta(U^n)} \leq M.$$

So $\left|g_j\right|_{\beta(U^n)} \leq M, \{g_j\}$ is bounded on $\beta(U^n)$.

For the compact subset E of U^n. If

$$f_j(z) = \log\left(1 - e^{-a(1-r_j)}z_1\right) - \log(1-z_1),$$

then

$$g_j(z) = \log\left(1 - e^{-a(1-r_j)}e^{-i\theta}z_1\right) - \log\left(1 - e^{-i\theta}z_1\right),$$

similar to $\{f_j\}, \{g_j\}$ tends to zero uniformly on compact subset E of U^n.

If

$$f_j(z) = \left(\sum_{l\neq k} e^{-i\theta_l^j}z_l\right)\sqrt{1-z_1}\left(\frac{1}{\sqrt{1-e^{-a(1-r_j)}z_1}} - \frac{1}{\sqrt{1-z_1}}\right),$$

$$g_j(z) = \left(\sum_{k=2}^{n} e^{-i\theta_k^j}z_k\right)\sqrt{1-e^{-i\theta}z_1}\left(\frac{1}{\sqrt{1-e^{-a(1-r_j)}e^{-i\theta}z_1}} - \frac{1}{\sqrt{1-e^{-i\theta}z_1}}\right),$$

similar to $\{f_j\}, \{g_j\}$ also tends to zero uniformly on compact subset E of U^n.

By (22), similar to $\{f_j\}$, we can prove that $\left\|C_\phi g_j\right\|_{\beta(U^n)} \nrightarrow 0$, as $j \to \infty$.

This contradicts the compactness of C_ϕ by Lemma (1.1.3). Now we complete the proof of Theorem (1.1.2).

Example (1.1.7)[1]: If $\psi \in Aut(U^n)$, then C_ψ is not compact on $\beta(U^n)$.In fact, it is easy to know from Theorem (1.1.2) and the well-known fact

$$H_{\psi(z)}(J\psi(z)u, J\psi(z)u) = H_z(u,u).$$

Section (1.2): μ-Bloch Spaces on the Unit Ball

For B denote the unit ball of C^n and D be the unit disc in the complex plane. The class of all holomorphic functions with a domain B will be denoted by $H(B)$.

A positive continuous function μ on $[0,1)$ is normal if there are two constants $0 < a < b$ such that (i) $\mu(r)(1-r)^{-a}$ is decreasing for $r \in [0,1)$ and $\mu(r)(1-r)^{-a} \to 0$ as $r \to 1^-$; (ii) $\mu(r)(1-r)^{-b}$ is increasing for $r \in [0,1)$ and $\mu(r)(1-r)^{-b} \to \infty$ as $r \to 1^-$.

Let μ be normal on $[0,1)$. $f \in H(B)$ is said to belong to the μ-Bloch space β_μ if

$$\|f\|_\mu = \sup_{\substack{u\in C^n-\{0\}\\ z\in B}} \frac{\mu(|z|)|<\nabla f(z),\bar{u}>|}{\sqrt{(1-|z|^2)|u|^2+|<z,u>|^2}} = \sup_{z\in B} \frac{u(|z|)}{1-|z|^2} Q_f(z) < \infty$$

and to the little μ-Bloch space $\beta_{\mu,0}$ if

$$\lim_{|z|\to 1} \sup_{\substack{u\in C^n-\{0\}\\ z\in B}} \frac{\mu(|z|)|<\nabla f(z),\bar{u}>|}{\sqrt{(1-|z|^2)|u|^2+|<z,u>|^2}} = 0$$

where ∇f is the complex gradient of f . It is well known that β_μ is a Banach space under the norm $\|f\|_{\beta_\mu} = |f(0)| + \|f\|_\mu$, and that $\beta_{\mu,0}$ is a closed subspace of β_μ. The normal function μ, as a weight, was usually used to define the mixed norm spaces before, for example [5]. When $\mu(r) = 1 - r^2$ and $\mu(r) = (1-r^2)^{1-\alpha}(0 < \alpha < \frac{1}{2})$– two typical normal functions, the induced spaces β_μ are the Bloch space β and Lipschitz space Λ_α respectively. And these spaces have been studied extensively.

Let X and Y be two Banach spaces of functions holomorphic on B and $\varphi : B \to B$ be holomorphic. For $\psi \in H(B)$, one can define the weighted composition operator $T_{\psi,\varphi}$ from X to Y by $T_{\psi,\varphi}(f) = \psi . f \circ \varphi (f \in X)$.

It is easy to see that an operator defined in this manner is linear. We can regard this operator as a generalization of a multiplication operator M_ψ and a composition operator C_φ. In the complex plane, the behaviors of the operators C_φ and $T_{\psi,\varphi}$ on β_{1-r^2} or $\beta_{(1-r^2)^p}$ were studied in refs. [2]–[10]; and the main results in [11] was to characterize the boundedness of C_φ on $\beta_{(1-r^2)\log(1-r^2)^{-1}}$. In several complex variables cases, Shi and Luo, Zhou and Zeng got that the characterization on φ for which C_φ is bounded or compact on β_{1-r^2} or $\beta_{(1-r^2)^p}$ in the unit ball in. [1]–[13] respectively. And in the polydiscs, Zhou [14]−[15] studied the same problems. Hu [16] discussed the boundedness and compactness of C_φ from β_μ to β_ν in the polydiscs. But the sufficient and necessary conditions for C_φ to be bounded or compact from β_μ to β_ν on the unit ball have not been obtained up to now, even in the simplest case $\mu(r) = (1-r^2)^p$, $\nu(r) = (1-r^2)^q$. The main purposes are to solve the problem and generalize the known corresponding results on composition operators and pointwise multipliers on the Bloch type space and the little Bloch type space.

We will use the symbol $c, c_1, \cdots$ to denote the finite positive numbers which do not depend on variables z, w and may depend on some bounded quantities, being not necessarily the same at each occurrence. “$E \approx F$” is comparable, that is, there exist two positive constants A_1 and A_2 such that $A_1 E \le F \le A_2 E$.

Lemma (1.2.1)[4]: Let μ be normal on $[0,1)$ (suppose $a > \frac{1}{2}$ when $n > 1$) and $f \in H(B)$. Then (a) $f \in \beta_\mu$ if and only if $\sup_{z\in B} \mu(|z|)|\nabla f(z)| < \infty$. Furthermore $\|f\|_{\beta_\mu} \approx |f(0)| + \sup_{z\in B} \mu(|z|)|\nabla f(z)|$. (b) $f \in \beta_{\mu,0}$ if and only if $\mu(|z|)|\nabla f(z)| \to 0$ as $|z| \to 1$.
Proof. The proof is similar to that of Theorem 7.2 in [17].
Lemma (1.2.2)[4]: Let μ be normal on $[0,1)$. (a) If $f \in \beta_\mu$, then

$$|f(z)| \le \left(1 + \int_0^{|z|} \frac{1}{\mu(t)} dt\right) \|f\|_{\beta_\mu} (z \in B).$$

(b) If $f \in \beta_{\mu,0}$ and $\int_0^1 \frac{1}{\mu(t)} dt = \infty$, then $\lim_{|z|\to 1} \frac{|f(z)|}{\int_0^1 [\mu(t)]^{-1} dt} = 0$.

Proof. By the definitions of β_μ and $\beta_{\mu,0}$, $u(|z|)\nabla f(z) \le \frac{v(|z|)}{1-|z|^2} Q_f(z) \le \|f\|_{\beta_\mu} (z \in B)$ and the equality

$$f(z) = f(0) + \int_0^1 < \nabla f(tz), \bar{z} > dt,$$

we can obtain the results easily.
Lemma (1.2.3)[4]: Let μ be normal on $[0,1)$ and $g(\xi) = 1 + \sum_{s=1}^{\infty} 2^s \xi^{n_s} (\xi \in D)$, where n_s is the intgral part of $(1 - r_s)^{-1}$, $\mu(r_s) = 2^{-s} (s = 1,2,\ldots)$. Then
(i) $g(r)$ is strictly increasing with $r \in [0,1)$ and $\inf_{r\in[0,1)} \mu(r)g(r) > 0$, $\sup_{\xi\in D} \mu(|\xi|)|g(\xi)| < \infty$.
(ii) $\mu(|z|)|g'(rz_1)| = O\left(\frac{1}{1 - r|z_1|}\right)$ for any $z = (z_1, \ldots, z_n) \in B$ and $0 \le r < 1$.
(iii) $\left|\int_0^{rz_1} g(t)dt\right| \le \int_0^r g(t)dt \le c\{g\left(\frac{1}{2}\right) + \int_0^{r^2} g(t)\, dt\}$ for any $z = (z_1, \ldots, z_n) \in B$ and $\sqrt{\frac{1}{2}} < r < 1$.
Proof. First, the definitions of μ and r_s show that r_s is strictly increasing and

$$\frac{\mu(r_{s+1})}{(1 - r_{s+1})^b} \ge \frac{\mu(r_s)}{(1 - r_s)^b}, \text{that is } \frac{1 - r_s}{1 - r_{s+1}} \ge 2^{\frac{1}{b}} (s = 1, 2, \ldots).$$

So

$$\lim_{s\to\infty} \inf \frac{n_{s+1}}{n_s} = \lim_{s\to\infty} \inf \frac{Int\left[\frac{1}{1 - r_{s+1}}\right]}{Int\left[\frac{1}{1 - r_s}\right]} = \lim_{s\to\infty} \inf \frac{\frac{1}{1 - r_{s+1}}}{\frac{1}{1 - r_s}} \ge 2^{\frac{1}{b}}.$$

This means that $\sum_{s=1}^{\infty} 2^s \xi^{n_s}$ is absolutely convergent on D.
(i) The result was proved by Hu [16].
(ii) If $\rho = r|z_1| > \mu^{-1}\left(\frac{1}{2}\right) = \rho_0 > 0$, then there exists $k \in \{1, 2, \ldots\}$ such that $r_k \le \rho < r_{k+1}$. By computation we have

$$\sup_{0\le x<\infty} \left\{x\rho^{\frac{x}{2}}\right\} = -\frac{2}{e\, log\rho} \text{ as } 0 < \rho < 1; \sup_{0\le x<1} x^{\frac{1}{1-x}} = \frac{1}{e}.$$

Thus

$$g'(\rho) = \sum_{s=1}^{\infty} n_s\, 2^s \rho^{n_s - 1} \le \frac{1}{\rho_0} \sum_{s=1}^{k} \frac{1}{1 - r_s} 2^s \rho^{n_s} + \frac{1}{\rho_0} \sum_{s=k+1}^{\infty} \frac{1}{1 - r_s} 2^s \rho^{n_s}$$

$$\le \frac{1}{(1-\rho)\rho_0}\sum_{s=1}^{k} 2^s\, \rho^{n_s} + \frac{1}{\rho_0}\sum_{s=k+1}^{\infty} \frac{1}{1-r_s}\; 2^s\, \rho^{\frac{1}{1-r_s}-1}$$

$$\le \frac{2^{k+1}}{(1-\rho)\rho_0} + \frac{1}{\rho_0^2}\sum_{s=k+1}^{\infty} \frac{\rho^{\frac{1}{2(1-r_s)}}}{1-r_s}\; 2^s\, \rho^{\frac{1}{2(1-r_s)}}$$

$$\le \frac{2^{k+1}}{(1-\rho)\rho_0} + \frac{2}{e\rho_0^2 \log 1/\rho}\sum_{s=k+1}^{\infty} 2^s\, \rho^{\frac{1}{2(1-r_s)}}$$

$$\le \frac{2^{k+1}}{(1-\rho)\rho_0} + \frac{2^{k+2}}{e\rho_0^2 \log 1/\rho}\sum_{s=k+1}^{\infty} 2^{s-(k+1)}\, r_{k+1}^{\frac{1}{2(1-r_s)}}$$

$$= \frac{2^{k+1}}{(1-\rho)\rho_0} + \frac{2^{k+2}}{e\rho_0^2 \log 1/\rho}\sum_{s=k+1}^{\infty} 2^{s-(k+1)}\, (r_{k+1}^{\frac{1}{2(1-r_{k+1})}})^{\frac{1-r_{k+1}}{1-r_{k+2}}\frac{1-r_{k+2}}{1-r_{k+3}}\cdots\frac{1-r_{s-1}}{1-r_s}}$$

$$\le \frac{2^{k+1}}{(1-\rho)\rho_0} + \frac{2^{k+2}}{e\rho_0^2 \log 1/\rho}\sum_{s=k+1}^{\infty} 2^{s-(k+1)}\, (\frac{1}{\sqrt{e}})^{(2^{\frac{1}{b}})^{s-(k+1)}}$$

$$= \frac{2^{k+1}}{(1-\rho)\rho_0} + \frac{2^{k+2}}{e\rho_0^2 \log 1/\rho}\sum_{s=0}^{\infty} 2^{s}\, \left(\frac{1}{\sqrt{e}}\right)^{(2^{\frac{1}{b}})^{s}}$$

$$= \frac{2^{k+1}}{(1-\rho)\rho_0} + \frac{c\, 2^{k+2}}{\log 1/\rho} \le \frac{c_1\, 2^k}{1-\rho}$$

This means that

$$\mu(|rz_1|)|g'(rz_1)| \le \mu(\rho)g'(\rho) \le \frac{c_1\, 2^k}{1-\rho}\mu(r_k) = \frac{c_1}{1-r|z_1|}.$$

If $\rho = r|z_1| \le \mu^{-1}(\frac{1}{2}) = \rho_0$, then

$$g'(\rho) = \sum_{s=1}^{\infty} n_s\, 2^s\, \rho_0^{n_s-1} \le \frac{m}{\rho_0^2}\sum_{s=1}^{\infty} 2^s\, \rho_0^{\frac{1}{2(1-r_s)}} \le \frac{m}{\rho_0^2}\sum_{s=1}^{\infty} 2^s\, (\rho_0^{\frac{1}{2(1-r_1)}})^{(2^{\frac{1}{b}})^{s-1}}$$

$$\le c \le \frac{c_1}{1-\rho}, \qquad \text{where } m = \max\left\{1, \frac{2}{e\, log 1/\rho_0}\right\}.$$

Thus

$$\mu(|rz_1|)|g'(rz_1)| \le \mu(\rho)g'(\rho) = \frac{c_1\mu(0)}{1-\rho} = \frac{c_1\mu(0)}{1-r|z_1|}.$$

(iii)

$$\left|\int_0^{rz_1} g(t)dt\right| = \left|\int_0^{r} z_1 g(sz_1)ds\right| \le \int_0^{r} |z_1 g(sz_1)|ds \le \int_0^{r} g(s)ds$$

$$= \int_0^{r^2} \frac{g(\sqrt{t})}{2\sqrt{t}}dt \le \int_0^{r^2} \frac{c_1}{2\sqrt{t}\,\mu(\sqrt{t})}dt \le \int_0^{r^2} \frac{2^b c_1}{2\sqrt{t}\,\mu(t)}dt$$

$$\le c_2 \int_0^{r^2} \frac{g(t)}{2\sqrt{t}}dt$$

$$= c_2\left(\int_0^{\frac{1}{2}} + \int_{\frac{1}{2}}^{r^2}\right)\frac{g(t)}{2\sqrt{t}}dt \le \frac{c_2}{\sqrt{2}}\{g(\frac{1}{2}) + \int_0^{r^2} g(t)\, dt\}.$$

Lemma (1.2.4)[4]: Let μ and ν be normal on $[0,1)$. Suppose φ is a holomorphic self-map of B and $\psi \in H(B)$. Then $T_{\psi,\varphi}$ is a compact operator from β_μ to β_ν if and only if for any bounded sequence $\{f_j\}$ in β_μ which converges to 0 uniformly on a compact subset of B, we have $\|T_{\psi,\varphi} f_j\|_{\beta_\nu} \to 0$ as $j \to \infty$.

Proof. The result can be proved by using Montel Theorem (1.2.7) and Lemma (1.2.2).

Lemma (1.2.5)[4]: Let μ be normal on $[0,1)$ and $\int_0^1 [\mu(t)]^{-1}\, dt < \infty$. If the sequence $\{f_j\}$ is bounded in β_μ and converges to 0 uniformly on a compact subset of B, then

$$\lim_{j\to\infty} \sup_{z\in B} |f_j(z)| = 0.$$

Proof. Let $\|f_j\|_{\beta_\mu} \le M_0$. Since $\int_0^1 [\mu(t)]^{-1}\, dt < \infty$, for any $\varepsilon > 0$, there is $0 < \eta < 1$ such that $\int_0^1 [\mu(t)]^{-1}\, dt < \varepsilon$. If $\eta < |z| < 1$, then

$$\left|f_j(z) - f_j(\frac{\eta}{|z|}z)\right| = \left|\int_{\frac{\eta}{|z|}}^1 < \nabla f_j(tz), \bar{z} > dt\right|$$

$$\le \int_{\frac{\eta}{|z|}}^1 \frac{cM_0|z|}{\mu(t|z|)}dt = \int_\eta^{|z|} \frac{cM_0}{\mu(t)}dt < cM_0\varepsilon.$$

We have $\sup_{\eta<|z|<1} |f_j(z)| \le cM_0\varepsilon + \sup_{|w|=\eta} |f_j(w)|$. Thus

$$\lim_{j\to\infty}\sup \sup_{z\in B} |f_j(z)| \le \lim_{j\to\infty}\sup \left\{\sup_{|z|\le\eta} |f_j(z)| + \sup_{\eta<|z|<1} |f_j(z)|\right\} \le cM_0\varepsilon.$$

This means that $\sup_{z\in B} |f_j(z)| \to 0$ as $j \to \infty$.

Lemma (1.2.6)[4]: Let ν be normal on $[0,1)$. A closed set K in $\beta_{\nu,0}$ is compact if and only if it is bounded and satisfies $\sup_{f\in K} \nu(|z|)|\nabla f(z)| \to 0$ as $|z| \to 1$.

Proof. The proof is similar to that of Lemma 1 in [2].

Theorem (1.2.7)[4]: Let μ and ν be normal on $[0,1)$ (suppose $a > \frac{1}{2}$ when $n > 1$), φ be a holomorphic self-map of B and $\psi \in H(B)$. (i) $T_{\psi,\varphi}$ is a bounded operator from β_μ to β_ν if and only if

$$\sup_{\substack{u\in C^n-\{0\}\\ z\in B}} \frac{\nu(|z|)|\psi(z)|}{\mu(|\varphi(z)|)}\left\{\frac{(1-|\varphi(|z|)|^2)|J\varphi(z)u|^2|< \varphi(z), J\varphi(z)u >|^2}{(1-|z|^2)|u|^2 + |< z, u >|^2}\right\}^{\frac{1}{2}} < \infty, \quad (23)$$

and

$$\sup_{z\in B} \frac{\nu(|z|)}{1-|z|^2} Q_\psi(z)\left(1 + \int_0^{|\varphi(z)|} \frac{1}{\mu(t)}dt\right) < \infty, \quad (24)$$

where $J\varphi(z)$ denotes a Jacobian matrix of $\varphi(z)$ as follows:

$$J\varphi(z) = \left(\frac{\partial\varphi_j(z)}{\partial z_k}\right)_{1\le j,\ k\le n} \quad \text{and } J\varphi(z)u = \left(\sum_{k=1}^n \frac{\partial\varphi_1(z)}{\partial z_k}u_k, \dots, \sum_{k=1}^n \frac{\partial\varphi_n(z)}{\partial z_k}u_k\right)^T.$$

(ii) $T_{\psi,\varphi}$ is a bounded operator from $\beta_{\mu,0}$ to $\beta_{\nu,0}$ if and only if (23) and (24) hold, $\psi \in \beta_{\nu,0}$ and $\psi\varphi_l \in \beta_{\nu,0}$ for all $l = 1, 2, \cdots, n$.

Proof. (i) Suppose (23) and (24) hold. For any $f \in \beta_\mu$, by Lemma (1.2.1) and Lemma (1.2.2) we obtain

$$\nu(|z|)\left|\nabla\left(T_{\varphi,\psi}f\right)(z)\right| \le \nu(|z|)\left(|\nabla\psi(z)|\left|f\left(\varphi(z)\right)\right| + |\psi(z)|\left|\nabla\left(C_\varphi f\right)(z)\right|\right)$$

$$\le c\,\nu(|z|)|\nabla\psi(z)|\left(1 + \int_0^{|\varphi(z)|}\frac{1}{\mu(t)}\,dt\right)\|f\|_{\beta_\mu}$$

$$+ \sup_{\substack{u\in C^n-\{0\}\\ z\in B}}\frac{c_1\,\nu(|z|)|\psi(z)|}{\mu(|\varphi(z)|)}\left\{\left(\frac{(1-|\varphi(z)|^2)|J\varphi(z)u|^2 + |<\varphi(z), J\varphi(z)u>|^2}{(1-|z|^2)|u|^2 + |<z,u>|^2}\right)^{\frac{1}{2}}\right.$$

$$\left.\times\frac{\mu(|\varphi(z)|)\left|<\nabla(f)\left(\varphi(z)\right), \overline{J\varphi(z)u}>\right|}{\sqrt{(1-|\varphi(|z|)|^2)|J\varphi(z)u|^2 + |<\varphi(z), J\varphi(z)u>|^2}}\right\} \le c_2\|f\|_{\beta_\mu}.$$

This means that $T_{\psi,\varphi}$ is a bounded operator from β_μ to β_ν .

Conversely, suppose $T_{\psi,\varphi}$ is a bounded operator from β_μ to β_ν. Then we can easily obtain $\psi \in \beta_\nu$ and $\psi\varphi_l \in \beta_\nu$ by taking $f(z) = 1$ and $f(z) = z_l (l = 1,\ldots,n)$ in β_μ respectively.

For any given $w \in B$ and $u \in C^n$-$\{0\}$, if $|\varphi(w)| \le \sqrt{2/3}$, it follows from $\psi \in \beta_\nu$ and $\psi\varphi_l \in \beta_\nu$ that

$$\frac{\nu(|w|)|\psi(w)|}{\mu(|\varphi(w)|)}\left\{\frac{(1-|\varphi(w)|^2)|J\varphi(w)u|^2 + |<\varphi(w), J\varphi(w)u>|^2}{(1-|w|^2)|u|^2 + |<w,u>|^2}\right\}^{\frac{1}{2}}$$

$$\le \frac{1}{\mu(\sqrt{2/3})}\nu(|w|)|\psi(w)|\frac{|J\varphi(w)u|}{\sqrt{(1-|w|^2)|u|^2 + |<w,u>|^2}}$$

$$\le \frac{1}{\mu(\sqrt{2/3})}\sum_{l=1}^{n}\nu(|w|)\frac{|<|\psi(w)|\nabla\varphi_l(w), \bar{u}>|}{\sqrt{(1-|w|^2)|u|^2 + |<w,u>|^2}}$$

$$\le \frac{1}{\mu(\sqrt{2/3})}\sum_{l=1}^{n}\nu(|w|)\frac{|<\nabla(\psi\varphi_l)\nabla(w), \bar{u}>||<\nabla\psi(w), \bar{u}>||\varphi_l(w)|}{\sqrt{(1-|w|^2)|u|^2 + |<w,u>|^2}}$$

$$\le \frac{1}{\mu(\sqrt{2/3})}\sum_{l=1}^{n}\left(\|\psi\varphi_l\|_{\beta_\nu} + \|\psi\|_{\beta_\nu}\right).$$

This shows that (23) and (24) hold.

In the following, we always assume that $|\varphi(w)| > \sqrt{2/3}$. First we suppose $\varphi(w) = r_w\, e_1$, where $r_w = |\varphi(w)|, e_1$ is a vector $(1, 0, \ldots, 0)$.

(i) If $\sqrt{(1 - r_w^2)(|\xi_2|^2 + \cdots + |\xi_n|^2)} \le |\xi_1|$ (where $(\xi_1, \ldots, \xi_n)^T = J\varphi(w)u$), we set

$$f_w(z) = \int_{r_w^2 z_1}^{r_w z_1^2} g(t)\,dt, \text{where } g \text{ is the function in Lemma (1.2.3).}$$

By the properties of normal function and Lemma (1.2.3), we have

$$\frac{\mu(|z|)}{1-|z|^2}Q_{f_w}(z)\leq c\left\{\frac{1-|z|}{1-|z_1|}\right\}^{a-\frac{1}{2}}\leq c\Rightarrow\|f_w\|_{\beta_\mu}\leq c.$$

It is clear that

$$\begin{aligned}c\|T_{\psi,\varphi}\|&\geq\|T_{\psi,\varphi}\|\|f_w\|_{\beta_\mu}\geq\|T_{\psi,\varphi}f_w\|_{\beta_v}\\&\geq\frac{v(|w|)\left|<f_w\big(\psi(w)\big)\nabla(\psi)(w)+\psi(w)\nabla\big(C_\varphi f_w\big)(w),\bar{u}>\right|}{\sqrt{(1-|w|^2)|u|^2+|<w,u>|^2}}\\&=\frac{v(|w|)|\psi(w)|\left|<\nabla(f_w)\big(\varphi(w)\big),\overline{J\varphi(w)u}>\right|}{\sqrt{(1-|w|^2)|u|^2+|<w,u>|^2}}\\&=\frac{v(|w|)|\psi(w)|r_w^2g(r_w^3)|\xi_1|}{\sqrt{(1-|w|^2)|u|^2+|<w,u>|^2}}.\end{aligned}\tag{25}$$

It follows from (25) and Lemma (1.2.3) that

$$\begin{aligned}&\frac{v(|w|)|\psi(w)|}{\mu(|\varphi(w)|)}\left\{\frac{(1-|\varphi(w)|^2)|J\varphi(w)u|^2+|<\varphi(w),J\varphi(w)u>|^2}{(1-|w|^2)|u|^2+|<w,u>|^2}\right\}^{\frac{1}{2}}\\&=\frac{v(|w|)|\psi(w)|}{\mu(|\varphi(w)|)}\left\{\frac{(1-r_w^2)(|\xi_2|^2+\cdots+|\xi_n|^2)+|\xi_1|^2}{(1-|w|^2)|u|^2+|<w,u>|^2}\right\}^{\frac{1}{2}}\\&\leq\frac{v(|w|)|\psi(w)|}{\mu(|\varphi(w)|)}\frac{\sqrt{2}|\xi_1|}{(1-|w|^2)|u|^2+|<w,u>|^2}\\&=\frac{v(|w|)|\psi(w)|r_w^2g(r_w^3)|\xi_1|}{\sqrt{(1-|w|^2)|u|^2+|<w,u>|^2}}\frac{\sqrt{2}}{r_w^2\,\mu(r_w)g(r_w^3)}\leq c_1\|T_{\psi,\varphi}\|.\end{aligned}$$

(ii) If $\sqrt{(1-r_w^2)(|\xi_2|^2+\cdots+|\xi_n|^2)}>|\xi_1|$, for $j=2,\ldots,n$, let $\theta_j=\arg\xi_j$ and $a_j=e^{-i\theta_j}$ as $\xi_j\neq 0$ or $a_j=0$ as $\xi_j=0$. We take $f_w(z)=(a_2z_2+\cdots+a_nz_n)g(r_wz_1)$. By Lemma (1.2.3) and the definition of μ,

$$\frac{\mu(|z|)}{1-|z|^2}Q_{f_w}(z)\leq c\left\{\frac{1-|z|}{1-|z|_1}\right\}^{a-\frac{1}{2}}\frac{1}{\sqrt{1-r_w^2}}\leq\frac{c}{\sqrt{1-r_w^2}}\Rightarrow\|f_w\|_{\beta_\mu}\leq c/\sqrt{1-r_w^2}$$

Similarly to the proof of (25) we can get

$$\begin{aligned}&\frac{v(|w|)|\psi(w)|}{\mu(|\varphi(w)|)}\frac{|\xi_2|+\cdots+|\xi_n|}{\sqrt{(1-|w|^2)|u|^2+|<w,u>|^2}}\\&\leq\frac{c_1\|T_{\psi,\varphi}\|}{g(r_w^2)\,\mu(r_w)\sqrt{1-r_w^2}}\leq\frac{c\|T_{\psi,\varphi}\|}{\sqrt{1-r_w^2}}.\end{aligned}$$

Thus

$$\begin{aligned}&\frac{v(|w|)|\psi(w)|}{\mu(|\varphi(w)|)}\left\{\frac{(1-|\varphi(w)|^2)|J\varphi(w)u|^2+|<\varphi(w),J\varphi(w)u>|^2}{(1-|w|^2)|u|^2+|<w,u>|^2}\right\}^{\frac{1}{2}}\\&\leq\frac{v(|w|)|\psi(w)|}{\mu(|\varphi(w)|)}\frac{\sqrt{2(1-r_w^2)(|\xi_2|^2+\cdots+|\xi_n|^2)}}{\sqrt{(1-|w|^2)|u|^2+|<w,u>|^2}}\\&\leq\sqrt{2}\frac{v(|w|)|\psi(w)|}{\mu(|\varphi(w)|)}\frac{\sqrt{1-r_w^2}(|\xi_2|+\cdots+|\xi_n|)}{\sqrt{(1-|w|^2)|u|^2+|<w,u>|^2}}\leq\sqrt{2}c\|T_{\psi,\varphi}\|\end{aligned}$$

This shows that (23) holds.

In a general situation, if $\varphi(w) \neq |\varphi(w)|e_1$, we use the unitary transformation U_w to make $\varphi(w) = \rho_w e_1 U_w$, where $\rho_w = |\varphi(w)| > \sqrt{2/3}$.
(i) If $(1 - |\varphi(w)|^2)|J\varphi(w)u|^2 \leq |<\varphi(w), J\varphi(w)u>|^2$, we set

$$g_w = f_w \circ U_w^{-1}, \qquad where\ f_w(z) = \int_{\rho_w^2 z_1}^{\rho_w z_1^2} g(t)dt.$$

By $\nabla g_w(z) = \nabla(f_w \circ U_w^{-1})(z) = (\nabla f_w)(zU_w^{-1})(U_w^{-1})^T$ and a simple computation we can get $\|g_w\|_{\beta_\mu} \leq c$. Thus

$$\begin{aligned} c\|T_{\psi,\varphi}\| &\geq \|T_{\psi,\varphi}\|\|g_w\|_{\beta_\mu} \geq \|T_{\psi,\varphi}\|_{\beta_\mu} \\ &\geq \frac{\nu(|w|)\left|<\psi(w)\nabla\left(C_\varphi g_w\right)(w) + g_w\left(\varphi(w)\right)\nabla(\psi)(w), \bar{u}>\right|}{\sqrt{(1-|w|^2)|u|^2 + |<w,u>|^2}} \\ &= \frac{\nu(|w|)|\psi(w)|\rho_w^2 g(\rho_w^3)\left|<e_1(U_w^{-1})^T, \overline{J\varphi(w)u}>\right|}{\sqrt{(1-|w|^2)|u|^2 + |<w,u>|^2}} \\ &= \frac{\nu(|w|)|\psi(w)|\rho_w^2 g(\rho_w^3)|<\varphi(w), J\varphi(w)u>|^2}{\sqrt{(1-|w|^2)|u|^2 + |<w,u>|^2}} \end{aligned}$$

So

$$\begin{aligned} &\frac{\nu(|w|)|\psi(w)|}{\mu(|\varphi(w)|)}\left\{\frac{(1-|\varphi(w)|^2)|J\varphi(w)u|^2 + |<\varphi(w), J\varphi(w)u>|^2}{(1-|w|^2)|u|^2 + |<w,u>|^2}\right\}^{\frac{1}{2}} \\ &\leq \frac{\nu(|w|)|\psi(w)|}{\mu(|\varphi(w)|)} \frac{\sqrt{2}|<\varphi(w), J\varphi(w)u>|^2}{\sqrt{(1-|w|^2)|u|^2 + |<w,u>|^2}} \leq \frac{\sqrt{2}c\|T_{\psi,\varphi}\|}{\rho_w g(\rho_w^3)\mu(\rho_w)} \leq c_1\|T_{\psi,\varphi}\|. \end{aligned}$$

(ii) If $(1-|\varphi(w)|^2)|J\varphi(w)u|^2 > |<\varphi(w), J\varphi(w)u>|^2$, let $(U_w^{-1})^T J\varphi(w)u = \left(|\xi_1|e^{i\theta_1}, \ldots, |\xi_n|e^{i\theta_n}\right)^T$. We take $g_w = f_w \circ U_w^{-1}$, where

$$f_w(z) = (e^{-i\theta_2}z_2 + \cdots. + e^{-i\theta_n}z_n)g(\rho_w z_1) \Rightarrow \|g_w\|_{\beta_\mu} \leq \frac{c}{\sqrt{1-\rho_w^2}}.$$

Similarly, we can get

$$\frac{\nu(|w|)|\psi(w)|g(\rho_w^2)(|\xi_2| + \cdots + |\xi_n|)}{\sqrt{(1-|w|^2)|u|^2 + |<w,u>|^2}} \leq \frac{c\|T_{\psi,\varphi}\|}{\sqrt{1-\rho_w^2}}. \tag{26}$$

Since $\rho_w > \sqrt{2/3}$, we have

$$\begin{aligned} |\xi_1| + \cdots + |\xi_n| &= \left|<\left(e^{-i\theta_1}, \cdots, e^{-i\theta_n}\right)(U_w^{-1})^T, \overline{J\varphi(w)u}>\right| \\ = \left|<\left(0, e^{-i\theta_2}, \cdots, e^{-i\theta_n}\right)(U_w^{-1})^T, \overline{J\varphi(w)u}> + <\left(e^{-i\theta_1}, 0, \cdots, 0\right)(U_w^{-1})^T, \overline{J\varphi(w)u}>\right| \\ &\leq |\xi_2| + \cdots + |\xi_n| + \frac{\left|<\varphi(w), \overline{J\varphi(w)u}>\right|}{\rho_w} \\ &\leq |\xi_2| + \cdots + |\xi_n| + \frac{\sqrt{1-\rho_w^2}|J\varphi(w)u|}{\rho_w} \Rightarrow |\xi_1| \leq \frac{\sqrt{1-\rho_w^2}}{\rho_w}|J\varphi(w)u| \\ &\Rightarrow |\xi_1|^2 \leq \frac{1-\rho_w^2}{2\rho_w^2 - 1}(|\xi_2|^2 + \cdots + |\xi_n|^2) \\ &\Rightarrow |\xi_1|^2 + \cdots + |\xi_n|^2 \leq \frac{\rho_w^2}{2\rho_w^2 - 1}(|\xi_2|^2 + \cdots + |\xi_n|^2) \\ &\leq 2(|\xi_2|^2 + \cdots + |\xi_n|^2). \end{aligned} \tag{27}$$

Using (26), (27), Lemma (1.2.3) and the property of unitary transformation,

$$\frac{\nu(|w|)|\psi(w)|}{\mu(|\varphi(w)|)}\left\{\frac{(1-|\varphi(w)|^2)|J\varphi(w)u|^2+|<\varphi(w),J\varphi(w)u>|^2}{(1-|w|^2)|u|^2+|<w,u>|^2}\right\}^{\frac{1}{2}}$$
$$\le \sqrt{2}\frac{\nu(|w|)|\psi(w)|}{\mu(|\varphi(w)|)}\frac{\sqrt{1-\rho_w^2}|J\varphi(w)u|}{\sqrt{(1-|w|^2)|u|^2+|<w,u>|^2}}$$
$$\le \frac{2\nu(|w|)|\psi(w)|}{\mu(|\varphi(w)|)}\frac{\sqrt{1-\rho_w^2}(|\xi_2|+\cdots+|\xi_n|)}{\sqrt{(1-|w|^2)|u|^2+|<w,u>|^2}}$$
$$\le \frac{\sqrt{2}c\|T_{\psi,\varphi}\|}{\mu(\rho_w)g(\rho_w^2)}\le c_1\|T_{\psi,\varphi}\|.$$
This shows that (23) holds.

In order to prove (24), we take

$$h_w(z) = 1 + \int_0^{<z,\varphi(w)>} g(t)\,dt\,.$$

Then $\|h_w\|_{\beta_\mu} \le c$.

Therefore,

$$c\|T_{\psi,\varphi}\| \ge \|T_{\psi,\varphi}h_w\|_{\beta_\nu} \ge c_1\nu(|w|)|\nabla\psi(w)|\,|h_w(\varphi(w))|$$
$$-\sup_{u\in C^n-\{0\}}\frac{\nu(|w|)|\psi(w)|\left|<\nabla(hw)(\varphi(w)),\overline{J\varphi(w)u}>\right|}{\sqrt{(1-|w|^2)|u|^2+|<w,u>|^2}}.$$

That is

$$\nu(|w|)|\nabla\psi(w)|\left(1+\int_0^{|\varphi(w)|^2}g(t)dt\right)$$
$$\le c_1\|T_{\psi,\varphi}\|+\sup_{u\in C^n-\{0\}}\frac{c_2\nu(|w|)|\psi(w)|g(|\varphi(w)|^2)|<\varphi(w),J\varphi(w)u>|}{\sqrt{(1-|w|^2)|u|^2+|<w,u>|^2}}$$
$$\le c_1\|T_{\psi,\varphi}\|+c_3\sup_{u\in C^n-\{0\}}\frac{\nu(|w|)|\psi(w)||<\varphi(w),J\varphi(w)u>|}{\mu(|\varphi(w)|)\sqrt{(1-|w|^2)|u|^2+|<w,u>|^2}} \qquad (28)$$

It follows from (23), (28) and Lemma (1.2.3) that

$$\nu(|w|)|\nabla\psi(w)|\left(1+\int_0^{|\varphi(w)|}\frac{1}{\mu(t)}dt\right)$$
$$\le \nu(|w|)|\nabla\psi(w)|\left(1+c\int_0^{|\varphi(w)|}g(t)dt\right)$$
$$\le \nu(|w|)|\nabla\psi(w)|\left\{1+c_1g\left(\frac{1}{2}\right)+c_1\int_0^{|\varphi(w)|^2}g(t)dt\right\}$$
$$\le c_2\|\psi\|_{\beta_\nu}+c_3\nu(|w|)|\nabla\psi(w)|\left(1+\int_0^{|\varphi(w)|^2}g(t)dt\right)$$
$$\le c_2\|\psi\|_{\beta_\nu}+c_4\|T_{\psi,\varphi}\|+c_5.$$

This shows that (24) holds.

(ii) Suppose (23) and (24) hold, $\psi \in \beta_{\nu,0}$ and $\psi\varphi_l \in \beta_{\nu,0}$ for all $l = 1,2,\dots,n$. Take any $\varepsilon > 0$. Let $f \in \beta_{\mu,0}$. Then there exists $0 < \delta_1 < 1$ such that $\mu(|z|)|\nabla f(z)| < \varepsilon, \nu(|z|)|\nabla\psi(z)| < \varepsilon$ and $\nu(|z|)|\nabla(\psi\varphi_l)(z)| < \varepsilon$ ($l = 1,2,\dots,n$) as $\delta_1 < |z| < 1$.

If $\int_0^1 [\mu(t)]^{-1}dt = \infty$, then, by Lemma (1.2.2), there exists $0 < \delta_2 < 1$ such that

$$\frac{|f(z)|}{\int_0^1 [\mu(t)]^{-1}dt} < \varepsilon \text{ as } \delta_2 < |z| < 1.$$

Thus for $|\varphi(z)| > \delta = \max(\delta_1, \delta_2)$ and $\int_0^1 [\mu(t)]^{-1}dt = \infty$, we get

$$\begin{aligned}
&\nu(|z|)\left|\nabla\left(T_{\psi,\varphi}f\right)(z)\right| \\
&\le \nu(|z|)|\nabla\psi(z)|\left(1+\int_0^{|\varphi(z)|}\frac{1}{\mu(t)}dt\right)\frac{\left|f\left(\varphi(z)\right)\right|}{\int_0^{|\varphi(z)|}[\mu(t)]^{-1}dt} \\
&+\sup_{\substack{u\in C^n-\{0\}\\ z\in B}}\frac{c\,\nu(|z|)|\psi(z)|}{\mu(|\varphi(z)|)}\left\{\left(\frac{(1-|\varphi(z)|^2)|J\varphi(z)u|^2+|<\varphi(z),J\varphi(z)u>|^2}{(1-|z|^2)|u|^2+|<z,u>|^2}\right)^{\frac{1}{2}}\right. \\
&\left.\times\frac{\mu(|\varphi(z)|)\left|<\nabla(f)\left(\varphi(z)\right),\overline{J\varphi(z)u}>\right|}{\sqrt{(1-|\varphi(z)|^2)|J\varphi(z)u|^2+|<\varphi(z),J\varphi(z)u>|^2}}\right\} \\
&\le \frac{c_1\left|f\left(\varphi(z)\right)\right|}{\int_0^{|\varphi(z)|}[\mu(t)]^{-1}dt}+c_2\varepsilon\|f\|_{\beta_\mu}\le c_3\varepsilon. \qquad (29)
\end{aligned}$$

If $|\varphi(z)| > \delta = \max(\delta_1, \delta_2)$ and $\int_0^1 [\mu(t)]^{-1}dt < \infty$, then

$$\begin{aligned}
&\nu(|z|)\left|\nabla\left(T_{\psi,\varphi}f\right)(z)\right| \le c\,\nu(|z|)|\nabla\psi(z)|\,\|f\|_{\beta_\mu} \\
&+\sup_{\substack{u\in C^n-\{0\}\\ z\in B}}\frac{c_1\nu(|z|)|\psi(z)|}{\mu(|\varphi(z)|)}\left\{\left(\frac{(1-|\varphi(z)|^2)|J\varphi(z)u|^2+|\langle\varphi(z),J\varphi(z)u\rangle|^2}{(1-|z|^2)|u|^2+|<z,u>|^2}\right)^{\frac{1}{2}}\right. \\
&\left.\times\frac{\mu(|\varphi(z)|)\left|<\nabla(f)\left(\varphi(z)\right),\overline{J\varphi(z)u}>\right|}{\sqrt{(1-|\varphi(z)|^2)|J\varphi(z)u|^2+|<\varphi(z),J\varphi(z)u>|^2}}\right\} \\
&\le c_2\varepsilon\|f\|_{\beta_\mu}. \qquad (30)
\end{aligned}$$

If $|\varphi(z)| \le \delta = \max(\delta_1, \delta_2)(|z| > \delta)$, then

$$\begin{aligned}
\nu(|z|)\left|\nabla\left(T_{\psi,\varphi}f\right)(z)\right| &\le c\,\nu(|z|)|\nabla\psi(z)|\left(1+\int_0^\delta\frac{1}{\mu(t)}dt\right)\|f\|_{\beta_\mu} \\
&+c_1\sum_{l=1}^{n}\left(\nu(|z|)|\nabla(\psi\varphi_l)(z)|+\nu(|z|)|\nabla\psi(z)|\right)\frac{\|f\|_{\beta_\mu}}{\mu(\delta)} \\
&< c\left(1+\int_0^\delta\frac{1}{\mu(t)}dt\right)\|f\|_{\beta_\mu}\varepsilon+\frac{2nc_1\|f\|_{\beta_\mu}}{\mu(\delta)}\varepsilon. \qquad (31)
\end{aligned}$$

By (29)– (31) we get $\nu(|z|)\left|\nabla\left(T_{\psi,\varphi}f\right)(z)\right| \to 0$ as $|z| \to 1$. This shows that $T_{\psi,\varphi}f \in \beta_{\nu,0}$.

Conversely, if $T_{\psi,\varphi}$ is a bounded operator from $\beta_{\mu,0}$ to $\beta_{\nu,0}$, then $\psi \in \beta_{\nu,0}$ and $\psi\varphi_l \in \beta_{\nu,0}$ by taking $f(z) = 1$ and $f(z) = z_l(l = 1, \dots, n)$ in $\beta_{\mu,0}$ respectively. The rest proof is similar to that of (a). The proof is complete.

Corollary (1.2.8)[4]: Let μ and ν be normal on $[0,1)$ (suppose $a > \frac{1}{2}$ when $n > 1$). Suppose φ is a holomorphic self-map of B. Then (a) C_φ is a bounded operator from β_μ to β_ν if and only if

$$\sup_{\substack{u\in C^n-\{0\}\\ z\in B}} \frac{\nu(|z|)}{\mu(|\varphi(z)|)} \left\{\frac{(1-|\varphi(z)|^2)|J\varphi(z)u|^2 + |< \varphi(z), J\varphi(z)u >|^2}{(1-|z|^2)|u|^2 + |< z, u >|^2}\right\}^{\frac{1}{2}} < \infty. \quad (32)$$

(b) C_φ is a bounded operator from $\beta_{\mu,0}$ to $\beta_{\nu,0}$ if and only if (32) holds and $\varphi_l \in \beta_{\nu,0}$ for all $l \in \{1,2,\dots,n\}$.

Corollary (1.2.9)[4]: Let μ and ν be normal on $[0,1)$ (suppose $a > \frac{1}{2}$ when $n > 1$). Suppose φ is a holomorphic self-map of B. (a) If

$$\sup_{z\in B} \frac{\nu(|z|)|\varphi'(z)|}{\mu(|\varphi(z)|)} < \infty. \quad (33)$$

then C_φ is a bounded operator from β_μ to β_ν , where

$$|\varphi'(z)| = \left\{\sum_{k=1}^{n}\sum_{l=1}^{n}\left|\frac{\partial\varphi_k}{\partial z_l}(z)\right|^2\right\}^{\frac{1}{2}}.$$

(b) If $\varphi_l \in \beta_{\nu,0}$ for all $l \in \{1,2,\dots,n\}$ and (33) holds, then C_φ is a bounded operator from $\beta_{\mu,0}$ to $\beta_{\nu,0}$.

Proof. (a) Suppose

$$\sup_{z\in B} \frac{\nu(|z|)|\varphi'(z)|}{\mu(|\varphi(z)|)} = M < \infty.$$

Then

$$\frac{\nu(|z|)}{\mu(|\varphi(z)|)} \left(\frac{(1-|\varphi(z)|^2)|J\varphi(z)u|^2 + |< \varphi(z), J\varphi(z)u >|^2}{(1-|z|^2)|u|^2 + |< z, u >|^2}\right)^{\frac{1}{2}}$$
$$\le \frac{1}{\mu(|\varphi(z)|)} \frac{\nu(|z|)|J\varphi(z)u|}{\sqrt{(1-|z|^2)|u|^2 + |< z, u >|^2}} \le \sum_{j=1}^{n} \frac{c\,\nu(|z|)\left|\nabla\varphi_j(z)\right|}{\mu(|\varphi(z)|)}$$
$$\le \frac{c\sqrt{n}\,\nu(|z|)\,|\varphi'(z)|}{\mu(|\varphi(z)|)} \le c\sqrt{n}M.$$

This shows that C_φ is a bounded operator from β_μ to β_ν by Corollary (1.2.8).

(b) Noting $\varphi_l \in \beta_{\nu,0}$, the proof is similar to that of (a).

Corollary (1.2.10)[4]: Let $n = 1$. Suppose μ and ν are normal on $[0,1)$, and φ is a holomorphic self-map of $D, \psi \in H(D)$. Then (a) $T_{\psi,\varphi}$ is a bounded operator from β_μ to β_ν if and only if

$$\sup_{z\in D} \frac{\nu(|z|)|\psi(z)||\varphi'(z)|}{\mu(|\varphi(z)|)} < \infty \quad (34)$$

and

$$\sup_{z\in D} \nu(|z|)|\psi'(z)|\left(1 + \int_0^{|\varphi(z)|}\frac{1}{\mu(t)}dt\right) < \infty. \quad (35)$$

(b) $T_{\psi,\varphi}$ is a bounded operator from $\beta_{\mu,0}$ to $\beta_{\nu,0}$ if and only if $\psi \in \beta_{\nu,0}, \psi\varphi \in \beta_{\nu,0}$, (34) and (35) hold.

Corollary (1.2.11)[4]: Let $n = 1$. Suppose μ and ν are normal on $[0,1)$, and φ is a holomorphic self-map of D. Then (a) C_φ is a bounded operator from β_μ to β_ν if and only if

$$\sup_{z\in D}\frac{\nu(|z|)|\varphi'(z)|}{\mu(|\varphi(z)|)}<\infty. \qquad (36)$$

(b) C_φ is a bounded operator from $\beta_{\mu,0}$ to $\beta_{\nu,0}$ if and only if $\varphi \in \beta_{\nu,0}$ and (36) holds.

Theorem (1.2.12)[4]: (i) Let μ and ν be normal on $[0,1)$ (suppose $a > \frac{1}{2}$ when $n > 1$), φ be a holomorphic self-map of B and $\psi \in H(B)$. (a) If $\int_0^1[\mu(t)]^{-1}dt < \infty$, then $T_{\psi,\varphi}$ is a compact operator from β_μ to β_ν if and only if $\psi \in \beta_\nu, \psi\varphi_l \in \beta_\nu$ for all $l \in \{1,2,\dots,n\}$ and

$$\lim_{|\varphi(z)|\to 1}\sup_{u\in C^n-\{0\}}\frac{v(|z|)|\psi(z)|}{\mu(|\varphi(z)|)}\left\{\frac{(1-|\varphi(|z|)|^2)|J\varphi(z)u|^2|<\varphi(z),J\varphi(z)u>|^2}{(1-|z|^2)|u|^2+|<z,u>|^2}\right\}^{\frac{1}{2}} = 0. \qquad (37)$$

(b) If $\int_0^1[\mu(t)]^{-1}dt = \infty$, then $T_{\psi,\varphi}$ is a compact operator from β_μ to β_ν if and only if $\psi \in \beta_\nu$, $\psi\varphi_l \in \beta_\nu$ for all $l \in \{1,2,\dots,n\}$, (37) holds and

$$\lim_{|\varphi(z)|\to 1}\frac{v(|z|)}{1-|z|^2}Q_\psi(z)\left(1+\int_0^{|\varphi(z)|}\frac{1}{\mu(t)}dt\right)=0. \qquad (38)$$

(c) $T_{\psi,\varphi}$ is a compact operator from $\beta_{\mu,0}$ to $\beta_{\nu,0}$ if and only if

$$\lim_{|z|\to 1}\sup_{u\in C^n-\{0\}}\frac{v(|z|)|\psi(z)|}{\mu(|\varphi(z)|)}\left\{\frac{(1-|\varphi(|z|)|^2)|J\varphi(z)u|^2+|<\varphi(z),J\varphi(z)u>|^2}{(1-|z|^2)|u|^2+|<z,u>|^2}\right\}^{\frac{1}{2}} = 0. \qquad (39)$$

and

$$\lim_{|z|\to 1}\frac{v(|z|)}{1-|z|^2}Q_\psi(z)\left(1+\int_0^{|\varphi(z)|}\frac{1}{\mu(t)}dt\right)=0. \qquad (40)$$

Proof. (a)–(b) Suppose (37) and (38) hold. Then for any $\varepsilon > 0$, there is $0 < \delta < 1$, when $|\varphi(z)|^2 > 1-\delta$, such that

$$\sup_{u\in C^n-\{0\}}\frac{\nu(|z|)|\psi(z)|}{\mu(|\varphi(z)|)}\left\{\frac{(1-|\varphi(z)|^2)|J\varphi(z)u|^2+|<\varphi(z),J\varphi(z)u>|^2}{(1-|z|^2)|u|^2+|<z,u>|^2}\right\}^{\frac{1}{2}}<\varepsilon \qquad (41)$$

and

$$\nu(|z|)|\nabla\psi(z)|(1+\int_0^{|\varphi(z)|}\frac{1}{\mu(t)}dt)<\varepsilon. \qquad (42)$$

Let $\{f_j\}$ be any a sequence $\{f_j\}$ which converges to 0 uniformly on compact subset of B satisfying $\|f_j\|_{\beta_\mu} \le 1$. Then $\{f_j\}$ and $\{|\nabla f_j|\}$ converges to 0 uniformly on $\{w : |w|^2 \le 1-\delta\}$.

If $|\varphi(z)|^2 > 1-\delta$ and $\int_0^1[\mu(t)]^{-1}dt < \infty$, then, by (41) we have

$$\sup_{u\in C^n-\{0\}}\frac{\nu(|z|)\left|<\nabla\left(T_{\psi,\varphi}f_j\right)(z),\bar{u}>\right|}{\sqrt{(1-|z|^2)|u|^2+|<z,u>|^2}}$$

$$\le \sup_{u\in C^n-\{0\}}\frac{\nu(|z|)\left|<\psi(z)\nabla\left(f_j\right)(\varphi(z)),\overline{J\varphi(z)u}>\right|+\left|<f_j(\varphi(z))\nabla\psi(z),\bar{u}>\right|}{\sqrt{(1-|z|^2)|u|^2+|<z,u>|^2}}$$

$$\leq \sup_{u\in C^n-\{0\}} \frac{\nu(|z|)|\psi(z)|}{\mu(|\varphi(z)|)} \left\{\frac{(1-|\varphi(z)|^2)|J\varphi(z)u|^2+|<\varphi(z),J\varphi(z)u>|^2}{(1-|z|^2)|u|^2+|<z,u>|^2}\right\}^{\frac{1}{2}}$$

$$\times \frac{\mu(|\varphi(z)|)\left|<\nabla(f_j)(\varphi(z)),\overline{J\varphi(z)u}>\right|}{\sqrt{(1-|\varphi(z)|^2)|J\varphi(z)u|^2+|<\varphi(z),J\varphi(z)u>|^2}} + \|\psi\|_{\beta_\nu}\left|f_j(\varphi(z))\right|$$

$$\leq \varepsilon\left\|f_j\right\|_{\beta_\nu} + \|\psi\|_{\beta_\nu}\left|f_j(\varphi(z))\right|. \qquad (43)$$

If $|\varphi(z)|^2 > 1-\delta$ and $\int_0^1[\mu(t)]^{-1}dt = \infty$, by (41), (42), Lemma (1.2.1) and Lemma (1.2.2) we get

$$\sup_{u\in C^n-\{0\}} \frac{\nu(|z|)\left|<\nabla\left(T_{\psi,\varphi}f_j\right)(z),\bar{u}>\right|}{\sqrt{(1-|z|^2)|u|^2+|<z,u>|^2}}$$

$$\leq \sup_{u\in C^n-\{0\}} \frac{\nu(|z|)|\psi(z)|}{\mu(|\varphi(z)|)} \left\{\frac{(1-|\varphi(z)|^2)|J\varphi(z)u|^2+|<\varphi(z),J\varphi(z)u>|^2}{(1-|z|^2)|u|^2+|<z,u>|^2}\right\}^{\frac{1}{2}}$$

$$\times \frac{\mu(|\varphi(z)|)\left|\langle\nabla\left(f_j\right)(\varphi(z)),\overline{J\varphi(z)u}\rangle\right|}{\sqrt{(1-|\varphi(z)|^2)|J\varphi(z)u|^2+|<\varphi(z),J\varphi(z)u>|^2}} + c\,\nu(|z|)|\nabla\psi(z)|\left|f_j(\varphi(z))\right|$$

$$\leq \varepsilon\left\|f_j\right\|_{\beta_\mu} + c_1\nu(|z|)|\nabla\psi(z)|\left(1+\int_0^{|\varphi(z)|}\frac{1}{\mu(t)}dt\right)\left\|f_j\right\|_{\beta_\mu}$$

$$\leq \varepsilon\left\|f_j\right\|_{\beta_\mu} + c_1\varepsilon\left\|f_j\right\|_{\beta_\mu} \leq c_2\varepsilon. \qquad (44)$$

If $|\varphi(z)|^2 \leq 1-\delta$, by $\psi \in \beta_\nu$ and $\psi\varphi_l \in \beta_\nu (l = 1,2,\dots,n)$ we have

$$\sup_{u\in C^n-\{0\}} \frac{\nu(|z|)\left|<\nabla\left(T_{\psi,\varphi}f_j\right)(z),\bar{u}>\right|}{\sqrt{(1-|z|^2)|u|^2+|<z,u>|^2}}$$

$$\leq \sup_{u\in C^n-\{0\}} \frac{\nu(|z|)|<\nabla\psi(z),\bar{u}>|\left|f_j(\varphi(z))\right|}{\sqrt{(1-|z|^2)|u|^2+|<z,u>|^2}} + \sup_{u\in C^n-\{0\}}$$

$$\times \frac{\nu(|z|)|\psi(z)|\left|\nabla f_j(\varphi(z))\right|(|<\nabla\varphi_1(z),\bar{u}>|+\cdots+|<\nabla\varphi_n(z),\bar{u}>|)}{\sqrt{(1-|z|^2)|u|^2+|<z,u>|^2}}$$

$$\leq \|\psi\|_{\beta_\mu} + \left|f_j(\varphi(z))\right|\sum_{l=1}^{n}\left(\|\psi\varphi_l\|_{\beta_\mu}+\|\psi\|_{\beta_\nu}\right)\left|\nabla f_j(\varphi(z))\right|. \qquad (45)$$

On the other hand,

$$\left\|T_{\psi,\varphi}f_j\right\|_{\beta_\nu} \leq \left|\psi(0)f_j\left(\varphi(0)\right)\right|$$

$$+\left(\sup_{|\varphi(z)|^2>1-\delta} + \sup_{|\varphi(z)|^2\leq 1-\delta}\right)\sup_{u\in C^n-\{0\}} \frac{\nu(|z|)\left|\langle\nabla\left(T_{\psi,\varphi}f_j\right)(z),\bar{u}\rangle\right|}{\sqrt{(1-|z|^2)|u|^2+|\langle z,u\rangle|^2}}. \qquad (46)$$

By (43)–(46), we have

$$\limsup_{j\to\infty}\left\|T_{\psi,\varphi}f_j\right\|_{\beta_\nu} \leq c_2\varepsilon. \qquad (47)$$

By (47) we obtain $\left\|T_{\psi,\varphi}f_j\right\|_{\beta_\nu} \to 0 \quad (j\to\infty)$. This means that $T_{\psi,\varphi}$ is a compact operator from β_μ to β_ν by Lemma (1.2.4).

Conversely, for any $l \in \{1,2,\dots,n\}$, we get $\psi \in \beta_\nu$ and $\psi\varphi_l = T_{\psi,\varphi}f \in \beta_\nu$ by taking $f(z) = 1$ or $f(z) = z_l \in \beta_\mu$.

Assume that (37) fails. Then there exist the sequence $\{z^j\} \subset B$ satisfying $r_j = |\varphi(z^j)| \to 1$ as $j \to \infty$, the sequence $\{u^j\} \subset C^n - \{0\}$ and a constant $\varepsilon_0 > 0$ such that

$$\frac{\nu(|z^j|)|\psi(z^j)|}{\mu(|\varphi(z^j)|)}\left\{\frac{\left(1-|\varphi(z^j)|^2\right)|J\varphi(z^j)u^j|^2+|<\varphi(z^j),J\varphi(z^j)u^j>|^2}{(1-|z^j|^2)|u^j|^2+|<z^j,u^j>|^2}\right\}^{\frac{1}{2}} \geq \varepsilon_0. \quad (48)$$

Since $r_j \to 1 (j \to \infty)$, we can pick out the subsequence $\{r_{j_k}\}$ satisfying

$$1-\frac{1}{2k^2} < r_{j_k} < 1.$$

To construct the sequence of functions $\{f_j\}$, we first assume that

$$\varphi(z^j) = r_j e_1 \text{ and } 1-\frac{1}{2j^2} < r_{j_k} < 1 \quad (j = 1,2,\dots,).$$

(i) For $\int_0^1 [\mu(t)]^{-1} dt < \infty$, if $\sqrt{(1-r_j^2)\left(|w_2^j|^2+\cdots+|w_n^j|^2\right)} < |w_1^j|$ (where $(w_1^j,\dots,w_n^j)^T = J\varphi(z^j)u^j$), we take

$$f_j(z) = \frac{1}{r_j}\int_0^{r_j z_1} g(t)dt - \frac{1}{r_j^j}\int_0^{r_j^j z_1} g(t)\, dt.$$

We can prove easily $\|f_j\|_{\beta_\mu} \leq c$.

Let E be any a compact subset of B. Then there exists $0 < r < 1$ such that $E \subseteq \{z : |z| \leq r\}$. Thus

$$\begin{aligned}\max_{z\in E}|f_j(z)| &= \max_{z\in E}\left|\frac{1}{r_j}\int_{r_j^j z_1}^{r_j z_1} g(t)dt - \left(\frac{1}{r_j^j}-\frac{1}{r_j}\right)\int_0^{r_j^j z_1} g(t)\, dt\right| \\ &\leq \max_{z\in E} 2|z_1|\left(1-r_j^{j-1}\right)\max_{t\in[0,r]} g(t) \\ &\leq 2\left\{1-\left(1-\frac{1}{2j^2}\right)^{j-1}\right\}\max_{t\in[0,r]} g(t) \to 0 \text{ as } j \to \infty.\end{aligned}$$

That is $\{f_j\}$ converges to 0 uniformly on compact subset of B. But, by (48) we have

$$\begin{aligned}&\|T_{\psi,\varphi}f_j\|_{\beta_\nu} \\ &\geq \frac{\nu(|z^j|)|\psi(z^j)||<\nabla(f_j)(r_j e_1),\overline{J\varphi(z^j)u^j}>|}{\sqrt{(1-|z^j|^2)|u^j|^2+|<z^j,u^j>|^2}} - c\|\psi\|_{\beta_\nu}|f_j(r_j e_1)| \\ &= \frac{\nu(|z^j|)|\psi(z^j)|}{\mu(|\varphi(z^j)|)}\left\{\frac{\left(1-|\varphi(z^j)|^2\right)|J\varphi(z^j)u^j|^2+|<\varphi(z^j),J\varphi(z^j)u^j>|^2}{(1-|z^j|^2)|u^j|^2+|<z^j,u^j>|^2}\right\}^{\frac{1}{2}} \\ &\times \frac{\mu(r_j)|w_1^j||g(r_j^2)-g(r_j^{j+1})|}{\sqrt{(1-r_j^2)\left(|w_2^j|^2+\cdots+|w_n^j|^2\right)+|w_1^j|^2}} - c\,\|\psi\|_{\beta_\nu}|f_j(r_j e_1)| \\ &\geq \frac{\varepsilon_0}{\sqrt{2}}\mu(r_j)(g(r_j^2)-g(r_j^{j+1})) - c\|\psi\|_{\beta_\nu}|f_j(r_j e_1)|\end{aligned}$$

$$\geq \frac{\varepsilon_0}{\sqrt{2}}\left\{\frac{1}{2^b}\mu(r_j^2)g(r_j^2) - \mu(r_j^{j+1})g(r_j^{j+1})\frac{\mu(r_j)}{\mu(r_j^{j+1})}\right\} - c\|\psi\|_{\beta_v}\left|f_j\left(r_j e_1\right)\right|$$

$$\geq \frac{\varepsilon_0}{\sqrt{2}}\left\{\frac{1}{2^b}\mu(r_j^2)g(r_j^2) - \mu(r_j^{j+1})g(r_j^{j+1})\left(\frac{1}{1+r_j+\cdots+r_j^j}\right)^a\right\} - c\|\psi\|_{\beta_v}\left|f_j\left(r_j e_1\right)\right|.$$

Since

$$r_j^j > \left(1-\frac{1}{2j^2}\right)^j \to 1 \ \ as\, j \to 1,$$

we can assume that $r_j^j \ > \ 1/2$. Thus, by Lemma (1.2.3) and Lemma (1.2.5) we have

$$\left\|T_{\psi,\varphi}f_j\right\|_{\beta_v} \geq \frac{\varepsilon_0}{\sqrt{2}}\left\{\frac{c_1}{2^b} - c_2\left(\frac{2}{j+2}\right)^a\right\} - c\|\psi\|_{\beta_v}\left|f_j\left(r_j e_1\right)\right| \to \frac{c_1\varepsilon_0}{2^{b+\frac{1}{2}}}\,(j\to\infty).$$

This means that

$$\lim_{j\to\infty}\left\|T_{\psi,\varphi}f_j\right\|_{\beta_v} \neq 0.$$

This contradicts the compactness of $T_{\psi,\varphi}$ by Lemma (1.2.4).

If $\sqrt{(1-r_j^2)(|w_2^j|^2+\cdots+|w_n^j|^2)} \geq \left|w_1^j\right|$, we let $\theta_k^j = \arg w_k^j$ $(k \ = \ 2,\ldots,n)$ and take

$$f_j(z) \ = \ \left(e^{-i\theta_2^j}z_2+\cdots+e^{-i\theta_n^j}z_n\right)\sqrt{1 \ - \ r_j^2}\,g\left(r_j\, z_1\right).$$

We have $\left\|f_j\right\|_{\beta_\mu} \leq c$ and $\{f_j\}$ converges to 0 uniformly on compact subset of B. But

$$\left\|T_{\psi,\varphi}f_j\right\|_{\beta_v}$$

$$\geq \frac{\nu\left(\left|z^j\right|\right)\left|\psi(z^j) < \nabla f_j\left(r_j e_1\right),\overline{J\varphi(z^j)u^j} > +f_j(\varphi(z^j)) < \nabla\psi(z^j),\bar{u} >\right|}{\sqrt{(1-|z^j|^2)|u^j|^2+|< z^j,u^j >|^2}}$$

$$= \frac{\nu\left(\left|z^j\right|\right)\left|\psi\left(z^j\right)\right|}{\mu(|\varphi(z^j)|)}\left\{\frac{\left(1-\left|\varphi\left(z^j\right)\right|^2\right)\left|J\varphi\left(z^j\right)u^j\right|^2+\left|<\varphi\left(z^j\right),J\varphi\left(z^j\right)u^j>\right|^2}{(1-|z^j|^2)|u^j|^2+|< z^j,u^j >|^2}\right\}^{\frac{1}{2}}$$

$$\times\frac{\mu\left(r_j\right)g\left(r_j^2\right)\left(\left|w_2^j\right|+\cdots+\left|w_n^j\right|\right)\sqrt{1-r_j^2}}{\sqrt{\left(1-r_j^2\right)\left(\left|w_2^j\right|^2+\cdots+\left|w_n^j\right|^2\right)+\left|w_1^j\right|^2}} > c\varepsilon_0. \tag{49}$$

This contradicts the compactness of $T_{\psi,\varphi}$ by Lemma (1.2.4).

(ii) For $\int_0^1[\mu(t)]^{-1}dt \ = \infty$, if $\sqrt{\left(1-r_j^2\right)\left(\left|w_2^j\right|^2+\cdots+\left|w_n^j\right|^2\right)} < \left|w_1^j\right|$, we set

$$f_j(z) = \frac{1}{\int_0^{r_j^2} g(t)dt}\int_0^{r_j z_1} g(t)dt\int_{r_j^2 z_1}^{r_j z_1^2} g(t)\, dt\,.$$

By Lemma (1.2.3) and $\int_0^1[\mu(t)]^{-1}dt \ = \ \infty$ we can get $\left\|f_j\right\|_{\beta_\mu} \leq c$ and $\{f_j\}$ converges to 0 uniformly on compact subset of B. But, by (48) and Lemma (1.2.3) we have

$$\|T_{\psi,\varphi}f_j\|_{\beta_\nu}$$
$$\geq \frac{\nu(|z^j|)|\psi(z^j) < \nabla f_j(r_j e_1), \overline{J\varphi(z^j)u^j} > + f_j(\varphi(z^j)) < \nabla\psi(z^j), \bar{u} >|}{\sqrt{(1-|z^j|^2)|u^j|^2 + |< z^j, u^j >|^2}}$$
$$= \frac{\nu(|z^j|)|\psi(z^j)|}{\mu(|\varphi(z^j)|)} \left\{ \frac{\left(1-|\varphi(z^j)|^2\right)|J\varphi(z^j)u^j|^2 + |< \varphi(z^j), J\varphi(z^j)u^j >|^2}{(1-|z^j|^2)|u^j|^2 + |< z^j, u^j >|^2} \right\}^{\frac{1}{2}}$$
$$\times \frac{r_j^2 g(r_j^3)\mu(r_j)|w_1^j|}{\sqrt{(1-r_j^2)\left(|w_2^j|^2 + \cdots + |w_n^j|^2\right) + |w_1^j|^2}} > c\varepsilon_0.$$

Thus

$$\lim_{j\to\infty}\|T_{\psi,\varphi}f_j\|_{\beta_\nu} \neq 0.$$

This contradicts the compactness of $T_{\psi,\varphi}$ by Lemma (1.2.4).

If $\sqrt{(1-r_j^2)\left(|w_2^j|^2 + \cdots + |w_n^j|^2\right) + |w_1^j|^2} \geq |w_1^j|$, by (49) we can get contradiction.

If there exists $\varphi(z^j)$ such that $\varphi(z^j) = |\varphi(z^j)|e_1$, then there is the unitary transformation U_j such that $\varphi(z^j) = \rho_j\, e_1 U_j\; j \in \{1, 2, \ldots, n\}$. Now $g_j = f_j \circ U_j^{-1}$ is the desired function sequence.

Next we prove that (38) holds. Assume that (38) fails, then there exist the sequence $\{z^j\} \subset B$ satisfying $|\varphi(z^j)| \to 1$as $j \to \infty$ and a constant $\varepsilon_0 > 0$ such that

$$\nu(|z^j|)|\nabla\psi(z^j)|(1 + \int_0^{|\varphi(z^j)|} \frac{1}{\mu(t)} dt) \geq \varepsilon_0. \tag{50}$$

We take again

$$f_j(z) = \frac{1}{\int_0^{|\varphi(z^j)|^2} g(t)dt} \left(\int_0^{<z,\varphi(z^j)>} g(t)dt\right)^2.$$

Then $\{f_j\}$ is bounded in β_μ and converges to 0 uniformly on compact subset of B. By Lemma (1.2.1) and Lemma (1.2.3) we have

$$\|T_{\psi,\varphi}f_j\|_{\beta_\nu} \geq c\,\nu(|z^j|)|\nabla\psi(z^j)||f_j(\varphi(z^j))|$$
$$- \sup_{u\in C^n - \{0\}} \frac{\nu(|z^j|)|\psi(z^j)||< \nabla f_j(\varphi(z^j)), \overline{J\varphi(z^j)u} >|}{\sqrt{(1-|z^j|^2)|u|^2 + |< z^j, u >|^2}}$$
$$= c\,\nu(|z^j|)|\nabla\psi(z^j)|\left(1 + \int_0^{|\varphi(z^j)|^2} g(t)dt\right) - c\,\nu(|z^j|)|\nabla\psi(z^j)|$$
$$- \sup_{u\in C^n - \{0\}} \frac{\mu(|\varphi(z^j)|)g(|\varphi(z^j)|^2)\nu(|z^j|)|\psi(z^j)||\langle\varphi(z^j), J\varphi(z^j)u\rangle|}{\mu(|\varphi(z^j)|)\sqrt{(1-|z^j|^2)|u|^2 + |< z^j, u >|^2}}$$
$$\geq c_1\,\nu(|z^j|)|\nabla\psi(z^j)|\left(1 + \int_0^{|\varphi(z^j)|} \frac{1}{\mu(t)} dt\right) - c\,\nu(|z^j|)|\nabla\psi(z^j)|$$

$$-\sup_{u\in C^n-\{0\}}\frac{c_2\,\nu(|z^j|)\left|\psi(z^j)\right|\left|<\varphi(z^j),J\varphi(z^j)u>\right|}{\mu(|\varphi(z^j)|)\sqrt{(1-|z^j|^2)|u|^2+|<z^j,u>|^2}} \tag{51}$$

We know that a compact operator is a bounded operator. Thus

$$\sup_{z\in B}\nu(|z|)|\nabla\psi(z)|\left(1+\int_0^{|\varphi(z)|}\frac{1}{\mu(t)}dt\right)<\infty.$$

If $\int_0^1[\mu(t)]^{-1}dt=\infty$, we have

$$\lim_{|\varphi(z)|\to 1}\nu(|z|)|\nabla\psi(z)|=\ 0. \tag{52}$$

By (50)–(52) and (37) we get

$$\lim_{j\to\infty}\left\|T_{\psi,\varphi}f_j\right\|_{\beta_\nu}\neq 0.$$

This contradicts the compactness of $T_{\psi,\varphi}$ by Lemma (1.2.4). This shows that (38) holds.

(c) For any $f\ \in\ \beta_{\mu,0}$, we have

$$\nu(|z|)\left|\nabla\left[T_{\psi,\varphi}(f)\right](z)\right|$$
$$\leq c\nu(z\,)|\nabla(z)|\left(1+\int_0^{|\varphi(z)|}\frac{1}{\mu(t)}dt\right)\|f\|_{\beta_\mu}+\sup_{z\in C^n-\{0\}}\frac{c_1\,\nu(|z|)|\psi(z)|}{\mu(|\varphi(z)|)}$$
$$\times\left\{\frac{(1-|\varphi(z)|^2)|J\varphi(z)u|^2+|<\varphi(z),J\varphi(z)u>|^2}{(1-|z|^2)|u|^2+|<z,u>|^2}\right\}^{\frac{1}{2}}\|f\|_{\beta_\mu}.$$

By (39) and (40), the above inequality implies that

$$\lim_{|z|\to 1}\sup_{\|f\|_{\beta_\mu}\leq 1}\nu(|z|)\left|\nabla\left[T_{\psi,\varphi}(f)\right](z)\right|=0$$

This shows that $T_{\psi,\varphi}$ is a compact operator from $\beta_{\mu,0}$ to $\beta_{\nu,0}$ from Lemma (1.2.6).

Conversely, if $T_{\psi,\varphi}$ is a compact operator from $\beta_{\mu,0}$ to $\beta_{\nu,0}$, then $\psi,\psi\varphi_l\ \in\ \beta_{\nu,0}(l\ =\ 1,2,\dots,n)$. We can prove that (37) and (38) hold as the proof of Theorem B. By (37), (38) and $\psi,\psi\varphi_l\ \in\ \beta_{\nu,0}$, this implies that (39) and (40) hold. The proof is complete.

Corollary (1.2.13)[4]: Let μ and ν be normal on $[0,1)$ (suppose $a>\frac{1}{2}$ when $n\ >\ 1$). Suppose φ is a holomorphic self-map of B. Then (a) C_φ is a compact operator from β_μ to β_ν if and only if

$$\lim_{|\varphi(z)|\to 1}\sup_{u\in C^n-\{0\}}\frac{\nu(|z|)}{\mu(|\varphi(z)|)}\left\{\frac{(1-|\varphi(z)|^2)|J\varphi(z)u|^2+|<\varphi(z),J\varphi(z)u>|^2}{(1-|z|^2)|u|^2+|<z,u>|^2}\right\}^{\frac{1}{2}}=0$$

and $\varphi_l\ \in\ \beta_\nu$ for all $l\ \in\ \{1,2,\dots,n\}$.

(b) C_φ is a compact operator from $\beta_{\mu,0}$ to $\beta_{\nu,0}$ if and only if

$$\lim_{|\varphi(z)|\to 1}\sup_{u\in C^n-\{0\}}\frac{\nu(|z|)}{\mu(|\varphi(z)|)}\left\{\frac{(1-|\varphi(z)|^2)|J\varphi(z)u|^2+|<\varphi(z),J\varphi(z)u>|^2}{(1-|z|^2)|u|^2+|<z,u>|^2}\right\}^{\frac{1}{2}}$$
$$=0.$$

Corollary (1.2.14)[4]: Let μ and ν be normal on $[0,1)$ (suppose $a>\frac{1}{2}$ when $n\ >\ 1$). Suppose φ is a holomorphic self-map of B. (a) If

$$\lim_{|z|\to 1}\frac{\nu(|z|)|\varphi'(z)|}{\mu(|\varphi(z)|)}=0$$

and $\varphi_l\ \in\ \beta_\nu$ for all $l\ \in\ \{1,2,\dots,n\}$, then C_φ is a compact operator from β_μ to β_ν.

(b) If

$$\lim_{|z|\to 1} \frac{\nu(|z|)|\varphi'(z)|}{\mu(|\varphi(z)|)} = 0$$

then C_φ is a compact operator from $\beta_{\mu,0}$ to $\beta_{\nu,0}$.

Corollary (1.2.15)[4]: Let $n = 1$. Suppose μ and ν are normal on $[0,1)$, and φ is a holomorphic self-map of $D, \psi \in H(D)$. Then (a) $T_{\psi,\varphi}$ is a compact operator from β_μ to β_ν if and only if

(a)

$$\psi \in \beta_\nu, \psi\varphi \in \beta_\nu \text{ and } \lim_{|\varphi(z)|\to 1} \frac{\nu(|z|)|\psi(z)||\varphi'(z)|}{\mu|\varphi(z)|} = 0 \text{ as } \int_0^1 \frac{1}{\mu(t)} dt < \infty,$$

(b)

$$\psi \in \beta_\nu, \psi\varphi \in \beta_\nu \text{ and } \lim_{|\varphi(z)|\to 1} \frac{\nu(|z|)|\psi(z)||\varphi'(z)|}{\mu|\varphi(z)|} = 0$$

and

$$\lim_{|\varphi(z)|\to 1} \nu(|z|)|\psi'(z)| \left(1 + \int_0^{|\varphi(z)|} \frac{1}{\mu(t)} dt\right) = 0 \text{ as } \int_0^1 \frac{1}{\mu(t)} dt = \infty,$$

(b) $T_{\psi,\varphi}$ is a compact operator from $\beta_{\mu,0}$ to $\beta_{\nu,0}$ if and only if

$$\lim_{|z|\to 1} \frac{\nu(|z|)|\psi(z)||\varphi'(z)|}{\mu|\varphi(z)|} = 0$$

and

$$\lim_{|z|\to 1} \nu(|z|)|\psi'(z)| \left(1 + \int_0^{|\varphi(z)|} \frac{1}{\mu(t)} dt\right) = 0.$$

Corollary (1.2.16)[4]: Let $n = 1$. Suppose μ and ν are normal on $[0,1)$, and φ is a holomorphic self-map of D. Then (a) C_φ is a compact operator from β_μ to β_ν if and only if $\varphi \in \beta_\nu$ and

$$\lim_{|\varphi(z)|\to 1} \frac{\nu(|z|)|\varphi'(z)|}{\mu|\varphi(z)|} = 0$$

(b) C_φ is a compact operator from $\beta_{\mu,0}$ to $\beta_{\nu,0}$ if and only if

$$\lim_{|z|\to 1} \frac{\nu(|z|)|\varphi'(z)|}{\mu|\varphi(z)|} = 0.$$

Section (1.3): Bergman Spaces of the Unit Ball

For any positive integer n we let

$$\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$$

denote the n-dimensional complex Euclidean space. For any two points $z = (z_1, \cdots, z_n)$ and $w = (w_1, \cdots, w_n)$ in $\mathbb{C}^n$ we write

$$\langle z, w\rangle = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n,$$

and

$$|z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$

The open unit ball in $\mathbb{C}^n$ is the set

$$\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}.$$

The space of holomorphic functions in $\mathbb{B}_n$ will be denoted by $H(\mathbb{B}_n)$.

Let dv be Lebesgue volume measure on $\mathbb{B}_n$, normalized so that $v(\mathbb{B}_n) = 1$. For any $\alpha > -1$ we let

$$dv_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha \, dv(z),$$
where c_α is a positive constant chosen so that $v_\alpha(\mathbb{B}_n) = 1$. The weighted Bergman space A^p_α, where $p > 0$, consists of functions $f \in H(\mathbb{B}_n)$ such that
$$\int_{\mathbb{B}_n} |f(z)|^p \, dv_\alpha(z) < \infty.$$
The space A^2_α is a Hilbert space with inner product
$$\langle f, g \rangle = \int_{\mathbb{B}_n} f(z)\overline{g(z)} dv_\alpha(z).$$
Every holomorphic $\varphi : \mathbb{B}_n \to \mathbb{B}_n$ induces a composition operator
$$C_\varphi : H(\mathbb{B}_n) \to H(\mathbb{B}_n),$$
namely, $C_\varphi f = f \circ \varphi$. When $n = 1$, it is well known that C_φ is always bounded on A^p_α; and C_φ is compact on A^p_α if and only if
$$\lim_{|z|\to 1^-} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0.$$
See [20] and [21].

When $n > 1$, not every composition operator is bounded on A^p_α. For example, it can easily be checked with Taylor coifficients that the composition operator C_φ is not bounded on A^2_α when
$$\varphi(z) = (\pi(z), 0, \cdots, 0),$$
Where
$$\pi(z) = \sqrt{n^n} z_1 \cdots z_n.$$
See [20] for more examples.

The main result is the following.

Theorem (1.3.1)[18]: Suppose $p > 0$ and $\alpha > -1$. If the composition operator C_φ is bounded on A^q_β for some $q > 0$ and $-1 < \beta < \alpha$, then C_φ is compact on A^p_α if and only if
$$\lim_{|z|\to 1^-} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0. \qquad (52)$$
Note that the compactness of C_φ on A^p_α always implies condition (52); we do not need any assumption on φ for this half of the theorem. The assumption that C_φ be bounded on A^q_β for some $\beta < \alpha$ is needed only for the other half the theorem. The exponents p and q are not important.

We begin with the notion of compact composition operators on A^p_α.

When $p > 1$, the Bergman space A^p_α is a reflexive Banach space (see [17] for more information about Bergman spaces), and all reasonalble definitions of compactness of C_φ on A^p_α are equivalent. In general, for any $p > 0$, we say that the composition operator C_φ is compact on A^p_α if
$$\lim_{k\to\infty} \int_{\mathbb{B}_n} \left|C_\varphi f_k\right|^p dv_\alpha = 0$$
whenever $\{f_k\}$ is a bounded sequence in A^p_α that converges to 0 uniformly on compact subsets of $\mathbb{B}_n$.
For any holomorphic $\varphi : \mathbb{B}_n \to \mathbb{B}_n$ we can define a positive Borel measure $\mu_{\varphi,\alpha}$ on $\mathbb{B}_n$ as follows. Given a Borel set E in $\mathbb{B}_n$, we set

$$\mu_{\varphi,\alpha}(E) = v_\alpha(\varphi^{-1}(E)) = c_\alpha \int_{\varphi^{-1}(E)} (1 - |z|^2)^\alpha \, dv(z).$$

Obviously, $\mu_{\varphi,\alpha}$ is the pullback measure of dv_α under the map φ. Therefore, we have the following change of variables formula:

$$\int_{\mathbb{B}_n} f(\varphi)\, dv_\alpha = \int_{\mathbb{B}_n} f d\mu_{\varphi,\alpha}, \tag{53}$$

where f is either nonnegative or belongs to $L^1(\mathbb{B}_n, d\mu_{\varphi,\alpha})$. In particular, the composition operator C_φ is bounded on A_α^p if and only if there exists a constant $C > 0$ such that

$$\int_{\mathbb{B}_n} |f|^p d\mu_{\varphi,\alpha} \leq C \int_{\mathbb{B}_n} |f|^p dv_\alpha \tag{54}$$

for all $f \in A_\alpha^p$. Measures satisfying this condition are called Carleson measures for the Bergman space A_α^p.

Similarly, a positive Borel measure μ on $\mathbb{B}_n$ is called a vanishing Carleson measure for the Bergman space A_α^p if

$$\lim_{k\to\infty} \int_{\mathbb{B}_n} |f_k|^p \, d\mu = 0 \tag{55}$$

whenever $\{f_k\}$ is a bounded sequence in A_α^p that converges to 0 uniformly on compact subsets of $\mathbb{B}_n$. In particular, a composition operator C_φ is compact on A_α^p if and only if the pullback measure $\mu_{\varphi,\alpha}$ is a vanishing Carleson measure for A_α^p.

It is well known that Carleson (and vanishing Carleson) measures for the Bergman space A_α^p is indenpendent of p. More precisely, the following result holds.

Lemma (1.3.2)[18]: Suppose $p > 0$ and $\alpha > -1$. Then the following conditions are equivalent for any positive Borel measure μ on $\mathbb{B}_n$.

(i) μ is a Carleson measure for A_α^p, that is, there exists a constant $C >$ 0 such that

$$\int_{\mathbb{B}_n} |f|^p \, d\mu \leq C \int_{\mathbb{B}_n} |f|^p \, dv_\alpha$$

for all $f \in A_\alpha^p$.

(ii) For some (or each) $R > 0$ there exists a constant $C > 0$ (depending on R and α but independent of a) such that

$$\mu(D(a,R)) \leq C v_\alpha(D(a,R))$$

for all $a \in \mathbb{B}_n$, where $D(a,R)$ is the Bergman metric ball at a with radius R.

Proof. See [24] for example.

A consequence of the above lemma is the following well-known result about composition operators; see [20].

Corollary (1.3.3)[18]: Suppose $p > 0, q > 0$, and $\alpha > -1$. Then C_φ is bounded on A_α^p if and only if C_φ is bounded on A_α^q.

A similar characterization of vanishing Carleson measures for A_α^p also holds.

Lemma (1.3.4)[18]: Suppose $p > 0$ and $\alpha > -1$. The following two conditions are equivalent for a positive Borel measure on $\mathbb{B}_n$.

(i) μ is a vanishing Carleson measure for A_α^p.

(ii) For some (or any) $R > 0$ we have

$$\lim_{|a|\to 1^-} \frac{\mu(D(a,R))}{v_\alpha(D(a,R))} = 0.$$

Proof. See [24] for example.

As a result of the above lemma we see that the compactness of C_φ on A_α^p is indenpendent of p. We state this as the following corollary which can be found in [20] as well.

Corollary (1.3.5)[18]: Suppose $p > 0, q > 0$, and $\alpha > -1$. Then C_φ is compact on A_α^p if and only if C_φ is compact on A_α^q.

We need two more technical lemmas. The first of which is called Schur's test and concerns the boundedness of integral operators on L^p spaces. Thus we consider a measure space (X, μ) and an integral operator

$$T\,f(x) = \int_X H(x,y) f(y) d\mu(y), \tag{56}$$

where H is a nonnegative measurable function on $X \times X$.

Lemma (1.3.6)[18]: Suppose that there exists a positive measurable function h on X such that

$$\int_X H(x,y)h(y)\, d\mu(y) \le Ch(x)$$

for almost all x and

$$\int_X H(x,y)h(x)\, d\mu(x) \le Ch(y)$$

for almost all y, where C is a positive constant. Then the integral operator T defined in (56) is bounded on $L^2(X, d\mu)$. Moreover, the norm of T on $L^2(X, d\mu)$ is less than or equal to the constant C.

Proof. See [23] or [17].

Lemma (1.3.7)[18]: Suppose $\alpha > -1$ and $t > 0$. Then there exists a constant $C > 0$ such that

$$\int_{\mathbb{B}_n} \frac{dv_\alpha(w)}{|1 - \langle z, w\rangle|^{n+1+\alpha+t}} \le \frac{C}{(1 - |z|^2)^t}$$

for all $z \in \mathbb{B}_n$.

Proof. See [22].

In order to understand the mild assumption made in the statement of the main theorem, we show how the boundedness and compactness of composition operators on Bergman spaces can be described in terms of Bergman type kernel functions.

Theorem (1.3.8)[18]: Suppose $p > 0, \alpha > -1$, and $t > 0$. Then the composition operator C_φ is bounded on A_α^p if and only if

$$\sup_{a\in\mathbb{B}_n} (1-|a|^2)^t \int_{\mathbb{B}_n} \frac{dv_\alpha(z)}{|1 - \langle \alpha, \varphi(z)\rangle|^{n+1+\alpha+t}} < \infty. \tag{57}$$

Proof. It follows from Lemma (1.3.7) that the boundedness of C_φ on A_α^p implies condition (57).

Next we assume that condition (57) holds. Then by the change of variables formula (53) there exists a constant $C > 0$ such that

$$(1-|a|^2)^t \int_{\mathbb{B}_n} \frac{d\mu_{\varphi,\alpha}(z)}{|1 - \langle \alpha, z\rangle|^{n+1+\alpha+t}} \le C$$

for all $a \in \mathbb{B}_n$. For any fixed positive radius R we have

$$(1-|a|^2)^t \int_{D(a,R)} \frac{d\mu_{\varphi,\alpha}(z)}{|1 - \langle \alpha, z\rangle|^{n+1+\alpha+t}} \le C$$

for all $a \in \mathbb{B}_n$. It is well known that

$$|1 - \langle \alpha, z\rangle| \sim 1 - |a|^2$$

for $z \in D(a,R)$, and it is also well known that

$$(1 - |a|^2)^{n+1+\alpha} \sim v_\alpha(D(a,R));$$

see [24]. It follows that there exists another positive constant C (independent of a) such that

$$\mu_{\varphi,\alpha}(D(a,R)) \le C v_\alpha(D(a,R))$$

for all $a \in \mathbb{B}_n$. By Lemma (1.3.2), the measure $\mu_{\varphi,\alpha}$ is Carleson for A_α^p, and so the composition operator C_φ is bounded on A_α^p.

This result is probably well known to experts in the field. The main point here is that t can be an arbitrary positive constant. This also tells us roughly how far away the boundedness of C_φ on A_α^p is from that of C_φ on A_β^p.

Corollary (1.3.9)[18]: Suppose $p > 0, q > 0$, and $-1 < \beta < \alpha$. If C_φ is bounded on A_β^q, then C_φ is bounded on A_α^p.

Proof. Write $\alpha = \beta + \epsilon$ with $\epsilon > 0$. Since

$$\frac{(1 - |z|^2)^\epsilon}{|1-\langle a, \varphi(z)\rangle|^\epsilon} \le C_1 \left(\frac{1 - |z|^2}{1-|\varphi(z)|^2}\right)^\epsilon \le C_2,$$

where the last inequality is an easy consequence of Schwarz lemma for the unit ball, we have

$$\int_{\mathbb{B}_n} \frac{(1-|w|^2)^\alpha \, dv(w)}{|1 - \langle \alpha, \varphi(w)\rangle|^{n+1+\alpha+t}} \le C_2 \int_{\mathbb{B}_n} \frac{(1-|w|^2)^\beta \, dv(w)}{|1 - \langle \alpha, \varphi(w)\rangle|^{n+1+\beta+t}}$$

This shows that

$$\sup_{a\in\mathbb{B}_n} (1-|a|^2)^t \int_{\mathbb{B}_n} \frac{dv_\beta(z)}{|1 - \langle \alpha, \varphi(z)\rangle|^{n+1+\beta+t}} < \infty$$

Implies

$$\sup_{a\in\mathbb{B}_n} (1-|a|^2)^t \int_{\mathbb{B}_n} \frac{dv_\alpha(z)}{|1 - \langle \alpha, \varphi(z)\rangle|^{n+1+\alpha+t}} < \infty.$$

The desired result then follows from Theorem (1.3.8).

A similar argument gives the following characterization of the compactness of C_φ on A_α^p.

Theorem (1.3.10)[18]: Suppose $p > 0, \alpha > -1$, and $t > 0$. Then C_φ is compact on A_α^p if and only if

$$\lim_{|a|\to 1^-} (1-|a|^2)^t \int_{\mathbb{B}_n} \frac{dv_\alpha(z)}{|1 - \langle \alpha, \varphi(z)\rangle|^{n+1+\alpha+t}} = 0 \qquad (58)$$

Corollary (1.3.11)[18]: Suppose $p > 0, q > 0$, and $-1 < \beta < \alpha$. Then the compactness of C_φ on A_β^q implies the compactness of C_φ on A_α^p.

We do not need Hardy spaces, we mention here that if C_φ is bounded (or compact) on a Hardy space H^q of the unit ball, then C_φ is bounded (or compact) on very Bergman space A^p_α. This result, along with Corollaries (1.3.9) and (1.3.11) above, can be found in [19].

Theorem (1.3.12)[18]: Suppose $p > 0$ and $\alpha > -1$. If C_φ is bounded on A^q_β for some $q > 0$ and $-1 < \beta < \alpha$, then C_φ is compact on A^p_α if and only if

$$\lim_{|z|\to 1^-} \frac{1-|z|^2}{|1-|\varphi(z)|^2} = 0. \tag{59}$$

Proof. According to Corollary (1.3.5), we may assume that $p = 2$.

The normalized reproducing kernels of A^2_α are given by

$$k_z(w) = \frac{(1 - |z|^2)^{(n+1+\alpha)/2}}{(1 - \langle w,z\rangle)^{n+1+\alpha}}.$$

Each k_z is a unit vector in A^2_α and it is clear that

$$\lim_{|z|\to 1^-} k_z(w) = 0, w \in \mathbb{B}_n.$$

Furthermore, the convergence is uniform when w is restricted to any compact subset of $\mathbb{B}_n$. A standard computation shows that

$$\int_{\mathbb{B}_n} \left|C^*_\varphi k_z\right|^2 dv_\alpha = \left(\frac{1 - |z|^2}{1-|\varphi(z)|^2}\right)^{n+1+\alpha},$$

so the compactness of C_φ on A^2_α (which is the same as the compactness of C^*_φ on A^2_α) implies condition (59).

We proceed to show that condition (59) implies the compactness of C_φ on A^2_α, provided that C_φ is bounded on A^q_β for some $\beta \in (-1,\alpha)$. An easy computation shows that the operator

$$C_\varphi C^*_\varphi : A^2_\alpha \to A^2_\alpha$$

admits the following integral representation:

$$C_\varphi C^*_\varphi f(z) = \int_{\mathbb{B}_n} \frac{f(w)\, dv_\alpha(w)}{|1 - \langle\varphi(z),\varphi(w)\rangle|^{n+1+\alpha}}, \qquad f \in A^2_\alpha. \tag{60}$$

We will actually prove the compactness of $C_\varphi C^*_\varphi$ on A^2_α, which is equivalent to the compactness of C_φ on A^2_α. In fact, our arguments will prove the compactness of the following integral operator on $L^2(\mathbb{B}_n, dv_\alpha)$:

$$Tf(z) = \int_{\mathbb{B}_n} \frac{f(w)\, dv_\alpha(w)}{|1 - \langle\varphi(z),\varphi(w)\rangle|^{n+1+\alpha}}. \tag{61}$$

For any $r \in (0,1)$ let χ_r denote the characteristic function of the set $\{z \in \mathbb{C}^n : r < |z| < 1\}$. Consider the following integral operator on $L^2(\mathbb{B}_n, dv_\alpha)$:

$$T_r f(z) = \int_{\mathbb{B}_n} H_r(z,w) f(w) dv_\alpha(w), \tag{62}$$

where

$$H_r(z,w) = \frac{\chi_r(z)\chi_r(w)}{|1 - \langle\varphi(z),\varphi(w)\rangle|^{n+1+\alpha}}$$

is a nonnegative integral kernel. We are going to estimate the norm of T_r on $L^2(\mathbb{B}_n, dv_\alpha)$ in terms of the quantity

$$M_r = \sup_{r<|z|<1} \frac{1-|z|^2}{1-|\varphi(z)|^2}.$$

We do this with the help of Schur's test.

Let $\alpha = \beta + \sigma$, where $\sigma > 0$, and consider the function

$$h(z) = (1 - |z|^2)^{-\sigma}, z \in \mathbb{B}_n.$$

We have

$$\int_{\mathbb{B}_n} H_r(z,w)h(w)dv_\alpha(w) = \frac{c_\alpha}{c_\beta}\int_{\mathbb{B}_n} \frac{\chi_r(z)\chi_r(w)dv_\beta(w)}{|1 - \langle\varphi(z),\varphi(w)\rangle|^{n+1+\beta+\sigma}}$$
$$\leq \frac{c_\alpha}{c_\beta}\int_{\mathbb{B}_n} \frac{\chi_r(z)dv_\beta(w)}{|1 - \langle\varphi(z),\varphi(w)\rangle|^{n+1+\beta+\sigma}}.$$

By the boundedness of C_φ on A_β^q, there exists a constant $C_1 > 0$, independent of r and z, such that

$$\int_{\mathbb{B}_n} H_r(z,w)h(w)dv_\alpha(w) \leq C_1\chi_r(z)\int_{\mathbb{B}_n} \frac{dv_\beta(w)}{|1 - \langle\varphi(z),\varphi(w)\rangle|^{n+1+\beta+\sigma}}.$$

We apply Lemma (1.3.7) to find another positive constant C_2, independent of r and z, such that

$$\int_{\mathbb{B}_n} H_r(z,w)h(w)dv_\alpha(w) \leq \frac{C_2\chi_r(z)}{(1-|\varphi(z)|^2)^\sigma}$$
$$= C_2\chi_r(z)\left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^\sigma h(z)$$
$$\leq C_2 M_r^\sigma h(z)$$

for all $z \in \mathbb{B}_n$. By the symmetry of $H_r(z,w)$, we also have

$$\int_{\mathbb{B}_n} H_r(z,w)h(z)dv_\alpha(z) \leq C_2 M_r^\sigma h(w)$$

for all $w \in \mathbb{B}_n$. It follows from Lemma (1.3.6) that the operator T_r is bounded on $L^2(\mathbb{B}_n, dv_\alpha)$ and the norm of T_r on $L^2(\mathbb{B}_n, dv_\alpha)$ does not exceed the constant $C_2 M_r^\sigma$.

Now fix some $r \in (0,1)$ and fix a bounded sequence $\{f_k\}$ in A_α^2 that converges to 0 uniformly on every compact subset of $\mathbb{B}_n$. In particular, $\{f_k\}$ converges to 0 uniformly on $|z| \leq r$. We use (60) to write

$$C_\varphi C_\varphi^* f_k(z) = F_k(z) + G_k(z), z \in \mathbb{B}_n,$$

where

$$F_k(z) = \int_{|w|\leq r} \frac{f_k(w)dv_\alpha(w)}{|1 - \langle\varphi(z),\varphi(w)\rangle|^{n+1+\alpha}}$$

and

$$G_k(z) = \int_{\mathbb{B}_n} \frac{\chi_r(w)f_k(w)dv_\alpha(w)}{|1 - \langle\varphi(z),\varphi(w)\rangle|^{n+1+\alpha}}$$

Since $\{f_k(w)\}$ converges to 0 uniformly on for $|w| \leq r$, we have

$$\lim_{k\to\infty}\int_{\mathbb{B}_n} |F_k(z)|^2 dv_\alpha(z) = 0$$

For any fixed $z \in \mathbb{B}_n$, the weak convergence of $\{f_k\}$ to 0 in $L^2(\mathbb{B}_n, dv_\alpha)$ implies that $G_k(z) \to 0$ as $k \to \infty$. In fact, by splitting the ball into $|z| \leq \delta$ and $\delta < |z| < 1$, it is easy to show that

$$\lim_{k\to\infty} G_k(z) = 0$$

uniformly for z in any compact subset of $\mathbb{B}_n$.

It follows from the definition of T_r that

$$\int_{\mathbb{B}_n} |G_k|^2 dv_\alpha \le \int_{|z|\le r} |G_k|^2 dv_\alpha + \int_{\mathbb{B}_n} |T_r(|f_k|)|^2 dv_\alpha.$$

Since $\{f_k\}$ is bounded in $L^2(\mathbb{B}_n, dv_\alpha)$, and since the norm of the operator T_r on $L^2(\mathbb{B}_n, dv_\alpha)$ does not exceed $C_2 M_r^\sigma$, we can find a constant $C_3 > 0$, independent of r and k, such that

$$\int_{\mathbb{B}_n} |T_r(|f_k|)|^2 dv_\alpha \le C_3 M_r^{2\sigma}$$

for all k. Combining this with

$$\lim_{k\to\infty} \int_{|z|\le r} |G_k|^2 dv_\alpha = 0$$

we obtain

$$\limsup_{k\to\infty} \int_{\mathbb{B}_n} |G_k|^2 dv_\alpha \le C_3 M_r^{2\sigma}.$$

This along with the estimates for F_k in the previous paragraph gives

$$\limsup_{k\to\infty} \int_{\mathbb{B}_n} \left|C_\varphi C_\varphi^* f_k\right|^2 dv_\alpha \le C_3 M_r^{2\sigma}.$$

Since r is arbitrary and $M_r \to 0$ as $r \to 1^-$ (which is equivalent to the condition in (59)), we conclude that

$$\lim_{k\to\infty} \int_{\mathbb{B}_n} \left|C_\varphi C_\varphi^* f_k\right|^2 dv_\alpha = 0.$$

So C_φ is compact on A_α^2, and the proof of the theorem is complete.

Note that when $n = 1, C_\varphi$ is bounded on every Bergman space A_β^q, so the characterization of compact composition operators on A_α^p does not need any extra assumption. However, our proof here still works. The idea of using Schur's test to prove the compactness of composition operators seems to be new even in the case $n = 1$.

A similar compactness result was proven in [19] for composition operators on A_α^p. But the condition in [19] involves the derivatives of φ and is much stronger than our condition here.

Chapter 2
Isometries and Weighted Composition Operators

We show that into isometries and disjointness preserving linear maps from $C_0(X)$ into $C_0(Y)$ are essentially weighted composition operators $Tf = h \cdot f \circ \phi$ for some continuous map ϕ and some continuous scalar-valued function h.

Section (2.1): Function Spaces

Let A and B be Banach spaces. By an isometry from A onto B we mean a linear, norm preserving and surjective map between these Banach spaces. The isometries of most of the well-known Banach spaces have been described. The classical Banach-Stone Theorem states that any isometry from $C(X)$ onto $C(Y)$ is induced by a homeomorphism of Y and X. This result has been extended to various other Banach spaces by Nagasawa for function algebras [41]; by Amir [26], Cambern [28] and Cengiz [32] for regular subspaces; by Cambern and Pathak [30], Pathak [42] and Pathak and Vasavada [44] for spaces of differentiable functions; by de Leeuw [40] and Roy [46] for Lipschitz functions; by Pathak [43], Vasavada [47] and Rao and Roy [45] for absolutely continuous functions. In all the above-mentioned the following situation was considered.

Let A be a subspace of a Banach space $C(X)$, which separates points of X, and let T_A be a linear map from A into a Banach space V. We assume the complete norm on A is given by one of the following formulas:

$$\|f\| = \max\left(\|f\|_\infty, \|T_A f\|\right) \text{ for } f \in A, \qquad (M)$$

where by $\|.\|_\infty$ we denote the usual sup-norm on $C(X)$;

$$\|f\| = \|f\|_\infty, \|T_A f\| \ \text{ for } f \in A, \qquad (\sum)$$

$$\|f\| = \sup\{|f(x)| + |T_A f(x)| : x \in X\} \text{for } f \in A, \qquad (C)$$

Where, in this case, we assume that $V = C(X)$.

For example the space $C^1(X), X \subset R$, is defined by a map $T: C^1(X) \longrightarrow C(X): Tf = f'$, via the formula $(M), (\sum) or (C)$. The space $AC[0,1]$ of absolutely continuous functions is defined by a map $T: AC[0,1] \longrightarrow L^1[0,1]: Tf = f'$. The space $Lip_\alpha(X), X$-metric space, $0 < \alpha \leq 1$, is defined by

$$T: Lip_\alpha(X) \longrightarrow C\big(\beta(X \times X\{(x,x): x \in X\})\big): Tf(x,y) = \frac{f(x) - f(y)}{\big(d(x,y)\big)^\alpha}$$

Assume next that $B$is a subspace of $C(Y)$, which separates points of Y, and that the norm on B is given by a map $T_B: B \longrightarrow U$, via the same formula as the norm of A. The question arises whether any isometry Φ from A onto B is of the canonical form

$$\Phi(f)(y) = \chi(y). f \circ \varphi(y), f \in A, y \in Y, \qquad (*)$$

Where φ is a homeomorphism from Y onto X and x is a scalar valued function defined on Y such that $|\chi| \equiv 1$.

We give a very simple, elementary scheme to verify the abovementioned problems. This scheme covers all the results we mentioned at the beginning of the introduction. The results hold in both real and complex cases.

We denote by ext V the set of all extreme points of the closed unit ball V_1 of the space V; by V^* we denote the space of all continuous linear forms on V. If A is a subspace of $C(X)$ then we identify a point x of X with a linear functional δ_x on A defined by $A \ni f \longmapsto f(x)$. By S we denote the set of all scalars of modulus one. If we do not specify the set of scalars, we mean that the result holds both in the real and in the complex case.

Let $\bar{A}$ be a normed subspace of a Banach space $C(W)$, both with the usual sup norm. By a standard result [33] every extreme point of A_1^* is of the form $\lambda\delta_w$, where $w \in W$ and $|\lambda| = 1$.

Note that if the norm on A is defined by the formula (M) then there is an isometric embedding $A \ni f \longmapsto \bar{f} \in \bar{A} \subset C(W)$, where $W = XUV_1^*$, defined by

$$\bar{f}(x) = f(x), x \in X,$$
$$\bar{f}(v^*) = v^*(T_A f), v^* \in V_1^*.$$

Hence any extreme point of A_1^* is of the form

$$\bar{f} \longmapsto \alpha f(x), \text{where } x \in X \text{ and } \alpha \in S$$

or of the form

$$f \longmapsto F \circ T_A(f), \text{where } F \in \text{ext}V^*.$$

If the norm on A is defined by $(\sum)$ then a map $A \ni f \longmapsto \bar{f} \in \bar{A} \subset C(X \times V_1^*)$ defined by

$$\bar{f}(x, v^*) = f(x) + v^*(T_A f), (x, v^*) \in X \times V_1^*,$$

is an isometry. Hence any extreme point of A_1^* is of the form

$$f \longmapsto \alpha f(x) + F \circ T_A(f), \text{where } x \in X, F \in \text{ext}V, \alpha \in S.$$

If the norm on A is defined by (C) then a suitable choice is $W = X \times S$ and

$$\bar{f}(x, \lambda) = f(x) + \lambda(T_A f)(x), x \in X, \lambda \in S.$$

Hence any extreme point of A_1^* is given by

$$f \longmapsto \alpha f(x) + \beta T_A f(x),$$

where $x \in X, \alpha, \beta \in S$.

Let A be a subspace of a space $C(X), X$ compact Hausdorff space. We say that A is an M-subspace of $X, \sum$-subspace of X or C-subspace of X if there is a Banach space V and a linear map $T_A: A \longrightarrow V$ such that

(i) the norm on A is given by the formula $(M), (\sum) or (C)$, respectively,

(ii) for any $x_1, x_2 \in X$, functionals δ_{x_1} and δ_{x_2} are linearly independent, and if the corresponding assumptions listed below are satisfied:

(iii$_M$) $X_0 = \{x \in X: \delta_x \in \text{ext } A^*\}$ is a dense subset $of\ X$,

(iii$_\sum$) There is an F_0 in extV^* such that

$$X_0 = \{x \in X; \forall\alpha \in S, \delta_x + \alpha F_0 \circ T_A \in \text{ext } A^*\}$$

is a dense subset of X,

(iii$_C$) $X_0 = \{x \in X: \forall\alpha, \beta \in S, \alpha\delta_x + \beta\delta_x \circ T_A \in \text{ext } A^*\}$ is a dense Subset of X;

(iv$_\sum$) if $\alpha\delta_x + F \circ T_A = \alpha'\delta_{x'} + \beta' F' \circ T_A$, where $F, F' \in \text{ext}V^*, x, x' \in X, \alpha\delta_x + F \circ T_A \in ext\ A^*$ and α', β' are scalars then $x = x'$ and $\alpha = \alpha'$,

(iv$_C$) if $\alpha\delta_x + \delta_x \circ T_A = \alpha'\delta_x + \beta'\delta_{x'} \circ T_A$, where $x, x' \in X, \alpha\delta_x + \delta_x \circ T_A \in$ext A^* and α', β' are scalars, then $x = x'$ and $\alpha = \alpha'$.

Theorem (2.1.1)[25]: Let A and B be M-subspaces of $C(X)$ and $C(Y)$, respectively. Put $X_0 = \{x \in X: \delta_x \in extA^*\}, \tilde{X} = \{\alpha\delta_x: x \in X_0, a \in S\}, Y_0 = \{y \in Y: \delta_y \in \text{ext}B^*\}, Y = \{\alpha\delta_y: y \in Y_0, \alpha \in S\}$. Then an isometry Φ from A onto B iscanonical, of the form $(*)$ if and only if $\Phi^*(\tilde{Y}) = \check{X}$.

The assumption of the above Theorem, that $\Phi^*(\check{Y}) = \check{X}$, looks very strong and so the Theorem seems to be almost trivial, and as a matter of fact it is true, but the advantage of this statement is that for all the classical function spaces, with M norm, this strong assumption can be easily verified. The method of verifying this. We define a property P concerning the points of $A^*(B^*)$ such that $F \in A^*$ $(G \in B^*)$ has this property if and only if

$F \in \check{X}$ $(G \in \check{Y})$. This property is defined by the weak-* topology of $A^*(B^*)$, the norms of A and A^* (B and B^*) and the linear structures of these Banach spaces.

The map Φ^* is a weak-* homeomorphism and a norm isometry from B^* onto A^* *so* $G \in B^*$ has the P property if and only if $F = \Phi^*(G) \in A^*$ has the same property and hence we get $\Phi^*(\check{Y}) = \check{X}$. A similar remark concerns the next Theorems also.

Proof. The "only if" part of Theorem (2.1.1) is trivial.

Assume that $\Phi^*(\check{Y}) = \check{X}$. Then there are two functions $\varphi_1: \check{Y} \longrightarrow \check{X}$ and $\varphi_2: \check{Y} \longrightarrow S$ such that

$$\Phi^*(\lambda\delta_y) = \varphi_2(\lambda, y) \,.\, \delta_{\varphi_1(\lambda,y)} \text{ for } \lambda\delta_y \in \check{Y}.$$

For any $y \in Y$ and $\lambda_1, \lambda_2 \in S$ functionals $\lambda_1\delta_y$ and$\lambda_2\delta_y$ are proportional.

Hence$\delta_{\varphi_1(\lambda_1 y)}$and $\delta_{\varphi_1(\lambda_2,y)}$ are also proportional and this means that$\varphi_2(\lambda_1, y) = \varphi_2(\lambda_2, y)$ So $\varphi_1$does not depend on the first coordinate and, by linearity of Φ^*, the map $\varphi_2$is linear with respect to λ. We get

$$\Phi^*(\lambda\delta_y) = \lambda\varphi_2(\lambda, y)\, \delta_{\varphi_1(y)} \text{ for } \lambda\delta_y \in \check{Y}. \quad (1)$$

Since Φ^* is a weak-* homeomorphism of ext B^* onto ext A^* as well as a homeomorphism of the closures of these sets, and moreover the weak-* topologies of $\{\delta_x: x \in X\} \subset A^*$ and $\{\delta_y: y \in Y\} \subset B^*$ coincide with the original topologies of X and$Y$, the map $(\varphi_1, \varphi_2): \tilde{Y} \longrightarrow \tilde{X}$ can be extended to a homeomorphism (φ, χ) from $\bar{\bar{Y}} = S \times Y$ onto $\bar{\bar{X}} = S \times X$. Hence from (1) we get (*) and we are done.

Theorem (2.1.2)[25]: Let A and B be $\sum$-subspaces of $C(X)$ and $C(Y)$, respectively.

Then an isometry φ from A onto B is canonical, of the form (*), if and only if for any $\alpha_1\delta_{y_i} + G_i \circ T_B \in \text{ext}\, B^*, i = 1,2$, the following two implications hold:

(a) $y_1 = y_2$ iff $x_1 = x_2$ and

(b) $G_1 \circ T_B$ and$G_2 \circ T_B$ are proportional iff$F_1 \circ T_A$ and$F_2 \circ T_A$ are proportional; here by x_i and F_i, we denote the elements of X and ext V^*, respectively, such that $\Phi^*(\alpha_1\delta_{y_i} + G_1 \circ T_B) = \beta_1\delta_{x_i} + F_1 \circ T_A$, for some scalars $\beta_i, i = 1,2$.

Proof. As before the "only if" part is trivial. To prove the "if" part assume that an isometry $\Phi: A \longrightarrow B$ satisfies our assumptions. Hence there are functions $\varphi_1, \varphi_2$and $\varphi_3$defined on the set ext $B^* \subset\ S \times Y \times \text{extU}^*$, with values in X, S and extV^*, respectively, such that

$$\Phi^*(\alpha\delta_y + G_1 \circ T_B) = \varphi_2(\alpha, y, G)\delta_{\varphi_1(\alpha,y,G)} + \varphi_3(\alpha, y, G) \circ T_A + \quad (2)$$

for $\alpha\delta_y + G \circ T_B \in \text{ext}B^*$.

By the assumption (a) the function tpi does not depend on a and G and we write $\varphi_1(y)$in place of $\varphi_2(\alpha, y, G)$. By (b) the map φ_3 is of the form $\varphi_3(\alpha, y, G)\ \varphi_4(G)$ where $\varphi_4(G) \in \text{ext}V^*$ and ip has values in S.

By the same argument as in the proof of Theorem (2.1.1) the map $\varphi_1$can be extended to a homeomorphism φ from Y onto X. So to end the proof we have to show that Φ^*maps $\alpha\delta_y$ onto $\varphi_2(\alpha, y, G)\delta_{\varphi(y)}$.

For any $H = \alpha\delta_y + G_1 \circ T_B \in \text{ext}\, B^*$ we define

$$\Omega_B(H) = \{\alpha'\delta_y + \beta' G \circ T_B \text{ext}\, B^*: \alpha', \beta' \in S\}.$$

By our assumptions we have $\Omega_B(H) = \Omega_A(\Phi^*(H))$for any $H \in \text{ext}B^*$. Let G_0 and Y_0 be as in assumption $(\text{iii}_\sum)$ for the space B. Notice that for any $y \in Y_0$the set$\Omega_B(\delta_y + G_0 \circ T_B)$ is homeomorphic to $S \times S$. Hence, by (2), the set

$\Omega_A\left(\varphi_2(1,y,G_0)\delta_{\varphi_1(y)}+\psi(1,y)\,\varphi_4(G_0)\right)$ is also homeomorphic to $S\times S$ and so

$$\alpha\delta_{\varphi_1(y)}+\beta\varphi_4(G_0)\in \text{ext}\,A^* \text{ for all } y\in Y_0,\alpha,\beta\in S.$$

Hence we have $X_0=\varphi_4(Y_0)$ and we can put $F_0=\varphi_4(G_0)$ in our assumption (iii_Σ). Notice that for any vector space E, for any functionals e_1^*,e_2^*. on E and for any $e\in E$ we have $\text{card}(\{|\alpha e_1^*(e)+\beta e_2^*(e)|:\alpha,\beta\in S\})=1$ iff $e_1^*(e).e_2^*(e)=0$.

Hence for any $f\in A$ and any $y\in Y_0$ we have the following implications:

$$f(\varphi_1(y))=0\ or\ F_0\circ T_A(f)=0 \text{ iff}$$
$$\text{Card}\left(\left\{|H(f)|:H\in\Omega_A\left(\delta_{\varphi_1(y)}+F_0\circ T_A\right)\right\}\right)=1 \text{ iff}$$
$$\text{Card}\left(\left\{|H(f)|:H\in\Omega_B\left(\delta_y+G_0\circ T_B\right)\right\}\right)=1 \text{ iff}$$
$$\Phi(f)(y)=0\ or\ G_0\circ T_B(f)=0.$$

This means that the union of $\delta_{\varphi_1(y)}$ and $\ker F_1\circ T_A$ is equal to the union of $\ker\Phi^*\left(\delta_y\right)$ and $\ker\Phi^*(G_0\circ T_B)$- By the assumption (4s) functionals $\delta_{\varphi_1(y)}$ and $F_1\circ T_A$are linearly independent so the above proves that for any $y\in Y_0$ we have two possibilities.

(i) $\Phi^*\left(\delta_y\right)=\ \chi(y)\delta_{\varphi_1(y)}$ and $\Phi^*(G_0\circ T_B)=\chi'(y)F_0\circ T_A$or

(ii) $\Phi^*\left(\delta_y\right)=\ \chi'(y)F_1\circ T_A$ and $\Phi^*(G_0\circ T_B)=\chi(y)\delta_{\varphi_1(y)}$,

where χ and χ' are scalar valued functions.

On the other hand by our assumption (b) we have

$$\Phi^*\left(\left\{\alpha'\delta_y+\beta' G\circ T_B:y\in Y,\alpha,\beta\in S\right\}\right)=\ \{\alpha\delta_x+\beta F_0\circ T_A:x\in X,\alpha,\beta\in S\}.$$

This shows that the second possibility does not hold for any $y\in Y_0$ (except in the trivial case when $\text{card}(Y)=1$); so to end the proof of (*) it is sufficient to notice that by (i), (2) and by the assumption $\left(\text{iv}_\Sigma\right)$ we have $|\chi|=|\varphi_2|\equiv 1$.

Theorem (2.1.3)[25]: Let A and B be C-subspaces of $C(X)$ and $C(Y)$, respectively.

Then an isometry Φ from A onto B is canonical, of the form {*), if and only if the following two conditions hold:

(a) for any $\alpha_i\delta_{y_i}+\beta_i\delta_{y_i}\circ T_B\in \text{ext}B^*, i=1,2$, we have $y_1=y_2=$ iff $x_1=x_2$ where by x_i we denote an element of X such that

$$\Phi^*\left(\alpha_i\delta_{y_i}+\beta_i\delta_{y_i}\circ T_B=\alpha_i'\delta_{x_i}+\beta_i'\delta_{x_i}\circ T_A\ \right)$$

for some scalars $\alpha_i',\beta_i',i=1,2$;

(b) $\Phi^*\left(\left\{\alpha\delta_y:y\in Y,\alpha\in S\right\}\right)\cap\{\alpha\delta_x\circ T_A:x\in X,\alpha\in S\}=\emptyset.$

Proof. The "only if" part is again trivial. Assume that Φ satisfies assumptions (a) and (b). Following what is by now a standard argument we get a homeomorphism φ from Y onto X and scalar valued functions $\varphi_2$and φ_3 defined on $S\times S\times Y$, with $|\varphi_2|\equiv 1\equiv|\varphi_3|$such that

$$\Phi^*\left(\alpha\delta_y+\beta\delta_y\circ T_B\right)=\varphi_2(\alpha,y,G)\delta_{\varphi(y)}+\varphi_3(\alpha,y,G).\delta_{\varphi(y)}\circ T_A, \tag{3}$$

Where $y\in Y$ and $\alpha,\beta\in S$. For any $H=\alpha\delta_x+\beta\delta_x\circ T_A\in \text{ext}\,A^*$ we define

$$\Omega_\alpha(H)=\left\{a'\delta_y+\beta'\delta_x\circ T_A\in \text{ext}\,A^*:\alpha,\beta\in S\right\}.$$

As in the proof of the preceding Theorem we show that for any $y\in Y$ we have two possibilities:

(i) $\Phi^*\left(\delta_y\right)=\ \chi(y)\delta_{\varphi(y)}$ and $\Phi^*\left(\delta_y\circ T_B\right)=\chi'(y).\delta_{\varphi(y)}\circ T_A or$

(ii) $\Phi^*\left(\delta_y\right)=\chi'\ (y)\delta_{\varphi(y)}\circ T_A$and $\Phi^*\left(\delta_y\circ T_B\right)=\chi(y).\delta_{\varphi(y)}$.

By the assumption (b) the second possibility never holds; by (iv_C) we then get $|x|\equiv|\varphi_2|=1$ and we are done.

To verify the assumptions of our schemes, given by Theorems (2.1.1)-(2.1.3), it is usually necessary to have at least partial description of the extreme functionals in the unit ball of the dual spaces. Hence in some cases it is easier to apply the following Theorem, which is an immediate consequence of the Theorem of [37].

Theorem (2.1.4)[25]: Let A be a complex subspace of $C(X), X$ compact Hausdorff Space, such that

(i) A is sup-norm dense in $C(X)$,

(ii) the norm on A is given by a map $T_A: A \longrightarrow V_A$, via the formula (M) or $(\sum)$,

(iii) A contains the constant function 1 and $T_A(1) = 0$.

Assume next that B is a complex subspace of$C(Y)$ which satisfies the analogous assumptions (i)-(iii). Then any isometry Φ from A onto B, such that $\Phi(1) = 1$ is of the form

$$\Phi(f) = f \circ \varphi \quad for\, f \in A,$$

Where φ is a homeomorphism from Y onto X.

Example (2.1.5)[25]: Let $A = C^1(X)$ and $B = C^1(Y)$ be algebras of continuously differentiable functions defined on the compact subsets X and Y of the real line,

We do not assume here that the sets X and Y do not contain isolated points but we understand that the derivative of a function $f \in C^1(X)$ is defined only on the set of nonisolated points of X. Assume that the norms on A and B are given by the formula (M); this means

$$\|f\| = \mathrm{m\,ax}(\|f\|_\infty, \|f'\|_\infty), f \in A(B).$$

We prove that A is an M-subspace of $C(X)$. The first two assumptions of the definition of M-subspace are evidently fulfilled. To show the last one notice that for any $x_0 \in X$ there is an $f \in C^1(X)$ such that $\|f\| = \|f\|_\infty = f(x_0) = 1 > \|f\|_\infty$and such that $|f(x)| < 1$ for $x \in X\backslash\{x_0\}$ Hence for any $x_0 \in X$ the functional δ_{X_0} is an extreme point of A_1^*, so $X_0 = X$. Notice also that for any x_0 G X there is a $g \in C^1(X)$ such that $\|g\| = \|g'\|_\infty = g'(x_0) = 1 > \|g\|_\infty$, Hslloo and such that $\backslash g'(x)\backslash < 1$for $x \in X\backslash\{x_0\}$. Hence

$$\text{ext}\, A^* = \{\alpha\delta_x: x \in X, \alpha \in S\}\, U\, \{\alpha\delta'_x: x \in X, \alpha \in S\}, \tag{4}$$

Where by δ'_x we denote a functional defined by $\delta'_x(f) - \delta_x \circ D(f) = f'(x)$.

Now let Φ be any isometry from A onto B. We prove that $\Phi^*(Y) = X$. To this end we define the following property concerning points of $A^*(B^*)$.

$F \in A^*$ has the P-property iff $F \in \text{ext}A^*$ and there is $a\, G$ in ext A^*, not proportional to F, such that for any weak-* open neighborhood $U \subset \text{ext}A^*$ of G and any $f \in A \subset CA^{**}$ we have: if $f|_U \equiv 0$ then $F(f) = 0$.

We check that $F \in A^*$ has the P-property if and only if $F \in \text{ext}\, A^*\backslash X$. Fix any Point $x_0 \in X$ and put $F = \alpha\delta'_{x_0}$ with $\alpha \in S$. By (4) $F \in \text{ext}A^{**}$ and for any weak-* open neighborhood $U \subset \text{ext}\, A^*$of the functional δ_{x_0} and for any $f \in A$ we have

$f|_U \equiv 0 \Rightarrow f \equiv 0$ on an open neighborhood of $x_0 \Longrightarrow f'(x_0) = 0$.

Hence any element of ext $A^*\backslash\tilde{X}$ has the P-property.

Now fix again $x_0 \in X$ and put $F = \alpha\delta'_x$. Let $G \in \text{ext}A^*\backslash\{\gamma F: \gamma \in S\}$. By (4)

We have two possibilities: 1. $G = \beta\delta_{x_1}, x_1 \neq x_0$ or 2. $G = \beta\delta'_{x_1}$. If the first one Holds then we put $U = \{\gamma\delta_x: |x - x_1| < |x_0 - x_1|/2, \gamma \in S\}$; evidently there is an $f$in $A \subset A^{**}$ such that $f|_U \equiv 0$ and $f(x_0) = 1$. If the second possibility holds we put$U = \{\gamma\delta'_x: x \in X, \gamma \in S\}$ and put $f \equiv 1$. Hence F does not have the P-property.

We have proved that $F \in A^*$ has the P-property if and only if $F \in \text{ext}\, A^*\backslash X$.

This property is defined by the weak-* topology of A^*, and the norm and the linear structure of A^* ; this means by the properties which are preserved by Φ^*. Hence $F = \Phi^*(G) \in A^*$ has this property if and only if $G \in B^*$ has, and this proves that$\Phi^*(Y) = X$.

By Theorem (2.1.1) we now get that any isometry Φ from $C^1(X)$ onto $C^1(Y)$ is of the form

$$\Phi(f)(y) = x(y).f.\varphi(y) \text{ for } f \in A, y \in Y,$$

Where $x \in C^1(X), |x| \equiv 1$ and φ is a homeomorphism from Y onto X. It is also easy to verify now that since Φ preserves both M-norm and sup-norm, it also preserves the sup-norm of the derivative and hence we get $|\varphi'| \equiv 1$.

We have assumed at the beginning of this example that the sets X and Y are compact. In fact this assumption is not essential and the same holds for arbitrary subsets of the real line not necessarily bounded and closed. We then consider A as a subset of $C(\beta X)$ and the proof is slightly more technical.

The general form of the isometries of complex $C^1(X)$ spaces, defined on a compact subset X of the real line, without isolated points, was investigated by Pathak and Vasavada [44].

Example (2.1.6)[25]: Let $A = AC(X)$ be the space of all absolutely continuous, scalar valued functions defined on a compact subset X of the real line, such that $X = \overline{\text{int} X}$. We define norm on A by

$$\|f\| = max(\|f\|_\infty, \|f'\|_1) \text{ for } f \in A,$$

Where $\|f'\|_1 = \int_x |f'|$ dm and m is the Lebesgue measure.

As in the preceding example it is easy to notice that for any $x_0 \in X$ there is an $f \in A$ such that $\|f\| = \|f\|_\infty = f(x_0) = 1 > \|f'\|_1$and $|f(x)| < 1$ for $x \in X \backslash \{x_0\}$. Hence $X_0 = X$ and A is an Af-subspace of $C(X)$. Notice also that for any $F \in ext\left(L^1(m)\right)^* \cong ext(L^\infty(m))$ and a function $f \in AC(X)$ defined by

$$f(t) = \frac{1}{m(X)} \int_{X \cap (-\infty, t)} \bar{F}(x)\, dm(x) - \frac{1}{2m[X]} \int_X \bar{F}(x)\, dm(x)$$

We have $\|f\| = 1, \|f\|_\infty \leq \frac{1}{2}$ and for any $G \in ext(L^\infty) = ext(L^1)$

$$G(f) := \int_X G f' dm = 1 \text{ iff } G = F.$$

This means that the function $f \in A \subset C(X \, U \, ext(L^\infty))$ peaks exactly at the point $F \in ext(L^\infty)$, so any such point F is an extreme point of A^* and we have

$$\text{ext } A^* = \{\alpha\delta_x : x \in X, \alpha \in S\} U \{FoD : F \in \text{ext}(L^\infty)\},$$

Where $D: AC(X) \to L^1$ is defined by $D(f) = f'$.

By Theorem (2.1.1) to prove that any isometry $\Phi$from $AC(X)$ onto $AC(Y)$ is canonical we have to, as before, define the set$\check{X} = \{\alpha\delta_x \in \text{ext } A^* : x \in X, \alpha \in S\}$ by the weak-* topology of A^* and the norms and the linear structures of A and A^*. We have $F \in \check{X}$ iff $F \in \text{ext } A^*$ and there is a weak-* open neighborhood of F in ext A^* which is homeomorphic to a subset of $S \times R, R$ the real line.

Example (2.1.7)[25]: If X is any compact metric space with metric d we let

$$Lip_\alpha(X) = \left\{ f \in C(X) : \|f\|_{d^\alpha} = \sup_{\substack{x,y \in Y \\ x \neq y}} \frac{|f(x) - f(y)|}{d^\alpha(x,y)} < \infty \right\}$$

and

$$Lip_\alpha(X) = \left\{ f \in Lip_\alpha(X) := \lim_{d(x,y)\to 0} \frac{f(x) - f(y)}{d^\alpha(x,y)} = 0 \right\}$$

both provided with the M-norm; this means with the norm defined by

$$\|f\| = \max(\|f\|_\infty, \|f\|_{d^\alpha}).$$

As before it is easy to check that both $Lip_\alpha(X)$and $Lip_\alpha(X)$ are M-subspaces of $C(X)$. Therefore to prove that any isometry from $Lip_\alpha(X)$ onto $Lip_\alpha(Y)$ or from $Lip_\alpha(X)$ onto $Lip_\alpha(Y)$ is canonical we have to define a property which "separates$\check{X} = \{\alpha\delta_x : \mathrm{x} \in \mathrm{X}, \alpha \in S\}$from the rest of the extreme points of the unit ball of $(Lip_\alpha(X))^*$ or $(Lip_\alpha(X))^*$ We have $F \in \operatorname{ext} A^* \backslash \check{X}$ iff $F \in \operatorname{ext} A^*$ and there are sequences $(F_1^n)_{n=1}^\infty$ and $(F_2^n)_{n=1}^\infty$1s uch that $(F_1^n - F_1^n)\backslash\|F_1^n - F_2^n\|$ tends to F in the norm topology.

The isometries of the complex $Lip_1(X)$spaces with the above M-norm were considered by Roy [46], when X is connected with diameter at most 1, and by Vasavada [47], when X satisfies certain separation conditions. A similar space defined on the real line was investigated by de Leeuw [40].

Example (2.1.8)[25]: Let $A = C^1(X) and\, B = C^1(Y)$ be the same vector spaces as in Example (2.1.5) but now with the $\sum$-norm; this means with the norm given by

$$\|f\| = \|f\|_\infty + \|f'\|_\infty, \qquad f \in A(B).$$

To prove that A is a $\sum$-subspace of $C(X)$ it is sufficient to notice that for any $x_1 \neq x_2$ in X and any $\alpha, \beta \in S$ there is an f in A such that $\|f\| = 1, f(x_1) = \bar{\alpha}/2, |f(x)| < \frac{1}{2}$ for $x \neq x_1, f'(x_2) = \bar{\beta}/2$ and $|f'(x)| < \frac{1}{2}$ for $x \neq x_2$- Hence

$$\{\alpha\delta_x + \beta\delta_y\, oD : x \neq y, \in X, \alpha, \beta \in S\}\; C\mathrm{ext}\, A^*.$$

Now let Φ be any isometry from A onto B. To prove that Φ is canonical, by Theorem (2.1.2), we have to show that the assumptions (a) and (b) of this Theorem are Satisfied. To see this we define two equivalence relations ~ 1 and ~ 2 on ext A^* by the following formulas:

$$\alpha_1\delta_{x_1} + F_1 oD_{\sim 1}\alpha_2\delta_{x_2} + F_2 oD \text{ iff } x_1 = x_2 \tag{5}$$

And

$$\alpha_1\delta_{x_1} + F_1 oD_{\sim 2}\alpha_2\delta_{x_2} + F_2 oD \;\text{ iff } x_1 = x_2 \text{ iff } F_1 oD \text{ and } F_2 oD \text{ are proportional.} \tag{6}$$

To verify (a) and (b) we just have to prove that the map Φ^* preserves both the above relations. We prove this by defining ~ 1 and ~ 2 in terms of the weak-* topology of A^*, the norms and the linear structures of A and A^*, so in the terms which are preserved by Φ^*. To this end we notice that a sequence $\left(\alpha_n\delta_{x_n} + \beta_n\delta_{y_n} \circ D\right)_{n=1}^\infty C\mathrm{ext}\, A^*$ tends to $\alpha_0\delta_{x_0} + \beta_0\delta_{y_0} \circ D$ in the weak-* topology iff an $\alpha_n \longrightarrow \alpha_0, \beta_n \longrightarrow \beta_0, x_n \longrightarrow x_0$ and $y_n \longrightarrow y_0$,and in the norm topology iff $\alpha_n \longrightarrow \alpha_0, \beta_n \longrightarrow \beta_0, x_n \longrightarrow x_0$and $y_n = y_0$ for all but finitely many n. We have $F_{\sim 1}\ G$ iff there are sequences $(F_n)_{n=1}^\infty$and $(G_n)_{n=1}^\infty$ in ext A^* which tend, in the norm topology, to F and G, respectively and there is a scalar a such that dim(span$\{F_n : n = 1,2,\dots\}) = \infty$ and $F_n + \alpha G_n = F + \alpha G$for all $n \in N$.

We also have $F_{\sim 2}\ G$ iff there are sequences $(F_n)_{n=1}^\infty$and $(G_n)_{n=1}^\infty$ in ext A^*which tend to F and G respectively, in the weak-* topology, but do not tend in the norm topology, and there is a scalar a such that $F_n + \alpha G_n = F + \alpha G$for all $n \in N$.

The isometries of the complex C^1 [0,1] space with the above $\sum$ -norm were described by Rao and Roy [45].

Example (2.1.9)[25]: Let $A = AC[0,1]$ be the spaces of all complex absolutely continuous functions defined on the unit interval with the E-norm; this means with the norm given by

$$\|f\| = \|f\|_\infty + \|f'\|_1, \qquad f \in A.$$

As we noticed any extreme functional on A is of the form

$$A \ni f \xrightarrow{\alpha\delta_x + FoD} \alpha f(x) + \int F . f' dm, \tag{7}$$

Where $x \in X, \alpha \in S, m$ is the Lebesgue measure and F is an extreme point of $(L^1)^*$, which we identify with a function $F \in L^\infty$ such that $|F| = 1$ a.e. To prove

That A is a $\sum$ -subspace of $C[0,1]$ we have to show that there is an F_0 in ext(L^∞) such that for any $x \in X$ and any $\alpha \in S$ the corresponding functional defined by (7) is an extreme point of A_1^*. To this end let K be a measurable subset of [0,1] such that for any open subset U of [0,1] we have $0 < m(K \cap U) < m(U)$. By Lemma 2.1 of [45] the desired function F_0 can be defined by

$$F_0(t) = \begin{cases} 1 & \text{if } t \in K, \\ -1 & \text{if } t \notin K. \end{cases}$$

To prove that any isometry from A onto itself is canonical we have to, as in the previous example, describe the equivalence relations defined by (5) and (6) by the weak-* topology of A^*, the norms and the linear structures of A and A^*. To this end notice that a sequence $\left(\alpha_n\delta_{x_n} + \beta_n\delta_{y_n} \circ D\right)_{n=1}^{\infty} C\text{ext}\, A^*$ tends to $\alpha\delta_x F \circ D$ in the weak-* topology iff $\alpha_n \longrightarrow \alpha, x_n \longrightarrow x$ and F tends to F in the weak-* topology of L^∞; and in the norm topology iff $\alpha_n \longrightarrow \alpha$, $x_n = x$ for all but finitely many $n \in N$ and $\|F_n - F\| \longrightarrow 0$.

By Lemma 2.1 of [45] we have and $F_{\sim_1} G$ iff F and G are contained in the same connected component of the set ext A^* equipped with the norm topology, and $F_{\sim_2} G$ iff there are sequences $(F_n)_{n=1}^\infty$ and $(G_n)_{n=1}^\infty$ in ext A^* which tend, in the norm topology, to F and G, respectively and a scalar a such that dim(span$\{F_n : n = 1,2,\dots\}) = \infty$ and$F_n + \alpha G_n = F + \alpha G$ for all $n \in N$.

The isometries of the spaces $AC[0,1]$ have been described by Cambern [28] and by Rao and Roy [45].

Example (2.1.10)[25]: Let $A = C^1(X)$ *and* $B = C^1(Y)$ be as in Example (2.1.5) but now with
the C-norm; this means with the norm given by

$$\|f\| = \sup\{|f(x)| + |f'(x)| : x \in X\}.$$

In order for $\|.\|$ to be a well-defined norm we have to assume now that X and Y
Do not contain isolated points.

As before, it is standard to verify (see [30], for the complex case) that

$$\text{ext}\, A^* = \{\alpha\delta_x + \beta\delta_x \circ D : x \in X, \alpha, \beta \in S\}.$$

Hence A and B are C-subspaces of $C(X)$ and $C(Y)$, respectively. To prove that any isometry from A onto B is canonical we have to check whether assumptions (a) And (b) of Theorem (2.1.3) are satisfied. To this end we describe the equivalence relation on ext A^* defined by

$$\alpha\delta_x + \beta\delta_x \circ D \sim \alpha'\delta_x + \beta'\delta_x \circ D \text{ iff } x = x'$$

In terms of the weak-* topology of A^*, the norms and the linear structures of A and A^*, and we also "separate" the set $\{\alpha\delta_x \in A^* : x \in X, \alpha \in S\}$ from the set $\{\alpha\delta_x \circ D \in A^* : x \in X, \alpha \in S\}$ by the same properties, which are preserved by Φ^*. We have $F \sim G$ iff F and G are contained in the same connected component of the set ext A^*, equipped with the norm topology.

We also have $F \in \{\alpha\delta_x: x \in X, \alpha \in S\}$is an open subset of the set $\{\alpha\delta_x: x \in X, \alpha \in S\} U \{\alpha\delta_x \circ D: x \in X, \alpha \in S\}$, equipped with the norm topology.

The isometries of the complex $C^1(X)$ spaces with the C-norm have first been considered by Cambern [28] for $X = [0,1]$ and then by Cambern and Pathak [30] for X any compact subset of the real line, without isolated points.

It can be proved that $A^1(K)$ gives $M-, \sum -$ and C-subspaces of $C(\partial K)$, where $\partial K$is the topological boundary of K, and that all the assumptions of Theorems (2.1.1), (2.1.2) and (2.1.3) are satisfied for $A = A^1(K), B = A^1(L)$. Hence any $M-, \sum -$ or C −isometry Φ from $A^1(K)$ onto $A^1(L)$ is of the form

$$\Phi f(z) = \chi(z). f \circ \varphi(z)\ , f \in A^1(K), z \in L,$$

Where φ is an analytic homeomorphism from L onto K, with $|\varphi'| \equiv 1$ and $x \in A^1(K)$, with $|\chi| \equiv 1$.

Example (2.1.11)[25]: Let X be a compact subset of the real line. For $1 \leq p \leq \infty$ we Define

$$AC^P(X) = \{f \in C(X): f' \text{exists } a.e., f' \in L^P(X)\}$$

and we define a norm on $AC^P(X)$ by

$$\|f\| = \|f\|_\infty + \|f'\|_p.$$

For $p = 1$ (resp. oo) we get the space $AC(X)$ (resp. $Lip(X)$). Rao and Roy [45] proved that any isometry from the complex $AC^p([0, 1])$ space, $p = 1$ or∞ onto itself is canonical (cf. Example (2.1.9)) and asked whether the same holds for $1 < p < \infty$. The answer is positive and is a consequence of the following more general proposition.

Proposition (2.1.12)[25]: Let A be a complex subspace of $C(X), X$ compact Hausdorff Space, such that

(i) A is sup-norm dense in $C(X)$,

(ii) The norm on A is given by a map $T_A: A \longrightarrow V_A$ a, via the formula $(\sum)$,

(iii) A Contains the constant function 1 and$T_A(1) = 0$,

(iv) $\dim(T_A(A)) \geq 2$,

(v) V is strictly convex,

(iv) For any unimodular function $\chi \in A$ such that $T_A(\chi) = 0$ a map

$$A \ni f \longrightarrow f/\chi \in A$$

Is a well defined surjective isometry.

Assume next that B is a complex subspace of$C\{Y), Y$ compact Hausdorff, which also satisfies assumptions (i)-(vi). Then any isometry Φ from A onto B is of the form

$$\Phi(f) = \chi. f \circ \varphi, \quad f \in A,$$

Where φ is a homeomorphism from Y onto X and $\chi \in B$ is a unimodular function on Y such that $T_B(x) = 0$.

Proof. By Theorem (2.1.4) and by our assumption (vi) it is sufficient to prove that $\Phi(1)$ is a unimodular function on Y such that$T_B\{\Phi(1)) = 0$. We say that an element g of A has the P-property if $\|g\| = 1$ and if for any f in A there is a $\beta \in S$ such that

$$\|g + \beta f\| = \|g\| + \|f\|.$$

It is evident that this property is preserved by our isometry Φ. By the definition of the norm on A for any $f \in A$ and $\beta \in S$ such that

$$\|f\|_\infty = \sup_{x \in X} Re(\beta f(x))$$

We have

$$\|1 + \beta f\| = \|1 + \beta f\|_\infty + \|T_A(f)\| = 1 + \|f\|_\infty + \|T_A(f)\| = 1 + \|f\|.$$

Hence to end the proof we have to show that if $g \in A$ has the property P then g is a unimodular function on X such that$T_A(g) = 0$.

We first prove that $|g| \equiv c$on X for some constant$c$, then we prove that $T_A(g) = 0$ and then from the definition of the norm on A we get $c = \|g\|_\infty = \|g\| = 1$.

Assume that there is an $x_0 \in X$ such that $|g(x_0)| < \|g\|_\infty$. by assumption (i) there is an f in A such that

$$\|f\|_\infty = \|g\|_\infty - |g(x_0)|$$

And

$$|f(x)| \leq \|g\|_\infty - |g(x)| + \frac{1}{2}(\|g\|_\infty - |g(x_0)|) \quad \text{for } x \in X$$

For any $\beta \in S$ we have

$$\|g + \beta f\|_\infty \leq \|g\|_\infty + \frac{1}{2}\|\text{f}\|_\infty;$$

So

$$\|g + \beta f\| = \|g + \beta f\|_\infty + \|T_A(g + \beta f)\|;$$
$$\leq \|g\|_\infty + \frac{1}{2}\|\text{f}\|_\infty + \|T_A(g)\| + \|T_A(f)\|$$
$$= \|g\| + \|f\| - \frac{1}{2}\|f\|_\infty < 1 + \|f\|;$$

Hence$|g| =$ const. Assume now that$T_A(g) \neq 0$; by assumption (iv) there is an $f$in A such that $T_A(f)$and $T_A(g)$ are not proportional. Since V_A is strictly convex
For any h, h' in V_A we have $\|h + h'\| = \|h\| + \|h'\|$ iff h and h' are proportional and hence for any $\beta \in S$ we get

$$\|g + \beta f\| = \|g + \beta f\|_\infty + \|T_A(g + \beta f)\|$$
$$< \|g\|_\infty + \|\text{f}\|_\infty + \|T_A(g)\| + \|T_A(f)\| = 1 + \|f\|.$$

Hence $T_A(g) = 0$ and we are done.

Example (2.1.13)[25]: Let X be a compact metric space, with metric d and let $Lip_\alpha(X)$, $Lip_\alpha(X)$ Be defined as in Example (2.1.7), but now with the $\sum$-norm; this is with the norm defined by

$$\|f\| = \|f\|_\infty + \|f\|_{d^\alpha};$$

Rao and Roy [45] proved that any isometry from the complex Lipi [0, 1] onto itself is canonical and asked whether the same holds in general. To answer this question let X, Y be compact metric spaces, let A be equal to a complex $Lip_\alpha(X)$ or $Lip_\alpha(X)$space and let B be equal to a complex $Lip_{\alpha'}(Y)$or $Lip_{\alpha'}(Y)$ space. We prove that any isometry Φ from A onto B is canonical and hence of the form

$$\Phi f(y) = cf \circ \varphi(y), f \in A, y \in Y,$$

Where $|c| = 1$ and φ is an isometry from Y onto X. To this end, by Theorem (2.1.4), it is sufficient to prove that for any such isometry $\Phi, \Phi(1) = \chi$¡s a constant function of norm one.

Put

$$y = \beta(Y \times Y - \{(y, y): y \in Y\}),$$

Define $T: B \to C(y)$ by

$$Tg(y_1, y_2) = \frac{g(y_1) - g(y_2)}{d^{\alpha'}(y_1, y_2)}$$

and for any $g \in B$ let us denote by g an element of $C(Y \times y \times S)$ defined by

$$\bar{g}(y, \omega, \beta) = g(y) + \beta Tg(\omega).$$

As we noticed in the introduction the map $g \to \bar{g}$ i s an isometric embedding from B onto $\bar{B} \subset C(Y \times y \times S)$ and any element G of ext B^* is of the form

$$G(g) = \gamma\tilde{g}(y, \omega, \beta) = \gamma g(y) + \gamma\beta Tg(\omega).$$

Fix $(y_0, \omega_0, \beta_0) \in Y \times \omega \times S$ and let μ, be any norm one measure on $Y \times y \times S$ which represents, on $\tilde{B}$, evaluation at the point (y_0, ω_0, β_0)- Considering any function $g_0 \in B$ which peaks exactly at the point y_0 it is easy to notice that μ, is concentrated on the set $\{y_0\} \times y \times S$ and that for any $0 \leq \vartheta < 2\pi$ a measure μ_ϑ defined on $Y \times y \times S$ by

$$\mu_\vartheta(E) = \mu\big(\{(y, \omega, \beta) : (y, \omega, e^{i\vartheta\beta}) \ \in E\}\big)$$

represents, on $\breve{B}$, the functional of the evaluation at the point $\left(y_0, w_0, e^{-i\vartheta\beta_0}\right)$.
Hence we can define a map Ψ_ϑ: ext $B^* \longrightarrow$ ext B^* by

$$\Psi\big(\Upsilon^\delta_{(y,\omega,\beta)}\big) = \Upsilon^\delta\big(y, \omega, e^{i\vartheta\beta}\big)$$

For any $F \in$ ext A^* we have $|F(1)| = 1$, hence for any $G \in$ extB^* we also have $|G(\chi)| = 1$. Let $G = \delta_y + \beta\delta_\omega \circ T \in$ ext B^*. For any $0 \leq \vartheta < 2\pi$ we have

$$1 = \ |\Psi_\vartheta(G)(\chi)| = \left|\chi(y) + e^{i\vartheta}\beta T(\chi)(\omega)\right|,$$

hence for any such G we have two possibilities:

$$\chi(y) = 0 \text{ and } |T(\chi)(\omega)| \ = \ 0.$$

or

$$|\chi(y)| = 1 \text{ and } T(\chi)(\omega) \ = \ 0.$$

Since $1 = \|\chi\| = \|\chi\|_\infty + \|T(\chi)\|_\infty$ we get that the second possibility always holds, this means that $T(\chi) \equiv 0$ and we are done.

Beside the extensions of the surjective Banach-Stone Theorem for various function spaces also considered injective isometries between classical Banach spaces (see [27], [34], [35], [36], [38]) and isomorphisms with a small bound (see [26], [27], [29], [31], [32], [36], [38], [44]). These problems in general seem to be much harder and we do not know whether the similar general schemes can be produced. The following example shows however that even for a very simple function space an injective isometry may be "very uncanonical." Let A be equal to $C^1[0,1]$ with the M-norm. Let φ be a continuous map from $[0,1]$ onto A_1^* equipped with the weak-* topology. We define $\Phi: A \longrightarrow A'$ by

$$\Phi(f)(t) = \int_0^t \varphi\,(x)(f)dx.$$

On the other hand, if we assume that the injective isometry from the above space A into itself preserves the constant function then it is automatically surjective and of the canonical form. This can be proven exactly by the same arguments as used in [37].

Section (2.2): Banach-Stone Theorem

For T be a compact Hausdorf space and let $B(T)$be the open unit ball of the compex Banach space $C(T)$of all complex-valued continuous functions on T, with the uniform norm. if $u \in C(T)$ satisfies the condition $|u(t)| = 1$ for all $t \in T$, and if $x \in C(T)$ is such that $u + \zeta x \in B(T)$the closure of $B(T)$ for all $\zeta \in \varDelta$(the open unit disc of C). Then necessarily $x = 0$. Hence u is a complex extreme point of $\overline{B(T)}$. on other hand, if , for $u \in \overline{B(T)}$, there is some point $t_0 \in T$ such that$|u(t_0)| < 1$. then $|u(t)| < 1 - \varepsilon$ for all t in some neighborhood U of in T and for some$\varepsilon > 0$. If $x \in C(T)\backslash\{0\}$ is such that supp $(x) \subset U$and$\|x\| < \varepsilon$, then for all $t \in T$and all $\zeta \in \varDelta$,

$$|u(t) + \zeta x(t)| \leq 1,$$

Showing that u is not a complex extreme point of $\overline{B(T)}$. if $x \in \partial\Delta. \beta \in C\backslash\{0\}$, and if $r > 0$, then either $x + r\beta \notin \overline{\Delta}$. hence a complex extreme point of $\overline{B(T)}$ is also a real extreme point. In conclusion, the following lemma holds.

Lemma (2.2.1)[48]: The set $\Gamma(T)$ of all $u \in C(T)$ such that $|u(t)| = 1$ for all $t \in T$ is the se of all (complex=real) extreme points of $\overline{B(T)}$.

Let S and T be compact Hausdorff spaces, and let A be a surjective isometry of $C(S)$ onto $C(T)$. if $A(0) = 0$. according to a theorem of S. Mazur and Ulam [53]. [49], $A \in y\big(C(S), C(T)\big)$. if this is the case. Then, by a classical theorem established by S. Banach [49] when $S = T$is the interval [0, 1], and by $M.H$. Stone [54] in the genral case (see also [33]), there exist a homeomorghism τ of T onto S and a function $x \in \Gamma(T)$ such that

$$(Ax)(t) = \alpha(t)(x.\tau)(t) \tag{22}$$

For all $x \in C(S)$ and all $t \in T$ hypothesis that the isometry A be surjective is essential for the validity of the Mazur-Ulam theorem. An example will be constructed of a non-surjective linear isometry A of $C(T)$ onto $C(T)$ whicis fixes 0 but non-linear. The fact that as will be shown-such an isometry does not map $\Gamma(T)$ into itself turns out to be crucial. Indeed the following theorem holds, which extends linear isometries.

Theorem (2.2.2)[48]: let $A \in y(C(S), C(T))$ be such that $\|A\| \leq 1$. if

$$A\Gamma(S) \subset \Gamma(T). \tag{23}$$

Then there exist a continuous map $\tau: S \to T$ and a function $\alpha \in \Gamma(T)$ for which (23) holds. Moreover, A is a linear isometry of $C(S)$ into $C(T)$if, and only if, A is injective. And that happens if, and only if, τ is surjctive.

For any $t \in T$, the map $C(S) \ni x \longmapsto (Ax)(t)$ is a continuous linear form on $C(T)$, by the Riesz representation theorem, there exists a uniqe regular Borel measure μ_t in Ssuch that

$$(Ax)(t) = \int x d\mu_t \coloneqq (x, \mu_t).$$

Then (23) reads

$$|(u, \mu_t)| = 1 \tag{24}$$

For all $u \in \Gamma(T)$and $t \in T$. it will be shown now that this equation holds for all u contained in the set $\Lambda(S)$ for all measureable complex-valued functions on S such that $|u(s)| = 1$a.e. $|\mu_t|$. Any $z \in \Lambda(S)$ can be written

$$z(s) = e^{i\lambda(s)},$$

Where $\lambda: S \longrightarrow R$ is measureable, by the Lusin theorem, for any $v = 1,2, \ldots$ there is a continuous function $\lambda_v: S \longrightarrow R$ such that

$$|\mu_t|(s \in S: \lambda_v(s) \neq) < \frac{1}{v}$$

And

$$|\lambda_v(s)| \leq |\lambda(s)|$$

a.e. $|\mu_t|$. Hence the sequence λ_v converges to λ in measure, and therefore continains a subsequence λ_v, which converges to λ a.e. $|\mu_t|$ letting $z_{v_j}(s) = e^{i\lambda_{v_j}(s)}$,then $z_{v_j} \in \Gamma(S)$ andz_{v_j} converges to z a.e. $|\mu_t|$. As a consequence of the dominated convergence theorem, the sequence $\left(z_{v_j}, \mu_t\right)$ converges to (z, μ_t), and therefore $|(z, \mu_t)| = 1$. in conclusion, if (24) holds for all $x \in \Gamma(S)$, it$\Gamma(T)$ holds also for all $x \in \Lambda(S)$.

Lemma (2.2.3)[48]: For every $t \in T$ there is a complex constant $\alpha(t)$, with $|\alpha(t)| = 1$. and a point $s \in S$, such that

$$\mu_t = \alpha(t)\delta_s, \tag{25}$$

Where δ_s is the measure with mass 1 covcentrated at the point s, i.e,

$$(x, \mu_t) = \alpha(t)x(s)$$

For all $x \in C(S)$.

Proof. If $K \subset S$ is the support of $|\mu_t|$, for any open neighborhood V of $K, |\mu_t|(V) > 0$. hence, if K contains as least two distinct points, $s_1$and s_2,and if U is any open neighborhood of $s_1$such that $s_2 \notin \overline{U}$, then

$$|\mu_t|(U) > 0, \quad |\mu_t|(S/U) > 0. \tag{19}$$

Let $x \in \Gamma(S)$. and let ξ and η be two real numbers. The function z defined on S by

$$z(s) = e^{i\eta}x(s)$$

If $s \in U$, and by

$$z(s) = e^{i\eta}x(s)$$

If $s \notin U$, is contained in $\Lambda(S)$.thus $|(z, \mu_t)| = 1$, i.e,

$$\left|\int_U e^{i\eta}\, x d\mu_t\right|^2 + \left|\int_{S/U} e^{i\eta}\, x d\mu_t\right|^2 + 2\Re\left(\int_U e^{i\eta}\, x d\mu_t \int_{S/U} e^{i\eta}\, x d\mu_t\right) = 1.$$

Differentiation with respect to $\xi$and to ηyields

$$\int_U e^{i\xi}\, x d\mu_t \overline{\int_{S/U} e^{\iota\eta}\, x d\mu_t} - \overline{\int_U e^{\iota\xi}\, x d\mu_t} \int_{S/U} e^{i\eta}\, x d\mu_t = 0.$$

$$\int_U e^{i\xi}\, x d\mu_t \overline{\int_{S/U} e^{\iota\eta}\, x d\mu_t} + \overline{\int_U e^{\iota\xi}\, x d\mu_t} \int_{S/U} e^{i\eta}\, x d\mu_t = 0.$$

Whence

$$\int_U e^{i\xi}\, x d\mu_t \overline{\int_{S/U} e^{\iota\eta}\, x d\mu_t} = 0. \tag{20}$$

For all $\xi, \eta \in \boldsymbol{R}$ and all $x \in \Lambda(S)$, let $d\mu_t = hd|\mu_t|$be the polar decomposition of the measure μ_t, with $h \in \Lambda(S)$. choosing $\xi = \eta = 0$and $x = 1/h$ then (20) yields

$$\int_U d\,|\mu_t| \int_{S/U} d\,|\mu_t| = 0,$$

Contradicting (19),

Setting $s = \tau(t)$, one defines a map $\tau: T \to S$, and (25) becomes

$$(Ax)(t) = x(t).x(\tau(t))$$

For all $x \in C(S)$ and all $t \in T$. choosing $x = 1$, one sees that the function $x: T \to \partial\varDelta$is continuous. To show that $\tau: T \to S$ is a continuous map,

Assume the contrary, i.e. that are $t_0 \in T$ and an neighborhood W of $\tau(t_0)$ in S, such that every neighborhood U of t_0 contains some t with $\tau(t) \notin W$. let W_0 be an open neighborhood of $\tau(t_0)$, whose closure is contained in W, and let $y \in C(S)$ be such that $y = 1$ on W_0 and $y = 0$ on S/W. for $0 < \varepsilon < 1$, there is a neighborhood U of $t_0$in Tsuch that

$$\left|\alpha(t)y(\tau(t)) - \alpha(t_0)y(\tau(t_0))\right| < \varepsilon$$

For all $t \in U$. if $t \in U$is such that $\tau(t) \notin W$, then this inequality becomes

$$1 = \left|y(\tau(t_0))\right| < \varepsilon < 1.$$

This contradiction shows that $\tau$is continuous.

If τsurjective and if $Ax = 0$ for some $x \in C(S)$, then $x(s) = 0$ for all $s \in S$, i.e., $x = 0$. Vice versa, if τ is not surjective , $\tau(T)$ is a closed, proper subset of S. if $x \in C(S)\backslash\{0\}$ is such that $x \in \tau(T) = 0$, then $Ax = 0$, and therefore $A$is not injective.

If $\tau$is surjective, then

$$\|Ax\| = \sup\{|\alpha(t)x(\tau(t))|: t \in T\} = \sup\{|\in x(s)|: s \in S\} = \|x\|$$

For all $x \in C(S)$, and A is an isometry.

This completes the proof of the theorem.

Clearly the isometry A is surjective if, and only if, τ is a homeomorphism. Thus, Theorem (2.2.2) extends the Banach-Stone theorem, and above considerations offer a differenent proof of this latter result.

Let $B(T)$and $B(S)$ be the open unit balls of the comblex Banach spaces $C(T)$and $C(S)$, and let F be a bi-holomorphism of $B(S)$ onto $B(T)$. was shown in [51] that there exist a unique function $\alpha \in \Gamma(T)$ and a unique homeormorphism $\tau$of$T$ onto Ssuch that, setting $x_0 = F\ (0)$,

$$F(x)(t) = \alpha(t)\frac{x(\tau(t)) - x_0(\tau(t))}{1 - x_0(\tau(t))x(\tau(t))} \tag{21}$$

For all $x \in B(S)$and all $t \in T$.

Let $K_{B(T)}(x; .)$be the Caratheodory-Kobayashi differential metric of $B(T)$ at the open $\in B(T)$ for all $v \in C(T)$ and $t \in T$. (21) yields

$$dF(x)v = \alpha\frac{|x_0 - \tau|^2}{1 - (x_0, \tau)(x, \tau)}(v, \tau), \tag{22}$$

And therefore

$$K_{B(T)}(x; v) = \left\|\alpha\frac{v}{1 - |x|^2}\right\|. \tag{23}$$

Let Iso $(B(S) > B(T))$be the set of all holomorphic maps of $B(S)$into $B(T)$ which are isometries for $K_{B(S)}$and $K_{B(T)}$,i.e,

$$K_{B(T)}(F(x); dF(x)v) = K_{B(S)}(x; v)$$

For all $x \in B(S)$ and all $v \in C(S)$. in view of (23), $K_{B(S)}(0, v) = \|v\|$, hence, if $A \in y(C(S), C(T))$is such that $A_{B(S)} \in Iso(B(S), B(T))$, then A is a linear isometry of $C(S)$into $C(T)$. Obvisously, if A is a surjective linear isometry, $A_{B(S)}$is a holomorphic homeomorphism of $B(S)$ onto $B(T)$, however, as will be shown, there are linear isometries of $C(S)$into $C(T)$ whose restrictions to $B(S)$ are not contained in Iso$(B(S), B(T))$. the following lemma proves that all the linear isometries described by Theorem (2.2.2) do define holomorphic isometries of $B(S)$ into $B(T)$.

Lemma (2.2.4)[48]: If $A \in y(C(S), C(T))$ is a linear isometry of $C(S)$ into $C(T)$ satisyfying (23), then $A_{B(S)} \in Iso(B(S), B(T))$.

Proof. Since, for $x \in B(S)$ and $v \in C(S) < dA(x)v = Av$, then, by (22) and by theorem 1,

$$K_{B(T)}(Ax, dA(x)v) = \left\|\frac{Av}{1 - |Ax|^2}\right\|$$
$$= \sup\left\{\frac{|v \circ \tau(t)|}{1 - |x \circ \tau(t)|^2} : t \in T\right\}$$
$$= \sup\left\{\frac{|v(s)|}{1 - |x(s)|^2} : s \in S\right\}$$

$$= \left\| \frac{v}{1-|x|^2} \right\| = K_{B(S)}(x,v).$$

Since any holomrphic map of $B(S)$ into $B(T)$ is a contraction for $K_{B(S)}$and $K_{B(T)}$if $F \in HOI\big(B(S), B(T)\big)$ is a homeorphism of $B(S)$ onto $B(T)$, then $F \in HOI\big(B(S), B(T)\big)$.

Two facts identify within $HOI\big(B(S), B(T)\big)$ the family of all holomorphic homeomorphisms F of $B(S)$ onto $B(T)$,

(a) First of all, (21) shows that, if $F(0) = 0$, then F is the restriction to $B(S)$ of linear isomorphism of $C(S)$ onto $C(T)$, this is particular case of the linearization theorem of H. Cartan (see, eg., [51]), whereby, if E E E and F are tow compex Banach spaces, if $U$and V are open, bounded, ciricled neighbor hoods of 0in E and in F, and if F is a bi-holomorphic homeomorphism of Uonto $V$with $F(0) = 0.$ then F is the restriction to$U$ of a linear isomorphism of E onto F

(b) Furthermore, a direct inspection of (21) shows that there are no non-trivial holomorghic families of bi-holomorphic homeomorphisms of $B(S)$ onto $B(T)$. also this fact turns out to be a conscequence of a general result [51] acoording to which, if D is a hyperbolic domain (in particular, a bounded domain) in E and $F \in Hol(\varDelta \times D$,F)is such that, for some $\zeta_0 \in \varDelta, F(\zeta_v, D)$is an open set in F, and $F(\zeta_0, .)$is a bi-holomorpgic homeomorphism of D onto $F(\zeta_0, D)$, then $F(\zeta, .)$ is independent of $\zeta \in \varDelta$, as an example which will be exhibited will show, these two facts do not hold, in general, when holomorphic isometries of $B(S)$ onto $B(T)$ replace bi-holomorphic homoemorphisms of $B(S)$ onto $B(T)$. The following proposition provides however a sufficient condition for a holomorphic isometry of $B(S)$ onto $B(T)$ to be the restriction to $B(S)$ of a linear isometry of $C(S)$ onto $C(T)$.

Proposition (2.2.5)[48]: Let $F \in HoI(B(C), B(T))$ be such that $F(0) = 0$. if for every $x \in \Gamma(S)$, there is some $\zeta \in \varDelta/\{0\}$for which

$$\frac{1}{\zeta} F(\zeta x) \in \Gamma(\mathrm{T}),$$

Or if $\lim_{\zeta \to 0} \frac{1}{\zeta} F(\zeta x) \in \Gamma(\mathrm{T})$ for all $x \in \Gamma(\mathrm{S})$, then F is the restriction of $dF(0) \in \mathrm{Y}$ $\big(C(S), C(T)\big)$ to $B(S)$. moreover, if $dF(0)$ is injective, then $F \in Iso\big(B(S), B(T)\big)$.

Proof. By Lemma (2.2.1) and by the strong maximum principle, the function

$$\zeta \longmapsto \frac{1}{\zeta} F(\zeta x)$$

Is independent of $\zeta \in \varDelta/\{0\}$, there fore

$$F(\zeta x) = \zeta dF(0)\mathrm{x}$$

For all $\zeta \in \varDelta$ and all $x \in \Gamma(\mathrm{S})$. hence

$$dF(0)\Gamma(\mathrm{S}) \subset \Gamma(\mathrm{T}). \tag{24}$$

Since $\Gamma(S)$ is a stable subset of $\overline{B(S)}$, by the Schwarz lemma [52], F is the restriction of $dF(0)$ to $B(S)$. In view of (24), Theorem (2.2.2) and Lemma (2.2.1) yield the conclusion.

The holomorphic map $F: (z_1, z_2) \longmapsto z_1$of $\varDelta \times \varDelta$ onto $\varDelta$provides an example showing that the assumption on the injectivity of $dF(0)$ in the proposition cannot be avoided in general.

On the other hand, since the groups aut $B(S)$and aut$B(T)$ of all holomorphic automorphisms of $B(S)$and $B(T)$act transitively on $B(S)$and $B(T)$, the condition $F(0) = 0$ can be removed.

Let $F \in HoI(B(S), B(T))$and, for $x_0 \in B(S)$, let $F(x_0) = z_0$. the maps $H \in$Aut $B(S)$and $K \in$ Aut$B(T)$defined by

$$H(x) = \frac{x + x_0}{1 + \overline{x_0}x}$$

And by

$$K(z) = \frac{z - z_0}{1 - \bar{z}_0 z}$$

On $x \in B(S)$ and$z \in B(T)$, are such that $H(0) = x_0, K(z_0) = 0$. hence the map defined by

$$L = K, F, H \in HoI(B(S), B(T)) \quad (25)$$

Is such that $L(0) = 0$. since, for any $v \in C(S)$,

$$dL(0)v = dK(z_0)dK(x_0)dH(0)v.$$
$$dH(0)v = (1 - |x_0|^2)v,$$
$$dK(z_0)w = \frac{w}{1 - |z_0|^2}.$$

For all $w \in C(T)$. then

$$dL(0) = (1 - |z_0|^2)dF(x_0)\big((1 - |x_0|^2)v\big).$$

Hence $dF(0)v \in \Gamma(\mathrm{T})$if, and only if, $(1 - |x_0|^2)dF(x_0)\big((1 - |x_0|^2)v\big) \in \Gamma(\mathrm{T})$.

In conclusion, Proposition (2.2.5) yields the following ‘’Schwarz lemma’’

Theorem (2.2.6)[48]: Let $F \in HoI(B(S), B(T))$ if there is $x \in B(S)$such that

$$(1 - |F(z_0)|^2)dF(x)\big((1 - |x|^2)v\big) \in \Gamma(\mathrm{T})$$

For all $v \in \Gamma(\mathrm{S})$, and if $dF(x)$ is injective, then $F \in HoI\big(B(S), B(T)\big)$.

Satisfies the hypotheses of Theorem (2.2.6), then, by Theorem (2.2.2), there a unique continuous surective map $\tau: T \to S$ and a unique function $\alpha \in \Gamma(\mathrm{T})$ such that $Lx(t) = \alpha(t)x\big(\tau(t)\big)$ for all $x \in B(S)$ and all $t \in T$, hence (25) implies the following ‘’Schwarz-pick lemma’’ for $Iso\big(B(S), B(T)\big)$ which extends a similar result established in [51] for the group Aut $B(T)$.

Theorem (2.2.7)[48]: For every $F \in Iso\big(B(S), B(T)\big)$ satisfiying the hypothesis of Theorem (2.2.6) there exist a unique continuous surjective map $\tau: T \to S$and a unique function $\alpha \in \Gamma(\mathrm{T})$such that, setting $x_0 = F^1(0), F$ represented by (21) for all $x \in B(S)$ and all $t \in T$.

The following example will exhibit a non-trivial holomorphic family of non-linear holomorphic isometries acting on $B(T)$, where T is the closed interval [0, 1].

For $x \in C(T)$, let $\check{x} \in C(T)$ be defined by

$$\tilde{x} = \begin{cases} x(1 - 2t) & if\ 0 \le t \le 1/2 \\ x(2t - 1) & if\ 1/2 \le t \le 1. \end{cases}$$

The map $x \longmapsto \tilde{x}$is a linear isometry of $C(T)$ into$C(T)$.

Let $\rho: [0,1] \to [0,1]$and $\lambda: [0,1] \to [0,1]$be continuous functions such that

$$\rho(t) = \begin{cases} 0 & if\ 0 \le t \le 1/3 \\ 0 < \rho(t) < 1 & if\ 1/3 < t < 1/2 \\ 1 & if\ 1/2 \le t \le 1 \end{cases}$$

and

$$\lambda(t) = \begin{cases} 0 & if\ 0 \le t \le 1/5 \\ 0 > \lambda(t) > 1 & if\ 1/5 < t < 1/4 \\ 1 & if\ 1/4 \le t \le 1. \end{cases}$$

Finally, let $F \in HoI(\varDelta \times C(T), C(T))$be defined by

$$F(\zeta, x) = \zeta\lambda x^2 + \rho\tilde{x} \tag{19}$$

Since

$$F(\zeta, x)(t) = \begin{cases} \zeta\lambda(t)x(t)^2 & if\ 0 \le t \le 1/3 \\ 0 & if\ 1/4 \le t \le 1/3 \\ \rho(t)\tilde{x}(t) & if\ 1/3 \le t \le 1/2 \\ \tilde{x}(t) & if\ 1/2 \le t \le 1 \end{cases} \tag{20}$$

Then

$$\|F(\zeta, x)\| = \|x\| \tag{21}$$

For all $\zeta \in \varDelta$ and all $x \in B(T)$.
For all $\zeta, \eta \in \varDelta$and $x \in B(T), y \in C(T)$.

$$F(\zeta, x + \eta\zeta) = F(\zeta, x) + \eta d_2 F(\zeta, x) + \zeta\eta^2\lambda y^2,$$

Where $d_2F(\zeta, x)y = 2\zeta\lambda xy + p\tilde{\gamma}$ is the image of γ by the Fréchet differential of F with respect to the second variable at the point (ζ, x).
Hence (22) yields

$$K_{B(T)}(F(\zeta, x); yd_2F(\zeta, x)y) = \left\| \frac{2\zeta\lambda xy + p\tilde{\gamma}}{1 - |F(\zeta, x)|^2} \right\|. \tag{22}$$

And

$$\frac{2\zeta\lambda(t)x(t) + p(t)\tilde{\gamma}(t)}{1 - |F(\zeta, x)|^2} = \begin{cases} \dfrac{2\zeta\lambda(t)x(t)y(t)}{1 - |\zeta(\lambda)x(t)|^2} & if\ 0 \le t \le 1/4 \\ 0 & if\ 1/4 \le t \le 1/3 \\ \dfrac{p(t)\tilde{\gamma}(t)}{1 - |p(t)\tilde{\gamma}(t)|^2} & if\ 1/3 \le t \le 1/2 \\ \dfrac{y(2t-1)}{1 - |x(2t-1)|^2} & if\ 1/4 \le t \le 1/3 \end{cases} \tag{23}$$

For $0 \le t \le 1/2$.

$$\frac{p(t)|\tilde{\gamma}(t)|}{1 - |p(t)\tilde{x}(t)|^2} \le \frac{|\tilde{\gamma}(t)|}{1 - |\tilde{x}(t)|^2} = \frac{|y(1-2t)|}{1 - |x(1-2t)|^2}$$

And

$$\frac{|2\zeta\lambda(t)x(t)y(t)|}{1 - |\zeta(\lambda)x(t)^2|^2} \le \frac{|2\zeta x(t)y(t)|}{1 - |\zeta x(t)^2|^2} \le \frac{|2\zeta x(t)y(t)|}{1 - |\zeta x(t)|^2} \le \frac{2|\zeta|}{1 - |\zeta|^2}|y(t)|.$$

In the interval [0, 1] the inequality $2t/(1-t^2)^2 \le 1$ is equivalent to $0 \le t \le \sqrt{2} - 1$. hence, if $|\zeta| < \sqrt{2} - 1$.

$$\frac{|2\zeta\lambda(t)x(t)y(t)|}{1 - |\zeta\lambda(t)x(t)^2|^2} \le |y(t)| \le \frac{|y(t)|}{1 - |x(t)|}$$

For all $t \in [0,1]$.
In view of (23), (20)and(22)yield

Proposition (2.2.8)[48]: Whenever$|\zeta| < \sqrt{2} - 1, F(\zeta, .) \in Iso\big(B(T), B(T)\big)$, see [50], [55].

Replace now (19) by the map $F(\zeta, .) \in$ y$(C(T) < C(T))$defined by

$$F(\zeta, x) = \zeta\lambda x + \rho\tilde{x}. \tag{24}$$

A similar (even easier) computation to that carried out above shows that, whenever $\zeta \in \varDelta$, (21) holds for all $x \in C(T)$, i.e., is a linear isometry of $C(T)$ into itself. Furthermore, (22) and (23) become, respectively,

$$K_{B(T)}(F(\zeta,x);yd_2F(\zeta,x)y)=\left\|\frac{F(\zeta,y)}{1-|F(\zeta,x)|^2}\right\|.$$

$$\frac{F(\zeta,y)(t)}{1-|F(\zeta,x)|^2}=\begin{cases}\dfrac{\zeta\lambda(t)y(t)}{1-|\zeta\lambda(t)x(t)|^2} & if\ 0\le t\le 1/4\\ 0 & if\ 1/4\le t\le 1/3\\ \dfrac{p(t)\tilde{y}(t)}{1-|p(t)\tilde{x}(t)|^2} & if\ 1/3\le t\le 1/2\\ \dfrac{\tilde{y}(t)}{1-|\tilde{x}(t)|^2} & if\ 1/2\le t\le 1.\end{cases}\qquad(25)$$

Showing, as before, that Proposition (2.2.8) holds also when F is given by (24), whenever $|\zeta|<(\sqrt{5}-1)/2$.

Choosing as x a real constant k, with $0<k<1$, the top equality in (25) becomes, for $0\le 1\le 1/5$,

$$\frac{\zeta\lambda(t)y(t)}{1-|\zeta\lambda(t)x(t)|^2}=\frac{\zeta y(t)}{1-|\zeta|^2k^2},$$

And, if $(\sqrt{5}-1)/2<|\zeta|<1$, then

$$\frac{|\zeta|}{1-|\zeta|^2k^2}>1.$$

As a consequence, if $|\zeta|\in((\sqrt{5}-1)/2,1)$, the linear isometry $F(\zeta,.)$ given by (24) does not define a holomorphic isometry in $B(T)$, i.e,. $F(\zeta,.)\notin Iso(B(T),B(T))$.

Exampling the function λ to $\boldsymbol{R}_+$by assuming $\lambda(t)=0$ when $t\ge 1$, let $F:C(T)\to C(T)$ be defined by

$$F(x)(t)=\lambda(t)\lambda(\|x(t)\|)x(t)+\rho(t)\tilde{x}(t)$$

For all $t\in T$. since

$$F(x)(t)=\begin{cases}\lambda(t)\lambda(|x(t)|)x(t) & if\ 0\le t\le 1/3\\ \rho(t)\tilde{x}(t) & if\ 1/3\le t\le 1/2\\ \tilde{x}(t) & if\ 1/2\le t\le 1\end{cases}$$

Then $\|F(x)\|=\|x\|$ for all $x\in C(T)$, the function F provids an example of a non-linear isometry of $C(T)$ into $C(T)$ fixing 0.

Section (2.3): Weighted Composition Operators

For x and y be locally compact Hausdorff spaces. Let $C_0(X)$ (resp. $C_0(Y)$) be the Banach space of continuous scalar-valued (i.e. real- or complex-valued) functions defined on X (resp. Y) vanishing at infinity and equipped with the supremum norm. The classical Banach-Stone theorem gives a description of surjective isometries from $C_0(X)$ onto $C_0(Y)$. They are all weighted composition operators $Tf=h.f\circ\varphi$ = (i.e.$Tf(y)=h(y)f(\varphi(y)),\forall y\in Y$) for some homeomorphism φ from Y onto X and some continuous scalar-valued function h on Y with $|h(y)|\equiv 1,\forall y\in Y$. Different generalizations (see e.g. [58], [59], [61], [25], [48]) of the Banach-Stone Theorem have been studied in many years. Some of them discuss the structure of into isometries and disjointness preserving linear maps (see e.g. [60], [62]). A linear map from $C_0(X)$ into $C_0(Y)$ is said to be disjointness preserving if $f\circ g=0$ in $C_0(X)$ implies $Tf.Tg=0$ in $C_0(Y)$. We shall discuss the structure of weighted composition operators from $C_0(X)$ into $C_0(Y)$.

We prove that every into isometry and every disjointness preserving linear map from $C_0(X)$ into $C_0(Y)$ is essentially a weighted composition operator.

Theorem (2.3.1)[57]: Let X and Y be locally compact Hausdorff spaces and T a bounded disjointness preserving linear map from $C_0(X)$ into $C_0(Y)$. Then there exist an open subset Y_1 of Y and a weighted composition operator T_1 from $C_0(X)$ into $C_0(Y_1)$ such that for all f in $C_0(X)$, Tf vanishes outside Y_1 and

$$Tf_{|Y_1} = T_1 f = h.f \circ \varphi$$

for some continuous map φ from Y_1 into X and some continuous scalar-valued function h defined on Y_1 with $h(y) \neq 0, \forall y \in Y$.

Since weighted composition operators from $C_0(X)$ into $C_0(Y)$ are disjointness preserving, Theorem (2.3.1) gives a complete description of all such maps. When X and Y are both compact, Theorems (2.3.5) and (2.3.1) reduce to the results of W. Holsztynski [60] and K. Jarosz [62], respectively. It is plausible to think that Theorems (2.3.5) and (2.3.1) could be easily obtained from their compact space versions by simply extending an into isometry (or a bounded disjointness preserving linear map) $T: C_0(X) \to C_0(Y)$to a bounded linear map of $T_\infty: C(X_\infty) \to C(Y_\infty)$ of the same type, where $X_\infty = X \cup \{\infty\}$ and $Y_\infty = Y \cup \{\infty\}$ are the one –point compactifications of the locally compact Hausdorff spaces X and Y , respectively.

However, the example given will show that this idea is sometimes fruitless because *T* can have *no* such extensions at all. We thus have to modify, and in some cases give new arguments to, the proofs of W. Holsztynski [60] and K. Jarosz [62] to fit into our more general settings.

Recall that for f in $C_0(X)$, the *cozero* of f is coz(*f*) = $\{x \in X: f(x) \neq 0$ and the *support* supp(*f*) of f is the closure of coz(*f*) in X_∞. A linear map $T: C_0(X) \to C_0(Y)$is disjointness preserving if T maps functions with disjoint cozeroes to functions with disjoint cozeroes. For *x* in *X*, δ_x denotes the point evaluation at *x*, that is, δ_x is the linear functional on $C_0(X)$ defined by $\delta_x(f) = f(x)$. For *y* in , let supp($\delta_y oT$) be the set of all *x* in X_∞ such that for any open neighborhood U of *x* in X_∞ there is an *f* in $C_0(X)$ with *Tf*(*y*) ≠0 and coz(*f*)⊂ U *U*. The kernel of a function *f* is denoted by ker *f*.

Definition (2.3.2) Let *X* and *Y* be locally compact Hausdorff spaces. A map φ from *Y* into *X* is said to be *proper* if preimages of compact subsets of *X* under φ are compact in *Y* .

It is obvious that φ is proper if and only if $lim_{y\to\infty}\varphi(y) = \infty$. As a consequence, a proper continuous map δ from a locally compact Hausdorff space Y onto a locally compact Hausdorff space *X* is a quotient map, i.e. $\varphi^{-1}(O)$ is open in *X* if and only if *O* is open in Y . A quotient map from a locally compact space onto another is, however, not necessarily proper. For example, the quotient map φ from $(-\infty, +\infty)$ onto $[0, +\infty)$defined by

$$\varphi(y) = \begin{Bmatrix} y, y > 0 \\ 0, y \leq 0 \end{Bmatrix}$$

is not proper.

Lemma (2.3.3)[57]: Let X and Y be locally compact Hausdorff spaces, φ a map from Y into X, and h a continuous scalar-valued function defined on Y with bounds $M, m > 0$ such that $m \leq |h(y)| \leq M, \forall y \in Y$ Then the weighted composition $Tf = h.f \circ \varphi$ defines a(necessarily bounded) linear map from $C_0(X)$ into $C_0(Y)$ if and only if ' is continuous and proper.

Proof. For the sufficiency, we need to verify that $h.f$ vanishes at 1 for all f in $C_0(X)$. For any $\epsilon > 0$, $|f(x)|<\epsilon/M$ outside some compact subset K of X. Since φ is proper, $\varphi^{-1}(K)$ is compact in Y . Now the fact that $\left|h(y).f(\varphi(y))\right| \le M\left|f(\varphi(y))\right| < \epsilon$ outside $\varphi^{-1}(K)$ indicates that $h.f \circ \varphi \in C_0(Y)$. The boundedness of T is trivial in this case.

For the necessity, we first check the continuity of φ. Suppose y_λ in Y.

We want to show that $x_\lambda = \varphi(y_\lambda) \to \varphi(y)$ in X. Suppose not, by passing to a subnet if necessary, we can assume that x_λ, either converges to some $x \neq \varphi(y)$ in X or ∞. If x_λ in X then for all f in $C_0(X)$,

$$\begin{aligned} h(y)f(x) &= \lim h(y_\lambda)f(x_\lambda) = \lim h(y_\lambda)f(\varphi(y_\lambda)) \\ &= \lim Tf(y_\lambda) = Tf(y) = h(y)f(\varphi(y)): \end{aligned}$$

As $h(y) \neq 0$, $f(x) = f(\varphi(y))$, $\forall f \in C_0(X)$. Consequently, we obtain a contradiction $x = \varphi(y)$! if $x_\lambda \longrightarrow \infty$ then a similar argument gives $f(\varphi(y)) = 0$ for all f in $C_0(X)$. Hence $\varphi(y) = 1$, a contradiction again! Therefore, φ is continuous from Y into X. Finally, let K be a compact subset of X and we are going to see that $\varphi^{-1}(K)$ is compact in Y , or equivalently, closed in $Y_\infty = Y \cup \{\infty\}$, the one-point.

We want $y \in \varphi^{-1}(K)$, i.e. $y \neq \infty$ and $\varphi(y) \in K$. Without loss of generality, we can assume that $x_\lambda \to x$ for some x in K. compactification of Y . To see this, suppose $y_\lambda \to y$ in Y_∞ and $x_\lambda = \varphi(y_\lambda) \in K$. Now,

$$\lim |Tf(y_\lambda) = \lim |h(y_\lambda)f(\varphi(y_\lambda))| \geq m \lim |f(x_\lambda)| = m|f(x)|$$

for all f in $C_0(X)$. This implies that $y \neq 1$ and then a similar argument gives $\varphi(y) = x \in K$. The assumption on the bounds of f in Lemma (2.3.3) is significant. For example, let $X = Y = R = (-\infty, +\infty)$ and define

$$h(y) = \begin{cases} e^y, & y < 0 \\ 1, & y \geq 0 \end{cases} \quad and \quad \varphi(y) = \begin{cases} y \\ \sin y, & y \geq 0 \end{cases}$$

Then the weighted composition operator $Tf = h.f \circ \varphi$ from $C_0(\mathbb{R})$ into $C_0(\mathbb{R})$ is well-defined. It is not difficult to see that $\varphi^{-1}([-\frac{1}{2}, \frac{1}{2}])$ is not compact in R.

On the other hand, if we redefine $h(y) = e^y$ and $\varphi(y) = y$ for all y in R then the weighted composition operator T is not well-defined from C_0(R) into C_0(R), even though φ is proper and continuous in this case.

Recall that a bounded linear map T from a Banach space E into a Banach space F is called an injection if there is an $m > 0$ such that $||Tx|| \geq m||x||$, $\forall x \in E$. It follows from the open mapping theorem that T is an injection if and only if T is one-to-one and has closed range.

Proposition (2.3.4)[57]: Let X and Y be locally compact Hausdorff spaces, φ a map from Y into X and h a continuous scalar-valued function defined on Y . The weighted composition operator $\mathrm{Tf} = \mathrm{h.f} \circ \varphi$ from $\mathrm{C_0(X)}$ into $\mathrm{C_0(Y)}$ is an injection if and only if φ is continuous, proper and onto and h has bounds M,m > 0 such that $\mathrm{m} \leq |\mathrm{h(y)}| \leq \mathrm{M}, \forall \mathrm{y} \in \mathrm{Y}$. In this case, φ is a quotient map and thus X is a quotient space of Y .

Proof. The sufficiency follows easily from Lemma (2.3.3) and the observation that $\left|\left|Tf\right|\right| = ||h.f \circ \varphi|| \geq m||f||, \forall f \in C_0(\mathrm{X})$ For the necessity, we first note that there are constants $M,m > 0$ such that $m||f|| \leq ||Tf|| \leq M||f||$ for all f in $C_0(X)$. It is then obvious that $m \leq |h(y)| \leq M$, $\forall y \in Y$. By Lemma (2.3.3), φ is continuous and proper. Finally, we check that φ is onto. It is not difficult to see that φ has dense range. In fact, if $\varphi(Y)$ were not dense in X, then there were an x in X and a neighborhood U of x in X such that $U \cap$

$\varphi(Y) = \emptyset$. Choose an f in $C_0(X)$ such that $f(x) = 1$ and f vanishes outside U. Then $Tf(y) = h(y)f(\varphi(y)) = 0$ for all y in Y , i.e. $Tf = 0$. Since T is an injection, we get a contradiction that $f = 0$! We now show that $\varphi(Y) = X$. Let $x \in X$ and K a compact neighborhood of x in X.

By the density of $\varphi(Y)$ in X, there is a net $\{y_\lambda\}$ in Y such that $x_\lambda = \varphi(y_\lambda)$in X. Without loss of generality, we can assume that x_λ belongs to K for all .

Since $\varphi^{-1}(K)$ is compact in Y , $\varphi(\varphi^{-1}(K))$ is a compact subset of X containing the net $\{x_\lambda\}$. Consequently, $x = \lim x_\lambda$ belongs to $\varphi\big(\varphi^{-1}(K)\big) \subset \varphi(Y)$.

Theorem (2.3.5)[57]: Let X and Y be locally compact Hausdorff spaces and T a linear isometry from $C_0(X)$ into $C_0(Y)$. Then there exist a locally compact subset Y_1 (i.e. $Y_1$is locally compact in the subspace topology) and a weighted composition operator T_1 from $C_0(X)$ into $C_0(Y_1)$ such that for all f in $C_0(X)$,

$$Tf_{|Y_1} = T_1 f = h.f \circ \varphi$$

for some quotient map ' fromY_1 onto X and some continuous scalar-valued function h defined on Y_1 with $|h(y)| \equiv 1, \forall y \in Y_1$.

Proof. We adopt some notations from W. Holsztynski [60] and K. Jarosz [62]. Let $X_1 = X \cup \{\infty\}$ and $Y_\infty = Y \cup \{\infty\}$ be the one-point compactifications of X and Y , respectively. For each x in X and y in Y, put

$$S_x = \{f \in C_0(X): |f(x)| = \|f\| = 1\}$$
$$R_y = \{g \in C_0(Y): |g(y)| = \|g\| = 1\} and$$
$$Q_x = \{y \in Y: T(S_x) \subset R_y\}$$

We first Claim (2.3.7) that $\{Q_x\}_{x\in X}$ is a disjoint family of non-empty subsets of Y. In fact, for $f_1, f_2, \dots, f_n\ in\ S_x$, let $h = \sum_{i=1}^{n} \overline{f_i(x) f_i}$.

Then $\|h\| = n$ and thus $\|Th\| = n$. Hence there is a y in Y such that $|\sum_{i=1}^{n} \overline{f_i(x)} Tf_i(y)| = |Th(y)| = n$. This implies $|Tf_i(y)| = 1$ for all $i = 1,2 \dots n$. In other words $y \in \cap_{i=1}^{n} (Tf_i)^{-1}(\Gamma)$, where $\Gamma = \{z: |z| = 1$. We have just proved that the family $\{(Tf)^{-1}(\Gamma): f \in S_x\}$ of closed subsets of the compact spaceY_∞ has finite intersection property. It is plain that $\infty \notin (Tf)^{-1}(\Gamma)$ for all f in S_x. Hence $Q_x = \cap_{f\in S_x} (Tf)^{-1}(\Gamma)$ is non-empty for all x in X. Moreover,$Q_{x_1} \cap Q_{x_2} = \emptyset$ if $x_1 \neq x_2$in X. In fact, f_1 in S_{x_1} and f_2 in S_{x_2} exist such that coz(f_1)\coz$(f_2) = \emptyset$.

If there is a y in $Q_{x_1} \cap Q_{x_2}$ then it follows from $Tf_1 \in R_y$ and $Tf_2 \in R_y$ that

$$1 = \|f_1 + f_2\| = \|T(f_1 + f_2)\| = |T(f_1 + f_2)(y)| = 2, \text{ a contradiction!}$$

Let $Y_1 = \cup_{x\in X} Q_x$. It is not difficult to see that supp$(\delta_y \circ T) = \{x\}$ whenever $y \in Q_x$. So we can define a surjective map $\varphi: Y_1 \to X$ by

$$\{\varphi(y)\} = supp(\delta_y \circ T)$$

Note that for all f in $C_0(X)$ and for all y in Y_1,

$$\varphi(y) \notin supp(f) \Longrightarrow T(f)(y) = 0 \tag{26}$$

In fact, if $Tf(y) \neq 0$, without loss of generality, we can assume $Tf(y) = r > 0$ and $\|f\| = 1$. Since $\varphi(y) \notin$ supp(f), there is a g in $C_0(X)$ such that coz$(f)\cap$coz$(g) = \emptyset$ and $Tg(y) = \|g\| = 1$. Hence $1+r = T(f+g)(y) > \|f+g\| = 1$, a contradiction!

Now, we want to show that ' is continuous. Suppose φ were not continuous at some y in Y_1, without loss of generality, let $\{y_\lambda\}$ be a net converging to y in Y_1 such that $\varphi(y_\lambda) \to x \neq \varphi(y)$ in X_∞. Then there exist disjoint neighborhoods U_1 and U_2 of x and $\varphi(y)$ in X_1, respectively, and a λ_1 such that $\varphi(y_\lambda) \in U_1, \forall_\lambda \geq \lambda_0$. Let $f \in C_0(X)$ such that coz$(f) \subset U_2$

and $T(f)(y) = ||f|| = 1$. As supp(f)∩ U_1 =∅, we have $\varphi(y_\lambda) \notin supp(f)$,∀ $\lambda \geq \lambda_0$ by (26)., T(f)(y_λ)=0 $\forall \geq \lambda_0$This implies $T(f)$ is not continuous at y, a contradiction!

For each y in Y_1, put

$$J_y = \{f \in C_0(X): \varphi(y) \notin supp(f)\}, and$$

$$K_y = \{f \in C_0(X): f(\varphi(y)) = 0\}$$

For f in K_y and $\epsilon > 0$, let $X_1 = \{x \in X: |f(x)| \geq \epsilon\}$ ¸ and $X_2 = \{x \in X: |f(x)| < \epsilon/2\}$. Let g be a continuous function defined on X such that $0 \leq g(x) \leq 1, \forall\, x \in X\,, g(x) = 1 \quad, \forall x \in X, g(x) = 1,\ \forall x \in X_1$ and $g(x) = 0\ ,\forall x \in X_2$. *Let* $f_\epsilon = g.f$ Then $f_\epsilon \in J_y$ and $||f_\epsilon - f|| \leq 2\epsilon$. One thus can show that J_y is a dense subset of K_y. By (26), $J_y \subset ker(\delta_y \circ T)$, and hence ker($\delta_{\varphi(y)}$) =$K_y$ ⊂ ker($\delta_y \circ T$).Consequently, there exists a scalar $h(y)$ such that $\delta_y \circ T = h(y).\delta_{\varphi(y)}$, i.e.

$$T(f)(y) = h(y).f(\varphi(y)), \quad \forall f \in C_0(X)$$

It follows from the definition of Y_1 that h is continuous on Y_1 and $|h(y)|$ = 1, $\forall y \in Y_1$.
It is the time to see that Y_1 is locally compact. For each y_1 in Y_1 and a neighborhood U_1 of y_1 in Y_1, we want to find a compact neighborhood $K_!$ of y_1 in Y_1 such that $y_1 \in K_1 \subset U_1$. Let $x_1 = \varphi(y_1)$ in X. Then

$$|Tf(y_1)| = |f(x_1)|\,, \forall f \in C_0(X)$$

Fix f_1 in S_{x_1} . Then $V_1 = \varphi^{-1}\left(\left\{x \in X: |f_1(x)| > \frac{1}{2}\right\}\right) U_1$ is an open neighborhood of y_1 in Y_1 and contained in U_1. Since $V_1 = W \cap Y_1$ for some neighborhood W of y_1 in , there exists a compact neighborhood K of y_1 in Y such that $y_1 \in K \subset W$.

We are going to verify that $K_1 = K \cap Y_1$ is a compact neighborhood of y1 in Y_1.

Let $\{y_\lambda\}$ be a net in $K_1 \subset Y_1$. By passing to a subnet, we can assume that y_λ converges to y in K and we want to show $y \in Y_1$. Let $x_\lambda = \varphi(y_\lambda)$ in X. Since X_1 is compact, by passing to a subnet again, we can assume that $x_¸$ converges to x in X or $x_\lambda \to \infty$, $|Tf_1(y)|$ = in X ,$|Tf(y_\lambda)$ = lim $|h(y_\lambda)|$ = lim $|h(y_\lambda)f(\varphi(y_\lambda)$ =lim $|f(x_\lambda)| = |f(x)|$, for all f in $C_0(X)$. Hence $y \in Q_x$, and thus $y \in Y_1$. If

$$x_\lambda \to \infty\ , |Tf_1(y)| = \lim |Tf_1(y_\lambda)| = \lim |f_1(x_\lambda)| = 0.$$

However, the fact that $y_\lambda \in V_1$ ensures $|Tf_1(y_\lambda)| = |f_1(x_\lambda)| > \frac{1}{2}$ for all λ¸, a contradiction! Hence Y_1 is locally compact.

Let $T_1 : C_0(X) \to C_0(Y_1)$ defined by $T_1 f = h.f \circ \varphi$. It is clear that T_1 is a linear isometry and $Tf_{|Y_1} = T_1 f$. By Proposition (2.3.4), the surjective continuous map φ is proper and thus a quotient map. The proof is complete.

In Theorem (2.3.5), Y_1 can be neither open nor closed in Y and φ may not be an open map. See the following examples.

It is clear that Theorem (2.3.1) follows from the following more general result in which discontinuity of the linear disjointness preserving map T is allowed. The payoff of the discontinuity is a finite subset F of X at which the behaviour of T is not under control.

Theorem (2.3.6)[57]: Let X and Y be locally compact Hausdorff spaces and T a disjointness preserving linear map from $C_0(X)$ into $C_0(Y)$. Then Y can be written as a disjoint union $Y = Y \cup_1 Y_2 \cup Y_3$, in which Y_2 is open and Y_3 is closed. A continuous map φ from $Y_1 \cup Y_2$ into X_∞ exists such that for every f in $C_0(X)$,

$$\varphi(y) \notin \mathrm{supp}(f) \Rightarrow T(f)(y) = 0 \qquad (27)$$

Moreover, a continuous bounded non-vanishing scalar-valued function h on Y_1 exists such that

$$Tf_{|Y_1} = h.f \circ \varphi, and$$
$$Tf_{|Y_3} = 0$$

Furthermore, F = $\varphi(Y_2)$ is a finite set and the functional $\delta_y \circ T$ are discontinuous on $C_0(X)$ for all y in Y_2.

Proof. We shall follow the plan of K. Jarosz in his compact space version [62].

Set

$$Y_3 = \{y \in Y \big| \delta_y \circ T \equiv 0\}$$
$$Y_2 = \{y \in Y \big| \delta_y \circ T \text{ is discontinuous}\}, and$$
$$Y_1 = Y \setminus (Y_2 \cup Y_3)$$

First, we Claim (2.3.7) that supp($\delta_y \circ T$) contains exactly one point for every y in $Y_1 \cup Y_2$. Suppose on the contrary that supp($\delta_y \circ T$) contains two distinct points x_1 and x_2 in X_1. Let U_1 and U_2 be neighborhoods of x_1 and x_2 in X_1, respectively, such that $U_1 \cap U_2 = \emptyset =$ Let f_1 and f_2 in $C_0(X)$ with coz(f_1) $\subset U_1$ and coz(f_2) $\subset U_2$ be such that $Tf_1(y) \neq 0$ and $Tf_2(y) \neq 0$. However, $f1f2$ = 0 implies $Tf_1Tf_2 = 0$, a contradiction! Suppose supp($\delta_y \circ T$) is empty. Then we can write the compact Hausdorff space X_∞ as a finite union of open sets $X_\infty = \cup_{i=1}^{n} U_i$ such that $Tf(y)$ whenever coz(f) $\subset U_i$ for some $i = 1,2,\ldots n$. Let 1 $=\sum_{i=1}^{n} f_i$ be a continuous decomposition of the identity coordinate to $\{U_i\}_{i=1}^{n}$. Then for all f in $C_0(X), Tf(y) = \sum_{i=1}^{n} T(ff_i)(y) = 0$. This says $\delta_y \circ T \equiv 0$ and thus $y \in Y_3$.

Next we define a map φ from $Y_1 \cup Y_2$ into X_∞ by

$$\{\varphi(y)\} = supp(\delta_y \circ T)$$

We now prove (27). Assume $\varphi(y) \notin supp(f)$. Then there is an open neighborhood U of $\varphi(y)$ disjoint from coz(f). Let $g \in C_0(X)$ such that coz(g) $\subset U$ and $Tg(y) \neq 0$. Since $fg = 0$ and T is disjointness preserving, $Tf(y)$ = 0 as asserted.

It then follows from (27) the continuity of $'$ as one can easily modify an argument of the proof of Theorem (2.3.5) for this goal. Similarly, it also follows from (27) the desired representation

$$Tf(y) = h(y)f\big(\varphi(y)\big), \forall f \in C_0(X), \forall y \in Y_1 \qquad (28)$$

where h is a continuous non-vanishing scalar-valued function defined on Y_1.

Claim (2.3.7)[57]: Let $\{y_n\}_{n=1}^{\infty}$ be a sequence in $Y_1 \cup Y_2$ such that $x_n = \varphi(y_2)'$s are distinct points of X. Then

$$\limsup \left|\left|\delta_{y_n} \circ T\right|\right| < \infty$$

In particular, only finitely many $\delta_y \circ T$ can have in finite norms.

Assume the contrary and, by passing to a subsequence if necessary, we have

$$\left|\left|\delta_{y_n} \circ T\right|\right| > n^4, n = 1,2,\ldots$$

Let $f_n \in C_0(X)$with $||f_n|| \leq 1$ such that

$$|Tf_n(y_n)| \geq n^3, n = 1,2,\ldots$$

Let V_n, W_n and U_n be open subsets of X such that $x_n \in V_n \subseteq \overline{V_n} \subseteq W_n \subseteq \overline{W_n} U_n$ and $U_n \cap U_m = \emptyset$ if $n \neq m, n, m = 1,2,\ldots$, and let $g_n \in C(X_\infty)$ such that $0 \leq g_n \leq 1, g_{n|V_n} \equiv 1$ and $g_{n|X_\infty \setminus W_n} \equiv 0, n = 1,2,\ldots$. Then (27) implies

$$Tf_n(y_n) = T(f_ng_n)(y_n) + T\big(f_n(1-g_n)\big)(y_n)$$
$$= T(f_ng_n)(y_n), n = 1,2,\ldots$$

Therefore, we can assume supp$f_n \subset U_n$. Let $f = \sum_{n=1}^{\infty} \frac{1}{n^2} f_n$ in $C_0(X)$ By (27) again, $|Tf(y_n)| = \left|\frac{1}{n^2} Tf_n(y_n)\right| \geq n$ for $n = 1, 2, \dots$. This conflicts with the boundedness of Tf in $C_0(Y)$, and the Claim (2.3.7) is thus verified.

The assertion $F = \varphi(Y_2)$ is a finite subset of X is clearly a consequence of the Claim (2.3.7) while the boundedness of h follows from the Claim (2.3.7) and (28). It is also plain that $Y_3 = \cap \{kerTf : f \in C_0(X)\}$ is closed in . Finally, to see that Y_2 is open, we consider for every f in $C_0(X)$,

$$\sup\{|Tf(y)| : y \in \overline{Y_1 \cup Y_3} = \sup\{|Tf(y)| : y \in Y_1 \cup Y_3\}\}$$
$$= \sup\{ |Tf(y) : y \in Y_1\}$$
$$= \sup\{|h(y)f(\varphi(y))| : y \in Y_1\} \leq M||f||,$$

where $M > 0$ is a bounded of h on Y_1. It follows that the linear functional $\delta_y \circ T$ is bounded for all y in $\overline{Y_1 \cup Y_3}$, and thus $Y_2 \cap \overline{Y_1 \cup Y_3} = \emptyset$. Hence, $Y_1 \cup Y_3 = \overline{Y_1 \cup Y_3}$ is closed. In other words, Y_2 is open.

Theorem (2.3.8)[57]: Let X and Y be locally compact Hausdoff spaces and T a bijective disjointness preserving linear map from C_0(X) onto C_0(Y). Then T is a bounded weighted composition operator, and X and Y are homeomorphic.

Proof. We adopt the notations used in Theorem (2.3.6). Since T is surjective, $Y_3 = \emptyset$. We are going to verify that $Y_2 = \emptyset$, too. First, we note that the finite set $F \setminus \{\infty\}$ consists of non-isolated points in X. In fact, if $y \in Y_2$ such that $x = \varphi(y)$ is an isolated point in X then it follows from (27) that for every f in $C_0(X)$, $f(x) = 0$ implies $\varphi(y) = x \in supp\ f$ and thus $Tf(y) = 0$. Hence, $\delta_y \circ T = \lambda\ \delta_x$ for some scalar λ. Therefore, $\delta_y \circ T$ is continuous, a contradiction to the assumption that $y \in Y_2$. We then Claim (2.3.7) that $\varphi(y) = \varphi(Y_1 \cup Y_2)$ is dense in X. In fact, if a nonzero f in $C_0(X)$ exists such that supp$f \cap \varphi(Y) = \emptyset$ then $Tf = 0$ by (27), conflicting with the injectivity of T. Since

$$X = \overline{\varphi(Y)} = \overline{\varphi(Y_1 \cup \varphi(Y_2)} = \overline{\varphi(Y_1) \cup F} = \overline{\varphi(Y_1)}\ or$$

for every f in $C_0(X)$,

$$Tf_{|_{Y_1}} = 0 \Longrightarrow f_{|_{\varphi(Y_1)}} = 0 \Longrightarrow f = 0 \Longrightarrow Tf_{|_{Y_2}} = 0$$

Therefore, the open set $Y_2 = \emptyset$ by the surjectivity of T. Theorem (2.3.6) then gives

$$Tf = h.(f \circ \varphi)\ , \forall f \in C_0(X)$$

This representation implies that T^{-1} is also a bijective disjointness preserving linear map from $C_0(Y)$ onto $C_0(X)$. The above discussion provides that

$$T^{-1}g = h_1.g \circ \varphi_1\ , \qquad \forall g \in C_0(Y)$$

for some continuous non-vanishing scalar-valued function h_1 on X and continuous function φ_1 from X into Y . It is plain that $\varphi_1 = \varphi^{-1}$ and thus X and Y are homeomorphic.

The following example shows that not every into isometry or bounded disjointness preserving linear map from $C_0(X)$ into $C_0(Y)$ can be extended to a bounded linear map from $C(X_1)$ into $C(Y_1)$ of the same type. Here X and Y are locally compact Hausdorff spaces with one-point compactifications X_1 and Y_1, respectively.

Example (2.3.9)[57]: Let $X = [0, +\infty)$, $Y = (-\infty, +\infty)$ and the underlying scalar field is the field R of real numbers. Let

$$h(y) = \begin{Bmatrix} 1\ , y > 2 \\ y - 1\ , 0 \leq y \leq 2 \\ -1\ , \quad y < 0 \end{Bmatrix}$$

and

$$\varphi(y) = \begin{cases} y\,, y \geq 0 \\ -y\,, y < 0 \end{cases}$$

Then the weighted composition operator $Tf = h.f \circ \varphi$ is simultaneously an into isometry and a bounded disjointness preserving linear map from $C_0([0, +\infty))$ into $C_0((-\infty, +\infty\ )$ .However, no bounded linear extension T_∞ from $C([0,\infty])$ into $C((-\infty,+\infty) \cup \{\infty\}$of T can be an into isometry or a disjointness preserving linear map.

Suppose, on the contrary, T_∞ were an into isometry. Consider f_n in $C_0([0,+\infty))$ defined by

$$f_n(x) \begin{cases} 1\,, 0 \leq x \leq n, \\ \dfrac{2n-x}{n}, n < x < 2n\ , n = 1,2,.. \\ 0, 2n \leq x \leq +\infty, \end{cases}$$

Note that $\delta_y \circ T_\infty$ can be considered as a bounded Borel measure *my* on $[0,+\infty]$ for all point evaluation δ_y at y in $(-\infty,+\infty)\ \cup\{\infty\}$ with total variation $\left|\left|m_y\right|\right| = ||\delta_y \circ T_\infty|| \leq 1$. Let 1 be the constant function 1(*x*)≡ 1 in $C([0,+\infty])$. For all y in $(-\infty,+\infty)$,

$$T_\infty 1(y) = \delta \circ T_\infty(1) = \int_{[0,+\infty]} 1\, dm_y$$

$$= \lim_{n\to\infty} \int_{[0,+\infty]} f_n\, dm_y + m_y(\{\infty\})$$

$$= \lim_{n\to\infty} \delta_y \circ T_\infty(f_n) + m_y(\{\infty\})$$

$$= \lim_{n\to\infty} Tf_n(y) + m_y(\{\infty\}) = \lim_{n\to\infty} h(y).f_n(\varphi(y)) + m_y(\{\infty\})$$

$$= h(y) + m_y(\{\infty\})$$

Let $g(y) = m_y(\{\infty\})$ for all y in $(-\infty,+\infty)$. Then $g(y) = T_\infty 1(y) - h(y)$ is continuous on $(-\infty,+\infty)$ and $|g(y)| = |m_y(\{\infty\})| \leq \left|\left|m_y\right|\right| \leq 1, \forall y \in (-\infty,+\infty)$.Note that $||T_\infty 1|| = 1$. Therefore, $g(y) = T_\infty 1(y) - 1 \leq 0$ when $y > 2$, and $g(y) = T_\infty 1(y) + 1 \geq 0$ ¸ when $y < -2$. We Claim (2.3.7) that $g(y)g(-y) = 0$ whenever $|y| > 2$. In fact, if for example $g(y_0) < -\delta$ for some $y_0 > 2$ and some $y_0 > 2$, then for each small $\epsilon > 0$, $0 \leq T_\infty 1(y) < 1 - \delta$ for all y I $(y_0 - \epsilon, y_0 + \epsilon)$. We can choose an f in $C_0([0,+\infty))$ satisfying that $f(y_0) = ||f|| = 1$ and f vanishes outside $(y_0 - \epsilon, y_0 + \epsilon) \subset (2,+\infty)$. Now,

$$T_\infty(1+\delta f)(y) = T_\infty(1)(y) + \delta T_\infty(f)(y)$$

$$= T_\infty(1)(y) + \delta T(f)(y)$$

$$= h(y) + g(y) + \delta h(y) f(\varphi(y))$$

$$= \begin{cases} 1 + g(y) + \delta f(y), y > 2 \\ T_\infty 1(y), -2 \leq y \leq 2 \\ -1 + g(y) - \delta f(-y), y < -2 \end{cases}$$

Since $||T_\infty(1+\delta f|| = \left||1+\delta f|\right| = 1 + \delta$ and$|T_\infty(1+\delta f)(y)| \leq 1$ · unless $-y \in (y_0 - \epsilon, y_0 + \epsilon)$, there is a y_1 in $(y_0 - \epsilon, y_0 + \epsilon)$ such tha $|-1 + g(-y_1) - \delta f(y_1)| = 1 + \delta$. It forces that $g(-y_1) = 0$. Since ϵ can be arbitrary small, we have $g(-y_0) = 0$ and our

Claim (2.3.7) that $g(y)g(-y) = 0$ whenever $|y| > 0$ has thus been verified. As$T_\infty 1$ is continuous on $(-\infty, +\infty) \cup \{\infty\}$, we must have

$$\lim_{y\to+\infty} T_\infty 1(y) = \lim_{y\to-\infty} T_\infty 1(y)$$

that is ,

$$\lim_{y\to+\infty} -1 + g(y) = \lim_{y\to-\infty} \ 1 + g(y)$$

Let L be their common (finite) limit. Then

$$\lim_{y\to+\infty} g(y) = L + 1 , \quad \lim_{y\to-\infty} g(y) = L - 1$$

Consequently,

$$0 = \lim_{y\to+\infty} g(y)g(-y) = L^2 - 1$$

It follows that $L = \pm 1$, and thus either $\lim_{y\to+\infty} g(y) = 2$ or $\lim_{y\to-\infty} g(y) = -2$.

Both of them contradicts the fact that $|g(y)| \leq 1, \forall y \in (-\infty, +\infty)$.

On the other hand, suppose T_∞ were disjointness preserving. Since $f_n(1 - f_{2n}) = 0$, we have $T_\infty f_n \,.\, T_\infty(1 - f_{2n}) = 0$. That is,

$$T_\infty f_n(y). T_\infty(1 - f_{2n})(y) = 0 \quad , \forall y \in (-\infty, +\infty) \cup \{\infty\}$$

When $|y| < n$ and $y \neq 1$, $T_\infty f_n$(y)= $T f_n(y) = h(y) \neq 0$ and hence $T_\infty(1)(y) = T_\infty(f_{2n})(y) = T(f_{2n})(y) = h(y)$. Since $T_\infty 1$ is continuous on $(-\infty, +\infty) \cup \{\infty\}$, we must have

$$+1 = \lim_{y\to+\infty} h(y) = \lim_{y\to-\infty} h(y) = -1$$

a Contradiction again.

Chapter 3
Properties of Approximation Numbers with Compact and Weakly Compact Composition Operators

We connect with Hardy–Orlicz and Bergman–Orlicz spaces H^{ψ} and B^{ψ}, and provide a negative answer to the question of knowing if all composition operators which are weakly compact on a non-reflexive space are norm-compact. The approximation numbers of a compact composition operator cannot decay more rapidly than exponentially, and this speed of convergence is only obtained for symbols which do not approach the unit circle. We also give an upper bound and explicit an example. As a crucial step of the argument we simplify the compactness criterion due to Smith for C_{ϕ} on BMOA and show that his condition on the Nevanlinna counting function alone characterizes compactness. Additional equivalent compactness criteria are established. We show the unexpected result that compactness of C_{ϕ} on VMOA implies compactness even from the Bloch space into VMOA.

Section (3.1): Composition Operators Associated with Lens Maps

As a continuation of [72], [73], [74], [77] and [78].

For $\mathbb{D}$ be the open unit disk of the complex plane and $\mathcal{H}(\mathbb{D})$ be the space of holomorphic functions on $\mathbb{D}$. To every analytic self-map $\varphi\colon \mathbb{D} \to \mathbb{D}$ (also called Schur function), a linear map $C_{\varphi} : \mathcal{H}(\mathbb{D}) \to \mathcal{H}(\mathbb{D})$ can be associated by $C_{\varphi}(f) = f \circ \varphi$. This map is called the composition operator of symbol φ. A basic fact of the theory ([83], page 13, or [67], Theorem 1.7) is Littlewood's subordination principle which allows one to show that every composition operator induces a bounded linear map from the Hardy space H^p into itself, $1 \le p < \infty$.

We are specifically interested in a one-parameter family (a semigroup) of Schur functions: lens maps $\varphi_{\theta}, 0 < \theta < 1$, whose definition is given below. They turn out to be very useful in the general theory of composition operators because they provide non-trivial examples (for example, they generate compact and even Hilbert–Schmidt operators on the Hardy space H^2 [83], page 27). We illustrate that fact by new examples.

We show that, as operators on H^2, the approximation numbers of $C_{\varphi_{\theta}}$ behave as $e^{-c_{\theta}\sqrt{n}}$. In particular, the composition operator $C_{\varphi_{\theta}}$ is in all Schatten classes $S_p, p > 0$. We show that, when one "spreads" these lens maps, their approximation numbers become greater, and the associated composition operator $C_{\tilde{\varphi}_{\theta}}$ is in S_p if and only if $p > 2\theta$. We answer in the negative a question of H.-O. Tylli: is it true that every weakly compact composition operator on a non-reflexive Banach function space is actually compact? We show that there are composition operators on a (non-reflexive) Hardy–Orlicz space, which are weakly compact and Dunford–Pettis, though not compact, and that there are composition operators on a non-reflexive Bergman– Orlicz space which are weakly compact but not compact. We also show that there are composition operators on a non-reflexive Hardy–Orlicz space which are weakly compact but not Dunford–Pettis.

We give now the definition of lens maps (see [83], page 27).

Definition (3.1.1)[63]: (Lens maps): The lens map $\varphi_{\theta} : \mathbb{D} \to \mathbb{D}$ with parameter $\theta, 0 < \theta < 1$, is defined by:

$$\varphi_{\theta}(z) = \frac{(1+z)^{\theta} - (1-z)^{\theta}}{(1+z)^{\theta} + (1-z)^{\theta}}, z \in \mathbb{D}. \qquad (1)$$

In a more explicit way, φ_θ is defined as follows. Let $\mathbb{H}$ be the open right half-plane, and $T : \mathbb{D} \to \mathbb{H}$ be the (involutive) conformal mapping given by

$$T(z) = \frac{1 - z}{1 + z}. \tag{2}$$

We denote by γ_θ the self-map of $\mathbb{H}$ defined by

$$\gamma_\theta(w) = w^\theta = e^{\theta \log w}, \tag{3}$$

where log is the principal value of the logarithm and finally $\varphi_\theta : \mathbb{D} \to \mathbb{D}$ is defined by

$$\varphi_\theta = T^{-1} \circ \gamma_\theta \circ T. \tag{4}$$

Those lens maps form a continuous curve of analytic self-maps from $\mathbb{D}$ into itself, and an abelian semi-group for the composition of maps since we obviously have from (4) and the rules on powers that $\varphi_\theta(0) = 0$ and

$$\varphi_\theta \circ \varphi_{\theta'} = \varphi_{\theta'} \circ \varphi_\theta = \varphi_{\theta\theta'}. \tag{5}$$

For every operator $A: H^2 \to H^2$, we denote by

$$a_n(A) = \inf_{rank\, R<n} \|A - R\|, \qquad n = 1, 2, \ldots$$

its n-th approximation number. See [65] for more details on those approximation numbers.

Recall ([86], page 18) that the Schatten class S_p on H^2 is defined by

$$S_p = \{A: H^2 \to H^2 ;\ \left(a_n(A)\right)_n \in \ell^p\}, p > 0;$$

S_2 is the Hilbert–Schmidt class and the quantity $\|A\|_p = \left(\sum_{n=1}^{\infty} \left(a_n(A)\right)^p\right)^{1/p}$ is a Banach norm on S_p for $p \geq 1$.

We can now state the following theorem:

The lower bound in (9) was proved in [78]. The fact that C_{φ_θ} lies in all Schatten classes was first proved in [84] under a qualitative form.

The upper bound will be obtained below as a consequence of a result of O. G. Parfenov ([80]). However, an idea of infinite divisibility, which may be used in other contexts, leads to a simpler proof, though it gives a worse estimate in (9): $\sqrt{n}$ is replaced by $n^{1/3}$. We shall begin by giving this proof, because it is quite short. It relies on the semi-group property (5) and on an estimate of the Hilbert–Schmidt norm $\|C_{\varphi_\alpha}\|_2$ in terms of α, as follows:

Lemma (3.1.2)[63]: There exist numerical constants K_1, K_2 such that

$$\frac{K_1}{1 - \alpha} \leq \|C_{\varphi_\alpha}\|_2 \leq \frac{K_2}{1 - \alpha}, \qquad \text{for all } 0 < \alpha < 1. \tag{6}$$

In particular, we have

$$a_n(C_{\varphi_\alpha}) \leq \frac{K_2}{\sqrt{n}(1 - \alpha)}. \tag{7}$$

Proof. The relation (7) is an obvious consequence of (6) since

$$n\left[a_n(C_{\varphi_\alpha})\right]^2 \leq \sum_{j=1}^{n} \left[a_j(C_{\varphi_\alpha})\right]^2 \leq \sum_{j=1}^{\infty} \left[a_j(C_{\varphi_\alpha})\right]^2 = \|C_{\varphi_\alpha}\|_2^2 \leq \frac{K_2^2}{(1 - \alpha)^2}.$$

For the first part, let $a = \cos(\alpha\pi/2) = \sin((1 - \alpha)\pi/2) \geq 1 - \alpha$ and let $\sigma = T\,(m)$ (m is the normalized Lebesgue measure $dm(t) = dt/2\pi$ on the unit circle) be the probability measure carried by the imaginary axis which satisfies

$$\int_{\mathbb{H}} f\, d\sigma = \int_{-\infty}^{\infty} f(iy) \frac{dy}{\pi(1 + y^2)}.$$

By definition, defined in (2), is a unitary operator from $H^2(\mathbb{D},m)$ into $H^2(\mathbb{H},\sigma)$, and we easily obtain, setting $\gamma(y) = \gamma_\alpha(iy) = e^{i(\pi/2)\alpha\, sign(y)}\,|y|^\alpha$ (where sign is the sign of y and γ_α is defined in (3)), that (see [83])

$$\|C_{\varphi_\alpha}\|_2^2 = \int_{\mathbb{T}} \frac{dm}{1 - |\varphi_\alpha|^2} = \int_{\mathbb{H}} \frac{d\sigma}{1 - \frac{|1-\gamma|}{|1+\gamma|^2}} = \int_{\mathbb{H}} \frac{|1 + \gamma|^2}{4\,\mathcal{R}e\,\gamma}\, d\sigma$$

$$= \int_{-\infty}^{+\infty} \frac{|1 + \gamma(y)|^2}{4a|y|^\alpha} \frac{dy}{\pi(1 + y^2)}$$

$$\leq \frac{K}{1 - \alpha} \int_0^{+\infty} \frac{1 + y^{2\alpha}}{y^\alpha} \frac{dy}{1 + y^2} = \frac{2K}{1 - \alpha} \int_0^{+\infty} \frac{y^\alpha}{1 + y^2} dy$$

$$\leq \frac{4K}{(1 - \alpha)^2},$$

where K is a numerical constant. This gives the upper bound in (6) and the lower one is obtained similarly.

We can now finish the first proof of Theorem (3.1.3). Let k be a positive integer and let

$$\alpha_k = \theta^{1/k},$$

so that $\alpha_k^k = \theta$.

Now use the well-known sub-multiplicativity $a_{p+q-1}(vu) \leq a_p(v)a_q(u)$ of approximation numbers ([81], page 61), as well as the semi-group property (5) (which implies $C_{\varphi_\theta} = C^k_{\varphi_{\alpha_k}}$), and (7). We see that

$$a_{kn}(C_{\varphi_\theta}) = a_{kn}\left(C^k_{\varphi_{\alpha_k}}\right) \leq \left[a_n\left(C_{\varphi_{\alpha_k}}\right)\right]^k \leq \left[\frac{K_2}{1 - \alpha_k\sqrt{n}}\right]^k.$$

Observe that

$$1 - \alpha_k \geq \frac{1 - \alpha_k^k}{k} = \frac{1 - \theta}{k}.$$

We then get, $c = c_\theta$ denoting a constant which only depends on θ:

$$a_{kn}(C_{\varphi_\theta}) \leq \left(\frac{k}{c\sqrt{n}}\right)^k.$$

Set $d = c/e$ and take $k = d\sqrt{n}$, ignoring the questions of integer part. We obtain

$$a_{dn^{3/2}}(C_{\varphi_\theta}) \leq e^{-k} = e^{-d\sqrt{n}}.$$

Setting $N = dn^{3/2}$, we get

$$a_N\left(C_{\varphi_\theta}\right) \leq a\, e^{-bN^{1/3}} \tag{8}$$

for an appropriate value of a and b and for any integer $N \geq 1$. This ends our first proof, with an exponent slightly smaller that the right one ($1/3$ instead of $1/2$), yet more than sufficient to prove that $C_{\varphi_\theta} \in \bigcap_{p>0} S_p$.

Theorem (3.1.3)[63]: Let $0 < \theta < 1$ and φ_θ be the lens map defined in (1). There are positive constants a, b, a', b' depending only on θ such that

$$a' e^{-b'\sqrt{n}} \leq a_n\left(C_{\varphi_\theta}\right) \leq a\, e^{-b\sqrt{n}}. \tag{9}$$

In particular, C_{φ_θ} lies in all Schatten classes $S_p, p > 0$.

Proof. This proof will give the correct exponent $1/2$ in the upper bound. Moreover, it works more generally for Schur functions whose image lies in polygons inscribed in the unit disk. This upper bound appears, in a different context and under a very cryptic form, in [80]. First note the following simple lemma.

Lemma (3.1.4)[63]: Suppose that $a, b \in \mathbb{D}$ satisfy $|a - b| \leq M \min(1 - |a|, 1 - |b|)$, where M is a constant. Then

$$d(a,b) \leq \frac{M}{\sqrt{M^2} + 1} := \chi < 1.$$

Here d is the pseudo-hyperbolic distance defined by

$$d(a,b) = \left|\frac{a - b}{1 - \bar{a}b}\right|, \quad a, b \in \mathbb{D}.$$

Proof. Set $\delta = \min(1 - |a|, 1 - |b|)$. We have the identity

$$\frac{1}{d^2(a,b)} - 1 = \frac{(1-|a|^2)(1-|b|^2)}{|a-b|^2} \geq \frac{(1-|a|)(1-|b|)}{|a-b|^2} \geq \frac{\delta^2}{M^2\delta^2} = \frac{1}{M^2},$$

hence the lemma.

The second lemma gives an upper bound for $a_N(C_\varphi)$. In this lemma, κ is a numerical constant, $S(\xi, h)$ the usual pseudo-Carleson window centred at $\xi \in \mathbb{T}$ (where $\mathbb{T} = \partial\mathbb{D}$ is the unit circle) and of radius h ($0 < h < 1$), defined by

$$S(\xi, h) = \{z \in \mathbb{D} \,;\, |z - \xi| \leq h\}, \tag{10}$$

and m_φ is the pull-back measure of m, the normalized Lebesgue measure on $\mathbb{T}$, by φ^*. Recall that if $f \in \mathcal{H}(\mathbb{D})$, one sets $f_r(e^{it}) = f(re^{it})$ for $0 < r < 1$ and, if the limit exists m-almost everywhere, one sets

$$f^*(e^{it}) = \lim_{r \to 1-} f(re^{it}). \tag{11}$$

Actually, we shall do write f instead of f^*. Recall that a measure μ on $\overline{\mathbb{D}}$ is called a Carleson measure if there is a constant $c > 0$ such that $\mu[\overline{S(\xi,h)}] \leq c\, h$ for all $\xi \in \mathbb{T}$. Carleson's embedding theorem says that μ is a Carleson measure if and only if the inclusion map from H^2 into $L^2(\mu)$ is bounded (see [67], Theorem 9.3, for example).

Lemma (3.1.5)[63]: Let B be a Blaschke product with less than N zeroes (each zero being counted with its multiplicity). Then, for every Schur function φ, one has

$$a_N^2 := \left[a_N\left(C_\varphi\right)\right]^2 \leq \kappa^2 \sup_{0<h<1, \xi \in \mathbb{T}} \frac{1}{h} \int_{\overline{S(\xi,h)}} |B|^2 \, dm_\varphi, \tag{12}$$

for some universal constant $\kappa > 0$.

Proof. The subspace BH^2 is of codimension $\leq N - 1$. Therefore, $a_N = c_N\left(C_\varphi\right) \leq \left\|C_\varphi|_{BH^2}\right\|$, where the c_N 's are the Gelfand numbers (see [65]), and where we used the equality $a_N = c_N$ occurring in the Hilbertian case (see [65]). Now, since $\|Bf\|_{H^2} = \|f\|_{H^2}$ for any $f \in H^2$, we have

$$\left\|C_\varphi|_{BH^2}\right\|^2 = \sup_{\|f\|_{H^2} \leq 1} \int_{\mathbb{T}} |B \circ \varphi|^2 \, |f \circ \varphi|^2 \, dm = \sup_{\|f\|_{H^2} \leq 1} \int_{\overline{\mathbb{D}}} |B|^2 |f|^2 dm_\varphi$$

$$= \left\|R_\mu\right\|^2,$$

where $\mu = |B|^2 m_\varphi$ and where $R_\mu : H^2 \to L^2(\mu)$ is the restriction map. Of course, μ is a Carleson measure for H^2 since $\mu \leq m_\varphi$. Now, Carleson's embedding theorem tells us that

$$\left\|R_\mu\right\|^2 \leq \kappa^2 \sup_{0<h<1, \xi \in \mathbb{T}} \frac{\mu[\overline{S(\xi,h)}]}{h}$$

(see [67], Remark after the proof of Theorem 9.3, at the top of page 163; actually, in that book, Carleson's windows $W(\xi,h)$ are used instead of pseudo-Carleson's windows $S(\xi,h)$, but that does not matter, since $W(\xi,h) \subseteq S(\xi,2h)$: if $r \geq 1-h$ and $|t-t_0| \leq h$, then $|re^{it}-e^{it_0}| \leq |re^{it}-e^{it}|+|e^{it}-e^{it_0}| \leq 2h$). That ends the proof of Lemma (3.1.5).

The following lemma takes into account the behaviour of $\varphi_\theta(e^{it})$, and will be useful. The notation $u(t) \approx v(t)$ means that $a\,u(t) \leq v(t) \leq b\,u(t)$, for some positive constants a, b.

Lemma (3.1.6)[63]: Set $\gamma(t) = \varphi_\theta(e^{it}) = |\gamma(t)|e^{iA(t)}$, with $-\pi \leq t \leq \pi$, and $-\pi \leq A(t) \leq \pi$. Then, for $0 \leq |t|, |t'| \leq \pi/2$, one has

$|1-\gamma(t)| \approx 1-|\gamma(t)| \approx |t|^\theta$ and $|\gamma(t)-\gamma(t')| \leq K\,|t-t'|^\theta$. (13)

Moreover, we have for $|t| \leq \pi/2$

$$A(t) \approx |t|^\theta \quad \text{and} \quad A'(t) \approx |t|^{\theta-1}. \tag{14}$$

Proof. First, recall that

$$\varphi_\theta(z) = \frac{(1+z)^\theta-(1-z)^\theta}{(1+z)^\theta+(1-z)^\theta},$$

so that $\varphi_\theta(\bar{z}) = \overline{\varphi_\theta(z)}$ and $\varphi_\theta(-z) = -\varphi_\theta(z)$. It follows that $\gamma(-t) = \overline{\gamma(t)}$ and $\gamma(t+\pi) = -\gamma(t)$, so that we may assume $0 \leq t, t' \leq \pi/2$. Then, we have more precisely, setting $c = e^{-i\theta\pi/2}, s = sin(\theta\pi/2)$ and $\tau = \left(\tan(t/2)\right)^\theta$,

$$\gamma(t) = \frac{(\cos t/2)^\theta - e^{-i\theta\pi 2}(\sin t/2)^\theta}{(\cos t/2)^\theta + e^{-i\theta\pi 2}(\sin t/2)^\theta} = \frac{1-c\tau}{1+c\tau} = \frac{1-\tau^2}{|1+c\tau|^2}+\frac{2is\tau}{|1+c\tau|^2},$$

after a simple computation, since $\left(1+e^{it}\right)^\theta = e^{it\theta/2}(2\,cos t/2)^\theta$ and $\left(1-e^{it}\right)^\theta = e^{-i\theta\pi/2}\,e^{it\theta/2}(2\sin t/2)^\theta$. Note by the way that

$$\varphi_\theta(1) = 1\,;\ \varphi_\theta(i) = i\tan(\theta\pi/4)\,;\ \varphi_\theta(-1) = -1\,;\ \varphi_\theta(-i) = -i\tan(\theta\pi/4).$$

Now, observe that $2 \geq |1+c\tau| \geq \Re e\,(1+c\tau) \geq 1$ and therefore that

$$|1-\gamma(t)| = \left|\frac{2c\tau}{1+c\tau}\right| \approx \tau \approx t^\theta,$$

and similarly for $1-|\gamma(t)|$ since

$$1-|\gamma(t)|^2 = \frac{4(\Re e\,c)\tau}{|1+c\tau|^2}.$$

The relation (13) clearly follows. To prove (14), we just have to note that, for $0 \leq t \leq \pi/2$, we have $A(t) = \arctan\frac{2s\tau}{1-\tau^2}$.

Now, we prove Theorem (3.1.3) in the following form (in which $q = q_\theta$ denotes a positive constant smaller than one), which is clearly sufficient:

$$a_{4N^2+1} \leq Kq^N. \tag{15}$$

The proof will come from an adequate choice of a Blaschke product of length $4N^2$, with zeroes on the curve $\gamma(t) = \varphi_\theta(e^{it}), -\pi \leq t \leq \pi$. Let $t_k = \pi 2^{-k}$ and $p_k = \gamma(t_k)$, with $1 \leq k \leq N$, so that the points p_k are all in the first quadrant. We reflect them through the coordinate axes, setting

$$q_k = \overline{p_k}, \quad r_k = -p_k, \quad s_k = -q_k, \qquad 1 \leq k \leq N.$$

Let now B be the Blaschke product having a zero of order N at each of the points p_k, q_k, r_k, s_k, namely

$$B(z) = \prod_{k=1}^{N} \left[\frac{z-p_k}{1-\overline{p_k}z} \cdot \frac{z-q_k}{1-\overline{q_k}z} \cdot \frac{z-r_k}{1-\overline{r_k}z} \cdot \frac{z-s_k}{1-\overline{s_k}z}\right]^N.$$

This Blaschke product satisfies, by construction, the symmetry relations

$$B(\bar{z}) = \overline{B(z)}, \quad B(-z) = B(z). \tag{16}$$

Of course, $|B| = 1$ on the boundary of $\mathbb{D}$, but $|B|$ is small on a large portion of the curve γ, as expressed by the following lemma.

Lemma (3.1.7)[63]: For some constant $\chi = \chi_\theta < 1$, the following estimate holds:

$$t_N \le t \le t_1 \Rightarrow \left|B(\gamma(t))\right| \le \chi^N. \tag{17}$$

Proof. Let $t_N \le t \le t_1$ and k be such that $t_{k+1} \le t \le t_k$. Let

$$B_k(z) = \frac{z-p_k}{1-\overline{p_k}z}.$$

Then, with the help of Lemma (3.1.6), we see that the assumptions of Lemma (3.1.4) are satisfied with $a = \gamma(t)$ and $b = \gamma(t_k)$, since $|t-t_k| \le t_k - t_{k+1} = \pi 2^{-k-1}$, so that $\min(1-|a|, 1-|b|) \approx t_k^\theta \approx 2^{-k\theta}$ and hence, for some constant M,

$$|a-b| \le K\,|t-t_k|^\theta \le K 2^{-k\theta} \le M \min(1-|a|, 1-|b|).$$

We therefore have, by definition, and by Lemma (3.1.4), where we set $\chi = M/\sqrt{M^2+1}$,

$$|B_k(\gamma(t))| = d(\gamma(t), p_k) \le \chi < 1.$$

It then follows from the definition of B that

$$|B(\gamma(t))| \le \left|B_k(\gamma(t))\right|^N \le \chi^N,$$

and that ends the proof of Lemma (3.1.7).

Now fix $\xi \in \mathbb{T}$ and $0 < h \le 1$. By interpolation, we may assume that $h = 2^{-n\theta}$. By symmetry, we may assume that $\Re e\,\xi \ge 0$ and $\Re e\,\gamma(t) \ge 0$, i.e., $|t| \le \pi/2$. Then, since $\varphi_\theta(\mathbb{D})$ is contained in the symmetric angular sector of vertex 1 and opening $\theta\pi < \pi$, there is a constant $K > 0$ such that $|1-\gamma(t)| \le K(1-|\gamma(t)|)$. The only pseudo-windows $S(\xi, h)$ giving an integral not equal to zero in the estimation (12) of Lemma (3.1.5) satisfy $|\xi - 1| \le (K+1)h$. Indeed, suppose that $|\gamma(t)-\xi| \le h$. Then $1-|\gamma(t)| \le |\gamma(t)-\xi| \le h$ and $|1-\gamma(t)| \le K(1-|\gamma(t)|) \le Kh.$ If $|\xi-1| > (K+1)h$, we should have $|\gamma(t)-\xi| \ge |\xi-1| - |\gamma(t)-1| > (K+1)h - Kh = h$, which is impossible. Now, for such a window, we have by definition of m_φ

$$\int_{S(\xi,h)} |B|^2\, dm_{\varphi_\theta} = \int_{|\gamma(t)-\xi|\le h} \left|B(\gamma(t))\right|^2 \frac{dt}{2\pi} \le \int_{|\gamma(t)-1|\le (K+2)h} \left|B(\gamma(t))\right|^2 \frac{dt}{2\pi}$$

$$\le \int_{|t|\le Dt_n} \left|B(\gamma(t))\right|^2 \frac{dt}{2\pi} \stackrel{\text{def}}{=} I_h,$$

since $|\gamma(t)-1| \le |\gamma(t)-\xi| + |\xi-1| \le h + (K+1)h$ and since $|\gamma(t)-1| \ge a|t|^\theta$ and $|\gamma(t)-1| \le (K+2)h$ together imply $|t| \le Dt_n$, where $D > 1$ is another constant (recall that $h = 2^{-n\theta} = (t_n/\pi)^\theta$).

To finish the discussion, we separate two cases.

(a) If $n \ge N$, we simply majorize $|B|$ by 1. We set $q_1 = 2^{\theta-1} < 1$ and get

$$\frac{1}{h} I_h \le \frac{1}{h}\int_{-Dt_n}^{Dt_n} \left|B(\gamma(t))\right|^2 \frac{dt}{2\pi} \le \frac{2Dt_n}{2\pi h} = Dq_1^n \le D\, q_1^N.$$

(b) If $n \le N-1$, we write

$$\frac{1}{h} I_h = \frac{2}{h}\int_0^{Dt_N} \left|B(\gamma(t))\right|^2 \frac{dt}{2\pi} + \frac{2}{h}\int_{Dt_N}^{Dt_n} \left|B(\gamma(t))\right|^2 \frac{dt}{2\pi} := J_N + K_N.$$

The term J_N is estimated above: $J_N \leq D\, q_1^N$. The term K_N is estimated through Lemma (3.1.7), which gives us

$$K_N \leq 2^{n\theta}\,\frac{2Dt_n}{2\pi}\,\chi^{2N} \leq D\,\chi^{2N},$$

since $t_n 2^{n\theta} \leq \pi$, due to the fact that $\theta < 1$.

If we now apply Lemma (3.1.5) with $q = \max(q_1, \chi^2)$ and with N changed into $4N^2 + 1$, we obtain (15), by changing the value of the constant K once more. This ends the proof of Theorem (3.1.3).

Theorem (3.1.3) has the following consequence (as in [83], page 29).

Proposition (3.1.8)[63]: Let φ be a univalent Schur function and assume that $\varphi(\mathbb{D})$ contains an angular sector centred on the unit circle and with opening $\theta\pi, 0 < \theta < 1$. Then $a_n(C_\varphi) \geq a\, e^{-b\sqrt{n}}, n = 1, 2, \ldots$, for some positive constants a and b, depending only on θ.

Proof. We may assume that this angular sector is centred at 1. By hypothesis, $\varphi(\mathbb{D})$ contains the image of the "reduced" lens map defined by $\tilde{\varphi}_\theta(z) = \varphi_\theta((1 + z)/2)$. Since φ is univalent, there is a Schur function u such that $\tilde{\varphi}_\theta = \varphi \circ u$. Hence $C_{\tilde{\varphi}_\theta} = C_u \circ C_\varphi$ and $a_n(C_{\tilde{\varphi}_\theta}) \leq \|C_u\| a_n(C_\varphi)$. Theorem (3.1.3) gives the result, since the calculations for $\tilde{\varphi}_\theta$ are exactly the same as for φ_θ (because they are equivalent as z tends to 1).

The same is true if φ is univalent and $\varphi(\mathbb{D})$ contains a polygon with vertices on $\partial\mathbb{D}$.

In [72], we studied the effect of the multiplication of a Schur function φ by the singular inner function $(z) = e^{\frac{-1+z}{1-z}}$, and observed that this multiplication spreads the values of the radial limits of the symbol and lessens the maximal occupation time for Carleson windows. In some cases this improves the compactness or membership to Schatten classes of C_φ. We proved the following result.

Theorem (3.1.9)[63]: ([72], Theorem (3.1.14)): For every $p > 2$, there exist two Schur functions φ_1 and $\varphi_2 = \varphi_1 M$ such that $|\varphi_1^*| = |\varphi_2^*|$ and $C_{\varphi 1} : H^2 \to H^2$ is not compact, but $C_{\varphi 2} : H^2 \to H^2$ is in the Schatten class S_p.

We will meet the opposite phenomenon: the symbol φ_1 will have a fairly big associated maximal function ρ_{φ_1} , but will belong to all Schatten classes since it "visits" a bounded number of windows (meaning that there exists an integer J such that, for fixed n, at most J of the $W_{n,j}$ are visited by $\varphi^*(e^{it})$). The spread symbol will have an improved maximal function, but will visit all windows, so that its membership in Schatten classes will be degraded. We will prove that

Theorem (3.1.10)[63]: Fix $0 < \theta < 1$. Then there exist two Schur functions φ_1 and φ_2 such that:

(i) $C_{\varphi 1} : H^2 \to H^2$ is in all Schatten classes $S_p, p > 0$, and even $a_n(C_{\varphi_1}) \leq a\, e^{-b\sqrt{n}}$;
(ii) $|\varphi_1^*| = |\varphi_2^*|$;
(iii) $C_{\varphi_2} \in S_p$ if and only if $p > 2\theta$;
(iv) $a_n(C_{\varphi_2}) \leq K \left(\log\frac{n}{n}\right)^{1/2\theta}, n = 2, 3, \ldots$.

Of course, it would be better to have a good lower bound for $a_n(C_{\varphi_2})$, but we have not yet succeeded in finding it.

Proof. First observe that $C_{\varphi_1} \in S_2$, so that $C_{\varphi_2} \in S_2$ too, since $|\varphi_1^*| = |\varphi_2^*|$ and since the membership of C_φ in S_2 only depends on the modulus of φ^* because it amounts to ([83], page 26)

$$\int_{-\pi}^{\pi} \frac{dt}{1 - |\varphi^*(e^{it})|} < \infty.$$

Theorem (3.1.10) says that we can hardly have more. We first prove a lemma. Recall (see [72], for example) that the maximal Carleson function ρ_φ of a Schur function φ is defined, for $0 < h < 1$, by

$$\rho_\varphi(h) = \sup_{|\xi|=1} m_\varphi[S(\xi, h)]. \tag{18}$$

Lemma (3.1.11)[63]: Let $0 < \theta < 1$. Then, the maximal function ρ_{φ_θ} of φ_θ satisfies $\rho_{\varphi_\theta}(h) \leq K^{1/\theta}(1 - \theta)^{-1/\theta} h^{1/\theta}$ and, moreover,

$$\rho_{\varphi_\theta}(h) \approx h^{1/\theta}. \tag{19}$$

Proof. Let $0 < h < 1$ and $\gamma(t) = \varphi_\theta(e^{it})$; K and δ will denote constants which can change from a formula to another. We have, for $|t| \leq \pi/2$,

$$1 - |\gamma(t)|^2 = \frac{4(\Re e\, c)\tau}{|1 + c\tau|^2} \geq \delta \cos(\theta\pi/2) \frac{\tau}{|1 + c\tau|^2} \geq \delta(1 - \theta) \frac{\tau}{|1 + c\tau|^2}$$
$$\geq \delta(1 - \theta)|t|^\theta.$$

Hence, we get, from Lemma (3.1.6),

$$\rho_{\varphi_\theta}(h) \leq 2\, m(\{1 - |\gamma(t)| \leq h \text{ and } |t| \leq \pi/2\}) \leq 2\, m(\{(1-\theta)\delta|t|^\theta \leq Kh\})$$
$$\leq K^{1/\theta}(1 - \theta)^{-1/\theta} h^{1/\theta}.$$

Similarly, we have

$$\rho_{\varphi_\theta}(h) \geq m_{\varphi_\theta}[S(1,h)] \geq m(\{|1 - \gamma(t)| \leq h\}) \geq m(\{|t|^\theta \leq Kh\}) \geq Kh^{1/\theta},$$

and that ends the proof of the lemma.

Going back to the proof of Theorem (3.1.10), we take $\varphi_1 = \varphi_\theta$ and $\varphi_2(z) = \varphi_1(z)M(z^2)$. We use $M(z^2)$ instead of $M(z)$ in order to treat the points -1 and 1 together.

The first two assertions are clear. For the third one, we define the dyadic Carleson windows, for $n = 1, 2, \ldots, j = 0, 1, \ldots, 2^n - 1$, by

$$W_{n,j} = \{z \in \mathbb{D}\,;\ 1 - 2^{-n} \leq |z| < 1 \text{ and } (2j\pi)2^{-n} \leq \arg(z) < \big(2(j+1)\big)\pi\big)2^{-n}\}.$$

Recall (see [72], Proposition (3.1.11)) the following proposition, which is a variant of Luecking's criterion ([79]) for membership in a Schatten class, and which might also be used to give a third proof of the membership of C_{φ_θ} in all Schatten classes $S_p, p > 0$, although the first proof turns out to be more elementary.

Proposition (3.1.12)[63]: ([79], [72]): Let φ be a Schur function and $p > 0$ a positive real number. Then $C_\varphi \in S_p$ if and only if

$$\sum_{n=1}^{\infty} \sum_{j=0}^{2^n-1} \big[2^n m_\varphi\big(W_{n,j}\big)\big]^{p/2} < \infty.$$

We apply this proposition with $\varphi = \varphi_2$, which satisfies, for $0 < |t| \leq \pi/2$, the following relation:

$$\varphi(e^{it}) = |\gamma(t)| e^{i[A(t) - \cot(t)]} \overset{\text{def}}{=} |\gamma(t)| e^{iB(t)},$$

where $\gamma(t) = \varphi_1(e^{it})$ and (using Lemma (3.1.6))

$$0<|t|\leq \pi/2 \Longrightarrow B(t)=\Gamma(t)-\frac{1}{t}\text{ ,with }\Gamma(t)\approx |t|^{\theta}\text{ and }\Gamma'(t)\approx |t|^{\theta-1}. \qquad (20)$$

It clearly follows from (20) that the function B is increasing on some interval $[-\delta,0[$ where δ is a positive numerical constant. Let us fix a positive integer q0 such that $-\pi/2 \leq t < 0$ and

$$B(t) \geq 2q_0\pi \quad \Longrightarrow \quad t \geq -\delta.$$

Fix a Carleson window $W_{n,j}$ and let us analyze the set $E_{n,j}$ of those t's such that $\varphi(e^{it})$ belongs to $W_{n,j}$. Recall that $m_{\varphi}(W_{n,j})=m(E_{n,j})$. The membership in $E_{n,j}$ gives two constraints.

(a) Modulus constraint. We must have $|\gamma(t)| \geq 1-2^{-n}$, and therefore $|t|\leq K2^{-n/\theta}$.

(b) Argument constraint. Let us set $\theta_{n,j}=(2j+1)\pi 2^{-n}, h=\pi 2^{-n}$ and $I_{n,j}=(\theta_{n,j}-h,\theta_{n,j}+h)$. The angular constraint arg $\varphi(e^{it}) \in I_{n,j}$ will be satisfied if $t<0$ and

$$B(t)\in \bigcup_{q\geq q_0}\left[\theta_{n,j}-h+2q\pi,\theta_{n,j}+h+2q\pi\right]:=\bigcup_{q\geq q_0} J_q(h) := F.$$

We have $F \subset [2q_0\pi,\infty[$, and so $B(t) \in F$ and $t<0$ imply $t \geq -\delta$. Set

$$E=\bigcup_{q\geq q_0}\left[B^{-1}\left(\theta_{n,j}-h+2q\pi\right),B^{-1}\left(\theta_{n,j}+h+2q\pi\right)\right]:=\bigcup_{q\geq q_0} I_q(h)\subset[-\delta,0[.$$

The intervals I_q's are disjoint, since $\theta_{n,j}+2(q+1)\pi-h>\theta_{n,j}+2q\pi+h$ and since B increases on $[-\delta,0[$. Moreover, $t\in E$ implies that $B(t)\in F$, which in turn implies that arg $\varphi(e^{it}) \in I_{n,j}$. Using Lemma (3.1.6), we can find positive constants c_1,c_2 such that

$$q\geq q_0 \Longrightarrow -c_1/q \leq \min I_q(h) \leq \max I_q(h) \leq -c_2/q.$$

Now, by the mean-value theorem, $I_q(h)$ has length $2h/|B'(t_q)|$ for some $t_q\in I_q(h)$. But, using (20), we get

$$B(t)\approx\frac{1}{t} \quad\text{and}\quad |B'(t)|\approx\frac{1}{t^2},$$

so that $I_q(h)$ has length approximately $ht_q^2\approx h/q^2$ since $|t_q|\approx 1/q$. Because of the modulus constraint, the only involved q's are those for which $q\geq q_1$, where $q_1\approx 2^{n/\theta}$. Taking n numerically large enough, we may assume that $q_1>q_0$. We finally see that, for any $0 \leq j \leq 2^n-1$, we have the lower bound

$$m_{\varphi}(W_{n,j})=m(E_{n,j})\gtrsim\sum_{q\geq q_1} m\left(I_q(h)\right)\gtrsim\sum_{q\geq q_1}\frac{h}{q^2}\gtrsim\frac{h}{q_1}\gtrsim 2^{-n(1+1/\theta)}.$$

It follows that

$$\sum_{n=1}^{\infty}\sum_{j=0}^{2^n-1} 2^n m_{\varphi}(W_{n,j})^{\frac{p}{2}}\sum_{n=1}^{\infty}\sum_{j=0}^{2^n-1}\left[2^n 2^{-\frac{n(1+1)}{\theta}}\right]^{\frac{p}{2}}=\sum_{n=1}^{\infty}\sum_{j=0}^{2^n-1}\left[2^{-\frac{np}{2\theta}}\right]=\sum_{n=1}^{\infty}2^{n\frac{1-p}{2\theta}}$$
$$=\infty,$$

$if\ p \leq 2\theta$. Hence $C_{\varphi 2} \notin S_p\ for\ p \leq 2\theta$ by Proposition (3.1.12).

A similar upper bound, and the membership of $C_{\varphi 2}\ in\ S_p\ for\ p > 2\theta$, would easily be proved along the same lines. But this will also follow from the more precise result on approximation numbers. To that effect, we shall borrow the following result from [78]. Theorem 3.5 ([78]): Let φbe a Schur function. Then the approximation numbers of C_{φ} : $H^2 \to H^2$ have the upper bound

$$a_n(C_\varphi) \le K \inf_{0<h<1}\left[(1-h)^n + \sqrt{\frac{\rho_\varphi(h)}{h}}\right], n = 1,2,\ldots. \quad (21)$$

Applying this theorem to φ_2, which satisfies $\rho_{\varphi 2}\ (h) \le Kh^{\frac{1+1}{\theta}}$ as is clear from the preceding computations, would provide upper bounds for $m_\varphi(W_{n,j})$ of the same order as the lower bounds obtained. Then choosing $h = H\, log\, n/n,$ where H is a large constant ($H = 1/2\theta\ will\ do$) and using $1 - \mathrm{h} \le \mathrm{e}^{-\mathrm{h}}$, we get from (21)

$$a_n(C_{\varphi 2}) \le K\left[n^{-H} + \left(\frac{log\, n}{n}\right)^{\frac{1}{2\theta}}\right] \le K\left(\frac{log\, n}{n}\right)^{\frac{1}{2\theta}}.$$

This ends the proof of Theorem (3.1.10).

Recall that an operator $T : X \to Y$ between Banach spaces is said to be Dunford–Pettis (in short DP) or completely continuous, if for any sequence (x_n) which is weakly convergent to 0, the sequence $(T\,x_n)$ is norm-convergent to 0. It is called weakly compact (in short w-compact) if the image $T\,(B_X)$ of the unit ball in X is (relatively) weakly compact in Y. The identity map $i_1 : \ell_1 \to \ell_1$is DP and not w-compact, by the Schur property of ℓ_1 and its non-reflexivity. $If\ 1 < p < \infty$, the identity map $i_p : \ell_p \to \ell_p$ is w-compact and not DP by the reflexivity of ℓ_p and the fact that the canonical basis (e_n) of ℓ_p converges weakly to 0, whereas $\|e_n\|_p = 1$. Therefore, the two notions, clearly weaker than that of compactness, are not comparable in general. When X is reflexive, any operator $T : X \to Y$ is w-compact and any Dunford–Pettis operator $T : X \to Y$ is compact.

Yet, composition operators $T = C_\varphi : X \to X,$ with X a non-reflexive Banach space of analytic functions, several results say that weak compactness of $C_\varphi$implies its compactness. Let us quote some examples

- $X = H^1$; this was proved by D. Sarason in 1990 ([82]);

$-\,X = H^\infty$ and the disk algebra $X = A(\mathbb{D})$ ($A.$ Ulger [85] and R. Aron, ¨ P. Galindo and M. Lindstr¨om [64], independently; the first-named of us also gave another proof in [70]);

- X is the little Bloch space $\mathcal{B}_0$ ($K.$ Madigan and A. Matheson [2]);

- X is the Hardy–Orlicz space $X = H^\psi$, when the Orlicz function ψ grows more rapidly than power functions, namely when it satisfies the condition Δ^0 ([74], Theorem (3.1.14), page 55);

$-\,X = BMOA$ and $X = V\,MOA$ (J. Laitila, P. J. Nieminen, E. Saksman and $H.-O.$ Tylli [69]).

Moreover, in some cases, C_φ is compact whenever it is Dunford–Pettis ([70] for $X = H^\infty$ and [74], Theorem (3.1.14), page 55, $for\ X = H^\psi$, when the conjugate function of ψ satisfies the condition Δ_2). The question naturally arises whether for any non-reflexive Banach space X of analytic functions on $\mathbb{D}$, every weakly compact (resp. Dunford–Pettis) composition operator $C_\varphi : X \to X$ is actually compact. The forthcoming theorems show that the answer is negative in general. Our spaces X will be Hardy–Orlicz and Bergman–Orlicz spaces, so we first recall some definitions and facts about Orlicz spaces ([74]). An Orlicz function is a nondecreasing convex function $\psi: \mathbb{R}^+ \to \mathbb{R}^+$ such that and $\psi(\infty) = \infty$. Such a function is automatically continuous on $\mathbb{R}^+$. If $\psi(x)$ is not equivalent to $\psi(0) = 0$an affine function, we must have $\psi(x)/x \longrightarrow x \to \infty\ \infty$.

The Orlicz function ψ is said to satisfy the Δ_2-condition if $\psi(2x)/\psi(x)$ remains bounded. The conjugate function $\psi^{\sim}$ of an Orlicz function ψ is the Orlicz function defined by

$$\psi^{\sim}(x) = \sup_{y\geq 0} (xy - \psi(y)) .$$

For the conjugate function, one has the following characterization of Δ_2 (*see* [74], page 7): $\psi^{\sim}$ has Δ_2 if and only if, for some $\beta > 1$ and $x_0 > 0$,

$$\psi(\beta x) \geq 2\beta\psi(x), for\ all\ x \geq x_0. \tag{22}$$

Let $(\Omega, A, \mathbb{p})$be a probability space, and L^0 the space of measurable functions $f : \Omega \to \mathbb{c}$. The Orlicz space $L^{\psi} = L^{\psi}(\Omega, A, \mathbb{p})$ is defined by

$$L^{\psi}(\Omega, A, \mathbb{p}) = \left\{ f \in L^0 ; \int_{\Omega} \psi(|f|/K)\, d\,\mathbb{p} < \infty\ for\ some\ K > 0 \right\}$$

This is a Banach space for the Luxemburg norm:

$$\|f\|L^{\psi} = inf \left\{ K > 0 ; \int_{\Omega} \psi(|f|/K)\, d\,\mathbb{p} \leq 1 \right\}.$$

The Morse– Transue space M^{ψ} (*see* [74], *page* 9) is the subspace of functions $f\ in\ L^{\psi}$ for which $\int_{\Omega} \psi(|f|/K)\, d\,\mathbb{p} < \infty\ for\ every\ K > 0$. It is the closure of L^{∞}. One always has $\left(M^{\psi}\right)^* = L^{\tilde{\psi}}\ and\ L^{\psi} = M^{\psi}$ if and only if $\psi\ has\ \Delta_2$. When the conjugate function $\tilde{\psi}$ of ψ has Δ_2, the bidual of M^{ψ} is then (isometrically isomorphic to) L^{ψ}.

Now, we can define the Hardy–*Orlicz space* H^{ψ} attached to ψ as follows. Take the probability space $(\mathbb{T}, B, m)$ and, recalling that $f_r(e^{it}) = f(re^{it})$,

$$H^{\psi} = \left\{ f \in H(\mathbb{D}); \sup_{0<r<1} \| f_T\|_{L_{\psi}}(m) := \| f\|_{H^{\psi}} < \infty \right\}.$$

See [74] for more information on H^{ψ}. Similarly, we define (see [74]) the Bergman–Orlicz space B^{ψ}, using this time the normalized area measure A, by $B^{\psi} = \{f \in \mathcal{H}(\mathbb{D}) ; \|f\|_{B^{\psi}} := \|f\|_{L^{\psi}(A)} < +\infty\}$. If $\psi(x) = x^p, p \geq 1$, we get the usual Hardy and Bergman spaces H^p and B^p. Those spaces are Banach spaces for any ψ, and Hilbert spaces for $\psi(x) = x^2$. The Hardy–Morse–Transue space HM^{ψ} and Bergman–Morse–Transue space BM^{ψ} are defined by $HM^{\psi} = H^{\psi} \cap M^{\psi}$ and $BM^{\psi} = B^{\psi} \cap M^{\psi}$. When the conjugate function of ψ has Δ_2, the bidual of HM^{ψ} is (isometrically isomorphic to) H^{ψ} ([74], page 10).

We can now state the following.

Theorem (3.1.13)[63]: There exists a Schur function φ and an Orlicz function ψ such that H^{ψ} is not reflexive and the composition operator $C_{\varphi} : H^{\psi} \to H^{\psi}$ is weaklycompact and Dunford–Pettis, but is not compact.

Proof. First take for φ the lens map $\varphi_{\frac{1}{2}}$ which in view of (19) of Lemma (3.1.11) satisfies, for some constant $K > 1$,

$$\rho_{\varphi}(h) \geq K^{-1}h^2, 0 < h < 1. \tag{23}$$

We now recall the construction of an Orlicz function made in [76]. Let (x_n) be the sequence of positive numbers defined as follows: $x_1 = 4$ and then, for every integer $n \geq 1, x_n + 1 > 2x_n$ is the abscissa of the second intersection point of the parabola $y = x^2$ with the straight line containing (x_n, x_n^2) and $(2x_n, x_n^4)$; equivalently, $x_{n+1} = x_n^3 - 2x_n$. We now define our Orlicz function ψ by $\psi(x) = 4x\ for\ 0 \leq x \leq 4\ and, for\ n \geq 1, by, \psi(x_n) = x_n^2$,

$$\psi \text{ affine between } x_n \text{ and } x_{n+1}, \text{so that } \psi(2x_n) = x_n^4. \qquad (24)$$

Observe that ψ does not satisfy the Δ_{2-} condition, since $\psi(2x_n) = [\psi(x_n)]^2$. It clearly satisfies (since ψ−1 is concave)

$$x^2 \le \psi(x) \le x^4 \text{ for } x \ge 4, \psi^{-1}(Kx) \le K\psi^{-1}(x) \text{for any } x > 0, K > 1. \qquad (25)$$

Therefore, it has a moderate growth, but a highly irregular behaviour, which will imply the results we have in view. Indeed, let $y_n = \psi(x_n)$ and $h_n = 1/y_n$. see from (23), (24) and (25) that

$$D(h_n) \stackrel{\text{def}}{=} \frac{\psi^{-1}\left(\frac{1}{h_n}\right)}{\psi^{-1}\left(\frac{1}{\rho_\varphi(hn)}\right)} \ge \frac{\psi^{-1}\left(\frac{1}{hn}\right)}{\psi^{-1}\left(\frac{K}{h_n^2}\right)} = \frac{\psi^{-1}(yn)}{\psi^{-1}(Ky_n^2)} \ge \frac{x_n}{2Kx_n} = \frac{1}{2K}. \qquad (26)$$

Thus, we have $\lim \sup h \to 0 + D(h) > 0$. By [74], Theorem (3.1.13) (see also [75], comment before Theorem 5.2), C_φ is not compact.

On the other hand, let $j_\psi, 2 : H^\psi \to H^2$ and $j_{4,\psi} : H^4 \to H^\psi$ be the natural injections, which are continuous, thanks to (25). We have the following diagram:

$$H^\psi \xrightarrow{j\psi,2} H^2 \xrightarrow{C_\varphi} H^4 \xrightarrow{j_{4,\psi}} H^\psi.$$

The second map is continuous as a consequence of (19) and of a result of P. Duren ([66]; see also [67], Theorem 9.4, page 163), which extends Carleson's embedding theorem (see also [74], Theorem (3.1.13)8). Hence $C_\varphi = j_{4,\psi} \circ C_\varphi \circ j_{\psi,2}$ factorizes through a reflexive space (H^2 or H^4) and is therefore w-compact. To prove that C_φ is Dunford–Pettis, we use the following result of [77] (Theorem (3.1.3)):

Theorem (3.1.14)[63]: ([77]): Let φ be a Schur function and Φ be an Orlicz function. Assume that, for some $A > 0$, one has

$$\sup_{0 < t \le h} \frac{\rho_\varphi(t)}{t^2} \le (\frac{1}{h^2})/(\Phi\left(A\Phi^{-1}\left(\frac{1}{h^2}\right)\right), 0 < h < 1. \qquad (27)$$

Then, the canonical inclusion $j_{\Phi,\phi} : B^\Phi \to L^\Phi(m_\varphi)$ is continuous.

In particular, it is continuous for any $Orlicz$ function Φ if $\rho_\varphi(h) = O(h^2)$.

Now, $let\ J_\psi : H^\psi \to B^\psi$ be the canonical inclusion, and consider the following diagram:

$$H^\psi \xrightarrow{J_\psi} B^\psi \xrightarrow{j\psi,\varphi} L^\psi(m_\varphi).$$

The first map is Dunford–Pettis, by [76], Theorem (3.1.13). The second map is continuous by (19) and (27). Clearly, being Dunford–Pettis is an ideal property (if either u or v is Dunford–Pettis, so is vu). Therefore, $j_{\psi,\varphi} \circ J_\psi$ is Dunford– Pettis, and this amounts to saying that $C_\varphi : H^\psi \to H^\psi$ is Dunford–Pettis. Now, the non-reflexivity of H^ψ follows automatically, since C_φ is Dunford– Pettis but not compact.

This ends the proof of Theorem (3.1.13).

Theorem (3.1.13) admits the following variant.

Theorem (3.1.15)[63]: There exist a Schur function $\varphi$and an Orlicz function χ such that H^χ is not reflexive and the composition operator $C_\varphi : H^\chi \to H^\chi$ is weakly compact and not Dunford–Pettis; in particular, it is not compact.

Proof. We use the same Schur function $\varphi = \varphi_{\frac{1}{2}}$, but we replace ψ by the function χ defined by $\chi(x) = \psi(x^2)$. $Let\ A > 1$. Observe that, in view of (25),

$$\frac{\chi(Ax)}{[\chi(x)]^2} = \frac{\psi(A^2x^2)}{[\psi(x^2)]^2} \leq \frac{A^8x^8}{x^8} = A^8.$$

By [76], Proposition (3.1.16), $J_\chi: H^\chi \to B^\chi$ is w-compact, and we can see $C_\varphi: H^\chi \to H^\chi$ as the canonical inclusion $j: H^\chi \to L^\chi(m_\varphi)$. Hence Theorem (3.1.14) and the diagram

$$j = j_{\chi,\varphi} \circ J_\chi : H^\chi \xrightarrow{J^\chi} B^\chi \xrightarrow{j_{\chi,\varphi}} L_\chi(m_\varphi)$$

show that $C_\varphi : H^\chi \to H^\chi$ is w-compact as well. Now, to prove that C_φ is not Dunford–Pettis, we cannot use [76], as in the proof of Theorem (3.1.13), but we follow the lines of Proposition (3.1.9) of [76]. We remark first that, by definition, the function χ satisfies, for $\beta = 2,$ the following inequality:

$$\chi(\beta x) = \psi(4x^2) \geq 4\psi(x^2) = 2\beta\chi(x);$$

hence, by (22), this implies that the conjugate function of χ verifies the Δ^2- condition. Let xn be as in (24), and set

$$u_n = \sqrt{x_n} \; and \; A = \sqrt{2}$$

so that

$$\chi(Au_n) = [\chi(u_n)]^2 = x_n^4. \tag{28}$$

Finally, let

$$r_n = 1 - \frac{1}{\chi(u_n)} \; and \; f_n(z) = u_n \left(\frac{1 - r_n}{1 - r_n z}\right)^2.$$

By ([74], Corollary (3.1.9)), $\|fn\|_{H\chi} \leq 1$ and fn tends to 0 uniformly on compact subsets of $\mathbb{D}$; that implies that $f_n \to 0$ weakly in H^χ since the conjugate function of χ has Δ_2 ([74], Proposition 3.7). On the other hand, $if\ K_n = \|fn\|_{L\chi\,(m_\varphi)}$, mimicking the computation of ([76], Proposition (3.1.9)), we get

$$1 = \int_{\mathbb{D}} \chi(|f_n|/K_n)\; dm_\varphi \geq (1 - r_n)^2\chi(\alpha u_n/4K_n) \tag{29}$$

for some $0 < \alpha < 1$ independent of n, where we used the convexity of χ and the fact that the lens map φ satisfies, by (23),

$$m_\varphi(\{z \in \mathbb{D};\ |1 - z| \leq 1 - r_n\}) \geq \alpha(1 - rn)^2.$$

In view of (28), (29) reads as well

$$\chi(\alpha u_n/4K_n) \leq \chi^2(u_n) = \chi(Au_n),$$

so that

$$\|j(fn)\|_{L\chi(m_\varphi)} = K_n \geq \alpha/4A. \tag{30}$$

This shows that $j: H^\chi \to L^\chi(m_\varphi)$ and therefore also $C_\varphi : H^\chi \to H^\chi$ are not Dunford–Pettis.

It remains to show that H^χ is not reflexive. We shall prove below a more general result, but here, the conjugate function $\tilde{\chi}$ of χ satisfies the Δ_2 condition, as we saw. Hence H^χ is the bidual of HM^χ. Since χ fails to satisfy the $\Delta_2 -$ condition, we know that $L^\chi \neq M^\chi. Let u \in L^\chi \setminus M^\chi,$ with $u \geq 1$. Let f be the associated outer function, namely

$$f(z) = exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z}\log u(t)\, dt\right).$$

One has $|f^*| = u$ almost everywhere, with the notation of (11), and hence $f \in H^\chi \setminus HM^\chi$. It follows that $H^\chi = HM^\chi$. Hence HMχ is not reflexive, and therefore H^χ is not reflexive either.

As promised, we give the general result on non-reflexivity

Proposition (3.1.16)[63]: Let ψ be an Orlicz function which does not satisfy the Δ_{2-} condition. Then neither H^{ψ} nor B^{ψ} is reflexive.

Proof. We only give the proof for B^{ψ} because it is the same for H^{ψ}. Since ψ does not satisfy Δ_2 there is a sequence (x_n) of positive numbers, tending to infinity, such that $\psi(2x_n)/\psi(x_n)$ tends to infinity. Let $r_n \in (0,1)$ such that $(1 - r_n)^2 = 1/\psi(2x_n)$ and set

$$q_n(z) = \frac{(1 - r_n)^4}{(1 - r_n z)^4}.$$

One has

$$\|qn\|_{\infty} = 1 \, and \, \|qn\|_1 = \frac{(1 - r_n)^2}{(1 + r_n)^2} \leq (1 - r_n)^2.$$

On the other hand, on the pseud$o - C$arleson window $S(1, 1 - r_n)$, one ha

$$|1 - r_n z| \leq (1 - r_n) + r_n|1 - z| \leq (1 - r_n) + r_n(1 - r_n) = 1 - r_n^2$$
$$\leq 2(1 - r_n);$$

Hence $|q_n(z)| \geq 1/16$. It follows that

$$1 = \int_D \psi\left(\frac{|qn|}{\|qn\|_{\psi}}\right) dA \geq \int_{S(1,1-r_n)} \psi\left(\frac{|qn|}{\|qn\|_{\psi}}\right) dA$$
$$\geq A[S(1, 1 - r_n)]\psi\left(\frac{1}{16\,\|qn\|_{\psi}}\right) \geq \frac{1}{3}(1 - r_n)^2\psi\left(\frac{1}{16\|\,qn\|_{\psi}}\right)$$
$$\geq (1 - r_n)^2\psi\left(\frac{1}{48\,\|qn\|_{\psi}}\right) = \frac{1}{\psi(2x_n)}\psi\left(\frac{1}{48\,\|qn\|_{\psi}}\right);$$

hence $\psi(1/[48\,\|q_n\|_{\psi}]) \leq \psi(2x_n), so\ 1/(48\,\|q_n\|_{\psi}) \leq 2x_n$ and $96\,x_n\,\|q_n\|_{\psi} \geq 1.$

Set now $f_n = \frac{q_n}{\|q_n\|_{\psi}}$; one has $\|f_n\|_{\psi} = 1$ and (using that $\psi(x_n\,|q_n(z)|) \leq |q_n(z)|\psi(x_n)$, by convexity, since $|q_n(z)| \leq 1$)

$$\int_D \psi\left(\frac{|f_n|}{96}\right) dA = \int_D \psi\frac{x_n\,|q_n|}{96\,x_n\,\|q_n\|\psi}\, dA \leq \int_D \psi(x_n\,|q_n|)dA$$
$$\leq \psi(x_n)\int_D |q_n|dA$$
$$\leq \psi(x_n)(1 - r_n)^2 = \frac{\psi(x_n)}{\psi(2x_n)} \xrightarrow[n\to\infty]{} 0.$$

By [71], Lemma 11, this implies that the sequence (f_n) has a subsequence equivalent to the canonical basis of c_0 and hence B^{ψ} is not reflexive.

We finish by giving a counterexample using Bergman–Orlicz spaces instead of Hardy–Orlicz spaces.

Theorem (3.1.17)[63]: There exists a Schur functionφ and an Orlicz function ψ such that the space B^{ψ} is not reflexive and the composition operator $C_{\varphi} : B^{\psi} \to B^{\psi}$ is weakly-compact but not compact.

Proof. We use again the Orlicz function ψ defined by (24) and the Schur function $\varphi = \varphi_{\frac{1}{2}}$.

The space B^{ψ} is not reflexive since ψ does not satisfy the condition Δ_2.

We now need an estimate similar to (19) for φ_{θ}, namely (31)

$$\rho_{\varphi,2}(h) \coloneqq \sup_{s_{|\xi|=1}} A[\{z \in \mathbb{D}\,; \varphi(z) \in S(\xi, h)\}] \approx h^{\frac{2}{\theta}}. \qquad (31)$$

The proof of (31) is best seen by passing to the right half-plane with the measure A_{γ_θ} which is locally equivalent to the Lebesgue planar measure A; we get $\rho_{\varphi,2}(h) \geq A(\{|z|^\theta \leq h\} \cap \mathbb{H}) \geq Kh^{\frac{2}{\theta}}$ and the upper bound in (31) is proved similarly.

We now see that $C_\varphi : B^\psi \to B^\psi$ is not compact as follows. We use the same x_n as in (24) and set $\mathcal{Y}_n = \psi(x_n), k_n = 1\sqrt{\mathcal{Y}_n}$. We notice that, since $\rho_{\varphi,2}(h) \geq K^{-1}h^4$ $(with\, K > 1)$ in view of (31), we have

$$E(k_n) \overset{\text{def}}{=} \frac{\psi^{-1}\left(\frac{1}{k_n^2}\right)}{\psi^{-1}\left(\frac{1}{\rho_{\varphi,2}(k_n)}\right)} \geq \frac{\psi^{-1}\left(\frac{1}{k_n^2}\right)}{\psi^{-1}\left(\frac{K}{k_n^4}\right)} = \frac{\psi^{-1}(\mathcal{Y}_n)}{\psi^{-1}(K\mathcal{Y}_n^2)} \geq \frac{x_n}{2Kx_n} = \frac{1}{2K},$$

so that

$$\limsup_{(k\to 0+)} E(k) > 0,$$

and this implies that $C_\varphi : B^\psi \to B^\psi$ is not compact ([77], Theorem (3.1.10)). To see that $C_\varphi : B^\psi \to B^\psi$ is w-compact, we use the diagram

$$B^\psi \xrightarrow{j\psi,2} B^2 \xrightarrow{C_\varphi} B^4 \xrightarrow{j4,\psi} B^\psi$$

as well as (31), which gives $\rho_{\varphi,2}(h) \leq Kh^4$. A result of W. Hastings ([68]) now implies the continuity of the second map. This diagram shows that C_φ factors through a reflexive space (B^2 or B^4), and is therefore w −compact.

Section (3.2): Approximation Numbers of Composition Operators

For $\mathbb{D}$ be the open unit disk of the complex plane, equipped with its normalized area measure $f(z) = \frac{dxdy}{\pi}$. For $\alpha > -1$, let $\mathcal{B}_\alpha$ be the weighted Bergman space of analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on $\mathbb{D}$ such that

$$\|f\|_\alpha^2 = (\alpha+1)\int_{\mathbb{D}} |f(z)|^2(1-|z|^2)^\alpha dA(z) = \sum_{n=0}^{\infty} \frac{n!\,\Gamma(2+\alpha)}{\Gamma(n+2+\alpha)}|a_n|^2 < \infty.$$

The limiting case, as $\alpha \overset{>}{\to} -1$, of those spaces is the usual Hardy space H^2 (indeed, if f is a polynomial, we have $\lim_{\alpha \overset{>}{\to} -1} \|f\|_\alpha^2 = \sum_{n=0}^{\infty}|a_n|^2 = \|f\|_{H^2}^2$), which we shall treat as $\mathcal{B}_{-1}$. Note that $\|f\|_\alpha^2 \approx \sum_{n=0}^{\infty} \frac{|a_n|^2}{(n+1)^{\alpha+1}}$ and that

$$dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$$

is a probability measure on $\mathbb{D}$.

Bergman spaces [86] are Hilbert spaces of analytic functions on $\mathbb{D}$ with reproducing kernel $K_a \in \mathcal{B}_\alpha$, given by $K_a(z) = \left(\frac{1}{1-\bar{a}_z}\right)^{\alpha+2}$, namely, for every $a \in \mathbb{D}$:

$$f(a) = \langle f, K_a\rangle, \qquad \forall\, f \in \mathcal{B}_\alpha; \text{ and } \|K_a\|^2 = K_a(a) = \left(\frac{1}{1-|a|^2}\right)^{\alpha+2}. \tag{32}$$

An important common feature of those spaces is that the multipliers of $\mathcal{B}_\alpha$ can be (isometrically) identified with the space H^∞ of bounded analytic functions on $\mathbb{D}$, that is:

$$\forall g \in H^\infty, \qquad \|g\|_\infty = \sup_{f\in\mathcal{B}_\alpha, \|f\|_\alpha \leq 1} \|f\,g\|_\alpha. \tag{33}$$

Indeed, $\|fg\|_\alpha \leq \|g\|_\infty \|f\|_\alpha$ is obvious, and if $\|fg\|_\alpha \leq C\|f\|_\alpha$ for all $f \in \mathcal{B}_\alpha$, testing this inequality successively on $f = 1, g, \dots, g^n, \dots$ easily gives $g \in H^\infty$ and $\|g\|_\infty \leq C$.

Let now φ be a non-constant analytic self-map (a so-called Schur function) of $\mathbb{D}$ and let $C_\varphi: \mathcal{B}_\alpha \to H(\mathbb{D})$ the associated composition operator:

$$C_\varphi(f) = f \circ \varphi.$$

It is well-known [20] that such an operator is always bounded from $\mathcal{B}_\alpha$ into itself, and we are interested in its approximation numbers.

Also recall that the approximation (or singular) numbers $a_n(T)$ of an operator $T \in \mathcal{L}(H_1, H_2)$, between two Hilbert spaces H_1 and H_2, are defined, for $n = 1, 2, \ldots$, by:

$$a_n(T) = inf\{\|T - R\|; rank(R) < n\}.$$

We have:

$$a_n(T) = c_n(T) = d_n(T),$$

where the numbers c_n (resp. d_n) are the Gelfand (resp. Kolmogorov) numbers of T ([65] respectively).

We shall need the following quantity:

$$\beta(T) = \liminf_{n\to\infty} [a_n(T)]^{\frac{1}{n}}. \quad (34)$$

Those approximation numbers form a non-increasing sequence such that

$$a_1(T) = \|T\|, \qquad a_n(T) = a_n(T^*) = \sqrt{a_n(T^*T)}$$

and verify the so-called “ideal” and “subadditivity” properties [101]:

$$a_n(AT\,B) \leq \|A\|\, a_n(T)\|B\|; \;\; a_{n+m-1}(S + T) \leq a_n(S) + a_m(T). \quad (35)$$

Moreover, the sequence $(a_n(T))$ tends to 0 iff T is compact. If $(a_n(T)) \in \ell_p$, we say that T belongs to the Schatten class S_p of index $p, 0 < p < \infty$. Taking for T a compact diagonal operator, we see that this sequence is non-increasing with limit 0, but otherwise arbitrary. But if we restrict ourselves to a specified class of operators, the answer is far from being so simple, although in some cases the situation is completely elucidated. For example, for the class of Hankel operators on H^2 (those operators H_φ whose matrix $(a_{i,j})$ on the canonical basis of H^2 is of the form $a_{i,j} = \hat{\phi}(i + j)$ for some function $\phi \in L^\infty$), it is known that H_ϕ is compact if and only if the conjugate $\bar{\phi}$ of the symbol ϕ belongs to $H^\infty + C$, where C denotes the space of continuous, 2π-periodic functions (Hartman’s theorem, [110]). For those Hankel operators, the following theorem, due to Megretskii et al. [108], [114], shows that the approximation numbers are absolutely arbitrary, under the following form.

Theorem (3.2.1)[78]: (Megretskii–Peller–Treil). Let $(\varepsilon_n)_{n\geq 1}$ be a non-increasing sequence of positive numbers. Then there exists a Hankel operator H_ϕ satisfying:

$$a_n(H_\phi) = \varepsilon_n, \qquad \forall n \geq 1.$$

Indeed, if we take a positive self-adjoint operator A whose eigenvalues s_n coincide with the ε_n’s and whose kernel is infinite-dimensional, it is easily checked that this operator A verifies the three necessary and sufficient conditions of Theorem 0.1, page 490 in [114] and is therefore unitarily equivalent to a Hankel operator H_ϕ which will verify, in view of (35):

$$a_n(H_\phi) = a_n(A) = \varepsilon_n, \qquad n = 1, 2, \ldots$$

In particular, if $\varepsilon_n \to 0$, the above Hankel operator will be compact, and in no Schatten class if $\varepsilon_n = 1/log(n + 1)$ for example. We also refer to [100] for the following slightly weaker form due to Khruscev and Peller, but with a more elementary proof based on interpolation sequences in the Carleson sense: for any $\delta > 0$, there exists a Hankel operator H_ϕ such that

$$\frac{1}{1+\delta}\varepsilon_n \leq a_n(\mathrm{H}_\phi) \leq (1+\delta)\varepsilon_n, \qquad n = 1,2,\dots$$

We prove analogous theorems for the class of composition operators (whose compactness was characterized in [107], [117]). But if we are able to obtain the Khruscev–Peller analogue for the lower bounds, we will only obtain sub exponential estimates for the upper bounds, a fact which is explained by our second result: the speed of convergence to 0 of the approximation numbers of a composition operator cannot be greater than geometric (and is geometric for symbols φ verifying $\|\varphi\|_\infty < 1$). Our first result involves a constant < 1 and is not as precise as the result of Megretskii–Peller–Treil or even that of Khruscev–Peller; this is apparently due to the non-linearity of the dependence with respect to the symbol for the class of composition operators, contrary to the case of the Hankel class. This latter lower bound improves several previously known results on "non-Schattenness" of those operators (see Corollary (3.2.18)) and also answers in the positive to a question which was first asked to us by Le Merdy [103], concerning the bad rate of approximation of compact composition operators. Those theorems are, the first individual results on approximation numbers a_n of composition operators (in Parfenov [80], some good estimates are given for the approximation numbers of the Carleson embedding operator in the case of the space $H^2 = \mathcal{B}_{-1}$, but they remain fairly implicit, and are not connected with composition operators), whereas all previous results where in terms of symmetric norms of the sequence (a_n), not on the behavior of each a_n.

Before describing our results, let us recall two definitions. For every ξ with $|\xi| = 1$ and $0 < h < 1$, the Carleson window $W(\xi, h)$ centered at ξ and of size h is the set

$$W(\xi,h) = \{z \in \overline{\mathbb{D}}; |z| \geq 1 - h \text{ and } |arg(z\bar{\xi})| \leq \pi h\}.$$

Let μ be a positive, finite, measure on $\overline{\mathbb{D}}$; the associated maximal function ρ_μ is defined by:

$$\rho_\mu(h) = \sup_{|\xi|=1} \mu(W(\xi,h)) \,. \tag{36}$$

The measure μ is called a Carleson measure for the Bergman space $\mathcal{B}_\alpha$, or an $(\alpha+2)$-Carleson measure (including the case $\mathcal{B}_{-1} = H^2$), if $\rho_\mu(h) = O(h^{2+\alpha})$ as $h \to 0$. For any Schur function φ, we shall denote by m_φ the image $\varphi^*(m)$ of the Haar measure m of the unit circle under the radial limits function $\varphi^*(u) = \lim_{r\to1^-} \varphi(ru)$ of $\varphi, |u| = 1$, and by $A_{\varphi,\alpha+2}$ the image of the probability measure $(\alpha+1)(1-|z|^2)^\alpha dA(z)$ under φ. The corresponding maximal function will be denoted by $\rho_{\varphi,\alpha+2}$. This notation is justified by the fact that $m_\varphi \stackrel{\text{def}}{=} A_{\varphi,1}$ is a 1-Carleson measure and $A_{\varphi,\alpha}$ an $(\alpha+2)$-Carleson measure for $\alpha > -1$, in view of the famous Carleson embedding theorem which, expressed under a quantitative and generalized form, states the following, implicit as concerns $\|j\|$ and with different notations, but fully proved in [118], for the case $\alpha > -1$ (see [110]).

Theorem (3.2.2)[78]: (Carleson's Theorem). For any $(\alpha+2)$-Carleson measure μ, the canonical inclusion mapping $j: \mathcal{B}_\alpha \to L^2(\mu)$ is defined and continuous, and its norm satisfies

$$C^{-1} \sup_{0<h<1} \sqrt{\frac{\rho_\mu(h)}{h^{2+\alpha}}} \leq \|j\| \leq C \sup_{0<h<1} \frac{\rho_\mu(h)}{h^{2+\alpha}}. \tag{37}$$

We show some preliminary lemmas. Our first theorems concern lower bounds. We show that the convergence of the approximation numbers $a_n(C_\varphi)$ of a composition operator $C_\varphi: \mathcal{B}_\alpha \to \mathcal{B}_\alpha$ cannot exceed an exponential speed: for some $r \in (0,1)$ and some constant $c > 0$, one has $a_n(C_\varphi) \geq c\, r^n$. With the notations (34) and (47), one has $(C_\varphi) \geq [\varphi]^2$. This speed of convergence is only attained if the values of φ do not approach the boundary of the unit disk: $\|\varphi\|_\infty < 1$ (Theorem (3.2.12)). On the other hand, the speed of convergence to 0 of $a_n(C_\varphi)$ can be arbitrarily slow; this is proved. The proof is mainly an adaptation of the one in [92], but is fairly technical at some points, and will require several additional explanations. We prove an upper estimate (Theorem (3.2.23)), and give three applications of this theorem. We test our general results against the example of lens maps, which are known to generate composition operators belonging to all Schatten classes.

We shall state several lemmas, which are either already known or quite elementary, but turn out to be necessary for the proofs of our Theorems (3.2.9) and (3.2.17).

For the proof of Theorem (3.2.9), we shall need the Weyl lemma [65].

Lemma (3.2.3)[78]: (Weyl Lemma). Let $T: H \to H$ be a compact operator. Suppose that $(\lambda_n)_{n\geq 1}$ is the sequence of eigenvalues of T rearranged in non-increasing order. Then, we have:

$$\prod_{k=1}^{n} a_k(T) \geq \prod_{k=1}^{n} |\lambda_k|.$$

We recall [91], [97], [109] that an interpolation sequence (z_n) with (best) interpolation constant C is a sequence (z_n) (necessarily Blaschke, i.e., $\sum_{n=1}^{\infty}(1-|z_n|) < \infty$) in the unit disk such that, for any bounded sequence (w_n) of scalars, there exists a bounded analytic function $f(i.e., f \in H^\infty)$ such that:

$$f(z_n) = w_n, \qquad \forall n \geq 1, \qquad \text{and } \|f\|_\infty \leq C \sup_{n\geq 1}|w_n|.$$

The Carleson constant δ of a Blaschke sequence (z_n) is defined as follows:

$$\delta_n = \prod_{j\neq n} \rho(z_n, z_j); \;\; \delta = inf\ \delta_n = \inf_{n\geq 1}(1-|z_n|^2)|B'(z_n)|, \quad (38)$$

where B is the Blaschke product with zeros $z_n, n \geq 1$ and $\rho(.,.)$ is the pseudo-hyperbolic distance, defined below in (45). The interpolation constant C is related to the Carleson constant δ by the following inequality [94], in which λ is a positive numerical constant:

$$\frac{1}{\delta} \leq C \leq \frac{\lambda}{\delta}\left(1 + \log\frac{1}{\delta}\right). \quad (39)$$

This latter inequality can be viewed as a quantitative form of the Carleson interpolation theorem. Interpolation sequences and reproducing kernels of $\mathcal{B}_\alpha$ are related as follows [109].

Lemma (3.2.4)[78]: Let $(z_n)_{n\geq 1}$ be an H^∞-interpolation sequence of the unit disk, with interpolation constant C. Then, the sequence $(f_n) = \left(\frac{K_{z_n}}{\|K_{z_n}\|}\right)$ of normalized reproducing kernels at z_n is C-equivalent to an orthonormal basis in $\mathcal{B}_\alpha$, namely we have for any finite sequence (λ_n) of scalars:

$$C^{-1}\left(\sum_n |\lambda_n|^2\right)^{\frac{1}{2}} \leq \left\|\sum_n \lambda_n f_n\right\|_\alpha \leq C\left(\sum_n |\lambda_n|^2\right)^{\frac{1}{2}}. \quad (40)$$

The proof in [109] is only for H^2 , therefore we indicate a simple proof valid for Bergman spaces $\mathcal{B}_\alpha$ as well. Let $S = \sum \lambda_n K_{z_n}$ be a finite linear combination of the kernels K_{z_n} , $\omega = (\omega_n)$ be a sequence of complex signs, $S_\omega = \sum \omega_n \lambda_n K_{z_n}$ and $g \in H^\infty$ an interpolating function for the sequence $(\bar{\omega}_n)$, i.e., $g(z_n) = \bar{\omega}_n$ and $\|g\|_\infty \le C$. If $f \in \mathcal{B}_\alpha$ and $\|f\|_\alpha \le 1$, we see that:

$$\langle S_\omega, f\rangle = \sum \omega_n \lambda_n \overline{f(z_n)} = \sum \lambda_n \overline{(fg)(z_n)} = \sum \lambda_n \langle K_{z_n}, fg\rangle = \langle S, fg\rangle,$$

so that using (33):

$$|\langle S_\omega, f\rangle| \le \|S\|_\alpha \|fg\|_\alpha \le \|S\|_\alpha \|g\|_\infty \|f\|_\alpha \le C\|S\|_\alpha$$

and passing to the supremum on f , we get $\|S_\omega\|_\alpha \le C\|S\|_\alpha$. Since the coefficients λ_n are arbitrary, this implies that (f_n) is C-unconditional, namely:

$$C^{-1}\left\|\sum \omega_n \lambda_n f_n\right\|_\alpha \le \left\|\sum \lambda_n f_n\right\|_\alpha \le C\left\|\sum \omega_n \lambda_n f_n\right\|_\alpha.$$

Now, squaring and integrating with respect to random, independent, choices of signs ω_n's, we get (40).

We also recall [97] that an increasing sequence (r_n) of numbers such that $0 < r_n < 1$ and $\frac{1-r_{n+1}}{1-r_n} \le \rho < 1$ (i.e., verifying the so-called Hayman–Newman condition) is an interpolation sequence (see also [110]). In the following, let (r_n) be such a sequence verifying moreover the backward induction relation:

$$\varphi(r_{n+1}) = r_n. \tag{41}$$

Set $f_n = K_{r_n}/\|K\_r_n\|$ and $W = \overline{\text{span}}(f_n)$. Let $(e_n)_{n\ge1}$ be the canonical basis of ℓ^2, φ a Schur function and $h \in H^\infty$ a function vanishing at r_1. Denote by $M_h: \mathcal{B}_\alpha \to \mathcal{B}_\alpha$ the operator of multiplication by h. Then, we have the following basic lemma, which shows that some compression of C_φ^* is a backward shift with controlled weights [92].

Lemma (3.2.5)[78]: Let $J: \ell^2 \to W$ be the isomorphism given by $J(e_n) = f_n$. Then, the operator $\boldsymbol{B} = J^{-1}C_\varphi^* M_h^* J: \ell^2 \to \ell^2$ is the weighted backward shift given by:

$$\boldsymbol{B}(e_{n+1}) = w_n e_n \quad \text{and} \quad \boldsymbol{B}(e_1) = 0, \qquad where\ w_n = \overline{h(r_{n+1})}\frac{\|K_{r_n}\|}{\|K_{r_{n+1}}\|}. \tag{42}$$

To exploit Lemma (3.2.5), we shall need the following simple fact on approximation numbers of weighted backward shifts.

Lemma (3.2.6)[78]: Let $(e_n)_{n\ge1}$ be an orthonormal basis of the Hilbert space H and $\boldsymbol{B} \in \mathcal{L}(H)$ the weighted backward shift defined by

$$\boldsymbol{B}(e_1) = 0 \quad \text{and} \quad \boldsymbol{B}(e_{n+1}) = w_n e_n, \qquad where\ w_n \to 0.$$

Assume that $|w_n| \ge \varepsilon_n$ for all $n \ge 1$, where (ε_n) is a non-increasing sequence of positive numbers. Then **B** is compact, and satisfies:

$$a_n(\boldsymbol{B}) \ge \varepsilon_n, \qquad \forall n \ge 1. \tag{43}$$

Proof. The compactness of **B** is obvious. Let R be an operator of $rank < n$. Then ker R is of codimension $< n$, and therefore intersects the n-dimensional space generated by $e_2, \dots, e_{n+1}$ in a vector $x = \sum_{j=1}^n x_j e_{j+1}$ of norm one. We then have:

$$\|\boldsymbol{B} - R\|^2 \ge \|\boldsymbol{B}_x - R_x\|^2 = \|\boldsymbol{B}_x\|^2 = \sum_{j=1}^n |w_j|^2 |x_j|^2$$

$$\ge \sum_{j=1}^n \varepsilon_j^2 |x_j|^2 \ge \varepsilon_n^2 \quad n\, j = 1\, |x_j|^2 = \varepsilon_n^2.$$

This ends the proof of Lemma (3.2.6).

Now, in view of (32) and (42), the weight w_n roughly behaves as $\frac{1-r_{n+1}}{1-r_n}$, so we shall need good estimates on that quotient, before defining the sequence (r_n) explicitly.

We first connect this estimate with the hyperbolic distance d in $\mathbb{D}$. We denote (see [96] or [99] for the definition) by $d(z,w;U)$ the hyperbolic distance of two points z,w of a simply connected domain U. It follows from the generalized Schwarz–Pick lemma [99] applied to the canonical injection $U \to V$ that the bigger the domain the smaller the hyperbolic distance, namely:

$$U \subset V \quad \text{and} \quad z,w \in U\ H \Rightarrow d\,(z,w;V) \leq\ d(z,w;U). \tag{44}$$

Moreover, as is well-known,

$$0 \leq r < 1 \Rightarrow\ d(0,r;\mathbb{D}) = \frac{1}{2}\log\frac{1+r}{1\ -\ r}.$$

Recall that the pseudo-hyperbolic and hyperbolic distances ρ and d on $\mathbb{D}$ are defined by:

$$\rho(a,b) = \left|\frac{a-b}{1-\bar{a}b}\right|, \qquad d(a,b) = \frac{1}{2}\log\frac{1+\rho(a,b)}{1-\rho(a,b)}, \qquad a,b \in \mathbb{D}. \tag{45}$$

We shall omit the symbol $\mathbb{D}$ as far as the open unit disk is concerned. For this unit disk, we have the following simple inequality [92].

Lemma (3.2.7)[78]: Let $a,b \in \mathbb{D}$ with $0 < a < b < 1$. Then:

$$e^{-2d(a,b)} \leq \frac{1\ -\ b}{1\ -\ a} \leq 2\,e^{2d(a,b)}. \tag{46}$$

Finally, before proceeding to the construction of our Schur function φ, it will be useful to note the following simple technical lemma.

Lemma (3.2.8)[78]: Let (ε_n) be a non-increasing sequence of positive numbers of limit 0. Then there exists a decreasing and logarithmically convex sequence (δ_n) of positive numbers, with limit 0, such that $\delta_n \geq \varepsilon_n$ for all $n \geq 1$.

Proof. Provided that we replace ε_n by $\varepsilon_n + \frac{1}{n}$, we may assume that (ε_n) is decreasing. Let us define our new sequence by the inductive relation:

$$\delta_1 = \varepsilon_1;\quad \delta_2 = \varepsilon_2;\qquad \delta_{n+1} = \max\left(\varepsilon_{n+1}, \frac{\delta_n^2}{\delta_{n-1}}\right).$$

This sequence is log-convex by definition, i.e., $\delta_n^2 \leq \delta_{n+1}\delta_{n-1}$. By induction, it is seen to be decreasing. Therefore, it has a limit $l \geq 0$. If $\delta_n = \varepsilon_n$ for infinitely many indices, $l = 0$. Otherwise, for n large enough, we have the inductive relation $\delta_{n+1} = \delta_n^2/\delta_{n-1}$, which implies that $\delta_n = exp(\lambda n + \mu)$ for some constants λ, μ. Since (δ_n) is decreasing, we must have $\lambda < 0$ and again we get $l = 0$.

We may and will thus assume, without loss of generality, that (ε_n) is decreasing and logarithmically convex.

If

$$\varphi^{\#}(z) = \lim_{w\to z}\frac{\rho\big(\varphi(w),\varphi(z)\big)}{\rho(w,z)} = \frac{|\varphi'(z)|(1-|z|^2)}{1-|\varphi(z)|^2}$$

is the pseudo-hyperbolic derivative of φ, we set:

$$[\varphi] = \sup_{z\in\mathbb{D}}\varphi^{\#}(z) = \left\|\varphi^{\#}\right\|_{\infty}. \tag{47}$$

In our first theorem, we get that the approximation numbers cannot supersede a geometric speed.

Theorem (3.2.9)[78]: For any Schur function φ, there exist positive constants $c > 0$ and $0 < r < 1$ such that, for $C_\varphi\colon \mathcal{B}_\alpha \to \mathcal{B}_\alpha$, we have:

$$a_n(C_\varphi) \geq c\, r^n, \qquad n = 1, 2, \ldots. \tag{48}$$

More precisely, one has $\beta(C_\varphi) \geq [\varphi]^2$ and hence, for each $\kappa < [\varphi]$, there exists a constant $c_\kappa > 0$ such that:

$$a_n(C_\varphi) \geq c_\kappa\, \kappa^{2n}. \tag{49}$$

We shall see in Proposition (3.2.31) that this estimate is actually rather crude in general because $[\varphi]$ may be arbitrarily small, though $a_n(C_\varphi)$ decays "slowly".

For the proof, we need the following lemma.

Lemma (3.2.10)[78]: Let $T: H \to H$ be a compact operator. Suppose that $(\lambda_n)_{n\geq 1}$, the sequence of eigenvalues of T rearranged in non-increasing order, satisfies, for some $\delta > 0$ and $r \in (0,1)$:

$$|\lambda_n| \geq \delta r^n, \qquad n = 1, 2, \ldots$$

Then there exists $\delta_1 > 0$ such that

$$a_n(T) \geq \delta_1 r^{2n}, \qquad n = 1, 2, \ldots$$

In particular $\beta(T) \geq r^2$.

Proof. By the Weyl inequality (Lemma (3.2.3)), we have

$$\prod_{k=1}^{n} a_k(T) \geq \prod_{k=1}^{n} |\lambda_k| \geq \delta^n\, r^{\frac{n(n+1)}{2}}.$$

Since $a_k(T)$ is non-increasing and $a_k(T) \leq \|T\|$ for every k, changing n into 2_n, we get:

$$\|T\|^n a_n(T)^n \geq \prod_{k=1}^{2n} a_k(T) \geq \delta^{2n} r^{n(2n+1)}$$

and therefore $a_n(T) \geq \frac{\delta^2 r}{\|T\|} r^{2n} = \delta_1 r^{2n}$, as claimed.

By applying this lemma to composition operators, we get the following result, which ends the proof of Theorem (3.2.9).

Proposition (3.2.11)[78]: For every composition operator $C_\varphi: \mathcal{B}_\alpha \to \mathcal{B}_\alpha$ of symbol $\varphi: \mathbb{D} \to \mathbb{D}$, we have $\beta(C_\varphi) \geq [\varphi]^2$.

Proof. For every $a \in \mathbb{D}$, let Φ_a be the (involutive) automorphism of the unit disk defined by

$$\Phi_a(z) = \frac{a - z}{1 - \bar{a}z}, \qquad z \in \mathbb{D}.$$

Observe that we have

$$\Phi_a(a) = 0, \qquad \Phi_a(0) = a, \qquad \Phi_a'(a) = \frac{1}{|a|^2 - 1}, \qquad \Phi_a'(0) = |a|^2 - 1.$$

Define now $\psi = \Phi_{\varphi(a)} \circ \varphi \circ \Phi_a$. We have that 0 is a fixed point of ψ, whose derivative is, in modulus, by the chain rule:

$$|\psi'(0)| = \left|\Phi'\varphi(a)\big(\varphi(a)\big)\varphi'(a)\Phi_a'(0)\right| = \frac{|\varphi'(a)|(1 - |a|^2)}{1 - |\varphi(a)|^2} \stackrel{\text{def}}{=} \varphi^{\#}(a). \tag{50}$$

By the Schwarz lemma, we know that $|\psi'(0)| \leq 1$ and so $\frac{|\varphi'(a)|(1-|a|^2)}{1-|\varphi(a)|^2} \leq 1$ (the Schwarz–Pick inequality).

Let us first assume that the composition operator C_φ is compact. Then, so is C_ψ, since we have

$$C_\psi = C_{\Phi_a} \circ C_\varphi \circ C_{\Phi_{\varphi(a)}}. \tag{51}$$

If $\psi'(0) \neq 0$, the sequence of eigenvalues of C_ψ is $([\psi'(0)]^n)_{n\geq 0}$ ([83]; the result given for the space H^2 holds for $\mathcal{B}_\alpha \subset H^2$, and would also hold for any space of analytic functions in $\mathbb{D}$ on which C_ψ is compact). Lemma (3.2.10) then gives us:

$$\beta(C_\psi) \geq |\psi'(0)|^2 = [\varphi^\#(a)]^2 \geq 0.$$

This trivially still holds if $\psi'(0) = 0$.

Now, since C_{Φ_a} and $C_{\Phi_{\varphi(a)}}$ are invertible operators, (51) clearly implies that $\beta(C_\varphi) = \beta(C_\psi)$, and therefore, with the notation of (50):

$$\beta(C_\varphi) \geq [\varphi^\#(a)]^2, \qquad \text{for all } a \in \mathbb{D}.$$

By passing to the supremum on $a \in \mathbb{D}$, we end the proof of Proposition (3.2.11), and that of Theorem (3.2.9) in the compact case. If C_φ is not compact, the proposition trivially holds. Indeed, in this case, we have $\beta(C_\varphi) = 1 \geq [\varphi]^2$.

Theorem (3.2.12)[78]: For every $\alpha \geq -1$, there exists, for any $0 < r < 1, s = s\,(r) < 1$, satisfying $\lim_{r\to 1^-} s(r) = 1$, such that, for $C_\varphi\colon \mathcal{B}_\alpha \to \mathcal{B}_\alpha$, one has, with the notation coined in (34):

$$\operatorname{diam}_\rho(\varphi(\mathbb{D})) > r \Rightarrow \beta(C_\varphi) \geq s^2. \tag{52}$$

In particular, the exponential speed of convergence to 0 of the approximation numbers of a composition operator C_φ of symbol φ takes place if and only if $\|\varphi\|_\infty < 1$; in other words, we have:

$$\|\varphi\|_\infty = 1 \Leftrightarrow \beta(C_\varphi) = 1. \tag{53}$$

Let us remark that one cannot replace $diam_\rho(\varphi(\mathbb{D})) > r$ by $\|\varphi\|_\infty > r$ in (52). In fact, if for every $t \in (0,1)$, one takes an automorphism $\psi_t\colon \mathbb{D} \to \mathbb{D}$ such that $\psi_t(0) = t$ and if one sets $\varphi_t(z) = \psi_t(z/2)$, then $\|\varphi_t\|_\infty \geq t$, but $\beta(C_{\varphi_t}) = 1/2$ (in fact, if $u(z) = z/2$, it is easy to see that $a_n(C_u) = 1/2^{n-1}$ and, since C_{ψ_t} is invertible, $\beta(C_{\varphi_t}) = \beta(C_u)$).

The proof will proceed through a series of lemmas. Observe that given two points $a, b \in \mathbb{D}$, with $r = \rho(a, b)$, there exists an automorphism ψ of $\mathbb{D}$ such that $\psi(a) = 0$ and $\psi(b) = r$. As $\beta(C_\varphi) = \beta(C_{\psi\circ\varphi})$, we may assume, without loss of generality, throughout that proof, that 0 and r belongs to $\varphi(\mathbb{D})$.

Lemma (3.2.13)[78]: Let K be a compact subset of $\varphi(\mathbb{D})$ and μ be a probability supported by K. Then, there exists a constant $\delta > 0$ such that, if $R_\mu\colon \mathcal{B}_\alpha \to L^2(\mu)$ denotes the restriction operator, we have:

$$a_n(C_\varphi) \geq \delta\, a_n(R_\mu).$$

In particular:

$$\beta(C_\varphi) \geq \beta(R_\mu).$$

Proof. Since φ is an open map, there exists a compact set $L \subset \mathbb{D}$ and a Borel subset $A \subset L$ such that $\varphi(A) = K$ and $\varphi\colon A \to K$ is a bijection (see [113]). Then $\mu = \varphi(\nu)$, where $\nu = \varphi^{-1}(\mu)$ is a probability measure supported by L, and we have automatically $\|R_\nu\| < \infty$. Then, for every $f \in \mathcal{B}_\alpha$:

$$\|f\|^2_{L^2(\mu)} = \int_K |f|^2 d\mu = \int_L |f\circ\varphi|^2 d\nu = \|C_\varphi f\|^2_{L^2(\nu)}.$$

This yields $\|R_\mu f\| = \|(R_\nu \circ C_\varphi) f\|$, so we have $R_\mu = j\, R_\nu C_\varphi$, where $j\colon L^2(\nu) \to L^2(\mu)$ is an isometry, and the lemma follows, since we have then:

$$a_n(R_\mu) = a_n(R_\nu \circ C_\varphi) \leq \|R_\nu\| a_n(C_\varphi)$$

for every $n \geq 1$.

Observe that this provides a new proof of Theorem (3.2.9). Indeed, if $K \subset \varphi(\mathbb{D})$ is a small closed ball of center 0 and radius $t > 0$, we can take as μ the normalized area measure on K; then Parseval's formula easily shows that $\beta(R_\mu) \geq t$ in that case.

The strategy of the proof of Theorem (3.2.12) will consist of refining this observation. We shall show that the situation can be reduced to the case $K = [0, r]$, and that an appropriate choice of μ can be made in that case, giving a sharp lower bound for $\beta(R_\mu)$. We begin with explaining that choice in the next two lemmas.

Lemma (3.2.14)[78]: For every $r \in (0,1)$ there exists $s = s(r) < 1$ and $f = f_r \in H^\infty$ with the following properties:

(i) $\lim_{r\to 1^-} s(r) = 1$;

(ii) $\|f\|_\infty \leq 1$;

(iii) $f\big((0, r]\big) = s\, \partial\mathbb{D}$ in a one-to-one way.

Proof. Let $\rho = \frac{1-\sqrt{1-r^2}}{r}$. Then $r = \frac{2\rho}{1+\rho^2}$ and the automorphism $\varphi_\rho(z) = \frac{\rho - z}{1-\rho z}$ maps $[0, r]$ onto $[-\rho\,, \rho]$. We define $\varepsilon = \varepsilon\,(r)$ and $s = s(r)$ by the following relations:

$$\varepsilon(r) = \frac{\pi}{\log\frac{1+\rho}{1-\rho}}, \qquad \text{and} \quad s = e^{-\frac{\varepsilon\pi}{2}}. \tag{54}$$

Let now

$$\chi(z) = \varepsilon \log\frac{1+\varphi_\rho(z)}{1-\varphi_\rho(z)} \tag{55}$$

and

$$f(z) = s\, e^{i\chi(z)}. \tag{56}$$

Note that $f = e^h$, where

$$h(z) = i\varepsilon\log\frac{1+\varphi_\rho(z)}{1-\varphi_\rho(z)} - \varepsilon\frac{\pi}{2}$$

is a conformal mapping from $\mathbb{D}$ onto a small vertical strip of the left-half plane. This function f fulfills all the requirements of the lemma. Indeed, we have $|f\,(z)| \leq 1$ for all $z \in \mathbb{D}$ and

$$h([0,r]) = \left[-i\varepsilon\log\frac{1+\rho}{1-\rho}, i\varepsilon\log\frac{1+\rho}{1-\rho}\right] - \varepsilon\frac{\pi}{2} = [-i\pi, i\pi] - \varepsilon\frac{\pi}{2},$$

so that $f\big((0,r]\big) = \{w = se^{i\theta};\ -\pi \leq \theta \leq \pi\}$, in a one-to-one way.

Lemma (3.2.14) allows a good choice of the measure μ as follows.

Lemma (3.2.15)[78]: Let f be as in Lemma (3.2.14). Then, there exists a probability measure $\mu = \mu_r$ supported by $[0, r]$ and a constant $\delta_r > 0$ such that, for any integer $n \geq 1$ and any choice of scalars $c_0, c_1, \ldots, c_{n-1}$, we have:

$$\left\|\sum_{j=0}^{n-1} c_j R_\mu\left(f^j\right)\right\|_{L^2(\mu)} \geq \frac{s^n}{\sqrt{n}}\left\|\sum_{j=0}^{n-1} c_j f^j\right\|_{H^2} \geq \frac{s^n}{\sqrt{n}}\left\|\sum_{j=0}^{n-1} c_j f^j\right\|_{\mathcal{B}_\alpha}.$$

As a consequence, we can claim that, for $C_\varphi: \mathcal{B}_\alpha \to \mathcal{B}_\alpha$:

$$\varphi(\mathbb{D}) \supset [0, r] \Rightarrow \beta(C_\varphi) \geq s = s(r). \tag{57}$$

Proof. With our previous notations, we know that χ is a bijective map from $]0,r]$ onto the interval $]-\pi,\pi]$. Let m be the normalized Lebesgue measure on $]-\pi,\pi]$ and $\mu=\chi-1(m)$ be the image of m by χ^{-1}. We have, thanks to (56) and by definition of μ:

$$\left\|\sum_{j=0}^{n-1} c_j R_\mu(f^j)\right\|_{L^2(\mu)}^2 = \int_0^r \left|\sum_{j=0}^{n-1} c_j f^j(x)\right|^2 d\mu(x) = \int_0^r \left|\sum_{j=0}^{n-1} c_j s^j e^{ij\chi(x)}\right|^2 d\mu(x)$$

$$= \int_{-\pi}^{\pi} \left|\sum_{j=0}^{n-1} c_j s^j e^{ij\theta}\right|^2 \frac{d\theta}{2\pi} = \sum_{j=0}^{n-1} |c_j|^2 s^{2j} \ge s^{2n} \sum_{j=0}^{n-1} |c_j|^2.$$

Now, $\|f^j\|_{H^2} \le \|f^j\|_\infty \le 1$, so that we have, using the Cauchy–Schwarz inequality:

$$\left\|\sum_{j=0}^{n-1} c_j f^j\right\|_{H^2} \le \sum_{j=0}^{n-1} |c_j| \|f^j\|_{H^2} \le \sum_{j=0}^{n-1} |c_j| \le \sqrt{n} \left(\sum_{j=0}^{n-1} |c_j|^2\right)^{\frac{1}{2}},$$

giving the first inequality, since $\| \|_{H^2} \ge \| \|_{\mathcal{B}_\alpha}$. Finally, let $R:\mathcal{B}_\alpha \to L^2(\mu)$ be an operator of rank$< n$. We can find a function $g=\sum_{j=0}^{n-1} c_j f^j$ such that $\|g\|_{\mathcal{B}_\alpha}=1$ and $R(g)=0$. The first part of the proof gives:

$$\|R_\mu - R\| \ge \|R_\mu(g) - R(g)\| = \|R_\mu(g)\| = \left\|\sum_{j=0}^{n-1} c_j f^j\right\|_{L^2(\mu)} \ge \frac{s^n}{\sqrt{n}} \left\|\sum_{j=0}^{n-1} c_j f^j\right\|_{\mathcal{B}_\alpha} = \frac{s^n}{\sqrt{n}}.$$

Therefore $a_n(R_\mu) \ge s^n/\sqrt{n}$ and, in view of Lemma (3.2.13), the last conclusion of Lemma (3.2.15) follows.

The next lemma explains how to reduce the situation to the case $K=[0,r]$ when we only know that 0 and r belongs to $\varphi(\mathbb{D})$. It was inspired to us by the proof of the Lindelof theorem ¨ that convergence along a curve implies non-tangential convergence for functions in Hardy spaces [115].

Lemma (3.2.16)[78]: Suppose that 0 and r belong to $\varphi(\mathbb{D})$, with $0<r<1$. Let μ be a probability measure carried by $[0,r]$. Then, there exists a probability measure ν carried by a compact set $K \subset \varphi(\mathbb{D})$ such that, for any $f \in H(\mathbb{D})$:

$$\int_{[0,r]} |f(x)|^2 d\mu(x) \le \frac{1}{2} \int_K (|f(z)|^2 + |f(\bar{z})|^2) d\nu(z). \tag{58}$$

Proof. Since $\varphi(\mathbb{D})$ is open and connected and $0, r \in \varphi(\mathbb{D})$, there is a curve with image $K \subset \varphi(\mathbb{D})$ connecting 0 and r. Put $K=\{\bar{z};\ z \in K\}$. Then, there exists a compact set L such that $[0,r] \subset L$ and whose boundary $\partial L \subset (K \cup \widetilde{K})$. 1 Now, the existence of ν carried by K will be provided by an appropriate application of the Pietsch factorization theorem. To that effect, let X be the real subspace of $C(L)$ formed by the real functions which are harmonic in the interior of L. By the maximum principle for harmonic functions, X can be viewed as a subspace of $C(K \cup \widetilde{K})$. Now, the inclusion map j of X into $L^2(\mu)$ has 2-summing norm less than one ([88], or [104]). Therefore, the Pietsch factorization theorem ([88], or [104]) implies the existence of a probability ` σ on $K \cup \widetilde{K}$ such that, for every $u \in X$:

$$\|u\|_{L^2(\mu)}^2 = \int_{[0,r]} u^2 d\mu \le \int_{K\cup\widetilde{K}} u^2 d\sigma. \tag{59}$$

For any harmonic function u on $\mathbb{D}$, we can apply (59) to $u(z)$ and $u(\bar{z})$ to get:

$$2\int_{[0,r]} u^2 d\mu \leq \int_{K\cup\tilde{K}} [u^2(z) + u^2(\bar{z})]d\sigma(z) = \int_{K\cup\tilde{K}} [u^2(z) + u^2(\bar{z})]d\tilde{\sigma}(z),$$

where $\tilde{\sigma}$ is the symmetric measure of σ, defined by $\tilde{\sigma}(E) = \sigma(\tilde{E})$. There is a probability ν on K such that $\nu + \tilde{\nu} = \sigma + \tilde{\sigma}$. For this probability ν, we thus have, for any real harmonic function u on $\mathbb{D}$:

$$2\|u\|_{L^2(\mu)}^2 \leq \int_K [u^2(z) + u^2(\bar{z})]d\nu(z). \tag{60}$$

Now, given $f \in H(\mathbb{D})$, we use (60) with u the real and imaginary parts of f, and sum up to get (58).

We can now finish the proof of Theorem (3.2.12) as follows.

Suppose that $diam_\rho(\varphi(\mathbb{D})) > r$. Then we may assume, as explained before Lemma (3.2.13), that $0, r \in \varphi(\mathbb{D})$. Let μ be as in Lemma (3.2.15). Using Lemma (3.2.16), we find a probability measure ν, compactly supported by $\varphi(\mathbb{D})$, such that (58) holds. This inequality shows that:

$$\|R_\mu f\|^2 \leq \frac{1}{2}(\|R_\nu f\|^2 + \|R_{\tilde{\nu}} f\|^2),$$

so that $R_\mu = A(R_\nu \oplus R_{\tilde{\nu}})$ with $\|A\| \leq 1/\sqrt{2} \leq 1$. Therefore, by the ideal and sub-additivity properties (35):

$$a_{2n}(R_\mu) \leq a_{2n}(R_\nu \oplus R_{\tilde{\nu}}) \leq a_n(R_\nu) + a_n(R_{\tilde{\nu}}) = 2\, a_n(R_\nu),$$

implying $\beta(R_\nu) \geq \beta(R_\mu)^2$. Finally, Lemmas (3.2.13) and (3.2.15) give:

$$\beta(C_\varphi) \geq \beta(R_\nu) \geq \beta\,(R_\mu)^2 \geq s(r)^2,$$

and this ends the proof of Theorem (3.2.12).

We shall see that the convergence to 0 of the approximation numbers of a compact composition operator can be as slow as one wants. This answers in the positive to a question which was first asked to us by Le Merdy [103] in the OT Conference 2008 of Timisoara.

Theorem (3.2.17)[78]: Let $(\varepsilon_n)_{n\geq 1}$ be a non-increasing sequence of positive real numbers of limit zero. Then, there exists an injective Schur function φ such that $\varphi(0) = 0$ and $C_\varphi: \mathcal{B}_\alpha \to \mathcal{B}_\alpha$ is compact, i.e., $a_n(C_\varphi) \to 0$, but:

$$\liminf_{n\to\infty} \frac{a_n(C_\varphi)}{\varepsilon_n} > 0. \tag{61}$$

Equivalently, we have for some positive number $\delta > 0$, independent of n:

$$a_n(C_\varphi) \geq \delta\, \varepsilon_n \quad \text{for all} \quad n \geq 1.$$

As in the case of Hankel operators, an immediate consequence of Theorem (3.2.17) is the following:

Corollary (3.2.18)[78]: There exists a composition operator $C_\varphi: H^2 \to H^2$ which is compact, but in no Schatten class.

This corollary, which Theorem (3.2.17) reinforces and précises, was an answer to a question of Sarason, and has been first proved in [92]. Other proofs appeared in [87], [98], [72], [73], [120] (for a positive result on Schattenness, we refer to [106]).

The construction of the symbol φ in Theorem (3.2.17) follows that given in [92], but we have to proceed to some necessary adjustments. In order to exploit (46), we shall use, as in [92], the following two results due to Hayman [96] concerning the hyperbolic distance $d(z, w; U)$ of two points z, w of a simply connected domain U (see also [99]), whose proof uses in particular the comparison principle (44):

Proposition (3.2.19)[78]: Suppose that U contains the rectangle

$$R = \{z \in \mathbb{C};\ a_1 - b < \mathcal{R}e\, z < a_2 + b, |\mathcal{J}m\, z| < b\},$$

where $a_1 < a_2$ and $b > 0$. Then, we have the upper estimate:

$$d(a_1, a_2; U) \leq \frac{\pi}{4b}(a_2 - a_1) + \frac{\pi}{2}. \tag{62}$$

Proposition (3.2.20)[78]: Suppose that U contains the rectangle

$$R = \{z \in \mathbb{C}; a_1 - c < \mathcal{R}e\, z < a_2 + c, |\mathcal{J}m\, z| < c\},$$

where $a_1 < a_2$ and $c > 0$, but that the horizontal sides

$$\{z \in \mathbb{C}; a_1 - c \leq \mathcal{R}e\, z \leq a_2 + c, |\mathcal{J}m\, z| < c\}$$

of that rectangle are disjoint from U. Then, we have the lower estimate:

$$d(a_1, a_2; U) \geq \frac{\pi}{4c}(a_2 - a_1) - \frac{\pi}{2}. \tag{63}$$

We now proceed to the construction of our Schur function φ.

We first define a continuous map $\psi: \mathbb{R} \to \mathbb{R}$ as follows. Let (A_n) be an increasing sequence of positive numbers, which is concave for $n \geq 1$, and which tends to ∞. Let $A: [0, \infty) \to [0, \infty)$ be the increasing piecewise linear function on the intervals $(0,1)$ and (e^{n-1}, e^n) such that

$$A(0) \stackrel{\text{def}}{=} A_0 = 0, \qquad A(e^{n-1}) \stackrel{\text{def}}{=} A_n \quad \text{for} \quad n \geq 1, \qquad \text{and} \quad 2K = 1/A(1).$$

The sequence of slopes $\frac{A_n - A_{n-1}}{e^n - e^{n-1}}$ is decreasing, since $A_{n+1} - A_n \leq A_n - A_{n-1} \leq e(A_n - A_{n-1})$. The function A is hence increasing and concave on $(0, \infty)$ and vanishes at 0. This implies that $A(t)/t$ is decreasing on $(0, \infty)$.

We set:

$$\psi(t) = \begin{cases} K(1 + |t|) & \text{if } |t| \leq 1 \\ \dfrac{|t|}{A(|t|)} & \text{if } |t| > 1. \end{cases}$$

By the previous discussion, ψ is increasing on $(1, \infty)$.

We then define a domain Ω of the complex plane by:

$$\Omega = \{w \in \mathbb{C}; |\mathcal{J}m\, w| < \psi(|\mathcal{R}e\, w|)\}. \tag{64}$$

Let $\sigma: \mathbb{D} \to \Omega$ be the unique Riemann map such that $\sigma(0) = 0$ and $\sigma'(0) > 0$. This map exists in view of the following simple fact.

Lemma (3.2.21)[78]: The domain Ω defined by (64) is star-shaped with respect to the origin and $\sigma: (-1, 1) \to \mathbb{R}$ is an increasing bijection such that $\sigma(-1) = -\infty$ and $\sigma(1) = \infty$.

Proof. The star-shaped character of Ω will follow from the implication:

$$|\mathcal{J}m\, w| < \psi(|\mathcal{R}e\, w|) \ \text{and} \ 0 < \lambda < 1 \ \Rightarrow |\mathcal{J}m\, (\lambda w)| < \psi(|\mathcal{R}e\, (\lambda w)|).$$

We may assume that both $\mathcal{R}$e w, Im w are positive, and it is enough to prove:

$$\lambda\psi(x) \leq \psi(\lambda x), \qquad \forall \lambda \in [0, 1], \forall x > 0. \tag{65}$$

This is easy to check separating three cases:

(i) $x \leq 1$; then $\lambda\psi(x) = \lambda K(1 + x) \leq K(1 + \lambda x) = \psi(\lambda x)$;

(ii) $\lambda x \leq 1 < x$; then, since $A(x) > A(1)$,

$$\lambda\psi(x) = \lambda \frac{x}{A(x)} < 2K\lambda x \leq K(1 + \lambda x) = \psi\,(\lambda x);$$

(iii) $\lambda x > 1$; we then have, since A increases,

$$\lambda\psi(x) = \lambda\frac{x}{A(x)} \le \frac{\lambda x}{A(\lambda x)} = \psi(\lambda x)$$

and this ends the proof of (65). Now, since σ is determined by the value of $\sigma(0)$ and the sign of $\sigma'(0)$, we have $\sigma(\bar{z}) = \overline{\sigma(z)}$ for all $z \in \mathbb{D}$, so that $\sigma[\,(-1,1)] \subset \mathbb{R}$. And since the derivative of an injective analytic function does not vanish and $\sigma'(0) > 0$, we get that σ is increasing on $(-1,1)$. Finally, if $w \in \mathbb{R}$ and $w = \sigma(z)$, we have $\bar{w} = w$, so that $\sigma(\bar{z}) = \sigma(z)$ and $\bar{z} = z$, which proves the subjectivity of $\sigma: (-1,1) \to \mathbb{R}$.

We now choose A_n as follows, $\eta > 0$ denoting a positive numerical constant to be specified later.

$$A_n = \eta \log\frac{1}{\varepsilon_n}, n \ge 1. \tag{66}$$

Observe that this is an increasing, concave sequence tending to ∞ since we assumed that (ε_n) is log-convex and decreasing to 0.

Finally, we define our Schur function φ and our sequence (r_n) under the form of the following lemma, in which the increasing character of ψ is important.

Lemma (3.2.22)[78]: Let φ be defined by

$$\varphi(z) = \sigma^{-1}\big(e^{-1}\sigma(z)\big),$$

and let $r_n = \sigma^{-1}(e^n)$. Then we have:

(a) φ is univalent and maps $\mathbb{D}$ to $\mathbb{D}$, (r_n) increases, and $\varphi(0) = 0$;

(b) $\varphi(r_{n+1}) = r_n$;

(c) $\frac{1-r_{n+1}}{1-r_n} \to 0$ and therefore (r_n) is an interpolation sequence;

(d) $C_\varphi: \mathcal{B}_\alpha \to \mathcal{B}_\alpha$ is compact.

Proof.

(a) Since Ω is star-shaped, $e^{-1}\sigma(z) \in \Omega$ when $z \in \mathbb{D}$, so φ is well-defined and maps $\mathbb{D}$ to itself in a univalent way. Moreover, $\varphi(0) = \sigma^{-1}(0) = 0$, and (r_n) increases since σ^{-1} increases on $\mathbb{R}$.

(b) We have $\varphi(r_{n+1}) = \sigma^{-1}\left(\frac{1}{e}\sigma(r_{n+1})\right) = \sigma^{-1}\left(\frac{1}{e}e^{n+1}\right) = \sigma^{-1}(e_n) = r_n$.

(c) This assertion is more delicate and relies on Proposition (3.2.20) as follows.

Set $d_n = \psi(e^n)$. We have clearly $e^{n+1} + d_{n+2} < e^{n+2}$ for large n (recall that $\psi(t) = o(t)$ as $t \to \infty$), so that $\psi(e^{n+1} + d_{n+2}) < \psi(e^{n+2}) = d_{n+2}$ since ψ is increasing. By the intermediate value theorem for the function $\psi(e^{n+1} + x) - x$, we can therefore find a positive number $c_n < d_{n+2}$ such that $\psi(e^{n+1} + c_n) = c_n$.

Now, consider the open sets:

$$R_n = \{z \in \mathbb{C}; e^n - c_n < \mathcal{Re}\, z < e^{n+1} + c_n \text{ and } |\mathcal{Jm}\, z| < c_n\}, \qquad U_n = R_n \cup \Omega.$$

Those sets U_n satisfy the assumptions of Proposition (3.2.20) in view of (64). Indeed, if z belongs to the horizontal sides of R_n, we have $z \notin U_n$ since

$$e^n - c_n \le \mathcal{Re}\, z \le e^{n+1} + c_n \Rightarrow \psi(\mathcal{Re}\, z) \le \psi\,(e^{n+1} + c_n) = c_n = |\mathcal{Jm}\, z|.$$

This proposition then gives, since $\Omega \subset U_n$ and $c_n < d_{n+2}$, and since the hyperbolic metric is conformally invariant,

$$d(r_n, r_{n+1}) = d(e^n, e^{n+1}; \Omega) \ge d(e^n, e^{n+1}; U_n) \ge \frac{\pi}{4c_n}(e^{n+1} - e^n) - \frac{\pi}{2} \ge c\frac{e^{n+2}}{\psi(e^{n+2})}$$
$$= cA\,(e^{n+2}) \ge cA_n,$$

where c is a positive constant. Now, we use Lemma (3.2.7) to obtain:

$$\frac{1-r_{n+1}}{1-r_n} \leq 2\, e^{-2d(r_n,r_{n+1})} \leq 2\, e^{-2cA_n},$$

which proves that $\frac{1-r_{n+1}}{1-r_n} \to 0$, and implies that (r_n) is an interpolation sequence.

(d) Since φ is univalent, the compactness of $C_\varphi\colon \mathcal{B}_\alpha \to \mathcal{B}_\alpha$ amounts to proving that $\lim\limits_{|z|\to 1} \frac{1-|\varphi(z)|}{1-|z|} = \infty$. For $\alpha > -1$, this follows from [21] and for $\alpha = -1$ from [83]. By the Julia–Caratheodory Theorem [83], this in turn is equivalent to proving that for any u, v on the unit circle, the quotient $\frac{\varphi(z)-v}{z-u}$ has no finite limit as z tends to u radially. This latter fact requires some precise justification.

First, we notice that σ extends continuously to an injective map of the open upper half of the unit circle onto the upper part of the boundary of **Ω** (and similarly for lower parts). This follows from the Caratheodory extension theorem [115], applied to the restriction of σ^{-1} to the Jordan region limited by $\partial\Omega$ and two vertical lines $\mathcal{R}e\, w = \pm R$ where $R > 0$ is arbitrarily large. Now, let $u \in \partial\mathbb{D}$ with $u \neq \pm 1$. Then, $\sigma(ru) \to\ w \in \partial\Omega$ as $r \to 1-$, so that $e^{-1}\sigma(ru) \to e^{-1}w = w' \in \Omega$ and that $\varphi(ru) \to \sigma^{-1}(w') \in \mathbb{D}$. Therefore the image of φ touches the unit circle only at ± 1, and the assumption of the Julia–Caratheodory Theorem is ´ fulfilled if $u \neq \pm 1$. By symmetry, it remains to test the point $u = 1$ for which we have:

$$\limsup_{r \overset{<}{\to} 1} \frac{1-\varphi(r)}{1-r} \geq \limsup_{n\to\infty} \frac{1-\ \varphi(r_{n+1})}{1\ -\ r_{n+1}} = \limsup_{n\to\infty} \frac{1-r_n}{1-r_{n+1}} = \infty$$

by the preceding point 3. Since $|v-\ \varphi(r)| \geq 1-\varphi(r)$, this ends the proof of Lemma (3.2.22).

We now want a good lower bound for the weights w_n appearing in (42). To that effect, we apply Proposition (3.2.19) with

$$U = \Omega, \qquad a_1 = e^n, \qquad a_2 = e^{n+1} \quad \text{and} \quad b_n = \psi(e^{n-1}),$$

as well as

$$R'_n = \{z \in \mathbb{C};\ e^n - b_n < \mathcal{R}e\, z < e^{n+1} + b_n \text{ and } |\mathcal{J}m\, z| < b_n\}.$$

We have $e^n - b_n > e^{n-1}$ for large n, since this amounts to

$$e^n - e^{n-1} > b_n = e^{n-1}A(e^{n-1})\,, \qquad or \quad e-1 > \frac{1}{A(e^{n-1})},$$

which holds for large n since $A(t)$ tends to ∞ with t. We then observe that $R'_n \subset \Omega$. Indeed, $z \in R'_n \Rightarrow \mathcal{R}e\, z > e^n - b_n > e^{n-1}$ and, since ψ is increasing, we have $\psi(\mathcal{R}e\, z) > \psi(e^{n-1}) = b_n > |\mathcal{J}m\, z|$. Therefore, we can apply (62) and get, for all $n \geq 1$:

$$d(e^n, e^{n+1}; \Omega) \leq \frac{\pi}{4\psi(e^{n-1})}(e^{n+1} -\ e^n) + \frac{\pi}{2} \leq C_0 A(e^{n-1}) = C_0 A_n,$$

where C_0 is a numerical constant. By conformal invariance, we have as well $d(r_n, r_{n+1}) \leq C_0 A_n$. It then follows from (46) that:

$$\frac{1-r_{n+1}}{1-r_n} \geq \exp\bigl(-2d(r_n, r_{n+1})\bigr) \geq \exp(-2C_0 A_n). \tag{67}$$

Now, we take $h(z) = z - r_1$ in Lemma (3.2.6) and use the ideal property (35) of the approximation numbers. We get, denoting by C the interpolation constant of the sequence (r_n), and using the fact that $\|M_h\| = \|h\|_\infty \leq 2$:

$$a_n(\boldsymbol{B}) \leq \|J^{-1}\| a_n(C_\varphi) \|M_h\| \|J\| \leq 2C^2 a_n(C_\varphi). \tag{68}$$

Next, we choose $\eta = 1/C_0$ in (66) and we set $d = (r_2 - r_1)/\sqrt{2}$. Using Lemma (3.2.5) and relations (32), (42) and (67), we see that the weights w_n associated with **B** verify:

$$|w_n| = h(r_{n+1}) \frac{\|K_{r_n}\|}{\|K_{r_{n+1}}\|} = h(r_{n+1}) \sqrt{\frac{1 - r_{n+1}^2}{1 - r_n^2}} \geq \frac{r_2 - r_1}{\sqrt{2}} \sqrt{\frac{1 - r_{n+1}}{1 - r_n}} \geq d\, exp(-C_0 A_n)$$
$$\geq d\varepsilon_n \text{ for all } n \geq 1. \quad (69)$$

Finally, using Lemma (3.2.6), (68) and (69):

$$a_n(C_\varphi) \geq \frac{1}{2C^2} a_n(\boldsymbol{B}) \geq \frac{1}{2C^2} d\, \varepsilon_n \overset{\text{def}}{=} \delta\varepsilon_n \text{ for all } n \geq 1.$$

We thus get the desired conclusion (61) of Theorem (3.2.17).

We do not obtain a fairly good upper bound, and we shall content ourselves with the following result, whose proof is quite simple and, for the case $\alpha = -1$, partly contained in [80], but under a very cryptic form which is not easy to decipher.

Theorem (3.2.23)[78]: Let φ be a Schur function and $\alpha \geq -1$. Then, we have for the approximation numbers of $C_\varphi : \mathcal{B}_\alpha \to \mathcal{B}_\alpha$ the upper bound:

$$a_n(C_\varphi) \leq C \inf_{0<h<1} \left[n^{\frac{\alpha+1}{2}} (1-h)^n + \sqrt{\frac{\rho_{\varphi,\alpha} + 2(h)}{h^{2+\alpha}}} \right], \qquad n = 1, 2, \ldots \quad (70)$$

where C is a constant. In particular, if $\frac{\rho_{\varphi,\alpha}+2(h)}{h^2+\alpha} \leq e^{-\frac{h}{A(h)}}$, where the function $A: [0,1] \to [0,1]$ is increasing, with $A(0) = 0$ and with inverse function A^{-1}, we have:

$$a_n(C_\varphi) \leq C n^{\frac{\alpha+1}{2}} e^{-nA^{-1}\left(\frac{1}{2n}\right)}, \qquad n = 1, 2, \ldots \quad (71)$$

The proof of (70) uses a contraction principle which was first proved for $\alpha = -1$ [74] and $\alpha = 0$ [77], but is also valid for any $\alpha \geq -1$, as follows from the forthcoming work [105]. To prove Theorem (3.2.23), it will be convenient to prove first the following simple lemma.

Lemma (3.2.24)[78]: Let n be a positive integer, $g \in \mathcal{B}_\alpha$ and $f(z) = z^n g(z)$. Then, we have:

$$\|g\|_\alpha \leq C n^{\frac{\alpha+1}{2}} \|f\|_\alpha. \quad (72)$$

Proof. Let $w_n = \frac{n\Gamma(2+\alpha)}{\Gamma(n+2+\alpha)}$. We first observe that

$$\frac{w_k}{w_{k+n}} \leq C n^{\alpha+1}, \qquad \forall k \geq 0, \qquad \forall n \geq 1. \quad (73)$$

Indeed, we have:

$$\frac{w_k}{w_{k+n}} = \frac{k!}{(k+n)!} \frac{\Gamma(k+\alpha+2+n)}{\Gamma(k+\alpha+2)} = \prod_{j=1}^{n} \frac{(k+j+\alpha+1)}{(k+j)} \leq \prod_{j=1}^{n} \frac{j+\alpha+1}{j}$$
$$= \prod_{j=1}^{n} \left(1 + \frac{\alpha+1}{j}\right) \leq \exp\left[(\alpha+1) \sum_{j=1}^{n} \frac{1}{j}\right] \leq C n^{\alpha+1},$$

which proves (73).

Now, if $f(z) = \sum_{k=n}^{\infty} a_k z^k$, we have $g(z) = \sum_{k=0}^{\infty} a_{k+n} z^k$ so that, using (73):

$$\|g\|_\alpha^2 = \sum_{k=0}^{\infty} |a_{k+n}|^2 w_k = \sum_{l=n}^{\infty} |a_l|^2 w_{l-n} \leq C n^{\alpha+1} \sum_{l=n}^{\infty} |a_l|^2 w_l = C n^{\alpha+1} \|f\|_\alpha^2,$$

proving (72).

We shall now majorize $a_{n+1}(C_\varphi)$, but provided that we change the constant C, this makes no difference with majorizing an (C_φ). The choice of the approximating operator R of rank $\leq n$ for C_φ is quite primitive, but in counterpart we shall estimate $\|C_\varphi - R\|$ rather sharply. We denote by P_n the projection operator defined by $P_n f = \sum_{k=0}^{n-1} \hat{f}(k) z^k$ and we take $R = C_\varphi \circ P_n$, i.e., if we have $f(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k \in \mathcal{B}_\alpha$, then $R(f) = \sum_{k=0}^{n-1} \hat{f}(k) \varphi^k$, so that $(C_\varphi - R)f = C_\varphi(r)$, with, making use of (72):

$$r(z) = \sum_{k=n}^{\infty} \hat{f}(k) z^k = z^n s(z), \qquad \text{with } \|s\|_\alpha^2 \leq C n^{\alpha+1} \|r\|_\alpha^2, \|r\|_\alpha \leq \|f\|_\alpha. \quad (74)$$

Assume that $\|f\|_\alpha \leq 1$, fix $0 < h < 1$ and denote by μ_h the restriction of the measure $A_{\varphi,\alpha+2}$ to the annulus $1 - h < |z| \leq 1$. Then, we have:

$$\begin{aligned}
\|(C_\varphi - R)f\|_\alpha^2 &= \|C_\varphi(r)\|_\alpha^2 = \int_{\overline{D}} |r(z)|^2 d\, A_{\varphi,\alpha+2}(z) \\
&\leq (1-h)^{2n} \int_{|z|\leq 1-h} |s(z)|^2 d\, A_{\varphi,\alpha+2}(z) + \int_{1-h<|z|\leq 1} |r(z)|^2 dA_{\varphi,\alpha+2}(z) \\
&\leq (1-h)^{2n} \int_{\mathbb{D}} |s(z)|^2 dA_{\varphi,\alpha+2}(z) + \int_{\overline{\mathbb{D}}} |r(z)|^2 d\mu_h(z) \\
&= (1-h)^{2n} \|C_\varphi(s)\|_\alpha^2 + \int_{\overline{\mathbb{D}}} |r(z)|^2 d\mu_h(z) \\
&\leq C \left[(1-h)^{2n} \|s\|_\alpha^2 + \int_{\overline{\mathbb{D}}} |r(z)|^2 d\mu_h(z) \right] \\
&\leq C \left[n^{\alpha+1} (1-h)^{2n} + \sup_{0<t\leq h} \frac{\rho_{\varphi,\alpha+2}(t)}{t^{2+\alpha}} \right]
\end{aligned}$$

if we use (74), as well as (37) under the form

$$\int_{\overline{\mathbb{D}}} |r(z)|^2 d\mu_h(z) \leq C \sup_{0<t\leq h} \frac{\rho_{\varphi,\alpha+2}(t)}{t^{2+\alpha}} \|r\|_\alpha^2$$

and we know that $\|r\|_\alpha \leq \|f\|_\alpha \leq 1$.

To get rid of the supremum with respect to t, we make use of the following inequality, which holds for $h \leq 1 - |\varphi(0)|$ and $0 < \varepsilon \leq 1$:

$$\rho_{\varphi,\alpha+2}(\varepsilon h) \leq C \varepsilon^{\alpha+2} \rho_{\varphi,\alpha+2}(h). \quad (75)$$

For $\alpha = 0$ or $\alpha = -1$, this follows respectively from [74], and from [77]. The general case is proved in [105]. Setting $t = \varepsilon h$ for $0 < t \leq h$, this also reads $\frac{\rho_{\varphi,\alpha+2}(t)}{t^{\alpha+2}} \leq C \frac{\rho_{\varphi,\alpha+2}(h)}{h^{\alpha+2}}$, and we can forget the supremum in t in the previous inequalities. Taking square roots, we get the relation (70).

When $\rho_{\varphi,\alpha+2}(h)/h^{2+\alpha} \leq e^{-\frac{h}{A(h)}}$, let us take for h the nearly optimal value $h = A^{-1}(1/2n)$, so that $h/A(h) = 2nh$. We then have from (70), since $(1-h)^{2n} \leq e^{-2nh}$:

$$a_{n+1}(C_\varphi)^2 \leq \|C_\varphi - R\|_\alpha^2 \leq C n^{\alpha+1} \left[e^{-2nh} + e^{-\frac{h}{A(h)}} \right] \leq 2C n^{\alpha+1} e^{-2nA^{-1}\left(\frac{1}{2n}\right)},$$

proving (71), and ending the proof of Theorem (3.2.23).

Let us now indicate three corollaries, which improve results of [72], [102], [77] respectively.

Corollary (3.2.25)[78]: Suppose that $\rho_{\varphi,\alpha+2}(h) \le Ch^{(2+\alpha)\beta}$ for some $\beta > 1$. Then:

$$a_n(C_\varphi) \le Cn^{-\frac{(\beta-1)(\alpha+2)}{2}}(\log n)^{\frac{(\beta-1)(\alpha+2)}{2}}.$$

In particular, C_φ belongs to the Schatten class $S_p = S_p(\mathcal{B}_\alpha)$ for each $p > \frac{2}{(\beta-1)(\alpha+2)}$.

Proof. Set $\gamma = (\beta-1)(\alpha+2)/2, a = (\alpha+1)/2$, and $c = a+\gamma$. If we apply (70) of Theorem (3.2.23) with the value $h = c\log n/n$ which satisfies $n^a e^{-nh} = n^{-\gamma}$, as well as the inequality $(1-h)^n \le e^{-nh}$, we get:

$$a_n(C_\varphi) \le C\left[n^{-\gamma} + \left(\frac{\log n}{n}\right)^\gamma\right] \le C\left(\frac{\log n}{n}\right)^\gamma,$$

ending the proof.

In [72], we had only the assertion on Schatten classes, for the single value $\alpha = -1$, and not the upper bound for the individual approximation numbers $a_n(C_\varphi)$.

In particular, we can get $a_n(C_\varphi) \le Ce^{-\frac{n}{\log(n+1)}}$ and C_φ is in every Schatten class $S_p(\mathcal{B}_\alpha), p > 0$.

Notice that the sequence (ε_n) in the statement cannot be dispensed with. Indeed, if φ is surjective, we surely have $\|\varphi\|_\infty = 1$! And we know from Theorem (3.2.12) that $\beta(C_\varphi) = 1$ in that case.

We begin with a lemma of independent interest.

Lemma (3.2.26)[78]: Let $\delta: (0,1] \to \mathbb{R}$ be a positive and non-decreasing function. Then there exists a Schur function φ with the following properties:

(a) $\varphi: \mathbb{D} \to \mathbb{D}$ is surjective and 4-valent;

(b) $\rho_{\varphi,\alpha+2}(h) \le \delta(h)$, for $h > 0$ small enough.

Proof. We begin with the case $\alpha = -1$. Set, for $a = 1/2$:

$$\Phi_a(z) = \frac{a-z}{1-az}, \qquad B = \Phi_a^2,$$

and $C = \frac{1+a}{2(1-a)} = 3/2$. Note that $B\left(\frac{2a}{a+1}\right) = B(0)$. Let now

$$b_n = \frac{1}{4n\pi}, \qquad \varepsilon(h) = \frac{1}{2}\delta\left(\frac{h}{C}\right), \qquad \varepsilon_n = \varepsilon(b_{n+1}).$$

In the proof of Theorem (3.2.17) of [102], using an argument of harmonic measure and of barrier, we have found a 2-valent symbol φ_1 with $\varphi_1(\mathbb{D}) = \mathbb{D}^*$ such that, noting ρ_φ for $\rho_{\varphi,1}$:

$$b_{n+1} < h \le b_n \Rightarrow \rho_{\varphi_1}(h) \le \varepsilon_n. \tag{76}$$

This gives $\rho_{\varphi_1}(h) \le \varepsilon(b_{n+1}) \le \varepsilon(h)$. Let now, as in [102], $\varphi = B \circ \varphi_1$. This Schur function is surjective (since $\varphi(\mathbb{D}) = B(\mathbb{D}^*) = B(\mathbb{D}) = \mathbb{D}$), and 4-valent. Moreover, if $I = (u,v)$ is an arc of $\mathbb{T}$ of length $h < \frac{1}{2}$ and $J = \left(\frac{u}{2},\frac{v}{2}\right)$, we have $B^{-1}(I) \subset \Phi_a(J) \cup \Phi a\,(-J) = I_1 \cup I_2$, where I_1, I_2 are two arcs of $\mathbb{T}$ of length at most $\|P_a\|_\infty\,(h/2) = Ch$, since Φ_a being an inner function, we have [111], P_a being the Poisson kernel at a:

$$m_{\Phi_a} = P_a m.$$

Hence, using (76), we obtain:

$$m_\varphi(I) = m_{\varphi_1}(B^{-1}(I)) \le m_{\varphi_1}(I_1) + m_{\varphi_1}(I_2) \le 2\rho_{\varphi_1}(Ch) \le 2\varepsilon(Ch) = \delta(h),$$

and $\rho_\varphi(h) \le \delta(h)$ for small h, by passing to the supremum on all I's.

For the general case $\alpha \geq -1$, we use the following extension of an inequality from [77] (which treats the case $\alpha = 0$, see Remark before Corollary (3.2.9)1):

Lemma (3.2.27)[78]: For small h, namely $0 < h < (1 - |\varphi(0)|)/4$, we have, for every $\alpha > -1$:

$$\rho_{\varphi,\alpha+2}(h) \leq C\left[\rho_\varphi(Ch)\right]^{\alpha+2}. \qquad (77)$$

Proof. Let us define, as in [117], the generalized Nevanlinna counting function $N_{\varphi,\alpha+2}$ by the formula

$$N_{\varphi,\alpha+2}(w) = \sum_{\varphi(z)=w} \left[\log\left(\frac{1}{|z|}\right)\right]^{\alpha+2}, \qquad w \in \mathbb{D}\backslash\{\varphi(0)\}.$$

The case $\alpha = -1$ corresponds to the usual Nevanlinna counting function, which will be denoted by N_φ. The partial Nevanlinna counting function $N_\varphi(r,w)$ is defined, for $0 \leq r \leq 1$, by:

$$N_\varphi(r,w) = \sum_{\varphi(z)=w} \log^+(r/|z|),$$

so that $N_\varphi(1,w) = N_\varphi(w)$.

Since $\alpha + 2 \geq 1$, we have the obvious but useful inequality:

$$N_{\varphi,\alpha+2}(w) \leq \left[N_\varphi(w)\right]^{\alpha+2}. \qquad (78)$$

We shall also make use of the following identity, due to Shapiro ([117], where a weight $1/r$ is missing), and which can easily be checked after two integrations by parts:

$$N_{\varphi,\alpha+2}(w) = (\alpha+2)(\alpha+1)\int_0^1 N_\varphi(r,w)\left[\log\left(\frac{1}{r}\right)\right]^\alpha \frac{dr}{r}. \qquad (79)$$

As it was noticed in [77], this formula reads, for w close to the boundary, as follows, for $0 < h < (1 - |\varphi(0)|)/4$ and $|w| > 1 - h$:

$$N_{\varphi,\alpha+2}(w) = (\alpha+2)(\alpha+1)\int_{\frac{1}{3}}^1 N_\varphi(r,w)\left[\log\left(\frac{1}{r}\right)\right]^\alpha \frac{dr}{r}. \qquad (80)$$

Under the same conditions on h and w, this obviously implies:

$$N_{\varphi,\alpha+2}(w) \geq \frac{1}{C}\int_{\frac{1}{3}}^1 N_\varphi(r,w)(1-r^2)^\alpha\, rdr = \frac{1}{C}\int_0^1 N_\varphi(r,w)(1-r^2)^\alpha\, rdr.$$

Now, using the same arguments as in [77] and in particular using (80) for $\varphi_r(z) = \varphi(r_z)$, the identity $N_\varphi(r,w) = N_{\varphi_r}(w)$ and an integration in polar coordinates, we get:

$$\sup_{|w|\geq 1-h} N_{\varphi,\alpha+2}(w) \geq \frac{1}{C}\rho_{\varphi,\alpha+2}\left(\frac{h}{C}\right). \qquad (81)$$

The end of the proof is easy: changing h into Ch and using successively (81) and (78), we get for small h, depending on φ:

$$\rho_{\varphi,\alpha+2}(h) \leq C \sup_{|w|\geq 1-Ch} N_{\varphi,\alpha+2}(w) \leq C \sup_{|w|\geq 1-Ch}\left[N_\varphi(w)\right]^{\alpha+2} \leq C\left[\rho_\varphi(Ch)\right]^{\alpha+2},$$

the last inequality coming from [75]. This ends the proof of (77).

Going back to the proof of Lemma (3.2.26), if we apply the already settled case $\alpha = -1$ to the function $\tilde{\delta}(h) = \left[\frac{\delta\left(\frac{h}{C}\right)}{C}\right]^{\frac{1}{\alpha+2}}$, we obtain a surjective and 4-valent Schur function φ such that:

$$\rho_{\varphi,\alpha+2}(h) \le C\big[\rho_\varphi(Ch)\big]^{\alpha+2} \le C\big[\tilde\delta(Ch)\big]^{\alpha+2} = \delta(h),$$

for h small enough.

Corollary (3.2.28)[78]: Let (ε_n) a sequence of positive numbers which tends to 0. Then, there exists a Schur function φ with the following properties:

(a) $\varphi: \mathbb{D} \to \mathbb{D}$ is surjective and 4-valent;

(b) $a_n(C_\varphi) \le Ce^{-n\varepsilon_n}, n = 1, 2, \dots$

Proof. Set $a = (\alpha + 1)/2$. Provided that we replace (ε_n) by the decreasing sequence (ε'_n) with $\varepsilon'_n = \frac{1}{n} + \sup_{k\ge n} \varepsilon_k \ge \varepsilon_n$, we can assume that (ε_n) decreases. Let $A: [0, 1] \to [0, 1]$ be a function such that $A(0) = 0$, and which increases (as well as $A(t)/t$) so slowly that $A(\varepsilon_n + a\ (\log n/n)) \le 1/2n$; therefore $A^{-1}(1/2n) \ge \varepsilon_n + a(\log n/n)$ and

$$n^a e^{-nA^{-1}\left(\frac{1}{2n}\right)} \le e^{-n\varepsilon_n}.$$

We now apply Lemma (3.2.26) to the non-decreasing function $\delta(h) = h^{2+\alpha} e^{-\frac{h}{A(h)}}$ to get the result, in view of (71) of Theorem (3.2.23).

Our last corollary involves Hardy–Orlicz spaces H^ψ and Bergman–Orlicz spaces $\mathcal{B}^\psi$. For the definitions, see [74].

Corollary (3.2.29)[78]: There exists a Schur function φ and an Orlicz function ψ such that $C_\varphi: H^\psi \to H^\psi$ is compact whereas $C_\varphi: \mathcal{B}^\psi \to \mathcal{B}^\psi$ is not compact. Moreover, the approximation numbers $a_n(C_\varphi)$ of $C_\varphi: \mathcal{B}_\alpha \to \mathcal{B}_\alpha$ satisfy the upper estimate $a_n(C_\varphi) \le ae^{-b\sqrt{n}}$ where a, b are positive constants independent of n, and therefore C_φ belongs to $\bigcap_{p>0} S_p(\mathcal{B}_\alpha)$.

Proof. Let $\alpha \ge -1$ be fixed. The Schur function constructed in the proof of Theorem (3.2.18) of [77] satisfies the two first assertions, as well as $\rho_\varphi(h)/h \le e^{-\frac{d}{h}}$ for some positive constant $d > 0$. We now apply (77) to get for small h:

$$\frac{\rho_{\varphi,\alpha+2}(h)}{h^{\alpha+2}} \le C\,\frac{\big[\rho_\varphi(Ch)\big]^{\alpha+2}}{h^{\alpha+2}} \le\ C^{\alpha+3} e^{-\frac{(\alpha+2)d}{Ch}} \le ae^{-\frac{b}{h}}$$

for positive constants a and b. We can thus apply (71) of Theorem (3.2.23), for some $\delta > 0$, with the increasing function $A(h) = h^2/\delta$ (hence $A^{-1}(x) = \sqrt{\delta x}$) to get the result, diminishing slightly b to absorb the power factor $n^{\frac{(\alpha+1)}{2}}$ (see [89], [101], [81]).

$$\sup_{1\le k\le n} [k^\alpha e_k(T)] \le C_\alpha \sup_{1\le k\le n} [k^\alpha a_k(T)], \qquad \forall\alpha > 0. \tag{82}$$

$$\big(a_n(T)\big) \in \ell_q \Rightarrow \big(e_n(T)\big) \in \ell_q, \qquad \forall q > 0. \tag{83}$$

The converse of (83) does not hold in Banach spaces, but it does for operators between Hilbert spaces, by polar decomposition. More precisely, we have [81] $a_n(T) \le 4e_n(T)$ and, in particular, $\big(e_n(T)\big) \in \ell_q$ if and only if $\big(a_n(T)\big) \in \ell_q$.

We now have the following improved version of Theorem (3.2.9). Recall that $\varphi^\#(z) = \frac{|\varphi'(z)|(1-|z|^2)}{1-|\varphi(z)|^2}$ and $[\varphi] = \|\varphi^\#\|_\infty$.

Theorem (3.2.30)[78]: Let $T = C_\varphi$ be a compact composition operator on $\mathcal{B}_\alpha$, and $\gamma(T) = \liminf_{n\to\infty} [e_n(T)]^{\frac{1}{n}}$. Then:

$$\gamma(T) \ge [\varphi]^{\frac{1}{2}}. \tag{84}$$

Proof. We proceed as in the proof of Theorem (3.2.9). First, recall that the entropy numbers $e_n(T)$ also have the ideal property [101], namely:

$$e_n(AT\,B) \leq \|A\| e_n(T) \|B\|.$$

Then, we use an improved Weyl-type inequality for entropy numbers, due to Carl and Triebel [90], in which $\left(\lambda_n(T)\right)_{n\geq 1}$ denotes the sequence of eigenvalues of T rearranged in non-increasing order of moduli and $C = \sqrt{2}$:

$$\left(\prod_{k=1}^{n} |\lambda_k(T)|\right)^{\frac{1}{n}} \leq Cen\,(T). \tag{85}$$

It should be noted that this inequality can itself be improved [95]:

$$\left(\prod_{k=1}^{n} a_k(T)\right)^{\frac{1}{n}} \leq Ce_n(T). \tag{86}$$

Yet, the tempting similar inequality $(\prod_{k=1}^{n} |\lambda_k(T)|)^{\frac{1}{n}} \leq Ca_n(T)$ is wrong (even the inequality $|\lambda_n(T)| \leq Ca_n(T)$ is wrong) as follows from an example of [101]. Note that (86) implies the following:

$$a_n(T) \geq \delta r^n \Rightarrow e_n(T) \geq \frac{\delta}{C} r^{\frac{1}{2}} r^{\frac{n}{2}}.$$

This might explain why a square root appears in (84), and tends to indicate that $[\varphi]$ should appear instead of $[\varphi]^2$ in Theorem (3.2.9).

Now, for every $a \in \mathbb{D}$, let again Φ_a be defined by $\Phi_a(z) = \frac{a-z}{1-az}$, for $z \in \mathbb{D}$. Set $b = \varphi(a)$ and define $\psi = \Phi_b \circ \varphi \circ \Phi_a$. We already know that 0 is a fixed point of ψ and that $C_\psi = C_{\Phi_a} \circ C_\varphi \circ C_{\Phi_b}$. We may assume that $|\psi'(0)| = \varphi^{\#}(a) \neq 0$. The sequence of eigenvalues of C_ψ is then, as we have seen, $(\psi'(0)^n)_{n\geq 0}$ [83]. The Eq. (85) then gives us, setting $r = |\psi'(0)| = \varphi^{\#}(a)$:

$$e_n(C_\psi) \geq \frac{1}{C}\left(\prod_{k=0}^{n-1} r^k\right)^{\frac{1}{n}} = \frac{1}{C} r^{\frac{n-1}{2}}.$$

This clearly gives us $\gamma(C_\psi) \geq \sqrt{r}$. Now, since C_{Φ_a} and C_{Φ_b} are invertible operators, the relation $C_\psi = C_{\Phi_a} \circ C_\varphi \circ C_{\Phi_b}$ and the ideal property of the numbers $e_n(T)$ imply that $\gamma(C_\varphi) = \gamma(C_\psi)$, and therefore, with the notation of (50), $\gamma(C_\varphi) \geq \left(\varphi^{\#}(a)\right)^{\frac{1}{2}}$, for all $a \in \mathbb{D}$. Passing to the supremum on $a \in \mathbb{D}$, we end the proof of Theorem (3.2.30).

We shall suppose that $\alpha = -1$, i.e., we are concerned with the Hardy space H^2. Fix $0 < \theta < 1$. Denote by $\mathbb{H} = \{z \in \mathbb{C}; \mathcal{R}e\, z > 0\}$ the right half-plane, by $T: \mathbb{D} \to \mathbb{C}\backslash\{-1\}$ the involutive transformation defined by $T(z) = \frac{1-z}{1+z}$, which maps $\mathbb{D}$ to $\mathbb{H}$, and by τ_θ the transformation $z \in \mathbb{H} \to z^\theta \in \mathbb{H}$. Recall that the associated lens map $\varphi_\theta: \mathbb{D} \to \mathbb{D}$ is:

$$\varphi_\theta = T \circ \tau_\theta \circ T.$$

It is known that the associated composition operator on H^2 is in all Schatten classes S_p [84]. Alternatively, one could use Luecking's criterion [79]. Therefore, its approximation numbers decrease rather quickly. Still more precisely, adapting techniques of Parfenov [80], we might show the following (where $\beta_\theta, \gamma_\theta, \ldots$ are positive constants):

$$a_n(C_{\varphi_\theta}) \le \gamma_\theta e^{-\beta_\theta\sqrt{n}}. \qquad (87)$$

We do not detail this adaptation of Parfenov's methods from Carleson embeddings to composition operators here (see [63]), but shall dwell on the converse inequality, which is not proved in [80]. First, we show that there is no converse to the inequality of Theorem (3.2.9).

Proposition (3.2.31)[78]: The value of $[\varphi_\theta]$ for the lens map is

$$[\varphi_\theta] = \theta. \qquad (88)$$

In particular, $[\varphi_\theta]$ can be as small as we wish, although $\beta(C_{\varphi_\theta}) = 1$.

Recall that β is defined in (34) and $[\varphi]$ in (47).

Proof. First note the simple

Lemma (3.2.32)[78]: Let $z \in \mathbb{D}$ and $v = T(z) \in \mathbb{H}$. Then:

$$|T'(z)|(1-|z|^2) = 2\mathcal{R}e\big(T(z)\big) \text{ and } \frac{|T'(v)|}{1-|T(v)|^2} = \frac{1}{2\mathcal{R}e\, v}.$$

The two equalities are the same because $|T'(v)| = \frac{1}{|T'(z)|}$ in view of $= T^{-1}$. For the first one, we have:

$$|T'(z)|(1-|z|^2) = \frac{2(1-|z|^2)}{|1+z|^2} = 2\mathcal{R}e\,\big(T(z)\big).$$

Let now $z \in \mathbb{D}$ and $w = T(z) \in \mathbb{H}$. By the chain rule, we have:

$$\varphi_\theta'(z) = T'\big(\tau_\theta(w)\big)\tau_\theta'(w)T'(z).$$

Taking moduli and using the lemma with z and $v = \tau_\theta(w)$, we obtain:

$$\frac{|\varphi_\theta'(z)|(1-|z|^2)}{1-|\varphi_\theta(z)|^2} = \frac{\big|T'\big(\tau_\theta(w)\big)\big|}{1-\big|T\big(\tau_\theta(w)\big)\big|^2}|\tau_\theta'(w)||T'(z)|(1-|z|^2) = \frac{|\tau_\theta'(w)|\mathcal{R}e\, w}{\mathcal{R}e\big(\tau_\theta(w)\big)}.$$

Now, setting $w = re^{it}$ with $r > 0$ and $-\pi/2 < t < \pi/2$, this writes as well:

$$\varphi_\theta^{\#}(z) = \frac{\theta r^{\theta-1} r\cos t}{r^\theta \cos\theta t} = \frac{\theta\cos t}{\cos\theta t}.$$

Using the fact that w runs over $\mathbb{H}$ as z runs over $\mathbb{D}$ and that the cosine decreases on $\left(0,\frac{\pi}{2}\right)$, we obtain (88) by taking $t = 0$.

The proof of the second assertion is obvious in view of Theorem (3.2.12) since $\|\varphi_\theta\|_\infty = 1$.

We now give the following more precise form (the small Roman and Greek letters $a_\theta, \dots, \beta_\theta, \dots$ will denote positive constants depending only on θ):

The upper bound is (87). For the lower bound, we shall need two simple lemmas.

Lemma (3.2.33)[78]: Let $0 < \sigma < 1$ and $u = (u_j)$ be a sequence of points of $\mathbb{D}$ such that $\frac{1-|u_{j+1}|}{1-|u_j|} \le \sigma$. Then, the Carleson constant δ_u of the sequence u satisfies:

$$\delta_u \ge \exp\left(-\frac{a}{1-\sigma}\right), \qquad with\ a = \frac{\pi^2}{2}.$$

Proof. We use the following fact [97]:

$$\delta_u \ge \prod_{j=1}^{\infty}\left(\frac{1-\sigma^j}{1+\sigma^j}\right)^2. \qquad (89)$$

This implies $\log\,\delta_u \ge 2\sum_{j=1}^{\infty}\log\left(\frac{1-\sigma^j}{1+\sigma^j}\right)$. Now, expanding the logarithm in power series and permuting sums, we note that:

$$2\sum_{j=1}^{\infty}\log\left(\frac{1+\sigma^j}{1-\sigma^j}\right)=4\sum_{k=0}^{\infty}\frac{\sigma^{2k+1}}{(2k+1)(1-\sigma^{2k+1})}\le 4\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2(1-\sigma)}=\frac{a}{1-\sigma},$$

where we used $1-\sigma^{2k+1}\ge (2k+1)(1-\sigma)\sigma^{2k+1}$ and $\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}=\pi^2/8$. So that $\delta_u\ge\exp\left(-\frac{a}{1-\sigma}\right)$, which was to be proved.

The second lemma is similar.

Lemma (3.2.34)[78]: Let $0<\sigma<1, u_j=1-\sigma^j, v_j=\varphi_\theta(u_j)$ and $v=(v_j)$. Then, the Carleson constant δ_v of the sequence v satisfies:

$$\delta_v\ge\exp\left(-\frac{a_\theta}{1-\sigma}\right),\qquad with\ \ a_\theta=\frac{\pi^2}{2^\theta\theta}$$

Proof. We first note that $1-\varphi_\theta(r)=\frac{2(1-r)^\theta}{(1+r)^\theta+(1-r)^\theta}$, and so

$$\frac{1-v_{j+1}}{1-v_j}=\sigma^\theta\,\frac{\sigma^{j\theta}+\left(2-\sigma^j\right)^\theta}{\sigma^{(j+1)\theta}+(2-\sigma^{j+1})^\theta}=\sigma_j,$$

with $\sigma_j\le\sigma'=1-\frac{\theta}{2}2^\theta(1-\sigma)$. To see this, observe that:

$$1-\sigma_j=\frac{\left(2-\sigma^{j+1}\right)^\theta-\left(2\sigma-\sigma^{j+1}\right)^\theta}{\sigma^{(j+1)\theta}+(2-\sigma^{j+1})^\theta}\overset{\text{def}}{=}\frac{N}{D}\ge\theta 2^{\theta-1}(1-\sigma)=1-\sigma'.$$

Indeed, the function $f(x)=x^\theta+(2-x)^\theta$ increases on [0, 1], so $\mathbb{D}\le f(1)=2$. On the other hand, the mean-value theorem gives $N=2(1-\sigma)\theta c^{\theta-1}\ge\ \theta(1-\sigma)2^\theta$ for some $c\in(0,2)$.Lemma (3.2.33) then gives the result for the sequence v.

Proposition (3.2.35)[78]: There exist constants $b_\theta,c_\theta,\beta_\theta,\gamma_\theta$ with $b_\theta=\pi\sqrt{\frac{2(1-\theta)}{\theta}}$ such that:

$$c_\theta e^{-b_\theta\sqrt{n}}\le a_n\left(C_{\varphi_\theta}\right)\le\gamma_\theta e^{-\beta_\theta\sqrt{n}}.\qquad (90)$$

In particular, we have $\beta\left(C_{\varphi_\theta}\right)=1$ and C_{φ_θ} is in all Schatten classes $S_p, p>0$ but its approximation numbers do not decrease exponentially.

Proof. Fix an integer $n\ge 1$, and take $(u_j),(v_j)$ as in Lemma (3.2.34). We have $\varphi_\theta(0)=0, |\varphi_\theta(z)|\le|z|$ and so for $0<r<1$:

$$\frac{1-\ r^2}{1-\varphi_\theta(r)^2}\ge\frac{1-r}{1-\varphi_\theta(r)}=\frac{(1-r)\left[(1-r)^\theta+(1+r)^\theta\right]}{2(1-r)^\theta}\ \ge\frac{(1-r)^{1-\theta}}{2},$$

implying

$$\frac{1-u_j^2}{1-v_j^2}\ge\frac{1}{2}\sigma^{n(1-\theta)},\qquad \text{for } 1\le j\le n.$$

Let now R be an operator of rank$<n$. There exists a function $f=\sum_{j=1}^{n}\lambda_j K_{u_j}\in\ H^2\cap ker\ R$ with $\|f\|\ =1$. We thus have, denoting by C_u and C_v the interpolation constants of the sequences u and v, and using Lemma (3.2.4) twice:

$$\left\|C_{\varphi_\theta}^*-R\right\|^2\ge\ \left\|C_{\varphi_\theta}^*(f)-R(f)\right\|^2\ =\ \left\|C_{\varphi_\theta}^*(f)\right\|^2=\left\|\sum_{j=1}^{n}\lambda_j K_{v_j}\right\|^2$$

$$\geq C_v^{-2}\sum_{j=1}^{n}|\lambda_j|^2\left\|K_{v_j}\right\|^2 = C_v^{-2}\sum_{j=1}^{n}\frac{|\lambda_j|^2}{1-v_j^2}$$

$$\geq \frac{1}{2}\frac{C^{-2}}{v}\sigma^{n(1-\theta)}\sum_{j=1}^{n}\frac{|\lambda_j|^2}{1-u_j^2} \geq \frac{1}{2}\; C_u^{-2}C_v^{-2}\sigma^{n(1-\theta)}\|f\|^2 \; = \frac{1}{2}\; C_u^{-2}C_v^{-2}\sigma^{n(1-\theta)}.$$

Therefore, $a_n\left(C_{\varphi_\theta}\right) = a_n\left(C_{\varphi_\theta}^*\right) \geq \frac{1}{2}C_u^{-1}C_v^{-1}\sigma^{\frac{n(1-\theta)}{2}}$. But it follows from (39), Lemmas (3.2.33) and (3.2.34) that C_u, C_v satisfy, provided that we now take the value $a_\theta = \frac{\pi^2}{\theta} > \frac{\pi^2}{2} + \frac{\pi^2}{2^\theta\theta}$, since $\theta + 2^{1-\theta} < 2$, to absorb the logarithmic factor of (39):

$$C_uC_v \leq c_\theta^{-1}\exp\left(\frac{a_\theta}{1-\sigma}\right).$$

The preceding now gives us (c_θ changing from line to line):

$$a_n\left(C_{\varphi_\theta}\right) \geq c_\theta \exp\left(-\frac{a_\theta}{1-\sigma}\right)\exp\left(\frac{n(1-\theta)}{2}\log\sigma\right).$$

Finally, adjust $\sigma = 1-\lambda n^{-\frac{1}{2}}$ so that $\frac{a_\theta}{\lambda} = \frac{1-\theta}{2}\lambda$, i.e., $\lambda = \sqrt{\frac{2a_\theta}{1-\theta}}$ and use $\log(1-x) \geq -x-x^2$ for $0 \leq x \leq 1/2$; this gives (90) with the value

$$b_\theta = \frac{2a_\theta}{\lambda} = \sqrt{2a_\theta(1-\theta)} = \pi\sqrt{\frac{2(1-\theta)}{\theta}},$$

and that ends the proof of Proposition (3.2.35).

We prove the existence of the compact L claimed in the proof of Lemma (3.2.16). Let $\gamma: [0,1] \to \varphi(\mathbb{D})$ be a simple curve joining 0 and r, i.e., $\gamma(0) = 0$ and $\gamma(1) = r$, and consisting of segments parallel to the coordinate axes. This is always possible since $\varphi(\mathbb{D})$ is open and connected (for example, if γ had self-intersection points, we may add them and obtain a graph going from 0 to r. Now, from any finite such graph, we can extract a maximal tree rooted at 0 and finishing at r, and this tree generates the required simple curve). Denote by $K = \gamma([0,1]) \subset \varphi(\mathbb{D})$ the image of this curve, and set $\gamma(t) = x(t) + iy(t)$. We define inductively a sequence $0 = t_0 < t_1 < \cdots < t_M = 1$ in the following way. Start from $t_0 = 0$ and $\gamma(t_0) = x(t_0) = 0$ and suppose that we have defined $t_0 < \cdots < t_j$ with $\gamma(t_j) = x(t_j)$. If $t_j = 1$, we set $M = j$ and we have finished. If $t_j < 1$, we define $t_{j+1} > t_j$ as follows:

(i) If the curve just after t_j is followed by a horizontal segment of the real axis, until the time t_{j+1}, we say that j is an index of horizontal type and we have just defined t_{j+1}.

(ii) If the curve just after t_j is followed by a vertical segment, we say that j is of vertical type and we denote by t_{j+1} the first value of $t > t_j$ for which $y(t) = 0$. Such a value exists since $\gamma(1) = r$, implying $y(1) = 0$. Set $I_j = [t_j, t_{j+1}]$. If γ_j is the restriction of γ to I_j, we complete it by symmetry with respect to the real axis in a closed, positively oriented Jordan curve δ_j with image in $K \cup \widetilde{K}$. This is possible since γ is simple and γ_j intersects the real axis only at $x(t_j)$ and $x(t_{j+1})$. The process must stop after a finite number $M \geq 1$ of steps, and we now set $L_j = \{z; Ind\,(z,\delta_j) \neq 0\}$ and $L = \cup_{j=0}^{M-1}\overline{L_j} \cup K$ (Ind denotes the winding number). We claim that this set L has the following properties, which are exactly those required in the proof of Lemma (3.2.16). First, L is obviously a compact subset of $\mathbb{D}$.

Then $[0,r] \subset L$. In fact, first observe that each segment $[x(t_j), x(t_{j+1})]$ is a subset of L. Indeed, if j is of horizontal type, this is obvious. If j is of vertical type, we may assume without loss of generality that $x(t_j) < x(t_{j+1})$. By definition, $x < x(t_j)$ implies that $\notin \delta_j$, so that $Ind\,(x, \delta_j) = 0$ by connection to $-\infty$. Therefore $Ind\,(x, \delta_j) = 1$ for $x(t_j) < x < x(t_{j+1})$ since the index changes by one when one crosses orthogonally the boundary of a simple curve [112] and since δ_j contains a vertical segment passing through $x(t_j)$. Now, by the intermediate value theorem, we see that
$[0,r] \subset x\,([0,1]) \subset \cup_{j=0}^{M-1} [x(t_j), x\,(t_{j+1})] \subset L$. Finally, $\subset E = K \cup \widetilde{K}$. Indeed, using the Jordan curve theorem, we see that $\subset \left(\cup_{j=0}^{M-1} \partial L_j\right) \cup K \subset \left(\cup_{j=0}^{M-1} \delta_j\right) \cup K \subset K \cup \widetilde{K}$, since $\delta_j \subset K \cup \widetilde{K}$ by definition.

Section (3.3): Compact Composition Operators on BMOA

For $\mathbb{D}$ be the open unit disc of the complex plane $\mathbb{C}$. The space BMOA consists of the analytic functions $f: \mathbb{D} \to \mathbb{C}$ whose boundary values have bounded mean oscillation on the unit circle $\mathbb{T}$. Equivalently, f belongs to BMOA if and only if the seminorm

$$|f|_* = \sup_{a \in \mathbb{D}} \|f \circ \sigma_a - f(a)\|_{H^2}$$

is finite, where $\|\cdot\|_{H^2}$ is the standard norm of the Hardy space H^2 and $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$ is the automorphism of $\mathbb{D}$ that exchanges the points 0 and a. Then BMOA becomes a Banach space under the norm $\|f\|_* = |f(0)| + |f|_*$. Furthermore, VMOA is the closed subspace of BMOA consisting of those functions f whose boundary values have vanishing mean oscillation, or equivalently, which satisfy

$$\lim_{|a| \to 1} \|f \circ \sigma_a - f(a)\|_{H^2} = 0.$$

See [94], [126] and [23] for more information on the spaces BMOA and VMOA.

If $\varphi: \mathbb{D} \to \mathbb{D}$ is an analytic map, then the composition operator C_φ induced by φ is the linear map defined by $C_\varphi f = f \circ \varphi$ for all analytic functions $f: \mathbb{D} \to \mathbb{C}$. It is well known that C_φ is always bounded from BMOA into itself and that C_φ preserves VMOA if and only if $\varphi \in VMOA$; see e.g. [121], [123], [138]. Composition operators have been intensively studied on various spaces of analytic functions, see [20] or [83] for more about the classical background.

Recall that a linear operator is compact if it takes bounded sets into sets having a compact closure. The compactness of a composition operator C_φ acting on BMOA (or on its subspace VMOA) has been investigated various kinds of characterizations are known; see e.g. [123], [128], [129], [133], [135], [139], [141]–[143]. In particular, Smith [135] proved that C_φ is compact on BMOA if and only if φ satisfies the following pair of conditions:

$$\lim_{|\varphi(a)| \to 1} \sup_{0<|w|<1} |w^2| N\left(\sigma_{\varphi(a)} \circ \varphi \circ \sigma_a, w\right) = 0, \qquad (S1)$$

And for all $0 < R < 1$,

$$\lim_{t \to 1} \sup_{\{a: |\varphi(a)| \le R\}} |\zeta \in \mathbb{T}: |(\varphi \circ \sigma_a)(\zeta)| > t| = 0. \qquad (S2)$$

Above $N(\psi, w) = -\sum_{\psi(z)=w} log|z|$ denotes the Nevanlinna counting function of an analytic self-map ψ of the disc, $\varphi(\zeta)$ is the radial limit of φ for a.e. ζ on the unit circle $\mathbb{T}$, and $|E|$ stands for the normalized Lebesgue measure of sets $E \subset T$. Recently the first author [129] showed that (S1) is equivalent to the condition

$$\lim_{|\varphi(a)|\to 1} \left\| \sigma_{\varphi(a)} \circ \varphi \circ \sigma_a \right\|_{H^2} = 0, \qquad (L)$$

which is technically more convenient for our later purposes.

A well-known open problem concerning composition operators is that of characterizing the weak compactness of C_φ on the non-reflexive spaces BMOA and VMOA. Recall that an operator is weakly compact provided it takes bounded sets into sets whose closure is compact in the weak topology of the space. For C_φ acting on VMOA this problem was explicitly posed in [139] and [123], and for the BMOA case it was stated in [127], [128]. Partial results for VMOA were obtained in [133] and [125]. For instance, if $\varphi \in VMOA$ and $\varphi(\mathbb{D})$ is contained in a polygon inscribed in $\overline{\mathbb{D}}$ [133], or if φ is univalent [125], then compactness and weak compactness are equivalent for C_φ on VMOA. It is natural to conjecture that the same equivalence should persist for arbitrary symbols φ even on BMOA, especially because a similar phenomenon is known to occur for composition operators on many other classical non-reflexive spaces, such as H^1 [82], H^∞ (see e.g. [64]) and Bloch spaces [132], [2].

We solve the above problem. The main result reads as follows:

Theorem (3.3.1)[69]: Let $\varphi: \mathbb{D} \to \mathbb{D}$ be an analytic map. Then the following conditions are
equivalent:

(i) $C_\varphi: BMOA \to BMOA$ is compact.

(ii) $C_\varphi: BMOA \to BMOA$ is weakly compact.

(iii) φ satisfies condition (S1).

(iv) φ satisfies condition (L).

A key ingredient of our argument is the surprising result that condition (L) (and consequently also (S1)) actually implies (S2). This result is proved. Thus our work substantially clarifies and simplifies the existing compactness criteria for composition operators on BMOA. The proof of Theorem (3.3.1) is then completed by verifying that (ii) implies (iv). This step is carried out, where the argument is based on an idea of Leibov [130] (cf. also [134]) on how to construct explicit isomorphic copies of the sequence space c_0 inside VMOA.

As a by-product the results answer a recent question of Wulan, Zheng and Zhu [143]. Namely, it follows that the condition $\lim_{|a|\to 1} |\sigma_a \circ \varphi|_* = 0$ is sufficient for the compactness of C_φ on BMOA. The necessity was earlier observed by Wulan [142].

We further reformulate (L) as a pseudo-hyperbolic mean oscillation condition for the boundary values of the symbol as follows:

$$\frac{1}{|I|^2} \int_I \int_I \rho\big(\varphi(\zeta), \varphi(\xi)\big)^2 \, |d\zeta||d\xi| \to 0 \quad as \quad \left| \frac{1}{|I|} \int_I \varphi(\zeta) |d\zeta| \right| \to 1. \qquad (A)$$

Here ρ denotes the pseudo-hyperbolic metric, $I \subset \mathbb{T}$ is a boundary arc and the integration is with respect to the normalized Lebesgue measure on $\mathbb{T}$.

Collects together some related new results in the VMOA setting. We observe that the analogue of Theorem (3.3.1) holds on VMOA (that is, for symbols $\varphi \in VMOA$), where (L) can be replaced by $\lim_{|a|\to 1} \left\| \sigma_{\varphi(a)} \circ \varphi \circ \sigma_a \right\|_{H^2} = 0$. Moreover, we prove that one may substitute the genuine hyperbolic metric for the pseudo-hyperbolic metric in the VMOA version of condition (A). We then get the unexpected corollary that C_φ is compact on VMOA if and only if it is compact from the Bloch space to VMOA.

We prove that condition (L) alone is enough to characterize the compactness of C_φ on the space BMOA. It is known [129] that (L) is equivalent to Smith's first condition (S1). Since this fact is central to our work, we first briefly recall the argument for the reader's convenience.

If we write $\varphi_a = \sigma_{\varphi(a)} \circ \varphi \circ \sigma_a$, then $\varphi_a(0) = 0$, so that Stanton's change-of-variable formula (see e.g. [83]) gives the identity

$$\|\varphi a\|_{H^2}^2 = 2\int_{\mathbb{D}} N(\varphi_a, w) d\, A(w), \qquad (91)$$

where A is the normalized area measure. By Littlewood's inequality [20] we also have $N(\varphi_a, w) \le -log|w|$. Since this implies that the functions $N(\varphi a,\cdot)$ are uniformly integrable, the implication from (S1) to (L) follows from (91) upon observing that for $|w| \ge \delta$ one has the estimate $N(\varphi_a, w) \le \delta^{-2}|w|^2 N(\varphi_a, w)$. In the converse direction, observe that $|w|^2 N(\varphi_a, w)$ is uniformly small for $|w|$ close to 1 (again by Littlewood's inequality). For $0 > |w| \le 1-\delta$, we apply the submean-value property of $N(\varphi_a,\cdot)$ (see e.g. [83]) on a maximal w-centred disc contained in $\mathbb{D}\backslash\{0\}$, and deduce that $N(\varphi_a, w) \le min(|w|, \delta)^{-2}\int_{\mathbb{D}} N(\varphi_a,\cdot)$ d A. Thus $|w|^2 N(\varphi_a, w) \le 2^{-1}\delta^{-2}\|\varphi_a\|_{H^2}^2$ by (91).

In view of Smith's compactness criterion consisting of the pair (S1) and (S2), our work below reduces to showing that (S2) is actually implied by (S1), or by (L):

We stress that (L) is used here because it is technically very convenient for our arguments and also allows for quite appealing reformulations in terms of the boundary values of φ. In particular, by expressing the H^2 norm as an L^2 norm on $\mathbb{T}$ and performing a change of variable using the automorphism σ_a, we get

$$\left\|\sigma_\varphi(a) \circ \varphi \circ \sigma_a\right\|_{H^2}^2 = \int_{\mathbb{T}} \rho\left(\varphi(\sigma_a(\zeta)), \varphi(a)\right)^2 |d\zeta| = \int_{\mathbb{T}} \rho(\varphi(\zeta), \varphi(a))^2 P_a(\zeta)|d\zeta| \quad (92)$$

where $P_a(\zeta) = (1-|a|^2)/|\zeta - a|^2$ is the Poisson kernel for $a \in \mathbb{D}$ and $\rho(z, w) = |z - w|/|1 - \bar{w}z|$ denotes the pseudo-hyperbolic distance in $\overline{\mathbb{D}}$ (observe that ρ extends to the boundary $\mathbb{T}$ in a natural way if we agree that $\rho(z, z) = 0$ for $z \in \mathbb{T}$). Thus (L) can be seen as a kind of vanishing mean oscillation condition with respect to the pseudo-hyperbolic metric. We will elaborate on this point further.

It is relevant to observe that if φ satisfies condition (L), or equivalently (S1), then one has $|\varphi| < 1$ a.e. on T. This can be checked by a straightforward density point argument. Below and elsewhere in the text we use the following notation for closed arcs of $\mathbb{T}$: when $re^{i\theta} \in \mathbb{D}$ with $0 \le r < 1$, set

$$I(re^{i\theta}) = \{e^{it}: |t-\theta| \le \pi(1-r)\}.$$

Thus $I(re^{i\theta})$ denotes the arc of $\mathbb{T}$ whose midpoint is $e^{i\theta}$ and (normalized) length $|I(re^{i\theta})| = 1 - r$. The proof of Theorem (3.3.3) applies a uniform density estimate for Lebesgue measurable sets on $\mathbb{T}$:

Lemma (3.3.2)[69]: Suppose that $E \subset \mathbb{T}$ is a measurable set with $|E| > 0$. Then there is a measurable set $E' \subset E$ such that $|E'| > 0$ and

$$\frac{|I(r\zeta) \cap E|}{|I(r\zeta)|} \ge \frac{1-\sqrt{1-|E|}}{4}$$

for every $0 \le r < 1$ and $\zeta \in E'$.

Proof Consider a standard dyadic decomposition of $\mathbb{T}$. If ζ is the midpoint of an arc $I \subset \mathbb{T}$, then I contains a dyadic arc J such that $\zeta \in J$ and$|J| \ge |I|/4$. Thus it is enough to verify that $|J \cap E|/|J| \geq 1-\sqrt{1-|E|}$ for all dyadic arcs $J \subset \mathbb{T}$ that contain ζ. Passing to the complement $F = \mathbb{T} \backslash E$, this amounts to finding a subset $E' \subset E$ of positive measure such that

$$M\chi_F(\zeta) \leq \sqrt{|F|} \qquad for \quad \zeta \in E', \tag{93}$$

where $M\chi_F$ is the dyadic maximal function of the characteristic function of F.
It is known that the dyadic maximal function satisfies a weak 1–1 inequality with constant 1 (see e.g. [137], or apply Doob's 1–1 inequality for martingales, see e.g. [122]), whence one has the estimate

$$|\{M\chi_F > \sqrt{|F|} \leq \frac{|F|}{\sqrt{|F|}} = \sqrt{|F|} < 1.$$

This yields the desired result since almost every point $\zeta \in \mathbb{T}$ satisfying (93) belongs to E by the Lebesgue density theorem.
Theorem (3.3.3)[69]: Condition (L) implies (S2) for any analytic map $\varphi: \mathbb{D} \to \mathbb{D}$. Hence $C_\varphi: BMOA \to BMOA$ is compact if and only if (L) holds.
Proof. As a preparatory step we first establish a Möbius-invariant version of condition (L). Let $\varphi_b = \varphi \circ \sigma_b$ for $b \in \mathbb{D}$. Then the following identity can be verified just by inspection and using the self-inverse property of the automorphisms:

$$\sigma_{\varphi_b}(a) \circ \varphi_b \circ \sigma_a = \left[\sigma_\varphi\left(\sigma_b(a)\right) \circ \varphi \circ \sigma_{\sigma_b(a)} \circ \sigma_{\sigma_b(a)}\right] \circ \left[\sigma_{\sigma_b(a)} \circ \sigma_b \circ \sigma_a\right].$$

Note that the composite mapping enclosed in the last brackets is a disc automorphism that fixes the origin, hence a rotation. Therefore

$$\left\|\sigma_{\varphi_b(a)} \circ \varphi_b \circ \sigma_a\right\|_{H^2} = \left\|\sigma_{\varphi(\sigma_b(a))} \circ \varphi \circ \sigma_{\sigma_b(a)}\right\|_{H^2}.$$

Now, in view of (92) and the fact that $Pa(\zeta) \geq \frac{1}{4}|I(a)|^{-1}$ for $\zeta \in I(a)$, condition (L) implies the following: Given $\varepsilon > 0$, there exists $\eta < 1$ such that

$$\frac{1}{|I(a)|}\int_{I(a)} \rho\left(\varphi_b(\zeta), \varphi_b(a)\right)^2 |d\zeta| \leq \varepsilon \tag{94}$$

whenever a and b satisfy $|\varphi_b(a)| \geq \eta$.

For the actual proof of Theorem (3.3.3) we argue by contradiction, assuming that (L) holds but (S2) does not. Since (S2) fails, there are constants $R < 1$ and $c > 0$, points $b_k \in \mathbb{D}$, and numbers $0 < t_k < 1$ with $t_k \to 1$ such that for all $k \geq 1$ we have $|\varphi(b_k)| \leq R$ and the sets

$$E_k = \{\zeta \in \mathbb{T}: \text{the radial limit } \varphi_k(\zeta) \text{ exists and } |\varphi_k(\zeta)| > t_k\}$$

satisfy $|E_k| \geq c$, where $\varphi_k = \varphi \circ \sigma_{b_k}$. By Lemma (3.3.2) we can further find sets $E_k' \subset E_k$ such that $|E_k| > 0$ and

$$\frac{|I(r\zeta) \cap E_k|}{|I(r\zeta)|} \geq \frac{1-\sqrt{1-c}}{4} \quad \text{for } 0 \leq r < 1, \zeta \in E_k'. \tag{95}$$

Let $\varepsilon = (1-\sqrt{1-c})/8 > 0$. We may choose η large enough so that $R < \eta < 1$ and (94) holds for $|\varphi_b(a)| \geq \eta$. Fix k such that $t_k \geq \eta$. Recall that by the definition of E_k we have $|\varphi_k(r\zeta)| \to |\varphi_k(\zeta)| > t_k$ as $r \to 1$ for each $\zeta \in E_k$. In particular, we can fix a point $\zeta_k \in E_k'$ with this property. Moreover, since $|\varphi_k(0)| = |\varphi(b_k)| \leq R$, it follows from continuity that there is a radius $0 < r_k < 1$ such that $|\varphi_k(r_k\zeta_k)| = \eta$. Let $a_k = r_k\zeta_k$. By elementary geometry it holds for each $\zeta \in E_k$ that $\rho\left(\varphi_k(\zeta), \varphi_k(a_k)\right) \geq \rho(t_k, \eta)$. Hence we can use (95) to obtain the estimate

$$\frac{1}{|I(a_k)|}\int_{I(a_k)}\rho\big(\varphi_k(\zeta),\varphi_k(a_k)\big)^2|\,d\zeta| \geq \frac{|I(a_k)\cap E_k|}{|I(a_k)|}\rho(t_k,\eta)^2 \geq 2\varepsilon\rho(t_k,\eta)^2.$$

Since this estimate holds for all sufficiently large k, we may let $k\to\infty$. In this case $\rho(t_k,\eta)\to 1$, which leads to a contradiction with (94) by the choice of ε.

We close by addressing a question recently posed by Wulan, Zheng and Zhu [143]. Based on an earlier work by Wulan [142], they showed that the single condition

$$\lim_{n\to\infty}|\varphi^n|_* = 0 \qquad (W1)$$

characterizes the compactness of C_φ on BMOA. The earlier result in [142] involved the additional condition

$$\lim_{|a|\to 1}|\sigma_a\circ\varphi|_* = 0, \qquad (W2)$$

and consequently it was asked in [143] whether (W2) alone would suffice to characterize when C_φ is compact on BMOA. This is indeed the case.

Corollary (3.3.4)[69]: Let $\varphi\colon\mathbb{D}\to\mathbb{D}$ be an analytic map. Then C_φ is compact on BMOA if and only if (W2) holds.

Proof: It is enough to observe that $\left|\sigma_{\varphi(a)}\circ\varphi\right|_* \geq \left\|\sigma_{\varphi(a)}\circ\varphi\circ\sigma_a\right\|_{H^2}$, whence (W2) trivially implies (L).

After the work of the preceding the only step that remains to be proved in Theorem (3.3.1) is that (ii) implies (iv). Equivalently, if the map φ fails to satisfy condition (L), then we must show that the composition operator C_φ is not weakly compact on BMOA. This will be accomplished separately in Proposition (3.3.6) below.

Our argument depends on the following result which is due to Leˇıbov [130] for VMO($\mathbb{T}$) and independently to Müller and Schechtman [134] for the linearly isomorphic setting of dyadic martingale VMO. As usual, here c_0 denotes the Banach space of complex sequences converging to zero endowed with the supremum norm $\|\cdot\|_\infty$. We sketch a self-contained argument of the formulation required here.

Proposition (3.3.5)[69]: Let (f_n) be a sequence in VMOA such that $\|f_n\|_* = 1$ for all n and $\|f_n\|_{H^2}\to 0$ as $n\to\infty$. Then there exists a subsequence (f_{nk}) which is equivalent to the natural basis of c_0; that is, for which the map $(\lambda_k)\longmapsto\sum_k\lambda_k f_{n_k}$ is an isomorphism from c_0 into VMOA.

Proof: For brevity write $\gamma(f,a)=\|f\circ\sigma_a - f(a)\|_{H^2}$ whenever $f\in H^2$ and $a\in\mathbb{D}$. Note that $\gamma(f,a)$ defines a seminorm with respect to f for each a. We also have $\gamma(f,a)\leq\|f\circ\sigma_a\|_{H^2}\leq c_a\,\|f\|_{H^2}$ for some $c_a>0$, where c_a is an increasing function of $|a|$. Therefore

$$\sup\{\gamma\,(f_n,a)\colon|a|\leq r\}\ \to\ 0 \quad as\ \ n\to\infty$$

for any $0<r<1$. On the other hand, the VMOA condition says that $\gamma(f_n,a)\to 0$ as $|a|\to 1$ for each n. Proceeding inductively, one may use these properties of (f_n) to find increasing sequences of integers $n_k\geq 1$ and numbers $0<r_k<1$ such that for each $k\geq 1$ one has $\left\|f_{n_k}\right\|_{H^2}>2^{-k-1}$ and

$$\sup_{|a|\leq r_k}\gamma\big(f_{n_k},a\big)>2^{-k-1},\ \sup_{|a|\geq r_{k+1}}\gamma\big(f_{n_k},a\big)>2^{-k-1}.$$

Then every $a\in\mathbb{D}$ satisfies $\gamma\big(f_{n_k},a\big)<2^{-k-1}$ for all except possibly one index k, for which $\gamma\big(f_{n_k},a\big)\leq 1$. Also, for each $k\geq 1$ there is $a\in\mathbb{D}$ such that $\gamma\big(f_{n_k},a\big)\geq\frac{3}{4}$ since

$\|f_{n_k}\|_* = 1$ and $|f_{n_k}(0)| \le \|f_{n_k}\|_{H^2} \le \frac{1}{4}$. It is then fairly straightforward to verify that the sequence (f_{n_k}) is equivalent in VMOA to the natural basis of c_0.

The following proposition completes the proof of Theorem (3.3.1).

Proposition (3.3.6)[69]: Let $\varphi: \mathbb{D} \to \mathbb{D}$ be an analytic map and suppose that condition (L) fails. Then the composition operator $C_\varphi: BMOA \to BMOA$ fixes a copy of c_0 and therefore it is not weakly compact.

Proof: Since (L) fails to hold, we can find points $a_n \in \mathbb{D}$ such that $|\varphi(a_n)| \to 1$ and

$$\|\sigma_{\varphi(a_n)} \circ \varphi \circ \sigma_{a_n}\|_{H^2} \ge c$$

for some $c > 0$. Put $f_n = \sigma_{\varphi(a_n)} - \varphi(a_n)$. Then $f_n(0) = 0$ and, for each $a \in \mathbb{D}$,

$$\|f_n \circ \sigma_a - f_n(a)\|_{H^2} = \|\sigma_{\varphi(a_n)} \circ \sigma_a - \sigma_{\varphi(a_n)}(a)\|_{H^2} = \sqrt{1 - |\sigma_{\varphi(a_n)}(a)|^2}.$$

The last equality can be seen by using the fact that $\sigma_{\varphi(a_n)} \circ \sigma_a$ is an inner function. Now it follows easily that $f_n \in VMOA$ and $\|f_n\|_* = 1$ for each n. By taking $a = 0$ we obtain that $\|f_n\|_{H^2} \to 0$ as $n \to \infty$. Moreover,

$$\|C_\varphi f_n\|_* \ge \|f_n \circ \varphi \circ \sigma_{a_n} - f_n(\varphi(a_n))\|_{H^2} = \|\sigma_{\varphi(a_n)} \circ \varphi \circ \sigma_{a_n}\|_{H^2} \ge c.$$

According to Proposition (3.3.5) there is a subsequence (f_{n_k}) which is equivalent in VMOA to the natural basis of c_0. In particular, $(C_\varphi f_{n_k})$ is a weak-null sequence in BMOA. By applying the Bessaga-Pełczynski selection principle (see e.g. [88]) to $(C_\varphi f_{n_k})$ we can pass to a further subsequence, still denoted (f_{n_k}), such that $(C_\varphi f_{n_k})$ is a seminormalized basic sequence in BMOA. It follows that there are constants $A, B > 0$ so that

$$A \cdot \|\lambda\|_\infty \le \left\|\sum_k \lambda_k C_\varphi f_{n_k}\right\|_* \le \|C_\varphi\| \cdot \left\|\sum_k \lambda_k f_{n_k}\right\|_* \le B \cdot \|C_\varphi\| \|\lambda\|_\infty$$

holds for any sequence $\lambda = (\lambda_k) \in c_0$. (To find A just apply the biorthogonal basis functionals to $\sum_k \lambda_k C_\varphi f_{n_k}$.) These estimates state that the restriction of C_φ to the closed subspace of BMOA spanned by the sequence (f_{n_k}) is an isomorphism on a linearly isomorphic copy of c_0, and we are done.

We examine the function-theoretic meaning of condition (L) by revisiting the point of view that we already touched upon. That is, (L) can be thought of as a kind of pseudo-hyperbolic vanishing mean oscillation condition for the boundary values of φ over certain arcs in $\mathbb{T}$; see Proposition (3.3.8) below.

When $\varphi: \mathbb{D} \to \mathbb{D}$ is an analytic map and I is an arc of $\mathbb{T}$, denote

$$\varphi_I = \frac{1}{|I|}\int_I \varphi = \frac{1}{|I|}\int_I \varphi(\zeta)|d\zeta|$$

for the integral average of φ over I. Here and elsewhere all integrals over subsets of $\mathbb{T}$ are calculated with respect to the normalized Lebesgue arc-length measure. Also recall from that $I(re^{i\theta}) = \{e^{it}: |t - \theta| \le \pi(1 - r)\}$ for $re^{i\theta} \in \mathbb{D}$.

Lemma (3.3.7)[69]: For $a \in \mathbb{D}$ we have $|\varphi(a)| \to 1$ if and only if $|\varphi_I(a)| \to 1$.

Proof: The left-to-right implication is easy to prove. In fact, assuming that $\varphi(a) \ge 0$ (as we may, after applying a rotation), we get by using (96) that

$$1 - |\varphi(a)| = \int_{\mathbb{T}} (1 - Re\, \varphi) P_a \ge \frac{1}{4|I(a)|}\int_{I(a)} (1 - Re\, \varphi) \ge \frac{1}{4}\left(1 - |\varphi_{I(a)}|\right).$$

This clearly shows that $|\varphi(a)| \to 1$ implies $\left|\varphi_{I(a)}\right| \to 1$.

For the reverse implication, we may assume that $\varphi_{I(a)} \geq 1-\delta$ for some $0<\delta<\frac{1}{2}$. Let $E=\{\zeta \in I(a): Re\,\varphi(\zeta) \geq 1-2\delta\}$. Since $Re\varphi \leq 1$, we must have $|E| \geq \frac{1}{2}|I(a)|$. Consider the positive harmonic function $u=\log(2/|1-\varphi|)$. It is geometrically obvious that $|1-\varphi| \leq c\sqrt{\delta}$ on E for some constant $c>0$. Hence

$$u(a) \geq \int_{\mathbb{T}} u\, P_a \geq \left(\log \frac{2}{c\sqrt{\delta}}\right) \int_E P_a \geq \frac{1}{8}\left(\log \frac{2}{c\sqrt{\delta}}\right).$$

Since $|1-\varphi(a)|=2e^{-u(a)}$, we deduce from this estimate that $1-|\varphi(a)| \leq |1-\varphi(a)| \to 0$ as $\delta \to 0$.

Proposition (3.3.8)[69]: For any analytic map $\varphi: \mathbb{D} \to \mathbb{D}$ condition (L) is equivalent to the following:

$$\frac{1}{|I|^2} \int_I \int_I \rho(\varphi(\zeta), \varphi(\xi))^2 |d\zeta||d\xi| \to 0 \quad as \quad |\varphi_I| \to 1, \qquad (A)$$

where $I \subset \mathbb{T}$ are arcs.

In the proof we will make use of the following simple estimates for the Poisson kernel: for every $a \in \mathbb{D}$,

$$\frac{1}{4|I(a)|} \leq P_a(\zeta) \leq \frac{2}{|I(a)|}, \qquad \zeta \in I(a). \qquad (96)$$

For convenience we first isolate a technical step towards Proposition (3.3.8).

Proof: We start by proving the necessity of (A). By the preceding lemma $|\varphi_I| \to 1$ implies that $|\varphi(a_I)| \to 1$. Hence (92) and the left-hand side of (96) yield

$$\frac{1}{|I|} \int_{\mathbb{T}} \rho(\varphi(\zeta), \varphi(a_I))^2 |d\zeta| \to 0 \quad as \quad |\varphi_I| \to 1, \qquad (A')$$

where $I \subset \mathbb{T}$ is an arc and $a_I \in \mathbb{D}$ is the unique point for which $I=I(a_I)$. Then (A) is obtained from (A') by a simple application of the triangle inequality $\rho(\varphi(\zeta), \varphi(\xi)) \leq \rho(\varphi(\zeta), \varphi(aI))+\rho(\varphi(\xi), \varphi(aI))$.

To prove the sufficiency of (A) we will show that

$$J(a)=\int_{\mathbb{T}} \int_{\mathbb{T}} \rho(\varphi(\zeta), \varphi(\xi))^2\, P_a(\zeta) P_a(\xi)|d\zeta||d\xi| \to 0 \quad as \quad |\varphi(a)| \to 1. \quad (97)$$

In view of (92) this actually implies (L), because the function $w \mapsto \rho(z, w)^2$ is subharmonic in $\mathbb{D}$ and therefore $\int_{\mathbb{T}} \rho(z, \varphi(\xi))^2 P_a(\xi)|d\xi| \geq \rho(z, \varphi(a))^2$ for every $z \in \overline{\mathbb{D}}$.

Let $\varepsilon>0$. For each $a \in \mathbb{D}$ we can choose a point aon the line segment between 0 and a such that $\int_{I(a')} P_a \geq 1-\varepsilon$ and $1-|a'| \leq c_\varepsilon(1-|a|)$ for some constant $c_\varepsilon>0$. For real a close to 1 this can be seen by integration the estimate $P_a(e^{it}) \geq (1-a^2)/[(1-a)^2+t^2]$ over an interval $|t| \leq c(1-a)$ and letting $c \to \infty$. Thus $\int_{\mathbb{T} \backslash I(a')} P_a \leq \varepsilon$, and since $\rho \leq 1$, we can estimate

$$\begin{aligned} J(a) &\leq 2\varepsilon+\int_{I(a')} \int_{I(a')} \rho(\varphi(\zeta), \varphi(\xi))^2 P_a(\zeta) P_a(\xi)|d\zeta||d\xi| \\ &\leq 2\varepsilon+\frac{4c_\varepsilon^2}{|I(a')|^2} \int_{I(a')} \int_{I(a')} \rho(\varphi(\zeta), \varphi(\xi))^2 |d\zeta||d\xi| \end{aligned}$$

by using the right-hand side of (96) in the last step. According to the Schwarz-Pick inequality we have $\rho(\varphi(a), \varphi(a')) \leq \rho(a, a') \leq c_\varepsilon'$ for some $c_\varepsilon'<1$ due to the fact that

$1-|a| \le c_\varepsilon(1-|a|)$. Thus $|\varphi(a)| \to 1$ implies that $|\varphi(a)| \to 1$, which, in turn, yields $|\varphi_{I(a')}| \to 1$ by Lemma (3.3.7). By applying (A) to the arcs $I(a')$ we then deduce from the above estimate that $\limsup J(a) \le 2\varepsilon$ as $|\varphi(a)| \to 1$. Since $\varepsilon > 0$ was arbitrary, this proves (97).

We summarize the principal function-theoretic compactness criteria for C_φ on BMOA in the following theorem. Criteria of a different nature are given in [123] and [141].

Theorem (3.3.9)[69]: Compactness and weak compactness of C_φ: BMOA $\to$ BMOA are equivalent to each of the conditions (S1), (L), (W1), (W2), (A) and (A').

We discuss the case where $\varphi \in$ VMOA. Here simplified compactness criteria are available and new phenomena occur. Recall first that if $\varphi \in$ VMOA then C_φ takes VMOA into itself and C_φ: BMOA $\to$ BMOA can be identified with the biadjoint of its restriction to VMOA; see [125].

Let τ denote the hyperbolic metric in the unit disc, that is,

$$\tau(z,w) = \frac{1}{2}\log\frac{1+\rho(z,w)}{1-\rho(z,w)},$$

where $\rho(z,w)$ is the pseudo-hyperbolic distance between z and w (see e.g. [94] or [23]). Contrary to the pseudo-hyperbolic metric, τ is unbounded in $\mathbb{D}$ and it is appropriate to define $\tau(z,w) = \infty$ if z and w are distinct points (at least) one of which lies on the boundary.

We collect the main new results in the case of VMOA as follows.

The Bloch space B consists of the analytic functions $f: \mathbb{D} \to \mathbb{C}$ for which $\sup_{z\in\mathbb{D}} |f\ (z)|(1-|z|^2) < \infty$. Then B becomes a Banach space equipped with the norm $|f(0)| + \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2)$. Composition operators C_φ acting from B into VMOA or BMOA have been studied in e.g. [124], [131], [133], [136], [139], [144]. As observed by Makhmutov and Tjani [133], it follows from the results of Choe, Ramey and Ullrich [124] combined with [145] that C_φ is bounded from B into BMOA if and only if (99) holds. In addition, it was proved in [133] that C_φ is compact from B into VMOA if and only if (iv') holds. Therefore Theorem (3.3.12) has the following surprising consequence.

Corollary (3.3.10)[69]: Let $\varphi: \mathbb{D} \to \mathbb{D}$ be an analytic map with $\varphi \in VMOA$. Then C_φ is compact $VMOA \to VMOA$ if and only if it is compact B $\to VMOA$.

This result was known earlier in the special case of boundedly valent symbols φ whose image $\varphi(\mathbb{D})$ is contained in a polygon inscribed in $\overline{\mathbb{D}}$; see [133]. Of course, in Corollary (3.3.10) the implication from right to left follows from the fact that VMOA is continuously embedded in B. Furthermore, it is relevant to note that C_φ is bounded B $\to VMOA$ if and only if it is compact B $\to VMOA$; see [136].

Towards the proof of Theorem (3.3.1) we make some preliminary remarks. It was already observed by the first author [129] that condition (iii) alone characterizes the compactness of $C_\varphi: VMOA \to VMOA$. At first sight (iii) might seem stronger than (L) because $|\varphi(a)| \to 1$ always implies $|a| \to 1$ by the Schwarz lemma. We include a direct function-theoretic argument proving the equivalence of these two conditions for symbols $\varphi \in$ VMOA.

Lemma (3.3.11)[69]: Let $\varphi: \mathbb{D} \to \mathbb{D}$ be an analytic map. Then condition (iii) of Theorem (3.3.1) holds if and only if $\varphi \in VMOA$ and (L) holds.

Proof: Let $\varphi_a = \sigma_{\varphi(a)} \circ \varphi \circ \sigma_a$. By the self-inverse property of $\sigma_{\varphi(a)}$ we may write $\varphi \circ \sigma_a = \sigma_{\varphi(a)} \circ \varphi_a$, from which it follows that

$$|(\varphi \circ \sigma_a)(z) - \varphi(a)| = \frac{1 - |\varphi(a)|^2}{\left|1 - \overline{\varphi(a)}\varphi_a(z)\right|} |\varphi_a(z)|. \tag{98}$$

This yields $\|\varphi \circ \sigma_a - \varphi(a)\|_{H^2} \leq 2\|\varphi_a\|_{H^2}$. Hence (iii) implies that $\varphi \in VMOA$.

Conversely note that if (L) holds but (iii) fails, then there exists a sequence (a_n) such that $|a_n| \to 1$ while $|\varphi(a_n)| \leq r < 1$ and $\|\varphi_{a_n}\|_{H^2} \geq c > 0$ for all n. Then (98) implies that $\|\varphi \circ \sigma_{a_n} - \varphi(a_n)\|_{H^2} \geq (1 - r)\|\varphi_{a_n}\|_{H^2} \geq (1 - r)c$, whence $\varphi \notin VMOA$. This proves the lemma.

Theorem (3.3.12)[69]: Let $\varphi: \mathbb{D} \to \mathbb{D}$ be an analytic map such that $\varphi \in$ VMOA. Then the following conditions are equivalent:

(i) C_φ: VMOA → VMOA is compact.

(ii) C_φ: VMOA → VMOA is weakly compact.

(iii) $\lim_{|a|\to 1} \|\sigma_{\varphi(a)} \circ \varphi \circ \sigma_a\|_{H^2} = 0.$

(iv) $\lim_{|a|\to 1} \int_{\mathbb{T}} \rho\left(\varphi(\sigma_a(\zeta)), \varphi(a)\right)^2 |d\zeta| = 0.$

(v) $\lim_{|I|\to 0} \frac{1}{|I|^2} \int_I \int_I \rho(\varphi(\zeta), \varphi(\xi))^2 |d\zeta||d\xi| = 0$,where $I \subset T$ are arcs.

Further, (iv) and (v) are equivalent to the following conditions involving the hyperbolic metric:

(iv') $\lim_{|a|\to 1} \int_{\mathbb{T}} \tau\left(\varphi(\sigma_a(\zeta)), \varphi(a)\right) |d\zeta| = 0.$

(iiv') $\lim_{|I|\to 0} \frac{1}{|I|^2} \int_I \int_I \tau(\varphi(\zeta), \varphi(\xi))|d\zeta||d\xi| = 0$, where $I \subset \mathbb{T}$ are arcs.

The main novelty of Theorem (3.3.12), as compared to Theorem (3.3.1), lies in conditions (iv') and (v'), which relate to vanishing mean oscillation with respect to the genuine hyperbolic metric. This also ties to earlier research on composition operators from the Bloch space to VMOA. Before embarking on the proof of Theorem (3.3.12) we discuss the interpretation of (iv') from the literature and draw some consequences.

First note that if the integral $\int_{\mathbb{T}} \tau\left(\varphi(\sigma_a(\zeta)), \varphi(a)\right) |d\zeta|$ is finite for some $a \in \mathbb{D}$, then $|\varphi| < 1$ a.e. on T. Moreover, the integral stays bounded as a varies on a compact subset of $\mathbb{D}$. Hence (iv') implies

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{T}} \tau\left(\varphi(\sigma_a(\zeta)), \varphi(a)\right) |d\zeta| < \infty, \tag{99}$$

saying that φ belongs to the hyperbolic BMOA class introduced by Yamashita [145]. Actually the fact that (iv) implies the finiteness of the integral in (99) for some $a \in \mathbb{D}$ is already non-trivial.

Proof. Recall that the operator $C_\varphi: BMOA \to BMOA$ is the biadjoint of the restriction $C_\varphi: VMOA \to VMOA$, since here $\varphi \in VMOA$. Hence, according to Theorem (3.3.1), conditions (i) and (ii) are both equivalent to (L). On the other hand, in this case (L) and (iii) are equivalent by Lemma (3.3.11). We refer below for an approach to the equivalences between conditions (i)–(iii) which does not depend.

Conditions (iii) and (iv) are restatements of each other according to (92). The equivalence of (iii) and (v) is proved in the same way as Proposition (3.3.8); instead of invoking Lemma (3.3.7) we just observe that for points $a \in \mathbb{D}$ one has $|a| \to 1$ if and only if $|I(a)| \to 0$.

Since $\tau \geq c\rho^2$ for a suitable $c > 0$, it is obvious that (v') implies (v). Moreover, (v') can be deduced from (iv') by making a change of variable, using the lower estimate from (96) for the Poisson kernel and applying the triangle inequality as in the first part of the proof of Proposition (3.3.8). The crucial remaining step in the proof of Theorem (3.3.12) consists of verifying the implication that the pseudo-hyperbolic condition (iv) implies the hyperbolic condition (iv'). We isolate this more technical result below, which then completes the proof of the theorem.

The argument will employ ideas of Wik [140] related to his elementary approach to the John-Nirenberg inequality for BMO functions. We will require the following one-dimensional special case of [140]:

Lemma (3.3.13)[69]: Suppose that $0 < \lambda < 1$ and $E \subset [0,1]$ is any measurable set having Lebesgue measure $|E| \leq \lambda$. Then there is a sequence $Q_1, Q_2, \ldots$ of closed dyadic intervals of $[0,1]$, having pairwise disjoint interiors, such that $\frac{1}{2}\lambda|Q_k| \leq |Q_k \cap E| \leq \lambda|Q_k|$ for $k \geq 1$ and $|E \setminus \cup_k Q_k| = 0$.

Proposition (3.3.14)[69]: Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be an analytic map. Then condition (iv) implies condition (iv') in Theorem (3.3.1).

Proof. Assuming that condition (iv) (and equivalently also (v)) holds, we split the proof into two steps. As the first step we show:

Claim (3.3.15)[69]:

$$\lim_{|a|\to 1} \frac{1}{|I(a)|} \int_{I(a)} \tau\big(\varphi(\zeta), \varphi(a)\big)|d\zeta| = 0.$$

To begin recall from that condition (iv) implies that $|\varphi| < 1$ a.e. on $\mathbb{T}$ (this fact can alternatively be deduced by observing that (i) implies the compactness of C_φ on H^2 by [123]). Towards the proof of Claim (3.3.15) we first deduce from (iv) by a change of variable and (96) that

$$\lim_{|a|\to 1} \frac{1}{|I(a)|} \int_{I(a)} \rho\left(\varphi(\zeta), \varphi(a)\right)^2 |d\zeta| = 0, \tag{100}$$

where $I(a) = \{e^{it}: |t - \theta| \leq \pi(1 - r)\}$ is the subarc of $\mathbb{T}$ associated to $a = re^{i\theta} \in \mathbb{D}$. Hence we may pick $\delta > 0$ small enough so that

$$\frac{1}{|I(a)|} \int_{I(a)} \rho\big(\varphi(\zeta), \varphi(a)\big)^2 |d\zeta| < \frac{1}{4} \tag{101}$$

whenever $a \in \mathbb{D}$ satisfies $|a| > 1 - \delta$.

Let $\varepsilon \in (0, 1/32)$. According to (v) we may decrease $\delta > 0$, if necessary, to ensure that for all $a \in \mathbb{D}$ with $|a| > 1 - \delta$ we also have

$$\frac{1}{|I(a)|^2} \int_{I(a)} \int_{I(a)} \rho\big(\varphi(\zeta), \varphi(\xi)\big)^2 |d\zeta||d\xi| < \varepsilon. \tag{102}$$

Fix such a point a and put

$$C_k = \left\{\zeta \in I(a): \tau\left(\varphi(\zeta), \varphi(a)\right) \geq k\right\}, \qquad k = 0, 1, 2, \ldots,$$

whence $I(a) = C_0 \supset C_1 \supset C_2 \supset \cdots$. Observe that if $\zeta \in C_1$, then the definition of the hyperbolic metric yields $\rho(\varphi(\zeta), \varphi(a)) \geq \beta$, where $\beta = \frac{e^2-1}{e^2+1} > 1/\sqrt{2}$. One gets from (101) that

$$\frac{\beta^2|C_1|}{|I(a)|} \le \frac{1}{|I(a)|}\int_{I(a)} \rho\big(\varphi(\zeta),\varphi(a)\big)^2 |d\zeta| > \frac{1}{4},$$

Whence $|C_1| \le \frac{1}{2}|I(a)|$.

Let $k \ge 1$ be fixed. Then we may apply Lemma (3.3.13) to the set C_k relative to the arc $I(a)$ with $\lambda = \frac{1}{2}$, which gives a sequence $J_1, J_2, \ldots$ of subarcs of $I(a)$ with disjoint interiors such that for each $\ell \ge 1$

$$|C_k \cap J_\ell| \ge \frac{1}{4}|J_\ell|, \qquad |C_k^c \cap J_\ell| \ge \frac{1}{2}|J_\ell| \tag{103}$$

and

$$C_k \bigcup_{\ell=1}^{\infty} J_\ell = 0. \tag{104}$$

Observe next that if $\zeta \in C_k^c$ and $\xi \in C_{k+1}$, then $\tau\big(\varphi(\zeta),\varphi(\xi)\big) \ge \tau\,(\varphi(\xi),\varphi(a)) - \tau\,\big(\varphi(\zeta),\varphi(a)\big) \ge 1$, so that $\rho\big(\varphi(\zeta),\varphi(\xi)\big)^2 \ge \beta^2 > 0$. Consequently we get from (102), (103) and the assumption on a that

$$\varepsilon\ > \frac{1}{|J_\ell|^2}\int_{J_\ell}\int_{J_\ell} \rho\big(\varphi(\zeta),\varphi(\xi)\big)^2 |d\zeta||d\xi| \ge \beta^2 \frac{|C_k^c \cap J_\ell|}{|J|}\cdot\frac{|C_{k+1} \cap\ J_\ell|}{|J_\ell|} \ge \frac{1}{4}\frac{|C_{k+1} \cap J_\ell|}{|J_\ell|}$$

Thus $|C_{k+1} \cap\ J_\ell| \le 4\varepsilon|J_\ell|$ for $\ell \ge 1$. We sum this inequality over ℓ and employ (103) and (104) together with the essential disjointness of the subarcs J_ℓ to obtain

$$|C_{k+1}| = \sum_{\ell=1}^{\infty}|C_{k+1} \cap J_\ell| \le 4\varepsilon\sum_{\ell=1}^{\infty}|J_\ell| \le 16\varepsilon\sum_{\ell=1}^{\infty}|C_k \cap J_\ell| = 16\varepsilon|C_k|. \tag{105}$$

In particular, since $\varepsilon > 1/32$, we get by induction that $|C_k| \le 2^{2-k}|C_2|$ for $k \ge 2$. Note that $k \le \tau\,\big(\varphi(\zeta),\varphi(a)\big) > k+1$ whenever $\zeta \in C_k \backslash C_{k+1}$ and $k \ge 0$. Employing the short-hand notation $\{\tau > 2\}$ for the set $\big\{\zeta \in I(a){:}\,\tau\big(\varphi(\zeta),\varphi(a)\big) > 2\big\} = C_0\backslash C_2$ we thus get that

$$\int_{I(a)} \tau\big(\varphi(\zeta),\varphi(a)\big)|d\zeta| = \int_{\{\tau<2\}}\big(\phi(\zeta),\phi(a)\big)|d\zeta| + \sum_{k=2}^{\infty}\int_{C_k\backslash C_{k+1}} \tau\big(\phi(\zeta\ ),\phi(a)\big)|d\zeta|$$
$$\le \int_{\{\tau<2\}} \tau\big(\phi(\zeta),\phi(a)\big)|d\zeta|\ + \sum_{k=2}^{\infty}(k+1)|C_k|.$$

After division by $|I(a)|$ the last term is less than $|C_2||I(a)|^{-1}\sum_{k=2}^{\infty}(k+1)2^{2-k} \le 128\varepsilon$, which tends to 0 as $\varepsilon \to 0$. On the other hand, in the set $\{\tau < 2\}$

we have $\tau(\varphi(\zeta),\varphi(a)) \le c\rho\big(\varphi(\zeta),\varphi(a)\big)^2$ with a universal constant $c > 0$, so that also

$$\lim_{|a|\to 1}\frac{1}{|I(a)|}\int_{\{\tau<2\}} \tau\big(\varphi(\zeta),\varphi(a)\big)|d\zeta| = 0$$

in view of (100). This finishes the proof of Claim (3.3.15).

As the final step we show that the condition of Claim (3.3.15) implies the desired hyperbolic condition (iv’) of Theorem (3.3.12). The required argument is quite standard but more technical than the analogous fact for the pseudo-hyperbolic distance ρ because the hyperbolic distance τ is unbounded. We omit some computational details.

Claim (3.3.16)[69]:

$$\int_{\mathbb{T}} \tau\left(\varphi(\sigma_a(\zeta)),\varphi(a)\right)|d\zeta| = \int_{\mathbb{T}} \tau(\varphi(\zeta),\varphi(a))P_a(\zeta)|d\zeta| \to 0, |a| \to 1.$$

For the proof we assume that $a \in \mathbb{D}$ satisfies $2^{-N} \le 1 - |a| > 2^{1-N}$ for some $N \ge 1$, and then let $N \to \infty$ in our estimates. Define for $k = 1,\ldots,N$ the radii r_k , points $a_k \in \mathbb{D}$ and arcs I_k through $1 - r_k = 2^{N-k}(1 - |a|)$, $a_k = r_k a/|a|$ and $I_k = I(a_k)$. Set also $a_0 = 0$ and $I_0 = \mathbb{T}$. Then $a = a_N$ and $I(a) = I_N \subset I_{N-1} \subset\ldots\subset I_0 = \mathbb{T}$. Moreover, $2^{-k} \le |I_k| > 2^{1-k}$. Observe that if $1 \le k < N$ and $\zeta \in I_k\setminus I_{k+1}$, then elementary trigonometry yields $|\zeta - a| \ge \frac{1}{2}|I_{k+1}| \ge 2^{-k-2}$. Hence the Poisson kernel satisfies $P_a(\zeta) \lesssim 2^{2k-N}$ for all $\zeta \in I_k\setminus I_{k+1}$, where $\lesssim$ indicates that the left-hand side is bounded above by a constant multiple of the right-hand side, the constant being independent of N and k. Consequently we may estimate the second integral appearing in Claim (3.3.16) as follows:

$$\int_{\mathbb{T}} \tau(\varphi(\zeta),\varphi(a))P_a(\zeta)|d\zeta| \sum_{k=0}^{N-1} 2^{2k-N}\int_{I_k\setminus I_{k+1}} \tau(\varphi(\zeta),\varphi(a))|d\zeta|$$

$$+ 2^N \int_{I(a)} \tau(\varphi(\zeta),\varphi(a))|d\zeta| \lesssim \sum_{k=0}^{N} \frac{2^{k-N}}{|I_k|}\int_{I_k} \tau(\varphi(\zeta),\varphi(a))|d\zeta|$$

$$\le \sum_{k=0}^{N} \frac{2^{k-N}}{|I_k|}\int_{I_k} \tau(\varphi(\zeta),\varphi(a_k))|d\zeta| + \sum_{k=0}^{N-1} 2^{k-N}\tau\left(\varphi(a_k),\varphi(a)\right) \equiv A_N + B_N$$

It will suffice to verify that the condition of Claim (3.3.15) implies that the terms A_N and B_N both tend to zero as $N \to \infty$. First of all (observe that now (99) holds),

$$AN \lesssim \left(\sum_{k=0}^{\left[\frac{N}{2}\right]} 2^{k-N} + \sum_{k=\left[\frac{N}{2}\right]+1}^{N} 2^{k-N}\right)\frac{1}{|I_k|}\int_{I_k} \tau(\varphi(\zeta),\varphi(a_k))|d\zeta|$$

$$\lesssim N \cdot 2^{-\frac{N}{2}} + \sup_{k>\left[\frac{N}{2}\right]} \frac{1}{|I_k|}\int_{I_k} \tau(\varphi(\zeta),\varphi(a_k))|d\zeta|.$$

Above the first term tends to zero trivially, and the second term by Claim (3.3.15),as $N \to \infty$.

In order to relate the term B_N to the averages in Claim (3.3.15) we introduce the shorthand $b_k = |I_k|^{-1}\int_{I_k} \tau(\varphi(\zeta),\varphi(a_k))|d\zeta|$. Let $1 \le k \le N$. By averaging over the arc I_k we get from the triangle inequality for τ that

$$\tau(\varphi(a_{k-1}),\varphi(a_k)) \le \frac{1}{|I_k|}\int_{I_k} \tau(\varphi(\zeta),\varphi(a_{k-1}))|d\zeta|$$

$$+\frac{1}{|I_k|}\int_{I_k} \tau(\varphi(\zeta),\varphi(a_k))|d\zeta| \le 2b_{k-1} + b_k,$$

Since $|I_{k-1}| \le 2|I_k|$. Because $= a_N$, we deduce that

$$\tau(\varphi(a_k),\varphi(a)) \lesssim \sum_{j=k}^{N} b_j \le (N - k + 1)\max_{k\le j\le N} b_j.$$

Put $E_k = \max_{k\le j\le N} b_j$, so that by combining the above estimates one has

$$B_N \lesssim \sum_{k=0}^{N-1} (N-k+1)2^{k-N} E_k,$$

where the E_k 's have a uniform upper bound (independent of a) and $E_{\left[\frac{N}{2}\right]} \to 0$ as $N \to \infty$. By splitting the preceding sum as before at the level $[N/2]$ we deduce that $B_N \to 0$ as $N \to \infty$. This completes the proof of Claim 2, and hence of Proposition (3.3.14).

Chapter 4
Compact Composition Operators and Atomic Decomposition

We characterize boundedness, closedness of the range and compactness for composition operators acting on μ-Bloch spaces, where μ is a positive continuous function defined on the interval $0 < t \leq 1$, that satisfy certain holomorphic extension properties. At the same time, we give the briefly sufficient and necessary condition that C_ϕ is compact on β_μ for $a > 1$.

Section (4.1): μ-Bloch Type Spaces

For $\mathbb{D}$ denote the unit disk in the complex plane $\mathbb{C}$ and let $H(\mathbb{D})$ be the space of all holomorphic functions on $\mathbb{D}$ with the topology of uniform convergence on compact subsets of $\mathbb{D}$. The Bloch space B consists of all functions $f \in H(\mathbb{D})$ for which

$$\|f\|_B := \sup_{z\in\mathbb{D}}(1-|z|^2) \ \ \|f'(z)\| < \infty.$$

B becomes a Banach space when it is equipped with the norm $\|f\| := |f\ (0)| + \|f\|_B$ (see, e. g., [147]).

For $\alpha > 0$, the α- Bloch space, denoted as B^α, consists of all holomorphic functions f on $\mathbb{D}$ such that

$$\|f\|_\alpha := \sup_{z\in\mathbb{D}}(1-|z|^2) \ \ \|f'(z)\| < \infty.$$

α-Bloch spaces have been introduced and studied by numerous authors. For the general theory of α- Bloch functions see [159]. Many have studied different classes of Bloch type spaces, where the typical weight function, $w(z) = 1 - |z|^2$ ($z \in \mathbb{D}$), is replaced by a continuous positive function μ defined on the interval $0 < t \leq 1$. A function $f \in H(\mathbb{D})$ is called a μ-Bloch function, denoted as $f \in B^\mu$, if

$$\|f\|_\mu := \sup_{z\in\mathbb{D}}(1-|z|^2) \ \ \|f'(z)\| < \infty.$$

Clearly, if $\mu(t) = t^\alpha$ with $\alpha > 0$, B^μ is just the α-Bloch space. It is readily seen that B^μ is a Banach space with the norm $\|f\|_{B\mu} := |\ f\ (0)| + \ \|\ f\|_\mu$.

B^μ spaces appear in a natural way when one studies properties of some operators in certain spaces of holomorphic functions; for instance, if $\mu_1(t) = t\log\frac{2}{t}$, with $t \in (0,1]$, K. Attele in [148] proved that the Hankel operator induced by a function f in the Bergman space is bounded if and only if $f \in B^{\mu_1}$. The space B^{μ_1} is also known as the Log-Bloch space or the weighted Bloch space. Quite recently in [157] was introduced, so called, the logarithmic Bloch type space with $\mu(t) = t^\alpha \ \ \ln^\beta \frac{e}{t}$, $\alpha > 0$ and $\beta \geq 0$, where some properties of this space are studied and applied in studying of a composition operator.

Let H_1 and H_2 be two linear subspaces of $H(\mathbb{D}). If\ \varphi$ is a holomorphic self-map of $\mathbb{D}$, such that $f \circ \varphi$ belongs to H_2 for all $f \in H_1$, then φ induces a linear operator C_φ : $H_1 \ \rightarrow\ H_2$ defined as

$$C_\varphi(\ f\ ) :=\ f \circ \varphi,$$

called the composition operator with symbol φ. Composition operators has been studied by numerous authors in many subspaces of $H(\mathbb{D})$ and in particular in Bloch-type spaces.

In [2], Madigan and Matheson characterized continuity and compactness for composition operators on the classical Bloch space B. In turn, their results have been extended by Xiao [158] to the α-Bloch spaces and by Yoneda [11] to the Log-Bloch space. On the other hand, Gathage, Zheng and Zorboska [153] characterized closed range composition operators on the Bloch space. This result has been extended by Chen and

Gauthier [149] to α-Bloch spaces. Composition eoperators between α-Bloch and/or Lipschitz spaces on the unit ball are studied in [150], while the case of the polydisk was thoroughly studied in [151].

Also, in [4], Zhang and Xiao have characterized boundedness and compactness of weighted composition operators that act between μ-Bloch spaces on the unit ball of $\mathbb{C}^n$. In this case it is required that μ be a normal function. The results of Zhang and Xiao have been extended by Chen and Gauthier to the μ-Bloch spaces being μ a positive and non-decreasing continuous function such that μ(t)→0 as $t \to 0$ and $\mu(t)/t^{\delta}$ is decreasing for small t and for some $\delta > 0$.

Other compactness criteria for composition operators onBloch spaces have been found by Tjani [139]. Wulan, Zheng and Zhu (see [143]) proved the following result.

Theorem (4.1.1)[146]: ([143]): Let φ be an analyitic self-map of $\mathbb{D}$. Then C_φ is compact on the Bloch space B if and only if

$$\lim_{n\to\infty} \|\varphi^n\|B = 0.$$

We characterize boundedness and closedness of the range for composition operators acting on certain μ-Bloch spaces. We will approach these problems from a slightly different point of view. We will consider only those functions μ that can be extended to non vanishing, complex valued holomorphic functions, that satisfy a reasonable, for the purpose of extending the results of [11], [153], [149], geometric condition on the Euclidean disk $D(1,1)$. We will consider the problem of compactness of composition operators acting on μ- Bloch spaces. We discuss extensions of the results in [2], [143]. In fact, we will see that a result similar to Theorem (4.1.1) holds for $C\varphi : B \to B^\mu$.

We obtain genuine extensions of the results in [11], [153], [149]. For that reason, we will assume that $\mu : (0, 1] \to \mathbb{R}$ is a positive continuous function satisfying $\mu(t) \to 0 \; as \; t \to 0^+$, and also that μ can be extended to a complex function $\widetilde{\mu}$ satisfying the following properties:

(a) $\widetilde{\mu} \in H(D(1,1))$,

(b) $\widetilde{\mu}(z) \neq 0 \; for \; all \; z \in D(1,1)$,

(c) there exists a constant $M_\mu > 0$ such that

$$\mu(1 - |1 - z|) \leq M_\mu \, |\widetilde{\mu}(z)| \tag{1}$$

for all $z \in D(1,1)$.

For instance, the functions $\mu_1(t) := t^\alpha$, with $\alpha > 0$; $\mu_2(t) := t log\frac{2}{t}$ and $\mu_3(t) := t^p \, log(1 + t)$ with $p > 1$, defined on the interval (0, 1], satisfy all three conditions stated above. Observe that B^{μ_1} is the α-Bloch space and B^{μ_2} is the weighted Bloch space.

Let φ be a holomorphic self-map of the unit disk $\mathbb{D}$. For a function μ that satisfies all the properties mentioned above, put

$$\tau_\varphi^\mu(z) := \frac{\mu(1 - |z|^2)}{\mu(1 - |\varphi(z)|^2)} |\varphi'(z)| , z \in \mathbb{D} \quad . \tag{2}$$

It is readily seen that if $\tau_\varphi^\mu(z)$ is bounded on $\mathbb{D}$, then the composition operator C_φ is well defined on B^μ. Moreover, this condition turns out to be necessary and sufficient for C_φ to be bounded on B^μ, as the following proposition shows.

Theorem (4.1.2)[146]: The composition operator C_φ is bounded on B^μ if and only if

$$\sup_{z\in\mathbb{D}} \tau_\varphi^\mu(z) < \infty. \tag{3}$$

Proof. Let us suppose first that

$$\sup_{z\in\mathbb{D}} \tau_{\varphi}^{\mu}(z) = L < \infty.$$

Then, for each $f \in B^{\mu}$, we have the following estimate

$$\|C_{\varphi}(f)\|_{\mu} = \| f \circ \varphi\|_{\mu} = \sup_{z\in\mathbb{D}} \mu\_(1-|z|^2 \left|f'(\varphi(z))\right||\varphi'(z)|$$
$$= \sup_{z\in\mathbb{D}} \mu(1-|\varphi(z)|^2)\left|f'(\varphi(z))\right|\tau_{\varphi}^{\mu}(z)$$
$$\leq L\|f\|_{\mu}.$$

Thus, the composition operator C_{φ} is bounded on B^{μ}.

Now, suppose that there exists a constant $C > 0$ such that

$$\|C_{\varphi}(f)\|_{\mu} \leq C\|f\|_{\mu}$$

for all functions $f \in B^{\mu}$; then, since the identity function $i(z) = z$ belongs to B^{μ}, it follows that $\varphi \in B^{\mu}$. Let us fix $w \in \mathbb{D}\backslash\{0\}$ and consider the function

$$f_w(z) := \int_0^z \frac{ds}{\tilde{\mu}\left(1-\frac{\overline{w}^2}{|w|^2 s^2}\right)}.$$

Since the function $\tilde{\mu}$ satisfies conditions (a) and (b), it is clear that $f_w \in H(\mathbb{D})$.Also, by condition (c)

$$\mu(1-|z|^2)|f_w'(z)| = \mu(1-|z|^2)(\frac{1}{\left|\tilde{\mu}\left(1-\frac{\overline{w}^2}{|w|^2}z^2\right)\right|} \leq M_{\mu} \tag{4}$$

for all $z \in \mathbb{D}$. This means that $f_w \in B^{\mu}$. Thus, by the hypothesis on C_{φ} and (4), there is a constant $K > 0$ (depending only on μ) such that $\|f_w \circ \varphi\|_{\mu} \leq K,$ which implies that

$$\frac{\mu(1-|z|^2)}{\left|\tilde{\mu}\left(1-\frac{\overline{w}^2}{|w|^2}|(\varphi(z)|^2\right)\right|}|\varphi'(z)| \leq K \tag{5}$$

for all $z \in \mathbb{D}$ and all $w \in \mathbb{D}\backslash\{0\}$. Therefore, if $z \in \mathbb{D}$ satisfies $\mathbb{D}(z) \neq 0$, the substitution $w = \varphi(z)$ into (5) yields

$$\sup_{(z\in\mathbb{D}):\varphi(z)\neq 0} \frac{\mu(1-|z|^2)}{\mu(1-|\varphi(z)|^2)}|\varphi'(z)| \leq K,$$

where we have used the fact that $\tilde{\mu}$ is an extension of μ. Finally, since

$$\sup_{(z\in\mathbb{D}):\varphi(z)=0} \frac{\mu(1-|z|^2)}{\mu(1-|\varphi(z)|^2)}|\varphi'(z)| = \frac{1}{\mu(1)} \sup_{(z\in\mathbb{D}):\varphi(z)\neq 0} \mu(1-|z|^2)|\varphi'(z)|$$
$$\leq \frac{1}{\mu(1)}\|\varphi\|_{\mu} < \infty;$$

we can write

$$\sup_{z\in\mathbb{D}} \tau_{\varphi}^{\mu}(z) \leq \sup_{(z\in\mathbb{D}):\varphi(z)\neq 0} \tau_{\varphi}^{\mu}(z) + \sup_{(z\in\mathbb{D}):\varphi(z)\neq 0} \tau_{\varphi}^{\mu}(z) \leq K + \frac{1}{\mu(1)}\|\varphi\|_{\mu} < \infty$$

and the proof of Theorem (4.1.2) is complete.

Now, we present a necessary and sufficient condition for a composition operator on B^{μ} to be bounded below (and therefore with closed range). The purpose here is to generalize the results in [153], [149] to the μ- Bloch space. To this end, for $\varepsilon > 0$, let us denote

$$\Omega_{\varepsilon} := \{z \in \mathbb{D} : \tau_{\varphi}^{\mu}(z)\ \varepsilon\}.$$

Based on the definition of sampling sets for the Korenblum space (see [154]) and for α-Bloch spaces (see [149]), the following definition is now natural.
Here, as before, $\mu : (0,1] \to \mathbb{R}$ will be a positive continuous function such that $\mu(t) \to 0$ as $t \to 0^+$ and we will use B^μ to denote the Bloch-type space associated to a such function μ.
Definition (4.1.3)[146]: A subset G of the unit disk $\mathbb{D}$ is said to be a sampling set for B^μ if there exists a positive constant $L > 0$ such that

$$\sup_{z\in G}(\mu 1 - |z|^2)|f'(z)| \geq L\,\|f\|_\mu \tag{6}$$

for all $f \in B^\mu$.

In the following propositionwe characterize closed range composition operators on B^μ in terms of sampling sets.
Theorem (4.1.4)[146]: Let C_φ be a bounded composition operator on B^μ. C_φ is bounded below on B^μ if and only if there exists $\varepsilon > 0$ such that $G_\varepsilon = \varphi(\Omega_\varepsilon)$ is a sampling set for B^μ.
Proof. Let us suppose first that there exists $\varepsilon > 0$ such that $G_\varepsilon = \varphi(\Omega_\varepsilon)$ is a sampling set for B^μ. In this case, we can find a constant $L > 0$ such that

$$\|f\|_\mu \leq L \sup_{z\in G_\varepsilon} \mu(1 - |z|^2)|f'(z)|$$

for all functions $f \in B^\mu$. Hence, we have that

$$\begin{aligned}\|f\|_\mu &\leq L \sup_{z\in\Omega_\varepsilon} \mu(1 - |\varphi(z)|^2)|f'(\varphi(z))| \\ &= L \sup_{z\in\Omega_\varepsilon} \frac{1}{\tau_\varphi^\mu(z)} \mu(1 - |z|^2)|(f\circ\varphi)'(z)| \\ &\leq \frac{L}{\varepsilon}\|f\circ\varphi\|_\mu .\end{aligned}$$

This readily implies that the operator C_φ is bounded below on B^μ.

To prove the converse, suppose that C_φ is bounded below on B^μ. Then there exists a constant $K > 0$, such that

$$\left\|C_\varphi(f)\right\|_\mu = \sup_{z\in\mathbb{D}} \mu(1 - |z|^2)\left|f'(\varphi(z))\right||\varphi'(z)| \geq K$$

for all functions $f \in B^\mu$ with $\|f\|_\mu = 1$. Thus, by definition of supremum, we can find $z_f \in \mathbb{D}$ such that

$$\mu(1 - |z_f|^2)\left|f'(\varphi(z_f))\right||\varphi'(z_f)| \geq \frac{K}{2},$$

which, in turn, implies that

$$\tau_\varphi^\mu(z_f)\mu(1 - |\varphi(z_f)|^2)f'|(\varphi(z_f))| \geq \frac{K}{2}. \tag{7}$$

Thus, since $\mu(1 - |\varphi(z_f)|^2)|f'(\varphi(z_f))| \leq 1$, it must be $\tau_\varphi^\mu(z_f) \geq \frac{K}{2}$. Therefore, putting $\varepsilon := \frac{K}{2}$, we have $z_f \in \Omega_\varepsilon$.

Now, since C_φ is bounded, Theorem (4.1.2) implies that there is a constant $M_\mu > 0$ such that

$$\tau_\varphi^\mu(z) \leq M_\mu$$

for all $z \in \mathbb{D}$. In particular, $\tau_\varphi^\mu(z_f) \leq M\mu$. From (7) we conclude that

$$\mu\,(1-|\varphi(z_f)|^2)|\,f'\varphi(z_f)| \geq \frac{K}{2M_\mu}.$$

Finally, since $\varphi(z_f) \in G_\varepsilon$, it must be

$$\sup_{z\in G_\varepsilon} \mu\,(1-|z|^2)|f'(z)| \geq \frac{K}{2M_\mu}.$$

Therefore Gε is a sampling set for Bμ. The proof of Theorem is complete.

Indeed, from [152] we observe that any sampling sequence for B^μ is an η- net, for some $\eta \in (0,1)$ and, conversely, if $\Gamma = \{z_k\}$ is a separated η-net for $\eta \in (0,1)$ small enough, then Γ is a sampling sequence for B^μ. Thus, Theorem (4.1.4) can be rephrased as

Corollary (4.1.5)[146]: Let C_φ be a bounded composition operator on B^μ. If C_φ is bounded below on B^μ, then there exists $\epsilon = 0$ such that G contain a separated η-net for some $\eta \in (0,1)$. Conversely, if for some $\epsilon > 0$ the set G_ϵ contain an η-net with $\eta \in (0,1)$ small enough, then C_φ is bounded below on B_μ.

We characterize compactness of composition operators that act from the Bloch space, B, to μ- Bloch spaces. We obtain extensions of the results in [2], [143]. In order to do that we will assume that $\mu : (0,1] \to \mathbb{R}$ is a positive continuous function satisfying $\mu(t) \to 0$ as $t \to 0^+$. We also will assume that there exists $t_0 \in (0,1)$ such that μ is increasing in $(0, t_0)$, and that μ can be extended to a non vanishing analytic function $\tilde{\mu}$ on the disk $D(1,1)$ that satisfies the following properties:

(i) $1/\tilde{\mu}(z) = \sum_{n=0}^{\infty} b_n(1-z)^n$, with $b_n \geq 0$ for all $n \in \mathbb{N}$,

(ii) there exists a constant $K_\mu > 0$ such that $\lim t_{t\to 0^+} \frac{t\mu'(t)}{\mu(t)} = K_\mu$.

For instance, the functions $\mu_1(t) := t^\alpha$, with $\alpha > 0$; $\mu_2(t) := t\log\frac{3}{t}$ and $\mu_3(t) := t^p\log(1+t)$ with $p > 1$, satisfy the required conditions.

Let φ be a holomorphic self-map of the unit disk $\mathbb{D}$ and let μ be a function that satisfies all the properties mentioned above. Arguing as in the proof of Theorem (4.1.2) we can show that $C_\varphi : B \to B^\mu$ is bounded if and only if

$$\sup_{z\in\mathbb{D}} \frac{\mu\,(1-|z|^2)}{1-|\varphi(z)|^2}|\varphi'(z)| < \infty.$$

Also, as a consequence of [139] we have the following result.

Lemma (4.1.6)[146]: The composition operator $C_\varphi : B \to B_\mu$ is compact if and only if given a bounded sequence $\{f_n\}$ in B such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, then $\|C_\varphi(f_n)\|_\mu \to 0$ as $n \to \infty$.

Next we establish the compactness of $C_\varphi : B \to B^\mu$. It extends a result of Madigan and Matheson in [2].

Theorem (4.1.7)[146]: The composition operator $C_\varphi : B \to B^\mu$ is compact if and only if $\varphi \in B^\mu$ and

$$\lim_{|\varphi(z)|\to 1^-} \frac{\mu(1-|z|^2)}{1-|\varphi(z)|^2}\,|\phi'(z)| = 0. \tag{8}$$

Proof. Let us suppose first that $\varphi \in B^\mu$ and that (8) holds. Let $\{f_n\}$ be a bounded sequence in B converging to 0 uniformly on compact subsets of $\mathbb{D}$. Then, by Lemma (4.1.6), it suffices to show that $\left\|C_\varphi(f_n)\right\|_\mu \to 0$ as $n \to \infty$. To this end, set $K = \sup_n \|f_n\|_B$. Then, for $\epsilon > 0$ we can find an $r \in (0,1)$ such that

$$\frac{\mu(1-|z|^2)}{1-|\varphi(z)|^2}\,|\varphi'(z)| < \frac{\epsilon}{K},$$

for any $z \in \mathbb{D}$ satisfying $r < |\varphi(z)| < 1$. From here, we have that

$$\mu(1-|z|^2)|(f_n\circ\varphi)'(z)| = \frac{\mu(1-|z|^2)}{1-|\varphi(z)|^2}\,|\varphi'(z)|(1-|\varphi(z)|^2)|f_n'\circ\varphi(z)|$$
$$\le \frac{\epsilon}{K}K = \epsilon$$

Whenever $r < |\varphi(z)| < 1$.

On the other hand, since $\{f_n\}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$, $f_n \circ \varphi(0) \to 0$ and $\sup_{|w|\le r} \mu(1-|w|^2)|f_n'(w)| \to 0$, as $n \to \infty$

Also, since $\varphi \in B^\mu$, we can find a constant $C_r > 0$, depending only on r, such that

$$\sup_{|\varphi(z)|\le r}\frac{\mu(1-|z|^2)}{1-|\varphi(z)|^2}\,|\varphi'(z)| \le C_r\,\|\varphi\|_\mu.$$

Therefore, for the $\epsilon > 0$ given, there exists an $N \in \mathbb{N}$ such that

$$\sup_{|\varphi(z)|\le r} \mu(1-|z|^2)|(f_n\circ\varphi)'(z)|$$
$$= \sup_{|\varphi(z)|\le r}\frac{\mu(1-|z|^2)}{1-|\varphi(z)|^2}\varphi'(z)(1-|\varphi(z)|^2)|\,f_n'\circ\varphi(z)|$$
$$\le C_r\|\varphi\|_\mu$$

whenever $n \ge N$. Thus, we conclude that

$$\|f_n\circ\varphi\|_\mu = |f_n\circ\varphi(0)| + \sup_{z\in\mathbb{D}} \mu(1-|z|^2)\,|(f_n\circ\varphi)'(z)| < \left(1 + C_r\|\varphi\|_\mu\right)\epsilon$$

whenever $n \ge N$, which means that $C_\varphi : B \to B^\mu$ is a compact operator.

To prove the converse, suppose that there exists an $\epsilon_0 > 0$ such that

$$\sup_{|\varphi(z)|\ge r}\frac{\mu(1-|z|^2)}{1-|\varphi(z)|^2}\,|\varphi'(z)| \ge \epsilon_0$$

for any $r \in (0,1)$. Then, given a sequence of real numbers $\{r_n\} \subset (0,1)$ such that $r_n \to 1$ as $n \to \infty$, we can find a sequence $\{z_n\} \subset \mathbb{D}$ such that $|\varphi(z_n)| > r_n$ and

$$\frac{\mu(1-|zn|^2)}{1-|w_n|}|\varphi'(z_n)| \ge \frac{1}{2}\epsilon_0.$$

By taking a subsequence,if it is necessary, we may suppose that $w_n = \varphi(z_n) \to w_0 \in \partial\mathbb{D}$. Now, for $n \in \mathbb{N}$ and $z \in \mathbb{D}$, we set $g_n(z) := f_{n,0}(z) - f_n(z)$, where

$$f_{n,0}(z) = \frac{\overline{w_n}}{|w_n|}\int_0^z \frac{ds}{1-\frac{\overline{w_n}}{|w_n|}s} \quad ,$$

$$f_n(z) = \overline{w_n}\int_0^z \frac{ds}{1-\overline{w_n}s} \quad .$$

Clearly $\{g_n\}$ is a bounded sequence in B. Furthermore, since

$$|g_n'(z)| \le (1-|z|)^{-2}(1-|w_n|)$$

and $|g_n(z)| \le \int_0^z |g_n'(s)|\,|ds|$ for all $z \in \mathbb{D}$, $\{g_n\}$ is a sequence converging to 0 uniformly on compact subsets of $\mathbb{D}$, and satisfying

$$\left\|C_\varphi(g_n)\right\|_\mu \ge \mu(1-|zn|^2)g_n'(w_n)||\varphi'(z_n)|$$

$$= \frac{\mu(1-|z_n|^2)}{1-|w_n|}\,|\varphi'(z_n)| \geq \frac{1}{2}\,\epsilon_0 > 0.$$

Therefore, $C_\varphi : B \to B^\mu$ is not a compact operator. This completes the proof of the theorem.

We collect alternative tests to determine the compactness of a composition operator $C_\varphi : B \to B^\mu$. We have modified some of the techniques used in [143]. Here, for $a \in \mathbb{D}$ fixed, let

$$\sigma_a^\mu(z) := \frac{1}{\bar{a}}\frac{\mu(1-|a|^2)}{\tilde{\mu}(1-\bar{a}z)} = \frac{1}{\bar{a}}\mu(1-|a|^2)\sum_{n=0}^{\infty} b_n a^{-n} z^n .$$

Theorem (4.1.8)[146]: Let φ be an analytic function on $\mathbb{D}$ into itself such that $\varphi \in B^\mu$. The following conditions are equivalent.

(a) $\frac{\mu(1-|z|2)|\varphi'(z)|}{1-|\varphi(z)|^2} \to 0$ as $|\varphi(z)| \to 1^-$,

(b) $\|\varphi^n\|_\mu \to 0$ as $n \to \infty$

(c) $\left\|\sigma_a^\mu \circ \varphi\right\|_\mu \to 0$ as $|a| \to 1^-$

Proof. If $\|\phi\|_\infty < 1$, then the theorem is clear. Hence assume that $\|\phi\|_\infty = 1$. Suppose that condition a) holds. Then by Theorem (4.1.7), the composition operator $C_\varphi : B \to B^\mu$ is compact. Thus, since the sequence $\{z^n\}$is bounded in the Bloch space and converges to 0 uniformly on compact subsets of $\mathbb{D}$, Lemma (4.1.6) implies that $\lim_{n\to\infty} \|\varphi^n\|_\mu = 0$.

Suppose now that condition b) holds and let $\epsilon > 0$. Then by hypothesis, there exists an $N \in \mathbb{N}$ such that $\|\varphi^n\|_\mu < \epsilon$ whenever $n \geq N$. Thus, we can write.

$$\begin{aligned}\left\|\sigma_a^\mu \circ \phi\right\|_\mu &\leq \frac{1}{|a|}\mu(1-|a|^2) \times \sum_{n=0}^{N} b_n|a|^n\|\varphi^n\|_\mu + \frac{1}{|a|}\mu(1-|a|^2)\sum_{N+1}^{\infty} b_n|a|^n\|\varphi^n\|_\mu \\ &< \frac{1}{|a|}\mu(1-|a|^2)\sum_{n=0}^{N} b_n|a|^n\|\varphi^n\|_\mu + \epsilon\frac{1}{|a|}\,\frac{\mu(1-|a|^2)}{\mu(1-|a|)} . \qquad (9)\end{aligned}$$

On the other hand, since $\lim_{t\to 0^+}\frac{t\mu'(t)}{\mu(t)} = K_\mu > 0$ and μ is increasing for $t > 0$ small enough, then there exists $t_0 \in (0,1)$ such that

$$\frac{t\mu'(t)}{\mu(t)} < \frac{3}{2}\,K_\mu$$

whenever $0 < t < t_0$. It follows, by integration, that

$$\frac{\mu(2t)}{\mu(t)} < 2^{\frac{3}{2}K_\mu}$$

whenever $0 < t < t_0$. Thus, since $t \leq (2t - t^2) \leq 2t$ for $t \in (0,1]$, we conclude that

$$1 \leq \frac{\mu(2t-t^2)}{\mu(t)} \leq \frac{\mu(2t)}{\mu(t)} \leq 2^{\frac{3}{2}K_\mu}$$

for $t > 0$ small enough. Now, the hypothesis on μ together with the latter innequality and (9) gives that

$$\left\|\sigma_a^\mu \circ \varphi\right\|_\mu \leq \frac{1}{|a|}\mu(1-|a|^2)\sum_{n=0}^{N} b_n|a|^n\|\varphi^n\|_\mu + \epsilon\frac{1}{|a|}\,2^{\frac{3}{2}K_\mu}$$

whenever $1-|a| > 0$ is small enough. The condition c) follows by taking limit as $|a| \to 1^-$.

Finally, suppose that condition c) holds. Then, given $\epsilon > 0$ there exists $r \in (0,1)$ such that $\|\sigma_a^\mu \circ \varphi\|_\mu < \epsilon$ whenever $r < |a| < 1$. In particular, if $z \in \mathbb{D}$ satisfies $|\varphi(z)| > r$,

$$\left\|\sigma_{\varphi(z)}^\mu(\varphi)\right\|_\mu = \sup_{w\in\mathbb{D}} \mu(1-|w|^2)| \frac{(\mu(1-|\varphi(z)|^2)|\tilde{\mu}'(1-\overline{\varphi(z)}\varphi(w)}{\left|\tilde{\mu}\left(1-\overline{\varphi(z)}\varphi(w)\right)\right|^2}|\varphi'(z)| < \epsilon$$

, thus, for $w = z$ we have

$$\frac{\mu(1-|z|^2)}{\mu(1-|\varphi(z)|^2)}|\tilde{\mu}'(1-|\varphi(z)|^2)||\varphi'(z)| < \epsilon \tag{10}$$

whenever $r < |\varphi(z)| < 1$. Furthermore, since $\lim_{t\to 0^+} \frac{t\mu'(t)}{\mu(t)} = K_\mu$, we can choose $r \in (0,1)$ in such a way that

$$\frac{K\mu}{2(1-|\varphi(z)|^2)} \le \frac{|\tilde{\mu}'(1-|\varphi(z)|^2)|}{\mu(1-|\varphi(z)|^2)}$$

whenever $r < |\varphi(z)| < 1$. From this latter fact and (10), we finally obtain $\frac{\mu(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2} < \frac{2\epsilon}{K_\mu}$
whenever $r < |\varphi(z)| < 1$, and so a) holds.

We want to conclude with a comment about compactness of the composition operator $C_\varphi : B_0 \to B_0^\mu$; namely, if μ is as above, by applying similar arguments to those given by Madigan and Matheson in [2], we may also establish the following:

(i) A subset K of B_0^μ is compact if and only if

$$\lim_{|z|\to 1^-} \sup_{f\in K} \mu(1-|z|^2)| \; f'(z)| = 0.$$

(ii) The composition operator $C_\varphi : B_0 \to B_0^\mu$ is compact if and only if

$$\lim_{|z|\to 1^-} \frac{\mu(1-|z|^2)}{(1-|\varphi(z)|^2|} \varphi'(z)| = 0.$$

Section (4.2): μ-Bergman Space in C^{n*}

For dv be the Lebesgue measure on the unit ball B, normalized so that $v(B) = 1$; $dv_\alpha(z) = c_\alpha(1-|z|^2)^\alpha dv(z)$ $(z \in B, \alpha > -1)$ such that $v_\alpha(B) = 1$. The class of all holomorphic functions on B is denoted by $H(B)$. In the following, $z = (z_1, \cdots, z_n), w = (w_1, \cdots, w_n)$ in C^n, and $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w_j}$.

A positive continuous function μ on $[0,1)$ is called as normal, if there are constants $0 < a < b$ such that

(i) $\frac{\mu(r)}{(1-r^2)^a}$ is decreasing for $0 \le r < 1$ and $\lim_{r\to 1-} \frac{\mu(r)}{(1-r^2)^a} = 0$;

(ii) $\frac{\mu(r)}{(1-r^2)^b}$ is increasing for $0 \le r < 1$ and $\lim_{r\to 1-} \frac{\mu(r)}{(1-r^2)^b} = \infty$.

Let $p > 0$, and μ be normal on $[0,1)$. f is said to belong to the space $L^p(\mu)$ if f is Lebesgue measurable function on B and

$$\|f\|_{L^p(\mu)} = \left\{\int_B |f(z)|^p \frac{\mu^p(|z|)}{1-|z|^2} dv(z)\right\}^{\frac{1}{p}} < \infty.$$

$A^p(\mu) = L^p(\mu) \cap H(B)$ is called as μ-Bergman space. In particular, $A^p(\mu)$ is the Bergman space A^p when $\mu(r) = (1 - r^2)^{\frac{1}{p}}$, and $A^p(\mu)$ is the weight Bergman space A^p_β when $\mu(r) = (1 - r^2)^{\frac{\beta+1}{p}} (\beta > -1)$.

It is well known that $A^p(\mu)$ is a Banach space with the norm $\|\,.\,\|_{L^p(\mu)}$ when $p \geq 1$, and
$A^p(\mu)$ is a Fréchet space with the distance $\|\,.\,\|_{L^p(\mu)}$ when $0 < p < 1$. At the same time, $(A^p(\mu), \|\,.\,\|_{L^p(\mu)})$ is a topological vector space when $p > 0$.

Let $p > 0$. The sequence space l^p is defined as following

$$l^p = \left\{ \{c_k\} : \|\{c_k\}\| = \left(\sum_{k=1}^{\infty} |c_k|^p\right)^{\frac{1}{p}} < \infty \text{ and each } c_k \text{ is complex number} \right\}.$$

We will discuss the atomic decomposition. Atomic decomposition was studied for a long time (For example, [161]–[162]). In [161], R.Coifman and R.Rochberg discussed the problem on the weighted Bergman space A^p_α. In [17], Kehe Zhu modified the proof for the following Theorem:

Theorem (4.2.1)[160]: Suppose $p > 0, \alpha > -1$, and $b > n \max\left\{1, \frac{1}{p}\right\} + \frac{\alpha+1}{p}$. Then, there exists a sequence $\{a_k\}$ in B such that A^p_α consists exactly of functions of the form

$$f(z) = \sum_{k=1}^{\infty} c_k \frac{(1 - |a_k|^2)^{b - \frac{n+1+\alpha}{p}}}{(1 - \langle z, a_k \rangle)^b} \quad (z \in B),$$

where $\{c_k\}$ belongs to the sequence space l^p and the series converges in the norm topology of A^p_α.

We extend the weight $(1 - |z|^2)^\alpha$ to the normal weight $\frac{\mu^p(|z|)}{1-|z|^2}$. We show that every function in the μ–Bergman space $A^p(\mu)$ can be decomposed into a series of very nice atoms. These atoms are defined in terms of kernel functions and in some sense act as basis for the space $A^p(\mu)$.

The radial derivative $Rf(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z)$ for $f \in H(B)$. We will use the symbols c, c', c'', and c''' to denote positive constants, independent of variables z, w, and functions. But they may depend on some parameters, with different values in different cases.

We say that E and F are equivalent (denoted by $E \approx F$ in the following) if there exist two positive constants A_1 and A_2 such that $A_1 E \leq F \leq A_2 E$.

We first give some lemmas.

Lemma (4.2.2) ([17])[160]: There exists a positive integer N such that for any $0 < r \leq 1$, we can find a sequence $\{a_k\}$ in B, and for each $k \in \{1, 2, \cdots\}$, there exists a Lebesgue measurable set
D_k satisfying the following conditions:

(i) $B = \bigcup_{k=1}^{\infty} D(a_k, r) = \bigcup_{k=1}^{\infty} D_k$;

(ii) $D_k \cap D_j = \emptyset$ for $k \neq j$ $(k, j \in \{1, 2, \cdots\})$;

(iii) $D(a_k, \frac{r}{4}) \subset D_k \subset D(a_k, r)$ for every $k \in \{1, 2, \cdots\}$ ($D(a_k, r)$ is Bergman ball);

(iv) Each point $z \in B$ belongs to at most N of the sets $D(a_k, 4r)$.

Lemma (4.2.3)[160]: Let μ be normal on $[0,1), 0 < r \leq 1$, and $w \in B$. Then,

(i) $D(z,r) \subset D(w,2r) \subset D(w,4r)$ for any $z \in D(w,r)$;

(ii) $\frac{\mu(|z|)}{\mu(|w|)} \leq \left(\frac{1-|z|^2}{1-|w|^2}\right)^a + \left(\frac{1-|z|^2}{1-|w|^2}\right)^b$ for any $z \in B$;

(iii) There exists a constant $c > 0$, independent of r, such that $c^{-1}\mu(|w|) \leq \mu(|z|) \leq c\mu(|w|)$ for any $z \in D(w,r)$;

(iv) There exists a constant $A > 0$, independent of r, such that

$$A^{-1}(\tanh r)^{2n} \leq \frac{v_{\mu,p}[D(w,r)]}{(1 - |w|^2)^n\mu^p(|w|)} \leq A(\tanh r)^{2n},$$

where $p > 0$ and $v_{\mu,p}[D(w,r)] = \int_{D(w,r)} \frac{\mu^p(|z|)}{1-|z|^2} dv(z)$.

Proof. (i) By the definition of distance function, we have $D(z,r) \subset D(w,2r)$.

(ii) By the definition of normal function, if $|z| \leq |w|$, then we have

$$\frac{\mu(|z|)}{(1 - |z|^2)^b} \leq \frac{\mu(|w|)}{(1 - |w|^2)^b};$$

if $|z| > |w|$, then

$$\frac{\mu(|z|)}{(1 - |z|^2)^a} \leq \frac{\mu(|w|)}{(1 - |w|^2)^a} \Rightarrow \frac{\mu(|z|)}{\mu(|w|)} \leq \left(\frac{1 - |z|^2}{1 - |w|^2}\right)^a + \left(\frac{1 - |z|^2}{1 - |w|^2}\right)^b.$$

(iii) If $z \in D(w,r)$, then $\frac{1-\tanh 1}{1+\tanh 1} \leq \frac{1-|w|^2}{1-|z|^2} \leq \frac{1+\tanh 1}{1-\tanh 1}$ by (12) in [163]. Using (2), we have

$$\frac{\mu(|z|)}{\mu(|w|)} \leq \left(\frac{1 - |z|^2}{1 - |w|^2}\right)^a + \left(\frac{1 - |z|^2}{1 - |w|^2}\right)^b \leq \left(\frac{1 + \tanh 1}{1 - \tanh 1}\right)^a + \left(\frac{1 + \tanh 1}{1 - \tanh 1}\right)^b$$

and

$$\frac{\mu(|w|)}{\mu(|z|)} \leq \left(\frac{1 - |w|^2}{1 - |z|^2}\right)^a + \left(\frac{1 - |w|^2}{1 - |z|^2}\right)^b \leq \left(\frac{1 + \tanh 1}{1 - \tanh 1}\right)^a + \left(\frac{1 + \tanh 1}{1 - \tanh 1}\right)^b$$

$$\Rightarrow c^{-1}\mu(|w|) \leq \mu(|z|) \leq c\mu(|w|).$$

(iv) By Lemma 1.23 in [17] and (3), (12) in [163], we have

$$v_{\mu,p}[D(w,r)] \leq \frac{c^p\mu^p(|w|)(1 - |w|^2)^n}{(1 + \tanh 1)^n(1 - \tanh 1)^{n+2}}(\tanh r)^{2n}$$

and

$$v_{\mu,p}[D(w,r)] \geq \frac{c^{-p}(1 - \tanh 1)\mu^p(|w|)(1 - |w|^2)^n}{1 + \tanh 1}(\tanh r)^{2n}$$
$$\geq \frac{c^{-p}(1 - \tanh 1)^{n+2}\mu^p(|w|)(1 - |w|^2)^n}{(1 + \tanh 1)^{-n}}(\tanh r)^{2n}.$$

Lemma (4.2.4)[160]: Let $0 < r \leq 1$ and b be any real number. Then, there exist constants $c > 0$

and $A > 0$, independent of r, such that

$$\left|\frac{(1 - \langle z,u\rangle)^b}{(1 - \langle z,v\rangle)^b} - 1\right| \leq c \tanh r \text{ and } A^{-1} \leq \left|\frac{1 - \langle z,u\rangle}{1 - \langle z,v\rangle}\right| \leq A$$

for any z, u, and v in B with $\gamma(u,v) \leq r$.

Proof. This Lemma is Lemma (4.2.3) in [17]. We mainly show that there exist constants $c > 0$ and $A > 0$, independent of r.

If u and v satisfy (u, v) ≤ r, then we can write $v = \varphi_u(w)$ with $|w| \leq \tanh r$. Let

$z' = \varphi_u(z)$. By Lemma 1.3 in [17], we have

$$\frac{(1-\langle z,u\rangle)^b}{(1-\langle z,v\rangle)^b}-1=\frac{(1-\langle u,w\rangle)^b-(1-\langle z',w\rangle)^b}{(1-\langle z',w\rangle)^b}. \qquad (11)$$

If $b = 0$, then the result is obvious. If $b \neq 0$, then,

$$\left|(1-\langle u,w\rangle)^b-(1-\langle z',w\rangle)^b\right|=\left|\int_0^1\frac{b(\langle u,w\rangle-\langle z',w\rangle)}{\{1-(1-t)\langle u,w\rangle-t\langle z',w\rangle\}^{1-b}}dt\right|$$
$$\leq 2|b|[\,2^{b-1}+(1-\tanh 1)^{b-1}]\tanh r. \qquad (12)$$

By (11) and (12), we have

$$\left|\frac{(1-\langle z,u\rangle)^b}{(1-\langle z,v\rangle)^b}-1\right|\leq\frac{2|b|[\,2^{b-1}+(1-\tanh 1)^{b-1}]}{[(1-\tanh 1)^{-b}+2^{-b}]^{-1}}\tanh r.$$

If we take $b = 1$, then $\left|\frac{(1-\langle z,u\rangle)^b}{(1-\langle z,v\rangle)^b}\right| \leq 1 + c\tanh 1$; If we take $b = -1$, then $\left|\frac{(1-\langle z,v\rangle)^b}{(1-\langle z,u\rangle)^b}\right| \leq 1 + c\tanh 1$.

For $0 < r \leq 1, \eta$ denotes a positive radius that much smaller than r. Fixed a finite sequence $\{z_1,\cdots,z_J\}$ in $D(0,r)$ such that $\{D(z_j,\eta)\}$ cover $D(0,r)$ and that $\{D(z_j,\frac{\eta}{4})\}$ are disjoint. Then, each set $D(z_j,\frac{\eta}{4}) \cap D(0,r)$ is enlarged to a Borel set $E_j \subset D(z_j,\eta)$ such that $D(0,r) = \bigcup_{j=1}^{J}E_j$. For $k \in \{1,2,\cdots\}$ and $j \in \{1,2,\cdots,J\}$, let $D_{kj} = D_k \cap \varphi_{a_k}(E_j)$ and $a_{kj} = \varphi_{a_k}(z_j)$ [17].

Lemma (4.2.5)[160]: Let $p > 0$ and μ be normal on $[0,1)$. For $t > b + n\max\left\{\frac{1}{p}-1,0\right\}$, there exists a constant $c > 0$, independent of r and η, such that

$$|f(z)-S_1f(z)|\leq\sum_{k=1}^{\infty}\frac{c\sigma(1-|a_k|^2)^{n+t-\frac{n}{p}}}{|1-\langle z,a_k\rangle|^{n+t}\mu(|a_k|)}\left\{\int_{D(a_k,2r)}\frac{|f(w)|^p\mu^p(|w|)}{1-|w|^2}dv(w)\right\}^{\frac{1}{p}}$$

for any $0 < r \leq 1, z \in B$, and $f \in A^p(\mu)$, where $\sigma = \eta + \frac{\tanh\eta}{[\tanh r]^{1-2n(1-1/p)}}$ and

$$S_1f(z)=\sum_{k=1}^{\infty}\sum_{j=1}^{J}\frac{v_{t-1}(D_{kj})f(a_{kj})}{(1-\langle z,a_{kj}\rangle)^{n+t}}.$$

Proof If $f \in A^p(\mu)$, then

$$\int_B(1-|z|^2)^{pb-1}|f(z)|^pdv(z)\leq\frac{1}{\mu^p(0)}\int_B\frac{|f(z)|^p\mu^p(|z|)}{1-|z|^2}dv(z)\Rightarrow f\in A^p_{pb-1}.$$

If $0 < p \leq 1$, when $t > b + n\left(\frac{1}{p}-1\right)$, by Lemma (4.2.2) in [17], we have

$$\int_B|f(z)|dv_{t-1}(z)\leq c_{t-1}\int_B|f(z)|(1-|z|^2)^{\frac{n+pb}{p}-(n+1)}dv(z)\leq\frac{c_{t-1}}{c_{pb-1}}\|f\|_{p,pb-1}.$$

If $p > 1$, then $p'(t-1-b+\frac{1}{p}) > -1$ when $t > b$. By the Hölder inequality, we have

$$\int_B|f(z)|dv_{t-1}(z)=c_{t-1}\int_B\left\{|f(z)|(1-|z|^2)^{b-\frac{1}{p}}\right\}(1-|z|^2)^{t-1-b+\frac{1}{p}}dv(z)$$

$$\leq \frac{c_{t-1}}{(c_{pb-1})^{\frac{1}{p}}} \int_B \left\{(1 - |z|^2)^{p'\left(t-1-b+\frac{1}{p}\right)} dv(z)\right\}^{\frac{1}{p'}} \|f\|_{p,pb-1}$$
$$\leq c\|f\|_{p,pb-1}.$$

That is $f \in A^1_{t-1}$. By Theorem (4.2.3) in [17], when $t > b + n \max\left\{\frac{1}{p} - 1, 0\right\}$,

$$f(z) = \int_B \frac{f(w)}{(1 - \langle z, w\rangle)^{n+t}} dv_{t-1}(w) \ (z \in B). \quad (13)$$

From the proof process of Lemma (4.2.3) in [17], the above integral representation is necessary.

For any $k \in \{1, 2, \cdots\}$, we write $A_1 = D(a_k, 2r) \cap \{w : |a_k| < |w| < 1\}$ and $A_2 = D(a_k, 2r) \cap \{w : |w| \leq |a_k|\}$. By the definition of normal function and (12) in [163], we have

$$\int_{D(a_k,2r)} \frac{|f(w)|^p \mu^p(|w|)}{1 - |w|^2} dv(w)$$
$$\geq \frac{\mu^p(|a_k|)}{(1 - |a_k|^2)^{pb}} \int_{A_1} \frac{|f(w)|^p dv(w)}{(1 - |w|^2)^{-pb+1}} + \frac{\mu^p(|a_k|)}{(1 - |a_k|^2)^{pa}} \int_{A_2} \frac{|f(w)|^p dv(w)}{(1 - |w|^2)^{-pa+1}}$$
$$\geq \left(\frac{1 - \tanh 2}{1 + \tanh 2}\right)^{pb-pa} \frac{\mu^p(|a_k|)}{(1 - |a_k|^2)^{pa}} \int_{D(a_k,2r)} |f(w)|^p (1 - |w|^2)^{pa-1} dv(w). \quad (14)$$

We take $b = n + t$ and $\alpha = pa - 1$ from Lemma (4.2.3) in [17] and (14). Then,

$$\sum_{k=1}^{\infty} \frac{(1 - |a_k|^2)^{n+t-\frac{n}{p}}}{|1 - \langle z, a_k\rangle|^{n+t} \mu(|a_k|)} \left\{\int_{D(a_k,2r)} \frac{|f(w)|^p \mu^p(|w|)}{1 - |w|^2} dv(w)\right\}^{\frac{1}{p}}$$
$$\geq c \sum_{k=1}^{\infty} \frac{(1 - |a_k|^2)^{n+t-\frac{n}{p}}}{|1 - \langle z, a_k\rangle|^{n+t}} \left\{\int_{D(a_k,2r)} |f(w)|^p (1 - |w|^2)^{pa-1} dv(w)\right\}^{\frac{1}{p}}$$
$$\geq \frac{c'|f(z) - S_1 f(z)|}{\sigma}.$$

Theorem (4.2.6)[160]: Let μ be normal on $[0, 1)$. Suppose $p > 0, t > b + n \max\left\{\frac{1}{p} - 1, 0\right\}$, and
$0 < r \leq 1$. Then, there exists a constant $c > 0$, independent of r, such that

$$|f(z) - Sf(z)| \leq \sum_{k=1}^{\infty} \frac{cr^{2n\left(1-\frac{1}{p}\right)} (1 - |a_k|^2)^{t+n\left(1-\frac{1}{p}\right)}}{|1 - \langle z, a_k\rangle|^{n+t} \mu(|a_k|)} \left\{\int_{D(a_k,2r)} \frac{|f(u)|^p \mu^p(|u|)}{1 - |u|^2} dv(u)\right\}^{\frac{1}{p}}$$

for any $f \in A^p(\mu)$ and $z \in B$,

$$Sf(z) = \sum_{k=1}^{\infty} \frac{f(a_k)}{(1 - \langle z, a_k\rangle)^{n+t}} \int_{D_k} dv_{t-1}(w) \ (z \in B).$$

Proof If $f \in A^p(\mu)$, by Lemma (4.2.2) and (13), we have

$$f(z) - Sf(z) = \sum_{k=1}^{\infty} \int_{D_k} \left\{\frac{f(w)}{(1 - \langle z, w\rangle)^{n+t}} - \frac{f(a_k)}{(1 - \langle z, a_k\rangle)^{n+t}}\right\} dv_{t-1}(w) \Rightarrow$$

$$|f(z) - Sf(z)| \leq \sum_{k=1}^{\infty} \frac{1}{(1 - \langle z, a_k \rangle)^{n+t}} \int_{D_k} |f(w) - f(a_k)| dv_{t-1}(w)$$
$$+ \sum_{k=1}^{\infty} \frac{1}{(1 - \langle z, a_k \rangle)^{n+t}} \int_{D_k} \left| \left(\frac{1 - \langle z, a_k \rangle}{1 - \langle z, w \rangle} \right)^{n+t} \right| |f(w)| dv_{t-1}(w). \quad (15)$$

For every $k \in \{1, 2, \cdots\}$, by Lemma (4.2.2), Proposition 1.13 in [17], and (12) in [163], we have

$$\int_{D_k} |f(w) - f(a_k)| dv_{t-1}(w)$$
$$\leq \int_{D(a_k, r)} |f(w) - f(a_k)| dv_{t-1}(w)$$
$$\leq c(1 - |a_k|^2)^{n+t} \int_{D(0,r)} \left| f \circ \varphi_{a_k}(w) - f \circ \varphi_{a_k}(0) \right| dv(w)$$
$$= c(1 - |a_k|^2)^{n+t} \left| \int_{D(0,r)} \int_0^1 \frac{1}{\rho} R[f \circ \varphi_{a_k}](\rho w) d\rho \right| dv(w). \quad (16)$$

We write $R_1 = \tanh r$ and $R_2 = \tanh 2r$. Let $g_k(z) = f \circ \varphi_{a_k}(R_1 z)$.

For any $\alpha > -1$, we have

$$g_k(z) = \int_B \frac{g_k(R_1^{-1} R_2 u)}{(1 - \langle R_2^{-1} R_1 z, u \rangle)^{n+1+\alpha}} dv_\alpha(u) \ \left(|z| < \frac{R_2}{R_1} \right) \Rightarrow$$
$$R_{g_k}(z) = \frac{(n + 1 + \alpha) R_1}{R_2} \int_B \frac{g_k(R_1^{-1} R_2 u) \langle z, u \rangle}{(1 - \langle R_2^{-1} R_1 z, u \rangle)^{n+2+\alpha}} dv_\alpha(u). \quad (17)$$

When $p > 1$, we take $\alpha > 0$. By the polar coordinates transformation and (17), the Fubini

Theorem (4.2.1) and Proposition 1.4.10 in [22], the Hölder inequality, Proposition 1.7 in [17], and (13) in [163], we have

$$\int_{D(0,r)} \left\{ \int_0^1 \frac{1}{\rho} R[f \circ \varphi_{a_k}](\rho w) d\rho \right\} dv(w)$$
$$= R_1^{2n} \int_0^1 \frac{1}{\rho} \left\{ \int_B \left| R[f \circ \varphi_{a_k}](R_1 \rho w) \right| dv(w) \right\} d\rho$$
$$= R_1^{2n} \int_0^1 \frac{1}{\rho} \left\{ \int_B \left| R_{g_k}(\rho w) \right| dv(w) \right\} d\rho$$
$$\leq \int_B |u| |g_k(R_1^{-1} R_2 u)| \left\{ \int_0^1 \left(\int_B \frac{(n + 1 + \alpha) R_1^{2n+1} |w| dv(w)}{R_2 |1 - \langle R_2^{-1} R_1 \rho w, u \rangle|^{n+2+\alpha}} \right) d\rho \right\} dv_\alpha(u)$$
$$\leq \frac{c R_1^{2n+1}}{R_2} \int_B |g_k(R_1^{-1} R_2 u)| \left\{ \int_0^1 \frac{|u|}{(1 - R_2^{-1} R_1 \rho |u|)^{\alpha+1}} d\rho \right\} dv_\alpha(u)$$
$$\leq c' R_1^{2n} \left(\frac{R_2}{R_2 - R_1} \right)^\alpha \int_B |g_k(R_1^{-1} R_2 u)| dv(u)$$
$$\leq c' R_1^{2n} \left(\frac{R_2}{R_2 - R_1} \right)^\alpha \left\{ \int_B |g_k(R_1^{-1} R_2 u)|^p dv(u) \right\}^{\frac{1}{p}}$$

$$= \frac{c' R_1^{2n}}{R_2^{\frac{2n}{p}}} \left(\frac{R_2}{R_2 - R_1}\right)^{\alpha} \left(\int_{D(0,2r)} \left|f \circ \varphi_{a_k}(u)\right|^p dv(u)\right)^{\frac{1}{p}}$$

$$\leq \frac{c'' R_1^{2n}}{R_2^{\frac{2n}{p}} (1 - |a_k|^2)^{\frac{n+1}{p}}} \left(\frac{R_2}{R_2 - R_1}\right)^{\alpha} \left\{\int_{D(a_k,2r)} |f(u)|^p dv(u)\right\}^{\frac{1}{p}}. \tag{18}$$

When $0 < p \leq 1$, we take $\alpha = \frac{n+1}{p} - (n + 1)$. By the polar coordinates transformation and (17), the Fubini Theorem (4.2.1) and Proposition 1.4.10 in [22], Lemma (4.2.2) and Proposition 1.7 in [17],
(13) in [163], we have

$$\int_{D(0,r)} \left\{\int_0^1 \frac{1}{\rho} R[f \circ \varphi_{a_k}](\rho w) d\rho\right\} dv(w)$$

$$\leq c R_1^{2n} \left(\frac{R_2}{R_2 - R_1}\right)^{\alpha} \int_B |g_k(R_1^{-1} R_2 u)| (1 - |u|^2)^{\frac{n+1}{p} - n - 1} dv(u)$$

$$\leq c' R_1^{2n} \left(\frac{R_2}{R_2 - R_1}\right)^{\alpha} \left\{\int_B |g_k(R_1^{-1} R_2 u)|^p dv(u)\right\}^{\frac{1}{p}}$$

$$= c R_1^{2n} \left(\frac{R_2}{R_2 - R_1}\right)^{\alpha} \left\{\int_B \left|f \circ \varphi_{a_k}(R_2 u)\right|^p dv(u)\right\}^{\frac{1}{p}}$$

$$\leq \frac{c' R_1^{2n} R_2^{\frac{-2n}{p}}}{(1 - |a_k|^2)^{\frac{n+1}{p}}} \left(\frac{R_2}{R_2 - R_1}\right)^{\alpha} \left\{\int_{D(a_k,2r)} |f(u)|^p dv(u)\right\}^{\frac{1}{p}}. \tag{19}$$

When $0 < r \leq 1$, we have

$$\frac{r}{e^2} < R_1 = \frac{e^{2r} - 1}{e^{2r} + 1} = \frac{r}{e^{2r} + 1} \left\{2 + \frac{2^2 r}{2!} + \frac{2^3 r^2}{3!} + \cdots\right\} < e^2 r;$$

$$\frac{r}{e^4} < R_2 = \frac{e^{4r} - 1}{e^{4r} + 1} = \frac{r}{e^{4r} + 1} \left\{4 + \frac{4^2 r}{2!} + \frac{4^3 r^2}{3!} + \cdots\right\} < e^4 r;$$

$$\frac{r}{e^4} < \frac{R_1}{e^{2r}} = \frac{2e^{2r} R_1}{e^{4r} + e^{4r}} < \frac{2e^{2r} R_1}{e^{4r} + 1} = R_2 - R_1 < R_2 < e^{4r}. \tag{20}$$

By (16), (18)–(20), and Lemma (4.2.3), we have

$$\int_{D_k} |f(w) - f(a_k)| dv_{t-1}(w)$$

$$\leq c r^{2n\left(1 - \frac{1}{p}\right)} (1 - |a_k|^2)^{n + t - \frac{n+1}{p}} \left\{\int_{D(a_k,2r)} |f(u)|^p dv(u)\right\}^{\frac{1}{p}}$$

$$\leq \frac{c' r^{2n\left(1 - \frac{1}{p}\right)} (1 - |a_k|^2)^{t + n\left(1 - \frac{1}{p}\right)}}{\mu(|a_k|)} \left\{\int_{D(a_k,2r)} \frac{|f(u)|^p \mu^p(|u|)}{1 - |u|^2} dv(u)\right\}^{\frac{1}{p}}. \tag{21}$$

By Lemma 2.2–2.3, (12) in [163], Lemma 1.23 and Lemma (4.2.3) in [17], and (20), we have

$$\int_{D_k}\left|\left(\frac{1-\langle z,a_k\rangle}{1-\langle z,w\rangle}\right)^{n+t}-1\right||f(w)|dv_{t-1}(w)$$
$$\le \int_{D(a_k,r)}\left|\left(\frac{1-\langle z,a_k\rangle}{1-\langle z,w\rangle}\right)^{n+t}-1\right||f(w)|dv_{t-1}(w)$$
$$\le cR_1(1-|a_k|^2)^{t-1}\int_{D(a_k,r)}|f(w)|\,dv(w)$$
$$\le cR_1(1-|a_k|^2)^{t-1}v[D(a_k,r)]\sup_{w\in D(a_k,r)}|f(w)|$$
$$\le c'R_1^{2n+1}(1-|a_k|^2)^{t+n}\sup_{w\in D(a_k,r)}\left\{\frac{1}{v[D(w,r)]}\int_{D(w,r)}|f(u)|^p dv(u)\right\}^{\frac{1}{p}}$$
$$\le c''R_1^{2n\left(1-\frac{1}{p}\right)+1}(1-|a_k|^2)^{t+n-\frac{n+1}{p}}\left\{\int_{D(a_k,2r)}|f(u)|^p dv(u)\right\}^{\frac{1}{p}}$$
$$\le \frac{c'''r^{2n\left(1-\frac{1}{p}\right)+1}(1-|a_k|^2)^{t+n\left(1-\frac{1}{p}\right)}}{\mu(|a_k|)}\left\{\int_{D(a_k,2r)}|f(u)|^p\frac{\mu^p(|u|)}{1-|u|^2}dv(u)\right\}^{\frac{1}{p}}. \quad (22)$$

By (15) and (21)–(22), we have

$$|f(z)-Sf(z)|\le\sum_{k=1}^{\infty}\frac{cr^{2n\left(1-\frac{1}{p}\right)}(1-|a_k|^2)^{t+n\left(1-\frac{1}{p}\right)}}{|1-\langle z,a_k\rangle|^{n+t}\mu(|a_k|)}\left\{\int_{D(a_k,2r)}\frac{|f(u)|^p\mu^p(|u|)}{1-|u|^2}dv(u)\right\}^{\frac{1}{p}}.$$

Theorem (4.2.7)[160]: Let μ be normal on $[0,1)$. Suppose $p>0$ and $t>b+n\max\left\{\frac{1}{p}-1,0\right\}$.

Then, there exists a sequence $\{w_k\}$ in B such that $A^p(\mu)$ consists exactly of functions of the form

$$f(z)=\sum_{k=1}^{\infty}\frac{d_k(1-|w_k|^2)^{t+n-\frac{n}{p}}}{\mu(|w_k|)(1-\langle z,w_k\rangle)^{n+t}}\quad (z\in B),$$

where $\{d_k\}$ belongs to the sequence space l^p and the series converges in the norm topology of $A^p(\mu)$.

Proof: We take the sequence $\{a_{kj}\}$ from Lemma (4.2.5). We may write

$$f(z)=\sum_{k=1}^{\infty}\sum_{j=1}^{J}\frac{c_{kj}\left(1-|a_{kj}|^2\right)^{t+n-\frac{n}{p}}}{\mu(|a_{kj}|)\left(1-\langle z,a_{kj}\rangle\right)^{n+t}}, \quad (23)$$

where J is a fixed positive integer and $\{c_{kj}\}$ belongs to the sequence space l^p. In fact,

$$w_1=a_{11},\cdots,w_J=a_{1J},w_{J+1}=a_{21},\cdots,w_{J+J}=a_{2J},w_{2J+1}=a_{31},\cdots;\ d_1=c_{11},\cdots,d_J=c_{1J},d_{J+1}=c_{21},\cdots,d_{J+J}=c_{2J},d_{2J+1}=c_{31},\cdots.$$

First, let f admit a representation given in (23).

First assume that $0<p\le 1$.

We write $f_{kj}(z)=\frac{\left(1-|a_{kj}|^2\right)^{t+n-\frac{n}{p}}}{\mu(|a_{kj}|)\left(1-\langle z,a_{kj}\rangle\right)^{n+t}}$ for any $k\in\{1,2,\cdots\}$ and $j\in\{1,2,\cdots,J\}$.

The assumption on t implies that $pt + pn - pa - n > pt + pn - pb - n > 0$ when $0 < p \leq 1$.
By Proposition 1.4.10 in [22] and Lemma (4.2.3), we have

$$\|f_{kj}\|^p_{L^p(\mu)} = \frac{\left(1 - |a_{kj}|^2\right)^{pt+pn-n}}{\mu^p(|a_{kj}|)} \int_B \frac{\mu^p(|z|)}{(1 - |z|^2)\left|1 - \langle z, a_{kj}\rangle\right|^{p(n+t)}} dv(z)$$
$$\leq \left(1 - |a_{kj}|^2\right)^{pt+pn-n} \int_B \frac{(1 - |z|^2)^{pa-1}}{\left(1 - |a_{kj}|^2\right)^{pa} \left|1 - \langle z, a_{kj}\rangle\right|^{p(n+t)}} dv(z)$$
$$+ \left(1 - |a_{kj}|^2\right)^{pt+pn-n} \int_B \frac{(1 - |z|^2)^{pb-1}}{\left(1 - |a_{kj}|^2\right)^{pb} \left|1 - \langle z, a_{kj}\rangle\right|^{p(n+t)}} dv(z) \leq c.$$

As $\{c_{kj}\} \in l^p$ and $\{f_{kj}\}$ is bounded in $A^p(\mu)$, then, we have

$$\|f\|^p_{L^p(\mu)} = \int_B \left|\sum_{k=1}^{\infty}\sum_{j=1}^{J} c_{kj} f_{kj}(z)\right|^p \frac{\mu^p(|z|)}{1 - |z|^2} dv(z)$$
$$\leq \sum_{k=1}^{\infty}\sum_{j=1}^{J} |c_{kj}|^p \|f_{kj}\|^p_{L^p(\mu)} \leq c \sum_{k=1}^{\infty}\sum_{j=1}^{J} |c_{kj}|^p < \infty.$$

Next assume that $p > 1$.

Let $\{D_{kj}\}$ denote the sets from Lemma (4.2.3)9 in [17], then $D_{kj} \subset D_k$ and $1 - |a_k|^2 \approx 1 - |a_{kj}|^2, \mu(|a_k|) \approx \mu(|a_{kj}|), |1 - \langle z, a_k\rangle| \approx |1 - \langle z, a_{kj}\rangle|$ $(z \in B)$ for all $j \in \{1, 2, \cdots, J\}$.
We consider the function

$$g(z) = \sum_{k=1}^{\infty}\sum_{j=1}^{J} |c_{kj}| \{v_{\mu,p}(D_k)\}^{-\frac{1}{p}} X_k(z) \quad (z \in B),$$

where X_k is the characteristic function of D_k $(k \in \{1, 2, \cdots\})$.

By Lemmas (4.2.2) and (4.2.3), we have

$$\|g\|^p_{L^p(\mu)} = \sum_{k=1}^{\infty} \int_{D_k} \frac{|g(z)|^p \mu^p(|z|)}{1 - |z|^2} dv(z)$$
$$= \sum_{k=1}^{\infty}\sum_{j=1}^{J} |c_{kj}|^p \{v_{\mu,p}(D_k)\}^{-1} \int_{D_k} \frac{\mu^p(|z|)}{1 - |z|^2} dv(z) = \sum_{k=1}^{\infty}\sum_{j=1}^{J} |c_{kj}|^p$$

The assumption on $t > b$ implies that the operator

$$Tg(z) = \int_B \frac{(1 - |w|^2)^{t-1} g(w)}{|1 - \langle z, w\rangle|^{n+t}} dv(w) \quad (z \in B)$$

is bounded on $L^p(\mu)$ by [164].
By Lemmas (4.2.2) and (4.2.4), (12) and (13) in [163], Lemma 1.23 in [17], and (20), we have

$$Tg(z) = \sum_{k=1}^{\infty} \int_{D_k} \sum_{j=1}^{J} |c_{kj}| \{v_{\mu,p}(D_k)\}^{-\frac{1}{p}} \frac{(1 - |w|^2)^{t-1}}{|1 - \langle z, w\rangle|^{n+t}} dv(w)$$

$$\geq c(\tanh r)^{-\frac{2n}{p}} \sum_{k=1}^{\infty} \sum_{j=1}^{J} \frac{|c_{kj}|(1 - |a_k|^2)^{t-1-\frac{n}{p}}}{\mu(|a_k|)} \int_{D(a_k,\frac{r}{4})} \frac{1}{|1 - \langle z,w \rangle|^{n+t}} dv(w)$$

$$\geq c'(\tanh r)^{-\frac{2n}{p}} \sum_{k=1}^{\infty} \sum_{j=1}^{J} \frac{|c_{kj}|(1 - |a_k|^2)^{t-1-\frac{n}{p}}}{\mu(|a_k|)|1 - \langle z,a_k \rangle|^{n+t}} v[D(a_k,\frac{r}{4})]$$

$$\geq c'' r^{2n\left(1-\frac{1}{p}\right)} \sum_{k=1}^{\infty} \sum_{j=1}^{J} \frac{|c_{kj}|\left(1 - |a_{kj}|^2\right)^{t+n-\frac{n}{p}}}{\mu(|a_{kj}|)(1 - \langle z,a_{kj} \rangle)^{n+t}}$$

$$\geq c'' r^{2n\left(1-\frac{1}{p}\right)} |f(z)| \text{ for any } z \in B$$

$$\Rightarrow \|f\|_{L^p(\mu)}^p \leq cr^{2n(1-p)} \|T g\|_{L^p(\mu)}^p \leq cr^{2n(1-p)} \|T\|^p \sum_{k=1}^{\infty} \sum_{j=1}^{J} |c_{kj}|^p$$

$$\Rightarrow f \in A^p(\mu).$$

It remains to show that every function f ∈ Ap(μ) admits a representation given in (23). When $p > 1$, we write

$$F(z) = \sum_{k=1}^{\infty} \frac{(1 - |a_k|^2)^{t+n\left(1-\frac{1}{p}\right)}}{|1 - \langle z,a_k \rangle|^{n+t} \mu(|a_k|)} \left\{ \int_{D(a_k,2r)} \frac{|f(u)|^p \mu^p(|u|)}{1 - |u|^2} dv(u) \right\}^{\frac{1}{p}} X_k(z) \ (z \in B),$$

where X_k is the characteristic function of D_k ($k \in \{1, 2, \cdots\}$). By (13) in [163], and Lemmas (4.2.2)–(4.2.3), we have

$$\|F\|_{L^p(\mu)}^p = \sum_{k=1}^{\infty} \frac{(1 - |a_k|^2)^{pt+pn-n}}{\mu^p(|a_k|)} \left\{ \int_{D(a_k,2r)} \frac{|f(u)|^p \mu^p(|u|)}{1 - |u|^2} dv(u) \right\}$$
$$\times \int_{D_k} \frac{\mu^p(|z|)}{(1 - |u|^2)|1 - \langle z,a_k \rangle|^{pn+pt}} dv(z)$$
$$\geq c(\tanh r)^{2n} \sum_{k=1}^{\infty} \int_{D(a_k,4r)} \frac{|f(u)|^p \mu^p(|u|)}{1 - |u|^2} dv(u)$$
$$\geq c(\tanh r)^{2n} N \int_B \frac{|f(u)|^p \mu^p(|u|)}{1 - |u|^2} dv(u) = c(\tanh r)^{2n} N \|f\|_{L^p(\mu)}^p. \quad (24)$$

By Lemma (4.2.2) and Lemma (4.2.4), (12)–(13) in [163], and Lemma 1.23 in [17], we have

$$TF(z) = \sum_{k=1}^{\infty} \frac{(1 - |a_k|^2)^{t+n\left(1-\frac{1}{p}\right)}}{\mu(|a_k|)} \left\{ \int_{D(a_k,2r)} \frac{|f(u)|^p \mu^p(|u|)}{1 - |u|^2} dv(u) \right\}^{\frac{1}{p}}$$
$$\times \int_{D_k} \frac{(1 - |w|^2)^{t-1}}{|1 - \langle w,a_k \rangle|^{n+t} |1 - \langle z,w \rangle|^{n+t}} dv(w)$$
$$\geq c(\tanh r)^{2n} \sum_{k=1}^{\infty} \frac{(1 - |a_k|^2)^{t+n\left(1-\frac{1}{p}\right)}}{|1 - \langle z,a_k \rangle|^{n+t} \mu(|a_k|)} \left\{ \int_{D(a_k,2r)} \frac{|f(u)|^p \mu^p(|u|)}{1 - |u|^2} dv(u) \right\}^{\frac{1}{p}}. \ (25)$$

By (24)–(25) and (20), Lemma (4.2.5), and the bounded-ness of T on $L^p(\mu)$, we have

$$\begin{aligned}\|(I - S_1)f\|_{L^p(\mu)}^p &= \int_B |f(z) - S_1 f(z)|^p \frac{\mu^p(|z|)}{1 - |z|^2} dv(z) \\ &\le c\sigma^p (\tanh r)^{-2pn} \int_B |TF(z)|^p \frac{\mu^p(|z|)}{1 - |z|^2} dv(z) \\ &\le c\sigma^p (\tanh r)^{-2pn} \|T\|^p \|F\|_{L^p(\mu)}^p \\ &\le c'\sigma^p (\tanh r)^{-2pn} N \|T\|^p \|F\|_{L^p(\mu)}^p. \qquad (26)\end{aligned}$$

By Proposition 1.4.10 in [22] and Lemma (4.2.3) (2), for any $k = 1, 2, \cdots$, we have

$$\begin{aligned}&\int_B \frac{(1 - |a_k|^2)^{pn+pt-n} \mu^p(|z|) dv(z)}{(1 - |z|^2)|1 - \langle z, a_k\rangle|^{pn+pt} \mu^p(|a_k|)} \\ &\le (1 - |a_k|^2)^{pn+pt-pb-n} \int_B \frac{(1 - |z|^2)^{pb-1}}{|1 - \langle z, a_k\rangle|^{pn+pt}} dv(z) \\ &+ (1 - |a_k|^2)^{pn+pt-pa-n} \int_B \frac{(1 - |z|^2)^{pa-1}}{|1 - \langle z, a_k\rangle|^{pn+pt}} dv(z) \le c. \qquad (27)\end{aligned}$$

When $0 < p \le 1$, by Lemma (4.2.5), Lemma (4.2.2), and (27), we may prove that

$$\begin{aligned}\|(I - S_1)f\|_{L^p(\mu)}^p &= \int_B |(I - S_1)f(z)|^p \frac{\mu^p(|z|)}{1 - |z|^2} dv(z) \\ &\le \sum_{k=1}^{\infty} \int_B \frac{c\sigma^p (1 - |a_k|^2)^{pn+pt-n} \mu^p(|z|) dv(z)}{(1 - |z|^2)|1 - \langle z, a_k\rangle|^{pn+pt} \mu^p(|a_k|)} \left\{\int_{D(a_k, 4r)} \frac{|f(w)|^p \mu^p(|w|)}{1 - |w|^2} dv(w)\right\} \\ &\le c' N \sigma^p \int_B \frac{|f(w)|^p \mu^p(|w|)}{1 - |w|^2} dv(w) = c'\sigma^p N \|f\|_{L^p(\mu)}^p. \qquad (28)\end{aligned}$$

By (26) and (28), if σ is small enough, then we have $\|I - S\| < 1$. In this case, it follows from standard functional analysis that the operator S is invertible on $A^p(\mu)$. Therefore, every $f \in A^p(\mu)$ admits a representation

$$f(z) = \sum_{k=1}^{\infty} \sum_{j=1}^{J} \frac{c_{kj} \left(1 - |a_{kj}|^2\right)^{t+n-\frac{n}{p}}}{\mu(|a_{kj}|)(1 - \langle z, a_{kj}\rangle)^{n+t}} \ (z \in B),$$

where $c_{kj} = \frac{v_{t-1}(D_{kj}) \mu(|a_{kj}|) g(a_{kj})}{\left(1 - |a_{kj}|^2\right)^{t+n-\frac{n}{p}}}$ $(k \in \{1, 2, \cdots\},\ j \in \{1, 2, \cdots, J\})$ and $g = S^{-1} f$.

By (12) in [163], Lemma 1.24 and Lemma 2.34 in [17], Lemmas (4.2.2) and (4.2.3), (20), we have

$$\begin{aligned}\sum_{k=1}^{\infty} \sum_{j=1}^{J} |c_{kj}|^p &\le c \sum_{k=1}^{\infty} \sum_{j=1}^{J} \left(1 - |a_{kj}|^2\right)^n \mu^p(|a_{kj}|) |g(a_{kj})|^p \\ &\le c' \sum_{k=1}^{\infty} \sum_{j=1}^{J} \frac{\left(1 - |a_{kj}|^2\right)^n \mu^p(|a_{kj}|)}{v[D(a_k, r)]} \int_{D(a_k, r)} |g(w)|^p dv(w) \\ &\le c'' J r^{-2n} \sum_{k=1}^{\infty} \int_{D(a_k, 4r)} \frac{|g(w)|^p \mu^p(|w|)}{1 - |w|^2} dv(w) \\ &\le c'' r^{-2n} J N \int_B \frac{|g(w)|^p \mu^p(|w|)}{1 - |w|^2} dv(w) < \infty.\end{aligned}$$

This show that $\{c_{kj}\} \in l^p$. This completes the proof of the Theorem.

Section (4.3): The μ-Bergman Space in the Unit Ball

For B denote the unit ball of $\mathbb{C}^n$, and D denote the unit disc of $\mathbb{C}$. Suppose the class of all holomorphic functions with domain B is denoted by $H(B)$. Let $\mathrm{d}v$ denote the volume measure on the unit ball B, normalized so that $v(B) = 1$. The surface measure on the boundary S_n of B is denoted by $\mathrm{d}\sigma$, normalized so that $\sigma(S_n) = 1$. When $\alpha > -1$, a finite measure $\mathrm{d}v_\alpha$ on B is defined by $\mathrm{d}v_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha \mathrm{d}v(z)$, where c_α is a normalizing constant so that $v_\alpha(B) = 1$. The class of all bounded holomorphic functions on B is denoted by H^∞.

A positive continuous function μ on [0, 1) is called normal if there are constants $0 < a \le b$ and $0 \le r_0 < 1$ such that

(i) $\mu(r)(1 - r^2)^{-a}$ is decreasing for $r \in [r_0, 1)$;

(ii) $\mu(r)(1 - r^2)^{-b}$ is increasing for $r \in [r_0, 1)$. For example,

$$\mu(r) = (1 - r^2)\alpha \log^\beta \frac{2}{1 - r^2} \text{ and } \mu(r) = \left\{\sum_{k=1}^{\infty} \frac{k^\alpha r^{2k-2}}{\log(k + 1)}\right\}^{-1} \quad (\alpha > 0, \beta \text{ is real})$$

are all normal functions. In the following, a and b are always the constants in the definition of the normal function μ. Without loss of generality, in this article, let $r_0 = 0$.

Let $p > 0$, and μ be normal on $[0, 1)$. The μ-Bergman space is defined as

$$A^p(\mu) = \left\{ f \in H(B): \|f\|_{A^p(\mu)} = \left\{\int_B |f(z)|^p \frac{\mu^p(|z|)}{1 - |z|^2} \mathrm{d}v(z)\right\}^{\frac{1}{p}} < \infty \right\}.$$

In particular, $A^p(\mu)$ is the Bergman space A^p when $\mu(r) = (1 - r^2)^{\frac{1}{p}}$, and $A^p(\mu)$ is the weight Bergman space A^p_β when $\mu(r) = (1 - r^2)^{\frac{\beta+1}{p}}$ $(\beta > -1)$. $A^p(\mu)$ is a Banach space with the norm $\|.\|_{A^p(\mu)}$ when $p \ge 1$, and $A^p(\mu)$ is a complete metric space with the distance $\rho(f, g) = \|f - g\|^p_{A^p(\mu)}$ when $0 < p < 1$. At the same time, $\left(A^p(\mu), \|.\|_{A^p(\mu)}\right)$ is a topological vector space when $p > 0$.

Let μ be normal on [0, 1). f is said to belong to the μ-Bloch space β_μ if $f \in H(B)$ and

$$\|f\|_{\mu,1} = \sup_{z\in B} \mu(|z|)\, |\nabla f(z)| < \infty,$$

where complex gradient

$$\nabla f(z) = \left(\frac{\partial f}{\partial z_1}(z), \cdots, \frac{\partial f}{\partial z_n}(z)\right).$$

β_μ is a Banach space with the norm $\|f\|_{\beta_\mu} = |f(0)| + \|f\|_{\mu,1}$.

Let X and Y be two holomorphic function spaces on B, and $\varphi = (\varphi_1, \cdots, \varphi_n)$ be a holomorphic self-map of B and $\psi \in H(B)$. The weighted composition operator $T_{\varphi,\psi}$ from X to Y is defined by

$$T_{\varphi,\psi}(f) = \psi \cdot f \circ \varphi \; (f \in X).$$

If $\psi = 1$, then the operator $T_{\varphi,\psi}$ is composition operator C_φ. For $w \in B$, we will use the φ_w to denote the involutions on B with interchanges the points 0 and w, and $\varphi_w = \varphi_w^{-1}$. The Bergman ball $D(w, r) = \{z : z \in B \text{ and } |\varphi_w(z)| < r\}$ for $r \in (0,1)$. We use the symbols c , c' , c'', and c''' to denote positive constants, independent of variables z, w. But they may depend on some parameters (for example, p, a, b, γ, σ, etc.) or some fixed values (for example, $\mu(0), \varphi(0)$, etc.), with different values in different cases. We call E and F are equivalent (denoted by $E \approx F$ in the following) if there exist two positive constants A_1 and A_2 such that $A_1 E \leq F \leq A_2 E$.

The radial derivative

$$Rf(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z) = \langle \nabla f(z), z\rangle \text{ for } f \in H(B).$$

Let $R_\varphi(z) = \big(R\varphi_1(z), \cdots, R\varphi_n(z)\big)$ and

$$H_z(u) = \frac{(1 - |z|^2)|u|^2 + |\langle z, u\rangle|^2}{(1 - |z|^2)^2} \text{ for any } z \in B \text{ and } u \in \mathbb{C}^n - \{0\}.$$

Then, we have $H_{\varphi(z)}(R\varphi(z)) \leq cH_z(z)$ when $\Omega = B$ by Theorem (4.3.6) in [3]. For $f \in H(B)$, we write

$$\|f\|_{\mu,2} = \sup_{z\in B} \mu(|z|)|Rf(z)|.$$

For $f \in H(B)$, we write

$$\|f\|_{\mu,3} = \sup_{z\in B} \sup_{u\in\mathbb{C}^n-\{0\}} \left\{\frac{\mu(|z|)|\langle \nabla f(z), u\rangle|^p}{\sqrt{1 - |z|^2|u|^2 + |\langle z, u\rangle|^2}}\right\}.$$

By [4], and Proposition 1.18 and Theorem (4.3.11) in [17], we have

$$\|f\|_{\mu,3} = \sup_{z\in B} \frac{\mu(|z|)}{1 - |z|^2} Qf(z) = \sup_{z\in B} \mu(|z|) \left\{\frac{|\nabla f(z)|^2 - |Rf(z)|^2}{1 - |z|^2}\right\}^{\frac{1}{2}}.$$

Therefore, $\|f\|_{\mu,3} = \|f\|_{\mu,1}$ when $n = 1$.

In the disc, by [20], [21], we know that C_φ is always bounded on A^p_α, and C_φ is a compact operator on A^p_α if and only if

$$\lim_{|z|\to 1-} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0.$$

Could the mentioned result as above be generalized to the setting of the unit ball? First, C_φ is not always bounded on A^p_α when $n > 1$; for example, we take

$$\varphi(z) = (n^{\frac{n}{2}} z_1 \cdots z_n, 0, \cdots, 0), w = \left(\frac{r}{\sqrt{n}}, \cdots, \frac{r}{\sqrt{n}}\right) \text{ for } r \in (0,1)$$

and

$$fw(z) = \frac{1 - |\varphi(w)|^2}{(1 - \langle z, \varphi(w)\rangle)^{\frac{n+\alpha+1}{p}+1}}.$$

By computation, there is constant c, independent of r, so that $\|fw\|^p_{A^p_\alpha} \leq c$, and

$$\left\|C_\varphi(fw)\right\|^p_{A^p_\alpha} \approx \frac{1}{(1 - r)^{\frac{n-1}{2}}}.$$

This means that C_φ is not bounded on A^p_α when $n > 1$.

Second, the condition for which C_φ is compact on A^p_α, Kehe Zhu gave the result in [18]:

Theorem (4.3.1)[169]: Suppose $p > 0$ and $\alpha > -1$. If C_φ is bounded on A^q_β for some $q > 0$ and $-1 < \beta < \alpha$, then C_φ is compact on A^p_α if and only if

$$\lim_{|z|\to 1-} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0.$$

Theorem (4.3.1) is partly extended to the normal weight Bergman space in [170]:

Theorem (4.3.2)[169]: For $p > 1$, let μ be a normal function on [0, 1).

(i) If C_φ is compact on A^p (μ), then

$$\lim_{|z|\to 1-} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0.$$

(ii) If C_φ is bounded on A^1_β for some $-1 < \beta < pa - 1$, and

$$\lim_{|z|\to 1-} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0,$$

then C_φ is compact on A^p (μ).

As C_φ is not always bounded on A^p (μ), we fail to get the analogous necessary and sufficient conditions for compactness on the unit ball directly. A natural problem is that if can we find some analogous conditions for compactness as in the case of the complex analysis with one variable provided that C_φ is mapped into a space Y which contains A^p (μ)? We will give and prove the following result:

Proposition (4.3.3)[169]: Suppose $p > 0$ and μ is a normal function on [0, 1), $\nu(r) = (1-r^2)^{\frac{n}{p}+1}\mu(r)$ for r ∈ [0, 1). Then,

(i) C_φ is always bounded from A^p (μ) to $\beta\nu$;

(ii) C_φ is compact from A^p (μ) to $\beta\nu$ if and only if

$$\lim_{|z|\to 1-} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0$$

In [171], if $a > \frac{1}{2}$ (when $n > 1$), then we have

$$G^{\mu}_{\varphi(z)}\left(R\varphi(z), R\varphi(z)\right) \approx \frac{(1 - |\varphi(z)|^2)^2}{\mu^2(|\varphi(z)|)} H_{\varphi(z)}\left(R\varphi(z)\right).$$

If $a > 1$, then $\int_0^1 \mu^{-1}(t)dt = \infty$ and $\varphi_l \in H^\infty \subset \beta_\mu$ for $l \in \{1, 2, \cdots, n\}$. In fact, X. J. Zhang ([171]) and H. H. Chen ([146]) gave the following result.

Theorem (4.3.4)[169]: Let μ be normal on [0, 1). If $a > 1$ and $\|\varphi\|_\infty = \sup_{z\in B} |\varphi(z)| = 1$, then C_φ is compact on β_μ if and only if

$$\lim_{|\phi(z)|\to 1-} \frac{\mu|z|}{\mu(|\varphi(z)|)} \{(1 - |\phi(z)|^2)|R\varphi(z)|^2 + |\langle\varphi(z), R\varphi(z)\rangle|^2\}^{\frac{1}{2}} = 0.$$

We will give and prove the following result.

Proposition (4.3.5)[169]: Let μ be normal on [0, 1). If $a > 1$, then C_φ is compact on βμ if and only if

$$\lim_{|z|\to 1-} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0.$$

We will first discuss the bounded and compact conditions of $T_{\varphi,\psi}$ from A^p (μ) to β_ν . As Corollary, we give Proposition (4.3.3).

We first give some lemmas.

Lemma (4.3.6)[169]: Suppose μ is a normal function on [0, 1) and $f \in H(B)$. Then, $\|f\|_{\mu,1}$, $\|f\|_{\mu,2}$, and $\|f\|_{\mu,3}$ are equ1valet for $a > 1/2$ (when $n > 1$), and the controlling constants are independent of f.

Proof. We may give the result by Lemma 3 and Theorem 2 in [146].

Lemma (4.3.7)[169]: Suppose $p > 0$, and μ is a normal function on $[0,1), v(r) = (1 - r^2)^{\frac{n}{p}+1}\mu(r)$ for $r \in [0,1)$. If $f \in A^p(\mu)$, then $f \in \beta_v$ and $\|f\|_{\beta_v} \le c\|f\|_{A^p(\mu)}$.

Proof. If $f \in A^p(\mu)$, then we have

$$\int_B (1 - |z|^2)^{pb-1}|f(z)|^p dv(z) \le \frac{\|f\|^p_{A^p(\mu)}}{\mu^p(0)} \Rightarrow f \in A^p_{pb-1}.$$

By Theorem (4.3.6) in [17], we have

$$|f(z)| \le \frac{c\|f\|_{A^p_{pb-1}}}{(1 - |z|^2)^{\frac{n+pb}{p}}} \le \frac{c'\|f\|_{A^p(\mu)}}{(1 - |z|^2)^{\frac{n+pb}{p}}}\ (z \in B).$$

This means that $f \in A^1_\gamma$ when $\gamma > b + \frac{n}{p} - 1$. By Theorem 2.2 in [17], we have

$$f(z) = \int_B \frac{f(w)}{(1 - \langle z,w\rangle)^{n+1+\gamma}}\, dv_\gamma(w)\ (z \in B).$$

By Lemma 2.2 in [163], Lemma 2.2 in [160], and Proposition 1.4.10 in [22], we have

$$\begin{aligned} v(|z|)|Rf(z)| &\le c''\|f\|_{A^p(\mu)} \int_B \frac{(1 - |z|^2)^{\frac{n}{p}+1}\mu(|z|)(1 - |w|^2)^{\gamma-\frac{n}{p}}}{|1 - \langle z,w\rangle|^{n+2+\gamma}}\mu(|w|)\ dv(w) \\ &\le c''\|f\|_{A^p(\mu)}(1 - |z|^2)^{\frac{n}{p}+1+a} \int_B \frac{(1 - |w|^2)^{\gamma-\frac{n}{p}-a}}{|1 - \langle z,w\rangle|^{n+2+\gamma}}\ dv(w) \\ &+ c''\|f\|_{A^p(\mu)}(1 - |z|^2)^{\frac{n}{p}+1+b} \int_B \frac{(1 - |w|^2)^{\gamma-\frac{n}{p}-b}}{|1 - \langle z,w\rangle|^{n+2+\gamma}}\ dv(w) \\ &\le c''\|f\|_{A^p(\mu)}. \end{aligned} \tag{29}$$

On the other hand, by the subharmonicity of $|f|^p$ on B, we have

$$\begin{aligned} \|f\|_{A^p(\mu)} &= \int_B \frac{|f(z)|^p\mu^p(|z|)}{1 - |z|^2}\ dv(z) \\ &= 2n\int_0^1 \frac{r^{2n-1}\mu^p(r)}{1 - r^2}\left\{\int_{S_n} |f(r\xi)|^p d\sigma(\xi)\right\}dr \\ &\ge 2n\int_0^1 \frac{r^{2n-1}\mu^p(r)|f(0)|^p}{1 - r^2}dr \ge \frac{n!\,\Gamma(pb)\mu^p(0)}{\Gamma(n + pb)}|f(0)|^p. \end{aligned} \tag{30}$$

By (29)–(30) and Lemma (4.3.6), we have $f \in \beta_v$ and $\|f\|_{\beta_v} \le c\|f\|_{A^p(\mu)}$.

Lemma (4.3.8)[169]: Let $d(z,w) = |\varphi_w(z)|\ (w,z \in B)$ be the Bergman metric on B. Given $0 < r < \frac{1}{2}$ and $0 < \delta < \frac{1}{2}$, if $|z| > 1 - \delta$ and $d(z,w) \le r$, then

$$|w| > 1 - \frac{\delta(1 + r)}{1 - r + \delta r}.$$

Proof. If $d(z,w) \le r$, by Lemma 1.2 in [17], then we have

$$\begin{aligned} \frac{(1 - |z|^2)(1 - |w|^2)}{(1 - |zkw|)^2} &\ge \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \langle z,w\rangle|^2} = 1 - |\varphi w(z)|^2 \ge 1 - r^2 \\ &\Rightarrow (1 - r^2|w|^2)|z|^2 - 2(1 - r^2)|wkz| + |w|^2 - r^2 \le 0 \\ &\Rightarrow |z| \le \frac{|w| + r}{1 + r|w|}. \end{aligned}$$

When $|z| > 1 - \delta$, we have

$$\frac{|w| + r}{1 + r|w|} > 1 - \delta \Rightarrow |w| > 1 - \frac{\delta(1 + r)}{1 - r + \delta r}.$$

Lemma (4.3.9)[169]: Let φ be a automorphism of B. Then,

$$\frac{1 - |\varphi(0)|}{1 + |\varphi(0)|} \leq \frac{1 - |z|^2}{1 - |\varphi(z)|^2} \leq \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}.$$

Proof. The result can be obtained by Theorem 1.4 and Lemma 1.2 in [17].

Lemma (4.3.10)[169]: Let μ be normal on [0, 1) and

$$g(\xi) = 1 + \sum_{s=1}^{\infty} 2^s \xi^{n_s} \ (\xi \in D),$$

where ns is the integral part of $(1 - r_s)^{-1}, \mu(r_s) = 2^{-s}$ $(s = 1, 2, \cdots)$. Then

(i) $g(r)$ is increasing for r ∈ [0, 1) and

$$\inf_{r\in[0,1)} \mu(r)g(r) > 0, \sup_{\xi\in D} \mu(|\xi|)|g(\xi)| < \infty;$$

(ii) There exists constant $c > 0$ such that

$$\mu(\rho)g'(\rho) \leq \frac{c}{1 - \rho} \text{ for all } \rho \in [0, 1).$$

Proof The results come from Theorem 1 in [16] and Lemma (4.3.9) in [171].

We give and prove the main results.

Theorem (4.3.11)[169]: Suppose $t \geq 0$, and μ is a normal function on [0, 1). Let φ be a holomorphic self-map of B and $h \in H(B)$.

(i) If $\sup_{z\in B} \mu\frac{(|z|)|h(z)|}{(1 - |\varphi(z)|^2)}t = M < \infty$, then $\sup_{z\in B} (1 - |z|^2)\mu\frac{(|z|)|Rh(z)|}{(1 - |\varphi(z)|^2)^t} \leq cM$.

(ii) When $\|\varphi\|_\infty = \sup_{z\in B} |\varphi(z)| = 1$, if $\lim_{|\varphi(z)|\to 1-} \frac{\mu(|z|)h(z)}{\mu(|\phi(z)|)} = 0$, then

$$\lim_{|\varphi(z)|\to 1-} \frac{(1 - |z|^2)\mu(|z|)Rh(z)}{\mu(|\varphi(z)|)} = 0.$$

Proof. (i) For any $w \in B$, we take

$$F_w(z) = \frac{h(z)}{1 - \langle\varphi(z), \varphi(w)\rangle^t} \ (z \in B).$$

The theorem condition implies that

$$\sup_{z\in B} \mu(|z|)|F_w(z)| \leq 2^t M. \quad (31)$$

By Theorem 2.2 in [17], we have

$$F_w(z) = \int_B \frac{F_w(\eta)}{(1 - \langle z, \eta\rangle)^{n+1+\gamma}} \, dv_\gamma(\eta) \text{ when } \gamma > b - 1 \ (z \in B).$$

By (31), Lemma 2.2 in [160], Proposition 1.4.10 in [22], we have

$$\begin{aligned}\mu(|z|)|RF_w(z)| &\leq \int_B \frac{cM(1 - |\eta|^2)^\gamma \mu|z|}{|1 - \langle z, \eta\rangle|^{n+2+\gamma}\mu(|\eta|)} \, dv(\eta) \\ &\leq cM \int_B \frac{(1 - |z|^2)^a (1 - |\eta|^2)^{\gamma-a}}{|1 - \langle z, \eta\rangle|^{n+2+\gamma}} \, dv(\eta) \\ &\quad + cM \int_B \frac{(1 - |z|^2)^b (1 - |\eta|^2)^{\gamma-b}}{|1 - \langle z, \eta\rangle|^{n+2+\gamma}} \, dv(\eta) \\ &\leq \frac{c'M}{1 - |z|^2}. \end{aligned} \quad (32)$$

$H_{\varphi(w)}(R\varphi(w)) \le cH_w(w)$ shows that

$$\frac{(1-|w|^2)|\langle R\varphi(w),\varphi(w)\rangle|}{(1-|\varphi(w)|^2)} \le c(1-|w|^2)\{H_{\varphi(w)}(R\varphi(w))\}^{\frac{1}{2}} \le c. \quad (33)$$

Taking $z = w$ in (32), by computation and (33), we have

$$\frac{(1-|w|^2)\mu(|w|)|Rh(w)|}{(1-|\varphi(w)|^2)^t} \le c''M + \frac{t|h(w)|(1-|w|^2)\mu(|w|)|\langle R\varphi(w),\varphi(w)\rangle|}{(1-|\varphi(w)|^2)^{t+1}} \le c'''M.$$

(ii) When $\|\varphi\|_\infty = 1$, if

$$\lim_{|\varphi(z)|\to 1-} \frac{\mu(|z|)h(z)}{\mu(|\varphi(z)|)} = 0,$$

then, for any $0 < \varepsilon < 1$, there exists $0 < \delta < 1/2$ such that

$$\frac{\mu(|z|)|h(z)|}{\mu(|\varphi(z)|)} < \varepsilon \text{ when } |\varphi(z)| > 1-2\delta.$$

For any sequence $\{z^j\} \subset B$ satisfying $\lim_{j\to\infty} |\varphi(z^j)| = 1$, we write

$$\overline{D}\left(z^j, \frac{1}{3}\right) = \left\{w : w \in B \text{ and } d(w,z^j) \le \frac{1}{3}\right\}.$$

As $|\varphi(z^j)| \to 1$ $(j\to\infty)$, then there exists positive integer N such that $|\varphi(z^j)| > 1-\delta$ when $j > N$. By Theorem 8.1.4 in [22], we have

$$d\left(\varphi(w),\varphi(z^j)\right) \le d(w,z^j) \le \frac{1}{3} \text{ when } j > N \text{ and } w \in \overline{D}\left(z^j, \frac{1}{3}\right).$$

We take $r = 1/3$ in Lemma (4.3.8), then

$$|\varphi(w)| > 1-\frac{4\delta}{2+\delta} > 1-2\delta \Rightarrow \frac{\mu(|w|)|h(w)|}{\mu(|\varphi(w)|)} < \varepsilon. \quad (34)$$

On the other hand, by Corollary 1.22 in [17], we have

$$d(\varphi_{z^j}(z),z^j) = d\left(\varphi_{z^j}(z),\varphi_{z^j}(0)\right) = d(z,0) = |z| \le \frac{1}{6} \text{ when } |z| \le \frac{1}{6}.$$

If $|z| \le 1/6$ and $w \in \overline{D}\left(\varphi_{z^j}(z), \frac{1}{6}\right)$, then

$$d(w,z^j) \le d\left(w,\varphi_{z^j}(z)\right) + d(\varphi_{z^j}(z),z^j) \le \frac{1}{3} \Rightarrow \overline{D}\left(\varphi_{z^j}(z),\frac{1}{6}\right) \subseteq \overline{D}\left(z^j,\frac{1}{3}\right). \quad (35)$$

We take $G_j = F_j \circ \varphi_{z^j}$, where $F_j(z) = \mu(|z^j|)g(\langle\varphi(z),\varphi(z^j)\rangle)h(z)$, and g is the function in Lemma (4.3.10).

By Lemma 2.2 and Lemma 2.2 in [17], Lemma 1.2 and Lemma 1.23 in [17], (34)–(35), Lemma 2.2 in [160] and Lemma (4.3.10), we have

$$\begin{aligned}
|G_j(z)| = |F_j[\phi(z)]| &\le \frac{c'}{(1-|\varphi_{z^j}(z)|^2)^{n+1}} \int_{\overline{D}\left(\varphi_{z^j}(z),\frac{1}{6}\right)} |F_j(w)| dv(w) \\
&\le \frac{c'}{(1-|\varphi_{z^j}(z)|^2)^{n+1}} \int_{\overline{D}\left(\varphi_{z^j}(z),\frac{1}{3}\right)} |F_j(w)| dv(w) \\
&\le c''\varepsilon\left(1-|z^j|^2\right)^{-n-1} \int_{\overline{D}\left(\varphi_{z^j}(z),\frac{1}{3}\right)} \mu(|\varphi(w)|)g(|\varphi(w)|)dv(w) \\
&\le c\varepsilon \text{ when } |z| \le 1/6 \text{ and } j > N.
\end{aligned}$$

This shows that $\{G_j(z)\}$ converges to 0 uniformly on $\{z : |z| \le 1/6\}$. Thus, $\{|\nabla G_j(z)|\}$ must converge to 0 uniformly on $|z| \le 1/12$. In particular,

$$\lim_{j\to\infty} |\nabla G_j(0)| = 0. \quad (36)$$

On the other hand, by Lemma (4.3.6)4 and (2.11) in [17], we have

$$\left|\nabla G_j\,(0)\right|^2 \;=\; \left|\overline{\nabla} F_j\left(z^j\right)\right|^2 \;\geq\; \left(1 \;-\; \left|z^j\right|^2\right)^2 \left|RF_j\left(z^j\right)\right|^2. \qquad (37)$$

(36) and (37) show that

$$\lim_{j\to\infty} \left(1 \;-\; \left|z^j\right|^2\right)\left|RF_j\left(z^j\right)\right| \;=\; 0. \qquad (38)$$

By computation, we have

$$RF_j\left(z^j\right) = \mu\left(\left|z^j\right|\right)g'\left(\left|\varphi\left(z^j\right)\right|^2\right)\langle R\varphi\left(z^j\right), \phi(z\, j\,)\rangle h\left(z^j\right)$$
$$+\mu\left(\left|z^j\right|\right)g\left(\left|\varphi\left(z^j\right)\right|^2\right)Rh\left(z^j\right). \qquad (39)$$

By (33) and (38)–(39), and Lemma (4.3.10) and Lemma 2.2 in [160], we have

$$\frac{\left(1 \;-\; \left|z^j\right|^2\right)\mu\left(\left|z^j\right|\right)\left|Rh\left(z^j\right)\right|}{\mu(|\varphi(z^j)|)} = \frac{1 \;-\; \left|z^j\right|^2}{\mu(|\varphi(z^j)|)g(|\varphi(z^j)|^2)}\mu\left(\left|z^j\right|\right)g\left(\left|\varphi\left(z^j\right)\right|^2\right) \left|\, Rh\left(z^j\,\right)\right|$$
$$\leq\; c\left(1 \;-\; \left|z^j\right|^2\right)\left|RF_j\left(z^j\right)\right| + \frac{c'\mu\left(\left|z^j\right|\right)\left|h\left(z^j\right)\right|}{\mu(|\varphi(z^j)|)}$$
$$\Rightarrow \lim_{j\to\infty} \left(1 \;-\; \left|z^j\right|^2\right)\mu(|z^j|)Rh(z^j)\;\mu\left(\left|\varphi\left(z^j\right)\right|\right) \;=\; 0.$$

This means that

$$\lim_{|\varphi(z)|\to 1-} \frac{(1 \;-\; |z|^2\,)\mu(|z|)Rh(z)}{\mu(|\varphi(z)|)} \;=\; 0.$$

The proof is completed.

Theorem (4.3.12)[169]: Suppose $p > 0$, and μ is a normal function on $[0,1), \nu(r) = (1-r^2\,)^{\frac{n}{p}+1}\mu(r)$ for $r \in [0,1)$. If φ is a holomorphic self-map of B and $\psi \in H(B)$, then

(i) $T_{\varphi,\psi}$ is bounded from $A^p(\mu)$ to β_ν if and only if

$$M_0 \;= \sup_{z\in B}\left\{|\psi(z)|\left(\frac{1 \;-\; |z|^2}{1 \;-\; |\varphi(z)|^2}\right)^{\frac{n}{p}}\frac{\mu(|z|)}{\mu(|\varphi(z)|)}\right\} < \infty; \qquad (40)$$

(ii) $T_{\varphi,\psi}$ is compact from A^p (μ) to β_ν if and only if $\psi \in \beta_\nu$ when $\|\varphi\|_\infty < 1$;

(iii) $T_{\varphi,\psi}$ is compact from A^p (μ) to β_ν if and only if $\psi \in \beta_\nu$ and

$$\lim_{|\varphi(z)|\to 1-} \left(\frac{1 \;-\; |z|^2}{1 \;-\; |\varphi(z)|^2}\right)^{\frac{n}{p}}\frac{\mu(|z|)}{\mu(|\varphi(z)|)} \;=\; 0 \qquad (41)$$

when $\|\varphi\|_\infty \;=\; 1$.

Proof. (i) For any $f \in A^p$ (μ), by Lemma 2.2 in [163], we have

$$|f(z)| \leq \frac{c\|f\|_{A^p(\mu)}}{(1 \;-\; |z|^2)^{\frac{n}{p}}\mu(|z|)}\,(z \in B). \qquad (42)$$

If (40) holds, then we take $t \;=\; 0, h \;=\; T_{\varphi,\psi}(f)$ and the normal function $(1 \;-\; r^2\,)^{\frac{n}{p}}\,\mu(r)$ in Theorem (4.3.11). By Lemma 2.2 in [160] and (42), for any $z \in B$, we have

$$(1 \;-\; |z|^2\,)^{\frac{n}{p}+1}\mu(|z|)\left|R\left[T_{\varphi,\psi}(f)\right](z)\right|$$
$$\leq\; c'\sup_{w\in B}\;(1 \;-\; |w|^2)^{\frac{n}{p}}\,\mu(|w|)\left|T_{\varphi,\psi}\,(f)(w)\right|$$

$$\leq c'' \sup_{w\in B} \left\{\frac{|\psi(w)|\mu(|w|)}{\mu(|\varphi(w)|)}\left(\frac{1-|w|^2}{1-|\varphi(w)|^2}\right)^{\frac{n}{p}}\right\} \|f\|_{A^p(\mu)}$$

$$\leq c'' M_0 \|f\|_{A^p(\mu)}. \quad (43)$$

By (42)–(43) and Lemma (4.3.6), then $T_{\varphi,\psi}$ is bounded from Ap (μ) to βν. Conversely, if $T_{\varphi,\psi}$ is bounded from A^p (μ) to β_ν, then, for any $w \in B$, we take

$$f_w(z) = \frac{(1-|\varphi(w)|^2)^{b+1}}{\mu(|\varphi(w)|)(1-\langle z,\varphi(w)\rangle)^{b+1+\frac{n}{p}}}.$$

By the definition of normal function, we have

$$\mu^p(|z|) \leq \frac{\mu^p(|\varphi(w)|)(1-|z|^2)^{pb}}{(1-|\varphi(w)|^2)^{pb}} + \frac{\mu^p(|\varphi(w)|)(1-|z|^2)^{pa}}{(1-|\varphi(w)|^2)^{pa}}. \quad (44)$$

By Proposition 1.4.10 in [22] and (44), we obtain

$$\|f_w\|^p_{A^p(\mu)} = \frac{(1-|\varphi(w)|^2)^{pb+p}}{\mu^p(|\varphi(w)|)}\int_B \frac{\mu^p(|z|)}{(1-|z|^2)|1-\langle z,\varphi(w)\rangle|^{pb+p+n}}\,\mathrm{d}v(z)$$

$$\leq (1-|\varphi(w)|^2)^p \int_B \frac{(1-|z|^2)^{pb-1}}{|1-\langle z,\varphi(w)\rangle|^{pb+p+n}}\,\mathrm{d}v(z)$$

$$+ (1-|\varphi(w)|^2)^{pb+p}\int_B \frac{(1-|z|^2)^{pb-1}}{|1-\langle z,\varphi(w)\rangle|^{pb+p+n}}\,\mathrm{d}v(z) \leq c.$$

On the other hand, by Lemma 2.2 in [4] and Lemma (4.3.6), taking $z = w$, we have

$$|T_{\varphi,\psi}(f_w)(w)| \leq c\left(1 + \int_0^{|w|}\frac{1}{\nu(t)}\,\mathrm{d}t\right)\|T_{\varphi,\psi}(f_w)\|_{\beta_\nu}$$

$$\Rightarrow \frac{|\psi(w)|\mu(|w|)}{\mu(|\varphi(w)|)}\left(\frac{1-|w|^2}{1-|\varphi(w)|^2}\right)^{\frac{n}{p}}$$

$$\leq c'(1-|w|^2)^{\frac{n}{p}}\mu(|w|)\left(1+\int_0^{|w|}\frac{1}{\nu(t)}\,\mathrm{d}t\right)\|T_{\varphi,\psi}\|$$

$$\leq c'\mu(0)\|T_{\varphi,\psi}\| + c'\|T_{\varphi,\psi}\|(1-|w|^2)^{\frac{n}{p}}\int_0^{|w|}\frac{\mu(|w|)}{(1-t^2)^{1+\frac{n}{p}}\mu(t)}\,\mathrm{d}t$$

$$\leq c'\mu(0)\|T_{\varphi,\psi}\| + c'\|T_{\varphi,\psi}\|(1-|w|^2)^{\frac{n}{p}+a}\int_0^{|w|}\frac{1}{(1-t^2)^{1+\frac{n}{p}+a}}\,\mathrm{d}t$$

$$+c'\|T_{\varphi,\psi}\|(1-|w|^2)^{\frac{n}{p}+b}\int_0^{|w|}\frac{1}{(1-t^2)^{1+\frac{n}{p}+b}}\,\mathrm{d}t$$

$$\leq c'\mu(0)\|T_{\varphi,\psi}\| + c''\|T_{\varphi,\psi}\|.$$

This shows that (40) holds.

(ii)–(iii) If $T_{\varphi,\psi}$ is compact from A^p (μ) to β_ν, then $\psi \in \beta_\nu$ by taking $f(z) = 1$. When $\|\varphi\|_\infty = 1$, we prove that (41) holds.

Let $\{z^j\} \subset B$ be any sequence satisfying $|\varphi(z^j)| \to 1$ $(j \to \infty)$. We take

$$f_j(z) = \frac{\left(1-\left|\varphi(z^j)\right|^2\right)^{b+1}}{\mu(|\varphi(z^j)|)(1-\langle z,\varphi(z^j)\rangle)^{b+1+\frac{n}{p}}}.$$

Then, $\|f_j\|_{A^p(\mu)} \le c$ and $\{f_j(z)\}$ converges to 0 uniformly on any compact subset of B. This means that

$$\lim_{j\to\infty} \|T_{\varphi,\psi}(f_j)\|_{\beta_\nu} = 0.$$

On the other hand, we have

$$\left|T_{\varphi,\psi}(f_j)(z^j)\right| \le c\left(1 + \int_0^{|z^j|} \frac{1}{\nu(t)}\mathrm{d}t\right) \|T_{\varphi,\psi}(f_j)\|_{\beta_\nu}$$

$$\Rightarrow \frac{|\psi(z^j)|\mu(|z^j|)}{\mu(|\varphi(z^j)|)} \left(\frac{1 - |z^j|^2}{1 - |\varphi(z^j)|^2}\right)^{\frac{n}{p}}$$

$$\le c\left(1 - |z^j|^2\right)^{\frac{n}{p}} \mu(|z^j|)\left(1 + \int_0^{|z^j|} \frac{1}{\nu(t)}\mathrm{d}t\right) \|T_{\varphi,\psi}(f_j)\|_{\beta_\nu}$$

$$\le c\mu(0)\|T_{\varphi,\psi}(f_j)\|_{\beta_\nu} + c' \|T_{\varphi,\psi}(f_j)\|_{\beta_\nu}.$$

This shows that

$$\lim_{j\to\infty} \frac{|\psi(z^j)|\mu(|z^j|)}{\mu(|\varphi(z^j)|)} \left(\frac{1 - |z^j|^2}{1 - |\varphi(z^j)|^2}\right)^{\frac{n}{p}} = 0.$$

This means that (41) holds.

Conversely, for all $l \in \{1, 2, \cdots, n\}$, we have $\varphi_l \in H^\infty \subset \beta$. This means that $(1 - |z|^2)|R\varphi_l(z)| \le \|\varphi_l\|_\beta$ holds for all $z \in B$.

Let $\{f_j(z)\}$ be any sequence which converges to 0 uniformly on any compact subset of B and $\|f_j\|_{A^p(\mu)} \le 1$. When $\|\varphi\|_\infty < 1$, we know that $\{|\nabla f_j(w)|\}$ converges to 0 uniformly on $\{w : |w| \le \|\varphi\|_\infty\}$ and $\{f_j(\varphi(0))\}$ converges to 0. If $\in \beta_\nu$, by Lemma (4.3.6), Lemma 2.2 in [4], then

$$\|T_{\varphi,\psi}(f_j)\|_{\beta_\nu} \le |\psi(0)| \cdot |f_j[\varphi(0)]| + c' \sup_{z\in B} \nu(|z|)\left|R[T_{\varphi,\psi}(f_j)](z)\right|$$

$$\le |\psi(0)| \cdot |f_j[\varphi(0)]| + c' \sup_{z\in B} \nu(|z|)|R\psi(z)| \cdot |f_j[\varphi(z)]|$$

$$+ c' \sup_{z\in B} \nu(|z|)|\psi(z)| \cdot |\langle \nabla f_j[\varphi(z)], R\varphi(z)\rangle|$$

$$\le |\psi(0)| \cdot |f_j[\varphi(0)]| + c' \|\psi\|_{\beta_\nu} \sup_{|w|\le\|\varphi\|_\infty} |f_j(w)|$$

$$+(c''\mu(0) + c''')\|\psi\|_{\beta_\nu}\left(\sum_{l=1}^{n} \|\varphi_l\|_\beta\right) \sup_{|w|\le\|\varphi\|_\infty} |\nabla f_j(w)| \to 0 \ (j \to \infty).$$

When $\|\varphi\|_\infty = 1$, if (41) holds, then we take the normal function $(1 - r^2)^{\frac{n}{p}}\mu(r)$ and $h = \psi$ in Theorem (4.3.11). This means that

$$\lim_{|\varphi(z)|\to 1^-} |R\psi(z)| \frac{(1 - |z|^2)^{\frac{n}{p}+1}}{(1 - |\varphi(z)|^2)^{\frac{n}{p}}} \frac{\mu(|z|)}{\mu(|\varphi(z)|)} = 0.$$

Therefore, for any $\varepsilon > 0$, there exists $0 < \delta < 1$ such that

$$|\psi(z)|\left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2}\right)^{\frac{n}{p}} \frac{\mu(|z|)}{\mu(|\varphi(z)|)} < \varepsilon \text{ and } \frac{|R\psi(z)|(1 - |z|^2)^{\frac{n}{p}+1}}{(1 - |\varphi(z)|^2)^{\frac{n}{p}}} \frac{\mu(|z|)}{\mu(|\varphi(z)|)} < \varepsilon \quad (45)$$

when $|\varphi(z)| > \delta$. As $\{f_j(z)\}$ converges to 0 uniformly on any compact subset of B, then $\{|\nabla f_j(w)|\}$ converges to 0 uniformly on $\{w : |w| \le \delta\}$ and $\{f_j(\varphi(0))\}$ converges to 0. Therefore, there exists positive integer N such that

$$\sup_{|w|\le\delta} |f_j(w)| < \varepsilon, \sup_{|w|\le\delta} |\nabla f_j (w)| < \varepsilon, \quad \text{and } |f_j(\varphi(0))| < \varepsilon \text{ when } j > N.$$

When $|\varphi(z)| \le \delta$ and $j > N$, we have

$$\nu(|z|)|R[T_{\varphi,\psi}(f_j)](z)|$$
$$\le \|\psi\|_{\beta_\nu} \sup_{|w|\le\delta} |f_j(w)|$$
$$+(c'\mu(0) + c'')\|\psi\|_{\beta_\nu}\left(\sum_{l=1}^{n} \|\varphi_l\|_\beta\right)\sup_{|w|\le\delta} |\nabla f_j(w)| < c\varepsilon. \quad (46)$$

If $j > N$, by Lemma (4.3.6)–2.2, (42), (45)–(46), and $H_{\varphi(z)}[R\varphi(z)] \le cH_z(z)$, then

$$\|T_{\varphi,\psi}(f_j)\|_{\beta_\nu} \le |\psi(0)| \cdot |f_j(\varphi(0))|$$
$$+c'\left(\sup_{|\varphi(z)|>\delta} + \sup_{|\varphi(z)|\le\delta}\right)\nu(|z|)|R[T_{\varphi,\psi}(f_j)](z)|$$
$$< c''\varepsilon + c'\varepsilon \sup_{|\varphi(z)|>\delta} \frac{(1 - |\varphi(z)|^2)^{\frac{n}{p}+1}\mu(|\varphi(z)|)\left|\langle\nabla f_j[\varphi(z)], \overline{R\varphi(z)}\rangle\right|(1 - |z|^2)}{\sqrt{(1 - |\varphi(z)|^2)|R\varphi(z)|^2 + |\langle R\varphi(z), \varphi(z)\rangle|^2}}$$
$$\sqrt{H_{\varphi(z)}[R\varphi(z)]} < c''\varepsilon + c'''\varepsilon\|f_j\|_{\beta_\nu} \le c''\varepsilon + c\varepsilon\|f_j\|_{A^p(\mu)} \le c''\varepsilon + c\varepsilon.$$

This shows that

$$\lim_{j\to\infty} \|T_{\varphi,\psi}(f_j)\|_{\beta_\nu} = 0.$$

This means that $T_{\varphi,\psi}$ is compact from $A^p(\mu)$ to β_ν. The proof is completed.

Corollary (4.3.13)[169]: Suppose $p > 0$, and μ is a normal function on $[0, 1), \nu(r) = (1 - r^2)^{\frac{n}{p}+1}\mu(r)$ for $r \in [0, 1)$. If φ is a automorphism of B and $\psi \in H(B)$, then

(i) $T_{\varphi,\psi}$ is bounded from $A^p(\mu)$ to β_ν if and only if $\psi \in H^\infty$;

(ii) $T_{\varphi,\psi}$ is compact from $A^p(\mu)$ to β_ν if and only if $\psi \equiv 0$.

Proof. By Lemma 2.2 in [160], Lemma (4.3.9), there exists constant $c > 0$ such that

$$\frac{1}{c} \le \left\{\frac{1 - |z|^2}{1 - |\varphi(z)|^2}\right\}^{\frac{n}{p}} \frac{\mu(|z|)}{\mu(|\varphi(z)|)} \le c. \quad (47)$$

On the other hand, if φ is a automorphism of B, then $|\varphi(z)| \to 1^-$ if and only if $|z| \to 1^-$. Therefore, by (47), Theorem (4.3.12), and the Maximum Modulus Principle, we can obtain the result.

Corollary (4.3.14)[169]: Suppose $p > 0$, and μ is a normal function on $[0, 1), \nu(r) = (1 - r^2)^{\frac{n}{p}+1}\mu(r)$ for $r \in [0, 1)$. If φ is a holomorphic self-map of B, then

(i) C_φ is always bounded from $A^p(\mu)$ to β_ν;

(ii) C_φ is compact from $A^p(\mu)$ to β_ν if and only if

$$\lim_{|z|\to 1^-} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0. \quad (48)$$

Proof. (i) By Lemma 2.2 in [160], we have

$$\left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{\frac{n}{p}}\frac{\mu(|z|)}{\mu(|\varphi(z)|)}\leq\left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{\frac{n}{p}+a}+\left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{\frac{n}{p}+b}$$
$$\leq\left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{\frac{n}{p}+a}+\left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{\frac{n}{p}+b}.$$

By Theorem (4.3.12), C_φ is always bounded from A^p (μ) to β_ν.

(ii) If (48) holds and $\|\varphi\|_\infty = 1$, then we have

$$\left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{\frac{n}{p}}\frac{\mu(|z|)}{\mu(|\varphi(z)|)}$$
$$\leq\left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{\frac{n}{p}+a}+\left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{\frac{n}{p}+b}\quad(|\varphi(z)|\to 1^-).$$

By Theorem (4.3.12), C_φ is compact from A^p (μ) to β_ν.

Conversely, if C_φ is compact from A^p (μ) to β_ν and $\|\varphi\|_\infty = 1$, then, by Theorem (4.3.12), we have

$$0\leftarrow\left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{\frac{n}{p}}\frac{\mu(|z|)}{\mu(|\varphi(z)|)}\geq\left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{\frac{n}{p}+b}\left\{1+\left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{b-a}\right\}^{-1}$$
$$\geq\left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{\frac{n}{p}+b}\left\{1+\left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{b-a}\right\}^{-1}\quad(|\varphi(z)|\to 1^-).$$

This shows that

$$\lim_{|\varphi(z)|\to1^-}\frac{1-|z|^2}{1-|\varphi(z)|^2}=0.$$

Therefore, for any $\varepsilon > 0$, there exists $0 < \delta < 1$ such that $\frac{1-|z|^2}{1-|\varphi(z)|^2} < \varepsilon$ when $|\varphi(z)| > \delta$. On the other hand,

$$\lim_{|z|\to1^-}\frac{1-|z|^2}{1-\delta^2}=0,$$

then there exists $0 < \delta_0 < 1$ such that $\frac{1-|z|^2}{1-\delta^2} < \varepsilon$ when $|z| > \delta_0$. When $|z| > \delta_0$, if $|\varphi(z)| \leq \delta$, then

$$\frac{1-|z|^2}{1-|\varphi(z)|^2}\leq\frac{1-|z|^2}{1-\delta^2}<\varepsilon;\ \text{if } |\varphi(z)|>\delta, \text{then } \frac{1-|z|^2}{1-|\varphi(z)|^2}<\varepsilon.$$

This shows that

$$\lim_{|z|\to1^-}\frac{1-|z|^2}{1-|\varphi(z)|^2}=0.$$

The proof is completed.

This is Proposition (4.3.3).

Theorem (4.3.15)[169]: Let μ be normal on [0, 1), and φ be a holomorphic self- map of B. If a > 1, then C_φ is compact on βμ if and only if

$$\lim_{|z|\to1^-}\frac{1-|z|^2}{1-|\varphi(z)|^2}=0.$$

Proof First, suppose that C_φ is compact on β_μ. If $\|\varphi\|_\infty < 1$, then we have

$$\lim_{|z|\to 1^-} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0.$$

When $\|\varphi\|_\infty = 1$, let $\{z^j\} \subset B$ be any sequence satisfying

$$\lim_{j\to\infty} \left|\varphi(z^j)\right| = 1.$$

We take $f_j(z) = \left(1 - \left|\varphi(z^j)\right|^2\right) g\left(\langle z, \varphi(z^j)\rangle\right)$, where g is the function in Lemma (4.3.10). By Lemma (4.3.10), we have

$$\mu(|z|)\left|\nabla f_j(z)\right| = \mu(|z|)\left(1 - \left|\varphi(z^j)\right|^2\right)\left|g'\left(\langle z, \varphi(z^j)\rangle\right)\overline{\varphi(z^j)}\right|$$
$$\leq 2\mu\left(\left|\langle z, \varphi(z^j)\rangle\right|\right)\left(1 - \left|\langle z, \varphi(z^j)\rangle\right|\right) g'\left(\left|\langle z, \varphi(z^j)\rangle\right|\right) \leq c.$$

Therefore, $\|f_j\|_{\beta_\mu} \leq 1 + c$ and $\{f_j(z)\}$ converges to 0 uniformly on any compact subset of B. This means that

$$\lim_{j\to\infty} \left\|C_\varphi(f_j)\right\|_{\beta_\mu} = 0.$$

On the other hand, by Lemma (4.3.10), Lemma (4.3.6), Lemma 2.2 in [4], and Lemma 2.2 in [160], we have

$$0 < c \leq \mu\left(\left|\varphi(z^j)\right|^2\right) g\left(\left|\varphi(z^j)\right|^2\right) = \frac{\mu\left(\left|\varphi(z^j)\right|^2\right)\left|C_\varphi(f_j)(z^j)\right|}{1 - |\varphi(z^j)|^2}$$
$$\leq \frac{c'\mu\left(\left|\varphi(z^j)\right|^2\right)}{1 - |\varphi(z^j)|^2}\left(1 + \int_0^{|z^j|} \frac{1}{\mu(t)}\,dt\right) \left\|C_\varphi(f_j)\right\|_{\beta_\mu}$$
$$\leq \left\{c'\mu(0)\left(1 - \left|\varphi(z^j)\right|^4\right)^{a-1} + \frac{c''\left(1 - \left|\varphi(z^j)\right|^2\right)^{a-1}}{(1 - |z^j|^2)^{a-1}}\right.$$
$$\left. + \frac{c''\left(1 - \left|\varphi(z^j)\right|^2\right)^{b-1}}{(1 - |z^j|^2)^{b-1}}\right\} \left\|C_\varphi(f_j)\right\|_{\beta_\mu}$$
$$\Rightarrow \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2}\right)^{b-1}$$

$1 - |z\, j\, |2\ 1 - |\phi(z\, j\,)|2\ b - 1$

$$\leq \frac{1}{c}\left\{\frac{c'''\left(1 - \left|\varphi(z^j)\right|^2\right)^{b-1}}{(1 - |\varphi(z^j)|^2)^{b-1}} + \frac{c'''\left(1 - \left|\varphi(z^j)\right|^2\right)^{b-a}}{(1 - |\varphi(z^j)|^2)^{b-a}} + c''\right\} \left\|C_\varphi(f_j)\right\|_{\beta_\mu}$$
$$\leq \frac{1}{c}\left\{\frac{c'''(1 + |\varphi(0)|)^{b-a}}{(1 - |\varphi(0)|)^{b-a}} + \frac{c''(1 + |\varphi(0)|)^{b-a}}{(1 - |\varphi(0)|)^{b-a}} + c''\right\} \left\|C_\varphi(f_j)\right\|_{\beta_\mu}.$$

By $b \geq a > 1$, this shows that

$$\lim_{j\to\infty 1} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0 \Rightarrow \lim_{|\varphi(z)|\to 1^-} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0.$$

By Proof of Corollary (4.3.13), we have $\lim_{|z|\to 1^-} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0.$

Conversely, first, for all $l \in \{1, \cdots, n\}$, we have $\varphi_l \in \beta_\mu$ when $a > 1$. If

$$\lim_{|z|\to 1^-} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0,$$

then for any $0 < \varepsilon < 1$, there exists $0 < r < 1$ such that

$$\frac{1 - |z|^2}{1 - |\varphi(z)|^2} < \varepsilon \text{ when } r < |z| < 1. \tag{49}$$

Let $\{f_j(z)\}$ be any sequence that converges to 0 uniformly on any compact subset of B and $\|f_j\|_{\beta_\mu} \leq 1$. Let $r_1 = \max\{|\varphi(z)|: |z| \leq r\}$. Then, there exists positive integer N such that

$$|f_j\,[\varphi(0)]| < \varepsilon \text{ and } \sup_{|w|\leq r_1} |\nabla f_j\,(w)| < \varepsilon \text{ when } j > N. \tag{50}$$

Therefore, by Lemma (4.3.6) and $H_{\varphi(z)}\,[R\varphi(z)] \leq cH_z(z)$, (49)–(50), Lemma 2.2 in [160], as long as $j > N$, we have

$$\begin{aligned}
\|C_\varphi\,(f_j)\|_{\beta_\mu} &\leq c\left\{|f_j\,[\varphi(0)]| + \sup_{z\in B} \mu(|z|)\left|\langle \nabla f_j\,[\varphi(z)], R\varphi\,(z)\rangle\right|\right\} \\
&\leq c|f_j\,[\varphi(0)]| \\
&\quad + c\left\{\left(\sum_{l=1}^{n} \|\varphi_l\|_{\beta_\mu}\right)\sup_{|w|\leq r_1} |\nabla f_j\,(w)| \right. \\
&\quad \left. + \|f_j\|_{\mu,3} \sup_{r<|z|<1} \frac{\mu(|z|)(1-|\varphi(z)|^2)\sqrt{H_{\varphi(z)}\,[R\varphi(z)]}}{\mu(|\varphi(z)|)}\right\} \\
&\leq c'\varepsilon + c'' \sup_{r<|z|<1} \frac{\mu(|z|)}{\mu(|\varphi(z)|)}\frac{1 - |\varphi(z)|^2}{1 - |z|^2} \\
&\leq c'\varepsilon + c'' \sup_{r<|z|<1} \left\{\left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2}\right)^{a-1} + \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2}\right)^{b-1}\right\} \\
&< c'\varepsilon + 2c''\varepsilon^{a-1}.
\end{aligned}$$

This shows that

$$\lim_{j\to\infty} \|C_\varphi(f_j)\|_{\beta_\mu} = 0.$$

This implies that C_φ is compact on β_μ.

The proof is completed.

This is Proposition (4.3.5).

Lemma (4.3.16)[245]: Suppose $\epsilon \geq 0$, and μ is a normal function on $[0,1), \nu(1-\epsilon) = (2\epsilon - \epsilon^2)^2\mu(1-\epsilon)$ for $0 < \epsilon < 1$. If $f^2 \in A^{1+\epsilon}(\mu)$, then $f^2 \in \beta_\nu$ and $\|f^2\|_{\beta_\nu} \leq c\|f^2\|_{A^{1+\epsilon}(\mu)}$.

Proof. If $f^2 \in A^{1+\epsilon}(\mu)$, then we have

$$\int_B (1 - |z|^2)^{-\frac{1}{2}+\frac{5}{2}\epsilon+2\epsilon^2} |f^2(z)|^{1+\epsilon} dv(z) \leq \frac{\|f^2\|^{1+\epsilon}_{A^{1+\epsilon}(\mu)}}{\mu^{1+\epsilon}(0)} \Rightarrow f^2 \in A^{1+\epsilon}_{\frac{1}{2}+\frac{5}{2}\epsilon+2\epsilon^2}.$$

By Theorem 2.1 in [149], we have

$$|f^2(z)| \leq \frac{c\|f^2\|_{A^{1+\epsilon}_{-\frac{1}{2}+\frac{5}{2}\epsilon+2\epsilon^2}}}{(1 - |z|^2)^{2(1+\epsilon)}} \leq \frac{c'\|f^2\|_{A^{1+\epsilon}(\mu)}}{(1 - |z|^2)^{2(1+\epsilon)}}\ (z \in B).$$

This means that $f^2 \in A^1_{\frac{1}{2}+3\epsilon}$ when $\epsilon > 0$. By Theorem 2.2 in [149], we have

$$f^2(z) = \int_B \frac{f^2(z+\epsilon)}{(1 - \langle z, z+\epsilon\rangle)^{\frac{5}{2}+4\epsilon}} \, dv_{\frac{1}{2}+3\epsilon}(z+\epsilon) \ (z \in B).$$

By Lemma 2.2 in [2], Lemma 2.2 in [155], and Proposition 1.4.10 in [156], we have

$$\begin{aligned}
&\nu(|z|)|Rf^2(z)| \\
&\le c''\|f^2\|_{A^{1+\epsilon}(\mu)} \int_B \frac{(1 - |z|^2)^2\mu(|z|)(1 - |z+\epsilon|^2)^{3\epsilon-\frac{1}{2}}}{|1 - \langle z, z+\epsilon\rangle|^{\frac{7}{2}+4\epsilon}} \mu(|z+\epsilon|) \, dv(z+\epsilon) \\
&\le c''\|f^2\|_{A^{1+\epsilon}(\mu)}(1 - |z|^2)^{\frac{5}{2}+\epsilon} \int_B \frac{(1 - |z+\epsilon|^2)^{2\epsilon-1}}{|1 - \langle z, z+\epsilon\rangle|^{\frac{7}{2}+4\epsilon}} \, dv(z+\epsilon) \\
&+ c''\|f^2\|_{A^{1+\epsilon}(\mu)}(1 - |z|^2)^{\frac{5}{2}+2\epsilon} \int_B \frac{(1 - |z+\epsilon|^2)^{\epsilon-1}}{|1 - \langle z, z+\epsilon\rangle|^{\frac{7}{2}+4\epsilon}} \, dv(z+\epsilon) \\
&\le c''\|f^2\|_{A^{1+\epsilon}(\mu)}. \qquad (51)
\end{aligned}$$

On the other hand, by the subharmonicity of $|f^2|^{1+\epsilon}$ on B, we have

$$\begin{aligned}
\|f^2\|_{A^{1+\epsilon}(\mu)} &= \int_B \frac{|f^2(z)|^{1+\epsilon}\mu^{1+\epsilon}(|z|)}{1 - |z|^2} \, dv(z) \\
&= 2(1+\epsilon) \int_0^1 \frac{(1-\epsilon)^{1+2\epsilon}\mu^{1+\epsilon}(1-\epsilon)}{2\epsilon-\epsilon^2} \left\{\int_{S_{1+\epsilon}} \left|f^2\big((1-\epsilon)\xi\big)\right|^{1+\epsilon} d\sigma(\xi)\right\} d(1-\epsilon) \\
&\ge 2(1+\epsilon) \int_0^1 \frac{(1-\epsilon)^{1+2\epsilon}\mu^{1+\epsilon}(1-\epsilon)|f^2(0)|^{1+\epsilon}}{2\epsilon-\epsilon^2} d(1-\epsilon) \\
&\ge \frac{(1+\epsilon)!\,\Gamma\left(\frac{1}{2}+\frac{5}{2}\epsilon+2\epsilon^2\right)\mu^{1+\epsilon}(0)}{\Gamma\left(\frac{3}{2}+\frac{7}{2}\epsilon+2\epsilon^2\right)} |f^2(0)|^{1+\epsilon}. \qquad (52)
\end{aligned}$$

By (51)–(52) and Lemma (4.3.6), we have $f^2 \in \beta_\nu$ and $\|f^2\|_{\beta_\nu} \le c\|f^2\|_{A^{1+\epsilon}(\mu)}$.

Lemma (4.3.17)[245]: Let $d(z, z+\epsilon) = |(\varphi_r)_{z+\epsilon}(z)|$ $(z+\epsilon, z \in B)$ be the Bergman metric on B. Given $0 < \epsilon < \frac{1}{2}$ and $0 < \delta < \frac{1}{2}$, if $|z| > 1 - \delta$ and $d(z, z+\epsilon) \le \frac{1}{2} - \epsilon$, then

$$|z+\epsilon| > 1 - \frac{\delta\left(\frac{3}{2}-\epsilon\right)}{-\frac{1}{2}-\epsilon + \delta(\frac{1}{2}-\epsilon)}.$$

Proof. If $d(z, z+\epsilon) \le \frac{1}{2} - \epsilon$, by Lemma 1.2 in [149], then we have

$$\frac{(1 - |z|^2)(1 - |z+\epsilon|^2)}{(1 - |zk(z+\epsilon)|)^2} \ge \frac{(1 - |z|^2)(1 - |z+\epsilon|^2)}{|1 - \langle z, z+\epsilon\rangle|^2} = 1 - |\varphi_r(z+\epsilon)(z)|^2 \ge \frac{3}{4} + \epsilon - \epsilon^2$$

$$\Rightarrow \left(\left(\frac{3}{4}+\epsilon-\epsilon^2\right)|z+\epsilon|^2\right)|z|^2 - 2\left(\frac{3}{4}+\epsilon-\epsilon^2\right)|(z+\epsilon)kz| + |z+\epsilon|^2 - \left(\frac{1}{4}-\epsilon+\epsilon^2\right) \le 0$$

$$\Rightarrow |z| \le \frac{|z+\epsilon| + \frac{1}{2}-\epsilon}{1 + \left(\frac{1}{2}-\epsilon\right)|z+\epsilon|}.$$

When $|z| > 1-\delta$, we have

$$\frac{|z+\epsilon| + \frac{1}{2}-\epsilon}{1 + \left(\frac{1}{2}-\epsilon\right)|z+\epsilon|} > 1-\delta \Rightarrow |z+\epsilon| > 1 - \frac{\delta\left(\frac{3}{2}-\epsilon\right)}{\delta\left(\frac{1}{2}-\epsilon\right)-\left(\frac{1}{2}+\epsilon\right)}.$$

Theorem (4.3.18)[245]: Suppose $\epsilon \ge 1$, and μ is a normal function on [0, 1). Let φ_r be a holomorphic self-map of B and $h^2 \in H(B)$.

(i)If $\sup_{z\in B} \mu\frac{(|z|)|h^2(z)|}{(1-|\varphi_r(z)|^2)}(1+\epsilon) = M < \infty$, then $\sup_{z\in B} (1-|z|^2)\mu\frac{(|z|)|Rh^2(z)|}{(1-|\varphi_r(z)|^2)^{1+\epsilon}} \le cM.$

(ii)When $\|\varphi_r\|_\infty = \sup_{z\in B} |\varphi_r(z)| = 1$, if $\lim_{|\varphi_r(z)|\to 1-} \frac{\mu(|z|)h^2(z)}{\mu(|\varphi_r(z)|)} = 0$, then

$$\lim_{|\varphi_r(z)|\to 1-} \frac{(1-|z|^2)\mu(|z|)Rh^2(z)}{\mu(|\varphi_r(z)|)} = 0.$$

Proof. (i) For any $(z+\epsilon) \in B$, we take

$$F_{z+\epsilon}(z) = \frac{h^2(z)}{1-\langle\varphi_r(z),\varphi_r(z+\epsilon)\rangle^{1+\epsilon}} \quad (z \in B).$$

The theorem condition implies that

$$\sup_{z\in B} \mu(|z|)|F_{z+\epsilon}(z)| \le 2^{1+\epsilon}M. \tag{53}$$

By Theorem 2.2 in [149], we have

$$F_{z+\epsilon}(z) = \int_B \frac{F_{z+\epsilon}(\eta)}{(1-\langle z,\eta\rangle)^{\frac{3}{2}+4\epsilon}}\, dv_{-\frac{1}{2}+3\epsilon}(\eta) \text{when } \epsilon > 0 \ (z \in B).$$

By (53), Lemma 2.2 in [155], Proposition 1.4.10 in [156], we have

$$\mu(|z|)|RF_{z+\epsilon}(z)| \le \int_B \frac{cM(1-|\eta|^2)^{-\frac{1}{2}+3\epsilon}\mu|z|}{|1-\langle z,\eta\rangle|^{\frac{5}{2}+4\epsilon}\mu(|\eta|)}\, dv(\eta)$$

$$\le cM\int_B \frac{(1-|z|^2)^{\frac{1}{2}+\epsilon}(1-|\eta|^2)^{2\epsilon-1}}{|1-\langle z,\eta\rangle|^{\frac{5}{2}+4\epsilon}}\, dv(\eta)$$

$$+cM\int_B \frac{(1-|z|^2)^{\frac{1}{2}+2\epsilon}(1-|\eta|^2)^{\epsilon-1}}{|1-\langle z,\eta\rangle|^{\frac{5}{2}+4\epsilon}}\, dv(\eta)$$

$$\le \frac{c'M}{1-|z|^2}. \tag{54}$$

$H_{\varphi_r(z+\epsilon)}(R\varphi_r(z+\epsilon)) \le cH_{z+\epsilon}(z+\epsilon)$ shows that

$$\frac{(1-|z+\epsilon|^2)|\langle R\varphi_r(z+\epsilon),\varphi_r(z+\epsilon)\rangle|}{(1-|\varphi_r(z+\epsilon)|^2)} \le c(1-|z+\epsilon|^2)\{H_{\varphi_r(z+\epsilon)}(R\varphi_r(z+\epsilon))\}^{\frac{1}{2}} \le c. \quad (55)$$

Taking $\epsilon = 0$ in (54), by computation and (55), we have

$$\frac{(1-|z+\epsilon|^2)\mu(|z+\epsilon|)|Rh^2(z+\epsilon)|}{(1-|\varphi_r(z+\epsilon)|^2)^{1+\epsilon}}$$
$$\le c''M + \frac{(1+\epsilon)|h^2(z+\epsilon)|(1-|z+\epsilon|^2)\mu(|z+\epsilon|)|\langle R\varphi_r(z+\epsilon),\varphi_r(z+\epsilon)\rangle|}{(1-|\varphi_r(z+\epsilon)|^2)^{2+\epsilon}}$$
$$\le c'''M.$$

(ii) When $\|\varphi_r\|_\infty = 1$, if

$$\lim_{|\varphi_r(z)|\to 1-} \frac{\mu(|z|)h^2(z)}{\mu(|\varphi_r(z)|)} = 0,$$

then, for any $0 < \varepsilon < 1$, there exists $0 < \delta < 1/2$ such that

$$\frac{\mu(|z|)|h^2(z)|}{\mu(|\varphi_r(z)|)} < \varepsilon \text{ when } |\varphi_r(z)| > 1 - 2\delta.$$

For any sequence $\{z^j\} \subset B$ satisfying $\lim_{j\to\infty} |\varphi_r(z^j)| = 1$, we write

$$\overline{D}\left(z^j, \frac{1}{3}\right) = \left\{z+\epsilon : (z+\epsilon) \in B \text{ and } d(z+\epsilon, z^j) \le \frac{1}{3}\right\}.$$

As $|\varphi_r(z^j)| \to 1$ $(j \to \infty)$, then there exists positive integer N such that $|\varphi_r(z^j)| > 1 - \delta$ when $j > N$. By Theorem 8.1.4 in [156], we have

$$d\left(\varphi_r(z+\epsilon), \varphi_r(z^j)\right) \le d(z+\epsilon, z^j) \le \frac{1}{3} \text{ when } j > N \text{ and } (z+\epsilon) \in \overline{D}\left(z^j, \frac{1}{3}\right).$$

We take $\epsilon = \frac{2}{3}$ in Lemma (4.3.17), then

$$|\varphi_r(z+\epsilon)| > 1 - \frac{4\delta}{2+\delta} > 1 - 2\delta \Rightarrow \frac{\mu(|z+\epsilon|)|h^2(z+\epsilon)|}{\mu(|\varphi_r(z+\epsilon)|)} < \varepsilon. \quad (56)$$

On the other hand, by Corollary 1.22 in [149], we have

$$d((\varphi_r)_{z^j}(z), z^j) = d\left((\varphi_r)_{z^j}(z), (\varphi_r)_{z^j}(0)\right) = d(z,0) = |z| \le \frac{1}{6} \text{ when } |z| \le \frac{1}{6}.$$

If $|z| \le 1/6$ and $(z+\epsilon) \in \overline{D}\left((\varphi_r)_{z^j}(z), \frac{1}{6}\right)$, then

$$d(z+\epsilon, z^j) \le d\left(z+\epsilon, (\varphi_r)_{z^j}(z)\right) + d((\varphi_r)_{z^j}(z), z^j) \le \frac{1}{3} \Rightarrow \overline{D}\left((\varphi_r)_{z^j}(z), \frac{1}{6}\right)$$
$$\subseteq \overline{D}\left(z^j, \frac{1}{3}\right). \quad (57)$$

We take $G_j = F_j \circ (\varphi_r)_{z^j}$, where $F_j(z) = \mu(|z^j|)g^2(\langle\varphi_r(z), \varphi_r(z^j)\rangle)h^2(z)$, and g^2 is the function in Lemma (4.3.10).

By Lemma 2.24 and Lemma 2.20 in [149], Lemma 1.2 and Lemma 1.23 in [149], (56)–(57), Lemma 2.2 in [155] and Lemma (4.3.10), we have

$$\left|G_j(z)\right| = \left|F_j[\varphi_r(z)]\right| \le \frac{c'}{(1-|(\varphi_r)_{z^j}(z)|^2)^{2+\epsilon}} \int_{\overline{D}\left((\varphi_r)_{z^j}(z),\frac{1}{6}\right)} \left|F_j(z+\epsilon)\right| dv(z+\epsilon)$$
$$\le \frac{c'}{(1-|(\varphi_r)_{z^j}(z)|^2)^{2+\epsilon}} \int_{\overline{D}\left((\varphi_r)_{z^j}(z),\frac{1}{3}\right)} \left|F_j(z+\epsilon)\right| dv(z+\epsilon)$$
$$\le c''\varepsilon\left(1 - \left|z^j\right|^2\right)^{-(2+\epsilon)} \int_{\overline{D}\left((\varphi_r)_{z^j}(z),\frac{1}{3}\right)} \mu(|\varphi_r(z+\epsilon)|) g^2(|\varphi_r(z+\epsilon)|) dv(z+\epsilon)$$
$\le c\varepsilon$ when $|z| \le 1/6$ and $j > N$.

This shows that $\{G_j(z)\}$ converges to 0 uniformly on $\{z : |z| \le 1/6\}$. Thus, $\{|\nabla G_j(z)|\}$ must converge to 0 uniformly on $|z| \le 1/12$. In particular,

$$\lim_{j\to\infty} \left|\nabla G_j(0)\right| = 0. \quad (58)$$

On the other hand, by Lemma 2.14 and (2.11) in [149], we have

$$\left|\nabla G_j(0)\right|^2 = \left|\overline{\nabla} F_j\left(z^j\right)\right|^2 \ge \left(1 - \left|z^j\right|^2\right)^2 \left|RF_j\left(z^j\right)\right|^2. \quad (59)$$

(58) and (59) show that

$$\lim_{j\to\infty} \left(1 - \left|z^j\right|^2\right)\left|RF_j\left(z^j\right)\right| = 0. \quad (60)$$

By computation, we have

$$RF_j(z^j) = \mu\left(\left|z^j\right|\right) g^{2\prime}\left(\left|\varphi_r\left(z^j\right)\right|^2\right)\langle R\varphi_r\left(z^j\right), \varphi_r\left(z^j\right)\rangle h^2(z^j) + \mu(|z^j|) g^2(\left|\varphi_r\left(z^j\right)\right|^2) Rh^2(z^j). \quad (61)$$

By (55) and (60)–(61), and Lemma (4.3.10) and Lemma 2.2 in [155], we have

$$\frac{\left(1 - \left|z^j\right|^2\right)\mu\left(\left|z^j\right|\right)\left|Rh^2\left(z^j\right)\right|}{\mu(|\varphi_r(z^j)|)} = \frac{1 - \left|z^j\right|^2}{\mu(|\varphi_r(z^j)|) g^2(|\varphi_r(z^j)|^2)} \mu\left(\left|z^j\right|\right) g^2\left(\left|\varphi_r\left(z^j\right)\right|^2\right) \left| Rh^2\left(z^j\right)\right|$$
$$\le c\left(1 - \left|z^j\right|^2\right)\left|RF_j\left(z^j\right)\right| + \frac{c'\mu\left(\left|z^j\right|\right)\left|h^2\left(z^j\right)\right|}{\mu(|\varphi_r(z^j)|)}$$
$$\Rightarrow \lim_{j\to\infty} \left(1 - \left|z^j\right|^2\right)\mu(|z^j|) Rh^2(z^j)\, \mu\left(\left|\varphi_r\left(z^j\right)\right|\right) = 0.$$

This means that

$$\lim_{|\varphi_r(z)|\to 1-} \frac{(1 - |z|^2)\mu(|z|) Rh^2(z)}{\mu(|\varphi_r(z)|)} = 0.$$

The proof is completed.

Theorem (4.3.19)[245]: Suppose $\epsilon \ge 0$, and μ is a normal function on $[0,1), \nu(1-\epsilon) = (2\epsilon - \epsilon^2)^2 \mu(1-\epsilon)$ for $0 < \epsilon < 1$. If φ_r is a holomorphic self-map of B and $\psi_r \in H(B)$, then

(i) T_{φ_r,ψ_r} is bounded from $A^{1+\epsilon}(\mu)$ to β_ν if and only if

$$M_0 = \sup_{z\in B} \left\{|\psi_r(z)|\left(\frac{1 - |z|^2}{1 - |\varphi_r(z)|^2}\right) \frac{\mu(|z|)}{\mu(|\varphi_r(z)|)}\right\} < \infty; \quad (62)$$

(ii) T_{φ_r,ψ_r} is compact from $A^{1+\epsilon}$ (μ) to β_ν if and only if $\psi_r \in \beta_\nu$ when $\|\varphi_r\|_\infty < 1$;

(iii) T_{φ_r,ψ_r} is compact from $A^{1+\epsilon}$ (μ) to β_ν if and only if $\psi_r \in \beta_\nu$ and

$$\lim_{|\varphi_r(z)|\to 1-} \left(\frac{1 - |z|^2}{1 - |\varphi_r(z)|^2}\right) \frac{\mu(|z|)}{\mu(|\varphi_r(z)|)} = 0 \quad (63)$$

when $\|\varphi_r\|_\infty = 1$.

Proof. (i) For any $f^2 \in A^{1+\epsilon}$ (μ), by Lemma 2.2 in [2], we have

$$|f^2(z)| \leq \frac{c\|f^2\|_{A^{1+\epsilon}(\mu)}}{(1 - |z|^2)\,\mu(|z|)} (z \in B). \quad (64)$$

If (62) holds, then we take $\epsilon = 1, h^2 = T_{\varphi_r,\psi_r}(f^2)$ and the normal function $(2\epsilon - \epsilon^2)\ \mu(1-\epsilon)$ in Theorem (4.3.11). By Lemma 2.2 in [155] and (64), for any $z \in B$, we have

$$\begin{aligned}
&(1 - |z|^2)^2\mu(|z|)\left|R\left[T_{\varphi_r,\psi_r}(f^2)\right](z)\right| \\
&\leq c' \sup_{z+\epsilon\in B} (1 - |z+\epsilon|^2)\ \mu(|z+\epsilon|)\left|T_{\varphi_r,\psi_r}(f^2)(z+\epsilon)\right| \\
&\leq c'' \sup_{z+\epsilon\in B} \left\{\frac{|\psi_r(z+\epsilon)|\mu(|z+\epsilon|)}{\mu(|\varphi_r(z+\epsilon)|)}\left(\frac{1 - |z+\epsilon|^2}{1 - |\varphi_r(z+\epsilon)|^2}\right)\right\} \|f^2\|_{A^{1+\epsilon}(\mu)} \\
&\leq c'' M_0\|f^2\|_{A^{1+\epsilon}(\mu)}. \quad (65)
\end{aligned}$$

By (64)–(65) and Lemma (4.3.6), then T_{φ_r,ψ_r} is bounded from $A^{1+\epsilon}(\mu)$ to β_ν. Conversely, if T_{φ_r,ψ_r} is bounded from $A^{1+\epsilon}$ (μ) to β_ν, then, for any $(z+\epsilon) \in B$, we take

$$f^2_{z+\epsilon}(z) = \frac{(1 - |\varphi_r(z+\epsilon)|^2)^{\frac{3}{2}+2\epsilon}}{\mu(|\varphi_r(z+\epsilon)|)(1 - \langle z, \varphi_r(z+\epsilon)\rangle)^{\frac{5}{2}+2\epsilon}}.$$

By the definition of normal function, we have

$$\begin{aligned}
\mu^{1+\epsilon}(|z|) \leq{}& \frac{\mu^{1+\epsilon}(|\varphi_r(z+\epsilon)|)(1 - |z|^2)^{\frac{1}{2}+\frac{5}{2}\epsilon+2\epsilon^2}}{(1 - |\varphi_r(z+\epsilon)|^2)^{\frac{1}{2}+\frac{5}{2}\epsilon+2\epsilon^2}} \\
&+ \frac{\mu^{1+\epsilon}(|\varphi_r(z+\epsilon)|)(1 - |z|^2)^{\frac{1}{2}+\frac{3}{2}\epsilon+\epsilon^2}}{(1 - |\varphi_r(z+\epsilon)|^2)^{\frac{1}{2}+\frac{3}{2}\epsilon+\epsilon^2}}. \quad (66)
\end{aligned}$$

By Proposition 1.4.10 in [156] and (66), we obtain

$$\begin{aligned}
&\|f^2_{z+\epsilon}\|^{1+\epsilon}_{A^{1+\epsilon}(\mu)} \\
&= \frac{(1 - |\varphi_r(z+\epsilon)|^2)^{\frac{3}{2}+\frac{7}{2}\epsilon+2\epsilon^2}}{\mu^{1+\epsilon}(|\varphi_r(z+\epsilon)|)}\int_B \frac{\mu^{1+\epsilon}(|z|)}{(1 - |z|^2)|1 - \langle z, \varphi_r(z+\epsilon)\rangle|^{\frac{5}{2}+\frac{9}{2}\epsilon+2\epsilon^2}} dv(z) \\
&\leq (1 - |\varphi_r(z+\epsilon)|^2)^{1+\epsilon}\int_B \frac{(1 - |z|^2)^{-\frac{1}{2}+\frac{5}{2}\epsilon+2\epsilon^2}}{|1 - \langle z, \varphi_r(z+\epsilon)\rangle|^{\frac{5}{2}+\frac{9}{2}\epsilon+2\epsilon^2}} dv(z) \\
&+ (1 - |\varphi_r(z+\epsilon)|^2)^{\frac{3}{2}+\frac{7}{2}\epsilon+2\epsilon^2}\int_B \frac{(1 - |z|^2)^{-\frac{1}{2}+\frac{5}{2}\epsilon+2\epsilon^2}}{|1 - \langle z, \varphi_r(z+\epsilon)\rangle|^{\frac{5}{2}+\frac{9}{2}\epsilon+2\epsilon^2}} dv(z) \leq c.
\end{aligned}$$

On the other hand, by Lemma 2.2 in [2] and Lemma (4.3.6), taking $\epsilon = 0$, we have

$$|T_{\varphi_r,\psi_r}(f^2_{z+\epsilon})(z+\epsilon)| \le c\left(1 + \int_0^{|z+\epsilon|} \frac{1}{\nu(1+\epsilon)}\, d(1+\epsilon)\right) \left\|T_{\varphi_r,\psi_r}(f^2_{z+\epsilon})\right\|_{(\epsilon-1)_\nu}$$

$$\Rightarrow \frac{|\psi_r(z+\epsilon)|\mu(|z+\epsilon|)}{\mu(|\varphi_r(z+\epsilon)|)} \left(\frac{1 - |z+\epsilon|^2}{1 - |\varphi_r(z+\epsilon)|^2}\right)$$

$$\le c'(1 - |z+\epsilon|^2)\ \mu(|z+\epsilon|)\left(1 + \int_0^{|z+\epsilon|} \frac{1}{\nu(1+\epsilon)}\, d(1+\epsilon)\right) \left\|T_{\varphi_r,\psi_r}\right\|$$

$$\le c'\mu(0)\left\|T_{\varphi_r,\psi_r}\right\| + c'\left\|T_{\varphi_r,\psi_r}\right\|(1 - |z+\epsilon|^2) \int_0^{|z+\epsilon|} \frac{\mu(|z+\epsilon|)}{(2\epsilon-\epsilon^2)^2\mu(1+\epsilon)}\, d(1+\epsilon)$$

$$\le c'\mu(0)\left\|T_{\varphi_r,\psi_r}\right\| + c'\left\|T_{\varphi_r,\psi_r}\right\|(1 - |z+\epsilon|^2) + \left(\frac{1}{2} + \epsilon\right)\int_0^{|z+\epsilon|} \frac{1}{(2\epsilon-\epsilon^2)^{\frac{5}{2}+\epsilon}}\, d(1+\epsilon)$$

$$+ c'\left\|T_{\varphi_r,\psi_r}\right\|(1 - |z+\epsilon|^2)^{\frac{3}{2}+2\epsilon} \int_0^{|z+\epsilon|} \frac{1}{(2\epsilon-\epsilon^2)^{\frac{5}{2}+2\epsilon}}\, d(1+\epsilon) \le c'\mu(0)\left\|T_{\varphi_r,\psi_r}\right\| + c''\left\|T_{\varphi_r,\psi_r}\right\|.$$

This shows that (62) holds.

(ii)–(iii) If T_{φ_r,ψ_r} is compact from $A^{1+\epsilon}(\mu)$ to β_ν, then $\psi_r \in \beta_\nu$ by taking $f^2(z) = 1$. When $\|\varphi_r\|_\infty = 1$, we prove that (63) holds.

Let $\{z^j\} \subset B$ be any sequence satisfying $|\varphi_r(z^j)| \to 1$ $(j \to \infty)$. We take

$$f_j^2(z) = \frac{\left(1 - \left|\varphi_r(z^j)\right|^2\right)^{\frac{3}{2}+2\epsilon}}{\mu(|\varphi_r(z^j)|)(1 - \langle z, \varphi_r(z^j)\rangle)^{\frac{5}{2}+2\epsilon}}.$$

Then, $\left\|f_j^2\right\|_{A^{1+\epsilon}(\mu)} \le c$ and $\{f_j^2(z)\}$ converges to 0 uniformly on any compact subset of B. This means that

$$\lim_{j\to\infty} \left\|T_{\varphi_r,\psi_r}(f_j^2)\right\|_{\beta_\nu} = 0.$$

On the other hand, we have

$$\left|T_{\varphi_r,\psi_r}(f_j^2)(z^j)\right| \le c\left(1 + \int_0^{|z^j|} \frac{1}{\nu(1+\epsilon)}\, d(1+\epsilon)\right) \left\|T_{\varphi_r,\psi_r}(f_j^2)\right\|_{\beta_\nu}$$

$$\Rightarrow \frac{|\psi_r(z^j)|\mu(|z^j|)}{\mu(|\varphi_r(z^j)|)} \left(\frac{1 - |z^j|^2}{1 - |\varphi_r(z^j)|^2}\right)$$

$$\le c\left(1 - |z^j|^2\right)\ \mu(|z^j|)\left(1 + \int_0^{|z^j|} \frac{1}{\nu(1+\epsilon)}\, d(1+\epsilon)\right) \left\|T_{\varphi_r,\psi_r}(f_j^2)\right\|_{\beta_\nu}$$

$$\le c\mu(0)\left\|T_{\varphi_r,\psi_r}(f_j^2)\right\|_{\beta_\nu} + c'\left\|T_{\varphi_r,\psi_r}(f_j^2)\right\|_{\beta_\nu}.$$

This shows that

$$\lim_{j\to\infty} \frac{|\psi_r(z^j)|\mu(|z^j|)}{\mu(|\varphi_r(z^j)|)} \left(\frac{1 - |z^j|^2}{1 - |\varphi_r(z^j)|^2}\right) = 0.$$

This means that (63) holds.

Conversely, for all $l \in \{1, 2, \cdots, 1+\epsilon\}$, we have $(\varphi_r)_l \in H^\infty \subset \epsilon - 1$. This means that $(1 - |z|^2)|R(\varphi_r)_l(z)| \le \|(\varphi_r)_l\|_{\epsilon-1}$ holds for all $z \in B$.

Let $\{f_j^2(z)\}$ be any sequence which converges to 0 uniformly on any compact subset of B and $\|f_j^2\|_{A^{1+\epsilon}(\mu)} \le 1$. When $\|\varphi_r\|_\infty < 1$, we know that $\{|\nabla f_j^2(z+\epsilon)|\}$ converges to 0 uniformly on $\{z+\epsilon : |z+\epsilon| \le \|\varphi_r\|_\infty\}$ and $\{f_j^2(\varphi_r(0))\}$ converges to 0. If $\psi_r \in \beta_\nu$, by Lemma (4.3.6), Lemma 2.2 in [2], then

$$\begin{aligned}\|T_{\varphi_r,\psi_r}(f_j^2)\|_{\beta_\nu} &\le |\psi_r(0)| \cdot |f_j^2[\varphi_r(0)]| + c' \sup_{z\in B} \nu(|z|)|R[T_{\varphi_r,\psi_r}(f_j^2)](z)| \\ &\le |\psi_r(0)| \cdot |f_j^2[\varphi_r(0)]| + c' \sup_{z\in B} \nu(|z|)|R\psi_r(z)| \cdot |f_j^2[\varphi_r(z)]| \\ &+ c' \sup_{z\in B} \nu(|z|)|\psi_r(z)| \cdot |\langle \nabla f_j^2[\varphi_r(z)], R\varphi_r(z)\rangle| \\ &\le |\psi_r(0)| \cdot |f_j^2[\varphi_r(0)]| + c' \|\psi_r\|_{\beta_\nu} \sup_{|z+\epsilon|\le\|\varphi_r\|_\infty} |f_j^2(z+\epsilon)| \\ &+ (c''\mu(0) + c''')\|\psi_r\|_{\beta_\nu} \left(\sum_{l=1}^{1+\epsilon} \|(\varphi_r)_l\|_\beta\right) \sup_{|z+\epsilon|\le\|\varphi_r\|_\infty} |\nabla f_j^2(z+\epsilon)| \\ &\to 0\ (j \to \infty).\end{aligned}$$

When $\|\varphi_r\|_\infty = 1$, if (63) holds, then we take the normal function $(2\epsilon - \epsilon^2)\ \mu(1-\epsilon)$ and $h^2 = \psi_r$ in Theorem (4.3.18). This means that

$$\lim_{|\varphi_r(z)|\to 1^-} |R\psi_r(z)| \frac{(1 - |z|^2)^2}{(1 - |\varphi_r(z)|^2)} \frac{\mu(|z|)}{\mu(|\varphi_r(z)|)} = 0.$$

Therefore, for any $\varepsilon > 0$, there exists $0 < \delta < 1$ such that

$$|\psi_r(z)|\left(\frac{1-|z|^2}{1-|\varphi_r(z)|^2}\right)\frac{\mu(|z|)}{\mu(|\varphi_r(z)|)} < \varepsilon \text{ and } \frac{|R\psi_r(z)|(1 - |z|^2)^2}{(1 - |\varphi_r(z)|^2)}\frac{\mu(|z|)}{\mu(|\varphi_r(z)|)} < \varepsilon \quad (67)$$

when $|\varphi_r(z)| > \delta$. As $\{f_j^2(z)\}$ converges to 0 uniformly on any compact subset of B, then $\{|\nabla f_j^2(z+\epsilon)|\}$ converges to 0 uniformly on $\{z+\epsilon : |z+\epsilon| \le \delta\}$ and $\{f_j^2(\varphi_r(0))\}$ converges to 0. Therefore, there exists positive integer N such that

$$\sup_{|z+\epsilon|\le\delta} |f_j^2(z+\epsilon)| < \varepsilon, \sup_{|z+\epsilon|\le\delta} |\nabla f_j^2(z+\epsilon)| < \varepsilon,$$

$$\text{and } |f_j^2(\varphi_r(0))| < \varepsilon \text{ when } j > N.$$

When $|\varphi_r(z)| \le \delta$ and $j > N$, we have

$$\begin{aligned}&\nu(|z|)|R[T_{\varphi_r,\psi_r}(f_j^2)](z)| \\ &\le \|\psi_r\|_{\beta_\nu} \sup_{|z+\epsilon|\le\delta} |f_j^2(z+\epsilon)| \\ &\quad + (c'\mu(0) + c'')\|\psi_r\|_{\beta_\nu}\left(\sum_{l=1}^{1+\epsilon} \|\varphi_{r_l}\|_\beta\right) \sup_{|z+\epsilon|\le\delta} |\nabla f_j^2(z+\epsilon)| < c\varepsilon. \quad (68)\end{aligned}$$

If $j > N$, by Lemma (4.3.6)–(4.3.16), (64), (67)–(68), and $H_{\varphi_r(z)}[R\varphi_r(z)] \le cH_z(z)$, then

$$\left\|T_{\varphi_r,\psi_r}\left(f_j^2\right)\right\|_{\beta_\nu} \leq |\psi_r(0)| \cdot \left|f_j^2\left(\varphi_r(0)\right)\right| + c'\left(\sup_{|\varphi_r(z)|>\delta} + \sup_{|\varphi_r(z)|\leq\delta}\right)\nu(|z|)\left|R\left[T_{\varphi_r,\psi_r}\left(f_j^2\right)\right](z)\right|$$

$$< c''\varepsilon + c'\varepsilon \sup_{|\varphi_r(z)|>\delta} \frac{(1-|\varphi_r(z)|^2)^2\,\mu(|\varphi_r(z)|)\left|\langle \nabla f_j^2[\varphi_r(z)], \overline{R\varphi_r(z)}\rangle\right|(1-|z|^2)}{\sqrt{(1-|\varphi_r(z)|^2)|R\varphi_r(z)|^2 + |\langle R\varphi_r(z), \varphi_r(z)\rangle|^2}} \sqrt{H_{\varphi_r(z)}\left[R\varphi_r(z)\right]}$$

$$< c''\varepsilon + c'''\varepsilon\left\|f_j^2\right\|_{\beta_\nu} \leq c''\varepsilon + c\varepsilon\left\|f_j^2\right\|_{A^{1+\epsilon}(\mu)} \leq c''\varepsilon + c\varepsilon.$$

This shows that

$$\lim_{j\to\infty} \left\|T_{\varphi_r,\psi_r}\left(f_j^2\right)\right\|_{\beta_\nu} = 0.$$

This means that T_{φ_r,ψ_r} is compact from $A^{1+\epsilon}$ (μ) to β_ν. The proof is completed.

Corollary (4.3.20)[245]: Suppose $\epsilon \geq 0$, and μ is a normal function on $[0,1), \nu(1-\epsilon) = (2\epsilon-\epsilon^2)^2\mu(1-\epsilon)$ for $0<\epsilon<1$. If φ_r is a automorphism of B and $\psi_r \in H(B)$, then

(i) T_{φ_r,ψ_r} is bounded from $A^{1+\epsilon}$ (μ) to β_ν if and only if $\psi_r \in H^\infty$;

(ii) T_{φ_r,ψ_r} is compact from $A^{1+\epsilon}$ (μ) to β_ν if and only if $\psi_r \equiv 0$.

Proof. By Lemma 2.2 in [155], Lemma (4.3.9), there exists constant $c > 0$ such that

$$\frac{1}{c} \leq \left\{\frac{1-|z|^2}{1-|\varphi_r(z)|^2}\right\}\frac{\mu(|z|)}{\mu(|\varphi_r(z)|)} \leq c. \tag{69}$$

On the other hand, if φ_r is a automorphism of B, then $|\varphi_r(z)| \to 1^-$ if and only if $|z| \to 1^-$. Therefore, by (69), Theorem (4.3.19), and the Maximum Modulus Principle, we can obtain the result.

Corollary (4.3.21)[245]: Suppose $\epsilon \geq 0$, and μ is a normal function on $[0,1), \nu(1-\epsilon) = (2\epsilon-\epsilon^2)^2\mu(1-\epsilon)$ for $0<\epsilon<1$. If φ_r is a holomorphic self-map of B, then

(i) C_{φ_r} is always bounded from $A^{1+\epsilon}$ (μ) to β_ν ;

(ii) C_{φ_r} is compact from $A^{1+\epsilon}$ (μ) to β_ν if and only if

$$\lim_{|z|\to 1^-}\frac{1-|z|^2}{1-|\varphi_r(z)|^2} = 0. \tag{70}$$

Proof. (i) By Lemma 2.2 in [155], we have

$$\left(\frac{1-|z|^2}{1-|\varphi_r(z)|^2}\right)\frac{\mu(|z|)}{\mu(|\varphi_r(z)|)} \leq \left(\frac{1-|z|^2}{1-|\varphi_r(z)|^2}\right)^{\frac{3}{2}+\epsilon} + \left(\frac{1-|z|^2}{1-|\varphi_r(z)|^2}\right)^{\frac{3}{2}+2\epsilon}$$
$$\leq \left(\frac{1+|\varphi_r(0)|}{1-|\varphi_r(0)|}\right)^{\frac{3}{2}+\epsilon} + \left(\frac{1+|\varphi_r(0)|}{1-|\varphi_r(0)|}\right)^{\frac{3}{2}+2\epsilon}.$$

By Theorem (4.3.19), C_{φ_r} is always bounded from $A^{1+\epsilon}$ (μ) to β_ν.

(ii) If (70) holds and $\|\varphi_r\|_\infty = 1$, then we have

$$\left(\frac{1-|z|^2}{1-|\varphi_r(z)|^2}\right)\frac{\mu(|z|)}{\mu(|\varphi_r(z)|)} \leq \left(\frac{1-|z|^2}{1-|\varphi_r(z)|^2}\right)^{\frac{3}{2}+\epsilon} + \left(\frac{1-|z|^2}{1-|\varphi_r(z)|^2}\right)^{\frac{3}{2}+2\epsilon} \quad (|\varphi_r(z)| \to 1^-).$$

By Theorem (4.3.19), C_{φ_r} is compact from $A^{1+\epsilon}$ (μ) to β_ν.

Conversely, if C_{φ_r} is compact from $A^{1+\epsilon}$ (μ) to β_ν and $\|\varphi_r\|_\infty = 1$, then, by Theorem (4.3.19), we have

$$0 \leftarrow \left(\frac{1 - |z|^2}{1 - |\varphi_r(z)|^2}\right) \frac{\mu(|z|)}{\mu(|\varphi_r(z)|)} \geq \left(\frac{1 - |z|^2}{1 - |\varphi_r(z)|^2}\right)^{\frac{3}{2}+2\epsilon} \left\{1 + \left(\frac{1 - |z|^2}{1 - |\varphi_r(z)|^2}\right)^{\epsilon}\right\}^{-1}$$
$$\geq \left(\frac{1 - |z|^2}{1 - |\varphi_r(z)|^2}\right)^{\frac{3}{2}+2\epsilon} \left\{1 + \left(\frac{1 - |z|^2}{1 - |\varphi_r(z)|^2}\right)^{\epsilon}\right\}^{-1} \quad (|\varphi_r(z)| \to 1^-).$$

This shows that

$$\lim_{|\varphi_r(z)| \to 1^-} \frac{1 - |z|^2}{1 - |\varphi_r(z)|^2} = 0.$$

Therefore, for any $\varepsilon > 0$, there exists $0 < \delta < 1$ such that $\frac{1-|z|^2}{1-|\varphi_r(z)|^2} < \varepsilon$ when $|\varphi_r(z)| > \delta$. On the other hand,

$$\lim_{|z| \to 1^-} \frac{1 - |z|^2}{1 - \delta^2} = 0,$$

then there exists $0 < \delta_0 < 1$ such that $\frac{1 - |z|^2}{1 - \delta^2} < \varepsilon$ when $|z| > \delta_0$. When $|z| > \delta_0$, if $|\varphi_r(z)| \leq \delta$, then

$$\frac{1 - |z|^2}{1 - |\varphi_r(z)|^2} \leq \frac{1 - |z|^2}{1 - \delta^2} < \varepsilon; \text{ if } |\varphi_r(z)| > \delta, \text{then } \frac{1 - |z|^2}{1 - |\varphi_r(z)|^2} < \varepsilon.$$

This shows that

$$\lim_{|z| \to 1^-} \frac{1 - |z|^2}{1 - |\varphi_r(z)|^2} = 0.$$

The proof is completed.

Theorem (4.3.22)[245]: Let μ be normal on [0, 1), and φ_r be a holomorphic self- map of B. If $\epsilon > 0$, then C_{φ_r} is compact on β_μ if and only if

$$\lim_{|z| \to 1^-} \frac{1 - |z|^2}{1 - |\varphi_r(z)|^2} = 0.$$

Proof. First, suppose that C_{φ_r} is compact on β_μ. If $\|\varphi_r\|_\infty < 1$, then we have

$$\lim_{|z| \to 1^-} \frac{1 - |z|^2}{1 - |\varphi_r(z)|^2} = 0.$$

When $\|\varphi_r\|_\infty = 1$, let $\{z^j\} \subset B$ be any sequence satisfying

$$\lim_{j \to \infty} |\varphi_r(z^j)| = 1.$$

We take $f_j^2(z) = \left(1 - |\varphi_r(z^j)|^2\right) g^2(\langle z, \varphi_r(z^j)\rangle)$, where g^2 is the function in Lemma (4.3.10). By Lemma (4.3.10), we have

$$\mu(|z|)|\nabla f_j^2(z)| = \mu(|z|)\left(1 - |\varphi_r(z^j)|^2\right)\left|g^{2\prime}(\langle z, \varphi_r(z^j)\rangle)\,\overline{\varphi_r(z^j)}\right|$$
$$\leq 2\mu(|\langle z, \varphi_r(z^j)\rangle|)(1 - |\langle z, \varphi_r(z^j)\rangle|)g^{2\prime}(|\langle z, \varphi_r(z^j)\rangle|) \leq c.$$

Therefore, $\|f_j^2\|_{\beta_\mu} \leq 1 + c$ and $\{f_j^2(z)\}$ converges to 0 uniformly on any compact subset of B. This means that

$$\lim_{j \to \infty} \|C_{\varphi_r}(f_j^2)\|_{\beta_\mu} = 0.$$

On the other hand, by Lemma (4.3.10), Lemma (4.3.6), Lemma 2.2 in [2], and Lemma 2.2 in [155], we have

$$0 < c \le \mu\left(\left|\varphi_r(z^j)\right|^2\right) g^2\left(\left|\varphi_r(z^j)\right|^2\right) = \frac{\mu\left(\left|\varphi_r(z^j)\right|^2\right)\left|C_{\varphi_r}(f_j^2)(z^j)\right|}{1 - |\varphi_r(z^j)|^2}$$

$$\le \frac{c'\mu\left(\left|\varphi_r(z^j)\right|^2\right)}{1 - |\varphi_r(z^j)|^2}\left(1 + \int_0^{|z^j|} \frac{1}{\mu(1+\epsilon)}\, d(1+\epsilon)\right) \left\|C_{\varphi_r}(f_j^2)\right\|_{\beta_\mu}$$

$$\le \left\{c'\mu(0)\left(1 - \left|\varphi_r(z^j)\right|^4\right)^\epsilon + \frac{c''\left(1 - \left|\varphi_r(z^j)\right|^2\right)^\epsilon}{(1 - |z^j|^2)^\epsilon}\right.$$

$$\left. + \frac{c''\left(1 - \left|\varphi_r(z^j)\right|^2\right)^{2\epsilon}}{(1 - |z^j|^2)^{2\epsilon}}\right\}\left\|C_{\varphi_r}(f_j^2)\right\|_{\beta_\mu} \Rightarrow \left(\frac{1 - |z|^2}{1 - |\varphi_r(z)|^2}\right)^{2\epsilon}$$

$$\le \frac{1}{c}\left\{\frac{c'''\left(1 - \left|\varphi_r(z^j)\right|^2\right)^{2\epsilon}}{(1 - |\varphi_r(z^j)|^2)^\epsilon} + \frac{c'''\left(1 - \left|\varphi_r(z^j)\right|^2\right)^\epsilon}{(1 - |\varphi_r(z^j)|^2)^\epsilon} + c''\right\}\left\|C_{\varphi_r}(f_j^2)\right\|_{\beta_\mu}$$

$$\le \frac{1}{c}\left\{\frac{c'''(1 + |\varphi_r(0)|)^\epsilon}{(1 - |\varphi_r(0)|)^\epsilon} + \frac{c''(1 + |\varphi_r(0)|)^\epsilon}{(1 - |\varphi_r(0)|)^\epsilon} + c''\right\}\left\|C_{\varphi_r}(f_j^2)\right\|_{\beta_\mu}.$$

By $\epsilon > 1$, this shows that

$$\lim_{j\to\infty\ 1} \frac{1 - |z|^2}{1 - |\varphi_r(z)|^2} = 0 \Rightarrow \lim_{|\varphi_r(z)|\to 1^-} \frac{1 - |z|^2}{1 - |\varphi_r(z)|^2} = 0.$$

By Proof of Corollary (4.3.20), we have $\lim_{|z|\to 1^-} \frac{1 - |z|^2}{1 - |\varphi_r(z)|^2} = 0.$

Conversely, first, for all $l \in \{1,\cdots,1+\epsilon\}$, we have $(\varphi_r)_l \in \beta_\mu$ when $\epsilon > 0$. If

$$\lim_{|z|\to 1^-} \frac{1 - |z|^2}{1 - |\varphi_r(z)|^2} = 0,$$

then for any $0 < \varepsilon < 1$, we have

$$\frac{1 - |z|^2}{1 - |\varphi_r(z)|^2} < \varepsilon \text{ when } 1 - \epsilon < |z| < 1. \tag{71}$$

Let $\{f_j^2(z)\}$ be any sequence that converges to 0 uniformly on any compact subset of B and $\left\|f_j^2\right\|_{\beta_\mu} \le 1$. Let $r_1 = \max\{|\varphi_r(z)|: |z| \le 1 - \epsilon\}$. Then, there exists positive integer N such that

$$\left|f_j^2\left[\varphi_r(0)\right]\right| < \varepsilon \text{ and } \sup_{|z+\epsilon|\le r_1} \left|\nabla f_j^2(z+\epsilon)\right| < \varepsilon \text{ when } j > N. \tag{72}$$

Therefore, by Lemma (4.3.6) and $H_{\varphi_r(z)}\left[R\varphi_r(z)\right] \le cH_z(z)$, (71)–(72), Lemma 2.2 in [155], as long as $j > N$, we have

$$\left\|C_{\varphi_r}(f_j^2)\right\|_{\beta_\mu} \le c\left\{\left|f_j^2\left[\varphi_r(0)\right]\right| + \sup_{z\in B} \mu(|z|)\left|\langle\nabla f_j^2\left[\varphi_r(z)\right], R\varphi_r(z)\rangle\right|\right\}$$

$$\le c\left|f_j^2\left[\varphi_r(0)\right]\right|$$

$$+ c\left\{\left(\sum_{l=1}^{1+\epsilon} \|(\varphi_r)_l\|_{\beta_\mu}\right) \sup_{|z+\epsilon|\le r_1} \left|\nabla f_j^2(z+\epsilon)\right|\right.$$

$$\left. + \left\|f_j^2\right\|_{\mu,3} \sup_{1-\epsilon<|z|<1} \frac{\mu(|z|)(1 - |\varphi_r(z)|^2)\sqrt{H_{\varphi_r(z)}\left[R\varphi_r(z)\right]}}{\mu(|\varphi_r(z)|)}\right\}$$

$$\leq c'\varepsilon + c'' \sup_{1-\epsilon<|z|<1} \frac{\mu(|z|)}{\mu(|\varphi_r(z)|)} \frac{1 - |\varphi_r(z)|^2}{1 - |z|^2}$$

$$\leq c'\varepsilon + c'' \sup_{1-\epsilon<|z|<1} \left\{ \left(\frac{1 - |z|^2}{1 - |\varphi_r(z)|^2} \right)^{\epsilon} + \left(\frac{1 - |z|^2}{1 - |\varphi_r(z)|^2} \right)^{2\epsilon} \right\}$$

$$< c'\varepsilon + 2c''\varepsilon^{\epsilon}.$$

This shows that

$$\lim_{j\to\infty} \left\| C_{\varphi_r}\left(f_j^2\right) \right\|_{\beta_\mu} = 0.$$

This implies that C_{φ_r} is compact on β_μ.

The proof is completed.

Chapter 5
Isometries Between Spaces and Maeda–Ogasawara Spaces

We show that any linear isometry T from $\mathrm{Lip}(X)$ into $\mathrm{Lip}(Y)$ satisfying that $L(T1_X) < 1$ is essentially a weighted composition operator $Tf(y) = \tau(y)f(\phi(y))$ ($f \in \mathrm{Lip}(X), y \in Y_0$), where Y_0 is a closed subset of Y, ϕ is a Lipschitz map from Y_0 onto X with $L(\phi) \le \max\{1, \mathrm{diam}(X)\}$ and τ is a function in $\mathrm{Lip}(Y)$ with $\|\tau\| = 1$ and $|\tau(y)| = 1$ for all $y \in Y_0$. We improve this representation in the case of onto linear isometries and we classify codimension 1 linear isometries in two types. We state a Lipschitz version of a theorem due to Cambern concerning into linear isometries between spaces of vector-valued continuous functions and deduce a Lipschitz version of a celebrated theorem due to Jerison concerning onto linear isometries between such space. We show that a generalised statement holds for Maeda–Ogasawara spaces and refine these results in case the homomorphism preserves order limits.

Section (5.1): Lipschitz Functions

A map $f : X \to Y$ between metric spaces is said to be a Lipschitz map if there exists a constant k such that $d(f(x), f(z)) \le kd(x,z)$ for all $x, z \in X$. We shall use the letter d to denote the distance in any metric space.

Let X be a compact metric space and let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. The space Lip(X) is the Banach space of all Lipschitz functions f from X into $\mathbb{K}$, with the norm $\|f\| = \max\{\|f\|_\infty, L(f)\}$, where $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$ is the supremum norm of f and $L(f) = \sup\{|f(x) - f(y)|/d(x,y) : x, y \in X, x \ne y\}$ is the Lipschitz constant of f.

The study of linear isometries of spaces $\mathrm{Lip}(X)$ goes back to the sixties when Roy [46] described the surjective linear isometries T of $\mathrm{Lip}(X)$ in the case that X is connected and its diameter $\mathrm{diam}(X)$ is at most 1. Namely, he proved that such an isometry T has the canonical form:

$$T_f(y) = \tau f(\varphi(y)) \; (f \in \mathrm{Lip}(X), y \in Y),$$

where φ is an isometry from Y onto X and τ is a scalar of $S_{\mathbb{K}}$, the set of all unimodular elements of $\mathbb{K}$. Novinger [182] extended Roy's result to the case of linear isometries of $\mathrm{Lip}(X)$ onto $\mathrm{Lip}(Y)$ when X and Y are connected with diameter at most 1. Vasavada's result in [47] generalizes the aforementioned results since it states that if X and Y are β-connected for some $\beta < 1$ with diameter at most 2, then any linear isometry from $\mathrm{Lip}(X)$ onto $\mathrm{Lip}(Y)$ arises from an isometry from Y onto X as in the aforementioned canonical form. Let us recall that a metric space X is β-connected if it cannot be decomposed into two nonempty subsets A and B such that $d(a, b) \ge \beta$ for every $a \in A$ and $b \in B$.

The surjective isometries of $\mathrm{Lip}(X)$ have a valuable literature. However little has been published about the into isometries of $\mathrm{Lip}(X)$, that is not necessarily surjective. This fact is also meaningful if we compare it to the formidable literature existing about into isometries in the context of the Banach spaces $C(X)$ of scalar-valued continuous functions on a compact Hausdorff space X with the supremum norm.

The classical Banach–Stone theorem states that if T is a linear isometry from $C(X)$ onto $C(Y)$, then there exists a homeomorphism from Y onto X and a continuous function τ from Y into $S_{\mathbb{K}}$ such that

$$T f(y) = \tau(y)f(\varphi(y)) (f \in C(X), y \in Y).$$

An important generalization of this theorem was given by Holsztyński in [60] by considering into isometries. He proved that if T is a linear isometry from $C(X)$ into $C(Y)$, then there exists a closed subset Y_0 of Y, a continuous map φ from Y_0 onto X and a function $\tau \in \mathcal{C}(Y)$ with $\|\tau\|_\infty = 1$ and $|\tau(y)| = 1$ for all $y \in Y_0$ such that

$$T_f(y) = \tau(y)f(\varphi(y)) \ (f \in \mathcal{C}(X), y \in Y_0).$$

This result has been extended in many directions. We can cite, for example, the generalizations obtained by Cambern [176] for spaces of vector-valued continuous functions, by Moreno and Rodríguez [181] with a bilinear version, by Jeang and Wong for spaces of scalar-valued continuous functions vanishing at infinity [57], by Araujo and Font for certain subspaces of scalar-valued continuous functions [174] and by many other authors. The object is to show that Holsztyński's theorem has a natural formulation in the context of the spaces $\mathrm{Lip}(X)$. We focus our attention on linear isometries T from $\mathrm{Lip}(X)$ into $\mathrm{Lip}(Y)$ for which T_{1X} is a contraction, where $1X$ denotes the function constantly equal 1 on X. We recall that a Lipschitz function f is a contraction if $L(f) < 1$.

The main theorem states that any linear isometry T from $\mathrm{Lip}(X)$ into $\mathrm{Lip}(Y)$ for which T_{1_X} is a contraction, is essentially a weighted composition operator

$$T_f(y) = \tau(y)f(\varphi(y)) (f \in \mathrm{Lip}(X), y \in Y_0),$$

where Y_0 is a closed subset of Y, φ is a Lipschitz map from Y_0 onto X with $L(\varphi) \leq \max\{1, \mathrm{diam}(X)\}$ and τ is a function of $\mathrm{Lip}(Y)$ with $\|\tau\| = 1$ and $|\tau(y)| = 1$ for all $y \in Y_0$. Namely, the weight function τ is T_{1X}. Moreover, we show (Corollary (5.1.5)) that Y_0 is the largest subset of Y on which we can define a Lipschitz map φ with values in X satisfying the equality above.

We use extreme point techniques to prove our theorem as they do in [180], [182], [46] and [47]. This method of proof is still used to obtain similar results [173] and seems to date from the known proof of the Banach–Stone theorem given by Dunford and Schwartz in [33].

The theorem is not true when $L(T_{1_X}) = 1$ (see [184]). We must point out also that, the connectedness condition imposed on the metric spaces is used to prove that T_{1X} is a constant function, in whose case $L(T_{1_X}) = 0$. On the other hand, we want to emphasize that Vasavada's reduction to metric spaces of diameter at most 2 is not restrictive because if (X, d) is a compact metric space and X' is the set X remetrized with the metric $d'(x, y) = \min\{d(x, y), 2\}$, then $\mathrm{diam}(X') \leq 2$ and $\mathrm{Lip}(X')$ is isometrically isomorphic to $\mathrm{Lip}(X)$ (see [184]).

Our theorem also provides some new information concerning the onto case. We show (Theorem (5.1.6)) that any linear isometry T from $\mathrm{Lip}(X)$ onto $\mathrm{Lip}(Y)$ such that T_{1_X} is a nonvanishing contraction, is a weighted composition operator

$$T_f(y) = \tau(y)f(\varphi(y)) (f \in \mathrm{Lip}(X), y \in Y),$$

where φ is a Lipschitz homeomorphism from Y onto X and τ is a Lipschitz function from Y into $S_{\mathbb{K}}$. Our approach is different from one which Novinger [182], Roy [46], Vasavada [47] and Weaver [184] present since they impose conditions of connectedness. Let us recall that a map between metric spaces $\varphi : X \to Y$ is a Lipschitz homeomorphism if φ is a bijection such that φ and φ^{-1} are both Lipschitz, and a function $f : X \to \mathbb{K}$ is said to be nonvanishing if $f(x) \neq 0$ for all $x \in X$.

In [184] Weaver obtained a noncompact version of Vasavada's result. He defined $\mathrm{Lip}(X)$ as the space of all bounded Lipschitz scalar-valued functions f on a metric space X with the norm $\|f\| = \max\{\|f\|_\infty, L(f)\}$, and showed that if X and Y are complete 1-

connected with diameter at most 2, then every linear isometry T from $\mathrm{Lip}(X)$ onto $\mathrm{Lip}(Y)$ is of the form

$$T_f(y) = \tau f\big(\varphi(y)\big)(f \in \mathrm{Lip}(X), y \in Y),$$

where φ is an isometry from Y onto X and τ is a unimodular constant. Theorem (5.1.6) can be improved with the aid of this Weaver's result. Namely, we show (Theorem (5.1.8)) that if X and Y are compact metric spaces with diameter at most 2 and $T : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ is a surjective linear isometry such that T_{1X} is a nonvanishing contraction, then there exists a surjective isometry $\varphi : Y \to X$ and a function $\tau : Y \to S_{\mathbb{K}}$ with $\tau(x) = \tau(y)$ whenever $d(x,y) < 2$ such that

$$T_f(y) = \tau(y) f(\varphi(y)), \forall f \in \mathrm{Lip}(X), \forall y \in Y.$$

We classify codimension 1 linear isometries between $\mathrm{Lip}(X)$- spaces in two types. If $T : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ is such an isometry with $L\big(T_{1_X}\big) < 1$, Theorem (5.1.9) asserts the existence of a closed subset Y_0 of Y such that either $Y_0 = Y \backslash \{p\}$, where p is an isolated point of Y, or $Y_0 = Y$; a surjective Lipschitz map $\varphi: Y_0 \to X$ and a unimodular Lipschitz function $\tau : Y_0 \to \mathbb{K}$ such that $T f(y) = \tau(y) f(\varphi(y))$ for all $y \in Y_0$. If $Y \backslash Y_0$ is just a single point or $Y_0 = Y$, we label these isometries as of type I and of type II, respectively. These two types are not disjoint. We give a method for constructing codimension 1 linear isometries which are simultaneously of types I and II (Proposition (5.1.11)). We also give examples of type I codimension 1 linear isometries which are not of type II (Proposition (5.1.12)), and vice versa (Example (5.1.13)). The remainder is devoted to study the properties of φ (Proposition (5.1.15)).

In the last years, several have investigated about codimension 1 linear isometries on the space $\mathcal{C}(X)$ ([175], [177], [179], among others many). However the key is due to Gutek, Hart, Jamison and Rajagopalan [178]. These studied shift operators on $\mathcal{C}(X)$ and classified these operators using the aforementioned Holszty´nski's theorem [60]. We have followed a similar way to study codimension 1 linear isometries between $\mathrm{Lip}(X)$-spaces applying now our Lipschitz version of the cited theorem.

We begin by recalling some results which describe partially the set of extreme points of the closed unit ball of the dual space of $\mathrm{Lip}(X)$.

For a Banach space E, we denote by B_E the closed unit ball of E, by S_E the unit sphere of E, by $\mathrm{Ext}(B_E)$ the set of extreme points of B_E and by E^* the dual space of E.

Given a compact metric space X, let $\tilde{X} = (x,y) \in X^2 : x \neq y$ and let the compact Hausdorff space W be the disjoint union of X with $\beta\tilde{X}$, where $\beta\tilde{X}$ is the Stone-Cech compactification of $\tilde{X}$. Consider the mapping $\Phi : \mathrm{Lip}(X) \to C(W)$ defined for each $f \in \mathrm{Lip}(X)$ by

$$\Phi f(w) = \begin{cases} f(w) \text{ if } w \in X, \\ (\beta f^*)(w) \text{ if } w \in \beta\tilde{X}, \end{cases}$$

where

$$f^*(x,y) = \frac{f(x) - f(y)}{d(x,y)}, \forall (x,y) \in \tilde{X},$$

and βf^* is its norm-preserving extension to $\beta\tilde{X}$. It is easily seen that Φ is a linear isometry from $\mathrm{Lip}(X)$ into $\mathcal{C}(W)$. For each w ∈ W, define the functionals $\delta_w \in \mathcal{C}(W)^*$ and $\tilde{\delta}_w \in \mathrm{Lip}(X)^*$ by $\delta_w(f) = f(w)$ and $\tilde{\delta}_w(f) = \Phi f(w)$, respectively. Clearly, $\left|\tilde{\delta}_w(f)\right| \leq \|f\|$ for all $f \in \mathrm{Lip}(X)$ and therefore $\tilde{\delta}_w \in B_{\mathrm{Lip}(X)^*}$. It is well known (see [33]) that the extreme points of $B_{\mathrm{Lip}(X)^*}$ are essentially of this form:

Lemma (5.1.1)[172]: Every extreme point of $B_{\text{Lip}(X)^*}$ must be either of the form $\tau\,\tilde{\delta}_x$ with $\tau \in S_{\mathbb{K}}$ and $x \in X$ or of the form $\tau\,\tilde{\delta}_w$ with $\tau \in S_{\mathbb{K}}$ and $w \in \beta\tilde{X}$. We also shall need the following fact which was proved by Roy [46] using a result of de Leeuw [180]:

Lemma (5.1.2)[172]: For each $x \in X, \tilde{\delta}_x$ is an extreme point of $B_{\text{Lip}(X)^*}$. An application of the Hahn–Banach and Krein–Milman theorems yields the following fact surely known.

Lemma (5.1.3)[172]: Let X be a normed space and M a vector subspace of X. For each $g \in \text{Ext}(B_{M^*})$ there exists $f \in \text{Ext}(B_{X^*})$ such that $f|_M = g$.

Finally, we present two families of Lipschitz functions which will be used frequently throughout.

Let X be a compact metric space. For each $x \in X$, the real function f_x, defined on X by $f_x(z) = d(z,x)$, belongs to $\text{Lip}(X)$ with $L(f_x) \leq 1$ and $\|f_x\|_\infty \leq \text{diam}(X)$. Also, for each $x \in X$ and $\delta > 0$, the function $h_x^\delta : X \to [0,1]$ given by

$$h_x^\delta(z) = \max\left\{0, 1 - \frac{d(z,x)}{\delta}\right\}$$

is also in $\text{Lip}(X)$ with $L(h_x^\delta) \leq 1/\delta$ and $\left\|h_x^\delta\right\|_\infty \leq 1$.

We formulate the main result which is a version for isometries of $\text{Lip}(X)$-spaces of a known Holszty´nski's theorem on isometries of $C(X)$-spaces [60].

Theorem (5.1.4)[172]: Let $T : \text{Lip}(X) \to \text{Lip}(Y)$ be a linear isometry and suppose T_{1X} is a contraction. Then there exists a closed subset Y_0 of Y, a surjective Lipschitz map $\varphi : Y_0 \to X$ with $L(\varphi) \leq \max\{1, \text{diam}(X)\}$ and a function $\tau \in \text{Lip}(Y)$ with $\|\tau\| = 1$ and $|\tau(y)| = 1$ for all $y \in Y_0$ such that

$$T_f(y) = \tau(y) f(\varphi(y)), \qquad \forall f \in \text{Lip}(X), \qquad \forall y \in Y_0.$$

Proof. Let $\tau = T_{1X}$. Evidently, $\tau \in \text{Lip}(Y)$ and $\|\tau\| = \|1X\| = 1$. Let $Z = T(\text{Lip}(X))$ and define

$$Y_0 = \left\{y \in Y : \overline{\tau(y)} T^*\,\tilde{\delta}_y|_Z \in \text{Ext}(B_{\text{Lip}(X)^*}), |\tau(y)| = 1\right\}.$$

We first prove that Y_0 is nonempty. Since T is a linear isometry from $\text{Lip}(X)$ onto Z, the adjoint map $T^* : \text{Lip}(Y)^* \to \text{Lip}(X)^*$ is also a linear isometry from Z^* onto $\text{Lip}(X)^*$ and therefore T^* induces a bijection from $\text{Ext}(B_{Z^*})$ onto $\text{Ext}(B_{\text{Lip}(X)^*})$. Let $x \in X$. By Lemma (5.1.2), $\tilde{\delta}_x \in \text{Ext}(B_{\text{Lip}(X)^*})$. Therefore $T^*\mu = \tilde{\delta}_x$ for some $\mu \in \text{Ext}(B_{Z^*})$. By Lemma (5.1.3), μ is the restriction to Z of an extreme point of $B_{\text{Lip}(Y)^*}$. Hence $\mu = \alpha\,\tilde{\delta}_w|_Z$ for some $\alpha \in S_{\mathbb{K}}$ and $w \in Y \cup \beta\tilde{Y}$ by Lemma (5.1.1) and so $\alpha T^*\,\tilde{\delta}_w|_Z(1_X) = \tilde{\delta}_x(1_X)$. We now see that $w \in Y$. If there were $w \in \beta\tilde{Y}$, e we should have $\left|\alpha T^*\,\tilde{\delta}_w|_Z(1_X)\right| < 1$ since

$$\left|\alpha T^*\,\tilde{\delta}_w|_Z(1_X)\right| = \left|\tilde{\delta}_w(T_{1_X})\right| = \left|\beta(T_{1_X})^*(w)\right| < 1.$$

It suffices to observe that

$$\left|(T_{1_X})^*(y,z)\right| = \left|T_{1_X}(y) - T_{1_X}(z)\right| d(y,z) \leq L(T_{1_X}) < 1$$

for all $(y,z) \in \tilde{Y}$ and that $\tilde{Y}$ is dense in $\beta\tilde{Y}$. But, on the other hand, $\tilde{\delta}_x(1X) = 1$. This contradiction gives us $w = y$ for some $y \in Y$. It follows that

$$1 = \tilde{\delta}_x(1_X) = \alpha\tilde{\delta}_y(T(1_X)) = \alpha T(1_X)(y) = \alpha\tau(y).$$

From this it is deduced that $|\tau(y)| = 1$ and $\alpha = \overline{\tau(y)}$. Hence $\overline{\tau(y)} T^*\,\tilde{\delta}_y|_Z = \tilde{\delta}_x \in \text{Ext}(B_{\text{Lip}(X)^*})$ and so $y \in Y_0$. We next show that for each $y \in Y_0$, there exists a unique point $x \in X$ such that $\overline{\tau(y)} T^*\,\tilde{\delta}_y|_Z = \tilde{\delta}_x$. Let $y \in Y_0$. Since $\overline{\tau(y)} T^*\,\tilde{\delta}_y|_Z \in$

$\operatorname{Ext}(B_{\operatorname{Lip}(X)^*})$ by definition of Y_0, Lemma (5.1.1) yields $\overline{\tau(y)}T^* \tilde{\delta}_y|_Z = \alpha\tilde{\delta}_w$ for suitable $\alpha \in S_{\mathbb{K}}$ and $w \in X \cup \beta\tilde{X}$. We show that $w \in X$. If $w \in \beta\tilde{X}$, it is clear that $\alpha\tilde{\delta}_w(1X) = 0$, but

$$\overline{\tau(y)}T^* \tilde{\delta}_y|_Z (1_X) = \overline{\tau(y)}\tilde{\delta}_y(T_{1_X}) = \overline{\tau(y)}T_{1_X}(y) = |\tau(y)|^2 = 1.$$

This contradiction proves that $w = x$ for some $x \in X$. Then

$$\alpha = \alpha\tilde{\delta}_x(1_X) = \overline{\tau(y)}T^* \tilde{\delta}_y|_Z (1_X) = 1$$

and so $\overline{\tau(y)}T^* (\tilde{\delta}_y|_Z) = \tilde{\delta}_x$. This proves the existence of x.

To show its uniqueness, assume there exists $x' \in X$ such that $\overline{\tau(y)}T^* \tilde{\delta}_y|_Z = \tilde{\delta}_{x'}$. Consider the function $f_x \in \operatorname{Lip}(X)$. If there were $x' \neq x$, we would have

$$\overline{\tau(y)}T^* \tilde{\delta}_y|_Z (fx) = \tilde{\delta}_{x'}(f_x) = f_x(x') = d(x', x) \neq 0,$$

but also

$$\overline{\tau(y)}T^* \tilde{\delta}_y|_Z (f_x) = \tilde{\delta}_x(f_x) = f_x(x) = 0.$$

This contradiction gives us $x' = x$. Let $\varphi : Y_0 \to X$ be the map defined by $\varphi(y) = x$ whenever $\overline{\tau(y)}T^* \tilde{\delta}_y|_Z = \tilde{\delta}_x$. Clearly, $T^* \tilde{\delta}_y|_Z = \overline{\tau(y)}\tilde{\delta}_{\varphi(y)}$ for each $y \in Y_0$ and therefore

$$T_f(y) = \tau(y)f(\varphi(y)), \qquad \forall f \in \operatorname{Lip}(X), \qquad \forall y \in Y_0.$$

The map $\varphi : Y_0 \to X$ is surjective. This is proved as the existence of points in Y_0. We now check that $\varphi : Y_0 \to X$ is Lipschitz. Notice that for each $x \in X, f_x \in \operatorname{Lip}(X)$ and $\|f_x\| \leq k$ where $k = \max\{1, \operatorname{diam}(X)\}$. Hence $\|f_{\varphi(y)}\| \leq k$ for all $y \in Y_0$. Since T is a linear isometry, it follows that $\|T f_{\varphi(y)}\| \leq k$ for all $y \in Y_0$. Then $L(T f_{\varphi(y)}) \leq k$ for all $y \in Y_0$. Let $y, z \in Y_0$. We have $|T f_{\varphi(y)}(y) - T f_{\varphi(y)}(z)| \leq kd(y, z)$. An easy calculation yields

$$T f_{\varphi(y)}(y) = \tau(y)f_{\varphi(y)}(\varphi(y)) = \tau(y)d(\varphi(y), \varphi(y)) = 0,$$
$$f_{\varphi(y)}(z) = \tau(z)f_{\varphi(y)}(\varphi(z)) = \tau(z)d(\varphi(y), \varphi(z)),$$

and thus $d(\varphi(y), \varphi(z)) \leq kd(y, z)$. Finally, we see that Y_0 is closed in Y. To prove this, given $x \in X$ and $y \in Y$, we notice that $y \in Y_0$ and $\varphi(y) = x$ if and only if $|T f(y)| = |f(x)|$ for all $f \in \operatorname{Lip}(X)$. Indeed, if $y \in Y_0$ and $\varphi(y) = x$, then $T^* \tilde{\delta}_y|_Z = \tau(y)\tilde{\delta}_x$; hence, for all $f \in \operatorname{Lip}(X)$, we have

$$T_f(y) = T^* \tilde{\delta}_y|_Z (f) = \tau(y)\tilde{\delta}_x(f) = \tau(y)f(x),$$

and so $|T f(y)| = |f(x)|$. Conversely, if $|T f(y)| = |f(x)|$ for all $f \in \operatorname{Lip}(X)$, then $|\tau(y)| = 1$ and $\ker T^* \tilde{\delta}_y|_Z = \ker \tilde{\delta}_x$. This last implies that $T^* \tilde{\delta}_y|_Z = \alpha\tilde{\delta}_x$ for some nonzero scalar α. In particular, we deduce that $\tau(y) = \alpha$ and thus $\overline{\tau(y)}T^* \tilde{\delta}_y|_Z = \tilde{\delta}_x$. This says us that $y \in Y_0$ and $\varphi(y) = x$. To show that Y_0 is closed in Y, let $\{y_n\}$ be a sequence in Y_0 converging to a point $y \in Y$. For each natural n, let $x_n = \varphi(y_n)$. By the compactness of $X, \{x_n\}$ has a subsequence $\{x_{\sigma(n)}\}$ which converges to a point $x \in X$. Let $f \in \operatorname{Lip}(X)$. We have $|T f(y_{\sigma(n)})| = |f(x_{\sigma(n)})|$ for all $n \in \mathbb{N}$. It follows that $|T(f)(y)| = |f(x)|$. Since f was arbitrary, the remark above gives us $y \in Y_0$ and so Y_0 is closed in Y.

The next result shows that the triple $\{Y_0, \tau, \varphi\}$ associated to the isometry T in Theorem (5.1.4) possesses a universal property.

Corollary (5.1.5)[172]: Let T be a linear isometry from $\mathrm{Lip}(X)$ into $\mathrm{Lip}(Y)$ for which T_{1X} is a contraction. Let Y_0, τ and φ be as in Theorem (5.1.4). If Y_0' is a subspace (not necessarily closed) of Y, and $\tau' : Y_0' \to S_{\mathbb{K}}$ and $\varphi' : Y_0' \to X$ are Lipschitz maps such that

$$T f(y) = \tau'(y) f(\varphi'(y)), \qquad \forall f \in \mathrm{Lip}(X), \qquad \forall y \in Y_0',$$

then $Y_0' \subset Y_0, \tau' = \tau|_{Y_0'}$ and $\varphi = \varphi|_{Y_0'}$.

Proof. Let $y \in Y_0'$. Taking $f = 1X$ above in the expression of $T f$, we have $\tau'(y) = T_{1_X}(y) = \tau(y)$, and so $\tau' = \tau|_{Y_0'}$. Now the expression of $T f$ reads as $\overline{\tau(y)} T^* \tilde{\delta}_y|_Z = \tilde{\delta}_{\varphi'(y)}$ where $Z = T(\mathrm{Lip}(X))$. Since $\tilde{\delta}_{\varphi'(y)} \in \mathrm{Ext}(B_{\mathrm{Lip}(X)^*})$ by Lemma (5.1.2), it follows that $\overline{\tau(y)} T^* \tilde{\delta}_y|_Z \in \mathrm{Ext}(B_{\mathrm{Lip}(X)^*})$. Moreover,

$$1 = \tilde{\delta}_{\varphi'(y)}(1X) = \overline{\tau(y)} \tilde{\delta}_y(T_{1X}) = \overline{\tau(y)} \tau(y) = |\tau(y)|^2 .$$

Hence $y \in Y_0$ and thus $Y_0' \subset Y_0$. Let $f \in \mathrm{Lip}(X)$. Since $T f(z) = \tau(z) f(\varphi(z))$ for all $z \in Y_0$ by Theorem (5.1.4), $Y_0' \subset Y_0$ and $\tau' = \tau|_{Y_0'}$, we have $T f(y) = \tau'(y) f(\varphi(y))$. Moreover, $T f(y) = \tau'(y) f(\varphi'(y))$ by hypothesis. Therefore $f(\varphi'(y)) = f(\varphi(y))$ for all $f \in \mathrm{Lip}(X)$. This implies $\varphi'(y) = \varphi(y)$, because otherwise we could take the function $f_{\varphi(y)}$ and $f_{\varphi(y)}(\varphi'(y)) = d(\varphi'(y), \varphi(y)) \neq 0 = f_{\varphi(y)}(\varphi(y))$. Thus $\varphi' = \varphi|_{Y_0'}$.

We shall apply Theorem (5.1.4) to study the onto case.

Theorem (5.1.6)[172]: Let $T : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ be a surjective linear isometry. Suppose T_{1X} is a nonvanishing contraction. Then there exists a Lipschitz homeomorphism $\varphi : Y \to X$ with $L(\varphi) \leq \max\{1, \mathrm{diam}(X)\}$ and $L(\varphi^{-1}) \leq \max\{1, \mathrm{diam}(Y)\}$ and a Lipschitz function $\tau : Y \to S_{\mathbb{K}}$ such that

$$T f(y) = \tau(y) f(\varphi(y)), \qquad \forall f \in \mathrm{Lip}(X), \qquad \forall y \in Y.$$

Proof. According to Theorem (5.1.4), there exists a closed subset Y_0 of Y, a Lipschitz map φ from Y_0 onto X with $L(\varphi) \leq \max\{1, \mathrm{diam}(X)\}$ and a function $\tau \in \mathrm{Lip}(Y)$ with $\|\tau\| = 1$ and $|\tau(y)| = 1$ for all $y \in Y_0$ such that

$$T f(y) = \tau(y) f(\varphi(y)), \qquad \forall f \in \mathrm{Lip}(X), \qquad \forall y \in Y_0.$$

Since now T is surjective, the set Y_0 comes given by

$$\left\{ y \in Y : \overline{\tau(y)} T^* \tilde{\delta}_y \in \mathrm{Ext}(B_{\mathrm{Lip}(X)^*}) \right\} o .$$

We next prove that $Y_0 = Y$. Let $\in Y$. Since $\tilde{\delta}_y \in \mathrm{Ext}(B_{\mathrm{Lip}(Y)^*})$ by Lemma (5.1.2), it follows that $T^* \tilde{\delta}_y \in \mathrm{Ext}(B_{\mathrm{Lip}(X)^*})$ because T^* is a linear isometry from $\mathrm{Lip}(Y)^*$ onto $\mathrm{Lip}(X)^*$. Then $T^* \tilde{\delta}_y = \alpha \tilde{\delta}_w$ for some $\alpha \in S_{\mathbb{K}}$ and $w \in X \cup \beta\tilde{X}$ by Lemma (5.1.1). We see that $w \in X$. Indeed, if $w \in \beta\tilde{X}$, then $\alpha \tilde{\delta}_w(1X) = 0$, but

$$T^* \tilde{\delta}_y(1_X) = \tilde{\delta}_y(T_{1_X}) = T_{1_X}(y) \neq 0$$

because T_{1_X} is nonvanishing. Hence $w = x \in X$. Then

$$\tau(y) = T_{1_X}(y) = T^* \tilde{\delta}_y(1_X) = \alpha \tilde{\delta}_x(1_X) = \alpha.$$

Hence $|\tau(y)| = 1$ and so $\overline{\tau(y)} \alpha = 1$. It follows that

$$\overline{\tau(y)} T^* (\tilde{\delta}_y) = \overline{\tau(y)} \alpha \tilde{\delta}_x = \tilde{\delta}_x \in \mathrm{Ext}(B_{\mathrm{Lip}(X)^*})$$

and so $y \in Y_0$.

To prove the injectivity of φ, let $y, y_0 \in Y$ be for which $\varphi(y) = \varphi(y')$ and let us suppose $y \neq y'$. Since $T f(z) = \tau(z) f(\varphi(z))$ for all $f \in \mathrm{Lip}(X)$ and $z \in Y$, it is clear that $|T f(y)| = |T f(y')|$ for all $f \in \mathrm{Lip}(X)$ and, since T is surjective, it follows that

$|h(y)| = |h(y')|$ for all $h \in \mathrm{Lip}(Y)$. However this can not be because $|f_y(y)| = 0 \neq d(y,y') = |f_y(y')|$.

On the other hand, T^{-1} is a linear isometry from $\mathrm{Lip}(Y)$ onto $\mathrm{Lip}(X)$. It comes given indeed by

$$T^{-1} g(x) = \overline{\tau\left(\varphi^{-1}(x)\right)} g(\varphi^{-1}(x)), \qquad \forall g \in \mathrm{Lip}(Y), \qquad \forall x \in X.$$

Using this we can deduce that φ^{-1} is Lipschitz with $L(\varphi^{-1}) \leq \max\{1, \mathrm{diam}(Y)\}$. The proof is similar to that given in Theorem (5.1.4) to prove that φ is Lipschitz.

Under the conditions of Theorem (5.1.6), if T is also unital, that is $T(1X) = 1Y$, then T is an algebra isomorphism. Namely, we have the following:

Corollary (5.1.7)[172]: Let $T : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ be a surjective linear isometry. Suppose T is unital. Then there exists a Lipschitz homeomorphism $\varphi : Y \to X$ with $L(\varphi) \leq \max\{1, \mathrm{diam}(X)\}$ and $L(\varphi^{-1}) \leq \max\{1, \mathrm{diam}(Y)\}$ such that

$$T f(y) = f\left(\varphi(y)\right), \qquad \forall f \in \mathrm{Lip}(X), \qquad \forall y \in Y.$$

Next we see that Theorem (5.1.6) can be improved. If $\tau : Y \to S_{\mathbb{K}}$ is a function such that $\tau(y) = \tau(y')$ whenever $d(y,y') < 2$, and $\varphi : Y \to X$ is a surjective isometry, it is easily seen that $T f = \tau (f \circ \varphi)$ $(f \in \mathrm{Lip}(X))$ is a linear isometry from $\mathrm{Lip}(X)$ onto $\mathrm{Lip}(Y)$. Conversely, we have the following improvement of Theorem (5.1.6).

Theorem (5.1.8)[172]: Let X and Y be compact metric spaces with diameter at most 2 and let $T : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ be a surjective linear isometry such that T_{1X} is a nonvanishing contraction. Then there exist a Lipschitz function $\tau : Y \to S_{\mathbb{K}}$ with $\tau(y) = \tau(y')$ whenever $d(y,y') < 2$ and a surjective isometry $\varphi : Y \to X$ such that T is of the form

$$T f(y) = \tau(y) f\left(\varphi(y)\right), \qquad \forall f \in \mathrm{Lip}(X), \qquad \forall y \in Y.$$

Proof. By Theorem (5.1.6), there exists a Lipschitz function $\tau : Y \to S_{\mathbb{K}}$ and a Lipschitz homeomorphism $\varphi : Y \to X$ such that

$$T f(y) = \tau(y) f\left(\varphi(y)\right), \qquad \forall f \in \mathrm{Lip}(X), \qquad \forall y \in Y.$$

Let us define an equivalence relation on X by setting $x \sim z$ $(x, z \in X)$ if and only if there is a finite sequence of points $x_1, \ldots, x_k$ in X such that $x_1 = x, x_k = z$ and $d(x_i, x_{i+1}) < 1$ for $1 \leq i < k$. The equivalence class of this relation are called the 1-connected components of X. Since X is compact, there exist $z_1, \ldots, z_m \in X$ such that $X = \cup_{i=1}^{m} B(z_i, 1)$ where $B(z_i, 1) = \{x \in X : d(x, z_i) < 1\}$. For each $i \in \{1, \ldots, m\}$, the ball $B(z_i, 1)$ is contained in the 1-connected component of X which contains to z_i and therefore the number of 1-connected components of X is $n \leq m$. Let $X_1, \ldots, X_n$ be the 1-connected components of X. Notice that $X_1, \ldots, X_n$ are pairwise disjoint closed sets.

Fix $1 \leq i \leq n$ and identify $\mathrm{Lip}(X_i)$ with the functions in $\mathrm{Lip}(X)$ which are supported on X_i. Let $Y_i = \varphi^{-1}(X_i)$. Clearly, T takes $\mathrm{Lip}(X_i)$ isometrically onto $\mathrm{Lip}(Y_i)$ and then, by [184], there exists a constant $\tau_i \in S_{\mathbb{K}}$ and an isometry φ_i from Y_i onto X_i such that

$$T f(y) = \tau_i f(\varphi_i(y)), \qquad \forall f \in \mathrm{Lip}(X_i), \qquad \forall y \in Y_i.$$

A simple verification shows that $\tau|_{Y_i} = \tau_i 1Y$ and $\varphi|_{Y_i} = \varphi_i$.

Fix $i, j \in \{1, \ldots, n\}$ with $i \neq j$. Suppose $d(x_0, x_0') < 2$ for some $x_0 \in X_i$ and $x_0' \in X_j$. Let $a = \inf\{d(x, x_0) : x \in X_i, x_0 \in X_j\}$ and consider $f : X \to \mathbb{R}$ defined by $f(x) = -a/2$ if $x \in X_i$, $f(x) = a/2$ if $x \in X_j$ and $f(x) = 0$ if $x \notin X_i \cup X_j$. We claim that $f \in \mathrm{Lip}(X)$ with $L(f) = 1$. Let $x, x_0 \in X$. If $x \in X_i$ and $x' \in X_j$, we have

$$\frac{|f(x) - f(x')|}{d(x,x')} = \frac{a}{d(x,x')} \leq 1.$$

If $x \in X_i \cup X_j$ and $x' \notin X_i \cup X_j$, we obtain

$$\frac{|f(x) - f(x')|}{d(x,x')} = \frac{a^2}{d(x,x')} \leq a^2 < 1.$$

It follows that $L(f) \leq 1$. Using the definition of a it is easy to see that $L(f) = 1$. Since $\|f\|_\infty = a/2 < 1$, we have $\|f\| = 1$ and thus $\|T f\| = 1$. Moreover, since $T f(y) = -\tau_i a/2$ if $y \in Y_i$, $T f(y) = \tau_j a/2$ if $y \in Y_j$ and $T f(y) = 0$ elsewhere, we have $\|T f\|_\infty = a/2 < 1$ and thus $L(T f) = 1$. Let

$$b = \inf\{d(y,y'): y \in Y_i, y' \in Y_j\}.$$

Since Y_i and Y_j are pairwise disjoint closed sets, we have b > 0. Next we check that $L(T f) \leq a/b$. Let $y, y_0 \in Y$. If $y \in Y_i$ and $y' \in Y_j$, we have

$$\frac{|T f(y) - T f(y')|}{d(y,y')} = \frac{\frac{a}{2}|\tau_i + \tau_j|}{d(y,y')} \leq \frac{a}{b}.$$

If $y \in Y_i \cup Y_j$ and $y' \notin Y_i \cup Y_j$, we get

$$\frac{|T f(y) - T f(y')|}{d(y,y')} = \frac{\frac{a}{2}}{d(y,y')} \leq \frac{a}{2} \leq \frac{a}{b}.$$

Therefore $L(T f) \leq a/b$ and so $1 \leq a/b$.

We now prove that $\tau_i = \tau_j$. Suppose $\tau_i \neq \tau_j$ and let $g : X \to \mathbb{K}$ be the function given by $g(x) = -\tau_i a/2$ if $x \in X_i$, $g(x) = \tau_j a/2$ if $x \in X_j$ and $g(x) = 0$ elsewhere. To see that $g \in \mathrm{Lip}(X)$, let $x, x_0 \in X$. If $x \in X_i$ and $x' \in X_j$, we have

$$\frac{|g(x) - g(x')|}{d(x,x')} = \frac{\frac{a}{2}|\tau_i + \tau_j|}{d(x,x')} \leq \frac{|\tau_i + \tau_j|}{2} < 1$$

since $\tau_i, \tau_j \in S_{\mathbb{K}}$ and $\tau_i \neq \tau_j$. If now $x \in X_i \cup X_j$ and $x' \notin X_i \cup X_j$, we obtain

$$\frac{|g(x) - g(x')|}{d(x,x')} = \frac{\frac{a}{2}}{d(x,x')} \leq \frac{a}{2} < 1.$$

Hence $g \in \mathrm{Lip}(X)$ with $L(g) \leq \max\{|\tau_i + \tau_j|/2, a/2\} < 1$. Since $\|g\|_\infty = a/2$, it follows that $\|g\| < 1$ and so $L(T g) \leq \|T g\| = \|g\| < 1$. On the other hand, since $T g(y) = -a/2$ if $y \in Y_i$ and $T g(y') = a/2$ if $y' \in Y_j$, we deduce

$$\frac{|T g(y) - T g(y')|}{d(y,y')} = \frac{a}{d(y,y')} \leq L(T g),$$

which implies $a/b \leq L(T g)$. Then $a/b < 1$, which contradicts that $1 \leq a/b$. This proves that $\tau_i = \tau_j$.

It is an easily checked fact, which is contained in [47], that $\|\tilde{\delta}_y - \tilde{\delta}_{y'}\| = d(y,y')$ whenever $d(y,y') \leq 2$. Using this fact we now prove that φ is an isometry. Let $y, y' \in Y$ be such that $d(\varphi(y), \varphi(y')) < 2$. Clearly, $y \in Y_i$ and $y' \in Y_j$ for some $i, j \in \{1, \dots, n\}$. By what has been proved above, we have $\tau(y) = \tau_i = \tau_j = \tau(y')$ and it follows that

$$\begin{aligned} d(\varphi(y), \varphi(y')) &= \|\tilde{\delta}_{\varphi(y)} - \tilde{\delta}_{\varphi(y')}\| = \|\tau(y)\tilde{\delta}_{\varphi(y)} - \tau(y')\tilde{\delta}_{\varphi(y')}\| \\ &= \|\tilde{\delta}_y \circ T - \tilde{\delta}_{y'} \circ T\| = \|T^*(\tilde{\delta}_y - \tilde{\delta}_{y'})\| = \|\tilde{\delta}_y - \tilde{\delta}_{y'}\| = d(y,y'). \end{aligned}$$

If now $y, y' \in Y$ with $d(\varphi(y), \varphi(y')) = 2$, then $d(y, y') = 2$. In contrary case, it would be $d\left(\varphi^{-1}\left(\varphi(y)\right), \varphi^{-1}\left(\varphi(y')\right)\right) < 2$ and, by applying to T^{-1} what has already been proved, we would have $d\left(\varphi^{-1}\left(\varphi(y)\right), \varphi^{-1}\left(\phi(y')\right)\right) = d(\varphi(y), \varphi(y'))$, that is, $2 = d(\varphi(y), \varphi(y')) = d(y, y') < 2$, a contradiction.

Applying Theorem (5.1.4), we can describe codimension 1 linear isometries between $\mathrm{Lip}(X)$-spaces as follows.

Theorem (5.1.9)[172]: Let T be a codimension 1 linear isometry from $\mathrm{Lip}(X)$ into $\mathrm{Lip}(Y)$. Suppose T_{1_X} is a contraction. Then there exists a closed subset Y_0 of Y where either $Y_0 = Y \backslash \{p\}$ being p an isolated point of Y or $Y_0 = Y$, a surjective Lipschitz map $\varphi : Y_0 \to X$ and a Lipschitz function $\tau : Y_0 \to S_{\mathbb{K}}$ such that

$$T f(y) = \tau(y) f(\varphi(y)), \qquad \forall f \in \mathrm{Lip}(X), \qquad \forall y \in Y_0.$$

Proof. By Theorem (5.1.4) there exists a nonempty closed subset Y_0 of Y, a Lipschitz map φ from Y_0 onto X and a Lipchitz function τ from Y_0 into $S_{\mathbb{K}}$ such that

$$T f(y) = \tau(y) f(\varphi(y)), \qquad \forall f \in \mathrm{Lip}(X), \qquad \forall y \in Y_0.$$

Suppose $Y \backslash Y_0$ has two distinct points y_1, y_2. For $i \in \{1, 2\}$, let

$$\delta_i = d(Y_0 \cup \{y_j : j \neq i\}, y_i).$$

Clearly, $\delta_i > 0$ and $h_{y_i}^{\delta_i} \in \mathrm{Lip}(Y)$ satisfies that $h_{y_i}^{\delta_i}(y_i) = 1$ and $h_{y_i}^{\delta_i}(y) = 0$ for all $y \in Y_0 \cup \{y_j : j \neq i\}$.

We see that $h_{y_1}^{\delta_1}$ and $h_{y_2}^{\delta_2}$ are linearly independent. Suppose $\alpha h_{y_1}^{\delta_1} + \beta h_{y_2}^{\delta_2} = 0$ for some scalars α, β. Since $h_{y_i}^{\delta_i}(y_j) = \delta_{ij}$ where δ_{ij} is the Kronecker's delta, it follows that $\alpha = \beta = 0$.

We now prove that no nonzero linear combination of $h_{y_1}^{\delta_1}$ and $h_{y_2}^{\delta_2}$ belongs to the range of T. Let $\alpha, \beta \in \mathbb{K}$ and suppose there exists a $f \in \mathrm{Lip}(X)$ such that $T f = \alpha h_{y_1}^{\delta_1} + \beta h_{y_2}^{\delta_2}$. Then, for all $y \in Y_0$, we have $T f(y) = 0$, but also $T f(y) = \tau(y) f(\varphi(y))$. In consequence, $\tau(y) f(\varphi(y)) = 0$ for all $y \in Y_0$. Then f is the zero function because τ is unimodular and φ is surjective, and thus $\alpha h_{y_1}^{\delta_1} + \beta h_{y_2}^{\delta_2} = 0$ by the linearity of T.

From the above it is deduced that $h_{y_1}^{\delta_1}$ is not in the range of T. Since the range of T has codimension 1, we have $h_{y_2}^{\delta_2} = \alpha h_{y_1}^{\delta_1} + T(g)$ for some $\alpha \in \mathbb{K}$ and $g \in \mathrm{Lip}(X)$. Then $h_{y_2}^{\delta_2} - \alpha h_{y_1}^{\delta_1} = 0$, a contradiction. Therefore $Y \backslash Y_0$ has at most a point. Then either $Y_0 = Y$ or $Y_0 = Y \backslash \{p\}$ for some point $p \in Y$. This point p must be isolated since $Y \backslash \{p\}$ is closed.

Theorem (5.1.9) allows us to classify codimension 1 linear isometries between $\mathrm{Lip}(X)$-spaces in two types:

Definition (5.1.10)[172]: Let $T : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ be a codimension 1 linear isometry such that T_{1X} is a contraction. We say:

(i) T is of type I when there exists an isolated point p of Y, a surjective Lipschitz map $\varphi : Y \backslash \{p\} \to X$ and a Lipschitz function τ: $Y \backslash \{p\} \to S_{\mathbb{K}}$ such that

$$T f(y) = \tau(y) f(\varphi(y)), \qquad \forall y \in Y \backslash \{p\}.$$

(ii) T is of type II if there is a surjective Lipschitz map $\varphi : Y \to X$ and a Lipschitz function $\tau : Y \to S_{\mathbb{K}}$ such that

$$T f(y) = \tau(y) f(\varphi(y)), \forall y \in Y.$$

These two types are not necessarily disjoint as the next result shows.

Proposition (5.1.11)[172]: Let Y be a metric compact space with an isolated point p. Let $X = Y \setminus\{p\}$ and suppose there exists a point $x' \in X$ such that $d(x,x') \leq d(x,p)$ for all $x \in X$ (in particular, this happens when $\text{diam}(X) \leq d(p,X)$). Then the map $T : \text{Lip}(X) \rightarrow \text{Lip}(Y)$ defined by

$$T f(y) = f(y), \quad \forall y \in X, \quad T f(p) = f(x'),$$

is a codimension 1 linear isometry with $L(T_{1X}) < 1$, which is simultaneously of types I and II.

Proof. Let $f \in \text{Lip}(X)$. Obviously, $\|T f\|_\infty = \|f\|_\infty$ and $L(f) \leq L(T f)$. Moreover, we check at once that

$$|T f(x) - T f(p)| = |f(x) - f(x')| \leq L(f) d(x,x') \leq L(f) d(x,p), \quad \forall x \in X.$$

Therefore $L(T f) \leq L(f)$ and so $\|T f\| = \|f\|$. Clearly, T is linear. However T is not surjective, since the function $h_p^r \in \text{Lip}(Y)$ with $r = d(p,X) > 0$ is not in $T(\text{Lip}(X))$. Moreover, $T_{1_X} = 1_Y$ and thus $L(T_{1_X}) = 0 < 1$. Finally, $T(\text{Lip}(X))$ is of codimension 1 since every $g \in \text{Lip}(Y)$ may be expressed as

$$g = T f + \big(g(p) - g(x')\big) h_p^r,$$

where f is the function in $\text{Lip}(X)$ defined by $f(x) = g(x)$ for all $x \in X$.

Hence T is of type I taking as φ the identity map on $Y \setminus\{p\}$ and as τ the function $1_Y \setminus\{p\}$. But also T is of type II if we now put $\tau = 1_Y$ and φ the function from Y to $Y \setminus\{p\}$ given by $\varphi(y) = y$ if $y \neq p$ and $\varphi(p) = x'$.

We next give a method for constructing type I codimension 1 linear isometries which are not of type II.

Proposition (5.1.12)[172]: Let X and Y be metric compact spaces. Let p be a point of Y such that $1 < d(p, Y\setminus\{p\})$, $\varphi : Y\setminus\{p\} \rightarrow X$ a surjective isometry and τ a unimodular constant. Then the map T : Lip(X) → Lip(Y) defined by

$$T f(y) = \tau f\big(\varphi(y)\big), \quad \forall y \in Y\setminus\{p\}, \quad T f(p) = 0,$$

for all $f \in \text{Lip}(X)$, is a codimension 1 linear isometry of type I with $L\big(T_{1_X}\big) < 1$, but it is not of type II.

Proof. Obviously, T is linear and preserves the supremum norm. Let f ∈ Lip(X). For all $x, w \in X$, we have

$$\begin{aligned}|f(x) - f(w)| &= T f\big(\varphi^{-1}(x)\big) - T f\big(\varphi^{-1}(w)\big)\\ &\leq L(T f) d\big(\varphi^{-1}(x), \varphi^{-1}(w)\big) = L(T f) d(x,w).\end{aligned}$$

Hence $L(f) \leq L(T f)$ and so $\|f\| \leq \|T f\|$. On the other hand, it is clear that

$$\begin{aligned}|T f(y) - T f(z)| &= \big|f\big(\varphi(y)\big) - f\big(\varphi(z)\big)\big|\\ &\leq L(f) d(\varphi(y), \varphi(z)) \leq \|f\|\, d(y,z)\end{aligned}$$

for all $y, z \in Y\setminus\{p\}$, and

$$\begin{aligned}|T f(y) - T f(p)| &= |T f(y)| = \big|f\big(\varphi(y)\big)\big| \leq \|f\|_\infty\\ &\leq \|f\|_\infty\, d(p, Y\setminus\{p\}) \leq \|f\|\, d(p,y)\end{aligned}$$

for all $y \in Y\setminus\{p\}$. This implies that $L(T f) \leq \|f\|$ and thus $\|T f\| \leq \|f\|$. Hence T is an isometry. Moreover, we have

$$|T_{1_X}(y) - T_{1_X}(p)| = |T_{1_X}(y)| = 1 < d(p, Y\setminus\{p\}) \leq d(p,y)$$

for all $y \in Y\setminus\{p\}$, which gives $L\big(T_{1_X}\big) < 1$.

We now claim that T has codimension 1. First observe that the function $h_p^1 \in \text{Lip}(Y)$ does not belong to $T(\text{Lip}(X))$, since if $h_p^1 = T f$ for some $f \in \text{Lip}(X))$, then $1 = h_p^1(p) = T f(p) = 0$, a contradiction. Then, given $g \in \text{Lip}(Y)$, we take $f = \tau(g \circ \varphi^{-1}) \in \text{Lip}(X)$ and it is clear that $g = T f + g(p)h_p^1$, which proves our claim. Evidently, T is of type I. However T is not of type II since, in contrary case, we could write $T f = \tau'(f \circ \varphi')$ for some Lipschitz surjection $\varphi' : Y \to X$ and some Lipschitz function $\tau' : Y \to S_{\mathbb{K}}$, and we would have $T_{1_X}(p) = \tau'(p) \neq 0$, which contradicts the definition of T.

We next provide a example of a type II codimension 1 linear isometry which is not of type I.

Example (5.1.13)[172]: For $X = [0,2]$ and $Y = [0,1] \cup [2,3]$, let $\varphi : Y \to X$ be the map defined by

$$\varphi(y) = \begin{cases} y \text{ } if \text{ } y \in [0,1], \\ y^{-1} \text{ } if \text{ } y \in [2,3], \end{cases}$$

and let $\tau : Y \to S_{\mathbb{C}}$ be the function given by

$$\tau(y) = \begin{cases} 1 \text{ } if \text{ } y \in [0,1], \\ \dfrac{2}{3} + \dfrac{\sqrt{5}}{3} i \text{ } if \text{ } y \in [2,3]. \end{cases}$$

We claim that $T : \text{Lip}(X) \to \text{Lip}(Y)$ defined by

$$T f(y) = \tau(y) f(\varphi(y)), \qquad \forall y \in Y, \qquad \forall f \in \text{Lip}(X),$$

is a codimension 1 linear isometry.

Since φ and τ are Lipschitz, T is well defined. Obviously, T is linear and, by the surjectivity of φ, T preserves the supremum norm.

We now check that $L(f) \leq L(T f)$ for all $f \in \text{Lip}(X)$. Let $f \in \text{Lip}(X)$ and $x_1, x_2 \in X$ with $x_1 \neq x_2$. If $x_1, x_2 \in [0,1]$or $x_1, x_2 \in [1,2]$, then

$$|f(x_1) - f(x_2)| = |T f(y_1) - T f(y_2)| \leq L(T f)|y_1 - y_2| = L(T f)|x_1 - x_2|$$

for suitable points y_1, y_2 in Y satisfying $\varphi(y_1) = x_1$ and $\varphi(y_2) = x_2$. If $x_1 \in [0,1]$ and $x_2 \in [1,2]$, suppose that $x_1 \neq 1 \neq x_2$ and

$$\frac{|f(1) - f(x_1)|}{1 - x_1} < \frac{|f(x_2) - f(x_1)|}{x_2 - x_1}, \frac{|f(x_2) - f(1)|}{x_2 - 1} < \frac{|f(x_2) - f(x_1)|}{x_2 - x_1}.$$

Then we arrive at the following contradiction:

$$|f(x_2) - f(x_1)| \leq |f(x_2) - f(1)| + |f(1) - f(x_1)|$$
$$< \frac{|f(x_2) - f(x_1)|}{x_2 - x_1} [(x_2 - 1) + (1 - x_1)] = |f(x_2) - f(x_1)|.$$

Therefore we have

$$\frac{|f(x_2) - f(x_1)|}{x_2 - x_1} \leq \frac{|f(1) - f(x_1)|}{1 - x_1}$$

or

$$\frac{|f(x_2) - f(x_1)|}{x_2 - x_1} \leq \frac{|f(x_2) - f(1)|}{x_2 - 1}.$$

Applying the above-proved gives $|f(x_2) - f(x_1)| / (x_2 - x_1) \leq L(T f)$ and so $L(f) \leq L(T f)$. As also $\|T f\|_\infty = \|f\|_\infty$, we deduce that $\|f\| \leq \|T f\|$.

On the other hand, let $y_1, y_2 \in Y$. A simple calculation yields

$$|T f(y_1) - T f(y_2)| = \left|\tau(y_1) f\big(\varphi(y_1)\big) - \tau(y_2) f\big(\varphi(y_2)\big)\right|$$

$$\leq \left|\tau(y_1)f(\varphi(y_1)) - \tau(y_1)f(\varphi(y_2))\right| + \left|\tau(y_1)f(\varphi(y_2)) - \tau(y_2)f(\varphi(y_2))\right|$$
$$\leq L(f)|\varphi(y_1) - \varphi(y_2)| + \|f\|_\infty |\tau(y_1) - \tau(y_2)|.$$
If $y_1, y_2 \in [0,1]$ or $y_1, y_2 \in [2,3]$, it follows that
$$|T f(y_1) - T f(y_2)| \leq L(f)|y_1 - y_2| \leq \|f\| |y_1 - y_2|,$$
whereas for $y_1 \in [0,1]$ and $y_2 \in [2,3]$, we obtain that
$$|T f(y_1) - T f(y_2)| \leq \|f\| \left[|y_1 - y_2 + 1| + \left|\frac{1}{3} - \frac{\sqrt{5}}{3} i\right|\right]$$
$$= \|f\| \left(y_2 - y_1 - 1 + \sqrt{\frac{2}{3}} \right) \leq \|f\| (y_2 - y_1) = \|f\| |y_1 - y_2|.$$
This proves that $L(T f) \leq \|f\|$ and therefore $\|T f\| \leq \|f\|$. Hence T is a linear isometry. Furthermore, $L(T_{1_X}) = \sqrt{2/3} < 1$.

Finally, we claim that T has codimension 1. Clearly, the function
$$g(y) = 1 \; if \; y \in [0,1], g(y) = 0 \; if \; y \in [2,3].$$
is in $\mathrm{Lip}(Y)$, but not in $T(\mathrm{Lip}(X))$ since $|g(1)| = 1 \neq 0 = |g(2)|$. Given $h \in \mathrm{Lip}(Y)$, we can take the scalar $\alpha = h(1) - h(2)\left(\frac{2}{3} - \frac{\sqrt{5}}{3} i\right)$ and the function
$$f(x) = \begin{cases} h(x) - \alpha \; if \; x \in [0,1], \\ \left(\frac{2}{3} - \frac{\sqrt{5}}{3} i\right) h(x+1) \; if \; x \in [1,2]. \end{cases}$$
It is readily seen that $f \in \mathrm{Lip}(X)$ with $L(f) \leq L(h)$. Taking into account that
$$T f(y) + \alpha g(y) = \tau(y) f(\varphi(y)) + \alpha g(y) = f(y) + \alpha = h(y)$$
for all $y \in [0,1]$, and
$$T f(y) + \alpha g(y) = \tau(y) f(\varphi(y)) + \alpha g(y) = \left(\frac{2}{3} + \frac{\sqrt{5}}{3} i\right) f(y-1)$$
$$= \left(\frac{2}{3} + \frac{\sqrt{5}}{3} i\right)\left(\frac{2}{3} - \frac{\sqrt{5}}{3} i\right) h(y) = h(y)$$
for all $y \in [2,3]$, we have $h = T f + \alpha g$, which proves our claim.

Observe that T is of type II, but not of type I since Y has not isolated points. Next we study the properties of the map φ and state some conditions under which φ is a Lipschitz homeomorphism.

Lemma (5.1.14)[172]: Let $T : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ be a codimension 1 linear isometry such that T_{1_X} is a contraction. For any $f \in \mathrm{Lip}(X)$ and $x \in X$, the function $|T f|$ is constant on $\varphi^{-1}(\{x\})$.

Proof. By Theorem (5.1.9), for all $y \in Y_0$ we have $T f(y) = \tau(y) f(\varphi(y))$ and since $(y) \in S_{\mathbb{K}}$, it follows that $|T f(y)| = |f(\varphi(y))|$. If now $y \in \varphi^{-1}(\{x\})$, we get $|T f(y)| = |f(x)|$.

Proposition (5.1.15)[172]: Let $T : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ be a codimension 1 linear isometry such that T_{1_X} is a contraction. We take Y_0, φ and τ as in Theorem (5.1.9). The following assertions hold:

(i) For each $x \in X, \varphi^{-1}(\{x\})$ has at most two elements.

(ii) If there exists a point $x_0 \in X$ and two distinct points $a, b \in Y_0$ such that $\varphi(a) = \varphi(b) = x_0$, then $\varphi^{-1}(\{x\})$ is a singleton for each $x \in X \backslash \{x_0\}$.

(iii) If T is of type I ($Y_0 \neq Y$), then φ is injective and hence a homeomorphism.

(iv) If T is of type I with $Y_0 = Y \setminus \{p\}$ and $T^* (\tilde{\delta}_p) \in \text{Lip}(X)^*$ is zero, then φ is a Lipschitz homeomorphism.

Proof. (i) Suppose that there exist three distinct points $y_1, y_2, y_3 \in Y_0$ such that $\varphi(y_1) = \varphi(y_2) = \varphi(y_3)$. Put $\rho = \min\{d(y_1, y_2), d(y_2, y_3), d(y_1, y_3)\}$ and consider the functions $h_{y_1}^{\rho}$ and $h_{y_3}^{\rho}$. Since the codimension of range of T is 1, there exist constants α and β, not both zero, such that $\alpha h_{y_1}^{\rho} + \beta h_{y_3}^{\rho} \in T(\text{Lip}(X))$. By Lemma (5.1.14) we obtain

$$\left|\alpha h_{y_1}^{\rho}(y_1) + \beta h_{y_3}^{\rho}(y_1)\right| = \left|\alpha h_{y_1}^{\rho}(y_2) + \beta h_{y_3}^{\rho}(y_2)\right| = \left|\alpha h_{y_1}^{\rho}(y_3) + \beta h_{y_3}^{\rho}(y_3)\right| .$$

This gives $|\alpha| = 0 = |\beta|$, a contradiction. Hence $\varphi^{-1}(\{x\})$ contains at most two points for all $x \in X$.

(ii) Suppose there are distinct points $x_0, x \in X$ and $a, b, p, q \in Y_0$ such that

$$a \neq b, \quad \varphi(a) = \varphi(b) = x_0, \quad p \neq q, \quad \varphi(p) = \varphi(q) = x.$$

Take the positive numbers

$$\epsilon = \min\{d(a,b), d(p,b), d(q,b)\},$$
$$r_1 = \min\{d(a,b), d(a,p), d(a,q)\},$$
$$r_2 = \min\{d(p,a), d(p,b), d(p,q)\},$$

and consider the functions $f_1 = h_b^{\epsilon}$ and $f_2 = h_a^{r^1} + h_p^{r^2}$ in $\text{Lip}(Y)$. Then

$$f_1(b) = 1,\ f_1(a) = f_1(p) = f_1(q) = 0,\ f_2(a) = f_2(p) = 1,\ f_2(b) = f_2(q) = 0.$$

Again, the codimension 1 of range of T provides two scalars α, β, not both zero, such that $\alpha f_1 + \beta f_2 \in T(\text{Lip}(X))$. By Lemma (5.1.14), it follows that

$$|\alpha f_1(p) + \beta f_2(p)| = |\alpha f_1(q) + \beta f_2(q)|$$

and

$$|\alpha f_1(a) + \beta f_2(a)| = |\alpha f_1(b) + \beta f_2(b)| .$$

Therefore $|\alpha| = |\beta| = 0$, a contradiction. Hence ii) is true.

(iii) Let us assume T is of type I ($Y_0 \neq Y$). Then $Y \setminus Y_0 = \{p\}$ for some isolated point $p \in Y$. Therefore we may take $r = d(p, Y_0) > 0$ and consider $h_p^r \in \text{Lip}(Y)$. If $h_p^r \in T(\text{Lip}(X))$, it is easy to show that $h_p^r = 0$, a contradiction. Hence h_p^r does not belong to the range of T.

Suppose there exist $y_0, y \in Y_0$ such that $y_0 \neq y$ and $\varphi(y_0) = \varphi(y)$. Let $\epsilon = \min\{d(y_0, y), d(y, p)\}$. Since h_y^{ϵ} satisfies $h_y^{\epsilon}(y) = 1 \neq 0 = h_y^{\epsilon}(y_0)$, Lemma (5.1.14) gives $h_y^{\epsilon} \notin T(\text{Lip}(X))$. As a consequence, there exists $\alpha \in \mathbb{K}$ and $f \in \text{Lip}(X)$ such that $h_p^r = Tf + \alpha h_y^{\epsilon}$. Then Lemma (5.1.14) gives $\left|h_p^r(y0) - \alpha h_y^{\epsilon}(y_0)\right| = \left|h_p^r(y) - \alpha h_y^{\epsilon}(y)\right|$, that is $0 = |\alpha|$, but then $h_p^r \in T(\text{Lip}(X))$, a contradiction. Hence φ is injective.

(iv) Let $y, z \in Y_0$ with $y \neq z$. Putting $\gamma = \min\{d(y,z), d(y,p)\}$, define $g = d(y,z) h_y^{\gamma} \in \text{Lip}(Y)$. A trivial verification yields $L(g) \leq \max\{1, \text{diam}(Y_0)/d(p, Y_0)\}$ and $\|g\|_{\infty} = g(y) = d(y,z) \leq \text{diam}(Y_0)$. As a consequence, $\|g\| \leq k$ where $k = \max\{\delta, \text{diam}(Y_0)\}$ and $\delta = \max\{1, \text{diam}(Y_0)/d(p, Y_0)\}$.

We now consider the function $h_p^r \in \text{Lip}(Y)$ with $r = d(p, Y_0)$. Since $h_p^r \notin T(\text{Lip}(X))$ (see iii)) and $T(\text{Lip}(X))$ has codimension 1, there exist $\alpha \in \mathbb{K}$ and $f \in \text{Lip}(X)$ such that $= Tf + \alpha h_p^r$. Since

$$0 = g(p) = Tf(p) + \alpha = T^* \tilde{\delta}_p(f) + \alpha = \alpha,$$

it follows that $g = Tf$. Then

$$d(y,z) = |g(y)| = \left|g(y) - \tau(y)\overline{\tau(z)}g(z)\right| = \left|Tf(y) - \tau(y)\overline{\tau(z)}Tf(z)\right|$$
$$= \left|\tau(y)f(\varphi(y)) - \tau(y)\overline{\tau(z)}\tau(z)f(\varphi(z))\right| = \left|f(\varphi(y)) - f(\varphi(z))\right|$$
$$\leq L(f)d(\varphi(y),\varphi(z)) \leq \|g\|\, d(\varphi(y),\varphi(z)) \leq kd(\varphi(y),\varphi(z)).$$

Hence φ^{-1} is Lipschitz and so φ is a Lipschitz homeomorphism.

Section (5.2): Vector-Valued Lipschitz Functions

Given a metric space (X,d) and a Banach space E, we denote by $\text{Lip}(X,E)$ the Banach space of all bounded Lipschitz functions $f: X \longrightarrow E$ with the norm $\|f\| = \max\{L(f), \|f\|_\infty\}$, where

$$L(f) = \sup\{\|f(x) - f(y)\|/d(x,y): x,y \in X, x \neq y\} .$$

If E is the field of real or complex numbers, we shall write simply $\text{Lip}(X)$.

The study of surjective linear isometries between spaces $\text{Lip}(X)$ was initiated by Roy [46] and Vasavada [47]. In [46], Roy proved that if (X,d) is a compact connected metric space with diameter at most 1, then a map T is a surjective linear isometry from $\text{Lip}(X)$ onto itself if and only if there exist a surjective isometry $\varphi: X \longrightarrow X$ and a scalar τ of modulus 1 such that

$$T(f)(y) = \tau f(\varphi(y)), \forall y \in Y, \qquad \forall f \in \text{Lip}(X).$$

In [182], Novinger improved slightly Roy's result by considering linear isometries from $\text{Lip}(X)$ onto $\text{Lip}(Y)$. Vasavada [47] proved it for linear isome tries from $\text{Lip}(X)$ onto $\text{Lip}(Y)$ when the metric spaces X, Y are compact with diameter at most 2 and β-connected for some $\beta < 1$. Weaver [189] developed a technique to remove the compactness assumption on X and Y and showed that the above-mentioned characterization holds if X, Y are complete and 1-connected with diameter at most 2 [189]. The reduction to metric spaces of diameter at most 2 is not restrictive since if (X,d) is a metric space and X' is the set X remetrized with the metric $d'(x,y) = \min\{d(x,y), 2\}$, then the diameter of X' is at most 2 and $\text{Lip}(X')$ is isometrically isomorphic to $\text{Lip}(X)$ [184]. We must also mention the complete research carried out on surjective linear isome tries between spaces of Holder functions [25], [186], [180], [188]. See Weaver's book Lipschitz Algebras [184]. This is essentially the history of the onto scalar-valued case. Recently, into linear isometries (that is, not necessarily surjective) and codimension 1 linear isometries between spaces $\text{Lip}(X)$ have been studied in [172].

We give a complete description of linear isometries between spaces of vector-valued Lipschitz functions. Little or nothing is known on the matter in the vector-valued case. Our approach to the problem is not based on extreme points. We have used here a different method which is influenced by that utilized by Cambern [176] to characterize into linear isometries between spaces $C(X,E)$ of continuous functions from a compact Hausdorff space X into a Banach space E with the supremum norm. In [187], Jerison extended to the vector case the classical Banach-Stone theorem about onto linear isometries between spaces $C(X)$, and Jerison's theorem was generalized by Cambern [176] by considering into linear isometries.

We show that Cambern's and Jerison's theorems have a natural formulation in the context of Lipschitz functions.

Given a Banach space E, S_E will denote its unit sphere and B_E its closed unit ball. Let us recall that a Banach space E is said to be strictly convex if every element of S_E is an extreme point of B_E. For Banach spaces E and F, $L(E,F)$ will stand for the Banach space of all bounded linear operators from E into F with the canonical norm of operators. In the case $E = F$, we shall write $L(E)$ instead of $L(E,F)$. Given a metric space (X,d), we shall denote by l_x the function constantly 1 on X and by $\text{diam}(X)$ the diameter of X. If $\varphi: X \longrightarrow Y$ is a Lipschitz map between metric spaces, $L(\varphi)$ will be its Lipschitz constant.

For any $f \in \text{Lip}(X)$ and $e \in E$, define $f \otimes e: X \to E$ by $(f \otimes e)(x) = f(x)e$. It is easy to check that $f \otimes e \in \text{Lip}(X,E)$ with $\|f \otimes e\|_\infty = \|f\|_\infty \|e\|$ and $L(f \otimes e) = L(f)\|e\|$, and thus $\|f \otimes e\| = \|f\|\|e\|$.

Theorem (5.2.1)[185]: Let X and Y be compact metric spaces and let E be a strictly convex Banach space. Let T be a linear isometry from $\text{Lip}(X,E)$ into $\text{Lip}(Y,E)$ such that $T(1_X \otimes e) = I_y \otimes e$ for some $e \in S_E$. Then there exists a Lipschitz map φ from a closed subset Y_o of Y onto X with $L(\varphi) \leq \max\{1, \text{diam}(X)/2\}$, and a Lipschitz map $y \longmapsto T_y$ from Y into $L(E)$ with $\|T_y\| = 1$ for all $y \in Y$, such that

$$T(f)(y) = T_y\Big(f\big(\varphi(y)\big)\Big), \qquad \forall y \in Y_o, \forall f \in \text{Lip}(X,E).$$

Proof. For each $x \in X$, define

$$F(x) = \{f \in \text{Lip}(X,E): f(x) = \|f\|_\infty e\}.$$

Clearly, $l_x \otimes e \in F(x)$. For each $\delta > 0$, the map $h_{x,\delta} \otimes e: X \to E$, defined by

$$h_{X,\delta}(z) = \max\ \{0, 1 - d(z,x)/\delta\} \quad (z \in X),$$

belongs to $F(x)$. Indeed, an easy verification shows that $h_{x,\delta} \otimes \text{Lip}(X)$ with $\|h_{x,\delta}\|_\infty = 1 = h_{x,\delta}(x)$. Hence $h_{x,\delta} \otimes e \in \text{Lip}(X,E)$ with $\|h_{x,\delta} \otimes e\|_\infty = 1$ and $\big(h_{x,\delta} \otimes e\big)(x) = e$. Then $\big(h_{x,\delta} \otimes e\big)(x) = \|h_{x,\delta} \otimes e\|_\infty e$ and thus $h_{x,\delta} \otimes e \in F(x)$.

We shall prove the theorem in a series of steps.

Step 1. Let $x \in X$. For each $f \in F(x)$, the set

$$P(f) = \{y \in Y: T(f)(y) = f(x)\}$$

is nonempty and closed.

Let $f \in F(x)$. If $f = 0$, then $P(f) = Y$ and there is nothing to prove. Suppose $f \neq 0$ and consider $g(x) = \|f\|_\infty f + \|f\|^2 (1_x \otimes e)$. Clearly, $g \in \text{Lip}(X,E)$ with $L(g) = \|f\|_\infty L(f)$ and $g(x) = (\|f\|_\infty^2 + \|f\|^2)\, e$. The latter equality implies $g \neq 0$. Since

$$L(g) \leq \|f\|_\infty \|f\| \leq \|f\|_\infty^2 + \|f\|^2 = \|g(x)\| \leq \|g\|,$$

it follows that $\|g\| = \|g\|_\infty$. Moreover, $\|g\|_\infty = \|g(x)\| + \|f\|_\infty^2 + \|f\|^2$ since

$$\|g\|_\infty = \|\|f\|_\infty f + \|f\|^2(1_X \otimes e)\|_\infty \leq \|f\|_\infty^2 + \|f\|^2 = \|g(x)\|_\infty.$$

We now claim that there exists a point $y \in Y$ such that $T(g/\|g\|)(y) = e$. Contrary to our claim, assume $e \neq T(g/\|g\|)(y)$ for all $y \in Y$. Let $\varepsilon > 0$ and take $h = g/\|g\| + \varepsilon(1_X \otimes e)$. Clearly, $h \in \text{Lip}(X,E)$ and $T(h) = T(g)/\|g\| + \varepsilon(1_y \otimes e)$. A simple calculation yields

$$L\big(T(h)\big) = L(T(g))/\|g\| \leq T(g)/\|g\| = 1.$$

Next we show that $\|T(h)\|_\infty < 1 + \varepsilon$. For any $y \in Y$, we have

$$\|T(h)(y)\| = \|T(g/\|g\|)(y) + \varepsilon e\| \leq 1 + \varepsilon$$

since $\|T(g/\|g\|)(y)\| \leq \|T(g)\|/\|g\| = 1$. Indeed,

$$\|T(g/\|g\|)(y) + \varepsilon e\| \leq 1 + \varepsilon.$$

Otherwise the vector $u = (1/(1+\varepsilon))(T(g/\|g\|)(y) + \varepsilon e)$ would be an extreme point of B_E by the strict convexity of E, and since u is a convex combination of $T(g/\|g\|)(y)$ and e, which are in B_E, we infer that $T(g/\|g\|)(y) = e$, a contradic tion. Hence $\|T(h)(y)\| < 1 + \varepsilon$ for all $y \in Y$. Since $\|T(h)\|_\infty = \|T(h)(y)\|$ for some $y \in Y$, we conclude that $\|T(h)\|_\infty < 1 + \varepsilon$. From what we have proved above it is deduced that $\|T(h)\| < 1 + \varepsilon$, but, on the other hand,

$$1+\varepsilon=\|g(x)/\|g\|+\varepsilon e\|=\|h(x)\|\leq\|h\|_\infty\leq\|h\|=\|T(h)\|,$$
which is impossible. This proves our claim.

Now, let $y\in Y$ be such that $T\,(g/\|g\|)(y)=e$. Since $e=g(x)/\|g\|, Tg(y)=g(x)$, that is,
$$\|f\|_\infty\,Tf(y)+\|f\|^2\,T(1_X\otimes e)(y)=(\|f\|_\infty^2+\|f\|^2)\,e$$
Since $T(1_X\otimes e)=1_Y\otimes e$, we have
$$(\|f\|_\infty T(f)(y)+\|f\|^2e=(\|f\|_\infty^2+\|f\|^2)\,e,$$
and thus $T(f)(y)=\|f\|_\infty e$, which is $T(f)(y)=f(x)$ since $f\in F(x)$. Hence $P(f)\neq\emptyset$. Moreover, $P(f)$ is closed in Y since $P(f)=T(f)^{-1}(\{f(x)\})$ and $T(f)$ is continuous.

Step 2. For each $x\in X$, the set
$$B(x)=\{y\in Y: T(f)(y)=f(x), \forall f\in F(x)\}$$
is nonempty and closed.

Let $x\in X$. For each $f\in F(x)$, $P(f)$ is a nonempty closed subset of Y by Step 1. Since $B(x)=\bigcap_{f\in F(x)}P(f), B(x)$ is closed. To prove that $B(x){:}\neq\emptyset$, since Y is compact and $B(x)=\bigcap_{f\in F(x)}P(f)$, it suffices to check that if $f_1,\dots,f_n\in F(x)$, then $\bigcap_{j=1}^{n}P(f_j)\neq\emptyset$.

We can suppose, without loss of generality, that $f_j\neq 0$ for all $j\in\{1,\dots,n\}$ since $P(f_j)=Y$ if $f_j=0$. For each $j\in\{1,\dots,n\}$ define $g_j=(\|f\|_\infty f_j+\|f_j\|^2+(1_X\otimes e)$. As in the proof of Step 1, $g_j\in \mathrm{Lip}(X,E)$ with $g_j(x)=\left(\|f_j\|_\infty^2+\|f_j\|^2\right)e$ and $\|g_j\|=\|f_j\|_\infty^2+\|f_j\|^2$. Hence $g_j\neq 0$ and we can define $h=(1/n)\sum_{j=1}^{n}(g_j/\|g_j\|)$ Clearly, $h\in\mathrm{Lip}(X,E), h(x)=e$ and $\|h\|_\infty=1$. Hence $h(x)=\|h\|_\infty e$ and thus $h\in F(x)$. Then, by Step 1, there exists a point $y\in Y$ such that $T(h)(y)=h(x)$. Since $T(h)(y)=(1/n)\sum_{j=1}^{n}\left(T(g_j)(y)\|g_j\|\right)$ and $h(x)=e$, it follows that $e=(1/n)\sum_{j=1}^{n}(T(g_j)(y)/\|g_j\|)$. Since E is strictly convex and $\|T(g_j)(y)\|/\|g_j\|\leq\|T(g_j)\|/\|g_j\|=1$ for all $j\in\{1,\dots,n\}$, we infer that $T(g_j)(y)=\|g_j\|e$ for all $j\in\{1,\dots,n\}$. Reasoning as in Step 1, we obtain $T(f_j)(y)=f_j(x)$ for all $j\in\{1,\dots,n\}$ and thus $y\in\bigcap_{j=1}^{n}P(f_j)$.

Step 3. Let $f\in\mathrm{Lip}(X,E), x\in X$ and $y\in B(x)$. If $f(x)=0$, then $T(f)(y)=0$. If $f=0$, then there is nothing to prove. Suppose $f\neq 0$ and let $\delta=\|f\|_\infty/\|f\|$. Clearly, $L(f)/\|f\|_\infty\leq 1/\delta$. Consider $h_{x,\delta}\otimes e\in F(x)$. We next prove that $f/\|f\|_\infty+(h_{x,\delta}\otimes e)$ belongs to $F(x)$. Since $f\|f\|_\infty+\left(h_{x,\delta}\otimes e\right)\in\mathrm{Lip}(X,E)$ and $f(x)/\|f\|_\infty+(h_{x,\delta}\otimes e)(x)=e$, it suffices to check that $\left\|f/\|f\|_\infty+\left(h_{x,\delta}\otimes e\right)\right\|_\infty=1$. Let $z\in X$. If $d(z,x)\geq\delta$, we have $(h_{x,\delta}\otimes e)(z)=0$ and so
$$\left\|f(z)/\|f\|_\infty+(h_{x,\delta}\otimes e)(z)\right\|=\|f(z)/\|f\|_\infty\|\leq 1.$$
If $d(z,x)\leq\delta$, then $(h_{x,\delta}\otimes e)(z)=(1-d(z,x)/\delta)\,e$, and therefore
$$\left\|f(z)/\|f\|_\infty+h_{x,\delta}\otimes e))\,(z)\right\|\leq\|f(z)\|/\|f\|_\infty+1-d(z,x)/\delta\leq 1,$$
since
$$\|f(z)/\|f\|_\infty\|=\|f(z)-f(x)\|\|f\|_\infty\leq L(f)d(z,x)/\|f\|_\infty\leq d(z,x)/\delta.$$
Hence $\left\|f(x)/\|f\|_\infty+(h_{x,\delta}\otimes e))\,(z)\right\|_\infty\leq 1$. Since
$$\left\|f(x)/\|f\|_\infty+(h_{x,\delta}\otimes e))\,(x)\right\|=\|e\|=1,$$
we obtain the desired condition.

By the definition of $B(x)$ it follows that

$$T(f\|f\|_\infty + (h_{x,\delta}\otimes e))(y) = (f\|f\|_\infty + (h_{x,\delta}\otimes e))(x),$$

that is, T(f)(y)/ Ilf IOO+T(hx,60e)(y) = e. Moreover, since $y \in B(x)$ and $h_{x,\delta}\otimes e \in F(x)$, we have $T(h_{x,\delta}\otimes e)(y) = (h_{x,\delta}\otimes e)(x)\, e$. Hence $T(f)/(y)\,\|f\|_\infty + e = e$ and thus $T(f)(y) = 0$.

Step 4. Let $x, x' \in X$ with $x \neq x'$. Then $B(x) \cap B(x')\emptyset$.

Suppose $y \in B(x) \cap B(x')$. Let $\delta = d(x, x') > 0$ and consider $h_{x,\delta}\otimes e$. Since $y \in B(x)$ and $h_{x,\delta}\otimes e \in F(x)$, we have $T(h_{x,\delta}\otimes e)(y) = (h_{x,\delta}\otimes e)(x) = e$ by Step 2, but Step 3 also yields $T(h_{x,\delta}\otimes e)(y) = 0$ since $y \in B(x')$ and $(h_{x,\delta}\otimes e)(x') = 0$. So we arrive at a contradiction. Hence $B(x)\cap B(x') = \emptyset$. Steps 3 and 4 motivate the following:

Definition (5.2.2)[185]: Let $Y_0 = \bigcup_{x\in X} B(x)$. Define $\varphi: Y_0 \to X$ by $\varphi(y) = x$ if $y \in B(x)$.

Clearly, φ is surjective. Moreover, given $y \in Y_0$, there exists $x \in X$ such that $y \in B(x)$, and hence $\varphi(y) = x$ and $T(f)(y) = f(x)$ for all $f \in F(x)$. We shall obtain the representation of T in terms of the following functions.

Definition (5.2.3)[185]: For each $y \in Y$, define $T_y: E \to E$ by $T_y(u) = T(1_X\otimes u)(y)$. It is easy to show that $T_y \in L(E)$ with $\|T_y\| = 1 = \|T_y(e)\|$ for all $y \in Y$.

Step 5. The map $y \longmapsto T_y$ from Y into $L(E)$ is Lipschitz.

Let $y, z \in Y$. Given $u \in E$, we have

$$\|(T_y - T_z)(u)\| \leq L(T(1_X\otimes u))d(y,z) \leq \|T(1_X\otimes u)\|d(y,z) = \|u\|\, d(y,z),$$

and thus $\|T_y - T_z\| \leq d(y,z)$.

Step 6. $T(f)(y) = T_y\left(f\left(\varphi(y)\right)\right)$ for all $f \in \mathrm{Lip}(X,E)$ and $y \in Y_0$

Let $f \in \mathrm{Lip}(X,E)$ and $y \in Y_0$. Let $x = \varphi(y) \in X$ and define $h = f - (1_X\otimes f(x))$. Obviously, $h \in \mathrm{Lip}(X,E)$ with $h(x) = 0$. From Step 3, we have $T(h)(y) = 0$ and therefore $T(f)(y) = T\big(1_X\otimes f(x)\big)(y) = T_y\big(f(x)\big) = T_y\left(f\big((y)\big)\right)$.

Step 7. Y_0 is closed in Y.

Let $y \in Y$ and let $\{y_n\}$ be a sequence in Y_0 which converges to y. Let $x_n = \varphi(y_n)$ for all $n \in \mathbb{N}$. Since X is compact, there exists a subsequence $\{x_{\sigma(n)}\}$ converging to a point $x \in X$. Let $f \in F(x)$. Clearly, $\{T(f)\big(y,(n)\big)\}$ converges to $T(f)(y)$, but also to $f(x)$ as we see at once. Indeed, for each $n \in \mathbb{N}$, we have

$$T(f)\big(y_{\sigma(n)}\big) = T_{y_{\sigma(n)}}\left(f\left(x_{\sigma(n)}\right)\right) = T\left(1_X\otimes f\left(x_{\sigma(n)}\right)\right)\left(y_{\sigma(n)}\right),$$

by Step 6, and

$$f(x) = \|f\|_\infty e = \|f\|_\infty \left(1_Y \bigotimes e\right)\left(Y_{\sigma(n)}\right)$$
$$= \|f\|_\infty T(1_X\otimes e)(y_{\sigma(n)}) = T(1_X\otimes f(x))(y_{\sigma(n)}),$$

since $f \in F(x)$. We deduce that

$$\|T(f)(Y_{\sigma(n)}) - f(X)\| = \|T(1_X\otimes(f\big(x_{\sigma(n)}\big) - f(x)))\,(y_{\sigma(n)})\|$$
$$\leq \|\|T(1_X\otimes(f(x_{\sigma(n)}) - f(x)))\| = \|1_X\otimes(f(x_{\sigma(n)}) - f(x))\|$$
$$= \|f(X_{\sigma(n)}) - f(x)\|$$

for all $n \in \mathbb{N}$. Since $\{f\big(X_{\sigma(n)}\big)\} \to f(x)$, we conclude that $\{T(f)/\big(y_{\sigma(n)}\big)\} \to f(x)$. Hence $T(f)(y) = f(x)$ and thus $y \in B(x) \subset Y_0$.

Step 8. The map $\varphi\; Y_0 \to X$ is Lipschitz and $L(\varphi) \leq \max\{1, \mathrm{diam}(X)/2\}$.

Let $y, z \in Y_0$ be such that $\varphi(y) \neq \varphi(z)$ and put $\delta = d(\varphi(y), \varphi(z))/2$. De fine $f_{y,z} = \delta(h_{\varphi(y),\delta} - h_{\varphi(z),\delta})$ on X. It is easy to see that $f_{y,z} \in \mathrm{Lip}(X)$ and $\|f_{y,z}\| \leq k := \max\{1, \mathrm{diam}(X)/2\}$. Since T is an isometry, $\|T(f_{y,z} \otimes e)\|) \leq k$.This inequality implies $L(T(f_{y,z} \otimes e) \leq k$. It follows that

$$\|T(f_{y,z} \otimes e)(y) - T(f_{y,z} \otimes e)(z)\| \leq kd(y,z).$$

Using Step 6 we get

$$T(f_{y,z} \otimes e)(y) = T_y((f_{y,z} \otimes e)(\varphi(y))) = T_y(\delta e) = \delta e,$$

$$T(f_{y,z} \otimes e)(z) = T_z\left((f_{y,z} \otimes e)(\varphi(z))\right) = T_z(-\delta e) = -\delta e.$$

We conclude that $d(\varphi(y), \varphi(z)) \leq kd(y,z)$.

The condition in Theorem (5.2.1), $T(1_X \otimes e)1_Y \otimes e$ for some $e \in S_E$, is not too restrictive if we analyse the known results in the scalar case. In this case our condition means $T(1_X) = 1_Y$; notice that the connectedness assumptions on the metric spaces in [46] and [189] yield a similar condition, namely,that $T(1_X)$ is a constant function.

Recall that a map between metric spaces $\varphi: X \to Y$ is said to be a Lipschitz homeomorphism if φ is bijective and φ and φ^{-1} are both Lipschitz.

Theorem (5.2.4)[185]: Let X, Y be compact metric spaces and let E be a strictly convex Banach space. Let T be a linear isometry from $\mathrm{Lip}(X, E)$ onto $\mathrm{Lip}(Y, E)$ such that $T(1_X \otimes e) = 1_Y \otimes e$ for some $e \in S_E$. Then there exists a Lipschitz homeomorphism $\varphi: Y \to X$ with $L(\varphi) \leq \max\{1, \mathrm{diam}(X)/2\}$ and $L(\varphi^{-1}) \leq \max\{1, \mathrm{diam}(Y)/2\}$, and a Lipschitz map $y \longmapsto T_y$ from Y into $L(E)$ where T_y is an isometry from E onto itself for all $y \in Y$ such that

$$T(f)(y) = T_y\left(f(\varphi(y))\right), \qquad \forall y \in Y, \qquad \forall f \in \mathrm{Lip}(X, E).$$

Proof. Let Y_0 and φ be as in Theorem (5.2.1). Since $T^{-1}: \mathrm{Lip}(Y, E) \to \mathrm{Lip}(X, E)$ is a linear isometry and $T^{-1}(1_Y \otimes e) = 1_X \otimes e$, applying Theorem (5.2.1) we have

$$T^{-1}(g)(x) = (T^{-1})_x\left(g(\psi(x))\right), \qquad \forall x \in X_0, \forall g \in \mathrm{Lip}(Y, E),$$

where ψ is a Lipschitz map from a closed subset X_0 of X onto Y with $L(\psi) \leq \max\ \{1, \mathrm{diam}(Y)/2\}$, and $x \longmapsto (T^{-1})_x$ is a Lipschitz map from X into $L(E)$. Namely, $X_0 = \bigcup_{y \in Y} B(y)$ where, for each $y \in Y$,

$$B(y) = \{\ x \in X: T^{-1}(g)(x) = g(y), \qquad \forall g \in F(y)\}$$

with

$$F(y) = \{g \in \mathrm{Lip}(Y, E) :\ g(y) = \|g\|_\infty e\},$$

and $\psi: X_0 \to Y$ is the Lipschitz map defined by $\psi(x) = y$ if $x \in B(y)$. Moreover, using the same arguments as in Step 3, the following can be proved:

Claim (5.2.5)[185]: Let $g \in \mathrm{Lip}(Y, E)$, $y \in Y$ and $x \in B(y)$. If $g(y) = 0$, then $T^{-1}(g)(x) = 0$.

After this preparation we proceed to prove the theorem. Fix $x \in X$ and let $y \in B(x)$. We first prove that $x \in B(y)$. Suppose that $x \notin B(y)$. Since $B(y) \notin \emptyset$, there exists $x' \in B(y)$ with $x' \neq x$. Take $f \in \mathrm{Lip}(X, E)$ for which $f(x) = 0$ and $f(x') \neq 0$. Since $y \in B(x)$ and $f(x) = 0$, we have $T(f)(y) = 0$ by Step 3. Then $T^{-1}(T(f))(x') = 0$ since $x' \in B(y)$ by Claim (5.2.5), and thus $f(x') = 0$, a contradiction. Therefore $x \in B(y) \subset X_0$ and thus $X_0 = X$. Next we see that $Y_0 = Y$. Let $y \in Y$. We can take a point $x \in B(y)$. As above it is proved that $y \in B(x)$ and thus $y \in Y_0$.

To see that φ is a Lipschitz homeomorphism, let $y \in Y$. Then $y \in B(x)$ for some $x \in X$, that is, $\varphi(y) = x$. Moreover, by what we have proved above, $x \in B(y)$ and so $\psi(x) = y$. As a consequence, $\psi(\varphi(y)) = y$. Since φ was surjective, φ is bijective with $\varphi^{-1} = 4$ and thus φ is a Lipschitz homeomorphism.

To check that T_y is an isometry from E into itself for every $y \in Y$, we first show that T sends nonvanishing functions of $\mathrm{Lip}(X, E)$ into nonvanishing functions of $\mathrm{Lip}(Y, E)$. Assume there exists $f \in \mathrm{Lip}(X, E)$ such that $f(x) \neq 0$ for all $x \in X$, but $T(f)(y) = 0$ for some $y \in Y$. By the surjectivity of ψ, there is a point $x \in X_0$ such that $\psi(x) = y$, that is, $x \in B(y)$. Since $T(f)(y) = 0$, by Claim (5.2.5)
we have $f(x) = T^{-1}(T(f))(x) = 0$, a contradiction. Hence T maps nonvanishing functions into nonvanishing functions. If, for some $y \in Y, T_y$ is not an isometry, then there exists a $u \in S_E$ such that $\|T_y(u)\| = \|T(1_X \otimes u)(y)\| < 1$. Since T is surjective, there is an $f \in \mathrm{Lip}(X, E)$ such that $T(f) = 1_Y \otimes T(1_X \otimes u)(y)$. Thus $\|f\|_\infty \leq \|f\| = \|T(f)\| = \|T(1_X \otimes u)(y) < 1\|$ and $(1_X \otimes u) - f$ never vanishes on X. As $T(1_X \otimes u)(y) = T(f)(y)$, we arrive at a contradiction.

Next we prove that $T_y : E \to E$ is surjective for every $y \in Y$. Fix $y \in Y$ and let $v \in E$. Since T is surjective, there exists $f \in \mathrm{Lip}(X, E)$ such that $T(f) = 1_Y \otimes v$. Let $u = (f \circ \varphi)(y) \in E$. Using Step 6, we have $T_y(u) = T_y\left(f(\varphi(y))\right) = T(f)(y) = v$. Hence T_y is surjective.

We obtain the following:

Corollary (5.2.6)[185]: Let X, Y be compact metric spaces with diameter at most 2 and let E be a strictly convex Banach space. Then every surjective linear isometry T from $\mathrm{Lip}(X, E)$ into $\mathrm{Lip}(Y, E)$ satisfying that $T(1_X \otimes e) = 1_Y \otimes e$ for some $e \in S_E$, can be expressed as $T(f)(y) = T_y\left(f(\varphi(y))\right)$ for all $y \in Y$ and $f \in \mathrm{Lip}(X, E)$, where $\varphi: Y \to X$ is a surjective isometry and $y \mapsto T_y$ is a Lipschitz map from Y *into* $L(E)$ such that T_y is an isometry from E onto E for all $y \in Y$.

In the special case that E is a Hilbert space, Theorems (5.2.1) and (5.2.4) can be improved as follows. For a Hilbert space E, let us recall that a unitary operator is a linear map $\Phi: E \to E$ that is a surjective isometry.

Corollary (5.2.7)[185]: Let X and Y be compact metric spaces and let E be a Hilbert space. Let T be a linear isometry from $\mathrm{Lip}(X, E)$ into $\mathrm{Lip}(Y, E)$ such that $T(1_X \otimes e)$ is a constant function for some $e \in S_E$. Then there exists a Lipschitz map φ from a closed subset Y_0 of Y onto X with $L(\varphi) \leq \max\ \{1, \mathrm{diam}(X)/2\}$ and a Lipschitz map $y \mapsto T_y$ from Y into $L(E)$ with $\|T_y\| = 1$ for all $y \in Y$ such that

$$T(f)(y) = T_y\left(f(\varphi(y))\right), \qquad \forall y \in Y_o, \qquad \forall f \in \mathrm{Lip}(X, E).$$

If, in addition, T is surjective, then $Y_o = Y, \varphi$ is a Lipschitz homeomorphism with $L(\varphi^{-1}) \leq \max\ \{1, \mathrm{diam}(Y)/2\}$ and, for each $y \in Y$, T_y is a unitary operator.

Proof. Assume that $T(1_X \otimes e) = 1_Y \otimes u$ for some $u \in E$. Obviously, $\|u\| = 1$. Since E is a Hilbert space, we can construct a unitary operator $\Phi: E \to E$ such that $\Phi(u) = e$. Define $S : \mathrm{Lip}(Y, E) \to \mathrm{Lip}(Y, E)$ by

$$S(g)(y) = \Phi\left(g(y)\right), \qquad \forall y \in Y, \qquad \forall g \in \mathrm{Lip}(Y, E).$$

It is easy to prove that S is a surjective linear isometry satisfying that $S(1_Y \otimes u) = 1_Y \otimes e$. Hence $R = S \circ T$ is a linear isometry from $\mathrm{Lip}(X, E)$ into $\mathrm{Lip}(Y, E)$ with $R(1_X \otimes e) =$

$1_Y \otimes e$. Then Theorem (5.2.1) guarantees the existence of a Lipschitz map φ from a closed subset Y_0 of Y onto X with $L(\varphi) \leq \max\ \{1, \operatorname{diam}(X)/2\}$ and a Lipschitz map $y \longmapsto R_y$ from Y into $L(E)$ with $\|R_y\| = 1$ for all $y \in Y$ such that

$$R(f)(y) = R_y\left(f\left(\varphi(y)\right)\right), \qquad \forall y \in Y_o, \qquad \forall f \in \operatorname{Lip}(X, E).$$

For each $y \in Y$, consider $T_y = \Phi^{-1}\, o\, R_y \in L(E)$. It is easily seen that the map $y \longmapsto T_y$. from Y into $L(E)$ is Lipschitz with $\|T_y\| = 1$ for all $y \in Y$. Moreover, for any $y \in Y_o$ and $f \in \operatorname{Lip}(X, E)$, we have

$$T(f)(y) = \Phi^{-1}\left(R_y\left(f\left(\varphi(y)\right)\right)\right) = T_y\left(f\left(\varphi(y)\right)\right).$$

If, in addition, T is surjective, the rest of the corollary follows by applying Theo rem (5.2.4) to R.

Section (5.3): Generalised Weighted Composition Operators

Any Riesz homomorphism (i.e. a linear lattice homomorphism) between spaces of continuous functions on compact Hausdorff spaces can be characterised as a so-called weighted composition operator. We reformulate Theorem (5.3.4) from [193] in our terminology.

Theorem (5.3.1)[190]: Let X, Y be compact Hausdorff spaces and let $T : C(X) \to C(Y)$ be a positive operator. Then T is a Riesz homomorphism if and only if there exist a map $\pi: Y \to X$ and a function $\eta \in C(Y)^+$ such that for all $f \in C(X)$: $T f = \eta(f \circ \pi)$.

In this case, $\eta = T\, \mathbf{1}_X$ and π is uniquely determined and continuous on $\{\eta > 0\}$.

This result has been generalised and extended in multiple ways: we mention $C_0(X)$ for X locally compact in [57], Lipschitz functions on metric spaces in [185], and pre-Riesz spaces, a concept introduced in [195], in [196]. We provide another generalisation, concerning Maeda–Ogasawara spaces of extended continuous functions on a Stonean space. We study the setting in which E, F are Archimedean Riesz spaces, which we characterise as order dense subspaces of their respective Maeda–Ogasawara spaces $C^\infty(X)$ and $C^\infty(Y)$, with a Riesz homomorphism $T : E \to F$. We prove that there is indeed a continuous map $\pi : Y \to X$ such that for all $f, u \in E$: $T f = T u \cdot \left(\frac{f}{u} \circ \pi\right)$ on $\operatorname{supp}(T u)$, the support of $T u$. Note that an exact meaning of the quotient f $u \in C^\infty(X)$ is given in Definition (5.3.8).

Let us first introduce the necessary definitions, see [197].

Definition (5.3.2)[190]: A Riesz space E is

(i) Dedekind complete if every non-empty subset which is bounded from above has a supremum;

(ii) laterally complete if every non-empty disjoint subset of E^+ has a supremum. Concerning Riesz subspaces and homomorphisms, we use the following concepts.

Definition (5.3.3)[190]: A Riesz subspace D of a Riesz space E is

(i) a Riesz ideal if $f \in D, g \in E, |g| \leq |f|$ together imply that $g \in D$;

(ii) order dense if for every non-zero $f \in E^+$, there exists a $g \in D^+$ such that $0 < g \leq f$.

A Riesz homomorphism T from E to a Riesz space F is

(i) ($\sigma-$)order continuous if it preserves order limits (of sequences);

(ii) order bounded if for every $u \in E^+$, the set $\{|T f \mid \in F \mid -u \leq f \leq u\}$ is a bounded subset of F^+.

Every Archimedean Riesz space E has an order dense embedding in some unique Dedekind complete Riesz space, called its Dedekind completion E^{δ} . We identify E with this embedding and write $\subset E^{\delta}$. Before we introduce the representation theorem of Maeda and Ogasawara, we have.

Definition (5.3.4)[190]: Let X be a topological space.

(i) A subset A of X is meagre if there exist closed subsets $C_1, C_2, \ldots \subset X$ such that $C_n^{\circ} = \emptyset$ for all n and $A \subset \bigcup_n C_n$.

(ii) X is extremally disconnected if the closure of every open set is open (hence clopen). An extremally disconnected compact Hausdorff space is called Stonean.

Definition (5.3.5)[190]: Let $\overline{\mathbb{R}} := [-\infty, \infty]$, topologised in the natural way such that $\overline{\mathbb{R}}$ is homeomorphic to $[-1, 1]$. The space of extended continuous functions on a Stonean space X consists of all continuous functions f : X → R such that $f^{-1}(\{\pm\infty\})$ is meagre, and is denoted by $C^{\infty}(X)$.

Lemma (5.3.6)[190]: Let B be a dense subset of a Stonean space X, and suppose $f : B \to \mathbb{R}$ is continuous. Then f has a unique extension in C∞(X). For f, g ∈ C ∞(X), the closed set $A := f^{-1}(\{\pm\infty\}) \cup g^{-1}(\{\pm\infty\})$ is meagre. Hence $X \setminus A$ is dense, and we define $f + g, f \cdot g : X \setminus A \to \mathbb{R}$ by $(f + g)(x) := f(x) + g(x)$ and $f \cdot g(x) = f(x)g(x)$. The previous lemma leads to unambiguous extensions of $f + g$ and $f \cdot g$ in $C^{\infty}(X)$.

Theorem (5.3.7)[190]: The space $C^{\infty}(X)$ with these operations, supplemented by the natural ordering and scalar multiplication, is a multiplicative, Dedekind complete, and laterally complete Riesz space. In addition, it is an f-algebra with unit $\mathbf{1}_X$.

Definition (5.3.8)[190]: For future purposes, we also define the quotient $f\,g$ of $f, g \in C^{\infty}(X)$. Set supp$(g) := \overline{\{x \in X \mid g(x) \neq 0\}}$, the support of g. First define $\frac{\mathbf{1}}{g}$ to be the unique continuous extension of

$$x \mapsto \begin{cases} \dfrac{\mathbf{1}}{g(x)} \text{ if } 0 < |g(x)| < \infty \\ \quad 0 \; if \; x \notin \operatorname{supp}(g), \end{cases} \tag{1}$$

to the whole X. Now set $\frac{f}{g} := f \cdot \frac{\mathbf{1}}{g}$.

For a clopen $Z \subset X$, we embed $C^{\infty}(Z) \subset C^{\infty}(X)$ by setting $h(x) := 0$ if $x \notin Z$ for all $h \in C^{\infty}(Z)$. We list a few properties that follow directly from the definitions.

Proposition (5.3.9)[190]: Let $f, g \in C^{\infty}(X)$ and suppose that π is a continuous map from a Stonean space Y to X such that $f \circ \pi, g \circ \pi \in C^{\infty}(Y)$. Then:

(i) supp$(f) \subset X$ is clopen;

(ii) $\frac{\mathbf{1}}{g} \cdot g = \mathbf{1}_{\operatorname{supp}(g)}$ and $f\,g \cdot g = f \cdot \mathbf{1}_{\operatorname{supp}(g)}$;

(iii) supp$(f) = $ supp$(\frac{\mathbf{1}}{f})$;

(iv) for all $x \in X : f(x) = \infty$ implies $\frac{\mathbf{1}}{f(x)} = 0$;

(v) $(f \circ \pi) \cdot (g \circ \pi) = (f \cdot g) \circ \pi$;

We now come to the main point: the next theorem clarifies why spaces of extended continuous functions are of interest.

Theorem (5.3.10)[190]: (Maeda–Ogasawara). Let E be an Archimedean Riesz space. Then there exists a unique Stonean space X such that E is Riesz isomorphic to an order dense subspace of $C^{\infty}(X)$. $C^{\infty}(X)$ is called the Maeda–Ogasawara space of E. We also identify E with the embedding in its Maeda–Ogasawara space $C^\infty(X)$ and write $E \subset C^\infty(X)$.

Let us first outline the setting to which we adapt Theorem (5.3.1). Motivated by the Maeda– Ogasawara Theorem, we consider Stonean spaces , an order dense Riesz ideal1 $E \subset C^\infty(X)$, and a Riesz homomorphism $T : E \to C^\infty(Y)$ such that the image $T(E) \subset C^\infty(Y)$ is order dense. The next example shows why we can in general not hope for T to be an ordinary weighted composition operator.

Example (5.3.11)[190]: Let $\beta\mathbb{N}$ be the Stone-Čech compactification of $\mathbb{N}$, which is Stonean. Define $j \in C^\infty(\mathbb{N})$ to be the unique continuous extension of $\beta(n) \mapsto n \in \mathbb{R}$. Fix $a \in \beta\mathbb{N} \setminus \beta(\mathbb{N})$. Let $E \subset C^\infty(\beta\mathbb{N})$ be given by $E := \{f \in C^\infty(\beta\mathbb{N}) \mid (f \cdot j)(a) \in \mathbb{R}\}$ and $S : E \to C^\infty(\{0\}) \cong \mathbb{R}$ by $T S = S f(0) := (f \cdot j)(a)$. Then $\frac{\mathbf{1}}{j} \in E$ and $S\frac{\mathbf{1}}{j} = 1$, so $S \neq 0$.

Suppose $S f = \eta(f \circ \pi)$ for some $\eta \in \mathbb{R}^+, \pi : \{0\} \to \beta\mathbb{N}$. For $g \in C(\beta\mathbb{N}), \frac{g}{j} \in E$. Then $g(a) = S\frac{g}{j} = \eta(\frac{g}{j} \circ \pi)$. We have $\pi(0) = a$, for if not, we can find an open $U \subset \beta\mathbb{N}$ with $a \in U \not\ni \pi(0)$. For $g := \mathbf{1}_U$, this means that $g(a) = 1$, while $g\, j\left(\pi(0)\right) = 0$, which is a contradiction. Hence $S f = \eta f(a)$, but $f(a) = 0$ for all $f \in E$, implying $S f = 0$. This is again a contradiction, from which we conclude such η and π do not exist.

Lemma (5.3.12)[190]: Let X, Y, E, and T be as above. Then there exists a unique map $\pi : Y \to X$ such that $T(f \cdot u) = T u \cdot (f \circ \pi)$ for all $f \in C(X)^+$ and $u \in E$. This π is continuous.

Proof. Fix $f \in C(X)^+$. We see that for all $u \in E$:

$$|T(f \cdot u)| \leq \|f\|_\infty |T u|. \tag{2}$$

From T we construct the Riesz homomorphism $S : T(E) \to T(E)$ by $T u \mapsto T(f \cdot u)$. S is band preserving and order bounded, so $S \in \operatorname{Orth}\left(T(E)\right)$ by the definition of orthomorphism. By Theorem 8.12 of [191], S then extends to a positive orthomorphism S^δ on the Dedekind completion E^δ of E. Note that S and S^δ are order continuous by Theorem (5.3.5) from [193]. As $E \subset E^\delta \subset C^\infty(Y)$ is order dense, we may apply Theorem (5.3.4) from [193] to get a positive orthomorphism $\tilde{S} : C^\infty(Y) \to C^\infty(Y)$. Every positive orthomorphism on an f -algebra with unit is given by multiplication with a positive element (Theorem 141.1 from [198]), so we have a unique $\tilde{f} \in C(Y)$ such that $T(f \cdot u) = \tilde{S}(T u) = \tilde{f} \cdot T u$ for all $u \in E$. Note that $\tilde{f}$ is bounded by Eq. (2).

Now we view $f \mapsto \tilde{f}$ as a Riesz homomorphism from $C(X)$ to $C(Y)$, for which we note that $\mathbf{1}_X$ is mapped to 1Y . Application of Theorem (5.3.1) yields a map $\pi : Y \to X$ such that $\tilde{f} = f \circ \pi$ and $T(f \cdot u) = T u \cdot (f \circ \pi)$ for $f \in C(X)^+, u \in E$. Moreover, this π is uniquely determined and continuous on Y, by the second part of the theorem.

We can now prove that T is a generalised weighted composition operator.

Theorem (5.3.13)[190]: Let X, Y, E, and T be as above. There exists a unique map $\pi : Y \to X$ such that for all $f, u \in E$:

(i) $\pi\left(\operatorname{supp}(T u)\right) \subset \operatorname{supp}(u)$;

(ii) $\frac{f}{u} \circ \pi|_{\operatorname{supp}(T u)} \in C^\infty\left(\operatorname{supp}(T u)\right)$;

(iii) $T f = T u \cdot (\frac{f}{u} \circ \pi)$ on supp$(T u)$.

Furthermore, π is continuous.

Proof. We apply the preceding lemma to T and get the unique map $\pi : Y \to X$ that satisfies $T(f \cdot u) = T u \cdot (f \circ \pi)$ for all $f \in C(X), u \in E$. We also have that π is continuous.

Proof of (i). Let $u \in E$. Substituting $1_{\mathrm{supp}(u)}$ for f in the expression above, we have $T u = T u \cdot (\mathbf{1}_{\mathrm{supp}(u)} \circ \pi) = T u \cdot \mathbf{1}_{\pi^{-1}_{(\mathrm{supp}(u))}}$, so $1_{\mathrm{supp}(T u)} \leq 1_{\pi^{-1}_{(\mathrm{supp}(u))}}$ and (i) follows.

Proof of (ii) and (iii) for all $f, u \in E^+$. Let f, u ∈ E +. For $n \in \mathbb{N}$, we define $h_n := \frac{f}{u} \wedge n\mathbf{1}_X \in C(X)$. By the lemma: $T(h_n \cdot u) = T u \cdot (h_n \circ \pi)$. Also $h_n \cdot u = f \cdot \mathbf{1}_{\mathrm{supp}(u)} \wedge nu = f \wedge nu$, so $T(h_n \cdot u) = T f \wedge nT u$, whence $T f \wedge nT u = T u \cdot (h_n \circ \pi)$. Then for $y \in Y$ such that $T f(y) < \infty$, $0 < T u(y) < \infty$:

$$T f(y) \wedge nT u(y) = T u(y)\big(f u\,(\pi(y)) \wedge n\big) \tag{3}$$

and by taking limits:

$$T f(y) = T u(y) f u\,(\pi(y))$$

$$\frac{Tf}{Tu}(y) = \frac{Tf(y)}{Tu(y)} = \frac{f}{u}(\pi(y)). \tag{4}$$

Both $\frac{Tf}{Tu}$ and $f u \circ \pi$ are continuous functions from this dense subset of supp$(T u)$ to $\mathbb{R}$. As they coincide on this subset, we must have $\frac{f}{u} \circ \pi = \frac{Tf}{Tu}$ on the whole supp$(T u)$. Furthermore, $\frac{Tf}{Tu} \in C^{\infty}(\mathrm{supp}(T u))$, so (ii) and (iii) follow.

Proof of (ii) for all $f, u \in E$. Let $f, u \in E$. We have $\left|\frac{f}{u}\right| \circ \pi|_{\mathrm{supp}(T u)} \in C^{\infty}(\mathrm{supp}(T |u|))$, $\mathrm{supp}(T u) = \mathrm{supp}(T |u|)$, and $|g \circ \pi| = |g| \circ \pi$ for all $g \in E$. Also observe that $(f u)^{-1}(\mathbb{R})$ is $\left|\frac{f}{u}\right|^{-1}(\mathbb{R})$, which is dense in supp$(T u)$. This together implies $\frac{f}{u} \circ \pi|_{\mathrm{supp}(T u)} \in C^{\infty}(\mathrm{supp}(T u))$.

Proof of (iii) for all $f, u \in E$. First, let $f \in E^+$ and $u \in E$. By (i), $u \circ \pi = u^+ \circ \pi$ on supp$(T u^+)$, and supp$(T u^+) \cap$ supp$(T u^-) = \emptyset$. By (3):

$$T f = T u^{\pm} \cdot \left(\frac{f}{u^{\pm}} \circ \pi\right) \text{ on supp}(T u^{\pm}), \tag{5}$$

hence $T f = T u \cdot \left(\frac{f}{u} \circ \pi\right)$ on supp$(T u)$. Dropping the assumption that $f \in E^+$, we write $f = f^+ - f^-$ and observe that

$$\begin{aligned} T f &= T(f^+) - T(f^-) \\ &= T u \cdot \left(\left(\frac{f^+}{u} \circ \pi\right) - \left(\frac{f^+}{u} \circ \pi\right)\right) \\ &= T u \cdot \left(\frac{f^+ - f^-}{u} \circ \pi\right) \\ &= T u \cdot \left(\frac{f}{u} \circ \pi\right), \end{aligned} \tag{6}$$

which completes the proof.

As a final step, we translate the theorem to the setting of Maeda–Ogasawara spaces to state the main result.

Theorem (5.3.14)[190]: Let E, F be Archimedean Riesz spaces, with a Riesz homomorphism $T : E \to F$. We denote the Maeda–Ogasawara space of E by $C^\infty(X)$ and the one of $T(E)$ by $C^\infty(Y)$. Then there exists a map $\pi : Y \to X$ such that for all f, $u \in E$:

(i) $\pi\big(\operatorname{supp}(T u)\big) \subset \operatorname{supp}(u)$;

(ii) $\frac{f}{u} \circ \pi|_{\operatorname{supp}(T u)} \in C^\infty\big(\operatorname{supp}(T u)\big)$;

(iii) $T f = T u \cdot (\frac{f}{u} \circ \pi)$ on $\operatorname{supp}(T u)$.

Furthermore, any such π is continuous.

If in addition either E is Dedekind complete or T is order continuous, then π is uniquely determined.

Proof. By the Lipecki–Luxemburg–Schep Theorem (Theorem (5.3.3) in [193]), T can be extended to the ideal E' generated by E. Then we apply the preceding theorem.

In case E is Dedekind complete, $E \in C^\infty(X)$ is already a Riesz ideal. Theorem (5.3.13) then yields a unique $\pi : Y \to X$. Assuming order continuity of T and observing that $E' = E^\delta$, we use that fact that T has a unique extension T^δ on E^δ .

Note that the Lipecki–Luxemburg–Schep Theorem does not entail uniqueness of the extension. The following example illustrates that these extensions can induce different composition maps.

Example (5.3.15)[190]: Let $T : C[0,1] \to C^\infty(\{0\}) \cong \mathbb{R}$ be given by $T f = f(a)$ for some $a \in [0,1]$. Application of the Maeda–Ogasawara Theorem yields a Stonean space X such that $C[0,1] \subset C(X) \subset C^\infty(X)$, where $C[0,1]^\delta = C(X)$. There is a surjective (but not injective) map $\lambda : X \to [0,1]$, from which it follows that T can be extended to $T' : C(X) \to C^\infty(\{0\})$ by $T' \tilde{f} := \tilde{a}$ for any $\tilde{a} \in \lambda^{-1}(\{a\})$. The resulting π from Theorem (5.3.13) is then of course given by $\pi(0) = \tilde{a}$.

We make some remarks concerning Theorem (5.3.13). The details can be found in [194]. It is well-known that for composition operators between spaces of continuous functions, injectivity and surjectivity of the operator and its composition map are connected. In the generalised setting, similar results hold. We mention Theorem (5.3.16) from [194].

Theorem (5.3.16)[190]: Let X, Y, E, and T as in Theorem (5.3.13). The following statements are equivalent:

(i) T is injective and $T(E) \subset C^\infty(Y)$ is a Riesz ideal;

(ii) π is a homeomorphism.

When applying these results to the setting of Theorem (5.3.14), we need to overcome the ambiguity as illustrated in Example (5.3.15). We therefore require either that E is Dedekind complete or that T preserves order limits. As an example of an application, we note that the Maeda–Ogasawara space of L^p $(1 \le p < \infty)$ on a σ-finite measure space is Riesz isomorphic to the space of measurable functions on that measure space. Theorem (5.3.14) then provides a set map between the σ-algebras of the respective measure spaces. A detailed exposure can be found in [194].

We consider the same setting as in Theorem (5.3.14). E, F are Archimedean Riesz spaces, with a Riesz homomorphism $T : E \to F$. We again denote the Maeda–Ogasawara space of E by $C^\infty(X)$ and the one of $T(E)$ by $C^\infty(Y)$. Let us present an example that motivates.

Example (5.3.17)[190]: Define $j \in C^\infty(\beta\mathbb{N})$ as in Example (5.3.11). Fix $a \in \beta\mathbb{N} \setminus \beta(\mathbb{N})$ and observe that $j(a) = \infty$. Define $S : C^\infty(\beta\mathbb{N}) \to C^\infty(\{0\}) \cong \mathbb{R}$ by $Sf := \frac{f}{j}(a)$. Then $\mathrm{supp}(S\mathbf{1}) = \emptyset \neq \{0\} = \mathrm{supp}(Sj)$, while $\mathrm{supp}(\mathbf{1}) = \mathrm{supp}(j)$. We conclude that inclusions of supports of elements are in general not preserved under T.

Proposition (5.3.18)[190]: Let E, F, T, X, and Y be as above. Suppose T is σ-order continuous. If $f, g \in E$ with $\mathrm{supp}(f) \subset \mathrm{supp}(g)$, then $\mathrm{supp}(Tf) \subset \mathrm{supp}(Tg)$.

Proof. Let $f, g \in E$ with $\mathrm{supp}(f) \subset \mathrm{supp}(g)$. Then $f = \bigvee_n (f \wedge ng)$, so $Tf = \bigvee_n (Tf \wedge nTg)$. We conclude $Tf = 0$ outside $\mathrm{supp}(Tg)$, so $\mathrm{supp}(Tf) \subset \mathrm{supp}(Tg)$.

For the last part, assume that either

(i) E is Dedekind complete and T is σ-order continuous, or

(ii) T is order continuous, and let π be the unique map from Theorem (5.3.14).

Lemma (5.3.19)[190]: Let E, F, T, X, Y, and π be as above. For all $f \in E : f \circ \pi \in C^\infty(Y)$ and $\mathrm{supp}(f \circ \pi) = \mathrm{supp}(Tf)$.

Proof. In case of (ii), T has a unique order continuous extension to E^δ, so we do not need the Lipecki–Luxemburg–Schep Theorem and may assume E is an ideal. Take $f \in E^+$.

Claim. $f \circ \pi|_{\mathrm{supp}(Tf)} \in C^\infty(\mathrm{supp}(Tf))$.

As π is continuous, it suffices to prove that the closed set $\{y \in \mathrm{supp}(Tf) \mid f(\pi(y)) = \infty\}$ is meagre. Let $C \subset \mathrm{supp}(Tf)$ therefore be clopen such that $f(\pi(C)) = \{\infty\}$. We have $\frac{f \wedge \mathbf{1}_X}{f} \circ \pi \leq \frac{\mathbf{1}_X}{f} \circ \pi = 0$ on C by Proposition (5.3.9)(iv). Using Theorem (5.3.14)(iii), we see that $T(f \wedge \mathbf{1}_X) = Tf \cdot \left(\frac{f \wedge \mathbf{1}_X}{f} \circ \pi\right)$ on $\mathrm{supp}(Tf)$. Hence $T(f \wedge \mathbf{1}_X) = 0$ on C. Observing that $C \subset \mathrm{supp}(Tf) = \mathrm{supp}(T(f \wedge \mathbf{1}_X))$, we conclude that $C = \emptyset$.

Claim. $f \circ \pi = 0$ on $Y \setminus \mathrm{supp}(Tf)$. Order denseness of $T(E) \subset C^\infty(Y)$ implies that $W := \bigcup_{g \in E^+} \mathrm{supp}(Tg)$ is dense in Y. By Proposition (5.3.18), W is $\bigcup_{g \in E^+ \cap C(X)} \mathrm{supp}(Tg)$. Hence we are done if for every $g \in E^+ \cap C(X)$: $f \circ \pi = 0$ on $U_g := \mathrm{supp}(Tg) \setminus \mathrm{supp}(Tf)$.

Fix $g \in E^+ \cap C(X)$. We prove $f \circ \pi = 0$ on U_g. Observe that $\frac{f}{g} \circ \pi, g \circ \pi \in C^\infty(U_g)$, respectively because $U_g \subset \mathrm{supp}(Tg)$ and because $g \in C(X)$. Using Proposition (5.3.9)(v), this implies that $f \circ \pi = \left(\frac{f}{g} \circ \pi\right) \cdot (g \circ \pi)$ in $C^\infty(U_g)$. By Theorem (5.3.14)(iii), $Tf = Tg \cdot (\frac{f}{g} \circ \pi)$ on $\mathrm{supp}(Tg) \supset U_g$, while $Tf = 0$ on U_g by definition. The set $\{y \in U_g \mid Tg(y) = 0\}$ is meagre in U_g, so $\frac{f}{g} \circ \pi = 0$ on U_g. Hence $f \circ \pi = 0$ on U_g. This holds for all $g \in E^+ \cap C(X)$, proving the claim.

Claim. $\mathrm{supp}(Tf) \subset \mathrm{supp}(f \circ \pi)$. According to the preceding, we already have $f \circ \pi \in C^\infty(Y)$ and $\mathrm{supp}(f \circ \pi) \subset \mathrm{supp}(Tf)$. For the reverse inclusion, set $g := f \wedge f^2$. As $g \leq f$, g must be in E^+. From Theorem (5.3.14)(iii), which states that $Tg = Tf \cdot \left(\frac{g}{f} \circ \pi\right)$ on $\mathrm{supp}(Tf)$, we deduce that $\mathrm{supp}(Tg) \cap \mathrm{supp}(Tf) \subset \mathrm{supp}\left(\frac{g}{f} \circ \pi\right)$.

Note that $\operatorname{supp}(g) = \operatorname{supp}(f)$, so by order continuity of T: $\operatorname{supp}(T g) = \operatorname{supp}(T f)$. This allows us to calculate:

$$\operatorname{supp}(T f) \subset \operatorname{supp}\left(\frac{f \wedge f^2}{f} \circ \pi\right) \subset \operatorname{supp}\left(\frac{f^2}{f} \circ \pi\right)$$
$$= \operatorname{supp}(f \circ \pi), \tag{7}$$

which finishes the argument.

Proposition (5.3.20)[190]: Let E, F, T, X, Y, and π be as above. For all $f, g \in E$: $\frac{Tf}{f \circ \pi} = \frac{Tg}{g \circ \pi}$ on $\operatorname{supp}(T f) \cap \operatorname{supp}(T g)$.

Proof. Let $f, g \in E$. We have $f \circ \pi, g \circ \pi \in C^\infty(Y)$ by Lemma (5.3.19). Theorem (5.3.14) implies $\frac{g}{f} \circ \pi \in C^\infty(Y)$. We start with Theorem (5.3.14)(iii) and use Proposition (5.3.9)(v) and the preceding lemma:

$$T g = T f \cdot \left(\frac{g}{f} \circ \pi\right) \text{ on } \operatorname{supp}(T f)$$
$$(f \circ \pi) \cdot T g = T f \cdot \left(\frac{g}{f} \circ \pi\right) \cdot (f \circ \pi) \text{ on } \operatorname{supp}(T f)$$
$$(f \circ \pi) \cdot T g = T f \cdot \left(f \cdot \frac{g}{f} \circ \pi\right) \text{ on } \operatorname{supp}(T f)$$
$$(f \circ \pi) \cdot T g = T f \cdot (g \circ \pi) \text{ on } \operatorname{supp}(T f), \tag{8}$$

so $\frac{Tg}{g \circ \pi} = \frac{Tf}{f \circ \pi}$ on $\operatorname{supp}(T f) \cap \operatorname{supp}(T g)$, as desired.

In contrast with Example (5.3.11), T turns out to be an ordinary weighted composition operator, in the sense of Theorem (5.3.1).

Theorem (5.3.21)[190]: Let E, F, T, X, Y, and π be as above. There exists a unique $\eta \in C^\infty(Y)$ such that for all $f \in E$: $T f = \eta \cdot (f \circ \pi)$.

Proof. For any $f \in E^+$, the set $W f := \{y \in Y \mid T f(y) > 0, \frac{Tf}{f \circ \pi}(y) < \infty\}$ is an open and dense subset of $\operatorname{supp}(T f)$. Set $W := \bigcup_{f \in E^+} W f$. By order denseness of $T(E) \subset C^\infty(Y)$, W is dense in Y. The preceding theorem then implies that there exists a function $\tilde{\eta}: W \to \mathbb{R}$ that satisfies $\tilde{\eta} = \frac{Tf}{f \circ \pi}$ on $W f$ for every $f \in E^+$. Each $W f$ is open and $\frac{Tf}{f \circ \pi}$ is continuous for every $f \in E^+$, so $\tilde{\eta}$ is continuous and extends to a unique $\eta \in C^\infty(Y)$. For any $f, u \in E$, this leads to:

$$T f = T u \cdot \left(\frac{f}{u} \circ \pi\right) = T u \cdot \frac{f \circ \pi}{u \circ \pi}$$
$$= \eta \cdot (f \circ \pi) \tag{9}$$

on $\operatorname{supp}(T f) \cap \operatorname{supp}(T u)$. As $\operatorname{supp}(T f) = \operatorname{supp}(f \circ \pi)$, we conclude that $T f = \eta \cdot (f \circ \pi)$ on the whole Y.

Chapter 6
Approximation Numbers with Strict Singularity and Hardy Space

We give estimates for the approximation numbers of composition operators on the H^p spaces, $1 \leq p < \infty$. We obtain a new proof for the equivalence of the compactness and the weak compactness of T_g on H^1. A non-compact T_g acting on the space BMOA fixes an isomorphic copy of c_0. We reinterpret these results in terms of norm-closed ideals of the bounded linear operators on H^p, which contain the compact operators $K(H^p)$. The class of composition operators on H^p does not reflect the quite complicated lattice structure of such ideals.

Section (6.1): Composition Operators on H^p

The study of approximation numbers of composition operators on H^2 has been initiated (see [78], [203], [63], [209], [204]), and (upper and lower) estimates have been given. Most of the techniques used there are specifically Hilbertian (in particular Weyl's inequality; see [78]). We consider the case of composition operators on H^p for $1 \leq p < \infty$.

We focus essentially on lower estimates, because the upper ones are similar, with similar proofs, as in the Hilbertian case. We give in Theorem (6.1.4) a minoration involving the uniform separation constant of finite sequences in the unit disk and the interpolation constant of their images by the symbol. We finish with some upper estimates.

Recall that if X and Y are two Banach spaces of analytic functions on the unit disk $\mathbb{D}$, and $\varphi : \mathbb{D} \to \mathbb{D}$ is an analytic self-map of $\mathbb{D}$, one says that φ induces a composition operator $C_\varphi : X \to Y$ if $f \circ \varphi \in Y$ for every $f \in X$; φ is then called the symbol of the composition operator. One also says that φ is a symbol for X and Y if it induces a composition operator $C_\varphi : X \to Y$.

For every $a \in \mathbb{D}$, we denote by $e_a \in (H^p)^*$ the evaluation map at a, namely:

$$e_a(f) = f(a), \qquad f \in H^p. \tag{1}$$

We know that ([86], p. 253):

$$\|e_a\| = \left(\frac{1}{1-|a|^2}\right)^{1/p} \tag{2}$$

and the mapping equation

$$C_\varphi^*(e_a) = e_{\varphi(a)} \tag{3}$$

still holds.

We denote by $\| . \|$, without any subscript, the norm in the dual space $(H^p)^*$.

Let us stress that this dual norm of $(H^p)^*$ is, for $1 < p < 1$, equivalent, but not equal, to the norm $\| . \|_q$ of H^q, and the equivalence constant tends to infinity when p goes to 1 or to ∞.

As usual, the notation $A \lesssim B$ means that there is a constant c such that $A \leq cB$ and $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

For an operator $T : X \longrightarrow Y$ between Banach spaces X and Y, its approximation numbers are defined, for $n \geq 1$, as:

$$a_n(T) = \inf_{\operatorname{rank} R<n} \|T - R\|. \tag{4}$$

One has $\|T\| = a_1(T) \geq a_2(T) \geq \cdots \geq a_n(T) \geq a_{n+1}(T) \geq \cdots$, and (assuming that Y has the Approximation Property), T is compact if and only if $a_n(T) \underset{n\to\infty}{\longrightarrow} 0$.

We will also need other singular numbers (see [65], p. 49).

The n-th Bernstein number $b_n(T)$ of T, defined as:

$$b_n(T) = \sup_{\substack{E\subseteq X \\ \dim E=n}} \inf_{x\in S_E} \|Tx\|, \tag{5}$$

where $S_E = \{x \in E;\ \|x\| = 1\}$ is the unit sphere of E. When these numbers tend to $0, T$ is said to be superstrictly singular, or finitely strictly singular (see [208]).
The n-th Gelfand number of T, defined as:

$$c_n(T) = \inf_{\substack{L\subseteq Y \\ \operatorname{codim} L<n}} \|T_{|L}\|, \tag{6}$$

One always has:

$$a_n(T) \geq c_n(T) \text{ and } \quad a_n(T) \geq b_n(T), \tag{7}$$

and, when X and Y are Hilbert spaces, one has $a_n(T) = b_n(T) = c_n(T)$ ([207], Theorem (6.1.1)).

We first show that, as in the Hilbertian case H^2 ([78], Theorem (6.1.12)), the approximation numbers of the composition operators on H^p cannot decrease faster than geometrically.

We cannot longer appeal to the Hilbertian techniques of [78], Weyl's inequality has the following generalization ([90], Proposition 2).

Proposition (6.1.1)[199]: (Carl-Triebel). Let T be a compact operator on a complex Banach space E and $\lambda_n(T)_{n\geq 1}$ be the sequence of its eigenvalues, indexed such that $|\lambda_1(T)| \geq |\lambda_2(T)| \geq \cdot\ \cdot\ \cdot$. Then, for $n = 1, 2, \ldots$ and $m = 0, 1, \ldots, n-1$, one has:

$$\prod_{j=1}^{n} |\lambda_j(T)| \leq 16^n \|T\|^m a_{m+1}(T)^{n-m}. \tag{8}$$

(see [200]). Then, we can state:

Theorem (6.1.2)[199]: For every non-constant analytic self-map $\varphi : \mathbb{D} \to \mathbb{D}$, there exist $0 < r \leq 1$ and $c > 0$, depending only on φ, such that the approximation numbers of the composition operator $C_\varphi : H^p \to H^p$ satisfy:

$$a_n(C_\varphi) \geq c\, r^n, \qquad n = 1, 2, \ldots$$

In particular $\liminf_{n\to\infty} \left[a_n(C_\varphi)\right]^{1/n} \geq r > 0$.

Proof. If C_φ is not compact, the result is trivial, with $r = 1$; so we assume that C_φ is compact.

Before carrying on, we first recall some notation used in [78]. For every $z \in \mathbb{D}$, let

$$\varphi^\#(z) = \frac{|\varphi'(z)|(1 - |z|^2)}{1 - |\varphi(z)|^2}$$

be the pseudo-hyperbolic derivative of φ at z, and

$$[\varphi] = \sup_{z\in\mathbb{D}} \varphi^\#(z).$$

By the Schwarz-Pick inequality, one has $[\varphi] \leq 1$. Moreover, since φ is not constant, one has $[\varphi] > 0$.

We also set, for every operator $T : H^p \to H^p$:

$$\beta^-(T) = \liminf_{n\to\infty} [a_n(T)]^{1/n}.$$

For every $a \in \mathbb{D}$, we are going to show that $\beta^-(C_\varphi) \geq \left(\varphi^\#(a)\right)^2$, which will give $\beta^-(C_\varphi) \geq [\varphi]^2$, by taking the supremum for $a \in \mathbb{D}$, and the stated result, with $0 < r < [\varphi]^2$.

If $\varphi^{\#}(a) = 0$, the result is obvious, so we assume that $\varphi^{\#}(a) > 0$.
We consider the automorphism Φ_a, defined by $\Phi_a(z) = \frac{a-z}{1-\bar{a}z}$, and set

$$\psi_a = \Phi_{\varphi(a)} \circ \varphi \circ \Phi_a.$$

One has $\psi_a(0) = 0$ and $|\psi_a'(0)| = \varphi^{\#}(a)$.
Since C_φ is compact on $H^p, C_{\psi_a} = C_{\Phi_a} \circ C_\varphi \circ C\Phi_{\varphi(a)}$ is also compact on H^p. But we know that this is equivalent to say that it is compact on H^2. Since $\psi_a(0) = 0$ and $\psi_a'(0) = \varphi^{\#}(a) \neq 0$, we know, by the Eigenfunction Theorem ([83], p. 94), that the eigenvalues of $C_{\psi_a} : H^2 \to H^2$ are the numbers $\left(\psi_a'(0)\right)^j, j = 0, 1, \ldots$, and have multiplicity one. Moreover, the proof given in [83], § 6.2 shows that the eigenfunctions σ^j are not only in H^2, but in all $H^q, 1 \leq q < 1$.

Hence $\lambda_j(C_{\psi_a}) = \left(\psi_a'(0)\right)^{j-1}$. We now use Proposition (6.1.1), with $2n$ instead of n and $m = n - 1$; we get:

$$|\psi_a'(0)|^{n(2n-1)} = \prod_{j=1}^{2n} |\lambda_j(C_{\psi_a})| \leq 16^{2n} \left\|C_{\psi_a}\right\|^{n-1} a_n\left(C_{\psi_a}\right)^{n+1}$$

$$\leq 16^{2n} \left\|C_{\psi_a}\right\|^{n} a_n\left(C_{\psi_a}\right)^{n},$$

since $a_n(C_{\psi_a}) \leq \left\|C_{\psi_a}\right\|$.
That implies that $\beta^-(C_{\psi_a}) \geq |\psi_a'(0)|^2 = \left(\varphi^{\#}(a)\right)^2$.
Since C_{Φ_a} and $C_{\Phi_{\varphi(a)}}$ are automorphisms, we have $\beta^-(C_\varphi) = \beta^-(C_{\psi_a})$, hence the result.

We use the fortunate fact that, though the evaluation maps at well-chosen points of $\mathbb{D}$ can no longer be said to constitute a Riesz sequence, they will still constitute an unconditional sequence in H^p with good constants, as we are going to see, which will be sufficient for our purposes.
Recall (see [94], p. 276) that the interpolation constant κ_σ of a finite sequence $\sigma = (z_1, \ldots, z_n)$ of points $z_1, \ldots, z_n \in \mathbb{D}$ is defined by:

$$\kappa_\sigma = \sup_{|a_1|,\ldots,|a_n| \leq 1} \inf\{\|f\|_\infty;\ f \in H^\infty \text{ and } f(z_j) = a_j, 1 \leq j \leq n\}. \tag{9}$$

Then:
Lemma (6.1.3)[199]: For every finite sequence $\sigma = (z_1, \ldots, z_n)$ of distinct points $z_1, \ldots, z_n \in \mathbb{D}$, one has:

$$\kappa_\sigma^{-1} \left\|\sum_{j=1}^{n} \lambda_j e_{z_j}\right\| \leq \left\|\sum_{j=1}^{n} \omega_j \lambda_j e_{z_j}\right\| \leq \kappa_\sigma \left\|\sum_{j=1}^{n} \lambda_j e_{z_j}\right\| \tag{10}$$

for all $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and all complex numbers numbers $\omega_1, \ldots, \omega_n$ such that $|\omega_1| = \cdot \cdot \cdot = |\omega_n| = 1$.
Proof. Set $L = \sum_{j=1}^{n} \lambda_j e_{z_j}$ and $L_\omega = \sum_{j=1}^{n} \omega_j \lambda_j e_{z_j}$. There exists $h \in H^\infty$ such that $\|h\|_\infty \leq \kappa_\sigma$ and $h(z_j) = \omega_j$ for every $j = 1, \ldots, n$. For every $g \in H^p$, one has $L_\omega(g) = \sum_{j=1}^{n} \omega_j \lambda_j g(z_j) = \sum_{j=1}^{n} h(z_j) \lambda_j g(z_j) = L(hg)$; hence:

$$|L_\omega(g)| \leq \|L\| \, \|hg\|_p \leq \|L\| \, \|h\|_\infty \|g\|_p \leq \kappa_\sigma \|L\| \, \|g\|_p$$

and we get $\|L_\omega\| \leq \kappa_\sigma \|L\|$, which is the right-hand side of (10). The left-hand side follows, by replacing $\lambda_1, \ldots, \lambda_n$ by $\overline{\omega_1}\lambda_1, \ldots, \overline{\omega_n}\lambda_n$.

We now prove the following lower estimate.

For the proof, we need to know some precisions on the constant in Carleson's embedding theorem. Recall that the uniform separation constant δ_σ of a finite sequence $\sigma = (z_1,\dots,z_n)$ in the unit disk $\mathbb{D}$, is defined by:

$$\delta_\sigma = \inf_{1\le j\le n} \prod_{k\ne j} \left|\frac{z_j - z_k}{1-\bar{z}_j z_k}\right| \tag{11}$$

Lemma (6.1.5)[199]: Let $\sigma = (z_1,\dots,z_n)$ be a finite sequence of distinct points in $\mathbb{D}$ with uniform separation constant δ_σ. Then:

$$\sum_{j=1}^{n} \left(1 - |z_j|^2\right) |f(z_j)|^p \le 12\left[1+\log\frac{1}{\delta_\sigma}\right] \|f\|_p^p \tag{12}$$

for all $f \in H^p$.

Proof. For $a \in \mathbb{D}$, let $k_a(z) = \frac{\sqrt{1-|a|^2}}{1-\bar{a}z}$ be the normalized reproducing kernel.

For every positive Borel measure μ on $\mathbb{D}$, let:

$$\gamma_\mu = \sup_{a\in \operatorname{supp}\mu} \int_{\mathbb{D}} |k_a(z)|^2 d\mu(z).$$

The so-called Reproducing Kernel Thesis (see [110], Lecture VII, pp. 151–158) says that there is an absolute positive constant A_1 such that:

$$\int_{\mathbb{D}} |f(z)|^p d\mu(z) \le A_1\, \gamma_\mu\, \|f\|_p^p$$

for every $f \in H^p$ (that follows from the case $p = 2$ in writing $f = Bh^{2/p}$ where B is a Blaschke product and $h \in H^2$). Actually, one can take $A_1 = 2e$ (see [206], Theorem 0.2). But when μ is the discrete measure $\sum_{j=1}^{n}\left(1-|z_j|^2\right)\delta_{z_j}$, it is not difficult to check (see [201], Lemma 1, p. 150, or [97], p. 201) that:

$$\gamma_\mu \le 1 + 2\log\frac{1}{\delta_\sigma}.$$

That gives the result since $4e \le 12$.

Theorem (6.1.4)[199]: Let $\varphi : \mathbb{D} \to \mathbb{D}$ and $C_\varphi : H^p \to H^p$, with $1 \le p < 1$. Let $u_1,\dots,u_n \in \mathbb{D}$ such that $v_1 = \varphi(u_1),\dots,v_n = \varphi(u_n)$ are distinct. Then, for some constant c_p depending only on p, we have:

$$a_n(C_\varphi) \le c_p \kappa_v^{-1}\left(1+\log\frac{1}{\delta_u}\right)^{-\frac{1}{\min(p,2)}} \inf_{1\le j\le n}\left(\frac{1-|u_j|^2}{1-|v_j|^2}\right)^{\frac{1}{p}}, \tag{13}$$

where δ_u is the uniform separation constant of the sequence $u = (u_1,\dots,u_n)$ and κ_v the interpolation constant of $v = (v_1,\dots,v_n)$.

Proof. We will actually work with the Bernstein numbers of C_φ^*. Recall that they are defined in (5). That will suffice since $a_n(C_\varphi) \ge a_n(C_\varphi^*)$ (one has equality if C_φ is compact: see [202] or [65], pp. 89–91) and $a_n(C_\varphi^*) \ge b_n(C_\varphi^*)$.

Take $u_1,\dots,u_n \in \mathbb{D}$ such that $v_1 = \varphi(u_1),\dots,v_n = \varphi(u_n)$ are distinct. The points $u_1,\dots,u_n$ are then also distinct and the subspace $E = \operatorname{span}\{e_{u_1},\dots,e_{u_n}\}$ of $(H^p)^*$ is n-dimensional. Let

$$L = \sum_{j=1}^{n} \lambda_j e_{u_j}$$

be in the unit sphere of E. We set, for $f \in H^p$ and for $j = 1,\dots,n$:

$$\Lambda_j = \lambda_j \left\|e_{u_j}\right\|, \qquad \text{and } F_j = \left\|e_{u_j}\right\|^{-1} f(u_j),$$

and finally:

$$\Lambda = (\Lambda_1,\dots,\Lambda_n) \text{ and } F = (F_1,\dots,F_n).$$

We will separate three cases.

Case 1: $1 < p \leq 2$.

One has $\left\|C_\varphi^*(L)\right\| = \left\|\sum_{j=1}^n \lambda_j e_{v_j}\right\|$. Using Lemma (6.1.3), we obtain for any choice of complex signs $\omega_1,\dots,\omega_n$:

$$\left\|C_\varphi^*(L)\right\| \geq \kappa_v^{-1} \left\|\sum_{j=1}^n \omega_j \lambda_j e_{v_j}\right\|. \tag{14}$$

Let now q be the conjugate exponent of p. We know that the space H^p is of type p as a subspace of L^p ([104], p. 169) and therefore its dual $(H^p)^*$ is of cotype q ([104], p. 165), with cotype constant $\leq \tau_p$, the type p constant of L^p (let us note that we might use that $(H^p)^*$ is isomorphic to the subspace H^q of L^q, but we have then to introduce the constant of this isomorphism). Hence, by averaging (14) over all independent choices of signs and using the cotype q property of $(H^p)^*$, we get:

$$\left\|C_\varphi^*(L)\right\| \geq \tau_p^{-1}\kappa_v^{-1}\left(\sum_{j=1}^n |\lambda_j|^q \left\|e_{v_j}\right\|^q\right)^{1/q} \geq \tau_p^{-1}\kappa_v^{-1}\mu_n\left(\sum_{j=1}^n |\lambda_j|^q \left\|e_{u_j}\right\|^q\right)^{1/q},$$

so that

$$\left\|C_\varphi^*(L)\right\| \geq \tau_p^{-1}\kappa_v^{-1}\mu_n \|\Lambda\|_q, \tag{15}$$

where:

$$\mu_n = \inf_{1\leq j\leq n} \frac{\left\|e_{v_j}\right\|}{\left\|e_{u_j}\right\|} = \inf_{1\leq j\leq n}\left(\frac{1 - |u_j|^2}{1 - |v_j|^2}\right)^{1/p}.$$

It remains to give a lower bound for $\|\Lambda\|_q$.

But, by Hölder's inequality:

$$|L(f)| = \left|\sum_{j=1}^n \lambda_j f(u_j)\right| = \left|\sum_{j=1}^n \lambda_j F_j\right| \leq \|\Lambda\|_q \|\mathrm{F}\|_p.$$

Since

$$\|F\|_p^p = \sum_{j=1}^n \left\|e_{u_j}\right\|^{-p} |f(u_j)|^p = \sum_{j=1}^n \left(1 - |u_j|^2\right)|f(u_j)|^p,$$

Lemma (6.1.5) gives:

$$|L(f)| \leq \|\Lambda\|_q \left[12\left(1 + \log\frac{1}{\delta_u}\right)\right]^{1/p} \|f\|_p.$$

Taking the supremum over all f with $\|f\|_p \leq 1$, we get, taking into account that $\|L\| = 1$:

$$\|\Lambda\|_q \geq \left[12\left(1 + \log\frac{1}{\delta_u}\right)\right]^{-1/p}. \tag{16}$$

By combining (15) and (16), we get:

$$\left\|C_\varphi^*(L)\right\| \geq (12)^{-1/p}\tau_p^{-1}\mu_n\kappa_v^{-1}\left(1 + \log\frac{1}{\delta_u}\right)^{-1/p}.$$

Therefore:

$$b_n\left(C_\varphi^*\right) \geq (12)^{-\frac{1}{p}}\tau_p^{-1}\mu_n\kappa_v^{-1}\left(1 + \log\frac{1}{\delta_u}\right)^{-\frac{1}{p}}.$$

Case 2: $2 < p < 1.$

We follow the same route, but in this case, H^p is of type 2 and hence $(H^p)^*$ is of cotype 2. Therefore, we get:

$$\left\|C_\varphi^*(L)\right\| \geq \tau_2^{-1}\kappa_v^{-1}\mu_n\|\Lambda\|_2 \tag{17}$$

and, using Cauchy-Schwarz inequality:

$$\|\Lambda\|_2 \geq \left[12\left(1 + \log\frac{1}{\delta_u}\right)\right]^{-1/2}; \tag{18}$$

so:

$$\left\|C_\varphi^*(L)\right\| \geq (12)^{-1/2}\tau_2^{-1}\mu_n\kappa_v^{-1}\left(1 + \log\frac{1}{\delta_u}\right)^{-1/2}. \tag{19}$$

Case 3: $p = 1.$

In this case $(H^1)^*$ (which is isomorphic to the space $BMOA$) has no finite cotype. But, for each $k = 1, \dots, n$, one has, using Lemma (6.1.3):

$$\begin{aligned}|\lambda_k|\left\|e_{v_k}\right\| &= \frac{1}{2}\left\|\left(\sum_{j\neq k}\lambda_j e_{v_j} + \lambda_k e_{v_k}\right) - \left(\sum_{j\neq k}\lambda_j e_{v_j} + \lambda_k e_{v_k}\right)\right\| \\ &\leq \frac{1}{2}\left(\left\|\sum_{j\neq k}\lambda_j e_{v_j} + \lambda_k e_{v_k}\right\| + \left\|\sum_{j\neq k}\lambda_j e_{v_j} + \lambda_k e_{v_k}\right\|\right) \leq \kappa_v\left\|\sum_{j=1}^{n}\lambda_j e_{v_j}\right\|;\end{aligned}$$

hence:

$$\left\|C_\varphi^*(L)\right\| \geq \kappa_v^{-1}\mu_n\|\Lambda\|_\infty. \tag{20}$$

Since $|L(F)| \leq \|\Lambda\|_\infty\|\mathrm{F}\|_1$, we get, as above, using Lemma (6.1.5):

$$\|\Lambda\|_\infty \geq \left[12\left(1 + \log\frac{1}{\delta_u}\right)\right]^{-1} \tag{21}$$

and therefore:

$$\left\|C_\varphi^*(L)\right\| \geq (12)^{-1}\mu_n\kappa_v^{-1}\left(1 + \log\frac{1}{\delta_u}\right)^{-1}. \tag{22}$$

and that finishes the proof of Theorem (6.1.4).

Example (6.1.6)[199]: We will now apply this result to lens maps. See [83] or [63] for their definition. For $\theta \in (0, 1)$, we denote:

$$\lambda_\theta(z) = \frac{(1 + z)^\theta - (1 - z)^\theta}{(1 + z)^\theta + (1 - z)^\theta}. \tag{23}$$

Proposition (6.1.7)[199]: Let λ_θ be the lens map of parameter θ acting on H^p, with $1 \leq p < \infty$. Then, for positive constants a and b, depending only on θ and p:

$$a_n(C_{\lambda_\theta}) \geq a\, e^{-b\sqrt{n}}.$$

Actually, this estimate is valid for polygonal maps as well.

Proof. Let $0 < \sigma < 1$ and consider $u_j = 1 - \sigma^j$ and $v_j = \lambda_\theta(u_j), 1 \leq j \leq n$. We know from [78], Lemma 6.4 and Lemma 6.5, that, for $\alpha = \frac{\pi^2}{2}$ and $\beta = \beta_\theta = \frac{\pi^2}{2^\theta \theta}$:

$$\delta_u \geq e^{-\alpha/(1-\sigma)} \quad \text{and} \quad \delta_v \geq e^{-\beta/(1-\sigma)}.$$

But we know that the interpolation constant κ_σ is related to the uniform separation constant δ_σ by the following inequality ([94] page 278), in which Λ is a positive numerical constant:

$$\frac{1}{\delta_\sigma} \leq \kappa_\sigma \leq \frac{\Lambda}{\delta_\sigma}\left(1 + \log\frac{1}{\delta_\sigma}\right). \tag{24}$$

Actually, S. A. Vinogradov, E. A. Gorin and S. V. Hrušcëv [210] (see [205], p. 505) proved that

$$\kappa_\sigma \leq \frac{2e}{\delta_\sigma}\left(1 + 2\log\frac{1}{\delta_\sigma}\right),$$

so we can take $\Lambda \leq 4e \leq 12$.
It follows that

$$\kappa_v^{-1} \geq \frac{1 - \sigma}{\Lambda(\beta + 1)} e^{-\beta/(1-\sigma)}. \tag{25}$$

Setting $\tilde{p} = \min(p, 2)$, we have:

$$\left(1 + \log\frac{1}{\delta_u}\right)^{-1/\tilde{p}} \geq \left(\frac{1 - \sigma}{\alpha + 1}\right)^{\frac{1}{\tilde{p}}}. \tag{26}$$

We now estimate μ_n.
Since $\lambda_\theta(0) = 0$, Schwarz's lemma says that $|\lambda_\theta(z)| \leq |z|$; hence $\frac{1-|z|^2}{1-|\lambda_\theta(z)|^2} \geq \frac{1-|z|}{1-|\lambda_\theta(z)|}$.
But $1 - v_j = 1 - \lambda_\theta(u_j) = \frac{2\sigma^{j\theta}}{(2-\sigma^j)^\theta + \sigma^{j\theta}}$; hence (since u_j and v_j are real):

$$\frac{1 - |u_j|^2}{1 - |v_j|^2} \geq \frac{1 - u_j}{1 - v_j} = \frac{\sigma^j}{2\sigma^{j\theta}}\left[(2 - \sigma^j)^\theta + \sigma^{j\theta}\right].$$

Since the function $f(x) = (2 - x)^\theta + x^\theta$ increases on $[0, 1]$, one gets:

$$\frac{1 - |u_j|^2}{1 - |v_j|^2} \geq \left(\frac{1}{2}\sigma^j\right)^{1-\theta},$$

and therefore:

$$\mu_n \geq \left(\frac{1}{2}\sigma^j\right)^{\frac{1-\theta}{p}}. \tag{27}$$

Applying now Theorem (6.1.4) and using (25), (26) and (27), we get:

$$a_n(C_{\lambda_\theta}) \geq \alpha_{p,\theta} e^{-\frac{\beta}{1-\sigma}} (1 - \sigma)^{\frac{1}{\tilde{p}}} \sigma^{n(1-\theta)/p}$$

with $\alpha_{p,\theta} = \frac{c_p}{\Lambda(\beta+1)(\alpha+1)^{1/\tilde{p}} 2^{(1-\theta)/p}}$.
Taking $\sigma = e^{-\varepsilon}$ where $0 < \varepsilon < 1$, we get, since $1 - e^{-\varepsilon} \geq \varepsilon/2$:

$$a_n(C_{\lambda_\theta}) \geq \alpha_{p,\theta} e^{-\frac{2\beta}{\varepsilon}} \left(\frac{\varepsilon}{2}\right)^{1/\tilde{p}} e^{-\varepsilon n(1-\theta)/p}.$$

Optimizing by taking $\varepsilon = \sqrt{\frac{3\beta_p}{1-\theta}} \frac{1}{\sqrt{n}}$ gives, for n large enough (in order to have $\varepsilon < 1$):

$$a_n(C_{\lambda_\theta}) \geq \alpha'_{p,\theta} n^{-\frac{1}{2\tilde{p}}} e^{-\beta_{p,\theta}\sqrt{n}} \tag{28}$$

with $\alpha'_{p,\theta} = \alpha_{p,\theta} \left(\frac{\beta_p}{2(1-\theta)}\right)^{1/(2\tilde{p})}$ and $\beta_{p,\theta} = \sqrt{\frac{2\beta(1-\theta)}{p}}$.

We get Theorem (6.1.7), with $b > \beta_{p,\theta}$.

Let us note that $\beta_{p,\theta} = \frac{2^{\frac{1-\theta}{2}}}{\sqrt{p}} \sqrt{\frac{1-\theta}{\theta}}$ tends to 0 when θ goes to 1 and tends to infinity when θ goes to 0.

We are using Theorem (6.1.4) to give, as in [203], Theorem (6.1.13), a lower bound for $a_n(C_\varphi)$ which depends on the behaviour of φ near $\partial\mathbb{D}$.

We recall first (see [203]) that an analytic self-map $\varphi: \mathbb{D} \to \mathbb{D}$ is said to be real if it takes real values on $]-1,1[$. If $\omega: [0,1] \to [0,2]$ is a modulus of continuity (meaning that ω is continuous, increasing, sub-additive, vanishing at 0, and concave), φ is said to be an ω-radial symbol if it is real and:

$$1 - \varphi(r) \leq \omega(1 - r), \qquad 0 \leq r < 1. \tag{29}$$

We have the following result.

Theorem (6.1.8)[199]: Let φ be an ω-radial symbol. Then, for $1 \leq p < \infty$, the approximation numbers of the composition operator $C_\varphi : H^p \to H^p$ satisfy:

$$a_n(C_\varphi) \geq c'_p \sup_{0<\sigma<1} \left[\left(\frac{\omega^{-1}(a\,\sigma^n)}{a\,\sigma^n} \right)^{\frac{1}{p}} (1 - \sigma)^{\frac{1}{\max(p^*,2)}} \exp\left(-\frac{5}{1-\sigma}\right) \right], \tag{30}$$

where c'_p is a constant depending only on p, p^* is the conjugate exponent of p, and $a = 1 - \varphi(0) > 0$.

Proof. As in [203], p. 556, we fix $0 < \sigma < 1$ and define inductively $u_j \in [0,1)$ by $u_0 = 0$ and, using the intermediate value theorem:

$$1 - \varphi(u_{j+1}) = \sigma[1 - \varphi(u_j)], \qquad \text{with } 1 > u_{j+1} > u_j.$$

We set $v_j = \varphi(u_j)$. We have $-1 < v_j < 1$ and $1 - v_n = a\,\sigma^n$. We proved in [203], p. 556, that:

$$\frac{1 - |u_j|^2}{1 - |v_j|^2} \geq \frac{1}{2} \frac{\omega^{-1}(a\,\sigma^n)}{a\,\sigma^n}. \tag{31}$$

Moreover, we proved in [203], p. 557, that the uniform separation constant of $v = (v_1, \ldots, v_n)$ is such that:

$$\delta_v \geq \exp\left(-\frac{5}{1-\sigma}\right). \tag{32}$$

Since $\delta_u \geq \delta_v$, we get, from (24), that:

$$\kappa_u \leq 12 \left(\frac{6-\sigma}{1-\sigma}\right) \exp\left(\frac{5}{1-\sigma}\right) \leq 60 \left(\frac{1}{1-\sigma}\right) \exp\left(\frac{5}{1-\sigma}\right). \tag{33}$$

Using now (13) of Theorem (6.1.4) and combining (31), (32) and (33), we get Theorem (6.1.8).

Example (6.1.9)[199]: Lens maps. Let us come back to the lens maps λ_θ for testing Theorem (6.1.8). We have $\omega^{-1}(h) \approx h^{1/\theta}$ (see [63], Lemma (6.1.5)) and $a = 1 - \lambda_\theta(0) = 1$. Setting $K = \frac{1}{10\sqrt{p}} \sqrt{\frac{1-\theta}{\theta}}$ and taking, for n large enough, $\sigma = 1 - \frac{1}{K\sqrt{n}}$, we have, using that $e^{-s} \leq 1 - \frac{4}{5}s$ for $s > 0$ small enough, $\sigma^n \geq \exp(-\frac{5}{4K}\sqrt{n})$ and hence:

$$a_n(C_{\lambda_\theta}) \geq c_{\theta,p}\, n^{-1/2\max(p^*,2)} \exp(-\frac{5}{\sqrt{p}}\sqrt{\frac{1-\theta}{\theta}}\sqrt{n}).$$

Note that the coefficient of $\sqrt{n}$ in the exponential is slightly different of that in (28), but of the same order.

Example (6.1.10)[199]: Cusp map. We refer to [203], for its definition and properties. It is the conformal mapping χ from $\mathbb{D}$ onto the domain represented on Fig. 1 such that $\chi(1) = 1, \chi(-1) = 0, \chi(i) = (1+i)/2$ and $\chi(-i) = (1-i)/2$.

We proved in [203], Lemma 4.2, that, for $0 \leq r < 1$, one has:

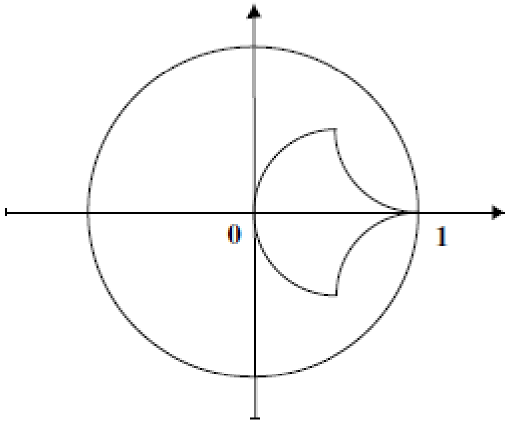

Figure (1)[199]: Cusp map domain

$$1 - \chi(r) = \frac{1}{1 + \frac{2}{\pi}\log\left[1/2 \arctan\left(\frac{1-r}{1+r}\right)\right]}$$

Since $1 - \frac{2}{\pi}\log 2 > 0$ and $\arctan x \leq x$ for $x \geq 0$, we get that:

$$1 - \chi(r) \leq \frac{\pi}{2}\frac{1}{\log\left(\frac{1+r}{1-r}\right)} \leq \frac{\pi}{2}\frac{1}{\log\left(\frac{1}{1-r}\right)} \leq 2\frac{1}{\log\left(\frac{1}{1-r}\right)}.$$

Hence χ is an ω-radial symbol with $\omega(x) = 2/\log(1/x)$. Then $\omega^{-1}(h) = e^{-2/h}$. By choosing $\sigma = 1 - \frac{\log n}{4n}$ in (30), we get, using that $\log(1-x) \geq -2x$ for $x > 0$ small enough, that, for n large enough, $\sigma^n \geq 1/\sqrt{n}$; hence:

$$a_n(C_\chi) \geq c''_p\left(\sqrt{n}\exp[-(2\,a)\sqrt{n}]\right)^{\frac{1}{p}}\left(\frac{\log n}{n}\right)^{1/\max(p^*,2)} \exp\left(-\frac{20n}{\log n}\right).$$

It follows that, for some constant $C_p > 0$ depending only on p, we have:

$$a_n(C_\chi) \geq C_p \exp\left(-\frac{20n}{\log n}\right). \tag{34}$$

It has to be stressed that the term in the exponential does not depend on p.

Example (6.1.11)[199]: Shapiro-Taylor's maps. These maps ς_θ, for $\theta > 0$, were defined in [84]. Let us recall their definition. For $\varepsilon > 0$, we set $V_\varepsilon = \{z \in \mathbb{C};\ \Re z > 0 \text{ and } |z| < \varepsilon\}$. For $\varepsilon = \varepsilon_\theta > 0$ small enough, one can define

$$f_\theta(z) = z(-\log z)^\theta, \tag{35}$$

for $z \in V_\varepsilon$, where $\log z$ will be the principal determination of the logarithm. Let now g_θ be the conformal mapping from $\mathbb{D}$ onto V_ε, which maps $T = \partial\mathbb{D}$ onto ∂V_ε, defined by $g_\theta(z) = \varepsilon\varphi_0(z)$, where φ_0 is the conformal map from $\mathbb{D}$ onto V_1, given by:

$$\varphi_0(z) = \frac{\left(\frac{z - i}{iz - 1}\right)^{\frac{1}{2}} - i}{-i\left(\frac{z - i}{iz - 1}\right)^{\frac{1}{2}} + 1}. \tag{36}$$

Then, we define:

$$\varsigma_\theta = \exp(-f_\theta \circ g_\theta). \tag{37}$$

We saw in [203], p. 560, that $\omega^{-1}(h) = K_\theta h\left(\log\left(\frac{1}{h}\right)\right)^{-\theta}$. Hence, choosing $\sigma = 1/(e\,\alpha_\theta^{\frac{1}{n}})$, where $\alpha_\theta = 1 - \varsigma_\theta(0)$, we get that:

$$a_n(C_{\varsigma_\theta}) \geq c_{p,\theta}.\frac{1}{n^{\frac{\theta}{2p}}}. \tag{38}$$

However, we already remarked in [203], that, even for $p = 2$, this result is not optimal.

For upper bounds, there is essentially no change with regard to the case $p = 2$. Hence we essentially only state some results.

We have the following upper bound, which can be obtained with the same proof as in [63].

Theorem (6.1.12)[199]: Let $C_\varphi : H^p \to H^p, 1 \leq p < \infty$, a composition operator, and $n \geq 1$. Then, for every Blaschke product B with (strictly) less than n zeros, each counted with its multiplicity, one has:

$$a_n(C_\varphi) \leq C\sqrt{n}\left(\sup_{\substack{0<h<1 \\ \xi\in\mathbb{T}}} \frac{1}{h}\int_{\overline{S(\xi,h)}} |B|^p\, dm_\varphi\right)^{1/p},$$

where m_φ is the pullback measure of m, the normalized Lebesgue measure on $\mathbb{T}$, under φ and $S(\xi,h) = \mathbb{D} \cap D(\xi,h)$ is the Carleson window of size h centered at $\xi \in \mathbb{T}$.

Proof. We first estimate the Gelfand number $c_n(C_\varphi)$ by restricting to the subspace BH^p which is of codimension $< n$. As in [63], Lemma (6.1.4):

$$c_n(C_\varphi) \lesssim \left(\sup_{\substack{0<h<1 \\ \xi\in\mathbb{T}}} \frac{1}{h}\int_{\overline{S(\xi,h)}} |B|^p\, dm_\varphi\right)^{1/p}.$$

Now (see [65], Proposition (6.1.4)), one has $a_n(C_\varphi) \leq \sqrt{2n}\,c_n(C_\varphi)$, hence the result.

We can then deduce, with the same proof, the following version of [203], Theorem (6.1.3). Recall ([203], Definition (6.1.2)) that a symbol $\varphi \in A(\mathbb{D})$ (i.e. $\varphi: \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ is continuous and analytic in $\mathbb{D}$) is said to be globally regular if $\varphi(\overline{\mathbb{D}}) \cap \partial\mathbb{D} = \{\xi_1, \dots, \xi_l\}$ and there exists a modulus of continuity ω (i.e. a continuous, increasing and sub-additive function $\omega: [0, A] \to \mathbb{R}^+$, which vanishes at zero, and that we may assume to be concave), such that, writing $E_{\xi_j} = \{t\,;\,\gamma(t) = \xi_j\}$, one has $\mathbb{T} = \bigcup_{j=1}^{l}\left(E_{\xi_j} + [-r_j, r_j]\right)$ for some $r_1, \dots, r_l > 0$, and for some positive constants $C, c > 0$:

$$|\gamma(t) - \gamma(t_j)| \leq C(1 - |\gamma(t)|) \tag{39}$$

$$c\,\omega(|t - t_j|) \leq |\gamma(t) - \gamma(t_j)| \tag{40}$$

for $j = 1,\dots,l$, all $t_j \in E_{\xi_j}$ with $|t - t_j| \le r_j$.

Theorem (6.1.13)[199]: Let φ be a symbol in $A(\mathbb{D})$ whose image touches $\partial\mathbb{D}$ exactly at the points $\xi_1,\dots,\xi_l$ and which is globally-regular. Then there are constants $\kappa, K, L > 0$, depending only on φ, such that, for every $k \ge 1$:

$$a_k(C_\varphi) \le K\left[\frac{\omega^{-1}(\kappa 2^{-N_k})}{\kappa 2^{-N_k}}\right]^{\frac{1}{p}}, \tag{41}$$

where N_k is the largest integer such that $lNd_N < k$ and d_N is the integer part of $\left[\log\frac{\kappa 2^{-N}}{\omega^{-1}(\kappa 2^{-N})}/\log(\chi^{-p})\right] + 1$, with $0 < \chi < 1$ an absolute constant.

As a corollary, we get for lens maps λ_θ (as well as for polygonal maps), in the same way as Theorem (6.1.4) in [203], p. 550 (recall that then $\omega(h) \approx h^\theta$), the following upper bound.

Theorem (6.1.14)[199]: Let $\varphi = \lambda_\theta$ be the lens map of parameter θ acting on $H^p, 1 < p < \infty$. Then, for positive constants b and c depending only on θ and p:

$$a_n(C_{\lambda_\theta}) \le c\, e^{-b\sqrt{n}}.$$

For the cusp map, we also have as in [203], Theorem 4.3 (here, $\omega(h) \approx 1/\log(1/h)$).

Theorem (6.1.15)[199]: Let $\varphi = \chi$ be the cusp map. For some positive constants b and c depend in g only on p, one has:

$$a_n(C_\chi) \le c\, e^{-bn/\log n}.$$

Section (6.2): Volterra-Type Integral Operator on H^p

For g be a fixed analytic function in the open unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$. We consider a linear integral operator T_g defined for analytic functions f in $\mathbb{D}$ by

$$T_g f(z) = \int_0^z f(\zeta) g'(\zeta) d\zeta, \qquad z \in \mathbb{D}.$$

Ch. Pommerenke consider the boundedness of the operator T_g on the Hardy space H^2 and he characterized it in connection to exponentials of $BMOA$ functions [224]. A systematic study of T_g was initiated by A. Aleman and A. G. Siskakis in [214], who showed that T_g is bounded (compact) on the Hardy spaces $H^p, 1 \le p < \infty$, if and only if $g \in BMOA$ ($g \in VMOA$). The same boundedness characterization of the operator T_g on $H^p, 0 < p < 1$, spaces was obtained by Aleman and J. Cima in [213]. Here $BMOA$ and $VMOA$ denote the spaces of analytic functions in $\mathbb{D}$ with boundary values of bounded mean oscillation and vanishing mean oscillation respectively. See [212] and [225].

A bounded operator $S : X \to Y$ between Banach spaces is strictly singular if its restriction to any infinite-dimensional closed subspace is not an isomorphism onto its range. This notion generalizes the concept of compact operators and it was introduced by T. Kato in [219]. Canonical examples of non-compact strictly singular operators are the inclusion mappings $i_{p,q} : \ell^p \hookrightarrow \ell^q$, where $1 \le p < q < \infty$.

There also exist non-compact strictly singular operators on H^p spaces for $1 \le p < \infty, p \ne 2$. To construct such an operator, one may consider cases $1 \le p < 2$ and $2 < p < \infty$ separately and use the fact that $H^p, 1 \le p < \infty$, contains complemented copies of ℓ^2 and ℓ^p; see e.g. [228] for $p = 1$. In the first case, one considers a bounded projection from H^p onto its closed subspace M, which is isomorphic to ℓ^p. Then one utilizes the inclusion mapping $i_{p,2}$ and the fact that ℓ^2 can be embedded in H^p. In the second case, one interchanges the roles of ℓ^p and ℓ^2 and repeats the reasoning above.

We show that every non-compact operator T_g acting on a Hardy space $H^p, 1 \leq p < \infty$, fixes an isomorphic copy of ℓ^p. In particular, this implies that T_g is strictly singular on H^p if and only if it is compact. This article was partly motivated by [221], where the same question was studied in connection to composition operators. For the case $p = \infty$; see also [216].

We should point out that there is a striking extrapolation result concerning strict singularity by Hernández, Semenov, and Tradacete in [217]. It states that if an operator S is bounded on $L^p_{\mathbb{R}}(E)$ and $L^q_{\mathbb{R}}(E)$ for some $1 < p < q < \infty$ and strictly singular on $L^r_{\mathbb{R}}(E)$ for some $p < r < q$, then it is compact on $L^s_{\mathbb{R}}(E)$ for all $p < s < q$. Here $L^p_{\mathbb{R}}(E)$ stands for the L^p space of real-valued functions on a finite measure space E. Taking the complex-valued counterpart of this result for granted, we may deduce the equivalence of strict singularity and compactness of T_g on H^p for $1 < p < \infty$ by using the Riesz projection in the following way:

Recall that strictly singular operators form a two-sided (closed) ideal in the space $\mathcal{L}(L^p)$ of bounded operators on $L^p = L^p_{\mathbb{C}}(\mathbb{T})$, where $\mathbb{T} = \partial\mathbb{D}$. Therefore the strict singularity of $T_g : H^p \to H^p$ implies that $T_g R : L^p \to L^p$ is strictly singular, where $R : L^p \to H^p$ is the Riesz projection and we have identified $T_g : H^p \to H^p$ with $T_g : H^p \to L^p$. Since the condition $g \in BMOA$ characterizes the boundedness of T_g on every H^q space and the Riesz projection is bounded on the scale $1 < q < \infty$, we get that $T_g R$ is bounded on every $L^q, 1 < q < \infty$, space. Now assuming that the complex version of the interpolation result is valid, it follows that $T_g R$ is compact on L^p and consequently the restriction $T_g R|_{H^p} = T_g$ is compact on H^p.

Theorem (6.2.6), however, makes a stronger statement: a non-compact operator T_g on H^p fixes an isomorphic copy of ℓ^p. This holds for $p = 1$ as well, in which case we obtain a new proof for the equivalence of the compactness and the weak compactness of T_g on H^1 : If $g \in BMOA \setminus VMOA$, i.e., the operator T_g is not compact, then by Theorem (6.2.6) T_g fixes an isomorphic copy of ℓ^1 and consequently it is not weakly compact.

We give the proof of Theorem (6.2.6). We consider the case of T_g acting on the space $BMOA$ and we make some remarks on strict singularity of T_g on other spaces.

We point out that the notions of strict singularity and compactness of an operator can also be defined in a more general setting, for example in quasi-Banach spaces; see [218]. Examples of such spaces are the Hardy spaces H^p when $0 < p < 1$. The same compactness and boundedness characterizations of T_g as in the case $1 \leq p < \infty$ also hold when $0 < p < 1$; see [213]. On the scale $0 < p < 1$, the triangle inequality is replaced by an inequality of the type $\|f+g\|_p \leq C(\|f\|_p + \|g\|_p)$ for some constant $C > 1$ for all $f, g \in H^p$. This brings only some (multiplicative) constants to the proofs of results and lemmas in this article. So a non-compact
$T_g : H^p \to H^p$ fixes an isomorphic copy of ℓ^p also in the case $0 < p < 1$.

We briefly recall some spaces of analytic functions that appear later.
Let $H(\mathbb{D})$ be the algebra of analytic functions in $\mathbb{D}$. We define the Hardy spaces

$$H^p = \left\{ f \in H(\mathbb{D}) : \|f\|_p = \left(\sup_{0 \le r < 1} \frac{1}{2\pi} \int_o^{2\pi} \left| f(re^{it}) \right|^p dt \right)^{1/p} < \infty \right\}.$$

The space $BMOA$ consists of functions $f \in H(\mathbb{D})$ with

$$\|f\|_* = \sup_{a \in \mathbb{D}} \|f \circ \sigma_a - f(a)\|_2 < \infty,$$

where $\sigma_a(z) = (a - z)/(1 - \bar{a}z)$ is the Möbius automorphism of $\mathbb{D}$ that interchanges the origin and the point $a \in \mathbb{D}$. Its closed subspace $VMOA$ consists of those $f \in BMOA$ with

$$\lim_{|a| \to 1} \sup \|f \circ \sigma_a - f(a)\|_2 = 0.$$

Every $BMOA$ function f satisfies "a reverse Hölder's inequality", which implies that for each $0 < p < \infty$ we have that

$$\|f\|_* \simeq \sup_{a \in \mathbb{D}} \|f \circ \sigma_a - f(a)\|_p < \infty, \tag{42}$$

where each side is bounded above by a constant multiple of the other. See e.g. [126] for more information on the spaces $BMOA$ and $VMOA$.

We show that a non-compact operator $T_g : H^p \to H^p, 1 \le p < \infty, g \in BMOA \backslash VMOA$, fixes an isomorphic copy of ℓ^p and hence the compactness and strict singularity are equivalent for T_g on H^p. This is done by constructing bounded operators $V : \ell^p \to H^p$ and $U : \ell^p \to H^p$ such that the diagram in Figure 1 commutes ($U = T_g V$), where $V(\ell^p) = M$ is the closed linear span of suitably chosen test functions $f_{a_k} \in H^p$ and the operator U is an isomorphism onto its range $U(\ell^p) = T_g(M)$.

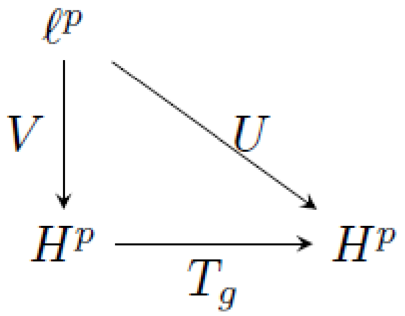

Figure (1)[211]: Operators U, V and T_g

The strategy for choosing the suitable test functions in Propositions (6.2.2) and (6.2.5) below is similar to the one used by Laitila et al. in [221], where they utilized these test functions to show that a non-compact composition operator $C_\varphi : H^p \to H^p$, where $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic, fixes an isomorphic copy of ℓ^p. However, due to the distinct nature of operators T_g and C_φ, a different kind of analysis is needed in our case.

Before proving our main result, we provide some preparatory material. We first state a localization lemma for the standard test functions in $H^p, 1 \le p < \infty$, defined by

$$f_a(z) = \left[\frac{1 - |a|^2}{(1 - \bar{a}z)^2} \right]^{1/p}, \qquad z \in \mathbb{D},$$

for each $a \in \mathbb{D}$. Observe that $\|f_a\|_p = 1$ for all $a \in \mathbb{D}$. The proof of the lemma is straightforward and therefore omitted.

Lemma (6.2.1)[211]: Let $1 \le p < \infty$ and m be the normalized Lebesgue measure on $\mathbb{T}$. Define

$$A_\varepsilon = \{e^{i\theta} : |e^{i\theta} - 1| < \varepsilon\}$$

for $\varepsilon > 0$. Then

$$(i)\ \lim_{a\to 1}\int_{\mathbb{T}\setminus A_\varepsilon}|f_a|^p dm = 0 \text{ for each } \varepsilon > 0.$$
$$(ii)\ \lim_{\varepsilon\to 0}\int_{A_\varepsilon}|f_a|^p dm = 0 \text{ for each } a \in \mathbb{D}.$$

Next, utilizing test functions $f_{a_k}, a_k \in \mathbb{D}$, for which $|a_k| \to 1$ sufficiently fast, we construct a certain type of bounded operator $V : \ell^p \to H^p$.

Proposition (6.2.2)[211]: Let $1 \le p < \infty$ and $(a_n) \subset \mathbb{D}$ be a sequence such that $a_n \to \omega \in \mathbb{T}$. Then there exists a subsequence (b_n) of (a_n) so that the mapping

$$V : \ell^p \to H^p, \qquad V(\alpha) = \sum_{n=1}^{\infty}\alpha_n f_{b_n},$$

where $\alpha = (\alpha_n) \in \ell^p$, is bounded.

Proof. We may assume that $\omega = 1$. For each $\varepsilon > 0$, we consider the set A_ε defined in Lemma (6.2.1). Using the fact that $\|f_a\|_p = 1$ for all $a \in \mathbb{D}$ and Lemma (6.2.1), we choose positive numbers ε_n with $\varepsilon_1 = 2\pi > \varepsilon_2 > \ldots > 0$ and a subsequence (b_n) of (a_n) such that the following conditions hold:

$$\text{(i)} \left(\int_{A_n}\left|f_{b_j}\right|^p dm\right)^{1/p} < 4^{-n}, \qquad j = 1, \ldots, n-1;$$
$$\text{(ii)} \left(\int_{\mathbb{T}\setminus A_n}\left|f_{b_n}\right|^p dm\right)^{1/p} < 4^{-n},$$

for every $n \ge 1$, where $A_n = A_{\varepsilon_n}$.

Using conditions (i)-(ii) and the fact that $\left(\int_{A_n}\left|f_{b_n}\right|^p dm\right)^{1/p} \le 1$ for all n, we show the upper bound $\|V\alpha\|_p \le C\|\alpha\|_{\ell^p}$ for all $\alpha = (\alpha_j) \in \ell^p$, where $C > 0$ may depend on p:

$$\|V\alpha\|_p^p = \int_{\mathbb{T}}\left|\sum_{j=1}^{\infty}\alpha_j f_{b_j}\right|^p dm = \sum_{n=1}^{\infty}\int_{A_n\setminus A_{n+1}}\left|\sum_{j=1}^{\infty}\alpha_j f_{b_j}\right|^p dm$$
$$\le \sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty}|\alpha_j|\left(\int_{A_n\setminus A_{n+1}}\left|f_{b_j}\right|^p dm\right)^{\frac{1}{p}}\right)^p$$
$$\le \sum_{n=1}^{\infty}\left(|\alpha_n|\left(\int_{A_n\setminus A_{n+1}}\left|f_{b_n}\right|^p dm\right)^{\frac{1}{p}} + \sum_{j\ne n}|\alpha_j|\left(\int_{A_n\setminus A_{n+1}}\left|f_{b_j}\right|^p dm\right)^{\frac{1}{p}}\right)^p,$$

where

$$\left(\int_{A_n\setminus A_{n+1}}\left|f_{b_j}\right|^p dm\right)^{\frac{1}{p}} \le \left(\int_{A_n}\left|f_{b_j}\right|^p dm\right)^{\frac{1}{p}} \le 4^{-n} \tag{43}$$

for $j < n$ by condition (i) and

$$\left(\int_{A_n\setminus A_{n+1}}\left|f_{b_j}\right|^p dm\right)^{\frac{1}{p}} \le \left(\int_{\mathbb{T}\setminus A_j}\left|f_{b_j}\right|^p dm\right)^{\frac{1}{p}} < 4^{-j} \tag{44}$$

for $j > n$ by condition (ii). Thus by estimates (43) and (44), we have that

$$\left(\int_{A_n\setminus A_{n+1}} \left|f_{b_j}\right|^p dm\right)^{\frac{1}{p}} < 2^{-n-j} \qquad (45)$$

for $j \neq n$. By using estimate (45) we get

$$\begin{aligned}\|V\,\alpha\|_p^p &\leq \sum_{n=1}^{\infty}\left(|\alpha_n|\left(\int_{A_n\setminus A_{n+1}} \left|f_{b_n}\right|^p dm\right)^{\frac{1}{p}} + \sum_{j\neq n}|\alpha_j|2^{-n-j}\right)^p \\ &\leq \sum_{n=1}^{\infty}(|\alpha_n| + \|\alpha\|_{\ell^p}2^{-n})^p \leq 2^p\left(\sum_{n=1}^{\infty}|\alpha_n|^p + \|\alpha\|_{\ell^p}^p\sum_{n=1}^{\infty}2^{-n}\right) \\ &\leq 2^{p+1}\|\alpha\|_{\ell^p}^p.\end{aligned}$$

The next result is an observation by Aleman and Cima [213] and it states that for a non-compact operator T_g on H^p we can find a sequence (f_{a_k}) of test functions so that the sequence $(\left\|T_g f_{a_k}\right\|_p)$ converges to a positive constant. Its proof is based on the fact that for all $0 < p < \infty$ and $t \in (0, p/2)$ there exists a constant $C = C(p,t) > 0$ such that

$$\left\|T_g f_a\right\|_p \geq C\|g\circ\sigma_a - g(a)\|_t$$

for all $a \in \mathbb{D}$, where $\sigma_a(z) = (a - z)/(1 - \bar{a}z)$; see [213].

Proposition (6.2.3)[211]: Let $g \in BMOA\setminus VMOA$ and $1 \leq p < \infty$. Then

$$c := \limsup_{|a|\to 1}\left\|T_g f_a\right\|_p > 0.$$

In particular, there exists a sequence $(a_k) \subset \mathbb{D}, a_k \to \omega \in \mathbb{T}$ so that

$$\lim_{k\to\infty}\left\|T_g f_{a_k}\right\|_p = c.$$

Next, we prove a localization result for the images $T_g f_a$ of the test functions f_a (cf. Lemma (6.2.1)).

Lemma (6.2.4)[211]: Let $g \in BMOA, 1 \leq p < \infty$, and $(a_k) \subset \mathbb{D}$ be a sequence such that $a_k \to \omega \in \mathbb{T}$. Define

$$A_\varepsilon = \{e^{i\theta} : |e^{i\theta} - \omega| < \varepsilon\}$$

for each $\varepsilon > 0$. Then

$$(i)\ \lim_{k\to\infty}\int_{\mathbb{T}\setminus A_\varepsilon}\left|T_g f_{a_k}\right|^p dm = 0 \text{ for every } \varepsilon > 0.$$

$$(ii)\ \lim_{\varepsilon\to 0}\int_{A_\varepsilon}\left|T_g f_{a_k}\right|^p dm = 0 \text{ for each } k.$$

Proof. (i) Let $\varepsilon > 0$. For the simplicity of notation, we may assume that $\omega = 1$. Also, we assume that $g(0) = 0$. We have that

$$|1 - \overline{a_k}se^{i\theta}| \geq \delta$$

for all $k, 0 \leq s \leq 1$, and $\varepsilon \leq |\theta| \leq \pi$, where $\delta > 0$. Thus for those s and θ we have

$$\left|f_{a_k}(se^{i\theta})\right|^p = \frac{1 - |a_k|^2}{|1 - \overline{a_k}se^{i\theta}|^2} \leq \frac{1 - |a_k|^2}{\delta^2}$$

and

$$\left|f'_{a_k}(se^{i\theta})\right|^p \leq C\frac{1 - |a_k|^2}{|1 - \overline{a_k}se^{i\theta}|^{2+p}} \leq C\frac{1 - |a_k|^2}{\delta^{2+p}}$$

for all k, where $C > 0$ may depend on p. For $\zeta \in \mathbb{T}\setminus A_\varepsilon$, we obtain

$$\begin{aligned}\left|T_g f_{a_k}(\zeta)\right|^p &= \left|\int_0^1 f_{a_k}(s\zeta)g'(s\zeta)\zeta ds\right|^p \\ &\leq C\left(\left(\left|f_{a_k}(\zeta)g(\zeta)\right|^p\right) + \left(\int_0^1 \left|f'_{a_k}(s\zeta)g(s\zeta)\right|ds\right)^p\right) \\ &\leq C\left(\frac{1-|a_k|}{\delta^2}|g(\zeta)|^p + \frac{1-|a_k|}{\delta^{2+p}}\left(\int_0^1 |g(s\zeta)|ds\right)^p\right),\end{aligned}$$

where constants $C > 0$ may depend on p and change from one instance to another. Since $g \in BMOA$, we have that $|g(z)| \leq C \log\left(\frac{1}{1-|z|}\right)\|g\|_*$ for some $C > 0$, and consequently $\int_0^1 |g(s\zeta)|ds \leq C\|g\|_*$. Therefore

$$\int_{\mathbb{T}\setminus A_\varepsilon} \left|T_g f_{a_k}\right|^p dm \leq \left(\frac{1-|a_k|}{\delta^2}\|g\|_p^p + \frac{1-|a_k|}{\delta^{2+p}}\|g\|_*^p\right) \to 0$$

as $k \to \infty$, where $\|g\|_p \leq \sup_{b\in\mathbb{D}}\left\|g\circ\sigma_b - g(b)\right\|_p \simeq \|g\|_* < \infty$; see (42).

(ii) If k is fixed, then it follows from the absolute continuity of the measure $B \mapsto \int_B \left|T_g f_{a_k}\right|^p dm$ that $\int_{A_\varepsilon}\left|T_g f_{a_k}\right|^p dm \to 0$ as $\varepsilon \to 0$.

As a final step towards the proof of Theorem (6.2.6), we construct an isomorphism from ℓ^p into H^p using a non-compact T_g and test functions.

Proposition (6.2.5)[211]: Let $g \in BMOA \setminus VMOA, 1 \leq p < \infty$, and $(a_n) \subset \mathbb{D}$ be the sequence from Proposition (6.2.3). Then there exists a subsequence (b_n) of (a_n) such that the mapping

$$U : \ell^p \to H^p, \qquad U(\alpha) = \sum_{n=1}^{\infty} \alpha_n T_g f_{b_n},$$

where $\alpha = (\alpha_n) \in \ell^p$, is an isomorphism onto its range.

Proof. By Proposition (6.2.2) there exists a subsequence (c_n) of (a_n) inducing a bounded operator

$$S : \ell^p \to H^p, \qquad S(\alpha) = \sum_{n=1}^{\infty} \alpha_n f_{c_n},$$

and for any subsequence (b_n) of (c_n) the operator $V : \ell^p \to H^p, V(\alpha) = \sum_{n=1}^{\infty} \alpha_n T_g f_{b_n}$ is bounded. After finding the suitable sequence (b_n), the operator $U = T_g V$ will be bounded as a composition of two bounded operators.

Before proving that U is bounded from below, we provide some preparatory material. Since (c_n) is a subsequence of (a_n), we have that $c_n \to \omega \in \mathbb{T}$ and there exists a number $c > 0$ such that

$$\lim_{n\to\infty}\left\|T_g f_{c_n}\right\|_p = c$$

by Proposition (6.2.3). Using Proposition (6.2.3) and Lemma (6.2.4), we choose positive numbers ε_n with $\varepsilon_1 > \varepsilon_2 > \ldots > 0$ and a subsequence (b_n) of (c_n) such that the following
conditions hold:

$$\text{(i)} \left(\int_{A_n}\left|T_g f_{b_j}\right|^p dm\right)^{1/p} < 4^{-n}\delta c, \qquad j = 1,\ldots,n-1,$$

(ii) $\left(\int_{\mathbb{T}\setminus A_n} \left|T_g f_{b_n}\right|^p dm\right)^{1/p} < 4^{-n}\delta c,$

(iii) $\frac{c}{2} \leq \left(\int_{A_n} \left|T_g f_{b_j}\right|^p dm\right)^{\frac{1}{p}} \leq 2c,$

for every $n \geq 1$, where

$$A_n = A_{\varepsilon_n} = \{e^{i\theta} : |e^{i\theta} - \omega| < \varepsilon_n\}$$

and $\delta > 0$ is a constant whose value is determined later.

Now we are ready to prove that U is bounded from below. Using conditions (ii) and (iii), we get

$$\|U\,\alpha\|_p^p = \int_{\mathbb{T}} \left|\sum_{j=1}^{\infty} \alpha_j T_g f_{b_j}\right|^p dm \geq \sum_{n=1}^{\infty} \int_{A_n\setminus A_{n+1}} \left|\sum_{j=1}^{\infty} \alpha_j T_g f_{b_j}\right|^p dm$$

$$\geq \sum_{n=1}^{\infty} \left| |\alpha_n| \left(\int_{A_n\setminus A_{n+1}} \left|T_g f_{b_n}\right|^p dm\right)^{\frac{1}{p}} - \left(\int_{A_n\setminus A_{n+1}} \left|\sum_{j\neq n} \alpha_j T_g f_{b_n}\right|^p dm\right)^{\frac{1}{p}} \right|^p$$

$$\geq \sum_{n=1}^{\infty} \left(2^{-p}|\alpha_n|^p \left(\frac{c}{2} - 4^{-n-1}\delta c\right)^p - \int_{A_n\setminus A_{n+1}} \left|\sum_{j\neq n} \alpha_j T_g f_{b_n}\right|^p dm \right)$$

$$\geq \sum_{n=1}^{\infty} \left(2^{-p}|\alpha_n|^p \left(\frac{c}{2} - 4^{-n-1}\delta c\right)^p - \left(\sum_{j\neq n} |\alpha_j| \left(\int_{A_n\setminus A_{n+1}} \left|T_g f_{b_n}\right|^p dm\right)^{\frac{1}{p}}\right)^p \right),$$

where

$$\left(\int_{A_n\setminus A_{n+1}} \left|T_g f_{b_j}\right|^p dm\right)^{1/p} \leq \left(\int_{A_n} \left|T_g f_{b_j}\right|^p dm\right)^{1/p} < 4^{-n}\delta c$$

for $j < n$ by condition (i) and

$$\left(\int_{A_n\setminus A_{n+1}} \left|T_g f_{b_j}\right|^p dm\right)^{1/p} \leq \left(\int_{\mathbb{T}\setminus A_n} \left|T_g f_{b_j}\right|^p dm\right)^{1/p} < 4^{-j}\delta c$$

for $j > n$ by condition (ii). Thus we have that

$$\left(\int_{A_n\setminus A_{n+1}} \left|T_g f_{b_j}\right|^p dm\right)^{1/p} < 4^{-n-j}\delta c$$

for $j \neq n$. Consequently, we can estimate

$$\|U\alpha\|_p^p \geq \sum_{n=1}^{\infty} \left(2^{-p}|\alpha_n|^p \left(\frac{c}{2} - 4^{-n-1}\delta c\right)^p - \left(\sum_{j=1}^{\infty} |\alpha_j| 2^{-n-j}\delta c\right)^p \right)$$

$$\geq \sum_{n=1}^{\infty} \left(2^{-p}|\alpha_n|^p \left(\frac{c}{2} - 4^{-n-1}\delta c\right)^p - 2^{-n}\delta^p c^p \|\alpha\|_{\ell^p}^p \right)$$

$$\geq 2^{-p}\sum_{n=1}^{\infty}|\alpha_n|^p\left(\frac{c}{2}-\frac{1}{16}\delta c\right)^p-\delta^p c^p\|\alpha\|_{\ell^p}^p$$
$$\geq 2^{-p}\left(\frac{7}{16}\right)^p c^p\|\alpha\|_{\ell^p}^p-\delta^p c^p\|\alpha\|_{\ell^p}^p$$
$$=\left(\left(\frac{7}{32}\right)^p-\delta^p\right)c^p\|\alpha\|_{\ell^p}^p=\left(\frac{1}{8}\right)^p c^p\|\alpha\|_{\ell^p}^p,$$

when we choose $0<\delta<1$ such that $\delta^p=\left(\frac{7}{32}\right)^p-\left(\frac{1}{8}\right)^p$, i.e., $\delta=\frac{(7^p-4^p)^{1/p}}{32}$.
Thus the bounded operator U is also bounded from below and consequently it is an isomorphism onto its range.

Now our main result follows.

Theorem (6.2.6)[211]: Let $g\in BMOA\setminus VMOA$ and $1\leq p<\infty$. Then the operator $T_g: H^p\to H^p$ fixes an isomorphic copy of ℓ^p inside H^p, that is, there exists a subspace $M\subset H^p$ which is isomorphic to ℓ^p and such that the restriction of T_g to M is an isomorphism onto its range. In particular, T_g is not strictly singular.

Proof. By Propositions (6.2.2) and (6.2.5), we can choose a sequence $(b_n)\subset\mathbb{D}$ that induces a bounded operator

$$V:\ell^p\to H^p,\qquad V(\alpha)=\sum_{n=1}^{\infty}\alpha_n f_{b_n},$$

where $\alpha=(\alpha_n)\in\ell^p$, and an isomorphism $U:\ell^p\to H^p, U=T_gV$ onto its range.
Define $M=\overline{\operatorname{span}\{f_{b_n}\}}$, where the closure is taken in H^p. Since U is bounded from below, we have that the restriction $T_g|_M$ is bounded from below. Thus $T_g|_M: M\to T_g(M)$ is an isomorphism and consequently M is isomorphic to ℓ^p. In particular, the operator T_g is not strictly singular.

We conclude by posing two questions concerning the ℓ^2-singularity of T_g. To be specific, define $S(H^p)$ to be the class of strictly singular operators on H^p and $S_r(H^p)$ to be the class of ℓ^r-singular operators on H^p, i.e., those bounded operators which do not fix a copy of the space ℓ^r. Then $S(H^p)=S_2(H^p)\cap S_p(H^p)$ for $1<p<\infty$; see [227]. For a non-compact operator T_g on H^p it follows from Theorem (6.2.6) that $T_g\notin S_p(H^p)$. However, we did not pursue the following questions.

Does a non-compact $T_g: H^p\to H^p$ satisfy $T_g\notin S_2(H^p)$? Can we characterize those g such that $T_g\in S_2(H^p)$, where $1\leq p<\infty, p\neq 2$?

We consider the strict singularity of T_g on $BMOA$, the Bergman spaces $A^p, 1\leq p<\infty$, and the Bloch space B. The case of $BMOA$ essentially follows from the reasoning done in [220], where we utilize an idea of Leĭbov [222] that there exist isomorphic copies of the space c_0 of null sequences inside $VMOA$.

This fact will imply that the strict singularity of T_g on $BMOA$ or on $VMOA$ is equivalent to the compactness of T_g on the same space. The sketch of the proof is the following.
We recall that the boundedness of T_g on $BMOA$ is characterized by the condition $g\in LMOA$, where $LMOA$ is the "logarithmic $BMOA$" space; see [226].
By the proof of Theorem 2 and Lemma 6 in [220] we can find a sequence (h_n) in $VMOA$ which is equivalent to the standard basis of c_0. If T_g is non-compact on $VMOA$, it follows

from the estimate (4.6) in [220] that $\|T_g h_n\|_* > c > 0$ for some constant c for all n. Since $g \in LMOA \subset BMOA$, the operator T_g is bounded on H^2 and consequently $\|T_g h_n\|_2 \to 0$, as $n \to \infty$. Now we apply Lemma 6 in [220] again to obtain (by passing to a subsequence, if needed) that $(T_g h_n)$ is equivalent to the standard basis of c_0. Hence $T_g|_M$, where $M = \overline{\text{span}\{h_n\}}$, is an isomorphism onto its range and T_g is not strictly singular on $VMOA$ (or on $BMOA$).

In the Bergman spaces, boundedness and compactness of T_g was characterized in [215]. It is known that $A^p, 1 \le p < \infty$, are isomorphic to ℓ^p; see e.g. [229]. Hence the strict singularity of the operator T_g coincides with the compactness, since all strictly singular operators on ℓ^p are compact; see [223] and a comment thereafter. The boundedness and compactness of T_g acting on Bloch spaces was investigated in [230]. In this case, we can deduce as follows: If T_g acting on B is strictly singular, then its restriction to the little Bloch space B_0 is also strictly singular. Since B_0 is isomorphic to c_0 and strictly singular operators on c_0 are compact, the restriction $T_g|_{B_0}$ is compact. Also, the bidual $(B_0)^{**}$ can be identified with B, so the operator T_g can be identified with the biadjoint operator $\left(T_g|_{B_0}\right)^{**}$. Therefore T_g is compact.

Section (6.3): Rigidity of Composition Operators

For $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in $\mathbb{C}$. For $0 < p < \infty$ the analytic function $f : \mathbb{D} \to \mathbb{C}$ belongs to the Hardy space H^p if

$$\|f\|_p^p = \sup_{0 \le r < \infty} \int_{\mathbb{T}} |f(r\xi)|^p < \infty, \tag{46}$$

where $T = \partial\mathbb{D}$ (identified with $[0, 2\pi]$) and $dm\left(e^{it}\right) = \frac{dt}{2\pi}$. Let $\phi : \mathbb{D} \to \mathbb{D}$ be an analytic self-map of $\mathbb{D}$. It is a well-known consequence of the Littlewood subordination principle, see e.g. [20], that the composition operator

$$f \to C_\phi(f) = f \circ \phi$$

is bounded $H^p \to H^p$ for any ϕ as above. Properties of these composition operators have been studied very extensively during the last 40 years on various Banach spaces of analytic functions on $\mathbb{D}$, see [20] and [83] for comprehensive expositions of the early developments of the area. The compactness of C_ϕ on H^p is well understood, and there are several equivalent characterisations in the literature. To exhibit a specific criterion recall that Shapiro [117] established that C_ϕ is a compact operator $H^p \to H^p$ if and only if

$$\lim_{|w| \to 1} \frac{N(\phi, w)}{\log(1/|w|)} = 0. \tag{47}$$

Above $N(\phi, w)$ is the Nevanlinna counting function of ϕ defined by $N(\phi, w) = \sum_{z \in \phi^{-1}(w)} \log(1/|z|)$ for $w \in \phi(\mathbb{D})$ (counting multiplicities). Finer gradations of compactness were obtained e.g. by Luecking and Zhu [106], who characterised the membership of C_ϕ in the Schatten p-classes on H^2. Moreover, the approximation numbers of C_ϕ on H^2 were estimated in [63], [78] and [203], as well as on H^p in [239].

We demonstrate that composition operators on H^p only allow a small variety of qualitative non-compact behaviour compared to that of arbitrary bounded operators on H^p. Let E, F and X be Banach spaces. It will be convenient to say that the bounded linear operator $U : E \to F$ fixes a copy of X in E if there is an infinite-dimensional subspace $M \subset E, M$ linearly isomorphic to X, for which $U|_M$ is bounded below on M, that is, there is $c > 0$ so that $\|Ux\| \geq c \cdot \|x\|$ for all $x \in M$. We use the standard notation $M \approx X$ for linearly isomorphic spaces M and X, and refer to [88], [223] and [229] for general background related to the theory of Banach spaces.

The trichotomy contained in Theorem (6.3.3) below is the main result. Let $E_\phi = \{e^{i\theta} : |\phi(e^{i\theta})| = 1\}$ be the boundary contact set of the analytic map $\phi : \mathbb{D} \to \mathbb{D}$. Here, and in the sequel, we use $\phi(e^{i\theta})$ to denote the a.e. radial limit function of ϕ on $\mathbb{T}$. It is part of the trichotomy that (47) together with the simple condition

$$m(E_\phi) = 0 \tag{48}$$

completely determine the composition operators which fix copies of the subspace ℓ^p or ℓ^2 in H^p. Recall that the known compactness results for C_ϕ on H^2 yield that (47) implies (48), but the class of symbols ϕ satisfying (48) is much larger than that of (47), see e.g. [83].

In the statement below we exclude the Hilbert space H^2, where the situation is known and much simpler, since part (ii) does not occur for $p = 2$ (cf. the discussion following Theorem (6.3.4)). We use $K(E)$ to denote the class of compact operators $E \to E$ for any Banach space E, and take into account the known characterisation of the composition operators $C_\phi \in K(H^p)$.

Theorem (6.3.3) is obtained by combining Theorems (6.3.4), (6.3.2) and (6.3.11) stated below, which also contain more precise information. We first recall some standard linear classes that classify the behaviour of non-compact operators. Let E, F and X be Banach spaces, and $\mathcal{L}(E,F)$ be the space of bounded linear operators from E to F. The operator $U \in \mathcal{L}(E,F)$ is called X-singular if U does not fix any copies of X in E. We denote

$$S_X(E,F) = \{U \in \mathcal{L}(E,F) : U \text{ is } X - \text{singular}\},$$

and put $S_p(E,F) = S_p\,(E,F)$ to simplify our notation in the case of $X = \ell^p$. Recall further that $U \in \mathcal{L}(E,F)$ is strictly singular, denoted by $U \in S(E,F)$, if U is not bounded below on any infinite-dimensional linear subspaces $M \subset E$. It is clear that $K(E,F) \subset S(E,F) \subset S_p(E,F)$ for any Banach spaces E and F, and it is known that the classes $S(E,F)$ and $S_p(E,F)$ define norm-closed operator ideals in the sense of Pietsch [241] for any $1 \leq p \leq \infty$ (cf. [227] for the case of S_p).

Part of Theorem (6.3.3) is contained in the following dichotomy, which we also relate to the known characterisation of the compact composition operators on H^p.

The above theorem holds for $p = 2$ because of the general fact due to Calkin that $K(H^2) = S(H^2) = S_2(H^2)$ for the Hilbert space H^2, see e.g. [241]. For $1 < p < \infty$ and $p = 2$ one has that

$$S(H^p) = S_2(H^p) \cap S_p(H^p). \tag{49}$$

This follows from the characterisation of $S(L^p)$ by Weis [227] combined with the wellknown fact that $H^p \approx L^p \equiv L^p(0,1)$, see e.g. [240]. By contrast, for $p = 2$ all the inclusions

$$K(H^p) \subsetneq S(H^p)\ , S(H^p) \subsetneq S_2(H^p), \qquad S(H^p) \subsetneq S_p(H^p) \tag{50}$$

are strict. This is easily deduced from the facts that $H^p \approx L^p$ contains complemented subspaces isomorphic to ℓ^p and ℓ^2, whereas any $U \in \mathcal{L}(\ell^p, \ell^q)$ is strictly singular for $p \neq q$, see e.g. [223]. Thus Theorem (6.3.4) states that for $p = 2$ the compactness of composition operators $C_\phi \in \mathcal{L}(H^p)$ is a fairly rigid property as compared to (49) and (50) for arbitrary operators. It is also convenient to rephrase this as follows:

Corollary (6.3.1)[231]: For $1 \leq p < \infty$ the following conditions are equivalent for any analytic map $\phi : \mathbb{D} \to \mathbb{D}$:

(i) ϕ satisfies (47),

(ii) $C_\phi \in K(H^p)$,

(iii) $C_\phi \in S(H^p)$,

(iv) $C_\phi \in S_p(H^p)$.

The first result (excluding the case H^2) in the direction of Theorem (6.3.4) and Corollary (6.3.1) is due to Sarason [82], who showed that C_ϕ is weakly compact $H^1 \to H^1$ if and only if it is compact. Jarchow [237] pointed out that as a consequence $C_\phi \in K(H^1)$ if and only if C_ϕ is weakly conditionally compact on H^1, that is, $C_\phi \in S_1(H^1)$ in view of Rosenthal's ℓ^1-theorem, see e.g. [223]. Hence the case $p = 1$ in Theorem (6.3.4) and Corollary (6.3.1) was known earlier. We refer for a list of further references to analogous rigidity results for composition operators on several (classical) Banach spaces of analytic functions on the unit disk $\mathbb{D}$.

The lattice structure of the operator norm-closed ideals of $\mathcal{L}(H^p) \approx \mathcal{L}(L^p)$ containing the compact operators is quite complicated for $1 < p < \infty$ and $p = 2$, see e.g. [241] and [243]. For instance, $S_p(H^p)$ and $S_2(H^p)$ are mutually incomparable classes, since $H^p \approx L^p$ contains complemented copies of ℓ^2 and ℓ^p. However, note that Corollary (6.3.1) implies that if $C_\phi \in \mathcal{L}(H^p)$ fixes a copy of ℓ^2 in H^p, then C_ϕ must also fix a copy of ℓ^p in H^p. These facts raise the problem whether it is possible to explicitly determine the ℓ^2-singular composition operators on H^p. In turns out in Theorem (6.3.2) below that condition (48) characterises this class, thus providing a finer classification of the non-compact $C_\phi \in \mathcal{L}(H^p)$ for $1 \leq p < \infty$ and $p = 2$. We stress that Theorem (6.3.2) (as well as the subsequent Theorem (6.3.11)) does not hold for H^2.

Theorem (6.3.2)[231]: Let $1 \leq p < \infty, p = 2$, and $\phi : \mathbb{D} \to \mathbb{D}$ be an analytic map. Then C_ϕ fixes a copy of ℓ^2 in H^p if and only if $m(E_\phi) > 0$. Equivalently, $C_\phi \in S_2(H^p)$ if and only if (48) holds. Cima and Matheson [233] have shown that (48) characterises the completely continuous composition operators $C_\phi \in \mathcal{L}(H^1)$. As a significant strengthening of Theorem (6.3.2) we are further able to show that for $p > 1$ (and $p = 2$) condition (48) actually describes the operators C_ϕ which belong to the class $S_{L^p}(H^p)$. Here $S_{L^p}(H^p)$ is the maximal non-trivial ideal of $\mathcal{L}(H^p)$, see [235]. To state the relevant result let h^p be the harmonic Hardy space consisting of the harmonic functions $f : \mathbb{D} \to \mathbb{C}$ normed by (46).

The proof of Theorem (6.3.4). The argument is based on explicit perturbation estimates, where the starting point is a known test function reformulation of the compactness criterion (47). The proofs of Theorems (6.3.2) and (6.3.11) are contained. Although these results are connected, we have stated them separately, since the argument for the ℓ^2-singularity in H^p also holds for $p = 1$. By contrast the proof of Theorem (6.3.11) relies on properties of $h^p = L^p(T, m)$ for $1 < p < \infty$, and it depends on the non-trivial fact due to Dosev et al. [235] that the class $S_{L^p}(L^p) \approx S_{L^p}(h^p)$ is additive. We contain a number of further comments and open problems. As an application we characterise the ℓ^2-singular compositions $C_\phi \in \mathcal{L}(VMOA)$. As an additional motivation we also indicate a connection between a weaker version of Corollary (6.3.1) and a general extrapolation result [217] for operators on L^p-spaces.

A starting point for was a question by Jonathan Partington about the strict singularity of composition operators on H^p for $p = 2$. We are indebted to Manuel González, Francisco Hernández and Dmitry Yakubovich for timely questions towards Theorems (6.3.2) and (6.3.11).

Theorem (6.3.3)[231]: Let $1 \le p < \infty, p = 2$, and ϕ be any analytic self-map of D. Then there are three mutually exclusive alternatives:

(i) C_ϕ is compact on H^p,

(ii) C_ϕ fixes a copy of ℓ^p in H^p, but does not fix any copies of ℓ^2 in H^p,

(iii) C_ϕ fixes a copy of ℓ^2 (as well as of ℓ^p) in H^p. In this case, if $1 < p < \infty$ and $p = 2$, then C_ϕ also fixes a copy of $L^p(0,1)$ in H^p.

Furthermore, regarding the above alternatives

(i) takes place if and only if Shapiro's condition (47) holds,

(ii) takes place if and only if (47) fails to hold but $m(E_\phi) = 0$,

(iii) takes place if and only if $m(E_\phi) > 0$.

In particular, $C_\phi \in K(H^p)$ if and only if C_ϕ does not fix any copies of ℓ^p in H^p.

Proof. For $a \in \mathbb{D}$ and fixed $0 < p < \infty$ let

$$g_a(z) = \frac{(1 - |a|^2)^{1/p}}{(1 - \bar{a}z)^{2/p}}, \qquad z \in \mathbb{D}.$$

Observe that if $\gamma_a(z) = \frac{(1-|a|^2)^{1/p}}{1-\bar{a}z}$ is the normalised reproducing kernel of H2 associated to $a \in \mathbb{D}$, then $|g_a(z)|^p = |\gamma_a(z)|^2$ for $z \in \mathbb{D}$, so that $\|g_a\|_p = 1$. The proof of Theorem (6.3.4) is based on the following criterion: $C_\phi \in K(H^p)$ if and only if

$$\lim_{|a|\to 1} \sup \left\|C_\phi(g_a)\right\|_p = 0. \tag{51}$$

This is a restatement using the test functions $(g_a) \subset H^p$ of a well-known characterisation of the compact operators $C_\phi \in \mathcal{L}(H^p)$ in terms of vanishing Carleson pull-back measures, see [20] (such a characterisation was first obtained by MacCluer [107] in the case of $H^p(B_N)$ for $N > 1$, where B_N is the open Euclidean ball in C_N). Alternatively, (51) is stated explicitly for $p = 2$ in e.g. [117], whereas the compactness of $C_\phi : H^p \to H^p$ is independent of $p \in (0,\infty)$ e.g. by [20]. Note that the compactness criterion (47) plays no explicit role, as we will mostly use condition (51). After these preparations we proceed to the proof itself.

Theorem (6.3.4)[231]: Let $1 \le p < \infty$ and let $\phi : \mathbb{D} \to \mathbb{D}$ be any analytic map. Then either $C_\phi \in K(H^p)$, or else $C_\phi \notin S_p(H^p)$. Equivalently, C_ϕ fixes a copy of ℓ^p in H^p if and only if (47) does not hold.

Proof. Suppose that $C_\phi \notin K(H^p)$, where $1 \le p < \infty$. We will show by an explicit perturbation argument that C_ϕ fixes a linearly isomorphic copy of ℓ^p in H^p. Since condition (51) fails there is $d > 0$ and a sequence $(a_n) \subset \mathbb{D}$ so that $|a_n| \to 1$ as $n \to \infty$ and

$$\left\|C_\phi\left(g_{a_n}\right)\right\|_p \ge d > 0 \tag{52}$$

for all $n \in \mathbb{N}$. We may further assume without loss of generality that $a_n \to 1$ as $n \to \infty$. Namely, we may pass to a convergent subsequence in $\mathbb{D}$ and compose ϕ with a suitable rotation of $\overline{\mathbb{D}}$ that defines a linear isomorphism of H^p. Our starting point is the phenomenon that (g_{a_n}) admits subsequences which are small perturbations of a disjointly supported sequence in $L^p(T,m)$, and hence span an isomorphic copy of ℓ^p. The crux of the argument is that this can be achieved simultaneously for further subsequences of $\left(C_\phi\left(g_{a_n}\right)\right)$, and the following claim actually contains the basic step of the argument:

Claim (6.3.5)[231]: There is a further subsequence of (a_n), still denoted by (a_n) for simplicity, for which there are constants $c_1, c_2 > 0$ so that

$$c_1 \cdot \left\|(b_j)\right\|_{\ell^p} \leq \left\|\sum_{j=1}^{\infty} b_j C_\phi\left(g_{a_j}\right)\right\|_p \leq c_2 \cdot \left\|(b_j)\right\|_{\ell^p} \text{ for all } (b_j) \in \ell^p. \quad (53)$$

Assuming Claim (6.3.5) momentarily, the proof of Theorem (6.3.4) is completed by using this claim a second time (formally in the case where $\phi(z) = z$ for $z \in \mathbb{D}$) to extract a further subsequence of $\left(g_{a_n}\right)$, still denoted by $\left(g_{a_n}\right)$, so that

$$d_1 \cdot \left\|(b_j)\right\|_{\ell^p} \leq \left\|\sum_{j=1}^{\infty} b_j g_{a_j}\right\|_p \leq d_2 \cdot \left\|(b_j)\right\|_{\ell^p} \text{ for all } (b_j) \in \ell^p, (54)$$

for suitable constants $d_1, d_2 > 0$. Then by combining (53) and (54) we get

$$\left\|\sum_{j=1}^{\infty} b_j C_\phi\left(g_{a_j}\right)\right\|_p \geq c_1 \left\|(b_j)\right\|_{\ell^p} \geq c_1 d_2^{-1} \left\|\sum_{j=1}^{\infty} b_j g_{a_j}\right\|_p,$$

so that the restriction of C_ϕ defines a linear isomorphism $M \to C_\phi(M)$, where $M = \text{span}\{g_{a_j} : j \in \mathbb{N}\} \approx \ell^p$.

Let $A = \{\xi \in T :$ the radial limit $\phi(\xi)$ exists$\}$ and

$$E\varepsilon = \{\xi \in A : |\phi(\xi) - 1| < \varepsilon\}$$

for $\varepsilon > 0$. Recall that $\mathbb{T} \setminus A$ has measure zero. The proof of Claim (6.3.5) is an argument of gliding hump type based on the following auxiliary observation.

Lemma (6.3.6)[231]: Let ϕ and $\left(g_{a_n}\right)$ be as above, where $a_n \to 1$ as $n \to \infty$. Then

(L1) $\int_{\mathbb{T}\setminus E\varepsilon} \left|C_\phi\left(g_{a_n}\right)\right|^p dm \to 0$ as $n \to \infty$ for each fixed $\varepsilon > 0$,

(L2) $\int_{E\varepsilon} \left|C_\phi\left(g_{a_n}\right)\right|^p dm \to 0$ as $\varepsilon \to 0$ for each fixed $n \in \mathbb{N}$.

Proof. Observe first that

$$\int_{E\varepsilon} \left|C_\phi(g)\right|^p dm \to 0$$

as $\varepsilon \to 0$ for any $g \in H^p$, since $\cap_{\varepsilon>0} E\varepsilon = \{\xi \in A : \phi(\xi) = 1\}$ has measure 0 as ϕ is not identically 1. Moreover, if $\varepsilon > 0$ is fixed and $\xi \in A \setminus E\varepsilon$, then there is $n\varepsilon$ such that

$$|1 - \overline{a_n}\phi(\xi)| = |1 - \phi(\xi) + \phi(\xi)(1 - \overline{a_n})| \geq |1 - \phi(\xi)| - |1 - a_n| > \varepsilon/2$$

for all $n \geq n_\varepsilon$. It follows that

$$\left|C_\phi\left(g_{a_n}\right)(\xi)\right|^p = \frac{1 - |a_n|^2}{|1 - \overline{a_n}\phi(\xi)|^2} \leq \frac{4(1 - |a_n|^2)}{\varepsilon^2},$$

so that (L1) holds as $n \to \infty$.

To continue the argument of Claim (6.3.5) recall that $\int_{\mathbb{T}} \left|C_\phi\left(g_{a_n}\right)\right|^p dm \geq d^p > 0$ by condition (52). We may then use Lemma (6.3.6) inductively to find indices $j_1 < j_2 <$ ... and a decreasing sequence $\varepsilon_j > \varepsilon_{j+1} \to 0$ so that

(i) $\left(\int_{E_{\varepsilon_n}} \left|C_\phi\left(g_{a_{j_k}}\right)\right|^p dm\right)^{1/p} < 4^{-n}\delta d$ for all $k = 1, \ldots, n - 1$,

(ii) $\left(\int_{\mathbb{T}\setminus E_{\varepsilon_n}} \left|C_\phi\left(g_{a_{j_n}}\right)\right|^p dm\right)^{1/p} < 4^{-n}\delta d,$

(iii) $\left(\int_{E_{\varepsilon_n}} \left|C_\phi\left(g_{a_{j_n}}\right)\right|^p dm\right)^{1/p} > d/2$

for all $n \in \mathbb{N}$.Here $\delta > 0$ is a small enough constant (to be chosen later). In fact, suppose that we have already found $a_{j_1},\ldots,a_{j_{n-1}}$ and $\varepsilon_1 >\ldots> \varepsilon_{n-1}$ satisfying (i)–(iii). Then property (L2) from Lemma (6.3.6) yields $\varepsilon_n < \varepsilon_{n-1}$ such that

$$\left(\int_{E_{\varepsilon_n}} \left|C_\phi\left(g_{a_{j_k}}\right)\right|^p dm\right)^{1/p} < 4^{-n}\delta d$$

for each $k = 1,\ldots,n-1$. After this use property (L1) from Lemma (6.3.6) together with (52) to find an index $j_n > j_{n-1}$ so that conditions (ii) and (iii) are satisfied for the set E_{ε_n} .

In the interest of notational simplicity we relabel ajn as an for $n \in \mathbb{N}$. The idea of the argument is that the sequence $\left(C_\phi\left(g_{a_n}\right)\right)$ essentially resembles disjointly supported peaks in $L^p(T,m)$ close to the point 1. We will next verify the left-hand inequality in (53) by a direct perturbation argument. Let $b = (b_j) \in \ell^p$ be arbitrary. Our starting point will be the identity

$$\left\|\sum_{j=1}^{\infty} b_j C_\phi\left(g_{a_j}\right)\right\|_p^p = \sum_{n=0}^{\infty} \int_{E_{\varepsilon_n}\setminus E_{\varepsilon_{n+1}}} \left|\sum_{j=1}^{\infty} b_j C_\phi\left(g_{a_j}\right)\right| dm, \qquad (55)$$

where we set $E_{\varepsilon_0} = T$.

Observe first that for each $n \in \mathbb{N}$ we get that

$$\left(\int_{E_{\varepsilon_n}\setminus E_{\varepsilon_{n+1}}} |C_\phi(g_{a_n})|^p dm\right)^{\frac{1}{p}} = \left(\int_{E_{\varepsilon_n}} |C_\phi(g_n)|^p dm - \int_{E_{\varepsilon_{n+1}}} |C_\phi(g_{a_n})|^p dm\right)^{\frac{1}{p}}$$

$$> \left(\left(\frac{d}{2}\right)^p - (4^{-n-1}\delta d)^p\right)^{1/p} \geq \frac{d}{2} - 4^{-n-1}\delta d$$

in view of (i) and (iii), where the last estimate holds because 0 < 1/p ≤ 1. Moreover, note that

$$\left(\int_{E_{\varepsilon_n}\setminus E_{\varepsilon_{n+1}}} \left|C_\phi\left(g_{a_j}\right)\right|^p dm\right)^{1/p} < 2^{-n-j}\,\delta d$$

for all $j \neq n$. In fact, $\left(\int_{E_{\varepsilon_n}\setminus E_{\varepsilon_{n+1}}} \left|C_\phi\left(g_{a_j}\right)\right|^p dm\right)^{1/p}$ is dominated by $4^{-n}\delta d$ for j_n in view of (i) and (ii). Thus we get from the triangle inequality in L^p, together with the preceding estimates, that for all $n \in \mathbb{N}$ one has

$$\left(\int_{E_{\varepsilon_n}\setminus E_{\varepsilon_{n+1}}} \left|\sum_{j=1}^{\infty} b_j C_\phi\left(g_{a_j}\right)\right|^p dm\right)^{1/p}$$

$$\geq |b_n|\left(\int_{E_{\varepsilon_n}\setminus E_{\varepsilon_{n+1}}} |C_\phi(g_{a_n})|^p dm\right)^{\frac{1}{p}} - \sum_{j\neq n} |b_j|\left(\int_{E_{\varepsilon_n}\setminus E_{\varepsilon_{n+1}}} \left|C_\phi\left(g_{a_j}\right)\right|^p dm\right)^{1/p}$$

$$\geq |b_n| \left(\frac{d}{2} - 4^{-n-1}\delta d\right) - 2^n \,\delta d\|b\|_d \geq \frac{d}{2} - 2^{-n+1}\delta d\|b\|_b.$$

By summing over n we get from the disjointness and the triangle inequality in ℓ^p that

$$\left\|\sum_{j=1}^{\infty} b_j C_\phi\left(g_{a_j}\right)\right\|_p \geq \left(\sum_{n=1}^{\infty} \left|\frac{d}{2}|b_n| - 2^{-n+1}\delta d\|b\|_p\right|^p\right)^{1/p}$$

$$\geq \frac{d}{2}\left(\sum_{n=1}^{\infty} |b_n|^p\right)^{\frac{1}{p}} - \delta d\|b\|_p\left(\sum_{n=1}^{\infty} 2^{(-n-1)}\right)^{1/p}$$

$$d\left(\frac{1}{2} - \delta\cdot(1 - 2^p)^{-1/p}\right)\|b\|_p \geq \frac{d}{4}\|b\|_p,$$

where the last estimate holds once we choose $\delta > 0$ small enough, so that $\delta\cdot(1 - 2^p)^{-1/p}$. The proof of the right-hand inequality in (53) is a straightforward variant of the preceding estimates. This inequality does not affect the choice of $\delta > 0$, and hence the details will be omitted here.

The proof of Theorem (6.3.2) is contained in the following three results. We first look separately at the case $p = 2$. Recall our notation $E_\phi = \{e^{i\theta} : |\phi(e^{i\theta})| = 1\}$ for analytic maps $\phi : \mathbb{D} \to \mathbb{D}$.

Lemma (6.3.7)[231]: Suppose that condition (48) fails, that is, $m(E_\phi) > 0$. Then there exist integers $0 \leq n_1 < n_2 < \cdots$ and a constant $K > 0$ such that

$$K^{-1} \cdot \|c\|_{\ell^2} \leq \left\|\sum_{k=1}^{\infty} c_k\phi^{n_k}\right\|_2 \leq K \cdot \|c\|_{\ell^2}$$

for all $c = (c_k) \in \ell^2$.

Proof. The upper estimate follows from the boundedness of C_ϕ on H^2 and the orthonormality of the sequence (z_n) in H^2.

To establish the lower estimate, note that $z_n \to 0$ weakly and therefore also $\phi_n = C_\phi(z_n) \to 0$ weakly in H^2 as $n \to \infty$. Hence we may set $n_1 = 0$ and then proceed inductively to pick increasing indices nk such that the inner-products satisfy $|\phi^{n_j}, \phi^{n_k}| \leq 2^{-2k}m(E_\phi)$ for all $1 \leq j < k$ and each $k \in \mathbb{N}$. Let $c = (c_k) \in \ell^2$ be arbitrary and note that

$$\left\|\sum_{k=1}^{\infty} c_k\phi^{n_k}\right\|_2^2 = \sum_{k=1}^{\infty} |c_k|^2\|\phi^{n_k}\|_2^2 + 2\mathrm{Re}\sum_{k=1}^{\infty}\sum_{j=1}^{k-1} c_j\,\overline{c}_k(\phi^{n_j}, \phi^{n_k}).$$

Obviously $\|\phi^{n_k}\|_2^2 \geq \int_{E_\phi} |\phi^{n_k}|^2 dm = m(E_\phi)$ for each k. Moreover, we get that

$$\sum_{k=1}^{\infty}\sum_{j=1}^{k-1} c_j\,\overline{c}_k(\phi^{n_j}, \phi^{n_k}) \leq \|c\|_{\ell^2}^2 \sum_{k=1}^{\infty}\sum_{j=1}^{k-1} 2^{-2k}m(E_\phi)$$

$$\leq \frac{1}{2}\|c\|_{\ell^2}^2 m(E_\phi)\sum_{k=1}^{\infty}\sum_{j=1}^{k-1} 2^{-2k} = \frac{1}{6}\|c\|_{\ell^2}^2 m(E_\phi)$$

By combining these estimates we obtain the desired lower bound

$$\left\|\sum_{k=1}^{\infty} c_k\phi^{n_k}\right\|_2^2 \geq \|c\|_{\ell^2}^2 m(E_\phi) - \frac{1}{3}\|c\|_{\ell^2}^2 m(E_\phi) = \left(\frac{2}{3}m(E_\phi)\right)\|c\|_{\ell^2}^2.$$

In order to treat general $p \in [1,\infty)$ recall that the analytic map $f: \mathbb{D} \to \mathbb{C}$ belongs to *BMOA* if

$$|f|_* = \sup_{a\in\mathbb{D}} \|f \circ \sigma_a - f(a)\|_2 < \infty,$$

where $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$ is the Möbius-automorphism of $\mathbb{D}$ interchanging 0 and a for $a \in \mathbb{D}$. The Banach space BMOA is normed by $\|f\|_{BMOA} = |f(0)| + |f|_*$. Moreover, $VMOA$ is the closed subspace of $BMOA$, where $f \in VMOA$ if

$$\lim_{|a|\to 1} \|f \circ \sigma_a - f(a)\|_2 = 0.$$

See [236] and [126] for background on $BMOA$. It follows readily from Littlewood's subordination theorem that C_ϕ is bounded $BMOA \to BMOA$ for any analytic map $\phi : \mathbb{D} \to \mathbb{D}$, see e.g. [123].

The following proposition establishes one implication of Theorem (6.3.2).

Proposition (6.3.8)[231]: Let $1 \leq p < \infty$ and suppose that $m(E_\phi) > 0$. Then there exist increasing integers $0 \leq n_1 < n_2 < \cdots$ such that the subspace

$$M = \overline{\text{span}}\{z^{n_k} : k \geq 1\} \subset H^p$$

is isomorphic to 2 and the restriction $C_\phi|_M$ is bounded below on M. Hence $C_\phi \notin S_2(H^p)$.

Proof. We start by choosing the increasing integers (n_k) as in Lemma (6.3.7). By passing to a subsequence we may also assume that (z^{n_k}) is a lacunary sequence, that is, $\inf_k (n_{k+1}/n_k) > 1$. Paley's theorem (see e.g. [201]) implies that for $1 \leq p < \infty$ the sequence (z^{n_k}) is equivalent in H^p to the unit vector basis of 2, that is,

$$\left\|\sum_{k=1}^{\infty} c_k z^{n_k}\right\|_p \sim \|c\|_{\ell^2} \tag{56}$$

for all $c = (c_k) \in \ell^2$. (Here, and in the sequel, we use ~ as a short-hand notation for the equivalence of the respective norms.) Case $p \geq 2$. By Hölder's inequality and Lemma (6.3.7) we have that

$$\left\|C_\phi \sum_{k=1}^{\infty} c_k z^{n_k}\right\|_p = \left\|\sum_{k=1}^{\infty} c_k\phi^{n_k}\right\|_p \geq \left\|\sum_{k=1}^{\infty} c_k\phi^{n_k}\right\|_2 \sim \|c\|_{\ell^2}.$$

According to (56) and the boundedness of C_ϕ this proves the claim for $p \geq 2$.

Case $1 \leq p < 2$. We start by invoking a version of Paley's theorem for $BMOA$ (see e.g. [126]), which together with the boundedness of C_ϕ on $BMOA$ ensures that

$$\left\|\sum_{k=1}^{\infty} c_k\phi^{n_k}\right\|_{BMOA} = \|C_\phi\| \cdot \left\|\sum_{k=1}^{\infty} c_k z^{n_k}\right\|_{BMOA} \leq K \cdot \|C_\phi\| \cdot \|c\|_{\ell^2}$$

for all $c = (c_k) \in \ell^2$ and a uniform constant $K > 0$. In view of Fefferman's $H^1 - BMOA$ duality pairing (see e.g. [126]) we may further estimate

$$\left\|\sum_{k=1}^{\infty} c_k\phi^{n_k}\right\|_{BMOA} = \left\|\sum_{k=1}^{\infty} c_k\phi^{n_k}\right\|_1$$

$$\leq \left|\left(\sum_{k=1}^{\infty} c_k\phi^{n_k}, \sum_{k=1}^{\infty} c_k\phi^{n_k}\right)\right| = \left\|\sum_{k=1}^{\infty} c_k\phi^{n_k}\right\|_2^2 \sim \|c\|_{\ell^2}$$

where we again use Lemma (6.3.7) at the final step. By applying Hölder's inequality and combining the preceding estimates we obtain that

$$\left\|\sum_{k=1}^{\infty} c_k\phi^{n_k}\right\|_p \geq \left\|\sum_{k=1}^{\infty} c_k\phi^{n_k}\right\|_1 \geq K'\|c\|_{\ell^2}$$

for some uniform constant $K' > 0$. In particular, $C_\phi \notin S_2(H^p)$ in view of (56), which completes the verification of the proposition for $1 \leq p < 2$.

The converse implication in Theorem (6.3.2) is contained in the following

Proposition (6.3.9)[231]: Let $1 \leq p < \infty, p = 2,$ and suppose that $m(E_\phi) = 0$. If (f_n) is any normalized sequence in H^p which is equivalent to the unit vector basis of ℓ^2, then C_ϕ is not bounded below on $\overline{\text{span}}\{f_n : n \in \mathbb{N}\} \subset H^p$. In particular, $C_\phi \in S_2(H^p)$.

Proof. Assume to the contrary that

$$\left\|\sum_{n=1}^{\infty} c_n C_\phi(f_n)\right\|_p \sim \left\|\sum_{n=1}^{\infty} c_k\phi^{n_k}\right\|_p \sim \|c\|_{\ell^2} \tag{57}$$

for all sequences $c = (c_n) \in \ell^2$. In particular, $C_\phi(f_n)^p \geq d > 0$ for all n and some constant d. We write $E_k = \{e^{i\theta} : |\phi(e^{i\theta})| \geq 1 - \frac{1}{k}\}$ for $k \geq 1$. Since $\lim_{k\to\infty} m(E_k) = m(E_\phi) = 0$, we get that

$$\lim_{k\to\infty} E_k \left|C_\phi(f_n)\right|^p dm = 0 \text{ for all } n.$$

On the other hand, $f_n \to 0$ weakly in H^p and hence $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$. This implies that

$$\lim_{n\to\infty} \int_{\mathbb{T}\setminus E_k} \left|C_\phi(f_n)\right|^p dm = 0 \text{ for all } k.$$

By using the above properties and proceeding recursively in a fashion similar to the argument for Theorem (6.3.4) we find increasing sequences of integers $0 \leq n_1 < n_2 < \cdots$ and $1 = k_1 < k_2 < \cdots$, such that

$$\left\|\sum_{n=1}^{\infty} c_n C_\phi(f_n)\right\|_p^p = \sum_{l=1}^{\infty} \int_{E_{k_l}\setminus E_{k+1}} \left|\sum_{j=1}^{\infty} c_n C_\phi(f_n)\right|^p dm \sim \|c\|_{\ell^2}$$

holds for all $c = (c_j) \in \ell^p$ with uniform constants. However, for $p = 2$ such estimates obviously contradict (57). Thus $C_\phi \in S_2(H^p)$, and this completes the proof of the proposition (and hence also of Theorem (6.3.2)).

We remind that Theorem (6.3.2) does not hold for $p = 2$. For $p \neq 2$, the result easily yields very explicit examples of operators $C_\phi \in S_2(H^p) \setminus S_p(H^p)$.

Example (6.3.10)[231]: Let $\phi(z) = \frac{1}{2}(1 + z)$ for $z \in \mathbb{D}$. Theorem (6.3.2) implies that C_ϕ does not fix any copies of ℓ^2 in H^p. On the other hand, it is well known that $C_\phi \notin K(H^p)$, see e.g. [83], so that C_ϕ does fix copies of ℓ^p in H^p by Theorem (6.3.4).

We next prepare for the proof of Theorem (6.3.11). This involves the harmonic Hardy space h^p, that is, the space of complex-valued harmonic functions $f : \mathbb{D} \to \mathbb{C}$ normed by (46). Recall that for $1 < p < \infty$ there is a well-known isometric identification $h^p =$

$L^p(T,m)$ as a complex Banach space. Here $f \in h^p$ corresponds to its a.e. radial limit function $f \in L^p(T,m)$, whereas conversely $g \in L^p(T,m)$ determines its harmonic extension $P[g] \in h^p$ through the Poisson integral. Moreover, $h^p = H^p \oplus \overline{H_0^p}$, where $\overline{H_0^p} = \{f \in H^p : f(0) = 0\}$ and $\overline{H_0^p} = \{\overline{f} : f \in H_0^p\}$.

Let $\phi : \mathbb{D} \to \mathbb{D}$ be any analytic map. The Littlewood subordination theorem for subharmonic functions (see e.g. [20]) implies that the composition operator $f \to f \circ \phi$ is also bounded $h^p \to h^p$ for $1 \le p < \infty$. It will be convenient in the argument to use the notation $\tilde{C}_\phi(f) = f \circ \phi$ for $f \in h^p$ to distinguish the composition operator on h^p from its relative on H^p. In particular, if in addition $\phi(0) = 0$, then we may decompose

$$\tilde{C}_\phi = \begin{pmatrix} C_\phi & 0 \\ 0 & C_\phi \end{pmatrix} \tilde{C}_\phi(f,g) = (f \circ \phi, g \circ \phi), \tag{58}$$

as a matrix direct sum with respect to the decomposition $h^p = H^p \oplus \overline{H_0^p}$. Here $\phi(0) = 0$ ensures that $g \circ \phi \in H_0^p$ for any $g \in H_0^p$.

Theorem (6.3.11)[231]: Let $1 < p < \infty, p = 2$, and $\phi : \mathbb{D} \to \mathbb{D}$ be an analytic map. Then the following conditions are equivalent:

(i) ϕ satisfies $m(E_\phi) = 0$,

(ii) $C_\phi \in S_{L^p}(H^p)$, that is, C_ϕ does not fix any copies of L^p in H^p,

(iii) $C_\phi \in S_{L^p}(h^p)$,

(iv) $C_\phi \in S_2(H^p)$.

Proof. We may assume during the proof that $\phi(0) = 0$. In fact, otherwise consider $\psi = \sigma_\phi(0) \circ \phi$, where $\sigma_\phi(0) : \mathbb{D} \to \mathbb{D}$ is the automorphism interchanging 0 and $\phi(0)$. Then $\psi(0) = 0$ and $\tilde{C}_\psi = \tilde{C}_\phi \circ \tilde{C}_{\sigma_\phi}(0)$, where $\tilde{C}_{\sigma_\phi}(0)$ is a linear isomorphism $h^p \to h^p$ (as well as $H^p \to H^p$), which does not affect any of the claims of the theorem.

The proof of the implication (iii) ⇒ (i) is contained in the following claim.

Claim (6.3.12)[231]: Let $1 < p < \infty$ and suppose that $m(E_\phi) > 0$. Then $\tilde{C}_\phi \notin S_{L^p}(h^p)$, that is, there is a subspace $M \subset h^p, M \approx L^p$, such that $\tilde{C}_\phi|_M$ is bounded below.

To prove the claim define the Borel measure ν on $\mathbb{T}$ by $\nu(A) = m\big(\phi^{-1}(A)\big)$. Then ν is absolutely continuous: if $A \subset T$ is a Borel set and $u_A = P[\chi_A]$ is the harmonic extension (i.e. the Poisson integral) of χ_A, we have that

$$\nu(A) = \int_{\phi^{-1}(A)} dm \le \int_{\mathbb{T}} u_A \circ \phi \, dm = u_A\big(\phi(0)\big) = u_A(0) = m(A).$$

Since $\nu(\mathbb{T}) = m\big(E_\phi\big) > 0$, it follows that the density $d\nu/dm \ge \delta$ for some $\delta > 0$ on a Borel set $F \subset \mathbb{T}$ of positive Lebesgue measure.

We may now choose $M = L^p(F,m)$. Indeed, given any $f \in L^p(F,m)$, we have

$$\|\tilde{C}_\phi f\|_{L^p}^p \ge \int_{E_\phi} |f \circ \phi|^p \, dm = \int_{\mathbb{T}} |f|^p \, d\nu \ge \delta \int_F |f|^p \, dm = \delta \|f\|_{L^p(F,m)}^p,$$

which establishes Claim (6.3.12), since $L^p(F,m) \approx L^p$.

The implication (ii) ⇒ (iii) follows from (58) and the non-trivial result that the class $S_{L^p}(L^p) \approx S_{L^p}(h^p)$ is additive, see [235]. In fact, if $C_\phi \in S_{L^p}(H^p)$, then

$$\tilde{C}_\phi = \begin{pmatrix} C_\phi & 0 \\ 0 & C_\phi \end{pmatrix} + \begin{pmatrix} 0 & C_\phi \\ C_\phi & 0 \end{pmatrix}$$

is the sum of two operators from S_{L^p} (hp), and hence L^p-singular by additivity. Here one applies the observation that if $M \subset L^p$ is a subspace isomorphic to L^p, then $\{f : f \in M\}$ is also linearly isomorphic to L^p.

Finally, the proof of the implication (i) ⇒ (ii) is already contained in that of Proposition (6.3.9). In fact, if there is a subspace $M \subset H^p, M \approx L^p$, so that C_ϕ is an isomorphism $M \to C_\phi(M)$, then C_ϕ also fixes the isomorphic copies of ℓ^2 contained in M. It was shown in Proposition (6.3.9) that the latter property is incompatible with condition (i) of Theorem (6.3.11).

We note that Claim (6.3.12) also holds for $p = 1$. However, there is no immediate analogue of Theorem (6.3.11) for H^1. In fact, $S_{L^1}(H^1) = L(H^1)$, since L^1 does not embed isomorphically into H^1, see e.g. [223]. In conclusion, recall that there are infinitely many norm-closed ideals I of $L(H^p)$ satisfying $S_2(H^p)\, I \subset S_{L^p}(H^p)$ for $1 < p < \infty$ and $p = 2$, see [241]. By contrast, Theorems (6.3.2) and (6.3.11) imply that there is no corresponding gradation for composition operators on H^p. In some cases the trichotomy of Theorem (6.3.3) can be sharpened by combining with known results about the subspaces of $H^p \approx L^p$. For instance, for $2 < p < \infty$ it follows from a result of Johnson and Odell [238] that if $C_\phi|_M$ is bounded below on an infinite-dimensional subspace $M \subset H^p$ that contains no isomorphic copies of ℓ^2, then M embeds isomorphically into p, whence $C_\phi \notin S_p(H^p)$.

We list some further examples of Banach spaces of analytic functions where composition operators have related rigidity properties, and draw attention to open problems. We also sketch another approach towards Theorem (6.3.4), though its conclusion is much weaker.

The weaker rigidity property

$$C_\phi \in S(E) \text{ if and only if } C_\phi \in K(E) \tag{59}$$

holds for many other Banach spaces E of analytic functions on D apart from the Hardy spaces. The following list briefly recalls some cases. Typically these results were not stated in terms of strict singularity, and as a rule they do not yield as precise information as our results for H^p.

(a) The following dichotomy in [216] is an explicit precursor of Theorem (6.3.4): either $C_\phi \in K(H_v^\infty)$ or $C_\phi \notin S_\infty(H_v^\infty)$. Here H_v^∞ is the weighted H^∞-space for a strictly positive weight function v on $\mathbb{D}$. It is also possible to deduce versions of (59) for H^∞ (the case $v \equiv 1$) from even earlier results. In fact, it follows from [85], [64] or [234] that $C_\phi \in L(H^\infty)$ is weakly compact if and only if $C_\phi \in K(H^\infty)$. Moreover, Bourgain [232] established that $W(H^\infty, X) = S_\infty(H^\infty, X)$ for any Banach space X, where W denotes the class of weakly compact operators. Here $K(H^\infty) S(H^\infty)$, since this holds for the complemented subspace ℓ^∞ of H^∞.

(b) The dichotomy in Theorem (6.3.4) holds for arbitrary bounded operators on the Bergman space A^p. In fact, $A^p \approx p$ for $1 \leq p < \infty$ by a result of Lindenstrauss and Pełczynski, see [229], whereas $S(\ell^p) = S_p(\ell^p) = K(\ell^p)$ by a result of Gohberg, Markus and Feldman, see [241].

(c) It is known that the Bloch space B is isomorphic to ℓ^∞, while $C_\phi \in W(B)$ if and only if $C_\phi \in K(B)$, see e.g. [132]. Moreover, any $U \notin W(\ell^\infty, X)$ fixes a copy of ℓ^∞ for any Banach space X, see [223]. Consequently either $C_\phi \in K(B)$ or $C_\phi \notin S_\infty(B)$.

(d) It follows from [69] that $C_\phi \in K(BMOA)$ if and only of $C_\phi \in S_{c_0}(BMOA)$. In fact, the argument shows that if $C_\phi \notin K(BMOA)$, then there is $M \subset VMOA, M \approx c_0$, so that $C_\phi|_M$ is bounded below. Here again $K(BMOA) \subsetneq S_{c_0}(BMOA)$, since $BMOA$ contains complemented subspaces isomorphic to ℓ^2 in view of Paley's theorem (see e.g. [126]).

Actually the results combined with [69] lead to a better understanding of the ℓ^2-singular composition operators on $VMOA$ and $BMOA$.

Proposition (6.3.13)[231]: (i) If $\phi : \mathbb{D} \to \mathbb{D}$ is an analytic map, and $C_\phi \in S_2(BMOA)$, then (48) holds (that is, $m(E_\phi) = 0$).

(ii) If $\phi \in VMOA$, then $C_\phi \in S_2(VMOA)$ if (and only if) (48) holds.

Proof. (i) The argument is essentially contained in that of Proposition (6.3.8). In fact, suppose that $m(E_\phi) > 0$, where $E_\phi = \{e^{i\theta} : |\phi(e^{i\theta})| = 1\}$. Then the proof of the case $1 \le p < 2$ of Proposition (6.3.8) gives a lacunary sequence (n_k) and constants $K_1, K_2 > 0$ so that in the $H^1 - BMOA$ duality pairing

$$\left\|\sum_{k=1}^{\infty} c_k \phi^{n_k}\right\|_{BMOA} \ge \left\|\sum_{k=1}^{\infty} c_k \phi^{n_k}\right\|_1 \ge K_1 \cdot \|c\|_{\ell^2}$$

as well as $\|\sum_{k=1}^{\infty} c_k \phi^{n_k}\|_1 \ge K_2 \cdot \|c\|_{\ell^2}$ for all $c = (c_k) \in \ell^2$. Since C_ϕ is bounded on $BMOA$ it follows as before from Paley's theorem in $BMOA$ that C_ϕ is bounded below on $\overline{\text{span}}\{z^{n_k} : k \in \mathbb{N}\} \approx 2$ in $BMOA$.

(ii) Recall that $C_\phi : VMOA \to VMOA$ if $\phi \in VMOA$, see e.g. [123]. Assume that $m(E_\phi) = 0$ and suppose to the contrary that there is a normalised sequence $(f_k) \subset VMOA$ equivalent to the unit vector basis of 2, for which

$$\left\|\sum_{k=1}^{\infty} c_k C_\phi(f_k)\right\|_{BMOA} \sim \cdot \|c\|_{\ell^2} \tag{60}$$

for all $c = (c_k) \in \ell^2$. In particular, $\|f_k \circ \phi\|_{BMOA} \ge d > 0$ for all k, while (f_k) is weak-null sequence in $VMOA$, so that $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. Moreover, by the John–Nirenberg inequality there is a uniform constant $d > 0$ so that

$$\|f_k \circ \phi\|_4 \le d\|f_k \circ \phi\|_{BMOA}, \qquad k \in \mathbb{N}.$$

Let $E_k = \{e^{i\theta} : |\phi(e^{i\theta})| \ge 1 - \frac{1}{k}\}$ for $k \in \mathbb{N}$. From the above estimates and Hölder's inequality we get that

$$\|f_n \circ \phi\|_2^2 = E_k\, |f_n \circ \phi|^2 dm + \int_{\mathbb{T}\setminus E_k} |f_n \circ \phi|^2 dm$$

$$\le \left(\int_{E_k} |f_n \circ \phi|^4 dm\right)^{1/2} \sqrt{m(E_k)} + \int_{\mathbb{T}\setminus E_k} |f_n \circ \phi|^2 dm.$$

Since $\int_{\mathbb{T}\setminus E_k} |f_n \circ \phi|^2 dm \to 0$ for each k as $n \to \infty$, we obtain that

$$\lim_{n\to\infty} \sup \|f_n \circ \phi\|_2^2 \le C\sqrt{m(E_k)}$$

for some constant $C > 0$ independent of $k \in \mathbb{N}$. By letting $k \to \infty$ and using that $m(E_\phi) = 0$ we deduce that $\lim_{n\to\infty} \|f_n \circ \phi\|_2 = 0$.

By [69] there is a subsequence $(f_{n_k} \circ \phi)$ such that

$$\left\|\sum_{k=1}^{\infty} c_k C_\phi(f_{n_k})\right\|_{BMOA} \sim \|c\|_{\ell^\infty}$$

holds for all $c = (c_k) \in c_0$. Obviously this contradicts (60).

We next indicate a different approach towards a weaker version of Theorem (6.3.4), which highlights a connection to the following general interpolation-extrapolation theorem for strictly singular operators on L^p-spaces due to Hernández et al. [217]: Let $1 \leq p < q \leq \infty$, and assume that the linear operator $\mathbb{T}$ is bounded $L^p \to L^p$ and $L^q \to L^q$. Moreover, suppose further that there is $r \in (p,q)$ for which $T \in S(L^r)$. Then $T \in K(L^s)$ for all $p < s < q$.

To apply the above result suppose that $C_\phi \in S(H^p)$, where $1 < p < \infty$. Recall from that the related operator $f \to \widetilde{C_\phi}(f) = f \circ \phi$ is bounded on the harmonic Hardy space h^p for $1 < p < \infty$, and that (58) holds with respect to $h^p = H^p \oplus \overline{H_0^p}$ provided $\phi(0) = 0$. It follows from (58) that $\widetilde{C_\phi} \in S(h^p)$, since $S(h^p)$ is a linear subspace. Fix q and r such that $1 < q < r < \infty$. Since $\widetilde{C_\phi}$ is bounded $h^t \to h^t$ for any $t \in (1,\infty)$ and $\widetilde{C_\phi} \in S(h^p)$, the above extrapolation result applied to $h^t = L^t(T,m)$ yields that $\widetilde{C_\phi} \in K(h^s)$ for any $q < r$. In particular, $C_\phi \in K(H^s)$ for any $q < r$ by restricting to $H^s \subset h^s$. Hence we have deduced by different means the following weak version of Theorem (6.3.4): if $C_\phi \in S(H^p)$, then $C_\phi \in K(H^p)$ for $1 < p < \infty$.

Above we do not address the technical issue that [217] only explicitly deals with real L^p-spaces, whereas the above application requires complex scalars. (We are indebted to Francisco Hernández for indicating that there is indeed also a complex version.) leave the above alternative here as an incomplete digression, because it is not possible to obtain the full strength of Theorem (6.3.4) in this way (cf. the following example).

Example (6.3.14)[231]: We point out for completeness that the extrapolation result [217] for $S(L^p) = S_p(L^p) \cap S_2(L^p)$ does not have an analogue for the classes $S_p(L^p)$ or $S_2(L^p)$. In fact, let (r_n) be the sequence of Rademacher functions on $[0,1]$ and $f \to Pf = \sum_{n=1}^{\infty} (f,r_n)r_n$ the canonical projection $L^p \to M$ for $1 < p < \infty$, where $M = \overline{\text{span}}\{r_n : n \in \mathbb{N}\}$. Since $M \approx \ell^2$ by the Khinchine inequalities, see e.g. [223], it follows that $P \in S_p(L^p)$ by the total incomparability of ℓ^p and ℓ^2 for $p \neq 2$. Furthermore, the results (in particular, see Example (6.3.10) and (58)) imply that for $p \neq 2$ there are composition operators $\widetilde{C_\phi} \in S_2(h^p)$ which fail to be compact.

Our results suggest several natural questions.

Problems (6.3.15)[231]: (i) Are there results corresponding to our main theorems for $C_\phi \in \mathcal{L}(H^p,H^q)$ in the case $p = q$? Note that the conditions for boundedness and compactness of $C_\phi : H^p \to H^q$ are different in the respective cases $p < q$ and $p,q > 2$ and $p = q$, then $S(L^p,L^q) = S_2(L^p,L^q)$ but $S_p(L^p,L^q) = \mathcal{L}(L^p,L^q)$. These equalities follow from the Kadec–Pełcynski dichotomy [88] and the total incomparability of ℓ^p and ℓ^q.

(ii) Is there an analogue of Theorem (6.3.11) for $p = 1$?

(iii) Is the converse of Proposition (6.3.13). (i) also true?

(iv) Is there a Banach space E of scalar-valued analytic functions on $\mathbb{D}$ and an analytic map $\phi : \mathbb{D} \to \mathbb{D}$, for which $C_\phi \in S(E) \setminus K(E)$? In this direction Lefèvre et al. [63] found a non-reflexive Hardy–Orlicz space H^ψ so that $C_\phi \in W(H^\psi) \setminus K(H^\psi)$, where ϕ is a lens map. The approach sketched suggests that weaker rigidity properties such as (59) are likely to hold for many other concrete classes of operators on H^p. Subsequently Miihkinen [211] has used similar techniques to show that the dichotomy of Theorem (6.3.4) remains valid for the class of analytic Volterra operators T_g on H^p, where

$$f \to (T_g(f))(z) = \int_0^z f(\tau)g'(\tau)d\tau, z \in \mathbb{D}.$$

See [212] or [225] for the conditions on the fixed analytic map $g : \mathbb{D} \to \mathbb{C}$ which characterise the boundedness or compactness of T_g.

Theorem (6.3.16)[245]: Let $0 \le \epsilon < \infty, \epsilon \ne 1$, and $(\varphi_1 + \varphi_2)$ be sum of any analytic self-maps of D. Then there are three mutually exclusive alternatives:

(i) $C_{(\varphi_1+\varphi_2)}$ is compact on $H^{1+\epsilon}$,

(ii) $C_{(\varphi_1+\varphi_2)}$ fixes a copy of $\ell^{1+\epsilon}$ in $H^{1+\epsilon}$, but does not fix any copies of ℓ^2 in $H^{1+\epsilon}$,

(iii) $C_{(\varphi_1+\varphi_2)}$ fixes a copy of ℓ^2 (as well as of $\ell^{1+\epsilon}$) in $H^{1+\epsilon}$. In this case, if $0 < \epsilon < \infty$ and $\epsilon \ne 1$, then $C_{(\varphi_1+\varphi_2)}$ also fixes a copy of $L^{1+\epsilon}(0,1)$ in $H^{1+\epsilon}$.

Furthermore, regarding the above alternatives

(i) takes place if and only if Shapiro's condition (47) holds,

(ii) takes place if and only if (47) fails to hold but $m(E_{(\varphi_1+\varphi_2)}) = 0$,

(iii) takes place if and only if $m(E_{(\varphi_1+\varphi_2)}) > 0$.

In particular, $C_{(\varphi_1+\varphi_2)} \in (1+\epsilon)(H^{1+\epsilon})$ if and only if $C_{(\varphi_1+\varphi_2)}$ does not fix any copies of $\ell^{1+\epsilon}$ in $H^{1+\epsilon}$.

Proof. For $a^2 \in \mathbb{D}$ and fixed $0 \le \epsilon < \infty$ let

$$g_{a^2}(z) = \left(1 - \frac{|a^2|^2}{\left(1 - \overline{a}^2 z\right)^2}\right)^{\frac{1}{1+\epsilon}}, \qquad z \in \mathbb{D}.$$

Observe that if $\gamma_{a^2}(z) = \frac{\left(1-|a^2|^2\right)^{\frac{1}{1+\epsilon}}}{1-\overline{a}^2 z}$ is the normalized reproducing kernel of H^2 associated to $a^2 \in \mathbb{D}$, then $|g_{a^2}(z)|^{1+\epsilon} = |\gamma_{a^2}(z)|^2$ for $z \in \mathbb{D}$, so that $\|g_{a^2}\|_{1+\epsilon} = 1$. The proof of Theorem (6.3.17) is based on the following criterion: $C_{(\varphi_1+\varphi_2)} \in (1+\epsilon)(H^{1+\epsilon})$ if and only if

$$\lim_{|a^2|\to 1} \sup \left\|C_{(\varphi_1+\varphi_2)}(g_{a^2})\right\|_{1+\epsilon} = 0. \tag{61}$$

Theorem (6.3.17)[245]: Let $0 \le \epsilon < \infty$ and let $(\varphi_1 + \varphi_2) : \mathbb{D} \to \mathbb{D}$ be sum of any analytic maps. Then either $C_{(\varphi_1+\varphi_2)} \in (1+\epsilon)(H^{1+\epsilon})$, or else $C_{(\varphi_1+\varphi_2)} \notin S_{1+\epsilon}(H^{1+\epsilon})$. Equivalently, $C_{(\varphi_1+\varphi_2)}$ fixes a copy of $\ell^{1+\epsilon}$ in $H^{1+\epsilon}$ if and only if (47) does not hold.

Proof. Suppose that $C_{(\varphi_1+\varphi_2)} \notin (1+\epsilon)(H^{1+\epsilon})$, where $0 \le \epsilon < \infty$. We will show by an explicit perturbation argument that $C_{(\varphi_1+\varphi_2)}$ fixes a linearly isomorphic copy of $\ell^{1+\epsilon}$ in $H^{1+\epsilon}$. Since condition (61) fails there is $\epsilon \ge 0$ and a sequence $(a_n^2) \subset \mathbb{D}$ so that $|a_n^2| \to 1$ as $n \to \infty$ and

$$\left\|C_{(\varphi_1+\varphi_2)}\left(g_{a_n^2}\right)\right\|_{1+\epsilon} \ge \epsilon \ge 0 \tag{62}$$

for all $n \in \mathbb{N}$. We may further assume without loss of generality that $a_n^2 \to 1$ as $n \to \infty$. Namely, we may pass to a convergent subsequence in $\mathbb{D}$ and compose $(\varphi_1 + \varphi_2)$ with a suitable rotation of $\overline{\mathbb{D}}$ that defines a linear isomorphism of $H^{1+\epsilon}$. The starting point is the phenomenon that $(g_{a_n^2})$ admits subsequences which are small perturbations of a disjointly supported sequence in $L^{1+\epsilon}(T,m)$, and hence span an isomorphic copy of $\ell^{1+\epsilon}$. The crux of the argument is that this can be achieved simultaneously for further subsequences of $\left(C_{(\varphi_1+\varphi_2)}\left(g_{a_n^2}\right)\right)$, and (see [231]):

Lemma (6.3.18)[245]: Let $(\varphi_1+\varphi_2)$ and $\left(g_{a_n^2}\right)$ be as above, where $a_n^2 \to 1$ as $n \to \infty$. Then

(L1) $\int_{\mathbb{T}\setminus E_\varepsilon} \left|C_{(\varphi_1+\varphi_2)}\left(g_{a_n^2}\right)\right|^{1+\epsilon} dm \to 0$ as $n \to \infty$ for each fixed $\varepsilon > 0$,

(L2) $\int_{E_\varepsilon} \left|C_{(\varphi_1+\varphi_2)}\left(g_{a_n^2}\right)\right|^{1+\epsilon} dm \to 0$ as $\varepsilon \to 0$ for each fixed $n \in \mathbb{N}$.

Proof. Observe first that

$$\int_{E_\varepsilon} \left|C_{(\varphi_1+\varphi_2)}(g)\right|^{1+\epsilon} dm \to 0$$

as $\varepsilon \to 0$ for any $g \in H^{1+\epsilon}$, since $\cap_{\varepsilon>0} E_\varepsilon = \{\xi \in A : (\varphi_1+\varphi_2)(\xi) = 1\}$ has measure 0 as $(\varphi_1+\varphi_2)$ is not identically 1. Moreover, if $\varepsilon > 0$ is fixed and $\xi \in A \setminus E_\varepsilon$, then there is n_ε such that

$$\left|1-\overline{a_n^2}(\varphi_1+\varphi_2)(\xi)\right| = \left|1-(\varphi_1+\varphi_2)(\xi)+(\varphi_1+\varphi_2)(\xi)\left(1-\overline{a_n^2}\right)\right|$$
$$\geq |1-(\varphi_1+\varphi_2)(\xi)| - |1-a_n^2| > \frac{\varepsilon}{2}$$

for all $n \geq n_\varepsilon$. It follows that

$$\left|C_{(\varphi_1+\varphi_2)}\left(g_{a_n^2}\right)(\xi)\right|^{1+\epsilon} = \frac{1-|a_n^2|^2}{\left|1-\overline{a_n^2}(\varphi_1+\varphi_2)(\xi)\right|^2} \leq \frac{4(1-|a_n^2|^2)}{\varepsilon^2},$$

so that (L1) holds as $n \to \infty$.

To continue the argument of Claim (6.3.5) recall that $\int_{\mathbb{T}} \left|C_{(\varphi_1+\varphi_2)}\left(g_{a_n^2}\right)\right|^{1+\epsilon} dm \geq (1+\epsilon)^{1+\epsilon} > 0$ by condition (62). We may then use Lemma (6.3.18) inductively to find indices $j_1 < j_2 < \ldots$ and a decreasing sequence $\varepsilon_j > \varepsilon_{j+1} \to 0$ so that

(i) $\left(\int_{E_{\varepsilon_n}} \left|C_{(\varphi_1+\varphi_2)}\left(g_{a_{j_k}^2}\right)\right|^{1+\epsilon} dm\right)^{\frac{1}{1+\epsilon}} < 4^{-n}\delta(1+\epsilon)$ for all $k = 1,\ldots,n-1$,

(ii) $\left(\int_{\mathbb{T}\setminus E_{\varepsilon_n}} \left|C_{(\varphi_1+\varphi_2)}\left(g_{a_{j_n}^2}\right)\right|^{1+\epsilon} dm\right)^{\frac{1}{1+\epsilon}} < 4^{-n}\delta(1+\epsilon)$,

(iii) $\left(\int_{E_{\varepsilon_n}} \left|C_{(\varphi_1+\varphi_2)}\left(g_{a_{j_n}^2}\right)\right|^{1+\epsilon} dm\right)^{\frac{1}{1+\epsilon}} > \frac{1+\epsilon}{2}$

for all $n \in \mathbb{N}$.Here $\delta > 0$ is a small enough constant (to be chosen later). In fact, suppose that we have already found $a_{j_1}^2,\ldots,a_{j_{n-1}}^2$ and $\varepsilon_1 > \ldots > \varepsilon_{n-1}$ satisfying (i)–(iii). Then property (L2) from Lemma (6.3.18) yields $\varepsilon_n < \varepsilon_{n-1}$ such that

$$\left(\int_{E_{\varepsilon_n}} \left|C_{(\varphi_1+\varphi_2)}\left(g_{a_{j_k}^2}\right)\right|^{1+\epsilon} dm\right)^{\frac{1}{1+\epsilon}} < 4^{-n}\delta(1+\epsilon)$$

for each $k = 1,\ldots,n-1$. After this use property (L1) from Lemma (6.3.18) together with (62) to find an index $j_n > j_{n-1}$ so that conditions (ii) and (iii) are satisfied for the set E_{ε_n} .

In the interest of notational simplicity we relabel $a_{j_n}^2$ as a_n^2 for $n \in \mathbb{N}$. The idea of the argument is that the sequence $\left(C_{(\varphi_1+\varphi_2)}\left(g_{a_n^2}\right)\right)$ essentially resembles disjointly supported peaks in $L^{1+\epsilon}(T,m)$ close to the point 1. We will next verify the left-hand inequality in (53) by a direct perturbation argument. Let $b = (b_j^2) \in \ell^{1+\epsilon}$ be arbitrary. Our starting point will be the identity

$$\left\|\sum_{j=1}^{\infty} b_j^2 C_{(\varphi_1+\varphi_2)}\left(g_{a_j^2}\right)\right\|_{1+\epsilon}^{1+\epsilon} = \sum_{n=0}^{\infty} \int_{E_{\varepsilon_n}\setminus E_{\varepsilon_{n+1}}} \left|\sum_{j=1}^{\infty} b_j^2 C_{(\varphi_1+\varphi_2)}\left(g_{a_j^2}\right)\right|^{1+\epsilon} dm, \quad (63)$$

where we set $E_{\varepsilon_0} = T$.

Observe first that for each $n \in \mathbb{N}$ we get that

$$\left(\int_{E_{\varepsilon_n}\setminus E_{\varepsilon_{n+1}}} \left| C_{(\varphi_1+\varphi_2)}(g_{a_n^2})\right|^{1+\epsilon} dm\right)^{\frac{1}{1+\epsilon}} = \left(\int_{E_{\varepsilon_n}} \left|C_{(\varphi_1+\varphi_2)}(g_n)\right|^{1+\epsilon} dm - \int_{E_{\varepsilon_{n+1}}} \left|C_{(\varphi_1+\varphi_2)}(g_{a_n^2})\right|^{1+\epsilon} dm\right)^{\frac{1}{1+\epsilon}}$$
$$> \left(\left(\frac{1+\epsilon}{2}\right)^{1+\epsilon} - (4^{-n-1}\delta(1+\epsilon))^{1+\epsilon}\right)^{\frac{1}{1+\epsilon}} \geq \frac{1+\epsilon}{2} - 4^{-n-1}\delta(1+\epsilon)$$

in view of (i) and (iii), where the last estimate holds because $0 \leq \epsilon$. Moreover, note that

$$\left(\int_{E_{\varepsilon_n}\setminus E_{\varepsilon_{n+1}}} \left|C_{(\varphi_1+\varphi_2)}\left(g_{a_j^2}\right)\right|^{1+\epsilon} dm\right)^{\frac{1}{1+\epsilon}} < 2^{-n-j}\,\delta(1+\epsilon)$$

for all $j \neq n$. In fact, $\left(\int_{E_{\varepsilon_n}\setminus E_{\varepsilon_{n+1}}} \left|C_{(\varphi_1+\varphi_2)}\left(g_{a_j^2}\right)\right|^{1+\epsilon} dm\right)^{\frac{1}{1+\epsilon}}$ is dominated by $4^{-n}\delta(1+\epsilon)$ for j_n in view of (i) and (ii). Thus we get from the triangle inequality in $L^{1+\epsilon}$, together with the preceding estimates, that for all $n \in \mathbb{N}$ one has

$$\left(\int_{E_{\varepsilon_n}\setminus E_{\varepsilon_{n+1}}} \left|\sum_{j=1}^{\infty} b_j^2 C_{(\varphi_1+\varphi_2)}\left(g_{a_j^2}\right)\right|^{1+\epsilon} dm\right)^{\frac{1}{1+\epsilon}}$$
$$\geq |b_n^2|\left(\int_{E_{\varepsilon_n}\setminus E_{\varepsilon_{n+1}}} \left| C_{(\varphi_1+\varphi_2)}(g_{a_n^2})\right|^{1+\epsilon} dm\right)^{\frac{1}{1+\epsilon}}$$
$$-\sum_{j\neq n} |b_j^2|\left(\int_{E_{\varepsilon_n}\setminus E_{\varepsilon_{n+1}}} \left|C_{(\varphi_1+\varphi_2)}\left(g_{a_j^2}\right)\right|^{1+\epsilon} dm\right)^{\frac{1}{1+\epsilon}}$$
$$\geq |b_n^2|\left(\frac{1+\epsilon}{2} - 4^{-n-1}\delta(1+\epsilon)\right) - 2^{n}\,\delta(1+\epsilon)\|b\|_{1+\epsilon}$$
$$\geq \frac{1+\epsilon}{2} - 2^{-n+1}\delta(1+\epsilon)\|b\|_{b}.$$

By summing over n we get from the disjointness and the triangle inequality in $\ell^{1+\epsilon}$ that

$$\left\|\sum_{j=1}^{\infty} b_j^2 C_{(\varphi_1+\varphi_2)}\left(g_{a_j^2}\right)\right\|_{1+\epsilon} \geq \left(\sum_{n=1}^{\infty} \left|\frac{1+\epsilon}{2}|b_n^2| - 2^{-n+1}\delta(1+\epsilon)\|b\|_{1+\epsilon}\right|^{1+\epsilon}\right)^{\frac{1}{1+\epsilon}}$$

$$\geq \frac{1+\epsilon}{2}\left(\sum_{n=1}^{\infty} |b_n^2|^{1+\epsilon}\right)^{\frac{1}{1+\epsilon}} - \delta(1+\epsilon)\|b\|_{1+\epsilon}\left(\sum_{n=1}^{\infty} 2^{(-n-1)}\right)^{\frac{1}{1+\epsilon}}$$

$$(1+\epsilon)\left(\frac{1}{2} - \delta \cdot (1 - 2^{1+\epsilon})^{-\frac{1}{1+\epsilon}}\right)\|b\|_{1+\epsilon} \geq \frac{1+\epsilon}{4}\|b\|_{1+\epsilon},$$

Note that: By combining (53) and (54) we have

(i) $$\left\|\sum_{j=1}^{\infty} b_j^2 C_{(\varphi_1+\varphi_2)}\left(g_{a_j^2}\right)\right\|_{1+\epsilon} = \left\|\sum_{j=1}^{\infty} b_j^2 g_{a_j^2}\right\|_{1+\epsilon} \geq \frac{1+\epsilon}{2} - 4^{-(n+1)}\delta(1+\epsilon)$$

Note that:

(ii) $$\left\|\sum_{j=1}^{\infty} b_j^2 g_{a_j^2}\right\|_{1+\epsilon}^{1+\epsilon} \geq \frac{1+\epsilon}{2} - 4^{-(n+1)}\delta(1+\epsilon)$$

(iii) $$\left\|\sum_{j=1}^{\infty} b_j^2 g_{a_j^2}\right\|_{1+\epsilon} \geq \frac{1+\epsilon}{4}\|b\|_{1+\epsilon}$$

(iv) By combining (ii) and (iii) we get

$$\|b\|_{1+\epsilon} \geq \left(2 - 4^{-n}\frac{\delta(1+\epsilon)}{1+\epsilon}\right)$$

where the last estimate holds once we choose $\delta > 0$ small enough, so that $\delta \cdot (1 - 2^{1+\epsilon})^{-\frac{1}{1+\epsilon}}$. The proof of the right-hand inequality in (53) is a straightforward variant of the preceding estimates. This inequality does not affect the choice of $\delta > 0$, and hence the details will be omitted here.

Lemma (6.3.19)[245]: Suppose that condition (48) fails, that is, $m(E_{(\varphi_1+\varphi_2)}) > 0$. Then there exist integers $0 \leq n_1 < n_2 < \cdots$ and a constant $\epsilon \geq 0$ such that

$$(1+\epsilon)^{-1} \cdot \|1+\epsilon\|_{\ell^2} \leq \left\|\sum_{\epsilon=0}^{\infty} (1+\epsilon)_k(\varphi_1+\varphi_2)^{n_k}\right\|_2 \leq (1+\epsilon) \cdot \|1+\epsilon\|_{\ell^2}$$

for all $1+\epsilon = ((1+\epsilon)_{1+2\epsilon}) \in \ell^2$.

Proof. The upper estimate follows from the boundedness of $C_{(\varphi_1+\varphi_2)}$ on H^2 and the orthonormality of the sequence (z_n) in H^2.

To establish the lower estimate, note that $z_n \to 0$ weakly and therefore also $(\varphi_1+\varphi_2)_n = C_{(\varphi_1+\varphi_2)}(z_n) \to 0$ weakly in H^2 as $n \to \infty$. Hence we may set $n_1 = 0$ and then proceed inductively to pick increasing indices $n_{1+2\epsilon}$ such that the inner-products satisfy $|(\varphi_1+\varphi_2)^{n_{1+\epsilon}}, (\varphi_1+\varphi_2)^{n_{1+2\epsilon}}| \leq 2^{-2(1+2\epsilon)}m\left(E_{(\varphi_1+\varphi_2)}\right)$ for all $\epsilon > 0$ and each $(1+2\epsilon) \in \mathbb{N}$. Let $1+\epsilon = ((1+\epsilon)_{1+2\epsilon}) \in \ell^2$be arbitrary and note that

$$\left\|\sum_{\epsilon=0}^{\infty} (1+\epsilon)_{1+2\epsilon}(\varphi_1+\varphi_2)^{n_{1+2\epsilon}}\right\|_2^2$$
$$= \sum_{\epsilon=0}^{\infty} |(1+\epsilon)_{1+2\epsilon}|^2 \|(\varphi_1+\varphi_2)^{n_{1+2\epsilon}}\|_2^2$$
$$+ 2\mathrm{Re}\sum_{\epsilon=0}^{\infty}\sum_{\epsilon=0}^{2\epsilon} (1+\epsilon)_{1+\epsilon}\,\overline{(1+\epsilon)}_{1+2\epsilon}((\varphi_1+\varphi_2)^{n_{1+\epsilon}},(\varphi_1+\varphi_2)^{n_{1+2\epsilon}}).$$

Obviously $\|(\varphi_1+\varphi_2)^{n_{1+2\epsilon}}\|_2^2 \geq \int_{E_{(\varphi_1+\varphi_2)}} |(\varphi_1+\varphi_2)^{n_{1+2\epsilon}}|^2 dm = m\big(E_{(\varphi_1+\varphi_2)}\big)$ for each $1+2\epsilon$. Moreover, we get that

$$\sum_{\epsilon=0}^{\infty}\sum_{\epsilon=0}^{2\epsilon} (1+\epsilon)_{1+\epsilon}\,\overline{(1+\epsilon)}_{1+2\epsilon}((\varphi_1+\varphi_2)^{n_{1+\epsilon}},(\varphi_1+\varphi_2)^{n_{1+2\epsilon}})$$
$$\leq \|1+\epsilon\|_{\ell^2}^2 \sum_{\epsilon=0}^{\infty}\sum_{\epsilon=0}^{2\epsilon} 2^{-2(1+2\epsilon)} m\big(E_{(\varphi_1+\varphi_2)}\big)$$
$$\leq \frac{1}{2}\|1+\epsilon\|_{\ell^2}^2 m\big(E_{(\varphi_1+\varphi_2)}\big)\sum_{\epsilon=}^{\infty}\sum_{\epsilon=0}^{2\epsilon} 2^{-2(1+2\epsilon)} = \frac{1}{6}\|1+\epsilon\|_{\ell^2}^2 m\big(E_{(\varphi_1+\varphi_2)}\big)$$

By combining these estimates we obtain the desired lower bound

$$\left\|\sum_{\epsilon=0}^{\infty} (1+\epsilon)_{1+2\epsilon}(\varphi_1+\varphi_2)^{n_{1+2\epsilon}}\right\|_2^2$$
$$\geq \|1+\epsilon\|_{\ell^2}^2 m\big(E_{(\varphi_1+\varphi_2)}\big) - \frac{1}{3}\|1+\epsilon\|_{\ell^2}^2 m\big(E_{(\varphi_1+\varphi_2)}\big)$$
$$= \left(\frac{2}{3} m\big(E_{(\varphi_1+\varphi_2)}\big)\right)\|1+\epsilon\|_{\ell^2}^2.$$

In order to treat general $0 \leq \epsilon < \infty$ recall that the analytic maps $f: \mathbb{D} \to \mathbb{C}$ belongs to *BMOA* if

$$|f|_* = \sup_{a^2\in\mathbb{D}} \|f \circ \sigma_{a^2} - f(a^2)\|_2 < \infty,$$

where $\sigma_{a^2}(z) = \frac{a^2-z}{1-\overline{a^2}z}$ is the Möbius-automorphism of $\mathbb{D}$ interchanging 0 and a^2 for $a^2 \in \mathbb{D}$. The Banach space BMOA is normed by $\|f\|_{BMOA} = |f(0)| + |f|_*$. Moreover, $VMOA$ is the closed subspace of $BMOA$, where $f \in VMOA$ if

$$\lim_{|a^2|\to 1} \|f \circ \sigma_{a^2} - f(a^2)\|_2 = 0.$$

We refer to e.g. [236] and [126] for background on $BMOA$. It follows readily from Littlewood's subordination theorem that $C_{(\varphi_1+\varphi_2)}$ is bounded $BMOA \to BMOA$ for sum of any analytic maps $(\varphi_1+\varphi_2): \mathbb{D} \to \mathbb{D}$, see e.g. [123].

Proposition (6.3.20)[245]: Let $0 \leq \epsilon < \infty$ and suppose that $m(E_{(\varphi_1+\varphi_2)}) > 0$. Then there exist increasing integers $0 \leq n_1 < n_2 < \cdots$ such that the subspace

$$M = \overline{\mathrm{span}}\{z^{n_{1+\epsilon}} : \epsilon \geq 0\} \subset H^{1+\epsilon}$$

is isomorphic to 2 and the restriction $C_{(\varphi_1+\varphi_2)}|_M$ is bounded below on M. Hence $C_{(\varphi_1+\varphi_2)} \notin S_2(H^{1+\epsilon})$.

Proof. We start by choosing the increasing integers $(n_{1+\epsilon})$ as in Lemma (6.3.19). By passing to a subsequence we may also assume that $(z^{n_{1+\epsilon}})$ is a lacunary sequence, that is, $\inf_{1+\epsilon} (n_{2+\epsilon}/n_{1+\epsilon}) > 1$. Paley's theorem (see e.g. [201]) implies that for $0 \leq \epsilon < \infty$ the sequence $(z^{n_{1+\epsilon}})$ is equivalent in $H^{1+\epsilon}$ to the unit vector basis of 2, that is,

$$\left\| \sum_{\epsilon=0}^{\infty} (1+\epsilon)_{1+\epsilon} z^{n_{1+\epsilon}} \right\|_{1+\epsilon} \sim \|1+\epsilon\|_{\ell^2} \quad (64)$$

for all $1+\epsilon = ((1+\epsilon)_{1+\epsilon}) \in \ell^2$. (Here, and in the sequel, we use $\sim$ as a short-hand notation for the equivalence of the respective norms.) Case $\epsilon \geq 0$. By Hölder's inequality and Lemma (6.3.19) we have that

$$\left\| C_{(\varphi_1+\varphi_2)} \sum_{\epsilon=0}^{\infty} (1+\epsilon)_{1+\epsilon} z^{n_{1+\epsilon}} \right\|_{2+\epsilon} = \left\| \sum_{\epsilon=0}^{\infty} (1+\epsilon)_{1+\epsilon} (\varphi_1+\varphi_2)^{n_{1+\epsilon}} \right\|_{2+\epsilon}$$
$$\geq \left\| \sum_{\epsilon=0}^{\infty} (1+\epsilon)_{1+\epsilon} (\varphi_1+\varphi_2)^{n_{1+\epsilon}} \right\|_{2} \sim \|1+\epsilon\|_{\ell^2} .$$

According to (64) and the boundedness of $C_{(\varphi_1+\varphi_2)}$ this proves the claim for $\epsilon \geq 0$.

Case $0 \leq \epsilon < 1$. We start by invoking a version of Paley's theorem for $BMOA$ (see e.g. [126]), which together with the boundedness of $C_{(\varphi_1+\varphi_2)}$ on $BMOA$ ensures that

$$\left\| \sum_{\epsilon=0}^{\infty} (1+\epsilon)_{1+\epsilon} (\varphi_1+\varphi_2)^{n_{1+\epsilon}} \right\|_{BMOA} = \left\| C_{(\varphi_1+\varphi_2)} \right\| \cdot \left\| \sum_{\epsilon=0}^{\infty} (1+\epsilon)_{1+\epsilon} z^{n_{1+\epsilon}} \right\|_{BMOA}$$
$$\leq (1+\epsilon) \cdot \left\| C_{(\varphi_1+\varphi_2)} \right\| \cdot \|1+\epsilon\|_{\ell^2}$$

for all $1+\epsilon = ((1+\epsilon)_{1+\epsilon}) \in \ell^2$ and a uniform constant $\epsilon \geq 0$. In view of Fefferman's $H^1 - BMOA$ duality pairing (see e.g. [126]) we may further estimate

$$\left\| \sum_{\epsilon=0}^{\infty} (1+\epsilon)_{1+\epsilon} (\varphi_1+\varphi_2)^{n_{1+\epsilon}} \right\|_{BMOA} = \left\| \sum_{\epsilon=0}^{\infty} (1+\epsilon)_{1+\epsilon} (\varphi_1+\varphi_2)^{n_{1+\epsilon}} \right\|_{1}$$
$$\leq \left| \left(\sum_{\epsilon=0}^{\infty} (1+\epsilon)_{1+\epsilon} (\varphi_1+\varphi_2)^{n_{1+\epsilon}}, \sum_{\epsilon=0}^{\infty} (1+\epsilon)_{1+\epsilon} (\varphi_1+\varphi_2)^{n_{1+\epsilon}} \right) \right|$$
$$= \left\| \sum_{\epsilon=0}^{\infty} (1+\epsilon)_{1+\epsilon} (\varphi_1+\varphi_2)^{n_{1+\epsilon}} \right\|_{2}^{2} \sim \|1+\epsilon\|_{\ell^2}$$

where we again use Lemma (6.3.19) at the final step. By applying Hölder's inequality and combining the preceding estimates we obtain that

$$\left\| \sum_{\epsilon=0}^{\infty} (1+\epsilon)_{1+\epsilon} (\varphi_1+\varphi_2)^{n_{1+\epsilon}} \right\|_{1+\epsilon} \geq \left\| \sum_{\epsilon=0}^{\infty} (1+\epsilon)_{1+\epsilon} (\varphi_1+\varphi_2)^{n_{1+\epsilon}} \right\|_{1} \geq (1+\epsilon)\|1+\epsilon\|_{\ell^2}$$

for some uniform constant $\epsilon \geq 0$. In particular, $C_{(\varphi_1+\varphi_2)} \notin S_2(H^{1+\epsilon})$ in view of (64), which completes the verification of the proposition for $0 \leq \epsilon < 1$.

Proposition (6.3.21)[245]: Let $0 \leq \epsilon < \infty, \epsilon \neq 1$, and suppose that $m(E_{\varphi_1+\varphi_2}) = 0$. If (f_n) is any normalized sequence in $H^{1+\epsilon}$ which is equivalent to the unit vector basis of ℓ^2, then $C_{(\varphi_1+\varphi_2)}$ is not bounded below on $\overline{\text{span}}\{f_n : n \in \mathbb{N}\} \subset H^{1+\epsilon}$. In particular, $C_{(\varphi_1+\varphi_2)} \in S_2(H^{1+\epsilon})$.

Proof. Assume to the contrary that

$$\left\|\sum_{n=1}^{\infty}(1+\epsilon)_n\, C_{(\varphi_1+\varphi_2)}(f_n)\right\|_{1+\epsilon} \sim \left\|\sum_{n=1}^{\infty}(1+\epsilon)_{1+\epsilon}(\varphi_1+\varphi_2)^{n_{1+\epsilon}}\right\|_{1+\epsilon} \sim \|1+\epsilon\|_{\ell^2} \quad (65)$$

for all sequences $1+\epsilon = ((1+\epsilon)_n) \in \ell^2$. In particular, $C_{(\varphi_1+\varphi_2)}(f_n)^{1+\epsilon} \geq 1+\epsilon > 0$ for all n and some constant $1+\epsilon$. We write $E_{1+\epsilon} = \{e^{i(\theta_1^2+\theta_2^2)} : \left|(\varphi_1+\varphi_2)\left(e^{i(\theta_1^2+\theta_2^2)}\right)\right| \geq \frac{\epsilon}{1+\epsilon}\}$ for $\epsilon \geq 0$. Since $\lim_{\epsilon\to\infty} m(E_{1+\epsilon}) = m(E_{(\varphi_1+\varphi_2)}) = 0$, we get that

$$\lim_{\epsilon\to\infty}\int_{\mathbb{T}\setminus E_{1+\epsilon}} E_{1+\epsilon}\left|C_{(\varphi_1+\varphi_2)}(f_n)\right|^{1+\epsilon} dm = 0 \text{ for all } n.$$

On the other hand, $f_n \to 0$ weakly in $H^{1+\epsilon}$ and hence $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$. This implies that

$$\lim_{n\to\infty}\int_{\mathbb{T}\setminus E_{1+\epsilon}}\left|C_{(\varphi_1+\varphi_2)}(f_n)\right|^{1+\epsilon} dm = 0 \text{ for all } 1+\epsilon.$$

By using the above properties and proceeding recursively in a fashion similar to the argument for Theorem (6.3.17) we find increasing sequences of integers $0 \leq n_1 < n_2 < \cdots$ and $0 = \epsilon < 2\epsilon < \cdots$, such that

$$\left\|\sum_{n=1}^{\infty}(1+\epsilon)_n C_{(\varphi_1+\varphi_2)}(f_n)\right\|_{1+\epsilon}^{1+\epsilon} = \sum_{l=1}^{\infty}\int_{E_{(1+\epsilon)_l}\setminus E_{2+\epsilon}}\left|\sum_{\epsilon=0}^{\infty}(1+\epsilon)_n C_{(\varphi_1+\varphi_2)}(f_n)\right|^{1+\epsilon} dm \sim \|1+\epsilon\|_{\ell^2}$$

holds for all $1+\epsilon = ((1+\epsilon)_{1+\epsilon}) \in \ell^{1+\epsilon}$ with uniform constants. However, for $\epsilon \neq 1$ such estimates obviously contradict (65). Thus $C_{(\varphi_1+\varphi_2)} \in S_2(H^{1+\epsilon})$, and this completes the proof of the proposition (and hence also of Theorem (6.3.4)).

Theorem (6.3.22)[245]: Let $0 < \epsilon < \infty, \epsilon \neq 1$, and $(\varphi_1+\varphi_2) : \mathbb{D} \to \mathbb{D}$ be sum of an analytic maps. Then the following conditions are equivalent:

(i) $(\varphi_1+\varphi_2)$ satisfies $m(E_{(\varphi_1+\varphi_2)}) = 0$,

(ii) $C_{(\varphi_1+\varphi_2)} \in S_{L^{1+\epsilon}}(H^{1+\epsilon})$, that is, $C_{(\varphi_1+\varphi_2)}$ does not fix any copies of $L^{1+\epsilon}$ in $H^{1+\epsilon}$,

(iii) $C_{(\varphi_1+\varphi_2)} \in S_{L^{1+\epsilon}}(h^{1+\epsilon})$,

(iv) $C_{(\varphi_1+\varphi_2)} \in S_2(H^{1+\epsilon})$.

Proof. We may assume during the proof that $(\varphi_1+\varphi_2)(0) = 0$. In fact, otherwise consider $\psi = \sigma_{(\varphi_1+\varphi_2)}(0) \circ (\varphi_1+\varphi_2)$, where $\sigma_{(\varphi_1+\varphi_2)}(0) : \mathbb{D} \to \mathbb{D}$ is the automorphism interchanging 0 and $(\varphi_1+\varphi_2)(0)$. Then $\psi(0) = 0$ and $\tilde{C}_\psi = \tilde{C}_{(\varphi_1+\varphi_2)} \circ \tilde{C}_{\sigma_{(\varphi_1+\varphi_2)}}(0)$, where $\tilde{C}_{\sigma_{(\varphi_1+\varphi_2)}}(0)$ is a linear isomorphism $h^{1+\epsilon} \to h^{1+\epsilon}$ (as well as $H^{1+\epsilon} \to H^{1+\epsilon}$), which does not affect any of the claims of the theorem.

Proposition (6.3.23)[245]: (i) If $(\varphi_1+\varphi_2) : \mathbb{D} \to \mathbb{D}$ is sum of an analytic maps, and $C_{(\varphi_1+\varphi_2)} \in S_2(BMOA)$, then (48) holds (that is, $m\left(E_{(\varphi_1+\varphi_2)}\right) = 0$).

(ii) If $(\varphi_1+\varphi_2) \in VMOA$, then $C_{(\varphi_1+\varphi_2)} \in S_2(VMOA)$ if (and only if) (48) holds.

Proof. (i) The argument is essentially contained in that of Proposition (6.3.20). In fact, suppose that $m\left(E_{(\varphi_1+\varphi_2)}\right) > 0$, where $E_{(\varphi_1+\varphi_2)} = \left\{e^{i(\theta_1^2+\theta_2^2)} : \left|(\varphi_1+\varphi_2)\left(e^{i(\theta_1^2+\theta_2^2)}\right)\right| = 1\right\}$. Then the proof of the case $0 \leq \epsilon < 1$ of Proposition (6.3.20) gives a lacunary sequence $(n_{1+\epsilon})$ and constants $\epsilon > 0$ so that in the $H^1 - BMOA$ duality pairing

$$\left\|\sum_{\epsilon=0}^{\infty} (1+\epsilon)_{1+\epsilon}(\varphi_1+\varphi_2)^{n_{1+\epsilon}}\right\|_{BMOA} \geq \left\|\sum_{\epsilon=0}^{\infty} (1+\epsilon)_{1+\epsilon}(\varphi_1+\varphi_2)^{n_{1+\epsilon}}\right\|_1 \geq (1+\epsilon)\cdot\|1+\epsilon\|_{\ell^2}$$

as well as $\|\sum_{\epsilon=0}^{\infty} (1+\epsilon)_{1+\epsilon}(\varphi_1+\varphi_2)^{n_{1+\epsilon}}\|_1 \geq (1+2\epsilon)\cdot\|1+\epsilon\|_{\ell^2}$ for all $1+\epsilon = ((1+\epsilon)_{1+\epsilon}) \in \ell^2$. Since $C_{(\varphi_1+\varphi_2)}$ is bounded on $BMOA$ it follows as before from Paley's theorem in $BMOA$ that $C_{(\varphi_1+\varphi_2)}$ is bounded below on $\overline{\text{span}}\{z^{n_{1+\epsilon}} : (1+\epsilon) \in \mathbb{N}\} \approx 2$ in $BMOA$.

(ii) Recall that $C_{(\varphi_1+\varphi_2)} : VMOA \to VMOA$ if $(\varphi_1+\varphi_2) \in VMOA$, see e.g. [123]. Assume that $m\left(E_{(\varphi_1+\varphi_2)}\right) = 0$ and suppose to the contrary that there is a normalised sequence $(f_{1+\epsilon}) \subset VMOA$ equivalent to the unit vector basis of (ii), for which

$$\left\|\sum_{\epsilon=0}^{\infty} (1+\epsilon)_{1+\epsilon}C_{(\varphi_1+\varphi_2)}(f_{1+\epsilon})\right\|_{BMOA} \sim \cdot\|1+\epsilon\|_{\ell^2}$$

for all $(1+\epsilon) = ((1+\epsilon)_{1+\epsilon}) \in \ell^2$. In particular, $\|f_{1+\epsilon} \circ (\varphi_1+\varphi_2)\|_{BMOA} \geq 1+\epsilon > 0$ for all $(1+\epsilon)$, while $(f_{1+\epsilon})$ is weak-null sequence in $VMOA$, so that $f_{1+\epsilon} \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $\epsilon \to \infty$. Moreover, by the John–Nirenberg inequality there is a uniform constant $\epsilon \geq 0$ so that

$$\|f_{1+\epsilon} \circ (\varphi_1+\varphi_2)\|_4 \leq (1+\epsilon)\|f_{1+\epsilon} \circ (\varphi_1+\varphi_2)\|_{BMOA}, \qquad (1+\epsilon) \in \mathbb{N}.$$

Let $E_{(1+\epsilon)} = \{e^{i(\theta_1^2+\theta_2^2)} : \left|(\varphi_1+\varphi_2)\left(e^{i(\theta_1^2+\theta_2^2)}\right)\right| \geq \frac{\epsilon}{1+\epsilon}\}$ for $(1+\epsilon) \in \mathbb{N}$. From the above estimates and Hölder's inequality we get that

$$\|f_n \circ (\varphi_1+\varphi_2)\|_2^2 = E_{(1+\epsilon)} |f_n \circ (\varphi_1+\varphi_2)|^2 dm + \int_{\mathbb{T}\setminus E_{(1+\epsilon)}} |f_n \circ (\varphi_1+\varphi_2)|^2 dm$$
$$\leq \left(\int_{E_{1+\epsilon}} |f_n \circ (\varphi_1+\varphi_2)|^4 dm\right)^{1/2} \sqrt{m\left(E_{(1+\epsilon)}\right)} + \int_{\mathbb{T}\setminus E_{1+\epsilon}} |f_n \circ (\varphi_1+\varphi_2)|^2 dm.$$

Since $\int_{\mathbb{T}\setminus E_{1+\epsilon}} |f_n \circ (\varphi_1+\varphi_2)|^2 dm \to 0$ for each $(1+\epsilon)$ as $n \to \infty$, we obtain that

$$\lim_{n\to\infty} \sup \|f_n \circ (\varphi_1+\varphi_2)\|_2^2 \leq (1+\epsilon)\sqrt{m(E_{(1+\epsilon)})}$$

for some constant $\epsilon \geq 0$ independent of $(1+\epsilon) \in \mathbb{N}$. By letting $\epsilon \to \infty$ and using that $m\left(E_{(\varphi_1+\varphi_2)}\right) = 0$ we deduce that $\lim_{n\to\infty} \|f_n \circ (\varphi_1+\varphi_2)\|_2 = 0$.

By [69] there is a subsequence $\left(f_{n_{1+\epsilon}} \circ (\varphi_1+\varphi_2)\right)$ such that

$$\left\|\sum_{\epsilon=0}^{\infty} (1+\epsilon)_{1+\epsilon}C_{(\varphi_1+\varphi_2)}(f_{n_{1+\epsilon}})\right\|_{BMOA} \sim \|1+\epsilon\|_{\ell^\infty}$$

holds for all $(1+\epsilon) = ((1+\epsilon)_{1+\epsilon}) \in (1+\epsilon)_0$. Obviously this contradicts (60).

List of Symbols

Symbol		Page
A_α^p:	Bergman space	1
sup:	supremum	1
dist:	distance	2
arg:	argument	2
inf:	infimum	4
L^1:	Lebesgue integral on the line	29
L^p:	Lebesgue space	30
L^2:	Hilbert space	30
H^q:	Hardy space	32
A_α^2:	Bergman space	32
lip:	Lipschitz	35
max:	maximum	35
ext:	extreme	36
card:	cardinal	38
ker:	kernel	38
int:	interior	40
L^∞:	essential Lebesgue space	40
a. e:	almost every where	42
dim:	dimension	42
Re:	Real	44
supp:	support	46
iso:	isometry	48
Hol:	Holomorphic	49
Aut:	Automorphism	50
BMOA:	The analytic function of the bounded mean oscillation	61
VMOA:	The analytic function of the vanishing mean oscillation	61
H^p:	Hardy space	61
H^2:	Hardy space	61
ℓ^p:	Banach space	62
min:	minimum	64
H^∞:	essential Hardy space	70
B^p:	Bergman space	71
HM^ψ:	Morse–Transue spaces	71
BM^ψ:	Bergman Morse–Transue space	71
diam:	diameter	82
$\otimes$:	tensor product	167
$\oplus$:	Direct sum	206

References

[1] Zhou, Z. H., Shi Jihuai, Composition operators on the Bloch space in polydiscs, Complex Variables, 2000, 46: 73–88.

[2] K. Madigan and A. Matheson, Compact composition operators on the Bloch space, Transactions of the American Mathematical Society 347 (1995), 2679–2687.

[3] Timoney, R. (1980). Bloch Function in Several Complex Variables, I, Bull. London Math. Soc., 12, 241 -267.

[4] Zhang X J, Xiao J B. Weighted composition operators between μ-Bloch spaces on the unit ball. Science in China, 2005, 48A: 1349–1368.

[5] Hu, Z. J., Extended Ces`aro operators on mixed norm spaces, Proc. Amer. Math. Soc., 2003, 131(7): 2171–2179.

[6] Zhao, R. H., Composition operators from Bloch type spaces to Hardy and Besov spaces, J. Math. Anal. Appl., 1999, 233: 749–766.

[7] Xiao, J., Composition operators associated with Bloch type spaces, Complex Variables Theory Appl., 2001, 46: 109–121.

[8] Zhang, X. J., The composition operator and weighted composition operator on p-Bloch space, Chin. Ann. of Math. (in Chinese), 2003, 24(6): 711–720.

[9] Madigan, K., Composition operators on analytic Lipschitz spaces, Proc. Amer. Math. Soc., 1993, 119: 465–473.

[10] Ohno, S., Zhao Ruhan, Weighted composition operators on the Bloch space, Bull. Austral. Math. Soc., 2001, 63: 177–185.

[11] Yoneda, R., The composition operators on weighted Bloch space, Arch. Math., 2002, 78: 310–317.

[12] Shi, J. H., Luo, L., Composition operators on the Bloch space of several complex variables, Acta Math. Sinica (Eng. Ser.), 2000, 16: 85–98.

[13] Zhou, Z. H., Zeng H. G., Composition operators between p-Bloch space and q-Bloch space in the unit ball, Progress in Natural Science, 2003, 13(3): 233–236.

[14] Zhou, Z. H., Shi, J. H., Compact composition operators on the Bloch space in polydiscs, Science in China, Ser. A, 2001, 44(3): 286–291.

[15] Zhou, Z. H., Zeng Shaobiao, Composition operators on the Lipschitz spaces in polydiscs, Science in China, Ser. A, 2003, 46(1): 33–38.

[16] Hu, Z. J., Composition operators between Blochtype spaces in the polydisc, Science in China, Ser. A, 2005, 48(supp): 268–282.

[17] Zhu, K. H., Spaces of Holomorphic Functions in the Unit Ball, New York: Springer-Verlag, 2004.

[18] Zhu K H. Compact composition operators on Bergman spaces of the unit ball. Houston J Math, 2007, 33(1): 273–283.

[19] D. D. Clahane, Compact composition operators on weighted Bergman spaces of the unit ball, J. Operator Theory 45 (2001), 335-355.

[20] C. Cowen and B. MacCluer, Composition Operators on Spaces of Analytic Functions, CRC Press, Boca Raton, 1994.

[21] B. MacCluer and J. Shapiro, Angular derivatives and compact composition operators on the Hardy and Bergman spaces, Canadian J. Math. 38 (1986), 878-906.

[22] W. Rudin, Function Theory in the Unit Ball of Cn, Springer-Verlag, New York, 1980.

[23] K. Zhu, Operator Theory in Function Spaces, Marcel Dekker, New York, 1990.
[24] K. Zhu, Positive Toeplitz operators on weighted Bergman spaces of bounded symmetric domains, J. Operator Theory 20 (1988), 329-357.
[25] K. Jarosz and V. D. Pathak, Isometries between function spaces, Trans. A.M.S. 305 (1988), 193-206.
[26] D. Amir, On isomorphisms of continuous function spaces, Israel J. Math. 3 (1965), 205-210.
[27] Y. Benyamini, Small into-isomorphisms between spaces of continuous functions, Proc. Amer. Math. Soc. 83 (1981), 479-485.
[28] M. Cambern, Isometries of certain Banach algebras, Studia Math. 25 (1965), 217-225.
[29] _, On isomorphisms with a small bound, Proc. Amer. Math. Sor. 18 (1967), 1062-1066.
[30] M. Cambern and V. D. Pathak, Isometries of spaces of differentiable. functions, Math. Japon. 26 (1981), 253-260.
[31] _, Isomorphisms of spaces of differentiate functions, Rev. Roumaine Math. Pures Appl. 27 (1982), 737-743.
[32] B. Cengiz, A generalization of the Banach-Stone theorem, Proc. Amer. Math. Soc. 40 (1973), 426-430.
[33] N. Dunford and J. T. Schwartz, Linear operators, Interscience, New York, 1958.
[34] M. el-Gebeily and J. Wolfe, Isometries of the disc algebra, Proc. Amer. Math. Soc. 93 (1985), 697-702.
[35] W. Holsztyñski, Continuous mapping induced by isometries of spaces of continuous functions, Studia Math. 26 (1966), 133-136.
[36] K. Jarosz, Into isomorphisms of spaces of continuous functions, Proc. Amer. Math. Soc. 94 (1985), 65-71.
[37] _, Isometries in semisimple, commutative Banach algebras, Proc. Amer. Math. Soc. 90 (1985), 65-71.
[38] _, Perturbations of Banach algebras, Lecture Notes in Math., vol. 1120, Springer-Verlag, 1985.
[39] _, H°°(D) is stable, Bull. London Math. Soc. (to appear).
[40] K. de Leeuw, Banach spaces of Lipschitz functions, Studia Math. 21 (1964), 55-66.
[41] M. Nagasawa, Isomorphisms between commutative Banach algebras with application to rings of analytic functions, Ködai Math. Sem. Rep. 11 (1959), 182-188.
[42] V. D. Pathak, Isometries of Cn [0, 1], Pacific J. Math. 94 (1981), 211-222.
[43] _, Linear isometries of spaces of absolutely continuous functions, Canad. J. Math. 34 (1982), 298-306.
[44] V. D. Pathak and M. H. Vasavada, Isometries and isomorphisms of Banach spaces of differentiable functions (submitted).
[45] N. V. Rao and A. K. Roy, Linear isometries of some function spaces, Pacific J. Math. 38 (1971), 177-192.
[46] A. K. Roy, Extreme points and linear isometries of the Banach spaces of Lipschitz functions, Canad. J. Math. 20 (1968), 1150-1164.
[47] M. H. Vasavada, Closed ideals and linear isometries of certain function spaces, Ph.D. thesis, Univ. of Wisconsin, 1969.

[48] E. Vesentini, On the Banach-Stone theorem, Advances in Math. 112 (1995), 135-146.
[49] S. BANACH, Théorie des opérations lInéaires, in "Monograije Matemaiyczne. PWN. Warsaw. 1932: Dover. New York. 1955.
[50] T. FRANZONI, The group of holomorphic automorphisms in certain J^*- algebras, Ann. Mat. Pura Appl. (4) 127 (1981), 51 65.
[51] T. FRANZONI AND E. VESNTK1, "Holomorphic Maps and Invariant Distances" North Holland, Amsterdam New York Oxford. 1980.
[52] L. A. HARRIS, Hounded symmetric homogeneous domains in infinite dimensional spaces. in "Lecture Notes in Mathematics" Vol. 634. pp. 1 3—40. Springer—Verlag. Berlin, 1974.
[53] S. MAZUR AND S. ULAM. Sur les transformations isométriques d'espaces vectoriels normés. C. R. Acad. Sci. Paris 194 (1932), 946 948.
[54] M. H. SIONE, Applications of the theory of Boolean rings lo general topology, Trans. Amer. Math. Soc. 41 (1937). 375—481.
[55] E. VISENIINI, Holomorphic families of holomorphic isometrics. in "1ecture Notes in Mathematics." Vol. 1277 pp. 290 302. Springer—Verlag. Berlin Heidelberg. New York. I 987.
[56] E. VISENIINI, Semigroups of holomorphic isometrics, in "Complex Potential Theory" (Gauthier—Sahidussi, Ed.) pp. 475 548. Kluwer Academic Publishers, Dordrecht Boston London, 1994.
[57] J.-S. Jeang and N.-C. Wong, Weighted composition operators of C0(X)0s, J. Math. Anal. Appl. 201 (1996), 981-993.
[58] E. Behrends and M. Cambern, An isomorphic Banach-Stone theorem, Studia Math. 90 (1988), 15-26.
[59] M. Cambern, Isomorphisms of C0(X) onto C0(Y), Pac. J. of Math. 35 (1970), 307-312.
[60] W. Holsztynski, Continuous mappings induced by isometries of spaces of continuous function, Studia Math. 26 (1966), 133-136.
[61] K. Jarosz, Into isomorphisms of spaces of continuous functions, Proc. Amer. Math. Soc. 90 (1984), 373-377.
[62] K. Jarosz, Automatic continuity of separating linear isomorphisms, Can. Math. Bull. 33 (1990), 139-144.
[63] P. Lefèvre, D. Li, H. Queffélec, L. Rodríguez-Piazza, Some new properties of composition operators associated to lens maps, Israel J. Math. 195 (2) (2013), 801–824.
[64] R. Aron, P. Galindo and M. Lindstr¨om, Compact homomorphisms between algebras of analytic functions, Studia Mathematica 123 (1997), 235–247.
[65] B. Carl and M. Stephani, Entropy, Compactness and the Approximation of Operators, Cambridge University Press, 1990.
[66] P. Duren, Extension of a theorem of Carleson, Bulletin of the American Mathematical Society 75 (1969), 143–146.
[67] P. Duren, Theory of Hp-spaces, second edition, Dover Publications, New York, 2000.
[68] W. H. Hastings, A Carleson measure theorem for Bergman spaces, Proceedings of the American Mathematical Society 52 (1975), 237–241.

[69] J. Laitila, P.J. Nieminen, E. Saksman, H.-O. Tylli, Compact and weakly compact composition operators on BMOA, Complex Anal. Oper. Theory 7 (2013) 163–181.
[70] P. Lef`evre, Some characterizations of weakly compact operators in H∞ and on the disk algebra. Application to composition operators, Journal of Operator Theory 54 (2005), 229–238.
[71] P. Lef`evre, D. Li, H. Queff´elec and L. Rodr´ıguez-Piazza, A criterion of weak compactness for operators on subspaces of Orlicz spaces, Journal of Function Spaces and Applications 6 (2008), 277–292.
[72] P. Lef`evre, D. Li, H. Queff´elec and L. Rodr´ıguez-Piazza, Some examples of compact composition operators on H2, Journal of Functional Analysis 255 (2008), 3098–3124.
[73] P. Lef`evre, D. Li, H. Queff´elec and L. Rodr´ıguez-Piazza, Compact composition operators on H2 and Hardy–Orlicz spaces, Journal of Mathematical Analysis and Applications 354 (2009), 360–371.
[74] P. Lef`evre, D. Li, H. Queff´elec and L. Rodr´ıguez-Piazza, Composition operators on Hardy–Orlicz spaces, Memoirs of the American Mathematical Society, Vol. 207, American Mathematical Society, Providence, RI, 2010.
[75] P. Lef`evre, D. Li, H. Queff´elec and L. Rodr´ıguez-Piazza, Nevanlinna counting function and Carleson function of analytic maps, Mathematische Annalen 351 (2011), 305–326.
[76] P. Lef`evre, D. Li, H. Queff´elec and L. Rodr´ıguez-Piazza, The canonical injection of the Hardy-Orlicz space Hψ into the Bergman-Orlicz space Bψ, Studia Mathematica 202 (2011), 123–144.
[77] P. Lef`evre, D. Li, H. Queff´elec and L. Rodr´ıguez-Piazza, Compact composition operators on Bergman–Orlicz spaces, Transactions of the American Mathematical Society, to appear; arXiv : 0910.5368.
[78] D. Li, H. Queff´elec and L. Rodr´ıguez-Piazza, On approximation numbers of composition, Journal of Approximation Theory 164 (2012), 413–450.
[79] D. H. Luecking, Trace ideal criteria for Toeplitz operators, Journal of Functional Analysis 73 (1987), 345–368.
[80] O. G. Parfenov, Estimates of the singular numbers of the Carleson embedding operator, Math. USSR Sbornik 59 (2) (1988), 497–514.
[81] G. Pisier, The Volume of Convex Bodies and Banach Space Geometry, Cambridge University Press, 1989.
[82] D. Sarason, Weak compactness of holomorphic composition operators on H1, in Functional Analysis and Operator Theory (New Delhi, 1990), Lecture Notes in Mathematics, Vol. 1511, Springer, Berlin, 1992, pp. 75–79.
[83] J. H. Shapiro, Composition Operators and Classical Function Theory, Universitext, Tracts in Mathematics, Springer-Verlag, New York, 1993.
[84] J. H. Shapiro and P. D. Taylor, Compact, nuclear, and Hilbert-Schmidt composition operators on H2, Indiana University Mathematics Journal 23 (1973), 471–496.
[85] A. ¨Ulger, Some results about the spectrum of commutative Banach algebras under the weak topology and applications, Monatshefte f¨ur Mathematik 121 (1996), 353–379.
[86] K. Zhu, Operator Theory in Function Spaces, Mathematical Surveys and Monographs, Vol. 138, American Mathematical Society, New York, 2007.

[87] J.R. Akeroyd, On Shapiro's compactness criterion for composition operators, J. Math. Anal. Appl. 379 (1) (2011) 1–7.
[88] F. Albiac, N. Kalton, Topics in Banach Space Theory, in: Graduate Texts in Mathematics, vol. 233, Springer, New York, 2006.
[89] B. Carl, Entropy numbers, s-numbers, and eigenvalue problems, J. Funct. Anal. 41 (1981) 290–306.
[90] B. Carl, H. Triebel, Inequalities between eigenvalues, entropy numbers, and related quantities of compact operators in Banach spaces, Math. Ann. 251 (1980) 129–133.
[91] L. Carleson, An interpolation problem for bounded analytic functions, Amer. J. Math. 80 (1958) 921–930.
[92] T. Carroll, C.C. Cowen, Compact composition operators not in the Schatten classes, J. Oper. Theory 26 (1991) 109–120.
[93] V.D. Erokhin, Best linear approximations of functions analytically continuable from a given continuum into a given region, Russian Math. Surveys 23 (1968) 93–135.
[94] J. Garnett, Bounded Analytic Functions, revised first ed., in: Graduate Texts in Mathematics, vol. 236, Springer, 2007.
[95] Y. Gordon, H. K¨onig, C. Sch¨utt, Geometric and probabilistic estimates for entropy and approximation numbers of operators, J. Approx. Theory 49 (1987) 219–239.
[96] W.K. Hayman, The logarithmic derivative of multivalent functions, Michigan Math. J. 27 (1980) 149–179.
[97] K. Hoffman, Banach Spaces of Analytic Functions, in: Prentice-Hall Series in Modern Analysis, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1962.
[98] M. Jones, Compact composition operators not in the Schatten classes, Proc. Amer. Math. Soc. 134 (2006) 1947–1953.
[99] L. Keen, N. Lakic, Hyperbolic Geometry from a Local Viewpoint, in: London Math. Student Texts, vol. 68, Cambridge University Press, Cambridge, 2007.
[100] S.V. Khrusc¨ev, V.V. Peller, Moduli of Hankel operators, past and future, Lecture Notes in Math. 1043 (1984) 92–97.
[101] H. K¨onig, Eigenvalue Distribution of Compact Operators, Operator Theory, in: Advances and Applications, vol. 16, Birkh¨auser, Basel, Boston, Stuttgart, 1986.
[102] P. Lef`evre, D. Li, H. Queff´elec, L. Rodr´ıguez-Piazza, Some revisited results about composition operators on Hardy spaces, Rev. Mat. Iberoam. 28 (1) (2012) 57–76.
[103] C. Le Merdy, The 22th International Conference on Operator Theory, Timisoara July, 2008.
[104] D. Li, H. Queff´elec, Introduction `a l' ´ Etude des Espaces de Banach. Analyse et Probabilit´es, in: Cours Sp´ecialis´es, vol. 12, Soci´et´e Math´ematique de France, Paris, 2004.
[105] D. Li, H. Queff´elec, L. Rodr´ıguez-Piazza, Infinitesimal Carleson property for weighted measures induced by analytic self-maps of the unit disk, Preprint.
[106] D.H. Luecking, K. Zhu, Composition operators belonging to the Schatten ideals, Amer. J. Math. 114 (1992) 878–906.
[107] B.D. MacCluer, Compact composition operators on H p(BN), Michigan Math. J. 32 (2) (1985) 237–248.

[108] A.V. Megretskii, V.V. Peller, S.R. Treil, The inverse spectral problem for self-adjoint Hankel operators, Acta Math. 174 (1995) 241–309.
[109] N.K. Nikolski, Operators, Functions and Systems: An Easy Reading, Volume 1, in: Math. Surveys and Monographs, vol. 92, Amer. Math. Soc., Providence, RI, 2002.
[110] N.K. Nikol'skii, Treatise on the Shift Operator. Spectral Function Theory, in: Grundl. Math. Wissenschaften, vol. 273, Springer-Verlag, Berlin, 1986.
[111] E.A. Nordgren, Composition operators, Canada. J. Math. 20 (1968) 442–449.
[112] B.P. Palka, An Introduction to Complex Function Theory, in: Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1991.
[113] K.R. Parthasarathy, Probability Measures on Metric Spaces, AMS Chelsea Publishing, Providence, RI, 2005.
[114] V.V. Peller, Hankel Operators and their Applications, in: Springer Monographs in Mathematics, Springer-Verlag, 2003.
[115] W. Rudin, Real and Complex Analysis, third ed., McGraw-Hill International Editions, 1987.
[116] E.B. Saff, V. Totik, Logarithmic Potentials with External Fields, in: Grundlehren der Mathematischen Wissenschaften, vol. 316, Springer-Verlag, Berlin, 1997.
[117] J.H. Shapiro, The essential norm of a composition operator, Ann. of Math. 125 (1987) 375–404.
[118] D.A. Stegenga, Multipliers of the Dirichlet space, Illinois J. Math. 24 (1980) 113–139.
[119] H. Widom, Rational approximation and n-dimensional diameter, J. Approx. Theory 5 (1972) 343–361.
[120] Y. Zhu, Geometric properties of composition operators belonging to Schatten classes, Int. J. Math. Sci. 26 (2001) 239–248.
[121] Arazy, J., Fisher, S.D., Peetre, J.: Möbius invariant function spaces. J. Reine Angew. Math. 363, 110–145 (1985).
[122] Bass, R.F.: Probabilistic Techniques in Analysis. Springer, Berlin (1995).
[123] Bourdon, P.S., Cima, J.A., Matheson, A.L.: Compact composition operators on BMOA. Trans. Am. Math. Soc. 351, 2183–2196 (1999).
[124] Choe, B.R., Ramey, W., Ullrich, D.: Bloch-to-BMOA pullbacks on the disk. Proc. Am. Math. Soc. 125, 2987–2996 (1997).
[125] Cima, J.A., Matheson, A.L.: Weakly compact composition operators on VMO. Rocky Mountain J. Math. 32, 937–951 (2002)
[126] Girela, D.: Analytic functions of bounded mean oscillation, Complex Function Spaces (Mekrijärvi, 1999), Univ. Joensuu Dept. Math. Rep. Ser., 4, Univ. Joensuu, pp. 61–170 (2001).
[127] Laitila, J.: Weakly compact composition operators on vector-valued BMOA. J. Math. Anal. Appl. 308, 730–745 (2005).
[128] Laitila, J.: Composition operators and vector-valued BMOA. Integr. Equ. Operator Theory 58, 487–502 (2007).
[129] Laitila, J.: Weighted composition operators on BMOA. Comput. Methods Funct. Theory 9, 27–46 (2009).
[130] Le˘ıbov, M.V.: Subspaces of the VMO space (Russian), Teor. Funktsi˘ı Funktsional. Anal. i Prilozhen. 46, 51–54 (1986); English transl. in J. Soviet Math. 48, 536–538 (1990).

[131] Lindström, M., Makhmutov, S., Taskinen, J.: The essential norm of a Bloch-to-Qp composition operator. Can. Math. Bull. 47, 49–59 (2004).
[132] Liu, P., Saksman, E., Tylli, H.-O.: Small composition operators on analytic vector-valued function spaces. Pac. J. Math. 184, 295–309 (1998)
[133] Makhmutov, S., Tjani, M.: Composition operators on some Möbius invariant Banach spaces. Bull. Aust. Math. Soc. 62, 1–19 (2000).
[134] Müller, P.F.X., Schechtman, G.: On complemented subspaces of H1 and VMO, Geometric Aspects of Functional Analysis (1987–88), Lecture Notes in Math., vol. 1376, pp. 113–125. Springer, Berlin (1989).
[135] Smith, W.: Compactness of composition operators on BMOA. Proc. Am. Math. Soc. 127, 2715–2725 (1999).
[136] Smith,W., Zhao, R.: Composition operators mapping into the Qp spaces. Analysis 17, 239–263 (1997).
[137] Stein, E.M.: Singular Integrals and Differentiability Properties of Functions. Princeton University Press, Princeton (1970).
[138] Stephenson, K.: Weak subordination and stable classes of meromorphic functions. Trans. Am. Math. Soc. 262, 565–577 (1980).
[139] Tjani, M.: Compact composition operators on some Möbius invariant Banach spaces, Ph.D. thesis, Michigan State University (1996).
[140] Wik, I.: On John and Nirenberg's theorem. Ark. Mat. 28, 193–200 (1990)
[141] Wirths, K.J., Xiao, J.: Global integral criteria for composition operators. J. Math. Anal. Appl. 269, 702–715 (2002).
[142] Wulan, H.: Compactness of composition operators on BMOA and VMOA. Sci. China Ser. A 50, 997–1004 (2007)
[143] Wulan, H., Zheng, D., Zhu, K.: Compact composition operators on BMOA and the Bloch space. Proc. Am. Math. Soc. 137, 3861–3868 (2009).
[144] Xiao, J.: Composition operators: Nα to the Bloch space to Qp. Stud. Math. 139, 245–260 (2000).
[145] Yamashita, S.: Holomorphic functions of hyperbolically bounded mean oscillation. Bol. UMI. 5-B, 983–1000 (1986).
[146] Chen, H., Gauthier, P.: Composition operators on μ-Bloch spaces, Rendiconti del Circolo Matematico di Palermo 59, 107 – 119 (2010).
[147] Anderson, J., Claunie, J., Pommerenke,Ch.: On the Bloch functions and normal functions, J. Reine Angew Math., 270 (1974), 12–37.
[148] Attele, K.: Toeplitz and Hankel operators on Bergman spaces, Hokkaido Math. J., 21 (1992), 279–293.
[149] Chen, H., Gauthier, P.: Boundedness from below of composition operators on α-Bloch spaces, Canad.Math. Bull., 51 (2008), 195–204.
[150] Clahane, D., Stevi´c, S.: Norm equivalence and composition operators between Bloch/Lipschitz spaces of the unit ball, J. Inequal. Appl., 2006, Art. ID 61018, 11 pages.
[151] Clahane,D., Stevi´c, S., Zhou, Z.: On composition operators between Bloch-type spaces in polydisk, (2005) arXiv:math/0507339v1.
[152] Domanski, P., Lindström, M.: Sets of interpolation and sampling for weighted Banach spaces of holomorphic functions, Ann. Polon. Math., 79 (2002), 233–264.
[153] Ghatage, P., Zheng, D., Zorboska, N.: Sampling sets and closed range composition operators on the Bloch space, Proc. Amer. Math. Soc., 133 (2004), 1371–1377.

[154] Hedenmalm,H., Korenblum, B., Zhu,K.: Theory of Bergman Spaces. New York: Springer Verlag (2000).
[155] Seip, K.: Beurling type density theorems in the unit disk, Invent. Math., 113 (1994), 21–39.
[156] Seip, K.: Interpolation and Sampling in Spaces of Analytic Functions. (University Lecture Series, vol. 33) Providence, R.I.: AmericanMathematical Society (2004).
[157] Stevi´c, S.: On new Bloch-type spaces, Appl. Math. Comput., 215 (2009), 841-849.
[158] Xiao, J.: Compact composition operators associated with Bloch-type spaces, Variables Theory Appl., 46 (2001), 109–121.
[159] Zhu, K.: Bloch type spaces of analytic functions, Rocky Mountain J. Math., 23 (1993), 1143–1177.
[160] Zhang X J, Xi L H, Fan H X, Li J F. Atomic Decomposition of μ-Bergman Space in Cn. Acta Mathematica Scientia, 2014, 34B(3): 779–789.
[161] Coifman R, Rochberg R. Representation theorems for holomorphic and harmonic function in Lp. Asterisque, 1980, 77: 11–66.
[162] Rochberg R, Semmes S. A decomposition theorem for BMO and applications. J Funct Anal, 1986, 67: 228–263.
[163] Zhang X J, Xiao J B, Hu Z J. The multipliers between the mixed norm space in Cn. J Math Anal Appl, 2005, 311(2): 664–674.
[164] Ren G B, Shi J H. Bergman type operator on mixed spaces with applications. Chin Ann Math, 1997, 18B(3): 265–276.
[165] Zhou Z H, Chen R Y. Weighted composition operator from F(p, q, s) to Bloch type spaces on the unit ball. Int J Math, 2008, 19: 899–926.
[166] Lou Z J. Composition operators on Bloch type spaces. Analysis, 2003, 23: 81–95.
[167] Wulan H S. Compactness of the composition operators from the Bloch space B to Qk spaces. Acta Math Sin (English Series), 2005, 21: 1415–1424.
[168] Ouyang C H, Yang W S, Zhao R H. Characterizations of Bergman spaces and Bloch space in the unit ball of Cn. Trans Amer Math Soc, 1995, 347: 4301–4319.
[169] S. LI, X. ZHANG, Si XU (Mg), THE COMPACT COMPOSITION OPERATOR ON THE μ-BERGMAN SPACE IN THE UNIT BALL∗, Acta Mathematica Scientia 2017, 37B (2): 425–438.
[170] Zhang X J, Yi K Y, Xiao J B. Composition operators on μ-Bergman spaces in the unit ball. Chin J Conte Math, 2009, 30(2): 185–194.
[171] Zhang X J, Li J X. Weighted composition operators between μ-Bloch spaces on the unit ball of Cn. Acta Mathematica Scientia, 2009, 29A(3): 573–583 (in Chinese).
[172] A. Jim?nez-Vargas and Mois?s Villegas-Vallecillos, Into linear isometries between spaces of Lipschitz functions, Houston J. Math. 34 (2008), 1165-1184.
[173] H. Al-Halees and R.J. Fleming, Extreme point methods and Banach-Stone theorems, J. Aust. Math. Soc. 75 (2003), 125-143.
[174] J. Araujo and J.J. Font, Linear isometries between subspaces of continuous functions, Trans. Amer. Math. Soc. 349 (1997), 413-428.
[175] J. Araujo and J.J. Font, Codimension 1 linear isometries on function algebras, Proc. Amer. Math. Soc. 127 (1999), 2273-2281.
[176] M. Cambern, A Holszty_nski theorem for spaces of continuous vector-valued functions, Studia Math. 63 (1978), 213{217.
[177] F.O. Farid and K. Varadajan, Isometric shift operators on C(X), Canad. J. Math. 46 (1994), 532-542.

[178] A. Gutek, D. Hart, J. Jamison and M. Rajagopalan, Shift operators on Banach spaces, J. Funct. Anal. 101 (1991), 97-119.
[179] A. Gutek and J. Norden, Type 1 shifts on C(X), Topology and its Applications. 114 (2001), 73-89.
[180] K. de Leeuw, Banach spaces of Lipschitz functions, Studia Math. 21 (1961/62), 55-66.
[181] A. Moreno and A. Rodr__guez, A bilinear version of Holszty_nski theorem on isometries of C(X)-spaces, Studia Math. 166 (2005), 83-91.
[182] W. Novinger, Linear isometries of subspaces of continuous functions, Studia Math. 53 (1975), 273-276.
[183] D. Sherbert, The structure of ideals and point derivations in Banach algebras of Lipschitz functions, Trans. Amer. Math. Soc. 111 (1964), 240{272.
[184] N. Weaver, Lipschitz Algebras, World Scientific, Singapore, 1999.
[185] A. Jimenez-Vargas, M. Villegas-Vallecillos, Linear isometries between spaces of vector-valued Lipschitz functions, Proc. Amer. Math. Soc. 37 (2009) 1381–1388.
[186] T. M. Jenkins, Banach Spaces of Lipschitz Functions on an Abstract Metric Space, Ph.D. Thesis, Yale University, 1968.
[187] M. Jerison, The space of bounded maps into a Banach space, Ann. of Math. (2) 52 (1950), 309-327. MR0036942 (12:188c).
[188] E. Mayer-Wolf, Isometries between Banach spaces of Lipschitz functions, Israel J. Math. 38 (1981), 58-74. MR599476 (82e:46036).
[189] N. Weaver, Isometries of noncompact Lipschitz spaces, Canad. Math, Bull. 38 (1995), 242 249. MR1335105 (96b:46032).
[190] T. Dings, A. C. M. van Rooij, Generalised weighted composition operators on Maeda–Ogasawara spaces, (2017).
[191] C.D. Aliprantis, O. Burkinshaw, Positive Operators, Academic Press, 1985.
[192] C.D. Aliprantis, O. Burkinshaw, Locally Solid Riesz Spaces with Applications to Economics, American Mathematical Society, 2003.
[193] C.D. Aliprantis, O. Burkinshaw, Positive Operators, Springer, 2006.
[194] T. Dings, Generalised Composition Multiplication Operators (master's thesis), Radboud University Nijmegen, 2016. Available via: slideshare.net/TeunDings/masterscriptie-t-dings-4029100.
[195] M.B.J.G. van Haandel, Completions in Riesz Space Theory (Ph.D. thesis), Radboud University Nijmegen, 1993. Available via: http://repository.ubn.ru.nl/handle/2066/145521.
[196] H. van Imhoff, Composition Multiplication Operators on Pre-Riesz Spaces (master's thesis), Leiden University, 2014. Available via: math.leidenuniv.nl/scripties/vanImhoffMaster.pdf.
[197] W.A.J. Luxemburg, A.C. Zaanen, Riesz Spaces I, North-Holland Publishing Company, 1971.
[198] A.C. Zaanen, Riesz Spaces II, North-Holland Publishing Company, 1983.
[199] D. Li, H. Queffélec, L. R. Piazza, Approximation numbers of composition operators on Hp, HAL Id: hal-01119589, https://hal-univ-artois.archives-ouvertes.fr/hal-01119589.
[200] B. Carl, A. Hinrichs, Optimal Weyl-type inequalities for operators in Banach spaces, Positivity 11 (2007), 41–55.
[201] P. L. Duren, Theory of Hp Spaces, Dover Public. (2000).

[202] C. V. Hutton, On the approximation numbers of an operator and its adjoint, Math. Ann. 210 (1974), 277–280.
[203] D. Li, H. Queffélec, L. Rodríguez-Piazza, Estimates for approximation numbers of some classes of composition operators on the Hardy space, Ann. Acad. Sci. Fenn. Math. 38 (2013), 547–564.
[204] D. Li, H. Queffélec, L. Rodríguez-Piazza, A spectral radius type formula for approximation numbers of composition operators, J. Funct. Anal., 267 (2014), no. 12, 4753–4774.
[205] R. Mortini, Thin interpolating sequences in the disk, Arch. Math. 92, no. 5 (2009), 504–518.
[206] S. Petermichl, S. Treil, B.D. Wick, Carleson potentials and the reproducing kernel thesis for embedding theorems, Illinois J. Math. 51, no. 4 (2007), 1249–1263.
[207] A. Pietsch, s-numbers of operators in Banach spaces, Studia Math. LI (1974), 201–223.
[208] A. Plichko, Rate of decay of the Bernstein numbers, Zh. Mat. Fiz. Anal. Geom. 9, no. 1 (2013), 59–72.
[209] H. Queffélec, K. Seip, Decay rates for approximation numbers of composition operators, J. Anal. Math., 125 (2015), no. 1, 371–399.
[210] S. A. Vinogradov, E. A. Gorin, S. V. Hruščëv, Free interpolation in H1 in the sense of P. Jones, J. Sov. Math. 22 (1983), 1838–1839.
[211] S. Miihkinen, Strict singularity of a Volterra-type integral operator on Hp, Proc. Amer. Math. Soc. 145 (2017) 165–175.
[212] Alexandru Aleman, A class of integral operators on spaces of analytic functions, Topics in complex analysis and operator theory, Univ. M´alaga, M´alaga, 2007, pp. 3–30. MR 2394654.
[213] Alexandru Aleman and Joseph A. Cima, An integral operator on Hp and Hardy's inequality, J. Anal. Math. 85 (2001), 157–176, DOI 10.1007/BF02788078. MR1869606.
[214] Alexandru Aleman and Aristomenis G. Siskakis, An integral operator on Hp, Complex Variables Theory Appl. 28 (1995), no. 2, 149–158. MR1700079.
[215] Alexandru Aleman and Aristomenis G. Siskakis, Integration operators on Bergman spaces, Indiana Univ. Math. J. 46 (1997), no. 2, 337–356, DOI 10.1512/iumj.1997.46.1373. MR1481594.
[216] Jos´e Bonet, Pawe_l Doma´nski, and Mikael Lindstr¨om, Essential norm and weak compactness of composition operators on weighted Banach spaces of analytic functions, Canad. Math. Bull. 42 (1999), no. 2, 139–148, DOI 10.4153/CMB-1999-016-x. MR1692002.
[217] Francisco L. Hern´andez, Evgeny M. Semenov, and Pedro Tradacete, Strictly singular operators on Lp spaces and interpolation, Proc. Amer. Math. Soc. 138 (2010), no. 2, 675–686, DOI 10.1090/S0002-9939-09-10089-8. MR2557184.
[218] N. J. Kalton, N. T. Peck, and James W. Roberts, An F-space sampler, London Mathematical Society Lecture Note Series, vol. 89, Cambridge University Press, Cambridge, 1984. MR808777.
[219] Tosio Kato, Perturbation theory for nullity, deficiency and other quantities of linear operators, J. Analyse Math. 6 (1958), 261–322. MR0107819.

[220] Jussi Laitila, Santeri Miihkinen, and Pekka J. Nieminen, Essential norms and weak compactness of integration operators, Arch. Math. (Basel) 97 (2011), no. 1, 39–48, DOI 10.1007/s00013-011-0272-z. MR2820586.
[221] J. Laitila, P. J. Nieminen, E. Saksman, and H.-O. Tylli, Structural rigidity of composition operators on Hp, (forthcoming).
[222] M. V. Leˇıbov, Subspaces of the space VMO (Russian), Teor. Funktsiˇı Funktsional. Anal. I Prilozhen. 46 (1986), 51–54, DOI 10.1007/BF01095622; English transl., J. Soviet Math. 48 (1990), no. 5, 536–538. MR865789.
[223] Joram Lindenstrauss and Lior Tzafriri, Classical Banach spaces. I, Springer-Verlag, Berlin-New York, 1977. Sequence spaces; Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 92. MR0500056.
[224] Ch. Pommerenke, Schlichte Funktionen und analytische Funktionen von beschr¨ankter mittlerer Oszillation (German), Comment. Math. Helv. 52 (1977), no. 4, 591–602. MR 0454017.
[225] Aristomenis G. Siskakis, Volterra operators on spaces of analytic functions—a survey, Proceedings of the First Advanced Course in Operator Theory and Complex Analysis, Univ. Sevilla Secr. Publ., Seville, 2006, pp. 51–68. MR2290748.
[226] Aristomenis G. Siskakis and Ruhan Zhao, A Volterra type operator on spaces of analytic functions, Function spaces (Edwardsville, IL, 1998), Contemp. Math., vol. 232, Amer. Math. Soc., Providence, RI, 1999, pp. 299–311, DOI 10.1090/conm/232/03406. MR1678342.
[227] L. Weis, On perturbations of Fredholm operators in Lp(μ)-spaces, Proc. Amer. Math. Soc. 67 (1977), no. 2, 287–292. MR0467377.
[228] P. Wojtaszczyk, The Banach space H1, Functional analysis: surveys and recent results, III (Paderborn, 1983), North-Holland Math. Stud., vol. 90, North-Holland, Amsterdam, 1984, pp. 1–33, DOI 10.1016/S0304-0208(08)71464-6. MR761370.
[229] P. Wojtaszczyk, Banach spaces for analysts, Cambridge Studies in Advanced Mathematics, vol. 25, Cambridge University Press, Cambridge, 1991. MR1144277.
[230] Rikio Yoneda, Integration operators on weighted Bloch spaces, Nihonkai Math. J. 12 (2001), no. 2, 123–133. MR1869743.
[231] J. Laitilaa, Pekka J. N., E. Saksmana, H. O. Tyllia,∗, Rigidity of composition operators on the Hardy space Hp, Advances in Mathematics 319 (2017) 610–629.
[232] J. Bourgain, New Banach space properties of the disc algebra and H∞, Acta Math. 152 (1984) 1–48.
[233] J.A. Cima, A.L. Matheson, Completely continuous composition operators, Trans. Amer. Math. Soc. 344 (1994) 849–856.
[234] M.D. Contreras, S. Díaz-Madrigal, Compact-type operators defined on H∞, in: Function Spaces, Edwardsville, IL, 1998, in: Contemp. Math., vol.232, 1999, pp.111–118.
[235] D. Dosev, W.B. Johnson, G. Schechtman, Commutators on Lp, 1 ≤p <∞, J. Amer. Math. Soc. 26 (2013) 101–127.
[236] J.B. Garnett, Bounded Analytic Functions, Academic Press, 1981.
[237] H. Jarchow, Compactness properties of composition operators, Rend. Circ. Mat. Palermo (2) Suppl. 56 (1998) 91–97.
[238] W.B. Johnson, E. Odell, Subspaces of Lpwhich embed into _p, Compos. Math. 28 (1974) 37–49.

[239] D. Li, H. Queffélec, L. Rodríguez-Piazza, Approximation properties of composition operators on Hp, Concr. Oper. 2 (2015) 98–109.
[240] J. Lindenstrauss, L. Tzafriri, Classical Banach Spaces II. Function Spaces, Ergeb. Math. Grenzgeb. (3), vol.97, Springer-Verlag, 1979.
[241] A. Pietsch, Operator Ideals, North-Holland, 1980.
[242] R. Riedl, Composition Operators and Geometric Properties of Functions, Thesis, Universität Zürich, 1994.
[243] T. Schlumprecht, A. Zsák, The algebra of bounded linear operators on _p⨁_qhas infinitely many closed ideals, J. Reine Angew. Math. (2017), http://dx.doi.org/10.1515/crelle-2015-0021, in press.
[244] W. Smith, Composition operators between Bergman and Hardy spaces, Trans. Amer. Math. Soc. 348 (1996) 2331–2348.
[245] Shawgy Hussein and Bushara Eisa Hamad Abd Alla, Compact and Weighted Composition Operators on μ-Bergman and Maeda-Ogasawara Spaces with Rigidity on Hardy Spaces, Ph.D. Thesis Sudan University of Science and Technology, Sudan (2020).