Sudan University of Science and Technology
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Operator of Logarithmic Weight Bloch Type Space and Weakly Compact Composition Operator

المؤثر للفضاء نوع بلوش المرجح اللوغريثمي ومؤثر التركيب المتراص الضعيف

A Thesis Submitted in Fulfillment for the Degree of Ph.D in Mathematics

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Dedication

To my family.

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Abstract

We deal with the integral–type operator from Bloch space, logarithmic Bloch space and Dirichlet space to the Bloch-type space on the unit ball. The norm of operators from logarithmic Bloch type spaces to weighted- type spaces are considered. We give the composition of Bloch with bounded analytic and inner functions, symmetric measures and biBloch mapping. The Bloch-to-BMOA compositions on complex balls, reverse estimates in logarithmic and weight Bloch spaces and quadratic integrals are established. The composition operators from Bloch type spaces to Hardy and Besov spaces are discussed with the compact and weakly compact composition operators from the Bloch space into Möbius invariant spaces are found.

الخلاصة

تعاملنا مع مؤثر نوع-التكامل من فضاء بلوش وفضاء بلوش اللوغريثمي وفضاء ديرشليت إلى الفضاء نوع-بلوش على كرة الوحدة. قمنا باعتبار النظيم للمؤثرات من الفضاءات نوع بلوش اللوغريثمي إلى فضاءات النوع المرجح. تم اعطاء التركيب لبلوش طبقاً للدوال التحليلية المحدودة والداخلة والقياسات المتماثلة وراسم بلوش الثنائي. تم تأسيس تركيبات بلوش-إلى-BMOA على الكرات المركبة والتقديرات العكسية في فضاءات بلوش اللوغريثمية والمرجحة وتكاملات الدرجة الثانية. قمنا بمناقشة مؤثرات التركيب من الفضاءات نوع بلوش إلى فضاءات هاردي وبيسوف مع ايجاد التراص ومؤثرات التركيب المتراصة الضعيفة من فضاء بلوش إلى الفضاءات اللامتغيرة موبيوس.

Introduction

From logarithmic Bloch-type and mixed-norm spaces to Blochtype spaces. We introduce the following integral-type operator on the space $H(\mathbb{B})$ of all holomorphic functions on the unit ball $\mathbb{B} \subset \mathbb{C}^n$, $P_\phi^g(f)(z) = \int_0^1 f\phi(tz)g(tz)\frac{dt}{t}$, $z \in \mathbb{B}$, where $g \in H(\mathbb{B}), g(0) = 0$ and ϕ is a holomorphic self-map of $\mathbb{B}$.

The construction of an inner function, decreases hyperbolic distances as much as desired. The problem of constructing functions f_1, f_2 analytic in the unit disc $\mathbb{D}$ of the complex plane satisfying $|f_1'(z) + f_2'(z)| = \psi\left(\frac{1}{1-|z|}\right), z \in \mathbb{D}$, is solved for a wide class of weights ψ that includes normal weights.

We give Necessary and sufficient conditions for a composition operator $C_\phi f = f \circ \phi$ to be compact on the Bloch space $\mathcal{B}$ and on the Little Bloch space $\mathcal{B}_0$. Weakly compact composition operators on $\mathcal{B}_0$ are shown to be compact. We express the essential norm of a composition operator on the Bloch space and the Little Bloch space as the asymptotic upper bound of a quantity involving the inducing map and the Pick-Schwarz Lemma.

We characterize the boundedness and compactness of the following integral-type operator $I_\phi^g(f)(z) = \int_0^1 \mathcal{R}f\big(\phi(tz)\big)g(tz)\frac{dt}{t}, z \in \mathbb{B}$, where g is a holomorphic function on the unit ball $\mathbb{B} \subset \mathbb{C}^n$ such that $g(0) = 0$, and ϕ is a holomorphic self-map of $\mathbb{B}$. Operator norm and essential norm of an integral-type operator, recently introduced. Operator norm of weighted composition operators from the iterated logarithmic Bloch space $\mathcal{B}_{\log_k}$, $k \in \mathbb{N}$, or the logarithmic Bloch-type space $\mathcal{B}_{\log^\beta}$, $\beta \in (0,1]$, to weighted-type spaces on the unit ball are calculated.

We obtain sharp reverse estimates for the logarithmic Bloch spaces on the unit disk.

Boundedness and compactness of composition operators from Bloch type spaces to Hardy spaces and analytic Besov spaces are characterized by function theoretic properties of their inducing maps. For the case of the Bloch space, the characterizations involve the hyperbolic versions of Hardy and Besov classes. For $\mathcal{N}_\alpha, \mathcal{B}$ and Q_β be the weighted Nevanlinna space, the Bloch space and the Q space, respectively. Note that $\mathcal{B}$ and Q_β are Möbius invariant, but $\mathcal{N}_\alpha$ is not. We obtain exhaustive results and treat in a unified way the question of boundedness, compactness, and weak compactness of composition operators from the Bloch space into any space from a large family of conformally invariant spaces that includes the classical spaces like BMOA, Q_α, and analytic Besov spaces B^p.

The Contents

Chapter 1
New Integral–Type Operators

We study the boundedness and compactness of the following integral-type operator, $P_\phi^g f(z) = \int_0^1 f(\phi(tz))g(tz)\frac{dt}{t}, z \in B$, where ϕ is a holomorphic self-map of the unit ball $\mathbb{B}$ in $\mathbb{C}^n$ and g is a holomorphic function on $\mathbb{B}$ such that $g(0) = 0$. The boundedness and compactness of the operator from the Bloch space $\mathcal{B}$ or the Little Bloch space $\mathcal{B}_0$ to the Bloch-type space $\mathcal{B}_\mu$ or the Little Bloch-type space $\mathcal{B}_{\mu,0}$, are characterized. We calculate the essential norm of the operators $P_\phi^g : \mathcal{B}$ (or $\mathcal{B}_0$) $\to \mathcal{B}_\mu$ (or $\mathcal{B}_{\mu,0}$) in an elegant way.

Section (1.1): From Logarithmic Bloch-Type and Mixed-Norm Spaces to Bloch-Type Spaces

For B = $\{z \in \mathbb{C}^n : |z| < 1\}$ be the open unit ball in $\mathbb{C}^n$, $\mathbb{D}$ the open unit disk in $\mathbb{C}$, $H.(\mathbb{B})$ / the class of all holomorphic functions on the unit ball and $H^\infty(\mathbb{B})$the space of all bounded holomorphic functions on $\mathbb{B}$ with the norm $\|f\|_\infty = sup_{z\in\mathbb{B}}|f(z)|$. Let $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ be points in $\mathbb{C}^n$ and $\mathbb{C}^n$ and $\langle z, w\rangle = \sum_{|\beta|\geq 0} a_\beta z^\beta$. For f

$$\Re f(z) = \sum_{|\beta|\geq 0} |\beta| a_\beta z^\beta$$

be the radial derivative of f, where $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ is a multi-index $|\beta| = (\beta_1 + \dots + \beta_n)$ and $z^\beta = z_1^{\beta_2} \dots z_n^{\beta_n}$.

Let μ be a strictly positive continuous function (weight) on the unit ball $\mathbb{B}$. A weight μ is called radial if $\mu(z) = \mu(|z|)$. For every $z \in \mathbb{B}$. Every radial weight μ which is non increasing with respect to $|z|$ and such that $\lim_{|z|\to 1-0} \mu(z) = 0$ is called typical.

The logarithmic Bloch-type space $\mathcal{B}_{log^\beta}^\alpha = \mathcal{B}_{log^\beta}^\alpha(\mathbb{B}), \alpha > 0, \beta \geq 0$, consists of all $f \in H(\mathbb{B})$ such that

$$b_{\alpha,\beta}(f) := sup_{z\in\mathbb{B}}(1-|z|)^\alpha \left(In\frac{e^{\beta/\alpha}}{1-|z|}\right)^\beta |\Re f(z)| < \infty.$$

the norm on $\mathcal{B}_{log^\beta}^\alpha$ is introduced ad follows

$$\|f\|_{\mathcal{B}_{log^\beta}^\alpha} = |f(0)| + b_{\alpha,\beta}(f) \tag{1}$$

When $\beta = 0, \mathcal{B}_{log^\beta}^\alpha$ becomes the α -Bloch space $\mathcal{B}^\alpha$ (see, [19]). For $\alpha = \beta = 1$, $\mathcal{B}_{log^\beta}^\alpha$ is the logarithmic = Bloch space [10], which appeared in characterizing the multipliers of the Bloch space (see [3] and [9]).

The Little logarithmic Bloch-type space $\mathcal{B}_{log^\beta}^\alpha = \mathcal{B}_{log^\beta}^\alpha(\mathbb{B}), \alpha > 0, \beta \geq 0$, consists of all $f \in \mathcal{B}_{log^\beta}^\alpha$ such that

$$\lim_{|z|\to 1-0}(1-|z|)^\alpha \left(In\frac{e^{\beta/\alpha}}{1-|z|}\right)|\Re f(z)| = 0$$

The Bloch-type space $\mathcal{B}_\mu = \mathcal{B}_\mu(\mathbb{B})$consists of all $f \in H(\mathbb{B})$ such that

$$\mathcal{B}_\mu(f) = \sup_{z\in\mathbb{B}} \mu(z)\, |\Re f(z)| < \infty$$

where μ is a weight. With the norm

$$\|f\|_{\mathcal{B}_\mu} = |f(0)| + \mathcal{B}_\mu(f)$$

the Bloch-type space becomes a Banach space.

The Little Bloch-type space $\mathcal{B}_{\mu,0} = \mathcal{B}_{\mu,0}(\mathbb{B})$ is a subspace of $\mathcal{B}_\mu$ consisting of all f such that

$$\lim_{|z|\to 1} \mu(z)\, |\Re f(z)| = 0$$

The weighted space (or weighted-type space) $H_\mu^\infty = H_\mu^\infty(\mathbb{B})$consists of all $f \in H(\mathbb{B})$ such that

$$\|f\|_{H_\mu^\infty} = \sup_{z\in\mathbb{B}} \mu(z)\, |f(z)| < \infty$$

where μ is a weight.

The Little weighted space $H_{\mu,0}^\infty = H_{\mu,0}^\infty(\mathbb{B})$ is a subspace of H_μ^∞ consisting of all $f \in H(\mathbb{B})$ such that

$$\lim_{|z|\to 1} \mu(z)\, |f(z)| = 0$$

A positive continuous function on ϕ on $[0,1]$ is called normal [11] if there are $\delta \in [0,1]$ and a and b, $0 < a < b$ such that

$$\frac{\phi(r)}{(1-r)^a} \text{ is decreasing on } [\delta, 1] \text{ and } \lim_{r\to 1} \frac{\phi(r)}{(1-r)^a} = 0\,;$$

$$\frac{\phi(r)}{(1-r)^b} \text{ is increasing on } [\delta, 1] \text{ and } \lim_{r\to 1} \frac{\phi(r)}{(1-r)^b} = 0\infty\,;$$

If we say that a function $v : \mathbb{B} \to [0,\infty]$ is normal we will also assume that it is radial on $\mathbb{B}$:

For $0 < p; q < \infty$ and ϕ normal, the mixed-norm space $H(p,q,\phi)$ $(\mathbb{B})$ consists of all functions $f \in H(\mathbb{B})$ such that

$$\|f\|_{H(p,q,\phi)} = \left(\int_0^1 M_q^p(f,r)\, \frac{\phi^p(r)}{1-r}\, dr\right)^{\frac{1}{p}} < \infty\,,$$

Where

$$M_q(f,r) = \left(\int_S |f(r\xi)|^q d\sigma\ (\zeta)\right)^{\frac{1}{q}},$$

For p = q and $\phi(r) = (1-r^2)^{\frac{\alpha+1}{p}}$ $,\alpha > -1.$ the mixed-norm space is equivalent with the weighted Bergman space $A_\alpha^p = A_\alpha^p(\mathbb{B})$ consisting of all $f \in H(\mathbb{B})$ such that

$$\int_{\mathbb{B}} |f(z)|^p (1-|z|^2)^\alpha\, dV(z) < \infty,$$

Where $dV(z)$ is the Lebesgue volume measure on $\mathbb{B}$.

Let φ be a holomorphic self-map of $\mathbb{B}$ (usually non-constant) and . For $f \in H(\mathbb{B})$ the corresponding weighted composition operator is defined by

$$\left(uC_\varphi\right)(f)(z) = u(z) f\left(\varphi(z)\right), z \in \mathbb{B}$$

It is of interest to provide function-theoretic characterizations for when φ and u induce bounded or compact weighted composition operators on spaces of holomorphic functions (see, e.g., [12]). For some results, in $\mathbb{C}^n$ or related to Bloch-type spaces, see, e.g., [4], [12], [30].

Let $g \in H(\mathbb{D})$ and φ be a holomorphic self-map of $\mathbb{D}$. Products of integral [31] and composition operators on $H(\mathbb{D})$ were introduced by Li and Stevi¢ (see [32], [36], as well as [37] and [38] for a related operator) as follows:

$$C_\varphi J_g f(z) = \int_0^{\varphi(z)} f(\zeta) g'(\zeta) d\zeta \ \ and \ \ J_g C_\varphi f(z) = \int_0^z f(\varphi(\zeta)) g'(\zeta) d\zeta. \quad (2)$$

In [39] (see also [10], [40], [41]) has extended the second operator in (2) to the unit ball setting as follows. Assuming that $g \in H(\mathbb{D}), g(0) = 0 \ and \ \varphi$ is a holomorphic self-map of $\mathbb{B}$, we define an operator on $\mathbb{B}$ in this way:

$$P_\varphi^g(f)(z) = \int_0^1 f(\varphi(tz)) g(tz) \frac{dt}{t}, \quad f \in H(\mathbb{B}), z \in \mathbb{B}. \quad (3)$$

If $n = 1$, then $g \in H(\mathbb{D})$ and $g(0) = 0$, so that $g(z) = zg_0(z)$, for some $g_0 \in H(\mathbb{D})$, by the change of variable $\xi = tz$, it follows that

$$P_\varphi^g(f)(z) = \int_0^1 f(\varphi(tz)) tz g_0 \frac{dt}{t}, \ \int_0^z f(\varphi(\zeta)) g_0(\zeta) d\zeta.$$

Thus the operator (3) is a natural extension of the operator $J_g C_{\varphi'}$ in (2).

For some results on related integral-type operators in $\mathbb{C}^n$ see, e.g., [6], [10], [42], [60]. The following research project was initiated in [40].

Let X and Y be two Banach spaces of holomorphic functions on the unit ball in $\mathbb{C}^n$ (e.g., the weighted Bergman space A_α^p , the Bloch-type space $\mathcal{B}_\mu$, the Hardy space H^p, the weighted space H_μ^∞, the Besov space B^p, BMOA, etc.) Characterize the boundedness, compactness, essential norms and other operator-theoretic properties of the operator $P_\varphi^g: X \to Y$ in terms of function-theoretic properties of the inducing functions φ and g.

We continue to study operator P_φ^g by investigating the boundedness and compactness of the operator from the logarithmic Bloch-type space $\mathcal{B}_{log^\beta}^\alpha$ or the Little logarithmic Bloch-type space $\mathcal{B}_{log^\beta,0}^\alpha$ to the Bloch-type space $\mathcal{B}_\mu$ or the Little Bloch-type space $\mathcal{B}_{\mu,0}$. Results complement those ones in [10]. We also extend some results in [40] by characterizing the boundedness and compactness of the operator P_φ^g from the mixed-norm space (p, q, ϕ) to the Bloch-type space $\mathcal{B}_\mu$ or the Little Bloch-type space $\mathcal{B}_{\mu,0}$.

We constant are denoted by C, they are positive and may differ from one occurrence to the other. The notation $a \preccurlyeq b$ means that there is a positive constant C such that $a \le Cb$. We say that $a \asymp b$, if both $a \preccurlyeq b$ and $b \preccurlyeq a$ hold.

We present several auxiliary results which will be used in the proofs of the main results.

Lemma (1.1.1)[1]: Assume $\alpha > 0, \beta \ge 0 \ and \ \gamma \ge \frac{\beta}{\alpha} + In2$.. Then the function

$$h(x) = x^\alpha \left(In \frac{e^\gamma}{x} \right)^\beta \quad (4)$$

Increasing on the interval $(0,2)$.

Proof: we have

$$h'(x) = x^{\alpha-1} \left(In \frac{e^\gamma}{x} \right)^{(\beta-1)} \left(\alpha \, In \frac{e^\gamma}{x} - \beta \right)$$

Now note that

$$x^{\alpha-1} \left(In \frac{e^\gamma}{x} \right)^{(\beta-1)} > 0.$$

When $x \in (0,2)$ and that the function

$$H(x) = \alpha\, In\frac{e^{\gamma}}{x} - \beta$$

Is decreasing on (0,2), (here we use that $\gamma \geq In2$).Hence

$$\alpha\, In\frac{e^{\gamma}}{x} - \beta > \alpha\, In\frac{e^{\gamma}}{2} - \beta = \alpha(\gamma - In2 - \beta/\alpha) \geq 0, x \in (0,2)$$

from which the lemma follows.

The following lemma can be proved similar to Lemma (1.1.1).

Lemma (1.1.2)[1]: Assume $\alpha > 0, \beta \geq 0$ and $\gamma \geq \frac{\beta}{\alpha}$. Then the function

$$h_1(x) = x^{\alpha}\left(\mathrm{In}\frac{e^{\gamma}}{x}\right)^{\beta}. \quad (5)$$

is increasing on the interval (0,1).

By using the L. Hopital rule, as well as some simple estimates, the following lemma can be proved.

Lemma (1.1.3)[1]: The following statements are true.

(a) Assume $\alpha > 1$ and $\beta \geq 0$. Then

$$\int_0^x \frac{\alpha}{(1-t)^{\alpha}\left(In\frac{e^{\beta/\alpha}}{1-t}\right)^{\beta}} \sim \frac{1}{(\alpha-1)(1-x)^{\alpha-1}\left(In\frac{e^{\beta/\alpha}}{1-x}\right)^{\beta}}, as\ x \to 1-0$$

(b) Assume $\alpha = 1$ and $\beta \in (0,1)$. Then

$$\int_0^x \frac{dt}{(1-t)^{\alpha}\left(In\frac{e^{\beta/\alpha}}{1-t}\right)^{\beta}} \sim \frac{1}{1-\beta}\left(In\frac{e^{\beta/\alpha}}{1-x}\right)^{1-\beta}, as\ x \to 1-0$$

(c) Assume $\alpha = 1$ and $\beta = 1$. Then

$$\int_0^x \frac{dt}{(1-t)^{\alpha}\left(In\frac{e^{\beta/\alpha}}{1-t}\right)^{\beta}} \sim In\frac{e^{\beta/\alpha}}{1-x}, as\ x \to 1-0$$

(d) if $\alpha = 1$ *and* $\beta > 1$, or $\alpha \in (0,1)$, Then the integral

$$\int_0^x \frac{dt}{(1-t)^{\alpha}\left(In\frac{e^{\beta/\alpha}}{1-t}\right)^{\beta}}.$$

is convergent.

Recall that the operator

$$\delta_z(f) = f(z)$$

where f are complex-valued functions defined on a domain Ω which belong to a Banach space X, is called the point evaluation functional on X at point z.

The next result gives some estimates for the point evaluation operator on the space $\mathcal{B}^{\alpha}_{log^{\beta}}$. As usual from these estimates it follows that the point evaluations are bounded functionals on $\mathcal{B}^{\alpha}_{log^{\beta}}(\mathbb{B})$.

Lemma (1.1.4)[1]: Let $f \in \mathcal{B}^{\alpha}_{log^{\beta}}(\mathbb{B})$.Then

$$|f(z)| \leq C \begin{cases} |f(0)| + \|f\|_{\mathcal{B}^{\alpha}_{log^{\beta}}} \quad \alpha \in (0,1) \; or \; \alpha = 1, \beta > 1, \\ |f(0)| + \|f\|_{\mathcal{B}^{\alpha}_{log^{\beta}}} max\left\{1, InIn\frac{e^{\frac{\beta}{\alpha}}}{1-|z|}\right\} \; \alpha = \beta = 1 \\ |f(0)| + \|f\|_{\mathcal{B}^{\alpha}_{log^{\beta}}}\left(In\frac{e^{\frac{\beta}{\alpha}}}{1-|z|}\right), \alpha = 1, \beta \in (0,1) \\ |f(0)| + \dfrac{\|f\|_{\mathcal{B}^{\alpha}_{log^{\beta}}}}{(1-|z|)^{\alpha-1}\left(In\frac{e^{\frac{\beta}{\alpha}}}{1-|z|}\right)^{\beta}} \; , \alpha > 1, \beta \geq 0 \end{cases} ,$$

For some $C > 0$ independent of f,

Proof. let $Z \in \mathbb{B}$. By the definition of the space $\mathcal{B}^{\alpha}_{log^{\beta}}$ and the change of variables $s = t|z|$, we have that

$$|f(z) - f(z/2)| = \left|\int_{1/2}^{1} \Re f(tz)\frac{dt}{t}\right| \leq b_{\alpha,\beta}(f)\int_{1/2}^{1}\frac{|z|dt}{(1-t|z|)^{\alpha}\left(In\frac{e^{\frac{\beta}{\alpha}}}{1-t|z|}\right)}$$

$$= b_{\alpha,\beta}(f)\int_{0}^{|z|}\frac{ds}{(1-s)^{\alpha}\left(In\frac{e^{\frac{\beta}{\alpha}}}{1-s}\right)^{\beta}} \tag{6}$$

$$\leq b_{\alpha,\beta}(f)\int_{0}^{1}\frac{ds}{(1-s)^{\alpha}\left(In\frac{e^{\frac{\beta}{\alpha}}}{1-s}\right)^{\beta}} \tag{7}$$

On the other hand, similar to Lemma (1.1.2). in [6] it can be proved that

$$M_{\infty}(F, 1/2) \leq |f(0)| + Cb_{\alpha,\beta}(f), \tag{8}$$

for each $\alpha > 0$ and $\beta \geq 0$, and for some C independent of f.

From (7), (8) and Lemma (1.1.3)(d), this lemma follows for the case $\alpha \in (0,1)$, or $\alpha = 1$ and $\beta > 1$. if $\alpha = \beta = 1$, then from (6) and by direct calculation we obtain

$$|f(z) - f(z/2)| \leq b_{1,1}(f)\int_{0}^{|z|}\frac{ds}{(1-s)\;\; In\frac{e}{1-s}} = b_{1,1}(f)\frac{e}{1-|z|}$$

from which along with (8) the third inequality in this lemma easily follows.

Finally If $\alpha = 1$ and $\beta \in (0,1)$ then we have

$$|f(z) - f(z/2)| \leq b_{1,\beta}(f)\int_{0}^{|z|}\frac{ds}{(1-s)\;\; In\frac{e^{\beta}}{1-s}} \leq \frac{b_{1,\beta}(f)}{1-\beta}\left(\frac{e^{\beta}}{1-|z|}\right)^{1-\beta}$$

From which a long with (8) the third inequality in this lemma easily follows.

Finally if $\alpha > 1 \; and \; \beta \geq 0$, then by Lemma (1.1.3)(a) and similarly as in the case $a > 1$ of Lemma (1.1.2). in [6] (see, also Lemma (1.1.1) in [10]), the estimate can be proved, finishing the proof of the lemma.

Lemma (1.1.5)[1]: Assume $\alpha > 0$ and $\beta \geq 0$. A closed set K in $\mathcal{B}^{\alpha}_{log^{\beta}}$ is compact if and only if it is bounded and

$$\lim_{|z|\to 1} \sup_{f\in K}(1-|z|)^{\alpha}\left(In\frac{e^{\beta/\alpha}}{1-|z|}\right)|\Re f(z)| = 0$$

Lemma (1.1.6)[1]: Assume that φ is a holomorphic self-map of $\mathbb{B}$, $g \in H(\mathbb{B})$ and $g(0) = 0$. Then for every $f \in H(\mathbb{B})$ it holds

$$\Re\left[P^{g}_{\varphi}(f)\right](z) = f\big(\varphi(z)\big)g(z),$$

The following characterization of compactness can be proved in a standard way (see, e.g., the proofs of the corresponding lemmas in [12], [52], [57], [58]).

Lemma (1.1.7)[1]: Assume that φ is a holomorphic self-map of $\mathbb{B}$, $g \in H(\mathbb{B})$ and $g(0) = 0$ and μ is a weight. Let X be one of the following spaces $\mathcal{B}^{\alpha}_{log^{\beta}}$,$\mathcal{B}^{\alpha}_{log^{\beta},0}$, H(p, q , ϕ) and Y one of the spaces $\mathcal{B}^{\alpha}_{log^{\beta}}$,$\mathcal{B}^{\alpha}_{log^{\beta},0}$. Then the o6perator $P^{g}_{\varphi}: X \to Y$ is compact if and only if $P^{g}_{\varphi}: X \to Y$ is bounded and for every 4bounded sequence $(f_k)_{k\in\mathbb{N}} \subset X$ converging to *0* uniformly on compacts of $\mathbb{B}$ we have

$$\lim_{k\to\infty}\left\|P^{g}_{\varphi} f_k\right\|_{\gamma} = 0$$

The following lemma gives us some concrete examples of the functions belonging to logarithmic Bloch-type spaces.

Lemma (1.1.8)[1]: The following statements are true.

(a) Assume that $\alpha \neq 1$ *and* $\beta \geq 0$. then

$$f_w(z) = \frac{1}{(1-\langle z,w\rangle)^{\alpha-1}\left(In\dfrac{e^{\gamma}}{1-\langle z,w\rangle}\right)^{\beta}}, \quad w \in \mathbb{B}, \tag{9}$$

where $\gamma \geq \frac{\beta}{\alpha} + In2$, is a nonconstant function belonging to $\mathcal{B}^{\alpha}_{log^{\beta}}$:

(b) Assume that $\alpha = 1$ and $\beta \neq 1$.then

$$f^{(1)}_w(z) = \left(In\frac{e^{\gamma}}{1-\langle z,w\rangle}\right)^{1-\beta}, w \in \mathbb{B}, \tag{10}$$

where $\gamma \geq \beta + In\ 2$, is a non constant function belonging to $\mathcal{B}^{\alpha}_{log^{\beta}}$.

(c) Assume that $\alpha = 1$ and $\beta = 1$. then

$$f^{(2)}_w(z) = InIn\frac{e^{\gamma}}{1-\langle z,w\rangle}, w \in \mathbb{B}, \tag{11}$$

Where $\gamma \geq 1 + In2$, is a non constant function belonging to $\mathcal{B}^{\alpha}_{log^{\beta}}$.

Moreover, for each w $\in \mathbb{B}$, it holds that $f_w, f^{(1)}_w, f^{(2)}_w$ belong to the corresponding $\mathcal{B}^{\alpha}_{log^{\beta}}$ space and for each fixed α and β

$$\sup_{w\in\mathbb{B}}\|f_w\|_{\mathcal{B}^{\alpha}_{log^{\beta}}} \leq C, \quad \sup_{w\in\mathbb{B}}\left\|f^{(1)}_w\right\|_{\mathcal{B}^{1}_{log^{\beta}}} \leq C, \quad \sup_{w\in\mathbb{B}}\left\|f^{(2)}_w\right\|_{\mathcal{B}^{1}_{log^{1}}} \leq C. \tag{12}$$

Proof: (a) Let $w \in \mathbb{B}$ be fixed. Then we have

$$(1-|z|)^{\alpha}\left(In\frac{e^{\beta/\alpha}}{1-|z|}\right)^{\beta}|\Re f_w(z)| = (1-|z|)^{\alpha}\left(In\frac{e^{\beta/\alpha}}{1-|z|}\right)^{\beta}$$

$$\times \left| \frac{(\alpha-1)-\langle z,w\rangle}{(1-\langle z,w\rangle)^{\alpha}\left(In\frac{e^{\gamma}}{1-\langle z,w\rangle}\right)^{\beta}} - \frac{\beta\langle z,w\rangle}{(1-\langle z,w\rangle)^{\alpha}\left(In\frac{e^{\gamma}}{1-\langle z,w\rangle}\right)^{\beta+1}} \right|$$

$$\leq |\alpha-1| \frac{(1-|z|)^{\alpha}\left(In\frac{e^{\frac{\beta}{\alpha}}}{1-|z|}\right)^{\beta}}{|1-\langle z,w\rangle|^{\alpha}\left(In\frac{e^{\frac{\beta}{\alpha\gamma}}}{1-\langle z,w\rangle}\right)^{\beta}} + \beta\frac{(1-|z|)^{\alpha}\left(In\frac{e^{\frac{\beta}{\alpha}}}{1-|z|}\right)^{\beta}}{|1-\langle z,w\rangle|^{\alpha}\left(In\frac{e^{\frac{\beta}{\alpha\gamma}}}{1-\langle z,w\rangle}\right)^{\beta+1}}$$

$$\leq \left(|\alpha-1| + \frac{\beta}{In\frac{e^{\gamma}}{2}}\right)\frac{\big|1-|z|\big|^{\alpha}\left(In\frac{e^{\gamma}}{1-|z|}\right)^{\beta}}{|1-\langle z,w\rangle|^{\alpha}\left(In\frac{e^{\gamma}}{|1-\langle z,w\rangle|}\right)^{\beta}} \quad (13)$$

$$\leq |\alpha-1| + \frac{\beta}{In\frac{e^{\gamma}}{2}} \quad (14)$$

where in (14) we have used the fact that the function in (4) is increasing on the interval $(0,2)$. From (13), since 1 - $|w| \leq |1-\langle z,w\rangle|, z,w \in \mathbb{B}$ and by Lemma (1.1.1), we have that

$$(1-|z|)^{\alpha}\left(In\frac{e^{\frac{\beta}{\alpha}}}{1-|z|}\right)^{\beta}|\Re f_w(z)|$$

$$\leq \left(|\alpha-1| + \frac{\beta}{In\frac{e^{\gamma}}{2}}\right)\frac{\big|1-|z|\big|^{\alpha}\left(In\frac{e^{\gamma}}{1-|z|}\right)^{\beta}}{|1-\langle z,w\rangle|^{\alpha}\left(In\frac{e^{\gamma}}{|1-\langle z,w\rangle|}\right)^{\beta}} \to 0$$

As $|z| \to 1-0$, from which it follows that

$$f_w \in \mathcal{B}^{\alpha}_{log^{\beta},0} \ , as\ desired$$

(b) for fixed $w \in \mathbb{B}$ we have

$$(1-|z|)\left(In\frac{e^{\beta}}{1-|z|}\right)^{\beta}\left|\Re f_w^{(1)}(z)\right| = (1-|z|)\left(In\frac{e^{\beta}}{1-|z|}\right)^{\beta}$$

$$\left|\frac{(1-\beta)\langle z,w\rangle}{(1-\langle z,w\rangle)\left(In\frac{e^{\gamma}}{|1-\langle z,w\rangle|}\right)^{\beta}}\right|$$

$$\leq |\beta-1|\frac{(1-|z|)\left(In\frac{e^{\gamma}}{1-|z|}\right)^{\beta}}{(1-\langle z,w\rangle)\left(In\frac{e^{\gamma}}{1-\langle z,w\rangle}\right)^{\beta}} \quad (15)$$

$$\leq |\beta-1| \quad (16)$$

Where (16) as in (a) we have used the fact that the function in (4) is increasing on the interval (0,2] from (15) and by the Lemma (1.1.1), we obtain

$$(1-|z|)\left(In\frac{e}{1-|z|}\right)^{\beta}\left|\Re f_w^{(1)}(z)\right| \le |\beta-1|\frac{(1-|z|)\left(In\frac{e^{\beta}}{1-|z|}\right)^{\beta}}{(1-|w|)\left(In\frac{e^{\gamma}}{1-|w|}\right)^{\beta}} \to 0$$

As $|z| \to 1-0 i.e.\ f_w^{(2)} \in \mathcal{B}^1_{log^1,0}$ finishing the proof in this case.

$$(1-|z|)\left(In\frac{e}{1-|z|}\right)\left|\Re f_w^{(2)}(z)\right| = (1-|z|)\left(In\frac{e}{1-|z|}\right)\left|\frac{\langle z,w\rangle}{(1-\langle z,w\rangle)In\frac{e^{\gamma}}{1-\langle z,w\rangle}}\right|$$

$$\le \frac{(1-|z|)In\frac{`e}{1-|z|}}{(1-\langle z,w\rangle)In\frac{e^{\gamma}}{1-\langle z,w\rangle}} \quad (17)$$

$$\le \frac{(1-|z|)In\frac{`e^{\gamma}}{1-|z|}}{(1-\langle z,w\rangle)In\frac{e^{\gamma}}{1-|z|}} \le 1. \quad (18)$$

Where again we have used the fact that function (4) is increasing in (0.2].

From (17), Lemma (1.1.1) and since $\gamma > 1$ we obtain

$$(1-|z|)\left(In\frac{e}{1-|z|}\right)\left|\Re f_w\ (z)\right| \le \frac{(1-|z|)\left(In\frac{e^{\gamma}}{1-|z|}\right)}{(1-|w|)\left(In\frac{e^{\gamma}}{1-|w|}\right)} \to 0,$$

As $|z| \to 1^-,\ i.e. f_w^{(2)} \in \mathcal{B}^1_{log^1,0.}$

Estimate (12) follows from (14), (16), and since

$$f_w(0) = \frac{1}{\gamma^{\beta}}, f_w^{(1)}(0) = \gamma^{1-\beta},, f_w^{(2)}(0) = In\gamma.$$

Finishing the proof of the lemma.

The following theorem summarizes some of the basic properties of the lograthmic Bloch- type space $\mathcal{B}^{\alpha}_{log^{\beta}}$ and the Little lograthmic Bloch- type space $\mathcal{B}^{\alpha}_{log^{\beta,0}}$. it can be proved.

Proposition (1.1.9)[1]: The following statements are true.

(a) The logarithmic Bloch- type space $\mathcal{B}^{\alpha}_{log^{\beta}}$ is Banach with the norm given in (1).

(b) $\mathcal{B}^{\alpha}_{log^{\beta},0}$ Is a closed subset of $\mathcal{B}^{\alpha}_{log^{\beta},}$

(c) Assume $f \in \mathcal{B}^{\alpha}_{log^{\beta}}$. then $f \in \mathcal{B}^{\alpha}_{log^{\beta,0}}$ if and only if $\lim_{r\to 1^-}\|f-f_r\|_{\mathcal{B}^{\alpha}_{log^{\beta},}} = 0$

(d) The set of all polynomials is dense in $\mathcal{B}^{\alpha}_{log^{\beta},0,}$

(e) Assume $f \in \mathcal{B}^{\alpha}_{log^{\beta}}$. Then for each $[0,1), f_r \in \mathcal{B}^{\alpha}_{log^{\beta,0}}$.

Moreover $\|f_r\|_{\mathcal{B}^{\alpha}_{log^{\beta},}} \le \|f\|_{\mathcal{B}^{\alpha}_{log^{\beta},}}$

Lemma (1.1.10)[1]: ([24]). Assume $0 < p, q < \infty$, and ϕ is normal. Then there is a positive constant C independent of f, such that

$$|f(z)| \leq C \frac{\|f\|_{H(p,q,\phi)}}{\phi(|z|)(1-|z|^2)^{\eta/q}}, \qquad z \in \mathbb{B}. \tag{19}$$

We characterize the boundedness and compactness of the operator $P_\varphi^g: \mathcal{B}_{log^\beta}^\alpha(\mathbb{B}) \left(or\ \mathcal{B}_{log^\beta,0}^\alpha(\mathbb{B})\right) \to \mathcal{B}_\mu(\mathbb{B}) \left(or\ \mathcal{B}_{\mu,0}(\mathbb{B})\right)$.

Case $\alpha > 1$ and $\beta \geq 0$

Theorem (1.1.11)[1]: assume that $\alpha < 1,\ \beta \geq 0, g \in H(\mathbb{B}), g(0) = 0,\ \mu$ is a weight, and φ is a holomorphic self-map self-map of $\mathbb{B}$, then $P_\varphi^g: \mathcal{B}_{log^\beta}^\alpha(\mathbb{B}) \left(or\ \mathcal{B}_{log^\beta,0}^\alpha(\mathbb{B})\right) \to \mathcal{B}_\mu(\mathbb{B})$ is bounded if and only if

$$M := \sup_{z\in\mathbb{B}} \frac{\mu(z)|g(z)|}{(1-|\varphi(z)|)^{\alpha-1}\left(In \frac{e^{\beta/\alpha}}{1-|\varphi(z)|}\right)^\beta} < \infty \tag{20}$$

And $g \in H_\mu^\infty$. Moreover, $P_\varphi^g: \mathcal{B}_{log^\beta}^\alpha(\mathbb{B}) \left(or\ \mathcal{B}_{log^\beta,0}^\alpha(\mathbb{B})\right) \to \mathcal{B}_\mu$ is bounded, then

$$\left\|P_\varphi^g\right\|_{\mathcal{B}_{log^\beta}^\alpha \to \mathcal{B}_\mu} \asymp \left\|P_\varphi^g\right\|_{\mathcal{B}_{log^\beta m,0}^\alpha \to \mathcal{B}_\mu} \asymp M + \|g\|_{H_\mu^\infty}. \tag{21}$$

Proof. Assume that (20) holds and $g \in H_\mu^\infty, . If\ \ f \in \mathcal{B}_{log^\beta}^\alpha(\mathbb{B}) \left(or\ \mathcal{B}_{log^\beta,0}^\alpha(\mathbb{B})\right)$, then by Lemma (1.1.6) and 4 we obtain

$$\left\|P_\varphi^g f\right\|_{\mathcal{B}_\mu} = sup_{z\in\mathbb{B}}\mu(z)\left|g(z)f(\varphi(z))\right| \leq C\|f\|_{\mathcal{B}_{log^\beta}^\alpha} sup_{z\in\mathbb{B}}\mu(z)|g(z)| \left(1 + \frac{1}{(1-|\varphi(z)|)^{\alpha-1}\left(In \frac{e^{\beta/\alpha}}{1-|\varphi(z)|}\right)^\beta}\right) \tag{22}$$

from which it follows that

$$\left\|P_\varphi^g\right\|_{\mathcal{B}_{log^\beta}^\alpha \to \mathcal{B}_\mu} \leq C\left(M + \|g\|_{H_\mu^\infty}\right) \tag{23}$$

Now assume that $P_\varphi^g: \mathcal{B}_{log^\beta,0}^\alpha \to \mathcal{B}_\mu$ is bounded by taking functions f_w in (9) which belong to $\mathcal{B}_{log^\beta,0}^\alpha$ and whose norms are bounded according to Lemma (1.1.8), and by using the boundedness of $P_\varphi^g: \mathcal{B}_{log^\beta,0}^\alpha \to \mathcal{B}_\mu$, we have

$$\begin{aligned} C\left\|P_\varphi^g\right\|_{\mathcal{B}_{log^\beta}^\alpha \to \mathcal{B}_\mu} &\geq \left\|f_{\varphi(w)}\right\|_{\mathcal{B}_{log}} \left\|P_\varphi^g\right\|_{\mathcal{B}_{log^\beta,0}^\alpha \to \mathcal{B}_\mu} \geq \left\|P_\varphi^g f_{\varphi(w)}\right\|_{\mathcal{B}_\mu} \\ &= sup_{z\in\mathbb{B}}\mu(z)|g(z)|\left|f_{\varphi(w)}(\varphi(w))\right| \\ &\geq \frac{\mu(w)|g(w)|}{(1-|\varphi(z)|)^{\alpha-1}\left(In \frac{e^{\beta/\alpha}}{1-|\varphi(z)|^2}\right)^{\beta(\ \ )}}. \end{aligned} \tag{24}$$

For every $w \in \mathbb{B}$, from which (20) direct follows in case

$$\beta = 0.$$

Now assume $\beta > 0$. Then from $\gamma > \frac{\beta}{\alpha}$ and $\alpha > 1$ we easily obtain

$$\left|1-|z|\right|^{\alpha-1}\left(in \frac{e^{\frac{\beta}{\alpha}}}{1-|z|}\right)^\beta \leq C(1-|z|^2)^{\alpha-1}\left(in \frac{e^\gamma}{1-|z|^2}\right)^\beta$$

$$\leq C2^{\alpha-1}\left(\frac{\gamma\alpha}{\beta}\right)^{\beta}(1-|Z|)^{\alpha-1}\left(in\frac{e^{\beta/\alpha}}{1-|z|}\right)^{\beta} \quad (25)$$

Hence from (24) and (25) we obtain

$$C\left\|P_{\varphi}^{g}\right\|_{\mathcal{B}_{log^{\beta},0}^{\alpha}\to\mathcal{B}_{\mu}} \geq \frac{\mu(z)|g(z)|}{(1-|\varphi(z)|)^{\alpha-1}\left(In\frac{e^{\beta/\alpha}}{1-|\varphi(z)|}\right)^{\beta}} \quad (26)$$

Thus (20) follows.

On the other hand, if we choose the function given by $h_0(z)\equiv 1\in\mathcal{B}_{log^{\beta,},0}^{\alpha}$ we obtain that

$$\left\|P_{\varphi}^{g}\right\|_{\mathcal{B}_{log^{\beta},0}^{\alpha}\to\mathcal{B}_{\mu}} = \|h_0\|_{\mathcal{B}_{log^{\beta},0}^{\alpha}}\left\|P_{\varphi}^{g}\right\|_{\mathcal{B}_{log^{\beta}}^{\alpha}\to\mathcal{B}_{\mu,0}} \geq \left\|P_{\varphi}^{g}(h_0)\right\|_{\mathcal{B}_{\mu}} = \|g\|_{H_{\mu}^{\infty}} \quad (27)$$

From (26) and (27) we obtain

$$C\left\|P_{\varphi}^{g}\right\|_{\mathcal{B}_{log^{\beta},0}^{\alpha}\to\mathcal{B}_{\mu}} \geq M+\|g\|_{H_{\mu}^{\infty}} \quad (28)$$

From (32), (28), and since

$$\left\|P_{\varphi}^{g}\right\|_{\mathcal{B}_{log^{\beta}}^{\alpha}\to\mathcal{B}_{\mu}} \geq \left\|P_{\varphi}^{g}\right\|_{\mathcal{B}_{log^{\beta}m,0}^{\alpha}\to\mathcal{B}_{\mu}}$$

The asymptotic relationships in (21) follow.

We characterize the compactness of the operator $P_{\varphi}^{g}:\mathcal{B}_{log^{\beta}}^{\alpha}\left(or\ \mathcal{B}_{log^{\beta},0}^{\alpha}\right)\to\mathcal{B}_{\mu}$

Theorem (1.1.12)[1]: Assume that $\alpha>1,\ \beta\geq 0, g\in H(\mathbb{B}), g(0)=0,\ \mu$ is a weight, and φ is a holomorphic self-map of $\mathbb{B}$, and $P_{\varphi}^{g}:\mathcal{B}_{log^{\beta}}^{\alpha}(\mathbb{B})\left(or\ \mathcal{B}_{log^{\beta},0}^{\alpha}\right)\to\mathcal{B}_{\mu}$ is bounded .Then $P_{\varphi}^{g}:\mathcal{B}_{log^{\beta}}^{\alpha}(\mathbb{B})\left(or\ \mathcal{B}_{log^{\beta},0}^{\alpha}\right)\to\mathcal{B}_{\mu}$ is compact if and only if

$$\lim_{|\varphi(z)|\to 1}\mu(z)\,|g(z)|\left(1+\frac{1}{(1-|\varphi(z)|)^{\alpha-1}\left(In\frac{e^{\beta/\alpha}}{1-|\varphi(z)|}\right)^{\beta}}\right)=0 \quad (29)$$

Proof. First assume that $P_{\varphi}^{g}:\mathcal{B}_{log^{\beta}}^{\alpha}(\mathbb{B})\left(or\ \mathcal{B}_{log^{\beta},0}^{\alpha}\right)\to\mathcal{B}_{\mu}$ is compact. If $\|\varphi\|_{\infty}<1$, then (29) is vacuously satisfied. Hence, assume $\|\varphi\|_{\infty}=1$ and let $\left(\varphi(z_m)\right)_{m\in\mathbb{N}}$.

Be a sequence in $\mathbb{B}$ such that $|\varphi(z_m)|\to 1$ as $m\to\infty$.

$$f_m(z)=\frac{\left(f_{\varphi(zm)}(z)\right)^{2}}{f_{\varphi(zm)}\left(\varphi(zm)\right)}\quad m\in\mathbb{N}. \quad (30)$$

Where f_w is defined in (9). As in Lemma (1.1.8) it can be seen that $(f_m)_{m\in\mathbb{N}}$ is bounded sequence in $\mathcal{B}_{log^{\beta},0}^{\alpha}$, and that it converges to zero uniformly on compact subsets of $\mathbb{B}$ as $m\to\infty$. Hence, by Lemma (1.1.7), it follows that

$$\lim_{m\to\infty}\left\|P_{\varphi}^{g}F_m\right\|_{\mathcal{B}_{\mu}} = 0. \quad (31)$$

On the other hand, for each $m\in\mathbb{N}$, we have

$$\begin{aligned}\left\|P_{\varphi}^{g}F_m\right\|_{\mathcal{B}_{\mu}} &= sup_{z\in\mathbb{B}}\mu(z)|g(z)||F_m\varphi(z)| \\ &\geq \mu(z_m)\left|\left(g(z_m)\right)F_m\varphi(z_m)\right|.\end{aligned} \quad (32)$$

Letting $m\to\infty$ in (32) and using (31), we obtain

$$\lim_{m\to\infty} sup \frac{\mu(z_m)|g(z_m)|}{(1-|\varphi(z)|^2)^{\alpha-1}\left(In\frac{e^{\gamma}}{1-|\varphi(z)|^2}\right)^{\beta(\ )}} = 0. \tag{33}$$

From (33), and since in this case

$$\lim_{m\to\infty} sup \frac{\mu(z_m)|g(z_m)|}{(1-|\varphi(z)|^2)^{\alpha-1}\left(In\frac{e^{\gamma}}{1-|\varphi(z)|^2}\right)^{\beta}} \geq \lim_{m\to\infty} sup\mu(z_m)|g(z_m)|.$$

We have that

$$\lim_{m\to\infty} \sup \mu(z_m)|g(z_m)| = 0. \tag{34}$$

From (25), (33) and (34) equality (29) easily follows.

Now assume that (29) holds. Since $P_{\varphi}^{g}: B_{\log^{\beta}}^{\alpha}$ $\left(\text{or } B_{\log^{\beta}\ ,0}^{\alpha}\right)$ is bounded, as in Theorem (1.1.11), we obtain $\epsilon > 0$ from (29) we have that for every there is a such that

$$\mu(z)\,|g(z)| \left(1 + \frac{1}{(1-|\varphi(z)|)^{\alpha-1}\left(In\frac{e^{\frac{\beta}{\alpha}}}{1-|\varphi(z)|}\right)^{\beta}} \right) < \varepsilon. \tag{35}$$

When $\rho < |\varphi(z)| < 1$.

Assume that $\|h_m\|_{m\in\mathbb{N}}$ is a bounded sequence in $\mathcal{B}_{log^{\beta}}^{\alpha}\left(or\ \mathcal{B}_{log^{\beta},0}^{\alpha}\right)$ say by Lm converging to 0 uniformly on compacts of B. then, by Lemma (1.1.6) and (1.1.4) and the fact that $g \in H_{\mu}^{\infty}$, we have

$$\begin{aligned}
\left\|P_{\varphi}^{g} h_m\right\|_{\mathcal{B}_{log^{\beta},}^{\alpha}\to\mathcal{B}_{\mu}} &= sup_{z\in\mathbb{B}}\mu(z)|g(z)||h_m\varphi(z)| \\
&\leq sup_{|\varphi(z)|\leq\rho}\mu(z)|g(z)||h_m\varphi(z)| \\
&\quad + sup_{|\varphi(z)|>\rho}\mu(z)|g(z)||h_m\varphi(z)| \\
&\leq \|g\|_{H_{\mu}^{\infty}}\, sup_{|w|\leq\rho}|h_m(w)| \\
&\quad + C\ sup_{m\in\mathbb{N}}\|h_m\|_{\mathcal{B}_{log^{\beta},}^{\alpha}} sup_{|\varphi(z)|>\rho}\ \mu(z)\ |g(z)|
\end{aligned}$$

$$\left(1 + \frac{1}{(1-|\varphi(z)|)^{\alpha-1}\left(In\frac{e^{\frac{\beta}{\alpha}}}{1-|\varphi(z)|}\right)^{\beta}} \right) \leq\ \|g\|_{H_{\mu}^{\infty}}\, sup_{|w|\leq\rho}|h_m(w)| + \varepsilon L. \tag{36}$$

Letting $m \to \infty$. in (36) using the assumption $sup_{|w|\leq\rho}|h_m(w)| \to 0$ as $m \to \infty$. The fact that ε is an arbitrary positive number and applying Lemma (1.1.7), the compactness of the operator. $P_{\varphi}^{g}: \mathcal{B}_{log^{\beta}}^{\alpha}(\mathbb{B})\left(or\ \mathcal{B}_{log^{\beta},0}^{\alpha}\right) \to \mathcal{B}_{\mu}$ follows.

The following theorem characterizes the boundedness of the operator $P_\varphi^g : \mathcal{B}^\alpha_{log^\beta,0} \to \mathcal{B}_{\mu,0}$ but for all $\alpha > 0$ *and* $\beta \geq 0$.

Theorem (1.1.13)[1]: assume that $\alpha > 0$, $\beta \geq 0, g \in H(\mathbb{B}), g(0) = 0$, μ is a weight, and φ is a holomorphic self-map of $\mathbb{B}$, then $P_\varphi^g : \mathcal{B}^\alpha_{log^\beta,0} \to \mathcal{B}_{\mu,0}$ is bounded if and only if $P_\varphi^g : \mathcal{B}^\alpha_{log^\beta,0} \to \mathcal{B}_\mu$ is bounded and $g \in H^\infty_{\mu,0}$.

Proof. Assume that $P_\varphi^g : \mathcal{B}^\alpha_{log^\beta,0} \to \mathcal{B}_{\mu,0}$ is bounded. Then clearly $P_\varphi^g : \mathcal{B}^\alpha_{log^\beta,0} \to \mathcal{B}_\mu$ is bounded. Taking the test function $\widehat{f}\,(z) \equiv 1 \in \mathcal{B}^\alpha_{log^\beta,0}$ we obtain $g \in H^\infty_{\mu,0}$.

Conversely, assume $P_\varphi^g : \mathcal{B}^\alpha_{log^\beta,0} \to \mathcal{B}_\mu$ is bounded and $g \in H^\infty_{\mu,0}$ then for every polynomial p, we have

$$\mu(z)\left|\Re\, P_\varphi^g p(z)\right| = \mu(z)\left|g(z)p\big(\varphi(z)\big)\right| \leq \mu(z)|g(z)|\|p\|_\infty \to 0, as|z| \to 1.$$

From which it follows that $P_\varphi^g p \in \mathcal{B}^\alpha_{log^\beta,0}$. Since by Proposition (1.1.9)*(d)* the set of all polynomials is dense in $\mathcal{B}^\alpha_{log^\beta,0}$, we have that for every $f \in \mathcal{B}^\alpha_{log^\beta,0}$ there is a sequence of polynomials $(p_m)_{m\in\mathbb{N}}$ such that

$$\lim_{m\to\infty} \|f - p_m\|_{\mathcal{B}^\alpha_{log^\beta}}, = \quad 0,$$

From this, and since the operator $P_\varphi^g : \mathcal{B}^\alpha_{log^\beta,0} \to \mathcal{B}_\mu$ is bounded, we have that

$$\left\| P_\varphi^g f - P_\varphi^g p_m\right\|_{\mathcal{B}_\mu} \leq \left\| P_\varphi^g\right\|_{\mathcal{B}^\alpha_{log^\beta,0}\to\mathcal{B}_\mu} \|f - p_m\|_{\mathcal{B}^\alpha_{log^\beta}}, \to 0$$

As $m \to \infty$. Hence $P_\varphi^g\left(\mathcal{B}^\alpha_{log^\beta,0}\right) \subseteq. \mathcal{B}_{\mu,0}$. Therefore the operator $P_\varphi^g : \mathcal{B}^\alpha_{log^\beta,0} \to \mathcal{B}_\mu$ is bounded.

We characterize the compactness of the operator. $P_\varphi^g : \mathcal{B}^\alpha_{log^\beta,}(\mathbb{B}) \left(or\ \mathcal{B}^\alpha_{log^\beta,0}\right) \to \mathcal{B}_{\mu,0}$.

Theorem (1.1.14)[1]: Assume that $\alpha > 1$, $\beta \geq 0, g \in H(\mathbb{B}), g(0) = 0$, μ is a weight, and φ is a holomorphic self-map of $\mathbb{B}$, and $P_\varphi^g : \mathcal{B}^\alpha_{log^\beta}(\mathbb{B}) \left(or\ \mathcal{B}^\alpha_{log^\beta,0}\right) \to \mathcal{B}_{\mu,0}$ is bounded. Then $P_\varphi^g : \mathcal{B}^\alpha_{log^\beta}(\mathbb{B}) \left(or\ \mathcal{B}^\alpha_{log^\beta,0}\right) \to \mathcal{B}_{\mu,0}$ is compact if and only if

$$\lim_{|z|\to 1} \mu(z)\,|g(z)| \left(1 + \frac{1}{(1 - |\varphi(z)|)^{\alpha-1}\left(In\dfrac{e^{\beta/\alpha}}{1 - |\varphi(z)|}\right)^\beta} \right) = 0. \qquad (37)$$

Proof: Assume that

$$P_\varphi^g : \mathcal{B}^\alpha_{log^\beta}(\mathbb{B}) \left(or\ \mathcal{B}^\alpha_{log^\beta,0}(\mathbb{B})\right) \to \mathcal{B}_{\mu,0} \quad is\ compact\ \ and\ g \in H^\infty_{\mu,0}.$$

By Theorem (1.1.12) we have that (29) holds.

By (29) we have that, for every $\varepsilon > 0$ there exists an $r \in (0,1)$ such that

$$\mu(z)|g(z)| = \left(1 + \frac{1}{(1 - \varphi|z|)^{\alpha-1}\left(In\dfrac{e^{\beta/\alpha}}{1 - \varphi|z|}\right)^\beta} \right), \quad < \varepsilon$$

When $r < \varphi|z| < 1$.

since $g \in H^\infty_\mu$, there exists $\rho \in (0,1)$ such that

$$\mu(z)|g(z)| < \varepsilon \left(1 + \frac{1}{\left(inf_{t\in[o,r]} 1 - t\right)^{\alpha-1} \left(In \frac{e^{\beta/\alpha}}{1-t}\right)^{\beta}} \right)^{-1}, \tag{38}$$

When $\rho < |z| < 1$.

Therefore, when $\rho < |z| < 1 \; and \;\; r < \varphi|z| < 1$, we have that

$$\mu(z)|g(z)| = \left(1 + \frac{1}{(1-\varphi|z|)^{\alpha-1} \left(In \frac{e^{\beta/\alpha}}{1-\varphi|z|}\right)^{\beta}} \right) < \varepsilon \tag{39}$$

On the other hand, if $\rho < |z| < 1 \; and \;\; \varphi|z| \leq 1$, from (98) we have

$$\mu(z)|g(z)| = \left(1 + \frac{1}{(1-\varphi|z|)^{\alpha-1} \left(In \frac{e^{\beta/\alpha}}{1-\varphi|z|}\right)^{\beta}} \right) < \varepsilon \tag{40}$$

Combining (39) and (40), we obtain that

$$\mu(z)|g(z)| = \left(1 + \frac{1}{(1-\varphi|z|)^{\alpha-1} \left(In \frac{e^{\beta/\alpha}}{1-\varphi|z|}\right)^{\beta}} \right) < \varepsilon \tag{41}$$

For $\rho < |z| < 1$ (42) the condition in (37) follows.

Now assume that (37) holds. Then (20) holds and by the Theorem (1.1.11) we have that $P_{\varphi}^{g}\left(\left\{f : \|f\|_{\mathcal{B}^{\alpha}_{log^{\beta}}} \leq 1\right\}\right)$ is a bounded set in $\mathcal{B}_{\mu}$.

From the following inequality

$\mu(z)\left|\Re P_{\varphi}^{g}(f)(z)\right| = \mu(z)\left|g(z) f \varphi |z|\right|$

$$\leq C\|f\|_{\mathcal{B}^{\alpha}_{log^{\beta}}} \mu(z)|g(z)| \left(1 + \frac{1}{(1-\varphi|z|)^{\alpha-1} \left(In \frac{e^{\frac{\beta}{\alpha}}}{1-\varphi|z|}\right)^{\beta}} \right) \tag{42}$$

And (37) we have more, namely that $P_{\varphi}^{g}\left(\left\{f : \|f\|_{\mathcal{B}^{\alpha}_{log^{\beta}}} \leq 1\right\}\right)$ is a bounded set in $\mathcal{B}_{\mu,0}$.

Taking the supremum in (42) over the unit ball in $\mathcal{B}^{\alpha}_{log^{\beta}}\left(or\, \mathcal{B}_{log_{k,0}}\right)$. Then letting $|z| \to 1$, using conditions (37) and employing Lemma (1.1.5), we obtain the compactness of the operator $P_{\varphi}^{g} : \mathcal{B}^{\alpha}_{log^{\beta}}\left(or\;\, \mathcal{B}^{\alpha}_{log^{\beta,0}}\right) \to \mathcal{B}_{\mu,0}$, as desired.

Case $\alpha = 1$ **and** $\beta \in (0,1)$

Theorem (1.1.15)[1]: Assume that $\alpha = 1,\;\; \beta \geq 0, g \in H(\mathbb{B}), g(0) = 0,\; \mu$ is a weight, and φ is a holomorphic self-map of $\mathbb{B}$, and $P_{\varphi}^{g} : \mathcal{B}^{\alpha}_{log^{\beta}}\left(or\, \mathcal{B}^{\alpha}_{log^{\beta},0}\right) \to \mathcal{B}_{\mu,}$ is bounded if and only if

$$M_1 := sup_{z\in\mathbb{B}}\mu(z)|g(z)|\left(In\frac{e^{\beta}}{1-\varphi|z|}\right)^{1-\beta} < \infty, \qquad (43)$$

$g \in H_{\mu}^{\infty}$.

Moreover, if $P_{\varphi}^{g}: \mathcal{B}_{log^{\beta}}^{1}\left(or\ \mathcal{B}_{log^{\beta},0}^{1}\right) \to \mathcal{B}_{\mu,}$ is bounded then

$$\left\|P_{\varphi}^{g}\right\|_{\mathcal{B}_{log^{\beta}}^{1}\to\mathcal{B}_{\mu,}} \asymp M_1 + \|g\|_{H_{\mu}^{\infty}}$$

Proof: assume that (43) holds and $g \in H_{\mu,0}^{\infty}$, if f $\in \mathcal{B}_{log^{\beta}}^{1}\left(or\ \mathcal{B}_{log^{\beta},0}^{1}\right)$ then by Lemma (1.1.5) and (1.1.4) we obtain

$$\left\|P_{\varphi}^{g}f\right\|_{\mathcal{B}_{\mu,}} = sup_{z\in\mathbb{B}}\mu(z)|g(z)f\varphi(z)|$$

$$\leq C\|f\|_{\mathcal{B}_{log^{\beta}}^{1}} = sup_{z\in\mathbb{B}}\ \mu(z)|g(z)|\ \left(1+\ \left(In\frac{e^{\beta}}{1-\varphi|z|}\right)^{1-\beta}\right) \qquad (45)$$

From which it follow that

$$\left\|P_{\varphi}^{g}\right\|_{\mathcal{B}_{log^{\beta}}^{1}\to\mathcal{B}_{\mu,}} \leq C\left(M_1+\|g\|_{H_{\mu,}^{\infty}}\right) \qquad (46)$$

From (47) and (48) we obtain

$$C\left\|P_{\varphi}^{g}\right\|_{\mathcal{B}_{log^{\beta},0}^{1}\to\mathcal{B}_{\mu,}} \geq sup_{z\in\mathbb{B}}\ \mu(z)|g(z)|\ \left(In\frac{e^{\beta}}{1-\varphi|z|}\right)^{1-\beta}$$

So that (43) holds.

On the other hand, if we choose the function given by $h_0(z) \equiv 1 \in \mathcal{B}_{log^{\beta,0}}^{1}$ we obtain $g \in H_{\mu,}^{\infty}$ and that (27) with holds $\alpha = 1$.

This along with (49) implies the following inequality

$$C\left\|P_{\varphi}^{g}\right\|_{\mathcal{B}_{log^{\beta,0}}^{1}\to\mathcal{B}_{\mu,}} \geq M_1+\|g\|_{H_{\mu,}^{\infty}} \qquad (50)$$

THE asymptotic relationship in (44) follow from (46), (50) and the inequality

$$\left\|P_{\varphi}^{g}\right\|_{\mathcal{B}_{log^{\beta}}^{1}\to\mathcal{B}_{\mu,}} \geq \left\|P_{\varphi}^{g}\right\|_{\mathcal{B}_{log^{\beta,0}}^{1}\to\mathcal{B}_{\mu,}}$$

Theorem (1.1.16)[1]: Assume that $\alpha = 1$, $\beta \in (0,1) g \in H(\mathbb{B}), g(0) = 0$, μ is a weight, and φ is a holomorphic self-map of $\mathbb{B}$, and $P_{\varphi}^{g}: \mathcal{B}_{log^{\beta}}^{\alpha}(\mathbb{B})\left(or\ \mathcal{B}_{log^{\beta},0}^{\alpha}\right) \to \mathcal{B}_{\mu}$ is bounded. Then $P_{\varphi}^{g}: \mathcal{B}_{log^{\beta}}^{\alpha}(\mathbb{B})\left(or\ \mathcal{B}_{log^{\beta},0}^{\alpha}\right) \to \mathcal{B}_{\mu}$ is compact, if and only if

$$\lim_{|\varphi(z)|\to 1}\mu(z)\,|g(z)|\left(1+\left(In\frac{e^{\beta}}{1-|\varphi(z)|}\right)^{1-\beta}\right) = 0 \qquad (51)$$

$P_{\varphi}^{g}: \mathcal{B}_{log^{\beta}}^{1}\left(or\ \mathcal{B}_{log^{\beta,0}}^{1}\right) \to \mathcal{B}_{\mu}$ is compact if $\|\varphi\|_{\infty} < 1$. such that and $g \in H_{\mu,0}^{\infty}$

$$\mathrm{F}_{\mathrm{m}}^{(1)}(\mathrm{z}) = \frac{\left(\mathrm{f}_{\varphi(\mathrm{zm})}^{(1)}(\mathrm{z})\right)^{2}}{\left(\mathrm{f}_{\varphi(\mathrm{zm})}^{(1)}(\varphi(\mathrm{z_m}))\right)}\mathrm{m} \in \mathbb{N} \qquad (52)$$

It can be seen that $\left(\mathrm{F}_{\mathrm{m}}^{(1)}\right)_{\mathrm{m}\in\mathbb{N}}$ is a bounded sequence in $\mathcal{B}_{log^{\beta,0}}^{1}$ and that it converges to zero

uniformly on compact subsets of $\mathbb{B}$ as $m \to \infty$.
Hence by Lemma (1.1.7), it follows that

$$\lim_{m\to\infty} \left\| P_\varphi^g \mathrm{F}_\mathrm{m}^{(1)} \right\|_{\mathcal{B}_\mu} = 0 \tag{53}$$

On the other hand, for each $\mathrm{m} \in \mathbb{N}$, we have

$$\left\| P_\varphi^g \mathrm{F}_\mathrm{m}^{(1)} \right\|_{\mathcal{B}_\mu} = sup_{z\in\mathbb{B}}\, \mu(z)|g(z)| \left|\mathrm{F}_\mathrm{m}^{(1)}\, \varphi(z)\right| \geq \mu(z_m) \left| g(z_m)\mathrm{F}_\mathrm{m}^{(1)}\, \varphi(z)\right| \tag{54}$$

Letting in (54) and using (53) we obtain

$$\lim_{m\to\infty} sup \mu(z_m)|g(z_m)| \left(In \frac{e^\gamma}{1-|\varphi(z)|^2}\right)^{1-\beta} = 0. \tag{55}$$

From (55), and since in this case

$$\lim_{m\to\infty} sup\; \mu(z_m)|g(z_m)| \left(In \frac{e^\gamma}{1-|\varphi(z)|^2}\right)^{1-\beta} \geq \lim_{m\to\infty} sup \mu(z_m)|g(z_m)|.$$

We have that

$$\lim_{m\to\infty} sup\; \mu(z_m)|g(z_m)| = 0. \tag{56}$$

From (55), (56) and (48), (51) follows.

Now assume that (51) holds, then for every $\varepsilon > 0$ there is a $\rho \in (0,1)$ such that

$$\mu(z)|g(z)| = \left(1 + \left(In \frac{e^\beta}{1-\varphi|z|}\right)^{1-\beta}\right), \quad < \varepsilon, \tag{57}$$

When $\rho < \varphi|z| < 1$. Assume that $(h_m)_{m\in\mathbb{N}}$ is a bounded sequence in: $\mathcal{B}^1_{log^\beta} \left(or\ \mathcal{B}^1_{log^{\beta,0}}\right)$ say by L_1 converging to 0 uniformly on compacts of $\mathbb{B}$ then by Lemma (1.1.6) and 4 and the fact that $g \in H_\mu^\infty$, we have

$$\begin{aligned}
\| P_\varphi^g h_m\|_{\mathcal{B}^1_{log^\beta} \to \mathcal{B}_\mu} &= sup_{z\in\mathbb{B}}\, \mu(z)\big|g(z)\big|\big|h_m\big(\varphi(z)\big)\big| \\
&\leq sup_{|\varphi(Z)|\leq\rho} \mu(z)|g(z)|\big|h_m\big(\varphi(z)\big)\big| \\
&\quad + sup_{|\varphi(Z)|>\rho} \mu(z)\big|g(z)\big|\big|h_m\big(\varphi(z)\big)\big| \\
\leq \|g\|_{H_\mu^\infty} sup_{|w|\leq\rho} |h_m(w)| &+ C sup_{m\in\mathbb{N}} \|h_m\|_{\mathcal{B}^1_{log^\beta}}\, sup_{|\varphi(Z)|>\rho} \mu(z)|g(z)| \\
&\left(1 + \left(In \frac{e^\beta}{1-\varphi|z|}\right)^{1-\beta}\right) \\
&\leq \|g\|_{H_\mu^\infty} sup_{|w|\leq\rho} |h_m(w)| + C\varepsilon L_1.
\end{aligned} \tag{58}$$

From (58) the compactness of the operator : $P_\varphi^g : \mathcal{B}^1_{log^\beta} \left(or\ \mathcal{B}^1_{log^{\beta,0}}\right) \to \mathcal{B}_\mu$ Follows as in the proof of Theorem (1.1.12).

Now we characterize the compactness of the operator $P_\varphi^g : \mathcal{B}^1_{log^\beta} \left(or\ \mathcal{B}^1_{log^{\beta,0}}\right) \to \mathcal{B}_{\mu,0}$

Theorem (1.1.17)[1]: Assume that $\beta \in (0,1)$, or $\alpha = 1$ and $g \in H(\mathbb{B}), g(0) = 0$, μ is a weight, and φ is a holomorphic self-map of $\mathbb{B}$, and $P_\varphi^g : \mathcal{B}^\alpha_{log^\beta}(\mathbb{B}) \left(or\ \mathcal{B}^\alpha_{log^\beta,0}\right) \to \mathcal{B}_{\mu,0}$ is compact if and only if

$$\lim_{|z|\to 1} \mu(z)|g(z)| = \left(1 + \left(In \frac{e^\beta}{1-\varphi|z|}\right)^{1-\beta}\right) = 0 \tag{59}$$

Proof. Assume that $P_\varphi^g : \mathcal{B}^1_{log^\beta}(\mathbb{B}) \left(or\ \mathcal{B}^1_{log^\beta,0}\right) \to \mathcal{B}_{\mu,0}$ is compact. Then the operator

$P_\varphi^g: \mathcal{B}^1_{log^\beta}(\mathbb{B})\left(or\ \mathcal{B}^1_{log^\beta,0}\right) \to \mathcal{B}_{\mu,0}$ is compact and $g \in H^\infty_{\mu,0}$. By Theorem (1.1.16) we have that (51) holds. Hence, for every $\varepsilon > 0$, there exists an $r \in (0,1)$ such that

$$lim_{|z|\to 1}\mu(z)|g(z)| = \left(1 + \left(In\frac{e^\beta}{1-\varphi|z|}\right)^{1-\beta}\right) < \varepsilon$$

Where $r < |\varphi(z)| < 1$.

Since $g \in H^\infty_{\mu,0}$, there exist $a\ \rho \in (0,1)$such that

$$\mu(z)|g(z)| < \varepsilon\left(1 + \left(In\frac{e^\beta}{1-r}\right)^{1-\beta}\right)^{-1} \tag{60}$$

When $\rho < |z| < 1$.

Therefore, when $\rho < |z| < 1\ and\ r < |\varphi(z)| < 1$, we have that

$$\mu(z)|g(z)|\left(1 + \left(In\frac{e^\beta}{1-|\varphi(z)|}\right)^{1-\beta}\right) < \varepsilon \tag{61}$$

On the other hand , if $\rho < |z| < 1$.and $|\varphi(z)| \le r$.from (60) we have

$$\mu(z)|g(z)|\left(1 + \left(In\frac{e^\beta}{1-|\varphi(z)|}\right)^{1-\beta}\right) < \varepsilon \tag{62}$$

Combining (61) and (62), (59) follows, as desired.

Now assume that condition (59) holds. Then (43) holds and by Theorem (1.1.15) we have that $P_\varphi^g\left(\left\{f: \|f\|_{\mathcal{B}^1_{log^\beta}} \le\right\}\right)$ is a bounded set in μ

From the following inequality $\mu(z)\left|\Re P_\varphi^g(f)(z)\right| = \mu(z)\left|g(z)f\varphi|z|\right|$

$$\le C\|f\|_{\mathcal{B}^1_{log^\beta,}}\mu(z)|g(z)|\left(1 + \left(In\frac{e^\beta}{1-\varphi|z|}\right)^{1-\beta}\right) \tag{63}$$

And (59) we have more, namely that $P_\varphi^g\left(\left\{f: \|f\|_{\mathcal{B}^1_{log^\beta,}} \le 1\right\}\right)$ is a bounded set in $\mathcal{B}_{\mu,0}$.

Taking the supremum in (63) over the unit ball in $\mathcal{B}^1_{log^\beta}\left(or\ \mathcal{B}^1_{log^\beta,}\right)$ Then letting $|z| \to 1$, using conditions (59) and employing Lemma (1.1.5), we obtain the compactness of the operator $P_\varphi^g : \mathcal{B}^1_{log^\beta}\left(or\ \mathcal{B}^1_{log^\beta,0}\right) \to \mathcal{B}_{\mu,0}$, as desired.

Case $\alpha \in (0,1)$ or $\alpha = 1$ and $\beta > 1$.

If $\alpha \in (0,1)\ or\ \alpha = 1 and\ \beta > 1$, then from the proofs of Theorems (1.1.11)- (1.1.17) and Lemma (1.1.4), it is easy to see that the following

Theorem (1.1.18)[1]: Assume that $\alpha \in (0,1)$, or $\alpha = 1$ and $\beta > 1\ g \in H(\mathbb{B}), g(0) = 0$, μ is a weight, and φ is a holomorphic self-map of $\mathbb{B}$, and $P_\varphi^g: \mathcal{B}^\alpha_{log^\beta}(\mathbb{B})\left(or\ \mathcal{B}^\alpha_{log^\beta,0}\right) \to \mathcal{B}_\mu$ is bounded. if and only if $g \in H^\infty_{\mu,0}$, if $P_\varphi^g: \mathcal{B}^\alpha_{log^\beta}(\mathbb{B})\left(or\ \mathcal{B}^\alpha_{log^\beta,0}\right)$ is bounded, then

$$\left\|P_\varphi^g\right\|_{\mathcal{B}^\alpha_{log^\beta}\to\mathcal{B}_\mu} \asymp \left\|P_\varphi^g\right\|_{\mathcal{B}^\alpha_{log^\beta}\to\mathcal{B}_\mu} \asymp \|g\|_{H^\infty_\mu}.$$

Theorem (1.1.19)[1]: Assume *that* $\alpha \in (0,1)$, or $\alpha = 1$ and $\beta > 1\ g \in H(\mathbb{B}), g(0) = 0$, μ is a weight, and φ is a holomorphic self-map of $\mathbb{B}$, and the $oP_\varphi^g: \mathcal{B}^\alpha_{log^\beta}(\mathbb{B})\left(or\ \mathcal{B}^\alpha_{log^\beta,0}\right) \to \mathcal{B}_\mu$ is bounded then the operator

$P_\varphi^g: \mathcal{B}_{log^\beta}^\alpha(\mathbb{B}) \left(or\ \mathcal{B}_{log^\beta,0}^\alpha\right)$ is compact if and only if $g \in H_{\mu,0}^\infty$,

Theorem (1.1.20)[1]: Assume $\alpha \in (0,1)$, or $\alpha = 1$ and $\beta > 1 g \in H(\mathbb{B}), g(0) = 0$, μ is a weight, and φ is a holomorphic self-map of $\mathbb{B}$, and $P_\varphi^g: \mathcal{B}_{log^\beta}^\alpha \left(or\ \mathcal{B}_{log^{\beta,0}}^\alpha\right) \to \mathcal{B}_{\mu,}$ is bounded. Then $P_\varphi^g: \mathcal{B}_{log^\beta}^\alpha \left(or\ \mathcal{B}_{log^{\beta,0}}^\alpha\right) \to \mathcal{B}_{\mu,}$ is compact

$$\lim_{|z|\to 1} \mu(z)\, |g(z)| = 0. \tag{64}$$

Theorem (1.1.21)[1]: Assume $g \in H(\mathbb{B}), g(0) = 0$, μ is a weight, and φ is a holomorphic self-map of $\mathbb{B}$, and the operator $P_\varphi^g: \mathcal{B}_{log^1}^1\left(or\ \mathcal{B}_{log^{1,0}}^1\right) \to \mathcal{B}_{\mu,0}$ is bounded. Then $P_\varphi^g: \mathcal{B}_{log^1}^1\left(or\ \mathcal{B}_{log^{1,0}}^1\right) \to \mathcal{B}_{\mu,0}$ is compact, if and only if

$$M := sup_{z\in\mathbb{B}}\mu(z)|g(z)|max\left\{1, InIn\frac{e}{1-|\varphi(z)|}\right\} < \infty$$

Moreover, if $P_\varphi^g: \mathcal{B}_{log^1}^1\left(or\ \mathcal{B}_{log^{1,0}}^1\right) \to \mathcal{B}_{\mu,0}$ is bounded, then

$$\left\|P_\varphi^g\right\|_{\mathcal{B}_{log^1}^1\to\mathcal{B}_\mu} \asymp \left\|P_\varphi^g\right\|_{\mathcal{B}_{log^1,0}^1\to\mathcal{B}_\mu} \asymp M_3$$

Theorem (1.1.22)[1]: Assume $g \in H(\mathbb{B}), g(0) = 0$, μ is a weight, and φ is a holomorphic self-map of $\mathbb{B}$, and the operator $P_\varphi^g: \mathcal{B}_{log^1}^1\left(or\ \mathcal{B}_{log^{1,0}}^1\right) \to \mathcal{B}_\mu$ is bounded. Then $P_\varphi^g: \mathcal{B}_{log^1}^1\left(or\ \mathcal{B}_{log\ \ ,0}^1\right) \to \mathcal{B}_\mu$ is compact, if and only if

$$\lim_{|z|\to 1} \mu(z)\, |g(z)| max\left(1, In\, In\frac{1}{1-|\varphi(z)|}\right) = 0.$$

Theorem (1.1.23)[1]: Assume $g \in H(\mathbb{B}), g(0) = 0$, μ is a weight, and φ is a holomorphic self-map of $\mathbb{B}$, and the operator $P_\varphi^g: \mathcal{B}_{log^1}^1\left(or\ \mathcal{B}_{log^{1,0}}^1\right) \to \mathcal{B}_{\mu,0}$ is bounded. Then $P_\varphi^g: \mathcal{B}_{log^1}^1\left(or\ \mathcal{B}_{log^{1,0}}^1\right) \to \mathcal{B}_{\mu,0}$ is compact, if and only if

$$\lim_{|z|\to 1} \mu(z)\, |g(z)| max\left(1, In\, In\frac{1}{1-|\varphi(z)|}\right) = 0.$$

Here we formulate and prove the results regarding the boundedness and compactness of the operators $P_\varphi^g: H(p,q,\phi) \to \mathcal{B}_\mu\left(or \mathcal{B}_{\mu,0}\right)$.

Theorem (1.1.24)[1]: Suppose $0 < p, q < \infty, g \in H(\mathbb{B}), g(0) = 0, \phi,$ is normal, μ is a weight, and φ is an analytic self-map of $\mathbb{B}$. Then the operator $P_\varphi^g: H(p,q,\phi) \to \mathcal{B}_\mu$ is bounded if and only if

$$M_4 := sup_{z\in\mathbb{B}}\frac{\mu(z)|g(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)^2|)^{n/q}} < \infty, \tag{65}$$

Moreover, if the operator $P_\varphi^g: H(p,q,\phi) \to \mathcal{B}_\mu$ is bounded, then the following asymptomatic relation holds

$$\left\|P_\varphi^g\right\|_{(p,q,\phi)\to\mathcal{B}_\mu} \asymp M_4 \tag{66}$$

Proof: Assume that $P_\varphi^g: H(p,q,\phi) \to \mathcal{B}_\mu$ is bounded, set

$$f_w(z) = \frac{(1-|w|^2)^\beta}{\phi(w)(1-\langle z,w\rangle)^{\frac{\eta}{q}+\beta}}, z \in \mathbb{B}, \tag{67}$$

Where $w \in \mathbb{B}$ $and\ \beta > b$.By Lemma (1.1.2) in [47] we have

$$sup_{w\in\mathbb{B}}\|f_w\|_{H(p,q,\phi)} \leq C.$$

For this, the boundeness of

$$P_\varphi^g : H(p,q,\phi) \to \mathcal{B}_\mu \ , \ P_\varphi^g f_w(0) = 0$$

and by using Lemma (1.1.6), we obtain

$$C\left\|P_\varphi^g\right\|_{H(p,q,\phi)\to\mathcal{B}_\mu} \geq \left\|P_\varphi^g f_{\varphi(w)}\right\|_{\mathcal{B}_\mu} = sup_{w\in\mathbb{B}}\mu(z)|g(z)|\left\|f_{\varphi(w)}(\varphi(Z))\right\|$$
$$\geq \mu(w)|g(w)| \left\|f_{\varphi(w)}(\varphi(w))\right\|$$
$$= \frac{\mu(w)|g(w)|}{\phi(|\varphi(w)|)(1-|\varphi(w)^2|)^{\eta/q}} \quad (68)$$

Taking the supremum over $w \in \mathbb{B}$ in (68) we obtain

$$M_4 \leq C \quad (69)$$

Now assume that (65) holds. By Lemma (1.1.6) and inequality (19), it follows that

$$\mu(z)\left|\Re\left(P_\varphi^g f\right)(z)\right| = \mu(z)\left|f(\varphi(z))\right||g(z)|$$
$$\leq C\|f\|_{H(p,q,\phi)} = \frac{\mu(z)|g(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)^2|)^{\frac{\eta}{q}}} \quad (70)$$

For every $z \in \mathbb{B} \ \ and \ f \in H(p,q,\phi)$

Using condition (65) in (70) and the fact $P_\varphi^g f(0) = (0)$, it follows that $H(p,q,\phi) \to \mathcal{B}_\mu$ is bounded and moreover

$$\left\|P_\varphi^g\right\|_{H(p,q,\phi)} \leq CM_4. \quad (71)$$

From (60) and (71), the asymptotic relationship in (66) follows, as desired.

Theorem (1.1.25)[1]: Suppose $0 < p,q < \infty, g \in H(\mathbb{B}), g(0) = 0, \phi,$ is normal, μ is a weight, and φ is an analytic self-map of $\mathbb{B}$. Then the operator $P_\varphi^g : H(p,q,\phi) \to \mathcal{B}_\mu$ is compact if and only if $P_\varphi^g : H(p,q,\phi) \to \mathcal{B}_\mu$ is bounded and

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|g(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)^2|)^{\frac{n}{q}}} = 0, \quad (72)$$

Proof: Assume $P_\varphi^g : H(p,q,\phi) \to \mathcal{B}_\mu$ is compact, then clearly $P_\varphi^g : H(p,q,\phi) \to \mathcal{B}_\mu$ is bounded, let $(z_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{B}$. such that $|\varphi(z_k)| \to 1$ as $k \to \infty$ if such a sequence does not exist then condition (72) is vacuously satisfies).

$$set \ \ \widehat{f_z}(z) = f_{\varphi(z_k)}(z), k \in \mathbb{N}, \quad (73)$$

Where f_w is defined in (67). from the proof of the Theorem (1.1.24) we know that

$$sup_{k\in\mathbb{N}}\left\|\widehat{f_k}\right\|_{H(p,q,\phi)} \leq C$$

on the other hand , since $\beta > b$, we have that

$$\lim_{k\to\infty} \frac{(1-|\varphi(z_k)|^2)^\beta}{\phi(\varphi(z_k))} = 0,$$

From which it follows that $\widehat{f_k}$converges to zero uniformly on compact of $\mathbb{B}$ $as \ k \to \infty$.

By using Lemma (1.1.7) it follows that

$$\lim_{k\to\infty}\left\| P_\varphi^g \widehat{f_k}\right\|_{\mathcal{B}_\mu} = 0, \quad (74)$$

We have

$$\left\| P_\varphi^g \widehat{f_k}\right\|_{\mathcal{B}_\mu} = sup_{z\in\mathbb{B}}\mu(z)\left|\Re\left(P_\varphi^g \widehat{f_k}\right)(z)\right|$$
$$\geq \mu(z_k)|g(z_k)| \left|\widehat{f_k}\varphi(z_k)\right|$$
$$= \frac{\mu(z_k)|g(z_k)|}{\phi(|\varphi(z_k)|(1-|\varphi(z_k)|^2)^{\eta/q}} \quad (75)$$

from (74) and (75), we obtain

$$\lim_{k\to\infty} \frac{\mu(z_k)|g(z_k)|}{\phi(|\varphi(z_k)|(1-|\varphi(z_k)|^2)^{\eta/q}} = 0.$$

From which (72) follows.

Now assume that $P_\varphi^g: H(p,q,\phi) \to \mathcal{B}_\mu$ is bounded and that condition (72) holds. Assume is a bounded sequence condition (72) implies that for every there is a, such that

$$\frac{\mu(z_k)|g(z)|}{\phi(|\varphi(z)|(1-|\varphi(z)|^2)^{\eta/q}} < \frac{\varepsilon}{L_2} \quad (76)$$

Wherever $\delta < |\varphi(2)| < 1$.

By using Lemma (1.1.6) and 9, and in equality (76), we obtain

$$\begin{aligned}\| P_\varphi^g \widehat{f}_k\|_{\mathcal{B}_\mu} &= sup_{z\in\mathbb{B}}\mu(z)\left|g(z)f_k\big(\varphi(z)\big)\right| \\ &\le sup_{\{z\in\mathbb{B}:|\varphi(Z)|\le\delta\}}\mu(z)|g(z)|\left|f_k\big(\varphi(z)\big)\right| \\ &\quad +sup_{\{z\in\mathbb{B}:\delta<|\varphi(Z)|<1\}}\mu(z)|g(z)|\left|f_k\big(\varphi(z)\big)\right| \\ &\le \|g\|_{H_\mu^\infty} sup_{|w|\le\delta}|f_k(w)| + C\|f_k\|_{H(p,q,\phi)} \\ sup_{\{z\in\mathbb{B}:\delta<|\varphi(Z)|<1\}} &\frac{\mu(z)|g(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\eta/Q}} \\ &\le \|g\|_{H_\mu^\infty} sup_{|w|\le\delta}|f_k(w)| + C_\varepsilon. \end{aligned} \quad (77)$$

Where $P_\varphi^g(1) \in \mathcal{B}_\mu$ implies $g \in H_\mu^\infty$, in view of the boundedness of the operator

$$P_\varphi^g: H(p,q,\phi) \to \mathcal{B}_\mu.$$

By letting $k\to\infty$ in (77), using the assumption

$$\lim_{k\to\infty} sup_{|w|\le\delta}|f_k(w)| = 0,$$

and since $\varepsilon$is an arbitrary positive number, we obtain

$$\lim_{k\to\infty}\| P_\varphi^g \widehat{f}_k\|_{\mathcal{B}_\mu} = 0,$$

Hence, by Lemma (1.1.7), the implication follows.

Theorem (1.1.26)[1]: Suppose $0 < p,q < \infty, g \in H(\mathbb{B}), g(0) = 0, \phi$ is normal, μ is a weight, and φ is an analytical self-map of $\mathbb{B}$, then :$p_\varphi^g: H(p,q,\phi) \to \mathcal{B}_{\mu,0}$ is bounded if and only if $p_\varphi^g: H(p,q,\phi) \to \mathcal{B}_\mu$, is bounded and $g \in H_{\mu,0}^\infty$.

Proof. The proof is similar to the proof of Theorem (1.1.13). It should be only noticed that the set of all polynomials is also dense in the space $H(p,q,\phi)$. We omit the result of the proof.

Theorem (1.1.27)[1]: Suppose $0 < p,q < \infty, g \in H(\mathbb{B}), g(0) = 0, \phi$ is normal, μ is a weight, and φ is an analytical self-map of $\mathbb{B}$, then: $p_\varphi^g: H(p,q,\phi) \to \mathcal{B}_{\mu,0}$ is compact if and only if

$$\lim_{\lceil z\rceil\to 1} \frac{\mu(z)|g(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{n}{q}}} = 0 \quad (78)$$

Proof: assume that (78) holds. Then Lemmas 6 and 9 imply

$$\mu(z)\left|\Re\big(P_\varphi^g f\big)(z)\right| \le C\|f\|_{H(p,q,\phi)} \frac{\mu(z)|g(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{n}{q}}}. \quad (79)$$

Taking the supremum in (79) over the set $\|f\|_{H(p,q,\phi)} \le 1$.then letting $\lceil z\rceil \to 1$ and employing (78) we obtain

$$\lim_{[z]\to 1} sup_{\|f\|_{H(p,q,\phi)}\le 1}\,\mu(z)\left|\Re\left(P_\varphi^g f\right)(z)\right| = 0 \tag{80}$$

From (80) and by using Lemma (1.1.5) the compactness of the operator $P_\varphi^g: H(p,q,\phi) \to \mathcal{B}_{\mu,0}$ follows.

Now assume that condition (78) does not hold. If it were, then it would exist $\varepsilon_0 < 0$ and a sequence $(z_k)_{k\in N} \in \mathbb{B}$, such that $\lim_{K\to\infty} |z_k| = 1$ and

$$\frac{\mu(z_k)|g(z_k)|}{\phi(|\varphi(z_k)|)(1-|\varphi(z_k)^2|)^{n/q}} \ge \varepsilon_0 < 0, \tag{81}$$

for sufficiently large k.

First assume that $sup_{k\in\mathbb{N}}|\varphi(z_k)| < 1$. Then by Theorem (1.1.26), we have that $g \in H_{\mu,0}^\infty$ and consequently

$$\lim_{k\to\infty} \mu(z_k)|g(z_k)| = 0$$

From this, (81) and the normality of ϕ we obtain a contradiction.

Now assume that $sup_{k\in\mathbb{N}}|\varphi(z_k)| < 1$. Then there is a subsequence of $\left(\varphi(z_k)\right)_{k\in\mathbb{N}}$ (which we may also denote by $\left(\varphi(z_k)\right)_{k\in\mathbb{N}}$) such that $\lim_{k\to\infty}|\varphi(z_k)| = 1$. Let $\left(\widehat{f_k}\right)_{k\in\mathbb{N}}$ be defined as in (73), where $\beta > b$. We know that $sup_{k\in\mathbb{N}}\left\|\widehat{f_k}\right\|_{H(p,q,\varphi)} \le C, and\ \widehat{f_k}$ converges to 0 uniformly on compact of $\mathbb{B}$ as $k \to \infty$, hence

$$\lim_{k\to\infty}\left\|P_\varphi^g \widehat{f_k}\right\|_{\mathcal{B}_{\mu,}} = 0. \tag{82}$$

On the other hand, from (75) and (81) we have

$$\left\|P_\varphi^g \widehat{f_k}\right\|_{\mathcal{B}_{\mu,}} \ge \frac{\mu(z_k)|g(z_k)|}{\phi(|\varphi(z_k)|)(1-|\varphi(z_k)^2|)^{n/q}} \ge \frac{\varepsilon_0}{2} < 0$$

for sufficiently large k, which contradicts to (82), finishing the proof of the theorem.

We characterize the boundedness and compactness of the operator

$$uC_\varphi: \mathcal{B}_{log^\beta}^\alpha(\mathbb{B})\left(or\ \mathcal{B}_{log^\beta,0}^\alpha(\mathbb{B})\right) \to H_\mu^\infty(\mathbb{B})\left(\quad orH_{\mu,0}^\infty(\mathbb{B})\right)$$

The proofs of these results are similar to those in the previous and the same test functions are used.

Theorem (1.1.28)[1]: Assume that $\alpha > 1$, $\beta \ge 0, u \in H(\mathbb{B}), \mu$ is a weight, φ is a holomorphic self-map of $\mathbb{B}$ then $u\mathrm{C}_\varphi: \mathcal{B}_{log^\beta}^\alpha\left(or\ \mathcal{B}_{log^\beta,0}^\alpha\right) \to H_{\mu,0}^\infty$ is bounded. Then the operator $u\mathrm{C}_\varphi: \mathcal{B}_{log^\beta}^\alpha\left(or\ \mathcal{B}_{log^\beta,0}^\alpha\right) \to H_{\mu,0}^\infty$ is compact if and only if

$$M_5 \coloneqq \sup_{z\in\mathbb{B}} \frac{\mu(z)|u(z)|}{\left(1-|\varphi(z)|\quad\right)^{\infty-1}\left(In\dfrac{e^{\beta/\alpha}}{1-|\varphi(z)|}\right)^\beta} < \infty$$

And $u \in H_\mu^\infty$,

Moreover .if $u\mathrm{C}_\varphi: \mathcal{B}_{log^\beta}^\alpha\left(or\ \mathcal{B}_{log^\beta,0}^\alpha\right) \to H_{\mu,0}^\infty\ is\ bounded\ , then$

$$\|\ u\mathrm{C}_\varphi\|_{\mathcal{B}_{log^\beta}^\alpha\to\mathcal{B}_\mu} \asymp \|\ u\mathrm{C}_\varphi\|_{\mathcal{B}_{log^\beta}^\alpha\to\mathcal{B}_\mu} \asymp M_5 + \|u\|_{H_{\mu,}^\infty}$$

Theorem (1.1.29)[1]: Assume that $\alpha > 1$, $\beta \ge 0, u \in H(\mathbb{B}), \mu$ is a weight, φ is a holomorphic self-map of $\mathbb{B}$, and $u\mathrm{C}_\varphi: \mathcal{B}_{log^\beta}^\alpha\left(or\ \mathcal{B}_{log^\beta,0}^\alpha\right) \to H_\mu^\infty$ is bounded. Then $u\mathrm{C}_\varphi: \mathcal{B}_{log^\beta}^\alpha\left(or\ \mathcal{B}_{log^\beta,0}^\alpha\right) \to H_{\mu,0}^\infty$ is compact if and only if

$$\lim_{|z|\to 1}\mu(z)|g(z)|\left(1+\left(In\frac{e^{\beta}}{1-|\varphi(z)|}\right)^{1-\beta}\right)= 0.$$

Theorem (1.1.30)[1]: Assume that $\alpha > 0$, $\beta \geq H(\mathbb{B}), \mu$ is a weight, and φ is a holomorphic self-map of $\mathbb{B}$, then $uC_{\varphi}: \mathcal{B}^{\alpha}_{log^{\beta},0} \to H^{\infty}_{\mu}$ is bounded if and only if.$uC_{\varphi}: \mathcal{B}^{\alpha}_{log^{\beta},0} \to H^{\infty}_{\mu}$ is bounded and $u \in H^{\infty}_{\mu,0}$.

Theorem (1.1.31)[1]: Assume that $\alpha > 1$, $\beta \geq 0, u \in H(\mathbb{B}), \mu$ is a weight, φ is a holomorphic self-map of $\mathbb{B}$, and the operator $uC_{\varphi}: \mathcal{B}^{\alpha}_{log^{\beta}}\left(or\ \mathcal{B}^{\alpha}_{log^{\beta},0}\right) \to H^{\infty}_{\mu}$ is compact if and only if

$$\lim_{|z|\to 1}\mu(z)|u(z)|\left(1+\frac{1}{1-(\varphi(z))^{\alpha-1}\left(In\frac{e^{\beta/\alpha}}{1-|\varphi(z)|}\right)^{\beta}}\right)= 0.$$

Theorem (1.1.32)[1]: Assume that $\alpha = 1$, $\beta \in (0,1), u \in H(\mathbb{B}), \mu$ is a weight, φ is a holomorphic self-map of $\mathbb{B}$. *Then* $uC_{\varphi}: \mathcal{B}^{\alpha}_{log^{\beta}}\left(or\ \mathcal{B}^{\alpha}_{log^{\beta},0}\right) \to H^{\infty}_{\mu,0}$ is bounded if and only if

$$M_6 \coloneqq sup_{z\in\mathbb{B}}\mu(z)|u(z)|\left(In\frac{e^{\beta}}{1-|\varphi(z)|}\right)^{1-\beta} < \infty$$

$u \in H^{\infty}_{\mu}$.

Moreover , if $uC_{\varphi}: \mathcal{B}^{1}_{log^{\beta}}\left(or\ \mathcal{B}^{1}_{log^{\beta},0}\right) \to H^{\infty}_{\mu}$ is bounded, them

$$\left\|uC_{\varphi}\right\|_{\mathcal{B}^{1}_{log^{\beta}}\to\mathcal{B}_{\mu}} \asymp \left\|uC_{\varphi}\right\|_{\mathcal{B}^{1}_{log^{\beta},0}\to\mathcal{B}_{\mu}} \asymp M_6 + \|u\|_{H^{\infty}_{\mu}}.$$

Theorem (1.1.33)[1]: Assume that $\alpha = 1$, $\beta \in (0,1), u \in H(\mathbb{B}), \mu$ is a weight, φ is a holomorphic self-map of $\mathbb{B}$, and the operator $uC_{\varphi}: \mathcal{B}^{\alpha}_{log^{\beta}}\left(or\ \mathcal{B}^{\alpha}_{log^{\beta},0}\right) \to H^{\infty}_{\mu,0}$ is bounded. Then the operator $uC_{\varphi}: \mathcal{B}^{\alpha}_{log^{\beta}}\left(or\ \mathcal{B}^{\alpha}_{log^{\beta},0}\right) \to H^{\infty}_{\mu,0}$ is compact if and only if

$$\lim_{|z|\to 1}\mu(z)|g(z)|\left(1+\left(In\frac{e^{\beta}}{1-|\varphi(z)|}\right)^{1-\beta}\right)= 0$$

Theorem (1.1.34)[1]: Assume that $\alpha = 1$, $\beta \in (0,1), u \in H(\mathbb{B}), \mu$ is a weight, φ is a holomorphic self-map of $\mathbb{B}$, and the operator $uC_{\varphi}: \mathcal{B}^{\alpha}_{log^{\beta}}\left(or\ \mathcal{B}^{\alpha}_{log^{\beta},0}\right) \to H^{\infty}_{\mu,0}$ is bounded. Then the operator $uC_{\varphi}: \mathcal{B}^{\alpha}_{log^{\beta}}\left(or\ \mathcal{B}^{\alpha}_{log^{\beta},0}\right) \to H^{\infty}_{\mu,0}$ is compact if and only if

$$\lim_{|z|\to 1}\mu(z)|g(z)|\left(1+\left(In\frac{e^{\beta}}{1-|\varphi(z)|}\right)^{1-\beta}\right)= 0$$

Theorem (1.1.35)[1]: $u \in H(\mathbb{B}), \mu$ is a weight, φ is a holomorphic self-map of $\mathbb{B}$, Then $uC_{\varphi}: \mathcal{B}^{1}_{log^{1}}\left(or\ \mathcal{B}^{1}_{log^{1},0}\right) \to H^{\infty}_{\mu}$ is bounded if and only if

$$M_7 \coloneqq sup_{z\in\mathbb{B}}\,\mu(z)|u(z)|\max\left\{1, InIn\frac{e}{1-|\varphi(z)|}\right\} < \infty$$

Moreover, if $uC_{\varphi}: \mathcal{B}^{1}_{log^{1}}\left(or\ \mathcal{B}^{1}_{log^{1},0}\right) \to H^{\infty}_{\mu}$ is bounded, then

$$\|u\mathrm{C}_{\varphi}\|_{\mathcal{B}^1_{log^1,}\to H^{\infty}_{\mu}} \asymp \|u\mathrm{C}_{\varphi}\|_{\mathcal{B}^1_{log^1,0}\to H^{\infty}_{\mu}} \asymp M_7$$

Theorem (1.1.36)[1]: Assume that $u \in H(\mathbb{B})$, μ is a weight, φ is a holomorphic self-map of $\mathbb{B}$, and the operator $u\mathrm{C}_{\varphi}: \mathcal{B}^1_{log^1}(or\ \mathcal{B}^1_{log^1,0}) \to H^{\infty}_{\mu,}$ is bounded. Then the operator $u\mathrm{C}_{\varphi}: \mathcal{B}^1_{log^1}(or\ \mathcal{B}^1_{log^1,0}) \to H^{\infty}_{\mu,0}$ is compact if and only if

$$\lim_{|\varphi(z)|\to 1} \mu(z)|u(z)|max\left\{1, InIn\frac{e}{1-|\varphi(z)|}\right\} = 0$$

Theorem (1.1.37)[1]: Assume that $u \in H(\mathbb{B})$, μ is a weight, φ is a holomorphic self-map of $\mathbb{B}$, and the operator $u\mathrm{C}_{\varphi}: \mathcal{B}^1_{log^1}(or\ \mathcal{B}^1_{log^1,0}) \to H^{\infty}_{\mu,0}$ is bounded. Then the operator $u\mathrm{C}_{\varphi}: \mathcal{B}^1_{log^1}(or\ \mathcal{B}^1_{log^1,0}) \to H^{\infty}_{\mu,0}$ is compact if and only if

$$\lim_{|z|\to 1} \mu(z)|u(z)|max\left\{1, InIn\frac{e}{1-|\varphi(z)|}\right\} = 0$$

Theorem (1.1.38)[1]: Assume that $\alpha \in (0,1)$ or $\alpha = 1$ $and\ \beta > 1$ $, u \in H(\mathbb{B}), \mu$ is a weight, φ is a holomorphic self-map of $\mathbb{B}$, then $u\mathrm{C}_{\varphi}: \mathcal{B}^{\alpha}_{log^{\beta}}\left(or\ \mathcal{B}^{\alpha}_{log^{\beta},0}\right) \to H^{\infty}_{\mu}$ is bounded if and only if $u \in H^{\infty}_{\mu,0}$.Morever, if $u\mathrm{C}_{\varphi}: \mathcal{B}^{\alpha}_{log^{\beta}}\left(or\ \mathcal{B}^{\alpha}_{log^{\beta},0}\right) \to H^{\infty}_{\mu}$ is bounded then

$$\|u\mathrm{C}_{\varphi}\|_{\mathcal{B}^{\alpha}_{log^{\beta}}\to H^{\infty}_{\mu}} \asymp \|u\mathrm{C}_{\varphi}\|_{\mathcal{B}^1_{log^1,0}\to H^{\infty}_{\mu}} \asymp \|u\|_{H^{\infty}_{\mu}}.$$

Theorem (1.1.39)[1]: Assume that $\alpha \in (0,1)$ or $\alpha = 1$ and $\beta > 1$ $, u \in H(\mathbb{B}), \mu$ is a weight, φ is a holomorphic self-map of $\mathbb{B}$, and the operator $u\mathrm{C}_{\varphi}: \mathcal{B}^{\alpha}_{log^{\beta}}\left(or\ \mathcal{B}^{\alpha}_{log^{\beta},0}\right) \to H^{\infty}_{\mu,0}$ is bounded. Then the operator $u\mathrm{C}_{\varphi}: \mathcal{B}^{\alpha}_{log^{\beta}}\left(or\ \mathcal{B}^{\alpha}_{log^{\beta},0}\right) \to H^{\infty}_{\mu,0}$ is compact if and only if $u \in H^{\infty}_{\mu,0}$,

Theorem (1.1.40)[1]: Assume that $\alpha \in (0,1)$ or $\alpha = 1$ and $\beta > 1$ $, u \in H(\mathbb{B}), \mu$ is a weight, φ is a holomorphic self-map of $\mathbb{B}$, and the operator $u\mathrm{C}_{\varphi}: \mathcal{B}^{\alpha}_{log^{\beta}}\left(or\ \mathcal{B}^{\alpha}_{log^{\beta},0}\right) \to H^{\infty}_{\mu,0}$ is bounded. Then the operator $u\mathrm{C}_{\varphi}: \mathcal{B}^{\alpha}_{log^{\beta}}\left(or\ \mathcal{B}^{\alpha}_{log^{\beta},0}\right) \to H^{\infty}_{\mu,0}$ is compact if

$$\lim_{(\varphi|z|)\to 1} \mu(z)|u(z)| = 0.$$

Theorem (1.1.41)[1]: Suppose $0 < p, q < \infty$, $u \in H(\mathbb{B}), \phi$ is an analytical self-map of $\mathbb{B}$ μ is a weight, and φ is an analytic self-map of $\mathbb{B}$. Then the operator $uC_{\varphi}: H(p, q.\phi) \to H^{\infty}_{\mu}$ is bounded if and only if

$$MS := sup_{z\in\mathbb{B}}\frac{\mu(z)|u(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{n}{q}}} < \infty$$

Moreover, if the operator $uC_{\varphi}: H(p, q.\phi) \to H^{\infty}_{\mu}$ is bounded then the following asymptotic relation holds

$$\|uC_{\varphi}\|_{H(p,q.\phi)\to H^{\infty}_{\mu}} \asymp M_S.$$

Theorem (1.1.42)[1]: Suppose $0 < p, q < \infty$, $u \in H(\mathbb{B}), \phi,$ is normal, μ is a weight, and φ is an analytic self-map of $\mathbb{B}$. Then the operator $uC_{\varphi}: H(p, q.\phi) \to H^{\infty}_{\mu}$ is compact if and only if $uC_{\varphi}: H(p, q.\phi) \to H^{\infty}_{\mu}$ is bounded and

$$\lim_{|z|\to 1}\frac{\mu(z)|u(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|)^{\frac{n}{q}}} = 0$$

Theorem (1.1.43)[1]: Suppose $0 < p;\ q < \infty, u \in H(\mathbb{B})$, ϕ is normal, μ is a weight, and φ is an analytic self-map of $\mathbb{B}$. Then $uC_{\varphi}: H(p,q,\phi) \to \mathcal{B}_{\mu,0}^{\infty}$ is bounded if and only if $uC_{\varphi}: H(p,q,\phi) \to H_{\mu}^{\infty}$ is bounded and $u \in H_{\mu,0}^{\infty}$.

Theorem (1.1.44)[1]: Suppose $0 < p;\ q < \infty, u \in H(\mathbb{B})$ is normal, μ is a weight, and φ is an analytic self-map of $\mathbb{B}$. Then $uC_{\varphi}: H(p,q,\phi) \to \mathcal{B}_{\mu,0}^{\infty}$ is compact if and only if

$$\lim_{[z]\to 1} \frac{\mu(z)|u(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|)^{\frac{n}{q}}} = 0.$$

Section (1.2): From the Bloch Space to Bloch –Type Space on the Unit Ball

For $\mathbb{B}$ be the open unit ball in $\mathbb{C}^n$, $\mathbb{D}$ the open unit disk in $\mathbb{C}$, $H.(\mathbb{B})$ the class of all holomorphic functions on the unit ball and $H^{\infty} = H^{\infty}(\mathbb{B})$ the space of all bounded holomorphic functions on $\mathbb{B}$ with the norm

$$\|f\|_{\infty} = sup_{z\in\mathbb{B}}|f(z)|.$$

Let $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ be points in $\mathbb{C}^n$

$$\langle z,w\rangle = \sum_{k=1}^{n} z_k\bar{w}_k \ and\ |z| = \sqrt{(z,z)} \sum_{|\beta|\geq 0} a_\beta z^\beta.$$

For $f \in H(\mathbb{B})$ with the Taylor expansion $f(z) = \sum_{|\beta|\geq 0} a_\beta z^\beta$, *let*

$$\Re f(z) = \sum_{|\beta|\geq 0} |\beta| a_\beta z^\beta$$

be the radial derivative of f, where $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ is a multi-index $|\beta| = (\beta_1 + \dots + \beta_n)$ and $z^\beta = z_1^{\beta_1} \dots z_n^{\beta_n}$. (see [77]).

A positive continuous function μ on [0, 1) is called normal [11] if there is $\delta \in [0,1)$ and a and b , $0 < a < b$ such that

$$\frac{(\mu)r}{(1-r)^a} \text{ is decreasing on } [\delta.1) and \lim_{r\to 1}\frac{\mu(r)}{(1-r)^a} = 0,$$

$$\frac{(\mu)r}{(1-r)^b} \ is\ decreasing\ on\ [\delta.1) and \lim_{r\to 1}\frac{\mu(r)}{(1-r)^b} = 0,$$

If we say that a function $\mu: \mathbb{B} \to [0,\infty)$ is normal we will also assume that it is radial , that is ,$\mu(z) = \mu(|z|), z \in \mathbb{B}$.

The weighted space $H_\mu^\infty = H_\mu^\infty(\mathbb{B})$ consists of all $f \in H(\mathbb{B})$ *such that*

$$Sup_{z\in\mathbb{B}}\ \mu(z)|f(z)| < \infty.$$

where μ is normal. For $\mu(z) = (1-|z|^2)^\beta$, $\beta > 0$ we obtain the (classical) weighted space $H_\beta^\infty = H_\beta^\infty(\mathbb{B})$. The Little weighted space $H_{\mu,0}^\infty = H_{\mu,0}^\infty(\mathbb{B})$ is a subspace of H_μ^∞ consisting of all $f \in H(\mathbb{B})$ such that

$$lim_{|z|\to 1}\ \mu(z)|f(z)| = 0.$$

The Bloch-type space, denoted by $\mathcal{B}_\mu = \mathcal{B}_\mu(\mu)$ consists of all $f \in H(\mathbb{B})$ such that

$$\mathcal{B}_\mu(f) = Sup_{z\in\mathbb{B}}\ \mu(z)|\Re f(z)| < \infty.$$

where μ is normal. With the norm

$$\|f\|_{\mathcal{B}_\mu} = |f(0)| + \mathcal{B}_\mu(f)$$

the Bloch-type space becomes a Banach space.

The α-Bloch space B^{α} is obtained for $\mu(z) = (1-|z|^2)^\alpha$, $\alpha \in (0,\infty)$ (see, e.g., [76],

[79], [9]). The Little Bloch-type space $B_{\mu,0}$ is a subspace of B_μ consisting of those f such that

$$lim_{|z|\to 1}\mu(z)|\Re f\ (z)| = 0.$$

Bearing in mind the following asymptotic relation from [60] (see also [4] for the case of the α-Bloch space)

$$b_\mu(f) := Sup_{z\in\mathbb{B}}\ \mu(z)|\nabla f(z)| \asymp Sup_{z\in\mathbb{B}}\ \mu(z)|\Re f(z)| \tag{83}$$

we see that B_μ can be defined as the class of all $f \in H(\mathrm{B})$ such that $b_\mu(f)$ is finite. Also the Little Bloch-type space is equivalent with the subspace of B_μ consisting of all $f \in H(\mathrm{B})$ such that

$$lim_{|z|\to 1}\mu(z)|\nabla f\ (z)| = 0.$$

From this observation and for some technical benefits, for the norm of the α-Bloch space we choose the second definition, that is, $f \in B^\alpha$ if and only if

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + Sup_{z\in\mathbb{B}}(1-|z|^2)^\alpha|\nabla f(0)| < \infty.$$

If $\mu(z) = (1-|z|^2)$, then the quantity $b_\mu(f)$ in (83)will be denoted $yb(f)$.

Let φ be a holomorphic self map of $\mathbb{B}$. for $f \in H(\mathbb{B})$ the composition operator is defined by $C_\varphi f(z) = f(\varphi(z))$ (see [47] or [62], [15], [17], [27]).

Let $g \in H(\mathbb{D})$ and ϕ be a holomorphic self-map of $\mathbb{D}$. Products of integral and composition operators on $H(\mathbb{D})$ were introduced by S. Li and S. Stević in a private communication (see, e.g., [57], [58] and [79], [27]) as follows

$$C_\varphi J_g f(z) = \int_0^{\varphi(z)} f(\zeta)g(\zeta)d\zeta \ \ and\ J_g\, C_\varphi f(z) = \int_0^{z} f(\varphi(\zeta))g(\zeta)d\zeta \tag{84}$$

Operators in (84) are extensions of the following integral operators

$$T_g(f)(z) = \int_0^z f(\zeta)\, g'(\zeta)d\zeta$$

which was introduced in [78]. Some other results on the operator T_g can be found, e.g., in [71]–[73], [78]. For some results on n-dimensional extensions of the operator, see [42]–[45], [74]–[52], [53], [54], [55], [6]–[58], [60].

One of the interesting questions is to extend operators in (84) in the unit ball settings and to study their function theoretic properties between spaces of holomorphic functions on the unit ball in terms of inducing functions.

Assume that $g \in H(\mathbb{B})$, $g(0) = 0$ and ϕ is a holomorphic self-map of $\mathbb{B}$, then we introduce the following operator on the unit ball

$$P_\varphi^g(f)(z) = \int_0^1 f(\varphi(tz))g(tz)\frac{dt}{t},\quad f \in H(\mathbb{B}), z \in \mathbb{B} \tag{85}$$

If $n = 1$, then $g \in H(\mathrm{D})$ and $g(0) = 0$, so that $g(z) = zg_0(z)$, for some $g_0 \in H(\mathrm{D})$. By the change of variable $\zeta = tz$, it follows that

$$P_\varphi^g f(z) = \int_0^1 f(\varphi(tz))tzg_0(tz)\frac{dt}{t} = \int_0^z f(\varphi(\zeta))g_0(\zeta)d\zeta.$$

Thus operator (85) is a natural extension of the second operator in (84).

Here we study the boundedness and compactness of operator P_ϕ^g from the Bloch space B or the Little Bloch space B_0 to the Bloch-type space B_μ or the Little Bloch-type space $B_{\mu,0}$.

We calculate the essential norm of the operators P_{ϕ}^{g} : B (or B_0) ⟶ B_μ (or $B_{\mu,0}$).

C will denote a positive constant not necessarily the same at each occurrence. The notation $A \asymp B$ means that there is a positive constant C such that $A/C \leq B \leq C\,A$.

The following lemmas are used in the proofs.

Lemma (1.2.1)[41]: Suppose g ∈ H(B), g(0) = 0, μ is normal and ϕ is a holomorphic self-map of B. Then the operator P_g^{ϕ} : B (or B_0) $\rightarrow B_\mu$ is compact if and only if P_g^{ϕ} : B (or B_0) $\rightarrow B_\mu$ is bounded and for any bounded sequence $(f_k)_{k\in N}$ in B (or B_0) converging to zero uniformly on compacts of $\mathbb{B}$, we have $\| P_{\varphi}^{g} f_k \|_{B\mu} \rightarrow 0$ as $k \rightarrow \infty$.

The proof of Lemma (1.2.1) follows by standard arguments (see, for example, the proofs of Proposition 3.11 in [47] and Lemma (1.2.3) in [58]).

Lemma (1.2.2)[41]: Suppose f, $g \in H(\mathrm{B})$ and $g(0) = 0$. Then

$$\Re P_{\varphi}^{g}(f)\,(f)(z) = f(\varphi(z))\,g(z).$$

Proof. Assume that the holomorphic function $f(\varphi(z))g(z)$. has the expansion $\sum_{\beta} a_\beta z^\beta$, since $g(0) = 0$, note that $a_0 = 0$, Then

$$\Re[P_{\varphi}^{g}(f)]\,(z) = \Re\int_0^1 \sum_{\beta\neq 0} a_\beta (tz)^\beta \frac{dt}{t} = \Re\left(\sum_{\beta\neq 0}\frac{a_\beta}{|\beta|}z^\beta\right) = \sum_{\beta\neq 0} a_\beta z^\beta.$$

Which is what we wanted to prove.

Lemma (1.2.3)[41]: Let, Then the following inequality holds

$$|f(z)| \leq \|f\|_{\mathcal{B}} max\left\{1.\frac{1}{2} In\frac{1+|z|}{1-|z|}\right\}. \tag{86}$$

Proof. The proof of the lemma follows from the following inequality

$$|f(z)-f(0)| = \left|\int_0^1 \langle \nabla f(tz), \bar{z}\rangle dt\right| \leq b(f)\int_0^1 \frac{|z|\,dt}{1-|z|^2t^2} = b(f)\frac{1}{2}In\frac{1+(z)}{1-(z)},$$

Where $b(f) = \sup_{z\in\mathbb{B}}(1-|z|^2)|\nabla f(z)|$.

We calculate the norm $\|P_{\varphi}^{g}\|_{\mathcal{B}\rightarrow B\mu}$ and $\|P_{\varphi}^{g}\|_{\mathcal{B}_0\rightarrow B\mu}$.

Theorem (1.2.4)[41]: Assume g ∈ H($\mathbb{B}$), g(0) = 0, μ is normal and ϕ is a holomorphic self-map of B. and P_{φ}^{g} : B (or B0) ⟶ Bμ is bounded then

$$\|P_{\varphi}^{g}\|_{\mathcal{B}\rightarrow B\mu} = \|P_{\varphi}^{g}\|_{\mathcal{B}_0\rightarrow B\mu} = sup_{z\in\mathbb{B}}\mu(z)|g(z)| max\left\{1.\frac{1}{2}\ In\frac{1+|\varphi(z)|}{1-|\varphi(z)|}\right\}. \tag{87}$$

Proof: if f ∈ $\mathcal{B}$, then by Lemma (1.2.2) and (86) we obtain

$$\|P_{\varphi}^{g}\|_{B\mu} = sup_{z\in\mathbb{B}}\,\mu(z)|g(z)f(\varphi(z))|$$

$$\leq \|f\|_{\mathcal{B}} sup_{z\in\mathbb{B}}\mu(z)|g(z)| \max\left\{1.\frac{1}{2}\ In\frac{1+|\varphi(z)|}{1-|\varphi(z)|}\right\}. \tag{88}$$

From which it follows that

$$\|P_{\varphi}^{g}\|_{B\rightarrow B\mu} \leq sup_{z\in\mathbb{B}}\mu(z)|g(z)| \max\left\{1.\frac{1}{2}\ In\frac{1+|\varphi(z)|}{1-|\varphi(z)|}\right\}. \tag{89}$$

The same inequality holds for $P_{\varphi}^{g} : B_0 \rightarrow B_\mu$

Now we prove the reverse inequality . By taking the function given by $f_0(z) \equiv 1 \in \mathcal{B}_0$ and using the boundrdness of $P_{\varphi}^{g} : B_0 \rightarrow B_\mu$ we obtain

$$\|P_{\varphi}^{g}\|_{B\rightarrow B\mu} = \|f_0\|\mathcal{B}\|P_{\varphi}^{g}\|_{B\rightarrow B\mu} \geq \|P_{\varphi}^{g} f_0\|_{B\mu}$$

$$= sup_{z\in\mathbb{B}}\mu(z)|g(z)||f_0(\varphi(z))| = sup_{z\in\mathbb{B}}\mu(z)|g(z)| \tag{90}$$

The same inequality holds for

$$P_\varphi^g : B0 \to B\mu.$$

for $w \in \mathbb{B}$, set

$$f_w(z) = \frac{1}{2} In \frac{1 + \langle z, w\rangle}{1 - \langle z, w\rangle} \tag{91}$$

with in $1 = 0$. Since $f_w(0) = 0$ and

$$(1 - |z|^2)\,|\nabla f_w(z)| = \frac{(1 - |z|^2)|w|}{(1 - |z, w|^2)} \le \frac{1 - |z|^2}{1 - |w|^2|z|^2} \le min\left\{1. \frac{1 - |z|^2}{1 - |w|^2}\right\}$$

it follows that $sup_{z\in\mathbb{B}}\ \|f_w\|_{\mathcal{B}} \le 1\ and\ f_w \in \mathcal{B}_0$. for each fixed $w \in \mathbb{B}$ from this and the boundedness of P_φ^g : B **(or** B_0) $\to B_\mu$ we have that when $\varphi(w) \neq 0$ and for every $t \in (0,1)$ the following inequality holds

$$\left\|P_\varphi^g\right\|_{B\ _0\to B\mu} \ge \left\|P_\varphi^g f_{r\varphi(w)/|\varphi(w)|}\right\|_{B\mu} =$$

$$sup_{z\in\mathbb{B}}\, \mu(z)|g(z)| \frac{1}{2}\left| In \frac{1 + t\,\langle\varphi(z). \dfrac{\varphi(w)}{|\varphi(w)|}\rangle}{1 - t\,\langle\varphi(z). \dfrac{\varphi(w)}{|\varphi(w)|}\rangle}\right|$$

$$\ge \frac{1}{2}\mu(w)|g(w)| In \frac{1 + t|\varphi(w)|}{1 - t|\varphi(w)|} \tag{92}$$

note that (92) obviously holds if $\varphi(w) = 0$.

Letting $t \to 1$ in (92), we obtain that for each $w \in \mathbb{B}$

$$\left\|P_\varphi^g\right\|_{B_0\to B\mu} \ge \frac{1}{2}\mu(w)|g(w)| In \frac{1 + |\varphi(w)|}{1 - |\varphi(w)|}.$$

from this and since w is an arbitrary element of $\mathbb{B}$, it follows that

$$\left\|P_\varphi^g\right\|_{B_0\to B\mu} \ge \frac{1}{2} sup_{z\in\mathbb{B}}\, \mu(z)|u(z)| In \frac{1 + |\varphi(z)|}{1 - |\varphi(z)|}. \tag{93}$$

note also that

$$\left\|P_\varphi^g\right\|_{B\to B\mu} \ge \left\|P_\varphi^g\right\|_{B_0\to B\mu} \tag{94}$$

from (90), (93) and (94) we obtain that

$$\left\|P_\varphi^g\right\|_{B\to B\mu} \ge \left\|P_\varphi^g\right\|_{B_0\to B\mu} \ge sup_{z\in\mathbb{B}}\, \mu(z)|g(z)|\, max\left\{1. \frac{1}{2} In \frac{1 + |\varphi(w)|}{1 - |\varphi(w)|}\right\} \tag{95}$$

from (89) and (95) , equalizes in (87) follows .

Corollary (1.2.5)[41]: Assume g ∈ H($\mathbb{B}$), g(0) = 0, μ is normal and ϕ is a holomorphic self-map of B. Then P_φ^g : B (or B_0) ⟶ B_μ is bounded if and only if

$$sup_{z\in\mathbb{B}}\mu(z)|g(z)| max\left\{1. \frac{1}{2}\ In \frac{1 + |\varphi(z)|}{1 - |\varphi(z)|}\right\} < \infty. \tag{96}$$

Proof. If $P_g^\phi : B\ (or\ B_0) \to B_\mu$ is bounded, then (96) follows from Theorem (1.2.4). If (96) holds, then the boundedness of $P_g^\phi : B\ (or\ B_0) \to B_\mu$ follows from (88).

Here we characterize the boundedness of the operator $\boldsymbol{P_\varphi^g}$: $\boldsymbol{\mathcal{B}_0\ B_{\mu.0}}$.

Theorem (1.2.6)[41]: Assume g ∈ H($\mathbb{B}$), g(0) = 0, μ is normal and ϕ is a holomorphic self-map of $\mathbb{B}$. then P_φ^g : B_0 ⟶ B_μ is bounded and g ∈ $H_{\mu.0}^\infty$.

Proof. Assume that P_φ^g: $\mathcal{B}_0$ →$B_{\mu.0}$ Is bounded , then clearly P_φ^g: $\mathcal{B}_0$ → B_μ is bounded. Taking the test function $f_0(z) = 1 \in \mathcal{B}_0$ we obtain $g \in H_{\mu.0}^\infty$.

Conversely, assume $P_\varphi^g: \mathcal{B}_0 \to B_\mu$. Is bounded and $g \in H_{\mu.0}^\infty$.Then, for every polynomial p, we have

$$\mu(z)\left|\Re P_\varphi^g p(z)\right| = \mu(z)\left|g(z)p(\varphi(z))\right| \le \mu(z)|g(z)|\|p\|_\infty \to 0, \ \ as\ |z| \to 1.$$

Since the set of all polynomial is dense in $\mathcal{B}_0$ for each $f \in \mathcal{B}_0$ there is a sequence of polynomials $(p_k)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \|f - p_k\|_{\mathcal{B}} = 0 \tag{97}$$

From (97) and since the operator $P_\varphi^g: \mathcal{B}_0 \to B_\mu$. is bounded, it follows that

$$\left\| P_\varphi^g f - P_\varphi^g p_k \right\|_{B\mu.} \le \left\| P_\varphi^g \right\|_{\mathcal{B}_0 \to B\mu.} \|f - p_k\|_{\mathcal{B}_0} \to 0.$$

As $k \to \infty$. Hence $P_\varphi^g(\mathcal{B}_0) \subset B\mu.0$. since $B_{\mu.0}$ is a closed sybset of $B\mu$ the boundedness of $P_\varphi^g: \mathcal{B}_0 \to B_{\mu,0}$ follows.

Let X and Y be Banach spaces, and $L : X \to Y$ be a bounded linear operator. The essential norm of the operator $L : X \to Y$, denoted by $\|L\|_{e,X\to Y}$, is defined as follows

$$\|L\|_{e,X\to Y} = \inf\ \{\|L+K\|_{X\to Y} : K \text{ is compact from } X \text{ to } Y\},$$

where $\|.\|_{X\to Y}$ denote the operator norm.

From this definition and since the set of all compact operators is a closed subset of the set of bounded operators it follows that operator L is compact if and only if $\|L\|_{e,X\to Y} = 0$..

We prove the main result, namely, we calculate the essential norm of the operator $P_\varphi^g: \mathcal{B}\ (or\ \mathcal{B}_0) \to B_\mu$.

Theorem (1.2.7)[41]: Assume g ∈ H(𝔹), g(0) = 0, μ is normal, ϕ is a holomorphic self-map of 𝔹 and $P_\varphi^g: \mathcal{B}\ (\text{or } \mathcal{B}_0) \to B\mu$. is bounded if $\|\varphi\|_\infty = 1$, then

$$\left\| P_\varphi^g \right\|_{e,\mathcal{B}\to\mathcal{B}_\mu} = \left\| P_\varphi^g \right\|_{e,\mathcal{B}_0\to\mathcal{B}_\mu} = \frac{1}{2}\lim_{|\varphi(z)|\to 1} \sup \mu(z)\, |g(z)|\, In \frac{1+|\varphi(z)|}{1-\left|\varphi|z|\right|}. \tag{98}$$

while $\|\varphi\|_\infty < 1$, *then*

$$\left\| P_\varphi^g \right\|_{e,\mathcal{B}\to\mathcal{B}_\mu} = \left\| P_\varphi^g \right\|_{e,\mathcal{B}_0\to\mathcal{B}_\mu} = 0. \tag{99}$$

Proof. First assume that $\|\varphi\|_\infty = 1$. Set the following family of test functions

$$f_w^\varepsilon(w) = \left(In\frac{(1+|w|)^2}{1-\langle z,w\rangle}\right)^{\varepsilon+1}\left(In\frac{1+|w|}{1-|w|}\right)^{-\varepsilon}, w \in \mathbb{B}/\ [0]$$

It is easy to see that

$$|f_w^\varepsilon(0)| \le (In(1+|w|)^2)^{\varepsilon+1}\left(In\frac{1+|w|}{1-|w|}\right)^{-\varepsilon} \le 2^{\varepsilon+1} In2$$

And

$$\lim_{|w|\to 1} |f_w^\varepsilon(0)| = 0. \tag{100}$$

Farther we have

$$(1-|z|^2)|\nabla f_w^\varepsilon(z)| = (\varepsilon+1)\frac{(1-|z|^2)|w|}{|1-\langle z,w\rangle|} \tag{101}$$

$$\le (\varepsilon+1)(1+|z|)|w|\left(In\frac{(1+|w|)^2}{1-|w|} + 2\pi\right)^{\varepsilon}\left(In\frac{1+|w|}{\left|1-|w|\right|}\right)^{-\varepsilon} \tag{102}$$

From (102) it follows that

$$\lim_{|w|\to 1} \sup b(f_w^\varepsilon) \le 2(\varepsilon+1) \tag{103}$$

And from (101) that m for each fixed $w \in \mathbb{B}/\{0\}. f_w^{\varepsilon} \in \mathcal{B}_0$.
Hence (100) and (103) imply

$$\lim_{|w|\to 1} \|f_w^{\varepsilon}\|_{\mathcal{B}} \leq 2(\varepsilon + 1) \tag{104}$$

Now, assume that $(\varphi(z_k))_{k\in\mathbb{N}}$ is a sequence in $\mathbb{B}$ such that $|\varphi(z_k)| \to 1$ *as* $k \to \infty$ note that from (104) it follows that the sequence $F_k(z) = f_{\varphi(z_k)}^{\varepsilon}(z),\quad k \in \mathbb{N}$ is such that

$$\lim_{k\to\infty} \|F_k\|_{\mathcal{B}} \leq 2(\varepsilon + 1) \tag{105}$$

and that F_k converges to zero uniformly on compacts of $\mathbb{B}$ as $k \to \infty$. By Theorem (1.2.7).16 in [9] it follows that $F_k \to 0$ weakly in B_0 as $k \to \infty$. Hence for every compact operator $K : B_0 \to B_\mu$ we have that

$$\lim_{k\to\infty} \|KF_k\|_{\mathcal{B}_\mu} = 0 \tag{106}$$

Assume that $K : B_0 \to B_\mu$ is an arbitrary compact operator. Then from the boundedness of $P_\varphi^g : B_0 \to B_\mu$ for each $k \in \mathbb{N}$

$$\|F_k\|_{\mathcal{B}} \left\|P_\varphi^g + K\right\|_{B0\to B\mu} \geq \left\|\left(P_\varphi^g + K\right)(F_k)\right\|_{B\mu} \geq \left\|P_\varphi^g F_k\right\|_{B\mu} - \|KF_k\|_{B\mu} \tag{107}$$

Letting $k \to \infty$ in (107) and using (106) we obtain

$$\begin{aligned}
\lim_{k\to\infty} \|F_k\|_{\mathcal{B}} \left\|P_\varphi^g + K\right\|_{B0\to B\mu} &\geq \lim_{k\to\infty} sup \left(\left\|P_\varphi^g F_k\right\|_{B\mu} - \|KF_k\|_{B\mu}\right) \\
&= \lim_{k\to\infty} sup \left\|P_\varphi^g F_k\right\|_{B\mu} \\
&= \lim_{k\to\infty} \sup sup_{z\in\mathbb{B}}\, \mu(z)|g(z)|\left|F_k(\varphi(z))\right| \\
&\geq \lim_{k\to\infty} sup\, \mu(z_k)\left|g(z_k)F_k(\varphi(z_k))\right| \\
&= \lim_{k\to\infty} sup\, \mu(z_k)|g(z_k)| In \frac{1 + |\varphi(z_k)|}{1 - |\varphi(z_k)|}
\end{aligned}$$

From this and (105) we have

$$2(\varepsilon + 1)\left\|P_\varphi^g + K\right\|_{B0\to B\mu} \geq \lim_{k\to\infty} sup\, \mu(z_k)|g(z_k)|\, In \frac{1 + |\varphi(z_k)|}{1 - |\varphi(z_k)|} \tag{108}$$

Taking the infimum in (108) over the set of all compact operators $K : B_0 \to B_\mu$ and since ε is an arbitrary positive

$$\left\|P_\varphi^g\right\|_{e,B0\to B\mu} \geq \lim_{k\to\infty} sup \frac{1}{2}\mu(z_k)g(z_k) In \frac{1 + |\varphi(z_k)|}{1 - |\varphi(z_k)|}$$

Which implies the inequality

$$\left\|P_\varphi^g\right\|_{e,B0\to B\mu} \geq \lim_{|\varphi(z)|\to 1} \frac{1}{2}\mu(z)g(z) In \frac{1 + |\varphi(z)|}{1 - |\varphi(z_k)|} \tag{109}$$

Now we prove the reverse inequality. Assume that $(r_l)l \in N$ is a sequence which increasingly converges to 1. Consider the operators defined by

$$P_{r_l\varphi}^g(f)(z) = \int_0^1 g(tz)\, f\left(r_l\varphi(tz)\right)\frac{dt}{t}. l \in \mathbb{N}. \tag{110}$$

By using the mean value theorem and the definition of the Bloch space, we obtain
$= C_\rho\,(1 - r_l) \to 0$ as $l \to \infty$.

Letting $l \to \infty$ in (111) and (112), using (114) and (115), and then letting $\rho \to 1$ we obtain the inequality both equalities in (98) follow.

$$sup_{f\in\mathcal{B},\|f\|_{\mathcal{B}}\leq 1}|f(z) - f(w)| = \frac{1}{2} In \frac{1+|\varphi_w(z)|}{1-|\varphi_w(z)|}. \quad z, w \in \mathbb{B}$$

(where φ_w is the involutive automorphism of B that interchanges 0 and w), We have

$$\left\|P_{\varphi}^{g}-P_{r_l\varphi}^{g}\right\|_{\mathcal{B}\to B\mu}=sup_{\|f\|_{\mathcal{B}}\le 1}\,up_{z\in\mathbb{B}}\,\mu(z)|g(z)|\left|f(\varphi(z))-f(r_l\varphi(z))\right|$$
$$\le sup_{\|f\|_{\mathcal{B}}\le 1}\,sup_{|\varphi(z)|\le\rho}\mu(z)|g(z)|\left|f(\varphi(z))-f(r_l\varphi(z))\right|$$
$$+sup_{\|f\|_{\mathcal{B}}\le 1}\,Sup_{|\varphi(z)|>\rho}\mu(z)|g(z)|\left|f(\varphi(z))-f(r_l\varphi(z))\right|$$
$$\le \|g\|_{H_{\mu}^{\infty}}\quad sup_{\|f\|_{\mathcal{B}}\le 1}\,sup_{|\varphi(z)|\le\rho}\left|f(\varphi(z))-f(r_l\varphi(z))\right| \tag{111}$$
$$+Sup_{|\varphi(z)|>\rho}\mu(z)|g(z)|\frac{1}{2}In\frac{\left|\varphi_{\varphi(z)}(r_l\varphi(z))\right|}{\left|\varphi_{\varphi(z)}(r_l\varphi(z))\right|}. \tag{112}$$

Since

$$\left|\varphi_{\varphi(z)}(r_l\varphi(z))\right| = \left|\frac{\varphi(z)-P_{\varphi(z)}(r_l\varphi(z))-S_qQ_{\varphi(z)}(r_l\varphi(z))}{1-\langle r_l\varphi(z),\varphi(z)\rangle,}\right|$$
$$=\frac{|\,\varphi(z)|(1-r_l)}{1-r_l|\,\varphi(z)|^2}\le |\,\varphi(z)|,$$

And since the function

$$h(x)=In\,\frac{1+x}{1-x}. \tag{113}$$

is increasing on the interval [0,1), we obtain

$$Sup_{|\varphi(z)|>\rho}\,\mu(z)|g(z)|In\frac{1+\left|\varphi_{\varphi(z)}(r_l\varphi(z))\right|}{1-\left|\varphi_{\varphi(z)}(r_l\varphi(z))\right|}$$
$$\le Sup_{|\varphi(z)|>\rho}\mu(z)|g(z)|\;In\frac{1+|\varphi(z)|}{1-|\varphi(z)|} \tag{114}$$

Now we estimate the quantity in (111). Let

$$I_l := \;sup_{\|f\|_{\mathcal{B}}\le 1}\,sup_{|\varphi(z)|\le\rho}\left|f(\varphi(z))-f(r_l\varphi(z))\right|$$

By using the mean value theorem and the definition of the Bloch space , we obtain

$$I_l\le\; sup_{\|f\|_{\mathcal{B}}\le 1}\,sup_{|\varphi(z)|\le\rho}(1-r_l)|\varphi(z)|sup_{|w|\le\rho}|\nabla f(w)|$$
$$\le\rho\,\frac{1-r_l}{1-\rho^2}sup_{\|f\|_{\mathcal{B}}\le 1}\|f\|_{\mathcal{B}}$$
$$=\;C_\rho(1-r_l)\to 0\quad as\; l\to\infty. \tag{115}$$

Letting $l\to\infty$. in (111) and (112) using (114) and (115), and then letting $\rho\to\infty$. we obtain the inequality

$$\left\|P_{\varphi}^{g}\right\|_{e,\mathcal{B}\to\mathcal{B}_\mu}\le\lim_{|\varphi(z)\to 1|}sup\frac{1}{2}\,\mu(z)g(z)\;In\frac{1+|\varphi(z)|}{1-|\varphi(z)|}. \tag{116}$$

From (109), (116) and since

$$\left\|P_{\varphi}^{g}\right\|_{e,\mathcal{B}\to\mathcal{B}_\mu}\ge\left\|P_{\varphi}^{g}\right\|_{e,\mathcal{B}_0\to\mathcal{B}_\mu}$$

Both equalities in (98) follow.

Now assume $\|\varphi\|\infty < 1$, then similar to operators in (110) it is proved that the operator P_{φ}^{g} : B (or B_0)⟶ B_μ is compact, which is equivalent with (99), finishing the proof of the theorem.

The following result regarding the compactness of the operator P_{φ}^{g} : B (or B_0)⟶ B_μ is a direct consequence of Theorem (1.2.7).

Corollary (1.2.8)[41]: Assume g ∈ H(B), g(0) = 0, μ is normal, ϕ is a holomorphic self-map of B such that $\|\varphi\|_\infty$= 1, and the operator P_{φ}^{g} : B (or B_0)⟶ B_μ is bounded. Then the operator

P_{φ}^{g} : B (or B_0)⟶ B_μ is compact if and only if

$$\lim_{|\varphi(z_k)|\to 1} \mu(z)|g(z)|In\frac{1+|\varphi(z)|}{1-|\varphi(z)|}=0. \quad (117)$$

We calculate the essential norm of the operator $P_{\varphi}^{g}: \mathcal{B}\ (or\ \mathcal{B}_0) \to B_{\mu,0}$.

Theorem (1.2.9)[41]: Assume g ∈ H(𝔹), g(0) = 0, μ is normal, ϕ is a holomorphic self-map of 𝔹 and the operator $P_{\varphi}^{g}: \mathcal{B}$ (or $\mathcal{B}_0$) → Bμ, 0 is bounded then

$$\| P_{\varphi}^{g}\|_{e,\mathcal{B}\to\mathcal{B}_{\mu,0}} = \| P_{\varphi}^{g}\|_{e,\mathcal{B}_0\to\mathcal{B}_{\mu,0}} = \frac{1}{2}\lim_{|z|\to 1} sup\ \mu(z)\,|g(z)|In\frac{1+|\varphi(z)|}{1-\big|\varphi|z|\big|}. \quad (118)$$

Proof. Since $P_{\varphi}^{g}: \mathcal{B}$ or $(\mathcal{B}_0)$ $B\mu, 0$ is bounded then, then for the test function $f_0(z) \equiv 1 \in \mathcal{B}_0$ we obtain that $g \in H_{\mu}^{\infty}$.

First assume $\|\infty\|_{\infty} < 1$. Then, similar to operator (110) it can be proved that is compact. Hence

$$\|P_{\varphi}^{g}\|_{e,\mathcal{B}(or\mathcal{B}_0)\to\mathcal{B}_{\mu,0}} = 0.$$

On the other hand , since $\|\varphi\|_{\infty} < 1$ and $g \in H_{\mu}^{\infty}$, it follow that

$$\lim_{|z|\to 1} sup\mu(z)|g(z)|In\frac{1+|\phi(z)|}{1-|\phi(z)|} \le In\frac{1+\|\varphi\|_{\infty}}{1-\|\varphi\|_{\infty}}\lim_{|z|\to 1}\mu(z)|g(z)| = 0,;$$

From which (118) follow in this case

Now assume $\|\varphi\|_{\infty} = 1$, it is clear that

$$\lim_{|z|\to 1} sup\ \mu(z)|g(z)|\ In\frac{1+|\phi(z)|}{1-|\phi(z)|} \ge \lim_{|\phi(z)|\to 1} sup\ \mu(z)|g(z)|In\ \frac{1+|\phi(z)|}{1-|\phi(z)|}$$

Assume that $(z_k)_{k\in\mathbb{N}}$ is such a sequence that

$$\lim_{|z|\to 1} sup\ \mu(z)|g(z)|\ In\frac{1+|\phi(z)|}{1-|\phi(z)|} = \lim_{k\to\infty}\mu(z_k)|g(z_k)|In\ \frac{1+|\phi(z_k)|}{1-|\phi(z_k)|} \quad (119)$$

If $sup_{k\in\mathbb{N}}\varphi(z_k) < 1,$ then in view of the fact $g \in H_{\mu}^{\infty}$, the last limit is zero.

And consequently the second limit in (119) is also zero. Otherwise, there is a subsequence $\left(\varphi\left(z_{k_l}\right)\right)_{l\in\mathbb{N}}$ such that $\left|\varphi\left(z_{k_l}\right)\right| \to 1 as\ l \to \infty$ so that both limit in (119) are equal, that is

$$\lim_{|z|\to 1} sup\ \mu(z)|g(z)|\ In\frac{1+|\phi(z)|}{1-|\phi(z)|} = \lim_{|\phi(z)|\to 1}\mu(z)|g(z)|In\ \frac{1+|\phi(z)|}{1-|\phi(z)|}\ .$$

From this and by Theorem (1.2.7) the result follows in this case, finishing the proof of the theorem.

Corollary (1.2.10)[41]: Assume g ∈ H(𝔹), g(0) = 0, μ is normal, ϕ is a holomorphic self-map of 𝔹 and the operator $P_{\varphi}^{g}: \mathcal{B}\ (or\ \mathcal{B}_0) \to \mathcal{B}_{\mu,0}$ is compact if and only if

$$\lim_{|z|\to 1}\mu(z)\,|g(z)|In\frac{1+|\varphi(z)|}{1-|\varphi(z)|} = 0.$$

Chapter 2
Composition of Blochs with Inner Function and BiBloch Mapping

We show that f is a holomorphic self-map of the open unit disc and $1 \le p < \infty$, then the following are equivalent. $h \circ f \in H^{2p}$ for all Bloch functions h, $\sup\limits_{r} \int_0^{2\pi} \left(\log \frac{1}{\left(1-|f(re^{i\theta})|^2\right)^p} d\theta \right)^p < \infty$, $\int_0^{2\pi} \left(\int_0^1 (f^{\#})^2 (re^{i\theta})(1-r)dr \right)^p d\theta < \infty$, where $f^{\#}$ is the hyperbolic derivative of f: $f^{\#} = |f'|/(1-|f|^2)$. We give several applications, we can generalize known characterizations on Bloch-BMO pullbacks.

Section (2.1): Bounded Analytic Functions

By P. Ahern and W. Rudin ([81], [82]), there is extensive research on Bloch-to-BMOA pullbacks, that is, research on those holomorphic maps f of the unit ball of C^n into the unit disc of C for which the composition operator defined by

$$C_f(h) = h \circ f$$

takes Bloch functions to functions of BMOA. See [87] for recent research on Bloch to BMOA pullbacks.

It is known (see [87]), when $n = 1$, that one of the necessary and sufficient conditions for C^n to take all Blochs to $BMOA$ is that f be a function of $BMOA_\sigma$, the Yamashita hyperbolic $BMOA$ class (see [84] and [90] for $BMOA$ and $BMOA_\sigma$).

Theorem (2.1.1)[80]: (Main Result). If f is a holomorphic self-map of the open unit disc and $1 \le p < \infty$, then the following are equivalent.

(i) C_f takes Blochs to H^{2p}, that is, $h \circ f \in H^{2p}$ for all Bloch functions h

(ii) f belongs to Yamashita's hyperbolic Hardy class H_σ^P, that is,

$$sup_r \int_0^{2\pi} \left(\log \frac{1}{1-|f(re^{i\theta})|^2} \right)^p d\theta < \infty.$$

(iii) $\int_0^{2\pi} \left(\int_0^1 (\mathrm{f}^{\#})^2 \left(re^{i\theta}\right)(1-r)dr \right)^p d\theta < \infty.$

where $f^{\#}$ is the hyperbolic derivative of f: $f^{\#} = |f'|/(1-|f|^2)$.

The Bloch space $\mathfrak{B}$ consists of those f holomorphic in the open unit disc D of the complex plane for which

$$\|f\|_{\mathfrak{B}} := sup_{\, z\in D} |f'(z)|(1-|z|^2) < \infty.$$

We let $1 \le p < \infty$ and set for f subharmonic in D

$$\|f\|_p := sup_{\ r} \left(\int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right|^p \frac{d\theta}{2\pi} \right)^{1/p}.$$

then $H^p = H^p(D)$ consists of those f holomorphic in D for which $\|f\|_p < \infty$. see [83] and [84] for Bloch and H^p spaces.

The Yamashita hyperbolic hardy class H_σ^p is defined as the set of those holomorphic self- map f of D for which

$$\|\sigma(f)\|_p < \infty,$$

where $\sigma(z)$ denotes the hyperbolic distance of z and 0 in D, namely,

$$\sigma(z) = \frac{1}{2} log \frac{1+|z|}{1-|z|}$$

Though H_σ^p is not a linear space, it has, as hyperbolic counterparts, many properties analogous to those of H^p. We let T be the boundary of D and set, following Yamashita,

$$\lambda(f) = \log\frac{1}{1-|f|^2} \quad and \quad f^\# = \frac{|f'|}{1-|f|^2}$$

for the holomorphic self-map f of D. Then $\sigma(f)^p;\ \lambda(f)^p, and\ (f\#)^p$ are subhar-monic functions, so that their integral means over rT are nondecreasing functions of r: for example,

$$\int_0^{2\pi} \lambda(f)^p\left(re^{i\theta}\right)\frac{d\theta}{2\pi} \nearrow \ \|\lambda(f)\|_p^p \ as\ r \nearrow 1$$

Also, there are corresponding maximal theorems for these Functions: Set

$$M_\lambda\,(f,\theta\,) \ = \ sup\ \left\{\lambda(f)\left(re^{i\theta}\right) :\ 0 \le\ r\ <\ 1\right\}\ ;$$

Then

$$\|M_\lambda\,(f,.)\,\|_{L^p} \le C_p\|\lambda(f)\|_P. \tag{1}$$

for $f \in H_\sigma^p$ ([88]). The left side of (1) means usual $L^p(T\,)$ norm. The function $f^\#$ is the hyperbolic counterpart of f' and it easily follows that

$$\frac{1}{2}\,\lambda(f) \le \sigma(f) < \frac{1}{2}\,\lambda(f) + log^2. \tag{2}$$

And

$$\Delta(\lambda(f)^p) = 4p\,\{(p-1)|f|^2 + \lambda(f)\}\lambda(f)^{p-2}(f^\#)^2, \tag{3}$$

Where Δ denotes the Laplacian:

$$\Delta\ = 4\ \frac{\partial^2}{\partial z\partial\overline{z}}.$$

From (2) and (3), it should be noted that

$$f \in H_\sigma^p \quad \text{if and only if} \quad \|\lambda(f)\|_p < \infty$$

$$\Delta(\lambda(f)^p){\sim}\Delta\lambda(f)^{p-1}(f^\#)^2 \tag{4}$$

Here and after $\psi{\sim}\phi$ means that either both sides are zero or the quotient ψ/ϕ lies between two positive uniform constants. See, for example, [85], [86], [88], and [89] for the theory of H_σ^p.

We show that f to being a holomorphic self-map of D and denote $f_r;\ 0 \le\ r \le\ 1$, as the function de ned by f_r (z) = $f(rz);\ z\ \in D$. Positive constants depending on p (or q) will be denoted by $C_p\ (or\ C_p)$, whose quantities may vary at each occurrence.

For h holomorphic in D, g-function of Paley defined by

$$g(\theta) \coloneqq g(h)(\theta) = \left(\int_0^1 |h'|^2\left(re^{i\theta}\right)(1-r)dr\right)^{\frac{1}{2}}, \qquad 0 \le \theta < 2\pi. \tag{5}$$

Satisfies $\|g(h)\|_{L^p}{\sim}\|h\|_p if\ h(0) = 0$, ([91]). Consider Green's theorem of the form

$$r\int_0^{2\pi}\frac{\partial\psi}{\partial r}d\theta = \iint_{|z|\le r}\Delta\psi\ dxdy$$

Valid for $\psi \in C^2(D)$. if we integrate both sides with respect to dr after dividing them by r and applying $\psi = \lambda(f)^p$, then we obtain , by use of (4)

$$\left\|\lambda(f_\rho)\right\|_p^p -\ \lambda(f)^p(0)$$

$$\sim \frac{1}{2\pi}\int_0^{\rho}\frac{dr}{r}\iint\limits_{|z|<\rho}\lambda(f)^{p-1}(f^{\#})^2(z)\,dx\,dy. \qquad (6)$$

$$= \frac{1}{2\pi}\iint\limits_{|z|<\rho}\lambda(f)^{p-1}(f^{\#})^2(z)log\frac{\rho}{|z|}dx\,dy. \qquad 0 \le \rho \le 1.$$

In particular, we see from (6) that $f \in H_\sigma^1$ if and only if

$$\infty > \iint_D (f^{\#})^2\,(z)log\frac{1}{|z|}dxdy.$$

This suggests we de ne the hyperbolic version of g-function using $f^{\#}$. We define

$$g_\sigma(\theta) := g_\sigma(f)(\theta) = \int_0^1 (f^{\#})^2\left(re^{i\theta}\right)(1-r)dr, \quad 0 \le \theta < 2\pi. \qquad (7)$$

It is not surprising to see the absence of the square root in the definition of g_σ in (7) when we compare it to that of g- function in (5), because there is a known parallelism (see [89]) between H^2 and H_σ^1 .

Theorem (2.1.2)[80]: $If \;\; 1 \le p < \infty$, then the following are equivalent.

(i) $f \in H_\sigma^p$

(ii) $g_\sigma(f) \in L^p(T)$:

In fact, $\|\lambda(f)\|_p \sim \|g_\sigma(f)\|_{L^p}$ $provided\, f(0) = 0.$

Proof. By (6) and (7), there is nothing to prove when $p = 1$. We assume $1 < p < \infty, and\ let\ \frac{1}{p} + \frac{1}{q} = 1.$

(i) ⟹ (ii) We begin with the identity

$$\|g_\sigma\|_{L^p} = sup\int_0^{2\pi} g_\sigma(\theta)\,h(\theta)\frac{d\theta}{2\pi}.$$

where the supremum is taken with respect to all nonnegative trigonometric poly-nomials h with $\|h\|_{L^q} \le 1.$. Since $(f^{\#})^2$is subharmonic, we have

$$(f^{\#})^2\left(r^2e^{i\theta}\right) \le \int_0^{2\pi} P(r,\theta-t)\,(f^{\#})^2\left(re^{i\theta}\right)\frac{d\theta}{2\pi}. 0 \le r < 1. \qquad (8)$$

where $P\,(r,\theta)$is the Poisson kernel:

$$P(r,\theta) = \frac{1-r^2}{1-2rcos\theta+r^{2.}}.$$

Let u be the Poisson integral of h. Then

$$\int_0^{2\pi} g_\sigma(\theta)\,h(\theta)d\theta = \int_0^{2\pi}\int_0^1 (f^{\#})^2\left(re^{i\theta}\right)\,h(\theta)(1-r)drd\theta$$

$$= \int_0^{2\pi}\int_0^1 (f^{\#})^2\left(r^2e^{i\theta}\right)\,h(\theta)(1-r^2)2rdrd\theta$$

$$\le 2\iint_D (f^{\#})^2(z)\,u(z)(1-|z|^2)dxdy \qquad (9)$$

$$\leq 4 \iint_D (f^{\#})^2(z)\, u(z) log \frac{1}{|z|} dxdy$$

where we changed the order of integration and used (8) in the first inequality.

If we denote $\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial x}\right)$ *and* $\overline{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial x}\right), z = x + iy,$, then it follows from elementary calculation and (3) that

$$4(f^{\#})^2 u = \Delta(\lambda(f)u) - 4\left(\partial\lambda(f)\overline{\partial}u + \bar{\partial}\lambda(f)\partial u\right),$$

So that by (9) we have

$$\int_0^{2\pi} g_\sigma(\theta)\, h(\theta) d\theta$$

$$\leq \left| \iint_D \Delta(\lambda(f)u)(z) log \frac{1}{|z|} dxdy \right| \tag{10}$$

$$+ 4 \iint_D \left|\partial\lambda(f)\overline{\partial}u + \bar{\partial}\lambda(f)\partial u\right| (z) log \frac{1}{|z|} dxdy = (I) + (II):$$

Now, using Green's Theorem (as in (6) with $p = 1$) with limiting process and Hölder's inequality, we obtain

$$(I) = \left| \lim_{\rho \to 1} \int_0^{2\pi} (\lambda(f)u)\left(\rho e^{i\theta}\right) d\theta - 2\pi(\lambda(f)u)(0) \right| \tag{11}$$

$$\leq 2\pi \, \|\lambda(f)\|_p \|u\|_q \leq 2\pi \|\lambda(f)\|_p.$$

On the other hand, if we let ϕ be a holomorphic function in D whose real part is u, it then follows from direct differentiation that

$$|\partial\lambda(f)| = \left|\bar{\partial}\lambda(f)\right| = |f|\, f^{\#}$$

and

$$|\partial u| = \left|\bar{\partial}u\right| = \frac{1}{2}|\phi'|$$

Hence

$$(II) \leq 4 \iint_D |\phi'(z)||f(z)| f^{\#}(z) log \frac{1}{|z|} dxdy$$

$$\leq 4 \int_0^{2\pi} \int_0^1 \left|\phi'\left(re^{i\theta}\right)\right| \left|f\left(re^{i\theta}\right)\right| f^{\#}\left(re^{i\theta}\right)(1-r) dr d\theta.$$

Since $\left|f\left(re^{i\theta}\right)\right| \leq \sqrt{\lambda(f)(re^{i\theta})} \leq M_\lambda^{\frac{1}{2}}(f,\theta)$,, we have, by the Schwarz inequality,

$$(II) \leq 4 \int_0^{2\pi} M_\lambda^{\frac{1}{2}}\left(f,\theta)\sqrt{g_\sigma(\theta)}\, g_\phi(\theta) d\theta\right). \tag{12}$$

where $g_\phi(\theta)$ is Paley g-function of ϕ . Applying Hölder's inequality with indices $2p$, $2p$, q to the right side of (12) and using maximal Theorem (2.1.1), we arrive at

$$(II) \leq C_p \, \|\lambda(f)\|_p^{\frac{1}{2}} \, \|g_\sigma(f)\|_{L^p}^{\frac{1}{2}} \left\|g_\phi\right\|_{L^q}.$$

From the theory of g-function, we know $\left\|g_\phi\right\|_{L^q} \leq C_q \|\phi\|_q$ and it follows from the theorem of M. Riesz ([91]) that $\|\phi\|_q \leq C_q \|u\|_q \leq C_q$. Thus

$$(II) \leq C_q \|\lambda(f)\|_p^{\frac{1}{2}} \|g_\sigma(f)\|_{L^p}^{\frac{1}{2}} \tag{13}$$

Hence, combining estimates (10), (11), and (13), we have

$$\int_0^{2\pi} g_\sigma(\theta) h(\theta) d\theta \leq (I) + (II) \leq 2\pi \|\lambda(f)\|_p + C_p \|\lambda(f)\|_p^{1/2} \|g_\sigma(f)\|_{L^p}^{1/2}$$

for all positive trigonometric polynomials h with $\|h\|_q \leq 1$. Therefore we obtain

$$\|g_\sigma(f)\|_{L^p} \leq \|\lambda(f)\|_p + C_p \|\lambda(f)\|_p^{\frac{1}{2}} \|g_\sigma(f)\|_{L^p}^{\frac{1}{2}}, \qquad f \in H_\sigma^p \tag{14}$$

Now we could from (14) that

$$\|g_\sigma(f)\|_{L^p} \leq C_p \|\lambda(f)\|_p. \tag{15}$$

In fact, if $f \equiv 0$, then there is nothing to prove; otherwise, setting

$$X(r) = X(f,p,r) = \left(\frac{\|g_\sigma(f)\|_{L^p}.}{\|\lambda(f_r)\|_p} \right)^{\frac{1}{2}}, 0 < r < 1,$$

with f_r the place of f becomes

$$X^2(r) \leq 1 + C_p X(r)$$

and this means, by comparing the order of $X(r)$, that $X(r)$, $0 < r < 1$, does not exceed the larger root of the equation $X^2 = 1 + C_p X$: This proves (15) with f_r, $0 \leq r < 1$, in place of f, and so (15) follows by the monotonicity of both sides.

(ii) $\Longrightarrow$ (i) It follows from (6) that

$$\|\lambda(f_r)\|_p^p - \lambda(f)^p(0) \sim C_p \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^r (f^\#)^2 \lambda(f)^{p-1}(\rho e^{i\theta}) log \frac{r}{\rho} \rho d\rho$$

$$\leq C_p \int_0^{2\pi} M_\lambda^{\frac{1}{2}}(f_r, \theta)\, g_\sigma(f)(\theta) \frac{d\theta}{2\pi}.$$

Hölder's inequality and the maximal theorem show the last quantity to be bounded

$$C_p(f)\, \|\lambda(f_r)\|_p^{p-1} \|g_\sigma(f)\|_{L^p}.$$

so that we have

$$\|\lambda(f_r)\|_p^p - \lambda(f)^p(0) \leq C_p(f) \|\lambda(f_r)\|_p^{p-1} \|g_\sigma(f)\|_{L^p}. \tag{16}$$

If $g_\sigma \equiv 0$, there remains nothing to prove. Otherwise, under the assumption $0 < \|g_\sigma\|_{L^p} < \infty$, by considering the order of

$$Y(r) = Y(f,p,r) = \|\lambda(f_r)\|_p / \|g_\sigma(f_r)\|_{L^p}\,);\ 0 < r < 1;$$

we conclude that $Y(r)$; $0 < r < 1$, does not exceed the largest root of the equation

$$Y^p - \frac{\lambda(f)^p(0)}{\|g_\sigma(f)\|_{L^p}^p.} = C_p Y^{p-1}$$

and from this follows

$$\|\lambda(f)\|_p \leq C_p(f)\, \|g_\sigma(f)\|_{L^p}.$$

Here, $C_p(f)$ denotes a constant depending on p and f.

The last assertion of Theorem (2.1.2) follows from (15) and (16).

We prove the main theorem, Theorem (2.1.1). By the aid of Theorem (2.1.2), we show the following.

Theorem (2.1.3)[80]: If $1 \leq p < \infty$, then the following are equivalent.

(i) $g_\sigma(f) \in L^p(T)$.

(ii) C_f takes Blochs to H^{2p}.

Proof. (i) $\Longrightarrow$ (ii) Let $h \in B$. Then

$$\left|(C_f h)'\right| |(h \circ f)'| \leq \|h\|_{\mathfrak{B}} f^{\#}. \tag{17}$$

Hence

$$\|g_{h\circ f}\|_{L^{2P}}^{2p} = \int_0^{2\pi} \left(\int_0^1 \left|(h \circ f)'(re^{i\theta})\right|^2 (1-r)dr \right)^p \frac{d\theta}{2\pi}$$

$$\leq \|h\|_{\mathfrak{B}}^{2p} \int_0^{2\pi} \left(\int_0^1 (f^{\#})^2 (re^{i\theta})(1-r)dr \right)^p \frac{d\theta}{2\pi}$$

$$= \|h\|_{\mathfrak{B}}^{2p} \|g_\sigma(f)\|_{L^p}^p \tag{18}$$

Therefore $\|h \circ f\|_{2p} < \infty \;\; if \;\; g_\sigma(f) \in L^p(T)$.

(ii) $\Longrightarrow$ (i) using g-function, (ii) says that

$$\int_0^{2\pi} \left(\int_0^1 (h \circ f)'|^2 (re^{i\theta})(1-r)dr \right)^p d\theta < \infty \text{ if } h \in \mathfrak{B}. \tag{19}$$

On the other hand, W. Ramey and D. Ullrich ([87], Proposition 5.4) constructed two Bloch functions $h_j, j = 1,2$, such that

$$(1 - |z|^2)(|h_1'(z)| + |h_2'(z)|) \geq 1; \;\; z \in D. \tag{20}$$

From (20) it follows that $((|h_1' \circ f| + |h_2' \circ f|) \geq (1-|f|^2)^{-1}$, so that

$$\left(\int_0^1 |(h_1 \circ f)'|^2 (1-r)dr \right)^p + \left(\int_0^1 |(h_2 \circ f)'|^2 (1-r)dr \right)^p$$

$$\geq 2^{-2p} \left(\int_0^1 |f'|^2 (|h_1' \circ f| + |h_2' \circ f|)^2 (1-r)dr \right)^2$$

$$\geq 2^{-2p} \left(\int_0^1 \frac{|f'|^2}{(1-|f|^2)^2} (1-r)dr \right)^p = 2^{-2p} g_\sigma^p(f) \tag{21}$$

Now, integrating (21) with respect to $d\theta$ and applying (19) with $h_j, j = 1,2$ in place of h, we obtain

$$\|g_\sigma(f)\|_p \leq C_p \left(\|h_1 \circ f\|_{2p} + \|h_2 \circ f\|_{2p} \right)$$

This completes the proof

H^∞ denotes, as usual, the space of bounded holomorphic functions on D. A well-known theorem of deLeeuw and Rudin ([83], Theorem 7.9) says that

$$\int_0^{2\pi} log \frac{1}{1-|f(e^{i\theta})|^2} \, d\theta = \infty.$$

is necessary and sufficient for a holomorphic $f \; with \; \|f\|_\infty = sup_{z\in D}|f(z)| = 1$ to be an extreme point of the closed unit ball of H^∞.The following is a direct corollary of Theorem 1.

Corollary (2.1.4)[80]: Let $f \in H^\infty$; $\|f\|_\infty = 1$. Then the following are equivalent.

(i) f is an extreme point of the closed unit ball of H^∞.
(ii) $h \circ f \notin H^2$ for some $h \in \mathcal{B}$.

Section (2.2): Bloch Spaces and Symmetric Measures

Let H^∞denote the algebra of bounded analytic functions in the unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$. The well-known Schwarz-Pick theorem asserts that if $f \in H^\infty$ with

$$\|f\|_\infty = sup\{|f(z)|: z \in \mathbb{D}\} \leq 1$$

Then f decreases hyperbolic distances; that is,

$$\left|\frac{f(z) - f(a)}{1 - \overline{f(a)}f(z)}\right| \leq \left|\frac{z - a}{1 - \bar{a}z}\right|$$

For all $z, a \in \mathbb{D}$ or, infinitesimally,

$$(1 - |z|^2)|f'(z)| \leq 1 - |f(z)|^2 \quad for\ z \in \mathbb{D}$$

A function $I \in H^\infty$ is called inner if it has radial limits of modulus 1 at almost every point of the unit circle $\mathbb{T}$. If $E \subset \mathbb{T}$ then $|E|$ denotes its normalized Lebesgue measure. We introduce several measures on $\mathbb{T}$, but the expression `almost every' always refers to Lebesgue measure. We assume a knowledge of inner function, such as is to be found in [84]. In particular, we may write *I* as *I=BS*

$$B(z) = \prod_{n=1}^{\infty} \frac{\overline{z_n}}{|z_n|}\left(\frac{z_n - z}{1 - \overline{z_n}\, z}\right)$$

Is the Blaschke product associated with the zero set $\{z_n\}$ of *I*, and

$$S = S[\mu](z) = exp\left\{-\int_{\mathbb{T}} \frac{\xi + z}{\xi - z} d\mu(\xi)\right\}$$

is the singular inner factor associated with the positive singular measure μ.
The first result is the construction of an inner function *I* which, in some sense, decreases hyperbolic distances as much as desired as $|z| \to 1$.

Theorem (2.2.1)[92]: Let $\phi: (0,1] \to (0, \infty)$be a continuous function with

$$\lim_{t \to 0} \phi(t) = 0.$$

Then there exists an inner function *I* such that

$$\left(\lim_{|z| \to 1^-} \frac{(1 - |z|^2)|I'(z)|}{\phi(1 - |I(z)|^2)}\right)$$

We apply this theorem to prove some results on composition operators, Zygmund functions and the existence of certain singular measures.

Recall that a function, analytic in $\mathbb{D}$, is called a Bloch function if the quantity

$$\|f\|_{\mathfrak{B}} = sup\{(1 - |z|^2)\, |f'(z)|: z \in \mathbb{D}\}$$

is finite. The Banach space of all such functions is the Bloch space, denoted by $\mathfrak{B}$ with $|f(0)| + \|f\|_{\mathfrak{B}}$ as norm. The Little Bloch space $\mathfrak{B}_0$ is the subspace of $\mathfrak{B}$ consisting of those $f \in \mathfrak{B}$ for which

$$\lim_{|z| \to 1^-} (1 - |z|^2)|f'(z)| = 0.$$

The Zygmund class $\Lambda^* = \Lambda^*(\mathbb{T})$ is the space of continuous functions F on $\mathbb{T}$ for which

$$\sup\{\left|F\left(e^{i(\theta+h)}\right) + F\left(e^{i(\theta-h)}\right) - 2F\left(e^{i\theta}\right)\right|: e^{i\theta} \in \mathbb{T}\} \leq K|h|$$

for some constant K. When the quantity above is $o(|h|)\ as\ h \to 0$ we say that F is in the small Zygmund class $\lambda^*(\mathbb{T})$. Roughly speaking, Zygmund functions are the primitives of functions in the Bloch space, namely an analytic function f is in $\mathbb{B}$ if and only if

$$F(z) = \int_0^z f(t)\, dt.$$

belongs to $\Lambda^*(\mathbb{T})$ for $|z| = 1$.. Analogous relations hold between $\mathfrak{B}_0$ and λ^* (see [108]).

Some consequences of Theorem (2.2.1) are as follows. Given a positive continuous function $w: [0,1) \to (0. +\infty)$ with

$$\lim_{t \to 1} w(t) = +\infty.$$

let $H(w)$ denote the Banach space of functions f, analytic in $\mathbb{D}$ such that

$$\|f\|_w = sup\{|f(z)|\, w(|z|)^{-1}: z \in \mathbb{D}\} < \infty.$$

Corollary (2.2.2)[92]: Let w be as above and $\varepsilon > 0$ be given. Then there exists a non-constant inner function I such that the composition operator $C(I)$, defined as

$$C(I)(f) = f \circ I$$

maps $H(w)$ into $\mathfrak{B}_0$. Moreover $C(I)$ is compact with $\|C(I)\| < \varepsilon$.

The argument leading from Theorem (2.2.1) to this corollary is very flexible and may be applied to obtain other results of a similar type. One such result is the following.

Corollary (2.2.3)[92]: Given any sequence $\{f_n\}$ of analytic function in $\mathbb{D}$, there exists an inner function I such that $f_n \circ I \in \mathfrak{B}_0$ $for\ n = 1,2,3, ...$

Another application of Theorem (2.2.1) is as follows.

Corollary (2.2.4)[92]: Let I be a non-constant inner function satisfying

$$\left(\lim_{|z| \to 1^-} \frac{(1 - |z|^2)|I'(z)|}{(1 - |I(z)|^2)^2}\right) = 0.$$

(That is, as the Theorem (2.2.1) with $\phi(t) = t^2$). Let J be a measurable subset of $\mathbb{T}$ and set

$$E = I^{-1}(J)$$

Then the function

$$F(x) = \int_0^x \chi_E(e^{it}) dt\ -$$

Belong to $\lambda^*(\mathbb{R})$

Lӧewner's lemma asserts, with the above notation, that $|E| = |J|$whenever $I(0) = 0$ and so, for any inner function

$$I, 0 < |E| < 1\ if\ 0 < |J| < 1.$$

the conclusion of Corollary (2.2.4) was considered in [103] where it was shown that if

$$F \in \lambda^*(\mathbb{R})\ \ then |E| = 0\ \ or\ |E| = 1\ or \dim(\partial E) = 1.$$

Thus, if I is as in Corollary (2.2.4), the boundary of the pre-image by I of any Borel set of positive measure has Hausdorff dimension 1. The inner function I has very wild behavior. The proof of Theorem (2.2.1) follows from the following two theorems.

Recall that a Blaschke product is called interpolating if

$$inf_n (1 - |z_n|^2)|B'(z_n)| > 0,$$

where $\{z_n\}$ is the zero sequence of B. Such a function cannot belong to $\mathfrak{B}_0$ except when it has a finite number of zeros.

The function B in Theorem (2.2.10) will in fact be a covering map. Theorem (2.2.10) permits us to establish Corollaries (2.2.2) and (2.2.3) with $\mathfrak{B}_0$replaced by $\mathcal{B}$, but with the extra conclusion that the corresponding inner function is an interpolating Blaschke product.

Functions in $\mathfrak{B}_0$map hyperbolic discs of a fixed diameter into euclidean discs of diameter tending to 0 as one approaches $\mathbb{T} = \partial\mathbb{D}$. The second step of our construction concerns inner functions which map hyperbolic discs of a fixed diameter into hyperbolic discs of diameter tending to 0 as one approaches $\mathbb{T}$.

Theorem (2.2.5)[92]: There exist a non- constant inner function I for which

$$\lim_{|z|\to 1^-} \frac{(1-|z|^2)|I'(z)|}{1-|I(z)|^2} = 0. \qquad (22)$$

Such an inner function I cannot extend analytically to any point of $\mathbb{T}$. Indeed, if has an angular derivative at the point $\xi \in \mathbb{T}$, that is, if the quotient

$$\frac{I(z)-I(\xi)}{z-\xi}$$

Has a limit when z approaches ξ non-tangentially, then the Julia-Caratheodory lemma asserts that

$$\lim_{z\to\xi} \frac{(1-|z|^2)|I'(z)|}{1-|I(z)|^2} > 0.$$

Although the inner functions of Theorem (2.2.5) are in $\mathfrak{B}_0$, they form a strict subclass of $\mathfrak{B}_0$. Because there exist inner functions in $\mathfrak{B}_0$ which can be extended analytically to almost every point of $\mathbb{T}$ (see [84]). Inner functions in $\mathfrak{B}_0$ have been considered by Bishop in [95] and we use some of his ideas.

It is worth mentioning also that the condition (22) in Theorem (2.2.5) has appeared in [19] in connection with composition operator from $\mathfrak{B}_0$ into itself, Indeed Theorem (2.2.5) in answer a question in [19] as whether there is a function ϕ in $\mathfrak{B}_0$ with $C(\phi)$ compact as an operator from $\mathfrak{B}_0$ to $\mathfrak{B}_0$ such that $\overline{\phi(\mathbb{D})} \cap \mathbb{T}$ is infinite. We may take $\phi(z)$ to be the inner function $I(z)$ of Theorem (2.2.5) for which $\overline{\phi(\mathbb{D})} = \overline{\mathbb{D}}$. Also, the completely opposite situation has been considered in [101].

Now suppose that $f \in H^\infty$, with $\|f\|_\infty \le 1$. For $\alpha \in \mathbb{T}$ the functions

$$H_\alpha(z) = \frac{\alpha + f(z)}{\alpha - f(z)} \qquad (23)$$

have positive real part. Hence there exist positive measures σ_α on T such that the Herglotz representation

$$ReH_\alpha(z) = \int_{\mathbb{T}} P(z,\xi)\,\sigma_\alpha(\xi)$$

holds for all $z \in \mathbb{D}$. Here,

$$P(z,\xi) = (1-|z|^2)\left|1-\bar{\xi}z\right|^{-2}$$

denotes the Poisson kernel. It is well known (and easy to prove) that the measure σ_α is singular for some $\alpha \in \mathbb{T}$ if and only if f is inner. Moreover if f and H_α are related by (23) then

$$\lim_{|z|\to 1^-} \frac{(1-|z|^2)|f'(z)|}{1-|f(z)|^2} = 0.$$

If and only if

$$\lim_{|z|\to 1} \frac{(1-|z|^2)|H_\alpha{}'(z)|}{ReH_\alpha(z)} = 0. \qquad (24)$$

So to prove Theorem (2.2.5) it is sufficient to construct a singular measure σ such that its Herglotz transform H satisfies (24).

To avoid endless repetition, J and J' will henceforth, denote adjacent arcs of $\mathbb{T}$ with

$|J| = |J'|$.

We have the following.

Theorem (2.2.6)[92]: Let H be analytic in $\mathbb{D}$ with $ReH(z) > 0 \ for \ z \in \mathbb{D}$. Let σ be the corresponding measure on $\mathbb{T}$ for which

$$ReH(z) = \int_{\mathbb{T}} P(z,\xi)\, d\sigma(\xi)$$

The following statements are equivalent:

(a) $$\lim_{|z|\to 1^-} \frac{(1-|z|^2)|H'(z)|}{ReH(z)} = 0$$

(b) $$\lim_{|J|\to 0} \frac{\sigma(J)}{\sigma(J')} = 1$$

Positive measures satisfying (b) are called symmetric (see [100]). Thus, to prove Theorem (2.2.5) it is sufficient to exhibit a positive singular symmetric measure. In fact, such measures were constructed by L. Carleson in [97] in connection with quasiconformal mappings. It is also possible to prove Theorem (2.2.5) using a construction of C. Bishop and the following result.

Here Q denotes the Carleson square

$$Q = \{z : z = re^{i\theta}, \theta \epsilon J, I - |J| \le |z| < 1\}.$$

Corresponding to an interval $J \subset \mathbb{T}$. $|Q| = |J| \ and \ Q'$ and is the corresponding Carleson square for J'.

As mentioned above, L, Carleson constructed singular symmetric measures. Indeed, let $w(t)$ be a continuous incresing function on [0, 1], with $w(0) = 0$, such that $t^{-1/2}w(t)$ is decreasing. Let σ be a positive measure on $\mathbb{T}$ such that

$$|\sigma(J) - \sigma(J')| \le w(|J|)\sigma(J).$$

For any arc J of the unit circle. L. Carleson showed that that the condition

$$\int_0 \frac{w^2(t)}{t} dt < \infty.$$

implies that σ is absolutely continuous and in fact, its derivative is in L^2. Conversely, if

$$\int_0 \frac{w^2(t)}{t} dt = \infty.$$

there exists a positive singular measure on $\mathbb{T}$ $such \ that$

$$|\sigma(J) - \sigma(J')| \le w(|J|)\sigma(J).$$

for any arc J of the unit circle.

A similar situation occurs when looking for the best decay one can have in Schwarz's Lemma. Given a positive increasing function $w \ on (0,1]$, consider

$$\widetilde{w}(t) = t\int_t^1 \frac{w(s)}{s^2} ds + tw(1) \ \ for \ t \in (0,1] \qquad (25)$$.

Observe that $\widetilde{w}(t) \ge w(t)$ for $0 < t < 1$, and $\widetilde{w}(t) \le c(\varepsilon)w(t)$ if $\frac{w(t)}{t^{1-\varepsilon}}$ is decreasing for some $\varepsilon > 0$.

Theorem (2.2.7)[92]: Let w be a positive continuous function on (0, 1].

a) Assume that

$$\int_0 \frac{w^2(t)}{t} dt < \infty.$$

Then there is no non-constant inner function I such that

$$(1-|z|^2)\frac{|I'(z)|}{1-|I(z)|^2} \le w(1-|z|).$$

For all $z \in \mathbb{D}$.

(b) Let w be increasing. Assume that there exist constants k and δ such that

$$w(t) \le kw(t) \ \ if \ 0 < t < \delta.$$

And

$$\int_0 \frac{w^2(t)}{t} dt = \infty.$$

Then there exists a non-constant inner function such that

$$(1-|z|^2)\frac{|I'(z)|}{1-|I(z)|^2} \le Cw(1-|z|). \qquad \text{For } z \in \mathbb{D}.$$

Where C is an absolute constant.

For instance the function $w(t) = |\log t|^{-\alpha}$ $satisfies\ (a) when\ \alpha > \frac{1}{2}\ and\ (b)\ when\ \alpha \le \frac{1}{2}$.

The construction of the inner function in part (b) of Theorem (2.2.7) uses symmetric singular measures. Actually, we need a refinement of the Carleson result, where we assume the integral condition and that $w(t)/t$decreases. This is done by means of Riesz products. Using Theorem (2.2.7), one can prove versions of Corollaries (2.2.2) and (2.2.3) with $\mathfrak{B}_0$ replaced by the space $\mathfrak{B}_0(w)$ of holomorphic functions f in the unit disc such that

$$\lim \frac{(1-|z|^2)|f'(z)|}{w(1-|z|^2)} = 0..$$

where w fulfills the conditions in part (b) of Theorem (2.2.7).

Corresponding to the Zygmund class and the Bloch space, there are the Zygmund measures, that is, positive measures μ in $\mathbb{T}$ for which

$$|\mu(J) - \mu(J')| = O(|J|) \ \ as\ |J| \to 0.$$

This condition is equivalent to the fact that the primitive of μ is in the Zygmund class. Piranian [107] and Kahane [104] constructed finite positive singular measures satisfying

$$|\mu(J) - \mu(J')| = o(|J|) \ \ as\ |J| \to 0.$$

We call such measures Kahane measures. Using Theorem (2.2.1) or Theorem (2.2.7) we will construct measures which are simultaneously symmetric and Kahane. In fact, as is to be expected from [97] and [104], one is able to replace the o (1) condition by a condition of the form $O\big(w(|J|)\big)$ where w fulfills the conditions in part (b) of Theorem (2.2.7). The point is that we do this in a new and uniform way. In private communications, A. Canton [96] and F. Nazarov showed us other ways of producing Kahane symmetric measures.

Also, one can establish the following sharp version of Corollary (2.2.4).

The hyperbolic metric in $\mathbb{D}$ is the Riemannian metric $\lambda_{\mathbb{D}}(z)|dz|$, where $\lambda_{\mathbb{D}}(z) = (1-|z|^2)$. Let Ω be a hyperbolic domain, that is, a domain in the complex plane whose complement has at least two points. Let $\pi\colon \mathbb{D} \to \Omega$ be a universal covering map. Then $\lambda_{\mathbb{D}}$ projects to the Poincar$é$ metric $\lambda_{\mathbb{D}}(z)|dz|$ of Ω , where

$$\lambda_{\Omega}\big(\pi(z)\big).|\pi'(z)| = \lambda_{\mathbb{D}}(z)$$

Schwarz's lemma asserts that holomorphic mapping f from $\mathbb{D}$ into Ω decrease hyperbolic distances, or infinitesimally,

$$(1-|z|^2)|f'(z)|\lambda_\Omega(f(z))$$

For all $z \in \mathbb{D}$.

A holomorphic function from the unit disc into is called inner (into) if

$$\left|\left\{e^{i\theta} : \lim_{r\to 1} f(e^{i\theta}) \; exists \; and \; belongs \; to \; \Omega = 0.\right\}\right|$$

If π is a holomorphic covering map from $\mathbb{D}$ into Ω, then π is inner; and as a matter of fact, if f is any holomorphic function from $\mathbb{D}$ into Ω which factorizes $f = \pi \circ b$, where $b\colon \mathbb{D} \to, \mathbb{D}$ then f is inner (into Ω) if and only if b is inner into $\mathbb{D}$ (see [99]).

The theorems stated have counterparts in this more general setting. For instance, Theorem (2.2.7) shows that if Ω is a hyperbolic domain and a positive weight satisfies

$$\int_0 \frac{w^2(t)}{t} dt < \infty.$$

then there is no non-constant inner function I into Ω such that

$$(1-|z|^2)|I'(z)|\lambda_\Omega(I(z)) \le w(1-|z|).$$

for all $z \in \mathbb{D}$. On the other hand, if w fulfills the conditions in part (b) of Theorem (2.2.7), there exists a non-constant inner function I into Ω such that

$$(1-|z|^2)|I'(z)|\lambda_\Omega(I(z)) \le w(1-|z|). \text{ for } z \in \mathbb{D}.]$$

We prove Theorem (2.2.10) and apply it to establish some results on composition operators. We contain two proofs of Theorem (2.2.5), using Theorems (2.2.6) and (2.2.28) respectively. Then we use Theorem (2.2.5) to establish Theorem (2.2.1) and the corollaries mentioned in this introduction, together with other related results. The proof of Theorem (2.2.6) and consists of a discretization procedure, which can be adapted to prove Theorem (2.2.28). As mentioned, this uses some of the ideas of [95]. We prove Theorem (2.2.7). This uses the existence of singular symmetric measures proved by L. Carleson and a refinement of Theorem (2.2.6), whose proof is different from the one. Also, several ways of constructing singular measures which are both symmetric and Kahane are mentioned. We construct singular symmetric measures using Riesz products.

We learned that Wayne Smith had previously obtained Theorem (2.2.7), and hence Theorem (2.2.5), by different methods [109].

The proof of Theorem (2.2.10) is based on an estimate of the density of the hyperbolic metric on plane domains, due to Beardon and Pommerenke [94]. We require only a crude estimate of this type, for which we present a proof.

Lemma (2.2.8)[92]: Let Ω be a domain in $\mathbb{D}$ and let f be an analytic function in $\mathbb{D}$ with $f(\mathbb{D}) \subset \Omega$. Then, for all $z \in \mathbb{D}$,

$$(1-|z|^2)|f'(z)| \le 6\, dist(f(z), \partial\Omega) \log \frac{e}{dist(f(z), \partial\Omega)}.$$

Proof. Let $a \in \partial\Omega$ be such that dist $(f(z), \partial\Omega) = |f(z) - a|$, and assume first that

$$|f(z) - a| \ge \frac{1}{4}(1-|f(z)|^2).$$

Then

$$(1-|z|^2)|f'(z)| \le (1-|f(z)|^2) \le 4|f(z)-a| \le 6|f(z)-a|\log\frac{e}{|f(z)-a|}.$$

If, on the other hand.

$$|f(z)-a| < \frac{1}{4}(1-|f(z)|^2). \tag{26}$$

Then $a \in \mathbb{D}$, that is $a \notin \mathbb{T}$. Since

$$S(z) = exp\left(-\frac{1+z}{1-z}\right)$$

is a universal covering map of the punctured unit disc $\frac{\mathbb{D}}{\{0\}}$, there exists a holomorphic mapping ϕ from $\mathbb{D}$ into $\mathbb{D}$ satisfying

$$\frac{f-a}{1-\bar{a}f} = S \circ \phi.$$

A simple calculation shows that

$$(1-|w|^2)|S'(w)| = 2|S(w)| \log|S(w)|^{-1}.$$

for $w \in \mathbb{D}$ and hence

$$\frac{(1-|z|^2)\,(1-|a|^2)}{|1-\bar{a}f(z)|^2}|f'(z)| \leq (1-|\phi(z)|^2)|S'\phi(z)|$$

$$= 2\left|\frac{f(z)-a}{1-\bar{a}f(z)}\right| \log \left|\frac{f(z)-a}{1-\bar{a}f(z)}\right|^{-1}.$$

Thus

$$(1-|z|^2)|f'(z)| \leq 2\frac{|1-\bar{a}f(z)|}{1-|a|^2}\,|f(z)-a| \log\frac{e}{|f(z)-a|}.$$

and the result follows from (26).

We also use the following elementary result.

Lemma (2.2.9)[92]: Let $h\colon (0,1] \to (0,1]$ be a continuous function. Then there exists a countable set $\Lambda \subset \mathbb{D}/\{0\}$ whose cluster set is contained in $\mathbb{T}$ such that, for all $z \in \mathbb{D}$,

$$\mathrm{dist}(z, \Lambda \cup \mathbb{T}) \leq h(1-|z|).$$

Theorem (2.2.10)[92]: Let $\phi\colon (0,1] \to (0,\infty)$ be a continuous function $\phi(0^+) = 0$. Then there exists an interpolating Blaschke product B such that

$$(1-|z|^2)|B'(z)| \leq \phi(1-|B(z)|)^2$$

For all $z \in \mathbb{D}$.

Proof. Given $\phi(t)$ consider a continuous function $h\colon (0,1] \to (0,1]$ satisfying

$$6h(t)log\frac{e}{h(t)} \leq \phi(t).$$

for all $t \in (0,1]$ For the set Λ of Lemma (2.2.9), let B be a holomorphic universal covering of $\mathbb{D}$ onto $\Omega = \frac{\mathbb{D}}{\Lambda}$. . Then Lemmas (2.2.8) and (2.2.9) show that

$$(1-|z|^2)|B'(z)| \leq \phi(1-|B(z)|^2)$$

as required and it remains to show that B is an interpolating Blaschke product. Since $B \in H^\infty$, its radial limit $B(\xi)$ for almost every $\xi \in \mathbb{T}$ Moreover, since B is a covering $B(\xi) \in \Lambda \cup \mathbb{T}$ and hence in fact $B(\xi) \in \mathbb{T}$ for almost every $\xi \in \mathbb{T}$ since Λ is countable. Thus B is inner.

If B had a singular inner factor then there would be at least one value of $\xi \in \mathbb{T}$, $\xi_0\ say, with$

$$\lim_{r \to 1^-} B\,(r\xi_0) = 0.$$

We have arranged that $0 \notin \Lambda$ and so this cannot happen. Thus B is a Blaschke product. To prove that it is interpolating it is sufficient to observe that the quantity

$(1-|z|^2)|B'(z)|$ depends only on $B(z)$. Indeed, if $B(a) = B(b)$, then there exists an automorphism ϕ of $\mathbb{D}$ such that $\phi(a) = b$ and $B \circ \phi \equiv B$. Hence

$$(1-|b|^2)|B'(b)| = (1-|a|^2)|\phi'(a)||B'(b)| = (1-|a|^2)|B'(a)|.$$

then

$$\inf_n\{|1-|z_n|^2)|B'(z_n)|: B'(z_n) = 0\} \geq \delta > 0.$$

for some suitable δ as required.

Now suppose that $B \in H^\infty$with $k\|B\|_{\infty!} \leq 1$ It was shown in [19] that the composition operator $C(B)$is compact in $\mathfrak{B}$ if and only if

$$(1-|z|^2)|B'(z)| = o(1)(1-|B(z)|^2) as |B(z)| \to 1.$$

Thus Theorem (2.2.10) has the following corollary.

Corollary (2.2.11)[92]: There exists an interpolating Blaschke product B such that the composition operator

$$C(B): \mathfrak{B} \to \mathfrak{B}, \ \ C(B)(f) = f \circ B.$$

is compact.

Next we consider the space $H(w)$of analytic functions in the unit disc such that the norm

$$\|f\|_w = sup\left\{\frac{f|z|}{w(|z|)}: z \in \mathbb{D}\right\} < \infty.$$

Here w denotes a positive continuous function on [0, 1) with $\lim_{t \to 1^-} w(t) = \infty$.

Corollary (2.2.12)[92]: For any function w as above and $\varepsilon > 0$, there exists an interpolating Blaschke product B such that the composition operator $C(B)$ maps $H(w)$ into the Bloch space $\mathfrak{B}$ and

$$\|C(B)(f)\|_{\mathfrak{B}} \leq \varepsilon \|f\|_w$$

Proof. Replacing w by $\varepsilon^{-1}w$, one can assume that $\varepsilon = 1,$ if $f \in H_w$ and $\|f\|_w = 1$, then, from Cauchy's inequality,

$$(1-|z|^2)|f'(z)| \leq 4w\left(|z| + \frac{1}{2}(1-|z|)\right).$$

If we choose $\phi(t)$ so that

$$w\left(t + \frac{1}{2}(1-t)\right)\phi(1-t^2) \leq 1.$$

for $0 \leq$ t < 1 then $\phi(t) \to$ as $t \to 0$. By Theorem (2.2.10), there exists an interpolating Blaschke product B such that

$$(1-|z|^2)|B'(z)| \leq \phi(1-|B(z)|^2)$$

for $z \in \mathbb{D}$. Hence for all $z \in \mathbb{D}$,

$$(1-|z|^2)(f \circ B)'(z) \leq 1.$$

Applying [19] or Corollary (2.2.11), one can arrange that the composition operator is compact.

$$C(B): H(w) \to \mathfrak{B}.$$

Corollary (2.2.13)[92]: Given a sequence $\{f_n\}$of functions analytic in $\mathbb{D}$, there exists an interpolating Blaschke product B such that $f_n \circ B \in \mathfrak{B}$ for $n = 1,2,3, ...$

Proof. It suffices to observe that there is a function $w(r)$ such that $f_n \in H(w) for$ $n =$ 1,2,3, ... instance, we may take

$$(r) = \sum\nolimits_{n<(1-r)^{-1}} sup\{|f_n(z)|: |z| = r\}.$$

We consider the case $(t) = ct^2$. for c > 0, in Theorem (2.2.10); that is, let I be an inner function satisfying

$$(1-|z|^2)\big(I'(z)\big) \le c(1-|I(z)|^2)^2 \tag{27}$$
For any $\alpha \in \mathbb{T}$ consider the holomorphic function
$$\|F_\alpha\|_{\mathfrak{B}} \le 8c.$$
Thus the measures σ_α satisfy the Zygmund condition uniformly in α. In other words, there is a constant C_1 such that
$$|\sigma_\alpha(j) - \sigma_\alpha(j)| \le C_1|J|$$
for all $\alpha \in \mathbb{T}$ and all $J,\ J'$..

Denote by $\mathcal{A}(I)$ the σ-algebra generated by the preimages under I of the Lebesgue measurable sets in $\mathbb{T}$ and the sets of measure 0.

Theorem (2.2.14)[92]: Let I be an inner function satisfying (27), and let $h \in L^1(\mathbb{T})$be measurable with respect to the σ-algebra $\mathcal{A}(I)$ Then the Cauchy transform of h, that is
$$F(z) = \int_T \frac{h(\xi)d\xi}{I - \bar{\xi}Z} \ \ for\ z\ \in \mathbb{D}.$$
Now there exists $g \in L^1(\mathbb{T})$ such that h = $g \circ I$ and it suffices to show that
$$\int_0^{2\pi} \frac{g(e^{i\theta}\ )}{I - e^{i\theta}I(z)}\ \ d\theta \in \mathfrak{B}.$$
We observe that the function
$$f(z) = \int_0^{2\pi} \frac{g(e^{i\theta}\ )}{I - e^{i\theta}z}\ \ d\theta.$$
belongs to $H(w)whereb\ w(t) = (1-t)^{-1}$. If I is an inner function satisfying (27) then the proof of Corollary (2.2.12) shows that $f \circ I \in \mathfrak{B}$ as required.

The following corollary is now immediate.

Corollary (2.2.15)[92]: Under the assumptions of Theorem (2.2.14), the function
$$F(x) = \int_0^x h\big(\ e^{it}\big)dt, \qquad with\ h \in L^1.$$
belongs to the Zygmund class $\Lambda^*(\mathbb{R})$.
$$ReH_\alpha(z) = Re\frac{\alpha + f(z)}{\alpha - f(z)} = \int_{\mathbb{T}} P(z,\xi)d\,\sigma_\alpha(\xi). \tag{28}$$
Where $\alpha \in \mathbb{T}, f \in H^\infty, With\ \|f\|_\infty \le 1\ and\ \sigma_\alpha(\xi)$ is thr associated positive probability measure on $\mathbb{T}$. The function is inner if and only if the measure σ_α is singular for some $\alpha \in \mathbb{T}$. In particular, if σ_α is singular for some $\alpha \in \mathbb{T}$ then σ_α is singular for all $\alpha \in \mathbb{T}$. Also, the support of σ_α is a finite set if and only if f is a finite Blaschke product. So this condition is also independent of $\alpha \in \mathbb{T}$. However, the fact that σ_α satisfies some property usually does not imply that σ_β satisfies the same property if $\beta \ne \alpha$. See [93], where some examples are considered.

The fact that f satisfies the conclusion of Theorem (2.2.5) can be rephrased in terms of σ_α, with $\alpha \in \mathbb{T}$.

Proposition (2.2.16)[92]: Suppose that with $f \in H^\infty\ \ with\ \|f\|_\infty \le 1.$. The following assertions are equivalent:

(a) $\lim\limits_{|z|\to 1} \frac{(1-|z|^2)|f'(z)|}{1-|f(z)|^2} = 0.$

(b) $\left|\int_T \frac{\bar{\xi} d\sigma_\alpha(\xi)}{(1-\bar{\xi}z)^2}\right| = \sigma(1) \int_T \frac{d\sigma_\alpha(\xi)}{(1-\bar{\xi}z)^2} \;\; as\, |z| \to 1$

(c) $\lim_{|z|\to 1^-} \frac{(1-|z|^2)|H'_\alpha(z)|}{ReH_\alpha(z)} = 0.$

Where f, H_α and σ_α are related by (28).

Proof. Fix $\alpha \in \mathbb{T}$. If $H_\alpha = (\alpha + f)(\alpha - f)^{-1}$, then $f = \alpha(H_\alpha - 1)(H_\alpha + 1)^{-1}$and

$$1 - f = \frac{4Re\, H_\alpha}{|1 + H_\alpha|^2}. \qquad f' = \frac{2\alpha H'_\alpha}{(H_\alpha + 1)^2}$$

Thus

$$\frac{|H'_\alpha|}{Re\, H_\alpha} = \frac{2|f'|}{1 - |f|^2},$$

Thus condition (a) may be written as

$$\lim_{|z|\to 1} \frac{(1 - |z|^2)|H'_\alpha(z)|}{ReH_\alpha(z)} \;=\; 0.$$

And since

$$H'_\alpha(z) = 2 \int_T \frac{\bar{\xi} d\sigma_\alpha(\xi)}{(1 - \bar{\xi}z)^2},$$

And

$$ReH_\alpha(z) = \int_T \frac{(1 - |z|^2) d\sigma_\alpha(\xi)}{|1 - \bar{\xi}z|^2}.$$

The result follows.

The proof of Theorem (2.2.5) now follows from Proposition (2.2.16), Theorem (2.2.6) and the existence of singular symmetric measures. We may also prove Theorem (2.2.5) from the following proposition.

Proposition (2.2.17)[92]: Let σ be a positive measure on $\mathbb{T}$ and set

$$S[\sigma](z) = \exp\left(-\int_{\mathbb{T}} \frac{\xi + z}{\xi - z}\, d\sigma(\xi\,) \right).$$

there σ is symmetric if and only if

$$\lim_{|z|\to 1^-} \frac{(1 - |z|^2)|S[\sigma](z)|}{|S[\sigma](z)| \log(S[\sigma](z)^{-1})} \;=\; 0.$$

Proof. If

$$H(z) = \int_T \frac{\xi + z}{\xi - z}\; d\sigma(\xi)\; z \in \mathbb{D}.$$

Then

$$\frac{(1 - |z|^2)|S[\sigma](z)|}{|S[\sigma](z)| \log(S[\sigma](z)^{-1})} = \frac{(1 - |z|^2)|H'(z)|}{ReH(z)}$$

and the result follows from Theorem (2.2.6).

Note that whenever σ is a singular symmetric measure, then Theorem (2.2.5) holds for $= S[\sigma]$.

There is yet another way of proving Theorem (2.2.5). In [95], Bishop has constructed a

Blaschke product in $\mathfrak{B}_0$. In fact, if

$$\mu = \sum_{z,B(z)=0} (1-|z|^2)\delta_z.$$

Then his construction satisfies

$$\lim_{|Q|\to 0} \frac{\mu(Q)}{\mu(Q')} = 1. \tag{29}$$

where, as before, Q and Q' are contiguous Carleson squares of the same size. Applying Theorem (2.2.28) one can easily show that (29) implies that

$$\lim_{|z|\to 1} \frac{(1-|z|^2)|B'(z)|}{1-|B(z)|^2} = 0.$$

Observe also that, by Proposition (2.2.16) and Theorem (2.2.6), the corresponding singular measures σ_α, with $\alpha \in \mathbb{T}$will be symmetric.

The next corollary follows from Theorem (2.2.5) and Theorem (2.2.1) in [19].

Corollary (2.2.18)[92]: There exists an inner function I such that the composition operator $C(I)$ *maps* $\mathfrak{B}$ into $\mathfrak{B}_0$compactly.

We set

$$I(z) = B(I_0(z)).$$

where B satisfies the hypotheses of Theorem (2.2.10) and I_0 the hypotheses of Theorem (2.2.5). Then

$$\frac{(1-|z|^2)|I'(z)|}{\phi(1-|I(z)|^2)} = \frac{(1-|z|^2)|B'(I_0(z))||I_0'(z)|}{\phi(1-|B(I_0(z))|^2)} \le \frac{(1-|z|^2)|I_0'(z)|}{1-|I_0(z)|^2} \to 0 \ \ as\ |z| \to 1.$$

Corollaries (2.2.2) and (2.2.3) then follow also from Corollaries (2.2.12) and (2.2.13) by composing with the same inner function I_0. Observe that in any of these results the inner function whose existence is asserted can be chosen to be singular or a Blaschke product. Moreover Corollary (2.2.12) and the Remark after Corollary (2.2.13) apply with $\mathfrak{B}$ replaced by $\mathfrak{B}_0$.

Let $\mathcal{D}$ be the set of inner functions I for which

$$\lim_{|z|\to 1^-} \frac{(1-|z|^2)|I'(z)|}{1-|I(z)|^2} = 0.$$

We note that $\mathcal{D}$ is an ideal in the space of inner functions with respect to composition from the left. In fact, if I $\in \mathcal{D}$ *and* $\phi \in H^\infty$ with $\|\phi\|_\infty \le 1$. then it follows from Schwarz's lemma that

$$\frac{(1-|z|^2)|\phi'(I(z))||I'(z)|}{1-\left|\phi\big(I(z)\big)\right|^2} \le \frac{(1-|z|^2)|I'(z)|}{1-|I(z)|^2}$$

This shows again that the inner function in Theorem (2.2.5) can be taken to be a singular inner function as well as a Blaschke product.

The next result asserts that the only primary ideals (with respect to left composition) of inner functions contained in $\mathfrak{B}_0$ are the ones given by functions in D.

Proposition (2.2.19)[92]: Let I be an inner function such that $\phi \circ I \in \mathfrak{B}_0$ for any inner function ϕ. Then $I \in \mathcal{D}$.

Proof. It is obvious that $I \in \mathfrak{B}_0$. If $I \in \mathcal{D}$ then there exists $\{z_n\} \subset \mathbb{D}$such that

$$\lim_{n\to\infty} I(z_n) = 1.$$

and

$$\frac{(1-|z_n|^2)|I'(z_n)|}{1-|I(z_n)|^2} \geq \delta > 0.$$

for $n = 1,2,3, \ldots$. Passing to a subsequence, if necessary, we may assume that $\{I(z_n)\}$ forms an interpolating sequence for H^∞. If ϕ is the corresponding interpolating Blaschke product, then for $n = 1,2,3, \ldots$ one has

$$(1-|I(z_n)|^2)\left|\phi'\big(I(z_n)\big)\right| \geq C.$$

And

$$(1-|I(z_n)|^2)|I'(z_n)|\left|\phi'\big(I(z_n)\big)\right| \geq C\frac{(1-|z_n|^2)|I'(z_n)|}{1-|I(z_n)|^2} \geq C\delta.$$

contradicting the fact that $\phi \circ I \in \mathfrak{B}_0$.

It is worth mentioning that there are no ideals with respect to composition from the right contained in $\mathfrak{B}_0$. Indeed if one consider the singular inner function

$$\phi(z) = exp\left[-\left(\frac{1+z}{1-z}\right)\right].$$

Then $I \circ \phi$ does not belong to $\mathfrak{B}_0$ for any non-constant analytic 1 quantity $(1-|z|^2)|I'(\phi(z))||\phi'(z)|$.

Cannot tend to zero, no matter what I is.

However, there do exist non-trivial right ideals. For instance, if $\alpha \geq 0$ then the set

$$\mathcal{D}_\alpha = \left\{f : f\ inner\ \frac{(1-|z|^2)|f'(z)|}{(1-|f(z)|)} = O(1) as\ |z| \rightarrow\right\}$$

Is a bilateral ideals. It is interesting to observe that if $f \in \mathcal{D}_\alpha$ and $g \in \mathcal{D}_\alpha$ then $f \circ g \in \mathcal{D}_{\alpha+\beta}$.

Let us next consider $\phi(t) = t^2$ in Theorem (2.2.1) so that I is an inner function satisfying

$$\lim_{|z|\rightarrow 1^-}\frac{(1-|z|^2)|I'(z)|}{1-|I(z)|^2} = 0. \tag{30}$$

Theorem (2.2.20)[92]: Let I be an inner function satisfying (30) and let σ_α, for $\alpha \in \mathbb{T}$, be the corresponding singular measures defined by (28). Then σ_α are (uniformly in $\alpha \in \mathbb{T}$) Kahan measures, that is

$$\lim_{|J|\rightarrow 0}\frac{1}{|J|}\big(\sigma_\alpha(J) - \sigma_\alpha(J')\big) = 0\,.$$

Uniformly for $\alpha \in \mathbb{T}$.

Proof. It is well known that the Herglotz integral of a positive measure is in $\mathfrak{B}$ if and only if the measure is Zygmund, and it is in $\mathfrak{B}_0$ if and only if the measure is in the small Zygmund class (see [108]). So it is sufficient to observe that the functions $(\alpha+1)(\alpha-1)^{-1}$are in $\mathfrak{B}_0$ and

$$\sup_\alpha \sup_{1>|z|\geq 1-r}(1-|z|^2)\left|\left(\frac{\alpha+I}{\alpha-I}\right)'(z)\right| \rightarrow 0\ \ as\ \ r \rightarrow 1.$$

Observe that Proposition (2.2.16) and Theorem (2.2.6) also show that σ_α are (uniformly in $\alpha \in \mathbb{T}$) symmetric measures.

The following theorem, is established in a similar manner to Theorem (2.2.14) and Corollary (2.2.15). Recall that given an inner function $I, \mathcal{A}(I)$ denotes the σ-algebra generated by the preimages under I of the Lebesgue measurable sets in $\mathbb{T}$ and the sets of measure 0.

Theorem (2.2.21)[92]: Let I be an inner function satisfying (30) and let $f \in L^1(\mathbb{T})$

measurable with respect to the σ -algebra $\mathcal{A}(I)$. Then
(a) the function

$$G(z) = \int_T \frac{f(\xi)\, d\xi}{1-\bar{\xi}z}.$$

Belong to $\mathfrak{B}_0$, and
(b) the function

$$F(x) = \int_0^x f(\, e^{it})dt..$$

Belong to $\lambda^*\mathbb{T}(\mathbb{R})$.
If one chooses f as the characteristic function of $I^{-1}(J)$, one obtains Corollary (2.2.4).

To prove Theorem (2.2.6) we restate condition (a) as

$$\int_T P\,(z,\xi\,)\frac{d\sigma(\xi)}{\tau(z,\xi\,)} = o(1)\int_T P\,(z,\xi\,)d\sigma(\xi) \quad as\ |z| \to 1^- \qquad (31)$$

Where

$$\tau(z,\xi\,) = \frac{\xi - z}{1-\bar{z}\xi}(\xi \in \mathbb{T}).$$

It is readily shown that this is equivalent to (a).

Given a point z = $re^{i\theta} \in \mathbb{D}$, denote by $J(z)$ the arc of $\mathbb{T}$ with center $e^{i\theta}$ and (normalized) length 1-r . Also, given an arc $J \subset \mathbb{T}$and M > 0 let MJ be the arc of the same centre and with $|MJ| = M|J|$

Part I: (b) $\Longrightarrow$ (a). Assume that (b) holds. We first prove the following.

Lemma (2.2.22)[92]: Given $\varepsilon > 0$ there exist $N > 0$ and $\delta > 0$ such that if $1-\delta < |z| < 1$, then

$$\int_{\mathbb{T}/nj(Z)} P(z,\xi)d\sigma(\xi) < \varepsilon \int_{\mathbb{T}} P(z,\xi)d\sigma(\xi)$$

The lemma states, roughly speaking, that contributions to the Poisson integral from far away do not matter.

Proof. Given $\varepsilon > 0$,, choose δ so that if J is an arc of $\mathbb{T}$ with $|J| < \delta\ \ then$

$$|\sigma(J) - \sigma(J')| < \varepsilon\sigma(J)$$

And hence

$$|\sigma(J{\cup}J') - \sigma(J')| < \varepsilon\sigma(J)$$

Hence, if $2^k|J| < \delta$, we have

$$\sigma(2^kJ) < (2+\varepsilon)^k\sigma(J).$$

We break the integral on the left into dyadic pieces. Let M denote the integer part of $log_2(\delta/(1-|z|))$, so that $2^M(1-|z|)\sim\delta$. Then, using crude estimates we obtain

$$\int_{\mathbb{T}/nj(Z)} P(z,\xi)d\sigma(\xi) \le C\left(\sum_{k-\log_2 N}^{M} \frac{\sigma\left(2^kJ(z)\right)}{2^{2k}(1-|z|^2)} + \sum_{k>M} \frac{\sigma\left(2^kJ(z)\right)}{2^{2k}(1-|z|^2)}\right).$$

Where C is an absolute constant.

The first sum is bounded by

$$\frac{\sigma\big(J(z)\big)}{|J(z)|}\sum_{k=log_2 N}^{\infty}\left(\frac{2+\varepsilon}{4}\right)^k < \varepsilon\int_{\mathbb{T}} P(z,\xi)d\sigma(\xi).$$

If N is sufficiently large.

Observe now that, for any $\varepsilon > 0$.

$$\frac{\sigma(J)}{|J|^2} \geq \left(\frac{4}{2+\varepsilon}\right)\frac{\mu(2J)}{|2J|^2}$$

If $|J|$ is sufficiently small. Iterating this inequality, we obtain

$$\frac{\sigma(J)}{|J|^2} > C\left(\frac{4}{2+\varepsilon}\right)^n \to \infty \quad as\ n \to \infty.$$

Thus

$$\lim_{|J|\to\mathbb{D}}\frac{\sigma(J)}{|J|^2} = \infty. \tag{32}$$

the second sum above can be estimated by

$$\frac{2\sigma(\mathbb{T})}{2^{2M}4(1-|z|)} \sim \frac{\sigma}{\delta^2}(1-|z|).$$

And from (32) if $(1-|z|)$ is sufficiently small, this does not exceed

$$\varepsilon\frac{\sigma\big(J(z)\big)}{1-|z|} < \varepsilon\int_{\mathbb{T}} P(z,\xi)d\sigma(\xi).$$

As required.

Now let l > 0 be a small number to be fixed later and divide $NJ(z)$ into N/l arcs each of length $l(1-|z|)$. Call these arcs J_k and let the center of each arc be $\xi_k = e^{i\theta}$. Then

$$\int_{J_k} P(z,\xi)\frac{d\sigma(\xi)}{\tau(z,\xi)} - P(z,\xi_k)\frac{\sigma(J_k)}{\tau(z,\xi_k)} \leq (1-|z|^2)\int_{J_k}\left|\frac{\xi}{(\xi-z)^2}-\frac{\xi_k}{(\xi_k-z)^2}\right|d\sigma(\xi)$$

$$\leq (1-|z|^2)\int_{J_k}\frac{|\xi-\xi_k|}{|\xi-z|^2}\frac{|\xi\xi_k-z^2|}{|\xi_k-z|^2}d\sigma(\xi)$$

$$\leq 4l\int_{J_k} P(z,\xi)d\sigma(\xi)..$$

Since $|\xi-\xi_k| < l(1-|z|)$and $|\xi\xi_k-z^2| \sim (\xi_k-z)$. now

$$\left|\int_{NJ(z)} P(z,\xi)\frac{d\sigma(\xi)}{\tau(z,\xi)} - \sum_{k=1}^{N/l} P(z,\xi_k)\frac{\sigma(J_k)}{\tau(z,\xi_k)}\right|$$

$$\leq 4l\int_{\mathbb{T}} P(z,\xi)d\sigma(\xi).$$

The estimate (31) follows on taking such that provided that we can show that

$$\left|\sum_{k=1}^{N/l} P(z,\xi_k)\frac{\sigma(J_k)}{\tau(z,\xi_k)}\right| \leq \frac{1}{N}\int_{\mathbb{T}} P(z,\xi)d\sigma(\xi). \tag{33}$$

For any $z \in \mathbb{D}$ such that $|z|$ is close enough to 1.
The number of arcs J_k is large but independent of z. Hence if $|z|$ is close enough to 1, we have

$$|\sigma(J_k) - \sigma(J_i)| < \frac{\epsilon}{2\pi}\sigma(J_k), \qquad \text{for } 1 \le k,j \le N/l.$$

We write

$$\sum_{k=1}^{\frac{N}{l}} P(z,\xi_k)\frac{\sigma(J_k)}{\tau(z,\xi)} = \sum_{k=1}^{\frac{N}{l}} P(z,\xi_k)\frac{\sigma(J_k)-\sigma(J_1)}{\tau(z,\xi)} + \sigma(J_1)\sum_{k=1}^{\frac{N}{l}} \frac{P(z,\xi_k)}{\tau(z,\xi)} = \mathbb{T}_1 + \mathbb{T}_2,$$

say.
Now

$$|\mathbb{T}_1| < \epsilon\frac{\sigma(J_1)}{|J_1|} < C\epsilon\int_{\mathbb{T}} P(z,\xi)d\sigma(\xi)\,,$$

where C is an absolute constant, while

$$\mathbb{T}_2 = \frac{\sigma(J_1)}{|J_1|}\sum_{k=1}^{\frac{N}{l}} \frac{1-|z|^2}{(\xi_k - z)^2}\xi_k|J\_k|$$

since $|J_k| = |J_1|$ for $1 \le k \le N/l$. The sum above is a Riemann sum of the integral

$$\int_{NJ(z)} \frac{1-|z|^2}{(\xi - z)^2}d\xi,$$

which an easy calculation shows to be bounded by $\frac{l}{1}/N$. The estimate (33) follows on taking N large enough since

$$\frac{\sigma(J_1)}{|J_1|} < 2\frac{\sigma\big(J(z)\big)}{|J(z)|} < C\int_{\mathbb{T}} P(z,\xi)d\sigma(\xi),$$

where C is an absolute constant.
Part II: (a) ⇒ (b). The proof follows closely the arguments of [95]. Consider the pseudo hyperbolic disc centred at z of radius $c < 1$, that is,

$$\{w : \rho(w,z) < c < 1\} \text{ where } \rho(w,z) = \left|\frac{w-z}{1-\bar{z}w}\right|.$$

Integrate (a) from z to w to obtain, for all $c < 1$,

$$\sup_{\rho(w,z)\le c} \frac{|\mathrm{Re}\, H(w) - \mathrm{Re}\, H(z)|}{\mathrm{Re}\, H(z)} \to 0 \text{ as } |z| \to 1.$$

Thus there exists a function $a(r)$ such that

(a) $a(r) \to 1$ as $r \to 1$,

(b) $$\sup\left\{\frac{|\mathrm{Re}\, H(w) - \mathrm{Re}\, H(z)|}{\mathrm{Re}\, H(z)} : \rho(w,z) < a(|z|)\right\} \to 0 \text{ as } |z| \to 1. \qquad (34)$$

Lemma (2.2.23)[92]: Suppose that (a) holds. Then, given $N > 1$ there exists $\delta = \delta(N) \in (0,1)$ such that if $1 - \delta < |z| < 1$, then

$$\int_{\mathbb{T}\setminus NJ(z)} P(z,\xi)\, d\sigma(\xi) < \frac{C}{N}\int_{\mathbb{T}} P(z,\xi)\, d\sigma(\xi),$$

where C is an absolute constant.
Proof. Let $\delta = \delta(N)$ be a small number, to be fxed later, with $\delta < 1/N$. Given $z \in \mathbb{D}$, with $1 - |z| < \delta$, consider the point

$$z_N = \left(1 - N(1-|z|)\right)(z/|z|).$$
So, $(z_N) \equiv NJ(z)$ and for $\xi \notin NJ(z)$ we have
$$|\xi - z_N| < C_0|\xi - z|,$$
where C_0 is an absolute constant. Hence
$$P(z_N, \xi) > C_0^{-2} NP(z, \xi)$$
for $\xi \notin NJ(z)$.
Now, if $\delta > 0$ is sufficiently small and $1 - \delta < |z| < 1$, we have
$$Re\, H(z) \geq \frac{1}{2} Re\, H(z_N)$$
and hence
$$\int_{\mathbb{T}} P(z,\xi)d\sigma(\xi) = Re\, H(z) \geq \frac{1}{2} Re\, H(z_N) \geq \frac{1}{2} C_0^{-2} N \int_{\mathbb{T}\backslash NJ(z)} P(z,\xi)d\sigma(\xi)$$
Lemma (2.2.24)[92]: With the above notation,
$$\left|\frac{\sigma\big(J(z)\big)}{|J(z)|} - Re\, H(z)\right| = o(1)\, Re\, H(z) as\ |z| \to 1^{-}.$$
Proof. For a given $z \in \mathbb{D}$, consider the arc
$$L = \{re^{i\theta} : |\theta - arg\, z| < \pi(1-\delta)(1-|z|)\}$$
where $r = r(z)\delta = \delta(z)$ will be chosen later to satisfy
$$r \to 1, \delta \to 0, \frac{1-r}{(1-|z|)\delta} \to 0, as\ |z| \to 1^{-}.$$
Given $\varepsilon > 0$, Lemma (2.2.23) shows that, for any $w \in L$,
$$\left|Re\, H(w) - \int_{J(z)} P(w,\xi)d\sigma(\xi)\right| < \varepsilon\, Re\, H(z)$$
provided that $(1 - r)/\delta(1-|z|)$ is sufficiently small. Thus
$$\sup_{w \in L} \frac{1}{ReH(z)} \left|Re\, H(z) \int_{J(z)} P(w,\xi)d\sigma(\xi)\right| \to 0 \quad \text{as} \quad |z| \to 1^{-}$$
Integrating along the arc L we obtain
$$\left||L|\, Re\, H(z) - \frac{1}{2\pi} \int_{J(z)} \int_{L} P(w,\xi)d\sigma(\xi)|dw|\right| = o(1)|L| ReH\,(z)\ as |z| \to 1^{-}.$$
Now $|J(z)| - |L| = \delta(1-|z|) \to 0$ and
$$\frac{1}{2\pi} \int_{L} P(w,\xi)|dw| \to 1\ as\ |z| \to 1^{-}$$
if $|\theta - arg\, z| < \pi\,(1 - c)(1-|z|)$. This shows that for any small number $c > 0$, we have
$$\lim inf_{|z|\to 1} \frac{\sigma\big(J(z)\big)}{|J(z)| Re\, H(z)} \geq 1 - c$$
and
$$\lim sup_{|z|\to 1} \frac{\sigma((1-c)J(z))}{|J(z)| Re\, H(z)} \leq 1 - c.$$

Consider the point w such that $J(w) = (1 - \ c)J(z)$, that is,

$$w = (1 - (1 - \ c)(1 - |z|))(z/|z|).$$

The second inequality gives

$$\lim sup_{|w|\to 1} \frac{\sigma\big(J(w)\big)}{\big|J(1 - c)^{-1}|J(w)|\big|Re\ H(w)} \leq \ 1 - \ c.$$

Thus,

$$1 - \ c \ \leq \ \ \lim inf_{|z|\to 1} \frac{\sigma\big(J(z)\big)}{|J(z)|Re\ H(z)} \leq \ \lim sup_{|z|\to 1} \frac{\sigma\big(J(z)\big)}{\big|J(z)^{-1}|J(z)|\big|Re\ H(z)} \ \leq \ 1,$$

for any small number $c \ > \ 0$. This proves the lemma.

The proof that $(a) \ \Longrightarrow \ (b)$ now follows immediately. For contiguous arcs J, J' with centres z and z' (and, as always, the same length),

$$\left|\frac{\sigma(J)}{|J|} - \frac{\sigma(J')}{|J'|}\right| \ \leq \left|\frac{\sigma(J)}{|J|} - Re\ H(z)\right| \ + \left|\frac{\sigma(J')}{|J|} - Re\ H(z')\right| + |Re\ H(z) - Re\ H(z')| \ .$$

Lemma (2.2.24) shows that the first two terms are bounded by $\varepsilon(Re\ H(z) + Re\ H(z'))$. Also z and z' are within a bounded hyperbolic distance of each other and hence by (34) the last term is also less than $\varepsilon(Re\ H(z))$. Summing up, we have

$$\left|\frac{\sigma(J)}{|J|} - \frac{\sigma(J')}{|J'|}\right| \qquad < \ 4\varepsilon\ Re\ H(z) < \ 5\varepsilon\ \frac{\sigma(J)}{|J|},$$

as required.

Theorem (2.2.25)[92]: Let $\{f_z : z \ \in \mathbb{D}\}$ be a family of positive continuous functions on $\mathbb{T}$. Assume that there exist constants $C, M \ > \ 0$ such that for all $z \ \in \mathbb{D}$ and all $\xi_1, \xi_2 \ \in \ \mathbb{T}$ we have

$$M^{-1} \leq \ f_z(\xi_1) \leq M, |\ f_z(\xi_1) - fz(\xi_2)| \leq \frac{C}{1 - |z|}\ |\xi_1 - \xi_2|.$$

Assume, further, that σ is a symmetric measure on $\mathbb{T}$. Then

$$\lim_{|z|\to 1} \left\{ \left(\frac{1}{\sigma\big(J(z)\big)} \int_{\mathbb{T}} \ \ f_z(\xi) P(z, \xi) d\sigma(\xi) \right) / \left(\frac{1}{|J(z)|} f_z(\xi) P(z, \xi) \frac{|d\xi|}{2\pi} \right) \right\} = 1. \quad (35)$$

Proof. (This is merely sketched.) As in Lemma (2.2.22) one may replace the integrals in (35) by integrals on $NJ(z)$ for large N. The Riemann sum argument used to prove that (b)) $\Longrightarrow \ (a)$ can now be applied.

Corollary (2.2.26)[92]: Let σ be a symmetric measure on $\mathbb{T}$ and suppose that f is a continuous function on $\mathbb{T}$. Then

$$\lim_{|z| \to 1} \frac{1 - |z|}{\sigma\big(J(z)\big)} \int_{\mathbb{T}} \ (f \circ \tau_z)(\xi) P(z, \xi) d\sigma(\xi) = \int_{\mathbb{T}} \ \ f_z(\xi) \frac{|d\xi|}{2\pi}$$

where, as before,

$$\tau_z(\xi) \ = \ \frac{\xi - z}{1 - \overline{z\xi}}.$$

Proof. Theorem (2.2.25) can be applied directly if the continuous function satisfies a Lipschitz condition,

$$|f(\xi_1) - f(\xi_2) \leq \ C|\xi_1 - \xi_2|$$

on $\mathbb{T}$. Moreover for $f \equiv \ 1$ one obtains

$$\lim_{|z|\to 1}\frac{1-|z|}{\sigma\big(J(z)\big)}\int_{\mathbb{T}} P(z,\xi)d\sigma(\xi)=1. \qquad (36)$$

Consequently,

$$\sup_{z\in\mathbb{D}}\frac{1-|z|}{\sigma\big(J(z)\big)}\int_{\mathbb{T}} P(z,\xi)\,d\sigma(\xi)<\infty.$$

Applying the Banach-Steinhaus theorem, we obtain the desired equality for any continuous function f .

Corollary (2.2.27)[92]: Let σ be a symmetric measure on $\mathbb{T}$ and f be a continuous function on $\mathbb{T}$. Then

$$\lim_{|z|\to 1}\frac{\int_{\mathbb{T}}(f\circ\tau_z)(\xi)P(z,\xi)d\sigma(\xi)}{\int_{\mathbb{T}}P(z,\xi)d\sigma(\xi)}=\int_{\mathbb{T}} f(\xi)\frac{|d\xi|}{2\pi}.$$

Proof. It suffices to apply (36) and Corollary (2.2.26).

Observe that by taking $f(z)=\bar{z}$, this corollary proves (b)) $\Longrightarrow$ (a) in Theorem (2.2.6).

Theorem (2.2.28)[92]: Given an inner function I, consider the positive measure in $\mathbb{D}\cup\mathbb{T}$,

$$\mu=\sum_{z:I(Z)=0}(1-|z|^2)\delta_z+2\sigma=0;$$

where δ_z denotes the Dirac mass at z, the sum takes into account the multiplicity of the zeros of I, and σ is the measure associated with the singular part of I. The following assertions are equivalent:

(a)
$$\lim_{z\to 1^-}\sum_{z:I(Z)=0}\frac{(1-|z|^2)|I'(z)|}{1-|I'(z)|^2}=0$$

(b)for any $\varepsilon>0$ the following two conditions hold:

$(1.b)$ $\qquad \lim_{\delta\to 0} sup_{|Q|<\delta}\left[\left|\frac{\mu(Q)}{\mu(Q)'}-1\right|:\frac{\mu(Q)}{|Q|}<\frac{1}{\varepsilon}\right]=0.$

$(2.b)$ $\qquad \lim_{N\to\infty} sup_Q\left\{\sum_{k=N}^{\infty}\frac{\mu(2^kQ/2^{k-1}Q)}{2^{2k}\mu(Q)}:\frac{\mu(Q)}{|Q|}<\frac{1}{\varepsilon}\right\}=0.$

Proof. This is similar to that of Theorem (2.2.6) and so is only sketched.

Part I: (b)) $\Longrightarrow$ (a). Using the characterization of the inner functions in $\mathcal{B}_0$ given by Bishop in [95] one can easily see that $I\in\mathcal{B}_0$. Hence in proving (a) one may assume that $|I(z)|\geq\frac{1}{2}$. A computation with logarithmic derivatives shows that

$$(1-|z|^2)\left|I'^{(z)}\right|=|I(z)|\left|\int_{\mathbb{D}} P(z,\xi)\frac{d\mu(\xi)}{r(z,\xi)}\right|. \qquad (37)$$

while

$$1-|I(z)|^2\sim log\,|I(z)|^{-2}\sim\int_{\mathbb{D}} P(z,\xi)d\mu(\xi)$$

and it is these last two integrals which one has to compare.

For fixed $\eta>0$, condition (2.b) of Theorem (2.2.28) yields an $N>0$ such that

$$\int_{\mathbb{D}\backslash NQ(z)} P(z,\xi)d\mu(\xi)<\eta\int_{\mathbb{D}} P(z,\xi)d\mu(\xi),$$

if $|z|$ is sufficiently close to 1. For such a z consider the $[N/\eta]$ disjoint Carleson squares, Q_k say, with $k = 1; 2; \ldots; [N/\eta]$, of size $\eta(1 - |z|)$ contained in $NQ(z)$.
Since $I \in \mathcal{B}_0$ and $|I(z)| \geq \frac{1}{2}$, the zeros of I are (hyperbolically) distant from z and we can assume that the zeros of I in $NQ(z)$ are contained in $\bigcup_k Q_k$. Thus

$$\mu\big(NQ(z)\big) = \mu\left(\bigcup_k Q_k\right).$$

As in the previous proof, the principal idea is to discretize the integral in (37) and compare it with an integral with respect to Lebesgue measure. If we write $A \sim B$ to mean

$$|A - B| \leq \eta \int_{\overline{\mathbb{D}}} P(z,\xi)d\mu(\xi),$$

then given points $\xi_k \in Q_k \cap \mathbb{T}$, one can show, as before, that

$$\sum_k \int_{Q_k} P(z,\xi)\frac{d\mu(\xi)}{\tau(z,\xi)} \sim \sum_k P(z,\xi_k)\frac{\mu(Q_k)}{\tau(z,\xi_k)} \sim \frac{\mu(z)}{|Q(z)|}\sum_k \frac{1-|z|^2}{(\xi_k - z)(1-\overline{z\xi})}|Q_k|$$

using (1.b) of Theorem (2.2.28) in the second estimate. Finally, one only has to observe that the last sum is a Riemann sum for the integral

$$\int_{NQ(z)\cap\mathbb{T}} \frac{1-|z|^2}{(\xi - z)^2}d\xi$$

and that this is bounded by $1/N$.
Part II: $(b)\,) \Longrightarrow (a)$. As in the proof of Theorem (2.2.6), one can show that, given $\eta > 0$, there exist $N > 0$ and $d > 0$ such that

$$\int_{NQ(z)\cap\mathbb{T}} P(z,\xi)d\mu(\xi) < \eta \int_{\overline{\mathbb{D}}} P(z,\xi)d\mu(\xi) \tag{38}$$

if $0 < 1-|z| < \delta$. To prove (1.b) of Theorem (2.2.28), it is sufficient to show that, for any $\varepsilon > 0$,

$$\sup_{z:|I|(z)>\varepsilon} \frac{\left|\mu\big(Q(z)\big)/|Q(z|) - \log|I(z)|^1\right|}{\log|I(z)|^1} \to 0\ as\ |z| \to 1^- \tag{39}$$

The estimate (39) can be proved with the same integration technique used in the corresponding implication in Theorem (2.2.6). Finally, to prove (2.b) of Theorem (2.2.28) we use (38) and (39) to show that

$$\int_{\overline{\mathbb{D}}\setminus NQ(z)} P(z,\xi)d\mu(\xi) < 2\eta\frac{\mu\big(Q(z)\big)}{|Q(z)|}$$

if $\mu\big(Q(z)\big) > \varepsilon|Q(z)|$. One now estimates the left-hand side dyadically to obtain (2.b). The details are omitted.

The existence of the function $H(z)$ of Theorem (2.2.6) as well as the existence of the inner function of Theorem (2.2.6) both depend ultimately on the existence of singular symmetric measures. In connection with the Beurling−Ahlfors extension theorem for quasi-conformal mappings, L. Carleson has shown [97] that such measures do exist. Indeed if $w(t)$ is a continuous increasing function on $[0,1]$ with $w(0) = 0$, such that $t^{-1/2}\ w(t)$ is decreasing and such that

$$\int_0 \frac{w^2(t)}{t}\,dt = \infty, \tag{40}$$

then there exists a singular measure σ on $\mathbb{R}$ such that

$$\sup_{x\in\mathbb{R}}\left|\frac{\sigma(x, x+h)}{\sigma(x-h, x)} - 1\right| \leq w(h)\ for\ h > 0. \tag{41}$$

Thus choosing, for instance, $w(t) = (\log(1/t))^{-a}$, with $a \leq \frac{1}{2}$, one obtains a singular symmetric measure. The integral condition (40) is also necessary for the existence of a singular measure satisfying (41), as was also established in [97]. Actually, if j is a measure satisfying (41) and

$$\int_0 \frac{w^2(t)}{t}\,dt < \infty,$$

then j is absolutely continuous and its derivative is in L^2_{loc}.

A similar situation occurs for inner functions.

Theorem (2.2.29)[92]: Let w be a positive continuous function on $(0, 1)$. Assume that

$$\int_0 \frac{w^2(t)}{t}\,dt < \infty.$$

Then, there is no non-constant inner function I such that

$$(1-|z|^2)\frac{|I'(z)|}{(1-|z|^2)} \leq w(1-|z\,|),$$

for all $z \in \mathbb{D}$.

Proof. Assume that such an inner function I exists. Consider a positive singular measure σ such that

$$H(z) = \frac{1+I(z)}{1-I(z)} \dagger \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\sigma(\theta) for\ z \in \mathbb{D}.$$

Then, for all $z \in \mathbb{D}$ we have

$$\frac{H'(z)}{H\,(z)} = \frac{2I'(z)}{1-I(z)^2}.$$

So

$$(1-|z|^2)\frac{|H'(z)|^2}{|H\,(z)|^2} \leq \frac{w^2(1-|z|)}{1-|z|^2}\ for\ z \in \mathbb{D}.$$

Therefore log H is an analytic function whose boundary values are of vanishing mean oscillation (see [84]). In particular, H belongs to the Hardy space H^p, for any $p < \infty$. Since σ is a singular measure, $Re\,H(e^{i\theta}) = 0$ for almost every $e^{i\theta} \in \mathbb{T}$, and this is a contradiction (see [84]).

Observe that the previous argument also shows, assuming the integral condition on w, that the only inner functions I satisfying

$$(1-|z|^2)I'(z) \leq w(1-|z\,|^2)\ for\ z \in \mathbb{D},$$

are the fnite Blaschke products.

The converse of Theorem (2.2.29) is the following.

We can then use the composition process. Let $\emptyset$ be a positive continuous function with $\emptyset(0^+) = 0$ as in Theorem (2.2.5), and let B_0 be the interpolating Blaschke product of Theorem (2.2.5).

Theorem (2.2.30)[92]: With $w, B_0, \emptyset$ and I as above, set $B = B_0 \circ I$. Then
$$\frac{(1-|z|^2|B'(z)|}{\emptyset(1-|B(z)|^2)} =\circ (w(1-|z|^2)\ as|z| \to 1^-.$$
This permits us to establish the analogues of Corollaries (2.2.2) and (2.2.3) with $\mathcal{B}_0$ replaced by
$$\mathcal{B}_0(w) = \left\{f : f\ analytic\ in\ \mathbb{D}, \lim_{|z|\to 1} \frac{(1-|z|^2)|f'(z)|}{w(1-|z|^2)} = 0\right\},$$
assuming always that w satisfies the conditions in Theorem (2.2.32).

As before, the case $\emptyset(t) = t^2$ in Theorem (2.2.30) is of special interest. If the inner function B is such that
$$\lim_{|z|\to 1-} \frac{(1-|z|^2|B'(z)|}{(1-|B(z)|^2)^2 w(1-|z|^2)} = 0,$$
then the corresponding family of positive singular measures σ_a, with $a \in \mathbb{T}$, satisfy, uniformly in a, the following two conditions simultaneously:
$$|\sigma_a(J) - \sigma_a(J')| \le w(|J|)\sigma_a(J) \tag{42}$$
$$|\sigma_a(J) - \sigma_a(J')| \le w(|J|)\ |J|,$$
The point is, however, that starting from a given symmetric measure σ, a whole family $\{\sigma_a : a \in \mathbb{T}\}$ of singular Kahane symmetric measures, with the additional property that σ_a and σ_β are mutually singular if $a \ne \beta$, can be obtained.

The condition (42) follows from the following refined version of (a)) $\Longrightarrow$ (b) of Theorem (2.2.6).

Theorem (2.2.31)[92]: Let H be analytic in $\mathbb{D}$ with $Re\ H(z) > 0$ for $z \in \mathbb{D}$. Let σ be the corresponding measure on $\mathbb{T}$ for which
$$Re\ H(z) = \int_{\mathbb{T}} P(z,\xi)d\sigma(\xi).$$
Assume that
$$\frac{(1-|z|^2|H'(z)|}{ReH(z)} \le a(1-|z|).$$
for all $z \in \mathbb{D}$, where a is a positive increasing function on $(0,\pi]$, with $a(0^+) = 0$. Then
$$|\sigma(J) - \sigma(J')| < Ca(\pi|J|)\sigma(J)$$
for any sufficiently small arc J of the unit circle.

Proof. We will use the following result due to N. G. Makarov. Given an arc J of the unit circle, denote by z_J the point $\tau(0)$ equidistant from the ends of J, where τ is the automorphism of the unit disc mapping the arc $\mathbb{T} \cap \{Re\ z > 0\}$onto J. Also, denote the domain $\tau(\{z \in \mathbb{D}: Re\ z > 0\})$by $\Delta(J)$

Lemma [105]. Let b be an analytic function in $\overline{\mathbb{D}}$, and J an arc of $\mathbb{T}$, and assume that
$$(1-|z|^2)b'(z) \le a\ for\ z \in \mathbb{D}\Delta(J),$$
for some $a < 2$. Then
$$\left|\frac{1}{|J|}\int_J \left[\exp\left(b(\xi) - b(z_J)\right) - 1\right]\frac{|d\xi|}{2\pi}\right| \le C(a).$$
Considering $H_r(z) = H(rz)$ with $r < 1$, we may assume that H is analytic in a neighbourhood of the unit disc. Given an arc J of the unit circle, replacing H by $H -$

$iIm\, H(z_J)$, we also may assume that $H(z_J) > 0$. Observe that the function $b = log\, H$ satisfies

$$(1-|z|^2)b'(z) \le a(1-|z|).$$

Since $1-|z_J| \le \pi|J|$, we obtain

$$\left|\frac{1}{|J|}\int_J Re\, H(\xi)\frac{|d\xi|}{2\pi} - ReH(z_J)\right| \le Ca(\pi|J|)ReH(z_J).$$

Hence,

$$\left|\frac{\sigma(J)}{|J|} - Re\, H(z_J)\right| \le C_a(\pi|J|)ReH(z_J).$$

Since

$$|Re\, H(z_J) - Re\, H(z'_J) \le C_2(a)(\pi|J|)ReH(z_J).$$

we deduce that

$$|\sigma(J) - \sigma(J')| \le C_3 a(\pi|J|)\sigma(J),$$

Theorem (2.2.32) follows from the following refined version of $(b)\,) \Rightarrow (a)$ of Theorem (2.2.6).

Theorem (2.2.32)[92]: Let w be a positive increasing function on $(0,1)$, with $w(0^+) = 0$. Assume that there exist constants k and δ such that

$$\widetilde{w}(t) \le kw(t) if\ |t| < \delta,$$

where $\widetilde{w}(t)$ is given by (25), and that

$$\int_0 \frac{w^2(t)dt}{t} = \infty.$$

Then, there exists an inner function I such that

$$(1-|z|^2)\frac{|I'(z)|}{1-|I(z)|^2} \le w(1-|z|)\ for\ z \in \mathbb{D},$$

Proof. By the Carleson Theorem, when $w(t)/t^{1/2}$ decreases, or applying Theorem (2.2.40) observing that $\widetilde{w}(t)/t$ decreases, we see that there exists a positive singular measure σ on $\mathbb{T}$ such that

$$|\sigma(J) - \sigma(J')| \le Cw(|J|)\sigma(J),$$

for any arc J of the unit circle.

Thus, Theorem (2.2.33) gives

$$(1-|z|^2)\frac{|H'(z)|}{ReH(z)} \le C_1\widetilde{w}(1-|z|) \le C_2 w(1-|z|),$$

for all $z \in \mathbb{D}$. So, one can choose $I = (H-1)(H+1)^{-1}$ or $I = \exp(-H)$.

Theorem (2.2.33)[92]: Let σ be a positive measure of the unit circle. Assume that

$$|\sigma(J) - \sigma(J')| \le a(\pi|J|)\sigma(J),$$

for any arc J of the unit circle, where a is a positive increasing function on $(0,1], a(0^+)\ 0$. Then, the function

$$H(z)\int_{\mathbb{T}} \frac{\xi+z}{\xi-z}\, d\sigma(\xi)$$

satisfies

$$\frac{(1-|z|^2|H'(z)|}{ReH(z)} \le C\tilde{a}(1-|z|).$$

for all $z \in \mathbb{D}$ where

$$\tilde{a}(t) = t \int_t^1 \frac{a(s)}{s^2}\, ds + ta(1).$$

Proof. Let J and Δ be arcs of the unit circle, with $J \subset \Delta$. L. Carleson observed in [97] that if $a(\Delta) < \frac{1}{2}$, one has

$$\left|\frac{\sigma(J)}{\sigma(\Delta)} - \frac{|J|}{|\Delta|}\right| \le Ca\left(\frac{1}{2}|\Delta|\right),$$

where C is an absolute constant. Actually, if a increases, then the argument of L. Carleson shows that $C = 1$. We need more information on the measure σ.

Lemma (2.2.34)[92]: Assume that the measure σ and the function a satisfy the conditions of Theorem (2.2.33). Let J and Δ be arcs of the unit circle, with $J \subset \Delta, |\Delta| \ge 2|J|$ and $a(\Delta) < \frac{1}{8}$. Then,

$$\frac{\sigma(J)}{|J|}\exp\left(-\int_{|J|}^{|\Delta|} \frac{4a(t)}{t}\, dt\right) \le \frac{\sigma(\Delta)}{|\Delta|} \le \frac{\sigma(J)}{|J|}\, exp\left(\int_{|J|}^{|\Delta|} \frac{4a(t)}{t}\, dt\right).$$

Proof. Choose a natural number n such that $2^n |J| < |\Delta| < 2^{n+1}|J|$, and arcs $J \subset K_0 \subset K_1 \subset \ldots \subset K_n = \Delta$, with $|K_{i+1}| = 2|K_i|$, for $i = 0; \ldots; n-1$, and $|K_0| \le 2|J|$. Then for $i = 0; \ldots; n-1$ we have

$$\frac{\sigma(K_i)}{|K_i|}\left(1+\frac{1}{2}\,a(|K_i|)\right)^{-1} \le \frac{\sigma(K_{i+1})}{|K_{i+1}|} \le \frac{\sigma(K_i)}{|K_i|}\,(1+\frac{1}{2}\,a(|K_i|))$$

and

$$\frac{\sigma(J)}{|J|}\left(1+2a(|J|)\right)^{-1} \le \frac{\sigma(K_0)}{|K_0|} \le \frac{\sigma(J)}{|J|}(1+\frac{17}{8}\,a(|J|))$$

Since,

$$1+\frac{1}{2}\,a(|K_i|) \le \exp\left(\int_{|K_i|}^{2|K_i|} \frac{a(t)}{t}\frac{dt}{2log2}\right)$$

and

$$1+\frac{17}{8}\,a(|J|) \le \exp\left(\int_{|J|}^{2|J|} \frac{17a(t)}{8(log2)t}\, dt\right)$$

the lemma follows.

The following result follows from Lemma (2.2.34).

Lemma (2.2.35)[92]: Under the assumptions of Lemma (2.2.34), one has

$$\left|\frac{\sigma(J)}{|J|} - \frac{\sigma(\Delta)}{|\Delta|}\right| \le min\left\{\frac{\sigma(J)}{|J|}, \frac{\sigma(\Delta)}{|\Delta|}\right\}\left[\exp\left(\int_{|J|}^{|\Delta|} \frac{4a(t)}{t}\, dt\right) - 1\right].$$

As in Theorem (2.2.6), to prove Theorem (2.2.33) it is sufficient to show the following estimate:

$$\int_{\mathbb{T}} \frac{\bar{\xi}(1-|z|^2)}{(1-\bar{\xi}z)^2}\, d\sigma(\xi) \le C\tilde{a}(|J|)\,\frac{\sigma(J)}{|J|},$$

where $J = J(z)$, for all $z \in \mathbb{D}$. Consequently, it is sufficient to prove that

$$\int_{\mathbb{T}} \frac{\bar{\xi}\, d\sigma(\xi)}{\left(1 - \bar{\xi} z\right)^2} \le C\tilde{a}(|J|\,)\, \frac{\sigma(J)}{|J|} \tag{43}$$

for all $z \in \mathbb{D}$. Consider the (signed) measure $\mu = \sigma - (2\pi +)^{-1} |J\,|^{-1}\, \sigma(J\,)|d\xi|$. It is clear that

$$\int_{\mathbb{T}} \frac{\bar{\xi}\, d\sigma(\xi)}{\left(1 - \bar{\xi} z\right)^2} = \int_{\mathbb{T}} \frac{\bar{\xi}\, d\mu(\xi)}{\left(1 - \bar{\xi} z\right)^2}$$

An integration by parts shows that the last integral is bounded by a multiple of $|\mu|(\mathbb{T}) + |J|^{-2} \int_0^{1/|J|} \min\{1, s^{-3}\}(|\mu((sJ)_+| + |\mu((sJ)_-)|)ds$.
Here if $z = re^{it}$, $(sJ)_+, (sJ)_-$ denote, respectively, the arcs,

$$(sJ)_+ = \left\{e^{i(t+\varphi)} :\ 0 \le \varphi \le \pi s(1 - |z|)\right\}, (sJ)_- = \{e^{i(t-\varphi)} :\ 0 \le \varphi \le \pi s(1 = |z|)\}$$

Hence (43) will follow if we prove the following two estimates:

$$|\mu|(\mathbb{T}) \le C\tilde{a}(|J\,|\frac{\sigma(J)}{|J|^2}, \tag{44}$$

$$\int_0^{1/|J|} \min\{1, s^{-3}\}\, |\mu(sJ)_+)|\, ds \le C\tilde{a}(|J\,|)\sigma(J). \tag{45}$$

Since $|\mu|(\mathbb{T}) \le \sigma(\mathbb{T}) + \sigma(\,J)/|J|$, (44) follows from the fact that

$$inf_J \left\{\frac{a(|J\,|)\sigma(J)}{|J\,|^2}\right\} > 0:$$

Actually, by Lemma (2.2.34), one has

$$\frac{\sigma(J)}{|J|} \ge C_1\, exp\left(-\int_{|J|}^1 \frac{4a(t)}{t}\, dt\right) \ge C_2\, \frac{|J|}{\tilde{a}(|J|)}$$

because

$$lim\ in_{t\to 0} \frac{\int_t^1 a(s)ds/s^2}{exp\,(\int_t^1 4a(s)ds/s)} > 0,$$

as a simple calculation shows.
Now let us prove (45). One can assume that $|J\,|$ is small. Observe that $\mu((sJ)_+) = \sigma(sJ)_+) - \frac{1}{2} s\sigma(J)$. Thus, for $0 < s < 1$, Lemma (2.2.35) gives $|\mu(sJ)_+)| \le |\sigma(sJ)_+) - s\sigma(J_+)\,| + s|\sigma(J_+) - \frac{1}{2}\sigma(J)| \le Cs\sigma(J)\left[exp\left(\int_{s|J|/2}^{|J|} \frac{4a(u)}{u}\, du\right) - 1\right] \le Cs\sigma(J)\left((2/s)^{4a(|\,J|)} - 1\right)$.
Consequently,

$$\int_0^1 |\mu((sJ)_+)|ds \le 3Ca(|J|)\sigma(J).$$

Also, using Lemma (2.2.35), for $1 < s < 2$ one has

$$|\mu((sJ)_+)| \le |\sigma((sJ)_+) - s\sigma(J_+)| + s|\sigma(J_+) - \frac{1}{2}\sigma(J)| \le 4Ca(|J|)\sigma(J)$$

and

$$\int_1^2 |\mu(sJ)_+)|ds \le 4Ca(|J|)\sigma(J).$$

Now, for $s > 2$, Lemma (2.2.35) gives

$$|\mu((sJ)_+| \le |\sigma((sJ)_+) - s\sigma(J)|_+ - s|\sigma(J_+) - \frac{1}{2}\sigma(J)|$$
$$\le s\sigma(J_+)\left[exp\left(\int_{|J|/2}^{s|J|/2} \frac{4a(t)}{t}\,dt\right) - 1\right] + sa(|J|)\sigma(J)$$
$$\le s\sigma(J_+)\left[\exp\left(\int_{|J|}^{s|J|} \frac{4a(t)}{t}\,dt\right) - 1\right] + sa(|J|)\sigma(J).$$

Set $s_0 = \tilde{a}(|J|)^{-1}$. Since

$$\int_{|J|}^{s_0|J|} \frac{a(t)}{t}\,dt \le \int_{|J|}^{s_0|J|} \frac{\tilde{a}(t)}{t}\,dt \le s_0|J|\,\frac{\tilde{a}(|J|)}{|J|} = 1 \tag{46}$$

we deduce that for $2 < s < s_0$,

$$|\mu((sJ)_+)| \le Cs\sigma(J) \int_{|J|}^{s|J|} \frac{a(t)}{t}\,dt. \tag{47}$$

Consequently,

$$\int_2^{s_0} |\mu((sJ)_+)|s^{-3}ds \le C\sigma(J) \int_2^{\infty} s^{-2} \int_{|J|/2}^{s|J|} \frac{a(t)}{t}\,dt\,ds$$

Observe that Lemma (2.2.34) and estimates (46) and (47) imply that $\sigma((s_0 J)_+) \le Cs_0\,\sigma(J)$. Take $\delta > 0$ such that $a(\delta) \le \frac{1}{8}$. For $s_0 < s < \delta/|J|$, Lemma (2.2.34) gives

$$\sigma((sJ)_+) \le \frac{2s}{s_0}\,\sigma((2s_0 J)_+\,exp\left(\int_{s_0|J|}^{s|J|} \frac{4a(t)}{t}\,dt\right) \le Cs\sigma(J)(2s/s_0)^{1/2} :$$

Consequently,

$$\int_{s_0}^{\delta/|J|} |\mu((sJ)_+)|s^{-3}\,ds \le C\,\frac{\sigma(J)}{s_0} = C\tilde{a}(|J|)\sigma(J).$$

Finally, applying (44), one has

$$\int_{\delta/|J|}^{1/|J|} |\mu((sJ)_+)|s^{-3}\,ds \le \frac{1}{\delta^2}\left(\sigma(\mathbb{T}) + \frac{\sigma(J)}{|J|}\right)|J|^2 \le \frac{C}{\delta^2}\bar{a}(|J|)\sigma(J).$$

To prove Corollary (2.2.37) stated in the introduction we will use the following version of Theorem (2.2.14).

Theorem (2.2.36)[92]: Let I be an inner function satisfying

$$\frac{(1-|z|^2)|I'(z)|}{1-|I(z)|^2} \le a(1-|z|),$$

for all $z \in \mathbb{D}$, where a is an increasing function on $(0,\pi)$, with $a(0^+) = 0$, such that $\tilde{a} \le Ca$, where $\tilde{a}$ is defined in Theorem (2.2.33). Let $h \in L^1(\mathbb{T})\dagger$ be a non-negative function, measurable with respect to the σalgebra $\mathcal{A}(I)$. Then

$$\left| \int_J h\,|d\xi| - \int_{J'} h\,|d\xi| \right| \le Ca(\pi|J|) \int_{J'} h\,|d\xi|,$$

for any arc J of the unit circle.

Proof. Take $g \in L^1(\mathbb{T})$ such that $h = g \circ I$ and consider

$$G(z) = \int_{\mathbb{T}} \frac{\xi + z}{\xi + z} g(\xi)|d\xi| \; for \; z \in \mathbb{D}.$$

Observe that

$$Re\, G(I)(z) = \int_{\mathbb{T}} P(z, \xi) h(\xi)|d\xi|.$$

Since $(1 - |z|^2 \, |G'(z) \le 2\, Re\, G(z)$, for all $z \in \mathbb{D}$, one deduces that

$$\frac{(1 - |z|^2)|(G\,{}^\circ I)'(z)|}{Re\, G(I)(z))} \le 2a(1 - |z|),$$

for all $z \in \mathbb{D}$. Now, one can apply Theorem (2.2.31).

Corollary (2.2.37)[92]: Let α be a positive increasing function on (0.1] with $\alpha(0^-) = 0$. assume that $\frac{\alpha(t)}{t^{1-\varepsilon}}$ is decreasing for some $\varepsilon > 0$. Then, the following assertions are equivalent:

a) there exist a measurable set $E \subset \mathbb{T}$, with $0 < |E| < 1$, such that the measure $\chi_E|d\xi|$ is α symmetric , that is

$$\big||E \cap J| - |E \cap J'|\big| \le \alpha(|J|)|E \cap J|$$

For any arc $J \subset \mathbb{T}$:

b) there exist a measurable set $E \subset \mathbb{T}$, with $0 < |E| < 1$, such that the measure $\chi_E|d\xi|$ is α- zygmund , that is

$$\big||E \cap J| - |E \cap J'|\big| \le \alpha(|J|)|J|$$

For any arc $J \subset \mathbb{T}$:.

c) $\int_0 \frac{\alpha^2(t)}{t} dt = \infty.$

Proof. Assume (b) holds. Consider the function

$$H(z) = \int_E \frac{\xi + z}{\xi + z} |d\xi| \; for \; z \in \mathbb{D}$$

Then $(1 - |z|)|H'(z) \le Ca(1 - |z|)$ for all $z \in \mathbb{D}$ and hence

$$(1 - |z||H'(z)|^2 \le C \frac{a^2 \, (1 - |z|)}{1 - |z|} \; for \; z \in \mathbb{D}.$$

Now, if (c) does not hold, one would deduce that H has vanishing mean oscillation, which is a contradiction.

Assume (c) holds. Apply Theorem (2.2.32) to get an inner function I such that

$$\frac{(1 - |z|^2)I'(z)|}{1 - |I(z)|^2} \le a(1 - |z|) \; for \; z \in \mathbb{D}.$$

Then, for any measurable set J of the unit circle, with $0 < |J| < 1$, let $E = I^{-1}(J)$ be its preimage. Now (a) follows from Theorem (2.2.36).

Given $f \in H^\infty$, with $|| f ||_\infty \le 1$, consider the family of positive measures $\{\sigma_a : a \in \mathbb{T}\}$ given by

$$Re\left(\frac{a+f(z)}{a-f(z)}\right)=\int_{\mathbb{T}} P(z,\xi)d\sigma_a(\xi).$$

Let w be an increasing function on $(0,1]$, with $w(0^+)=0$. Assume that for some $a_0\in\mathbb{T}$, the measure σ_{a_0} satisfies

$$\left|\sigma_{a_0}(J)-\sigma_{a_0}(J')\right|\leq w(|J|)\sigma_{a_0}(J)$$

for any arc J. Then, there exists a constant C such that

$$|\sigma_a(J)-\sigma_a(J')|\leq C\widetilde{w}(|J|)\sigma_a(J),$$

for any arc J and for any $a\in\mathbb{T}$. In particular, if $\widetilde{w}\leq Cw$, the above condition does not depend on $a\in\mathbb{T}$.

Another way of constructing a singular symmetric measure is by means of Riesz products. These are defined on $\mathbb{T}$ as the w $-$limit of the measures

$$\prod_{j=1}^{N}\left(1+Re\left(a_j\,\xi^{n_j}\right)\right)\frac{|d\xi|}{2\pi}$$

as $N\to\infty$. Here a_j are complex numbers, $\left|a_j\right|\leq 1$ for $j=1;\,2;\,\ldots;$ and the integers n_j satisfy $n_{j+1}/n_j\geq 3$. It is well known that the corresponding measure is singular if $\sum_{j=1}^{\infty}\left|a_j\right|^2=\infty$. See [102] for information on Riesz products.

Theorem (2.2.38)[92]: With the above notation assume $\left|a_j\right|<1$ for all j and $\lim\limits_{j\to\infty}a_j=0$. Then the measure

$$\sigma=\lim_{N\to\infty}\sum_{j=1}^{N}\left(1+Re\left(a_j\,\xi^{n_j}\right)\right)\frac{|d\xi|}{2\pi}$$

is symmetric.

Proof. Set

$$F_k(\xi)=\prod_{j=1}^{k}\left(1+\,Re\left(a_j\,\xi^{n_j}\right)\right),F_1\equiv 1$$

and

$$f_k(\xi)=\frac{1}{2}\,a_k\,\xi^{n_k}F_{k-1}(\xi).$$

It is clear that f_k is an analytic polynomial whose non-vanishing Fourier coefficients lie in the interval $[2^{-1}\,n_k,2^{-1}\,3n_k]$. Also $F_k-F_{k-1}=f_k+\bar{f}_k$.

If f is a continuous function in the unit circle, set

$$||\,f||_{l^1}=\sum_{n\in\mathbb{Z}}\left|\hat{f}(n)\right|$$

where

$$\hat{f}(n)=\int_{\mathbb{T}} f(\xi)\bar{\xi}^n\,\frac{|d\xi|}{2\pi}$$

are the Fourier coeffcients.

We have

$$||f_k||_{l^1}\leq\frac{1}{2}\,|a_k|\prod_{j=1}^{k-1}(1+\left|a_j\right|\leq 2^{k-2}\,|a_k|\,. \tag{48}$$

Lemma (2.2.39)[92]: Let J be a closed arc of the unit circle and $k \in \mathbb{N}$. Then the following estimates hold:

$$\frac{\max_J |F_k|}{\min_J |F_k|} \leq \exp\left(2\pi|J| \sum_{j=1}^{k} \frac{|a_j|n_j}{1-|a_j|}\right)$$

$$\left|\int_J F_k^{-1}\, d\sigma - |J|\right| \leq \frac{6}{\pi n_{k+1}} \sup_{j \geq k+1} |a_j|.$$

Proof. Considering logarithmic derivatives one gets

$$\left|\frac{d}{dt} \log F_k(e^{it})\right| \leq \sum_{j=1}^{k} \frac{|a_j|n_j}{1-|a_j|}.$$

Now, an integration proves the first estimate.

Replacing σ by the Riesz product $F_k^{-1}\sigma$, one shows that it is sufficient to prove the second inequality when $k = 0$. Let x_J be the characteristic function of J. Applying the inequality

$$|\hat{x}_J(k)| \leq \frac{1}{\pi|k|} \leq \ with\ k \neq 0,$$

and (48), one deduces that

$$|\sigma(J) - |J\,|| \leq \sum_{k\neq 0} |\hat{\sigma}(k)|\, |\hat{x}_J(k) \leq| \frac{4}{\pi}\sum_{j=1}^{\infty} \frac{\|f_j\|_{l^1}}{n_j} \leq \frac{1}{\pi}\sum_{j=1}^{\infty} \frac{2^j|a_j|}{n_j} \leq \frac{6}{\pi n_1} \sup_{j\geq 1} |a_j|.$$

A similar argument can be found in [106].

Now, let J be an arc of the unit circle and let ξ be the common end of J and $'$. Take k such that $n_{k+1}^{-1} \leq |J\ | < n_k^{-1}$. Applying Lemma (2.2.39), one has

$$\frac{\sigma(J)}{|J|} = \frac{1}{|J\,|} \int_J F_k F_k^{-1}\, d\sigma \cong F_k(\xi).$$

Here $A_k \cong B_k$ means that $A_k / B_k \to 1$ as $k \to \infty$. Similarly,

$$\sigma(J')/|J'\,| \cong F_k(\xi).$$

Hence σ is symmetric.

Assume that (a_j) satisfy the hypothesis of Theorem (2.2.38) and $\sum |a_j|^2 = \infty$. Let σ be the corresponding singular symmetric measure. Observe that the measures

$$\sigma_t = \prod_{j=1}^{\infty} [1 + Re(e^{it} a_j\, \xi^{n_j}) \frac{|d\xi|}{2\pi}\ \text{,where}\ t \in [0, 2\pi),$$

are also singular and symmetric. Actually the proof of Theorem (2.2.38) shows that

$$\lim_{|J|\to 0} \frac{\sigma_t(J\,)}{\sigma_t(J')} = 1,$$

uniformly in $t \in [0, 2\pi)$. Moreover, if $t \neq s$, the measures jt and js are mutually singular. Given a singular symmetric measure σ, we can use our composition process to obtain families of Kahane symmetric measures. If, on the other hand, one attempts to construct a Kahane measure by means of a Riesz product with $n_{j+1}/n_j \geq 3$ for all j, then P. Duren showed that $\sum |a_j|^2 < \infty$ so the measure is absolutely continuous [98].

Minor modifications of the proof of Theorem (2.2.38), show that, essentially, the measures constructed by L. Carleson can also be obtained as Riesz products.

Theorem (2.2.40)[92]: Let w be a positive increasing function on $[0,1]$ such that $w(t)/t$ is decreasing and

$$\int_0 \frac{w^2(t)}{t}\, dt = \infty.$$

Then there exists a sequence of non-negative numbers $\{r_k\}$, with $\sum_{k=0}^{\infty} r_k^2 = \infty$, such that for any sequence a_k of complex numbers, $|a_k| \leq r_k$ where $k = 0, 1;\ 2;\ \ldots;$ the measure j associated with the Riesz product

$$\prod_{(j=1)}^{\infty} \left(1 + Re\left(a_j \xi^{3^j}\right)\right) \frac{|d\xi|}{2\pi}$$

satisfies

$$\left|\frac{\sigma(J')}{\sigma(J)} - 1\right| \leq w(|J|),$$

for any $arc\ J$ of the unit circle. Moreover if $|a_k| = r_k$ for $k = 0;\ 1;\ 2;\ \ldots;$ the measure j is singular.

Proof. We may assume $\lim_{t\to 0} w(t) = 0$. Consider $\varepsilon_k = 20^{-1} w(3^{-k-1})$ with $k > 0$. The integral condition on w gives

$$\sum_{k=0}^{\infty} \varepsilon_k^2 = \infty.$$

Choose $r_k = \varepsilon_k - 3^{-1} \varepsilon_{k-1}$ with $k \geq 1$. Observe that $r_k \geq 0$ because $w(t)/t$ decreases. Also, $\sum_{k=1}^{\infty} r_k^2 = \infty$. Let J be an arc of the unit circle, $3^{-k-1} \leq |J| < 3^{-k}$. We now use the notation of the proof of Theorem (2.2.38). There exists a point $\xi_k \in J$ such that

$$\frac{\sigma(J)}{|J|} = \frac{1}{|J|} \int_J F_k F_k^{-1}\, d\sigma = F_k(\xi_k) \frac{1}{|J|} \int_J F_k^{-1}\, d\sigma.$$

Now, Lemma (2.2.39) gives

$$\left|\frac{\sigma(J)}{|J|} - F_k \xi_k\right| \leq F_k(\xi_j) \frac{6}{\pi} \sup_{j \geq k+1} |a_j| \leq 2\varepsilon_{k+1} F_k(\xi_k).$$

Similarly, there exists $\xi'_k \in J'$ such that

$$\left|\frac{\sigma(J')}{|J|} - F_k(\xi'_k)\right| \leq 2\varepsilon_{k+1} F_k(\xi'_k)$$

Writing $t = 4\pi|J| \sum_{j=1}^{k} |a_j| 3^j \left(1 - |a_j|\right)^{-1}$, we find that the first estimate of Lemma (2.2.39) gives

$$\left|F_k(\xi_k) - F_k\left(\xi'_k\right)\right| \leq F_k(\xi_k)(e^t - 1) \leq 15 F_k(\xi_k) \sum_{j=1}^{k} r_j 3^{j-k} \leq 15\varepsilon_k F_k(\xi_k).$$

Thus, if k is sufficiently large, one gets

$$|\sigma J - \sigma(J')| \leq 19\varepsilon_k F_k(\xi_k) |J| \leq 20\varepsilon_k \sigma(J) \leq w(|J|)\sigma(J).$$

Replacing r_k by $r'_k = r_{k-N}$, for $k > N$, where N is sufficiently large, and $r'_k = 0$ if $k < N$, we see that the last inequality holds for any arc J of the unit circle.

Section (2.3): Composition Operators from Bloch Type Spaces to BMOA

The existence of critical Bi Bloch mappings and its applications to Bloch-BMO pullback problems.

A real function h is said to be almost increasing (resp. almost decreasing) if there is a constant $C > 0$ such that $y > x$ implies $h(x) \leq Ch(y)$ (resp. $h(y) \leq Ch(x)$). A positive almost increasing function $\psi : [0, \infty) \mapsto (0, \infty)$ will be called *almost subnormal* if there is $\beta > 0$ such that $\psi(x)/x^{\beta}$, $x \geq 1$, is almost decreasing, and

$$\lim_{x\to\infty}\psi(x) = \infty. \tag{49}$$

If ψ is (strictly) increasing and (49) holds with "almost decreasing" replaced by "non-increasing", then ψ is called subnormal. If in addition there is $\alpha > 0$ such that

$$\frac{\psi(x)}{x^{\alpha}}, \quad x \geq 1, \text{ is almost increasing}, \tag{50}$$

then ψ is called *almost normal*. If ψ is subnormal and (50) holds (for some $\alpha > 0$) with "almost increasing" replaced by"non-decreasing", then ψ is called *normal*. The notion of a normal function was introduced by Shields and Williams [11].

Theorem (2.3.1)[110]: If ψ is an almost subnormal function, then there exist functions f_1, f_2 analytic in the unit disk D of the complex plane
such that

$$\|f_1'(z)\| + |f_2'(z)| \asymp \psi\left(\frac{1}{1 - |z|}\right), \quad z \in D. \tag{51}$$

As usual, the notation $A \asymp B$ means $B/C \leq A \leq CB$ for some constant C. The first result of this kind was proved by Ramey and Ullrich in [121], where the case $\psi(x) = x$ was considered. The term *BiBloch* comes from the fact that the Ramey–Ullrich theorem can be reformulated in the following way: There is a mapping $F : \mathbb{D} \to \mathbb{C}^2$ such that

$$\left|\left(F'(z)\right)\right| \asymp (1 - |z|)^{-1}, \quad z \in \mathbb{D}.$$

The existence of such a critical Bi Bloch mappings, as was shown in [121], plays an important role in characterizing composition operators from the Bloch space to, e.g., BMOA. An extension of the Ramey–Ullrich theorem to the case where $(x) = x^{\beta}$, $\beta > 0$, was proved by Gauthier and Xiao [113] (see also Xiao [124]). In [112], in connection with a problem on composition operators, Galanopulos considered the case where $\psi(x) = x(1 + \log x)$, which was extended to $\psi(x) = x^{\gamma}(1 + \log x), \gamma > 0$, by Liu and Li [114]. However in all these cases the function ψ is normal. Theorem (2.3.1) covers the case of normal functions, for example

$$\psi(x) = x^{\alpha}(1 + \log x)^{\gamma}, \ \alpha > 0, \gamma \in \mathbb{R},$$

as well as the case of non-normal functions such as

$$\psi(x) = (1 + \log x)^{\gamma}, \quad \gamma > 0.$$

The hypothesis that ψ is (almost) normal makes the proof (of (51)) almost identical to the proof of Ramey–Ullrich Theorem. In the general case, we also start with the idea to represent f as a sum of two series with Hadamard gaps but it seems that these series must heavily depend on ψ.

ψ is assumed to be almost subnormal, that is, it satisfies (49). C or c in the inequalities denotes a positive constant which is independent of the variables under consideration.

Let $H(\mathbb{D})$ denote the space of all functions analytic in the unit disc $\mathbb{D}$ and let $H(\mathbb{D}, \mathbb{D}) := \{\varphi \in H(\mathbb{D}) : \varphi(\mathbb{D}) \subset \mathbb{D}\}$. For a function $\varphi \in H(\mathbb{D}, \mathbb{D})$, the composition operator C_{φ} is defined on $H(\mathbb{D})$ by $C_{\varphi}(f)(z) = f(\varphi(z))$. For two function spaces X and Y , let us

$denote\ C(X, Y) = \{\varphi: C_\varphi(X) \subset Y\}$. Let $\mathfrak{B}(\psi)$ denote the space of those $f \in H(\mathbb{D})$ endowed with the norm

$$\|f\|_{\mathfrak{B}(\psi)} := |f(0)| + sup_{z\in\mathbb{D}} \frac{|f'(z)|}{\psi(1/(1 - |z|^2))} < \infty,^3$$

which reduces to the Bloch space $\mathfrak{B}$ if $\psi(x) = x$. BMOA is the space of $f \in H(\mathbb{D})$ endowed with the Garsia norm

$$\|f\|_*^2 := |f(0)|^2 + sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 \log \frac{1}{|\sigma_a(z)|} \ dA(z),$$

where dA denotes the normalized Lebesgue measure on $\mathbb{D}$ and

$$\sigma_a(z) = \frac{a - z}{1 - \bar{a}z}.$$

The space BMOA can be equivalently defined by the requirement $f \in H^2$ satisfying

$$sup_{a\in\mathbb{D}}(P[|f_*|^2])(0) - |f(a)|^2) < \infty,$$

where

$$P[u](a) = \frac{1}{2\pi} \int_{\partial\mathbb{D}} u(\zeta) \frac{1 - |a|^2}{|a - \zeta|^2} \ |d\zeta|, \ a \in \mathbb{D}, u \in L^1(\partial\mathbb{D}). \qquad (52)$$

The problem of characterizing those $\varphi\ in\ C(\mathfrak{B}, BMOA)$, so-called Bloch-BMO pullback problem, has been considered extensively. See [116]. We consider and resolve an extended version of the problem on the settings of $\mathfrak{B}\ (\psi)$.

If ψ is continuous on $[1, \infty)$, we define the function $\tilde{\ }\psi$ by

$$\bar{\psi}(r) = 1 + \int_0^r \psi^2\left(\frac{1}{1 - x}\right) \log \frac{r}{x} dx.$$

$$= 1 + \int_0^r \frac{1}{\rho} d\rho \int_0^\rho \psi^2\left(\frac{1}{1 - x}\right) dx, \quad 0 \le r \le 1.$$

Note that if $\psi(x) = x$, then

$$\bar{\psi}(x) = \log \frac{e}{1 - r}$$

Recall that the function $P[u]$ in (52) is harmonic in $\mathbb{D}$ and $\lim_{r\to 1-} P[u](r\zeta) = u(\zeta)\ for\ almost\ all\ \zeta \in \partial D$. We use (52) to define the Poisson integral of a measurable function with values in [0, ∞]. It can be easily checked for such a function u that $u \notin L^1(\partial D)$ if and only if $P[u](a) = \infty$ for all (for some) $a \in \mathbb{D}$. Theorem (2.3.8) directly gives

Corollary (2.3.2)[110]: If

$$\int_{\partial\mathbb{D}} \tilde{\psi}(|\varphi * (\zeta)|^2)|d\zeta| = \infty,$$

then φ does not map $\mathfrak{B}$(ψ) into BMOA. In particular, if

$$\int_0^1 \psi^2\left(\frac{1}{1 - x}\right)(1 - x) dx = \infty \qquad (53)$$

and φ maps $\mathfrak{B}(\psi)$ into BMOA, then $|\varphi_*| < 1$ a.e. on $\partial\mathbb{D}$. Also, if (53) holds and φ is an inner function, then φ does not map $\mathfrak{B}(\psi)$ into BMOA.

Recall that $\varphi \in H(\mathbb{D},\mathbb{D})$ is called inner if $|\varphi_*| = 1$ $a.e. on$ $\partial\mathbb{D}$. Noting the inequality

$$P\left[\tilde{\psi}(|\varphi_* I_*|^2)\right](a) - \tilde{\psi}(|\varphi(a)|^2|I(a)|^2|) = P\left[\tilde{\psi}|\varphi_*|^2\right](a) - \tilde{\psi}$$
$$(|\varphi(a)|^2|I(a)|^2) \geq P\left[\tilde{\psi}|\varphi_*|^2\right](a) - \tilde{\psi}(|\varphi(a)|^2),$$

the following also is an immediate consequence of Theorem (2.3.8).

Corollary (2.3.3)[110]: If $\varphi \in H(\mathbb{D},\mathbb{D})$ and $\varphi I \in C(\mathfrak{B}(\psi), \text{BMOA})$, where I is an inner function, then $\varphi \in C(\mathfrak{B}(\psi), \text{BMOA})$.

In Havin [115], Corollary (2.3.3) says that the class $C(\mathfrak{B}(\psi), BMOA)$ has the f-property. We can omit the hypothesis that ψ is continuous because there is an equivalent, continuous function ϕ (see Lemma (2.3.6)).

As a further application of Theorem (2.3.8), we have

Corollary (2.3.4)[110]: The inclusion $\mathfrak{B}(\psi) \subset$ BMOA is necessarily strict.

Let us denote, for $\gamma \in \mathbb{R}$,

$$\psi_\gamma(x) = \frac{x}{(1+log\, x)^\gamma}, 1 \leq x < \infty.$$

Then as a special case of Theorem (2.3.10) we have

Corollary (2.3.5)[110]: The following conditions are equivalent:

(i) $\mathfrak{B}(\psi_\gamma) \subset$ BMOA.

(ii) $\gamma > 1/2$.

(iii) $H(\mathbb{D},\mathbb{D}) \subset C(\mathfrak{B}(\psi_\gamma)$,BMOA).

The following theorem explains what is happened in the case $\gamma \leq 1/2$.

The special case $\gamma = 0$ of this theorem was proved by the first author [116].

After assigning to the proofs of Theorems (2.3.1)–(2.3.11), we consider the boundedness of the composition operators between Lipschitz spaces having general weights. Little “oh” version of the pullback problem is considered.

It is enough to $find$ $g_1, g_2 \in H(\mathbb{D})$ such that

$$g_1(z) + g_2(z) \asymp \psi\frac{1}{1-|z|}, z \in \mathbb{D}. \tag{54}$$

Also, we may assume $\beta = 1$ in (49): choose an integer $M > 0$ such that the function $\psi(x)^{1/M}/x$ is almost decreasing and put $\varphi = \psi^{1/M}$. We will find $h_1, h_2 \in H(\mathbb{D})$ such that

$$h_1(z) + h_2(z) \asymp \varphi\frac{1}{1-|z|}, z \in \mathbb{D}. \tag{55}$$

Then the functions $g_1 = h_1^M$ and $g_2 = h_2^M$ satisfy (54).

In order to prove (55) we want to replace φ by a function that behaves more regularly.

Lemma (2.3.6)[110]: Let ψ satisfy (49) with β = 1, and let

$$\varphi_1(x) = inf_{t\geq x}\psi(t),$$
$$\varphi_2(x) = x\, sup_{t\geq x}\frac{\varphi_1(x)}{t} = sup_{t\geq x}\frac{\varphi_1(tx)}{t}, \qquad x \geq 1.$$

Then φ_2 satisfies:

(i) $\varphi_2(x) \asymp \psi(x)$ and hence $lim_{x\to\infty}\varphi_2(x) = \infty$.

(ii) φ_2 is increasing.

(iii) $\varphi_2(x)/x$ is decreasing.

(iv) φ_2 is absolutely continuous.

Proof. The proof is straightforward.

It follows that in proving (55) we can assume that φ satisfies the above four properties. Also we can assume that $\varphi(1) = 1$ and that φ is strictly increasing since otherwise we can replace $\varphi(x)\ by\ \varphi(x) + x/(x+1)$.

Let $q > 4$ be a sufficiently large number, which will be chosen later on. Choose numbers $\lambda'_j, j \geq 1, so\ that$

$$\phi(\lambda'_j) = q^j \ , j \geq 1,$$

and then define the sequences $\{\lambda_j\}\ and\ \{\mu_j\}$:

$$\lambda_j = [\lambda'_j] + 1,$$

$$\mu_j = \left[\sqrt{\lambda_j \lambda_j + 1}\right] + 1,$$

where $[x], x \in \mathbb{R}$, denotes the unique integer such that $x - 1 < [x] \leq x$. These sequences have the properties:

$$\frac{\lambda_{j+1}}{\lambda_j} \geq \frac{q}{2}, \qquad \frac{\mu_{j+1}}{\mu_j} \geq \frac{q}{4} \quad , for\ j \geq 1. \tag{56}$$

To verify this, observe that $\lambda'_j \leq \lambda_j = [\lambda'_j] + 1 \leq 2\lambda'_j$ so that

$$q = \frac{\varphi(\lambda'_{j+1})}{\varphi(\lambda'_j)} \leq \frac{\lambda_j \varphi\,(\lambda_{j+1})}{\lambda'_j \varphi(\lambda_j)} \leq 2\frac{\varphi\,(\lambda_{j+1})}{\varphi(\lambda_j)} \leq 2\frac{(\lambda_{j+1})}{(\lambda_j)}$$

where we have used (ii) and (iii) of Lemma (2.3.6). In the case of μ_j we have

$$\frac{\mu_{j+1}}{\mu_j} \geq \frac{\sqrt{\lambda_{j+1}\lambda_{j+2}}}{2\sqrt{\lambda_{j+1}\lambda_j}} = \frac{1}{2}\sqrt{\frac{\lambda_{j+2}}{\lambda_j}} \geq \frac{q}{4}.$$

From (56) we get

$$\frac{\lambda_{j+n}}{\mu_n} \geq \left(\frac{q}{2}\right)^{j-1}\sqrt{\frac{q}{8}}, \qquad \frac{\mu_{j+n}}{\lambda_{n+1}} \geq \left(\frac{q}{4}\right)^{j-1}\sqrt{\frac{q}{2}}, \qquad for\ j, n \geq 1, \tag{57}$$

and

$$\lambda_n < \mu_n < \lambda_{n+1}, \qquad for\ n \geq 1. \tag{58}$$

Indeed,

$$\frac{\lambda_{j+n}}{\mu_n} = \frac{\lambda_{j+n}}{\lambda_{n+1}}\frac{\lambda_{j+n}}{\mu_n}$$

$$\left(\frac{q}{2}\right)^{j-1}\frac{\lambda_{n+1}}{\mu_{n+1}} = \left(\frac{q}{2}\right)^{j-1}\frac{\lambda_{j+n}}{1+\sqrt{\lambda_n\lambda_{n+1}}} \geq \left(\frac{q}{2}\right)^{j-1}\frac{\lambda_{n+1}}{2\sqrt{\lambda_n\lambda_{n+1}}}$$

$$= \left(\frac{q}{2}\right)^{j-1}\frac{1}{2}\sqrt{\frac{\lambda_{n+1}}{\lambda_n}} \geq \left(\frac{q}{4}\right)^{j-1}\sqrt{\frac{q}{8}},$$

And

$$\frac{\mu_{n+j}}{\lambda_{n+1}} = \frac{\mu_{n+j}}{\mu_{n+1}}\frac{\mu_{n+1}}{\lambda_{n+1}} \geq \left(\frac{q}{4}\right)^{j-1}\frac{\mu_{n+j}}{\lambda_{n+1}} \geq \left(\frac{q}{2}\right)^{j-1}\frac{\sqrt{\lambda_{n+1}\lambda_{n+2}}}{\lambda_{n+1}}$$

$$\geq \left(\frac{q}{4}\right)^{j-1}\sqrt{\frac{\lambda_{n+2}}{\lambda_{n+1}}} \geq \left(\frac{q}{4}\right)^{j-1}\sqrt{\frac{q}{2}},$$

This proves (57). The first inequality of (58) follows simply from

$$\lambda_n < \sqrt{\lambda_n\lambda_{n+1}} < \left[\sqrt{\lambda_n\lambda_{n+1}}\right] + 1 = \mu_n$$

while the second inequality of (58) follows from the first of (57) by taking $q > 8$. Now we $define\ h_1 and\ h_2$ by.

$$h_1(z) = 1 + \sum_{j=1}^{\infty} q^j z^{\lambda_j} \ and\ h_2(z) = 1 + \sum_{j=1}^{\infty} q^j z^{\mu_j} .$$

We shall first prove for each $n \geq 1$ that

$$|h_1(z)| \geq c\varphi\left(\frac{1}{1-|z|}\right), \ for\ 1 - \frac{1}{\lambda_n} \leq |z| \leq 1 - \frac{1}{\mu_n}, \tag{59}$$

And

$$|h_2(z)| \geq c\varphi\left(\frac{1}{1-|z|}\right), \ for\ 1 - \frac{1}{\mu_n} \leq |z| \leq 1 - \frac{1}{\lambda_{n+1}}, \tag{60}$$

which implies one directional validity of (55) in the annulus $1 - 1/\lambda_1 \leq |z| \leq 1$. Since the functions $h_1\ and\ h_2$ have finitely many zeroes in the disk $|z| < 1 - 1/\lambda_1$ with $h_1(0) = 0$, $h_2(0) = 0$, we can choose $\theta \in$ R so that the functions $z \mapsto h_1(e^{i\theta} z)$ and h_2 have no common zeroes in this disk. Thus the desired functions will be $h_1(e^{i\theta} z)$ and h_2.
Since $\varphi(\lambda_n) \asymp q^n$, we see that (59) and (60) are equivalent to

$$|h_1(z)| \geq cq^n, \qquad \text{for } 1 - \frac{1}{\lambda_n} \leq |z| \leq 1 - \frac{1}{\mu_n}, \tag{61}$$

$$|h_2(z)| \geq cq^n, \qquad \text{for } 1 - \frac{1}{\mu_n} \leq |z| \leq 1 - \frac{1}{\lambda_n + 1,} \tag{62}$$

Respectively.

We have, for $1 - 1/\lambda_n \leq |z| \leq 1 - 1/\mu_n$,

$$|h_1(z)| \geq cq^n|z| - 1 - \sum_{j=1}^{n-1} q^j - \sum_{j=n+1}^{\infty} q^j |n|^{\lambda_j}$$

$$\geq q^n\left(1 - \frac{1}{\lambda_n}\right)^{\lambda_n} - 1 - \frac{q^n}{q-1} - \sum_{j=n+1}^{\infty} q^j \left(1 - \frac{1}{\mu_n}\right)^{\lambda_j.}$$

Since the function $x \mapsto (1 - 1/x)^x, x > 1$, is increasing and $\lambda_1 = [\lambda' 1] + 1 \geq 2$, we have $(1 - 1/\lambda_n)^{\lambda_n} \geq (1 - 1/\lambda_1)^{\lambda_1} \geq 1/4$ and

$$\left(1 - \frac{1}{\mu_n}\right)^{\lambda_j} \leq exp\left(-\frac{\lambda_j}{\mu_n}\right)$$

so that

$$|h_1(z)| \geq q^n\left(\frac{1}{4} - \frac{1}{q^n} - \frac{1}{q-1}\right) - \sum_{j=n+1}^{\infty} q^j\, exp\left(-\frac{\lambda_j}{\mu_n}\right)$$

Using (57), we bound the last sum as follows

$$\sum_{j=n+1}^{\infty} q^j\, exp\left(-\frac{\lambda_j}{\mu_n}\right) = q^n \sum_{j=1}^{\infty} q^j exp\left(-\frac{\lambda_{j+n}}{\mu_n}\right) \leq q^n \sum_{j=1}^{\infty} q^j\, exp\left\{-\left(\frac{q}{2}\right)^{j-1}\sqrt{\frac{9}{8}}\right\}.$$

Since

$$\lim_{q\to\infty} \sum_{j=1}^{\infty} q^j\, exp\left\{-\left(\frac{q}{2}\right)^{j-1}\sqrt{\frac{9}{8}}\right\} = 0.$$

we can choose $q > 101$ so that

$$\sum_{j=1}^{\infty} q^j \, exp\left\{-\left(\frac{q}{2}\right)^{j-1}\sqrt{\frac{9}{8}}\right\} < \frac{1}{100}.$$

Since $1/q^n < 1/100$, we have

$$\frac{1}{4} - \frac{1}{q^n} - \frac{1}{q-1} > \frac{1}{4} - \frac{1}{100} - \frac{1}{100}.$$

Combining all these estimates we get (61).

In the case of (62) we have, for $1 - \frac{1}{\mu_n} \le |z| \le 1 - \frac{1}{\lambda_{n+1}}$,

$$|h_2(z)| \ge q^n |z|^{\mu_n} - \sum_{j=1}^{n=1} q^j - \sum_{j=n+1}^{\infty} q^j \, |z|^{\mu_j}$$

$$\ge q^n \left(1 - \frac{1}{\mu_n}\right)^{\mu_n} - \frac{q^n}{q-1} - \sum_{j=n+1}^{\infty} q^j \left(1 - \frac{1}{\lambda_{n+1}}\right)^{\mu_j}$$

$$\ge q^n \left(\frac{1}{4} - \frac{1}{q-1}\right) - \sum_{j=n+1}^{\infty} q^j \, exp\left(-\frac{\mu_j}{\lambda_{n+1}}\right)$$

$$= \; q^n \left(\frac{1}{4} - \frac{1}{q-1}\right) - q^n \sum_{j=1}^{\infty} q^j \, exp\left(-\frac{\mu_{j+n}}{\lambda_{n+1}}\right)$$

$$\ge q^n \left(\frac{1}{4} - \frac{1}{q-1}\right) - q^n \sum_{j=1}^{\infty} q^j \, exp\left\{-\left(\frac{q}{4}\right)^{j-1}\sqrt{\frac{q}{2}}\right\}$$

Now the proof of (62) can be completed as in the case of (61).

Finally, we have to prove that

$$|h(z)| \le C\varphi\left(\frac{1}{1-|z|}\right) \text{ for } |z| < 1, \qquad \text{for } \; |z| \; < \; 1,$$

where $h = h_1$ or h_2. This can be done in a similar way as in the case of (59) and (60). In fact, the proof is simpler in that it is valid for all $q > 1$; namely, we have:

Theorem (2.3.7)[110]: If $q > 1$, then there is an increasing sequence $\{\lambda_j\}$ of positive integers such that $\psi(\lambda_j) \asymp q^j$, and, moreover, the function

$$f_\psi(z) = \infty_j = 1 q^j \, z^{\lambda_j}$$

satisfies

$$|f\psi(z)| \le f_\psi(|z|) \; \le C\psi\left(\frac{1}{1 \; - \; |z|}\right), |z| < 1.$$

We also have

$$\sum_{j=1}^{\infty} \; q^j \, r^{\lambda_j} \ge \; c\psi \left(\frac{1}{1 \; - \; r}\right) \; , 1/2 \; < \; r \; < \; 1.$$

Consequently, if

$$g\psi, a(z) = \sum_{j=1}^{\infty} \; \frac{q^j}{\lambda_j \; + \; 1} \; (\bar{a}z \,)^{\lambda_j+1}, |z| \; < \; 1, |a| \; \le 1,$$

then $g_{\psi,a}$ belongs to $\mathcal{B}(\psi)$ and $\sup\limits_{|a|\le 1} \, g_{\psi,a} \, \mathcal{B}(\psi) \; < \; \infty.$

Proof. See [120], where an integrated version is proved as well. For the case of a normal ψ, see [117].

Theorem (2.3.8)[110]: Let $\varphi \in H(\mathbb{D},\mathbb{D})$ and let ψ be continuous. Then $\varphi \in C(\mathfrak{B}(\psi), \text{BMOA})$ if and only if

$$sup_{a\in D}\{P[\tilde{\psi} \circ |\varphi_*|^2](a) - \tilde{\psi}(|\varphi(a)|^2)\} < \infty, \quad (63)$$

Where

$$\varphi * (\zeta) = lim_{r\to 1-} \varphi(r\zeta), \ |\zeta| = 1.$$

Proof. Let $\emptyset \in H(\mathbb{D},\mathbb{D})$. A standard application of Theorem (2.3.1) (see [116], [121], [124]) shows that $\emptyset \in C(\mathcal{B}(\psi), \text{BMOA})$ if and only if

$$Q(\emptyset) := \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \emptyset'(z)^2 \psi^2 \left(\frac{1}{1-|\emptyset(z)|^2}\right) log \frac{1}{|\sigma_a(z)|}\, d A(z) < \infty,$$

as well as that

$||C_\emptyset|| \asymp \sqrt{Q(\emptyset)}$, if $Q(\emptyset), (0) = 0$.

Therefore Theorem (2.3.8) is a consequence of the following assertion.

Proposition (2.3.9)[110]: If $Q(\emptyset), \in H(\mathbb{D},\mathbb{D})$ and ψ is continuous, then

$$2\int_{\mathbb{D}} |\emptyset'(z)|^2 \psi^2 \left(\frac{1}{1-|\emptyset(z)|^2}\right) log \frac{1}{|\sigma_a(z)|}\, d A(z)$$

$$= P\left[\tilde{\psi} \circ \left|\emptyset *\right|^2\right](a) - \tilde{\psi}(|\emptyset(a)|^2), a \in \mathbb{D}. \quad (64)$$

Proof. Our proof is based on the Green theorem which, in its simplest form, says that if $u \in C^2(\mathbb{D})$, then

$$\frac{1}{2}\int_{|z|<\varepsilon} \Delta u(z) \log \frac{\varepsilon}{|z|}\, d A(z) = \frac{1}{2\pi}\int_{\partial\mathbb{D}} u(\varepsilon\zeta)|d\zeta| - u(0), 0 < \varepsilon < 1 \quad (65)$$

Also, it is not difficult to check that the function $t \mapsto \tilde{\psi}(t^2), -1 < t < 1$, is C^2, and that

$$\Delta(\tilde{\psi} \circ |\emptyset|^2 = 4\psi^2 (\frac{1}{1-|\emptyset|^2}) |\emptyset'|^2 . \quad (66)$$

Let $\psi_n(x) = min\{\psi(x), n\}$. Note that (66) holds with ψ replaced by ψ_n. Thus, by (65) and (66),

$$2\int_{|z|<\varepsilon} |\emptyset'(z)|^2 \psi^2 \left(\frac{1}{1-|\emptyset(z)|^2}\right) log \frac{1}{|\sigma_a(z)|}$$

$$= \frac{1}{2\pi}\int_{\partial\mathbb{D}} \tilde{\psi}_n(|\emptyset(\varepsilon\xi)|^2)|d\zeta| - \tilde{\psi}_n(|\emptyset(0)|^2). \quad (67)$$

Now fix n and let $\varepsilon \to 1^-$. We apply the monotone convergence theorem on the left-hand side and the dominated convergence theorem on the right (the function $\tilde{\psi}_n$ is bounded) to conclude that (67) holds for $\varepsilon = 1$ and all n. Now let $n \to \infty$ and apply the monotone convergence theorem on both sides of (67) (ε = 1) (which is possible because ψ_n and $\tilde{\psi}_n$ increase with n to ψ and $\tilde{\psi}$ respectively) to show that (64) holds for $a = 0$. To complete the proof we only have to apply this special case to the function $\emptyset \circ \sigma a$, and then use the substitutes $z \mapsto \sigma_a(z)$ and $\zeta \mapsto \sigma_a(\zeta)$.

Theorem (2.3.10)[110]: The following conditions are equivalent:

(i) $\mathfrak{B}(\psi) \subset$ BMOA.

(ii) $\int_0^1 \psi^2\left(\frac{1}{1-x}\right)(1-x)dx = \infty$.

(iii) $H(\mathbb{D},\mathbb{D}) \subset C(\mathfrak{B}\,(\psi)\mathrm{BMOA})$

A consequence of Theorem (2.3.10) says that there is no ψ such that $\mathfrak{B}\,(\psi)$ = BMOA: If $\mathfrak{B}\,(\psi)$ = BMOA, then the function $\log \frac{1}{1-z}$ belongs to $\mathfrak{B}\,(\psi)$ .which implies $\left|\frac{1}{1-z}\right| \le C\psi\left(\frac{1}{1-z}\right)$ so that

$$\int_0^1 \psi^2\left(\frac{1}{1-x}\right)(1-x)dx = \infty.$$

Proof. We may assume that ψ is continuous because $\mathcal{B}(\psi) = \mathcal{B}(\varphi)$ if $\psi \asymp \emptyset$. The validity of the implication (iii) ⇒ (i) is a consequence of the fact that the function $\emptyset(z) = z$ belongs to $H(\mathbb{D},\mathbb{D})$. That (ii) implies (iii) follows from Theorem (2.3.8) and the fact that the Poisson integral of a bounded function (in our case $\tilde{\psi}$) is bounded. Thus it remains to prove that (i) implies (ii).

By Theorem (2.3.7), there is an increasing sequence $\{\lambda_j\}$ of positive integers such that $\psi(\lambda_j) \asymp 2^j$. Let $\rho_n = 1 - 1/\lambda_{n+1}$. Then

$$I := \int_0^1 \psi^2\left(\frac{1}{1-x}\right)(1-x)dx = \left(\int_0^{\rho_2} + \sum_{n=1}^{\infty}\int_{\rho_n}^{\rho_{n+1}}\right)\psi^2\left(\frac{1}{1-x}\right)(1-x)dx$$

and we have

$$\int_{\rho_n}^{\rho_{n+1}} \psi^2\left(\frac{1}{1-x}\right)(1-x)dx\, \psi^2(\lambda_{n+2})\int_{p_n}^{1}(1-x)dx \le C2^{2n}\frac{1}{\lambda_{n+1}^2}.$$

It thus follows that

$$I \le C + C\sum_{n=1}^{\infty}\frac{2^{2n}}{\lambda_{n+1}^2}. \tag{68}$$

Now we consider the function

$$f(z) = \sum_{j=1}^{\infty}\frac{2^j}{\lambda_{j+1}+1}z^{\lambda_{j+1}+1}.$$

By Theorem (2.3.7), this function belongs to $\mathcal{B}(\psi)$ and hence, by the hypothesis (i), it belongs to BMOA. On the other hand, f is represented by a lacunary series, which implies that

$$\sum_{n=1}^{\infty}\left(\frac{2^n}{\lambda_{n+1}}\right)^2 \asymp ||f||_*^2 < \infty$$

(see [124]). This and (68) prove that (i) implies (ii) see [91].

Theorem (2.3.11)[110]: Let $\varphi \in H(\mathbb{D},\mathbb{D})$ and $\gamma \le 1/2$. Then $\varphi \in \mathfrak{B}\,(\psi_\gamma)$, BMOA) if and only *i*

$$\sup_{a\in\mathbb{D}}\left(P\left[F_\gamma|\phi_*|^2\right](a) - F_\gamma|\phi(a)|^2\right) < \infty,$$

where

$$F_\gamma(r) = \begin{cases}\left(\log\frac{e}{1-r}\right)^{1-2\gamma}, & \gamma < \frac{1}{2},\\ log\left(1+\log\frac{e}{1-r}\right), & \gamma = 1/2.\end{cases}$$

Proof. Observe that the function $\tilde{\psi}$ is the unique solution g of the Cauchy problem

$$g''(r)r + g'(r) = \psi^2\left(\frac{1}{1-r}\right), g(0) = 1, g'(0+) = \psi(1)^2\ (0 \le r < 1). \quad (69)$$

The positivity of $g''(r)r + g'(r)$ means that $g(r)$ is convex of $\log r$, $i.e.$, that the function $x \longmapsto g(e-x), x \ge 0$, is convex. Set $\alpha = 1 - 2\gamma$. If $\alpha > 0$ take

$$g(r) = \left(log\frac{e}{1-r}\right)^{\alpha}.$$

Then

$$g''(r)r + g'(r) = \alpha\frac{1}{(1-r)^2}\left(log\frac{e}{1-r}\right)^{\alpha-1} h(r),$$

where $h(r) = 1 + r + (\alpha - 1)r\left(log\frac{e}{1-r}\right)^{-1}$. Noting that $\alpha > 0$ and that $log\frac{e}{1-r} \ge 1$, we see $h(r) \asymp 1$. Thus,

$$\psi_0^2\left(\frac{1}{1-r}\right) := g''(r)r + g'(r) \asymp \frac{1}{(1-r)^2}\left(log\frac{e}{1-r}\right)^{\alpha-1}$$
$$= \psi_\gamma^2\left(\frac{1}{1-r}\right). \quad (70)$$

By the uniqueness of the solution of (69) it follows that $g = \tilde{\psi}_0$. By (70) $\psi_\gamma \asymp \psi_0$ so that $\mathcal{B}(\psi_\gamma) = \mathcal{B}(\psi_0)$. We therefore conclude from Theorem (2.3.8) that $\emptyset \in \mathcal{C}(\mathcal{B}(\psi_0)$, BMOA) if and only if

$$\sup_{a\in\mathbb{D}} \left(P[g \circ |\emptyset_*|^2 - g(|\emptyset(a)|^2)\right) < \infty.$$

This proves the theorem in the case $\gamma < 1/2$.

If $\gamma = 1/2$, we start from the function

$$g(r) = log\left(1 + log\frac{e}{1-r}\right)$$

and proceed as above to complete the proof of Theorem (2.3.11).

The space $\mathcal{B}(\psi)$ is closely related to the space

$$H_\infty(\psi) = \left\{f \in H(\mathbb{D}): f(z) = O\left(\psi\left(\frac{1}{1-r}\right)\right)\right\}$$

introduced and studied in [11], [122], [123], [117], [118], [120], [119].

If ψ satisfies (49) with $\beta < 1$, then $\mathcal{B}(\psi)$ is contained in the disk algebra $A(\mathbb{D})$ and can be identified with a space of analytic functions satisfying a Lipschitz condition on $\mathbb{D}$ or $\partial\mathbb{D}$.

Theorem (2.3.12)[110]: Let ψ satisfy (49) with $\beta < 1$, let $\omega(t) = t\psi(2/t)$, and let $f \in H(\mathbb{D})$. Then the following conditions are equivalent:

(i) $f \in \mathcal{B}(\psi)$.

(ii) $f \in A(\mathbb{D})$ and $|f(\zeta) - f(\eta)| \le C\omega(|\zeta - \eta|), \zeta, \eta \in \partial\mathbb{D}$.

(iii) $|f(z) - f(w)| \le C\omega(|z - w|), z, w \in \mathbb{D}$.

See [118], [119] for a general result that involves higher order derivatives. See also [111] for the case where ψ is almost normal.

The existence of critical Bloch functions as in (51) joined with Theorem (2.3.12) immediately gives the following, where $Lip_\omega(\partial\mathbb{D})$ and $Lip_\omega(\mathbb{D})$ denote the function spaces consisting f satisfying (ii) and (iii) of Theorem (2.3.12) respectively.

Theorem (2.3.13)[110]: Let $\psi_{j,j} = 1, 2$ satisfy (49) with $\beta < 1$, let $\omega_j(t) = t\psi_j(2/t)$, and let $f \in H(\mathbb{D})$. Then the following conditions are equivalent:

(i) $f \in \mathcal{C}(Lip_{\omega_1}(\partial\mathbb{D}), Lip_{\omega_2}(\partial\mathbb{D}))$.

(ii) $f \in \mathcal{C}(Lip_{\omega_1}(\mathbb{D}), Lip_{\omega_2}(\mathbb{D}))$.

(iii) $\sup_{z\in\mathbb{D}} f(z)\dfrac{\psi_1\left(\frac{1}{1-|f(z)|^2}\right)}{\psi_2(\frac{1}{1-|z|^2})} < \infty$.

(iv) $\sup_{z\in\mathbb{D}} |f'(z)| \dfrac{1-|z|^2\omega_1(1-|f(z)|^2)}{1-|f(z)|^2\omega_2(1-|z|^2)} < \infty$.

The subspace of those $f \in \mathcal{B}(\psi)$ for which

$$|f'(z)| = o\left(\psi\left(\frac{1}{1-|z|^2}\right)\right), |z| \to 1^-,$$

is denoted by $\mathcal{B}_0(\psi)$. In the case $\psi(x) = x$, the space reduces to the Little Bloch space, $\mathcal{B}_0$. It is known and easy to see that $\mathcal{B}_0(\psi)$ coincides with the closure in $\mathcal{B}(\psi)$ of the set of all polynomials.

The space VMOA (of functions of vanishing mean oscillation) is the subspace of BMOA defined by the requirement

$$l\lim_{|a|\to 1} - \int_{\mathbb{D}} |f'((z))|^2 \log\frac{1}{|\sigma_a(z)|}\, dA(z) = 0,$$

or equivalently that $f \in H^2$ and

$$\lim_{|a|\to 1} (P[|f_*|^2](a) - |f(a)|^2) = 0.$$

It is known that VMOA coincides with the closure in BMOA of the set of all polynomials.

Theorem (2.3.14)[110]: Let $\emptyset \in H(\mathbb{D},\mathbb{D})$. Then $\emptyset \in \mathcal{C}(\mathcal{B}_0(\psi),\text{VMOA})$ if and only if $\emptyset$ satisfies (63) and $\emptyset \in$ VMOA.

Proof. Assume that $C_\emptyset$ maps $\mathcal{B}_0(\psi)$ into VMOA. Since the function $z \mapsto z$ belongs to $\mathcal{B}_0(\psi)$ we have that $\emptyset \in$ VMOA. Let $f \in \mathcal{B}(\psi)$. Then $f_\rho \in \mathcal{B}_0(\psi)$, where $f_\rho(z) = f(\rho z)$ and $0 < \rho < 1$, and therefore, by the hypothesis that $C_\emptyset$ maps $\mathcal{B}_0(\psi)$ into VMOA, we have

$$\left|f\left(\rho_\emptyset(0)\right)\right|^2 + \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|f'\left(\rho_\emptyset(z)\right)\right|^2 \rho^2 |\emptyset'(z)^2 \log\frac{1}{|\sigma_a(z)|}\, dA(z)$$
$$\le C\left|\left|f_\rho\right|\right|^2_{\mathcal{B}(\psi)}. \qquad (71)$$

On the other hand, using the hypothesis that ψ is almost increasing, one shows that $\left|\left|f_\rho\right|\right|_{\mathcal{B}(\psi)} \le C\,||f||_{\mathcal{B}(\psi)}$. Combining this with (71) and using Fatou's lemma, we find that (71) holds for $\rho = 1$, which means that C_φ acts from $\mathcal{B}(\psi)$ to BMOA. Now we use Theorem (2.3.8) to conclude that (63) holds.

Assume, conversely, that $\emptyset \in$ VMOA and that (63) is satisfied. Since $\emptyset$ is VMOA and is bounded we see from the definition of VMOA that $\emptyset^n \in$ VMOA, for every integer $n \ge 0$. This implies that C_φ maps polynomials into VMOA. Since polynomials are dense in $\mathcal{B}_0(\psi)$, it follows that C_φ maps $\mathcal{B}_0(\psi)$ into VMOA. This completes the proof.

The argument in the above proof involving (71) actually proves the following:

Theorem (2.3.15)[110]: $\mathcal{C}(\mathcal{B}_0(\psi), BMOA) = \mathcal{C}(\mathcal{B}(\psi), BMOA)$.

Chapter 3
Bloch Pull-Backs

We investigate the assertion that if $\phi \in \mathcal{B}_0$ is a conformai mapping of the unit disk $\mathbb{D}$ into itself whose image $\phi(\mathbb{D})$ approaches the unit circle $\mathbb{T}$ only in a finite number of nontangential cusps, then C_ϕ is compact on $\mathcal{B}_0$. On the other hand if there is a point of $\mathbb{T} \cap \overline{\phi(\mathbb{D})}$ at which $\phi(\mathbb{D})$ doses not have a cusp, then C_ϕ is not compact. As a consequence, we obtain a new proof of a recently obtained characterization of the compact composition operators on Bloch spaces.

Section (3.1): Bounded Mean Oscillation

We consider holomorphic maps:

$$F: B_n \to D,$$

Where $B_n$denote the open unit ball in C. We will say that F has the pull-back property if $f \circ FBMOA, (B_n)$ wherever f belong to the Bloch space of D. The pull-back property was first studied in [125], where Ahern showed that the map $F(z) = n^{\frac{n}{2}} z_1 z_2 \dots z_n$ has the property. Ahren was interested in the Fatou theorem: Because the above F has the pull back property , there exist a function in $BMOA, (B_n)$ with a radial limit at no point of the n-torus $|z_1| = |z_2| = \dots = |z_n| = 1/\sqrt{n}(H^\infty$-functions must have limits in a set of full n-dimensional measure on the torus).)

Although the pull -back property now appears less useful than other techniques in studying the Fatou theorem for $BMOA$(see [136]). In [82], Ahern and Rudin posed the problem of characterizing the maps F having the pull-back property. It seemed puzzling that the pull-back property was difficult to verify, even for maps as spmple as Ahern's. Unit now, only certain homogenous polynomials were known to have the property [125], [82], [132], [127], [126]. Most of these results are based on the fact that $BMOA, (B_n)$ is the dual space of $H^1(B_n)$, and all involve somewhat intricate calculations that depend on the symmetry of the maps F considered.

We go well beyond the previous results by showing that if $F \in Lip_1(B_n)$, then F has the pull-back property. This theorem should be contrasted with a result of Tomaszewski [135], which shows that there exist maps F failing to have the pull-back property even though F$\in Lip_\alpha(B_n)$ for some $\alpha > 0$.

We take a different approach to the pull-back problem: The:' bounded mean oscillation'' definition of $BMOA$ on ∂B_n with respect to the usual non isotropic metric) is used directly, as is the conformal invariance of the Bloch space. This leads to a suggestive geometrical picture that, while nowhere present in statements or proofs of theorems, was the starting point for our investigation.

The result that makes everything work is Theorem (3.1.4)(b), which gives an estimate on who fast the complex tangential derivative of a holomorphic function tends to zero as a point of maximum modulus set.)

We discuss several necessary and sufficient conditions for a map F to have the pull back property. We show how some of the techniques can be used to prove a theorem related to a result of Rudin.

Any unexplained notation will be as in [77], [133].

We summarize a few facts about the Bloch space that we need in the sequel, most of which are well known. The Bloch space $\mathfrak{B}(D)$ consists of those holomorphic functions f on D for which

$$\|f\|_{\mathfrak{B}} = \ sup_{z\in D}|f'(z)|(1-|z|^2) < \infty. \quad (1)$$

This is a Banach space if the norm of $f \in \mathfrak{B}(D)$ is defined to be.

The invariant form of the Schwarz lemma [G.P.2] shows that if $\varphi: D \rightarrow \ \ D$ is holomorphic, then

$$\|f \circ \varphi\|_{\mathfrak{B}} \leqq \|f\|_{\mathfrak{B}} \quad (2)$$

For all $f \in \mathfrak{B}(D)$. Equality in (2) hold in the case $\varphi = \ \varphi_\alpha$ where φ_α.

Is the automorphism of D defined by

$$\varphi_\alpha(z) = \frac{\alpha - z}{1-\bar{\alpha}z}$$

Let $f \in \mathfrak{B}(D)$, integrating f' from 0 to z, one easily see that

$$|f(z) - f(0)| \leqq \|f\|_{\mathfrak{B}} log\frac{1}{1-|z|} \leqq \|f\|_{\mathfrak{B}} \log\frac{2}{1-|z|^2} \quad (3)$$

For all (3) $z \in D$. The identity

$$1\text{-} \ |\varphi_\alpha(\beta)|^2 = \frac{|1-\overline{\alpha}\beta|^2}{(1-|\alpha|^2)(1-|\beta|^2)}$$

Together with (2) and (3), show that

$$|f(\alpha) - f(\beta)| = \ \left|f \circ \varphi_\beta\big(\varphi_\beta(\alpha) - \varphi_\beta(0)\ \big)\right| \quad (4)$$

For all $\alpha, \beta \in D$.

Membership in $\mathfrak{B}(D)$ is equivalent to a bounded mean oscillation condition with respect to area measure. We let dA denote Lebseque area measure on C, normalized so that $A(D) = 1$.For $0 < \delta \leqq 2$ and $\zeta \in T$.

The unit circle, put

$$\Omega_\delta(\zeta) = \{Z \in D: |Z-\zeta| < \delta\}$$

The average of any $f \in L^1(D, dA)$ over $\Omega_\delta(\zeta) = \Omega$ is defined by

$$f_\Omega = \frac{1}{A(\Omega)}\int_\Omega f dA.$$

Note that (3) implies that every $\mathfrak{B}(D)$ $belong\ to$ $L^1(D, dA)$.

Throughout c and C will denote numerical constant whose values may change from line to line.

The expression $A(f) \approx B(f)$ will mean that therte exist positive constant C and c such that $cB(f) \leqq A(f) \leqq CB(f)$ for all functions f under consideration.

The following proposition is essentially contained in [128].We supply a proof that does not depend on the machinery developed.

Proposition (3.1.1)[121]: Suppose f is holomorphic in D and $f \in L^1(D, dA)$. Then

$$\|f\|_{\mathfrak{B}} \approx sup\frac{1}{A(\Omega)}\int_\Omega (f - f_\Omega\ )dA < \infty,$$

Where the supremum is taken over all region $\Omega = \Omega_\delta(\zeta)$.

Proof. A straightforward estimate shows that

$$\frac{1}{A(\Omega_\delta)}\int_{\Omega_\delta} log\frac{1}{(1-|z|^2)} dA(z) = \log\frac{1}{\delta} + o(1)\big(0 < \delta \leqq 2\big) \quad (5)$$

Suppose $f \in \mathfrak{B}(D)$. Setting $\Omega = \Omega_\delta(1)$, we have by (4) that

$$\frac{1}{A(\Omega)}\int_{\Omega}(f-f_{\Omega})dA \leqq \frac{2}{A(\Omega)}\int_{\Omega} |f-f(1-\delta)|dA$$
$$\leq \frac{2}{A(\Omega)}\|f\|_{\mathfrak{B}}\int_{\Omega} \log\frac{2|1-(1-\delta)_z|^2}{\delta(2-\delta)(1-|z|^2)}dA(z).$$
Since $|1-(1-\delta)_z| \leqq 2\delta$ in Ω, (5) shows that the last line is less than or equal to a constant times $\|f\|_{\mathfrak{B}}$ for $\delta \in (0,1)$. The fact that the L^1-norm of $of\ \ f-f(0)$ is less than a constant times $\|f\|_{\mathfrak{B}}$ handles the case $\delta \geqq 1$.

For the other direction, note than
$$|f'(0)| \leqq \int_D |f|\, dA \tag{6}$$
For all f holomorphic in. [Recall that $A(D)\ = 1$] Applying the appropriate translated and dilated version of (6) to the function $f-f_{\Omega}$, we have
$$(1-|z|)|f'(z)| \leqq \frac{1}{(1-|z|)^2}\int_{D(z,1-|z|)}|f-f_{\Omega}|\, dA$$
$$\leqq \frac{4}{A(\Omega)}\int_{\Omega} (f-f_{\Omega})dA,$$
Where $D(z,1-|z|)$ is the disc centered at z of radius $1-|z|$ and $\Omega=\Omega_{\delta}(\zeta)$, with $\delta=2(1-|z|), \zeta=z/|z|$.

Another BMO-type condition that characterizes the Bloch space involves H^1-norms over circle internally tangent to the unit circle T. For $\alpha \in d$, define the circle maps $\gamma_{\alpha}: \overline{D} \to \overline{D}\ by$
$$\gamma_{\alpha}(z) = \alpha + (1-|\alpha|)z$$
We let $\sigma_1=\sigma$ denote arc length measure on T, normalized so that $\sigma(T)=1$.

Proposition (3.1.2)[121]: If $0<\varepsilon<1$ and f is holomorphic in D, then
$$\|f\|_{\mathfrak{B}} \approx sup_{\varepsilon<|\alpha|<1}\|f\circ\gamma_{\alpha}-f(\alpha)\|_{H^1(D)} \tag{7}$$
Proof. We may take $\alpha>0$. Noting that $|1-\alpha\gamma_{\alpha}| \leqq |1-\alpha|+|(1-\gamma_{\alpha})| \leqq 3(1-\alpha)$ and that 1- $\left|\gamma_{\alpha}(e^{i\theta})\right|^2 = 2\alpha(1-\alpha)(1-cos\theta)$, we have
$$\int_T |f\circ\gamma_{\alpha}-f(\alpha)|\, d\sigma \leqq \|f\|_{\mathfrak{B}}\int_{\mathrm{T}} \log\frac{2|1-\alpha\gamma_{\alpha}|^2}{(1-\alpha^2)(1-|\gamma_{\alpha}|^2)}d\sigma.$$
$$\leq \|f\|_{\mathfrak{B}}\int_0^{2\pi} log\frac{9}{\varepsilon(1-cos\theta)}d\theta$$
Let C denote the value of the integral in the last line. The above shows that
$$\sup_{\varepsilon<|\alpha|<1}\|f_R\circ\gamma_{\alpha}-f_r(\alpha)\|_{H^1(D)} \leqq C_{\varepsilon}\|f_r\|_{\mathfrak{B}} \tag{8}$$
Where $f_r(z)=f(rz)$ for $0<r<1$, Since $\|f_r\|_{\mathfrak{B}} \to \|f\|_{\mathfrak{B}}\ as\ r \to 1$, (1,10) hold with f in place of f_r.

For the other direction, apply the inequality
$$|f'(0)| \leqq \int_T |f|\, d\sigma$$
[valid for all $f \in H^1(D)$] to the functions $f\circ\gamma_{\alpha}-f_r(\alpha)$.

Although Proposition (3.1.2) will not explicitly be used in the proof of the main theorem, we have included it because it may be helpful to the reader in understanding our approach to the pull-back problem' see (13) and (14).

The above proof can be easily modified to show that Proposition (3.1.2) remains valid if the H^1-norms in (7) are replaced by H^p-norms , for any $p \in (0,\infty)$. The following corollary is the case $p=2$, which may be of some independent intrest.

Corollary (3.1.3)[121]: If $0<\varepsilon<1$ and f is holomorphic in D, then

$$\|f\|_{\mathfrak{B}} \approx sup_{\varepsilon<|\alpha|<1}\left(\sum_{k=1}^{\infty}\left(\frac{f^{(k)}(\alpha)}{k!}(1-|\alpha|)^k\right)^2\right)^{1/2}$$

Rotation –invariant Lebesgue measure on ∂B_n normalized to have total mass 1. Will be denoted by σ_n . We write $\sigma_n = \sigma$ *when the* dimension is clear from context. For $\zeta \in \partial B_n$ and $0<\delta \leqq 2$, *but* $Q_\delta(\zeta) = \{\eta \in \partial B_n: |1-\langle\eta,\zeta\rangle|<\delta\}$. Notice that the z_1-projection of $Q_\delta(e_1)$ *into* D *is* $\Omega_\delta(1)$ where $e_1 = (1,0,\dots,0)$.

The class BMOA (B_n) consists of the functions $g \in H^1(B_n)$ for which

$$\|g\|_{BMO} = \sup\frac{1}{\sigma(Q)}\int_Q |g-g_Q|\,dQ < \infty,$$

Where g_Q denotes the average of g over Q and the supremum is taken over all $Q=Q_\delta(\zeta)$ (we have identified g with its boundary.)

BMOA (B_n) is a Banach space under the norm BMOA (B_n), and, as is well known [128], can be identified with the dual space of $H^1(B_n)$.

This duality relation is not important to our approach to the pull-back problem, and will appear only as a technical devise in extending the $n=2$ case of Theorem (3.1.6) to higher dimensions.

Most of the work will be done for the case $n=2$. For any $g \in L^1(\partial B_2)$ we have [77]

$$\int_{\partial B_2} g d\sigma = \int_D \int_T g_\alpha d\sigma_1 dA(\alpha), \tag{9}$$

Where $g_\alpha(w) = g\left(\alpha,(1-|\alpha|^2)^{\frac{1}{2}}w\right)$ for $w \in T$, when $g \in H^1(B_n)$, (9) and the mean value property give

$$g_Q = \frac{2}{A(\Omega)}\int_\Omega g(\alpha,0)dA(\alpha), \tag{10}$$

where

$$Q = Q_\delta(e_1) \quad and \quad \Omega = \Omega_\delta(1).$$

For $\mathrm{K}=1,\dots,\mathrm{n},\mathrm{D_k}$ will denote the holomorphic partial derivative $/\partial_{Z_K}$. The class Lip 1(B) is the set of functions g on $B=B_n$ for which

$$\|g\|_{Lip\,1} = sup\frac{|g(z)-g(w)|}{|z-w|} < \infty,$$

The supremum being taken over all $\mathrm{z},\mathrm{w} \in \mathrm{B}$ with $\mathrm{z}\neq\mathrm{w}$. Note that $\|\mathrm{g}\|_{\mathrm{Lip}\,1} = \|\nabla\mathrm{g}\|_\infty$ whenever g is holomorphic in , where $\|\quad\|_\infty$ denoyes the supremum norm on B and $\nabla\mathrm{g} = (\mathrm{D_1g},\dots\mathrm{D_ng})$.

When $\mathrm{n}=2$ we can define a canonical complex tangential derivative by setting

$$\mathrm{D_r g(r\zeta)} - \bar{\zeta}_1\mathrm{D_2g(r\zeta)} - \bar{\zeta}_2\mathrm{D_1g(r\zeta)}$$

For $0<r<1$ *and* $\zeta \in \partial\mathrm{B}_{2:}\,\mathrm{D_rg(z)}$ is the complex derivative of g in the direction orthogonal to z.

Recall that F always denotes a holomorphic map from B_n, into D.

Theorem (3.1.4)[121]: If $F \in \mathrm{Lip}_1\ (B_2)$ then

- As $r \to 1$, $D_T F(r\zeta)$ converges uniformly on ∂B_2 to a continuous function $D_T F(\zeta)$
- There exists a constant C, depending only on $\|F\|_{\mathrm{Lip}_1}$ such that

$$|D_T F(\zeta)| \leqq C(1 - |F(\zeta)|)^{1/2}$$

For all $\zeta \in \partial B_2$.

Proof. We will show first that

$$|D_T F(s\zeta) - D_T F(r\zeta)| \leqq 2(1 - r)^{1/2} \|F\|_{\mathrm{Lip}\ 1} \quad (11)$$

for all $\zeta \in \partial B_2$, whenever $0 \leqq r < s < 1$. . For simplicity, we take $\zeta = e_1$, so that $D_T = D_2$.

Because $\|D_1 F\|_\infty \leqq \|F\|_{\mathrm{Lip}\ 1}$, we may apply Cauchy's estimates in the z_2-direction to obtain.

$$D_2\, D_1.F(re_1)$$

Reversing the order of differentiation and then integrating, we find this proves (11) and hence (a) of the theorem to prove (11) and hence (a) of the theorem.

$r \in [0,1]$setting $r = |F(e_1)|$, we arrive at

$$\Omega = \Omega_\delta(\zeta)$$
$$\delta = 2(1 - |z|), \zeta = z/|z|$$
$$\gamma_\alpha: \overline{D} \to \overline{D}$$
$$\gamma_\alpha(z) = \alpha + (1 - |\alpha|)z$$
$$\sigma_1 = \sigma \quad \sigma(T) = 1$$

To prove (b), observe that the invariant Schwarz lemma and the Lipschitz condition on F imply

$$|D_2 F(e_1)| \leqq |D_2 F(e_1) - D_2 F(re_1)| + D_2 F(re_1)$$
$$\leqq 4\|F\|_{\mathrm{Lip}\ 1}(1 - r)^{1/2} + 2(1 - |F(re_1)|)(1 - r)^{-1/2}$$

For all $r \in [0,1]$, thus by [3.2],

$$|D_2 F(e_1)| \leqq |D_2 F(e_1) - D_2 F(re_1)| + D_2 F(re_1)$$
$$\leqq 4\|F\|_{\mathrm{Lip}\ 1}(1 - r)^{1/2} + 2(1 - |F(re_1)|)(1 - r)^{-1/2}$$

For all $r \in [0,1]$. setting $r = |F(e_1)|$, we arrive at

$$|D_2 F(e_1)| \leqq \left(4\|F\|_{\mathrm{Lip}1} + 2\right)(1 - |F(e_1)|)^{1/2}$$

Completing the proof of (b).

(3.1.4) (a) is not new but is included for the sake of completeness. Note that it implies that the restriction of F to any complex tangential curve in ∂B^2 is continuously differentiable. In fact, F is a good deal smoother than this on such curve according to Stein [134] (see also [77] or [130]).

Given a complex-valued function g defined on the set E, define

$$\mathrm{osc}_E g = \sup_{\zeta,\eta \in E} |g(\zeta) - g(\eta)|$$

Corollary (3.1.5)[121]: If $F \in Lip_1(B_2)$ then there exists a constant C, Depending only on $\|F\|_{\mathrm{Lip}1}$, such that

$$\mathrm{osc}_{Q_\delta(\zeta)} F \leqq C\left(\delta + \delta^{1/2}(1 - |F(\zeta)|)^{1/2}\right)$$

For all $Q_\delta(\zeta) \subset \partial B_2$.

Proof. For convenience we take $\zeta = e_1$. Define the complex tangential curve $\gamma_\theta(t) = \gamma(t) = \left(\cos t, e^{i\theta} \sin t\right)(t \in R)$, where $\theta \in R$ is fixed. Because F is continuously differentiable along γ, Theorem (3.1.4) (b) shows

$$\left|F(\gamma(b))-F(\gamma(e_1))\right| \leqq \int_0^b \left|D_T F(\gamma(t))\right||\gamma'(t)|\,dt \leqq C\int_0^b \left(1-\left|F(\gamma(t))\right|^2\right)^{1/2} dt$$

Setting $s(t) = \left(1-\left|F(\gamma(t))\right|^2\right)^{1/2}$, we have

$$|s'(t)| \leqq \frac{\left|D_T F(\gamma(t))\right|}{\left(1-\left|F(\gamma(t))\right|^2\right)^{1/2}} \leqq C.$$

Whenever $\left|F(\gamma(t))\right| < 1$. it follows that $s(t) \leq s(0) + Ct$, giving

$$\left|F(\gamma(b))-F(e_1)\right| \leqq C\left(b^2 + b(1-|F(e_1)|^{\frac{1}{2}}\right)$$

For small δ every point $Q_\delta(e_1)$ has Euclidean distance less than δ from one of the curve $\gamma_\theta([0,2\delta^{1/2}])$ since $F \in Lip_1(B_2)$, since the proof of the corollary is complete,

The main result the following theorem.

Theorem (3.1.6)[121]: Suppose that $F \in Lip_1(B_n)$ and $F(0) = 0$. Then there exists a constant C, depending only on $\|F\|_{Lip_1}$, such that

$$\|f\circ F\|_{BMOA(B)} \leq \|f\|_{\mathfrak{B}(D)}$$

For all $f \in \mathfrak{B}(D)$.

The assumption $F(0) = 0$ is merely a convenience normalization. [If $F \in Lip_1(B_n)$ and $F(0) \neq 0$, Theorem (3.1.6) may be applied to the composition of F with an appropriate automorphism of D. It then follows that F has the pull-back property, with a constant depending both on $\|F\|_{Lip_1}$ and $F(0)$].

Until further notice we taken $n = 2$. The higher-dimensional case will follow from this by a slicing argument. (See Proposition (3.1.10))

To prove Theorem (3.1.6). we need to show the averages

$$\frac{1}{\sigma(Q)}\left|f\circ F-(f\circ F)_Q\right|_{dQ} \tag{12}$$

are bounded by a constant times $\|f\|_{\mathfrak{B}(D)}$ where $Q = Q_\delta(\zeta)$. We also need to know that $f \circ F \in H^1(B_2)$. To avoid this latter technical detail at the beginning, we assume until further notice that $f \in C(\overline{D})$.

The proof of Theorem (3.1.6) comes in two parts, one dealing with "small Q's", the other with "large Q's". We start with the smaller Q's"., which are easier to handle .

Let C denote the constant associated with F by Corollary (3.1.5). Letting $c = 1/(9C)$, we see that if $0 < \delta \leq c(1-|F(\zeta)|)\ then\ F(Q_\delta(\zeta))$ is contained in the disk Δ with center $F(\zeta)$ and radius $(1-F|\zeta|)/2$, setting $Q = Q_\delta(\zeta)$we have

$$\frac{1}{\sigma(Q)}\int_Q \left|f\circ F-(f\circ F)_Q\right| d\sigma \leqq \frac{2}{\sigma(Q)}\int_Q \left|f\circ F - f(F(\zeta))\right| d\sigma \leqq 2\, osc_\Delta f$$

The estimate (1) on f' shows that the oscillation of f over Δ is bounded by an absolute constant times the Bloch norm of f.

The case $\delta > c(1-|F(\zeta)|)$ is substantially more involved, but there is an intermediate case that is easily handled. Suppose we know the average (12) are bounded by a constant (depending only on $\|F\|_{Lip_1}$) times the Bloch norm of f , whenever $\delta > 4(1-|F(\zeta)|)$. Then for the range $c(1-|F(\zeta)|) \leq \delta \leq 4(1-|F(\zeta)|)$ we obtain

$$\frac{1}{\sigma(Q)}\int_Q \left|f\circ F-(f\circ F)_Q\right| d\sigma \leqq \frac{\sigma(Q_1)}{\sigma(Q)}\frac{2}{\sigma(Q_1)}\int_{Q_1}\left|f\circ F-(f\circ F)_{Q_1}\right| d\sigma,$$

Where $Q = Q_\delta(\zeta)$ and $Q_1 = Q_{4(1-|F(\zeta)|)}(\zeta)$. The last expression is then bounded by a constant (depending only on $\|F\|_{Lip_1}$ times the Bloch norm of f.

It is thus the range $\delta > 4(1-|F(\zeta)|)$ that is trouble some. From now on we take $\zeta = e_1$ *with* $Q = Q_\delta(e_1)$ *and* $\Omega = \Omega_\delta(1)$ (11), and (12) show that

$$\frac{1}{\sigma(Q)}\int_Q |f\circ F-(f\circ F)_Q|\,d\sigma_2 \leqq \frac{1}{A(\Omega)}\int_\Omega\int_T |f\circ F_\alpha - f\circ F_\alpha 0)|\,d\sigma_1 dA(\alpha)$$

$$+\frac{1}{A(\Omega)}\int_Q |fF(\alpha,0)-(f\circ F)(.,0)_\Omega|\,dA(\alpha) \qquad (13)$$

[recall the notation $F_\alpha(w) = F\left(\alpha,(1-|\alpha|^2)^{\frac{1}{2}}w\right)$ w] Proposition (3.1.1) and (2) show that the second summand on the right is bounded by an absolute constant times the Bloch norm of f.

It is thus the Ω-averages of

$$\int_T |f\circ F_\alpha - f\circ F_\alpha(0)|\,d\sigma_1 \qquad (14)$$

That we must control. Note the similarity between (14) and (7). In fact, it is not difficult to convince oneself that $\alpha \to 1, F_\alpha(T)$looks more and` more like a circle with center $F_\alpha(0)$. This is the geometrical picture mentioned. (it is tempting to think that something like Littlewood's subordination principle, combined with Proposition (3.1.2), would now finish the proof, but we were not able to make this idea work.)

By (4), (14) is less than or equal to

$$\|f\|_{\mathfrak{B}}\int_{\mathrm{T}} \log\frac{2|1-\overline{F_\alpha}(0)F_\alpha|^2}{,(1-|F_\alpha(0)|^2)\ \ (1-|F_\alpha|^2)}d\sigma. = \|f\|_{\mathfrak{B}}\int_{\mathrm{T}} \log\frac{2|1-F_\alpha(0)|^2}{(1-|F_\alpha|^2)}d\sigma. \quad (15)$$

The equality following because $\log 2|1-\overline{F_\alpha}(0)F_\alpha|^2$ is harmonic on $\overline{D}$.

Lemma (3.1.7)[121]: If $F \in Lip_1(B_2)$, then there exist a constant C, depending only on $\|F\|_{Lip_1}$, such that

$$\int_{\mathrm{T}} \log\frac{1}{(1-|F_\alpha|^2)}d\sigma \leqq C + \log\frac{1}{|D_1F(\alpha,0)|^2(1-|\alpha|^2)}$$

For all $\alpha \in D$.

Proof. By Theorem (3.1.4) (b) and Fatou's lemma,

$$\int_{\mathrm{T}} \log(1-|F_\alpha|)^2 d\sigma \geqq \log c + \int_{\mathrm{T}} \log|(D_TF)_\alpha|^2 d\sigma =$$

$$\log c + \int_{\mathrm{T}} \lim_{\mathrm{r}\to 1}\log\left|D_TF(r\alpha, r(1-|\alpha|^2)^{1/2}w\right|^2 d\sigma(w) \geqq$$

$$\log c + \lim_{\mathrm{r}\to 1} sup\int_T \log|D_TF(r\alpha, r(1-|\alpha|^2)|^{1/2}w|^2 d\sigma(w)$$

Using the definition of D_T and multiplying by w with the absolute values, the integrand in the last line becomes

$$\log\left|wr\bar{\alpha}D_2F\left(r\alpha, r(1-|\alpha|^2)^{1/2}w\right) - r((1-|\alpha|^2)^{1/2})D_1F\left(r\alpha, r(1-|\alpha|^2)^{1/2}w\right)\right|^2$$

Which is a subharmonic function of w. It follows that

$$\int_T log(1-|F_\alpha|^2)d\sigma \geq logc + \lim_{R\to 1}\sup\ log\left|(1-|\alpha|^2)^{1/2}D_1F(\alpha,0)\right|^2$$

$$= \log c + \log\left|(1-|\alpha|^2)^{1/2}D_1F(\alpha,0)\right|^2$$

Completing the proof of the lemma.

The next two lemmas will be applied to the one variable function $g(\alpha) = F(\alpha, 0)$.

Lemma (3.1.8)[121]: Suppose g is holomorphic in D, $g \in Lip_{\,1}(D)$ and $\|g\|_\infty \leqq 1$, , then there exists a constant C depending only on $\|g\|_{Lip_{\,1}}$, such that

$$\frac{1}{A(\Omega)}\int_\Omega \log(1-|g(\alpha)|^2)dA(\alpha) \leqq log\delta + C\ .$$

For all $\Omega = \Omega_\delta(1)$ $with\ \delta > 4\big(1-|g(1)|\ \big)$

Proof. If $\alpha \in \Omega_\delta(1)$, then

$$1-|g(\alpha)|^2 \leqq 2(1-|g(\alpha)|) \leqq 2(1-|g(1)|) + |g(1)-g(\alpha)|$$

$$\leqq 2\left(\left(\frac{\delta}{4}\right) + \|g\|_{Lip\,1}|1-\alpha| \leqq 2\delta\left(\frac{1}{4} + \|g\|_{Lip\,1}\right)\right)$$

Lemma (3.1.9)[121]: Suppose g is holomorphic in D, $g \in Lip_{\,1}(D)$ and $\|g\|_\infty \leqq 1$, and $g(0) = 0$, then there exists a constant $C > -\infty$ depending only on $\|g\|_{Lip_{\,1}}$, such that

$$\frac{1}{A(\Omega)}\int_\Omega \log(g')dA \geqq C\ .$$

For all $\Omega = \Omega_\delta(1)$ $with\ \delta > 4\big(1-|g(1)|\ \big)$

Proof. First observe that if u is subharmonic and non positive in D, then

$$sup_{0\leqq|z|\leqq r}u(z) \leqq \left(\frac{1-r}{1+r}\right)^2\int_D udA. \tag{16}$$

Whenever $0 \leqq r < 1$. Inequality (16) is clear when $r = 0$.

The general case follows by applying the case $r = 0\ to\ u$ composed with automorphism of D and then changing variables.

Assume to begin with that $0 < \delta < 1$. Set L = $\|g\|_{Lip_{\,1}}$, and put , $x = 1-\delta, y = 1-\frac{\delta}{2(L+1)}$. By the Schwarz lemma [recall g(0) = 0]$|g(x)| \leqq x, Thus$

$$|g(y)-g(x)| \geqq |g(1)-g(1)-g(y)| - x \geqq \delta/4$$

Where we have used the hypothesis $\delta > 4(1-|g(1)|)$. It follows that $|g'| \geqq 1/4$ at some point of $[x, y]$.

Define the automorphism $|g'| \geqq 1/4$.

$$\varphi(z) = \frac{1-\delta+z}{1+(1-\delta)_z}$$

Note that $\varphi(0) = 1-\delta$ and $\varphi(1) = 1$; φ is a " dilation" that pull G in towards 1.setting $r = 1-\frac{1}{(2(L+1))}$ and $v = \{\ z \in D\colon \mathrm{Re}z > 0\}$, it is not hard to verify that

$$[x,y]\varphi([0.r]), \varphi^{-1}\big(\Omega_\delta(1)\big) \subset V, \text{and } |\varphi'| \leqq 2\delta \text{ in } V.$$

Now put $h = \frac{g'}{\|g'\|_\infty}$ and $\Omega = \Omega_\delta(1)$ we use the remarks above and (16) (with $u = \log|h|$) to conclude

$$\frac{1}{A(\Omega)}\int_{\Omega} log|h|\,dA \geqq \frac{1}{A(\Omega)}\int_{V} log|h\circ\varphi||\varphi'|^2\,dA$$

$$\geqq C\int_{D} log|h\circ\varphi|\,dA \geqq C\left(\frac{1-r}{1+r}\right)^2 \sup_{(0,r)} log|h\circ\varphi|$$

$$\geqq C\left(\frac{1+r}{1-r}\right)^2\left[\log\frac{1}{4} - log\ L\right] .$$

(Recall that $L = \|g'\|_{\infty}$) .We are done in the case $0 < \delta < 1$.

The case $0 \leqq \delta \leqq 1$ is similar but easier. We need only show

$$\int_{D} \log|g'|dA \geqq C .$$

Here we know $|g(1)| > 1/2$,, which implies that $|g(r)| > 1/4$, where it $r = 1 - 1/(4(L+1)$. It follows that that $|g'|$ is at least ¼ somewhere in $[0, r]$ and now we apply (16) as before. The proof of the lemma is complete.

Finishing the proof of Theorem (3.1.6) is now a matter tying up some loose ends. We need to show

$$\frac{1}{A(\Omega)}\int_{\Omega}\int_{T} \log\frac{(1-|F_{\alpha}(0)|^2)}{(1-|F_{\alpha}|^2)} d\sigma dA(\alpha).$$

Is uniformly bounded provided $\delta > 4(1-|F(e_1)|)$, where of course $\Omega = \Omega_{\delta}(1)$ Lemma (3.1.7) and (3.1.8) show that the above is less than

$$C + \log\delta + \frac{1}{A(\Omega)}\int_{\Omega} \log\frac{1}{1-|\alpha|^2}\, dA(\alpha) + \frac{1}{A(\Omega)}\int_{\Omega} log\frac{1}{|D_1F(\alpha,0)|^2}dA(\alpha).$$

By (5) and Lemma (3.1.9), this expression is bounded by a constant depending only on $\|f\|_{Lip1}$

We have thus shown

$$\|f\circ F\|_{BMOA(B_2)} \leqq C\|f\|_{\mathfrak{B}(D)} \qquad (17)$$

For a constant C depending only on $\|f\|_{Lip1}$, at least for holomorphic $f \in C(\bar{D})$. For the general $f \in \mathfrak{B}(D)$, apply (17) to the dilates f_r and take limit s as before (using the fact that $\|f_r\|_{\mathfrak{B}} \to \|f\|_{\mathfrak{B}}$as $r \to 1$).

The proof of Theorem (3.1.6) in the case $n = 2$ is complete, The next proposition shows that Theorem (3.1.6) for $n > 2$ follows from this case.

Proposition (3.1.10)[121]: For $n > 2$ there exists a constant $C_n$with the following property: if g is holomorphic in $B_n$and the BMOA(B_2) norm of g on two-dimensional slices of $B_n$through the origin are bounded by the constant C, then$\|g\|_{BMOA(B_n)} \leqq C_nC$.

Proof. Here we use the fact that BMOA is the dual space 0F H^1 : See [128], theorem v]. Thus for any k.

$$. \|g\|_{BMOA(B_k)} \approx sup\left|\int_{\partial B_k} g\bar{P}d\sigma_k\right|$$

Where the supremum is taken over all holomorphic polynomials P of $H^1 - norm$ at most 1 such that $P(0) = 0$. . The proposition now follows the formula

$$\int_{\partial B_n} hd\sigma_n = \int_{\mathfrak{U}(n)}\int_{\partial B_n} h\circ U(\zeta_1,\zeta_2,0,\ldots,0)\,d\sigma_2(\zeta_1,\zeta_2)dU, \qquad (18)$$

Valid for all integrable on $h\ on\ \partial B_n$. Here $\mathcal{U}(n)$ is the complex unitary group on C^n and dU denotes Haar measure on $\mathcal{U}(n)$, (formula (18) follows from Fubini's theorem and proposition 1.4.7 (3) in [77].)

For $z, w \in D$, define

$$\varrho\,(z, w) = \log \frac{|a - \bar{z}w|^2}{(1-|z|^2)\,(1-|w|^2)} = \log \frac{1}{1-|\varphi_z(w)|^1}$$

Suppose $F: B_2 \to D$ is holomorphic. It follows from (13)-(15) that if

$$\text{Sup}\ \frac{1}{\sigma(Q)} \int_Q \varrho(F(\zeta), F(\zeta_1, 0) d\sigma_2(\zeta) < \infty). \tag{19}$$

Then

$$\text{Sup}\ \frac{1}{\sigma(Q)} \int_Q \left(\left|f \circ F - (f \circ F)_Q\right| d\sigma_2 < \infty\right)$$

For every $f \in \mathfrak{B}(D)$ where the suprema are taken over all $Q = Q_\delta(e_1)$.

Now if $f \in \mathfrak{B}(D)$, then $|f(z) - f(w)|$ is in fact much smaller than $\varrho\,(z, w)$ for most $z, w \in D$.

We find that (19) is a necessary condition for the pull-back problem.

We will give the proof of Theorem (3.1.11) for the case $n = 2$, when $n > 2$, the argument is essentially the same but is somewhat less convenient because of the Jacobian factor $(1-|\lambda|^2)^{N-2}$ that appears in the higher dimensional analogue of (9) (see [77]).

Theorem (3.1.11)[121]: If $f: B_n \to (D)$ is a holomorphic , then F has the pull-back property if and only if

$$\text{Sup}\ \frac{1}{\sigma\left(Q_\eta(\eta)\right)} \int_{Q_\eta(\eta)} \varrho(F(\zeta), F(\langle\zeta, \eta\rangle\eta) d\sigma(\zeta) < \infty) \tag{20}$$

Where the supremum is taken over all $Q_\eta(\eta) \subset \partial B_n$

Proof. (n =2). We have already seen that (20) is sufficient for the pull-back property. To show the necessity of (20), first observe that for any $g \in BMOA(B_2)$, Bessel's inequality shows that

$$\begin{aligned}
&\frac{1}{A(\Omega)} \int_\Omega \int_T |g_\alpha - g_\alpha(0)|^2\, d\sigma_1 dA(\alpha) \\
&\leqq \frac{1}{A(\Omega)} \int_\Omega \int_T \left|g_\alpha - g_Q(0)\right|^2 d\sigma_1 dA(\alpha) \\
&= \frac{1}{\sigma(\Omega)} \int_\Omega \left|g - g_Q\right|^2 d\sigma_2 \leqq C(\|g\|_{BMO})^2
\end{aligned} \tag{21}$$

Now we have made use of the L^2 - criterion for membership in BMO: see [84].

Now suppose $f: B_2 \to D$ has the pull-back property. Proposition (3.1.12) shows there exists a holomorphic map $f: D \to C^2$ such that

$$\frac{1}{1-|z|} \leqq |f'(z)| \leqq \frac{C}{1-|z|}$$

For all $z \in D$. It follows that $f \circ F$ is a C^2-valued element of $FBMOA, (B_2)$.

Take $\eta = e_1$ for convenience, and set $g = f \circ F.A$. A classical Littlewood-paley identity [84] shows that

$$\int_T |g_\alpha - g_\alpha(0)|^2\, d\sigma = 2\int_D |g'_\alpha(w)|^2 log\frac{1}{w}\, dA(w)$$
$$= 2\int_D \left|f'\big(F_\alpha(w)\big)\right|^2 |(D_2F)_\alpha(w)|^2 \quad (1-|\alpha|^2) log\frac{1}{|w|}\, dA(w)$$
$$\geqq 2\int_D \frac{|(D_2F)_\alpha(w)|^2}{(1-|F_\alpha(w)|^2)^2}(1-|\alpha|^2) log\frac{1}{|w|}\, dA(w)$$
$$= \frac{1}{2}\int_D \Delta h_\alpha(w)\, log\frac{1}{|w|}\, dA(w) \qquad (22)$$
Where $h_\alpha(w) = \log\left|\frac{1}{1-|F_\alpha(w)|^2}\right|$ because $\log|w|$ is the fundamental solution for the Laplacian and $h_\alpha(w)$ is a subharmonic, a simple dialation argument using Fatou's lemma shows that
$$\int_D \Delta h_\alpha(w)\, log\frac{1}{|w|}\, dA(w) \geqq 2\int_T \Big(h_\alpha - h_\alpha(0))d\sigma = 2\int_T \varrho\big(F_\alpha, F_\alpha(0)\big)\Big)\, d\sigma,$$
The last equality above following as in (15) because $Log|1-\bar{F}_\alpha(0)F_\alpha|^2$
Is harmonic on $\bar{D}$, thus (21) and (22) show that
$$\frac{1}{\sigma(Q)}\int_Q \varrho\big(F(\zeta), F(\zeta_1, 0)\big)\, d\sigma_2(\zeta) = \frac{1}{AQ}\int_\Omega \int_T \varrho\big(F_\alpha, F_\alpha(0)\big)\, d\sigma_1 dA(\alpha)$$
$$\leqq \frac{1}{A\Omega}\int_\Omega \int_T |g_\alpha, g_\alpha(0)|^2\, d\sigma_1 dA(\alpha)$$
$$\leqq C(\|g\|_{BMO})^2 \leqq C.$$
The proof of Theorem (3.1.11) (for the case $n = 2$) is completed.

Proposition (3.1.12)[121]: There exist $f, g \in \mathfrak{B}(D)$ such that
$$|f'(z)| + |g'(z)| \geq \frac{1}{1-|z|}.$$
For all $z \in D$.

Proof. Let $f(z) = \sum_{j=0}^{\infty} z^{q^j}$, where q is a large positive integer to be determined. because f is a lacunary power series with bounded coefficients, $f \in \mathfrak{B}(D)$ (See [131]). We first show that
$$|f'(z)| \geqq \frac{c}{1-|z|} \quad if\ 1-q^{-k} \leqq |z| \leqq 1-q^{-(k+1/2)}, k = 1,2,.. \qquad (23)$$
We have
$$|f'(z)| \geqq q^k|z|^{q^k} - \sum_{j=0}^{k-1} q^j|z|^{qj} - \sum_{k+1}^{\infty} q^j|z|^{qj} = \mathrm{I} - \mathrm{II} - \mathrm{III}$$
for all $z \in D$. Fix a z as in (23), and put $x = |z|^{q^k}$. Then
$$(I-q^{-k})^{q^k} \leqq X \leqq \left[\big(1-q^{-(k+1/2)}\big)^{q^{k+1/2}}\right]^{q^{-1/2}},$$
which implies
$$1/3 \leqq x \leqq (1/2)^{q^{-1/2}} \qquad for\ \ k \geqq 1$$
if q is large enough.
We thus have $I \geqq q^k/3$, and we easily estimate that

$$\mathrm{II} \leqq \sum_{j=0}^{k-1} q^j \leqq \frac{q^k}{q-1}.$$

In III, note that because the ratio of two successive terms is no larger than the ratio of the first two terms, the series is dominated by the geometric series having the same first two terms. Thus

$$\mathrm{III} \leqq q^{k+1}|z|^{q^{k+1}} \sum_{j=0}^{\infty} \left(q|z|^{\left(q^{k+2}-q^{k+1}\right)}\right)^j$$

$$= \frac{q^{k+1}|z|^{q^{k+1}}}{1-q|z|^{\left(q^{k+2}-q^{k+1}\right)}} = q^k \frac{qx^q}{1-qx^{q^2-q}}$$

$$\leqq q^k \frac{q(1/2)^{q^{1/2}}}{1-q(1/2)^{q^{3/2}-q^{1/2}}}.$$

It follows that

$$|f'(z)| = \frac{q^k}{4} = \frac{q^{k+1/2}}{4q^{1/2}} \geqq \frac{1}{4q^{1/2}(1-|z|)}$$

if q is large enough, for the ranges of k and z specified in (23).

A similar argument shows that if q is a large positive integer and $g(z) = \sum_{j=0}^{\infty} z^{n_j}$, where n_j is the integer closest to $q^{J+1/2}$, then

$$|g'(z)| \geqq \frac{c}{1-|z|} \quad \text{if} \quad 1-q^{-(k+1/2)} \leqq |Z| \leqq 1-q^{-(k+1)}, \quad k = 1, 2, \ldots.$$

We are done unless it happens that f' and g' have a common zero in $\{|z| < 1-q^{-1}\}$, in which case we can replace $g(z)$ with $g(\alpha z)$ for a suitable g with $|\alpha| = 1$. Note that $f'(0) = 1$.] The proof of Proposition (3.1.12) is complete.

Proposition (3.1.12) may be used to give various other characterizations of the pullback property. We mention two, omitting the proofs:

(i) A "Garsia-norrn characterization": F has the pull-back property if and only if

$$\sup_{\alpha \in B} P[Q(F, F(a))]\,(a) < \infty.$$

Here $P[\cdot]$ denotes the Poisson-Szegö integral, or the "invariant Poisson integral" as in [77].

(ii) A "Carleson-measure characterization": F has the pull-back property if and only if

$$\sup \delta^{-n} \int_{\Omega_\delta(\xi)} \frac{|\nabla_T F|^2}{(1-|F|^2)} dV_n < \infty,$$

where V_n denotes volume measure on C^n, ∇_T is the complex tangential gradient, and the supremum is taken over all $\Omega_\delta(\xi) = \{z \in B_n : |1 - \langle z, \xi \rangle < \delta\}$. (This follows from Proposition (3.1.12) together with the Cadeson-measure characterization of BMOA in terms of ∇_{T} given in I-CC].)

The techniques developed can be used to prove a theorem related to a result of Rudin (see Theorem 11.4.7 in [77]). We still assume $F: B_n \to D$ is holomorphic. Given $\xi \in \partial B_n$, the function on D defined by $F_\xi(\lambda) = F(\lambda\xi)$ $(\lambda \in D)$ is called a slice function of F. In the next theorem we identify F_ξ and F_n whenever $\xi = e^{i\theta}\eta$.

Theorem (3.1.13)[121]: Suppose $F \in \mathrm{Lip}_\alpha(B_n)$ for some $\alpha > 1/2$, and that $\nabla F(0) \neq 0$. Then at most one slice function of F is an extreme point of the closed unit ball of $H^\infty(D)$.

A result of de-Leuw and Rudin I-D, Theorem 7.9] shows that if $g \in H^{\infty}(D)$ and $\|g\|_{\infty} = 1$, then g is an extreme point of the closed unit ball of $H^{\infty}(D)$ if and only if

$$\int_T \log(1 - |g|)d\sigma = -\infty. \quad (24)$$

Thus the modulus of the function F of Theorem (3.1.13) is severely constrained to stay away from 1, not only in terms of the size of the set where $|F| = 1$, but also in terms of the rate at which $|F|$ can tend to 1 in the real tangential directions at a point of the maximum modulus set of F.

It is easy to see from (24) that if g is not an extreme point of the closed unit ball of $H^{\infty}(D)$ and $g \in \text{Lip}_{\alpha}(D)$ for some $\alpha > 0$, then the subset of ∂D where $|g| = 1$ is a "Carleson set" (see [77]). Thus Theorem (3.1.13) implies that, except for possibly one complex line through the origin, the intersection of the set where $|F| = 1$ with any complex line through the origin is a Carleson set. Rudin's result yields even more information along these lines, but apparently our observation about extreme points is new.

We sketch the proof of Theorem (3.1.13).

Assume $n = 2$ for the moment. Using the fact that $F \in \text{Lip}_{\alpha}(B_2)$ if and only if $|\nabla F(z)| = O(l - |z|)^{\alpha-1}$ (see [77]), the proof of Theorem (3.1.4) shows that Theorem (3.1.4) (a) is still true as stated, and that $|D_T F(\xi) \leqq C(1 - |F(\xi)|^{\alpha-1/2}$ holds in place of (b). It follows that $\log(l - |F|) \geqq C_1 + C_2 1 \log|D_T F|^2$ on ∂B_2, where $C_1 > -\infty$ and $C_2 > 0$. Thus the argument given for Lemma (3.1.7) shows

$$\int_T \log(1 - |F_{\alpha}|)\, d\sigma \geqq C_1 + C_2 \log\left(|D_1 F(\alpha, 0)|^2 (1 - |\alpha|^2)\right). \quad (25)$$

Suppose now $\alpha = 0$ and that F_o is an extreme point of the closed unit ball of $H^{\infty}(D)$ Then (24) shows the left side of (25) equals $-\infty$, which implies $D_1 F(0,0) = 0$.

Now taking $n \geqq 2$, the above argument shows that if any slice function of F is an extreme point of the closed unit ball of $H^{\infty}(D)$ then the first derivatives of F in complex directions orthogonal to this slice vanish at the origin. Because $\nabla F(0) \neq 0$, there can be at most one such slice.

1. A result of Tomaszewski [135] implies that for every n there exists a positive number $\alpha = \alpha(n)$ and a holomorphic map $F: B_n \to D$ with $F \in \text{Lip}_{\alpha}(B_n)$ and $|F| = 1$ on a subset of ∂B_n having positive σ_n-measure. Because there exist Bloch functions that fail to have a finite limit along any curve in D tending to a point of ∂D, such an F trivially fails to have the pull-back property.

Section (3.2): Compact Composition Operators

For $\mathbb{D}$ denote the unit disk in the complex plane. A function f holomorphic in $\mathbb{D}$ is said to belong to the Bloch space $\mathfrak{B}$ if

$$sup_{z\in\mathbb{D}}(1 - |z|^2)|f'(z)| < \infty.$$

And to the Bloch space $\mathfrak{B}_0$. If

$$\lim_{|z|\to 1}(1 - |z|^2)|f'(z)| = 0.$$

It is well known that $\mathfrak{B}$ is a Banach space under the norm

$$\|f\|_{\mathfrak{B}} = |f(0)| + sup_{z\in\mathbb{D}}(1 - |z|^2)|f'(z)|.$$

and that $\mathfrak{B}_0$ is a closed subspace of $\mathfrak{B}$. Furthermore, $\mathfrak{B}$ is isometrically isomorphic to the second dual of $\mathfrak{B}_0$ and the inclusion $\mathfrak{B}_0 \subset \mathfrak{B}$ corresponds to the canonical imbedding of $\mathfrak{B}_0$ into $\mathfrak{B}_0$** [138]. It is a simple consequence of the Schwarz-Pick lemma [137] that a holomorphic mapping ϕ of the unit disk into itself induces a bounded composition operator $C_\varphi f \in f \circ \varphi$ $on\, \mathfrak{B}$. Indeed, if $f \in \mathfrak{B}$, then

$$(1 - |z|^2)|(f \circ \varphi)'(z)| = (1 - |z|^2)|f'(\varphi(z))||\varphi'(z)|$$
$$= \frac{1 - |z|^2}{1 - |\varphi(z)|^2}|\varphi'(z)|(1-|\varphi(z)|^2)f'(\varphi(z)). \quad (26)$$

And the Schwarz-Pick lemma guarantees that

$$\frac{1 - |z|^2}{1 - |\varphi(z)|^2}|\varphi'(z)| \le 1 \quad (27)$$

Since the identity function $f(z) = z$ belongs to $\mathfrak{B}_0$, it is clear that $\phi \in \mathfrak{B}_0$ if C_ϕ maps $\mathfrak{B}_0$ into itself. Conversely, if $\phi \in \mathfrak{B}_0$ and $f \in \mathfrak{B}_0$, it follows from (26) and (27) that $f \circ \phi \in \mathfrak{B}_0$. Indeed, if $\epsilon > 0$, there exists $\delta > 0$ such that $(1-|z|^2)|f'(z)| < \epsilon$ whenever $|z|^2 >$ l- δ. In particular, $(1-|z|^2)|\,(f \circ \phi)'(z)| < \epsilon$ whenever $|\phi(z)|^2 >$ l$-$ δ. On the other hand, if $|\phi(z)|^2 \le$ l$-$ δ,

$$(1 - |z|^2)|(f \circ \varphi)'(z)| \le \frac{\|f\|_{\mathfrak{B}}}{\delta}(1 - |z|^2)|\varphi'(z)|$$

and the right-hand side tends to 0 as $|z| \to 1$.

The compact composition operators on $\mathfrak{B}_0$ and on $\mathfrak{B}$ will be characterized in terms of the quotient $\frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi'(z)|$, A bounded linear operator $T: X \to Y$ from the Banach space X to the Banach space Y is weakly compact if T takes bounded sets in X into relatively weakly compact sets in Y. Gantmacher's theorem [139] asserts that T is weakly compact if and only if $T^{**}(X^{**}) \subset Y$ where T** denotes the second adjoint of T. This theorem and the characterization of compact operators on $\mathfrak{B}_0$ will be used to show that every weakly compact composition operator on $\mathfrak{B}_0$ is compact.

To certain univalent functions ϕ which map $\mathbb{D}$ into itself. It is known that such functions belong to $\mathfrak{B}_0$ [141]; and it will be clear from that if $\|\phi\|_\infty < 1$, then C_ϕ is compact on $\mathfrak{B}_0$. On the other hand if $\|\phi\|_\infty = 1$ and there is a point of $\mathbb{T} \cap \phi(\mathbb{D}) at\ which\ \phi(\mathbb{D})$ does not have a cusp, then C_ϕ is not compact. However if $\mathbb{T} \cap \phi(\mathbb{D})$ consists of only one point at which $\phi(\mathbb{D})$ has a nontangential cusp, then $C_\phi$is compact on $\mathfrak{B}_0$.

Theorem (3.2.2) Gives a precise description of those ϕ which induce compact composition operators on $\mathfrak{B}_0$. It will be useful first to give a criterion for compactness in $\mathfrak{B}_0$.

Lemma (3.2.1)[19]: A closed set K in $\mathfrak{B}_0$ is compact if and only if it is bounded and satisfies

$$\lim_{|z|\to 1} sup_{f\in K}(1-|z|^2)|f'(z)| = 0 \quad (28)$$

Proof. First suppose that K is compact and let $\epsilon > 0$ choose an ϵ /2-Net $f_1, \quad f_2, \dots, \quad f_n\ in\ K.T$here is an r, .$0 < r < 1$, such that $(1 - |z|^2)|f_1'(z)| < \frac{\epsilon}{2}\ if\ |z| > r, 1 \le i \le n. If\ f \in K, \|f - f_i\|_{\mathfrak{B}} < \epsilon$ /2 for some f_i and so

$$(1 - |z|^2)|f'(z)| \le \|f - f_i\|_{\mathfrak{B}} + (1 - |z|^2)|f_1'(z)| < \epsilon.$$

Whenever $|z| > r$. This establishes (28).

On the other hand if K is a closed bounded set which satisfies (28) and (f_n) is a sequence in K, then by Montel's theorem there is a subsequence (f_{n_k}) which converges uniformly on compact subsets of $\mathbb{D}$ to some holomorohic function f. Then also (f'_{n_k}) converges uniformly to f' on compact subsets of $\mathbb{D}$. By (28), if $\epsilon > 0$ there is an $r, 0 < r < 1$, such that for all $g \in K$, It follows that $(1 - |z|^2)|g'(z)| < \epsilon/2$. if $|z| > r$ .Since (f_{n_k}) converges uniformly to f and (f'_{n_k}) converges uniformly to f' on $|z| \le r, it$ follows that $\lim sup_{k\to\infty}\|f_{n_k} - f\|_{\mathfrak{B}} \le \epsilon$. . Since $\epsilon > 0$, $\lim_{k\to\infty}\|f_{n_k} - f\|_{\mathfrak{B}} = 0$ and so K is compact.

Theorem (3.2.2)[19]: If ϕ is a holomorphic mapping of the unit disk $\mathbb{D}$ into itself , then ϕ induces a compact composition operator on $\mathfrak{B}_0$ if and only if

$$\lim_{|z|\to1} \frac{1 - |z|^2}{1 - |\varphi(z)|^2}|\varphi'(z)| = 0. \tag{29}$$

Proof. It follows from Lemma (3.2.1) that C_φ is compact on $\mathfrak{B}_0$ if and only if

$$\lim_{|z|\to1} sup_{\|f\|_{\mathfrak{B}}\le1}(1 - |z|^2)|(f \circ \varphi)'(z)| = 0.$$

But

$$(1 - |z|^2)|(f \circ \varphi)'(z)| = \frac{1 - |z|^2}{1 - |\varphi(z)|^2}|\varphi'(z)|(1 - |\varphi(z)|^2)|f'\varphi(z)|,$$

And

$$sup_{\|f\|_{\mathfrak{B}}\le1}(1 - |\omega|^2)|f'(\omega)| = 1.$$

$for\ each\ \omega \in \mathbb{D}$ .The theorem follows.

It should be remarked that (29) implies $\phi \in \mathfrak{B}_0$. A similar condition characterizes compact composition operators on $\mathfrak{B}$.

Theorem (3.2.3)[19]: If ϕ is a holomorphic mapping of the unit disk $\mathbb{D}$ into itself, then ϕ induces a compact composition operator on on $\mathfrak{B}$ if and only if for every $\epsilon > 0$ there exists r, $0 < r < 1$, such that

$$\frac{1 - |z|^2}{1 - |\varphi(z)|^2}|\varphi' z| < \epsilon. \tag{30}$$

Whenever $|\varphi(z)| > r$,

Proof. First assume that (30) holds. In order to prove that C_ϕ is compact on $\mathfrak{B}$ it is enough to show that if (f_n) is a bounded sequence in $\mathfrak{B}$ which converges to 0 uniformly on compact subsets of $\mathbb{D}$, then $\|f_n \circ \varphi\|_{\mathfrak{B}} \to 0$. . Let $M = sup_n\|f_n\|_{\mathfrak{B}}$. Given $\epsilon > 0$, there $r, 0 < r < 1$, such that $\frac{1-|z|^2}{1-|\varphi(z)|^2}|\phi' z| < \frac{\epsilon}{2M}$ IF $|\phi(z)| > r. Since$

$$(1 - |z|^2)|(f_n \circ \phi)'(z)| = \frac{1 - |z|^2}{1 - |\phi(z)|^2}|\phi'(z)|(1 - |\phi(z)|^2)\left|f_n'(\phi(z))\right|$$

$$\le M\frac{1 - |z|^2}{1 - |\varphi(z)|^2}|\varphi'(z)|.$$

It follows that $(1 - |z|^2)\ (f_n \circ \phi)'(z) < \frac{\epsilon}{2}\ if\ |\phi(z)| > r$.

On the other hand. $f_n \circ \phi(0) \to 0$ And $(1 - |w|^2)\ |f_n'(\omega)| \to 0$ uniformly for $|\omega| \le r$. Since

$$(1 - |z|^2)\ (f_n \circ \phi)'(z) \le (1 - |\phi(z)|^2)\left|f_n'(\phi(z))\right|.$$

It follows that for large enough $n, |f_n \circ \phi(0)| < \frac{\epsilon}{2}$ and $(1 - |z|^2)\ (f_n \circ \phi)'(z) < \frac{\epsilon}{2}$ if $|\phi(z)| \le r$ .Hence $\|f_n \circ \varphi\|_{\mathfrak{B}} < \epsilon$ for large n.

Now assume that (30) fails, Then there exists a subsequence $(z_n)\ in\ \mathbb{D}$ in and $\epsilon > 0$ an such that $|z_n| \to 1$ and $\frac{1-|z_n|^2}{1-|\phi(z_n)|^2}|\phi'(z)| > \epsilon$ for all n. Passing to a subsequence if necessary it may be assumed that $\omega_n = \phi(z_n) \to \omega_0 \in \mathbb{T}$, Let $f_n(z) = log\frac{1}{1-\bar{\omega}_n z}$. Then (f_n) converges to f_{0n}uniformly on compact subsets of $\mathbb{D}$. On other hand,

$$\|C_\phi f_n - C_\phi f_0\|_{\mathfrak{B}} \geq (1-|z_n|^2)\left|(C_\phi f_n)'(z_n) - (C_\phi f_0)'(z_n)\right|$$
$$= (1-|z_n|^2)|\phi'(z_n)|\left|\frac{\bar{w}_n}{1-|w_n|^2} - \frac{\bar{w}_0}{1-\bar{w}_0 w_n}\right|$$
$$= \frac{(1-|z_n|^2)}{1-|w_n|^2}|\phi'(z_n)|\left|\frac{\bar{w}_n - \bar{w}_0}{1-\bar{w}_0 w_n}\right| > \epsilon.$$

For all n, so $C_\phi f_n$ does not converge to $C_\phi f_n$in norm. Hence C_ϕ is not compact.

It is important to note that although (29) implies (30), since in this case C_ϕ on $\mathfrak{B}$ is the second adjoint of C_ϕ on $\mathfrak{B}_0$, the two conditions are not equivalent, Condition (29) implies that $\phi \in \mathfrak{B}_0$, while there certainly exist functions $\phi \notin \mathfrak{B}_0$ which satisfy (30). Indeed, any ϕ for which $\|\phi\|_\infty < 1$ satisfies (30) trivially.

A sequence (ω_n) in $\mathbb{D}$ is said to be η -seperated if $\rho(\omega_n, \omega_m) = \left|\frac{\omega_m - \omega_n}{1-\bar{\omega}_m \omega_n}\right| > \eta$ whenever $m \neq n$. Thus an η -seperated sequence consists of points which are uniformly far apart in the pseudohyperbolic metric on $\mathbb{D}$ or Equivalently, the hyperbolic balls $\Delta(\omega_n, r) = \{z | \rho(z, \omega_n) < r\}$ are pairwise disjoint for some $r > 0$. Evidently any sequence $(\omega_n) in\ \mathbb{D}$ in which satisfies $|\omega_n| \to 1$ possesses an η -seperated subsequence for any $\eta > 0$. In particular, if the sequence (ω_n) in the proof of Theorem (3.2.3) is η -seperated, then the calculation in the proof shows that $\|C_\phi f_m - C_\phi f_n\| > \epsilon\eta$ whenever $m \neq n,$ so $(C_\phi f_n)$ has no norm convergent subsequences.

Another property of separated sequences is contained in the next proposition. This proposition is related to some interpolation results of Rochberg [142], [143]. Since the method of proof is precisely the same as Rochberge's, a proof will only be sketched.

Proposition (3.2.4)[19]: There is an absolute constant $R > 0$ such that if (ω_n) is R-separated, then for every bounded sequence (λ_n) there is an $f \in \mathfrak{B}$ such that $(1-|\omega_n|^2)f'(\omega_n) = \lambda_n$ for all n.

The idea of the proof is to consider two operators $S: \mathfrak{B} \to l^\infty$ given by

$$S(f)_n = (1-|w_n|^2)f'(w_n)$$

And $T: l^\infty \to \mathfrak{B}$ given by

$$T(\lambda)(z) = \sum_{n=1}^{\infty} \lambda_n \frac{1}{3\bar{w}_n}\frac{(1-|w_n|^2)^3}{(1-\bar{w}_n z)^3}.$$

Where $\lambda = (\lambda_n) \in l^\infty$. The proposition will follow if it can be shown that,

$$\|I - ST\| < 1,$$

for then ST will be invertible and so S will be onto the symbol C will denote a constant whose value changes from place to place but does not depend on R. Now

$$(ST - I)(\lambda)_n = (1-|w_n|^2)\sum_{m\neq n} \lambda_m \frac{(1-|w_n|^2)^3}{(1-\bar{w}_m w_n)^4}.$$

And so it will be enough to estimate

$$\text{sup}_\text{n}(1-|w_n|)\sum_{\text{m}\neq\text{n}} \frac{(1-|w_m|^2)^3}{(1-\bar{w}_m w_n)^4}$$

If $R > \frac{i}{2}$, say, then there is a fixed $\delta > 0$ such that the Euclidean disk D_m of center ω_m and radius $\delta(1-|\omega_m|^2)$.

Is containedin the hyperbolic disk and is disjoint from the hyperbolic disk $\Delta_m = \Delta(\omega_m, R)$ and is disjoint from the hyperbolic disks Δ_n for $n \neq m$. . Since $|1-\bar{z}\,\omega_n|^{-4}$ is subharmonic and the radius of $D_m$is comparable to 1-$|\omega_m|^2$.

$$\frac{(1-|w_n|^2)^3}{(1-\bar{w}_m w_n)^4} \le C \iint_{D_m} \frac{1-|w_m|^2}{(1-\bar{z}\,w_n)^4} dx\,dy;$$

And since $|1-\bar{w}_n z|$ dominates $1-|w_m|^2$ on D_m , it follows that

$$\frac{(1-|w_m|^2)^3}{(1-\bar{w}_m w_n)^4} \le C \iint_{D_m} \frac{1}{(1-\bar{w}_n z)^3} dx\,dy;$$

And hence

$$\sup_n(1-|w_n|) \sum_{m\neq n} \frac{(1-|w_m|^2)^3}{(1-\bar{w}_m w_n)^4} \le C \iint_{\cup_{m\neq n} D_m} \frac{1-|w_n|^2}{(1-\bar{w}_n z)^3} dx\,dy;$$

$$\le \iint_{\mathbb{D}/\Delta n} \frac{1-|w_n|^2}{(1-\bar{w}_n z)^3} dx\,dy;$$

The change of variables $z = \frac{w_n+\zeta}{1+\bar{w}_n\zeta}$. turns this into

$$\sup_n(1-|w_n|) \sum_{m\neq n} \frac{(1-|w_m|^2)^3}{(1-\bar{w}_m w_n)^4} \le C \iint_{|\zeta|>R} \frac{1}{|1+\bar{w}_n\zeta|} d\xi\,d\eta;$$

And the last integral can be made arbitrarily small uniformly in n if R is shosen close enough to 1. This provides the desired estimate.

Since every sequence (w_n) with $|w_n| \to 1$ contains an R-seperated subsequence $\left(w_{n_k}\right)$, it follows that there is an $f \in \mathfrak{B}$ such that $\left(1-\left|w_{n_k}\right|^2\right) f'\left(w_{n_k}\right) = 1$ for all k, This will be used in the proof of the next theorem.

Theorem (3.2.5)[19]: Every weakly compact composition operator C_ϕ on $\mathfrak{B}_0$ is compact.

Proof. The composition operator $C_\phi: \mathfrak{B}_0 \to \mathfrak{B}_0$ is compact if and only if

$$\lim_{|z|\to 1} \frac{1-|z|^2}{1-|\phi(z)|^2}\, |\phi'(z)| = 0.$$

And, according to Gantmacher's theorem, weakly compact if and only if $C_\phi f \in \mathfrak{B}_0$for every $f \in \mathfrak{B}$. If C_ϕ is not compact, there is an $\epsilon > 0$ and a sequence $(z_n), |z_n| \to 1$, such that

$$\frac{1-\ |z_n|^2}{1-|\phi(z_n)|^2}\, |\phi'(z_n)| \ge \epsilon,$$

For all n, since $\phi \in \mathfrak{B}_0, |\phi(z_n)| \to 1,$, and by passing to a sub-sequence it may be assumed that $(\phi(z_n))$ is R-seperated. If $f \in \mathfrak{B}.$,

$$(1-\ |z_n|^2)\left|\left(C_\phi f\right)'(z_n)\right| = \frac{1-\ |z_n|^2}{1-|\phi(z_n)|^2}\, |\phi'(z_n)|\, (1-|\phi(z_n)|^2)\left|f'\left(\phi(z_n)\right)\right|$$

$$\ge \epsilon\,(1-|\phi(z_n)|^2)|f'(\phi(z_n))|.$$

Since $(\phi(z_n))$ is R-seperated, an application of Proposition (3.2.4) produces an $f \in \mathfrak{B}$ such that $(1-|\phi(z_n)|^2)\left|\left(C_\phi f\right)'(z_n)\right| = 1,$, for all $n,$ Since $(1-$

$|z_n|^2)\left|\left(C_\phi f\right)'(z_n)\right| \geq \epsilon$ *and* $|z_n| \to 1,\ C_\phi f \notin \mathfrak{B}_0$ *and* so C_ϕ is not weakly compact.

A slight refinement of these arguments will show that a non compact composition operator on $\mathfrak{B}_0$ must be an isomorphism on a subspace isomorphic to the sequence space c_0. This is not surprising since $\mathfrak{B}_0$ is known to be isomorphic to c_0 .

As remarked any holomorphic mapping ϕ of the unit disk into itself satisfying $\|\phi\|_\infty < 1$ induces a compact composition operator on $\mathfrak{B}$ and also on $\mathfrak{B}_0$ if $\phi \in \mathfrak{B}_0$. On the other hand it is easy to see that if ϕ has a finite angular derivative at some point of $\mathbb{T}$, then C_ϕ cannot be compact. Indeed, ϕ has an angular derivative at $\zeta \in \mathbb{T}$ if the non tangential limit $\omega = f(\zeta) \in \mathbb{T}$ exists and if the quotient $\frac{f(z)-f(\zeta)}{z-\zeta}$ converges to some complex number μ *as* $z \to \zeta$ nontangentially. It is known that $\mu \neq 0$, and the Julia-Carathe ́odory lemma shows that $\frac{1-|z|^2}{1-|\phi(z)|^2}|\phi'(z_n)|$ converges to $\zeta\bar{\omega}\mu \neq 0$ non tangentially. Applying Theorem (3.2.2) or (3.2.3) as appropriate shows that C_ϕ is not compact.

It turns out, however, that ϕ can push the disk much more sharply into itself and still induce a non-compact composition operator. The easiest way to see this is to consider the functions $\phi_{\lambda,\alpha}(z) = 1 - \lambda(1-z)^\alpha, 0 < \lambda, \alpha < 1.$ *It is* easy to see that $\phi_{\lambda,\alpha} \in \mathfrak{B}_0$ and that $\phi_{\lambda,\alpha}$ maps $\mathbb{D}$ onto a region which behaves at 1 like a Stolz angle of opening $\pi\alpha.$ If C_ϕ were compact on $\mathfrak{B}_0$, composition with $\log\frac{1}{1-z}$ would yield a function in $\mathfrak{B}_0$ *, but* an easy calculation shows that this is not so. This leads to the consideration of cusps,

Throughout the reminder of ϕ will denote a univalent mapping of the unit disk $\mathbb{D}$ into itself with image $G = \phi(\mathbb{D})$.

For simplicity it will be assumed that $\bar{G}\cap\mathbb{T} = \{1\}$.

The region G is said to have a cusp at 1 [141] if

$$dist\ (\omega, \partial G) = o(\ |1-\omega|) \tag{31}$$

As $\omega \to 1$ *in* G. Otherwise G does not have a cusp at 1. The cusp is said to be non tangential if G lies inside a Stolz angle near l, i.e., there exist $r, M > 0$ such that

$$|1-\omega| \leq M(|1-\omega|^2) \tag{32}$$

If $|1-\omega| < r, \omega \in G.$ Finally the following geometric property of the conformal mapping ϕ will be needed. If ϕ is a conformal mapping with domain $\mathbb{D}$.

$$\frac{1}{4}(1-|z|^2)|\phi'(z)| \leq dist\ (\phi(z), \partial G) \leq (1- \ |z|^2)|\phi'(z)|. \tag{33}$$

This inequality, known as the Koeba distortion theorem, is an elementary consequence of the Schwarz lemma and Koeb;s one-quarter theorem [140]. It can be used to prove that bounded univalent functions lie in $\mathfrak{B}_0$. Indeed, if $\phi \notin \mathfrak{B}_0$, there is a $\delta > 0$ and a sequence (z_n) *in* $\mathbb{D}$ in with $|z_n| \to 1$ and $(1-|z_n|)$ $|\phi'(z_n)| > \delta$ for all n. Hence dist $(\phi(z_n), \partial G) > \frac{\delta}{4}$ *so* $\phi(z_n)$ has a cluster point in G, contradicting the fact that is a proper map.

Theorem (3.2.6)[19]: If ϕ is univalent and $G = \phi(\mathbb{D})$ satisfies $\bar{G}\cap\mathbb{T} = \{1\}$ but does not have a cusp at 1, then C_ϕ is not compact on $\mathfrak{B}_0$.

Proof. Since G does not have a cusp at 1, (31) fails. Hence there is a $\delta > 0$ and a sequence (z_n) in $\mathbb{D}$ such that $|z_n| \to 1$, but

$$dist(\phi(z_n), \partial G) \geq \delta|1-\phi(z_n)|$$

Hence

$$\delta(1-|\phi(z_n)|^2) \leq 2\delta\big(1-\phi(z_n)\big)$$
$$\leq 2dist(\phi(z_n),\partial G) \leq 2(1-|z_n|^2)|\phi'(z_n)|$$

So

$$\frac{(1-|z_n|^2)}{|1-\phi(z_n)|^2}|\phi'(z_n)| \geq \frac{\delta}{2},$$

Since $|z_n| \to 1$, Theorem (3.2.2) shows that C_ϕ is not compact.

The next theorem shows how to produce compact c omposition operators on from univalent mapping ϕ with $\|\phi\|_\infty = 1$.

Theorem (3.2.7)[19]: If ϕ is univalent and if G has a nontangential cusp at 1 and touches the unit circle at no other point, then C_ϕ is a compact operator on $\mathfrak{B}_0$.

Proof. As . $\phi \in \mathfrak{B}_0$, it will be enough to show that

$$\lim_{|z|\to 1}\frac{1-|z|^2}{1-|\phi(z)|^2}\,|\phi'(z)| = 0.$$

Since the theorem will then follow from Theorem (3.2.2). Since G has a non-tangential cusp at 1, there exist r, M > 0 such that

$$|1-\omega| \leq M(|1-\omega|^2)$$

If $|1-\omega| < r, \omega \in G$, Let $\epsilon > 0$. Since G has a cusp at 1, there is a $\delta > 0$ such that

$$dist(\omega,\partial G) \leq \frac{\epsilon}{4M}|1-\omega|.$$

If $|1-\omega| < \delta\ , \omega \in G$. Let $\eta = \min(\delta, r)\ if\ |1-\phi(z)| < \eta$. It follows that

$$\frac{1-|z|^2}{1-|\phi(z)|^2}\ |\phi'(z)| \leq \frac{4dist(\phi(z),\partial G)}{1-|\phi(z)|^2} \leq \frac{\epsilon}{M}\frac{|1-\phi(z)|}{1-|\phi(z)|^2} \leq \epsilon.$$

On the other hand if $|1-\phi(z)| \geq \eta$, there is a constant $N > 0$ such that $|\phi'(z)| \leq N$ by the smoothness assumption and a $\rho > 0$ such that $1-|\phi(z)|^2 \geq \rho$. In this case

$$\frac{1-|z|^2}{1-|\phi(z)|^2}\ |\phi'(z)| \leq \frac{N}{\rho}(1-|z|^2).$$

And this is less than $\epsilon\ if\ |z|^2 > 1-\frac{\rho\epsilon}{N}$. That complete the proof.

It is possible to describe region G with tangential cusp such that the Riemann mapping $\phi: \mathbb{D} \to G$ admits either possibility. Indeed, suppose that $h(\theta) and\ , k(\theta)$ are positive continuous functions on $[0,\theta_0]$ with $h(\theta) = o(\theta)$ and $k(\theta) = o(\theta)$. Let

$$G = \{re^{i\theta} | 0 < \theta < \theta_0, h(\theta) < 1-r < h(\theta)+k(\theta)\}$$

Then clearly G has a tangential cusp at 1. If $k(\theta) = o\big(h(\theta)\big)$, then for $\omega = re^{i\theta} = \phi(z)$,

$$(1-|z|^2)\,|\phi'(z)| \leq dist(\omega,\partial G) \leq k(\theta).$$

and

$$1-|\omega|^2 \geq 1-|\omega| > h(\theta).$$

So $\frac{1-|z|^2}{1-|\phi(z)|^2}\ |\phi'(z)| \to 0\ as\ |\phi(z)| \to 1$, Since ϕ is a univalent , the argument of Theorem (3.2.7) shows that C_ϕ is compact.

On the other hand if $k(\theta) = 2h(\theta)$ and $\omega(\theta) = \big(1-2h(\theta)\big)e^{i\theta} = \phi(z(\theta))$, then evidently $dist(\omega(\theta),\partial G) > ch(\theta)$. For some constant c, and since $\big(1-|z|^2)\big)|\phi'(z)| \geq dist(\phi(z),\partial G)$, it follows that $\frac{1-|z(\theta)|^2}{1-|\omega(\theta)|^2}\ \big|\phi'\big(z(\theta)\big)\big| \geq \frac{c}{4}$. and so C_ϕ is not compact.

Although the condition of Theorem (3.2.2) and (3.2.3) provide succinct analytic conditions on a function ϕ in order that it induce compact composition operators, it is desirable to have more geometric condition. For example, it is clear from that if ϕ is a conformal mapping which has only a finite number of nontangental cusps on the unit circle $\mathbb{T}$ and no other points of contact, then C_φ will be compact on $\mathfrak{B}_0$. This raises the question of whether or not there is a $\phi \in \mathfrak{B}_0$ such that $\phi(\mathbb{D}) \cap \mathbb{T}$ is infinite and C_ϕ is compact on $\mathfrak{B}_0$. In this regard, it is known that if ϕ has nontangantial limit of modulus one on a set of positivemeasure, then ϕ has an angular derivative at some point and so C_ϕ is not compact [144]. Further information about compact operators considered from a geometric point of view, especially on H^2, can be found in [144] and [145].

Finally, if $\phi \in \mathfrak{B}_0$ and C_ϕ is compact, then $\operatorname{Log} \frac{1}{1-\bar{\omega}\phi(z)} \in \mathfrak{B}_0$ for all $\omega \in \mathbb{T}$. Is the converse of this true?

Section (3.3): The Essential Norm of a Composition Operator

For $\mathbb{D}$ denote the unit disk in the complex plane. A function f analytic on the unit disk is said to belong to the Bloch space $\mathfrak{B}$ if

$$sup_{\mathbb{D}}(1 - |z|^2)|f'(z)| < \infty.$$

and to the Little Bloch space $\mathfrak{B}_0$ if

$$\lim_{|z|\to 1^-}(1 - |z|^2)|f'(z)| = 0.$$

It is well known and easy to prove that $\mathfrak{B}$ is a Banach space under the norm

$$\|f\| = |f(0)| + sup_{\mathbb{D}}(1 - |z|^2)|f'(z)|.$$

And that $\mathfrak{B}_0$ is a closed subspace of $\mathfrak{B}$.

If φ is an analytic function on $\mathbb{D}$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$, then the equation $C_\varphi f \in f \circ \varphi$ defines a composition operator on C_φ the space of all holomorphic functions on $\mathbb{D}$. The Pick-Schwarz Lemma (see [12]) for instances asserts that

$$\frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi'(z)| \le 1. \tag{34}$$

As noticed in [19] this and the chain rule give an easy proof of the fact that C_φ acts boundedly on the Bloch space. In fact we have

$$\begin{aligned}(1 - |z|^2)|(f \circ \varphi)'(z)| &= (1 - |z|^2)\left|f'\big(\varphi(z)\big)\right||\varphi'(z)| \\ &= \frac{1 - |z|^2}{1 - |\varphi(z)|^2}|\varphi'(z)|(1-|\varphi(z)|^2)f'\big(\varphi(z)\big). \\ &\le sup_{\mathbb{D}}(1-|\varphi(z)|^2)|f'\big(\varphi(z)\big)| \\ &= sup_{\varphi(\mathbb{D})}(1-|\omega|^2)|f'(\omega)| \\ &\le sup_{\mathbb{D}}(1-|z|^2)|f'(z)|\end{aligned}$$

In addition, if C_φ acts boundedly on $\mathfrak{B}_0$ then φ must belong to $\mathfrak{B}_0$. This follows from the fact that $C_\varphi z = \varphi$. Conversely, if $\varphi \in \mathfrak{B}_0$, then from the estimates above it is easy to show that φ induces a continuous operator on $\mathfrak{B}_0$ (see [19]). The main goal is to compute the essential norm of $C_\varphi$in terms of an asymptotic bound involving the quantity.

We recall that the essential norm of a continuous linear operator T is the distance from T to the compact operators, that is,

$$\frac{1 - |z|^2}{1 - |\varphi(z)|^2}|\varphi'(z)|.$$

We recall that the essential norm of a continuous linear operator is the distance from T to the compact operators, that is

$$\|T\|e = inf\{\|T - K\| : K \text{ is compact}\}.$$

Notice that $\|T\|_e$= 0 if and only if T is compact, so that estimates on $\|T\|_e$lead to conditions for T to be compact. Thus we will obtain a different proof of a recent result of Madigan and Matheson [19] in which they characterize those φ which induces compact composition operators on $\mathfrak{B}$ and $\mathfrak{B}_0$. The fundamental ideas of the proof are those used by J.H. Shapiro [153] to obtain the essential norm of a composition operator on Hilbert spaces of analytic functions (Hardy and weighted Bergman spaces) in terms of natural counting functions associated with φ. However, since neither $\mathfrak{B}$ and $\mathfrak{B}_0$ are Hilbert spaces our method differs in some interesting details from those of Shapiro.

We want to say a word about the well-known heuris-tic principle which states that if a “big-oh” condition describes a class of bounded operators, then the corresponding “Little-oh” condition picks out the subclass of compact operators. An excellent example of this principle in action can be seen in J.H. Shapiro [153] mentioned above. The “big-oh” condition on Bloch spaces is given by (34). Madigan and Mathe-son were able to prove the “Little-oh” condition, that is, that a composition operator C_φ on $\mathfrak{B}_0$ is compact if and only if

$$\lim_{|z|\to 1^-} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\varphi'(z)| = 0$$

They also obtained (with a different proof) that C_φ is compact on $\mathfrak{B}$ if and only if for every ε > 0 there exists r, 0 < r < 1, such that

$$sup_{|\varphi(z)|>r} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\varphi' z| < \varepsilon.$$

As we will see later the conditions of compactness on $\mathfrak{B}$ and $\mathfrak{B}_0$ are actually the same. In fact, the essential norm of a composition operator is indepen-dent of the underlying space $\mathfrak{B}$ or $\mathfrak{B}_0$. This should not cause any surprise. The fact that $\mathfrak{B}$ is isometrically isomorphic to the second dual of $\mathfrak{B}_0$ and the inclusion $\mathfrak{B}_0 \subset \mathfrak{B}$ corresponds to the canonical imbedding of $\mathfrak{B}_0$ into ${\mathfrak{B}_0}^{**}$ (see [138]) does not affect the computation of the essential norm. This is exactly what happens if we consider a bounded diagonal operator defined by a bounded sequence $\{a_n\}$ on the sequence spaces l^∞ and c_0, respectively.

Then its essential norm equals $lim\ sup\ a_n$ and this quantity is independent of the underlying space. In fact the proof of the main result is done simultaneously for both $\mathfrak{B}$ and $\mathfrak{B}_0$.

Main Theorem (3.3.1). Suppose that C_φ defines a continuous operator on $\mathfrak{B}$(or on $\mathfrak{B}_0$). Then

$$\left\|C_\varphi\right\|_e = \lim_{s\to 1^-} sup_{|\varphi(z)|>r} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\varphi'(z)|. \tag{35}$$

In particular, $C_\varphi\ is$ compact on $\mathfrak{B}$ (or $\mathfrak{B}_0$). if and only if

$$\lim_{s\to 1^-} sup_{|\varphi(z)|>s} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\varphi'(z)| = 0.$$

It is understood that if {z : | $\varphi(z)$| > s} is the empty set for some 0 < s < 1 the supremum equals zero. This happens when $\varphi(\mathbb{D})$is a relatively compact subset of $\mathbb{D}$ and in this case it is easy to show that $C_\varphi$is a compact operator.

If φ has an angular derivative at a point $\xi \in \partial\mathbb{D}$, then we can apply the Julia Carath´eodory Theorem (see [144]) and the Pick-Schwarz
Lemma to obtain

$$1 = \lim_{z\to\xi} inf \frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi'(z)| \le \lim_{s\to 1^-} sup_{|\varphi(z)|>s}\frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi'(z)| \le 1.$$

Thus, as an immediate consequence of Theorem (3.3.1) we have $\|C_\varphi\|_e = 1$ whenever φ has a finite angular derivative.

Before proving Theorem (3.3.1) let us show that for the Little Bloch space B_0 there is an equivalent formula in terms of another quantity. This a simple consequence of the following proposition:

Proposition (3.3.2)[146]: Suppose that C_{φ} defines a continuous operator on $\mathfrak{B}_0$. Then

$$\lim_{s\to 1^-} sup_{|\varphi(z)|>s}\frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi'(z)| = \lim_{|z|\to 1^-} sup \frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi'(z)|. \qquad (36)$$

Proof. As remarked in the introduction the fact that C_φ acts boundedly on $\mathfrak{B}_0$ implies that $\varphi \in \mathfrak{B}_0$. If $\varphi(\mathbb{D})$ is a relatively compact subset of $\mathbb{D}$, then both limits in (35) are zero and coincide. So we may suppose that $\varphi(\mathbb{D})$ is not a relatively compact subset of $\mathbb{D}$. Let $0 < s_n < 1$ be any increasing sequence tending to 1. We set $t_n = inf\{t : |\varphi(z)| > s_n$ for some z with $|z| > t\}$. By continuity $\{t_n\}$ also tends to 1. Since $\{z : |z| > t_n\} = \{z : |\varphi(z)| > s_n$ and $|z| > t_n\} \cup \{z : |\varphi(z)| \le s_n$ and $|z| > t_n\}$ we find that the left hand side of (35) is less than or equal to the right hand side of (35). On the other hand, we can always find a sequence $\{z_n\}$ for which

$$\lim_{n\to\infty}\frac{1-|z_n|^2}{1-|\varphi(z_n)|^2}|\varphi' z_n| = \lim_{s\to 1} sup_{|z|>s}\frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi' z|$$

$$= \lim_{|z|\to 1^-} sup \frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi' z|. \qquad (37)$$

Then either there is a subsequence $\{z_{n_k}\}$ such that $\{|\varphi(z_{n_k})|\} \to 1$ as $k \to \infty$, or for every positive integer n we have $|\varphi(z_{n_k})| \le s_0$ for some $0 < s_0 < 1$.

Clearly, in the former case both limits in (35) coincide. For the latter case we find that the limit in (36) is zero because $\varphi \in \mathfrak{B}_0$. Since this limit is greater than or equal to the limit on the left hand side of (35), we find that they are the same again. The proof is now finished. The lower estimate. First we show that:

$$\|C_\varphi\|_e \ge \lim_{s\to 1^-} sup_{|\varphi(z)|\ge s}\frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi' z|. \qquad (38)$$

Instead of the reproducing kernels used by Shapiro for the Hardy and Berg-man spaces we will use the sequence $\{z^n\}_{n\ge 2}$. This sequence converges uniformly on compact subsets of the unit disk. An elementary computation shows that

$$\|z^n\| = \max_{\mathbb{D}} (1 - |z|^2)|nz^{n-1}| = \frac{2n}{n+1}\left(\frac{n-1}{n+1}\right)^{(n-1)/2}.$$

Observe that for each $n \ge 2$ the above maximum is attained at any point on the circle centered at the origin and of radius $r_n = \left(\frac{(n-1)}{n+1}\right)^{1/2}$. These maxima form a decreasing sequence which tends to $2/e$. Therefore, the sequence $\{z_n\}_{n\ge 2}$ is bounded away from zero. Now we consider the normalized sequence $\left\{f_n = \frac{z^n}{\|z^n\|}\right\}$ which also tends to zero uniformly on compact subsets of the unit disk. For each $n \ge 2$ we define the closed annulus $A_n = \{z \in \mathbb{D} : r_n \le |z| \le r_n + 1\}$ and compute

$$\min_{A_n} (1 - |z|^2)|f_n'(z)| = (1 - r_{n+1}^2)|f_n'(r_{n+1})|$$
$$= \left(\frac{n + 1}{n + 2}\right)\left(\frac{n^2 + n}{n^2 + n - 2}\right)^{(n-1)/2}. \quad (38)$$
Observe that these minima tend to 1 as $n \to \infty$ and for each $n \geq 2$ the minimum above is attained at any point of the circle centered at the origin and of radius r_{n+1}. For the moment fix any compact operator K on $\mathcal{B}_0$ or $\mathcal{B}$. The uniform convergence on compact subsets of the sequence $\{f_n\}$ to zero and the compactness of K imply that $\|Kf_n\| \to 0$. It is easy to show that if a bounded sequence that is contained in $\mathcal{B}_0$ converges uniformly on compact subsets of the unit disk, then it also converges weakly to zero in $\mathcal{B}_0$ as well as in $\mathcal{B}$. Thus
$$\|C_\varphi - K\|k \geq \limsup_n \|(C_\varphi - K)f_n\|$$
$$\geq \limsup_n (\|C_\varphi f_n\| - \|Kf_n\|)$$
$$= \limsup_n \|C_\varphi f_n\|.$$
Upon taking the infimum of both sides of this inequality over all compact operators K, we obtain the lower estimate:
$$\|C_\varphi\|_e \geq \limsup_n \|C_\varphi f_n\|$$
$$= \limsup_n \sup_{\mathbb{D}} (1 - |z|^2)|f_n'(\varphi(z))||\varphi'(z)|$$
$$= \limsup_n \sup_{\mathbb{D}} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\varphi'(z)|(1 - |\varphi(z)|^2)|f_n'(\varphi(z))|. \quad (39)$$
Now (39) is greater than or equal to
$$\limsup_n \sup_{\varphi(z)\in A_n} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\varphi'(z)|(1 - |\varphi(z)|^2)|f_n'(\varphi(z))| \quad (40)$$
and (40) is greater than or equal to
$$\limsup_n \sup_{\varphi(z)\in A_n} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\varphi'(z)| \sup_{\varphi(z)\in A_n} (1 - |\varphi(z)|^2)|f_n'(\varphi(z))|. \quad (41)$$
If $\varphi(\mathbb{D})$ is a relatively compact subset of $\mathbb{D}$ both sides of (37) are zero and there is nothing to prove. Otherwise we find that $\min_{\varphi(z)\in A_n} (1 - |\varphi(z)|^2)|f_n'(\varphi(z))|. = \sup_{\varphi(z)\in A_n} (1 - |z|^2)|f_n'(z)|.$ (z)| because the minimum in (38) is attained at any point on the circle centered at the origin and of radius r_{n+1}. Since these minima tend to 1 as $n \to \infty$, it follows that (41) is equal to
$$\limsup_n \sup_{\varphi(z)\in A_n} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\varphi'(z)|. \quad (42)$$
Finally, an easy exercise shows that (42) coincides with the right hand side of (37). To obtain the upper estimate in the case of the Hardy and Bergman spaces, Shapiro [153] used the operators P_n which take f to the nth partial sum of its Taylor series. On the Hardy space these operators satisfy: i) Each P_n is compact, ii) $(I - P_n)f$ tends to zero uniformly on compact subsets for any f in the Hardy space, and iii) for each n the norm in the Hardy space of $I - P_n$ equals 1. Although each P_n is also compact in the Bloch space, and $(I - P_n)f$ tends to zero uniformly on compact subsets for each function $f \in \mathcal{B}$, this sequence does not satisfy anything analogous to iii) above. In fact, $\|P_n\| \geq C \log n$ where C is a universal constant (see [138]). Therefore, by the reverse triangle inequality $\|I - P_n\| \geq C \log n -$

1. One of the issues here is that in general it is not easy to compute exactly either the norms of Bloch functions, or the norms of operators defined on Bloch spaces. To obtain the upper estimate we need the operators $K_n, n \geq 2$, which take each function $f(z)$ to $f\left(\frac{n-1}{n} z\right)$. Every operator K_n is compact on $\mathcal{B}$ (or $\mathcal{B}_0$). We also have that $(I - K_n)f$ tends to zero uniformly on compact subsets of the unit disk for every $f \in \mathcal{B}$, and (although we do not know if $\lim_{n\to\infty} \|I - K_n\| = 1$) we have the following proposition, whose proof is delayed. This will be accomplished by applying Proposition (3.3.5). Since each L_n is compact so is $C_\varphi L_n$. Therefore

$$\|C_\varphi\|_e \leq \|C_\varphi - C_\varphi L_n\| = \|C_\varphi (I - L_n)\|.$$

On the other hand, we have

$$\|C_\varphi (I - L_n)\| = \sup_{\|f\|=1} \|C_\varphi (I - L_n) f\|$$

$$= \sup_{\|f\|=1} \sup_{|z|<1} (1 - |z|^2)\big((I - L_n)f'\big)(\varphi(z))|\varphi'(z)|$$

$$= \sup_{\|f\|=1} \sup_{|z|<1} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\varphi'(z)|(1 - |\varphi(z)|^2)\left|\big((I - L_n)f\big)'(\varphi(z))\right| |\varphi'(z)| \quad (43)$$

Now fix $0 < s < 1$.Then the right hand side of (43) is less than or equal to

$$\sup_{\|f\|=1} \sup_{|\varphi(z)|\leq s} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\varphi'(z)|(1 - |\varphi(z)|^2)\left|\big((I - L_n)f\big)'(\varphi(z))\right|$$
$$+ \sup_{\|f\|=1} \sup_{|\varphi(z)|\leq s} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\varphi'(z)|(1 - |\varphi(z)|^2)$$
$$\times \left|\big((I - L_n)f\big)'(\varphi(z))\right|. \quad (44)$$

By applying the Pick-Schwarz Lemma in the first term, and the fact that for f in the unit ball

$$\sup_{|\varphi(z)|>s} (1 - |\varphi(z)|^2)\left|\big((I - L_n)f\big)'(\varphi(z))\right|$$
$$\leq \sup_{|z|<1} (1 - |z|^2)\left|\big((I - L_n)f\big)'(z)\right| \leq \|I - L_n\|$$

to the second term, we find that (44) is less than or equal to

$$\sup_{\|f\|=1} \sup_{|w|\leq s} (1 - |w|^2)\left|\big((I - K_m)f\big)'(w)\right|$$
$$+\|I - L_n\| \sup_{|z|<1} \sup_{|\varphi(z)|\leq s} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\varphi'(z)|. \quad (45)$$

Let us prove that the first term in (45) tends to zero as $n \to \infty$. By the triangle inequality we have that the first term in (45) is less than or equal to

$$\sum_{m\geq n} c_{n,m} \sup_{\|f\|=1} \sup_{|w|\leq s} (1 - |w|^2)\left|\big((I - K_m)f\big)'(w)\right|. \quad (46)$$

By the triangle inequality again we find that $(1 - |w|^2)\left|\big((I - K_m)f\big)'(w)\right|$ is less than or equal to

$$\sup_{\|f\|=1} \sup_{|w|\le s} (1 - |w|^2)\left|f'(w) - f'\left(\left(1 - \frac{1}{m}\right) w\right)\right|$$
$$+\frac{1}{m}\sup_{\|f\|=1} \sup_{|w|\le s} (1 - |w|^2)\left|f'\left(\left(1 - \frac{1}{m}\right) w\right)\right|. \quad (47)$$

By integrating f'' along the radial segment $[(1 - 1/m)w, w]$ it is easy to see that the first term in (47) is less than or equal to

$$\frac{1}{m}\sup_{\|f\|=1} \sup_{|w|\le s} (1 - |w|^2)|w|\left|f''\big(\xi(w)\big)\right|, \quad (48)$$

where $\xi(w)$ belongs to the radial segment $[(1 - 1/m)w, w]$ that is still contained in the closed disk of radius s. The Cauchy inequalities applied to a circle $C(\xi(w))$ centered at $\xi(w)$ and of any fix radius $0 < R < 1 - s$ yields that (48) is less than or equal to

$$\frac{1}{mR}\sup_{\|f\|=1} \sup_{|w|\le s} (1 - |w|^2)|w| \max_{|z|=s+R} |f'(z)|. \quad (49)$$

On the other hand, on the unit ball of $\mathcal{B}$ (or $\mathcal{B}_0$) we have $\max_{|z|=s+R} |f'(z)| \le \frac{1}{1-(s+R)^2}$. So we find that (49) is less than or equal to

$$\frac{1}{mR}\sup_{|w|\le s} (1 - |w|^2)|w|\frac{1}{1 - (s + R)^2} \le \frac{1}{mR}\frac{s}{1 - (s + R)^2}.$$

Since the second term in (47) is less than $1/m$ we find that (47) is $\le C/m$, where C only depends on s. Therefore, we find that (46) is less than or equal to

$$\sum_{m\ge n} c_{n,m}\frac{C}{m} \le \sum_{m\ge n} c_{n,m}\frac{C}{n} = \frac{C}{n}$$

which tends to zero as $n \to \infty$. Hence, letting $n \to \infty$ in (45), applying Proposition (3.3.5) and putting everything together, the following inequality follows

$$\left\|C_\varphi\right\|_e \le \sup_{|\varphi(z)|>s} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\varphi'(z)|.$$

Since s was arbitrary inequality (50) holds.

The proof of Theorem (3.3.1) will be completed once we have proved Proposition (3.3.5). In order to do this we need some basic facts about Bloch spaces. Recall that dual space $\mathcal{B}_0^*$ of $\mathcal{B}_0$ is isomorphic to the space $A^1(\mathbb{D})$ of analytic functions on the unit disk such that

$$\int_{\mathbb{D}} |g(z)|\, dA(z) < \infty$$

where $dA(z)$ is Lebesgue area measure on $\mathbb{D}$, normalized to have total mass 1, that is, $dA(z) = \frac{1}{\pi}\, dxdy = \frac{1}{\pi}\, \rho d\theta d\rho$ for $z = x + iy = \rho ei\theta$. This duality is realized by the integral pairing

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)g(z)\, dA(z)$$

(see [154]). Let $0 < r < 1$ be fixed and let $Kr : \mathcal{B}_0 \to \mathcal{B}_0$ be the operator which assigns to each function f the function $f(rz)$. Now, for any $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{B}_0$ and any $g(z) = \sum_{n=0}^{\infty} b_n z^n \in A^1(\mathbb{D})$ a straightforward computation shows that

$$\langle f(rz), g(z)\rangle = \sum_{n=0}^{\infty} \frac{r^n}{n + 1} a_n \bar{b}_n = \langle f(z), g(rz)\rangle.$$

Thus, the adjoint operator $K_r^* : A^1(\mathbb{D}) \to A^1(\mathbb{D})$ acts in the same way as does K_r. We also have that the Bloch space $\mathcal{B}$ is the dual of $A^1(\mathbb{D})$ under the same integral pairing. Thus in a similar way, it can be shown that the bi-adjoint operator $K_r^{**} : \mathcal{B} \to \mathcal{B}$ of K_r is the operator that assigns to each function $f(z)$ the function $f(rz)$. Thus, we denote K_r^* and K_r^{**} by K_r. With this we may observe that if we have constructed the sequence $\{L_n\}$ required by Proposition (3.3.5) for $\mathcal{B}_0$, then just considering the bi-adjoint sequence the result follows for the Bloch space B. This is trivial because $L_n^{**} = \left(\sum_{m\geq n} c_{n,m} K_n\right)^{**} = \sum_{m\geq n} c_{n,m} K_n$ and $\|(I - L_n)^{**}\| = \|I - L_n\|$.

To prove Proposition (3.3.5) we also need the following proposition about the compact operators K_r.

Proposition (3.3.3)[146]: For any $g \in A^1(\mathbb{D})$ we have $\|K_r g - g\| \to 0$ as $r \to 1^-$.

Proof. Let $\varepsilon > 0$ be fixed. By the continuity of the integral we can find an $s, 1 > s > 0$, such that

$$\int_{|z|>s} |g(z)| dA(z) < \frac{\varepsilon}{3}.$$

Now $rs \to s$ and $1/r \to 1$ as $r \to 1$. Therefore, the change of variables $w = rz$ and the above display show that

$$\int_{|z|>s} |g(rz)| dA(z) = \frac{1}{r} \int_{rs<|w|\leq r} |g(w)|\, dA(w) \leq \frac{1}{r} \int_{rs<|w|} |g(w)|\, dA(w) < \frac{\varepsilon}{3}$$

for r near enough to 1. On the other hand, since $K_r g$ tends to g uniformly on comp act subsets of the unit disk as $r \to 1^-$, we have

$$\max_{|z|\leq s} |g(rz) - g(z)| < \frac{\varepsilon}{3}$$

for r near enough to 1. Thus for r close to 1 we have

$$\begin{aligned}\|g(rz) - g(z)\| &= \int_{|z|\leq s} |g(rz) - g(z)| dA(z) + \int_{|z|>s} |g(rz) - g(z)| dA(z) \\ &< \frac{\varepsilon}{3} s^2 + \int_{|z|>s} |g(z)| dA(z) + \int_{|z|>s} |g(rz)| dA(z) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Since ε was arbitrary, the result follows.

Given two Banach spaces X and Y we denote by $L(X, Y)$ the Banach space of bounded operators from X into Y and by $K(X, Y)$ the Banach space of compact operators from X into Y. We need a theorem of Mazur that asserts that if a sequence in a Banach space converges weakly, then some sequence of convex combinations converges in norm (see [139]). We begin with the following theorem, whose proof was provided by Joel H. Shapiro (alternatively, in the proof of Proposition (3.3.5), we can use Theorem 1 in [149]).

Theorem (3.3.4)[146]: Suppose X and Y are Banach spaces and $\{T_n\}$ is a sequence of compact linear operators from X to . Suppose further that for every $y^* \in Y^*$ and $x^{**} \in X^{**}$ we have: $\langle T_n^* y^*, x^{**}\rangle \to 0$. Then there is a sequence $\{S_n\}$ of convex combinations of the original T_n such that $\|S_n\| \to 0$.

Proof. Let Q denote the cartesian product of the closed unit ball of Y^* and the closed unit ball of X^{**}, where each ball has its respective weak star topology. Thus Q is a compact Hausdorff space. For $T \in K(X, Y)$ the function $\widehat{T^*} : Q \to \mathbb{C}$ defined by:

$$\widehat{T^*}((y^*, x^{**})) = \langle T^* y^*, x^{**}\rangle \left(= x^{**}\left(T^*(y^*)\right)\right) \ (x^{**} \in X^{**} \text{ and } y^* \in Y^*)$$

belongs to $C(Q)$ (see [149]), and the map $T^* \to \widehat{T^*}$ is an isometry taking a certain closed subspace of $K(Y^*, X^*)$ (namely the weak-star continuous compacts) onto a closed subspace of $C(Q)$.

By this correspondence and the Hahn-Banach theorem, $T_n^* \to 0$ weakly in $L(Y^*, X^*)$ if and only if $\widehat{T_n^*}$ tends weakly in $C(Q)$. By the Riesz Representation Theorem and the Lebesgue bounded convergence theorem, a sequence of functions in $C(Q)$ converges weakly to zero if and only if it converges pointwise to zero. But the hypothesis on $\{T_n^*\}$ is just the statement that $\widehat{T_n^*} \to 0$ pointwise on Q. In addition, it follows from the Uniform Boundedness Principle that supn $\|T_n\| < \infty$, hence because $\|T_n\| = \|T_n^*\|$, the sequence $\widehat{T_n^*}$ is also bounded. Thus $\widehat{T_n^*} \to 0$ weakly in $L(Y^*, X^*)$ and so by Mazur's theorem, there is a sequence of convex combinations $\|S_n^*\|$ of the original operators $\{T_n^*\}$, such that $\|S_n^*\| \to 0$. Thus also $\|S_n\| \to 0$, which is the desired result.

To prove Proposition (3.3.5) we will use the fact that $\mathcal{B}_0$ is isomorphic to the sequence space c_0. For completeness we include a proof of this fact. Let us consider the function $\phi(r) = 1 - r^2$ defined on the interval [0, 1] and let $h_\infty(\phi)$ be the Banach space of complex-valued functions, u harmonic in the unit disk with the norm

$$\|u\|_\phi = \sup_{\mathbb{D}} |u(z)|\phi(z)$$

and let $h_0(\phi)$ be the closed subspace of functions u for which $|u(z)|\phi(z) \to 0$ as $|z| \to 1^-$. The space $h_0(\phi)$ is isomorphic to the sequence space c_0 (see [122]). Finally, we denote by $H_0(\phi)$ the closed subspace of those functions in $h_0(\phi)$ that are analytic on the unit disk. Now, observe that $h_0(\phi)$ is self-conjugate, that is, $u \in h_0(\phi)$ if and only if its conjugate $\bar{u} \in h_0(\phi)$. This fact along with the Closed Graph Theorem implies that the Riesz projection $P: h_0(\phi) \to H_0(\phi)$ defined by

$$P\,u = \frac{1}{2}(u + i\bar{u}) + \frac{1}{2}u(0)$$

is bounded. Thus we can express $h_0(\phi) = H_0(\phi) \oplus kerP$. Now, a famous theorem of Pelczy´nski (see [152]) asserts that if F is a complemented subspace of c_0, then either F is isomorphic to c_0 or F is of finite dimension. Since $H_0(\phi)$ is complemented in a space isomorphic to c_0, it follows that $H_0(\phi)$ is isomorphic to c_0. Finally, since $H_0(\phi)$ is isometrically isomorphic to $\mathcal{B}_0$ (consider the map $\to f'$), it follows that $\mathcal{B}_0$ is isomorphic to c_0.

As mentioned, the following argument was indicated by N. J. Kalton. Some parts of this argument already appear in [149] (see also [150] and [148]).

Proposition (3.3.5)[146]: There exists a sequence of convex combinations L_n of K_n ($L_n = \sum_{m\ge n} c_{n,m}K_m$ with $c_{m,n} > 0$ and $\sum_{m\ge n} c_{n,m} = 1$) such that $\lim_{n\to\infty} \|I - L_n\| = 1$.

The upper estimate. The goal now is to show that

$$\|C_\varphi\|_e \le \lim_{s\to 1^-} \sup_{|\varphi(z)|>s} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\varphi'(z)| . \tag{50}$$

Proof. As pointed out before it is enough to prove the result for the Little Bloch space. It will be sufficient to show that for any $\varepsilon > 0$ there exists a convex linear combination L_n of $\{K_m\}_{m\ge n}$ with $\|I - L_n\| < 1 + \varepsilon$. Once this is done the proof can be completed by a simple diagonal argument.

Since $\mathcal{B}_0$ is isomorphic to the sequence space c_0, James's Theorem (see [151]) can be applied to find that there exists a Banach subspace $X_0 \subset c_0$ such that the Banach-Mazur distance from $\mathcal{B}_0$ to X_0 is strictly less than $\sqrt{1 + \varepsilon}$. That is, there is an isomorphism T :

$\mathcal{B}_0 \to X_0$ such that $||T||||T^{-1}|| < \sqrt{1 + \varepsilon}$. We define $T_n = TK_nT^{-1} : X_0 \to X_0$. Upon applying Proposition (3.3.3) we find that

$$\lim_{n\to\infty} ||T_n^* x^* - x^*|| = 0 \tag{51}$$

for each $x^* \in X_0^*$. If P_n is the sequence of coordinate projections on c_0, then we also have

$$\lim_{n\to\infty} ||P_n^* x^* - x^*|| = 0 \tag{52}$$

for each $x^* \in l\,1 = c_0^*$ the dual space of c_0. Now, if J denotes the inclusion from X_0 into c_0, then $JT_n - P_nJ \in K(X_0, c_0)$. Furthermore, by applying (51) and (52) the sequence $\langle (JT_n - P_nJ)^* x^*, y^{**} \rangle$ tends to zero for $y^{**} \in X_0^{**}$ and $x^* \in l^1$. Thus we may apply Theorem (3.3.4) to see that there exist a sequence of convex combinations of $\{JT_n - P_nJ\}$ that tends to zero in norm. This implies that there are sequences $\{T_n^c\}$ and $\{P_n^c\}$ of convex combinations of $\{T_m\}_m \geq n$ and $\{P_m\}_{m \geq n}$, respectively, such that $JT_n^c - P_n^cJ$ tends to zero in norm. Therefore, we have for all sufficiently large n:

$$||I - T_n^c|| = ||J(I - T_n^c)|| \leq ||(I - P_n^c)J|| + ||JT_n^c - P_n^cJ|| \leq \sqrt{1 + \varepsilon},$$

where we have used successively: The fact that $J : X_0 \to c_0$ is the inclusion map, the triangle inequality, and the inequality $||(I - P_n^c)J|| \leq 1$. Finally, if we set $L_n = T^{-1}T_n^cT$, then

$$||I - L_n|| = ||T^{-1}(I - T_n^c)T|| \leq ||T^{-1}||||I - T_n^c||||T|| < 1 + \varepsilon$$

that is what we had to prove. The proof of Proposition (3.3.5), and therefore that of Theorem (3.3.1), is now completed.

Chapter 4
Integral Operator with Norm and Essential Norm of Some Operators

We study acting from α-Bloch spaces to Bloch-type spaces on $\mathbb{B}$. The Dirichlet space to the Bloch-type space on the unit ball in $\mathbb{C}^n$ are calculated here. It is calculated norm of the product of differentiation and composition operators among these spaces on the unit disk.

Section (4.1): Bloch–Type Spaces on the Unit Ball

For $\mathbb{B}$ be the open unit ball in C^{n}, $\mathbb{D}$ the open unit disk in C, $H(\mathbb{B})$ the class of all holomorphic functions on $\mathbb{B}$ and $H^{\infty}(\mathbb{B})$ the class of all bounded holomorphic functions on $\mathbb{B}$ with the norm

$$\|f\|_{\infty} = \sup_{z\in\mathbb{B}}|f(z)|.$$

Let $z = (z_1, \ldots, n_n)$ and $w = (w_1, \ldots, w_n)$ ve points in $\mathrm{C}^{\mathrm{n}}, \langle z, w\rangle = \sum_{k=1}^{n} z_k \overline{w}_k$ and $|z| = \sqrt{(z, z)}$ for $f \in H(\mathbb{B})$ with the taylor expansion

$$F(z) = \sum_{\beta\geq 0} a_\beta z^\beta,$$

Let $\mathfrak{R}F(z) = \sum_{\beta\geq 0}|\beta| a_\beta z^\beta$.

Be the radial derivative of f, where $\beta = (\beta_1, \beta_2 \ldots, \beta_n)$ is multi index, $|\beta| = (\beta_1 + \cdots, +\beta_n)$ and $z^\beta = z_1^{\beta_1},,, z_n^{\beta_n}$. It is well known [77] that

$$\mathfrak{R}F(z) = \sum_{j=1} z_j \frac{\partial f}{\partial zj}(z) = (\nabla f(z), \overline{z})$$

A positive continuous function φ on $[0, 1)$ is called normal [11] if there is $\delta \in [0, 1)$ and a and $b, 0 < a < b$

$$\frac{\phi(r)}{(1-r)^a} \text{ is decreasing on } [\delta, 1) \text{ and } \lim_{r\to 1} \frac{\phi(r)}{(1-r)^a} = 0;$$
$$\frac{\phi(r)}{(1-r)^b} \text{ is increasing on } [\delta, 1) \text{ and } \lim_{r\to 1} \frac{\phi(r)}{(1-r)^b} = \infty;$$

The Bloch-type space, denoted by $\mathfrak{B}_\mu = \mathfrak{B}_\mu(\mathbb{B}))$, consists of all $f \in H(\mathrm{B})$ such that

$$\mathfrak{B}_\mu(f) = sup_{z\in\mathbb{B}}\, \mu(z)|\mathfrak{R}f(z)| < \infty,$$

Where $\mu(z) = \mu(|z|)$ and μ is normal on [0, 1) [26], [60] with the norm

$$\|f\|_{\mathfrak{B}_\mu} = |f(0)| + \mathfrak{B}_\mu(f)$$

the Bloch-type space becomes a Banach space. When $\mu(z) = (1 - |z|^2)^\alpha$, $\alpha \in (0, \infty)$, the space becomes the α-Bloch space $\mathfrak{B}^\alpha$ (see, e.g., [2], [76], [6], [7], [8], [9]). Some other weighted spaces re-lated to Bloch-type spaces, can be found, for example, in [62], [20], [159].

The Little Bloch-type space $\mathfrak{B}_{\mu,0}$ is a subspace of $\mathfrak{B}_\mu$ consisting of those $f \in \mathfrak{B}_\mu$ such that

$$\lim_{|z|\to 1} \mu(z)\, |\mathfrak{R}f(z)| = 0$$

Bearing in mind the dollowing asymptotic relation from [60]

$$b_\mu(f) \coloneqq sup_{z\in\mathbb{B}}\mu(z)|\nabla f(z)| \asymp sup_{z\in\mathbb{B}}\mu(z)|\mathfrak{R}f(z)| \qquad (1)$$

(for the case $\mu(z) = (1 - |z|^2)^\alpha$, $\alpha > 0$, see, e.g., [4]) we see that $\mathfrak{B}_\mu$ can be defined as the class of all $f \in H(\mathbb{B})$ such that $b_\mu(f)$ is finite. Also the Little Bloch-type space is equivalent with the subspace of $\mathfrak{B}_{\mu}$ consisting of all $f \in H(\mathbb{B})$ such that

$$\lim_{|z|\to 1} \mu(z)\,|\nabla f(z)| = 0$$

Assume $g \in H(\mathbb{B})$, $g(0) = 0$ and ϕ is a holomorphic self-map of $\mathbb{B}$. We introduce the follow-ing integral-type operator on $H(\mathbb{B})$

$$I_{\varphi}^{g}(f)(z) = \int_0^1 \Re f(\varphi(tz))g(tz)\frac{dt}{t}, z \in \mathbb{B} \qquad \mathbf{(2)}$$

Operator (2) is related to operators

$$T_g(f)(z) = \int_0^1 f(tz)\Re g(tz)\frac{dt}{t},$$

and

$$I_g(f)(z) = \int_0^1 \Re f(tz)g(tz)\frac{dt}{t}$$

acting on $H(\mathbb{B})$introduced in [48] and [52], as well as the operator T_g introduced in [57] acting on holomorphic functions on the unit polydisk (see, also [58], as well as [44] for a particular case of the operator). One of motivations for introducing operator I_{φ}^{g} stems from the operator introduced in [37]. Some characterizations of the boundedness and compactness of these and some other integral-type operators mostly in C^n, can be found, for example, in [43]–[46], [47]–[55], [6]–[58], [40]–[158].

Recall that a linear operator $L : X \to Y$, where X and Y are Banach spaces, is compact if for every bounded sequence $(x_k)_{k\in\mathbb{N}}$ in X, the sequence $(L(x_k))_{k\in\mathbb{N}}$ has a convergent subse-quence. The operator L is said to be weakly compact if for every bounded sequence $(x_k)_{k\in\mathbb{N}}$ in X, $(L(x_k))_{k\in\mathbb{N}}$has a weakly convergent subsequence, i.e., there is a subsequence $(x_{k_m})_{k\in\mathbb{N}}$such that for every $\Lambda \in Y^*$, the sequence $\left(\Lambda\left(L(x_{km})\right)\right)_{m\in N}$ converges. A useful characterization for an operator to be weakly compact is the following Gantmacher's theorem: L is weakly com-pact if and only if $L^{**}(X^{**}) \subset Y$, where L^{**} is the second adjoint of L (see, for example, [156]).

We characterize the boundedness and compactness of I_{φ}^{g} from the α-Bloch space (or the Little α-Bloch space) to the Bloch-type space (or the Little Bloch-type space).

We constant are denoted by C, they are positive and may differ from one occurrence to the other. If we say that a function $\mu: \mathbb{C} \to [0, \infty)$ is normal we will also assume that it is radial, that is, $\mu(z) = \mu(|z|), z \in \mathbb{B}$. The notation $a \preccurlyeq b$ means that there is a positive constant C such that $a \le Cb$. We say that $a \asymp b$ if both $a \preccurlyeq b$ and $b \preccurlyeq a$ hold.

Several auxiliary results are given. They will be used in the proofs of the main results.

The following lemma follows by standard arguments (see, for example, the corresponding lemmas in [52], [57], [58]).

Lemma (4.1.1)[155]: Suppose $\alpha \in (0, \infty)$, μ is normal, $g \in H(\mathbb{B})$, $g(0) = 0$ and ϕ is an analytic self-map of B. Then $I_{\varphi}^{g}: \mathfrak{B}^{\alpha}(\text{OR } \mathfrak{B}_0^{\alpha}) \to \mathfrak{B}_{\mu}$is compact if and only if $I_{\varphi}^{g}: \mathfrak{B}^{\alpha}(\text{OR } \mathfrak{B}_0^{\alpha}) \to \mathfrak{B}_{\mu}$is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathfrak{B}^{\alpha}(\text{OR } \mathfrak{B}_0^{\alpha})$ converging to zero uniformly on compacts of $\mathbb{B}$ as $k \to \infty$, we have $\|I_{\varphi}^{g} f_k\|_{\mathfrak{B}_{\mu}} \to 0 \text{ as } k \to \infty$.

Lemma (4.1.2)[155]: Suppose μ is normal. A closed set K in $B_{\mu,0}$ is compact if and only if it is bounded and satisfies

$$\lim_{|z|\to 1} sup_{f\in k}\, \mu(z)\, |\Re F(z)| = 0$$

The proof of Lemma (4.1.2) follows the lines of the proof of Lemma (4.1.2) in [19], hence is omitted.

Lemma (4.1.3)[155]: Assume that f, g ∈ H (𝔹) and g(0) = 0. Then

$$\Re I_\varphi^g(f)(z) = \Re f(\varphi(z))g(z)$$

Proof. assume that the holomorphic function $\Re f(\varphi(z))g(z)$ has the expansion $\sum_\alpha a_\alpha z^\alpha$. Since $\alpha \neq 0$, we obtain.

$$\Re[I_\varphi^g(f)](z) = \Re \int_0^1 \sum_\alpha a_\alpha (tz)^\alpha \frac{dt}{t} = \Re\left(\sum_\alpha \frac{a_\alpha}{|\alpha|} z^\alpha\right) = \sum_\alpha a_\alpha z^\alpha$$

As claimed.

Let $A^1 = A^1(\mathbb{B})$, denote the Bergman space, i.e., the space of all $f \in H(\mathbb{B})$ such that

$$\int_{\mathbb{B}} |f(z)| dV(z) < \infty$$

where $dV\,(z)$ is the Lebesgue volume.

The next lemma can be found, for example, in Theorems 7.5 and 7.6 in [9].

Lemma (4.1.4)[155]: Suppose α ∈ (0, ∞). Then, the following statements are true.

x $(\mathfrak{B}_0^\alpha)^* = A^1$.

x $(A^1)^* = B^\alpha$

x *The second dual of* B_α^0 *is* B^α.

Recall that the duality $(\mathfrak{B}_0^\alpha)^* = A^1$ is given by the following integral pairing

$$\langle f, g\rangle_{\alpha-1} = \lim_{r\to 1-0} c_{\alpha-1} \int_{\mathbb{B}} f(rz)\,\overline{g(rz)}(1-|z|^2)^{\alpha-1} dV(Z)$$

where $f \in \mathfrak{B}_0^\alpha$, $g \in A^1$, and where $c_{\alpha-1}$ is chosen such that

$$c_{\alpha-1} \int_{\mathbb{B}} (1-|z|^2)^{\alpha-1}\; dV(Z) = 1,$$

while the duality $(A^1)^* = \mathfrak{B}^\alpha$ is given by the same integral pairing, where $f \in A^1$ and $g \in (A^1)^* = \mathfrak{B}^\alpha$.

Lemma (4.1.5)[155]: Suppose $0 < \alpha < \infty$, , $g \in H(\mathbb{B})$, $g(0) = 0$, ϕ is an analytic self-map of 𝔹 and X is A Banana space . then I_φ^g: $\mathfrak{B}_0^\alpha \to X$ is compact if and only if I_φ^g: $\mathfrak{B}_0^\alpha \to X$is is weakly compact.

Proof. By Lemma (4.1.4) we know that $(\mathfrak{B}_0^\alpha)^* = A^1$. Assume that I_φ^g: $\mathfrak{B}_0^\alpha \to X$ is compact . By a well–known theorem then this is equivalent with the operator $\left(I_\varphi^g\right)^*: X^* \to A^1$ is compact. Now recall that A^1 has the Schur property, that is every weakly convergent sequence in A^1 is norm- convergent (see, for example, [156]). Hence, this is equivalent with $\left(I_\varphi^g\right)^*: X^* \to A^1$ is weakly compact, which is equivalent with $\left(I_\varphi^g\right)^*: X^* \to A^1$ is weakly com-pact.

Based on a result from [157], in [113] proved the following result.

Lemma (4.1.6)[155]: Suppose α ∈ (0, ∞). Then there exist two holomorphic functions f_1, $f_2 \in B^\alpha$ (D) such that

$$(1-|z|^2)^\alpha(|f_1'(z)|)+(|f_2'(z)|)\asymp 1. \tag{3}$$

Now we are in a position to formulate and prove the main results.

Theorem (4.1.7)[155]: Suppose α > 0, μ is normal, g ∈ H (𝔹), g(0) = 0 and ϕ is an analytic self-map of 𝔹. Then the following statements are equivalent.

(i) $I_\varphi^g: \mathfrak{B}^\alpha \to \mathfrak{B}_\mu$ is bounded.

(ii) $I_\varphi^g: \quad \mathfrak{B}_0^\alpha \to \mathfrak{B}_\mu$ is bounded

(iii)

$$M := sup_{z\in\mathbb{B}}\frac{\mu(z)|g(z)|\varphi(z)|}{(1-|\varphi(z)|^2)^\alpha}<\infty. \tag{4}$$

Moreover if, if $I_\varphi^g: \mathfrak{B}^\alpha \to \mathfrak{B}_\mu$ is bounded, then

$$\left\|I_\varphi^g\right\|_{\mathfrak{B}^\alpha\to\mathfrak{B}_\mu}\asymp M \tag{5}$$

Proof. (iii) ⇒ (i) By Lemma (4.1.3), the definition of the α-Bloch space and asymptotic relation-ship (1), we have

$$\mu(z)\left|\Re\left(I_\varphi^g f\right)(z)\right| = \mu(z)\left|\Re f\big(\varphi(z)\big)\right||g(z)|\le C\|f\|_{\mathfrak{B}^\alpha}\frac{\mu(z)|g(z)|\varphi(z)|}{(1-|\varphi(z)|^2)^\alpha}.$$

For every $z\in$ 𝔹 and $f\in\mathfrak{B}^\alpha$. From this, by using (4) and since $I_\varphi^g f(0)=0$, it follows that $I_\varphi^g: \mathfrak{B}^\alpha \to \mathfrak{B}_\mu$ is bounded and that

$$\left\|I_\varphi^g\right\|_{\mathfrak{B}^\alpha\to\mathfrak{B}_\mu}\asymp CM \tag{6}$$

(i)⇒(ii) The implication is obvious.

(ii)⇒(iii) Using the following test functions.

$$f_l(z)=\ z_l\in\mathfrak{B}_0^\alpha,\qquad l\in\{1,\dots,n\} \tag{7}$$

We obtain $I_\varphi^g f_l\in\mathfrak{B}_\mu$ for $l\in\{1,\dots,n\}$, that is

$$\left\|I_\varphi^g f_l\right\|_{\mathfrak{B}_\mu} =\ sup_{z\in\mathbb{B}}\,\mu(z)|g(z)|\varphi_l(z)|\le\left\|I_\varphi^g\right\|_{\mathfrak{B}^\alpha\to\mathfrak{B}_\mu}\|f_l\|_{\mathfrak{B}^\alpha}<\infty$$

For each $l\in\{1,\dots,n\}$ and consequently.

$$sup_{z\in\mathbb{B}}\,\mu(z)|g(z)|\varphi(z)|\le\sum_{l=1}^{n}sup_{z\in\mathbb{B}}\,\mu(z)|g(z)|\varphi_l(z)|$$
$$\le\left\|I_\varphi^g\right\|_{\mathfrak{B}^\alpha\to\mathfrak{B}_\mu}\sum_{l=1}^{n}\|f_l\|_{\mathfrak{B}^\alpha}<\infty. \tag{8}$$

Set

$$\widehat{f_a}(z)=\ \frac{1-|a|^2}{(1-\langle z,a\rangle)^\alpha},a\in\mathbb{B} \tag{9}$$

It is easy to see $\widehat{f_a}\in\ \mathfrak{B}_0^\alpha$. Morever

$$M_1 := sup_{a\in\mathbb{B}}\left\|\widehat{f_a}\right\|_{\mathfrak{B}^\alpha}\le\alpha 2^{\alpha+1}+1.$$

From this and the boundedness of $I_\varphi^g: \mathfrak{B}^\alpha \to \mathfrak{B}_\mu$ it follows that

$$M_1\left\|I_\varphi^g\right\|_{\mathfrak{B}_0^\alpha\to\mathfrak{B}_\mu}\ge\left\|I_\varphi^g\,\tilde f_{\varphi(a)}\right\|_{\mathfrak{B}_\mu}=\ sup_{z\in\mathbb{B}}\,\mu(z)|g(z)|\Re\tilde f_{\varphi(a)}\varphi(z)|$$
$$\ge\frac{\alpha\mu(z)|g(a)|\varphi(a)|^2}{(1-|\varphi(a)|^2)^\alpha}$$

From which it follows that

$$sup_{|\varphi(z)|\geq\frac{1}{2}}\frac{\mu(z)|g(z)|\varphi(z)|}{(1-|\varphi(z)|^2)^\alpha}\leq sup_{|\varphi(z)|\geq\frac{1}{2}}\frac{2\mu(z)|g(z)|\varphi(z)|^2}{(1-|\varphi(z)|^2)^\alpha}$$
$$\leq\frac{M_1}{\alpha}\left\|I_\varphi^g\right\|_{\mathfrak{B}_0^\alpha\to\mathfrak{B}_\mu}<\infty \qquad (10)$$

On the other hand if , $|\varphi(z)\leq 1/2$ by using (8) we obtain

$$\frac{\mu(z)|g(z)|\varphi(z)|}{(1-|\varphi(z)|^2)^\alpha}\leq\frac{4^\alpha}{3^\alpha}sup_{z\in\mathbb{B}}\,\mu(z)|g(z)||\varphi(z)|$$
$$\leq\frac{4^\alpha}{3^\alpha}\left\|I_\varphi^g\right\|_{\mathfrak{B}_0^\alpha\to\mathfrak{B}_\mu}\sum_{l=1}^{n}\|f_l\|_{\mathfrak{B}^\alpha}<\infty. \qquad (11)$$

Condition (4) as well as the inequality

$$M\leq C\left\|I_\varphi^g\right\|_{\mathfrak{B}_0^\alpha\to\mathfrak{B}_\mu} \qquad (12)$$

Is direct consequence of (10) and (11).

The asymptotic relation in (5). Follows from (6) and (12).

Theorem (4.1.8)[155]: Suppose α > 0, μ is normal, g ∈ H (𝔹), g(0) = 0 and ϕ is an analytic self-map of $\mathfrak{B}_0^\alpha\to\mathfrak{B}_\mu$. Then the following statements are equivalent.

(i) $I_\varphi^g:\mathfrak{B}^\alpha\to\mathfrak{B}_\mu$ is compact

(ii) $I_\varphi^g:\ \mathfrak{B}_0^\alpha\to\mathfrak{B}_\mu$ *is compact*

(iii) $I_\varphi^g:\ \mathfrak{B}_0^\alpha\to\mathfrak{B}_\mu$ Is bounded and

$$\lim_{|\varphi(z)|\to1}\frac{\mu(z)|g(z)|\varphi(z)|}{(1-|\varphi(z)|^2)^\alpha}=0\,; \qquad (13)$$

(iv) $I_\varphi^g:\mathfrak{B}^\alpha\to\mathfrak{B}_\mu$ Is bounded and condition 13 holds.

Proof. First note that in view of Theorem (4.1.8) it follows that (iii) and (iv) are equivalent.

(i) ⇒ (ii) this implication is obvious.

(ii) ⇒ (iii) since $I_\phi{}^g : B_0{}^\alpha\to B_\mu$ is compact then clearly $I_\phi{}^g : B_0{}^\alpha\to B_\mu$ is bounded. Let $(zk)_{k\in\mathbb{N}}$be a sequence in 𝔹 such that $|\varphi(zk)|\to 1\ as\ k\to\infty$ if such a sequence does not exist then condition (13) is vacuously satisfied.

Set

$$F_k(z)=f_{\varphi(zk)},k\in\mathbb{N}. \qquad (14)$$

where $\tilde{f}_w$is defined in (9). From the proof of Theorem (4.1.7) we see that $sup_{\,k\in\mathbb{N}}\|F_k\|_{\mathcal{B}^\alpha}<\infty$. beside this this F_k converges to zero uniformly on compacts of B as $k\to\infty$.

Lemma (4.1.1) implies

$$\lim_{k\to\infty}\left\|I_\varphi^gF_k\right\|_{\mathcal{B}_\mu} \qquad (15)$$

$$\left\|I_\varphi^gF_k\right\|_{\mathcal{B}_\mu}=\ sup_{\,z\in\mathbb{B}}\mu(z)\left|\Re\left(I_\varphi^gF_k\right)(z)\right|$$
$$\geq\mu(z_k)|g(z_k)|\left|\Re\left(I_\varphi^gF_k\right)\left(\varphi(z_k)\right)\right|$$
$$=\frac{\alpha\mu(z_k)|g(z_k)|\varphi(z_k)|^2|}{1-|\varphi(z_k)|^2} \qquad (16)$$

From (15), (16) and by using the assumption $|\phi(zk\,)|\ \to\ 1\ as\ k\ \to\ \infty$, we obtain

$$\lim_{k\to\infty} \frac{\mu(z_k)\left|g(z_k)|\varphi(z_k)|\ \ \right|}{(1-|\varphi(z_k)|^2)^\alpha} = 0$$

from which (13) follows.

(iii) ⇒ (i) Since $I_\phi{}^g : B_0{}^\alpha \to B_\mu$ is bounded then condition (8) holds. Let $(f_k)_{k\in N}$ be a sequence in B^α such that $sup_{k\in\mathbb{N}}\|f_k\|_{\mathcal{B}^\alpha} =: L < \infty$ and $f_k \to 0$ uniformly on compacts of B as $k \to \infty$.

From (13) for every $\varepsilon > 0$, there is a $\delta \in (0, 1)$, such that

$$\frac{\mu(z)\left|g(z)|\varphi(z)|\ \ \right|}{(1-|\varphi(z)|^2)^\alpha} < \frac{\varepsilon}{L}, \tag{17}$$

Whenever

Lemma (4.1.3), 1, 17 and 8 yeild.

$$\begin{aligned}\left\|I_\varphi^g F_k\right\|_{\mathcal{B}_\mu} &= sup_{z\in\mathbb{B}}\mu(z)\left|g(z)\Re f_k(\varphi(z))\ \ \right| \\ &\le sup_{\{z\in\mathbb{B}:|\varphi(z)|\le\delta\}}\mu(z)|g(z)|\left|\Re f_k(\varphi(z))\ \ \right| \\ &+ sup_{\{z\in\mathbb{B}:\delta<|\varphi(z)|<1\}}\mu(z)|g(z)|\left|\Re f_k(\varphi(z))\ \ \right| \\ &\le L_g sup_{|w|\le\delta}|\nabla f_k(w)| + C\|f_k\|_{\mathcal{B}^\alpha} \\ & sup_{\{z\in\mathbb{B}:\delta<|\varphi(z)|<1\}}\frac{\mu(z)|g(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^\alpha} \\ &\le L_g sup_{|w|\le\delta}|\nabla f_k(w)| + C_\varepsilon \end{aligned} \tag{18}$$

where

$$L_g = sup_{z\in\mathbb{B}}\mu(z)|g(z)||\varphi(z)|$$

The uniform convergence of $(f_k)_{k\in N}$ on compacts of B along with Cauchy's estimate implies$(|\nabla f_k|)_{k\in\mathbb{N}}$ also converges to zero on compacts of $\mathbb{B}$ $as\ k \to \infty, hence$

$$\lim_{k\to\infty.} sup_{|w|\le\delta}|\nabla f_k(w)| = 0. \tag{19}$$

Letting $k \to \infty$. In (18) and using (19) we obtain

$$\lim_{k\to\infty.} sup\left\|I_\varphi^g f_k\right\|_{\mathcal{B}_\mu} \le C_\varepsilon$$

for each positive ε. Hence the limit is equal to zero, from which by Lemma (4.1.1) it follows that the operator $I_\varphi^g : B^\alpha \to \mathcal{B}_\mu$ is compact.

Theorem (4.1.9)[155]: Suppose $\alpha > 0$, μ is normal, $g \in H(B)$, $g(0) = 0$ and ϕ is an analytic self-map of B. Then $I_\phi{}^g : B_0{}^\alpha \to B_{\mu,0}$ is bounded if and only if $I_\varphi^g : B_0^\alpha \to \mathcal{B}_\mu$is bounded and

$$\lim_{|z|\to1}\mu(z)|g(z)||\varphi(z)| = 0 \tag{20}$$

Proof. First assume $I_\varphi^g : B^\alpha \to \mathcal{B}_\mu$ is bounded and that condition (20) holds. Then, for each polynomial p, which obviously belongs to $B_0{}^\alpha$, we obtain

$$\begin{aligned}\mu(z)\left|\Re I_\varphi^g p(z)\ \ \right| &\le \mu(z)|g(z)|\left|\Re p(\varphi(z))\ \ \right| \\ &\le \mu(z)|g(z)||\varphi(z)|\|\nabla p\|_\infty \to 0.\end{aligned}$$

As $|z| \to 1$ hence $I_\varphi^g p \in \mathcal{B}_{\mu,0.}$

Since the set of all polynomials is dense in $\mathcal{B}_0^\alpha$, for each $f \in \mathcal{B}_0^\alpha$ there a sequence of polynomials $(p_k)_{k\in\mathbb{N}}$.

Such that $|\boldsymbol{f} - \boldsymbol{p_k}|_{\boldsymbol{\mathcal{B}^\alpha\to 0}}$ **as** $k \to \infty$, from this and since the operator I_φ^g:

Theorem (4.1.10)[155]: Suppose $\alpha > 0$, μ is normal, $g \in H(B)$, $g(0) = 0$ and ϕ is an analytic self map of $\mathbb{B}$ Then the following statements are equivalent.

(i) $I_{\varphi}^{g}: \mathfrak{B}^{\alpha} \to \mathfrak{B}_{\mu,0}$ is bounded ;

(ii) $I_{\varphi}^{g}: \mathfrak{B}^{\alpha} \to \mathfrak{B}_{\mu,0}$ *is compact*;

(iii) $I_{\varphi}^{g}: \mathfrak{B}_{0}^{\alpha} \to \mathfrak{B}_{\mu,0}$ Is *compact*

(iv) $I_{\varphi}^{g}: \mathfrak{B}_{0}^{\alpha} \to \mathfrak{B}_{\mu,0}$; Is *weakly compact*

(v) $I_{\varphi}^{g}(\mathfrak{B}^{\alpha}) \subset \mathfrak{B}_{\mu,0}$

(vi)

$$\lim_{|z|\to 1} \mu(z)|g(z)|\varphi(z)| = 0. \tag{21}$$

And

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|g(z)|\varphi(z)|}{(1-|\varphi(z)|^2)^{\alpha}} = 0 \tag{22}$$

(vii)

$$\lim_{|z|\to 1} \frac{\mu(z)|g(z)|\varphi(z)|}{(1-|\varphi(z)|^2)^{\alpha}} = 0\,; \tag{23}$$

Proof. (vii) $\Longrightarrow (ii)$ assume that (23) holds . By Lemma (4.1.3) and (1) we have

$$\mu(z)\left|\Re\left(I_{\varphi}^{g} f\right)(z)\right| \le C\|f\|_{\mathfrak{B}^{\alpha}} \frac{\mu(z)|g(z)|\varphi(z)|}{(1-|\varphi(z)|^2)^{\alpha}} \tag{24}$$

From this and (23) it follows that the set $I_{\varphi}^{g}(\{f: \|f\|_{\mathfrak{B}^{\alpha}} \le 1\})$ is bounded in $\mathfrak{B}_{\mu}$, moreover in $\mathfrak{B}_{\mu,0}$. Taking the supremum in (24) over the unit ball of the space $\mathfrak{B}^{\alpha}$, then letting $|z| \to 1$ and using (23), we obtain

$$\lim_{|z\to 1|} sup_{\|f\|_{\mathfrak{B}^{\alpha}\le 1}} \mu(z)\left|\Re\left(I_{\varphi}^{g} f\right)(z)\right| = 0 \tag{25}$$

From (25) and by using Lemma (4.1.2) the compactness of the operator $I_{\varphi}^{g}: \mathfrak{B}^{\alpha} \to \mathfrak{B}_{\mu,0}$ follows.

[D] ⇒ (iii) This implication is obvious.

[E] ⇒ (iv) Just recall that every compact operator is weakly compact.

[F] ⇒ (v) By Lemma (4.1.4) we know that $(\mathfrak{B}_{0}^{\alpha})^{**} = \mathfrak{B}^{\alpha}$. Since I_{φ}^{g} maps $\mathfrak{B}_{0}^{\alpha}$ into $\mathfrak{B}_{\mu,0}$ and $(\mathfrak{B}_{0}^{\alpha})^{*} = A^{1}$, we have that $(I_{\varphi}^{g})^{*} : (\mathfrak{B}_{\mu,0})^{*} \to A^{1}$. Hence

$$\langle I_{\varphi}^{g}(f)h\rangle = \langle f, (I_{\varphi}^{g}) * (h)\rangle$$

For every $f \in \mathfrak{B}_{0}^{\alpha}$ *and* $h \in \left(\mathfrak{B}_{\mu,0}\right)^{*}$.

On the other hand, by Lemma (4.1.4) we have $(A^{1})^{*} = \mathfrak{B}^{\alpha}$ which implies that $\left(I_{\varphi}^{g}\right)^{**}: \mathfrak{B}^{\alpha} \to \mathfrak{B}_{\mu,0}$. hence every $f \in \mathfrak{B}_{0}^{\alpha}$ can be viewed as an element of the space $(A^{1})^{*}$and

$$\langle f\left(I_{\varphi}^{g}\right)^{*}(h)\rangle = \langle\left(I_{\varphi}^{g}\right)^{**}(f), h\rangle$$

Hence

$$\langle I_{\varphi}^{g}(f)(h)\rangle = \langle\left(I_{\varphi}^{g}\right)^{**}(f), h\rangle$$

For every $h \in \left(\mathfrak{B}_{\mu,0}\right)^{*}$ by a well–known consequence of Hann- Banach theorem we obtain $\left(I_{\varphi}^{g}\right)^{**}(f) = I_{\varphi}^{g}(f)\, for\, every\, f \in \mathfrak{B}_{0}^{\alpha}$.

Since $\mathfrak{B}_{0}^{\alpha}$ is w^{*} dense in $\mathfrak{B}^{\alpha}$ it follows that $\left(I_{\varphi}^{g}\right)^{**}(f) = I_{\varphi}^{g}(f)$ for every $f \in \mathfrak{B}^{\alpha}$. Gantmacher's theorem implies that $I_{\varphi}^{g}(\mathfrak{B}^{\alpha}) \subset \mathfrak{B}_{\mu,0}$ as desired .

[G] ⇒ (vi) By using the test functions in (7), as in Theorem (4.1.9), it follows that (21) holds. If $\|\varphi\|_\infty < 1$ then (22) is vacuously satisfied. Now assume $\|\varphi\|_\infty = 1$, and assume to the contrary that the condition (22) does not hold. If it were, then it would exist $\varepsilon_0 > 0$ and a sequence$(z_k)_{k\in\mathbb{N}} \subset \mathbb{B}$ such that $\lim_{k\to\infty} |\varphi(z_k)| = 1$ *and*

[H]

$$\frac{\mu(z_k)|g(z_k)|\varphi(z_k)|}{(1-|\varphi(z)|^2)^\alpha} \geq \varepsilon_0 > 0 \tag{26}$$

For sufficiently large k.

We may also assume that $\varphi(z_k) \to (1,0,\dots,0) as\ k \to 0$.

By Lemma (4.1.6) there are two functions $f_1 f_2 \mathfrak{B}^\alpha(\mathbb{D})$ such that asymptotic relation (3) holds,

Let

$$F_1(z) = f_1(z_1)\ and\ F_2(z) = f_2(z_1)\ z \in \mathbb{B}$$

Clearly $F_1, F_2 \in \mathfrak{B}^\alpha(\mathbb{B})$ and consequently

$$I_\varphi^g F_1, I_\varphi^g F_2 \in \mathfrak{B}_{\mu,0}\,, \tag{27}$$

On the other hand, by Lemma (4.1.3) and (3) we have

$$\begin{aligned}
&\mu(z_k)\left|\Re\left(I_\varphi^g F_1\right)(z_k)\right| + \mu(z)\left|\Re\left(I_\varphi^g F_2\right)(z_k)\right| \\
&= \mu(z_k)|g(z_k)||\Re F_1\varphi(z_k)| + \mu(z_k)|g(z_k)||\Re F_2\varphi(z_k)| \\
&= \mu(z_k)|g(z_k)|\left|\varphi_1(z_k) f_1'\left(\varphi_1(z_k)\right)\right| \\
&+ \mu(z_k)\left|g(z_k)\right|\left|\varphi_1(z_k) f_2'\left(\varphi_1(z_k)\right)\right| \asymp \frac{\mu(z_k)|g(z_k)||\varphi_1(z_k)|}{(1-|\varphi(z_k)|^2)^\alpha} \\
&\asymp \frac{C\mu(z_k)|g(z_k)||\varphi(z_k)|}{(1-|\varphi(z_k)|^2)^\alpha} \geq \frac{C\varepsilon_0}{2} > 0 \dots
\end{aligned}$$

for sufficiently large k, which is a contradiction with (27).

[I] ⇒ (vii) From (22) it follows that for every $\varepsilon > 0$ there is an $r \in (0, 1)$ such that

$$\frac{\mu(z)|g(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^\alpha} < \varepsilon. \tag{28}$$

Whenever $r \leq |\varphi(z)| < 1$.

From (21) it follows that there is a $\sigma \in (0,1)$ *such that*

$$\mu(z)|g(z)|\varphi(z)| \leq \varepsilon(1-r^2)^\alpha. \tag{29}$$

When $\sigma < |z| < 1$,

If $|\varphi(z)| \leq r\ \ and\ \sigma < |z| < 1$, then from (29) we have

$$\frac{\mu(z)|g(z)|\varphi(z)|}{(1-|\varphi(z)|^2)^\alpha} \leq \frac{\mu(z)|g(z)|\varphi(z)|}{(1-r^2)^\alpha} < \varepsilon. \tag{30}$$

Now note that (28) holds on the set $r\ |\phi(z)| < 1$ and $\sigma < |z| < 1$. From this and (30) the implication follows.

(i) ⇒ (v) This implication is obvious.

(vii) ⇒ (i) From (23) it follows that condition (4) holds. Hence by Theorem (4.1.7) it follows that the operator $I_\varphi^g: \mathfrak{B}^\alpha \to \mathfrak{B}_\mu$is bounded. On the other hand from (23) and (24) it follows that for every $f \in \mathfrak{B}^\alpha, I_\varphi^g f \in\ \mathfrak{B}_{\mu,0}$from which the boundedness of $I_\varphi^g: \mathfrak{B}^\alpha \to \mathfrak{B}_{\mu,0}$ follows, finishing the proof of the theorem.

Section (4.2): An Integral-Type Operator from the Dirichlet Space to the Bloch-Type Space on the Unit Ball

For $\mathbb{B}$ be the open unit ball in C^n, $\mathbb{D}$ the open unit disk in C, $H(\mathbb{B})$ the class of all holomorphic functions on $\mathbb{B}$, and $H^\infty(\mathbb{B})$ the space consisting of all $f \in H(\mathbb{B})$ such that $\|f\|_\infty = \sup_{z\in\mathbb{B}}|f(z)| < \infty$

For an $f \in H(\mathbb{B})$ with the Taylor exoansion $f(z) = \sum_\alpha a_\alpha z^\alpha$, let

$$\Re F(z) = \sum_\alpha |\alpha| a_\alpha z^\alpha \tag{31}$$

Be the radial derivative of f, where $\alpha(\alpha_1, \dots, \alpha_n)$ is a multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $z^\alpha = z_1^{\alpha_1} \dots z_n^{\alpha_n}$ $.Let\ \alpha! = \alpha_1! \dots \alpha_n!$

The Dirichlet space $\mathcal{D}^2(\mathbb{B}) = \mathcal{D}^2$ contains all $f(z)\ \sum_\alpha a_\alpha z^\alpha \in H(\mathbb{B})$

Such that

$$\|f\|_{\mathcal{D}^2}^2 := \left|(f(0))\right|^2 + \sum_a |\alpha| \frac{\alpha!}{|\alpha|!} |a_a|^2 < \infty \tag{32}$$

The quantity $\|f\|_{\mathcal{D}^2}$ is a norm on $\mathcal{D}^2$ which for $n = 1$is equal to usual norm

$$\|f\|_{\mathcal{D}^2(\mathbb{D})} \left(|f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dA(Z)\right)^{\frac{1}{2}} \tag{33}$$

Where $dA(Z) = \left(\frac{1}{\pi}\right) r\ dr\ d\theta$ is the normalized area measure on $\mathbb{D}$.

The inner product, between two functions

$$f(z) \sum_\alpha a_\alpha z^\alpha \qquad g(z) \sum_\alpha b_\alpha z^\alpha \tag{34}$$

On $\mathcal{D}^2$ is defined by

$$(f,g)| := f(0)\overline{g(0)} \quad \sum_\alpha |\alpha| \frac{\alpha!}{|\alpha|!} a_\alpha \bar{b}_\alpha \tag{35}$$

For $\alpha \neq 0$, let

$$e_\alpha(z) \sqrt{\frac{|\alpha|!}{|\alpha|\alpha!}} z^\alpha, \quad z \in \mathbb{B} \tag{36}$$

and $e_0(z) \equiv 1$, then it is easy to see that the family $\{e_\alpha\}$ is an orthonormal basis for $\mathcal{D}^2$, and hence the reproducing kernel $k_w(z)$_ for $\mathcal{D}^2$ is given by ([1])as follows:

$$k_w(z) = 1 + \sum_{\alpha\neq 0} \sqrt{\frac{|\alpha|!}{|\alpha|\alpha!}} z^\alpha w^\alpha = 1 + In \quad \frac{1}{(1-(z,w))} \tag{37}$$

where $\langle z,w\rangle\ \sum_{j=1}^n z_j \overline{w_j}$ is the inner product in $\mathbb{C}^n$ clearly for each $f \in \mathcal{D}^2$ $and\ w \in \mathbb{B}^m$ the next producing formula holds:

$$f(w) = \langle f, k_w\rangle \tag{38}$$

note that for $f = k_w(1.8)$, we obtain

$$k_w(w) = \|k_w\|_{\mathcal{D}^2}^2 = In \quad \frac{e}{1-|w|^2} \tag{39}$$

Also, by the Cauchy-Schwarz inequality and (39), we have that, for each $f \in \mathcal{D}^2$ $and\ w \in \mathbb{B}$

$$|f(w)| = |\langle f, k_w\rangle| \le \|f\|_{\mathcal{D}^2}\|k_w\|_{\mathcal{D}^2} = \|f\|_{\mathcal{D}^2}\left(In\ \frac{e}{1-|w|^2}\right)^{\frac{1}{2}} \quad (40)$$

Note that inequality (40) is exact since it is attained for

$$f = K_w.$$

The weighted-type space $H_\mu^\infty(\mathbb{B}) = H_\mu^\infty([2,3]\ )$consists of all $f \in H(\mathbb{B})$ such that

$$\|F\|_{H_\mu^\infty} := sup_{z\in\mathbb{B}}\ \mu(z)|f(z)| < \infty \quad (41)$$

Where μ is a positive continuous function on $\mathbb{B}$ (wright).

The Bloch- type space $\mathcal{B}_\mathcal{M}(\mathbb{B}) = \mathcal{B}_\mathcal{M}$ consists of all $f \in H(\mathbb{B})$ such that

$$\|F\|_{\mathcal{B}_\mathcal{M}} := |f(0)| + sup_{z\in\mathbb{B}}\mu(z)|\Re f(z)| < \infty \quad (42)$$

Where μ is a (wright).

Let $f \in H(\mathbb{D}), g(0) = 0\ and$, ϕ be a holomorphic self-map of $\mathbb{B}$, then the following integral-type operator:

$$P_\varphi^g(f)(z) = \int_0^1 f(\varphi(tz))g(tz)\frac{dt}{t}, z \in \mathbb{B},\ \ f \in H(\mathbb{B}) \quad (43)$$

has been recently introduced in [163] and considerably studied (see, e.g [49]-[164]). For some related operators, see also [165]–[168] and the references therein.

We provide function-theoretic characterizations for when ϕ and g induce bounded or compact integral-type operator on spaces of holomorphic functions. Majority of only find asymptotics of operator norm of linear operators. Somewhat concrete but perhaps more interesting problem is to calculate operator norm of these operators between spaces of holomorphic functions on various domains. Some results on this problem can be found, for example, in [26], [169]–[67] (see also) [41], [25]–[178]. In [26], we started with systematic investigation of methods for calculating operator norms of concrete operators between spaces of holomorphic function.

We calculate the operator norm as well as the essential norm of the operator P_φ^g : $\mathcal{D}^2 \to \mathcal{B}_\mathcal{M}$, considerably extending our recent result in [179].

We quote several auxiliary results which are used in the proofs of the main results.

Lemma (4.2.1)[160]: (see [163]). Let $g \in H(B), g(0) = 0$, and ϕ be a holomorphic self-map of $\mathbb{B}$, then

$$\Re P_\varphi^g(f)(z) = g(z)f(\varphi(z)) \quad (44)$$

The next Schwartz-type Lemma ([180]) can be proved in a standard way. Hence, we omit its proof.

Lemma (4.2.2)[160]: Assume that $g \in H(\mathbb{B}), g(0) = 0, \mu$ is a weight, and φ is an analytic self-map of $\mathbb{B}$, then $P_\varphi^g : \mathcal{D}^2 \to \mathcal{B}_\mathcal{M}$ is compact if and only if $P_\varphi^g : \mathcal{D}^2 \to \mathcal{B}_\mathcal{M}$ is bounded and for any bounded sequence $(f_k)_{k\in N}$ in $\mathcal{D}^2$ converging to zero uniformly on compacts of $\mathbb{B}$ as $k \to \infty$, one has

$$\lim_{k\to\infty}\left\|P_\varphi^g(f_k)\right\|_{\mathcal{B}_\mu} = 0. \quad (45)$$

Lemma (4.2.3)[160]: Assume that $g \in H(\mathbb{B}), g(0) = 0, \mu$ is a weight, and ϕ is an analytic self-map of $\mathbb{B}$,such that $\|\varphi\|_\infty < 1$ and the operator $P_\varphi^g : \mathcal{D}^2 \to \mathcal{B}_\mathcal{M}$ is bounded, then $P_\varphi^g : \mathcal{D}^2 \to \mathcal{B}_\mathcal{M}$ is compact.

Proof. first note that since $P_\varphi^g : \mathcal{D}^2 \to \mathcal{B}_\mathcal{M}$ is bounded and $f_0(z) \equiv 1 \in \mathcal{D}^2$ by Lemma (4.2.1), it follows that

$$\Re P_\varphi^g(f_0) = g \in H_\mu^\infty$$

Now assume that $(f_k)_{k\in\mathbb{N}}$ is bounded sequence in $\mathcal{D}^2$ converging to zero on compacts of B as k → ∞, then we have

$$\left\|P_\varphi^g(f_k)\right\|_{\mathcal{B}_\mathcal{M}} \leq \|g\|_{H_\mu^X} SUP_{W\in\varphi(\mathbb{B})}|f_k(W)| \to 0 \tag{46}$$

As $K \in \infty$, since $\varphi(\mathbb{B})$ is contained in the ball $|W| \leq \|\varphi\|_\infty$ which is a compact subset of $\mathbb{B}$, according to the assumption $\|\varphi\|_\infty < 1$.

Hence by Lemma (4.2.2) the operator $P_\varphi^g : \mathcal{D}^2 \to \mathcal{B}_\mathcal{M}$ is compact.

We calculate the operator norm of $P_\varphi^g : \mathcal{D}^2 \to \mathcal{B}_\mathcal{M}$

Theorem (4.2.4)[160]: Assume that $g \in H(B), g(0) = 0,$ and φ is a holomorphic self-map of $\mathbb{B}$, then

$$\left\|P_\varphi^g\right\|_{\mathcal{D}^2\to\mathcal{B}_\mathcal{M}} = sup_{z\in\mathbb{B}}\, \mu(z)|g(z)|\left(In\ \frac{e}{1-|w|^2}\right)^{\frac{1}{2}} =: L \tag{47}$$

Proof. Using Lemma (4.2.1), reproducing formula (38) , the Cauchy – Schwarz inequality , and finally (1.0). We get that, for each $f \in \mathcal{D}^2$ *and* $w \in \mathbb{B}$

$$\begin{aligned}\mu(w)\left|\Re P_\varphi^g(w)\right| &= \mu(w)|g(w)|\left|f\left(\varphi(w)\right)\right| \\ &= \mu(w)|g(w)|\left|\langle f, k_{\varphi(w)}\rangle\right| \\ &\leq \mu(w)|g(w)|\|f\|_{\mathcal{D}^2}\left\|k_{\varphi(w)}\right\|_{\mathcal{D}^2} \\ &\|f\|_{\mathcal{D}^2}\, \mu(w)|g(w)|\left(In\ \frac{e}{1-|w|^2}\right)^{1/2}\end{aligned} \tag{48}$$

Taking the supremum in (48) over $w \in$ B as well as the supremum over the unit ball in $\boldsymbol{\mathcal{D}^2}$ and using the fact $P_\varphi^g(f)(0) = 0 = 0$ for each $f \in H(\mathbb{B})$, which follows from the assumption $g(0) = 0$, we get

$$\left\|P_\varphi^g\right\|_{\mathcal{D}^2\to\mathcal{B}_\mathcal{M}} \leq L. \tag{49}$$

Now assume that the operator

$$P_\varphi^g : \mathcal{D}^2 \to \mathcal{B}_\mathcal{M}$$

is bounded. From (19) we obtain that, for each $w \in$ B

$$\begin{aligned}\left(In\ \frac{e}{1-|w|^2}\right)^{\frac{1}{2}}\left\|P_\varphi^g\right\|_{\mathcal{D}^2\to\mathcal{B}_\mathcal{M}} &= \left\|k_{\varphi(w)}\right\|_{\mathcal{D}^2}\left\|P_\varphi^g\right\|_{\mathcal{D}^2\to\mathcal{B}_\mathcal{M}} \\ &\geq \left\|P_\varphi^g k_{\varphi(w)}\right\|_{\mathcal{B}_\mathcal{M}} \\ &= sup_{z\in\mathbb{B}}\ \mu(z)|g(z)|\left|k_{\varphi(w)}\left(\varphi(z)\right)\right| \\ &\geq \mu(z)|g(w)|\left|k_{\varphi(w)}\left(\varphi(w)\right)\right|\end{aligned} \tag{50}$$

From (39) and (50) it follows that

$$L \leq \left\|P_\varphi^g\right\|_{\mathcal{D}^2\to\mathcal{B}_\mathcal{M}} \tag{51}$$

Hence if $P_\varphi^g : \mathcal{D}^2 \to \mathcal{B}_\mathcal{M}$ is bounded, then from (49) and (51) we obtain (47).

In the case $P_\varphi^g : \mathcal{D}^2 \to \mathcal{B}_\mathcal{M}$ is unbounded, the result follows from inequality (49). Let X and Y be Banach spaces, and let $L : X \to Y$ be a bounded linear operator. The essential norm of the operator $L: X \to Y$, $\|L\|_{e,X\to Y}$, is defined as follows:

$$\|L\|_{e,\ X\to\ Y} = \inf\{\|L + K\|_{X\to\ Y} : k \text{ is compact from } X \text{ to } Y\} \tag{52}$$

where $\|.\|_{X\to\ Y}$denote the operator norm.

From this and since the set of all compact operators is a closed subset of the set of bounded operators, it follows that L is compact if and only if

$$\|L\|_{e,\ X\to\ Y} = 0 \tag{53}$$

We calculate the essential norm of the operator $P_{\varphi}^{g}: \mathcal{D}^2 \to \mathcal{B}_{\mu}$

Theorem (4.2.5)[160]: Assume that g ∈ H (B), g(0) = 0, μ is a weight and ϕ is a holomorphic self-map of $\mathbb{B}$ and $P_{\varphi}^{g}: \mathcal{D}^2 \to \mathcal{B}_{\mu}$ is bounded. If $\|\varphi\|_{\infty} < 1$, then $\|P_{\varphi}^{g}\|_{e,\mathcal{D}^2\to\mathcal{B}_{\mu}} = 0$, and if $\|\varphi\|_{\infty} = 1$, then

$$\|P_{\varphi}^{g}\|_{e,\mathcal{D}^2\to\mathcal{B}_{\mu}} = \lim_{|\varphi(z)|\to 1} sup\, \mu(z)|g(z)| \left(In\ \frac{e}{1-|\varphi(z)|^2}\right)^{1/2} \quad (54)$$

Proof. Since $P_{\varphi}^{g}: \mathcal{D}^2 \to \mathcal{B}_{\mu}$ is bounded, for the test function $f(z) \equiv 1$, we get $g \in H_{\mu}^{\infty}$ (B). If $\|\varphi\|_{\infty} < 1$, then from Lemma (4.2.3) it follows that $P_{\varphi}^{g}: \mathcal{D}^2 \to \mathcal{B}_{\mu}$is compact which is equivalent with $\|P_{\varphi}^{g}\|_{e,\mathcal{D}^2\to\mathcal{B}_{\mu}} = 0$. On the other hand, it is clear that in this case the condition $|\phi(z)| \to 1$ is vacuous, so that (54) is vacuously satisfied.

Now assume that $\|\varphi\|_{\infty} = 1$, and that $(\varphi(z_k))_{k\in\mathbb{N}}$is a sequence in $\mathbb{B}$ such that $|\phi(z_k)| \to 1$ as $k \to \infty$. For $w \in \mathbb{B}$ fixed, set

$$f_w(z) = \frac{In\big(e/(1-\langle z,w\rangle)\big)}{In\big(e/(1-|w|^2)\big)^{1/2}} \quad z \in \mathbb{B} \quad (55)$$

By (39), we have that $\|f_w\|_{\mathcal{D}^2} = 1$, for each $w \in$ B. Hence, the sequence $(f_{\varphi(z_k)})_{k\in\mathbb{N}}$ is such that $(f_{\varphi(z_k)})_{\mathcal{D}^2} = 1$, for each $k \in$ N, and clearly it converges to zero uniformly on compacts of $\mathbb{B}$. From this and by [9], it easily follows that $(f_{\varphi(z_k)})_{-} \to 0$ weakly in $\mathcal{D}^2$, as $k \to \infty$. Hence, for every compact operator $K : \mathcal{D}^2 \to \mathcal{B}_{\mu}$, we have that

$$lim_{k\to\infty}\|Kf_{\varphi(z_k)}\|_{\mathcal{B}_{\mu}} = 0. \quad (56)$$

Thus, for every such sequence and for every compact operator $K: \mathcal{D}^2 \to \mathcal{B}_{\mu}$, we have that

$$\|P_{\varphi}^{g} + k\|_{\mathcal{D}^2\to\mathcal{B}_{\mathcal{M}}} \geq lim_{k\to\infty}sup \frac{\|P_{\varphi}^{g} f_{\varphi(z_k)}\|_{\mathcal{B}_{\mathcal{M}}} - \|kf_{\varphi(z_k)}\|_{\mathcal{B}_{\mathcal{M}}}}{\|f_{\varphi(z_k)}\|_{\mathcal{D}^2}}$$

$$= lim_{k\to\infty}sup\|P_{\varphi}^{g} f_{\varphi(z_k)}\|_{\mathcal{B}_{\mathcal{M}}} \geq lim_{k\to\infty}sup(z_k)\big|g(z_k)f_{\varphi(z_k)}(\varphi(z_k))\big| \quad (57)$$

$$= lim_{n\to\infty}\sup\ \mu(z_k)\ |g(z_k)| \left(In\ \frac{e}{1-|w|^2}\right)^{\frac{1}{2}}$$

Taking the infimum in (57) over the set of all compact operators

$$K: \mathcal{D}^2 \to \mathcal{B}_{\mathcal{M}}$$

$$\|P_{\varphi}^{g}\|_{e,\mathcal{D}^2\to\mathcal{B}_{\mathcal{M}}} \geq \lim_{n\to\infty} sup\, \mu(z_k)|g(z_k)| \left(In\ \frac{e}{1-|w|^2}\right)^{1/2} \quad (58)$$

from which an inequality in (54) follows

We prove the reverse inequality. Assume that $(r_1)_{l\in\mathbb{N}}$ is a sequence of positive numbers which increasingly converges to 1. Consider the operators defined by

$$\big(P_{r_l\varphi}^{g} f\big)(z) = \int_0^1 f\big(r_l\varphi(tz)\big)\, g(tz)\frac{dt}{t}, \qquad l \in \mathbb{N}. \quad (59)$$

Since $\|r_l\varphi\|_{\infty} < 1$, by Lemma (4.2.3) , we have that these operators are compact.

Since $P_{\varphi}^{g}: \mathcal{D}^2 \to \mathcal{B}_{\mu}$ is bounded, then $g \in H_{\mu}^{\infty}. let$ $\rho \in (0,1)$, be fixed for a moment . By Lemma (4.2.1), we get

$$\|P_{\varphi}^{g} - P_{r_l\varphi}^{g}\|_{\mathcal{D}^2\to\mathcal{B}_{\mathcal{M}}} = \sup_{\|f\|_{\mathcal{D}^2}\leq 1} sup_{z\in\mathbb{B}}\, \mu(z_k)|g(z_k)|f\varphi(z_k) - f\big(r_l\varphi(tz)\big)|$$

$$\le \sup_{\|f\|_{\mathcal{D}^2}\le 1} sup_{|\varphi(z)|\le\rho}\, \mu(z)|g(z)|f\varphi(z) - f\big(r_l\varphi(tz)\big)\big|$$
$$+\sup_{\|f\|_{\mathcal{D}^2}\le 1}\ sup_{|\varphi(z)|\le\rho}\, \mu(z)|g(z)|f\varphi(z) - f\big(r_l\varphi(tz)\big)\big| \quad (60)$$
$$\le \|g\|_{H_\mu^X} \sup_{\|f\|_{\mathcal{D}^2}\le 1} sup_{|\varphi(z)|\le\rho}|f\varphi(z) - f\big(r_l\varphi(tz)\big)\big|$$
$$+\sup_{\|f\|_{\mathcal{D}^2}\le 1}\ sup_{|\varphi(z)|\le\rho}\, \mu(z)|g(z)|f\varphi(z) - f\big(r_l\varphi(tz)\big)\big| \quad (61)$$

Further we have

$$\|f - fr\,\|_{\mathcal{D}^2}^2 = \sum_\alpha |\alpha| \frac{\alpha!}{|\alpha|!} |a_\alpha|^2 \left(1 - r^{|\alpha|}\right)^2$$
$$\le \sum_\alpha |\alpha| \frac{\alpha!}{|\alpha|!} |a_\alpha|^2 \le \|f\|_{\mathcal{D}^2} \quad (62)$$

From (40), (62) and the fact $|f(z) - f(rz)| \in \mathcal{D}^2$

$$|f(z) - f(rz)| \le \|f\|_{\mathcal{D}^2} \left(In\ \ \frac{e}{1-|z|^2}\right)^{\frac{1}{2}} \quad (63)$$

In particular

$$|f\varphi(z) - f\big(r_l\varphi(z)\big)| \le \|f\|_{\mathcal{D}^2} \left(In\ \ \frac{e}{1-|\varphi(z)|^2}\right)^{\frac{1}{2}} \quad (64)$$

Let

$$I_l \coloneqq \sup_{\|f\|_{\mathcal{D}^2}\le 1} sup_{|\varphi(z)|\le\rho}|f\varphi(z) - f\big(rl\varphi(z)\big)\big| \quad (65)$$

The mean value theorem along with the subharmonicity of the moduli of partial derivatives of f, well-known estimates among the partial derivatives of analytic functions, Theorem 6.2, and Proposition 6.2 in [9] , yield

$$I_l \le \sup_{\|f\|_{\mathcal{D}^2\le 1}} sup_{|\varphi(z)|\le\rho}$$
$$(1 - r_1)|\varphi(z)| sup_{|w|\le\rho}|\nabla f(w)| \le C_\rho$$
$$(1 - r_l)\sup_{\|f\|_{\mathcal{D}^2}\le 1}\left(\sum_{\mathrm{j=1}}^{[\mathrm{n/p}]} \left|\nabla^{\mathrm{j}}\mathrm{f}(0)\right| + sup_{|w|\le(1+\rho)/2}\left|\nabla^{|\mathrm{n/p}|+1}\mathrm{f(w)}\right|\right)$$
$$\le C_\rho(1 - r_l)\sup_{\|f\|_{\mathcal{D}^2}\le 1}\left(\sum_{\mathrm{j=1}}^{[\mathrm{n/p}]} \left|\nabla^{\mathrm{j}}\mathrm{f}(0)\right|\right.$$
$$\left. + \int_{|\mathrm{w}|+(3+\rho/4))} \left|\nabla^{[\mathrm{n/p}]+1} f(w)\right|^2 (1 - |\mathrm{w}|^2)^{2([\mathrm{n/p}]+1)} \mathrm{d}\tau(\mathrm{w})\right)^{1/2}$$
$$\le C_\rho(1 - r_l) \to 0, \ \text{ as } l \to \infty, \quad (66)$$

where $dr(z) = dV(z)/(1 - |z|^2)^{n+1}$ and $dV(z)$ is the Lebesgue volume measure on $\mathbb{B}$.

Using (64) in (61), letting $l \to \infty$ in (60), using (66), and then letting $\rho \to 1$, the reverse inequality follows, finishing the proof of the theorem.

Section (4.3): From Logarithmic Bloch Type Spaces to Weighted-Type Spaces

For $\mathbb{B}^n = \mathbb{B}$ B be the open unit ball in the complex vector space $\mathbb{C}^n, \mathbb{B}^1 = \mathbb{D}$ the unit disk in $\mathbb{C}, H(X)$ the class of all holomorphic functions on set X and $S(X)$ the class of all holomorphic self-maps of X. The expression $a \asymp b$ means that there is a positive constant C such that $C^{-1}\ \ a \leq b \leq ca$

For an $f \in H(\mathbb{B})$ with the Taylor expansion $f(z) = \sum_{|\beta|\geq 0} a_\beta z^\beta$, let

$$\Re f(z) = \sum_{|\beta|\geq 0} |\beta| a_\beta z^\beta$$

be the radial derivative of f, where $\beta = (\beta_1, \beta_2 \dots, \beta_n)$ is a multi-index, $|\beta| = \beta_1 + \cdots + \beta_n$ and $z^\beta = z_1^{\beta_1} \dots z_n^{\beta_n}$. [11]. It is easy to see that

$$\Re f(z) = \langle \nabla f(z), \bar{z} \rangle$$

Where ∇f is the complex gradient of function f, that is

$$\nabla f = \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\right)$$

Let $k \in \mathbb{N}$, the iterated logarithmic Bloch space $\mathfrak{B}_{\log_k} = \mathfrak{B}_{\log_k}(\mathbb{B})$, which was introducedin [184], consists of all $f \in H(\mathbb{B})$ such that

$$b_{\log_k}(f) = sup_{z\in\mathbb{B}}(1 - |z|^2)\left(\prod_{j=1}^{k} In^{[j]} \frac{e^{[k]}}{1 - |z|^2}\right)$$
$$\Re f(z) < \infty.$$

Where $e^{[k]}$ is defined inductively by $e^{[1]} = e, e^{[k]} = e^{[k-1]}$ *and*

$$In^{[j]} z = \underbrace{In \dots Inz}_{\text{j times}}$$

The norm on $\mathfrak{B}_{\log_k}$ is given by

$$\|f\|_{\mathfrak{B}_{\log_k}} = [f(0)] + \mathrm{b}_{\log_k}(f) \qquad (67)$$

For k = 1, we obtain the logarithmic Bloch space $\mathfrak{B}_{\log_1} = \mathfrak{B}_{\log}$.

The logarithmic Bloch space on D appeared in characterizing the multipliers of the Bloch space (see [3]). For the case of the unit ball see [9].

The Little iterated logarithmic Bloch space $\mathfrak{B}_{\log_{k,0}} = \mathfrak{B}_{\log_{k,0}}(\mathbb{B})$ consist of all $f \in \mathfrak{B}_{\log_k}$ such that

$$\lim_{|z|\to 1}(1 - |z|^2)\left(\prod_{j=1}^{k} \mathrm{In}^{[j]} \frac{e^{[k]}}{1 - |z|^2}\right) \Re f(z) = 0$$

A positive continuous function ϕ on the interval [0, 1} is called normal [11] if there are $\delta \in [0,1]$ *and* a *and* $, b\ \ 0 < a < b$ such that

$$\frac{\phi(r)}{(1-r)^a} \quad [\delta, 1]\ \ lim_{r\to 1} \frac{\phi(r)}{(1-r)^a} = 0.$$
$$\frac{\phi(r)}{(1-r)^b} \quad [\delta, 1]\ \ lim_{r\to 1} \frac{\phi(r)}{(1-r)^b} = \infty$$

Since the function

$$w(r) = (1-r^2)\prod_{j=1}^{k} In^{[j]}\frac{e^{[k]}}{1-r^2}.$$

is normal, by Theorem (4.3.3). in [60] we have that

$$\|f\|_{\mathfrak{B}_{\log_k}} \asymp |f(0)|+sup_{z\in\mathbb{B}}(1-|z|^2)\left(\prod_{j=1}^{k} In^{[j]}\frac{e^{[k]}}{1-|z|^2}\right)|\nabla f(z)| \qquad (68)$$

On the other hand, by Lemma (4.3.1) in [184] we know that the function

$$h_k(x) = x\prod_{j=1}^{k} In^{[j]}\frac{e^{[k]}}{x} \qquad (69)$$

$$h_k\left(\frac{x}{2}\right) \quad h_k(x) \asymp h_k\left(\frac{x}{2}\right) \;\; x\in(0,2]$$

Is increasing on the interval (0.1], from which it easily follows that $h_k\left(\frac{x}{2}\right)$ is increasing on the interval (0, 2] and $h_k(x) \asymp h_k\left(\frac{x}{2}\right), x\in(0,2]$.

From this, (68) and some simple estimates we have also that

$$\|f\|_{\mathfrak{B}_{\log_k}} \asymp |f(0)|+sup_{z\in\mathbb{B}}(1-|z|)\left(\prod_{j=1}^{k} In^{[j]}\frac{e^{[k]}}{1-|z|}\right)|\nabla f(z)| =: \|f\|'_{\mathfrak{B}_{\log_k}}$$

From now on the quantity $\|f\|'_{\mathfrak{B}_{\log_k}}$ will be used as the norm on $\mathfrak{B}_{\log_k}(\mathbb{B})$ and we will regard that an $f\in\mathfrak{B}_{\log_k}(\mathbb{B})$ belongs to the Little iterated logarithmic bloch space $\mathfrak{B}_{\log_k,0}(\mathbb{B})$ if

$$\lim_{|z|\to 1^-}(1-|z|)\left(\prod_{j=1}^{k} In^{[j]}\frac{2e^{[k]}}{1-|z|}\right)|\nabla f(z)| = 0$$

The weighted- type space $H_\mu^\infty = H_\mu^\infty(\mathbb{B})\, consist\ of\ f\in H(\mathbb{B})$such that

$$\|f\|_{H_\mu^\infty} := sup_{z\in\mathbb{B}}\, \mu(z)|f(z)| < \infty,$$

Where μ is a weight , that is , a positive continuous function on $\mathbb{B}$.

Assume $u\in H(\mathbb{B})\quad and\ \varphi\in S(\mathbb{B})$, the weighted composition operator induced by u and ϕ is defined on $H(\mathbb{B})$ by

$$(uC_\varphi f)(z) = u(z)f(\varphi(z))$$

A typical problem is to provide function theoretic characterizations when u and ϕ induce bounded or compact weighted composition operators between two given spaces of holomorphic functions. It is also of some interest to calculate operator norm of weighted composition operators. Some recent results in the area can be found, e.g., in [169], [170], [182], [183], [37], [64], [26], [65]–[41], [172]–[188], [67], [28], [69], [189].

Motivated by [14], [15], [16], [23] (see also [170], [70]), in [26] we calculated operator norm of $uC_\varphi: \mathfrak{B}(\mathbb{B})(or\mathfrak{B}_0(\mathbb{B})) \to H_\mu^\infty$.

Namely, the following formula was proved

$$\|uC_\varphi\|_{\mathfrak{B}(\mathbb{B})(or\mathfrak{B}_0(\mathbb{B}))\to H_\mu^\infty} = \max\left\{\|u\|_{H_\mu^\infty}, \frac{1}{2}sup_{z\in\mathbb{B}}\, \mu(z)|u(z)|\frac{1+|\varphi(z)|}{1-|\varphi(z)|}\right\}$$

$$\|f\|'_{\mathfrak{B}^\alpha} := |f(0)|+sup_{z\in\mathbb{B}}(1-|z|^2)^\alpha\, |\nabla f(z)| \qquad (70)$$

$uC_\varphi: \mathfrak{B}(\mathbb{B})(or\mathfrak{B}_0(\mathbb{B})) \to H_\mu^\infty$, when

$\|uC_\varphi\|_{\mathfrak{B}^\alpha(or\mathfrak{B}_0{}^\alpha)\to H_\mu^\infty}$

$$= \max\left\{\|u\|_{H_\mu^x}, \frac{1}{2} sup_{z\in\mathbb{B}} \frac{\mu(z)|u(z)|}{\alpha-1} \frac{1}{(1-|\varphi(z)|)^{\alpha-1}} - 1\right\}, \quad (71)$$

but instead of the norm in (70) we have used the following norm

$$\|f\|_{\mathfrak{B}^\alpha} := |f(0)| + sup_{z\in\mathbb{B}}(1-|z|)^\alpha\, |\nabla f(z)|,$$

on space B^a. As it was noticed in [61], this slight change of the definition of norm k _ $k^0{}_B$a on space B^a enabled us to calculate norm in (71), which is difficult if norm on B ^{a} is k _ $k^0{}_B$a. This shows that calculating operator norms depend much on the choice of the norms on the spaces which we deal with. There are general formulae for operator norm of an operator from a general Banach space to a weighted-type space (see, e.g. [183]). However, they are not proved for any weight, but for a specific type of weights, such as associated weights (see, e.g. [162]). Hence, it is of some interest to calculate operator norms when function l in the image space $H^1{}_l$ is a weight. Motivated by this line of research here we calculate operator norms of some operators.

Here we calculate operator norm of $uC_\varphi\colon \mathfrak{B}_{\log_k}(\mathrm{B})\left(\mathfrak{B}_{\log_{k,0}}(\mathrm{B})\right) \to H_\mu^\infty(\mathrm{B})$. Before we calculate it we prove an auxiliary result .

Lemma (4.3.1)[181]: Let $k\in\mathbb{N}$ and $f\in\mathfrak{B}_{\log_k}(\mathbb{B})$Then the following inequality holds

$$f|z| \le f|0| + \mathrm{b}'_{\log_{k,}}(\mathrm{f})\left(\mathrm{In}^{[k=1]}\frac{2\mathrm{e}^{[k]}}{1-|z|} - \mathrm{In}^{[k=1]}2\mathrm{e}^{[k]}\right), \quad (72)$$

where

$$\mathrm{b}'_{\log_{k,}}(f) = sup_{z\in\mathbb{B}}(1-|z|)\left(\prod_{j=1}^{k} In^{[j]}\frac{2e^{[k]}}{1-|z|}\right)|\nabla f(z)|$$

Proof. Using the definition of norm $\|.\|'_{\mathfrak{B}\log_{k,}}$, we have

$$|f(z) - f(0)| = \left|\int_0^1 \frac{d}{dt}\big(f(tz)\big)dt\right| = \left|\int_0^1 \langle\nabla f(tz), \bar{z}\rangle\right|$$

$$\le \mathrm{b}'_{\log_{k,}}(\mathrm{f})\int_0^1 \frac{|\mathrm{z}|\mathrm{dt}}{(1-|tz|)\left(\prod_{j=1}^k In^{[j]}\frac{2e^{[k]}}{1-t|z|}\right)}$$

$$= \mathrm{b}'_{\log_{k,}}(\mathrm{f})\left(\mathrm{In}^{[k+1]}\frac{2e^{[k]}}{1-|z|} - \mathrm{In}^{[k+1]}\,2\mathrm{e}^{[k]}\right) \quad (73)$$

From which the lemma easily follows.

Theorem (4.3.2)[181]: Assume $k\in\mathbb{N}, u\in H(\mathbb{B})$ and $\varphi\in S(\mathbb{B}), \mu$ is a weight and $uC_\varphi\colon \mathfrak{B}\log_{k,}(\mathbb{B}) \to H_\mu^\infty$ is bounded. Then

$\|uC_\varphi\|_{\mathfrak{B}_{\log_{k,}}\left(or\mathfrak{B}_{\log_{k,0,}}\right)\to H_\mu^\infty}$

$$= \max\left\{\|u\|_{H_\mu^\infty}, \frac{1}{2} sup_{z\in\mathbb{B}}\, \mu(z)|u(z)|\left(In^{[k+1]}\frac{2e^{[k]}}{1-|\varphi(z)|} - In^{[k+1]}2\mathrm{e}^{[k]}\right)\right\} \quad (74)$$

Proof: if $f\in\mathfrak{B}_{\log_{k,}}$, by Lemma (4.3.1) and the definition of $\|.\|'_{\mathfrak{B}\log_{k,}}$. we get

$$\|uC_\varphi f\|_{H_\mu^x} = sup_{z\in\mathbb{B}}\, \mu(z)\big|u(z)f(\varphi(z))\big| \le sup_{z\in\mathbb{B}}\, \mu(z)|u(z)||f(0)|$$

$$+\mathrm{b}'_{\log_{k,}}(\mathrm{f})\left(\mathrm{In}^{[k+1]}\frac{2e^{[k]}}{1-|\varphi(z)|} - \mathrm{In}^{[k+1]}\,2\mathrm{e}^{[k]}\right)$$

$$\leq \|f\|_{\mathfrak{B}_{\log_k,}} \max\left\{\begin{matrix} \|u\|_{H^\infty_\mu}, \frac{1}{2} sup_{z\in\mathbb{B}}\, \mu(z)|u(z)| \\ \left(\mathrm{In}^{[k+1]} \frac{2e^{[k]}}{1-|\varphi(z)|} - \mathrm{In}^{[k+1]} 2e^{[k]}\right)\end{matrix}\right\};$$

from which it follows that

$$\|uC_\varphi\|_{\mathfrak{B}_{\log_k,} \to H^\infty_\mu} \leq \max\left\{\|u\|_{H^\infty_\mu}, sup_{z\in\mathbb{B}}\, \mu(z)|u(z)| \left(\mathrm{In}^{[k+1]} \frac{2e^{[k]}}{1-|\varphi(z)|} - \mathrm{In}^{[k+1]} 2e^{[k]}\right)\right\} \quad (75)$$

Let $f_0(z) \equiv 1$ then $\|f_0\|'_{\mathfrak{B}_{\log_k}}=1$ and $f \in \mathfrak{B}_{\log_{k,.0}}$. Hence

$$\|uC_\varphi\|_{\mathfrak{B}_{\log_{k,0}} \to H^\infty_\mu} = \|f_0\|'_{\mathfrak{B}\log_{k,0}} \|uC_\varphi\|_{\mathfrak{B}_{\log_{k,0}} \to H^\infty_\mu} \geq \|uC_\varphi f_0\|_{H^x_\mu} = \|u\|_{H^x_\mu} \quad (76)$$

For a fixed $w \in \mathbb{B}$ set

$$f_w(z) = \mathrm{In}^{[k+1]} \frac{2e^{[k]}}{1-\langle z,w\rangle} - \mathrm{In}^{[k+1]} 2e^{[k]} \quad (77)$$

Since the function $h_k(x/2)$ is increasing on the interval (0, 2] we have that

$$(1-|z|)\left(\prod_{j=1}^{k} In^{[j]} \frac{2e^{[k]}}{1-|z|}\right) |\nabla f_w(z)| = \frac{|w|(1-|z|)\prod_{j=1}^{k} In^{[j]} \frac{2e^{[k]}}{1-|z|}}{(1-|z,w|)\prod_{j=1}^{k} In^{[j]} \frac{2e^{[k]}}{1-|z,w|}}. \quad (78)$$

$$\leq \frac{|w|(1-|z|)\prod_{j=1}^{k} In^{[j]} \frac{2e^{[k]}}{1-|z|}}{\left(1-\big|z||w|\big|\right)\prod_{j=1}^{k} In^{[j]} \frac{2e^{[k]}}{1-|z|,|w|}} \cdot \frac{\left(1-\big|z||w|\big|\right)\prod_{j=1}^{k} In^{[j]} \frac{2e^{[k]}}{1-|z||w|}}{|1-\langle z,w\rangle|\prod_{j=1}^{k} In^{[j]} \frac{2e^{[k]}}{|1-\langle z,w\rangle|}} \leq 1. \quad (79)$$

From this and since $f_w(0) = (0)$ it follows that $sup_{w\in\mathbb{B}}\|f_w\|'_{\mathfrak{B}_{\log_k}} \leq 1$, while by letting $|z| \to 1^-$ in (78) we get $f_w \in \mathfrak{B}_{\log_{k,.0}}$ for each $w \in \mathbb{B}$.

This along with the boundedness of $uC_\varphi : \mathfrak{B}_{\log_{k,0}} \to H^\infty_\mu$, for $\varphi(w) \neq 0$ and every $r \in (0,1)$ implies

$$\|uC_\varphi\|_{\mathfrak{B}_{\log_{k,0}} \to H^\infty_\mu} \geq \left\|uC_\varphi f_{\frac{r\varphi(w)}{|\varphi(w)|}}\right\|_{H^\infty_\mu} = sup_{z\in\mathbb{B}}\, \mu(z)|u(z)| \left|\mathrm{In}^{[k+1]} \frac{2e^{[k]}}{1-\frac{r\langle\varphi(z),\varphi(w)\rangle}{|\varphi(w)|}} - \mathrm{In}^{[k+1]} 2e^{[k]}\right| \geq \mu(w)|u(w)| \left(\mathrm{In}^{[k+1]} \frac{2e^{[k]}}{1-r|\varphi(w)|} \mathrm{In}^{[k+1]} 2e^{[k]}\right) \quad (80)$$

If $\varphi(w) = 0$, then (80) obviously holds.

Letting $r \to 1^-$ in (80), then talking the supermum over the unit ball $\mathbb{B}$ in such obtained inequality, we get

$$\|uC_{\varphi}\|_{\mathfrak{B}_{\log_{k,},0}\to H_{\mu}^{\infty}}$$
$$\geq sup_{z\in\mathbb{B}}\,\mu(z)|u(z)|\left(\mathrm{In}^{[k+1]}\frac{2\mathrm{e}^{[k]}}{1-|\varphi(z)|}-\mathrm{In}^{[k+1]}2e^{[k]}\right) \qquad (81)$$

From (76) and (81) it follows that

$$\|uC_{\varphi}\|_{\mathfrak{B}_{\log_{k,},0}\to H_{\mu}^{\infty}}$$
$$\geq \max\left\{\|u\|_{H_{\mu}^{\infty}}, sup_{z\in\mathbb{B}}\,\mu(z)|u(z)|\left(\mathrm{In}^{[k+1]}\frac{2\mathrm{e}^{[k]}}{1-|\varphi(z)|}-\ \mathrm{In}^{[k+1]}2\mathrm{e}^{[k]}\right)\right\} \quad (82)$$

From (75) and (82) and the inequality

$$\|uC_{\varphi}\|_{\mathfrak{B}_{\log_{k,},0}\to H_{\mu}^{\infty}}\leq\|uC_{\varphi}\|_{\mathfrak{B}_{\log_{k,}}\to H_{\mu}^{x}}$$

Formula (74) follows, as desired.

We calculate the norm of the operator $\mathrm{DC}_{\varphi}:\mathfrak{B}_{\log_{k,}}(D)\ \left(OR\ \mathfrak{B}_{\log_{k,},0}(D)\right)\to H_{\mu}^{\infty}(D)$.

Theorem (4.3.3)[181]: Assume $k\in\mathbb{N},\mu$ is a weight, $\varphi\in s(\mathbb{D})$, and that the operator $\mathrm{DC}_{\varphi}:\mathfrak{B}_{\log_{k,}}(D)\ \left(OR\ \mathfrak{B}_{\log_{k,},0}(D)\right)\to H_{\mu}^{\infty}(D)$ is bounded. Then the following formulae true hold

$$\|\mathrm{DC}_{\varphi}\|_{\mathfrak{B}_{\log_{k,},0}\to H_{\mu}^{\infty}}=\|\mathrm{DC}_{\varphi}\|_{\mathfrak{B}_{\log_{k,}}\to H_{\mu}^{X}}$$
$$=\ sup_{z\in\mathbb{B}}\ \frac{\mu(z)|\varphi'(z)|}{1-|\varphi(z)|\prod_{j=1}^{k}In^{[j]}\frac{2e^{[k]}}{1-|\varphi(z)|}} \qquad (83)$$

Proof: for every $f\in\mathfrak{B}_{\log_{k,}}$ and $z\in\mathbb{D}m,$ we have

$$\mu(z)\left|\left(DC_{\varphi}f\right)(z)\right|=\ \mu(z)|\varphi'(z)||f'\varphi(z)|$$
$$\leq\frac{\mu(z)|\varphi'(z)|}{(1-|\varphi(z)|)\prod_{j=1}^{k}In^{[j]}\frac{2e^{[k]}}{1-|\varphi(z)|}}\|f\|'_{\mathfrak{B}_{\log_{k,}}}$$

Hence, by taking the supremum over $z\in\mathbb{D}$ and the unit ball in $\mathfrak{B}_{\log_{k,}}$. we obtain

$$\|\mathrm{DC}_{\varphi}\|_{\mathfrak{B}_{\log_{k,},0}\to H_{\mu}^{\infty}}\leq\ sup_{z\in\mathbb{B}}\ \frac{\mu(z)|\varphi'(z)|}{1-|\varphi(z)|\prod_{j=1}^{k}In^{[j]}\frac{2e^{[k]}}{1-|\varphi(z)|}} \qquad (84)$$

Since $\mathrm{DC}_{\varphi}:\mathfrak{B}_{\log_{k,},0}\to H_{\mu}^{\infty}$ is bounded , and by using the test functions in (77) for the case $n=1$, we get

$$\|\mathrm{DC}_{\varphi}\|_{\mathfrak{B}_{\log_{k,},0}\to H_{\mu}^{\infty}}\geq\left\|\mathrm{DC}_{\varphi}\left(\mathrm{f}_{\mathrm{r}\frac{\varphi(w)}{|\varphi(w)|}}\right)\right\|_{H_{\mu}^{\infty}}=sup_{z\in\mathbb{B}}\,\mu(z)\left|\varphi'(z)\mathrm{f}_{\mathrm{r}\frac{\varphi(w)}{|\varphi(w)|}}\varphi(z)\right|$$
$$\geq\frac{\mu(w)|\varphi'(w)|r}{(1-\mathrm{r}|\varphi(w)|)\prod_{j=1}^{k}In^{[j]}\frac{2e^{[k]}}{1-r|\varphi(w)|}} \qquad (85)$$

for each $\varphi(w)=0,$ for some $w\in\mathbb{D}$, then since $\varphi\in H(\mathbb{D})$, for $\varphi(z)\not\equiv0$, there is a sequence $(w_m)_{m\in\mathbb{N}}\subset\mathbb{D}$, such that $w_m\to w$ as $m\to\infty$ and $\varphi(w_m)\neq0$ for every $m\in\mathbb{N}$ consequently , we have that $\varphi(w_m)\to\varphi(w),\varphi'(w_m)\to\varphi'(w)$, as $m\to\infty$ and from (85) that

$$\left\|DC_{\varphi}\right\|_{\mathfrak{B}_{\log_{k,0}}\to H_{\mu}^{\infty}} \geq \frac{\mu(w_m)|\varphi'(w_m)|r}{(1-r|\varphi(w_m)|)\prod_{j=1}^{k} In^{[j]}\frac{2e^{[k]}}{1-r|\varphi(w_m)|}} \tag{86}$$

For every $m \in \mathbb{N}$

By letting $m \to \infty \quad in\ (20) we\ get$

$$\left\|DC_{\varphi}\right\|_{\mathfrak{B}_{\log_{k,0}}\to H_{\mu}^{\infty}} \geq \frac{\mu(w)|\varphi'(w)|r}{\prod_{j=1}^{k} In^{[j]}2e^{[k]}}.$$

For each $w \in \mathbb{D}$ such that $\varphi(w) = 0$. Hence, we have that (85) holds for every $w \in \mathbb{D}$ letting $r \to 1^{-}$ in (85) we obtain

$$\frac{\mu(w)|\varphi'(w_m)|}{(1-|\varphi(w)|)\prod_{j=1}^{k} In^{[j]}\frac{2e^{[k]}}{1-|\varphi(w)|}} \leq \left\|DC_{\varphi}\right\|_{\mathfrak{B}_{\log_{k,0}}\to H_{\mu}^{\infty}} \tag{87}$$

For every $w \in \mathbb{D}$, from (84), (87) and since $\left\|DC_{\varphi}\right\|_{\mathfrak{B}_{\log_{k,0}}\to H_{\mu}^{\infty}} \leq \left\|DC_{\varphi}\right\|_{\mathfrak{B}_{\log_{k,}}\to H_{\mu}^{\infty}}$ (83) follows.

The logarithmic Bloch-type space $\mathfrak{B}_{\log^{\beta}}^{\alpha} = \mathfrak{B}_{\log^{\beta}}^{\alpha}(\mathbb{B}), \alpha > 0, \beta \geq 0,$ which was introduced in [1] consists of all $f \in H(\mathbb{B})$ such that

$$b_{\alpha,\beta}(f) \coloneqq sup_{z\in\mathbb{B}}(1-|z|)^{\alpha} In\frac{e^{\beta/\alpha}}{1-|z|}|\mathbb{R}f(z)| < \infty.$$

The norm on $\mathfrak{B}_{\log^{\beta}}^{\alpha}$ $\quad can\ be\ introduced\ as$

$$\|f\|_{\mathfrak{B}_{\log^{\beta}}^{\alpha}} = |f(0)| + b_{\alpha,\beta}(f),$$

but we will here use the following equivalent norm

$$\|f\|'_{\mathfrak{B}_{\log^{\beta}}^{\alpha}} = |f(0)| + b'_{\alpha,\beta}(f)$$

Where $\quad b'_{\alpha,\beta}(f) \coloneqq sup_{z\in\mathbb{B}}(1-|z|)^{\alpha}\left(In\frac{2e}{1-|z|}\right)^{\beta}|\nabla f(z)| < \infty.$

In the proof of the next result we will need two auxiliary results which are incorporated in the lemmas which follow.

Lemma (4.3.4)[181]: Let $f \in \mathfrak{B}_{\log^{\beta}}(B),\ \beta \in (1,0)$, then the following inequality holds

$$|f(z)| \leq |f(0)| + \frac{b'_{\alpha,\beta}(f)}{1-\beta}\left(\left(In\frac{2e}{1-|z|}\right)^{1-\beta} - (In2e)^{1-\beta}\right). \tag{88}$$

Where $\quad b'_{\alpha,\beta}(f) = sup_{z\in\mathbb{B}}(1-|z|)\left(In\frac{2e}{1-|z|}\right)^{\beta}|\nabla f(z)|.$

Proof: using the definition of space $\mathfrak{B}_{\log^{\beta}}$ in the second equality in (73) we get

$$|f(z)-f(0)| \leq b'_{\alpha,\beta}(f)\int_0^1 \frac{|z|dt}{(1-t|z|)\left(In\frac{2e}{1-t|z|}\right)^{\beta}}$$

$$= \frac{b'_{1,\beta}(f)}{1-\beta}\left(\left(In\frac{2e}{1-|z|}\right)^{1-\beta} - (In2e)^{1-\beta}\right),$$

From which (88) easily follows.

The following lemma was proved in [1].

Lemma (4.3.5)[181]: Assume $\alpha > 0, \beta \geq 0\ and\ \gamma \geq \frac{\beta}{\alpha}. then\ the\ function$

$$h_{\alpha,\beta,\gamma}(x) = x^{\alpha}\left(In\frac{e^{\gamma}}{x}\right)^{\beta}.$$

Is increasing on the interval (0,1]

Theorem (4.3.6)[181]: $k \in \mathbb{N},\ \beta \in (0,1), u \in H(\mathbb{B}), \varphi \in S(\mathbb{B})M$, $uC_{\varphi}: \mathfrak{B}_{\log^{\beta}}(B) \to H_{\mu}^{\infty}(B)$ is bounded then

$$\|uC_{\varphi}\|_{\mathfrak{B}_{\log\beta}\ or\left(\mathfrak{B}_{\log^{\beta},0}\right)\to H_{\mu}^{\infty}} = \max\left\{\|u\|_{H_{\mu}^{\infty}}, sup_{z\in\mathbb{B}}\ \frac{\mu(z)|u(z)|}{1-\beta}\left(\left(\text{In}\frac{2\text{e}}{1-|\varphi(z)|}\right)^{1-\beta} - (\text{In}2\text{e})^{1-\beta}\right)\right\}. \quad (89)$$

Proof. If $f \in \mathfrak{B}_{\log\beta,}$ by Lemma (4.3.4) and the definition of the norm $\|.\|'_{\mathfrak{B}_{\log\beta}}$ we get

$$\|uC_{\varphi}f\|_{\mu} = sup_{z\in\mathbb{B}}\ \mu(z)|u(z)f(\varphi(z))|$$
$$\leq sup_{z\in\mathbb{B}}\ \mu(z)|u(z)|\left(|f(0)|\frac{b'_{1,\beta}(f)}{1-\beta}\left(In\frac{2e}{1-|z|}\right)^{1-\beta} - (In2e)^{1-\beta}\right)$$
$$\leq \|f\|'_{\mathfrak{B}_{\log^{\beta}}} \max\left\{\|u\|_{H_{\mu}^{\infty}}, sup_{z\in\mathbb{B}}\ \frac{\mu(z)|u(z)|}{1-\beta}\left(\left(\text{In}\frac{2\text{e}}{1-|\varphi(z)|}\right)^{1-\beta} - (\text{In}2\text{e})^{1-\beta}\right)\right\}$$

from which it follows that

$$\|uC_{\varphi}\|_{\mathfrak{B}_{\log\beta}\ \leq\to H_{\mu}^{\infty}} \leq \max\left\{\|u\|_{H_{\mu}^{\infty}}, sup_{z\in\mathbb{B}}\ \frac{\mu(z)|u(z)|}{1-\beta}\left(\left(\text{In}\frac{2\text{e}}{1-|\varphi(z)|}\right)^{1-\beta} - (\text{In}2\text{e})^{1-\beta}\right)\right\} \quad (90)$$

Let $f_0(z) \equiv 1\ .\|f_0\|'_{\mathfrak{B}_{\log^{\beta}}} = 1\ \ and\ \ f \in \mathfrak{B}_{\log^{\beta},0}$. Hence we have

$$\|uC_{\varphi}\|_{\mathfrak{B}_{\log^{\beta},0}\to H_{\mu}^{\infty}} = \|f_0\|'_{\log^{\beta},0}\|uC_{\varphi}\|_{\mathfrak{B}_{\log^{\beta},0}\to H_{\mu}^{\infty}} \geq \|uC_{\varphi}f_0\|_{H_{\mu}^{\infty}} = \|u\|_{H_{\mu}^{\infty}} \quad (91)$$

For a fixed $w \in \mathbb{B}$

$$\text{f}_{\text{w}}(\text{z}) = \frac{1}{1-\beta}\left(\left(\text{In}\frac{2\text{e}}{1-\langle\text{z},\text{w}\rangle}\right)^{1-\beta} - (\text{In}2\text{e})^{1-\beta}\right) \quad (92)$$

Since by Lemma (4.3.5) the function $h_{1,\beta,1}(x/2), \beta \in (0,1)$ is increasing on the interval (0, 2] we have

$$(1-|\text{z}|)\left(\text{In}\frac{2\text{e}}{1-\langle\text{z},\text{w}\rangle}\right)^{\beta}|\nabla\text{f}_{\text{w}}(\text{z})| = \frac{|\text{w}|(1-|\text{z}|)\left(\text{In}\frac{2\text{e}}{1-\text{z}}\right)^{\beta}}{|1-\langle\text{z},\text{w}\rangle|\left(\text{In}\frac{2\text{e}}{1-\text{z}}\right)^{\beta}}, \quad (93)$$

$$\leq \frac{|\text{w}|(1-|\text{z}|)\left(\text{In}\frac{2\text{e}}{1-\text{z}}\right)^{\beta}}{\big|1-|\text{z}||\text{w}|\big|\left(\text{In}\frac{2\text{e}}{1-\text{z}}\right)^{\beta}}.\frac{(1-|\text{z}||\text{w}|)\left(\text{In}\frac{2\text{e}}{1-|\text{z}||\text{w}|}\right)^{\beta}}{|1-\langle\text{z},\text{w}\rangle|\left(\text{In}\frac{2\text{e}}{|1-\langle\text{z},\text{w}\rangle|}\right)^{\beta}} \leq 1. \quad (94)$$

from this and since $f_w(0)=0$. $sup_{w\in\mathbb{B}}\|f_w\|'_{\mathfrak{B}_{\log^\beta}}\le 1$, while by letting $|z|\to 1^-$ in (93) we have that $f_w\in\mathfrak{B}_{\log^\beta,0}$ for a fixed $w\in\mathbb{B}$.

This along with the boundedness of $uC_\varphi:\mathfrak{B}_{\log^\beta,0}\to H_\mu^\infty$, for $\varphi(w)\neq 0$ and every $r\in(0,1)$ implies

$$\|uC_\varphi\|_{\mathfrak{B}_{\log\beta,0}\ \le\to H_\mu^\infty}\ge\left\|uC_\varphi f_{\frac{r\varphi(w)}{|\varphi(w)|}}\right\|_{H_\mu^\infty},$$

$$sup_{z\in\mathbb{B}}\frac{\mu(z)|u(z)|}{1-\beta}\left|\left(\mathrm{In}\frac{2\mathrm{e}}{1-\frac{\mathrm{r}\langle\varphi(\mathrm{z}),\varphi(\mathrm{w})\rangle}{|\varphi(\mathrm{w})|}}\right)^{1-\beta}-(\mathrm{In}2\mathrm{e})^{1-\beta}\right|$$

$$\ge\frac{\mu(z)|u(z)|}{1-\beta}\left(\left(\mathrm{In}\frac{2\mathrm{e}}{1-\mathrm{r}\,|\varphi(\mathrm{w})|}\right)^{1-\beta}-(\mathrm{In}2\mathrm{e})^{1-\beta}\right)\tag{95}$$

If $\varphi(w)=0$, then (95) obviously holds. Letting $r\to 1^-$ in (95), then taking the supremum over the unit ball in such obtained inequality. We get

$$\|uC_\varphi\|_{\mathfrak{B}_{\log^\beta,0}\to H_\mu^\infty}\ge sup_{z\in\mathbb{B}}\frac{\mu(z)|u(z)|}{1-\beta}\left(\left(\mathrm{In}\frac{2\mathrm{e}}{1-|\varphi(\mathrm{z})|}\right)^{1-\beta}-(\mathrm{In}2\mathrm{e})^{1-\beta}\right)\tag{96}$$

From (91) and (96) it follows that

$$\|uC_\varphi\|_{\mathfrak{B}_{\log^\beta,0}\to H_\mu^\infty}$$

$$\ge\max\left\{\|u\|_{H_\mu^\infty},sup_{z\in\mathbb{B}}\frac{\mu(z)|u(z)|}{1-\beta}\left(\mathrm{In}\frac{2\mathrm{e}}{1-|\varphi(\mathrm{z})|}\right)^{1-\beta}-(\mathrm{In}2\mathrm{e})^{1-\beta}\right\}\tag{97}$$

From (90), (97) and the inequality

$$\|uC_\varphi\|_{\mathfrak{B}_{\log^\beta,0}\to H_\mu^\infty}\le\|uC_\varphi\|_{\mathfrak{B}_{\log^\beta}\to H_\mu^\infty}$$

We calculate norm of $\mathrm{DC}_\varphi:\mathfrak{B}_{\log^\beta}(D)\ \ OR\left(\mathfrak{B}_{\log^\beta,0}(D)\right)\to H_\mu^\infty(D)$.

Theorem (4.3.7)[181]: Assume, $\beta\in(0,1),\mu$ is a weight, $\varphi\in S(\mathbb{D})$ and the operator, $\mathrm{DC}_\varphi:\mathfrak{B}_{\log^\beta}(\mathbb{D})\left(or\ \mathfrak{B}_{\log^\beta,0}(\mathbb{D})\right)\to H_\mu^\infty(\mathbb{D})$ is bounded, then the following formulae true hold

$$\|DC_\varphi\|_{\mathfrak{B}_{\log^\beta,0}\to H_\mu^\infty}=\|\mathrm{DC}_\varphi\|_{\mathfrak{B}_{\log^\beta}\to H_\mu^\infty}$$

$$=\sup_{z\in\mathbb{D}}\frac{\mu(z)|\varphi'(z)|}{(1-|\varphi(\mathrm{z})|)\left(\mathrm{In}\frac{2\mathrm{e}}{1-|\varphi(\mathrm{z})|}\right)^{\beta}}\tag{98}$$

Proof: for every $f\in\log^\beta$ and $\mathrm{z}\in\mathbb{D}$, we have

$$\mu(z)\left|\left(\mathrm{DC}_\varphi\mathrm{f}\right)(\mathrm{z})\right|=\mu(z)|\varphi'(z)||f'\varphi(\mathrm{z})|$$

$$\le\frac{\mu(z)|\varphi'(z)|}{(1-|\varphi(\mathrm{z})|)\left(\mathrm{In}\frac{2\mathrm{e}}{1-|\varphi(\mathrm{z})|}\right)^{\beta}}\|f\|_{\mathfrak{B}_{\log^\beta}}$$

Hence by taking supreme over $z\in\mathbb{D}$ and the unit ball in $\mathfrak{B}_{\log^\beta}$, we obtain

$$\left\|DC_{\varphi}\right\|_{\mathfrak{B}_{\log^{\beta}}\to H_{\mu}^{\infty}} \leq sup_{z\in\mathbb{D}} \frac{\mu(z)|\varphi'(z)|}{(1-|\varphi(z)|)\left(\mathrm{In}\frac{2e}{1-|\varphi(z)|}\right)^{\beta}} \tag{99}$$

By using the test function in (92) for the case n=1 , we get

$$\left\|DC_{\varphi}\right\|_{\mathfrak{B}_{\log^{\beta},0}\to H_{\mu}^{\infty}} \geq \left\|f_{r\frac{\varphi(w)}{|\varphi(w)|}}\right\|_{H_{\mu}^{\infty}} = sup_{z\in\mathbb{D}}\, \mu(z)\left|\varphi'(z)f'_{r\frac{\varphi(w)}{|\varphi(w)|}}(\varphi(z))\right|$$

$$\geq \frac{\mu(w)|\varphi'(w)|r}{(1-r|\varphi(w)|)\left(\mathrm{In}\frac{2e}{1-|\varphi(w)|}\right)^{\beta}} \tag{100}$$

For each $\varphi(w) \neq 0$, if $\varphi(w) = 0$ for some $w \in \mathbb{D}$, then since $\varphi \in H(\mathbb{D})$, for $\varphi(z) \not\equiv 0$, there is a sequence $(w_m)_{m\in\mathbb{N}} \subset \mathbb{D}$, such that $w_m \to w$ as $m \to \infty$ and $\varphi(w_m) \neq 0$, for every $m \in \mathbb{N}$, thus $\varphi(w_m) \to \varphi(w), \varphi'(w_m) \to \varphi'(w)$ as $m \to \infty$ and from (100) we have that

$$\left\|DC_{\varphi}\right\|_{\mathfrak{B}_{\log^{\beta},0}\to H_{\mu}^{\infty}} \geq sup_{z\in\mathbb{D}} \frac{\mu(w_m)|\varphi'(w_m)|r}{(1-r|\varphi(w_m)|)\left(\mathrm{In}\frac{2e}{1-r|w_m|}\right)^{\beta}} \tag{101}$$

For every $m \in \mathbb{N}$, by letting $m \to \infty$ in (101) we get

$$\left\|DC_{\varphi}\right\|_{\mathfrak{B}_{\log^{\beta},0}\to H_{\mu}^{\infty}} \frac{\mu(w)|\varphi'(w)|r}{(\mathrm{In}2e)^{\beta}},$$

for each $w \in \mathbb{D}, \varphi(w) = 0$, hence, we have that (100)holds for every $w \in \mathbb{D}$ letting $r \to 1^{-}$ in (100) it follows that

$$\frac{\mu(w)|\varphi'(w)|r}{(1-|\varphi(w_m)|)\left(\mathrm{In}\frac{2e}{1-|\varphi(w)|}\right)^{\beta}} \leq \left\|DC_{\varphi}\right\|_{\mathfrak{B}_{\log^{\beta},0}\to H_{\mu}^{\infty}} \tag{102}$$

for each $w \in \mathbb{D}$, from (99) and (102) and the inequality

$$\left\|DC_{\varphi}\right\|_{\mathfrak{B}_{\log^{\beta},0}\to H_{\mu}^{\infty}} \leq \left\|DC_{\varphi}\right\|_{\mathfrak{B}_{\log^{\beta}}\to H_{\mu}^{\infty}},$$

Formula in (98) follow.

Chapter 5
Bloch-to-BMOA with Reverse Estimates and Weighted Bloch Spaces

We characterize those ϕ for which the composition operator $f \to f \circ \phi$ maps the Bloch space into BMOA. As an application, we study composition operators with values in the space BMOA. For $\mathcal{B}^{\omega}(B_d)$ denote the ω-weighted Bloch space in the unit ball B_d of $\mathbb{C}^d, d \geq 1$. We show that the quadratic integral $\int_x^1 \frac{\omega^2(t)}{t} dt, 0 < x < 1$, governs the radial divergence and integral reverse estimates in $\mathcal{B}^{\omega}(B_d)$.

Section (5.1): Compositions on Complex Balls

For $H(B_m)$ denote the space of holomorphic functions in the unit ball B_m of $\mathbb{C}^m, m \geq 1$.

The Bloch space $\mathfrak{B}(B_m)$ consists of those functions $f \in H(B_m)$ for which

$$\|f\|_{\mathfrak{B}(B_m)} = |f(0)| + sup_{\omega \in B_m} |\mathcal{R}f(\omega)|(1 - |\omega|^2) < \infty,$$

Where

$$\mathcal{R}f(\omega) = \sum_{j=1}^{m} \omega_j \frac{\partial f}{\partial \omega_j}(\omega), \qquad \omega \in B_m,$$

is the radial derivative of f. The Hardy space $H^p(B_n), p > 0, n \geq 1$, consists of functions $f \in H(B_n)$ such that

$$\|f\|_{H^p(B_n)}^p = sup_{0<r<1} \int_{\partial B_n} |f(r\zeta)|^p \, d\sigma_n(\zeta) < \infty,$$

where σ_n is the normalized Lebesgue measure on the sphere ∂B_n. Also, we consider BMOA(B_n), the space of holomorphic functions that have bounded mean oscillation on ∂B_n. Equivalent definitions of BMOA(B_n).

Given a holomorphic map $\varphi : B_n \to B_m$, the composition operator $C_\varphi : H(B_m) \to H(B_n)$ is defined by the following identity:

$$(C_\varphi f)(z) = f(\varphi(z)), \qquad f \in H(B_m), \qquad z \in B_n.$$

Various properties of C_φ are presented in the monographs [12], [144]. We describe those φ for which C_φ maps $\mathfrak{B}(B_m)$ $into$ $BMOA(B_n)$.

There is a series of results about the operators under consideration. In particular, characterizations of the bounded operators $C_\varphi : \mathfrak{B}(B_1) \to BMOA(B_n)$ were obtained in [121]; see also [195], [196], [198]. The cases $n = 1$ and $n \geq 2$ are rather deferent. Indeed, let $\varphi : B_n \to B_1$ be a holomorphic Lipschitz function of order 1. Then C_φ does not map $\mathfrak{B}(B_1)$ into $BMOA(B_n)$ when $\|\varphi\|_\infty = 1$ and $n = 1$, but C_φ maps $\mathfrak{B}(B_1)$ into $BMOA(B_n)$ when $n \geq 2$ (see [194] and [121], respectively). See also [193], [121].

For arbitrary $n, m \in \mathbb{N}$, the problem in question was considered only in [193], where the bounded and compact composition operators $C_\varphi : \mathfrak{B}(B_m) \to BMOA(B_n)$ are characterized under an additional regularity assumption about φ. Namely, the operator C_φ is bounded if and only if

$$\frac{(1 - |z|^2)|\mathcal{R}\varphi(z)|^2}{(1 - |\varphi(z)|^2)^2} dv_n(z) \text{ is a carelson mesure,} \qquad (1)$$

where v_n *is* Lebesgue measure on B_n and $v_n(B_n) = 1$.

We use the Möbius-invariance of the spaces $BMOA(B_n)$ and $\mathfrak{B}(B_m)$. So, for $z \in B_n$, let φ_z denote the involution of $B_n$such that φ_z (0) = *z*. Let B_m denote the Bergman metric on the ball B_m. The main result is the following theorem.

Recall that the Garsia seminorm on $BMOA(B_n)$ is defined by the identity

$$\|f\|_{G_{1(B_n)}} = sup_{z\in B_n}\ sup_{0<r<1} \int_{\partial B_n} |f\ (\phi_z\ (r\zeta))\ -\ f\ (z)|\ d\sigma_n(\zeta).$$

Therefore, (11) reduces to the property $\|f\|_{G_{1(B_n)}} < \infty$ when φ is replaced by $f \in H(B_n)$ and β_m is replaced by the Euclidean metric. So, as in [196], [90] for $n = m = 1$, we say that (11) defines the hyperbolic BMOA class. However, other names have been used for this class; see [195].

As observed in [193], the implication (1)⇒ (10) holds for all holomorphic maps $\varphi :$ $B_n \rightarrow B_m$. Hence, Theorem (5.1.3) guarantees that (1) implies (11) for arbitrary φ So, one could expect that (11) implies (1) for all φ. If this is the case, then it would be interesting to find a direct proof of the implication in question.

The classical seminorm on $BMOA(B_n)$ is defined by the identity

$$\|f\|_{\mathrm{BMOA}(B_n)} = \sup \frac{1}{\sigma_n(Q)} \int_Q |f^* - f_Q^*|\ d\sigma_n.$$

where f^* is the boundary function of f , f_Q^* is the average of f^* over Q, and the supremum is taken over all quasi-balls $Q = Q_{r(\eta)} = \{\xi \in \partial B_n : |1 - \langle \eta, \xi \rangle | < r\}, \eta \in \partial B_n$.

The hyperbolic analog of the property $\|f\|_{BMOA(B_n)} < \infty$ is the following one:

$$sup_{r>0,\eta\in\partial B_n} \frac{1}{Q_r(\eta)} \int_{Q_r(\eta)} \beta_m\left(\varphi^*(\zeta), \varphi\left(\eta\sqrt{1-r^2}\right)\right) d\sigma_n(\zeta) < \infty. \quad (2)$$

The relations between (11), (2) and similar properties will be considered else-where.
Basic properties of $\mathfrak{B}(B_m)$ and $BMOA(B_n)$ are collected. Further details are given in [191], [9]; see also [84] for *n* = *m* = 1.

The automorphism group of B_n denoted by Aut(B_n), consists of all biholomorphic mappings from B_n onto B_n. Given $z \in B_n$, the involution (or the Möbius transform) $\phi_z \in Aut(B_n)$ is defined for $\lambda \in B_n$ *as* follows:

$$\phi_z(\lambda) = -\lambda \ \ when\ z = 0,$$

$$\phi_z(\lambda) = \frac{z - P_z\lambda - \sqrt{1-|z|^2} Q_z\lambda}{1 - \langle \lambda - z \rangle} \quad when\ z \in B_n/\{0\},$$

where $P_z\lambda = |z|^{-2}\langle \lambda - z \rangle z$, $Q_z\lambda = \lambda - P_z\lambda$. To distinguish the involutions of B_m, we write ψ_ω, $\omega \in B_m$ in the place of φ_z , $z \in B_n$.

The hyperbolic BMOA is defined by (11) in terms of the Bergman metric β_m *on* B_m. Note that

$$\beta_m(\omega_1,\omega_2) = C\ log\ \frac{1+|\psi_{\omega_1}(\omega_2)|}{1-|\psi_{\omega_1}(\omega_2)|}, \omega_1,\omega_2 \in B_m$$

So, a holomorphic map $\varphi : B_n \to B_m$ is in the hyperbolic BMOA if and only if

$$sup_{z\in B_n} sup_{0<r<1} \int_{\partial B_n} log \frac{1}{1-\left|\psi_{\varphi(z)}\left(\varphi(\phi_z(r\zeta))\right)\right|^2}\ d\sigma_n(\zeta) < \infty. \qquad (3)$$

For $f \in H(B_m)$, put

$$\left|\tilde{\nabla} \mathrm{f}\,(\omega)\right|^2 = (1-|\omega|^2)(|\nabla \mathrm{f}\,(\omega)|^2 - |\mathcal{R}\,\mathrm{f}\,(\omega)|^2),\quad w \in B_m,$$

Where

$$\nabla \mathrm{f}\,(\omega) = \left(\frac{\partial f}{\partial \omega_1}(\omega), \dots, \frac{\partial f}{\partial \omega_m}(\omega)\right)$$

is the complex gradient of f .

Let $\tilde{\mathfrak{B}}(B_m)$ denote the quotient of $\mathfrak{B}(B_m)$ by the space of constant functions.

Then $\tilde{\mathfrak{B}}(B_m)$ is a Banach space with respect to the following norms:

$$sup_{\omega\in B_m} |Rf\ (\omega)|(1-|\omega|^2);$$
$$sup_{\omega\in B_m} |\nabla f\ (\omega)|(1-|\omega|^2);$$
$$\|f\|_{\tilde{\mathfrak{B}}(B_m)} = sup_{\omega\in B_m}\left|\tilde{\nabla}\mathrm{f}\,(\omega)\right|$$

Clearly, the above expressions are seminorms on $\mathfrak{B}(B_m)$; these seminorms degenerate exactly on the constant functions. The main advantage of $\|.\|_{\tilde{\mathfrak{B}}(B_m)}$ is its Möbius-invariance. Namely,

$$\|f\circ\psi\|_{\tilde{\mathfrak{B}}(B_m)} = \|f\|_{\tilde{\mathfrak{B}}(B_m)}$$

for all $\psi \in Aut(B_m), f \in \mathfrak{B}(B_m)$.

Also, a function $f \in H(B_m)$ belongs to $\mathfrak{B}(B_m)$if and only if there exists a Constant $C > 0$ such that

$$|f(\omega_1) - f(\omega_2)| \le C\beta_m(\omega_1,\omega_2)\ \ for\ all\ \omega_1,\omega_2 \in B_m. \qquad (4)$$

For $\zeta \in \partial B_{n.}$ and $r > 0$, put

$$Q_r(\zeta) = \{\xi \in \partial B_{n.}: |1-\langle\zeta,\xi\rangle| < r\}$$

Recall that the radial limits $|f^*(\zeta)| = \lim_{r\to1^-}|f(r\zeta)|$ are defined for σ_n– almost all for every $f \in H^1(B_n)$. Let denote the space of functions $f \in H^1(B_{n.})$ such that

$$|f(0)|^p + sup_{\zeta\in\partial B_n, r>0}\frac{1}{\sigma_n(Q)}\int_Q \left|f^* - f_Q^*\right|^p d\sigma_n < \infty. \qquad (5)$$

Where $p = 1, Q = Qr(\zeta)$ and

$$f_Q^* = \frac{1}{\sigma_n(Q)}\int_Q f^*\ d\sigma_n$$

The norm $\|f\|_{BMOA(B_n)}$ is defined as the left-hand side of (5) with $p = 1$. By the John-Nirenberg theorem, there exist constants $A(n) > 0\ and\ C(n) > 0$ such that

$$\int_{\partial B_n} exp(|f^*(\zeta)|)\, d\sigma_n(\zeta) \le C(n) \qquad (6)$$

for all $f \in BMOA(B_n)$ with $\|f\|_{BMOA(B_n)} \le A(n)$. The John-Nirenberg inequality guarantees that $f \in BMOA(B_n)$ if and only if $f \in H^2(B_n)$ and (5) holds with $p = 2$.

The proof of Theorem (5.1.3) will be based on the following fact: (5) holds for $p = 1$ or for $p = 2$ if and only if

$$\|f\|_{G^p(B_n)} = sup_{z\in B_n}\|f \circ \phi_z - f(z)\|_{H^p(B_n)} < \infty \qquad (7)$$

For $p = 1$ or for $p = 2$. The above seminorms degenerate exactly on the constant functions. Let $BMOA(B_n)$ denote the quotient of $BMOA(B_n)$ by the space of constant functions.

Then $BMOA(B_n)$ is a Banach space with respect to the Garsia norm $\|\cdot\|_{G^p(B_n)}, p = 1$ or $p = 2$.

Ryll and Wojtaszczyk [197] constructed holomorphic polynomials which proved to be very useful for many problems of function theory in the unit ball. We need the following improvement of the Ryll–Wojtaszczyk theorem.

Theorem (5.1.1)[190]: ([192]). Let $m \in \mathbb{N}$. Then there exists $\delta = \delta(m) \in (0,1)$ and $J = J(m) \in \mathbb{N}$ with the following property: For every $d \in \mathbb{N}$, there exist holomorphic homogeneous polynomials Wj [d] of degree d, $1 \le j \le J$, such that

$$\left\|W_j[d]\right\|_{L^\infty(\partial B_m)} \le 1 \quad and \qquad (8)$$

$$max_{1\le j\le J}\left|W_j[d](\zeta)\right| \ge \delta \quad for\ all\ \zeta \in \partial B_m. \qquad (9)$$

For $m = 1$, it is known that the following lemma holds with $J(1) = 1$; see [116].

Lemma (5.1.2)[190]: Let $m \in \mathbb{N}$and let $0 < p < \infty$. Then there exist constants $J = J(m) \in \mathbb{N}, \tau_{m,p} > 0$ and there exist functions $F_{j,x} \in \mathfrak{B}(B_m), 1 \le j \le J, 0 \le x \le 1$, such that $\|F_{j,x}\|_{\mathfrak{B}(B_m)} \le 1, F_{j,x}(0) = 0$, and

$$\sum_{j=1}^{J}\int_0^1 \left|F_{\ j,x}(\omega)\right|^P dx \ge \tau_{m,p}\left(log\frac{1}{1-|\omega|^2}\right)^{\frac{p}{2}}$$

for all $w \in B_m$.

Proof. Let the constant $\delta \in (0, 1)$ and the polynomials $W_j\ [d], 1 \le j \le J, d \in \mathbb{N}$, be those provided by Theorem (5.1.1). $For\ k \in \mathbb{Z}_+$, let $R_k$denote the Rademacher function:

$$R_k(x) = sign\sin(2^{k+1}\pi x), \qquad x \in [0,1]$$

For each non- dyadic $x \in [0,1]$, consider the functions

$$F_{\ j,x}(\omega) = \frac{1}{4}\sum_{k=0}^{\infty} R_k(x)\, W_j[2^k](\omega), \quad \omega \in B_m, 1 \le j \le J.$$

Estimate (8) guarantees that

$$(1-|\omega|^2)\left|(\mathcal{R}F_{\ j,x})(\omega)\right| \le \frac{1-|\omega|^2}{4}\sum_{k=0}^{\infty} 2^k|\omega|^{2^k} \le \frac{1-|\omega|^2}{4}\sum_{n=1}^{\infty}|\omega|^n \le 1.$$

For all $\omega \in B_m$. Observe that $F_{j,x}(0) = 0;$ hence , $\|F_{j,x}\|_{\mathfrak{B}(B_m)} \leq 1.$ next

$$C_p \int_0^1 |F_{j,x}(\omega)|^p dx \geq \left(\sum_{k=0}^{\infty} |W_j[2^k](\omega)|^2\right)^{\frac{p}{2}}$$

by [91]. Given positive numbers a_j , $1 \leq j \leq J = J(m)$, we have

$$\left(\sum_{j=1}^{J} a_j\right)^{\frac{p}{2}} \leq C_{p,m} \sum_{j=1}^{J} a_j^{p/2}.$$

Hence,

$$C_{p,m} \sum_{j=1}^{J} \int_0^1 |F_{j,x}(\omega)|^p dx \geq \left(\sum_{k=0}^{\infty} \sum_{j=1}^{J} |W_j[2^k](\omega)|^2 \right)^{\frac{p}{2}}$$

Recall that $W_j[2^k]$ is a homogeneous polynomial of degree 2^k ; thus

$$\sum_{k=0}^{\infty} \sum_{j=1}^{J} |W_j[2^k](\omega)|^2 \geq \delta^2 \sum_{k=0}^{\infty} |\omega|^{2^{k+1}}$$

$\geq \delta^2 \sum_{n=1}^{\infty} \frac{|\omega|^{2n}}{n} = \delta^2 \log \frac{1}{1-|\omega|^2}, \ \omega \in B_{m,}$
By (9). So,

$$\sum_{j=1}^{J} \int_0^1 |F_{j,x}(\omega)|^p dx \geq \left(\frac{\delta^2}{C_{m,p}} \log \frac{1}{1-|\omega|^2}\right)^{\frac{p}{2}},$$

As required.

Theorem (5.1.3)[190]: Let $\varphi : B_n \to B_m$ be a holomorphic map, then the following properties are equivalent:

$$C_\varphi : \mathfrak{B}(B_1) \to BMOA(B_n) \ is\ a\ bounded\ operator; \quad (10)$$

$$sup_{z \in B_n} sup_{0<r<1} \int_{\partial B_n} \beta_m\left(\varphi\left(\phi_z(r\zeta)\right), \varphi(z)\right) d\sigma_n(\zeta) < \infty, (11)$$

For $m = 1$, Theorem (5.1.3) was proved in [195]; see also [121].

Proof. Assume that (10) holds. Note that $C_\varphi 1 = 1$; hence, $C_\varphi: \widetilde{\mathfrak{B}}(B_m) \to \widetilde{BMOA}\ (B_n)$ is a bounded operator. Using (7) with $p = 2$, we obtain

$$sup_{z \in B_n} sup_{0<r<1} \int_{\partial B_n} |f \circ \varphi \circ \phi_z(r\zeta) - f \circ \varphi(z)|^2 d\sigma_n(\zeta) \leq C\|f\|^2_{\widetilde{\mathfrak{B}}(B_m)}. \quad (12)$$

Let the constant $\tau = \tau_{m,2} > 0$ and the functions $F_{j,x}$, $1 \leq j \leq J, 0 \leq x \leq 1$. Be those provided by Lemma (5.1.2) for $p = 2$.Note that $\|F_{j,x}\|_{\widetilde{\mathfrak{B}}(B_m)} \leq C\|F_{j,x}\|_{\mathfrak{B}(B_m)} \leq C.$

Recall that $\|.\|_{\widetilde{\mathfrak{B}}(B_m)}$ is Mo ̈bius-invariant hence, $\|F_{j,x} \circ \psi_{\varphi(z)}\|_{\widetilde{\mathfrak{B}}(B_m)} \leq C$ where the constant $C > 0$ does not depend on $z \in B_n$. Also, we have $F_{j,x} \circ \psi_{\varphi(z)}(\varphi(z)) = F_{j,x}(0) = 0$. Thus, by (12) with $f = F_{j,x} \circ \psi_{\varphi(z)}$,

$$\int_{\partial B_n} \left|F_{j,x} \circ \psi_{\varphi(z)}\left(\varphi(\phi_z(r\zeta))\right)\right|^2 d\sigma_n(\zeta) \leq C,$$

for all $z \in B_n$, $0 < r < 1$, $o \leq x \leq 1$. Hence,

$$\sum_{j=1}^{J} \int_0^1 \int_{\partial B_n} \left|F_{j,x} \circ \psi_{\varphi(z)}\left(\varphi(\phi_z(r\zeta))\right)\right|^2 d\sigma_n(\zeta)\, dx \leq C,$$

$$z \in B_n, \quad 0 < r < 1.$$

Therefore, Fubini's theorem and Lemma (5.1.2) guarantee that

$$\tau \int_{\partial B_n} \log \frac{1}{1 - \left|\psi_{\varphi(z)}\left(\varphi(\phi_z(r\zeta))\right)\right|^2} \quad d\sigma_n(\zeta) \leq C, \quad z \in B_n,$$

$0 < r < 1$.

So, we obtain (3) or, equivalently, (11).

To prove the converse implication, assume that (11) holds, that is,

$$sup_{z\in B_n}\ sup_{0<r<1} \int_{\partial B_n} \beta m\left(\phi(\varphi z(r\zeta)), \phi(z)\right) d\sigma_n(\zeta) < \infty.$$

Let $f \in \mathfrak{B}(Bm)$. Then

$$|f\,(\varphi(\phi_z(r\zeta))) - f\,(\varphi(z))| \leq C\beta_m(\varphi(\phi_z\,(r\zeta)), \varphi(z))$$

By (4). Hence,

$$sup_{z\in B_n}\ sup_{0<r<1} \int_{\partial B_n} \left|f(\varphi(\phi_z\,(r\zeta)) - f(\varphi(z))\right| d\sigma_n(\zeta)$$

$$\leq C sup_{z\in B_n}\ sup_{0<r<1} \int_{\partial B_n} \beta_m(\varphi(\phi_z\,(r\zeta)), \varphi(z))\, d\sigma_n(\zeta) < \infty.$$

Using (7) with $p = 1$, we have $f \circ \varphi \in BMOA(B_n)$. So, (10) holds by the closed graph theorem. The proof of Theorem (5.1.3) is complete.

The space $BMOA(B_n)$ is not a lattice, so it is not expected that (10) is equivalent to a restriction on $|\phi * (\zeta)|, \zeta \in \partial B_n$. However, applying Lemma (5.1.2), we obtain a related explicit condition, which is necessary for (10).

Proposition (5.1.4)[190]: Let $\varphi : B_n \to B_m$ be a holomorphic map. Assume that $C_\varphi : \mathfrak{B}(B_m) \to BMOA(B_n)$ is a bounded operator. Then there exist constants $\varepsilon = \varepsilon(n, m, \|C_\varphi\|_{\mathfrak{B} \to BMOA}) > 0$ and $C = C(n) > 0$ such that

$$\int_{\partial B_n} exp\left(\varepsilon \log \frac{1}{1 - |\varphi^*(\zeta)|^2}\right)^{\frac{1}{2}} d\sigma_n(\zeta) \leq C.$$

Proof. The operator C_φ *maps* $\mathfrak{B}(B_m)$ to the Hardy space $H^2(B_n)$. So, arguing as in the proof of the implication (10)⇒(11), we obtain

$$sup_{0<r<1} \int_{\partial B_n} \log \frac{1}{1-|\varphi(r\zeta)|^2} d\sigma_n(\zeta) < \infty.$$

Hence,

$$\int_{\partial B_n} \log \frac{1}{1-|\varphi^*(\zeta)|^2} d\sigma_n(\zeta) < \infty.$$

By Fatou's. In particular,

$$|\varphi^*(\zeta)| < 1. \tag{13}$$

For σ_n- almosy every $\zeta \in \partial B_n$. If (13) holds, then

$$f(\varphi^*(\zeta)) = lim_{r\to 1^-}\ f(\varphi(r\zeta)) = \ (f \circ \varphi)^*(\zeta). \tag{14}$$

For any $f \in H(B_m)$.

Now, let the constant $\tau = \tau_{m.1} > 0$ and the functions $F_{j,x} \in \mathfrak{B}(B_m)$, $1 \le j \le J, 0 \le x \le 1$, be those provided by Lemma (5.1.2) with $p = 1$.we have, be those provided by Lemma (5.1.2) with $p = 1$. We have

$$\|F_{j,x} \circ \varphi\|_{BMOA(B_n)} \le \|C_\varphi\|_{\mathfrak{B}\to BMOA}.$$

Thus, for $\delta = A(n)\|C_\varphi\|^{-1}_{\mathfrak{B}\to BMOA}$ (14) and (6) guarantee that

$$\int_{\partial B_n} \exp(\delta|F_{j,x}(\varphi * (\zeta))|)\, d\sigma_n(\zeta) = \int_{\partial B_n} exp(\delta|F_{j,x} \circ \varphi)^*(\zeta)|)\ d\sigma_n(\zeta) \le\ C(n),$$
$$1\ \le\ j\ \le\ J, \qquad 0\ \le\ x\ \le\ 1,$$

where the constants $A(n)\ >\ 0$ *and* $C(n)\ >\ 0$ are those provided by the John– Nirenberg theorem for $BMOA(B_n)$. Therefore,

$$\begin{aligned} C(n) &\ge \int_{\partial B_n} \frac{1}{J}\sum_{j=1}^{J}\int_0^1 exp(\delta|F_{j,x}(\varphi^*(\zeta))\ dx\ d\sigma_n(\zeta) \\ &\ge \int_{\partial B_n} exp\left(\frac{\delta}{J}\sum_{j=1}^{J}\int_0^1 |F_{j,x}(\varphi^*(\zeta))\ dx\right) d\sigma_n(\zeta) \\ &\ge \int_{\partial B_n} exp\left(\frac{\tau\delta}{J}\sqrt{log\frac{1}{1-|(\varphi^*(\zeta)|^2}}\right) d\sigma_n(\zeta). \end{aligned}$$

By Fubini's theorem. Jensen's inequality and Lemma (5.1.2).

Section (5.2): Logarithmic Bloch Spaces

For H ($\mathbb{D}$) denote the space of holomorphic functions on the unit disk $\mathbb{D}\ =\ \{z\ \in\ \mathbb{C}: |z|\ <\ 1\}$.

The question about reverse estimates naturally arises in the study of the growth spaces A^v ($\mathbb{D}$), where v is a weight function, that is, v: [0, 1) → (0, +∞) a non-decreasing, continuous, unbounded function. By definition, the growth space $A^v(\mathbb{D})$, consists $of\ f \in H(\mathbb{D})$ such that

$$|f(z)| \leq Cv(|z|), \qquad z \in \mathbb{D}, \tag{15}$$

for some constant $C > 0$.

In applications, it is useful to have test functions $f \in A^v(\mathbb{D})$, for which the reverse of estimate (15) holds, in an appropriate sense. As shown in [200], the required test functions exist for a sufficiently large class of v. Namely, recall that a weight function $v : [0, 1) \to (0, +\infty)$ is called doubling if

$$v(1 - s/2) < Av(1 - s), \ \ 0 < s \leq 1.$$

For some constant $A > 1$.

Theorem (5.2.1)[199]: ([200]) Let $v: [0,1) \to (0, +\infty)$ be adoupling weight function. There exist functions $f_1, f_2 \in A^v(\mathbb{D})$, such that

$$|f_1(z)| + |f_2(z)| \geq v(|z|), z \in \mathbb{D}. \tag{16}$$

An assertion similar to Theorem (5.2.1) was also obtained in [110]. The first result of the above type was proved by Ramey and Ullrich [121] for $v(t) = (1 - t^2)^{-1}$. See [200], [110] for further references.

Clearly, estimate (16) is sharp since the same weight function v is used in both (15) and (16). So, it is interesting to find those spaces $X \subset H(\mathbb{D})$ for which the corresponding lower and upper estimates are sharp, but different. To the best knowledge of the author, the only known example of such a space X is the Bloch space $\mathfrak{B}(\mathbb{D})$.

The Bloch space $\mathfrak{B}(\mathbb{D})$. consists of those $f \in H(\mathbb{D})$ for which

$$\|f\|_{\mathfrak{B}(\mathbb{D})} = |f(0)| + sup_{z\in\mathbb{D}} |f'(z)|(1 - |z|^2) < \infty.$$

On the one hand , if $f \in \mathfrak{B}(\mathbb{D}), \|f\|_{\mathfrak{B}(\mathbb{D})} \leq 1$, then

$$|f(z)| \leq C \log \frac{e}{1 - |z|^2}, \qquad z \in \mathbb{D}, \tag{17}$$

for an absolute constant $C > 0$. On the other hand, the following integral reverse estimate is known:

Theorem (5.2.2)[199]: (see, e.g., [116]) Let $0 < p < \infty$. Then there exist functions $F_x \in \mathfrak{B}(\mathbb{D}), 0 \leq x \leq 1$, such that $\|F_x\|_{\mathfrak{B}(\mathbb{D})} \leq 1$ and

$$\left(\int_0^1 |F_x(z)|^p dx\right)^{\frac{1}{P}} \geq \tau_p \left(\log \frac{1}{1 - |z|^2}\right)^{\frac{1}{2}}, z \in \mathbb{D}, \tag{18}$$

For a constant $\tau_p > 0$.

While one has $\log \frac{e}{1-|z|^2}$ in the upper estimate (17) and $\left(\log \frac{1}{1-|z|^2}\right)^{\frac{1}{2}}$ in the lower estimate (18), both (17) and (18) are known to be sharp. To find similar examples, it is natural to consider the weighted Bloch spaces $\mathfrak{B}^\omega(\mathbb{D})$. Given a weight function w, the space $\mathfrak{B}^\omega(\mathbb{D})$ consists of $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathfrak{B}^\omega(\mathbb{D})} = |f(0)| + sup_{z\in\mathbb{D}} \frac{|f'(z)|}{\omega|(z)|} < \infty.$$

If $w(t) = (1 - t^2)^{-1}$, then $\mathfrak{B}^\omega(\mathbb{D})$ coincides with $\mathfrak{B}(\mathbb{D})$ So, for $w_\alpha(t) = (1 - t^2)^{-\alpha}, \alpha > 0$, one may consider the spaces $\mathfrak{B}^{\omega_\alpha}(\mathbb{D})$ as possible analogs of $\mathfrak{B}(\mathbb{D})$. However, if $0 < \alpha < 1$, then $\mathfrak{B}^{\omega_\alpha}(\mathbb{D})$ coincides with the Lipschitz space $\Lambda^{1-\alpha}(\mathbb{D})$; hence, there are no reverse estimates in this case. If $\alpha > 1$, then $\mathfrak{B}^{\omega_\alpha}(\mathbb{D})$ coincides with the growth space $A^{v_\alpha}(\mathbb{D}), v_\alpha(t) = (1 - t^2)^{1-\alpha}$.

Therefore, to find appropriate analogs of $\mathfrak{B}(\mathbb{D})$, we have to consider sufficiently weak, say logarithmic, multiplicative perturbations of the weight function $w(t) = (1 - t^2)^{-1}$.

For $\alpha \in \mathbb{R}$, the logarithmic Bloch space $L^{\alpha}\mathfrak{B}(\mathbb{D})$ consists of those $f \in H(\mathbb{D})$ for which

$$\|f\|_{L^{\alpha}\mathfrak{B}(\mathbb{D})} = |f(0)| + sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2)\left(log\frac{e}{1-|z|^2}\right)^{\alpha} < \infty.$$

Note that the function $\omega_{\alpha}(t) = \frac{1}{1-t^2}\left(log\frac{e}{1-t^2}\right)^{-\alpha}$ is increasing on [0, 1) When $\alpha \leq 1. If\ \alpha = 0$,, then $L^{\alpha}\mathfrak{B}(\mathbb{D})$ coincides with the Bloch space $\mathfrak{B}(\mathbb{D})$.

IF $\alpha > \frac{1}{2}$, then $L^{\alpha}\mathfrak{B}(\mathbb{D})$ is a rather small space, in particular $L^{\alpha}\mathfrak{B}(\mathbb{D}) \subset$ BMOA($\mathbb{D}$) (see, e.g., [201]). So, there are no appropriate reverse estimates in the spaces $L^{\alpha}\mathfrak{B}(\mathbb{D}), \alpha < \frac{1}{2}$, . The main technical result is the following integral reverse estimate for $L^{\alpha}\mathfrak{B}(\mathbb{D}), \alpha < \frac{1}{2}$.

As shown, the exponent $\frac{1}{2} - \alpha$ in estimate (30) is sharp. Also if $\alpha < 1, f \in L^{\alpha}B(\mathbb{D})$, and $\|f\|_{L^{\alpha}\mathfrak{B}(\mathbb{D})} \leq 1$, then we have the following sharp upper estimate:

$$|f(z)| \leq C_{\alpha}\left(log\frac{e}{1-|z|^2}\right)^{1-\alpha}, \qquad z \in \mathbb{D}, \tag{19}$$

for a constant $C_{\alpha} > 0$. Therefore, the spaces $L^{\alpha}B(\mathbb{D}), \alpha < \frac{1}{2}$, provide examples of the phenomenon discussed above for $\mathfrak{B}$ ($\mathbb{D}$): while the exponents in (30) and (19) are different, both (30) and (19) are sharp.

Reverse estimates are known to be useful in the study of concrete linear operators on the corresponding spaces of holomorphic functions. We consider composition operators.

Given a holomorphic function $\varphi: \mathbb{D} \to \mathbb{D}$, the composition operator $\varphi: H(\mathbb{D}) \to H(\mathbb{D})$ s defined by the formula

$$C_{\varphi} f(z) = f(\varphi(z)), \ f \in H(\mathbb{D}), \ z \in \mathbb{D}.$$

We study the composition operators from $L^{\alpha}\mathfrak{B}(\mathbb{D})$ into the Hardy space $H^2(\mathbb{D})$. As a by-product, we deduce that the reverse estimate (30) is sharp, up to a multiplicative constant. Also, we consider the composition operators from $L^{\alpha}\mathfrak{B}(\mathbb{D})$ into the space BMOA ($\mathbb{D}$).

We devoted to the proof of Theorem (5.2.5). Composition operators on the spaces $L^{\alpha}\mathfrak{B}(\mathbb{D}), \alpha < \frac{1}{2}$.

To prove Theorem (5.2.5), we need two auxiliary lemmas.

Lemma (5.2.3)[199]: (cf. [202]). Let $\beta > 0$, and let $t \in [0, 1)$. Then there exists a constant $C_{\beta} > 0$ such that

$$\sum_{k=0}^{\infty}(k+1)^{\beta-1}\, t^{2^k-1} \geq C_{\beta}\left(log\frac{1}{1-t}\right)^{\beta} \tag{20}$$

Proof. First, let $t \in \left[0, \frac{1}{2}\right]$. Then

$$\sum_{k=0}^{\infty}(k+1)^{\beta-1}\, t^{2^k-1} \geq 1 \geq \left(log_2\frac{1}{1-t}\right)^{\beta}$$

$$= (\log 2)^{-\beta}\left(log\frac{1}{1-t}\right)^{\beta}. \tag{21}$$

Second, let $t \in [\frac{1}{2}, 1)$. Select $n \in$N such that $1 - \frac{1}{2^n} \leq t \leq 1 - \frac{1}{2^{n+1}}$, then we have

$$\sum_{k=0}^{\infty}(k+1)^{\beta-1}\, t^{2^k-1}$$
$$\geq \sum_{k=0}^{n}(k+1)^{\beta-1}\left(1-\frac{1}{2^n}\right)^{2^k-1}$$
$$\geq \left(1-\frac{1}{2^n}\right)^{2^n-1}\sum_{k=0}^{n}(k+1)^{\beta-1} \tag{22}$$
$$\geq \frac{1}{e}\sum_{k=0}^{n}(k+1)^{\beta-1}.$$

Put $S_n = \frac{1}{e}\sum_{k=0}^{n}(k+1)^{\beta-1}$. Continuation of estimate (2.3) depends on β and uses the inequality $t \leq 1-\frac{1}{2^{n+1,}}$, which is equivalent to

$$\left(log_2\frac{1}{1-t}\right)^{\beta} \leq (n+1)^{\beta} \tag{23}$$

If $0<\beta\leq 1$, then, by (23)

$$S_n \geq \frac{(n+1)^{\beta}}{e} \geq \frac{1}{e}\left(log_2\frac{1}{1-t}\right)^{\beta} = \frac{1}{e}(\log 2)^{-\beta}\left(log\frac{1}{1-t}\right)^{\beta}. \tag{24}$$

If $\beta \geq 1$, then, by (23)

$$S_n = \frac{1}{2e}\sum_{k=0}^{n}\left((k+1)^{\beta-1} + (n+1-k)^{\beta-1}\right)$$
$$\geq \frac{1}{2e}\sum_{k=0}^{n}\frac{(n+2)^{\beta-1}}{2^{\beta-1}} \geq \frac{(n+1)^{\beta}}{e2^{\beta}} \tag{25}$$
$$\geq \frac{1}{e}\left(\frac{1}{2}log_2\frac{1}{1-t}\right)^{\beta} = \frac{1}{e}(2\log 2)^{-\beta}\left(log\frac{1}{1-t}\right)^{\beta}$$

Finally (21), (24) and (25) imply (20) with $C_\beta = \frac{1}{e}(2\log 2)^{-\beta}$.

Lemma (5.2.4)[199]: Let $\alpha\in\mathbb{R}$. Then there exists a constant $C_\alpha>0$ such that

$$\sum_{K=1}^{\infty}\frac{2^K-1}{(k+1)^{\alpha}}t^{2^{k-1}-1} \leq C_\alpha(1-t)^{-1}\left(log\frac{e}{1-t}\right)^{-\alpha}, t\in[0,1) \tag{26}$$

Proof. Put

$$G_\alpha(t) = (1-t)\left(log\frac{e}{1-t}\right)^{\alpha}\sum_{K=1}^{\infty}\frac{2^K-1}{(k+1)^{\alpha}}t^{2^{k-1}-1}, \alpha\in\mathbb{R}\ , t\in[0,1)$$

For $n\in\mathbb{N}$ and $t\in\left[1-\frac{1}{2^n}, 1-\frac{1}{2^{n+1}}\right]$, we have

$$G_\alpha(t) \leq C_\alpha\left(\sum_{k=1}^{\infty}\left(\frac{n}{k}\right)^{\alpha}2^{k-n}t^{2^{k-1}}-1\right.$$
$$\leq C_\alpha\left(\sum_{k=1}^{\infty}\left(\frac{n}{k}\right)^{\alpha}2^{k-n} + \sum_{k=n+1}^{\infty}\left(\frac{n}{k}\right)^{\alpha}2^{k-n}\left(1-\frac{1}{2^{n+1}}\right)^{2^{k-1}-1}\right)$$

$$\leq C_\alpha\left(\sum_{k=1}^{n}\left(\frac{n}{k}\right)^{\alpha}2^{k-n}+\sum_{k=n+1}^{\infty}\left(\frac{n}{k}\right)^{\alpha}2^{k-n}q^{2^{k-n}},\right). \quad (27)$$

Where $q = e^{-\frac{1}{8}} \in (0,1)$.

Continuation of estimate (27) depends on α. If $\alpha \leq 0$, then $\left(\frac{n}{k}\right)^{\alpha} \leq e^{-\alpha\frac{k-n}{n}} \leq e^{-\alpha\,(k-n)}$, so

$$G_\alpha(t) \leq C_\alpha\left(\sum_{k=1}^{n}2^{k-n}+\sum_{k=n+1}^{\infty}(2e^{-\alpha})^{k-n}q^{2^{k-n}} \leq C_\alpha\left(2+\sum_{s=1}^{\infty}(2e^{-\alpha})^{s}q^{2^{s}}\right) = C'_\alpha. \quad (28)$$

If $\alpha \geq 0,\ then$,

$$G_\alpha(t) \leq C_\alpha\left(\sum_{1\leq k\leq\frac{n}{2}}n^{\alpha}2^{-\frac{n}{2}}+2^{\alpha}\sum_{\frac{n}{2}\leq k\leq n}2^{k-n}+\sum_{s=1}^{\infty}2^{s}q^{2^{s}}\right)$$

$$\leq C_\alpha\left(n^{\alpha+1}2^{-\frac{n}{2}-1}+2^{\alpha+1}+\sum_{s=1}^{\infty}2^{s}q^{2^{s}}\right) = C''_\alpha. \quad (29)$$

It remains to remark that (26) follows from (28) and (29).

We are in position to prove the reverse estimates in the logarithmic Bloch spaces $L^{\alpha}\mathfrak{B}(\mathbb{D}), \alpha < \frac{1}{2}$.

Theorem (5.2.5)[199]: Let $\alpha < \frac{1}{2}$ and let $0 < p < \infty$ then there exist functions $F_x \in L^{\alpha}\mathfrak{B}(\mathbb{D}), 0 \leq x \leq 1$ such that $\|F_x\|_{L^{\alpha}\mathfrak{B}(\mathbb{D})} \leq 1$ and

$$\left(\int_0^1 |F_x(z)|^p\,dx\right)^{\frac{1}{p}} \geq \tau_{p,\alpha}\left(log\frac{e}{1-|z|^2}\right)^{\frac{1}{2}-\alpha} \quad z \in \mathbb{D}, \quad (30)$$

for a constant $\tau_{p,\alpha} > 0$.

Proof. Let the constant $C_\alpha > 0$ be that provided by Lemma (5.2.4) for $x \in [0,1]$ consider the following functions :

$$F_x(z) = \frac{1}{1+C_\alpha}\sum_{K=0}^{\infty}\frac{R_k(x)}{(k+1)^{\alpha}}z^{2^{k}-1}, \qquad z \in \mathbb{D}.$$

Where $R_k(x) =$ sign in $(2^{k+1}\pi x)$ are the Rademacher functions. First , we have $F_x \in H(\mathbb{D})$ and

$$\|F_x\|_{L^{\alpha}\mathfrak{B}(\mathbb{D})} \leq \frac{1}{1+C_\alpha}(1+sup_{z\in\mathbb{D}}(1-|z|^2)\left(log\frac{e}{1-|z|^2}\right)^{\alpha}\sum_{k=1}^{\infty}\frac{2^{k}-1}{(k+1)^{\alpha}}|z|^{2k-2}\right) \leq 1.$$

By Lemma (5.2.4), with $t = |z|^2$.Second, by [91]

$$\int_0^1 |F_x(z)|^p\,dx \geq A_{p,\alpha}\left(\sum_{k=0}^{\infty}\frac{|z|^{2(2^{k}-1)}}{(k+1)^{2\alpha}}\right)^{\frac{p}{2}},$$

Applying Lemma (5.2.3) with $\beta = 1-2\alpha$ and $t = |z|^2$, we obtain

$$\sum_{k=0}^{\infty}\frac{|z|^{2(2^k-1)}}{(k+1)^{2\alpha}}\geq C_{1-2\alpha}\left(log\frac{1}{1-|z|^2}\right)^{1-2\alpha},\quad z\in\mathbb{D}$$

Therefore,

$$\int_0^1|F_x(z)|^p\,dx\geq\tau_{p,\alpha}^p\left(log\frac{1}{1-|z|^2}\right)^{\left(\frac{1}{2}-\alpha\right)p},z\in\mathbb{D}$$

As require.

As mentioned, reverse estimates are known to be useful in the study of composition operators (see, e.g., [200], [80], [116]–[121]). We consider operators with values in the Hardy space $H^2(\mathbb{D})$ and in the space BMOA($\mathbb{D}$).

Let σ denote the normalized Lebesgue measure on the unit circle $\mathbb{T}=\{\zeta\in\mathbb{C}:|\zeta|=1\}$; $\sigma(\mathbb{T})=1$.For $0<p<\infty$, the Hardy space $H^P(\mathbb{D})$ consists of $f\in H(\mathbb{D})$, such that

$$\|f\|_{H^P(\mathbb{D})}^p= sup_{0<r<1}\int_{\mathbb{T}}|f(r\zeta)|^p\,d\sigma(\zeta)<\infty.$$

Given $f\in H(\mathbb{D})$, the Littlewood–Paley g-function is defined as follows:

$$g(f)(\zeta)=\left(\int_0^1|f'(r\zeta)|^2(1-r)dr\right)^{\frac{1}{2}},\zeta\in\mathbb{T}.$$

It is known that $f\in H^P(\mathbb{D})$ if and only if $g(f)\in L^p(\mathbb{T})$ (see, e.g., [91] for $p>1$).

Also, recall the definition of the hyperbolic Hardy class $H_h^p(\mathbb{D})$ (see, for example, [204]). For $0<p<\infty, H_h^p(\mathbb{D})$ consists of those holomorphic functions $\varphi:\mathbb{D}\to\mathbb{D}$ for which

$$sup_{0<r<1}\int_{\mathbb{T}}\left(log\frac{1}{1-|\varphi(r\zeta)|^2}\right)^p d\sigma(\zeta)<\infty$$

Remark that $H_h^{p2}(\mathbb{D})\subsetneq H_h^{p1}(\mathbb{D})\, for\ 0<p_1<p_2<\infty.$

For $>\frac{1}{2}$, we have $L^\alpha\mathfrak{B}(\mathbb{D})\subset BMOA(\mathbb{D})\subset H^2(\mathbb{D})$, hence, the composition operator C_φ maps $L^\alpha\mathfrak{B}(\mathbb{D})$ into $H^2(\mathbb{D})$ for any symbol φ. For $\alpha=0$, a description of the bounded operators $C_\varphi:L^\alpha\mathfrak{B}(\mathbb{D})\to H^2(\mathbb{D})$ is given in [80]. For arbitrary $\alpha<\frac{1}{2}$, we have the following characterization:

Theorem (5.2.6)[199]: Let $\alpha<\frac{1}{2}$, and let $\varphi:\mathbb{D}\to\mathbb{D}$ be a holomorphic mapping. Then the following properties are equivalent:

$$C_\varphi\ :\ L^\alpha\mathfrak{B}(\mathbb{D})\to H^2(\mathbb{D}).\tag{31}$$

is a bounded operator.(31)

$$\int_0^1\frac{|\varphi'(r\zeta)|^2}{(1-|\varphi(r\zeta)|^2)^2}\left(log\frac{e}{1-|\varphi(r\zeta)|^2}\right)^{-2\alpha}(1-r)dr\in L^1(\mathbb{T});\tag{32}$$

$$\varphi\in H_h^{1-2\alpha}(\mathbb{D}).\tag{33}$$

Proof. Let (31) hold. Applying Theorem (5.2.1), choose $f_1, f_2\in L^\alpha\mathfrak{B}(\mathbb{D})$ that

$$|f_1'(z)|^2+|f_2'(z)|^2\geq(1-|z|^2)^{-2}\left(log\frac{e}{1-|z|^2}\right)^{-2\alpha},\qquad z\in\mathbb{D}.$$

By (31), we have $C_\varphi f_j\in H^2(\mathbb{D}),\ j=1,2$. Thus,

$$\infty > \left\| g(C_\varphi f_1) \right\|_{L^2(\mathbb{T})}^2 + \left\| g(C_\varphi f_2) \right\|_{L^2(\mathbb{T})}^2$$

$$= \int_{\mathbb{T}} \int_0^1 \left(\left| f_1'(\varphi(r\zeta)) \right|^2 + \left| f_2'(\varphi(r\zeta)) \right|^2 \right) |\varphi'(r\zeta)|^2 (1-r) dr d\sigma(\zeta)$$

$$\geq \int_{\mathbb{T}} \int_0^1 \frac{|\varphi'(r\zeta)|^2}{(1-|\varphi(r\zeta)|^2)^2} \left(log \frac{e}{(1-|\varphi(r\zeta)|^2)^2} \right)^{-2\alpha} (1-r) dr d\sigma(\zeta).$$

So, (31) implies (32).

To prove the converse implication, assume that (32) holds.

Given $f \in L^\alpha \mathfrak{B}(\mathbb{D})$ and $\zeta \in$T, we have

$$g^2(C_\varphi f)(\zeta) \leq \|f\|_{L^\alpha \mathfrak{B}(\mathbb{D})}^2 \int_0^1 \frac{|\varphi'(r\zeta)|^2}{(1-|\varphi(r\zeta)|^2)^2} \left(log \frac{e}{1-|\varphi(r\zeta)|^2} \right)^{-2\alpha} (1-r) dr.$$

Whence, $g(C_\varphi f) \in L^2(\mathbb{T})$ by (32).

Therefore, $C_\varphi f \in H^2(\mathbb{D})$. So, (33)implies (32).

Finally, properties (32) and (33) are equivalent by [203].

Remark that the implication (31) ⇒(33) also follows from Theorem (5.2.5). In fact, a related argument guarantees that the estimate (30) is sharp.

Assume that there exist functions $F_x \in L^\alpha \mathfrak{B}(\mathbb{D}), 0 \leq x \leq 1$, such that (30) holds with $\beta \geq \frac{1}{2} - \alpha$ in the place of $\frac{1}{2} - \alpha$. *For every* $\varphi \in H_h^{1-2\alpha}(\mathbb{D})$, the composition operator $C_\varphi : L^\alpha \mathfrak{B}(\mathbb{D}) \to H^2(\mathbb{D})$ is bounded by Theorem (5.2.6). Hence,

$$\left\| C_\varphi \right\|_{L^\alpha \mathfrak{B}(\mathbb{D}) \to H^2(\mathbb{D})}^2 \geq \int_0^1 \int_{\mathbb{T}} \left| F_x(\varphi(r\zeta)) \right|^2 d\sigma(\zeta) dx$$

$$= \int_{\mathbb{T}} \int_0^1 \left| F_x(\varphi(r\zeta)) \right|^2 dx\, d\sigma(\zeta) \geq C_\beta \int_{\mathbb{T}} \left(log \frac{1}{1-|\varphi(r\zeta)|^2} \right)^{2\beta} \quad d\sigma(\zeta)$$

for all $r \in (0,1)$. In other words, $\varphi \in H_h^{2\beta}(\mathbb{D})$. Hence, $H_h^{1-2\alpha}(\mathbb{D}) = H_h^{2\beta}(\mathbb{D})$ and $2\beta = 1 - 2\alpha$.

The space BMOA($\mathbb{D}$) consists of those $f \in H^2(\mathbb{D})$ for which

$$\|f\|_{\mathrm{BMOA}(\mathbb{D})}^2 = |f(0)|^2 + sup_{\alpha \in \mathbb{D}} \int_{\mathbb{T}} |f^*(\zeta) - f(a)|^2 \frac{1-|a|^2}{|\zeta - a|^2} d\sigma(\zeta) < \infty,$$

where the radial limits $f^*(\zeta) = lim_{r \to 1-} f(r\zeta)$ are defined $\sigma - a.e.$

By the John–Nirenberg theorem, there exist constants $A > 0$ and $C > 0$. Such that

$$\int_{\mathbb{T}} exp(|f^*(\zeta)|)\, d\sigma(\zeta) \leq C. \qquad (34)$$

For all $f \in \mathrm{BMOA}(\mathbb{D})$ *with* $\|f\|_{\mathrm{BMOA}(\mathbb{D})} \leq A$.

Recall that $L^\alpha \mathfrak{B}(\mathbb{D}) \subset \mathrm{BMOA}(\mathbb{D})$ for $\alpha > \frac{1}{2}$. *So, if* $\varphi : \mathbb{D} \to \mathbb{D}$ is an arbitrary holomorphic function and $\alpha > \frac{1}{2}$, then the composition operator C_φ maps $L^\alpha \mathfrak{B}(\mathbb{D})$ into $BMOA(\mathbb{D})$.

For $\leq \frac{1}{2}$, a theoretical characterization of the bounded composition operators $C_\varphi : L^\alpha\,\mathfrak{B}(\mathbb{D}) \to BMOA(\mathbb{D})$ is given in [110]. Applying Theorem (5.2.5), we obtain an explicit condition that is necessary for the boundedness of $C_\varphi : L^\alpha\,\mathfrak{B}(\mathbb{D}) \to BMOA(\mathbb{D}), \alpha < \frac{1}{2}$,

Proposition (5.2.7)[199]: (cf. [190]) Let $\alpha < \frac{1}{2}$, and let $\varphi : \mathbb{D} \to \mathbb{D}$ be a holomorphic function. Assume that $C_\varphi : L^\alpha\,\mathfrak{B}(\mathbb{D}) \to BMOA(\mathbb{D})$ is a bounded operator. Then there exist constants $\varepsilon = \varepsilon(\alpha, \|C_\varphi\|_{L^\alpha\,\mathfrak{B}(\mathbb{D}) \to BMOA(\mathbb{D})} > 0$ and $C > 0$ such that

$$\int_{\mathbb{T}} \exp\left(\varepsilon \log \frac{1}{1-|\varphi^*(\zeta)|^2}\right)^{\frac{1}{2}-\alpha} d\sigma(\zeta) \leq C.$$

Proof. Since $\alpha < \frac{1}{2}$ and the operator $C_\varphi : L^\alpha\,\mathfrak{B}(\mathbb{D}) \to BMOA(\mathbb{D})$ is bounded, we have $|\phi * (\zeta)| < 1\, for\, \sigma - a.e. \zeta \in \mathbb{T}$ (see [110]). Therefore, for every $f \in H(\mathbb{D})$,

$$f(\varphi^*(\zeta)) = \lim_{r\to 1^-} f(\varphi(r\zeta)) = (f \circ \varphi)^*(\zeta). \quad \sigma - a.e. \zeta \in \mathbb{T} \tag{35}$$

Let the functions $F_x \in L^\alpha\,\mathfrak{B}(\mathbb{D}), 0 \leq x \leq 1$, be those provided by Theorem (5.2.5) with $p = 1. Then$

$$\|f_x \circ \varphi\|_{BMOA(D)} \leq \|C_\varphi\|_{L^\alpha\,\mathfrak{B}(\mathbb{D}) \to BMOA(D)}.$$

$Put\ \delta = A\|C_\varphi\|^{-1}_{L^\alpha\,\mathfrak{B}(\mathbb{D}) \to BMOA(D)}$, where A is the constant provided by the John– Nirenberg theorem . By (35) and (34), we have

$$\int_{\mathbb{T}} exp(\delta|F_x(\varphi^*(\zeta))|)\, d\sigma(\zeta) = \int_{\mathbb{T}} \exp(\delta|(F_x \circ \varphi)^*(\zeta)|)\, d\sigma(\zeta) \leq C, \quad 0 \leq x \leq 1.$$

Finally, applying Fubini's theorem, Jensen's inequality, and Theorem (5.2.5), we obtain

$$\begin{aligned} C &\geq \int_{\mathbb{T}} \int_0^1 exp\left(\delta|F_x(\varphi^*(\zeta))|\right) dx\, d\sigma(\zeta) \\ &\geq \int_{\mathbb{T}} \exp\left(\delta \int_0^1 |F_x(\varphi^*(\zeta))|\, dx\right) d\sigma(\zeta) \\ &\geq \int_{\mathbb{T}} \exp\left(\delta\tau_{1,\alpha}\left(log \frac{1}{1-|\varphi^*(\zeta)|^2}\right)^{\frac{1}{2}-\alpha} dx\right) d\sigma(\zeta). \end{aligned}$$

As required.

Corollary (5.2.8)[276]: (cf. [202]). Let $\epsilon \geq 0$, and let $t \in [0, 1)$. Then there exists a constant $C_{1+\epsilon} > 0$ such that

$$\sum_{k=0}^{\infty} (k+1)^\epsilon\, t^{2^k - 1} \geq C_{1+\epsilon}\left(log \frac{1}{1-t}\right)^{1+\epsilon} \tag{36}$$

Proof. First, let $t \in \left[0, \frac{1}{2}\right]$. Then

$$\sum_{k=0}^{\infty} (k+1)^\epsilon\, t^{2^k - 1} \geq 1 \geq \left(log_2 \frac{1}{1-t}\right)^{1+\epsilon}$$

$$= (\log 2)^{-(1+\epsilon)} \left(log \frac{1}{1-t} \right)^{1+\epsilon} . (37)$$

Second, let $t \in [\frac{1}{2}, 1)$. Select $n \in \mathbb{N}$ such that $1 - \frac{1}{2^n} \le t \le 1 - \frac{1}{2^{n+1}}$, then we have

$$\sum_{k=0}^{\infty} (k+1)^{\epsilon} t^{2^k - 1} \ge \sum_{k=0}^{n} (k+1)^{\epsilon} \left(1 - \frac{1}{2^n}\right)^{2^k - 1} \ge \left(1 - \frac{1}{2^n}\right)^{2^n - 1} \sum_{k=0}^{n} (k+1)^{\epsilon}$$

$$\ge \frac{1}{e} \sum_{k=0}^{n} (k+1)^{\epsilon}. \quad (38)$$

Put $S_n = \frac{1}{e} \sum_{k=0}^{n} (k+1)^{\epsilon}$. Continuation of estimate (38) depends on $(1+\epsilon)$ and uses the inequality $t \le 1 - \frac{1}{2^{n+1}},$ which is equivalent to

$$\left(log_2 \frac{1}{1-t} \right)^{1+\epsilon} \le (n+1)^{1+\epsilon} \quad (39)$$

If $\epsilon < 1$, then, by (39)

$$S_n \ge \frac{(n+1)^{1-\epsilon}}{e} \ge \frac{1}{e} \left(log_2 \frac{1}{1-t} \right)^{1-\epsilon} = \frac{1}{e} (\log 2)^{\epsilon - 1} \left(log \frac{1}{1-t} \right)^{1-\epsilon}. \quad (40)$$

If $\epsilon \ge 0$, then, by (39)

$$S_n = \frac{1}{2e} \sum_{k=0}^{n} ((k+1)^{\epsilon} + (n+1-k)^{\epsilon})$$

$$\ge \frac{1}{2e} \sum_{k=0}^{n} \frac{(n+2)^{\epsilon}}{2^{\epsilon}} \ge \frac{(n+1)^{1+\epsilon}}{e 2^{1+\epsilon}} \quad (41)$$

$$\ge \frac{1}{e} \left(\frac{1}{2} log_2 \frac{1}{1-t} \right)^{1+\epsilon} = \frac{1}{e} (2 \log 2)^{-(1+\epsilon)} \left(log \frac{1}{1-t} \right)^{1+\epsilon}$$

Finally (37), (40) and (41) imply (36) with $C_{1+\epsilon} = \frac{1}{e} (2 \log 2)^{-(1+\epsilon)}$.

Corollary (5.2.9)[276]: Let $\epsilon \in \mathbb{R}$. Then there exists a constant $C_{\frac{1}{2}-\epsilon} > 0$ such that

$$\sum_{K=1}^{\infty} \frac{2^K - 1}{(k+1)^{\frac{1}{2}-\epsilon}} t^{2^{k-1}-1} \le C_{\frac{1}{2}-\epsilon} (1-t)^{-1} \left(log \frac{e}{1-t} \right)^{-(\frac{1}{2}-\epsilon)}, t \in [0,1) \quad (42)$$

Proof. Put

$$G_{\frac{1}{2}-\epsilon}(t) = (1-t) \left(log \frac{e}{1-t} \right)^{\frac{1}{2}-\epsilon} \sum_{K=1}^{\infty} \frac{2^K - 1}{(k+1)^{\frac{1}{2}-\epsilon}} t^{2^{k-1}-1}, \qquad \epsilon \in \mathbb{R}, t \in [0,1)$$

For $n \in \mathbb{N}$ and $t \in \left[1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}}\right]$, we have

$$G_{\frac{1}{2}-\epsilon}(t) \le C_{\frac{1}{2}-\epsilon} \left(\sum_{k=1}^{\infty} \left(\frac{n}{k}\right)^{\frac{1}{2}-\epsilon} 2^{k-n} t^{2^{k-1}} - 1 \right.$$

$$\left. \le C_{\frac{1}{2}-\epsilon} \left(\sum_{k=1}^{\infty} \left(\frac{n}{k}\right)^{\frac{1}{2}-\epsilon} 2^{k-n} + \sum_{k=n+1}^{\infty} \left(\frac{n}{k}\right)^{\frac{1}{2}-\epsilon} 2^{k-n} \left(1 - \frac{1}{2^{n+1}}\right)^{2^{k-1}-1} \right) \right.$$

$$\leq C_{\frac{1}{2}-\epsilon}\left(\sum_{k=1}^{n}\left(\frac{n}{k}\right)^{\frac{1}{2}-\epsilon}2^{k-n}+\sum_{k=n+1}^{\infty}\left(\frac{n}{k}\right)^{\frac{1}{2}-\epsilon}2^{k-n}q^{2^{k-n}},\right). \quad (43)$$

Where $q = e^{-\frac{1}{8}} \in (0,1)$.

Continuation of estimate (43) depends on $1-\epsilon$. If $\epsilon \leq 1$, then $\left(\frac{n}{k}\right)^{1-\epsilon} \leq e^{(\epsilon-1)\frac{k-n}{n}} \leq e^{(\epsilon-1)(k-n)}$, so

$$G_{1-\epsilon}(t) \leq C_{1-\epsilon}\left(\sum_{k=1}^{n}2^{k-n}+\sum_{k=n+1}^{\infty}(2e^{\epsilon-1})^{k-n}q^{2^{k-n}}\right)$$

$$\leq C_{1-\epsilon}\left(2+\sum_{\epsilon=0}^{\infty}(2e^{\epsilon-1})^{1-\epsilon}q^{2^{1-\epsilon}}\right)$$

$$= C'_{1-\epsilon}. \quad (44)$$

If $\epsilon \geq 0$, then

$$G_{1+\epsilon}(t) \leq C_{1+\epsilon}\left(\sum_{1\leq k\leq\frac{n}{2}}n^{1+\epsilon}2^{-\frac{n}{2}}+2^{1+\epsilon}\sum_{\frac{n}{2}\leq k\leq n}2^{k-n}+\sum_{\epsilon=0}^{\infty}2^{1-\epsilon}q^{2^{1-\epsilon}}\right)$$

$$\leq C_{1+\epsilon}\left(n^{2+\epsilon}2^{-\frac{n}{2}-1}+2^{2+\epsilon}+\sum_{\epsilon=0}^{\infty}2^{1-\epsilon}q^{2^{1-\epsilon}}\right) = C''_{1+\epsilon}. \quad (45)$$

It remains to remark that (42) follows from (44) and (45).

Corollary (5.2.10)[276]: Let $\epsilon < \frac{1}{2}$ and let $0 \leq \epsilon < \infty$ then there exist functions $F_x \in L^{\epsilon+\frac{1}{2}}\mathfrak{B}(\mathbb{D}), 0 \leq x \leq 1$ such that $\|F_x\|_{L^{\epsilon+\frac{1}{2}}\mathfrak{B}(\mathbb{D})} \leq 1$ and

$$\sum_s\left(\int_0^1|F_x(z_s)|^{1+\epsilon}\,dx\right)^{\frac{1}{1+\epsilon}} \geq \sum_s \tau_{1+\epsilon,\frac{1}{2}+\epsilon}\left(log\frac{e}{1-|z_s|^2}\right)^{-\epsilon}, \qquad z_s \in \mathbb{D}, \quad (46)$$

for a constant $\tau_{1+\epsilon,\frac{1}{2}+\epsilon} > 0$.

Proof. Let the constant $C_{\frac{1}{2}-\epsilon} > 0$ be that provided by Corollary (5.2.9) for $x \in [0,1]$ consider the following functions :

$$\sum_s F_x(z_s) = \frac{1}{1+C_{\frac{1}{2}-\epsilon}}\sum_{K=0}^{\infty}\sum_s\frac{R_k(x)}{(k+1)^{\frac{1}{2}-\epsilon}}z_s^{2^k-1}, \qquad z_s \in \mathbb{D}.$$

Where $R_k(x) =$ sign in $(2^{k+1}\pi x)$ are the Rademacher functions. First , we have $F_x \in H(\mathbb{D})$ and

$$\|F_x\|_{L^{-\epsilon+\frac{1}{2}}\mathfrak{B}(\mathbb{D})}$$

$$\leq \frac{1}{1+C_{\frac{1}{2}-\epsilon}}\left(1+\sup_{z_s\in\mathbb{D}}\sum_s(1-|z_s|^2)\left(log\frac{e}{1-|z_s|^2}\right)^{\frac{1}{2}-\epsilon}\sum_{k=1}^{\infty}\frac{2^k-1}{(k+1)^{\frac{1}{2}-\epsilon}}|z_s|^{2k-2}\right)$$

$$\leq 1.$$

by Corollary (5.2.9), with $t = |z_s|^2$. Second, by [91]

$$\int_0^1 \sum_s |F_x(z_s)|^{1+\epsilon}\, dx \geq A_{1+\epsilon,\frac{1}{2}-\epsilon} \sum_s \left(\sum_{k=0}^{\infty} \frac{|z_s|^{2(2^k-1)}}{(k+1)^{2(\frac{1}{2}-\epsilon)}} \right)^{\frac{1+\epsilon}{2}},$$

Applying Corollary (5.2.8) with $\epsilon = 1$ and $t = |z_s|^2$, we obtain

$$\sum_{k=0}^{\infty} \sum_s \frac{|z_s|^{2(2^k-1)}}{(k+1)^{2(\frac{1}{2}-\epsilon)}} \geq C_{2\epsilon} \sum_s \left(\log \frac{1}{1-|z_s|^2} \right)^{2\epsilon}, \qquad z_s \in \mathbb{D}$$

Therefore,

$$\int_0^1 \sum_s |F_x(z_s)|^{1+\epsilon}\, dx \geq \tau^{1+\epsilon}_{1+\epsilon,\frac{1}{2}-\epsilon} \sum_s \left(\log \frac{1}{1-|z_s|^2} \right)^{\epsilon(1+\epsilon)}, \qquad z_s \in \mathbb{D}$$

As require.

Corollary (5.2.11)[276]: Let $\epsilon < \frac{1}{2}$, and let $\varphi : \mathbb{D} \to \mathbb{D}$ be a holomorphic mapping. Then the following properties are equivalent:

$$C_\varphi \ : \ L^{-\epsilon+\frac{1}{2}} \mathfrak{B}(\mathbb{D}) \to H^2(\mathbb{D}). \tag{47}$$

is a bounded operator (47)

$$\int_0^1 \sum_s \frac{|\varphi'(r\zeta_s)|^2}{(1-|\varphi(r\zeta_s)|^2)^2} \left(\log \frac{e}{1-|\varphi(r\zeta_s)|^2} \right)^{-2(\frac{1}{2}-\epsilon)} (1-r)dr \in L^1(\mathbb{T}); \tag{48}$$

$$\varphi \in H_h^{2\epsilon}(\mathbb{D}). \tag{49}$$

Proof. Let (47) hold. Applying Theorem (5.2.1), choose $f_1, f_2 \in L^{-\epsilon+\frac{1}{2}} \mathfrak{B}(\mathbb{D})$ that

$$\sum_s (|f_1'(z_s)|^2 + |f_2'(z_s)|^2) \geq \sum_s (1-|z_s|^2)^{-2} \left(\log \frac{e}{1-|z_s|^2} \right)^{2\epsilon-1}, \qquad z_s \in \mathbb{D}.$$

By (47), we have $C_\varphi f_j \in H^2(\mathbb{D})$, $j = 1, 2$. Thus,

$$\infty > \|g(C_\varphi f_1)\|^2_{L^2(\mathbb{T})} + \|g(C_\varphi f_2)\|^2_{L^2(\mathbb{T})}$$

$$= \int_{\mathbb{T}} \int_0^1 \sum_s \left(\left|f_1'(\varphi(r\zeta_s))\right|^2 + \left|f_2'(\varphi(r\zeta_s))\right|^2 \right) |\varphi'(r\zeta_s)|^2 (1-r) dr d\sigma(\zeta_s)$$

$$\geq \int_{\mathbb{T}} \int_0^1 \sum_s \frac{|\varphi'(r\zeta_s)|^2}{(1-|\varphi(r\zeta_s)|^2)^2} \left(\log \frac{e}{(1-|\varphi(r\zeta_s)|^2)^2} \right)^{2\epsilon-1} (1-r) dr d\sigma(\zeta_s).$$

So, (47) implies (48).

To prove the converse implication, assume that (48) holds.

Given $f \in L^{-\epsilon+\frac{1}{2}} \mathfrak{B}(\mathbb{D})$ and $\zeta_s \in T$, we have

$$\sum_s g^2(C_\varphi f)(\zeta_s)$$

$$\leq \|f\|^2_{L^{-\epsilon+\frac{1}{2}} \mathfrak{B}(\mathbb{D})} \int_0^1 \sum_s \frac{|\varphi'(r\zeta_s)|^2}{(1-|\varphi(r\zeta_s)|^2)^2} \left(\log \frac{e}{1-|\varphi(r\zeta_s)|^2} \right)^{-2()} (1-r)dr.$$

Whence, $g(C_\varphi f) \in L^2(\mathbb{T})$ by (48).

Therefore, $C_\varphi f \in H^2(\mathbb{D})$. So, (48)implies (47).

Finally, properties (48) and (49) are equivalent by [203].

Corollary (5.2.12)[276]: (cf. [190]) Let $\epsilon < \frac{1}{2}$, and let $\varphi : \mathbb{D} \to \mathbb{D}$ be a holomorphic function. Assume that $C_\varphi : L^{-\epsilon+\frac{1}{2}}\mathfrak{B}(\mathbb{D}) \to \mathrm{BMOA}(\mathbb{D})$ is a bounded operator. Then there exist constants $\varepsilon = \varepsilon(\frac{1}{2}-\epsilon, \|C_\varphi\|_{L^{-\epsilon+\frac{1}{2}}\mathfrak{B}(\mathbb{D})\to BMOA(\mathbb{D})} > 0$ and $\epsilon \geq 0$ such that

$$\int_{\mathbb{T}} \sum_s \exp\left(\varepsilon \log \frac{1}{1-|\varphi^*(\zeta_s)|^2}\right)^{\epsilon} d\sigma(\zeta_s) \leq 1+\epsilon.$$

Proof. Since $\epsilon < \frac{1}{2}$ and the operator $C_\varphi : L^{-\epsilon+\frac{1}{2}}\mathfrak{B}(\mathbb{D}) \to \mathrm{BMOA}(\mathbb{D})$ is bounded, we have $\sum_s |\varphi^*(\zeta_s)| < 1$ for $\sigma - a.e. \zeta_s \in \mathbb{T}$ (see [110]). Therefore, for every $f \in H(\mathbb{D})$,

$$\sum_s f(\varphi^*(\zeta_s)) = \lim_{r\to 1^-} \sum_s f(\varphi(r\zeta_s)) = \sum_s (f\circ\varphi)^*(\zeta_s). \quad \sigma - a.e. \zeta_s \in \mathbb{T} \quad (50)$$

Let the functions $F_x \in L^{-\epsilon+\frac{1}{2}}\mathfrak{B}(\mathbb{D}), 0 \leq x \leq 1$, be those provided by Corollary (5.2.10) with $\epsilon = 0$. Then

$$\|f_x \circ \varphi\|_{BMOA(D)} \leq \|C_\varphi\|_{L^{-\epsilon+\frac{1}{2}}\mathfrak{B}(\mathbb{D})\to BMOA(D)}.$$

Put $\delta = (1+\epsilon)\|C_\varphi\|^{-1}_{L^{-\epsilon+\frac{1}{2}}\mathfrak{B}(\mathbb{D})\to BMOA(D)}$, where $(1+\epsilon)$ is the constant provided by the John– Nirenberg theorem. By (50) and (34), we have

$$\int_{\mathbb{T}} \exp\sum_s \left(\delta|F_x(\varphi^*(\zeta_s))|\right) d\sigma(\zeta_s) = \int_{\mathbb{T}} \exp\sum_s \left(\delta|(F_x\circ\varphi)^*(\zeta_s)|\right) d\sigma(\zeta_s) \leq 1+\epsilon,$$
$$0 \leq x \leq 1.$$

Finally, applying Fubini's theorem, Jensen's inequality, and Corollary (5.2.10), we obtain

$$\begin{aligned}1+\epsilon &\geq \int_{\mathbb{T}} \int_0^1 \exp\sum_s \left(\delta|F_x(\varphi^*(\zeta_s))|\right) dx\, d\sigma(\zeta_s)\\ &\geq \int_{\mathbb{T}} \exp\sum_s \left(\delta \int_0^1 |F_x(\varphi^*(\zeta_s))|\, dx\right) d\sigma(\zeta_s)\\ &\geq \int_{\mathbb{T}} \exp\sum_s \left(\delta\tau_{1,\frac{1}{2}-\epsilon}\left(\log\frac{1}{1-|\varphi^*(\zeta_s)|^2}\right)^{\epsilon} dx\right) d\sigma(\zeta_s).\end{aligned}$$

As required.

Section (5.3): Quadratic Integrals

For $H(B_d)$ denote the space of holomorphic functions on the unit ball B_d of $\mathbb{C}^d$, $d \geq 1$. Given a gauge function $\omega : (0, 1] \to (0, +\infty)$, the weighted Bloch space $\mathfrak{B}^\omega(B_d)$ consists of those $f \in H(B_d)$ for which

$$\|f\|_{\mathfrak{B}^\omega(B_d)} = |f(0)| + sup_{z\in Bd} \frac{|\mathcal{R}f(z)||(1-|z|)|}{\omega(1-|z|)} < \infty \quad (51)$$

Where

$$\mathcal{R}f(z) = \sum_{j=1}^{d} z_j \frac{\partial f}{\partial z_j}(z), \qquad z \in B_d$$

is the radial derivative of f. $\mathfrak{B}^\omega(B_d)$ is a Banach space with respect to the norm defined by (51). If $\omega \equiv 1$, then $\mathfrak{B}^\omega(B_d)$ is the classical Bloch space $\mathfrak{B}(B_d)$. Usually we suppose that the gauge function ω is increasing; hence, we have $\mathfrak{B}^{\omega}(B_d) \subset \mathfrak{B}(B_d)$.

The above notation is not completely standard: often the weight $t/\omega(t)$ is attributed to $\mathfrak{B}(B_d)$.

Assuming that ω is sufficiently regular, we show that the quadratic integral

$$I(x) = I_x(x) = \int_x^1 \frac{w^2(t)}{t}\, dt, \qquad 0 < x < 1,$$

governs the radial divergence and integral reverse estimates in $\mathfrak{B}^{\omega}(B_d)$. In both cases, the solutions are based on the classical Hadamard gap series.

Given $f \in H(B_d)$ and $\zeta \in \partial B_d$, we say that f has a radial limit at ζ if there exists a *finite* limit $f^*(\zeta) = \lim_{r\to 1^-} f(r\zeta)$.

Let σ_d denote the normalized Lebesgue measure on the unit sphere ∂B_d. The radial convergence or divergence in $\mathfrak{B}^{\omega}(B_d)$ is described in terms of $(I(0^+))$ by the following dichotomy:

Remark that the condition $I(0+) = \infty$ was previously used by Dyakonov [201] to construct a non-BMO function lying in $\mathfrak{B}^{\omega}(B_1)$ and in all Hardy spaces $H^p(B_1)$, $0 < p < \infty$.

Given an unbounded decreasing function $v : (0, 1] \to (0, +\infty)$, typical reverse estimates are obtained in the growth space $A^v(B_d)$, which consists of $f \in H(B_d)$ such that $|f(z)| \le C_v(1 - |z|)$ for all $z \in B_d$. Namely, under appropriate restrictions on v, there exists a finite family

$$\{f_j\}_{j=1}^{J} \subset A^v(B_d)$$

such that

$$|f_1(z)| + \cdots + |f_J(z)| \ge C_v(1 - |z|)$$

for all $z \in B_d$ (see, for example, [200]).

For the weighted Bloch space $\mathfrak{B}^{\omega}(B_d)$ the following result provides integral reverse estimates related to the function $\Phi^{\frac{1}{2}}(1 - |z|), z \in B_d$, where

$$\Phi(x) = \Phi_\omega(x) = 1 + \int_x^1 \frac{w^2(t)}{t}\, dt, \quad 0 < x < 1.$$

For $\omega \equiv 1$ and for logarithmic functions ω, the above estimates were obtained in [190] and [199], respectively.

We devoted to the radial divergence problem. We prove Theorem (5.3.5) and we show that estimate (56) is sharp, up to a multi-plicative constant. Applications of Theorem (5.3.5).

Proposition (5.3.1)[205]: Let $\omega : (0, 1] \to (0, +\infty)$ be an increasing function.

i. Let $I(0+) < \infty$. If $f \in \mathfrak{B}^{\omega}(B_d)$, then f has radial limits σ_d-almost everywhere.

ii. Let $I(0+) = \infty$ and let $\omega(t)/t^{1-\varepsilon}$ be decreasing for some $\varepsilon > 0$. Then the space $\mathfrak{B}^{\omega}(B_d)$ contains a function with no radial limits σ_d-almost everywhere.

Proof. (i) is a known fact. Indeed, if $I(0+) < \infty$ and $f \in \mathfrak{B}^{\omega}(B_d)$, then $|Rf(z)|^2(1 - |z|)$ is a Carleson measure, hence, $f \in \mathrm{BMOA}(B_d)$. In particular, f has radial limits $\sigma_d - a.e.$

(ii) for $d = 1$ Put

$$f(z) = \sum_{k=0}^{\infty} w(2^{-k})\, z^{2^k}, z \in B_1.$$

Standard arguments guarantee that $f \in \mathfrak{B}^{\omega}(B_1)$. For example, let $t \in (0, 1]$ and let $\tau = \frac{1}{t} \ge 1$. Observe that

$$\frac{\tau\omega\left(\frac{1}{\tau}\right)}{\tau} \quad is \ \ a \, decreasing \ \ function \, of \, \tau \geq 1.$$

because $\omega(t)$ is increasing. Also,

$$\tau\omega\left(\frac{1}{\tau}\right)/\tau^{\varepsilon} \quad is \, an \ \ incresing \, function \, of \, \tau \geq 1.$$

$because \frac{\omega(t)}{t^{1-\varepsilon}} is \, decreasing \ \ therefore \, , \tau\omega\left(\frac{1}{\tau}\right), \tau \ \geq \ 1,$ is a normal weight in the sense of [11]. The derivative f' is represented by a Hadamard gap series, hence, $f \in \mathfrak{B}^{\omega}$ (B_1) (see, e.g., [206]).

Since ω is increasing, we have

$$\sum_{k=0}^{\infty} \omega^2(2^{-k}) \geq I(0+) = \infty. \tag{52}$$

Thus, f has no radial limits $\sigma_1 - a.e.$ by [91].

(ii) $d \geq 2$ Fix a Ryll–Wojtaszczyk sequence $\{W\,[n]\}_{n=1}^{\infty}$ (see [197]). By definition, $W\,[n]$ is a holomorphic homogeneous polynomial of degree n, $\|W\,[n]\|_{L^{\infty}(\partial B_d)} = 1$ and $\|W\,[n]\|L^2(\partial B_d) \ \geq \ \delta$ for a universal constant $\delta \geq 0$. In particular, (52) guarantees that

$$\sum_{k=0}^{\infty} \left\|\omega(2^{-k})W[2^k]\right\|_{L^2(\partial B_d)}^2 = \infty.$$

Hence, by [77], there exists a sequence $\{U_k\}_{k=1}^{\infty}$of unitary operators on $\mathbb{C}^d$ such that

$$\sum_{k=0}^{\infty} \omega^2(2^{-k})\left|W[2^k] \circ U_k(\zeta)\right|^2 = \infty. \tag{53}$$

for σ_d-almost all $\zeta \in \partial B_d$. Put

$$f(z) = \sum_{k=0}^{\infty} \omega(2^{-k})W[2^k] \circ U_k(z) \quad , z \in B_d$$

First, fix a point $\zeta \ \in \ \partial B_d$ with property (53). Consider the slice-function $f_{\zeta}\,(\lambda) \ = \ f\,(\lambda\zeta), \lambda \ \in \ B_1$. Note that

$$f_{\zeta}\,(\lambda) \ = \sum_{k=0}^{\infty} a_k \lambda^{2^k} \ \ , \lambda \in B_{1.}$$

Where $a_k = w(2^{-k})W[2^k] \circ U_k(\zeta). \ \ By\,(4), we \ have \ \{a_k\}_{k=1}^{\infty} \notin \ell^2.$

Thus , f_{ζ} has no radial limits $\sigma_1 - a.e\,.by$ [91]. Since the latter property holds for σ_d $-$almost all $\zeta \ \in \ \partial B_d$, Fubini's theorem guarantees that f has no radial limits $\sigma_d - a.e.$.

Second, recall that $\left\|W\,[2^k] \circ \ U_k\right\|_{L^{\infty}(\partial B_d)} \ = \ 1.$ So, we deduce that $f \ \in \ \mathfrak{B}^{\omega}\ (B_d)$, applying the argument to the slice-functions f_{ζ} , $\zeta \in \partial B_d$.. This ends the proof of Proposition (5.3.1).

If $\omega\,(0+) \ > \ 0$, then $\mathfrak{B}^{\omega}(B_d)$ coincides with $\mathfrak{B}\,(B_d)$, hence, $\mathfrak{B}^{\omega}(B_d)$ contains a function with no radial limits everywhere (see [207], [136]).However, if $\omega(0+) \ = \ 0$, then Proposition (5.3.1)(ii) is not improvable in this direction. Indeed, if$\omega\,(0+) \ = \ 0 \, and \, f \ \in \ \mathfrak{B}^{\omega}(B_1)$, then f has radial limits on a set of Hausdorff dimension one (see [208]).

To obtain the hyperbolic analog of $\mathfrak{B}^{\omega}(B_d)$, replace $\mathcal{R}f\,(z)$ by

$$\frac{\mathcal{R}\varphi\ (z),}{1\ -\ |\varphi(z)|2\ ,}$$

where $\varphi : B_n \to B_m, m, n \in \mathbb{N}$, is a holomorphic mapping. The radial limit $\varphi^*(\zeta)$ is defined at $\sigma_n$almost every point of ∂B_n, hence, it is natural to replace the radial divergence condition by the following property: $|\varphi^*| = 1\ \sigma_d - a.e.$, that is, φ is inner. While the problem in the hyperbolic setting is more sophisticated, the following analog of Proposition (5.3.1) is known, at least for $n = m = 1$.

Theorem (5.3.2)[205]: ([13)], [92]). Let $\omega : (0,1] \to (0,+\infty)$ be an increasing function

(i) Assume that $I(0+) < \infty$ and $\varphi : B_1 \to B_1$ is a holomorphic function such that

$$\frac{|\varphi'(z)|(1-|z|)}{1-|\varphi(z)|} \le \omega(1-|z|), z \in B_{1.}$$

Then φ is not inner.

(ii) Assume that $I(0+) = \infty$ and $\omega(t)/t^{1-\varepsilon}$ decreases for some $\varepsilon > 0$. Then there exists an inner function $\varphi : B_1 \to B_1$ such that

$$\frac{|\varphi'(z)|(1-|z|)}{1-|\varphi(z)|} \le \omega(1-|z|), z \in B_{1.}$$

We apply Theorem (5.3.5) to obtain quantitative versions of Theorem (5.3.2)(i).

Lemma (5.3.3)[205]: Let $\omega : (0,1] \to (0,+\infty)$ be an increasing function. Put

$$\Psi(r) = \sum_{k=0}^{\infty} \omega^2(2^{-k}) r^{2^k-1}, \qquad 0 \le r < 1.$$

Then

$$\Psi(r) \ge C\Phi(1-r)\ for\ a\ constant\ C = C_\omega > 0.$$

Proof. Let $2^{-n-1} \le 1-r < 2^{-n}\ for\ some\ n \in \mathbb{Z}_+$. Then

$$2\Psi(r) \ge 2\omega^2(1) + \sum_{k=1}^{n} \omega^2\ (2^{-k})(1-2^{-n})^{2^k-1}$$

$$\ge \omega^2(1) + \frac{1}{e}\sum_{k=0}^{n} \omega^2\ (2^{-k}) \ge C\Phi(2^{-n-1}) \ge C\Phi(1-\mathrm{r}),$$

since ω is increasing and Φ is decreasing.

Also, we need the following improvement of the Ryll–Wojtaszczyk theorem used.

Theorem (5.3.4)[205]: ([192]). Let $d \in \mathbb{N}$. Then there exist $\delta = \delta(d) \in (0,1)$ and $J = J(d) \in \mathbb{N}$ with the following property: For every $n \in \mathbb{N}$, there exist holomorphic homogeneous polynomials $W_j\ [n]$ of degree n, $1 \le j \le J$, such that

$$\left\|W_j[n]\right\|_{L^\infty(\partial B_d)} \le 1\ and \tag{54}$$

$$max_{1\le j\le J}\left|W_j[n](\xi)\right| \ge \zeta\quad for\ all\ \xi \in \partial B_{d.} \tag{55}$$

Probably, it is worth mentioning that $J(1) = 1$.

Theorem (5.3.5)[205]: Let $d \in \mathbb{N}$ and let $0 < p < \infty$. Assume that $\omega : (0, 1] \to (0, +\infty)$ increases and $\omega\ (t) / t^{1-\varepsilon}$ decreases for some $\varepsilon > 0$. Then there exists a constant $\tau_{d,p,\omega} > 0$ and functions $F_y \in \mathfrak{B}^\omega\ (B_d)$, $0 \le y \le 1$, such that $\left\|F_y\right\|_{\mathfrak{B}^\omega(B_d)} \le 1$ and

$$\int_0^1 \left|F_y(z)\right|^{2p} dy \ge \tau_{d,p,\omega}\ \Phi^p\ (1-|z|) \tag{56}$$

For all $z \in B_d$

Proof. Let the constant $\delta \in (0, 1)$ and the polynomials $W_j\ [n]$, $1 \le j \le J$, $n \in \mathbb{N}$, be those provided by Theorem (5.3.4).

For each non-dyadic $y \in [0, 1]$, consider the following functions:

$$F_{jy}(z) = \sum_{k=0}^{\infty} R_K\,(y) w(2^{-k}) W_j[2^k - 1](z), \qquad z \in B_d, 1 \leq j \leq J,$$

Where

$$R_k(y) sign\ sin(2^{k+1}\pi y), \qquad y \in [0,1]$$

Is the Rademacher function.

First, arguing as using estimate (54), we deduce that

$$\|F_{j,y}\|_{\mathfrak{B}^{\omega}(B_d)} \leq C.$$

Second, we obtain

$$C_P \int_0^1 |F_{jy}(z)|^{2p} dy \geq \left(\sum_{k=0}^{\infty} |\omega(2^{-k}) W_j[2^k - 1](z)|^2\right)^p$$

by [91]. Given positive numbers aj, $1 \leq j \leq J = J(d)$, we have

$$\left(\sum_{j=1}^{J} a_j\right)^p \leq C_{d,p} \sum_{j=1}^{J} a_j^p.$$

Hence,

$$C_{d,p} \sum_{j=1}^{j} \int_0^1 |F_{j,y}(z)|^{2p}\, dy \geq \left(\sum_{k=0}^{\infty} \sum_{j=1}^{J} \omega^2\,(2^{-k}) |W_j[2^k - 1](z)|^2\right)^p$$

Since $W_j[2^k - 1], 1 \leq j \leq J$, are homogeneous polynomials of degree $2^k - 1$, we obtain

$$\sum_{k=0}^{\infty} \sum_{j=1}^{j} \omega^2\,(2^{-k}) |W_j[2^k - 1](z)|^2 \geq \delta^2 \sum_{k=0}^{\infty} \omega^2\,(2^{-k}) |z|^{2^{k+2}-2}$$
$$\geq \delta^2 C_\omega \Phi(1 - |z|^2), \quad z \in B_d.$$

By (55) and Lemma (5.3.3) with $r = |z|^2$. So,

$$C_{d,p} \sum_{j=1}^{J} \int_0^1 |F_{j,y}\, z|^{2p}\, dy \geq \left(\delta^2 C_\omega \Phi\,(1 - |z|^2)\right)^p, \quad z \in B_d,$$

Changing the indices of the functions $F_{j,y}$ and using a new variable of integration, we may reduce the above sum of integrals to one integral over [0, 1]. So, it remains to verify that

$$C\Phi(1 - r^2) \geq \Phi(1 - r), \qquad 0 \leq r < 1.$$

First, if $0 \leq r \leq \frac{2}{3}$, then $\Phi(1 - r) \leq C_\omega \leq C_\omega \Phi(1 - r^2)$ for a constant $C_\omega > 0$. Second, if $0 < \varepsilon < \frac{1}{3}$, then $\Phi(\varepsilon) - \Phi(2\varepsilon) \leq \omega^2(2\varepsilon) \leq 3\Phi\,(2\varepsilon)$, because ω is increasing. Thus

$$\Phi(1 - r) \leq 4\Phi(1 - r^2) for\ \frac{2}{3} < r < 1.$$

The proof of Theorem (5.3.5) is finished.

To show that inequality (56) is sharp, we estimate the integral means

$$M_p(f,r) = \left(\int_{\partial B_d} |f\,(r\zeta)|^p \, d\sigma_d(\zeta) \right)^{\frac{I}{p}}. \quad 0 < r < 1,$$

for the functions $f \in \mathfrak{B}^{\omega}(B_d)$.

For $\omega \equiv 1$, the following result was obtained in [209] and [210].

Proposition (5.3.6)[205]: Let $0 < p < \infty$ and let $f \in \mathfrak{B}^{\omega}(B_d)$ Then

$$M_p(f,r) \le C\|f\|_{\mathfrak{B}^{\omega}(B_d)} \Phi^{\frac{1}{2}}(1-r), \quad 0 < r < 1 \tag{57}$$

for a constant $C > 0$.

Proof. For $f \in H(B_d)$ *and* $0 < r < 1$, we have

$$M_p(f,r) \le C|f(0)| + C\left(\int_{\partial B_d} \left(\int_0^1 r^2 \; |\mathcal{R}f(rt\zeta)|^2 (1-t)\, dt \right)^{\frac{p}{2}} d\sigma_d(\zeta) \right)^{1/p}$$

for a constant $C > 0$; see, for example, [211].

If $f \in \mathfrak{B}^{\omega}(B_d)$, then, using the defining property (51), we obtain

$$\int_0^1 r^2\, |\mathcal{R}f(rt\zeta)|^2 (1-t) dt = \int_0^r |\mathcal{R}f(t\zeta)|^2 (r-t) dt \le \|f\|^2_{\mathfrak{B}^{\omega}(B_d)} \int_0^r \frac{\omega^2(1-t)}{1-t} dt$$
$$\le \|f\|^2_{\mathfrak{B}^{\omega}(B_d)} \Phi(1-r).$$

Since $|f\,(0)| \le \|f\|_{\mathfrak{B}^{\omega}(B_d)}$ in sum we obtain the required estimate.

Comparing Proposition (5.3.6) and Theorem (5.3.5), we conclude that the direct estimate (57) and the reverse estimate (56) are not improvable, up to multiplicative constants.

Given a gauge function ω, the weighted Hardy–Bloch space $\mathfrak{B}^{\omega}(B_d), 0 < p < \infty$, consists of those $f \in H(B_d)$ for which

$$\|f\|_{\mathfrak{B}^{\omega}_p(B_d)} = |f(0)| + sup_{0<r<1} \frac{M_p(\mathcal{R}f,r)(1-r)}{\omega(1-r)} < \infty. \tag{58}$$

Clearly we have $\mathfrak{B}^{\omega}(B_d) \subset \mathfrak{B}^{\omega}_p(B_d),\ 0 < p < \infty,$. So , it is interesting that estimate (57) is sharp for $f \in \mathfrak{B}^{\omega}(B_d)$ and holds for all $f \in \mathfrak{B}^{\omega}_p(B_d)$ with $p \ge 2.$ Namely, we have the following proposition that was proved in [202] for $\omega \equiv 1$.

Proposition (5.3.7)[205]: Let $2 \le p < \infty$ and let . Then $f \in \mathfrak{B}^{\omega}_p(B_d)$, Then

$$M_p(f,r) \le C\|f\|_{\mathfrak{B}^{\omega}(B_d)} \Phi^{\frac{1}{2}}(1-r), \quad 0 < r < 1, \tag{59}$$

For a constant $C > 0$.

Proof. For $f \in H(B_d)$ and $0 < r < 1$, , we have

$$M_p(f,r) \le C|f(0)| + C\left(\int_0^1 \left(\int_{\partial B_d} |\mathcal{R}f(rt\zeta)|^P \, d\sigma_d(\zeta) \right)^{\frac{2}{p}} r^2(1-t)dt \right)^{\frac{1}{2}} \tag{60}$$

For a constant $C > 0$ (see [212] for $d = 1$: integration by slices gives the result for $d \ge$

1). Now, we argue as in the proof of Proposition (5.3.6). Namelym for $f \in \mathfrak{B}^{\omega}(B_d)$, the defining property (58) guarantees that

$$\int_0^1 \left(\int_{\partial B_d} |\mathcal{R}f(rt\zeta)|^P \, d\sigma_d(\zeta) \right)^{\frac{2}{p}} r^2(1-t)dt = \int_0^r M_p^2(\mathcal{R}f, t)(r-t)dt$$

$$\leq \|f\|^2_{\mathfrak{B}^{\omega}(B_d)} \int_0^r \frac{\omega^2(1-t)}{1-t} \, dt \leq \|f\|^2_{\mathfrak{B}^{\omega}(B_d)} \Phi \, (1-r)$$

Since $|f(0)| \leq \|f\|_{\mathfrak{B}^{\omega}(B_d)}$, the proof is finished.

We assume that $\omega: (0,1] \to (0,+\infty)$ is an increasing function.

Given a space $X \subset H(B_d)$ and $0 < q < \infty$, recall that a positive Borel measure μ on B_d is called q-Carleson for X if $X \subset L^q(B_d, \mu)$.

Suppose that $\omega(t)/t^{1-\varepsilon}$ decreases for some $\varepsilon > 0$. A direct application of Theorem (5.3.5) gives the following result:

Corollary (5.3.8)[205]: Let $0 < q < \infty$ and let μ be a q-Carleson measure for $B^{\omega}(B_d)$. Then

$$\int_{B_d} \Phi^{\frac{q}{2}}(1-|z|)d\mu(z) < \infty.$$

If μ is a radial measure, then the above corollary is reversible. Moreover, the corresponding result holds for all spaces $B_p^{\omega}(B_d)$, $p \geq 2$.

Proposition (5.3.9)[205]: Let $0 < q < \infty$ and let ρ be a positive measure on [0, 1). Then the following properties are equivalent:

$$\int_0^1 \int_{\partial B_d} |f(r\zeta)|^q d\sigma_d(\zeta)d\rho(r) < \infty \qquad for\ all\ f \in \mathfrak{B}_p^{\omega}(B_d), p \geq 2; \tag{61}$$

$$\int_0^1 \int_{\partial B_d} |f(r\zeta)|^q d\sigma_d(\zeta)d\rho(r) < \infty \qquad for\ all\ f \in \mathfrak{B}^{\omega}(B_d); \tag{62}$$

$$\int_0^1 \Phi^{\frac{q}{2}}(1-r)d\rho(r) < \infty. \tag{63}$$

Proof. The implication (61)⇒ (62) is trivial, because $B_{\omega}(B_d) \subset B_p^{\omega}(B_d)$. Next, (62) implies (63) by Corollary (5.3.8). Finally, Proposition (5.3.7) guaran-tees that (63) implies (62).

Let $I_{\omega}(0+) < \infty$. As observed in [213], the conclusion of Theorem (5.3.2)(i) remain true if the restriction

$$\frac{|\varphi'(z)||1-|z||}{1-\varphi(z)} \leq \omega(1-|z|). z \in B_1,$$

is replaced by the following weaker assumption:

$$\frac{|\varphi'(z)||1-|z||}{1-|\varphi(z)|} \Omega\big(1-\varphi(z)\big) \leq \omega(1-|z|), z \in B_{1,}$$

where $\Omega : (0, 1] \to (0, +\infty)$ is a bounded measurable function such that

$$I_\Omega = \int_0^1 \frac{\Omega^2(t)}{t}\, dt = \infty.$$

To obtain quantitative results of the above type, we apply Theorem (5.3.5). Also, we make weaker assumptions about φ.

So, suppose that Ω is increasing and $\Omega(t)/t^{1-\varepsilon}$ is decreasing for some $\varepsilon > 0$. Put

$$\Phi_\Omega(x) = I + \int_x^1 \frac{\Omega^2(t)}{t}\, dt, \qquad 0 < x < 1$$

Corollary (5.3.10)[205]: Let ϕ : B1 $\rightarrow$ B1 be a holomorphic mapping and let $1 \le p < \infty$. Assume that $I_\omega\,(0+) < \infty, 1_\Omega = \infty$ and

$$(1-r)\left(\int_{\partial B_1}\left(\frac{|\varphi'(r\zeta)|}{1-|\varphi(r\zeta)|}\Omega(1-|\varphi(r\zeta)|)\right)^{2P} d\sigma_1(\zeta)\right)^{\frac{1}{2P}} \le \omega(1-r) \qquad (64)$$

For $0 < r < 1$. then

$$sup_{0<r<1} \int_{\partial B_1} \Phi_\Omega^P\,(1-|\varphi(r\zeta)|)d\sigma_1(\zeta) < \infty.$$

in particular, $|\varphi^*| < 1\ \sigma_1 - a.e.$

Proof. Let the constant $\tau = \tau_{1,p,\Omega} > 0$ and the function $F_y \in \mathfrak{B}^\Omega(B_1)$, $0 \le y \le 1$, by those provided by Theorem (5.3.5) for d =1 and for Ω in place of ω.

Since $\|F_y\|_{\mathfrak{B}^\Omega(B_1)} \le 1$, , we have

$$\left|\left(F_y \circ \varphi\right)'(z)\right| \le \left|F_y'\left(\varphi(z)\right)\right|\left|\varphi'(z)\right| \le \frac{\left|\varphi'(z)\right|}{1-|\varphi(z)|}\Omega(1-|\varphi(z)|), z \in B_1.$$

So, using (64) and the hypothesis $I_\omega(0+) < \infty$, we obtain

$$\int_0^1 M_{2P}^2\left(\left(F_y \circ \varphi\right)', t\right)(1-t)dt \le \int_0^1 \frac{\omega^2(1-t)}{1-t}\, dt < \infty.$$

We further observe that $|F_y \circ \varphi(0)| \le C_\varphi \|F_y\|_{B^\Omega(B1)} \le C$, and so estimate (60) guarantees that

$$\int_{\partial B_1} \left|F_y \circ \varphi(r\zeta)\right|^{2p} d\sigma_1(\zeta) \le C, \quad 0 \le y \le 1, \ 0 < r < 1$$

for a universal constant $C > 0$. Hence, applying Fubini's theorem and Theorem (5.3.5), we obtain

$$C \ge \int_{\partial B_1}\int_0^1 \left|F_y \circ \varphi(r\zeta)\right|^{2P} dy\, d\sigma_1(\zeta) \ge \int_{\partial B_1} \Phi_\Omega^p\,(1-|\varphi(r\zeta)|)\, d\sigma_1(\zeta),$$

as required.

Corollary (5.3.11)[276]: Let $\omega : (0,1] \rightarrow (0,+\infty)$ be an increasing function.

(i) Let $I\,(0+) < \infty$. If $f \in \mathfrak{B}^\omega\,(B_{1+\epsilon})$, then f has radial limits $\sigma_{1+\epsilon}$-almost everywhere.

(ii) Let $I(0+) = \infty$ and let $\frac{\omega(t)}{t^{1-\varepsilon}}$ be decreasing for some $\varepsilon > 0$. Then the space

$\mathfrak{B}^{\omega}(B_{1+\epsilon})$ contains a function with no radial limits σ_d-almost everywhere.

Proof. (i) is a known fact. Indeed, if $I\ (0+)\ <\ \infty$ and $f\ \in\ \mathfrak{B}^{\omega}(B_{1+\epsilon})$, then $|Rf(z_r)|^2(1-|z_r|)$ is a Carleson measure, hence, $f\ \in$ BMOA($B_{1+\epsilon}$). In particular, f has radial limits $\sigma_{1+\epsilon}-a.e.$

(ii) for $\epsilon=0$ (see [276]).

Put

$$\sum_r f(z_r)=\sum_{k=0}^{\infty}\sum_r \omega(2^{-k})\,z_r^{2^k}, z_r\in B_1.$$

Standard arguments guarantee that $f\ \in\ \mathfrak{B}^{\omega}(B_1)$. For example, let $t\in(0,1]$ and let $t=\frac{1}{1+\epsilon}$, $\epsilon\geq 0$. Observe that

$$\omega\left(\frac{1}{1+\epsilon}\right)\text{ is a decreasing function of }\epsilon\geq 0.$$

because $\omega(t)$ is increasing. Also,

$$(1+\epsilon)^{1-\epsilon}\omega\left(\frac{1}{1+\epsilon}\right)\text{ is an incresing function of }\epsilon\geq 0.$$

because $\frac{\omega(t)}{t^{1-\varepsilon}}$is decreasing therefore , $(1+\epsilon)\omega\left(\frac{1}{1+\epsilon}\right), \epsilon\geq 0,$ is a normal weight in the sense of [11]. The derivative f' is represented by a Hadamard gap series, hence, $f\in\mathfrak{B}^{\omega}(B_1)$ (see, e.g., [206]).

Since ω is increasing, we have

$$\sum_{k=0}^{\infty}\omega^2(2^{-k})\geq I(0+)=\infty. \tag{65}$$

Thus, f has no radial limits $\sigma_1-a.e.$ by [91].

(ii) for $\epsilon\geq 0$ (see [276]).

Fix a Ryll–Wojtaszczyk sequence $\{W\ [n]\}_{n=1}^{\infty}$ (see [197]). By definition, $W\ [n]$ is a holomorphic homogeneous polynomial of degree n, $\|W\ [n]\|_{L^{\infty}(\partial B_{2+\epsilon})}=1$ and $\|W\ [n]\|L^2(\partial B_{2+\epsilon})\ \geq\ \delta$ for a universal constant $\delta\ \geq 0$. In particular, (65) guarantees that

$$\sum_{k=0}^{\infty}\left\|\omega(2^{-k})W[2^k]\right\|_{L^2(\partial B_{2+\epsilon})}^2=\infty.$$

Hence, by [77], there exists a sequence $\{U_k\}_{k=1}^{\infty}$of unitary operators on $\mathbb{C}^{2+\epsilon}$ such that

$$\sum_{k=0}^{\infty}\sum_r \omega^2(2^{-k})\left|W[2^k]\circ U_k(\zeta_r)\right|^2=\infty. \tag{66}$$

for $\sigma_{2+\epsilon}$-almost all $\zeta_r\ \in\ \partial B_{2+\epsilon}$. Put

$$\sum_r f(z_r)=\sum_{k=0}^{\infty}\sum_r \omega(2^{-k})W[2^k]\circ U_k(z_r)\quad , z_r\in B_{2+\epsilon}$$

First, fix a point $\zeta_r\ \in\ \partial B_{2+\epsilon}$ with property (66). Consider the series of slice-function $\sum_r\ f_{\zeta_r}(\lambda)=\sum_r\ f(\lambda\zeta_r), \lambda\ \in\ B_1$. Note that

$$\sum_r f_{\zeta_r}(\lambda)=\sum_{k=0}^{\infty}a_k\lambda^{2^k}\ ,\lambda\in B_{1.}$$

Where $a_k=\sum_r\ \omega(2^{-k})W[2^k]\circ U_k(\zeta_r)$. By (66), we have $\{a_k\}_{k=1}^{\infty}\notin\ell^2$,

Thus , f_{ζ_r} has no radial limits σ_1-a.e .by [91]. Since the latter property holds for

$\sigma_{2+\epsilon}$ –almost all $\zeta_r \in \partial B_{2+\epsilon}$, Fubini's theorem guarantees that f has no radial limits $\sigma_{2+\epsilon} - a.e.$.

Second, recall that $\|W\ [2^k] \circ\ U_k\|_{L^\infty(\partial B_{2+\epsilon})} = 1$. So, we deduce that $f \in \mathfrak{B}^\omega\ (B_{2+\epsilon})$, applying the argument to the slice-functions $f_{\zeta_r}, \zeta_r \in \partial B_{2+\epsilon}$.. This ends the proof of Corollary (5.3.11).

Corollary (5.3.12)[276]: Let $\omega :\ (0,1] \rightarrow (0,+\infty)$ be an increasing function. Put

$$\Psi(1-\epsilon) = \sum_{k=0}^{\infty} \omega^2(2^{-k})(1-\epsilon)^{2^k-1}, \qquad \epsilon \leq 1.$$

Then

$$\Psi(1-\epsilon) \geq (1+\epsilon)\Phi(\epsilon) \text{ for a constant } 1+\epsilon = C_\omega > 0.$$

Proof. Let $2^{-(n+1)} \leq \epsilon < 2^{-n}$ for some $n \in \mathbb{Z}_+$. Then

$$2\Psi(1-\epsilon) \geq 2\omega^2(1) + \sum_{k=1}^{n} \omega^2\ (2^{-k})(1-2^{-n})^{2^k-1} \geq \omega^2(1) + \frac{1}{e}\sum_{k=0}^{n} \omega^2\ (2^{-k})$$
$$\geq (1+\epsilon)\Phi\left(2^{-(n+1)}\right) \geq (1+\epsilon)\Phi(\epsilon),$$

since ω is increasing and Φ is decreasing.

Corollary (5.3.13)[276]: Let $1+\epsilon \in \mathbb{N}$ and let $0 \leq \epsilon < \infty$. Assume that $\omega :\ (0,1] \rightarrow (0,+\infty)$ increases and $\frac{\omega(t)}{t^{1-\varepsilon}}$ decreases for some $\varepsilon > 0$. Then there exists a constant $(1+\epsilon)_{1+\epsilon,1+\epsilon,\omega} > 0$ and functions $F_{y_r} \in \mathfrak{B}^\omega(B_{1+\epsilon})$, $0 \leq y_r \leq 1$, such that $\sum_r \|F_{y_r}\|_{\mathfrak{B}^\omega(B_{1+\epsilon})} \leq 1$ and

$$\int_0^1 \sum_r \left|F_{y_r}(z_r)\right|^{2(1+\epsilon)} dy_r \geq (1+\epsilon)_{1+\epsilon,1+\epsilon,\omega} \sum_r \Phi^{1+\epsilon}\ (1-|z_r|) \tag{67}$$

For all $z_r \in B_{1+\epsilon}$.

Proof. Let the constant $\delta \in (0,1)$ and the polynomials $W_{1+\epsilon}[n], \epsilon \geq 0, n \in \mathbb{N}$, be those provided by Theorem (5.3.4).

For each non-dyadic $y_r \in [0,1]$, consider the series of the following functions:

$$\sum_r F_{1+\epsilon,y_r}(z_r) = \sum_{k=0}^{\infty}\sum_r R_K\ (y_r)\omega(2^{-k})W_{1+\epsilon}[2^k-1](z_r), \qquad z_r \in B_{1+\epsilon}, \qquad \epsilon \geq 0,$$

where

$$\sum_r R_k(y_r) = \sum_r \text{ sign} \sin(2^{k+1}\pi y_r), \qquad y_r \in [0,1],$$

is the series of Rademacher functions.

First, arguing estimate (5), we deduce that

$$\sum_r \|F_{1+\epsilon,y_r}\|_{\mathfrak{B}^\omega(B_{1+\epsilon})} \leq 1+\epsilon.$$

Second, we obtain

$$C_{1+\epsilon}\int_0^1 \sum_r \left|F_{1+\epsilon,y_r}(z_r)\right|^{2(1+\epsilon)} dy_r \geq \sum_r \left(\sum_{k=0}^{\infty}\left|\omega(2^{-k})W_{1+\epsilon}[2^k-1](z_r)\right|^2\right)^{1+\epsilon}$$

by [91]. Given positive numbers $a_{1+\epsilon}, \epsilon = 0$, we have

$$\left(\sum_{\epsilon=0}^{1+2\epsilon} a_{1+\epsilon}\right)^{1+\epsilon} \leq C_{1+\epsilon,1+\epsilon} \sum_{\epsilon=0}^{1+2\epsilon} a_{1+\epsilon}^{1+\epsilon}.$$

Hence,

$$C_{1+\epsilon,1+\epsilon} \sum_{\epsilon=0}^{1+2\epsilon} \int_0^1 \sum_r \left|F_{1+\epsilon,y_r}(z_r)\right|^{2(1+\epsilon)} dy_r$$

$$\geq \sum_r \left(\sum_{k=0}^{\infty} \sum_{\epsilon=0}^{1+2\epsilon} \omega^2 (2^{-k}) \left|W_{1+\epsilon}[2^k - 1](z_r)\right|^2\right)^{1+\epsilon}$$

Since $W_{1+\epsilon}[2^k - 1], \epsilon \geq 0$, are homogeneous polynomials of degree $2^k - 1$, we obtain

$$\sum_{k=0}^{\infty} \sum_{\epsilon=0}^{1+2\epsilon} \sum_r \omega^2 (2^{-k}) \left|W_{1+\epsilon}[2^k - 1](z_r)\right|^2 \geq \delta^2 \sum_{k=0}^{\infty} \sum_r \omega^2 (2^{-k}) |z_r|^{2^{k+2}-2}$$

$$\geq \delta^2 C_\omega \sum_r \Phi(1 - |z_r|^2), \; z_r \in B_{1+\epsilon}.$$

By (6) and Corollary (5.3.12) with $1 - \epsilon = |z_r|^2$. So,

$$C_{1+\epsilon,1+\epsilon} \sum_{\epsilon=0}^{1+2\epsilon} \int_0^1 \sum_r \left|F_{1+\epsilon,y_r} z_r\right|^{2(1+\epsilon)} dy_r \geq \sum_r \left(\delta^2 C_\omega \Phi\,(1 - |z_r|^2)\right)^{1+\epsilon}, \quad z_r \in B_{1+\epsilon},$$

Changing the indices of the functions $F_{1+\epsilon,y_r}$ and using a new variable of integration, we may reduce the above sum of integrals to one integral over [0, 1]. So, it remains to verify that

$$(1+\epsilon)\Phi(2\epsilon - \epsilon^2) \geq \Phi(\epsilon), \qquad \epsilon \leq 1.$$

First, if $\epsilon \leq \frac{2}{3}$, then $\Phi(\frac{1}{3} + \epsilon) \leq C_\omega \leq C_\omega \Phi(\frac{5}{9} + \frac{4}{3}\epsilon - \epsilon^2)$ for a constant $C_\omega > 0$. Second, if $0 < \varepsilon < \frac{1}{3}$, then $\Phi(\varepsilon) - \Phi(2\varepsilon) \leq \omega^2(2\varepsilon) \leq 3\Phi\,(2\varepsilon)$, because ω is increasing. Thus $\Phi(\epsilon) \leq 4\Phi(2\epsilon - \epsilon^2)$ for $\epsilon < \frac{1}{3}$.

The proof of Corollary (5.3.13) is finished.

Corollary (5.3.14)[276]: Let $0 \leq \epsilon < \infty$ and let $f \in \mathfrak{B}^\omega(B_{1+\epsilon})$ Then

$$M_{1+\epsilon}(f, 1-\epsilon) \leq (1+\epsilon)\|f\|_{\mathfrak{B}^\omega(B_{1+\epsilon})} \Phi^{\frac{1}{2}}(\epsilon), \qquad \epsilon \leq 1 \tag{68}$$

for a constant $\epsilon \geq 0$.

Proof. For $f \in H(B_{1+\epsilon})$ and $\epsilon \leq 1$, we have

$$M_{1+\epsilon}(f, 1-\epsilon) \leq (1+\epsilon)|f(0)| + (1+\epsilon) \sum_r \left(\int_{\partial B_{1+\epsilon}} \left(\int_0^1 (1-\epsilon)^2 \; |\mathcal{R}f((1-\epsilon)t\zeta_r)|^2 (1-t)\, dt\right)^{\frac{1+\epsilon}{2}} d\sigma_{1+\epsilon}(\zeta_r)\right)^{\frac{1}{1+\epsilon}}$$

for a constant $\epsilon \geq 0$; see, for example, [211].

If $f \in \mathfrak{B}^\omega(B_{1+\epsilon})$, then, using the defining property (1), we obtain

$$\int_0^1 \sum_r (1-\epsilon)^2 \, |\mathcal{R}f((1-\epsilon)t\zeta_r)|^2 (1-t)dt = \int_0^{1-\epsilon} \sum_r |\mathcal{R}f(t\zeta_r)|^2 (1-\epsilon-t)dt$$
$$\leq \|f\|^2_{\mathfrak{B}^{\omega}(B_{1+\epsilon})} \int_0^{1-\epsilon} \frac{\omega^2(1-t)}{1-t} dt \leq \|f\|^2_{\mathfrak{B}^{\omega}(B_{1+\epsilon})} \Phi(\epsilon).$$

Since $|f(0)| \leq \|f\|_{\mathfrak{B}^{\omega}(B_{1+\epsilon})}$ in sum we obtain the required estimate.

Corollary (5.3.15)[276]: Let $0 \leq \epsilon < \infty$ and let . Then $f \in \mathfrak{B}^{\omega}_{2+\epsilon}(B_{1+\epsilon})$, Then

$$M_{2+\epsilon}(f, 1-\epsilon) \leq (1+\epsilon)\|f\|_{\mathfrak{B}^{\omega}(B_{1+\epsilon})} \Phi^{\frac{1}{2}}(\epsilon), \quad \epsilon \leq 1, \tag{69}$$

For a constant $\epsilon \geq 0$.

Proof. For $f \in H(B_{1+\epsilon})$ and ≤ 1, , we have

$$M_{2+\epsilon}(f, 1-\epsilon) \leq (1+\epsilon)|f(0)|$$
$$+ (1+\epsilon)\sum_r \left(\int_0^1 \left(\int_{\partial B_{1+\epsilon}} |\mathcal{R}f((1-\epsilon)t\zeta_r)|^{2+\epsilon} d\sigma_{1+\epsilon}(\zeta_r) \right)^{\frac{2}{2+\epsilon}} (1-\epsilon)^2(1-t)dt \right)^{\frac{1}{2}} \tag{70}$$

For a constant $\epsilon \geq 0$ (see [212] for $\epsilon = 0$: integration by slices gives the result for $\epsilon \geq 0$). Now, we argue as in the proof of Corollary (5.3.14). Namely, for $f \in \mathfrak{B}^{\omega}(B_{1+\epsilon})$, the defining property (8) guarantees that

$$\int_0^1 \sum_r \left(\int_{\partial B_{1+\epsilon}} |\mathcal{R}f((1-\epsilon)t\zeta_r)|^{2+\epsilon} d\sigma_{1+\epsilon}(\zeta_r) \right)^{\frac{2}{2+\epsilon}} (1-\epsilon)^2(1-t)dt$$
$$= \int_0^{1-\epsilon} M^2_{2+\epsilon}(\mathcal{R}f, t)(1-\epsilon-t)dt$$
$$\leq \|f\|^2_{\mathfrak{B}^{\omega}(B_{1+\epsilon})} \int_0^{1-\epsilon} \frac{\omega^2(1-t)}{1-t} dt \leq \|f\|^2_{\mathfrak{B}^{\omega}(B_{1+\epsilon})} \Phi(\epsilon)$$

Since $|f(0)| \leq \|f\|_{\mathfrak{B}^{\omega}(B_{1+\epsilon})}$, the proof is finished.

Corollary (5.3.16)[276]: Let $0 \leq \epsilon < \infty$ and let ρ be a positive measure on [0, 1). Then the following properties are equivalent:

$$\int_0^1 \int_{\partial B_{1+\epsilon}} \sum_r |f((1-\epsilon)\zeta_r)|^{1+\epsilon} d\sigma_{1+\epsilon}(\zeta_r) d\rho(1-\epsilon) < \infty \text{ for all } f \in \mathfrak{B}^{\omega}_{2+\epsilon}(B_{1+\epsilon}),$$
$$\epsilon \geq 0; \tag{71}$$

$$\int_0^1 \int_{\partial B_{1+\epsilon}} \sum_r |f((1-\epsilon)\zeta_r)|^{1+\epsilon} d\sigma_{1+\epsilon}(\zeta_r) d\rho(1-\epsilon) < \infty \text{ for all } f \in \mathfrak{B}^{\omega}(B_{1+\epsilon}); \tag{72}$$

$$\int_0^1 \Phi^{\frac{1+\epsilon}{2}}(\epsilon) d\rho(1-\epsilon) < \infty. \tag{73}$$

Proof. The implication (71)⇒ (72) is trivial, because $B_{\omega}(B_{1+\epsilon}) \subset B^{\omega}_{2+\epsilon}(B_{1+\epsilon})$. Next, (72) implies (73) by Corollary (5.3.16). Finally, Corollary (5.3.15) guarantees that (73) implies

(72).

Corollary (5.3.17)[276]: Let $\varphi :\ B_1 \ \rightarrow\ B_1$ be a holomorphic mapping and let $0 \leq \epsilon < \infty$. Assume that $I_\omega\ (0+)\ <\ \infty, 1_\Omega\ =\ \infty$ and

$$\epsilon \sum_r \left(\int_{\partial B_1} \left(\frac{|\varphi'((1-\epsilon)\zeta_r)|}{1-|\varphi((1-\epsilon)\zeta_r)|} \Omega(1-|\varphi((1-\epsilon)\zeta_r)|) \right)^{2(1+\epsilon)} d\sigma_1(\zeta_r) \right)^{\frac{1}{2(1+\epsilon)}} \leq \omega(\epsilon) \ (74)$$

For $\epsilon \leq 1$. Then

$$\sup_{\epsilon \leq 1} \int_{\partial B_1} \sum_r \ \Phi_\Omega^{1+\epsilon} (1-|\varphi((1-\epsilon)\zeta_r)|) d\sigma_1(\zeta_r) < \infty.$$

in particular, $|\varphi^*| < 1\ \sigma_1 - a.e.$

Proof. Let the constant $1+\epsilon = (1+\epsilon)_{1,1+\epsilon,\Omega} > 0$ and the function $F_{y_r} \in \mathfrak{B}^\Omega(B_1),\ 0\ \leq\ y_r\ \leq\ 1$, by those provided by Corollary (5.3.13) for $\epsilon = 0$ and for Ω in place of ω. Since $\sum_r\ \|F_{y_r}\|_{\mathfrak{B}^\Omega(B_1)} \leq 1,$, we have

$$\sum_r \left|(F_{y_r} \circ \varphi)'(z_r)\right| \leq \sum_r \ |F_{y_r}{}'(\varphi(z_r))||\varphi'(z_r)| \leq \sum_r \ \frac{|\varphi'(z_r)|}{1-|\varphi(z_r)|} \Omega(1-|\varphi(z_r)|),$$

$$z_r \in B_1.$$

So, using (74) and the hypothesis $I_\omega(0+)\ <\ \infty$, we obtain

$$\int_0^1 \sum_r \ M_{2(1+\epsilon)}^2 \left((F_{y_r} \circ \varphi)', t \right) (1-t) dt \leq \int_0^1 \frac{\omega^2(1-t)}{1-t}\ dt < \infty.$$

We further observe that $\sum_r\ |F_{y_r} \circ \varphi(0)|\ \leq\ C_\varphi \sum_r\ \|F_{y_r}\|_{B^\Omega(BI)}\ \leq\ 1+\epsilon$, and so estimate (70) guarantees that

$$\int_{\partial B_1} \sum_r \ |F_{y_r} \circ \varphi((1-\epsilon)\zeta_r)|^{2(1+\epsilon)} d\sigma_1(\zeta_r) \leq 1+\epsilon, \quad 0\ \leq\ y_r\ \leq\ 1, \qquad \epsilon \leq 1$$

for a universal constant $\epsilon \geq 0$. Hence, applying Fubini's theorem and Corollary (5.3.13), we obtain

$$1+\epsilon \geq \int_{\partial B_1} \int_0^1 \sum_r \ \left|F_{y_r} \circ \varphi\big((1-\epsilon)\zeta_r\big)\right|^{2(1+\epsilon)} dy_r\, d\sigma_1(\zeta_r)$$

$$\geq \int_{\partial B_1} \sum_r \ \Phi_\Omega^{1+\epsilon} \left(1-\left|\varphi\big((1-\epsilon)\zeta_r\big)\right|\right) d\sigma_1(\zeta_r),$$

as required.

Chapter 6
Compact and Weakly Compact Composition Operators

We show that the conditions for the hyperbolic Besov classes are then interpreted geometrically when the symbols are univalent, and strict inclusion between different hyperbolic Besov classes is shown by an example. We characterize, in function-theoretic terms, when the composition operator $C_\phi f = f \circ \phi$ induced by an analytic self-map ϕ of the unit disk defines an operator $C_\phi: \mathcal{N}_\alpha \to \mathcal{B},\ \mathcal{B} \to Q_\beta, \mathcal{N}_\alpha \to Q_\beta$ which is bounded resp. compact. In particular, by combining techniques from both complex and functional analysis, we show that weak compactness is equivalent to compactness. For the operators into the corresponding "small" spaces we also characterize the boundedness and show that it is equivalent to compactness.

Section (6.1): Bloch Type Spaces to Hardy and Besov Spaces

For $H(D)$be the space of all analtytic function on the unit disk D. very analytic self-map $\varphi: D \to D$ of the unit disk induces through composition a linear composition operator C_φ from $H\ (D)$· to itself. Thus C_φ is defined by $C_\varphi\ (f\ ) = \ f \circ \varphi$ for $f\ \in\ H\ (D)$. The study of composition operators lies at the interface of analytic functions and operator theory. Many interesting results have been found for composition operators on Hardy and Bergman spaces see [12], [224], [180], [153], [144] for only a few examples, while the study of composition operators on other Banach spaces, such as the Bloch space and BMOA, is just in the beginning. Recently, K. Madigan and A. Matheson characterized the boundedness and the compactness of C_φ on the Bloch space and the Little Bloch space in [19]. After this work, boundedness and compactness of composition operators from the Bloch space to some other function spaces, such as Hardy spaces, BMOA, the spaces $\mathcal{Q}_p$which are introduced in [216], and analytic Besov spaces are studied almost simultaneously by [194], [222], [80], [225], [229], [230]·. A common feature is that these results involve the hyperbolic versions of the corresponding spaces. We consider composition operators from Bloch type spaces $\mathfrak{B}^\alpha$ to Hardy spaces $H^{\ p}$and analytic Besov type spaces B_p. We recall the definitions of these spaces here. Let $0 < \alpha < \infty$ An analytic function f on D is said to be in the a-Bloch space $\mathfrak{B}^\alpha$, if

$$\|f\|_{\mathfrak{B}^\alpha} = sup_{z\in D}|f'(z)|(1-|z|^2)^\alpha < \infty.$$

Correspondingly, f is in the Little $\boldsymbol{\alpha}$ -Bloch space $\mathfrak{B}_0^\alpha$, if

$$\lim_{|z|\to 1}|f'(z)|\,(1-|z|^2)^\alpha = 0.$$

Note that for the case $\alpha = 1$, we have $\mathfrak{B}^1 = \mathfrak{B}$, the Bloch space and $\mathfrak{B}_0^1 = \mathfrak{B}_0$, the Little Bloch space. When $0 < \alpha < 1$, the spaces $\mathfrak{B}^\alpha$ and $\mathfrak{B}_0^\alpha$. can be identified with the analytic Lipschitz space $lip_{1-\alpha}$ and the Little Lipschitz space $lip_{1-\alpha}$.

For $1 \le p < \infty$ we say that an analytic function f on D is in the Hardy space H^p if

$$\|f\|_{H^p} = sup_{1<r<1}\left(\frac{1}{2\pi}\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^p d\theta\right)^{1/p} < \infty.$$

Finally, let $0 < \mathrm{p} < \infty$ and $-1 < \mathrm{q} < \infty$. We say that f in the Besov type space $\mathrm{B}_{\mathrm{p,q}}$, if

$$\|f\|_{p,q} = \left(\int_D |f'(z)|^p(1-|z|^2)^q dm(z)\right)^{1/p} < \infty.$$

Where dm(z) denotes the Lebesque area measure on D. We note that $B_{p,p-2} = B_p$, the analytic Besov spaces (see [154] and [234])·, for $1 < p < \infty$; $B_{2,q} = D_q$. , the weighted Dirichlet spaces see, for example, [12]·; and $B_{p,p} = L_a^p$, the Bergman spaces, for $1 \le p < \infty$. (see [154]).

We have the following

When $1 < p < \infty$, we may define the hyperbolic Besov class B_p^h as the set of analytic self maps $\varphi: D \to D$ such that

$$\int_D \left(\varphi^h(z)\right)^p (1-|z|^2)^{p-2} dm(z) < \infty.$$

Where $\varphi^h(z) = \frac{|\varphi'(z)|}{(1-\varphi(z)^2)}$.

Using B_p^h we can state the special case of Theorem (6.1.4) for $\beta = 1$ and $q = p - 2$ in the following form

Corollary (6.1.1)[214]: $\varphi: D \to D$ be an analytic self map, and let $1 < p < \infty$, . Then the following statements are equivalent:

(i) $\mathfrak{B} \to B_p$ Is bounded

(ii) $C_\varphi: \mathfrak{B} \to B_p$ Is compact;

(iii) $C_\varphi: \boldsymbol{\mathfrak{B}_0} \to B_p$ Is bounded.

(iv) $C_\varphi: \boldsymbol{\mathfrak{B}_0} \to B_p$ Is compact

(v) $\varphi \in B_p^h$.

We note here that the equivalence of (i)·,(ii) ·, and $(v.)$ of Corollary (6.1.1) was independently proved by S. Makhmutov in [225] and M. Tjani in [230], with different methods.

Theorem (6.1.4). We consider the composition operators from the Bloch space and the Little Bloch space to the minimal Besov space B_1. The equivalence of boundedness and compactness for composition operators from Bloch type spaces to Besov type spaces which appears in Theorem (6.1.4) is not an accidental phenomenon. It can be derived from general Banach space theory. An explanation for the equivalence of (i), (iv) in Theorem (6.1.4) from the point of view of the general Banach space theory, as well as some more general results will be given.

We consider composition operators from Bloch type spaces to Hardy spaces. We assume that the symbol φ is univalent and give a geometric criterion for boundedness and compactness of C_φ from $\mathfrak{B}$ and $\mathfrak{B}_0$ to the Besov space B_p, for $1 < p < \infty$ We then construct an example for which $C_\varphi: \mathfrak{B} \to B_{p_2}$ is compact, but $C_\varphi: \mathfrak{B} \to B_{p_1}$ is not compact, provided $1 < p_1 < p_2 < \infty$.

In the following, ‘‘ $A \sim B$’’ means that there are two absolute positive constants C_1 and C_2 such that $C_1 A \le B \le C_2 B$.

First of all, let us generalize a result concerning Carleson type measure by J. Arazy, S. D. Fisher, and J. Peetre in [215]. Let μ be a positive Borel measure on the unit disk. For $0 < p < \infty$ we denote $D_p(\mu)$· as the space of analytic functions on D satisfying

$$\|f\|_{D_p(\mu)} = \left(\int_D |f'(z)|^p d\mu(z)\right)^{1/p} < \infty.$$

The following is the result of Theorem (6.1.4) of [215]:

Theorem (6.1.2)[214]: Let μ be a positive Borel measure on D. and let $0 < p < \infty$. Then the inclusion map $i: \mathfrak{B} \to D_p(\mu)$ is bounded if and only if

$$\int_D \frac{d\mu(z)}{(1-|z|^2)^p} < \infty .$$

We generalize this result to the $\alpha - Bloch\ spaces\ for\ 0 < \alpha \le 1$

For a proof, we require the following result on gap series, see [91].

Lemma (6.1.5)[214]: Suppose that (n_k) is an increasing sequience of positive integers with Hadamard gaps. That is, $\frac{n_{k+1}}{n_k} \ge \lambda > 1$ for all k.

Let $0 < p < \infty$.

Then there is a constant $M > 0$ depending on p and λ such that

$$M^{-1}\left(\sum_{k=1}^{N}|a_k|^2\right)^{1/2} \le \left(\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{k=1}^{N}a_k e^{in_k\theta}\right|^p d\theta\right)^{1/p} \le M\left(\sum_{k=1}^{N}|a_k|^2\right)^{1/}$$

For any scalars $a_1 \dots . a_N$ and $N = 1,2, \dots$

Theorem (6.1.3)[214]: Let μ be a positive Borel measure on D. Let $0 < p < \infty$ and $1 \le \beta < \infty$. Then the following statements are equivalent:

(i) $\mathfrak{B}^{1/\beta} \to D_{p\beta}(\mu)$ is bounded.

(ii) $\mathfrak{B}^{1/\beta} \to D_{p\beta}(\mu)$ is compact.

(iii) $\mathfrak{B}_0{}^{1/\beta} \to D_{p\beta}(\mu)$ is bounded.

(iv) $\mathfrak{B}_0{}^{1/\beta} \to D_{p\beta}(\mu)$ is bounded.

(v) $\int_D \frac{d\mu(z)}{(1-|z|^2)^p} < \infty$.

Note that the absence of β in (v) means that if μ is independent of β and one of (i)-(iv)holds for some $\beta, 1 \le \beta < \infty$, then it will hold for all $\beta, 1 \le \beta < \infty$.

Proof. Since it is obvious that $(ii) \Longrightarrow (i) \Rightarrow (iii)$ and $(ii) \Longrightarrow (iv) \Rightarrow (iii)$ we need only to prove that $(iii) \Longrightarrow (v) \Rightarrow (ii)$ suppose (iii) is true, Let $r_n \in (0,1)$ satisfy $r_n \to 1$, and let

$$f_{n,l}(z) = \sum_{k=1}^{\infty} a_k z^{2^k} = \frac{1}{r_n}\sum_{k=1}^{\infty} 2^{k(1/\beta-1)}\left(r_n e^{il} z\right)^{2^k}.$$

Since $2^{(k(1-1)/\beta)}|a_k| \to 0$ as $k \to \infty$ [8] (see also [235]) we see that $f_{n.l} \in \mathfrak{B}_0^{1/\beta}$ and $\|f_{n.l}\|_{\mathfrak{B}^{1/\beta}} \le K < \infty$, , where $K > 0$ is a constant independent of n and . Since $i: \mathfrak{B}_0^{1/\beta} \to D_{P\beta}(\mu)$ is bounded, we know that

$$\int |f'_{n.l}(z)|^{P\beta}\, d\mu(z) = \|f_{n.l}\|^{P\beta}_{D_{p\beta}(\mu)} \le \|f_{n.l}\|^{P\beta}_{\mathfrak{B}^{\frac{1}{\beta}}}\|i\|^{P\beta} \le K^{P\beta}\|i\|^{P\beta}. \quad (1)$$

Integrating this inequality with respect to t, applying Fubini’s theorem. Lemma (6.1.5) and Hӧlder’s inequality we get that

$$K^{P\beta}\|i\|^{P\beta} \ge \int_D \left(\sum_{K=1}^{\infty} 2^{2K}(r_n|z|)^{2(2^k-1)\beta}\right)^{p/2} d\mu(z).$$

Since

$$\sum_{k=1}^{\infty} 2^{2k}\left((r_n|z|)^{2(2^k-1)\beta}\right) \geq \frac{1}{2}\frac{1}{(1-(r_n|z|)^{2\beta})^2}$$

(see, [222]) and $1-(r_n|z|)^{2\beta}\sim 1-(r_n|z|)^2$ we get that

$$\int_D \frac{d\mu(z)}{(1-(r_n|z|)^2)^p} \leq 2^{P/2}\, K^{P\beta}\|i\|^{P\beta}.$$

Thus (v) is obtained from thae above inequality and Fatou's Lemma.
To prove (v) ⟹ (ii), suppose (v) is true. Then

$$\|f_{n.l}\|_{D_{p\beta}(\mu)}^{P\beta} = \int |f'(z)|^{P\beta}\, d\mu(z) \leq \|f_{n.l}\|_{\mathfrak{B}^{1/\beta}}^{P\beta} \int_D \frac{d\mu(z)}{(1-|z|^2)^p}.$$

Thus: $\mathfrak{B}^{1/\beta} \to D_{p\beta}(\mu)$ is bounded. To see that y_j is operator is moreover compact, let $\{f_n\} \subset \mathfrak{B}^{1/\beta}$ be such that $\|f_n\|_{\mathfrak{B}^{1/\beta}} \leq 1$. We must show that $\{C_\varphi f_n\}$ has a subsequence that converges in $D_{p\beta}(\mu)$. It is easy to show that for every $f \in \mathfrak{B}^{1/\beta}$.

$$|f(z)| \leq |f(0)| + \|f\|_{\mathfrak{B}^{1/\beta}}(1-|z|)^{-1/\beta}$$

Thus there is a subsequence of $\{f_n\}$ that converges uniformly on compact subsets of D to an analytic function f. By passing to this subsequence, we may assume that the sequence $\{f_n\}$ itself converges to f. We get also that $f \in \mathfrak{B}^{1/\beta}$ and $\|f\|_{\mathfrak{B}^{1/\beta}} \leq 1$. Thus $f = if \in D_{p\beta}(\mu)$ and it suffices to show that

$$\lim_{n\to\infty} \|f_n - f\|_{D_{p\beta}(\mu)} = 0.$$

This is consequence of the Lebesgue Dominated convergence theorem, since
$(f_n - f\ \ )'(z) \to 0$ pointwise in D and

$$|(f_n - f\ \ )'(z)|^{p\beta} \leq 2^{p\beta}\left(\|f_n\|_{\mathfrak{B}^{1/\beta}}^{P\beta} + \|f\|_{\mathfrak{B}^{1/\beta}}^{P\beta}\right)(1-|z|^2)^{-p} \leq 2^{p\beta+1}(1-|z|^2)^{-p}.$$

Thus we have shown that (v) implies (ii) and the proof is complete.

Now we can derive our main theorem from Theorem (6.1.3).

Theorem (6.1.4)[214]: Let $\varphi: D \to D$ be an analytic self map, let $0 < p < \infty, -1 < q < \infty$ and $1 \leq \beta < \infty$. Then the following statements are equivalent:

(i) $C_\varphi: \mathfrak{B}^{1/\beta} \to B_{p\beta,q}$ Is bounded

(ii) $C_\varphi: \mathfrak{B}^{1/\beta} \to B_{p\beta,q}$ Is compact;

(iii) $C_\varphi: \mathfrak{B}_0^{1/\beta} \to B_{p\beta,q}$ Is bounded.

(iv) $C_\varphi: \mathfrak{B}_0^{1/\beta} \to B_{p\beta,q}$ Is compact

(v) $\int_D \frac{|\varphi'(z)|^{p\beta}(1-|z|^2)^q}{(1-\varphi(z)^2)^p dm(z) < \infty}$.

Proof. We make the following change of variables

$$\int_D |(f\circ\varphi)'(z)|^{p\beta}\,(1-|z|^2)^q dm(z) = \int_D |f'(w)|^{p\beta}\, G_\varphi(w) dm(z). \qquad (2)$$

where

$$G_\varphi(w) = \sum_{\varphi(z)=w} |\varphi'(z)|^{p\beta-2}\,(1-|z|^2)^q.$$

Let μ_φ be the measure on $I)$ defined by $\mu_\varphi(E) = \int_E G_\varphi(w)dm(w)$. Then, by (2),$C_\varphi$ is bounded from $\mathfrak{B}^{1/\beta}$ to $B_{p\beta,q}$ if and only if the inclusion map $:\mathfrak{B}^{1/\beta} \to D_{p\beta}(\mu_\varphi)$ is a bounded operator. By Theorem (6.1.3), this is equivalent to

$$\int_D \frac{d\mu_\varphi(w)}{(1-|w|^2)^p} < \infty.$$

If we change variables back, we get

$$\int_D \frac{d\mu_\varphi(w)}{(1-|w|^2)^p} = \int_D \frac{|\varphi'(z)|^{p\beta}}{(1-|\varphi(z)|^2)^p} (1-|z|^2)^q \, dm(z) < \infty.$$

Thus Theorem (6.1.4) is a direct consequence of Theorem (6.1.3).

We extend the result of Corollary (6.1.1) to the case $p = 1$. Let H^∞ be the space of all bounded analytic functions f on the unit disk with norm $\| f\|_{H^\infty} = \sup_{z\in D} |f(z)|$. We may define the hyperbolic Hardy class H^∞ as the set of all analytic self-maps $\varphi: D \to D$ satisfying $\|\varphi\|_{H^\infty} < 1,$ which means that the hyperbolic distance from $\varphi(z)$. to 0 is uniformly bounded.

Recall that the minimal Besov space B_1 is defined as the set of analytic functions f on D which are of the forms

$$f(z) = \sum_{k=1}^{\infty} \lambda_k \, \sigma_{a_k}(z), \tag{3}$$

where $|a_k| \le 1, \sigma_{a_k}(z) = (a_k - z)/\ (1-\bar{a}_k z)$. and $\sum_{k=1}^{\infty} |\lambda_k| < \infty$.The norm of f is given by

$$\|f\|_{B_1} = \inf \left\{\sum |\lambda_k|: (3) \text{ holds}\right\}$$

(see [215]). It is known that, if $f(0) = f'(0) = 0$, then

$$\|f\|_{B_1} \sim \int_D |f''(z)|dm(z)$$

(see [215]. It is easy to see that $B_1 \subset B_p \subset \mathfrak{B}$ for $1 < 0p < \infty$. In fact, B_1 is the minimal Möbius invariant Banach space and the Bloch space $\mathfrak{B}$ is the largest Möbius invariant Banach space under some reasonable assumptions (see [215] and [227]). We give the following result:

Theorem (6.1.5). Let $\varphi: D \to D$ be an analytic self map. Then the following statements are equivalent:

(i) $C_\varphi : \mathfrak{B} \to B_1$ is bounded;

(ii) $C_\varphi : \mathfrak{B} \to B_1$ is compact;

(iii) $C_\varphi : \mathfrak{B} \to B_1$ is bounded;

(iv) $C_\varphi :\ B$ ªB is compact;

(v) $\varphi \in b_2^h and \int_D |\varphi''(z)|/\ (1-|\varphi(z)|)^2 \, dm\,(z) < \infty$;

(vi) $\varphi \in B_1 \cap H_h^\infty$.

Proof. As before, it is obvious that (ii)⟹(i) ⟹(iii) and (ii) ⟹(iv) ⟹ (iii). Thus the proof will be complete if we prove (iii) ⟹(vi) ⟹ (vi) ⟹ (ii).

Suppose (iii) is true. Since $f_0\,(z) = z \in \mathfrak{B}_0$, we get $\varphi = f_0 \circ \varphi \in B_1$. It is obvious that B_1 ;H^∞. Thus Cw : $B_0 \cap H_h^\infty$ is bounded. Thus, from the Closed Graph Theorem (6.1.2) and the fact that B_0 contains unbounded functions it is easy to see that $\varphi \in H_h^\infty$. Thus we have got (iii) implies (vi).

To prove (vi) ⟹(v), let $\varphi \in B_1 \cap H_h^\infty$. Thus $\|\varphi\|_{H^\infty} = \delta < 1$. Since $\varphi = B_1 \subset B_2$, we have

$$\int_D \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}\, dm(z) < \frac{1}{(1-\delta^2)^2} \int_D |\varphi'(z)|^2\, dm\,(z) \le \frac{M_1}{(1-\delta^2)^2} \le \infty,$$

and

$$\int_D \frac{|\varphi''(z)|}{1-|\varphi(z)|^2} dm(z) < \frac{1}{1-\delta^2} \int_D |\varphi''(z)|\ dm\,(z) \le \frac{M_2}{1-\delta^2} \le \infty.$$

So we get that (vi) implies (v).

Finally, suppose that (v) is true. Let $f \in \mathfrak{B}_0$. Without loss of generality, we may suppose that $f(0) = f'(0) = 0$. Since $\| f\|_{\mathfrak{B}} \backsim \sup_{z\in D} |f''(z)|(1-|z|^2)^2$ (see for example, [154]), by (v) we get

$$\begin{aligned}\|C_\varphi f\|_{B_1} &\le C \int_D |(f\circ\varphi)''(z)|\, dm\,(z)\\ &\le C\left(\int_D |f''(\varphi(z))||\varphi'(z)|^2 dm(z) + \int_D |f'(\varphi(z))||\varphi''(z)|dm(z)\right)\\ &\le C\|f\|_{\mathfrak{B}}\left(\int_D \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}\right) dm(z) + \int_D \left|\frac{\varphi''(z)}{1-|\varphi(z)|^2} dm(z)\right)\\ &\le CM\|f\|_{\mathfrak{B}}.\end{aligned}$$

Thus $C_\varphi : \mathfrak{B}_0 \to B_1$ is bounded. Now by a similar argument as in the proof of Theorem (6.1.3) we can prove that C_φ is moreover compact from $\mathfrak{B}_0$ to B_1 .We left the details. Thus (v) implies (ii) and the proof is complete.

The equivalence of boundedness and compactness for composition operators from Bloch type spaces to Besov type spaces which appears in Theorem (6.1.4) and Theorem (6.1.5) is not an accidental phenomenon. It can be explained from general Banach space theory. To see this, let us first look at some basic facts on the a-Bloch spaces.

Lemma (6.1.6). For $0 < \alpha < \infty$, the dual space of the Little a-Bloch space $\mathfrak{B}_0^\alpha$ is isomorphic with the Bergman space L_a^1, and the dual space of L_a^1 is isomorphic with the α-Bloch space $\mathfrak{B}^a$.

For a proof, see [235].

Lemma (6.1.7). The Bergman space L_a^1 has the Schur Property, that is, every weakly con¨ergent sequence is norm-convergent.

This is because that the Bergman space L_a^1 is isomorphic to the sequence space l^1 (see [231] and the later one has the Schur Property (see, for example, [231] or [233]).

Since $\mathfrak{B}^a$ is the dual of L_a^1, and the point evaluation are continuous in $\mathfrak{B}^a$, obviously we have the following result.

Lemma (6.1.8). In the space $\mathfrak{B}^a$, weak-star convergence implies pointwise convergence.

Since $\mathfrak{B}_0^\alpha$ is weak-star dense in $\mathfrak{B}^\alpha$ (see, for example, [232]), we can easily prove the following result using Lemma (6.1.8).

Theorem (6.1.9). (‘‘Big-oh’’ vs ‘‘Little-oh’’). If Y is a Banach space of analytic functions and if $C_\varphi : \mathfrak{B}_0^\alpha \to Y$ is bounded, then $C_\varphi^{**} = C_\varphi$ on $\mathfrak{B}^\alpha$.

The details of the proof are left. As a direct consequence of Theorem (6.1.9) we easily get

Corollary (6.1.10). Let Y be a Banach space of analytic functions on the unit disk D.

(i) If $C_\varphi : \mathfrak{B}_0^\alpha \to Y$ is bounded, then $C_\varphi: \mathfrak{B}^\alpha \to Y^{**}$ is bounded.

(ii) If $C_\varphi : \mathfrak{B}_0^\alpha \to Y$ is compact, then $C_\varphi : \mathfrak{B}^\alpha \to Y^{**}$ is compact.

If in the above corollary, Y is a reflexive space, then $Y^{**} = Y$ and we get that $C_\varphi : \mathfrak{B}_0^\alpha \to Y$ is bounded (compact) if and only if $C_\varphi : \mathfrak{B}^\alpha \to Y$ is bounded (compact). Since for $1 < p\beta < \infty$, the Besov space $B_{p\beta,q}$ is reflexive, being a closed subspace of a reflexive space $L^{p\beta}(\mu)$, the equivalence of (i) and (iii) and the equivalence of (ii) and (iv) in Theorem (6.1.4) are direct consequences of this corollary.

To see how to get the equivalence between boundedness and compactness, we give the following

Theorem (6.1.11). General compactness theorem. Suppose X or Y is a reflexive space and Y is a space with the Schur Property. Then every bounded operator $T: X \to Y$ is compact.

Proof. If $T: X \to Y$ is bounded and X or Y is a reflexive Banach space then T is weakly compact see, for example, [218].

Since Y has the Schur property, it is clear that T is compact.

As an application we give the following:

Corollary (6.1.12). Let $0 < a < \infty$ and let Y be a reflexive Banach space of analytic functions.

(i) If $T: \mathcal{B}_0^a \to Y$ is a bounded operator, then T is also compact.

(ii) If $T: \mathcal{B}^a \to Y$ is a weak-star continuous operator, then T is also compact.

Proof. (i) Suppose $T: \mathcal{B}_0^a \to Y$ is bounded. By Lemma (6.1.6), $T^*: Y^* \to (\mathcal{B}_0^a)^* = L_a^1$ is bounded. Lemma (6.1.7) and Theorem (6.1.11) implies that T^* is 0 a compact, and so is T. (ii) Suppose $T: \mathcal{B}^a \to Y$ is a weak-star continuous operator. Then it is the adjoint of some bounded operator $S: Y^* \to (\mathcal{B}_0^a)^* = L_a^1$ (see [218].) Note that since. Y is reflexive, the predual space of Y is same as the dual space Y^*. Again, by Theorem (6.1.11), S is compact, and so is T. Let $1 < p < \infty$ and $-1 < q < \infty$. As a closed subspace of a reflexive space $L_p(\mu), B_{p,q}$ is reflexive. Thus, for the case $1 < p\beta < \infty$, the equivalence of (iii) and (iv) in Theorem (6.1.4) is a direct consequence of Corollary (6.1.12)(i). The equivalence of (i) and (ii) in Theorem (6.1.4) for $1 < p\beta < \infty$, follows from Corollary (6.1.12) ii and the following lemma.

Lemma (6.1.13). Let $0 < a < \infty$ and let Y be a reflexive Banach space of analytic functions. If $C_\varphi : \mathcal{B}^a \to Y$ is bounded, then it is weak-star continuous.

Proof. Let $C_\varphi : \mathcal{B}^a \to Y$ be bounded. Since $\mathcal{B}_0^a \subset \mathcal{B}^a$, we see that $C_\varphi : \mathcal{B}^a \to Y$ is bounded. Theorem (6.1.9) implies that $C_\varphi^{**} = C_\varphi$ on $\mathcal{B}^a$. Let $f_n \to f$ in the weak-star topology of $\mathcal{B}^a$ and let $h \in Y^*$. Because $C_\varphi : \mathcal{B}_0^a \to Y$ is bounded, we get that $C_\varphi^* : Y^* \to (\mathcal{B}_0^a)^*$ is bounded and so $C_\varphi^* h \in (\mathcal{B}_0^a)^* = L_a^1$. Thus

$$\lim_{n\to\infty} \left|\langle h, C_\varphi (f - f)\rangle\right| = \lim_{n\to\infty} \left|\langle h, C^{**}{}_\varphi (f_n - f)\rangle\right| = \lim_{n\to\infty} \left|\langle h, C_\varphi^* f_n - f\rangle\right| = 0$$

Thus C_φ is weak-star continuous.

Similar reasoning also applies to Theorem (6.1.5), and to Theorem (6.1.4) for the case $p\beta = 1$ though the range spaces there are not reflexive. To see this, let c_0 denote the space of sequences $\{a_n\}$ for which $a_n \to 0$, and, for $0 \le p < \infty$,let l denote the space of sequence an n such that $\left||\{a_n\}|\right|_p^p = \sum_{n=1}^\infty |a_n|^p < \infty$. The norm of a sequence $\{a_n\}$ is given by $\left||\{a_n\}|\right|_\infty = \sup_n |a_n|$in c_0 and $\left||\{a_n\}|\right|_p$in l^p.

Theorem (6.1.14). Let $1 \le p < q < \infty$. Then every bounded operator from l_q to l_pp is compact. The same is true for bounded operators from c_0 to l_p .

For a proof, see [223].

Since the Besov spaces B_1 and $B_{1,q}$ are isomorphic to the Bergman space L_a^1, which is isomorphic to l^1, and, as the pre-dual space of L_a^1, the a a Little a −Bloch space $\mathcal{B}_0^a$ is isomorphic to c_0, we see that the equivalences of (iii) and (iv) in Theorem (6.1.5) and Theorem (6.1.4) in the case $p\beta = 1$ are direct consequences of Theorem (6.1.14).

The other equivalences of Theorem (6.1.5) and Theorem (6.1.4) in the case $p\beta = 1$ can be also derived from Theorem (6.1.14), although more considerations are needed here.

We discuss results concerning Hardy spaces. Recently, E. G. Kwon characterized in [80] the boundedness of the composition operators from the Bloch space B to Hardy spaces $H^{2p}, 1 \le p < \infty$. His result involved the hyperbolic Hardy classes H_h^p. Following S. Yamashita [204] and [89], an analytic self-map φ on the unit disk is said to be in the hyperbolic Hardy class H_h^p, if

$$\sup_{0<r<1} \int_0^{2\pi} \left(\log \frac{1}{1-|\varphi(re^{i\theta})|^2}\right)^{p} d\theta < \infty.$$

Let $\varphi^h = |\varphi'|/(1-|\varphi|^2)$ be the hyperbolic derivative of the analytic self-map $\varphi: D \to D. E. G.$ Kwon proved the following result in [80].

Theorem (6.1.15). If $\varphi: D \to D$ is analytic and $1 \le p < \infty$ then the following statements are equivalent:

(i)C_φ is bounded from the Bloch space $\mathcal{B}$ to H^{2p};

(ii) $\varphi \in H_h^p$

(iii) $\int_0^{2\pi} \left(\int_0^1 \left(\varphi^h(re^{i\theta})\right)^2 (1-r)dr\right)^p d\theta < \infty.$

By Theorem (6.1.9) and Corollary (6.1.12), we immediately get the following result.

Theorem (6.1.16). Let $\varphi: D \to D$ be an analytic self map and let $1 \le p < \infty$. Then the following statements are equivalent:

(i) $C_\varphi : \mathcal{B} \to H^{2p}$ *is bounded*;

(ii) $C_\varphi : \mathcal{B} \to H^{2p}$ *is compact*;

(iii) $C_\varphi : \mathcal{B}_0 \to H^{2p}$ *is bounded*;

(iv) $C_\varphi : \mathcal{B} \to H^{2p}$ *is compact*;

(v) $\varphi \in H_h^p$;

(vi) $\int_0^{2\pi} \left(\int_0^1 \varphi^h(re^{i\theta}))^2(1-r)\, dr\right)^p d\theta < \infty.$

We note that the case $p = 1$ has been proved by Wayne Smith and [229].

In [222], H. Jarchow and R. Riedl got a criterion for the composition operators bounded from an a-Bloch space to a Hardy space. We restate Corollary 4 of [222] as follows:

Theorem (6.1.17). Let $\varphi: D \to D$ be an analytic self map, let $1 \le \beta < \infty$ and $0 < p < \infty$. Then $C_\varphi : \mathcal{B}^{1+1/\beta} \to H^{p\beta}$ is bounded if and only if

$$\sup_{0<r<1} \int_0^{2\pi} \left(\frac{1}{1-|\varphi(re^{i\theta})|^2}\right)^p d\theta < \infty$$

This result can be improved as follows:

Theorem (6.1.18). Let $\varphi: D \to D$ be an analytic self-map, let $1 \le \beta < \infty$ and $0 < p < \infty$. Then the following statements are equivalent:

(i) $C_\varphi: \mathcal{B}^{1+1/\beta} \to H^{p\beta}$ is bounded;

(ii) $C_\varphi: \mathcal{B}^{1+1/\beta} \rightarrow H^{p\beta}$is compact;

(iii) $C_\varphi: \mathcal{B}_0^{1+1/\beta} \rightarrow H^{p\beta}$ s bounded;

(iv) $C_\varphi: \mathcal{B}_0^{1+1/\beta} \rightarrow H^{p\beta}$ is compact;

(v) $\sup\limits_{0<r<1} \int_0^{2\pi} \left(1/1-\left|\varphi\left(re^{i\theta}\right)\right|^2\right)^p d\theta < \infty.$

This can be proved either by a similar method as in the proof of Theorem (6.1.4), or, in the case $1 < p\beta < \infty$, directly from Theorem (6.1.4), Theorem (6.1.9) and Corollary (6.1.12). Note that, however, unlike the Bergman space L_a^1 , the Hardy space H^1 is not isomorphic to the sequence space l^1 Žsee [231]. Thus Theorem (6.1.14) cannot be used in the case H^1 .

The results of Theorem (6.1.16) can be viewed as the limiting case $\beta \rightarrow \infty$ of Theorem (6.1.18). Note that the missing of β in (v) means that if one of $(i)-(iv)$ holds for some β ,$1 < \beta < \infty$then it will hold for all $\beta, 1 \leq \beta < \infty$.

By Theorem (6.1.5), we see that, if a composition operator $C_\varphi : \mathcal{B}_0 \rightarrow \mathcal{B}_1$is compact, then $\left||\varphi|\right|_{H^\infty} < 1.$ Since for $1 < p < \infty, B_1 \subset B_p \subset \mathcal{B}_0$, we may ask a natural question: for $1 < p < \infty$, are there some analytic self-maps w: $D \rightarrow D$ such that $\|\varphi\|_{H^\infty} = 1$ and $C_\varphi : \mathcal{B} \rightarrow B_p$ are compact? The answer is positive. In fact, for $1 < p_1 < p_2 < \infty$, we will construct an analytic and univalent self map φ of D such that $\|\varphi\|_{H^\infty} = 1, C_\varphi : \mathcal{B} \rightarrow B_{p_2}$ is compact, while $C_\varphi : \mathcal{B} \rightarrow B_{p_1}$ is not compact.

We first give a geometric criterion for a univalent function $\varphi \in B_p^h$

Theorem (6.1.19). Let φ: $D \rightarrow D$ be an analytic and univalent self map, let $G = \varphi(D)$ and let $1 < p < \infty$. Then $\varphi \in B_p^h$ if and only if.

$$\int_G \frac{\delta_G(w)^{p-2}}{(1-|w|^2)} dm(w) < \infty, \tag{4}$$

where $\delta_G(w)$ is the Euclidean distance between w and the boundary of G.

Proof. The result is easily obtained from Corollary (6.1.1) by using the Koebe distortion theorem which says $\delta_G\left(\varphi(z)\right) \sim (1-|z|^2)|\varphi'(z)|$ see, for exmaple, [144] and changing of variables $\varphi(z) = w$.

Example (6.1.20). Let $1 < p_1 < p_2 < \infty$. Then there exists an analytic and univalent self-map φ of D such that

(i) $\|\varphi\|_{H^\infty} = 1$;

(ii) $C_\varphi: \mathcal{B} \rightarrow B_{p_2}$ (or $\mathcal{B} \rightarrow B_{p_2}$) is compact;

(iii) $C_\varphi: \mathcal{B} \rightarrow B_{p_1}$(or $\mathcal{B} \rightarrow B_{p_2}$) is not compact.

Proof. Suppose $1 < p_1 < p_2 < \infty$. For any integers $k \geq 1$, let $r_k = 1 - 2^{-k}, \theta_k = 2^{-k} k^{-1/(p_1-1)}$, and let $E\ (k)$ be the following polar rectangulars in D:

$$E(k) = \{w = re^{i\theta} \in D: r_k \leq r \leq r_{k+1}, -\theta_k \leq \theta \leq \theta_k$$

Let E be the interior of $\bigcup_{k=1}^\infty E\ (k)$ Then E is a simple connected region of zoom lens shape along the real axis. Let φ be a Riemann map from D onto E. We claim that φ is the required map. Obviously, we have $\|\varphi\|_{H^\infty} = 1.$ Thus, by Corollary (6.1.1), we need only check the condition (4) in Theorem (6.1.19) for p_1 and p_2 we first note that, if $w = re^{i\theta} \in E \cap E\ (k)$. and $\theta \geq 0,$ then clearly we have

$$\delta_E(w) \leq r(\theta_k - \theta).$$

Since for $re^{i\theta} \in E(k)$ we have $1/2 < r < 1$ and $1 - r^2 \sim 2^{-k}$, we get that for every $p, 1 < p < \infty$,

$$\int_{r_k}^{r_{k+1}} \frac{r^{p-1}}{(1-r^2)^p} dr \sim 2^{kp} \int_{r_k}^{r_{k+1}} dr = 2^{kp}(2^{-k} - 2^{-k-1}) = 2^{kp-k-1} \quad . \tag{5}$$

Thus, by symmetricity and (5),

$$I_E := \int_E \frac{\delta_E^{p-2}(w)}{(1-|w|^2)^p} dm(w) = \sum_{k=0}^{\infty} \int_{E(k)} \frac{\delta_E^{p-2}(w)}{(1-|w|^2)^p} dm(w)$$
$$\leq 2 \sum_{k=0}^{\infty} \int_{r_k}^{r_{k+1}} \int_0^{\theta_k} \frac{\left(r(\theta_k - \theta)\right)^{p-2}}{(1-r^2)^p} r d\theta \sim \sum_{k=1}^{\infty} 2^{kp-k} \int_0^{\theta_k} (\theta_k - \theta)^{p-2} d\theta$$
$$= \frac{1}{p-1} \sum_{k=1}^{\infty} 2^{k(p-1)} \theta_k^{p-1} = \frac{1}{p-1} \sum_{k=1}^{\infty} k^{-(p-1)/(p_1-1)}. \tag{6}$$

For completing the proof, we need a lower estimate of $I_E(p)$. For this E purpose, let $E'(k)$ be the subset of $E\ (k)$ defined by

$$E'\ (k) = \left\{ re^{i\theta} \in D : r_k \leq r \leq r_{k+1}, -\theta_{k+2} \leq \theta \leq \theta_{k+2} \right\}$$

and let E' be the interior of $\bigcup_{k=1}^{\infty} E'(k)$ It is obvious that, for any $w = reire^{i\theta} \in E'(k)$

$$\delta_E(w) \geq rsin(\theta_{k+2} - \theta) \sim r(\theta_{k+2} - \theta).$$

Using this estimate and (5), by a similar calculation as in (6), we have, for any $p, 0 < p < \infty$,

$$I_E(p) \geq \int_{E'} \frac{\delta_E^{p-2}(w)}{(1-|w|^2)} dm(w)$$
$$\geq \frac{C}{p-1} \sum_{k=1}^{\infty} (k+2)^{-(p-1)/(p_1-1)} \sim \sum_{k=1}^{\infty} k^{-(p-1)/(p_1-1)}. \tag{7}$$

Now (6) and (7) mean that for any $p, 0 < p < \infty$,

$$I_E(p) \sim \sum_{k=1}^{\infty} k^{-(p-1)/(p_1-1)}.$$

It follows that $I_E\ (p)$ is finite if $p = p_2 > p_1$ and infinite if $p = p_1$. Thus Corollary (6.1.1) and Theorem (6.1.19) implies that $C_\varphi : \mathcal{B} \to \mathcal{B}_{p_2}$ is compact but $C_\varphi : \mathcal{B} \to \mathcal{B}_{p_1}$ is not compact.

Let $E\ (a_n) = \bigcup_{k=1}^{\infty} E(n,k)$. Then each $E\ (a_n)$ is a zoom lens shape region in D along the radial direction t_n . It is easy to see that, for $n_1 \neq n_2$, $E\ (a_n) \cap E\ (a_n) = \emptyset$ .Set

$$G^* = \overline{D}_{1/2} \cup \left(\bigcup_{n=1}^{\infty} E(a_n) \right)$$

and let G be the interior of G^*. Then G is a simple connected domain in D and $\eth G \cap \eth D = \{a_n\}$, which is infinite. Let φ be a Riemann map from D onto G, then a similar calculation as in the Example shows that

$$\int_{E(a_n)} \frac{\delta_G^{p_2-2}(w)}{(1-|w|^2)^{p_2}} dm(w) \sim n^{-\beta(p_2-1)}$$

Since $\beta = max\ (2, 2/(p_2 - 1))$, we have $\beta\ (p_2 - 1) \geq 2.$ Note that the integral over $\overline{D_{\frac{1}{2}}}$ is finite. Thus taking the sum of above integrals over n we easily see that

$$\int_{G} \frac{\delta_G^{p_2-2}(w)}{(1-|w|^2)^{p_2}} dm(w) < \infty,$$

and so $C_\varphi : \mathcal{B} \to \mathcal{B}_{p_2}$ is compact. But as in the Example we see that for any n,

$$\int_{E(a_n)} \frac{\delta_G^{p_2-2}(w)}{(1-|w|^2)^{p_1}} dm(w) = \infty$$

Thus $C_\varphi : \mathcal{B} \to \mathcal{B}_{p_1}$ is not compact.

Wayne Smith pointed out to the author that it is very easy to construct examples for answering the above question by using [19]. All that is required is to construct a simply connected subset G of D such that

$$\frac{\delta_G(w)}{(1-|w|)} \to 0 \qquad (8)$$

and take a Riemann map φ onto G. Theorem (6.1.4) of [19] shows that C_φ is w compact on $\mathcal{B}_0$, and clearly G can intersect ∂D in infinitely many points.

In fact, a much stronger example is given by Wayne Smith in [228]. It is proved that there is an analytic and univalent self-map φ of D such that C_φ is compact on $\mathcal{B}_0$, while $\overline{\varphi(D)} \cap \eth D$ is the whole unit circle $\eth D$.

Section (6.2): $\mathcal{N}_\alpha$ to the Bloch Space to $\mathcal{Q}_\beta$

For Δ be the unit disk $\{z \in \mathbb{C}: |z| < 1\}$ in the complex plane, and let $\mathcal{H}\ (\Delta)$ be the space of all analytic functions on Δ.

Any analytic map $\phi: \Delta \to \Delta$ gives rise to an operator $C_\phi: \mathcal{H}\ (\Delta) \to \mathcal{H}\ (\Delta)$ defined $C_\phi f = f \circ \phi$, the composition operator induced by ϕ .

One of the central problems on composition operators is to know when C_ϕ maps between two subclasses of $\mathcal{H}\ (\Delta)$ and in fact to relate function theoretic properties of ϕ to operator- theoretic properties of C_ϕ . This problem is addressed here for the weighted Nevanlinna, the Bloch and the $\mathcal{Q}$ spaces with respect to boundedness and compactness of the operator. (See for example [245], [247], [121], [198], [232], and [253]).

For each $\alpha \in (-1, \infty)$, let $\mathcal{N}_\alpha$ be the space of all functions $f \in \mathcal{H}\ (\Delta)$ satisfying

$$T_\alpha(f) = \frac{1+\alpha}{\pi} \int_\Delta \left[log^+[f(z)]\right] (1+|z|^2)^\alpha\, dm(z) < \infty.$$

Here and after wards, dm means the usual element of the area measure on Δ, and $log^+ x\ is \log x\ if\ x > 1\ and\ 0\ if\ 0 \leq x \leq 1.$

From $log^+ x \leq$ log(1+x) $\leq 1 + log^+ x\ for\ x \geq 0$ we see that a function $f \in \mathcal{H}\ (\Delta)$ belongs to $\mathcal{N}_\alpha$ if and only if

$$\|f\|_{\mathcal{N}_\alpha} = \int_\Delta [log\big(1+f(z)\big)\ ](1-|z|^2)^\alpha\, dm(z) < \infty.$$

Obviously,max $\{\|f+g\|_{\mathcal{N}_\alpha}, \|fg\|_{\mathcal{N}_\alpha}\} \leq \|f\|_{\mathcal{N}_\alpha}+\|g\|_{\mathcal{N}_\alpha}$.

For all $f, g \in \mathcal{N}_\alpha$. Consequently, $\mathcal{N}_\alpha$ is not only a vector space but even an algebra. Further, by setting

$$d_\alpha(f,g) = \|f+g\|_{\mathcal{N}_\alpha}.$$

For $f, g \in \mathcal{N}_\alpha$, we obtain a translation invariant metric on $\mathcal{N}_\alpha$. More is true: $\|.\|_{\mathcal{N}_\alpha}$ is an F-norm, and under this norm, $\mathcal{N}_\alpha$ is an F-space, i.e. a complete metrizable topological vector space (cf.[244]).

The Bloch space B consists of all functions $f \in \mathcal{H}(\Delta)$ obeying

$$\|f\|_B = |f(0)| + sup_{z\in\Delta} (1-|z|^2)|f'(z)| < \infty.$$

$\|.\|_{\boldsymbol{B}}$ Is a norm and makes B a Banach space.

Given $\boldsymbol{\omega} \in \Delta$, let

$$\varphi_\omega(z) = \frac{\omega - z}{1-\bar{\omega}z}.$$

Be a Möbius transformation which exchange ω *and* 0. Stroethoff's idea = in the proof of theorems 4.1 and 4.2 in [251] yield that $f \in \mathcal{H}(\Delta)$ lies in B if and only if

$$sup_{\omega\in\Delta} T_\alpha\left(C_{\varphi_\omega} f - f(\omega)\right) < \infty.$$

That is to say, B is the Möbius bounded subspaces of $\mathcal{N}_\alpha$.

For $\beta \in (-1,\infty)$, let $\mathcal{Q}_\beta$ be the class of all functions $f \in \mathcal{H}(\Delta)$ with

$$\|f\|_{\mathcal{Q}_\beta} = |f(0)| + sup_{\omega\in\Delta} [\int_\Delta |\left(C_{\varphi_\omega} f\right)'(z)|^2 (1-|z|^2)^\beta dm(z)]^{\frac{1}{2}} < \infty.$$

Observe that if $\beta \in (-1,0), \beta = 0, \beta = 1$ *and* $\beta \in (1,\infty)$, then $\mathcal{Q}_\beta = \mathbb{C}, \mathcal{D}$ (the classical Dirichlet space), BMOA and B respectively (cf. [248], [Ba], [240], [237], [252]. Of course, $\mathcal{Q}_\beta$ is the Möbius bounded subspaces of the weighted Dirichlet space (see also [238], [239], [243]. The spaces $\mathcal{N}_{\alpha,} B$ *AND* $\mathcal{Q}_\beta$ are linked by the inclusions $\mathcal{N}_{\alpha,} \supset B \supset \mathcal{Q}_\beta$. Notice that B and $\mathcal{Q}_\beta$ are Möbius invariant, but $\mathcal{N}_{\alpha,}$ is not.

We are going to work with the composition operators sending 'big' spaces to 'small' spaces since the converse is clear. In fact, $C_\phi: B \to \mathcal{N}_{\alpha,}$ and $C_\phi: \mathcal{Q}_\beta \to \mathcal{N}_{\alpha,}$ are always compact ([253]), while $C_\phi: \mathcal{Q}_\beta \to B_,$ is compact if and only if $\lim_{|\phi(z)|\to 1} (1-|z|^2)$ $|\phi'(z)| / (1 = |\phi(z)|^2) = 0$. (cf. [247] and [198].

The main results are the next three theorem. The first concerns boundedness and compactness of C_ϕ: $\mathcal{N}_{\alpha,} \to \mathcal{Q}_\beta$.

Arcs in the unit circle $\partial\Delta$ are sets of the form $I = \{z \in \partial\Delta : \theta_1 \leq \arg z < \theta_2\}$ where $\theta_1, \theta_2 \in [0,2\pi)$ *and* $\theta_1 < \theta_2$. The length of an arc $I \subset \partial\Delta$will be denoted by [I]. The Carleson box based on an arc I is the set

$$S(I) = \left\{z \in \Delta : 1 - \frac{|I|}{2\pi} \leq |Z| < 1, \frac{z}{|z|} \in I\right\} \tag{9}$$

Also for an $r \in (0,1)$ and an analytic self-map ϕ of Δ , put $\Omega_r = \{z \in \Delta : |\phi(z)| > r\}$. The characteristic function of a set $\mathbb{E} \subset \Delta$ is denoted by $1_{\mathbb{E}}$

The third theorem deals with boundedness and compactness of C_ϕ: $\mathcal{N}_{\alpha,} \to \mathcal{Q}_\beta$. This requite the Möbius invariant version of the generalized Nevanlinna counting function (cf. [232]). For $\beta \in (0,\infty!)$ and an analytic map $\phi: \Delta \to \Delta$.

Let

$$N(\beta,\omega,z,\phi)=\begin{cases}\sum_{\phi(v)=z}[1-|\phi_\omega(v)|^2]^\beta, & z\in\phi(\Delta),\\ 0, & z\in\Delta/\phi(\Delta)\end{cases}$$

We denote positive constants by $M, M_0, M_1, , M_2, \dots$ those constants depend only on some parameters such as and unless a special remark is made . Also given two families $x=\left(x(\omega)\right)_{\omega\in\Omega}$ and $y=\left(y(\omega)\right)_{\omega\in\Omega}$ of non negative two families real numbers (or functions) on the given domain Ω , we write x ≍ y if (there exists constant $M_1, M_2>0$ such that) , $M_1x(\omega)\le y(\omega)\le M_2x(\omega)$ for all $\omega\in\Omega$.

1. C_ϕ: $\mathcal{N}_{\alpha,}\to B$ The space $\mathcal{H}$ (Δ) is a Fréchet space with respect to the compact-open toplpogy, that is, the topology of uniform convergence on compact subsets of Δ ; in fact $\mathcal{H}$ (Δ) is even a Fréchet algebra. By Montel's theorem, bounded sets in $\mathcal{H}$ (Δ) are relatively compact; accordingly, bounded sequences in $\mathcal{H}$ (Δ) admit convergent subsequences. Convergence in the space will be reffered to as locally uniform $(l,u.)$ convergence.

Recall that $\mathcal{N}_{\alpha,}$ is a linear subspace (even a subalgebra) of $\mathcal{H}$ (Δ) . Note that $\mathcal{N}_{\alpha,}$ is a topological vector space with respect to the F-norm $\|.\|_{\mathcal{N}_{\alpha,}}$. This is in marked contrast to the situation for the classical Nevanlinna class which is not a topological vector space [249]. Under $\|.\|_{\mathcal{N}_{\alpha,}}$, the topology of $\mathcal{N}_{\alpha,}$ is stronger than that of locally uniform convergence. This is a simple consequence of the following estimate:

$$\log\left(1+f(z)\right)\le\frac{M_0\|f\|_{\mathcal{N}_{\alpha,}}}{(1-|z|^2)^{2+\alpha}},\qquad f\in\ \mathcal{N}_\alpha, \tag{10}$$

Where $M_0>0$ is a constant depending only on α.

As in [251], $\mathcal{N}_\alpha$, has B as its Möbius bounded subspace.

Proposition (6.2.1)[236]: Let $\alpha\in(-1,\infty)$ and $f\in\mathcal{H}$ (Δ) . Then the following are equivalent:

(i) f belong to B.

(ii) $sup_{\omega\in\Delta}T_\alpha\left(C_{\varphi\omega}\Delta f-f(\omega)\right)<\infty$.

(iii) $sup_{\omega\in\Delta}\left\|C_{\varphi\omega}f-f(\omega)\right\|_{\mathcal{N}_{\alpha,}}<\infty$.

Proof. It suffices to show (i) ⇔ (iii), for (i) ⇔ (ii) can be verified in a similar manner to proving Theorem 4.1 and 4.2 of [251].

Observe that if f is a Bloch function with $\|f\|_B>0$ then for $z\in\Delta$

$$\left|C_{\varphi\omega}f(z)-f(\omega)\right|\le\frac{\|f\|_B}{2}\ log\frac{1+|z|}{1-|z|},$$

It follows that for each $t>0$,

$$m_\alpha[t]=\ m_\alpha\{z\in\Delta{:}\left|C_{\varphi\omega}f(z)-f(\omega)\right|>t\}\le M_1exp\left[-\frac{2(\alpha+1)t}{\|f\|_B}\right]$$

Let now be a Bloch function. We may assume that $\|f\|_B>0$. There is a constant $M_2>0$ depending only on α such that for each $\omega\in\Delta$

$$\left\|C_{\varphi\omega}f(z)-f(\omega)\right\|_{\mathcal{N}_{\alpha,}}=\int_0^\infty\frac{m_\alpha[t]}{1+t}dt\le M_2\|f\|_B \tag{11}$$

Which proves (iii).

Suppose conversely that (iii) is true, Let $r \in (0,1)$. If $z \in \Delta$ is such that $|\varphi_\omega(z)| < r$ then, by (10) and since φ_ω is an analytic automorphism of Δ with $\varphi_\omega^{-1} = \varphi_\omega$,

$$\log(1 + |f(z) - f(\omega)|) \leq \frac{M_0 \left\| C_{\varphi\omega} f - f(\omega) \right\|_{\mathcal{N}_\alpha}}{(1-r)^{2+\alpha}}. \quad (12)$$

An application in [251]. The proof is complete.

Note that B has a closed subspace, the Little Bloch space B_0 of all function obeying $f \in B$ obeying

$$\lim_{|z| \to 1} (1 - |z|^2)|f'(z)| = 0.$$

It is well known that the polynomials are dense in B_0 under $\|.\|_B$. We have

Corollary (6.2.2)[236]: Let $\alpha \in (-1, \infty)$ and $f \in \mathcal{H}\ (\Delta)$. Then the following are equivalent:

(i) f belong to B_0-

(ii) $\lim_{|\omega| \to 1} T_\alpha \left(\varrho^{-1} \left(C_{\varphi\omega} f - f(\omega) \right) \right) = 0 \quad for\ every\ \varrho > 0,$

(iii) $\lim_{|\omega| \to 1} \left\| C_{\varphi\omega} f - f(\omega) \right\|_{\mathcal{N}_{\alpha,}} = 0.$

Proof. As in Proposition (6.2.1), it is enough to verify $(i) \Leftrightarrow (iii)$. Suppose that f belong to B_0. By density, given any $\varepsilon \in (0,1)$, there is a polynomial P such that $\|f - P\|_B < \varepsilon$. Consequently, by (11),

$$\left\| C_{\varphi\omega}(f - P) - (f - P)(\omega) \right\|_{\mathcal{N}_{\alpha,}} \leq M_2 \quad \|f - P\|_B < M_2 \varepsilon.$$

This implies (iii), owing to $\lim_{|\omega| \to 1} \left\| C_{\varphi\omega} P - P(\omega) \right\|_{\mathcal{N}_{\alpha,}} = 0.$

The converse follows easily from (12) and from Theorem (6.2.7) of [251].

A subset E of $\mathcal{N}_{\alpha,}$ is called *bounded* if it is bounded for the defining F-norm $\|.\|_{\mathcal{N}_{\alpha,}}$. Given a Banach space Y , we say that a linear map T: $\mathcal{N}_{\alpha,} \to Y$ is bounded if $T(E) \subset Y$ is bounded for every bounded subset $E\ OF\ \mathcal{N}_{\alpha,}$. In addition , we say that T is compact if $T(E) \subset Y$ is relatively compact for every bounded set. $E \subset \mathcal{N}_{\alpha,}$. A useful tool is the following compactness criterion which follows readily in [245] and [232].

Lemma (6.2.3)[236]: Let $\alpha \in (-1, \infty)$ and Y be a Banach subspace of $\mathcal{H}\ (\Delta)$. with norm $\|.\|_{\mathcal{N}_{\alpha,}}$. Then C_φ : $\mathcal{N}_{\alpha,} \to Y$ is compact if and onlu if for every $s > 0$ and every sequences $\{f_n\}$ satisfies $\|f_n\|_{\mathcal{N}_{\alpha,}} \leq s$ and converges to $0.l.u, lim_{n\to\infty} \left\| C_\varphi f_n \right\|_Y = 0.$

Theorem (6.2.4)[236]: Let $\alpha \in (-1, \infty)$ and let $\phi: \Delta \to \Delta$ be analytic . Then the following are equivalent:

(i) C_ϕ: $\mathcal{N}_{\alpha,} \to \mathcal{Q}_\beta$ exists as a bounded operator.

(ii) C_ϕ: $\mathcal{N}_{\alpha,} \to \mathcal{Q}_\beta$ Exists as a comact operator.

(iii) For all $c > 0.$,

Before giving the second assertion on boundedness and compactness of $C_\phi: B \to \mathcal{Q}_p$, we explain the necessary notation.

Proof. It suffices to check two implication $(i) \Leftrightarrow (iii)$ and $(iii) \Leftrightarrow (ii)$

(i) $\Leftrightarrow$ (iii). Let (i) hold. For any $c > 0$ and $\omega = \phi(z_0)$ (where $z_0 \in \Delta$ is fixed), consider the test function.

$$f_\omega(z) = exp\left[c \left(\frac{1 - |\omega|^2}{(1 - \bar{\omega} z)^2} \right)^{2+\alpha} \right] \quad (13)$$

Since

$$\log(1+x) \leq 1 + log^+ x \qquad for\ x \geq 0,$$

$$\|f_\omega\|_{\mathcal{N}_{\alpha,}} \leq \frac{\pi}{1+\alpha} + \int_\Delta \left[log^+[f_\omega(z)]\right](1+|z|^2)^\alpha \, dm(z)$$

$$\leq \frac{\pi}{1+\alpha} + c\int_\Delta \left(\frac{1-|\omega|^2}{(1-\bar{\omega}z)^2}\right)^{2+\alpha} (1-|z|^2)^\alpha \, dm(z) \leq M_{3.}$$

Where $M_{3.} > 0$ done not depend on ω and it comes from Lemma 4.2.2 of [154]. Because $C_\phi\colon \mathcal{N}_{\alpha,} \to B$ is bounded and

$$f'_\omega(z) = \frac{2(2+\alpha)c\bar{\omega}(1-|\omega|^2)^{2+\alpha}}{(1-\bar{\omega}z)^{2(2+\alpha)+1}} \exp\left[c\left(\frac{1-|\omega|^2}{(1-\bar{\omega}z)^2}\right)^{2+\alpha}\right]$$

There is a constant $M_4 > 0$ depending only on $c\ and\ \alpha$ such that

$$M_4 \geq (1-|z|^2)\left|f'_\omega(\phi(z))\right|.|\phi'(z)|$$

$$\geq \frac{2(1-|z|^2)|\phi'(z)|(1-|\omega|^2)^{2+\alpha}}{\left(1-\bar{\omega}\phi(z)\right)^{2(2+\alpha)+1}} \exp\left[c\left(\frac{1-|\omega|^2}{\left(1-\bar{\omega}\phi(z)\right)^2}\right)^{2+\alpha}\right]$$

This estimate leads to

$$\frac{(1-|z_0|^2)|\phi'(z_0)|}{1-|\phi(z_0)|^2}\ exp\left[\frac{c}{(1-|\phi(z_0)|^2)^{2+\alpha}}\right] \leq \frac{M_4(1-|\phi(z_0)|^2)^{2+\alpha}}{c|\phi(z_0)|} \qquad (14)$$

Where forces (iii) to hold.

(ii) $\Longrightarrow$ (ii) Assume that (iii) is valid for all $c > 0$. Note that if $f \in \mathcal{N}_{\alpha,}$ then by (10) and Cauchy's formula

$$(1-|z|^2)|f'(z)| \leq \frac{2}{\pi}\int_{\partial\Delta} |f(z+2^{-1}(1-|z|)\zeta|\,|d\zeta| \leq exp\left[\frac{4^{2+\alpha}M_0\|f_\omega\|_{\mathcal{N}_{\alpha,}}}{(1-|z|^2)^{2+\alpha}}\right] \quad (15)$$

To demonstrate that $C_\phi\colon \mathcal{N}_{\alpha,} \to B$ is compact, we choose, for $s > 0$, any sequence $\{f_n\}$ in $\mathcal{N}_{\alpha,}$ such that $\|f_n\|_{\mathcal{N}_{\alpha,}} \leq s$ and $f_n \to 0$ l. u. on Δ . Then for each $\delta \in (0,1)$,

$$sup_{|\phi(z)|\leq\delta}(1-|z|^2)\left|\left(C_\phi f_n\right)'(z)\right| \leq sup_{|\phi(z)|\leq\delta}(1-|\phi(z_0)|^2)\left|f'_n(\phi(z))\right| \to 0, n \to \infty.$$

On the other hand, from (15) and (iii) is turns out that whenever $\delta \to 1$.

$$sup_{|\phi(z)|>\delta}(1-|z|^2)\left|\left(C_\phi f_n\right)'(z)\right|$$

$$\leq sup_{|\phi(z)|>\delta}\frac{(1-|z|^2)|\phi'(z)|}{1-|\phi(z)|} \exp\ \frac{4^{2+\alpha}M_0 s}{(1-|\phi(z)|^2)^{2+\alpha}} \to 0.$$

Combining the above estimates we that $\left\|C_\phi f_n\right\|_{B\to 0}$ AS $n \to \infty$. Hence (ii) follows from Lemma (6.2.3). The proof is complete. There is an: analogue of Theorem (6.2.4) for the Little Bloch space B_0.

Corollary (6.2.5)[236]: Let $\alpha \in (-1,\infty)$ and let $\phi : \Delta \to \Delta$ be analytic. Then the following are equivalent:

- $C_\phi\colon \mathcal{N}_{\alpha,} \to B_0$ exist as a bounded operator.
- $C_\phi\colon \mathcal{N}_{\alpha,} \to B_0$ exist as a compact operator.
- for all $c > 0$.

$$\lim_{|z|\to 1}\frac{(1-|z|^2)|\phi'(z)|}{1-|\phi(z)|^2} \exp\left[\frac{c}{(1-|\phi(z)|^2)^{2+\alpha}}\right] = 0 \qquad (16)$$

Proof. It suffices to demonstrate $(iii) \Longrightarrow (ii)$ *and* $(i) \Longrightarrow (iii)$. The first implication follows easily from the proof of the corresponging case of Theorem (6.2.4). The second will be verified by contradiction, Suppose that C_ϕ: $\mathcal{N}_{\alpha,} \to B_0$ is bounded. so $\phi \in B_0$ Now, if (16) is not true for all, $c > 0$ then there are $c_{0,}$, $\varepsilon_{0,}$and a sequence $\{z_n\}$ tending to $\partial\Delta$ such that

$$\frac{(1-|z_n|^2)|\phi'(z_n)|}{1-|\phi(z_n)|^2}\exp\left[\frac{c_0}{(1-|\phi(z_n)|^2)^{2+\alpha}}\right] \geq \varepsilon_0, \tag{17}$$

Since $\phi \in B_0$ (17) indicate that $\{z_n\}$ has a subsequence $\{z_{n_k}\}$ with $|\phi(z_{n_k})| \to 1$. Also since C_ϕ: $\mathcal{N}_{\alpha,} \to B_0$ is bounded, one has (9) (for all c > 0, which, in particular, produces the following limit:

$$\frac{\left(1-\left|z_{n_k}\right|^2\right)\left|\phi'(z_{n_k})\right|}{1-\left|\phi(z_{n_k})\right|^2}\exp\left[\frac{c_0}{\left(1-\left|\phi(z_{n_k})\right|^2\right)^{2+\alpha}}\right] \to 0. \tag{18}$$

It is evident that (18) contradicts (17). We are done.

(ii)$C_\varphi \to B \to \mathcal{Q}_\beta$. We prove Theorem (6.2.8). The proof will borrow a technique from [242]. Before proceeding, we need an inverse inquality for B due to Ramey and Ullrich [121].

Lemma (6.2.6)[236]: There are two functions $f_1, f_2 \in B$ such that

$$\inf_{z\in\Delta}(1-|z|^2)|f_1'(z)| + |f_{21}'(z)| \geq 1. \tag{19}$$

For $\beta \in (0,\infty)$we say that a positive Borel measure on $d\mu$ on Δ is a β −Carelson measure provided $\frac{sup_{I\subset\partial\Delta}\, \mu(S(I))}{|I|^\beta} < \infty$. This definition was introduced by [239] to characterize the $\mathcal{Q}_\beta$ space.

Lemma (6.2.7)[236]: Let $\beta \in (0,\infty)$ and let $f \in \mathcal{H}$ (Δ) with

$$d\mu_{f,\beta}(z) = |f'(z)|^2(1-|z|^2)^\beta d\mu\,(z).$$

Then $f \in \mathcal{Q}_\beta$ if and only if $d\mu_{f,\beta}$ is a β - Carleson measure. Moreover,

$$\|f\|_{\mathcal{Q}_\beta} \asymp |f(0)| + \left[sup_{I\subset\partial\Delta}\frac{\mu_{f,\beta}(S(I))}{|I|^\beta}\right]^{1/2}. \tag{20}$$

Theorem (6.2.8)[236]: Let $\beta \in (0,\infty)$ and let $\phi: \Delta \to \Delta$ be analytic. Then

(i) $C_\phi: B \to \mathcal{Q}_\beta$ exists as a bounded operator if and only if

$$sup_{I\subset\partial\Delta}|I|^{-\beta}\int_{S(I)}\left[\frac{(1-|Z|^2)^{\beta/2}|\phi'(Z)|}{1-|\phi(Z)|^2}\right]^2 dm(z) < \infty \tag{21}$$

(ii) $C_\phi: B \to \mathcal{Q}_\beta$ exists as a compact operator if and only if $\phi \in \mathcal{Q}_\beta$ and

$$\lim_{r\to1} sup_{I\subset\partial\Delta}|I|^{-\beta}\int_{S(I)}\left[\frac{(1-|Z|^2)^{\frac{\beta}{2}}|\phi'(Z)|}{1-|\phi(Z)|^2}\right]^2 1_{\Omega_r}(z)dm(z) = 0\,. \tag{22}$$

Note that (i) of Theorem (6.2.8) is essentially known (cf. [198]) and is listed here only for, the sake of completeness. However, (ii) is new and is just what Smith-Zhao did not figure out. Moreover, if $\beta > 1$ then (22) is equivalent to $\lim_{|\phi(z)|1}(1-|z|^2)\,|\phi'(z)|/(1-|\phi(z)|^2) = 0$, (cf. [247]).

Proof. From now on, $\mathbb{B}_X$ stands for the unit ball of a given Banach space $(X, \|.\|_X)$.

i. Follows obviously from Lemmas (6.2.6) and (6.2.7). The key is to infer (ii). Sufficiency of (ii). Let $\phi \in \mathcal{Q}_\beta$ and let (22) Δhold. We have to show that if $\{f_n\} \subset \mathbb{B}_B$

converges to 01.u. on Δ then $\left\{\|C_\phi f_n\|_{\mathcal{Q}_\beta}\right\}$ then converges to 0. for each $r \in (0,1)$ set $\breve{\Omega}_r = \frac{\Delta}{\Omega_r.} so \{f_n'(\phi)\}$ tends to 0 uniformly on $\breve{\Omega}_r$. Hence by Lemma (6.2.7), for every $\varepsilon > 0$ there is an integer $N > 1$ such that for $n \geq N$,

$$sup_{I\subset\partial\Delta} |I|^\beta \int_{S(I)} \left|\left(C_\phi f_n\right)'(z)\right|^2 (1-|z|^2)^\beta I_{\breve{\Omega}_r}(z)\, d\mu\,(z) \leq \varepsilon\, M\|\phi\|^2_{\mathcal{Q}_\beta}$$

On the other hand, from (22) and the growth of the derivatives of B-function one derivatives that for every $\varepsilon > 0$ there exists a $\delta \in (0,1)$ such that for $r \in [\delta, 1)$,

$$sup_{I\subset\partial\Delta} |I|^\beta \int_{S(I)} \left|\left(C_\phi f_n\right)'(z)\right|^2 (1-|z|^2)^\beta I_{\Omega_r}(z)\, dm\,(z) < \varepsilon.$$

Combining the previous inequalities with Lemma (6.2.7) , we obtain $\|C_\phi f_n\|_{\mathcal{Q}_\beta}$.

Necessity of (ii) . This part is more difficult. Let $C_\varphi \to B \to \mathcal{Q}_\beta$ be compact. It is clear that $\phi \in \mathcal{Q}_\beta$. So, we must show (22) . Since $\{z^n\}$ is norm bounded in B and it converges to 01.u. on Δ , we have $\|\phi^n\|_{\mathcal{Q}_\beta} \to 0$. Applying by Lemma (6.2.7), we find that for every $\varepsilon > 0$, there is an integer $N > 1$ such that for $n \geq N$.

$$n^2 sup_{I\subset\partial\Delta} |I|^{-\beta} \int_{S(I)} |\phi(z)|^{2n-2}\, |\phi'(z)|^2 (1-|z|^2)^\beta\, dm\,(z) < \varepsilon.$$

Thus for each $r \in (0,1)$

$$N^2 r^{2N-2} sup_{I\subset\partial\Delta} |I|^{-\beta} \int_{S(I)} |\phi'(z)|^2 (1-|z|^2)^\beta 1_{\Omega_r} dm\,(z) < \varepsilon.$$

Taking $r \geq N^{-1/(N-1)}$, we get

$$sup_{I\subset\partial\Delta} |I|^{-\beta} \int_{S(I)} |\phi'(z)|^2 (1-|z|^2)^\beta 1_{\Omega_r} dm\,(z) < \varepsilon. \qquad (23)$$

Keeping (23) in mind we show that for every $f \in \mathbb{B}_B$ and for every $\varepsilon > 0$, , there is a $\delta = \delta(f, \varepsilon)$ such that for$r \in [\delta.1)$

$$T(f, \phi.\beta.r) = \quad sup_{I\subset\partial\Delta} |I|^{-\beta} \int_{S(I)} \left|(C_\phi f)'(z)\right|^2 (1-|z|^2)^\beta 1_{\Omega_r} dm\,(z) < \varepsilon. \qquad (24)$$

As a matter of fact, if we let $f_t(z) = f(tz)\, for\ \ f \in \mathbb{B}_B\ and\ t\ \in (0,1)$ then $f_t \to f\ 1, u, on\ \Delta\ as\ t \to 1$. Since $C_\varphi \to B \to \mathcal{Q}_\beta$ is compact, $\|f_t \circ \phi - f \circ \phi\|_{\mathcal{Q}_\beta} \to 0\ as\ t \to 1$. Furthermore, Lemma (6.2.7) yields that for every $\varepsilon > 0$ there is a $t \in (0,1)$ such that

$$sup_{I\subset\partial\Delta} |I|^{-\beta} \int_{S(I)} \left|(C_\phi f_t)'(z) - (C_\phi f_t)'(z)\ \right|^2 (1-|z|^2)^\beta dm\,(z) < \varepsilon.$$

Accordingly, by (23),

$$T(f,\phi,\beta,r) \leq 2\varepsilon + 2\, sup_{I\subset\partial\Delta} |I|^{-\beta} \int_{S(I)} \left|(C_\phi f_t)'(z)\right|^2 (1-|z|^2)^\beta 1_{\Omega_r}(z) dm\,(z)$$

$$\leq \varepsilon + 2\|f_t'\|_\infty^2 \quad sup_{I\subset\partial\Delta} |I|^{-\beta} \int_{S(I)} |\phi'(z)|^2 (1-|z|^2)^\beta 1_{\Omega_r}(z) dm\,(z)$$

$$\leq 2\varepsilon(1+\|f_t'\|_\infty^2).$$

Since C_ϕ sends $\mathbb{B}_B$ to a relatively compact subset of $\mathcal{Q}_\beta$, there exists, for every $\varepsilon > 0$, a finite collection of functions $f_1, \dots, f_N\ in\ \mathbb{B}_B$ such that for each $f \in \mathbb{B}_B$ there is a $k \in \{1, \dots, N\}$ with

$$sup_{I\subset\partial\Delta} |I|^{-\beta} \int_{S(I)} \left|(C_\phi f)'(z) - (C_\phi f_k)'(z)\right|^2 (1-|z|^2)^\beta dm\,(z) < \varepsilon.$$

Now (24) is used to deduce that for $\delta = max_{1\leq k\leq N}\, \delta(f_k, \varepsilon)\ and\ r \in [\delta, 1)$,

$$sup_{I\subset\partial\Delta} |I|^{-\beta} \int_{S(I)} \left|(C_\phi f_k)'(z)\right|^2 (1-|z|^2)^\beta 1_{\Omega_r}(z) dm\,(z) \leq \varepsilon;$$

Thus

$$sup_{f\in\mathbb{B}_B} \quad sup_{I\subset\partial\Delta} |I|^{-\beta} \int_{S(I)} \left|(C_\phi f)'(z)\right|^2 (1-|z|^2)^\beta 1_{\Omega_r}(z) dm\,(z) \leq 4\varepsilon; \qquad (25)$$

An application of Lemma (6.2.6) to (25) implies (22). This concludes the proof.

The space $\mathcal{Q}_\beta$, like B, has a closed subspace $\mathcal{Q}_{\beta,0}$ which consists of those $f \in \mathcal{Q}_\beta$ satisfying

$$\lim_{|\omega|\to 1} \int_\Delta \left|(C_{\phi\omega} f)'(z)\right|^2 (1-|z|^2)^\beta dm(z) = 0.$$

It is known that $\mathcal{Q}_{\beta,0} = \mathbb{C}, VMOA$ and B_0 whenever $\beta \in (-1.0], \beta = 1\ and\ \beta \in (1,\infty)$ and, respectively (c.f. [248], [237]). Moreover, the $\mathcal{Q}_{\beta,0}$-version of Lemma (6.2.7) states that $f \in \mathcal{Q}_{\beta,0}$ if and only if $d\mu_{f,\beta}$ is a vanishing β-Carelson measure, i.e. $\frac{\lim_{|I|\to 0} \mu_{f,\beta}(S(I))}{|I|^\beta} = 0$ uniformly for all Carelson boxes S(I)(cf. [239]).

The purpose of mentioning $\mathcal{Q}_{\beta,0}$ is to solve another problem in [198]: ‘’ when is $C_\varphi \to B_0 \to \mathcal{Q}_\beta$ or $\mathcal{Q}_{\beta,0}$compact? “ the method of treating Theorem (6.2.8) can be adopted to provide an answer to this question.

For convenience, let $\Delta_r = \{z \in \Delta : |z| > r\}$ where $r \in (0,1]$. We have

Corollary (6.2.9)[236]: Let $\beta \in (0,\infty]$ and let $\phi: \Delta \to \Delta$ be analytic. Then

(i) $\mathcal{Q}_\beta$ exists as a compact operator if and only if $\phi \in \mathcal{Q}_\beta$ and (22) holds.

(ii) $\mathcal{Q}_{\beta,0}$ exists as a compact operator if and only if $\phi \in \mathcal{Q}_\beta$ and

$$sup_{I\subset\partial\Delta} |I|^{-\beta} \int_{S(I)} \left[\frac{(1-|z|^2)^{\frac{\beta}{2}} |\phi'(z)|^2}{1-|\phi(z)|^2}\right]^2 1_{\Delta r}(z)\, dm(z) = 0. \qquad (26)$$

Proof. (i) sufficiency. It follows from Theorem (6.2.8)(ii).

Necessity. Suppose that $C_\varphi \to B_0 \to \mathcal{Q}_{\beta,}$ is compact. Then $\phi \in \mathcal{Q}_\beta$ follows right away. Note that if $f \in \mathbb{B}_B$ then $\|f_t\|_B \le \|f\|_B \le 1$. Now for a fixed $t \in (0,1)$, , put $\mathbb{B}_B^t = \{f_t : f \in \mathbb{B}_B\}$. Then $\mathbb{B}_B^t$ is a subset of $\mathbb{B}_{B_0}$. By compactness of C_φ is a relatively compact subset of $\mathcal{Q}_{\beta,}$. The proof of Theorem (6.2.8) (ii) actually shows that for every $\varepsilon > 0$ there is a $\delta \in (0,1)$ (independent of t) (such that for $r \in [\delta, 1)$.

$$sup_{I\subset\partial\Delta} |I|^{-\beta} \int_{S(I)} \left|(C_\phi f_t)'(z)\right|^2 (1-|z|^2)^\beta \, 1_{\Omega_r}(z) dm(z) < \varepsilon.$$

This estimate and Lemma (6.2.6) result in

$$sup_{I\subset\partial\Delta} |I|^{-\beta} \int_{S(I)} \left[\frac{t|\phi'(z)|(1-|z|^2)^{\beta/2}}{1-t^2|\phi(z)|^2}\right]^2 1_{\Omega_r}(z)\, dm(z) < 2\varepsilon;$$

And so (26) follows, by Fatou's lemma.

(ii) Sufficiency. Let $\phi \in \mathcal{Q}_\beta$ and let ϕ satisfy (26). Suppose that $\{f_n\} \subset \mathbb{B}_{B_0}$ is a dequence which converges to 01.u.on Δ . To prove that $C_\varphi \to B_0 \to \mathcal{Q}_{\beta,0,}$is compact, it suffices to verify that $\lim_{n\to\infty} \|C_\varphi f_n\|_{\mathcal{Q}_\beta} = 0.$ For each $r \in (0,1)$ put $\widetilde{\Delta_r} = \Delta/\Delta_r$ Since $\widetilde{\Delta_r}$ is a compact subset of Δ . $\{f_n'(\phi)\}$tend to 0 uniformly on $\widetilde{\Delta_r}$. From $\phi \in \mathcal{Q}_\beta$ and Lemma (6.2.7) it is seen that

$$\lim_{n\to\infty} \quad sup_{I\subset\partial\Delta} |I|^{-\beta} \int_{S(I)} \left|(C_\phi f_t)'(z)\right|^2 (1-|z|^2)^\beta 1_{\widetilde{\Delta_r}}(z)\, dm(z) = 0.$$

This limit, together with (26), gives $\lim_{n\to\infty} \|C_\varphi f_n\|_{\mathcal{Q}_\beta} = 0$

Let $C_\varphi \to B_0 \to \mathcal{Q}_{\beta,0}$ be compact. It is trivial to deduce that $\phi \in \mathcal{Q}_\beta$ and $C_\varphi(\mathbb{B}_{B_0})$is a relatively compact subset of $\mathcal{Q}_{\beta,0}$. Given an $\varepsilon > 0$ for every $f \in \mathbb{B}_{B_0}$ there are finitely many functions $g_k \in \mathcal{Q}_{\beta,0}$ such that

$$sup_{I\subset\partial\Delta} |I|^{-\beta} \int_{S(I)} \left|(C_\phi f_t)'(z) - g'_k(z)\right|^2 (1-|z|^2)^\beta dm(z) < \varepsilon$$

Where we have used Lemma (6.2.7). Consequently, for all $r \in (0,1)$

$$sup_{I\subset\partial\Delta} |I|^{-\beta} \int_{S(I)} \left|(C_\phi f_t)'(z) - g'_k(z)\right|^2 (1-|z|^2)^\beta 1_{\widetilde{\Delta_r}}(z)\, dm(z) < \varepsilon$$

Since $g_k \in \mathcal{Q}_{\beta,0}$ there is $r \in (0,1)$ such that for $r \in [\delta, 1)$,

$$sup_{I\subset\partial\Delta} |I|^{-\beta} \int_{S(I)} \left|g'_k(z)\right|^2 (1-|z|^2)^\beta 1_{\Delta_r}(z)\, dm(z) < \varepsilon$$

Which implies

$$\sup_{f\in\,\mathbb{B}_{B_0}} sup_{I\subset\partial\Delta} |I|^{-\beta} \int_{S(I)} \left|(C_\phi f_t)'(z)\right|^2 (1-|z|^2)^\beta 1_{\Delta_r}(z)\, dm(z) < 2\varepsilon$$

A careful inspection of the above argument for the necessity of (i) shows that (26) follows immediately from another application of Lemma (6.2.6) and Fatou's lemma to the last inequality. The proof is complete.

We close by an observation on the condition (22). it is clear that (22) holds if

$$\int_{\Delta} \left[\frac{\phi'(z)}{1-|\phi(z)|^2}\right]^2 dm(z) < \infty. \qquad (27)$$

Shapiro- Taylor [250] showed that (27) force s $C_\phi: \mathcal{D} \to \mathcal{D}$ to be a Hilbert- Schmidt operator. Tjani [232] pointed out.

That (27) ensures that $C_\phi: B \to \mathcal{D}$ is compact. Since $\mathcal{D} \subset \mathcal{Q}_\beta \subset B$, our conditions (22) and $\phi \in \mathcal{Q}_\beta$ fill up the gap between $\mathcal{D}$ and B in the sense of the Hilbert- Schmidt property and compactness.

4. $C_\phi: \mathcal{N}_\alpha \to \mathcal{Q}_\beta$. We show Theorem (6.2.10). A dyadic division of Δ, quite different from the one used for Theorem (6.2.4), will be involved to control Theorem (6.2.10).

Following [72], we divide Δ into dyadic boxes. Let I denote the family of dyadic arcs in $\partial\Delta$, that is, the family of all arcs of the form

$$\{z \in \partial\Delta: 2\pi k/2^l \le \arg z < 2\pi(1+k)/2^l\}, k = 0,1,\ldots,2^l-1, l = 0.1,\ldots$$

Given an arc $I \subset \partial\Delta$, let $H(I)$ denote the half of $S(I)$ which is closest to the origin, namely,

$$H(I) = \left\{z \in S(I): 1 - \frac{I}{2\pi} \le |z| < 1 - |I|/4\pi\right\}.$$

Note that the $H(I)'s$ for $I \in \mathcal{I}$ are pair wise disjoint and cover Δ. Fix any enumeration $\{H_j: j = 1,2,\ldots\}$ of these sets and select a point a_j in each H_j. Almost any point would work, but in order to simplify some parts later on let us agree that a_j is the " center" of H_j in the sense that $|a_j|$ and arg a_j bisect the interval of abslute values and the interval of arguments, respectively , of points in $H_j. If\ H_j =$ then $H(I)\ then\ |I| \asymp 1 - |a_j|$.

Theorem (6.2.10)[236]: Let $\alpha \in (-1,\infty), \beta \in (0,\infty)$ and let $\phi: \Delta \to \Delta$ be analytic. Then the following are equivalent:

(i) $C_\phi: \mathcal{N}_{\alpha,} \to \mathcal{Q}_\beta$ exists as a bounded operator .

(ii) $C_\phi: \mathcal{N}_{\alpha,} \to \mathcal{Q}_\beta$ exists as a compact operator

(iii) $\phi \in \mathcal{Q}_\beta$ $\quad and\ for\ all\ c > 0.$

$$sup_{\omega\in\Delta} sup_{I\subset\partial\Delta} \frac{|I|^{-2(\alpha+3)}}{\exp(c|I|^{2+\alpha})} \int_{S(I)} N(\beta,\omega,z,\phi)dm(z) < \infty. \qquad (28)$$

Comparing Theorem (6.2.10) with Theorem (6.2.4) we find that (28) ⇔(9) when $\beta > 1$.

We devote to the proof of Theorem (6.2.4) and its consequences. The proof of Theorem (6.2.8). We devoted to proving Theorem (6.2.10) and a further discussion.

Proof. It is enough to verify the implications (i) ⇒ (iii) ⟹ (ii).

Put

$$dm_{\beta,\omega,\phi}(z) = N(\beta,\omega,z,\phi)dm(z).$$

With this choice, we establish

$$\|C_\phi f\|_{\mathcal{Q}_\beta} = |f(\phi(0))| + sup_{\omega\in\Delta}\left[\int_{\Delta} |f'(z)|^2 dm_{\beta,\omega,\phi}(z)\right]^{1/2} \qquad (29)$$

(i) ⇒ (iii), Suppose that $C_\phi: \mathcal{N}_\alpha \to \mathcal{Q}_\beta$ is bounded, Then clearly ϕ is a member of $\mathcal{Q}_\beta$. In order to show that $dm_{\beta,\omega,\phi}$ satisfies (28), fix $\theta \in [0,2\pi)$ and $u = \left[1 + (2\pi)^{-1}[I]\right]e^{i\theta}$. Consider, for any $c > 0$, the test function

$$g_u(z) = \exp\left[\frac{c(1-|u|^2)^{m-2-\alpha}}{(1-\bar{u}z)^m}\right],$$

Where m is the smallest integer greater than $2+\alpha$. Then

$$g'_u(z) = \exp\frac{cm\bar{u}(1-|u|^2)^{m-2-\alpha}}{(1-\bar{u}z)^{m+1}}\exp\left[\frac{c(1-|u|^2)^{m-2-\alpha}}{(1-\bar{u}z)^m}\right],$$

Since $\log(1+x) \le 1 + log^+x \quad for\ x \ge 0$.

$$\|g_u\|_{\mathcal{N}_\alpha} \le \frac{\pi}{1+\alpha} + c\int_\Delta \frac{(1-|u|^2)^{m-2-\alpha}(1-|z|^2)^\alpha}{(1-\bar{u}z)^m} dm(z) \le M, \tag{30}$$

Once again, this constant $M > 0$ is independent of u and it is determined by Lemma 4.2.2 of [154]. Let I be the arc centered at $e^{i\theta}$. Then there is $\delta \in (0,1)$ such that for $|I| < \delta$,

$$sup_{z\in S(I)}|1\bar{u}z| \le M_1|I|, \qquad inf_{z\in S(I)}Re[(1-u\bar{z})^m] \ge M_2|I|^m$$

And hence

$$inf_{z\in S(I)}\left|\left(g'_u(z)\right)\right| \ge \frac{M_3|I|^{-(3+\alpha)}}{\exp(M_4|I|^{2+\alpha)})}$$

Where $M_1 > 0\ and\ M_2 > 0$ rely upon $\delta\ and\ \alpha\ only$ but also give

$$M_3 = \frac{cm}{2(2\pi)^{m-2-\alpha}M_1^{m+1}}. \quad M_4 = \frac{cM_2}{2(2\pi)^{m-2-\alpha}M_1^{2m}}$$

By (29) and since $log^+x \le log(1+x) on\ [0,\infty)$,

$$\left\|C_\phi g_u\right\|_{\mathcal{Q}_\beta}^2 \ge \frac{M_3^2 m_{\beta,\omega,\phi}\left(S(I)\right)}{|I|^{2(3+\alpha)}\exp(2M_4|I|^{2+\alpha)})}. \tag{31}$$

Appealing to the closed graph theorem, (31) and (30), one obtain (28) at once. On the other hand, if $|I| \ge \delta$, then (29) and $\phi \in \mathcal{Q}_\beta$ easily imply (28) too.

(iii) $\Longrightarrow$ (ii) Assume now that $\phi \in \mathcal{Q}_\beta$ and $dm_{\beta,\omega,\phi}$ is such that (28) is valid for all $\boldsymbol{c} > 0$. For every $\boldsymbol{s} > 0$ we choose a sequence $\{f_n\}$ in $\mathcal{N}_\alpha$ so that $\|f_n\|_{\alpha\in} \le s$ and $\{f_n\}$ converges to $01, u. on\, \Delta$ on . With the help og the dyadic division of Δ , for $f_n \in \mathcal{N}_\alpha\ let\ a_j^* \in \bar{H}_j$ let (closure of $\bar{H}_j$) be a point where $|f'_n|$ attain its maximum on $\bar{H}_j$. If l is the integer such that H_j is contained in

$$A_l := \left\{z \in \Delta: 1 - 2^{-l} \le |z| < 1 - 2^{-(l+2)}\right\},$$

then the set

$$S_j := \left\{z \in \Delta: 1 - 2^{-(l+1)} \le |z| < 1 - 2^{-(l+2)}, \left|\arg z - arga_j^*\right| < 2^{-l-1}\right\}$$

Contains a disc Δ_j with center a_j^* and radius comparable to 2^{-l} . Note that S_j intersects at most 6 of the sets H_k and that $1 - |z|^2 \asymp 2^{-l}$ whenever $z \in S_j$. Using these observations, (15) snd the submean value property of $|f'_n|$, we find that to every $\varepsilon \in (0,1)$ there corresponds an $r \in (0,1)$ such that for all f_n and all $\omega \in \Delta$.

$$\int_{\tilde{\Delta}_r} |f'_n|^2\, dm_{\beta,\omega,\phi} \le \sum_j sup_{z\in H_j \cap \Delta r}|f'_n(z)|^2 m_{\beta,\omega,\phi}\left(H_j \cap \Delta_r\right)$$

$$\leq \varepsilon^{2(1+\alpha)} M_5 \sum_j \left|f_n'\left(a_j^*\right)\right|^2 \left(1-\left|a_j\right|^2\right)^4 exp\left[-cM_6\left(1-\left|a_j\right|^2\right)^{2+\alpha}\right]$$

$$\leq \varepsilon^{2(1+\alpha)} M_7 \sum_j \int_{\Delta j} |f_n'(z)|^2(1-|z|^2)^2\, exp[-cM_8(1-|z|^2)^{2+\alpha}]dm(z)$$

$$\leq \varepsilon^{2(1+\alpha)} M_7 \sum_j \int_{Hj} [|f_n'(z)(1-|z|^2)]^2\, exp[-cM_8(1-|z|^2)^{2+\alpha}]dm(z)$$

$$\leq \varepsilon^{2(1+\alpha)} M_9 \int_{\Delta} exp[-|cM_8 - 4^{2+\alpha}M_0 s)\ (1-|z|^2)^{2+\alpha})] \quad dm(z).$$

Since (28) holds for all $c > 0$, it follows from picking $c > 4^{2+\alpha}$ s M_0 / M_s in the above estimates that

$$\int_{\tilde{\Delta}_r} |f_n'|^2\, dm_{\beta,\omega,\phi} < \varepsilon^{2(1+\alpha)} M_{10} \tag{32}$$

Also since $\phi \in \mathcal{Q}_\beta$, and $f_n' \to 0$uniformly on $\tilde{\Delta}_r$to the above ε and $r$there corresponds an integer $N > 0$ such that for $n \geq N$.

$$\int_{\tilde{\Delta}_r} |f_n'|^2\, dm_{\beta,\omega,\phi} < \varepsilon \|\phi\|_{\mathcal{Q}_\beta}^2 \tag{33}$$

Putting (29), (32) and (33) together produces that $\left\|C_\phi f_n\right\|_{\mathcal{Q}_{\beta,}} \to 0.$ as $n \to \infty$.

We present a $\mathcal{Q}_{\beta,0}$ version of Theorem (6.2.10).

Corollary (6.2.11)[236]: Let $\alpha \in (-1, \infty)$ $\beta \in (0, \infty)$ and let $\phi: \Delta \to \Delta$ and let be analytic. Then the following are equivalent:

(i) $C_\phi: \mathcal{N}_\alpha \to \mathcal{Q}_{\beta,0}$ exist as a bounded operator.

(ii) $C_\phi: \mathcal{N}_\alpha \to \mathcal{Q}_{\beta,0}$ exist as a compact operator.

(iii) $\phi \in \mathcal{Q}_{\beta,0}$ and (28) holds for all $c > 0$

Proof. It suffices to show (iii) $\Longrightarrow$ (ii) because (ii) $\Longrightarrow$ (i) is trivial and (i) $\Longrightarrow$ (iii) follows from Theorem (6.2.10). So let (iii) be true. Since the poly nomias are dense in $\mathcal{N}_\alpha$ and in $\mathcal{Q}_{\beta,0}$ (this is easily verified via the triangle inequality), if $f \in \mathcal{N}_\alpha$ then for every $\varepsilon > 0$ there is a polynomial P such that $\|f - P\|_{\mathcal{N}_\alpha} < \varepsilon$. Observe that (iii) asserts boundedness of $C_\phi: \mathcal{N}_\alpha \to \mathcal{Q}_\beta$. So, there is a constant $M > 0$ such that $\left\|C_\phi f - C_\phi P\ \right\|_{\mathcal{Q}_\beta} < \varepsilon\, M$ also since $\phi \in \mathcal{Q}_{\beta,0}$, it follws from the $\mathcal{Q}_{\beta,0}$ -version of Lemma (6.2.7) that $\phi^n \in \mathcal{Q}_{\beta,0}$ for every integer $n > 0$. As a result, $C_\phi P \ \in \mathcal{Q}_{\beta,0}$the triangle inequality and the density of the polynomials in $\mathcal{Q}_{\beta,0}$ yield $C_\phi P \ \in \mathcal{Q}_{\beta,0}$. In other words, $C_\phi maps\, \mathcal{N}_\alpha$ into $\mathcal{Q}_{\beta,0}$. Furthermore, the last part of the proof of Theorem (6.2.10) shows that $C_\phi: \mathcal{N}_\alpha \to \mathcal{Q}_\beta$ is compact, that is, (ii) holds.

Section (6.3): The Bloch Space into Möbius Invariant Spaces

By a self-map of the unit disk $\mathbb{D}$ we will mean an analytic function φ from the unit disk $\mathbb{D}$ into itself. Every self-map of $\mathbb{D}$ induces the composition operator C_φ with *symbol* φ by the formula $C_\varphi(f) = f \circ \varphi$ on the set of all analytic functions in $\mathbb{D}$ but it is often of interest to consider C_φ as an operator between Banach spaces of analytic functions. For several classical spaces of analytic functions such as a Hardy space H^p, a Bergman space A^P, or the Bloch space $\mathfrak{B}$, any symbol φ gives rise to a bounded operator $C_\varphi$from the space into itself. However, this is not the case for the Dirichlet space or for more general analytic Besov spaces B^p, so the question of deciding which φ induce a bounded operator C_φ is of interest. The situation becomes more complicated if we consider composition operator acting between two deferent spaces.

A related problem is to characterize all compact or weakly compact operators C_φ between two given spaces in terms of the symbols.

Criteria for compactness of $C_\varphi$when acting on Hardy and Bergman spaces (due to J.H. Shapiro and B. MacCluer) are now already considered a classical knowledge; see [12], [144]. For compact operators acting on $\mathfrak{B}$ and on the Little Bloch space B_0 see [19], [146], and [259]. Related results regarding composition operators from $\mathfrak{B}$ into the Dirichlet space D or the more general analytic Besov spaces B^p can be found in [227], [267], and [214]. Composition operators from $\mathfrak{B}$ into Hardy spaces were treated in [268] while those from the Bloch space into the conformally invariant subspaces *BMOA* and *V MOA* of Hardy spaces and other spaces were studied in [232] and [196]. For composition operators from $\mathfrak{B}$ into Q_P-type spaces see [198]. Weak compactness of composition operators on vector-valued versions of classical spaces of analytic functions have been considered in [257], for Example (6.3.9) (6.3.4).

Obviously, there are quite a few on the subject but it turns out that many similar setups are treated in an isolated way and many proofs are essentially repeated while it looks desirable to show the “bigger picture”. One purpose is precisely to treat such questions globally, for those C_φ that map the Bloch space into other spaces. We would like to underline that our work also provides new results in the case when the target space is one of the many rather classical Banach spaces.

We consider the spaces X which are Möbius invariant, *i.e.*, those whose seminorm s has the following property: $s(f \circ \sigma) \leq C\, s(f), f \in X,$ for some fixed constant C and all disk automorphisms σ. These spaces were given a systematic treatment in [215] which was also pioneering in the theory of composition operators acting on them. This family of spaces includes the Bloch space $\mathfrak{B}$, the Little Bloch space $\mathfrak{B}_0$, and analytic Besov spaces denoted B^P. We also mention the important spaces *BMOA* (a variant of the classical John-Nirenberg space *BMO*) and *V MOA* (introduced by Sarason; see [263]), both Möbius -invariant subspaces of the Hardy space H^2. The classical Hardy and Bergman spaces, however, do not satisfy the requirements for belonging to this family.

The question whether the weak compactness of a composition operator acting between two conformally invariant spaces of analytic functions is actually equivalent to its compactness has generated considerable interest among the experts. For the composition operators on *BMOA* or *V MOA* or between these spaces, this question was posed (in its deferent versions) by Bourdon, Cima, and Matheson [244], [258], by Laitila [264], and also by Tjani. An affirmative answer has been given recently by Laitila, Nieminen, Saksman, and Tylli [258], where they used some functional analysis tools such as the Bessaga-Pelczy´nski selection principle.

It is important to notice that there exist weakly compact composition operators acting on other function spaces which are not compact. An Example (6.3.9) (6.3.4) of such C_φ induced by a lense map φ, was given in [265].

The idea of considering the largest conformally invariant subspace of a given Banach space of analytic functions has already been considered. Two relevant sources are [256] and [275]. Significant motivation for the work comes from the approach adopted by Aleman and Simbotin (Persson); see [255] or [272].

We consider three fairly large families of spaces of analytic functions: the spaces D^p_μ defined in terms of integrability of the derivative of a function with respect to a certain Borel measure μ, their conform ally invariant subspaces $M\ (D^p_\mu)$, and the small subspaces $M_0(D^p_\mu)$. We defer their precise definitions which coincide with those given in [255] or [272]. These families include various types of well-known spaces:

(i) the Hardy space H^2 and all weighted Bergman and Dirichlet-type spaces,

(ii) their M$\ddot{o}$bius invariant subspaces such as *BMOA*, $\mathfrak{B}$, analytic Besov spaces, and Q_P spaces, and

(iii) the small subspaces of the above spaces such as *V MOA*, $\mathfrak{B}_0$ or $Q_{p,0.}$ It should also be remarked that families of "large" and "small" spaces defined by means of oscillation and density of polynomials in them (which is also discussed here) were considered in Perfekt's [269].

We present a unified approach to charac-terizing all bounded, compact, and weakly compact composition operators from $\mathfrak{B}$ into any of the spaces belonging to the family mentioned above. Our principal result shows that every weakly compact composition operator from $\mathfrak{B}$ into any space$M\ (D^p_\mu)$, is actually compact. We also generalize a number of existing but scattered results and add some new results. For instance, we characterize the compact and weakly compact operators from the Bloch space into the space *BMOA*. We do this by using a combination of complex analysis arguments and Banach space techniques.

Part of the motivation for our approach to compactness comes from Xiao's treatment [124].

First of all, we characterize completely and in terms of the hyperbolic derivative of the symbol φ all bounded and compact composition operators C_φ from the Bloch space $\mathfrak{B}$ into any of the general spaces $D^p_\mu, M\ (D^p_\mu), and\ M_0(D^p_\mu)$ considered. It turns out that when-ever $C_\varphi : \mathfrak{B} \to D^p_\mu\ or\ C_\varphi : \mathfrak{B} \to M_0(D^p_\mu)$, the compactness of C_ϕ follows "for free" (after some work).

Our Theorem (6.3.7) describes the compact composition oper-ators from $\mathfrak{B}$ into the invariant space $M\ (D^p_\mu)$ and shows that, in this case, weak compactness is equivalent to compactness. The proof is based on a theorem of Banach-Saks type from functional analysis and techniques from function spaces. The result is accompanied by appropriate Example (6.3.9) (6.3.4)'s.

Another relevant point (Theorem (6.3.13)) is a rigorous and detailed proof that, for all natural radial measures of certain type, the polynomials are dense in the small subspace $M_0(D^p_\mu)$ of the conform ally invariant space $M\ (D^p_\mu)$, in analogy with the classical cases. This provides a wide range of Examples (6.3.9), (6.3.4) where the separability hypothesis of our Theorem (6.3.15) is satisfied. This last result characterizes the bounded and compact composition operators from the Bloch space $\mathfrak{B}$ into the small spaces $M_0(D_\mu^p)$.

In what follows, $\mathbb{D}$ will denote the unit disk in the complex Plane: $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ *and* dA will denote the normalized Lebesgue area measure on $\mathbb{D}$:

$$dA(z) = \frac{1}{\pi}\, dx\, dy = \frac{1}{\pi}\, r\, dr\, d\theta, \quad z = x + yi = re^{i\theta.}$$

By a *disk automorphism*, we will mean a one-to-one analytic mapping of $\mathbb{D}$ onto itself. The set of all such maps, $Aut(\mathbb{D})$, is a transitive group under composition. As is well known, every $\sigma \in Aut(\mathbb{D})$ has the form

$$\sigma(z) = \lambda \frac{a - z}{1 - \bar{a}z}, \quad |\lambda| = 1, \quad |a| < 1. \tag{34}$$

An important property of disk automorphisms is that they yield equality in the Schwarz-Pick lemma:

$$(1 - |z|^2)|\sigma'(z)| = 1 - |\sigma(z)|^2, \qquad z \in \mathbb{D}. \tag{35}$$

We shall always consider a positive Borel measure μ on $\mathbb{D}$. A typical Example (6.3.9) (6.3.4) is

$$d\mu(z) = (1 - |z|^2)^\alpha dA(z),$$

a measure which is finite if and only if $-1 < \alpha < \infty$. Another Example (6.3.9) (6.3.4) is

$$d\mu(z) = log^\alpha \frac{1}{|z|}\, dA(z),$$

Note that for z near the unit circle the function $log^\alpha \frac{1}{|z|}$ behaves asymptotically like $(1 - |z|^2)^\alpha$. In principle, our measures are not assumed to be of the form $h(|z|)\, dA(z)$, where h is some integrable positive function on [0, 1) like in the above Example (6.3.9) (6.3.4)'s. However, the result will mostly be displayed for measures that satisfy this assumption.

We will use $\mathcal{H}(\mathbb{D})$ to denote the set of all functions analytic in $\mathbb{D}$. A function $f \in \mathcal{H}(\mathbb{D})$ is said to belong to the Bloch space $\mathfrak{B}$ if its invariant derivative:$(1 - |z|^2)|f'(z)|$ *is* bounded in $\mathbb{D}$. The name comes from the fact that this quantity does not change under a composition with any $\sigma \in Aut(\mathbb{D})$ in view of our formula (35). The Bloch space becomes a Banach space when equipped with the norm

$$\|f\|_{\mathfrak{B}} = |f(0)| + sup_{z\in\mathbb{D}}(1 - |z|^2)|f'(z)|.$$

every function in $\mathfrak{B}$ satisfies the standard growth condition :

$$|f(z)| \le \left(1 + \frac{1}{2}\log\frac{|1+z|}{|1-z|}\right)\|f\|_{\mathfrak{B}}, \qquad z \in \mathbb{D}. \tag{36}$$

Given a positive Borel measure μ on $\mathbb{D}$ and $p \in [1, \infty)$, we can define the, weighted Girichlet-types space D_μ^p in the usual way:

$$D_\mu^p = \left\{ f \in \mathcal{H}(\mathbb{D}): \|f\|_{D_\mu^p}^p := |f(0)|^p + \int_{\mathbb{D}} |f'|^p\, d\mu < \infty \right\}.$$

Consider the point evaluation functional φ_ζ *, defined by* $\varphi_\zeta(f) = f(\zeta)$ *, for* $\zeta \in \mathbb{D}$. It is natural to require the following axioms to hold:

D_μ^p is a Banach space;

The point-evaluation functional φ_ζ is bounded on D_μ^p for each $\zeta \in \mathbb{D}$.

In view of the uniform boundedness principle, these two requirements can be summarized in one single axiom:

The point-evaluation functional are uniformly bounded on D_μ^p on compact subsets of $\mathbb{D}$.

Following the notation used, for Example (6.3.9) (6.3.4), in [255], we define the *Möbius invariant subspace* $M\ (D_\mu^p)$ as the space of all functions $f\ in\ \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{M\ (D_\mu^p)}^p := \ |f(0)|^p + sup_{\sigma\in Aut(\mathbb{D})} \int\limits_{\mathbb{D}} |(f\circ\sigma)'|^p\ d\mu < \infty.$$

We also define the corresponding *Little invariant subspaces*:

$$M_0\big(D_\mu^p\big) = \left\{\mathrm{f}\in M\big(D_\mu^p\big) : \lim_{|\sigma(0)|\to 1, \sigma\in Aut(\mathbb{D})} \int\limits_{\mathbb{D}} |(f\circ\sigma)'|^p d\mu = 0\right\}.$$

A few remarks are in order:

It is routine to verify that $s(\mathrm{f}) \ = \ sup_{\sigma\in Aut(\mathbb{D})} \left(\int_{\mathbb{D}} |(f\circ\sigma)'|^p d\mu\right)^{1/p}$
Defines a seminorm on $M\big(D_\mu^p\big)$ and $\|.\|_{M(D_\mu^p)}$ has all the properties of a norm.

Since $\{\tau\ \circ\circ\sigma : \sigma\in Aut(\mathbb{D})\} = Aut(\mathbb{D})$ holds for any fixed $\tau\in Aut(\mathbb{D})$, it follows that the $s\ (f\circ\tau\ ) = s\ (f\ )$. In other words, this seminorm is conform ally invariant.

Since the identity map of $\mathbb{D}$ is trivially a disk automorphism, it is immediate that $M\big(D_\mu^p\big) \subset D_\mu^p$. It actually follows from our previous comment that $M\big(D_\mu^p\big)$ is the largest conformally invariant subspace of D_μ^p.

Note that we actually require that $f\in M\ \big(D_\mu^p\big)$is in the definition of $M_0\ \big(D_\mu^p\big)$is since it is not obvious, even for somewhat special measures
μ, that the assumption

$$\lim_{|\sigma(0)|\to 1, \sigma\in Aut(\mathbb{D})} \int\limits_{\mathbb{D}} |(f\circ\sigma)'|^p d\mu = 0$$

Implies that

$$sup_{\sigma\in Aut(\mathbb{D})} \int\limits_{\mathbb{D}} |(f\circ\sigma)'|^p d\mu < \infty.$$

Assuming the uniform boundedness of point evaluations in D_μ^p on compact subsets of $\mathbb{D}$, by a standard normal families argument and Fatou's lemma one can deduce the completeness of $M\ (D_\mu^p)$. It is easily checked that $M_0(D_\mu^p)$ is a closed subspace of $M\ (D_\mu^p)$, so it is also complete.

It is not difficult to see that each one of the spaces defined above contains sufficiently many functions for most "reasonable" measures μ. For Example (6.3.9) (6.3.4), if μ is a finite measure then every function analytic in a disk larger than $\mathbb{D}$ and centered at the origin is readily seen to belong to $M\ (D_\mu^p)$. We shall discuss the membership and density of the polynomials in $M_0\ (D_\mu^p)$.

In several only the involutive automorphisms are considered: $\sigma_a(z) = (a - z)/(1 - \bar{a}z),\ a\in\mathbb{D}, requiring\ that\ |a| \to 1$ in the definitions of the special small spaces. Here we have opted for the full generality and for considering the entire automorphism group, which adds certain technical difficulties to some proofs.

An appropriate choice of μ in the above definitions of our spaces

$D_\mu^p, M\ (D_\mu^p), and\ M_0(D_\mu^p)$ yields a number of well-known spaces of analytic functions in the disk as special cases. Here is a list of some important Example (6.3.9) (6.3.4)s.

(A) In view of the well-known Littlewood-Paley identity [144]:

$$\|f\|_{H^2}^2 := |f(0)|^2 + 2\int_{\mathbb{D}} |f'(z)|^2\ log\frac{1}{|z|} dA(z)\ ,$$

the Hardy space H^2 can be seen as a D_μ^2 space by choosing $d\,\mu\ (z)$ = log $\frac{1}{|z|^2}$ $dA(z)$. Its conformally invariant subspace $M\ (D_\mu^2)$ is the well-known *BMOA* space of analytic functions of bounded mean oscillation and the corresponding space $M_0(D_\mu^2)$ = *V MOA*, the space of functions of vanishing mean oscillation; see [263] for more about these space. It should be remarked that in this case our definition involving all possible disk automorphisms coincides with the usual one that takes into account only the involutive automorphisms σ_a mentioned above in view of rotation invariance of the measure μ.

(B) The analytic Besov spaces $B^p, 1 < p < \infty$, are obtained as D_μ^p spaces by choosing $d\mu(z) = (p-1)(1-|z|^2)^{p-2}dA(z), 1 < p < \infty$. See [215] or [154] for more about these spaces. Note that, in this case, combining the simple change of variable $w = \sigma(z), dA(w) = |\sigma'(z)|^2 dA(z)$ with (35) shows that

$$\int_{\mathbb{D}} |(f\circ\sigma)'(z)|^p d\mu(z) = \int_{\mathbb{D}} |f'(\omega)|^p\, d\mu(\omega).$$

So it is immediate that here while the corresponding space is trivial (consisting only of the constant functions).

(c) The Bergman spaces $A^p, 1 \le p \le \infty$, can be obtained by taking $d\mu(z) = (1-|z|^2)^p dA(z)$. Well- known (but too lengthy to repeat here) arguments using the Cauchy integral formula and Minkowski's inequality in its integral form as in [262] show that the norm in our definition is equivalent to the standard Bergman norm:

$$|f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p\,(1-|z|^2)^p dA(z) \asymp \int_{\mathbb{D}} |f(z)|^p\, dA(z).$$

(meaning that each of the two sides is bounded by a constant multiple of the other, this multiple being independent of f). In this case it turns out that $M\ (D_\mu^p) = \mathfrak{B}$ and $M_0(D_\mu^p) = \mathfrak{B}_0$, the Little Bloch space (the closure of polynomials in $\mathfrak{B}$), as was shown by Axler [256].

(D) The Q_α spaces, defined by Aulaskari, Xiao, and Zhao [216] and studied by other as well (see [124] for an extensive account), can be seen as $M\ (D_\mu^p)$ spaces by taking $p = 2, d\mu(z) = log^\alpha \frac{1}{|z|}\ dA(z), 0 < \alpha < \infty$. An equivalent norm is obtained by choosing $d\mu = (1-|z|)^\alpha\ dA(z)$ instead. (Note that we will use the notation $Q_\alpha$rather than the traditional Q_p because here $p = 2$ is fixed and the exponent α from the weight is the one that determines the space.) It is well known that Q_α coincides as a set with $\mathfrak{B}$ (but is, of course, endowed with a deferent norm) whenever $\alpha > 1$ and with *BMOA* when $\alpha = 1$, while it is an entirely deferent space when $0 < \alpha < 1$. The corresponding small space $M_0(D_\mu^p)$ is the space usually denoted as $Q_{\alpha,0}$ and

$$Q_{1,0} = V\ MOA.$$

The following lemma in the case $p = 1$ has been proved explicitly by Ramey and Ullrich

[121] although the argument can probably be traced back to Ahern and Rudin.

Lemma (6.3.1)[254]: Let $1 \leq p < \infty$. There exist two functions f and g in the Bloch space $\mathfrak{B}$ and a positive constant C such that

$$|f'(z)|^p + |g'(z)|^p \geq \frac{C}{(1-|z|^2)^p}$$

for all z in $\mathbb{D}$.

The proof follows by [121]. The key point is to select a partition of the disk into two sets of concentric annuli centered at the origin and two lacunary series, one of which takes on large enough values:

$$|f'(z)| \geq \frac{A}{1-|z|^2}$$

on the odd-numbered annuli and the other does the same on the even-numbered annuli. This takes care of the case $p = 1$, for arbitrary $p \geq 1$ the statement follows readily by the standard inequality $(a + b)^P \leq 2^P(a^p + b^p)$, where $a, b \geq 0$.

It will be convenient to use the following version of the hyperbolic derivative of an analytic self-map φof $\mathbb{D}$:

$$\varphi^{\#}(z) = \frac{|\varphi'(z)|}{1-|\varphi(z)|^2}$$

It should be noted that there is another related quantity also called hyperbolic derivative but only the above expression will be useful for our purpose.

We state two basic facts which characterize the bounded composition operators from $\mathfrak{B}$ $into\ D_\mu^p\ and\ into\ M\ (D_\mu^p)$ respectively. The proofs of such facts are relatively straightforward and have by now become standard. We record them here only for the sake of completeness.

Proposition (6.3.2)[254]: The following statements are equivalent:

(a) C_φ: B $\to D_\mu^p$;

(b) C_φ is bounded operator from B into D_μ^p;

(c) $D|\phi\#|p\ d\mu < \infty$.

Proof. It suffices to verify the following short chain of implications: (a) ⇒(c) ⇒(b) ⇒(a).

$[(a) \Rightarrow (c).]$Suppose that $f \circ \varphi \in D_\mu^p for\ each\ f\ in$ $\mathfrak{B}$. Choose two functions f, $g \in \mathfrak{B}$ and the constant $C > 0$ as in Lemma (6.3.1). Evaluate them at $\varphi(z)$ to get that

$$\frac{C}{(1-|\varphi(z)|^2)^p} \leq \left|f'(\varphi(z))\right|^p + \left|g'(\varphi(z))\right|^p, \quad z \in \mathbb{D}.$$

This yields

$$C\int_{\mathbb{D}} \frac{|\varphi'|^p}{(1-|\varphi|^2)^p} d\mu \leq \int_{\mathbb{D}} |f' \circ \varphi|^p\, |\varphi'|^p d\mu + \int_{\mathbb{D}} |g' \circ \varphi|^p\, |\varphi'|^p d\mu.$$

$$\leq \|f \circ \varphi\|_{D^p}^p + \|g \circ \varphi\|_{D^p}^p < \infty.$$

This proves.

Suppose that let f be an arbitrary function In $\mathfrak{B}$. Then

$$\left|f'(\varphi(z))\right|(1-|\varphi(z)|^2) \leq \|f\|_{\mathfrak{B}}$$

for every $z \in \mathbb{D}$. This readily implies that

$$\int_{\mathbb{D}} |f' \circ \varphi|^p\, |\varphi'|^p d\mu \leq$$

$$\int_{\mathbb{D}} \left|\varphi^{\#}\right|^p \ d\mu \cdot \|f\|_{\mathfrak{B}}^p < \infty.$$

Also, from the growth estimate (36) we obtain

$$\left|f(\varphi(0))\right| \leq \left(1+\frac{1}{2}\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)\|f\|_{\mathfrak{B}}.$$

By summing up the last two inequalities, it follows that C_{φ} is bounded as an operator from $\mathfrak{B}$ *into* $M\,(D_\mu^p)$.

[(b) ⇒ (a)]is trivial.

Proposition (6.3.3)[254]: The following statements are equivalent:

(a) $C_\varphi : \mathfrak{B} \to M\,(D_\mu^p)$;

(b) C_φ is a bounded operator from $\mathfrak{B}$ into $M\,(D_\mu^p)$;

(c) $\sup_{\sigma\in Aut(D)} \int_{\mathbb{D}} \left|(\varphi\circ\sigma)^{\#}\right|^P d\mu < \infty$.

Proof. The proof can be worked out along the same lines as that of Proposition (6.3.2), with the necessary modifications.

It is important to make sure that we are not dealing with trivial situations by displaying Example (6.3.9) that work in a large number of cases. Here is a very simple Example (6.3.9) showing that very simple symbols may or may not yield bounded composition operators from $\mathfrak{B}_0$ our spaces.

Example (6.3.4)[254]: Let $1 \leq p < \infty, d\mu(z) = (1-|z|^2)^\alpha\, dA(z), and\ let\ \varphi(z) \equiv z$.

Then the following statements are equivalent:

(a) C_φ is bounded as an operator from $\mathfrak{B}$ into D_μ^p.

(b) C_φ is bounded as an operator from $\mathfrak{B}$ into $M(D_\mu^p)$.

(c) $p-\alpha < 1$.

The case of $M(D_\mu^p)$ is slightly more involved, but still easy to check, in view of the identity (35):

$$\int_{\mathbb{D}} \left|(\varphi\circ\sigma)^{\#}\right|^p d\mu = \int_{\mathbb{D}} \left|\sigma^{\#}(z)\right|^p d\mu(z) = \int_{\mathbb{D}} \frac{d\mu(z)}{(1-|z|^2)^p} = \int_{\mathbb{D}} \frac{dA(z)}{(1-|z|^2)^{p-\alpha}}.$$

Trivial integration in polar coordinates shows that the last integral converges only for the range indicated in (c). (Note that this is really a statement about the containments $\mathfrak{B} \subset M(D_\mu^p)$ but is at the same time an Examples (6.3.9), (6.3.4) for composition operators.)

The case of bounded operators from $\mathfrak{B}$ into the Little Möbius invariant subspaces $M^0(D_\mu^p)$ will be considered together with the compactness question. This is done because the two turn out to be equivalent and the proof requires other results to be obtained first.

We probably noticed some differences in the formulation of the results and those by other pertaining to the cases like *BMOA* or Q_α. The reason for this is very simple: these spaces are obtained in the special case $p = 2$ when some of our results above can be rewritten in a different language.

To this end, denote by N_φ the counting function of φ:

$$N_\varphi(\omega) = |\{z \in \mathbb{D} : \varphi(z) = \omega\}|,$$

understanding 0, 1, 2,. . . ,∞ as its possible values. Let us also agree to write A_h for the hyperbolic area of a subset of the disk:

$$A_h(S) = \int_S \frac{dA(z)}{(1-|z|^2)^2.}$$

Lemma (6.3.5)[254]: For arbitrary positive measure μ, we have

$$\frac{|\varphi'|^2}{(1-|\varphi|^2)^2} = \Delta \log \frac{1}{1-|\varphi|^2}. \tag{37}$$

When $d\mu = dA$, we also have

$$\int_{\mathbb{D}} \frac{|\varphi'|^2}{(1-|\varphi|^2)^2}\, dA = \int_{\mathbb{D}} N_\varphi dA_h, \tag{38}$$

Formula (37) is a simple consequence of the identity $\Delta(u \circ \varphi) = (\Delta u \circ \varphi)|\varphi'|^2$ while (38) follows from the well-known formula for non-univalent change of variable (see [144] or [12]).

Taking into account the equivalent forms of writing $|\varphi\#|$ from Lemma (6.3.5), it becomes obvious how the condition (c) in Proposition (6.3.2) and Proposition (6.3.3) can be rewritten. For Example (6.3.9) (6.3.4), in two special cases we could state our Proposition (6.3.2) or Proposition (6.3.3) as follows:

For arbitrary μ, the composition operator C_φ is bounded from $\mathfrak{B}$ into $M(D_\mu^p)$ if and only if

$$sup_{\sigma \in Aut(\mathbb{D})} \int_{\mathbb{D}} \Delta\, log \frac{1}{1-|\varphi \circ \sigma|^2} d\mu < \infty.$$

C_φ is bounded from $\mathfrak{B}$ into D_A^2 (A being the area measure) if and only if

$$sup_{\sigma \in Aut(\mathbb{D})} \int_{\mathbb{D}} N_{\varphi\circ\sigma}\, dA_{h.} = \int_{\mathbb{D}} N_\varphi dA_h < \infty.$$

in view of conformal invariance.

Recall that a bounded linear operator between two Banach spaces is said to be *compact* if it takes bounded sets into sets whose closure is compact; equivalently, if for every bounded sequence in the space the sequence of images has a convergent subsequence in the norm topology. A bounded operator is *weakly compact* if it takes bounded sets into sets whose closure is weakly compact; equivalently, if for every bounded sequence in the space the sequence of images has a subsequence that converges in the weak topology. Compactness obviously implies weak compactness.

We will now show that every composition operator from the Bloch space B into any of our spaces $M(D_\mu^p)$ is compact if and only if it is weakly compact and will also give a characterization of this property in terms of the symbol φ which unifies all previously obtained results for concrete spaces. In the special case of composition operators from $\mathfrak{B}$ to Q_α, the equivalence of (a) and (c) in Theorem (6.3.7) below has been proved before by Smith and Zhao [198]; see also [275] or [273]. However, weak compactness was not considered in these works.

The main novelty of our approach consists of the use of certain techniques usually employed by the experts in Banach space theory, the main one being a version of the Banach-Saks theorem. We formulate below the statement needed as a lemma but remark that it proof relies on some rather non-trivial results. It should be observed that the lemma is no longer true (even for composition operators) if we only assume boundedness of the operator.

Lemma (6.3.6)[254]: Suppose that T is a weakly compact operator from $\mathfrak{B}$ into an arbitrary

Banach space Y . Then every bounded sequence $(f_n)_n$ in $\mathfrak{B}$ has a subsequence $(f_{n_k})_k$ such that the arithmetic means of the images $T f_{n_k}$ converge to some element in the norm of Y.

Proof. Recall that the Bloch space is isomorphic to the space of all bounded complex sequences l^∞; see Lusky's [266]. On the other hand, l^∞ is a unital commutative C^*-algebra endowed with the usual operations of coor-dinatewise multiplication and conjugation. The Gelfand-Naimark Theorem (see [271] where uses the term B^* −algebra instead) now implies that l^∞ is isomorphic to a space of continuous functions on its maximal ideals (which is a compact Hausdorff space by [271]). Thus, we are allowed to apply a Banach-Saks type theorem proved in 1979 by Diestel - Seifert and (see [260]) which establishes that any weakly compact linear operator from a space of continuous functions on a compact Hausdorff space into an arbitrary Banach space has the Banach-Saks property.

Alternatively, we could have deduced the statement from a more general result of $Jarchow$ referring directly to the $C^* - algebras$ (see also [260]).

Note that the measure μ in the theorem below is not required to be of any special form. In particular, it need not be finite.

For the sake of brevity, throughout the rest we will write simply $\{|\varphi \circ \sigma| > r\}$ to denote the set $\{z \in \mathbb{D} : |(\varphi \circ \sigma)(z)| > r\}$.

Theorem (6.3.7)[254]: Let $1 \le p < \infty$. Suppose that C_φ is a bounded operator from $\mathfrak{B}$ to $M(D_\mu^p)$. Then the following statements are equivalent:

(a) C_φ is a compact operator $\mathfrak{B}$ to $M\left(D_\mu^p\right)$.

(b) C_φ is a weakly compact operator from $\mathfrak{B}$ to $M\left(D_\mu^p\right)$.

(c) $lim_{r\to 1}\, sup_{\sigma\in Aut(\mathbb{D})} \int_{\{|\phi\circ\circ\sigma|>r\}} |(\varphi \circ \circ \ \sigma)^\#|\,^p d\mu = 0.$

Proof. We proceed to prove the statement by proving the implications (c) ⇒ (a) ⇒ (b) ⇒ (c).

[(c) ⇒ (a).] Suppose (c) holds. It is clear that if $(f_n)_n$ is a bounded sequence in the Bloch space that converges to zero uniformly on compact sets, then $lim_{n\to\infty} f_n(\varphi(0)) = 0$. Thus, let us concentrate on the second term that appears in the norm. Fix an arbitrary $\varepsilon > 0$. Then there exists $r_0 \in (0, 1)$ such that

$$sup_{\sigma\in Aut(\mathbb{D})} \int_{\{|\phi\circ\circ\sigma|>r_0\}} |(\varphi \circ \circ \ \sigma)^\#|\,^p d\mu < \frac{\varepsilon}{2^{p+1}}. \tag{39}$$

Let $(f_n)_n$ be an arbitrary sequence in $\mathfrak{B}$ $with\ \|f_n\|_{\mathfrak{B}} \le 1\ for\ all\ n$. By a normal families argument, there exists a subsequence which we denote by $(g_n)_n$ which converges uniformly on compact sets to an analytic function g. From

$$|g_n'(z)| \le \frac{\|g_n\|_{\mathfrak{B}}}{1-|z|^2} \le \frac{1}{1-|z|^2}, \quad z \in \mathbb{D}.$$

It readily follows that g enjoys the same estimate , hence $g \in \mathfrak{B}$ and $\|g\|_{\mathfrak{B}} \le 1$. Moreover, we also have that $|g_n'(\omega) - g'(\omega)| \le \frac{2}{1-|\omega|^2}$ for all ω in $\mathbb{D}$, hence

$$|(g_n - g)' \circ (\varphi \circ \sigma)|^p |(\varphi \circ \sigma)'|^p \le 2^p \left|(\varphi \circ \sigma)^\#\right|^p \tag{40}$$

holds throughout $\mathbb{D}$. In order to show that $C_\varphi(g_n) \to C_\varphi(g)$ in the $M\left(D_\mu^p\right)$ norm, we need to show that the integrals

$$\int_{\mathbb{D}} |(g_n - g)' \circ (\varphi \circ \sigma)|^p \ |(\varphi \circ \sigma)'|^p \, d\mu.$$

are uniformly small independently of σ as $n \to \infty$. $for\ this\ purpose$, is convenient to split the above integral into two (omitting the integrals below):

$$\int_{\mathbb{D}} = \int_{\{|\varphi\circ\sigma|\le r_0\}} + \int_{\{|\varphi\circ\sigma|> r_0\}} . \tag{41}$$

By assumption, C_φ is bounded from B to $M\left(D_\mu^p\right)$ so in view of Proposition (6.3.3), we have

$$sup_{\sigma\in Aut(\mathbb{D})} \int_{\mathbb{D}} |(\varphi \circ \circ \ \sigma)^{\#}|^{\,p} d\mu \ \le M..$$

for some fixed positive constant M. Given $\varepsilon > 0$, by virtue of the uniform convergence: $(g_n \ - \ g)' \ \to \ 0$ on the compact set $\{z \ : \ |z| \ \le \ r_0\}$, for large enough n we have

$$\int_{\{|\varphi\circ\sigma|\le r_0\}} |(g_n - g)' \circ (\varphi \circ \sigma)|^p \, |(\varphi \circ \sigma)'|^p \, d\mu$$

$$\le \int_{\{|\varphi\circ\sigma|\le r_0\}} |(g_n - g)' \circ (\varphi \circ \sigma)|^p \frac{|(\varphi \circ \sigma)'|^p}{1 - |(\varphi \circ \sigma)^2|^p} d\mu$$

$$< \frac{\varepsilon}{2M} \int_{\{|\varphi\circ\sigma|\le r_0\}} \frac{|(\varphi \circ \sigma)'|^p}{1 - |(\varphi \circ \sigma)^2|^p} d\mu \le \frac{\varepsilon}{2}.$$

Thus, for n sufficiently large, the first integral in (41) can be made smaller than $< \varepsilon/2$.

The second integral in (41) is smaller than $\varepsilon/2$ in view of the inequalities (39) and (40). This implies that

$$\left\|C_\varphi(g_n) - C_\varphi(g)\right\|_{M\left(D_\mu^p\right)} < \varepsilon.$$

for n large enough, as asserted.

(a) $\Longrightarrow$ (b) is obvious.

(b) $\Longrightarrow$ (c) is the most intricate part of the proof. We follow the steps indicated below.

<u>Step 1</u>: We first show that the weak compactness assumption on C_φ implies

$$\lim_{r\to 1} \ sup_{\sigma\in Aut(\mathbb{D})} \int_{\{|\varphi\circ \ \circ \ \sigma|>r\}} |(\varphi \circ \circ \ \sigma)'|^{\,p} d\mu = 0. \tag{42}$$

This condition alone is apparently much weaker than (c). However, we will eventually show that, together with the weak compactness of C_φ it actually implies the desired condition (c). Thus, suppose that C_φ is weakly compact from B into $M\left(D_\mu^p\right)$ but (42) does not hold. Then we can find a positive number δ, an increasing sequence $(\rho_j)_j$ of numbers in (0, 1) such that $\lim_{n\to\infty} \rho_j = 1$, and a sequence of disk automorphisms $(\tau_j)_j$ such that

$$\int_{\{|\varphi\circ\circ\ \tau_j|>\rho_j\}} |(\varphi\circ\circ\ \tau_j)'|^p d\mu \geq \delta. \quad (43)$$

For a positive integer k, let us agree to write

$$C_k = \frac{k(k+1)}{2}$$

Next, choose recursively a subsequence $(m_n)_n$ of the integers in such a way that

$$m_0 = 1\ , m_n > C_{m_{n-1}} + n.$$

Once the sequence $(m_n)_n$ has been fixed, let us choose the subsequence (r_n) of the sequence $(\rho_j)_j$ so that

$$m_n r_n^{m_n-1} > C_{m_{n-1}} + n, \qquad n \geq 1$$

This is possible since $\lim_{r\to 1} m_n r_n^{m_n-1} = m_n$. Note that then

$$m_n r_n^{m_n-1} > C_{m_{n-1}} + n \geq m_{n-1} + n > m_{n-1} \quad (44)$$

Also, let us choose the subsequence $(\sigma_n)_n\ of\ (\tau_j)_j$ with the same indices as those of $(r_n)_n$ with respect to $(\rho_j)_j$.

By applying Lemma (6.3.6) to our weakly compact operator $C_\varphi : B \to M\left(D_\mu^p\right)$ and observing that the sequence $(z^{m_n})_n$ is bounded in the Bloch space, we conclude that there exists a subsequence $(m_{n_k})_k$ of $(m_n)_n$ for which the arithmetic means

$$\frac{1}{N}\sum_{k=1}^{N} \varphi^{m_{n_k}}$$

converge in the norm of $M\left(D_\mu^p\right)$. They actually must tend to zero since they converge to zero uniformly on compact sets. Hence,

$$\left\|\frac{1}{N}\sum_{k=1}^{N}\varphi^{m_{n_k}}\right\|_{M(D_\mu^p)}^p \geq sup_\sigma \int_{\mathbb{D}} \left|\frac{1}{N}\sum_{k=1}^{N} m_{n_k}\left(\varphi^{m_{n_k}-1}\circ\sigma\right)\right|^P |(\varphi\circ\circ\ \sigma)'|^p d\mu.$$

Hence

$$\lim_{N\to\infty} sup_\sigma \int_{\mathbb{D}} \left|\frac{1}{N}\sum_{k=1}^{N} m_{n_k}(\varphi^{m_{n_{k-1}}}\circ\sigma)\right|^P |(\varphi\circ\circ\ \sigma)'|^p d\mu = 0.$$

$$sup_\sigma \int_{\mathbb{D}} \left|\frac{1}{N}\sum_{k=1}^{N} m_{n_k}(\varphi^{m_{n_{k-1}}}\circ\sigma)\right|^P |(\varphi\circ\circ\ \sigma)'|^p d\mu < \varepsilon.$$

$$\int_{\{|(\varphi\circ\circ\ \sigma_{n_N})|>r_{n_N}\}} \left|\frac{1}{N}\sum_{k=1}^{N} m_{n_k}\left(\varphi^{m_{n_k}-1}\circ\sigma_{n_N}\right)\right|^P \left|\left(\varphi\circ\circ\ \sigma_{n_N}\right)'\right|^p d\mu < \varepsilon. \quad (45)$$

For an arbitrary but fixed z such that $|(\varphi\circ\circ\ \sigma_{n_N})(z)| > r_{n_N}$, let us use the shorthand $x = |(\varphi\circ\circ\ \sigma_{n_N})(z)|$. Then, using the triangle inequality for complex numbers, the obvious inequalities $r_{n_N} < x < 1$, the elementary identity for the sum of the first $N-1$ positive integers, and (44), together with the fact that the sequence $(C_k)_k$ is increasing and the obvious inequalities $m_{n_{N-1}} \geq m_{n_{N-1}}$ and $n_N \geq N$, it follows that

$$\frac{1}{N}\left|\sum_{k=1}^{N} m_{n_k}(\varphi^{m_{n_{k-1}}}\circ\sigma_{n_N})(z)\right| \geq \frac{1}{N}\left(m_{n_k}\, x^{m_{n_{N-1}}} - \sum_{k=1}^{N} m_{n_k} x^{m_{n_{k-1}}}\right)$$
$$\geq \frac{1}{N}\left(m_{n_k}\, x^{m_{n_{N-1}}} - \sum_{j=1}^{m_{n_{N-1}}} j x^{j-1}\right) \geq \frac{1}{N}\left(m_{n_k}\, x^{m_{n_{N-1}}} - \sum_{j=1}^{m_{n_{N-1}}} j\right)$$
$$\geq \frac{1}{N}\left(m_{n_k}\, x^{m_{n_N}-1} - \frac{m_{n_{N-1}}(m_{n_{N-1}}+1)}{2}\right) \geq \frac{1}{N}\left(m_{n_k}\, x^{m_{n_N}-1} - C_{m_{n_{N-1}}}\right)$$
$$\geq \frac{1}{N}\left(m_{n_k}\, x^{m_{n_N}-1} + n_N - C_{m_{n_{N-1}}}\right) \geq \frac{1}{N} n_N \geq 1.$$
Together with (12) , this yields

$$\int_{\{|(\varphi\circ\sigma_{n_N})|>r_{n_N}\}} \left|(\varphi\circ\sigma_{n_N})'\right|^P d\mu < \varepsilon.$$

Since this must hold for an arbitrary choice of ε, it contradicts our assumption (43). This completes the proof that (42) holds.

Step 2: Next, we show that the above condition (42), together with the weak compactness of C_φ, implies the following condition

$$\lim_{r\to 1}\ sup_{\sigma\in Aut(\mathbb{D})} \int_{\{|\varphi\circ\ \circ\ \sigma|>r\}} |(f\circ\varphi\circ\ \circ\ \sigma)'|\ ^p d\mu = 0, \quad f\in\mathfrak{B}. \tag{46}$$

For any constant function the above condition is trivially fulfilled so let f be an arbitrary but fixed non-constant function in *B*. Pick an increasing sequence (r_n) convergent to 1. Let us agree to denote by f_r the dilations of *f* defined in the usual way:

$$f_r(z) := f(rz), \qquad 0<r<1. \tag{41}$$

In view of the obvious inequality:

$$(1-|z|^2)r_n|f'(r_n z)| \leq (1-|r_n z|^2)|f'(r_n z)| \leq \|f\|_{\mathfrak{B}}.$$

the sequence $(f_{r_n}\ )$ is a bounded in the Bloch space. Also, it converges to *f* uniformly on compact sets. Since the operator $C_\varphi : B \to M\left(D_\mu^p\right)$ has the Banach-Saks property (in reality, it s$u$ffices to use the fact that it is weakly compact), there exists a subsequence of (r_n), denoted in the same way by an abuse of notation, such that

$$\lim_{N\to\infty}\left\|\frac{1}{N}\sum_{k=1}^{N}(f_{r_k}\circ\varphi) - (f\circ\varphi)\right\|_{M(D_\mu^p)} = 0.$$

Our axiom on boundedness of the point evaluations on D_μ^p implies uniform convergence of $\frac{1}{N}\sum_{k=1}^{N}(f_{r_k}\circ\varphi) to\ (f\circ\varphi)$. We are, of course, not interested in the trivial case when the symbol φ is a constant function. Since neither of the functions f and φ is identically constant, the same is true of the function $f\circ\ \varphi$. Thus, we may select a further subsequence, denoted again by (r_n) in order not to burden the notation, so that $\frac{1}{N}\sum_{k=1}^{N}(f_{r_k}\circ\varphi)$ is not identically constant, and since is not ide4ntically constant, this implies

$$\left\|\frac{1}{N}\sum_{k=1}^{N} f'_{r_k}\right\|_{H^\infty} \neq 0.$$

Given $\varepsilon > 0$, there exists a positive integer N such that

$$\left\| \frac{1}{N} \sum_{k=1}^{N} (f_{r_k} \circ \varphi) - (f \circ \varphi) \right\|_{M(D_\mu^p)} < \frac{\varepsilon}{2}.$$

Moreover, by (42) there exists $r_0 \in (0,1)$ such that if $r_0 \le r < 1$ then

$$sup_\sigma \int_{\{|\varphi \circ \circ \sigma| > r\}} |(\varphi \circ \circ \sigma)'| d\mu < \frac{\varepsilon}{2 \left\| \frac{1}{N} \sum_{k=1}^{N} f'_{r_k} \right\|_{H^\infty}}.$$

Hence for $r_0 \le r < 1$ and for every disk automorphism σ we have

$$\int_{\{|\varphi \circ \circ \sigma| > r\}} |(f \circ \varphi \circ \sigma)'| d\mu$$

$$\le \int_{\{|\varphi \circ \circ \sigma| > r\}} \left| \left(\frac{1}{N} \sum_{k=1}^{N} f_{r_k} \circ \varphi \circ \sigma \right)' - (f \circ \varphi \circ \sigma)' \right| d\mu$$

$$+ \int_{\{|\varphi \circ \circ \sigma| > r\}} \left| \left(\frac{1}{N} \sum_{k=1}^{N} f_{r_k} \circ \varphi \circ \sigma \right)'^{-} \right| d\mu$$

$$\le \frac{\varepsilon}{2} + \int_{\{|\varphi \circ \circ \sigma| > r\}} \left| \left(\frac{1}{N} \sum_{k=1}^{N} f'_{r_k} \circ \varphi \circ \sigma \right) (\varphi \circ \sigma)' \right| d\mu$$

$$\le \frac{\varepsilon}{2} + \left\| \frac{1}{N} \sum_{k=1}^{N} f'_{r_k} \right\|_{H^\infty} \int_{\{|\varphi \circ \circ \sigma| > r\}} |(\varphi \circ \sigma)'| d\mu \le \varepsilon.$$

Taking the supremum over all automorphisms σ, we obtain (46).

Step 3: Finally, in order to see that our condition (46) implies

$$\lim_{r \to 1} sup_{\sigma \in Aut(\mathbb{D})} \int_{\{|\varphi \circ \circ \sigma| > r\}} |(\varphi \circ \circ \sigma)^{\#}|^{p} d\mu = 0,$$

which is (c), it *suffices* to recall Lemma (6.3.1): there exist functions f and g in B such that

$$|f'(z)|^p + |g'(z)|^p \ge \frac{C}{(1 - |z|^2)^p} \quad , \quad z \in \mathbb{D}.$$

By applying this inequality at the point $\varphi(\sigma(z))$ instead of z and then using (46), we see that (c) follows immediately.

Example (6.3.8)[254]: Let $1 \le p < \infty$ and let μ be an arbitrary measure (not necessarily finite) that satisfies our axioms. Then for any analytic symbol φ such that $\varphi(\mathbb{D})$ is a compact subset of $\mathbb{D}$, the operator C_φ is compact (equivalently, weakly compact) from B into M (D_μ^p). Indeed, condition (c) in Theorem (6.3.7) is trivially verified. An obvious Examples (6.3.9), (6.3.8) is $\varphi(z) = az + b$ with $|a| + |b| < 1, a \neq 0$.

It should be made clear that not every bounded composition operator from B into M (D_μ^p) will be compact so the above theorem describes a non-trivial situation. Here is a very simple Examples (6.3.9), (6.3.8).

Example (6.3.9)[254]: Let $d\mu = (1 - |z|^2)^p \, dA(z), 1 \le p = \alpha < \infty$. Then the

symbol $\varphi(z) = (1 + z)/2$ induces a bounded composition operators from B into M (D_μ^p) which is not compact. Indeed, recall first that in this case M (D_μ^p) = B and that every self-map of the disk induces a bounded com-position operator on B. On the other hand, our operator is not compact because the sequence $(z^n)_n$ is bounded in B and converges to zero uniformly
On compact subsets of the unit disk but evaluation at the points

$$\frac{n-1}{n+1}\ yeilds \left\|C_\varphi(z^n)\right\|_{\mathfrak{B}} \geq sup_{z\in\mathbb{D}}(1-|z|^2)\left|\left(\left(\frac{1+z}{2}\right)^n\right)'\right| \geq 2$$

$$\left(\frac{n}{2}\right)^{n+1} \to \frac{2}{e} \neq 0,$$

Related Example (6.3.9)s of this kind in the specific context of operators from into of $\mathcal{Q}_\alpha$ type can be found in [258] or [198].

Corollary (6.3.10)[254]: Suppose that the operator $C_\varphi : B \to M\left(D_\mu^p\right)$ is bounded. Then the following assertions are equivalent.

(i) $C_\varphi : B \to M\left(D_\mu^p\right)$ Is not compact.

(ii) There exist a subspace X of B isomorphic to I^∞ sucgh that the restriction of C_φ to X is an isomorphism.

Proof. It suffices to apply the fact already mentioned that B is isomorphic to I^∞ as a Banach space and the following result (ii): An operator T defined in I^∞ is not weakly compact if and only if there exists a subspace X of I^∞isomorphic to I^∞ and such that is an isomorphism.

It is natural to ask whether there is a version of Proposition (6.3.2) for

Compact operators into D_μ^p such a result can be easily proved by following and simplyifying an easy part of the proof of Theorem (6.3.7). It also shows that in this case Example (6.3.9) like the last one are not possible,

Theorem (6.3.11)[254]: Let $1 \leq p < \infty$. If the composition operator C_φ is bounded from B to D_μ^p it is also compact

Proof: Let C_φ be bounded from B into D_μ^p. By Proposition (6.3.2), we know that

$$\int_{\mathbb{D}} \left|\varphi^\#\right|^p d\mu < \infty.$$

Let $(r_n)_n$ be an increasing sequence with $\lim_n r_n = 1$ and define

$$\boldsymbol{\Omega_n} = \{z \in \mathbb{D}: |\varphi(z)| \leq r_n\}$$

Note that $\boldsymbol{\Omega_n}$ is an ascending chain in the sense of inclusion whose union is $\mathbb{D}$. Denoting by $\mathcal{X}\Omega_n$ the characteristic function of $\boldsymbol{\Omega_n}$, it is clear that $\left|\boldsymbol{\varphi}^\#\right|^{\boldsymbol{p}} \mathcal{X}\Omega_n$converges to $\left|\boldsymbol{\varphi}^\#\right|^{\boldsymbol{p}}$ pointwise and $\left|\boldsymbol{\varphi}^\#\right|^{\boldsymbol{p}} \mathcal{X}_{\boldsymbol{\Omega_n}}$. by the Lebesgue domainated convergence theorem, we have

$$\lim_{n\to\infty} \int_{\Omega_m} \left|\varphi^\#\right|^p d\mu = \lim_{n\to\infty} \int_{\mathbb{D}} \left|\varphi^\#\right|^p\ \mathcal{X}\Omega_n d\mu = \int_{\mathbb{D}} \left|\varphi^\#\right|^p\ d\mu.$$

This shows that

$$\lim_{n\to\infty} \int_{\{|\varphi|>r_n\}} \left|\varphi^\#\right|^p d\mu = 0.$$

Now one can just retrace the steps of the proof of the *implication* $(c) \Rightarrow (a)$ in Theorem (6.3.7) and simplify them (without taking the supremum and working only with the identity automorphism) to see that the last condition is *sufficient* for compactness of C_φ.

We characterize the bounded and compact operators from B into the small spaces $M_0(D_\mu^p)$. The proof of this characterization will require some "obvious" properties such as separability of $M_0(D_\mu^p)$which are well known to hold in the classical "Little spaces" like VMOA, B_0, and $Q_{\alpha,0}$. For Example (6.3.9), this property is fulfilled whenever the polynomials are dense in the space. However, in our general context separability has to be checked and it turns out that a complete and rigorous proof of this fact is somewhat involved.

In what follows, we shall typically (but not exclusively) consider positive measures μ of the form $d\mu(z) = h(|z|)\, dA(z), where\ h \in [0,1) \to [0,\infty)$ is an integrable function. Moreover, we shall assume that there exist positive constants α and C such that

$$h(|\sigma(z)|) \le Ch(|z|)|\sigma'(z)|^\alpha \tag{48}$$

for all $z \in \mathbb{D}\ and\ all\ \sigma \in Aut(\mathbb{D})$. Then the induced measure μ is finite. We will refer to such μ as the *radial measure induced by h.* We remind the reader that the definitions of all classical conformally invariant spaces involve measures of this type.

Let us agree to write

$$h_\sigma := \quad (h \circ |\sigma|)|\sigma'|^{2-p} \tag{49}$$

Using the standard change of variable: $z = \sigma(\omega), dA(z) = |\sigma'(\omega)|^2 dA(\omega)$ it is easy to verify the identity

$$\int_{\mathbb{D}} |(f\circ\sigma^{-1})'(z)|^p\, d\mu(z) = \int_{\mathbb{D}} |f'(\omega)|^p\ h\circ(\omega)dA(\omega) \tag{50}$$

for every function f in $M(D_\mu^p)$.

The first natural question is: when does $M_0(D_\mu^p)$. contain the polynomials?

Proposition (6.3.12)[254]: Let μ be a radial measure induced by an integrable, non-negative, radial function h. Then the following statements are equivalent:

(a) The identity function, given by $\mathrm{f}(\mathrm{z}) = \mathrm{z}$, belongs to $\mathrm{M}_0(\mathrm{D}_\mu^p)$.

(b) All polynomials belong to $\mathrm{M}_0(\mathrm{D}_\mu^p)$.

(c) The following two conditions hold simultaneously

$$\sup_\sigma \int_{\mathbb{D}} \mathrm{h}\circ(\omega)\mathrm{dA}(\omega) < \infty, \tag{51}$$

$$\lim_{\sigma\in Aut(\mathbb{D}),|\sigma(0)|\to 1} \int_{\mathbb{D}} h_\sigma(\omega)dA(\omega) = 0.$$

Proof. Formula (50) readily that implied the identity function, given by f (z) = z belong to $M_0(D_\mu^p)$ if and only if and (51) holds**.**

Trivially, if all polynomials belong to $M_0\left(D_\mu^p\right)$ then so does $f(z) = z$.

If the identity is in $M_0\left(D_\mu^p\right)$ then (51) holds, and choosing $f(z) = z^n$ we get

$$sup_\sigma \int_{\mathbb{D}} |(f\circ\sigma^{-1})'|^p\, d\mu = sup_\sigma \int_{\mathbb{D}} |f'|^p\, h_\sigma dA \le n^p sup_\sigma \int_{\mathbb{D}} h_\sigma dA < \infty.$$

and

$$\lim_{|\sigma(0)|\to 1} \int_{\mathbb{D}} |(f\circ\sigma^{-1})'|^p \, d\mu = \lim_{|\sigma(0)|\to 1} \int_{\mathbb{D}} |f'(\omega)|^p \, h_\sigma(\omega) dA(\omega)$$

$$\le n^p \lim_{|\sigma(0)|\to 1} \int_{\mathbb{D}} h_\sigma(\omega) dA(\omega) = 0.$$

It is easy to see that $|\sigma^{-1}(0)| \to 1$ if and only if $|\sigma(0)| \to 1$. Recalling also the obvious fact that $\{\sigma^{-1} : \sigma \in Aut(\mathbb{D})\} = Aut(\mathbb{D})$, the statement follows.

Theorem (6.3.13)[254]: Let μ be a radial measure induced by an integrable, non-negative, radial function h that satisfies (48); suppose also that the identity function belongs to $M_0\left(D_\mu^p\right)$. Let the dilations f_r be defined as in (47). Then the following statements are equivalent:

(a) The function f belongs to $M_0\left(D_\mu^p\right)$;

(b) $\lim\limits_{r\to 1} \|f - f_r\|_{M(D_\mu^p)}$

(c) f belongs to the closure of the polynomials in $M_0\left(D_\mu^p\right)$.

Proof. Our proof will consist of proving the chain of implications (a) ⇒ (b) ⇒(c) ⇒ (a).

$[(a) \Rightarrow (b)]$. Let $f \in M_0\left(D_\mu^p\right)$. The key points is that, by assumption, the identity belongs to $M_0\left(D_\mu^p\right)$ Also, all dilations f_r have a continuous derivative in the closed disk. However, we will need a uniform bound on their norms in terms of f. Using the Poisson's kernel, we can rewrite the function f_r as

$$fr(z) = \frac{1}{2\pi} \int_{\mathbb{T}} f(z\xi) \frac{1-r^2}{\left|1-r\bar{\xi}\right|^2} |d\xi|.$$

Thus

$$f'r(z) = \frac{1}{2\pi} \int_{\mathbb{T}} \xi f'(z\xi) \frac{1-r^2}{\left|1-r\bar{\xi}\right|^2} |d\xi|.$$

Fix $\sigma \in Aut(\mathbb{D}). Given\ \lambda \in T,$ let us define $\sigma_\lambda(z) := \sigma(\bar{\lambda} z), z \in \mathbb{D}$. Then, applying the equality (50) and Minkowski's inequality,

$$\left(\int_{\mathbb{D}} |(f_r\circ\sigma^{-1})'|^p \, d\mu\right)^{\frac{1}{p}} = \left(\int_{\mathbb{D}} |f_r'|^p \, h_\sigma dA\right)^{\frac{1}{p}}$$

$$= \frac{1}{2\pi} \left(\int_{\mathbb{D}} \left| \int_{\mathbb{T}} \xi f'(z\xi) \frac{1-r^2}{\left|1-r\bar{\xi}\right|^2} |d\xi| \right|^p h_\sigma(z) dA(z)\right)^{\frac{1}{p}}$$

$$\le \frac{1}{2\pi} \left(\int_{\mathbb{T}} \left(\int_{\mathbb{D}} |\xi f'(z\xi)|^p \, h_\sigma(z) dA(z)\right)^{\frac{1}{p}} \frac{1-r^2}{\left|1-r\bar{\xi}\right|^2} |d\xi|\right.$$

$$= \frac{1}{2\pi}\left(\int_{\mathbb{T}}\left(\int_{\mathbb{D}}|f'(\omega)|^{p}\, h_{\sigma_\xi}(\omega)dA(z)\right)^{\frac{1}{p}}\frac{1-r^2}{\left|1-r\bar{\xi}\right|^2}|d\xi|\right.$$

$$\leq \frac{1}{2\pi}\int_{\mathbb{T}} sup_{\lambda\in\mathbb{T}}\left(\int_{\mathbb{D}}|f'(\omega)|^{p}\quad h_{\sigma_\lambda}(\omega)dA(\omega)\right)^{\frac{1}{p}}\frac{1-r^2}{\left|1-r\bar{\xi}\right|^2}|d\xi|$$

$$= sup_{\lambda\in\mathbb{T}}\left(\int_{\mathbb{D}}|f'(\omega)|^{p}h_{\sigma_\lambda}(\omega)dA(\omega)\right)^{1/p}\frac{1}{2\pi}\int_{\mathbb{T}}\frac{1-r^2}{\left|1-r\bar{\xi}\right|^2}$$

$$= sup_{\lambda\in\mathbb{T}}\left(\int_{\mathbb{D}}|f'(\omega)|^{p}h_{\sigma_\lambda}(\omega)dA(\omega)\right)^{1/p}.$$

That is, for all $\sigma \in Aut(\mathbb{D})$ and all we have that

$$\int_{\mathbb{D}}|(f_r\circ\sigma^{-1})'|^{p}\,d\mu \leq sup_{\lambda\in\mathbb{T}}\int_{\mathbb{D}}|(f_r\circ\sigma^{-1})'|^{p}\,d\mu \qquad (52)$$

It now follows that $\|f_r\|_{M(D_\mu^p)} \leq \|f\|_{M(D_\mu^p)}$ $since\ f_r(0) = f(0)$.

In the special case when σ is chosen to be the identity, a close inspection of the above long chain of inequalities shows that

$$\int_{\mathbb{D}}|f_r'|^{p}\,d\mu \leq \int_{\mathbb{D}}|f'|^{p}\,d\mu \qquad (53)$$

Using the description (34) of the group of disk automorphisms, it is easy to see that for every function $f \in M_0(D_\mu^p)$ we have

$$\lim_{|\sigma(0)|\to 1}\left(sup_{\lambda\in\mathbb{T}}\int_{\mathbb{D}}\left|\left(f\circ\sigma_\lambda^{-1}\right)'\right|^{p}d\mu\right) = 0. \qquad (54)$$

Assume the contrary of what we want to prove: $lim_{r\to 1}\|f-f_r\|_{M_0(D_\mu^p)} \neq 0$.

Then there exist a constant $\delta > 0$, a sequence of positive numbers $rn \nearrow 1$, and a sequence of automorphisms of the unit disk $(\sigma_n)_n$ such that

$$\left(\int_{\mathbb{D}}|(f\circ\sigma_n^{-1})' - (f\circ\sigma_n^{-1})'|^{p}\,d\mu\right)^{1/p} \geq \delta. \qquad (55)$$

For all n.

After passing to a subsequence, we may assume that the sequence $(\sigma_n)_n$ converges uniformly on compact subsets of the unit disk. By a corollary to Hurwitz's theorem, it converges either to a $constant\ \lambda \in \mathbb{T}$ or to an automorphism σ. We analyze the two cases separately.

<u>Case 1</u>. Suppose $that\ \sigma_n(0) \to \lambda\ with\ |\lambda| = 1$; then $|\sigma_n(0)| \to 1$. By (54), we can find $n_0 \in N$ such that

$$sup_{\lambda\in\mathbb{T}}\left(\int\limits_{\mathbb{D}} \left|\left(f\circ\sigma_{n,\lambda}^{-1}\right)'\right|^{p}\, d\mu\right)^{1/p} < \frac{\delta}{4}.$$

for all $n \geq n_0$, where $\sigma_{n,\lambda}(z) = \sigma_n(\bar{\lambda}z)$ as before. From here we deduce by (52) that

$$\left(\int\limits_{\mathbb{D}} \left|(f\circ\sigma_n^{-1})' - \left(f_{r_n}\circ\sigma_n^{-1}\right)'\right|^{p} d\mu\right)^{\frac{1}{p}}$$

$$\leq \left(\int\limits_{\mathbb{D}} \left|\left(f_{r_n}\circ\sigma_n^{-1}\right)'\right|^{p} d\mu\right)^{\frac{1}{p}} + \left(\int\limits_{\mathbb{D}} |(f\circ\sigma_n^{-1})'|^{p}\, d\mu\right)^{\frac{1}{p}}$$

$$\leq 2\, sup_{\lambda\in\mathbb{T}}\left(\int\limits_{\mathbb{D}} \left|\left(f\circ\sigma_{n,\lambda}^{-1}\right)'\right|^{p} d\mu\right)^{1/p} < \frac{\delta}{2},$$

which is in contradiction with (55).

<u>Case 2</u>. Suppose $\sigma_n \to \sigma$ uniformly on compact sets, $\sigma \in Aut(D)$. Then there exists $r \in (0,1)$ such that $|\sigma_n^{-1}(0)| \leq r$ for all $n \in \mathcal{U}$. Thus,

$$\frac{1-r}{1+r} \leq |(\sigma_n^{-1})(z)| \leq \frac{1-r}{1+r}$$

holds for all $z \in \mathbb{D}$. Taking α to be the same constant as in (48), denote by D the following finite positive constant:

$$D = \max\left\{\left(\frac{1-r}{1+r}\right)^{2-p+\alpha}, \left(\frac{1-r}{1+r}\right)^{2-p+\alpha}\right\}$$

Extend h to $\mathbb{D}$ radially by defining $h(z) = h(|z|)$. By formula (50) and our hypotheses on h, we have

$$\int\limits_{\mathbb{D}} \left|(f\circ\sigma_n^{-1})' - \left(f_{r_n}\circ\sigma_n^{-1}\right)'\right|^{p} d\mu = \int\limits_{\mathbb{D}} \left|f' - f_{r_n}'\right|^{P} h_{\sigma_{\mathrm{n}}}\, \mathrm{dA}$$

$$\leq C\int\limits_{\mathbb{D}} \left|f'(\omega) - f_{r_n}'(\omega)\right|^{P} \mathrm{h}(|\omega|)|\sigma_n'(\omega)|^{2-\mathrm{p}+\alpha}\, \mathrm{dA}(\omega)$$

$$\leq CD\int\limits_{\mathbb{D}} \left|f'(\omega) - f_{r_n}'(\omega)\right|^{P} \mathrm{h}(|\omega|)\, \mathrm{dA}(\omega)$$

$$\leq CD_p\int\limits_{\mathbb{D}} \left(|f'(\omega)|^{p} - \left|f_{r_n}'(\omega)^{P}\right)\mathrm{h}(|\omega|)\, \mathrm{dA}(\omega),$$

Notice $that$ $|f'|^p h$ belongs to $L^1(\mathbb{D}, dA)$ because $f \in M_0\left(D_\mu^p\right)$ and $\left|f_{r_n}\right|^p h$ also belongs to $L^1(\mathbb{D}, dA)$ by (53). Since $|f_{r_n}'|$ converges pointwise to $|f'|$, Fatou's lemma and again (53) together imply that

$$\int_{\mathbb{D}} |f'|^p d\mu \leq \lim_n \inf \int_{\mathbb{D}} \left|f'_{r_n}\right|^p d\mu \leq \lim_n \sup \int_{\mathbb{D}} \left|f'_{r_n}\right|^p d\mu = \int_{\mathbb{D}} |f'|^p\, d\mu\,,$$

Thus

$$\lim_n \int_{\mathbb{D}} \left|f'_{r_n}\right|^p d\mu = \int_{\mathbb{D}} |f'|^p\, d\mu, \tag{56}$$

In summary, we know the following:

(i) $-\left|f'_{r_n} - f'\right|^p h_{\sigma_n} \leq CD_p\left(\left|f'_{r_n}\right|^p + |f'|^p\right)h$ holds throughout $\mathbb{D}$;

(ii) $\left|f'_{r_n} - f'\right|^p h_{\sigma_n} \to 0$ pointwise in $\mathbb{D}$ as $n \to \infty$:

(iii) $\left(\left|f'_{r_n}\right|^p + |f'|^p\right)h \to 2|f'|^p h$ poinywise in $\mathbb{D}$;

(iv) $\int_{\mathbb{D}} \left(\left|f'_{r_n}\right|^p + |f'|^p\right)h\, dA \to \int_{\mathbb{D}} 2|f'|^p h\, dA$ by (23).

Thus, we are allowed to apply the well-known generalization of the Lebesgue dominated convergence theorem usually called Pratt's lemma [270], obtaining

$$\int_{\mathbb{D}} \left|(f \circ \sigma_n^{-1})' - \left(f_{r_n} \circ \sigma_n^{-1}\right)'\right|^p d\mu = \int_{\mathbb{D}} \left|f' - f'_{r_n}\right|^P h_{\sigma_n}\, \mathrm{dA} \to 0, \quad \mathrm{n} \to \infty.$$

which again yields a contradiction. This concludes our proof that (a) ⇒ (b).

[(b) ⇒ (c).] Fix $r \in (0,1)$. Since f_r is analytic in a larger disk centered at the origin, there exists a sequence of polynomials $(p_n)_n$ *such that* $p_n(0) = f(0)$ and

$$M_n \coloneqq \sup\{|f_r' z - p_n'(z)| : z \in \mathbb{D}\} \to 0 \quad as\ \mathrm{n} \to \infty.$$

thus,

$$sup_\sigma \int_{\mathbb{D}} |(f \circ \sigma^{-1})' - (p_n \circ \sigma^{-1})'|^p\, d\mu = sup_\sigma \int_{\mathbb{D}} \left|f_r' - p'_n\right|^P \mathrm{h}_\sigma\, \mathrm{dA}$$

$$\leq \mathrm{M}_\mathrm{n}^\mathrm{p} \sup_\sigma \int_{\mathbb{D}} h_\sigma\, dA = M_n^p \|z\|^p_{M(D^p_\mu)}.$$

Therefore, $lim_n \|\ f_r - p_n\|_{M(D^p_\mu)} = 0$ *and* f_r belongs to the closure of the polynomials in $M\left(D^p_\mu\right))$ *for all* $r > 1$.

Since

$$lim_{r\to 1}\|f - f_r\|_{M(D^p_\mu)} = 0,$$

the function f also belongs to the closure of the polynomials

We would like to remark that here our inspiration for the above result comes from [274]. Since we are considering compositions with all automorphisms, there are some technical difficulties involved. The radial character of the measure μ does not seem to guarantee all rotation-invariant properties to hold so it appears necessary to consider the supremum over all compositions with rotations as we have done above or to use some similar argument in order to complete the proof.

Corollary (6.3.14)[254]: Let μ a radial measure induced by a positive, radial, inte-grable function h that satisfies (48). If the identity function is in $\mathrm{M}_0\left(\mathrm{D}^\mathrm{p}_\mu\right)$then the polynomials are dense in $\mathrm{M}_0\left(\mathrm{D}^\mathrm{p}_\mu\right)$. In particular, the Banach space $\mathrm{M}_0\left(\mathrm{D}^\mathrm{p}_\mu\right)$is separable.

We a characterization of bounded and compact com-position operators from the Bloch space into the small spaces.

Theorem (6.3.15)[254]: Let $1 \le p < \infty$. Assume that the Banach space $M_0(D_\mu^p)$ is separable. Let φbe an analytic self-map of the unit disk. Then the following statements are equivalent

(a) $C_\varphi: \mathfrak{B} \to M_0(D_\mu^p)$ is a bounded operator.

(b) $C_\varphi: \mathfrak{B} \to M_0(D_\mu^p)$ is a compact operator.

(c) Both conditions.

$$\lim_{|\sigma(0)|\to 1} \int_{\mathbb{D}} \left|(\varphi \circ \sigma)^\#\right|^p d\mu = 0 \quad and \quad \sup_{\sigma \in Aut\,(\mathbb{D})} \int_{\mathbb{D}} \left|(\varphi \circ \sigma)^\#\right|^p d\mu < \infty.$$

Are fulfilled.

Proof. We will first show that the conditions (a) and (b) are equivalent and then also that (a) is equivalent to (c).

[(b) ⇒ (a)] is trivial.

[(a) ⇒ (b).] As was already remarked, $\mathfrak{B}$ is isomorphic to l^∞see, *e.g.*, [266]. Therefore every bounded operator from B into a separable Banach space is weakly compact (see [261], for Example (6.3.9)). In Particular, every bounded operator from $\mathfrak{B}$ into $M_0(D_\mu^p)$ is weakly compact.

Since $M_0(D_\mu^p)$ *is a subspace of* $M(D_\mu^p)$, *it follow that* $C_\varphi: \mathfrak{B} \to M(D_\mu^p)$ is also a weakly compact operator. By Theorem (6.3.7), it is compact.

[(a) ⇒ (c)] By Lemma (6.3.1), there exist $f, g \in \mathfrak{B}$ and a positive constant C such that

$$|f'(z)|^p + |g'(z)|^p \ge \frac{C}{(1-|z|^2)^p}, \qquad for\ all\ z \in \mathbb{D}. \tag{57}$$

Thus, given $\sigma \in Aut\,(\mathbb{D})$, we have that

$$\begin{aligned} |(f\circ\varphi\circ\sigma)'|^p + |(g\circ\varphi\circ\sigma)'|^p &= (|f'\circ\varphi\circ\sigma|^p + |g'\circ\varphi\circ\sigma|^p)|(\varphi\circ\sigma)'|^p \\ &\ge \frac{C}{(1-|(\varphi\circ\ \sigma)|^2)^p}, \\ |(\varphi\circ\sigma)'|^p &= C\left|(\varphi\circ\sigma)^\#\right|^p. \end{aligned}$$

Since C_φ maps the Bloch space into $M_0(D_\mu^p)$. The functions $f \circ \varphi$ and $g \circ \varphi$ belong to $M_0(D_\mu^p)$ so from the above inequalities we deduce that

$$\lim_{|\sigma(0)|\to 1} \int_{\mathbb{D}} \left|(\varphi \circ \sigma)^\#\right|^p d\mu = 0.$$

By (a), the operator $C_\varphi: \mathfrak{B} \to M(D_\mu^p)$ is bounded and, by Proposition (6.3.3), we have

$$sup_\sigma \int_{\mathbb{D}} \left|(\varphi \circ \sigma)^\#\right|^p d\mu < \infty.$$

[(c)⟹(a)]. Applying again Proposition (6.3.3), we conclude that the composition operator $C_\varphi$is bounded from the Bloch space into $M_0(D_\mu^p)$. It is only left to prove that the range of C_φ is contained in the Little space $M_0(D_\mu^p)$. To this end, suppose $f \in \mathfrak{B}$. Then

$$\begin{aligned} |(f\circ\varphi\circ\sigma)'|^p &= |(f\circ\varphi\circ\sigma)'|^p|(\varphi\circ\sigma)'|^p \le \frac{\|f\|_{\mathfrak{B}}^p}{(1-|\varphi\circ\sigma|^2)^p}|(\varphi\circ\sigma)'|^p \\ &= \|f\|_{\mathfrak{B}}^p\left|(\varphi\circ\sigma)^\#\right|^p \end{aligned}$$

By our assumption (c), it follows that

$$\lim_{|\sigma(0)|\to 1} \int_{\mathbb{D}} |(f \circ \varphi \circ \sigma)'|^p \, d\mu \; = \; 0.$$

Hence $f \circ \varphi \in M_0\left(D_\mu^p\right)$, This complete the proof.

List of Symbols

References

[1] S. Stevic´ , On an integral-type operator from logarithmic Bloch-type and mixed-norm spaces to Bloch-type spaces, Nonlinear Anal. TMA 71 (2009) 6323–6342.

[2] K.L. Avetisyan, Hardy_Bloch type spaces and lacunary series on the polydisk, Glasg. Math. J. 49 (2) (2007) 345_356.

[3] L. Brown, A.L. Shields, Multipliers and cyclic vectors in the Bloch space, Michigan Math. J. 38 (1991) 141_146.

[4] D. Clahane, S. Stevi¢, Norm equivalence and composition operators between Bloch/Lipschitz spaces of the unit ball, J. Inequal. Appl. 2006 (2006) Article ID 61018, 11 pages.

[5] S. Li, S. Stevi¢, Some characterizations of the Besov space and the _-Bloch space, J. Math. Anal. Appl. 346 (2008) 262_273.

[6] S. Stevi¢, On an integral operator on the unit ball in Cn, J. Inequal. Appl. 1 (2005) 81-88.

[7] S. Stevi¢, On _-Bloch spaces with Hadamard gaps, Abstr. Appl. Anal. 2007 (2007) Article ID 39176, 7 pages.

[8] S. Yamashita, Gap series and α-Bloch functions, Yokohama Math. J. 28 (1980) 31–36.

[9] K. Zhu, Spaces of Holomorphic Functions in the Unit Ball, in: Graduate Texts in Mathematics, Springer, New York, 2005.

[10] S. Stevi¢, On a new operator from the logarithmic Bloch space to the Bloch-type space on the unit ball, Appl. Math. Comput. 206 (2008) 313-320.

[11] A.L. Shields, D.L. Williams, Bounded projections, duality, and multipliers in spaces of analytic functions, Trans. Amer. Math. Soc. 162 (1971) 287-302.

[12] C.C. Cowen, B.D. MacCluer, Composition Operators on Spaces of Analytic Functions, CRC Press, Boca Raton, FL, 1995.

[13] S. Li, S. Stevi¢, Weighted composition operators from Bergman-type spaces into Bloch spaces, Proc. Indian Acad. Sci. Math. Sci. 117 (3) (2007) 371-385.

[14] S. Li, S. Stevic´ , Weighted composition operators from a-Bloch space to H1 on the polydisk, Numer. Func. Anal. Optimiz. 28 (7) (2007) 911–925.

[15] S. Li, S. Stevi¢, Weighted composition operators from H∞ to the Bloch space on the polydisc, Abstr. Appl. Anal. 2007 (2007) Article ID 48478, 12 pages.

[16] S. Li, S. Stevi¢, Weighted composition operators between H1 and _-Bloch spaces in the unit ball, Taiwanese J. Math. 12 (2008) 1625-1639.

[17] L. Luo, S.I. Ueki, Weighted composition operators between weighted Bergman spaces and Hardy spaces on the unit ball of Cn, J. Math. Anal. Appl. 326 (1) (2007) 88-100.

[18] B.D. MacCluer, R. Zhao, Essential norms of weighted composition operators between Bloch-type spaces, Rocky Mountain J. Math. 33 (4) (2003) 1437_1458.

[19] K. Madigan, A. Matheson, Compact composition operators on the Bloch space, Trans. Amer. Math. Soc. 347 (7) (1995) 2679_2687.

[20] A. Montes-Rodriguez, Weighted composition operators on weighted Banach spaces of analytic functions, J. London Math. Soc. 61 (3) (2000) 872-884.

[21] S. Ohno, Weighted composition operators between H1 and the Bloch space, Taiwanese J. Math. 5 (2001) 555-563.

[22] S. Ohno, K. Stroethoff, R. Zhao, Weighted composition operators between Bloch-type spaces, Rocky Mountain J. Math. 33 (2003) 191-215.
[23] S. Stevic´ , Composition operators between H1 and the a-Bloch spaces on the polydisc, Z. Anal. Anwendungen 25 (4) (2006) 457–466.
[24] S. Stevi¢, Weighted composition operators between mixed norm spaces and H1_ spaces in the unit ball, J. Inequal. Appl. 2007 (2007) Article ID 28629, 9 pages.
[25] S. Stevi¢, Essential norms of weighted composition operators from the _-Bloch space to a weighted-type space on the unit ball, Abstr. Appl. Anal. 2008 (2008) Article ID 279691, 11 pages.
[26] S. Stevi¢, Norm of weighted composition operators from Bloch space to H1_ on the unit ball, Ars. Combin. 88 (2008) 125-127.
[27] S.I. Ueki, L. Luo, Compact weighted composition operators and multiplication operators between Hardy spaces, Abstr. Appl. Anal. 2008 (2008) 11 pages. Article ID 196498.
[28] E. Wolf, Weighted composition operators between weighted Bergman spaces and weighted Banach spaces of holomorphic functions, Rev. Mat. Complut. 21 (2) (2008) 475-480.
[29] E. Wolf, Weighted composition operators between weighted Bergman spaces and weighted Bloch type spaces, J. Comput. Anal. Appl. 11 (2) (2009) 317-321.
[30] X. Zhu, Weighted composition operators from F .p; q; s/ spaces to H1_ spaces, Abstr. Appl. Anal. 2009 (2009) Article ID 290978, 12 pages.
[31] Ch. Pommerenke, Schlichte funktionen und analytische funktionen von beschränkter mittlerer oszillation, Comment. Math. Helv. 52 (1977) 591-602.
[32] S. Li, Volterra composition operators between weighted Bergman space and Bloch type spaces, J. Korea Math. Soc. 45 (2008) 229-248.
[33] S. Li, S. Stevi¢, Products of composition and integral type operators from H1 to the Bloch space, Complex Var. Elliptic Equ. 53 (5) (2008) 463-474.
[34] S. Li, S. Stevi¢, Products of Volterra type operator and composition operator from H1 and Bloch spaces to the Zygmund space, J. Math. Anal. Appl. 345 (2008) 40-52.
[35] S. Li, S. Stevi¢, Products of integral-type operators and composition operators between Bloch-type spaces, J. Math. Anal. Appl. 349 (2009) 596-610.
[36] S. Stevi¢, Products of integral type operators and composition operators from the mixed norm space to Bloch-type spaces, Sibirsk. Mat. Zh. 50 (4) (2009) (in press).
[37] S. Li, S. Stevi¢, Generalized composition operators on Zygmund spaces and Bloch type spaces, J. Math. Anal. Appl. 338 (2008) 1282-1295.
[38] S. Stevi¢, Generalized composition operators from logarithmic Bloch spaces to mixed-norm spaces, Util. Math. 77 (2008) 167-172.
[39] S. Stevi¢, On a new operator from H1 to the Bloch-type space on the unit ball, Util. Math. 77 (2008) 257-263.
[40] S. Stevi¢, On a new integral-type operator from the weighted Bergman space to the Bloch-type space on the unit ball, Discrete Dyn. Nat. Soc. 2008 (2008) Article ID 154263, 14 pages.
[41] S. Stevi¢, On a new integral-type operator from the Bloch space to Bloch-type spaces on the unit ball, J. Math. Anal. Appl. 354 (2009) 426-434.
[42] K. Avetisyan, S. Stevi¢, Extended Cesàro operators between different Hardy spaces, Appl. Math. Comput. 207 (2009) 346-350.

[43] D.C. Chang, S. Li, S. Stevi¢, On some integral operators on the unit polydisk and the unit ball, Taiwanese J. Math. 11 (5) (2007) 1251-1286.
[44] D.C. Chang, S. Stevi¢, Estimates of an integral operator on function spaces, Taiwanese J. Math. 7 (3) (2003) 423-432.
[45] D.C. Chang, S. Stevi¢, The generalized Cesàro operator on the unit polydisk, Taiwanese J. Math. 7 (2) (2003) 293-308.
[46] D.C. Chang, S. Stevi¢, Addendum to the paper ``A note on weighted Bergman spaces and the Cesàro operator", Nagoya Math. J. 180 (2005) 77-90.
[47] Z. Hu, Extended Cesàro operators on mixed norm spaces, Proc. Amer. Math. Soc. 131 (7) (2003) 2171-2179.
[48] Z. Hu, Extended Cesàro operators on the Bloch space in the unit ball of Cn, Acta Math. Sci. Ser. B Engl. Ed. 23 (4) (2003) 561-566.
[49] S. Krantz, S. Stevi¢, On the iterated logarithmic Bloch space on the unit ball, Nonlinear Anal. TMA 71 (2009) 1772-1795.
[50] S. Li, S. Stevi¢, Integral type operators from mixed-norm spaces to _-Bloch spaces, Integral Transform. Spec. Funct. 18 (7) (2007) 485-493.
[51] S. Li, S. Stevi¢, Riemann_Stieltjes operators on Hardy spaces in the unit ball of Cn, Bull. Belg. Math. Soc. Simon Stevin 14 (2007) 621-628.
[52] S. Li, S. Stevi¢, Riemann_Stieltjes type integral operators on the unit ball in Cn, Complex Var. Elliptic Equ. 52 (6) (2007) 495-517.
[53] S. Li, S. Stevi¢, Compactness of Riemann_Stieltjes operators between F .p; q; s/ and _-Bloch spaces, Publ. Math. Debrecen 72 (1_2) (2008) 111-128.
[54] S. Li, S. Stevi¢, Riemann_Stieltjes operators between different weighted Bergman spaces, Bull. Belg. Math. Soc. Simon Stevin 15 (4) (2008) 677-686.
[55] S. Li, S. Stevi¢, Riemann_Stieltjes operators between mixed norm spaces, Indian J. Math. 50 (1) (2008) 177-188.
[56] S. Li, S. Stevi¢, Cesàro type operators on some spaces of analytic functions on the unit ball, Appl. Math. Comput. 208 (2009) 378-388.
[57] S. Stevi¢, Boundedness and compactness of an integral operator on a weighted space on the polydisc, Indian J. Pure Appl. Math. 37 (6) (2006) 343-355.
[58] S. Stevi¢, Boundedness and compactness of an integral operator on mixed norm spaces on the polydisc, Sibirsk. Mat. Zh. 48 (3) (2007) 694-706.
[59] S. Stevi¢, Boundedness and compactness of an integral operator between H1 and a mixed norm space on the polydisc, Siberian J. Math. 50 (3) (2009) 495-497.
[60] X. Tang, Extended Cesàro operators between Bloch-type spaces in the unit ball of Cn, J. Math. Anal. Appl. 326 (2) (2007) 1199-1211.
[61] S. Stevi´c, "Norm of weighted composition operators from α-Bloch spaces to weighted-type spaces," Applied Mathematics and Computation, vol. 215, no. 2, pp. 818–820, 2009.
[62] X. Fu, X. Zhu, Weighted composition operators on some weighted spaces in the unit ball, Abst. Appl. Anal. 2008 (Article ID 605807) (2008) 8p.
[63] D. Gu, Weighted composition operators from generalized weighted Bergman spaces to weighted-type spaces, J. Inequal. Appl. 2008 (Article ID 619525) (2008) 14p.
[64] S. Stevic´ , Norms of some operators from Bergman spaces to weighted and Bloch-type space, Util. Math. 76 (2008) 59–64.
[65] S. Stevic´ , Norm and essential norm of composition followed by differentiation from a-Bloch spaces to H1l, Appl. Math. Comput. 207 (2009) 225–229.

[66] S. Stevic´ , Weighted composition operators from weighted Bergman spaces to weighted-type spaces on the unit ball, Appl. Math. Comput. 212 (2009) 499–504.
[67] S.-I. Ueki, "Weighted composition operators on some function spaces of entire functions," Bulletin of the Belgian Mathematical Society. Simon Stevin, vol. 17, no. 2, pp. 343–353, 2010.
[68] S.I. Ueki, L. Luo, Essential norms of weighted composition operators between weighted Bergman spaces of the ball, Acta Sci. Math. (Szeged) 74 (2008) 829–843.
[69] C. Xiong, Norm of composition operators on the Bloch space, Bull. Aust. Math. Soc. 70 (2004) 293–299.
[70] W. Yang, Weighted composition operators from Bloch-type spaces to weighted-type spaces, Ars. Combin., in press.
[71] A. Aleman, J.A. Cima, An integral operator on Hp and Hardy's inequality, J. Anal. Math. 85 (2001) 157–176.
[72] A. Aleman, A.G. Siskakis, An integral operator on Hp , Complex Var. Theory Appl. 28 (1995) 149–158.
[73] A. Aleman, A.G. Siskakis, Integral operators on Bergman spaces, Indiana Univ. Math. J. 46 (1997) 337–356.
[74] X. Guo, G. Ren, Cesàro operators on Hardy spaces in the unit ball, J. Math. Anal. Appl. 339 (1) (2008) 1–9.
[75] Z. Hu, Extended Cesàro operators on Bergman spaces, J. Math. Anal. Appl. 296 (2004) 435–454.
[76] S. Li, H. Wulan, Characterizations of α-Bloch spaces on the unit ball, J. Math. Anal. Appl. 343 (1) (2008) 58–63.
[77] W. Rudin, Function Theory in the Unit Ball of Cn, Springer-Verlag, Berlin, Heidelberg, New York, 1980.
[78] A. Siskakis, R. Zhao, A Volterra type operator on spaces of analytic functions, Contemp. Math. 232 (1999) 299–311.
[79] S. Stevi´ c, On Bloch-type functions with Hadamard gaps, Abstr. Appl. Anal. 2007 (2007), Article ID 39176, 8 pp.
[80] E. G. Kwon, Composition of Blochs with bounded analytic functions, Proc. Amer. Math. Soc. 124 (1996), 1473–1480.
[81] Patrick R. Ahern, On the behavior near torus of functions holomorphic in the ball, Paci_c J. Math. 107 (1983), 267{278. MR 84i: 32023.
[82] Patrick R. Ahern and Walter Rudin, Bloch functions, BMO, and boundary zeros, Indiana Univ. Math. J. 36 (1987), 131-148. MR 88d: 42036.
[83] Peter. L. Duren, The theory of Hp functions, Academic Press, New York, 1970. MR 42: 3552.
[84] John. B. Garnett, Bounded analytic functions, Academic Press, New York, 1981. MR 83g: 30037.
[85] E. G. Kwon, Fractional integration and the hyperbolic derivative, Bull. Austral. Math. Soc. 38 (1988), 357{364. MR 90a: 30096.
[86] Mean growth of the hyperbolic Hardy class functions, Math. Japonica 35 (1990), 451-460. MR 91e: 30064.
[87] Wade Ramey and David Ullrich, Bounded mean oscillations of Bloch pullbacks, Math. Ann. 291 (1991), 591{606. MR 92i: 32004.
[88] Shinji Yamashita, Functions with Hp hyperbolic derivative, Math. Scand. 13 (1983), 238-244. MR 85f: 30055.

[89] S. Yamashita, Hyperbolic Hardy classes and hyperbolically Dirichlet-finite functions, Hokkaido Math. J. 10 _1981 Sp., 709-722.
[90] S. Yamashita, Holomorphic functions of hyperbolically bounded mean oscillation, Boll. Un. Mat. Ital. B (6) 5 (1986), no. 3, 983–1000.
[91] A. Zygmund, Trigonometric series, Cambridge Univ. Press, London, 1959. MR 21: 6498.
[92] A. B. Aleksandrov, J. M. Anderson, A. Nicolau, Inner functions, Bloch spaces and symmetric measures, Proc. London Math. Soc. (3) 79 (2) (1999) 318–352.
[93] A. B. Aleksandrov, `Multiplicity of boundary values of inner functions', Izv. Akad. Nauk Arm. SSR 22 (1987) 490±503 (Russian).
[94] A. F. Beardon and Ch. Pommerenke, `The PoincareÂ metric of plane domains', J. London Math. Soc. (2) 18 (1978) 475±483.
[95] C. J. Bishop, `Bounded functions in the Little Bloch space', Pacific J. Math. 142 (1990) 209±225.
[96] A. Canton, `Singular measures and the Little Bloch space', Publ. Mat. 42 (1998) 211±222.
[97] L. Carleson, `On mappings, conformal at the boundary', J. Anal. Math. 19 (1967) 1±13.
[98] P. L. Duren, `Smoothness of functions generated by Riesz products', Proc. Amer. Math. Soc. 16 (1965) 1263±1268.
[99] J. L. FernaÂndez and A. Nicolau, `Boundary behaviour of inner functions and holomorphic functions', Math. Ann. 310 (1998) 423±445.
[100] F. P. Gardiner and D. P. Sullivan, `Symmetric structures on a closed curve', Amer. J. Math. 114 (1992) 683±736.
[101] J. Garnett and M. Papadimitrakis, `Almost isometric maps of the hyperbolic plane', J. London Math. Soc. (2) 43 (1991) 269±282.
[102] V. Havin and B. JoÈricke, The uncertainty principle in harmonic analysis (Springer, Berlin, 1994).
[103] G. Hungerford, `Boundary of smooth sets and singular sets of Blaschke products in the Little Bloch class', thesis, California Institute of Technology, 1988.
[104] J. P. Kahane, `Trois notes sur les ensembles parfait lineÂares', Enseign. Math. 15 (1969) 185±192.
[105] N. G. Makarov, `Probability methods in the theory of conformal mappings', Leningrad Math. J. 1 (1990) 1±56.
[106] A. Peyriere, `E Â tudes de quelques proprieÂteÂs des produits de Riesz', Ann. Inst. Fourier 25 (1975) 127±169.
[107] G. Piranian, `Two monotonic, singular, uniformly almost smooth functions', Duke Math. J. 33 (1966) 255±262.
[108] C. Pommerenke, Boundary behaviour of conformal mappings (Springer, Berlin, 1994).
[109] W. Smith, `Inner functions in the hyperbolic Little Bloch class', Michigan Math. J. 45 (1998) 103±114.
[110] R. Zhao, Composition operators from Bloch type spaces to Hardy and Besov spaces, J. Math. Anal. Appl. 233 (1999), no. 2, 749–766.
[111] O. Blasco, G.S. De Souza, Spaces of analytic functions on the disc where the growth of Mp (F , r) depends on a weight, J. Math. Anal. Appl. 147 (2) (1990) 580–598.

[112] P. Galanopulos, On Blog to Q p log pullbacks, J. Math. Anal. Appl. 337 (2008) 712–725.
[113] P.M. Gauthier, J. Xiao, BiBloch-type maps: existence and beyond, Complex Var. Theory Appl. 47 (8) (2002) 667–678.
[114] Haiying Li, Peide Liu, Composition operators between generally weighted Bloch space and Q q log space, Banach, J. Math. Anal. 3 (1) (2009) 99110.
[115] V.P. Havin, On the factorization of holomorphic functions smooth up to the boundary, Zap. Nauchn. Sem. LOMI 22 (1971) 202–205 (in Russian).
[116] E.G. Kwon, Hyperbolic g-function and Bloch pullback operators, J. Math. Anal. Appl. 309 (2005) 626–637.
[117] M. Mateljevi ´ c, M. Pavlovi ´ c, Lp behaviour of the integral means of analytic functions, Studia Math. 77 (1984) 219–237.
[118] M. Pavlovi ´ c, Lipschitz spaces and spaces of harmonic functions in the unit disc, Michigan Math. J. 35 (2) (1988) 301–311.
[119] M. Pavlovi ´ c, On the moduli of continuity of Hp -functions with 0 < p < 1, Proc. Edinb. Math. Soc. (2) 35 (1) (1992) 89–100.
[120] M. Pavlovi ´ c, Mixed norm spaces of analytic and harmonic functions. I, Publ. Inst. Math. (Beograd) (N.S.) 40 (54) (1986) 117–141.
[121] W. Ramey, D. Ullrich, Bounded mean oscillation of Bloch pull-backs, Math. Ann. 291 (1991) 591–606.
[122] A.L. Shields, D.L. Williams, Bounded projections, duality, and multipliers in spaces of harmonic functions, J. Reine Angew. Math. 299/300 (1978) 265–279.
[123] A.L. Shields, D.L. Williams, Bounded projections and the growth of harmonic conjugates in the unit disc, Michigan Math. J. 29 (1982) 3–25.
[124] J. Xiao, Holomorphic Q Classes, Lecture Notes in Math., vol. 1767, Springer, 2001.
[125] Ahem, P.: On the behavior near a torus of functions holomorphic in the ball. Pac. J. Math. 107, 267-278 (1983).
[126] Choa, J., Choe, B.R.: Composition with a homogeneous polynomial. (Preprint).
[127] Choe, B.R.: Cauchy integral equalities and applications. Trans. Am. Math. Soc. 315, 337-352 (1989).
[128] Coifman, R., Rochberg, R., Weiss, G.: Factorization theorems for Hardy spaces in several variables. Ann. Math. 103, 611-635 (1976).
[129] Duren, P.: Theory of HP-spaces. New York: Academic Press 1970.
[130] Krantz, S.: Function theory of several complex variables. New York: Wiley 1982.
[131] Pommerenke, Ch.: On Bloch functions. J. Lond. Math. Soc. 2, 689-695 (1970).
[132] Russo, P.: Boundary behavior of BMOA(B,,). Trans. Am. Math. Soc. 292, 733-740 (1985).
[133] Stanton, C.: H p and BMOA pullback properties of smooth maps. (Preprint).
[134] Stein, E.M.: Singular integral integrals and estimates for the Cauchy- Riemann equations. Bull. Am. Math. Soc. 79, 440-445 (1973).
[135] Tomaszewski, B.: Interpolation by Lipschitz holomorphic functions. Ark. Mat. 23, 327-338 (1985).
[136] D. C. Ullrich, Radial divergence in BMOA, Proc. London Math. Soc. (3) 68 (1) (1994) 145–160.
[137] Lars V. Ahlfors, Conformai invariants, McGraw-Hill, New York, 1973.
[138] J. M. Anderson, J. Clunie and Ch. Pommerenke, On Bloch functions and normal functions, J. Reine Angew. Math. 270 (1974), 12-37.

[139] Joseph Diestel, Sequences and series in Banach spaces, Springer-Verlag, New York, 1984.
[140] John Garnett, Applications of harmonie measure, Univ. Arkansas Lecture Notes in the Math. Sei., vol. 8, John Wiley & Sons, New York, 1986.
[141] Ch. Pommerenke, Boundary behaviour of conformai maps, Springer-Verlag, Berlin, 1992.
[142] R. Rochberg, Decomposition theorems for Bergman spaces and their applications, Operators and Function Theory (S. C. Power, ed.), Reidel, Dordrecht, 1985, pp. 225-277.
[143] R. Rochberg, Interpolation by functions in the Bergman spaces, Michigan Math. J. 29 (1982), 229-236.
[144] J. Shapiro, Composition operators and classical function theory, Springer-Verlag, New York, 1993.
[145] J. H. Shapiro, W. Smith, and D. A. Stegenga, Geometric models and compactness of composition operators, J. Funct. Anal, (to appear).
[146] A. Montes-Rodr´ıguez, The essential norm of a composition operator on Bloch spaces, Pacific J. Math. 188 (1999), 339–351.
[147] J.L. Fern´andez, On the coefficients of Bloch functions, J. London Math. Soc., 29 (2) (1984), 94-102.
[148] P. Harmand, D. Werner and W. Werner, M-ideals in Banach Spaces and Banach Algebras, Springer-Verlag, Berlin-Heidelberg-New York, 1993.
[149] N.J. Kalton, Spaces of compact operators, Math. Ann., 208(2) (1974), 267-278.
[150] M-ideals of compact operators, Illinois J. Math., 37 (1993), 147-169.
[151] J. Lindenstrauss and L. Tzafriri, Classical Banach Spaces I, Springer, Berlin-Heidelberg-New York, 1977.
[152] A. Pelczynski, Projections in certain Banach spaces, Studia Math., 19 (1960), 209-228.
[153] J.H. Shapiro, The essential norm of a composition operator, Annals of Math., 125 (1987), 375-404.
[154] K. Zhu, Operator Theory in Function Spaces, Marcel Dekker, I.N.C., 1990.
[155] S. Stevi´c, “On an integral operator between Bloch-type spaces on the unit ball,” Bulletin des Sciences Mathematiques, vol. 134, no. 4, pp. 329–339, 2010.
[156] N. Dunford, J.T. Schwartz, Linear Operators I, Interscience Publishers, Jon Willey and Sons, New York, 1958.
[157] W. Ramey, D. Ullrich, Bounded mean oscillation on Bloch pull-back, Math. Ann. 291 (1991) 591–606.
[158] J. Xiao, Riemann–Stieltjes operators on weighted Bloch and Bergman spaces of the unit ball, J. London Math. Soc. (2) 70 (2) (2004) 199–214.
[159] S. Ye, Weighted composition operators between the Little α-Bloch space and the logarithmic Bloch, J. Comput. Anal. Appl. 10 (2) (2008) 243–252.
[160] S. Stevic´ , Norm and essential norm of an integral-type operator from the Dirichlet space to the Bloch-type space on the unit ball, Abstr. Appl. Anal. 2010 (2010) 9. Article ID 134969.
[161] K. Zhu, “Distances and Banach spaces of holomorphic functions on complex domains,” Journal of the London Mathematical Society, vol. 49, no. 1, pp. 163–182, 1994.

[162] K. D. Bierstedt and W. H. Summers, “Biduals of weighted Banach spaces of analytic functions,” Journal of Australian Mathematical Society A, vol. 54, no. 1, pp. 70–79, 1993.

[163] S. Stevi´c, “On a new operator from H∞ to the Bloch-type space on the unit ball,” Utilitas Mathematica, vol. 77, pp. 257–263, 2008.

[164] W. Yang, “On an integral-type operator between Bloch-type spaces,” Applied Mathematics and Computation, vol. 215, no. 3, pp. 954–960, 2009.

[165] D. Gu, “Extended Ces`aro operators from logarithmic-type spaces to Bloch-type spaces,” Abstract and Applied Analysis, vol. 2009, Article ID 246521, 9 pages, 2009.

[166] S. Stevi´c, “Integral-type operators from a mixed norm space to a Bloch-type space on the unit ball,”Siberian Mathematical Journal, vol. 50, no. 6, pp. 1098–1105, 2009.

[167] S. Stevi´c, “Products of integral-type operators and composition operators from the mixed norm space to Bloch-type spaces,” Siberian Mathematical Journal, vol. 50, no. 4, pp. 726–736, 2009.

[168] S. Stevi´c, “Compactness of the Hardy-Littlewood operator on some spaces of harmonic functions,” Siberian Mathematical Journal, vol. 50, no. 1, pp. 167–180, 2009.

[169] R. F. Allen and F. Colonna, “Weighted composition operators from H∞ to the Bloch space of a bounded homogeneous domain,” Integral Equations and Operator Theory, vol. 66, no. 1, pp. 21–40, 2010.

[170] R. F. Allen and F. Collona, “Weighted composition operators on the Bloch space of a bounded homogeneous domain,” Operator Theory: Advances and Applications, vol. 202, pp. 11–37, 2010.

[171] M. H. Shaabani and B. K. Robati, “On the norm of certain weighted composition operators on the Hardy space,” Abstract and Applied Analysis, vol. 2009, Article ID 720217, 13 pages, 2009.

[172] S. Stevi´c, “Weighted composition operators between Fock-type spaces in CN,” Applied Mathematics and Computation, vol. 215, no. 7, pp. 2750–2760, 2009.

[173] S. Stevi´c, “Norms of some operators on bounded symmetric domains,” Applied Mathematics and Computation, vol. 216, no. 1, pp. 187–191, 2010.

[174] S. Stevi´c, “Composition operators from the Hardy space to the Zygmund-type space on the upper half-plane,” Abstract and Applied Analysis, vol. 2009, Article ID 161528, 8 pages, 2009.

[175] S. Stevi´c, “Essential norm of an operator from the weighted Hilbert-Bergman space to the Bloch-type space,” Ars Combinatoria, vol. 91, pp. 123–127, 2009.

[176] S. Stevi´c, “Norms of some operators on the Bergman and the Hardy space in the unit polydisk and the unit ball,” Applied Mathematics and Computation, vol. 215, no. 6, pp. 2199–2205, 2009.

[177] S.-I. Ueki, “Hilbert-Schmidt weighted composition operator on the Fock space,” International Journal of Mathematical Analysis, vol. 1, no. 13–16, pp. 769–774, 2007.

[178] S.-I. Ueki, “Weighted composition operators on the Bargmann-Fock space,” International Journal of Modern Mathematics, vol. 3, no. 3, pp. 231–243, 2008.

[179] S. Stevi´c, “Norm of an integral-type operator from Dirichlet to Bloch space on the unit disk,” Utilitas Mathematica, vol. 83, pp. 301–303, 2010.

[180] H. J. Schwartz, Composition operators on Hp, Thesis, University of Toledo, 1969.
[181] Stevo Stevic´, Norm of some operators from logarithmic Bloch-type spaces to weighted-type spaces, Applied Mathematics and Computation 218 (2012) 11163–11170.
[182] F. Colonna, G.R. Easley, D. Singman, Norm of the multiplication operators from H1 to the Bloch space of a bounded symmetric domain, J. Math. Anal. Appl. 382 (2) (2011) 621–630.
[183] P. Galindo, M. Lindström, S. Stevic´ , Essential norm of operators into weighted-type spaces on the unit ball, Abstr. Appl. Anal. 2011 (2011) 13. Article ID 939873.
[184] S. Stevic´ , Bloch-type functions with Hadamard gaps, Appl. Math. Comput. 208 (2009) 416–422.
[185] S. Stevic´ , Extended Cesàro operators between mixed-norm spaces and Bloch-type spaces in the unit ball, Houston J. Math. 36 (3) (2010) 843–858.
[186] S. Stevic´ , Norms of multiplication operators on Hardy spaces and weighted composition operators from Hardy spaces to weighted-type spaces on bounded symmetric domains, Appl. Math. Comput. 217 (2010) 2870–2876.
[187] S. Stevic´ , A.K. Sharma, A. Bhat, Essential norm of products of multiplication composition and differentiation operators on weighted Bergman spaces, Appl. Math. Comput. 218 (2011) 2386–2397.
[188] S. Stevic´ , A.K. Sharma, Composition operators from the space of Cauchy transforms to Bloch and the Little Bloch-type spaces on the unit disk, Appl. Math. Comput. 217 (2011) 10187–10194.
[189] X. Zhu, Generalized weighted composition operators from Bloch-type spaces to weighted Bergman spaces, Indian J. Math. 49 (2) (2007) 139–149.
[190] E. Doubtsov, Bloch-to-BMOA compositions on complex balls, Proc. Amer. Math. Soc. 140 (2012), 4217–4225.
[191] A. B. Aleksandrov, Function theory in the ball, Current problems in mathematics. Fundamental directions, Vol. 8, Itogi Nauki i Tekhniki, Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1985, pp. 115–190, 274 (Russian); English transl., Encyclopaedia Math. Sci., vol. 8, Springer–Verlag, Berlin, 1994, pp. 107–178. MR850487 (88b:32002)
[192] A. B. Aleksandrov, Proper holomorphic mappings from the ball to the polydisk, Dokl. Akad. Nauk SSSR 286 (1986), no. 1, 11–15 (Russian); English transl.: Soviet Math. Dokl. 33 (1986), no. 1, 1–5. MR822088 (87g:32029)
[193] O. Blasco, M. Lindström, and J. Taskinen, Bloch-to-BMOA compositions in several complex variables, Complex Var. Theory Appl. 50 (2005), no. 14, 1061–1080. MR2175841 (2006f:47027).
[194] B. R. Choe, W. Ramey, and D. Ullrich, Bloch-to-BMOA pullbacks on the disk, Proc. Amer. Math. Soc. 125 (1997), no. 10, 2987–2996. MR1396971 (97m:47039)
[195] E. G. Kwon, On analytic functions of Bergman BMO in the ball, Canad. Math. Bull. 42 (1999), no. 1, 97–103. MR1695858 (2001a:32005).
[196] S. Makhmutov and M. Tjani, Composition operators on some Möbius invariant Banach spaces, Bull. Austral. Math. Soc. 62 (2000), no. 1, 1–19. MR1775882 (2001i:47041)
[197] J. Ryll and P. Wojtaszczyk, On homogeneous polynomials on a complex ball, Trans. Amer. Math. Soc. 276 (1983), no. 1, 107–116. MR684495 (84f:32004).
(144)

[198] W. Smith and R. Zhao, Composition operators mapping into the Qp spaces, Analysis 17 (1997), no. 2-3, 239–263. MR1486367 (98j:47075).
[199] A. N. Petrov, Reverse estimates in logarithmic Bloch spaces, Arch. Math. (Basel) 100 (6) (2013) 551–560.
[200] E. Abakumov and E. Doubtsov, Reverse estimates in growth spaces, Math. Z. 271 (2012), 399–413.
[201] K. M. Dyakonov, Weighted Bloch spaces, Hp, and BMOA, J. London Math. Soc. (2) 65 (2002), 411–417.
[202] D. Girela, M. Pavlovi´c, and J. ´A. Pel´aez, Spaces of analytic functions of Hardy-Bloch type, J. Anal. Math. 100 (2006), 53–81.
[203] E. G. Kwon, Hyperbolic mean growth of bounded holomorphic functions in the ball, Trans. Amer. Math. Soc. 355 (2003), 1269–1294.
[204] S. Yamashita, Hyperbolic Hardy class H1, Math. Scand. 45 (1979), 261–266.
[205] Evgueni Doubtsov1, Weighted Bloch spaces and quadratic integrals, Preprint submitted to Journal of Mathematical Analysis and ApplicationsOctober 25, 2013.
[206] M. Pavlovi´c, Lacunary series in weighted spaces of analytic functions, Arch. Math. (Basel) 97 (5) (2011) 467–473.
[207] D. C. Ullrich, A Bloch function in the ball with no radial limits, Bull. London Math. Soc. 20 (4) (1988) 337–341.
[208] N. G. Makarov, On the radial behavior of Bloch functions, Dokl. Akad. Nauk SSSR 309 (2) (1989) 275–278 (Russian); English transl.: Soviet Math. Dokl. 40 (3) (1990), 505–508.
[209] J. G. Clunie, T. H. MacGregor, Radial growth of the derivative of univalent functions, Comment. Math. Helv. 59 (3) (1984) 362–375.
[210] N. G. Makarov, On the distortion of boundary sets under conformal mappings, Proc. London Math. Soc. (3) 51 (2) (1985) 369–384.
[211] P. Ahern, J. Bruna, Maximal and area integral characterizations of Hardy-Sobolev spaces in the unit ball of Cn, Rev. Mat. Iberoamericana 4 (1) (1988) 123–153.
[212] G. H. Hardy, J. E. Littlewood, Notes on the theory of series. XX: Generalizations of a theorem of Paley, Quart. J. Math. (Oxford Ser.) 8 (1937) 161–171.
[213] E. Doubtsov, Inner mappings, hyperbolic gradients and composition operators, Integral Equations Operator Theory 73 (4) (2012) 537–551.
[214] Ruhan Zhao†, Composition Operators from Bloch Type Spaces to Hardy and Besov Spaces*, Journal of Mathematical Analysis and Applications 233, 74766 _1999.
[215] J. Arazy, S. D. Fisher, and J. Peetre, Möbius invariant function spaces, J. Reine Angew. Math. 363 _1985., 11145.
[216] R. Aulaskari, J. Xiao, and R. Zhao, On subspaces and subsets of BMOA and UBC, Analysis 15 _1995., 10121.
[217] B. Bollob´as, ‘‘Linear Analysis,’’ Cambridge Univ. Press, Cambridge, UK, 1990.
[218] N. Dunford and J. Schwartz, ‘‘Linear Operators,’’ Vol. 1, Interscience, New York, 1958.
[219] P. L. Duren, ‘‘Theory of Hp Spaces,’’ Academic Press, New York, 1970.
[220] G. H. Hardy, and J. E. Littlewood, Some properties of fractional integrals, II, Math. Z. 34, 1932., 40439.
[221] H. Jarchow, and R. Riedl, Factorization of composition operators through Bloch type spaces, Illinois J. Math. 39 _1995., 43440.

[222] E. G. Kwon, Composition of Blochs with bounded analytic functions, Proc. Amer. Math. Soc. 124 (1996), 1473–1480.
[223] J. Lindenstrauss and L. Tzafriri, ‘‘Classical Banach Spaces,’’ Lecture Notes in Math., 338, Springer-Verlag, Berlin, 1973.
[224] B. D. MacCluer, and J. H. Shapiro, Angular derivatives and compact composition operators on the Hardy and Bergman spaces, Can. J. Math. 38 _1986., 87906.
[225] S. Makhmutov, On Bloch-to-Besov composition operators, Proc. Japan. Acad. 72(A) _1996., 23234.
[226] Ch. Pommerenke, ‘‘Boundary Behaviour of Conformal Maps,’’ Springer-Verlag, Berlin, 1992.
[227] L. A. Rubel and R. M. Timoney, An extremal property of the Bloch space, Proc. Amer. Math. Soc. 75 _1979., 449.
[228] W. Smith, Composition operators between Bergman and Hardy spaces, Trans. Amer. Math. Soc. 348 _1996., 2332348.
[229] W. Smith and R. Zhao, Composition operators mapping into the Qp spaces, Analysis 17 (1997), no. 2-3, 239–263. MR1486367 (98j:47075).
[230] M. Tjani, Compact composition operators on some Möbius invariant Banach spaces, Dissertation, Michigan State University _1996..
[231] P. Wojtaszczyk, ‘‘Banach Spaces for Analysts,’’ Cambridge Studies in Advanced Mathematics, 25, Cambridge Univ. Press, Cambridge, UK, 1991.
[232] J. Xiao, On Bergman, Bloch, Little Bloch spaes and Ces´aro arithmetic means, preprint, Centre de Recerca Matem`atica 317 _1996..
[233] K. Yosida, ‘‘Functional Analysis,’’ Springer-Verlag, Berlin, 1968.
[234] K. Zhu, Analytic Besov spaces, J. Math. Anal. Appl. 157 _1991., 31336.
[235] K. Zhu, Bloch type spaces of analytic functions, Rocky Mountain J. Math. 23 _1993., 1141177.
[236] J. JEXIAO, Composition Operators: $\mathcal{N}_\alpha$ To the Bloch Space to $\mathcal{Q}_\beta$, STUDIA MATHEMATICA 139 (3) (2000).
[237] R. Aulaskari and P. Lappan, Criteria for an analytic function to be Bloch and a harmonic or meromorphic function to be normal, in: Complex Analysis and its Applications, Pitman Res. Notes in Math. 306, Longman, 1994, 136—146.
[238] R. Aulaskari1 M. Norwaic and R. Zhao, The n-the derivative characterizations of Möbius invariant Dirichiet spaces, Bull. Austral. Math. Soc. 58 (1998), 43—56
[239] R. Aulaskari, D. StegengaandJXiao, Some subclasses of BMOA and their characterization in terms of Carleson measures, Rocky Mountain J. Math. 26 (1996), 485—506.
[240] R. Au1askari, J. Xiao and R. Zhao, On subspaces and subclasses of BMOA and UBO, Analysis 15 (1995), 101—121.
[241] A. Bacrustein II, Analytic ftnctions of bounded mean oscillation, in.: Aspects of Contemporary Complex Analysis, Academic Press, London, 1980, 2—26.
[242] P. 9. Bourdon, J. A. Cima and A. L. Matheson, Compact composition operators on BMOA, ‘:&ans. Amer. Math. Soc. 351 (1999), 2183—2196.
[243] M. Essén and J. Xiao, Some results on Q spaces, O <p < 1, J. Reine Angew. Math. 485 (1997), 173—195.
[244] H. Jarchow, Locally Convex Spaces, Teubuer, 1981.
[245] H. Jarchow and J. Xiao, Composition operators between Nevanlinna classes and Bergman spaces with weights, J. Operator Theory, to appear.

[246] D. H. Luecking, Thace ideal criteria for Toeplitz operators, J. Punet. AnaL 73 (1987), 345—368.
[247] K. Madigan and A. Matheson, Compact composition operators on the Block space, ‘Thans. Amer. Math. Soc. 347 (1995), 2679—2687.
[248] A. Nicolau and J. Xiao, Bounded functions in Möbius invariant Dirìchlet spaces, J. Funct. Anal. 150 (1997), 383—425.
[249] J. H. Shapiro and A. L. Shields, Unusual topological properties of the Neva.nlinna class, Amer. J. Math. 97 (1976), 915—936.
[250] J. H. Shapiro and P. 0. Taylor, Compact, nuclear, and Hubert—Schmidt com position operators on H2, Indiana Univ. Math. J, 23 (1973), 471—496.
[251] K. Stroeth off, Nevanlinna-type characterizations for the Block space and related spaces, Proc. Edinburgh Math1 Soc. 33 (1990), 123—142.
[252] J. Xiao, Carleson measure, atomic decomposition and free interpolation from Block space, Ann. Acad. Sd. Penn. Ser. A.I. Math. 19 (1994), 35—44.
[253] J. Xiao, Compact composition operators on the area-Nevanlinna class, Exposition. Math. 17 (1999), 255—264.
[254] J. Laitila, P. Nieminen, E. Saksman, H.-O. Tylli, Compact and Weakly Compact Composition Operators on BMOA, Complex Anal. Oper. Theory 7 (2013), no. 1, 163—181.
[255] A. Aleman and A.-M. Simbotin (A.-M. Persson), Estimates in Möbius invariant spaces of analytic functions, Complex Var. Theory Appl. 49 (2004), no. 7-9, 487—510.
[256] S. Axler, The Bergman space, the Bloch space, and commutators of multiplication operators, Duke Math. J. 53 (1986), no. 2, 315—332.
[257] J. Bonet, P. Doma´nski, M. Lindström, Weakly compact composition operators on analytic vector-valued function spaces, Ann. Acad. Sci. Fenn. Math. 26 (2001), no. 1, 233—248.
[258] J.A. Cima, A.L. Matheson, Weakly compact composition operators on VMO, Rocky Mountain J. Math. 32 (2002), Number 3, 937–951.
[259] M.D. Contreras, A.G. Hern´andez-D´ıaz, Weighted composition operators in weighted Banach spaces of analytic functions, J. Austral. Math. Soc. (Ser. A) 69 (2000), no. 1, 41—60.
[260] J. Diestel, H. Jarchow, A. Tonge, Absolutely summing operators, Cambridge Studies in Advanced Mathematics 43, Cambridge University Press, Cambridge 1995.
[261] J. Diestel, J.J. Uhl, Jr., Vector Measures, Mathematical Surveys 15, American Mathematical Society, Providence, R.I. 1977.
[262] P.L. Duren, Theory of Hp Spaces, Academic Press, New York-London 1970. Reprint: Dover, Mineola, New York 2000.
[263] D. Girela, Analytic functions of bounded mean oscillation, in: Complex Function Spaces, Mekrij¨arvi 1999, Report Series Vol. 4 (University of Joensuu, Department of Mathematics, Joensuu, 2001), pp. 61—170.
[264] J. Laitila, Weakly compact composition operators on vector-valued BMOA, J. Math. Anal. Appl. 308 (2005), no. 2, 730—745.
[265] P. Lef`evre, D. Li, H. Queff´elec, L. Rodr´ıguez-Piazza, Some new properties of composition operators associated with lens maps, Israel J. Math. 195 (2013), no. 2, 801—824.

[266] W. Lusky, On the isomorphic classification of weighted spaces of holomorphic functions, Acta Univ. Carolin. Math. Phys. 41 (2000), 51–60.
[267] S. Makhmutov, A hyperbolic version of the fundamental theorem of Nevanlinna (Russian), Dokl. Akad. Nauk 370 (2000), no. 3, 309–312.
[268] F. P´erez-Gonz´alez and J. Xiao, Bloch-Hardy pullbacks, Acta Sci. Math. (Szeged) 67 (2001), 709–718.
[269] K.-M. Perfekt, Duality and distance formulas in spaces defined by means of oscillation, Ark. Mat. 51 (2013), no. 2, 345—361.
[270] H. Royden, Real Analysis, Second edition, Macmillan Publishing Company, New York 1968.
[271] W. Rudin, Functional Analysis, McGraw-Hill, New York 1973.
[272] A.-M. Simbotin (A.-M. Persson), Estimates and Duality in Q-Spaces, Licentiate Thesis, Lund University 2004.
[273] W. Smith, Composition operators between some classical spaces of analytic functions, In: Proceedings of the International Conference on Function Theory, Seoul, Korea (2001), 32–46.
[274] K-J. Wirths, J. Xiao, Recognizing Qp,0 functions per Dirichlet space structure, Bull. Belg. Math. Soc. Simon Stevin 8 (2001), no. 1, 47—59.
[275] J. Xiao, Composition operators: Nα to the Bloch space to Qβ, Studia Math. 139 (2000), no. 3, 245—260.
[276] Shawgy Hussein and Hekmat Mansour, Operator of Logarithmic Weight Bloch Type Space and Weakly Compact Composition Operator, Ph.D. Thesis Sudan University of Science and Technology, Sudan (2019).