Sudan University of Science and Technology
College of Graduate Studies

كلية الدراسات العليا

Eigenfunctions Restriction Estimates on Riemannian Manifolds and Bilinear Kakeya–Nikodym Averages with Principal Eigenvalue of Nonlocal Operators

تقديرات قصر الدوال الذاتية على متعدد طيات ريمان ومتوسطات كاكيا-نيكوديم ثنائية الخطية مع القيمة الذاتية الأساسية للمؤثرات غير الموضعية

A Thesis Submitted in Fulfillment for the Degree of Ph.D

in Mathematics

By:

Kawther Bashir Mohamed Elhassin

Supervisor:

Prof. Dr. Shawgy Hussein AbdAlla

January-2020

Dedication

To My Family.

Acknowledgements

I would like to express my deep gratitude to my supervisor Prof. Dr. Shawgy Hussein AbdAlla for many helpful suggestions and good following.

Abstract

The L^p norm estimates and an improvement of eigenfunctions restricted to submanifolds, for compact boundaryless Riemannian manifolds with nonpositive sectional curvature and constant negative curvature are studied. We show the refined, microlocal and bilinear Kakeya-Nikodym averages bounds for eigenfunctions in two dimensions, on compact Riemannian surfaces and lower bounded for nodal sets of eigenfunctions in higher dimensions with L^p –norms. Simple criterion for the existence and properties of principal eigenvalue of the elliptic operators in Euclidean space and principal eigenfunctions and spectrum points of some nonlocal dispersal operators, and applications are considered.

الخلاصة

قمنا بدراسة تقديرات نظيم L^p والتحسينات للدوال الذاتية قصراً إلى متعددات الطيات الجزئية ولأجل متعدد طيات ريمان غير المحدودة المتراصة مع الانحناء المقطعي غير الموجب والانحناء السالب الثابت. أوضحنا حديات متوسطات كاكيا-نيكوديم المشتقة والموضعية الصغيرة وثنائي الخطية للداليات الذاتية في بعدين على سطوح ريمان المتراصة والمحدودة السفلى لأجل الفئات العقدية للدوال الذاتية في الابعاد اللانهائية مع نظائم-L^p. تم أعتبار المعيار البسيط لأجل الوجود والخصائص للقيمة الذاتية الأساسية للمؤثرات الناقصية في الفضاء الاقليدي والدوال الذاتية الأساسية ونقاط الطيف لبعض مؤثرات تشتيت الانتشار غير الموضعي والتطبيقات.

Introduction

We study the relationship between the extrinsic geometry of the submanifolds and the canonical relations associated to the oscillatory integral operators. Let (M, g) be an n-dimensional compact boundaryless Riemannian manifold with nonpositive sectional curvature. Then we can give improved estimates for the L^p norms of the restrictions of eigenfunctions of the Laplace-Beltrami operator to smooth submanifolds of dimension k, for $p > \frac{2n}{n-1}$ when $k = n-1$ and $p > 2$ when $k \leq n-2$, compared to the general results of Burq, G´erard and Tzvetkov. Earlier, B´erard gave the same improvement for the case when $p = \infty$, for compact Riemannian manifolds without conjugate points for $n = 2$, or with nonpositive sectional curvature for $n \geq 3$ and $k = n-1$. We give the improved estimates for $n = 2$, the L^p norms of the restrictions of eigenfunctions to geodesics. The proof uses the fact that the exponential map from any point in $x \in M$ is a universal covering map from $\mathbb{R}^2 \simeq T_x M$ to M, which allows us to lift the calculations up to the universal cover $(\mathbb{R}^2, \tilde{g})$, where $\tilde{g}$ is the pullback of g via the exponential map.

We provide a necessary and sufficient condition that L^p-norms, $2 < p < 6$, of eigenfunctions of the square root of minus the Laplacian on two-dimensional compact boundaryless Riemannian manifolds M are small compared to a natural power of the eigenvalue λ. The condition that ensures this is that their L^2-norms over $O(\lambda^{-\frac{1}{2}})$ neighborhoods of arbitrary unit geodesics are small when λ is large. The proof exploits Gauss' lemma and the fact that the bilinear oscillatory integrals in Hörmander's proof of the Carleson-Sjölin theorem become better and better behaved away from the diagonal. If (M, g) be a two-dimensional compact boundaryless Riemannian manifold with nonpostive curvature, then we shall give improved estimates for the L^2 -norms of the restrictions of eigenfunctions to unit-length geodesics, compared to the general results of Burq, Gerard and Tzvetkov. By earlier results of Bourgain, they are equivalent to improvements of the general L^p-estimates for $n = 2$ and $2 < p < 6$. The proof uses the fact that the exponential map from any point in $x_0 \in M$ is a universal covering map from $\mathbb{R}^2 \simeq T_{x_0} M$ to M (the Cartan-Hadamard- von Mangolt theorem), which allows us to lift the necessary calculations up to the universal cover $(\mathbb{R}^2, \tilde{g})$ where $\tilde{g}$ is the pullback of g via the exponential map. We extend a result to dimensions $d \geq 3$ which relates the size of L^p-norms of eigenfunctions for $2 < p < \frac{2(d+1)}{d-1}$ to the amount of L^2 -mass in shrinking tubes about unit-length geodesics. The proof uses bilinear oscillatory integral estimates of Lee and a variable coefficient variant of an "ε removal lemma" of Tao and Vargas. We also use Hörmander's

L^2 oscillatory integral theorem and the Cartan-Hadamard theorem to show that, under the assumption of nonpositive curvature, the L^2-norm of eigenfunctions e_λ over unit length tubes of width $\lambda^{-\frac{1}{2}}$ goes to zero.

Two generalizations of the notion of principal eigenvalue for elliptic operators in $\mathbb{R}^N$. We show several results comparing these two eigenvalues in various settings: general operators in dimension one; self-adjoint operators; and "limit periodic" operators. We are interested in the existence of a principal eigenfunction of a nonlocal operator which appears in the description of various phenomena ranging from population dynamics to micro-magnetism. We study the following eigenvalue problem: $\Omega \int J\left(\frac{x-y}{g(y)}\right)\frac{\varphi(y)}{g^n(y)}dy + a(x)\varphi = \rho\varphi$, where $\Omega \subset \mathbb{R}^n$ is an open connected set, J a non-negative kernel and g a positive function.

For (M, g) be a compact, boundaryless manifold of dimension n with the property that either (i) $n = 2$ and (M, g) has no conjugate points, or (ii) the sectional curvatures of (M, g) are nonpositive. Let Δ be the positive Laplacian on M determined by g. We study the $L^2 \to L^p$ mapping properties of a spectral cluster of $\sqrt{\Delta}$ of width $1/\log\lambda$. We show that one can obtain logarithmic improvements of L^2 geodesic restriction estimates for eigenfunctions on 3-dimensional compact Riemannian manifolds with constant negative curvature. We obtain a $(\log\lambda)^{-\frac{1}{2}}$ gain for the L^2-restriction bounds, which improves the corresponding bounds of Burq, Gérard and Tzvetkov, Hu, Chen and Sogge. We achieve this by adapting the approaches developed by Chen and Sogge, Blair and Sogge. We derive an explicit formula for the wave kernel on 3D hyperbolic space, which improves the kernel estimates from the Hadamard parametrix in Chen and Sogge.

We obtain some improved essentially sharp Kakeya–Nikodym estimates for eigenfunctions in two dimensions. We obtain an improvement of the bilinear estimates of Burq, Gérard and Tzvetkov in the spirit of the refined Kakeya–Nikodym estimates of Blair.

We investigate the dependence of the principal spectrum points of nonlocal dispersal operators on underlying parameters and to consider its applications. We study the effects of the spatial inhomogeneity, the dispersal rate, and the dispersal distance on the existence of the principal eigenvalues, the magnitude of the principal spectrum points, and the asymptotic behavior of the principal spectrum points of nonlocal dispersal operators with Dirichlet type, Neumann type, and periodic boundary conditions in a unified way. We study some spectral properties of the linear operator $\mathcal{L}_\Omega + a$ defined on the space $C(\bar{\Omega})$ by : $\mathcal{L}_\Omega\,[\phi] + a\phi := \int_\Omega K(x,y)\phi(y)\,dy + a(x)\phi(x)$ where $\Omega \subset \mathbb{R}^N$ is a domain, possibly unbounded, a is a continuous bounded function and K is a continuous, non negative kernel satisfying an integrability condition. We focus our analysis on the properties of the generalized principal eigenvalue $\lambda_p(\mathcal{L}_\Omega + a)$ defined by $\lambda_p(\mathcal{L}_\Omega + a) := sup\{\lambda \in \mathbb{R} \mid \exists\phi \in C(\bar{\Omega}), \phi > 0$, such that $\mathcal{L}_\Omega[\phi] + a\phi + \lambda\phi \leq 0$ in $\Omega\}$. We establish some new properties of this generalized principal eigenvalue λ_p.

The Contents

Chapter 1
L^p Norm Estimates and Improvement on Eigenfunction Restriction Estimates

We study the growth rate of L^p norms of eigenfunctions of the Laplace-Beltrami operator restricted to submanifolds of compact C^∞ Riemannian manifolds. The spectral projection operators can be expressed as oscillatory integral operators, so the question reduces to oscillatory integral operator norm estimates. We show the main estimates by using the Hadamard parametrix for the wave equation on $(\mathbb{R}^2, \tilde{g})$, the stationary phase estimates, and the fact that the principal coefficient of the Hadamard parametrix is bounded, by observations of Sogge and Zelditch. The improved estimates also work for $n \geq 3$, with $p > \frac{4k}{n-1}$. We can then get the full result by interpolation.

Section (1.1): Eigenfunctions Restricted to Submanifolds

Measurements of concentration of eigenfunctions of the Laplace-Beltrami operator on a manifold have been studied in several ways. One way is by describing their associated semi-classical (Wigner) measures (e.g., see Shnirelman [14], Zelditch [18], etc.). Another way is by studying the growth of the L^p norms of the eigenfunctions on the manifold (see Sogge [15], [16], Sogge-Zelditch [17], Burq, Gerard, and Tzvetkov ´ [3], [4], [2]). A recently introduced third way is to consider the possible growth of the L^p norm $(2 \leq p \leq +\infty)$ of the restrictions of the eigenfunctions to submanifolds. Burq, Gerard, ´ and Tzvetkov [5] obtained L^p norm estimates of the eigenfunctions restricted to a curve on a Riemannian surface and extended the estimates to some cases in higher dimension. Also see Reznikov [13] in two dimensions.

Here we consider this last question, i.e., the growth of the L^p norm of restrictions of the eigenfunctions to submanifolds, with the help of the estimates of oscillatory integral operators whose canonical relations have fold type singularities. Suppose that (M, g) is a compact smooth Riemannian manifold (without boundary) of dimension n and Δ is the Laplace-Beltrami operator on M associated to the metric g. Let $\{\varphi\lambda_j, \lambda_j \geq 0\}$ be the eigenfunctions of Δ such that $-\Delta\varphi\lambda_j = \lambda_j^2\varphi\lambda_j$ and $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$. For the simplest case when $\dim M = 2$ and γ is a closed smooth curve in M, [5] obtained estimates, stated in the following Theorem (1.1.3) and Theorem (1.1.7).

Turning to higher dimensions, we have the following:

The estimates in Theorem (1.1.16) can be improved in the hypersurface case, i.e., $k = n - 1$, if the hypersurfaces have curvature.

We will focus on the structure of the canonical relation of the phase function of the specified oscillatory integral operators and the geometric structure of the submanifold to get the desired estimates.

We condensed version, supervised by Professor Allan Greenleaf at the University of Rochester.

Consider the first order pseudo-differential operator $\sqrt{-\Delta}$ defined by the spectral theorem: $\sqrt{-\Delta} = \sum_{j=0}^{\infty} \lambda_j E_j$, $I = \sum_{j=0}^{\infty} E_j$, where E_j are projection operators into the finitedimensional eigenspace $\mathcal{E}_j$ with eigenvalue λ_j. Denote: $e^{it\sqrt{-\Delta}} = \sum_{j=0}^{\infty} e^{it\lambda_j} E_j$. Let $\chi \in S(\mathbb{R})$ such that $\chi(0) = 1$ and the support of its Fourier transformation $\operatorname{supp}\hat{\chi}(t) \subset \left(\frac{\varepsilon_0}{2}, \varepsilon_0\right)$ for some $\varepsilon_0 > 0$. Then we can define approximate projection operators as:

$$\chi_\lambda = \chi(\sqrt{-\Delta} - \lambda) = \sum_j \chi(\lambda_j - \lambda)E_j .$$

Clearly $\chi(\sqrt{-\Delta} - \lambda)\varphi_\lambda = \varphi_\lambda$ for all $\lambda = \lambda_0, \lambda_1, \ldots$. On the other hand,

$$\chi(\sqrt{-\Delta} - \lambda) = \sum_j \chi(\lambda_j - \lambda)E_j$$

$$= \sum_j \frac{1}{2\pi} e^{it}(\lambda_j - \lambda)\hat{\chi}(t)dtE_j$$

$$= \frac{1}{2\pi} \int e^{-it\lambda}e^{it\sqrt{-\Delta}}\hat{\chi}(t)\, dt.$$

Choosing ε_0 small enough, in local coordinates we can represent $e^{it\sqrt{-\Delta}}$ as a Fourier integral operator (see, e.g., Hörmander [11]). Applying a stationary phase argument, we have the following theorem (see Sogge [16]):

Theorem (1.1.1)[1]: In local coordinates,

$$\chi_\lambda(f) = \chi(\sqrt{-\Delta} - \lambda)f = \lambda^{\frac{n-1}{2}} \int_{y\in D_0} e^{i\lambda\psi(x,y)} a_\lambda(x,y)f(y)dy + R_\lambda(f),$$

where $D_0 = \left\{y : \frac{\varepsilon_0}{2C_0} |x - y| 2C_0\varepsilon_0\right\}$. $\psi(x,y) = -d_g(x,y)$ is the geodesic distance with respect to metric g between x and y. Furthermore, $a_\lambda \in C_0^\infty$ has uniform bounds $\left|\partial_{x,y}^\alpha a_\lambda(x,y)\right| C_\alpha$, and R_λ is an operator such that $\|R_\lambda\|_{L^2\to L^q} \le C\lambda^{-N}$ for $2 \le q \le +\infty$. Because of this property of R_λ, we may henceforth ignore it and denote

$$T_\lambda(f) = \int_{y\in D_0} e^{i\lambda\psi(x,y)} a_\lambda(x,y)f(y)dy.$$

We only need to focus on the operator

$$\lambda^{\frac{n-1}{2}} T_\lambda(f) = \lambda^{\frac{n-1}{2}} \int_{y\in D_0} e^{i\lambda\psi(x,y)} a_\lambda(x,y)f(y)dy.$$

For any $x_0 \in M$, we may choose the geodesic normal coordinate system about x_0 such that for $x \in U = \{x : |x| \le c\varepsilon\}$,

$$T_\lambda(f) = \int_{y\in D} e^{i\lambda\psi(x,y)} a_\lambda(x,y)f(y)dy,$$

where $D = \{y : c_1\varepsilon\, |y| \le c_2\varepsilon\}$ and $a_\lambda(x,y)$ is supported on the set $\{(x,y) \in U \times D : |x| \le c\varepsilon < c_1\varepsilon\, |y| \le c_2\varepsilon\}$.

Since $\chi_\lambda(\varphi_\lambda) = \chi(\sqrt{-\Delta} - \lambda)\varphi_\lambda = \varphi_\lambda$ and $\chi_\lambda = \lambda^{\frac{1}{2}} T_\lambda + R_\lambda$, it suffices to consider the operator norm estimates of T_λ to get the eigenfunction estimates, as R_λ satisfies much better bounds than we want to prove. We will focus on the structure of the canonical relation of the phase function of T_λ and the geometric structure of the submanifold to get the estimates with the help of oscillatory integral operator estimates.

Suppose that (M,g) is a compact smooth Riemannian manifold (without boundary) of dimension 2. Let γ be a smooth closed curve in M and $\left\{\varphi_{\lambda_j}, \lambda_j \ge 0\right\}$ be the eigenfunctions of Laplace-Beltrami operator Δ on M, so that $-\Delta\varphi_{\lambda_j} = \lambda_j^2\, \varphi_{\lambda_j}$ and $0 = \lambda_0 < \lambda_1 \le \lambda_2 \cdots$. We will refer to a general eigenfunction as φ_λ.

Since $\chi_\lambda(\varphi_\lambda) = \chi(\sqrt{-\Delta} - \lambda)\varphi_\lambda = \varphi_\lambda$, it suffices to show the operator norm estimates:

$$\|\chi_\lambda(f)\|_{L^p(\gamma)}\ C(1+\lambda)^{\delta(p)}\|f\|_{L^2(M)}.$$

Taking into account that $\chi_\lambda = \lambda^{\frac{1}{2}}T_\lambda + R_\lambda$ and R_λ is an operator whose bounds are rapidly decreasing in λ, it remains to show the norm estimates of T_λ:

$$\|T_\lambda(f)\|_{L^p(\gamma)}\ C(1+\lambda)^{\delta(p)-\frac{1}{2}}\|f\|_{L^2(M)}.$$

Assume that we are in the geodesic normal coordinate system about $x_0 \in M$, and γ is parameterized by arc length s and passes through x_0. Partition of unity allows us to assume that γ is contained in the coordinate patch U, i.e., $|x(s)| \le c\varepsilon$ and $x(0) = 0$. Therefore, it is enough to show that

$$\|T_\lambda(f)\|_{L^p(\gamma)}\ C(1+\lambda)^{\delta(p)-\frac{1}{2}}\|f\|_{L^2(D)}.$$

Since y stays in the annulus with inner radius $c_1\varepsilon$ and outer radius $c_2\varepsilon$, we can restrict y to the circle with radius r. In fact, we can represent y in polar coordinates as $y = r\omega_0$ (i.e., geodesic polar coordinates on M), $c_1\varepsilon \le r \le c_2\varepsilon, \omega_0 = (\omega_1^0, \omega_2^0) \in \mathbb{S}^1$ and denote: $\psi_r(x,\omega_0) = \psi(x,y), f_r(\omega_0) = f(y), a_r(x,\omega_0) = ra(x,y)$ and

$$(T_\lambda^r f_r)(x) = \int_{\mathbb{S}^1} e^{i\lambda\psi_r(x,\omega_0)}\, a_r(x,\omega_0) fr(\omega_0)d\omega_0.$$

Then, $(T_\lambda f)(x) = \int_{c_1\varepsilon}^{c_2\varepsilon} (T_\lambda^r f_r)(x)d\,r$. If we have the estimates on the unit circle:

$$\|T_\lambda^r(f_r)\|_{L^p(\gamma)}\ C(1+\lambda)^{\delta(p)-\frac{1}{2}}\|f_r\|_{L^2(\mathbb{S}^1)},$$

then, by Minkowski's integral inequality,

$$\begin{aligned}\|T_\lambda f\|_{L^p(\gamma)} &\le \int_{c_1\varepsilon}^{c_2\varepsilon} \|T_\lambda^r f_r\|_{L^p(\gamma)}\,d\,r\\ &\le C\lambda^{\delta(p)-\frac{1}{2}}\int_{c_1\varepsilon}^{c_2\varepsilon}\|f_r\|_{L^2(\mathbb{S}^1)}\,d\,r\\ &\le C\lambda^{\delta(p)-\frac{1}{2}}\|f\|_{L^2(D)}.\end{aligned}$$

So it suffices to prove

$$\|T_\lambda^r(f)\|_{L^p(\gamma)}\ C(1+\lambda)^{\delta(p)-\frac{1}{2}}\|f\|_{L^2(\mathbb{S}^1)}\ , \tag{1}$$

where $(T_\lambda^r f)(x) = \int_{\mathbb{S}^1} e^{i\lambda\psi_r(x,\omega_0)}\, a_r(x,\omega_0) f(\omega_0)d\omega_0$, and $f \in L^2(\mathbb{S}^1)$.

Because we are using geodesic normal coordinates, $g(0) = \mathrm{Id}$. Suppose that $\omega(x,y) \in T_xM$ denotes the unit vector such that $\exp_x(-\psi(x,y)\,\omega(x,y)) = y$, and $u(x,y) \in T_yM$ denotes the unit vector such that $\exp_y\big(-\psi(x,y)u(x,y)\big) = x$. Clearly $\omega(0,y) = \omega_0 = (\omega_1^0, \omega_2^0) \in \mathbb{S}^1$. Then we have

Lemma (1.1.2)[1]: $\frac{\partial\psi}{\partial s} = \nabla x\psi(x(s),y)\cdot\dot{x} = g(\omega(x(s),y),\dot{x})$, where $x(s) = \big(x_1(s), x_2(s)\big) \in U \subset \mathbb{R}^2$.

Proof. Differentiating the identity $\exp_y\big(-\psi(x,y)u(x,y)\big) = x$ with respect to s, by the chain rule, we get

$$\begin{aligned}D_{-\psi(x,y)u(x,y)}\big(\exp_y\big)\cdot\big[-\nabla_x\psi(x(s),y)\cdot\dot{x}u\big(x(s)\big) - \psi(x(s),y)D_xu(x(s),y)\cdot\dot{x}\big]&\\ = \dot{x}.&\end{aligned} \tag{2}$$

On the other hand, differentiating the identity $\exp_y\big(tu(x,y)\big) = \exp_x\big((-\psi(x,y)-t)\omega(x,y)\big)$ with respect to t and evaluating at $t = -\psi_r(x,y)$, we have

$$D_{-\psi(x,y)u(x,y)}\big(\exp_y\big)\cdot u(x,y) = D_0(\exp_x)\cdot\big(-\omega(x,y)\big) = -\omega(x,y). \tag{3}$$

Taking the scalar product of (2) and (3), on the left hand side we have the following by Gauss' lemma:

$$-\nabla_x\psi(x(s),y)\ \cdot\ \dot{x}u(x(s),y)\ -\ \psi(x(s),y)D_x\,u(x(s),y)\ \cdot\dot{x},u(x,y).$$

Noticing that $\langle D_x u(x(s),y)\ \cdot\ \dot{x},u(x,y)\rangle\ =\ 0$ and $\langle u(x,y),u(x,y)\rangle\ =\ 1$, we only have $-\nabla_x\psi(x(s),y)\ \cdot\dot{x}$ left. On the other hand, the scalar product of the right hand sides of (2) and (3) gives us $-g(\omega(x,y),\dot{x})$.Therefore, we have

$$\nabla_x\psi(x(s),y)\cdot\ \dot{x}\ =\ g(\omega(x(s),y),\dot{x}).$$

Theorem (1.1.3)[1]: Let $\gamma\ :\ [a,b]\rightarrow\ M$ be a smooth curve parametrized by arc length. Then, for all φ_λ

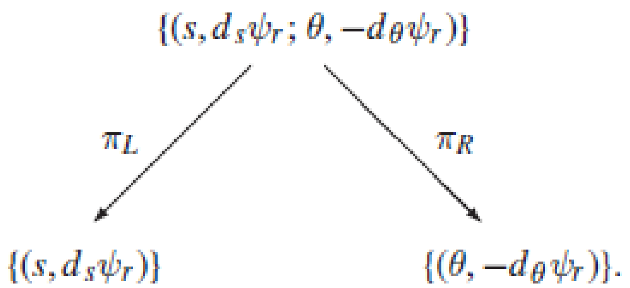
$$\|\varphi_\lambda\|_{L^p(\gamma)}\,C(1\ +\ \lambda)^{\delta(p)}\,\|\varphi_\lambda\|_{L^2(M)},$$

where

$$\delta(p)\ =\begin{cases}\dfrac{1}{4} & \text{if } 2\le p\le 4\\ \dfrac{1}{2}-\dfrac{1}{p} & if\ \ 4\le p\le+\infty\end{cases}$$

For $2\le\ p\le\ 4$, the estimates in Theorem (1.1.3) can be improved for curves with non-vanishing geodesic curvature.

Proof. Clearly when $s\ =\ 0$, Lemma (1.1.2) gives us $d_s\psi r|s=0\ =\ \nabla_x\psi_r(0,\omega_0)\cdot\ \dot{x}\ =\ g_0(\omega_0,\dot{x})$. Suppose $\omega_0\ =\ (\omega_1^0\,,\omega_2^0)\ =\ (\cos\theta,\sin\theta)\ \in\ \mathbb{S}^1$. Consider the phase function $\psi_r(x(s),\omega_0)$, the canonical relation associated with ψr is



When $s\ =\ 0$, $d_s\psi_r\ =\ \dot{x}_1(0)\cos\theta\ +\ \dot{x}_2(0)\sin\theta$ and $d_{s\theta}\psi_r\ =\ -\,\dot{x}_1(0)\sin\theta\ +\ \dot{x}_2(0)\cos\theta$. So the critical point set of π_L denoted by Ω restricted to $s\ =\ 0$ is

$$\Omega_0\ =\ \Omega|_{s=0}=\ \{(s,\theta)\colon d_{s\theta}\psi_r\ =\ 0\}|_{s=0}$$
$$=\ \{(0,\theta)\colon -\ \dot{x}_1(0)\sin\theta\ +\ \dot{x}_2(0)\cos\theta\ =\ 0\}.$$

A nonvanishing kernel vector field of π_L is $V_L\ =\dfrac{\partial}{\partial\theta}$. So, at $s\ =\ 0$,

$$V_L\,(d_{s\theta}\psi_r)\ =\ d_{s\theta\theta}\psi_r\ =\ -\,\dot{x}_1(0)\cos\theta\ -\ \dot{x}_2(0)\sin\theta.$$

It is easy to see that $V_L\,(d_{s\theta}\psi_r)|_{\Omega_0}\ =\ 0$. Hence Ω_0 is the set of fold points for π_L. By the stability of fold singularities, π_L has at most fold singularities near $s\ =\ 0$ (and so we may assume in the whole coordinate chart U) as long as we choose ε small enough. Then, by Theorem (1.1.1) in [10],

$$\|T_\lambda^r\|_{L^2\to L^p}\le C(1\ +\ \lambda)^{-\frac{1}{4}},\qquad \text{if}\ \ 2\le\ p\le\ 4,$$
$$\|T_\lambda^r\|_{L^2\to L^p}\le C(1\ +\ \lambda)^{-\frac{1}{p}},\qquad \text{if}\ \ 2\le\ p\le\infty.$$

This means that (1) holds which completes the proof of Theorem (1.1.3).

The estimates in Theorem (1.1.3) can be improved if the curve has nonzero geodesic curvature.

For Theorem (1.1.7), similarly to Theorem (1.1.3), it suffices to show

$$\|T_\lambda^r(f)\|_{L^p(\gamma)}\, C(1+\lambda)^{\tilde{\delta}(p)-\frac{1}{2}}\|f\|_{L^2(\mathbb{S}^1)}\ . \qquad (4)$$

Lemma (1.1.4)[1]: We have $\frac{\partial\omega(x(s),y)}{\partial s}|_{s=0} = 0$ if $\langle\frac{d\omega_0}{d\theta}, \dot{x}(0)\rangle = 0$.

Proof. Since $\langle\omega_0,\omega_0\rangle = 1$, taking the derivative with respect to θ, we have $\langle\frac{d\omega_0}{d\theta}, \omega_0\rangle = 0$. On the other hand, $\langle\frac{d\omega_0}{d\theta}, \dot{x}(0)\rangle = 0$, so $\dot{x}(0) = c\omega_0$ for some constant c. Consider $g(\omega(x,y),\omega(x,y)) = 1$, i.e.,

$$(\omega_1,\omega_2)\begin{pmatrix} g_{11}(x) & g_{12}(x) \\ g_{21}(x) & g_{22}(x)\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} = 1. \qquad (5)$$

Since we are working in geodesic normal coordinate system, all the first derivatives of the metric vanish at the origin and $(g_{ij})|_{x=0} = \text{Id}$ (see [9]). In (5), take the derivative with respect to x_1 and x_2 respectively; then, for $x = 0$,

$$\begin{pmatrix}\omega_{1x_1} & \omega_{2x_1}\\ \omega_{1x_2} & \omega_{2x_2}\end{pmatrix}\begin{pmatrix}\omega_1^0\\ \omega_2^0\end{pmatrix} = 0. \qquad (6)$$

It follows from Lemma (1.1.2) that for arbitrary $(\dot{x}_1,\dot{x}_2)$,

$$\left(\frac{\partial\psi_r}{\partial x_1},\frac{\partial\psi_r}{\partial x_2}\right)\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = (\omega_1,\omega_2)\begin{pmatrix} g_{11}(x) & g_{12}(x) \\ g_{21}(x) & g_{22}(x)\end{pmatrix}\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}.$$

So

$$\frac{\partial\psi_r}{\partial x_1} = \omega_1 g_{11}(x) + \omega_2 g_{21}(x), \qquad (7)$$

and

$$\frac{\partial\psi_r}{\partial x_2} = \omega_1 g_{12}(x) + \omega_2 g_{22}(x). \qquad (8)$$

Taking the derivatives with respect to x_2 in (7) and x_1 in (8), for $x = 0$, we see that $\frac{\partial^2\psi_r}{\partial x_1 x_2} = \omega_{1x_2}\, g_{11}(0) + \omega_{2x_2}\, g_{21}(0)$ and $\frac{\partial^2\psi_r}{\partial x_1 x_2} = \omega_{1x_1}\, g_{12}(0) + \omega_{2x_1}\, g_{22}(0)$. Then $\omega_{1x_2} = \omega_{2x_1}$ since $g_{ij}(0) = \delta_{ij}$. That means the 2×2 matrix in (6) is symmetric. Therefore for $x = 0$,

$$\begin{pmatrix}\omega_{1x_1} & \omega_{2x_1}\\ \omega_{1x_2} & \omega_{2x_2}\end{pmatrix}\begin{pmatrix}\omega_1^0\\ \omega_2^0\end{pmatrix} = 0.$$

So

$$\begin{aligned}\frac{d\omega(x(s),y)}{ds}|_{s=0} &= \begin{pmatrix}\dot{\omega}_1\\ \dot{\omega}_2\end{pmatrix}\\ &= \begin{pmatrix}\omega_{1x_1} & \omega_{2x_1}\\ \omega_{1x_2} & \omega_{2x_2}\end{pmatrix}\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}\\ &= c\begin{pmatrix}\omega_{1x_1} & \omega_{2x_1}\\ \omega_{1x_2} & \omega_{2x_2}\end{pmatrix}\begin{pmatrix}\omega_1^0\\ \omega_2^0\end{pmatrix}\\ &= 0.\end{aligned}$$

as desired.

Lemma (1.1.5)[1]: $\dot{x}(0), \ddot{x}(0) = 0$.

Proof. Recall that all the first derivatives of the metric vanish at the origin in geodesic normal coordinates. The lemma follows immediately from differentiating $g(\dot{x},\dot{x}) = 1$ with respect to s and letting $s = 0$.

Lemma (1.1.6)[1]: $|\ddot{x}(0)|^2 = g\left(\frac{D}{ds}\gamma'(0), \frac{D}{ds}\gamma'(0)\right)$.

Proof. In geodesic normal coordinates, $\gamma(s) = \sum_{j=1}^{2} \dot{x}_j(s)\left(\frac{\partial}{\partial x_j}\right)$. So the covariant derivative of γ' at $s = 0$ is

$$\frac{D}{ds}\gamma'|_{s=0} = \sum_{i=1}^{2}\left(\ddot{x}_i(0) + \sum_{j,k=1}^{2} \Gamma_{jk}^{i}(0)\dot{x}_j(0)\dot{x}_k(0)\right)\left(\frac{\partial}{\partial x_i}\right),$$

where Γ_{jk}^{i} are the Christoffel symbols of the metric g. In the geodesic normal coordinates, we have $\Gamma_{jk}^{i}(0) = 0$, therefore, $|\ddot{x}(0)|^2 = g\left(\frac{D}{ds}\gamma'(0), \frac{D}{ds}\gamma'(0)\right)$ as desired.

Theorem (1.1.7)[1]: Suppose $g\left(\frac{D}{ds}\gamma', \frac{D}{ds}\gamma'\right) \neq 0$. Then, for all φ_λ

$$\|\varphi_\lambda\|_{L^p(\gamma)} C(1+\lambda)^{\tilde{\delta}(p)} \|\varphi_\lambda\|_{L^2(M)},$$

where $\tilde{\delta}(p) = \frac{1}{3} - \frac{1}{3p}$, $2 \le p \le 4$.

Proof. From the proof of Theorem (1.1.3) we know that πL has at most fold singularities in U. Now let us take a look at πR, which has the same critical point set Ω as π_L. At $s = 0$,

$$\frac{\partial^2\psi_r}{\partial s^2}|_{s=0} = \frac{\partial g(\omega, \dot{x})}{\partial s}|_{s=0} = \frac{\partial \omega}{\partial s}|_{s=0}, \dot{x}(0) + \omega_0, \ddot{x}(0) = \omega_0, \ddot{x}(0).$$

A nonvanishing kernel vector field of π_R is $V_R = \frac{\partial}{\partial s}$. Note again that all the first derivatives of the metrics vanish at the origin in the geodesic normal coordinates. Thus,

$$V_R(d_{s\theta}\psi_r) = d_{ss\theta}\psi_r|_{s=0} = \langle\frac{d\omega_0}{d\theta}, \ddot{x}(0)\rangle = -\ddot{x}_1(0)\sin\theta + \ddot{x}_2(0)\cos\theta.$$

In Ω_0,$\langle\frac{d\omega_0}{d\theta}, \dot{x}(0)\rangle = 0$ together with $\langle\dot{x}(0), \ddot{x}(0)\rangle = 0$ (Lemma (1.1.5)) yields $\langle\frac{d\omega_0}{d\theta}, \ddot{x}(0)\rangle = 0$ as long as $\ddot{x}(0) \neq 0$. Therefore, if $\ddot{x}(0) = g\left(\frac{D}{ds}\gamma'(0), \frac{D}{ds}\gamma'(0)\right) = 0$ (Lemma (1.1.6)), π_R has at most fold singularities at $s = 0$ and furthermore in U. By Theorem (1.1.7) in [7] (also see [12]), $\|T_\lambda^r(f)\|_{L^2(\gamma)} C(1+\lambda)^{-\frac{1}{2}+\frac{1}{6}} \|f\|_{L^2(\mathbb{S}^1)}$. By interpolation, $\|T_\lambda^r(f)\|_{L^p(\gamma)} C(1+\lambda)^{-\frac{1}{3p}-\frac{1}{6}}\|f\|_{L^2(\mathbb{S}^1)}$ which is exactly (4).

If the metric g is locally the standard Euclidian metric, e.g., in the case of the flat torus $\mathbb{T} = \mathbb{R}^2/(2\pi\mathbb{Z})^2$, we can show the following:

Theorem (1.1.8)[1]: Let Σ be a smooth curve $\gamma : [a,b] \to T$ parametrized by arc length. Suppose that the curvature of γ vanishes to at most lth order, i.e., $\forall t \in [a,b], \left|\gamma^{(2)}(t)\right| = \ldots = \left|\gamma^{(j)}(t)\right| = 0$, but $\left|\gamma^{(j+1)}(t)\right| = 0$ where $j \le l$. Here $l \le 1$, and $l = 1$ simply means $\left|\gamma^{(2)}(t)\right| = 0$. Then

$$\|T_\lambda^r(f)\|_{L^p(\gamma)} C(1+\lambda)^{\delta(p)} \|f\|_{L^2(\mathbb{S}^1)},$$

where

$$(T_\lambda^r f)(x) = \int_{\mathbb{S}^1} e^{i\lambda\psi_r(x,\omega_0)} a_r(x,\omega_0) f(\omega_0) d\omega_0,$$

and

$$\delta(p) = -\frac{1}{(2l+1)p} - \frac{l}{4l+2}, 2 \le p \le 4.$$

Proof. Since the metric is Euclidian metric, the phase function of T_λ^r is $\psi_r(x(s),\omega_0) = -\sqrt{(x_1(s) - r\cos\theta)^2 + (x_2(s) - r\sin\theta)^2}$. By calculation, $d_{s\theta}\psi_r = \frac{r(x_1\cos\theta + x_2\sin\theta - r)(\dot{x}_1 x_2 - x_1\dot{x}_2 - \dot{x}_1 r\sin\theta + \dot{x}_2 r\cos\theta)}{\psi_r^3}$. Since $|x| \le c\varepsilon$ and $c_1\varepsilon \le r \le c_2\varepsilon$, we can choose small c such that $\frac{r(x_1\cos\theta + x_2\sin\theta - r)}{\psi_r^3} \ne 0$. So the critical point set of the projection π_R is

$$\Omega = \{(s,\theta) : \dot{x}_1 x_2 - x_1\dot{x}_2 - \dot{x}_1 r\sin\theta + \dot{x}_2 r\cos\theta = 0\}.$$

A kernel vector field of π_R is $V_R = \frac{\partial}{\partial s}$. Denote the defining function of Ω by $F = \dot{x}_1 x_2 - x_1\dot{x}_2 - \dot{x}_1 r\sin\theta + \dot{x}_2 r\cos\theta$.We hope to apply Theorem (1.1.7) in [7] and it is equivalent to check $V_R(F)$ instead of $V_R(\det D(d_{s\theta}\psi_r))$. Noting that $x(s)$ is the parameterization of γ', from the assumption, we know that the worst situation we have is that: $|\ddot{x}(0)| = \ldots = |x^{(l)}(0)| = 0$, but $|x^{(l+1)}(0)| = 0$. This means that $V_R(F)|_{s=0} = \ldots = V_R^{l-1}(F)|_{s=0} = 0$, and

$$V_R^l(F)|_{s=0} = -x_1^{(l+1)}(0) r\sin\theta + x_2^{(l+1)}(0) r\cos\theta = r\langle x^{(l+1)}(0), \omega_0^\perp\rangle,$$

where $\omega_0^\perp = (-\sin\theta, \cos\theta)$. From the assumption of these derivatives and $\langle \dot{x}(s), \dot{x}(s)\rangle = 1$, it is easy to see that $\langle \dot{x}(0), x^{(l+1)}(0)\rangle \ne 0$.At $s = 0$, the critical points satisfy $\langle \dot{x}(0), \omega^\perp\rangle = 0$. Thus $\langle x^{(l+1)}(0), \omega^\perp\rangle = 0$ which means $V_R^l(F)|_{s=0} = 0$. Therefore, π_R has at most type l singularities. From the proof of Theorem (1.1.3), we know that π_L has at most fold singularities. The desired estimates follow from Theorem (1.1.7) in [7] and interpolation. From Theorem (1.1.8), we can immediately obtain the following eigenfunction estimates on the flat torus $\mathbb{T} = \mathbb{R}^2/(2\pi\mathbb{Z})^2$:

$$\|\varphi_\lambda\|_{L^p(\gamma)}\, C(1 + \lambda)^{\delta(p)+\frac{1}{2}}\, \|\varphi_\lambda\|_{L^2(M)}.$$

Clearly, these estimates agree with Theorem (1.1.7) when $l = 1$ and Theorem (1.1.3) when $l = +\infty$, respectively.

Unfortunately, the estimates above or the estimates in Theorem (1.1.3) and (1.1.7) are far from being sharp for the flat torus. In fact, $\|\varphi_\lambda\|_{L^2(\lambda)}\, C_\varepsilon(1 + \lambda)^\varepsilon \|\varphi_\lambda\|_{L^2(\mathbb{T}^2)}$. In other words, we have a better bound, $C_\varepsilon(1 + \lambda)^\varepsilon$ (see [5]).

Suppose that (M,g) is a compact smooth Riemannian manifold (without boundary) and $\dim M = n$. Let Σ be a smooth submanifold in M with $\dim\Sigma = k$. Suppose $\{\varphi_{\lambda_j}, \lambda_j \ge 0\}$ are the eigenfunctions of Laplace-Beltrami operator Δ on M such that $-\Delta\varphi_{\lambda_j} = \lambda_j^2\, \varphi_{\lambda_j}$ and $0 = \lambda_0 < \lambda_1 \le \lambda_2 \cdots$.

As in §3, since $\chi_\lambda(\varphi_\lambda) = \chi(\sqrt{-\Delta} - \lambda)\varphi_\lambda = \varphi_\lambda$ and $\chi_\lambda = \lambda^{\frac{n-1}{2}} T_\lambda + R_\lambda$, where R_λ is an operator which satisfies rapid decay in λ, it suffices to show the norm estimates of T_λ. Assume that in the geodesic normal coordinate system about $x_0 \in M, \Sigma$ is parameterized by $x(u_1, u_2, \ldots, u_k)$, $x(0) = 0$ and

$$x_u = \begin{pmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \cdots & \frac{\partial x_1}{\partial u_k} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \cdots & \frac{\partial x_2}{\partial u_k} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial x_n}{\partial u_1} & \frac{\partial x_n}{\partial u_2} & \cdots & \frac{\partial x_n}{\partial u_k} \end{pmatrix}_{n\times k}$$

has maximal rank. By means of a partition of unity, we can assume that Σ is contained in the coordinate patch U, i.e., $|x(u)| \leq c\varepsilon, x(0) = 0$ which forces $c_1\varepsilon \leq |y| \leq c_2\varepsilon$. We may introduce the polar coordinates for y (i.e., geodesic polar coordinates on M), $y = r\omega_0, c_1\varepsilon \leq r \leq c_2\varepsilon, \omega_0 = (\omega_1^0, \omega_2^0, \dots, \omega_n^0) \in \mathbb{S}^{n-1}$. It is enough to consider one projection coordinate patch where $\omega_n^0 = \sqrt{1-(\omega_1^0)^2 - \cdots - (\omega_1^0)^2} = 0$. Then it suffices to estimate the L^2 to L^p operator norm of

$$(T_\lambda^r f)(x) = \int_{\mathbb{S}^{n-1}} e^{i\lambda\psi_r(x,\omega_0)} a_r(x,\omega_0) f(\omega_0) d\omega_0,$$

where $f \in L^2(\mathbb{S}^{n-1}), \psi_r(x,\omega_0) = \psi(x,y)$ and $a_r(x,\omega_0) = r^{n-1}a(x,y)$. Similar to Lemma (1.1.2), we have the following:

Lemma (1.1.9)[1]:

$$\begin{aligned}\nabla_u\psi &= \nabla_x\psi(x(u),y)\cdot(x_u)\\ &= \left(g\big(\omega(x(u),y),x_{u_1}\big), g\big(\omega(x(u),y),x_{u_2}\big),\dots, g\big(\omega(x(u),y),x_{u_k}\big)\right),\end{aligned}$$

where $x(u) = \big(x_1(u), x_2(u), \dots, x_n(u)\big) \in U \subset \mathbb{R}^2$.

Lemma (1.1.10)[1]: Denote $\frac{\partial x_i}{\partial u_j}$ by x_iu_j. If the $k \times (n-1)$ matrix A is given by

$$A = \begin{pmatrix} -x_{1u_1}\omega_n^0 + x_{nu_1}\omega_1^0 & -x_{2u_1}\omega_n^0 + -x_{nu_1}\omega_2^0 & \dots & -x_{n-1u_1}\omega_n^0 + x_{nu_1}\omega_{n-1}^0 \\ -x_{1u_2}\omega_n^0 + x_{nu_2}\omega_1^0 & -x_{2u_1}\omega_2^0 + x_{nu_2}\omega_2^0 & \dots & -x_{n-1u_2}\omega_n^0 + x_{nu_2}\omega_{n-1}^0 \\ \vdots & \vdots & \vdots & \vdots \\ -x_{1u_k}\omega_n^0 + x_{nu_k}\omega_1^0 & -x_{2u_k}\omega_n^0 + x_{nu_k}\omega_2^0 & \dots & -x_{n-1u_k}\omega_n^0 + x_{nu_k}\omega_{n-1}^0 \end{pmatrix},$$

then, rank $(A) \geq k-1$.

Proof. We perform the following elementary column operations on the matrix x_u^{T}:

$$x_u^{\mathsf{T}} = \begin{pmatrix} x_{1u_1} & x_{2u_1} & \cdots & x_{nu_1} \\ x_{1u_2} & x_{2u_1} & \cdots & x_{nu_2} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1u_k} & x_{2u_k} & \dots & x_{nu_k} \end{pmatrix}$$

$$\to \begin{pmatrix} -x_{1u_1}\omega_n^0 & -x_{2u_1}\omega_n^0 & \dots & -x_{n-1u_1}\omega_n^0 & x_{nu_1} \\ -x_{1u_2}\omega_n^0 & -x_{2u_1}\omega_2^0 & \dots & -x_{n-1u_2}\omega_n^0 & x_{nu_2} \\ \vdots & \vdots & \vdots & \vdots \\ -x_{1u_k}\omega_n^0 & -x_{2u_k}\omega_n^0 & \dots & -x_{n-1u_k}\omega_n^0 & x_{nu_k} \end{pmatrix}$$

$$\to \begin{pmatrix} -x_{1u_1}\omega_n^0 + x_{nu_1}\omega_1^0 & -x_{2u_1}\omega_n^0 + -x_{nu_1}\omega_2^0 & \dots & -x_{n-1u_1}\omega_n^0 + x_{nu_1}\omega_{n-1}^0 & x_{nu_1} \\ -x_{1u_2}\omega_n^0 + x_{nu_2}\omega_1^0 & -x_{2u_1}\omega_2^0 + x_{nu_2}\omega_2^0 & \dots & -x_{n-1u_2}\omega_n^0 + x_{nu_2}\omega_{n-1}^0 & x_{nu_2} \\ \vdots & \vdots & \vdots & \vdots \\ -x_{1u_k}\omega_n^0 + x_{nu_k}\omega_1^0 & -x_{2u_k}\omega_n^0 + x_{nu_k}\omega_2^0 & \dots & -x_{n-1u_k}\omega_n^0 + x_{nu_k}\omega_{n-1}^0 & x_{nu_k} \end{pmatrix}$$

The last matrix has rank k since x_u^{T} does. Matrix A comes from the last matrix by deleting the last column, so rank$(A) \geq k-1$.

Lemma (1.1.11)[1]: Suppose that rank $(A) = k-1$ and the top left $(k-1)\times(k-1)$ block of A, denoted by B, is nondegenerate, i.e., rank $(B) = k-1$, where

$$B = \begin{pmatrix} -x_{1u_1}\omega_n^0 + x_{nu_1}\omega_1^0 & \dots & -x_{k-1u_1}\omega_n^0 + x_{nu_1}\omega_{k-1}^0 \\ \vdots & \vdots & \vdots \\ -x_{1u_{k-1}}\omega_n^0 + x_{nu_{k-1}}\omega_1^0 & \dots & -x_{k-1u_{k-1}}\omega_n^0 + x_{nu_{k-1}}\omega_{k-1}^0 \end{pmatrix}.$$

Replace the j th column in B by the $(i+k-1)$th column in A (first $k-1$ components in that column) and denote it by B_{ij} where $i=1,\dots,n-k$ and $j=1,\dots,k-1$. If we denote $\Delta=(\omega_n^0)^{-(k-2)}\det B$ and $\Delta_{ij}=(\omega_n^0)^{-(k-2)}\det B_{ij}$, then the solution space of the linear system of equations $Az=0$ is spanned by $\{v_i\}$ where

$$v_i=\left(\Delta_{i_1},\Delta_{i_2},\dots,\Delta_{i_{k-1}},0,\dots,0,\overbrace{i+k-1}^{-\Delta},0,\dots,0\right).$$

Proof. Since $\operatorname{rank}(A)=k-1$ and $\operatorname{rank}(B)=k-1$, $Az=0$ is equivalent to

$$B\begin{pmatrix}z_1\\ \vdots\\ z_2\end{pmatrix}=-\begin{pmatrix}-x_{1u_1}\omega_n^0+x_{nu_1}\omega_1^0 & \dots & -x_{k-1u_1}\omega_n^0+x_{nu_1}\omega_{k-1}^0\\ \vdots & \vdots & \vdots\\ -x_{1u_{k-1}}\omega_n^0+x_{nu_{k-1}}\omega_1^0 & \dots & -x_{k-1u_{k-1}}\omega_n^0+x_{nu_{k-1}}\omega_{k-1}^0\end{pmatrix}\begin{pmatrix}z_k\\ \vdots\\ z_{n-1}\end{pmatrix}.$$

On the right hand side, fix $z_{i+k-1}=1$ and all other components to be zeros in $(z_k,\dots,z_{n-1})$, by Cramer' rule,

$$z=\left(\frac{\Delta_{i_1}}{\Delta},\frac{\Delta_{i_2}}{\Delta},\dots,\frac{\Delta_{i_{k-1}}}{\Delta},0,\dots,0,\overbrace{i+k-1}^{1},0,\dots,0\right),$$

which yields

$$v_i=\left(\Delta_{i_1},\Delta_{i_2},\dots,\Delta_{i_{k-1}},0,\dots,0,\overbrace{i+k-1}^{-\Delta},0,\dots,0\right)$$

as desired.

Lemma (1.1.12)[1]: Suppose that Δ and Δ_{ij} are defined as in Lemma (1.1.11). Then,

$$\Delta=\begin{vmatrix}\omega_1^0 & \dots & \omega_{k-1}^0 & \omega_n^0\\ x_{1u_1} & \dots & x_{k-1u_1} & x_{nu_1}\\ \vdots & \vdots & \vdots & \vdots\\ x_{1u_{k-1}} & \dots & x_{k-1u_{k-1}} & x_{nu_{k-1}}\end{vmatrix}_{k\times k}$$

and

$$\Delta_{ij}=\begin{vmatrix}\omega_1^0 & \dots & \omega_{j-1}^0 & \omega_{i+k-1}^0 & \omega_{j+1}^0 & \dots & \omega_{k-1}^0 & \omega_n^0\\ x_{1u_1} & \dots & x_{j-1u_1} & x_{i+k-1u_1} & x_{j+1u_1} & \dots & x_{k-1u_1} & x_{nu_1}\\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots\\ x_{1u_{k-1}} & \dots & x_{j-1u_{k-1}} & x_{i+k-1u_{k-1}} & x_{j+1u_{k-1}} & \dots & x_{k-1u_{k-1}} & x_{nu_{k-1}}\end{vmatrix}_{k\times k}.$$

Proof.

$$(\omega_n^0)^{k-2}\Delta=\begin{vmatrix}-x_{1u_1}\omega_n^0+x_{nu_1}\omega_1^0 & \dots & -x_{k-1u_1}\omega_n^0+x_{nu_1}\omega_{k-1}^0\\ -x_{1u_2}\omega_n^0+x_{nu_2}\omega_1^0 & \dots & -x_{k-1u_2}\omega_n^0+x_{nu_2}\omega_{k-1}^0\\ \vdots & \vdots & \vdots\\ -x_{1u_{k-1}}\omega_n^0+x_{nu_{k-1}}\omega_1^0 & \dots & -x_{k-1u_{k-1}}\omega_n^0+x_{nu_{k-1}}\omega_{k-1}^0\end{vmatrix}$$

$$=\begin{vmatrix}-x_{1u_1}\omega_n^0 & \dots & -x_{k-1u_1}\omega_n^0\\ -x_{1u_2}\omega_n^0 & \dots & -x_{k-1u_2}\omega_n^0\\ \vdots & \vdots & \vdots\\ -x_{1u_{k-1}}\omega_n^0 & \dots & -x_{k-1u_{k-1}}\omega_n^0\end{vmatrix}$$

$$+\begin{vmatrix} -x_{nu_1}\omega_1^0 & -x_{2u_1}\omega_n^0 & \dots & -x_{k-1u_1}\omega_n^0 \\ -x_{nu_2}\omega_1^0 & -x_{2u_2}\omega_n^0 & \dots & -x_{k-1u_2}\omega_n^0 \\ \vdots & \vdots & & \vdots \quad \vdots \\ -x_{nu_{k-1}}\omega_1^0 & -x_{2u_{k-1}}\omega_n^0 & \dots & -x_{k-1u_{k-1}}\omega_n^0 \end{vmatrix}$$

$$\vdots$$

$$+\begin{vmatrix} -x_{1u_1}\omega_n^0 & \dots & -x_{k-2u_1}\omega_n^0 & -x_{nu_1}\omega_{k-1}^0 \\ -x_{1u_2}\omega_n^0 & \dots & -x_{k-2u_2}\omega_n^0 & -x_{nu_2}\omega_{k-1}^0 \\ \vdots & \vdots & \vdots & \vdots \\ -x_{1u_{k-1}}\omega_n^0 & \dots & -x_{k-2u_{k-1}}\omega_n^0 & -x_{nu_{k-1}}\omega_{k-1}^0 \end{vmatrix}$$

$$= (-1)^{k-1}(\omega_n^0)^{k-1}\begin{vmatrix} -x_{1u_1} & \dots & -x_{k-1u_1} \\ -x_{1u_2} & \dots & -x_{k-1u_2} \\ \vdots & \vdots & \vdots \\ -x_{1u_{k-1}} & \dots & -x_{k-1u_{k-1}} \end{vmatrix}$$

$$+\omega_1^0(-1)^{k-2}(\omega_n^0)^{k-2}\begin{vmatrix} -x_{nu_1} & \dots & -x_{k-1u_1} \\ -x_{nu_2} & \dots & -x_{k-1u_2} \\ \vdots & \vdots & \vdots \\ -x_{nu_{k-1}} & \dots & -x_{k-1u_{k-1}} \end{vmatrix}$$

$$\vdots$$

$$+\omega_{k-1}^0(-1)^{k-2}(\omega_n^0)^{k-2}\begin{vmatrix} -x_{1u_1} & \dots & -x_{k-2u_1} & x_{nu_1} \\ -x_{1u_2} & \dots & -x_{k-2u_2} & x_{nu_2} \\ \vdots & \vdots & \vdots & \vdots \\ -x_{1u_{k-1}} & \dots & -x_{k-2u_{k-1}} & x_{nu_{k-1}} \end{vmatrix}$$

$$= (\omega_n^0)^{k-2}\begin{vmatrix} \omega_n^0 & \dots & \omega_{k-1}^0 & \omega_n^0 \\ x_{1u_1} & \dots & x_{k-1u_1} & x_{nu_1} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1u_{k-1}} & \dots & x_{k-1u_{k-1}} & x_{nu_{k-1}} \end{vmatrix}_{k\times k}$$

as desired.

Similarly we can show that

$$(\omega_n^0)^{k-2}\Delta_{ij}$$

$$= (\omega_n^0)^{k-2}\begin{vmatrix} \omega_1^0 & \dots & \omega_{j-1}^0 & \omega_{i+k-1}^0 & \omega_{j+1}^0 & \dots & \omega_{k-1}^0 & \omega_n^0 \\ x_{1u_1} & \dots & x_{j-1u_1} & x_{i+k-1u_1} & x_{j+1u_1} & \dots & x_{k-1u_1} & x_{nu_1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ x_{1u_{k-1}} & \dots & x_{j-1u_{k-1}} & x_{i+k-1u_{k-1}} & x_{j+1u_{k-1}} & \dots & x_{k-1u_{k-1}} & x_{nu_{k-1}} \end{vmatrix}_{k\times k}$$

Lemma (1.1.13)[1]: Suppose that

$$f_i = (\omega_n^0)^{-(k-1)}\begin{vmatrix} B & -x_{i+k-1u_1}\omega_n^0 + x_{nu_1}\omega_{i+k-1}^0 \\ & \vdots \\ -x_{1u_k}\omega_n^0 + x_{nu_k}\omega_1^0 \dots - x_{k-1u_k}\omega_n^0 + x_{nu_k}\omega_{k-1}^0 & -x_{i+k-1u_k}\omega_n^0 + x_{nu_k}\omega_{i+k-1}^0 \end{vmatrix}$$

i.e., the determinant of the $k \times k$ matrix which comes from the first $k - 1$ columns and the $(i + k - 1)$th column of A. Here $i = 1, \dots, n - k$. Then

$$f_i = \begin{vmatrix} \omega_1^0 & \dots & \omega_{k-1}^0 & \omega_{i+k-1}^0 & \omega_n^0 \\ x_{1u_1} & \dots & x_{k-1u_1} & x_{i+k-1u_1} & x_{nu_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{1u_k} & \dots & x_{k-1u_k} & x_{i+k-1u_k} & x_{nu_k} \end{vmatrix}_{(k+1)\times(k+1)}.$$

Proof. Similarly to the proof of Lemma (1.1.12),

$$(\omega_n^0)^{k-1} f_i$$

$$= \begin{vmatrix} -x_{1u_1}\omega_n^0 + x_{nu_1}\omega_1^0 & \dots & -x_{k-1u_1}\omega_n^0 + -x_{nu_1}\omega_{k-1}^0 & -x_{i+k-1u_1}\omega_n^0 + x_{nu_1}\omega_{i+k-1}^0 \\ -x_{1u_2}\omega_n^0 + x_{nu_2}\omega_1^0 & \dots & -x_{k-1u_2}\omega_n^0 + x_{nu_2}\omega_{k-1}^0 & -x_{i+k-1u_2}\omega_n^0 + x_{nu_2}\omega_{i+k-1}^0 \\ \vdots & \vdots & \vdots & \vdots \\ -x_{1u_k}\omega_n^0 + x_{nu_k}\omega_1^0 & \dots & -x_{k-1u_k}\omega_n^0 + x_{nu_k}\omega_{k-1}^0 & -x_{i+k-1u_k}\omega_n^0 + x_{nu_k}\omega_{i+k-1}^0 \end{vmatrix}$$

$$= \begin{vmatrix} -x_{nu_1}\omega_1^0 & \dots & -x_{k-1u_1}\omega_n^0 & -x_{i+k-1u_1}\ \omega_n^0 \\ -x_{nu_2}\omega_1^0 & \dots & -x_{k-1u_2}\omega_n^0 & -x_{i+k-1u_2}\ \omega_n^0 \\ \vdots & \vdots & \vdots & \vdots \\ -x_{nu_k}\omega_1^0 & \dots & -x_{k-1u_k}\omega_n^0 & -x_{i+k-1u_k}\ \omega_n^0 \end{vmatrix}$$

$$+ \begin{vmatrix} x_{nu_1}\omega_1^0 & \dots & -x_{k-1u_1}\omega_n^0 & -x_{i+k-1u_1}\ \omega_n^0 \\ x_{nu_2}\omega_1^0 & \dots & -x_{k-1u_2}\omega_n^0 & -x_{i+k-1u_2}\ \omega_n^0 \\ \vdots & \vdots & \vdots & \vdots \\ x_{nu_k}\omega_1^0 & \dots & -x_{k-1u_k}\omega_n^0 & -x_{i+k-1u_k}\ \omega_n^0 \end{vmatrix}$$

$$\vdots$$

$$+ \begin{vmatrix} -x_{nu_1}\omega_1^0 & \dots & -x_{k-1u_1}\omega_n^0 & x_{nu_1}\ \omega_{i+k-1}^0 \\ -x_{nu_2}\omega_1^0 & \dots & -x_{k-1u_2}\omega_n^0 & x_{nu_2}\ \omega_{i+k-1}^0 \\ \vdots & \vdots & \vdots & \vdots \\ -x_{nu_k}\omega_1^0 & \dots & -x_{k-1u_k}\omega_n^0 & x_{nu_k}\ \omega_{i+k-1}^0 \end{vmatrix}$$

$$= (-1)^k(\omega_n^0)^k \begin{vmatrix} x_{1u_1} & \dots & x_{k-1u_1} & x_{i+k-1u_1} \\ x_{1u_2} & \dots & x_{k-1u_2} & x_{i+k-1u_2} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1u_k} & \dots & x_{k-1u_k} & x_{i+k-1u_k} \end{vmatrix}$$

$$+\omega_1^0(-1)^{k-1}(\omega_n^0)^{k-1} \begin{vmatrix} x_{nu_1} & \dots & x_{k-1u_1} & x_{i+k-1u_1} \\ x_{nu_2} & \dots & x_{k-1u_2} & x_{i+k-1u_2} \\ \vdots & \vdots & \vdots & \vdots \\ x_{nu_k} & \dots & x_{k-1u_k} & x_{i+k-1u_k} \end{vmatrix}$$

$$\vdots$$

$$+\omega_{i+k-1}^0(-1)^{k-1}(\omega_n^0)^{k-1} \begin{vmatrix} x_{1u_1} & \dots & x_{k-1u_1} & x_{nu_1} \\ x_{1u_2} & \dots & x_{k-1u_2} & x_{nu_2} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1u_k} & \dots & x_{k-1u_k} & x_{nu_k} \end{vmatrix}$$

$$= (\omega_n^0)^{k-1} \begin{vmatrix} \omega_1^0 & \dots & \omega_{k-1}^0 & \omega_{i+k-1}^0 & \omega_n^0 \\ x_{1u_2} & \dots & x_{k-1u_1} & x_{i+k-1u_1} & x_{nu_1} \\ \vdots & \vdots & & \vdots & \vdots \\ x_{1u_k} & \dots & x_{k-1u_k} & x_{i+k-1u_k} & x_{nu_k} \end{vmatrix}_{(k+1)\times(k+1)}$$

as desired.

Lemma (1.1.14)[1]: If $\Delta \neq 0$ and $f_i = 0, i = 1, \ldots, n-k$, we have $\omega_0 = \sum_{l=1}^{k} a_l x_{u_l}$, where $a_l \in \mathbb{R}$. Furthermore, $i_j = (-1)^j a_k b_{i_j}, j = 1, \ldots, k-1$, where

$$b_{ij} = \begin{vmatrix} x_{1u_1} & \cdots & x_{j+1u_1} & x_{k-1u_1} & x_{i+k-1u_1} & x_{nu_1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \quad \vdots \\ x_{1u_{k-1}} & \cdots & x_{j+1u_{k-1}} & x_{k-1u_{k-1}} & x_{i+k-1u_{k-1}} & x_{nu_{k-1}} \\ x_{1u_k} & \cdots & x_{j+1u_k} & x_{k-1u_k} & x_{i+k-1u_k} & x_{nu_k} \end{vmatrix}_{k\times k},$$

and $= (-1)^{k-1} a_k b$ where

$$b = \begin{vmatrix} x_{1u_1} & \cdots & x_{k-1u_1} & x_{nu_1} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1u_{k-1}} & \cdots & x_{k-1u_{k-1}} & x_{nu_{k-1}} \\ x_{1u_k} & \cdots & x_{k-1u_k} & x_{nu_k} \end{vmatrix}_{k\times k},$$

$a_k \neq 0$ and $b \neq 0$ since $\Delta \neq 0$.

Proof. $\Delta \neq 0$ and for all $i, f_i = 0$ implies that the following matrix has rank k:

$$\begin{pmatrix} \omega_1^0 & \cdots & \omega_n^0 \\ x_{u_1} & \cdots & x_{nu_1} \\ \vdots & \vdots & \vdots \\ x_{1u_k} & \cdots & x_{nu_k} \end{pmatrix}_{(k+1)\times n} \tag{9}$$

However,

$$\begin{pmatrix} x_{1u_1} & \cdots & x_{nu_1} \\ \vdots & \vdots & \vdots \\ x_{1u_k} & \cdots & x_{nu_k} \end{pmatrix}_{k\times n}$$

has maximal rank, therefore, in (9), the first row can be expressed as the linear combination of the other rows, i.e., $\omega_0 = \sum_{l=1}^{k} a_l x_{u_l}$. $a_k = 0$ follows from $\Delta \neq 0$. Furthermore,

$$\Delta_{ij} = \begin{vmatrix} \omega_1^0 & \cdots & \omega_{j-1}^0 & \omega_{i+k-1}^0 & \omega_{j+1}^0 & \cdots & \omega_{k-1}^0 & \omega_n^0 \\ x_{1u_1} & \cdots & x_{j-1u_1} & x_{i+k-1u_1} & x_{j+1u_1} & \cdots & x_{k-1u_1} & x_{nu_1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{1u_k} & \cdots & x_{j-1u_k} & x_{i+k-1u_k} & x_{j+1u_{k-1}} & \cdots & x_{k-1u_{k-1}} & x_{nu_{k-1}} \end{vmatrix}$$

$$= a_k \begin{vmatrix} \omega_1^0 & \cdots & \omega_{j-1}^0 & \omega_{i+k-1}^0 & \omega_{j+1}^0 & \cdots & \omega_{k-1}^0 & \omega_n^0 \\ x_{1u_1} & \cdots & x_{j-1u_1} & x_{i+k-1u_1} & x_{j+1u_1} & \cdots & x_{k-1u_1} & x_{nu_1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{1u_k} & \cdots & x_{j-1u_k} & x_{i+k-1u_k} & x_{j+1u_{k-1}} & \cdots & x_{k-1u_{k-1}} & x_{nu_{k-1}} \end{vmatrix}$$

$$= (-1)^{k-1} a_k \begin{vmatrix} \omega_1^0 & \cdots & \omega_{j-1}^0 & \omega_{i+k-1}^0 & \omega_{j+1}^0 & \cdots & \omega_{k-1}^0 & \omega_n^0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ x_{1u_{k-1}} & \cdots & x_{j-1u_{k-1}} & x_{i+k-1u_{k-1}} & x_{j+1u_{k-1}} & \cdots & x_{k-1u_{k-1}} & x_{nu_{k-1}} \\ x_{1u_k} & \cdots & x_{j-1u_k} & x_{i+k-1u_k} & x_{j+1u_k} & \cdots & x_{k-1u_k} & x_{nu_k} \end{vmatrix}.$$

Since $(-1)^{(k-1)+(k-j-1)} = (-1)^{-j} = (-1)^j$, the proof for ij is complete. A similar argument applies for Δ.

Lemma (1.1.15)[1]: Suppose that $\omega_0 = \sum_{l=1}^{k} a_l x_{u_l}$, then

$$-(\omega_n^0)^{-1}\left(\Delta_{i1}\omega_1^0 + \cdots + \Delta_{ik-1}\omega_{k-1}^0 - \Delta\omega_{i+k-1}^0\right) = (-1)^{k+1} a_k b_{in},$$

where

$$b_{in} = \begin{vmatrix} x_{1u_1} & \dots & x_{k-1u_1} & x_{i+1k-1u_1} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1u_k} & \dots & x_{k-1u_k} & x_{nu_1} \end{vmatrix}.$$

Proof.

$$-(\omega_n^0)^{-1}\left(\Delta_{i1}\omega_1^0 + \cdots + \Delta_{ik-1}\omega_{k-1}^0 - \Delta\omega_{i+k-1}^0\right)$$

$$a_k\,(\omega_n^0)^{-1}\left((-1)^{1+1}\omega_1^0 b_{i1} + \cdots + (-1)^k\omega_{k-1}^0 b_{ik-1} + (-1)^{k+1}\omega_{i+k-1}^0 b\right)$$

$$= a_k\,(\omega_n^0)^{-1}\begin{vmatrix} \omega_1^0 & \dots & \omega_{k-1}^0 & \omega_{i+k-1}^0 & \omega_n^0 \\ x_{1u_2} & \dots & x_{k-1u_1} & x_{i+k-1u_1} & x_{nu_1} \\ \vdots & \vdots & & \vdots & \vdots \\ x_{1u_k} & \dots & x_{k-1u_k} & x_{i+k-1u_k} & x_{nu_k} \end{vmatrix}$$

$$-a_k\,(\omega_n^0)^{-1}\begin{vmatrix} 0 & \dots & 0 & 0 & \omega_n^0 \\ x_{1u_2} & \dots & x_{k-1u_1} & x_{i+k-1u_1} & x_{nu_1} \\ \vdots & \vdots & & \vdots & \vdots \\ x_{1u_k} & \dots & x_{k-1u_k} & x_{i+k-1u_k} & x_{nu_k} \end{vmatrix}$$

$$= (-1)^{k+1}a_k\begin{vmatrix} x_{1u_1} & \dots & x_{k-1u_1} & x_{i+1k-1u_1} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1u_k} & \dots & x_{k-1u_k} & x_{nu_1} \end{vmatrix}$$

$$= (-1)^{k+1}a_k b_{in}.$$

Theorem (1.1.16)[1]: Let (M, g) be a smooth compact Riemannian manifold of dimension n and Σ be a smooth submanifold of dimension k $(1 \le k < n)$.

(i) If $k = n - 1$, we have, for all φ_λ,

$$\|\varphi_\lambda\|_{L^p(\Sigma)}\, C(1 + \lambda)^{\rho_{n-1}(p,n)}\, \|\varphi_\lambda\|_{L^2(M)},$$

where

$$\rho_{n-1}(p,n) = \begin{cases} \dfrac{n-1}{4} - \dfrac{n-2}{2p} & \text{if } 2 \le p \le \dfrac{2n}{n-1} \\ \dfrac{n-1}{2} - \dfrac{n-1}{p} & \text{if } \dfrac{2n}{n-1} \le p \le +\infty. \end{cases}$$

(ii) If $k = n - 2$, we have, for all φ_λ,

$$\|\varphi_\lambda\|_{L^2(\Sigma)}\, C(1 + \lambda)^{\frac{1}{2}}\left(\log(3 + \lambda)\right)^{\frac{1}{2}}\|\varphi_\lambda\|_{L^2(M)},$$

$$\|\varphi_\lambda\|_{L^p(\Sigma)}\, C(1 + \lambda)^{\rho_{n-2}(p,n)}\|\varphi_\lambda\|_{L^2(M)},$$

where $\rho_{n-2}(p,n) = \frac{n-1}{2} - \frac{n-2}{p}$, $2 < p \le +\infty$.

(iii) If $1 \le k\, n - 3$, we have, for all

$$\varphi_\lambda, \|\varphi_\lambda\|_{L^p(\Sigma)}\, C(1 + \lambda)^{\rho_k(p,n)}\, \|\varphi_\lambda\|_{L^2(M)},$$

where $\rho_k(p,n) = \frac{n-1}{2} - \frac{k}{p}$, $2 \le p \le +\infty$.

Proof. Consider the left projection of the canonical relation $\mathcal{C}$:

$$\pi_L : (u, \omega_1^0, \dots, \omega_{n-1}^0) \to \left(u, d_{u_1}\psi_r, \dots, d_{u_k}\psi_r\right).$$

Since the projection in the u variable is the identity,

$$D\pi_L \sim D_{u\omega_0}\psi_r = \begin{pmatrix} d_{u_1\omega_1^0}\psi_r & d_{u_1\omega_2^0}\psi_r & \cdots & d_{u_1\omega_{n-1}^0}\psi_r \\ d_{u_2\omega_1^0}\psi_r & d_{u_2\omega_2^0}\psi_r & \cdots & d_{u_2\omega_{n-1}^0}\psi_r \\ \vdots & \vdots & \vdots & \vdots \\ d_{u_k\omega_1^0}\psi_r & d_{u_k\omega_2^0}\psi_r & \ldots & d_{u_k\omega_{n-1}^0}\psi_r \end{pmatrix}.$$

Lemma (1.1.9) yields

$$\begin{aligned} \nabla_u\psi_r|_{u=0} &= \nabla_x\psi_r(0,\omega_0)\cdot x_u \\ &= \left(g_0(\omega_0, x_{u_1}), g_0(\omega_0, x_{u_2}), \ldots, g_0(\omega_0, x_{u_k})\right) \\ &= \left(\langle\omega_0, x_{u_1}\rangle, \langle\omega_0, x_{u_2}\rangle, \ldots, \langle\omega_0, x_{u_k}\rangle\right) \\ &= (\omega_1^0, \ldots, \omega_n^0)\begin{pmatrix} x_{1u_1} & x_{2u_1} & \cdots & x_{1u_k} \\ x_{1u_2} & x_{2u_1} & \cdots & x_{2u_k} \\ \vdots & \vdots & \vdots & \vdots \\ x_{nu_1} & x_{nu_2} & \ldots & x_{nu_k} \end{pmatrix}. \end{aligned} \qquad (10)$$

So

$$D\pi_L|_{u=0} = \left(\begin{pmatrix} 1 & 0 & \ldots & 0 & -\frac{\omega_1^0}{\omega_n^0} \\ 0 & 1 & \ldots & 0 & -\frac{\omega_2^0}{\omega_n^0} \\ & & & & \vdots \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & \ldots & 1 & -\frac{\omega_{n-1}^0}{\omega_n^0} \end{pmatrix}\begin{pmatrix} x_{1u_1} & x_{2u_1} & \cdots & x_{1u_k} \\ x_{1u_2} & x_{2u_1} & \cdots & x_{2u_k} \\ \vdots & \vdots & \vdots & \vdots \\ x_{nu_1} & x_{nu_2} & \ldots & x_{nu_k} \end{pmatrix}\right)^{\mathsf{T}}$$

$$= -\frac{1}{\omega_n^0}\begin{pmatrix} -x_{1u_1}\omega_n^0 + x_{nu_1}\omega_1^0 & -x_{2u_1}\omega_n^0 + -x_{nu_1}\omega_2^0 & \ldots & -x_{n-1u_1}\omega_n^0 + x_{nu_1}\omega_{n-1}^0 & x_{nu_1} \\ -x_{1u_2}\omega_n^0 + x_{nu_2}\omega_1^0 & -x_{2u_1}\omega_2^0 + x_{nu_2}\omega_2^0 & \ldots & -x_{n-1u_2}\omega_n^0 + x_{nu_2}\omega_{n-1}^0 & x_{nu_2} \\ \vdots & \vdots & \vdots & & \vdots \\ -x_{1u_k}\omega_n^0 + x_{nu_k}\omega_1^0 & -x_{2u_k}\omega_n^0 + x_{nu_k}\omega_2^0 & \ldots & -x_{n-1u_k}\omega_n^0 + x_{nu_k}\omega_{n-1}^0 & x_{nu_k} \end{pmatrix}$$

$$= -\frac{1}{\omega_n^0}A.$$

From Lemma (1.1.10), $\operatorname{rank}(A) \geq k-1$, which means dim ker $(D\pi_L)^{n-k}$.The critical points of πL are the points where $\operatorname{rank}(A) = k-1$. Assume that the top left $(k-1)\times(k-1)$ block of A is nondegenerate, i.e., $\Delta \neq 0$ (see Lemma (1.1.11)). Then the critical point set $\overline{\Omega}$ at $u = 0$ is $\overline{\Omega_0} = \{(0,\ldots,0,\omega_1^0,\ldots,\omega_{n-1}^0) : f_i = 0, i = 1,\ldots,n-k\}$, where f_i is defined in Lemma (1.1.12). Furthermore, by Lemma (1.1.11), the kernel vector field is spanned by $V_i, i = 1,\ldots,n-k$,

where $V_i = \Delta_{i1}\frac{\partial}{\partial\omega_1^0} + \Delta_{i2}\frac{\partial}{\partial\omega_2^0} + \cdots + \Delta_{ik-1}\frac{\partial}{\partial\omega_{k-1}^0} - \Delta\frac{\partial}{\partial\omega_{i+k-1}^0}$. If $i \neq j$,

$$V_i f_j = \begin{vmatrix} \Delta_{i1} & \ldots & \Delta_{ik-1} & 0 & -(\omega_n^0)^{-1}\left(\Delta_{i1}\omega_1^0 + \cdots + \Delta_{ik-1}\omega_{k-1}^0 - \Delta\omega_{i+k-1}^0\right) \\ x_{1u_1} & \ldots & x_{k-1u_1} & x_{j+k-1u_1} & x_{nu_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{1u_k} & \ldots & x_{k-1u_k} & x_{j+k-1u_k} & x_{nu_k} \end{vmatrix}$$

$$= -a_k \begin{vmatrix} b_{i1} & \dots & (-1)^{k-2} b_{ik-1} & (-1)^k b_{in} \\ x_{1u_1} & \dots & x_{k-1u_1} & x_{j+k-1u_1} & x_{nu_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{1u_k} & \dots & x_{k-1u_1} & x_{j+k-1u_k} & x_{nu_k} \end{vmatrix}$$

$$= -a_k \left(b_i 1 b_{j\ 1} + \dots + b_{ik-1} b_{j\ k-1} + b_{in} b_{j\ n} \right).$$

If $i = j$,

$$V_i f_i = \begin{vmatrix} \Delta_{i1} & \dots & \Delta_{ik-1} - \Delta & -(\omega_n^0)^{-1}\left(\Delta_{i1}\omega_1^0 + \dots + \Delta_{ik-1}\omega_{k-1}^0 - \Delta\omega_{i+k-1}^0\right) \\ x_{1u_1} & \dots & & x_{k-1u_1} & x_{j+k-1u_1} & x_{nu_1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ x_{1u_k} & \dots & & x_{k-1u_k} & x_{j+k-1u_k} & x_{nu_k} \end{vmatrix}$$

$$= -a_k \begin{vmatrix} b_{i1} & \dots & (-1)^{k-2} b_{ik-1} & (-1)^k b_{in} \\ x_{1u_1} & \dots & x_{k-1u_1} & x_{j+k-1u_1} & x_{nu_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{1u_k} & \dots & x_{k-1u_1} & x_{j+k-1u_k} & x_{nu_k} \end{vmatrix}$$

$$= -a_k \left(b_i 1 b_{j\ 1} + \dots + b_{ik-1} b_{j\ k-1} + b_{in} b_{j\ n} \right).$$

Therefore, $(V_i f_j) = -a_k\, (\mathcal{A}\mathcal{A}^\intercal + b^2 I)$, where $a_k \neq 0$ and

$$\mathcal{A} = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1k-1} & b_{in} \\ \vdots & \vdots & & \vdots\ \vdots\ \vdots & \\ b_{n-k1} & b_{n-k2} & \dots & b_{n-kk-1} & b_{n-kn} \end{pmatrix}$$

Clearly $\left(V_i f_j\right)$ is positive definite or negative definite (up to the sign of $-a_k$), so $\det\left(V_i f_j\right) = 0$. Thus, the singularities of π_L are at most submersions with folds when $u = 0$ (or in the coordinate chart by continuity and stability). For $k \leq n - 2$, the desired estimates in [10] ($r \leq 1$) and the relation $\chi_\lambda = \frac{n-1}{2} T_\lambda + R_\lambda$. For $k = n - 1$, we are trying to apply Theorem 2.2 in [10]. It suffices to check that $L_0 = \left\{\left(d_{u_1}\psi_r, \dots, d_{u_k}\psi_r\right) : u = 0 \text{ and } f_i = 0, i = 1, \dots, n - k\right\}$ is a hypersurface with nonzero principal curvatures. In fact, (10) implies that at $u = 0, d_{u_i}\psi_r = \langle \omega_0, x_{u_i} \rangle$. By Lemma (1.1.14), $d_{u_i}\psi_r = \sum_{l=1}^{k} a_l x_{u_l}, x_{u_i}$. So

$$\begin{pmatrix} d_{u_1}\psi_r \\ \vdots \\ d_{u_k}\psi_r \end{pmatrix} = \mathcal{B} \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix},$$

Where

$$\mathcal{B} = \begin{pmatrix} \langle x_{u_1}, x_{u_1} \rangle & \dots & \langle x_{u_1}, x_{u_k} \rangle \\ \vdots & \vdots & \vdots \\ \langle x_{u_k}, x_{u_1} \rangle & \dots & \langle x_{u_k}, x_{u_k} \rangle \end{pmatrix}.$$

On the other hand, $\langle \omega_0, \omega_0 \rangle = 1$ and $\omega_0 = \sum_{l=1}^{k} a_l x_{u_l}$ yield

$$(a_1, \dots, a_k)\mathcal{B} \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix} = 1.$$

Therefore,

$$(d_{u_1}\psi_r,\dots,d_{u_k}\psi_r)\mathcal{B}^{-1}\mathcal{B}\mathcal{B}^{-1}\begin{pmatrix} d_{u_1}\psi_r \\ \vdots \\ d_{u_k}\psi_r \end{pmatrix} = 1.$$

This implies that $\mathcal{L}_0$ is a hyperellipsoid and hence all its $n-2$ principal curvatures are nonzero (by continuity, one get the same result nearby). $\frac{n-1}{2}$ $T_\lambda + R_\lambda$, we get the desired estimates for $k = n-1$.

When $k = n-1$, the estimates in Theorem (1.1.16) can be improved if Σ satisfies some curvature condition. For Theorem (1.1.21), similarly it suffices to show

$$\|T_\lambda^r(f)\|_{L^p(\gamma)} C(1+\lambda)^{\rho(p,n)-\frac{n-1}{2}}\|f\|_{L^2(\mathbb{S}^{n-1})}. \quad (11)$$

Lemma (1.1.17)[1]: Consider the linear equation system $A^\intercal z = 0$ where

$$A = \begin{pmatrix} -x_{1u_1}\omega_n^0 + x_{nu_1}\omega_1^0 & -x_{2u_1}\omega_n^0 + -x_{nu_1}\omega_2^0 & \dots & -x_{n-1u_1}\omega_n^0 + x_{nu_1}\omega_{n-1}^0 & x_{nu_1} \\ -x_{1u_2}\omega_n^0 + x_{nu_2}\omega_1^0 & -x_{2u_1}\omega_2^0 + x_{nu_2}\omega_2^0 & \dots & -x_{n-1u_2}\omega_n^0 + x_{nu_2}\omega_{n-1}^0 & x_{nu_2} \\ \vdots & \vdots & \vdots & \vdots & \\ -x_{1u_k}\omega_n^0 + x_{nu_k}\omega_1^0 & -x_{2u_k}\omega_n^0 + x_{nu_k}\omega_2^0 & \dots & -x_{n-1u_k}\omega_n^0 + x_{nu_k}\omega_{n-1}^0 & x_{nu_k} \end{pmatrix}$$

as in Lemma (1.1.10). Suppose $\det A = 0$ and the top left $(n-2)\times(n-2)$ block of A is nondegenerate, i.e., rank $(B) = n-2$ where

$$B = \begin{pmatrix} -x_{1u_1}\omega_n^0 + x_{nu_1}\omega_1^0 & \dots & -x_{1u_{n-2}}\omega_n^0 + x_{nu_{n-2}}\omega_1^0 \\ \vdots & \vdots & \vdots \\ -x_{m-2u_1}\omega_n^0 + x_{nu_1}\omega_{n-2}^0 & \dots & -x_{n-2u_{n-2}}\omega_n^0 + x_{nu_{n-2}}\omega_{n-2}^0 \end{pmatrix},$$

Replace the j th column in $\overline{B}$ by the $n-1$st column in A (first $n-2$ components in that column) and denote it by $\overline{B_j}$ where $j = 1,\dots,n-2$. If we denote $= (\omega_n^0)^{-(n-3)}\det\overline{B_j}$, then the solution space to the linear equation system $A^\intercal z = 0$ is spanned by $\{w\}$ where $w = (\overline{\Delta_1},\overline{\Delta_2},\dots,\overline{\Delta_{n-2}},-\overline{\Delta})$.

Proof. Since $\det A = 0$ and rank $(B) = n-2, A^\intercal z = 0$ is equivalent to

$$\overline{B}\begin{pmatrix} z_1 \\ \vdots \\ z_2 \end{pmatrix} = -\begin{pmatrix} -x_{1u_{n-1}}\omega_n^0 + x_{nu_{n-1}}\omega_1^0 \\ \vdots \\ -x_{n-2u_{n-1}}\omega_n^0 + x_{nu_{n-1}}\omega_{n-2}^0 \end{pmatrix} z_{n-1}.$$

On the right hand side, fix $z_{n-1} = 1$, by Cramer' rule, $z = \left(-\frac{\overline{\Delta}_1}{\overline{\Delta}}, -\frac{\overline{\Delta}_2}{\overline{\Delta}}, \dots, \frac{\overline{\Delta}_{n-2}}{\overline{\Delta}}, 1\right)$ which yields $w = (\overline{\Delta}_1,\overline{\Delta}_2,\dots,\overline{\Delta}_{n-2},-\overline{\Delta})$, as desired.

Lemma (1.1.18)[1]: Suppose $\overline{\Delta}$ and $\overline{\Delta}_j$ as in Lemma (1.1.17) where $j = 1,\dots,n-2$. If $\det A = 0$ when $u = 0$, we have $\omega 0 = \sum_{l=1}^{n-1} a_l x_{u_l}(0)$, where $a_l \in \mathbb{R}$. Furthermore, $\overline{\Delta}_j = (-1)^{n-2}a_j b$ and $\overline{\Delta} = (-1)^{n-2}a_{n-1}b$ where

$$b = \begin{vmatrix} x_{1u_1} & \dots & x_{n-2u_1} & x_{nu_1} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1u_{n-2}} & \dots & x_{n-2u_{n-2}} & x_{nu_{n-2}} \\ x_{1u_{n-1}} & \dots & x_{n-2u_{n-1}} & x_{nu_{n-1}} \end{vmatrix}$$

as in Lemma (1.1.14).

Proof. If $\det A = 0$, we have

$$\det A = (\omega_n^0)^{n-2} \begin{vmatrix} \omega_1^0 & \dots & \omega_{n-2}^0 & \omega_{n-1}^0 & \omega_n^0 \\ x_{1u_2} & \dots & x_{n-2u_1} & x_{n-1u_1} & x_{nu_1} \\ \vdots & \vdots & & \vdots & \vdots \\ x_{1u_{n-1}} & \dots & x_{n-2u_{n-1}} & x_{n-1u_{n-1}} & x_{nu_{n-1}} \end{vmatrix} = 0. \qquad (12)$$

Since

$$\begin{pmatrix} x_{1u_1} & \dots & x_{nu_1} \\ \vdots & \vdots & \vdots \\ x_{1u_{n-1}} & \dots & x_{nu_{n-1}} \end{pmatrix}_{(n-1)\times n}$$

has rank $n - 1$, the first row in (12) can be expressed as the linear combination of the other rows, i.e., $\omega_0 = \sum_{l=1}^{n-1} a_l x_{u_l}$, where $a_l \in \mathbb{R}$.

Similarly to the proof of Lemma (1.1.12), we obtain

$$\overline{\Delta} = \begin{vmatrix} \omega_1^0 & \dots & \omega_{n-2}^0 & \omega_n^0 \\ x_{1u_2} & \dots & x_{n-2u_1} & x_{nu_1} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1u_{n-1}} & \dots & x_{n-2u_{n-1}} & x_{nu_{n-1}} \end{vmatrix}$$

$$= a_{n-1} \begin{vmatrix} x_{1u_{n-1}} & \dots & x_{n-2u_{n-1}} & x_{nu_{n-1}} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1u_{n-2}} & \dots & x_{n-2u_{n-2}} & x_{nu_{n-2}} \end{vmatrix}$$

$$= (-1)^{n-2}\, a_{n-1}\, b$$

and

$$\overline{\Delta}_j = \begin{vmatrix} \omega_1^0 & \dots & \omega_{n-2}^0 & \omega_n^0 \\ x_{1u_1} & \dots & x_{n-2u_1} & x_{nu_1} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1u_{j-1}} & \vdots & x_{n-2u_{j-1}} & x_{nu_{j-1}} \\ x_{1u_{n-1}} & \dots & x_{n-2u_{n-1}} & x_{nu_{n-1}} \\ x_{1u_{j+1}} & \dots & x_{n-2u_{j+1}} & x_{nu_{j+1}} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1u_{n-2}} & \dots & x_{n-2u_{n-2}} & x_{nu_{n-2}} \end{vmatrix} = (-1)^{n-2} a_j\, b.$$

Lemma (1.1.19)[1]: Let $W = \sum_{l=1}^{n-1} a_l \frac{\partial}{\partial u_l}$. If $\det A = 0$ when $u = 0$, we have $W(\omega(x(u), y))|_{u-0} = 0$.

Proof. Consider $g(\omega(x,y), \omega(x,y)) = 1$, i.e.,

$$(\omega_1, \dots, \omega_n) \begin{pmatrix} g_{11}(x) & \dots & g_{1n}(x) \\ \vdots & \vdots & \vdots \\ g_{n1}(x) & \dots & g_{nn}(x) \end{pmatrix} \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_n \end{pmatrix} = 1. \qquad (13)$$

Since we are working in geodesic normal coordinate system, all the first order derivatives of the metric vanish at the origin and $(g_{ij})|_{x=0} = \mathrm{Id}$. In (13), taking the derivatives with respect to $x_1, \dots, x_n$ respectively, then for $x = 0$,

$$\begin{pmatrix} \omega_{1x_1} & \dots & \omega_{nx_1} \\ \vdots & \vdots & \vdots \\ \omega_{1x_n} & \dots & \omega_{nx_n} \end{pmatrix} \begin{pmatrix} \omega_1^0 \\ \vdots \\ \omega_n^0 \end{pmatrix} = 0. \qquad (14)$$

It follows from Lemma (1.1.9) that

$$\left(\frac{\partial\psi_r}{\partial x_1},\dots,\frac{\partial\psi_r}{\partial x_n}\right) = (\omega_1,\dots,\omega_n)\begin{pmatrix} g_{11}(x) & \dots & g_{1n}(x) \\ \vdots & \vdots & \vdots \\ g_{n1}(x) & \dots & g_{nn}(x)\end{pmatrix}. \tag{15}$$

In (15), taking the derivatives with respect to $x_1,\dots,x_n$ respectively and keeping in mind that all the first order derivatives of the metric vanish at the origin and $(g_{ij})|_{x=0} = \mathrm{Id}$, we have for $x = 0$,

$$\begin{pmatrix} \frac{\partial^2\psi_r}{\partial x_1\partial x_1} & \dots & \frac{\partial^2\psi_r}{\partial x_1\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial^2\psi_r}{\partial x_n\partial x_1} & \dots & \frac{\partial^2\psi_r}{\partial x_n\partial x_n}\end{pmatrix} = \begin{pmatrix} \omega_{1x_1} & \dots & \omega_{nx_1} \\ \vdots & \vdots & \vdots \\ \omega_{1x_n} & \dots & \omega_{nx_n}\end{pmatrix}.$$

So the matrix

$$\begin{pmatrix} \omega_{1x_1} & \dots & \omega_{nx_1} \\ \vdots & \vdots & \vdots \\ \omega_{1x_n} & \dots & \omega_{nx_n}\end{pmatrix} \tag{16}$$

is symmetric when u = 0. (14) and (16) yield

$$\begin{pmatrix} \omega_{1x_1} & \dots & \omega_{nx_1} \\ \vdots & \vdots & \vdots \\ \omega_{1x_n} & \dots & \omega_{nx_n}\end{pmatrix}\begin{pmatrix} \omega_1^0 \\ \vdots \\ \omega_n^0\end{pmatrix} = 0.$$

Therefore, at $u = 0$,

$$W(\omega(x(u),y)) = \sum_{l=1}^{n-1} a_l\frac{\partial\omega}{\partial u_l} = \sum_{l=1}^{n-1} a_l\begin{pmatrix} \omega_{1x_1} & \dots & \omega_{nx_1} \\ \vdots & \vdots & \vdots \\ \omega_{1x_n} & \dots & \omega_{nx_n}\end{pmatrix}x_{u_l}$$
$$= \begin{pmatrix} \omega_{1x_1} & \dots & \omega_{nx_1} \\ \vdots & \vdots & \vdots \\ \omega_{1x_n} & \dots & \omega_{nx_n}\end{pmatrix}\begin{pmatrix} \omega_1^0 \\ \vdots \\ \omega_n^0\end{pmatrix} = 0$$

as desired.

Lemma (1.1.20)[1]: Suppose that the second fundamental form of Σ is (positive or negative) definite. Then, at $u = 0$, the matrix A_1 is (positive or negative) definite where

$$A_1 = \begin{pmatrix} \langle x_{u_1u_1}, N\rangle & \dots & \langle x_{u_1u_{n-1}}, N\rangle \\ \vdots & \vdots & \vdots \\ \langle x_{u_{n-1}u_1}, N\rangle & \dots & \langle x_{u_{n-1}u_{n-1}}, N\rangle\end{pmatrix},$$

and N is the unit normal vector of Σ.

Proof. If the second fundamental form of Σ is definite, we have that

$$\begin{pmatrix} g\left(D_{x_{u_1}x_{u_1}}, N\right) & \dots & g\left(D_{x_{u_1}x_{u_{n-1}}}, N\right) \\ \vdots & \vdots & \vdots \\ g\left(D_{x_{u_{n-1}}x_{u_1}}, N\right) & \dots & g\left(D_{x_{u_{n-1}}x_{u_{n-1}}}, N\right)\end{pmatrix}$$

is definite. It suffices to show that at $u = 0$, $g\left(D_{x_{u_i}} x_{u_j}, N\right) = x_{u_iu_j}, N, i = 1,\dots,n-1$ and $j = 1,\dots,n-1$. Since $g(0) = \mathrm{Id}$, it is enough to show that

$$D_{x_{u_i}} x_{u_j} = \sum_{l=1}^{n} x_{l_{u_iu_j}}\frac{\partial}{\partial x_l}. \tag{17}$$

In fact,

$$D_{x_{u_i}} x_{u_j} = D_{x_{u_i}} \left(\sum_{l=1}^{n} x_{l_{u_j}} \frac{\partial}{\partial x_l} \right) = \sum_{l=1}^{n} \left(\left(D_{x_{u_i}} x_{l_{u_j}} \right) \frac{\partial}{\partial x_l} + x_{l_{u_j}} D_{x_{u_i}} \frac{\partial}{\partial x_l} \right)$$

$$= \sum_{l=1}^{n} \left(x_{l_{u_i u_j}} \frac{\partial}{\partial x_l} + x_{l_{u_j}} D_{x_{u_i}} \frac{\partial}{\partial x_l} \right).$$

Since we are working in the geodesic normal coordinate system, all the Christoffel symbols vanish at $u = 0$, so that $D_{x_{u_i}} \frac{\partial}{\partial x_l} = 0$. Therefore, (17) follows.

Theorem (1.1.21)[1]: Let (M, g) be a smooth compact Riemannian manifold of dimension n and Σ be a smooth submanifold of dimension $n - 1$. Suppose that the second fundamental form of Σ is (positive or negative) definite. Then, we have, for all φ_λ

$$\|\varphi_\lambda\|_{L^p(\Sigma)} C(1 + \lambda)^{\tilde{\rho}(p,n)} \|\varphi_\lambda\|_{L^2(M)},$$

where $\tilde{\rho}(p, n) = \frac{n-1}{3} - \frac{2n-3}{3p}$ if $2 \leq p \leq \frac{2n}{n-1}$.

Proof. From the proof of Theorem (1.1.16), we know that πL has at most fold singularities in U. Now consider π_R which has the same critical point set $\overline{\Omega}$ as $\pi_L : (u, \omega_0) \to (\omega_0, -d\omega_1^0 \psi_r, \dots, -d\omega_{n-1}^0 \psi_r)$. Since the projection on ω_0 part is identity,

$$D_{\pi_R} \sim D_{\omega_0 u} \psi_r = \begin{pmatrix} d_{\omega_1^0 u_1} \psi_r & d_{\omega_1^0 u_2} \psi_r & \dots & d_{\omega_1^0 u_{n-1}} \psi_r \\ d_{\omega_2^0 u_1} \psi_r & d_{\omega_2^0 u_2} \psi_r & \dots & d_{\omega_2^0 u_{n-1}} \psi_r \\ \vdots & \vdots & \vdots & \vdots \\ d_{\omega_{n-1}^0 u_1} \psi_r & d_{\omega_{n-1}^0 u_2} \psi_r & \dots & d_{\omega_{n-1}^0 u_{n-1}} \psi_r \end{pmatrix}.$$

From (10), it is easy to see that at $u = 0$,

$$D_{\pi_R} = \begin{pmatrix} 1 & 0 & \dots & 0 & -\frac{\omega_1^0}{\omega_n^0} \\ 0 & 1 & \dots & 0 & -\frac{\omega_2^0}{\omega_n^0} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -\frac{\omega_{n-1}^0}{\omega_n^0} \end{pmatrix} \begin{pmatrix} x_{1u_1} & x_{2u_1} & \dots & x_{1u_{n-1}} \\ x_{1u_2} & x_{2u_1} & \dots & x_{2u_{n-1}} \\ \vdots & \vdots & \vdots & \vdots \\ x_{nu_1} & x_{nu_2} & \dots & x_{nu_{n-1}} \end{pmatrix}$$

$$= -\frac{1}{\omega_n^0} \begin{pmatrix} -x_{1u_1}\omega_n^0 + x_{nu_1}\omega_1^0 & \dots & -x_{n-1u_1}\omega_n^0 + x_{nu_1}\omega_{n-1}^0 \\ -x_{1u_2}\omega_n^0 + x_{nu_2}\omega_1^0 & \dots & -x_{n-1u_2}\omega_n^0 + x_{nu_2}\omega_{n-1}^0 \\ \vdots & \vdots & \vdots \\ -x_{1u_{n-1}}\omega_n^0 + x_{nu_{n-1}}\omega_1^0 & \dots & -x_{n-1u_k}\omega_n^0 + x_{nu_k}\omega_{n-1}^0 \end{pmatrix}$$

$$-\frac{1}{\omega_n^0} A^{\intercal}. \tag{18}$$

When $u = 0$, the critical point set is $\overline{\Omega_0} = \{(0, \dots, 0, \omega_1^0, \dots, \omega_{n-1}^0) : \det A = 0\}$. WLOG, assume that $\overline{\Delta} \neq 0$. By Lemma (1.1.17) and (1.1.18), a nonvanishing kernel vector field of π_R is $= \sum_{l=1}^{n-1} w_l(x, y) \frac{\partial}{\partial u_l}$, where $w_l = a_l$ when $u = 0$. Note again that all the first order derivatives of the metrics vanish at the origin in geodesic normal coordinates. Thus, in $\overline{\Omega_0}$,

$$W\left(\det\left(D_{\omega_0 u}\psi_r\right)\right) = W\left(\left\|\begin{pmatrix}\dfrac{\partial\omega}{\partial\omega_1^0}\\ \vdots \\ \dfrac{\partial\omega}{\partial\omega_{n-1}^0}\end{pmatrix}\begin{pmatrix} g_{11}(x) & \dots & g_{1n}(x)\\ \vdots & \vdots & \vdots \\ g_{n1}(x) & \dots & g_{nn}(x)\end{pmatrix} x_u\right\|\right)$$

$$= \left(\left\|\begin{pmatrix}\dfrac{\partial\omega}{\partial\omega_1^0}\\ \vdots \\ \dfrac{\partial\omega}{\partial\omega_{n-1}^0}\end{pmatrix} x_u\right\|\right)$$

$$\vdots$$

$$+\left(\left\|\begin{pmatrix}\dfrac{\partial\omega}{\partial\omega_1^0}\\ \vdots \\ \dfrac{\partial\omega}{\partial\omega_{n-1}^0}\end{pmatrix}\begin{pmatrix} g_{11}(x) & \dots & g_{1n}(x)\\ \vdots & \vdots & \vdots \\ g_{n1}(x) & \dots & g_{nn}(x)\end{pmatrix} x_u\right\|\right) + W\left(\det\left(-\frac{1}{\omega_n^0}A^{\intercal}\right)\right)$$

$$= \left(-\frac{1}{\omega_n^0}\right)^{n-1} W\left(\det(A^{\intercal})\right)$$

So we only need to focus on $W(\det A)$ when we stay in $\overline{\Omega_0}$. Recall that

$$\det A = (\omega_n^0)^{n-2}\begin{vmatrix} \omega_1^0 & \dots & \omega_{n-2}^0 & \omega_{n-1}^0 & \omega_n^0 \\ x_{1u_2} & \dots & x_{n-2u_1} & x_{n-1u_1} & x_{nu_1} \\ \vdots & \vdots & & \vdots & \vdots \\ x_{1u_{n-1}} & \dots & x_{n-2u_{n-1}} & x_{n-1u_{n-1}} & x_{nu_{n-1}} \end{vmatrix}.$$

Denote $N = (N_1, \dots, N_n) = \frac{1}{M}(M_{11}, \dots, M_{1n})$, where M_{1j} is the corresponding minor of $\det A$ and $M = \sqrt{M_{11}^2 + \dots + M_{1n}^2}$. Clearly $M = 0$ since

$$\begin{pmatrix} x_{1u_1} & \dots & x_{nu_1} \\ \vdots & \vdots & \vdots \\ x_{1u_{n-1}} & \dots & x_{nu_{n-1}} \end{pmatrix}_{(n-1)\times n}$$

has maximal rank. So $\det A = (\omega_n^0)^{n-2}\widetilde{M}\langle N, \omega_0\rangle$ and $\det A = 0 \Leftrightarrow \langle N, \omega_0\rangle = 0$. Since we have $\det A = 0$ in $\overline{\Omega_0}$,

$$(\omega_n^0)^{n-2} W(\det A) = W(\widetilde{M})\langle N, \omega_0\rangle + \widetilde{M}\langle W(N), \omega_0\rangle$$

$$= \widetilde{M}\ \langle \sum_{l=1}^{n-1} a_l N_{u_l}, \omega\rangle =$$

$$\widetilde{M}\ (a_1, \dots, a_{n-1})\begin{pmatrix} \langle x_{u_1u_1}, N\rangle & \dots & \langle x_{u_1u_{n-1}}, N\rangle \\ \vdots & \vdots & \vdots \\ \langle x_{u_{n-1}u_1}, N\rangle & \dots & \langle x_{u_{n-1}u_{n-1}}, N\rangle \end{pmatrix}\begin{pmatrix} a_1 \\ \vdots \\ a_{n-1}\end{pmatrix}$$

$$\widetilde{M}\ (a_1, \dots, a_{n-1})A_1\begin{pmatrix} a_1 \\ \vdots \\ a_{n-1}\end{pmatrix},$$

where A_1 is defined in Lemma (1.1.20) since it is easy to see that $\langle N_{u_i}, x_{u_j}\rangle = -\langle N, x_{u_iu_j}\rangle$. Clearly $\overline{\Delta} \neq 0$ implies that $a_{n-1} \neq 0$. Lemma (1.1.20) guarantees that A_1

is positive definite or negative definite in Ω_0, then $W(\det A) \neq 0$ which means that $W\left(\det\left(D_{\omega_0 u}\psi_r\right)\right) = 0$. So $W\left(\det\left(D_{\omega_0 u}\psi_r\right)\right) = 0$ in Ω as long as we choose ε small enough. Therefore, π_R has at most fold singularities in U. By Theorem (1.1.7) in [7] (also see [12]), $\|T_\lambda^r(f)\|_{L^2(\Sigma)}\ C(1+\lambda)^{-\frac{n-1}{2}+\frac{1}{6}}\|f\|_{L^2(\mathbb{S}^{n-1})}$. (11) follows by interpolation.

If dim $M = 3$ and dim $\Sigma = 2$, with a locally Euclidian metric g, we can show Theorem (1.1.23).

Lemma (1.1.22)[1]: If a regular surface in $\mathbb{R}^3$ is parameterized by $x(u) = x(u_1, u_2) = (u_1, h(u_1, u_2), u_2)$, then the differential matrix of the Gauss map N is

$$d_N = \frac{1}{\tilde{\Delta}}\begin{pmatrix} h_{u_1u_1} & h_{u_1u_2} \\ h_{u_2u_2} & h_{u_2u_2} \end{pmatrix}\begin{pmatrix} \tilde{g}_{11} & \tilde{g}_{12} \\ \tilde{g}_{12} & \tilde{g}_{22} \end{pmatrix}^{-1}$$

where $\tilde{\Delta} = \sqrt{-1 + h_{u_1}^2 + h_{u_2}^2}$, $\tilde{g}_{11} = \langle x_{u_1}, x_{u_1}\rangle$, $\tilde{g}_{12} = x_{u_1}, x_{u_2}$ and $\tilde{g}_{22} = \langle x_{u_2}, x_{u_2}\rangle$.

Proof. This follows simply by calculations from differential geometry (e.g., see [6] §3.3).

Theorem (1.1.23)[1]: Let Σ be a smooth surface in M. Suppose that either the second fundamental form of Σ is definite or Σ has exactly one zero principal curvature. Let K be the Gauss curvature of Σ and $\tilde{\gamma} = \{P \in \Sigma : K(P) = 0\}$. Suppose $\nabla K \neq 0$ so that $\tilde{\gamma}$ is a C^∞ curve. If the tangent vector corresponding to the zero principal curvature is not tangent to $\tilde{\gamma}$, then for all φ_λ

$$\|\varphi_\lambda\|_{L^p(\Sigma)}\ C(1+\lambda)^{\overline{\rho}(p)}\|\varphi_\lambda\|_{L^2(M)},$$

where $\rho(p) = \frac{3}{5} - \frac{4}{5p}$, $2 \leq p \leq 3$.

Proof. The proof of Theorem (1.1.16) shows that π_L has at most fold singularities. So we only need to consider π_R:

$$\pi_R : (u_1, u_2, \omega_0) \to \left(\omega_0, -d_{\omega_1^0}\psi_r, -d_{\omega_2^0}\psi_r\right)$$
$$\sim (u_1, u_2) \to \left(-d_{\omega_1^0}\psi_r, -d_{\omega_2^0}\psi_r\right).$$

Since our metric is Euclidian, WLOG, we may choose coordinate system such that $x(u) = x(u_1, u_2) = (u_1, h(u_1, u_2), u_2)$ for some smooth function h where $h(0,0) = 0$ and $y = r(\omega_1^0, \omega_2^0, \omega_3^0)$ where $\omega_3^0 = \sqrt{1 - (\omega_1^0)^2 - (\omega_2^0)^2} = 0$. The phase function of T_λ^r is \$\psi_r(x(u_1, u_2), \omega_0) = -\sqrt{(u_1 - r\omega_1^0)^2 + (h(u_1, u_2) - r\omega_2^0)^2 + (u_2 - r\omega_3^0)^2}$.

So by Lemma (1.1.9),

$$D_{\pi_R} = \begin{pmatrix} -d_{\omega_1^0 u_1}\psi_r & -d_{\omega_1^0 u_2}\psi_r \\ -d_{\omega_2^0 u_1}\psi_r & -d_{\omega_{12}^0 u_2}\psi_r \end{pmatrix} = \begin{pmatrix} \dfrac{\partial\omega}{\partial\omega_1^0} \\ -\dfrac{\partial\omega}{\partial\omega_2^0} \end{pmatrix}\left(x_{u_1}, x_{u_2}\right)$$

$$= \begin{pmatrix} -\langle \dfrac{\partial\omega}{\partial\omega_1^0}, x_{u_1}\rangle & -\langle \dfrac{\partial\omega}{\partial\omega_1^0}, x_{u_2}\rangle \\ -\langle \dfrac{\partial\omega}{\partial\omega_2^0}, x_{u_1}\rangle & -\langle \dfrac{\partial\omega}{\partial\omega_2^0}, x_{u_2}\rangle \end{pmatrix}.$$

In the critical point set $\overline{\Omega}$, $\operatorname{rank}\left(D_{\pi_R}\right) = 1$. WLOG, assume that $-\langle \frac{\partial\omega}{\partial\omega_1^0}, x_{u_1}\rangle \neq 0$; then, a nonvanishing kernel vector field is

$$W = -\omega_3^0 d_{\omega_1^0 u_2}\, \psi_r \frac{\partial}{\partial u_1} + \omega_3^0 d_{\omega_1^0 u_1}\, \psi_r \frac{\partial}{\partial u_2}$$
$$= -\omega_3^0 \langle \frac{\partial \omega}{\partial \omega_1^0}, x_{u_1} \rangle \frac{\partial}{\partial u_1} + \omega_3^0 \langle \frac{\partial \omega}{\partial \omega_1^0}, x_{u_1} \rangle \frac{\partial}{\partial u_2}.$$

Note that $x_{u_1} = (1, h_{u_1}, 0), x_{u_2} = \big(0, h_{u_2}, 1\big), x_{u_1u_1} = \big(0, h_{u_1u_1}, 0\big), x_{u_1u_2} = \big(0, h_{u_1u_2}, 0\big)$ and $x_{u_2u_2} = \big(0, h_{u_2u_2}, 0\big)$. By calculation, at $u = 0, W = (\omega_1^0) \frac{\partial}{\partial u_1} + (\omega_3^0) \frac{\partial}{\partial u_2}$ and $W^2 = (\omega_1^0)^2 \frac{\partial^2}{\partial u_1^2} + 2\omega_1^0\omega_3^0 \frac{\partial^2}{\partial_{u_1}\partial_{u_2}} + (\omega_3^0)^2 \frac{\partial^2}{\partial u_2^2}$. Another calculation tells us that

$$\det\big(D_{\pi_R}\big) = \frac{1}{\omega_3^0\psi_r^2} \det G - \frac{u_1 - \omega_1^0 + h_{u_1} h - h_{u_1\omega_2^0}}{\omega_3^0\psi_r^4} \det G_1$$
$$\frac{-u^2 - \omega_3^0 + h_{u_2} h - h_{u_2\omega_2^0}}{\omega_3^0\psi_r^4} \det G_2,$$

Where

$$G = \begin{pmatrix} \omega_1^0 & \omega_2^0 & \omega_3^0 \\ 1 & h_{u_1} & 0 \\ 0 & h_{u_2} & 1 \end{pmatrix}, G_1 = \begin{pmatrix} \omega_1^0 & \omega_2^0 & \omega_3^0 \\ u_1 & h_{u_1} & u_2 \\ 0 & h_{u_2} & 1 \end{pmatrix}, \; G_2 = \begin{pmatrix} \omega_1^0 & \omega_2^0 & \omega_3^0 \\ 1 & h_{u_1} & 0 \\ u_1 & h_{u_2} & u_2 \end{pmatrix}$$

If $\det\big(D_{\pi_R}\big) = 0$ at $u = 0$, we have $\det(G) = 0$ which means $\omega_1^0 = a_1$ and $\omega_3^0 = a^2$ where $a^2 \neq 0$ by Lemma (1.1.18). So, at $u = 0, W = a_1 \frac{\partial}{\partial u_1} + a_2 \frac{\partial}{\partial u_2}$, and $W_2 = a_1^2 \frac{\partial^2}{\partial u_1^2} + 2a_1a_2 \frac{\partial^2}{\partial u_1 \partial u_2} + a_2^2 \frac{\partial^2}{\partial u_2^2}$.

Then, $W\left(\det\big(D_{\pi_R}\big)\right) = \frac{1}{\omega_3^0\psi_r^2} W(\det(G))$ when $u = 0$ and $\det(G) = 0$ since $W\big(\det(G_1)\big) = W\big(\det(G_2)\big) = \det(G_1) = \det(G_2) = 0$ at u = 0. If the second fundamental form of Σ is definite at $u = 0$, then $W\big(\det(G)\big) \neq 0$ at $u = 0$ (and hence nearby) as in Theorem (1.1.21), therefore, there are at most fold singularities for π_R which means that we can obtain even better estimates than we want to prove. At $u = 0$, when $\det(G) = W\big(\det(G)\big) = 0, W_2\left(\det\big(D_{\pi_R}\big)\right) = \frac{1}{\omega_3^0\psi_r^2} W_2\big(\det(G)\big)$ since $W\big(\det(G_1)\big) = W\big(\det(G_2)\big) = W_2\big(\det(G_1)\big) = W_2\big(\det(G_2)\big) = 0$ at $u = 0$.

We focus on $W_2\big(\det(G)\big)$ when $\det(G) = W(\det(G)) = 0$ at u = 0. Calculation shows us that $W(\det(G)) = a_1^2 h_{u_1u_1} + 2a_1a_2 h_{u_1u_2} + a_2^2 h_{u_2u_2}$. If Σ has exactly one zero principal curvature, Lemma (1.1.22) implies that $h_{u_1u_2}^2 - h_{u_1u_1} h_{u_2u_2} = 0$ and one of $h_{u_1u_1}$ and $h_{u_2u_2}$ must be nonzero, otherwise, det $(N) = 0$ would imply that $N = 0$ which yields two zero principal curvatures. Assume $h_{u_1u_1} \neq 0$. Then

$$W\big(\det(G)\big) = h_{u_1u_1}^{-1} \Big(\big(a_1 h_{u_1u_1} + a_2 h_{u_1u_2}\big)^2$$
$$- a_2^2\big(h_{u_1u_2}^2 - h_{u_1u_1} h_{u_2u_2}\big)\Big). W(\det(G))|_{u=0} = 0$$

forces $a_1 h_{u_1u_1} + a_2 h_{u_1u_2} = 0$ at $u = 0$ which means that $W = a_1 \frac{\partial}{\partial u_1} + a_2 \frac{\partial}{\partial u_2}$ is an eigenvector corresponding to the zero eigenvalue at $u = 0$. Since the eigenvector corresponding to the eigenvalue 0 is not tangent to $\tilde{\gamma}$, i.e., $W(h_{u_1u_2}^2 - h_{u_1u_1} h_{u_2u_2}) \neq 0$, keeping in mind that $a_1 h_{u_1u_1} + a_2 h_{u_1u_2} = 0$ at $u = 0$ and $a_2 \neq 0$, we have that at $u = 0$,

$$W_2(\det(G)) = W\left(h_{u_1u_1}^{-1}\left(\left(a_1 h_{u_1u_1} + a_2 h_{u_1u_2}\right)^2 - a_2^2\left(h_{u_1u_2}^2 - h_{u_1u_1} h_{u_2u_2}\right)\right)\right)$$
$$= h_{u_1u_1}^{-1} a_2^2 W\left(h_{u_1u_2}^2 - h_{u_1u_1} h_{u_2u_2}\right) \neq 0.$$

So $W_2(\det\pi_R) \neq 0$, which implies that π_R has at most type 2 singularities at u = 0 (and thus nearby). By Theorem (1.1.1) in [7],

$$\|T_\lambda^r(f)\|_{L^2(\Sigma)}\, C(1 + \lambda)^{-1+\frac{1}{5}}\|f\|_{L^2(\mathbb{S}^2)},$$

and Theorem (1.1.23) follows from interpolation. From Theorem (1.1.23), we immediately get the following eigenfunction estimates:

$$\|\varphi_\lambda\|_{L^p(\Sigma)}\, C(1 + \lambda)^{\overline{\rho}(p)+1}\|\varphi_\lambda\|_{L^2(M)}.$$

Section (1.2): Compact Boundaryless Riemannian Manifolds with Nonpositive Sectional Curvature

For (M, g) be a compact, smooth n-dimensional boundaryless Riemannian manifold with nonpositive sectional curvature. Denote Δ_g the Laplace-Beltrami operator associated to the metric g, and $d_g(x, y)$ the geodesic distance between x and y associated with the metric g. We know that there exist $\lambda \geq 0$ and $\phi_\lambda \in L^2(M)$ such that $-\Delta_g\phi_\lambda = \lambda^2\phi_\lambda$, and we call ϕ_λ an eigenfunction corresponding to the eigenvalue λ. Let $\{e_j(x)\}_{j\in\mathbb{N}}$ be an $L^2(M)$-orthonormal basis of eigenfunctions of $\sqrt{-\Delta_g}$, with eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$, and $\{E_j(x)\}_{j\in\mathbb{N}}$ be the projections onto the j-th eigenspace, restricted to Σ, i.e. $E_jf(x) = e_j(x)\int_M e_j(y)f(y)dy$, for any $f \in L^2(M), x \in \Sigma$. We may consider only the positive λ's as we are interested in the asymptotic behavior of the eigenfunction projections. We have the following.

Theorem (1.2.1)[19]: Let (M, g) be a compact smooth n-dimensional boundaryless Riemannian manifold with nonpositive curvature, and Σ be a k-dimensional smooth submanifold on M. Let $\{E_j(x)\}_{j\in\mathbb{N}}$ be the projections onto the j-th eigenspace, restricted to Σ. Given any $f \in L^2(M)$, we have the following estimates: When $k = n - 1$,

$$\left\|\sum_{|\lambda_j-\lambda|\leq(\log\lambda)^{-1}} E_jf\right\|_{L^p(\Sigma)} \lesssim \frac{\lambda^{\delta(p)}}{(\log\lambda)^{\frac{1}{2}}}\|f\|_{L^2(M)}, \qquad \forall p > \frac{2n}{n-1}. \tag{19}$$

When $k \leq n - 2$,

$$\left\|\sum_{|\lambda_j-\lambda|\leq(\log\lambda)^{-1}} E_jf\right\|_{L^p(\Sigma)} \lesssim \frac{\lambda^{\delta(p)}}{(\log\lambda)^{\frac{1}{2}}}\|f\|_{L^2(M)}, \qquad \forall p > 2, \tag{20}$$

where $\delta(p) = \frac{n-1}{2} - \frac{k}{p}$.

Note that we may assume that (M, g) is also simply connected in the proof.

The following corollary is an immediate consequence of this theorem.

Corollary (1.2.2)[19]: Let (M, g) be a compact smooth n-dimensional boundaryless Riemannian manifold with nonpositive curvature, and Σ be a k-dimensional smooth sub manifold on M. For any eigenfunction ϕ_λ of Δ_g s.t. $-\Delta_g\phi_\lambda = \lambda^2\phi_\lambda$, we have the following estimate:

When $k = n - 1$,

$$\|\phi_\lambda\|_{L^p(\Sigma)} \lesssim \frac{\lambda^{\delta(p)}}{(\log \lambda)^{\left(\frac{1}{2}\right)}} \|\phi_\lambda\|_{L^2(M)}, \qquad \forall p > \frac{2n}{n-1}. \tag{21}$$

When $k \leq n - 2$,

$$\|\phi_\lambda\|_{L^p(\Sigma)} \lesssim \frac{\lambda^{\delta(p)}}{(\log \lambda)^{\left(\frac{1}{2}\right)}} \|\phi_\lambda\|_{L^2(M)}, \qquad \forall p > 2. \tag{22}$$

where $\delta(p) = \frac{n-1}{2} - \frac{k}{p}$.

In [26], Reznikov achieved weaker estimates for hyperbolic surfaces, which inspired this current line of research. In [5], Theorem 3, Burq, G´erard and Tzvetkov showed that given any k-dimensional sub manifold Σ of an n-dimensional compact boundaryless manifold M, for any $p > \frac{2n}{n-1}$ when $k = n - 1$ and for any $p > 2$ when $k \leq n - 2$, one has

$$\|\phi_\lambda\|_{L^p(\Sigma)} \lesssim \lambda^{\delta(p)} \|\phi_\lambda\|_{L^2(M)}, \tag{23}$$

while for $p = \frac{2n}{n-1}$ when $k = n - 1$ and for $p = 2$ when $k = n - 2$ one has

$$\|\phi_\lambda\|_{L^p(\Sigma)} \lesssim \lambda^{\delta(p)} (\log \lambda)^{\frac{1}{2}} \|\phi_\lambda\|_{L^2(M)}. \tag{24}$$

Later on, Hu improved the result at one endpoint in [1], so that one has (23) for $p = \frac{2n}{n-1}$ when $k = n - 1$. It is very possible that one can also improve the result at the other endpoint, where $p = 2, k = n - 2$, so that we also have (23) there. Our Theorem (1.2.8) gives an improvement for (23) of $(\log \lambda)^{-\frac{1}{2}}$ for $p \geq 2$ for certain small k's (see the remark after Theorem (1.2.8)).

Note that their proof of Theorem 3 in [5] indicates that for any $f \in L^2(M)$,

$$\left\| \sum_{|\lambda_j - \lambda| < 1} E_j f \right\|_{L^p(\Sigma)} \lesssim \lambda^{\delta(p)} \|f\|_{L^p(M)}, \tag{25}$$

for any $p \geq \frac{2n}{n-1}$ when $k = n - 1$ and $p \geq 2$ when $k \leq n - 2$ except that there is an extra $(\log \lambda)^{\frac{1}{2}}$ on the right hand side when $p = 2$ and $k = n - 2$. In the proof, they constructed $\chi_\lambda = \chi\left(\sqrt{-\Delta_g} - \lambda\right)$ from $L^2(M)$ to $L^p(\Sigma)$, where $\chi \in S(\mathbb{R})$ such that $\chi(0) = 1$, and showed that $\chi_\lambda (\chi_\lambda)^*$ is an operator from $L^p(\Sigma)$ to $L^{p'}(\Sigma)$ with norm $O\left(\lambda^{2\delta(p)}\right)$. That means there exists at least an $\varepsilon > 0$ such that

$$\left\| \sum_{|\lambda_j - \lambda| < \varepsilon} E_j f \right\|_{L^p(\Sigma)} \lesssim \lambda^{\delta(p)} \|f\|_{L^p(M)}, \tag{26}$$

The reason why (26) is true can be seen in the following way. Consider the dual form of

$$\left\| \chi\left(\lambda - \sqrt{-\Delta_g}\right) f \right\|_{L^p(\Sigma)} \lesssim \lambda^{\delta(p)} \|f\|_{L^2(M)}, \tag{27}$$

which says

$$\left\| \sum_j \chi(\lambda - \lambda_j) E_j^* g \right\|_{L^2(M)} \lesssim \lambda^{\delta(p)} \|g\|_{L^p(\Sigma)}, \tag{28}$$

where E_j^* is the conjugate operator of E_j such that $E_j^* g(x) = e_j(x) \int_\Sigma e_j(y) g(y) dy$, for any $g \in L^2(\Sigma)$ and $x \in M$. There exists an $\varepsilon > 0$ such that $\chi(t) > \frac{1}{2}$ when $|t| < \varepsilon$ because we assumed that $\chi(0) = 1$. Therefore, the square of the left hand side of (28) is

$$\sum_{|\lambda-\lambda_j|<\varepsilon} \left\|\chi(\lambda-\lambda_j)E_j^* g\right\|_{L^2(M)}^2 + \sum_{|\lambda-\lambda_j|>\varepsilon} \left\|\chi(\lambda-\lambda_j)E_j^* g\right\|_{L^2(M)}^2$$

$$\geq \frac{1}{4} \sum_{|\lambda-\lambda_j|<\varepsilon} \left\|E_j^* g\right\|_{L^2(M)}^2. \tag{29}$$

That means

$$\left\| \sum_{|\lambda-\lambda_j|<\varepsilon} E_j^* g \right\|_{L^2(M)} \lesssim \lambda^{\delta(p)} \|g\|_{L^p(\Sigma)}, \tag{30}$$

which is the dual version of (26).

If we divide the interval $(\lambda - 1, \lambda + 1)$ into $\frac{1}{\varepsilon}$ sub-intervals whose lengths are 2ε, and apply the last estimate $\frac{1}{\varepsilon}$ times, we get (25). Thinking in this way, our estimates (19) and (20) are equivalent to the estimates for

$$\left\| \sum_{|\lambda_j-\lambda|<\varepsilon \log^{-1}\lambda} E_j \right\|_{L^2(M)} \to L^p(\Sigma), \tag{31}$$

for some number $\varepsilon > 0$, which is equivalent to estimating

$$\left\|\chi\left(T\left(\lambda - \sqrt{-\Delta g}\right)\right)\right\|_{L^2(M)} \to L^p(\Sigma), \tag{32}$$

for $T \approx \log^{-1}\lambda$.

The estimates (23) and (24) are sharp (except for the $(\log\lambda)^{\frac{1}{2}}$ loss) when M is the standard sphere $\mathbb{S}^n$ and Σ is any submanifold of dimension k, when it is saturated by the zonal spherical harmonics. It is natural to try to improve it on Riemannian manifolds with nonpositive sectional curvature. Recently, Sogge and Zelditch in [30] showed that for any 2-dimensional compact boundaryless Riemannian manifold with nonpositive curvature one has

$$\sup_{\gamma\in\Pi} \|\phi_\lambda\|_{L^p(\gamma)} / \|\phi_\lambda\|_{L^2(M)} = o\left(\lambda^{\frac{1}{4}}\right), \qquad for\ 2 \leq p < 4, \tag{33}$$

where Π denotes the space of all unit-length geodesics in M. Our result implies that on 2-dimensional manifolds, we have $\sup_{\gamma\in\Pi} \|\phi_\lambda\|_{L^p(\gamma)} / \|\phi_\lambda\|_{L^2(M)} = O\left(\frac{\lambda^{\frac{1}{2}-\frac{1}{p}}}{(\log\lambda)^{\frac{1}{2}}}\right)$ for $p > 4$. This together with (33) improves (23) for the whole range of p in dimension 2 except for $p = 4$. Note that (25) is sharp for any compact manifold, in the sense that we fix the scale of the spectral projection (see [5]), but if we are allowed to consider a smaller scale of spectral projection, then our Theorem (1.2.1) is an improvement of $(\log\lambda)^{\frac{1}{2}}$ for (25), with the extra assumption that M has nonpositive curvature, and the corollary is an improvement of (23).

Theorem (1.2.1) is related to certain L^p-estimates for eigenfunctions. For example, for 2-dimensional Riemannian manifolds, Sogge showed in [28] that

$$\frac{\|\phi_\lambda\|_{L^p(M)}}{\|\phi_\lambda\|_{L^2(M)}} = o\left(\lambda^{\frac{1}{2}\left(\frac{1}{2}-\frac{1}{p}\right)}\right) \tag{34}$$

for some $2 < p < 6$ if and only if

$$\sup_{\gamma\in\Pi}\|\phi_\lambda\|_{L^p(\gamma)}/\|\phi_\lambda\|_{L^2(M)} = o\left(\lambda^{\frac{1}{4}}\right). \tag{35}$$

This indicates relations between the restriction theorem and the L^p-estimates for eigenfunctions in [15] by Sogge, which showed that for any compact Riemannian manifold of dimension n, one has

$$\|\phi_\lambda\|_{L^p(M)} \lesssim \lambda^{\frac{n-1}{2}\left(\frac{1}{2}-\frac{1}{p}\right)}\|\phi_\lambda\|_{L^2(M)}, \qquad for\ 2 \le p \le \frac{2(n+1)}{n-1}, \tag{36}$$

And

$$\|\phi_\lambda\|_{L^p(M)} \lesssim \lambda^{n\left(\frac{1}{2}-\frac{1}{p}\right)-\frac{1}{2}}\|\phi_\lambda\|_{L^2(M)}, \qquad for\ \frac{2(n+1)}{n-1} \le p \le \infty. \tag{37}$$

There have been several results showing that (37) can be improved for $p > \frac{2(n+1)}{n-1}$ (see [29] and [17]) to bounds of the form

$$\|\phi_\lambda\|_{L^p(M)}/\|\phi_\lambda\|_{L^2(M)} = o\left(\lambda^{n\left(\frac{1}{2}-\frac{1}{p}\right)-\frac{1}{2}}\right)$$

for fixed $p > 6$. Recently, Hassell and Tacey [23], following B´erard's [20] estimate for $p = \infty$, showed that for fixed $p > 6$, this ratio is $O(\lambda^{n\left(\frac{1}{2}-\frac{1}{p}\right)-\frac{1}{2}}/\sqrt{\log\lambda})$ on Riemannian manifolds with constant negative curvature.

We first analyze the situation for any dimension n.

Take a real-valued multiplier operator $\chi \in S(\mathbb{R})$ such that $\chi(0) = 1$, and $\hat{\chi}(t) = 0$ if $|t| \ge \frac{1}{2}$. Let $\rho = \chi^2$; then $\hat{\rho}(t) = 0$ if $|t| \ge 1$. Here, $\hat{\chi}$ is the Fourier transform of χ.

For some number T, which will be determined later, and is approximately $\log\lambda$, we have $\chi\left(T\left(\lambda - \sqrt{-\Delta_g}\right)\right)\varphi_\lambda = \varphi_\lambda$. The theorem is proved if we can show that for any $f \in L^2(M)$,

$$\left\|\chi_T^\lambda f\right\|_{L^p(\Sigma)} \lesssim \frac{\lambda^{\delta(p)}}{(\log\lambda)^{\frac{1}{2}}}\|f\|_{L^2(M)}, \tag{38}$$

where $\chi_T^\lambda = \chi\left(T\left(\lambda - \sqrt{-\Delta_g}\right)\right)$ is an operator from $L^2(M)$ to $L^p(\Sigma)$.

This is equivalent to saying that for any $g \in L^{p'}(\Sigma)$,

$$\left\|\chi_T^\lambda\left(\chi_T^\lambda\right)^* g\right\|_{L^p(\Sigma)} \lesssim \frac{\lambda^{2\delta(p)}}{\log\lambda}\|g\|_{L^{p'}(\Sigma)}, \tag{39}$$

where p' is the conjugate number of p such that $\frac{1}{p} + \frac{1}{p'} = 1$, and $\left(\chi_T^\lambda\right)^*$ is the conjugate operator of χ_T^λ, which maps $L^{p'}(\Sigma)$ into $L^2(M)$.

If $\{e_j(x)\}_{j\in\mathbb{N}}$ is an $L^2(M)$ orthonormal basis of eigenfunctions of $\sqrt{-\Delta_g}$, with eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$, and $\{E_j(x)\}_{j\in\mathbb{N}}$ is the projection onto the j-th eigenspace restricted to Σ, then $I|_\Sigma = \sum_{j\in\mathbb{N}} E_j$, and $\sqrt{-\Delta_g}|_\Sigma = \sum_{j\in\mathbb{N}} \lambda_j E_j$. If we set $\rho_T^\lambda = \rho\left(T\left(\lambda - \sqrt{-\Delta_g}\right)\right)$: $L^2(M) \to L^p(\Sigma)$, then the kernel of $\chi_T^\lambda\left(\chi_T^\lambda\right)^*$ is the kernel of ρ_T^λ, which is restricted to $\Sigma \times \Sigma$. This can be seen in the following way.

Expand χ_T^λ and $\left(\chi_T^\lambda\right)^*$,

$$\chi_T^\lambda f(x) = \sum_{j\in\mathbb{N}} \chi\left(T(\lambda-\lambda_j)\right) e_j(x)\int_M e_j(y)f(y)dy, \qquad \forall f\in L^2(M), \quad (40)$$

and

$$(\chi_T^\lambda)^* g(x) = \sum_{j\in\mathbb{N}} \chi\left(T(\lambda-\lambda_j)\right) e_j(x)\int_\Sigma e_j(y)f(y)dy, \forall f\in p'(\Sigma), \quad (41)$$

Then

$$\begin{aligned}\chi_T^\lambda(\chi_T^\lambda)^* g(x) &= \sum_{i,j\in\mathbb{N}} \chi(T(\lambda-\lambda_i))\chi\left(T(\lambda-\lambda_j)\right) e_j(x)\int_M e_j(y)e_i(y)\int_\Sigma e_i(z)g(z)dzdy\\ &= \sum_{j\in\mathbb{N}} \chi\left(T(\lambda-\lambda_j)\right)^2 e_j(x)\int_\Sigma e_j(z)g(z)dz\\ &= \sum_{j\in\mathbb{N}} \rho\left(T(\lambda-\lambda_j)\right) e_j(x)\int_\Sigma e_j(z)g(z)dz. \qquad (42)\end{aligned}$$

On the other hand,

$$\begin{aligned}\rho_T^\lambda &= \sum_{j\in\mathbb{N}} \rho\left(T(\lambda-\lambda_j)\right)E_j = \sum_{j\in\mathbb{N}} \frac{1}{2\pi}\int_{-1}^{1}\hat\rho(t)e^{it[T(\lambda-\lambda_j)]}E_j dt\\ &= \sum_{j\in\mathbb{N}} \frac{1}{2\pi T}\int_{-T}^{T}\hat\rho\left(\frac{t}{T}\right)e^{it(\lambda-\lambda_j)}E_j dt = \frac{1}{2\pi T}\int_{-T}^{T}\hat\rho\left(\frac{t}{T}\right)e^{it(\lambda-\sqrt{-\Delta_g})}dt\\ &= \frac{1}{\pi T}\int_{-T}^{T}\hat\rho\left(\frac{t}{T}\right)\cos\left(t\sqrt{-\Delta_g}\right)e^{it\lambda}dt - \rho\left(T\left(\lambda+\sqrt{-\Delta_g}\right)\right). \qquad (43)\end{aligned}$$

Here, $\rho\left(T(\lambda+\sqrt{-\Delta_g})\right)$ is an operator whose kernel is $O(\lambda^{-N})$, for any $N\in\mathbb{N}$, so that we only have to estimate the first term. We are not going to emphasize the restriction to Σ until we get to the point when we take the L^p norm on Σ.

Denote the kernel of $\cos(t\sqrt{-\Delta_g})$ as $\cos(t\sqrt{-\Delta_g})(x,y)$, for $x,y\in M$. Then for any $g\in L^{p'}(\Sigma)$,

$$\chi_T^\lambda(\chi_T^\lambda)^* g(x) = \frac{1}{\pi T}\int_\Sigma\int_{-T}^{T}\hat\rho\left(\frac{t}{T}\right)\cos\left(t\sqrt{-\Delta_g}\right)(x,y)e^{it\lambda}g(y)dtdy + O(1). \quad (44)$$

Take the $L^p(\Sigma)$ norm on both sides,

$\|\chi_T^\lambda(\chi_T^\lambda)^* g\|\_(L^p(\Sigma)$

$$\leq \frac{1}{\pi T}\left(\int_\Sigma\left|\int_\Sigma\int_{-T}^{T}\hat\rho\left(\frac{t}{T}\right)\cos\left(t\sqrt{-\Delta_g}\right)(x,y)e^{it\lambda}g(y)dtdy\right|^p dx\right)^{\frac{1}{p}} + O(1). \qquad (45)$$

We are going to use Young's inequality (see [16]), with $\frac{1}{r} = 1-\left[\left(1-\frac{1}{p}\right)-\frac{1}{p}\right] = \frac{2}{p}$, and

$$K(x,y) = \frac{1}{\pi T}\int_{-T}^{T}\hat\rho\left(\frac{t}{T}\right)\left(cost\sqrt{-\Delta_g}\right)(x,y)e^{it\lambda}dt. \qquad (46)$$

Denote K as the operator with the kernel $K(x,y)$ from now on.

Since $K(x,y)$ is symmetric in x and y, once we have

$$\sup_{x\in\Sigma}\|K(x,\cdot)\|_{L^r(\Sigma)} \lesssim \frac{\lambda^{2\delta(p)}}{\log\lambda}, \qquad (47)$$

where $r = p/2$, then by Young's inequality, the theorem is proved.

We can use the same argument as in [30] to lift the manifold to $\mathbb{R}^n$. As stated in Theorem IV.1.3 in [25], for (M, g) with nonpositive curvature, considering x to be a fixed point on Σ, there exists a universal covering map $p = \exp_x: \mathbb{R}^n \to$. In this way, (M, g) is lifted to $(\mathbb{R}^n, \tilde{g})$, with the metric $\tilde{g} = (\exp_x\ )^* g$ being the pullback of g via $\exp_x$. $\tilde{g}$ is a complete Riemannian metric on $\mathbb{R}^n$. Define an automorphism for $(\mathbb{R}^n, \tilde{g}), \alpha: \mathbb{R}^n \to \mathbb{R}^n$, to be a deck transformation if

$$p \circ \alpha = p,$$

when we shall write $\alpha \in Aut(p)$. If $\tilde{x} \in \mathbb{R}^n$ and $\alpha \in Aut(p)$ (let us call $\alpha(\tilde{x})$ the translate of $\tilde{x}$ by α), then we call a simply connected set $D \subset \mathbb{R}^n$ a fundamental domain of our universal cover p if every point in $\mathbb{R}^n$ is the translate of exactly one point in D. We can then identify our submanifold Σ in (M, g) uniquely with a submanifold in $D \subset \mathbb{R}^n$ with one-to-one correspondence. Likewise, a function $f(x)$ in M is uniquely identified by one $f_D(\tilde{x})$ on D if we set $f_D(\tilde{x}) = f(x)$, where $\tilde{x}$ is the unique point in $D \cap p^{-1}(x)$. Using f_D we can define a "periodic extension", $\tilde{f}$, of f to $\mathbb{R}^n$ by defining $\tilde{f}(\tilde{y})$ to be equal to $f_D(\tilde{x})$ if $\tilde{x} = \tilde{y}$ modulo $Aut(p)$, i.e. if $(\tilde{x}, \alpha) \in D \times Aut(p)$ are the unique pair so that $\tilde{y} = \alpha(\tilde{x})$.

We shall exploit the relationship between solutions of the wave equation on (M, g) of the form

$$\begin{cases} (\partial_t^2 - \Delta_g) u(t, x) = 0, (t, x) \in \mathbb{R}_+ \times M, \\ u(0,\cdot) = f, \qquad \partial_t u(0,\cdot) = 0, \end{cases} \tag{48}$$

and certain ones on $(\mathbb{R}^n, \tilde{g})$,

$$\begin{cases} (\partial_t^2 - \Delta \tilde{g}) \tilde{u}(t, \tilde{x}) = 0, (t, \tilde{x}) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ \tilde{u}(0,\cdot) = \tilde{f}, \partial_t \tilde{u}(0,\cdot) = 0. \end{cases} \tag{49}$$

If $(f(x), 0)$ is the Cauchy data in (48) and $(\tilde{f}(\tilde{x}), 0)$ is the periodic extension to $(\mathbb{R}^n, \tilde{g})$, then the solution $\tilde{u}(t, \tilde{x})$ to (49) must be a periodic function of $\tilde{x}$ since $\tilde{g}$ is the pullback of g via p and $p \circ \alpha = p$. As a result, we have that the solution to (48) must satisfy $u(t, x) = \tilde{u}(t, \tilde{x})$ if $\tilde{x} \in D$ and $p(\tilde{x}) = x$. Thus, periodic solutions to (49) correspond uniquely to solutions of (48). Note that $u(t, x) = \cos\left(t\sqrt{-\Delta_g}\right) f(x)$ is the solution of (48), so that

$$\cos\left(t\sqrt{-\Delta_g}\right)(x, y) = \sum_{\alpha \in Aut(p)} \cos\left(t\sqrt{-\Delta_{\tilde{g}}}\right)\left(\tilde{x}, \alpha(\tilde{y})\right), \tag{50}$$

if $\tilde{x}$ and $\tilde{y}$ are the unique points in D for which $p(\tilde{x}) = x$ and $p(\tilde{y}) = y$.

While we can prove Theorem (1.2.1) for any dimension n, we will prove the case when $n = 2$ first separately, as it is the simplest case, and does not involve interpolation or various subdimensions. Here is what it says.

Theorem (1.2.3)[19]: Let (M, g) be a compact smooth boundaryless Riemannian surface with nonpositive curvature, and γ be a smooth curve with finite length. Then for any $f \in L^2(M)$, we have the following estimate:

$$\left\| \sum_{|\lambda_j - \lambda| < (\log \lambda)^{-1}} E_j f \right\|_{L^p(\gamma)} \lesssim \frac{\lambda^{\frac{1}{2} - \frac{1}{p}}}{(\log \lambda)^{\frac{1}{2}}} \|f\|_{L^2(M)}, \forall p > 4. \tag{51}$$

We will prove Theorem (1.2.3). By a partition of unity, we can assume that we fix x to be the midpoint of γ, and parametrize γ by its arc length centered at x so that

$$\gamma = \gamma[-1, 1] and\ \gamma(0) = x, \tag{52}$$

and we may assume that the geodesic distance between any x and $y \in \gamma$ is comparable to the arc length between them on γ. We need to estimate the $L^r(\gamma)$ norm of

$$\int_{-T}^{T} \hat{\rho}\left(\frac{t}{T}\right)\left(cost\sqrt{-\Delta_g}\right)(x,y)e^{it\lambda}dt$$
$$= \sum_{\alpha \in Aut(p)} \int_{-T}^{T} \hat{\rho}\left(\frac{t}{T}\right)\left(cost\sqrt{-\Delta_{\tilde{g}}}\right)(\tilde{x},\alpha(\tilde{y}))e^{it\lambda}dt. \quad (53)$$

We should have the following estimates:

Up to an error of $O(\lambda^{-1}) \exp\left(O\big(d\tilde{g}(\tilde{x},\tilde{y})\big)\right) + O(e^{dT})$ or $O(\lambda^{-1}) \exp\left(O\left(d\tilde{g}\big(\tilde{x},\alpha(\tilde{y})\big)\right)\right) + O(e^{dT})$ respectively,

$$\int_{-T}^{T} \hat{\rho}\left(\frac{t}{T}\right)\left(cost\sqrt{-\Delta_{\tilde{g}}}\right)(\tilde{x},\tilde{y})e^{it\lambda}dt = O(\lambda) when\ d_{\tilde{g}}(\tilde{x},\tilde{y}) < \frac{1}{\lambda}, \quad (54)$$

$$\int_{-T}^{T} \hat{\rho}\left(\frac{t}{T}\right)\left(cost\sqrt{-\Delta_{\tilde{g}}}\right)(\tilde{x},\tilde{y})e^{it\lambda}dt = O\left(\left(\frac{\lambda}{d_{\tilde{g}}(\tilde{x},\tilde{y})}\right)^{\frac{1}{2}}\right) when\ d_{\tilde{g}}(\tilde{x},\tilde{y}) \geq \frac{1}{\lambda}, (55)$$

$$\alpha \neq Id, \qquad \int_{-T}^{T} \hat{\rho}\left(\frac{t}{T}\right)\left(cost\sqrt{-\Delta_{\tilde{g}}}\right)(\tilde{x},\alpha(\tilde{y}))e^{it\lambda}dt = O\left(\left(\frac{\lambda}{d_{\tilde{g}}\big(\tilde{x},\alpha(\tilde{y})\big)}\right)^{\frac{1}{2}}\right). (56)$$

To prove (55) and (56), we need the following lemma.

Lemma (1.2.4)[19]: Assume that $w(\tilde{x},\tilde{x}')$ is a smooth function from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^n$,and $\Theta \in \mathbb{S}^{n-1}$. Then there exist constants $a_\pm$ such that

$$\int_{\mathbb{S}^{n-1}} e^{iw(\tilde{x},\tilde{x})\cdot\Theta}d\Theta = \sqrt{2\pi}^{n-1}\sum_{\pm} a_\pm \frac{e^{\pm i|w(\tilde{x},\tilde{x})|}}{|w(\tilde{x},\tilde{x}')|^{\frac{n-1}{2}}} + O\left(|w(\tilde{x},\tilde{x})|^{\frac{-n-1}{2}-1}\right), \quad (57)$$

when $|w(\tilde{x},\tilde{x})| \geq 1$. The proof can be found in Chapter 1 of [16]. Let us return to estimating the kernel $K(x,y)$. Applying the Hadamard Parametrix,

$$\cos\left(t\sqrt{-\Delta_{\tilde{g}}}\right)(\tilde{x},\alpha(\tilde{y})) = \frac{w_0\big(\tilde{x},\alpha(\tilde{y})\big)}{(2\pi)^n}\sum_{\pm}\int_{\mathbb{R}^n} e^{i\Phi(\tilde{x},\alpha(\tilde{y}))\cdot\xi \pm it|\xi|}d\xi$$
$$+ \sum_{\nu=1}^{N} w_\nu\big(\tilde{x},\alpha(\tilde{y})\big)\mathcal{E}_\nu\left(t, d_{\tilde{g}}\big(\tilde{x},\alpha(\tilde{y})\big)\right) + R_N\left(t,x,\tilde{\alpha}(\tilde{y})\right), \quad (58)$$

where $\left|\Phi\big(\tilde{x},\alpha(\tilde{y})\big)\right| = d_{\tilde{g}}(\tilde{x},\alpha(\tilde{y})), \mathcal{E}_\nu, \nu = 1,2,3,\ldots$, are defined recursively by $2\mathcal{E}_\nu(t,r) = -t\int_0^t \mathcal{E}_{\nu-1}(s,r)ds$, where $\mathcal{E}_0(t,x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}cos(t|\xi|)d\xi$, 3 and $w_\nu\big(\tilde{x},\alpha(\tilde{y})\big)$ equals some constant times $u_\nu\big(\tilde{x},\alpha(\tilde{y})\big)$ that satisfies

$$\begin{cases} u_0\big(\tilde{x},\alpha(\tilde{y})\big) = \Theta^{-\frac{1}{2}}\big(\alpha(\tilde{y})\big), \\ u_{\nu+1}\big(\tilde{x},\alpha(\tilde{y})\big) = \Theta\big(\alpha(\tilde{y})\big)\int_0^1 s^\nu \Theta^{\frac{1}{2}}(\tilde{x}_s)\Delta_{\tilde{g}}u_\nu(\tilde{x},\tilde{x}_s)ds, \nu \geq 0, \end{cases} \quad (59)$$

where $\big(\alpha(\tilde{y})\big) = \left(det\ g_{ij}\big(\alpha(\tilde{y})\big)\right)^{\frac{1}{2}}$, and $(\tilde{x}_s)_{s\in[0,1]}$ is the minimizing geodesic from $\tilde{x}$ to $\alpha(\tilde{y})$ parametrized proportionally to arc length (see [20] and [30]).

First note that for $N \geq n + \frac{3}{2}$, by using the energy estimates (see [27] Theorem (1.2.3).5), one can show that $\left|R_N\left(t, \tilde{x}, \alpha(\tilde{y})\right)\right| = O(e\wedge dt)$, for some constant $d > 0$, which is at most $O(e^{dT}) = O\left(\lambda^{d\beta}\right)$ after we choose T to be approximately $\beta \log \lambda$, so that it is small compared to the first N terms, since we may choose β as close to 0 as possible.

Theorem (1.2.5)[19]: Given an n-dimensional compact Riemannian manifold (M, g) with nonpositive curvature, and let $(\mathbb{R}^n, \tilde{g})$ be the universal covering of (M, g). Then if $N \geq n + \frac{3}{2}$, in local coordinates,

$$\left(cost\sqrt{-\Delta_{\tilde{g}}}\right) f(\tilde{x}) = \int K_N\,(t, \tilde{x}; \tilde{y}) f(\tilde{y}) dV_{\tilde{g}}(\tilde{y}) + \int R_N\,(t, \tilde{x}; \tilde{y}) f(\tilde{y}) dV_{\tilde{g}}(\tilde{y}), \quad (60)$$

where

$$K_N(t, \tilde{x}; \tilde{y}) = \sum_{\nu=0}^{N} w_\nu(\tilde{x}, \tilde{y}) \mathcal{E}_\nu\left(t, d_{\tilde{g}}(\tilde{x}, \tilde{y})\right), \quad (61)$$

with the remainder kernel R_N satisfying

$$|R_N\,(t, \tilde{x}; \tilde{y}))| = O(e^{dt}) \quad (62)$$

for some number $d > 0$.

This comes from equation (42) in [20]. The proof can be found in [20].

By this theorem,

$$\int_{-T}^{T} \left|R_N\left(t, \tilde{x}, \alpha(\tilde{y})\right)\right| dt \leq C \int_0^T e^{dt} dt = O(e^{dT}). \quad (63)$$

Moreover, for $\nu = 1, 2, 3, \ldots$, we have the following estimate for $\mathcal{E}_\nu(t, r)$.

Theorem (1.2.6)[19]: For $\nu = 0, 1, 2, \ldots$ and $\mathcal{E}_\nu(t, r)$ defined above, we have

$$\left|\int \hat{\rho}(t) e^{it\lambda} \mathcal{E}_\nu(t, r) dt\right| = O(\lambda^{n-1-2\nu}), \qquad \lambda \geq 1. \quad (64)$$

Proof. Recall that

$$\mathcal{E}_0(t, r) = \frac{H(t)}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\Phi(\tilde{x}, \tilde{y}) \cdot \xi} cost|\xi| d\xi, \quad (65)$$

so that

$$\begin{aligned} \left|\int \hat{\rho}(t) e^{it\lambda} \mathcal{E}_0(t, r) dt\right| &= \left|\frac{1}{2(2\pi)^n} \int \int_{\mathbb{R}^n} \hat{\rho}(t) e^{it(\lambda \pm |\xi|) + i\Phi(\tilde{x}, \tilde{y}) \cdot \xi} d\xi dt\right| \\ &\approx \left|\int_{\mathbb{R}^n} [\rho(\lambda + |\xi|) + \rho(\lambda - |\xi|)] e^{i\Phi(\tilde{x}, \tilde{y}) \cdot \xi} d\xi\right| \\ &\leq \int_{\mathbb{R}^n} |\rho(\lambda + |\xi|)| + |\rho(\lambda - |\xi|)| d\xi = O(\lambda^{n-1}). \end{aligned} \quad (66)$$

By the definition of $\mathcal{E}_\nu$ such that $\frac{\partial \mathcal{E}_\nu}{\partial t} = \frac{t}{2} \mathcal{E}_{\nu-1}$ and integrating by parts, we get that for any $\nu = 1, 2, 3, \ldots$,

$$\int \hat{\rho}(t) e^{it\lambda} \mathcal{E}_\nu(t, r) dt = O(\lambda^{n-1-2\nu}). \quad (67)$$

The following theorem has been shown by Berard in [20] about the size of the coefficients $u_k(\tilde{x}, \tilde{y})$.

Theorem (1.2.7)[19]: Let (M, g) be a compact n-dimensional Riemannian manifold and let σ be its sectional curvature (hence, there is a number Γ such that $-\Gamma^2 \leq \sigma$). Assume that either

(i) $n = 2$, and M does not have conjugate points;

or

(ii) $-\Gamma^2 \leq \sigma \leq 0$; i.e. M has nonpositive sectional curvature.

Let $(\mathbb{R}^n, \tilde{g})$ be the universal covering of (M, g), and let $\tilde{u}_\nu, \nu = 0, 1, 2, \ldots$, be defined by the relations (59). Then for any integers l and ν

$$\Delta_{\tilde{g}}^l \tilde{u}_\nu(\tilde{x}, \tilde{y}) = O\left(\exp\left(O\left(d_{\tilde{g}}(\tilde{x}, \tilde{y})\right)\right)\right). \tag{68}$$

The proof can be found in [20], Appendix: Growth of the Functions $u_k(x, y)$.

Since $w_\nu(\tilde{x}, \alpha(\tilde{y}))$ is a constant times $\tilde{u}_\nu(\tilde{x}, \alpha(\tilde{y}))$, this theorem tells us that $|w_\nu(\tilde{x}, \alpha(\tilde{y}))| = O\left(\exp\left(c_\nu d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y}))\right)\right)$, for some constant c_ν depending on ν.

Moreover, denote that $\psi(t) = \hat{\rho}\left(\frac{t}{T}\right)$, and $\tilde{\psi}$ is the inverse Fourier transform of ψ. Thus we have $\tilde{\psi} \in S(\mathbb{R})$ such that

$$|\tilde{\psi}(t)| \leq T(1 + T|t|)^{-N}, \qquad for\ all\ N \in \mathbb{N}. \tag{69}$$

Therefore,

$$\begin{aligned}
\sum_{\nu=1}^{N} \left| w_\nu(\tilde{x}, \alpha(y)) \int_{-T}^{T} \hat{\rho}\left(\frac{t}{T}\right) e^{it\lambda} \mathcal{E}_\nu\left(t, d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y}))\right) dt \right| \\
= \sum_{\nu=1}^{N} O\left(T(T\lambda)^{n-1-2\nu} exp\left(c_\nu d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y}))\right)\right) \\
= O\left(T^{n-2}\lambda^{n-3} \exp\left(C_N\, d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y}))\right)\right),
\end{aligned} \tag{70}$$

for some C_N depending on $c_1, c_2, \ldots, c_{N-1}$.

All in all, taking $n = 2$, and disregarding the integral of the remainder kernel,

$$\begin{aligned}
\left| \int_{-T}^{T} \hat{\rho}\left(\frac{t}{T}\right) \cos\left(t\sqrt{-\Delta_{\tilde{g}}}\right)(\tilde{x}, \alpha(\tilde{y})) e^{it\lambda} dt \right| \\
= \left| \int_{-T}^{T} \hat{\rho}\left(\frac{t}{T}\right) \frac{w_0(\tilde{x}, \alpha(\tilde{y}))}{4\pi^2} \sum_{\pm} \int_{\mathbb{R}^n} e^{i\Phi(\tilde{x}, \alpha(\tilde{y}))\cdot\xi \pm it|\xi|} e^{it\lambda} d\xi dt \right| \\
+ O\left(\lambda^{-1} \exp\left(C_N\, d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y}))\right)\right).
\end{aligned} \tag{71}$$

On the other hand, $w_0(\tilde{x}, \tilde{y})$ has a better estimate. By applying Günther's Comparison Theorem [21], with the assumption of nonpositive curvature, we can show that $|w_0(\tilde{x}, \tilde{y})| = O(1)$. The proof is given by Sogge and Zelditch in [30] for $n = 2$. Let's see the case for any dimension n. In the geodesic polar coordinates we are using, $t\Theta, t > 0, \Theta \in \mathbb{S}^{n-1}$, for $(\mathbb{R}^n, \tilde{g})$, the metric $\tilde{g}$ takes the form

$$ds^2 = dt^2 + \mathrm{A}^2(t, \Theta) d\Theta^2, \tag{72}$$

where we may assume that $\mathrm{A}(t, \Theta) > 0$ for $t > 0$. Consequently, the volume element in these coordinates is given by

$$dV_g(t, \theta) = \mathrm{A}(t, \Theta) dt d\Theta, \tag{73}$$

and by Gunther's Comparison Theorem [21] if the curvature of (M, g), which is the same as that of $(\mathbb{R}^n, \tilde{g})$, is nonpositive, we have

$$\mathrm{A}(t, \theta) \geq t^{n-1}, \tag{74}$$

where t^{n-1} is the volume element of the Euclidean space. In geodesic normal coordinates about x, we have

$$w_0(x,y) = \ det\ g_{ij}(y)^{-\frac{1}{4}}$$

(see [20], [22] or §2.4 in [27]). If y has geodesic polar coordinates (t,Θ) about x, then $t = d_{\tilde{g}}(x,y)$, so that $w_0(x,y) = \sqrt{t^{n-1}/\mathrm{A}(t,\Theta)} \leq 1$.

Therefore,

$$\left|\sum_{\pm}\int_{\mathbb{R}^n}\int_{-T}^{T} e^{i\Phi(\tilde{x},\tilde{y})\cdot\xi \pm it|\xi|+it\lambda}\hat{\rho}\left(\frac{t}{T}\right)dtd\xi\right|$$
$$= \left|\int_{\mathbb{R}^n} e^{i\Phi(\tilde{x},\tilde{y})\cdot\xi}\left(\tilde{\psi}(\lambda+|\xi|)+\tilde{\psi}(\lambda-|\xi|)\right)d\xi\right|$$
$$\leq \int_{\mathbb{R}^n}\left|\tilde{\psi}(\lambda+|\xi|)\right|d\xi + \int_{\mathbb{R}^n}\left|\tilde{\psi}(\lambda-|\xi|)\right|d\xi. \qquad (75)$$

Note that $\tilde{\psi}(\lambda+|\xi|) = O(T(1+\lambda+|\xi|)-N)$, for any $N \in \mathbb{N}$, so $\int_{\mathbb{R}^n}\left|\tilde{\psi}(\lambda+|\xi|)\right|d\xi$ can be arbitrarily small, while $\tilde{\psi}(\lambda-|\xi|) = O\left(T\left(1+T\left|\lambda-|\xi|\right|\right)^{-N}\right)$, for any $N \in \mathbb{N}$, so that $\int_{\mathbb{R}^n}\left|\tilde{\psi}(\lambda-|\xi|)\right|d\xi \lesssim T\int_{\lambda-1\leq|\xi|\leq\lambda+1}\left(1+T\left|\lambda-|\xi|\right|\right)^{-N}d\xi = O(\lambda)$, provided that $\lambda \geq 1$. So

$$\int_{-T}^{T}\hat{\rho}\left(\frac{t}{T}\right)\left(cost\sqrt{-\Delta_{\tilde{g}}}\right)(\tilde{x},\tilde{y})e^{it\lambda}dt = O(\lambda) + O\left(\lambda^{-1}\exp\left(C_N d_{\tilde{g}}(\tilde{x},\tilde{y})\right)\right), \qquad (76)$$

disregarding the integral of the remainder kernel.

However, this estimate can be improved when $d_{\tilde{g}}(\tilde{x},\tilde{y}) \geq \frac{1}{\lambda}$. As we can see, the main term of

$$\cos\left(t\sqrt{-\Delta_{\tilde{g}}}\right)(\tilde{x},\tilde{y}) = \frac{w_0(\tilde{x},\tilde{y})}{4\pi^2} \pm \int_{\mathbb{R}^n} e^{i\Phi(\tilde{x},\tilde{y})\cdot\xi\pm it|\xi|}d\xi$$
$$+\sum_{\nu=1}^{N} w_\nu(\tilde{x},\tilde{y})\mathcal{E}_\nu\left(t, d_{\tilde{g}}(\tilde{x},\tilde{y})\right) + R_N\ (t,\tilde{x},\tilde{y}) \quad (77)$$

is the first term, and the corresponding term in $\int_{-T}^{T}\hat{\rho}\left(\frac{t}{T}\right)\left(cost\sqrt{-\Delta_{\tilde{g}}}\right)(\tilde{x},\tilde{y})e^{it\lambda}dt$ is bounded by

$$C\left|\sum_{\pm}\int_{-T}^{T}\int_{\mathbb{R}^n}\hat{\rho}\left(\frac{t}{T}\right)e^{i\Phi(\tilde{x},\tilde{y})\cdot\xi\pm it|\xi|}e^{it\lambda}dtd\xi\right|$$
$$= C\left|\sum_{\pm}\int_{-T}^{T}\int_{0}^{\infty}\int_{0}^{2\pi}\hat{\rho}\left(\frac{t}{T}\right)e^{ir\Phi(\tilde{x},\tilde{y})\cdot\Theta\pm itr+it\lambda}rdtdrd\theta\right|. \qquad (78)$$

Integrate with respect to t first; then the quantity above is bounded by a constant times

$$\sum_{\pm}\int_{0}^{\infty}\int_{0}^{2\pi}\tilde{\psi}(\lambda\pm r)e^{ir\Phi(\tilde{x},\tilde{y})\cdot\Theta}rd\theta dr. \qquad (79)$$

Because $\tilde{\psi}(\lambda\ \pm\ r) \lesssim T(1+T|\lambda\ \pm\ r|)^{-N}$ for any $N > 0$, the term with $\tilde{\psi}(\lambda+r)$ in the sum is $O(1)$, while the other term with $\tilde{\psi}(\lambda-r)$ is significant only when r is comparable to λ, say $c_1\lambda < r < c_2\lambda$ for some constants c_1 and c_2. In this case, as we assumed that $d_{\tilde{g}}(\tilde{x},\tilde{y}) \geq \frac{1}{\lambda}$; we can also assume that $d_{\tilde{g}}(\tilde{x},\tilde{y}) \gtrsim \frac{1}{r}$.

By Lemma (1.2.4), $\int_0^{2\pi} e^{iw\cdot\Theta}d\theta = \sqrt{2\pi}|w|^{-\frac{1}{2}}\sum_{\pm} a_{\pm}e^{\pm i|w|} + O\left(|w|^{-\frac{3}{2}}\right), |w| \geq 1$, where $w = r\Phi(\tilde{x},\tilde{y})$. Integrating up θ, the above quantity is then controlled by a constant times

$$\left|\sum_{\pm}\int_{c_1\lambda}^{c_2\lambda}\tilde{\psi}(\lambda-r)\left|rd_{\tilde{g}}(\tilde{x},\tilde{y})\right|^{-\frac{1}{2}}e^{\pm ird_{\tilde{g}}(\tilde{x},\tilde{y})}rdr + \int_{c_1\lambda}^{c_2\lambda}\tilde{\psi}(\lambda-r)\left|rd_{\tilde{g}}(\tilde{x},\tilde{y})\right|^{-\frac{3}{2}}\right|$$

$$\leq d_{\tilde{g}}(\tilde{x},\tilde{y})^{-\frac{1}{2}}\int_{c_1\lambda}^{c_2\lambda}\left|\tilde{\psi}(\lambda-r)\right|r^{\frac{1}{2}}dr + d_{\tilde{g}}(\tilde{x},\tilde{y})^{\left(-\frac{3}{2}\right)}\int_{c_1\lambda}^{c_2\lambda}\left|\tilde{\psi}(\lambda-r)\right|r^{-\frac{1}{2}}dr$$

$$= d_{\tilde{g}}(\tilde{x},\tilde{y})^{-\frac{1}{2}}O\left(\lambda^{\frac{1}{2}}\right) + O\left(d_{\tilde{g}}(\tilde{x},\tilde{y})^{-1}\right) = O\left(\left(\frac{\lambda}{d_{\tilde{g}}(\tilde{x},\tilde{y})}\right)^{\frac{1}{2}}\right). \tag{80}$$

Note that these two equalities are still valid when c_1 and c_2 are changed to 0 and ∞. Therefore, when $d_{\tilde{g}}(\tilde{x},\tilde{y}) \geq \frac{1}{\lambda}$,

$$\left|\frac{w_0(\tilde{x},\tilde{y})}{4\pi^2}\sum_{\pm}\int_{\mathbb{R}^n}\hat{\rho}\left(\frac{t}{T}\right)e^{i\Phi(\tilde{x},\tilde{y})\cdot\xi\pm it|\xi|}e^{it\lambda}d\xi\right| = O\left(\left(\frac{\lambda}{d_{\tilde{g}}(\tilde{x},\tilde{y})}\right)^{\frac{1}{2}}\right). \tag{81}$$

We have finished the estimates for $\alpha = Id$. For $\alpha \neq Id$, note that we can find a constant C_p that is different from 0, depending on the universal cover, p, of the manifold M, such that

$$d_{\tilde{g}}\left(\tilde{x},\alpha(\tilde{y})\right) > C_p, \tag{82}$$

for all $\alpha \in Aut(p)$ different from Id. The constant C_p comes from the fact that if we assume that the injectivity radius of M is greater than a number, say 1, and that x is the center of some geodesic ball with radius one contained in M, then we can choose the fundamental domain D such that x is at least some distance, say $C_p > 1$, away from any translation of D, which we denote as $\alpha(D)$, for any $\alpha \in Aut(p)$ that is not identity. Therefore, we may use the estimates for $d_{\tilde{g}}(\tilde{x},\tilde{y}) \geq \frac{1}{\lambda}$ before assuming λ is larger than $1/C_p$. Use the Hadamard parametrix (see [30]), similarly as before, estimating only the main term:

$$\left|\int_{-T}^{T}\hat{\rho}\left(\frac{t}{T}\right)\left(cost\sqrt{-\Delta_{\tilde{g}}}\right)\left(\tilde{x},\alpha(\tilde{y})\right)e^{it\lambda}dt\right|$$

$$\lesssim \left|(2\pi)^{-2}\int_{\mathbb{R}^n}\int_{-T}^{T}\hat{\rho}\left(\frac{t}{T}\right)e^{i\Phi\left(\tilde{x},\alpha(\tilde{y})\right)\cdot\xi}\cos(t|\xi|)\,e^{it\lambda}dt\right|$$

$$\lesssim \sum_{\pm}\left|\int_0^{2\pi}\int_0^{\infty}\int_{-T}^{T}e^{ir\Phi\left(\tilde{x},\alpha(\tilde{y})\right)\cdot\Theta\pm itr+it\lambda}\hat{\rho}\left(\frac{t}{T}\right)rdtdrd\theta\right|$$

$$= \sum_{\pm}\left|\int_0^{\infty}\int_0^{2\pi}\tilde{\psi}(\lambda-r)e^{ir\Phi\left(\tilde{x},\alpha(\tilde{y})\right)\cdot\Theta}rd\theta dr\right|$$

$$\lesssim \sum_{\pm}\left|\int_0^{\infty}\tilde{\psi}(\lambda-r)\left|rd_{\tilde{g}}\left(\tilde{x},\alpha(\tilde{y})\right)\right|^{-\frac{1}{2}}e^{ird_{\tilde{g}}\left(\tilde{x},\alpha(\tilde{y})\right)}rdr\right.$$

$$\left. + \int_0^{\infty}\tilde{\psi}(\lambda-r)\left|rd_{\tilde{g}}\left(\tilde{x},\alpha(\tilde{y})\right)\right|^{-\frac{3}{2}}rdr\right|$$

$$= O\left(\left(\frac{\lambda}{d_{\tilde{g}}\left(\tilde{x},\alpha(\tilde{y})\right)}\right)^{\frac{1}{2}}\right). \tag{83}$$

Now we have shown all the estimates (54), (55), and (56). Totally, $K(x,y)$ is

$$O(\frac{1}{T}\left(\frac{\lambda}{\left(\lambda^{-1}+d_{\tilde{g}}(\tilde{x},\tilde{y})\right)^{\frac{1}{2}}}\right)+\sum_{Id=\alpha\in Aut(p)}\left[O\left(\frac{1}{T}\left(\frac{\lambda}{d_{\tilde{g}}\left(\tilde{x},\alpha(\tilde{y})\right)}\right)^{\frac{1}{2}}\right)+O\left(\frac{e^{ET}}{T}\right)\right] \tag{84}$$

where $E = \max\{C_N\,,d\}+1$.

Note that, by the finite propagation speed of the wave operator $\partial_t^2 - \Delta_{\tilde{g}}$, $d_{\tilde{g}}\left(\tilde{x},\alpha(\tilde{y})\right) \leq T$ in the support of $\cos\left(t\sqrt{-\Delta_g}\right)\left(\tilde{x},\alpha(\tilde{y})\right)$. While M is a compact manifold with nonpositive curvature, the number of terms of α's such that $d_{\tilde{g}}\left(\tilde{x},\alpha(\tilde{y})\right) \leq T$ is at most e^{cT}, for some constant c depending on the curvature, by the Bishop Comparison Theorem (see [25], [30]).

We take the $L^r(\gamma)$ norms of each individual term first; then by Minkowski's inequality, $\|K(x,\cdot)\|_{L^r(\gamma[-1,1])}$ is bounded by the sum. Also note that we may consider the geodesic distance to be comparable to the arc length of the geodesic. The first term is simple, and it is controlled by a constant times

$$\frac{1}{T}\left(\int_0^1\left(\frac{\lambda}{\lambda^{-1}+\tau}\right)^{\frac{r}{2}}d\tau\right)^{\frac{1}{r}} = O\left(\frac{\lambda^{\frac{p-2}{p}}}{T}\right). \tag{85}$$

Accounting for the number of terms of those α's, the second term is bounded by a constant times

$$e^{cT}\cdot\frac{\lambda^{\frac{1}{2}}}{T}\left(\int_0^1\left(\frac{1}{C_p}\right)^{\frac{r}{2}}d\tau\right)^{\frac{1}{r}} = O\left(e^{cT}\frac{\lambda^{\frac{1}{2}}}{T}\right). \tag{86}$$

Therefore,

$$\begin{aligned}\|K(x,\cdot)\|_{L^r(\gamma[-1,1])} &= O\left(\frac{\lambda^{\frac{p-2}{p}}}{T}\right)+O\left(e^{cT}\frac{\lambda^{\frac{1}{2}}}{T}\right)+O\left(\frac{e^{(c+E)T}}{T}\right)\\ &= I+II+III.\end{aligned} \tag{87}$$

Now take $T=\beta\log\lambda$, where $\beta \leq \frac{p-4}{2(c+E)p}$. (Note that we can assume that $c \neq 0$; otherwise, there is only one α that we are considering, which is $\alpha = Id$.) Then

$$I = II = O\left(\frac{\lambda^{\frac{p-2}{p}}}{\log\lambda}\right), \tag{88}$$

and

$$III = o\left(\frac{\lambda^{\frac{p-2}{p}}}{\log\lambda}\right) \tag{89}$$

Summing up, we get that

$$\|K(x,\cdot)\|_{L^r(\gamma[-1,1])} = O\left(\frac{\lambda^{\frac{p-2}{p}}}{\log\lambda}\right). \tag{90}$$

Now applying Young's inequality with $r = \frac{p}{2}$, we get that

$$\forall f \in L^{p'}(\gamma), \left\|\chi_T^\lambda\left(\chi_T^\lambda\right)^* f\right\|_{L^p(\gamma)} \lesssim \frac{(1+\lambda)^{1-\frac{2}{p}}}{\log\lambda}\|f\|_{L^{p'}(\gamma)}.$$

Therefore, Theorem (1.2.3) is proved.

We move on to the case for $n \geq 3$. While we want to show Theorem (1.2.1) for the full range of p directly, we can only show it under the condition that $p > \frac{4k}{n-1}$ using the same method. Although we only need $p = \infty$ later to interpolate and get to the full version of Theorem (1.2.1), we will show the most as we can for the moment.

Theorem (1.2.8)[19]: Let (M, g) be a compact smooth n-dimensional boundaryless Riemannian manifold with nonpositive curvature, and Σ be a k-dimensional compact; smooth sub manifold on M. Then for any $f \in L^2(M)$, we have the following estimate:

$$\left\|\sum_{|\lambda_j-\lambda|\leq(\log\lambda)^{-1}} E_j f\right\|_{L^p(\Sigma)} \lesssim \frac{\lambda^{\delta(p)}}{(\log\lambda)^{\frac{1}{2}}}\|f\|_{L^2(M)}, \qquad \forall p > \frac{4k}{n-1}, \tag{91}$$

where

$$\delta(p) = \frac{n-1}{2} - \frac{k}{p}. \tag{92}$$

For $n \geq 3$, for the sake of using interpolation later, we need to insert a bump function. Take $\varphi \in C_0^\infty(\mathbb{R})$ such that $\varphi(t) = 1$ when $|t| \leq \frac{1}{2}$ and $\varphi(t) = 0$ when $|t| > 1$. Then we only have to consider the following kernel:

$$K(x,y) = \frac{1}{\pi T}\int_{-T}^{T}(1-\varphi(t))\hat{\rho}\left(\frac{t}{T}\right)\left(cost\sqrt{-\Delta_g}\right)(x,y)e^{it\lambda}dt, \tag{93}$$

which is nonzero only when $|t| > \frac{1}{2}$. In the following discussion, we may sometimes only show estimates for $K(x,y)$ when $t > \frac{1}{2}$, as the part for $t < -\frac{1}{2}$ can be done similarly.

The reason why we only consider the above kernel $K(x,y)$ is because of the following lemma.

We show Theorem (1.2.8), which is essentially the same as the lower dimension case, and what we need to show is (47). By a partition of unity, we may choose some point $x \in \Sigma$, and consider Σ to be within a ball with geodesic radius 1 centered at x, and under the geodesic normal coordinates centered at x, parametrize Σ as

$$\Sigma = \{(t,\Theta)\big| y = \exp_x(t\Theta) \in \Sigma, t \in [-1,1], \Theta \in \mathbb{S}^{k-1}\}.$$

Applying the Hadamard parametrix, for any $\alpha \in Aut(p)$,

$$\cos\left(t\sqrt{-\Delta_{\tilde{g}}}\right)(\tilde{x},\alpha(\tilde{y})) = \frac{w_0(\tilde{x},\alpha(\tilde{y}))}{(2\pi)^n}\sum_{\pm}\int_{\mathbb{R}^n} e^{i\Phi(\tilde{x},\alpha(\tilde{y}))\cdot\xi\pm it|\xi|}d\xi$$

$$+\sum_{\nu=1}^{\infty} w_\nu(\tilde{x},\tilde{y})\mathcal{E}_\nu\left(t, d_{\tilde{g}}(\tilde{x},\alpha(\tilde{y}))\right) + R_N(t,\tilde{x},\alpha(\tilde{y})), \tag{94}$$

where $|\Phi(\tilde{x},\alpha(\tilde{y}))| = d_{\tilde{g}}(\tilde{x},\alpha(\tilde{y}))$, and $\mathcal{E}_\nu, \nu = 1,2,3,\cdots$, are those described.

By Theorem (1.2.7),

$$\int_{-T}^{T} \left|R_N\left(t,\tilde{x},\alpha(\tilde{y})\right)\right| dt \lesssim \int_0^T e^{dt} dt = O(e^{dT}). \tag{95}$$

Moreover, by (64), for $\nu = 1, 2, 3, \cdots$,

$$\left|\int_{-T}^{T} (1-\varphi(t))\hat{\rho}\left(\frac{t}{T}\right) e^{it\lambda} \mathcal{E}_\nu\left(t, d_{\tilde{g}}\left(\tilde{x},\alpha(\tilde{y})\right)\right) dt\right| = O(T(T\lambda)^{n-1-2\nu}). \tag{96}$$

Since $\left|w_\nu\left(\tilde{x},\alpha(\tilde{y})\right)\right| = O\left(\exp\left(c_\nu d_{\tilde{g}}\left(\tilde{x},\alpha(\tilde{y})\right)\right)\right)$ by [20], for some constant c_ν depending on ν,

$$\sum_{\nu=1}^{N} \left| w_\nu\left(\tilde{x},\alpha(\tilde{y})\right) \int_{-T}^{T} (1-\phi(t))\hat{\rho}\left(\frac{t}{T}\right) e^{it\lambda} \mathcal{E}_\nu\left(t, d_{\tilde{g}}\left(\tilde{x},\alpha(\tilde{y})\right)\right) dt \right|$$

$$= \sum_{\nu=1}^{N} O\left(T(T\lambda)^{n-1-2\nu} \exp\left(c_\nu d_{\tilde{g}}\left(\tilde{x},\alpha(\tilde{y})\right)\right)\right)$$

$$= O\left(T^{n-2}\lambda^{n-3} \exp\left(C_N d_{\tilde{g}}\left(\tilde{x},\alpha(\tilde{y})\right)\right)\right), \tag{97}$$

for some C_N depending on $c_1, c_2, \ldots, c_{N-1}$.

All in all, disregarding the integral of the remainder kernel,

$$\left|\int_{-T}^{T} (1-\varphi(t))\hat{\rho}\left(\frac{t}{T}\right) \cos\left(t - \sqrt{\Delta_{\tilde{g}}}\right)\left(\tilde{x},\alpha(\tilde{y})\right) e^{it\lambda} dt\right|$$

$$= \left|\int_{-T}^{T} (1-\varphi(t))\hat{\rho}\left(\frac{t}{T}\right) \frac{w_0(\tilde{x},\tilde{y})}{(2\pi)^n} \sum_{\pm} \int_{\mathbb{R}^n} e^{i\Phi\left(\tilde{x},\alpha(\tilde{y})\right)\cdot\xi \pm it|\xi|} e^{it\lambda} d\xi dt\right|$$

$$+ O\left(T^{n-2}\lambda^{n-3} \exp\left(C_N d_{\tilde{g}}\left(\tilde{x},\alpha(\tilde{y})\right)\right)\right). \tag{98}$$

On the other hand, $|w_0(\tilde{x},\tilde{y})| = O(1)$ (see [30]) by applying Gunther's Comparison Theorem in [21], and for

$$\left|\sum_{\pm} \int_{\mathbb{R}^n} \int_{-T}^{T} (1-\varphi(t)) e^{i\Phi\left(\tilde{x},\alpha(\tilde{y})\right)\cdot\xi \pm it|\xi| + it\lambda} \hat{\rho}\left(\frac{t}{T}\right) dt d\xi\right|, \tag{99}$$

as we may assume as before that $d_{\tilde{g}}\left(\tilde{x},\alpha(\tilde{y})\right) > \frac{1}{2}$ by the stationary phase estimates in [16].

Denote that $\psi(t) = \left(1-\varphi(t)\right)\hat{\rho}\left(\frac{t}{T}\right)$, and $\tilde{\psi}$ is the inverse Fourier transform of ψ. Again we have

$$\tilde{\psi}(\lambda + |\xi|) = O(T(1+\lambda+|\xi|)^{-N}),$$

for any $N \in \mathbb{N}$, so $\int_{\mathbb{R}^n} \left|\tilde{\psi}(\lambda+|\xi|)\right| d\xi$ can be arbitrarily small, while $\tilde{\psi}(\lambda - |\xi|) = O\left(T\left(1+T\left|\lambda - |\xi|\right|\right)^{-N}\right)$.

Integrate (99) with respect to t first; then it is bounded by a constant times

$$\sum_{\pm} \int_0^\infty \int_{\mathbb{S}^{n-1}} \tilde{\psi}(\lambda \pm r) e^{ir\Phi\left(\tilde{x},\alpha(\tilde{y})\right)\cdot\Theta} r^{n-1} d\Theta dr. \tag{100}$$

Because $\tilde{\psi}(\lambda \pm r) \le T(1+T|\lambda \pm r|)^{-N}$ for any $N > 0$, the term with $\tilde{\psi}(\lambda + r)$ in the sum is $O(1)$, while the other term with $\tilde{\psi}(\lambda - r)$ is significant only when r is comparable to λ, say $c_1\lambda < r < c_2\lambda$ for some constants c_1 and c_2. In this case, as we assumed that $d_{\tilde{g}}\left(\tilde{x},\alpha(\tilde{y})\right) \ge D$; we can also assume that $d_{\tilde{g}}\left(\tilde{x},\alpha(\tilde{y})\right) \gtrsim \frac{1}{r}$ for large λ.

By Lemma (1.2.4), $\int_{\mathbb{S}^{n-1}} e^{iw\cdot\Theta}d\Theta = \sqrt{2\pi}^{n-1}|w|^{-\frac{n-1}{2}}\sum_{\pm} a_{\pm}e^{\pm i|w|} + O\left(|w|^{-\frac{n+1}{2}}\right)$, $|w| \geq 1$, where $w = r\Phi\big(\tilde{x},\alpha(\tilde{y})\big)$. Integrating up to Θ, the above quantity is then controlled by a constant times

$$\left|\sum_{\pm}\int_{c_1\lambda}^{c_2\lambda}\tilde{\psi}(\lambda-r)\left|rd_{\tilde{g}}\big(\tilde{x},\alpha(\tilde{y})\big)\right|^{-\frac{n-1}{2}}e^{\pm ird_{\tilde{g}}(\tilde{x},\tilde{y})}r^{n-1}dr\right.$$
$$\left.+\int_{c_1\lambda}^{c_2\lambda}\tilde{\psi}(\lambda-r)\left|rd_{\tilde{g}}\big(\tilde{x},\alpha(\tilde{y})\big)\right|^{-\frac{n+1}{2}}r^{n-1}dr\right|$$
$$\leq d_{\tilde{g}}(x,y)^{-\frac{n-1}{2}}\int_{c_1\lambda}^{c_2\lambda}\tilde{\psi}(\lambda-r)|r^{\frac{n-1}{2}}dr$$
$$+\,d_{\tilde{g}}\big(\tilde{x},\alpha(\tilde{y})\big)^{-\frac{n+1}{2}}\int_{c_1\lambda}^{c_2\lambda}|\tilde{\psi}(\lambda-r)|r^{\frac{n-3}{2}}dr$$
$$= O\left(\left(\frac{\lambda}{d_{\tilde{g}}\big(\tilde{x},\alpha(\tilde{y})\big)}\right)^{\frac{n-1}{2}}\right). \tag{101}$$

Therefore, disregarding the integral of the remainder kernel,

$$\int_{-T}^{T}\big(1-\varphi(t)\big)\hat{\rho}\left(\frac{t}{T}\right)\left(cost\sqrt{-\Delta_{\tilde{g}}}\right)\big(\tilde{x},\alpha(\tilde{y})\big)e^{it\lambda}dt = O\left(\left(\frac{\lambda}{d_{\tilde{g}}\big(\tilde{x},\alpha(\tilde{y})\big)}\right)^{\frac{n-1}{2}}\right)$$
$$+O\left(T^{n-2}\lambda^{n-3}\exp\left(C_N d_{\tilde{g}}\big(\tilde{x},\alpha(\tilde{y})\big)\right)\right). \tag{102}$$

Now $K(x,y)$ is

$$\sum_{\alpha\in Aut(p)}\left[O\left(\frac{1}{T}\left(\frac{\lambda}{d_{\tilde{g}}\big(\tilde{x},\alpha(\tilde{y})\big)}\right)^{\frac{n-1}{2}}\right)+O\left(\frac{e^{ET}}{T}\right)\right], \tag{103}$$

where $E = \max\{C_N, d\} + 1$.

We still have: the number of terms of α's such that $d_{\tilde{g}}\big(\tilde{x},\alpha(\tilde{y})\big) \leq T$ is at most e^{cT}, for some constant c depending on the curvature, and there exists a constant C_p such that $d_{\tilde{g}}\big(\tilde{x},\alpha(\tilde{y})\big) > C_p$ for any $\alpha \in Aut(p)$ different from identity.

We take the $L^r(\Sigma)$ norms of each individual term. By (82), and accounting for the number of terms of those α's, the first one is bounded by a constant times

$$\frac{e^{cT}\lambda^{\frac{n-1}{2}}}{T}\left(\int_0^1 C_p^{-\frac{(n-1)}{2}\cdot r}\tau^{k-1}d\tau\right)^{\frac{1}{r}} = O\left(\frac{e^{cT}\lambda^{\frac{n-1}{2}}}{T}\right). \tag{104}$$

Therefore,

$$\|K(x,\cdot)\|_{L^r(\Sigma)} = O\left(\frac{e^{cT}\lambda^{\frac{n-1}{2}}}{T}\right)+O\left(\frac{e^{(c+E)T}}{T}\right) = I + II. \tag{105}$$

Now take $T = \beta\log\lambda$, where $\beta = \frac{\frac{n-1}{2}-\frac{2k}{p}-\delta}{c+E}$, where δ satisfies $0 < \delta < \frac{n-1}{2}-\frac{2k}{p}$. Note that $\frac{n-1}{2}-\frac{2k}{p} > 0$ when $p > \frac{4k}{n-1}$. Then

$$I=O\left(\frac{\lambda^{\beta c+\frac{n-1}{2}}}{\log\lambda}\right)=O\left(\frac{\lambda^{\frac{n-1}{2}-\frac{2k}{p}-\delta+\frac{n-1}{2}}}{\log\lambda}\right)=o\left(\frac{\lambda^{n-1-\frac{2k}{p}}}{\log\lambda}\right), \tag{106}$$

and

$$II=O\left(\frac{\lambda^{\beta(c+E)}}{\log\lambda}\right)=O\left(\frac{\lambda^{\frac{n-1}{2}-\frac{2k}{p}-\delta}}{\log\lambda}\right)=o\left(\frac{\lambda^{n-1-\frac{2k}{p}}}{\log\lambda}\right). \tag{107}$$

Summing up, we get that

$$\|K(x,\cdot)\|_{L^r(\Sigma)}=o\left(\frac{\lambda^{n-1-\frac{2k}{p}}}{\log\lambda}\right). \tag{108}$$

Now applying Young's inequality, with $r=\frac{p}{2}$, together with the estimate in Lemma (1.2.9), we have

$$\forall f\in L^{p'}(\Sigma), \left\|\chi_T^\lambda\left(\chi_T^\lambda\right)^* f\right\|_{L^p(\Sigma)}\lesssim\frac{\lambda^{n-1-\frac{2k}{p}}}{\log\lambda}\|f\|_{L^p(\Sigma)}. \tag{109}$$

Therefore, Theorem (1.2.8) is proved.

Lemma (1.2.9)[19]: For $\phi\in C_0^\infty(\mathbb{R})$ such that $\varphi(t)=1$ when $|t|\le\frac{1}{2}$ and $\varphi(t)=0$ when $|t|>1$, let

$$\widetilde{K}(x,y)=\frac{1}{\pi T}\int_{-1}^{1}\varphi(t)\hat{\rho}\left(\frac{t}{T}\right)\left(cost\sqrt{-\Delta_g}\right)(x,y)e^{it}\lambda dt. \tag{110}$$

Then

$$\sup_x\left\|\widetilde{K}(x,\cdot)\right\|_{L^r(\Sigma)}=O\left(\frac{\lambda^{2\delta(p)}}{\log\lambda}\right). \tag{111}$$

Proof. With similar approaches as in the previous discussions, we can show that $\widetilde{K}(x,y)$ is

$$O\left(\frac{1}{T}\left(\frac{\lambda}{\lambda^{-1}+d_{\tilde{g}}(\tilde{x},\tilde{y})}\right)^{\frac{n-1}{2}}\right)+\sum_{I\,d\neq\alpha\in Aut(p)}\left[O\left(\frac{1}{T}\left(\frac{\lambda}{d_{\tilde{g}}\left(\tilde{x},\alpha(\tilde{y})\right)}\right)^{\frac{n-1}{2}}\right)+O\left(e^{ET}\right)\right], \tag{112}$$

where $E=\max\{C_N,d\}+1$.

Note that $|t|\le 1$ for $\varphi(t)=0$, and the number of terms such that $d_{\tilde{g}}\left(\tilde{x},\alpha(\tilde{y})\right)\le 1$ is at most e^c, so that

$$\left\|\widetilde{K}(x,y)\right\|_{L^r(\Sigma)}=O\left(\frac{\lambda^{2\delta(p)}}{\log\lambda}\right), \tag{113}$$

if we take $T=\log\lambda$ and calculate as before.

To show Theorem (1.2.1), we need to use interpolation. Recall that

$$\begin{aligned}K(x,y)&=\frac{1}{\pi T}\int_{-T}^{T}(1-\varphi(t))\hat{\rho}\left(\frac{t}{T}\right)\left(cost\sqrt{-\Delta_g}\right)(x,y)e^{it\lambda}dt\\&=\frac{1}{2\pi T}\int_{-T}^{T}(1-\varphi(t))\hat{\rho}\left(\frac{t}{T}\right)\left(e^{it\sqrt{-\Delta_g}}+e^{-t\sqrt{-\Delta_g}}\right)(x,y)e^{it\lambda}dt\end{aligned} \tag{114}$$

is the kernel of the operator

$$\frac{1}{2\pi T}\left[\sum_j \tilde{\psi}(\lambda-\lambda_j)E_j + \sum_j \tilde{\psi}(\lambda-\lambda_j)E_j\right] = \frac{1}{2\pi T}\left[\sum_j \tilde{\psi}(\lambda-\lambda_j)E_j\right] + O(1)$$

$$= \frac{1}{2\pi T}\tilde{\psi}(\lambda - \sqrt{-\Delta_g}) + O(1), \qquad (115)$$

where $\tilde{\psi}(t)$ is the inverse Fourier transform of $(1-\varphi(t))\hat{\rho}\left(\frac{t}{T}\right)$ so that $|\tilde{\psi}(t)| \le T(1+|t|)^{-N}$ for any $N \in \mathbb{N}$.

We have the following estimate for $\tilde{\psi}(\lambda - \sqrt{-\Delta_g})$.

Theorem (1.2.10)[19]: For $k \neq n-2$,

$$\|\tilde{\psi}(\lambda - P)g\|_{L^2(\Sigma)} \lesssim T\lambda^{2\delta(2)}\|g\|_{L^2(\Sigma)}, \qquad for\ any\ g \in L^2(\Sigma), \qquad (116)$$

and for $k = n-2$,

$$\|\tilde{\psi}(\lambda - P)g\|_{L^2(\Sigma)} \lesssim T\lambda^{2\delta(2)}\log\lambda\|g\|_{L^2(\Sigma)}, \qquad for\ any\ g \in L^2(\Sigma), \quad (117)$$

where $P = \sqrt{-\Delta_g}$.

Proof. Recall the proof of the corresponding restriction theorem in [5]. They showed that for $\chi \in S(\mathbb{R})$, and defining

$$\chi_\lambda = \chi\left(\sqrt{-\Delta_g} - \lambda\right) = \sum_j \chi(\lambda_j - \lambda)E_j, \qquad (118)$$

we have

$$\|\chi_\lambda\|_{L^2(M)\to L^2(\Sigma)} = O(\lambda^{\delta(2)}), \qquad (119)$$

for $k \neq n-2$, and

$$\|\chi_\lambda\|_{L^2(M)\to L^2(\Sigma)} = O(\lambda^{\delta(2)(\log\lambda)^{\frac{1}{2}}}, \qquad (120)$$

for $k = n-2$.

Now consider $\tilde{\psi}(\lambda - P)$ as $S\tilde{S}^*$, where

$$S = \sum_j (1+|\lambda_j - \lambda|)^{-M}E_j \qquad (121)$$

and

$$\tilde{S} = \sum_j (1+|\lambda_j - \lambda|)^{M}\tilde{\psi}(\lambda_j - \lambda)E_j, \qquad (122)$$

where M is some large number.

Recall that $|\tilde{\psi}(\tau)| \le T(1+|\tau|)^{-N}$ for any $N \in \mathbb{N}$. We then have

$$\left|(1+|\lambda_j-\lambda|)^M\tilde{\psi}(\lambda_j-\lambda)\right| \le T(1+|\lambda_j-\lambda|)^{-N} \qquad (123)$$

for any N.

By (25), which we deduced from the proof of Theorem 3 in [5], for a given λ,

$$\left\|\sum_{\lambda_j\in(\lambda-1,\lambda+1)} E_j\right\|_{L^2(M)\to L^2(\Sigma)} = O(\lambda^{\delta(2)}), \qquad if\ k \neq n-2 \qquad (124)$$

and

$$\left\|\sum_{\lambda_j\in(\lambda-1,\lambda+1)} E_j\right\|_{L^2(M)\to L^2(\Sigma)} = O\left(\lambda^{\delta(2)}(\log\lambda)^{\frac{1}{2}}\right), \qquad if\ k \neq n-2 \qquad (125)$$

so that for any $f \in L^2(M)$,

$$\left\|\sum_j \left(1+\left|\lambda_j-\lambda\right|^{-M}\right) E_j f\right\|_{L^2(\Sigma)} \leq \left\|\sum_{\lambda_j\in(\lambda-1,\lambda+1)} E_j f\right\|_{L^2(\Sigma)}$$

$$+\left\|\sum_{\lambda_j\notin(\lambda-\delta,\lambda+\delta)} \left(1+\left|\lambda_j-\lambda\right|^{-M}\right) E_j f\right\|_{L^2(\Sigma)}$$

$$\begin{cases} \lambda^{\delta(2)}\|f\|_{L^2(M)} + \sum_{\lambda_j\notin(\lambda-1,\lambda+1)} \left(1+\left|\lambda_j-\lambda\right|\right)^{-M}\left\|E_j f\right\|_{L^2(\Sigma)}, if\ k\neq n-2, \\ \lambda^{\delta(2)}(\log\lambda)^{\frac{1}{2}}\|f\|_{L^2(M)} + \sum_{\lambda_j\notin(\lambda-1,\lambda+1)} \left(1+\left|\lambda_j-\lambda\right|\right)^{-M}\left\|E_j f\right\|_{L^2(\Sigma)}, if\ k=n-2. \end{cases} \qquad (126)$$

As

$$\sum_{\lambda_j\notin(\lambda-1,\lambda+1)} \left(1+\left|\lambda_j-\lambda\right|\right)^{-M}\left\|E_j f\right\|_{L^2(\Sigma)}$$

$$\leq \begin{cases} \sum_{\lambda_j\notin(\lambda-1,\lambda+1)} \lambda_j^{\delta(2)}\left(1+\left|\lambda_j-\lambda\right|\right)^{-M}\left\|E_j f\right\|_{L^2(M)}, if\ k\neq n-2, \\ \sum_{\lambda_j\notin(\lambda-1,\lambda+1)} \lambda_j^{\delta(2)}\left(\log\lambda_j\right)^{\frac{1}{2}}\left(1+\left|\lambda_j-\lambda\right|\right)^{-M}\left\|E_j f\right\|_{L^2(M)}, if\ k=n-2, \end{cases} \qquad (127)$$

which can be made arbitrarily small when M is sufficiently large,

$$\left\|\sum_j \left(1+\left|\lambda_j\ -\lambda\right|^{-M}\right) E_j f\right\|_{L^2(\Sigma)} \leq \begin{cases} \lambda^{\delta(2)}\|f\|_{L^2(M)}, if\ k\neq n-2, \\ \lambda^{\delta(2)}(\log\lambda)^{\frac{1}{2}}\|f\|_{L^2(M)}, if\ k=n-2. \end{cases} \qquad (128)$$

Similarly, we have

$$\left\|\sum_j \left(1+\left|\lambda_j\ -\lambda\right|^{M}\right) \tilde{\phi}(\lambda_j-\lambda) E_j f\right\|_{L^2(\Sigma)}$$

$$\leq \begin{cases} \lambda^{\delta(2)}\|f\|_{L^2(M)}, if\ k\neq n-2, \\ \lambda^{\delta(2)}(\log\lambda)^{\frac{1}{2}}\|f\|_{L^2(M)}, if\ k=n-2. \end{cases} \qquad (129)$$

Therefore,

$$\left\|\tilde{\psi}(\lambda-P)g\right\|_{L^2(\Sigma)} = \left\|S\tilde{S}*g\right\|_{L^2(\Sigma)} \leq \|S\|_{L^2(M)\to L^2(\Sigma)}\left\|\tilde{S}^*\right\|_{L^2(\Sigma)\to L^2(M)}\|g\|_{L^2(\Sigma)}$$

$$= \|S\|_{L^2(M)\to L^2(\Sigma)}\left\|\tilde{S}\right\|_{L^2(M)\to L^2(\Sigma)}\|g\|_{L^2(\Sigma)}$$

$$\lesssim \begin{cases} T\lambda^{\delta(2)}\|g\|_{L^2(M)}, if\ k\neq n-2, \\ T\lambda^{\delta(2)}\log\lambda\,\|f\|_{L^2(M)}, if\ k=n-2. \end{cases} \qquad (130)$$

Now we may finish the proof of Theorem (1.2.1).

Recall that we denote K as the operator whose kernel is $K(x,y)$. The above theorem tells us that

$$\|K\|_{L^2(\Sigma)\to L^2(\Sigma)} \le \begin{cases} O\left(\lambda^{2\delta(2)}\right), & for\ k \ne n-2; \\ O\left(\lambda^{2\delta(2)}\log\lambda\right), for\ k = n-2. \end{cases} \tag{131}$$

Interpolating this with

$$\|K\|_{L^1(\Sigma)\to L^\infty(\Sigma)} = O\left(\frac{e^{cT}\lambda^{\frac{n-1}{2}}}{T}\right) \tag{132}$$

by Theorem (1.2.8) respectively, we get that for any p and $k \ne n-2$,

$$\begin{aligned} \|K\|_{L^{p'}(\Sigma)\to L^p(\Sigma)} &= O\left(\frac{\lambda^{\frac{n-1}{2}\left(1-\frac{2}{p}\right)} e^{cT\left(1-\frac{2}{p}\right)} \lambda^{2\delta(2)\cdot\frac{2}{p}}}{T^{1-\frac{2}{p}}}\right) \\ &= O\left(\frac{\lambda^{\frac{n-1}{2}-\frac{n-1}{p}+\frac{4\delta(2)}{p}} e^{cT\left(1-\frac{2}{p}\right)}}{T^{1-\frac{2}{p}}}\right), \end{aligned} \tag{133}$$

and for $k = n-2$,

$$\begin{aligned} \|K\|_{L^{p'}(\Sigma)\to L^p(\Sigma)} &= O\left(\frac{\lambda^{\frac{n-1}{2}-\frac{n-1}{p}+\frac{4\delta(2)}{p}} e^{cT\left(1-\frac{2}{p}\right)} (\log\lambda)^{\frac{2}{p}}}{T^{1-\frac{2}{p}}}\right) \\ &= O\left(\frac{\lambda^{\frac{n-1}{2}-\frac{n-1}{p}+\frac{4\delta(2)}{p}} e^{cT\left(1-\frac{2}{p}\right)} (\log\lambda)^{\frac{2}{p}}}{T^{1-\frac{2}{p}}}\right). \end{aligned} \tag{134}$$

If $k = n-1$, then $\delta(2) = \frac{1}{4}$. Thus

$$\|K\|_{L^{p'}(\Sigma)\to L^p(\Sigma)} = O\left(\frac{\lambda^{\frac{n-1}{2}-\frac{n-2}{p}} e^{cT\left(1-\frac{2}{p}\right)}}{T^{1-\frac{2}{p}}}\right). \tag{135}$$

Since $\frac{n-1}{2} - \frac{n-2}{p} < 2\delta(p)$ if $> \frac{2n}{n-1}$, say $\frac{n-1}{2} - \frac{n-2}{p} + \delta < 2\delta(p)$ for some small number $\delta > 0$, then taking $\beta = \frac{\delta}{c\left(1-\frac{2}{p}\right)}$, and $T = \beta\ log\ \lambda$, we have

$$\|K\|_{L^p(\Sigma)\to L^p(\Sigma)} = O\left(\frac{\lambda^{2\delta(p)-\delta}}{T^{1-\frac{2}{p}}}\right) = O\left(\frac{\lambda^{2\delta(p)-\delta^{1-\frac{2}{p}}}}{\log\lambda}\right) = o\left(\frac{\lambda^{2\delta(p)}}{\log\lambda}\right), \tag{136}$$

which indicates Theorem (1.2.1).
If $k = n-2$,

$$\begin{aligned} \|K\|_{L^{p'}(\Sigma)\to L^p(\Sigma)} &= O\left(\frac{\lambda^{\frac{n-1}{2}-\frac{n-1}{p}+\frac{4\delta(2)}{p}} e^{cT\left(1-\frac{2}{p}\right)} (\log\lambda)^{\frac{2}{p}}}{T^{1-\frac{2}{p}}}\right) \\ &= O\left(\frac{\lambda^{\frac{n-1}{2}-\frac{n-1}{p}+\frac{2}{p}} e^{cT\left(1-\frac{2}{p}\right)} (\log\lambda)^{\frac{2}{p}}}{T^{1-\frac{2}{p}}}\right). \end{aligned} \tag{137}$$

Now since $\frac{n-1}{2}-\frac{n-1}{p}+\frac{2}{p}<(n-1)-\frac{2(n-2)}{p}$ when $p>2$, we can take $\delta>0$ such that $\frac{n-1}{2}-\frac{n-1}{p}+\frac{2}{p}+\delta<(n-1)-\frac{2(n-2)}{p}$, and take $\beta=\delta/c\left(1-\frac{2}{p}\right), T=\beta\log\lambda$. Then

$$\|K\|_{L^{p'}(\Sigma)\to L^p(\Sigma)}=O\left(\frac{\lambda^{2\delta(p)-\delta}(\log\lambda)^{\frac{2}{p}}}{(\log\lambda)^{1-\frac{2}{p}}}\right)=o\left(\frac{\lambda^{2\delta(p)}}{\log\lambda}\right), \tag{138}$$

which is the what we need.

If $k\le n-3, \delta(2)=\frac{n-1}{2}-\frac{k}{2}$, then

$$\begin{aligned}\|K\|_{L^p(\Sigma)\to Lp(\Sigma)}&=O\left(\frac{\lambda^{\frac{n-1}{2}-\frac{n-1}{p}+\frac{4\delta(2)}{p}}e^{cT\left(1-\frac{2}{p}\right)}}{T^{1-\frac{2}{p}}}\right)\\&=O\left(\frac{\lambda^{\frac{n-1}{2}-\frac{n-1}{p}+\frac{2(n-1)-2k}{p}}e^{cT\left(1-\frac{2}{p}\right)}}{T^{1-\frac{2}{p}}}\right).\end{aligned} \tag{139}$$

Since $\frac{n-1}{2}-\frac{n-1}{p}+\frac{2(n-1)-2k}{p}<(n-1)-\frac{2k}{p}=2\delta(p)$ for $p>2$, we can take $\delta>0$ such that $\frac{n-1}{2}-\frac{n-1}{p}+\frac{2(n-1)-2k}{p}+\delta<(n-1)-\frac{2k}{p}$, and take $\beta=\delta/c\left(1-\frac{2}{p}\right), T=\beta\log\lambda$. Then

$$\|K\|_{L^{p'}(\Sigma)\to L^p(\Sigma)}=O\left(\frac{\lambda^{2\delta(p)-\delta}}{(\log\lambda)^{1-\frac{2}{p}}}\right)=o\left(\lambda^{2\delta(p)}(\log\lambda)\right), \tag{140}$$

which finishes Theorem (1.2.1).

Corollary (1.2.11)[246] For $\nu=0,1,2,\ldots$ and $\mathcal{E}_\nu\left(t,\frac{1+\epsilon}{2}\right)$ defined above, we have

$$\left|\int\hat\rho(t)e^{it\lambda}\mathcal{E}_\nu\left(t,\frac{1+\epsilon}{2}\right)dt\right|=O(\lambda^{n-1-2\nu}),\qquad \lambda\ge 1. \tag{141}$$

Proof. Recall that

$$\mathcal{E}_0\left(t,\frac{1+\epsilon}{2}\right)=\frac{H(t)}{(2\pi)^n}\int_{\mathbb{R}^n}\sum_r e^{i\Phi_r(\tilde x,\tilde x+\epsilon)\cdot\xi}cost|\xi|d\xi, \tag{142}$$

so that

$$\begin{aligned}\left|\int\hat\rho(t)e^{it\lambda}\mathcal{E}_0\left(t,\frac{1+\epsilon}{2}\right)dt\right|&=\left|\frac{1}{2(2\pi)^n}\int\int_{\mathbb{R}^n}\sum_r \hat\rho(t)e^{it(\lambda\pm|\xi|)+i\Phi_r(\tilde x,\tilde x+\epsilon)\cdot\xi}d\xi dt\right|\\&\approx\sum_r\left|\int_{\mathbb{R}^n}[\rho(\lambda+|\xi|)+\rho(\lambda-|\xi|)]e^{i\Phi_r(\tilde x,\tilde x+\epsilon)\cdot\xi}d\xi\right|\\&\le\int_{\mathbb{R}^n}(|\rho(\lambda+|\xi|)|+|\rho(\lambda-|\xi|)|)d\xi=O(\lambda^{n-1}).\end{aligned} \tag{143}$$

By the definition of $\mathcal{E}_\nu$ such that $\frac{\partial\mathcal{E}_\nu}{\partial t}=\frac{t}{2}\mathcal{E}_{\nu-1}$ and integrating by parts, we get that for any $\nu=1,2,3,\ldots$,

$$\int \hat{\rho}(t)e^{it\lambda}\mathcal{E}_\nu\left(t,\frac{1+\epsilon}{2}\right)dt = O(\lambda^{n-1-2\nu}). \quad (144)$$

Corollary (1.2.12)[246]: For $\varphi_r \in C_0^\infty(\mathbb{R})$ such that $\sum_r \varphi_r(t) = 1$ when $|t| \leq \frac{1}{2}$ and $\sum_r \varphi_r(t) = 0$ when $|t| > 1$, let

$$\widetilde{K}(x,x+\epsilon) = \frac{1}{\pi T}\int_{-1}^{1}\sum_r \varphi_r(t)\hat{\rho}\left(\frac{t}{T}\right)\left(cost\sqrt{-\Delta_{g_r}}\right)(x,x+\epsilon)e^{it}\lambda dt. \quad (145)$$

Then

$$\sup_x\left\|\widetilde{K}(x,\cdot)\right\|_{L^{\frac{4+\epsilon}{2}}(\Sigma)} = O\left(\frac{\lambda^{2\delta(2+\epsilon)}}{\log\lambda}\right). \quad (146)$$

Proof. [19] With similar approaches as in the previous discussions, we can show that $\widetilde{K}(x,x+\epsilon)$ is

$$\sum_r O\left(\frac{1}{T}\left(\frac{\lambda}{\lambda^{-1}+(1+\epsilon)_{\tilde{g}_r}(\tilde{x},\tilde{x}+\epsilon)}\right)^{\frac{n-1}{2}}\right) + \sum_{I\,d\neq\alpha\in Aut\left(\frac{\epsilon(n+3)+4}{n-1}\right)}\left[O\left(\frac{1}{T}\left(\frac{\lambda}{(1+\epsilon)_{\tilde{g}_r}\left(\tilde{x},\alpha(\tilde{x}+\epsilon)\right)}\right)^{\frac{n-1}{2}}\right) + O(e^{ET})\right], \quad (147)$$

where $E = \max\{C_{1+\epsilon}, 1+\epsilon\}+1$.

Note that $|t| \leq 1$ for $\sum_r \varphi_r(t) = 0$, and the number of terms such that $(1+\epsilon)_{\tilde{g}_r}\left(\tilde{x},\alpha(\tilde{x}+\epsilon)\right) \leq 1$ is at most e^c, so that

$$\left\|\widetilde{K}(x,x+\epsilon)\right\|_{L^{\frac{4+\epsilon(3+n)}{2(n-1)}}(\Sigma)} = O\left(\frac{\lambda^{2\delta\left(\frac{\epsilon(n+3)+4}{n-1}\right)}}{\log\lambda}\right), \quad (148)$$

if we take $T = \log\lambda$ and calculate as before.

Corollary (1.2.13)[246]: For $\epsilon \neq 0$,

$$\sum_r \left\|\tilde{\psi}_r(\lambda-P)g_r\right\|_{L^2(\Sigma)} \lesssim T\lambda^{2\delta(2)}\sum_r \|g_r\|_{L^2(\Sigma)}, \qquad \text{for any } g_r \in L^2(\Sigma), \quad (149)$$

and for $\epsilon = 0$,

$$\sum_r \left\|\tilde{\psi}_r(\lambda-P)g_r\right\|_{L^2(\Sigma)} \lesssim T\lambda^{2\delta(2)}\sum_r \log\lambda\|g_r\|_{L^2(\Sigma)}, \qquad \text{for any } g_r \in L^2(\Sigma), \quad (150)$$

where $P = \sqrt{-\Delta_{g_r}}$.

Proof. Recall the proof of the corresponding restriction theorem in [5]. They showed that for $\chi \in S(\mathbb{R})$, and defining

$$\chi_\lambda = \chi\left(\sqrt{-\Delta_{g_r}} - \lambda\right) = \sum_j \chi(\lambda_j - \lambda)E_j, \tag{151}$$

we have

$$\|\chi_\lambda\|_{L^2(M)\to L^2(\Sigma)} = O(\lambda^{\delta(2)}), \tag{152}$$

for $\epsilon \neq 0$, and

$$\|\chi_\lambda\|_{L^2(M)\to L^2(\Sigma)} = O(\lambda^{\delta(2)(\log\lambda)^{\frac{1}{2}}}, \tag{153}$$

for $\epsilon = 0$.

Now consider $\tilde{\psi}_r(\lambda - P)$ as $S\tilde{S}^*$, where

$$S = \sum_j \left(1 + |\lambda_j - \lambda|\right)^{-M} E_j \tag{154}$$

and

$$\tilde{S} = \sum_j \sum_r \left(1 + |\lambda_j - \lambda|\right)^M \tilde{\psi}_r(\lambda_j - \lambda)E_j, \tag{155}$$

where M is some large number.

Recall that $\sum_r \left|\tilde{\psi}_r(\tau)\right| \leq T(1 + |\tau|)^{-(1+\epsilon)}$ for any $(1+\epsilon) \in \mathbb{N}$. We then have

$$\sum_r \left|\left(1 + |\lambda_j - \lambda|\right)^M \tilde{\psi}_r(\lambda_j - \lambda)\right| \leq T\left(1 + |\lambda_j - \lambda|\right)^{-(1+\epsilon)} \tag{156}$$

for any $1 + \epsilon$.

By (25), which we deduced from the proof of Theorem 3 in [5], for a given λ,

$$\left\|\sum_{\lambda_j \in (\lambda-1,\lambda+1)} E_j\right\|_{L^2(M)\to L^2(\Sigma)} = O(\lambda^{\delta(2)}), \qquad \text{if } \ \epsilon \neq 0 \tag{157}$$

and

$$\left\|\sum_{\lambda_j \in (\lambda-1,\lambda+1)} E_j\right\|_{L^2(M)\to L^2(\Sigma)} = O\left(\lambda^{\delta(2)}(\log\lambda)^{\frac{1}{2}}\right), \qquad \text{if } \epsilon \neq 0 \tag{158}$$

so that for any $f_r \in L^2(M)$,

$$\left\| \sum_j \sum_r \left(1 + |\lambda_j - \lambda|^{-M}\right) E_j f_r \right\|_{L^2(\Sigma)}$$

$$\leq \left\| \sum_{\lambda_j \in (\lambda-1, \lambda+1)} \sum_r E_j f_r \right\|_{L^2(\Sigma)}$$

$$+ \left\| \sum_{\lambda_j \notin (\lambda-\delta, \lambda+\delta)} \sum_r \left(1 + |\lambda_j - \lambda|^{-M}\right) E_j f_r \right\|_{L^2(\Sigma)}$$

$$\begin{cases} \sum_r \lambda^{\delta(2)} \|f_r\|_{L^2(M)} + \sum_{\lambda_j \notin (\lambda-1, \lambda+1)} \sum_r \left(1 + |\lambda_j - \lambda|\right)^{-M} \|E_j f_r\|_{L^2(\Sigma)}, & \text{if } \epsilon \neq 0, \\ \sum_r \lambda^{\delta(2)} (\log \lambda)^{\frac{1}{2}} \|f_r\|_{L^2(M)} + \sum_{\lambda_j \notin (\lambda-1, \lambda+1)} \sum_r \left(1 + |\lambda_j - \lambda|\right)^{-M} \|E_j f_r\|_{L^2(\Sigma)}, & \text{if } \epsilon = 0. \end{cases} \tag{159}$$

as

$$\sum_{\lambda_j \notin (\lambda-1, \lambda+1)} \sum_r \left(1 + |\lambda_j - \lambda|\right)^{-M} \|E_j f_r\|_{L^2(\Sigma)}$$

$$\leq \begin{cases} \sum_{\lambda_j \notin (\lambda-1, \lambda+1)} \sum_r \lambda_j^{\delta(2)} \left(1 + |\lambda_j - \lambda|\right)^{-M} \|E_j f_r\|_{L^2(M)}, & \text{if } \epsilon \neq 0, \\ \sum_{\lambda_j \notin (\lambda-1, \lambda+1)} \sum_r \lambda_j^{\delta(2)} (\log \lambda_j)^{\frac{1}{2}} \left(1 + |\lambda_j - \lambda|\right)^{-M} \|E_j f_r\|_{L^2(M)}, & \text{if } \epsilon = 0, \end{cases} \tag{160}$$

which can be made arbitrarily small when M is sufficiently large,

$$\left\| \sum_j \sum_r \left(1 + |\lambda_j - \lambda|^{-M}\right) E_j f_r \right\|_{L^2(\Sigma)}$$

$$\leq \begin{cases} \sum_r \lambda^{\delta(2)} \|f_r\|_{L^2(M)}, & \text{if } \epsilon \neq 0, \\ \sum_r \lambda^{\delta(2)} (\log \lambda)^{\frac{1}{2}} \|f_r\|_{L^2(M)}, & \text{if } \epsilon = 0. \end{cases} \tag{161}$$

Similarly, we have

$$\left\| \sum_j \sum_r \left(1 + |\lambda_j - \lambda|^{M}\right) \tilde{\varphi}_r(\lambda_j - \lambda) E_j f_r \right\|_{L^2(\Sigma)}$$

$$\leq \begin{cases} \sum_r \lambda^{\delta(2)} \|f_r\|_{L^2(M)}, \text{if } \epsilon \neq 0, \\ \sum_r \lambda^{\delta(2)} (\log \lambda)^{\frac{1}{2}} \|f_r\|_{L^2(M)}, \text{if } \epsilon = 0. \end{cases} \tag{162}$$

Therefore,

$$
\begin{aligned}
\sum_r \left\|\tilde{\psi}_r(\lambda - P) g_r\right\|_{L^2(\Sigma)} &= \sum_r \left\|S\tilde{S} * g_r\right\|_{L^2(\Sigma)} \\
&\leq \|S\|_{L^2(M)\to L^2(\Sigma)} \left\|\tilde{S}^*\right\|_{L^2(\Sigma)\to L^2(M)} \sum_r \|g_r\|_{L^2(\Sigma)} \\
&= \|S\|_{L^2(M)\to L^2(\Sigma)} \left\|\tilde{S}\right\|_{L^2(M)\to L^2(\Sigma)} \sum_r \|g_r\|_{L^2(\Sigma)} \\
&\lesssim \begin{cases} \sum_r T\lambda^{\delta(2)} \|g_r\|_{L^2(M)}, & \text{if } \epsilon \neq 0, \\ \sum_r T\lambda^{\delta(2)} \log\lambda \, \|f_r\|_{L^2(M)}, & \text{if } \epsilon = 0. \end{cases}
\end{aligned} \tag{163}
$$

Chapter 2
Kakeya-Nikodym Averages and Eigenfunction Restriction Estimates

We show that the results are related to a recent work of Bourgain who showed that L^2-averages over geodesics of eigenfunctions are small compared to a natural power of the eigenvalue λ provided that the $L^4(M)$ norms are similarly small. Our results imply that QUE cannot hold on a compact boundaryless Riemannian manifold (M, g) of dimension two if L^p-norms are saturated for a given $2 < p < 6$. We also show that eigenfunctions cannot have a maximal rate of L^2-mass concentrating along unit portions of geodesics that are not smoothly closed. We show the main estimates by using the Hadamard parametrix for the wave equation on $(\mathbb{R}^2, \tilde{g})$ and the fact that the classical comparison theorem of Gunther for the volume element in spaces of nonpositive curvature gives us desirable bounds for the principal coefficient of the Hadamard parametrix, allowing us to prove the main result. Using by an estimate, we deduce that, the L^p-norms of eigenfunctions for the above range of exponents is relatively small. We can slightly improve the known lower bounds for nodal sets in dimensions $d \geq 3$ of Colding and Minicozzi in the special case of (variable) nonpositive curvature.

Section (2.1): L^p-Norms of Eigenfunctions

We slightly sharpen a recent result of Bourgain [36] concerning two-dimensional compact boundaryless Riemannian manifolds. We shall be able to provide a natural necessary and sufficient condition concerning the growth rate of L^p-norms of eigenfunctions for $2 < p < 6$ and their L^2-concentration about geodesics.

There are different ways of measuring the concentration of eigenfunctions. One is by means of the size of their L^p-norms for various values of $p > 2$. If M is a compact boundaryless manifold with Riemannian metric $g = g_j k(x)$ and if $\triangle_g$ is the associated LaplaceBeltrami operator, then the eigenfunctions solve the equation $-\triangle_g e_{\lambda j}(x) = \lambda_j^2 e_{\lambda j}(x)$ for a sequence of eigenvalues $0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \ldots$. Thus, we are normalizing things so that λ_j are the eigenvalues of the first-order operator $\sqrt{-\triangle_g}$. We shall also usually assume that the $e_{\lambda j}$ have L^2-norm one, in which case $\{e_{\lambda j}\}$ provides an orthonormal basis of $L^2(M, dx)$ where dx is the volume element coming from the metric. Earlier, in the two-dimensional case, we showed in [15] that if M is fixed then there is a uniform constant C so that for $2 \leq p \leq \infty$ and $j = 1, 2, 3, \ldots$

$$\left\|e_{\lambda j}\right\|L^p(M) \leq C\lambda_j^{\delta(p)} e_{\lambda j} \; L^2(M) \qquad (1)$$

with

$$\delta(p) = \begin{cases} \frac{1}{2}\left(\frac{1}{2} - \frac{1}{p}\right), 2 \leq p \leq 6, \\ \frac{1}{2} - \frac{2}{p}, 6 \leq p \leq \infty. \end{cases}$$

These estimates are sharp for the round sphere S^2, and in this case they detect two types of concentration of eigenfunctions that occur there. Recall that on S^2 with the canonical metric the distinct eigenvalues are $\sqrt{k^2 + k}, k = 0, 1, 2, \ldots$, which repeat with multiplicity $d_k = 2k + 1. If\ \mathcal{H}_k$, the space of spherical harmonics of degree k, is the space of all eigenfunctions with eigenvalue $\sqrt{k^2 + k}$, and if $H_k(x, y)$ is the kernel of the projection operator onto $\mathcal{H}_k$, then the k-th zonal function at $x_0 \in S^2$ is $Z_k(y) =$

$\left(H_k(x_0,x_0)\right)^{-\frac{1}{2}}H_k(x_0,y)$. Its L^2- norm is one but its mass is highly concentrated at $\pm x_0$ where it takes on the value $\sqrt{\frac{d_k}{4\pi}}$. Explicit calculations show that $\|Z_k\|_{L_p(S^2)} \approx k^{\delta(p)}$ for $p \geq 6$ (see e.g. [53]), which shows that in the case of $M = S^2$ with the round metric (1) cannot be improved for this range of exponents. Another extreme type of concentration is provided by the highest weight spherical harmonics which have mass concentrated on the equators of S^2, which are its geodesics. The ones concentrated on the equator $\gamma_0 = \{(x_1,x_2,0);\ x_1^2 + x_2^2 = 1\}$ are the functions Q_k , which are the restrictions of the $\boldsymbol{R}^3$ harmonic polynomials $k^{\frac{1}{4}}(x_1 + ix_2)^k\ to\ S^2 = \{x\ ;\ |x| = 1\}$.
One can check that the Q_k have L^2-norms comparable to one and L^p-norms comparable to $k^{\frac{1}{2}\left(\frac{1}{2}-\frac{1}{p}\right)}$ when $2 \leq p \leq 6$ (see e.g. [53]). Notice also that the Q_k have Gaussian type concentration about the equator γ_0. Specifically, if $\mathcal{T}_{k^{-\frac{1}{2}}}(\gamma_0)$ denotes all points on S^2 of distance smaller than $k^{-\frac{1}{2}}$ from γ0 then one can check that

$$lim\inf_{k\to\infty} \int_{\mathcal{T}_{k^{-\frac{1}{2}}(\gamma_0)}} |Qk(x)|^2 dx > 0\,. \tag{2}$$

Obviously the Q_k also have the related property that

$$\int_{\gamma_0} |Qk|^2 ds \approx k^{\frac{1}{2}}, \tag{3}$$

if ds is the measure on γ_0 induced by the volume element.

Thus, the sequence of highest weight spherical harmonics shows that the norms in (1) (for $2 < p < 6$), (2) and (3) are related. We show that this is true for general two-dimensional compact manifolds without boundary.

We remark that, although the estimates (1) are sharp for the round sphere, one expects that it should be the case that, for generic manifolds, and L^2-normalized eigenfunctions one has

$$lim\sup_{j\to\infty} \lambda_j^{-\delta(p)} \left\|e_{\lambda_j}\right\| L^p = 0 \tag{4}$$

for every $2 < p \leq \infty$. This was verified for exponents $p > 6$ by Zelditch and [17] by showing that if there are no points x through which a positive measure of geodesics starting at x loop back through x then $\|e_\lambda\|_\infty = o\left(\lambda^{\frac{1}{2}}\right)$. By interpolating with the estimate (1) for $p = 6$, this yields (4) for all $p > 6$. Corresponding results were also obtained in [17] for higher dimensions. Recently, these results were strengthened by Toth and Zelditch [29] to allow similar results for quasimodes under the weaker condition that at every point x the set of recurrent directions for the first return map for geodesic flow has measure zero in the cosphere bundle S_x^*M over x.

Other than the partial results in Bourgain [36], there do not seem to be any results addressing when (4) holds for a given $2 < p < 6$ (although Zygmund [37] showed that on the torus L^2-normalized eigenfunctions have uniformly bounded L^4-norms). Furthermore, there do not seem to be results addressing the interesting endpoint case of $p = 6$, where one expects both types of concentration mentioned before to be relevant. Recently have studied the L^2 norms of eigenfunctions over unit-length geodesics. Burq, Gérard and Tzvetkov [5] showed that if Π is the collection of all unit length geodesics then

$$\sup_{\gamma \in \Pi} \int_{\gamma} \left|e_{\lambda_j}\right|^2 \; ds \lesssim \lambda_j^{\frac{1}{2}} \left\|e_{\lambda_j}\right\|^2 \; L^2(M), \quad \mathrm{j} = 1,2,3,\ldots, \tag{5}$$

which is sharp in view of (3). Related results for hyperbolic surfaces were obtained earlier by Reznikov [26], who opened up the present line of investigation. The proof of (5) boils down to bounds for certain Fourier integral operators with folding singularities (cf. Greenleaf and Seeger [42], Tataru [56]). We shall use ideas from [42], [56], and [40], [46], [29], [17] to show that if $\gamma \in \Pi$ and

$$\limsup_{\mathrm{j}\to\infty} \lambda_{\mathrm{j}}^{-\frac{1}{2}} \int_{\gamma} \left|\mathrm{e}_{\lambda_{\mathrm{j}}}\right|^2 \mathrm{ds} > 0\,,$$

then the geodesic extension of γ must be a smoothly closed geodesic. Presumably it also has to be stable, but we cannot prove this. Further recent work on L^2-concentration along curves can be found in Toth [57].

In [36], Bourgain proved an estimate that partially links the norms in (1) and (5), namely that for all $p \geq 2$

$$\sup_{\gamma \in \Pi} \int_{\gamma} \left|\mathrm{e}_{\lambda_{\mathrm{j}}}\right|^2 ds \; \lambda_j^{\frac{1}{p}} \; \left\|e_{\lambda_j}\right\|^2_{Lp(M)} \tag{6}$$

For $p = 2$, this is just (5); however, an interesting feature of (6) is that the estimate for a given $2 < p \leq 6$ combined with (1) yields (5). Thus, if $e_{\lambda_{j_k}}$ is a sequence of eigenfunctions with (relatively) small $L^p(M)$ norms for a given $2 < p \leq 6$, it follows that its L^2-norms over unit geodesics must also be (relatively) small. Bourgain [36] also came close to establishing the equivalence of these two things by showing that given $\varepsilon > 0$ there is a constant C_ε so that for $j = 1,2,\ldots$

$$\left\|e_{\lambda_j}\right\|_{L^4(M)} \leq C_\varepsilon \left(\lambda_j^{\frac{1}{8}+\varepsilon} \; \left\|e_{\lambda_j}\right\|_{L2(M)} \right)^{\frac{3}{4}} \left[\lambda_j^{-\frac{1}{2}} \sup_{\gamma \in \Pi} \int_{\gamma} \left|\mathrm{e}_{\lambda_{\mathrm{j}}}\right|^2 ds \right]^{\frac{1}{8}}. \tag{7}$$

Since $\delta(4) = 1/8$ in (1), if the preceding inequality held for $\varepsilon = 0$ one would obtain the linkage of the size of the norms in (5) for large energy with the size of the $L^4(M)$ norms.

Our main estimate in Theorem (2.1.1) is that a variant of (7) holds, which is strong enough to complete the linkage.

Bourgain's approach in proving (7) was to employ ideas going back to Córdoba [39] and Fefferman [41] that were used to give a proof of the Carleson-Sjölin Theorem [37]. The key object that arose in Córdoba's work [39] was what he called the Kakeya maximal function in $\boldsymbol{R}^2$, namely,

$$\mathcal{M}f\,(x) = \sup\_(x \in T_{\lambda^{-\frac{1}{2}}} \left|\mathcal{T}_{\lambda^{-\frac{1}{2}}}\right|^{-1} \int_{T_{\lambda^{-\frac{1}{2}}}} |f\,(y)|dy\,, f \in L^2\boldsymbol{R}^2, \tag{8}$$

with the supremum taken over all $\lambda^{-\frac{1}{2}}$-neighborhoods $\mathcal{T}_{\lambda^{-\frac{1}{2}}}$ of unit line segments containing x, and $|\mathcal{T}_{\lambda^{-\frac{1}{2}}}| \approx \lambda^{-\frac{1}{2}}$ denoting its area. The above maximal operator is now more commonly called the Nikodym maximal operator as this is the terminology in Bourgain's important [33]–[35] which established highly nontrivial progress towards establishing the higher dimensional version of the Carleson-Sjölin Theorem for Euclidean spaces $\boldsymbol{R}^n, n \geq 3$. One could also consider variable coefficient versions of the maximal operators in (8). In the present context if $\gamma \in \Pi$ is a unit geodesic, one could consider the $\lambda^{-\frac{1}{2}}$-tube about it given by

$$\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma) = \left\{ y \in M ; \inf_{x\in\gamma} d_g(x,y) < \lambda^{-\frac{1}{2}} \right\}.$$

with $d_g(x,y)$ being the geodesic distance between x and y. Then if Volg $\left(\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)\right)$ denotes the measure of this tube, the analog of (8) would be

$$\mathcal{M} f(x) = \sup_{x\in\gamma\in\Pi} \frac{1}{Vol_g\left(\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)\right)} \int_{\mathcal{T}_{\lambda^{-\frac{1}{2}}}} |f(y)| dy .$$

These operators have been studied before because of their applications in harmonic analysis on manifolds. See e.g. [48], [54]. As was shown in [47], following the earlier [35], they are much better behaved in 2-dimensions compared to higher dimensions.

As (7) suggests, it is not the size of the L^2-norm of Mf for $f \in L^2(M)$ that is relevant for estimating $L^4(M)$ −norms of eigenfunctions but rather the sup-norm of this quantity with $f = \left|e_{\lambda_j}\right|^2$, which up to the normalizing factor in front of the integral is the quanitity

$$\operatorname{Sup}_{\gamma\in\Pi} \int_{\mathcal{T}_{\lambda^{-\frac{1}{2}}(\gamma)}} \left|e_{\lambda_j}(x)\right|^2 dx .$$

If the e_{λ_j} are L^2-normalized this is trivially bounded by one. In rough terms our results say that beating this trivial bound is equivalent to beating the bounds in (1) for a given $2 < p < 6$.

Let us now state our variant of (7):

Theorem (2.1.1)[28]: Fix a two-dimensional compact boundaryless Riemannian manifold (M,g). Then given $\varepsilon > 0$ there is a constant C_ε so that for eigenfunctions e_λ of $\sqrt{-\Delta_g}$ with eigenvalues $\lambda \geq 1$ we have

$$\|e_\lambda\|_{L4(M)}^4 \leq \varepsilon\lambda^{\frac{1}{2}}\|e_\lambda\|_{L4(M)}^4 + C_\varepsilon\lambda^{\frac{1}{2}}\|e_\lambda\|_{L2(M)}^2 \sup_{\gamma\in\Pi} \int_{\mathcal{T}_{\lambda^{-\frac{1}{2}}(\gamma)}} |e_\lambda(x)|^2 dx$$
$$+ C\|e_\lambda\|_{L^2(M)}^4 , \quad (9)$$

with C being a fixed constant which is independent of λ and ε.

We shall prove this not by adapting Córdoba's [39] proof of the Carleson-Sjölin Theorem but rather that of Hörmander [45]. He obtained sharp oscillatory integral bounds in $\boldsymbol{R}^2$ that provided sharp Böchner-Riesz estimates for $L^4(\boldsymbol{R}^2)$ (*i.e.*, the Carleson-Sjölin Theorem), which turns out to be the endpoint case for this problem in 2-dimensions. Hörmander's approach was to turn this L^4-problem into an L^2-problem by squaring the oscillatory integrals and then estimating their L^2-norms. As his proof shows, the resulting bilinear operators that arise are better and better behaved away from the diagonal, and this fact is what allows us to take the constant in front of the first term in the right side of (9) to be arbitrarily small (at the expense of the 2nd term).

Stein [55] provided a generalization of Hörmander's oscillatory integral Theorem to higher dimensions in a way that proved to be sharp because of a later construction of Bourgain [35]. Bourgain's example and related ones in [47] suggest that extending the results to higher dimensions (where the range of exponents would be $2 < p < 2(n +$

$1)/(n-1))$ could be subtle. On the other hand, since the constructions tend to involve concentration about hypersurfaces as opposed to geodesics, their relevance is not plain.

We shall prove Theorem (2.1.1) by estimating an oscillatory integral operator, which up to a remainder term, reproduces eigenfunctions. The remainder term in this reproducing formula accounts for the last term in (9), which we could actually take to $be \leq C_N \lambda^{-N} \|e_\lambda\|_2^4$ for any N, but this is not important for our applications. Also, we remark that the proof of the Theorem will show that the constant C_ε in (9) can be taken to be $O(\varepsilon^{-2})$ as $\varepsilon \to 0$.

We state an immediate consequence of Theorem (2.1.1) which states that the size of L^4-norms of eigenfunctions is equivalent to size of L^2-mass near geodesics.

Corollary (2.1.2)[28]: Let $e_{\lambda_{j_k}}$ be a sequence of eigenfunctions with eigenvalues $\lambda_{j_1} \leq \lambda_{j_2} \leq \ldots$ and unit $L^2(M)$-norms. Then

$$\limsup_{k\to\infty} \sup_{\gamma\in\Pi} \int_{\mathcal{T}_{\lambda_{i_k}^{-\frac{1}{2(\gamma)}}}} \left|e_{\lambda_{j_k}}(x)\right|^2 = 0 \tag{10}$$

if and only if

$$\limsup_{k\to\infty} \lambda_{jk}^{-\frac{1}{8}} \left\|e_{\lambda_{j_k}}\right\|_{L4(M)} = 0. \tag{11}$$

To prove this, we first notice that if we assume (10), then (11) must hold because of (9). Also, by Hölder's inequality

$$\left(\int_{\mathcal{T}_{\lambda^{-\frac{1}{2}}(\gamma)}} |e_\lambda(x)|^2\, dx\right)^{\frac{1}{2}} \leq \left(Vol_g\left(\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)\right)\right)^{\frac{1}{4}} \|e_\lambda\|_{L4(M)}$$
$$\lesssim \lambda^{-\frac{1}{8}} \|e_\lambda\|_{L4(M)},$$

and so (11) trivially implies (10).

If we use Bourgain's estimate (6) and (1) we can say a bit more.

Corollary (2.1.3)[28]: Let $\left\{e_{\lambda_{j_k}}\right\}_{k=1}^{\infty}$ be as above and suppose that $2 < p < 6$. Then the following are equivalent

$$lim \sup_{k\to\infty} \lambda_{jk}^{-\frac{1}{2}} \sup_{\gamma\in\Pi} \int_\gamma \left|e_{\lambda_{j_k}}(s)\right|^2 = 0 \tag{12}$$

$$\limsup_{k\to\infty} \sup_{\gamma\in\Pi} \int_{\mathcal{T}_{\lambda_{jk}^{-\frac{1}{2}}(\gamma)}} \left|e_{\lambda_{j_k}}(x)\right|^2 dx = 0 \tag{13}$$

$$\limsup_{k\to\infty} \lambda_{jk}^{-\delta(p)} \left\|e_{\lambda_{j_k}}\right\|_{Lp\,(1.14)(M)} = 0. \tag{14}$$

To prove this result, we first note that, by the M. Riesz interpolation Theorem and (1) for $p = 2$ and $p = 6$, (14) holds for a given $2 < p < 6$ if and only if it holds for $p = 4$, which we just showed is equivalent to (13). Clearly (12) implies (13). Finally, since Bourgain's estimate (6) shows that (14) implies (12), the proof of Corollary (2.1.3) is complete.

By describing one more application. Recall that a sequence of L^2-normalized eigenfunctions $\left\{e_{\lambda_{j_k}}\right\}_{k=1}^{\infty}$ satisfies the quantum unique ergodicity property (QUE) if the associated Wigner measures $\left|e_{\lambda_{j_k}}\right|^2 \, dx$ tend to the Liouville measure on S^*M. If this is the case, then one certainly cannot have

$$\lim_{k\to\infty}\sup \; \sup_{\gamma\in\Pi} \int_{\mathcal{T}_{\lambda_{jk}^{-\frac{1}{2}}(\gamma)}} \left|e_{\lambda_{j_k}}(x)\right|^2 dx > 0,$$

since the tubes are shrinking.

In the case where M has negative sectional curvature Schnirelman's [14] Theorem says there is a density one subsequence $\left\{e_{\lambda_{j_k}}\right\}_{k=1}^{\infty}$ of all the $\{e_{\lambda_j}\}$ satisfying QUE. Rudnick and Sarnak [50] conjectured that in the negatively curved case there should be no exceptional subsequences violating QUE, i.e., in this case QUE should hold for the full sequence $\{e_{\lambda_j}\}$ of L^2-normalized eigenfunctions. On the other hand, by Corollary (2.1.3), we have the following.

Corollary (2.1.4)[28]: Let M be a two-dimensional compact boundaryless Riemannian manifold. Then QUE cannot hold for M if for a given $2 < p < 6$ there is saturation of L^p norms, i.e.,

$$\lim_{j\to\infty}\sup \; \lambda_j^{-\delta(p)} \left\|e_{\lambda_j}\right\| L^p(M) > 0,$$

with e_{λ_j} being the L^2 -n ormalized eigenfunctions.

See e.g. [59] for connections between QUE and the Lindelöf hypothesis, and see [38] for recent developments regarding the QUE conjecture.

As in [36] and [5] we shall prove our estimate by using certain convenient operators that reproduce eigenfunctions. Specifically, we shall use a slight variant of a result from [16], Chapter 5 that was presented in [5].

Lemma (2.1.5)[28]: Let $\delta > 0$ be smaller than half of the injectivity radius of (M, g). Then there is a function $\chi \in S(\boldsymbol{R})$ with $\chi(0) = 1$ so that if dg (x, y) is the geodesic distance between $x, y \in M$

$$\chi_\lambda f(x) = \chi\left(\sqrt{-\Delta_g} - \lambda\right) f(x)$$
$$= \lambda^{\frac{1}{2}} \int_M e^{i\lambda dg(x,y)} \alpha(x, y, \lambda) f(y) dy + R_\lambda f(x), \quad (15)$$

Where

$$\|R_\lambda f\| L^\infty(M) \le C_{N^{\lambda^{-N}}} \|f\| L^1(M) \text{ for all } N = 1, 2, \dots,$$

and $\alpha \in C^\infty$ has the property that

$$\left|\partial_{x,y}^\alpha \alpha(x, y, \lambda)\right| \le C_\alpha \text{ for all } \alpha,$$

and, moreover,

$$\alpha(x, y, \lambda) = 0 \text{ if } d_g(x, y) \notin \left(\frac{\delta}{2}, \delta\right). \quad (16)$$

Since $\chi_\lambda e_\lambda = e_\lambda$ and since the 4th power of the L^4-norm of $R_\lambda e_\lambda$ is dominated by the last term in (9), we conclude that in order to prove Theorem (2.1.1) it is enough to show that, given $\varepsilon > 0$ there is a constant C_ε so that when $\lambda \ge 1$

$$\int_M \left|\lambda^{\frac{1}{2}} \int_M e^{i\lambda dg(x,y)} \alpha(x, y, \lambda) f(y) dy\right|^2 |f(x)|^2 dx$$
$$\le \varepsilon \lambda^{\frac{1}{4}} \|f\|_{L^2(M)}^2 \|f\|_{L^4(M)}^2 \quad (17)$$

$$+ C_\varepsilon \lambda^{\frac{1}{2}} \|f\|^2_{L^2(M)} \sup_{\gamma \in \Pi} \int_{\mathcal{T}_{\lambda^{-\frac{1}{2}}(\gamma)}} |f(x)|^2 dx ,$$

for, if $f = e_\lambda$, the first term in the right is bounded by a fixed constant times $\varepsilon\lambda^{\frac{1}{2}}\|e_\lambda\|^4_{L2(M)}$, because of (1).

After applying a partition of unity, we may assume that in addition to (16), $\alpha(x, y, \lambda)$ vanishes unless x is in a small neighborhood of some $x_0 \in M$ and y is in a small neighborhood of some $y_0 \in M$ with $\delta/2 < d_g(x_0, y_0) < 10\delta\}$. As mentioned before, we are also at liberty to take $\delta > 0$ to be small. To simplify the calculations to follow, it is convenient to choose a natural coordinate system. Specifically, we shall choose Fermi normal coordinates about the geodesic γ_0 which passes through x_0 and is perpendicular to the geodesic connecting x_0 and y_0. These coordinates will be well defined on $B(x_0, 10\delta)$ if δ is small. Furthermore, we may assume that the

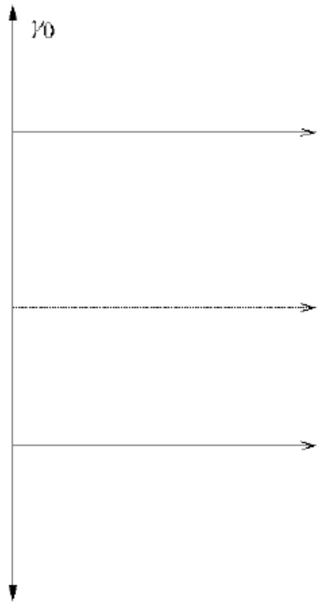


Figure (1)[28]: Fermi normal coordinates about γ_0.

image of $\gamma_0 \cap B(x_0, 10\delta)$ in the resulting coordinates is a line segment which is parallel to the 2nd coordinate axis and that all horizontal line segments $s \to \{(s, t_0)\}$ are geodesic with the property that $d_g((s_1, t_0), (s_2, t_0)) = |s_1 - s_2|$.

If we use these coordinates and apply Schwarz's inequality, we conclude that, in order to prove (17), it suffices to show that given $\varepsilon > 0$ we can find $C_\varepsilon < \infty$ so that when $\lambda \geq 1$

$$\int \left(\int \left| \lambda^{\frac{1}{2}} \int e^{i\lambda d_g(x,(s,t))} \alpha(x, (s,t), \lambda) f(s,t) dt \right|^2 |f(x)|^2 dx \right) ds$$
$$\leq \varepsilon \lambda^{\frac{1}{4}} \|f\|^2_{L2(M)} \|f\|^2_{L^4(M)} + C_\varepsilon \lambda^{\frac{1}{2}} \|f\|^2_{L2(M)} \sup_{\gamma \in \Pi} \int_{\mathcal{T}_{\lambda^{-\frac{1}{2}}(\gamma)}} |f(x)|^2 dx .$$

This, in turn would follow if we could show that given $\varepsilon > 0$

$$\int \left| \lambda^{\frac{1}{2}} \int e^{i\lambda d_g(x,(s,t))} \alpha(x, (s,t), \lambda) h(t) dt \right|^2 |f(x)|2dx$$
$$\leq \varepsilon \lambda^{\frac{1}{4}} \|h\|^2_{L2(dt)} \|f\|^2_{L4(M)}$$

$$+ C_\varepsilon \lambda^{\frac{1}{2}} \|h\|^2_{L2(dt)} \sup_{\gamma \in \Pi} \int_{\mathcal{T}_{\lambda^{-\frac{1}{2}}(\gamma)}} |f(x)|^2 dx, \quad (18)$$

with C_ε depending on $\varepsilon > 0$ but not on s or on $\lambda \geq 1$.

We shall establish this estimate for a particular value of s, which, after relabeling, we may assume to be $s = 0$. Since the proof of (18) for this case relies only on Gauss' lemma and the related Carleson-Sjölin condition, it also yields the uniformity in s, assuming, as we may, that α has small support.

To prove this inequality, let us choose a function $\eta \in C_0^\infty(\boldsymbol{R})$ satisfying $\eta(t) = 0, |t| > 1$, and $\sum_{j=-\infty}^{\infty} \eta(t - j) \equiv 1$. Given $\lambda \geq 1$ fixed, we shall then set

$$\eta_j(t) = \eta_{\lambda,j}(t) = \eta(\lambda^{\frac{1}{2}} t - j).$$

Then, given $N = 1, 2, \ldots$, we have that

$$\left| \lambda^{\frac{1}{2}} \int e^{i\lambda d_g(x,(0,t))} \alpha(x,(0,t),\lambda) h(t) dt \right|^2$$
$$\leq N \sum_j \left| \lambda^{\frac{1}{2}} \int e^{i\lambda d_g(x,(0,t))} \eta_j(t) \alpha(x,(0,t),\lambda) h(t) dt \right|^2 \quad (19)$$
$$+ \left| \lambda \int \int e^{i\lambda(d_g(x,(0,t)) d_g(x,(0,t')))} aN(x,t,t') h(t) h(t') dt dt' \right|,$$

Here

$$aN(x,t,t') = \sum_{|j-k|>N} \eta_j(t) \alpha(x,(0,t),\lambda) \eta_k(t') \alpha(x,(0,t'),\lambda)$$

anishes when $|t - t'| \leq (N - 1)\lambda^{-\frac{1}{2}}$. The first term in the right side of the preceding inequality comes from applying Young's inequality to handle the double-sum over indices with $|j - k| \leq N$. Because of (19), we onclude that (18) would follow if we could show that there is a constant independent of $\lambda \geq 1$ and $N = 2, 3, 4 \ldots$ so that

$$\left\| \lambda \int \int e^{i\lambda[d_g(x,(0,t)) d_g(x,(0,t'))]} aN(x,t,t') h(t) h(t') dt dt' \right\|_{L2(dx)}$$
$$\leq \lambda^{\frac{1}{4}} N^{-\frac{1}{2}} \|h\|^2_{L^2(dt)}, \quad (20)$$

and also that there is a constant C ndependent of $j \in Z$ and $\lambda \geq 1$ so that

$$\int \left| \lambda^{\frac{1}{2}} \int e^{i\lambda d_g(x,(0,t))} \eta_j(t) \alpha(x,(0,t),\lambda) h(t) dt \right|^2 |f(x)|^2 dx$$
$$\leq C \lambda^{\frac{1}{2}} \|h\|^2_{L^2(dt)} \sup_{\gamma \in \Pi} \int_{\mathcal{T}_{\lambda^{-\frac{1}{2}}(\gamma)}} |f(x)|^2 dx. \quad (21)$$

Indeed, by using the finite overlapping of the supports of the η_j, if we set $\varepsilon = CN^{-\frac{1}{2}}$, then we see that these two inequalities and (19) imply (18) with $C_\varepsilon \approx \varepsilon^{-2}$. Since the proof of (21) only uses Gauss' lemma and the fact that coordinates have been chosen so that $s \to (s, t_0)$ are unit speed geodesics for fixed t_0, we shall just verify (21) for $j = 0$, as the argument for this case will yield the other cases as well.

The next step is to see that these two inequalities are consequences of the following two propositions.

We need to introduce one more coordinate system, which finally explains where the L^2 norms over small tubular neighborhoods of geodesics comes into play. Since we are proving (21) with $j = 0$ and since η_0 is supported in the mall interval $[-\lambda^{-\frac{1}{2}}, \lambda^{-\frac{1}{2}}]$, it is natural to take geodesic normal coordinates bout $(0,0)$. If we recall that the 1st coordinate axis is a unit-speed eodesic in our original Fermi normal coordinates, we shall naturally choose the geodesic normal coordinates $x \to \kappa(x)$ that preserve this axis (and its orientation). Such a system is unique up to reflection about this axis, and we shall just fix one of these two choices.

Proposition (2.1.6) would imply (20) if $\phi(x,t) = d_g\,(x,(0,t))$ satisfies the CarlesonSjölin condition. The fact that this is the case is well known. See [16]. It follows from our choice of coordinates and the fact that if $x_0 \in M$ is fixed then the set of points $\{\nabla_x\, d_g\,(x,y);\ x = x_0, d_g\,(x0,y) \in (\delta/2,\delta)\}$ is the cosphere at $x_0, S^*_{x_0} M = \{\xi\,; \sum g^{j^k}\ (x_0)\xi_j\,\xi_k = 1\}$, where $g^{j^k}\ (x)$ is the cometric (inverse to $g_{jk}(x)$). If we choose geodesic normal coordinates $\kappa(y)$ vanishing at x_0 then the gradient becomes $\kappa(y)$. This turns out to be equivalent to the usual formulation of Gauss' lemma, saying that this exponential map $y \to \kappa(y)$ is a local radial isometry. It says that small geodesic spheres centered at x_0 get sent to spheres centered at the origin and small geodesic rays through x_0 intersect these geodesic spheres orthogonally and get sent to rays through the origin, which is what allows Proposition (2.1.7) to be true.

We see that Proposition (2.1.7) implies (21) for $j = 0$. If we take $\rho(t;\, x) = \eta_0(t)\alpha(x,(0,t),\lambda)$, then ρ satisfies (27). Also, if we let

$$S_j = \left\{y\,;\ \theta\,(y) \in \left(\lambda^{-\frac{1}{2}} j, \lambda^{-\frac{1}{2}}\,(j + 1)\right]\right\},$$

where $\theta\,(y) \in [0,2\pi\,)$ is defined so that $y = |y|(cos\,\theta\,(y), sin\,\theta\,(y))$, then, if $y = \kappa(x)$ are the geodesic normal coordinates about $(0,0)$ in the Proposition (2.1.7), then the left side of (21) is dominated by

$$\sum_j \left\| \lambda^{\frac{1}{2}} \int e^{i\lambda\psi(x,t\,)}\rho(t;\, x)h(t)dt\ \right\|^2_{L^\infty\left(\kappa^{-1}(S_j\,)\right)} \|f\|^2_{L^2(\kappa^{-1}(S_j\,)\cap K)}$$

$$\leq \sup_k\ \|f\,\|^2_{L^2(\kappa^{-1}(S_k)\cap K)} \sum_j \left\|\ \lambda^{\frac{1}{2}} \int\ e^{i\lambda\psi(x,t\,)}\rho(t;\, x)h(t)dt\right\|^2_{L^\infty\left(\kappa^{-1}(S_j\,)\right)},$$

where K is the x-support of ρ. Since the first factor on the right is dominated by the last factor in the right-hand side of (21) (the sup can just be taken over $(0,0) \in \gamma \in \Pi$ here), we conclude that we would obtain this inequality if we could show that there is a uniform constant so that for all choices of $x_j \in \kappa^{-1}(S_j\,)$

$$\lambda^{\frac{1}{2}} \sum_j \left|\ \int\ e^{i\lambda\psi(x_j\,,t\,)}\rho\big(t;\, x_j\,\big)h(t)dt\right|^2 \leq C\|h\|^2_{L^2(dt\,)}. \quad (22)$$

This inequality is an estimate for an operator from $L^2(dt) \to \ell^2$. The dual operator is the one in Proposition (2.1.7). Therefore since, by duality, (22) follows from (29) we get (21). To verify this assertion, we use the fact that if ρ has small support then the terms in (22) with $\rho\big(t;\, x_j\,\big) \neq 0$ will fulfill the hypotheses in Proposition (2.1.7).

To finish the proof of Theorem (2.1.1) we must prove the two propositions. Let us start with the first one since it is pretty standard. It is based on the well known fact that the bilinear oscillatory integrals arising in Hörmander's [45] proof of the Carleson-Sjölin [37] Theorem become better and better behaved away from the diagonal.

Proposition (2.1.6)[28]: Let $a(x,t,t'), x \in \boldsymbol{R}^2, t,t' \in \boldsymbol{R}$ satisfy $|\partial_x^\alpha a| \le C\alpha$ for all multiindices α and $a(x,t,t') = 0\, if\, |x| > \delta\, or\, |t - t'| > \delta$ here $\delta > 0$ is small. Suppose also that $\phi \in C^\infty(\boldsymbol{R}^2 \times \boldsymbol{R})$ is real and satisfies the Carleson-Sjölin condition on the support of $a, i.e.$,

$$det\begin{pmatrix} \phi''_{x1t} & \phi_{x2t} \\ \phi_{1tt} & \phi_{x2tt} \end{pmatrix} \neq 0. \tag{23}$$

Then if the $\delta > 0$ above is sufficiently small, there is a uniform constant C so that when $\lambda, N \ge 1$

$$\left\| \iint_{|t-t|\ge N\lambda^{-\frac{1}{2}}} e^{i\lambda[\phi(x,t)+\phi(x,t')]}\, a(x,t,t')F(t,t')dtdt' \right\|_{L^2(\boldsymbol{R}^2)}^2 \le \lambda^{-\frac{3}{2}} N^{-1} \|F\|_{L^2(\boldsymbol{R}^2)}^2. \tag{24}$$

Proof. Let $\phi(x;t,t') = \phi(x,t') + \phi(x,t')$ be the phase function in (24). Then Φ is a symmetric function in the (t,t') variables. So if we make the change of variables

$$u = (t - t', t + t'),$$

then since $|du/d(t,t')| = 2$, we see that (23) implies that the Hessian determinant of Φ satisfies

$$\left| det\left(\frac{\partial^2 \Phi}{\partial x \partial u}\right) \right| \ge c|u_1|,$$

for some $c > 0$ on the support of a, if the latter is small. Since $\Phi(x;\, u)$ is an even function of the diagonal variable u_1, it must be a C^∞ function of u_1^2. So if we make the final change of Variables

$$v = \left(\frac{1}{2} u_1^2, u_2\right),$$

then since $|dv/du| = |u_1|$, it follows that

$$\left| det \frac{\partial^2 \Phi}{\partial x \partial v} \right| \ge c,$$

for some $c > 0$. This in turn implies that if v and $\tilde{v}$ are close then

$$|\nabla_x[\Phi(x,v) - \Phi(x,\tilde{v})]| \ge c'|v - \tilde{v}|,$$

for some $c' > 0$, and since $x, v \to \Phi$ is smooth, we also have hat

$$|\partial_x^\alpha [\Phi(x,v) - \Phi(x,\tilde{v})]| \le C_\alpha |v - \tilde{v}|,$$

for all multi-indices α. Therefore, if we let

$$K_\lambda(v,\tilde{v}) = \int_{\boldsymbol{R}^2} a(x,t,t')\overline{a(x,\tilde{t},\tilde{t}')}\; e^{i\lambda[\Phi(x,v)-\Phi(x,\tilde{v})]}\, dx,$$

then by integrating by parts, we find that if the number $\delta > 0$ in the statement of the Proposition is small then for $j = 1,2,3,\ldots$

$$\begin{aligned} |K_\lambda(v,\tilde{v})| &\le C_j (1 + \lambda|v - \tilde{v}|)^{-2j} \\ &\le C_j (1 + \lambda|(t + t') - (\tilde{t} + \tilde{t}')|)^{-j} \\ &\quad (1 + \lambda|(t - t')^2 - (\tilde{t} - \tilde{t})^2|)^{-j}. \end{aligned} \tag{25}$$

Note that the left side of (24) equals

$$\int \cdots \int_{|t-t'|,|\tilde{t}-\tilde{t}'|\ge N\lambda^{-\frac{1}{2}}} K_\lambda(t,t';\tilde{t},\tilde{t}')F(t,t')\overline{F(\tilde{t},\tilde{t}')}dtdt'd\tilde{t}d\tilde{t}.$$

We next laim that there is a uniform constant C so that for $\lambda, N \ge 1$

$$\sup_{\tilde{t},\tilde{t}'} \int_{|t-t'\geq N\lambda^{-\frac{1}{2}}} |K_\lambda| dt dt', \sup_{t,t'} \int_{|\tilde{t}-\tilde{t}'|\geq N\lambda^{-\frac{1}{2}}} |K_\lambda| d\tilde{t} d\tilde{t}$$
$$\leq C\lambda^{-2}\left(\frac{\lambda^{\frac{1}{2}}}{N}\right). \quad (26)$$

This follows from (25) and the fact that if $\tau = s^2$ then 2sds $=$ τ and so, given $\tau_0 \in$ **R**, we have

$$\int_{s\geq N\lambda^{-\frac{1}{2}}} (1+\lambda|s^2-\tau 0|)^{-2} ds = \frac{1}{2}\int_{\sqrt{\tau}\geq N\lambda^{-\frac{1}{2}}} (1+\lambda|\tau-\tau_0|)^{-2}\frac{d\tau}{\sqrt{\tau}}$$
$$\leq \left(\frac{\lambda^{\frac{1}{2}}}{N}\right)\int_{-\infty}^{+\infty} (1+\lambda|\tau|)^{-2} d\tau \leq \lambda^{-1}(\lambda^{\frac{1}{2}}/N).$$

Since (26) and Young's inequality yield (24), the proof s complete.

We need to prove the other Proposition, which is a straightforward application of Gauss' lemma.

Proposition (2.1.7)[28]: *Let* $\psi(x,t) = d_g(x,(0,t))$, and suppose that $\rho \in C_0^\infty(\boldsymbol{R}\times\boldsymbol{R}^2)$ satisfies

$$|\partial_t^m \rho(t;x)| \leq C_m\left(\lambda^{\frac{1}{2}}\right)^m, and\ \rho(t;x) = 0, |t| \geq \lambda^{-\frac{1}{2}}. \quad (27)$$

Suppose also that ρ vanishes when x is outside of a small neighborhood $\mathcal{N}$ of a fixed point $(-s_0, 0)$ (in the Fermi normal coordinates) with $s_0 > 0$. If $x \to \kappa(x) = (\kappa_1(x), \kappa_2(x))$ are the coordinates described above, assume that points $x_j \in \mathcal{N}$ are chosen so that

$$\left|\frac{\kappa_2(x_j)}{|\kappa(x_j)|} - \frac{\kappa_2(x_k)}{|\kappa(x_k)|}\right| \geq c\lambda^{-\frac{1}{2}}|j-k|, if\ |j-k| \geq 10, \quad (28)$$

with $c > 0$ fixed. It then follows that, if $\mathcal{N}$ is sufficiently small, then there is a uniform constant C, which is independent of the $\{x_j\}$ chosen as above, so that

$$\lambda^{\frac{1}{2}}\int\left|\sum_j e^{i\lambda\psi(xj,t)}\rho(t;xj)aj\right|^2 dt \leq C\sum|aj|^2. \quad (29)$$

Proof. The support assumptions on the amplitude will allow us to linearize the function $t \to \psi$ in the proof, which is a tremendous help. Specifically,

$$\psi(x,t) = \psi(x,0) + t\left(\partial_t\psi(x,0)\right) + r(x,t),$$

where

$$|\partial_t^m r(x,t)| \leq C_m|t|^{2-m}, 0 \leq m \leq 2, and\ |\partial_t^m r| \leq C_m, m \geq 2. \quad (30)$$

Our choice of coordinates implies that

$$\partial_t\psi(x,0) = \langle\nu, \frac{\kappa(x)}{|\kappa(x)|}\rangle,$$

where the inner-product is the euclidean one and $\nu \in \boldsymbol{R}^2$ is chosen so that $\langle\nu,\nabla\rangle$ is the pushforward of $\partial/\partial x_2$ at $(0,0)$ under the $map\ x \to \kappa(x)$— $i.e.$, tangent vector to the curve $t \to \kappa((0,t))$. Since the pushforward of $\partial/\partial x_1$ is itself under this map, it follows that the second coordinate of ν is nonzero. (See Figure 2 below.) Therefore, if $\mathcal{N}$ $(s_0, 0)$ is small enough, then our assumption (28) implies that

$\left|\partial_t\psi(x_j,0) - \partial_t\psi(x_k,0)\right| \geq c'\lambda^{-\frac{1}{2}}$ (2.17)$|j - k|, if\ |j - k| \geq 10$, and $x_j, x_k \in \mathcal{N}$, for some constant $c > 0$.

It is easy now to finish the proof of (29). If we get

$$\rho(x_j, x_k; t) = \rho(t; x_j)\overline{\rho(t; x_k)}e^{(i\lambda(\psi(x_j,0))+r(x_j,t)}e^{-i\lambda(\psi(x_k,0)+r(x_k,t))},$$

it follows from (27) and (30) that

$$\left|\partial_t^m \rho(x_j, x_k; t)\right| \leq C_m\lambda^{\frac{m}{2}},$$

and $(x_j\ x_k; t) = 0, if\ |t| \geq \lambda^{-\frac{1}{2}}, x_j \notin \mathcal{N}, or\ x_k \notin \mathcal{N}$.

We can use this since the left ide of (29) equals

$$\lambda^{\frac{1}{2}}\sum_{j,k} |a_j\, a_k| \left(\int e^{it\lambda\left(\partial_t\psi(x_j,0)-\partial_t\psi(x_k,0)\right)}\rho(x_j, x_k; t)dt\right),$$

which, after integrating by parts $N = 1, 2, 3 \ldots$ times, we onclude is dominated by a fixed constant C_N times

$$\sum_{j,k} |a_j a_k|(1 + |j - k|)^{-N}.$$

ince, by Young's inequality, this is dominated by the right side of (29) hen $N = 2$, the proof is complete.

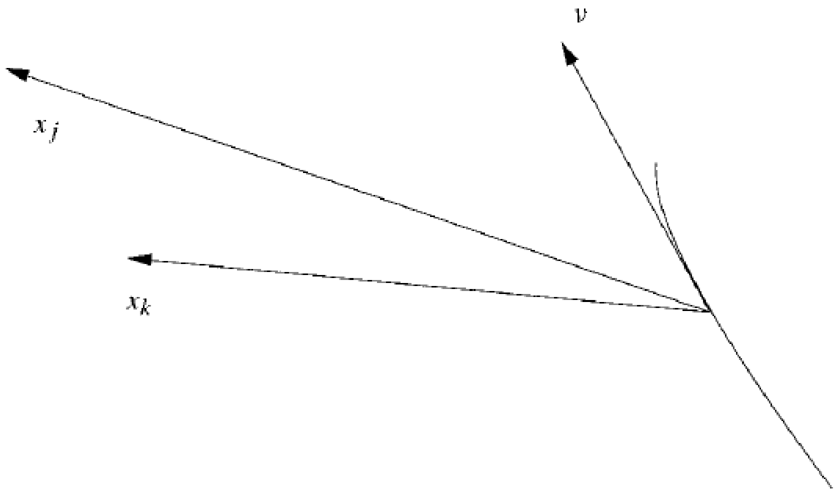


Figure (2)[28]: Image of $\{(0, t)\}$ in geodesic normal coordinates about $(0, 0)$.

We have shown above that if $\left\{e_{\lambda_{j_k}}\right\}_{k=1}^{\infty}$ is a sequence of L^2-normalized eigenfunctions satisfying

$$lim \sup_{k\to\infty} \sup_{\gamma \in \Pi} \lambda_{j_k}^{-\frac{1}{2}} \int_\gamma \left|e_{\lambda_{j_k}}\right|^2 ds = 0, \tag{31}$$

then $\lambda_{jk}^{-\delta(p)} \left\|e_{\lambda_{j_k}}\right\|_{L^p(M)} = 0, 2 < p < 6$. While it seems difficult to determine when this holds, one can show the following.

Proposition (2.1.8)[28]: Suppose that $\gamma \in \Pi$ is not contained in a smoothly closed geodesic. Then if $\{e_{\lambda_j}\}$ is the full sequence of L^2-normalized eigenfunctions, we have

$$\lim\sup_{j\to\infty} \lambda_j^{-\frac{1}{2}} \int_\gamma \left|e_{\lambda_j}\right|^2 ds = 0. \tag{32}$$

In proving this proposition we may assume, after possible multiplying the metric by a constant, that the injectivity radius is more than 10. This will allow us to write down Fourier integral operators representing the solution of the wave equation up to times $|t| \leq 10$. More important, though, is that we shall use an observation of Tataru [56] that the map from Cauchy data to the solution of the wave equation restricted to $\gamma \times \boldsymbol{R}$ is a Fourier integral operator with a one-sided fold. Using this fact and the standard method of long-time averages (see e.g. [40], [46], [17], [29]), we shall be able to prove Proposition (2.1.8).

To set up our proof, let us choose Fermi normal coordinates about γ so that, in these coordinates, γ becomes $\{(s,0);\ 0 \leq s \leq 1\}$. Note that in these coordinates the metric takes the form $g_{11}(x)dx_1^2 + dx_2^2$. As a consequence if $p(x,\xi) = \sqrt{\sum g^{jk}(x)\xi_j\,\xi_k}$ is the principal symbol of $P = \sqrt{-\Delta_g}$ then $p((s,0),\xi) = \sqrt{g_{11}\big((s,0)\big)\xi_1^2 + \xi_2^2}$ is an even function of ξ_2.

To proceed, let us fix a real-valued function $\chi \in S(\mathcal{R})$ with $\chi(0) = 1$ and $\hat{\chi}(t) = 0, |t| > \frac{1}{2}$. Then if e_λ is an eigenfunction with eigenvalue λ it follows that $\chi\big(N(P-\lambda)\big)e_\lambda = e_\lambda$. Thus, in order to prove (32), it would suffice to prove that given $\lambda, N \geq 1$

$$\left\|\chi\big(N(P-\lambda)\big)f\right\|_{L^2(\gamma)} \leq CN^{-\frac{1}{2}}\lambda^{\frac{1}{4}}\|f\|_{L^2(M)} + C_N\|f\|_{L^2(M)}. \quad (33)$$

Note that

$$\chi\big(N(P-\lambda)\big)f(x) = N^{-1}\int \hat{\chi}\left(\frac{t}{N}\right)e^{-it\lambda}\big(e^{itP}f\big)(x)dt. \quad (34)$$

and because of the support properties of the $\hat{\chi}$ the integrand vanishes when $|t| \geq N/2$.

The operator

$$f \to \big(e^{itP}f\big)(x)$$

is a Fourier operator with canonical relation

$$\{(x,t,\xi,\tau;\ y,\eta);\ \Phi_t(x,\xi) = (y,\eta), \pm\tau = p(x,\xi)\},$$

with $\Phi_t : T^*M \to T^*M$ being geodesic flow on the cotangent bundle and $p(x,\xi)$, as above, being the principal symbol of $\sqrt{-\Delta_g}$. Given that we want to restrict the operator in (34) to $\gamma = (s,0), 0 \leq s \leq 1$, we really need to also focus on the the Fourier integral operator

$$f \to \big(e^{itP}f\big)\ (s,0).$$

Given the above, its canonical relation is

$$C = \big\{\Pi_{\gamma\times R}(x,t,\xi,\tau;\ y,\eta) \in T^*(\gamma \times R) \times T^*M;\ \Phi_t(bx,\xi) = (y,\eta), \pm\tau = p(x_1,0,\xi)\big\},$$

with $\Pi_{\gamma\times\boldsymbol{R}}$ being the projection map from $T^*(M \times \boldsymbol{R})\ to\ T^*(\gamma \times \boldsymbol{R})$. Note that the projection from the latter canonical relation to $T^*(\gamma \times \boldsymbol{R})$ is the map

$$(s,t,\xi) \to \Big(s,t,\xi_1,p\big((s,0),\xi\big)\Big),$$

which has a fold singularity when $\xi_2 = 0$ but has surjective differential away from this set (given the aforementioned properties of p).

Because of this, given the xplicit formula in Fermi coordinates, if we choose $\psi \in C_0^\infty(M)$ equal to ne on γ and $\alpha \in C_0^\infty(\boldsymbol{R})$ satisfying $\alpha = 1\ on\ [-1/2, 1/2]\ but\ \alpha(\tau) = 0, |\tau|$, then

$$b_\varepsilon(x,\xi) = \psi(x)\alpha\left(\frac{\xi_2}{\varepsilon|\xi|}\right)$$

equals one on a conic neighborhood of the set that projects onto the set where the left projection of C has a folding ingularity. This means that

$$B_\varepsilon(x,\xi) = \psi(x)\left(1 - \alpha\left(\frac{\xi_2}{\varepsilon|\xi|}\right)\right)$$

has symbol anishing in a conic neighborhood of this set and consequently the map

$$f \to \left(B\varepsilon\ e^{itP} f\right)\left((s,0)\right), 0 \leq s \leq 1$$

is a nondegenerate Fourier integral operator f order zero. Therefore, Hörmander's Theorem [44] about the L^2 oundedness of Fourier integral operators yields

$$\int_{(-N)}^{(N)} \int_0^1 \left|\left(B_\varepsilon\ e^{itP} f\right)\ (s,0)\right|^2 dsdt \leq CN, B\varepsilon \|f\|^2_{L2(M)}.$$

Therefore, an pplication of Schwarz's inequality yields

$$\chi_\lambda^{N,B_\varepsilon} f\, L^2(\gamma) \leq C'_{N,B_\varepsilon} \|f\|_{L^2(M)},$$

Therefore, an application of Schwarz's inequality yields

$$\left\|\chi_\lambda^{N,B_\varepsilon} f\right\|_{L^2(\gamma)} \leq C'_{N,B_\varepsilon} \|f\|_{L^2(M)},$$

if

$$\chi_\lambda^{N,B_\varepsilon} f = B_\varepsilon\, o\, \chi\left(N(P - \lambda)\right)f = N^{-1} \int \hat{\chi}\left(\frac{t}{N}\right) e^{-it\lambda}\left(B_\varepsilon\ e^{itP}\right) f dt\,.$$

Therefore if we similarly define $\chi_\lambda^{N,b_\varepsilon} f = b_\varepsilon\, o\, \chi\left(N(P - \lambda)\right)f$, then $\chi_\lambda^{N,B_\varepsilon} f + \chi_\lambda^{N,b_\varepsilon} f = \psi\chi\left(N(P - \lambda)\right)f$ and since $\psi = 1$ on γ , the proof of (33) would be complete if we could show that if $\varepsilon > 0$ is small enough (depending on N) then for $\lambda \geq 1$ we have for a constant C independent of ε, N and $\lambda \geq 1$

$$\left\|\chi_\lambda^{N,b_\varepsilon} f\right\|_{L^2(\gamma)} \leq CN^{-\frac{1}{2}}\lambda^{\frac{1}{4}}\|f\|_{L^2(M)} + C_N, b_\varepsilon \|f\|_{L^2(M)}. \quad (35)$$

In addition to taking $\varepsilon > 0$ to be small, we shall also take the support of ψ about γ to be small.

It is in proving (35) of course where we shall use our assumption that γ is not part of a smoothly closed geodesic. A consequence of this is that, given fixed N, if ε and the support of ψ are small enough then

$$b_\varepsilon(y,\eta) = 0 \text{ whenever}$$
$$(y,\eta) = \Phi_t(x,\xi), (x,\xi) \in \sup b_\varepsilon, 2 \leq |t| \leq N. \quad (36)$$

In what follows, we shall assume that ε and ψ have been chosen so that this is the case. The point here is that if $\gamma(s), s \in \boldsymbol{R}$, is the geodesic starting at $(0,0)$ and containing $\{\gamma(s) = (s,0);\ 0 \leq s \leq 1\}$, points on the curve $\gamma(s), |s| \leq N + 1$ might intersect γ, but the intersection must be transverse as $s \to \gamma(s)$ is not a smoothly closed geodesic. Then if ε is chosen to be a small multiple of the smallest angle of intersection and if ψ has small enough support about γ , then we get (36). Using the canonical relation for eitP , we can deduce from this that

$$b_\varepsilon\, o\, e^{itP}\, o\, b_\varepsilon^* \text{ is a smoothing operator when} 2 \leq |t| \leq N + 1. \quad (37)$$

i.e., for such times this operator's kernel is smooth.

Let T be the operator $\chi_\lambda^{N,b_\varepsilon} f\ |_\gamma$, $i.e.$, the truncated approximate spectral projection operator restricted to γ . Our goal is to show (35) which says that

$$\|T\|_{L^2(M)\to L^2(\gamma)} \le CN^{-\frac{1}{2}}\lambda^{\frac{1}{4}} + C_{N,b_\varepsilon}.$$

This is equivalent to saying that the dual operator $T^* : L^2(\gamma) \to L^2(M)$ with the same norm, and since

$$\|T^*g\|^2_{L^2(M)} = \int_M T^*g\overline{T^*g}\,dx = \int_\gamma T\,T^*g\,\bar{g}ds \le \|T\,T^*g\|_{L^2(\gamma)}\|g\|_{L^2(\gamma)},$$

we would be done if we could show that

$$\|T\,T^*g\|_{L^2(\gamma)} \le \left(CN^{-1}\lambda^{\frac{1}{2}} + C_{N,b_\varepsilon}\right)\|\,g\|_{L^2(\gamma)}. \qquad (38)$$

But the kernel of $T\,T^*$ is $K(\gamma(s),\gamma(s'))$, where $(x,y), x,y \in M$ is the kernel of the operator $b_\varepsilon\, o\, \rho\big(N(P-\lambda)\big)\, o\, b_\varepsilon^*$ with $\big(\tau\ \ \chi(\tau)\big)^2$ being the square of χ. Its Fourier transform, $\hat{\rho}$, is the convolution of $\hat{\chi}$ with itself, and thus $\hat{\rho}(t) = 0, |t| \ge 1$. Consequently, we can write

$$b_\varepsilon\, o\, \rho\big(N(P-\lambda)\big)\, o\, b_\varepsilon^* = N^{-1}\int \hat{\rho}\left(\frac{t}{N}\right)e^{-it\lambda}\left(b_\varepsilon\, o\, e^{itP}\, o\, b_\varepsilon^*\right)dt \qquad (39)$$

Thus, if $\alpha \in C_0^\infty(\boldsymbol{R})$ is as above, then by (36) and (37), the difference of the kernel of the operator in (39) and the kernel of the operator given by

$$N^{-1}\int \alpha\left(\frac{t}{10}\right)\hat{\rho}\left(\frac{t}{N}\right)e^{-it\lambda}\left(b_\varepsilon\, o\, e^{itP}\, o\, b_\varepsilon^*\right)dt \qquad (40)$$

is $O(\lambda^{-J})$for any J. Thus, if we restrict the kernel of the difference to $\gamma \times \gamma$, it contributes a portion of $T\,T^*$ that maps $L^2(\gamma) \to L^2(\gamma)$ with norm $\le C_{N,b_\varepsilon}$.

To finish, we need to estimate the remaining piece, which has the kernel of the operator in (40) restricted to $\gamma \times \gamma$. Since we are assuming that the injectivity radius of M is 10 or more one can use the Hadamard parametrix for the wave equation and standard stationary phase arguments (similar to ones in [16], Chapter 5, or the proof of Lemma 4.1 in [5]) to see that the kernel $K(x,y)$ of the operator in (40) satisfies

$$|K(x,y)| \le CN^{-1}\lambda^{\frac{1}{2}}\left(d_g(x,y)\right)^{-\frac{1}{2}} + C_{b_\varepsilon}.$$

The first term comes from the main term in the stationary phase expansion for the kernel and the other one is the resulting remainder term in the one-term expansion. Since this kernel restricted to $\gamma \times \gamma$ gives rise to an integral operator satisfying the estimates in (38), the proof is complete.

While as we explained before the condition that for the L^2- normalized eigenfunctions

$$\limsup_{j\to\infty}\ \sup_{\gamma\in\Pi}\ \lambda_j^{-\frac{1}{2}}\int_\gamma \left|e_{\lambda_j}\right|^2 ds = 0$$

is a natural one to quantify non-concentration, it would be interesting to formulate a geometric condition involving the long-time dynamics of the geodesic flow that would imply it and its equivalent version that $\lambda_j^{-\delta(p)}\left\|e_{\lambda_j}\right\|_p \to 0, 2 < p < 6$. Presumably if $\gamma \in \Pi$ and

$$\limsup_{j\to\infty}\ \lambda_j^{-\frac{1}{2}}\int_\gamma \left|e_{\lambda_j}\right|^2 ds > 0 \qquad (41)$$

then γ would have to be part of a stable smoothly closed geodesic, and not just a closed geodesic as we showed above. Toth and Zeldtich made a similar conjecture to this in [58], saying that, in n-dimensions, if γ is a closed stable geodesic then one should be able to find a sequence of eigenfunctions on which sup-norms are blowing up like $\lambda^{\frac{n-1}{2}}$. In [32], [49], it was shown that there is a sequence of quasimodes blowing up at this rate.

It would also be interesting to formulate a condition that would ensure that $\|e_\lambda\|_{L^6(M)} = o(\lambda^{\delta(6)}) = o\left(\lambda^{\frac{1}{6}}\right), for\ L^2$-normalized eigenfunctions. Presumably, such a condition would have to involve both ones like those and conditions of the type in [29], [17]. Since L^6 is an endpoint for (1) one expects that one would need a condition that both guarantees that L^p bounds for $2 < p < 6$ and $p > 6$ be small. Formally, the proof of Theorem (2.1.1) suggests that L^4-norms over geodesics might be relevant for the problem of determining when the $L^6(M)$ norms of eigenfunctions are small. This is interesting because the L^4-norm is the unique L^p-norm taken over geodesics that captures both the concentration of the highest weight spherical harmonics on geodesics and the concentration of zonal functions at points. Indeed, the highest weight spherical harmonics saturate these norms for $2 \le p \le 4$, while the zonal functions saturate them for $p \ge 4$ (see [5]).

Also, it would be interesting to see whether the results here generalize to the case of two dimensional compact manifolds with boundary. Recently, Smith and [52] were able to obtain sharp eigenfunction estimates in this case. In this case, the critical estimate was an L^8 one. So the results here suggest that size estimates for the Kakeya-Nikodym maximal operator associated with broken unit geodesics and applied to squares of eigenfunctions could be relevant for improving the bounds in [52], which are known to be sharp in the case of the disk (see [43]). An observation of Grieser [43] involving the Rayleigh whispering gallery modes suggests that in order to obtain a variant of Corollary (2.1.2) for compact domains one would have to consider L^2-norms over $\lambda_j^{-\frac{2}{3}}$-neighborhoods of broken geodesics. Smith and [51] also showed that for compact manifolds with geodesically concave boundary one has better estimates than one does for compact domains in $\boldsymbol{R}^n$. For example, when $n = 2(1)$ holds. Based on this and the better behavior of the geodesic flow, it seems reasonable that the analog of Corollary (2.1.2) might hold (with the same scales) in this setting.

Finally, as mentioned before it would be interesting to see to what extent the results for the boundaryless case extend to higher dimensions. The arguments given here and in [36], though, rely very heavily on special features of the two-dimensional case.

Section (2.2): L^4-Bounds for Compact Surfaces with Nonpositive Curvature

For (M, g) be a compact two-dimensional Riemannian manifold without boundary. We shall assume throughout that the curvature of (M, g) is everywhere nonpositive. If Δ_g is the Laplace-Beltrami operator associated with the metric g, then we are concerned with certain size estimates for the eigenfunctions

$$-\Delta_g e_\lambda(x) = \lambda^2 e_\lambda(x), x \in M.$$

Thus we are normalizing things so that e_λ is an eigenfunction of the first order operator $\sqrt{-\Delta_g}$ with eigenvalue λ. If e_λ is also normalized to have L^2 -norm one, we are interested in various size estimates for the e_λ which are related to how concentrated they may be along geodesics. If Π denotes the space of all unit-length geodesics in M then our main result is the following "restriction theorem" for this problem.

Theorem (2.2.1)[30]: Assume that (M, g) is as above. Then given $\varepsilon > 0$ there is a $\lambda(\varepsilon) < \infty$ so that

$$\sup_{\gamma\in\Pi}\left(\int_{\gamma}|e_{\lambda}|^{2}\,ds\right)^{1/2} \leq \varepsilon\lambda^{\frac{1}{4}}\,\|e_{\lambda}\|_{L^{2}(M)}\,,\lambda>\lambda(\varepsilon), \qquad (42)$$

with ds denoting arc-length measure on γ, and $L^2(M)$ being the Lebesgue space with respect to the volume element dV_g for (M, g).

Earlier, Burq, G´erard and Tzvetkov [5] showed that for any 2-dimensional compact boundaryless Riemannian manifold one has

$$\left(\int_{\gamma}|e_{\lambda}|^{2}\,ds\right)^{1/2} \leq\ C\lambda^{\frac{1}{4}}\,\|e_{\lambda}\|_{L^{2}(M)}, \qquad (43)$$

ith C independent of $\gamma\ \in\ \Pi$. The first such estimates were somewhat weaker ones of Reznikov [26] for hyperbolic surfaces, which inspired this current line of research. The estimate (43) is sharp for the round sphere S^2 because of the highest weight spherical harmonics (see [5], [28]). Burq, G´erard and Tzvetkov [5] also showed that

$$\left(\int_{\gamma}|e_{\lambda}|^{4}\,ds\right)^{1/4} \leq\ C\lambda^{\frac{1}{4}}\,\|e_{\lambda}\|_{L^{2}(M)},,\gamma\ \in\ \Pi,$$

and so by interpolating with this result and (42) one concludes that when M has nonpositive curvature $\sup_{\gamma\in\Pi}\|e_{\lambda}\|_{Lp(\gamma)}/\|e_{\lambda}\|_{L^{2}(M)}=o(\lambda^{\frac{1}{4}})$ for $2\ \leq\ p\ <\ 4$. An interesting but potentially difficult problem would be to show that this remains true under this hypothesis for the endpoint $p=4$.

Theorem (2.2.1) is related to certain L^p -estimates for eigenfunctions. [15] proved that for any compact Riemannian manifold of dimension 2 one has for $\lambda\geq 1$,

$$\|e_{\lambda}\|_{L^{p}(M)}\leq\ C\lambda^{\frac{1}{2}\left(\frac{1}{2}-\frac{1}{p}\right)}\,\|e_{\lambda}\|_{L^{2}(M)}, 2\leq p\leq 6, \qquad (44)$$

and

$$\|e_{\lambda}\|_{L^{p}(M)}\leq\ C\lambda^{2\left(\frac{1}{2}-\frac{1}{p}\right)-\frac{1}{2}}\,\|e_{\lambda}\|_{L^{2}(M)},\ 6\ \leq\ p\ \leq\ \infty. \qquad (45)$$

These estimates are also sharp for the round sphere S^2 (see [53]). The first estimate, (44), is sharp because of the highest weight spherical harmonics, and thus, like (42) or (43), it measures concentration of eigenfunction mass along geodesics. The second estimate, (45), is sharp due to the zonal functions on S^2, which concentrate at points. The sharp variants of (44) and (45) (with different exponents) for manifolds with boundary were obtained by H. Smith and [52], and it would be interesting to obtain analogues of the results for this setting, but this appears to be difficult.

In the last decade there have been several results showing that, for typical (M, g), (45) can be improved for $p > 6$ (see [29], [17]) to bounds of the form $\|e_{\lambda}\|_{L^{p}(M)}/\ \|e_{\lambda}\|_{L^{2}(M)}=o(\lambda^{\frac{1}{2}\left(\frac{1}{2}-\frac{1}{p}\right)})$ for fixed $p > 6$. Recently, Hassell and Tacey [23], following B´erard's [61] earlier estimate for $p=\infty$, showed that for fixed $p>6$ this ratio is $O(\lambda^{2\left(\frac{1}{2}-\frac{1}{p}\right)-\frac{1}{2}}/\sqrt{log\ \lambda})$, which influenced the present work. Also, in [66] the authors showed that if the geodesic flow is ergodic, which is automatically the case if the curvature of M is negative, then (42) holds for a density one sequence of eigenfunctions.

Except for some special cases of an arithmetic nature (e.g. Zygmund [60] or Spinu [67]) there have been few cases showing that (44) can be improved for Lebesgue exponents with $2 < p < 6$. In [28], using in part results from Bourgain [36], it was shown that

$$\|e_\lambda\|_{L^p(M)} / \|e_\lambda\|_{L^2(M)} = o(\lambda^{\frac{1}{2}\left(\frac{1}{2}-\frac{1}{p}\right)})$$
for some $2 < p < 6$ if and only if
$$\sup_{\gamma\in\Pi} \|e_\lambda\|_{L^2(\gamma)} / \|e_\lambda\|_{L^2(M)} = o(\lambda^{\frac{1}{4}}).$$
Thus, we have the following corollary to Theorem (2.2.1).

Corollary (2.2.2)[30]: As above, let (M, g) be a compact 2-dimensional manifold with nonpositive curvature. Then, if $\varepsilon > 0$ and $2 < p < 6$ are fixed there is $a\ \lambda(\varepsilon, p) < \infty$ so that
$$\|e_\lambda\|_{L^p(M)} \le \varepsilon\lambda^{\frac{1}{2}\left(\frac{1}{2}-\frac{1}{p}\right)} \|e_\lambda\|_{L^p(M)}, \lambda > \lambda(\varepsilon, p).$$
We remark that an interesting open problem would be to obtain this type of result for the case of $p = 6$. It is valid for the standard torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ since Zygmund [60] showed that there one has $\|e_\lambda\|_{L^4(\mathbb{T}^2)}/\|e_\lambda\|_{L^2(\mathbb{T}^2)} = O(1)$ and the classical theorem of Gauss about lattice points in the plane yields $\|e_\lambda\|_{L^\infty(\mathbb{T}^2)}/\|e_\lambda\|_{L^2(\mathbb{T}^2)} = O(\lambda^{\frac{1}{4}})$ Since $p = 6$ is the exponent for which concentration at points and concentration along geodesics are both relevant, proving a general result along the lines of Corollary (2.2.2) would presumably have to take into account both of these phenomena. One expects, though, such a result for $p = 6$ should be valid when M has negative curvature. This result seems to be intimately related to the problem of trying to determine when one has the endpoint improvement for the restriction problem, i.e., $\sup_{\gamma\in\Pi}\|e_\lambda\|_{L^4(\gamma)}/\|e_\lambda\|_{L^2(M)} = O(\lambda^{\frac{1}{4}})$.
[28] showed that if $\gamma_0 \in \Pi$ is not part of a periodic geodesic then
$$\|e_\lambda\|_{L^2(\gamma_0)}/\|e_\lambda\|_{L^2(M)} = O(\lambda^{\frac{1}{4}}).$$
The proof involved an estimate involving the wave equation associated with Δg and a bit of microlocal (wavefront) analysis. The main step in proving Theorem (2.2.1) is to see that this remains valid as well if γ_0 is part of a periodic orbit under the above curvature assumptions. We shall be able to do this by lifting the wave equation for (M, g) up to the corresponding one for its universal cover, which by a classical theorem of Hadamard [63] and von Mangolt [68], is $(\mathbb{R}^2, \tilde{g})$, with the metric $\tilde{g}$ being the pullback of g via a covering map, which can be taken to be $\exp_{x_0}$ for any $x_0 \in M$. By identifying solutions of wave equations for (M, g) with "periodic" ones for $(\mathbb{R}^2, \tilde{g})$we are able to obtain the necessary bounds using a bit of wavefront analysis and the Hadamard parametrix for $(\mathbb{R}^2, \tilde{g})$. Fortunately for us, by a classical volume comparison theorem of Gunther [6], the leading coefficient of the Hadamard parametrix has favorable size estimates under our curvature assumptions. (It is easy to see that the contribution of the lower order terms in the Hadamard parametrix to (42) are straightforward to handle.)

Since the space of all unit-length geodesics is compact, in order to prove (42), it suffices to show that, given $\gamma_0 \in \Pi$ and $\varepsilon > 0$, one can find a neighborhood $N(\gamma_0, \varepsilon)$ of γ_0 in Π and a number $\lambda(\gamma_0, \varepsilon)$ so that
$$\int_\gamma |e_\lambda|^2 ds \le \varepsilon\lambda^{\frac{1}{2}} \|e_\lambda\|^2_{L^2}(M), \gamma \in N(\gamma_0, \varepsilon), \lambda > \lambda(\gamma_0, \varepsilon). \tag{46}$$
In proving this we may assume that the injectivity radius of (M, g) is ten or more. We recall also that, given $x_0 \in M$, the exponential map at x_0, $\exp_{x_0} : T_{x_0}M \simeq \mathbb{R}^2 \to M$ is a universal covering map. We shall take x_0 to be the midpoint of our unit-length geodesic γ_0. We also shall work in geodesic polar coordinates about x_0.

If $\tilde{g}$ is the pullback to $\mathbb{R}^2$ of the metric g via the covering map then $(\mathbb{R}^2, \tilde{g})$ is a Riemannian universal cover of (M, g). Like (M, g) it also has nonpositive curvature. Additionally, rays $t \to t(\cos\theta, \sin\theta), t \geq 0$, through the origin are geodesics for $\tilde{g}$. Such a ray is the lift of the unit speed geodesic starting at x_0, which in our local coordinate system has the initial tangent vector $(\cos\theta, \sin\theta)$. Note that in these coordinates vanishing at $x_0, t \to t(\cos\theta, \sin\theta), |t| \leq 10$ are also geodesics for g. We may assume further that we have

$$\gamma_0 = \left\{(t, 0): -\frac{1}{2} \leq t \leq \frac{1}{2}\right\}. \quad (47)$$

To prove (46) it will be convenient to fix a real-valued even function $\chi \in S(R)$ having the property that $\chi(0) = 1$ and $\hat{\chi}(t) = 0, |t| \geq \frac{1}{4}$, where $\hat{\chi}$ denotes the Fourier transform of χ. We then have that for $T > 0$

$$\chi\left(T\left(\sqrt{-\Delta_g} - \lambda\right)\right) e_\lambda = e_\lambda,$$

and, therefore, to prove (46), it suffices to show that if T is large and fixed then there is a neighborhood $N = N(\gamma_0, T)$ of γ_0 so that

$$\int_\gamma \left|\chi\left(T\left(\sqrt{-\Delta_g} - \lambda\right)\right) f\right|^2 d \leq CT^{-1}\lambda^{\frac{1}{2}} \|f\|^2_{L^2}(M) + C'_{T,N} \|f\|^2_{L^2}(M), \quad \gamma \in N, (48)$$

where C (but not $C'_{T,N}$) is a uniform constant depending on (M, g) but independent of T and N . To prove (48), we shall be able to use the wave equation as

$$\chi\left(T\left(\sqrt{-\Delta_g} - \lambda\right)\right) f = \frac{1}{2\pi T} \int_{\mathbb{R}} \hat{\chi}\left(\frac{t}{T}\right) e^{-it\lambda} e^{it\sqrt{-\Delta_g}} f dt$$

$$= \frac{1}{\pi T} \int_{-T/4}^{T/4} \hat{\chi}(t/T) e^{-it\lambda} \cos t\sqrt{-\Delta_g} f\, dt + \chi(T(\sqrt{-\Delta_g} + \lambda))f, (49)$$

using the fact that $\hat{\chi}(t)$ is even and supported in $|t| \leq \frac{1}{4}$. Since the kernel of the last term satisfies

$$\left|\partial^\alpha_{x,y} \chi\left(T\left(\sqrt{-\Delta_g} + \lambda\right)\right)(x, y)\right| \leq C_{T,N} \lambda^{-N} \quad (50)$$

for any N in compact subsets of any local coordinate system, to prove (48) it suffices to show that

$$\int_\gamma \left|\frac{1}{\pi T} \int_{-T/4}^{T/4} \hat{\chi}(t/T) e^{-it\lambda} \cos t \sqrt{-\Delta_g} f\, dt\right|^2 ds$$

$$\leq CT^{-1}\lambda^{\frac{1}{2}} + C'_{T,N} \|f\|_{L^2(M)}, \gamma \in N(\gamma_0, T). \quad (51)$$

If γ_0 is not part of a periodic geodesic of period $\leq T$, then we can easily prove (51) just by using wavefront analysis and arguments that are similar to the proof of the Duistermaat-Guillemin theorem [40]. This was done in [28], but we shall repeat the argument here for the sake of completeness and since it motivates what is needed to handle the argument when γ_0 is a portion of a periodic geodesic of period $\leq T$.

To handle the latter case we shall exploit the relationship between solutions of the wave equation on (M, g) of the form

$$\begin{cases}(\partial_t^2 - \Delta_g)u(t,x) = 0, (t,x) \in \mathbb{R}_+ \times M \\ \quad u(0,\cdot) = f, \partial_t u(0,\cdot) = 0,\end{cases} \tag{52}$$

and certain ones on $(\mathbb{R}, \tilde{g})$

$$\begin{cases}(\partial_t^2 - \Delta_{\tilde{g}})\tilde{u}(t,\tilde{x}), (t,\tilde{x}) \in \mathbb{R}_+ \times \mathbb{R}^2 \\ \quad \tilde{u}(0,\cdot) = \tilde{f}, \partial_t \tilde{u}(0,\cdot) = 0,\end{cases} \tag{53}$$

Note that $u(t,x) = (cos(t\sqrt{-\Delta_g})f)\,(x)$ is the solution of (52).

To describe the relationship between the two equations we shall use the deck transformations associated with our universal covering map

$$p = \exp_{x_0} \mathbb{R}^2 \to M. \tag{54}$$

Recall that an automorphism for $(\mathbb{R}^2, \tilde{g}), \alpha : \mathbb{R}^2 \to \mathbb{R}^2$, is a deck transformation if

$$p \circ \alpha = p.$$

In this case we shall write $\alpha \in Aut(p)$. In the case where $\mathbb{T}^2$ is the standard two-torus, each α would just be translation in $\mathbb{R}^2$ with respect to some $\in \mathbb{Z}^2$. Motivated by this if $\tilde{x} \in \mathbb{R}^2$ and $\alpha \in Aut(p)$, let us call $\alpha(\tilde{x})$ the translate of $\tilde{x}$ by α. then we recall a set $D \subset \mathbb{R}^2$ is called a fundamental domain of our universal covering p if every point in $\mathbb{R}^2$ is the translate of exactly one point in D. Of course there are infinitely many fundamental domains, but we may assume that ours is relatively compact, connected and contains the ball of radius 2 centered at the origin in view of our assumption about the injectivity radius of (M, g). We can then think of our unit geodesic $\gamma_0 = \{(t,0) : |t| \le \frac{1}{2}\}$ (written in geodesic polar coordinates as above) both as one in (M, g) and one in the fundamental domain which is of the same form. Likewise, a function $f(x)$ on M is uniquely identified by one $f_D(\tilde{x})$ on D if we set $f_D(\tilde{x}) = f(x)$, where $\tilde{x}$ is the unique point in $D \cap p^{-1}(x)$. Using f_D we can define a "periodic extension", $\tilde{f}$, of f to $\mathbb{R}^2$ by defining $\tilde{f}(\tilde{y})$ to be equal to $fD(\tilde{x})$ if $\tilde{x} = \tilde{y}$ modulo Aut(p), i.e. if $(\tilde{x}, \alpha) \in D \times Aut(p)$ are the unique pair so that $\tilde{y} = \alpha(\tilde{x})$. Note then that $\tilde{f}$ is periodic with respect to Aut(p) since we necessarily have that $\tilde{f}(\tilde{x}) = \tilde{f}(\alpha(\tilde{x}))$ for every $\alpha \in Aut(p)$. We can now describe the relationship between the wave equations (52) and (53). First, if $(f(x), 0)$ is the Cauchy data in (52) and $(\tilde{f}(\tilde{x}), 0)$ is the periodic extension to $(\mathbb{R}^2, \tilde{g})$, then the solution $\tilde{u}(t, \tilde{x})$ to (53) must also be a periodic function of $\tilde{x}$ since $\tilde{g}$ is the pullback of g via p and $p = p \circ \alpha$. As a result, we have that the solution to (52) must satisfy $u(t,x) = \tilde{u}(t,\tilde{x})$ if $\tilde{x} \in D$ and $p(\tilde{x}) = x$. Another way of saying this is that if $\tilde{f}$ is the pullback of f via p and t is fixed then $\tilde{u}(t,\cdot)$ solving (53) must be the pullback of $u(t,\cdot)$ in (52). Thus, periodic solutions to (53) correspond uniquely to solutions of (52). In other words, we have the important formula for the wave kernels

$$cos(t\sqrt{-\Delta_g}(x,y) = \sum_{\alpha \in Aut(p)} \cos(t\sqrt{-\Delta_{\tilde{g}}}\;(\tilde{x}, \alpha(\tilde{y})), \tag{55}$$

if $\tilde{x}$ and $\tilde{y}$ are the unique points in D for which $p(\tilde{x}) = x$ and $p(\tilde{y}) = y$.

Note that the sum in (55) only has finitely many nonzero terms for a given (x, y, t) since, by the finite propagation speed for $\tilde{g} = \partial_t^2 - \Delta_{\tilde{g}}$, the summands in the in the right all vanish when $d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y})) > t$. For instance, if $x = y = x_0$ the number of nontrivial terms would equal the cardinality of $p^{-1}(x_0) \cap \{\tilde{x} \in \mathbb{R}^2 : |\tilde{x}| \le t\}$ where $|\tilde{x}|$ denotes the Euclidean length, due to the fact that $d_{\tilde{g}}(0, \tilde{x}) = |\tilde{x}|$. Despite this, the number of nontrivial terms will grow exponentially in t if the curvature is bounded from above by a fixed negative constant.

We review one last thing before focusing more closely on the proof of our restriction estimate. As we shall see, even though there can be an exponentially growing number of nontrivial terms in the right hand side of (55), which could create havoc for our proofs if we are not careful, this turns out to be related to something that will actually be beneficial for our calculations.

These facts are related to the fact that in the geodesic polar coordinates we are using, $(t\cos\theta, t\sin\theta), t > 0, \theta \in (-\pi, \pi]$, for $(\mathbb{R}^2, \tilde{g})$, the metric $\tilde{g}$ takes the form

$$ds^2 = dt^2 + A^2(t,\xi)d\theta^2, \tag{56}$$

where we may assume that $A(t,\theta) > 0$ for $t > 0$. Consequently, the volume element in these coordinates is given by

$$dV_g(t,\theta) = A(t,\theta)dtd\theta, \tag{57}$$

and by Günther's [6] comparison theorem if the curvature of (M, g) and hence that of $(\mathbb{R}^2, \tilde{g})$ is nonpositive, we have

$$A(t,\theta) \geq t. \tag{58}$$

Furthermore, if one assumes that the curvature is $\leq -\kappa^2$, with $\kappa > 0$ then one has

$$A(t,\theta) \geq \frac{1}{\kappa} \sinh(\kappa t). \tag{59}$$

Since the volume element for two-dimensional Euclidean space in polar coordinates is $tdtd\theta$ and that of the hyperbolic plane with constant curvature $-\kappa^2$ is $\frac{1}{\kappa}\sinh(\kappa t)\, dtd\theta$, Günther's volume comparison theorem says that in geodesic polar coordinates the volume element for spaces of nonpositive curvature is at least that of $\mathbb{R}^2$ with the flat metric, while if the curvature is bounded above by $-\kappa^2$ the volume element is at least that of the hyperbolic plane of constant curvature $-\kappa^2$. In the latter case, as we warned, the number of nontrivial terms in the sum in the right side of (55) will be at least bounded below by a multiple of $e^{\kappa t}$ as $t \to +\infty$. Let us now turn to the proof of (51) and hence Theorem (2.2.1). Given $\gamma \in \Pi$ we let $T^*\gamma \subset T^*M$ and $S^*\gamma \subset S^*M$ be the cotangent and unit cotangent bundles over γ, respectively. Thus, if $(x, \xi) \in T^*\gamma$ then $\xi_\sharp$ is a tangent vector to γ at x if $T^*M \ni \xi \to \xi_\sharp \in TM$ is the standard musical isomorphism, which, in local coordinates, sends $\xi = (\xi_1, \xi_2) \in T_x^*M$ to $\xi_\sharp = (\xi_\sharp^1, \xi_\sharp^2)$ with $\xi_\sharp^j = \sum_k g^{jk}(x)\ \xi_k$. Then if $\Phi_t: S^*M \to S^*M$ denotes geodesic flow in the unit cotangent bundle over M, and $(x,\xi) \in S^*\gamma$ we let $L(x,\xi)$ be the minimal $t > 0$ so that $\Phi_t(x,\xi) = (x,\xi)$ and define it to be $+\infty$ if no such time t exists. Then if γ is not part of a periodic geodesic this quantity is $+\infty$ on $S^*\gamma$, and if it is then it is constant on $S^*\gamma$ and equal to the minimal period of the geodesic, $\ell(\gamma)$ (which must be larger than ten because of our assumptions). Note also that $L(x,\xi)$ can also be thought of as a function on S^*M, and that, in this case, it is lower semicontinuous. Recall that we are working in geodesic polar coordinates vanishing at x_0, the midpoint of γ_0, and that γ_0 is of the form (47) in these coordinates. Let us choose $\beta \in C_0^\infty(R)$ equal to one on $[-\frac{3}{4}, \frac{3}{4}]$ but 0 outside $[-1,1]$. We then let $b_\varepsilon(x,D)$ and $B_\varepsilon(x,D)$ be zero-order pseudodifferential operators which in the above local coordinates have symbols

$$b_\varepsilon(x,\xi) = \beta(|x|)\beta(\xi_2/\varepsilon|\xi|), \text{and } B_\varepsilon(x,\xi) = \beta(|x|)(1 - \beta(\xi_2/\varepsilon|\xi|)),$$

respectively.

Our first claim is that if $\varepsilon > 0$ and $\gamma \in \Pi$ are fixed, then we can find a neighborhood $N(\gamma_0, \varepsilon)$ of γ_0 so that

$$\int_{-T/4}^{T/4}\int_{\gamma}\left|B_{\varepsilon}\circ cos(t\sqrt{-\Delta_g})f\right|^2\ dsdt\le C_{T,\varepsilon}\|f\|_{L^2}^2(M),\gamma\in N\ (\gamma_0,\varepsilon),\quad (60)$$

which, by an application of the Schwartz inequality, would yield part of (51), namely,

$$\int_{\gamma}\left|\frac{1}{\pi T}\ \int_{-T/4}^{T/4}\hat{\chi}(t/T)e^{-i\lambda t}\,B_{\varepsilon}\circ\cos\left(t\sqrt{-\Delta_g}\right)f\ dt\right|^2 d\le C'_{T,\varepsilon}\|f\|_{L^2}^2(M),$$

$$\gamma\in N\ (\gamma_0,\varepsilon).\quad (61)$$

If R_{γ} denotes the restriction to $\gamma\in\Pi$, then (60) follows from the fact that the operator

$$f\to\ R_{\gamma}(A\circ cos(t\sqrt{-\Delta_g})f),$$

regarded as an operator from $C^{\infty}(M)\to C^{\infty}(\gamma\times[-T\ /4,T\ /4])$, is a Fourier integral operator of order zero which is locally a canonical graph (i.e., nondegenerate) if supp $A(x,\xi)\ \cap\ S^*\gamma=\emptyset$, and hence a bounded operator from $L^2(M)$ to $L^2(\gamma\times[-T\ /4,T\ /4])$. since $B_{\varepsilon}(x,\xi)$ vanishes on a neighborhood of $S^*\gamma_0$, we conclude that this is the case $A=B_{\varepsilon}$ for $\gamma\in\Pi$ close to γ_0, which gives us (60). The L^2 -boundedness of nondegenerate Fourier integrals is a theorem of Hörmander [44], while the observation about $R_{\gamma}(A\circ cos(t\sqrt{-\Delta_g}))$ is one of Tataru [56]. It is also easy to check the latter, because, for fixed t,e it$\sqrt{-\Delta_g}$: $C^{\infty}(M)\ \to C^{\infty}(M)$ is a nondegenerate Fourier integral operator, and, therefore, one needs only to verify the assertion when $t=0$, in which case it is an easy calculation using any parametrix for the half-wave operator.

The estimate (61) holds for any $\gamma_0\in\Pi$. Let us now argue that if $\ell(\gamma_0)$, the period of γ_0, is larger than T or if γ_0 is not part of a periodic geodesic, then we have also have favorable bounds if B_{ε} is replaced by b_{ε}, with $\varepsilon>0$ sufficiently small. To do this, we recall that the wave front set of the kernel of $b_{\varepsilon}\circ cos(t\sqrt{-\Delta_g})\circ b_{\varepsilon}^*$ is contained in

$$\{(x,t,\xi,\tau;\ y,-\eta)\colon\Phi_{\pm t}(x,\xi)=(y,\eta),\tau^2=\sum g^{jk}(x)\xi_j\xi_k,(x,\xi),(y,\eta)\in\ supp\ b_{\varepsilon}\}.(62)$$

To exploit this, let W_{γ} be the operator

$$W_{\gamma}f=R_{\gamma}\left(\tfrac{1}{\pi T}\int_{-T/4}^{T/4}\hat{\chi}(t/T)e^{-i\lambda t}b_{\varepsilon}\cos\left(t\sqrt{-\Delta_g}\right)f\ dt\right).\quad (63)$$

Our goal then is to show, that under the present assumption that $\ell(\gamma_0)>\ T$

$$\left\|W_{\gamma}\right\|_{L^2(M)\to L^2(\gamma)}\le CT^{-\frac{1}{2}}\lambda^{\frac{1}{4}}+C_{T,b_{\varepsilon}}$$

for $\gamma\in\Pi$ belonging to some neighborhood $N\ (\gamma_0,T,\varepsilon)$ of γ_0. This is equivalent to showing that the dual operator $W_{\gamma}^*\ :L^2(\gamma)\to L^2(M)$ with the same norm, and since

$$\left\|W_{\gamma}^*g\right\|_{L^2(M)}^2=\int_{\gamma}W_{\gamma}W_{\gamma}^*\ g\bar{g}\ ds\le\left\|W_{\gamma}W_{\gamma}^*\ g\right\|_{L^2(\gamma)}\|g\|_{L^2(\gamma)}\ ,$$

we would be done if we could show that

$$\left\|W_{\gamma}^*g\right\|_{L^2(M)}^2\le\left(CT^{-1}\lambda^{\frac{1}{2}}+C_{T,b_{\varepsilon}}\right)\|g\|_{L^2(\gamma)}.\quad (64)$$

But, by Euler's formula, the kernel of $4W_{\gamma}W_{\gamma}^*$ is $K|_{\gamma\times\gamma}$, where $K(x,y),x,y\in M$ is the kernel of the operator $b_{\varepsilon}\circ\rho(T\ (\sqrt{-\Delta_g}-\lambda))\circ b_{\varepsilon}^*+b_{\varepsilon}\circ\rho(T(\sqrt{-\Delta_g}+\lambda))\circ b_{\varepsilon}^*+2b_{\varepsilon}\circ\chi(T(\sqrt{-\Delta_g}-\lambda))\chi(T(\sqrt{-\Delta_g}\ +\lambda))\circ b_{\varepsilon}^*\,,if\ \rho(\tau)=\left(\chi(\tau)\right)^2$. The last two terms satisfy bounds like those in (50) (with constant depending on T and b_{ε}), and the first term is

$$\tfrac{1}{\pi T}\int_{-T/2}^{T/2}\hat{\rho}(t/T)e^{-i\lambda t}\left(b_{\varepsilon}\circ cos(t\sqrt{-\Delta_g})\circ b_{\varepsilon}^*\right)\ \ (x,y)dt.\quad (65)$$

We are using the fact that $\hat{\rho} = \hat{\chi} * \hat{\chi}$ is supported in $[-\frac{1}{2}, \frac{1}{2}]$. In view of (62), if $\varepsilon > 0$ is sufficiently small, since we are assuming that $\ell(\gamma_0) > T$, it follows that we can find a neighborhood N of γ_0 in M so that $(b_\varepsilon \circ cos(t\sqrt{-\Delta_g}) \circ b_\varepsilon^*)(x, y)$ is smooth on $N \times N$ when $t \geq 2$. Thus, on $N \times N$ the difference between (64) and

$$K(x,y) = \frac{1}{\pi T} \int_{-T/2}^{T/2} \beta(t/5)p(t/T)e^{-i\lambda t}\left(b_\varepsilon \circ cos(t\sqrt{-\Delta_g}) \circ b_\varepsilon^*\right) \quad (x,y)\, dt$$

is $O_{T,b_\varepsilon}(1)$. But, by using the Hadamard parametrix (see below) one finds that

$$|K(x,y)| \leq CT^{-1}\lambda^{\frac{1}{2}}\left(d_g(x,y)\right)^{-\frac{1}{2}}$$
$$+C_{b_\varepsilon,T}\left(1 + \lambda\left(1+\lambda d_g(x,y)\right)^{-\frac{3}{2}}\right), x, y \in N, \quad (66)$$

for some uniform constant C, which is independent of ε, T and λ. Since, by Young's inequality, the integral operator with kernel $K|_{\gamma\times\gamma}$ is bounded from $L^2(\gamma) \to L^2(\gamma)$ with norm bounded by $CT^{-1}\lambda^{\frac{1}{2}} + C_{b_\varepsilon,T}$ if $\subset N$, we get (64), which finishes the proof that (48) holds provided that $\ell(\gamma_0) > T$.

The above argument used the fact that if $\ell(\gamma_0) > T$, with T fixed, then if $\varepsilon > 0$ is small enough and $(x, \xi) \in supp\ b_\varepsilon$ with $x \in \gamma_0$ then $\Phi_t(x,\xi) \notin supp\ b_\varepsilon$ for $2 < |t| \leq T/2$. In effect, this allowed us to cut the effect of loops though γ_0 of its extension of length T from our main calculuation, since they were all transverse. If $\gamma_0 \in \Pi$ is part of a periodic geodesic of period $\leq T$, i.e., $\ell(\gamma_0) \leq T$, then this need not be true. On the other hand, if T is fixed and (x, ξ) is as above, then for sufficiently small ε we will have

$$\Phi_{\pm t}(x,\xi) \notin supp\ b_\varepsilon, if\ x \in \gamma_0, and\ t \notin \cup_{j\in\mathbb{Z}}[j\ell(\gamma_0) - 2, j\ell(\gamma_0) + 2]. \quad (67)$$

Note that our assumption that the injectivity radius of (M, g) is 10 or more implies that

$$\ell(\gamma_0) \geq 10.$$

To exploit this, we shall use (55) which relates the wave kernel for (M, g) with the one for its universal cover using the covering map given by $p = \exp_{x_0}$ with x_0 being the midpoint of γ_0. Note that the points $\alpha(0), \alpha \in Aut(p)$ exactly correspond to geodesic loops through x_0, with looping time being equal to the distance from $\alpha(0)$ to the origin in $\mathbb{R}^2$. Just a few of these correspond to smooth loops through x_0 along the periodic geodesic containing γ_0. Since we are assuming that we are working with local coordinates on (M, g) and global geodesic polar ones on $(\mathbb{R}^2, \tilde{g})$ so that γ_0 is of the form (47), the automorphisms with this property are exactly the $\alpha_j \in Aut(p), j \in \mathbb{Z}$ for which

$$\alpha_j(0) = j\ell(\gamma_0, o). \quad (68)$$

Note that $G_{\gamma_0} = \{\alpha_j\}_{j\in\mathbb{Z}}$ is a cyclic subgroup of Aut$(p)$with generator α_1, which is the stabilizer group for the lift of periodic geodesic containing γ_0. Consequently, we can choose $\varepsilon > 0$ small enough and a neighborhood N of γ_0 in M so that

$$\left(b_\varepsilon \circ \cos\left(t\sqrt{-\Delta_{\tilde{g}}}\right)\right) \circ b_\varepsilon^*(\tilde{x}, \alpha(\tilde{y})) \in C^\infty N \times N$$
$$\times [j\ell(\gamma_0) - 2, j\ell(\gamma_0) + 2], \quad (69)$$

if Aut$(p) \ni \alpha \notin G\gamma_0$.

Therefore, by (67)–(69), if we repeat the arguments that were used to prove (64), we conclude that we would have

$$\int_{\gamma}\left|\frac{1}{\pi T}\int_{-\infty}^{\infty}\hat{\chi}(t/T)e^{-i\lambda t}\ \ b_{\varepsilon}\circ\cos\left(t\sqrt{-\Delta_g}\right)f\ dt\right|^2\ ds$$
$$\leq\left(CT^{-\frac{1}{4}}\lambda^{\frac{1}{4}}+C_{T,b_{\varepsilon}}\right)^2\ \|f\|^2_{L^2(M)},\gamma\in N(\gamma_0,T), \qquad (70)$$
for some neighborhood $N(\gamma_0,T)$ in Π, if we could show that if the α_j are as in (68) and
$$K(x,y)=\frac{1}{\pi T}\sum_{\{j\in\mathbb{Z}_+:\ jt(\gamma_0)\leq T/2\}}\int_{-\infty}^{\infty}\beta\big((s-j\ell(\gamma_0))/5\big)\,\hat{\rho}(s/T)e^{-is\lambda}$$
$$\times\left(b_{\varepsilon}\circ cos\ s\sqrt{-\Delta_{\tilde{g}}}\circ b_{\varepsilon}^*\right)\left(\tilde{x},\alpha_j(\tilde{y})\right)ds, \qquad (71)$$
Then
$$|K(x,y)|\leq CT^{-1}\lambda^{\frac{1}{2}}\left(d_g(x,y)\right)^{-\frac{1}{2}}+T^{-\frac{1}{2}}\lambda^{\frac{1}{2}}$$
$$+C_{T,b_{\varepsilon}}(1+\lambda\left(1\ +\lambda d_g(x,y)\right)^{-\frac{3}{2}},x,y\in N, \qquad (72)$$
with N being some neighborhood in M of γ_0 (depending on T). The second term in the right side of this inequality did not occur in the previous steps. It comes from the terms in (71) with $j\neq 0$. Also, the fact that (72) yields (70) just follows from an application of Young's inequality. To prove (72), it suffices to see that we can find N as above so that
$$\int\beta\left((s-j\ell(\gamma_0))/5)\hat{\rho}(s/T)e^{-is\lambda}\,(b_{\varepsilon}\circ\cos s\sqrt{-\Delta_{\tilde{g}}}\circ b_{\varepsilon}^*\right)\left(\tilde{x},\alpha_j(\tilde{y})\right)ds$$
$$\leq\ C\lambda^{\frac{1}{2}}(\ \max\{d_g\left(\tilde{x},\alpha_j(\tilde{y})\right),e^{\kappa d_g}\left(\tilde{x},\alpha_j(\tilde{y})\}\right)^{-\frac{1}{2}}+C_{T,b_{\varepsilon}},$$
$$x,y\ \in N,0\neq|j|,\ell(\gamma_0)\leq \text{T}. \qquad (73)$$
Cassuming that the curvature of (M,g) is everywhere $\leq -\kappa^2,\kappa\geq 0$, while for $j=0$, we have
$$\int_{\beta}(s/5)\hat{\rho}(s/T)e^{-is\lambda}\left(b_{\varepsilon}\circ cos\ s\ \sqrt{-\Delta_g}\circ b_{\varepsilon}^*\right)(x,y))\ ds$$
$$\leq C\lambda^{\frac{1}{2}}\left(d_g(x,y)\right)^{-\frac{1}{2}}+\ C\_(T,b\_\varepsilon)\ (1+\lambda\ \left(1+\lambda d_g(x,y)\right)^{-\frac{3}{2}},$$
$$x,y\ \in\ N. \qquad (74)$$
Note that $d_{\tilde{g}}(\tilde{x},\alpha_j(\tilde{y}))\in[j\ell(\gamma_0)-1,j\ell(\gamma_0)+1]$ when $x,y\in\gamma_0$ and hence $d\tilde{g}(\tilde{x},\alpha_j(\tilde{y}))\geq|j|$ when $x,y\in N$ with N being a small neighborhood of γ_0 in M. We shall assume that this is the case in what follows. We then get (72) by summing over j. (Observe that if the curvature is assumed to be bounded below by a negative constant, we get something a bit stronger than (72) where in the second term we may replace $T^{-\frac{1}{2}}$ by T^{-1} .)

Both (73) and (74) are routine consequences of stationary phase and the Hadamard parametrix for the wave equation.

To prove (74) let $\varphi(x,y)$ denote geodesic normal coordinates of y about x. Then if $|t|\leq 5,$ by the Hadamard parametrix (see [64] or [27]) and the composition calculus for Fourier integral operators (see Chapter 6 in [16])
$$\left(b_{\varepsilon}\circ\cos\left(t\ \sqrt{-\Delta_g}\right)\circ b_{\varepsilon}^*\right)(x,y)=\sum_{\pm}\int_{\mathbb{R}^2}e^{i\emptyset(x,y)\cdot\xi\pm it|\xi|}\,a_{\varepsilon}(x,y,\xi)d\xi\ +\ O_{\varepsilon}(1), \qquad (75)$$
where $a_{\varepsilon}\in S^0_{1,0}$ depends on $-\Delta_g$ and b_{ε} but satisfies

$$|a_\varepsilon| \le C, \text{and } \left|\partial^\alpha_{x,y}\partial^\sigma_\xi\, a_\varepsilon\right| \le C_{\varepsilon\alpha\sigma}(1+|\xi|)^{-|\sigma|}. \qquad (76)$$

The first constant is independent of C and only depends on the size of the symbol of b_ε, which is $\le \|\beta\|^4_{L^\infty(\mathbb{R})}$. Recall (see [16]) the following fact about the Fourier transform of a density times Lebesgue measure on the circle $S^1 = \{\Theta = (\cos\theta, \sin\theta)\}$,

$$\int_0^{2\pi} e^{iw\cdot\Theta} a_\varepsilon\,(x,y,\Theta)d\theta \;=|2\pi w|^{-\frac{1}{2}}\sum_{\pm} e^{\pm i|w|}\, a_\varepsilon(x,y,\pm w)$$

$$+O_\varepsilon(|w|^{-\frac{3}{2}}), |w| \ge 1, \qquad (77)$$

where the constants for the last term depend on the size of finitely many constants in (77). Since $|\varphi(x,y)| = d_g(x,y)$, if we combine (75) and (76), we find that, modulo a $O_\varepsilon(1)$ term, if $\psi(s) = \beta(s/5)\hat{\rho}(s/T)$, then when $d_g(x,y) \ge \lambda^{-1}$, the quantity in (74) is the sum over $\pm$ of a fixed multiple of

$$\left(d_g(x,y)\right)^{-\frac{1}{2}}\int_0^\infty \Bigg(\hat{\psi}(\lambda-r)+\hat{\psi}(\lambda+r)e^{\pm ird_g(x,y)}\, a_\varepsilon\big(x,y,\pm r\emptyset(x,y)\big)r^{\frac{1}{2}}\,dr$$

$$+\,O_\varepsilon\left(\left(d_g(x,y)\right)^{-\frac{3}{2}}\int_0^\infty\left(\left|\hat{\psi}(\lambda-r)\right|+\left|\hat{\psi}(\lambda+r)\right|\right)\,(1+r)^{-\frac{1}{2}}\,dr\right)\Bigg)$$

By (76), the first term in is $O(\|a_\varepsilon\|_\infty\left(\lambda d_g(x,y)\right)^{-\frac{1}{2}})$, since $|\hat{\psi}(\tau)| \le C_N(1+|\tau|)^{-N}$ for any N. Since the last term is $O_\varepsilon(\lambda^{-1/2}\left(d_g(x,y)\right)^{-\frac{3}{2}})$, we have established (74) when $d_g(x,y) \ge \lambda^{-1}$. The fact that it is also $O(\lambda) + O_\varepsilon(1)$ is a simple consequence of (75) and (76) which gives the bounds for $d_g(x,y) \le \lambda^{-1}$ and concludes the proof of (74).

To prove (74) we can exploit the fact that, unlike the case of $t = 0$, if $t \ne 0$ then cost $\sqrt{-\Delta_g}\colon C^\infty(M) \to C^\infty(M)$ is a conormal Fourier integral operator with singular support of codimension one. Based on this and (62) we deduce that if $(x,t,\xi,\tau;\, y,\eta)$ is in the wave front set of

$$(cos(t\sqrt{-\Delta_{\tilde{g}}}))(\tilde{x},\alpha_j(\tilde{y})), j \ne 0,$$

and both x and y are on γ_0 then both ξ and η must be on the first coordinate axis. Therefore, since the symbol, $b_\varepsilon(x,\xi)$, of bε equals one when $x \in \gamma_0$ and ξ is in a conic neighborhood of this axis (depending on ε), we conclude that there must be a neighborhood N of γ_0 in M so that

$$(b_\varepsilon \circ \cos(t\,c) \circ b^*_\varepsilon)\left(\tilde{x},\alpha_j(\tilde{y})\right) - cost\sqrt{-\Delta_{\tilde{g}}}\,\left(\tilde{x},\alpha_j(\tilde{y})\right) \in C^\infty(N\times N\,),$$

$$0 \ne |j|\ell(\gamma_0) \le T.$$

Because of this, we would have the remaining inequality, (73), if we could show that

$$\int_\beta (s-j\ell(\gamma_0)/5)\,\hat{\rho}(s/T)e^{-is\lambda}\cos s\sqrt{-\Delta_g}\,\left(\tilde{x},\alpha_j\,(\tilde{y})\right)ds$$

$$\le\; C\lambda^{\frac{1}{2}}(\max\{d_g(\tilde{x},\alpha_j\,(\tilde{y})), e^{\kappa d_g\left(\tilde{x},\alpha_j(\tilde{y})\right)}\})^{-\frac{1}{2}} + C_T$$

$$x,y \in N\,, 0 \ne |j|\ell(\gamma_0) \le T. \qquad (78)$$

To prove this, we shall use the fact that on $(\mathbb{R}^2, \tilde{g})$ we can use the Hadamard parametrix even for large times. Recall that the Hadamard parametrix says that if we set

$$\varepsilon_0(t,x) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{(ix\cdot\xi)}\ cos(t|\xi|)\, d\xi,$$

and define $\varepsilon_\nu, \nu = 1,2,3,\ldots$ recursively by $2\varepsilon_\nu(t,x) = t\int_0^t \varepsilon_{\nu-1}(s,x)ds,\ \nu = 1,2,3,\ldots$, then there are functions $w_\nu \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ so that we have

$$(cos(t\sqrt{-\Delta_{\tilde{g}}})(x,y) = \sum_{\nu=0}^{N} w_\nu(x,y)\varepsilon_\nu\ \ (t, d_{\tilde{g}}(x,y)) + R_N(t,x,y),$$

where for $n = 2, R_N \in L^\infty_{loc}(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2)$ if $N \geq 10$. We are abusing the notation a bit by putting $\varepsilon_\nu(t,r)$ equal to the radial function $\varepsilon_\nu(t,x)$ for some $|x| = r$. The $\varepsilon_\nu, \nu = 1,2,3,\ldots$, are Fourier integrals of order $-\nu$; for instance,

$$\varepsilon_1(t,x) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{ix\cdot\xi}\ \frac{tsin\ t|\xi|}{2|\xi|}\ d\xi\ .$$

As a result of this, we would have (78) if we could show that

$$\left|w_0\left(\tilde{x}, \alpha_j(\tilde{y})\right) \int\int \beta(s - j\ell(\gamma_0)/5)\hat{\rho}(s/T)e^{-i\lambda s}e^{i\left(\tilde{x}-\alpha_j(\tilde{y})\right)\cdot\xi} \cos(s|\xi|)\, d\xi\, ds\right|$$
$$\leq C\lambda^{\frac{1}{2}}(\max\{d_g(\tilde{x}, \alpha_j\ (\tilde{y})), e^{\kappa d_g\left(\tilde{x},\alpha_j(\tilde{y})\right)}\})^{-\frac{1}{2}}, j = 1,2,\ldots, \quad (79)$$

as well as

$$\left|\int \beta(s - j\ell(\gamma_0)/5)\rho(s/T)e^{-is\lambda}\varepsilon_\nu(s, d_g(\tilde{x}, \alpha_j(\tilde{y}))ds\right| \leq\ C_\nu\ , 0 \neq j\ell(\gamma_0)$$
$$\leq\ T, \nu\ =\ 1,2,3,\ldots. \quad (80)$$

Here we are using the fact that $|w_\nu(x,y)| \leq C_T$ for $|x|,|y| \leq T$. If we repeat the stationary phase argument that was used to prove (74), we see that the left side of (79) is dominated by a fixed constant times

$$\lambda^{\frac{1}{2}}\, w_0\left(\tilde{x}, \alpha_j(\tilde{y})\right)\left(d_{\tilde{g}}\left(\tilde{x}, \alpha_j(\tilde{y})\right)\right)^{-\frac{1}{2}},$$

and, consequently, we would have (74) if

$$w_0\left(\tilde{x}, \alpha_j(\tilde{y})\right)\left(d_{\tilde{g}}\left(\tilde{x}, \alpha_j(\tilde{y})\right)\right)^{-\frac{1}{2}} \leq\ C(\max\{d_g(\tilde{x}, \alpha_j\ (\tilde{y})), e^{\kappa d_g\left(\tilde{x},\alpha_j(\tilde{y})\right)}\})^{-\frac{1}{2}} \quad (81)$$

assuming, as above, that the curvature of M is $\leq -\kappa^2, \kappa \geq 0$. The last inequality comes from the fact that in geodesic normal coordinates about x, we have

$$w_0(x,y) = \left(det\ g_{ij}(y)\right)^{-\frac{1}{4}},$$

(see [61], [22] or §2.4 in [27]). If y has geodesic polar coordinates (t,θ) about x, then $t = d_{\tilde{g}}(x,y)$, and if $A(t,\theta)$ is as in (57), we conclude that $w_0(x,y) = \sqrt{t/A(t,\theta)}$, and therefore (81) follows from Günther's comparison estimate (59) if $-\kappa^2 < 0$ and (58) if $\kappa = 0$.

The second estimate (80) is elementary and left for the reader, who can check that the terms are actually $O(\lambda^{\frac{1}{2}-\nu})$. (This is also just a special case of Lemma 3.5.3 in [27].) This completes the proof of (78), and, hence, that of Theorem (2.2.1).

We see that the proof of Theorem (2.2.1) shows that one can strengthen our main estimate (42) in a natural way. Specifically, if γ_0 is a periodic geodesic of length $\ell(\gamma_0)$ and if we define the δ-tube about γ to be

$$T_\delta(\gamma_0) = \{y \in M: dist_g(y, \gamma_0) < \delta\},$$

with $\delta > 0$ fixed, then there is a uniform constant C_δ so that whenever $\varepsilon > 0$ we have for large λ

$$\frac{1}{\ell(\gamma_0)}\int_{\gamma_0}|e_\lambda|^2 ds \le \varepsilon\lambda^{\frac{1}{2}}\|e_\lambda\|^2_{L^2(T_\delta(\gamma_0))} + C_{\gamma_0,\delta,\varepsilon}\|e_\lambda\|^2_{L^2(M)}. \quad (82)$$

Thus, (42) essentially lifts to the cylinder $\mathbb{R}^2/G_{\gamma_0}$, with, as above, G_{γ_0} , being the stabilizer group for the lift of γ_0 to the universal cover $(\mathbb{R}^2, \tilde{g})$. To prove this, we as before write $I = B_\varepsilon + b_\varepsilon$, with $b_\varepsilon(x,\xi)$ equal to one near $T^*\gamma_0$ but supported in a small conic neighborhood of this set. Since the analog of (61) is valid, i.e.,

$$\int_{\gamma_0}\left|\frac{1}{\pi T}\int_{-T/4}^{T/4}\hat{\chi}(t/T)e^{-i\lambda t}\; B_\varepsilon\circ cos(t\sqrt{-\Delta_g})f\,dt\right|^2 ds \le C'_{T,\varepsilon,\gamma_0}\|f\|^2_{L^2(M)}, (83)$$

it suffices to show that

$$\frac{1}{\ell(\gamma_0)}\int_{\gamma_0}\left|\frac{1}{T}\int_{-T/4}^{T/4}\hat{\chi}(t/T)e^{-i\lambda t}\; b_\varepsilon\circ cos\left(t\sqrt{-\Delta_g}\right)e_\lambda dt\right|^2 ds$$

is dominated by the right side of (82).

If $K_\varepsilon(x,s), x \in M, s \in \gamma_0$ denotes the kernel of this operator then, if $\delta > 0$ and T are fixed, it follows that

$$|K_\varepsilon(x,s)| \le C_{\gamma_0,T,\delta}, x \notin T_\delta(\gamma_0), \quad (84)$$

provided that b_ε is supported in a sufficiently small conic neighborhood of $T^*\gamma_0$. This is a simple consequence of the fact that when b_ε is as above, by (62), $b_\varepsilon \circ cost\sqrt{-\Delta_g}\ (x,s)$ is smooth when $x \notin T_\delta(\gamma_0), s \in \gamma_0$ and $|t| \le T$. Since (66) is valid, we conclude that there is a uniform constant C so that for large λ we have

$$\frac{1}{\ell(\gamma_0)}\int_{\gamma_0}\left|\frac{1}{T}\int_{-T/4}^{T/4}\hat{\chi}(t/T)e^{-i\lambda t}\; b_\varepsilon\circ cos\left(t\sqrt{-\Delta_g}\right)e_\lambda dt\right|^2 ds$$

$$\le CT^{-1}\lambda^{\frac{1}{2}}\|e_\lambda\|^2_{L^2(T_\delta(\gamma_0))} + C_{T,\delta,\gamma_0}\|e_\lambda\|^2_{L^2(M)} \quad (85)$$

which along with (83) gives us (82). This is because we can dominate the quantity in (85) by the sum of the corresponding expression where e_λ is replaced by $1_{T_\delta}(\gamma_0)^{e_\lambda}$ and $1_{T^c_\delta}(\gamma_0)^{e_\lambda}$ and use (66) and our earlier arguments to show that the first of these terms is dominated by the first term in the right side of (85) if λ is large, while the second such term is dominated by last term in the right side of (85) on account of (84). We would also like to point out that it seems likely that one should be able to take the parameter T in the proof of either (42) or (82) to be a function of λ. This would also require that the parameter ε to also be a function of λ, and thus the argument would be more involved. It would not be surprising if, as in B´erard [61] or Hassell and Tacey [23], one could take T to be $\approx\ log\,\lambda$, in which case the L^2 -restriction bounds in Theorem (2.2.1) and the L^4 -estimates in Corollary (2.2.2) could also be improved to be $O(\lambda^{\frac{1}{4}}(log\,\lambda)^{-\delta_1}$ and $O(\lambda^{\frac{1}{8}}\,(log\,\lambda)^{-\delta_2}$, respectively, for some $\delta_j > 0$. It is doubtful that these bounds would be optimal, though–indeed if a difficult conjecture of Rudnick and Sarnak [50] were valid, both would be $O(\lambda^\varepsilon)$ for any $\varepsilon > 0$. One of the main technical issues in carrying out the analysis when T depends on λ would be to determine the analog of (60) in this case. One would also have to take into account more carefully size estimates for the coefficients $w_\nu, \nu > 0$, in the Hadamard parametrix, but B´erard [61] carried out an analysis of these that would seem to be sufficient if $T \approx \log\lambda$. On the other hand, we have argued here that the w_0 coefficient is very well behaved, and so perhaps there could be further grounds for improvement.

Section (2.3): L^p -Norms and Lower Bounds for Nodal Sets of Eigenfunctions in Higher Dimensions

For (M, g) be a smooth, compact boundaryless Riemannian manifold of dimension $d \geq 3$. Let Δ_g be the nonnegative Laplace-Beltrami operator and consider eigenfunctions e_λ satisfying $\Delta_g e_\lambda = \lambda^2 e_\lambda$ with $\lambda \geq 0$. If Π denotes the space of unit length geodesics and dz the volume element associated with the metric g, then our main result is the following generalizations of [28]:

Theorem (2.3.1)[69]: Let $e_\lambda, \lambda \geq 1$, be an eigenfunction and $\frac{2(d+2)}{d} < q < \frac{2(d+1)}{d-1}$. Then there is a uniform constant $C < \infty$ so that given $\varepsilon > 0$ we can find a constant C_ε so that

$$\|e_\lambda\|^q_{L^q(M)} \leq \varepsilon\lambda^{q\left(\frac{d-1}{2}\right)\left(\frac{1}{2}-\frac{1}{q}\right)} \|e_\lambda\|^q_{L^2(M)} + C\|e_\lambda\|^q_{L^2(M)}$$
$$+ C_\varepsilon\lambda^{q\left(\frac{d-1}{2}\right)\left(\frac{1}{2}-\frac{1}{q}\right)}\|e_\lambda\|^2_{L^2(M)} \sup_{\gamma\in\Pi}\left(\int_{\mathcal{T}_{\lambda^{-\frac{1}{2}}(\gamma)}} |e_\lambda(z)|^2\, dz\right)^{\frac{q-2}{2}}, \quad (86)$$

if

$$\mathcal{T}_{\lambda^{-\frac{1}{2}}(\gamma)} = \left\{x \in M : d_g(x,\gamma) \leq \lambda^{-\frac{1}{2}}\right\} \quad (87)$$

denotes the $\lambda^{-\frac{1}{2}}$ -tube about γ, with $d_g(\cdot\,,\cdot)$ being the Riemannian distance function.

Corollary (2.3.2)[69]: The following are equivalent for any subsequence of L^2 -normalized eigenfunctions $\left\{e_{\lambda_{j_k}}\right\}_{k=1}^{\infty}$:

$$\limsup_{k\to\infty}\ \sup_{\gamma\in\Pi} \int_{\mathcal{T}_{\lambda_{jk}^{-\frac{1}{2}}(\gamma)}} \left|e_{\lambda_{j_k}}(z)\right|^2 dz = 0 \quad (88)$$

$$\limsup_{k\to\infty}\ \lambda_{jk}^{-\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{p}\right)} \left\|e_{\lambda_j}\right\|_{L^p(M)} = 0 \text{ for any } 2 < p < \frac{2(d+1)}{d-1}. \quad (89)$$

Proof. Given Theorem (2.3.1), it is routine to verify that (88) implies (89) for $\frac{2(d+2)}{d} < p < \frac{2(d+1)}{d-1}$. The remaining values of p then follow from interpolation. For the converse, observe that Hölder’s inequality gives

$$\int_{T_{\lambda^{-\frac{1}{2}}(\gamma)}} |e_\lambda(z)|^2\, dz \lesssim \lambda^{-\left(\frac{d-1}{2}\right)\left(1-\frac{2}{p}\right)}\|e_\lambda\|^2_{L^p(M)},$$

and the implication follows.

In the case when (M, g) has nonpositive sectional curvatures, we shall be able to show that (89) holds for the full sequence of eigenvalues and hence extend the two-dimensional results of the second author and Zelditch [30] to higher dimensions:

Theorem (2.3.3)[69]: Let (M, g) be a compact boundaryless manifold of dimension $d \geq 2$. Assume further that (M, g) has everywhere nonpositive sectional curvatures. Then if $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \ldots$ are the eigenvalues of $\sqrt{\Delta_g}$ we have

$$\limsup_{\lambda_j \to \infty} \left(\sup_{\gamma \in \Pi} \int_{\mathcal{T}_{\lambda_j^{-\frac{1}{2}}(\gamma)}} \left| e_{\lambda_j} \right|^2 \, dx \right) = 0. \tag{90}$$

Consequently, if $2 < p < \frac{2(d+1)}{d-1}$, we have, in this case,

$$\limsup_{\lambda_j \to \infty} \lambda_j^{-\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{p}\right)} \left\| e_{\lambda_j} \right\|_{L^q(M)} = 0. \tag{91}$$

In [15] the first author showed that $\|e_\lambda\|_{L^q(M)} = O\left(\lambda^{\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{p}\right)}\right)$ when $2 \leq p \leq \frac{2(d+1)}{d-1}$, and that these estimates are sharp on the standard sphere S^d because of the highest weight spherical harmonics. We should point out that for the complementary range $p > \frac{2(d+1)}{d-1}$ improved L^p -estimates under the above curvature assumptions follow, by interpolation from the aforementioned $p = \frac{2(d+1)}{d-1}$ and an improved L^∞-estimate which is implicit in Berard [61] (see also Zelditch [17] and [28]). Hassell and Tacy [82] have recently obtained further results for this range exponents. Improvements for $p > \frac{2(d+1)}{d-1}$ are a bit more straightforward than (91) due to the fact that everything follows from pointwise estimates, while, to obtain (90) and consequently (91), we have to use oscillatory integrals and a finer analysis involving the deck transforms of the universal cover. We should point out that there are no general L^p-improvements for the endpoint $p = \frac{2(d+1)}{d-1}$ of the results in [15], which on the sphere are saturated by eigenfunctions concentrating at points as well as ones concentrating along geodesics.

The special case of $d = 2$ of Theorem (2.3.3) is in [30]. When $d = 3$, if one assumes constant nonpositive curvature, (90) follows from recent work of Chen and [73], who showed that if ds denotes arc length measure on γ, then

$$\sup_{\gamma \in \Pi} \int_\gamma |e_\lambda|^2 \, ds = o(\lambda) \text{as } \lambda \to \infty. \tag{92}$$

In dimensions $d \geq 4$, Burq, Gerard and Tzvetkov [5] showed that one has the following bounds for geodesic restrictions

$$\int_\gamma |e_\lambda|^2 \, ds = O(\lambda^{d-2}). \tag{93}$$

Improving this to $o(\lambda^{d-2})$ bounds as in (92) for $d = 3$ is not strong enough to obtain (90) when $d \geq 4$. This comes as no surprise since, in these dimensions, (93) is saturated on the round sphere S^d not by the highest weight spherical harmonics which concentrate along geodesics, but rather zonal spherical harmonics, which concentrate at points. By our main result, Theorem (2.3.1), we know that (90) is relevant for measuring the size of L^p -norms in the range $2 < p < \frac{2(d+1)}{d-1}$, which are saturated on S^d by highest weight spherical harmonics. These eigenfunctions saturate the Kakeya-Nikodym averages in (90), by which we mean that the left side of (90) is $\Omega(1)$, but they do not saturate the restriction estimates (93) for $d \geq 4$.

Fortunately, we can adopt the proof of the aforementioned improvement (92) of Chen and the second author [73] to obtain (90) in all dimensions under the assumption of nonpositive curvature. Additionally, even for $d = 3$, unlike the stronger estimate (92), our techniques do not require that we assume constant sectional curvature.

By recording some applications of Theorems (2.3.1) and (2.3.3). First, using (90) we can improve the lower bounds for L^1 -norms of Zelditch [85] under the above assumptions:

Corollary (2.3.4)[69]: Let (M, g) be a d-dimensional compact boundaryless manifold with $d \geq 2$. Then

$$\liminf_{\lambda\to\infty} \lambda^{\frac{d-1}{4}} \|e_\lambda\|_{L^1(M)} = \infty. \tag{94}$$

As pointed out in [85], no such improvement is possible for the sphere. The proof of (94) is very simple. For, by Hölder's inequality, if $p > 2$,

$$1 = \|e_\lambda\|_{L^2} \leq \|e_\lambda\|_{L^1}^{\frac{p-2}{2(p-1)}} \|e_\lambda\|_{L^p}^{\frac{p}{2(p-1)}},$$

whence

$$\|e_\lambda\|_{L^p}^{-\frac{p}{p-2}} \leq \|e_\lambda\|_{L^1}, \qquad p > 2.$$

As a result,

$$\left(\lambda^{-\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{p}\right)} \|e_\lambda\|_{L^p}\right)^{-\frac{p}{p-2}} \leq \lambda^{\frac{d-1}{4}} \|e_\lambda\|_{L^1},$$

meaning that (91) implies (94).

Let us now see how (94), along with an estimate of Hezari and [83] improves the known lower bounds for the Hausdorff measure of eigenfunctions on manifolds of variable nonpositive curvature.

To this end, for a given real eigenfunction, e_λ, we let

$$Z_\lambda = \{x \in M : e_\lambda(x) = 0\}$$

denote its nodal set and $\mathcal{H}^{d-1}(Z_\lambda)$ its $(d-1)$-dimensional Hausdorff measure. Yau [91] conjectured that $\mathcal{H}^{d-1}(Z_\lambda) \approx \lambda$. This was verified by Donnelly and Fefferman [78] in the real analytic case and so, in particular, if (M, g) has constant sectional curvature. The lower bound $\mathcal{H}^{d-1}(Z_\lambda) \geq c_\lambda$ was verified in the C^∞ case when $d = 2$ by Bruning [72] and Yau, but much less is known in this case. An upper bound $\mathcal{H}^{d-1}(Z_\lambda) = O(\lambda^{\frac{3}{2}})$ is also known by Dong [77] and Donnelly and Fefferman [79] when $d = 2$, but the best known upper bounds for $d \geq 3$ are $\mathcal{H}^{d-1}(Z_\lambda) = O\left((c\lambda)^{(c\lambda)}\right)$, which are due to Hardt and Simon [81].

Until recently, in higher dimensions for the C^∞ case, the best known lower bounds for $\mathcal{H}^{d-1}(Z_\lambda)$ were also of an exponential nature (see [80]). Recently, Colding and Minicozzi [74] and the second author and Zelditch [85] proved lower bounds of a polynomial nature. Specifically, the best known lower bounds for $d \geq 3$ in the C^∞ case are those of Colding and Minicozzi [74] who showed that

$$c\lambda^{1-\frac{d-1}{2}} \leq \mathcal{H}^{d-1}(Z_\lambda). \tag{95}$$

Subsequent proofs of this using the original approach of the second author and Zeldtich [85] were obtained by Hezari and [83] and Zelditch [86]. The latter works and the earlier one [85] were based on a variation of an identity of Dong [77]. The proof of (95) in [83] was based on the following lower bound

$$c\lambda\left(\int_M |e_\lambda| dx\right)^2 \leq \mathcal{H}^{d-1}(Z_\lambda). \tag{96}$$

Indeed, simply combining (96) and the L^1 -lower bound of Zelditch [85]

$$c\lambda^{-\frac{d-1}{4}} \leq \|e_\lambda\|_{L^1} \tag{97}$$

yields (95). Similarly, by using the improvement (94) of (97), we can improve1 the known lower bounds (95) under our assumptions:

Corollary (2.3.5)[69]: Let (M, g) be a compact boundaryless Riemannian manifold of dimension $d \geq 3$ with nonpositive sectional curvatures. Then

$$\liminf_{\lambda\to\infty} \lambda^{-1+\frac{d-1}{2}} \mathcal{H}^{d-1}(Z_\lambda) = \infty. \tag{98}$$

In particular, when $d = 3, \mathcal{H}^2(Z\lambda)$ becomes arbitrarily large as $\lambda \to \infty$.

By a simple argument (see [28]) one always has (88) and consequently (89) as λ ranges over a subsequence of eigenvalues $\{\lambda_{jk}\}$ if the resulting eigenfunctions form a quantum ergodic system (i.e. $\left|e_{\lambda_{jl}}\right|^2 dx$ converges in the weak* topology to the uniform probability measure $dx/\mathrm{Vol}_g(M)$). Consequently, by the above proof, we also have the following

Corollary (2.3.6)[69]: Let $\left\{e_{\lambda_{jk}}\right\}$ be a quantum ergodic system on a compact Riemannian manifold of dimension $d \geq 3$. We then have

$$\lim_{k\to\infty} \lambda_{jk}^{-1+\frac{d-1}{2}} \mathcal{H}^{d-1}\left(Z_{\lambda_{jk}}\right) = \infty. \tag{99}$$

In particular, if the geodesic flow is ergodic, we have (99) as $\{\lambda_{jk}\}$ ranges over a subsequence of eigenvalues of density one. The last part of the corollary follows from the quantum ergodic theorem of Snirelman [14] / Zelditch [18] / Colin de Verdiere [75] (see also [28]).

We shall present the proof of our main result, Theorem (2.3.1). We shall go through the essentially routine step of reducing matters to proving certain bilinear estimates, and this step is very similar to the argument for the two-dimensional case of one of us [28]. It gives partial control of the left side of (86) by the last term in the right. The needed bilinear estimates, which lead to the first term in the right side of (86). We show the bilinear estimate we require follows, up to an ε loss, from one of Lee [84]. We then are able to remove this loss using a variable coefficient version of the "ε- removal lemma" of Tao and Vargas [88] (see also Bourgain [70]). We prove Theorem (2.3.3) which says that we have o(1) bounds for L^2 -norms over shrinking tubes under the assumption of nonpositive curvature, and consequently, by Theorem (2.3.1), improved $L^p(M)$-norms for $2 < p < \frac{2(d+1)}{d-1}$ of the estimates in [15].

We begin the proof of Theorem (2.3.1), reducing matters to estimates on oscillatory integral operators. Let χ_λ denote the operator $\chi(\sqrt{\Delta_g} - \lambda)$, where χ is a smooth bump function with $\chi(0) = 1$ and sufficiently small compact support. Hence $\chi_\lambda e_\lambda = e_\lambda$. Recall (see Sogge, Chapter 5 [16]) that the kernel of this operator can be written as

$$\chi_\lambda f(z) = \chi\left(\sqrt{\Delta_g} - \lambda\right) f(z) = \lambda^{\frac{d-1}{2}} \int_M e^{i\lambda d_g(z,y)} \alpha_\lambda(z, y) f(y)\, dy + R_\lambda f(z)$$

where $\alpha_\lambda(z, y)$ is supported in $\delta \leq d_g(z, y) \leq 2\delta$ for some $\delta > 0$ sufficiently small and less than half the injectivity radius of (M, g). Moreover, $\|R_\lambda f\|_{L^q(M)} \lesssim \|f\|_{L^2(M)}$.

Using a sufficiently fine partition of unity, we may assume that the support of α_λ is sufficiently small. In particular, we may assume that $\mathrm{supp}(\alpha_\lambda) \subset \{|z - z_0| + |y - y_0| \ll \varepsilon_0\}$ for some points $z_0, y_0 \in M$ with $|z_0 - y_0| \approx \delta$. Let γ_0 denote the geodesic connecting z_0, y_0 and suppose that Σ is a suitable codimension 1 submanifold passing

through y_0 such that γ_0 is orthogonal to Σ. Now let $(t,s) \in \mathbb{R}^{d-1} \times \mathbb{R}$ denote Fermi coordinates for Σ with $(0,0) = y_0, (0,s)$ parameterizing γ_0, and $(t,0)$ parameterizing Σ. This means that for any fixed $t_0, (t_0,s)$ locally parameterizes the geodesic passing through $(t0,s)$ orthogonal to Σ.
It suffices to prove that

$$\int \left(\int \left|\lambda^{\frac{d-1}{2}} \int e^{i\lambda d_g(z,y)}\alpha_\lambda(z,(t,s))f(t,s)dt\right|^2 |f(z)|^{q-2}\, dz\right) ds$$

$$\leq \varepsilon \left(\lambda^{\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)}\|f\|_{L^2(M)}\right)^2 \|f\|_{L^q(M)}^{q-2}$$

$$+ C_\varepsilon \lambda^{q\left(\frac{d-1}{2}\right)\left(\frac{1}{2}-\frac{1}{q}\right)}\|f\|_{L^2(M)}^2 \sup_{\gamma\in\Pi}\left(\int_{\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)} |f(z)|^2\, dz\right)^{\frac{q-2}{2}}$$

Indeed, using Young's inequality for products applied to the Hölder conjugates $\frac{q}{2}, \frac{q}{q-2}$, we may absorb the contribution of $\varepsilon^{\frac{(q-2)}{q}}\|f\|_{L^q(M)}^{q-2}$ from the first term into the left hand side, for ε sufficiently small, yielding (86) when $f = e_\lambda$. It suffices to prove that for each s the expression in parentheses on the left hand side is bounded by the right hand side. For convenience, we will show this for $s = 0$ as the argument below works for any value of s and does not use the structure of Σ once Fermi coordinates are given.

Fix λ and let $Th(z) = \int e^{i\lambda\psi(z,t)}\alpha_\lambda(z,(t,0))h(t)\, dt$ where $\psi(z,t) = d_g(z,(t,0))$. We will show that

$$\int \left|\lambda^{\frac{d-1}{2}} Th(z)\right|^2 |f(z)|^{q-2}\, dz \leq \varepsilon \left(\lambda^{\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)}\|h\|_{L_t^2}\right)^2 \|f\|_{L_z^q}^{q-2}$$

$$+ C_\varepsilon \lambda^{\frac{d-1}{2}} \|h\|_{L_t^2}^2 \sup_{\gamma\in\Pi} \int\_(\mathcal{T}_{\lambda^{-\frac{1}{2}}(\gamma)} |f(z)|^{q-2}\, dz. \qquad (100)$$

Hölder's inequality with conjugates $\frac{2}{q-2}, \frac{2}{4-q}$ will then imply that

$$\lambda^{\frac{d-1}{2}} \int_{\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)} |f(z)|^{q-2}\, dz \lesssim \lambda^{\frac{d-1}{2}-\left(\frac{d-1}{2}\right)\left(2-\frac{q}{2}\right)} \left(\int_{\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)} |f(z)|^2\, dz\right)^{\frac{q-2}{2}}$$

and it is verified that the exponent of λ on the right is the same as the one in (86). Observe that

$$\left(T\, h(z)\right)^2 = \int e^{i\lambda\left(\psi(z,t)+\psi(z,t')\right)}\alpha_\lambda(z,t)\alpha\lambda(z,t')h(t)h(t')\, dt\, dt'.$$

Suppose ε_0 is a small dyadic number such that $\operatorname{supp}(\alpha_\lambda(z,\cdot)) \subset [-\varepsilon_0,\varepsilon_0]^d$ for all z. Let $N > 0$ be a sufficiently large dyadic number (which will essentially play the same role as the integer N in [28]) and let j_0 be the largest integer such that $2^{-j_0} \geq \lambda^{-\frac{1}{2}}$. Take a Whitney-type decomposition of $[-\varepsilon_0,\varepsilon_0]^d \times [-\varepsilon_0,\varepsilon_0]^d$ away from its diagonal D into almost disjoint cubes

$$[-\varepsilon_0,\varepsilon_0]^d \times [-\varepsilon_0,\varepsilon_0]^d \setminus D$$
$$= \left(\bigcup_{\varepsilon_0 \geq 2^j > N2^{-j_0}} \bigcup_{d\left(Q_\nu^j, Q_{\nu'}^j\right) \approx 2^{-j}} Q_\nu^j \times Q_{\nu'}^j \right)$$
$$\cup \left(\bigcup_{d\left(Q_\nu^{j_0}, Q_{\nu'}^{j_0}\right) \leq N2^{-j_0}} Q_\nu^{j_0} \times Q_{\nu'}^{j_0} \right)$$

where each Q_ν^j has sidelength 2^{-j} and is centered at a point $\nu \in 2^{-j}\mathbb{Z}^{d-1}$. Set $h_\nu^j(t) = 1_{Q_\nu^j}(t)h(t)$ where the first factor denotes the indicator of the cube Q_ν^j. Hence

$$\left(Th(z)\right)^2 = \sum_{\varepsilon_0 \geq 2^{-j} > N2^{-j_0}} \sum_{(\nu,\nu') \in \Xi_j} Th_\nu^j(z)\, Th_{\nu'}^j(z)$$
$$+ \sum_{(\nu,\nu') \in \Xi_{j_0}} Th_\nu^j(z) Th_{\nu'}^j(z) \tag{101}$$

where Ξ_j denotes the collection of (ν, ν') indexing the cubes satisfying $d\left(Q_\nu^j, Q_{\nu'}^j\right) \approx 2^{-j}$ (or $\leq N2^{-j_0}$ when $j = j_0$).

Theorem (2.3.7)[69]: Suppose $T = T_\lambda$ is the oscillatory integral operator defined by

$$Th(z) := \int e^{i\lambda\phi(z,s,t)}\, a_\lambda(z,s,t)h(t)\, dt$$

where a_λ is smooth and supp(a_λ) is contained in a sufficiently small uniform compact set and whose derivative bounds can be taken uniform in λ. Assume further that $\phi(x,s,t)$ satisfies a Carleson-Sjölin type condition that $\nabla_{xt}^2\phi$ is invertible and that if θ is a unit vector for which $\nabla_t\langle\nabla(x,s)\phi,\theta\rangle = 0$, then

$$\nabla_{tt}^2\langle\nabla(x,s)\phi,\theta\rangle \text{ has eigenvalues of the same sign.} \tag{102}$$

Then

$$\left\| \sum_{(\nu,\nu') \in \Xi_j} Th_\nu^j\, Th_{\nu'}^j \right\|_{L_x^{\frac{q}{2}}} \lesssim 2^{j\left(\frac{2(d+1)}{q} - (d-1)\right)} \lambda^{-\frac{2d}{q}} \|h\|_{L_t^2}^2 . \tag{103}$$

It can be verified that setting $z = (x,s) \in \mathbb{R}^{d-1} \times \mathbb{R}$, the phase function in question $\phi(x,s,t) := d_g\big((x,s),(t,0)\big) = \psi\big((x,s),t\big)$ satisfies the Carleson-Sjölin condition given here. Moreover, our assumption that $q < \frac{2(d+1)}{d-1}$ ensures that the exponent of 2^j in (103) is positive. Hence this estimate yields

$$\sum_{\varepsilon_0^{-1} \leq 2^j < N^{-1}2^{j_0}} \left\| \sum_{(\nu,\nu') \in \Xi_j} Th_\nu^j\, Th_{\nu'}^j \right\|_{L_z^{\frac{q}{2}}} \lesssim N^{-\left(\frac{2(d+1)}{q} - (d-1)\right)} \lambda^{-\frac{d-1}{q} - \frac{d-1}{2}} \|h\|_{L_t^2}^2 .$$

Since Hölder's inequality with conjugates $\frac{q}{2}, \frac{q}{q-2}$, and the triangle inequality yield

$$\lambda^{d-1}\int\left|\sum_j\sum_{(\nu,\nu')\in\Xi_j} Th_\nu^j\, Th_{\nu'}^j\right|^2 |f|^{q-2}\,dz$$
$$\leq \lambda^{d-1}\sum_j\left\|\sum_{(\nu,\nu')\in\Xi_j} Th_\nu^j\, Th_{\nu'}^j\right\|_{L_z^{\frac{q}{2}}} \|f\|_{L_z^q}^{q-2}$$

the contribution of this sum is bounded by the first term on the right hand side of (100) by taking N suitably large. This estimate can be considered as analogous to [28].

Our main tool in proving (103) will be a bilinear estimate due to Lee [84] along with a refinement of arguments of that same work. Indeed, the estimate (103) should be compared with [84]. [84], prove bilinear estimates which can be thought of as a variable coefficient versions of bilinear restriction estimates due to Tao [87] for elliptic surfaces (inspired by prior work of Wolff [90] and Tao-Vargas-Vega [89]). Lee then showed that these bilinear estimates in turn implied linear estimates on oscillatory integral operators whose phase function satisfies the Carleson-Sjölin type condition (102) (more generally called the "Hormander problem"). However, his estimates suffer losses when compared to the optimal estimate predicted by scaling.

We cannot afford such losses. Hence one of the central tasks is to prove a variable coefficient version of the ε-removal lemma for bilinear estimates in [88] (see also Bourgain [70]) and refine the almost orthogonality arguments in [84].

We now turn to the second sum in (101); since $2^{-j_0} \approx \lambda^{-\frac{1}{2}}$ it will be treated essentially the same way as in [28]. Observe that

$$\left|\sum_{(\nu,\nu')\in\Xi_{j_0}} Th_\nu^j(z)\, Th_{\nu'}^j(z)\right| \lesssim N^{d-1}\sum_\nu\left|Th_\nu^{j_0}(z)\right|^2$$

The main estimate for this term is then

$$\int\left|\lambda^{\frac{d-1}{2}}\, Th_\nu^{j_0}(z)\right|^2 |f(z)|^{q-2}\,dz$$
$$\lesssim \lambda^{\frac{d-1}{2}}\left\|h_\nu^{j_0}\right\|_{L_t^2}^2 \sup_{\gamma\in\Pi}\int_{\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)} |f(z)|^{q-2}\,dz \tag{104}$$

Since $\sum_\nu \left\|h_\nu^{j_0}\right\|_{L_t^2}^2 = \|h\|_{L_t^2}^2$, we may sum in ν to see that the contribution of these terms is bounded by the last term in (100).

To see (104), we will use geodesic normal coordinates centered at the point on M corresponding to $(\nu, 0)$ in the Fermi coordinates (recall that $\nu \in 2^{-j_0}\mathbb{Z}^{d-1}$) and let $x \mapsto \kappa(x)$ denote the diffeomorphism which makes this change of coordinates. We may assume that $\kappa(\nu, s) = (0, s)$ (parameterizing the geodesic orthogonal to Σ through $(\nu, 0)$). We now let $\{\omega_l\}_l$ denote a $\lambda^{-\frac{1}{2}}$-separated collection of points in a neighborhood of (0, . . . , 0, 1) on S^{d-1} indexed by a subset of $\mathbb{Z}^{d-1}$ so that

$$|\omega_l - \omega_k| \gtrsim \lambda^{-\frac{1}{2}}|l - k|.$$

Now let

$$S_l := \left\{z : \left|\frac{z}{|z|} - \omega_l\right| \leq \lambda^{-\frac{1}{2}}\right\}$$

and observe that the left hand side of (104) can be dominated by

$$\sum_l \left\| \lambda^{\frac{d-1}{2}} T\left(h_\nu^{j_0}\right) \right\|^2_{L^\infty\left(\kappa^{-1}(S_l)\right)} \|f\|^{q-2}_{L^{q-2}(\kappa^{-1}(S_l)\cap K)}$$

$$\le \sup_k \|f\|^{q-2}_{L^{q-2}(\kappa^{-1}(S_k)\cap K)} \sum_l \left| \lambda^{\frac{d-1}{2}} T\left(h_\nu^{j_0}\right)(z_l) \right|^2$$

where the z_l are chosen to maximize $\left|T\left(h_\nu^{j_0}\right)(z)\right|$ as z ranges over $\kappa^{-1}(S_l)$ and K is a small set containing the x-support of $\alpha_\lambda(x, y)$. It thus suffices to see that for some suitable bump function ψ,

$$\sum_l \left| \lambda^{\frac{d-1}{2}} \int e^{i\lambda\psi(z_l,t)} \alpha_\lambda(z_l, (0,t)) \psi\left(\lambda^{\frac{1}{2}}(t - \nu)\right) h_\nu^{j_0}(t) dt \right|^2 \lesssim \lambda^{\frac{d-1}{2}} \left\|h_\nu^{j_0}\right\|^2_{L_t^2}.$$

After a translation in t, it suffices to assume that $\nu = 0$ and the desired $L^2 \to \ell^2$ estimate follows from the one dual to (106) below.

Theorem (2.3.8)[69]: Suppose $\psi(z, t)$ is as defined above and $\rho(z, t)$ is a smooth bump function satisfying $|\partial_t^\alpha \rho(z, t)| \lesssim_\alpha \lambda^{\frac{|\alpha|}{2}}$ and $\mathrm{supp}(\rho(\cdot, z)) \subset \{|t|. \lambda^{-\frac{1}{2}}\}$. Assume also that ρ vanishes when z is outside of a small neighborhood N of $(s_0, 0)$ with $s_0 \approx \delta$ with $\delta > 0$ (in the Fermi coordinates described above). Let z_l be a collection of points in N indexed by $\mathbb{Z}^{d-1}$ such that whenever $|l - k|$ is sufficiently large,

$$\frac{\left(\kappa_1(z_l), \ldots, \kappa_{d-1}(z_l)\right)}{|\kappa(z_l)|} - \frac{\left(\kappa_1(z_k), \ldots, \kappa_{d-1}(z_k)\right)}{|\kappa(z_k)|} \gtrsim \lambda^{-\frac{1}{2}} |l - k|. \quad (105)$$

Then

$$\lambda^{\frac{d-1}{2}} \int \left| \sum_l e^{i\lambda\psi(z_l,t)} \rho(z_l, t) a_l \right|^2 dt \lesssim \sum_l |a_l|^2. \quad (106)$$

The proof of (106) is the same as the one in [28], once it is observed that

$$|\nabla_t \psi(z_l, 0) - \nabla_t \psi(z_k, 0)| \gtrsim \lambda^{-\frac{1}{2}} |l - k|.$$

But since the pushforward of $\partial/\partial z_d$ under $z \longmapsto \kappa(z)$ is itself, this is a consequence of (105) and the identity

$$\partial_{t_i} \psi(z, 0) = \langle \nu_i, \kappa(z)/|\kappa(z)| \rangle, \ i = 1, \ldots, d-1$$

where ν_i is the pushforward of $\partial/\partial z_i$.

We begin the proof of Theorem (2.3.7). We first appeal to [84] (which follows results of Bourgain [35] and Hörmander [45]) and the ensuing remark, which states that after a change of coordinates and multiplying Th, h by harmless functions of modulus one, we may assume

$$\phi(x, s, t) = x \cdot t + \frac{1}{2} s|t|^2 + \mathcal{E}(x, s, t) \quad (107)$$

where

$$\mathcal{E}(x, s, t) = O(|x| + |s|)^2 |t|^2 + O((|x| + |s|)|t|^3). \quad (108)$$

Let ψ be a smooth bump function supported in $[-1, 1]^{d-1}$ satisfying $\sum_{k\in\mathbb{Z}^{d-1}} \psi^2(x - k) = 1$ and set $A_\mu(x) = \psi^2(2^j(x - \mu))$ with $\mu \in 2^{-j}\mathbb{Z}^{d-1}$.

Lemma (2.3.9)[69]: Suppose $1 \le p \le 2$ and that T is as in Theorem (2.3.7). There exist amplitudes $a_{\nu,\mu}, a_{\nu',\mu}$ both with x-support contained in $\mathrm{supp}(A_\mu)$ and satisfying derivative bounds of the form

$$\left|\partial_x^\alpha\, a_{\nu,\mu}(x,s,t)\right| \lesssim_\alpha\, 2^{j|\alpha|} \tag{109}$$

such that if $T_{\nu,\mu}$ is the oscillatory integral operator with phase ϕ and amplitude $a_{\nu,\mu}$

$$T_{\nu,\mu}(h)(x,s) \;=\; \int_{\mathbb{R}^{d-1}} e^{i\lambda\phi(x,s,t)} a_{\nu,\mu}(x,s,t)h(t)\,dt$$

then

$$\left\|A_\mu \sum_{\nu,\nu'\in\Xi_j} T(h_\nu^j)T(g_{\nu'}^j)\right\|_{L^p(\mathbb{R}^d)}^p \lesssim \sum_{\nu,\nu'\in\Xi_j} \left\|T_{\nu,\mu}(h_\nu^j)T_{\nu',\mu}(g_{\nu'}^j)\right\|_{L^p(\mathbb{R}^d)}^p$$

Proof. For a given s, consider the slice of $T(h)$ at s $T^s\,(h)(x) \;=\; T(h)x,r|_{r=s}$. It suffices to show that

$$\left\|A_\mu \sum_{\nu,\nu'\in\Xi_j} T^s\,(h_\nu^j)T^s\,(g_{\nu'}^j)\right\|_{L^p(\mathbb{R}^{d-1})}^p \lesssim \sum_{\nu,\nu'\in\Xi_j} \left\|\,T_{\nu,\mu}^s(h_\nu^j)T_{\nu',\mu}^s\,(g_{\nu'}^j)\right\|_{L^p(\mathbb{R}^{d-1})}^p,$$

and hence we shall assume that s is fixed throughout the proof. Now let $\Phi(x,t,t') \;=\; \phi(x,s,t) \;+\; \phi(x,s,t')$ and observe that $A_\mu T^s\,(h_\nu^j)T^s\,(g_{\nu'}^j)$ can be written as

$$A_\mu(x)\int e^{i\lambda\Phi(x,t,t')}\, a(x,s,t)a(x,s,t')h_\nu^j\,(t)g_{\nu'}^j\,(t')\,dtdt',$$

Treating $D_x \;=\; -i\nabla_x$ as a vector-valued differential operator we want to write

$$\left(1 \;+\; \left(\lambda^{-1}2^j\right)^2 |\lambda\nabla_x\Phi(\mu,\nu,\nu') \;-\; D_x|^2\right)^N A_\mu T^s\,(h_\nu^j)T^s\,(g_{\nu'}^j)$$
$$= \;T_{\nu,\mu}^s(h_\nu^j)T_{\nu'}^s\,,\mu(g_{\nu'}^j) \tag{110}$$

for some N large based on d and each operator on the right satisfies (109). It thus suffices to see that this can be done for any monomial of

$$\lambda^{-1}\,2^j\,(\lambda\nabla_x\Phi(\mu,\nu,\nu') - \;D_x),$$

which in turn will follow by induction. To this end, observe that products of functions satisfying (109) satisfy the same condition as do weighted derivatives $(c\partial_x)^\alpha$ of such functions provided $|c| \;\le\; 2^{-j}$. On $\operatorname{supp}(A_\mu) \;\times\; Q_\nu^j \;\times\; Q_{\nu'}^j$ we have that

$$\lambda^{-1}\,2^j\left(\lambda\partial_k\Phi(\mu,\nu,\nu') \;-\; \lambda\partial_k\Phi(x,t,t')\right)$$

satisfies (109). Moreover, since $\lambda^{-1}2^j \;\le\; 2^{-j}$, it is seen that for any α, $\left(\lambda^{-1}\,2^j\partial_x\right)^\alpha A_\mu^{\frac{1}{2}}$ satisfies (109). The claim then follows.

It now suffices to see that if $P_{\nu,\nu'}$ is the Fourier multiplier

$$P_{\nu,\nu'}\,(D_x) \;=\; \left(1 \;+\; \left(\lambda^{-1}\,2^j\right)^2 |\lambda\nabla_x\Phi(\mu,\nu,\nu') \;-\; D_x|^2\right)^{-N},$$

then for any sequence of $\{f_{\nu,\nu'}\}$ of Schwartz class functions defined on $\mathbb{R}^{d-1}$,

$$\left\|\sum_{\nu,\nu'\in\Xi_j} P_{\nu,\nu'}f_{\nu,\nu'}\right\|_{L^2(\mathbb{R}^{d-1})}^2 \lesssim \sum_{\nu,\nu'\in\Xi_j} \left\|f_{\nu,\nu'}\right\|_{L^2(\mathbb{R}^{d-1})}^2,$$

$$\left\|\sum_{\nu,\nu'\in\Xi_j} P_{\nu,\nu'}f_{\nu,\nu'}\right\|_{L^1(\mathbb{R}^{d-1})} \lesssim \sum_{\nu,\nu'\in\Xi_j} \left\|f_{\nu,\nu'}\right\|_{L^1(\mathbb{R}^{d-1})}.$$

The latter follows from the triangle inequality and Young's inequality for convolutions, so it suffices to treat the former. But $\nabla x\Phi(\mu,\nu,\nu') = 2\nabla_x\phi(\mu,s,\nu) + O(2^{-j})$, so the invertibility of $\nabla^2\phi_{x,t}$ gives

$$2^j\,|\nabla_x\Phi(\mu,\nu,\nu') - \nabla x\Phi(\mu,\tilde{\nu},\tilde{\nu}')| \approx 2^j\,|\nu - \tilde{\nu}|.$$

Recall that for each ν, the number of ν' such that $(\nu,\nu') \in \Xi_j$ is $O(1)$. Therefore since the ν range over a regularly spaced 2^{-j} lattice, the desired bound follows from a routine computation using Plancherel's identity.

Returning to the proof of Theorem (2.3.7), fix a pair $(\nu,\nu') \in \Xi_j$. Set $h_1(t) = h_\nu^j\,(2^{-j}\,t), a_{j,\nu,\mu}(x,s,t) = a_{\nu,\mu}(x,s,2^{-j}\,t), \phi_j\,(x,s,t) = 2^j\phi(x,s,2^{-j}\,t)$ so that rescaling variables $t \mapsto 2^{-j}\,t$ in the integral defining $T_{\nu,\mu}(h_\nu^j\,)(x,s)$ yields

$$T_{j,\nu,\mu}(h_1)(x,s) := \int e^{i\lambda 2^{-j}\phi_j(x,s,t)}\,a_{\nu,\mu}(x,s,t)h_1(t)dt$$
$$= 2^{j(d-1)}T_{\nu,\mu}\big(h_\nu^j\big)(x,s).$$

Also set $h_2(t) = h_{\nu'}^j\,(2^{-j}\,t)$ and define $T_{j,\nu',\mu}(h_2)(x,s)$ analogously, noting that ϕ_j remains independent of , ν' . Moreover, we may assume that $a_{j,\nu,\mu}(x,s,\cdot)$ (resp. $a_{j,\nu',\mu}(x,s,\cdot)$) is supported in a slightly larger cube containing $\operatorname{supp}(h_1)$ (resp. $\operatorname{supp}(h_2)$). It is helpful to observe that given (107), (108)

$$\phi j\,(x,s,t) = x \cdot t + 2-j-1\,s|t|\,2 + 2j\,E(x,s,2\,-j\,t).$$

Lemma (2.3.10)[69]: There exists an amplitude $\tilde{a}_{j,\nu,\mu}(x,s,t)$ satisfying bounds of the form (109) such that

$$\left(1 + 2^{2j}\left|\lambda^{-1}\,2^jD_t - \mu\right|^2\right)^N e^{i\lambda 2^{-j}\phi_j\,(x,s,t)}a_{j,\nu,\mu}(x,s,t) = e^{i\lambda 2^{-j}\phi_j(x,s,t)}\tilde{a}_{j,\nu,\mu}(x,s,t).$$

Proof. Observe that

$$e^{-i\lambda 2^{-j}\phi_j}\,2^j\left(\lambda^{-1}\,2^jD_{t_k} - \mu_k\right)e^{i\lambda 2^{-j}\phi_j}\,a_{j,\nu,\mu}$$
$$= 2^j\left(\partial_{t_k}\phi_j - \mu_k\right)a_{j,\nu,\mu} + \lambda^{-1}\,2^{2j}D_{t_k}\,a_{j,\nu,\mu}.$$

Since $\lambda^{-1}2^{2j} \leq 1$, second term satisfies (109). Moreover, by (107), (108)

$$\left(\partial_{t_k}\phi_j\,(x,s,t) - \mu_k\right) = x_k - \mu_k + O\big(2^{-j}\big)$$

and thus by the support properties of $a_{j,\nu,\mu}$ the first term satisfies (109) as well. The lemma then follows by an inductive argument a_k in to that in Lemma (2.3.9).

Given this lemma we let $P_\mu = P_\mu(D_t)$ be the Fourier multiplier with symbol $P_\mu(\zeta) = \left(1 + 2^{2j}\left|\lambda^{-1}2^j\,\zeta + \mu\right|^2\right)^{-N}$ and observe that by self-adjointness of $P_\mu(-D_t)$, we have

$$T_{j,\nu,\mu}(h_1)(x,s) = \int e^{i\lambda 2\,-j\phi_j\,(x,s,t)}\,\tilde{a}_{j,\nu,\mu}(x,s,t)(P_\mu h_1)(t)\,dt$$

Thus if we can show that

$$\left\|T_{j,\nu,\mu}(h_1)T_{j,\nu',\mu}(h_2)\right\|_{L^{\frac{q}{2}}(\mathbb{R}^d)}$$
$$\lesssim \lambda^{-\frac{2d}{q}}\,2^{\frac{2j(d+1)}{q}}\,\|P_\mu h_1\|_{L^2(\mathbb{R}^{d-1})}\|P_\mu h_2\|_{L^2(\mathbb{R}^{d-1})} \qquad (111)$$

taking a sum with respect to μ and applying Cauchy-Schwarz will give

$$\sum_\mu \left\|T_{j,\nu,\mu}(h_1)T_{j,\nu',\mu}(h_2)\right\|^{\frac{q}{2}}_{L^{\frac{q}{2}}(\mathbb{R}^d)} \lesssim \left(\lambda^{-\frac{2d}{q}}\,2^{\frac{2j(d+1)}{q}}\right)^{\frac{q}{2}}\prod_{i=1}^{2}\left(\sum_\mu \|P_\mu\,h_i\|^q_{L^2(\mathbb{R}^{d-1})}\right)^{\frac{1}{2}}$$

and by almost orthogonality of the $P_\mu h_i$, $\left(\sum_\mu \|P_\mu h_i\|_{L^2}^q\right)^{\frac{1}{2}} \lesssim \|h_i\|_{L^2}^{\frac{q}{2}}$. Rescaling therefore yields

$$\sum_\mu \|T_{\nu,\mu}(h_\nu^j)T_{\nu',\mu}(h_{\nu'}^j)\|_{L^{\frac{q}{2}}(\mathbb{R}^d)}^{\frac{q}{2}}$$

$$\lesssim \left(\lambda^{-\frac{2d}{q}} 2^{j\left(\frac{2(d+1)}{q}\right)-(d-1)}\right)^{\frac{q}{2}} \|h_\nu^j\|_{L^2(\mathbb{R}^{d-1})}^{\frac{q}{2}} \|h_{\nu'}^j\|_{L^2(\mathbb{R}^{d-1})}^{\frac{q}{2}}. \qquad (112)$$

Hence Lemma (2.3.9) and Cauchy-Schwarz mean that the left hand side of (103) is dominated by

$$\lambda^{-\frac{2d}{q}} 2^{j\left(\frac{2(d+1)}{q}-(d-1)\right)} \left(\sum_\nu \|h_\nu^j\|_{L^2(\mathbb{R}^{d-1})}^q\right)^{\frac{1}{q}} \left(\sum_\nu \|h_{\nu'}^j\|_{L^2(\mathbb{R}^{d-1})}^q\right)^{\frac{1}{q}}.$$

The desired estimate (103) now follows from the embedding $\ell^2 ,\to \ell^q$.

We are left to show (111). At this stage, $d(\mathrm{supp}(h1), \mathrm{supp}(h2)) \approx 1$, but we want to exhibit the uniformity of the phases and amplitudes. To this end, observe that

$$\phi(x,s,t+\nu) = (x+s\nu)\cdot t + \frac{1}{2} s|t|^2$$
$$+\mathcal{E}(x,s,t+\nu) + s^2|\nu|^2 + x\cdot\nu.$$

The last two terms here can be neglected. A Taylor expansion gives

$$\mathcal{E}(x,s,t+\nu) = \mathcal{E}(x,s,\nu) + \nabla_t\mathcal{E}(x,s,\nu)\cdot t$$
$$+\frac{1}{2}\sum_{|\alpha|=2} \partial_t^\alpha \mathcal{E}(x,s,\nu) t\,\alpha + R_\nu(x,s,\nu).$$

As observed in [84], we may change variables $y = x + s\nu + \nabla_t\mathcal{E}(x,s,\nu)$ and, neglecting terms which can be absorbed into either $T(h_i)$ or h_i, we can write

$$\phi(y,s,t+\nu) = y\cdot t + 1\,2\,s|t|\,2 + E\nu(y,s,t),$$

where $\mathcal{E}_\nu(y,s,t)$ will also satisfy (108) (with y replacing x). Hence

$$\phi_j\left(y,s,t+2^j\nu\right) = 2^j\phi\left(y,s,2^{-j}t+\nu\right)$$
$$= y\cdot t + 2^{-j-1}s|t|^2 + 2^j\mathcal{E}_\nu(y,s,2^{-j}t).$$

Also define $\sigma_s = \mu + s\nu + \nabla_t\mathcal{E}(\mu,s,\nu)$ (recalling that μ is the center of the x-support of $\tilde{a}_{j,\nu,\mu}, \tilde{a}_{j,\nu',\mu}$) and observe that linearizing the change of coordinates gives that if $|x-\mu| \lesssim 2^{-j}$, then $|y-\sigma_s| \lesssim 2^{-j}$. We next set

$$\tilde{\phi}(y,s,t) = 2^{2j}\phi_j\left(2^{-j}y+\sigma_s,s,t\right)$$
$$= y\cdot t + \frac{1}{2}s|t|^2 + 2^{2j}\mathcal{E}_\nu(2^{-j}y+\sigma_s,s,2^{-j}t+\nu)$$

and define

$$\tilde{T}_1(g_1)(y,s) = \int e^{i\lambda 2^{-2j}\tilde{\phi}(y,s,t)}\,\tilde{a}_{j,\nu,\mu}\left(2^{-j}y+\sigma_s,s,t\right)g_1(t)\,dt$$

and $\tilde{T}_2(g_2)$ in the same way except with amplitude $\tilde{a}_{j,\nu',\mu}(2^{-j}y+\sigma_s,s,t)$. The bound (111) will then follow from

$$\left\|\tilde{T}_1(g_1)\tilde{T}_2(g_2)\right\|_{L^{\frac{q}{2}}(\mathbb{R}^d)} \lesssim \left(\lambda 2^{-2j}\right)^{-\frac{2d}{q}} \|g_1\|_{L^2(\mathbb{R}^{d-1})}\|g_2\|\_(L^2(\mathbb{R}^{d-1}). \qquad (113)$$

This estimate in turn follows from one of Lee [84] along with ε-removal lemmas. We state this using his hypotheses.

For i = 1, 2, let T_i be oscillatory integral operators

$$T_i f(z) = \int e^{i\lambda\phi_i(z,\xi)} a_i(z,\xi) f(\xi) d\xi$$

$$z = (x,s) \in \mathbb{R}^{d-1} \times \mathbb{R}, \qquad \xi \in \mathbb{R}^{d-1}$$

with a_i smooth and of sufficiently small compact support. Assume that $\nabla^2_{x\xi}\phi_i$ has rank $d - 1$ and that $\xi \longmapsto \nabla_x\phi_i(x,s,\xi)$ is a diffeomorphism on supp(a_i). Take $q_i(x,s,\xi) = \partial_s\phi_i(x,s,[\nabla_x\phi_i(x,s,\cdot)]^{-1}(\xi))$ so that $\partial_s\phi_i(x,s,\xi) = q_i(x,s,\nabla_x\phi_i(x,s,\xi))$. Suppose further that $\nabla^2_{\xi\xi}q_i(z,\nabla_x\phi_i(z,\xi_i))$ is nonsingular for $(z,\xi_i) \in$ supp(a_i).

Theorem (2.3.11)[69]: For $i = 1, 2, a_i, \phi_i$ satisfy the hypotheses outlined in the preceding discussion. Set $u_i = \nabla_x\phi(z,\xi_i)$ and $\delta(z,\xi_1,\xi_2) = \nabla_\xi q_1(z,u_1) - \nabla_\xi q_2(z,u_2)$. Then if

$$\left|\langle\nabla_x^2\xi\phi(z,\xi_i)\delta(z,\xi_1,\xi_2), |\nabla_x^2\xi\phi(z,\xi i)] - 1 \left[\nabla^2_{\xi\xi}q_i(z,u_i)\right]^{-1} \delta(z,\xi_1,\xi_2)\rangle\right| \geq c > 0 \qquad (114)$$

for i = 1, 2, then for any $\frac{d+2}{d} < p$

$$\|T_1f_1T_2f_2\|_{L^p(\mathbb{R}^d)} \lesssim \lambda^{-\frac{d}{p}} \|f_1\|_{L^2(\mathbb{R}^{d-1})}\|f_2\|_{L^2(\mathbb{R}^{d-1})}\,. \qquad (115)$$

Moreover, if T_1, T_2 are members of a family of operators whose phase and amplitude functions satisfy these hypotheses uniformly and are uniformly bounded in C^∞ with amplitudes supported in a set of uniform size, then the implicit constant in (115) can be taken independent of each operator in the family.

We postpone the proof of this theorem until. It is then verified (see [84]) that if one takes $\xi = t, z = (x,s), \tilde{\phi}(x,s,t) = \phi_1(x,s,t) = \phi_2(x,s,t)$ and a_1, a_2 as the amplitudes in $\tilde{T}_1, \tilde{T}_2$ respectively, then the left hand side of (114) satisfies

$$|\xi_1 - \xi_2| + O(\varepsilon_0) + O(2^{-j}).$$

Therefore since $|t_1 - t_2| \approx 1$, the desired bound follows by taking (115) with λ replaced by $\lambda 2^{-2j}$.

Remark (2.3.12)[69]: As a consequence of Theorem (2.3.11) and the almost orthogonality arguments, we obtain the bound

$$\|Th\|_{L^q(\mathbb{R}^d)} \lesssim \lambda^{-\frac{d}{q}} \|h\|_{L^p(\mathbb{R}^{d-1})} \text{ when } q > \frac{2(d+2)}{d} \text{ and } \frac{d+1}{q} < \frac{d-1}{p'} \qquad (116)$$

for operators T satisfying the hypotheses of Theorem (2.3.7). In other words, we obtain Lee's estimate [84] without the ε-loss. Indeed, the Whitney-type decomposition of $(Th)^2$ in the previous is essentially the same as that in his work, and the estimate over the $(\nu,\nu')\in \Xi_{j'}$ is treated on p. 85 there. Since Hölder's inequality gives $\|h^j_\nu\|_{L^2(\mathbb{R}^{d-1})} \lesssim 2^{-\frac{j(d-1)}{2}\left(\frac{1}{2}-\frac{1}{p}\right)} \|h^j_\nu\|_{L^p(\mathbb{R}^{d-1})}$, (112) and the almost orthogonality arguments above yield the following variation on (103)

$$\left\|\sum_{(\nu,\nu')\in\Xi_j} Th^j_\nu\, Th^j_{\nu'}\right\|_{L^{\frac{q}{2}}(\mathbb{R}^d)} \lesssim 2^{j\frac{2(d+1)}{q} - 2(d-1)\left(1-\frac{1}{p}\right)} \lambda^{-\frac{2d}{q}} \|h\|^2_{L^p(\mathbb{R}^{d-1})}$$

(since it suffices to treat the cases where $q \geq p$). Taking a sum in j then yields (116).

We also note that when $p = \infty$, the estimate in (116) is valid for a larger range of q by a recent work of Bourgain and Guth [71].

Turning to the proof of (115), the estimate

$$\|T_1 f_1 T_2 f_2\|_{L^q(\mathbb{R}^d)} \le C_\alpha \lambda^{-\frac{d}{q}+\alpha} \|f_1\|_{L^2(\mathbb{R}^{d-1})} \|f_2\|_{L^2(\mathbb{R}^{d-1})} \qquad (117)$$

for arbitrary $\alpha > 0$ and $\frac{d+2}{d} \le q$ is due to Lee [84]. Moreover, as observed in [84], the constant C_α is stable under small perturbations in ai and ϕ_i . In particular, if families of amplitudes and phase functions are considered and these functions are uniformly bounded in C^∞ then C_α can be taken uniform within the family of operators. The rest will be dedicated to the following lemma, a generalization of [88] which completes the proof of Theorem (2.3.11).

Lemma (2.3.13)[69]: Suppose T_1, T_2 satisfy the hypotheses of the previous theorem and that they satisfy the estimate (117) for some $1 < q < \frac{d+1}{d-1}$ and some α > 0. Assume further that

$$\frac{1}{p}\left(1 + \frac{8\alpha}{d-1}\right) \le \frac{1}{q} + \frac{4\alpha}{d+1}. \qquad (118)$$

Then the scale-invariant estimate

$$\|T_1 f_1 T_2 f_2\|_{L^r(\mathbb{R}^d)} \lesssim \lambda^{-\frac{d}{r}} \|f_1\|_{L^2(\mathbb{R}^{d-1})} \|f_2\|_{L^2(\mathbb{R}^{d-1})} .$$

is also valid for any $r > p$.

The hypothesis (118) is stronger than the one appearing in [88] (corresponding to $\sigma = \frac{d-1}{2}$ there)

$$\frac{1}{p}\left(1 + \frac{4\alpha}{d-1}\right) < \frac{1}{q} + \frac{2\alpha}{d+1}, \qquad (119)$$

but is sufficient for our purposes. Let f_1, f_2 be unit normalized functions in $L^2(\mathbb{R}^{d-1})$. By a Marcinkiewicz interpolation argument, it suffices to see that

$$|\{x : |T_1 f_1(x) T_2 f_2(x)| > \beta\}| \lesssim \lambda^{-d} \beta^{-p} .$$

Denote the set on the left by E. Observe that since $\|T_1 f_1 T_2 f_2\|_\infty \lesssim 1$, it suffices to assume that $\beta \lesssim 1$. Hence we may assume that $|E| \gtrsim \lambda^{-d}$ throughout since the desired bounds are guaranteed otherwise. Moreover, we know from (117) and Tchebychev's inequality

$$|E| \lesssim \lambda^{-d+q\alpha} \beta^{-q} .$$

Consequently it suffices to assume that $\beta > \lambda^{-\frac{q\alpha}{p-q}}$. This gives the a priori bound

$$|E| \lesssim \lambda^{-d+q\alpha\left(1+\frac{q}{p-q}\right)} . \qquad (120)$$

Since $\beta|E| \lesssim \|1_E T_1 f_1 T_2 f_2\|_{L^1}$, it suffices to show that

$$\|1_E T_1 f_1 T_2 f_2\|_{L^1} \lesssim \lambda^{-\frac{d}{p}} |E|^{\frac{1}{p'}} .$$

We deduce this by showing that for any unit vectors g_1, g_2 in $L^2(\mathbb{R}^{d-1})$

$$\|1_E T_1 g_1 T_2 g_2\|_{L^1} \lesssim \lambda^{-\frac{d}{p}} |E|^{\frac{1}{p'}} ,$$

where it should be stressed that E is dependent on f_1, f_2 above, but that g_1, g_2 are completely independent of these functions. Fix g_2 and let $T = T_{E,g_2}$ be the linear operator $Tg_1 = 1_E T_2 g_2 T_1 g_1$. It suffices to show that

$$\|T * F\|_{L^2(\mathbb{R}^{d-1})} \lesssim \lambda^{-\frac{d}{p}} |E|^{\frac{1}{p'}} \|F\|_{L^\infty(\mathbb{R}^d)} ,$$

since duality then implies that $\|Tg_1\|_{L^1} \lesssim \lambda^{-\frac{d}{p}}|E|^{\frac{1}{p'}}$. We may assume $\|F\|$_(L^∞) $\lesssim$ 1. Set $\tilde{F} := 1_E T_2 g_2 F$. By a duality argument, we square the left hand side of the previous inequality to see that it suffices to show that

$$\left|\langle T_1 T_1^* \tilde{F}, \tilde{F}\rangle\right| \lesssim \lambda^{-\frac{2d}{p}}|E|^{\frac{2}{p'}} = \lambda^{-2d}(\lambda^d|E|)^{\frac{2}{p'}}, \qquad (121)$$

where the inner product on the left is with respect to $L^2(\mathbb{R}^d)$. The integral kernel of $T^1 T_1^*$ is

$$K(w,z) = \int e^{i\lambda(\phi_1(w,\xi)-\phi_1(z,\xi))} a_1(w,\xi)\overline{a_1(z,\xi)}\, d\xi.$$

and satisfies estimates

$$|K(w,z)| \lesssim (1+\lambda|w-z|)^{-\frac{d-1}{2}}.$$

This bound follows from the invertibility of $\nabla^2_{\xi\xi}\partial_s\phi_1$ when $w-z$ is inside a small cone about $(0,\ldots,0,1)$. Otherwise, stronger estimates result from integration by parts and the invertibility of $\nabla_{x\xi}\phi_1$. We now let $R \geq \lambda^{-1}$ be a parameter to be determined shortly and write $K(w,z) = K^R(w,z) + K^R(w,z)$ where $K^R(w,z)$ is smoothly truncated to $|w-z| \geq R$ and $K^R(w,z)$ is supported in $|w-z| \leq 2R$. Observe that by Stein's generalization of Hormander's variable coefficient oscillatory integral theorem (see [55] or [16])

$$\|\tilde{F}\|_{L^1(\mathbb{R}^d)} \lesssim |E|^{\frac{d+3}{2(d+1)}}\|T_2 g_2\|_{L^{\frac{2(d+1)}{d-1}}(\mathbb{R}^d)}\|F\|_{L^\infty(\mathbb{R}^d)} \lesssim |E|^{\frac{d+3}{2(d+1)}}\lambda^{-\frac{d(d-1)}{2(d+1)}}.$$

Thus the contribution of K^R to $\langle T_1 T_1^* \tilde{F}, \tilde{F}\rangle$ is bounded by

$$(\lambda R)^{-\frac{d-1}{2}}|E|^{\frac{d+3}{d+1}}\lambda\hat{}(-\frac{d(d-1)}{d+1} = (\lambda R)^{-\frac{d-1}{2}}(\lambda^d|E|)^{\frac{d+3}{d+1}}\lambda^{-2d}.$$

It is now verified that taking

$$R = \lambda^{-1}(\lambda^d|E|)\hat{}(\frac{2}{d-1}(\frac{d+3}{d+1}-\frac{2}{p'}) \geq \lambda^{-1},$$

ensures that the contribution of K^R is acceptable towards proving (121) (by scaling, this is consistent with the choice of R in [88]). We also remark that another computation reveals that (120) along with the hypothesis (118) ensures that $R \lesssim \lambda^{-\frac{1}{2}}$.

It remains to control the contribution of K^R to (121). Let $\{\psi_k\}_k$ be a partition of unity over $[-\varepsilon_0,\varepsilon_0]^d$ such that supp(ψ_k) is contained in a cube of sidelength $2R$ centered at a point $w_k \in R\mathbb{Z}^d$. Let P_R be the operator determined by the integral kernel K^R and observe that its contribution to the left hand side of (121) is dominated by

$$\sum_{k,k'} \left|\langle P_R(\psi_k\tilde{F}), \psi_{k'}\tilde{F}\rangle\right|. \qquad (122)$$

Given a fixed k, the number of k' for which $\langle P_R(\psi_k\tilde{F}), \psi_{k'}\tilde{F}\rangle \neq 0$ is $O(1)$ and satisfies d(supp(ψ_k),supp($\psi_{k'}$))$\lesssim R$. Hence we will restrict attention to the sum over the diagonal $k = k'$, as a slight adjustment of the argument below will handle the off-diagonal terms.

At this stage it will be convenient to use the semiclassical Fourier transform with $h = \frac{1}{\lambda}$ (cf. [92])

$$F_{\frac{1}{\lambda}}(G)(\eta) = \int_{\mathbb{R}^d} e^{-i\lambda w\cdot\eta}G(w)dw,$$

$$F_{\frac{1}{\lambda}}^{-1}(g)(w) = \frac{\lambda^d}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\lambda w\cdot\eta}G(\eta)\, d\eta. \qquad (123)$$

Since $F_{\frac{1}{\lambda}}$ is related to the usual Fourier transform by $F_{\frac{1}{\lambda}}(G)(\eta) = F(G)(\lambda\eta)$, we have the Plancherel identity $(2\pi)^d \|G\|_{L^2}^2 = \lambda^d \left\|F_{\frac{1}{\lambda}}(G)\right\|_{L^2}^2$ (cf. [92]). We now have

$$(2\pi)^d \langle P_R(\psi_k\tilde{F}), \psi_k\tilde{F}\rangle = \lambda^d \langle \mathcal{F}_{\frac{1}{\lambda}} P_R(\psi_k\tilde{F}), \mathcal{F}_{\frac{1}{\lambda}}(\psi_k\tilde{F})\rangle, \qquad (124)$$

and the right hand side can be written as

$$\frac{1}{(2\pi)^d}\int\int J_k(\eta,\zeta)\mathcal{F}_{\frac{1}{\lambda}}(\psi_k\tilde{F})(\zeta)\overline{\mathcal{F}_{\frac{1}{\lambda}}(\psi_k\tilde{F})(\eta)}\,d\zeta\, d\eta,$$

where

$$J_k(\eta,\zeta) = \lambda^{2d}\int\int\int e^{-i\lambda(\eta\cdot w-\phi_1(w,\xi)+\phi_1(z,\xi)-z\cdot\eta)}\tilde{\psi}_k(z,w)a_1(w,\xi)\overline{a_1(z,\xi)}\,dz\,dw\,d\xi,$$

for some $\tilde{\psi}_k$ supported in $|z-w_k|, |w-w_k| \lesssim R$ satisfying $\left|\partial_{w,z}^{\alpha}\tilde{\psi}_k\right| \lesssim_\alpha R^{-|\alpha|}$. Strictly speaking, one needs to justify the use of Fubini's theorem here, but this can be done by passing to Schwartz class approximations to $\tilde{F}$ and employing crude L^2 continuity bounds for P_R. Therefore over supp($\tilde{\psi}_k$),

$$|\nabla_w\phi_1(w_k,\xi) - \nabla_w\phi_1(w,\xi)| + |\nabla_z\phi_1(w_k,\xi) - \nabla z\phi_1(z,\xi)| \lesssim R \lesssim (\lambda R)^{-1},$$

where we use that $R \lesssim \lambda^{-\frac{1}{2}}$ in the second inequality. Hence integration by parts gives for any N and some uniform cube $Q \subset \mathbb{R}^{d-1}$

$$|J_k(\eta,\zeta)| \lesssim N\,(\lambda R)^{2d}\int_Q (1 + \lambda R|\eta - \nabla_w\phi_1(w_k,\xi)| + \lambda R|\zeta - \nabla_w\phi_1(w_k,\xi)|) - N\, d\xi,$$

as the domain of integration in (w,z) is of volume R^{2d}. Let S_k^1 denote the hypersurface $\{\nabla\phi_1(w_k,\xi) : \xi \in Q\}$. This in turn allows us to deduce that
$|J_k(\eta,\zeta)| \lesssim_N (\lambda R)^{d+1}(1 + \lambda R\, d(\eta, S_k^1) + \lambda R^d(\zeta, S_k^1) + \lambda R|\zeta - \eta|)^{-N}$.
Consequently, by using Cauchy-Schwarz in (124) we have that

$$\left|\langle P_R(\psi_k\tilde{F}), \psi_k\tilde{F}\rangle\right| \lesssim_N \lambda R\int \left(1 + \lambda R\left|d(\eta, S_k^1)\right|\right)^{-N}\left|\mathcal{F}_{\frac{1}{\lambda}}\psi_k\tilde{F}(\eta)\right|^2 d\eta.$$

Now let $S_{k,l}^1$ denote the $(\lambda R)^{-1}2^l$ neighborhood of S_k^1. We have that

$$\sum_k \left|\langle P_R(\psi_k\tilde{F}), \psi_k\tilde{F}\rangle\right| \lesssim_N \sum_k\sum_{l=0}^{\infty} \lambda R 2^{-lN}\left\|\mathcal{F}_{\frac{1}{\lambda}}(\psi_k\tilde{F})\right\|_{L^2(S_{k,l}^1)}^2.$$

We examine the case $l = 0$, the other cases are similar and aided by the factor of 2^{-lN}. Let $\{\tilde{g}_{1,k}\}_k$ be a sequence of functions with supp($\tilde{g}_{1,k}$) $\subset S_{k,0}^1$ for each k and $\sum_k \left\|\tilde{g}_{1,k}\right\|_{L^2(S_{k,0}^2)}^2 = 1$. To finish the proof and show that (124) is dominated by the right side of (121), it suffices to see that

$$\sum_k \langle \psi_k\tilde{F}, \lambda^{-d}\mathcal{F}_{\frac{1}{\lambda}}^{-1}(\tilde{g}_{1,k})\rangle \lesssim \lambda^{-\frac{d}{p}}(\lambda R)^{-\frac{1}{2}}|E|^{\frac{1}{p'}}\|F\|_{L^\infty(\mathbb{R}^d)},$$

which in turn follows from

$$\sum_k\int \psi_k 1_E\lambda^{-d}F_{\frac{1}{\lambda}}^{-1}(\tilde{g}_{1,k})T_2 g_2 \lesssim \lambda^{-\frac{d}{p}}(\lambda R)^{-\frac{1}{2}}|E|^{\frac{1}{p'}}.$$

Now reverse the roles of g_1 and g_2 from the previous step, treating $\{\tilde{g}_{1,k}\}_k$ as a fixed sequence and redefine $T = T_{E},\{g_{1,k}\}$ by $Tg_2 = \left\{\psi_k 1_E \mathcal{F}_{\frac{1}{\lambda}}^{-1}(\tilde{g}_{1,k}) T_2 g_2\right\}_k$ so that it suffices to show

$$\|T\|_{L^2(\mathbb{R}^{d-1})\to l_k^1 L^1(\mathbb{R}^d)} \lesssim \lambda^{d-\frac{d}{p}} (\lambda R)^{-\frac{1}{2}} |E|^{\frac{1}{p'}}.$$

Let $\{F_k\}_k$ be any sequence of functions satisfying $\sup_k \|F_k\|_{L^\infty(\mathbb{R}^d)} \leq 1$ and set $\tilde{F}_k = \psi_k 1_E \lambda^{-d} \mathcal{F}_{\frac{1}{\lambda}}^{-1}(\tilde{g}_{1,k}) F_k$. By duality, the desired bound on T will follow from

$$\left\| T^*\left(\sum_k F_k\right)\right\|_{L^2(\mathbb{R}^{d-1})} \lesssim \lambda^{-\frac{d}{p}} (\lambda R)^{-\frac{1}{2}} |E|^{\frac{1}{p'}}$$

or equivalently

$$\sum_{k,k'} \langle T_2 T_2^* (\tilde{F}_k), \tilde{F}_{k'}\rangle \lesssim \left(\lambda^{-\frac{d}{p}} (\lambda R)^{-\frac{1}{2}} |E|^{\frac{1}{p'}}\right)^2.$$

Observe that

$$\sum_k \|\tilde{F}_k\|_{L^1(\mathbb{R}^d)} \leq \sum_k \|F_k\|_{L^\infty(\mathbb{R}^d)} \int \left|\psi_k 1_E \lambda^{-d} \mathcal{F}_{\frac{1}{\lambda}}^{-1}(\tilde{g}_{1,k})\right| dw$$

$$\leq \left(\int_E \sum_k \psi_k^{\frac{2(d+1)}{d+3}}\right)^{\frac{d+3}{2(d+1)}} \left\|\lambda^{-d} \mathcal{F}_{\frac{1}{\lambda}}^{-1}(\tilde{g}_{1,k})\right\|_{\ell_k^{\frac{2(d+1)}{d-1}} L^{\frac{2(d+1)}{d-1}}(\mathbb{R}^d)}. \quad (125)$$

By finite overlap of the supp(ψ_k), the first factor on the right is seen to be bounded by $|E|^{\frac{d+3}{2(d+1)}}$. Similar to before, an application of the Stein-Tomas theorem for S_k^1 gives

$$\left\|\lambda^{-d} \mathcal{F}_{\frac{1}{\lambda}}^{-1}(\tilde{g}_{1,k})\right\|_{\ell_k^{\frac{2(d+1)}{d-1}} L^{\frac{2(d+1)}{d-1}}(\mathbb{R}^d)} \lesssim \lambda^{-\frac{d(d-1)}{2(d+1)}} (\lambda R)^{-\frac{1}{2}} \|\tilde{g}_{1,k}\|_{\ell_k^2 L^2(S_{k,0}^1)},$$

(cf. the formula for $\mathcal{F}_{\frac{1}{\lambda}}^{-1}$ in (123)) where we use that $\ell_k^2 \hookrightarrow \ell_k^{\frac{2(d+1)}{d-1}}$. Decomposing the integral kernel of $T^2 T_2^*$ as a sum $K^R + K_R$ as before, we may handle the contribution of K^R by using (125) to reason analogously to the argument above. We are thus reduced to handling the contribution of K_R and denote the corresponding operator as P_R. As before, we restrict attention to the diagonal terms, and are thus reduced to seeing that

$$\sum_{\mathrm{k}} \langle \lambda^{\mathrm{d}} \mathcal{F}_{\frac{1}{\lambda}}(P_R(\tilde{\mathrm{F}}_{\mathrm{k}})), \mathcal{F}_{\frac{1}{\lambda}}(\tilde{\mathrm{F}}_{\mathrm{k}})\rangle \lesssim \left(\lambda^{-\frac{\mathrm{d}}{\mathrm{p}}} (\lambda_R)^{-\frac{1}{2}} |\mathrm{E}|^{\frac{1}{\mathrm{p}'}}\right)^2.$$

Since supp($\tilde{F}_k$) $\subset$ supp(ψ_k), this analogously reduces to showing that

$$\sum_k \sum_{l=0}^{\infty} 2^{-lN} \left\|\mathcal{F}_{\frac{1}{\lambda}}(\tilde{F}_k)\right\|_{L^2(S_{k,l}^2)}^2 \lesssim \left(\lambda^{-\frac{d}{p}} (\lambda R)^{-1} |E|^{\frac{1}{p'}}\right)^2,$$

where this time $S_{k,l}^2$ denotes the $(\lambda R)^{-1} 2^l$ neighborhood of the hypersurface $S_k^2 = \{\nabla\_\phi_2(w_{k,\xi}) : \xi \in Q\}$. We again restrict attention to the $l = 0$ case, and let $\{\tilde{g}_{2,k}\}_k$ be a sequence such that supp($\tilde{g}_{2,k}$) $\subset S_{k,0}^2$ and $\sum_k \|\tilde{g}_{2,k}\|_{L^2(S_{k,0}^2)}^2 = 1$. Observe that

$$\sum_k \left|\langle \tilde{F}_k, \lambda^{-d}\mathcal{F}_{\frac{1}{\lambda}}^{-1}(\tilde{g}_{2,k})\rangle\right| \lesssim \sum_k \ \|F_k\|_{L^\infty(\mathbb{R}^d)} \int \ \left|\lambda^{-2d}\psi_k 1_E \mathcal{F}_{\frac{1}{\lambda}}^{-1}(\tilde{g}_{1,k})\mathcal{F}_{\frac{1}{\lambda}}^{-1}(\tilde{g}_{2,k})\right| \, dw,$$

and it suffices to show that the right hand side is bounded by $\lambda^{-\frac{d}{p}} \, (\lambda R)^{-1} \, |E|^{\frac{1}{p'}}$. But each term on the right is bounded by

$$|E|^{\frac{1}{q'}} \left(\int \ \left|\lambda^{-2d}\psi_k \mathcal{F}_{\frac{1}{\lambda}}^{-1}(\tilde{g}_{1,k})\mathcal{F}_{\frac{1}{\lambda}}^{-1}(\tilde{g}_{2,k})\right|^q \, dw\right)^{\frac{1}{q}}.$$

Rescaling $w \mapsto Rw$ and applying the bilinear estimates (117) (or even those in [87]) shows the preceding term is bounded by

$$|E|^{\frac{1}{q'}} R^{\frac{d}{q}} (\lambda R)^{-\frac{d}{q}+\alpha} \ (\lambda R)^{-1} \left\|\tilde{g}_{1,k}\right\|_{L^2(S^1_{k,0})}\left\|\tilde{g}_{2,k}\right\|_{L^2(S^2_{k,0})}.$$

Taking the sum in k and applying Cauchy-Schwarz completes the proof once we observe that

$$|E|^{\frac{1}{q'}} \lambda^{-\frac{d}{q}+\alpha} R^\alpha \ \lesssim \ |E|^{\frac{1}{p'}}\lambda^{-\frac{d}{p}}.$$

Recalling that $R \approx \lambda^{-1}\,(\lambda^d\,|E|)^{\frac{2}{d-1}\left(\frac{d+3}{d+1}-\frac{2}{p'}\right)} = \lambda^{-1}(\lambda^d\,|E|)^{\frac{2}{d-1}\left(\frac{2}{p}-\frac{d-1}{d+1}\right)}$ this inequality is equivalent to

$$|E|^{-\frac{1}{q}} \lambda^{-\frac{d}{q}} (\lambda^d\,|E|)^{\frac{2\alpha}{d-1}\left(\frac{2}{p}-\frac{d-1}{d+1}\right)} \lesssim |E|^{-\frac{1}{p}} \lambda^{-\frac{d}{p}},$$

which in turn can be rearranged as

$$(\lambda^d\,|E|)^{\frac{2\alpha}{d-1}\left(\frac{2}{p}-\frac{d-1}{d+1}\right)} \lesssim (\lambda^d\,|E|)^{\frac{1}{q}-\frac{1}{p}}.$$

But since $\lambda^d\,|E| \gtrsim 1$, this follows once it is observed that (119) is equivalent to

$$\frac{2\alpha}{d-1}\left(\frac{2}{p}-\frac{d-1}{d+1}\right) < \frac{1}{q}-\frac{1}{p}.$$

Even though we only need the weaker condition (119) to conclude the argument, the stronger hypothesis (118) is used above in a significant way to ensure that $\lambda R^2 \le 1$.

By Corollary (2.3.2), (91) follows from (90). Therefore, if , as before, Π denotes the space of unit length geodesics, we must show that if (M,g) has nonpositive sectional curvature, then if $\varepsilon > 0$ is fixed there is a $\Lambda_\varepsilon < \infty$ so that

$$\int_{\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)} |e_\lambda|^2 \, dx \le \varepsilon, \qquad \lambda \ge \Lambda_\varepsilon, \gamma \in \Pi. \tag{126}$$

Here, as before, we are denoting the volume element associated to the metric simply by dx.

We shall first fix $\gamma \in \Pi$ and prove the special case

$$\int_{\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)} |e_\lambda|^2 \, dx \le \varepsilon, \qquad \lambda \ge \Lambda_\varepsilon. \tag{127}$$

After doing this we shall see that we can adapt its proof using the compactness of Π to obtain the estimates (126) which are uniform as γ ranges over this space.

To prove these estimates, we shall want to use a reproducing operator which is similar to the local one, χ_λ, that was used to prove Theorem (2.3.1). This operator was a local one, but to able to take advantage of our curvature assumptions and make use of the method of time-averaging, it will be convenient to use a variant that is in effect scaled in the spectral parameter.

To this end, let us fix a real-valued function $\rho \in \mathcal{S}(\mathbb{R})$ satisfying

$$\rho(0) = 1, \qquad \hat{\rho}(t) = 0 \text{ if } |t| \geq \frac{1}{4} \text{ and } \hat{\rho}(t) = \hat{\rho}(-t). \qquad (128)$$

Then for a given fixed $T \gg 1$ we have

$$\rho\left(T\left(\lambda - \sqrt{\Delta_g}\right)\right) e_\lambda = e_\lambda.$$

As a result, we would have (127) if we could show that there is a uniform constant $A = A(M, g)$ so that whenever $T \gg 1$ is fixed there is a constant $A_T < \infty$ so that for $\lambda \geq 1$ we have

$$\left\| \rho\left(T\left(\lambda - \sqrt{\Delta_g}\right)\right) f \right\|_{L^2\left(T_{\lambda^{-\frac{1}{2}}}(\gamma)\right)} \leq \left(AT^{-\frac{1}{4}} + A_T \lambda^{-\frac{1}{8}}\right) \|f\|_{L^2(M)}. \qquad (129)$$

Since $\rho\left(T\left(\lambda - \sqrt{\Delta_g}\right)\right) : L^2(M) \to L^2(M)$ is self-adjoint, by duality, (129) is equivalent to the following

$$\left\| \rho\left(T\left(\lambda - \sqrt{\Delta_g}\right)\right) h \right\|_{L^2(M)} \leq \left(AT^{-\frac{1}{4}} + A_T \lambda^{-\frac{1}{8}}\right) \|h\|_{L^2(M)},$$

$$\text{if supp } h \subset T_{\lambda^{-\frac{1}{2}}}(\gamma). \qquad (130)$$

If we now let

$$m(\tau) = \left(\rho(\tau)\right)^2, \qquad (131)$$

we can square the right side of (130) to see that whenever h is supported in the tube $T_{\lambda^{-\frac{1}{2}}}(\gamma)$ we have

$$\left\| \rho\left(T\left(\lambda - \sqrt{\Delta_g}\right)\right) h \right\|^2_{L^2(M)} = \langle m(T(\lambda - \sqrt{\Delta_g}))h, \rangle$$

$$\leq \left\| m\left(T\left(\lambda - \sqrt{\Delta_g}\right)\right) h \right\|_{L^2\left(T_{\lambda^{-\frac{1}{2}}}(\gamma)\right)} \|h\|_{L^2\left(T_{\lambda^{-\frac{1}{2}}}(\gamma)\right)}.$$

Whence we deduce that our desired inequalities (127), (129) and (130) would all follow from

$$\left\| m\left(T\left(\lambda - \sqrt{\Delta_g}\right)\right) h \right\|_{L^2\left(T_{\lambda^{-\frac{1}{2}}}(\gamma)\right)} \leq (CT^{-\frac{1}{2}} + C_T \lambda^{-\frac{1}{4}} \|h\|_{L^2(M)},$$

$$\text{if supp } h \subset T_{\lambda^{\frac{1}{2}}}(\gamma), \qquad (132)$$

with C and C_T being equal to A^2 and A_T^2, respectively.

Since, by (128),

$$\hat{m}(\tau) = (2\pi)^{-1} (\hat{\rho} * \hat{\rho})(\tau)$$

is supported in $|\tau| < 1$, we can write

$$m\left(T\left(\lambda - \sqrt{\Delta_g}\right)\right) = \frac{1}{2\pi T} \int_{-T}^{T} \hat{m}\left(\frac{\tau}{T}\right) e^{i\lambda\tau} e^{-i\tau\sqrt{\Delta_g}} \, d\tau.$$

After perhaps multiplying the metric, we may assume that the injectivity radius of the manifold is larger than 10. Let us then fix an even bump function $\beta \in C_0^\infty(\mathbb{R})$ satisfying

$$\beta(\tau) = 1, \quad |\tau| \le \frac{3}{2} \text{ and } \beta(\tau) = 0, \quad |\tau| \ge 2.$$

We then can split

$$m\left(T\left(\lambda - \sqrt{\Delta_g}\right)\right) = R_\lambda + W_\lambda$$

where (suppressing the T-dependence)

$$W_\lambda = \frac{1}{2\pi T}\int_{-T}^{T} (1 - \beta(\tau))\hat{m}\left(\frac{\tau}{T}\right) e^{i\lambda\tau} e^{-i\tau\sqrt{\Delta_g}}\, d\tau,$$

and, if $r_T(\tau)$denotes the inverse Fourier transform of $\tau \to \beta(\tau)\hat{m}\left(\frac{\tau}{T}\right)$,

$$R_\lambda h = T^{-1} r_T\left(\lambda - \sqrt{\Delta_g}\right) h.$$

Clearly, $|r_T(\tau)| \le B$ for some B independent of $T \ge 1$, and therefore,

$$\|R_\lambda f\|_{L^2(M)} \le BT^{-1}\|f\|_{L^2(M)}, T \ge 1.$$

As a result, we would obtain (132) if we could show that (133)

$$\|W_\lambda h\|_{L^2\left(T_{\lambda^{-\frac{1}{2}}}(\gamma)\right)} \le \left(CT^{-\frac{1}{2}} + C_T\lambda^{-\frac{1}{4}}\right)\|h\|_{L^2}, \text{if supp } h \subset T_{\lambda^{-\frac{1}{2}}}(\gamma). \quad (133)$$

By Euler’s formula, if $\widetilde{m}_T$ denotes the inverse Fourier transform of $T^{-1}\left(1 - \beta(\tau)\right)\hat{m}\left(\frac{\tau}{T}\right)$, we have

$$W_\lambda = \frac{1}{2\pi T}\int_{-T}^{T} (1 - \beta(\tau))\hat{m}\left(\frac{\tau}{T}\right) e^{i\lambda\tau} \cos\left(\tau\sqrt{\Delta_g}\right) d\tau + \widetilde{m}_T\left(\lambda + \sqrt{\Delta_g}\right).$$

Since $\widetilde{m}_T(\lambda + \sqrt{\Delta_g})$ has a kernel which, for $T \ge 1$, is $O_T((1+\lambda)^{-N})$ for every $N = 1, 2, \ldots$ as $\widetilde{m}_T \in \mathcal{S}(\mathbb{R})$, we conclude that we would obtain (133) if we could prove that

$$\|S_\lambda h\|_{L^2\left(T_{\lambda^{-\frac{1}{2}}}(\gamma)\right)} \le \left(CT^{-\frac{1}{2}} + C_T\lambda^{-\frac{1}{4}}\right)\|h\|_{L^2}, \text{if supp } h \subset T_{\lambda^{-\frac{1}{2}}}(\gamma), \quad (134)$$

With

$$S_\lambda = \frac{1}{2\pi T}\int_{-T}^{T} (1 - \beta(\tau))\hat{m}\left(\frac{\tau}{T}\right) e^{i\lambda\tau} \cos\left(\tau\sqrt{\Delta_g}\right) d\tau. \quad (135)$$

It is at this point that we shall finally use our hypothesis that (M, g) has nonpositive sectional curvature. By the Cartan-Hadamard theorem (see [25], [76]), for each point $P \in M$, the exponential map at P, $\exp_P$, sending T_PM, the tangent space at P, to M is a universal covering map. It is natural to take P to be the center of our unit-length geodesic segment γ. Thus, with this choice, if we identify $\mathbb{R}^d$ with T_PM, we have that

$$\kappa = \exp_P : \mathbb{R}^d \backsimeq T_PM \to M \quad (136)$$

is a covering map.

If $\tilde{g} = \kappa^* g$ denotes the pullback via κ of the metric g to $\mathbb{R}^d$, it follows that κ is a local isometry. We let $d\tilde{g}(y, z)$ denote the Riemannian distance with respect to $\tilde{g}$ of $, z \in \mathbb{R}^d$. By the Cartan-Hadamard theorem there are no conjugate points for either (M, g) or $(\mathbb{R}^d, \tilde{g})$. Also, the image under κ of any geodesic in $(\mathbb{R}^d, \tilde{g})$ is one in (M, g). If $\{\gamma(t) : t \in \mathbb{R}\}$ denotes the parameterization by arc length of the extension of our geodesic segment

$\gamma \in \Pi$, let $\gamma = \{\tilde{\gamma}(t): t \in \mathbb{R}\}$ denote the lift of this extension, which is the unique geodesic in $(\mathbb{R}^d, \tilde{g})$ that passes through the origin and satisfies $\kappa(\tilde{\gamma}(t)) = \gamma(t), t \in \mathbb{R}$.

Next we recall that the deck transforms are the set of diffeomorphisms $\alpha : \mathbb{R}^d \to \mathbb{R}^d$ for which

$$\kappa \circ \alpha = \kappa.$$

The collection of these maps form a group Γ. Since $\alpha^* \tilde{g} = \tilde{g}, \alpha \in \Gamma$, any deck transform preserves angles and distances. Consequently, the image of any geodesic in $(\mathbb{R}^d, \tilde{g})$ under a deck transform is also a geodesic in this space. As a result, the collection of all $\alpha \in \Gamma$ for which $\alpha(\tilde{\gamma}) = \tilde{\gamma}$ is a subgroup of Γ, which is called the stabilizer subgroup of $\tilde{\gamma}$ that we denote by Stab $(\tilde{\gamma})$. If $\{\gamma(t) : t \in \mathbb{R}\}$ is not a periodic geodesic, i.e., if there is no $t_0 > 0$ so that $\gamma(t + t_0) = \gamma(t), t \in \mathbb{R}$, then Stab$(\tilde{\gamma})$ is just the identity element in Γ. If the extension of $\gamma \in \Pi$ is periodic with minimal period $t_0 > 0$ then Stab$(\tilde{\gamma})$ is a cyclic subgroup which we can write as $\{\alpha_\ell : \ell \in \mathbb{Z}\}$, where α_ℓ is determined by $\alpha_\ell(\tilde{\gamma}(t)) = \tilde{\gamma}(t + \ell t_0), \ell = 0, \pm 1, \pm 2, \ldots$. Thus, restricted to $\tilde{\gamma}, \alpha_\ell$ just involves shifting the geodesic $\tilde{\gamma}(t)$ by ℓ times its period, and Stab$(\tilde{\gamma})$ is generated by α_1. Next, let

$$D_{Dir} = \{\tilde{y} \in \mathbb{R}^d : d_{\tilde{g}}(0, \tilde{y}) < d\tilde{g}(0, \alpha(\tilde{y})), \quad \forall \alpha \in \Gamma, \alpha \neq Identity\}$$

be the Dirichlet domain for $(\mathbb{R}^d, \tilde{g})$. We can then add to D_{Dir} a subset of $\partial D_{Dir} = \overline{D_{Dir}} \backslash \mathrm{Int}\ (D_{Dir})$ to obtain a natural fundamental domain D, which has the property that $\mathbb{R}^d$ is the disjoint union of the $\alpha(D)$ as α ranges over Γ and $\{\tilde{y} \in \mathbb{R}^d : d_{\tilde{g}}(0, \tilde{y}) < 10\} \subset D$ since we are assuming that the injectivity radius of (M, g) is more than ten. Given $x \in M$, let $\tilde{x} \in D$ be the unique point in our fundamental domain for which $\kappa(\tilde{x}) = x$. We then have (see e.g. [28]) that the kernel of cos $(\tau \sqrt{\Delta_g})$ can be written as

$$\cos\left(\tau \sqrt{\Delta_g}\right)(x, y) = \sum_{\alpha \in \Gamma}\left(\cos\ \tau \sqrt{\Delta_{\tilde{g}}}\right)(\tilde{x}, \alpha(\tilde{y})), \tag{137}$$

where $cos(\tau \sqrt{\Delta_{\tilde{g}}}): L^2(\mathbb{R}^d, \tilde{g}) \to L^2(\mathbb{R}^d, \tilde{g})$ is the cosine transform associated with $\tilde{g}$. Thus, if $dV_{\tilde{g}}$ is the associated volume element, we have that when $f \in C_0^\infty(\mathbb{R}^d)$

$$u(\tau, \tilde{x}) = \int_{\mathbb{R}^d}\left(\cos\ \tau \sqrt{\Delta_{\tilde{g}}}\right)(\tilde{x}, \tilde{z}) f(\tilde{z}) dV_{\tilde{g}}(\tilde{z})$$

is the solution of the Cauchy problem (with $D_\tau = -i\partial_t$)

$$(D_\tau^2 - \Delta_{\tilde{g}})u = 0, \quad u|_{\tau=0} = f, \partial_\tau u|_{\tau=0} = 0.$$

Therefore, by Huygens principle,

$$\left(\cos\ \tau \sqrt{\Delta_{\tilde{g}}}\right)(\tilde{x}, \tilde{z}) = 0 \text{ if } d_{\tilde{g}}(\tilde{x}, \tilde{z}) > |\tau|. \tag{138}$$

Also, this kernel is smooth when $d_{\tilde{g}}(\tilde{x}, \tilde{z}) \neq |\tau|, i.e.,$

sing

$$\mathrm{supp}(\cos\ \tau \sqrt{\Delta_{\tilde{g}}}\ (\cdot, \cdot) \subset \{(\tilde{x}, \tilde{z}) \in \mathbb{R}^d \times \mathbb{R}^d : d_{\tilde{g}}(\tilde{x}, \tilde{z}) = |\tau|\}. \tag{139}$$

To proceed, we need a result which follows from the Hadamard parametrix and stationary phase:

This lemma is standard and can essentially be found in [73], [30] or [28]. So let us postpone its proof and focus now on using it to help us to prove (134).

If we combine (135) and (137), we can write the kernel of our operator as

$S_\lambda(x, y)$

$$= \frac{1}{2\pi T} \sum_{\alpha \in \Gamma} \int_{-T}^{T} \left(1 - \beta(\tau)\right) \hat{m}\left(\frac{\tau}{T}\right) e^{i\lambda\tau} \left(\cos \tau \sqrt{\Delta_{\tilde{g}}}\right) \left(\tilde{x}, \alpha(\tilde{y})\right) d\tau, \quad (140)$$

with, as in (137), $\tilde{x}$ and $\tilde{y}$ being the unique points in our fundamental domain having the property that $x = \kappa(\tilde{x})$ and $y = \kappa(\tilde{y})$, respectively. In view of (138) the number of nonzero summands in (140) is finite, but, if the sectional curvatures of (M, g) are strictly negative, the number of such terms grows exponentially with T. Therefore, as in [30] and [73], it is convenient and natural to split the sum into the terms in the stabilizer group for $\tilde{\gamma}$ and everything else. So let us write

$$S_\lambda(x,y) = S_\lambda^{Stab}(x,y) + S_\lambda^{Osc}(x,y), \quad (141)$$

where

$$S_\lambda^{Stab}(x,y) = \frac{1}{2\pi T} \sum_{\alpha \in Stab(\tilde{\gamma})} \int_{-T}^{T} \left(1 - \beta(\tau)\right) \hat{m}\left(\frac{\tau}{T}\right) e^{i\lambda\tau} \left(\cos \tau \sqrt{\Delta_{\tilde{g}}}\right) \left(\tilde{x}, \alpha(\tilde{y})\right) d\tau, \quad (142)$$

and

$$S_\lambda^{Osc}(x,y) = \frac{1}{2\pi T} \sum_{\alpha \in \Gamma \backslash Stab} (\tilde{\gamma}) \int_{-T}^{T} \left(1 - \beta(\tau)\right) \hat{m}\left(\frac{\tau}{T}\right) e^{i\lambda\tau} \left(\cos \tau \sqrt{\Delta_{\hat{g}}}\right) \left(\tilde{x}, \alpha(\tilde{y})\right) d\tau, \quad (143)$$

We shall also call the operator associated with the second term in the right side of (141) S_λ^{Osc} since we shall be able to use oscillatory integral operator bounds to control it. The other piece is very easy to estimate. We claim that

$$\left\|S_\lambda^{Stab} h\right\|_{L^2\left(T_{\lambda^{-\frac{1}{2}}}(\gamma)\right)} \leq \left(CT^{-\frac{1}{2}} + C_T \lambda^{-2}\right) \|h\|_{L^2}, \quad \text{if } \operatorname{supp} h \subset T_{\lambda^{-\frac{1}{2}}}(\gamma). \quad (144)$$

By Young's inequality, this would be a consequence of the following estimate for the kernel

$$\left|S_\lambda^{Stab}(x,y)\right| \leq CT^{-\frac{1}{2}} \lambda^{\frac{d-1}{2}} + C_T \lambda^{-2+\frac{d-1}{2}}, \quad (145)$$

since we may restrict to $(x,y) \in T_{\lambda^{-\frac{1}{2}}}(\gamma) \times T\_(\lambda^{-\frac{1}{2}}(\gamma)$. If our $\gamma \in \Pi$ is not a segment of a periodic geodesic in (M, g) then $Stab(\tilde{\gamma})$ is just the identity element, in which case (145) follows trivially from Lemma (2.3.16). Otherwise, if the geodesic has period $t_0 > 0$ then as noted before $Stab(\tilde{\gamma}) = \{\alpha_\ell\}_{\ell \in \mathbb{Z}}$ where $\alpha_\ell(\tilde{\gamma}(t)) = \tilde{\gamma}(t + \ell t_0)$. Since $d_{\tilde{g}}(\alpha(\tilde{w}), \alpha(\tilde{z}))$ is uniformly bounded as $\tilde{w}$ and $\tilde{z}$ range over D and α over Γ, Lemma (2.3.16) also yields, in this case,

$$\left|S_\lambda^{Stab}(x,y)\right| \leq CT^{-1} \sum_{1 \leq \ell t_0 \leq 2T} \lambda^{\frac{d-1}{2}} (1 + \ell)^{-\frac{d-1}{2}} + C_T \lambda^{\frac{d-1}{2}-2}, \quad (146)$$

using (156) (with $j = 0$) to obtain the first term in the right and (157) to obtain the other term. Since $d \geq 2$, (146) implies (145). For later use, note that, since the period t_0 must be larger than 10, in view of our assumption on the injectivity radius of (M, g), the constants in (144) can be chosen to be independent of $\gamma \in \Pi$. In view of (144), the proof of (134) would be complete if we could show that

$$\|S_\lambda^{Osc}h\|_{L^2\left(T_{\lambda^{-\frac{1}{2}}}(\gamma)\right)} \le C_T\,\lambda^{-\frac{1}{4}}\,\|h\|_{L^2}\,,\text{if supp } h \subset T_{\lambda^{-\frac{1}{2}}}(\gamma). \quad (147)$$

By Lemma (2.3.16),

$$S_\lambda^{Osc}(x,y) = \rho(x,y)\frac{\lambda^{\frac{d-1}{2}}}{T}\sum_{\alpha\in\Gamma\setminus Stab(\tilde{\gamma})} a\pm\left(\lambda,T;\ d_{\tilde{g}}(\tilde{x},\alpha(\tilde{y}))\right) e^{\pm i\lambda d\tilde{g}(\tilde{x},\alpha(\tilde{y}))} + R\lambda(x,y), \quad (148)$$

where, with bounds independent of $\gamma \in \Pi$,

$$|R_\lambda(x,y)| \le C_T\,\lambda^{-2+\frac{d-1}{2}}. \quad (149)$$

By invoking Young's inequality one more time, we find that by (148) and (149) we would have (147) if we could show that

$$\left(\int_{\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)} \left|\int_{\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)} \rho(x,y)a\pm\left(\lambda,T;\ d_{\tilde{g}}(x,\alpha(y))\right)e^{\pm i\lambda d_{\tilde{g}}(x,\alpha(y))}\,h(y)dy\right|^2 dx\right)^{\frac{1}{2}} \le C_\alpha\lambda^{-\frac{d-1}{2}-\frac{1}{4}}\,\|h\|_{L^2}\,,\alpha \in \Gamma\setminus Stab(\tilde{\gamma}). \quad (150)$$

Here, to simplify the notation to follow, as we may, we are identifying $\mathcal{T}_{\lambda^{-\left(\frac{1}{2}\right)}}(\gamma)$ with its preimage in D via κ. So we have lifted our calculation to $\mathbb{R}^d$, and dy denotes the volume element coming from the metric $\tilde{g}$.

To prove this we shall use the following result which is an immediate consequence of Hormander's L^2 -oscillatory integral theorem in [45] (see also [16]).

Lemma (2.3.14)[69]: Let

$$\phi(z;\ x,y) \in C^\infty(\mathbb{R}^m \times \mathbb{R}^{d-1} \times \mathbb{R}^d)$$

be real and

$$a(z;\ x,y) \in C_0^\infty(\mathbb{R}^m \times \mathbb{R}^{d-1} \times \mathbb{R}^d).$$

Assume that the mixed Hessian in the (x,y) variables of ϕ satisfies

$$Rank\left(\frac{\partial^2}{\partial x_j\partial y_k}\phi(z;\ x,y)\right) \equiv d-1 \ on\ supp\ a.$$

Then there is a uniform constant C so that for $\lambda \ge 1$

$$\left(\int_{\mathbb{R}^{d-1}}\left|\int_{\mathbb{R}^d} e\ i\lambda\phi(z;x,y)a(z;\ x,y)f(y)dy\right|^2 dx\right)^{\frac{1}{2}} \le C\lambda^{-\frac{d-1}{2}}\|f\|_{L^2(\mathbb{R}^d)},$$

where all the integrals are taken with respect to Lebesgue measure.

We also require the following simple geometric lemma so that we can use Lemma (2.3.14) to exploit the fact that our tubes are only have width $\lambda^{-\frac{1}{2}}$ to obtain (150).

Lemma (2.3.15)[69]: Suppose that $\alpha \in \Gamma\setminus Stab(\tilde{\gamma})$ and that $x_0, y_0 \in \tilde{\gamma}\cap D$. Then either $\alpha(y_0) \notin \tilde{\gamma}$ or $\alpha^{-1}(x_0) \notin \tilde{\gamma}$ or both.

Proof. Since $\alpha \in \Gamma\setminus Stab(\tilde{\gamma})$, it follows that $\tilde{\gamma}$ and $\alpha(\tilde{\gamma})$ are distinct or intersect at a unique point $P = P(\tilde{\gamma},\alpha)$ (by the Cartan-Hadamard theorem). In the first case both $\alpha(y_0) \notin \tilde{\gamma}$ and $\alpha^{-1}(x_0) \notin \tilde{\gamma}$. We also have the desired conclusion if $P \neq \alpha(y_0)$, for then we must have $\alpha(y_0) \notin \tilde{\gamma}$ as $\alpha(y_0) \in \alpha(\tilde{\gamma})$.

Suppose that we are in the remaining case where $\tilde{\gamma} \cap \alpha(\tilde{\gamma}) = \{\alpha(y_0)\}$. Since $x_0, y_0 \in D$ and $D \cap \alpha(D) = \emptyset$, it follows that $x_0 \neq \alpha(y_0)$. Therefore, as $x_0 \in \tilde{\gamma}$, we must have that

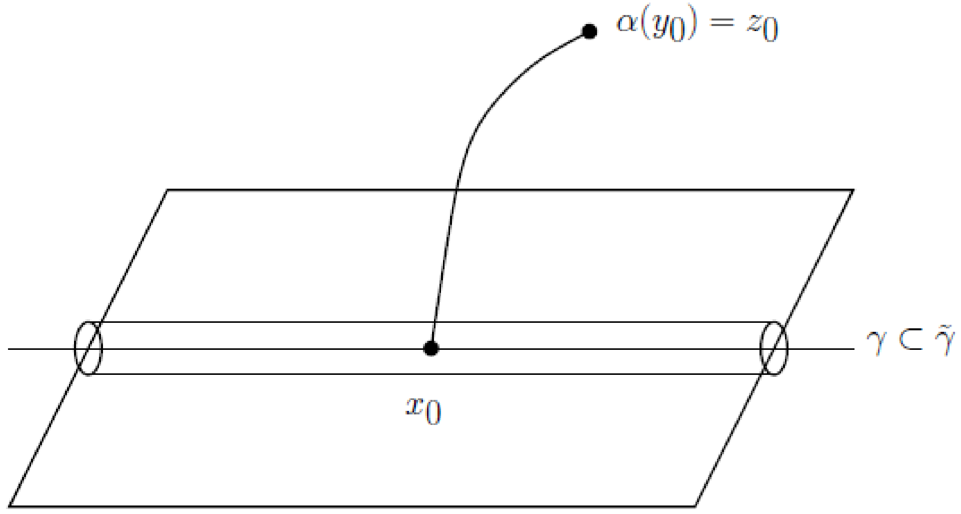


Figure (1)[69]: Transversal intersections

$x_0 \notin \alpha(\tilde{\gamma})$. Thus, in this case, we must have $\alpha^{-1}(x_0) \notin \tilde{\gamma}$, meaning that we have the desired conclusion for this case as well.

To use these two lemmas we require some simple facts about the Riemannian distance function $d_{\tilde{g}}(x, z)$. We recall that $(\mathbb{R}^d, \tilde{g})$ has no conjugate points. Thus, the $d \times d$ Hessian $\frac{\partial^2}{\partial x_j \partial z_k} d_{\tilde{g}}(x, z)$ has rank identically equal to $d - 1$ away from the diagonal.

With this in mind, let us fix points x_0 and y_0 on our unit geodesic segment $\gamma \subset D$. We shall now prove a local version of our remaining estimate (150). By Lemma (2.3.15), for our given $\alpha \in \Gamma \backslash Stab(\tilde{\gamma})$, we know that either $\alpha(y_0) \notin \tilde{\gamma}$ or $\alpha^{-1}(x_0) \notin \tilde{\gamma}$. For the moment, let us assume the former, i.e.,

$$\alpha(y_0) \notin \tilde{\gamma}. \tag{151}$$

We then have that the geodesic passing through $z_0 = \alpha(y_0)$ and $x_0 \in \gamma \subset \tilde{\gamma}$ intersects γ transversally. We may therefore choose geodesic normal coordinates in $\mathbb{R}^d$ vanishing at x_0 so that $\tilde{\gamma}$ is the first coordinate axis, i.e.

$$\tilde{\gamma} = \{(t, 0, \ldots, 0) : t \in \mathbb{R}\},$$

and, moreover, if $x_0 = (x_1, \ldots, x_{d-1})$ are the first $d - 1$ coordinates of x in this coordinate system then

$$Rank\left(\frac{\partial^2}{\partial x'_j \partial z_k} d_{\tilde{g}}\left((x', 0), z\right)\right) = d - 1 \text{ at } x' = 0 \text{ and } z = z_0 = \alpha(y_0). \tag{152}$$

By Gauss' lemma this will be the case if the geodesic through the origin and z_0 intersects the hyperplane $\{x : x_n = 0\}$ transversally as shown in Figure (1) below, which can be achieved after performing a rotation fixing the first coordinate axis if needed. Since $\alpha : \mathbb{R}^d \to \mathbb{R}^d$ is a diffeomorphism it follows that in given our fixed points $x_0, y_0 \in \gamma \subset \tilde{\gamma} \cap D$, we can find $\delta > 0$ so that, in the above coordinates,

$$Rank\left(\frac{\partial^2}{\partial x'_j \partial y_k} d_{\tilde{g}}\left((x', x_n), \alpha(y)\right)\right) = d - 1,$$

$$\text{if } x \in B_\delta(x_0) \text{ and } y_0 \in B_\delta(y_0),$$

with $B_\delta(w)$ denoting the geodesic ball of radius δ about $x \in \mathbb{R}^d$.

Next, it follows from (148) and Lemma (2.3.14) that, in our coordinates, for each fixed value of x_n, we have

$$\left(\int_{\left\{x':(x',x_n)\in\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)\cap B_\delta(x_0)\right\}}\left|\int_{\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)\cap B_\delta(y_0)}\rho(x,\alpha(y))\,a_\pm\left(\lambda,T;\,d_{\tilde{g}}(x,\alpha(y))\right)\right.\right.$$

$$\left.\left.\times\ e^{\pm i\lambda d_{\tilde{g}}}\,(x,\alpha(y))\,h(y)dy\right|dx'\right)^{\frac{1}{2}} \le C_\alpha\lambda^{-\frac{d-1}{2}}\left(\int |h(y)|^2\,dy\right)^{\frac{1}{2}}.$$

Since $|x_n| \lesssim \lambda^{-\frac{1}{2}}$ in $\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)$, from this, we deduce that, under our assumption (151), we have that

$$\left(\int_{\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)}\cap B_\delta(x_0)\left|\int_{\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)\cap B_\delta(y_0)}\rho(x,\alpha(y))a_\pm\left(\lambda,T;\,d_{\tilde{g}}(x,\alpha(y))\right)\right.\right.$$

$$\left.\left.\times\ e^{\pm i\lambda d_{\tilde{g}}(x,\alpha(y))}\,h(y)dy\right|dx\right)^{\frac{1}{2}}$$

$$\le C_\alpha\lambda^{-\frac{d-1}{2}}\lambda^{-\frac{1}{4}}\left(\int |h(y)|^2\,dy\right)^{\frac{1}{2}}. \qquad (153)$$

Lemma (2.3.15) tells us that if we do not have (151) then

$$\alpha^{-1}(x_0)\notin\gamma. \qquad (154)$$

We claim that for our fixed points $x_0, y_0 \in \gamma$ we can find $\delta > 0$ so that (153) remains valid for this case as well. To do this, we just use the fact that our $\alpha \in \Gamma\backslash Stab(\tilde{\gamma})$ is an isometry and therefore

$$d_{\tilde{g}}(x,\alpha(y)) = d_{\tilde{g}}(\alpha^{-1}(x),y).$$

Consequently, since $\alpha^{-1} \in \Gamma\backslash Stab(\tilde{\gamma})$, we obtain (153) under the assumption (154) since it is essentially just the dual version of the case we just handled, and so follows from the above argument after taking adjoints.

Since we have shown that (153) holds either under assumption (151) or (154), Lemma (2.3.14) tells us that given any two fixed points $x_0, y_0 \in \gamma$ we can find a $\delta > 0$ so that (153) is valid. By the compactness of our unit geodesic segment γ, this implies (150), which completes the proof of the estimate (127) for our fixed $\gamma \in \Pi$.

It is straightforward to see how to obtain the stronger estimate (126), which involves uniform bounds over Π, by using the proof of (127). We use the fact that if $T \gg 1$ is fixed and if $\gamma \in \Pi$ is fixed then there is a neighborhood $\mathcal{N}(\gamma)$ of γ in Π so that if $\alpha \in \Gamma\backslash Stab(\tilde{\gamma})$ and the geodesic distance between our fundamental domain D and its image $\alpha(D)$ is $\le 2T$, then we also have that $\alpha \,/\!\in \Gamma\backslash Stab\,(\tilde{\gamma}_0)$ for any $\gamma_0 \in \mathcal{N}(\gamma)$. This follows from the fact that there are only finitely many $\alpha \in \Gamma$ for which the distance between D and $\alpha(D)$ is $\le 2T$, and if α is not a stabilizer for ˜γ then it is also not a stabilizer for nearby geodesics.

Because of this and the uniform dependence on the smooth parameter z in Lemma (2.3.14), if we define $S_\lambda^{Osc,\gamma}$ to be the operator whose kernel is given by (143), we have the uniform bounds

$$\left\|S_\lambda^{Osc,\gamma}\, h\right\|_{L^2\left(T_{\lambda^{-\frac{1}{2}}}(\gamma)\right)} \le C_T\, \lambda^{-\frac{1}{4}}\, \|h\|_{L^2}\,,$$

$$\text{if } \gamma_0 \in \mathcal{N}(\gamma) \text{ and supp } h \subset T\_(\lambda^{-\frac{1}{2}}(\gamma_0).$$

If then $S_\lambda^{Stab,\gamma} = S_\lambda - S_\lambda^{Osc,\gamma}$ is then defined using γ, then the proof of (144) clearly also yields the following variant

$$\left\|S_\lambda^{Stab,\gamma}\, h\right\|_{L^2\left(\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)\right)} \le CT^{-\frac{1}{2}} + C_T\lambda^{-2}\, \|h\|_{L^2}\,,$$

$$\text{if } \gamma_0 \in \mathcal{N}(\gamma) \text{ and supp } h \subset \mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma_0).$$

Together these two estimates imply the analog of (126) where, instead of having the geodesic segments range over Π, we have them range over $\mathcal{N}(\gamma)$ and $\Lambda_\varepsilon = \Lambda_\varepsilon(\mathcal{N}(\gamma))$ depends on $\mathcal{N}(\gamma)$. By the compactness of Π, this in turn yields (126).

We need to prove Lemma (2.3.16).

Lemma (2.3.16)[69]: Let m be as in (128) and (131), and, as above, assume that $\beta \in C_0^\infty(\mathbb{R})$ satisfies $\beta(\tau) = 1, |\tau| < \frac{3}{2}$ and $\beta(\tau) = 0, |\tau| \ge 2$. Then if $\lambda, T \ge 1$ and $\tilde{x}, \tilde{y} \in \mathbb{R}^d$, we have

$$\begin{aligned}\frac{1}{2\pi T}\int_{-T}^{T} &\left(1 - \beta(\tau)\right)\hat{m}\left(\frac{\tau}{T}\right) e^{i\lambda\tau}\left(\cos\tau\sqrt{\Delta_{\tilde{g}}}\right)(\tilde{x},\tilde{y})d\tau \\ &= \rho(x,y)\frac{\lambda^{\frac{d-1}{2}}}{T}\sum_{\pm} a\pm\left(\lambda, T;\, d_{\tilde{g}}(\tilde{x},\tilde{y})\right) e^{\pm i\lambda d\tilde{g}}(\tilde{x},\tilde{y}) \\ &\quad + R(\lambda, T, \tilde{x}, \tilde{y}),\end{aligned} \tag{155}$$

where $\rho \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d) \cap C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$,

$$a_\pm(\lambda, T;\, r) = 0, r \notin [1, T], \qquad \partial_r^j\, a_\pm(\lambda, T;\, r) \le C_j\, r^{-\frac{d-1}{2}-j}\,, \tag{156}$$

with constants C_j independent of $T, \lambda \ge 1$, and

$$R(\lambda, T;\, \tilde{x}, \tilde{y}) = 0 \text{ if } d_{\tilde{g}}(\tilde{x},\tilde{y}) > T, \text{and}$$

$$|R(\lambda, T;\, \tilde{x}, \tilde{y})| \le C_{T,K}\lambda^{-2-\frac{d-1}{2}}, \text{if } \tilde{x}, \tilde{y} \in K \Subset \mathbb{R}^d. \tag{157}$$

Proof. Since $\hat{m}(\tau) = 0$ when $|\tau| > 1/2$ it follows that the left side of (155),

$$\frac{1}{2\pi T}\int_{-T}^{T}\left(1 - \beta(\tau)\right)\hat{m}\left(\frac{\tau}{T}\right) e^{i\lambda\tau}\left(\cos\ \tau\sqrt{\Delta_{\hat{g}}}\right)\ (\tilde{x},\tilde{y})\, d\tau, \tag{158}$$

vanishes when $d_{\tilde{g}}(\tilde{x},\tilde{y}) > T$. Since $\beta(\tau) = 1$ for $|\tau| \le 3/2$, by (139), it is $O_{N,T}\left((1+\lambda)^{-N}\right)$ for any $N = 1, 2, 3, \ldots$ if $d_{\tilde{g}}(\tilde{x},\tilde{y}) \le 1$. Therefore, we need only to prove the assertions in Lemma (2.3.16) when $1 \le d\_\tilde{g}(\tilde{x},\tilde{y}) \le T$.

To prove this, we shall use the Hadamard parametrix (see e.g. [64] and [28]). Since $(\mathbb{R}^d, \tilde{g})$ has nonpositive curvature, for $0 \le \tau \le T$ we can write

$$\left(\cos\tau\sqrt{\Delta_{\tilde{g}}}\right)(\tilde{x},\tilde{y}) = \rho(\tilde{x},\tilde{y})(2\pi)^{-d}\int_{\mathbb{R}}^{d} e^{id_{\tilde{g}}(\tilde{x},\tilde{y})\xi_1}\cos\ \ \tau|\xi|d\xi$$

$$= \sum_{\pm} \int_{\mathbb{R}^d} e^{i d_{\tilde g(\tilde x,\tilde y)}\xi_1} \alpha_\pm(\tau,\tilde x,\tilde y,|\xi|) e^{\pm i\tau|\xi|}\, d\xi \; + \; R(\tau,\tilde x,\tilde y), \quad (159)$$

where the leading Hadamard coefficient, ρ, is smooth and uniformly bounded (by the curvature hypothesis), and if $m \in N$ is fixed we can have $\partial_\tau^j R(\tau,\tilde x,\tilde y) \in L^\infty\ loc, 0 \le j \le m$, and also

$$\left|\partial^\beta_{\tau,\tilde x,\tilde y}\, \partial_r^j \alpha_\pm(\tau,\tilde x,\tilde y,r)\right| \le C_{T,K,\beta,j}\, r^{-2-j},$$

$$\text{if } r \ge 1, 0 \le \tau \le T, j = 0,1,2,\ldots, \text{and } \tilde x,\tilde y \in K \Subset \mathbb{R}^d. \quad (160)$$

We also recall (see e.g. [16]) that we can write the Fourier transform of Lebesgue measure on the sphere in $\mathbb{R}^d$ as

$$\int_{S^{d-1}} e^{ix\cdot\omega}\, d\sigma(\omega) = |x|^{-\frac{d-1}{2}} \left(c_+(|x|)e^{i|x|} + c_-(|x|)e^{-i|x|}\right), (161)$$

where for each $j = 0,1,2,\ldots$, we have

$$\left|\partial_r^j c_+(r)\right| + \left|\partial_r^j c_-(r)\right| \le C_j\, r^{-j}, r \ge 1. \quad (162)$$

If in (158) we replace $(\cos \ \tau\sqrt{\Delta_{\tilde g}})(\tilde x,\tilde y)$ by the first term in (159), the resulting expression equals $\rho(\tilde x,\tilde y)$ times a fixed multiple of

$$\frac{1}{2\pi T}\int_{-T}^{T}\int_{\mathbb{R}^d} \hat m\left(\frac{\tau}{T}\right) e^{i\lambda\tau} \cos\ (\tau|\xi|) e^{i d_{\tilde g}(\tilde x,\tilde y)\xi_1}\, d\xi d\tau$$

$$= \sum_\pm \frac{1}{2\pi T}\int_{-T}^{T}\int_0^\infty \hat m\left(\frac{\tau}{T}\right) e^{i\lambda\tau}\cos\ (\tau\, r)$$

$$e^{\pm ir d_{\tilde g}}(\tilde x,\tilde y)\, c_\pm\left(d_{\tilde g}(\tilde x,\tilde y) r\right) \frac{r^{\frac{d-1}{2}}}{\left(d_{\tilde g(\tilde x,\tilde y)}\right)^{\frac{d-1}{2}}}\, dr d\tau \quad (163)$$

minus

$$\sum_\pm \frac{1}{2\pi T}\int_{-2}^{2}\int_0^\infty \beta(\tau)\hat m\left(\frac{\tau}{T}\right) e^{i\lambda\tau}\cos\ (\tau\, r) e\hat{}(\pm ir dg\tilde{}(\tilde{}x,y\tilde{})$$

$$c_\pm(dg\tilde{}(\tilde{}x,y\tilde{})r) \frac{r^{\frac{d-1}{2}}}{\left(dg\tilde{}(\tilde{}x,y\tilde{})\right)^{\frac{d-1}{2}}} dr d\tau. \quad (164)$$

If we replace $\cos\ (\tau\, r)$ by $e^{-i\tau r}$ in the right side of (163), the resulting expression equals the sum over $\pm$ of

$$\int_0^\infty m\left(T(\lambda - r)\right) c_\pm\left(d_{\tilde g}(\tilde x,\tilde y) r\right) e^{\pm ir d_{\tilde g}(\tilde x,\tilde y)} \frac{r^{\frac{d-1}{2}}}{\left(d_{\tilde g}(\tilde x,\tilde y)\right)^{\frac{d-1}{2}}}\, dr$$

$$= \frac{\lambda^{\frac{d-1}{2}}}{T} e^{\pm i\lambda d_{\tilde g}(\tilde x,\tilde y)}\, a_\pm\left(\lambda, T;\ d_{\tilde g}(\tilde x,\tilde y)\right),$$

where, using the fact that $m \in \mathcal{S}(\mathbb{R})$ and (162), $a_\pm$ satisfies (156). If in (163) we replace $\cos\ (\tau r)$ by $e^{i\tau r}$, then this argument also implies that the resulting expression is $O_{N,T}\left((1 + \lambda)^{-N}\right)$ for any $= 1,2,3,\ldots$. Thus, modulo such an error ρ times the terms in (163) can be written as the first term in the right side of (155) with (156) being valid. Since this argument shows that the same is the case for (164), we conclude that the first term in the right side of (159), up to $O_{N,T}\ ((1\ +\ \lambda)^{-N}\ )$ errors, gives us the first term in the right side of (155).

This argument and (160) also shows that if in (155) we replace $(\cos\quad \tau\sqrt{\Delta_g})(\tilde{x},\tilde{y})$ by the second term in the right side of (159), then we get a term obeying the bounds in (157). Since, as noted we can take the remainder term in (159) to satisfy for a given $m\in\mathbb{N}, \partial_\tau^j R(\tau,\tilde{x},\tilde{y})\ \in\ L_{loc}^\infty, j\ =\ 0,1,\ldots,m$, we also see that if we choose m large enough, the same is true for it.

Corollary (2.3.17)[246]: (See [69]) The following are equivalent for any subsequence of L^2 -normalized eigenfunctions $\left\{e^n_{(1+\epsilon)_{j_k}}\right\}_{k=1}^\infty$:

$$\limsup_{k\to\infty}\ \sup_{\gamma\in\Pi}\ \int_{\mathcal{T}_{(1+\epsilon)_{jk}^{-\frac{1}{2}}(\gamma)}} \sum_n \left|e^n_{(1+\epsilon)_{j_k}}(z)\right|^2\ dz\ =\ 0 \tag{165}$$

$$\limsup_{k\to\infty}\ (1+\epsilon)_{jk}^{-\left(\frac{\epsilon}{4}\right)}\sum_n \left\|e^n_{(1+\epsilon)_j}\right\|_{L^{2+\epsilon}(M)}\ =\ 0 \text{ for any } 0<\epsilon<\frac{4}{2+\epsilon}. \tag{166}$$

Proof. Given Theorem (2.3.1), it is routine to verify that (165) implies (166) for $e>0$. The remaining values of $2+\epsilon$ then follow from interpolation. For the converse, observe that Hölder's inequality gives

$$\int_{T_{(1+\epsilon)^{-\frac{1}{2}}(\gamma)}} \sum_n |e^n_{1+\epsilon}(z)|^2\,dz\ \lesssim\ (1+\epsilon)^{-\left(\frac{2+\epsilon}{2}\right)\left(\frac{4+3\epsilon+\epsilon^2}{10+5\epsilon+\epsilon^2}\right)}\sum_n\ \|e^n_{1+\epsilon}\|^2_{L^{\frac{10+5\epsilon+\epsilon^2}{3+\epsilon}}(M)},$$

and the implication follows.

Corollary (2.3.18)[246]: Suppose $0\le\epsilon\ \le 1$ and that T is as in Theorem (2.3.7). There exist amplitudes $a_{\nu,\mu}, a_{\nu',\mu}$ both with x-support contained in $\text{supp}(A_\mu)$ and satisfying derivative bounds of the form

$$\left|\partial_x^{1+\epsilon}\ a_{\nu,\mu}(x,s,t)\right|\lesssim_{1+\epsilon}\ 2^{j|1+\epsilon|} \tag{167}$$

such that if $T_{\nu,\mu}$ is the oscillatory integral operator with phase ϕ and amplitude $a_{\nu,\mu}$

$$\sum_n\ T_{\nu,\mu}(h^n)(x,s)\ =\int_{\mathbb{R}^{2+\epsilon}}\sum_n\ e^{i(1+\epsilon)\phi(x,s,t)n}a_{\nu,\mu}(x,s,t)h^n(t)\,dt$$

then

$$\left\|A_\mu\sum_{\nu,\nu'\in\Xi_j}\sum_n\ T\left(h_\nu^{jn}\right)T\left(g_{\nu'}^{jn}\right)\right\|^{1+\epsilon}_{L^{1+\epsilon}(\mathbb{R}^{3+\epsilon})}\ \lesssim\ \sum_{\nu,\nu'\in\Xi_j}\sum_n\ \left\|T_{\nu,\mu}\left(h_\nu^{jn}\right)T_{\nu',\mu}\left(g_{\nu'}^{jn}\right)\right\|^{1+\epsilon}_{L^{1+\epsilon}(\mathbb{R}^{3+\epsilon})}$$

Proof. For a given s, consider the slice of $T(h^n)$ at $\sum_n\ s\,T^s\ (h^n)(x)=\sum_n\ T(h^n)x,r|_{r=s}$. It suffices to show that

$$\left\|A_\mu\sum_{\nu,\nu'\in\Xi_j}\sum_n\ T^s\left(h_\nu^{jn}\right)T^s\left(g_{\nu'}^{jn}\right)\right\|^{1+\epsilon}_{L^{1+\epsilon}(\mathbb{R}^{2+\epsilon})}$$
$$\lesssim\sum_{\nu,\nu'\in\Xi_j}\ \left\|\ T^s_{\nu,\mu}\left(h_\nu^{jn}\right)T^s_{\nu',\mu}\left(g_{\nu'}^{jn}\right)\right\|^{1+\epsilon}_{L^{1+\epsilon}(\mathbb{R}^{2+\epsilon})},$$

and hence we shall assume that s is fixed throughout the proof. Now let $\Phi(x,t,t')\ =\ \phi(x,s,t)\ +\ \phi(x,s,t')$ and observe that $A_\mu T^s\left(h_\nu^{jn}\right)T^s\left(g_{\nu'}^{jn}\right)$ can be written as

$$A_{\mu}(x)\int\sum_{n} e^{i(1+\epsilon)\Phi(x,t,t')n}\, a(x,s,t)a(x,s,t')h_{\nu}^{jn}(t)g_{\nu'}^{jn}(t')\,dtdt',$$

Treating $D_x = -i\nabla_x$ as a vector-valued differential operator we want to write

$$\left(1+\left((1+\epsilon)^{-1}2^j\right)^2|(1+\epsilon)\nabla_x\Phi(\mu,\nu,\nu')-D_x|^2\right)^N\sum_n A_\mu T^s\left(h_\nu^{jn}\right)T^s\left(g_{\nu'}^{jn}\right)$$

$$=\sum_n T_{\nu,\mu}^s\left(h_\nu^{jn}\right)T_{\nu'},_\mu^s\left(g_{\nu'}^{jn}\right) \tag{168}$$

for some N large based on $(3+\epsilon)$ and each operator on the right satisfies (167). It thus suffices to see that this can be done for any monomial of

$$(1+\epsilon)^{-1}\,2^j\,((1+\epsilon)\nabla_x\Phi(\mu,\nu,\nu')-D_x),$$

which in turn will follow by induction. To this end, observe that products of functions satsifying (167) satisfy the same condition as do weighted derivatives $(c\partial_x)^{1+\epsilon}$ of such functions provided $|c|\le 2^{-j}$. On $\operatorname{supp}(A_\mu)\times Q_\nu^j\times Q_{\nu'}^j$, we have that

$$(1+\epsilon)^{-1}\,2^j\left((1+\epsilon)\partial_k\Phi(\mu,\nu,\nu')-(1+\epsilon)\partial_k\Phi(x,t,t')\right)$$

satisfies (167). Moreover, since $(1+\epsilon)^{-1}2^j\le 2^{-j}$, it is seen that for any $1+\epsilon$, $\left((1+\epsilon)^{-1}\,2^j\partial_x\right)^{1+\epsilon}A_\mu^{\frac{1}{2}}$ satisfies (167). The claim then follows.

It now suffices to see that if $P_{\nu,\nu'}$ is the Fourier multiplier

$$P_{\nu,\nu'}(D_x)=\left(1+\left((1+\epsilon)^{-1}\,2^j\right)^2|(1+\epsilon)\nabla_x\Phi(\mu,\nu,\nu')-D_x|^2\right)^{-N},$$

then for any sequence of $\{f_{\nu,\nu'}^n\}$ of Schwartz class functions defined on $\mathbb{R}^{2+\epsilon}$,

$$\left\|\sum_{\nu,\nu'\in\Xi_j}\sum_n P_{\nu,\nu'}f_{\nu,\nu'}^n\right\|_{L^2(\mathbb{R}^{2+\epsilon})}^2\lesssim\sum_{\nu,\nu'\in\Xi_j}\sum_n\|f_{\nu,\nu'}^n\|_{L^2(\mathbb{R}^{2+\epsilon})}^2,$$

$$\left\|\sum_{\nu,\nu'\in\Xi_j}\sum_n P_{\nu,\nu'}f_{\nu,\nu'}^n\right\|_{L^1(\mathbb{R}^{2+\epsilon})}\lesssim\sum_{\nu,\nu'\in\Xi_j}\sum_n\|f_{\nu,\nu'}^n\|_{L^1(\mathbb{R}^{2+\epsilon})}.$$

The latter follows from the triangle inequality and Young's inequality for convolutions, so it suffices to treat the former. But $\nabla x\Phi(\mu,\nu,\nu')=2\nabla_x\phi(\mu,s,\nu)+O\left(2^{-j}\right)$, so the invertibility of $\nabla^2\phi_{x,t}$ gives

$$2^j\,|\nabla_x\Phi(\mu,\nu,\nu')-\nabla x\Phi(\mu,\tilde{\nu},\tilde{\nu}')|\approx 2^j\,|\nu-\tilde{\nu}|.$$

Recall that for each ν, the number of ν' such that $(\nu,\nu')\in\Xi_j$ is $O(1)$. Therefore since the ν range over a regularly spaced 2^{-j} lattice, the desired bound follows from a routine computation using Plancherel's identity.

Corollary (2.3.19)[246]: (See [69]) There exists an amplitude $\tilde{a}_{j,\nu,\mu}(x,s,t)$ satisfying bounds of the form (167) such that

$$\left(1+2^{2j}\left|(1+\epsilon)^{-1}\,2^jD_t-\mu\right|^2\right)^N\sum_n e^{i(1+\epsilon)2^{-j}\phi_j(x,s,t)n}a_{j,\nu,\mu}(x,s,t)$$

$$=\sum_n e^{i(1+\epsilon)2^{-j}\phi_j(x,s,t)n}\tilde{a}_{j,\nu,\mu}(x,s,t).$$

Proof. Observe that

$$\sum_n e^{-i(1+\epsilon)n2^{-j}\phi_j} 2^j \left((1+\epsilon)^{-1} 2^j D_{t_k} - \mu_k\right) e^{i(1+\epsilon)n2^{-j}\phi_j} a_{j,\nu,\mu}$$
$$= 2^j \left(\partial_{t_k} \phi_j - \mu_k\right) a_{j,\nu,\mu} + (1+\epsilon)^{-1} 2^{2j} D_{t_k} a_{j,\nu,\mu}.$$
Since $(1+\epsilon)^{-1} 2^{2j} \leq 1$, second term satisfies (167). Moreover, by (3.1), (3.2)
$$\left(\partial_{t_k} \phi_j (x,s,t) - \mu_k\right) = x_k - \mu_k + O\left(2^{-j}\right)$$
and thus by the support properties of $a_{j,\nu,\mu}$ the first term satisfies (167) as well. The lemma then follows by an inductive argument akin to that in Corollary (2.3.18).

Corollary (2.3.20)[246]: Suppose that $\alpha \in \Gamma \backslash Stab(\tilde{\gamma})$ and that $x_0, y_0 \in \tilde{\gamma} \cap D$. Then either $\alpha(y_0) \notin \tilde{\gamma}$ or $\alpha^{-1}(x_0) \notin \tilde{\gamma}$ or both.

Proof. (See [69]) Since $\alpha \in \Gamma \backslash Stab(\tilde{\gamma})$, it follows that $\tilde{\gamma}$ and $\alpha(\tilde{\gamma})$ are distinct or intersect at a unique point $P = P(\tilde{\gamma}, \alpha)$ (by the Cartan-Hadamard theorem). In the first case both $\alpha(y_0) \notin \tilde{\gamma}$ and $\alpha^{-1}(x_0) \notin \tilde{\gamma}$. We also have the desired conclusion if $P \neq \alpha(y_0)$, for then we must have $\alpha(y_0) \notin \tilde{\gamma}$ as $\alpha(y_0) \in \alpha(\tilde{\gamma})$.

Suppose that we are in the remaining case where $\tilde{\gamma} \cap \alpha(\tilde{\gamma}) = \{\alpha(y_0)\}$. Since $x_0, y_0 \in D$ and $D \cap \alpha(D) = \emptyset$, it follows that $x_0 \neq \alpha(y_0)$. Therefore, as $x_0 \in \tilde{\gamma}$, we must have that

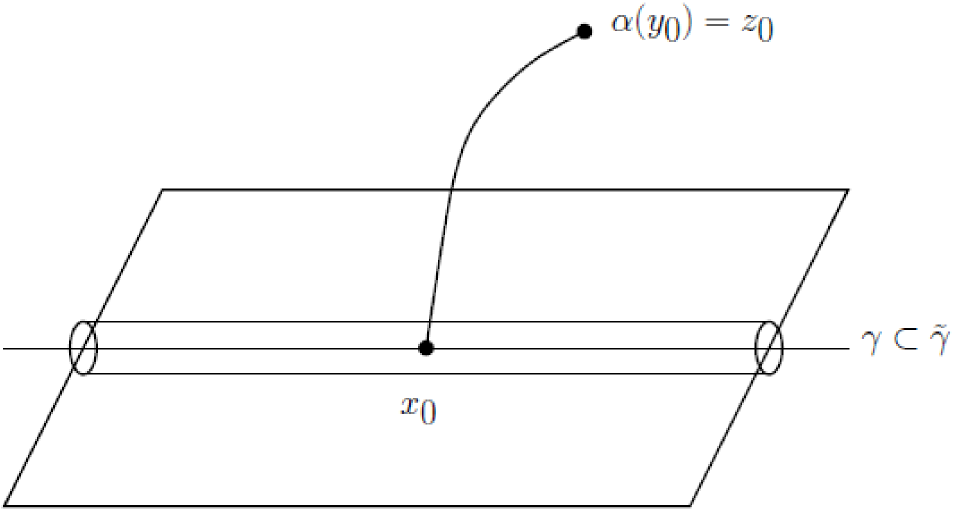


Figure (2)[246]: Transversal intersections

$x_0 \notin \alpha(\tilde{\gamma})$. Thus, in this case, we must have $\alpha^{-1}(x_0) \notin \tilde{\gamma}$, meaning that we have the desired conclusion for this case as well.

Corollary (2.3.21)[246]: Let m be as in (128) and (131), and, as above, assume that $\beta \in C_0^\infty(\mathbb{R})$ satisfies $\beta(\tau) = 1, |\tau| < \frac{3}{2}$ and $\beta(\tau) = 0, |\tau| \geq 2$. Then if $1+\epsilon, T \geq 1$ and $\tilde{x}, \tilde{y} \in \mathbb{R}^{3+\epsilon}$, we have
$$\frac{1}{2\pi T} \int_{-T}^{T} \sum_n \left(1 - \beta(\tau)\right) \hat{m}\left(\frac{\tau}{T}\right) e^{i(1+\epsilon)\tau n} \left(\cos \tau \sqrt{\Delta_{\tilde{g}^n}}\right)(\tilde{x}, \tilde{y}) d\tau$$
$$= \rho(x,y) \frac{(1+\epsilon)^{\frac{2+\epsilon}{2}}}{T} \sum_\pm \sum_n a$$
$$\pm \left(1+\epsilon, T;\ d_{\tilde{g}^n}(\tilde{x}, \tilde{y})\right) e^{\pm i(1+\epsilon)n d_{\tilde{g}^n}}(\tilde{x}, \tilde{y}) + R\,(1+\epsilon, T, \tilde{x}, \tilde{y}), \qquad (169)$$
where $\rho \in L^\infty(\mathbb{R}^{3+\epsilon} \times \mathbb{R}^{3+\epsilon}) \cap C^\infty(\mathbb{R}^{3+\epsilon} \times \mathbb{R}^{3+\epsilon})$,
$$a_\pm(1+\epsilon, T;\ r) = 0, r \notin [1, T], \qquad \partial_r^j a_\pm(1+\epsilon, T;\ r) \leq C_j r^{-\frac{2+\epsilon}{2} - j}, \qquad (170)$$

with constants C_j independent of $T, (1+\epsilon) \geq 1$, and

$$R(1+\epsilon, T;\ \tilde{x},\tilde{y}) = 0 \text{ if } d_{\tilde{g}^n}(\tilde{x},\tilde{y}) > T, \text{and } |R(1+\epsilon, T;\ \tilde{x},\tilde{y})| \leq C_{T,K}(1+\epsilon)^{-\frac{6+\epsilon}{2}}, \quad \text{if } \tilde{x},\tilde{y} \in K \Subset \mathbb{R}^{3+\epsilon}. \tag{171}$$

Proof. Since $\widehat{m}(\tau) = 0$ when $|\tau| > 1/2$ it follows that the left side of (169),

$$\frac{1}{2\pi T}\int_{-T}^{T}\sum_n \left(1 - \beta(\tau)\right)\widehat{m}\left(\frac{\tau}{T}\right)e^{i(1+\epsilon)\tau n}\left(\cos\ \tau\sqrt{\Delta_{\tilde{g}^n}}\right)(\tilde{x},\tilde{y})\,d\tau,$$

vanishes when $d_{\tilde{g}^n}(\tilde{x},\tilde{y}) > T$. Since $\beta(\tau) = 1$ for $|\tau| \leq 3/2$, by (5.14), it is $O_{N,T}\left((2+\epsilon)^{-N}\right)$ for any $N = 1,2,3,\ldots$ if $d_{\tilde{g}^n}(\tilde{x},\tilde{y}) \leq 1$. Therefore, we need only to prove the assertions in Corollary (2.3.21) when $1 \leq d_{\tilde{g}^n}(\tilde{x},\tilde{y}) \leq T$.

To prove this, we shall use the Hadamard parametrix (see e.g. [64] and [28]). Since $(\mathbb{R}^{2+\epsilon}, \tilde{g}^n)$ has nonpositive curvature, for $0 \leq \tau \leq T$ we can write

$$\sum_n \left(\cos\tau\sqrt{\Delta_{\tilde{g}^n}}\right)(\tilde{x},\tilde{y}) = \rho(\tilde{x},\tilde{y})(2\pi)^{-(2+\epsilon)}\int_{\mathbb{R}}^{2+\epsilon}\sum_n e^{id_{\tilde{g}^n}(\tilde{x},\tilde{y})\xi_1 n}\cos\ \tau|\xi|\,d\xi$$
$$= \sum_{\pm}\int_{\mathbb{R}^{2+\epsilon}}\sum_n e^{id_{\tilde{g}^n}(\tilde{x},\tilde{y})\xi_1 n}\alpha_{\pm}(\tau,\tilde{x},\tilde{y},|\xi|)e^{\pm i\tau n|\xi|}\,d\xi + R(\tau,\tilde{x},\tilde{y}), \tag{172}$$

where the leading Hadamard coefficient, ρ, is smooth and uniformly bounded (by the curvature hypothesis), and if $m \in N$ is fixed we can have $\partial_\tau^j R(\tau,\tilde{x},\tilde{y}) \in L^\infty_{\text{loc}}, 0 \leq j \leq m$, and also

$$\left|\partial^{1-\epsilon}_{\tau,\tilde{x},\tilde{y}}\,\partial^j_{1+\epsilon}\alpha_{\pm}(\tau,\tilde{x},\tilde{y},1+\epsilon)\right| \leq C_{T,K,1-\epsilon,j}\,(1+\epsilon)^{-2-j}, \text{if } \epsilon \geq 0, 0 \leq \tau \leq T, j = 0,1,2,\ldots, \quad \text{and } \tilde{x},\tilde{y} \in K \Subset \mathbb{R}^{2+\epsilon}. \tag{173}$$

We also recall (see e.g. [16]) that we can write the Fourier transform of Lebesgue measure on the sphere in $\mathbb{R}^{2+\epsilon}$ as

$$\int_{S^{1+\epsilon}}\sum_n e^{ix\cdot\omega n}\,d\sigma(\omega) = |x|^{-\frac{1+\epsilon}{2}}\sum_n\left(c_+(|x|)e^{i|x|n} + c_-(|x|)e^{-i|x|n}\right), \tag{174}$$

where for each $j = 0,1,2,\ldots$, we have

$$\left|\partial^j_{1+\epsilon}\,c_+(1+\epsilon)\right| + \left|\partial^j_{1+\epsilon}\,c_-(1+\epsilon)\right| \leq C_j\,(1+\epsilon)^{-j}, \epsilon \geq 0. \tag{175}$$

If in (158) we replace $(\cos\ \tau\sqrt{\Delta_{\tilde{g}^n}})(\tilde{x},\tilde{y})$ by the first term in (172), the resulting expression equals $\rho(\tilde{x},\tilde{y})$ times a fixed multiple of

$$\frac{1}{2\pi T}\int_{-T}^{T}\int_{\mathbb{R}^{2+\epsilon}}\sum_n \widehat{m}\left(\frac{\tau}{T}\right)e^{i(1+\epsilon)\tau n}\cos\ (\tau|\xi|)e^{id_{\tilde{g}^n}(\tilde{x},\tilde{y})\xi_1 n}\,d\xi d\tau$$
$$= \sum_{\pm}\frac{1}{2\pi T}\int_{-T}^{T}\int_0^\infty\sum_n \widehat{m}\left(\frac{\tau}{T}\right)e^{i(1+\epsilon)\tau n}\cos\left(\tau(1+\epsilon)\right)e^{\pm i(1+\epsilon)d_{\tilde{g}^n}n}(\tilde{x},\tilde{y})\,c_{\pm}\left(d_{\tilde{g}^n}(\tilde{x},\tilde{y})(1+\epsilon)\right)\frac{(1+\epsilon)^{\frac{1+\epsilon}{2}}}{\left(d_{\tilde{g}^n(\tilde{x},\tilde{y})}\right)^{\frac{1+\epsilon}{2}}}\,d(1+\epsilon)d\tau$$

minus

$$\sum_{\pm} \frac{1}{2\pi T}\int_{-2}^{2} \int_{0}^{\infty}\sum_{n} \beta(\tau)\widehat{m}\left(\frac{\tau}{T}\right) e^{i(1+\epsilon)\tau n}\cos (\tau (1$$

$$+ \epsilon))e^{\pm i(1+\epsilon)d_{\tilde{g}^n}(\tilde{x},\tilde{y})n} c_{\pm}\left(d_{\tilde{g}^n}(\tilde{x},\tilde{y})(1+\epsilon)\right)\frac{(1+\epsilon)^{\frac{1+\epsilon}{2}}}{d_{\tilde{g}^n}(\tilde{x},\tilde{y})^{\frac{1+\epsilon}{2}}} d(1+\epsilon)d\tau.$$

If we replace cos $(\tau(1+\epsilon))$ by $e^{-i\tau n(1+\epsilon)}$ in the right side of (163), the resulting expression equals the sum over ± of

$$\int_{0}^{\infty}\sum_{n} m\left(T(0)\right) c_{\pm}\left(d_{\tilde{g}^n}(\tilde{x},\tilde{y})(1+\epsilon)\right)e^{\pm i(1+\epsilon)d_{\tilde{g}^n}(\tilde{x},\tilde{y})n} \frac{(1+\epsilon)^{\frac{1+\epsilon}{2}}}{\left(d_{\tilde{g}^n}(\tilde{x},\tilde{y})\right)^{\frac{1+\epsilon}{2}}} d(1+\epsilon)$$

$$= \frac{(1+\epsilon)^{\frac{1+\epsilon}{2}}}{T}\sum_{n} e^{\pm i(1+\epsilon)d_{\tilde{g}^n}(\tilde{x},\tilde{y})n} a_{\pm}\left(1+\epsilon, T;\ d_{\tilde{g}^n}(\tilde{x},\tilde{y})\right),$$

where, using the fact that $m \in \mathcal{S}(\mathbb{R})$ and (175), $a_{\pm}$ satisfies (170). If in (163) we replace cos $(\tau(1+\epsilon))$ by $e^{i\tau n(1+\epsilon)}$, then this argument also implies that the resulting expression is $O_{N,T}\left((2+\epsilon)^{-N}\right)$ for any $N = 1, 2, 3, \ldots$. Thus, modulo such an error ρ times the terms in (5.38) can be written as the first term in the right side of (169) with (170) being valid. Since this argument shows that the same is the case for (134), we conclude that the first term in the right side of (172), up to $O_{N,T}\left((2+\epsilon)^{-N}\right)$ errors, gives us the first term in the right side of (169).

This argument and (173) also shows that if in (169) we replace $(\cos\ \tau\sqrt{\Delta_{g^n}})(\tilde{x},\tilde{y})$ by the second term in the right side of (172), then we get a term obeying the bounds in (171). Since, as noted we can take the remainder term in (172) to satisfy for a given $m \in \mathbb{N}, \partial_\tau^j R(\tau,\tilde{x},\tilde{y}) \in L^\infty_{loc}, j = 0, 1, \ldots, m$, we also see that if we choose m large enough, the same is true for it.

Chapter 3
Principal Eigenvalue and Simple Criterion

We indicate several outstanding open problems and formulate some conjectures. We establish a criterion for the existence of a principal eigenpair (λ_p, φ_p). We also explore the relation between the sign of the largest element of the spectrum with a strong maximum property satisfied by the operator. As an application of these results we construct and characterise the solutions of some nonlinear nonlocal reaction diffusion equations.

Section (3.1): Elliptic Operators in $\mathbb{R}^N$ and Applications

The principal eigenvalue is a basic notion associated with an elliptic operator. For instance, the study of semilinear elliptic problems in bounded domains often involves the principal eigenvalue of the associated linear operator. To motivate the results of the present, we recall some classical properties of a class of semilinear elliptic problems in bounded domains.

Let L be a linear elliptic operator acting on functions defined on a bounded and smooth domain $\subset \mathbb{R}^N$:

$$Lu = a_{ij}(x)\partial_{ij} u + b_i(x)\partial_i u + c(x)u$$

(the summation convention on repeated indices is used).

Consider the Dirichlet problem

$$\begin{cases} -Lu = g(x,u), & x \in \Omega, \\ u = 0 & on\, \Omega. \end{cases} \quad (1)$$

We are interested in positive solutions of (1.1). Assume that g is a C_1 function such that

$$g(x,s) < g'_s(x,0)s, \qquad \forall x \in \Omega$$

and

$$\exists M > 0 \text{ such that } g(x,s) + c(x)s \leq 0\,, \forall_s \geq M$$

Then existence of positive solutions of (1) is determined by the principal eigenvalue μ_1 of the problem linearized about u = 0:

$$\begin{cases} -L_\varphi - g_s(x,0)_\varphi = \mu_{1\varphi} \text{ in } \Omega, \\ \varphi = 0 \text{ on } \partial\Omega. \end{cases} \quad (2)$$

Recall that μ_1 is characterized by the existence of an associated eigenfunction $\varphi > 0$ of (2). It is known indeed that (1) has a positive solution if and only if $\mu_1 < 0$ (see e.g. [96]). Under the additional assumption that $s \mapsto g(x,s)/s$ is decreasing, one further obtains a uniqueness result [96]. Problems of the type (1) arise in several contexts, in particular in population dynamics.

The problem is set in an unbounded domain, often in $\mathbb{R}^N$.

Clearly, extensions to unbounded domains of the previous result, as well as others of the same type, require one to understand the generalizations and properties of the notion of principal eigenvalue of elliptic operators in unbounded domains. We indicate some new results about such a semilinear problem, extending the result for (1).

Another example of use of principal eigenvalue is the characterization of the existence of the Green function for periodic linear operators (see Agmon [94]). See [114] and to its bibliography for details on the subject. In [111], Kuchment and Pinchover derived an integral representation formula for the solutions of linear elliptic equations with periodic coefficients in the whole space, provided that an associated generalized principal eigenvalue is positive. It can be seen that the generalized eigenvalue in [111] coincides with (6) here.

This result yields in particular a Liouville type theorem extending those of [95], [112] for periodic self-adjoint operators. Moreover, the principal eigenvalue of an elliptic operator has been shown to play an important role in some questions in branching processes (see Englander and Pinsky [103], Pinsky [115]). Very recently, the principal eigenvalue of an elliptic operator in RN is being introduced in the context of economic models [105].
Some definitions of the notion of principal eigenvalue in unbounded domains have emerged in the works of Agmon [94], Berestycki, Nirenberg and Varadhan [100], Pinsky [114] and others. With a view to applications to semilinear equations, in particular two definitions have been used in [97], [98], [99]. We will recall these definitions. We examine these definitions and further investigate their properties.

We are interested in understanding when the two definitions coincide or for which classes of operators one or the other inequality holds. We also further explain the choice of definition. We review the relevant results from [99].

We define the class of elliptic operators (in nondivergence form) as the elliptic operators $-L$ with

$$Lu = a_{ij}(x)\,\partial_{ij}u + b_i(x)\,\partial_i u + c(x)u \quad \text{in } \mathbb{R}^N$$

Self-adjoint elliptic operators $-L$ are defined by

$$Lu = \partial_i\big(a_{ij}(x)\,\partial_j u\big) + c(x)u \ \text{ in } \mathbb{R}^N.$$

Throughout, $(\partial_{ij})_{ij}$ will denote an $N \times N$ symmetric matrix field such that

$$\forall_x, \xi \in \mathbb{R}^N, \underline{a}|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \bar{a}|\xi|^2 \qquad (3)$$

where a and a are two positive constants, $(b_i)_i$ will denote an N-dimensional vector field and c a real-valued function. We always assume that there exists $0 < \alpha \le 1$ such that

$$a_{ij}, b_i, c \in C_b^{0,\alpha}(\mathbb{R}^N) \qquad (4)$$

in the case of general operators, and

$$a_{ij} \in C_b^{1,\alpha}(\mathbb{R}^N), c \in C_b^{0,\alpha}(\mathbb{R}^N) \qquad (5)$$

in the self-adjoint case. $C_b^{k,\alpha}$ By $(\mathbb{R}^N)$, we mean the class of functions $C^k(\mathbb{R}^N)$ such that ϕ and the derivatives of ϕ up to order k are bounded and uniformly Hölder continuous with exponent α Notice that every self-adjoint operator satisfying (5) can be viewed as a particular case of a general elliptic operator satisfying (4).
It is well known that any elliptic operator $-L$ as defined above admits a unique principal eigenvalue, both in bounded smooth domains associated with Dirichlet boundary conditions, and in $\mathbb{R}^N$ provided that its coefficients are periodic in each variable. This principal eigenvalue is the bottom of the spectrum of $-L$ in the appropriate function space, and it admits an associated positive principal eigenfunction. This result follows from the Krein–Rutman theory and from compactness arguments (see [108] and [107]).

We examine some properties of two different generalizations of the principal eigenvalue in unbounded domains. The first one, originally introduced in [100], reads:
Definition (3.1.1)[93]: Let $-L$ be a general elliptic operator defined in a domain $\Omega \subseteq \mathbb{R}^N$. We set

$$\lambda_1(-L,\Omega) := \sup\{\lambda \big| \exists_\emptyset \in C^2(\Omega)C^1_{loc}\ (\bar{\Omega}), \big|\phi > 0 \text{ and } (L+\lambda)\phi \le 0 \text{ in} \Omega\}. \quad (6)$$

Here, $C^1_{loc}(\bar{\Omega})$ denotes the set of functions $\phi \in C^1(\Omega)$ for which ϕ and $\nabla_\emptyset$ can be extended by continuity on $\partial\Omega$, but which are not necessarily bounded. The generalized principal eigenvalue λ_1 given by (6) is the same as the one used in [111]. Indeed, in [111], the eigenvalue is defined with equality in formula (6). Using the existence of a generalized principal eigenfunction (which follows from the same arguments as in [99]) one sees that the two notions actually coincide. Berestycki, Nirenberg and Varadhan showed that this is a natural generalization of the principal eigenvalue. Indeed, if Ω is bounded and smooth, then $\lambda_1(-L,\Omega)$ coincides with the principal eigenvalue of $-L$ in Ω with Dirichlet boundary conditions. As we will see later, the eigenvalue λ_1 does not suffice to completely describe the properties of semilinear equations in the whole space, in contrast to the Dirichlet principal eigenvalue in bounded domains for problem (1).

We also require another generalization, whose definition is similar to that of λ_1.

This generalization has been introduced in [97], [99] and reads:

Definition (3.1.2)[93]: Let $-L$ be a general elliptic operator defined in a domain $\Omega \subseteq \mathbb{R}^N$. We set

$$\lambda_1'(-L,\Omega) := \inf\{\lambda \mid \exists_\phi \in C^2(\Omega) \cap C^1_{loc}(\overline{\Omega}) \cap W^{2,\infty}(\Omega)$$
$$\phi > 0 \text{ and } -(L+\lambda)\phi \leq 0 \text{in } \Omega, \phi = 0 \text{ on } \boldsymbol{\partial\Omega} \neq \emptyset\} \qquad (7)$$

Several other generalizations are possible, starting from Definition (3.1.1) and playing on the space of functions or the inf and sup inequalities. We will explain why Definition (3.1.2) is relevant.

If L is periodic (in the sense that its coefficients are periodic in each variable, with the same period) then $\lambda_1(-L,\mathbb{R}^N) \geq \lambda_1'(-L,\mathbb{R}^N)$, as is shown by taking ϕ equal to a positive periodic principal eigenfunction in (6) and (7). If there exists a bounded positive eigenfunction φ, then $\lambda_1 \geq \lambda_1'$. But in general, if the operator L is not self-adjoint, equality need not hold between λ_1 and λ_1' even if L is periodic. It is then natural to ask about the relations between λ_1 and λ_1' in the general case. We review a list of statements, most of them given in [99], which answer this question in some particular cases. We state our new main results as well as some problems which are still open. We motivate our choice of taking (6) and (7) as generalizations of the principal eigenvalue.

We describe how the eigenvalues λ_1 and λ_1' are involved in the study of the following class of nonlinear problems:

$$-a_{ij}(x)\,\partial_{ij}u(x)\,\partial u(x) = f\big(x,u(x)\big) \quad \text{in } \mathbb{R}^N. \qquad (8)$$

This type of problem arises in particular in biology and in population dynamics. Here and in what follows, the function $f(x,s)\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is assumed to be in $C_b^{0,\alpha}(\mathbb{R}^N)$ with respect to the variable x, locally uniformly in $x \in \mathbb{R}^N$, and to be locally Lipschitz continuous in the variable s, uniformly in $x \in \mathbb{R}^N$. Furthermore, we always assume that

$$\forall x \in \mathbb{R}^N \quad f(x,0) = 0,$$

$\exists\delta > 0$ such that $s \longmapsto f(s,x)$ belongs to $C^1([0,\delta])$, uniformly in $x \in \mathbb{R}^N$,

$$f_s(x,0) \in C_b^{0,\alpha}(\mathbb{R}^N)$$

We will always denote by L_0 the linearized operator around the solution $u \equiv 0$ associated to the equation (8), that is,

$$L_0u = a_{ij}(x)\partial_{ij}u + b_i(x)\partial_iu + f_s(x,0)u \quad in\ \mathbb{R}^N$$

In [97] it is proved that, under suitable assumptions on f, if L_0 is self-adjoint and the functions

a_{ij} and $x \longmapsto f(s,x)$ are periodic (in each variable) with the same period, then (8) admits a unique positive bounded solution if and only if the periodic principal eigenvalueof $-L_0$ is negative (see Theorems (3.1.3) and (3.1.6) in [97]). This result has been extended in [99] to nonperiodic, non-self-adjoint operators, by using $\lambda_1(-L_0, \mathbb{R}^N)$ and $\lambda'_1(-L_0, \mathbb{R}^N)$ instead of the periodic principal eigenvalue of $-L_0$. The assumptions required are:

$$\exists M > 0 , \forall x \in \mathbb{R}^N \ , \forall_s \geq M , f(x, s) \leq 0 \tag{9}$$

$$\forall x \in \mathbb{R}^N , \forall s \geq 0 , f(x, s) \leq f_s(x, 0)s \tag{10}$$

The existence result of [99] is:

Theorem (3.1.3)[93]: Let L_0 be the linearized operator around zero associated to equation (8).

(i) If (9) holds and either $\lambda_1(-L_0, \mathbb{R}^N) < 0$ or $\lambda'_1(-L_0, \mathbb{R}^N) < 0$, then there exists at least one positive bounded solution of (8).

(ii) If (10) holds and $\lambda'_1(-L_0, \mathbb{R}^N) > 0$, then there is no nonnegative bounded solution of (8) other than the trivial one $u \equiv 0$. Theorem (3.1.3) follows essentially from Definitions (3.1.1), (3.1.2) and a characterization of λ_1 (see [99]). In [103], Engl¨ander and Pinsk proved a similar existence result for a class of solutions of minimal growth (which they define there) for nonlinearities of the type $f(x, u) = b(x)u - a(x)u^2$ with inf $a > 0$ (see also [102], [115]). Since the theorem involves both λ_1 and λ'_1, one does not have a simple necessary and sufficient condition. This is one of the motivations to investigate the properties of these two generalized eigenvalues. In particular, it is useful to determine conditions which yield equality between them or at least an ordering.

From the results we can deal in particular with the case that the operator is self-adjoint and limit periodic. The notion of limit periodic operator is defined precisely below. Essentially, it means that the operator is the uniform limit of a sequence of periodic operators. In this case, we still have a condition, extending that in Theorem (3.1.3), which is nearly necessary and sufficient.

Theorem (3.1.4)[93]: Let $-L_0$ be a self-adjoint limit periodic operator.

(i) If (9) holds and $\lambda_1(-L_0, \mathbb{R}^N) < 0$, then there exists at least one positive bounded solution of (8). If, in addition, (11) below holds, then such a solution is unique.

(ii) If (10) holds and $\lambda'_1(-L_0, \mathbb{R}^N) > 0$, then there is no nonnegative bounded solution of (8) other than the trivial one $u \equiv 0$ The same result holds in dimension $N = 1$ if L_0 is an arbitrary self-adjoint operator. The case of equality: $\lambda_1(-L_0, \mathbb{R}^N) = 0$ is open.

For uniqueness, in unbounded domains, one needs to replace the classical assumption that s $s \mapsto f(x, s)/s$ is decreasing by the following one:

$$\forall 0 < s_1 < s_2, \qquad \inf_{x \in \mathbb{R}^N} \left(\frac{f(x, s_1)}{s_1} - \frac{f(x, s_2)}{s_2} \right) > 0 \tag{11}$$

The uniqueness result of [99] is more delicate and involves the principal eigenvalue of some limit operators defined there. It becomes simpler to state in case the coefficients in (8) are almost periodic, in the sense of the following definition:

Definition (3.1.5)[93]: A function $g: \mathbb{R}^N \to \mathbb{R}$ is said to be almost periodic (a.p.) if from any sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^N$ one can extract a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $g(x_{n_k} + x)$ converges uniformly in $x \in \mathbb{R}^N$.

Theorem (3.1.6)[93]: (Theorem 1.5 in [99]). Assume that the functions a_{ij} , b_i and $f_s(\cdot, 0)$ are a.p. If (11) holds and $\lambda_1(-L_0, \mathbb{R}^N) < 0$, then (8) admits at most one nonnegative bounded solution besides the trivial one $u \equiv 0$. Theorems (3.1.3) and (3.1.6) essentially contain the results in the periodic self-adjoint framework (which hold under the same assumptions (9), (10) and (11)). In that case, in fact,

$$\lambda_1(-L_0, \mathbb{R}^N)$$

$\lambda_1'(-L_0,\mathbb{R}^N)$ coincide with the periodic principal eigenvalue of $-L_0$ (see Proposition (3.1.9) below) and then the only case which is not covered is when the periodic principal eigenvalue is equal to zero.

Unless otherwise specified, $-L$ denotes a general elliptic operator. When we say that L is periodic, we mean that there exist N positive constants $l_1,\ldots,l_N$ such that

$$\forall x\in\mathbb{R}^N,\forall k\in\{1,\ldots,N\}, a_{ij}(x+l_ke_k)=a_{ij}(x)$$
$$b_i(x+l_ke_k)=b_i(x),\quad c(x+l_ke_k)=c(x)$$

where $(e_1,\ldots,e_N)$ is the canonical basis of $\mathbb{R}^N$. The following are some of the known results concerning λ_1 and λ_1' Actually, in some statements of [99], the coefficients of L were in $C^{0,\alpha}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, and the "test functions" $\emptyset$ in the definition of λ_1' were taken in $C^2(\mathbb{R}^N)\cap W^{1,\infty}(\mathbb{R}^N)$instead of $C^2(\mathbb{R}^N)\cap W^{2,\infty}(\mathbb{R}^N)$. However, one can check that the following results—as well as Theorem (3.1.3)—can be proved arguing exactly as in the proofs of the corresponding results in [99].

Proposition (3.1.7)[93]: ([100] and Proposition (3.1.12) in [99])**.** Let Ω be a general domain in $\mathbb{R}^N$ and
$(\Omega_n)_n$ be a sequence of nonempty open sets such that

$$\Omega_n\subset\Omega_{n+1},\quad \bigcup_{n\in\mathbb{N}}\Omega_n=\Omega$$

Then $\lambda_1(-L,\Omega_n)\searrow\lambda_1(-L,\Omega)$ asn $\to\infty$.

Proposition (3.1.7) yields $\lambda_1(-L,\mathbb{R}^N)<\infty$. Furthermore, taking $\emptyset\equiv 1$ as a test function in (6), we see that $\lambda_1(-L,\mathbb{R}^N)\geq -||c||_\infty$. Thus, λ_1 is always a well defined real number.

In the case of L periodic, the periodic principal eigenvalue of $-L$ is defined as the unique real number λ_p such that there exists a positive periodic $\varphi\in C^2(\mathbb{R}^N)$ satisfying $(L+\lambda_p)_\varphi$ in $\mathbb{R}^N$. Its existence follows from the Krein–Rutman theory.

Proposition (3.1.8)[93]: (Proposition 6.3 in [99])**.** If L is periodic, then its periodic principal eigenvalue λ_p coincides with $\lambda_1'(-L,\mathbb{R}^N)$ It is known that, in the general non-self-adjoint case, $\lambda_1=\lambda_1'$. Indeed, as an example, consider the one-dimensional operator $-Lu=-u''\quad+u'$, which is periodic with arbitrary positive period. Then it is easily seen that

$$\lambda_1'(-L,\mathbb{R})=0<\frac{1}{4}=\lambda_1(-L,\mathbb{R}^N)$$

In fact, since $\emptyset\equiv 1$ satisfies $-L_\varphi=0$, it follows that the periodic principal eigenvalue of $-L$ is 0 and then, by Proposition (3.1.8), $\lambda_1'(-L,\mathbb{R})$= 0. On the other hand, for any $R>0$, the function

$$\varphi_R(x):=\cos\left(\frac{\pi}{2R}x\right)e^{x/2}$$

satisfies $-L_{\varphi R}$=$(1/4+\pi^2/4R^2)_{\varphi_R}$, which shows that φ_R is a principal eigenfunction of $-L$ in $(-R,R)$, under Dirichlet boundary conditions. Therefore, by Proposition (3.1.7),

$$\lambda_1(-L,\mathbb{R})=\lim_{R\to\infty}\left(\frac{1}{4}+\frac{\pi^2}{4R^2}\right)=\frac{1}{4}>\lambda_1'(-L,\mathbb{R})$$

Proposition (3.1.9)[93]: (Proposition 6.6 in [99])**.** If the elliptic operator $-L$ is self-adjoint and periodic, then $\lambda_1(-L,\mathbb{R}^N)=\lambda_1'(-L,\mathbb{R}^N)$, where $\lambda_p$is the periodic principal eigenvalue of $-L$. For the rest it is useful to recall the proof of the last statement.

Proof. First, from Proposition (3.1.8) one knows that $\lambda_p=\lambda_1'(-L,\mathbb{R})$.

Now, let 'p be a positive periodic principal eigenfunction of $-L$ in $\mathbb{R}^N$. Taking $\emptyset = \varphi_p$ in (6), it is straightforward to see that $\lambda_1(-L,\mathbb{R}) \geq \lambda_p$ To show the reverse inequality, consider a family $(\chi_R)_R \geq 1$ of cutoff functions in $C^2(\mathbb{R}^N)$, uniformly bounded in $W^{2,\infty}(\mathbb{R}^N)$ such that $0 < \chi_R < 1$ supp $\chi_R \subset \bar{B}R$ and $\chi_R = 1$ in $B_R - 1$. Fix $R > 1$ and let $\lambda_R$be the principal eigenvalue of $-L$in B_R. It is obtained by the following variational formula:

$$\lambda_R = \min\left\{\frac{\int_{B_R}\left(a_{ij}(x)\,\partial_i v\,\partial_j v - c(x)v^2\right)}{\int_{B_R} v^2} v \in H_0^1(B_R), v \neq 0\right\}. \tag{12}$$

Taking $v = \varphi_P\chi_R$ as a test function in (12), and writing $C_R = B_R \setminus B_{R-1}$, we find

$$\lambda_R \leq -\frac{\int_{B_R}\left(L(\varphi_P\chi_R)\right)\varphi_P\chi_R}{\int_{B_R}\varphi_P^2\chi_R^2} = \frac{\lambda_P\int_{B_{R-1}}\varphi_P^2 - \int_{C_R}(L(\varphi_P\chi_R))\varphi_P\chi_R}{\int_{B_R}\varphi_P^2\chi_R^2}$$

$$= \lambda_P - \frac{\lambda_P\int_{C_R}\varphi_P^2\chi_R^2 + \int_{C_R}(L(\varphi_P\chi_R))\ \varphi_P\chi_R}{\int_{B_R}\varphi_P^2\chi_R^2}$$

Since $\min \varphi_P > 0$, it follows that there exists $K > 0$, independent of R, such that

$$\int_{B_R}\varphi_P^2\chi_R^2 \geq \int_{B_{R\_-1}}\varphi_P^2 \geq K(R-1)^N$$

Consequently,

$$\lambda_R \leq \lambda_P + K'\frac{R^{N-1}}{(R-1)^N}$$

where K' is a positive constant independent of R. Letting R go to infinity and using Proposition (3.1.7), we get $\lambda_1(-L,\mathbb{R}^N) \leq \lambda_P$, and therefore $\lambda_1(-L,\mathbb{R}^N) = \lambda_P$.

The next result is an extension of the previous proposition. It is still about periodic operators, but which are not necessarily self-adjoint. A gradient type assumption on the first order coefficients is required.

Theorem (3.1.10)[93]: (Theorem 6.8 in [99])**.** Consider the operator

$$Lu := \partial_i\left(a_{ij}(x)\,\partial_j u\right) + b_i(x)\,\partial_i u + c(x)u\,, \qquad x \in \mathbb{R}^N$$

where a_{ij} , b_i , c are periodic in x with the same period $(L_1, \dots L_N)$, the matrix field $A(x) = (a_{ij}(x))_{1\leq i,j\leq N}$ is in $C^{1,\propto}(\mathbb{R}^N)$, elliptic and symmetric, the vector field $b = (b_1, \dots b_N)$ is in $C^{1,\propto}(\mathbb{R}^N)$, and $c \in C^{0,\propto}(\mathbb{R}^N)$, . Assume that there is a function $B \in C^{2,\propto}(\mathbb{R}^N)$, such that a_{ij} $\partial_j B = b_i$ for all $i = 1, \dots, N$ and the vector field $A^{-1}b$ has zero average on the periodicity cell $C = (0,l_1) \times \dots \times (0,l_N)$. Then $\lambda_1(-L,\mathbb{R}^N) = \lambda_P = \lambda_1'(-L,\mathbb{R}^N)$, where λ_P is the periodic principal eigenvalue of $-L$ in $\mathbb{R}^N$.

Next, the natural question is to ask what happens when we drop the periodicity assumption. Up to now, the only available result has been obtained in [99] in the case of dimension one. It states:

Proposition (3.1.11)[93]: ([99])**.** Let $-L$ be a self-adjoint operator in dimension one. Then $\lambda_1(-L,\mathbb{R}) \leq \lambda_1'(-L,\mathbb{R})$, This type of result will be extended below.

We will examine three main classes: self-adjoint operators in low dimension, limit periodic operators and general operators in dimension one.We seek to identify classes of operators for which either equality or an inequality between $\lambda_1$and λ_1' holds.

Our first result is an extension of the comparison result of Proposition (3.1.11) to dimensions $N = 2,3$ in the self-adjoint framework.

Next, we examine the class of limit periodic operators which extends that of periodic operators. In a sense, this class is intermediate between periodic and a.p. here is the definition:

Definition (3.1.12)[93]: (i) We say that a general elliptic operator $-L$ is general limit periodic if there exists a sequence of general elliptic periodic operators

$$-L_n u := -a_{ij}^n \partial_{ij} u - b_i^n \partial_i u - c^n u$$

such that $a_{ij}^n \to a_{ij}$, $b_i^n \to b_i$ and $c^n \to c$ in $C_b^{0,\alpha}(\mathbb{R}^N)$ as n goes to infinity.

(ii) We say that a self-adjoint elliptic operator $-L$ is self-adjoint limit periodic if there exists a sequence of self-adjoint elliptic periodic operators

$$-L_n u := -\partial_i\left(a_{ij}^n \partial_j u\right) - c^n u$$

such that $a_{ij}^n \to a_{ij}$ in $C_b^{1,\alpha}(\mathbb{R}^N)$ as n goes to infinity.

Clearly, if all the coefficients of the operators L_n in Definition (3.1.12) have the same period $(l_1, \dots, L_N)$ then L is periodic too. It is immediate to show that the coefficients of a limit periodic operator are in particular a.p. in the sense of Definition (3.1.5). One of the results we obtain is:

We make use of the Schauder interior estimates and the Harnack inequality. One can find a treatment of these results in [104], or consult [109], [110] and [117] for the original proofs of the Harnack inequality.

Going back to the nonlinear problem, owing to Theorem (3.1.19), the existence and uniqueness results in the limit periodic case can be expressed in terms of λ_1(or, equivalently, λ_1' only, which is the statement of Theorem (3.1.4).

We establishe a comparison between λ_1 and λ_1' for general elliptic operators in dimension one:

The notions of generalized principal eigenvalue raise several questions which still need an answer. Some of them are:

Conjecture (3.1.13)[93]: If $-L$ is a self-adjoint elliptic operator, then $\lambda_1(-L, \mathbb{R}^N) \leq \lambda_1'(-L, \mathbb{R}^N)$.. in any dimension N. Note that should the answers, then we would have $\lambda_1 = \lambda_1{}'$ in the self-adjoint case, in arbitrary dimension.

We present various definitions which one could consider as generalizations of the principal eigenvalue in the whole space. Then we explain the choice of (6) and (7) as the most relevant extensions. Here, $-L$ will always denote a general elliptic operator (satisfying (3) and (4)). The quantity λ_1 given by (6) is often called the “generalized” principal eigenvalue. It is considered the “natural” generalization of the principal eigenvalue because, as already mentioned, it coincides with the Dirichlet principal eigenvalue in bounded smooth domains. Also, the sign of λ_1 determines the existence or nonexistence of a Green function for the operator (see [113]). The constant λ_1' has been introduced, more recently, in [97]. If Ω is bounded and smooth, then. $\lambda_1'(-L, \Omega) \leq \lambda_1(-L, \Omega)$ Moreover, as we have seen in Proposition (3.1.8), in the periodic case λ_1' coincides with the periodic principal eigenvalue. The quantity λ_1' is the largest constant for which $-(L + \lambda)$ admits a positive subsolution. The definition of λ_1' is based on that of λ_1, with two changes: first, we take subsolutions instead of supersolutions (and we replace the sup with inf); second, we take test functions in $W^{2,\alpha}$. If we introduce only one of these changes, we obtain the following definitions:

$$\mu_1(-L, \Omega) := \sup\{\lambda | \exists_{\emptyset} \in C^2(\Omega) \cap C^1(\Omega) W^{2,\alpha}(\Omega)$$
$$\phi > 0 \text{ and } (L + \lambda)\phi \leq 0 \text{in } \Omega, \phi = 0 \text{ and } \partial\Omega \neq 0 \tag{13}$$

or

$$\mu_1'(-L,\Omega) := \inf\{\lambda | \exists_\emptyset \in C^2(\Omega)C^1_{loc}(\overline{\Omega}), \phi > 0 \text{ and } -(L+\lambda)\phi \le 0 \text{ in } \Omega \quad (14)$$

The quantity μ_1 is not interesting for us because, as is shown by Remark 6.2 in [99], if we replace λ_1' with μ_1 , then the necessary condition given by Theorem (3.1.3)(ii) fails to hold.

The proof of Theorem (3.1.15) consists in a not so immediate adaptation of the proof of Proposition (3.1.9). It makes use of the following observation, which holds in any dimension N.

Lemma (3.1.14)[93]: Let $\phi \in C^2(\mathbb{R}^N)$ be a nonnegative function. Let A(x) be the largest eigenvalue of the matrix $(\partial_{ij}\phi(x)_{ij}$ and assume that $A := \sup_{x\in\mathbb{R}^N} \Lambda(x) < \infty$. Then

$$\forall x \in \mathbb{R}^N \quad , \left|\nabla_\phi(x)\right|^2 \le 2\Lambda_\phi(x) \quad (15)$$

Proof. First, if $\Lambda \le 0$ then $\partial_{ij}\phi \le 0$ for every $i = 1, \dots, N$. This shows that is concave in every direction xi and hence, being nonnegative, it is constant. In particular, (15) holds. Consider the case $\Lambda > 0$. The Taylor expansion of ϕ at the point $x \in \mathbb{R}^N$ gives

$$\forall y \in \mathbb{R}^N, \quad \phi(y) = \phi(x) + \nabla_\phi(x)(y-x) + \frac{1}{2}\boldsymbol{\partial_{ij}\phi(z)(y-x)_i(y-x)_j}$$

where z is a point on the segment connecting x and y. Hence,

$$0 \le \phi(y) \le \phi(x) + \nabla\phi(x)(y-x) + \frac{1}{2}\Lambda|y-x|^2$$

If we take in particular $y = x - \nabla\phi(x)/\Lambda$ we obtain

$$0 \le \phi(x) - \frac{\left|\nabla_\phi(x)\right|^2}{2\Lambda}$$

and the statement is proved. Note that if ϕ is a positive function in $W^{2,\infty}(\mathbb{R}^N)$, then Lemma (3.1.14) shows that its gradient is controlled by the square root of ϕ. Actually, this is the reason why in (7) we take test functions in $W^{2,\infty}(\mathbb{R}^N)$.

Theorem (3.1.15)[93]: Let $-L$ be a self-adjoint elliptic operator in $\mathbb{R}^N$, with $N \le 3$. Then

$$\lambda_1(-L,\mathbb{R}^N) \le \lambda_1'(-L,\mathbb{R}^N).$$

The assumption $N \le 3$ in Theorem (3.1.15) seems to be only technical, as was the assumption $N = 1$ in Proposition (3.1.11). That is why we believe that the above result holds in any dimension N. But the problem is open at the moment.

Proof. Let $\lambda \in \mathbb{R}$ be such that there exists a positive function $\phi \in C^2(\mathbb{R}^N) \cap W^{2,\infty}(\mathbb{R}^N)$ satisfying $-(L+\lambda)_\phi \le 0$. We would like to proceed as in the proof of Proposition (3.1.9), with $\varphi_P$replaced by ϕ, and obtain $(-L,\mathbb{R}^N) \le \lambda$ This is not possible because, in general, φ is not bounded from below away from zero. Lemma (3.1.14) allows us to overcome this difficulty. Consider in fact the same type of cutoff functions $(\chi_R)_{R\ge1}$ as in Proposition (3.1.9) and let λ_R be the principal eigenvalue of $-L$ in B_R with Dirichlet boundary conditions. The representation formula (12) yields, for

$$\lambda_R \le \frac{\int_{B_R}\left[a_{ij}(x)\,\partial_i(\phi\chi R)\,\partial_j(\phi\chi R) - c(x)\phi^2\chi_R^2\right]}{\int_{B_R}\phi^2\chi_R^2}$$

Hence, since $\lambda_R = 1$ on B_{R-1}, we get

$$\lambda_R \le \lambda - \frac{\int_{c_R}[2a_{ij}(x)(\partial_i\phi\left(\partial_j\chi_R\right)\phi\chi_R + \partial_i\left(a_{ij}(x)\,\partial_j\chi_R\right)\phi^2\chi_R]}{\int_{B_R}\phi^2\chi_R^2}$$

Our aim is to prove that by appropriately choosing the cutoff functions $(\chi_R)_{R\ge1}$ we get

$$\limsup_{R\to\infty} \frac{\int_{C_R}\left[2a_{ij}(x)(\partial_i\phi)(\partial_j\chi_R)\phi\chi_R + \partial_i(a_{ij})(x)\,\partial_j\chi_R)\phi^2\chi_R\right]}{\int_{B_R}\phi^2\chi_R^2} \geq 0 \qquad (16)$$

Choose χ_R so that

$$\forall x \in B_R/B_{R-1/2},\ \ \chi_R(x) = \exp\left(\frac{1}{|x|-R}\right)$$

$$\forall\, x \in B_{R-1/2}\,, \chi_R(x) \geq e^{-1/2}$$

By direct computation, we see that, for $x \in B_R/B_{R-1/2}$

$$\nabla\chi_R(x) = -\frac{x}{|x|}(R-|x|)^{-2}\exp\left(\frac{1}{|x|-R}\right)$$

and

$$\partial_{ij}\chi_R(x) = \left[\left(\frac{x_ix_j}{|x|^3} - \frac{\delta_{ij}}{|x|}\right)(|x|-R)^2 + 2\frac{x_ix_j}{|x|^2}(|x|-R) + \frac{x_ix_j}{|x|^2}\right](|x|-R)^{-4}\exp\left(\frac{1}{|x|-R}\right)$$

Consequently, using the usual summation convention, we have

$$\forall x \in B_R/B_{R-\frac{1}{2}}, \partial_i(a_{ij}(x)\,\partial_j\chi_R) \geq \left[\underline{a} - C(|x|-R)\right](|x|-R)^{-4}\exp\left(\frac{1}{|x|-R}\right)$$

where C is a positive constant depending only on N and the $W^{1,\infty}$ norm of the a_{ij} (and not on R) and a is given by (3). Therefore, there exists h independent of R with $0 < h \leq 1/2$ and such that $\partial_i(a_{ij}(x)\,\partial_j\chi_R \geq 0$ in$B_R\backslash B_{R-h}$. Since $\chi_R > \exp(-h^{-1})$ in B_{R-h}, it is possible to choose C' large enough, independent of R, such that $\partial_i(a_{ij}(x)\boldsymbol{\partial_j\chi_R} \geq -\boldsymbol{C'\chi_R}$ in B_R. On the other hand, owing to Lemma (3.1.14), we can find another constant $C'' > 0$, depending only on N,$||a_{ij}||_{L^\infty(\mathbb{R}^N)}$, $||\phi||_{W^{2,\infty}(\mathbb{R}^N)}$and $||\chi_R||_{W^{2,\infty}(\mathbb{R}^N)}$) (which does not depend on R, such that

$$a_{ij}(x)(\partial_i\phi)(\partial_j\chi_R) \geq -C''\phi^{1/2}\chi_R^{\frac{1}{2}}$$

Assume, by way of contradiction, that (16) does not hold. Then there exist $\varepsilon > 0$ and$R_0 \geq 1$ such that, for $R \geq R_0$,

$$-\varepsilon\int_{B_R}\phi^2\chi_R^2 \geq \int_{C_R}\left[2a_{ij}(x)(\partial_i\phi)(\partial_j\chi_R)\phi\chi_R + \partial_i(a_{ij}(x)\,\partial_j\chi_R)\phi^2\chi_R\right]$$

$$\geq -\int_{C_R}(C'\chi_R^2\,\phi^2 + 2C''\phi^{\frac{3}{2}}\chi_R^{\frac{3}{2}})$$

Since ϕ and χ_R are bounded, the above inequalities yield the existence of a positive constant k such that, for $R \geq R_0$,

$$k\int_{B_R}\phi^2\chi_R^2 \leq \int_{C_R}\phi^{3/2}\chi_R^{\frac{3}{2}}$$

Notice that, since $\phi > 0$, we can choose $k > 0$ in such a way that the above inequality holds for any $R \geq 1$. Using the Hölder inequality with $p = 4/3$ and $p' = 4$, we then obtain

$$\forall R \geq 1 \quad \int_{B_R}\phi^2\chi_R^2 \leq k^{-1}\left(\int_{C_R}\phi^2\chi_R^2\right)^{\frac{3}{4}}|C_R|^{\frac{1}{4}} \leq k^{-1}\,R^{\frac{N-1}{4}}\left(\int_{C_R}\phi^2\chi_R^2\right)^{3/4}$$

where k is another positive constant. For $n \in \mathbb{N}$ set $\alpha_n := (\int_{C_n}\phi^2\chi_n^2)^{\frac{3}{4}}$. Since for $n \in \mathbb{N}$, we have

$$\int_{B_n} \phi^2\chi_n^2 = \sum_{j=1}^{n-1}\int_{C_j}\phi^2 + \int_{C_n}\phi^2\chi_n^2 \geq \sum_{j=1}^{n}\int_{C_j}\phi^2\chi_n^2$$

it follows that

$$\alpha_n \geq Kn^{\frac{1-N}{4}}\sum_{j=1}^{n}\alpha_j^{\frac{4}{3}}. \tag{17}$$

We claim that the sequence $(\alpha_n)_{n\in\mathbb{N}}$ grows faster than any power of n. This contradicts the definition of α_n, because

$$\alpha_n = (\int_{C_n}\phi^2\chi_n^2)^{3/4} \leq ||\phi||_{L^\infty(\mathbb{R}^N)}^{\frac{3}{2}}|C_n|^{3/4} \leq Hn^{3(N-1)/4}$$

for some positive constant H. To prove our claim, we use (17) recursively. At the first step we have $\alpha_n \geq K_0 n^{\beta_0}$, where $K_0 = K\alpha_1^{\frac{4}{3}}$ and $\beta_0 = (1-N)/4$. At the second step we get $\alpha_n \geq KK_0^{\frac{4}{3}}n^{(1-N)/4}\sum_{j=1}^{n} j^{4\beta_0/3}$ If $\beta_0 > -3/4$ (i.e. if N < 4) then $\sum_{j=1}^{n} j^{4\beta_0/3} \sim n^{4\beta_0/3+1}$. Hence, in this case there exists $K_1 > 0$ such that $\alpha_n \geq K_m n^{\beta_m}$, where. $\beta_1 = 4\beta_0/3 + (5-N)/4$ Proceeding in the same way we find, after m steps, that$\alpha_n \geq K_m m^{\beta_m}$, where K_m is a positive constant and $\beta_m = 4\beta_{m-1}/3 + (5-N)/4$, provided that $\beta_0, \dots, \beta_{m-1} > -3/4$ if $\beta_{m-1} > -3/4$, we have

$$\beta_m > \beta_{m-1} \Leftrightarrow \beta_{m-1} > 3/4(N-5)$$

Since

$$\beta_0 > \frac{3}{4(N-5)} \Leftrightarrow N < 4$$

it follows that for $N_r \leq 3$ the sequence $(\beta_m)_{m\in\mathbb{N}}$ is strictly increasing. Thus, $\lim_{m\to\infty}\beta_m = +\infty$ if N, because if the sequence had a finite limit, it would have to be $3(N-5)/4$ which is less than β_0. Therefore, as $n \to \infty, \alpha_n$ goes to infinity faster than any polynomial in n.

We consider limit periodic elliptic operators $-L$. According to Definition (3.1.12), we let either

$$L_n u = a_{ij}^n(x)\,\partial_{ij}u + b_i^n(x)\,\partial_i u + c^n(x)u$$

if $-L$ is a general operator, or

$$L_n u = \partial_i\big(a_{ij}^n(x)\,\partial_j u\big) + c^n(x)u$$

if $-L$ is self-adjoint. We denote by λ_n and φ_n respectively the periodic principal eigenvalue and a positive periodic principal eigenfunction of $-L_n$ in $\mathbb{R}^N$. Our results make use of the following lemma.

Lemma (3.1.16)[93]: The sequence $(\lambda_n)_{n\in\mathbb{N}}$ is bounded and

$$\lim_{n\to\infty}||\frac{(L-L_n)\varphi_n}{\varphi_n}||_{L^\infty(\mathbb{R}^N)} = 0.$$

Proof. We can assume, without loss of generality, that the operators $-L, -L_n$ are general elliptic. Since the operators L_n are periodic, from Proposition (3.1.8) it follows that

$$-||c_n||_\infty \leq \lambda_1'(-L,\mathbb{R}^N) = \lambda_n \leq ||c_n||_\infty.$$

Hence, the sequence $(\lambda_n)_{n\in\mathbb{N}}$ is bounded because $c^n \to c$ in $C_b^{0,\alpha}$. For all $n \in \mathbb{N}$ the functions λ_n satisfy $-(L_n + \lambda_n)_{\varphi_n}$. Then, using interior Schauder estimates, we can find a constant $C_n > 0$ such that

$$\forall x \in \mathbb{R}^N, ||\varphi_n||_{C_b^{2,\alpha}(B_1(x))} \le C_n ||\varphi_n||_{L^\infty(B_2(x))}$$

where the C_n are controlled by λ_n and $||a_{ij}^n||_{C_b^{0,\alpha}(\mathbb{R}^N)}$. $||b_i^n||_{C_b^{0,\alpha}(\mathbb{R}^N)}$, $||c^n||_{C_b^{0,\alpha}(\mathbb{R}^N)}$. We know that the are λ_n bounded in $n \in \mathbb{N}$, and the same is true for the $C_b^{0,\alpha}$ norms of a_{ij}^n and c^n because they converge in the $C_b^{0,\alpha}$ norm to a_{ij}, b_i and c respectively. Thus, there exists a positive constant C such that $C \ge C_n$ for every $n \in \mathbb{N}$. Moreover, applying the Harnack inequality for the operators $-(L_n + \lambda_n)$, we can find another positive constant C' which is again independent of n (and x), such that

$$\forall x \in \mathbb{R}^N, \quad ||\varphi_n||_{L^\infty(B_2(x))} \le C'\varphi_n(x)$$

Therefore,

$$\sup_{x\in\mathbb{R}^N}\left|\frac{(L-L_n)\varphi_n(x)}{\varphi_n(x)}\right| \sup_{x\in\mathbb{R}^N}\frac{\left(||a_{ij}-a_{ij}^n||_\infty + ||b_i - b_i^n||_\infty + ||c-c^n||_\infty\right)||\varphi_n||_{C_b^{2,\alpha}(B_1(x))}}{\varphi_n(x)} -$$

$$\le CC'(||a_{ij}-a_{ij}^n||_\infty + ||b_i - b_i^n||_\infty + ||c-c^n||_\infty$$

which goes to zero as n goes to infinity.

Theorem (3.1.17)[93]: Let $-L$ be a general limit periodic operator. Then $\lambda_1'(-L,\mathbb{R}^N) \le \lambda_1(-L,\mathbb{R}^N)$.

Another result obtained concerns self-adjoint limit periodic operators. It extends Proposition (3.1.9).

Proof. For $n \in \mathbb{R}^N$ define

$$H_n := \left|\left|\frac{(L-L_n)_{\varphi_n}}{\varphi_n}\right|\right|_{L^\infty(\mathbb{R}^N)}. \tag{18}$$

By Lemma (3.1.16), we know that $\lim_{n\to\infty} H_n = 0$. Since $|(L+\lambda_n)\varphi_n| \le H_n\varphi_n$, it follows that $(L+\lambda_n - H_n)\varphi_n \le 0$ and $-(L+\lambda_n + H_n)\varphi_n \le 0$. Hence, using φ_n as a test function in (6) and (7), we infer that $\lambda_1(-L,\mathbb{R}^N) \ge \lambda_n - H$ and $\lambda_1'(-L,\mathbb{R}^N) \le \lambda_n + H_n$ for every $n \in \mathbb{N}$. The proof is complete because, passing to the lim inf and lim sup as n goes to infinity in the above inequalities, we get

$$\lambda_1'(-L,\mathbb{R}^N) \le \lim_{n\to\infty} \inf \lambda_n \le \lim_{n\to\infty} \lim\sup \lambda_n \le \lambda_1(-L,\mathbb{R}^N). \tag{19}$$

The proof of Theorem (3.1.19) is divided into two parts, the first one being the next lemma.

Lemma (3.1.18)[93]: The sequence $(\lambda_n)_{n\in\mathbb{N}}$ converges to $\lambda_1'(-L,\mathbb{R}^N)$ as n goes to infinity.

Proof. Proceeding as in the proof of Theorem (3.1.17), we derive (19). So, we only need to show that $\lim\sup_{n\to\infty} \lambda_n \le \lambda_1'(-L,\mathbb{R}^N)$ To this end, consider a constant $\lambda \ge \lambda_1'(-L,\mathbb{R}^N)$ such that there exists a positive function $\phi \in C^2(\mathbb{R}^N) \cap W^{2,\infty}(\mathbb{R}^N)$ satisfyin $-(L+\lambda)\phi \le 0$. Fix $n \in \mathbb{N}$ and define $\psi_n = 0 := k_n\varphi_n - \phi$ where K_n is the positive constant (depending on n) such that $\inf \psi_n = 0$ (such a constant always exists and it is unique because 'n is bounded from below away from zero and _ is bounded from above). From the inequalities

$$-(L+\lambda)_{\psi_n} \ge -k_n(L+\lambda)_{\varphi_n} = k_n(L_n - L)\varphi_n + k_n(\lambda_n - \lambda)\varphi_n$$

and defining H_n as in (18), we find that

$$-(L+\lambda)\psi_n \ge k_n(\lambda_n - \lambda - H_n)\varphi_n. \tag{20}$$

Since $\inf \psi_n = 0$, there exists a sequence $(x_m)_{m\in N}$ in $\mathbb{R}^N$ such that $\lim_{m\to 1} n(x_m) = 0$.

For $m \in N$, define the functions

$$\theta_m(x) := \psi_n(x) + \psi_n(x_m)|x - x|^2 \ , x \in \mathbb{R}^N$$

Since $\theta_m(x) := \psi_n(x) + \psi_n(x_m)$ and $\theta_m(x) \ge \psi_n(x_m)$ for $x \in \partial B_1(x_m)$, for any $m \in \mathbb{N}$ there exists a point $y_m \in B_1(x_m)$ of local minimum of θ_m . Hence,

$$0 = \nabla\theta_m(y_m) = \nabla\psi_n(y_m) + 2\psi_n(x_m)(y_m - x_m)$$

and

$$0 \le \left(\partial_{ij}\theta(y_m)\right)_{ij} = \left(\partial_{ij}\psi_n(y_m)\right)_{ij} + 2\psi_n(x_m)I$$

where I denotes the $N \times N$ identity matrix. Thanks to the ellipticity of $-L$, we then get

$$-(L+\lambda)\psi_n(y_m) \le 2\psi_n(x_m)a_{ii}(y_m) + 2\psi_n(x_m)b_i(y_m)(y_m - x_m)_i \\ -(c(y_m)+\lambda)\psi_n(y_m) \qquad (21)$$

Furthermore, since $\theta_m(y_m) = \psi_n(y_m) + \psi_n(x_m)|y_m - x_m|^2 \le \theta_m(x_m) = \psi_n(x_m)$, we see that $\psi_n(y_m) \le \psi_n(x_m)$. Consequently, taking the limit as m goes to infinity in (21), we derive $\limsup_{m\to\infty} -(L+\lambda)\psi_n(y_m) \le 0$ Therefore, by (20), $\lim\sup_{m\to\infty} \psi_n(\lambda_n - \lambda - H_n)\varphi_n(y_m) \le 0$ which implies that $\lambda_n - \lambda - H_n \le 0$ because $\inf_{\mathbb{R}^N}\varphi_n > 0$. Since by Lemma (3.1.16) we know that H_n goes to zero as n goes to infinity, it follows that

$$\lambda \ge \lim_{m\to\infty} \sup(\lambda_n - H_n) = \lim_{n\to\infty} \sup\lambda_n$$

Taking the infimum over λ we finally get $\lambda_1'(-L,\mathbb{R}^N) \ge \limsup_{n\to\infty}\lambda_n$

Theorem (3.1.19)[93]: Let $-L$ be a self-adjoint limit periodic operator. Then $\lambda_1(-L,\mathbb{R}^N\ ) = \lambda_1'(-L,\mathbb{R}^N)$.

Proof. Owing to Theorem (3.1.17), it only remains to show that $\lambda_1(-L,\mathbb{R}^N) \le \lambda_1'(-L,\mathbb{R}^N)$. To do this, we fix $R > 1$ and $n \in \mathbb{N}$ and proceed as in the proof of Proposition (3.1.9), replacing the test function 'p by 'n. We thus get

$$\lambda_1(-L,B_R) \le -\frac{\int_{B_R}\left(L(\varphi_n\chi_R)\right)\varphi_n\chi_R}{\int_{B_R}\varphi_n^2\chi_R^2}$$

$$= \frac{\int_{B_{R-1}}((\lambda_n + L_n - L)\,\varphi_n)\,\varphi_n - \int_{C_R}\left(L(\varphi_n\chi_R)\right)\varphi_n\,\chi_R}{\int_{B_R}\varphi_n^2\chi_R^2}$$

$$= \lambda_n - \frac{\int_{B_{R-1}}\left((L-L_n)\varphi_n\right)\varphi_n + \int_{C_R}\left((L+\lambda_n)\varphi_n\chi_R\right)\varphi_n\chi_R}{\int_{B_R}\varphi_n^2\chi_R^2}$$

Setting H_n as in (18), we get

$$\lambda_1(-L,B_R) \le \lambda_n + \frac{H_n\int_{B_{R-1}}\varphi_n^2 + K_n|C_R|}{\int_{B_R}\varphi_n^2\chi_R^2}$$

where $|C_R|$ denotes the measure of the set C_R and K_n is a positive constant (independent of R because the λ_R are uniformly bounded in $W^{2,\infty}(\mathbb{R}^N)$). Therefore, since $\min_{\mathbb{R}^n\varphi_n} > 0$ there exists another constant $\widetilde{K}_n > 0$ such that

$$\lambda_1(-L,B_R) \le \lambda_n + H_n + \frac{\widetilde{K}_n}{R}$$

Letting R go to infinity in the above inequality and using Proposition (3.1.7) shows that$\lambda_1(-L,\mathbb{R}^N) \le \lambda_n + H_n$. By Lemmas (3.1.16) and (3.1.18), we know that $H_n \to 0$ and $\lambda_n \to \lambda_1'(-L,\mathbb{R}^N)$ as $n \to \infty$. Thus, we conclude that $\lambda_1(-L,\mathbb{R}^N) \le \lambda_1'(-L,\mathbb{R}^N)$.

We are concerned with general elliptic operators in dimension one, that is, operators of the type

$$-Lu = -a(x)u'' - b(x)u' - c(x)u \ , \quad x \in \mathbb{R}$$

with the usual regularity assumptions on a, b, c. The ellipticity condition becomes $\underline{a} \le a(x) \le \bar{a}$ for some constants $0 \le \underline{a} \le a$.

Theorem (3.1.20)[93]: Let $-L$ be a general elliptic operator in dimension one. Then $\lambda_1'(-L, \mathbb{R}^N) \le \lambda_1(-L, \mathbb{R}^N)$.Notice that, by Theorems (3.1.17) and (3.1.20), if $-L_0$ is limit periodic or $N = 1$, then we can state Theorem (3.1.3) without mentioning λ_1. Hence, only the sign of $\lambda_1' 1$ is involved in the existence result.

Proof. Fix $R > 0$ and denote by λ_R and φ_R the principal eigenvalue and eigenfunction respectively of $-L$ in $(-R, R)$, with the Dirichlet boundary condition.

Then define

$$\psi_R(x) := \frac{h}{k} e^{-k(x-R)}, x \in \mathbb{R}$$

where h, k are two positive constants that will be chosen later. The function ψ_R satisfies

$$-(L + \lambda_R)\psi_R = \left(-a(x)k + b(x) - (c(x) + \lambda_R)\frac{1}{k}\right) he^{-k(x-R)}$$

There exists $k_0 > 0$ (independent of h) such that $-(L + \lambda_R)\psi_R < 0$ in $\mathbb{R}$ for any choice of $k \ge k_0$. Our aim is to connect smoothly the functions φ_R and ψ_R in order to obtain a function $\phi_R \in C^2([0, \infty)) \cap W^{2,\infty}([0, \infty))$ satisfying $-(L + \lambda_R)_{\phi_R} \le 0$. To this end, we set $g_R(x) := \eta(x - R + \delta)^3$, with $\eta, \delta > 0$ to be chosen. Since

$$-(L + \lambda_R)_{gR} = [-6a(x) - 3b(x)(x - R + \delta) - (c(x) + \lambda_R)(x - R + \delta)^2]\eta(x - R + \delta)$$

we can find a constant $\delta_0 > 0$ such that $-(L + \lambda_R)g_R \le 0$ in $(R - \delta, R)$, for any choice of $0 < \delta \le \delta_0$. Then we define

$$\phi_R(x) := \begin{cases} \varphi_R(x) & \text{for } 0 \le x \le R - \delta, \\ \varphi_R(x) + g_R(x) & \text{for } R - \delta < x \le R, \\ \psi_R(x) & \text{for } x > R. \end{cases} \tag{22}$$

It follows that if $k \ge k_0$ and $\delta \le \delta_0$, then $-(L + \lambda_R)\phi_R \le 0$ in $(0, R - \delta) \cup (R - \delta, R) \cup (R, +\infty)$. In order to ensure the C^2 regularity of ϕ_R, we need to solve the following system in the variables h, η, δ:

$$\begin{cases} \eta\delta^3 = h/k \\ \varphi_R'(R) + 3\eta\delta^2 = -h \\ \varphi_R''(R) + 6\eta\delta = hk \end{cases}$$

One can see that if $h < -\varphi_R'(R)$ (notice that $\varphi_R'(R) < 0$ by the Hopf lemma), the previous system becomes, after some simple algebra,

$$\begin{cases} \gamma(h) = \varphi_R''(R)\delta \\ \delta k = \dfrac{3h}{-\varphi_R'(R) - h} \\ \eta = \dfrac{hk - \varphi_R''(R)}{6\delta} \end{cases} \tag{23}$$

Where

$$\gamma(h) := \frac{3h^2}{-\varphi_R'(R) - h} + 2(h + \varphi_R'(R))$$

We want to show that there exists δ small enough such that the system (23) admits positive solutions $\delta, h_\delta, k_\delta, \eta_\delta$ satisfying

$$\delta \le \delta_0 \ , h_\delta < -\varphi_R'(R) \ , k_\delta \ge k_0 \tag{24}$$

Let $0 < \delta_1 \le \delta_0$ be such $|\varphi_R''(R)|\delta_1 < -\varphi_R'(R)$. Thus, if $\delta \le \delta_1$, the first equation of*23))* yields $|\gamma(h)| < -\varphi_R'$. Since $\gamma(0) = 2\varphi_R'(R)$ and $\lim_{h\to\varphi_R'(R)} -\gamma(h) = +\infty$ there exists a constant $0 < -\varphi_R'(R)$ such that, for any choice of $\delta \in (0, \delta_1)$, the first equation of (23) admits a solution $h_\delta \in [h_1 - \varphi_R'(R)$. For $\delta \in (0, \delta_1)$ and $h = h_\delta$, the second equation of (23) gives

$$k_\delta = \frac{3h_\delta}{-\varphi_R'(R) - h_\delta}\delta^{-1} \ge \frac{3h_1}{-\varphi_R'(R) - h_1}\delta^{-1}. \tag{25}$$

Hence, for δ small enough, we have $k_\delta \ge k_0$. Finally, by the last equation of (23), for $\delta \in (0, \delta_1)$, we have

$$\eta\delta = \frac{h_\delta k_s - \varphi_R''(R)}{6\delta} \ge \frac{h_1 k_\delta - \varphi_R''(R)}{6\delta},$$

and so, since k_δ satisfies (25) $\eta_\delta > 0$ for δ small enough. Therefore, there exist four positive constants h, k, η, δ solving (23) and satisfying (24).With this choice of h, k, η, δ the function ϕ_R is in $C^2([0,\infty)) \cap W^{2,\infty}([0,\infty))$.
Proceeding as above, we can extend $\varphi_R(x)$ for x negative, and get a function $C^2(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$. such that $-(L + \lambda_R)\,\phi_R \le 0$ in $\mathbb{R}$. U sing $\phi_R$as a test function in (7), we find that $\lambda_1'(-L, \mathbb{R}) \le \lambda_R$ Thus, passing to the limit as $R \to \infty$, by Proposition (3.1.7), we derive $\lambda_1'(-L, \mathbb{R}) \le \lambda_1(-L, \mathbb{R})$. The proof is thereby complete.

Hence, by uniqueness of the principal eigenfunction up to a constant factor, it follows that $\varphi_R(x) \equiv \varphi_R(Mx)$ that is, φ_R is a radial function. Since for any radial function $u = u(|x|)$ the expression of Lu reads

$$Lu = a(|x|)u'' + \left(b(|x|) + \frac{N-1}{|x|}a(|x|)\right)u' + c(|x|)u,$$

we can proceed as in the one-dimensional case and build a radial function $\phi_R \in C^2(\mathbb{R}^N) \cap W^{2,\infty}(\mathbb{R}^N)$ such that $-(L + \lambda_R)\phi_R \le 0$. Therefore, $\lambda_1'(-L, \mathbb{R}^N) \le \lambda_R$ and then, passing to the limit as $R \to \infty$, we obtain the stated inequality between $\lambda_1$and λ_1'.

Section (3.2): Existence of a Principal Eigenfunction of Some Nonlocal Operators

Much attention has been drawn to the study of nonlocal reaction diffusion equations, where the usual elliptic diffusion operator is replaced by a nonlocal operator of the form

$$\mathcal{M}[u] := \int_\Omega k(x,y)u(y)dy - b(x)u, \tag{26}$$

where $\Omega \subset \mathbb{R}^n, k \ge 0$ satisfies $\int_{\mathbb{R}^n} k(y,x)dy < \infty$ for all $x \in \mathbb{R}^n$ and $b(x) \in C(\Omega)$; see [119]–[121], [125]–[127], [129]–[131], [134], [137]–[139], [148], [149], [153]. Such type of diffusion process has been widely used to describe the dispersal of a population through its environment in the following sense. As stated in [144], [145], [147] if $u(y,t)$ is thought of as a density at a location y at a time t and $k(x,y)$ as the probability distribution of jumping from a location y to a location x, then the rate at which the individuals from all other places are arriving to the location x is

$$\int_\Omega k(x,y)u(y,t)dy.$$

On the other hand, the rate at which the individuals are leaving the location x is $-b(x)u(x,t)$. This formulation of the dispersal of individuals finds its justification in many ecological problems of seed dispersion; see = [124], [128], [140], [148], [149], [153].

We study the properties of the principal eigenvalue of the operator $\mathcal{M}$, when the kernel $k(x,y)$ takes the form

$$k(x,y) = J\left(\frac{(x - y)}{g(y)}\right)\frac{1}{g^n(y)}, \quad (27)$$

where J is a continuous probability density and the function g is bounded and positive. That is to say, we investigate the following eigenvalue problem:

$$\int_{\Omega} J\left(\frac{x - y}{g(y)}\right)\frac{u(y)}{g^n(y)}\,dy - b(x)u = -\lambda u \quad \text{in } \Omega. \quad (28)$$

Such type of diffusion kernel was recently introduced by Cortázar et al. [129] in order to model a nonhomogeneous dispersal process. Along, with no further specifications, we will always make the following assumptions on Ω, J, g and b:

$\Omega \subset \mathbb{R}^n$ is an open connected set, $(H1)$

$J \in C_c(\mathbb{R}^n), J \geq 0, J(0) > 0,$ $(H2)$

$g \in L^{\infty}(\Omega), 0 \leq \alpha \leq g \leq \beta,$ $(H3)$

$b \in C(\Omega) \cap L^{\infty}(\Omega),$ $(H4)$

where $C_c(\mathbb{R}^n)$ denotes the set of continuous functions with compact support.

The existence and a variational characterisation of the principal eigenvalue λ_p of $\mathcal{M}$ is known from a long time, see for example Donsker and Varadhan [141]. However, as Donsker and Varadhan [141] have already noticed, λ_p is in general not an eigenvalue, that is to say, there exists no positive function ϕ_p such that (λ_p, ϕ_p) is a solution of (28). We are interested in finding some conditions on $\mathcal{M}$ ensuring the existence of a principal eigenpair (λ_p, ϕ_p) of (28) such that $\phi_p \in C(\Omega)$ and $\phi_p > 0$. Such type of solution is commonly used to analyse the long-time behaviour of some nonlocal evolution problems [125], [129] and had proven to be a very efficient tool in the analysis of nonlinear integrodifferential problems; see [136], [146].

Besides some particular situations the existence of an eigenpair (λ_p, ϕ_p) for Eq. (28) is still an open question and many of the known results concern these two cases:

(i) $b(x) \equiv$ Constant.

(ii) The operator $\mathcal{M}$ satisfies a mass preserving property, i.e. $\forall u \in C(\Omega)$,

$$\int_{\Omega}\int_{\Omega} J\left(\frac{x - y}{g(y)}\right)\frac{u(y)}{g^n(y)}\,dy\,dx - \int_{\Omega} b(x)u(x)dx = 0.$$

In both cases, the principal eigenvalue problem (28) is either reduced to the analysis of the spectrum of the positive operator $\mathcal{L}_{\Omega}$ defined below:

$$\mathcal{L}_{\Omega}[u] := \int_{\Omega} J\left(\frac{(x - y)}{g(y)}\right)\frac{u(y)}{g^n(y)}\,dy$$

or the principal eigenvalue is explicitly known, i.e. $\lambda_p = 0$ and the principal eigenfunction ϕ_p is also the positive solution of the following eigenvalue problem

$$\int_{\Omega} J\left(\frac{x - y}{g(y)}\right)\frac{\psi(y)}{g^n(y)}\,dy = \rho b(x)\psi.$$

Note that even in these two simplified cases, showing the existence of an eigenfunction is still a difficult task when the domain Ω is unbounded. As observed in [134], Eq. (26) shares many properties with the usual elliptic operators

$$\mathcal{E} := \sigma_{ij}(x)\partial_{ij} + \beta_i(x)\partial_i + c(x).$$

In particular, acting on smooth functions, we can rewrite $\mathcal{M}$

$$\mathcal{M}[u] = \mathcal{E}[u] + \mathcal{R}[u]$$

with $\mathcal{R}$ an operator involving derivatives of higher order that in $\mathcal{E}$.

Indeed, we have

$$\mathcal{M}[u] = \int_{\Omega} k(x,y)\ u(y) - u(x)\, dy - c(x)u,$$

with $c(x) := b(x) - \int_{\Omega} k(x,y)dy$. Using the change of variables $z = x - y$ and performing a formal Taylor expansion of u in the integral, we can rewrite the nonlocal operator as follows

$$\int_{x-\Omega} k(x, x - z)[u(x - z) - u(x)]\, dy = \sigma_{ij}(x)\partial_{ij}u + \beta_i(x)\partial_i u + \mathcal{R}[u]$$

where we use the Einstein summation convention and $\sigma_{ij}(x), \beta_i(x)$, and $\mathcal{R}$ are defined by the following expressions

$$\sigma_{ij}(x) = \frac{1}{2}\int_{x-\Omega} k(x, x - z) z_i\, z_j\, dz,$$

$$\beta_i(x) = \int_{x-\Omega} k(x, x - z) z_i\, dz,$$

$$R[u] := \int_0^1 \int_0^1 \int_0^1 \int_{x-\Omega} k(x, x - z) z_i\, z_j t^2 s \partial_{ijk} u(x + ts\tau z) dt\, ds\, d\tau\, dz.$$

For the second order elliptic operator $\mathcal{E}$, the existence of a principal eigenpair (λ_p, ϕ_p) is well known and various variational formulas characterising the principal eigenvalue exist, see for example [100], [141], [143], [151]–[152]. In particular, Berestycki, Nirenberg and Varadhan [100] give a very simple and general definition of the principal eigenvalue of $\mathcal{E}$ that we recall below. Namely, they define the principal eigenvalue of the elliptic operator $\mathcal{E}$ by the following quantity:

$$\lambda_1 := \sup\ \{\lambda \in \mathbb{R}\, |\exists \phi \in C(\Omega), \phi > 0, \text{such that } E[\phi] + \lambda\phi \leq 0\}. \quad (29)$$

We adopt the definition of Berestycki, Nirenberg and Varadhan for the definition of the principal eigenvalue of the operator $\mathcal{M}$. The principal eigenvalue of the operator $\mathcal{M}$ is then given by the following quantity:

$$\lambda_p(\mathcal{M}) \coloneqq \sup\ \{\lambda \in \mathbb{R}\, |\exists \phi \in C(\Omega), \phi > 0, \text{such that } M[\phi] + \lambda\phi \leq 0\}.$$

To make more explicit the dependence of the different parameters and to simplify the presentation of the results, we shall adopt the following notations:

- Let A and B be two sets, we denote $A \Subset B$ the compact inclusion $A \subset\subset B$.
- $a(x) := -b(x)$.
- $\sigma := \sup\limits_{\Omega}\ a(x)$.
- $d\mu$ is the measure defined by $:= \frac{dx}{g^n(x)}$.
- $\mathcal{L}_\Omega\,[u] := \int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{u(y)}{g^n(y)}\, dy = \int_\Omega J\left(\frac{x-y}{g(y)}\right) u(y) d\mu.$
- $\mathcal{M} := \mathcal{M}_\Omega := \mathcal{L}_\Omega + a(x)Id.$

With this new notation the principal eigenvalue of $\mathcal{M}_\Omega$ can be rewritten as follows

$$\lambda_p(\mathcal{M}_\Omega) := \sup\ \{\lambda \in \mathbb{R}\, |\exists \phi \in C(\Omega), \phi > 0, \text{such that } \mathcal{L}_\Omega\,[\phi] + (a(x) + \lambda)\phi \leq 0\}. \quad (30)$$

Under the assumptions $(H1)$–$(H4)$, the principal eigenvalue $\lambda_p(\mathcal{M}_\Omega)$ is well defined.

Obviously, λ_p is monotone with respect to the domain, the zero order term $a(x)$ and J. Moreover, λ_p is a concave function of its argument and is Lipschitz continuous with respect to $a(x)$. We have

Theorem (3.2.1)[118]: (Sufficient condition). Assume that Ω, J, g and a satisfy $(H1)$– $(H4)$. Let us denote $\sigma := \sup_{\bar{\Omega}} a(x)$ and assume further that the function $a(x)$ satisfies $\frac{1}{\sigma - a(x)} \notin L^1_{d\mu}(\Omega_0)$ for some bounded domain $\Omega_0 \subset \bar{\Omega}$. Then there exists a principal eigenpair (λ_p, ϕ_p) solution of (28). Moreover, $\phi_p \in C(\Omega), \phi_p > 0$ and we have the following estimate

$$-\sigma' < \lambda_p < -\sigma,$$

where $\sigma := \sup_{x \in \Omega} \left[a(x) + \int_\Omega J\left(\frac{y-x}{g(x)}\right) dy\, g^n(x)\right]$.

Note that the theorem holds true whenever Ω is bounded or not.

The condition $\frac{1}{\sigma - a(x)} \notin L^1_{d\mu}(\Omega_0)$ is sharp in the sense that if $\frac{1}{\sigma - a(x)} \in L^1_{d\mu,loc}(\Omega)$ then we can construct an operator $\mathcal{M}_\Omega$ such that Eq. (28) does not have a principal eigenpair. This is discussed, where such an operator is constructed. We want also to stress that the boundedness of the open set Ω does not ensure the existence of an eigenfunction. In contrast with the elliptic case, the sufficient condition has nothing to do with the regularity of the functions $a(x)$, J or g. This means that in general improving the regularity of the coefficients does not ensure at all the existence of an eigenpair. However, in low dimension of space $n = 1, 2$ the condition $\frac{1}{\sigma - a(x)} \notin L^1_{d\mu}(\Omega_0)$ can be related to a regularity condition on the coefficient $a(x)$. Indeed, in one dimension if a is Lipschitz continuous and achieves a maximum in Ω then the condition $\frac{1}{\sigma - a(x)} \notin L^1_{d\mu}(\Omega_0)$ is automatically satisfied. Similarly, when $n = 2$ the non-integrability condition is always satisfied when $a(x) \in C^{1,1}(\Omega)$ and achieves a maximum in Ω. We have the following:

Theorem (3.2.2)[118]: Assume that Ω, J, g and a satisfy $(H1)$– $(H4)$, that a achieves a global maximum at some point $x_0 \in \Omega$. Then there exists a principal eigenpair $(\lambda p, \phi_p)$ solution of (28) in the following situations

(a) $n = 1, a(x) \in C^{0,1}(\Omega)$,

(b) $n = 2, a(x) \in C^{1,1}(\Omega)$,

(c) $n \geq 3, a(x) \in C^{n-1,1}(\Omega), \forall k < n, \partial^k a(x_0) = 0$.

One of the most interesting properties of the principal eigenvalue for an elliptic operator $\mathcal{E}$ is its relation with the existence of a maximum principle for E. Indeed, Berestycki et al. [100] have shown that there exists a strong relation between the sign of this principal eigenvalue and the existence of a maximum principle for the elliptic operator $\mathcal{E}$. They have proved the following theorem

Theorem (3.2.3)[118]: (BNV). Let Ω be a bounded open set, then $\mathcal{E}$ satisfies a refined maximum principle if and only if $\lambda_1 > 0$.

It turns out that when the principal eigenpair exists for $\mathcal{M}$, we can also obtain a similar relation between the sign of the principal eigenvalue of $\mathcal{M}$ and some maximum principle property. We define the maximum principle property satisfied by M:

Definition (3.2.4)[118]: (Maximum principle). When Ω is bounded, we say that the maximum principle is satis-fied by an operator $\mathcal{M}_\Omega$ if for all function $u \in C(\bar{\Omega})$ satisfying

$$\mathcal{M}_\Omega [u] \leq 0 \text{ in } \Omega,$$
$$u \geq 0 \text{ in } \partial\Omega,$$

then $u \geq 0$ in Ω.

With this definition of maximum principle, we show

Theorem (3.2.5)[118]: Assume that Ω is a bounded set and let J, g and a be as in Theorem (3.2.1). Then the maximum principle is satisfied by $\mathcal{M}_\Omega$ if and only if $\lambda_p(\mathcal{M}_\Omega) \geq 0$.

Note that there is a slight difference between the criteria for elliptic operators and for nonlocal ones. To have a maximum principle for nonlocal operator it is sufficient to have a non-negative principal eigenvalue, which is untrue for an elliptic operator where a strict sign of λ_p is required.

The last result is an application of the sufficient condition for the existence of a principal eigenpair to obtain a simple criterion for the existence/non-existence of a positive solution of the following semilinear problem:

$$\mathcal{M}_\Omega[u] + f(x,u) = 0 \text{ in } \Omega, \tag{31}$$

where f is a KPP type nonlinearity. Such type of equation naturally appears in some ecological problems when in addition to the dispersion of the individuals in the environment, the birth and death of these individuals are also modelled, see [146]–[149]. On f we assume that:

$$\begin{cases} f \in C(\mathbb{R} \times [0,\infty) \text{ and is differentiable with respect to } u, \\ f_u(\cdot,0) \text{ is Lipschitz}, \\ f(\cdot,0) \equiv 0 \text{ and } f(x,u)/u \text{ is decreasing with respect to } u, \\ \text{there exists } M > 0 \text{ such that } f(x,u) \leq 0 \text{ for all } u \geq M \text{ and all } x. \end{cases} \tag{32}$$

The simplest example of such a nonlinearity is

$$f(x,u) = u(\mu(x) - u),$$

where $\mu(x)$ is a Lipschitz function.

Such type of problem has received recently a lot of attention, see [122], [147]–[149]. In particular, for Ω bounded and for a symmetric kernel J Hutson et al. [147] have shown that there exists a unique non-trivial stationary solution (31) provided that some principal eigenvalue of the linearised operator around the solution 0 is positive. This result can be extended to more general kernel J using the definition of principal eigenvalue (30). We show that:

Theorem (3.2.6)[118]: Assume Ω, J, g and a satisfy $(H1)$– $(H4)$, Ω is bounded, $a(x) \leq 0$ and f satisfies (32). Then there exists a unique non-trivial solution of (31) when

$$\lambda_p\left(\mathcal{M}_\Omega + f_u(x,0)\right) < 0,$$

where λ_p is the principal eigenvalue of the linear operator $\mathcal{M}_\Omega + f_u(x,0)$. Moreover, if $\lambda_p \geq 0$ then any non-negative uniformly bounded solution of (31) is identically zero.

As a consequence, we can derive the asymptotic behaviour of the solution of the evolution problem associated to (31):

$$\frac{\partial u}{\partial t} = \mathcal{M}_\Omega[u] + f(x,u) \text{in } \mathbb{R}^+ \times \Omega, \tag{33}$$

$$u(0,x) = u_0(x) \text{in } \Omega. \tag{34}$$

Namely, the asymptotic behaviour of u(t, x) as t → +∞ is described in the following theorem:

Note that this criterion involves only the sign of λ_p and does not require any conditions on the function $f_u(x,0)$ ensuring the existence of a principal eigenfunction. Therefore, even in a situation where no principal eigenfunction exists for the operator $\mathcal{M}_\Omega + f_u(x,0)$ we still have information on the survival or the extinction of the considered species. Observe also that the condition obtained on the principal eigenvalue of the linearised operator is sufficient and necessary for the existence of a non-trivial solution.

The above results can be easily extended to the case of a dispersal kernel $k(x,y)$ which satisfies the following conditions:

$$k(x,y) \in C_c(\Omega \times \Omega), k \geq 0, \int_\Omega k(x,y)dy < +\infty, \qquad \forall x \in \Omega, (\widetilde{H}1)$$

$$\exists c_0 > \epsilon_0, 0 > 0 \text{ such that} \min_{x\in\Omega} \left(\min_{y\in B(x,\epsilon_0)} k(x,y)\right) > c_0. (\widetilde{H}2)$$

An example of such kernel is given by

$$k(x,y) = J\left(\frac{x_1 - y_1}{g_1(y)}; \frac{x_2 - y_2}{g^2(y)}; \dots; \frac{x_n - y_n}{g^n(y)}\right)\frac{1}{\prod_{i=1}^n g_i(y)},$$

with $0 < \alpha_i \leq g_i \leq \beta_i$.

We want also to emphasize that the condition that J or k has a compact support is only needed to construct an eigenpair when Ω is unbounded. For a bounded domain, all the results will also hold true if J is not assume compactly supported in Ω.

Note that the assumption $J(0) > 0$ implies that the operator $\mathcal{L}_\Omega$ is not trivial on any open subset $\omega \subset \Omega$, i.e. $\forall \omega \subset \Omega$, $\forall u \in C(\Omega)$, $\mathcal{L}_\Omega[u] = 0$ for $x \in \omega$. This condition makes sure that the principal eigenfunction ϕ_p is positive in Ω, which is a necessary condition for the existence of such principal eigenfunction. Indeed, when there exists an open subset $\omega \subset \Omega$ such that $\mathcal{L}_\Omega$ is trivial, there is no guarantee that a principal eigenpair exists. For example, this is the case for the operator $\mathcal{M}_\Omega$ where $\Omega := (-1,1)$, J is such that $\text{supp}(J) \subset (\frac{1}{2},1)$ and $3 \leq g \leq 4$. In this situation, we easily see that for any $x \in \left(-\frac{1}{4},\frac{1}{4}\right)$ and for any function $u \in C(\Omega)$, we have $\mathcal{L}_\Omega[u](x) = 0$. Therefore, the existence of an eigenfunction will strongly depend on the behaviour of the function $a(x)$ on this subset, i.e. $\left(\lambda_p + a(x)\right)\phi \equiv 0$ for $x \in \left(-\frac{1}{4},\frac{1}{4}\right)$. If $\left(\lambda_p + a(x)\right) \neq 0$ then $\phi \equiv 0$ in $\left(-\frac{1}{4},\frac{1}{4}\right)$. In this situation there is clearly no existence of a positive principal eigenfunction. However, the condition $J(0) > 0$ can still be relaxed and the above theorems hold also true if we only assume that the kernel J is such that there exists a positive integer $p \in \mathbb{N}_0$ such that the following kernel $J_p(x,y)$ satisfies $(\widetilde{H}\,2)$ where $J_p(x,y)$ is defined by the recursion

$$J_1(x,y) := J\left(\frac{x - y}{g(y)}\right)\frac{1}{g^n(y)},$$

$$J_{p+1}(x,y) := \int_\Omega J_1(x,z)J_p(z,y)dz \text{ for } p \geq 1.$$

The above condition is slightly more general that $J(0) > 0$ and we see that $J(0) > 0$ implies that J_1 satisfies $(\widetilde{H}2)$. In particular, as showed for example in [132], for a convolution operator $K(x,y) := J(x - y)$, this new condition is optimal and can be related to a geometric condition on the convex hull of $\{y \in \mathbb{R}^n \mid J(y) > 0\}$:

There exists $p \in \mathbb{N}^*$, such that J_p satisfies $(\widetilde{H}2)$ if and only if the convex hull of $\{y \in \mathbb{R}^n \mid J(y) > 0\}$ contains 0.

We also want to stress that we can easily extend the results of Theorems (3.2.5) and (3.2.6) to a periodic setting using the above generalisation on general non-negative kernel. If we consider the following problem

$$\frac{\partial u}{\partial t} = \mathcal{M}\,\mathbb{R}^n\,[u] + f\,(x,u)\text{in } \mathbb{R}^n \times \mathbb{R}^+, \quad (35)$$

where g and $f\,(.,u)$ are assumed to be periodic functions then the existence of a unique non-trivial periodic solution of (35) is uniquely conditioned by the sign of the periodic principal eigenvalue $\lambda_{p,per}\big(\mathcal{M}\mathbb{R}^n + f_u(x,0)\big)$, where $\lambda_{p,per}$ is defined as follows:

$\lambda_{p,per}(M) := \sup\ \{(\lambda \in \mathbb{R}) | \exists\psi > 0, \psi \in C_{per}(\mathbb{R}^n)$ such that $\mathcal{M}\mathbb{R}^n\,[\psi] + \lambda\psi \leq 0\}$.

Using the periodicity, we have

$$\lambda_{p,per}(\,\mathcal{M}\mathbb{R}^n + f_u(x,0) \;= \lambda_p\,\big(\mathcal{L}_Q + f_u(x,0), Q\big),$$

where Q is the unit periodic cell and $\mathcal{L}_Q\,[\psi] := \int_Q k(x,y)u(y)dy$ with k a positive kernel satisfying $(\widetilde{H}1)$ and $(\widetilde{H}\,2)$. Hence the analysis of the existence/non-existence of stationary solutions of (35) will be handled through the analysis of the existence/non-existence of stationary solutions of a semilinear KPP problem defined on a bounded domain.

Finally, along the analysis, provided a more restrictive assumption on the coefficient $a(x)$ is made, we also observe that Theorem (3.2.1) holds as well when we relax the assumption on the function g and allow g to touch 0. Assuming that g satisfies

$$g \in L^\infty(\Omega), 0 \leq g \leq \beta, \frac{1}{g^n} \in L^p_{loc}(\bar{\Omega}) \text{ with } p > 1\ (\widetilde{H}\,3)$$

then for a bounded domain Ω, we have the following result:

As a consequence the criterion on the survival/extinction of a species obtained in Theorems (3.2.5) and (3.2.6) can be extended to such type of dispersal kernel. We have

Theorem (3.2.7)[118]: Assume Ω, J and g satisfy (H1), $(\widetilde{H}2)$, $(\widetilde{H}3)$, Ω is bounded and f satisfies (32). Then there exists a unique non-trivial solution of (31) if

$$\lambda_p\,\big(\mathcal{M}_\Omega + f_u(x,0)\big) < 0,$$

where λ_p is the principal eigenvalue of the linear operator $\mathcal{M}_\Omega + f_u(x,0)$. Moreover, if $\lambda_p \geq 0$ then any non-negative uniformly bounded solution is identically zero.

Theorem (3.2.8)[118]: Let Ω, J, g, b and f be as in Theorem (3.2.7). Let u_0 be an arbitrary bounded and continuous function in Ω such that $u_0 \geq 0, u_0 \not\equiv 0$. Let $u(t,x)$ be the solution of (33) with initial datum $u(0,x) = u_0(x)$. Then, we have:

(i) If 0 is an unstable solution of (31) (that is $\lambda_p < 0$), then $u(t,x) \to p(x)$ pointwise as $t \to \infty$, where p is the unique positive solution of (31) given by Theorem (3.2.7).

(ii) If 0 is a stable solution of (31) (that is $\lambda_p \geq 0$), then $u(t,x) \to 0$ pointwise in Ω as $t \to +\infty$.

The existence of a simple sufficient condition for the existence of a principal eigenpair when Ω is an unbounded domain is more involved and we have to make a technical assumption on the set $\Sigma := \{x \in \Omega \mid g(x) = 0\}$. We show

Theorem (3.2.9)[118]: Assume that Ω, J and a satisfy $(\widetilde{H}1), (\widetilde{H}2), (\widetilde{H}4)$ and g satisfies $(\widetilde{H}3)$. Let us denote $\sigma := \sup_{\bar{\Omega}} a(x)$ and let Γ, Σ be the following sets

$$\Gamma := \{x \in \bar{\Omega} | \, a(x) = \sigma\},$$
$$\Sigma := (\, x \in \bar{\Omega} | \, g(x) = 0\,\}.$$

Assume further that $\Omega \cap \Sigma \Subset \Omega$ and $\dot{\Gamma} = \emptyset$. Then there exists a principal eigenpair $\big(\lambda_p, \phi_p\big)$ solution of (28). Moreover, $\phi_p > 0$ and we have the following estimate

$$-\sigma' < \lambda_p < -\sigma,$$

where $\sigma := \sup_{x\in\Omega} \left[a(x) + \int_\Omega J\left(\frac{y-x}{g(x)}\right)\frac{dy}{g^n(x)}\right]$.

We review some spectral theory of positive operators and we recall some Harnack's inequalities satisfied by a positive solution of integral equation. Then, we prove Theorems (3.2.25) and (3.2.24). The relation between the maximum principle and the sign of the principal eigenvalue Theorem (3.2.5) and a counterexample to the existence of a principal eigenpair are obtained respectively. We devoted to the derivation of the survival/extinction criteria (Theorems (3.2.5), (3.2.6), (3.2.20)).

We first recall some results on the spectral theory of positive operators and some Harnack's inequalities satisfied by a positive solution of

$$\mathcal{L}_\Omega[u] - b(x)u = 0, \tag{36}$$

where $\mathcal{L}_\Omega$ is defined as above and $b(x)$ is a positive continuous function in Ω. Let us start with the spectral theory.

Let us recall some basic spectral results for positive operators due to Edmunds, Potter and Stuart [142] which are extensions of the Krein–Rutman theorem for positive non-compact operators. A cone in a real Banach space X is a non-empty closed set K such that for all $x, y \in K$ and all $\alpha \geq 0$ one has $x + \alpha y \in K$, and if $x \in K, -x \in K$ then $x = 0$. A cone K is called reproducing if $X = K - K$. A cone K induces a partial ordering in X by the relation $x \leq y$ if and only if $x - y \in K$. A linear map or operator $T : X \to X$ is called positive if $T(K) \subseteq K$. The dual cone K^* is the set of functional $x^* \in X^*$ which are positive, that is, such that $x^*(K) \subset [0, \infty)$.

If $T : X \to X$ is a bounded linear map on a complex Banach space X, its essential spectrum (according to Browder [123]) consists of those λ in the spectrum of T such that at least one of the following conditions holds: (1) the range of $\lambda I - T$ is not closed, (2) λ is a limit point of the spectrum of A, (3) $\bigcup_{n=1}^{\infty} ker((\lambda I - T)^n)$ is infinite dimensional. The radius of the essential spectrum of , denoted by $r_e(T)$, is the largest value of |λ| with λ in the essential spectrum of T. For more properties of $r_e(T)$ see [150].

Theorem (3.2.10)[118]: (Edmunds, Potter, Stuart). Let K be a reproducing cone in a real Banach space X, and let $T \in \mathcal{L}(X)$ be a positive operator such that $T^p(u) \geq cu$ for some $u \in K$ with $\|u\| = 1$, some positive integer p and some positive number c. Then if $c^{\frac{1}{p}} > r_e(T_c)$, T has an eigenvector $v \in K$ with associated eigenvalue $\rho \geq c^{\frac{1}{p}}$ and T^* has eigenvector $v^* \in K^*$ corresponding to the eigenvalue ρ. Moreover, ρ is unique.

A proof of this theorem can be found in [142].

We present some Harnack's inequality satisfied by any positive continuous solution of the nonlocal equation (36).

Theorem (3.2.11)[118]: (Harnack inequality). Assume that Ω, J, g and $b > 0$ satisfy $(H1), (\widetilde{H}2), (H3), (H4)$. Let $\omega \Subset \Omega$ be a compact set. Then there exists $C(J, \omega, b, g)$ such that for all positive continuous bounded solutions u of (36) we have

$$u(x) \leq Cu(y) \text{ for all } x, y \in \omega.$$

When the assumption on g is relaxed the above Harnack's estimate does not hold any more but a uniform estimate still holds.

Theorem (3.2.12)[118]: (Local uniform estimate). Assume that Ω, J, g and $b > 0$ satisfy $(H1), (\widetilde{H}2), (\widetilde{H}3), (H4)$. Assume that $\Omega \cap \Sigma \Subset \Omega$ and let $\omega \subset \Omega$ be a compact set. Let $\Omega(\omega)$ denote the following set

$$\Omega(\omega) := \bigcup_{x\in\omega} B(x,\beta).$$

Then there exists a positive constant η^* such that, for any $0 < \eta \le \eta^*$, there exist a compact set $\omega' \Subset \Omega(\omega) \cap \Omega$ and a constant $C(J,\omega,\Omega,\omega',b,g,\eta)$ such that the following assertions are verified:

(i) $\left\{x \in \Omega(\omega) \cap W_\eta \mid d\left(x, \partial\left(\Omega(\omega) \cap W_\eta\right)\right) > \eta\right\} \subset \omega'$, where $W_\eta := \{x \in \Omega \mid g(x) > \eta\}$,

(ii) for all positive continuous solution u of (36), the following inequality holds:

$$u(x) \le Cu(y) \text{ for all } x \in \omega, y \in \omega' \cap \omega.$$

We present a contraction lemma which guarantees that when Ω is bounded then any continuous positive solution u of Eq. (36) is bounded in Ω.

Lemma (3.2.13)[118]: (Contraction lemma). Let $\Omega \subset \mathbb{R}^n$ and $u \in C(\Omega)$ be respectively an open set and a positive solution of (36). Then there exists $\epsilon^* > 0$ such that for all ϵ^*, there exists Ω and $C(\alpha,\beta,J,\epsilon,b)$ such that

$$\int_{\Omega_\epsilon} u(y)dy \ge C \int_\Omega u(y)dy.$$

Moreover, Ω satisfies the following chain of inclusion

$$\{x \in \Omega \mid d(x,\partial\Omega) > \alpha\epsilon\} \subset \Omega_\epsilon \subset \left\{x \in \Omega \mid d(x,\partial\Omega) > \frac{\alpha\epsilon}{2}\right\}.$$

A proof of these results can be found in [134].

We prove the criterion of existence of a principal eigenpair (Theorems (3.2.25), (3.2.24) and (3.2.8)). That is, we prove the existence of a solution (λ_p,ϕ_p) of the equation

$$\mathcal{L}_\Omega\left[\phi_p\right] + a(x)\phi_p = -\lambda_p\phi_p \text{ in } \Omega \tag{37}$$

with $\phi_p > 0, \phi_p \in C(\Omega)$ and λ_p is the principal eigenvalue of $\mathcal{L}_\Omega + a(x)$ defined by (30). We first restrict our analysis to the case of a bounded domain Ω and then prove the criterion for unbounded domains. Each of them dedicated to one situation.

We will first concentrate our attention on the construction of a principal eigenpair when J, g, b satisfy the assumptions (H2)–(H4) (Theorem (3.2.1)). Then we provide an argumentation for the construction of a principal eigenpair when the assumptions on g are relaxed (Theorem (3.2.20)).

In a first step, let us show that the eigenvalue problem (37) admits a positive solution, i.e. there exists $(\mu_1, 0, \phi_1)$ with $\phi_1 > 0, \phi_1 \in L^\infty(\Omega) \cap C(\Omega)$ solution of (37). We show

Theorem (3.2.14)[118]: Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and assume that J, g, and $a(x)$ satisfy (H1)–(H4). Let us denote $\sigma := \sup_{\overline{\Omega}} a(x)$ and $\Omega_\theta := \{x \in \Omega \mid d(x,\partial\Omega) > \theta\}$. Assume further that the function $a(x)$ satisfies $\frac{1}{\sigma - a(x)} \notin L^1_{d\mu}(\bar{\Omega})$. Then there exists $\theta_0 > 0$ such that for all $\theta \le \theta_0$ the operator $\mathcal{L}_{\Omega_\theta} + a(x)$ has a unique eigenvalue $\mu_{1,\theta}$ in $C(\Omega_\theta)$, that is to say, there is a unique $\mu_{1,\theta} \in \mathbb{R}$ such that

$$\mathcal{L}_{\Omega_\theta}[\phi_1] + a(x)\phi_1 = -\mu_{1,\theta}\phi_1 \text{ in } \Omega_\theta \tag{38}$$

admits a positive solution $\phi_1 \in C(\bar{\Omega}_\theta)$. Moreover, $\mu_{1,\theta}$ is simple (i.e. the space of $C(\bar{\Omega}_\theta)$ solutions to (37) is one-dimensional) and satisfies

$$\mu_{1,\theta} < -\max_{\Omega_\theta} a(x).$$

To conclude the proof of Theorem (3.2.1) which establishes the criterion of existence of an eigenpair, we are left to show that the principal eigenvalue defined by (30) is the same as the one obtained in Theorem (3.2.14) for $\theta = 0$. Namely, we are reduced to prove of the following results.

Lemma (3.2.15)[118]: Let $a(x)$ be as in Theorem (3.2.14) then we have $\lambda_p = \mu_{1,0}$ where λ_p and $\mu_{1,0}$ are respectively the principal eigenvalue of $\mathcal{L}_\Omega + a(x)$ defined by (30) and the eigenvalue of $\mathcal{L}_\Omega + a(x)$ obtained in Theorem (3.2.14).

Before proving Theorem (3.2.14), let us prove the above lemma.

Proof. First, let us define the following quantity

$$\lambda'_p := \sup \{\lambda \in \mathbb{R} | \exists \phi > 0, \phi \in C(\Omega) \text{so that} \mathcal{L}_\Omega [\phi] + a(x)\phi + \lambda\phi \leq 0 \; in \; \bar{\Omega}\}.$$

Obviously λ'_p is well defined and is sharing the same properties than λ_p . Moreover, we have $\lambda'_p \leq \lambda_p$. Let us now show that $\lambda'_p = \mu_{1,0}$. First by definition of λ'_p we easily have $\lambda'_p \geq \mu_{1,0}$. Now to obtain the equality $\lambda'_p = \mu_{1,0}$ we argue by contradiction. Assume that $\lambda'_p > \mu_{1,0}$. By definition of λ'_p there exists $\psi > 0, \psi \in C(\bar{\Omega})$ such that

$$\mathcal{L}_\Omega [\psi] + (a(x) + \lambda) \; \psi \leq 0 \text{ in } \bar{\Omega}. \tag{39}$$

Observe that we can rewrite $\mathcal{L}_\Omega [\phi_1] + a(x)\phi_1$ as follows

$$\mathcal{L}_\Omega [\phi_1] + a(x)\phi_1 = \int_\Omega J \left[\frac{x - y}{g(y)}\right] \frac{\phi_1(y)}{g(y)} dy + a(x)\phi_1$$

$$= \int_\Omega J \left[\frac{x - y}{g(y)}\right] \frac{\psi(y)\phi_1(y)}{\psi(y)g(y)} dy + a(x)\frac{\phi_1(x)}{\psi(x)} \psi(x).$$

From (39), we find that

$$a(x)\psi - \mathcal{L}_\Omega [\psi] - \lambda\psi$$

and it follows that

$$\mathcal{L}_\Omega [\phi_1] + a(x)\phi_1 \leq \int_\Omega J \left[\frac{x - y}{g(y)}\right] \frac{\psi(y)}{g(y)} \left[\frac{\phi_1(y)}{\psi(y)} - \frac{\phi_1(x)}{\psi(x)}\right] dy - \lambda \frac{\phi_1(x)}{\psi(x)} \psi(x).$$

By using the definition of $\mu_{1,0}$, we end up with the following inequality

$$\int_\Omega J \left[\frac{x - y}{g(y)}\right] \frac{\psi(y)}{g(y)} \left[\frac{\phi_1(y)}{\psi(y)} - \frac{\phi_1(x)}{\psi(x)}\right] dy \geq (\lambda - \mu_{1,0})\phi_1 > 0. \tag{40}$$

Let us denote $w := \frac{\phi_1}{\psi}$. Observe that by (39) $w \in L^\infty \cap C(\bar{\Omega})$, therefore w achieves a global maximum somewhere in Ω, say at x. By using the inequality (40) at the point x, we find the following contradiction

$$0 < \int_\Omega J \left[\frac{x - y}{g(y)}\right] \frac{\psi(y)}{g(y)} \; [w(y) - w(\bar{x})]dy \leq 0.$$

Thus $\mu_{1,0} = \lambda'_p$.

Observe now that if there exists a positive eigenfunction $\psi \in C(\Omega) \cap L^\infty(\Omega)$ associated to the principal eigenvalue λ_p , i.e. $\mathcal{L}_\Omega [\psi] + (a(x) + \lambda_p)\psi = 0$, then we have $\psi \in C(\bar{\Omega})$. Therefore, using the definition of λ'_p it follows that $\lambda_p \leq \lambda'_p = \mu_{1,0} \leq \lambda_p$. To conclude the proof, we are left to show that such bounded function ψ exists.

So let $(\theta_n)_{n\in\mathbb{N}}$ be a positive sequence which converges to 0 and consider the sequence of set $(\Omega_{\theta_n})_{n\in\mathbb{N}}$ defined in Theorem (3.2.14). By construction, using the monotonicity property of the principal eigenvalue with respect to the domain ((i) of Proposition (3.2.25)) we deduce that $\left(\lambda'_p\left(\mathcal{L}_{\Omega_{\theta_n}} + a(x)\right)\right)_{n\in\mathbb{N}}$ is a non-increasing bounded sequence. Namely, we have for all $n \in \mathbb{N}$

$$\lambda_p(\mathcal{L}_\Omega + a(x)) \le \lambda_p' \left(\mathcal{L}_{\Omega_{\theta_{n+1}}} + a(x)\right) \le \lambda_p' \left(\mathcal{L}_{\Omega_{\theta_n}} + a(x)\right).$$

Thus, as n goes to infinity $\lambda_p'(\mathcal{L}_{\Omega_{\theta_n}} + a(x))$ converges to some $\bar{\lambda} \ge \lambda_p$.

On another hand since θ_n tends to 0, by Theorem (3.2.14), there exists n_0 so that for all $n \ge n_0$, a principal eigenpair $(\mu_{1,\theta_n}, \phi_n)$ exists for the operator $\mathcal{L}_{\Omega_{\theta_n}} + a(x)$. Arguing as above, we conclude that $\mu_{1,\theta_n} = \lambda_p'(\mathcal{L}_{\Omega_{\theta_n}} + a(x))$. We claim that:

Assume for the moment that the claim holds. Then the final argumentation goes as follows. Next, let us normalised ϕ_n so that $\sup_{\Omega_{\theta_n}} \phi_n = 1$. With this normalisation $(\phi_n)_{n\in\mathbb{N}}$ is a uniformly bounded sequence of continuous functions. So by a standard diagonal extraction argument, there exists a subsequence still denoted $(\phi_n)_{n\in\mathbb{N}}$ such that $(\phi_n)_{n\in\mathbb{N}}$ converges locally uniformly to a non-negative bounded continuous function ψ. Furthermore, ψ satisfies

$$\mathcal{L}_\Omega [\psi] + \left(a(x) + \bar{\lambda}\right)\psi = 0.$$

Now recall that $\left(\mu_{1,\theta_n}, \phi_n\right)$ satisfies

$$\mathcal{L}_{\Omega_{\theta_n}} [\phi_n] + a(x)\phi_n + \mu_{1,\theta_n} \phi_n = 0.$$

Using the above claim, we have $\mu_{1,\theta_n} < -\sigma = -\sup_\Omega a(x) - \sup_{\Omega_{\theta_n}} a(x)$ for n big enough, so $\sup_{\Omega_{\theta_n}} \left(a(x) + \mu_{1,\theta_n}\right) < 0$ and the uniform estimates i.e. Theorem (3.2.12) applies to ϕ_n. Thus we have for $\eta > 0$ small fixed independently of n

$$1 \le C(\eta)\phi_n(x) \text{for all } x \in \left\{x \in \Omega_{\theta_n} \middle| \, d\left(x, \partial\Omega_{\theta_n}\right) > \eta\right\}.$$

Therefore ψ is non-trivial and (λ, ψ) solves the eigenvalue problem (37). Using once again the equation satisfied by ψ and the definition of λ_p , we easily obtain that $\bar{\lambda} \le \lambda_p \le \bar{\lambda}$ which proves that ψ is our desired eigenfunction associated to λ_p .

Let us turn our attention to the proof of Claim (3.2.17). But before proving the claim let us establish the following useful estimate.

Lemma (3.2.16)[118]: There exist positive constants r and c_0 so that

$$\forall x \in \bar{\Omega}, \int_{B_r(x)\cap\bar{\Omega}} J\frac{x - y}{g(y)} \, u(y)d\mu(y) \ge c_0 \int_{B_r(x)\cap\bar{\Omega}} u(y)d\mu(y).$$

Proof. Since J is continuous and $J(0) > 0$, there exist $\delta > 0$ and $c_0 > 0$ so that for all $z \in B(0,\delta)$ we have $J(z)c_0$.

Observe that for all $(x,y) \in \bar{\Omega} \times B_r(x)$ with $< \frac{\delta\alpha}{2}$, using that $g \ge \alpha > 0$, we have

$$\left\|\frac{x - y}{g(y)}\right\| \le \frac{2r}{\alpha} \le \delta.$$

Thus, for $r < \frac{\delta\alpha}{2}$ and $y \in B_r(x)$ we have $J\left(\frac{x-y}{g(y)}\right) > c_0$, and the estimate follows.

Claim (3.2.17)[118]: There exists $n_1 \in \mathbb{N}$ such that for all $n \ge n_1$ we have $\mu_{1,\theta_n} < -\sigma = -\sup_\Omega a(x)$.

Proof. Let us denote by σ the maximum of $a(x)$ in $\bar{\Omega}$. By assumption, we have $\frac{1}{\sigma - a(x)} \notin L^1_{d\mu,loc}(\bar{\Omega})$. So there exists $x_0 \in \bar{\Omega}$ such that $\frac{1}{\sigma - a(x)} \notin L^1_{d\mu}(B_r(x_0) \cap \bar{\Omega})$ and for small enough, say 0, we have

$$c_0 \int_{\Omega \cap B(x_0,r)} \frac{d\mu}{-(a(x) - \sigma + \epsilon)} \geq 4.$$

Choose n_1 big enough, so that for all $n \geq n_1, B_r(x_0) \cap \Omega_{\theta_n} = \emptyset$. For 0, since $\Omega_{\theta_n} \to \Omega$, we can increase n_1 if necessary to achieve for all $n \geq n_1$

$$c_0 \int_{\bar{\Omega}_{\theta_n} \cap B(x_0,r)} \frac{d\mu}{-(a(x) - \sigma - \epsilon)} \geq 2. \tag{41}$$

Recall now that for n big enough, say $n \geq n_2$, there exists $(\mu_{1,\theta_n}, \phi_n)$ that satisfies the equation

$$\mathcal{L}_{\Omega_{\theta_n}}[\phi_n] + a(x)\phi_n + \mu_{1,\theta_n}\phi_n = 0.$$

Since ϕ_n is positive we have

$$\mathcal{L}_{\bar{\Omega}_{\theta_n} \cap B(x_0,r)}[\phi_n] - a(x) + \mu_{1,\theta_n} \phi_n.$$

Using Lemma (3.2.16), we see that

$$\frac{c_0}{-(a(x) + \mu_{1,\theta_n})} \int_{\bar{\Omega}_{\theta_n} \cap B(x_0,r)} \phi_n(y) d\mu \leq \phi_n(x).$$

Integrating the above inequality on $\Omega_{\theta_n} \cap B(x_0,r)$ it follows that

$$\int_{\bar{\Omega}_{\theta_n} \cap B(x_0,r)} \left(\frac{c_0}{-(a(x) + \mu_{1,\theta_n})} \int_{\bar{\Omega}_{\theta_n} \cap B(x_0,r)} \phi_n(y)\, d\mu \right) d\mu \leq \int_{\Omega_{\theta_n} \cap B(x_0,r)} \phi_n(x) d\mu,$$

$$\int_{\bar{\Omega}_{\theta_n} \cap B(x_0,r)} \left(\frac{c_0}{-(a(x) + \mu_{1,\theta_n})} \right) d\mu \int_{\bar{\Omega}_{\theta_n} \cap B(x_0,r)} \phi_n(y)\, d\mu \leq \int_{\Omega_{\theta_n} \cap B(x_0,r)} \phi_n(x) d\mu.$$

Thus,

$$\int_{\bar{\Omega}_{\theta_n} \cap B(x_0,r)} \left(\frac{c_0}{-(a(x) + \mu_{1,\theta_n})} \right) d\mu \leq 1.$$

From (41), it follows that for all $n \geq \sup\ (n_1, n_2)$ we have

$$\mu_{1,\theta_n} \leq -\sigma - \epsilon\,.$$

Let us now turn our attention to the proof of Theorem (3.2.14).

For convenience, in this proof we write the eigenvalue problem

$$\mathcal{L}_{\Omega_\theta}[u] + a(x)u = -\mu u$$

in the form

$$\mathcal{L}_{\Omega_\theta}[u] + \bar{a}(x)u = \rho u \tag{42}$$

where

$$\bar{a}(x) = a(x) + k, \qquad \rho = -\mu + k$$

and $k > 0$ is a constant such that $\inf_{\Omega_\theta} a > 0$.

Let us now prove the following useful result:

Observe that the proof of Theorem (3.2.14) easily follows from the above lemma. Indeed, if the lemma holds true, since under the assumptions (H1)–(H4) the operator $\mathcal{L}_\Omega : C(\bar{\Omega}_\theta) \to C(\bar{\Omega}_\theta)$ is compact, we have $r_e\left(\mathcal{L}_{\Omega_\theta} + \bar{a}(x)\right) = r_e(\bar{a}(x)) = \bar{\sigma}(\theta)$. Thus $(\bar{\sigma}(\theta) + \delta) > r_e(\mathcal{L}_{\Omega_\theta} + a(x))$ and the existence theorem of Edmunds et al. (Theorem (3.2.10)) applies.

Finally we observe that the principal eigenvalue is simple since for a bounded domain Ω the cone of positive continuous functions has a non-empty interior and, for a sufficiently large p, the operator $\left(\mathcal{L}_{\Omega_{\theta_n}} + \bar{a}\right)^p$ is strongly positive, that is, it maps $u \geq 0, u \not\equiv 0$ to a strictly positive function, see [154].

Lemma (3.2.18)[118]: Let Ω, J, g and a be as in Theorem (3.2.14). Then there exists $\theta_0 > 0$ so that for all $\theta \leq \theta_0$ there exist $\delta > 0$ and $u \in C(\bar{\Omega}_\theta), u \geq 0, u \equiv 0$, such that in $\bar{\Omega}_\theta$

$$\mathcal{L}_{\Omega_\theta}[u] + \bar{a}(x)u \geq (\bar{\sigma} + \delta)u,$$

where $\bar{\sigma}(\theta) := \max_{\bar{\Omega}_\theta} \ a(x)$.

Proof. Let us denote by Γ the closed set where the continuous function a takes its maximum $\bar{\sigma}$ in $\bar{\Omega}$:

$$\Gamma := \{z \in \bar{\Omega} |\, \bar{a}(z) = \bar{\sigma}\} .$$

Since $\bar{a}$ is a continuous function and Ω is bounded, Γ is a compact set. Therefore Γ can be covered by a finite number of balls of radius r, i.e. $\Gamma \subset \bigcup_{i=1}^{\mathbb{N}} B_r(x_i)$ with $x_i \in \Gamma$. By construction, we have $\frac{1}{\sigma - a(x)} = 1/(\sigma - a(x)) \notin L^1_{d\mu,loc}(\Omega)$. Therefore $\frac{1}{\sigma - a(x)} \notin L^1_{d\mu}(\bigcup_{i=1}^{\mathbb{N}} B_r(x_i) \cap \bar{\Omega})$ and there exists $-\lambda_0 > \sigma$ so that for some x_i we have

$$\int_{B_r(x_i)\cap\bar{\Omega}} \frac{c_0}{-\lambda_0 - \bar{a}(x)}\, d\mu \geq 4. \tag{43}$$

Since $\Omega_\theta \rightarrow \Omega$ as θ tends to 0 there exists θ_0 so that for all $\theta \leq \theta_0$ we have

e

$$\int_{B_r(x_i)\cap\bar{\Omega}_\theta} \frac{c_0}{-\lambda_0 - \bar{a}(x)}\, d\mu \geq 2. \tag{44}$$

Let us fix x_i such that (44) holds true and let us denote $\omega_\theta := B_r(x_i) \cap \bar{\Omega}_\theta$. We consider now the following eigenvalue problem

$$c_0 \int_{\omega_\theta} u(y)d\mu(y) + \bar{a}(x)u(x) + \lambda u(x) = 0, \tag{45}$$

where c_0 is the constant obtained in Lemma (3.2.16). We claim that:

Observe that by proving this claim we end the proof of the lemma. Indeed, fix $\theta < \theta_0$ and assume for the moment that this claim holds true. Then there exists (λ_1, ϕ_1) such that

$$c_0 \int_{\omega_\theta} \varphi_1(y)d\mu(y) + \bar{a}(x)\phi_1(x) + \lambda_1\phi_1(x) = 0. \tag{46}$$

Obviously, for any positive constant $\rho, (\lambda_1, \rho\phi_1)$ is also a solution of Eq. (46). Therefore without any loss of generality we can assume that ϕ_1 is such that $\phi_1 \preccurlyeq 1$. Set $\tilde{c}_0 := c_0 \int_{\omega_\theta} \varphi_1(y)d\mu(y)$. From Eq. (46), since$\theta < \phi_1 \leq 1$ we see easily that

$$-\left(\lambda_1 + \bar{a}(x)\right) > \tilde{c}_0.$$

Therefore there exists a positive constant d_0 such that

$$\phi_1 \geq d_0 \text{ in } \omega \tag{47}$$

and

$$-\left(\lambda_1 + \bar{\sigma}(\theta)\right) \geq \tilde{c}_0 > 0. \tag{48}$$

Let us now consider a set $\omega_\epsilon \Subset \omega_\theta$ which verifies

$$\int_{\omega_\theta \backslash \omega_\epsilon} d\mu \frac{d_0|\lambda_1 + \sigma(\theta)|}{2c_0} . \tag{49}$$

Since by construction $\bar{\Omega}_\theta \setminus \omega_\theta$ and ω are two disjoint closed subsets of $\bar{\Omega}_\theta$, the Urysohn's lemma applies and there exists a positive continuous function η such that $0 \le \eta \le 1, \eta(x) = 1$ in ω, $\eta(x) = 0$ in $\bar{\Omega}_\theta \setminus \omega_\theta$.

Next, we define $w := \phi_1 \eta$ and we compute $\mathcal{L}_{\Omega_\theta}[w] + b(x)w$.

Since $w \equiv 0$ in $\bar{\Omega}_\theta \setminus \omega_\theta$, we have

$$\mathcal{L}_{\Omega_\theta}[w] + \bar{a}(x)w = \int_{\omega_\theta} J\left(\frac{x-y}{g(y)}\right) w(y)d\mu(y) \ge (\bar{\sigma}(\theta) + \delta)\, w = 0$$

for any $\delta > 0$.

On another hand, in ω_θ, by using Lemma (3.2.16) we see that

$$\mathcal{L}_{\Omega_\theta}[w] + \bar{a}(x)w = \int_{\omega_\theta} J\left(\frac{x-y}{g(y)}\right) w(y)d\mu(y) + \bar{a}(x)w \quad (50)$$

$$\ge c_0 \int_{\omega_\theta} w(y)d\mu(y) + \bar{a}(x)w \quad (51)$$

$$\ge c_0 \int_{\omega_\epsilon} \phi_1(y)d\mu(y) + \bar{a}(x)w. \quad (52)$$

Since ϕ_1 satisfies Eq. (46), using the estimates (47), (48) and (49) we deduce from the inequality (52) that

$$\mathcal{L}_{\Omega_\theta}[w] + \bar{a}(x)w \ge -\left(\lambda_1 + \bar{a}(x)\right)\phi_1 + \bar{a}(x)w$$

$$- c_0 \int_{\omega_\theta \setminus \omega_\epsilon} \phi_1(y)d\mu(y) \quad (53)$$

$$\ge \frac{|\lambda_1 + \bar{\sigma}(\theta)|}{2}\phi_1 + \left(\bar{\sigma}(\theta) - \bar{a}(x)\right)\phi_1 + \bar{a}(x)w$$

$$+ \frac{d_0|\lambda_1 + \bar{\sigma}(\theta)|}{2} - c_0 \int_{\omega_\theta \setminus \omega_\epsilon} \phi_1(y)d\mu(y) \quad (54)$$

$$\ge \left(\frac{|\lambda_1 + \bar{\sigma}(\theta)|}{2}\right)\phi_1 + \left(\bar{\sigma}(\theta) - \bar{a}(x)\right)\phi_1 + \bar{a}(x)w, \quad (55)$$

where we use in the last inequality, that $\phi_1 \le 1$ and the estimate (49).

Since $(\bar{\sigma}(\theta) - a(x))$ and $\frac{|\lambda_1 + \bar{\sigma}(\theta)|}{2}$ are two positive quantities and $\phi_1 \ge w$, we conclude that

$$\mathcal{L}_{\Omega_\theta}[w] + \bar{a}(x)w\left(\frac{|\lambda_1 + \bar{\sigma}(\theta)|}{2} + \bar{\sigma}(\theta)\right) w. \quad (56)$$

Hence, in $\bar{\Omega}_\theta$, w satisfies

$$\mathcal{L}_{\Omega_\theta}[w] + \bar{a}(x)w \ge (\bar{\sigma}(\theta) + \delta)\, w,$$

with $= \frac{|\lambda_1 + \bar{\sigma}(\theta)|}{2}$, which proves the lemma.

Let us now prove Claim (3.2.19).

Claim (3.2.19)[118]: There exists (λ_1, ϕ_1) solution of (45) so that $\phi_1 \in L^\infty(\omega_\theta) \cap C(\omega_\theta)$ and $\phi_1 > 0$.

Proof. Fix $\theta \le \theta_0$. For $\lambda < -\bar{\sigma}(\theta)$, consider the positive function $\phi_\lambda := \frac{c_0}{-\lambda - a(x)}$. Let us substitute ϕ_λ into Eq. (45), then we have

$$c_0 \int_{\omega_\theta} \phi_\lambda \, d\mu - c_0 = 0.$$

Therefore, we end the proof of Claim (3.2.19) by finding λ such that $\int_{\omega_\theta} \phi_\lambda \, d\mu = 1$. Observe that the functional $F(\lambda) := \int_{\omega_\theta} \phi_\lambda$ dμ is continuous and monotone increasing with respect to λ in $(-\infty, -\bar{\sigma})$. Moreover, by construction, we have:

$$\lim_{\lambda \to -\infty} F(\lambda) = 0 \text{ and } F(\lambda_0) \geq 2.$$

Hence by continuity there exists a λ_1 such that $F(\lambda_1) = 1$.

Now we expose the argumentation for the construction of a principal eigenpair when the assumptions on g are relaxed and prove Theorem (3.2.20). To show Theorem (3.2.20) we follow the scheme of the argument developed above.

Theorem (3.2.20)[118]: Assume that Ω, J and a satisfy $(H1), (H2), (\tilde{H}3), (H4)$, Ω is bounded and g satisfies $(\tilde{H}3)$. Let us denote $\sigma := \sup_{\bar{\Omega}} a(x)$ and let Γ be the following set

$$\Gamma := \{x \in \bar{\Omega} | \ a(x) = \sigma\} .$$

Assume further that $\dot{\Gamma} = \emptyset$. Then there exists a principal eigenpair (λ_p, ϕ_p) solution of (28). Moreover, $\phi_p \in C(\Omega), \phi_p > 0$ and we have the following estimate

$$-\sigma' < \lambda_p < -\sigma,$$

where $\sigma' := \sup_{x \in \Omega} \left[a(x) + \int_\Omega J\left(\frac{y-x}{g(x)}\right) \frac{dy}{g^n(x)} \right]$.

Proof. As above, we can rewrite the eigenvalue problem (37) as follows

$$\mathcal{L}_{\Omega_\theta}[u] + \bar{a}(x)u = \rho u \tag{57}$$

with

$$\bar{a}(x) = a(x) + k, \qquad \rho = -\mu + k$$

and $k > 0$ is a constant such that $\inf_{\Omega_\theta} \bar{a} > 0$.

Observe that under the assumptions (H1), (H2), (H̃3), (H4) the following family

$$\mathcal{L}_{\Omega_\theta}(B_1) := \{L_{\Omega_\theta}[f] / f : \Omega \to \mathbb{R}, \|f\|_\infty \leq 1\}$$

is equicontinuous. Indeed, let $\epsilon > 0$ be fixed. Since $\frac{1}{g^n} \in L^p_{loc}(\bar{\Omega}_\theta)$, there exists $\eta > 0$ such that

$$\int_{\Omega_\theta \cap \{g < \eta\}} \frac{dy}{g^n(y)} < \frac{\epsilon}{4\|J\|_\infty}. \tag{58}$$

From the uniform continuity of J in the unit ball $B(0,1)$, we deduce that there exists $\gamma > 0$ such that for $|w - \bar{w}| < \gamma / \eta$,

$$|J(w) - J(\bar{w})| < \epsilon \eta^n / 2|\Omega_\theta|. \tag{59}$$

A short computation using (58) and (59) shows that for $|x - z| < \gamma$

$$\begin{aligned} \left|\mathcal{L}_{\Omega_\theta}[f](x) - \mathcal{L}_{\Omega_\theta}[f](z)\right| &\leq \int_{\Omega_\theta} \left| J\left[\frac{x-y}{g(y)}\right] - J\left[\frac{z-y}{g(y)}\right] \right| \left|\frac{f(y)}{g^n(y)}\right| dy \\ &\leq 2\|J\|_\infty \int_{\Omega_\theta \cap \{g < \eta\}} \frac{1}{g^n(y)} dy \\ &\quad + \frac{1}{\delta^n} \int_{\Omega_\theta \cap \{g \geq \eta\}} |J\left[\frac{x-y}{g(y)}\right] - J\left[\frac{z-y}{g(y)}\right] dy \leq \epsilon \end{aligned}$$

Hence, $\mathcal{L}_{\Omega_\theta}(B_1)$ is equicontinuous and $\mathcal{L}_{\Omega_\theta} : C(\bar{\Omega}_\theta) \to C(\bar{\Omega}_\theta)$ is a compact operator.

We show the following

Lemma (3.2.21)[118]: Let Ω, J, g and a be as in Theorem (3.2.20). Then there exists θ_0 so that for all $\theta \leq \theta_0$ there exists $\delta > 0$ and $u \in C(\bar{\Omega}_\theta), u \geq 0, u \not\equiv 0$, such that in $\bar{\Omega}_\theta$

$$\mathcal{L}_{\Omega_\theta}[u] + \bar{a}(x)u \geq (\bar{\sigma} + \delta)u.$$

As above the existence of a positive eigenpair (ρ, φ) easily follows from Lemma (3.2.21). Arguing as above, we see that $\mu_{1,0} = \lambda_p(\mathcal{L}_\Omega + a(x))$, which concludes the proof of Theorem (3.2.20).

Proof. First let us recall that by assumption $\dot{\Gamma} \neq \emptyset$ where $\Gamma := \{x \in \bar{\Omega} \mid a(x) = \sigma\}$ and let us define the following set $\Sigma_\eta := \{x \in \Omega \mid g(x) \geq \eta\}$.

By construction, we easily see that $\dot{\Gamma}' \neq \emptyset$ where $\Gamma' := \{x \in \bar{\Omega} \mid \bar{a}(x) = \bar{\sigma}\}$. Therefore, there exist $x_0 \in \Omega$ and $\epsilon > 0$ such that $B_\epsilon(x_0) \subset (\dot{\Gamma}' \cap \Omega)$. Moreover, for θ small, say $\theta \leq \theta_0$ we have $B_\epsilon(x_0) \subset (\dot{\Gamma} \cap \Omega_\theta)$.

Let us define $\omega_\eta := B_\epsilon(x_0) \cap \Sigma_\eta$. By assumption we have $1/g^n \in L^p(\Omega)$, so for η small enough ω_η is a non-void open subset of Ω_θ for $\theta \leq \theta_0$.

Let us now consider the eigenvalue problem (57) with $\Omega = \omega_\eta$, i.e.

$$\mathcal{L}_{\omega_\eta}[u] + a(x)u = \rho u \text{ in } \omega_\eta.$$

By construction, in $B(x_0)$ we have $\bar{a}(x) \equiv \bar{\sigma}$. So the above equation reduces to:

$$\mathcal{L}_{\omega_\eta}[u] = \rho u \text{ in } \omega_\eta, \qquad (60)$$

where $\bar{\rho} = (\rho - \bar{\sigma})$.

Since $\mathcal{L}_{\omega_\eta}$ is a compact strictly positive operator in $C(\bar{\omega}_\eta)$, using Krein–Rutman theorem there exists a positive eigenvalue $\bar{\rho}_1 > 0$ and a positive eigenfunction $\phi_1 \in C(\bar{\omega}_\eta)$ such that (ρ_1, ϕ_1) satisfies (60), i.e.

$$\mathcal{L}_{\omega_\eta}[\phi_1] = \bar{\rho}\phi_1.$$

Arguing as in Lemma (3.2.18), for all $\theta \leq \theta_0$ we can construct a non-negative test function u such that in $\bar{\Omega}_\theta$

$$\mathcal{L}_{\Omega_\theta}[u] + \bar{a}(x)u \geq (\delta + \bar{\sigma})u,$$

for a $\delta > 0$ small enough.

In particular, we can extend the criterion of existence of a principal eigenpair for an operator $\mathcal{T} + a(x)$ where $\mathcal{T}$ is an integral operator with a kernel $k(x,y)$ that only satisfies that there exists a positive integer N, so that the kernel $k_N(x,y)$ satisfies $(\widetilde{H}2)$ where k_N is defined by the recursion:

$$k_1(x,y) := k(x,y),$$
$$k_{N+1}(x,y) := \int_\Omega k_1(x,z)k_N(z,y)dz \text{ for } N \geq 1.$$

Indeed, in this situation the construction of a test function u (Lemma (3.2.18) or Lemma (3.2.21)) holds also for the operator $\mathcal{T}^N + \bar{a}^N(x)$. Using that $\bar{a} \geq 0$, we deduce

$$\left(\mathcal{T} + \bar{a}(x)\right)^N[u] \geq \mathcal{T}^N u + \bar{a}^N(x)u \geq (\bar{\sigma}^N + \delta)\, u.$$

Since in this situation $\mathcal{T}$ is a compact operator, we also have $r_e\left(\left(\mathcal{T} + \bar{a}(x)\right)^N\right) = r_e(a(x)^N)$. Thus $(\bar{\sigma}^N + \delta) > r_e\left(\left(\mathcal{T} + \bar{a}(x)\right)^N\right)$ and Theorem (3.2.10) applies. Hence, there exists a unique principal eigenpair (λ_p, ϕ_p) of the following problem

$$\left(\mathcal{T} + \bar{a}(x)\right)^N \phi_p = -\lambda_p \phi_p.$$

To obtain a principal eigenpair for $\mathcal{T} + a$ we argue as follows. Applying $\mathcal{T} + a(x)$ to the above equation it follows that

$$\left(\mathcal{T} + \bar{a}(x)\right)^{N+1} \phi_p = -\lambda_p\left(\mathcal{T} + a(x)\right)\phi_p,$$
$$\left(\mathcal{T} + \bar{a}(x)\right)^N \psi = -\lambda_p \psi$$

with $\psi := \left(\mathcal{T} + a(x)\right)\phi_p$. Since $(\mathcal{T} + \bar{a})^N$ is positive operator in $C(\bar{\Omega})$, λ_p is simple, we have $\psi = \rho\phi_p$. Hence, $\left(\left(-\lambda_p\right)^{\frac{1}{N}}, \phi_p\right)$ is the principal eigenpair of $\mathcal{T} + \bar{a}(x)$.

For simplicity in the presentation of the arguments and since the proof of the existence of a principal eigenpair under the relaxed assumptions does not significantly differ, we will only present the case where Ω, J, g and a satisfy the assumptions $(H1)$– $(H4)$.

To construct an eigenpair $\left(\lambda_p, \phi_p\right)$ in this situation, we proceed using a standard approximation scheme.

First let us recall that, by assumption, there exists $\Omega_0 \subset \bar{\Omega}$ a bounded subset such that $\frac{1}{\sigma - a(x)} \notin L^1_{d\mu}(\bar{\Omega}_0)$. Let $(\omega_n)_{n\in\mathbb{N}}$ be a sequence of bounded increasing connected set which covers Ω, i.e.

$$\omega_n \subset \omega_{n+1}, \qquad \bigcup_{n\in\mathbb{N}} \omega_n = \Omega.$$

Without loss of generality, we can also assume that $\Omega_0 \subset \omega_0$ and therefore $\frac{1}{\sigma - a(x)} \notin L^1_{d\mu}(\omega_n)$ for all $n \in \mathbb{N}$. Observe that for each ω_n Theorem (3.2.14) and Lemma (3.2.15) apply. Therefore for each n there exists a principal eigenpair $(\lambda_{p,n}, \phi_{p,n})$ to the eigenvalue problem (37) with ωn instead of Ω.

By construction, using the monotonicity of the sequence of $(\omega_n)_{n\in\mathbb{N}}$ and the assertion (i) of Proposition (3.2.25) we deduce that $\left(\lambda_{p,n}\right)_{n\in\mathbb{N}}$ is a monotone non-increasing sequence which is bounded from below. Thus $\lambda_{p,n}$ converges to some $\bar{\lambda} \geq \lambda_p(\mathcal{L}_\Omega + a(x))$. Moreover, we also have that for all $n \in \mathbb{N}$

$$\lambda_p\left(\mathcal{L}_\Omega + a(x)\right) \leq \bar{\lambda} \leq \lambda_{p,n} < \lambda_{p,0} < -\sup_{\bar{\Omega}} a(x) = \sigma.$$

Let us now fix $x_1 \in \omega_0 \cap \Omega$. Observe that since for each integer n the eigenvalue $\lambda_{p,n}$ is simple we can normalise $\phi_{p,n}$ by $\phi_{p,n}(x_1) = 1$.

Let us now define $b_n(x) := -\lambda_{p,n} - a(x)$. Then $\phi_{p,n}$ satisfies

$$\mathcal{L}_{\omega_n}\left[\phi_{p,n}\right] = b_n(x)\phi_{p,n} \text{ in } \omega_n. \tag{61}$$

By construction for all $n \in \mathbb{N}$ we have $b_n(x) - \lambda_{p,0} - \sigma > 0$, therefore the Harnack inequality (Theorem (3.2.11)) applies to $\phi_{p,n}$. Thus for n fixed and for all compact set $\omega' \Subset \omega_n$ there exists a constant $C_n(\omega')$ such that

$$\phi_{p,n}(x) \leq C_n(\omega')\phi_{p,n}(y) \qquad \forall x, y \in \omega'.$$

Moreover, the constant $C_n(\omega')$ only depends on $\cup_{x\in\omega} B(x,\beta)$ and is monotone decreasing with respect to $\inf_{x\in\omega_n} b_n(x)$. For all n, the function $b_n(x)$ being uniformly bounded from below by a constant independent of n, the constant C_n is bounded from above independently of n by a constant $C(\omega')$. Thus we have

$$\phi_{p,n}(x) \leq C(\omega')\phi_{p,n}(y) \quad \forall x, y \in \omega'.$$

From a standard argumentation, using the normalisation $\phi_{p,n}(x_1) = 1$, we deduce that the sequence $\left(\phi_{p,n}\right)_{n\in\mathbb{N}}$ is bounded in $C_{loc}(\Omega)$ topology. Moreover, from a standard diagonal extraction argument, there exists a subsequence still denoted $\left(\phi_{p,n}\right)_{n\in\mathbb{N}}$ such that $\left(\phi_{p,n}\right)_{n\in\mathbb{N}}$ converges locally uniformly to a continuous function ϕ. Furthermore, ϕ is a non-negative non-trivial function and $\phi(x_1) = 1$.

Since J has a compact support we can pass to the limit in Eq. (61) using the Lebesgue monotone convergence theorem and get

$$\int_{\Omega} J\left(\frac{x-y}{g(y)}\right) \phi(y)d\mu(y) + \left(\bar{\lambda} + a(x)\right) \phi(x) = 0 \text{ in } \Omega.$$

As above using the equation, we deduce that $\phi > 0$ in Ω. Lastly, from the definition of λ_p using (λ, ϕ) as a test function, we see that $\bar{\lambda} \leq \lambda_p \leq \bar{\lambda}$. Hence, (λ, ϕ) is our desired eigenpair.

We explore the relation between a maximum principle property satisfied by an operator $\mathcal{M}$ and the sign of its principal eigenvalue. Namely, we prove Theorem (3.2.5) that we recall below.

Theorem (3.2.22)[118]: Assume that Ω is a bounded set and let J , g and a be as in Theorem (3.2.1). Then the maximum principle is satisfied by $\mathcal{M}_{\Omega}$ if and only if $\lambda_p(\mathcal{M}_{\Omega}) \geq 0$.

Proof. Assume first that the operator satisfies the maximum principle. From Theorem (3.2.1), there exists $\left(\lambda_p, \phi_p\right)$ such that $\phi_p \in C(\bar{\Omega}), \phi_p > 0$ and

$$\mathcal{L}_{\Omega}\left[\phi_p\right] + a(x)\phi_p + \lambda_p\phi_p = 0.$$

We can normalise ϕ_p so that we have $1 \geq \phi_p \geq c_0$. Furthermore, there exists $\delta > 0$ so that $-\lambda_p - \sigma \geq \delta > 0$ where σ denotes the maximum of a in Ω.

Assuming by contradiction that $\lambda_p < 0$ we have

$$\mathcal{L}_{\Omega}[\phi_p] + a(x)\phi_p = -\lambda_p\phi_p > 0.$$

Let us choose $\omega \Subset \Omega$ such that

$$\int_{\Omega\backslash\omega} d\mu(y) \leq \frac{c_0 \inf \{\delta, |\lambda_p|\}}{2\|J\|_{\infty}}.$$

We can construct a continuous function η such that $0 \leq \eta \leq 1, \eta(x) = 1$ in $\omega, \eta(x) = 0$ in $\partial\Omega$. Consider now $\phi_p\eta$ and let us compute $\mathcal{L}_{\Omega}[\phi_p\eta] + a(x)\phi_p\eta$. Then we have

$$\begin{aligned}\mathcal{L}_{\Omega}\left[\phi_p\eta\right] + a(x)\phi_p\eta &\geq -\lambda_p\phi_p - \|J\| \int_{\Omega\backslash\omega} d\mu(y) - a(x)\phi_p(1-\eta) \\ &\geq -\lambda_p\phi_p - \frac{c_0 \inf \{\delta, |\lambda_p|\}}{2} - a(x)\phi_p(1-\eta) \\ &\geq -\lambda_p\phi_p - \frac{c_0 \inf \{\delta, |\lambda_p|\}}{2} - \max\{\sigma, 0\}\phi_p \\ &\geq -\left(\lambda_p + max\{\sigma, 0\}\right)\phi_p - \frac{c_0 \inf \{\delta, |\lambda_p|\}}{2}.\end{aligned}$$

Since by assumption $-\lambda_p > 0$ and $-\lambda_p - \sigma \geq 0$ it follows from the above inequality that

$$\mathcal{L}_{\Omega}[\phi_p\eta] + a(x)\phi_p\eta - (\lambda_p + \max\{\sigma, 0\} c_0 - \frac{c_0 \inf \{\delta, |\lambda_p|\}}{2} \geq \frac{c_0 \inf \{\delta, |\lambda_p|\}}{2} \geq 0.$$

By construction we have $\phi_p\eta \in C(\Omega)$ that satisfies

$$\mathcal{L}_{\Omega}\left[\phi_p\eta\right] + a(x)\phi_p\eta \geq 0 \text{ in } \Omega,$$
$$\phi_p\eta = 0 \text{ on } \partial\Omega.$$

Therefore, by the maximum principle 1.4, $\phi_p\eta \leq 0$ in Ω which is a contradiction. Hence, $\lambda_p \geq 0$.

Let us now show the converse implication. Assume that $\lambda_p\big(\mathcal{L}_\Omega + a(x)\big) \geq 0$, then we will show that the operator satisfies the maximum principle. Let $u \not\equiv 0, u \in C(\bar{\Omega})$ such that $u \geq 0$ on $\partial\Omega$ and

$$\mathcal{L}_\Omega [u] + a(x)u \leq 0.$$

Let us show that $u > 0$ in Ω. By Theorem (3.2.1), there exists $\phi_p > 0$ such that

$$\mathcal{L}_\Omega [\phi_p] + a(x)\phi_p = -\lambda_p\phi_p \leq 0.$$

Let us rewrite $\mathcal{L}_\Omega [u] + a(x)u$ as follows

$$\begin{aligned}\mathcal{L}_\Omega [u] + a(x)u &= \int_\Omega J \left[\frac{x - y}{g(y)}\right]\frac{\phi_p(y)}{g(y)}\frac{u(y)}{\phi_p(y)}\, dy + a(x)\phi_p(x)\frac{u(x)}{\phi_p(x)} \\ &= \int_\Omega J \left[\frac{x - y}{g(y)}\right]\frac{\phi_p(y)}{g^n(y)}\frac{u(y)}{\phi_p(y)} - \frac{u(x)}{\phi_p(x)} dy - \lambda_p\phi_p\frac{u(x)}{\phi_p(x)}.\end{aligned}$$

Let us set $:= \frac{u}{\phi_p}$, then we have the following inequality in Ω

$$\int_\Omega J \left[\frac{x - y}{g(y)}\right]\frac{\phi_p(y)}{g^n(y)} \big(w(y) - w(x)\big)\, dy - \lambda_p\phi_p\, w(x) \leq 0.$$

From the above inequality we deduce that w cannot achieve a non-positive minimum in Ω without being constant. Therefore it follows that either $w > 0$ in Ω or $w \equiv 0$. Since $u \not\equiv 0$, we have $w > 0$. Hence, $\frac{u}{\phi_p} > 0$ which implies that $u > 0$.

We provide an example of nonlocal equation where no positive bounded eigenfunction exists. Let Ω be a bounded domain and let us consider the following principal eigenvalue problem:

$$\rho \int_\Omega u\, dx + a(x)u = \lambda u, \tag{62}$$

where $\sigma = a(x_0) = \max_\Omega a(x), \rho$ is a positive constant and $a(x) \in C_0(\Omega)$ satisfies the condition $\frac{1}{\sigma - a(x)} \in L^1_{loc}(\Omega)$. For this eigenvalue problem, we show the following result:

Theorem (3.2.23)[118]: If ρ is so that $\rho \int_\Omega \frac{dx}{\sigma - a(x)} < 1$, then there exists no bounded continuous positive principal eigenfunction ϕ to (62).

Proof. We argue by contradiction. Let us assume that there exists a bounded positive continuous eigenfunction ϕ associated with λ_p that we normalise by $\int_\Omega \phi(x)dx = 1$. By substituting ϕ into Eq. (62) it follows that

$$\rho = \big(\lambda_p - a(x)\big)\phi.$$

Since $\rho > 0$, from the above equation we conclude that $\lambda_p - \sigma \geq \tau > 0$. Therefore

$$\phi = \frac{\rho}{\lambda_p - a(x)}.$$

Next, using the normalisation we obtain

$$1 = \rho \int_\Omega \frac{dx}{\lambda_p - a(x)}.$$

By construction $\lambda_p \geq \sigma$, therefore we have

$$1 = \rho \int_\Omega \frac{dx}{\lambda_p - a(x)} \leq \rho \int_\Omega \frac{dx}{\sigma - a(x)}.$$

Since $\rho \int_\Omega \frac{dx}{\sigma - a(x)} < 1$ we end up with the following contradiction

$$1 = \rho \int_{\Omega} \frac{dx}{\lambda_p - a(x)} \rho \int_{\Omega} \frac{dx}{\sigma - a(x)} < 1.$$

Hence there exists no positive bounded eigenfunction ϕ associated to λ_p .

We prove Theorem (3.2.6). That is to say, we investigate the existence/non-existence of solution of the following problem:

$$\mathcal{M}_{\Omega}[u] + f(x,u) = 0 \text{ in } \Omega \quad (63)$$

where f is of KPP type. We show that the existence of a non-trivial solution of (31) is governed by the sign of the principal eigenvalue of the following operator $\mathcal{M}_{\Omega} + f_u(x,0)$. Moreover, when a non-trivial solution exists, then it is unique. To show the existence/non-existence of solutions of (31) and their properties, we follow and adapt the arguments developed in [97], [99], [135].

Let us assume that

$$\lambda_p\big(\mathcal{M}_{\Omega} + f_u(x,0)\big) < 0.$$

Then we will show that there exists a non-trivial solution to (31).

Before going to the construction of a non-trivial solution, let us first define some quantities. First let us denote $a(x) := f_u(x,0) - b(x)$ and $\sigma := \sup_{\Omega} a(x)$. Observe that with this notation, we have $\lambda_p(\mathcal{M}_{\Omega} + f_u(x,0)) = \lambda_p(\mathcal{L}_{\Omega} + a(x))$.

From the definition of σ there exists a sequence of points $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in \Omega$ and $|\sigma - a(x_n)| \le \frac{1}{n}$.

Then by continuity of $a(x)$, for each n there exists η_n such that for all $x \in B_{\eta_n}(x_n)$ we have $|\sigma - a(x)| \le \frac{2}{n}$.

Now let us consider a sequence of real numbers $(\epsilon_n)_{n\in\mathbb{N}}$ which converges to zero such that $\epsilon_n \le \frac{\eta_n}{2}$.

Next, let $(\chi_n)_{n\in\mathbb{N}}$ be the following sequence of cut-off functions: $\chi_n(x) := \chi\left(\frac{\|x-x_n\|}{n}\right)$ where χ is a smooth function such that $0 \le \chi \le 1$, $\chi(x) = 0$ for $|x| \ge 2$ and $\chi(x) = 1$ for $|x| \le 1$.

Finally, let us consider the following sequence of continuous functions $(a_n)_{n\in\mathbb{N}}$, defined by $a_n(x) := \sup\,\{a(x), \sigma\chi_n\}$. Observe that by construction the sequence $(a_n)_{n\in\mathbb{N}}$ is such that $\|a(x) - a_n(x)\|_\infty \to 0$.

Let us now proceed to the construction of a non-trivial solution.

By construction, for each n, the function a_n satisfies $\sup_{\Omega} a_n = \sigma$ and $a_n \equiv \sigma$ in $B_{\frac{\epsilon_n}{2}}(x_n)$. Therefore, the sequence an satisfies $\frac{1}{\sigma - a_n} \notin L^1_{loc}(\Omega)$ and by Theorem (3.2.1) there exists a principal eigenpair $\big(\lambda_{n\,p}, \phi_n\big)$ solution of the eigenvalue problem:

$$\mathcal{L}_{\Omega}[\phi] + a_n(x)\phi + \lambda\phi = 0,$$

such that $\phi_n \in L^\infty(\Omega) \cap C(\Omega)$.

Next, using that $\|a_n(x) - a(x)\|_\infty \to 0$ as $n \to \infty$, from (iii) of Proposition (3.2.25) it follows that for n big enough, say $n \ge n_0$, we have

$$\lambda_p^n < \frac{\lambda_p\big(\mathcal{L}_{\Omega} + a(x)\big)}{2} < 0.$$

Moreover, by choosing n_0 bigger if necessary, we achieve for $n \ge n_0$

$$\lambda_p^n + \|a_n(x) - a(x)\|_\infty \le \frac{\lambda_p\big(\mathcal{L}_{\Omega} + a(x)\big)}{4}.$$

Let us now compute $\mathcal{M}_\Omega\,[\,\phi_n] \,+\, f\,(x,\phi_n)$. For $n \geq n0$, we have

$$\mathcal{M}_\Omega\,[\,\epsilon\phi_n] \,+\, f\,(x,\epsilon\phi_n) \,=\, f(x,\epsilon\phi_n) \,-\, \big(b(x) + \,a_n(x)\big)\epsilon\phi_n \,-\, \lambda_p^n\phi_n$$
$$= \Big(f_u(x,0) - \big(\,a_n(x) + \,b(x)\big)\Big)\,\epsilon\phi_n \,-\epsilon\lambda_p^n\phi_n \,+\, o(\epsilon\phi_n)$$
$$\geq \big(-\,\|a(x) - \,a_n(x)\|_\infty \,-\, \lambda_p^n\,\big)\phi_n \,+\, o(\epsilon\phi_n)$$
$$\geq (-\frac{\lambda^p\big(\mathcal{M}_\Omega \,+\, f_u(x,0)\big)}{4}\epsilon\phi_n \,+\, o(\epsilon\phi_n) \,>\, 0.$$

Therefore, for $\epsilon > \,0$ sufficiently small and n big enough, $\epsilon\phi_n$ is a sub-solution of (31). By definition of , any large enough constant M is a super-solution of (31). By choosing M so large that $\epsilon\phi_n \leq M$ and using a basic iterative scheme we obtain the existence of a positive non-trivial solution u of (31).

Let now turn our attention to the non-existence result. Let us prove that when $\lambda_p\big(\mathcal{M}_\Omega \,+\, f_u(x,0)\big) \geq 0$ then there exists no non-trivial solution to (31).

Assume by contradiction that $\lambda_p\big(\mathcal{M}_\Omega \,+\, f_u(x,0)\big) \geq \,0$ and there exists a positive bounded solution u to Eq. (31).

Obviously, since u is non-negative and bounded, using (31) we have for all $x \,\in\, \bar{\Omega}$

$$0 \leq \mathcal{L}_\Omega\,[u] \,= \left(b(x) \,- \frac{f(x,u)}{u}\right) u. \tag{64}$$

Let us denote $h(x) := \mathcal{L}_\Omega[u]$. By construction, h is a non-negative continuous function in Ω. Therefore, since Ω is compact, h achieves at some point $x_0 \,\in\, \bar{\Omega}$ a non-negative minimum. A short argument shows that $h(x_0) \,>\, 0$. Indeed, otherwise we have

$$\int_\Omega J\left(\frac{x_0 \,-\, y}{g(y)}\right)\frac{u(y)}{g^n(y)}\,dy \,=\, 0.$$

Thus, since J, g and u are non-negative quantities, from the above equality we deduce that $u(y) \,=\, 0$ for almost every $y \,\in\, \{z \,\in\, \bar{\Omega}\,|\, \frac{x_0 - z}{g(z)} \,\in\, \text{supp}\,(J)\}$. By iterating this argument and using the assumption $J(0) \,>\, 0$, we can show that $u(y) \,=\, 0$ for almost every $y \,\in\, \bar{\Omega}$, which implies that $u \,\equiv\, 0$ since u is continuous.

As a consequence $\inf_{x\in\Omega}\,\left(b(x) - \frac{f(x,u)}{u}\right) \geq \delta$ for some $\delta \,>\, 0$ and there exists a positive constant c_0 so that $u \,>\, c_0$ in Ω. From the monotone properties of $f\,(x,.)$, we deduce that $\frac{f(x,u)}{u} \leq \frac{f(x,c_0)}{c_0} < fu(x,0)$.

Let us now denote $\gamma\,(x) = \frac{f(x,c_0)}{c_0} \,-\, b(x)$. By construction, we have $\gamma\,(x) \,<\, a(x)$ and therefore by (ii) of Proposition (3.2.25),

$$\lambda_p\big(\mathcal{L}_\Omega \,+\, \gamma\,(x)\big) \,>\, \lambda_p\,\big(\mathcal{L}_\Omega \,+\, a(x)\big) \geq 0.$$

Moreover, since u is a solution of (31), we have

$$\mathcal{L}_\Omega\,[u] + \,\gamma\,(x)u \geq \mathcal{M}_\Omega\,[u] \,+\, f(x,u) \,=\, 0.$$

By definition of $\lambda_p(\mathcal{L}_\Omega \,+\, \gamma\,(x))$, for all positive $\lambda_p(\mathcal{L}_\Omega \,+\, a(x)) \,<\, \lambda \,<\, \lambda_p(\mathcal{L}_\Omega \,+\, \gamma\,(x))$ there exists a positive continuous function ϕ_λ such that

$$\mathcal{L}_\Omega\,[\phi_\lambda] \,+\, \gamma\,(x)\phi_\lambda \leq -\lambda\phi_\lambda \leq\, 0.$$

Arguing as above, we can see that $\phi_\lambda \geq \delta$ for some positive δ. Let us define the following quantity

$$\tau^* := \inf\,\{\tau \,>\, 0\,|u \leq \tau\phi_\lambda\}\,.$$

Obviously, we end the proof of the theorem by proving that $\tau^* = 0$. Assume that $\tau^* > 0$. Then by definition of τ^*, there exists $x_0 \in \bar{\Omega}$ such that $\tau^*\phi_p(x_0) = u(x_0) > 0$. At this point x_0, we have

$$0 \leq \mathcal{L}_\Omega [w](x_0) = \mathcal{L}_\Omega[(\tau^*\phi_\lambda - u)](x_0) \leq 0.$$

Therefore, since $w \geq 0$, using a similar argumentation as above, we have $w(y) = 0$ for almost every $y \in \bar{\Omega}$. Thus, we end up with $\tau^*\phi_1 \equiv u$ and we get the following contradiction,

$$0 \leq \mathcal{L}_\Omega [u] + \gamma(x)u = \mathcal{L}_\Omega [\tau^*\phi_\lambda] + \gamma(x)\tau^*\phi_\lambda < 0.$$

Hence $\tau^* = 0$.

Lastly, we show that when a solution of (31) exists then it is unique. The proof of the uniqueness of the solution is obtained as follows.

Let u and v be two non-negative bounded solutions of (31). We see that there exist two positive constants c_0 and c_1 such that

$$u \geq c_0 \text{ in } \bar{\Omega},$$
$$v \geq c_1 \text{ in } \bar{\Omega}.$$

Since u and v are bounded and strictly positive, the following quantity is well defined

$$\gamma^* := \inf \{\gamma > 0 \,|\gamma u \geq v\}.$$

We claim that $\gamma^* \leq 1$. Indeed, assume by contradiction that $\gamma^* > 1$. From (31) we see that

$$\mathcal{M}_\Omega [\gamma^* u] + f(x,\gamma^* u) = f(x,\gamma^* u) - \gamma^* f(x,u) \tag{65}$$
$$= \gamma^* u \left(\frac{f(x,\gamma^* u)}{\gamma^* u} - \frac{f(x,u)}{u}\right) \leq 0. \tag{66}$$

Now, by definition of γ^*, there exists $x_0 \in \bar{\Omega}$ so that $\gamma u(x_0) = v(x_0)$ and from (31) we can easily see that

$$\mathcal{M}_\Omega [\gamma^* u](x_0) + f(x,\gamma^* u(x_0)) = \mathcal{L}_\Omega (\gamma^* u - v] \geq 0. \tag{67}$$

From (66) and (67) we deduce that

$$\mathcal{L}_\Omega[\gamma^* u - v](x_0) = 0.$$

Therefore, arguing it follows that $\gamma^* u = v$. Using now (66), we deduce that

$$0 = \mathcal{M}_\Omega [v] + f(x,v) = \mathcal{M}_\Omega [\gamma^* u] + f(x,\gamma^* u)$$
$$= \gamma^* u \,(\frac{f(x,\gamma^* u)}{\gamma^* u} - \frac{f(x,u)}{u} \leq 0,$$

which implies that for all $x \in \Omega$ $f(x,\gamma^* u) \equiv f(x,u)$. This later is impossible since $\gamma^* > 1$. Hence, $\gamma^* \leq 1$ and as a consequence $u \geq v$. Observe that the role of u and v can be interchanged in the above argumentation. So we also have $v \geq u$, which shows the uniqueness of the solution.

We prove Theorem (3.2.24) which establishes the asymptotic behaviour of the solution of

$$\frac{\partial u}{\partial t} = \mathcal{M}_\Omega [u] + f(x,u) \text{ in } \mathbb{R}^+ \times \Omega,$$
$$u(0,x) = u_0(x) \text{ in } \Omega.$$

Theorem (3.2.24)[118]: Let Ω, J, g, b and f be as in Theorem (3.2.6). Let u_0 be an arbitrary bounded and continuous function in Ω such that $u_0 \geq 0, u_0 \not\equiv 0$. Let $u(t,x)$ be the solution of (33) with initial datum $u(0,x) = u_0(x)$. Then, we have:

(i) If 0 is an unstable solution of (31) (that is $\lambda_p < 0$), then $u(t,x) \to p(x)$ pointwise as $t \to \infty$, where p is the unique positive solution of (31) given by Theorem (3.2.6).

(ii) If 0 is a stable solution of (31) (that is $\lambda_p \geq 0$), then $u(t,x) \to 0$ pointwise in Ω as $t \to +\infty$.

Proof. The existence of a solution defined for all time t follows from a standard argument and will not be exposed. Moreover, since $u_0 \geq 0$ and $u_0 \not\equiv 0$, using the parabolic maximum principle, there exists a positive constant δ such that $u(1,x) > \delta$ in Ω. Let us first assume that $\lambda_p < 0$. By following the argument developed, we can construct a bounded continuous function ψ so that ψ is a sub-solution of (33) for small enough. Since, $u(1,x)$ δ and ψ is bounded, by choosing smaller if necessary we achieve also that ψ $u(1,x)$. Now, let us denote by $\underline{\Psi}(x,t)$ the solution of evolution problem (33) with initial datum $\epsilon\psi$. By construction, using a standard argument, $\underline{\Psi}(t,x)$ is a non-decreasing function of the time and $\Psi(t,x) \leq u(t+1,x)$. On the other hand, since for M big enough M is a super-solution of (33) and u_0 is bounded, we have also $u(t,x) \leq \overline{\Psi}(t,x)$, where $\overline{\Psi}(x,t)$ denotes the solution of evolution problem (33) with initial datum $\overline{\Psi}(0,x) = M \geq u_0$. A standard argument using the parabolic comparison principle shows that $\overline{\Psi}$ is a non-increasing function of t. Thus we have for all time t

$$\epsilon\psi \leq \underline{\Psi}(t,x) \leq u(t+1,x) \leq \overline{\Psi}(t+1,x).$$

Since $\underline{\Psi}(t,x)$ (resp. $\overline{\Psi}(t,x)$) is a uniformly bounded monotonic function of t, $\underline{\Psi}$ (resp. $\overline{\Psi}$) converges pointwise to $\underline{p}$ (resp. $\overline{p}$) which is a solution of (31). From $\underline{\Psi}(t,x) \not\equiv 0$, using the uniqueness of a nontrivial solution (Theorem (3.2.6)), we deduce that $\underline{p} \equiv \overline{p} \not\equiv 0$ and therefore, $u(x,t) \to p$ pointwise in Ω, where p denotes the unique non-trivial solution of (31).

In the other case, when $\lambda_p \geq 0$ we argue as follows. As above, we have $0 \leq u(t,x) \leq \overline{\Psi}(t,x)$ and $\overline{\Psi}$ converges pointwise to $\overline{p}$ a solution of (31). By Theorem (3.2.6) in this situation we have $\overline{p} \equiv 0$, hence $u(x,t) \to 0$ pointwise in Ω.

We first prove Proposition (3.2.25). Then we recall the method of sub- and supersolution to obtain solution of the semilinear problem:

$$\mathcal{M}_\Omega[u] = f(x,u) \text{ in } \Omega. \tag{68}$$

Before going to the proof of Proposition (3.2.25), let us show that $\lambda_p(\mathcal{L}_\Omega + a(x))$ is well defined. Let us first show that the set $\Lambda := \{\lambda \mid \exists\phi \in C(\Omega), \phi > 0$ such that $\mathcal{L}_\Omega[\phi] + \lambda_\phi \leq 0\}$ is non-empty. Indeed, as observed in [133] (Theorem (3.2.20)), for Ω, J, g and a satisfying the assumptions (H1)–(H4) there exists a continuous positive function ψ satisfying

$$\int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{\psi(y)}{g^n(y)}\,dy = c(x)\psi(x),$$

where $c(x)$ is defined by

$$c(x) := \begin{cases} 1 \text{ if } x \in \{x \in \bar{\Omega} \mid g(x) = 0\}, \\ \displaystyle\int_\Omega J\left(\frac{y-x}{g(x)}\right)\frac{dy}{g^n(x)} \text{ otherwise.} \end{cases}$$

Obviously $c(x) \in L^\infty$ and for any $\lambda \leq (|a\|_\infty + \|c\|_\infty)$ we have

$$\begin{aligned} \mathcal{L}_\Omega[\psi] + (a(x) + \lambda)\psi &= (a(x) + c(x) + \lambda)\psi \\ &\leq (a(x) + c(x) - \|a\|_\infty - \|c\|_\infty) \leq \psi\, 0. \end{aligned}$$

Therefore, the set Λ is non-empty.

Observe now that since J, g are non-negative functions and $a(x) \in L^{\infty}$, for any continuous positive function ϕ we have

$$\mathcal{L}_{\Omega}[\phi] + (a(x) + \|a(x)\|_{\infty})\,\phi \geq 0.$$

Therefore, the set Λ has an upper bound and λ_p is well defined.

Let us now prove Proposition (3.2.25).

Proposition (3.2.25)[118]:

(i) Assume $\Omega_1 \subset \Omega_2$, then

$$\lambda_p\left(\mathcal{L}_{\Omega_1} + a(x)\right) \geq \lambda_p\left(L_{\Omega_2} + a(x)\right).$$

(ii) Fix Ω and assume that $a_1(x) \geq a_2(x)$, then

$$\lambda_p\left(\mathcal{L}_{\Omega} + a_2(x)\right) \geq \lambda_p\left(\mathcal{L}_{\Omega} + a_1(x)\right).$$

Moreover, if $a_1(x) \geq a_2(x) + \delta$ for some $\delta > 0$ then

$$\lambda_p\left(\mathcal{L}_{\Omega} + a_2(x)\right) > \lambda_p\left(\mathcal{L}_{\Omega} + a_1(x)\right).$$

(iii) $\lambda_p\left(\mathcal{L}_{\Omega} + a(x)\right)$ is Lipschitz continuous in $a(x)$. More precisely,

$$\left|\lambda_p\left(\mathcal{L}_{\Omega} + a(x)\right) - \lambda_p\left(\mathcal{L}_{\Omega} + b(x)\right)\right| \leq \| a(x) - b(x)\|_{\infty}.$$

(vi) Let $J_1 \leq J_2$ be two positive continuous integrable functions and let us denote respectively by $\mathcal{L}_{1,\Omega}$ and $\mathcal{L}_{2,\Omega}$ the corresponding operators. Then we have

$$\lambda_p\left(\mathcal{L}_{1,\Omega} + a(x)\right) > \lambda_p\left(L_{2,\Omega} + a(x)\right).$$

Let us state our first result concerning a sufficient condition for the existence of a principal eigenpair (λ_p, ϕ_p) for the operator $\mathcal{M}$.

Proof. (i) easily follows from the definition of λ_p. First, let us observe that to obtain

$$\lambda_p\left(\mathcal{L}_{\Omega_2} + a(x)\right) \leq \lambda_p\left(\mathcal{L}_{\Omega_1} + a(x)\right)$$

it is sufficient to prove the inequality

$$\lambda \leq \lambda_p\left(\mathcal{L}_{\Omega_1} + a(x)\right)$$

for any $\lambda < \lambda_p\left(\mathcal{L}_{\Omega_2} + a(x)\right)$.

Let us fix $\lambda < \lambda_p\left(\mathcal{L}_{\Omega_2} + a(x)\right)$. Then by definition of $\lambda_p\left(L_{\Omega_2} + a(x)\right)$ there exists a positive function $\phi \in C(\Omega_2)$ such that

$$\mathcal{L}_{\Omega_2}[\phi] + (a(x) + \lambda)\phi \leq 0.$$

Since $\Omega_1 \subset \Omega_2$, an easy computation shows that

$$\mathcal{L}_{\Omega_1}[\phi] + (a(x) + \lambda)\phi \leq \mathcal{L}_{\Omega_2}[\phi] + (a(x) + \lambda)\phi \leq 0.$$

Therefore, by definition of $\lambda_p\left(\mathcal{L}_{\Omega_1} + a(x)\right)$ we have $\lambda < \lambda_p\left(\mathcal{L}_{\Omega_1} + a(x)\right)$. Hence,

$$\lambda_p\left(\mathcal{L}_{\Omega_2} + a(x)\right) \leq \lambda_p\left(\mathcal{L}_{\Omega_1} + a(x)\right).$$

To show (ii), we argue as above. By definition of $\lambda_p\left(\mathcal{L}_{\Omega} + a_1(x)\right)$ for any $\lambda < \lambda_p\left(\mathcal{L}_{\Omega} + a_1(x)\right)$ there exists a positive $\phi \in C(\Omega)$ such that

$$\mathcal{L}_{\Omega}[\phi] + (a_1(x) + \lambda)\phi \leq 0$$

and we have

$$\mathcal{L}_{\Omega}[\phi] + (a_2(x) + \lambda)\phi \leq \mathcal{L}_{\Omega}[\phi] + (a_1(x) + \lambda)\phi \leq 0.$$

Therefore $\lambda \leq \lambda_p(\mathcal{L}_{\Omega} + a_2(x))$. Hence (ii) holds true.

Let us now prove (iii). Again we fix $\lambda < \lambda_p\left(\mathcal{L}_{\Omega_1} + a(x)\right)$ For this λ, there exists $\phi \in C(\Omega), \phi > 0$ such that

$$\mathcal{L}_\Omega [\phi] + (a(x) + \lambda)\phi \leq 0. \tag{69}$$

An easy computation shows that we rewrite the above equation as follows

$$\begin{aligned}\mathcal{L}_\Omega [\phi] + (a(x) + \lambda)\phi &= \mathcal{L}_\Omega [\phi] + (b(x) + \lambda)\phi + \big(a(x) - b(x)\big)\phi \\ &\geq \mathcal{L}_\Omega [\phi] + (b(x) + \lambda - \|a(x) - b(x)\|_\infty)\phi.\end{aligned}$$

Using that (λ, ϕ) satisfies (69), it follows that

$$\mathcal{L}_\Omega [\phi] + (b(x) + \lambda - \|a(x) - b(x)\|_\infty)\phi \leq 0.$$

Therefore, $\lambda - \|a(x) - b(x)\|_\infty \leq \lambda_p(\mathcal{L}_\Omega + b(x))$ and we have

$$\lambda \leq \lambda_p\big(\mathcal{L}_\Omega + b(x)\big) + \|a(x) - b(x)\|_\infty.$$

The above computation being valid for any $\lambda < \lambda_p\left(\mathcal{L}_{\Omega_1} + a(x)\right)$ we end up with

$$\lambda_p(\mathcal{L}_\Omega + a(x) - \lambda_p(\mathcal{L}_\Omega + b(x) \leq \|a(x) - b(x)\|_\infty.$$

Note that the role of $a(x)$ and $b(x)$ can be interchanged in the above argumentation. So, we also have

$$\lambda_p\big(\mathcal{L}_\Omega + b(x)\big) - \lambda_p(\mathcal{L}_\Omega + a(x) \leq \|a(x) - b(x)\|_\infty.$$

Hence

$$\big|\lambda_p\big(\mathcal{L}_\Omega + a(x)\big) - \lambda_p(\mathcal{L}_\Omega + b(x)\}\big| \leq \|a(x) - b(x)\|_\infty,$$

which proves (iii).

The proof of (iv) being similar to the proof of (ii), it will be omitted.

Before recalling the sub/super-solution method, let us introduce some definitions and notations. We call a bounded continuous function $\overline{u}$ (resp. $\underline{u}$) a super-solution (resp. a sub-solution) if $\overline{u}$ (resp. $\underline{u}$) satisfies the following inequalities:

$$\mathcal{M}_\Omega [u] \leq (\geq) f(x,u) \text{ in } \Omega. \tag{70}$$

Let us now state the theorem.

Theorem (3.2.26)[118]: Assume $f(x,.)$ is a Lipschitz function uniformly in x and let $\overline{u}$ and $\underline{u}$ be respectively a supersolution and a sub-solution of (68) continuous up to the boundary. Assume further that $\underline{u} \leq \overline{u}$. Then there exists a solution $u \in C(\overline{\Omega})$ solution of (68) satisfying $\underline{u} \leq u \leq \overline{u}$.

Proof. Let us first choose $k > \left|\lambda_p(\mathcal{M}_\Omega)\right|$ big enough such that the function $-ks + f(x,s)$ is a decreasing function of s uniformly in x. We can increase further k if necessary to ensure that $k \in \rho(\mathcal{M}_\Omega)$, where $\rho(\mathcal{M}_\Omega)$ denotes the resolvent of the operator $\mathcal{M}_\Omega$.

Note that by this choice of k, by Theorem (3.2.5) the operator $\mathcal{M}_\Omega - k$ satisfies a comparison principle.

Now, let u_1 be the solution of the following linear problem

$$\mathcal{M}_\Omega [u_1] - ku_1 = -k\underline{u} + f(x,u) \text{ in } \Omega. \tag{71}$$

u_1 always exists, since by construction the continuous operator $\mathcal{M}_\Omega - k$ is invertible. We claim that $\underline{u} \leq u_1 \leq \overline{u}$. Indeed, since $\underline{u}$ and $\overline{u}$ are respectively a sub- and super-solution of (68), we have

$$\mathcal{M}_\Omega \left[u_1 - \underline{u}\right] - k\big(u_1 - \underline{u}\big)0 \text{ in } \Omega,$$

$$\mathcal{M}_\Omega [u_1 - \overline{u}] - k(u_1 - \overline{u}) \geq -k\big(\underline{u} - \overline{u}\big) + f\big(x,\underline{u}\big) - f(x,\overline{u}) \geq 0 \text{ in } \Omega.$$

So, the inequality $\underline{u} \leq u_1 \leq \overline{u}$ follows from the comparison principle satisfied by the operator $\mathcal{M}_\Omega - k$. Now let u_2 be the solution of (71) with u_1 instead of $\underline{u}$. From the monotonicity of $-ks + f(x,s)$ and using the comparison principle, we have $\underline{u} \leq u_1 \leq$

$u_2 \leq \overline{u}$. By induction, we can construct an increasing sequence of function $(u_n)_{n\in\mathbb{N}}$ satisfying $\underline{u} \leq u_n \leq \overline{u}$ and

$$\mathcal{M}_{\Omega}[u_{n+1}] - ku_{n+1} = -ku_n + f(x, u_n) \quad \text{in } \Omega. \tag{72}$$

Since the sequence is increasing and bounded, $u^{-}(x) := \sup_{n\in\mathbb{N}} u_n(x)$ is well defined. Moreover, passing to the limit in Eq. (72) using Lebesgue's theorem it follows that u^{-} is a solution of (68).

Chapter 4
Improvement of Critical Eigenfunctions Restriction Estimates

We extend Bérard logarithmic improvement for the remainder term of the eigenvalue counting function which directly leads to a $(\log \lambda)^{\frac{1}{2}}$ improvement for Hörmander's estimate on the L^1 norms of eigenfunctions. To the L^p estimates for all $p > \frac{2(n+1)}{n-1}$. We show detailed oscillatory integral estimates with fold singularities by Phong and Stein and use the Poincaré half-space model to establish bounds for various derivatives of the distance function restricted to geodesic segments on the universal cover $\mathbb{H}^3$.

Section (4.1): Nonpositive Curvature

We say the norm of a Banach space $(X, || \cdot ||)$ is C^k smooth if its kth Fréchet derivative exists and is continuous at every point of X $\{0\}$. The norm is C^∞ smooth if this holds for all $k \in \mathbb{N}$. We concerned with the problem of establishing sufficient conditions for when a Banach space has a C^k smooth renorming, for $k \in \mathbb{N} \cup \{\infty\}$.

Definition (4.1.1)[82]: A subset $B \subseteq B_{X^*}$ is a called a boundary if for each $x$in the unit sphere S_X, there exists $f \in B$ such that $f(x) = 1$.

Example (4.1.2)[82]: The following will be boundaries for any Banach space X.

(i) The dual unit sphere S_{X^*} . This is a consequence of the HahnBanach Theorem.

(ii) The set of extreme points of the dual unit ball, $\text{Ext}(B_{X^*})$. This follows from the proof of the Krein-Milman Theorem ([155]).

Given $\varepsilon > 0$ and norms $|| \cdot ||$ and $||| \cdot |||$ on a Banach space X, say $||| \cdot |||$ ε-approximates $|| \cdot ||$ if, for all $x \in X$,

$$(1 - \varepsilon)\big||x|\big| \leq \Big|\big||x|\big|\Big| \leq (1 + \varepsilon)\big||x|\big|.$$

The notion of a boundary plays an important role in this area of study. Frequently, the existence of a boundary with certain properties gives rise to the desired renormings, as seen in the following result of H´ajek, which is part of a more general theorem.

H´ajek and Haydon provided another sufficient condition for when this property holds, namely when $X = C(K)$ and K is a compact Hausdorff σ-discrete space. We call a topological space K σ-discrete if $K = \bigcup_{n=0}^{\infty} D_n$, where each D_n is relatively discrete: given $x \in D_n$, there exists Ux open in K such that $U_x \cap D_n = \{x\}$.

In [36], it is shown that if X has a countable boundary then X has an equivalent analytic norm which ε-approximates the original norm. Moreover, if $C(K)$ admits an analytic renorming, then K is countable [11]. For a norm $|| \cdot ||$ to be analytic we mean it is a real valued analytic function on $X \setminus \{0\}$. Analytic functions on Banach spaces are defined and explored [17].

The Orlicz functions M for which the corresponding Orlicz sequence spaces l_M and Orlicz function spaces $l_M(0,1), l_M(0,\infty)$ have an equivalent C^∞ smooth norm were characterised in [28]. Futhermore, the Orlicz sequence spaces h_M with equivalent analytic norm were characterised in [15].

The main result, Theorem (4.1.7), generalises these results as corollaries. It also takes into account smoothness of injective tensor products, in a manner similar to that of [16]. As in the proof of [158], the proof of Theorem (4.1.7) makes use of two lemmas ([158])

concerning the so-called generalised Orlicz norm, denoted by $\|\cdot\|_{\phi}$. The first lemma provides a condition where $\|\cdot\|_{\phi}$ is equivalent to $\|\cdot\|$.

Definition (4.1.3)[82]: Let B be a set. Suppose for every element $t \in B$ there exists a convex function ϕ_t on $[0,\infty)$ with $\phi_t(0) = 0$ and $\lim_{\alpha\to\infty} \phi_t(\alpha) = \infty$ (such functions are called Orlicz functions). Define $\|\cdot\|_{\phi}$ $on\ l_\infty(B) by$

$$\|f\|_{\phi} = \inf\left\{\rho > 0 : \sum_{t\in B} \phi_t \left(\frac{|f(t)|}{\rho}\right) \leq 1\right\}.$$

and define ℓ_{ϕ}(B) as the set of f $\in$ $\ell_\infty(B)$ satisfying $\|f\|_{\varphi} < \infty$.

Lemma (4.1.4)[82]: [158]. Let $\|\cdot\|_{\phi}$ be as in Definition (4.1.3). Suppose there exist $\beta > \alpha > 0$ with the property $\phi_t(\alpha) = 0$ and $\phi_t(\beta) \geq 1$ for all $t \in B$. Then $\ell_{\varphi}(B)\backslash\cong \ell_\infty(B)$ and

$$\alpha\|\cdot\|_{\phi} \leq \|\cdot\|_\infty \leq \beta\|\cdot\|_{\phi}.$$

We use $\|\cdot\|_{\phi}$ to define another norm on a more general space X, which we also denote by $\|\cdot\|_{\phi}$. The second lemma gives a sufficient condition for when $\|\cdot\|_{\phi}$on X is C^k smooth. It uses the $\|\cdot\|_{\phi}$notion of local dependence on finitely many coordinates and generalises [158].

Lemma (4.1.5)[82]: Let $\|\cdot\|_{\phi}$ be as in Lemma (4.1.4) and let $\Pi : X \to \ell_{\phi}(B)$ be an embedding (non-linear in general), where the map $x \to \Pi(x)(t)$ is a seminorm which is C^k smooth on the set where it is non-zero, for all $t \in B$. Assume the assignment $\|x\|_{\phi} = \|\Pi(x)\|_{\phi}$ defines an equivalent norm on X. Suppose for each $x \in X$, with $\|x\|_{\Phi} = 1$, there exists an open U $\subseteq$ X containing x, and finite $F \subseteq B$, such that $\phi_t(|\mathcal{Y}(t)|) = 0$ when $\mathcal{Y} \in U$ and t $\in$ B\F. Finally, assume that each ϕ_t is C^∞ smooth. Then $\|\cdot\|_{\phi}$is C^k smooth on X.

As Lemma (4.1.5) appears in [158], X is taken to be a closed subspace of $\ell_\infty(B)$ and Π is the identity. The proof uses the fact that
each coordinate map $x \to |x(t)|$ is C^∞ smooth on the set where it is non-zero and uses the implicit function theorem to show that $\|\cdot\|_{\phi}$ is also C^∞ smooth. In our case, each coordinate map is C^k smooth on the set where it is non-zero and the same argument guarantees that $\|\cdot\|_{\phi}$is C k smooth. The first part of the proof of Theorem (4.1.7) is concerned with setting up the necessary framework to apply these lemmas. The remainder uses a series of claims to prove they do in fact hold. Theorems (4.1.14) and (4.1.15) are obtained as corollaries of Theorem (4.1.7), along with some other results and applications. Before proceeding to the statement of Theorem (4.1.7), a key notion of w^*-locally relatively compact sets (w^* -LRC for short) needs to be introduced. This property is first studied in [156], in the context of polyhedral norms.

Definition (4.1.6)[82]: ([156]). Let X be a Banach space. We call $E \subseteq X^*$ w^* -LRC if given y ∈ E, there exists a w^* -open set U such that $y \in U$ and $E \cap U$ $\|\cdot\|$ is norm compact. Example 1.9 ([156]). The following sets are w^*-LRC.

(i) Any norm compact or w^*-relatively discrete subset of a dual space.

(ii) Given X with an unconditional basis $(e_i)_{i\in I}$ and $f \in X^*$, define

$$supp(f) = \{i \in I : f(e_i)6 = 0\}.$$

Let $E \subset X^*$ have the property that if f, g $\in E$, then $|supp(f)| = |supp(\mathrm{g})| < \infty$. E is w^* -LRC. Indeed, take $f \in E$ and define the w^*-open set $U = \{\mathrm{g} \in X^* : 0 <$

$|g(e_i)| < |f(e_i)| + 1 : i \in supp(f)\}$. Clearly, if $g \in U \cap E$, then $supp(g) = supp(f)$. Thus $U \cap E$ is a norm bounded subset of a finite dimensional space.

The main result is concerned with renorming injective tensor products. Given Banach spaces X and , the injective tensor product $X \otimes_\varepsilon Y$ is the completion of the algebraic tensor product $X \otimes Y$ with respect to the norm

$$||\sum_{i=1}^{\infty} x_i \otimes y_i || = \sup\left\{\sum_{i=1}^{\infty} f(x_i)g(y_i) : f \in B_{X^*}, g \in B_{Y^*}\right\}.$$

Also note the following facts. If I_Y is the identity operator on , then given $f \in X^*$ we define $f^Y = f \otimes I_Y$ on $X \otimes Y$ by $f^Y(\sum_{i=1}^{\infty} x_i \otimes y_i) = \sum_{i=1}^{\infty} f(x_i)y_i$. We have $||f^Y|| = ||f||$ and extend to the completion. Similarly define g^X for $g \in Y^*$. A useful fact is $f \otimes g = g\, o f^Y = f\, o g^X$. Given two boundaries $N \subseteq X^*$ and $\subseteq Y^*$, the set $\{f \otimes g : f \in N, g \in M\}$ is a boundary for $X \otimes_\varepsilon Y$. To see this, take $u \in X \otimes_\varepsilon Y$. There exists $f \in BX^*$ and $g \in BY^*$ such that $||u|| = (f \otimes g)(u) = ||g^X(u)||$. Then there exists $\hat{f} \in N$ such that $\hat{f}(g^X(u)) = ||u|| = ||\hat{f}\, Y(u)||$. Finally, there exists $\hat{g} \in M$ such that $\hat{g}(\hat{f}\, Y(u)) = (\hat{f} \otimes \hat{g})(u) = ||u||$.

Given a Banach space Y with a C^k smooth renorming, Haydon gave a sufficient condition on X for $X \otimes_\varepsilon Y$ to have a C^k smooth renorming ([16]). This condition involves a type of operator that are now known as Talagrand operators. Another sufficient condition is given in the main result below. It is worth noting that these conditions are incomparable. For example, the space $C[0, \omega_1]$ satisfies Haydon's condition but not that of Theorem (4.1.7). On the other hand, if we take K to be the Ciesielski-Pol space as seen in [20], then $C(K)$ satisfies the hypothesis of Theorem (4.1.7) but not Haydon's condition.

Theorem (4.1.7)[82]: Let X and Y be Banach spaces and let (E_n) be a sequence of w^*-LRC subsets of X^*, such that $E = \bigcup_{n=0}^{\infty} E_n$ *is* $\sigma - w^*$- compact and contains a boundary of X. Suppose further that Y has a C^k smooth norm $||\cdot||$Y for some $k \in \mathbb{N} \cup \{\infty\}$. Then $X \otimes_\varepsilon Y$ admits a C^k smooth renorming that $\varepsilon - \alpha$ pproximates the canonical injective tensor norm.

The proof of this theorem is based to some degree on that of [156]. Given its technical nature, some of that proof is repeated here for clarity.

Proof. To begin, we can assume E is a boundary and $\bar{E}_n^{w^*} \subseteq E$ for all $n \in \mathbb{N}$. Indeed, if neccessary, taking $E = \bigcup_{m=0}^{\infty} km$, where Km is w^*-compact, we can consider for all $n, m \in \mathbb{N}$,

$$E_n \cap K_m \cap B_{X^*}.$$

By [156] there exist w^*-open sets V_n such that if we set $A_n = \overline{E_n}^{w^*} \cap V_n$, then

$$E_n \subseteq A_n \subseteq \overline{E_n}^{\|\cdot\|} \text{ and } A_n \text{ is } w^* - LRC.$$

Each A_n is both norm F_σ and norm G_δ. So for each $n \in \mathbb{N}$, $A_n \setminus \bigcup_{k<n} A_k$ will in particular be norm $F\sigma$. Now we write

$$A_n \setminus \bigcup_{k<n} A_k = \bigcup_{m=0}^{\infty} Hn, m,$$

where each $H_{n,m}$ is norm closed. By arrangement, we assume $H_{n,m} \subseteq H_{n,m+1}$ for all $m \in \mathbb{N}$ and, for convenience, we set $H_{n,-1} = \emptyset$. Let $\pi : \mathbb{N}^2 \longrightarrow \mathbb{N}$ be a bijection and for all $i, j \in \mathbb{N}$, define

$$L_{\pi(i,j)} = H_{i,j} \setminus H_{i,j-1}.$$

Clearly E is the disjoint union of the L_n and $\overline{L_n}^{w^*} \subseteq \overline{E_p}^{w^*} \subseteq E$, where $n = \pi(p,q)$. Given $f \in E$, let

$$I(f) = \left\{ n \in \mathbb{N} : f \in \overline{L_n}^{w^*} \right\} and\ n(f) = \min I(f).$$

Now fix $\varepsilon > 0$. We define $\psi : E \longrightarrow (1, 1 + \varepsilon)$ by

$$\psi(f) = 1 + \frac{1}{2} \varepsilon \cdot 2^{-n(f)} \left(1 + \frac{1}{4} \sum_{i \in I(f)} 2^{-i} \right).$$

Set $\varepsilon_n = \frac{1}{96} \varepsilon \cdot 4^{-n}$. Fix n. As $\psi(L_n) \subseteq (1, 1 + \varepsilon)$, there is a finite partition of L_n into sets J, such that $\operatorname{diam}(\psi(J)) \leq \varepsilon_n$.

Let $P = \{I \subseteq J : I$ is ε_n-separated$\}$. This set is non-empty because any singleton is in P. For a chain $T \subseteq P$ we have $\bigcup_{N \in T} N \in P$, so we can apply Zorn's Lemma to get $\Gamma \subseteq J$, a maximal ε_n-separated subset of J. By maximality, Γ is also an ε_n-net. And by the ε_n-separation, for a totally bounded set $M \subseteq J$, the intersection $M \cap \Gamma$ is finite. By considering the finite union of these Γ, there exists $\Gamma_n \subseteq L_n$, with the property that given $f \in L_n$ there exists $h \in \Gamma_n$ so that

$$|\psi(f) - \psi(h)| \leq \varepsilon_n \text{ and } \|f - h\| \leq \varepsilon_n.$$

Moreover, if $M \subseteq L_n$ is totally bounded, $M \cap \Gamma_n$ is finite. Now define $B = \bigcup_{n=0}^{\infty} \Gamma_n$. We are now ready to define $[\![\cdot]\!]_\phi$ $on\ \ell_\infty(B)$.

For each $f \in B$ we pick a C^∞ Orlicz function ϕ_f so that

$$\phi_f(\alpha) = 0 \ if\ \alpha \leq \frac{1}{\psi(f)},$$

$$\phi_f(\alpha) > 1 \ if\ \alpha \geq \frac{1}{\theta(f)}, \text{ where } \theta(f) = \psi(f) - \varepsilon_n.$$

We define $\| \cdot \|_\phi$ with respect to these functions, as per Definition (4.1.3). By taking $(1 + \varepsilon)^{-1}$ and 1 as the constants in the hypothesis of Lemma (4.1.4) we have $l_\phi(B) \cong l_\infty(B)$ and $\| \cdot \|_\infty \leq \| \cdot \|_\phi \leq (1 + \varepsilon)\| \cdot \|_\phi$. We embed $X \otimes_\varepsilon Y$ into $\ell_\infty(B)$ by setting $\Pi(u)(f) = \|f^Y(u)\|_Y, f \in B$. The coordinate map $u \to \|f^Y(u)\|$ is a seminorm which is C^k smooth on the set where it is non-zero for each $f \in B$. Since $\|\Pi(u)\|_\infty = \|u\|$, it follows that $\| \cdot \| \leq \| \cdot \|_\phi \leq (1 + \varepsilon)\| \cdot \|$ on X.

Suppose for the sake of contradiction that the remaining hypothesis of Lemma (4.1.5) does not hold. Then we can find $u \in X \otimes_\varepsilon Y$ with $\| u \|_\phi = 1, (u_n) \subseteq X \otimes_\varepsilon Y$ with $u_n \to u$ and distinct $(f_n) \subseteq B$ such that $\phi_{fn}\left(\|f_n^Y(u_n)\|\right) > 0$, for all n. Then $\psi(f_n)\|f_n^Y(u_n)\| > 1$ for all n. Take a subsequence of (f_n), again called (f_n), such that $\psi(f_n) \to \alpha$ for some $\alpha \in \mathbb{R}$. Now take $(g_n) \subseteq S_{Y^*}$ such that $\|f_n^Y(u_n)\| = g_n(f_n^Y(u_n))$. Let $(f, g) \in B_{X^*} \times B_{Y^*}$ be an accumulation point of (f_n, g_n) in the product of the w^*-topologies. Then $f \otimes g$ is a w^*-accumulation point of $(f_n \otimes g_n)$ and $\alpha(f \otimes g)(u) \geq 1$.

The remainder of the proof is concerned with obtaining the contradiction $\alpha(f \otimes g)(u) < 1$.

Case 1: $\alpha = 1$. With $\alpha = 1$, it is evident that $\alpha(f \otimes g)(u) = (f \otimes g)(u) \leq \|u\|$. The following claim ensures $\|u\| < 1$.

Claim (4.1.8)[82]: If $v \neq 0$, then $\|v\| < \| v \|_\phi$.

Let $||v|| = 1$ and pick $p \in E, q \in S_{Y^*}$ such that $1 = (p \otimes q)(v)$. As noted above, this is possible because E and S_{Y^*} are boundaries of X and Y, respectively. By (1) above, let $r \in B$ such that $||p - r|| \leq \varepsilon_n$ for an appropriate n. Observe that $\theta(r)\big((r \otimes q)(v)\big) \leq \| v \|_\phi$ holds. Indeed,

$$\sum_{l \in B} \phi_l \left(\frac{\left| |l^Y(v)| \right|}{\theta(r)(r \otimes q)(v)} \right) \geq \phi_r \left(\frac{\left| |r^Y(v)| \right|}{\theta(r) q\left(r^Y(v)\right)} \right) \geq \phi_r \left(\frac{1}{\theta(r)} \right) > 1.$$

Now to prove the claim,

$$\begin{aligned} 1 &= (p \otimes q)(v) \\ &= (r \otimes q)(v) + \big((p - r) \otimes q\big)(v) \\ &= \theta(r)(r \otimes q)(v) + \big(1 - \theta(r)\big)(r \otimes q)(v) + \big((p - r) \otimes q\big)(v) \\ &\leq \| v \|_\phi + \big(1 - \theta(r)\big)(r \otimes q)(v) + \big((p - r) \otimes q\big)(v). \end{aligned}$$

So we are done if $(\theta(r) - 1)(r \otimes q)(v) + ((r - p) \otimes q)(v) > 0$. Indeed,

$$\begin{aligned} \theta(r) - 1 &= \psi(r) - \varepsilon_n - 1 \\ &\geq \frac{1}{2} \varepsilon \cdot 2^{-n}(r) - \varepsilon_n \\ &\geq \frac{1}{2} \varepsilon \cdot 2^{-n} - \varepsilon_n. \end{aligned}$$

Also, $(r \otimes q)(v) = r\big(q^X(v)\big) \geq 1 - \big||p - r|\big| \cdot \big||q^X(v)|\big| \geq \frac{1}{2}$. Thus,

$$\begin{aligned} (\theta(r) - 1)(r \otimes q)(v) + \big((r - p) \otimes q\big)(v) &\geq \frac{1}{4} \varepsilon \cdot 2^{-n} - \frac{1}{2} \varepsilon_n - \varepsilon_n \\ &= \frac{1}{4} \varepsilon \cdot 2^{-n} - \frac{3}{2} \varepsilon_n \\ &= \frac{1}{4} \varepsilon \cdot 2^{-n} - \frac{1}{64} \varepsilon \cdot 4^{-n} > 0. \end{aligned}$$

And the claim is proven.

Case 2: $\alpha > 1$.

We'll first prove $f \in E$.

Fix N large enough so that $1 + \varepsilon \cdot 2^{-N} < \frac{1}{2}(1 + \alpha)$. Because $\psi(f_n) \to \alpha$ we have $\psi(f_m) > \frac{1}{2}(1 + \alpha)$for all m large enough. Hence, $n(f_m) < N$. Therefore, $f_m \in \bigcup_{k > \alpha} . \overline{L_k}^{w^*}$ for all such m. By w^*closure, $f \in \bigcup_{k < n} . \overline{L_k}^{w^*} \subseteq E$.

Now the aim is to prove $\psi(f) > \alpha$.

We can assume $f_n \neq f$ for all $n \in \mathbb{N}$, because the fn are distinct. Now fix the unique m such that $f \in L_m$ and let

$$J = I(f) \cup \{k \in \mathbb{N} : k \geq m + 2\}.$$

Clearly $m \in I(f)$. Let $(p, q) \in \mathbb{N}^2$ such that $m = \pi(p, q)$. We have $L_m \subseteq A_p$. Since A_p is w^* -LRC, there exists a w^* -open set $U \ni f$, such that $A_p \cap U$ is relatively norm compact.

From before, $\Gamma_{\pi(p,k)} \cap U$ is finite for all $k \in \mathbb{N}$, since $\Gamma_{\pi(p,k)} \subseteq A_p$. So the set

$$V = U \left(\bigcup_{i \in \mathbb{N} \setminus J} \overline{L_i}^{w^*} \cup \left(\bigcup_{k=0}^{q} \Gamma_{\pi(p,k)} \{f\} \right) \right)$$

is w^* -open. Moreover, because $f \overline{\in} \bigcup_{i\in\mathbb{N}\setminus J} \overline{L_i}^{w^*}$, we have $f \in V$. We assume from now on that $f_n \in V$.

Claim (4.1.9)[82]: $m \notin I(f_n)$.

If $m \in I(f_n)$, then

$$f_n \in \overline{L_m}^{w^*} \cap V \subseteq \overline{L_m \cap V}^{w^*} = \overline{L_m \cap V}^{\|\cdot\|} \subseteq \overline{L_m}^{\|\cdot\|} \subseteq H_{p,q}.$$

It follows that $f_n \in H_{p,k}\setminus H_{p,k-1} = L_{\pi(p,k)}$ for some $k \leq q$. On the other hand, $f_n \in B$, so $f_n \in L_{\pi(p,k)} \cap B = \Gamma_{\pi(p,k)}$. However, this cannot be the case, since $f_n \in V\setminus\{f\}$.

Claim (4.1.10)[82]: $I(fn) \subseteq J$.

Let $i \in I(f_n)$. If $i \notin J$, then $f_n \in \bigcup_{j\in\mathbb{N}\setminus J} \overline{L_j \cap V}^{w^*}$, but this contradicts $f_n \in V$.

Claim (4.1.11)[82]: $\psi(f) - \psi(f_n) \geq \frac{1}{16}\, \varepsilon \cdot 4^{-m} = 6\varepsilon_m$.

First note $n(f_n) \geq n(f)$, using Claim (4.1.10) and $n(f) = min\, I(f) = min\, J$. There are two cases to consider. If $n(f_n) > n(f)$, then
$\psi(f) - \psi(f_n) \geq 1 + \frac{1}{2}\, \varepsilon \cdot 2^{-n(f)} - (1 + \frac{3}{4}\, \varepsilon \cdot 2^{-n(fn)} \geq \frac{1}{8}\, \varepsilon \cdot 2^{-n(f)} \geq \frac{1}{8}\, \varepsilon \cdot 2^{-m}$. And if $n(f_n) = n(f)$, then

$$\begin{aligned}\psi(f) - \psi(f_n) &\geq \frac{1}{8}\, \varepsilon \cdot 2^{-n(f)} \left(\sum_{i\in I(f)} 2^{-i} - \sum_{i\in I(f_n)} 2^{-i} \right) \\ &= \frac{1}{8}\, \varepsilon \cdot 2^{-n(f)} \left(\sum_{i\in I(f)\setminus I(f_n)} 2^{-i} - \sum_{i\in I(f_n)\setminus I(f)} 2^{-i} \right) \\ &\geq \frac{1}{8}\, \varepsilon \cdot 2^{-n(f)} \left(2^{-m} - \sum_{i\in J\setminus I(f)} 2^{-i} \right) \\ &\geq \frac{1}{8}\, \varepsilon \cdot 2^{-n(f)} \cdot 2^{-m-1} \geq \frac{1}{16}\, \varepsilon \cdot 4^{-m}\end{aligned}$$

Claim (4.1.12)[82]: For $h \in B$, $\left\|h \otimes g\right\|_\phi \leq \frac{1}{\theta(h)}$.

If $|(h \otimes g)(v)| > \frac{1}{\theta(h)}$, then

$$\sum_{l\in B} \phi_l\left(\left\|l^Y(v)\right\|\right) \geq \phi_h\left(\left\|h^Y(v)\right\|\right)$$
$$\geq \phi_h\big((h \otimes g)(v)\big) > 1 \Longrightarrow \left\|v\right\|_\phi > 1.$$

So, $\left\|h \otimes g\right\|_\phi = sup\left\{|(h \otimes g)(v)| : \left\|v\right\|_\phi \leq 1\right\} \leq \frac{1}{\theta(h)}$.

We can now prove $\alpha(f \otimes g)(x) < 1. By$ (1), take $h \in B$ such that $||f - h|| \leq \varepsilon_n$ and $|\psi(f) - \psi(h)| \leq \varepsilon_n$. We then have

$$\begin{aligned}\alpha\left\|f \otimes g\right\|_\phi &\leq \alpha\left(\left\|h \otimes g\right\|_\phi + \left\|(f - h) \otimes g\right\|_\phi\right) \\ &\leq \alpha\left(\left\|h \otimes g\right\|_\phi + \left\|(f - h) \otimes g\right\|\right) \\ &\leq \left(\alpha \frac{1}{\theta(h)} + \varepsilon_n\right).\end{aligned}$$

So we are done if $\alpha(\frac{1}{\theta(h)} + \varepsilon_n) < 1$. Well,

$1-\frac{\alpha}{\theta(h)} - \alpha\varepsilon_n > 0$

$$\iff \theta(h) - \alpha - \varepsilon_n\theta(h)\alpha > 0$$
$$\iff \psi(h) - \varepsilon_n - \alpha - \varepsilon_n\theta(h)\alpha > 0.$$

By claim $2c$, we have $\psi(h) - \varepsilon_n - \alpha \geq 4\varepsilon_n$ and since $\theta(h), \alpha < 2$, it follows that $\varepsilon_n\theta(h)\alpha < 4\varepsilon_n$.

And so, $\alpha\|f \otimes g\|_\phi < 1 \implies \alpha(f \otimes g)(u) < 1$.

Corollary (4.1.13)[82]: Suppose X has $a\ \sigma - w^*$ -LRC and σ-w^* -compact boundary. Then X has a C^∞ renorming.

Proof. Apply Theorem (4.1.7) to $X \otimes_\varepsilon \mathbb{R} = X$.

We can now prove Theorems (4.1.14) and (4.1.15) as corollaries of Corollary (4.1.13).

Theorem (4.1.14)[82]: ([157]). If $(X, ||\cdot||)$ admits a boundary contained in $a\ ||\cdot|| - \sigma$-compact subset of B_{X^*} , then X admits an equivalent C^∞ smooth norm that ε − approximates $||\cdot||$.

Proof. Any norm compact subset of X^* is trivially w^* - LRC. The result follows from Corollary (4.1.13).

Theorem (4.1.15)[82]: ([158]). Let K be $a\ \sigma$-discrete compact space. Then, given $\varepsilon > 0$, $C(K)$ admits an equivalent C^∞ smooth norm that ε −approximates $\|\cdot\|_\infty$.

Proof. Let $K = \bigcup_{n=0}^\infty D_n$, where each D_n is relatively discrete. Let δ_t be the usual evaluation functionals, $\delta_t(f) = f(t)$. Then $E_n = \{\pm\delta_t : t \in D_n\}$ is w^* -relatively discrete and so w^* -LRC. Moreover, $E = \bigcup_{n=0}^\infty E_n$ is a w^* -compact boundary of $C(K)$ because given any $f \in C(K)$, there exists $t \in K$ such that $\|f\|_\infty = |f(t)|$, by compactness.

The corollaries below are new results. Before presenting them, a definition and a theorem appearing in [156] are needed.

Definition (4.1.16)[82]: ([156]). Let X be a Banach space. We say a set $F \subseteq X^*$ is a relative boundary if, whenever $x \in X$ satisfies $sup\{f(x) : f \in F\} = 1$, there exists $f \in F$ such that $f(x) = 1$.

Example (4.1.17)[82]: Any boundary and any w^* -compact set will be a relative boundary.

Theorem (4.1.18)[82]: ([156]). Let X be a Banach space and suppose we have sets $S_n \subseteq S_X$ and an increasing sequence $H_n \subseteq B_{X^*}$ of relative boundaries, such that $S_X = \bigcup_{n=0}^\infty S_n$ and the numbers

$$b_n = \inf\{\sup\{h(x): h \in H_n\} : x \in S_n\}$$

are strictly positive and converge to 1. Then for a suitable sequence $(a_n)_{n=0}^\infty$ of numbers the set $F = \bigcup_{n=0}^\infty a_n(H_n \backslash H_{n-1})$ is a boundary of an equivalent norm.

Given a Banach space with an unconditional basis $(e_i)_{i\in I}$ and $x = \sum_{i\in I} x_i e_i$ $, let\ e_i^*(x) = x_i$. For $\sigma \subseteq I$, let $P\sigma$ denote the projection given by $P_\sigma(x) = \sum_{i\in\sigma} e_i^*(x)e_i$.

Corollary (4.1.19)[82]: Let X have a monotone unconditional basis $(e_i)_{i\in I}$, with associated projections $P_\sigma, \sigma \subseteq I$, and suppose we can write $S_X = \bigcup_{n=1}^\infty S_n$ in such a way that the numbers

$$c_n = \inf\{\sup\{\|P_\sigma(x)\| : \sigma \subseteq I, |\sigma| = n\} : x \in S_n\}$$

are strictly positive and converge to 1. Then X admits an equivalent C^∞ smooth norm.

Proof. Let $H_n = \{h \in B_{X^*} : |supp(h)| \leq n\}$. Each H_n is a relative boundary because it is w^* -compact. Note that given $x \in S_n$ and $\sigma \subseteq I$, with $|\sigma| = n$,

$$\|P_\sigma(x)\| = \sup\{f(P_\sigma(x)): f \in B_{X^*}\}$$
$$= \sup\{P_\sigma^* f(x): f \in B_{X^*}\}.$$

Of course, $|supp(P_\sigma^* f)| \leq n$, for all $f \in B_{X^*}$. And by monotonicity, $||P_\sigma^* |f| = 1. So\ P_\sigma^* (f) \in H_n$. Therefore,

$$0 < c_n = \inf\{\sup\{\|P_\sigma(x)\|: \sigma \subseteq I, |\sigma| = n\}: x \in S_n\}$$
$$= \inf\{\sup\{P_\sigma^* f(x): f \in B_{X^*}, \sigma \subseteq I, |\sigma| = n\}: x \in S_n\}$$
$$= \inf\{\sup\{h(x): h \in H_n\}: x \in S_n\} = b_n.$$

Thus, (b_n) is a strictly positive sequence converging to 1. The set $H_n \backslash H_{n-1}\ is\ w^*\text{-}LRC$.

By Theorem (4.1.18), there exists a sequence $(a_n)_{n=0}^\infty$, where the set $F = \bigcup_{n=0}^\infty a_n(H_n \backslash H_{n-1})$ is a $\sigma - w^*$ -LRC and σ-w^* -compact boundary for an equivalent norm $|||\cdot|||$. By Corollary (4.1.13), X will admit an equivalent C^∞ −smooth that ε-approximates $|||\cdot|||$.

Corollary (4.1.20)[82]: Let X be a Banach space with a monotone unconditional basis $(e_i)_{i\in I}$ and suppose for each $x \in S_X$ there exists $\sigma \subset I, |\sigma| < \infty$, so that $||P_\sigma(x)|| = 1$. Then X admits an equivalent $C^\infty -$ smooth norm that ε-approximates the original norm.

Proof. Let $H_n = \{h \in B_{X^*} : |supp(h)| \leq n\}$. As mentioned in the proof of Corollary (4.1.19), each H_n is w^* -compact and the finite union of w^* -LRC sets. Now take $x \in S_X$ and σ such that $||P_\sigma(x)|| = 1$. Then there is $f \in B_{X^*}$ such that

$$1 = \|P_\sigma(x)\| = f(P_\sigma(x)) = P_\sigma^* f(x).$$

Because $(e_i)_{i\in I}$ is monotone, $||P_\sigma^*|| = 1$ and so $P_\sigma^* f \in H_{|\sigma|}$. Therefore, the set $H = \bigcup_{n=0}^\infty H_n$ is a boundary satisfying the hypothesis of Corollary (4.1.13).

Using Corollary (4.1.19) we can obtain new examples of spaces with equivalent C^∞ smooth renormings.

Example (4.1.21)[82]: Let $\mathbb{N} = \bigcup_{n=0}^\infty A_n$, where each A_n is finite, and let $p = (p_n)$ be an unbounded increasing sequence of real numbers with $p_n \geq 1$. For each sequence of real numbers $x = (x_n)$ define

$$\Phi(x) = \sup\left\{\sum_{n=0}^{\infty} \sum_{k\in B_n} |x(k)|^{pn} : B_n \subset A_n\ and\ Bn\ are\ pairwise\ disjoint.\right\}$$

Proof. We define $\ell_{A,p}$ as the space of sequences x where $\Phi(x/\lambda) < \infty$ for some $\lambda > 0$, with norm $||x|| = inf\{\lambda > 0 : \Phi(x/\lambda) \leq 1\}$. Define the subspace $h_{A,p}$ as the norm closure of the linear space generated by the basis $e_n(k) = \delta_{n,k}$. [156] provides an appropriate sequence of subsets (S_n) of S_X so that Corollary (4.1.19) holds.

Example (4.1.22)[82]: Let M be an Orlicz function with

$$M(t) > 0\ for\ all\ t > 0, \text{and} \lim_{t\to 0} (\frac{M(K(t))}{M(t)} = +\infty,$$

for some constant $K > 0$. Let hM (Γ) be the space of all real functions x defined on Γ with $\sum_{\gamma\in\Gamma} M(x_\gamma/\rho) < \infty$ for all $\rho > 0$, with the norm

$$\|x\| = inf\left\{\rho > 0 : \sum_{\gamma\in\Gamma} M\left(\frac{x_\gamma}{\rho}\right) \leq 1\right\}.$$

Proof. The canonical unit vector basis $(e_\gamma)_{\gamma\in\Gamma}$ of functions $e_\gamma(\beta) = \delta_{\gamma,\beta}$ is unconditionally monotone. [156] provides suitable subsets of S_X to ensure the hypothesis of

Corollary (4.1.19) holds. The final example concerns the predual of a Lorentz sequence space $d(w,1,A)$, for an arbitrary set A.

Let $w = (w_n) \in c_0\backslash \ell_1$ with each w_n strictly positive and $w_0 = 1$. We define $d(w,1,A)$ as the space of $x: A \longrightarrow \mathbb{R}$ for which

$$||x|| = \sup \sum_{j=0}^{\infty} w_j \, |x(a_j)| : (a_j) \subseteq A \text{ is a sequence of distinct points}) < \infty .$$

The canonical predual $d_*(w,1,A)$ of $d(w,1,A)$ is given by the space of $y : A \longrightarrow \mathbb{R}$ for which $\overline{y} = (\overline{y}_k) \in c_0$, where

$$\overline{y}_k = sup\left\{\frac{\sum_{i=0}^{k-1}|y(a_i)|}{\sum_{i=0}^{k-1} w_i} : a_0, a_1, \ldots, a_{k-1} \text{ are distinct points of } A\right\},$$

with norm $||y|| = \left\|\overline{y}\right\|_{\infty}$. We can see that $(e_a)_{a\in A}$ is a monotone unconditional basis for both $d(w,1,A)$ and $d_*(w,1,A)$. The separable version of $d_*(w,1,A)$ was first introduced in [29].

Example (4.1.23)[82]: $X = d_*(w,1,A)$ has a C^∞ smooth equivalent renorming that ε-approximates the original norm.

Proof. Let $y \in S_X$. Since $\overline{y} \in c_0$, there exists $k \in \mathbb{N}$ such that $\overline{y}^k = 1$. It can also be shown $y \in c_0(A)$ and thus the supremum in the definition of $\overline{y}^k$ is attained. Following this, there exists $a_0, a_1, \ldots, a_{k-1} \in A$ such that

$$1 = \overline{y}^k = \frac{\sum_{i=0}^{k-1}|y(a_i)|}{\sum_{i=0}^{k-1} w_i}.$$

Setting $\sigma = \{a_0, a_1, \ldots, a_{k-1}\}$, we have $||P_\sigma(y)|| = 1$. By Corollary (4.1.20), X has a C^∞ smooth equivalent renorming that ε-approximates the original norm.

Section (4.2): Riemannian Manifolds with Constant Negative Curvature

For (M,g) be a compact n-dimensional Riemannian manifold and let Δ_g be the associated Laplace–Beltrami operator. Let e_λ denote the L^2-normalized eigenfunction

$$-\Delta_g e_\lambda = \lambda^2 e_\lambda,$$

so that $\lambda \geq 0$ is the eigenvalue of the operator $-\Delta_g$. A classical result on the L^p-estimates of the eigenfunctions is due to Sogge [15]:

$$\|e_\lambda\|_{L^p(M)} \leq C\lambda^{\delta}(p), \tag{1}$$

where $2 \leq p \leq \infty$ and

$$\delta(p) = \begin{cases} \dfrac{n-1}{2} \, (\dfrac{1}{2} - \dfrac{1}{p}), 2 \leq p \leq p_c, \\ n\left(\dfrac{1}{2} - \dfrac{1}{p}\right) - \dfrac{1}{2}, p_c \leq p \leq \infty, \end{cases}$$

if we set $p_c = \frac{2n+2}{n-1}$. These estimates (1) are saturated on the round sphere Sn by zonal functions for $p \geq p_c$ and for $2 < p \leq p_c$ by the highest weight spherical harmonics. However, it is expected that (1) can be improved for generic Riemannian manifolds. It was known that one can get log improvements for $\|e_\lambda\|_{L^p(M)}, pc < p \leq \infty$, when M has nonpositive sectional curvature. Indeed, Bérard's results [20] on improved remainder term bounds for the pointwise Weyl law imply that

$$\|e_\lambda\|_{L^\infty(M)} \leq C\lambda^{\frac{n-1}{2}} (\log \lambda)^{-\frac{1}{2}} \|e_\lambda\|_{L^2(M)}.$$

Recently, Hassell and Tacy [82] obtained a similar $(\log \lambda)^{-\frac{1}{2}}$ gain for all $p > p_c$.

Similar L^p-estimates have been established for the restriction of eigenfunctions to geodesic segments. Let Π denotes the space of all unit-length geodesics. The works [163], [1], [156] (see also [26] for earlier results on hyperbolic surfaces) showed that

$$\sup_{\gamma \in \Pi} \left(\int_\gamma |e_\lambda|^p \, ds \right)^{\frac{1}{p}} \leq C\lambda^{\sigma(n,p)} \, \|e_\lambda\|_{L^2(M)}, \tag{2}$$

where

$$\sigma(2,p) = \begin{cases} \frac{1}{4}, 2 \leq p \leq 4, \\ \frac{1}{2} - \frac{1}{p}, 4 \leq p \leq \infty, \end{cases} \tag{3}$$

$$\sigma(n,p) = \frac{n-1}{2} - \frac{1}{p}, \text{if } p \geq 2 \text{ and } n \geq 3. \tag{4}$$

It was known that these estimates are saturated by the highest weight spherical harmonics when $n \geq 3$ on round sphere Sn, as well as in the case of $2 \leq p \leq 4$ when $n = 2$, while in this case the zonal functions saturate the bounds for $p \geq 4$.

There are considerable works towards improving (2) for the 2-dimensional manifolds with nonpositive curvature. Chen [19] proved a $(\log \lambda)^{-\frac{1}{2}}$ gain for all $p > 4$. Sogge and Zelditch [30] and Chen and Sogge [156] showed that one can improve (2) for $2 \leq p \leq 4$, in the sense that

$$\sup_{\gamma \in \Pi} \left(\int_\gamma |e_\lambda|^p \, ds \right)^{\frac{1}{p}} = o\left(\lambda^{\frac{1}{4}}\right). \tag{5}$$

Recently, using the Toponogov's comparison theorem, Blair and Sogge [162] obtained log improvements for $p = 2$:

$$\sup_{\gamma \in \Pi} \left(\int_\gamma |e_\lambda|^2 \, ds \right)^{\frac{1}{2}} \leq C\lambda^{\frac{1}{4}} (\log \lambda)^{-\frac{1}{4}} \, \|e_\lambda\|_{L^2(M)}. \tag{6}$$

Inspired by the works [162], [156], [168], X_i and [160] was able to deal with the other endpoint $p = 4$ and proved a $(\log \log \lambda)^{-\frac{1}{8}}$ gain for surfaces with nonpositive curvature and a $(\log \lambda)^{-\frac{1}{4}}$ gain for hyperbolic surfaces

$$\sup_{\gamma \in \Pi} \left(\int_\gamma |e_\lambda|^4 \, ds \right)^{\frac{1}{4}} \leq C\lambda^{\frac{1}{4}} (\log \lambda)^{-\frac{1}{4}} \, \|e_\lambda\|_{L^2(M)}. \tag{7}$$

In the 3-dimensional case, under the assumption of nonpositive curvature, Chen [19] also proved a $(\log \lambda)^{-\frac{1}{2}}$ gain for all $p > 2$. With the assumption of constant negative curvature, Chen and Sogge [156] showed that

$$\sup_{\gamma \in \Pi} \left(\int_\gamma |e_\lambda|^2 \, ds \right)^{\frac{1}{2}} = o\left(\lambda^{\frac{1}{2}}\right). \tag{8}$$

Hezari and Rivière [165] and Hezari [164] used quantum ergodic methods to get logarithmic improvements at critical exponents in the cases above on negatively curved manifolds for a density one subsequence.

We prove a $(\log\lambda)^{-\frac{1}{2}}$ gain for the L^2 geodesic restriction bounds on 3-dimensional compact Riemannian manifolds with constant negative curvature. We mainly follow the approaches developed in [162], [156], [160]. We derive an explicit formula for the wave kernel on $\mathbb{H}^3$, which is one of the key steps to get the $(\log\lambda)^{-\frac{1}{2}}$ gain. We shall lift all the calculations to the universal cover $\mathbb{H}^3$ and then use the Poincaré half-space model to derive the explicit formulas of the mixed derivatives of the distance function restricted to the unit geodesic segments. Then we decompose the domain of the distance function and compute the bounds of various mixed derivatives explicitly, since it was observed in [156] and [160] that the desired kernel estimates follow from the oscillatory integral estimates and the estimates on the mixed derivatives. Moreover, whether one can get similar logarithmic improvements on 3-dimensional manifolds with nonpositive curvature is still an interesting open problem. One of the technical difficulties is that these manifolds may not have sufficiently many totally geodesic submanifolds (see [156]). We shall assume that the injectivity radius of M is sufficiently large, and fix γ to be a unit length geodesic segment parameterized by arclength.

Theorem (4.2.1)[160]: Let (M,g) be a 3-dimensional compact Riemannian manifold of constant negative curvature, let $\gamma\subset M$ be a fixed unit-length geodesic segment. Then for $\lambda\gg 1$, there is a constant C such that

$$\|e_\lambda\|_{L^2(\gamma)}\le C\lambda^{\frac{1}{2}}(\log\lambda)^{-\frac{1}{2}}\|e_\lambda\|_{L^2(M)}. \tag{9}$$

Moreover, if Π denotes the set of unit-length geodesics, there exists a uniform constant $C=C(M,g)$ such that

$$\sup_{\gamma\in\Pi}\left(\int_\gamma |e_\lambda|^2\,ds\right)^{\frac{1}{2}}\le C\lambda^{\frac{1}{2}}(\log\lambda)^{-\frac{1}{2}}\|e_\lambda\|_{L^2(M)}. \tag{10}$$

We start with some standard reductions. Since the uniform bound (10) follows from a standard compactness argument in [156], we only need to prove (9). Let $T\gg 1$. Let $\rho\in S(\mathbb{R})$ such that $\rho(0)=1$ and $\operatorname{supp}\hat\rho\subset[-1/2,1/2]$, then it is clear that the operator $\rho\left(T\left(\lambda-\sqrt{-\Delta_g}\right)\right)$ reproduces eigenfunctions, namely $\rho\left(T\left(\lambda-\sqrt{-\Delta_g}\right)\right)e_\lambda=e_\lambda$. Let $\chi=|\rho|^2$. After a standard TT^* argument, we only need to estimate the norm

$$\left\|\chi\left(T\left(\lambda-\sqrt{-\Delta_g}\right)\right)\right\|_{L^2(\gamma)\to L^2(\gamma)}. \tag{11}$$

Choose a bump function $\beta\in C_0^\infty(\mathbb{R})$ satisfying

$$\beta(\tau)=1 \text{ for } |\tau|\le 3/2, \text{and } \beta(\tau)=0, |\tau|\ge 2.$$

By the Fourier inversion formula, we may represent the kernel of the operator $\chi\left(T\left(\lambda-\sqrt{-\Delta_g}\right)\right)$ as an operator valued integral

$$\begin{aligned}\chi\left(T\left(\lambda-\sqrt{-\Delta_g}\right)\right)&(x,y)\\ &=\frac{1}{2\pi T}\int \beta(\tau)\hat\chi(\tau/T)e^{i\lambda\tau}\left(e^{-i\tau\sqrt{-\Delta_g}}\right)(x,y)d\tau\\ &+\frac{1}{2\pi T}\int\left(1-\beta(\tau)\right)\hat\chi(\tau/T)e^{i\lambda\tau}\left(e^{-i\tau\sqrt{-\Delta_g}}\right)(x,y)d\tau\\ &=K_0(x,y)+K_1(x,y).\end{aligned}$$

Then one may use a parametrix to estimate the norm of the integral operator associated with the kernel $K_0\big(\gamma(t),\gamma(s)\big)$ (see [156])

$$\|K_0\|_{L^2[0,1]\to L^2[0,1]} \leq C\lambda T^{-1}. \tag{12}$$

Since the kernel of $\chi(T(\lambda + \sqrt{-\Delta_g}))$ is $O(\lambda^{-N})$ with constants independent of T, by Euler's formula we are left to consider the integral operator S_λ:

$$S_\lambda h(t) = \frac{1}{\pi T}\int_{-\infty}^{\infty}\int_0^1 \big(1 - \beta(\tau)\big)\hat{\chi}(\tau/T)e^{i\lambda\tau}\left(\cos\tau\sqrt{-\Delta_g}\right)\big(\gamma(t),\gamma(s)\big)h(s)\,ds\,d\tau. \tag{13}$$

As in [162], [156], [160], we use the Hadamard parametrix and the Cartan–Hadamard theorem to lift the calculations up to the universal cover $(\mathbb{R}^3,\tilde{g})$ of (M,g). Let Γ denote the group of deck transformations preserving the associated covering map $\kappa : \mathbb{R}^3 \to M$ coming from the exponential map from $\gamma(0)$ associated with the metric g on M. The metric $\tilde{g}$ is its pullback via κ. Choose also a Dirichlet fundamental domain, $D \simeq M$, for M centered at the lift $\tilde{\gamma}(0)$ of $\gamma(0)$. Let $\tilde{\gamma}(t), t \in \mathbb{R}$, satisfy $\kappa\big(\tilde{\gamma}(t)\big) = \gamma(t)$, where γ is the unit speed geodesic containing the geodesic segment $\{\gamma(t): t \in [0,1]\}$. Then $\tilde{\gamma}(t)$ is also a geodesic parameterized by arclength. We measure the distances in $(\mathbb{R}^3,\tilde{g})$ using its Riemannian distance function $d\tilde{g}(\cdot,\cdot)$. Moreover, we recall that if $\tilde{x}$ denotes the lift of $x \in M$ to D, then (

$$\left(\mathrm{cost}\sqrt{-\Delta_g}\right)(x,y) = \sum_{\alpha\in\Gamma}\left(\mathrm{cost} - \sqrt{\Delta_{\tilde{g}}}\right)\big(\tilde{x},\alpha(\tilde{y})\big).$$

Hence for $t \in [0,1]$,

$$S_\lambda h(t) = \frac{1}{\pi T}\sum_{\alpha\in\Gamma}\int_{\mathbb{R}}\int_0^1 \big(1 - \beta(\tau)\big)\hat{\chi}(\tau/T)e^{i\lambda\tau}\left(\cos\tau\sqrt{-\Delta_{\tilde{g}}}\right)\Big(\tilde{\gamma}(t),\alpha\big(\tilde{\gamma}(s)\big)\Big)h(s)\,ds\,d\tau.$$

As in [162] and [160], we denote the R-tube about the infinite geodesic $\tilde{\gamma}$ by

$$T_R(\tilde{\gamma}) = \big\{(x,y,z) \in \mathbb{R}^3 : d_{\tilde{g}}\big((x,y,z),\tilde{\gamma}\big) \leq \mathbb{R}\big\} \tag{14}$$

and

$$\Gamma_{\mathrm{T}_R}(\tilde{\gamma}) = \{\alpha \in \Gamma : \alpha(D)\cap T_R(\tilde{\gamma}) = \emptyset\}.$$

From now on we fix $R \approx \mathrm{Inj}M$. We will see that $\mathrm{T}_R(\tilde{\gamma})$ plays a key role in the proof of Lemma (4.2.5). Then we decompose the sum

$$S_\lambda h(t) = S_\lambda^{tube}\,h(t) + S_\lambda^{osc}\,h(t) = \sum_{\alpha\in\Gamma_{\mathrm{T}_R}(\tilde{\gamma})} S_{\lambda,\alpha}^{tube}\,h(t) + \sum_{\alpha\notin\Gamma_{\mathrm{T}_R}(\tilde{\gamma})} S_{\lambda,\alpha}^{osc}h(t), t \in [0,1].$$

Then by the finite propagation speed property and $\hat{\chi}(\tau) = 0$ if $|\tau| \geq 1$, we have

$$d_{\tilde{g}}\Big(\tilde{\gamma}(t),\alpha\big(\tilde{\gamma}(s)\big)\Big) \leq T, s,t \in [0,1].$$

As observed in [162],

$$\#\{\alpha \in \Gamma_{\mathrm{T}_R}(\tilde{\gamma}) : d_{\tilde{g}}(0,\alpha(0)) \in [2k,2k+1]\} \leq C2^k. \tag{15}$$

Thus the number of nonzero summands in $S_\lambda^{tube}\,h(t)$ is $O(T)$ and in $S_\lambda^{osc}\,h(t)$ is $O(e^{CT})$. Given $\alpha \in \Gamma$ set with $s,t \in [0,1]$

$$K\alpha(t,s) = \frac{1}{\pi T}\int_{-T}^{T} \left(1 - \beta(\tau)\right)\hat{\chi}(\tau T)e^{i\lambda\tau}\left(\cos\tau\sqrt{-\Delta_{\tilde{g}}}\right)\left(\tilde{\gamma}(t), \alpha\left(\tilde{\gamma}(s)\right)\right)d\tau.$$

When $\alpha = I$ dentity, one can use the Hadamard parametrix to prove the same bound as (12) (see e.g. [19], p. 9)

$$\|K_{\mathrm{Id}}\|_{L^2[0,1]\to L^2[0,1]} \le C\lambda T^{-1}. \tag{16}$$

If $\alpha \neq I$dentity, we set $\phi(t,s) = d_{\tilde{g}}\left(\tilde{\gamma}(t), \alpha\left(\tilde{\gamma}(s)\right)\right), s,t \in [0,1]$. Then by finite propagation speed and $\alpha \neq I$dentity, we have

$$2 \le \phi(t,s) \le T, \text{if } s,t \in [0,1]. \tag{17}$$

As in [156], one may use the Hadamard parametrix and stationary phase to show that $|K_\alpha(t,s)| \le C\lambda T^{-1}r^{-1} + e^{CT}$, where $r = d_{\tilde{g}}\left(\tilde{\gamma}(t), \alpha\left(\tilde{\gamma}(s)\right)\right)$. However, we may get a much better estimate for K_α. To see this, we need to derive the explicit formula of the wave kernel on hyperbolic space. We may assume that (M,g) has constant negative curvature -1, which implies that the covering manifold $(\mathbb{R}^3, \tilde{g})$ is the hyperbolic space $\mathbb{H}^3$. If we denote the shifted Laplacian operator by

$$L = \Delta_{\tilde{g}} + \frac{(n-1)^2}{4} = \Delta_{\tilde{g}} + 1 \text{ (for } n = 3),$$

which has the property $\mathrm{Spec}(-L) = [0,\infty)$, then there are exact formulas for various functions of L (see e.g. [169]). Indeed,

$$h\left(\sqrt{-L}\right)\delta_y(x) = -\frac{1}{(2\pi)^{3/2}}\frac{1}{\sinh r}\frac{\partial}{\partial r}\hat{h}(r),$$

where $\hat{h}$ is the Fourier transform defined by

$$\hat{h}(r) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h(k)e^{-irk}dk.$$

If $h(k) = \frac{\sin(tk)}{k}$, then $\hat{h}(r) = \frac{\sqrt{2\pi}}{2}\,1_{\{r\le|t|\}}$. Hence, for $t > 0$,

$$\frac{\sin t\sqrt{-L}}{\sqrt{-L}}\,\delta_y(x) = \frac{\delta(t-r)}{4\pi\sinh r}, \tag{18}$$

where $x, y \in \mathbb{H}^3$ and $r = d_{\tilde{g}}(x,y)$. Differentiating it yields

$$\cos t\sqrt{-L}\,\delta_y(x) = \frac{\delta'(t-r)}{4\pi\sinh r}. \tag{19}$$

Recall the following relation between L and $\Delta_{\tilde{g}}$ (see e.g. [166])

$$\cos t\sqrt{-\Delta_{\tilde{g}}} = \cos t\sqrt{-L} - t\int_0^t \frac{J_1\left(\sqrt{t^2-s^2}\right)}{\sqrt{t^2-s^2}}\cos s\sqrt{-L}ds, \tag{20}$$

where $J_1(v)$ is the Bessel function

$$J_1(v) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,(k+1)!}\left(\frac{v}{2}\right)^{2k+1}.$$

We plug (19) into the relation (20) to see that for $t > 0$,

$$\cos t\sqrt{-\Delta_{\tilde{g}}}\,\delta_y(x) = \frac{\delta'(t-r)}{4\pi\sinh r} - t\int_0^t \frac{J_1\left(\sqrt{t^2-s^2}\right)}{\sqrt{t^2-s^2}}\frac{\partial_s\delta(s-r)}{4\pi\sinh r}ds.$$

Thus, integrating by parts and noting that $\cos t\sqrt{-\Delta_{\tilde{g}}}$ is even in t, we get the following explicit formula for the wave kernel "$\cos t\sqrt{-\Delta_{\tilde{g}}}(x,y)$" on $\mathbb{H}^3$

$$\cos t\sqrt{-\Delta_{\tilde{g}}}\,\delta_y(x)$$
$$= \frac{1}{4\pi\sinh r}\left[\delta'(|t| - r) - J_1'(0)|t|\delta(|t| - r) - \frac{r|t|G'\left(\sqrt{t^2 - r^2}\right)}{\sqrt{t^2 - r^2}}\, 1_{\{r\le|t|\}}\right], \quad (21)$$
where $t \in \mathbb{R}\setminus\{0\}$, and $G(v) = J_1(v)/v$ is an entire function of v_2, satisfying
$$G(v) \sim Cv^{-3/2}\cos\left(v - \frac{3\pi}{4}\right) + \cdots, \text{as } v \to +\infty. \quad (22)$$
Lemma (4.2.2)[160]: If $\alpha \neq Identity$, we have
$$|K_\alpha(t,s)| \le C\lambda T^{-1}e^{-r/2}, \text{for } t,s \in [0,1],$$
where $r = d_{\tilde{g}}\left(\tilde{\gamma}(t), \alpha\left(\tilde{\gamma}(s)\right)\right) \ge 1$ and C is a constant independent of T and r.

Using this lemma and (15), we get
$$\sum_{\alpha\in\Gamma_{T_R}(\tilde{\gamma})\setminus\{\mathrm{Id}\}} K_\alpha(t,s) \le C\lambda T^{-1}\sum_{1\le 2^k\le T} 2^k e - 2^{\frac{k}{2}} \le C\lambda T^{-1}. \quad (23)$$
Consequently, by Young's inequality and the estimate on K_{Id} (16) we have
$$\left\|S_\lambda^{tube}\right\|_{L^2[0,1]\to L^2[0,1]} \le C\lambda T^{-1}. \quad (24)$$
Proof. Since the formula of the wave kernel (21) consists of 3 terms, we should estimate their contributions separately. Integrating by parts yields
$$\left|\int_{-T}^{T}\left(1 - \beta(\tau)\right)\hat{\chi}(\tau/T)e^{i\lambda\tau}\,\delta(|\tau| - r)d\tau\right| \le \sum_{\tau=\pm r}\left|\frac{d}{d\tau}\left(1 - \beta(\tau)\right)\hat{\chi}(\tau/T)e^{i\lambda\tau}\right|$$
$$\le C\lambda, \quad (25)$$
since $\beta, \hat{\chi} \in S(\mathbb{R})$. Similarly,
$$\left|\int_{-T}^{T}\left(1 - \beta(\tau)\right)\hat{\chi}(\tau/T)e^{i\lambda\tau}\,|\tau|\delta(|\tau| - r)d\tau\right| = \left|\sum_{\tau=\pm r}\left(1 - \beta(\tau)\right)\hat{\chi}(\tau/T)e^{i\lambda\tau}\,|\tau|\right|$$
$$\le Cr. \quad (26)$$
Noting that $J_1(v), J_1'(v)$ are uniformly bounded for $v \in \mathbb{R}$ and $G(v)$ is an entire function of v^2, we see that $G(v)/v$ is also uniformly bounded for $v \in \mathbb{R}$. Moreover, by (22), there is some $N \gg 1$ such that
$$|G'(v)/v| \le Cv^{-5/2}, \text{for } v > N.$$
This gives
$$\left|\int_{-T}^{T}\left(1 - \beta(\tau)\right)\hat{\chi}(\tau/T)e^{i\lambda\tau}\,\frac{r|\tau|G'\left(\sqrt{\tau^2 - r^2}\right)}{\sqrt{\tau^2 - r^2}}\,1_{\{r\le|\tau|\}}d\tau\right|$$
$$\le Cr\int_{|\tau|\ge r}|\tau|\left|\frac{G'\left(\sqrt{\tau^2 - r^2}\right)}{\sqrt{\tau^2 - r^2}}\right|d\tau$$
$$\le Cr\left(\int_0^N |\rho + r|d\rho + \int_N^\infty |\rho + r||\rho|^{-5/2}d\rho\right)$$
$$\le Cr(C + Cr), \quad (27)$$
where $\rho = |\tau| - r$. Hence
$$|K_\alpha(t,s)| \le \frac{C\lambda + Cr + Cr^2}{T\sinh r} \le C\lambda T^{-1}e^{-r/2}.$$

We estimate the kernels $K_\alpha(t,s)$ with $\alpha \notin \Gamma_{T_R}(\tilde{\gamma})$. From now on, we assume that $\alpha \notin \Gamma_{T_R}(\tilde{\gamma})$. We need a slight variation of the oscillatory integral theorem in [160]. Indeed, it is a detailed version of the estimates by Phong and Stein [167] on the oscillatory integrals with fold singularities.

Proposition (4.2.3)[160]: Let $a \in C_0^\infty(\mathbb{R}^2)$, let $\phi \in C^\infty(\mathbb{R}^2)$ be real valued and $\lambda > 0$, set

$$T_\lambda f(t) = \int_{-\infty}^{\infty} e^{i\lambda\phi(t,s)} a(t,s) f(s) ds, \qquad f \in C_0^\infty(\mathbb{R}).$$

If $\phi''_{st} = 0$ on supp a, then

$$\|T_\lambda f\|_{L^2(\mathbb{R})} \leq C_{a,\phi} \lambda^{-\frac{1}{2}} \|f\|_{L^2(\mathbb{R})},$$

where

$$C_{a,\phi} = \text{Cdiam}(\text{supp } a)^{\frac{1}{2}} \left\{ \|a\|_\infty + \frac{\sum_{0\leq i,j\leq 2} \left\|\partial_t^i a\right\|_\infty \left\|\partial_t^j \phi''_{st}\right\|_\infty}{\inf |\phi''_{st}|^2} \right\}. \tag{28}$$

Assume supp a is contained in some compact set $F \subseteq \mathbb{R}^2$. Denote the ranges of t and s in F by $F_t \subseteq \mathbb{R}$ and $F_s \subseteq \mathbb{R}$ respectively. If for any s ∈ Fs, there is a unique $t_c = t_c(s) \in F_t$ such that $\phi'''_{st}(t_c, s) = 0$, and if $\phi'''_{stt}(t_c, s) \neq 0$ on F_s, then

$$\|T_\lambda f\|_{L^2(\mathbb{R})} \leq C'_{a,\phi} \lambda^{-\frac{1}{4}} \|f\|_{L^2(\mathbb{R})},$$

where

$$C''_{a,\phi} = \text{Cdiam}(\text{supp } a)^{\frac{1}{4}} \left\{ \|a\|_\infty + \frac{\sum_{0\leq i,j\leq 2} \left\|\partial_t^i a\right\|_\infty \left\|\partial_t^j \phi''_{st}\right\|_\infty}{\inf \left|\phi''_{st}/\left(t - t_c(s)\right)\right|^2} \right\}. \tag{29}$$

Dually, if for any $t \in F_t$, there is a unique $s_c = s_c(t) \in F_s$ such that $\phi''_{st}(t, s_c) = 0$, and if $\phi''_{tss}(t, s_c) = 0$ on F_t, then

$$\|T_\lambda f\|_{L^2(\mathbb{R})} \leq C''_{a,\phi} \lambda^{-\frac{1}{4}} \|f\|_{L^2(\mathbb{R})},$$

where

$$C''_{a,\phi} = \text{Cdiam}(\text{supp } a)^{\frac{1}{4}} \left\{ \|a\|_\infty + \frac{\sum_{0\leq i,j\leq 2} \left\|\partial_s^i a\right\|_\infty \left\|\partial_t^j \phi''_{st}\right\|_\infty}{\inf \left|\phi''_{st}/\left(s - s_c(s)\right)\right|^2} \right\}. \tag{30}$$

The L^∞-norm and the infimum are taken on supp a. The constant C > 0 is independent of λ, a, ϕ and F.

Proof. Noting that the first part is due to non-stationary phase (see [160]) and the third part simply follows from duality, we only need to prove the second part. As in [160], by a TT^* argument, it suffices to estimate the kernel of $T_\lambda^* T_\lambda$

$$K(s,s) = \int e^{i\lambda}\big(\phi(t,s) - \phi(t,s)\big) a(t,s) a(t,s) dt.$$

Let

$$\varphi(t,s,s) = \frac{\phi(t,s) - \phi(t,s)}{s - s'}, \qquad \text{for } s \neq s', \qquad \text{and } \varphi(t,s,s) = \phi'_s(t,s),$$

$$\tilde{a}(t,s,s') = a(t,s)\overline{a(t,s')}.$$

Then the kernel has the form

$$K(s,s) = \int e^{i\lambda(s-s)\varphi(t,s,s')} \tilde{a}(t,s,s') dt. \tag{31}$$

Using the mean value theorem, we have $\varphi_t'(t,s,s') = \phi_{st}''(t,s'')$, where sis a number between s and s . By our assumptions, we see that there is a unique point $t_c(s'') \in F_t$ such that $\phi_{st}''(t_c(s''),s'') = 0$, and $\phi_{stt}'''(t_c(s''),s'') = 0$. Let $\theta > 0$. Select $\eta \in C_0^\infty(\mathbb{R})$ satisfying $\eta(t) = 1, |t| \le 1$, and $\eta(t) = 0, |t| \ge 2$. Then we decompose the oscillatory integral into two parts. First,

$$\left|\int e^{i\lambda(s-s')\varphi}\tilde{a}\eta\left(\left(t - t_c(s)\right)/\theta\right)dt\right| \le 4\theta\|a\|_\infty^2.$$

Then integrating by parts yields if $s \neq s'$,

$$\left|\int e^{i\lambda(s-s')\varphi}\tilde{a}\eta\left(\left(t - t_c(s'')\right)/\theta\right)dt\right|$$

$$\le (\lambda|s - s'|)^{-2}\int_{|t-t_c(s'')|>\theta}\left|\frac{\partial}{\partial t}\left(\frac{1}{\varphi_t'}\frac{\partial}{\partial t}\left(\frac{\tilde{a}\left(1 - \eta\left(\left(t - t_c(s'')\right)/\theta\right)\right)}{\varphi_t'}\right)\right)\right| dt$$

$$\le \frac{C\left(\sum_{0\le i,j\le 2}\left\|\partial_s^i a\right\|_\infty\left\|\partial_t^j \phi_{st}''\right\|_\infty\right)^2}{(\lambda|s-s'|)^2\cdot\inf\left(\left|\phi_{st}''\left(s - s_c(s)\right)\right|\right)^4}$$

$$\int_{|t-t_c(s'')|>\theta}\left(|t - tc(s)|^{-4} + \theta^{-2}|t - t_c(s'')|^{-2}\right)dt$$

$$\le C\theta^{-3}(\lambda|s-s'|)^2\frac{\left(\sum_{0\le i,j\le 2}\left\|\partial_s^i a\right\|_\infty\left\|\partial_t^j \phi_{st}''\right\|_\infty\right)^2}{\cdot\inf\left(\left|\phi_{st}''/\left(s - s_c(s)\right)\right|\right)^4},$$

where C is a constant independent of λ, a, ϕ and F. If we set $\theta = (\lambda|s - s'|)^{-\frac{1}{2}}$, then

$$|K(s,s')| \le C\left\{\|a\|_\infty^2 + \frac{\left(\sum_{0\le i,j\le 2}\left\|\partial_s^i a\right\|_\infty\left\|\partial_t^j \phi_{st}''\right\|_\infty\right)^2}{\cdot\inf\left(\left|\phi_{st}''/\left(s - s_c(s)\right)\right|\right)^4}\right\}(\lambda|s - s'|)^{-\frac{1}{2}}, \text{if } s \neq s'.$$

Hence,

$$|K(s,s')|ds \le C_{a,\phi}'^2\lambda^{-\frac{1}{2}},$$

which completes the proof by Young's inequality.

We will use C to denote various positive constants independent of T. Using the Hadamard parametrix and stationary phase [156], we can write

$$K_\alpha(t,s) = w\left(\tilde{\gamma}(t),\alpha\left(\tilde{\gamma}(s)\right)\right)\sum_{\pm} a_\pm\left(T,\lambda;\phi(t,s)\right)e^{\pm i\lambda\phi(t,s)} + R_\alpha(t,s),$$

where $|w(x,y)| \le C$, and for each $j = 0,1,2,\ldots$, there is a constant C_j independent of $T, \lambda \ge 1$ so that

$$\left|\partial_r^j a_\pm(T,\lambda;r)\right| \le C_j T_\lambda^{-1} r^{-1-j}, \qquad r \ge 1. \tag{32}$$

From the Hadamard parametrix with an estimate on the remainder term (see [27]), we see that with a uniform constant C

$$|R_\alpha(t,s)| \le e^{CT}.$$

Noting that diam(supp $a_\pm$) ≤ 2 and we have good control on the size of a± and its derivatives by (32), it remains to estimate the size of ϕ_{st}'' and its derivatives. We may assume that (M,g) is a compact 3-dimensional Riemannian manifold with constant curvature equal to −1. As in [160], we will compute the various mixed derivatives of the distance function explicitly on its universal cover $\mathbb{H}^3$. We consider the Poincaré half-space model

$$\mathbb{H}^3 = \{(x, y, z) \in \mathbb{R}^3 : z > 0\},$$

with the metric $ds^2 = z^{-2}(dx^2 + dy^2 + dz^2)$. Recall that the distance function for the Poincaré half-space model is given by

$$\operatorname{dist}\big((x_1, y_1, z_1), (x_2, y_2, z_2)\big) = \operatorname{arcosh}\left(1 + \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}{2z_1 z_2}\right),$$

where arcosh is the inverse hyperbolic cosine function

$$\operatorname{arcosh}(x) = \ln\left(x + \sqrt{x^2 - 1}\right), x \geq 1.$$

Moreover, the geodesics are the straight vertical rays normal to the $z = 0$-plane and the half-circles normal to the $z = 0$-plane with origins on the $z = 0$-plane. Without loss of generality, we may assume that $\tilde{\gamma}$ is the z-axis. Let $\tilde{\gamma}(t) = (0, 0, e^t), t \in \mathbb{R}$, be the infinite geodesic parameterized by arclength. Our unit geodesic segment is given by $\tilde{\gamma}(t), t \in [0, 1]$. Then its image $\alpha(\tilde{\gamma}(s)), s \in [0, 1]$, is a unit geodesic segment of $\alpha(\tilde{\gamma})$. As before, we denote the distance function $d_{\tilde{g}}\left(\tilde{\gamma}(t), \alpha(\tilde{\gamma}(s))\right)$ by $\phi(t, s)$. Since we are assuming $\alpha \notin \Gamma_{T_R}(\tilde{\gamma})$, we have

$$2 \leq \phi(t, s) \leq T, \qquad \text{if } s, t \in [0, 1]. \tag{33}$$

If $\tilde{\gamma}$ and $\alpha(\tilde{\gamma})$ are contained in a common plane, it is reduced to the 2-dimensional case. We recall the following lemma from [160], where $\tilde{\gamma}(t) = (0, e^t)$ in the Poincarè half-plane model.

Lemma (4.2.4)[160]: Let $\alpha \notin \Gamma_{T_R}(\tilde{\gamma})$. If $\alpha(\tilde{\gamma}) \cap \tilde{\gamma} = \emptyset$, we have

$$\inf |\phi''_{st}| \geq e^{-CT}.$$

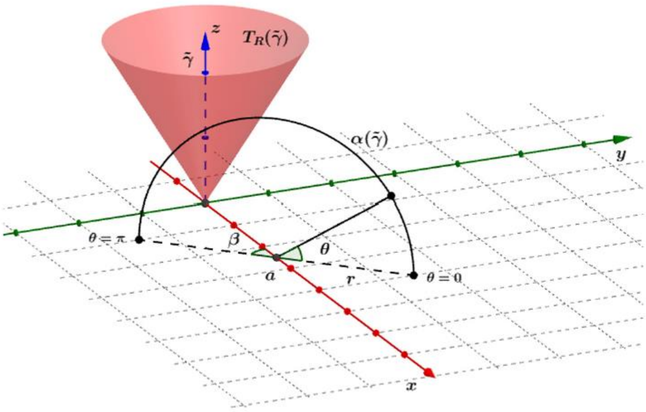


Fig. (1)[160] Pioncarè half-space model.

Assume that $\alpha(\tilde{\gamma})$ is a half-circle intersecting $\tilde{\gamma}$ at the point $(0, e^{t_0}), t_0 \in \mathbb{R}$. If $t_0 \notin [-1, 2]$, which means the intersection point $(0, e^{t_0})$ is outside some neighborhood of the geodesic segment $\{\tilde{\gamma}(t) : t \in [0, 1]\}$, then we also have

$$\inf |\phi''_{st}| \geq e^{-CT}.$$

If $t_0 \in [-1, 2]$, then

$$\inf |\phi''_{st}/(t - t_0)| \geq e^{-CT}.$$

Moreover,

$$\|\phi''_{st}\|_\infty + \|\phi'''_{stt}\|_\infty + \|\phi''''_{sttt}\|_\infty \leq e^{CT},$$

where $C > 0$ is independent of T. The infimum and the norm are taken on the unit square $\{(t, s) \in \mathbb{R}^2 : t, s \in [0, 1]\}$.

We assume that $\alpha \notin \Gamma_{T_R}(\tilde{\gamma})$, and $\tilde{\gamma}$ and $\alpha(\tilde{\gamma})$ are not contained in a common plane. Without loss of generality, we set $a \geq 0, r > 0$, and $\beta \in (0, \frac{\pi}{2}]$. Indeed, one can properly choose a coordinate system to achieve this. Let $\gamma_1(t) = (0, 0, e^t)$, and $\gamma_2(s) = \left(a + \frac{1-e^{2s}}{1+e^{2s}} \text{ r}\cos\beta, \frac{1-e^{2s}}{1+e^{2s}} r\sin\beta, \frac{2re^s}{1+e^{2s}}\right)$. It is not difficult to verify that both of them are parameterized by arclength. Assume that $\{\tilde{\gamma}(t) : t \in [0,1]\} = \{\gamma_1(t) : t \in [0,1]\}, \{\alpha(\tilde{\gamma}(s)) : s \in [0,1]\} = \{\gamma_2(s): s \in I\}$, where I is some unit closed interval of $\mathbb{R}$. Here $\gamma_2(s), s \in \mathbb{R}$, is a half circle centered at $(a, 0, 0)$ with radius $r. \beta$ is the angle between the y-axis and the normal vector of the plane containing the half circle. Moreover, these two geodesics are contained in a common plane when $\beta = 0$. See Fig. (1).

We are ready to compute φst explicitly and analyze its zero set. For simplification, we denote

$$d_1 = \sqrt{a^2 + r^2 - 2\text{a}r\cos\beta} \text{ and } d_2 = \sqrt{a_2 + r_2 + 2\text{a}r\cos\beta}.$$

Direct computation gives

$$\phi(t,s) = d_{\tilde{g}}(\gamma_1(t), \gamma_2(s)) = \operatorname{arccosh}\left(\frac{A}{4re^{s+t}}\right), t \in [0,1], s \in I,$$

where $A = e^{2s+2t} + e^{2t} + d_1^2 e^{2s} + d_2^2$. Taking derivatives yields

$$\phi''_{st} = \frac{16re^{2s+2t}[(\text{a}\cos\beta - r)(e^{2s+2t} + d_2^2) + (\text{a}\cos\beta + r)(e^{2t} + d_1^2 e^{2s})]}{(A^2 - 16r^2 e^{2s+2t})^{3/2}}. \quad (34)$$

The computation is technical. To see (34), we write

$$e^{s+t}\cosh\phi = \frac{A}{4r}.$$

Taking derivatives on both sides, we obtain

$$(\phi'_t + \phi'_s + \phi''_{ts})\sinh\phi + (1 + \phi'_t\phi'_s)\cosh\phi = e^{s+t}/r. \quad (35)$$

Denote $P = e^{s+t}, Q = d_1^2 e^{s-t}, R = e^{t-s}$, and $S = d_2^2 e^{-s-t}$. Since

$$4\text{r}\cosh\phi = P + Q + R + S,$$

taking derivatives yields

$$4r\phi'_t\sinh\phi = P - Q + R - S, 4r\phi'_s\sinh\phi = P + Q - R - S.$$

Then we multiply both sides of (35) by $4r^2(\sinh\phi)^2$ and use the hyperbolic trigonometric identity $(\sinh\phi)^2 = (\cosh\phi)^2 - 1$ to obtain

$$4r^2(\sinh\phi)3\phi''_{st} = (a\cos\beta - r)(P + S) + (a\cos\beta + r)(Q + R).$$

This gives our desired expression (34).

We denote the zero set of ϕ''_{st} by Z. Clearly, if $r \leq \text{a}\cos\beta$, then $Z = \emptyset$. Assume that $r > \text{a}\cos\beta$. In the interesting special case $\beta = \frac{\pi}{2}$,

$$Z = \{(t,s) \in \mathbb{R}^2 : t = t_0 \text{ or } s = s_0\},$$

where $e^{2t_0} = a^2 + r^2$ and $e^{2s_0} = 1$. See Fig. (2). In this case, we can easily see that ϕ'''_{stt} and ϕ'''_{tss} vanish at the point (t_0, s_0), as observed in [156]. In general, if $0 < \beta \leq \frac{\pi}{2}$,

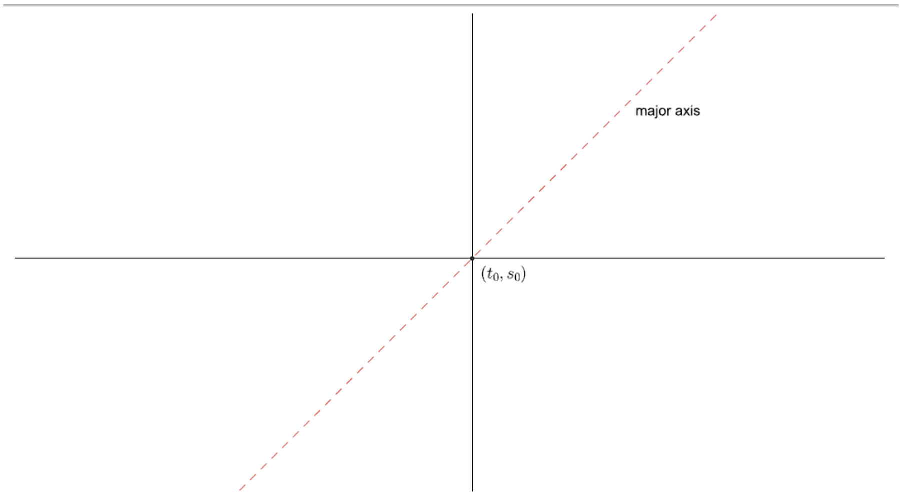


Fig. (2)[160]: Zero set of $\phi''_{st}, \beta = \frac{\pi}{2}$.

we have

$$Z = \{(t,s) \in \mathbb{R}^2 : (e^{2t} - X_0)(e^{2s} - Y_0) = B\}, \quad (36)$$

where

$$Y_0 = \frac{r + a\cos\beta}{r - a\cos\beta}, X_0 = d_1^2 Y_0, \quad B = \frac{4a^3 r\cos\beta\sin^2\beta}{(r - a\cos\beta)^2}, \quad (37)$$

and

$$X_0 Y_0 - B = d_2^2. \quad (38)$$

When $\beta \in (0, \frac{\pi}{2})$, the set Z consists of two disconnected curves. See Fig. (3). It has four different asymptotes:

$$l_1 : t = \ln\sqrt{X_0}, \quad l_2 : t = \ln\sqrt{X_0 - B/Y_0},$$
$$l_3 : s = \ln\sqrt{Y_0}, \quad l_4 : s = \ln\sqrt{Y_0 - B/X_0}.$$

They intersect at four points, which constitute the "central square" in Fig. (3). Clearly, the "central square" converges to the point (t_0, s_0) as $\beta \to \frac{\pi}{2}$. We set

$$e^{2t_\pm} = X_0 \pm \frac{\sqrt{BX_0}}{Y_0} \text{ and } e^{2s_\pm} = Y_0 \pm \frac{\sqrt{BY_0}}{X_0}. \quad (39)$$

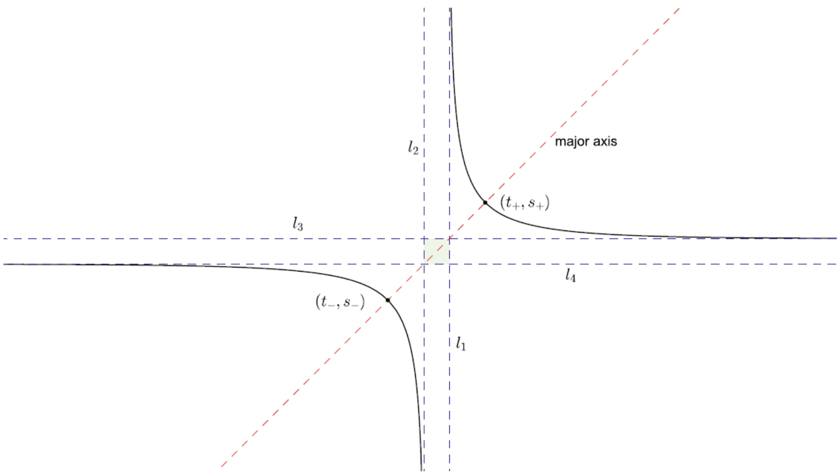


Fig. (3)[160]: Zero set of $\phi''_{st}, \beta \in \left(0, \frac{\pi}{2}\right)$.

The points (t_+, s_+) and (t_-, s_-) are a pair of vertices of Z in Fig. (3). They both converge to (t_0, s_0) as $\beta \to \frac{\pi}{2}$. A simple computation shows that the straight line passing through these two vertices, namely the “major axis”, is parallel to the straight line $t - s = 0$. This fact makes the “restriction trick” work in the proof of Lemma (4.2.5). Moreover, if $s > s_+$ or $s < s_-$, there is a unique $t_c = t_c(s)$ such that $(t_c, s) \in Z$. If $t > t_+$ or $t < t_-$, there is a unique $s_c = s_c(t)$ such that $(t, s_c) \in Z$. These two facts are related to the oscillatory integral estimates in Proposition (4.2.3). Indeed, one can see from (36) that

$$e^{2t_c(s)} = X_0 + \frac{B}{e^{2s} - Y_0}, e^{2s_c(t)} = Y_0 + \frac{B}{e^{2t} - X_0}. \tag{40}$$

Given $0 <$ 1, we denote the -neighborhood of Z by

$$Z = \{(t, s) \in \mathbb{R}^2 : \operatorname{dist}((t, s), Z) \leq \epsilon\}.$$

In particular, we set $Z = \emptyset$ if $Z = \emptyset$. See Figs. (4) and (5). We decompose the domain [0, 1] of the phase function into 4 parts:

(i) Non-stationary phase part: $[0,1]^2 \setminus Z$;

(ii) Left folds part: $[0,1]^2 \cap \{(t, s) \in Z : s > s_+ + \epsilon \text{ or } s < s_- - \epsilon\}$;

(iii) Right folds part: $[0,1]^2 \cap \{(t, s) \in Z : t > t_+ + \epsilon \text{ or } t < t_- - \epsilon\}$;

(iv) Young’s inequality part: $[0,1]^2 \cap Z \cap ([t_- - \epsilon, t_+ + \epsilon] \times [s_- - \epsilon, s_+ + \epsilon])$.

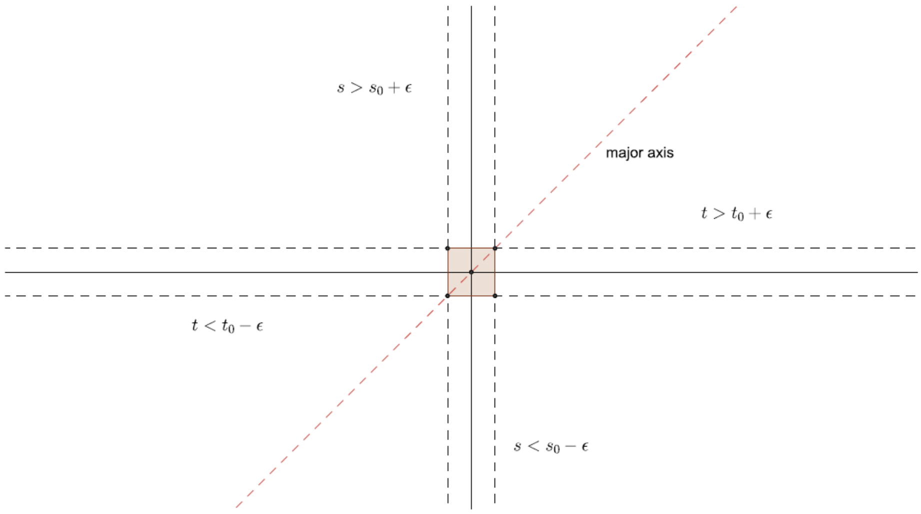


Fig. (4)[160]: Z and its decomposition, $\beta = \frac{\pi}{2}$.

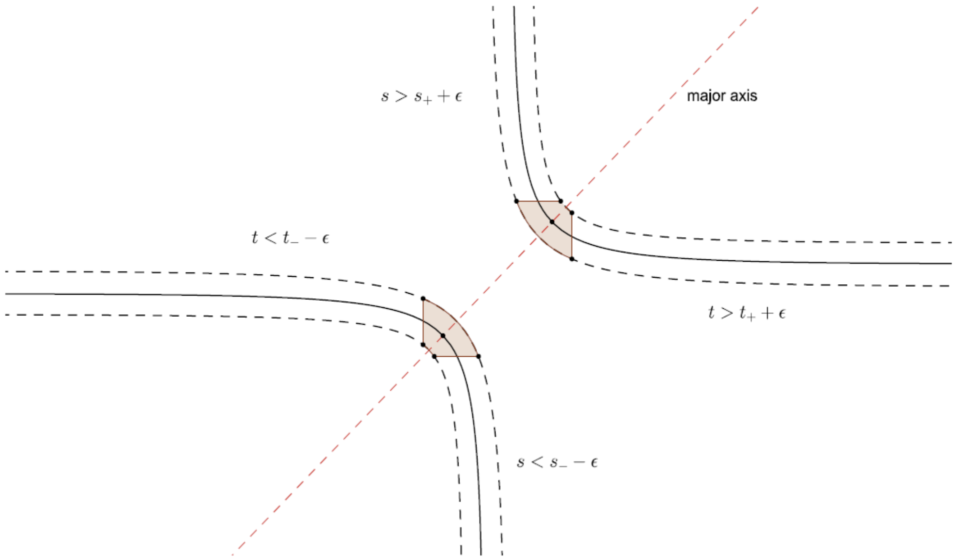


Fig. (5)[160]: Z and its decomposition, $\beta \in \left(0, \frac{\pi}{2}\right)$.

We postpone the proof of the lemmas and finish proving Theorem (4.2.1). We always use C to denote various positive constants independent of and T. Recall that there are at most $O(e^{CT})$ summands with $\alpha \notin \Gamma_{T_R}(\tilde{\gamma})$. We claim that the kernel $K_\lambda^{osc}(t,s)$ of the operator S_λ^{osc} is bounded by $e^{CT}\left(\epsilon\lambda + \epsilon^{-2}\lambda^{\frac{3}{4}} + -4\lambda^{\frac{1}{2}}\right)$. Indeed, one can properly choose some smooth cutoff functions to decompose the domain $[0,1]^2$ and then apply Proposition (4.2.3), Lemma (4.2.4) and Young's inequality to the corresponding parts (1)–(4). Recall that Proposition (4.2.3) consists of "non-stationary phase", "left folds" and "right folds". Since the estimate (32) on the amplitude holds, it is not difficult to see that λ comes from Young's inequality, $-2\lambda^{\frac{3}{4}}$ comes from one-side folds (or stationary phase), and $-4\lambda^{\frac{1}{2}}$ comes from non-stationary phase. Then Young's inequality gives

$$\|S_\lambda^{osc}\|_{L^2[0,1]\to L^2[0,1]} \le e^{CT}\left(\epsilon\lambda + \epsilon^{-2}\lambda^{\frac{3}{4}} + -4\lambda^{\frac{1}{2}}\right). \quad (41)$$

Taking $T = c\log\lambda$ and $\epsilon = e^{-CT}T^{-1}$, where $c > 0$ is a small constant $(c < (12C)^{-1})$, and combining (41) with the estimates on S_λ^{tube} (24) and K_0 (12), we finish the proof.

Before proving the lemmas, we remark that in the Poincaré half-space model

$$T_R(\tilde{\gamma}) = \left\{(x,y,z) \in \mathbb{R}^3 : z > 0 \text{ and } z \ge \sqrt{x^2 + y^2}/\sqrt{(\cosh R)^2 - 1}\right\}.$$

See Fig. (1). Indeed, the distance between $(0,0,e^t)$ and $(x,y,z), z > 0$, is

$$f(t) = \operatorname{arcosh}\left(1 + \frac{x^2 + y^2 + (z - e^t)^2}{2ze^t}\right) = \operatorname{arcosh}\left(\frac{x^2 + y^2 + z^2 + e^{2t}}{2ze^t}\right).$$

Setting $f'(t) = 0$ gives $t = \ln\sqrt{x^2 + y^2 + z^2}$, which must be the only minimum point. Thus the distance between (x,y,z) and the infinite geodesic $\tilde{\gamma}$ is

$$\operatorname{dist}\big((x,y,z),\tilde{\gamma}\big) = \operatorname{arcosh}\left(\sqrt{1 + (x/z)^2 + (y/z)^2}\right).$$

Since $\operatorname{dist}\big((x,y,z),\tilde{\gamma}\big) \le R$ in $T_R(\tilde{\gamma})$, it follows that $z \ge \sqrt{x^2 + y^2}/\sqrt{(\cosh R)^2 - 1}$.

Lemma (4.2.5)[160]: Let $\alpha \notin \Gamma_{T_R}(\tilde{\gamma})$. Assume that $\tilde{\gamma}$ and $\alpha(\tilde{\gamma})$ are not contained in a common plane. Then we have

$$\inf|\phi''_{st}| \ge \epsilon^2 e^{-CT},$$

where the infimum is taken on $[0,1]^2 \setminus Z$. If $Z = \emptyset$, then we have

$$\inf|\phi''_{st}|/|t - t_c(s)| \ge e^{-CT},$$

where the infimum is taken on $[0,1]^2 \cap \{(t,s) \in Z : s > s_+ + \epsilon \text{ or } s < s_- - \epsilon\}$, and

$$\inf|\phi''_{st}|/|s - s_c(t)| \ge e^{-CT},$$

where the infimum is taken on $[0,1]^2 \cap \{(t,s) \in Z : t > t_+ + \epsilon \text{ or } t < t_- - \epsilon\}$. The constant $C > 0$ is independent of and T.

Proof. First of all, we need to derive some useful results from the condition that $\phi(t,s) \le T$. Namely,

$$(e^{2t} + d_1^2)e^{2s} - 4r(\cosh T)e^t e^s + e^{2t} + d_2^2 \le 0, t \in [0,1], s \in I. \quad (42)$$

olving the quadratic inequality (42) about e^s, we have

$$\frac{r}{4\cosh T} \le e^s \le 4r\cosh T. \quad (43)$$

The discriminant of (42) has to be nonnegative:

$$16r^2(\cosh T)2e^{2t} - 4(e^{2t} + d_1^2)(e^{2t} + d_2^2) \ge 0,$$

from which we see that

$$\frac{a}{r} \le 2e\cosh T, \quad (44)$$

$$d_1 \le 2e\cosh T, \tag{45}$$
$$r \ge \frac{1}{2}\cosh T, \tag{46}$$

which are similar to the observations in [160]. Moreover, to get the lower bounds of the derivatives, we need the condition that $\alpha \notin \Gamma_{\mathrm{T}_R}(\tilde{\gamma})$. We claim that there exists some constant C independent of T such that

$$\alpha \notin \Gamma_{\mathrm{T}_R}(\tilde{\gamma}) \Rightarrow r \le C\cosh T \text{ or } d_1 \ge \frac{1}{C\cosh T}. \tag{47}$$

Indeed, we are going to prove the contrapositive:

$$r \ge C\cosh T \text{ and } d_1 \le 1\, C\cosh T \Rightarrow \alpha \in \Gamma_{\mathrm{T}_R}(\tilde{\gamma}). \tag{48}$$

We obtain this by showing that under the above assumptions on r and d_1, the segment $\gamma_2(s), s \in [-\ln(4\mathrm{r}^{-1}\cosh\mathrm{T}), \ln(4r\cosh T)]$ is completely included in $T_R(\tilde{\gamma})$, which implies $\alpha \in \Gamma_{\mathrm{T}_{R(\tilde{\gamma})}}$ by (43). The argument is generalized from [160]. Solving the polynomial system

$$z = \begin{cases} \sqrt{x^2 + y^2} / \sqrt{(\cosh R)^2 - 1} \\ (x, y, z) = \left(a + \dfrac{1 - e^{2s}}{1 + e^{2s}}\, r\cos\beta, \dfrac{1 - e^{2s}}{1 + e^{2s}}\, r\sin\beta, \dfrac{2re^s}{1 + e^{2s}}\right) \end{cases}$$

we can see that

$$\{\gamma_2(s) : s \in \mathbb{R}\} \cap T_R(\tilde{\gamma}) = \{\gamma_2(s) : d_1^2 e^{4s} + 2(a^2 + r^2 - 2(\cosh R)^2 r^2)e^{2s} + d_2^2 \le 0\}. \tag{49}$$

Note that

$$\begin{cases} r \ge C\cosh T \\ d_1 \le (C\cosh T)^{-1} \end{cases} \Rightarrow a/r \le 1 + (C\cosh T)^{-2} \le (\cosh R)^2 - 1.$$

This implies

$$\frac{a}{r} \le \sqrt{\frac{(\cosh R)^2 - 1}{(\cosh R)^2 - \cos^2\beta}}\,\cosh R,$$

which is equivalent to

$$(a^2 + r^2 - 2(\cosh R)^2 r^2)^2 - d_1^2 d_2^2 \ge 0.$$

This means that the discriminant of the quadratic polynomial in terms of e^{2s} in (49) is nonnegative. Thus when $d_1 > 0$, the RHS of (49) becomes

$$\{\gamma_2(s) : u_- \le e^{2s} \le u_+\}, \tag{50}$$

where

$$u_\pm = \frac{2(\cosh R)^2 r^2 - r^2 - a^2 \pm \sqrt{(a^2 + r^2 - 2(\cosh R)^2 r^2)^2 - d_1^2 d_2^2}}{d_1^2}. \tag{51}$$

It is easy to see that

$$u_- \le \frac{d_2^2}{2(\cosh R)^2 r^2 - r^2 - a^2} \le \frac{d_2^2}{(\cosh R)^2 r^2} \le \frac{(\cosh R)^2 + 2\cosh R}{(\cosh R)^2}, \tag{52}$$
$$u_+ \ge \frac{(2(\cosh R)^2 - 1)r^2 - a^2}{d_1^2} \ge \frac{(\cosh R)^2 r^2}{d_1^2}. \tag{53}$$

So if we choose $C = 4\sqrt{\cosh R + 2}/\sqrt{\cosh R}$, we see that

$$d_1 > 0 \text{ and } \begin{cases} r \ge C\cosh T \\ d_1 \le (C\cosh T)^{-1} \end{cases} \begin{cases} u_- \le r^2(4\cosh T)^{-2} \\ u_+ \ge (4\cosh T)^2 \end{cases} \Rightarrow \alpha \in \Gamma_{\mathrm{T}_{R(\tilde{\gamma})}}. \tag{54}$$

In the easier case $d_1 = 0$, we have $u_+ = +\infty$. Consequently, we obtain (48), which is equivalent to our claim (47).

Moreover, we notice that by $\phi \leq T$,

$$|\phi''_{st}| \geq |\phi''_{st}| \left(\frac{A}{4re^s + t\cosh T}\right)^2$$
$$\geq \frac{|(a\cos\beta - r)(e^{2s+2t} + d_2^2) + (a\cos\beta + r)(e^{2t} + d_1^2 e^{2s})|}{(\cosh T)^2 rA}. \quad (55)$$

Now we need to consider two cases: $(I)\ r \leq a\cos\beta$; $(II)\ r > a\cos\beta$.

Case (I): ϕ''_{st} has no zeros and it is not difficult to obtain the lower bound of $|\phi''_{st}|$. Indeed, if $d_1 \geq 1$, by (55) and (43)–(44), we get

$$|\phi''_{st}| \geq \frac{C(a\cos\beta + r)d_1^2 r 2(\cosh T)^{-2}}{(\cosh T)^2 r(d_1^2 r^2 (\cosh T)^2)} \geq Ce^{-6T}.$$

If $d_1 \leq 1$, the claim (47) is needed. We assume that $r \leq C\cosh T$. Then by (55) and (43)–(46), we obtain

$$|\phi''_{st}| \geq \frac{C(a\cos\beta + r)e^{2t}}{(\cosh T)^2 r(r^2(\cosh T)^2)} \geq Ce^{-6T}.$$

Otherwise, we assume that $d_1 \geq (C\cosh T)^{-1}$. Then similarly we have

$$|\phi''_{st}| \geq \frac{C(a\cos\beta + r)d_1^2 r^2 (\cosh T)^{-2}}{(\cosh T)^2 r(r^2(\cosh T)^2)} \geq Ce^{-8T}.$$

Case (II): Since ϕ''_{st} has zeros, we prove the lower bound of $|\phi''_{st}|$ on $([0,1] \times I) \setminus Z$ first. The claim (47) is essential here. However, for technical reasons we only need a slightly weaker but useful version of the claim:

$$\alpha \notin \Gamma_{T_{R(\tilde{\gamma})}} \Rightarrow r \leq C(\cosh T)^7 \text{ or } d_1 \geq (C\cosh T)^{-1}, r \geq C(\cosh T)^7. \quad (56)$$

(i) Assume that $r \leq C(\cosh T)^7$.

In this case, we use a "restriction trick" to reduce it to a one-variable problem. Let $\delta \in \mathbb{R}$. We restrict $\phi''_{st}(t,s)$ on the straight line $s - t = \delta$ and obtain a uniform lower bound independent of δ. Indeed,

$$\begin{aligned}
&|(a\cos\beta - r)(e^{2s+2t} + d_2^2) + (a\cos\beta + r)(e^{2t} + d_1^2 e^{2s})| \\
&= (r - a\cos\beta)|e^{2s+2t} - Y_0 e^{2t} - X_0 e^{2s} + d_2^2| \\
&= (r - a\cos\beta)\left|e^{4t} - (X_0 + Y_0 e^{-2\delta})e^{2t} + d_2^2 e^{-2\delta}\right| e^{2\delta} \\
&= (r - a\cos\beta)|(e^{2t} - e^{2\tau_-})(e^{2t} - e^{2\tau_+})|e^{2\delta},
\end{aligned}$$

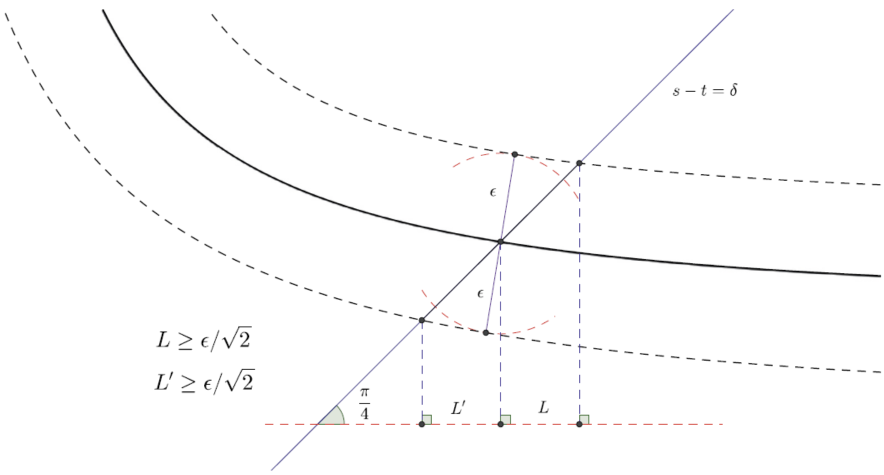


Fig. (6)[160]: Restriction on $s - t = \delta$.

where

$$2e^{2\tau_\pm} = X_0 + Y_0 e^{-2\delta} \pm \sqrt{(X_0 - Y_0 e^{-2\delta})^2 + 4Be^{-2\delta}}.$$

If $r - a\cos\beta \le \frac{r + a\cos\beta}{100} e^{-2\delta}$, then

$$2e^{2\tau_+} \ge Y_0 e^{-2\delta} \ge 100.$$

But $t \in [0,1]$ implies that

$$|e^{2t} - e^{2\tau_+}| \ge \frac{1}{2} e^{2\tau_+} \ge \frac{1}{4} Y_0 e^{-2\delta}.$$

Let $Z,\delta = \{t \in \mathbb{R} : (t, t + \delta) \in Z_\epsilon\}$. Since the straight line $s - t = \delta$ is parallel to the "major axis" of Z, we have

$$\mathrm{dist}(\tau_\pm, [0,1] \setminus Z, \delta) \ge \epsilon/\sqrt{2}. \tag{57}$$

See Fig. (6). This implies

$$|e^{2t} - e^{2\tau_-}| \ge 1 - e^{-\epsilon\sqrt{2}} \ge \epsilon/10, \text{for } t \in [0,1] \setminus Z_\epsilon, \delta.$$

Thus

$$(r - a\cos\beta)|(e^{2t} - e^{2\tau_-})(e^{2t} - e^{2\tau_+})|e^{2\delta} \ge 40(r + a\cos\beta).$$

If $r - a\cos\beta \ge \frac{r + a\cos\beta}{100} e^{-2\delta}$, then we use (57) again to see that

$$|e^{2t} - e^{2\tau_\pm}| \ge 1 - e^{-\epsilon\sqrt{2}} \ge \epsilon/10, \qquad \text{for } t \in [0,1] \setminus Z_\epsilon, \delta,$$

which gives

$$(r - a\cos\beta)|(e^{2t} - e^{2\tau_-})(e^{2t} - e^{2\tau_+})|e^{2\delta} \ge \frac{\epsilon^2}{10000}(r + a\cos\beta).$$

So we can use (55), (43)–(46) and our assumption $r \le C(\cosh T)^7$ to obtain the lower bound of $|\phi''_{st}|$, namely

$$|\phi''_{st}| \ge \frac{C\epsilon^2(r + a\cos\beta)}{(\cosh T)^2 r(r^2(\cosh T)^4)} \ge C\epsilon^2 e^{-20T}. \tag{58}$$

(ii) Assume that $d_1 \ge (C\cosh T)^{-1}$ and $r \ge C(\cosh T)^7$.

If $|r - a\cos\beta| \le 1$, we can use (43)–(46) and our assumption to get

$$|(a\cos\beta - r)(e^{2s+2t} + d_2^2) + (a\cos\beta + r)(e^{2t} + d_1^2 e^{2s})| \ge Cr^3(\cosh T)^{-4},$$

since $(r + a\cos\beta)(d_1^2 e^{2s} + e^{2t}) \ge Cr^3(\cosh T)^{-4}$ and $(r - a\cos\beta)(e^{2s+2t} + d_2^2) \le Cr^2(\cosh T)^2$.

If $|r - a\cos\beta| \ge 1$, then $d_1 \ge |r - a\cos\beta| \ge 1$. Thus, $(r + a\cos\beta)(d_1^2 e^{2s} + e^{2t}) \ge Cr^3(\cosh T)^{-2}$ and $(r - a\cos\beta)(e^{2s+2t} + d_2^2) \le Cr^2(\cosh T)^3$, which imply

$$|(a\cos\beta - r)(e^{2s+2t} + d_2^2) + (a\cos\beta + r)(e^{2t} + d_1^2 e^{2s})| \ge Cr^3(\cosh T)^{-2}.$$

Therefore, we use (55) and (43)–(46) to get

$$|\phi''_{st}| \ge \frac{Cr^3(\cosh T)^{-4}}{(\cosh T)^2 r(r^2(\cosh T)^4)} \ge Ce^{-10T}, \tag{59}$$

which is better than the bound $\epsilon^2 e^{-CT}$. Since the lower bounds in (58) and (59) are independent of δ, we finish the proof of the lower bound of $|\phi''_{st}|$ on $([0,1] \times I) \setminus Z_\epsilon$. Now we are ready to give the proof of the lower bounds of $|\phi''_{st}/(t - t_c)|$ and $|\phi''_{st}/(s - s_c)|$. Denote

$$\epsilon_0 = \frac{1}{2}\ln\left(1 + \sqrt{\frac{B}{X_0 Y_0}}\right) + \epsilon.$$

Part 1: Assume that

$$([0,1] \times I) \cap \{(t,s) \in Z_\epsilon : s > s_+ + \epsilon \text{ or } s < s_- - \epsilon\} \ne \emptyset. \tag{60}$$

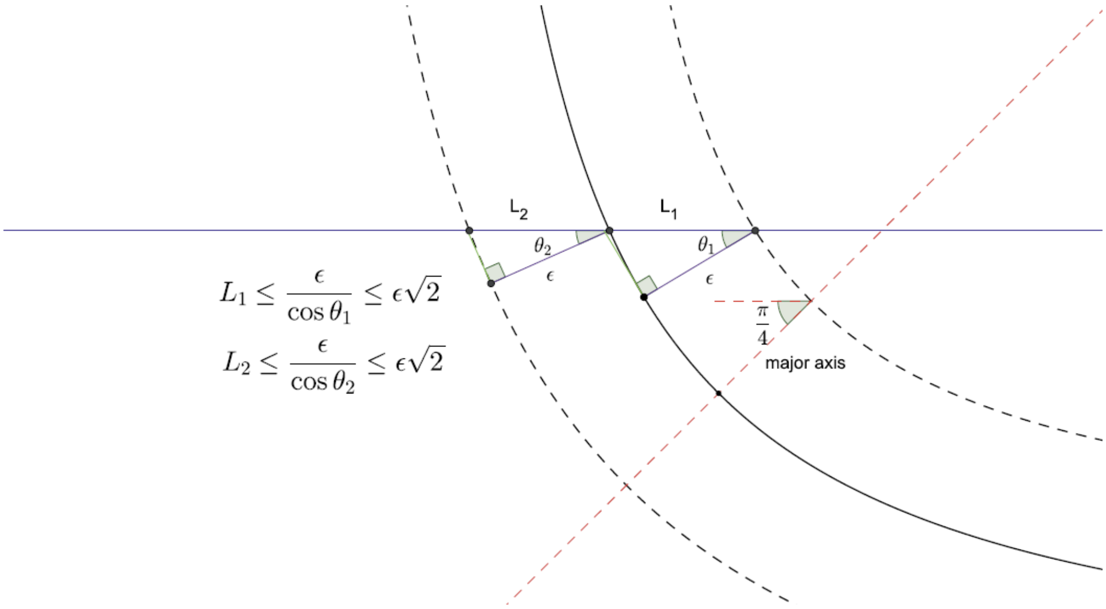


Fig.(7)[160]: $\operatorname{dist}(t_c,[0,1]) \le \epsilon/\sqrt{2}$.

We need to obtain the lower bound of $|\phi''_{st}/(t - t_c)|$ on this set. A simple computation using (37)–(39) shows that

$$s > s_+ + \epsilon \Leftrightarrow e^{2s} > Y_0 e^{2\epsilon_0},$$

$$s < s_- - \epsilon \Leftrightarrow s < \left(Y_0 - \frac{B}{X_0}\right) e^{-2\epsilon_0}. \qquad (61)$$

Hence

$$|e^{2s} - Y_0| \ge (1 - e^{-2\epsilon_0})Y_0.$$

Since the "major axis" of Z is parallel to the straight line $s - t = 0$, by our assumption (60) we have $t_c \in [-\epsilon\sqrt{2}, 1 + \epsilon\sqrt{2}]$. See Fig. (7). Thus,

$$|(a\cos\beta - r)(e^{2s+2t} + d_2^2) + (a\cos\beta + r)(e^{2t} + d_1^2 e^{2s})|/|t - t_c|$$

$$= (r - a\cos\beta)\left|\frac{(e^{2t} - X_0)(e^{2s} - Y_0) - B}{t - t_c}\right|$$

$$= (r - a\cos\beta)\left|\frac{(e^{2t} - e^{2t_c})(e^{2s} - Y_0)}{t - t_c}\right|$$

$$= (r - a\cos\beta)\cdot 2e^{2t'}\cdot |e^{2s} - Y_0|$$

$$\ge (r - a\cos\beta)\cdot 2e^{-2\epsilon\sqrt{2}}\cdot (1 - e^{-2\epsilon_0})Y_0$$

$$\ge \frac{\epsilon}{100}(r + a\cos\beta),$$

where we use the mean value theorem and $\epsilon_0 \ge \epsilon$.

First, we assume that $r \le C(\cosh T)^7$. Then using (55) and (44)–(46), we obtain

$$\left|\frac{\phi''_{st}}{t - t_c}\right| \ge \frac{C\epsilon(r + a\cos\beta)}{(\cosh T)^2 r (r^2(\cosh T)^4)} \ge C\epsilon e^{-20T}.$$

Under the other assumption that "$d_1 \ge (C\cosh T)^{-1}$ and $r \ge C(\cosh T)^7$", since $|t - t_c| \le 1 + \epsilon\sqrt{2}$ and the lower bound (59) of $|\phi''_{st}|$ is still applicable here, we get

$$\left|\frac{\phi''_{st}}{t - t_c}\right| \ge Ce^{-10T}.$$

Part 2: Assume that

$$([0,1]\times I) \cap \{(t,s) \in Z_\epsilon : t > t_+ + \epsilon \text{ or } t < t_- - \epsilon\} \ne \emptyset \qquad (62)$$

We need to get the lower bound of $|\phi''_{st}/(s-s_c)|$ on this set. It is also difficult to see from (37)-(39) that

$$t > t_+ + \epsilon \Leftrightarrow e^{2t} > X_0 e^{2\epsilon_0}, \quad t < (X_0 - B/Y_0)e^{-2\epsilon_0}. \tag{63}$$

Hence

$$|e^{2t} - X_0| \geq (1 - e^{-2\epsilon_0})\max\{X_0, 1\}.$$

If $t > t_+ + \epsilon$, clearly we have $e^{2s_c} \geq Y_0$. See Fig. (3). If $B = 0$, we have $e^{2s_c} = Y_0$. If $t < t_- - \epsilon$ and $B > 0$, then from (40) we get

$$\begin{aligned} e2sc &= Y_0 - \frac{B}{X_0 - e^{2t}} \geq Y_0 - \frac{B}{e^{2t+2\epsilon_0} + B/Y_0 - e^{2t}} \\ &= Y_0 - \frac{B}{e^{2t}(e^{2\epsilon} - 1) + e^{2t+2\epsilon}\sqrt{\frac{B}{(X_0Y_0)}} + \frac{B}{Y_0}} \\ &\geq Y_0 - \frac{B}{\sqrt{B(X_0Y_0)} + BY_0} \\ &= Y_0 - \frac{X_0Y_0}{\sqrt{X_0Y_0/B} + X_0} \geq Y_0 - \frac{X_0Y_0}{1 + X_0} \\ &= \frac{Y_0}{X_0 + 1}, \end{aligned}$$

where we use $X_0Y_0/B > 1$ from (38). Since the "major axis" of Z is parallel to the straight line $s - t = 0$, by the assumption (62) we get $\text{dist}(s_c, I) \leq \epsilon\sqrt{2}$. See Fig. (8).

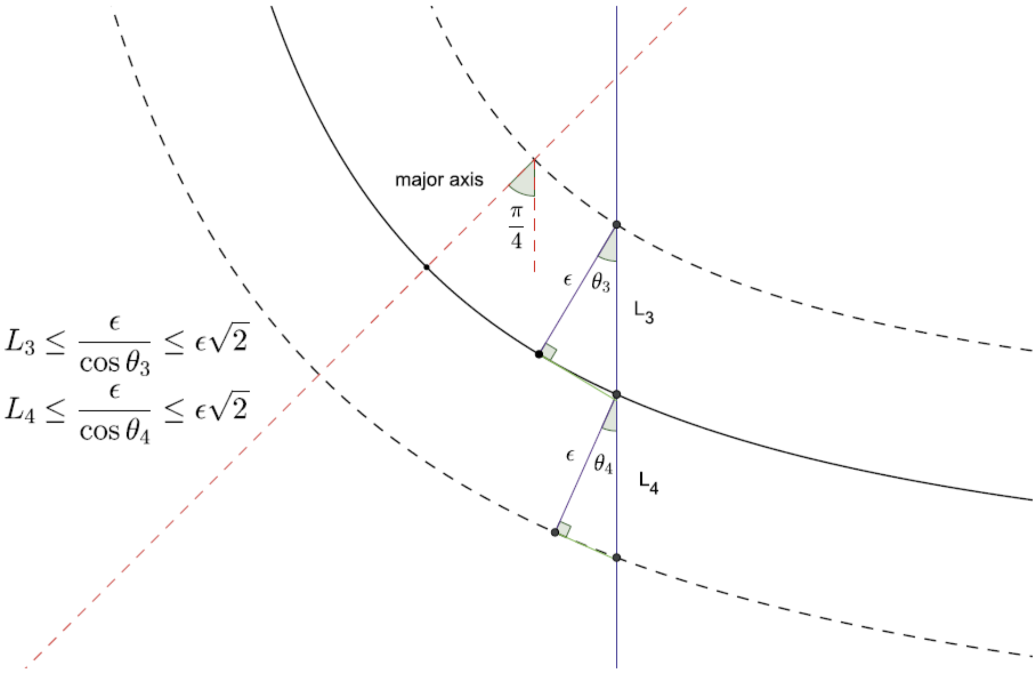


Fig. (8)[160]: $\text{dist}(s_c, I) \leq \epsilon\sqrt{2}$.

Therefore,

$$|(a\cos\beta - r)(e^{2s+2t} + d_2^2) + (a\cos\beta + r)(e^{2t} + d_1^2 e^{2s})|/|s - s_c| = (r - a\cos\beta)\left|\frac{(e^{2t} - X_0)(e^{2s} - Y_0) - B}{s - s_c}\right|$$

$$= (r - a\cos\beta)\left|\frac{(e^{2t} - X_0)(e^{2s} - e^{2s_c})}{s - s_c}\right|$$
$$= (r - a\cos\beta)|e^{2t} - X_0| \cdot 2e^{2s'}$$
$$\geq (r - a\cos\beta)|e^{2t} - X_0| \cdot 2e^{2(s_c-1-\epsilon\sqrt{2})}$$
$$\geq (r - a\cos\beta)(1 - e^{-2\epsilon_0})\max\{X_0, 1\} \cdot \frac{2Y_0}{X_0 + 1} e^{-2-2\epsilon\sqrt{2}}$$
$$\geq \frac{\epsilon}{100}(r + a\cos\beta),$$

where we use the mean value theorem and $\epsilon_0 \geq \epsilon$. Then we can obtain the lower bound of $|\phi''_{st}/(s - s_c)|$ in the same way as Part 1. First, under the assumption that $r \leq C(\cosh T)^7$, we have

$$\left|\frac{\phi''_{st}}{s - s_c}\right| \geq \frac{C\epsilon(r + a\cos\beta)}{(\cosh T)^2 r(r^2(\cosh T)^4)} \geq C\epsilon e^{-20T}.$$

Under the other assumption that "$d_1 \geq (C\cosh T)^{-1}$ and $r \geq C(\cosh T)^7$", noting that $|s - s_c| \leq 1 + \epsilon\sqrt{2}$ and the bound (59) is still valid here, we get

$$\left|\frac{\phi''_{st}}{s - s_c}\right| \geq Ce^{-10T}.$$

So far we have finished the proof of all the lower bounds.

Lemma (4.2.6)[160]: For every multi-index $\alpha = (\alpha_1, \alpha_2)$,

$$\|D^\alpha \phi\|_\infty \leq e^{C_\alpha T},$$

where the norm is taken on the unit square $[0, 1]^2$. The constant C_α is independent of T.

Proof. We only need to prove the upper bounds of mixed derivatives when $\alpha \neq I$dentity, since the bounds for pure derivatives are well known in [20], [162] and we do not use them. For convenience, we denote

$$G(t, s) = (a\cos\beta - r)(e^{2s+2t} + d_2^2) + (a\cos\beta + r)(e^{2t} + d_1^2 e^{2s}),$$
$$E(t, s) = A^2 - 16r^2 e^{2s+2t}.$$

Recalling the formula (34), we have $\phi''_{st} = 16re^{2s+2t} GE^{-3/2}$. By induction it is not difficult to see that for any multi-index $\alpha = (\alpha_1, \alpha_2)$

$$D^\alpha\left(\frac{G}{E^\gamma}\right) = E^{-\gamma-|\alpha|} \sum_{0\leq|\beta_0|+\cdots+|\beta_{|\alpha|}|\leq|\alpha|} C_{\gamma,\alpha,\beta_0,\ldots,\beta_{|\alpha|}} D^{\beta_0}G \cdot D^{\beta_1}E \cdots D^{\beta_{|\alpha|}}E,$$

where $|\alpha| = \alpha_1 + \alpha_2$, and $C_{\gamma,\alpha,\beta_0,\ldots,\beta_{|\alpha|}}$ are constants independent of G and E. Thus,

$$D^\alpha \phi''_{st} = \frac{re^{2s+2t}}{E^{3/2+|\alpha|}} \sum_{0\leq|\beta_0|+\cdots+|\beta_{|\alpha|}|\leq|\alpha|} C_{\alpha,\beta_0,\ldots,\beta_{|\alpha|}} D^{\beta_0}G \cdot D^{\beta_1}E \cdots D^{\beta_{|\alpha|}}E.$$

From the condition that $\phi(t, s) \geq 2$, we have $A \geq 4(\cosh 2)re^{s+t}$. Thus,

$$A - 4re^{s+t} \geq (4\cosh 2 - 4)re^{s+t}.$$

If $r \geq C\cosh T$, then by (43)–(46),

$$E \geq (A - 4re^{s+t})^2 \geq Cr^2 e^{2s+2t} \geq Cr^4(\cosh T)^{-2},$$
$$|D^\alpha E| \leq C_\alpha r^4(\cosh T)^8, |D^\alpha G| \leq C_\alpha r^3(\cosh T)^5.$$

Hence,

$$|D^\alpha \phi''_{st}| \leq \frac{C_\alpha r(r\cosh T)^2}{(r^4(\cosh T)^{-2})^{3/2+|\alpha|}} r^3(\cosh T)^5(r^4(\cosh T)^8)^{|\alpha|} \leq C_\alpha e^{(10|\alpha|+10)T}.$$

If $r \leq C\cosh T$, then by (43)–(46),

$$E \geq (A - 4re^{s+t})^2 \geq Cr^2 e^{2s+2t} \geq C(\cosh T)^{-6},$$

$$|D^{\alpha}E| \leq C_{\alpha}(\cosh T)^{12}, |D^{\alpha}G| \leq C_{\alpha}(\cosh T)^{8}.$$

Therefore,

$$|D^{\alpha}\phi''_{st}| \leq \frac{C_{\alpha}r(r\cosh T)^{5}}{((\cosh T)^{-6})^{3/2+|\alpha|}}(\cosh T)^{8}((\cosh T)^{12})^{|\alpha|} \leq C_{\alpha}e^{(18_{|\alpha|}+22)T}.$$

Chapter 5
Refined and Microlocal with Bilinear Kakeya–Nikodym Bounds and Averages

We show that stronger related microlocal estimates involving a natural decomposition of phase space are adapted to the geodesic flow. We do this by using microlocal techniques and a bilinear version of Hörmander's oscillatory integral theorem.

Section (5.1): Eigenfunctions in Two Dimensions

Suppose that (M, g) is a two-dimensional compact Riemannian manifold and $\{e_\lambda\}$ are the associated eigenfunctions. That is, if Δ_g is the Laplace–Beltrami operator, we have

$$-\Delta_g e_\lambda(x) = \lambda^2 e_\lambda(x),$$

and we assume throughout that the eigenfunctions are normalized to have L^2 -norm one, i.e.,

$$\int_M |e_\lambda|^2 \, dV_g = 1,$$

where dV_g is the volume element.

We obtain essentially sharp estimates that link, in two dimensions, the size of L^p -norms of eigenfunctions with $2 < p < 6$ to their L^2 -concentration near geodesics. Specifically, we have the following:

Theorem (5.1.1)[170]: For every $0 < \varepsilon_0 \le \frac{1}{2}$, we have

$$\|e_\lambda\|_{L^4(M)} \lesssim_{\varepsilon_0} \lambda^{\varepsilon_0/4} \|e_\lambda\|^{1/2}_{L^2(M)} \times |||e\lambda|||^{1/2}_{KN(\lambda,\varepsilon_0)} \tag{1}$$

If

$$|||e\lambda|||_{KN(\lambda,\varepsilon_0)} = \left(\sup_{\gamma \in \Pi} \lambda^{1/2-\varepsilon_0} \int_{\mathcal{T}_{\lambda^{-1/2+\varepsilon_0}}(\gamma)} |e_\lambda|^2 \, dV \right)^{1/2}. \tag{2}$$

Equivalently, if $\varepsilon_0 > 0$, then there is a $C = C(\varepsilon_0, M)$ such that

$$\|e_\lambda\|_{L^4} \le C\lambda^{\frac{1}{8}} \|e_\lambda\|^{\frac{1}{2}}_{L^2(M)} \times \left(\sup_{\gamma \in \Pi} \int_{\mathcal{T}_{\lambda^{-\frac{1}{2}+\varepsilon_0}}(\gamma)} |e_\lambda|^2 \, dV \right)^{\frac{1}{4}}, \tag{3}$$

and therefore if $\int_M \; |e_\lambda|^2 \, dV = 1$, for any $\varepsilon > 0$ there is a $C = C(\varepsilon, M)$ such that

$$\begin{aligned} \|e_\lambda\|_{L^4(M) \le C\lambda^{\frac{1}{8}+\varepsilon}} \sup_{\gamma \in \Pi} \|e_\lambda\|^{\frac{1}{2}}_{L^2(\mathcal{T}_{\lambda}-1/2(\gamma))} \\ \le C\lambda^{1/16+\varepsilon} \sup \gamma \in \Pi \, \|e_\lambda\|^{1/2}_{L^4(\mathcal{T}_{\lambda_{jk}^{-1/2}}(\gamma))}. \end{aligned} \tag{4}$$

Here Π denotes the space of unit-length geodesics in M and the last factor in (2) involves averages of $|e_\lambda|^2$ over $\lambda^{-1/2+\varepsilon_0}$ tubes about $\gamma \in \Pi$. Also, for simplicity, we are only stating things here and throughout for eigenfunctions, but the results easily extend to quasimodes using results from [173].

Note that if $\varepsilon_0 = 1\,2$, then (1) is equivalent to the eigenfunction estimates from [15]

$$\|e_\lambda\|_{L^4(M)} \lesssim \lambda^{1/8} \|e_\lambda\|_{L^2(M)},$$

which are saturated by highest weight spherical harmonics on the standard two-sphere. We also remark that, up to the factor $\lambda^{\varepsilon_0/4}$, the estimate (1) is saturated by both the highest weight spherical harmonics and zonal functions on S^2. This is because the highest weight spherical harmonics are given by the restriction of the harmonic polynomials $\lambda^{1/4(x_1+i x_2)^k}$, $\lambda = \sqrt{k(k+1)}\, to$ the unit sphere, while the L^2-normalized zonal functions centered about the north pole on S^2 behave like $\left(\lambda^{-1} + dist\left(x, \pm(0,0,1)\right)\right)^{-1/2}$. See [33].

In [36] (with a slight loss) and in [28], inequalities of the form (1) and (3) were proved, where the first norm on the right is raised. The inequalities in [28] were not formulated in this way but easily lead to this result. The approach in [28] made inefficient use of the Cauchy–Schwarz inequality to handle the "easy" term (not the bilinear one), which led to the loss. The strategy for proving (1) will be to make an angular dyadic decomposition of a bilinear expression and pay close attention to the dependence of the bilinear estimates in terms of the angles, which we shall exploit using a multilayered microlocal decomposition of phase space.

Before turning to the details of the proof, let us record a few simple corollaries of our main estimate. If $\left\{a_{\lambda_{jk}}\right\}_{k=0}^{\infty}$ is a sequence depending on a subsequence $\{\lambda_{jk}\}$ of the eigenvalues of Δ_g , then we say that

$$a_\lambda = o-(\lambda^\sigma)$$

if there are some $\varepsilon > 0$ and $C < \infty$ such that

$$|a_\lambda| \le C(1+\lambda)^{\sigma-\varepsilon}.$$

Then using Theorem (5.1.1), we get:

Corollary (5.1.2)[170]: The following are equivalent:

$$\left\|e_{\lambda_{jk}}\right\|_{L^4(M)} = o-\left(\lambda_{jk}^{1/8}\right), \tag{5}$$

$$sup_{\gamma\in\Pi}\left\|e_{\lambda_{jk}}\right\|_{L^4(\mathcal{T}_{\lambda_{jk}^{-1/2}}(\gamma))} = o-(\lambda_{jk}^{1/8}), \tag{6}$$

$$\sup_{\gamma\in\Pi}\left\|e_{\lambda_{jk}}\right\|_{L^2(\mathcal{T}_{\lambda_{jk}^{-1/2}}(\gamma))} = o-(1). \tag{7}$$

Also, if either

$$\sup_{\gamma\in\Pi}\int_\gamma |e_\lambda|^2\,ds = O(\lambda_{jk}^{\varepsilon}), for\ all\ \varepsilon > 0 \tag{8}$$

or

$$\sup_{\gamma\in\Pi}\|e_{\lambda jk}\|_{L^2(\mathcal{T}\_(\lambda_{jk}^{-1/2}(\gamma))} = O(\lambda_{jk}^{-1/4+\varepsilon}),\quad for\ all\ \varepsilon > 0, \tag{9}$$

then

$$\left\|e_{\lambda_{jk}}\right\|_{L^4(M)} = O(\lambda_{jk}^{\varepsilon}), for\ all\ \varepsilon > 0. \tag{10}$$

Here, ds denotes the arc length measure on γ.

Clearly (5) implies (6). Also, (7) follows from (6) and Hölder's inequality. Since (1) shows that (7) implies (5), the last part of the corollary is also an easy consequence of Theorem (5.1.1). Note also that (4) says that if $e_{\lambda_{jk}}$ is a sequence of eigenfunctions with

$$\left\|e_{\lambda_{jk}}\right\|_{L^4(M)} = \Omega(\lambda_{jk}^{1/8}),$$

then for any ε, there must be a sequence of shrinking geodesic tubes $\{\mathcal{T}_{\lambda_{jk}^{-1/2}}(\gamma k)\}$ for which, for some $c = c\varepsilon > 0$, we have

$$\left\|e_{\lambda_{jk}}\right\|_{L^4(\mathcal{T}_{\lambda_{jk}^{-1/2}}(\gamma k))} \ge c\,\lambda_{jk}^{1/8-\varepsilon}.$$

In other words, up to a factor of $\lambda - \varepsilon$ for any $\varepsilon > 0$, they fit the profile of the highest weight spherical harmonics by having maximal L^4 -mass on a sequence of shrinking $\lambda^{-1/2}$ tubes.

Like in Bourgain's estimate, (1) involves a slight loss, but this is not so important in view of the above application. In a later work we hope to show that (1) holds without this loss (in other words with $\varepsilon_0 = 0$), which should mainly involve refining the $S_{1/2,1/2}$ microlocal arguments that are to follow. Note that, because of the zonal functions on S^2 , this result would be sharp.

We shall introduce a microlocal Kakeya–Nikodym norm and an inequality involving it, (24), which implies (1). This norm is associated to a decomposition of phase space which is naturally associated to the geodesic flow on the cosphere bundle. In particular, each term in the decomposition will involve bump functions which are supported in tubular neighborhoods of unit geodesics in S^*M. This decomposition and the resulting square function arguments are similar to the earlier ones of Mockenhaupt, Seeger and [48], but there are some differences and new technical issues that must be overcome. We do this and prove our microlocal Kakeya–Nikodym estimate. There, after some pseudodifferential arguments, we reduce matters to an oscillatory integral estimate which is a technical variation on the classical one in Hörmander [45], which was the main step in his proof of the Carleson–Sjölin theorem [37]. The result which we need does not directly follow from the results in [45]; however, we can prove it by adapting Hörmander's argument and using Gauss's lemma. We shall see how our results are in some sense related to Zygmund's theorem [60] saying that in two dimensions, eigenfunctions on the standard torus have bounded L^4 −norms. Specifically, we shall see there that if we could obtain the endpoint version of (1), we would be able to recover Zygmund's theorem with no loss if we also knew a conjectured result that arcs on λS^1 of length $\lambda^{1/2}$ contain a uniformly bounded number of lattice points [66].

As in [28]; [16], we use the fact that we can use a reproducing operator to write $e_\lambda = \chi_\lambda f = \rho(\lambda - \sqrt{\Delta_g})e_\lambda$, for $\rho \in \mathcal{S}$ satisfying $\rho(0) = 1$, where, if supp $\hat{\rho} \subset (1,2)$, we also have modulo $O(\lambda^{-N})$ errors (see [16])

$$\chi_\lambda f(x) = \frac{1}{2\pi}\int_{\hat{\rho}} (\,t)e^{i\lambda t}\left(e^{-it}\sqrt{\Delta_g f}\,\right)(x)dt$$

$$= \lambda^{\frac{1}{2}}\int \; e^{i\lambda\psi(x,y)}a_\lambda(x,y)\,f\,(y)\,dV(y), \qquad (11)$$

where

$$\psi(x,y) = d_g(x,y) \qquad (12)$$

is the Riemannian distance function, and if, as we may, we assume that the injectivity radius is 10 or more, a_λ belongs to a bounded subset of C^∞ and satisfies

$$a_\lambda(x,y) = 0, if\ d_g(x,y) \notin (1,2). \qquad (13)$$

Thus, in order to prove (1), it suffices to work in a local coordinate patch and show that if a is smooth and satisfies the support assumptions in (13), if $0 < \delta < 1\ 10$ is small but fixed, and if

$$x_0 = (0,y_0), \frac{1}{2} < y_0 < 4$$

is also fixed, then

$$\left\|\lambda^{\frac{1}{2}} \int e^{i\lambda\psi}(x,y)a(x,y)f(y)dy\right\|^2_{L^4(B(0,\delta))}$$
$$\lesssim_{\varepsilon_0} \lambda^{\varepsilon_0/2} \| f \|_{L^2} \times ||| f ||| _{KN(\lambda,\varepsilon_0)},\ if\ supp\ f \subset B(x_0,\delta). \tag{14}$$
Here $B(x,\delta)$ denotes the δ-ball about x in our coordinates. We may assume that in our local coordinate system the line segment $(0,y), |y| < 4$ is a geodesic.

In order to prove (14) we also need to define a microlocal version of the above Kakeya–Nikodym norm. We first choose $0 \le \beta \in C_0^\infty(\mathbb{R}^2)$ satisfying
$$\sum_{\nu\in\mathbb{Z}^2} \beta(z+\nu) = 1\ and\ supp\,\beta \subset \{x \in \mathbb{R}^2 : |x| \le 2\}. \tag{15}$$
To use this bump function, let $\phi_t(x,\xi) = (x(t),\xi(t))$ denote the geodesic flow on the cotangent bundle.

Then if (x,ξ) is a unit cotangent vector with $x \in B(x_0,\delta)\ and\ |\xi_1| < \delta$, with δ small enough, it follows that there is a unique $0 < t < 10$ such that $x(t) = (s,0)$ for some $s(x,\xi). If\ \xi(t) = (\xi_1(t),\xi_2(t))$ for this t, it follows that $\xi_2(t)$ is bounded from below. Let us then set $\ell(x,\xi) = (s(x,\xi),\xi_1(t)/|\xi(t)|)$. Note that φ then is a smooth map from such unit cotangent vectors to $\mathbb{R}^2$. Also, φ is constant on the orbit of ϕ. Therefore, $|\varphi(x,\xi) - \varphi(y,\eta)|$ can be thought of as measuring the distance from the geodesic in our coordinate patch through (x,ξ) to that of the one through (y,η). Let $\alpha(x)$ be a nonnegative C_0^∞ function which is one in $B\left(x_0,\frac{3}{2}\delta\right)$ and zero outside of $B(x_0,2\delta)$. Given $\theta = 2^{-k}$ with $\lambda^{-1/2} \le \theta \le 1$ and $\nu \in \mathbb{Z}^2, let\ \Upsilon \in C^\infty(\mathbb{R})$ satisfy
$$\Upsilon(s) = 1, s \in [c,c^{-1}], \Upsilon(s) = 0, s \notin \left[\frac{c}{2}, 2c^{-1}\right], \tag{16}$$
for some $c > 0$ to be specified later. We then put
$$Q_\theta^\nu(x,\xi) = \alpha(x)\beta(\theta^{-1}\varphi(x,\xi)+\nu)\Upsilon(|\xi|/\lambda). \tag{17}$$
This is a function of unit cotangent vectors, and we also denote its homogeneous of degree zero extension to the cotangent bundle with removed by $Q_\theta^\nu(x,\xi), \xi \ne 0$, and the resulting pseudodifferential operator by $Q_\theta^\nu(x,D)$. Then if f is as in (2-4), we define its microlocal Kakeya–Nikodym norm corresponding to frequency λ and angle $\theta_0 = \lambda^{-1/2+\varepsilon_0}$ to be
$$\left|||f|||\right._{MKN(\lambda,\varepsilon_0)} = \sup_{\theta_0\le\theta\le1}\left(\sup_{\nu\in\mathbb{Z}^2} \theta^{-\frac{1}{2}}\|Q\|_\theta^\nu(x,D)\|f\|_{L^2(\mathbb{R}^2)}\right) + \|f\|_{L^2(\mathbb{R}^2)},$$
$$\theta_0 = \lambda^{-1/2+\varepsilon_0}. \tag{18}$$
Note that
$$\sup_{\nu\in\mathbb{Z}^2} \theta^{-1/2} \|Q_\theta^\nu(x,D)f\|_{L^2(\mathbb{R}^2)}$$
measures the maximal microlocal concentration of f about all unit geodesics in the scale of θ. This is because if we consider the restriction of Q_θ^ν to unit cotangent vectors and if $Q_\theta^\nu(x,\xi) \ne 0$, then supp Q_θ^ν is contained in an $O(\theta)$ tube in the space of unit cotangent vectors about the orbit $t \to \phi_t(x,\xi)$. Let us collect a few facts about these pseudodifferential operators. First, the Q_θ^ν belong to a bounded subset of $S^0_{1/2+\varepsilon_0,1/2-\varepsilon_0}$ (pseudodifferential operators of order zero and type $(\frac{1}{2}+\varepsilon_0,\frac{1}{2}-\varepsilon_0)$), if $\lambda^{-1/2+\varepsilon_0} \le \theta \le 1$, with $\varepsilon_0 > 0$ fixed. Therefore, there is a uniform constant C_{ε_0} such that
$$\|Q_\theta^\nu(x,D)g\|_{L^2} \le C_{\varepsilon_0}\|g\|_{L^2}, \lambda^{-\frac{1}{2}+\varepsilon_0} \le \theta \le 1. \tag{19}$$

Similarly, if $P_\theta^\nu = (Q_\theta^\nu)^* \circ Q_\theta^\nu$ for such θ, then by (15), $\sum_\nu P_\theta^\nu$ belongs to a bounded subset of $S^0_{1/2+\varepsilon 0,1/2-\varepsilon_0}$, and so we also have the uniform bounds

$$\left\| \sum_{\nu\in\mathbb{Z}^2} P_\theta^\nu(x,D)g \right\|_{L^2} \le C_{\varepsilon_0} \|g\|_{L^2}, \lambda^{-\frac{1}{2}+\varepsilon_0} \le \theta \le 1. \quad (20)$$

We can relate the microlocal Kakeya–Nikodym norm to the Kakeya–Nikodym norm if we realize that if the $\delta > 0$ above is small enough, then there is a unit length geodesic γ_ν such that $Q_\theta^\nu(x,\xi) = 0\ for\ x \notin \mathcal{T}_{C\theta_\nu}(\gamma)$, with C a uniform constant. As a result, since $Q_\theta^\nu(x,\xi) = 0$ if $|\xi|$ is not comparable to λ, we can improve (19) and deduce that for every $N = 1,2,\ldots$, there is a uniform constant C' such that

$$\|Q_\theta^\nu(x,D)g\|_{L^2} \le C_{\varepsilon_0}\left(\int_{\mathcal{T}_{C'\theta}(\gamma_\nu)} |g|^2\,dy\right)^{1/2} + C_N\lambda^{-N}\|g\|_{L^2}, \lambda^{-1/2+\varepsilon_0} \le \theta \le 1, \quad (21)$$

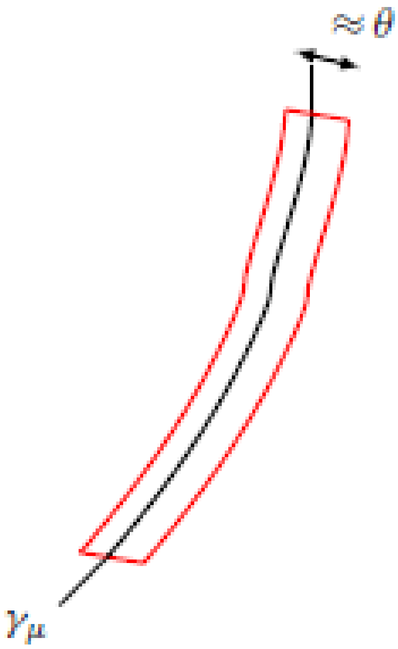


Figure (1)[170]: $\mathcal{T}_{C'\theta(\gamma_\nu)}$.

since the kernel $K_\theta^\nu(x,y)$of $Q_\theta^\nu(x,D)$ is $O(\lambda^{-N})$for any N if y is not in $\mathcal{T}_{C'\theta}(\gamma_\nu)$, with C' sufficiently large but fixed. (See Figure 1.) Since

$$\theta^{-\frac{1}{2}}\left(\int_{\mathcal{T}_{C'\theta}(\gamma_\nu)} |g|^2\,dy\right)^{\frac{1}{2}} \lesssim \sup_{\gamma\in\Pi} \theta_0^{-1}\left(\int_{\mathcal{T}_{\theta_0}(\gamma)} |g|^2\,dy\right)^{\frac{1}{2}}, \quad \lambda^{-1/2+\varepsilon_0} = \theta_0 \le \theta \le 1,$$

we have

$$\sup_{\nu\in\mathbb{Z}^2} \theta^{-1/2}\|Q_\theta^\nu(x,D)f\|_{L^2(\mathbb{R}^2)} \le C_{\varepsilon_0}\left|\left|\left|g\right|\right|\right|_{KN(\lambda,\varepsilon_0)}, \lambda^{-1/2+\varepsilon_0} \le \theta \le 1, \quad (22)$$

meaning that we can dominate the microlocal Kakeya–Nikodym norm by the Kakeya–Nikodym norm. From this, we conclude that we would have (14) if we could show

$$\left\|\int \lambda^{\frac{1}{2}} e^{i\lambda\psi}(x,y)a(x,y)f(y)dy\right\|^2_{L^4(B(0,\delta))} \lesssim_{\varepsilon 0} \lambda^{\frac{\varepsilon_0}{2}}\|f\|_{L^2} \times |||f|||_{MKN(\lambda,\varepsilon_0)}, \quad if\ supp\ f \subset B(x_0,\delta). \quad (23)$$

We note also that since $\chi_\lambda e_\lambda = e_\lambda$, this inequality of course yields the following microlocal strengthening of Theorem (5.1.1):

Theorem (5.1.3)[170]: For every $0 < \varepsilon_0 \le 1\ 2$, we have

$$\|e_\lambda\|_{L^4}(M) \lesssim_{\varepsilon_0} \lambda^{\varepsilon_0/4} \|e_\lambda\|^{1/2}_{L^2(M)} \times \left|||e_\lambda|||\right|^{1/2}_{MKN(\lambda,\varepsilon_0)}, \qquad (24)$$

if $\left|||e_\lambda|||\right|_{MKN(\lambda,\varepsilon_0)}$ is as in (18).

We prove the estimates in (23). We shall follow arguments from §6 of [48].

We first note that if supp $f \subset B(x_0,\delta)$ as in (24), and if

$$\theta_0 = \lambda^{-1/2+\varepsilon_0} \qquad (25)$$

with $\varepsilon_0 > 0$ fixed,

$$\chi_\lambda f = \sum_{\nu\in\mathbb{Z}^2} \chi_\lambda Q^\nu_{\theta_0}(x,D) f + R_\lambda f,$$

where, if $c > 0$ in (16) is small enough, and $N = 1,2,3,\ldots$,

$$\|R_\lambda f\|_{L^\infty} . \lambda^{-N} \|f\|_{L^2}.$$

Therefore, in order to prove (14), it suffices to show that

$$\left\|\sum_{\nu,\nu'\in\mathbb{Z}^2} \chi_\lambda Q^\nu_{\theta_0} f\, \chi_\lambda Q^{\nu'}_{\theta_0} f\right\|_{L^2} \lesssim_{\varepsilon_0} \lambda^{\varepsilon_0/2} \|f\|_{L^2} \times |||f|||_{MKN(\lambda,\varepsilon_0)}. \qquad (26)$$

We split the sum on the left based on the size of $|\nu - \nu'|$. Indeed, the left side of (26) is dominated by

$$\left\|\sum_\nu \left(\chi_\lambda Q^\nu_{\theta_0} f\right)^2\right\|_{L^2} + \sum_{\ell=1}^{\infty} \left\|\sum_{|\nu-\nu 0|\in[2^\ell,2^{\ell+1})} \chi_\lambda Q^\nu_{\theta_0} f \chi_\lambda Q^{\nu'\theta}_0 f\right\|_{L^2}. \qquad (27)$$

The square of the first term in (27) is

$$\sum_{\nu,\nu'} \int \left(\chi_\lambda Q^\nu_{\theta_0} f\right)^2 \overline{\left(\chi_\lambda Q^{\nu'}_{\theta_0} f\right)^2}\, dx.$$

Next we need an orthogonality result, similar to Lemma 6.7 in [48], which says that if A is large enough we have

$$\sum_{(|\nu-\nu'|\ge A)} \left|\int \left(\chi_\lambda Q^\nu_{\theta_0} f\right)^2 \overline{\left(\chi_\lambda Q^{\nu'}_{\theta_0} f\right)^2}\, dx\right| \lesssim_{\varepsilon_0,N} \lambda^{-N} \|f\|^4_{L^2}. \qquad (28)$$

We shall postpone the proof of this result until, when we will have recorded the information about the kernels of $\chi_\lambda Q^\nu_\theta$ that will be needed for the proof. Since by [15],

$$\|\chi_\lambda\|_{L^2\to L^4} = O(\lambda^{1/8}),$$

if we use (28) we conclude that the first term in (27) is majorized by (20) and (22):

$$\lambda^{\frac{1}{2}} \sum_\nu \left\|Q^\nu_{\theta_0} f\right\|^2_{L^2} \left\|Q^\nu_{\theta_0} f\right\|^2_{L^2} + \lambda^{-N} k\|f\|^4_{L^2} . \lambda^{\frac{1}{2}} \|f\|^2_{L^2}$$

$$\times \sup_{\nu\in\mathbb{Z}^2} \left\|Q^\nu_{\theta_0} f\right\|^2_{L^2} + \lambda^{-N} \|f\|^4_{L^2}$$

$$= \lambda^{\varepsilon_0} \|f\|^2_{L^2} \times \lambda^{1/2-\varepsilon_0} \sup_{\nu\in\mathbb{Z}^2} \left\|Q^\nu_{\theta_0} f\right\|^2_{L^2} + \lambda^{-N} \|f\|^4_{L^2}. \qquad (29)$$

Therefore, the first term in (27) satisfies the desired bounds

Using (22) again, the proof of (23) and hence (14) would be complete if we could estimate the other terms in (27) and show that

$$\sum_{|\nu-\nu'|\in[2^\ell,2^{\ell+1})} \chi_\lambda Q^\nu_{\theta_0} f\, \chi_\lambda Q^\nu_{\theta_0} f\Big\|^2_{L^2} \lesssim_{\varepsilon_0} \|f\|^2_{L^2} \times \left(2^\ell\theta_0\right)^{-1}$$

$$\sup_{\nu\in\mathbb{Z}^2} kQ^\nu\, 2^\ell\theta_0\, \|f\|^2_{L^2} + \lambda^{-N}\, \|f\|^4_{L^2}. \qquad (30)$$

Note that if $2^\ell\theta_0 \gg 1$, the left side of (30) vanishes and thus, as in (22), we are just considering $\ell \in \mathbb{N}$ satisfying $1 \le 2^\ell \le \lambda^{1/2-\varepsilon_0}$. In proving this, we may assume that ℓ is larger than a fixed constant, since the bound for small ℓ (with an extra factor of λ^{ε_0} on the right) follows from what we just did. We can handle the sum over ℓ in (27) due to the fact that the right side of (30) does not include a factor λ^{ε_0}. We now turn to estimating the nondiagonal terms in (27). We first note that by (15),

$$\chi_\lambda Q^\nu_{\theta_0} f = \sum_{\mu\in\mathbb{Z}^2} \chi_\lambda Q^\mu_\theta\, Q^\nu_{\theta_0} f + O_N(\lambda^{-N}\|f\|_2), if\ supp\, f \subset B(x_0,\delta).$$

Furthermore, if, as we may, we assume that $\ell \in N$ is sufficiently large, then given $N_0 \in \mathbb{N}$, there are fixed constants $c_0 > 0$ and $N_1 < \infty$ (with c_0 depending only on N_0 and the cutoff β in the definition of these pseudodifferential operators) such that if

$$\theta_\ell = \theta_0 2^\ell,$$

then

$$\sum_{(|\nu-\nu'|\in[2^\ell,2^{\ell+1})} \chi_\lambda Q^\nu_{\theta_0} f\, \chi_\lambda Q^{\nu'}_{\theta_0} f$$

$$= \sum_{\{\mu,\mu^\ell\in\mathbb{Z}^2 : N_0\le|\mu-\mu^\ell|\le N_1\}\, X\, |\nu-\nu'|\in[2\ell,2^{\ell+1})} \chi_\lambda Q^\mu_{c_0\theta_\ell}\, Q^\nu_{\theta_0} f\, \chi_\lambda Q^{\mu'}_{c_0\theta_\ell}\, Q^{\nu'}_{\theta_0} f$$

$$+ O_N(\lambda^{-N}\|f\|^2_{L^2}), \qquad (31)$$

for each $N \in \mathbb{N}$. Also, given $\mu \in \mathbb{Z}^2$, there is a $\nu_0(\mu) \in \mathbb{Z}^2$ such that

$$kQ^\mu_{c_0\theta_\ell}\, Q^\nu_{\theta_0}\, \|f\|^{L^2} \le C_N\, \lambda^{-N}\|f\|_{L^2}, if\ |\nu - \nu_0(\mu)| \ge C2^\ell,$$

for some uniform constant C. If $|\mu - \mu'| \le N_1$, then $|\nu_0(\mu) - \nu_0(\mu')| \le C2^\ell$ for some uniform constant C.

Since $\left\|\left(Q^{\nu'}_\theta\right)^* \circ Q^\nu_\theta\right\|_{L^2\to L^2} = O(\lambda^{-N})$ for every N if $|\nu - \nu'|$ is larger than a fixed constant, it follows that

$$\int\int \left|\sum_{\nu_0(\mu)-\nu|,|\nu_0(\mu')-\nu'|\le C2^\ell}\ \sum_{|\nu-\nu'|\in[2^\ell,2^{\ell+1})} Q^\nu_{\theta_0} f(x) Q^{\nu'}_{\theta_0} f(y)\right|^2 dx\, dy$$

$$\lesssim \sum_{|\nu-\nu'(\mu)|,|\nu'-\nu_0(\mu)|\le C'2^\ell} \left\|Q^\nu_{\theta_0} f\right\|^2_{L^2} \left\|Q^{\nu'}_{\theta_0} f\right\|^2_{L^2} + O_N\left(\lambda^{-N}\|f\|^2_{L^2}\right),$$

$$if\ |\mu - \mu'| \le C_0, \qquad (32)$$

for every N if C' is a sufficiently large but fixed constant. Also, using (20), we deduce that

$$\sum_{(\mu\in\mathbb{Z}^2}\ \sum_{|\nu_0(\mu)-\nu|\le C'2^\ell} \left\|Q^\nu_{\theta_0} f\right\|^2_{L^2} . \|f\|^2_{L^2}.$$

We clearly also have

$$\sum_{|\nu(\mu)-\nu'|\le C'2^\ell} \left\|Q^{\nu'}_{\theta_0} f\right\|^2_{L^2} \lesssim \sup_{\mu\in\mathbb{Z}^2} \left\|Q^\mu_{2^\ell\theta} f\right\|^2_{L^2}.$$

Using these two inequalities and (32), we deduce that

$$\sum_{|\mu-\mu'|\le N_1}\left\|\sum_{|\nu_0(\mu)-\nu|,|\nu_0(\mu')-\nu'|<C2^\ell}\ \sum_{|\nu-\nu'|\in[2^\ell,2^{\ell+1})} \mathcal{Q}^{\nu}_{\theta_0}f(x)\mathcal{Q}^{\nu'}_{\theta_0}f(y)\right\|_{L^2(dx\,dy)}$$
$$\lesssim \|f\|_{L^2}\times\sup_{\mu\in\mathbb{Z}^2}\left\|\mathcal{Q}^{\mu}_{2^\ell\theta}f\right\|_{L^2}+O_N(\lambda^{-N}\|f\|^2_{L^2}). \qquad (33)$$

In addition to (28), we shall need another orthogonality result whose proof we postpone until, which says that whenever θ is larger than a fixed positive multiple of θ_0 in (25) and N_1 is fixed,

$$\int \left|\left(\chi_\lambda\mathcal{Q}^{\mu}_{\theta}\,g_1\chi_\lambda\mathcal{Q}^{\mu'}_{\theta}\,g_2\right)\overline{\left(\chi_\lambda\mathcal{Q}^{\tilde\mu}_{\theta}\,g_3\chi_\lambda\mathcal{Q}^{\tilde\mu'}_{\theta}\,g_4\right)}\,dx\right|\lesssim_N \lambda^{-N}\prod_{j=1}^{4}\|g_j\|_{L^2},$$
$$\text{if } |\mu-\tilde\mu|+|\mu'-\tilde\mu'|\ge C \text{ and } |\mu-\mu'|,|\tilde\mu-\tilde\mu'|\le N_1, \qquad (34)$$

for every $N = 1,2,\dots$, with C being a sufficiently large uniform constant (depending on N_1 of course).

Using (33) and (34), we conclude that we would have (30) (and consequently (14)) if we prove the following:

Proposition (5.1.4)[170]: Let

$$(T^{\mu,\mu'}_{\lambda,\theta}F)(x) = \iint (\chi_\lambda\mathcal{Q}^{\mu}_{\theta})(x,y)(\chi_\lambda\mathcal{Q}^{\mu'}_{\theta})(x,y')F(y,y')\,dy\,dy', \qquad (35)$$

Where

$$(\chi_\lambda\mathcal{Q}^{\mu}_{\theta})(x,y)$$

denotes the kernel of $\chi_\lambda\mathcal{Q}^{\mu}_{\theta}$. Then if $\delta > 0$ is sufficiently small and if θ is larger than a fixed positive constant times θ_0 in (3-1) and if $N_0 \in \mathbb{N}$ is sufficiently large and if $N_1 > N_0$ is fixed, we have

$$\left\|T^{\mu,\mu'}_{\lambda,\theta}F\right\|_{L^2(B(0,\delta))}\lesssim_{\varepsilon_0}\theta^{-1/2}\|F\|_{L^2},\quad if\ N_0\le|\mu-\mu'|\le N_1,$$
$$F(y,y')=0,\qquad if\ (y,y')\notin B(x_0,2\delta)\times B(x_0,2\delta). \qquad (36)$$

To prove this we shall need some information about the kernel of $\chi_\lambda\mathcal{Q}^{\mu}_{\theta}$. By (17), the kernel is highly concentrated near the geodesic in M

$$\gamma_\mu=\{x_\mu(t):\ -2\le t\le 2, \Phi_t(x_\mu,\xi_\mu)=(x_\mu(t),\xi_\mu(t)),$$
$$\theta^{-1}\varphi(x_\mu,\xi_\mu)+\mu=0\}, \qquad (37)$$

which corresponds to $\mathcal{Q}^{\mu}_{\theta}$. We also will exploit the oscillatory behavior of the kernel near γ_μ.

Specifically, we require the following:

Lemma (5.1.5)[170]: Let $\theta\in[C_0\lambda^{-1/2+\varepsilon_0},\frac{1}{2}]$, where C_0 is a sufficiently large fixed constant, and, as above, $\varepsilon_0 > 0$. Then there is a uniform constant C such that for each $N = 1,2,3,\dots$, we have

$$|(\chi_\lambda\mathcal{Q}^{\mu}_{\theta})(x,y)|\le C_N\lambda^{-N},\ \ if\ x\notin\mathcal{T}_{C\theta}(\gamma_\mu) \text{ or } y\notin\mathcal{T}_{C\theta}(\gamma_\mu). \qquad (38)$$

Furthermore,

$$(\chi_\lambda\mathcal{Q}^{\mu}_{\theta})(x,y)=\lambda^{1/2}e^{i\lambda d_g(x,y)}a_{\mu,\theta}(x,y)+O_N(\lambda^{-N}), \qquad (39)$$

where one has the uniform bounds

$$\left|\nabla^\alpha_y a_{\mu,\theta}(x,y)\right|\le C_\alpha\theta^{-|\alpha|}, \qquad (40)$$
$$\left|\partial^j_t\,a_{\mu,\theta}\left(x,x_\mu(t)\right)\right|\le C_j,\qquad x\in\gamma_\mu, \qquad (41)$$

if , as in (37), $\{x_\mu(t)\} = \gamma_\mu$.

Proof. To prove the lemma it is convenient to choose Fermi normal coordinates so that the geodesic becomes the segment $\{(0,s) : |s| \le 2\}$. Let us also write θ as

$$\theta = \lambda^{-1/2+\delta},$$

where, because of our assumptions, $c_1 \le \delta \le \frac{1}{2}$ for an appropriate $c_1 > 0$. Then in these coordinates, $Q_\theta^\mu(x,D)$ has symbol satisfying

$$q_\theta^\mu(x,\xi) = 0, \text{ if } |\xi_1/|\xi|| \ge C\lambda^{-1/2+\delta}, |x_1| \ge C\lambda^{-1/2+\delta}, \text{or } |\xi|/\lambda \notin [C^{-1}, C], \quad (42)$$

for some uniform constant C, and, additionally,

$$|\partial_{x_1}^j \partial_{x_2}^k \partial_{\xi_1}^l \partial_{\xi_2}^m q_\theta^\mu(x,\xi) \le C_{j,k,l,m}(1+|\xi|)^{j(1/2-\delta)-l(1/2+\delta)-m}. \quad (43)$$

Next we recall that $\chi_\lambda = \rho(\lambda - p\sqrt{-\Delta_g})$, where $\rho \in \mathcal{S}(R)$ satisfies $\hat\rho \subset (1,2)$, and that the injectivity radius of (M,g) is ten or more. Therefore, we can use Fourier integral parametrices for the wave equation to see that the kernel of χ_λ is of the form

$$\chi_\lambda(x,y) = \iint e^{iS(t,x,\xi) - iy\cdot\xi + it\lambda}\hat\rho(t)\alpha(t,x,y,\xi)\, d\xi\, dt,$$

where $\alpha \in S_{1,0}^1$, and S is homogeneous of degree one in ξ and is a generating function for the canonical relation for the half wave group $e^{-it\sqrt{-\Delta_g}}$. Thus,

$$\partial_t S(t,x,\xi) = -p\big(x, \nabla_x S(t,x,\xi)\big), \;\; S(0,x,\xi) = x\cdot\xi. \quad (44)$$

Let $\widetilde\Phi_t(x,\xi)$ denote the Hamiltonian flow generated by $p(x,\xi)$, which is homogeneous of degree one in ξ and agrees with the geodesic flow $\Phi_t(x,\xi)$ when restricted to unit cotangent vectors. The phase $S(t,x,\xi)$ also satisfies

$$\widetilde\Phi_t(x,\nabla_x S) = (\nabla_\xi S, \xi). \quad (45)$$

Furthermore,

$$\det \frac{\partial S}{\partial x \partial \xi} \ne 0. \quad (46)$$

By (42), (43), and the proof of the Kohn–Nirenberg theorem, we have that

$$\big(\chi_\lambda Q_\theta^\mu\big)(x,y) = \int\int e^{iS(t,x,\xi)-iy\cdot\xi+i\lambda t}\hat\rho(t)q(t,x,y,\xi)d\xi\,dt + O(\lambda^{-N}),$$
$$= \lambda^2 \int\int e^{i\lambda(S(t,x,\xi)-y\cdot\xi+t)}\hat\rho(t)q(t,x,y,\lambda\xi)\,d\xi\,dt + O(\lambda^{-N}), \quad (47)$$

where for all t in the support of

$$\hat\rho, q(t,x,y,\xi) = 0 \; if \; |\xi_1/|\xi|| \ge C\lambda^{-\frac{1}{2}+\delta}, |x_1| \ge C\lambda^{-\frac{1}{2}+\delta}, \; or\; |\xi|/\lambda \notin [C^{-1}, C], \quad (48)$$

with C as in (43), and also

$$\left|\partial_{x_1}^j \partial_{x_2}^k \partial_{\xi_1}^l \partial_{\xi_2}^m q(t,x,y,\xi)\right| \le C_{j,k,l,m}(1+|\xi|)j^{(1/2-\delta)-l(1/2+\delta)-m}. \quad (49)$$

Let us now prove (38). We have the assertion if $y \notin \mathcal{T}_{C\lambda-1/2+\delta\,(\gamma_\mu)}$ by (48). To prove that remaining part of (48) which says that this is also the case when x is not in such a tube, we note that by (45), if $d_g(x_0,y_0) = t_0$ and $x_0, y_0 \in \gamma_\mu$, then

$$\nabla_\xi\,(S(t_0,x_0,\xi) - y_0\cdot\xi) = 0, if\; \xi_1 = 0.$$

By (46), we then have

$$\left|\nabla_\xi\,(S(t_0,x,\xi) - y_0\cdot\xi)\right| \approx d_g(x,x_0), if\; \xi_1 = 0.$$

We deduce from this that if $|\xi_1|/|\xi| \le C^{\lambda-\frac{1}{2}+\delta}, |y_1| \le C^{\lambda-1/2+\delta}$,and $|\xi| \in [C^{-1}, C]$, then there are a $c_0 > 0$ and a $C_0 < \infty$ such that

$$\left|\nabla_\xi \left(S(t_0,x,\xi\,) - y \cdot \xi\,\right)\right| \geq c_0\lambda^{-1/2+\delta}, if\ x \notin \mathcal{T}_{C_0\,\lambda^{-1/2+\delta}\,(\gamma_\mu)}.$$

From this we obtain the remaining part of (38) via a simple integration by parts argument if we use the support properties (48) and size estimates (49) of $q(t,x,y,\xi\,)$. We note that every time we integrate by parts in ξ we gain by $\lambda^{-2\delta}$, which implies (38) since q vanishes unless $|\xi\,| \approx \lambda$ and δ is bounded below by a fixed positive constant.
To finish the proof of the lemma and obtain (39)–(41), we note that if we let

$$\Psi(t,x,y,\xi\,) = S(t,x,\xi\,) - y \cdot \xi + t$$

denote the phase function of the second oscillatory integral in (47), then at a stationary point where

$$\nabla_{\xi\,,t}\Psi = 0,$$

we must have $\Psi = d_g(x,y)$, due to the fact that $S(t,x,\xi\,) - y \cdot \xi = 0$ and $t = d_g(x,y)$ at points where the ξ -gradient vanishes. Additionally, it is not difficult to check that the mixed Hessian of the phase satisfies

$$\det\left(\frac{\partial^2\Psi}{\partial(\xi\,,t)\partial(\xi\,,t)}\right) \neq 0$$

on the support of the integrand. This follows from the proof of Lemma 5.1.3 of [16]. Moreover, since modulo $O(\lambda^{-N}\,)$ error terms $(\chi_\lambda\mathcal{Q}_\theta^\mu\,)(x,y)$ equals

$$\lambda^2 \iint e^{i\lambda\Psi}\hat{\rho}(t)q(t,x,y,\lambda\xi\,)\,d\xi\,dt, \tag{50}$$

we obtain (3-15)–(3-16) by the proof of this result if we use the stationary phase and (48)–(49). Indeed, by (45), (50) has a stationary phase expansion (see [24]), where the leading term is a fixed constant times

$$\lambda^{1/2}\,e^{i\lambda t}\,q(t,x,y,\lambda\xi\,), \text{if } t = d_g(x,y) \text{and } \widetilde{\Phi}_{-t}(y,\xi\,) = \left(x,\nabla_x\,S(t,x,\xi\,)\right). \tag{51}$$

From this, we see that the leading term in the asymptotic expansion must satisfy (40), and subsequent terms in the expansion will satisfy better estimates, where the right-hand side involves increasing negative powers of $\lambda^{2\delta}$ (by [24] and (49)), from which we deduce that (40) must be valid. Since $\xi_1 = 0$ and $p(y,\xi\,) = 1$ (by (45)) in (51) when $x,y \in \gamma_\mu$, we similarly deduce from (49) that the leading term in the stationary phase expansion must satisfy (41), and since the other terms satisfy better bounds involving increasing powers of $\lambda^{-2\delta}$, we similarly obtain (41), which completes the proof of the lemma. Let us now collect some simple consequences of Lemma (5.1.5). First, in addition to (38), the kernel $(\chi_\lambda\mathcal{Q}_\theta^\mu\,)(x,y)$ is also $O(\lambda^{-N}\,)$ unless the distance between x and y is comparable to one by (2-3). From this we deduce that if $N_0 \in \mathbb{N}$ is sufficiently large,

$$(\chi_\lambda\mathcal{Q}_\theta^\mu)(x,y)(\chi_\lambda\mathcal{Q}_\theta^{\mu'})(x,y') = O(\lambda^{-N}),$$

unless $\text{Angle}(x;\ y,y') \in [\theta\,,C_2\theta]$ and $x,y,y' \in \mathcal{T}_{C_2\theta}(\gamma_\mu)$, if $|\mu - \mu'| \in [N_0,N_1]$, (52)
if $\text{Angle}(x,y,y')$ denotes the angle at x of the geodesic connecting x and y and the one connecting x and y', and where $C_2 = C_2(N_1)$.

This is because in this case, if $x \in \mathcal{T}_{C\theta}\,(\gamma_\mu) \cap \mathcal{T}_{C\theta}\,(\gamma_{\mu'})$, then the tubes must be disjoint at a distance bounded below by a fixed positive multiple of θ if N_0 is large enough, and in this region their separation is bounded by a fixed constant times θ if N_1 is fixed; see Figure 2.
To exploit this key fact, as above, let us choose Fermi normal coordinates (see [172]) about γ_μ so that the geodesic becomes the segment $\{(0,s) : |s| \leq 2\}$. Then, as in (12), let

$$\psi(x;\ y)\ =\ d_g\ (x_1,x_2),(y_1,y_2)$$

be the Riemannian distance function written in these coordinates. Then if x,y,y' are close to this segment and if the distances between x and y and x and y' are both comparable to 1 and if, as well, y is close to y', it follows from Gauss's lemma that

$$Angle\ \left(x;(y_1,y_2),(y_1',y_2')\right) \approx \left|\frac{\partial}{\partial y_1}\frac{\partial}{\partial x_2}\ \psi(x,y) - \frac{\partial}{\partial y_1}\frac{\partial}{\partial x_2}\ \psi(x,y')\right|. \qquad (53)$$

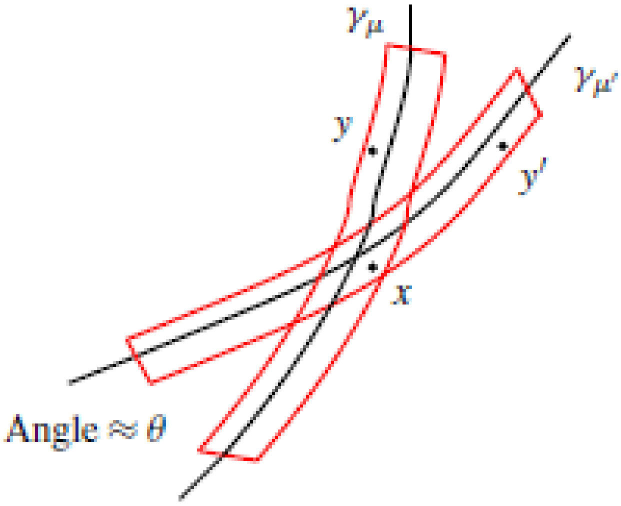


Figure (2)[170]: θ-tubes intersecting at angle $\geq\ N_0\theta$.

As a result, by (52), there must be a constant $c_0\ >\ 0$ such that

$$(\chi_\lambda Q_\theta^\mu)(x,y)\left(\chi_\lambda Q_\theta^{\mu'}\right)(x,y') = O(\lambda^{-N}),$$

$$\text{if}\left|\frac{\partial}{\partial y_1}\frac{\partial}{\partial x_2}\ \psi(x,y) - \frac{\partial}{\partial y_1}\frac{\partial}{\partial x_2}\ \psi(x,y')\right| \leq c_0\theta \text{ and } |\mu-\mu'| \in [N_0,N_1], \quad (54)$$

with, as above, $N_0\ \in \mathbb{N}$ sufficiently large and N_1 fixed. Another consequence of Gauss's lemma is that if x and y as in (53) are close to this segment and at a distance from each other which is comparable to one, then

$$\frac{\partial}{\partial x_1}\frac{\partial}{\partial y_1}\ \psi(x,y) \neq 0. \qquad (55)$$

We shall also need to make use of the fact that, in these Fermi normal coordinates, we have

$$\frac{\partial}{\partial x_2}\frac{\partial}{\partial y_1}\ \psi\ (0,x_2),(0,y_2)\ = \frac{\partial}{\partial x_1}\ \psi\ (0,x_2),(0,y_2)$$
$$= 0, if\ d_g\ (0,x_2),(0,y_2)\ \approx\ 1. \qquad (56)$$

Next, by (41)–(39), modulo terms which are $O(\lambda^{-N})$ we can write

$$(\chi_\lambda Q_\theta^\mu)(x,y)(\chi_\lambda Q_\theta^{\mu'})(x,y')\ =\ \lambda e^{i\lambda\left(\psi(x,y)+\psi(x,y')\right)}b_\mu(x;\ y,y'),$$

where, by (52) and (54),

$$b_\mu(x;\ y,y') = 0, if\ d_g(x,y) or\ d_g(x,y') \notin\ [1,2],$$
$$\text{or } |x_1| + |y_1| + |y_1'| \geq\ c_0^{-1}\ \theta,$$
$$\text{or}\left|\frac{\partial}{\partial y_1}\frac{\partial}{\partial x_2}\ \psi(x,y) - \frac{\partial}{\partial y_1}\ \frac{\partial}{\partial x_2}\psi(x,y')\right| \leq c_0\theta, \qquad (57)$$

and, since we are working in Fermi normal coordinates,

$$\left|\frac{\partial^j}{\partial x_1^j}\ \frac{\partial^k}{\partial x_2^k}\ b_\mu(x,y,y')\right| \leq\ C_0\theta^{-j}, 0\ \leq\ j,k\ \leq\ 3. \qquad (58)$$

The constants C_0 and c_0 can be chosen to be independent of $\mu \in \mathbb{Z}^2$ and $\theta \geq \lambda^{-1/2+\varepsilon_0}$ if $\varepsilon_0 > 0$. But then, by (57) and (58) if y_2 and y_2' are fixed and close to one another, and if we set

$$\psi(x; s, t) = \psi\big(x, (s + t, y_2)\big) + \psi\big(x, (s - t, y_2')\big) \text{ and}$$
$$b(x; s, t) = b_\mu(x; s + t, y_2, s - t, y_2'),$$

there is a fixed constant C such that

$$b(x; s, t) = 0 \; if \; |x_1| + |s| + |t| \geq C_\theta \,,$$
$$\text{and} \left|\frac{\partial^j}{\partial x_1^j}\frac{\partial^k}{\partial x_2^k} b(x; s, t)\right| \leq C\theta^{-j}, 0 \leq j, k \leq 3, \tag{59}$$

while, by (55) and (56),

$$\frac{\partial}{\partial x_2}\frac{\partial}{\partial s}\Psi(0, x_2;\, 0, 0) = \frac{\partial}{\partial x_2}\frac{\partial}{\partial t}\Psi(0, x_2;\, 0, 0) = \frac{\partial}{\partial x_1}\Psi(0, x_2;\, 0, 0) = 0,$$
$$\text{but} \frac{\partial}{\partial x_1}\frac{\partial}{\partial s}\Psi(0, x_2;\, 0, 0) \neq 0 \text{ if } b(0, x_2;\, 0, 0) \neq 0, \tag{60}$$

and, moreover, by (57),

$$\left|\frac{\partial}{\partial x_2}\frac{\partial}{\partial t}\Psi(x; s, t)\right| \geq c\theta, \;\text{ if } b(x; s, t) \neq 0. \tag{61}$$

Also, if we assume that $|y_2 - y_2'| \leq \delta$, as we may because of the support assumption in (36), then

$$\left|\frac{\partial}{\partial x_1}\frac{\partial}{\partial t}\Psi(x; s, 0)\right| \leq C\delta, \quad \text{if } b(x; s, t) \neq 0, \tag{62}$$

since the quantity on the left vanishes identically when $y_2 = y_2'$.

Another consequence of Gauss's lemma is that if y, y', x are close to the second coordinate axis and if the distances between x and each of y and y' are comparable to 1, then if θ above is bounded below, the 2 × 2 mixed Hessian of the function $(x;\, y_1, y_1') \to \psi(x, y) + \psi(x, y')$ has nonvanishing determinant. Thus, in this case (36) just follows from Hörmander's nondegenerate L^2 -oscillatory integral lemma [45] (see [16]). Therefore, it suffices to prove (36) when θ is bounded above by a fixed positive constant, and so Proposition (5.1.4), and hence Theorem (5.1.1), is a consequence of the following:

Lemma (5.1.6)[170]: Suppose that $b \in C_0^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ vanishes when $|(s, t)| \geq \delta$. Then if $\Psi \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ is real and (59)–(62) are valid, there is a uniform constant C such that if $\delta > 0$ and $\theta > 0$ are smaller than a fixed positive constant and

$$T_\lambda F(x) = \iint e^{i\lambda\Psi(x;s,t)} b(x; s, t) F(s, t)\, ds\, dt,$$

then we have

$$\|T_\lambda F\|_{L^2(\mathbb{R}^2)} \leq C\lambda^{-1}\,\theta^{-1/2}\,\|F\|_{L^2(\mathbb{R}^2)}\,. \tag{63}$$

We shall include the proof of this result for the sake of completeness even though it is a standard result. It is a slight variant of the main lemma in Hörmander's proof [45] of the Carleson–Sjölin theorem (see [16]). Hörmander's proof gives this result in the special case where $y_2 = y_2'$, and, as above, Ψ is defined by two copies of the Riemannian distance function. The case where y_2 and y_2' are not equal to each other introduces some technicalities that, as we shall see, are straightforward to overcome.

Proof. Inequality (63) is equivalent to the statement that $\|T_\lambda^*\, T_\lambda\|_{L^2\to L^2} \leq C\lambda^{-2}\theta^{-1}$. The kernel of $T_\lambda^* T_\lambda$ is

$$K(s,t;s',t') = \iint e^{i\lambda\left(\Psi(x;s,t)-\Psi(x;s',t')\right)} a(x;s,t,s',t')\,dx_1\,dx_2,$$
$$\text{if } a(x;s,t,s',t') = b(x,s,t)\overline{b(x;s',t')}.$$
Therefore, we would have this estimate if we could show that
$$|K(s,t;s',t')| \le C\theta^{1-N}(1+\lambda|s-s',t-t'|)^{-N} + C\theta(1+\lambda\theta|s-s',t-t'|)^{-N},$$
$$N = 0,1,2,3, \tag{64}$$
for then by using the $N = 0$ bounds for the regions where $|(s-s',t-t')| \le (\lambda\theta)^{-1}$ and the $N = 3$ bounds in the complement, we see that
$$\sup_{s,t} \iint |K|ds'\,dt', \sup_{s',t'} \iint |K|\,ds\,dt \le C\lambda^{-2}\theta^{-1},$$
which means that by Young's inequality, $\|T_\lambda^* T_\lambda\|_{L^2\to L^2} \le C\lambda^{-2}\theta^{-1}$, as desired.

The bound for $N = 0$ follows from the first part of (59). To prove the bounds for $N = 1, 2, 3$, we need to integrate by parts.

Let us first handle the case where
$$|s-s'| \ge A^{-1}|t-t'|, \tag{65}$$
where $A \ge 1$ is a possibly fairly large constant which we shall specify in the next step. By the second part of (60) and by (62), we conclude that if $\delta > 0$ is sufficiently small (depending on A), we have
$$\left|\frac{\partial}{\partial x_1}\left(\Psi(x;s,t)-\Psi(x;s',t')\right)\right| \ge c|s-s'|, |s-s'| \ge A^{-1}|t-t'|, \tag{66}$$
for some uniform constant $c > 0$.
Since $|K|$ is trivially bounded by the second term on the right side of (3-40) when $|s-s'| \le (\lambda\theta)^{-1}$ and (65) is valid, we shall assume that $|s-s'| \ge (\lambda\theta)^{-1}$.

If we then write
$$e^{i\lambda\left(\Psi(x;s,t)-\Psi(x;s\,0,t\,0)\right)} = Le^{i\lambda\left(\Psi(x;s,t)-\Psi(x;s\,0,t\,0)\right)},$$
$$\text{where } L(x,D) = \frac{1}{i\lambda\left(\Psi'_{x_1}(x;s,t)-\Psi'_{x_1}(x;s',t')\right)}\frac{\partial}{\partial_{x_1}}, \tag{67}$$
then we obtain
$$|K| \le \int\int \left|\left(L^*(x,D)\right)^N a(x;s,t,s',t')\right| dx.$$
Note that
$$\left|\lambda\,\psi'_{x_1}(x;s,t) - \psi'_{x_1}(x;s',t')\right|^N |(L^*)^N a| \le C_N \sum_{0\le j+k\le N} \left|\frac{\partial_j}{\partial x_1^j} a\right|$$
$$\times \sum_{\alpha_1+\cdots+\alpha_k\le N} \frac{\prod_{m=1}^k \left|\frac{\partial^{\alpha_m}}{\partial x_1^{\alpha_m}}\left(\psi'_{x_1}(x;s',t')\right)\right|}{\psi'_{x_1}(x;s,t) - \psi'_{x_1}(x;s',t')^k}. \tag{68}$$
Clearly,
$$\prod_{m=1}^{k} \left|\frac{\partial^{\alpha_m}}{\partial x_1^{\alpha_m}}\left(\psi'_{x_1}(x;s,t) - \psi'_{x_1}(x;s',t')\right)\right| \le C_k\,|s-s',t-t'|k, \tag{69}$$
and consequently, by (65) and (66),

$$\frac{\prod_{m=1}^{k}\left|\frac{\partial^{\alpha_m}}{\partial x_1^{\alpha_m}}\left(\psi'_{x_1}(x;s,t)-\psi'_{x_1}(x;s',t')\right)\right|}{\left|\left(\psi'_{x_1}(x;s,t)-\psi'_{x_1}(x;s',t')\right)\right|^k}\leq C_{A,k}. \qquad (70)$$

Since by (59), we have that $|\partial_{x_1}^j a| \leq C\theta^{-j}, j = 0,1,2,3,$ and (59) also says that a vanishes when $|x_1|$ is larger than a fixed multiple of θ, we conclude from (66)–(70) that if (65) holds, then $|K|$ is dominated by the first term on the right side of (64).

We now turn to the remaining case, which is

$$|t - t'| \geq A|s - s'|, \qquad (71)$$

and where the parameter $A \geq 1$ will be specified. By the first part of (60) and by (61) and the fact that $|s|,|s'|,|t|,|t'|$ are bounded by a fixed multiple of θ in the support of a, it follows that we can fix A (independent of θ small) so that if (71) is valid, then

$$\left|\frac{\partial}{\partial x_2}\left(\psi(x;s,t)-\psi(x;s',t')\right)\right| \geq c\theta|t - t'|, on\ supp\ a,$$

for some uniform constant $c > 0$. Then since (56) implies that

$$\prod_{m=1}^{k}\left|\frac{\partial^{\alpha m}}{\partial x_2^{\alpha m}}(\Psi'_{x_2}(x;s,t) - \Psi'_{x_2}(x;s',t')\right| \leq C_k\theta^k|s-s',t-t'|^k, \text{on supp } a,$$

and since, by (59),

$$|\partial_{x_2}^j a| \leq C_N, 1 \leq j \leq N,$$

we conclude that if we repeat the argument just given but now integrate by parts with respect to x_2 instead of x_1, then $|K|$ is bounded by the second term on the right side of (64), which completes the proof of Lemma (5.1.6).

To see this, we note that by Lemma (5.1.5), if $(\chi_\lambda Q_\theta^{\mu})(x,y)$ denotes the kernel of $\chi_\lambda Q_\theta^{\mu}$, then

$$(\chi_\lambda Q_\theta^{\mu})(x,y)\left(\chi_\lambda Q_\theta^{\mu'}\right)(x,y')\overline{(\chi_\lambda Q_\theta^{\tilde{\mu}})(x,\tilde{y})\left(\chi_\lambda Q_\theta^{\tilde{\mu}'}\right)(x,\tilde{y}')} = O_N(\lambda^{-N}),$$
$$\text{if } x \notin \mathcal{T}_{C\theta}(\gamma_\mu)\cap\mathcal{T}_{C\theta}(\gamma_{\mu'})\cap\mathcal{T}_{C\theta}(\gamma_{\tilde{\mu}})\cap\mathcal{T}_{C\theta}(\gamma_{\tilde{\mu}'}),$$

with C sufficiently large and the geodesics defined by (37). On the other hand, if x is in the above intersection of tubes, then the condition on $(\mu,\mu',\tilde{\mu},\tilde{\mu}')$ in (34) ensures that if the constant C there is large enough, we have

$$\left|\nabla_x d_g(x,y) + d_g(x,y') - d_g(x,\tilde{y}) - d_g(x,\tilde{y}')\right| \geq c_0\theta,$$
$$if\ y \in \mathcal{T}_{C\theta}(\gamma_\mu), y' \in \mathcal{T}_C\theta(\gamma_{\mu'}), \tilde{y} \in \mathcal{T}_{C\theta}(\gamma_{\tilde{\mu}}), and\ \tilde{y}' \in \mathcal{T}_{C\theta}(\gamma_{\tilde{\mu}'}),$$

for some uniform $c_0 > 0$. Thus, (34) follows from Lemma (5.1.5) and a simple integration by parts argument since we are assuming that $\theta \geq \theta_0 = \lambda^{-1/2+\varepsilon_0}$ with $\varepsilon_0 > 0$.

Recall that for $\mathbb{T}^2$, Zygmund [60] showed that if e_λ is an eigenfunction on $\mathbb{T}^2$, i.e.,

$$e_\lambda(x) = \sum_{\{\varepsilon\in\mathbb{Z}^2:\,|\ell|=\lambda\}} a_\ell e^{ix\cdot\ell}, \qquad (72)$$

then

$$\|e_\lambda\|_{L^4(\mathbb{T}^2)} \leq C,$$

for some uniform constant C.

As observed in [5], using well-known pointwise estimates in two dimensions, one has

$$\sup_{\gamma\in\Pi}\int_\gamma |e_\lambda|^2\,ds = O_\varepsilon(\lambda^\varepsilon)$$

for all $\varepsilon > 0$. This of course implies that one also has

$$\sup_{\gamma \in \Pi} \int_{\mathcal{T}_{\lambda^{-1/2}}(\gamma)} |e_\lambda|^2 \, dx = O_\varepsilon(\lambda^{-1/2+\varepsilon})$$

for any $\varepsilon > 0$.

Sarnak (unpublished) made an interesting observation that having $O(1)$ geodesic restriction bounds for $\mathbb{T}^2$ is equivalent to the statement that there is a uniformly bounded number of lattice points on arcs of λS^1 of aperture $\lambda^{-1/2}$. (Cilleruelo and Córdoba [171] showed that this is the case for arcs of aperture $\lambda^{-1/2-\delta}$ for any $\delta > 0$.)

Using (1) we can essentially recover Zygmund's bound and obtain $\|e_\lambda\|_{L^4(\mathbb{T}^2)} = O_\varepsilon(\lambda^\varepsilon)$ for every $\varepsilon > 0$. (Of course this just follows from the pointwise estimate, but it shows how the method is natural too.)

If we could push the earlier results to include $\varepsilon_0 = 0$ and if we knew that there were uniformly bounded restriction bounds, then we would recover Zygmund's estimate.

Section (5.2): Eigenfunctions on Compact Riemannian Surfaces

For (M, g) be a two-dimensional compact boundaryless Riemannian manifold with Laplacian Δ_g. If e_λ are the associated eigenfunctions of $\sqrt{-\Delta_g}$ such that $-\Delta_g e_\lambda = \lambda^2 e_\lambda$, then it is well known that

$$\|e_\lambda\|_{L^4(M)} \le C \lambda^{\frac{1}{8}} \|e_\lambda\|_{L^2(M)}, \tag{73}$$

which was proved in [15] using approximate spectral projectors $\chi_\lambda = \chi(\lambda - \sqrt{-\Delta_g})$ and showing

$$\|\chi_\lambda f\|_{L^4(M)} \le C \lambda^{\frac{1}{8}} \|f\|_{L^2(M)}. \tag{74}$$

If $0 < \lambda \le \mu$ and e_λ, e_μ are two associated eigenfunctions of $\sqrt{-\Delta_g}$ as above, Burq et al. [3] proved the following bilinear L^2-refinement of (73)

$$\|e_\lambda e_\mu\|_{L^2(M)} \le C \lambda^{\frac{1}{4}} \|e_\lambda\|_{L^2(M)} \|e_\mu\|_{L^2(M)}, \tag{75}$$

as a consequence of a more general bilinear estimate on the reproducing operators

$$\|\chi_\lambda f \chi_\mu g\|_{L^2(M)} \le C \lambda^{\frac{1}{4}} \|f\|_{L^2(M)} \|g\|_{L^2(M)}. \tag{76}$$

The bilinear estimate (75) plays an important role in the theory of nonlinear Schrödinger equations on compact Riemannian surfaces and it is sharp in the case when $M = \mathbb{S}^2$ endowed with the canonical metric and $e_\lambda(x) = h_p(x), e_\mu(x) = h_q(x)$ are highest weight spherical harmonic functions of degree p and q, concentrating along the equator

$$\{x = (x_1, x_2, x_3) : x_1^2 + x_2^2 = 1, x_3 = 0\}$$

with $\lambda^2 = p(p+1), \mu^2 = q(q+1)$. Indeed, one may take $h_k(x) = (x_1 + ix_2)^k$ to see $\|h_k\|_2 \approx k^{-1/4}$ by direct computation.

We will construct a generic example to show the optimality of (76) and exhibit that the mechanism responsible for the optimality seems to be the existence of eigenfunctions concentrating along a tubular neighborhood of a segment of a geodesic. As observed in [16], (74) is saturated by constructing an oscillatory integral which highly concentrates along a geodesic. The dynamical behavior of geodesic flows on M accounts for the analytical properties of eigenfunctions exhibits the transference of mathematical theory from classical mechanics to quantum mechanics (see [27]).

That the eigenfunctions concentrating along geodesics yield sharp spectral projector inequalities leads naturally to the refinement of (73) in [36] and [28], where it is proved for an L^2 normalized eigenfunction e_λ, its L^4-norm is essentially bounded by a power of

$$\sup_{\gamma\in\Pi} \frac{1}{|T_{\lambda^{-1/2}}(\gamma)|} \int_{T_{\lambda^{-12}}(\gamma)} |e_\lambda(x)|^2\,dx, \tag{77}$$

where Π denotes the collection of all unit geodesics and $T_\delta(\gamma)$ is a tubular δ-neighborhood about the geodesic γ. This fact motivates the Kakeya–Nikodym maximal average phenomena measuring the size and concentration of eigenfunctions.

This result was refined by Blair and Sogge [170], proved for every $0 < \varepsilon \le 1/2$, there is a $C = C(\varepsilon, M)$ so that

$$\|e_\lambda\|_{L^4(M)} \le C\,\lambda^{\frac{1}{8}}\|e_\lambda\|^{\frac{1}{2}}_{L^2(M)} \times \left(\sup_{\gamma\in\Pi} \int_{T_{\lambda^{-\frac{1}{2}+\varepsilon}}(\gamma)} |e_\lambda(x)|^2 dx\right)^{\frac{1}{4}}. \tag{78}$$

We shall assume throughout that our eigenfunctions are L^2-normalized, but we shall formulate our main estimates as in (78) to emphasize the difference between the norms over all of M and over shrinking tubes.

As mentioned in [170] it would be interesting to see whether the ε-loss in (78) can be eliminated. Further results for higher dimensions are in [69] and [175]. These results played a crucial role in obtaining improved L^p eigenfunction estimates under certain curvature assumptions, see [162] and [30].

Inspired by [28], we are interested in the bilinear version of the main result in [28], searching for the essentially appropriate control of $\|e_\lambda e_\mu\|_2$ by means of Kakeya– Nikodym maximal averages. In fact, we will obtain a better result by establishing the microlocal version of Kakeya–Nikodym average in the spirit of [170], and our main result reads

Theorem (5.2.1)[174]: Assume $0 < \lambda \le \mu$ and e_λ, e_μ are two eigenfunctions of $\sqrt{-\Delta_g}$ associated to the frequencies λ and μ respectively. Then for every $0 < \varepsilon \le \frac{1}{2}$, we have a $C_\varepsilon > 0$ such that

$$\|e_\lambda e_\mu\|_{L^2(M)} \le C_\varepsilon \lambda^{\frac{\varepsilon}{2}} \|e_\mu\|_{L^2(M)} |||e_\lambda|||_{KN(\lambda,\varepsilon)}, \tag{79}$$

and

$$\|e_\lambda e_\mu\|_{L^2(M)} \le C_\varepsilon \lambda^{\frac{\varepsilon}{2}} \|e_\lambda\|_{L^2(M)} |||e_\mu|||_{KN(\lambda,\varepsilon)}, \tag{80}$$

where the Kakeya–Nikodym norm is defined by

$$|||f|||_{KN(\lambda,\varepsilon)} = \left(\sup_{\gamma\in\Pi} \lambda^{\frac{1}{2}-\varepsilon} \int_{T_{\lambda^{-\frac{1}{2}+\varepsilon}(\gamma)}} |f(x)|^2\,dx\right)^{\frac{1}{2}}. \tag{81}$$

Note also that we can reformulate our main estimates as follows

$$\|e_\lambda e_\mu\|_{L^2(M)} \le C_\varepsilon \lambda^{\frac{1}{4}} \|e_\mu\|_{L^2(M)} \times \left(\sup_{\gamma\in\Pi} \lambda^{\frac{1}{2}-\varepsilon} \int_{T_{\lambda^{-\frac{1}{2}+\varepsilon(\gamma)}}} |e_\lambda|^2\,dx\right)^{\frac{1}{2}}, \tag{82}$$

and

$$\|e_\lambda e_\mu\|_{L^2(M)} \le C_\varepsilon \lambda^{\frac{1}{4}} \|e_\lambda\|_{L^2(M)} \times \left(\sup_{\gamma\in\Pi} \lambda^{\frac{1}{2}-\varepsilon} \int_{T_{\lambda^{-\frac{1}{2}+\varepsilon}(\gamma)}} |e_\mu|^2 \, dx \right)^{\frac{1}{2}}, \quad (83)$$

both of which are bilinear variants of (78). Also, by taking the geometric means of (79) and (80) one of course has that

$$\|e_\lambda e_\mu\|_{L^2(M)} \le C_\varepsilon \lambda^{\frac{\varepsilon}{2}} \|e_\lambda\|_{L^2(M)}^{\frac{1}{2}} \|e_\mu\|_{L^2(M)}^{\frac{1}{2}} |||e_\lambda|||_{KN(\lambda,\varepsilon)}^{\frac{1}{2}} |||e_\mu|||_{KN(\lambda,\varepsilon)}^{\frac{1}{2}}. \quad (84)$$

Note that it is the geodesic tubes corresponding to the lower frequency that accounts for the optimal upper bound of $\|e_\lambda e_\mu\|_2$. We point out that in (80) one cannot take the $KN(\mu,\varepsilon)$-norm of e_μ. For on $\mathbb{T}^n \approx (-\pi,\pi]^n$ if $e_\lambda = e^{ij\cdot x}, |j| = \lambda$, and $e_\mu = e^{ik\cdot x}, |k| = \mu$, the analog of (80) involving $|||e_\mu|||_{KN(\mu,\varepsilon)}$ is obviously false for small $\varepsilon > 0$ if $\mu \gg \lambda$. Note also that if eμ is replaced by a subsequence, $e_{\mu_{j_k}}$ of quantum ergodic eigenfunctions (see [27]) then (80) implies that $\left\|e_\lambda e_{\mu_{j_k}}\right\|_{L^2(M)} \to 1$ as $\mu_{j_k} \to \infty$. This is another reason why it would be interesting to know whether the analog of (80) is valid with $\varepsilon = 0$ there.

We construct an example to show the sharpness of (76). We introduce some basic preliminaries and reduce the proof of Theorem (5.2.1) to the situation, where the strategy in [170] can be applied. We employ the orthogonality argument to conclude the theorem by assuming a specific bilinear oscillatory integral inequality. Finally, we prove this inequality based on the instrument in [3], which provides a bilinear version of Hörmander's oscillatory integral theorem [45]. We shall assume $0 < \lambda \le \mu$.

We construct an example showing the optimality of the universal bounds (76). We use approximate spectral projectors χ_λ and χ_μ which reproduce eigenfunctions and can be written as proper Fourier integral operators up to a smooth error.

We may assume the injectivity radius of M is sufficiently large. Take a Schwartz function $\chi \in S(\mathbb{R})$ with $\chi(0) = 1$ and χ supported, so that the spectral projectors are represented by

$$\chi_\lambda f(x) = \lambda^{1/2} T_\lambda f(x) + R_\lambda f(x), \chi_\mu g(x) = \mu^{1/2} T_\mu g(x) + R_\mu g(x),$$

where

$$\|R_\lambda f\|_{L^\infty(M)} \le C_N \lambda^{-N} \|f\|_{L^1(M)}, \|R_\mu g\|_{L^\infty(M)} \le C_N \mu^{-N} \|g\|_{L^1(M)},$$

for all $N = 1, 2, \ldots$, and the main terms read

$$T_\lambda f(x) = \int_M e^{i\lambda d_g(x,y)} a(x,y,\lambda) f(y)\, dy, \quad (85)$$

$$T_\mu g(x) = \int_M e^{i\mu d_g(x,z)} a(x,z,\mu) g(z) dz. \quad (86)$$

Here $d_g(x,y)$ is the geodesic distance between $x, y \in M$, and the amplitudes $a(x,y,\lambda), a(x,z,\mu) \in C^\infty$ have the following property

$$\left|\partial_{x,y}^\alpha a(x,y,\lambda)\right| + \left|\partial_{x,z}^\alpha a(x,z,\mu)\right| \le C_\alpha, \text{for all } \alpha.$$

Moreover $a(x,y,\lambda) = 0$ if $d_g(x,y) \notin (1,2)$ and likewise for $a(x,z,\mu)$. (See [16].)

After applying a partition of unity, for small δ fixed, we may fix three points $x_0, y_0, z_0 \in M$ with $1 \le d_g(x_0,y_0) \le 2, 1 \le d_g(x_0,z_0) \le 2$, and assume that $a(x,y,\lambda)$ vanishes outside the region $\{(x,y)|x \in B(x_0,\delta), y \in B(y_0,\delta)\}$, $a(x,z,\mu)$

vanishes outside the region $\{(x,z)|x \in B(x_0,\delta), z \in B(z_0,\delta)\}$. To see the sharpness of (76), we will prove the following result.

We will choose suitable f and g concentrating along a segment of the geodesic γ_0 connecting x_0 and y_0 with appropriate oscillations. The explicit expression of f and g will yield automatically upper bounds on $\|f\|_2 \, \|g\|_2$. On the other hand, we will see there is a strip region Ω_μ containing x_0 such that $\|T_\lambda f T_\mu g\|_{L^2(\Omega_\mu)}$ is bounded below by $(\lambda_\mu)^{-1/2}$ times the upper bound of $\lambda^{\frac{1}{4}} \|f\|_2 \|g\|_2$.

Recall first the geodesic normal coordinate centered at y_0. Let $\{e_1,e_2\}$ be the orthonormal basis in $T_{y_0}M$ such that e_1 is the tangent vector of γ_0, pointing to x_0. The exponential map $\exp_{y_0}$ is a smooth diffeomorphism between the ball $\{Y \in T_{y_0}M : Y = Y_1e_1 + Y_2e_2, |Y| < 10\}$ and $B(y_0,10)$. Let $\{\omega_1,\omega_2\}$ be the dual basis of $\{e_1,e_2\}$ and set $y_j = \omega_j \circ \exp_{y_0}^{-1}$ for $j = 1,2$. Then $\{y_1,y_2\}$ is the Riemannian geodesic normal coordinates such that $y_0 = 0$ and

$$\begin{cases} g_{ij}(0) = \delta_{ij}, \\ dg_{ij}(0) = 0, \end{cases} \text{for all } i,j = 1,2.$$

In particular, $\Gamma_{ij}^k(0) = 0, \forall\, i,j,k = 1,2$, and $dG(0) = 0$ with $G = det(g_{ij})$. In this coordinate system, γ_0 is parameterized by $t \to \{(t,0)\}$.

Lemma (5.2.2)[174]: If we denote by $\phi(x,y) = d_g(x,y)$, then in these coordinates $\phi(x,0) = |x|$. Moreover, if we set $x = (x_1,x_2), y = (y_1,y_2)$ and assume $0 < y_1 < x_1$, then $\phi(x,y) = x_1 - y_1 + O((x^2 - y^2)^2)$.

Proof. See p. 144 in [16].

With Lemma (5.2.2) at hand, we are ready to prove Proposition (5.2.3).

Proposition (5.2.3)[174]: There exist f and g such that for some $C > 0$,

$$\|T_\lambda f\, T_\mu g\|_{L^2} \geq C\,\lambda^{-1/4}\mu^{-1/2}\|f\|_{L^2}\,\|g\|_{L^2}. \tag{87}$$

Proof. We work in the above coordinates and let

$$\Omega_\mu = \{x : \delta/2C_0 \leq x_1 \leq 2C_0\delta, |x_2| \leq \varepsilon_1\mu^{-1/2}\}, 0 < \varepsilon_1\,\delta,$$

where $C_0 > 0$ is chosen as on p. 144 in [16]. The region Ω_λ is defined similarly. Take $\alpha \in C_0^\infty(-1,1)$ and set

$$f(y) = \alpha(y_1/\varepsilon_1)\alpha\left(\lambda^{\frac{1}{2}}\, y_2/\varepsilon_1\right)e^{i\lambda y_1}, \tag{88}$$

$$g(z) = \alpha(z_1/\varepsilon_1)\alpha\left(\mu^{\frac{1}{2}}\, z_2/\varepsilon_1\right)e^{i\mu z_1}. \tag{89}$$

Denote by $\epsilon = \lambda/\mu$. Then similar to Chapter 5 in [16], we estimate

$$\Omega_\mu \left|T_\lambda f(x) T_\mu g(x)\right|^2 dx.$$

Indeed, for $x \in \Omega_\mu$, we have

$$|T_\lambda f(x)|^2 = \iint_{\Omega_\lambda^2} \overline{e^{i\lambda(d_g(x,y)-d_g(x,y')-[(x_1-y_1)-(x_1-y_1')])}\, a(x,y,\lambda)\alpha(y_1/\varepsilon_1)\alpha\left(\lambda^{\frac{1}{2}}\, y_2/\varepsilon_1\right)}$$

$$\times\; a(x,y',\lambda)\,\alpha(y_1/\varepsilon_1)\,\alpha(\lambda^{\frac{1}{2}}\, y_2'/\varepsilon_1)\,dydy.$$

Notice that by Lemma (5.2.2), the phase function equals $O(|x_2 - y_2|^2) + O(|x_2 - y_2'|^2)$. Since $|x_2| \leq \varepsilon_1\mu^{-\frac{1}{2}}$ and $|y_2|,|y_2'| \leq \varepsilon_1\lambda^{-\frac{1}{2}}$, we see that the phase in the exponent is of order ε_1^2 on Ω_μ, and the oscillation is eliminated in the integrand by choosing ε_1 small. Thus on Ω_μ

$$|T_\lambda f(x)|^2\,|\Omega_\lambda|^2 \;=\; \lambda^{-1}.$$

Similarly,

$$|T_\lambda g(x)|^2\,\left|\Omega_\mu\right|^2 \;=\; \mu^{-1},$$

Thus, $\left\|T_\lambda f T_\mu g\right\|_{L^2(\Omega_\mu)}$ is bounded below by $\mu^{-\frac{3}{4}}\lambda^{-\frac{1}{2}}$. On the other hand, $\|f\|_2\|g\|_2 \le c\,(\lambda_\mu)^{-\frac{1}{4}}$ for f and g given by (88) and (89), we have

$$(\lambda_\mu)^{\frac{1}{2}}\left\|T_\lambda f\, T_\mu g\right\|_2/(\,\|f\|_2\,\|g\|_2\,) \;\ge\; C\varepsilon_1\,\lambda^{1/4}.$$

This example exhibits the concentration of eigenfunctions along a tubular neighborhood of a geodesic leading to the sharpness of the bilinear spectral projector estimate (76), where our bilinear generalization of the main result in [28] is motivated.

Remark (5.2.4)[174]: Comparing this example with (84), one may suspect that (84) can be further refined. Indeed, one may observe that the example suggests the possibility of refining (84) by strengthening the L^2-norm of the eigenfunction corresponding to the higher frequency on the right side to a $\lambda^{-\frac{1}{2}}$ -neighborhood of the same geodesic segment for the lower frequency eigenfunction. An interesting problem would be to see if the following refinement of (84) is valid:

$$\left\|e_\lambda e_\mu\right\|_{L^2(M)} \le\; C\varepsilon_0\,\lambda^{\frac{1}{4}}\sup_{\gamma\in\Pi}\left[\left(\int_{T_{\lambda^{-\frac{1}{2}+\varepsilon_0}(\gamma)}}\left|e_\mu(x)\right|^2 dx\right)\left(\int_{T_{\lambda^{-\frac{1}{2}+\varepsilon_0}(\gamma)}}|e_\lambda(x)|^2\,dx\right)\right]^{\frac{1}{4}}. \qquad (90)$$

In view of $\chi_\lambda e_\lambda \;=\; e_\lambda$ and $\chi_\mu e_\mu \;=\; e_\mu$, we are reduced to estimating $\left\|T_\lambda f T_\mu g\right\|_{L^2}$. By scaling, we may assume the injectivity radius of M is large enough, say inj $M \;>\; 10$. We use partitions of unity on M to reduce the L^2 integration of $T_\lambda f T_\mu g$ on the geodesic ball $B(x_0,\delta)$ with $\delta \;>\; 0$ small. In view of the property of supp a, we may apply partition of unity once more and assume supp $f \;\subset\; B(y_0,\delta)$ and supp $g \;\subset\; B(z_0,\delta)$ for some y_0 and z_0 satisfying

$$1 \;\le\; d_g(x_0,y_0), d_g(x_0,z_0) \;\le\; 2.$$

Next, we need to choose a suitable coordinate system to simplify the calculations on a larger ball $B(x_0,10)$. As in [28] and [175], we shall use Fermi coordinate system about the geodesic γ connecting x_0 and y_0. Let $\gamma_\perp$ be the geodesic through x_0 perpendicular to γ. The Fermi coordinates about γ is defined on the ball $B(x_0,10)$, where the image of $\gamma_\perp \,\cap\, B(x_0,10)$ in the resulting coordinate system is parameterized by $s \,\to\, \{(s,0)\}$. All the horizontal segments are parameterized by $s \;\to\; (s,t_0)$ and we have

$$d_g\big((s_1,t_0),(s_2,t_0)\big) \;=\; |s_1 \;-\; s_2|.$$

Clearly, in our coordinate system, y_0 is on the 2nd coordinate axis, and z_0 is a point satisfying $1 \;\le\; d_g\big(z_0,(0,0)\big) \;\le\; 2$.

Therefore, if we set $y \;=\; (s,t), z \;=\; (s',t')$ in this coordinate system, we may write $T_\lambda f$ and $T_\mu g$ locally as

$$T_\lambda f(x) = \int_{\mathbb{R}^2} e^{i\lambda d_g(x,(s,t))} a(x,(s,t),\lambda)\, f(s,t)\, dsdt, \tag{91}$$
$$T_\mu g(x) = \int_{\mathbb{R}^2} e^{i\mu d_g(x,(s,t))} a(x,(s',t'),\mu)\, g(s',t') ds\, dt\,. \tag{92}$$
Moreover, by noting that $1 \le d_g(x_0,y_0), d_g(x_0,z_0) \le 2$ and $y \in B(y_0,\delta), z \in B(z_0,\delta)$, we shall assume
$$\max\{|s|, |t - d_g(y_0,x_0)|, |dg((s',t'),z_0)|\} \le \delta.$$
We remark that we are at liberty to take δ to be small when necessary.

We deal with the case when the angle between γ and the geodesic γ' connecting x_0 and z_0 is bounded below by some $\varepsilon_2 > 0$. To do this, we shall use the geodesic normal coordinates around x_0. Set $\{e_1, e_2\}$ to be the orthonormal basis in $T_{x_0}M$, where the metric g at x_0 is normalized, such that e1 is the tangent vector of $\gamma_\perp$ at x_0 and $-e_2$ is the tangent vector of γ at x_0 if γ is oriented from x_0 to y_0. Let $\{\omega_1,\omega_2\}$ be the dual basis of $\{e_1,e_2\}$ and set $\{x_j = \omega_j \circ \exp_0^{-1}\} j = 1,2$ to be the Riemannian geodesic normal coordinate system on $B(x_0, 10)$, where $x_0 = 0$ and γ is parameterized by $x_2 \to \{(0,x_2)\}$, whereas $\gamma^\perp$ is parameterized by $x_1 \to \{(x_1,0)\}$ with $|x_1| \le 5$. Let $\theta_0 = \theta(z_0)$ be such that $z_0 = d_g(x_0,z_0)(\cos\theta_0, \sin\theta_0)$, where the angular variable is oriented in clockwise direction. It follows that $\gamma'^\perp$ is given by $r \to \exp_0((r\cos\varphi_0, r\sin\varphi_0))$ with $\varphi_0 = \theta_0 + \frac{\pi}{2}$ and $|r| < 5$.

Writing
$$y = (r_1\cos\theta_1, r_1\sin\theta_1), \qquad z = (r_2\cos\theta_2, r_2\sin\theta_2) \tag{93}$$
in geodesic normal coordinates, we have
$$T_\lambda f(x) = \iint e^{i\lambda d_g(x,(r_1,\theta_1))} a(x,(r_1,\theta_1),\lambda)\, f(r_1,\theta_1)\, dr_1 d\theta_1, \tag{94}$$
$$T_\mu g(x) = \iint e^{i\mu d_g(x,(r_2,\theta_2))} a(x,(r_2,\theta_2),\mu)\, g(r_2,\theta_2)\, dr_2 d\theta_2. \tag{95}$$
We recall the following fact.

Proposition (5.2.5)[174]: Let $\varepsilon_2 > 0$ be a small parameter. Assume $\left|\theta(z_0) + \frac{\pi}{2}\right| \ge \varepsilon_2$ and $\left|\theta(z_0) - \frac{\pi}{2}\right| \ge \varepsilon_2$. If we choose δ small enough depending on ε_2, there exists C such that
$$\|T_\lambda f\, T_\mu g\|_2 \le C(\lambda_\mu)^{-12}\|f\|_2\|g\|_2. \tag{96}$$
Thus in order to prove Theorem (5.2.1), it suffices to consider either $\left|\theta(z_0) + \frac{\pi}{2}\right| \le \varepsilon_2$ or $\left|\theta(z_0) - \frac{\pi}{2}\right| \le \varepsilon_2$. This confines z_0 in a small neighborhood of the geodesic γ by compressing γ and γ' to be almost parallel with each other.

Essentially, this proposition is proved in [3] based on the following lemma.

Lemma (5.2.6)[174]: Let $y = \exp_0(r(\cos\theta, \sin\theta))$ and $\phi_r(x,\theta) = d_g(x,y)$. For every $0 < \varepsilon_2 < 1$, there exists $c > 0, \delta_1 > 0$ such that for every $|x| < \delta_1$,
$$|\det(\nabla_x\partial_\theta\phi_r(x,\theta), \nabla_x\partial_\theta\phi_r(x,\theta)| \ge c, \tag{97}$$
if $|\theta - \theta'| \ge \varepsilon_2$ and $|\theta + \pi - \theta| \ge \varepsilon_2$. In addition, for every $\theta \in [0,2\pi]$,
$$\left|\det\nabla_x\partial_\theta\phi r(x,\theta), \nabla_x\partial_\theta^2\phi_r(x,\theta)\right| \ge c. \tag{98}$$
This is an immediate consequence of the following fact.

Lemma (5.2.7)[174]: Let $y \to \kappa(y) = \exp_0^{-1}(y)$ be the geodesic normal coordinates vanishing at x_0, as described above. Then we have

$$\nabla_x d_g(x,y) \ |_{x=x_0} = \kappa(y)/|\kappa(y)|. \tag{99}$$

Proof. Relation (99) is equivalent to Gauss' lemma. See [28] and [3].
The map $y \to \kappa(y)$ is a local radial isometry. See [28].

We sketch the proof of Proposition (5.2.5) briefly for completeness. In our situation, we have $(y_0) = -\frac{\pi}{2}$. Fixing a parameter $\varepsilon_2 > 0$, we assume $\left|\theta(z_0) + \frac{\pi}{2}\right| \geq \varepsilon_2$ and $\left|\theta(z_0) - \frac{\pi}{2}\right| \geq \varepsilon_2$. Since $y \in B(y_0,\delta), z \in B(z_0,\delta)$ given by (93), we may choose δ 0. By Schur's test, it suffices to show

$$K(\theta_1,\theta_2,\theta_1',\theta_2') \leq C(\mu|\theta_2 - \theta_2'| + \lambda|\theta_1 - \theta_1'|) - 10, \tag{100}$$

where

$$K(\theta_1,\theta_2,\theta_1',\theta_2') = \int e^{i\Psi_{\lambda,\mu}(x;\,\theta_1,\theta_2,\theta_1',\theta_2')} A(x;\,\theta_1,\theta_1',\theta_2,\theta_2')\,dx,$$

$$A(x;\,\theta_1,\theta_1',\theta_2,\theta_2') = a(x,(r_1,\theta_1),\lambda)\overline{a(x,(r_1,\theta_1'),\lambda)}a(x,(r_2,\theta_2),\mu)\overline{a(x,(r_2,\theta_2'),\mu)},$$

$$\Psi_{\lambda,\mu}(x;\,\theta_1,\theta_2,\theta_1',\theta_2') = \lambda\left(\phi_{r_1}(x,\theta_1) - \phi_{r_1}(x,\theta_1')\right) + \mu\left(\phi_{r_2}(x,\theta_2) - \phi_{r_2}(x,\theta_2')\right).$$

For all multi-index $\alpha, |\alpha| \leq 10$, Lemma (5.2.6) and the above formula give

$$|\nabla_x\Psi_{\lambda,\mu}| \geq C(\lambda|\theta_1 - \theta_1'| + \mu|\theta_2 - \theta_2'|), |\partial_x^\alpha \Psi_{\lambda,\mu}| \leq C(\lambda|\theta_1 - \theta_1'| + \mu|\theta_2 - \theta_2'|).$$

Now (100) follows from integration by parts.

We will employ the strategy introduced by [170] (see also [48]), where a microlocal refinement of Kakeya–Nikodym averages are exploited. From now on, we shall always assume

$$\left|\theta(z_0) + \frac{\pi}{2}\right| \leq \varepsilon_2 \ll 1$$

where $\frac{\pi}{2} = -\theta(y_0)$. Recall that we may write, modulo trivial errors,

$$\chi_\lambda f(x) \approx \lambda^{\frac{1}{2}} \int_{\mathbb{R}^2} e^{i\lambda d_g(x,y)} a_\lambda(x,y) f(y)dy, \tag{101}$$

$$\chi_\mu g(x) \approx \mu^{\frac{1}{2}} \int_{\mathbb{R}^2} e^{i\mu d_g(x,z)} a_\mu(x,z)\, g(z)\,dz, \tag{102}$$

with supp $f \subset B(y_0,\delta)$, supp $g \subset B(z_0,\delta)$ and $x \in B(0,\delta)$.

We may choose $\varepsilon_2 > 0$ sufficiently small to make z_0 to be within an fixed small neighborhood of γ.

To decompose the phase space, we shall use the geodesic flow $\Phi_\tau(y,\xi)$ on the cosphere bundle S^*M, which starts from y in direction of $\xi \in S_y^*M$. We use the Fermi coordinates around γ to write

$$y(\tau), \xi(\tau) = \Phi_\tau(y,\xi), \big(y(0),\xi(0)\big) = (y,\xi),$$

where $\xi(\tau)$ is the unit cotangent vector in $T_y^*(\tau)M$. Define $\Theta : (y,\xi) \in S^*M \to \mathbb{R} \times \mathbb{R}$ by

$$\Theta(y,\xi) = \left(\Pi_{y_1}\Phi_{\tau_0}(y,\xi), \frac{\Pi_{\xi_1}\Phi_{\tau_0}(y,\xi)}{|\Pi_\xi\Phi_{\tau_0}(y,\xi)|}\right),$$

where τ_0 is chosen so that $y_2(\tau_0) = \Pi_{y_2}\Phi_{\tau_0}(y,\xi) = 0$. By $\Pi_\blacklozenge$, we mean the projection to the component of ♦-variable.

Remark (5.2.8)[174]: As in [170], we require $|\xi_1| < \delta$ with δ small enough with $y \in B(y_0, C_0\delta)$. Moreover, Θ is constant on the orbit of Φ and $|\Theta(y,\xi) - \Theta(z,\eta)|$ can be used as a natural distance function between geodesics passing respectively through (y,ξ) and (z,η). Next, we microlocalize $\chi_\lambda f$ and $\chi_\mu g$ by introducing smooth functions $\alpha_1(y)$ and $\alpha_2(z)$ adapted respectively to the ball $B(y_0,2\delta)$ and $B(z_0,2\delta)$ and setting

$$Q_\theta^\nu(y,\xi) = \alpha_1(y)\,\beta(\theta^{-1}\Theta(y,\xi) + \nu)\,\Upsilon(|\xi|/\lambda) \quad (103)$$

$$P_\theta^\upsilon(z,\eta) = \alpha_2(z)\beta(\theta^{-1}\Theta(z,\eta) + \upsilon)\,\Upsilon(|\eta|/\mu) \quad (104)$$

where $\lambda^{-1/2} \le \theta \le 1, \nu, \upsilon \in \mathbb{Z}^2$, with β smooth such that

$$\sum_{\nu\in\mathbb{Z}^2} \beta(\cdot + \nu) = 1, \operatorname{supp}\beta \subset \{x \in \mathbb{R}^2 : |x| \le 2\}, \quad (105)$$

and $\Upsilon \in C_0^\infty(\mathbb{R})$ is supported in $[c, c-1]$ for some $c > 0$. Let us take a look at the symbols $Q_\theta^\nu(y,\xi)$ and $P_\theta^\upsilon(z,\eta)$. First, we define $\beta(\theta^{-1}\Theta(y,\xi) + \nu)$ and $\beta(\theta^{-1}\Theta(z,\eta) + \upsilon)$ on the cosphere bundle. Since these two functions are of degree zero in the cotangent variables, we then extend them homogeneously to the cotangent bundle. The above $Q_\theta^\nu(y,\xi)$ and $P_\theta^\upsilon(z,\eta)$ are well-defined for $\xi \ne 0, \eta \ne 0$. Given $\xi, \beta(\theta^{-1}\Theta(y,\xi) + \nu) = 0$ unless y belongs to a tubular neighborhood of γ_ν, where

$$\gamma_\nu = \{y(\tau) : -2 \le \tau \le 2, (y(\tau), \xi(\tau)) = \Phi_\tau(y,\xi), \Theta(y,\xi) + \theta_\nu = 0\}.$$

Moreover, if we set $\nu = (\nu_1,\nu_2)$, the direction of γ_ν at $y(\tau_0)$ is determined by $\theta\nu_2$ and is independent of λ. Since $(y,\xi) = \Phi_{\tau_0}^{-1}(y(\tau_0),\xi(\tau_0))$ and $y(\tau_0) = (y_1(\tau_0),0)$ with $y_1(\tau_0) = \theta\nu_1 + O(\theta)$, one easily finds that $y \in TC_1\theta(\gamma_\nu)$, for some $C_1 \ge 1$. Similar statements hold for $P_\theta^\upsilon(z,\eta)$.

Let $Q_\theta^\nu(x,D), P_\theta^\upsilon(x,D)$ be the pseudo-differential operators associated to the symbols defined in (103) and (104) respectively. We next record some properties of $Q_\theta^\nu(y,D)$ and $P_\theta^\upsilon(z,D)$. The first lemma indicates that these two kinds of operators provide a natural microlocal wave-packet decomposition in the phase space for 2-dimensional manifolds.

Lemma (5.2.9)[174]: If $\lambda^{-1/2+\varepsilon} \le \theta \le 1$ with $\varepsilon > 0$ fixed, the symbols Q_θ^ν and P_θ^υ belong to a bounded subset of $S^0_{1/2+\varepsilon,1/2-\varepsilon}$. Then there is C_ε and $C_2 \ge C_1$ such that for $\lambda^{-1/2+\varepsilon} \le \theta \le 1$, we have

$$\|Q_\theta^\nu(x,D)f\|_{L^2} \le C_\varepsilon\|f\|_{L^2\left(T_{C_2\theta}(\gamma_\nu)\right)} + C_N\,\lambda^{-N}\,\|f\|_2 \quad (106)$$

$$\|P_\theta^\upsilon(x,D)g\|_{L^2} \le \|C_\varepsilon g\|_{L^2\left(T_{C_2\theta}(\gamma_\nu)\right)} + C_N\,\mu^{-N}\,\|g\|_2. \quad (107)$$

Moreover, for any integer $N \ge 0$, one may write

$$\chi_\lambda f = \sum_{\nu\in\mathbb{Z}^2} \chi_\lambda \circ Q_\theta^\nu(x,D)\,f + R_\lambda f, \text{if } \operatorname{supp} f \subset B(y_0,\delta), \quad (108)$$

$$\chi_\mu g = \sum_{\nu\in\mathbb{Z}^2} \chi_\mu \circ P_\theta^\upsilon(x,D)\,g + R_\mu g, \text{if } \operatorname{supp} g \subset B(z_0,\delta), \quad (109)$$

with $\|R_\lambda\|_{L^2\to L^\infty} \lesssim \lambda^{-N}, \|R_\mu\|_{L^2\to L^\infty}\,\mu^{-N}$.

Proof. That $Q_\theta^\nu(y,\xi) \in S^0_{1/2+\varepsilon,1/2-\varepsilon}$ has already been proved in [170]. If we use $\mu \ge \lambda$, we get $\mu^{-1}\lambda^{1/2-\varepsilon} \le \mu^{-1/2-\varepsilon}$, and the same calculation as for Q_θ^ν yields that the $P_\theta^\upsilon(z,D)$ belong to a bounded subset of pseudodifferential operators of order zero and type $(1/2+\varepsilon, 1/2-\varepsilon)$. To see (106), one observes that the kernel $K_\theta^\nu(x,y)$ of the operator Q_θ^ν is bounded by $O(\lambda^{-N})$ if y does not belong to $T_{C_2\theta}(\gamma_\nu)$ for some large $C_2 > C_1$ by using integration by parts. We can deduce (108) from (105). In fact, if we recall the process of constructing parametrix for the half wave operator $e^{it\sqrt{-\Delta_g}}$ in [16], we may use integration by parts to see that in (108), one may assume f

$(\xi) = 0$ if $|\xi| \notin [c_\lambda, C_\lambda]$ up to some terms of the form $R_\lambda f$. It suffices to see the difference of $f(x)$ and $\sum_\nu Q_\theta^\nu(x,D)f(x)$ is of the form $R_\lambda f(x)$. This is easy due to the fact that $\Upsilon(|\xi|/\lambda) = 1$ on the support of f by choosing suitable c, C and $\big(1 - \alpha_1(x)\big)f(x) = 0$. Now (105) yields

$$\alpha_1(x)f(x) = \sum_\nu Q_\theta^\nu(x,D)f(x).$$

Similar argument yields (107) and (109).

Now, we recall the microlocal Kakeya–Nikodym norm in [170], corresponding to frequency λ and $\theta_0 = \lambda^{-1/2+\varepsilon_0}$

$$|||f|||_{MKN(\lambda,\varepsilon_0)} = \sup_{\theta_0 \le \theta \le 1} \left(\sup_{\nu \in \mathbb{Z}^2} \theta^{-1/2} \|Q_\theta^\nu(x,D)f\|_{L^2(\mathbb{R}^2)} \right) + \|f\|_{L^2(\mathbb{R}^2)}. \quad (110)$$

As pointed out in [170], the maximal microlocal concentration of f about all unit geodesics in the scale of θ amounts to the quantity

$$\sup_{\nu \in \mathbb{Z}^2} \theta^{-1/2} \|Q_\theta^\nu(x,D)f\|_{L^2(\mathbb{R}^2)}.$$

From Lemma (5.2.9), one can prove $|||f|||_{MKN(\lambda,\varepsilon_0)} \le C_{\varepsilon_0} |||f|||_{KN(\lambda,\varepsilon_0)}$. We refer to [170] for more details. Similarly, for the same θ_0, we can define

$$|||g|||_{MKN(\lambda,\varepsilon_0)} = \sup_{\theta_0 \le \theta \le 1} \left(\sup_{\nu \in \mathbb{Z}^2} \theta^{-1/2} \|P_\theta^\nu(x,D)g\|_{L^2(\mathbb{R}^2)} \right) + \|g\|_{L^2(\mathbb{R}^2)}, \quad (111)$$

again by Lemma (5.2.9), we see that $|||g|||_{MKN(\lambda,\varepsilon_0)} \le C_{\varepsilon_0} |||g|||_{KN(\lambda,\varepsilon_0)}$.

Lemma (5.2.10)[174]: For any $\varepsilon > 0$, there exists some $C_\varepsilon > 0$ such that for all $\lambda^{-1/2+\varepsilon} \le \theta \le 1$,

$$\left\| \sum_\nu (Q_\theta^\nu)^* \circ Q_\theta^\nu f \right\|_{L^2} \le C_\varepsilon \|f\|_{L^2}, \qquad \left\| \sum_\upsilon (P_\theta^\upsilon)^* \circ P_\theta^\upsilon g \right\|_{L^2} \le C_\varepsilon \|g\|_{L^2}. \quad (112)$$

Proof. The L^2-estimates (112) are valid thanks to (105) and the classical calculus of pseudo-differential operators of type $(1/2 + \varepsilon, 1/2 - \varepsilon)$ with $\varepsilon > 0$.

We describe next the kernels of the operators $\chi_\lambda Q_\theta^\nu := (\chi_\lambda \circ Q_\theta^\nu)(x,D)$ and $\chi_\mu P_\theta^\upsilon := (\chi_\mu \circ P_\theta^\upsilon)(x,D)$ following [170].

Lemma (5.2.11)[174]: Denote by $(\chi_\lambda Q_\theta^\nu)(x,y)$ and $\big(\chi_\mu P_\theta^\upsilon\big)(x,z)$ the kernels of the pseudodifferential operators $\chi_\lambda Q_\theta^\nu(x,D)$ and $\chi_\mu P_\theta^\upsilon(x,D)$ respectively. Assume $\theta \in [C_0\theta_0, 1]$ with $\theta_0 = \lambda^{-1/2+\varepsilon}$ and $C_0 \gg 1$. We can find a uniform constant C so that for each $= 1, 2, 3, \ldots$, we have

$$|(\chi_\lambda Q_\theta^\nu)(x,y)| \le C_N \lambda^{-N}, \text{if } x \notin T_{C_\theta}(\gamma_\nu) \text{or } y \notin T_{C_\theta}(\gamma_\nu), \quad (113)$$

and

$$\big|(\chi_\mu P_\theta^\upsilon)(x,z)\big| \le C_N \mu^{-N}, \text{if } x \notin T_{C_\theta}(\gamma_\upsilon) \text{ or } z \notin T_{C_\theta}(\gamma_\upsilon). \quad (114)$$

Furthermore,

$$(\chi_\lambda Q_\theta^\nu)(x,y) = \lambda^{\frac{1}{2}} e^{i\lambda d_g(x,y)} a_{\nu,\theta}(x,y) + O_N(\lambda^{-N}), \quad (115)$$

$$(\chi_\mu P_\theta^\upsilon)(x,z) = \mu^{\frac{1}{2}} e^{i\mu d_g(x,z)} b_{\upsilon,\theta}(x,z) + O_N(\mu^{-N}), \quad (116)$$

where we have the uniform bounds

$$\big|(\nabla_x^\perp)^\alpha a_{\nu,\theta}(x,y)\big| \le C_\alpha \theta^{-|\alpha|}, |(\nabla_x^\perp)^\alpha b_{\upsilon,\theta}(x,z)| \le C_\alpha \theta^{-|\alpha|}, \quad (117)$$

and

$$\big|\partial_t^j a_{\nu,\theta}\big(x, x_\nu(t)\big)\big| \le C_j, x \in \gamma_\nu = \{x_\nu(t)\}, \quad (118)$$

$$\left|\partial_t^\ell b_{\upsilon,\theta}\big(x,x_\upsilon(t)\big)\right| \le C, x \in \gamma_\upsilon = \{x_\upsilon(t)\}, \tag{119}$$

where $\nabla_x^\perp$ denotes the directional derivative along the direction perpendicular to the geodesics $\{x_\nu(t)\}$ with $\nu = \nu$ or υ and

$\gamma_\nu = \{z_\nu(\tau): -2 \le \tau \le 2, (z_\nu(\tau), \eta_\nu(\tau)) = \Phi_\tau(z_\nu,\eta_\nu), \theta^{-1}\Theta(z_\nu,\eta_\nu) + \nu = 0\}$.

Proof. The properties for $(\chi_\lambda Q_\theta^\nu)(x,y)$ are exactly the same as in [170], and the proof is identical to that of Lemma (5.2.6) in [170]. Since $\theta \ge \mu^{-\frac{1}{2}+\varepsilon}$, the properties for $(\chi_\lambda P_\theta^\upsilon)(x,z)$ follows from the same proof.

We have the following.

Lemma (5.2.12)[174]: Assume $\theta \ge \theta_0$ and N_1 is fixed. Then there exists $C_0 \gg 1$, when $|\nu - \tilde{\nu}| + |\upsilon - \tilde{\upsilon}| \ge C_0$ and $|\nu - \upsilon|, |\tilde{\nu} - \tilde{\upsilon}| \le N_1$, we have

$$\left|\iint \chi_\lambda Q_\theta^\nu h_1(x)\chi_\mu P_\theta^\upsilon\, h_2(x)\overline{\chi_\lambda Q_\theta^{\tilde{\nu}} h_3(x)}\;\overline{\chi_\mu P_\theta^{\tilde{\upsilon}}\, h_4(x)}\,dx\right| \le C_N\,\mu^{-N}\prod_{j=1}^{4}\|h_j\|_2.$$

Proof. To get $O_N(\mu^{-N})$ decay as claimed, we need to split into two cases depending on the size of μ. Assume first $\mu \ge \lambda^2$.

It suffices to consider the kernel

$$K(y,z,\tilde{y},\tilde{z}) = \int \chi_\lambda Q_\theta^\nu(x,y)\chi_\mu P_\theta^\upsilon(x,z)\chi_\lambda Q_\theta^{\tilde{\nu}}(x,\tilde{y})\,\chi_\mu P_\theta^{\tilde{\upsilon}}(x,\tilde{z})dx.$$

Indeed, by Lemma (5.2.11), up to a $O_N(\mu^{-N})$ error, we can restrict the domain of integration here to $\Omega = T_{C_\theta}(\gamma_\upsilon) \cap T_{C_\theta}(\gamma_{\tilde{\upsilon}})$.

Plugging (116) into the expression of $K(y,z,\tilde{y},\tilde{z})$, we get

$$K(y,z,\tilde{y},\tilde{z}) = \mu\int_\Omega b(x,y,z,\tilde{y},\tilde{z})e^{i\mu\left(d_g(x,z)-d_g(x,\tilde{z})\right)}dx + O_N(\mu^{-N}),$$

where

$$b(x,y,z,\tilde{y},\tilde{z}) = \chi_\lambda Q_\theta^\nu(x,y)\chi_\lambda Q_\theta^{\tilde{\nu}}(x,\tilde{y})\,b_{\upsilon,\theta}(x,z)b_{\tilde{\upsilon},\theta}(x,\tilde{z}).$$

It is easy to see that $b(x,y,z,\tilde{y},\tilde{z})$ satisfies

$$|\nabla_x^\alpha b(x,y,z,\tilde{y},\tilde{z})| \le C\lambda^{|\alpha|+1}.$$

Now we consider the phase function

$$\mu\left(d_g(x,z) - d_g(x,\tilde{z})\right).$$

The gradient reads

$$\mu\nabla_x\left(d_g(x,z) - d_g(x,\tilde{z})\right).$$

We claim that for C_0 big enough, there exists some $c_0 > 0$, such that

$$\left|\nabla_x\left(d_g(x,z) - d_g(x,\tilde{z})\right)\right| \ge c_0\theta,$$

then our lemma follows from simple integration by parts argument.

Indeed, since $x \in T_{C_\theta}(\gamma_\upsilon) \cap T_{C_\theta}(\gamma_{\tilde{\upsilon}}), z \in T_{C_\theta}(\gamma_\upsilon)$ and $\tilde{z} \in T_{C_\theta}(\gamma_{\tilde{\upsilon}})$, we see that

$$\left|\nabla_x\left(d_g(x,z) - d_g(x,\tilde{z})\right)\right| |\upsilon - \tilde{\upsilon}|\theta,$$

noticing that

$$|\upsilon - \tilde{\upsilon}| \ge |\nu - \tilde{\nu}| - |\nu - \upsilon| - |\tilde{\nu} - \tilde{\upsilon}| \ge |\nu - \tilde{\nu}| - 2N_1,$$

thus for C_0 big enough,

$$|\upsilon - \tilde{\upsilon}| \ge \frac{1}{2}(|\upsilon - \tilde{\upsilon}| + |\nu - \tilde{\nu}|) - N_1 \ge \frac{1}{2}C_0 - N_1 \ge c_0,$$

finishes the proof for the case $\mu \ge \lambda^2$.

Now we assume $\mu \le \lambda^2$, then again by Lemma (5.2.11), up to a $O_N(\mu^{-N}) = O_{2N}(\lambda^{-2N})$ error, we can further restrict the domain of integration in this case to $\Omega' = T_{C_\theta}(\gamma_\upsilon) \cap T_{C_\theta}(\gamma_{\tilde{\upsilon}}) \cap T_{C_\theta}(\gamma_\nu) \cap T_{C_\theta}(\gamma_{\tilde{\upsilon}})$.

Similarly as above, by plugging (115) and (116) into the expression of $K(y, z, \tilde{y}, \tilde{z})$, we see that the resulting phase function is given by

$$\lambda\left(d_g(x,y) - d_g(x,\tilde{y})\right) + \mu\left(d_g(x,z) - dg(x,\tilde{z})\right).$$

The gradient reads

$$\lambda\nabla_x\left(d_g(x,y) - d_g(x,\tilde{y})\right) + \mu\nabla_x\left(d_g(x,z) - d_g(x,\tilde{z})\right).$$

Let us denote $\nabla_x\left(d_g(x,y)\right) = Y$, here Y is a unit vector in T_xM, similarly denote $\nabla_x\left(d_g(x,\tilde{y})\right) = Y, \nabla_x(d_g(x,z)) = Z$ and $\nabla_x\left(d_g(x,\tilde{z})\right) = Z$. By the separation conditions we have, it is easy to see that $\angle(Y,Z), \angle(\tilde{Y},\tilde{Z}) \le N_1\theta$ and $\angle(Y,\tilde{Y}) + \angle(Z,\tilde{Z}) \ge C_0\theta$. We claim that

$$\left|Y - \tilde{Y} + \frac{\mu}{\lambda}(Z - \tilde{Z})\right| = \left|\left(Y + \frac{\mu}{\lambda}Z\right) - \left(\tilde{Y} + \frac{\mu}{\lambda}\tilde{Z}\right)\right| \ge c\frac{\mu}{\lambda}\theta, \quad (120)$$

which implies the desired result using integration by parts. Indeed, it suffices to show that $\angle\left(Y + \frac{\mu}{\lambda}Z, \tilde{Y} + \frac{\mu}{\lambda}\tilde{Z}\right)$ is bounded below by some uniform constant times θ. Note that $\angle\left(Y + \frac{\mu}{\lambda}Z, Y\right), \angle\left(\tilde{Y}, \tilde{Y} + \frac{\mu}{\lambda}\tilde{Z}\right), \angle\left(Y + \frac{\mu}{\lambda}Z, Z\right), \angle\left(\tilde{Z}, \tilde{Y} + \frac{\mu}{\lambda}\tilde{Z}\right) \le N_1\theta$, we have

$$\angle\left(Y + \frac{\mu}{\lambda}Z, \tilde{Y} + \frac{\mu}{\lambda}\tilde{Z}\right) \ge \angle(Y, \tilde{Y}) - 2N_1\theta,$$

similarly,

$$\angle\left(Y + \frac{\mu}{\lambda}Z, \tilde{Y} + \frac{\mu}{\lambda}\tilde{Z}\right) \ge \angle\left(Z, \tilde{Z}\right) - 2N_1\theta.$$

Thus for C_0 large enough,

$$\angle\left(Y + \frac{\mu}{\lambda}Z, \tilde{Y} + \frac{\mu}{\lambda}\tilde{Z}\right) \ge \frac{1}{2}\left(\angle\left(Y,\tilde{Y}\right) + \angle\left(Z,\tilde{Z}\right)\right) - 2N_1\theta \ge \frac{1}{2}C_0\theta - 2N_1\theta \ge c_0\theta,$$

finishes the proof.

We use orthogonality argument to reduce the proof of Theorem (5.2.1) to a specific bilinear estimate. We use Lemma (5.2.9) and Minkowski's inequality to estimate $\|\chi_\lambda f \chi_\mu g\|_2$ by

$$\left\|\sum_{|\nu-\upsilon|\le M} \chi_\lambda Q^\nu_{\theta_0}\ f\ \chi_\mu P^\upsilon_{\theta_0}\ g\right\|_2 \quad (121)$$

$$+ \sum_{\ell=\log M/\log 2}^{O(\log\lambda)} \left\|\sum_{2^\ell\le|\nu-\upsilon|<2^{\ell+1}} \chi_\lambda Q^\nu_{\theta_0} f\ \chi_\mu P^\upsilon_{\theta_0} g\right\|_2, \quad (122)$$

for certain dyadic M large enough. The square of (121) is estimated by

$$\left[\sum_{|\upsilon-\upsilon'|+|\upsilon'-\upsilon|\le C_0}\ \sum_{|\upsilon-\upsilon'|+|\upsilon'-\upsilon|\ge C_0}\right]\int \chi_\lambda Q^\nu_{\theta_0}(x)\chi_\mu P^\upsilon_{\theta_0}(x)\overline{\chi_\lambda Q^{\upsilon'}_{\theta_0}(x)}\ \overline{\chi_\mu P^{\upsilon'}_{\theta_0}(x)}\,dx, \quad (123)$$

where $|\nu - \upsilon|, |\nu\_ - \upsilon\_| \le M$.

By Lemma (5.2.9), the second term of (123) is negligible by choosing C_0 sufficiently large.

We can estimate the contribution of the first term as

$$\sum_{\upsilon\in\mathbb{Z}^2}\sum_{\nu:|\nu-\upsilon|\le M}\left\|\chi_\lambda Q^\nu_{\theta_0}f\,\chi_\mu P^\upsilon_{\theta_0}g\right\|_2^2.$$

If we use the bilinear estimate (76), we can estimate this sum by

$$\lambda^{\frac{1}{2}}\sum_{\upsilon\in\mathbb{Z}^2}\left\|P^\upsilon_{\theta_0}g\right\|_2^2\sum_{\nu:|\nu-\upsilon|\le M}\left\|Q^\nu_{\theta_0}f\right\|_2^2.$$

By the L^2-orthogonality, we see the contribution of (121)is

$$\lambda^{\frac{\varepsilon_0}{2}}\|g\|_2\times\left(\lambda^{\frac{1}{2}-\varepsilon_0}\sup_\nu\left\|Q^\nu_{\theta_0}f\right\|_2^2\right)^{\frac{1}{2}},$$

which corresponds to (79). Similarly, since the sum is symmetric, we can also bound (121) by

$$\lambda^{\frac{\varepsilon_0}{2}}\|f\|_2\times\left(\lambda^{\frac{1}{2}-\varepsilon_0}\sup_\nu\left\|P^\nu_{\theta_0}g\right\|_2^2\right)^{\frac{1}{2}},$$

which corresponds to (80).

The second microlocalization. For the off diagonal part (122), we will reduce the matters to a bilinear oscillatory integrals as in [170]. Fixing $\ell\ \ge \log M/\log 2$, we see that if $2^\ell\le|\nu-\upsilon|<2^{\ell+1}$ then the distance between γ_ν and γ_υ in the sense of Remark (5.2.8) is approximately $2^\ell\theta_0$. To explore this and use orthogonality argument, one naturally employs wider tubes to collect thinner tubes by making use of the second microlocalization. Precisely, up to some negligible terms, we may write for $\theta_\ell = 2^\ell\theta_0$ with c_0 to be specified later

$$\chi_\lambda Q^\nu_{\theta_0}f(x)\approx\sum_{\sigma_1\in\mathbb{Z}^2}\left(\chi_\lambda Q^{\sigma_1}_{c_0\theta_\ell}\right)\circ Q^\nu_{\theta_0}f(x),\chi_\mu P^\upsilon_{\theta_0}g(x)$$
$$\approx\sum_{\sigma_2\in\mathbb{Z}^2}\left(\chi_\mu P^{\sigma_2}_{c_0\theta_\ell}\right)\circ P^\upsilon_{\theta_0}g(x).$$

Noting that the kernels of the operators $\left(\chi_\lambda Q^{\sigma_1}_{c_0\theta_\ell}\right)\circ Q^\nu_{\theta_0}$ and $\left(\chi_\mu P^{\sigma_2}_{c_0\theta_\ell}\right)\circ P^\upsilon_{\theta_0}$ decrease rapidly unless $T_{C_1c_0\theta_\ell}(\gamma_{\sigma_1})\cap T_{C_1\theta_0}(\gamma_\nu)\ne\emptyset$ and $T_{C_1c_0\theta_\ell}(\gamma_{\sigma_2})\cap T_{C_1\theta_0}(\gamma_\upsilon)\ne\emptyset$, we have by choosing M large enough, there are $N_0 = N_0(c_0,M)$ and N_1 such that up to some negligible terms

$$\sum_{2^\ell\le|\nu-\upsilon|<2^{\ell+1}}\chi_\lambda Q^\nu_{\theta_0}f(x)\chi_\mu P^\upsilon_{\theta_0}g(x)\tag{124}$$
$$\sum_{\sigma_1,\sigma_2\in\mathbb{Z}^2,N_0\le|\sigma_1-\sigma_2|\le N_1}\ \sum_{2^\ell\le|\nu-\upsilon|<2^{\ell+1}}\left(\chi_\lambda Q^{\sigma_1}_{c_0\theta_\ell}\right)\circ Q^\nu_{\theta_0}f(x)\left(\chi_\mu P^{\sigma_2}_{c_0\theta_\ell}\right)\circ P^\upsilon_{\theta_0}g(x).$$

Moreover, we may find a $C_3>0$ having the property that for every σ_1 and σ_2, there are $\nu(\sigma_1)$ and $\upsilon(\sigma_2)$ such that $|\nu-\nu(\sigma_1)|,|\upsilon-\upsilon(\sigma_2)|\ \ge\ C_3 2^\ell$ implies

$$\left\|\left(\chi_\lambda Q^{\sigma_1}_{c_0\theta_\ell}\right)\circ Q^\nu_{\theta_0}f\right\|_{L^\infty}\lesssim_N\lambda^{-N},\left\|\left(\chi_\mu P^{\sigma_2}_{c_0\theta_\ell}\right)\circ P^\upsilon_{\theta_0}g\right\|_{L^\infty}\lesssim_N\mu^{-N},$$

for all $N = 1,2,\ldots$. Therefore, we may estimate (124) as follows

$$\left\|\sum_{2^\ell\le|\nu-\upsilon|<2^{\ell+1}}\chi_\lambda Q^\nu_{\theta_0}f\chi_\mu P^\upsilon_{\theta_0}g\right\|_2^2\lesssim\sum_{\substack{N_0\le|\sigma_1-\sigma_2|\le N_1\\|\sigma_1-\sigma_1|+|\sigma_2-\sigma_2|\ge C}}\int T^{\sigma_1,\sigma_2}_{\lambda,\mu,\theta_\ell}F(x)\overline{T^{\sigma_1,\sigma_2}_{\lambda,\mu,\theta_\ell}F(x)}\,dx$$

$$+\sum_{\substack{N_0\le|\sigma_1-\sigma_2|\le N_1\\ |\sigma_1-\sigma_1|+|\sigma_2-\sigma_2|\ge C}} \int T^{\sigma_1,\sigma_2}_{\lambda,\mu,\theta_\ell}F(x)\overline{T^{\sigma_1,\sigma_2}_{\lambda,\mu,\theta_\ell}F(x)}\,dx$$

where N_0 can be sufficiently large by choosing c_0 small and

$$T^{\sigma_1,\sigma_2}_{\lambda,\mu,\theta_\ell}F(x)=\iint\left(\chi_\lambda\circ Q^{\sigma_1}_{c_0\theta_\ell}\right)(x,y)\left(\chi_\mu\circ P^{\sigma_2}_{c_0\theta_\ell}\right)(x,z)F(y,z)dydz,\qquad(125)$$

$$F(y,z)=\sum_{\substack{2^\ell\le|\nu-\upsilon|<2^{\ell+1}\\ |\nu(\sigma_1)-\nu|+|\upsilon(\sigma_2)-\upsilon|\le C_3 2^\ell}} Q^\nu_{\theta_0}f(y)P^\upsilon_{\theta_0}g(z),\qquad(126)$$

with $F(y,z)=0$ if $(y,z)\notin B(y_0,C_0\delta)\times B(z_0,C_0\delta)$. It follows again from Lemma (5.2.12) that if we choose C large enough, the second term in the expression preceding (125) is negligible.

To evaluate the first term there, we are reduced to estimating

$$\sum_{N_0\le|\sigma_1-\sigma_2|\le N_1}\left\|T^{\sigma_1,\sigma_2}_{\lambda,\mu,\theta_\ell}F\right\|_{2L^2(B(0,\delta))}.\qquad(127)$$

We shall need the following proposition whose proof is postponed.

Proposition (5.2.13)[174]: Let

$$T^{\sigma_1,\sigma_2}_{\lambda,\mu,\theta_\ell}F(x)=\iint\left(\chi_\lambda\circ Q^{\sigma_1}_{c_0\theta_\ell}\right)(x,y)\left(\chi_\mu\circ\circ P^{\sigma_2}_{c_0\theta_\ell}\right)(x,z)\,F(y,z)\,dydz.\,(128)$$

Assume as before that $\delta>0$ is sufficiently small and θ is larger than a fixed positive constant times θ_0. Then if N_0 is sufficiently large and $N_1>N_0$ is fixed, there exists a positive constant $C=C_{\varepsilon_0}$ such that

$$\left\|T^{\sigma_1,\sigma_2}_{\lambda,\mu,\theta_\ell}F\right\|_{2L^2(B(0,\delta))}\le C\,\theta^{-12}\|F\|_2,\text{if } N_0\le|\sigma_1-\sigma_2|\le N_1.\quad(129)$$

Assuming (129), we can now complete the proof of Theorem (5.2.1). In fact, we have

$$\left\|\sum_{2^\ell\le|\nu-\upsilon|<2^{\ell+1}}\chi_\lambda Q^\nu_{\theta_0}f\,\chi_\mu P^\upsilon_{\theta_0}g\right\|_2^2$$

$$\le C\left(2^\ell\theta_0\right)^{-1}\sum_{N_0\le|\sigma_1-\sigma_2|\le N_1}\iint\left|\sum_{\substack{2^\ell\le|\nu-\upsilon|<2^{\ell+1}\\ |\nu(\sigma_1)-\nu|+|\upsilon(\sigma_2)-\upsilon|\le C_3 2^\ell}} Q^\nu_{\theta_0}\,f(y)P^\upsilon_{\theta_0}\,g(z)\right|^2 dydz.$$

Notice that

$$\iint\left|\sum_{\substack{2^\ell\le|\nu-\upsilon|<2^{\ell+1}\\ |\nu(\sigma_1)-\nu|+|\upsilon(\sigma_2)-\upsilon|\le C_3 2^\ell}} Q^\nu_{\theta_0}\,f(y)P^\upsilon_{\theta_0}\,g(z)\right|^2 dydz$$

$$=\sum_{\substack{2^\ell\le|\nu-\upsilon|<2^{\ell+1}\\ |\nu(\sigma_1)-\nu|+|\upsilon(\sigma_2)-\upsilon|\le C_3 2^\ell}}\ \sum_{\substack{2^\ell\le|\nu'-\upsilon'|<2^{\ell+1}\\ |\nu(\sigma_1)-\nu'|+|\upsilon(\sigma_2)-\upsilon'|\le C_3 2^\ell}}\langle\left(Q^{\nu'}_{\theta_0}\right)^*\circ Q^\nu_{\theta_0}f,f\rangle\langle\left(P^{\upsilon'}_{\theta_0}\right)^*\circ P^\upsilon_{\theta_0}g,g\rangle,$$

where $\left\|\left(Q^{\nu'}_{\theta_0}\right)^*\circ Q^\nu_{\theta_0}\right\|_{L^2\to L^2}=O(\lambda^{-N})$ and $\left\|\left(P^{\upsilon'}_{\theta_0}\right)^*\circ P^\upsilon_{\theta_0}\right\|_{L^2\to L^2}=O(\mu^{-N})$ if $|\nu-\nu'|+|\upsilon-\upsilon'|\ge C$ for Clarge. Consequently, we have up to some negligible terms

$$\|(124)\|_2^2 \leq C\left(2^{\ell}\theta_0\right)^{-1} \sum_{N_0 \leq |\sigma_1-\sigma_2| \leq N_1} \sum_{|\nu-\nu(\sigma_1)|+|v-v(\sigma_2)| \leq C_3\, 2^{\ell}} \left\|Q_{\theta_0}^{\nu} f\right\|_2^2 \left\|P_{\theta_0}^{v} g\right\|_2^2$$

$$\leq C\left(2^{\ell}\theta_0\right)^{-1} \left(\sup_{\sigma_1} \sum_{|\nu-\nu(\sigma_1)| \leq C_3\, 2^{\ell}} \left\|Q_{\theta_0}^{\nu} f\right\|_2^2\right) \cdot \left(\sum_{\sigma_2} \sum_{|v-v(\sigma_2)| \leq C_3\, 2^{\ell}} \left\|P_{\theta_0}^{v} g\right\|_2^2\right)$$

$$\leq C\left(2^{\ell}\theta_0\right)^{-1} \|g\|_2^2 \sup_{\nu \in \mathbb{Z}^2} \left\|Q_{2^{\ell}\theta_0}^{\nu} f\right\|_2^2.$$

Thanks to the fact that we are allowed to have an extra small power of λ, we may sum over $1 \lesssim \ell \lesssim \log \lambda t$ o finish the proof of (79). To get (80), one notes that the above sum is again symmetric, thus we may interchange the role of $Q_{\theta_0}^{\nu} f$ and $P_{\theta_0}^{v} g$ to get

$$\left\|\sum_{2^{\ell} \leq |\nu-v| < 2^{\ell+1}} \chi_\lambda Q_{\theta_0}^{\nu} f(x)\, \chi_\mu P_{\theta_0}^{v} g(x)\right\|_2^2 \leq C\left(2^{\ell}\theta_0\right)^{-1} \|f\|_2^2 \sup_{\nu \in \mathbb{Z}^2} \left\|P_{2^{\ell}\theta_0}^{\nu} g\right\|_2^2.$$

summing over ℓ finishes the proof of (80).

We take $\theta = c_0\theta_\ell$ and prove Proposition (5.2.13). We work in the geodesic normal coordinates about a fixed point $\tilde{x} \in T_{C_\theta}(\gamma_{\sigma_1}) \cap T_{C_\theta}(\gamma_{\sigma_2})$. Without loss of generality, we may assume $\tilde{x} \in \gamma_{\sigma_1}$ and the geodesic γ_{σ_1} is parameterized by $\{(0,s): |s| \leq 2\}$. In the following, we denote by $\phi(x,y) = d_g\big((x_1,x_2),(y_1,y_2)\big)$ the geodesic distance between x and y.

In order to estimate the $L^2\big(B(0,\delta)\big)$ norm of

$$T_{\lambda,\mu,\theta}^{\sigma_1,\sigma_2} F(x) = \iint \left(\chi_\lambda \circ Q_\theta^{\sigma_1}\right)(x,y) \left(\chi_\mu \circ P_\theta^{\sigma_2}\right)(x,z) F(y,z)\, dydz. \quad (130)$$

we shall need the following lemma to further restrict the domain of x, y, z.

Lemma (5.2.14)[174]: There exists a constant C, such that if we set $\Omega_1 = T_{C_\theta}(\gamma_{\sigma_1})$ and $\Omega_2 = T_{C_\theta}(\gamma_{\sigma_2})$ we have

$$\left\|\iint_{y \notin \Omega_1} \left(\chi_\lambda \circ Q_\theta^{\sigma_1}\right)(x,y) \left(\chi_\mu \circ P_\theta^{\sigma_2}\right)(x,z) F(y,z)\, dydz\right\|_{L^2(B(0,\delta))} \leq C_N \lambda^{-N} \|f\|_2 \|g\|_2,$$

and

$$\left\|T_{\lambda,\mu,\theta}^{\sigma_1,\sigma_2} F\right\|_{L^2(B(0,\delta))} \leq C_N \lambda^{-N} \|f\|_2 \|g\|_2,$$

Similarly, we have

$$\left\|\iint_{z \notin \Omega_1} \left(\chi_\lambda \circ Q_\theta^{\sigma_1}\right)(x,y) \left(\chi_\mu \circ P_\theta^{\sigma_2}\right)(x,z) F(y,z)\, dydz\right\|_{L^2(B(0,\delta))} \leq C_N \lambda^{-N} \|f\|_2 \|g\|_2,$$

and

$$\left\|T_{\lambda,\mu,\theta}^{\sigma_1,\sigma_2} F\right\|_{L^2(B(0,\delta) \setminus \Omega_2)} \leq C_N \lambda^{-N} \|f\|_2 \|g\|_2.$$

Proof. Since we know there are at most $O(\lambda_2)$ many terms in the sum

$$F(y,z) = \sum_{v} \sum_{v: |v-v| \in [2^{\ell}, 2^{\ell+1})} Q_{\theta_0}^{v} f(y)\, P_{\theta_0}^{v} g(z),$$

it suffices to show the $L^2\big(B(0,\delta)\big)$ norm of

$$\iint \left(\chi_\lambda \circ Q_\theta^{\sigma_1}\right)(x,y) \left(\chi_\mu \circ P_\theta^{\sigma_2}\right)(x,z) f(y) g(z)\, dydz$$

satisfies our claim.

Indeed, by Lemma (5.2.7), we can find C such that if $x \notin T_{C_\theta}(\gamma_{\sigma_1})$ or $y \notin T_{C_\theta}(\gamma_{\sigma_1})$,

$$|\chi_\lambda Q_\theta^{\nu}(x,y)| \le C_N \lambda^{-N}.$$

Thus

$$\left\| \int \left(\chi_\lambda \circ Q_\theta^{\sigma_1}\right)(x,y) f(y)\, dy \right\|_{L^\infty(dx)} \le C_N \lambda^{-N} \|f\|_{L^2},$$

while we know χ_μ has $L^2 \to L^2$ norm 1, so

$$\left\| \int \left(\chi_\mu \circ P_\theta^{\sigma_2}\right)(x,y) g(z)\, dz \right\|_{L^2(dx)} \le C_N \lambda^{-N} \|g\|_{L^2}.$$

Therefore

$$\left\| \iint \left(\chi_\lambda \circ\ Q_\theta^{\sigma_1}\right)(x,y) \left(\chi_\mu \circ\ P_\theta^{\sigma_2}\right)(x,z) f(y) g(z) dy dz \right\|_{L^2} \le C_N \lambda^{-N} \|f\|_{L^2} \|g\|_{L^2}$$

as claimed.

The second part of our lemma follows from the exact same proof.

Remark (5.2.15)[174]: By the above lemma, we see that we can assume in (130), $y \in T_{C_\theta}(\gamma_{\sigma_1}), z \in T_{C_\theta}(\gamma_{\sigma_2})$, and $x \in T_{C_\theta}(\gamma_{\sigma_1}) \cap T_{C_\theta}(\gamma_{\sigma_2})$. Moreover, if $N_0 \le |\sigma_1 - \sigma_2| \le N_1$, then we may assume the angle $\operatorname{Ang}(x;\ y,z)$ between the geodesic connecting x and y and the one connecting x and z belongs to $[\tilde{\theta}, C_4\theta]$. This geometric assumption yields $x, y, z \in TC_4\theta(\gamma_{\sigma_1})$ for some large constant C_4. Moreover, we also have $\angle(\gamma_{\sigma_1}, \gamma_{\sigma_2}) \ge N_0\theta$. Noticing that $d_g(x,y)$ and $d_g(x,z)$ are comparable to 1, we claim that for N_0 sufficiently large, we can find $c > 0$ such that

$$|y_1 - z_1| > c\theta. \tag{131}$$

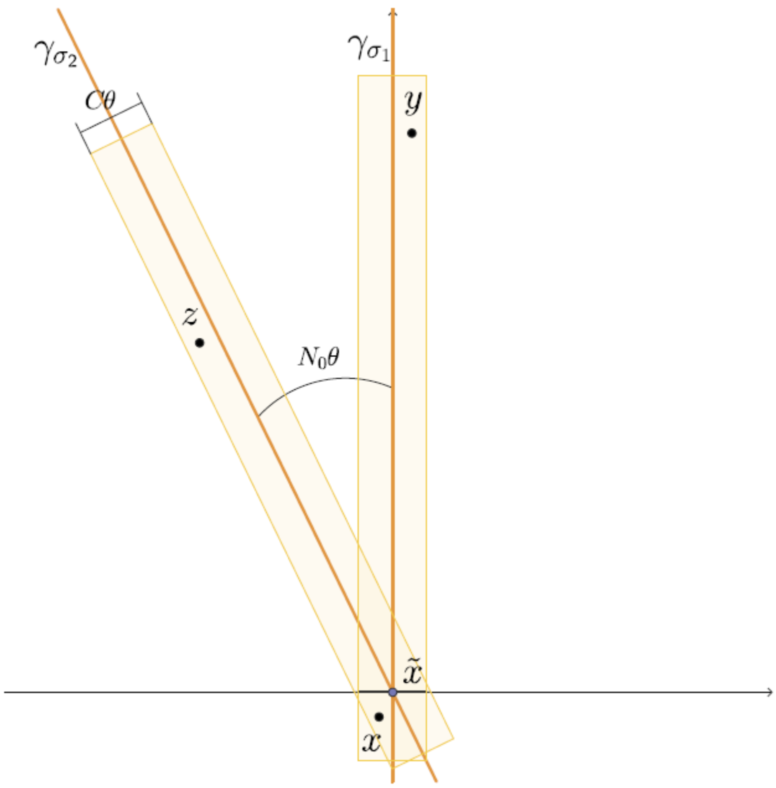


Fig.(3)[174]: The geodesic tubes $T_{C_\theta}(\gamma_{\sigma_1})$ and $T_{C_\theta}(\gamma_{\sigma_2})$.

Indeed, it is easy to see that $|y_1| \le C\theta$ and $d_g(z, \gamma_{\sigma_2}) \le C\theta$. Since the constant Chere is a uniform constant, we can choose $N_0 \gg C$. Then we have $|z_1| \ge N_0\theta - C\theta$, see Fig. (3). Therefore $|y_1 - z_1| \ge N_0\theta - 2C\theta \ge c\theta$ as claimed.

Returning to $T^{\sigma_1,\sigma_2}_{\lambda,\mu,\theta}F\ (x)$, we have from Cauchy–Schwarz

$$\left\|T^{\sigma_1,\sigma_2}_{\lambda,\mu,\theta}F\right\|_2^2 \lesssim \lambda\mu \iint \left|\iint e^{i\mu\Phi_\epsilon(x;\,(y_1,y_2),(z_1,z_2))} a^{\sigma_1,\sigma_2}_{\lambda,\mu,\theta}(x,y,z)F(y,z)dy_1dz_1\right|^2 dxdy_2dz_2,$$

where $\epsilon = \lambda/\mu$ and

$$\Phi_\epsilon(x;\ y,z) = \epsilon\phi(x,y) + \phi(x,z),$$
$$a^{\sigma_1,\sigma_2}_{\lambda,\mu,\theta}(x,y,z) = a_{\sigma_1,\theta}(x,y)b_{\sigma_2,\theta}(x,z).$$

Fix y_2 and z_2 , it suffices to prove

$$\int_{\mathbb{R}^2}\left|\int_{\mathbb{R}^2} e^{i\mu\Phi_\epsilon(x;\,(y_1,y_2),(z_1,z_2))} a^{\sigma_1,\sigma_2}_{\lambda,\mu,\theta}(x,y,z)G(y_1,z_1)dy_1dz_1\right|^2 dx \le C(\lambda\mu\theta)^{-1}\|G\|^2_{L^2}, \tag{132}$$

uniformly with respect to y_2, z_2 where we set $G(y_1,z_1) = F(y,z)$ for brevity.
Squaring the left side of (132) shows that we need to estimate

$$\iint e^{i\mu\Psi(x;\,y_1,y_1',z_1,z_1')} A^{\sigma_1,\sigma_2}_{\lambda,\mu,\theta}(x;\ y_1,y_1',z_1,z_1')\overline{G(y_1',z_1')}dxdy_1dz_1dy_1'dz_1', \tag{133}$$

where

$$A^{\sigma_1,\sigma_2}_{\lambda,\mu,\theta}(x;\ y_1,y_1',z_1,z_1') = a^{\sigma_1,\sigma_2}_{\lambda,\mu,\theta}\big(x,(y_1,y_2),(z_1,z_2)\big)\overline{a^{\sigma_1,\sigma_2}_{\lambda,\mu,\theta}\big(x,(y_1',y_2),(z_1',z_2)\big)}$$
$$\Psi = \Psi_{\epsilon,y_2,z_2}(x;\ y_1,y_1',z_1,z_1') = \Phi_\epsilon\big(x,(y_1,y_2),(z_1,z_2)\big)\Phi_\epsilon\big(x,(y_1',y_2),(z_1',z_2)\big)$$

Set

$$K^{\sigma_1,\sigma_2}_{\lambda,\mu,\theta}(x;\ y_1,y_1',z_1,z_1') = \int_{\mathbb{R}^2} e^{i\mu\Psi(x;\,y_1,y_1',z_1,z_1')} A^{\sigma_1,\sigma_2}_{\lambda,\mu,\theta}(x;\ y_1,y_1',z_1,z_1')dx. \tag{134}$$

Then by Schur test, we are reduced to proving

$$\sup_{y_1',z_1'}\int_{\mathbb{R}^2}\left|K^{\sigma_1,\sigma_2}_{\lambda,\mu,\theta}(x;\ y_1,y_1',z_1,z_1')\right|dy_1,dz_1 \sup_{y_1,z_1}\int_{\mathbb{R}^2}\left|K^{\sigma_1,\sigma_2}_{\lambda,\mu,\theta}(x;\ y_1,y_1',z_1,z_1')\right|dy_1',dz_1' \le C/\lambda\mu\theta.$$

By symmetry, we shall only deal with the first one.

By Remark (5.2.15), we have

$$|y_1 - z_1| \ge c\theta, |y_1' - z_1'| \ge c\theta. \tag{135}$$

This would allow us to study the oscillatory integral (134) using the strategy of [28] and a change of variables argument similar to the one in pp.217–218of [3]. In fact, if we let $\psi(x, y1) = \phi\big(x,(y_1,y_2)\big)$, then ψ is a Carleson–Sjölinphase for fixed y_2, i.e.

$$\det\begin{pmatrix} \psi''_{x_1,y_1} & \psi''_{x_2,y_1} \\ \psi'''_{x_1,y_1,y_1} & \psi'''_{x_2,y_1,y_1}\end{pmatrix} \ne 0, \tag{136}$$

see [16], [28]. Changing variables $(y_1,z_1) \mapsto (\tau,\tau'), (y_1',z_1') \mapsto (\tilde\tau,\tilde\tau')$ by

$$\begin{cases}\tau = \dfrac{\lambda}{2\mu}(y_1 - z_1)^2, \\ \tau' = z_1 + \dfrac{\lambda}{\mu}y_1\end{cases}, \begin{cases}\tilde\tau = \dfrac{\lambda}{2\mu}(y_1' - z_1')^2, \\ \tilde\tau' = z_1' + \dfrac{\lambda}{\mu}y_1'\end{cases},$$

where we may assume $y_1 > z_1$ by symmetry. It is clear that the above bijective mapping sends variables from $\{y_1 - z_1 \ge c\theta\}$ to $\{(\tau,\tau') : \tau \ge c_\lambda\theta_2/2\mu\}$, whose Jacobian reads

$$\frac{D(\tau\,,\tau')}{D(y_1,z_1)}=(1+\epsilon)(2\epsilon\tau)^{1/2}$$

The phase function in (134) goes to

$$\widetilde{\Psi}(x;\tau,\tilde{\tau},\tau',\tilde{\tau}')\ =\Psi(x;y_1,y_1',z_1,z_1'),$$

under the change of variables. The Carleson–Sjölincondition allows us to obtain as in [3]

$$\left|\nabla_x\widetilde{\Psi}(x;\tau,\tilde{\tau},\tau',\tilde{\tau}')\right|\approx|\tau-\tilde{\tau}|+|\tau'-\tilde{\tau}'|$$

$$\left|\partial_x^\alpha\widetilde{\Psi}(x;\tau,\tilde{\tau},\tau',\tilde{\tau}')\right|\leq C_\alpha(|\tau-\tilde{\tau}|+|\tau'-\tilde{\tau}'|),|\alpha|\leq 5$$

In view of integration by parts and relation (135), we have for fixed (y_1',z_1') hence fixed $(\tilde{\tau},\tilde{\tau}')$, thus

$$\iint_{y_1-z_1\geq c\theta}\left|K_{\lambda,\mu,\theta}^{\sigma_1,\sigma_2}(x;\ y_1,y_1',z_1,z_1')\right|dy_1,dz_1$$

$$\leq C\iint_{r\geq c\lambda\theta^2/2\mu}(1+\mu|\tau-\tilde{\tau}|+\mu|\tau'-\tilde{\tau}'|)^{-5}\left(\frac{\lambda}{\mu}\tau\right)^{-1/2}d\tau d\tau'$$

$$\leq C/\lambda\mu\theta,$$

finishes the proof.

Chapter 6
Principal Spectrum and Properties of the Principal Eigenvalue

We discuss the applications of the principal spectral theory of nonlocal dispersal operators to the asymptotic dynamics of two species competition systems with nonlocal dispersal. We show the equivalence of different definitions of the principal eigenvalue. We also study the behaviour of $\lambda_p(\mathcal{L}_\Omega + a)$ with respect to some scaling of K. For kernels K of the type, $K(x,y) = J(x - y)$ with J a compactly supported probability density, we also establish some asymptotic properties of $\lambda_p\left(\mathcal{L}_{\sigma,m,\Omega} - \frac{1}{\sigma^m} + a\right)$ where $\mathcal{L}_{\sigma,m,\Omega}$ is defined by $\mathcal{L}_{\sigma,2,\Omega}[\phi] := \frac{1}{\sigma^{2+N}}\int_\Omega J\left(\frac{x-y}{\sigma}\right)\phi(y)\,dy$. We show that $\lim_{\sigma\to 0} \lambda_p\,(\mathcal{L}_{\sigma,2,\Omega} - 1/\sigma^2 + a = \lambda_1\left(\frac{D_2(J)}{2N}\Delta + a\right)$, where $D_2(J) := \int_{\mathbb{R}^N} J(z)|z|^2 dz$ and λ_1 denotes the Dirichlet principal eigenvalue of the elliptic operator. We obtain some convergence results for the corresponding eigenfunction $\phi_{p,\sigma}$.

Section (6.1): Principal Eigenvalues of Nonlocal Dispersal Operators and Applications

We devoted to the study of principal spectrum of the following three eigenvalue problems associated to nonlocal dispersal operators,

$$\nu_1\left[\int_D k(y-x)u(y)dy - u(x)\right] + a_1(x)u = \lambda u(x), \qquad x \in \bar{D}, \quad (1)$$

where $D \subset \mathbb{R}^N$ is a smooth bounded domain,

$$\nu_2\int_D k(y-x)[u(y) - u(x)]dy + a_2(x)u(x) = \lambda u(x), \quad x \in \bar{D}, (2)$$

where $D \subset \mathbb{R}^N$ is as in (1), and

$$\begin{cases}\nu_3\left[\int_{\mathbb{R}^N} k(y-x)u(y)dy - u(x)\right] + a_3(x)u(x) = \lambda u(x), & x \in \mathbb{R}^N, \\ u(x + p_j e_j) = u(x), & x \in \mathbb{R}^N,\end{cases} \quad (3)$$

where $p_j > 0, e_j = (\delta_j 1, \delta_{j2}, \cdots, \delta_{jN})$ with $\delta_{jk} = 1$ if $j = k$ and $\delta_{jk} = 0$ if $j \neq k$, and $a_3(x + p_j e_j) = a_3(x), j = 1, 2, \cdots, N$. In (1), (2), and (3), $k(\cdot)$ is a nonnegative C^1 function with compact support, $k(0) > 0$, and $\mathbb{R}\,\mathbb{R}^N\, k(z)dz = 1$.

Observe that the nonlocal dispersal operators in (1), (2), and (3), that is, $u(x) \mapsto D\,k(y-x)u(y)dy - u(x), u(x) \mapsto D\,k(y-x)[u(y) - u(x)]dy, and\, u(x) \mapsto \mathbb{R}^N\,k(y-x)u(y)dy - u(x)$, can be viewed as $u(x) \mapsto \mathbb{R}^N\,k(y-x)[u(y) - u(x)]dy$ with Dirichlet type boundary condition $\mathbb{R}^N \setminus D\,k(y-x)u(y)dy = 0\, for\, x \in \bar{D}, u(x) \mapsto \mathbb{R}^N\,k(y-x)[u(y) - u(x)]dy$ with Neumann type boundary condition $\mathbb{R}^N \setminus D\,k(y-x)[u(y) - u(x)]dy = 0\, for\, x \in \bar{D}, and\, u(x) \mapsto \mathbb{R}^N\,k(y - x)[u(y) - u(x)]dy$ with periodic boundary condition $u(x + p_j e_j) = u(x)\, for\, x \in \mathbb{R}^N$, respectively.

Observe also that the eigenvalue problems (1), (2), and (3) can be viewed as the nonlocal counterparts of the following eigenvalue problems associated to random dispersal operators,

$$\begin{cases}(\nu_1\Delta u(x) + a_1(x)u(x) = \lambda u(x), & x \in D, \\ u(x) = 0, & x \in \partial D,\end{cases} \quad (4)$$

$$\begin{cases}(\nu_2\Delta u(x) + a_2(x)u(x) = \lambda u(x), & x \in D, \\ \dfrac{\partial u}{\partial n}(x) = 0, & x \in \partial D,\end{cases} \quad (5)$$

and

$$\begin{cases} \nu_3 \Delta u(x) + a_3(x)u(x) = \lambda u(x), & x \in \mathbb{R}^N, \\ u(x + p_j e_j) = u(x), & x \in \mathbb{R}^N, \end{cases} \quad (6)$$

respectively. [199], explore the relations between (1) and (4) (resp. (2) and (5), (3) and (6)) and prove that the principal eigenvalues of (4), (5), and (6) can be approximated by the principal spectrum points of (1), (2), and (3) with properly rescaled kernels, respectively (see Definition (6.1.1) for the definition of principal spectrum points of (1), (2), and (3)).

The nonlocal dispersal operator $u(x) \mapsto \int_{\mathbb{R}^N} k(y-x)[u(y)-u(x)]dy$ with Dirichlet type or Neumann type or periodic boundary condition and the random dispersal operator $u(x) \mapsto \Delta u(x)$ with Dirichlet or Neumann or periodic boundary condition are widely used to model diffusive systems in applied sciences. In particular, the random dispersal operator $u(x) \mapsto \Delta u(x)$ with proper boundary condition is usually adopted when the organisms in a diffusive system move randomly between the adjacent spatial locations.

Nonlocal dispersal operator such as $u(x) \mapsto \mathbb{R}^N k(y-x)[u(y)-u(x)]dy$ is applied when diffusive systems exhibit long range internal interactions (see [183], [184], [147]). Here if there is $\delta > 0$ such that $\text{supp}(k(\cdot)) \subset B(0,\delta) := \{z \in \mathbb{R}^N \mid \|z\| < \delta\}$ and for any $0 < \tilde{\delta} < \delta, \text{supp}(k(\cdot)) \cap \left(B(0,\delta) \setminus B(0,\tilde{\delta}) \right) \neq \emptyset, \delta$ is called the dispersal distance of the nonlocal dispersal operator $u(x) \mapsto \mathbb{R}^N k(y-x)[u(y)-u(x)]dy$. As a basic technical tool for the study of nonlinear evolution equations with random and nonlocal dispersals, it is of great importance to investigate aspects of spectral theory for random and nonlocal dispersal operators.

The eigenvalue problems (4), (5), and (6), and in particular, their associated principal eigenvalue problems, are well understood. For example, it is known that the largest real part, denoted by $\lambda_{R,1}(\nu_1, a_1)$, of the spectrum set of (4) is an isolated algebraically simple eigenvalue of (4) with a positive eigenfunction, and for any other λ in the spectrum set of (4), $Re\lambda < \lambda_{R,1}(\nu_1, a_1)$ ($\lambda_R, 1(\nu_1, a_1)$ is called the principal eigenvalue of (4)). Similar properties hold for the largest real parts, denoted by $\lambda_{R,2}(\nu_2, a_2)$ and $\lambda_{R,\nu_3}(\nu_3, a_3)$, of the spectrum sets of (5) and (6).

The principal eigenvalue problems (1), (2), and (3) have also been studied recently by many people (see [118], [146], [187], [189], [200], [199]). Let $\tilde{\lambda}_1(\nu_1, a_1)$ $(resp. \tilde{\lambda}_2(\nu_2, a_2), \tilde{\lambda}_3(\nu_3, a_3))$ be the largest real part of the spectrum set of (1) (resp. (2), (3)). $\tilde{\lambda}_1(\nu_1, a_1)$ $(resp. \tilde{\lambda}_2(\nu_2, a_2), \tilde{\lambda}_3(\nu_3, a_3))$ is called the principal spectrum point of (1) (resp. (2), (3)). $\tilde{\lambda}_1(\nu_1, a_1)$ $(resp. \tilde{\lambda}_2(\nu_2, a_2), \tilde{\lambda}_3(\nu_3, a_3))$ is also called the principal eigenvalue of (1) (resp. (2),(3)) if it is an isolated algebraically simple eigenvalue of (1) (resp. (2), (3)) with a positive eigenfunction. It is known that a nonlocal dispersal operator may not have a principal eigenvalue (see [118], [200]), which reveals some essential difference between nonlocal and random dispersal operators. Some sufficient conditions are provided in [118], [189], and [200] for the existence of principal eigenvalues of (1), (2), and (3) (the conditions in [118] apply to (1) and (2), the conditions in [189] apply to (1), and the conditions in [200] apply to (3)). Such sufficient conditions have been found important in the study of nonlinear evolution equations with nonlocal dispersals (see [118], [185], [187], [189], [190], [191], [200], [201], [202]). However, the understanding is still little to many interesting questions regarding the principal spectrum points/principal eigenvalues of nonlocal dispersal operators, including the dependence of principal spectrum points or principal eigenvalues (if exist) of nonlocal dispersal operators on the underlying parameters.

The objective of the current is to investigate the dependence of the principal spectrum points of nonlocal dispersal operators on the underlying parameters. We study the effects of the spatial inhomogeneity, the dispersal rate, and the dispersal distance on the existence of principal eigenvalues, on the magnitude of the principal spectrum points, and on the asymptotic behavior of the principal spectrum points of nonlocal dispersal operators with different types of boundary conditions in a unified way. Among others, we obtain the following:

(a) criteria for $\tilde{\lambda}_1(\nu_1, a_1)$ $(resp. \tilde{\lambda}_2(\nu_2, a_2), \tilde{\lambda}_3(\nu_3, a_3))$ to be the principal eigenvalue of (1) (resp. (2), (3)) (see Theorem (6.1.14) (i), (ii), Theorem (6.1.15) (iii), and Theorem (6.1.16) (iii) for detail);

(b) lower bounds of $\tilde{\lambda}_i(\nu_i, a_i)$ in terms of $\hat{a}_i$, where $\hat{a}_i$ is the spacial average of $a_i(x)$ $(i = 2, 3)$ (see Theorem (6.1.14) (iv) for detail);

(c) monotonicity of $\tilde{\lambda}_i(\nu_i, a_i)$ with respect to $a_i(x)$ and ν_i $(i = 1, 2, 3)$ (see Theorem (6.1.14) (v) and Theorem (6.1.15) (i) for detail);

(d) limits of $\tilde{\lambda}_i(\nu_i, a_i)\ as\ \nu_i \to 0\ and\ \nu_i \to \infty\ (i = 1, 2, 3)$ (see Theorem (6.1.15) (iv), (v) for detail);

(e) limits of $\tilde{\lambda}_i(\nu_i, a_i, \delta)\ as\ \delta \to 0\ and\ \delta \to \infty$ in the case $k(z) = \frac{1}{\delta N} \tilde{k}(z\delta)$ and $\tilde{k}(z) \geq 0$, $\mathrm{supp}(\tilde{k}) = B(0, 1)$, $\mathbb{R}^N\, \tilde{k}(z)dz = 1$, where $\tilde{\lambda}_i(\nu_i, a_i, \delta) = \tilde{\lambda}(\nu_i, a_i)$ $(i = 1, 2, 3)$ (see Theorem (6.1.16) (i), (ii) for detail).

We also investigate the applications of principal spectrum point properties of nonlocal dispersal operators to the asymptotic dynamics of the following two species competition system,

$$\begin{cases} u_t = \nu[\int_D k(y-x)u(t,y)dy - u(t,x)] + uf(x, u+v), x \in \bar{D} \\ v_t = \nu \int_D k(y-x)[u(t,y) - u(t,x)]dy + vf(x, u+v), x \in \bar{D}, \end{cases} \tag{7}$$

where D and $k(\cdot)$ are as in (1) with $k(-z) = k(z)$ and $f(\cdot,\cdot)\ is\ a\ C^1$ function satisfying that $\tilde{\lambda}_1(\nu, f(\cdot, 0)) > 0, f(x, w) < 0\ for\ w \gg 1$, and $\partial_2 f(x, w) < 0\ for\ w > 0$. (7) models the population dynamics of two competing species with the same local population dynamics (i.e. the same growth rate function $f(\cdot, \cdot)$), the same dispersal rate $(i.e. \nu)$, but one species adopts nonlocal dispersal with Dirichlet type boundary condition and the other adopts nonlocal dispersal with Neumann type boundary condition, where $u(t, x)$ and $v(t, x)$ are the population densities of two species at time t and space location x. We show

(f) the species diffusing nonlocally with Neumann type boundary condition drives the species diffusing nonlocally with Dirichlet type boundary condition extinct (see Theorem (6.1.19) for detail).

Nonlocal evolution equations have been attracting more and more attentions due to the presence of nonlocal interactions in many diffusive systems in applied sciences. See [177], [122], [125], [179], [180], [181], [131], [137], [135], [146], [188], [189], [190], [192], [194], [196], [198], etc. for the study of various aspects of nonlocal dispersal equations.

We investigate the effects of spatial variation on the principal spectrum points of nonlocal dispersal operators and prove Theorem (6.1.14). We consider the effects of dispersal rate on the principal spectrum points of nonlocal dispersal operators and prove Theorem (6.1.15). We explore the effects of dispersal distance on the principal spectrum points of nonlocal dispersal operators and prove Theorem (6.1.16). We consider the asymptotic dynamics of (7) by applying some of the principal spectrum point properties of nonlocal dispersal operators and prove Theorem (6.1.19).

Let

$$X_i = C(\bar{D}) \tag{8}$$

with norm $\|u\|_{X_i} = \max_{x\in\bar{D}} |u(x)| \, for \, i = 1,2,$

$$X_i^+ = \{u \in X_i \mid u(x) \geq 0, x \in \bar{D}\}, i = 1,2, \tag{9}$$

and

$$X_i^{++} = Int(X_i^+) = \{u \in X + i \mid u(x) > 0, x \in \bar{D}\}, i = 1,2. \tag{10}$$

Let

$$X_3 = \{u \in C(\mathbb{R}^N, \mathbb{R}) \mid u(x + p_j e_j) = u(x), x \in \mathbb{R}^N, j = 1,2,\cdots,N\} \tag{11}$$

with norm $\|u\|_{X_3} = \max_{x\in\mathbb{R}^N} |u(x)|$,

$$X_3^+ = \{u \in X_3 \mid u(x) \geq 0, x \in \mathbb{R}^N\}, \tag{12}$$

and

$$X_3^{++} = Int(X_3^+) = \{u \in X_3^+ \mid u(x) > 0, x \in \mathbb{R}^N\}. \tag{13}$$

Let

$$\mathcal{K}_i : X_i \to X_i, (\mathcal{K}_{iu})(x) = D\, k(y - x)u(y)dy \,\forall u \in X_i, i = 1,2, \tag{14}$$

and

$$\mathcal{K}_3 : X_3 \to X_3, (\mathcal{K}_{3u})(x) = \mathbb{R}^N\, k(y - x)u(y)dy \,\forall u \in X_3. \tag{15}$$

Observe that $X_2 = X_1$ and $K_2 = K_1$. The introduction of X_2 and $\mathcal{K}_2$ is for convenience. We denote the identity map in the space under consideration.

Let

$$\begin{cases} h_1(x) = -\nu_1 + a_1(x), \\ h_2(x) = -\nu_2 \int_D k(y - x)dy + a_2(x), \\ h_3(x) = -\nu_3 + a_3(x). \end{cases} \tag{16}$$

So, we have

$$h_i(\cdot)I : X_i \to X_i, (h_i(\cdot)\mathcal{T}u)(x) = h_i(x)u(x) \,\forall u \in X_i, i = 1,2,3, \tag{17}$$

where $a_i \in X_i, i = 1,2,3$ and $a_i(\cdot)I$ has the same meaning as in (17) with $h_i(\cdot)$ being replaced by $a_i(\cdot)$.

In the following, for (iii), we put

$$D = [0,p_1] \times [0,p_2] \times \cdots \times [0,p_N]. \tag{18}$$

For given $a_i \in X_i$, let

$$\hat{a}_i = \frac{1}{|D|} \int_D a_i(x)dx, i = 1,2,3, \tag{19}$$

where $|D|$ is the Lebesgue measure of D. Let

$$a_{i,\max} = \max_{x\in\bar{D}} a_i(x), \qquad a_{i,min} = \min_{x\in\bar{D}} a_i(x),$$

and

$$h_{i,max} = \max_{x\in\bar{D}} h_i(x), \qquad h_{i,\min} = \min_{x\in\bar{D}} h_i(x).$$

Let $\sigma(\nu_i\mathcal{K}_I + h_i(\cdot)I)$be the spectrum of $\nu_i\mathcal{K}_I + h_i(\cdot)\mathcal{T}$ for $I = 1,2,3$ and

$$\tilde{\lambda}_i(\nu_i, a_i) = \sup\{Re\mu \mid \mu \in \sigma(\nu_i\mathcal{K}_i + h_i(\cdot)\mathcal{T})\}, \qquad i = 1,2,3. \tag{20}$$

Definition (6.1.1)[176]: Let $1 \leq i \leq 3$ be given.

(i) $\tilde{\lambda}_i(\nu_i, a_i)$ defined in (20) is called the principal spectrum point of $\nu_i\mathcal{K}_i + h_i(\cdot)\mathcal{T}$.

(ii) A real number $\lambda_i(\nu_i, a_i) \in \mathbb{R}$ is called the principal eigenvalue of $\nu_i\mathcal{K}_i + h_i(\cdot)\mathcal{T}$ if it is an isolated algebraically simple eigenvalue of $\nu_i\mathcal{K}_i + h_i(\cdot)\mathcal{T}$ with a positive eigenfunction and for any $\mu \in \sigma(\nu_i\mathcal{K}_i + h_i(\cdot)\mathcal{T}) \setminus \{\lambda_i(\nu_i, a_i)\}, Re\mu < \lambda_i(\nu_i, a_i)$.
Observe that $\tilde{\lambda}_i(\nu_i, a_i) \in \sigma(\nu_i\mathcal{K}_i + h_i(\cdot)\mathcal{T})$ (see Proposition (6.1.6)). Observe also that if $\lambda_i(\nu_i, a_i)$ exists $(1 \le i \le 3)$, then

$$\lambda_i(\nu_i, a_i) = \tilde{\lambda}_i(\nu_i, a_i).$$

Consider (7). By general semigroup theory, for any $(u_0, v_0) \in X_1 \times X_2$, (7) has a unique (local) solution $(u(t,x;\, u_0, v_0), v(t,x;\, u_0, v_0))$ with $(u(0,x;\, u_0, v_0), v(0,x;\, u_0, v_0)) = (u_0(x), v_0(x))$. The main results are stated in the following four theorems.

Corollary (6.1.2)[176]: (Criteria for the existence of principal eigenvalues). *Let* $1 \le i \le 3$ be given.

(i) $\lambda_i(\nu_i, a_i)$ exists provided that $\max_{x\in\bar{D}} a_i(x) - \min_{x\in\bar{D}} a_i(x) < \nu_i \inf_{x\in\bar{D}} \int_D k(y - x)dy$ in the case $i = 1, 2$ and $\max_{x\in\bar{D}} a_i(x) - \min_{x\in\bar{D}} a_i(x) < \nu_i$ in the case $i = 3$.

(ii) $\lambda_i(\nu_i, a_i)$ exists provided that $h_i(\cdot)$ is in $C^N(\bar{D})$, there is some $x_0 \in Int(D)$ satisfying that $h_i(x_0) = h_{i,\max}$, and the partial derivatives of $h_i(x)$ up to order $N - 1$ at x_0 are zero.

(iii) There is $\nu_i^0 > 0$ such that the principal eigenvalue $\lambda_i(\nu_i, a_i)$ *of* $\nu_i\mathcal{K}_i + h_i(\cdot)I$ exists for $\nu_i > \nu_i^0$.

(iv) Suppose that $k(z) = \mathcal{K}_\delta(z)$, where $k_\delta(z)$ is defined as in (44) and $\tilde{k}(\cdot)$ is symmetric with respect to 0. Then there is $\delta_0 > 0$ such that the principal eigenvalue $\lambda_i(\nu_i, a_i)$ *of* $\nu_i\mathcal{K}_i + h_i(\cdot)I$ exists for $0 < \delta < \delta_0$.

Proof. (i) and (ii) are Theorem (6.1.14) (i) and (ii), respectively.
(iii) is Theorem (6.1.15)(iii).
(iv) is Theorem (6.1.16)(iii).

We first present some basic properties of the solutions to the following evolution equations associated to the eigenvalue problems (i), (ii), and (iii),

$$\partial_t u(t,x) = \nu_1\left[\int_D k(y - x)u(t,y)dy - u(t,x)\right] + a_1(x)u(t,x), x \in \bar{D}, \qquad (21)$$

$$\partial_t u(t,x) = \nu_2 \int_D k(y - x)[u(t,y) - u(t,x)]dy + a_2(x)u(t,x), \quad x \in \bar{D}, \qquad (22)$$

and

$$\begin{cases} (\partial_t u(t,x) = \nu_3[\int_{\mathbb{R}^N} k(y - x)u(t,y)dy - u(t,x)] + a_3(x)u(t,x), & x \in \mathbb{R}^N, \\ u(t, x + p_j e_j) = u(t,x), & x \in \mathbb{R}^N, \end{cases} \qquad (23)$$

respectively.

By general semigroup theory, for any given $u_0 \in X_1$ (resp. $u_0 \in X_2, u_0 \in X_3$), (21) (resp. (22), (23)) has a unique solution $u_1(t,\cdot;\, u_0, \nu_1, a_1) \in X_1$ (resp. $u_2(t,\cdot;\, u_0, \nu_2, a_2) \in X_2, u_3(t,\cdot;\, u_0, \nu_3, a_3) \in X_3$) with $u_i(0,x;\, u_0, \nu_i, a_i) = u_0(x)$ $(i = 1,2,3)$. As mentioned before, by general semigroup theory, for any given $(u_0, v_0) \in X_1 \times X_2$, (7) also has a unique (local) solution $(u(t,\cdot;\, u_0, v_0), v(t,\cdot;\, u_0, v_0))$ with $(u(0,x;\, u_0, v_0), v(0,x;\, u_0, v_0)) = (u_0(x), v_0(x))$. For given $u^1, u^2 \in X_i$, we define

$$u^1 \le u^2, if\ u^2 - u^1 \in X_i^+,$$

and

$$u^1 \ll u^2, if\ u^2 - u^1 \in X_i^{++}.$$

Definition (6.1.3)[176]: A continuous function $u(t,x)$ on $[0,\tau) \times \bar{D}$ is called a super-solution (or sub-solution) of (21) if for any $x \in \bar{D}$, $u(t,x)$ is differentiable on $[0,\tau)$ and satisfies that

$$\partial_t u(t,x) \geq (or \leq) \nu_1 h \left[\int_D k(y-x)u(t,y)dy - u(t,x)i\right] + a_1(x)u(t,x)$$

for $t \in [0,\tau)$.

Super-solutions and sub-solutions of (22) and (23) are defined in an analogous way.

Proposition (6.1.4)[176]: (Comparison principle).

(i) If $u^1(t,x)$ and $u^2(t,x)$ are bounded sub- and super-solution of (21) (resp. (22), (23)) on [0, τ), respectively, and $u^1(0,\cdot) \leq u^2(0,\cdot)$, then $u^1(t,\cdot) \leq u^2(t,\cdot)$ for $t \in [0,\tau)$.

(ii) For given $1 \leq i \leq 3$, if $u^1, u^2 \in X_i, u^1 \leq u^2$ and $u^1 \not\equiv u^2$, then $u_i(t,\cdot; u^1, \nu_i, a_i) \ll u_i(t,\cdot; u^2, \nu_i, a_i)$ for all $t > 0$.

(iii) For given $1 \leq i \leq 3, u_0 \in X_i^+$, and $a_i^1, a_i^2 \in X_i$, if $a_i^1 \leq a_i^2$, then $u_i(t,\cdot; u_0, \nu_i, a_i^1) \leq ui(t,\cdot; u_0, \nu_i, a_i^2)$ for $t \geq 0$.

Proof. (i) It follows from the arguments in [200].

(ii) It follows from the arguments in [200].

(iii) We consider the case $i = 1$. Other cases can be proved similarly.

Note that $u_1(t,\mathcal{X}; \nu_1, a_1^2)$ is a supersolution of (21) with $a_1(\cdot)$ being replaced by $a_1^1(\cdot)$. Then by (i),

$$u_1(t,\cdot; u_0, \nu_1, a_1^1) \leq u_1(t,\cdot; u_0, \nu_1, a_1^2) \; \forall t \geq 0.$$

Next, we consider (7) and present some basic properties for solutions of the two species competition system.

For given $(u^1, v^1), (u^2, v^2) \in X_1 \times X_2$, we define

$$(u^1, v^1) \leq 1 (u^2, v^2), if\; u^1(x) \leq u^2(x), v^1(x) \leq v^2(x),$$

and

$$(u^1, v^1) \leq 2 (u^2, v^2), if\; u^1(x) \leq u^2(x), v^1(x) \geq v^2(x).$$

Let $T > 0$ and $(u(t,x), v(t,x)) \in C([0,T) \times \bar{D}, \mathbb{R}^2)$ with $(u(t,\cdot), v(t,\cdot)) \in X_1^+ \times X_2^+$. Then $(u(t,x), v(t,x))$ is called a super-solution (sub-solution) of (7) on $[0,T)$ if

$$\begin{cases} (\partial_t u(t,x) \geq (\leq) \nu[\int_D k(y-x)u(t,y)dy - u(t,x)] + u(t,x)f(x, u(t,x) + v(t,x)), x \in \bar{D}, \\ \partial_t v(t,x) \leq (\geq) \nu \int_D k(y-x)[v(t,y) - v(t,x)]dy + v(t,x)f(x, u(t,x) + v(t,x)), x \in \bar{D}, \end{cases}$$

for $t \in [0,T)$.

Proposition (6.1.5)[176]: (i) If $(0,0) \leq_1 (u_0, v_0)$, then $(0,0) \leq 1 (u(t,\cdot; u_0, v_0), v(t,\cdot; u_0, v_0))$ for all $t > 0$ at which $(u(t,\cdot; u_0, v_0), v(t,\cdot; u_0, v_0))$ exists.

(ii) If $(0,0) \leq_1 (u_i, v_i)$, for $i = 1,2, \left(u_1(0,\cdot), v_1(0,\cdot)\right) \leq_2 (u_2(0,\cdot), v_2(0,\cdot))$, and $(u_1(t,x), v1(t,x))$ and $(u_2(t,x), v_2(t,x))$ are a sub-solution and a super-solution of (7) on $[0,T)$ respectively, then $\left(u_1(t,\cdot), v_1(t,\cdot)\right) \leq_2 (u_2(t,\cdot), v_2(t,\cdot))$ for $t \in (0,T)$.

(iii) If $(0,0) \leq 1 (u_i, v_i)$, for $i = 1,2$ and $(u_1, v_1) \leq_2 (u_2, v_2)$, then $(u(t,\cdot; u_1, v_1), v(t,\cdot; u_1, v_1)) \leq 2 (u(t,\cdot; u_2, v_2), v(t,\cdot; u_2, v_2))$ for all $t > 0$ at which both $(u(t,\cdot; u_1, v_1), v(t,\cdot; u_1, v_1))$ and $(u(t,\cdot; u_2, v_2), v(t,\cdot; u_2, v_2))$ exist.

(iv) Let $(u_0, v_0) \in X_1^+ \times X_2^+$, then $(u(t,\cdot; u_0, v_0), v(t,\cdot; u_0, v_0))$ exists for all $t > 0$.

Proof. It follows from the arguments in Proposition (6.1.4) in [185].

We prove some basic properties of principal spectrum points/principal eigenvalues of nonlocal dispersal operators. First of all, we derive some properties of the principal spectrum points of nonlocal dispersal operators by using the spectral radius of the solution operators of the associated evolution equations. To this end, define $\Phi_i(t;\ \nu_i, a_i) : X_i \to X_i$ by

$$\Phi_i(t;\ \nu_i, a_i)u_0 = u_i(t,\cdot;\ u_0, \nu_i, a_i), u_0 \in X_i, i = 1,2,3. \quad (24)$$

Let $r(\Phi_i(t;\ \nu_i, a_i))$ be the spectral radius of $\Phi_i(t;\ \nu_i, a_i)$. We have the following propositions.

Proposition (6.1.6)[176]: Let $1 \le i \le 3$ be given.

(i) For given $t > 0, e^{\tilde{\lambda}_i(\nu_i,a_i)t} = r(\Phi_i(t;\ \nu_i, a_i))$.

(ii) $\tilde{\lambda}_i(\nu_i, a_i) \in \sigma(\nu_i\mathcal{K}_i + h_i(\cdot)\mathcal{T})$.

Proof. Observe that $\nu_i\mathcal{K}_i + h_i(\cdot)\mathcal{T} : X_i \to X_i$ is a bounded linear operator. Then by spectral mapping theorem,

$$e^{\sigma(\nu_i\mathcal{K}_i+h_i(\cdot)\mathcal{T})t} = \sigma(\Phi_i(t;\ \nu_i, a_i)) \setminus \{0\}\ \forall\, t > 0. \quad (25)$$

By Proposition (6.1.4),

$$\Phi_i(t;\ \nu_i, a_i)X_i^+ \subset X_i^+ \quad \forall\, t > 0. \quad (26)$$

Hence $\Phi_i(t;\ \nu_i, a_i)$ is a positive operator on X_i . Then by [195], $r(\Phi_i(t;\ \nu_i, a_i) \in \sigma(\Phi_i(t;\ \nu_i, a_i))$ for any $t > 0$. By (25),

$$e^{\tilde{\lambda}_i(\nu_i,a_i)t} = r(\Phi_i(t;\ \nu_i, a_i)) \quad \forall\, t > 0,$$

and hence $\tilde{\lambda}_i(\nu_i, a_i) \in \sigma(\nu_i\mathcal{K}_i + h_i(\cdot)\mathcal{T})$.

Proposition (6.1.7)[176]: (i) $\tilde{\lambda}_1(\nu_1, 0) < 0$.

(ii) $\tilde{\lambda}_2(\nu_2, 0) = 0$.

(iii) $\tilde{\lambda}_3(\nu_3, 0) = 0$.

Proof. (i) Let $u_0(x) \equiv 1$. Observe that

$$\int_D k(y-x)u_0(y)dy - u_0(x) \le 0,$$

and there is $x_0 \in D$ such that

$$\int_D k(y - x_0)u_0(y)dy - u_0(x) < 0.$$

By Proposition (6.1.4) (ii),

$$0 \ll \Phi_1(t;\ \nu_1, 0)u_0 \ll u_0 \quad \forall\, t > 0,$$

and then

$$\|\Phi_1(t;\ \nu_1, 0)u_0\|k < 1 \ \forall\, t > 0.$$

Note that for any $u_0 \in X_1$ with $\|\tilde{u}_0\| \le 1$, by Proposition (6.1.4) (ii) again,

$$\|\Phi_1(t;\ \nu_1, 0)\tilde{u}_0\| \le \|\Phi_1(t;\ \nu_1, 0)u_0\| < 1\ \forall\, t > 0.$$

This implies that

$$r(\Phi_1(t;\ \nu_1, 0)) < 1\ \forall\, t > 0,$$

and then $\tilde{\lambda}_1(\nu_1, 0) < 0$.

(ii) Let $u_0(\cdot) \equiv 1$. Observe that

$$\Phi_2(t;\ \nu_2, 0)u_0 = u_0 \ \ \forall\, t \ge 0,$$

And

$$\|\Phi_2(t;\ \nu_2, 0)\tilde{u}_0\| \le \|\Phi_2(t;\ \nu_2, 0)u_0\|k = 1$$

for all $t \ge 0$ and $\tilde{u}_0 \in X_2$ with $\|\tilde{u}_0\| \le 1$. It then follows that

$$r\big(\Phi_2(t;\ \nu_2, 0)\big) = 1 \quad \forall\, t \ge 0,$$

and then $\tilde{\lambda}_2(\nu_2, 0) = 0$.

(iii) It can be proved by the similar arguments as in (ii).

Next, we prove some properties of principal spectrum points of nonlocal dispersal operators by using the spectral radius of the induced nonlocal operators $U^i_{a_i,\nu_i,\alpha_i}$ and $V^i_{a_i,\nu_i,\alpha_i}(i = 1,2,3)$, where $\alpha_i > \max_{x\in D} h_i(x)$ $(i = 1,2,3)$,

$$(U^i_{a_i,\nu_i,\alpha_i}u)(x) = \int_D \frac{\nu_i k(y-x)u(y)}{\alpha_i - h_i(y)}\,dy, \quad i = 1,2, \tag{27}$$

$$(U^3_{a_3,\nu_3,\alpha_3}u)(x) = \int_{\mathbb{R}^N} \frac{\nu_3 k(y-x)u(y)}{\alpha_3 - h_3(y)}\,dy, \tag{28}$$

And

$$\left(V^i_{a_i,\nu_i,\alpha_i}u\right)(x) = \frac{\nu_i \int_D k(y-x)u(y)dy}{\alpha_i - h_i(x)} = \frac{\nu_i(\mathcal{K}_i u)(x)}{\alpha_i - h_i(x)}, \quad i = 1,2, \tag{29}$$

$$\left(V^3_{a_3,\nu_3,\alpha_3}u\right)(x) = \frac{\nu_3 \int_{\mathbb{R}^N} k(y-x)u(y)dy}{\alpha_3 - h_3(x)} = \frac{\nu_3(\mathcal{K}_3 u)(x)}{\alpha_3 - h_3(x)}. \tag{30}$$

Observe that $U^i_{a_i,\nu_i,\alpha_i}$ and $V^i_{a_i,\nu_i,\alpha_i}$ are positive and compact operators on X_i $(i = 1,2,3)$. Moreover, there is $n \geq 1$ such that

$$\left(U^i_{a_i,\nu_i,\alpha_i}\right)^n (X_i^+ \setminus \{0\}) \subset X_i^{++}, \qquad i = 1,2,3,$$

And

$$\left(V^i_{a_i,\nu_i,\alpha_i}\right)^n (X_i^+ \setminus \{0\}) \subset X_i^{++}, \qquad i = 1,2,3.$$

Then by Krein-Rutman Theorem,

$$r(U^i_{a_i,\nu_i,\alpha_i}) \in \sigma(U^i_{a_i,\nu_i,\alpha_i}), r(V^i_{a_i,\nu_i,\alpha_i}) \in \sigma(V^i_{a_i,\nu_i,\alpha_i}), \tag{31}$$

and $r(U^i_{a_i,\nu_i,\alpha_i})$ and $r(V^i_{a_i,\nu_i,\alpha_i})$ are isolated algebraically simple eigenvalues of $U^i_{a_i,\nu_i,\alpha_i}$ and $V^i_{a_i,\nu_i,\alpha_i}$ with positive eigenfunctions, respectively.

Proposition (6.1.8)[176]: (i) $\alpha_i > h_{i,\max}$ is an eigenvalue of $\nu_i\mathcal{K}_i + h_i(\cdot)\mathcal{I}$ with $\phi(x)$ being an eigenfunction iff 1 is an eigenvalue of $U^i_{a_i,\nu_i,\alpha_i}$ with $\psi(x) = \left(\alpha_i - h_i(x)\right)\phi(x)$ being an eigenfunction.

(ii) $\alpha_i > h_{i,\max}$ is an eigenvalue of $\nu_i\mathcal{K}_i + h_i(\cdot)\mathcal{I}$ with $\phi(x)$ being an eigenfunction iff 1 is an eigenvalue of $V^i_{a_i,\nu_i,\alpha_i}$ with $\phi(x)$ being an eigenfunction.

Proof. It follows directly from the definitions of $U^i_{a_i,\nu_i,\alpha_i}$ and $V^i_{a_i,\nu_i,\alpha_i}$.

Proposition (6.1.9)[176]: Let $1 \leq i \leq 3$ be given.

(a) $r(U^i_{a_i,\nu_i,\alpha_i})$ is continuous in $\alpha_i(> h_{i,\max})$, strictly decreases as α_i increases, and $r(U^i_{a_i,\nu_i,\alpha_i}) \to 0$ as $\alpha_i \to \infty$.

(b) $r(V^i_{a_i,\nu_i,\alpha_i})$ is continuous in $\alpha_i(> h_{i,\max})$, strictly decreases as α_i increases, and $r(V^i_{a_i,\nu_i,\alpha_i}) \to 0$ as $\alpha_i \to \infty$.

Proof. We prove (a) in the case $i = 1$. The other cases can be proved similarly. First, note that $r(U^1_{a1,\nu_1,\alpha_1})$ is an isolated algebraically simple eigenvalue of $U^1_{a_1,\nu_1,\alpha_1}$. It then follows from the perturbation theory of the spectrum of bounded operators that $r(U^1_{a_1,\nu_1,\alpha_1})$ is continuous in $\alpha_1(> h_{1,\max})$.

Next, we prove that $r(U^1_{a_1,\nu_1,\alpha_1})$ is strictly decreasing as α_1 increases. To this end, fix any $\alpha_1 > h_{1,max}$. Let $\phi_1(\cdot)$ be a positive eigenfunction of $U^1_{a_1,\nu_1,\alpha_1}$ corresponding to the eigenvalue $r(U^1_{a_1,\nu_1,\alpha_1})$. Note that for any given $\tilde{\alpha}_1 > \alpha_1$, there is $\delta_1 > 0$ such that

$$\frac{\tilde{\alpha}_1 - \alpha_1}{\alpha_1 - h_1(x)} > \delta_1 \,\forall\, x \in \bar{D}.$$

This implies that

$$U^1_{a_1,\nu_1,\tilde{\alpha}_1}\ \phi_1\ (x) = \int_D \frac{\nu_1 k(y-x)\phi_1(y)}{\tilde{\alpha}_1 - h_1(y)dy}$$

$$= \int_D \frac{\nu_1 k(y-x)\phi_1(y)}{\alpha_1 - h_1(y)} \cdot \frac{1}{1+\dfrac{\tilde{\alpha}_1-\alpha_1}{\alpha_1-h_1(y)}}\,dy$$

$$\leq \frac{1}{1+\delta_1}\int_D \frac{\nu_1 k(y-x)\phi_1(y)}{\alpha_1 - h_1(y)}\,dy$$

$$= \frac{r\left(U^1_{a_1,\nu_1,\alpha_1}\right)}{1+\delta_1}\,\phi_1(x)\ \forall\, x \in \bar{D}.$$

It then follows that

$$r\left(U^1_{a_1,\nu_1,\tilde{\alpha}_1}\right) \leq \frac{r\left(U^1_{a_1,\nu_1,\alpha_1}\right)}{1+\delta_1} < r(U^1_{a_1,\nu_1,\alpha_1}),$$

and hence $r(U^1_{a_1,\nu_1,\alpha_1})$ is strictly decreasing as α_1 increases.

Finally, we prove that $r\left(U^1_{a_1,\nu_1,\alpha_1}\right) \to 0\ as\ \alpha_1 \to \infty.$ Note that for any $\in > 0$, there is $\alpha_1^* > 0$ such that for $\alpha_1 > \alpha_1^*$,

$$\int_D \frac{\nu_1 k(y-x)}{\alpha_1 - h_1(y)}\,dy < \epsilon\ \forall\, x \in \bar{D}.$$

This implies that

$$\left\|U^1_{a_1,\nu_1,\alpha_1}\right\| < \epsilon\ \ \forall\, \alpha_1 > \alpha_1^*.$$

Hence $r(U^1_{a_1,\nu_1,\alpha_1}) \to 0$ as $\alpha_1 \to \infty$.

Proposition (6.1.10)[176]: Let $1 \leq i \leq 3$ be given.

(a) If there is $\alpha_i > h_{i,\max}$ such that $r(U^i_{a_i,\nu_i,\alpha_i}) > 1$, then $\tilde{\lambda}_i(\nu_i, a_i) > h_{i,\max}$.

(b) If there is $\alpha_i > h_{i,\max}$ such that $r(V^i_{a_i,\nu_i,\alpha_i}) > 1$, then $\tilde{\lambda}_i(\nu_i, a_i) > h_{i,\max}$.

Proof. We prove (b). (a) can be proved similarly.

Fix $1 \leq i \leq 3$. Suppose that there is $\alpha_i > h_{i,\max}$ such that $r(V_i\, a_i, \nu_i, \alpha_i) > 1$. Then By Proposition (6.1.9), there is $\alpha_0 > h_{i,\max}$ such that

$$r(V^i_{a_i,\nu_i,\alpha_0}) = 1. \qquad (32)$$

By Proposition (6.1.8), $\alpha_0 \in \sigma(\nu_i\mathcal{K}_i + h_i(\cdot)I)$. This implies that $\tilde{\lambda}_i(\nu_i, a_i) \geq \alpha_0 > h_{i,\max}$.

Proposition (6.1.11)[176]: (Necessary and sufficient condition). For given $1 \leq i \leq 3, \lambda_i(\nu_i, a_i)$ exists if and only if $\tilde{\lambda}_i(\nu_i, a_i) > h_{i,\max}$

Proof. For $1 \leq i \leq 3, \nu_i\mathcal{K}_i$ is a compact operator. Hence $\nu_i\mathcal{K}_i + h_i(\cdot)I$ can be viewed as compact perturbation of the operator $h_i(\cdot)I$. Clearly, the essential spectrum $\sigma_{ess}(h_iI)$ of $h_i(\cdot)I$ is given by

$$\sigma_{ess}(h_iI) = [h_{i,\min}, h_{i,\max}].$$

Since the essential spectrum is invariant under compact perturbations (see [182]), we have

$$\sigma_{ess}(\nu_i\mathcal{K}_i + h_iI) = [h_{i,\min}, h_{i,\max}],$$

where $\sigma_{ess}(\nu_i\mathcal{K}_i + h_iI)$ is the essential spectrum of $\nu_i\mathcal{K}_i + h_i(\cdot)I. Let$

$$\sigma_{disc}(\nu_i\mathcal{K}_i + h_iI) = \sigma(\nu_i\mathcal{K}_i + h_iI)\backslash\sigma_{ess}(\nu_i\mathcal{K}_i + h_iI).$$

Note that if $\lambda \in \sigma_{disc}(\nu_i\mathcal{K}_i + h_iI)$, then it is an isolated eigenvalue of finite multiplicity. On the one hand, if $\tilde{\lambda}_i(\nu_i, a_i) > h_{i,\max}(x), then\ \tilde{\lambda}_i(\nu_i, a_i) \in \sigma_{disc}(\nu_i\mathcal{K}_i + h_iI)$. By Proposition (6.1.8), $1 \in \sigma\ U^i_{a_i,\nu_i,\tilde{\lambda}_i}(\nu_i, a_i)$. Hence

$$r\left(U^i_{a_i,\nu_i,\tilde{\lambda}_i(\nu_i,a_i)}\right) \geq 1.$$

By Proposition (6.1.9), there is $\tilde{\tilde{\lambda}} \geq \tilde{\lambda}_i(\nu_i, a_i)$ such that

$$r\left(U^i_{a_i,\nu_i,\tilde{\tilde{\lambda}}}\right) = 1.$$

This together with Proposition (6.1.8) implies that $\tilde{\tilde{\lambda}}$ is an isolated algebraically simple eigenvalue of $\nu_i\mathcal{K}_i + h_i(\cdot)I$ with a positive eigenfunction. By Definition (6.1.1) (ii), $\lambda_i(\nu_i, a_i)$ exists.

On the other hand, if $\lambda_i(\nu_i, a_i)$ exists, then $\tilde{\lambda}_i(\nu_i, a_i) = \lambda_i(\nu_i, a_i) \in \sigma_{disc}(\nu_i\mathcal{K}_i + h_iI)$. This implies that $\tilde{\lambda}_i(\nu_i, a_i) > h_{i,\max}(x)$.

Finally, we present some variational characterization of the principal spectrum points of nonlocal dispersal operators when the kernel function is symmetric. We assume that $k(\cdot)$ is symmetric with respect to 0. Recall

$$K_3 : X_3 \to X_3, (\mathcal{K}_3u)(x) = \int_{\mathbb{R}^N} k(y - x)u(y)dy \, \forall\, u \in X_3.$$

For given $a \in X_3$, let

$$\hat{k}(z) = \sum_{j_1,j_2,\cdots,j_N \in \mathcal{Z}} k(z + (j_1p_1, j_2p_2,\cdots, j_N p_N)), \quad (33)$$

where $p_1, p_2,\cdots p_N$ are periods of a(x). Then $\hat{k}(\cdot)$ is also symmetric with respect to 0 and

$$(\mathcal{K}_3u)(x) = \int_D \hat{k}(y - x)u(y)dy \, \forall\, u \in X_3, \quad (34)$$

where $D = [0, p_1] \times [0, p_2] \times\cdots\times [0, p_N]$ (see (18)).

Proposition (6.1.12)[176]: Assume that $k(\cdot)$ is symmetric with respect to 0. Then

$$\tilde{\lambda}_i(\nu_i, a_i) = \sup_{u\in L^2(D), \|u\|L^2(D)=1} \int_D [\nu_i(K_iu)(x)u(x) + h_i(x)u^2(x)]dx \; (i = 1,2,3)$$

Proof. First of all, note that $\nu_i\mathcal{K}_i + h_i(\cdot)I$ is also a bounded operator on $L^2(D)$ and $\nu_i\mathcal{K}_i$ is a compact operator on $L^2(D)$, where $\mathcal{K}_i$ is defined as in (34) when $i = 3$. Let $\sigma(\nu_i\mathcal{K}_i + h_iI, L^2(D))$ be the spectrum of $\nu_i\mathcal{K}_i + h_i(\cdot)I$ considered on $L^2(D)$ and

$$\tilde{\lambda}(\nu_i, a_i, L^2(D)) = \sup\{Re\lambda \mid \lambda \in \sigma(\nu_i\mathcal{K}_i + h_iI, L^2(D))\}.$$

Then we also have

$\tilde{\lambda}(\nu_i, a_i, L^2(D)) \in \sigma(\nu_i\mathcal{K}_i + h_iI, L^2(D)), [h_{i,\min}, h_{i,\max}] \subset \sigma(\nu_i\mathcal{K}_i + h_iI, L^2(D))$, and

$$\tilde{\lambda}(\nu_i, a_i, L^2(D)) \geq h_{i,\max}.$$

Moreover, if $\tilde{\lambda}_i(\nu_i, a_i) > h_{i,\max}$ $(resp. \tilde{\lambda}_i(\nu_i, a_i, L^2(D)) > h_{i,\max})$, then $\tilde{\lambda}_i(\nu_i, a_i)$ $(resp. \tilde{\lambda}_i(\nu_i, a_i, L^2(D)))$ is an eigenvalue of $\nu_i\mathcal{K}_i + h_iI$ considered on $L^2(D)$ $(resp. C(\bar{D}))$ and hence $\tilde{\lambda}_i(\nu_i, a_i, L^2(D)) \geq \tilde{\lambda}_i(\nu_i, a_i)$ $(resp. \tilde{\lambda}_i(\nu_i, a_i) \geq \tilde{\lambda}_i(\nu_i, a_i, L^2(D)))$. We then must have

$$\tilde{\lambda}_i(\nu_i, a_i) = \tilde{\lambda}_i(\nu_i, a_i, L^2(D)).$$

Assume now that $k(\cdot)$ is symmetric with respect to 0, that is, $k(-z) = k(z)$ $for\ any\ z \in \mathbb{R}^N$. Then for any $u, v \in L^2(D)$, in the case $i = 1, 2$,

$$\begin{aligned}\int_D (\mathcal{K}_iu)(x)v(x)dx &= \int_D\int_D k(y - x)u(y)v(x)dydx \\ &= \int_D\int_D k(x - y)u(x)v(y)dxdy\end{aligned}$$

$$= \int_D \int_D k(y - x)v(y)u(x)dydx$$
$$= \int_D (\mathcal{K}_i v)(x)u(x)dx$$
and in the case $i = 3$,
$$\int_D (\mathcal{K}_3 u)(x)v(x)dx = \int_D \int_D \hat{k}(y - x)u(y)v(x)dydx$$
$$= \int_D \int_D \hat{k}(x - y)u(x)v(y)dxdy$$
$$= \int_D \int_D \hat{k}(y - x)v(y)u(x)dydx$$
$$= \int_D (\mathcal{K}_3 v)(x)u(x)dx.$$
Therefore $\mathcal{K}_i : L^2(D) \to L^2(D)$ is self-adjoint. By classical variational formula (see [141]), we have
$$\tilde{\lambda}_i\big(\nu_i, a_i, L^2(D)\big) = \sup_{u\in L^2(D), \|u\|_{L^2(D)}=1} \int_D [\nu_i(\mathcal{K}_i u)(x)u(x) + h_i(x)u^2(x)]dx.$$
The proposition then follows.

We provide a useful technical lemma.

Lemma (6.1.13)[176]: Let $1 \le i \le 3$ and $a_i \in X_i$ be given. For any $\epsilon > 0$, there is $a_i^\epsilon \in X_i$ such that
$$\|a_i - a_i^\epsilon\| < \epsilon,$$
$h_i^\epsilon(x) = -\nu_i + a_i^\epsilon(x)$ for $i = 1$ or 3 and $h_i^\epsilon(x) = -\nu_i \int_D k(y - x)dy + a_i^\epsilon(x)$ for $i = 2$ is in C^N, and satisfies the following vanishing condition: there is $x_0 \in Int(D)$ such that $h_i^\epsilon(x_0) = \max_{x\in\bar{D}} h_i^\epsilon(x)$ and the partial derivatives of $h_i^\epsilon(x)$ up to order $N - 1$ at x_0 are zero.

Proof. We prove the case $i = 2$. Other cases can be proved similarly.

First, let $\tilde{x}_0 \in \bar{D}$ be such that
$$h_2(\tilde{x}_0) = \max_{x\in\bar{D}} h_2(x).$$
For any $\epsilon > 0$, there is $\tilde{x}_\epsilon \in Int(D)$ such that
$$h_2(\tilde{x}_0) - h_2(\tilde{x}_\epsilon) < \frac{\epsilon}{3}. \tag{35}$$
Let $\tilde{\sigma} > 0$ be such that
$$B(\tilde{x}_\epsilon, \tilde{\sigma}) \Subset D,$$
where $B(\tilde{x}_\epsilon, \tilde{\sigma})$ denotes the open ball with center $\tilde{x}_\epsilon$ and radius $\tilde{\sigma}$.

Note that there is $\xi(\cdot) \in C(\bar{D})$ such that $0 \le \xi(x) \le 1, \xi(\tilde{x}_\epsilon) = 1$, and $\text{supp}(\xi) \subset B(\tilde{x}_\epsilon, \tilde{\sigma})$.
$$h_{2,\epsilon}(x) = h_2(x) + \frac{\epsilon}{3}\xi(x). \tag{36}$$
Then $h_{2,\epsilon}(\cdot)$ is continuous on D and $h_{2,\epsilon}(\cdot)$ attains its maximum in Int(D).

Let $\tilde{D} \subset \mathbb{R}^N$ be such that $\Subset \tilde{D}$. Note that $h_{2,\epsilon}(\cdot)$ can be continuously extended to $\tilde{D}$. Without loss of generality, we may then assume that $h_{2,\epsilon}(\cdot)$ is a continuous function on $\tilde{D}$ and there is $x_0 \in \text{Int}(D)$ such that $h_{2,\epsilon}(x_0) = \sup_{x\in\tilde{D}} h_{2,\epsilon}(x)$. Observe that there is $\sigma > 0$ and $\bar{h}_{2,\epsilon}(\cdot) \in C(\tilde{D})$ such that $B(x_0, \sigma) \Subset D$,

$$0 \leq \bar{h}_{2,\epsilon}(x) - h_{2,\epsilon}(x) \leq \frac{\epsilon}{3} \quad \forall\, x \in \widetilde{D}, \tag{37}$$

And

$$\bar{h}_{2,\epsilon}(x) = h_{2,\epsilon}(x_0) \qquad \forall\, x \in B(x_0,\sigma).$$

Let

$$\eta(x) = \begin{cases} C \exp(\frac{1}{\|x\|^2 - 1}) & \text{if } \|x\| < 1, \\ 0 & \text{if } \|x\| \geq 1, \end{cases}$$

where $C > 0$ is such that $\int_{\mathbb{R}^N} \eta(x)dx = 1$. For given $\delta > 0$, set

$$\eta_\delta(x) = \frac{1}{\delta^N}\, \eta(\frac{x}{\delta}).$$

Let

$$h_{2,\epsilon,\delta}(x) = \int_{\widetilde{D}} \eta_\delta(y - x)\bar{h}_{2,\epsilon}(y)dy.$$

By [143], $h_{2,\epsilon,\delta}(\cdot)$ is in $C^\infty(\widetilde{D})$ and when $0 < \delta \ll 1$,

$$\left|h_{2,\epsilon,\delta}(x) - \bar{h}_{2,\epsilon}(x)\right| < \frac{\epsilon}{3} \quad \forall\, x \in \bar{D}. \tag{38}$$

It is not difficulty to see that for $0 < \delta \ll 1$,

$$h_{2,\epsilon,\delta}(x) \leq \bar{h}_{2,\epsilon}(x_0) \quad \forall x \in B(x_0,\sigma),$$

And

$$h_{2,\epsilon,\delta}(x) = \bar{h}_{2,\epsilon}(x_0) \quad \forall x \in B(x_0,\sigma/2).$$

Fix $0 < \delta \ll 1$. Let

$$h_2^\epsilon(x) = h_{2,\epsilon,\delta}(x).$$

Then $h_2^\epsilon(\cdot)$ attains its maximum at some $x_0 \in \text{Int}(D)$, and the partial derivatives of $h_2^\epsilon(\cdot)$ up to order $N - 1$ at x_0 are zero. Let

$$a_2^\epsilon(x) = h_2^\epsilon(x) + \nu_2 \int_D k(y - x)dy \quad \forall\, x \in \bar{D}.$$

Then $a_2^\epsilon \in X_2, -\nu_2 \int_D k(y - x)dy + a_2^\epsilon(x) = h_2^\epsilon(x)$, and

$\|a_2 - a_2^\epsilon\| = \|h_2^\epsilon - h_2\| \leq \|h_2^\epsilon - \bar{h}_{2,\epsilon}\| + \|\bar{h}_{2,\epsilon} - h_{2,\epsilon}\| + \|h_{2,\epsilon} - h_2\| < \epsilon.$

The lemma is thus proved.

We investigate the effects of spatial variations on the principal spectrum points/principal eigenvalues of nonlocal dispersal operators and prove Theorem (6.1.14). First of all, for given $1 \leq i \leq 3$ and $c_i \in \mathbb{R}$, let

$$X_i(c_i) = \{a_i \in X_i \mid \hat{a}_i = c_i\}$$

(see (19) for the definition of $\hat{a}_i$). For given $x_0 \in \mathbb{R}^N$ and $\sigma > 0$, let

$$B(x_0,\sigma) = \{y \in \mathbb{R}^N \mid \|y - x_0\| < \sigma\}.$$

Theorem (6.1.14)[176]: (Effects of spatial variation). (i) (Existence of principal eigenvalues) For given $1 \leq i \leq 2, \lambda_i(\nu_i, a_i)$ exists if $a_{i,max} - a_{i,min} < \nu_i \inf_{x \in \bar{D}} \int_D k(y - x)dy$.

(ii) (Existence of principal eigenvalues) For given $1 \leq i \leq 2, \lambda_i(\nu_i, a_i)$ exists if $h_i(\cdot)$ is in $C^N(\bar{D})$, there is some $x_0 \in Int(D)$ satisfying that $h_i(x_0) = h_{i,max}$, and the partial derivatives of $h_i(x)$ up to order $N - 1$ at x_0 are zero.

(iii) (Upper bounds) For given $1 \leq i \leq 3$ and $c_i \in \mathbb{R}$, $\sup\{\tilde{\lambda}_i(\nu_i, a_i) \mid a_i \in X_i, \hat{a}_i = c_i\} = \infty$.

(iv) (Lower bounds) Assume that $k(\cdot)$ is symmetric with respect to 0 (i.e. $k(-z) = k(z)$) and $i = 2$. For given $c_i \in \mathbb{R}$, $\inf\{\tilde{\lambda}_i(\nu_i, a_i) | a_i \in X_i, \hat{a}_i = c_i\} = \lambda\_i(\nu_i, c_i)(= c_i)$ (hence $\tilde{\lambda}_i(\nu_i, a_i) \geq \tilde{\lambda}_i(\nu_i, \hat{a}_i)$). If the principal eigenvalue of $\nu_i\mathcal{K}_i + h_i(\cdot)\mathcal{T}$ exists, then " = " holds if and only if $a_i(\cdot)$ is a constant function, that is $a_i(\cdot) \equiv \hat{a}_i$.

(v) (Monotonicity) For given $a_i^1, a_i^2 \in X_i, if\ a_i^1(x) \leq a_i^2(x), then\ \tilde{\lambda}_i(a_i^1, \nu_i) \leq \tilde{\lambda}_i(a_i^2, \nu_i)$ $(i = 1, 2, 3)$.

Proof. (i) We first prove the case $i = 1$. Let $x_0 \in \bar{D}$ be such that

$$h_1(x_0) = h_{1,\max}.$$

Note that there is $\epsilon_0 > 0$ such that

$$0 \leq a_1(x_0) - a_1(x) < \nu_1 \inf_{x \in \bar{D}} \int_D k(y - x)dy - \epsilon_0$$
$$\leq \nu_1 \int_D k(y - x)dy - \epsilon_0 \quad \forall x \in \bar{D}.$$

For any $0 < \epsilon < \epsilon_0$, put

$$\lambda_\epsilon = h_1(x_0) + \epsilon (= -\nu_1 + a_1(x_0) + \epsilon).$$

Then

$$\frac{\nu_1 \int_D k(y - x)dy}{\lambda_\epsilon - h_1(x)} = \frac{\nu_1 \int_D k(y - x)dy}{a_1(x_0) - a_1(x) + \epsilon}$$
$$\geq \frac{\nu_1 \int_D k(y - x)dy}{\nu_1 \int_D k(y - x)dy + \epsilon - \epsilon_0}$$
$$> 1 \ \forall x \in \bar{D}.$$

This implies

$$r(V^1_{a_1,\nu_1,\lambda_\epsilon}) > 1 \ \forall\ 0 < \epsilon \ll 1.$$

Then by Proposition (6.1.10) (b), $\tilde{\lambda}_1(\nu_1, a_1) > h_{1,\max}$. By Proposition (6.1.11), $\lambda_1(\nu_1, a_1)$ exists. We now prove the case $i = 2$. Similarly, let $x_0 \in \bar{D}$ be such that

$$h_2(x_0) = h_{2,\max}.$$

Note that there is $\epsilon_0 > 0$ such that

$$0 \leq a_2(x_0) - a_2(x) < \nu_2 \inf_{x \in \bar{D}} \int_D k(y - x)dy - \epsilon_0$$
$$\leq \nu_2 \int_D k(y - x_0)dy - \epsilon_0.$$

For any $0 < \epsilon < \epsilon_0$, put

$$\lambda_\epsilon = h_2(x_0) + \epsilon(= -\nu_2 \int_D k(y - x_0)dy + a_2(x_0) + \epsilon).$$

Then

$$\nu_2 \int_D k(y - x)dy\, \lambda_\epsilon$$
$$- h_2(x) = \nu_2 \int_D k(y - x)dy\, a_2(x_0) - \nu_2 \int_D k(y - x_0)dy$$
$$+ \nu_2 \int_D k(y - x)dy - a_2(x) + \epsilon \geq \nu_2 \int_D k(y - x)dy\, \nu_2$$
$$\int_D k(y - x)dy + \epsilon - \epsilon_0 > 1 \ \forall x \in \bar{D}.$$

This again implies that

$$r\left(V^2_{a_2,\nu_2,\lambda_\epsilon}\right) > 1 \,\forall\, 0 < \epsilon \ll 1.$$

Then by Proposition (6.1.10) (b), $\tilde{\lambda}_2(\nu_2, a_2) > h_{2,\max}$. By Proposition (6.1.11), $\lambda_2(\nu_2, a_2)$exists. (ii) It can be proved by the similar arguments as in [200]. For the completeness, we provide a proof in the following. Let $x_0 \in Int(D)$ be such that $h_i(x_0) = h_{i,\max}$ and the partial derivatives of $h_i(x)$ up to order $N - 1$ at x_0 are zero. Then there is $M > 0$ such that

$$h_i(x_0) - h_i(y) \leq M\left||x_0 - y|\right|^N \,\forall\, y \in D.$$

Fix $\sigma > 0$ such that $B(x_0, 2\sigma) \subset D\ and\ B(0, 2\sigma) \Subset supp(k(\cdot)). Let\ v^* \in X + i$ be such that $v^*(x) = (1 \,\forall\, x \in B(x_0, \sigma), 0 \,\forall\, x \in D\backslash B(x_0, 2\sigma)$.

Clearly, for every $x \in D\backslash B(x_0, 2\sigma)$ and $\gamma > 1$, we have

$$(U^i_{a_i,\nu_i,h_i}(x_0) + \epsilon\, v^*)(x) \geq \gamma v^*(x) = 0 \,\forall\, \epsilon > 0. \tag{39}$$

Note that there is $\widetilde{M} > 0$ such that for any $x \in B(x_0, 2\sigma), k(y - x) \geq \widetilde{M} \forall\, y \in B(x_0, \sigma)$.

It then follows that for $x \in B(x_0, 2\sigma)$

$$\left(U^i_{a_i,\nu_i,h_i(x_0)} + \epsilon\, v^*\right)(x) = \int_D \frac{\nu_i k(y - x) v^*(y)}{h_i(x_0) + \epsilon - h_i(y)}\, dy$$

$$\geq \int_{B(x_0,\sigma)} \frac{\nu_i k(y - x)}{M\left||x_0 - y|\right|N + \epsilon}\, dy$$

$$\geq \int_{B(x_0,\sigma)} \frac{\nu_i \widetilde{M}}{M||x_0 - y||N + \epsilon}\, dy.$$

Notice that $\int_{B(x_0,\sigma)} \frac{\widetilde{M}}{M\,||x_0 - y||^N}\, dy = \infty$. This implies that for $0 < \epsilon \ll 1$, there is $\gamma > 1$ such that

$$\left(U^i_{a_i}, \nu_i, h_i(x_0)\right) + \epsilon^{v^*}\ )(x) > \gamma v^*(x) \,\forall\, x \in B(x_0, 2\sigma). \tag{40}$$

By (39) and (40),

$$\left(U^i_{a_i}, \nu_i, h_i(x_0)\right) + \epsilon^{v^*}\ (x) \geq \gamma v^*(x) \,\forall\, x \in D.$$

Hence, $r\left(U^i_{a_i,\nu_i,h_i(x_0)+\epsilon}\right) > 1$. By Proposition (6.1.10)(a), $\tilde{\lambda}\, i(\nu_i, a_i) > h_i(x_0) = h_{i,\max}$. By Proposition (6.1.11), the principle eigenvalue $\lambda_i(\nu_i, a_i)$ exists.

(iii) Recall that $\tilde{\lambda}_i(\nu_i, \tilde{a}) = sup\{Re\mu | \mu \in \sigma(\nu_i \mathcal{K}_i + \tilde{h}_i(\cdot) I)\}$ with $\tilde{h}_i(x) = -\nu_i + \tilde{a}(x)$ for $i = 1, 3$ and $\tilde{h}_i(x) = -\nu_2 \int_D k(y - x) dy + \tilde{a}(x)$ for $i = 2$. By the arguments of Proposition (6.1.11),

$$\sigma_{ess}(\nu_i \mathcal{K}_i + \tilde{h}_i I) = [\min_{x\in\bar{D}} \tilde{h}_i(x), \max_{x\in\bar{D}} \tilde{h}_i(x)].$$

Note that

$$\sup_{\tilde{a}\in X_i(c_i)} (\max_{x\in\bar{D}} \tilde{a}(x)) = \infty.$$

Then

$$\sup_{\tilde{a}\in X_i(c_i)} \tilde{\lambda}_i(\nu_i, \tilde{a}) \geq \sup_{\tilde{a}\in X_i(c_i)} (\max_{x\in D} \tilde{h}_i(x)) \geq -\nu_i + \sup_{\tilde{a}\in X_i(c_i)} (\max_{x\in D} \tilde{a}(x)) = \infty.$$

(iv) We first assume that the principal eigenvalue $\lambda_2(\nu_2, a_2)$ exists. Suppose that $u_2(x)$ is a strictly positive principal eigenfunction with respect to the eigenvalue $\lambda_2(\nu_2, a_2)$. We divide both sides of (ii) by $u_2(x)$ and integrate with respect to x over D to obtain

$$\int_D \left[\frac{\nu_2\left[\int_D\ k(y-x)\big(u_2(y)-u_2(x)\big)dy\right]+\ a_2(x)u_2(x)}{u_2(x)}\right]\ dx\ =\ \int_D\ \lambda_2(\nu_2,a_2)dx,$$
Or
$$\lambda_2(\nu_2,a_2)=\frac{\nu_2}{|D|}\int_D\int_D\ k(y-x)u_2(y)-\ u_2(x)u_2(x)dydx\ +\frac{1}{|D|}\int_D\ a_2(x)dx$$
$$=\frac{\nu_2}{|D|}\int_D\int_D\ k(y-x)u_2(y)\ -\ u_2(x)u_2(x)\,dydx\ +\ \hat{a}_2.$$
By the symmetry of $k(\cdot)$,
$$\int_D\int_D\ k(y-x)u_2(y)-\ u_2(x)u_2(x)dydx$$
$$=\frac{1}{2}\int\int_{D\times D}\ k(y-x)u_2(y)-\ u_2(x)u_2(\,x)dydx$$
$$+\frac{1}{2}\int\int_{D\times D}\ k(y-x)u_2(y)-\ u_2(x)u_2(x)dydx$$
$$=\frac{1}{2}\int\int_{D\times D}\ k(y-x)u_2(y)-\ u_2(x)u_2(x)dydx$$
$$+\frac{1}{2}\int\int_{D\times D}\ k(y-x)u_2(x)-\ u_2(y)u_2(y)dydx$$
$$=\frac{1}{2}\int\int_{D\times D}\ k(y-x)\frac{\big(u_2(y)-\ u_2(x)\big)^2}{u_2(x)u_2(y)}\ dydx\ \geq\ 0. \tag{41}$$
So,
$$inf\{\lambda_2(\nu_2,a_2)|a_2\ \in\ X_2,\hat{a}_2\ =\ c_2\}\ \geq\ \hat{a}_2\ =\ c_2.$$
And clearly, $\lambda_2(\nu_2,\hat{a}_2)\ =\ \hat{a}_2$. Together, we get
$$inf\{\lambda_2(\nu_2,a_2)|a_2\ \in\ X_2,\hat{a}_2\ =\ c_2\}\ =\ \lambda_2(\nu_2,\hat{a}_2)\ =\ c_2.$$
Second, by Lemma (6.1.13), for any $\epsilon\ >\ 0$, there is $a_2^\epsilon\ \in\ X_2\ \cap\ C^N$, such that
$$\|a_2\ -\ a_2^\epsilon\|\ <\ \epsilon,$$
and $h_2^\epsilon(\cdot)\ \in\ C^N\,(=-\nu_2\ \int_D\ k(y-x)dy\ +\ a_2^\epsilon)$ satisfies the vanishing condition in Theorem (6.1.14) (ii). So, the principal eigenvalue $\lambda_2(\nu_2,a_2^\epsilon)$ exists and $\tilde{\lambda}_2(\nu_2,a_2^\epsilon)\ =\ \lambda_2(\nu_2,a_2^\epsilon)$. By the above arguments,
$$\tilde{\lambda}_2(\nu_2,a_2^\epsilon)\ =\ \lambda_2(\nu_2,a_2^\epsilon)\ \geq\ \lambda_2(\nu_2,\hat{a}_2^\epsilon)\ =\ \hat{a}_2^\epsilon. \tag{42}$$
We claim that
$$\lim_{\epsilon\to 0}\ \ \tilde{\lambda}_2(\nu_2,a_2^\epsilon)\ =\ \tilde{\lambda}_2(\nu_2,a_2).$$
In fact, $\|a_2^\epsilon\ -\ a_2\|\ \leq\ \epsilon$, that is
$$a_2(x)\ -\ \epsilon\ \leq\ a_2^\epsilon(x)\ \leq\ a_2(x)\ +\ \epsilon\,\forall\,x\ \in\ \bar{D}.$$
Note that $\Phi_2(t;\ \nu_2,a_2\ +\ \epsilon)u_0\ =\ e^{\epsilon t}\Phi_2(t;\ \nu_2,a_2)u_0$, where $\Phi_2(t;\ \nu_2,a_2)u_0$ is the solution of (22) with the initial value $u_0(\cdot)$. Similarly, we have $\Phi_2(t;\ \nu_2,a_2-\epsilon)u_0\ =\ e^{-\epsilon t}\Phi_2(t;\ \nu_2,a_2)u_0$. So
$$r(\Phi_2(t;\ \nu_2,a_2\ \pm\ \epsilon))\ =\ e^{\pm\epsilon t}r(\Phi_2(t;\ \nu_2,a_2)).$$
Hence
$$\tilde{\lambda}_2(\nu_2,a_2\ \pm\ \epsilon)\ =\ \tilde{\lambda}_2(\nu_2,a_2)\ \pm\ \epsilon. \tag{43}$$
By Proposition (6.1.4), we have
$$\Phi_2(t;\ \nu_2,a_2\ -\ \epsilon)u_0\ \leq\ \Phi_2(t;\ \nu_2,a_2^\epsilon)u_0\ \leq\ \Phi_2(t;\ \nu_2,a_2\ +\ \epsilon)u_0.$$
Hence

$$r(\Phi_2(t;\ \nu_2\ , a_2 - \epsilon))\ \le\ r(\Phi_2(t;\ \nu_2, a_2^{\epsilon}\ ))\ \le\ r(\Phi_2(t;\ \nu_2, a_2\ +\ \epsilon)).$$

By(43),

$$\tilde{\lambda}_2(\nu_2, a_2 - \epsilon) \le\ \tilde{\lambda}_2(\nu_2, a_2^{\epsilon}\ ) \le\ \tilde{\lambda}_2(\nu_2, a_2\ +\ \epsilon).$$

Taking the limit of (42) as $\epsilon\ \to\ 0$, we have

$$\tilde{\lambda}_2(\nu_2, a_2) \ge\ \hat{a}_2$$

So, $\inf\{\tilde{\lambda}_2(\nu_2, a_2) | a_2\ \in\ X_2, \hat{a}_2\ =\ c_2\} =\ \lambda_2(\nu_2, c_2)(=\ c_2)$.

When the principal eigenvalue exists, it is not difficult to prove that the " $=$ " holds if and only if $a_2(\cdot) \equiv\ c_2$. In fact, suppose that $\lambda_2(\nu_2, a_2)$ exists and $u_2(\cdot)$ is a corresponding positive eigenfunction. By (41), $\lambda_2(\nu_2, a_2) = \hat{a}_2(=\ c_2) iff\ u_2(x) =\ u_2(y) for\ all\ x, y\ \in \bar{D}$. Hence $\lambda_2(\nu_2, a_2) =\ \hat{a}_2(=\ c_2) iff\ u_2(\cdot) \equiv$constant, which implies that $a_2(x) = \lambda_2(\nu_2, a_2) = \hat{a}_2$.

(v) Suppose that $a_i^1\ , a_i^2\ \in\ X_i$ and $a_i^1\ \le\ a_i^2$. By Proposition (6.1.4), for any $u_0\ \in\ X_i^+$ and $t\ \ge\ 0$,

$$\Phi_i(t;\ \nu_i\ , a_i^1\ )u_0\ \le\ \Phi_i\big(t;\ \nu_i\ , a_i^2\ \big)u_0.$$

This implies that

$$r(\Phi_i(t;\ \nu_i\ , a_1\ i\ ))\ \le\ r(\Phi_i(t;\ \nu_i\ , a_i^2\ )).$$

By Proposition (6.1.6), we have

$$\tilde{\lambda}_i(\nu_i\ , a_i^1\ )\ \le\ \tilde{\lambda}_i(\ \nu_i\ , a_i^2\ ).$$

We give a proof for the Neumann boundary case. Let $\psi(x)$ be the eigenvalue function of the operator $\Delta\ +\ a_2(\cdot)I$ defined on C^2 ($[0, L]$) with Neumann boundary condition. So $\psi(x)\ >\ 0$ and we have

$$\begin{cases} (\ \psi\ ''(x)\ +\ a_2(x)\psi(x)\ =\ \lambda_{R,2}\psi(x), & x\ \in\ (0, L), \\ \dfrac{\partial_\psi}{\partial_n}\ (x)\ =\ 0, & x\ =\ 0\ or\ L. \end{cases}$$

Multiplying this by $\psi(x)$ and integrating it from 0 to L, we have

$$-\ \int_0^L\ \psi'^2\ (x)dx\ +\ \int_0^L\ a_2(x)\psi^2\ (x)dx\ =\ \lambda_{R,2}\int_0^L\ \psi^2\ (x)dx.$$

Hence

$$\lambda_{R,2}\ =\ \frac{-\ \int_0^L\ \psi'^2\ (x)dx\ +\ \int_0^L\ a_2(x)\psi^2(x)dx}{\int_0^L\ \psi^2(x)dx}.$$

Take $x_1, x_2\ \in\ [0, L)$, we have

$$\psi^2\ (x_2) -\ \psi^2\ (x_1) = \int_{x_1}^{x_2}\ 2\psi(x)\psi'\ (x)dx.$$

Hence, for any positive number $k\ >\ 0$,

$$\psi^2\ \ (x_2) -\ \psi^2\ (x_1) \le \frac{1}{k}\int_0^L\ \psi'^2\ (x)dx\ +\ k\int_0^L\ \psi^2\ (x)dx.$$

Multiplying the above inequality by $a_2(x_2)$ and integrating it with respect to $x_1\ \in [0, L)\ and\ x_2\ \in\ [0, L)$, we get

$$L\int_0^L\ a_2(x_2)\psi^2\ (x_2)dx_2\ -\ c_2L\int_0^L\ \psi^2\ (x_1)dx_1$$
$$\le\ c_2L^2\left(\frac{1}{k}\int_0^L\ \psi'^2\ (x)dx\ +\ k\int_0^L\ \psi^2\ (x)dx\right).$$

This is equivalent to

$$L \int_0^L a_2(x)\psi^2(x)dx - c_2 L \int_0^L \psi^2(x)dx$$
$$\leq c_2 L^2 \left(\frac{1}{k}\int_0^L \psi'^2(x)dx + k\int_0^L \psi^2(x)dx!\right).$$

Letting $k = c_2 L$, we obtain

$$-\int_0^L \psi'^2(x)dx + \int_0^L a_2(x)\psi^2(x)dx \leq (c_2 + c_2^2 L^2) \int_0^L \psi^2(x)dx.$$

So, we have

$$\lambda_{R,2} \leq c_2 + c_2^2 L^2 .$$

(ii) Theorem (6.1.14) (iv) may not be true for the Dirichlet type boundary condition. That is, $\tilde{\lambda}_1(\nu_1, a_1) \geq \lambda_1(\nu_1, \hat{a}_1)$ may not be true, where $a_1 \in X_1$.

In the random dispersal case, There is an example in [199] which shows that the principal eigenvalue $\lambda_{R,1}(\nu_1, a_1)$ of (iv) is smaller than the principal eigenvalue $\lambda_{R,1}(\nu_1, c_1)$ of (iv) with $a_1(x)$ being replaced by $c_1 (= \hat{a}_1)$. It is prove in [186] that

$$\tilde{\lambda}_1(\nu_1, a_1, \delta) \to \lambda_{R,1}(\nu_1, a_1)$$

as $\delta \to 0$. So, for any $0 < \delta \ll 1$, $\tilde{\lambda}_1(\nu_1, a_1, \delta)$ is close to $\lambda_{R,1}(\nu_1, a_1)$, and $\tilde{\lambda}_1(\nu_1, c_1, \delta)$ is close to $\lambda_{R,1}(\nu_1, c_1)$. Hence $\tilde{\lambda}_1(\nu_1, a_1, \delta)$ can be smaller than $\tilde{\lambda}_1(\nu_1, c_1, \delta) = \lambda_1(\nu_1, c_1, \delta)\ for\ \delta \ll 1$.

(iii) Theorem (6.1.14) (iv) holds for periodic case (see [202]). When $\lambda_i(\nu_i, a_i)$ does not exist ($i = 2, 3$), we may have $\tilde{\lambda}_i(\nu_i, a_i) = \hat{a}_i$, but $a_i(\cdot)$ is not a constant function. For example, let $X_3 = \{u(x) \in C(\mathbb{R}^N, \mathbb{R}) | u(x + e_j) = u(x)), x \in \mathbb{R}^N, j = 1, 2, \cdots, N\}$, and $q \in X_3$ with

$$q(x) = \begin{cases} e\dfrac{\|x\|^2}{\|x\|^2 - \sigma^2} & if\ \|x\| < \sigma, \\ 0 & if\ \sigma \leq \|x\| \leq \dfrac{1}{2}. \end{cases}$$

Then $\mathcal{K}_3 + h_3(\cdot)I$ with $k(z) = k_\delta(z)$ has no principal eigenvalue for $M > 1, 0 < \sigma \ll 1, \delta \gg 1$ and $h_3(x) = -1 + M\,q(x)$ where $x \in \mathbb{R}^N$ and $N \geq 3$ (see [200]). Hence $\tilde{\lambda}_3 = \max_{x\in\bar{D}} h_3(x) = -1 + M \max_{x\in\bar{D}} q(x) = -1 + M$. Choosing $M = \frac{1}{1-\hat{q}}$, we have $M\hat{q} = -1 + M$, that is $\hat{a}_3 = \tilde{\lambda}_3$, but $a_3(x) = M\,q(x)$ is not a constant function.

We investigate the effects of the dispersal rates on the principal spectrum points and the existence of principal eigenvalues of nonlocal dispersal operators and prove Theorem (6.1.15).

Theorem (6.1.15)[176]: (Effects of dispersal rate). Assume that $1 \leq i \leq 3$ and $k(\cdot)$ is symmetric with respect to 0.

(i) (Monotonicity) Assume $a_i(\cdot) \not\equiv$ constant. If $\nu_i^1 < \nu_i^2$, then $\tilde{\lambda}_i(\nu_i^1, a_i) > \tilde{\lambda}_i(\nu_i^2, a_i)$.

(ii) (Existence of principal eigenvalue) If $i = 1$ or 3 and $\lambda_i(\nu_i, a_i)$ exists for some $\nu_i > 0$, then $\tilde{\lambda}_i(\nu_i, a_i)$ exists for all $\tilde{\nu}_i > \nu_i$.

(iii) (Existence of principal eigenvalue) There is $\nu_i^0 > 0$ such that the principal eigenvalue $\lambda_i(\nu_i, a_i)$ of $\nu_i K_i + h_i(\cdot)I$ exists for $\nu_i > \nu_i^0$.

(iv) (Limits as the dispersal rate goes to 0) $\lim_{\nu i\to 0+} \tilde{\lambda}_i(\nu_i, a_i) = a_{i,max}$.

(v) (Limits as the dispersal rate goes to ∞) $\lim_{\nu i \to \infty} \tilde{\lambda}_i(\nu_i, a_i) = -\infty$ for i = 1 and $\lim_{\nu i \to \infty} \tilde{\lambda}_i(\nu_i, a_i) = \hat{a}_i$ for $i = 2$ and 3.

For given $\delta > 0$ and $\tilde{k}(\cdot): \mathbb{R}^N \to \mathbb{R}^+$ satisfying that $\sup(\tilde{k}) = B(0,1) := \{z \in \mathbb{R}^N \mid \|z\| < 1\}$ and $\int_{\mathbb{R}^N} \tilde{k}(z)dz = 1$, let

$$k\delta(z) = \frac{1}{\delta N}\tilde{k}\left(\frac{z}{\delta}\right). \tag{44}$$

When $k(z) = k\delta(z)$, to indicate the dependence of $\tilde{\lambda}\, i(\nu_i, a_i)$ on δ, put

$$\tilde{\lambda}_i(\nu_i, a_i, \delta) = \tilde{\lambda}_i(\nu_i, a_i).$$

Proof. (i) Assume that $k(\cdot)$ is symmetric. Observe that for any $u(\cdot) \in L^2(D)$,

$$\begin{aligned}
&\int\int_{D\times D} k(y-x)u(x)u(y)dydx - \int_D u^2(x)dx \\
&\le \int_D\int_D k(y-x)u(y)u(x)dydx - \int_D\int_D k(y-x)dyu^2(x)dx \\
&= \int_D\int_D k(y-x)\big(u(y)-u(x)\big)u(x)dydx \\
&= \frac{1}{2}\int\int_{D\times D} k(y-x)\big(u(y)-u(x)\big)u(x)dydx \\
&\quad + \frac{1}{2}\int Z\int_{D\times D} k(y-x)\big(u(y)-u(x)\big)u(x)dydx \\
&= \frac{1}{2}\int\int_{D\times D} k(y-x)\big(u(y)-u(x)\big)u(x)dydx \\
&\quad + \frac{1}{2}\int\int_{D\times D} k(y-x)\big(u(x)-u(y)\big)u(y)dydx \\
&= -\frac{1}{2}\int\int_{D\times D} k(y-x)\big(u(y)-u(x)\big)^2 dydx \le 0.
\end{aligned}$$

Then (i) follows from the following facts: $\forall\, \nu_i > 0$,

$$\tilde{\lambda}_i(\nu_i, a_i) = \sup_{u\in L^2(D), \|u\|_{L^2(D)}=1}$$

$$\left[\nu_i\left(\int_D\int_D k(y-x)u(y)u(x)dydx - \int_D u^2(x)dx + \right)\int_D a_i(x)u^2(x)dx\right]$$

in the case $i = 1$,

$$\tilde{\lambda}_i(\nu_i, a_i) = \sup_{u\in L^2(D), \|u\|_{L^2(D)}=1}$$

$$\left[-\frac{\nu_i}{2}\int\int_{D\times D} k(y-x)\big(u(y)-u(x)\big)^2 dydx + \int_D a_i(x)u^2(x)dx\right]$$

in the case $i = 2$, and

$$\tilde{\lambda}_i(\nu_i, a_i) = \sup_{u\in L^2(D), \|u\|_{L^2(D)}=1}$$

$$\left[\nu_i\left(\int_D\int_D \hat{k}(y-x)u(y)u(x)dydx - \int_D u^2(x)dx\right) + \int_D a_i(x)u^2(x)dx\right]$$

in the case $i = 3$ (see (34)).

(ii) We prove the case $i = 1$. The case $i = 3$ can be proved similarly.

Without loss of generality, assume $a_1(x) > 0$ for $\in \overline{D}$. Assume that $\nu_1 > 0$ is such that $\lambda_1(\nu_1, a_1)$ exists and $\tilde{\nu}_1 > \nu_1$. By Proposition (6.1.11), $\lambda_1(\nu_1, a_1) > \max_{x\in\overline{D}} h_1(x)$, that is,

$$\lambda_1(\nu_1, a_1) > \max_{x\in\overline{D}} (-\nu_1 + a_1(x)).$$

Let $\phi_1(\cdot)$ be a positive principal eigenfunction with $||\phi_1||_{L^2(D)} = 1$. Then

$$\tilde{\lambda}_1(\tilde{\nu}_1, a_1) = \nu_1 \int\int_{D\times D} k(y - x)\phi_1(y)\phi_1(x)dydx - \nu_1 + \int_D a_1(x)\phi_1^2(x)dx > \max_{x\in\overline{D}} (-\nu_1 + a_1(x)).$$

By Proposition (6.1.12),

$$\tilde{\lambda}_1(\tilde{\nu}_1, a_1) \geq \tilde{\nu}_1 \int\int_{D\times D} k(y - x)\phi_1(y)\phi_1(x)dydx - \tilde{\nu}_1 + \int_D a_1(x)\phi_1^2(x)dx$$

$$= \lambda_1(\nu_1, a_1) + (\tilde{\nu}_1 - \nu_1)\int\int_{D\times D} k(y - x)\phi_1(y)\phi_1(x)dydx + \nu_1 - \tilde{\nu}_1$$

$$> \max_{x\in\overline{D}} \left(-\nu_1 + a_1(x)\right) + \nu_1 - \tilde{\nu}_1 + (\tilde{\nu}_1 - \nu_1) \int\int_{D\times D} k(y - x)\phi_1(y)\phi_1(x)dydx$$

$$> \max_{x\in\overline{D}} (-\tilde{\nu}_1 + a_1(x)).$$

By Proposition (6.1.11) again, $\lambda_1(\tilde{\nu}_1, a_1)$ exists.

(iii) It follows from Theorem (6.1.14)(i) and can also be proved as follows.

To show $\lambda_i(\nu_i, a_i)$ exists, we only need to show $\tilde{\lambda}_i(\nu_i, a_i) > \max_{x\in\overline{D}} h_i(x)$, where $h_i(x) = -\nu_i + a_i(x)$ for $i = 1$ and 3 and $h_i(x) = -\nu_i \int_D k(y - x)dy + a_i(x)$ for $i = 2$. In the case $i = 2$ or 3, $\tilde{\lambda}_i(\nu_i, a_i) \geq \hat{a}_i$ by Theorem (6.1.14)(iv). This implies that

$$\tilde{\lambda}_i(\nu_i, a_i) > h_{i,max} \quad \forall\, \nu_i \gg 1.$$

In the case $i = 1$, note that $\lambda_1(1, 0)$ exists and

$$-1 < \lambda_1(1, 0) < 0.$$

This implies that $\lambda_1(1, \frac{a_1}{\nu_1})$ exists for $\nu_1 \gg 1$ and then $\lambda_1(\nu_1, a_1)$ exists for $\nu_1 \gg 1$.

(iv) On the one hand, we have

$$\tilde{\lambda}_i(\nu_i, a_i) \geq h_{i,\max} \geq -\nu_i + a_{i,\max}.$$

On the other hand, for any $\lambda > a_{i,max}, \lambda\mathcal{I} - a_i(\cdot)\mathcal{I}$ has bounded inverse. This implies that

$$a_{i,\max} + \epsilon > \tilde{\lambda}_i(\nu_i, a_i) \quad \forall\, 0 < \nu_i \ll 1.$$

Therefore,

$$\lim_{\nu_i\to 0} \tilde{\lambda}_i(\nu_i, a_i) = a_{i,max.}$$

(v) We prove the cases $i = 1$ and $i = 2$. The case $i = 3$ can be proved by the similar arguments as in the case $i = 2$.

First of all, we prove the case $i = 1$. By Proposition (6.1.7),

$$\tilde{\lambda}_1(1, 0) < 0.$$

Observe that

$$\tilde{\lambda}_1(\nu_1, a_1) = \nu_1\tilde{\lambda}_1 \left(1, \frac{a_1}{\nu_1}\right) \; and \; \tilde{\lambda}_1 \left(1, \frac{a_1}{\nu_1}\right) \to \tilde{\lambda}_1(1, 0)$$

as $\nu_1 \to \infty$. It then follows that

$$\tilde{\lambda}_1(\nu_1, a_1) \leq \frac{\nu_1}{2} \tilde{\lambda}_1(1,0) \ \forall \ \nu_1 \gg 1.$$

This implies that

$$\lim_{\nu_1 \to \infty} \tilde{\lambda}_1(\nu_1, a_1) = -\infty.$$

Second of all, we prove the case $i = 2$. By (iii), $\lambda_2(\nu_2, a_2)$ exists for $\nu_2 \gg 1$. In the following, we assume $\nu_2 \gg 1$ such that $\lambda_2(\nu_2, a_2)$ exists. Let $\phi_2, \nu_2\ (x)$ be a positive principal eigenfunction with $\int_D \ \phi_2^2, \nu_2\ (x)dx = 1$.

Note that

$$\tilde{a}_2 \leq \lambda_2(\nu_2, a_2) \leq a_{2,max,}$$

and

$$\nu_2 \int_D \int_D k(y - x)\Big(\phi_{2,\nu_2}(y) - \phi_{2,\nu_2}(x)\Big)\phi_{2,\nu_2}(x)dydx$$
$$+ \int_D a_2(x)\phi_{2,\nu_2}(x)dx = \lambda_2(\nu_2, a_2).$$

This implies that

$$\frac{\nu_2}{2}\int_D \int_D k(y - x)\Big(\phi_{2,\nu_2}(y) - \phi_{2,\nu_2}(x)\Big)^2 dydx$$
$$= \int_D a_2(x)\phi_{2,\nu_2}(x)dx - \lambda_2(\nu_2, a_2) \leq a_{2,\max} - \hat{a}_2,$$

and then

$$\int_D \int_D k(y - x)\Big(\phi_{2,\nu_2}(y) - \phi_{2,\nu_2}(x)\Big)^2 dydx \leq \frac{2\big(a_{2,\max} - \hat{a}_2\big)}{\nu_2}. \quad (45)$$

Let $\psi_2, \nu_2\ (x) = \phi_{2,\nu_2}(x) - \phi_{2,\nu_2}$. Then

$$\nu_2 \int_D \int_D k(y - x)\Big(\phi_{2,\nu_2}(y) - \emptyset_{2,\nu_2}(x)\Big)dydx$$
$$+ \int_D a_2(x)\phi_{2,\nu_2}(x)dx = \int_D a_2(x)(\psi_{2,\nu_2}(x) + \widehat{\emptyset}_{2,\nu 2})dx,$$

and hence

$$\lambda_2(\nu_2, a_2) \int_D \phi_{2,\nu_2}(x)dx = \hat{\phi}_{2,\nu_2} \int_D a_2(x)dx + \int_D a_2(x)\psi_{2,\nu_2}(x)dx.$$

This implies that

$$\lambda_2(\nu_2, a_2)\widehat{\emptyset}_{2,\nu 2} = \hat{a}_2\widehat{\emptyset}_2, \nu_2 + \frac{1}{|D|}\int_D a_2(x)\psi_{2,\nu_2}(x)dx. \quad (46)$$

To show $\lambda_2(\nu_2, a_2) \to \hat{a}_2\ as\ \nu_2 \to \infty$, we first show that $\int_D \ a_2(x)\psi_{2,\nu 2}(x)dx \to 0\ as\ \nu_2 \to \infty$. Note that $\tilde{\lambda}_2(1,0) = 0\ and\ \tilde{\lambda}_2(1,0)$ is the principal eigenvalue of $\mathcal{K}_2 + b_0(\cdot)I$ with $\emptyset(\cdot) \equiv 1$ being a principal eigenfunction, where

$$b_0(x) = -\int_D k(y - x)dy.$$

Moreover, $\tilde{\lambda}_2(1,0)$ is also an isolated algebraically simple eigenvalue of $\mathcal{K}_2 + b_0(\cdot)I$ on $L^2\ (D)$. Note also that

$$\int_D ((-\mathcal{K}_2 - b_0 I)u)\ (x)u(x)dx$$

$$= \frac{1}{2} \int_D \int_D k(y - x)\big(u(y) - u(x)\big)^2 \, dydx \geq 0 \qquad (47)$$

for any $u(\cdot) \in L^2(D)$ and $-\mathcal{K}_2 - b_0(\cdot)I$ is a self-adjoint operator on $L^2(D)$. Then there is a bounded linear operator $A : L^2(D) \to L^2(D)$ such that

$$\int_D (-\mathcal{K}_2 - b_0 I)u(x)u(x)dx = \int_D (Au)(x)(Au)(x)dx \ \forall\, u \in L^2(D). \quad (48)$$

Let

$$E_1 = span\{\varphi(\cdot)\},$$

and

$$E_2 = \{u(\cdot) \in L^2(D) | \int_D u(x)dx = 0\}.$$

Then

$$L^2(D) = E_1 \oplus E_2$$

and

$$\mathcal{K}_2 + b_0(\cdot)I \ (E_2) \subset E_2.$$

Moreover, $(\mathcal{K}_2 + b_0(\cdot)I)|_{E_2}$ is invertible. We claim that there is $C > 0$ such that

$$\int_D (Au)(x)(Au)(x)dx \geq C \int_D u^2(x)dx \ \forall\, u \in E_2. \qquad (49)$$

For otherwise, there is $u_n \in E_2$ with $\int_D u_n^2(x)dx = 1$ such that

$$\int_D (Au_n)(x)(Au_n)(x)dx \to 0$$

as $n \to \infty$. It then follows that $0 \in \sigma((\mathcal{K}_2 + b_0(\cdot)I)|E_2)$, a contradiction. Hence (49) holds.

By (47), (48) and (49), for any $\nu_2 \gg 1$,

$$\int_D \psi_{2,\nu_2}^2(x)dx \leq \frac{1}{2C}\int_D \int_D k(y - x)\left(\psi_{2,\nu_2}(y) - \psi_{2,\nu_2}(x)\right)^2 dydx. \quad (50)$$

Observe that

$$\int_D \int_D k(y - x)\left(\phi_{2,\nu_2}(y) - \phi_{2,\nu_2}(x)\right)^2 dydx$$
$$= \int_D \int_D k(y - x)\left(\psi_{2,\nu_2}(y) - \psi_{2,\nu_2}(x)\right)^2 dydx.$$

This together with (45) and (50) implies that

$$\int_D \psi_{2,\nu_2}^2(x)dx \to 0 \ as\ \nu_2 \to \infty,$$

and then

$$\int_D a_2(x)\psi_{2,\nu_2}(x)dx \to 0 \ as\ \nu_2 \to \infty$$

Second, assume $\lambda_2(\nu_2, a_2) \nrightarrow \hat{a}_2 \ as\ \nu_2 \to \infty. By$ (46), we must have $\hat{\phi}_{2,\nu_2,n} \to 0$ for some sequence $\nu_{2,n} \to \infty$. This and (45) implies that

$$\int_D \phi_{2,\nu_{2,n}}^2(x)dx \leq C_0 \int_D \int_D k(y - x)\phi_{2,\nu_{2,n}}^2(x)dydx$$
$$= C_0 \int_D \int_D k(y - x)\left(\phi_{2,\nu_{2,n}}^2(x) - \phi_{2,\nu_{2,n}}^2(x)\phi_{2,\nu_{2,n}}^2(y)\right)dydx$$

$$+ C_0 \int_D \int_D k(y - x)\phi^2_{2,\nu_{2,n}}(y)\phi^2_{2,\nu_{2,n}}(x)dydx$$
$$\leq \frac{C_0}{2} \int_D \int_D k(y - x)\left(\phi^2_{2,\nu_{2,n}}(y) - \phi^2_{2,\nu_{2,n}}(x)\right)^2 dydx$$
$$+ |D|^2 C_0 M \hat{\phi}_{2,\nu_{2,n}} \hat{\phi}_{2,\nu_{2,n}}$$
$$\leq \frac{C_0\left(a_{2,\max} - \hat{a}_2\right)}{\nu_2} + |D|^2 C_0 M \varphi\hat{\ } 2, \nu_{2,n}\, \varphi\hat{\ } 2, \nu_{2,n}$$
where $C_0 = (\min_{x\in\bar{D}} \int_D k(y - x)dy) - 1$ and $M = \sup_{x,y\in\bar{D}} k(y - x)$. That is
$$\int_D \phi^2_{2,\nu_{2,n}}(x)dx \to 0\ as\ \nu_{2,n} \to \infty.$$
This is a contradiction. Therefore
$$\lambda_2(\nu_2, a_2) \to \hat{a}_2\ as\ \nu_2 \to \infty.$$
We investigate the effects of the dispersal distance on the principal spectrum points and the existence of principal eigenvalues and prove Theorem (6.1.16).

Theorem (6.1.16)[176]: (Effects of dispersal distance). Suppose that $k(z) = k_\delta(z)$, where $k_\delta(z)$ is defined as in (44) and $\tilde{k}(z) = \tilde{k}(-z)$. Let $1 \leq i \leq 3$.

(i) (Limits as dispersal distance goes to 0) $\lim_{\delta\to 0} \tilde{\lambda}_i(\nu_i, a_i, \delta) = a_{i,\max}$.

(ii) (Limits as dispersal distance goes to ∞) $\lim_{\delta\to\infty} \tilde{\lambda}_1(\nu_1, a_1, \delta) = -\nu_1 + a_{1,\max}$, $\lim_{\delta\to\infty} \tilde{\lambda}_2(\nu_2, a_2, \delta) = a_{2,\max}$, $and\ \lim_{\delta\to\infty} \tilde{\lambda}_3(\nu_3, a_3, \delta) = \tilde{\lambda}_3(\nu_3, a_3)$, where
$$\tilde{\lambda}_3(\nu_3, a_3) = max\{Re\lambda \mid \lambda \in \sigma(\nu_3\bar{I} + h_3(\cdot)I)\},$$
and
$$\bar{I}u = \frac{1}{|D|}\int_D u(x)dx.$$
(iii) (Existence of principal eigenvalue) There is $\delta_0 > 0$ such that the principal eigenvalue $\lambda_i(\nu_i, a_i)$ of $\nu_i\mathcal{K}_i + h_i(\cdot)I$ exists for $0 < \delta < \delta_0$.

Proof. (i) As mentioned, the cases $i = 1$ and 3 are proved in [189]. The case $i = 2$ can be proved by the similar arguments as in [189]. For completeness, we provide a proof for the case $i = 2$ in the following. By Proposition (6.1.12),
$$\tilde{\lambda}_i(\nu_i, a_i, \delta) = \sup_{u\in L^2(D), \|u\|L^2(D)=1} \int_D \nu_i \int_D k_\delta(y - x)\big(u(y) - u(x)\big)dy + a_i(x)u(x)\ u(x)dx.$$
On the one hand,
$$\tilde{\lambda}_i(\nu_i, a_i, \delta) = \sup_{u\in L^2(D), \|u\|L^2(D)=1} \left[-\frac{\nu_i}{2}\int_D \int_D k_\delta(y - x)\big(u(y) - u(x)\big)^2 dydx + \int_D \int_D a_i(x)u^2(x)dx\right] \leq a_{i,\max}.$$
On the other hand, assume that $x_0 \in \bar{D}$ is such that $a_i(x_0) = a_{i,max.}$ Then for any $0 < \epsilon < 1$, there are $\sigma_0^* > 0$ and $x_0^* \in IntD$ such that $B(x_0^*, \sigma_0^*) \subset \bar{D}$ and
$$a_i(x_0) - a_i(x) < \epsilon/2\ for\ x \in B(x_0^*, \sigma_0^*).$$
Let $u_0(\cdot)$ be a smooth function with $\text{supp}(u_0(\cdot)) \cap D \subset B(x_0^*, \sigma_0^*)$ and $\|u_0\|L^2(D) = 1$. Then

$$\tilde{\lambda}_i\,(\nu_i\,, a_i\,, \delta) \geq \int_D \left(\nu_i \int_D k_{\,\delta}(y - x)\big(u_0(y) - u_0(x)\big)dy + a_i(x)u_0(x)\right)$$
$$u_0(x)dx \geq \nu_i \int_D \left(\int_D k_\delta(y - x)\big(u_0(y) - u_0(x)\big)dy\right) u_0(x)dx$$
$$+ \left(a_{i,\max} - \frac{\epsilon}{2}\right).$$

Note that

$$\int_D k_\delta(y - x)(u_0(y) - u_0(x))dy \to 0\ \forall\, x \in Int(D)$$

$as\ \delta \to 0$. And

$$\left|\int_D k_\delta(y - x)\big(u_0(y) - u_0(x)\big)dy\right| \leq 2\,max_{y\in\bar{D}}\ |u_0(y)|\ \forall\, x \in D.$$

Hence, there exists $\delta_0 > 0$, such that for any $\delta < \delta_0$, we have

$$\nu_i \int_D \int_D k_\delta(y - x)\big(u_0(y) - u_0(x)\big)dy\, u_0(x)dx \leq \frac{\epsilon}{2}$$

It then follows that

$$a_{i,\max} \geq \tilde{\lambda}_i\,(\nu_i\,, a_i\,, \delta) \geq a_{i,\max} - \epsilon$$

This implies that $\tilde{\lambda}_i(\nu_i\,, a_i\,, \delta) \to a_{i,\max}$ as $\delta \to 0$.

(ii) First, for $i = 1$,

$$\left|\int_D k_\delta(y - x)u(y)dy\right| \leq \|u\| \int_D k_\delta(y - x)dy \to 0$$

as $\delta \to \infty$ uniformly in $u \in X_1$ with $\|u\| \leq 1$. Therefore,

$$\tilde{\lambda}_1(\nu_1, a_1, \delta) \to \sup\left\{Re\lambda\middle|\lambda \in \sigma\Big(\big(-\nu_1 + a_1(\cdot)\big)I\Big)\right\} = -\nu_1 + a_1,$$

As $\delta \to \infty$. For $i = 2$,

$$\int_D k_\delta(y - x)\big(u(y) - u(x)\big)dy \leq 2\|u\| \int_D k_\delta(y - x)dy \to 0$$

as $\delta \to \infty$ uniformly in $u \in X_2$ with $\|u\| \leq 1$. Hence

$$\tilde{\lambda}_2(\nu_2, a_2, \delta) \to sup\{Re\lambda|\lambda \in \sigma(a_2(\cdot)I)\} = a_{2,\max}\ as\ \delta \to \infty.$$

For $i = 3$, recall that

$$\bar{\lambda}_3(\nu_3, a_3) = sup\{Re\lambda \mid \lambda \in \sigma(\nu_2 \bar{I} + h_3(\cdot)I)\},$$

where

$$\bar{I}u = \frac{1}{p_1 p_2 \cdots p_N} \int_0^{p_1} \int_0^{p_2} \cdots \int_0^{pN} u(x)dx.$$

We first assume that $a_3(\cdot)$ satisfies the conditions. Then by similar arguments as in Theorem (6.1.14) (ii), $\bar{\lambda}_3(\nu_3, a_3)$ is the principal eigenvalue of $\nu_3 \bar{I} + h_3(\cdot)I. Let\ \phi_3(\cdot)$ be the positive principal eigenfunction of $\nu_3 \bar{I} + h_3(\cdot)I$ with $\hat{\phi}_3 = \frac{1}{|D|}\int_D \phi_3(x)dx = 1.$ We then have $\bar{\lambda}_3(\nu_3, a_3) > h_{3,\max}$ and

$$\frac{1}{|D|}\int_D \nu_3 \psi_3(x)\bar{\lambda}_3(\nu_3, a_3) + \nu_3 - a_3(x)\, dx = 1, \qquad (51)$$

where

$$\psi_3(x) = \Big(\bar{\lambda}_3(\nu_3, a_3) + \nu_3 - a_3(x)\Big)\phi_3(x).$$

$Fix\ 0 < \epsilon < \bar{\lambda}_3(\nu_3, a_3) - hi, max.$ Then

$$\frac{1}{|D|}\int_D \frac{\nu_3\psi_3(x)}{\bar{\lambda}_3(\nu_3,a_3)-\epsilon+\nu_3-a_3(x)}dx > 1. \quad (52)$$

Observe that for any $k = (k_1,k_2,\cdots,k_N) \in \mathbb{Z}^N \ \{0\}$,

$$\int_{\mathbb{R}^N} \tilde{k}(z)\, cos\left(\sum_{i=1}^N k_ip_ix_i + \delta\sum_{i=1}^N k_ip_iz_i\right)dz \to 0,$$

and

$$\int_{\mathbb{R}^N} \tilde{k}(z)\, sin\left(\sum_{i=1}^N k_ip_ix_i + \delta\sum_{i=1}^N k_ip_iz_i\right)\ dz \to 0$$

as $\delta \to \infty$. This implies that for any $a \in X_3$,

$$\int_{\mathbb{R}^N} \tilde{k}(z)a(x+\delta_z)dz \to \hat{a}$$

$as\ \delta \to \infty$ and then

$$\int_{\mathbb{R}^N} \frac{\nu_3k_\delta(y-x)\psi_3(y)}{\bar{\lambda}_3(\nu_3,a_3)-\epsilon+\nu_3-a_3(y)}dy$$

$$= \int_{\mathbb{R}^N} \nu_3\frac{\tilde{k}(z)\psi_3(x+\delta_z)}{\bar{\lambda}_3(\nu_3,a_3)-\epsilon+\nu_3-a_3(x+\delta_z)}d_z$$

$$\to \frac{1}{|D|}\int_D \frac{\nu_3\psi_3(x)}{\bar{\lambda}_3(\nu_3,a_3)-\epsilon+\nu_3-a_3(x)}\ dx$$

As $\delta \to \infty$uniformly in $x \in \mathbb{R}^N$. This together with (52) implies that

$$\int_{\mathbb{R}^N} \frac{\nu_3k_\delta(y-x)\psi_3(y)}{\bar{\lambda}_3(\nu_3,a_3)-\epsilon+\nu_3-a_3(y)}dy > 1\,\forall\, x \in \mathbb{R}^N, \delta \gg 1.$$

It then follows that

$$\tilde{\lambda}_3(\nu_3,a_3,\delta) > \bar{\lambda}_3(\nu_3,a_3) - \epsilon > h_{i,\max}\ \forall\, \delta \gg 1 \quad (53)$$

and $\lambda_3(\nu_3,a_3,\delta)$ exists for $\delta \gg 1$.

Now for any $\epsilon > 0$, by (51),

$$\frac{1}{|D|}\int_D \nu_3\psi_3(x)\bar{\lambda}_3(\nu_3,a_3) + \epsilon + \nu_3 - a_3(x)\,dx < 1. \quad (54)$$

Then by the similar arguments in the above,

$$\tilde{\lambda}_3(\nu_3,a_3,\delta) < \bar{\lambda}_3(\nu_3,a_3) + \epsilon\,\forall\,\delta \gg 1. \quad (55)$$

By (53) and (55),

$$\tilde{\lambda}_3(\nu_3,a_3,\delta) \to \bar{\lambda}_3(\nu_3,a_3)\ as\ \delta \to \infty.$$

Now for general $a_3 \in X_3$, and for any $\epsilon > 0$, there is $a_3,\epsilon \in X_3$ such that

$$\|a_3 - a_{3,\epsilon}\| < \epsilon\,\forall\, x \in \mathbb{R}^N,$$

and $a_3,\epsilon(\cdot)$ satisfies the conditions. By Theorem (6.1.14) (v),

$$\tilde{\lambda}_3(\nu_3,a_3,\epsilon,\delta) - \epsilon \le \tilde{\lambda}_3\,(\nu_3,a_3,\delta) \le \tilde{\lambda}_3(\nu_3,a_3,\epsilon,\delta) + \epsilon.$$

By the above arguments,

$$\bar{\lambda}_3(\nu_3,a_3) - 3\epsilon \le \tilde{\lambda}_3(\nu_3,a_{3,\epsilon}) - 2\epsilon$$
$$\le \tilde{\lambda}_3(\nu_3,a_3,\delta) \le \bar{\lambda}_3(\nu_3,a_{3,\epsilon}) + 2\epsilon$$
$$\le \bar{\lambda}_3(\nu_3,a_3) + 3\epsilon\,\forall\,\delta \gg 1.$$

We hence also have

$$\tilde{\lambda}_3(\nu_3,a_3,\delta) \to \bar{\lambda}_3(\nu_3,a_3) as\ \delta \to \infty.$$

(iii) By (i), for any $\epsilon > 0$,

$$\tilde{\lambda}_i(\nu_i, a_i, \delta) > a_{i,\max} - \epsilon \;\forall\, 0 < \delta \ll 1.$$

This implies that there is $\delta_0 > 0$ such that

$$\tilde{\lambda}_i(\nu_i, a_i, \delta) > h_{i,\max} \;\forall\, 0 < \delta < \delta_0.$$

Then by Proposition (6.1.12), $\lambda_i(\nu_i, a_i)$ exists for $0 < \delta < \delta_0$.

We consider the asymptotic dynamics of the two species competition system (7) and prove Theorem (6.1.19) by applying some of the principal spectrum properties developed. We assume that $k(-z) = k(z), \tilde{\lambda}_1(\nu, f(\cdot, 0)) > 0, f(x,w) < 0 \; for \; w \gg 1$, and $\partial_2 f(x,w) < 0$ for $w \geq 0$.

We first present two lemmas.

Lemma (6.1.17)[176]: For any given $\nu > 0$ and $a \in X_1(= X_2)$,

$$\tilde{\lambda}_1(\nu, a) \leq \tilde{\lambda}_2(\nu, a)$$

and if $\lambda_1(\nu, a)$ exists, then

$$\tilde{\lambda}_1(\nu, a)(= \lambda_1(\nu, a)) < \tilde{\lambda}_2(\nu, a)$$

Proof. First, assume that $\lambda_1(\nu, a)$ exists. Let $\phi(\cdot)$ be the positive principal eigenfunction of $\nu\mathcal{K}_1 - \nu I + a(\cdot)\mathrm{I}$ with $\|\phi\| = 1$. Then

$$\Phi_1(t;\, \nu, a)\phi = e^{\lambda_1(\nu,a)t}\phi, \text{ and } \Phi_2(t;\, \nu, a)\phi = e^{\tilde{\lambda}_2(\nu,a)t}\phi \;\forall\, t > 0.$$

By Proposition (6.1.4),

$$\Phi_2(t;\, \nu, a)\phi \gg \Phi_1(t;\, \nu, a)\phi \;\forall\, t > 0.$$

This implies that

$$\tilde{\lambda}_2(\nu, a) > \lambda_1(\nu, a).$$

In general, by Lemma (6.1.13) and Theorem (6.1.14) (ii), for any $\epsilon > 0$, there is $a_\epsilon \in X_1$ such that $\lambda_1(\nu, a_\epsilon)$ exists and

$$a_\epsilon(x) - \epsilon \leq a(x) \leq a_\epsilon(x) + \epsilon.$$

By the above arguments,

$$\tilde{\lambda}_2(\nu, a_\epsilon) > \lambda_1(\nu, a_\epsilon).$$

Observe that

$$\tilde{\lambda}_2(\nu, a) \geq \tilde{\lambda}_2\,(\nu, a_\epsilon) - \epsilon \; and \; \lambda_1(\nu, a_\epsilon) \geq \tilde{\lambda}_1(\nu, a) - \epsilon.$$

Hence

$$\tilde{\lambda}_2(\nu, a) \geq \tilde{\lambda}_1(\nu, a) - 2\epsilon.$$

Letting $\epsilon \to 0$, we have

$$\tilde{\lambda}_2(\nu, a) \geq \tilde{\lambda}_1(\nu, a).$$

Consider

$$u_t = \nu\left[\int_D k(y - x)u(t,y)dy - u(t,x)\right] + u(t,x)g(x, u(t,x)), x \in \bar{D} \quad (56)$$

and

$$v_t = \nu \int_D k(y - x)[v(t,y) - v(t,x)]dy + v(t,x)g(x, v(t,x)), x \in \bar{D}, \quad (57)$$

where g is a C^1 function, $g(x,w) < 0 \; for \; w \gg 1, and \; \partial_{2g}(x,w) < 0$ for $w \geq 0$.

Lemma (6.1.18)[176]: (i) If $\lambda_1\big(\nu, g(\cdot,0)\big) > 0$, then there is $u^* \in X_1^{++}$ such that $u = u^*$ is a stationary solution of (56) and for any solution $u(t,x)$of (56) with $u(0,\cdot) \in X_1^+ \;\; \{0\}, u(t,\cdot) \to u^*\,(\cdot) in \; X_1$.

(ii) If $\lambda_2\big(\nu, g(\cdot,0)\big) > 0$, then there is $v^* \in X_2^{++}$ such that $v = v^*$ is a stationary solution of (57) and for any solution $v(t,x)$of (57) with $v(0,\cdot) \in X_2^+ \;\{0\}, v(t,\cdot) \to v^* \;\; (\cdot) in \; X_2$.

Proof. It follows from [197].

Theorem (6.1.19)[176]: (i) There are $u^*(\cdot) \in X_1^{++}$ *and* $v^*(\cdot) \in X_2^{++}$ such that $(u^*(\cdot), 0)$ and $(0, v^*(\cdot))$ are stationary solutions of (7). Moreover, for any $(u_0, v_0) \in X_1^+ \times X_2^+$ with $u_0 \neq 0$ and $v_0 = 0$ $(resp. u_0 = 0$ and $v_0 \neq 0)$, $(u(t,\cdot; u_0, v_0), v(t,\cdot; u_0, v_0)) \to (u^*(\cdot), 0)$ $(resp. (u(t,\cdot; u_0, v_0), v(t,\cdot; u_0, v_0)) \to (0, v^*(\cdot)))$ $as\ t \to \infty$.
(ii) For any $(u_0, v_0) \in (X_1^+ \setminus \{0\}) \times (X_2^+ \setminus \{0\})$, $\lim_{t\to\infty} (u(t,\cdot; u_0, v_0), v(t,\cdot; u_0, v_0)) = (0, v^*(\cdot))$.

Proof. (i) By $\tilde{\lambda}_1(\nu, f(\cdot, 0)) > 0$ and Lemma (6.1.17), we have $\tilde{\lambda}_2(\nu, f(\cdot, 0)) > 0$. Then by Lemma (6.1.18), there are $u^* \in X_1^{++}$ and $v^* \in X_2^{++}$ such that $(u^*, 0)$ and $(0, v^*)$ are stationary solutions of (7). Moreover, for any $(u_0, v_0) \in X_1^+ \times X_2^+$ with $u_0 \neq 0$ and $v_0 = 0$ (resp. $u_0 = 0$ and $v_0 \neq 0$), $(u(t,\cdot; u_0, v_0), v(t,\cdot; u_0, v_0)) \to (u^*(\cdot), 0)$ $(resp. (u(t,\cdot; u_0, v_0), v(t,\cdot; u_0, v_0)) \to (0, v^*(\cdot)))$ as $t \to \infty$.
(ii) Observe that

$$\nu\left[\int_D k(y-x)u^*(y)dy - u^*(x)\right] + f(x, u^*(x))u^*(x) = 0, x \in \bar{D}. \quad (58)$$

This implies that $\lambda_1\left(\nu, f\left(\cdot, u^*(\cdot)\right)\right)$ exists and $\lambda_1\left(\nu, f\left(\cdot, u^*(\cdot)\right)\right) = 0$. By Lemma (6.1.17), we have

$$\tilde{\lambda}_2\left(\nu, f\left(\cdot, u^*(\cdot)\right)\right) > 0.$$

By Lemma (6.1.13), there are $\epsilon > 0$ and $a \in X_1$ such that $\lambda_2(\nu, a)$ exists,

$$a(x) \leq f(x, u^*(x)) - \epsilon, \lambda_2(\nu, a) > 0,$$

and

$$\tilde{\lambda}_2(\nu, f(\cdot, u^*(\cdot) + \epsilon)) > 0.$$

Let $\phi(\cdot)$ be the positive eigenfunction of $\nu\mathcal{K}_2 - \nu b(\cdot)I + a(\cdot)I$ with $\|\phi\| = 1$, where $b(x) = \int_D k(y-x)dy$. Let

$$u_\delta(x) = u^*(x) + \delta^2 \ and\ v_\delta(x) = \delta_\phi(x).$$

Then

$$\begin{aligned}
0 &= \nu\left[\int_D k(y-x)u^*(y)dy - u^*(x)\right] + u^*(x)f(x, u^*(x)) \\
&= \nu\left[\int_D k(y-x)u_\delta(y)dy - u_\delta(x)\right] + u_\delta(x)f(x, u_\delta(x) + v_\delta(x)) \\
&\quad + \nu\delta^2\left(1 - \int_D k(y-x)dy\right) - \delta^2 f(x, u^*(x)) \\
&\quad + u_\delta\left[f(x, u^*(x)) - f(x, u_\delta(x) + v_\delta(x))\right] \\
&\geq \nu\left[\int_D k(y-x)u_\delta(y)dy\right] - u_\delta(x) + u_\delta(x)f(x, u_\delta(x) + v_\delta(x))
\end{aligned}$$

for $0 < \delta \ll 1$, and

$$\begin{aligned}
0 &\leq \lambda_2(\nu, a)v_\delta(x) = \nu\int_D k(y-x)[v_\delta(y) - v_\delta(x)]dy + a(x)v_\delta(x) \\
&\leq \nu\int_D k(y-x)[v_\delta(y) - v_\delta(x)]dy + \left[f(x, u^*(x)) - \epsilon\right]v_\delta(x) \\
&= \nu\int_D k(y-x)[v_\delta(y) - v_\delta(x)]dy + v_\delta(x)f(x, u_\delta(x) + v_\delta(x)) \\
&\quad + v_\delta(x)\left[f(x, u^*(x)) - f(x, u_\delta(x) + v_\delta(x)) - \epsilon\right]
\end{aligned}$$

$$\leq \nu \int_D k(y-x)[v_\delta(y) - v_\delta(x)]dy + v_\delta(x) f(x, u_\delta(x) + v_\delta(x))$$

for $0 < \delta \ll 1$. It then follows that for $0 < \delta \ll 1, (u_\delta(x), v_\delta(x))$ is a super-solution of (7). By Proposition (6.1.5),

$$(u(t_2,\cdot;\, u_\delta, v_\delta), v(t_2,\cdot;\, u_\delta, v_\delta)) \leq 2\, (u(t_1,\cdot;\, u_\delta, v_\delta), v(t_1,\cdot;\, u_\delta, v_\delta))\ \forall\, 0 < t_1 < t_2. \tag{59}$$

Let

$$\left(u_\delta^{**}(x), v_\delta^{**}(x)\right) = \lim_{t\to\infty} (u(t,x;\, u_\delta, v_\delta), v(t,x;\, u_\delta, v_\delta))\ \forall\, x \in \bar{D}$$

(this pointwise limit exists because of (59)).

We claim that $(u_\delta^{**}(\cdot), v_\delta^{**}(\cdot)) = (0, v^*(\cdot))$. Observe that $u_\delta^{**}(\cdot)$ and $v_\delta^{**}(\cdot)$ are semi-continuous and $(u_\delta^{**}(\cdot), v_\delta^{**}(\cdot))$ satisfies that

$$\begin{cases} (\nu[\int_D k(y-x) u_\delta^{**}(y)dy - u_\delta^{**}(x)] + u_\delta^{**}(x) f(x, u_\delta^{**}(x) + v_\delta^{**}(x)) = 0, x \in \bar{D}, \\ \nu \int_D k(y-x)[v_\delta^{**}(y) - v_\delta^{**}(x)]dy + v_\delta^{**}(x) f(x, u_\delta^{**}(x) + v_\delta^{**}(x)) = 0, x \in \bar{D} \end{cases} \tag{60}$$

(see the arguments in [185]). Multiplying the first equation in (60) by $v_\delta^{**}(x)$, second equation by $u_\delta^{**}(x)$, and integrating over D, we have

$$\int_D u_\delta^{**}(x) v_\delta^{**}(x)dx = \int_D \int_D k(y-x)dy\, u_\delta^{**}(x) v_\delta^{**}(x)dx.$$

This together with $v_\delta^{**}(x) \geq \delta_\phi(x) > 0$ implies that

$$\left[1 - \int_D k(y-x)dy\right] u_\delta^{**}(x) = 0\ \forall\, x \in \bar{D}.$$

Note that $\int_D k(y-x)dy < 1$ for x near ∂D. This together with the first equation in (60) implies that $u_\delta^{**}(x) = 0$ for all $x \in \bar{D}$. We then must have $v_\delta^{**}(x) = v^*(x)$ for all $x \in \bar{D}$. Moreover, by (59) and Dini's theorem,

$$\lim_{t\to\infty} (u(t,\cdot;\, u_\delta, v_\delta), v(t,\cdot;\, u_\delta, v_\delta)) = (0, v^*(\cdot))\ in\ X_1 \times X_2. \tag{61}$$

Now, for any $(u_0, v_0) \in (X_1^+ \setminus \{0\}) \times (X_2^+ \setminus \{0\})$, there is $M_0 > 0$ such that

$$(u_0, v_0) \leq 2\, (M, 0).$$

Then by Proposition (6.1.5),

$$(u(t,\cdot;\, u_0, v_0), v(t,\cdot;\, u_0, v_0)) \leq 2\, (u(t,\cdot;\, M, 0), v(t,\cdot;\, M, 0))\ \forall\, t > 0.$$

Since $(u(t,\cdot;\, M, 0), v(t,\cdot;\, M, 0)) \to (u^*(\cdot), 0)\ in\ X_1 \times X_2$ for $0 < \delta \ll 1$, there is $T > 0$ such that $(u(t,\cdot;\, u_0, v_0), v(t,\cdot;\, u_0, v_0)) \leq 2\, (u_\delta(\cdot), 0)\ \forall\, t \geq T$. Then $v(t,\cdot;\, u_0, v_0)$ satisfies

$$v_t(t,x) \geq \nu \int_D k(y-x)[v(t,y) - v(t,x)]dy$$
$$+ v(t,x) f(x, u^*(x) + \epsilon + v(t,x))\, for\ t \geq T.$$

Note that $\tilde{\lambda}_2(\nu, f(\cdot, u^*(\cdot) + \epsilon)) > 0$. By Lemma (6.1.18), for $0 < \delta \ll 1$, there is $\tilde{T} \geq T$ such that $v(t,\cdot;\, u_0, v_0) \geq v_\delta(\cdot)\ \forall\, t \geq 0$. We then have

$$(u(t+\tilde{T},\cdot;\, u_0, v_0), v(t+\tilde{T},\cdot; u_0, v_0)) \leq 2(u(t,\cdot; u_\delta, v_\delta), v(t,\cdot; u_\delta, v_\delta))\ \forall\, t \geq 0.$$

By (61),

$$\lim_{t\to\infty} (u(t,\cdot;\, u_0, v_0), v(t,\cdot;\, u_0, v_0)) = (0, v^*(\cdot)).$$

The theorem is thus proved.

Section (6.2): Some Nonlocal Operators

The principal eigenvalue of an operator is a fundamental notion in modern analysis. In particular, this notion is widely used in PDE's literature and is at the source of many profound results especially in the study of elliptic semi linear problems. For example, the principal eigenvalue is used to characterise the stability of equilibrium of a reaction-diffusion equation enabling the definition of persistence criteria [215], [216], [217], [97], [102], [232], [115]. It is also an important tool in the characterisation of maximum principle properties satisfies by elliptic operators [209], [100] and to describe continuous semi-groups that preserve an order [204], [141], [234]. It is further used in obtaining Liouville type results for elliptic semi-linear equations [93], [99].

We are interested in such notion for linear operators $\mathcal{L}_{\Omega}+a$ defined on the space of continuous functions $C\,(\bar{\Omega})$ by:

$$\mathcal{L}_{\Omega}[\varphi]+a\varphi:=\int_{\Omega}K\,(x,y)\varphi(y)dy+\;a(x)\varphi(x)$$

Where $\Omega\subset\mathbb{R}^N$ is a domain, possibly unbounded, a is a continuous bounded function and K is a non negative kernel satisfying an integrability condition. The precise assumptions on Ω, K and a will be given later on.

For most of positive operators, the principal eigenvalue is a notion related to the existence of an eigen-pair, namely an eigenvalue associated with a positive eigen-element. For 2 the operator $\mathcal{L}_{\Omega}+a$, when the function a is not constant, for any real λ, neither $\mathcal{L}_{\Omega}+a+\lambda$ nor its inverse are compact operators. Moreover, as noticed in [118], [224], [229], [200], the operator $\mathcal{L}_{\Omega}+a$ may not have any eigenvalues in the space $L^p(\Omega)\;or\;C\,(\bar{\Omega})$. for such operator, the existence of an eigenvalue associated with a positive eigenvector is then not guaranteed. Studying quantities that can be used as surrogates of a principal eigenvalue and establishing their most important properties are therefore of great interest for such operators.

In this perspective, we are interested in the properties of the following quantity:

$$\lambda_p(\mathcal{L}_{\Omega}+a):=\sup\{\lambda\in\mathbb{R}|\exists_{\varphi}\in C\,(\bar{\Omega}),\varphi>0,\text{such that }\mathcal{L}_{\Omega}[\varphi]+a(x)\varphi+\lambda_{\varphi}\le 0\;in\;\Omega\},\quad(62)$$

Which can be expressed equivalently by the sup inf formula:

$$\lambda_p(\mathcal{L}_{\Omega}+a)=\sup_{\substack{\varphi\in C\,(\bar{\Omega})\\ \varphi>0}}\inf_{x\in\Omega}\left(-\frac{\mathcal{L}_{\Omega}\,|\varphi|(x)+a(x)\varphi(x)}{\varphi(x)}\right).\qquad(63)$$

This number was originally introduced in the Perron-Frobenius Theory to characterise the eigenvalues of an irreducible positive matrix [218], [245]. Namely, for a positive irreducible matrix A, the eigenvalue $\lambda_1(A)$ associated with a positive eigenvector can be characterised as follows:

$$\lambda_p(A):=\sup_{\substack{x\in\mathbb{R}^N\\ x>0}}\inf_{i\in\{1,\dots,N\}}\left(-\frac{A_{\mathrm{x}}}{x_i}\right)=\lambda_p(A):=\inf_{\substack{x\in\mathbb{R}^N\\ \mathrm{x}\ge0,\mathrm{x}\ne0}}\sup_{i\in\{1,\dots,N\}}\left(-\frac{(A_{\mathrm{x}})_i}{x_i}\right)=:\lambda'_P(A),\;(64)$$

also known as the Collatz-Wieldandt characterisation.

Numerous generalisation of these types of characterisation exist. Generalisations of the characterisation of the principal eigenvalue by variants of the Collatz-Wielandt characterization (i.e. (64)) were first obtained for positive compact operators in $L^p(\Omega)$ [230], [231], [242] and later for general positive operators that posses an eigen-pair [226].

In parallel with the generalisation of the Perron-Frobenius Theory, several inf sup formulas have been developed to characterise the spectral properties of elliptic operators satisfying a maximum principle, see the fundamental works of Donsker and Varadhan [141], Nussbaum, Pinchover [151], Berestycki, Nirenberg, Varadhan [100] and Pinsky [238], [113]. In particular, for an elliptic operator defined in a bounded domain $\Omega \subset \mathbb{R}^N$ and with bounded continuous coefficients, $\mathcal{E} := a_{ij}(x)\partial_{ij} + b_i(x)\partial_i + c(x)$, several notions of principal eigenvalue have been introduced. On one hand, Donsker and Varadhan [141] have introduced a quantity $\lambda_V(\mathcal{E})$, called principal eigenvalue of $\mathcal{E}$, that satisfies

$$\lambda_V(\mathcal{E}) \coloneqq \lambda_p(A) := \inf_{\substack{\varphi \in dom(\mathcal{E}) \\ \varphi > 0}} \sup_{x \in \Omega} \left(-\frac{\mathcal{E}|\varphi|(x)}{\varphi(x)} \right)$$

$=\inf\{\lambda \in \mathcal{E} | \exists \varphi \in dom(\mathcal{E}), \varphi > 0$ such that $\mathcal{E}[\varphi](x) + \lambda_\varphi(x) \geq 0$ in $\Omega\}$, where $dom(\mathcal{E}) \subset C\,(\bar{\Omega})$ denotes the domain of definition of $\mathcal{E}$. On the other hand Berestycki, Nirenberg and Varadhan [100] have introduced $\lambda_1(\mathcal{E})$ defined by:

$$\lambda_1(\mathcal{E}) := \sup\{\lambda \in \mathbb{R} | \exists\, \varphi \in W^{2,N}(\Omega), \varphi > 0 \text{such that} \mathcal{E}[\varphi](x) + \lambda_\varphi(x) \leq 0 \text{ in} \Omega\}$$

$$= \sup_{\substack{\varphi \in W^{2,N} \\ \varphi > 0}} \inf_{x \in \Omega} \left(-\frac{\mathcal{E}|\varphi|(x)}{\varphi(x)} \right)$$

as another possible definition for the principal eigenvalue of $\mathcal{E}$. When Ω is a smooth bounded domain and $\mathcal{E}$ has smooth coefficients, both notions coincide (i.e. $\lambda_V(\mathcal{E}) = \lambda_1(\mathcal{E})$). The equivalence of this two notions has been recently extended for more general elliptic operators, in particular the equivalence holds true in any bounded domains Ω and in any domains when $\mathcal{E}$ is an elliptic self-adjoint operator with bounded coefficients [209]. It is worth mentioning that the quantity $\lambda_V(\mathcal{E})$ was originally introduced by Donsker and Varadhan [141] to obtain the following variational characterisation of $\lambda_1(\mathcal{E})$ in a bounded domain:

$$\lambda_1(E) = \sup_{d\mu \in \mathbb{P}(\Omega)} \inf_{\substack{\varphi \in dom(\mathcal{E}) \\ \varphi > 0}} \int_{\Omega} \left(-\frac{\mathcal{E}|\varphi|(x)}{\varphi(x)} \right) d\mu(x),$$

where $\mathbb{P}(\Omega)$ is the set of all probability measure on Ω. Such characterisation is still valid when Ω is unbounded, see Nussbaum and Pinchover [151].

Lately, the search of Liouville type results for semilinear elliptic equations in unbounded domains [93], [241] and the characterisation of spreading speed [206], [236] have stimulated the studies of the properties of $\lambda_1(E)$ and several other notions of principal eigenvalue have emerged. For instance, several new notions of principal eigenvalue have been introduced for general elliptic operators defined on (limit or almost) periodic media [99], [93], [237], [241]. See [209], for a review and a comparison of the different notions of principal eigenvalue for an elliptic operator defined in a unbounded domain.

For the operator $\mathcal{L}_\Omega + a$, much less is known and only partial results have been obtained when Ω is bounded [118], [223], [141], [146], [228], [229] or in a periodic media [135], [224], [200], [202]. $\lambda_p(\mathcal{L}_\Omega + a)$ has been compared to one of the following definitions:

$\lambda'_p(\mathcal{L}_\Omega + a)$
$=: \inf\{\lambda \in \mathbb{R} | \exists\, \varphi \in C(\Omega) \cap L^\infty(\Omega), \varphi \geq\neq 0, a.t. \mathcal{L}_\Omega[\varphi](x) + (a(x) + \lambda)\varphi(x) \geq 0\ in \Omega\}$
or when $\mathcal{L}_\Omega + a$ is a self-adjoint operator:

$$\lambda_V(\mathcal{L}_\Omega + a) := \inf_{\varphi\in L^2(\Omega),\varphi\neq 0} -\frac{\langle \mathcal{L}_{\Omega}[\varphi] + a\varphi, \varphi\rangle}{\|\varphi\|^2_{L^2(\Omega)}}$$
$$= \inf_{\substack{\varphi\in L^2(\Omega)\\ ,\varphi\neq 0}} \frac{\frac{1}{2}\int\int_{\Omega\times\Omega} K(x,y)\,[\varphi(x)-\varphi(y)]^2 dxdy - \int_\Omega\left[a(x) + \int_\Omega K(x,y)\,dy\right]\varphi^2(x)\,dx.}{\|\varphi\|^2_{L^2(\Omega)}}$$

where $\langle , \rangle$denotes the scalar product of $L^2(\Omega)$. For $\Omega \subset \mathbb{R}^N$a bounded domain and for particular kernels K, an equality similar to $\lambda_V(\mathcal{E}) = \lambda_1(\mathcal{E})$, has been obtained in [118], provided that $K \in C(\overline{\Omega}\times\overline{\Omega})$ satisfies some non-degeneracy conditions. The author shows that

$$\lambda_p(\mathcal{L}_\Omega + a) = \lambda_p'(\mathcal{L}_\Omega + a). \tag{65}$$

In a periodic media, an extension of this equality was obtained in [135], [224] for kernels K of the form $K(x,y) := J(x-y)$ with J a symmetric positive continuous density of probability. In such case, they prove that

$$\lambda_p(\mathcal{L}_{\mathbb{R}^N} + a) = \lambda_p'(\mathcal{L}_{\mathbb{R}^N} + a) = \lambda_V(\mathcal{L}_{\mathbb{R}^N} + a). \tag{66}$$

We pursue the works begun in [118], [224], [223] we investigate more closely the properties of$\lambda_p(\mathcal{L}_\Omega + a)$. Namely, we first look whether $\lambda_p(\mathcal{L}_\Omega + a)$ can be characterised by other notions of principal eigenvalue and under which conditions on $\Omega, K$and a the equality (65) or (66) holds true. We introduce a new notion of principal eigenvalue, $\lambda_p''(\mathcal{L}_\Omega + a)$ defined by: $\lambda_p''(\mathcal{L}_\Omega + a) := \sup\{\lambda \in \mathbb{R} |\, \exists\, \varphi \in C_c(\Omega), \varphi \geq\neq 0$ such that $\mathcal{L}_\Omega[\varphi](x) + (a(x) + \lambda)\varphi(x) \geq 0$ in$\Omega\}$and we compare this new quantity with λ_p, λ_p' and λ_V.

Another natural question is to obtain a clear picture on the dependence of λ_p with respect to all the parameters involved. If the behaviour of $\lambda_p(\mathcal{L}_\Omega + a)$ with respect to a or Ω can be exhibited directly from the definition, the impact of scalings of the kernel is usually unknown and has been largely ignored in the literature except in some specific situations involving particular nonlocal dispersal operators defined in a bounded domain [180], [180], [229], [176].

For a particular type of K and a, we establish the asymptotic properties of λ_p with respect to some scaling parameter. Let $K(x,y) = J(x-y)$ and let us denote $J_\sigma(z) := \frac{1}{\sigma^N} J\left(\frac{Z}{\sigma}\right)$when J is a non negative function of unit mass, we study the properties of the principal eigenvalue of the perator $\mathcal{L}_{\sigma,m,\Omega} - \frac{1}{\sigma^m} + a$, where the operator $\mathcal{L}_{\sigma,m,\Omega}$ is defined by:

$$\mathcal{L}_{\sigma,m,\Omega} := \int_\Omega J_\sigma(x-y)\varphi(y)dy.$$

In this situation, the operator $\mathcal{L}_{\sigma,m,\Omega} - \frac{1}{\sigma^m}$ refers to a nonlocal version of the standard diffusion operator with a homogeneous Dirichlet boundary condition. Such type of operators has appeared recently in the literature to model a population that have a constrained dispersal [205], [145], [147], [229], [176].

The pre-factor $\frac{1}{\sigma^m}$ is interpreted as a frequency at which the events of dispersal occur.

For $m \in [0, 2]$ and a large class of J, we obtain the asymptotic limits of $\lambda_p\left(\mathcal{L}_{\sigma,m,\Omega} - \frac{1}{\sigma^m} + a\right)$ and as $\sigma \to 0$ and as $\sigma \to +\infty$.

Our interest in studying the properties of $\lambda_p(\mathcal{L}_\Omega + a)$ stems from the recent studies of populations having a long range dispersal strategy [118], [223], [205], [229], [200]. For such a population, a commonly used model that integrates such long range dispersal is the following nonlocal reaction diffusion equation ([145], [184], [147], [235], [244]):

$$\partial_t u(t,x) = \int_\Omega J(x-y)u(t,y)dy - u(t,x)$$

$$\int_\Omega J(x-y)u(t,y)dy + f\big(x,u(t,x)\big) in\ \mathbb{R}^+ \times \Omega. \quad (67)$$

$u(t,x)$is the density of the considered population, J is a dispersal kernel and $f(x,s)$ is a KPP type non-linearity describing the growth rate of the population. When Ω is a bounded domain [122], [118], [223], [227], [229], [202], an optimal persistence criteria has been obtained using the sign of $\lambda_p(\mathcal{M}_\Omega + \partial_u f(x,0))$, where $\mathcal{M}_\Omega$ stands for the operator:
$\mathcal{M}_\Omega [\varphi]: = \int_\Omega J(x-y)\varphi(y)dy - j(x)\varphi(x)$,
where $j(x) := \int_\Omega J(y-x)\, dy$.
In such model, a population will persists if and only if $\lambda_p(\mathcal{M}_\Omega + \partial_u f(x,0)) < 0$. We can easily checkthat $\lambda_p(\mathcal{M}_\Omega + \partial_u f(x,0)) = \lambda_p(\mathcal{L}_\Omega - j(x) + \partial_u f(x,0))$.

When $\Omega = \mathbb{R}^N$ and in periodic media, adapted versions of λ_p have been recently used to define an optimal persitence criteria [135], [224], [200], [176]. The extension of such type of persistence criteria for more general environments is currently investigated by ourself [205] by means of our findings on the properties of λ_p.

The understanding of the effect of a dispersal process conditioned by a dispersal budget is another important question. The idea introduced by Hutson, Martinez, Mischaikow and Vickers [147], is simple and consists in introducing a cost function related to the amount of energy an individual has to use to produce offspring, that jumps on a long range. When a long range of dispersal is privileged, the energy consumed to disperse an individual is large and so very few offsprings are dispersed. On the contrary, when the population chooses to disperse on a short range, few energy is used and a large amount of the offsprings is dispersed. In $\mathbb{R}^N$, to understand the impact of a *dispersal budget* on the range of dispersal, we are led to consider the family of dispersal operator:

$$\mathcal{M}_{\sigma,\mathrm{m}} [\varphi](x): = \frac{1}{\sigma^m} \big(J_\sigma * \varphi(x) - \varphi(x)\big),$$

Where $J_\sigma(z) := \frac{1}{\sigma^N} J\left(\frac{z}{\sigma}\right)$is the standard scaling of the probability density$J$. For such family, the study of the dependence of $\lambda_p\big(\mathcal{M}_{\sigma,\mathrm{m}} + a\big)$with respect to σ and m is a first step to analyze the impact of the range of the dispersal σ on the persistence of the population. In particular the asymptotic limits $\sigma \to +\infty$ and $\sigma \to 0$ are of primary interest.

Let us now state the precise assumptions we are making on the domain Ω, the kernel K and the function a. Here, throughout, $\Omega \subset \mathbb{R}^N$is a domain (open connected set of $\mathbb{R}^N$) and for a and K we assume the following:

$$a \in C(\bar{\Omega}) \cap L^\infty(\Omega), \quad (68)$$

and K is a non-negative Caratheodory function, that is $K \geq 0$ and,
$\forall x \in \Omega K(x, \cdot)$ is measurable, $K(\cdot, y)$ is uniformly continuous
for almost every $y \in \Omega$. (69)
For our analysis, we also require that K satisfies the following non-degeneracy condition:

There exist positive constants $r_0 \geq r_1 > 0, C_0 \geq c_0 > 0$ such that K satisfies:

$$C_0 1_{\Omega \cap} B_{r_0}(x)(y) \geq K(x,y) \geq c_0 1_{\Omega \cap B_{r_1(x)}}(y) \text{for all} x, y \in \Omega, \qquad (70)$$

where 1_A denotes the characteristic function of the set $A \subset \mathbb{R}^N$ and $Br(x)$ *is* the ball centred *at* x of radius r. These conditions are satisfied for example for kernels like $K(x,y) = J\left(\frac{x-y}{g(y)h(x)}\right)$with $h$and g positive and bounded in Ω and $J \in C(\mathbb{R}^N), J \geq 0$, a compactly supported function such that $J(0) > 0$. Note that when Ω is bounded, any kernel $K \in C(\bar{\Omega} \times \bar{\Omega})$ which is positive on the diagonal,6 satisfies all theses assumptions. Under this assumptions, we can check that the operator $\mathcal{L}_\Omega + a$ is continuous in $C(\bar{\Omega})$, [233].

We start by investigating the case of a bounded domain Ω.

In this situation, we prove that λ_p, λ_p', and λ_p'' represent the same quantity. Namely, we show the following

Theorem (6.2.1)[203]: Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and assume that K and a satisfy (68) – (70).Then, the following equality holds:

$$\lambda_p(\mathcal{L}_\Omega + a) = \lambda_p'(\mathcal{L}_\Omega + a) = \lambda_p''(\mathcal{L}_\Omega + a).$$

In addition, if K is symmetric, then

$$\lambda_p(\mathcal{L}_\Omega + a) = \lambda_v(\mathcal{L}_\Omega + a).$$

When Ω is an unbounded domain, the equivalence of $\lambda_p, \lambda_p'$and λ_p'' is not clear for general kernels. Namely, let consider $\Omega = \mathbb{R}, K(x,y) = J(x - y)$ with J a density of probability with acompact support and such that$\int_{\mathbb{R}} J(z)z\,dz > 0$. For the operator $\mathcal{L}_{\mathbb{R}}$, which corresponds to the standard convolution by J, by using $e^{\lambda x}$ and constants as test functions, we can easily check that $\lambda_p'(\mathcal{L}_{\mathbb{R}}) \leq -1 < \min_{\lambda>0} \int_{\mathbb{R}} J(z)e^{\lambda z}\,dz \leq \lambda p(\mathcal{L}_{\mathbb{R}})$. < However some inequalities remain true in general and the equivalence of the three notions holds for self-adjoint operators. We prove here the following

Another striking property of λ_p refers to the invariance of λ_p under a particular scaling of the kernel K. We show

Proposition (6.2.2)[203]: Let $\Omega \subset \mathbb{R}^N$ *be* a domain and assume that a and K satisfy (68) – (70). For all$\sigma > 0$, *let* $\Omega_\sigma := \sigma\Omega, a_\sigma(x) := a\left(\frac{x}{\sigma}\right)$ and

$$L_{\sigma,\Omega\sigma}[\varphi](x) := \frac{1}{\sigma^N} \int\limits_{\Omega\sigma} K\left(\frac{x}{\sigma}, \frac{y}{\sigma}\right) \varphi(y)\,dy.$$

Then for all $\sigma > 0$, one has

$$\lambda_p(\mathcal{L}_\Omega + a) = \lambda_p(\mathcal{L}_{\sigma,\Omega_\sigma} + a).$$

Observe that no condition on the domain is imposed. Therefore, the invariance of λ_p is still valid for $\Omega = \mathbb{R}^N$. In this case, since $\mathbb{R}^N$ is invariant under the scaling, we get

$$\lambda_p(\mathcal{L}_{\mathbb{R}^N} + a) = \lambda_p(\mathcal{L}_{\sigma,\mathbb{R}^N} + a_\sigma).$$

Next, for particular type of kernel K, we investigate the behaviour of λ_p with respect of some scaling parameter. Let $K(x,y) = J(x - y)$ and let $J_\sigma(z) := \frac{1}{\sigma^N} J\left(\frac{z}{\sigma}\right)$.

We consider the following operator

$$\mathcal{L}_{\sigma,\mathrm{m},\Omega}[\varphi] := \frac{1}{\sigma^m} \int\limits_{\Omega} J_\sigma(x-y)\varphi(y)\,dy.$$

For J is a non negative function of unit mass, we study the asymptotic properties of the principal eigenvalue of the operator $\mathcal{L}_{\sigma,\mathrm{m},\Omega} - \frac{1}{\sigma^N} + a$ when $\sigma \to 0$ and $\sigma \to +\infty$.

To simplify the presentation of our results, let us introduce the following notation. We denote by $\mathcal{M}_{\sigma,\mathrm{m},\Omega}$, the following operator:

$$\mathcal{M}_{\sigma,\mathrm{m},\Omega}[\varphi] := \frac{1}{\sigma^m}\left(\frac{1}{\sigma^N}\int_{\Omega} J\left(\frac{x-y}{\sigma}\right)\varphi(y)dy - \varphi(x)\right). \qquad (71)$$

For any domains Ω, we obtain the limits of $\lambda_p(\mathcal{M}_{\sigma,\mathrm{m},\Omega} + a)$ when σ tends either to zero or to $+\infty$.

Let us denote the second moment of J by

$$D2(J) := \int_{\mathbb{R}^N} J(z)|z|^2\, dz,$$

the following statement describes the limiting behaviour of $\lambda_p(\mathcal{M}_{\sigma,\mathrm{m},\Omega} + a)$:

Theorem (6.2.3)[203]: Let Ω be a domain and assume that J and a satisfy (68) – (70). Assume further that J is even and of unit mass. Then, we have the following asymptotic behaviour:

(i) When $0 < m \le 2$, $\lim_{\sigma\to+\infty} \lambda_p(\mathcal{M}_{\sigma,\mathrm{m},\Omega} + a) = -\sup_{\Omega} a$

(ii) When $m = 0$ $\lim_{\sigma\to+\infty} \lambda_p(\mathcal{M}_{\sigma,\mathrm{m},\Omega} + a) = 1 - \sup_{\Omega} a$.

In addition, when $\Omega = \mathbb{R}^N$ and if a is symmetric $(a(x) = a(-x)$ for all $x)$ and the map $t \to a(tx)$ is non increasing for all x, $t > 0$ then $\lambda_p(\mathcal{M}_{\sigma,0,\mathbb{R}^N} + a)$ is monotone non decreasing with respect to σ.

(iii) When $0 \le m < 2$, $\lim_{\sigma\to 0} \lambda_p(\mathcal{M}_{\sigma,\mathrm{m},\Omega} + a) = -\sup_{\Omega} a$

(iv) When $m = 2$ and $a \in C^{0,\alpha}(\Omega)$ for some $\alpha > 0$, then

$$\lim_{\sigma\to 0} \lambda_p(\mathcal{M}_{\sigma,2,\Omega} + a) = \lambda_1\left(\frac{D_2(J)}{2N}\Delta + a\right)$$

And

$$\lambda_1\left(\frac{D_2(J)}{2N}\Delta + a\right) := \inf_{\varphi\in H_0^1(\Omega),\varphi\neq 0} \frac{D_2(J)}{2N}\frac{\int_\Omega |\nabla\varphi|^2(x)\,dx}{\|\varphi\|_2^2} - \frac{\int_\Omega a(x)\varphi^2(x)\,dx}{\|\varphi\|_2^2}.$$

Note that the results hold for any domains Ω, so the results holds true in particular for $\Omega = \mathbb{R}^N$.

Having established the asymptotic limits of the principal eigenvalue $\lambda_p(\mathcal{M}_{\sigma,2,\Omega} + a)$, it is natural to ask whether similar results hold for the corresponding eigenfunction $\varphi_{\sigma,p}$ when it exists. In this direction, we prove that for $m = 2$, such convergence does occur:

Theorem (6.2.4)[203]: Let Ω be any domain and assume that J and a satisfy (68) – (70). Assume further that J is even and of unit mass. Then there exists σ_0 such that for all $\sigma \le \sigma_0$, there exists a positive principal eigenfunction $\varphi_{\sigma,p}$ associated to $\lambda_p(\mathcal{M}_{\sigma,2,\Omega} + a)$. In addition, when $\varphi_{\sigma,p} \in L^2(\Omega)$ for all $\sigma \le \sigma_0$, we have

$$\varphi_{p,\sigma} \to \varphi_1 \text{ in } L^2_{\mathrm{loc}}(\Omega),$$

where $\varphi_1 \in H_0^1(\Omega)$ is a positive principle eigenfunction associated to $\lambda_1\left(\frac{D_2(J)}{2N}\Delta + a\right)$.

First, we can notice that the quantity λ_v defined by Donsker and Varadhan [141] for elliptic operators can also be defined for the operator $\mathcal{L}_\Omega + a$ and is equivalent to the quantity λ_p. The equality (65) can then be seen as the nonlocal version of the equality $\lambda_1 = \lambda_V$ where λ_1 is the notion introduced by Berestycki-Nirenberg-Varadhan [100].

Next, we would like to emphasize, that unlike the classical elliptic operators, due to the lack of a regularising effect of the operator $\mathcal{L}_{\Omega} + a$, the quantity $\lambda_p(\mathcal{L}_{\Omega} + a)$ may not be an eigenvalue, i.e.the spectral problem

$$\mathcal{L}_{\Omega}[\varphi](x) + a(x)\,\varphi(x) + \lambda_{\varphi}(x) = 0 \ in \ \Omega,$$

may not have a solution in spaces of functions like $L^p(\Omega), C(\Omega)$[135], [222], [141], [229]. As a consequence, even in bounded domains, the relations between $\lambda_p, \lambda'_p, \lambda''_p$, and λ_v are quite delicate to obtain.

Another difficulty inherent to the study of nonlocal operators in unbounded domains concerns the lack of natural *a* priori estimates for the positive eigenfunction thus making standard approximation schemes difficult to use in most case.

Lastly, we make some additional comments on the assumptions we have used on the dispersal kernel *K*. The non-degeneracy assumption (70) we are using, is related to the existence of Local Uniform Estimates [219], [220] (Harnack type estimates) for a positive solution of a nonlocal equation:

$$\mathcal{L}_{\Omega}[\varphi] + b(x)\,\varphi = 0 \ \ in \ \ \ \Omega. \qquad (72)$$

Such type of estimates is a key tool in our analysis, in particular in unbounded domains, where we use it to obtain fundamental properties of the principal eigenvalue $\lambda_p(\mathcal{L}_{\Omega} + a)$, such as the limit:

$$\lambda_p(\mathcal{L}_{\Omega} + a) = \lim_{n\to\infty} \lambda_p(\mathcal{L}_{\Omega n} + a),$$

where Ω_n is a sequence of set converging to Ω. As observed in [221], some local uniform estimates can also be obtained for some particular kernels *K* which does not satisfies the non-degeneracy condition (70). For example, for kernels of the form $K(x,y) = \frac{1}{g^N(y)} J\left(\frac{x-y}{g(y)}\right)$with J satisfying (69) and (70) and $g \geq 0$ a bounded function such that $\{x|g(x) = 0\}$ is a bounded set and with Lebesgue measure zero, some local uniform estimates can be derived for positive solutions of (72). As a consequence, the Theorems (6.2.1) and (6.2.14) hold true for such kernels. We have also observed that the condition (70) can be slightly be relaxed and the Theorems (6.2.1) and (6.2.14) hold true for kernels *K* such that, for some positive integer *p*, the kernel K_p defined recursively by:

$$K_1(x,y) := K(x,y),$$

$$K_{n+1}(x,y) := \int_{\Omega} K_n\,(x,z) K_1(z,y) dz. \ for \ n \ \geq 1,$$

satisfies the non-degeneracy condition (70).

For a convolution operator, i.e. $K(x,y) := J(x-y)$, this last condition is optimal. It is related to a geometric property of the convex hull of $\{y \in \mathbb{R}^N | J(y) > 0\}$:
K_p satisfies (70) for some $p \in \mathbb{N}$ if and only if the convex hull of$\{y \in \mathbb{R}^N | J(y) > 0\}$ contains 0.

Note that if a relaxed assumption on the lower bound of the non-degeneracy condition satisfied by *K* appears simple to find, the condition on the support of *K* seems quite tricky to relax. To tackle this problem, it is tempting to investigate the spectrum of linear operators involving the Fractional Laplacian, Δ^{α}:

$$\Delta^{\alpha}\varphi := C_{N,\alpha} P.V.\left(\int_{\Omega} \frac{\varphi(y) - \varphi(x)}{|x-y|^{N+2\alpha}}\, dy.\right)\varphi \equiv 0 \text{in} \mathbb{R}^N \setminus \Omega$$

That is, to look for the properties of the principal eigenvalue of the spectral problem:

$$\Delta^{\alpha}\varphi + (a + \lambda)\varphi = 0 \qquad in \qquad \Omega. \tag{73}$$

As for elliptic operators and $\mathcal{L}_{\Omega} + a$, analogues of λ_1, λ_1', and $\lambda_0\lambda 0$ can be defined for $\Delta^{\alpha} + a$ and the relations between all possible definitions can be investigated. When Ω is bounded or a is periodic, the different definitions are equivalent [207]. However, in the situations considered in [207] the operator $\Delta^{\alpha} + a$ has a compact resolvent enabling the use of the Krein Rutmann Theory. Thus, the corresponding λ_p is associated with a positive eigenfunction, rendering the relations much more simpler to obtain.

Moreover, in this analysis, the regularity of the principal eigenfunction and a Harnack type inequality [213], [214], [243] for some non negative solution of (73) are again the key ingredients in the proofs yielding to the inequality

$$\lambda_p'(\Delta^{\alpha} + a, \Omega) \le \lambda p(\Delta^{\alpha} + a, \Omega)$$

for any smooth domain Ω.

Such Harnack type inequalities are not known for operators $\mathcal{L}_{\Omega} + a$ involving a continuous kernel K with unbounded support. Furthermore, it seems that most of the tools used to establish these Harnack estimates in the case of the Fractional Laplacian [213], [243] do not apply when we consider an operator $\mathcal{L}_{\Omega} + a$. Thus, obtaining the inequality

$$\lambda_p'(\mathcal{L}_{\Omega} + a) \le \lambda_p(\mathcal{L}_{\Omega} + a)$$

With a more general kernel requires a deeper understanding of Harnack type estimates and/or the development of new analytical tools for such type of nonlocal operators.

In this direction and in dimension one, for some kernels with unbounded support, we could obtain some inequalities between the different notions of principal eigenvalue.

Proposition (6.2.5)[203]: Assume $N = 1$ and let $\Omega \subset \mathbb{R}$ be a unbounded domain. Assume that K and asatisfy (68)–(69). Assume further that K is symmetric and there exists $C > 0$ and $\alpha > \frac{3}{2}$ such that $K(x, y) \le C(1 + |x - y|)^{-\alpha}$. Then we have

$$\lambda_p(\mathcal{L}_{\Omega} + a) \le \lambda_v(\mathcal{L}_{\Omega} + a) \le \lambda_p'(\mathcal{L}_{\Omega} + a) \le \lambda_p''(\mathcal{L}_{\Omega} + a)$$

We recall some known results and properties of the principal eigenvalue $\lambda_p(\mathcal{L}_{\Omega} + a)$. The relations between the different definitions of the principal eigenvalue, $\lambda_p, \lambda_p', \lambda_p''$ and λ_v (Theorems (6.2.1), (6.2.14) and Proposition (6.2.5)). We derive the asymptotic behaviour of λ_p with respect to the different scalings of K (Proposition (6.2.2) and Theorems (6.2.3) and (6.2.4)).

To simplify the presentation of the proofs, we introduce some notations and various linear operator:

(i) $B_R(x_0)$ denotes the standard ball of radius R centred at the point x_0

(ii) 1_R will always refer to the characteristic function of the ball $B_R(0)$.

(iii) $S(\mathbb{R}^N)$ denotes the Schwartz space,[212]

(iv) C(Ω) denotes the space of continuous function in Ω,

(v) $C_c(\Omega)$ denotes the space of continuous function with compact support in Ω.

(vi) For a positive integrable function $J \in S(\mathbb{R}^N)$, the constant $\int_{\mathbb{R}^N} J(z)|z|^2 \mathrm{dz}$ will refer to

$$\int_{\mathbb{R}^N} J(z)|z|^2 \mathrm{dz} := \int_{\mathbb{R}^N} J(z)\left(\sum_{t=1}^{N} z_1^2\right)\mathrm{dz}$$

(vii) For a bounded set $\omega \subset \mathbb{R}^N$, $|\omega|$ will denotes its Lebesgue measure

(viii) For two L^2 functions $\varphi, \psi, \varphi, \psi$ denotes the L^2 scalar product of ψ and φ

(ix) For $J \in L^1(\mathbb{R}^N), J_\sigma(z) := \frac{1}{\sigma^N} J\left(\frac{z}{\sigma}\right)$

(x) We denote by $\mathcal{L}_{\sigma,m,\Omega}$ the continuous linear operator

$$\mathcal{L}_{\sigma,m,\Omega}: \mathrm{C}(\overline{\Omega}) \to \mathrm{C}(\overline{\Omega})$$

$$\varphi \to \frac{1}{\sigma^m} \int_\Omega J_\sigma(x - y)u(y)dy, \tag{74}$$

where $\Omega \subset \mathbb{R}^N$.

(xi) We denote by $\mathcal{M}_{\sigma,m,\Omega}$ the operator $\mathcal{M}_{\sigma,m,\Omega} := \mathcal{L}_{\sigma,m,\Omega} - \frac{1}{\sigma^{\mathrm{m}}}$

We recall some standard results on the principal eigenvalue of the operator $\mathcal{L}_\Omega + a$. Since the early work [141] on the variational formulation of the principal eigenvalue, an intrinsic difficulty related to the study of these quantities comes from the possible non-existence of a positive continuous eigenfunction associated to the definition of $\lambda_p, \lambda'_p, \lambda''_p$, or to λ_v. This means that there is not always a positive continuous eigenfunction associated to $\lambda_p, \lambda'_p, \lambda''_p$, or λ_v. A simple illustration of this fact can be found in [118], [222]. Recently, some progress have been made in the understanding of λ_p. In particular, some flexible criteria have been found to guarantee the existence of a positive continuous eigenfunction [118], [229], [202].

Theorem (6.2.6)[203]: (Sufficient condition [118]). Let $\Omega \subset \mathbb{R}^N$ be a domain, $a \in C(\Omega) \cap L^\infty(\Omega)$ and $K \in C(\mathrm{C}(\overline{\Omega} \times \overline{\Omega})$ non negative, satisfying the condition (70). Let us denote $\nu := sup\overline{\Omega}$ a and assume furtherthat the function a satisfies $\frac{1}{\nu-a} \in L^1(\Omega_0)$ for some bounded domain $\Omega_0 \subset \overline{\Omega}$. Then there exists aprincipal eigen-pair (λ_p, φ_p) solution of

$$\mathcal{L}_\Omega[\phi](x) + (a(x) + \lambda)\phi(x) = 0 \; in \; \Omega.$$

Moreover, $\varphi_p \in \mathrm{C}(\overline{\Omega}), \varphi_p > 0$ and we have the following estimate

$$-\nu' < \lambda_p < -\nu,$$

where $\nu' := \sup_{x\in\Omega}\left[a(x) + \int_\Omega K(x,y)\mathrm{dy}\right]$.

This criteria is almost optimal, in the sense that we can construct example of operator $\mathcal{L}_\Omega + a$ with Ω bounded and a such that $\frac{1}{\nu-a} \in L^1(\Omega)$ and where $\lambda_p(\mathcal{L}_\Omega + a)$ is not an eigenvalue in $\mathrm{C}(\overline{\Omega})$, see [118], [229], [202].

When Ω is bounded, sharper results have been recently derived in [222] where it is proved that$\lambda_p(\mathcal{L}_\Omega + a)$is always an eigenvalue in the Banach space of positive measure, that is, we can always find a positive measure $d\mu_p$ that is solution in the sense of measure of

$$\mathcal{L}_\Omega\left[d\mu_p\right](x) + a(x)d\mu_p(x) + \lambda_p d\mu_p(x) = 0. \tag{75}$$

We have the following characterisation of λ_p:

Theorem (6.2.7)[203]: ([224], [222]). $\lambda_p(\mathcal{L}_\Omega + a)$ is an eigenvalue in $\mathrm{C}(\overline{\Omega})$ if and only if $\lambda_p(\mathcal{L}_\Omega + a) < -\sup_{x\in\Omega}(x)$.

See [222] for a more complete description of the positive solution associated to λ_p when the domain Ω is bounded.

We recall some properties of λ_p.

Proposition (6.2.8)[203]: (i) Assume $\Omega_1 \subset \Omega_2$, then for the two operators $\mathcal{L}_{\Omega_1} + a$and $\mathcal{L}_{\Omega_2} + a$ arespectively defined on $C(\Omega_1)$ and $C(\Omega_2)$, we have :

$$\lambda_p(\mathcal{L}_{\Omega_1} + a) \geq \lambda_p(\mathcal{L}_{\Omega_2} + a).$$

(ii) For a fixed Ω and assume that $a_1(x) \geq a_2(x)$, for all $x \in \Omega$. Then

$$\lambda_p(\mathcal{L}_\Omega + a_2) \geq \lambda_p(\mathcal{L}_\Omega + a_1).$$

(iii) $\lambda_p(\mathcal{L}_\Omega + a)$ is Lipschitz continuous with respect to a. More precisely,

$$\left|\lambda_p(\mathcal{L}_\Omega + a) - \lambda_p(\mathcal{L}_\Omega + b)\right| \leq \|a - b\|_\infty$$

(iv) The following estimate always holds

$$-\sup_{x\in\Omega}\left(a(x) + \int_\Omega K(x,y)\mathrm{dy}\right) \leq \lambda_p(\mathcal{L}_\Omega + a) \leq -\sup_\Omega a.$$

See [118], [223] for the proofs of (i) − (iv).

We prove some limit behaviour of $\lambda_p(\mathcal{L}_\Omega + a)$ with respect to the domain Ω. We show

Lemma (6.2.9)[203]: Let Ω be a domain and assume that a and K satisfy (68)–(70). Let $(\Omega_n)_{n\in N}$ be a sequence of subset of Ω so that $\lim_{n\to\infty} \Omega_n = \Omega, \Omega_n \subset \Omega_{n+1}$. Then we have

$$\lim_{n\to\infty} \lambda_p\left(\mathcal{L}_{\Omega_n} + a\right) = \lambda_p(\mathcal{L}_\Omega + a)$$

Proof. By a straightforward application of the monotone properties of λp with respect to the domain

((i) of Proposition (6.2.8)) we get the inequality

$$\lambda_p(\mathcal{L}_\Omega + a) \leq \lim_{n\to\infty} \lambda_p\left(\mathcal{L}_{\Omega_n} + a\right). \tag{76}$$

To prove the equality, we argue by contradiction. So, let us assume

$$\lambda_p(\mathcal{L}_\Omega + a) < \lim_{n\to\infty} \lambda_p\left(\mathcal{L}_{\Omega_n} + a\right). \tag{77}$$

and choose $\lambda \in \mathbb{R}$ such that

$$\lambda_p(\mathcal{L}_\Omega + a) < \lambda < \lim_{n\to\infty} \lambda_p\left(\mathcal{L}_{\Omega_n} + a\right). \tag{78}$$

We claim

Claim (6.2.10)[203]: There exists $\varphi > 0, \varphi \in C(\Omega)$so that (λ, ϕ) is an adequate test function. That is, ϕ satisfies

$$\mathcal{L}_\Omega[\varphi](x) + (a(x) + \lambda)\varphi(x) \leq 0 \ in \ \Omega.$$

Assume for the moment that the above claim holds. By definition of $\lambda_p(\mathcal{L}_\Omega + a)$, we get a straightforward contradiction

$$\lambda_p(\mathcal{L}_\Omega + a) < \lambda \leq \lambda_p(\mathcal{L}_\Omega + a).$$

Hence,

$$\lim_{n\to\infty} \lambda_p\left(\mathcal{L}_{\Omega_n} + a\right) = \lambda_p(\mathcal{L}_\Omega + a)$$

Let us now prove Claim (6.2.10).

Proof. By definition of $\nu := \sup_\Omega a$, there exists a sequence of points $(x_k)_{k\in\mathbb{N}}$ such that $x_k \in \Omega \ and \ |a(x_k) - \nu| < \frac{1}{k}$. By continuity of a, for each k, there exists $\eta_k > 0$ such that

$$B_{\eta_k}(x_k) \subset \Omega, \text{ and } \sup_{B_{\eta_k}} B_{\eta_k}|a - \nu| \leq \frac{2}{k}.$$

Now, let χ_k be the following cut-off” functions: $\chi_k(x) := \chi\left(\frac{\|xk-x\|}{\varepsilon_k}\right)$ where $\varepsilon_k > 0$ is to be chosen later on and χ is a smooth function such that $0 \leq \chi \leq 1, \chi(z) = 0$ for $|z| \geq 2$ and χ(z) = 1 for $|z| \leq 1$. Finally, let us consider the continuous functions $a_k(\cdot)$, defined by $a_k(x) := sup\{a, (\nu - inf\Omega\ a)\chi_k(x) + \inf_\Omega a\}$. By taking a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ so that $\varepsilon_k \leq \frac{\eta k}{2}$, $\varepsilon_k \to 0$, we have

$$a_k(x) = \begin{cases} a & for\ x \in \Omega \setminus B_{2\varepsilon_k}(\chi_k) \\ \nu & for\ x \in \Omega \cap B_{\varepsilon_k}(\chi_k) \end{cases}$$

and therefore

$$\|a - a_k\|_\infty \le \nu := \sup_{B_{\eta_k}(\chi_k)} |\nu - a| \to 0\ as\ k \to \infty.$$

By construction, for k large enough, say $\geq k_0$, we get for all $k \geq k_0$

$$\|a - a_k\|_\infty \le \inf| \left\{ \frac{|\lambda_p(\mathcal{L}_\Omega + a) - \lambda|}{2} \frac{\left| \lim_{n\to\infty} \lambda_p(\mathcal{L}_{\Omega_n} + a) - \lambda \right|}{2} \right\}.$$

Since $\Omega_n \to \Omega$ when n→∞, there exists $n_0 := n(k_0)$ so that

$$B_{\eta_{k_0}}(x_{k_0}) \subset \Omega_n\ for\ all\ n \geq n_0.$$

On the othre hand, from the Lipschitz continuity of $\lambda_p(\mathcal{L}_\Omega + a)$ with respect to a ((iii) Proposition (6.2.8)), inequality (78) yields

$$\lambda_p(\mathcal{L}_\Omega + a_{k_0}) < \lambda < \lim_{n\to\infty} \lambda_p(\mathcal{L}_{\Omega_n} + a_{k_0}) \tag{79}$$

Now, by construction we see that for $n \geq n_0$, $\sup_{\Omega_n} a_{k_0} = \sup_{\Omega_n} a_{k_0} = \nu$ and since $a_{k_0} \equiv \nu$ ν in$\frac{B_{\varepsilon_{k_0}}}{2}(x_{k_0})$, for all $n \geq n_0$ the function $\frac{1}{\nu - ak0} \notin L^1_{loc}(\bar{\Omega}_n)$. Therefore, by Theorem (6.2.6), for all $n \geq n_0$there exists $\varphi_n \in C(\bar{\Omega}_n), \varphi_n > 0$ associated with $\lambda_p(\mathcal{L}_{\Omega_n} + a_{k_0})$.

Moreover, since $x_{k_0} \in \cap_{n \geq n_0} \Omega_n$, for all $n \geq n_0$, we can normalize $\varphi_n$by $\varphi_n x_{k_0} = 1$. Recall that for all $n \geq n_0, \varphi_n$satisfies

$$\mathcal{L}_{\Omega_n}[\varphi_n](x) + (a_{k_0}(x) + \lambda_p(\mathcal{L}_{\Omega_n} + a_{k_0}(x)))\varphi_n(x) = 0\ in\ \Omega_n,$$

so from (79), it follows that (φ_n, λ) satisfies

$$\mathcal{L}_{\Omega_n}[\varphi_n](x) + (a_{k_0}(x) + \lambda) < \mathcal{L}_{\Omega_n}[\varphi_n](x) + (a_{k_0}(x) + \lambda_p(\mathcal{L}_{\Omega_n} + a_{k_0})\varphi_n(x) = 0\ in\ \Omega_n \tag{80}$$

Let us now define $b_n(x) := -\lambda_p(\mathcal{L}_{\Omega_n} + a_{k_0}(x)) - a_{k_0}(x)$, then for all $n \geq n_0, \varphi_n$n satisfies

$$\mathcal{L}_{\Omega_n}[\varphi_n](x) = b_n(x)\varphi_n(x) = 0\ in\ \Omega_n. \tag{81}$$

Construction, for $n \geq n_0$, we have $b_n(x) \geq -\lambda_p(\mathcal{L}_{\Omega_n} + a_{k_0}) - \nu > 0$. Therefore, since Ksatisfies the condition (70), the Harnack inequality (Theorem (6.2.3) in [221]) applies to φ_n. Thus, for $n \geq n_0$ fixed and for any compact set $\omega \subset\subset \Omega_n$ there exists a constant $C_n(\omega)$ such that

$$\varphi_n(x) \le C_n(\omega)\varphi_n(y) \quad \forall\, x, y \in \omega.$$

Moreover, the constant $C_n(\omega)$ only depends on $\delta_0 < \frac{d(\omega,\partial\Omega)}{4}$, $c_0 \cup_{x\in\omega} B_{\delta_0}(x)$, and $\inf_{\Omega_n} b_n$. Furthermore, this constant is decreasing with respect to $\inf_{\Omega_n} b_n$ bn. Notice that for all $n \geq n_0$, the function $b_n(x)$ being uniformly bounded from below by a constant independent of n, the constant C_n is bounded from above independently of n by a constant $C(\omega)$. Thus, we have

$$\varphi_n(x) \le C(\omega)\varphi_n(y)\ \forall\, x, y \in \omega.$$

From a standard argumentation, using the normalization $\varphi_n(x_{k_0}) = 1$, we deduce that the sequence $(\varphi_n)_{n \geq n_0}$is uniformly bounded in $C_{loc}(\Omega)$ topology and is locally uniformly equicontinuous.

Therefore, from a standard diagonal extraction argument, there exists a subsequence, still denoted$(\varphi_n)_{n \geq n_0}$, such that converges locally uniformly to a continuous function ϕ which is nonnegative, non trivial function and satisfies $\varphi_n(x_{k_0}) = 1$.
Since K satisfies the condition (70), we can pass to the limit in the Equation (80) using the Lebesgue monotone convergence theorem and we get

$$\mathcal{L}_\Omega[\varphi] + \left(a_{k_0}(x) + \lambda\right)\varphi(x) \leq 0 \ in\ \Omega\ .$$

Hence, we have

$$\mathcal{L}_\Omega[\varphi](x) + (a(x) + \lambda)\varphi(x) \leq 0 \ in\ \Omega,$$

since $a \leq a_{k_0}$.

We investigate the relations between the quantities $\lambda_p, \lambda'_p, \lambda''_p$ and λ_v and prove Theorems (6.2.1) and (6.2.14).
First, remark that, as consequences of the definitions, the monotone and Lipschitz continuity properties satisfied by λ_p ((i)–(iii) of Proposition (77)) are still true for λ_p and λ_v. We investigate now the relation between λ'_p and λ_p:
Lemma (6.2.11)[203]: Let $\Omega \subset \mathbb{R}^N$ be a domain and assume that K and a satisfy (68)–(70). Then,

$$\lambda'_p(\mathcal{L}_\Omega + a) \ \leq \lambda_p(\mathcal{L}_\Omega + a).$$

Proof. Observe that to get inequality $\lambda'_p(\mathcal{L}_\Omega + a) \ \leq \lambda_p(\mathcal{L}_\Omega + a)$, it is sufficient to show that for any $\delta > 0$:

$$\lambda'_p(\mathcal{L}_\Omega + a) \ \leq \lambda_p(\mathcal{L}_\Omega + a) + \delta.$$

For $\delta > 0$, let us consider the operator $\mathcal{L}_\Omega + \delta_p$ where $\delta_p := a + \lambda_p(\mathcal{L}_\Omega + a) + \delta$. We claim that
Claim (6.2.12)[203]: For all $\delta > 0$, there exists $\varphi_\delta \in C_c(\Omega)$ such that $\varphi_\delta \geq 0$ and φ_δ satisfies

$$\mathcal{L}_\Omega[\varphi_\delta](x) + b_\delta(x)\varphi_\delta(x) \geq 0 \ in\ \Omega\ .$$

By proving the claim, we prove the Lemma. Indeed, assume for the moment that the claim holds.
Then, by construction, $\left(\varphi_\delta, \lambda_p(\mathcal{L}_\Omega + a) + \delta\right)$ satisfies

$$\mathcal{L}_\Omega[\varphi_\delta](x) \ + \ [a(x) \ + \lambda_p(\mathcal{L}_\Omega + a) + \delta]\varphi_\delta(x) \ \geq 0 \ in\ \Omega.$$

Thus, by definition of $\lambda'_p(\mathcal{L}_\Omega + a)$, we have $\lambda'_p(\mathcal{L}_\Omega + a) \leq \lambda_p(\mathcal{L}_\Omega + a) + \delta$. The constant δ being arbitrary, we get for all $\delta > 0$:

$$\lambda'_p(\mathcal{L}_\Omega + a) \leq \lambda_p(\mathcal{L}_\Omega + a) + \delta.$$

Proof. Let $\delta > 0$ be fixed. By construction $\lambda_p(\mathcal{L}_\Omega + b_\delta) < \ 0$, so by Lemma (6.2.9), there exists a bounded open set ω such that $\lambda_p(\mathcal{L}_\omega + b_\delta) < \ 0$. For any $\varepsilon > 0$ small enough, by taking ω larger if necessary, arguing as in the proof of Claim (6.2.10), we can find b_ε such that

$$\|b_\delta - b_\varepsilon\|_{\infty,\omega} = \|b_\delta - b_\varepsilon\|_{\infty,\Omega} \leq \ \varepsilon$$
$$\lambda_p(\mathcal{L}_\omega + b_\varepsilon(x)) \ + \varepsilon \ < \ 0,$$

and there is $\varphi_p \in C(\overline{\omega}), \varphi_p > 0$ associated to $\lambda_p(\mathcal{L}_\omega + b_\varepsilon(x))$. That is ϕp satisfies

$$L\omega\ \left[\varphi_p\right](x) + \ b_\varepsilon(x)\varphi_p(x) = -\lambda_p\left(\mathcal{L}_\omega + b_\varepsilon(x)\right)\varphi_p(x) \ \ in \ \ \ \omega. \qquad (82)$$

Without loss of generality, assume that $\varphi_p \leq 1$.
Let ν denotes the maximum ofb_εε in $\overline{\omega}$, then by Proposition (6.2.8), there exists $\tau > 0$ such that

$$-\lambda_p(\mathcal{L}_\omega + b_\varepsilon(x)) - \varepsilon - \nu \geq \tau > 0.$$

Moreover, since φ_p satisfies (82), there exists $d_0 > 0$ so that $\inf_{\omega\varphi_p} \geq d_0$.

Let us choose $\omega' \subset\subset \omega$ such that

$$\frac{|\omega \setminus \omega'| \leq d_0 inf\{\tau, -\lambda_p(\mathcal{L}_\omega + b_\varepsilon) - \varepsilon\}}{2\|K\|_\infty},$$

where for a set $A, |A|$ denotes the Lebesgue measure of A.

Since $\omega' \subset\subset \omega$ and $\partial\omega$ are two disjoint closed sets, by the Urysohn's Lemma there exists a continuous function η such that $0 \leq \eta \leq 1, \eta = 1$ in $\omega, \eta = 0$ in $\partial\omega$. Consider now $\varphi_p\eta$ and let us compute $\mathcal{L}_\omega[\varphi_p\eta] + b_\delta\varphi_p\eta$. Then, we have

$$\begin{aligned}\mathcal{L}_\omega[\varphi_p\eta] + b_\delta\varphi_p\eta \geq & -\lambda_p(\mathcal{L}_\omega + b_\varepsilon)\varphi_p - \|K\|\,|\omega \ \ \omega| \\ & -b_\varepsilon\varphi_p(1-\eta) - (b_\varepsilon - b_\delta)\varphi_p\eta, \\ \geq & -\left(\lambda_p(\mathcal{L}_\omega + b_\varepsilon) + \|b_\delta - b_\varepsilon\|_{\infty,\omega}\right)\varphi_p \\ & - \frac{d_0 \inf\{\tau, -\lambda_p(\mathcal{L}_\omega + b_\varepsilon) - \varepsilon\}}{2} - b_\varepsilon(x)\varphi_p(1-\eta), \\ \geq & -\left(\lambda_p(\mathcal{L}_\omega + b_\varepsilon) + \varepsilon\right)\varphi_p \\ & - \frac{d_0 \inf\{\tau, -\lambda_p(\mathcal{L}_\omega + b_\varepsilon) - \varepsilon\}}{2} - \max\{\nu, 0\}_{\varphi_p}, \\ \geq & -\lambda_p(\mathcal{L}_\omega + b_\varepsilon) + \varepsilon + \max\{\nu, 0\}_{\varphi_p} \\ & - \frac{d_0 \inf\{\tau, -\lambda_p(\mathcal{L}_\omega + b_\varepsilon) - \varepsilon\}}{2}.\end{aligned}$$

Since $-\lambda_p(\mathcal{L}_\omega + b_\varepsilon) - \varepsilon > 0$ and $-\lambda_p(\mathcal{L}_\omega + b_\varepsilon) - \varepsilon - \nu \geq \tau > 0$, from the above inequality, we infer that

$$\begin{aligned}\mathcal{L}_\omega[\varphi_p\eta] + b_\delta\varphi_p\eta \geq & -\left(\lambda_p(\mathcal{L}_\omega + b_\varepsilon) + \varepsilon + \max\{\nu, 0\}\right)d_0 \\ & - \frac{d_0 \inf\{\tau, -\lambda_p(\mathcal{L}_\omega + b_\varepsilon) - \varepsilon\}}{2}, \\ \geq & \frac{d_0 \inf\{\tau, -\lambda_p(\mathcal{L}_\omega + b_\varepsilon) - \varepsilon\}}{2} \geq 0.\end{aligned}$$

By construction, we have $\varphi_p\eta \in C(\omega)$ satisfying

$$\mathcal{L}_\omega\,[\varphi_p\eta] + b_\delta\varphi_p\eta \geq 0 \ in\ \omega,$$
$$\varphi_p\eta = 0 \ on\ \partial\omega.$$

By extending $\varphi_p\eta$ by 0 outside ω and denoting φ_δ this extension, we get

$$\mathcal{L}_\Omega[\varphi_\delta](x) + b_\delta(x)\varphi_\delta(x) = \mathcal{L}_\omega[\varphi_\delta](x) + b_\delta(x)\varphi(x) \geq 0 \ in\ \omega,$$
$$\mathcal{L}_\Omega[\varphi_\delta](x) + b_\delta(x)\varphi_\delta(x) = \mathcal{L}_\omega[\varphi_\delta](x) \geq 0 \ in\ \Omega\backslash\omega.$$

Hence, $\varphi_\delta \geq 0, \varphi \in C_c(\Omega)$ is the desired test function.

Assume for the moment that Ω is a bounded domain and let us show that the three definitions $\lambda_p, \lambda'_p, \lambda''_p$ andλ_p are equivalent and if in addition K is symmetric, λ_v is equivalent to λ_p. We start by the case $\lambda'_p = \lambda_p$. We show.

Lemma (6.2.13)[203]: Let Ω be a bounded domain of $\mathbb{R}^N$ and assume that a and K satisfy (68)–(70). Then,

$$\lambda_p(\mathcal{L}_\Omega + a) = \lambda'_p(\mathcal{L}_\Omega + a).$$

In addition, when $\mathcal{L}_\Omega + a$ is self adjoined, we have

$$\lambda_p(\mathcal{L}_\Omega + a) = \lambda_v(\mathcal{L}_\Omega + a).$$

The proof of Theorem (6.2.1) is a straightforward consequence of the above Lemma. Indeed, the definition of λ''_p we have

$$\lambda_p'(\mathcal{L}_\Omega + a) \le \lambda_p''(\mathcal{L}_\Omega + a) \le \lambda_p(\mathcal{L}_\Omega + a)$$

Thus, from the above Lemma we get

$$\lambda_p(\mathcal{L}_\Omega + a) = \lambda_p'(\mathcal{L}_\Omega + a) \le \lambda_p''(\mathcal{L}_\Omega + a) \le \lambda_p(\mathcal{L}_\Omega + a) = \lambda_p'(\mathcal{L}_\Omega + a).$$

Let us now turn to the proof of Lemma (6.2.13)

Proof. By Lemma (6.2.11), we already have

$$\lambda_p'(\mathcal{L}_\Omega + a) \le \lambda_p(\mathcal{L}_\Omega + a).$$

So, it remains to prove the converse inequality. Let us assume by contradiction that

$$\lambda_p'(\mathcal{L}_\Omega + a) \le \lambda_p(\mathcal{L}_\Omega + a).$$

Pick now $\lambda \in (\lambda_p(\mathcal{L}_\Omega + a), \lambda_p(\mathcal{L}_\Omega + a)$, then, by definition of λ_p and λ_p'', there exists φ and ψ non negative continuous functions such that

$$\mathcal{L}_\Omega[\varphi](x) + (a(x) + \lambda)\varphi(x) \le 0 \; in \; \Omega,$$
$$\mathcal{L}_\Omega[\psi](x) + (a(x) + \lambda)\psi(x) \ge 0 \; in \; \Omega.$$

Moreover,

$\varphi > 0 \; in \; \bar{\Omega}$. By taking λ smaller if necessary, we can assume that ϕ satisfies

$$\mathcal{L}_\Omega[\varphi](x) + (a(x) + \lambda)\varphi(x) < 0 \; in \; \Omega.$$

A direct computation yields

$$\int_\Omega K(x,y)\,\varphi(y)\left(\frac{\psi(y)}{\varphi(y)} - \frac{\psi(x)}{\varphi(x)}\right) dy > 0.$$

Since $\frac{\psi}{\varphi} \in C(\bar{\Omega})$, the function $\frac{\psi}{\varphi}$ achieves a maximum at some point $x_0 \in \bar{\Omega}$, evidencing thus the contradiction:

$$0 < \int_\Omega K(x,y)\,\varphi(y)\left(\frac{\psi(y)}{\varphi(y)} - \frac{\psi(x_0)}{\varphi(x_0)}\right) dy \le 0.$$

Thus,

$$\lambda_p'(\mathcal{L}_\Omega + a) = \lambda_p(\mathcal{L}_\Omega + a).$$

In the self-adjoined case, it is enough to prove that

$$\lambda_p'(\mathcal{L}_\Omega + a) = \lambda_v(\mathcal{L}_\Omega + a).$$

From the definitions of λ_p' and λ_v, we easily obtain that $\lambda_v \le \lambda_p'$. Indeed, let $\lambda > \lambda_p'(\mathcal{L}_\Omega + a)$, then by definition of λ_p' there exists ψ ≥ 0 such that $\psi \in C(\Omega) \cap L^\infty(\Omega)$ and

$$\mathcal{L}_\Omega[\psi](x) + (a(x) + \lambda)\psi(x) \ge 0 \; in \; \Omega. \tag{83}$$

Since Ω is bounded and $\psi \in L^\infty(\Omega), \psi \in L^2(\Omega)$. So, multiplying (83) by $-\psi$ and integrating over Ω we get

$$-\int_\Omega \int_\Omega K(x,y)\,\psi(x)\psi(y)dxdy - \int_\Omega a(x)\,\psi(x)^2 dx \le \lambda \int_\Omega \psi^2(x)dx,$$

$$\frac{1}{2}\int_\Omega \int_\Omega K(x,y)\left(\psi(x) - \psi(y)\right)^2 dxdy$$

$$-\int_\Omega (a(x) + k(x))\,\psi(x)^2 dx \le \lambda \int_\Omega \psi^2(x)dx,$$

$$\lambda_v(\mathcal{L}_\Omega + a) \le \lambda \int_\Omega \psi^2(x)dx \le \lambda \int_\Omega \psi^2(x)dx.$$

Therefore, $\lambda_v(\mathcal{L}_\Omega + a) \le \lambda_p'(\mathcal{L}_\Omega + a)$.

Let us prove now the converse inequality. Again, we argue by contradiction and let us assume that

$$\lambda_v(\mathcal{L}_\Omega + a) < \lambda_p'(\mathcal{L}_\Omega + a). \tag{84}$$

Observe first that by density of $C(\bar{\Omega})$ in $L^2(\Omega)$, we easily check that

$$\begin{aligned}
&-\lambda_v(\mathcal{L}_\Omega + a)\\
&= -\inf_{\varphi\in L^2(\Omega),\varphi\neq 0} \frac{\frac{1}{2}\int_\Omega \int_\Omega K(x,y)\left(\psi(x)-\psi(y)\right)^2 dydx - \int_\Omega\left(a(x)+k(x)\right)\psi(x)^2 dx}{\|\varphi\|^2_{L^2(\Omega)}},\\
&-\inf_{\varphi\in L^2(\Omega),\varphi\neq 0} \frac{-\int_\Omega \int_\Omega K(x,y)\left(\psi(x)\psi(y)\right)^2 dydx - \int_\Omega\left(a(x)+k(x)\right)\psi(x)^2 dx}{\|\varphi\|^2_{L^2(\Omega)}},\\
&= \sup_{\varphi\in L^2(\Omega),\varphi\neq 0} -\frac{\langle \mathcal{L}_\Omega[\varphi] + a\varphi, \varphi\rangle}{\|\varphi\|^2_{L^2(\Omega)}},\\
&= \sup_{\varphi\in C(\Omega),\varphi\neq 0} -\frac{\langle \mathcal{L}_\Omega[\varphi] + a\varphi, \varphi\rangle}{\|\varphi\|^2_{L^2(\Omega)}}.
\end{aligned}$$

By (iv) of Proposition (6.2.8), since $\lambda_p'(\mathcal{L}_\Omega + a) = \lambda_p(\mathcal{L}_\Omega + a)$, from (84) we infer that λ_+ defined by

$$\lambda_+ = \sup_{\varphi\in C(\bar{\Omega})} -\frac{\langle \mathcal{L}_\Omega[\varphi] + a\varphi, \varphi\rangle}{\int_\Omega \varphi^2} \tag{85}$$

ct computation yields

$$\int_\Omega K(x,y)\,\varphi(y)\left(\frac{\psi(y)}{\varphi(y)} - \frac{\psi(x)}{\varphi(x)}\right)dy > 0.$$

Since $\frac{\psi}{\varphi} \in C(\bar{\Omega})$, the function $\frac{\psi}{\varphi}$ achieves a maximum at some point $x_0 \in \bar{\Omega}$, evidencing thus the contradiction:

$$0 < \int_\Omega K(x,y)\,\varphi(y)\left(\frac{\psi(y)}{\varphi(y)} - \frac{\psi(x_0)}{\varphi(x_0)}\right)dy \leq 0.$$

satisfies

$$\lambda_+ > -\lambda_p(\mathcal{L}_\Omega + a) \geq \max_{\bar{\Omega}} a. \tag{86}$$

Using the same arguments as in [135], [147], we infer that the supremum in (85) is achieved. Indeed, it is a standard fact [212] that the spectrum of $\mathcal{L}_\Omega + a$a is at the left of λ_+ and that there exists a sequence $\varphi_n \in C(\bar{\Omega})$ such that $\|\varphi_n\|_{L^2(\Omega)} = 1$ and $\|(\mathcal{L}_\Omega + a - \lambda_+)\varphi_n\|_{L^2(\Omega)} \to 0$ as $n \to +\infty$.

By compactness of $\mathcal{L}_\Omega : L^2(\Omega) \to C(\Omega)$, for a subsequence, $\lim_{n\to+\infty} \mathcal{L}_\Omega[\varphi_n]$ exists in $C(\bar{\Omega})$. Then, using (86), we see that $\varphi_n \to \varphi$ in $L^2(\Omega)$ for some $\varphi$and $(\mathcal{L}_\Omega + a)\varphi = \lambda_+\varphi$. This equation implies $\varphi \in C(\bar{\Omega})$, and λ_+ is an eigenvalue for the operator $\mathcal{L}_\Omega + a$. Moreover, $\varphi \geq 0$, since φ^+ is also aminimizer. Indeed, we have

$$\begin{aligned}
\lambda_+ &= \frac{\int_\Omega [\mathcal{L}_\Omega[\varphi](x) + a(x)\varphi(x)]\,\varphi(x)^+ dx}{\|\varphi^+\|^2_{L^2(\Omega)}},\\
&= \frac{\int_\Omega [\mathcal{L}_\Omega[\varphi^+](x) + a(x)(\varphi^+)(x)]\,\varphi(x)^+ dx}{\|\varphi^+\|^2_{L^2(\Omega)}}
\end{aligned}$$

$$+\frac{\frac{1}{2}\int_{\Omega}\int_{\Omega} K(x,y)\,\varphi^{-}(x)\varphi^{+}(y)dydx}{\|\varphi^{+}\|^{2}_{L^{2}(\Omega)}},$$
$$\leq \frac{\int_{\Omega}[\mathcal{L}_{\Omega}[\varphi^{+}]a\varphi^{+}(x)]\,\varphi(x)^{+}dx}{\|\varphi^{+}\|^{2}_{L^{2}(\Omega)}} \leq \lambda_{+}.$$
Thus, there exists a non-negative continuous φ so that
$$\mathcal{L}_{\Omega}[\varphi](x) + (a(x) + \lambda_{v})\varphi(x) = 0 \ in\ \Omega.$$
Since $\lambda_v < \lambda_p$, we can argue as above and get the desired contradiction. Hence, $\lambda_v = \lambda_+ = \lambda_p = \lambda'_p$.

Now let Ω be an unbounded domain. From Lemma (6.2.11), we already know that
$$\lambda'_p(\mathcal{L}_{\Omega} + a) \leq \lambda''_p(\mathcal{L}_{\Omega} + a) \leq \lambda_p(\mathcal{L}_{\Omega} + a).$$
To complete the proof of Theorem (6.2.14), we are then left to prove that
$$\lambda'_p(\mathcal{L}_{\Omega} + a) = \lambda''_p(\mathcal{L}_{\Omega} + a) = \lambda_v(\mathcal{L}_{\Omega} + a),$$
when $\mathcal{L}_{\Omega} + a$ is self-adjoined and the kernel K is such that $p(x) := \int_{\Omega} K(x,y)\,dy$ is a bounded function in Ω. To do so, we prove the following inequality:

Assume that Lemma (6.2.16) holds and let us end the proof of Theorem (6.2.14).

Theorem (6.2.14)[203]: Let $\Omega \subset \mathbb{R}^N$ be an unbounded domain and assume that K and a satisfy (68) – (70).

Then the following inequalities hold:
$$\lambda'_p(\mathcal{L}_{\Omega} + a) \leq \lambda''_p(\mathcal{L}_{\Omega} + a) \leq \lambda_p(\mathcal{L}_{\Omega} + a).$$
When K is symmetric and such that $p(x) := \int_{\Omega} K(x,y)\,dy \in L^{\infty}(\Omega)$ then the following equality holds:
$$\lambda_v(\mathcal{L}_{\Omega} + a) = \lambda''_p(\mathcal{L}_{\Omega} + a) = \lambda_p(\mathcal{L}_{\Omega} + a).$$
Proof. From Lemma (6.2.11) and (6.2.16), we get the inequalities:
$$\lim_{n\to+\infty} \lambda_v(\mathcal{L}_{\Omega_n} + a) \leq \lambda'_p(\mathcal{L}_{\Omega} + a) \leq \lambda''_p(\mathcal{L}_{\Omega} + a) \leq \lambda_p(\mathcal{L}_{\Omega} + a),$$
$$\lambda_p(\mathcal{L}_{\Omega} + a) \leq \lim_{n\to+\infty} \lambda_v(\mathcal{L}_{\Omega_n} + a) \leq \lambda'_p(\mathcal{L}_{\Omega} + a) \leq \lambda''_p(\mathcal{L}_{\Omega} + a).$$
with $\Omega_n := \Omega \cap B_n(0)$. Therefore,
$$\lim_{n\to\infty} \lambda_v(\mathcal{L}_{\Omega_n} + a) = \lambda'_p(\mathcal{L}_{\Omega} + a) = \lambda''_p(\mathcal{L}_{\Omega} + a) = \lambda_p(\mathcal{L}_{\Omega} + a),$$
It remains to prove that $\lambda_v(\mathcal{L}_{\Omega} + a) = \lambda_p(\mathcal{L}_{\Omega} + a)$.

By definition of $\lambda''_p(\mathcal{L}_{\Omega} + a)$, we check that
$$\lambda_v(\mathcal{L}_{\Omega} + a) \leq \lambda_p(\mathcal{L}_{\Omega} + a) = \lambda_p(\mathcal{L}_{\Omega} + a).$$
On the other hand, by definition of $\lambda_v(\mathcal{L}_{\Omega} + a)$, for any $\delta > 0$ there exists $\phi_\delta \in L^2(\Omega)$ such that
$$\frac{\frac{1}{2}\int\int_{\Omega\times\Omega} K(x,y)\left(\varphi_\delta(x) - \varphi_\delta(y)\right)^2 dydx - \int_{\Omega}\left(a(x) + p(x)\right)\varphi_\delta^2(x)dx}{\|\varphi_\delta\|^2_{L^2(\Omega)}}$$
$$\leq \lambda_v(\mathcal{L}_{\Omega} + a) + \delta.$$
Define
$$\mathcal{T}_R(\varphi_\delta) := \frac{\frac{1}{2}\int\int_{\Omega_R\times\Omega_R} K(x,y)\left(\varphi_\delta(x) - \varphi_\delta(y)\right)^2 dydx - \int_{\Omega_R}\left(a(x) + p_R(x)\right)\varphi_\delta^2(x)dx}{\|\varphi_\delta\|^2_{L^2(\Omega_R)}},$$

with $p_R(x) := \int_{\Omega_R} K(x,y)\,dy$. Since$\lambda_p(\mathcal{L}_\Omega + a) \le \lim_{R\to\infty} p_R\,(x) = p(x)$ for all $x \in \Omega, a \in L^\infty$ and $\varphi_\delta \in L^2(\Omega)$, by Lebesgue's monotone convergence Theorem we get for R large enough

$$-\int_{\Omega_R} \big(a(x) + p_R(x)\big)\varphi_\delta^2(x)dx \le \|\varphi_\delta\|^2_{L^2(\Omega_R)} - \int_{\Omega} \big(a(x) + p_R(x)\big)\varphi_\delta^2(x)dx.$$

Thus, we have for R large enough

$$\mathcal{T}_R(\varphi_\delta) \le \frac{\frac{1}{2}\int \int_{\Omega\times\Omega} K(x,y)\left(\varphi_\delta(x) - \varphi_\delta(y)\right)^2 dydx - \int_\Omega \big(a(x) + p(x)\big)\varphi_\delta^2(x)dx}{\|\varphi_\delta\|^2_{L^2(\Omega)}},$$

$$\le \mathcal{T}_R(\varphi_\delta) := \frac{\|\varphi^+\|^2_{L^2(\Omega)}}{\|\varphi_\delta\|^2_{L^2(\Omega_R)}}(\lambda_v(\mathcal{L}_\Omega + a) + \delta) + \delta,$$

$$\le \lambda_v(\mathcal{L}_\Omega + a) + C\delta,$$

for some universal constant $C > 0$.

By definition of $\lambda_v(\mathcal{L}_\Omega + a)$, we then get
$\lambda_v\big(\mathcal{L}_{\Omega_R} + a\big)\mathcal{T}_R(\varphi_\delta) \le \lambda_v(\mathcal{L}_\Omega + a) \ \le C\delta$ for R large enough.
Therefore,

$$\lim_{R\to\infty} \lambda_v\left(\mathcal{L}_{\Omega_R} + a\right) \le \lambda_v(\mathcal{L}_\Omega + a) + C\delta. \tag{87}$$

Since (87) holds true for any δ, we get

$$\lim_{R\to\infty} \lambda_v\left(\mathcal{L}_{\Omega_R} + a\right) \le \lambda_v(\mathcal{L}_\Omega + a).$$

As a consequence, we obtain

$$\lambda_p(\mathcal{L}_\Omega + a) \le \lim_{R\to\infty} \lambda_v\left(\mathcal{L}_{\Omega_R} + a\right) \le \lambda_v(\mathcal{L}_\Omega + a) \le \lambda_p''(\mathcal{L}_\Omega + a) = \lambda_v(\mathcal{L}_\Omega + a),$$

which enforces

$$\lambda_v(\mathcal{L}_\Omega + a) = \lambda_p(\mathcal{L}_\Omega + a).$$

We can now turn to the proof of Lemma (6.2.16). But before proving this Lemma, we start by showing some technical Lemma in the spirit of Lemma 2.6 in [207]. Namely, we prove

Lemma (6.2.15)[203]: Assume Ω is unbounded and let $g \in L^\infty(\Omega)$ be a non negative function, then for any $R_0 > 0$, we have

$$\lim_{R\to\infty} \frac{\int_{\Omega\cap(B_{R_0+R}/B_R)} g}{\int_{\Omega\cap B_R} g} = 0.$$

Proof. Without loss of generality, by extending g by 0 outside Ω we can assume that $\Omega = \mathbb{R}^N$. For any $R_0, R > 0$ fixed, let us denote the annulus $C_{R_0,R} := B_{R_0+R} \setminus B_R$. Assume by contradiction that

$$\lim_{R\to\infty} \frac{\int_{C_{R_0,R}} g}{\int_{B_R} g} = 0.$$

Then there exists $\varepsilon > 0$ and $R_\varepsilon > 1$ so that

$$\forall R \ge R_\varepsilon, \frac{\int_{C_{R_0,R}} g}{\int_{B_R} g} \ge \ \varepsilon.$$

Consider the sequence $(R_n)_{n\in N}$ defined by $R_n := R_\varepsilon + nR_0$ and set $a_n := \int_{C_{R_0,R_n}} g$.
For all n, wehave$C_{R_0,R_n} = B_{R_{n+1}}/B_{R_n}$ and

$$B_{R_{n+1}} = B_{R_\varepsilon} \cup \left(\bigcup_{k=1}^{n} C_{R_0,R_k} \right).$$

From the last inequality, for $n \geq 1$ we deduce that $a_n \geq \varepsilon \int_{B_{R_n}} g \geq \varepsilon \sum_{k=0}^{n} a_k$.

Arguing now as in [207], by a recursive argument, the last inequality yields

$$\forall\, n \geq 1, a_n \ \geq \varepsilon a_0 (1+\varepsilon)^{n-1}. \tag{88}$$

On the other hand, we have

$$a_n = \int_{C_{R_0,R_n}} g \leq \|g\|_\infty |C_{R_0,R_n}| \leq d_0 n^N,$$

with d_0 a positive constant, contradicting thus (88).

We are now in a position to prove Lemma (6.2.16).

Lemma (6.2.16)[203]: Let Ω be an unbounded domain and assume that a and K satisfies (68)–(70). Assume further that K is symmetric and $p(x) := \int_\Omega K(x,y)\,dy \in L^\infty(\Omega)$. Then, we have

$$\lambda_p(\mathcal{L}_\Omega + a) \leq \lim_{n\to+\infty} \inf \lambda_v\left(\mathcal{L}_{\Omega_n} + a\right).$$

where $\Omega_n := (\Omega \cap B_n)_{n\in\mathbb{N}}$ and B_n is the ball of radius n centred at 0.

Proof. The proof follows some ideas developed in [208], [207], [135], [225]. To simplify the presentation, let us call $\lambda_p = \lambda_p(\mathcal{L}_\Omega + a)$ and$\lambda_p' = \lambda_p'(\mathcal{L}_\Omega + a)$.

First recall that for a bounded domain Ω, we have

$$\lambda_p = \lambda_p' = \lambda_v.$$

Let $(B_n)_{n\in N}$ be the increasing sequence of balls of radius n centred at 0 and let $\Omega_n := \Omega \cap B_n$. By monotonicity of λ_p with respect to the domain, we have

$$\lambda_p(\mathcal{L}_\Omega + a) \leq \lambda_p\left(\mathcal{L}_{\Omega_n} + a\right) = \lambda_v\left(\mathcal{L}_{\Omega_n} + a\right)$$

Therefore

$$\lambda_p(\mathcal{L}_\Omega + a) \leq \lim_{n\to\infty} \inf \lambda_v\left(\mathcal{L}_{\Omega_n} + a\right).$$

Thanks to the last inequality, we obtain the inequality $\lambda_p(\mathcal{L}_\Omega + a) \leq \lambda_p'(\mathcal{L}_\Omega + a)$ by proving that

$$\lim_{n\to\infty} \inf \lambda_v\left(\mathcal{L}_{\Omega_n} + a\right) \leq \lambda_p'(\mathcal{L}_\Omega + a). \tag{89}$$

To prove (89), it is enough to show that for any $\delta > 0$

$$\lim_{n\to\infty} \inf \lambda_v\left(\mathcal{L}_{\Omega_n} + a\right) \leq \lambda_p'(\mathcal{L}_\Omega + a) + \ \delta. \tag{90}$$

Let us fix $\delta > 0$ and let us denote $\mu := \lambda_p'(\mathcal{L}_\Omega + a) + \delta$. By definition of$\lambda_p'(\mathcal{L}_\Omega + a)$ there exists a function $\varphi \in C(\Omega\ \cap L^\infty(\Omega), \varphi \geq 0$ satisfying

$$\mathcal{L}_\Omega[\varphi](x) + a(x)\varphi(x) + \mu\varphi(x)\ \geq 0 \ in\ \Omega. \tag{91}$$

Without loss of generality, we can also assume that $\|\varphi\|_{L^\infty(\Omega)} = 1$.

Let 1_{Ω_n} be the characteristic function of $\Omega_n = \Omega \cap B_n$ and let $\mathrm{w_n} = \varphi 1_{\Omega_n}$. By definition of$\lambda_v\left(\mathcal{L}_{\Omega_n} + a\right)$ and since $\mathrm{w_n} \in \mathrm{L}^2(\Omega_n)$, we have

$$\lambda_v\left(\mathcal{L}_{\Omega_n} + a\right)\|\omega_\mathrm{n}\|^2_{\mathrm{L}^2(\Omega_n)} \leq \int_{\Omega_n} \left(-\mathcal{L}_{\Omega_n}[w_\mathrm{n}](\mathrm{x}) - \mathrm{a(x)w_n(x)}\right) w_\mathrm{n}(\mathrm{x})\mathrm{dx}. \tag{92}$$

Since $L_\Omega[\varphi]_{w_n}\ \in \mathrm{L}^1(\Omega_n)$, from (92) and by using (91) we get

$$\lambda_v\left(\mathcal{L}_{\Omega_n} + a\right)\|\omega_\mathrm{n}\|^2_{\mathrm{L}^2(\Omega_n)} \leq \int_{\Omega_n} \left(-\mathcal{L}_{\Omega_n}[w_\mathrm{n}](\mathrm{x}) - \mathrm{a(x)w_n(x)} - \mu w_\mathrm{n} + \mu w_n\right) w_\mathrm{n}(\mathrm{x})\mathrm{dx}.$$

$$\leq \mu\|\omega_n\|^2_{L^2(\Omega_n)} + \int_{\Omega_n}\left(-\mathcal{L}_{\Omega_n}[w_n](x) + \mathcal{L}_{\Omega}[\varphi](x)\right)w_n(x)dx,$$
$$\leq \mu\|\omega_n\|^2_{L^2(\Omega_n)} + \int_{\Omega_n}\left(\int_{\Omega\backslash\Omega_n} K(x,y)\varphi(y)dy\right)w_n(x)dx,$$
$$\leq \mu\|\omega_n\|^2_{L^2(\Omega_n)} + I_n,$$
where I_n denotes
$$I_n := \int_{\Omega_n}\left(\int_{\Omega\backslash\Omega_n} K(x,y)\varphi(y)dy\right)\varphi(x)dx.$$
Observe that we achieve (90) by proving
$$\lim_{n\to\infty}\inf\frac{I_n}{\|\varphi\|^2_{L^2(\Omega_n)}} = 0, \tag{93}$$
Recall that K satisfies (70), therefore there exists $C > 0$ and $R_0 > 0$ such that $K(x,y) \leq C1_{R_0}(|x-y|))$. So, we get
$$I_n \leq \int_{\Omega_n}\left(\int_{\Omega\cap(B_{R_{0+n}}\backslash B_n)} K(x,y)\varphi(y)dy\right)\varphi(x)dx. \tag{94}$$
By Fubini's Theorem, Jensen's inequality and Cauchy-Schwarz's inequality, it follows that
$$I_n \leq \int_{\Omega_n}\left(\int_{\Omega\cap(B_{R_{0+n}}\backslash B_n)}\varphi^2(y)dy\right)^{1/2}\left(\int_{\Omega\cap(B_{R_{0+n}}\backslash B_n)}\left(\int_{\Omega_n} K(x,y)\varphi(y)dx\right)^2 dy\right)^{1/2},$$
$$\leq \|\varphi\|_{L^2(\Omega\cap(B_{R_{0+n}}\backslash B_n))}\left(\int_{\Omega\cap(B_{R_{0+n}}\backslash B_n)}\left(\int_{\Omega\cap B_n} K^2(x,y)\varphi^2(y)dx\right)dy\right)^{1/2},$$
$$\leq \|\varphi\|_{L^2(\Omega\cap(B_{R_{0+n}}\backslash B_n))}\left(\int_{\Omega\cap B_n}\left(\int_{\Omega\cap(B_{R_{0+n}}\backslash B_n)} K^2(x,y)dy\right)\varphi^2(x)\,dx\right)^{1/2}.$$
Since K and p are bounded functions, we obtain
$$I_n \leq \|K\|_\infty\|p\|_\infty \leq \|\varphi\|_{L^2(\Omega\cap(B_{R_{0+n}}\backslash B_n))}\|\varphi\|_{L^2(\Omega\cap B_n)}. \tag{95}$$
Dividing (95) by $\leq \|\varphi\|^2_{L^2(\Omega_n)}$, we then get
$$\frac{I_n}{\|\varphi\|^2_{L^2(\Omega_n)}} \leq C\frac{\|\varphi\|_{L^2(\Omega\cap(B_{R_{0+n}}\backslash B_n))}}{\|\varphi\|^2_{L^2(\Omega\cap B_n)}}.$$
Thanks to Lemma (6.2.15), the right hand side of the above inequality tends to 0 as $n\to\infty$. Hence, we get
$$\lim_{n\to\infty}\inf\lambda_v(\mathcal{L}_{\Omega_n}+a) \leq \mu + \lim_{n\to\infty}\inf\frac{I_n}{\|\varphi\|^2_{L^2(\Omega_n)}} = \lambda'_p(\mathcal{L}_\Omega + a) + \delta. \tag{96}$$
Since the above arguments holds true for any arbitrary $\delta > 0$, the Lemma is proved.

When $N = 1$, the decay restriction imposed on the kernel can be weakened, see [225]. In particular, we have

Lemma (6.2.17)[203]: Let Ω be an unbounded domain and assume that a and K satisfy (68)–(69). Assume further that K is symmetric and K satisfies $0 \leq K(x,y) \leq C(1+|x-y|)^{-\alpha}$ for some $\alpha > \frac{3}{2}$. Then one has

$$\lambda_p(\mathcal{L}_\Omega + a) \leq \lim_{n\to+\infty} \inf \lambda_v(\mathcal{L}_{\Omega_n} + a) \leq \lambda_p'(\mathcal{L}_\Omega + a),$$

where $\Omega_n := \Omega \cap (-n, n)$.

Proof. By arguing as in the above proof, for any $\delta > 0$ there exists $\varphi \in C(\Omega) \cap L^\infty(\Omega)$ such that

$$\mathcal{L}_\Omega[\varphi] + (a + \lambda_p(\mathcal{L}_\Omega + a) + \delta)\varphi(x) \geq 0 \; in \; \Omega.$$

and

$$\lambda_v(\mathcal{L}_{\Omega_n} + a) \leq \|\omega_n\|^2_{L^2(\Omega_n)} \leq \mu\|\omega_n\|^2_{L^2(\Omega_n)} + I_n,$$

where $\mu := \lambda_p(\mathcal{L}_\Omega + a) + \delta)$.

$\lambda_p(\mathcal{L}_\Omega + a), w_n := \varphi 1_{(-n,n)}$ and I_n denotes

$$I_n := \int_{\Omega_n} \left(\int_{\Omega/_{\Omega_n}} K(x,y)\varphi(y)dy \right) \varphi(x)dx. \tag{97}$$

As above, we end our proof by showing

$$\lim_{n\to\infty} \inf \frac{I_n}{\|\varphi\|^2_{L^2(\Omega_n)}} = 0. \tag{98}$$

Let us now treat two cases independently:

Case 1: $\varphi \in L^2(\Omega)$

In this situation, again by using Cauchy-Schwarz's inequality, Jensen's inequality and Fubini's Theorem, the inequality (97) yields

$$I_n \leq \|\varphi\|_{L^2(\Omega_n)} \left[\int_{\Omega\setminus\Omega_n} \left(\int_{\Omega_n} K^2(x,y)\varphi(y)dy \right) \varphi^2(x)dx \right]^{\frac{1}{2}}.$$

Recall that K satisfies $K(x,y) \leq C(1 + |x - y|)^{-\alpha}$ for some $C > 0$ and α>3/2, therefore $p(y) := \int_\Omega K(x,y)\, dx$ is bounded and from the latter inequality we enforce

$$I_n \leq C\|\varphi\|_{L^2(\Omega\setminus\Omega_n)}\|\varphi\|_{L^2(\Omega_n)}.$$

Thus,

$$\lim_{n\to\infty} \inf \frac{I_n}{\|\varphi\|^2_{L^2(\Omega_n)}} \leq \lim_{n\to\infty} \inf \frac{\|\varphi\|_{L^2(\Omega\setminus\Omega_n)}}{\|\varphi\|^2_{L^2(\Omega_n)}} = 0.$$

Case 2: $\varphi \notin L^2(\Omega)$.

Assume now that $\varphi \notin L^2(\Omega)$, then we argue as follows. Again, applying Fubini's Theorem and Cauchy-Schwarz's inequality in the inequality (97) yields

$$I_n \leq \|\varphi\|_{L^2(\Omega_n)} \left[\int_{(\Omega\cap R-)\setminus\Omega_n} \left(\left(\int_{\Omega_n} K(x,y)^2 dx \right)^{\frac{1}{2}} \right) \varphi(y)\, dy \right.$$

$$\left. + \int_{(\Omega\cap R+)\setminus\Omega_n} \left(\left(\int_{\Omega_n} K(x,y)^2 dx \right)^{\frac{1}{2}} \right) \varphi(y)\, dy \right]$$

$$\leq \|\varphi\|_{L^2(\Omega_n)}[I_n^- + I_n^+] \tag{99}$$

Recall that by assumption there exists C>0 such that $K(x,y) \le C(1+|x-y|)^{-\alpha}$ with α >32 .So, we have

$$\bar{I}_n^- \le C\int_{(\Omega\cap R-)\setminus\Omega_n}\left(\left(\int_{\Omega_n}(1+|x-y|)^{-2\alpha}\,\mathrm{dx}\right)^{\frac{1}{2}}\right)\varphi(y)\,\mathrm{dy}$$

$$\tilde{I}_n^+ \le C\int_{(\Omega\cap R-)\setminus\Omega_n}\left(\left(\int_{-\mathrm{n}}^{\mathrm{n}}(1+|x-y|)^{-2\alpha}\,dx\right)^{\frac{1}{2}}\right)\varphi(y)\,\mathrm{dy}.$$

To complete our proof, we have to show that $\frac{\tilde{I}_n^+}{\|\varphi\|^2_{\mathrm{L}^2(\Omega_n)}} \to 0$. The proof being similar in both cases,so we only prove that $\frac{\tilde{I}_n^+}{\|\varphi\|^2_{\mathrm{L}^2(\Omega_n)}} \to 0$. We claim that

Claim (6.2.18)[203]: There exists $C > 0$ so that for all $n \in N$,

$$\int_{(\Omega\cap R+)\setminus\Omega_n}\left(\left(\int_{\Omega_n}(1+|x-y|)^{-2\alpha}\,dx\right)^{\frac{1}{2}}\right)\varphi(y)\,\mathrm{dy}.$$

Assume for the moment that the claim holds true, then from (99), we deduce that

$$\frac{\mathrm{I_n}}{\|\varphi\|^2_{\mathrm{L}^2(\Omega_n)}} \le C\frac{C}{\|\varphi\|^2_{\mathrm{L}^2(\Omega_n)}} \to 0 \text{ when } n \to \infty.$$

Hence, in both situation, we get

$$\lim_{n\to+\infty}\inf\lambda_v\left(\mathcal{L}_{\Omega_n}+a\right) \le \mu\lim_{n\to\infty}\inf\frac{\mathrm{I_n}}{\|\varphi\|^2_{\mathrm{L}^2(\Omega_n)}} = \lambda_p'(\mathcal{L}_\Omega + a)+\delta,$$

Since δ > 0 can be chosen arbitrary, the above inequality is true for any δ > 0 and the Lemma is proved.

Proof. Since $\varphi \in L^\infty(\Omega)$ and y ≥ n then x ≤ y and we have

$$\bar{I}_n^+ \le \|\varphi\|_\infty\int_{(\Omega\cap R+)\setminus\Omega_n}\left(\left(\int_{\Omega_n}(1+y-x)^{-2\alpha}\,\mathrm{dx}\right)^{\frac{1}{2}}\right)\mathrm{dy},$$

$$\le \|\varphi\|_\infty\int_{\mathrm{n}}^{+\infty}\left(\left(\int_{-\mathrm{n}}^{\mathrm{n}}(1+y-x)^{-2\alpha}\,\mathrm{dx}\right)^{\frac{1}{2}}\right)\mathrm{dy},$$

$$\le \frac{\|\varphi\|_\infty}{\sqrt{2\alpha-1}}\int_{\mathrm{n}}^{-\infty}(1+y-n)^{-\alpha+\frac{1}{2}}\,\mathrm{dy},$$

$$\le C\int_0^{+\infty}(1+z)^{-\alpha+\frac{1}{2}}\,dz.$$

We investigate further the properties of the principal eigenvalue $\lambda_p(\mathcal{L}_\Omega + a)$ and in particular its behaviour with respect to some scaling of the kernel K ((Proposition (6.2.2)) and Theorem (6.2.3)). For simplicity, one dedicated to the proof of Proposition (6.2.2) and the other one dealing with the proof of Theorem (6.2.3). Let us start with the scaling invariance of $\mathcal{L}_\Omega + a$, (Proposition (6.2.2)).

This invariance is a consequence of the following observation. By definition of $\lambda_p(\mathcal{L}_\Omega + a)$, we have for all $\lambda < \lambda_p(\mathcal{L}_\Omega + a)$,

$$\mathcal{L}_\Omega[\varphi](x) + (a(x) + \lambda)\varphi(x) \le 0 \; in \;\; \Omega,$$

for some positive $\varphi \in C(\Omega)$. Let $X = \sigma x, \Omega_\sigma := \frac{1}{\sigma}\Omega$ and $\psi(X) := \varphi(\sigma X)$ then we can rewrite the above inequality as follows

$$\int_\Omega K\left(\frac{X}{\sigma}, y\right)\varphi(y)dy + \left(a\left(\frac{X}{\sigma}\right) + \lambda\right)\varphi\left(\frac{X}{\sigma}\right) \le 0 \; for \; any \; X \in \Omega_\sigma,$$

$$\int_\Omega K\left(\frac{X}{\sigma}, y\right)\varphi(y)dy + (a_\sigma(X) + \lambda)\varphi(X) \le 0 \text{ for any } X \in \Omega_\sigma,$$

$$\int_{\Omega_\sigma} K_\sigma(X, Y)\psi(Y)dY + (a_\sigma(X) + \lambda)\psi(X) \le 0 \text{ for any } X \in \Omega_\sigma,$$

with $K_\sigma(x, y) := \frac{1}{\sigma^N} K(\frac{x}{\sigma}, \frac{y}{\sigma})$ and $a_\sigma(x) := a\left(\frac{x}{\sigma}\right)$. Thus ψ is a positive continuous function that satisfies

$$\mathcal{L}_{\sigma,\Omega_\sigma}[\psi](x) + (a_\sigma(x) + \lambda)\psi(x) \le 0 \; in \; \Omega_\sigma.$$

Therefore, $\lambda \le \lambda_p(\mathcal{L}_{\sigma,\Omega_\sigma} + a_\sigma)$ and as a consequence

$$\lambda_p(\mathcal{L}_\Omega + a) \le \lambda_p(\mathcal{L}_{\sigma,\Omega_\sigma} + a_\sigma).$$

Interchanging the role of $\lambda_p(\mathcal{L}_\Omega + a)$ and $\lambda_p(\mathcal{L}_{\sigma,\Omega_\sigma} + a_\sigma)$ in the above argument yields

$$\lambda_p(\mathcal{L}_\Omega + a) \ge \lambda_p(\mathcal{L}_{\sigma,\Omega_\sigma} + a_\sigma).$$

Hence, we get

$$\lambda_p(\mathcal{L}_\Omega + a) = \lambda_p\left(\mathcal{L}_{\sigma,\Omega_\sigma} + a_\sigma\right).$$

Let us focus on the behaviour of the principal eigenvalue of the spectral problem

$$\mathcal{M}_{\sigma,m,\Omega}[\varphi] + (a + \lambda)\varphi = 0 \; in \; \Omega,$$

Where

$$\mathcal{M}_{\sigma,m,\Omega}[\varphi] := \frac{1}{\sigma^m}\int_\Omega J_\sigma(x - y)\varphi(y)dy - \varphi(x),$$

with $J_\sigma(z) := \frac{1}{\sigma^N} J\left(\frac{z}{\sigma}\right)$. Assuming that $0 \le m \le 2$, we obtain here the limits of $\lambda_p(\mathcal{M}_{\sigma,m,\Omega} + a)$, when $\sigma \to 0$ and $\sigma \to \infty$. But before going to the study of these limits, we recall a known inequality.

Lemma (6.2.19)[203]: Let $J \in C(\mathbb{R}^N), J \ge 0, J$ symmetric with unit mass, such that $|z|^2 J(z) \in L^1(\mathbb{R}^N)$.

Then for all $\varphi \in H_0^1(\Omega)$ we have

$$-\int_\Omega \left(\int_\Omega J(x - y)\varphi(y)\, dy - \varphi(x)\right)\varphi(x)\, dx \le \frac{1}{2}\int_{\mathbb{R}^N} J(z)|z|^2\, dz \|\nabla\varphi\|^2_{L^2(\Omega)}.$$

Proof. Let $\in C_c^\infty$, then by applying the standard Taylor expansion we have

$$\varphi(x+z)-\varphi(x)=\int_0^1 z_i\partial_i\varphi\,(x+tz)dt \tag{100}$$

$$=z_i\partial_i\varphi(x)+\int_0^1 t\left(\int_0^1 z_iz_j\partial_{ij}\varphi\,(x+tsz)ds\right)dt \tag{101}$$

where use the Einstein summation convention $a_ib_i=\sum_{i=1}^N a_ib_i$.

Let us denote

$$\mathcal{T}(\varphi):=-\int_\Omega\left(\int_\Omega J(x-y)\varphi(y)dy-\varphi(x)\right)\varphi(x)\,dx.$$

Then, for any $\varphi\in C_c(\Omega),\varphi\in C_c(\mathbb{R}^N)$ and we can easily see that

$$\mathcal{T}(\varphi)=\frac{1}{2}\int\int_{\mathbb{R}^{2N}} J(x-y)\big(\varphi(x)-\varphi(y)\big)^2\ \ dxdy.$$

By plugging the Taylor expansion of φ (100) in the above equality we see that

$$\frac{1}{2}\int\int_{\mathbb{R}^{2N}} J(z)\big(\varphi(x+z)-\varphi(x)\big)^2\ \ dzdx=\frac{1}{2}\int_{\mathbb{R}^N} J(z)\left(\int_{\mathbb{R}^N} z_i\partial_i\varphi(x+tz)\ dt\right)^2\ dzdx,$$

$$\leq\frac{1}{2}\int_{\mathbb{R}^{2N}} J(z)\left(|z_i|\left[\int_0^1|\partial_i\varphi(x+tz)|^2\right]^{\frac{1}{2}}dt\right)^2\ dzdx$$

$$\leq\frac{1}{2}\int_{\mathbb{R}^{2N}} J(z)\left(\sum_{i=1}^N z_i^2\right)\left[\sum_{i=1}^N\int_0^1|\partial_i\varphi(x+tz)|^2\ dt\right]dzdx$$

where we use in the last inequality the standard inequality $\left(\sum_{i=1}^N a_ib_i\right)^2\leq\left(\sum_{i=1}^N a_1^2\right)\left(\sum_{i=1}^N b_1^2\right)$

So, by Fubini's Theorem and by rearranging the terms in the above inequality, it follows that

$$\mathcal{T}(\varphi)\leq\frac{1}{2}\left(\int_{\mathbb{R}^N} J(z)|z|^2dz\right)\|\nabla\varphi\|^2_{L^2(\Omega)}.$$

By density of $C_c^\infty(\Omega)$ in $H_0^1(\Omega)$, the above inequality holds true for $\varphi\in H_0^1(\Omega)$, since obviously the functional I(ϕ) is continuous in $L^2(\Omega)$.

Let us also introduce the following notation

$$J_\sigma(z):=\frac{1}{\sigma^N}\,J\,\left(\frac{z}{\sigma}\right),\qquad p_\sigma(x):=-\int_\Omega J_\sigma(x-y)dy,$$

$$D_2(J)\coloneqq\int_{\mathbb{R}^N} J(z)|z|^2\,dz,$$

$$\mathcal{A}(\varphi)\coloneqq\frac{-\int_\Omega a\varphi^2(x)\,dx}{\|\varphi\|^2_{L^2(\Omega)}},\mathcal{R}_{\sigma,m}(\varphi)\coloneqq\frac{1}{\sigma^m}\frac{\int_\Omega(p_\sigma(x)-1)\varphi^2(x)\,dx}{\|\varphi\|^2_{L^2(\Omega)}},$$

$$\mathcal{T}_{\sigma,m}(\varphi):=\frac{\frac{1}{\sigma^m}\left(-\int_\Omega\left(\int_\Omega J(x-y)\varphi(y)dy-\varphi(x)\right)\varphi(x)\,dx\right)}{\|\varphi\|^2_{L^2(\Omega)}}-\mathcal{A}(\varphi)$$

$$\mathcal{J}(\varphi)\coloneqq\frac{D_2(J)}{2}\,\frac{-\int_\Omega|\nabla\varphi|^2(x)\,dx}{\|\varphi\|^2_{L^2(\Omega)}}.$$

With this notation, we see that

$$\lambda_v\left(\mathcal{M}_{\sigma,m,\Omega}+a\right)=\inf_{\varphi\in L^2(\Omega)}\mathcal{T}_{\sigma,m}(\varphi),$$

and by Lemma (6.2.19), for any $\varphi\in H_0^1(\Omega)$ we get

$$\mathcal{T}_{\sigma,m}(\varphi)\le\sigma^{2-m}\,\mathcal{J}(\varphi)-\mathcal{A}(\varphi).\tag{102}$$

We are now in position to obtain the different limits of $\lambda_p\left(\mathcal{M}_{\sigma,m,\Omega}+a\right)$ as σ → 0 and σ → ∞.

For simplicity, we analyse three distinct situations: $m=0, 0<m<2$ and m = 2. We will see that $m=0$ and $m=2$ are ,indeed, two critical situations.

Let us first deal with the easiest case, that is, when 0 < m < 2.

In this situation, we claim that

Claim (6.2.20)[203]: Let Ω be any domain and let $J\in C(\mathbb{R}^N)$ be positive, symmetric and such that $|z|^2J(z)\in L^1(\mathbb{R}^N)$. Assume further that J satisfies (68)–(70) and $0<m<2$ then

$$\lim_{\sigma\to0}\lambda_p\left(\mathcal{M}_{\sigma,m,\Omega}+a\right)=-\sup_\Omega a$$

$$\lim_{\sigma\to+\infty}\lambda_p\left(\mathcal{M}_{\sigma,m,\Omega}+a\right)=-\sup_\Omega a$$

Proof. First, let us look at the limit of λ_p when σ → 0. Up to adding a large positive constant to the function a, without any loss of generality, we can assume that the function a is positive somewhere in Ω.

Since $\mathcal{M}_{\sigma,m,\Omega}+a$ is a self-adjoined operator, by Theorem (6.2.14) and (102), for any $\varphi\in H_0^1(\Omega)$ we have

$$\lambda_p\left(\mathcal{M}_{\sigma,m,\Omega}+a\right)a)=\lambda_v\left(\mathcal{M}_{\sigma,m,\Omega}+a\right)\le\mathcal{T}_{\sigma,m}(\varphi)\le\sigma^{2-m}J(\varphi)-\mathcal{A}(\varphi).$$

Define $\nu:=\sup_\Omega a$, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence of point such that $|\nu-a(x_n)|<\frac{1}{n}$. Since a is positive somewhere, we can also assume that for all $n, x_n\in\Gamma:=\{x\in\Omega|a(x)>0\}$.

By construction, for any $n>0$, there exists ρ_n such that $B_\rho(x_n)\subset\Gamma$ for any positive $\rho\le\rho_n$. Fix now n, for any $0<\rho\le\rho_n$ there exists $\varphi_\rho\in H_0^1(\Omega)$ such that $\operatorname{supp}(\varphi_\rho)\subset B_\rho(x_n)$ and therefore,

$$\limsup_{\sigma\to0}\lambda_p(\mathcal{M}_{\sigma,m,\Omega}+a)\le-\mathcal{A}(\varphi_\rho)=\int_{B_\rho(x_n)}\frac{a-\varphi_\rho^2(x)dx}{\left\|\varphi_\rho\right\|^2_{L^2(\Omega)}}\le-\min_{B_\rho(x_n)}a^+(x).$$

By taking the limit $\rho\to0$ in the above inequality, we then get

$$\limsup_{\sigma\to0}\lambda_p(\mathcal{M}_{\sigma,m,\Omega}+a)\le-a(x_n).$$

Thus,

$$\limsup_{\sigma\to0}\lambda_p(\mathcal{M}_{\sigma,m,\Omega}+a)\le-\nu+\frac{1}{n}.$$

By sending now $n\to\infty$ in the above inequality, we obtain

$$\limsup_{\sigma\to0}\lambda_p(\mathcal{M}_{\sigma,m,\Omega}+a)\le-\nu.$$

On the other hand, by using the test function $(\varphi, \lambda) = (1, -\nu)$ we can easily check that for any $\sigma > 0$

$$\lambda_p(\mathcal{M}_{\sigma,m,\Omega} + a) \geq -\nu.$$

Hence,

$$-\nu \leq \lim_{\sigma\to 0} \sup\ \lambda_p(\mathcal{M}_{\sigma,m,\Omega} + a) \leq \lim_{\sigma\to 0} \sup\ \lambda_p(\mathcal{M}_{\sigma,m,\Omega} + a) \leq -\nu.$$

Now, let us look at the limit of $\lambda_p(\mathcal{M}_{\sigma,m,\Omega} + a)$ when $\sigma \to +\infty$. This limit is a straightforward consequence of (iv) of the Proposition (6.2.8). Indeed, as remarked above, for any σ by using the test function $(\varphi, \lambda) = (1, -\nu)$, we have

$$-\nu \leq \lambda_p(\mathcal{M}_{\sigma,m,\Omega} + a)$$

whereas from (iv) of the Proposition (6.2.8) we have

$$\lambda_p(\mathcal{M}_{\sigma,m,\Omega} + a) \leq -\sup_{\Omega}\left(-\frac{1}{\sigma^m} + a\right).$$

Therefore, since $m > 0$ we have

$$-\nu| \leq \lim_{\sigma\to 0} \sup\ \lambda_p(\mathcal{M}_{\sigma,m,\Omega} + a) \leq -\nu.$$

Indeed, the analysis of the limit of $\lambda_p(\mathcal{M}_{\sigma,m,\Omega} + a)$ when $\sigma \to 0$ holds true as soon as < 2. Thus,

$$\lambda_p\left(\mathcal{M}_{\sigma,m,\Omega} + a\right) \to -\sup_{\Omega} a \ as\ \sigma \to 0.$$

On the other hand, the analysis of the limit of $\lambda_p(\mathcal{M}_{\sigma,m,\Omega} + a)$ when $\sigma \to +\infty$ holds true as soon as $m > 0$. Therefore,

$$\lambda_p\left(\mathcal{M}_{\sigma,2,\Omega} + a\right) \to -\sup_{\Omega} a \ as\ \sigma \to +\infty.$$

In this situation, one of the above argument fails and one of the expected limits is not $-\nu$ any more. Indeed, we have

Lemma (6.2.21)[203]: Let Ω be any domain and let $J \in C(\mathbb{R}^N)$ be positive, symmetric and such that $|z|^2 J(z) \in L^1(\mathbb{R}^N)$. Assume further that J satisfies (68)–(70) and $m = 0$ then

$$\lim_{\sigma\to 0} \lambda_p\ \left(\mathcal{M}_{\sigma,0,\Omega} + a\right) = -\sup_{\Omega} a$$

$$\lim_{\sigma\to +\infty} \lambda_p\ \left(\mathcal{M}_{\sigma,0,\Omega} + a\right) = 1 - \sup_{\Omega} a$$

Proof. As already noticed, the limit of $\lambda_p(\mathcal{M}_{\sigma,m,\Omega} + a)$ when $\sigma \to 0$ can be obtained by following the arguments developed in the case $0 < m < 2$.

Therefore, it remains only to establish the limit of $\lambda_p(\mathcal{M}_{\sigma,m,\Omega} + a)$ when $\sigma \to \infty$.

As above, up to adding a large positive constant to a, without any loss of generality, we can assume that a is positive somewhere in Ω and we denote $\nu := \sup_{\Omega} a > 0$. By using constant test functions and (iv) of the Proposition (6.2.8), we observe that

$$-\nu \leq \lambda_p(\mathcal{M}_{\sigma,0,\Omega} + a) \leq 1 - \nu, \text{for all } \sigma > 0.$$

So, we have

$$\lim_{\sigma\to\infty} \sup\ \lambda_p(\mathcal{M}_{\sigma,0,\Omega} + a) \leq 1 - \nu.$$

On the other hand, for any $\varphi \in C_c(\Omega)$ we have for all σ,

$$\mathcal{J}_{\sigma,0}(\varphi) \int_{\Omega} \varphi^2(x) = -\int_{\Omega} \left(\int_{\Omega} J_\sigma(x-y)\varphi(y)dy - \varphi(x)\right)\varphi(x)dx - \int_{\Omega} a\varphi^2(x)\,dx,$$

$$= -\int\int_{\Omega\times\Omega} J_\sigma(x-y)\varphi(x)\varphi(y)dxdy - \int_{\Omega} \varphi^2(x)\,dx - \int_{\Omega} a\varphi^2(x)\,dx,$$

$$\geq -\|\varphi\|_{L^2(\Omega)}\left(\int_\Omega\left(\int_\Omega J_\sigma(x-y)\varphi(x)\varphi(y)dx\right)^2 dy\right)^{1/2}$$

$$+\int_\Omega \varphi^2(x)\,dx - \sup_\Omega a\int_\Omega \varphi^2(x)\,dx,$$

$$\geq \sqrt{\|J_\sigma\|_\infty} - \|\varphi\|_{L^2(\Omega)} + \int_\Omega \varphi^2(x)\,dx - v\int_\Omega \varphi^2(x)\,dx,$$

$$\geq \left(-\frac{\sqrt{\|J\|_\infty}}{\sigma^{N/2}} + 1 - v\right)\int_\Omega \varphi^2(x)\,dx.$$

Thus, for all σ we have

$$\mathcal{I}_{\sigma,0}(\varphi) \geq \left(-\frac{\sqrt{\|J\|_\infty}}{\sigma^{\frac{N}{2}}} + 1 - v\right).$$

By density of $C_c(\Omega)$ in $L^2(\Omega)$, the above inequality holds for any $\varphi \in L^2(\Omega)$. Therefore, by Theorem (6.2.14) for all σ

$$\lambda_p(\mathcal{M}_{\sigma,0,\Omega} + a) = \lambda_v(\mathcal{M}_{\sigma,0,\Omega} + a) \geq -\frac{\sqrt{\|J\|_\infty}}{\sigma^{\frac{N}{2}}} + 1 - v,$$

and

$$\lim_{\sigma\to+\infty}\inf\, \lambda_p(\mathcal{M}_\sigma + a) \leq 1 - \nu.$$

Hence,

$$1 - \nu \leq \lim_{\sigma\to+\infty}\inf\, \lambda_p(\mathcal{M}_{\sigma,0,\Omega} + a) \leq \lim_{\sigma\to+\infty}\sup\, \lambda_p(\mathcal{M}_{\sigma,0,\Omega} + a) \leq 1 - \nu.$$

We analyse the monotonic behaviour of $\lambda_p(\mathcal{M}_{\sigma,0,\Omega} + a)$ with respect to σ in the particular case $\Omega = \mathbb{R}^N$.

Proposition (6.2.22)[203]: Let $\Omega = \mathbb{R}^N$, $a \in C(\mathbb{R}^N)$ and $J \in C(\mathbb{R}^N)$ be positive, symmetric and such that $|z|^2 J(z) \in L^1(\mathbb{R}^N)$. Assume further that J satisfies (68)–(70), m = 0 and a is symmetric ($a(x) = a(-x)$ for all x) and the map $t \to a(tx)$ is non increasing for all $x, t > 0$. Then the map $\sigma \to \lambda_p(\sigma)$ is monotone non decreasing.

Proof. When $\Omega = \mathbb{R}^N$, thanks to Proposition (6.2.2), we have

$$\lambda_p\big(\mathcal{M}_{\sigma,0,\mathbb{R}^N} + a\big) = \lambda_p\big(\mathcal{M}_{1,0,\mathbb{R}^N} + a_\sigma(x)\big).$$

Since the function aσ(x) is monotone non increasing with respect to σ, by (i) of Proposition (6.2.8), for all $\sigma \geq \sigma^*$ we have

$$\lambda_p\big(\mathcal{M}_{\sigma^*,0,\mathbb{R}^N} + a\big) = \lambda_p\left(\mathcal{M}_{1,0,\mathbb{R}^N} + a\sigma^*(x)\right)$$

$$\leq \lambda_p\big(\mathcal{M}_{1,0,\mathbb{R}^N} + a\sigma(x)\big) = \lambda_p\big(\mathcal{M}_{\sigma,0,\mathbb{R}^N} + a\big).$$

Finally, let us study the case $m = 2$ and end the proof of Theorem (6.2.3). In this situation, we claim that

Lemma (6.2.23)[203]: Let Ω be a domain, $a \in C(\Omega)$ and let $J \in C(\mathbb{R}^N)$ be positive, symmetric and such that $|z|^2 J(z) \in L^1(\mathbb{R}^N)$. Assume further that J satisfies (68)–(70), $a \in C^{0,\alpha}(\Omega)$ with $\alpha > 0$ and $m = 2$ then

$$\lim_{\sigma\to+\infty} \lambda_p(\mathcal{M}_{\sigma,2,\Omega} + a) \leq -\sup_\Omega a,$$

$$\lim_{\sigma\to 0}\lambda_p(\mathcal{M}_{\sigma,2,\Omega}+a) = \lambda_1\left(\frac{D_2(J)K_{2,N}}{2}\Delta + a,\Omega\right), \tag{103}$$

where

$$K_{2,N} := \frac{1}{|S^N-1|}\int_{S^{N-1}} (s.e_1)^2\ ds = \frac{1}{N}$$

and

$$\lambda_1\left(K_{2,N}D_2(J)\Delta + a,\Omega\right) := \inf_{\varphi\in H_0^1(\Omega),\varphi\neq 0} K_{2,N}\mathcal{J}(\varphi)-\mathcal{A}(\varphi).$$

Proof. In this situation, as already noticed, by following the arguments used in the case $2 > m > 0$, we can obtain the limit of $\lambda_p(\mathcal{M}_{\sigma,2,\Omega}+a)$ as $\sigma\to\infty$.So, it remains to prove (103). Let us rewrite $\mathcal{I}_{\sigma,2}(\varphi)$ in a more convenient way. Let $\rho_\sigma(z) := \frac{1}{\sigma^2 D_2(J)}J_\sigma(z)|z|^2$, then for $\varphi\in H_0^1(\Omega)$, we have

$$\mathcal{I}_{\sigma,2}(\varphi) = \frac{1}{\|\varphi\|^2_{L^2(\Omega)}}\left(\frac{1}{2\sigma^2}\int\int_{\Omega\times\Omega} J_\sigma(x-y)\big(\varphi(x)-\varphi(y)\big)^2 dxdy\right) -\mathcal{R}_\sigma(\varphi)-\mathcal{A}(\varphi), \tag{104}$$

$$= \frac{1}{\|\varphi\|^2_{L^2(\Omega)}}\left(\frac{D_2(J)}{2}\int\int_{\Omega\times\Omega}\rho_\sigma(x-y)\frac{\big(\varphi(x)-\varphi(y)\big)^2}{|x-y|^2}dxdy\right)-\mathcal{R}_\sigma(\varphi)-\mathcal{A}(\varphi). \tag{105}$$

We are now is position to prove (103). Let us first show that

$$\limsup_{\sigma\to 0}\ \lambda_p(\mathcal{M}_{\sigma,1,\Omega}+a)\le\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a,\Omega\right). \tag{106}$$

This inequality follows from the two following observations.
First, for any $\omega\subset\Omega$ compact subset of Ω, we have for σ small enough

$$p_\sigma(x) = \int_\Omega J_\sigma(x-y)dy = 1 \quad \text{for all } x\in\omega.$$

Therefore, for$\varphi\in C_c^\infty(\Omega)$ and σ small enough,

$$\mathcal{R}_{\sigma,2} = \frac{1}{\sigma^2\|\varphi\|^2_{L^2(\Omega)}}\int_\Omega p_\sigma(x-1)\,\varphi^2(x)\,dx = 0. \tag{107}$$

Secondly, by definition, p_σ is a continuous mollifier such that

$$\begin{cases} p_\sigma \ge 0\ \ in\mathbb{R}^N \\ \int_{\mathbb{R}^N} p_\sigma(z)dz = 1, \forall\sigma>0, \\ \lim_{\sigma\to 0}\int_{|z|\ge\delta} p_\sigma(z)dz = 1, \forall\sigma>0, \end{cases}$$

which, from the characterisation of Sobolev spaces in [210], [211], [240], enforces that

$$\lim_{\sigma\to 0}\int_{\Omega\times\Omega}\rho_\sigma(x-y)\frac{\big(\varphi(x)-\varphi(y)\big)^2}{|x-y|^2}dxdy = K_{2,N}\|\varphi\|^2_{L^2(\Omega)}, \text{for any } \phi\in H_0^1(\Omega). \tag{108}$$

Thus, for any $\varphi\in C_c^\infty(\Omega)$

$$\lim_{\sigma\to 0}\lambda_p(\mathcal{M}_{\sigma,2,\Omega}+a)\le\lim_{\sigma\to 0}\mathcal{T}_{\sigma,2}(\varphi) = K_{2,N}\mathcal{J}(\varphi)-\mathcal{A}(\varphi).$$

From the above inequality, by definition of $\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\,\Delta + a,\Omega\right)$, it is then standard to obtain

$$\limsup_{\sigma\to 0}\ \lambda_p(\mathcal{M}_{\sigma,2,\Omega} + a) \le \lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\,\Delta + a,\Omega\right).$$

To complete our proof, it remains to establish the following inequality

$$\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\,\Delta + a,\Omega\right) \le \liminf_{\sigma\to 0}\ \lambda_p(\mathcal{M}_{\sigma,2,\Omega} + a).$$

Observe that to obtain the above inequality, it is sufficient to prove that

$$\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\,\Delta + a,\Omega\right) \le \liminf_{\sigma\to 0}\ \lambda_p(\mathcal{M}_{\sigma,2,\Omega} + a) +\ 2\delta \text{ for all } \delta > 0. \quad (109)$$

Let us fix $\delta > 0$. Now, to obtain (109), we construct adequate smooth test functions φ_σ and estimate $K_{2,N}\mathcal{J}(\varphi_\sigma) - \mathcal{A}(\varphi_\sigma)$ in terms of$\lambda_p(\mathcal{M}_{\sigma,2,\Omega} + a), \delta$ and some reminder $R(\sigma)$ that converges to 0 as $\sigma \to 0$. Since our argument is rather long, we decompose it into three steps.

We first claim that, for all $\sigma > 0$, there exists $\varphi_\sigma \in C_c^\infty(\Omega)$ such that

$$\mathcal{M}_{\sigma,2,\Omega}[\varphi_\sigma](x) + \big(a(x) + \lambda_p(\mathcal{M}_{\sigma,2,\Omega} + a) + 2\delta\big)\varphi_\sigma(x) \ge 0 \text{ for all } x \in \Omega.$$

Indeed, by Theorem (6.2.14), we have $\lambda_p\big(\mathcal{M}_{\sigma,2,\Omega} + a\big) = \lambda_p''(\mathcal{M}_{\sigma,2,\Omega} + a)$, therefore for all σ, there exists $\psi_\sigma \in C_c(\Omega)$ such that

$$\mathcal{M}_{\sigma,2,\Omega}[\psi_\sigma](x) + \big(a(x) + \lambda_p(\mathcal{M}_{\sigma,2,\Omega} + a) + \delta\big)\psi_\sigma(x) \ge 0 \text{ for all } x \in \Omega.$$

Since $\psi_\sigma \in C_c(\Omega)$, we can easily check that

$$\mathcal{M}_{\sigma,2,\mathbb{R}^N}[\psi_\sigma](x) + \big(a(x) + \lambda_p(\mathcal{M}_{\sigma,2,\Omega} + a) + \delta\big)\psi_\sigma(x) \ge 0 \text{ for all } x \in \mathbb{R}^N.$$

Now, let η be a smooth mollifier of unit mass and with support in the unit ball and consider $\eta_\tau := \frac{1}{\tau^N}\,\eta\left(\frac{z}{\tau}\right)$or $\tau > 0$.

By taking $\bar{\varphi}_\sigma := \eta\tau * \psi_\sigma$ and observing that $\mathcal{M}_{\sigma,2,\mathbb{R}^N}[\bar{\varphi}_\sigma](x) = \eta_\tau\left(\mathcal{M}_{\sigma,2,\mathbb{R}^N}[\psi_\sigma]\right)(x)$ for any $x \in \mathbb{R}^N$we deduce that

$$\eta_\tau * \big(\mathcal{M}_{\sigma,2,\mathbb{R}^N}[\psi_\sigma] + \big(a(x) + \lambda_p\big(\mathcal{M}_{\sigma,2,\Omega} + a\big) + \delta\big)\psi_\sigma\big) \ge 0 \ for\ all\ x \in \mathbb{R}^N,$$

$$\mathcal{M}_{\sigma,2,\mathbb{R}^N}[\bar{\varphi}_\sigma](x) + \big(\lambda_p(\mathcal{M}_{\sigma,2,\Omega} + a) + \delta\big)\bar{\varphi}_\sigma(x) + \eta_\tau * (a\psi_\sigma)(x) \ge 0 \text{for all} x \in \mathbb{R}^N.$$

By adding and subtracting a, we then have, for all $x \in \mathbb{R}^N$,

$$\mathcal{M}_{\sigma,2,\mathbb{R}^N}[\bar{\varphi}_\sigma](x) + \big(a(x) + \lambda_p\big(\mathcal{M}_{\sigma,2,\Omega} + a\big) + \delta\big)\bar{\varphi}_\sigma(x) + \int_{\mathbb{R}^N} \eta_\tau(x - y)\psi_\sigma(y)\big(a(y) - a(x)\big)dy \ge 0.$$

For τ small enough, say $\tau \le \tau_0$, the function $\bar{\varphi}_\sigma \in C_c^\infty(\Omega)$ and for all $x \in \Omega$ we have

$$\mathcal{M}_{\sigma,2,\mathbb{R}^N}[\bar{\varphi}_\sigma](x) = \frac{1}{\sigma^2}\left(\int_{\mathbb{R}^N} J_\sigma(x - y)\bar{\varphi}_\sigma(y)dy - \bar{\varphi}_\sigma(x)\right),$$

$$= \frac{1}{\sigma^2}\left(\int_{\Omega} J_\sigma(x - y)\bar{\varphi}_\sigma(y)dy - \bar{\varphi}_\sigma(x)\right)\mathcal{M}_{\sigma,2,\Omega}[\bar{\varphi}_\sigma](x).$$

Thus, from the above inequalities, for $\tau \le \tau_0$, we get for all $x \in \Omega$,

$$\mathcal{M}_{\sigma,2,\Omega}[\bar{\varphi}_\sigma](x) + \big(a(x) + \lambda_p\big(\mathcal{M}_{\sigma,2,\Omega} + a\big) + \delta\big)\bar{\varphi}_\sigma(x)$$

$$+ \int_{\mathbb{R}^N} \eta_\tau (x - y)\psi_\sigma(y)\big(a(y) - a(x)\big)dy \geq 0.$$

Since a is Hölder continuous, we can estimate the integral by

$$\left| \int_{\mathbb{R}^N} \eta_\tau (x - y)\psi_\sigma(y)\big(a(y) - a(x)\big)dy \right| \leq \int_{\mathbb{R}^N} \eta_\tau (x - y)\psi_\sigma(y) \left| \frac{\big(a(y) - a(x)\big)}{|y - x|^\alpha} \right| dy,$$
$$\leq \kappa\tau^\alpha \bar{\varphi}_\sigma(x),$$

where κ is the Hölder semi-norm of a. Thus, for τ small, says $\tau \leq \inf\{ \left(\frac{\delta}{2\kappa}\right)^{1/\alpha}, \tau_0\}$, we have

$$\mathcal{M}_{\sigma,2,\Omega}[\bar{\varphi}_\sigma](x) + (a(x) + \lambda_p(\mathcal{M}_{\sigma,2,\Omega} + a) + 2\delta)\, \bar{\varphi}_\sigma(x) \geq 0 \text{ for all } x \in \Omega. \quad (110)$$

Let us consider now $\varphi_\sigma := \gamma\, \bar{\varphi}_\sigma$, where γ is a positive constant to be chosen. From (110), we obviously have

$$\mathcal{M}_{\sigma,2,\Omega}[\varphi_\sigma](x) + (a(x) + \lambda_p(\mathcal{M}_{\sigma,2,\Omega} + a) + 2\delta)\varphi_\sigma(x) \geq 0 \text{ for all } x \in \Omega. \quad (111)$$

By taking $\gamma := \frac{\int_{\mathbb{R}^N} \psi_\sigma^2(x)\, dx}{\int_{\mathbb{R}^N} \varphi_\sigma^2\, (x)\, dx}$, we get

$$\frac{\int_{\mathbb{R}^N} \psi_\sigma^2(x)\, dx}{\int_{\mathbb{R}^N} \varphi_\sigma^2\, (x)\, dx} = 1. \quad (112)$$

Step Two: A first estimate of λ_1.

Now, by multiplying $\mathcal{M}_{\sigma,2,\Omega}[\varphi_\sigma]$ by $\bar{\varphi}_\sigma$and integrating over Ω, we then get

$$- \int_\Omega \mathcal{M}_{\sigma,2,\Omega}\, [\varphi_\sigma]\varphi_\sigma(x)dx$$
$$= - \int \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{1}{\sigma^2} J_\sigma\, (x - y)\big(\varphi_\sigma(y)dy - \varphi_\sigma(x)\big)\; \varphi_\sigma(x)\; dydx, \quad (113)$$
$$= \frac{1}{2\sigma^2} \int \int_{\mathbb{R}^N\times\mathbb{R}^N} J_\sigma\, (x - y)\big(\varphi_\sigma(y)dy - \varphi_\sigma(x)\big)^2\; dxdy, \quad (114)$$
$$= \frac{D_2(J)}{2} \int \int_{\mathbb{R}^N\times\mathbb{R}^N} \rho_\sigma(z) \frac{\big(\varphi_\sigma(x+z) - \varphi_\sigma(x)\big)^2}{|z|^2}\; dzdx. \quad (115)$$

By combining (110) and (115) we therefore obtain

$$= \frac{D_2(J)}{2} \int \int_{\mathbb{R}^N\times\mathbb{R}^N} \rho_\sigma(z) \frac{\big(\varphi_\sigma(x+z) - \varphi_\sigma(x)\big)^2}{|z|^2}\; dzdx$$
$$- \int_{\mathbb{R}^N} a(x)\psi_\sigma^2(x)dx \leq \big(\lambda_p\big(\mathcal{M}_{\sigma,2,\Omega} + a\big) + 2\delta\big) \int_{\mathbb{R}^N} \psi_\sigma^2(x)dx. \quad (116)$$

On the other hand, inspired by the proof of Theorem 2 in [211], since $\varphi_\sigma \in C_c^\infty\, (\mathbb{R}^N)$, by Taylor's expansion, for all $x, z \in \mathbb{R}^N$, we have

$$|\varphi_\sigma(x + z) - \varphi_\sigma(x) - z \cdot \nabla_{\varphi_\sigma}(x)| \leq \sum_{i,j} |z_i z_j| \int_0^1 t \left(\int_0^1 |\partial_{ij}\varphi_\sigma(x + tsz)|\, ds \right) dt.$$

Therefore,

$$|z\cdot \nabla_{\varphi_\sigma}(x)| \le \sum_{i,j}|z_iz_j|\int_0^1 t\left(\int_0^1 |\partial_{ij}\varphi_\sigma(x\ +\ tsz)|\,ds\right)dt.+|\varphi_\sigma(x+z)-\varphi_\sigma(x)|,$$

and for every $\theta > 0$ we have

$$\left|z\cdot \nabla_{\varphi_\sigma}(x)\right|^2 \ \le C_\theta\left[\sum_{i,j}\left|z_iz_j\right|\int_0^1 t\left(\int_0^1\left|\partial_{ij}\varphi_\sigma(x\ +\ tsz)\right|ds\right)dt\right]^2$$
$$+(1+\theta)|\varphi_\sigma(x+z)-\varphi_\sigma(x)|^2,$$
$$\le C_\theta\sum_{i,j}|z_iz_j|^2\int\int_{[0,1]^2} t^2|\partial_{ij}\varphi_\sigma(x+tsz)|^2\,ds\,dt+(1+\theta)|\varphi_\sigma(x+z-\varphi_\sigma(x)|^2.$$

Thus, by integrating in x and z over $\mathbb{R}^N\times\mathbb{R}^N$, we get

$$\int\int\frac{\rho_\sigma(|z|)}{|z|^2}|z\cdot\nabla\varphi_\sigma(x)|^2dzdx$$
$$\le C_\theta\int\int_{[0,1]^2}\rho_\sigma(|z|)\sum_{i,j}\frac{\left|z_iz_j\right|^{2^2}}{|z|^2}\left(\int\int_{[0,1]^2}t^2\left|\partial_{ij}\varphi_\sigma(x\ +\ tsz)\right|^2dzdx\right)$$
$$+(1+\theta)\int\int\rho_\sigma(|z|)\frac{|\varphi_\sigma(x+z)-\varphi_\sigma(x)|^2}{|z|^2}dzdx.$$

For σ small, $\mathrm{supp}(\rho_\sigma)\subset B_1(0)$, and we have for all $x\in\mathbb{R}^N$,

$$\int_{\mathbb{R}^N}\frac{\rho_\sigma(|z|)}{|z|^2}|z\cdot\nabla\varphi_\sigma(x)|^2dz = K_{2,N}|\nabla\varphi_\sigma(x)|^2,$$

Whence,

$$K_{2,N}\int_{\mathbb{R}^N}|\nabla\varphi_\sigma(x)|^2dz$$
$$\le C_\theta\int\int\rho_\sigma(|z|)\sum_{i,j}\frac{\left|z_iz_j\right|^2}{|z|^2}\left(\int\int_{[0,1]^2}t^2\left|\partial_{ij}\varphi_\sigma(x\ +\ tsz)\right|^2dsdt\right)dzdx$$
$$+(1+\theta)\int\int\rho_\sigma(|z|)\frac{|\varphi_\sigma(x+z)-\varphi_\sigma(x)|^2}{|z|^2}dzdx. \qquad (117)$$

Dividing (117) by $\|\varphi_\sigma\|^2_{L^2(\Omega)}$ and then subtracting $\mathcal{A}(\varphi_\sigma)$ on both side, we get

$$K_{2,N}\mathcal{J}(\varphi_\sigma)-\mathcal{A}(\varphi_\sigma)\le R(\sigma)+(1+\theta)\mathcal{I}_{\sigma,2}(\varphi_\sigma), \qquad (118)$$

where $R(\sigma)$ is defined by

$$R(\sigma) := \frac{C_\theta}{\|\varphi_\sigma\|^2_{L^2(\Omega)}}\int\int\rho_\sigma(|z|)\sum_{i,j}\frac{|z_iz_j|^2}{|z|^2}\left(\int\int_{[0,1]^2}t^2|\partial_{ij\varphi_\sigma}(x\ +\ tsz)|^2\,dsdt\right)dzdx.$$

By combining now (118) with (116), by definition of $\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\,\Delta+a,\Omega\right)$, we obtain

$$\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\,\Delta+a,\Omega\right)\le R(\sigma)+(1+\theta)\left[\lambda_p\left(\mathcal{M}_{\sigma,2,\Omega}+a\right)+2\delta\right]. \qquad (119)$$

Let us now estimate $R(\sigma)$ and finish our argument.

By construction, we have $\partial_{ij\varphi_\sigma} = \partial_{ij\eta_\tau} * \psi_\sigma$. So, by Fubini's Theorem and standard convolution estimates, we get for σ small

$$R(\sigma) \le \sum_{i,j} \int_{|z|\le 1} \int \int_{[0,1]^2} \rho_\sigma(|z|) \sum_{i,j} \frac{|z_i z_j|^2}{|z|^2} t^2$$

$$\left(\int_{\mathbb{R}^N} \left|\partial_{ij\eta_\tau} * \psi_\sigma(x + tsz)\right|^2 dx \right) dtdsdz,$$

$$\le \left(\int_{|z|\le 1} \int_{[0,1]^2} \rho_\sigma(|z|) \sum_{i,j} \frac{|z_i z_j|^2}{|z|^2} t^2 dtdz \right) \left\|\nabla^2{}_{\eta_\tau}\right\|_{L^1(\mathbb{R}^N)} \|\psi_\sigma\|^2_{L^2(\mathbb{R}^N)},$$

$$\le \frac{2}{3} \left\|\nabla^2{}_{\eta_\tau}\right\|_{L^1(\mathbb{R}^N)} \|\psi_\sigma\|^2_{L^2(\mathbb{R}^N)} \int_{|z|\le 1} \rho_\sigma\,(|z|)|z|^2 dz.$$

Combining this inequality with (119), we get

$$\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\,\Delta + a, \Omega\right) \le (1+\theta)\left[\lambda_p\left(\mathcal{M}_{\sigma,2,\Omega} + a\right) + 2\delta\right]$$

$$+ \frac{2C_\theta}{3} \left\|\nabla^2{}_{\eta_\tau}\right\|_{L^1(\mathbb{R}^N)} \frac{\|\psi_\sigma\|^2_{L^2(\mathbb{R}^N)}}{\|\varphi_\sigma\|^2_{L^2(\Omega)}} \int_{|z|\le 1} \rho_\sigma\,(|z|)|z|^2 dz.$$

Since $\varphi_\sigma \in C_c^\infty\,(\Omega)$, $\|\varphi_\sigma\|^2_{L^2(\Omega)} = \|\varphi_\sigma\|^2_{L^2(\mathbb{R}^N)}$ and thanks to (112), the above inequality reduces to

$$\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\,\Delta + a, \Omega\right) \le (1+\theta)\left[\lambda_p\left(\mathcal{M}_{\sigma,2,\Omega} + a\right) + 2\delta\right]$$

$$+ \frac{2C_\theta}{3} \left\|\nabla^2{}_{\eta_\tau}\right\|_{L^1(\mathbb{R}^N)} \int_{|z|\le 1} \rho_\sigma\,(|z|)|z|^2 dz. \quad (120)$$

Now, since $\int_{|z|\le 1} \rho_\sigma(|z|)|z|^2\, dz \le \sigma^2$, letting $\sigma \to 0$ in (120) yields

$$\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a, \Omega\right) \le (1+\theta)\left[2\delta + \liminf_{\sigma\to 0} \lambda_p\left(\mathcal{M}_{\sigma,2,\Omega} + a\right)\right]. \quad (121)$$

Since (121) holds for every θ, we obtain

$$\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a, \Omega\right) \le \liminf_{\sigma\to 0} \lambda_p\left(\mathcal{M}_{\sigma,2,\Omega} + a\right) + 2\delta.$$

We investigate the existence of a positive continuous eigenfunction $\varphi_{p,\sigma}$ associated to the principal eigenvalue $\lambda_p\left(\mathcal{M}_{\sigma,2,\Omega} + a\right)$.

The existence of such a $\varphi_{p,\sigma}$ is a straightforward consequence of the existence criteria in bounded domain (Theorem (6.2.7)) and the asymptotic behaviour of the principal eigenvalue (Theorem (6.2.3)).

Indeed, assume first that Ω is bounded, then since $a \in L^\infty(\bar{\Omega})$, there exists σ_0 such that for all $\sigma \le \sigma_0$,

$$\frac{1}{\sigma^2} - \sup_\Omega a > 1 + \left|\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a, \Omega\right)\right|.$$

Now, thanks to $\lambda_p\left(\mathcal{M}_{\sigma,2,\Omega} + a\right) \to \lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a, \Omega\right)$, for σ small enough, says $\sigma \le \sigma_1$, we get

$$\lambda_p(\mathcal{M}_{\sigma,2,\Omega} + a) \leq 1 + \left|\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a, \Omega\right)\right|.$$

Thus, for $\sigma \leq \inf\{\sigma_1, \sigma_0\}$,

$$\lambda_p(\mathcal{M}_{\sigma,2,\Omega} + a) < \frac{1}{\sigma^2} - \sup_{\Omega} a,$$

which, thanks to Theorem (6.2.7), enforces the existence of a principal positive continuous eigenfunction $\varphi_{p,\sigma}\sigma$ associated with $\lambda_p(\mathcal{M}_{\sigma,2,\Omega} + a)$.

From the above argument, we can easily obtain the existence of eigenfunction when Ω is unbounded.

Indeed, let Ω_0 be a bounded sub-domain of Ω and let $\gamma := sup\{|\lambda_1(\Omega_0)|\,;\,|\lambda_1(\Omega)|\}$. Since a is bounded in Ω, there exists σ_0 such that for all $\sigma \leq \sigma_0$,

$$\frac{1}{\sigma^2} - \sup_{\Omega} a > 2 + \gamma.$$

As above, since $\lambda_p(\mathcal{M}_{\sigma,2,\Omega_0} + a) \to \lambda_1(\Omega_0)$, there exists σ_1 such that for all $\sigma \leq \sigma_1$ we have

$$\lambda_p(\mathcal{M}_{\sigma,2,\Omega_0} + a) \leq 1 + \gamma.$$

For any bounded domain Ω' such that $\Omega_0 \subset \Omega' \subset \Omega$, by monotonicity of $\lambda_p(\mathcal{M}_{\sigma,2,\Omega} + a)$ with respect to Ω', for all $\sigma \leq \sigma_1$ we have

$$\lambda_p(\mathcal{M}_{\sigma,2,\Omega_0} + a) \leq 1 + \gamma.$$

Therefore, for all $\sigma \leq \sigma_2 := \inf\{\sigma_0, \sigma_1\}$, we have

$$\lambda_p(\mathcal{M}_{\sigma,2,\Omega'} + a) + 1 \leq \frac{1}{\sigma^2} - \sup_{\Omega'} a,$$

and thus, thanks to Theorem (6.2.7), for all $\sigma \leq \sigma^2$ there exists $\varphi_{p,\sigma}$ associated to $\lambda_p(\mathcal{M}_{\sigma,2,\Omega'} + a)$.

To construct a positive eigenfunction $\varphi_{p,\sigma}$ associated to $\lambda_p(\mathcal{M}_{\sigma,2,\Omega} + a)$, we then argue as follows.

Let $(\Omega_n)_{n\in\mathbb{N}}$ be an increasing sequence of bounded sub-domain of Ω that converges to Ω. Then, for all $\sigma \leq \sigma_2$, for each n there exists a continuous positive function $\varphi_{n,\sigma}$ associated to $\lambda_p(\mathcal{M}_{\sigma,2,\Omega_n} + a)$.

Without any loss of generality, we can assume that φ_n is normalised by $\varphi_n(x_0) = 1$ for some fixed $x_0 \in \Omega_0$. Since for all $n, \lambda_p(\mathcal{M}_{\sigma,2,\Omega_n} + a) + 1 \leq \frac{1}{\sigma^2} - \sup_{\Omega_n} a$, the Harnack inequality applies to φ_n and thus the sequence $(\Omega_n)_{n\in\mathbb{N}}$ is locally uniformly bounded in C_0 topology. By a standard diagonal argument, there exists a subsequence, still denoted $(\Omega_n)_{n\in\mathbb{N}}$, that converges point-wise to some nonnegative function φ. Thanks to the Harnack inequality, φ is positive. Passing to the limit in the equation satisfied by φ_n, thanks to the Lebesgue dominated convergence Theorem, φ satisfies

$$M\ \sigma, 2, \Omega[\varphi](x) + (a(x) + \lambda_{p,\sigma}(\mathcal{M}_{\sigma,2,\Omega} + a)\ \varphi(x) = 0\ for\ all\ x \in \Omega.$$

Since a is continuous and $\left((a(x) + \lambda_{p,\sigma}(\mathcal{M}_{\sigma,2,\Omega} + a) - \frac{1}{\sigma^2}\right) < 0$, we deduce that ϕ is also continuous.

Hence, φ is a positive continuous eigenfunction associated with $\lambda_{p,\sigma}(\mathcal{M}_{\sigma,2,\Omega} + a)$.

Finally, let us complete the proof of Theorem (6.2.4) by obtaining the asymptotic behaviour of $\varphi_{p,\sigma}$when $\sigma \to 0$ assuming that $\varphi_{p,\sigma} \in L^2(\Omega)$. We first recall the following useful identity:

Proposition (6.2.24)[203]: Let $\rho \in C_c(\mathbb{R}^N)$ be a radial function, then for all $u \in L^2(\mathbb{R}^N), \varphi \in C_0^\infty(\mathbb{R}^N)$ we have

$$\int\int_{\mathbb{R}^N\times\mathbb{R}^N} \rho(z)[u(x+z)-u(x)]\varphi(x)\,dzdx = \frac{1}{2}\int\int_{\mathbb{R}^N\times\mathbb{R}^N} \rho(z)u(x)\Delta_z[\varphi](x)\,dzdx$$

Where

$$\Delta_z[\varphi](x) := \varphi(x+z) - 2\varphi(x) + \varphi(x-z).$$

Proof. Set

$$I := \int\int_{\mathbb{R}^N\times\mathbb{R}^N} \rho(z)[u(x+z)-u(x)]\varphi(x)dzdx\,.$$

By standard change of variable, thanks to the symmetry of ρ, we get

$$I = \frac{1}{2}\int\int_{\mathbb{R}^N\times\mathbb{R}^N} \rho(z)[u(x+z)-u(x)]\varphi(x)$$
$$+\frac{1}{2}\int\int_{\mathbb{R}^N\times\mathbb{R}^N} \rho(-z)[u(x+z)-u(x)]\varphi(x)\,,$$
$$= \frac{1}{2}\int\int_{\mathbb{R}^N\times\mathbb{R}^N} \rho(z)[u(x+z)-u(x)]\varphi(x)$$
$$+\frac{1}{2}\int\int_{\mathbb{R}^N\times\mathbb{R}^N} \rho(z)[u(x)-u(x+z)]\varphi(x+z)\,,$$
$$= -\frac{1}{2}\int\int_{\mathbb{R}^N\times\mathbb{R}^N} \rho(z)[u(x+z)-u(x)][\varphi(x+z)-\varphi(x)]\,,$$
$$= -\frac{1}{2}\int\int_{\mathbb{R}^N\times\mathbb{R}^N} \rho(z)u(x)[\varphi(x)-\varphi(x-z)] + \frac{1}{2}\int\int_{\mathbb{R}^N\times\mathbb{R}^N} \rho(z)u(x)[u(x+z)-\varphi(x)]\,,$$
$$= \frac{1}{2}\int\int_{\mathbb{R}^N\times\mathbb{R}^N} \rho(z)[u(x+z)-u(x)]\varphi(x)dzdx$$

Consider now $\sigma \le \sigma_2(\Omega)$ and let $\varphi_{p,\sigma}$ be a positive eigenfunction associated with $\lambda_{p,\sigma}$. That is $\varphi_{p,\sigma}$satisfies

$$\mathcal{M}_{\sigma,2,\Omega}[\varphi_{n,\sigma}](x) + (a(x) + \lambda_{p,\sigma})\varphi_{p,\sigma}(x) = 0 \text{ for all } x \in \Omega. \qquad (122)$$

Let us normalize $\varphi_{p,\sigma}$ by $\|\varphi_{p,\sigma}\|_{L^2(\Omega)} = 1$.

Multiplying (122) by $\varphi_{p,\sigma}$ and integrating over Ω, we get

$$\frac{D_2(J)}{2}\int\int_{\Omega\times\Omega} \rho_\sigma(x-y)\frac{|\varphi_{p,\sigma}(y)-\varphi_{p,\sigma}(x)|^2}{|x-y|^2}dxdy$$
$$\le \int_\Omega (a(x) + \lambda_{p,\sigma})\varphi_{p,\sigma}^2(x)\,dx \le C.$$

Since a and $\lambda_{p,\sigma}$ are bounded independently of $\sigma \le \sigma_2(\Omega)$, the constant C stands for all $\sigma \le \sigma_2(\Omega)$.

Therefore for any bounded sub-domain $\bar{\Omega} \subset \Omega$,

$$\int\int_{\Omega'\times\Omega'} \rho_\sigma\,(x-y)\ \frac{\left(\varphi_{p,\sigma}(y)-\varphi_{p,\sigma}(x)\right)^2}{|x-y|^2}dxdy\ < C.$$

Therefore by the characterisation of Sobolev space in [240], [239], for any bounded sub-domain $\bar{\Omega} \subset \Omega$, along a sequence, $\varphi_{p,\sigma} \to \varphi in\ L^2(\bar{\Omega})$. Moreover, by extending $\varphi_{p,\sigma}$ by 0 outside Ω, we have $\varphi_{p,\sigma} \in L^2(\mathbb{R}^N)$ and for any $\psi \in C_c^2\,(\Omega)$ by Proposition (6.2.24) it follows that

$$\frac{D_2(J)}{2}\int\int_{\Omega\times\mathbb{R}^N} \frac{\rho_\sigma(z)}{|z|^2}\ \varphi_{p,\sigma}(x)\Delta_z[\psi]\,dxdz$$

$$= \int_\Omega \Big(a(x) +\ \lambda_{\mathrm{p},\sigma} - 1 + p_\sigma(x)\Big)\,\psi\varphi_{p,\sigma}(x)dx. \qquad (123)$$

Recall that $\psi \in C_c^\infty(\mathbb{R}^N)$, so there exists $C(\psi)$ and $R(\psi)$ such that for all $x \in \mathbb{R}^N$ $|\Delta_z[\psi](x) - t_z(\nabla^2\psi(x))z\ < C(\psi)|z|^3\mathbb{I}_{B_{R(\psi)}}(x)$.

Therefore, since $\phi p,\sigma$ is bounded uniformly in $L^2(\Omega)$,

$$\frac{D_2(J)}{2}\int\int_{\Omega\times\mathbb{R}^N} \frac{\rho_\sigma(z)}{|z|^2}\ \varphi_{p,\sigma}(x)[\Delta_z]\big[\psi - t_z\big(\nabla^2\psi(x)\big)z\big]\ dxdz$$

$$\leq CC(\psi)\ \int_{\mathbb{R}^N} p_n(z)|z| \to 0. \qquad (124)$$

On the other hand, $\psi \in C_c^2(\Omega)$ enforces that for σ small enough $\mathrm{supp}(1 - p_\sigma(x)) \cap \mathrm{supp}(\psi) = \emptyset$.

Thus passing to the limit along a sequence in (123), thanks to (124), we get

$$\frac{D_2(J)K_{2,N}}{2}\int_\Omega \varphi(x)\Delta\psi(x)dx + \int_\Omega \varphi(x)\psi(x)(a(x)+\lambda_1)\,dx = 0. \qquad (125)$$

being true for any ψ, it follows that φ is the smooth positive eigenfunction associated to $\lambda_1$normalised by $\|\varphi\|_{L^2(\Omega)}1 = \lim_{\sigma\to 0}\|\varphi_{p,\sigma}\|_{L^2(\Omega)}$ .The normalised first eigenfunction being uniquely defined, we get $\varphi = \varphi_1$ and $\varphi_{p,\sigma} \to \varphi_1$ in $L^2_{loc}(\Omega)$ when $\sigma\ \to\ 0$.

List of Symbols

Symbol		Page
L^p	Lebesgue space	1
dim:	dimension	1
det:	determinant	7
L^2:	Hilbert space	8
Δ_g:	Laplace-Beltrami operator	23
sup:	supremum	26
det:	determinant	28
max:	maximum	38
QUE:	Quantum unique ergodicity	47
inf:	infimum	48
supp:	support	69
L^∞:	Essential Lebesgue space	75
L^1:	Lebesgue space on the line	77
ℓ^2:	Hilbert space of sequences	81
ℓ^q:	Dual of Banach space of sequence	84
stab:	stabilizer	93
Dir:	Dirichlet	93
int:	interior	93
Osc:	oscillatory	94
loc:	local	107
a. p:	almost periodic	108
$W^{2,\infty}$:	Sobolev space	109
min:	minimum	123
per:	periodic	124
ker:	kernel	125
LRC:	locally Relative compact	145
ℓ_∞:	Essential Banach space	145
$\otimes$:	tensor product	146
diam:	diameter	147
dist:	distance	167
inj:	injectivity	190
Re:	Real	204
int:	interior	206
ess:	essential	211
dom:	domain	231

References

[1] R. Hu, L^p norm estimates of eigenfunctions restricted to submanifolds, Forum Math. 6 (2009) 1021–1052.

[2] Burq N., Gérard P., Tzvetkov N.: Multilinear estimates for the Laplace spectral projector on compact manifolds. C. R. Math. Acad. Sci. Paris 338 (2004), 359–364

[3] Burq N., Gérard P., TzvetkovN.: Bilinear eigenfunction estimates and the nonlinear Schrödinger equation on surfaces. Invent. Math. 159 (2005), 187–223

[4] Burq N., Gérard P.,TzvetkovN.: Multilinear eigenfunction estimates and global existence for the three dimensional nonlinear Schrödinger equations. Ann. Sci. École Norm. Sup. (4) 38 (2005), 255–301

[5] Burq N., Gérard P., Tzvetkov N.: Restrictions of the Laplace-Beltrami eigenfunctions to submanifolds. Duke Math. J. 138 (2007), 445–486

[6] Do CarmoM. P.: Differential Geometry of Curves and Surfaces. Prentice-Hall, Inc., Englewood Cliffs, N. J. 1976

[7] ComechA.: Optimal regularity for fourier integral operatorswith one-sidedfolds. Comm. Partial Differential Equations 24 (1999), 1263–1281

[8] Erdös L., Salmhofer M.: Decay of the Fourier transform of surfaces with vanishing curvature. Math. Z. 257 (2007), 261–294

[9] Gallot S., Hulin D., Lafontaine J.: Riemannian Geometry. Springer-Verlag 1987

[10] GreenleafA., SeegerA.: Fourier integral operatorswith fold singularities. J. Reine Angew. Math. 455 (1994), 35–56

[11] Hörmander L.: The spectral function of an elliptic operator. Acta Math. 121 (1968), 193–218

[12] PanY., Sogge C.: Oscillatory integrals associated to folding canonical relations. Colloq. Math. 60/61 (1990), 413–419

[13] ReznikovA.: Norms of geodesic restrictions on hyperbolic surfaces and representation theory. arXiv: math.AP/0403437, 2004

[14] Shnirelman A. I.: Ergodic properties of eigenfunctions. Uspehi Mat. Nauk 29 180 (1974), 181–182

[15] Sogge C.: Concerning the Lp norm of spectral clusters for second order elliptic operators on compact manifolds. J. Funct.Anal. 77 (1988), 123–138

[16] Sogge C.: Fourier Integrals in Classical Analysis. Cambridge Tracts in Mathematics 105. Cambridge University Press, Cambridge, 1993

[17] Sogge C., Zelditch S.: Riemannian manifolds with maximal eigenfunction growth. Duke Math. J. 114 (2002), 387–437

[18] Zelditch S.: Uniform distribution of eigenfunctions on compact thyperbolic surfaces. DukeMath. J. 55 (1987), 919–941

[19] X. Chen, An improvement on eigenfunction restriction estimates for compact boundaryless Rieman-nian manifolds with nonpositive sectional curvature, Trans. Amer. Math. Soc. 367 (2015) 4019–4039.

[20] Pierre H. B´erard, On the wave equation on a compact Riemannian manifold without conjugate points, Math. Z. 155 (1977), no. 3, 249–276. MR0455055 (56 #13295)

[21] Paul G¨unther, Einige S¨atze ¨uber das Volumenelement eines Riemannschen Raumes (German), Publ. Math. Debrecen 7 (1960), 78–93. MR0141058 (25 #4471)

[22] Jacques Hadamard, Lectures on Cauchy’s problem in linear partial differential equations, Dover Publications, New York, 1953. MR0051411 (14,474f)

[23] A. Hassell and M. Tacey, personal communication.

[24] L. Hörmander, The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis. (Reprint of the second (1990) edition), Springer-Verlag, Berlin, 2003. MR1996773
[25] Isaac Chavel, Riemannian geometry, 2nd ed., Cambridge Studies in Advanced Mathematics, vol. 98, Cambridge University Press, Cambridge, 2006. A modern introduction. MR2229062 (2006m:53002)
[26] A. Reznikov, Norms of geodesic restrictions for eigenfunctions on hyperbolic surfaces and representation theory, arXiv:math.AP/0403437.
[27] C.D. Sogge, Hangzhou Lectures on Eigenfunctions of the Laplacian, Ann. of Math. Stud., vol.188, Princeton University Press, Princeton, NJ, 2014.
[28] C. D. Sogge, Kakeya-Nikodym averages and Lp-norms of eigenfunctions, Tohoku Math. J. (2) 63 (2011), no. 4, 519–538. MR2872954
[29] Christopher D. Sogge, John A. Toth, and Steve Zelditch, About the blowup of quasimodes on Riemannian manifolds, J. Geom. Anal. 21 (2011), no. 1, 150–173, DOI 10.1007/s12220-010-9168-6. MR2755680 (2012c:58054)
[30] C. Sogge and S. Zelditch, On eigenfunction restriction estimates and L4-bounds for compact surfaces with nonpositive curvature.
[31] Elias M. Stein, Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy; Monographs in Harmonic Analysis, III. MR1232192 (95c:42002)
[32] V. M. BABIˇC AND V. F. LAZUTKIN, The eigenfunctions which are concentrated near a closed geodesic, (Russian) Zap. Nauˇcn. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI) 9 (1967), 15–25.
[33] J. BOURGAIN, Besicovitch type maximal operators and applications to Fourier analysis, Geom. Funct. Anal. 1 (1991), 147–187.
[34] J. BOURGAIN, Some new estimates on oscillatory integrals, Essays on Fourier analysis in honor of Elias M. Stein (Princeton, NJ, 1991), 83–12, Princeton Math. Ser. 42, Princeton Univ. Press, Princeton, NJ, 1995.
[35] J. BOURGAIN, Lp-estimates for oscillatory integrals in several variables, Geom. Funct. Anal. 1 (1991), 321–374.
[36] J. BOURGAIN, Geodesic restrictions and Lp-estimates for eigenfunctions of Riemannian surfaces, Linear and complex analysis, 27–35, Amer. Math. Soc. Tranl. Ser. 2, 226, Amer. Math. Soc., Providence, RI, 2009.
[37] L. CARLESON AND P. SJÖLIN, Oscillatory integrals and a multiplier problem for the disc, Sudia Math. 44 (1972), 287–299.
[38] Y. COLIN DE VERDIÈRE, Semi-classical measures and entropy [after Nalini Anantharaman and Stéphane Nonnenmacher], (English summary) Séminaire Bourbaki. Vol. 2006/2007, Astrisque No. 317 (2008), Exp. No. 978, ix, 393–414.
[39] A. CÓRDOBA, A note on Bochner-Riesz operators, Duke Math. J. 46 (1979), 505–511.
[40] J. J. DUISTERMAAT AND V. W. GUILLEMIN, The spectrum of positive elliptic operators and periodic bicharacteristics, Invent. Math. 29 (1975), 39–79.
[41] C. FEFFERMAN, A note on spherical summation operators, Israel J. Math. 15 (1973), 44–52.
[42] A. GREENLEAF AND A. SEEGER, Fourier integrals with fold singularities, J. Reine Angew. Math. 455 (1994), 35–56.

[43] D. GRIESER, Lp bounds for eigenfunctions and spectral projections of the Laplacian near concave boundaries, Ph. D. Thesis, University of California, Los Angeles, 1992.
[44] L. HÖRMANDER, Fourier integral operators. I, Acta Math. 127 (1971), 79–183.
[45] L. HÖRMANDER, Oscillatory integrals and multipliers on FLp, Ark. Math. II (1973), 1–11.
[46] V. IVRII, The second term of the spectral asymptotics for a Laplace-Beltrami operator on manifolds with boundary, (Russian) Funktsional. Anal. i Prilozhen. 14 (1980), 25–34.
[47] W. P. MINICOZZI AND C. D. SOGGE, Negative results for Nikodym maximal functions and related oscillatory integrals in curved space, Math. Res. Lett. 4 (1997), 221–237.
[48] G. MOCKENHAUPT, A. SEEGER AND C. D. SOGGE, Local smoothing of Fourier integral operators and Carleson-Sjölin estimates, J. Amer. Math. Soc. 6 (1993), 65–130.
[49] J. V. RALSTON, On the construction of quasimodes associated with stable periodic orbits, Comm. Math. Phys. 51 (1976), 219–242.
[50] Z. RUDNICK AND P. SARNAK, The behaviour of eigenstates of arithmetic hyperbolic manifolds, Comm. Math. Phys 161 (1994), 195–213.
[51] H. SMITH AND C. D. SOGGE, On the critical semilinear wave equation outside convex obstacles, J. Amer. Math. Soc. 8 (1995), 879–916.
[52] H. SMITH AND C. D. SOGGE, On the Lp norm of spectral clusters for compact manifolds with boundary, Acta Math. 198 (2007), 107–153.
[53] C. D. SOGGE, Oscillatory integrals and spherical harmonics, Duke Math. J. 53 (1986), 43–65.
[54] C. D. SOGGE, Concerning Nikodym-type sets in 3-dimensional curved spaces, J. Amer. Math. Soc. 12 (1999), 1–31.
[55] E. M. STEIN, Oscillatory integrals in Fourier analysis, Beijing lectures in harmonic analysis (Beijing, 1984), 307–355, Ann. of Math. Stud.,112, Princeton Univ. Press, Princeton, NJ, 1986.
[56] D. TATARU, On the regularity of boundary traces for the wave equation, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 26 (1998), 185–206.
[57] J. TOTH, L2-restriction bounds for eigenfunctions along curves in the quantum completely integrable case, Comm. Math. Phys. 288 (2009), 379–401.
[58] J. TOTH AND S. ZELDITCH, Lp norms of eigenfunctions in the completely integrable case, Ann. Henri Poincaré 4 (2003), 343–368.
[59] S. ZELDITCH, Mean Lindelöf hypothesis and equidistribution of cusp forms and Eisenstein series, J. Funct. Anal. 97 (1991), 1–49.
[60] A. ZYGMUND, On Fourier coefficients and transforms of two variables, Studia Math. 50 (1974), 189–201.
[61] P. H. B´erard, On the wave equation on a compact manifold without conjugate points, Math. Z. 155 (1977), 249–276.
[62] E. Cartan, Le¸cons sur la Geom´etrie de Espaces de Riemann (2nd ed.) Paris, Gauthier-Villars, 1946.
[63] J. Hadamard, Les surfaces `a courbures oppose´es et leurs g´eod´esiques. J. Math. Pures Appl. 4 (1898), 27–73.
[64] Hörmander, The analysis of linear partial differential operators. III. Pseudodifferential operators, Springer-Verlag, Berlin, 1985.

[65] P. Kröger, An extension of G¨unther's volume comparison theorem. Math. Ann. 329 (2004), 593–596.
[66] C. D. Sogge and S. Zelditch, "Concerning the L4 norms of typical eigenfunctions on compact surfaces", pp. 407–423 in Recent developments in geometry and analysis, edited by Y. Dong et al., Adv. Lect. Math. (ALM) 23, Int. Press, Somerville, MA, 2012. MR 3077213
[67] F. Spinu, The L4 norm of the Eisenstein series, Thesis, Princeton University 2003, www.math.jhu.edu/˜fspinu.
[68] H. von Mangolt, ¨Uber diejenigen Punkte auf positiv gekrummten Fl¨achen, welche die Eigenschaft haben, dass die von ihnen ausgehended geod¨atischen Linien nie aufhörn, k¨urzeste linien zu sien, J. Reine Angew. Math. 91 (1881), 23–52.
[69] M. Blair, C.D. Sogge, On Kakeya–Nikodym averages, Lp-norms and lower bounds for nodal sets of eigenfunctions in higher dimensions, J. Eur. Math. Soc. (JEMS) 17 (2015) 2513–2543.
[70] Bourgain, J. Estimates for cone multipliers, Operator Theory: Advances and Applications 77 (1995), 41{60.
[71] Bourgain, J.; Guth, L. Bounds on oscillatory integral operators based on multilinear estimates, Geom. Funct. Anal. 21 (2011), 1239{1295.
[72] Bruning, J. • Uber Knoten Eigenfunktionen des Laplace-Beltrami Operators, Math. Z. 158 (1978), 15{21.
[73] Chen, X.; Sogge, C.D. A few endpoint restriction estimates for eigenfunctions, arXiv:1210.7520.
[74] Colding, T.H.; Minicozzi, W.P. II Lower bounds for nodal sets of eigenfunctions, Comm. Math. Phys. 306 (2011), 777{784.
[75] Colin de Verdi_ere, Y.: Ergodicit_e et fonctions propres du laplacien, Comm. Math. Phys., 102 (1985), 497{502.
[76] Do Carmo, M. Riemannian geometry, Birkh • auser, Basel, Boston, Berlin, 1992.
[77] Dong, R.T. Nodal sets of eigenfunctions on Riemann surfaces, J. Di_erential Geom. 36 (1992), 493{506.
[78] Donnelly, H.; Fe_erman, C. Nodal sets of eigenfunctions on Riemannian manifolds, Invent. Math. 93 (1988), 161{183.
[79] Donnelly, H.; Fe_erman, C. Nodal sets for eigenfunctions of the Laplacian on surfaces, J. Amer. Math. Soc. 3(2) (1990), 332{353.
[80] Han, Q; Lin F.H. Nodal sets of solutions of Elliptic Di_erential Equations, book in preparation (online at http://www.nd.edu/qhan/nodal.pdf).
[81] Hardt, R.; Simon, L. Nodal sets for solutions of elliptic equations, J. Di_erential Geom. 30 (1989), 505{522.
[82] A. Hassell, M. Tacy, Improvement of eigenfunction estimates on manifolds of nonpositive curvature, Forum Math. 27(3) (2015) 1435–1451.
[83] Hezari, H; Sogge, C.D. A natural lower bound for the size of nodal sets, Analysis and PDE 5 (2012), 1133{1137.
[84] Lee, S. Linear and bilinear estimates for oscillatory integral operators related to restriction to hypersurfaces. J. Funct. Anal. 241 (2006), 56{98.
[85] Sogge, C.D.; Zelditch, S. Lower bounds on the Hausdorff measure of nodal sets, Math. Res. Lett. 18 (2011), 25{37.
[86] Sogge, C.D.; Zelditch, S. Lower bounds on the Hausdorff measure of nodal sets II, arXiv:1208.2045.

[87] Tao, T. A sharp bilinear restriction estimates for paraboloids. Geom. Funct. Anal. 13 (2003), 1359{1384.
[88] Tao, T.; Vargas, A. A bilinear approach to cone multipliers I: Restriction estimates. Geom. Funct. Anal. 10 (2000), 185{215.
[89] Tao, T.; Vargas, A.; Vega, L. A bilinear approach to the restriction and Kakeya conjectures. J. Amer. Math. Soc. 11 (1998), 967{1000.
[90] Wolff, T. A sharp cone restriction estimate. Ann. of Math. 153 (2001), 661{698.
[91] Yau, S.T. Survey on partial differential equations in differential geometry. Seminar on Differential Geometry, pp. 3{71, Ann. of Math. Stud., 102, Princeton Univ. Press, Princeton, N.J., 1982.
[92] Zworski, M. Semiclassical Analysis, Graduate Studies in Mathematics 138, Amer. Math. Soc., Providence, 2012.
[93] H. Berestycki and L. Rossi, On the principal eigenvalue of elliptic operators in RN and applications, J. Eur. Math. Soc. (JEMS) 8 (2006), no. 2, 195–215. MR MR2239272 (2007d:35076)
[94] Agmon, S.: On positive solutions of elliptic equations with periodic coefficients in RN, spectral results and extensions to elliptic operators on Riemannian manifolds. In: Proc. Internat. Conf. on Differential Equations, I. Knowles and R. Lewis (eds.), North-Holland Math. Stud. 92, North-Holland, Amsterdam, 7–17 (1984) Zbl 0564.35033 MR 0799327
[95] Avellaneda, M., Lin, F.-H.: Un th´eor`eme de Liouville pour des ´equations elliptiques `a coefficients p´eriodiques. C. R. Acad. Sci. Paris S´er. I Math. 309, 245–250 (1989) Zbl 069135022 MR 1010728
[96] Berestycki, H.: Le nombre de solutions de certains probl`emes semi-lin´eaires elliptiques. J. Funct. Anal. 40, 1–29 (1981) Zbl 0452.35038 MR 0607588
[97] Berestycki, H., Hamel, F., Roques, L.: Analysis of the periodically fragmented environment model: I—Influence of periodic heterogeneous environment on species persistence. J. Math. Biology 51, 75–113 (2005) Zbl 1066.92047
[98] Berestycki, H., Hamel, F., Roques, L.: Analysis of the periodically fragmented environment model: II—Biological invasions and pulsating travelling fronts. J. Math. Pures Appl. (9) 84, 1101–1146 (2005) Zbl pre02228673 MR 2155900
[99] Berestycki, H., Hamel, F., Rossi, L.: Liouville type results for semilinear elliptic equations in unbounded domains. Preprint (2006)
[100] Berestycki, H., Nirenberg, L., Varadhan, S. R. S.: The principal eigenvalue and maximum principle for second-order elliptic operators in general domains. Comm. Pure Appl. Math. 47, 47–92 (1994) Zbl 0806.35129 MR 1258192
[101] Bochner, S.: A new approach to almost periodicity. Proc. Nat. Acad. Sci. USA 48, 2039–2043 (1963) Zbl 0112.31401 MR 0145283
[102] Engl¨ander, J., Kyprianou, A. E.: Local extinction versus local exponential growth for spatial branching processes. Ann. Probab. 32, 78–99 (2004) Zbl 1056.60083 MR 2040776
[103] Engl¨ander, J., Pinsky, R. G.: On the construction and support properties of measure-valued diffusions on D _ Rd with spatially dependent branching. Ann. Probab. 27, 684–730 (1999) Zbl 0979.60078 MR 1698955
[104] Gilbarg, D., Trudinger, N. S.: Elliptic Partial Differential Equations of Second Order. Springer, Berlin (1983) Zbl 0562.35001 MR 0737190
[105] Hansen, L. P., Scheinkman, J. A.: Long term risk: an operator approach. Preprint.

[106] Kolmogorov, A. N., Petrovskiĭ, I. G., Piskunov, N. S.: Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique. Bull. Univ. d'Etat à Moscou (Byul. Moskov. Gos. Univ.), Série Internat. A1, 1–26 (1937) Zbl 0018.32106
[107] Krasnosel'skij, M. A., Lifshits, Je. A., Sobolev, A. V.: Positive Linear Systems. The Method of Positive Operators, Heldermann, Berlin (1989) Zbl 0674.47036 MR 1038527
[108] Krein, M. G., Rutman, M. A.: Linear operators leaving invariant a cone in a Banach space. Uspekhi Mat. Nauk. 3, no. 1, 3–95 (1948) (in Russian); English transl. in Amer. Math. Soc. Transl. 1950, no. 26 MR 0038008
[109] Krylov, N. V., Safonov, M. V.: An estimate of the probability that a diffusion process hits a set of positive measure. Soviet Math. Dokl. 20, 253–255 (1979) Zbl 0459.60067
[110] Krylov, N. V., Safonov, M. V.: A certain property of solutions of parabolic equations with measurable coefficients. Izv. Akad. Nauk SSSR 44, 161–175 (1980) Zbl 0464.35035 MR 0563790
[111] Kuchment, P., Pinchover, Y.: Integral representations and Liouville theorems for solutions of periodic elliptic equations. J. Funct. Anal. 181, 402–446 (2001) Zbl 0986.35028 MR 1821702
[112] Moser, J., Struwe, M.: On a Liouville-type theorem for linear and nonlinear elliptic differential equations on a torus. Bol. Soc. Brasil. Mat. (N.S.) 23, 1–20 (1992) Zbl 0787.35028 MR 1203171
[113] Pinsky, R. G.: Positive Harmonic Functions and Diffusion. Cambridge Univ. Press (1995) Zbl 0858.31001 MR 1326606
[114] Pinsky, R. G.: Second order elliptic operators with periodic coefficients: criticality theory, perturbations, and positive harmonic functions. J. Funct. Anal. 129, 80–107 (1995) Zbl 0826.35030 MR 1322643
[115] Pinsky, R. G.: Transience, recurrence and local extinction properties of the support for supercritical finite measure-valued diffusions. Ann. Probab. 24, 237–267 (1996) Zbl 0854.60087 MR 1387634
[116] Protter, M. H., Weinberger, H. F.: Maximum Principles in Differential Equations. Prentice-Hall, Englewood Cliffs, NJ (1967) Zbl 0153.13602 MR 0219861
[117] Safonov, M. V.: Harnack's inequality for elliptic equations and the Hölder property of their solutions. J. Soviet Math. 21, 851–863 (1983) Zbl 0511.35029
[118] J. Coville, On a simple criterion for the existence of a principal eigenfunction of some nonlocal operators, J. Differential Equations 249 (2010), pp. 2921-2953.
[119] Giovanni Alberti, Giovanni Bellettini, A nonlocal anisotropic model for phase transitions. I. The optimal profile problem, Math. Ann. 310 (3) (1998) 527–560.
[120] Peter W. Bates, Adam Chmaj, An integrodifferential model for phase transitions: stationary solutions in higher space dimensions, J. Stat. Phys. 95 (5–6) (1999) 1119–1139.
[121] Peter W. Bates, Paul C. Fife, Xiaofeng Ren, Xuefeng Wang, Traveling waves in a convolution model for phase transitions, Arch. Ration. Mech. Anal. 138 (2) (1997) 105–136.
[122] Peter W. Bates, Guangyu Zhao, Existence, uniqueness and stability of the stationary solution to a nonlocal evolution equation arising in population dispersal, J. Math. Anal. Appl. 332 (1) (2007) 428–440.

[123] Felix E. Browder, On the spectral theory of elliptic differential operators. I, Math. Ann. 142 (1960/1961) 22–130.
[124] Michael L. Cain, Brook G. Milligan, Allan E. Strand, Long-distance seed dispersal in plant populations, Am. J. Bot. 87 (9) (2000) 1217–1227.
[125] Emmanuel Chasseigne, Manuela Chaves, Julio D. Rossi, Asymptotic behavior for nonlocal diffusion equations, J. Math. Pures Appl. (9) 86 (3) (2006) 271–291.
[126] Xinfu Chen, Existence, uniqueness, and asymptotic stability of traveling waves in nonlocal evolution equations, Adv. Differential Equations 2 (1) (1997) 125–160.
[127] Adam J.J. Chmaj, Xiaofeng Ren, The nonlocal bistable equation: stationary solutions on a bounded interval, Electron. J. Differential Equations (2002), No. 02, 12 pp. (electronic).
[128] James S. Clark, Why trees migrate so fast: Confronting theory with dispersal biology and the paleorecord, Amer. Natural. 152 (2) (1998) 204–224.
[129] C. Cortázar, J. Coville, M. Elgueta, S. Martínez, A nonlocal inhomogeneous dispersal process, J. Differential Equations 241 (2) (2007) 332–358.
[130] Carmen Cortazar, Manuel Elgueta, Julio D. Rossi, A nonlocal diffusion equation whose solutions develop a free boundary, Ann. Henri Poincaré 6 (2) (2005) 269–281.
[131] Jérôme Coville, On uniqueness and monotonicity of solutions of non-local reaction diffusion equation, Ann. Mat. Pura Appl. (4) 185 (3) (2006) 461–485.
[132] Jérôme Coville, Remarks on the strong maximum principle for nonlocal operators, Electron. J. Differential Equations (2008), No. 66, 10 pp.
[133] Jérôme Coville, Harnack's inequality for some nonlocal equations and application, preprint du MPI, February 2008.
[134] Jérôme Coville, Travelling fronts in asymmetric nonlocal reaction diffusion equation: The bistable and ignition case, preprint du CMM, July 2006.
[135] Jérôme Coville, Juan Dávila, Salomé Martínez, Existence and uniqueness of solutions to a nonlocal equation with monostable nonlinearity, SIAM J. Math. Anal. 39 (5) (2008) 1693–1709.
[136] Jérôme Coville, Juan Dávila, Salomé Martínez, Nonlocal anisotropic dispersal with monostable nonlinearity, J. Differential Equations 244 (12) (2008) 3080–3118.
[137] Jérôme Coville, Louis Dupaigne, Propagation speed of travelling fronts in non local reaction–diffusion equations, Nonlinear Anal. 60 (5) (2005) 797–819.
[138] Jerome Coville, Louis Dupaigne, On a non-local equation arising in population dynamics, Proc. Roy. Soc. Edinburgh Sect. A 137 (4) (2007) 727–755.
[139] A. De Masi, T. Gobron, E. Presutti, Travelling fronts in non-local evolution equations, Arch. Ration. Mech. Anal. 132 (2) (1995) 143–205.
[140] E. Deveaux, C. Klein, Estimation de la dispersion de pollen à longue distance à l'echelle d'un paysage agicole : une approche expérimentale, Publication du Laboratoire Ecologie, Systématique et Evolution, 2004.
[141] Monroe D. Donsker, S.R.S. Varadhan, On a variational formula for the principal eigenvalue for operators with maximum principle, Proc. Natl. Acad. Sci. USA 72 (1975) 780–783.
[142] D.E. Edmunds, A.J.B. Potter, C.A. Stuart, Non-compact positive operators, Proc. R. Soc. London Ser. A 328 (1572) (1972) 67–81.
[143] Lawrence C. Evans, Partial Differential Equations, Grad. Stud. Math., vol. 19, American Mathematical Society, Providence, RI, 1998.

[144] Paul C. Fife, Mathematical Aspects of Reacting and Diffusing Systems, Lecture Notes in Biomath., vol. 28, Springer-Verlag, Berlin, 1979.
[145] Paul C. Fife, An integrodifferential analog of semilinear parabolic PDEs, in: Partial Differential Equations and Applications, in: Lect. Notes Pure Appl. Math., vol. 177, Dekker, New York, 1996, pp. 137–145.
[146] Jorge García-Melián, Julio D. Rossi, On the principal eigenvalue of some nonlocal diffusion problems, J. Differential Equations 246 (1) (2009) 21–38.
[147] V. Hutson, S. Martinez, K. Mischaikow, G.T. Vickers, The evolution of dispersal, J. Math. Biol. 47 (6) (2003) 483–517.
[148] Jan Medlock, Mark Kot, Spreading disease: integro-differential equations old and new, Math. Biosci. 184 (2) (2003) 201–222.
[149] J.D. Murray, Mathematical Biology, second ed., Biomathematics, vol. 19, Springer-Verlag, Berlin, 1993.
[150] Roger D. Nussbaum, The radius of the essential spectrum, Duke Math. J. 37 (1970) 473–478.
[151] Roger D. Nussbaum, Yehuda Pinchover, On variational principles for the generalized principal eigenvalue of second order elliptic operators and some applications, J. Anal. Math. 59 (1992) 161–177, Festschrift on the occasion of the 70th birthday of Shmuel Agmon.
[152] Carlo Pucci, Maximum and minimum first eigenvalues for a class of elliptic operators, Proc. Amer. Math. Soc. 17 (1966) 788–795.
[153] Frank M. Schurr, Ofer Steinitz, Ran Nathan, Plant fecundity and seed dispersal in spatially heterogeneous environments: models, mechanisms and estimation, J. Ecol. 96 (4) (2008) 628–641.
[154] Eberhard Zeidler, Nonlinear Functional Analysis and Its Applications. I: Fixed-Point Theorems, Springer-Verlag, New York, 1986, translated from the German by Peter R. Wadsack.
[155] J. Bourgain, P. Shao, C. D. Sogge and X. Yao, On Lp-resolvent estimates and the density of eigenvalues for compact Riemannian manifolds, preprint (2012), http://arxiv. org/ abs/ 1204. 3927.
[156] X. Chen, C.D. Sogge, A few endpoint geodesic restriction estimates for eigenfunctions, Comm. Math. Phys. 329(3) (2014) 435–459.
[157] C. Guillarmou, A. Hassell and A. Sikora, Restriction and spectral multiplier theorems on asymptotically conic manifolds, preprint (2012), http://arxiv.org/ abs/1012.3780.
[158] H. Koch, D. Tataru and M. Zworski, Semiclassical Lp estimates, Ann. Henri Poincaré 8 (2007), 885–916.
[159] M. Tacy, Semiclassical Lp estimates of quasimodes on submanifolds, Comm. Partial Differential Equations 35 (2010), 1538–1562.
[160] Y. Xi, C. Zhang, Improved critical eigenfunction restriction estimates on Riemannian surfaces with nonpositive curvature, Comm. Math. Phys. (2016), http://dx.doi.org/10.1007/ s00220-016-2721-9, arXiv: 1603.01601.
[161] M.D. Blair, On logarithmic improvements of critical geodesic restriction bounds in the presence of nonpositive curvature, arXiv:1607.03174.
[162] M.D. Blair, C.D. Sogge, Concerning Toponogov’s Theorem and logarithmic improvement of esti-mates of eigenfunctions, arXiv:1510.07726.
[163] N. Burq, P. Gérard, N. Tzvetkov, Restriction of the Laplace–Beltrami eigenfunctions to submani-folds, Duke Math. J. 138 (2007) 445–486.

[164] H. Hezari, Quantum ergodicity and Lpnorms of restrictions of eigenfunctions, arXiv:1606.08066.
[165] H. Hezari, G. Rivière, Lpnorms, nodal sets, and quantum ergodicity, Adv. Math. 290 (2016) 938–966.
[166] J. Metcalfe, M. Taylor, Nonlinear waves on 3d hyperbolic space, Trans. Amer. Math. Soc. 363(7) (2011) 3489–3529.
[167] D.H. Phong, E.M. Stein, Radon transforms and torsion, Int. Math. Res. Not. IMRN 1991(4) (1991) 49–60.
[168] C.D. Sogge, Improved critical eigenfunction estimates on manifolds of nonpositive curvature, arXiv:1512.03725.
[169] M. Taylor, Partial Differential Equations II: Qualitative Studies of Linear Equations, vol. 116, second edition, Springer, New York, 2011.
[170] M. Blair, C.D. Sogge, Refined and microlocal Kakeya–Nikodym bounds for eigenfunctions in two dimensions, Anal. PDE 8 (2015) 747–764.
[171] J. Cilleruelo and A. Córdoba, "Trigonometric polynomials and lattice points", Proc. Amer. Math. Soc. 115:4 (1992), 899–905. MR 92j:11116 Zbl 0777.11035
[172] A. Gray, Tubes, 2nd ed., Progress in Mathematics 221, Birkhäuser, Basel, 2004. MR 2004j:53001 Zbl 1048.53040
[173] C. D. Sogge and S. Zelditch, "A note on L p-norms of quasi-modes", preprint, 2014.
[174] Changxing Miao a, Christopher D. Sogge b, Yakun Xi b,∗, Jianwei Yang c,d, Bilinear Kakeya–Nikodym averages of eigenfunctions on compact Riemannian surfaces ☆, Journal of Functional Analysis ••• (••••) •••–•••.
[175] M. Blair, C.D. Sogge, Refined and microlocal Kakeya–Nikodym bounds of eigenfunctions in higher dimensions, arXiv:1510.07724.
[176] W. Shen and X. Xie, On principal spectrum points/principal eigenvalues of nonlocal dispersal operators and applications, Discrete and Continuous Dynamical Systems 35 (2015), no. 4, 1665–1696.
[177] P. Bates and F. Chen, Spectral analysis of traveling waves for nonlocal evolution equations, SIAM J. Math. Anal. 38 (2006), pp. 116-126.
[178] Reinhard B¨urger, Perturbations of positive semigroups and applications to population genetics, Math. Z. 197 (1988), pp. 259-272.
[179] F. Chen, Stability and uniqueness of traveling waves for system of nonlocal evolution equations with bistable nonlinearity, Discrete Contin. Dyn. Syst. 24 (2009), pp. 659-673.
[180] C. Cortazar, M. Elgueta, J. D. Rossi, and N. Wolanski, How to approximate the heat equation with Neumann boundary conditions by nonlocal diffusion problems, Arch. Ration. Mech. Anal. 187 (2008), pp. 137-156.
[181] C. Cortazar, M. Elgueta, and J. D. Rossi, Nonlocal diffusion problems that approximate the heat equation with Dirichlet boundary conditions, Israel J. of Math., 170 (2009), pp. 53-60.
[182] Edmunds, D. E. and Evans, W. D., Spectral theory and differential operators, The Clarendon Press Oxford University Press, New York, 1987.
[183] P. Fife, Some nonclassical trends in parabolic and parabolic-like evolutions, Trends in nonlinear analysis, 153-191, Springer, Berlin, 2003.
[184] M. Grinfeld, G. Hines, V. Hutson, K. Mischaikow, and G. T. Vickers, Non-local dispersal, Differential Integral Equations, 18 (2005), pp. 1299-1320.

[185] G. Hetzer, T. Nguyen, and W. Shen, Coexistence and extinction in the Volterra-Lotka competition model with nonlocal dispersal, Communications on Pure and Applied Analysis, 11 (2012), pp. 1699-1722.
[186] G. Hetzer, T. Nguyen, and W. Shen, Effects of small variation of the reproduction rate in a two species competition model, Electron. J. Differential Equations , (2010), No. 160, 17 pp.
[187] G. Hetzer, W. Shen, and A. Zhang, Effects of spatial variations and dispersal strategies on principal eigenvalues of dispersal operators and spreading speeds of monostable equations, Rocky Mountain Journal of Mathematics, 43 (2013), pp. 489-513.
[188] V. Hutson, W. Shen and G.T. Vickers, Spectral theory for nonlocal dispersal with periodic or almostperiodic time dependence, Rocky Mountain Journal of Mathematics 38 (2008), pp. 1147-1175.
[189] C.-Y. Kao, Y. Lou, and W. Shen, Random dispersal vs non-Local dispersal, Discrete and Continuous Dynamical Systems, 26 (2010), no. 2, pp. 551-596
[190] C.-Y. Kao, Y. Lou, and W. Shen, Evolution of mixed dispersal in periodic environments, Discrete and Continuous Dynamical Systems, Series B, 17 (2012), pp. 2047-2072.
[191] L. Kong andW. Shen, Positive stationary solutions and spreading speeds of KPP equations in locally spatially inhomogeneous media, Methods and Applications of Analysis, 18 (2011), pp. 427-456.
[192] W.-T. Li, Y.-J. Sun, Z.-C. Wang, Entire solutions in the Fisher-KPP equation with nonlocal dispersal, Nonlinear Analysis, 11 (2010), pp. 2302-2313.
[193] X. Liang, X. Lin, and H. Matano, A variational problem associated with the minimal speed of travelling waves for spatially periodic reaction-diffusion equations, Trans. Amer. Math. Soc., 362 (2010), no. 11, pp. 5605-5633.
[194] G. Lv and M. Wang, Existence and stability of traveling wave fronts for nonlocal delayed reaction diffusion systems, J. Math. Anal. Appl. 385 (2012), pp. 1094-1106.
[195] P. Meyre-Nieberg, Banach Lattices, Springer-Verlag, 1991.
[196] S. Pan, W.-T. Li, and G. Lin, Existence and stability of traveling wavefronts in a nonlocal diffusion equation with delay, Nonlinear Analysis: Theory, Methods & Applications, 72 (2010), pp. 3150-3158.
[197] Nar Rawal and W. Shen, Criteria for the Existence and Lower Bounds of Principal Eigenvalues of Time Periodic Nonlocal Dispersal Operators and Applications, J. Dynam. Differential Equations, 24 (2012), pp. 927-954.
[198] W. Shen and G. T. Vickers, Spectral theory for general nonautonomous/random dispersal evolution operators, J. Differential Equations, 235 (2007), pp. 262-297.
[199] W. Shen and X. Xie, Approximations of random dispersal operators/equations by nonlocal dispersal operators/equations, in preparation.
[200] W. Shen and A. Zhang, Spreading speeds for monostable equations with nonlocal dispersal in space periodic habitats, Journal of Differential Equations 249 (2010), pp. 747-795.
[201] W. Shen and A. Zhang, Traveling wave solutions of spatially periodic nonlocal monostable equations, Communications on Applied Nonlinear Analysis, 19 (2012), pp. 73-101.

[202] W. Shen and A. Zhang, Stationary solutions and spreading speeds of nonlocal monostable equations in space periodic habitats, Proc. AMS, 140 (2012), pp. 1681-1696.
[203] Henri Berestycki ∗, J´er^ome Coville †, Hoang-Hung Vo ‡, On the definition and the properties of the principal eigenvalue of some nonlocal operators, J. Funct. Anal. (2016), http://dx.doi.org/10.1016/j.jfa.2016.05.017.
[204] Marianne Akian, St´ephane Gaubert, and Roger Nussbaum, A collatz-wielandt characterization of the spectral radius of order-preserving homogeneous maps on cones, arXiv preprint arXiv:1112.5968 (2014).
[205] H. Berestycki, J. Coville, and H.-H. Vo, Persistence criteria for populations with non-local dispersion, ArXiv e-prints (2014).
[206] H. Berestycki and G. Nadin, Spreading speeds for one-dimensional monostable reaction-diffusion equations, J. Math. Phys. 53 (2012), 115619.
[207] H Berestycki, J M Roquejoffre, and L. Rossi, The periodic patch model for population dynamics with fractional diffusion, Discrete Contin. Dyn. Syst. S 4 (2011), no. 1, 1–13.
[208] Reaction-diffusion equations for population dynamics with forced speed. I. The case of the whole space, Discrete Contin. Dyn. Syst. 21 (2008), no. 1, 41–67. MR 2379456 (2009f:35173)
[209] Henri Berestycki and Luca Rossi, Generalizations and properties of the principal eigenvalue of elliptic operators in unbounded domains, Communications on Pure and Applied Mathematics 68 (2015), no. 6, 1014–1065.
[210] J. Bourgain, H. Brezis, and P. Mironescu, Another look at sobolev spaces, in Optimal Control and Partial Differential Equations, Citeseer, 2001.
[211] H. Brezis, How to recognize constant functions. connections with sobolev spaces, Russian Mathematical Surveys 57 (2002), no. 4, 693.
[212] , Functional analysis, sobolev spaces and partial differential equations, Universitext Series, Springer, 2010.
[213] Xavier Cabr´e and Yannick Sire, Nonlinear equations for fractional laplacians, i: Regularity, maximum principles, and hamiltonian estimates, Annales de l'Institut Henri Poincare (C) Non Linear Analysis 31 (2014), no. 1, 23–53.
[214] L. Caffarelli and L. Silvestre, Regularity theory for fully nonlinear integro-differential equations, Comm. Pure Appl. Math. 62 (2009), no. 5, 597–638. MR 2494809 (2010d:35376)
[215] R. S. Cantrell and C. Cosner, Diffusive logistic equations with indefinite weights: population models in disrupted environments, Proc. Roy. Soc. Edinburgh, Section: A Mathematics 112 (1989), 293–318.
[216] R. S. Cantrell and C. Cosner, The effects of spatial heterogeneity in population dynamics, J. Math. Biol. 29 (1991), no. 4, 315–338. MR MR1105497 (92b:92049)
[217] , On the effects of spatial heterogeneity on the persistence of interacting species, J.Math. Biol. 37 (1998), no. 2, 103–145. MR MR1649516 (99m:92025)
[218] Lothar Collatz, Einschließungssatz f¨ur die charakteristischen zahlen von matrizen, Mathematische Zeitschrift 48 (1942), no. 1, 221–226.
[219] A. Cornea, Finiteness principle and Harnack principle, ICPT '91 (Amersfoort, 1991), Kluwer Acad. Publ., Dordrecht, 1994, pp. 203–216. MR 1293764 (95m:31006)

[220] A. Cornea and J. Vesely, Martin compactification for discrete potential theory and the mean value property, Potential Analysis 4 (1995), 547–569, 10.1007/BF01048068.
[221] Harnack type inequality for positive solution of some integral equation, Annali di Matematica Pura ed Applicata 191 (2012), no. 3, 503–528 (English).
[222] Singular measure as principal eigenfunction of some nonlocal operators, Applied Mathematics Letters (2013), no. 26, 831–835.
[223] Nonlocal refuge model with a partial control, Discrete and Continuous Dynamical Systems 35 (2015), no. 4, 1421–1446.
[224] J. Coville, J. Davila, and S. Martinez, Pulsating fronts for nonlocal dispersion and kpp nonlinearity, Ann. I. H. Poincare – AN (2013), no. 30, 179–223.
[225] J. Coville and L. Rossi, Relations between λ1 and λ_1 for nonlocal operators in 1d, private communication.
[226] Shmuel Friedland, Characterizations of the spectral radius of positive operators, Linear Algebra and its Applications 134 (1990), 93–105.
[227] J. Garcia-Melian and J. D. Rossi, A logistic equation with refuge and nonlocal diffusion, Commun. Pure Appl. Anal. 8 (2009), no. 6, 2037–2053. MR 2552163 (2010k:45002)
[228] L. I. Ignat, J. D. Rossi, and A. San Antolin, Lower and upper bounds for the first eigenvalue of nonlocal diffusion problems in the whole space, Journal of Differential Equations 252 (2012), no. 12, 6429 – 6447.
[229] C-Y. Kao, Y. Lou, and W. Shen, Random dispersal vs. nonlocal dispersal, Discrete and Continuous Dynamical Systems 26 (2010), no. 2, 551–596.
[230] Samuel Karlin, Positive operators, Journal of Mathematics and Mechanics 8 (1959), no. 6.
[231] , The existence of eigenvalues for integral operators, Transactions of the American Mathematical Society 113 (1964), no. 1, pp. 1–17 (English).
[232] K. Kawasaki and N. Shigesada, Biological invasions: Theory and practice, Oxford University Press, 1997.
[233] 1920 Krasnosel'skii, M. A. (Mark Aleksandrovich), Integral operators in spaces of summable functions, Leyden : Noordhoff International Publishing, 1976, 1976.
[234] Bas Lemmens and Roger Nussbaum, Nonlinear perron-frobenius theory, no. 189, Cambridge University Press, 2012.
[235] F. Lutscher, E. Pachepsky, and M. A. Lewis, The effect of dispersal patterns on stream populations, SIAM Rev. 47 (2005), no. 4, 749–772 (electronic). MR MR2212398 (2006k:92082)
[236] G. Nadin, Existence and uniqueness of the solution of a space–time periodic reaction–diffusion equation, Journal of Differential Equations 249 (2010), no. 6, 1288 – 1304.
[237] Gregoire Nadin, The principal eigenvalue of a space–time periodic parabolic operator, Annali di Matematica Pura ed Applicata 188 (2009), no. 2, 269–295.
[238] R. Pinsky, Second order elliptic operator with periodic coefficients: criticality theory, perturbations, and positive harmonic functions, Journal of Functional Analysis 129 (1995), 80–107.
[239] A. C Ponce, An estimate in the spirit of poincar´e's inequality, J. Eur. Math. Soc.(JEMS) 6 (2004), no. 1, 1–15.
[240] , A new approach to sobolev spaces and connections to Γ-convergence, Calculus of Variations and Partial Differential Equations 19 (2004), no. 3, 229–255.

[241] Luca Rossi, Liouville type results for periodic and almost periodic linear operators, 26 (2009), no. 6, 2481–2502.
[242] HH Schaefer, A minimax theorem for irreducible compact operators inl p-spaces, Israel Journal of Mathematics 48 (1984), no. 2-3, 196–204.
[243] Jinggang Tan and Jingang Xiong, A harnack inequality for fractional laplace equations with lower order terms, Discrete and Continuous Dynamical Systems A 31 (2011), no. 3, 975–983.
[244] P. Turchin, Quantitative analysis of movement: Measuring and modeling population redistribution in animals and plants, Sinauer Associates, 1998.
[245] Helmut Wielandt, Unzerlegbare, nicht negative matrizen, Mathematische Zeitschrift 52 (1950), no. 1, 642–648.
[246] Shawgy Hussein and Kawther Bashier Mohammed Al-Hussain, Eigenfunction Restriction Estimates on Riemannian Manifolds and Bilinear Kakeya–Nikodym Averages with Principal Eigenvalue of Nonlocal Operators, Ph.D. Thesis Sudan University of Science and Technology, Sudan 2020.