Sudan University of Science and Technology
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The $(G'/G)-$expansion method for solving the time dependent boundary layer

طريقة مفكوك $(G'/G)-$ لحل الطبقة الحدية المعتمدة علي الزمن

A Thesis Submitted in Fulfillment for the Degree of Ph.D in Mathematics

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February 12, 2020

Dedication

This research is dedicated to my parents, my brothers and to my all family.

Acknowledgments

Praise be to Allah, who has helped me to complete my research and blessed me with the grace of health and wellness.
My thanks, gratitude and profound appreciation to my Supervisor: prof. Osam Mohammed ELMakki for his perseverance as a supervisor over time and also Dr: Emad eldeen Abd Allah, who stood beside as a brother with his knowledge, Guidance bright until the completion of this research .
I am grateful to all my family who helped and motivated me to overcome the difficulties and their unlimited support .
I must also thank many brothers, friends, colleagues and others for their contribution and good support.
Special gratitude are due to my beloved mother for her understanding encouraging, spiritual and emotional support, love and unconditional care.

Abstract

The problem of the time dependent boundary layer is important for its applications in several areas, and many methods have been used to solve it. The objective of this research is to apply the (G'/G) -expansion method to this problem, it is shown that the proposed method is direct, effective and can be used for many other nonlinear evolution equations in mathematical physics. It represents the main objective of the problem itself. From the solutions we have found, we observed several for the wave solutions of nonlinear partial differential equations hers presented (G'/G) -expansion method. As a result, we obtained several new exact solutions including hyperbolic functions, the trigonometric functions and the fractional functions which might have significant impact on future researches. The obtained solutions with free parameters may be important to explain some physical phenomena. It is shown that the performance of this method is productive, effective and well-built mathematical tool for solving nonlinear evolution equations most from the methods used by former researchers in the field. We have verified all the obtained solutions with the aid of Maple.

الخلاصة

تعد مشكلة الطبقة الحدية المعتمدة علي الزمن مهمة لتطبيقاتها في العديد من المجالات وقد تم استخدام العديد من الطرق لحلها. الهدف من هذا البحث هو تطبيق طريقة المفكوك $-(G'/G)$ علي هذه المشكلة وحيث تبين أن الطريقة المقترحة مباشرة وفعّالة ويمكن استخدامها في العديد من معادلات التطور غير الخطية الأخرى في الفيزياء الرياضية وانه يمثل الهدف الرئيسي للمشكلة نفسها. من الحلول التي تم الحصول عليها لوُحظ تعدد للحلول الموجية من المعادلات التفاضلية غير الخطية لها عبر طريقة المفكوك $-(G'/G)$. نتيجة لذلك وتحصلنا علي حلول دقيقة جديدة ومتعددة بما في ذلك الدوال الزائدية والدوال المثلثية والدوال الكسرية التي قد يكون لها تأثير كبير علي الأبحاث المستقبلية. الحلول التي تم الحصول عليها مع المعلمات الحرة قد تكون مهمة لشرح بعض الظواهر الفيزيائية. من الواضح أن أداء هذه

الطريقة مُثمر وفعّال وأداة رياضية جيّدة مبنية لحل معادلات التطور غير الخطية أكثر من الأساليب المستخدمة من قِبل الباحثين السابقين في هذا المجال. لقد تحققنا من جميع الحلول التي تم الحصول عليها بمساعدة مابل.

Contents

Chapter 1

The Generalized (G'/G)-Expansion Method

1.1 Introduction

In mathematics, a partial differential equation (PDE) is a differential equation that contains unknown multi variable functions and their partial derivatives. (A special case are ordinary differential equations (ODEs), which deal with functions of a single variable and their derivatives).
PDEs are used to formulate problems involving functions of several variables, and are either solved by hand, or used to create a relevant computer model.
In order to apply mathematical method to physical or real life problem , we must formulate the problem in mathematical model for problem . non-linear wave phenomena appears in various sci-

entific and engineering fields such as sound, heat, electrostatics, electrodynamics, fluid dynamics, elasticity, or quantum mechanics , plasma physics , optical fibers ,biophysics , geochemistry ,propagation of shallow water waves ,high-energy physics , condensed matter physics , elastic media , biology solid state physics , chemical kinematics , chemical physics and so on .

This is also noticed to arise in engineering , chemical and biological applications .

The application of non-linear traveling waves has been brought prosperity in the field of applied science.

The analytical solutions of non-linear PDEs play an important role in non-linear science and engineering [9].

In order to understand better, as well as further applications in practical life , it is important to generate exact traveling wave solution .

In the recent past , a diverse group of scientists presented a variety of methods to construct analytical and numerical solution. For instance , the Hirotas bilinear transformation method [1], the truncated Painleve expansion method [2],the Backlund transformation method [3], the Weirstrass elliptic function method [4], the inverse scattering method [5], the tanh-coth method [6, 7], the Riccati equation method [8, 10], the Jacobi elliptic function expansion method [11], the F-expansion method [12, 13], the Exp-function method [14, 15], the sine-cosine method [16] and others [17, 18]. Recently, Wang et al. [19] presented one of the powerful methods and called the (G'/G) expansion method for constructing traveling wave solutions of some non-linear evolution equations (NLEEs).

In this method, they implemented the second order linear ordinary differential equation (ODE) .

The history of the korteweg - de Varies (KdV) equation started with experiments,by John Scott Russel in 1843, when he observed solitary waves on the Union Canal. This observation had no theoretical explanation at that time . followed by theoretical investigations by Lord Rayleigh Boussesq around in 1870. Finally in 1895 Diederick Korteweg and his student Gustav de Varies derived a nonlinear PDE that is now called the KdV equation.

The KdV equation was not studied much after this until Zabusky and Kruskal 1965,discovered numerically that it's solution seemed to decompose at large times into a collection of solution":well separated solitary waves.

Moreover the solution seems to be almost unaffected in shape by passing through each other (though this could cause a change in their position). They also made the connection to earlier numerical experiments by Fermi ,Pasta,Ulam,and Tsingou by showing that the KdV equation was the continuum limit of the FPU system .

Development of the analytic solution by means of the inverse scattering transform was done in 1967 by Gardner,Greene ,Kruskal and Miura .[36,37].

The KdV equation has several connections to physical problems. In addition to being the governing equation of the string in the Fermi-Pasta-Ulam problem in the continuum limit, it approximately describes the evolution of long ,one-dimensional waves in many physical settings,including: shallow-waves weakly non-linear resting forces,long internal waves in a density-stratified ocean ,ion acoustic waves in a plasma, a coustic waves on a crystal lattice . Wang et al.(2008) [15] introduced the most direct and effective method for obtaining exact solution of non-linear partial differential equations (PDEs).

Called he (G'/G) expansion method, where $G = G(\xi)$ satisfies the second order linear ODE

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0,$$

λ and μ are arbitrary constants .

In the present research we will be solve the KdV equation with the (G'/G) expansion method,and then we will give a new approach of (G'/G) expansion method for constructing more general exact solutions of KdV equation .

1.2 The Description of The Generalized(G'/G)-Expansion Method :

Let us consider a general non-linear PDE in the form

$$P(u, u_x, u_y, u_t, u_{xx}, u_{yy}, u_{xy}, u_{txy}, \cdots) \qquad (1.2.1)$$

where $u(x, y, t)$ is an unknown function , P is polynomial in $u(x, y, t)$ and it's derivatives where in the highest order the partial derivatives.

step 1

We combine the real variables x,y and t by a compound variable ξ :

$$u(x, y, t) = u(\xi) : \qquad \xi = k_1 x + k_2 y + k_3 t, \qquad (1.2.2)$$

the traveling wave transformation (1.2.2) converts Eq.(1.2.1) into an ordinary differential equation (ODE)for :

$$Q(u, u', u'', \cdots) = 0, \qquad (1.2.3)$$

where Q is a polynomial of u and it's derivatives and the superscripts indicate the ordinary derivatives with respect to ξ .

step 2

According to possibility Eq.(1.2.3)can be integrated term by term one or more time yields constant(s) of integration. The integral constant may be zero, for simplicity.

step 3

Suppose the traveling wave solution of Eq.(1.2.3) can be expressed as follows :

$$u(\xi) = \sum_{i=0}^{N} a_i \left(\frac{G'(\xi)}{G(\xi)} \right)^i \tag{1.2.4}$$

Where $(a_i(i = 0, 1, 2, \cdots, N))$ are arbitrary constants to be determined later and $G = G(\xi)$ satisfies the following auxiliary equation :

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{1.2.5}$$

λ and μ are arbitrary constants.

step 4

To determine the positive integer N, taking the homogeneous balance between the highest order nonlinear terms and the derivatives of highest order appearing in Eq.(1.2.3)

step 5

Substitute Eq.(1.2.4) and Eq.(1.2.5) into Eq.(1.2.3) with the value of N obtained in step 4, we obtain polynomials in

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^i \quad (i = 0, 1, 2, \cdots, N),$$

we collect each coefficient of the resulted polynomials and setting them to zero yields a set of algebraic equation for k_1, k_2, k_3 , μ , λ , and a_i $(i = 0, 1, 2, \cdots, N)$. can be found by solving the algebraic equations obtained in step 5. Since the general of Eq.(5) is well known to us, inserting the values of k_1 , k_2 , k_3 , μ, λ, and a_i $(i = 0, 1, 2, \cdots, N)$ into Eq. (1.2.4),we obtain more general type new exact travailing wave solution of the nonlinear partial differential equation (1.2.1) using the general solution of

Eq.(1.2.5), we have the following ratios:

$$\frac{G'(\xi)}{G(\xi)} = \begin{cases} \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}.\frac{c_1\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)+c_2\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}{c_1\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)+c_2\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}; \lambda^2-4\mu>0 \\ \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}.\frac{-c_1\sin(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)+c_2\cos(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}{c_1\cos(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)+c_2\sin(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}; \lambda^2-4\mu<0 \\ \frac{-\lambda}{2} + \frac{c_2}{c_1+c_2\xi}; \qquad \lambda^2-4\mu=0 \end{cases} \tag{1.2.6}$$

where C_1 and C_2 are arbitrary constants

Chapter 2

Application of The(G'/G)-Expansion Method to KdV Equations

2.1 Introduction

The investigation of the traveling wave solutions for nonlinear partial differential equations plays an important role in the study of nonlinear physical phenomena. Nonlinear wave phenomena appear in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics and geochemistry. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations. In the past several decades, there have been significant improvements in the study of exact solutions.

2.2 Exact Solution of The KdV Equation

We will apply the generalized (G'/G)-expansion method to consruct the exact solution of the following KdV equation.
Then

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0 \tag{2.2.1}$$

Example 1: the cvKdV equation

We start with the cvKdV equation in the form:

$$u_t + f(t)uu_x + g(t)uu_{xxx} = 0 \tag{2.2.2}$$

Where $f(t) \neq 0$, $g(t) \neq 0$ are some given functions.
This equation is well-known as a model equation describing the propagation of weakly-nonlinear weakly-dispersive waves in homogeneous. Obtaining exact solutions for non-linear differential equations have long been one of the central themes of perpetual interest in mathematics and physics.
To study the traveling wave solutions of Eq.(2.2.2), we take the following transformation

$$u(x,t) = u(\xi) \qquad , \qquad \xi = x + \frac{\omega}{\alpha}\int_0^t g(t)d(t) \tag{2.2.3}$$

Where ω the wave is speed and α is a constant.

By using Eq. (2.2.2) and Eq.(2.2.3) is converted into an ODE

$$\frac{\omega}{\alpha}u' + 2uu' + u''' = 0 \tag{2.2.4}$$

Where the functions $f(t)$ and $g(t)$ in Eq.(2.2.2) should satisfy the condition

$$f(t) = 2g(t), \tag{2.2.5}$$

integrating Eq.(2.2.4) with respect to ξ once and taking the constant of integration to be zero, we obtain

$$\frac{\omega}{\alpha}u + u^2 + u'' = 0, \tag{2.2.6}$$

suppose that the solution of ODE (2.2.6) can be expressed by polynomial in terms of $(\frac{G'}{G})$ as follows:

$$u(\xi) = \sum_{i=0}^{n} a_i \left(\frac{G'(\xi)}{G(\xi)}\right)^i, \tag{2.2.7}$$

where $a_i(i = 0, 1, ...)$ are arbitrary constants, while $G(\xi)$ satisfies the second order linear ODE (2.2.1). Considering the homogeneous balance between the highest order derivatives and the nonlinear terms in Eq. (2.2.6), we get $n = 2$.

Thus we have

$$u(\xi) = a_0 + a_1\left(\frac{G'(\xi)}{G(\xi)}\right) + a_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2, \tag{2.2.8}$$

where a_0, a_1 and a_2 are constants to be determined later.
Substituting Eq.(2.2.8) with Eq.(2.2.1) into Eq.(2.2.6) and collecting all terms with the same power of $\left(\frac{G'(\xi)}{G(\xi)}\right)$.
Setting each coefficients of this polynomial to be zero,we have the following system of algebraic equations:

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^0 : \frac{\omega a_0}{\alpha} + a_0^2 + \lambda\mu a_1 + 2\mu^2 a_2 = 0$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^1 : \lambda^2 a_1 + 2\mu a_1 + \frac{\omega a_1}{\alpha} + 2a_0 a_1 + 6\lambda\mu a_2 = 0$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^2 : 3\lambda a_1 + a_1^2 + 4\lambda^2 a_2 + 8\mu a_2 + \frac{\omega a_2}{\alpha} + 2a_0 a_2 = 0$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^3 : 2a_1 + 10\lambda a_2 + 2a_1 a_2 = 0$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^4 : 6a_2 + a_2^2 = 0, \tag{2.2.9}$$

on solving the above algebraic Eq.(2.2.9)
By using the Maple, we have

$$a_0 = -6\mu, \qquad a_1 = -6\lambda, \qquad a_2 = -6, \qquad \frac{\omega}{\alpha} = \frac{-a_1^2 + 144\mu}{36}, \tag{2.2.10}$$

substituting Eq. (2.2.10) into Eq.(2.2.8) yields

$$u_i(\xi) = -6\mu - 6\lambda\left(\frac{G'(\xi)}{G(\xi)}\right) - 6\left(\frac{G'(\xi)}{G(\xi)}\right)^2, \tag{2.2.11}$$

$$(i = 1, 2, ..., n \qquad solutions).$$

where

$$\xi = x + \frac{(-a_1^2 + 144\mu)}{36} \int_0^t g(t)d(t) \tag{2.2.12}$$

Consequently, we have the following three types of exact solution of equation (2.2.10).

Case 1

When $(\lambda^2 - 4\mu) > 0$, we obtain the hyperbolic solution in the form:

$$\frac{G'(\xi)}{G(\xi)} = \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} . \frac{c_1 \cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2 \sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}{c_1 \sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2 \cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)} \tag{2.2.13}$$

$$u_1(\xi) = -6\mu - 6\lambda \left(\frac{G'(\xi)}{G(\xi)}\right) - 6\left(\frac{G'(\xi)}{G(\xi)}\right)^2 \tag{2.2.14}$$

In particular,if we set:$\mu = 0, \qquad \lambda = 2, \qquad c_1 \neq 0, \qquad c_2 = 0.$ we find:

$$u_1(\xi) = 6(1 - \coth^2(\xi)) \tag{2.2.15}$$

Case 2

When $(\lambda^2 - 4\mu) < 0$, we have the trigonometric function solution in the form

$$\left(\frac{G'}{G}\right) = \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} . \frac{c_1 \sin(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2 \cos(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}{c_1 \cos(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2 \sin(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}$$

$$u_2(\xi) = -6\mu - 6\lambda\left(\frac{G'(\xi)}{G(\xi)}\right) - 6\left(\frac{G'(\xi)}{G(\xi)}\right)^2 \tag{2.2.16}$$

In particular ,if we set

$\mu = 0, \qquad \lambda = 2, \qquad c_2 \neq 0, \qquad c_1 = 0.$

we find:

$$u_2(\xi) = 6(1 - \cot^2(\xi)) \tag{2.2.17}$$

Case 3

When $(\lambda^2 - 4\mu) = 0$,we get the rational function solution in the form

$$\left(\frac{G'(\xi)}{G(\xi)}\right) = \frac{-\lambda}{2} + \frac{c_2}{c_1 + c_2\xi} \tag{2.2.18}$$

$$u_3(\xi) = -6\mu - 6\lambda\left(\frac{G'(\xi)}{G(\xi)}\right) - 6\left(\frac{G'(\xi)}{G(\xi)}\right)^2 \tag{2.2.19}$$

In particular ,if we set

$\lambda > 0, \qquad \mu = 0, \qquad c_2 \neq 0, \qquad c_1 = 0.$

we find:

$$u_3(\xi) = \frac{3\lambda^2}{2} - \frac{6}{\xi^2} \tag{2.2.20}$$

Example 2: The MDWW Equation

We study the MDWW equations

$$u_t = -\frac{1}{4}v_{xx} + \frac{1}{2}(uv)_x \tag{2.2.21}$$

$$v_t = -u_{xx} - 2uu_x + \frac{3}{2}vv_x \tag{2.2.22}$$

The traveling wave variables below

$$u(x,t) = u(\xi), \qquad v(x,t) = v(\xi), \qquad \xi = k_0(x + \omega t)$$

permit us converting the equations (2.2.21) and (2.2.22) into ODEs for $u(x,t)$=$u(\xi)$ and $v(x,t)$ =$v(\xi)$ as follows:

$$-\frac{1}{4}k_0 v'' + \frac{1}{2}(uv)' + \omega u' = 0 \tag{2.2.23}$$

$$-k_0 u'' - 2uu' + \frac{3}{2}vv' - \omega v' = 0, \tag{2.2.24}$$

where k_0 and ω are the wave number and the wave speed, respectively. On integrating Eqs.(2.2.23) and (2.2.24) with respect to ξ once, yields

$$k_1 - \frac{1}{4}k_0 v' + \frac{1}{2}(uv) + \omega u = 0 \tag{2.2.25}$$

$$k_2 - k_0 u' - u^2 + \frac{3}{2}v^2 - \omega v = 0, \tag{2.2.26}$$

where k_1 and k_2 is an integration constants.

Suppose that the solutions of the ODEs (2.2.25) and (2.2.26) can be expressed by polynomials in terms of $\left(\frac{G'(\xi)}{G(\xi)}\right)$ as follows:

$$u(\xi) = \sum_{i=0}^{n} a_i \left(\frac{G'(\xi)}{G(\xi)}\right)^i \tag{2.2.27}$$

$$v(\xi) = \sum_{i=0}^{m} b_i \left(\frac{G'(\xi)}{G(\xi)}\right)^i, \tag{2.2.28}$$

where $a_i(i = 0, 1, ...)$and $(b_i = 0, 1, , m)$ are arbitrary constants, while $\left(\frac{G'(\xi)}{G(\xi)}\right)$ satisfies the second order linear ODE (2.2.1).

Considering the homogeneous balance between the highest order derivatives and the nonlinear terms in Eqs.(2.2.27) and (2.2.28), we get $n = m = 1$.

Thus, we have

$$u(\xi) = a_0 + a_1 \left(\frac{G'(\xi)}{G(\xi)}\right), \qquad a_1 \neq 0 \tag{2.2.29}$$

$$v(\xi) = b_0 + b_1 \left(\frac{G'(\xi)}{G(\xi)}\right), \qquad b_1 \neq 0, \tag{2.2.30}$$

where a_0,a_1 , b_0 *and* b_1 are arbitrary constants to be determined later.

Substituting Eqs.(2.2.29) , (2.2.30) with Eq.(2.2.1) into Eqs.(2.2.25) and (2.2.26).

collecting all terms with the same power of $\left(\frac{G'(\xi)}{G(\xi)}\right)$ and setting them to zero, we have the following system of algebraic equations:

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^0 : \omega a_0 + \frac{a_0 b_0}{2} + \frac{1}{4}k\mu b_1 + k_1 = 0,$$
$$\left(\frac{G'(\xi)}{G(\xi)}\right)^1 : \omega a_1 + \frac{a_1 b_0}{2} + \frac{1}{4}k_0\lambda b_1 + \frac{a_0 b_1}{2} = 0,$$
$$\left(\frac{G'(\xi)}{G(\xi)}\right)^2 : \frac{k_0 b_1}{4} + \frac{a_1 b_1}{2} = 0,$$
$$\left(\frac{G'(\xi)}{G(\xi)}\right)^0 : -a_0^2 + k_0\mu a_1 + \omega b_0 + \frac{3b_0^2}{4} + k_2 = 0,$$
$$\left(\frac{G'(\xi)}{G(\xi)}\right)^1 : k_0\lambda a_1 - 2a_0 a_1 + \omega b_1 + \frac{3b_0 b_1}{2} = 0,$$
$$\left(\frac{G'(\xi)}{G(\xi)}\right)^2 : k_0 a_1 - a_1^2 + \frac{3b_1^2}{4} = 0, \qquad (2.2.31)$$

Solving this system by using the Maple gives:

The first of the solution:

$$a_0 = \frac{1}{2}(\lambda a_1 + \omega), \qquad a_1 = a_1, \qquad b_0 = -\lambda a_1 - \omega, \qquad b_1 = -2a_1,$$
$$k_0 = -2a_1, \qquad k_1 = \frac{1}{4}(\omega^2 + \lambda^2 a_1^2 - 4\mu a_1^2), \qquad k_2 = \frac{1}{2}(\omega^2 - \lambda^2 a_1^2 + \mu a_1^2) \qquad (2.2.32)$$

The second of the solution:

$$a_0 = \frac{1}{2}(\lambda a_1 - \omega), \qquad a_1 = a_1, \qquad b_0 = \lambda a_1 - \omega, \qquad b_1 = 2a_1,$$
$$k_0 = -2a_1, \qquad k_1 = \frac{1}{4}(\omega^2 - \lambda^2 a_1^2 + 4\mu a_1^2), \qquad k_2 = \frac{1}{2}(\omega^2 - \lambda^2 a_1^2 + \mu a_1^2) \qquad (2.2.33)$$

Substituting Eq.(2.2.33) into Eqs.(2.2.29) and (2.2.30) we obtain

Case 1

When $(\lambda^2 - 4\mu) > 0$, we obtain the hyperbolic solution in the form

$$\frac{G'(\xi)}{G(\xi)} = \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} . \frac{c_1 \cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2 \sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}{c_1 \sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2 \cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)} \tag{2.2.34}$$

$$u_1(\xi) = \frac{1}{2}(\lambda a_1 + \omega) + a_1 \left(\frac{G'(\xi)}{G(\xi)}\right) \tag{2.2.35}$$

$$v_1(\xi) = -\lambda a_1 - \omega - 2a_1 \left(\frac{G'(\xi)}{G(\xi)}\right) \tag{2.2.36}$$

In particular ,if we set

$\mu = 0, \qquad \lambda = 2, \qquad c_1 \neq 0, \qquad c_2 = 0.$

we find:

$$u_1(\xi) = \frac{\omega}{2} - a_1 \coth(\xi) \tag{2.2.37}$$

$$v_1(\xi) = -\omega - 2a_1 \coth(\xi) \tag{2.2.38}$$

Where

$$\xi = -2a_1(x + \omega t) \tag{2.2.39}$$

and

$$u_4(\xi) = \frac{1}{2}(\lambda a_1 - \omega) + a_1 \left(\frac{G'(\xi)}{G(\xi)}\right) \tag{2.2.40}$$

$$v_4(\xi) = \lambda a_1 - \omega + 2a_1 \left(\frac{G'(\xi)}{G(\xi)} \right) \quad (2.2.41)$$

In particular ,if we set

$\mu = 0, \qquad \lambda = 2, \qquad c_1 \neq 0, \qquad c_2 = 0.$

we find:

$$u_4(\xi) = \frac{-\omega}{2} + a_1 \coth(\xi) \quad (2.2.42)$$

$$v_4(\xi) = -\omega + 2a_1 \coth(\xi) \quad (2.2.43)$$

Where

$$\xi = -2a_1(x + \omega t) \quad (2.2.44)$$

Case 2

When $(\lambda^2 - 4\mu) < 0$, we have the trigonometric function solution in the form

$$\left(\frac{G'}{G} \right) = \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} \left(\frac{c_1 \sin(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2 \cos(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}{c_1 \cos(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2 \sin(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)} \right) \quad (2.2.45)$$

$$u_2(\xi) = \frac{1}{2}(\lambda a_1 + \omega) + a_1 \left(\frac{G'(\xi)}{G(\xi)} \right) \quad (2.2.46)$$

$$v_2(\xi) = -\lambda a_1 - \omega - 2a_1 \left(\frac{G'(\xi)}{G(\xi)} \right) \quad (2.2.47)$$

In particular ,if we set

$\mu = 0, \qquad \lambda = 2, \qquad c_1 \neq 0, \qquad c_2 = 0.$

we find:

$$u_2(\xi) = \frac{\omega}{2} + a_1 \cot(\xi) \tag{2.2.48}$$

$$v_2(\xi) = -\omega - 2a_1 \cot(\xi) \tag{2.2.49}$$

Where

$$\xi = -2a_1(x + \omega t) \tag{2.2.50}$$

$$u_5(\xi) = \frac{1}{2}(\lambda a_1 - \omega) + a_1 \left(\frac{G'(\xi)}{G(\xi)}\right) \tag{2.2.51}$$

$$v_5(\xi) = \lambda a_1 - \omega + 2a_1 \left(\frac{G'(\xi)}{G(\xi)}\right) \tag{2.2.52}$$

In particular ,if we set

$\mu = 0, \qquad \lambda = 2, \qquad c_1 \neq 0, \qquad c_2 = 0.$

we find:

$$u_5(\xi) = \frac{-\omega}{2} + a_1 \cot(\xi) \tag{2.2.53}$$

$$v_5(\xi) = -\omega + 2a_1 \cot(\xi) \tag{2.2.54}$$

Where

$$\xi = -2a_1(x + \omega t) \tag{2.2.55}$$

Case 3

When $(\lambda^2 - 4\mu) = 0$,we get the rational function solution in the form

$$\left(\frac{G'(\xi)}{G(\xi)}\right) = \frac{-\lambda}{2} + \frac{c_2}{c_1 + c_2\xi} \tag{2.2.56}$$

$$u_3(\xi) = \frac{1}{2}(\lambda a_1 + \omega) + a_1 \left(\frac{G'(\xi)}{G(\xi)}\right) \tag{2.2.57}$$

$$v_3(\xi) = -\lambda a_1 - \omega - 2a_1 \left(\frac{G'(\xi)}{G(\xi)}\right) \tag{2.2.58}$$

In particular ,if we set

$\lambda > 0, \qquad \mu = 0, \qquad c_2 \neq 0, \qquad c_1 = 0.$

$$u_3(\xi) = \frac{\omega}{2} + \frac{a_1}{\xi} \tag{2.2.59}$$

$$v_3(\xi) = -\omega - \frac{2a_1}{\xi} \tag{2.2.60}$$

Where

$$\xi = -2a_1(x + \omega t) \tag{2.2.61}$$

$$u_6(\xi) = \frac{1}{2}(\lambda a_1 - \omega) + a_1 \left(\frac{G'(\xi)}{G(\xi)}\right) \tag{2.2.62}$$

$$v_6(\xi) = \lambda a_1 - \omega - 2a_1 \left(\frac{G'(\xi)}{G(\xi)}\right) \tag{2.2.63}$$

In particular ,if we set

$\lambda > 0, \qquad \mu = 0, \qquad c_2 \neq 0, \qquad c_1 = 0.$

$$u_6(\xi) = \frac{-\omega}{2} + \frac{a_1}{\xi} \tag{2.2.64}$$

$$u_6(\xi) = -\omega + \frac{2a_1}{\xi} \tag{2.2.65}$$

Where

$$\xi = -2a_1(x + \omega t) \tag{2.2.66}$$

Example 3:

The Symmetrically Coupled KdV Equations

We consider the symmetrically coupled KdV equations

$$u_t = u_{xxx} + v_{xxx} + 6uu_x + 4uv_x + 2u_x v = 0, \qquad (2.2.67)$$

$$v_t = u_{xxx} + v_{xxx} + 6vv_x + 4vu_x + 2v_x u = 0. \qquad (2.2.68)$$

The traveling wave variables below

$$u(x,t) = u(\xi), \qquad v(x,t) = v(\xi), \qquad \xi = k(x + \omega t) \quad (2.2.69)$$

permits us converting Eqs.(2.2.67) and (2.2.68) into the following ODEs:

$$-\omega u' + k^2(u''' + v''') + 6vv' + 4vu' + 2v'u = 0 \qquad (2.2.70)$$

$$-\omega v' + k^2(u''' + v''') + 6vv' + 4uv' + 2u'v = 0 \qquad (2.2.71)$$

Considering the homogeneous balance between highest order derivatives and nonlinear terms in Eqs.(2.2.69) and (2.2.70), we have

$$u(\xi) = a_0 + a_1\left(\frac{G'(\xi)}{G(\xi)}\right) + a_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2, \qquad a_2 \neq 0 \quad (2.2.72)$$

$$v(\xi) = b_0 + b_1\left(\frac{G'(\xi)}{G(\xi)}\right) + b_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2, \qquad b_2 \neq 0, \quad (2.2.73)$$

where a_0, a_1, a_2, b_0, b_1 and b_2 are arbitrary constants to be determined later.

Substituting Eqs.(2.2.72) and (2.2.73) with Eq.(2.2.1) into Eqs.(2.2.69) and (2.2.70), collecting all terms with the same power of $\left(\frac{G'(\xi)}{G(\xi)}\right)$ and setting them to zero, we have the following system of algebraic equations:

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^0 : -k^2\lambda^2\mu a_1 - 2k^2\mu^2 a_1 + \mu\omega a_1 - 6\mu a_0 a_1 - 6k^2\lambda\mu^2 a_2$$
$$-k^2\lambda^2\mu b_1 - 2k^2\mu^2 b_1 - 4\mu a_0 b_1 + 6k^2\lambda\mu^2 b_2 = 0$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^1 : -k^2\lambda^3 a_1 - 8k^2\lambda\mu a_1 + \lambda\omega a_1 - 6\lambda a_0 a_1 - 6\mu a_1^2 - 14k^2\lambda^2\mu a_2$$
$$-12\mu a_0 a_2 - k^2\lambda^3 b_1 - 8k^2\lambda\mu b_1 - 16k^2\mu a_2 + 2\mu\omega a_2$$
$$-4\lambda a_0 b_1 - 6\mu a_1 b_1 - 14k^2\lambda^2\mu b_2 - 16k^2\mu^2 b_2 - 8\mu a_0 b_2 = 0$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^2 : -7k^2\lambda^2 a_1 - 8k^2\mu a_1 + \omega a_1 - 6a_0 a_1 - 6\lambda a_1^2 - 8k^2\lambda^3 a_2 - 52k^2\lambda\mu a$$
$$+2\lambda\omega a_2 - 12\lambda a_0 a_2 - 18\mu a_1 a_2 - 7k^2\lambda^2 b_1 - 8k^2\mu b_1 - 4a_0 b_1$$
$$-6\lambda a_1 b_1 - 8\mu a_2 b_1 - 8k^2\lambda^3 b_2 - 52k^2\lambda\mu b_2 - 8\lambda a_0 b_2 - 10\mu a_1 b_2 = 0$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^3 : -12k^2\lambda a_1 - 6a_1^2 - 38k^2\lambda^2 a_2 - 40k^2\mu a_2 + 2\omega a_2 - 12a_0 a_2$$

$$-18\lambda a_1 a_2 - 12\mu a_2^2 - 12k^2\lambda a_2^2 - 12k^2\lambda b_1 - 6a_1 b_1 - 8\lambda a_2 b_1$$

$$-38k^2\lambda^2 b_2 - 40k^2\mu b_2 - 8a_0 b_2 - 10\lambda a_1 b_2 - 12\mu a_2 b_2 = 0$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^4 : -6k^2 a_1 - 54k^2\lambda a_2 - 18a_1 a_2 - 12\lambda a_2^2 - 6k^2 b_1 - 8a_2 b_1$$

$$-54k^2\lambda b_2 - 10a_1 b_2 - 12\lambda a_2 b_2 = 0$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^5 : -24k^2 a_2^2 - 12a_2^2 - 24k^2 b_2 - 12a_2 b_2 = 0,$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^0 : -k^2\lambda^2\mu a_1 - 2k^2\mu^2 a_1 - 6k^2\lambda\mu^2 a_2 - k^2\lambda^2\mu b_1 - 2k^2\mu^2 b_1$$

$$+\mu\omega b_1 - 2\mu a_0 b_1 - 6k^2\lambda\mu^2 b_2 = 0$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^1 : -k^2\lambda^3 a_1 - 8k^2\lambda\mu a_1 - 14k^2\lambda^2\mu a_2 - 16k^2\mu^2 a_2 - k^2\lambda^3 b_1$$

$$-8k^2\lambda\mu b_1 + \lambda\omega b_1 - 2\lambda a_0 b_1 - 6\mu a_1 b_1 - 6\mu b_1^2$$

$$-14k^2\lambda^2\mu b_2 - 16k^2\mu^2 b_2 + 2\mu\omega b_2 - 4\mu a_0 b_2 = 0,$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^2 : -7k^2\lambda^2 a_1 - 8k^2\lambda\mu a_1 - 8k^2\lambda^3 a_2 - 52k^2\lambda\mu a_2 - 7k^2\lambda^2 b_1$$

$$-8k^2\mu b_1 + \omega b_1 - 2a_0 b_1 - 6\lambda b_1^2 - 8k^2\lambda^3 b_2 - 52k^2\lambda\mu b_2$$

$$+2\lambda\omega b_2 - 4\lambda a_0 b_2 - 8\mu a_1 b_2 - 18\mu b_1 b_2 = 0,$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^3 : -12k^2\lambda a_1 - 38k^2\lambda^2 a_2 - 40k^2\mu a_2 - 12k^2\lambda b_1 - 6a_1 b_1$$

$$-10\lambda a_2 b_1 - 6b_1^2 - 38k^2\lambda^2 b_2 - 40k^2\mu b_2 + 2\omega b_2 - 4a_0 b_2$$

$$-8\lambda a_1 b_2 - 12\mu a_2 b_2 - 18\lambda b_1 b_2 - 12\mu b_2^2 = 0,$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^4 : -6k^2 a_1 - 54k^2\lambda a_2 - 6k^2 b_1 - 10a_2 b_1 - 54k^2\lambda b_2 - 8a_1 b_2$$

$$-12\lambda a_2 b_2 - 18b_1 b_2 - 12\lambda b_2^2 = 0,$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^5 : -24k^2 a_2 - 24k^2 b_2 - 12a_2 b_2 - 12b_2^2 = 0. \quad (2.2.74)$$

Solving the above algebraic Eqs.(2.2.74) by using the Maple, we have

$$a_0 = b_0 = 0, \qquad a_1 = -\frac{\omega\lambda}{\lambda^2+8\mu}, \qquad a_2 = -\frac{\omega}{\lambda^2+8\mu},$$

$$b_1 = -\frac{\omega\lambda}{\lambda^2+8\mu}, \qquad b_2 = -\frac{\omega}{\lambda^2+8\mu}, \qquad k = \pm\sqrt{\frac{\omega}{2\lambda^2+16\mu}} \tag{2.2.75}$$

Substituting Eq.(2.2.75) into Eqs.(2.2.72) and (2.2.73) yields. we deduce the following three types of traveling wave solutions:

case 1

if $\lambda^2 - 4\mu > 0$,we obtain the hyperbolic function traveling wave solutions

$$\frac{G'(\xi)}{G(\xi)} = \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2} . \frac{c_1 \cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2 \sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}{c_1 \sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2 \cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)} \tag{2.2.76}$$

$$u_1(\xi) = -\frac{\omega\lambda}{\lambda^2+8\mu}\left(\frac{G'(\xi)}{G(\xi)}\right) - \frac{\omega}{\lambda^2+8\mu}\left(\frac{G'(\xi)}{G(\xi)}\right)^2 \tag{2.2.77}$$

$$v_1(\xi) = -\frac{\omega\lambda}{\lambda^2+8\mu}\left(\frac{G'(\xi)}{G(\xi)}\right) - \frac{\omega}{\lambda^2+8\mu}\left(\frac{G'(\xi)}{G(\xi)}\right)^2 \tag{2.2.78}$$

Where

$$\xi = \pm\sqrt{\frac{\omega}{2\lambda^2+16\mu}}\,(x+\omega t) \tag{2.2.79}$$

In particular ,if we set

$\mu = 0, \qquad \lambda = 2 \qquad \omega = 8, \qquad c_1 \neq 0, \qquad c_2 = 0.$

we find:

$$u_1(\xi) = 2 - 2coth^2(\xi) \tag{2.2.80}$$

$$v_1(\xi) = 2 - 2coth^2(\xi) \tag{2.2.81}$$

where

$$\xi = \pm(x + 8t)$$

case 2

if $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function traveling wave solutions

$$\left(\frac{G'}{G}\right) = \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} . \frac{c_1 \sin(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2 \cos(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}{c_1 \cos(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2 \sin(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)} \tag{2.2.82}$$

$$u_2(\xi) = -\frac{\omega\lambda}{\lambda^2 + 8\mu}\left(\frac{G'(\xi)}{G(\xi)}\right) - \frac{\omega}{\lambda^2 + 8\mu}\left(\frac{G'(\xi)}{G(\xi)}\right)^2 \tag{2.2.83}$$

$$v_2(\xi) = -\frac{\omega\lambda}{\lambda^2 + 8\mu}\left(\frac{G'(\xi)}{G(\xi)}\right) - \frac{\omega}{\lambda^2 + 8\mu}\left(\frac{G'(\xi)}{G(\xi)}\right)^2 \tag{2.2.84}$$

In particular ,if we set

$\mu = 0, \qquad \lambda = 2 \qquad \omega = 8, \qquad c_2 \neq 0, \qquad c_1 = 0.$

we find:

$$u_2(\xi) = 2 - 2cot^2(\xi) \tag{2.2.85}$$

$$v_2(\xi) = 2 - 2cot^2(\xi) \tag{2.2.86}$$

where

$$\xi = \pm(x + 8t)$$

case 3

if $\lambda^2 - 4\mu = 0$,

we obtain the trigonometric function traveling wave solutions

$$\left(\frac{G'(\xi)}{G(\xi)}\right) = \frac{-\lambda}{2} + \frac{c_2}{c_1 + c_2\xi} \tag{2.2.87}$$

$$u_3(\xi) = -\frac{\omega\lambda}{\lambda^2 + 8\mu}\left(\frac{G'(\xi)}{G(\xi)}\right) - \frac{\omega}{\lambda^2 + 8\mu}\left(\frac{G'(\xi)}{G(\xi)}\right)^2 \tag{2.2.88}$$

$$v_3(\xi) = -\frac{\omega\lambda}{\lambda^2 + 8\mu}\left(\frac{G'(\xi)}{G(\xi)}\right) - \frac{\omega}{\lambda^2 + 8\mu}\left(\frac{G'(\xi)}{G(\xi)}\right)^2 \tag{2.2.89}$$

In particular ,if we set

$\mu = 0, \qquad \lambda = 2 \qquad \omega = 8, \qquad c_2 \neq 0, \qquad c_1 = 0.$

we find:

$$u_3(\xi) = 2 - \frac{2}{\xi^2} \tag{2.2.90}$$

$$v_3(\xi) = 2 - \frac{2}{\xi^2} \tag{2.2.91}$$

where

$$\xi = \pm(x + 8t)$$

Chapter 3

Application of The(G'/G)-Expansion Method

Application of the (G'/G) -expansion method for the Modified Equal Width Wave Equation, Burgers,Burgers-Huxley and modified Burgers-KdV equations

3.1 Introduction

In this chapter, we apply the (G'/G) - expansion method to solve the Burgers, Burgers-Huxley and modified Burgers-KdV equations (mBKdV).

The Burgers equation appears in various areas of applied mathematics, such as modeling of fluid dynamics, turbulence, boundary layer behavior, shock wave formation and traffic flow.

The Modified Equal Width Wave Equation and Burgers-Huxley equation can be regarded as a model to describe the interaction between reaction mechanisms, convection effects and diffusion transports [105-106]. Many physical problems can be described by Burger-KdV and mBKdV equations. Typical examples are provided by the behavior of long waves in shallow water and waves in plasmas. Mcintosh [106] demonstrated how to describe the average behavior of traveling wave solution of mBKdV in the case of small dissipation.

3.2 Applications

We apply the $(G'/G)-$expansion method to solve the Modified Equal Width Wave Equation, Burgers, Burgers-Huxley and modified Burgers-KdV equations.
Then

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0 \tag{3.2.1}$$

Example 1: The Modified Equal Width Wave Equation (MEW)

$$u_t + \alpha u^3 + \beta u_{xxt} = 0 \tag{3.2.2}$$

Using the transformation

$$u = U(\xi), \qquad \xi = x - kt, \tag{3.2.3}$$

where w is a constant to be determined later, Eq. (3.2.2) becomes an ordinary differential equation, as follows:

$$-ku' + \alpha u^3 - k\beta u'' = 0 \tag{3.2.4}$$

balancing u'' with u^3 gives n=1

$$k_2 u'' + \alpha k_1 u'' + \beta u + \gamma u^3 = 0 \tag{3.2.5}$$

$$u(\xi) = \sum_{i=0}^{N} a_i \left(\frac{G'(\xi)}{G(\xi)} \right)^i \tag{3.2.6}$$

Then the solution is :

$$u(\xi) = a_0 + a_1 \left(\frac{G'(\xi)}{G(\xi)} \right), \qquad a_1 \neq 0 \tag{3.2.7}$$

where a_0, a_1, λ, and μ are arbitrary constants.

by eq(3.2.5) and eq(3.2.7) we derive

$$u^3 = a_0^3 + 3a_0^2 a_1 \left(\frac{G'(\xi)}{G(\xi)} \right) + 3a_0 a_1^2 \left(\frac{G'(\xi)}{G(\xi)} \right)^2 + a^3 \left(\frac{G'(\xi)}{G(\xi)} \right)^3 \tag{3.2.8}$$

$$\begin{aligned} u'' = \lambda\mu a_1 + \lambda^2 a_1 \left(\frac{G'(\xi)}{G(\xi)} \right) + 2\mu a_1 \left(\frac{G'(\xi)}{G(\xi)} \right) \\ + 3\lambda a_1 \left(\frac{G'(\xi)}{G(\xi)} \right)^2 + 2a_1 \left(\frac{G'(\xi)}{G(\xi)} \right)^3 \end{aligned} \tag{3.2.9}$$

Substituting eqs.(3.2.5),(3.2.8) and(3.2.9) into (3.2.4) and equating the coefficients of $\left(\frac{G'(\xi)}{G(\xi)}\right)^i$ to zero, we obtain a system of algebraic equation in a_0, a_1, a_2, λ, and μ as following

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^0 := ka_1\mu + \alpha a_0^3 - k\beta\lambda\mu a_1 = 0 \qquad ,$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^1 := k\lambda a_1 + 3\alpha a_0^2 a_1 - k\beta\lambda^2 a_1 - 2k\beta\mu a_1 = 0 \qquad ,$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^2 := ka_1 + 3\alpha a_0 a_1^2 - k\beta a_1 = 0 \qquad ,$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^3 := \alpha a_1^3 - 2k\beta\alpha a_1 = 0 \qquad .$$

Solving this system by Maple gives

$$a_0 = \pm\sqrt{\frac{2\beta k}{\alpha}}\left(\frac{\beta - 1}{6\beta}\right), \qquad a_1 = \pm\sqrt{\frac{2\beta w}{\alpha}}, \qquad k = k,$$
$$\mu = \frac{-1}{12\beta^2}\left(-6\lambda - 1 + 2\beta - \beta^2 + 6\lambda^2\beta^2\right), \qquad \beta = \beta. \tag{3.2.10}$$

$$u_i(\xi) = a_0 + a_1\left(\frac{G'(\xi)}{G(\xi)}\right), \qquad (i = 1, 2, ..., n \qquad solutions) \tag{3.2.11}$$

$$\xi = x - 8t,$$

Consequently, we have the following three types of exact solution of equation (3.2.10).

Case 1:

When $(\lambda^2 - 4\mu) > 0$, we obtain the hyperbolic solution in the form

$$\frac{G'(\xi)}{G(\xi)} = \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} . \frac{c_1 \cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2 \sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}{c_1 \sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2 \cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)} \tag{3.2.12}$$

$$u_1(\xi) = \pm\sqrt{\frac{2\beta w}{\alpha}}\left(\frac{\beta - 1}{6\beta}\right) \pm \sqrt{\frac{2\beta w}{\alpha}}\left(\frac{G'(\xi)}{G(\xi)}\right)$$

In particular ,if we set

$$\lambda = 2, \qquad \beta = 1, \qquad k = 5, \qquad c_1 = 0, \qquad c_2 \neq 0.$$

$$\omega = 4, \qquad \alpha = 8, \qquad \mu = -1.$$

we find:

$$u_{(1:1)}(\xi) = -1 + coth(\sqrt{2}\xi) \tag{3.2.13}$$

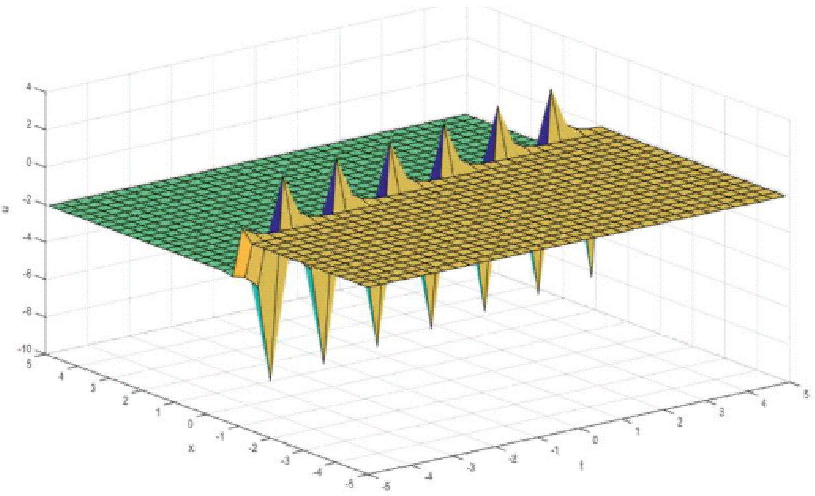

Figure 3.2.1: The graph of exact travelling wave solution of eq. (2.2.13)

$$u_{(1:2)}(\xi) = 1 - coth(\sqrt{2}\xi) \tag{3.2.14}$$

and

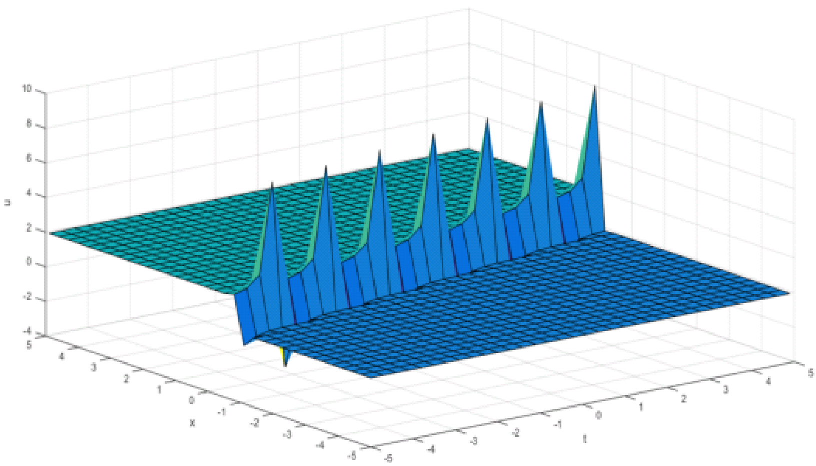

Figure 3.2.2: The graph of exact travelling wave solution of eq. (3.2.14)

Case 2

When $(\lambda^2 - 4\mu) < 0$, we have the trigonometric function solution in the form

$$\left(\frac{G'}{G}\right) = \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} \cdot \frac{-c_1 \sin(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2 \cos(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}{c_1 \cos(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2 \sin(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)} \tag{3.2.15}$$

$$u_2(\xi) = \pm\sqrt{\frac{2\beta w}{\alpha}} \left(\frac{\beta - 1}{6\beta}\right) \pm \sqrt{\frac{2\beta w}{\alpha}} \left(\frac{G'(\xi)}{G(\xi)}\right)$$

In particular ,if we set

$$\lambda = 2, \qquad \beta = 1, \qquad k = 5, \qquad c_1 = 0, \qquad c_2 \neq 0.$$

$$\omega = 4, \qquad \alpha = 8, \qquad \mu = -1.$$

we find:

$$u_{(2:1)}(\xi) = -1 + cot(\sqrt{2}\xi) \tag{3.2.16}$$

and

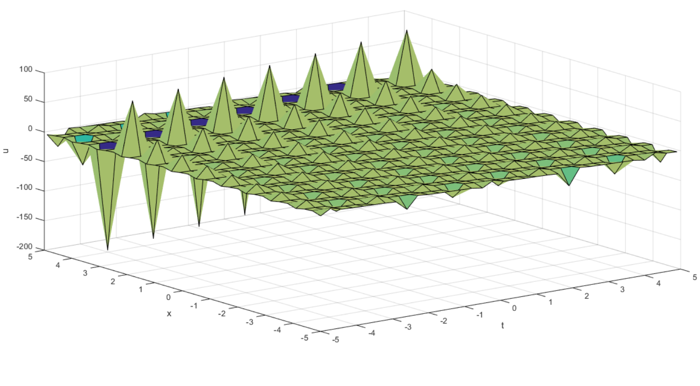

Figure 3.2.3: The graph of exact travelling wave solution of eq. (3.2.16)

$$u_{(2:2)}(\xi) = 1 - cot(\sqrt{2}\xi) \tag{3.2.17}$$

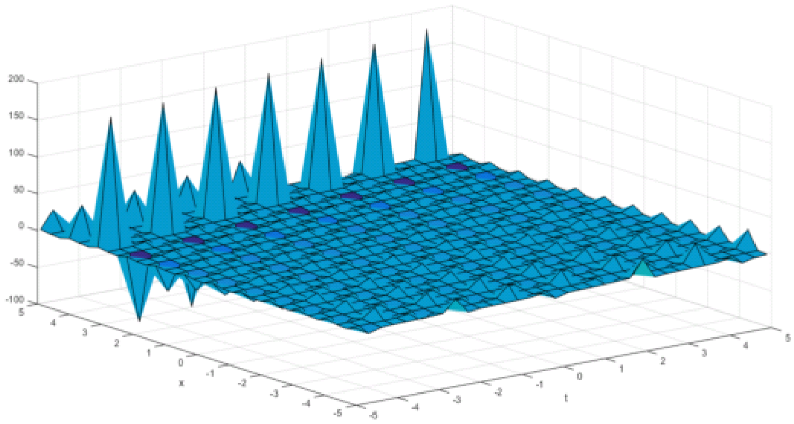

Figure 3.2.4: The graph of exact travelling wave solution of eq. (3.2.17)

Case 3

When $(\lambda^2 - 4\mu) = 0$,we get the rational function solution in the form

$$\left(\frac{G'(\xi)}{G(\xi)}\right) = \frac{-\lambda}{2} + \frac{c_2}{c_1 + c_2\xi} \tag{3.2.18}$$

$$u_3(\xi) = \pm\sqrt{\frac{2\beta w}{\alpha}}\left(\frac{\beta - 1}{6\beta}\right) \pm \sqrt{\frac{2\beta w}{\alpha}}\left(\frac{G'(\xi)}{G(\xi)}\right)$$

In particular ,if we set

$$\lambda = 2, \qquad \beta = 1, \qquad k = 5, \qquad c_1 = 0, \qquad c_2 \neq 0.$$

$$\omega = 4, \qquad \alpha = 8, \qquad \mu = -1.$$

we find:

$$u_{(3)}(\xi) = \pm\left(-1 + \frac{1}{\xi}\right)$$

Example 2: The Burgers Equation

The Burgers equation is presented as

$$u_t + uu_x = u_{xx} \tag{3.2.19}$$

We make the transformation

$$u(x,t) = U(\xi), \qquad \xi = x - ct. \tag{3.2.20}$$

Then we get

$$-cU' + UU' - U'' = 0 \tag{3.2.21}$$

By one time integrating with respect to (ξ) , eq. (3.2.17) becomes

$$-cU + \frac{1}{2}U^2 - U' + D = 0 \tag{3.2.22}$$

where D is the integration constant.

Balancing U' with U^2 gives $N = 1$.

Therefore, we can write the solution of eq. (3.2.18) in the form

$$U(\xi) = a_0 + a_1\left(\frac{G'(\xi)}{G(\xi)}\right), \qquad a_0 \neq 0 \tag{3.2.23}$$

By eqs (3.2.1) and (3.2.19) we derive

$$U^2(\xi) = a_1^2\left(\frac{G'(\xi)}{G(\xi)}\right)^2 + 2a_0a_1\left(\frac{G'(\xi)}{G(\xi)}\right) + a_0^2 \tag{3.2.24}$$

$$U'(\xi) = -a_1 \left(\frac{G'(\xi)}{G(\xi)}\right)^2 - a_1\lambda \left(\frac{G'(\xi)}{G(\xi)}\right) - a_1\mu \qquad (3.2.25)$$

substituting eqs (3.2.5)-(3.2.7) into eq. (3.2.4), setting the coefficients of $\left(\frac{G'(\xi)}{G(\xi)}\right)^i (i = 0, 1, 2)$
to zero, we obtain a system of algebraic equations for a_0, a_1, c, λ and μ as follows:

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^0 : -ca_0 + \frac{1}{2}a_0^2 + a_1\mu + D = 0, \qquad (3.2.26)$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^1 : -ca_1 + a_0a_1 + a_1\lambda = 0, \qquad (3.2.27)$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^2 : \frac{1}{2}a_1^2 + a_1 = 0 \qquad (3.2.28)$$

Solving this system by Maple gives:

$$a_0 = c - \lambda, \qquad a_1 = -2, \qquad , c = c \qquad D = \frac{1}{2}(c^2 - \lambda^2 + 4\mu). \qquad (3.2.29)$$

Substituting the solution set (3.2.25) and the corresponding solutions of (3.2.1) into (3.2.19),
we have the solutions of eq. (3.2.18) as follows:

case 1

if $\lambda^2 - 4\mu > 0$,we obtain the hyperbolic function traveling wave solutions

$$\frac{G'(\xi)}{G(\xi)} = \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} . \frac{c_1 \cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2 \sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}{c_1 \sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2 \cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)} \tag{3.2.30}$$

$$U_1(\xi) = c - \lambda - 2\left(\frac{G'(\xi)}{G(\xi)}\right) \tag{3.2.31}$$

In particular ,if we set

$\mu = 0, \qquad \lambda = 2 \qquad c = 6, \qquad c_1 \neq 0, \qquad c_2 = 0.$

we find:

$$U_1(\xi) = 6 - 2coth(\xi) \tag{3.2.32}$$

where $\xi = x - 6t$

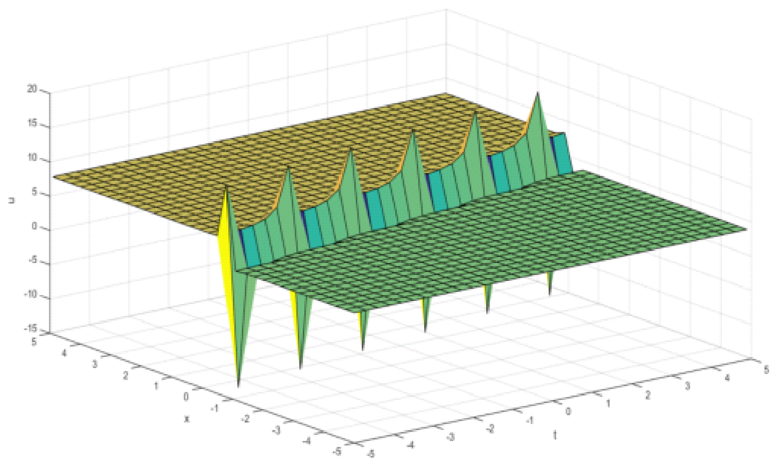

Figure 3.2.5: The graph of exact travlling wave solution of eq. (3.2.32)

case 2

if $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function traveling wave solutions

$$\left(\frac{G'}{G}\right) = \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} . \frac{c_1 \sin(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2 \cos(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}{c_1 \cos(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2 \sin(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)} \tag{3.2.33}$$

$$U_2(\xi) = c - \lambda - 2\left(\frac{G'(\xi)}{G(\xi)}\right) \tag{3.2.34}$$

In particular ,if we set

$\mu = 0, \quad \lambda = 2 \quad c = 6, \quad c_2 \neq 0, \quad c_1 = 0.$

we find:

$$U_2(\xi) = 6 - 2cot(\xi) \tag{3.2.35}$$

where $\xi = x - 6t$

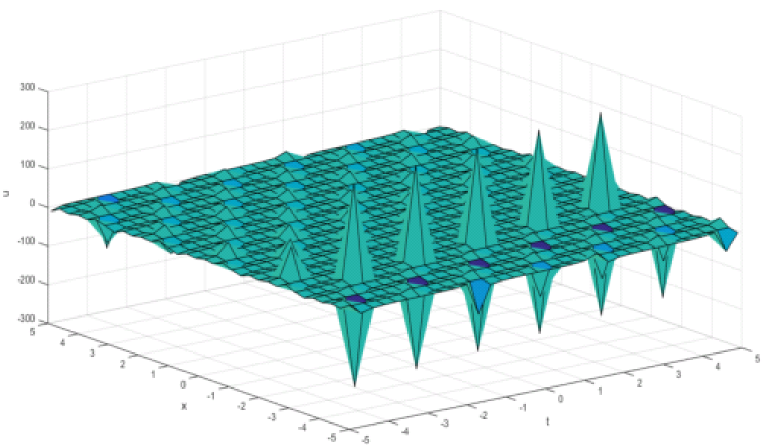

Figure 3.2.6: The graph of exact travlling wave solution of eq. (3.2.35)

case 3

if $\lambda^2 - 4\mu = 0$, we obtain the trigonometric function traveling wave solutions

$$\left(\frac{G'(\xi)}{G(\xi)}\right) = \frac{-\lambda}{2} + \frac{c_2}{c_1 + c_2\xi} \tag{3.2.36}$$

$$U_3(\xi) = c - \lambda - 2\left(\frac{G'(\xi)}{G(\xi)}\right) \tag{3.2.37}$$

In particular ,if we set

$\lambda = 2, \qquad \mu = 0, \qquad c = 6, \qquad c_2 \neq 0, \qquad c_1 = 0.$

in Eq(3.2.19) then we get:

$$u_3(\xi) = 6 - \frac{2}{\xi} \tag{3.2.38}$$

where $\xi = x - 6t$

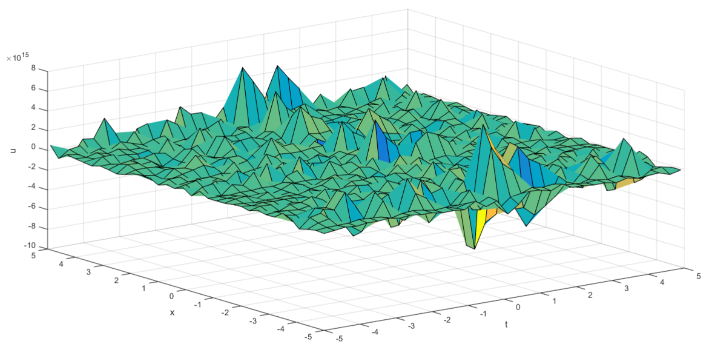

Figure 3.2.7: The graph of exact travlling wave solution of eq. (3.2.38)

In solutions $U_i(\xi)(i = 1, 2, 3)$, C_1 and C_2 are left as free parameters.

It is obvious that hyperbolic, trigonometric and rational solutions were obtained by using the $\left(\frac{G'}{G}\right)$ – expansion method, whereas only hyperbolic solutions and hyperbolic and trigonometric solutions.
We observe that the results are particular cases of our results. Then our solutions are more general.

Example 3: The Burgers-Huxley Equation

Now, let us consider the following Burgers-Huxley equation in the form

$$u_t = u_{xx} + uu_x + u(k-u)(u-1), \qquad k \neq 0 \qquad (3.2.39)$$

We make the transformation

$$u_(x,t) = U(\xi), \qquad \xi = x - ct. \qquad (3.2.40)$$

Then we get

$$cU' + UU' + U'' + U(k-U)(U-1) = 0 \qquad (3.2.41)$$

where prime denotes the derivative with respect to ξ.
Balancing U'' with U^3 gives $N = 1$. Therefore, we can write the solution of eq. (3.2.37) in the form

$$U(\xi) = a_0 + a_1 \left(\frac{G'(\xi)}{G(\xi)}\right), \qquad a_1 \neq 0 \qquad (3.2.42)$$

By using eqs (3.2.1) and (3.2.37) we have

$$U'(\xi) = -a_1 \left(\frac{G'(\xi)}{G(\xi)}\right)^2 - a_1\lambda \left(\frac{G'(\xi)}{G(\xi)}\right) - a_1\mu \qquad (3.2.43)$$

$$U''(\xi) = -a_1 \left(\frac{G'(\xi)}{G(\xi)}\right)^3 + 3a_1\lambda \left(\frac{G'(\xi)}{G(\xi)}\right)^2 + (a_1\lambda^2 + 2a_1\mu) \left(\frac{G'(\xi)}{G(\xi)}\right) + a_1\lambda\mu, \qquad (3.2.44)$$

$$U^2(\xi) = a_1^2 \left(\frac{G'(\xi)}{G(\xi)}\right)^2 + 2a_0a_1 \left(\frac{G'(\xi)}{G(\xi)}\right) + 2a_0^2, \qquad (3.2.45)$$

$$U^3(\xi) = a_1^3 \left(\frac{G'(\xi)}{G(\xi)}\right)^3 + 3a_0a_1^2 \left(\frac{G'(\xi)}{G(\xi)}\right)^2 + 3a_0^2a_1 \left(\frac{G'(\xi)}{G(\xi)}\right) + a_0^3. \qquad (3.2.46)$$

Substituting eqs (3.2.38)-(3.2.42) into (3.2.37), setting coefficients of $\left(\frac{G'(\xi)}{G(\xi)}\right)^i$ $(i = 0, 1, 2, 3)$ to zero, we obtain a system of nonlinear algebraic equations a_0, a_1, c, λ and μ as follows:

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^0 : -ca_1\mu - a_0a_1\mu - a_1\lambda\mu - ka_0 + ka_0^2 + a_0^2 - a_0^3 = 0,$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^1 : -ca_1\lambda - a_0a_1\lambda - a_1^2\mu + a_1\lambda^2 + 2a_1\mu - ka_1$$

$$+2ka_0a_1 + 2a_0a_1 - 3a_0^2a_1 = 0,$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^2 : -ca_1 - a_0a_1 - a_1^2\lambda + 3a_1\lambda + ka_1^2 + a_1^2 - 3a_0a_1^2 = 0$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^3 : -a_1^2 + 2a_1 - a_1^3 = 0 \qquad (3.2.47)$$

Solving this system by Maple gives

The first of the solution:

$$a_0 = \frac{2\lambda + k - c + 1}{4}, \qquad a_1 = 1 \qquad ,$$

$$\mu = a_0\lambda^2 + 2a_0 - 3a_0^2 - ka_0, \qquad \lambda = \lambda. \qquad (3.2.48)$$

The second of the solution:

$$a_0 = \frac{c + 2k - 5\lambda + 2}{5}, \qquad a_1 = -2,$$

$$\mu = \frac{1}{4}a_0\lambda^2 + \frac{3}{2}a_0^2 + \frac{1}{2}ka_0 - a_0 \qquad \lambda = \lambda. \qquad (3.2.49)$$

where

$$U_i(\xi) = a_0 + a_1\left(\frac{G'(\xi)}{G(\xi)}\right), \qquad (i = 1, 2, ..., n \qquad \textit{solutions}) \qquad (3.2.50)$$

case 1

if $\lambda^2 - 4\mu > 0$,we obtain the hyperbolic function traveling wave solutions

$$\frac{G'(\xi)}{G(\xi)} = \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} . \frac{c_1 \cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2 \sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}{c_1 \sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2 \cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)} \tag{3.2.51}$$

$$U_1(\xi) = \left(\frac{2\lambda + k - c + 1}{4}\right) + \left(\frac{G'(\xi)}{G(\xi)}\right) \tag{3.2.52}$$

In particular ,if we set

$\mu = 0, \quad \lambda = 2 \quad k = 8, \quad c_1 \neq 0, \quad c_2 = 0 \quad c = 1.$

we find:

$$U_1(\xi) = 3 + coth(\xi) \tag{3.2.53}$$

where $\xi = x - t$

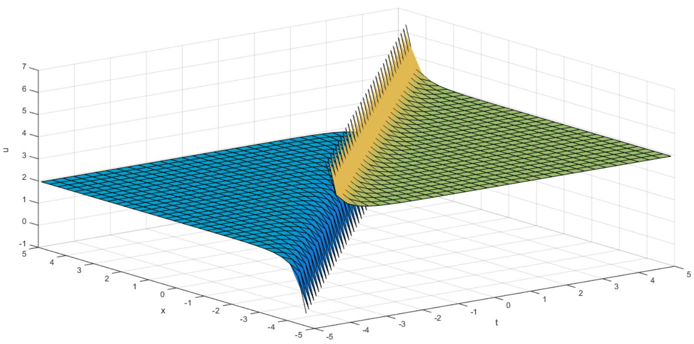

Figure 3.2.8: The graph of exact travlling wave solution of eq. (3.2.53)

and

$$U_4(\xi) = \left(\frac{c + 2k - 5\lambda + 2}{5}\right) - 2\left(\frac{G'(\xi)}{G(\xi)}\right) \tag{3.2.54}$$

In particular ,if we set

$\mu = 0, \quad \lambda = 2 \quad k = 8, \quad c_1 \neq 0, \quad c_2 = 0 \quad c = 1.$

we find:

$$U_4(\xi) = \frac{19}{5} - 2coth(\xi) \tag{3.2.55}$$

where $\xi = x - t$

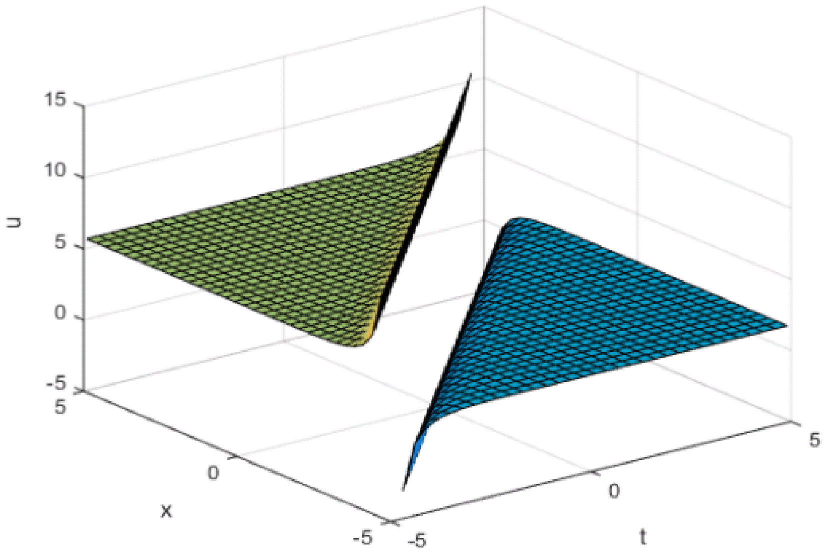

Figure 3.2.9: The graph of exact travlling wave solution of eq. (3.2.55)

case 2

if $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function traveling wave solutions

$$\left(\frac{G'}{G}\right) = \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} . \frac{c_1 \sin(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2 \cos(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}{c_1 \cos(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2 \sin(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)} \tag{3.2.56}$$

$$U_2(\xi) = \left(\frac{2\lambda + k - c + 1}{4}\right) + \left(\frac{G'(\xi)}{G(\xi)}\right) \tag{3.2.57}$$

In particular ,if we set

$\mu = 0, \qquad \lambda = 2 \qquad k = 8, \qquad c_2 \neq 0, \qquad c_1 = 0 \qquad c = 1.$

we find:

$$U_2(\xi) = 3 + cot(\xi) \tag{3.2.58}$$

where $\xi = x - t$ and

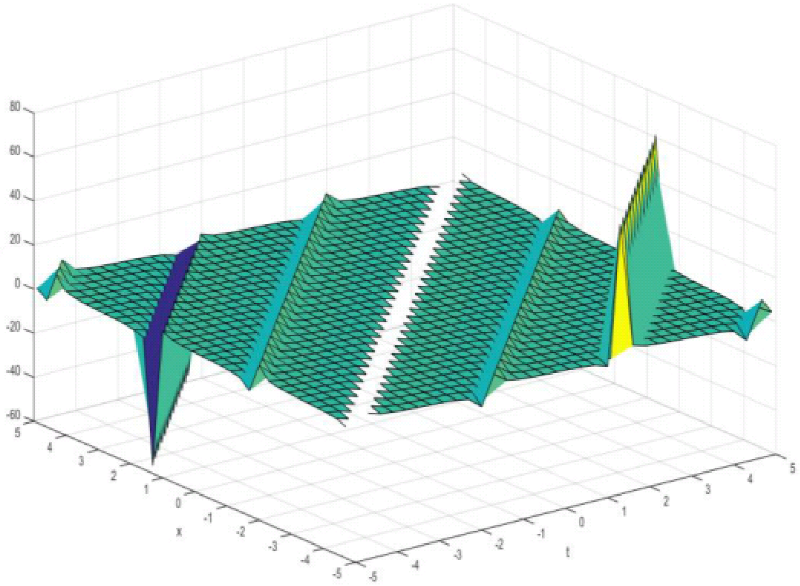

Figure 3.2.10: The graph of exact travlling wave solution of eq. (3.2.60)

$$U_5(\xi) = \left(\frac{c + 2k - 5\lambda + 2}{5}\right) - 2\left(\frac{G'(\xi)}{G(\xi)}\right) \tag{3.2.59}$$

In particular ,if we set

$\mu = 0, \qquad \lambda = 2 \qquad k = 8, \qquad c_2 \neq 0, \qquad c_1 = 0 \qquad c = 1.$

we find:

$$U_4(\xi) = \frac{19}{5} - 2cot(\xi) \tag{3.2.60}$$

where $\xi = x - t$

case 3

if $\lambda^2 - 4\mu = 0$, we obtain the trigonometric function traveling wave solutions

$$\left(\frac{G'(\xi)}{G(\xi)}\right) = \frac{-\lambda}{2} + \frac{c_2}{c_1 + c_2\xi} \tag{3.2.61}$$

$$U_3(\xi) = \left(\frac{2\lambda + k - c + 1}{4}\right) + \left(\frac{G'(\xi)}{G(\xi)}\right) \tag{3.2.62}$$

In particular ,if we set

$\mu = 0,\quad \lambda = 2\quad k = 8,\quad c_2 \neq 0,\quad c_1 = 0\quad c = 1.$

we find:

$$u_3(\xi) = 2 + \frac{1}{\xi} \tag{3.2.63}$$

where $\xi = x - t$

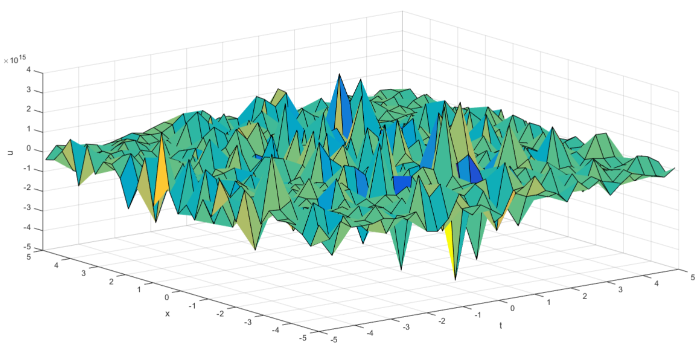

Figure 3.2.11: The graph of exact travlling wave solution of eq. (3.2.63)

and

$$U_6(\xi) = \left(\frac{c + 2k - 5\lambda + 2}{5}\right) - 2\left(\frac{G'(\xi)}{G(\xi)}\right) \quad (3.2.64)$$

In particular ,if we set

$\mu = 0, \qquad \lambda = 2 \qquad k = 8, \qquad c_2 \neq 0, \qquad c_1 = 0 \qquad c = 1.$

we find:

$$u_6(\xi) = 3 + \frac{1}{\xi} \quad (3.2.65)$$

where

$$\xi = x - t \quad (3.2.66)$$

Example 4: The Modified Burgers-KdV Equation

We next consider the modified Burgers –KdV equation

$$u_t + pu^2u_x + qu_{xx} - ru_{xxx} = 0 \quad (3.2.67)$$

where p, q and r are real constants. When $q = 0$, the modified Burgers - KdV equation reduces to the modified KdV equation. During the past several years, many have done research on traveling wave solution of the KdV equation. Macintosh [104] demonstrated how to describe the average behavior of traveling wave solution of eq.(3.2.63) during small dissipation. Jacobs and co -workers investigated the limit when r and q approached zero and the ratio r/q^2 remained constant, thus balancing the dissipation and dispersion in balance . In the limit, it was shown that the traveling wave solutions of eq. (3.2.63) approach a shock wave solution.

To determine the traveling wave solution of eq. (3.2.63), we make the transformation

$$u(x,t) = U(\xi), \qquad \xi = x - ct. \tag{3.2.68}$$

Then we get

$$-cU' + pU^2 + qU'' - rU''' = 0 \tag{3.2.69}$$

By integration with respect to (ξ) in eq. (3.2.39), we get

$$-cU + \frac{p}{3}U^3 + qU' - rU'' = 0 \tag{3.2.70}$$

Balancing U'' with U^3 gives $N = 1$. Therefore, we can write the solution of eq. (3.2.66) in the form

$$U(\xi) = a_0 + a_1\left(\frac{G'(\xi)}{G(\xi)}\right), \qquad a_1 \neq 0 \tag{3.2.71}$$

By using eqs (3.2.1) and (3.2.66) we have

$$U'(\xi) = -a_1\left(\frac{G'(\xi)}{G(\xi)}\right)^2 - a_1\lambda\left(\frac{G'(\xi)}{G(\xi)}\right) - a_1\mu \tag{3.2.72}$$

$$U''(\xi) = 2a_1\left(\frac{G'(\xi)}{G(\xi)}\right)^3 + 3a_1\lambda\left(\frac{G'(\xi)}{G(\xi)}\right)^2 + (a_1\lambda^2 + 2a_1\mu)\left(\frac{G'(\xi)}{G(\xi)}\right) + a_1\lambda\mu \tag{3.2.73}$$

$$U^3(\xi) = a_1^3\left(\frac{G'(\xi)}{G(\xi)}\right)^3 + 3a_0a_1^2\left(\frac{G'(\xi)}{G(\xi)}\right)^2$$
$$+ 3a_0^2a_1\left(\frac{G'(\xi)}{G(\xi)}\right) + a_0^3 \qquad (3.2.74)$$

Substituting eqs (3.2.67)-(3.2.70) into (3.2.66), setting coefficients of $\left(\frac{G'(\xi)}{G(\xi)}\right)^i$ $(i = 0, 1, 2, 3)$ to zero, we obtain a system of nonlinear algebraic equations a_0, a_1,c,λ and μ as follows:

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^0 : -ca_0 + \frac{1}{3}pa_0^3 - qa_1\mu - ra_1\lambda\mu = 0,$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^1 : -ca_1 + pa_0^2a_1 - qa1\lambda - ra_1\lambda^2 - 2ra_1\mu = 0,$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^2 : pa_0a_1^2 - qa_1 - 3ra_1\lambda = 0,$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^3 : \frac{1}{3}pa_1^3 - 2ra_1 = 0 \qquad (3.2.75)$$

Solving this system by Maple gives:

The first of the solution:

$$a_0 = \frac{q}{p}\pm\frac{\lambda}{2}\sqrt{\frac{6r}{p}}, \qquad a_1 = \pm\sqrt{\frac{6r}{p}}, \qquad q = \frac{3r(2a_0 - a_1\lambda)}{a_1^2} \qquad \lambda = \lambda,$$

$$\mu = \frac{a_0(a_0 - a_1\lambda)}{a_1^2}, \qquad rp > 0, \qquad c = \left(\frac{2r(4a_0^2 - 4a_0a_1\lambda + a_1^2\lambda)}{a_1^2}\right)$$
$$(3.2.76)$$

The second of the solution:

$$a_0 = \pm\frac{\lambda}{2}\sqrt{\frac{6r}{p}}, \qquad a_1 = \pm\sqrt{\frac{6r}{p}}, \qquad \lambda = \lambda,$$

$$c = \frac{-r}{2}\left(4\mu - \lambda^2\right), \qquad rp > 0 \qquad \mu = \mu \qquad r = r. \quad (3.2.77)$$

Substituting the eq. (3.2.73) and the corresponding solutions of (3.2.1) into (3.2.67), we have the solutions of eq. (3.2.66) as follows: When $\lambda^2 - 4\mu$, we obtain the hyperbolic function traveling wave solutions

case 1

if $\lambda^2 - 4\mu > 0$,we obtain the hyperbolic function traveling wave solutions

$$\left(\frac{G'}{G}\right) = \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2}\left(\frac{-c_1\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}{c_1\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}\right) \quad (3.2.78)$$

$$U_1(\xi) = \left(\frac{q}{p} \pm \frac{\lambda}{2}\sqrt{\frac{6r}{p}}\right) \pm \sqrt{\frac{6r}{p}}\left(\frac{G'(\xi)}{G(\xi)}\right) \quad (3.2.79)$$

In particular ,if we set

$$\lambda = 2, \qquad q = -9, \qquad p = -3, \qquad c_2 \neq 0,$$

$$c_1 = 0, \qquad r = -2, \qquad \mu = 0, \qquad c = 4.$$

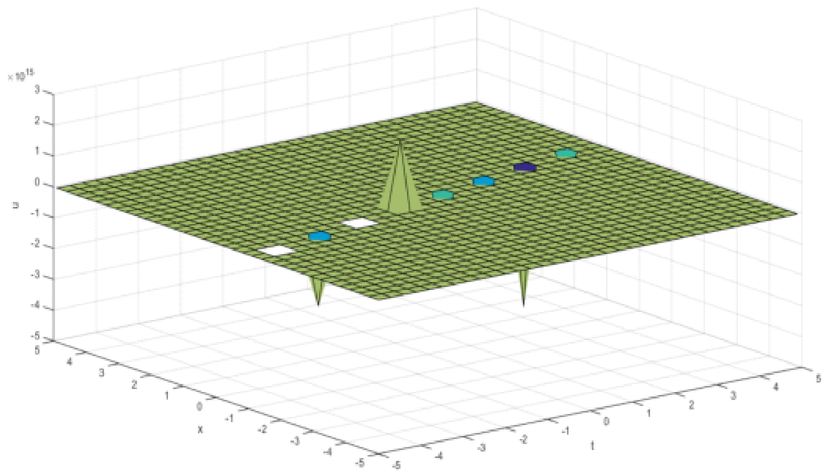

Figure 3.2.12: The graph of exact travlling wave solution of eq. (3.2.80)

we find:

$$U_1(\xi) = 2 + 2coth(\xi) \tag{3.2.80}$$

where $\xi = x - 4t$ and

$$U_4(\xi) = \left(\pm\frac{\lambda}{2}\sqrt{\frac{6r}{p}}\right) \pm \sqrt{\frac{6r}{p}}\left(\frac{G'(\xi)}{G(\xi)}\right) \tag{3.2.81}$$

In particular ,if we set

$$\lambda = 2, \qquad p = -3, \qquad c_1 \neq 0,$$

$$c_2 = 0, \qquad r = -2, \qquad \mu = 0, \qquad c = 4.$$

we find:

$$U_4(\xi) = -2coth(\xi) \tag{3.2.82}$$

where $\xi = x - 4t$

case 2

if $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function traveling wave solutions

$$\left(\frac{G'}{G}\right) = \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} . \frac{c_1 \sin(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2 \cos(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}{c_1 \cos(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2 \sin(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)} \tag{3.2.83}$$

$$U_2(\xi) = \left(\frac{q}{p} \pm \frac{\lambda}{2}\sqrt{\frac{6r}{p}}\right) \pm \sqrt{\frac{6r}{p}} \left(\frac{G'(\xi)}{G(\xi)}\right) \tag{3.2.84}$$

In particular ,if we set

$$\lambda = 2, \qquad q = -9, \qquad p = -3, \qquad c_2 \neq 0,$$

$$c_0 = 0, \qquad r = -2, \qquad \mu = 0, \qquad c = 4.$$

we find:

$$U_2(\xi) = 2 + 2cot(\xi) \tag{3.2.85}$$

where

$$\xi = x - 4t$$

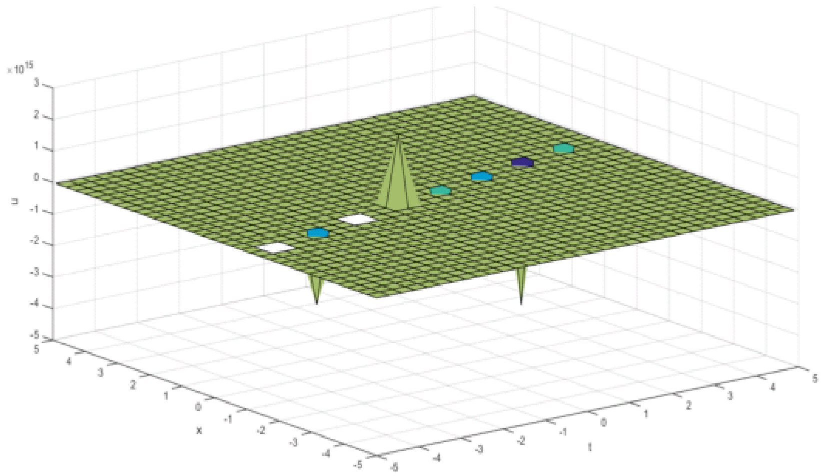

Figure 3.2.13: The graph of exact travlling wave solution of eq. (3.2.85)

and

$$U_5(\xi) = \left(\pm\frac{\lambda}{2}\sqrt{\frac{6r}{p}}\right) \pm \sqrt{\frac{6r}{p}}\left(\frac{G'(\xi)}{G(\xi)}\right) \tag{3.2.86}$$

In particular ,if we set

$$\lambda = 2, \qquad p = -3, \qquad c_2 \neq 0,$$

$$c_1 = 0, \qquad r = -2, \qquad \mu = 0, \qquad c = 4.$$

we find:

$$U_5(\xi) = -2cot(\xi) \tag{3.2.87}$$

where

$$\xi = x - 4t$$

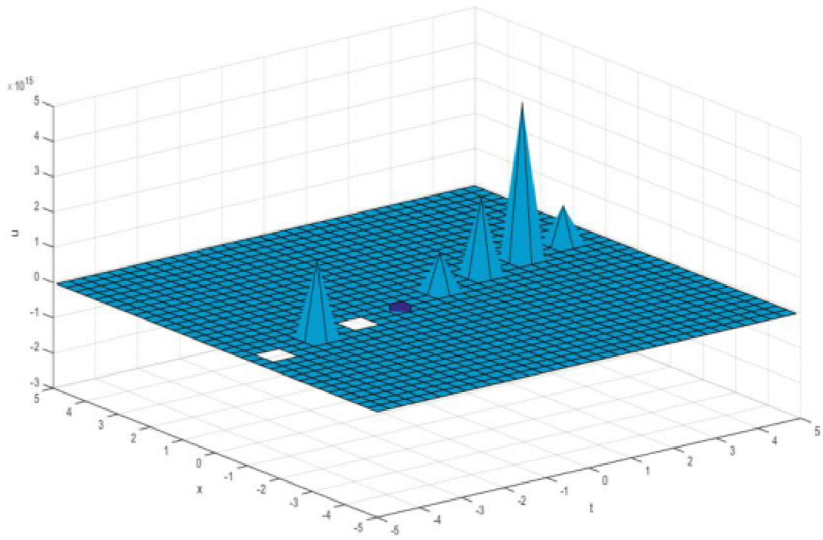

Figure 3.2.14: The graph of exact travlling wave solution of eq. (3.2.87)

case 3

if $\lambda^2 - 4\mu = 0$, we obtain the trigonometric function traveling wave solutions

$$\left(\frac{G'(\xi)}{G(\xi)}\right) = \frac{-\lambda}{2} + \frac{c_2}{c_1 + c_2\xi} \tag{3.2.88}$$

$$U_3(\xi) = \left(\frac{q}{p} \pm \frac{\lambda}{2}\sqrt{\frac{6r}{p}}\right) \pm \sqrt{\frac{6r}{p}}\left(\frac{G'(\xi)}{G(\xi)}\right) \tag{3.2.89}$$

In particular ,if we set

$$\lambda = 2, \qquad q = -9, \qquad p = -3, \qquad c_2 \neq 0,$$

$$c_0 = 0, \qquad r = -2, \qquad \mu = 0, \qquad c = 4.$$

we find:

$$U_3(\xi) = 2 + \frac{2}{\xi} \tag{3.2.90}$$

where $\xi = x - 4t$

and

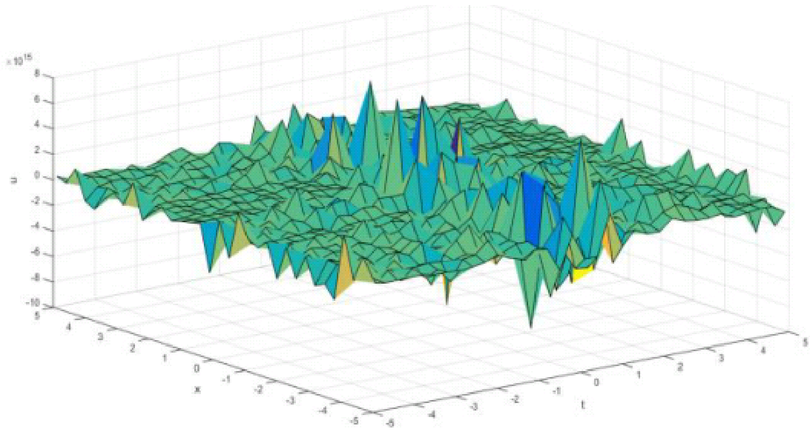

Figure 3.2.15: The graph of exact travlling wave solution of eq. (3.2.90)

$$U_6(\xi) = \left(\pm\frac{\lambda}{2}\sqrt{\frac{6r}{p}}\right) \pm \sqrt{\frac{6r}{p}}\left(\frac{G'(\xi)}{G(\xi)}\right) \tag{3.2.91}$$

In particular ,if we set

$$\lambda = 2, \qquad p = -3, \qquad c_1 \neq 0,$$

$$c_2 = 0, \qquad r = -2, \qquad \mu = 0, \qquad c = 4.$$

we find:

$$U_6(\xi) = -2cot(\xi) \tag{3.2.92}$$

where

$$\xi = x - 4t$$

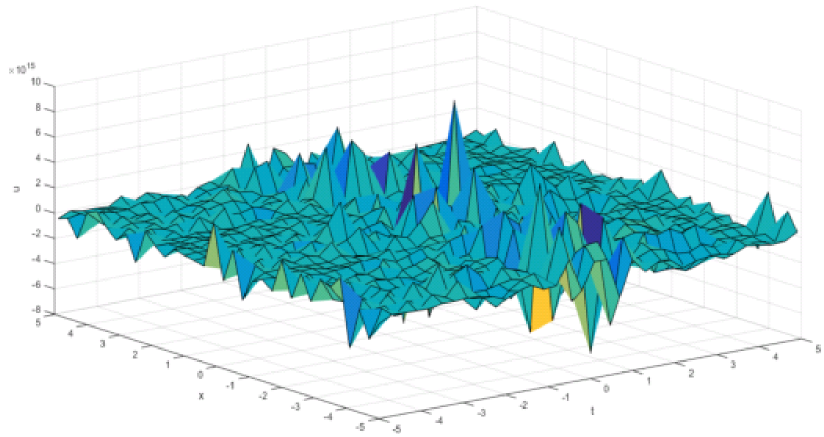

Figure 3.2.16: The graph of exact travlling wave solution of eq. (3.2.92)

Chapter 4

Application of The(G'/G)-Expansion Method of Blasius Equation

4.1 Introduction

Laminar boundary layers have long been the subject of numerous studies, since they play an important role in understanding the main physical features of boundary layer phenomena.
Generally, no closed-form solutions are available for laminar boundary value problems. Therefore, many researchers have resorted to various numerical or semi-analytical methods to solve such problems.
However, it is not an easy task to solve, numerically, such types of problem.
The main issue is how to model such problems (with infinite or semi-infinite domains) by a method of approximation with finite

grid spacing.

To tackle this issue in mathematical modeling of the problem, one can apply the infinite boundary condition at a finite boundary placed at a large distance from the object (i.e., truncated boundary). This,however, begs the question of what is a 'large distance' and, obviously, substantial errors may arise if the boundary is not placed far enough away.

On the other hand, pushing this out excessively far necessitates the introduction of a large number of grids to model regions of relatively little interest to the analyst. Obviously , when a low - order numerical method is used for the solution of boundary layer problems , many calculations should be done to accurately predict the location of the truncated boundary. Therefore, to accurately predict the location of the truncated boundary and to reduce the computational time, higher-order numerical methods should be used to model the boundary layer problems.

The Blasius boundary layer is an example of two dimensional boundary layer problems.

The Blasius problem models the behavior of a two-dimensional steady state laminar viscous of an incompressible fluid over a semi-infinite at plate [31],[32],[33].

The governing differential equation of the problem is

$$2f''' + ff'' = 0$$

The Blasius Equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Generally, a similarity solution is one in which number of variables can be reduced by one or more by some analytical means usually by coordinate transformation.

By applying a coordinate transformation and change of variables, Blasius reduced the partial differential equations to ordinary differential equation which he was able to solve.

Partial differentiation of the velocity components is done to obtain $\frac{\partial u}{\partial x}$,$\frac{\partial u}{\partial y}$ and $\frac{\partial^2 u}{\partial y^2}$ .These are substituted back in the simplified x-momentum equation and the following non-linear third order ordinary differential equation is obtained.

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = v\frac{\partial^2 u}{\partial y^2}$$

In dimensionless form, the boundary layer velocity profiles on a flat plate should be similar regardless of the location along the plate. For time-dependent flows, the time derivative of the velocity fields in the equations is not zero for one-dimensional flows $v = 0$. Thus the continuity equation reduces to $\frac{\partial u}{\partial x} = 0$. The x-momentum equation reduces to

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{-1}{p}\frac{\partial p}{\partial x} + v\frac{\partial^2 u}{\partial y^2}$$

Without the source term, this equation represents the fundamental equation for all transient diffusion problems and can be solved by analytical methods. For this purpose it is useful to consider the equation without the source term and the following equation holds for the molecular momentum transport.

$$\frac{\partial p}{\partial x} = 0$$

Consider an unsteady, one-dimensional fluid flow problem due to the oscillatory movement of a plate in a such way that the fluid movement created in the immediate vicinity of the plate is communicated to the fluid above the plate, by molecular momentum diffusion. The movement of the fluid above the plate is thus governed by the following partial differential equation [66]. The partial differential equation representing the motion is

$$\frac{\partial u}{\partial t} = v\frac{\partial^2 u}{\partial y^2}$$

$$u^2 + uu_x - u_t = 0$$

4.2 The Exact Solution of Blasius Equation By (G'/G) - Expansion Method

We will apply the generalized (G'/G) expansion method to construct the exact solution of the following KdV equation.

$$u^2 + uu_x - u_t = 0 \tag{4.2.1}$$

$$u(x,t) = u(\xi) \qquad where \qquad \xi = k_1 x + k_2 t.$$

the partial differential equation (PDE) is reduced to an OPE

$$u^2 + k_1 uu' - k_2 u' = 0 \tag{4.2.2}$$

Balancing u^2 and u' given n=1

$$u(\xi) = a_0 + a_1 \left(\frac{G'(\xi)}{G(\xi)}\right), \qquad a_0 \neq 0 \tag{4.2.3}$$

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{4.2.4}$$

where a_0, a_1, λ, and μ are arbitrary constants.

by eq.(4.2.3) and eq. (4.2.4) we derive

$$u^2 = a_0^2 + 2a_0 a_1 \left(\frac{G'(\xi)}{G(\xi)}\right) + a_1^2 \left(\frac{G'(\xi)}{G(\xi)}\right)^2 \tag{4.2.5}$$

$$u' = -\mu a_1 - \lambda a_1 \left(\frac{G'(\xi)}{G(\xi)}\right) - a_1 \left(\frac{G'(\xi)}{G(\xi)}\right)^2 \tag{4.2.6}$$

$$uu' = -\mu a_0 a_1 - \lambda a_0 a_1 \left(\frac{G'(\xi)}{G(\xi)}\right) - \mu a_1^2 \left(\frac{G'(\xi)}{G(\xi)}\right) - a_0 a_1 \left(\frac{G'(\xi)}{G(\xi)}\right)^2$$

$$- \lambda a_1^2 \left(\frac{G'(\xi)}{G(\xi)}\right)^2 - a_1^2 \left(\frac{G'(\xi)}{G(\xi)}\right)^3 \tag{4.2.7}$$

Substituting eqs.(4.2.3)-(4.2.7) into (4.2.2) and equating the coefficients of $\left(\frac{G'(\xi)}{G(\xi)}\right)^i$ to zero, we obtain a system of algebraic equation in a_0, a_1, a_2, λ, and μ as following

$$\begin{aligned}
a_0^2 - k_2\mu a_1 - k_1\mu a_0 a_1 + 2a_0 a_1 \left(\frac{G'(\xi)}{G(\xi)}\right)^1 \\
-k_1\lambda a_0 a_1 \left(\frac{G'(\xi)}{G(\xi)}\right) - k_2\lambda a_1 \left(\frac{G'(\xi)}{G(\xi)}\right) - k_1\mu a_1^2 \left(\frac{G'(\xi)}{G(\xi)}\right) \\
+a_1^2 \left(\frac{G'(\xi)}{G(\xi)}\right)^2 - k_1\lambda a_1^2 \left(\frac{G'(\xi)}{G(\xi)}\right)^2 - k_1 a_0 a_1 \left(\frac{G'(\xi)}{G(\xi)}\right)^2 \\
- k_2 a_1 \left(\frac{G'(\xi)}{G(\xi)}\right)^2 - k_1 a_1^2 \left(\frac{G'(\xi)}{G(\xi)}\right)^3 = 0.
\end{aligned} \tag{4.2.8}$$

$$\begin{aligned}
\left(\frac{G'(\xi)}{G(\xi)}\right)^0 &:= a_0 - k_1\mu a_0 a_1 + k_2\mu a_1 = 0 \quad , \\
\left(\frac{G'(\xi)}{G(\xi)}\right)^1 &:= 2a_0 a_1 - k_1\lambda a_0 a_1 - k_1\mu a_1^2 + k_2\lambda a_1 \quad , \\
\left(\frac{G'(\xi)}{G(\xi)}\right)^2 &:= a_1^2 - k_1 a_0 a_1 - k_1 a_1^2 + k_2 a_1 \quad , \\
\left(\frac{G'(\xi)}{G(\xi)}\right)^3 &:= -k_1 a_1^2 = 0 \quad .
\end{aligned} \tag{4.2.9}$$

Solving this system by Maple gives

$$a_0 = \frac{-\lambda}{2k_2}, \qquad a_1 = -k_2, \qquad k_2 = k_2, \qquad \mu = \mu, \qquad k_1 = 0. \tag{4.2.10}$$

$$u_i(\xi) = \left(\frac{-\lambda}{2k_2}\right) - k_2\left(\frac{G'(\xi)}{G(\xi)}\right), \qquad (i = 1, 2, ..., n \qquad solutions) \tag{4.2.11}$$

Case 1

When $(\lambda^2 - 4\mu) > 0$, we obtain the hyperbolic solution in the form

$$\frac{G'(\xi)}{G(\xi)} = \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} . \frac{c_1 \cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2 \sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}{c_1 \sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2 \cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)} \tag{4.2.12}$$

$$u_1(\xi) = \left(\frac{-\lambda}{2k_2}\right) - k_2\left(\frac{G'(\xi)}{G(\xi)}\right). \tag{4.2.13}$$

In particular ,if we set

$$\mu = 0, \qquad \lambda = 2, \qquad c_1 \neq 0, \qquad c_2 = 0, \qquad k_1 = 0, \qquad k_2 = 3. \tag{4.2.14}$$

we find:

$$u_1(\xi) = \frac{8}{3} - 3coth(\xi) \tag{4.2.15}$$

where

$$\xi = 3t$$

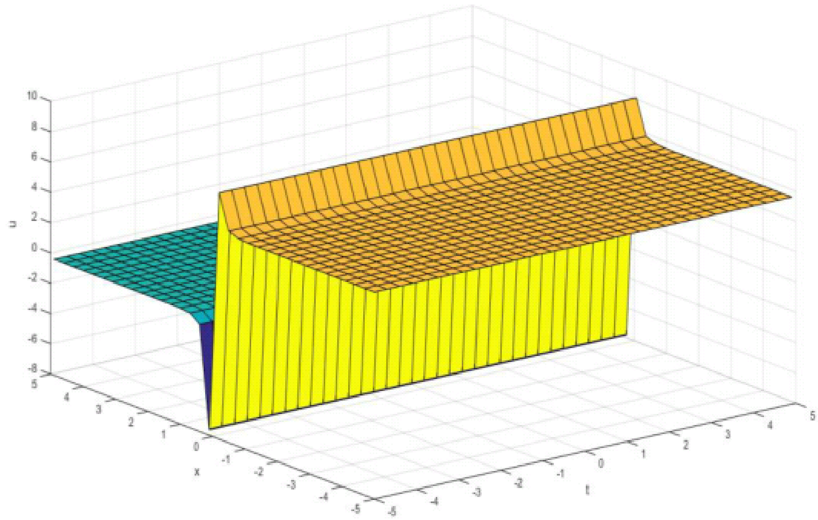

Figure 4.2.1: he graph of exact travelling wave solution of eq. (4.2.15)

Case 2

When $(\lambda^2-4\mu) < 0$, we have the trigonometric function solution in the form

$$\left(\frac{G'}{G}\right) = \frac{-\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}}{2}.\frac{-c_1\sin(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)+c_2\cos(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}{c_1\cos(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)+c_2\sin(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)} \tag{4.2.16}$$

$$u_2(\xi) = \left(\frac{-\lambda}{2k_2}\right) - k_2\left(\frac{G'(\xi)}{G(\xi)}\right). \tag{4.2.17}$$

In particular ,if we set

$$\mu = 0, \qquad \lambda = 2, \qquad c_2 \neq 0, \qquad c_1 = 0, \qquad k_1 = 0, \qquad k_2 = 3. \tag{4.2.18}$$

we find:

$$u_2(\xi) = \frac{8}{3} - 3cot(\xi) \tag{4.2.19}$$

where

$$\xi = 3t$$

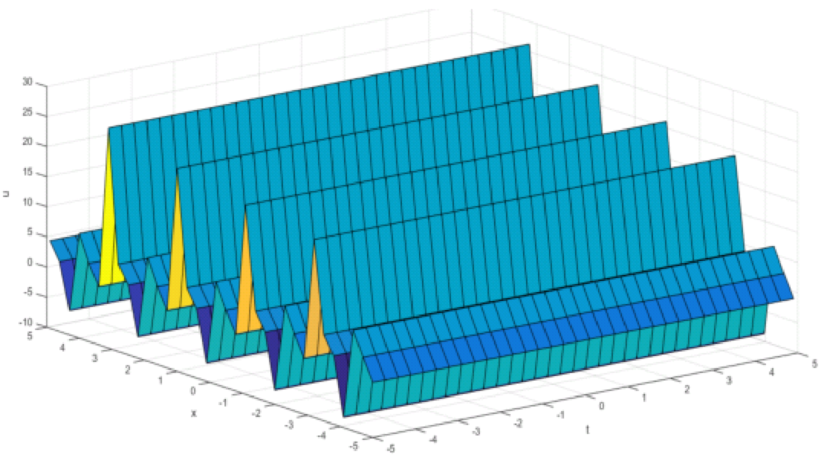

Figure 4.2.2: The graph of exact travelling wave solution of eq. (4.2.19)

Case 3

When $(\lambda^2 - 4\mu) = 0$,we get the rational function solution in the form:

$$\left(\frac{G'(\xi)}{G(\xi)}\right) = \frac{-\lambda}{2} + \frac{c_2}{c_1 + c_2\xi} \tag{4.2.20}$$

$$u_3(\xi) = \left(\frac{-\lambda}{2k_2}\right) - k_2\left(\frac{-\lambda}{2} + \frac{c_2}{c_1 + c_2\xi}\right) \tag{4.2.21}$$

In particular ,if we set

$$\lambda = 2, \qquad c_1 = 0, \qquad c_2 \neq 0 \qquad , \qquad , k_2 = 3. \tag{4.2.22}$$

we find:

$$\begin{cases} u_3(\xi) = \frac{4+8\xi}{1+\xi} \\ \\ \xi = x + 2t \end{cases} \tag{4.2.23}$$

Chapter 5

Application of The(G'/G)-Expansion Method of Schr$\ddot{o}$dinger Equation

5.1 Introduction

In mathematical physics, the Schr$\ddot{o}$dinger equation and the closely related Heisenberg equation are the most fundamental equations in non-relativistic quantum mechanics, playing the same role as Hamilton's laws of motion and the closely related Poisson equation in non-relativistic classical mechanics.
In relativistic quantum mechanics, it is the equations of quantum field theory which replace the role of Heisenberg's equation, while Schr$\ddot{o}$dinger equation does not directly have a natural analogue.
In pure mathematics, the Schr$\ddot{o}$dinger equation and its variants is one of the basic equations studied in the field of partial differ-

ential equations, and has applications to geometry, to spectral and scattering theory, and to integrable systems.

There are actually two (closely related) variants of Schr$\ddot{o}$dinger equation, the time dependent Schr$\ddot{o}$dinger equation and the time independent Schr$\ddot{o}$dinger equation, we will begin with the discussion of the time-dependent equation.

The time-independent Schr$\ddot{o}$dinger equation $H\psi = E\psi$ is of course an eigenvalue equation for the operator H.

If H was a self-adjoint transformation on a infinite - dimensional space then as is well known, there would only be a finite number of eigenvalues E for which the equation $H\psi = E\psi$ had a non-trivial solution; however since H acts on an infinite-dimensional space, the situation can be more complicated.

Indeed, H can have eigenfunctions ψ which lie in the domain $L^2(Rn^n)$ of H (consisting of square-integrable complex-valued functions), but it is also possible to have solutions $H\psi = E\psi$ which do not decay at infinity, but instead are bounded or grow at infinity; in fact, the behavior at infinity depends crucially on the value of E, and in particular whether it lies in one or more components of the spectrum of H, defined as the set of energies E for which the operator H - E fails to be invertible with a bounded inverse.

This leads to the spectral theory of Schr$\ddot{o}$dinger operators and their variants, which is a vast and active area of current research. Closely related to spectral theory is scattering theory, which is of importance in both physics and mathematics.

If the potential function U decays sufficiently quickly at infinity, and $k \in Rn^n$ is a non-zero frequency vector, then setting the energy level as $E := \frac{h^2|k|^2}{2m}$, the time-dependent Schr$\ddot{o}$dinger equation $H\psi = E\psi$ admits solutions $\psi(q)$ which behave asymptotically as $|q| \longrightarrow \infty$ as

$$\psi(q) \approx e^{ik.q} + f(\frac{q}{|q|}, k)\frac{e^{i|k|q|}}{r^{(\frac{n-1}{2})}}$$

for some canonical function $f : S^{n1} * Rn \longrightarrow C$, known as the scattering amplitude function.

This scattering amplitude depends (in a non-linear fashion) on the potential U , and the map from U to f is known as the scattering transform, and can be viewed as a non-linear variant of the Fourier transform.

A major area of study in scattering theory is understanding the relationship between properties of the potential U and properties of the scattering amplitude f, and in particular whether one can reconstruct the potential U from the scattering amplitudes.

This theory is important not just for the study of the Schr$\ddot{o}$dinger equation, but is (rather surprisingly) also useful for studying a number of integrable systems, for instance one can explicitly write the solution to the Korteweg - de Vries equation

$$\frac{\partial U}{\partial t} + \frac{\partial^3 U}{\partial t^3} = 6U\frac{\partial U}{\partial x}$$

by means of the one-dimensional scattering and inverse scattering transforms.

There are many generalizations and variants of the Schr$\ddot{o}$dinger equation; one can generalize to many-particle systems, or add other forces such as magnetic fields or even non-linear terms.

One can also couple this equation to other physical equations such as Maxwells equations of electromagnetism, or replace the domain Rn^n by another space such as a torus, a discrete lattice, a manifold, or alternatively one could place some impenetrable obstacles in the domain (thus effectively removing those regions of space from the domain).

The study of all of these variants is a vast and diverse field in both pure mathematics and in mathematical physics [108].

5.2 The Exact Solution of Schr$\ddot{o}$dinger Equation By (G'/G) - Expansion Method

We will apply the generalized (G'/G) expansion method to construct the exact solution of the following KdV equation.

$$u_t + u_{ttt} - 6uu_x = 0 \tag{5.2.1}$$

Using the transformation $u(x,t) = u(\xi)$ Where $\xi = k_1 x + k_2 t$ the PDE is reduced to an ODE

$$k_2 u' + k_2^3 u''' - 6k_1 uu' = 0 \tag{5.2.2}$$

Where the primes denot the derivativ with respect to ξ .
Balancing uu' and u''' i.e $n + n + 1 = n + 3$. There for, we assume the solution of (5.2.2) in the form

$$U(\xi) = a_0 + a_1 \left(\frac{G'(\xi)}{G(\xi)}\right) + a_2 \left(\frac{G'(\xi)}{G(\xi)}\right)^2 \tag{5.2.3}$$

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{5.2.4}$$

where a_0, a_1, a_2, λ, and μ are arbitrary constants.
by eq(5.2.1) and eq(5.2.2) we derive

$$U' = -\mu a_1 - \lambda a_1 \left(\frac{G'(\xi)}{G(\xi)}\right) - 2\mu a_2 \left(\frac{G'(\xi)}{G(\xi)}\right) - a_1 \left(\frac{G'(\xi)}{G(\xi)}\right)^2$$
$$- 2\lambda a_2 \left(\frac{G'(\xi)}{G(\xi)}\right)^2 - 2a_2 \left(\frac{G'(\xi)}{G(\xi)}\right)^3 \tag{5.2.5}$$

$$UU' = -\mu a_0 a_1 - \lambda a_1 a_2 \left(\frac{G'(\xi)}{G(\xi)}\right) - 2\mu a_0 a_2 \left(\frac{G'(\xi)}{G(\xi)}\right)$$

$$-\mu a_1^2 \left(\frac{G'(\xi)}{G(\xi)}\right) - \lambda a_1 a_2 \left(\frac{G'(\xi)}{G(\xi)}\right) - a_0 a_1 \left(\frac{G'(\xi)}{G(\xi)}\right)^2$$

$$-2\lambda a_0 a_2 \left(\frac{G'(\xi)}{G(\xi)}\right)^2 - \lambda a_1^2 \left(\frac{G'(\xi)}{G(\xi)}\right)^2 - \mu a_1 a_2 \left(\frac{G'(\xi)}{G(\xi)}\right)^2$$

$$-2a_0 a_2 \left(\frac{G'(\xi)}{G(\xi)}\right)^3 - a_1 a_2 \left(\frac{G'(\xi)}{G(\xi)}\right)^3 - \mu 2a_1 a_2 \left(\frac{G'(\xi)}{G(\xi)}\right)^3$$

$$-2\lambda a_1 a_2 \left(\frac{G'(\xi)}{G(\xi)}\right)^3 - 2\mu a_2^2 \left(\frac{G'(\xi)}{G(\xi)}\right)^3 - a_1 a_2 \left(\frac{G'(\xi)}{G(\xi)}\right)^4$$

$$- 2\lambda a_2^2 \left(\frac{G'(\xi)}{G(\xi)}\right)^4 - 2a_2^2 \left(\frac{G'(\xi)}{G(\xi)}\right)^5 \qquad (5.2.6)$$

Substituting eqs.(5.2.3)-(5.2.6) into (5.2.2) and equating the coefficients of $\left(\frac{G'(\xi)}{G(\xi)}\right)^i$ to zero, we obtain a system of algebraic equation in a_0, a_1, a_2, λ, and μ as following

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^0 : -k_2 a_1 u^2 - 6a_2 k_2^3 \lambda \mu^2 - k_2^3 a_1 \lambda^2 \mu + 6k_1 a_0 a_1 \mu^2 = 0$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^1 : -16a_2 k_2^3 \mu^2 + 6k_1 a_1^2 \mu + 12k_1 a_0 a_2 \mu - k_2 a_1 \lambda - 14a_2 k_2^3 \lambda^2 \mu$$

$$-2k_2 a_2 \mu - k_2^3 a_1 \lambda^3 + 6k_1 a_0 a_1 \lambda - 8k_2^3 \lambda \mu = 0$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^2 : -2k_2a_2\lambda - 8k_2^3a_2\lambda^3 + 6k_1a_0a_1 - k_2a_1 + 6k_1\lambda a_1^2 - 8\mu a_1k_2^3$$

$$+6k_1a_0a_1 - k_2a_1 + 6k - 1\lambda a_1^2 - 8\mu a_1k_2^3 - 52\lambda\mu a_2k_2^3$$

$$-7a_1\lambda^2k_2^3 + 12k_1a_0a_2\lambda + 18k_1a_1a_2\mu = 0$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^3 : 12\mu k_1a_2^2 - 2k_2a_2 - 40a_2\mu k_2^3 + 12k_1a_0a_2 + 18k_1a_1a_2\lambda$$

$$+6k_1a_1^2 - 38k_2^3a_2\lambda^2 - 12k_2^3a_1\lambda = 0$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^4 : -6a_1k_2^3 + 18k_1a_1a_2 + 12\lambda k_1a_2^2 - 54k_2^3a_2\lambda = 0,$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^5 : 12k_1a_2^2 - 24a_2k_2^3 = 0. \tag{5.2.7}$$

Solving this system by Maple gives:

The first of the solution:

$$\begin{cases} a_0 = \frac{1}{24}\frac{a_1^2+32k_2^4\mu+4k_2^2}{k_2^2}, \qquad a_1 = 2\lambda k_2^2, \\ a_2 = 2k_2^2, \qquad \mu = \mu, \qquad k_1 = k_2, \qquad k_2 = k_2. \end{cases} \tag{5.2.8}$$

The second of the solution:

$$\begin{cases} a_0 = \frac{1}{6} + \frac{1}{6}k_2^2\lambda^2, \qquad a_1 = 2\lambda k_2^2, \\ a_2 = 2k_2^2, \qquad \lambda = \lambda, \qquad \mu = \mu, \qquad k_1 = k_2. \end{cases} \tag{5.2.9}$$

$$U_i(\xi) = a_0 + a_1\left(\frac{G'(\xi)}{G(\xi)}\right) + a_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2$$
$$(i = 1, 2, ..., n \qquad solutions)$$

Case 1

When $(\lambda^2 - 4\mu) > 0$, we obtain the hyperbolic solution in the form:

$$\frac{G'(\xi)}{G(\xi)} = \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2}.\frac{c_1\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}{c_1\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)} \tag{5.2.10}$$

$$u(\xi)_1 = \frac{1}{24}\frac{a_1^2 + 32k_2^4\mu + 4k_2^2}{k_2^2} + 2\lambda k_2^2\left(\frac{G'(\xi)}{G(\xi)}\right) + 2k_2^2\left(\frac{G'(\xi)}{G(\xi)}\right)^2 \tag{5.2.11}$$

In particular ,if we set

$$\lambda = 2, \qquad \mu = 0, \qquad k_1 = 1, \qquad k_2 = 1, c_1 = 0, \qquad c_2 \neq 0.$$

$$\begin{cases} u_1(\xi) = \frac{-28}{24} + 2coth^2(\xi) \\ \\ \xi = x + t \end{cases} \tag{5.2.12}$$

and

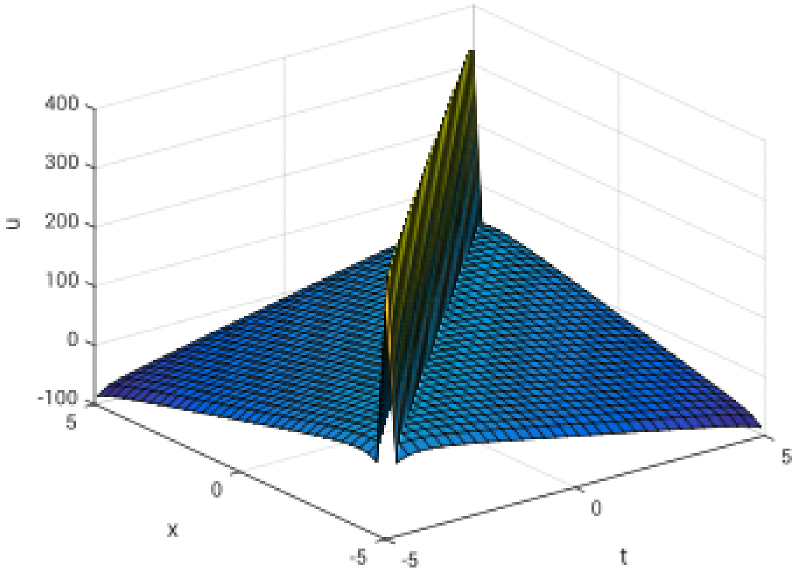

Figure 5.2.1: The graph of exact travelling wave solution of eq. (5.2.12)

$$u_2(\xi) = \frac{1}{6} + \frac{1}{6}k_2^2\lambda^2 + 2\lambda k_2^2\left(\frac{G'(\xi)}{G(\xi)}\right) + 2k_2^2\left(\frac{G'(\xi)}{G(\xi)}\right)^2 \tag{5.2.13}$$

In particular ,if we set

$k_1 = 2, \qquad k_2 = 1, \qquad \mu = 0, \qquad \lambda = 2, \qquad c_1 = 0, \qquad c_2 \neq 0.$

we find:

$$\begin{cases} u_2(\xi) = \frac{-7}{6} + 2tanh^2(\xi) \\ \\ \xi = x + t \end{cases} \tag{5.2.14}$$

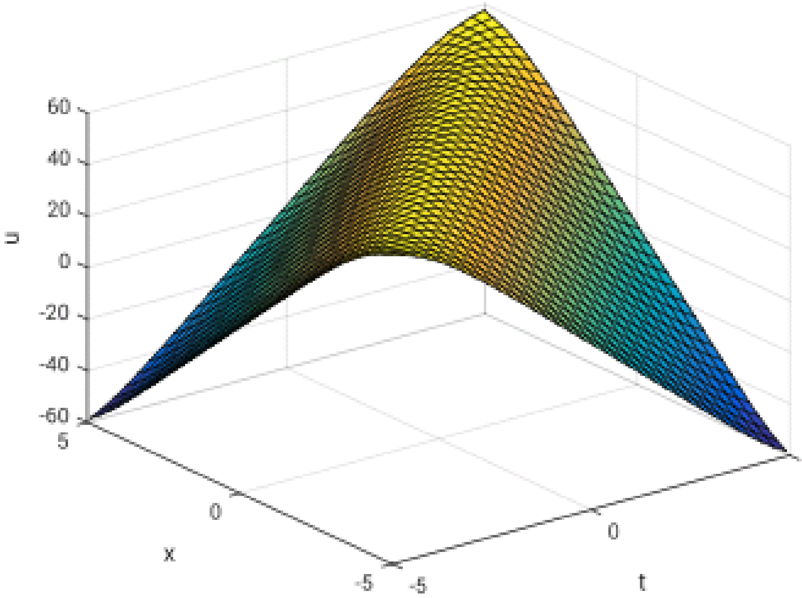

Figure 5.2.2: The graph of exact travelling wave solution of eq. (5.2.14)

Case 2

When $(\lambda^2 - 4\mu) < 0$, we have the trigonometric function solution in the form

$$\left(\frac{G'(\xi)}{G(\xi)}\right) = \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} \cdot \frac{-c_1 \sin(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2 \cos(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}{c_1 \cos(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2 \sin(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)} \tag{5.2.15}$$

$$u_3(\xi) = \frac{1}{24}\frac{a_1^2 + 32k_2^4\mu + 4k_2^2}{k_2^2} + 2\lambda k_2^2 \left(\frac{G'(\xi)}{G(\xi)}\right) + 2k_2^2 \left(\frac{G'(\xi)}{G(\xi)}\right)^2 \tag{5.2.16}$$

In particular ,if we set

$$\lambda = 2, \qquad \mu = 2, \qquad k_1 = 2, k_2 = 2, \qquad c_1 \neq 0, \qquad c_2 = 0 \tag{5.2.17}$$

we find:

$$\begin{cases} u_3(\xi) = 224 + 8cot^2(\xi) \\ \\ \xi = 2x + 2t \end{cases} \tag{5.2.18}$$

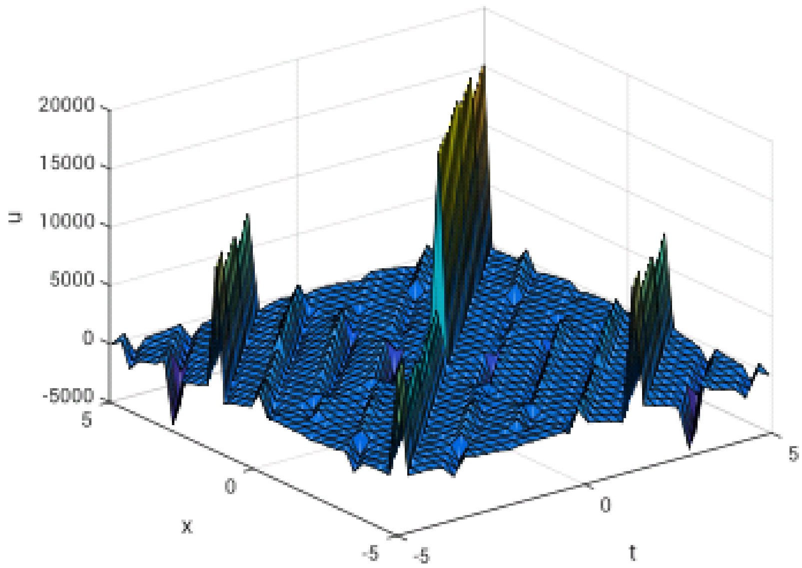

Figure 5.2.3: The graph of exact travelling wave solution of eq. (5.2.18)

and

$$u_4(\xi) = \frac{1}{6} + \frac{1}{6}k_2^2\lambda^2 + 2\lambda k_2^2\left(\frac{G'(\xi)}{G(\xi)}\right) + 2k_2^2\left(\frac{G'(\xi)}{G(\xi)}\right)^2 \tag{5.2.19}$$

In particular ,if we set

$$k_1 = 2, \qquad k_2 = 2, \qquad \mu = 1, \lambda = 2, \qquad c_1 = 0, \qquad c_2 \neq 0.$$

we find:

$$\begin{cases} u_4(\xi) = \frac{-31}{6} + 8tan^2(\xi) \\ \\ \xi = 2x + 2t \end{cases} \tag{5.2.20}$$

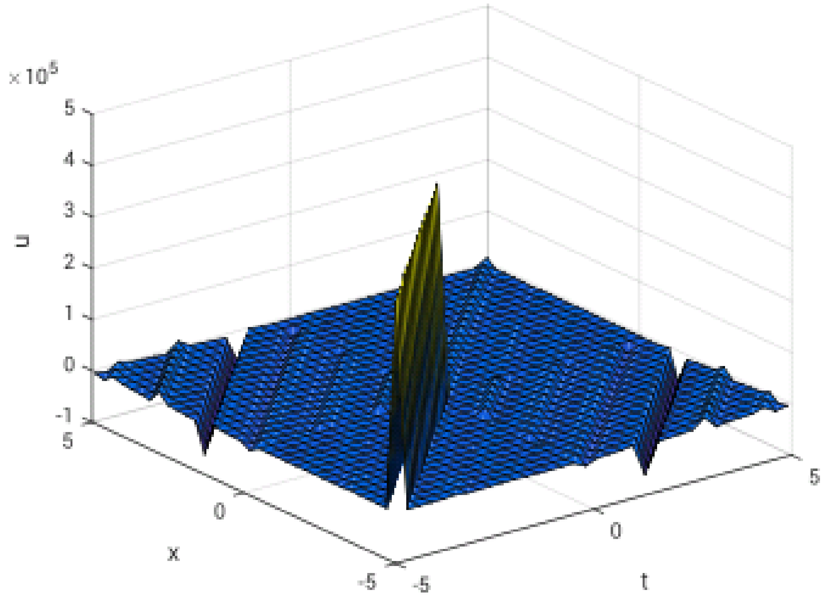

Figure 5.2.4: The graph of exact travelling wave solution of eq. (5.2.20)

Case 3

When $(\lambda^2 - 4\mu) = 0$,we get the rational function solution in the form

$$\left(\frac{G'(\xi)}{G(\xi)}\right) = \frac{-\lambda}{2} + \frac{c_2}{c_1 + c_2\xi} \tag{5.2.21}$$

$$u_5(\xi) = \frac{1}{24}\frac{a_1^2 + 32k_2^4\mu + 4k_2^2}{k_2^2} + 2\lambda k_2^2\left(\frac{G'(\xi)}{G(\xi)}\right) + 2k_2^2\left(\frac{G'(\xi)}{G(\xi)}\right)^2 \tag{5.2.22}$$

$$\xi = k_1 x + k_2 t,$$

In particular ,if we set

$$\lambda = 2, \qquad \mu = 1, \qquad k_1 = 1, k_2 = 1, \qquad c_1 = 1, \qquad c_2 = 1$$

we find:

$$\begin{cases} u_5(\xi) = \frac{1}{6} + \frac{2}{(1+\xi)^2} \\ \\ \xi = x + t \end{cases} \tag{5.2.23}$$

and

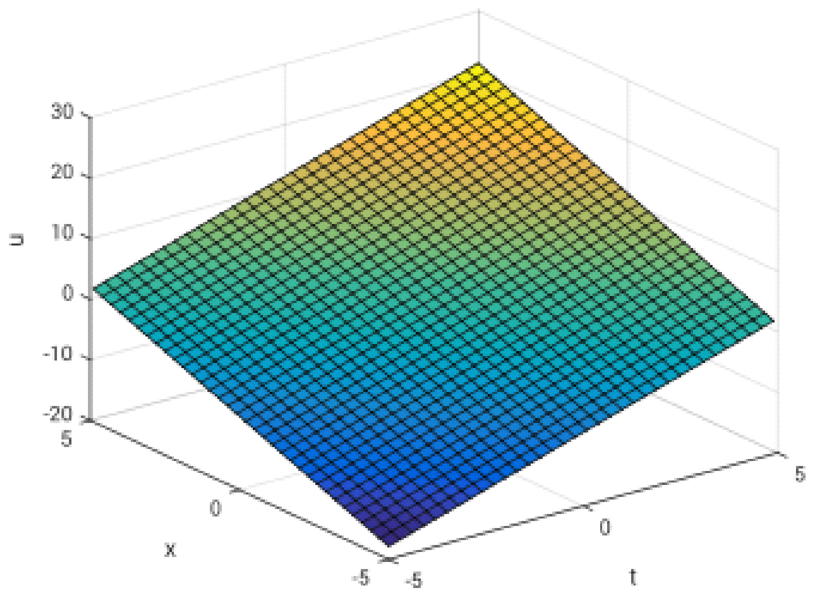

Figure 5.2.5: The graph of exact travelling wave solution of eq. (5.2.23)

$$u_6(\xi) = \frac{1}{6} + \frac{1}{6}k_2^2\lambda^2 + 2\lambda k_2^2 \left(\frac{G'(\xi)}{G(\xi)}\right) + 2k_2^2 \left(\frac{G'(\xi)}{G(\xi)}\right)^2 \tag{5.2.24}$$

$$\xi = k_1 x + k_2 t,$$

In particular ,if we set

$$\lambda = 2, \qquad \mu = 1, \qquad k_1 = 1, k_2 = 1, \qquad c_1 = 1, \qquad c_2 = 1$$

we find:

$$\begin{cases} u_6(\xi) = \frac{29}{6} + \frac{2}{(1+\xi)^2} \\ \\ \xi = x + t \end{cases} \tag{5.2.25}$$

Chapter 6

Application of The(G'/G)-Expansion Method of Klein-Gordon Equation

6.1 Introduction

Nonlinear evolution equations are widely used in a variety of fields such as fluid mechanics, quantum mechanics, solid state physics, plasma physics, population dynamics, chemical kinetics, nonlinear optics etc. [88],[97]. Exact analytic solutions of nonlinear equations are always in great demand to explain complex dynamics of the underlying physical systems.
In past, considerable efforts have been made to obtain exact analytical solutions of nonlinear equations with varying degree of success. For this purpose, a number of methods have been developed for obtaining explicit traveling wave solutions of nonlinear evolution equations [109].

Many complex real world problems in nature are due to nonlinear phenomena.

Nonlinear processes are one of the biggest challenges and not easy to control because the nonlinear characteristic of the system abruptly changes due to some small changes of valid parameters including time.

Seeking the exact solutions of nonlinear partial differential equations plays an significant role, when we want to understand the physical mechanism of the phenomena such as the wave phenomena observed in fluid dynamics [110], plasma and elastic media [111]and optical fibers [112] etc.

6.2 The Exact Solution of The Klein-Gordon Equation By $(G'/G)-$Expansion Method

We will apply the generalized (G'/G) expansion method to construct the exact solution of the following KdV equation.

$$u_{tt} + \alpha u'' + \beta u + \gamma u^3 = 0, \tag{6.2.1}$$

where α, β, γ are arbitrary constants balancing u'' with u^3 gives n=1

$$k_2 u'' + \alpha k_1 u'' + \beta u + \gamma u^3 = 0 \tag{6.2.2}$$

$$u(\xi) = \sum_{i=0}^{N} a_i \left(\frac{G'(\xi)}{G(\xi)}\right)^i$$

Then the solution is :

$$u(\xi) = a_0 + a_1 \left(\frac{G'(\xi)}{G(\xi)}\right), \qquad a_1 \neq 0 \tag{6.2.3}$$

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{6.2.4}$$

where a_0, a_1, λ, and μ are arbitrary constants.

by eq(6.2.4) and eq(6.2.5) we derive

$$u^3 = a_0^3 + 3a_0^2 a_1 \left(\frac{G'(\xi)}{G(\xi)}\right) + 3a_0 a_1^2 \left(\frac{G'(\xi)}{G(\xi)}\right)^2 + a^3 \left(\frac{G'(\xi)}{G(\xi)}\right)^3 \tag{6.2.5}$$

$$u'' = \lambda\mu a_1 + \lambda^2 a_1 \left(\frac{G'(\xi)}{G(\xi)}\right) + 2\mu a_1 \left(\frac{G'(\xi)}{G(\xi)}\right)$$
$$+ 3\lambda a_1 \left(\frac{G'(\xi)}{G(\xi)}\right)^2 + 2a_1 \left(\frac{G'(\xi)}{G(\xi)}\right)^3 \tag{6.2.6}$$

Substituting eqs.(6.2.3)-(6.2.6) into (6.2.2) and equating the coefficients of $\left(\frac{G'(\xi)}{G(\xi)}\right)^i$ to zero, we obtain a system of algebraic equation in a_0, a_1, a_2, λ, and μ as following

$$\gamma a_0 + \beta a_0^3 + k_1^2 \alpha\lambda\mu a_1 + k_2^2 \lambda\mu a_1 + \gamma a_1 \left(\frac{G'(\xi)}{G(\xi)}\right)$$
$$+k_2^2 \lambda^2 a_1 \left(\frac{G'(\xi)}{G(\xi)}\right) + 2k_2^2 \mu a_1 \left(\frac{G'(\xi)}{G(\xi)}\right) + 2k_1^2 \alpha\mu a_1 \left(\frac{G'(\xi)}{G(\xi)}\right)$$
$$+3\beta a_0^2 a_1 \left(\frac{G'(\xi)}{G(\xi)}\right) + k_1^2 \alpha\lambda^2 a_1 \left(\frac{G'(\xi)}{G(\xi)}\right) + 3\beta a_0 a_1^2 \left(\frac{G'(\xi)}{G(\xi)}\right)^2$$

$$+3\lambda a_1\left(\frac{G'(\xi)}{G(\xi)}\right)^2+\beta a^3\left(\frac{G'(\xi)}{G(\xi)}\right)^3+2a_1\left(\frac{G'(\xi)}{G(\xi)}\right)^3=0 \quad , \tag{6.2.7}$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^0 : \alpha a_0+\beta a_0^3+k_2^2\lambda\mu a_1+k_1^2\alpha\lambda\mu a_1=0,$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^1 := k_2^2\lambda^2 a_1+2k_2^2\mu a_0+k_1^2\alpha\lambda^2 a_1+2k_1^2\alpha\mu a_1$$

$$+\gamma a_1+3\beta a_0^2 a_1=0,$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^2 := 3k_2^2\lambda a_1+3k_1^2\alpha\lambda a_1+3\beta a_0 a_1^2=0,$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^3 := 2k_2^2 a_1+2k_1^2\alpha a_1+\beta a_1^3=0.$$

Solving this system by Maple gives:

The first of the solution:

$$a_0=a_0; \qquad a_1=1; \qquad k_1=k_1; \qquad k_2=k_2; \qquad \mu=\mu,$$

$$\gamma=k_2\left(\frac{6a_0^4k_2+\mu^2-4\mu^2a_0k_2-3a_0^2\mu}{a_0^2}\right); \qquad \lambda=\lambda,$$

$$\alpha=\frac{1}{2}k_2\left(\frac{-\mu+4a_0^2k_2}{a_0k_1^2}\right); \qquad \beta=-k_2\left(\frac{6a_2^2k_2-\mu}{a_0^2}\right) \tag{6.2.8}$$

The second of the solution:

$$a_0=0, \qquad a_1=-\alpha k_1^2, \qquad k_1=k_1, \qquad k_2=-\alpha k_1^2 \qquad \mu=\mu,$$

$$\gamma=k_2\left(-3k_2\lambda^2+2\mu\right), \qquad \lambda=\lambda, \qquad \beta=-2k_2+2 \tag{6.2.9}$$

Consequently, we have the following three types of exact solution of equation (6.2.2).

$$u_i(\xi) = a_0 + \left(\frac{G'(\xi)}{G(\xi)}\right), \qquad (i = 1, 2, ..., n \qquad solutions)$$

Case 1

When $(\lambda^2 - 4\mu) > 0$, we obtain the hyperbolic solution in the form:

$$\frac{G'(\xi)}{G(\xi)} = \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} . \frac{c_1 \cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2 \sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}{c_1 \sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2 \cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)} \tag{6.2.10}$$

$$u_1(\xi) = a_0 + \left(\frac{G'(\xi)}{G(\xi)}\right), \tag{6.2.11}$$

$$\xi = k_1 x + k_2 t,$$

In particular ,if we set

$$k_1 = 1, \qquad k_2 = 2, \qquad \lambda = 2, \qquad \mu = 0, \qquad c_2 = 0, \qquad c_1 \neq 0 \tag{6.2.12}$$

we find:

$$\begin{cases} u_1(\xi) = 1 + tanh(\xi) \\ \\ \xi = x + 2t \end{cases} \tag{6.2.13}$$

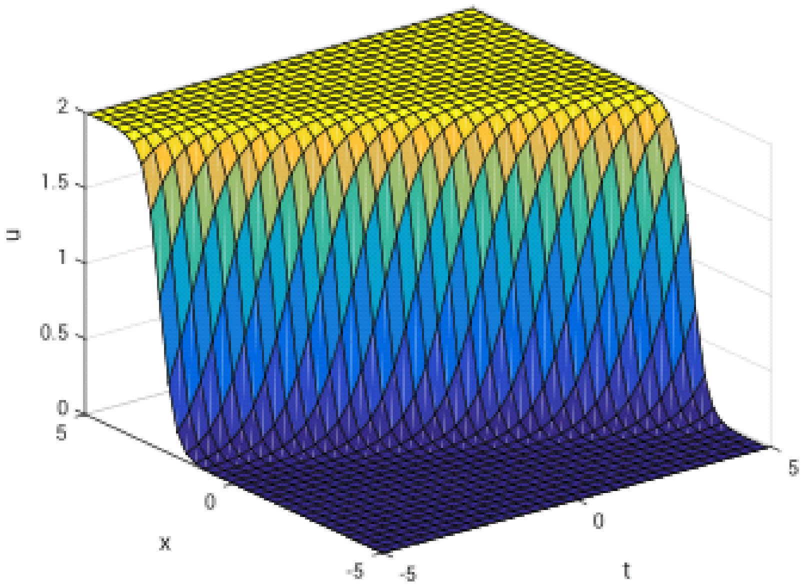

Figure 6.2.1: The graph of exact travelling wave solution of eq. (6.2.14)

and

$$u_2(\xi) = -\alpha k_1^2 \left(\frac{G'(\xi)}{G(\xi)} \right), \tag{6.2.14}$$

In particular ,if we set

$$k_1 = 2, \qquad k_2 = 1, \qquad \mu = 0, \qquad \lambda = 2, \qquad c_1 = 0, \qquad c_2 \neq 0.$$

we find:

$$\begin{cases} u_2(\xi) = 2tanh(\xi) - 1 \\ \\ \xi = 2x + t \end{cases} \tag{6.2.15}$$

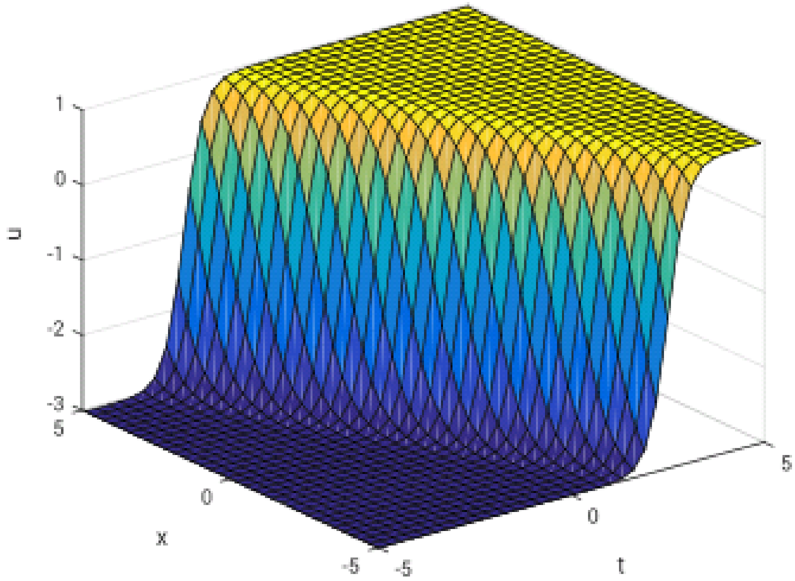

Figure 6.2.2: The graph of exact travelling wave solution of eq. (6.2.15)

Case 2

When $(\lambda^2 - 4\mu) < 0$, we have the trigonometric function solution in the form

$$\left(\frac{G'(\xi)}{G(\xi)}\right) = \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} \cdot \frac{-c_1 \sin(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2 \cos(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}{c_1 \cos(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2 \sin(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)} \tag{6.2.16}$$

$$u_3(\xi) = a_0 + \left(\frac{G'(\xi)}{G(\xi)}\right), \tag{6.2.17}$$

In particular ,if we set

$$\mu = 2, \qquad \lambda = 2, \qquad c_1 = 0, \qquad c_2 \neq 0, \qquad k_1 = 1, \qquad k_2 = 4.$$

we find:

$$\begin{cases} u_3(\xi) = 2\left(1 - tanh(\xi)\right) \\ \\ \xi = x + 4t \end{cases} \tag{6.2.18}$$

and

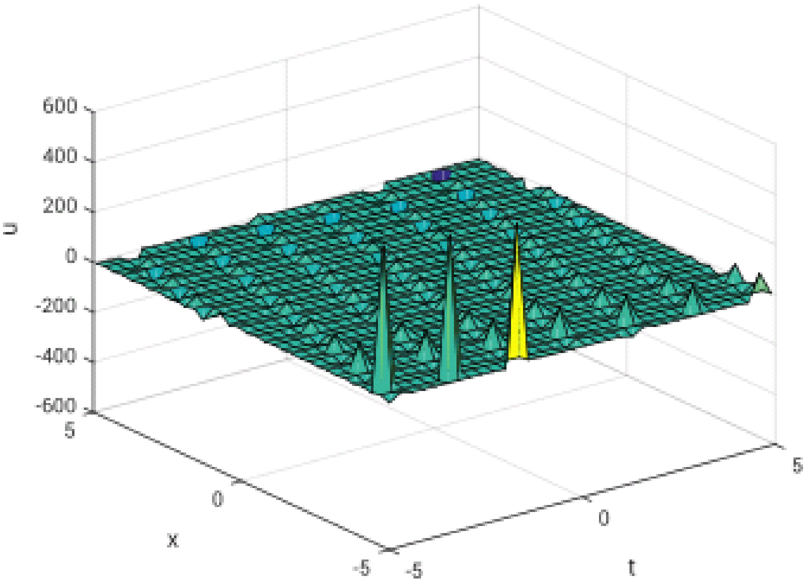

Figure 6.2.3: The graph of exact travelling wave solution of eq. (6.2.18)

$$u_4(\xi) = -\alpha k_1^2 \left(\frac{G'(\xi)}{G(\xi)}\right), \tag{6.2.19}$$

In particular ,if we set

$$k_1 = 2, \qquad k_2 = 1, \qquad \mu = 2, \qquad \lambda = 2, \qquad c_1 \neq 0, \qquad c_2 = 0.$$

we find:

$$\begin{cases} u_4(\xi) = 1 - 2cot(\xi) \\ \\ \xi = 2x + t \end{cases} \tag{6.2.20}$$

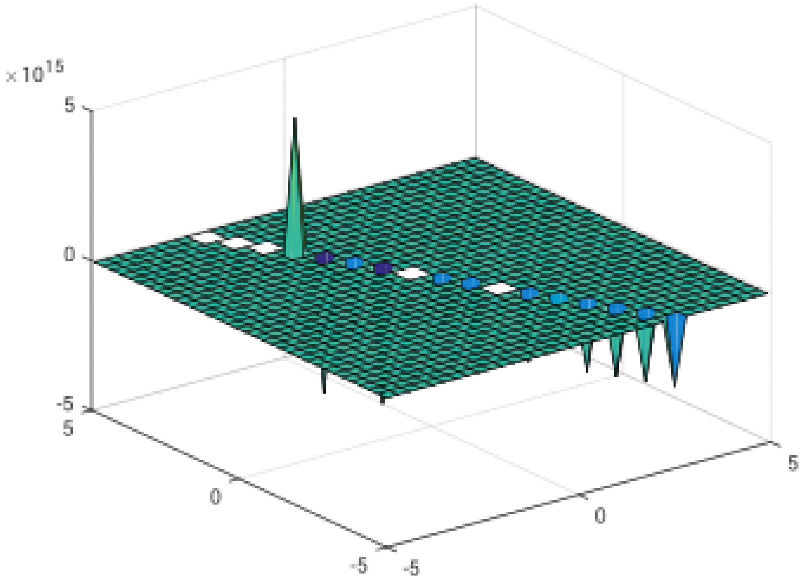

Figure 6.2.4: The graph of exact travelling wave solution of eq. (6.2.20)

Case 3

When $(\lambda^2 - 4\mu) = 0$,we get the rational function solution in the form

$$\left(\frac{G'(\xi)}{G(\xi)}\right) = \frac{-\lambda}{2} + \frac{c_2}{c_1 + c_2\xi} \tag{6.2.21}$$

$$u_5(\xi) = a_0 + \left(\frac{-\lambda}{2} + \frac{c_2}{c_1 + c_2\xi}\right) \tag{6.2.22}$$

$$\xi = k_1 x + k_2 t,$$

In particular ,if we set

$$\mu = 1, \qquad \lambda = 2, \qquad c_1 = 1, \qquad c_2 = 1, \qquad k_1 = 2, \qquad k_2 = 4.$$

we find:

$$\begin{cases} u_5(\xi) = \frac{\xi+2}{\xi+1} \\ \\ \xi = 2x + 4t \end{cases} \tag{6.2.23}$$

and

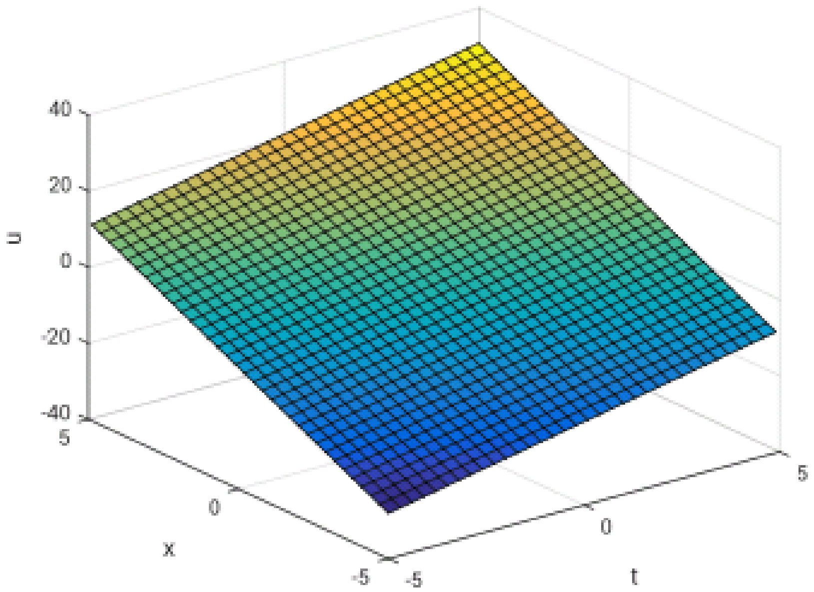

Figure 6.2.5: The graph of exact travelling wave solution of eq. (6.2.23)

$$u_6(\xi) = -\alpha k_1^2 \left(\frac{G'(\xi)}{G(\xi)} \right), \tag{6.2.24}$$

In particular ,if we set

$$k_1 = 1, \qquad k_2 = 1, \qquad \mu = 1, \qquad \lambda = 2, \qquad c_1 = 1, \qquad c_2 = 1.$$

we find:

$$\begin{cases} u_6(\xi) = \frac{-\xi}{1+\xi} \\ \\ \xi = x + t \end{cases} \tag{6.2.25}$$

Chapter 7

Comparison Between $(G'/G)-$ Expansion and Tanh-Methods

7.1 Introduction

The nonlinear equations of mathematical physics are major subjects in physical science, and various powerful methods have been presented, such as tanh method [7],[27],[29] sine-cosine method [16], Adomian decomposition method [19], Exp function method [12], (G'/G) expansion method [14], and many others [87].
We consider tanh and (G'/G) expansion methods.
The tanh method is one of most direct and effective algebraic method for finding exact solutions of nonlinear diffusion equations. This method presented by Malfliet [61] for the computation of exact traveling wave solutions. Malfliet used the tanh technique by introducing tanh as a new variable, since all deriva-

tive of a tanh are presented by a tanh itself.
This method has been applied by several researchers to obtain solution of different PDE, So far.
The second method, (G'/G) expansion method, has been proposed by Wang et al. [48] for the first time, to look for traveling wave solutions of nonlinear evolution equations.
The (G'/G) - expansion method is based on the assumptions that the travelling wave solutions can be expressed by a polynomial in (G'/G), and that (G'/G) satisfies a second order linear ordinary differential equation (ODE) [47],[48].
In this unit, it will be try to prove that (G'/G)-expansion method is a general form of the tanh-method.

7.2 The Comparison Application of (G'/G)-Expansion Method and Tanh-Methods

For more explanation and clarification of the new idea, we stated one example.

apply

Let's consider Dodd - Bullough - Mikhailov equation as the following form:

$$u_{xt} + e^{\mu} + e^{-\mu} = 0. \qquad (7.2.1)$$

We first use that will carry out the DBM into the following form

$$-\,wu'' + e^{u} + e^{-2u} = 0. \qquad (7.2.2)$$

Applying the transformation u = ln(v) Eq. (7.2.1) terns to

$$-\,w(VV'' - V'^2) + V^3 + 1 = 0 \qquad (7.2.3)$$

The (G'/G) - expansion method

To apply (G'/G) - expansion method, we consider the homogeneous balance between vv'' and v^3 in Eq. (7.2.3).
So we drive m = 2, and Eq. (7.2.3) turns to the following simple form

$$V(\xi) = a_0 + a_1\left(\frac{G'(\xi)}{G(\xi)}\right) + a_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2, \qquad a_2 \neq 0 \quad (7.2.4)$$

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \qquad (7.2.5)$$

by eq(7.2.4) and eq(7.2.5) we derive

$$V' = -\mu a_1 - \lambda a_1\left(\frac{G'(\xi)}{G(\xi)}\right) - 2\mu a_2\left(\frac{G'(\xi)}{G(\xi)}\right) - a_1\left(\frac{G'(\xi)}{G(\xi)}\right)^2$$

$$- 2\lambda a_2 \left(\frac{G'(\xi)}{G(\xi)}\right)^2 - 2a_2 \left(\frac{G'(\xi)}{G(\xi)}\right)^3 \qquad (7.2.6)$$

$$V'' = \lambda\mu a_1 + 2\mu^2 a_2 + \lambda^2 a_1 \left(\frac{G'(\xi)}{G(\xi)}\right) + 2\mu a_1 \left(\frac{G'(\xi)}{G(\xi)}\right)$$
$$+6\lambda\mu a_2 \left(\frac{G'(\xi)}{G(\xi)}\right) + 3\lambda a_1 \left(\frac{G'(\xi)}{G(\xi)}\right)^2 + 4\lambda^2 a_2 \left(\frac{G'(\xi)}{G(\xi)}\right)^2$$
$$+8\mu a_2 \left(\frac{G'(\xi)}{G(\xi)}\right)^2 + 2a_1 \left(\frac{G'(\xi)}{G(\xi)}\right)^3 + 10\lambda a_2 \left(\frac{G'(\xi)}{G(\xi)}\right)^3$$
$$+ 6a_2 \left(\frac{G'(\xi)}{G(\xi)}\right)^4 \qquad (7.2.7)$$

Substituting eqs.(7.2.4)-(7.2.7) into (7.2.3) and equating the coefficients of $\left(\frac{G'(\xi)}{G(\xi)}\right)^i$ to zero, we obtain a system of algebraic equation in a_0, a_1, a_2, λ, and μ as following

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^0 := 1 - 2wa_0a_2u^2 + a_0^3 - wa_0a_1\lambda\mu + wa_1\mu^2 = 0,$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^1 := 2wa_1a_2\mu^2 + wa_1\lambda\mu - 2wa_0a_1\mu - wa_0a_1\lambda^2$$
$$-6wa_0a_2\lambda\mu + 3a_0^2a_1 = 0,$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^2 := 2wa_2^2\mu^2 - 8wa_0a_1\mu - 3a_0a_1\lambda - 4wa_0a_1\lambda^2$$
$$+wa_1a_2\lambda\mu + 3a_0a_2 + 3a_0a_1^2 = 0,$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^3 := -wa_1\lambda - 2wa_0a_1 + a_1^3 - 2wa_1a_2\mu + wa_0a_2\lambda^2$$

$$+wa_2\lambda\mu - 10wa_0a_2\lambda + 6a_0a_1a_2 = 0,$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^4 := -6wa_0a_2 - wa_1^2 - wa_2^2 - 5wa_1a_2\lambda + 3a_1a_2$$

$$+3a_0a_2^2 = 0,$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^5 := -2wa_2^2 - 4wa_1a_2 + 3a_1a_2^2 = 0,$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^6 := -2wa_2^2 + a_2^3 = 0. \qquad (7.2.8)$$

Solving this algebraic equations, reads two sets of the solutions:

The first of the solution:

$$a_0 = \frac{2\mu + \lambda^2}{4\mu - \lambda^2} \quad , a_1 = \frac{6\lambda}{4\mu - \lambda^2}$$

$$a_2 = \frac{6}{4\mu - \lambda^2} \quad , w = \frac{3}{4\mu - \lambda^2}. \qquad (7.2.9)$$

The second of the solution:

$$a_0 = \frac{2\mu + \lambda^2}{4\mu - \lambda^2}\left(\frac{-1}{2} \pm \frac{\sqrt{3}}{2}i\right), \qquad a_1 = \frac{6\lambda}{4\mu - \lambda^2}\left(\frac{-1}{2} \pm \frac{\sqrt{3}}{2}i\right),$$

$$a_2 = \frac{6}{4\mu - \lambda^2}\left(\frac{-1}{2} \pm \frac{\sqrt{3}}{2}i\right), \qquad w = \frac{3}{4\mu - \lambda^2}\left(\frac{-1}{2} \pm \frac{\sqrt{3}}{2}i\right), \qquad (7.2.10)$$

where λ and μ are arbitrary constants.

By substituting (7.2.9) into (7.2.4), we have

$$v(\xi) = \frac{2\mu + \lambda^2}{4\mu - \lambda^2} + \frac{6\lambda}{4\mu - \lambda^2}\left(\frac{G'(\xi)}{G(\xi)}\right) + \frac{6}{4\mu - \lambda^2}\left(\frac{G'(\xi)}{G(\xi)}\right)^2 \qquad (7.2.11)$$

where

$$\xi = x - \frac{3}{4\mu - \lambda^2}t \tag{7.2.12}$$

Consequently, we have the following three types of exact solution of equation (7.2.2).

Case 1

When $(\lambda^2 - 4\mu) > 0$, we obtain the hyperbolic solution in the form

$$\frac{G'(\xi)}{G(\xi)} = \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} . \frac{c_1 \cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2 \sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}{c_1 \sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2 \cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)} \tag{7.2.13}$$

$$V_1(\xi) = a_0 + a_1\left(\frac{G'(\xi)}{G(\xi)}\right) + a_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2, \qquad a_2 \neq 0 \tag{7.2.14}$$

$$V_1(\xi) = -\frac{3}{2}\left(\frac{c_1 sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + c_2 cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{c_1 cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + c_2 sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}\right) - \frac{6}{16\mu - 4\lambda^2} + \frac{2\mu + \lambda^2}{4\mu - \lambda^2} \tag{7.2.15}$$

In particular, if

$$c_1 \neq 0, \qquad c_2 = 0, \qquad \lambda = 0 \qquad \sqrt{-\mu} = \eta, \qquad .$$

$$V_1(x,t)=\frac{1}{2}\left(1-3tanh\left(\frac{1}{2}\sqrt{\frac{-3}{w}}(x-wt)\right)\right)^2, \qquad w<0. \tag{7.2.16}$$

In particular, if

$$c_1=0, \qquad c_2\neq 0, \qquad \lambda=0 \qquad \sqrt{-\mu}=\eta.$$

$$V_1(x,t)=\frac{1}{2}\left(1-3coth\left(\frac{1}{2}\sqrt{\frac{-3}{w}}(x-wt)\right)\right)^2, \qquad w<0. \tag{7.2.17}$$

Case 2

When $(\lambda^2-4\mu)<0$, we have the trigonometric function solution in the form

$$V_2(\xi)=-\frac{3}{2}\left(\frac{c_1sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+c_2cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{c_1cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+c_2sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}\right) -\frac{6}{16\mu-4\lambda^2}+\frac{2\mu+\lambda^2}{4\mu-\lambda^2} \tag{7.2.18}$$

Case 3

When $(\lambda^2-4\mu)=0$,we get the rational function solution in the form

$$V_3(\xi)=\frac{2\mu+\lambda^2}{4\mu-\lambda^2}-\frac{6\lambda^2}{16\mu-4\lambda^2}+\frac{6}{4\mu-\lambda^2}\left(\frac{2c_2}{c_1+c_2\xi}\right)^2 \tag{7.2.19}$$

where c_1 and c_2 are arbitrary constants.

The Tanh - Method

Balancing vv'' with v^3 in Eq. (7.2.3), gives m = 2, hence we set the tanh assumption by

$$V(x,t) = s(Y) = \beta_0 + \beta_1 Y^1 + \beta_2 Y^2, \qquad Y = tanh(\eta\xi). \tag{7.2.20}$$

Substituting (7.2.20) into (7.2.3), collecting the coefficients of each power of Y , and using any symbolic computation program such as Mathematica:

$$\beta_0 = \frac{1}{2}, \qquad \beta_1 = \frac{-3}{2}, \qquad \eta = \frac{1}{2}\sqrt{\frac{-3}{w}}, \qquad w < 0, \tag{7.2.21}$$

This in turn gives the following solitary wave solutions:

$$V_1(x,t) = \frac{1}{2}\left(1 - 3tanh\left(\frac{1}{2}\sqrt{\frac{-3}{w}}(x - wt)\right)\right)^2, \qquad w < 0. \tag{7.2.22}$$

and

$$V_1(x,t) = \frac{1}{2}\left(1 - 3coth\left(\frac{1}{2}\sqrt{\frac{-3}{w}}(x - wt)\right)\right)^2, \qquad w < 0. \tag{7.2.23}$$

where Eqs. (7.2.22) and (7.2.23) are the same as Eqs. (7.2.16) and(7.2.17).

If we take $\lambda = 0$ and $\sqrt{-\mu} = \eta$, (7.2.9) turns to

$$a_0 = \frac{1}{2}, \qquad a_1 = 0, \qquad a_2 = \frac{-3}{2\eta^2}, \qquad w = \frac{-3}{4\eta^2}. \qquad (7.2.24)$$

So

$$a_i \eta^i = \beta_i, \qquad i = 0, 1, 2. \qquad (7.2.25)$$

Therefore, the obtained parameter in tanh-method can be calculated by applying(G'/G) - expansion method and substituting eq. (7.2.25).
As a result,(G'/G) - expansion method will compute all obtained solution in tanh-method.

Chapter 8

Conclusion and Future Outlook

In this research we have applied the (G'/G)-expansion method for solving the time dependent boundary layer.
This method has successfully solved the problem for the following reasons:
1- The generalized (G'/G) - expansion method is simple but it's results are very cumbersome.
The results of thees method contain many arbitrary constants compare to the results of other method.
The performance of generalized (G'/G)-expansion method is reliable, simple, direct, concise and gives more new exact solutions compared to the other method.
This method allowed us to solve more complicated PDEs in the mathematical physics.
2- An implementation of the(G'/G) -expansion method is given by applying it to three nonlinear equations to illustrate the validity and advantages of the method.
As a result,we have seen that three types of traveling wave solutions in terms of hyperbolic function solutions, trigonometric function solutions and rational function solutions with parameters are obtained.
The obtained solutions with free parameters may be important to explain some physical phenomena.

In this research shows that the devised algorithm is effective and can be used for many other NLEEs in mathematical physics.
3- Some of these solutions presented in this latter have been checked with Maple by butting them back into the original equations.
The obtained solutions of these nonlinear evolution equations have many potential applications in mathematical physics and engineering.
The performance of this method is reliable, simple and gives many new exact solutions.
Though the obtained solutions represent only a small part of the large variety of possible solutions for the equations considered, they might serve as seeding solutions for a class of localized structures existing in the physical systems.
The obtained solutions are in more general forms, and many known solutions to these equations are only special cases of them.
The solutions contain free parameters and therefore might be useful in different physical applications where the equations arise.
We observed that the new generalized (G'/G) -expansion method changes the given intricate problems into simple problems which can be solved easily.
The method will be functional for further studies of nonlinear evolution equations in applied sciences.

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