Sudan University of Science & Technology.

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Symmetry Methods in Lagrangian and Hamiltonian Systems

طـرق الـتـمـاثـل فـي نـــظـــم لاجـرانـج و هـمـلـتـون

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By:

Idris Hussein Gamar Ali

Supervisor:

Prof. Mohamed Ali Bashir

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Contents

2 Symmetries 30

Dedication

I would like to dedicate this dissertation to my father and my mother the best teachers in my life; they have made me a real person. Also to my brothers and sisters for their forever moral support and to my children always make good to me.

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ABSTRACT

The aim of this research is to discuss and study symmetries of Lagrangian and Hamiltonian systems using Lie algebra of the symmetry Lie groups. In particular conservation laws for invariant variational based on Noether theorem. We introduced analytical and geometrical formulation of Lagrangian and Hamiltonian systems that contain symmetry rules on the vector space by using classical variational calculus. Also we obtained the reduction of controlled Lagrangian and Hamiltonian systems with symmetry. Finally we classify the symmetry groups of Hamiltonian system with degrees of freedom and we provided some application of symmetries of Lagrangian and Hamiltonian systems.

الخلاصة

الهدف من هذا البحث هو المناقشة و دراسة نظم التماثلات لأنظمة لاجرانج وهاملتون مستخدمين جبر لي للتغايرية لزمر لي المتماثلية. بصفة خاصة قوانين البقاء اللامتغيرة علي اساس مبرهنة نوثير. قمنا بإدخال الصيغة التحليلية والهندسية لأنظمة لاجرانج وهاملتون التي تحتوي قواعد التماثل علي فضاء المتجه بواسطة إستخدام حسبان التغاير التقليدي. وأيضا تحصلنا علي الاختزالية للأنظمة الخاضعة للاجرنج و هاملتون مع التماثل. واخيرا صنفنا زمر التماثل لنظام هاملتون مع درجات حرية واشترطنا بعض التطبيقات للتماثلات لأنظمة لاجرانج وهاملتون.

Introduction

Conservation laws and symmetries have always been of considerable interest in science. They are important in formulation and investigation of many mathematical models. They were used for providing global existence theorems. Roughly speaking, symmetry of a geometrical - object is a transformation whose action leaves the object apparently unchanged. We consider the local one-parameter point transformations to move for local one-parameter transformation group to obtain conserved quantities associated to an infinitesimal symmetry. What was important in these discoveries are:

In chapter one we study the Lie groups and their Lie algebra as essential tools in the study of general mathematical fields. These include topological manifolds, differential manifolds, partial differential equations, differential forms in general. Invariance, vector fields and tangent space at the identity and action of groups are presented. Finally we introduce exponential map and representation of Lie group.

In chapter two we study the symmetries of differential equations which is helpful to consider symmetries of simpler objects. We introduce the local one-parameter point transformations, local one-parameter transformation of group and we use group operator to introduce the prolongation formulas. Invariant functions, generate point symmetries of dependents and independents variables, Lie group of heat equation is used. Also we used invariant solution to solve the differential equations.

In chapter three, we introduce Lagrangian and Hamiltonian systems that contain symmetry rules on the vector space by using classical variational calculus, the Euler-Lagrange equations and Hamilton's equations are obtained. We divided this chapter in to analytical and geometrical formulation with the continuous system of N degrees of freedom, state as Noether's theorems, also we use symplectic form to obtained a symmetry of Hamiltonian system.

Mechanics has two main system Lagrangian and Hamiltonian mechanical system.

In Lagrangian mechanical system is based on variational principles and it generalized directly to the general relativistic context. Hamiltonian mechanics is based directly on the energy concept and it's closely connected to quantum mechanics.

In chapter four the quantum will be based on our chapter and we deal with the concepts of the Classical mechanics. An important concept is that the equations of motion of the Classical Mechanics which is based on a variational principle, that along a path describing classical motion the action integral (Hamiltonian Principle of Least Action).

The results of variational calculus derived allow us to formulate the Hamiltonian principle of Least Action of Classical Mechanics and study its equivalence to the Newtonian equations of motion. We used Gauge symmetry to obtain transformation for Lagrangian. The Euler operator is importance and we obtain continuous symmetries and conservation laws of Noether's, also we obtain the reduction of controlled Lagrangian system with symmetry and we will present reduction of controlled Hamiltonian systems with symmetry.

In chapter five, we discussed the relation between one parameter continuous symmetries of dynamics, defined on physical grounds, and conservation laws. In the Hamiltonian formulation, such symmetries of the dynamics in general leave the Hamiltonian invariant only up to a total derivative, $dG(q)/dt$. And we study the infinitesimal symmetries, New- tonoid vector fields, infinitesimal Noether symmetries and conservation laws of Hamiltonian systems. Finally we classify the symmetry groups of an autonomous Hamiltonian system with two degrees of freedom. With the exception of the harmonic oscillator or a free particle where the dimension is 15, we obtain all dimensions between 1 and 7. For each system in the classification we examine integrability. In chapter six we obtain some applications of symmetries for Lagrangian and Hamiltonian systems.

Chapter One

Lie Groups

1.1Introduction

Lie group are named after the nineteenth century by Norwegian mathematician Sophus Lie who laid the function of the theory of continuous transformation group

Lie group represent the best developed theory of continuous symmetry of mathematical objects and structure, which makes them in dispensable tools for many parts of contemporary mathematic, as well as for modern theoretical physics. They provide a natural frame work for analyzing the continuous symmetries of differential equations, in much the same way as permutation groups are used in Galois Theory for analyzing the discrete symmetries of algebraic equations .An extension of Galois Theory to the case of continuous symmetry groups was one of Lie principle motivations.

The circle of center 0 and radius 1 in the complex plane is a Lie group with complex multi- plication.

A lie groups and their Lie algebras are essential tools in the study of general mathematical field. These include partial differential equations, physical fields and their classification, homogenous spaces, symmetric spaces and differential geometry in general.

In fact Lie groups are endowed with two structures.

1. Group structure

2. Differentiable structures

1.2Topological Manifolds

A manifold M of dimension n or n-manifold is a topological space with following properties

1- M is Hausdorff
2- M is a locally Euclidean of dimension n
3- M has a countable basis of open sets

The dim M is used for dimension of M, when dim M = 0, then M is countable space with the distance topology. It follows from the homeomorphism of U and U' that locally Euclidean is equivalent to the requirement that each point p have a neighborhood U homeomorphic to an n- ball in R^n . Thus a manifold of dimension 2 is locally homeomorphic to an open disk.

1.3 Chart:

Definition1.3.1 A local chart on M is the pair $(U_i , \emptyset)$ consisting of :

1. An open U_i of M

2. A homeomophric ∅ of U_i on to an open subset $\emptyset(U_i)$

The open U_i is called domain of the chart or we can define a chart is a pair (U, ∅) where U is an open set in X and $\emptyset: U \to R^n$ is homeomorphism on to its image. The components of

$\emptyset = (x^1, x^2, \dots, x^n)$ are called coordinates.

Given, two charts $(U_1 , \emptyset_1)$and $(U_2 , \emptyset_2)$ then we get overlap or transition maps

$\emptyset_2 \circ \emptyset_1^{-1} : \emptyset_1 (U_1 \cap U_2) \rightarrow \emptyset_2 (U_1 \cap U_2)$ and $\emptyset_1 \circ \emptyset_2^{-1} : \emptyset_2 (U_1 \cap U_2) \rightarrow \emptyset_1(U_1 \cap U_2)$.

Definition1.3.2 Two charts ($U_1, \emptyset_1$) and ($U_2, \emptyset_2$) are called compatible if the overlap maps are smooth.

Definition1.3.3 Two charts ($U_j, \emptyset_j$)and ($U_k, \emptyset_k$) on M , such that $U_j \cap U_k \neq 0$ are called C^q-compatible ($q \geq 1$) if the overlap mapping $\emptyset_{kj}$ = $\emptyset_k \circ \emptyset_j^{-1} / U_j \cap U_k$ is a C^q-diffeomorphism between the open $\emptyset_j(U_j \cap U_k)$ And $\emptyset_k(U_j \cap U_k)$ of R^n

1.4 Local Coordinates

The local coordinate x^i of a point p belong to the domain U of chart (U, $\emptyset$) of M is the co- ordinates of points $\varphi(p)$ of R^n

1.5 Differential Manifolds

A differentiable or C^∞(or smooth) structure on a topological manifold M is a family

u = (U_α , φ_α) of coordinate neighborhoods such that

1. The U_α cover M

2. For any α, β the neighborhood U_α , φ_α and U_β , φ_β are C^∞ -compatible.

Any co ordinate neighborhood V, ψ compatible with every U_α ,$\varphi_\alpha \in u$ is itself in u

A C^∞ manifold is a topological manifold together with C^∞ -(differentiable) structure.

Definition1.5.1 A differentiable manifold is a pair of Hausdorff space with countable basis and atlas and also it's a manifold if for every point of space there exist an admissible local chart (U ,∅) such that (U ,∅) $\subset$ R^n

For instance:

sphere S^n , in R^{n+1} let us consider the n-sphere

$$S^n = \{x = (x^1, \ldots, x^{n+1}) \mid \textstyle\sum_{i=1}^{n+1}(x^i)=1\}$$

Answer:

To provide S^n With a differentiable manifold structure we define an atlas consisting of 2n +2charts (1$\leq i \leq n+1$)

$$U_i^+ = \{x \in S^n \mid x^i > 0\} : \text{and } U_i^- = \{x \in S^n \mid x^i < 0\}$$

the sphere S^n is really covered with such charts.

Now we must construct transformations between charts (change of charts) which are C^∞ diffeomorphism. Let us consider

$$\varphi_1^+ : U_i^+ \to R^n : x = (x^1, \ldots, x^{n+1}) \to x = (x^1, \ldots, x^{\wedge i}, \ldots, x^{n+1})$$

Where the symbol ^ means ith coordinates are removed. It is a way the orthogonal projection of the “positive hemisphere” onto the corresponding equatorial “plane”. That is really bi continuous bijection. Analogically we define

$$\varphi_1^- : U_i^- \to R^n : x = (x^1, \ldots, x^{n+1})$$

For instance, let us consider any point x of $U_i^+ \cap U_j^+$ such that the ith and jth coordinates are positive. The following mapping between opens of R^n

$\varphi_j^+ \circ (\varphi_i^+)^{-1} : \varphi_i^+ (U_i^+ \cap U_j^+) \rightarrow \varphi_j^+(U_i^+ \cap U_j^+) :(x^1, \dots, x^{\wedge i}, \dots, x^{n+1})$

$\rightarrow (x^1, \dots, x^{j-1}, \sqrt{1 - \sum_{\{k=1 \atop k \neq i}^{n+1} (x^k)^2} , \dots, , \dots, x^{\wedge i}, \dots, x^{n+1})$

is actually a diffeomorphism. A difficulty could have occurred because of the square root but the expression under the radical sign is always positive.

1.6 Lie Groups

A Lie group G is a differentiable manifold which is also a group such that the product and inverse operations are differentiable this mean that for any x , y in G we have.

$m: G \times G \rightarrow G , m(x, y) = xy$ (Multiplication) is a differentiable

$Inv: \quad G \rightarrow G , \ I(x) = x^{-1}$ (Inversion) is a differentiable "C^{∞}"

$I(x_1, x_2) \rightarrow x_1 \ x_2^{-1}$

Example1.6.1 The $GL(n, R)$ is a Lie group called the real general linear group, completely analogously we gave the Lie group is a group under the operations

$$\mathrm{m(A\,,B) = AB\,, and\ I(A) = \ A^{-1} = \frac{adjA}{detA}}$$

Example1.6.2 The complex general linear group $\mathrm{GL}(n , ₵) = \{\mathrm{A} = M_n(₵)/\mathrm{detA} \neq 0\}$.

Example1.6.3 The orthogonal group $\mathrm{O}(n) = \{\mathrm{A} \in M_n(R)/\mathrm{AA^T} = 1\}$ is a Lie group as a sub group and sub manifold of $GL(n , \mathrm{R})$

Example 1.6.4 The following are examples of Lie groups.

1. R^n With the group operation given by addition

$(R^n ,*) and \ (R^+,*)$, $S^1 = \{Z \ \in ₵ : |Z| = 1\}$.

2.$GL(n, \mathrm{R}) \subset \mathrm{R}^{n^2}$, many of the groups we will consider will be sub groups of

$GL(n , \mathrm{R})$ or $(n , ₵)$.

3.$\mathrm{SU}(2) = \{\mathrm{A} \in \mathrm{GL}(2, ₵)/\mathrm{A}\overline{\mathrm{A}}^{\mathrm{t}} = 1 , \det \mathrm{A} \ = 1\}$, indeed one can easily see that.

$$SU(2) = \left\{ \begin{pmatrix} -\frac{\propto}{\beta} & \frac{\beta}{\propto} \end{pmatrix} : \propto, \beta \ \in ₵ , |\propto|^2 + |\beta|^2 = 1 \right\}$$

Writing $\propto = \ x_1 + ix_2, \beta = x_3 + ix_4, x_i \in R$

We see that *SU (2)* is diffeomorphic to

$$S^3 = \{x_1^2 + \ x_2^2 + \ x_3^2 + \ x_4^2\} \ \subset R^4$$

Recall that a space is simply connected it every closed curve (a loop) can be contracted to a point.

Clearly, this is not true for a curve that wraps a round S^1.

A general (topological) space is compact if each open cover contains a finite cover this are rather abstract (through important) notion. Luckily, for subset of R^n , there is a simpler criterion: They are if they are closed and bounded.

Clearly, SO (2) and $SU(2)$ are compact (note that we did not need to introduce parameters for $So(2)$ to see this).

Example of a non compact would be $So(1,1)$, the Lorentz group in two dimensions. It is defined as the group of linear transformations of R^2 which leave the indefinite inner product.

$$\langle \bar{v}^{\rightarrow}, \bar{u}^{\rightarrow} \rangle = \ v_1 u_1 - v_2 u_2$$

Invariant, and have determinant one, it can be written similarly to $SO(2)$ as

$$\Lambda = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, a^2 - b^2 = 1$$

And parameterized by $x \in R$ as

$$\Lambda(x) = \begin{pmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{pmatrix}$$

Hence, as manifold, so $\mathrm{So}\,(1,1) \cong R$ .Actually since $\Lambda(x)\Lambda(\mathcal{E}) = \Lambda(x + \xi)$,

This isomorphism hold for the group as will.

Theorem1.6.5 (close sub group) Let G be a Lie group and H<G a closed subgroup of G, then, H is a Lie group in the induce topology as an embedded sub manifold of G.

Theorem 1.6.6 If H is a regular sub manifold and sub group of a Lie group G, Then H is a closed as a sub set of G.

Proof: It is enough to show that whenever a sequence $\{h_n\}$ of the elements of H has a limit $g \in \mathrm{G}$, then g is in H.

Let U, φ be a preferred coordinate neighborhood of the identity e relative to the regular sub manifold H, then:

$\varphi(\mathrm{U}) = C_E^m(o)$ is a cube with $\varphi(\mathrm{e}) = \mathrm{o}$, $\mathrm{V} = \mathrm{H} \cap \mathrm{U}$ consists exactly of those point whose last *m* – *n* coordinates are zero , and $\varphi' = \varphi|_{\mathrm{v}}$ maps V homeomorphically on to this slice to the cube if $\{h_n^{\sim}\}$ is a sequence in V = H ∩ U and $\lim h_n^{\sim} = \tilde{g}$ with $\tilde{g} \in \mathrm{U}$, then last *m* – *n* coordinates of $\tilde{g}$ are also zero so $\tilde{g} \in \mathrm{H} \cap \mathrm{U} \subset \mathrm{H}$.

Now let $\{h_n\}$ be any sequence of H with $\lim h_n = g$ and let W be a neighborhood of e small enough so that $\mathrm{W}^{-1}\mathrm{W} \subset \mathrm{U}$, where $\mathrm{W}^{-1}\mathrm{W} = \{x^{-1}y \in G|_{x,y} \in W\}$ such W exist be continuity of the group operations.

There exist *N* such that for $n \geq N, h_n \in g$ W, in particular $h_N \in g$ W, using group operations, we may verify that

(1) $\tilde{g} = g^{-1}h_N \in$ W and setting $h_n = h_n^{-1}h_N$, we have.

(2) $\lim h_{\tilde{n}} = \tilde{g}$. But for $n \geq N, h_{\tilde{n}} = h_n^{-1}h_N$ lies in $(\mathrm{gW})^{-1}\mathrm{gW} = \mathrm{W}^{1-}\mathrm{W} \subset$ U.Thus $\breve{g} \in \mathrm{H}$, and hence $g = h_N\tilde{g}^{-1} \in \mathrm{H}$ which was to be proved.

Corollary1. 6.7 If G and G[1] are Lie groups and $\emptyset: G \rightarrow G^1$ is a continuous homomorphism then $\emptyset$ is smooth.

From the close sub group theorem we can generate quite a few more examples of Lie groups.

Examples 1.6.8. The real special linear group $SL(n\,,\mathrm{R})$ where $\mathrm{SL}(n\,,\mathrm{R}) = \{\mathrm{A} \in \mathrm{CL}(n,\mathrm{R}) / \det \mathrm{A} = 1\}$

The complex special linear group $SL(n\,,₵) = \{A \in GL(n\,,₵) / \, detA \; = 1\}$

The special orthogonal group, $So(n\,,R) = SL(n,R) \cap O(n)$.

The unitary group $\mathrm{U}(n) = \{\mathrm{A} \in \mathrm{GL}(n\mathrm{n}\,,\mathrm{R}) / \mathrm{AA}^* = 1\}$ (where A^* denotes the Harmitian transpose of A)

The special unitary group, $U(N) = U(n) \cap SL(n\,,₵)$.

1.7 Some Differential Geometry

Since Lie groups are analytic manifolds, we can apply the apparatus of differential geometry. In particular will turn out that almost all information about the Lie group is a contained in its tangent space at the identity, the Lie algebra, intuitively, the tangent space is just that: the space of all tangent vectors, i.e. all possible "directions" at the given point .When considering sub manifolds, the tangent space can visualized as a plane touching the manifold at the point g. Mathematically, the notion of a tangent vector is formalized as a differential operator, this makes intuitive sense since a tangent vector corresponds to "going ''in to a particular direction with a certain "speed", i.e. the length of the vector, we notice that we move because things around we change. Hence it is reasonable that tangent vectors measure changes, i.e. they are derivatives.

1.8 Tangent Vectors

We introduce a bit of machinery: A curve is a differentiable map

K: R $\beth I \to G$.

Where I is some open interval (note that the map itself is the curve, not just the image).

Definition 1.8.1 Let k :$(-\varepsilon, \varepsilon) \to G$ be a curve with k (0) = g. the tangent vector of k in g is the operator that maps each differentiable function f: G $\to$ K to it's directional derivative along.

X: f$\to X[f] = \frac{d}{dt} f(k(t)) \Big|_{t=0}$

The set of all tangent vectors in g is called the tangent space of G in g, $T_g G$.

This is naturally a vector space: for two tangent vectors X and Y and a real number λ, define the sum and multiple by

(X+Y)[f] = X[f] + Y[f], (X λ)[f] = $\lambda X[f]$

One can find curves that realize the vectors on the right- hand side, but we only care about the vectors.

Tangent vectors are defined independently of coordinates. Practically, one often needs to calculate a tangent vector in a given coordinate system, i.e. a particular map $\emptyset_i$.then we have

$X[f] = \frac{d}{dt}(f \circ k(t)) \Big|_{t=0}$ = $\frac{d}{dt}\left(f \circ \emptyset_i^{-1} \circ \emptyset_i \circ k(t)\right) \Big|_{t=0}$
= $\frac{d}{dt}(f \circ \emptyset_i^{-1}) \Big|_g . \frac{d}{dt} \emptyset_i(kt)) \Big|_{t=0}$.

Even more practically: if the element of V_i, i.e. the coordinates around g, are given by x^a,then it is a common abuse of notation to write the curve as $\emptyset\big(k(t)\big) = x^a(t)$ and the

Function $(f \circ \emptyset_i^{-1})(x^a) = f(\emptyset_i^{-1}(x)\, as\, f(x)$. thus we get $X[f] = \frac{\partial}{\partial x^a} f(x). \frac{d}{dt} x^a(t)$

Here we again use the summation convention: an index that appears twice (the a) is summed over. The nice thing about this way of writing the tangent vector is that we have separated the f-dependent pieces, and we can even write the tangent without referring to f as the differential operator

X = $\frac{d}{dt} x^a(t) \Big|_{t=0} \cdot \frac{\partial}{\partial x^a} = X^a \partial_a$

Hence, the partial derivatives along the coordinate directions provide a basis for the tangent space at any given point, called the coordinate basis. Clearly, the dimension of the tangent apace is equal to the dimension of manifold. the X^a are called the component of X. this way of writing a vector comes at the price of introducing a coordinates system, and the components of the vector.

Will depend on the chosen coordinates (as it should be: components depend on the basis). However, so do the partial derivatives, and the vector itself is entirely independent of the coordinates. Hence one often speaks of "the vectorX^a".

1.9 Tangent Space at The Identity

We define the tangent space $T_p(M) to\, M$ at p to be the set of all mappings $X_p: C^\infty \rightarrow R$ satisfying for all $\alpha, \beta \in R\; and\; f, g \in C^\infty(p)$ the two conditions

1. $X_P(\propto f + \beta g) = \propto (X_P f) + \beta(X_P g)$ (Linearity)

2. $X_P(\propto f) = (X_P F)g(P) + f(P)(X_P g)$ (Leibniz rule).

With the vector space operations $T_P(M)$ defined by

1. $(X_P + Y_P)f = X_p F + Y_P F$

2. $(\propto X_P)F = \propto \left(X_p F\right)$, a tangent vector to *M* at *P* is any $X_p \in T_P(M)$

Theorem 1.9.1 Let F: M →N be a C^{∞}map of manifolds. Then for $p \in M$ $the\ map$ $F^*: C^{\infty}(F(p)) \to C^{\infty}(p)$ identityby $F^*(f) = f \circ F$ isa homeomorphism of algebras and induces a dual vector space homeomorphism $F_*: T_p(M) \to T_{F(p)}(N)$. define by $F_*(X_p)f = X_p(F^*f)$. which gives $F_*(X_p)$ as a map of $C^{\infty}(F(p))$ to R. when F: M $\to M$is the identity, both $F^* and\ F_*$ are the identity isomorphism. If H= $G \circ F$ is a composition of C^{∞}maps. Then

$H^* = F^* \circ G^*$ and $H_* = G_* \circ F_*$

Proof: The proof consists of routinely checking the statements against definitions. We omit the verification that F^* is a homomorphism and consider F_* only. Let $X_p \in T_p(M)$ and.$g \in C^{\infty}(F(p))$; We must prove that the map $F_*(X_p): C^{\infty}(F(p)) \to R$ is a vector $at\ F(p)$.that is a linear map satisfying the Leibniz rule. We have

$\boldsymbol{F}_*(X_p)(fg) = X_p F^*(fg) = X_p[f \circ F(g \circ F)] = X_p(f \circ F) g(F(p)) + f(F(p)) X_p(g \circ F)$.

And so we obtain

$\boldsymbol{F}_*(X_p)(fg) = (F_*(X_p)f) g(F(p)) + f(F(p)) \boldsymbol{F}_*(X_p) g$

(Linearity is even simpler).thus$F_*: T_p(M) \to T_{F(p)}(N).\ further, F_*$is a homomorphism.

$F_*(\alpha X_p + \beta Y_p)f = ((\alpha X_p + \beta Y_p)(f \circ F) = \alpha X_p(f \circ F) + \beta Y_p(f \circ F)$

$= \alpha F_*(X_p)f + \beta F_*(Y_p)f = [\alpha F_*(X_p) + \beta F_*(Y_p)]f$. [2]

Notation: The homomorphism $F_*: T_p(M) \to T_{F(p)}(M)$is often called the differential of map.

Corollary: 1.9.2 If F: M $\rightarrow$ N is a diffeomorphism of M onto an open set $U \subset N and p \in M$, then $F_{*}: T_p(M) \rightarrow T_{F(p)}(M)$ is an isomorphism onto.This follow at once from the last statement of the theorem and note that if we suppose G is inverse to. Then both $G_{*} \circ F_{*}: T_p(M) \rightarrow T_p(M)$and $F_{*} \circ G_{*}: T_p(M) \rightarrow T_{F(p)}(N)$ are the identity isomorphism on the corresponding vector space.

Remembering that any open subset of a manifold is a (sub) manifold of the same dimension. we see that if U, φ is a coordinate neighborhood on M, then the coordinate map φ induces an isomorphism $\varphi_{*}: T_p(M) \rightarrow T_{\varphi(p)}(R^n)$ of the tangent space at each point $p \in U\ onto\ T_{\alpha}(R^n) = \varphi(p)$. the map φ_{*}^{-1}, on the other hand , maps $T_{\alpha}(R^n)$ isomorphically onto $T_p(M)$.

The images $E_{ip} = \varphi_{*}^{-1}\left(\partial/_{\partial x^i}\right), i = 1, \dots, \dots, \dots n.$of the natural basis $\partial/_{\partial x^1}, \dots, \partial/_{\partial x^n}$ at each $a \in \varphi(U) \subset R^n$ determine at $p = \varphi^{-1}(a) \in M$ a basis$E_{1p}, \dots, \dots, E_{ap}\ of\ T_p(M)$, we call bases the coordinate frames.

Corollary 1.9.3 To each coordinate neighborhood U on M there corresponds a natural basis $E_{1p}, \dots, \dots, E_{np}$ of $T_p(M)$ for every $p \in U$; in particular, $T_p(M) = dimM$. let f be a$C^{\infty} function$ defined in a neighborhood of p, and $f \grave{=} f \circ \varphi^{-1}$ it's expression in local coordinates relative to φ, then $E_{ip} f = (\partial y^j / \partial x^i)_{\varphi(p)}$.

In particular, if $x^i(q)$is the ith coordinate function, $X_p X^i$ is the ith component of $X_p$in this basis, that is, $X_p = \sum_{i=1}^{n} (X_p X^i) E_{ip}$.

The last statement of the c0rorllary is a restatement of the definition in theorem (1.9.2) for

$E_{ip} = \varphi_*^{-1}\left(\partial/_{\partial x^i}\right)$, namely $E_{ip}f = \left(\varphi_*^{-1}\left(\partial/_{\partial x^i}\right)\right)f = \frac{\partial}{\partial x^i}(f\circ\varphi^{-1})\Big|_{v=\varphi(p)}$

If we take f to be ith coordinate function. $f(q) = x^i(q) and X_p = \sum \alpha^i E_{ip}$, Then

$$X_p x^i = \sum_i \alpha^i\left(E_{ip}x^i\right) = \sum \alpha^i\left(\frac{\partial x^i}{\partial x^i}\right)_{\varphi(p)} = \alpha.$$

We may use this to derive a standard formula which gives the matrix of the linear map F_* relative to local coordinate systems. Let F: M→ N is a smooth map. And let $U, \varphi\ and\ V, \psi\ be$ coordinate neighborhood on M and N with

F (U) ⊂V. suppose that in these local coordinates F is given by

$y^i = f^i(x^1, \dots \dots\dots, x^n), i = 1, \dots\dots\dots, m$

And that p is a point with coordinates = $(a^1, \dots\dots\dots, a^n)$. Then F(p) has y coordinate determined by these functions. Further let $\partial y^i/_{\partial x^i}\ denote\ \partial f^i/_{\partial x^i}$.

Theorem 1.9.4 Let

$E_{ip} = \varphi_*^{-1}\left(\partial/_{\partial x^i}\right)$ and $\tilde{E}_{F(p)} = \psi_*^{-1}\left(\partial/_{\partial y^i}\right)$,$i = 1, \dots\dots\dots, n\ and$

$i = 1, \dots\dots\dots. m$, be the basis of $T_p(M)$ and $T_{F(p)}(N)$ respectively, determined by the given coordinate neighborhoods. Then

$$F_*\left(E_{ip}\right) = \sum_{j=1}^{m}\left(\frac{\partial y^i}{\partial x^i}\right)\tilde{E}_{F(p)}.\ i = 1, \dots\dots\dots, n$$

In the terms of components, if $X_p = \sum \alpha^i \, E_{ip}$ maps to $F_*(X_{p)} = \sum \beta^i \, \tilde{E}_{F(p)}$. then we have

$$\beta^i = \sum_{j=1}^{m} \alpha^i \left(\frac{\partial y^i}{\partial x^i}\right)_a . \; j = 1, \dots \dots \dots, m$$

The partial derivatives in these formulas are evaluated of

$P: a = (a^1, \dots \dots \dots, a^n) = \varphi(P)$.

Proof: We have $F_*\left(E_{ip}\right) = F_{*\circ}\varphi_*^{-1}\left(\partial/\partial x^i\right)_{\varphi(P)}$ and according to corollary (1.9. 3), to compute its components relative to $\tilde{E}_{jF(P),}$ we must apply this vector as an operator on $C^\infty(F(P))$ to the coordinate functions y^j

$$F_*(E_{iP})y^j = \left(F_{*\circ}\varphi_*^{-1}\left(\frac{\partial}{\partial x^i}\right)\right)y^j = \frac{\partial}{\partial x^i} y^j (F_\circ \varphi_*^{-1})(x) = \frac{\partial f^j}{\partial x^i}.$$

These derivatives being evaluate at the coordinates of P, that is, at $\varphi(P)$;

They could be also written $\left(\partial y^j \big/ \partial x^i\right)_{\varphi(P)}$.

Example 1.9.5 Suppose M to be a two-dimensional sub-manifold ofR^3 that is a surface. Let W be an open subset, say a rectangle in the $(u, v) - plane\ R^2\ and$

$\theta: W \to R^3$ a parameterization of apportion M.

Namely, suppose $\theta$is an imbedding whose image is an open subset $V\ of; V,\ \theta^{-1}$is a coordinate neighborhood on M. Suppose

$\theta(u_o, v_0) = (x_0, y_{o,} z_0)$,

where we now use (x, y, z) as the natural coordinate in R^3.we may assume that θ is given by coordinate functions

$x = f(u, v), \qquad y = g\ (u, v), \quad z = h(u, v).$

Since θ is embedding, the Jacobian matrix $\partial(f,g,h)/\partial(u,v)$ has rank 2 at each point of W. we consider the image of the basis vectors $\partial/\partial u$ and $\partial/\partial v$ at (u_0, v_0), we denote these by $(X_u)_0$ *and* $(X_v)_0$, according to the first formula of pervious theorem(1.23). They are given by

$$(X_u)_0 = \theta_*(\partial/\partial u) = \frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial}{\partial z}$$

$$(X_v)_0 = \theta_*(\partial/\partial v) = \frac{\partial x}{\partial v}\frac{\partial}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial}{\partial z}$$

Where we have written $\partial x/\partial u, \partial x/\partial v$ *for* $\partial f/\partial u, \partial f/\partial v$ *and so on*.. these derivatives being evaluated at (u_0, v_0)since θ has rank 2, these are linearity independent vectors. And they span a two-dimensional subspace of $T_{(x_0, y_0, z_0)}(R^{n)}$, this subspace is what we have by our identification then we use the tangent at this point $(x_0, y_{o,} z_0)$:it consists of all vectors of the form

$$\alpha\theta_*(\partial/\partial u) + \beta\theta_*(\partial/\partial v) = \alpha(X_u)_0 + \beta(X_v)_0, \alpha, \beta \in R;$$

their initial point of course always is $(x_0, y_{o,} z_0)$, it is easily to seen that this subspace is the usual tangent plane to a surface, as we would naturally expect it to be. We use one of standard descriptions of the tangent plane at a point P of surface M in R^n; the collection of all tangent vectors at p to curves through P which lie on M. in fact let I an open interval about $t = t_0$ and let consider a curve on M through $(x_0, y_{o,} z_0)$. It is no loss of generality to suppose the curve given by

$F: I \to W$ Composed with $\theta: W \to R^3$; thus u, v are functions of t *with* $u(t_0) = u_0$ *and*

$v(t_0) = v_0$ and the curve is given by

$\theta\big(f(t)\big) = (x\big(u(t), v(t)\big), y\big(u(t), v(t)\big), z(u(t), v(t))$.

The tangent to the curve at $\left(x_0, y_{o,} z_0\right)$is given by

$$(\theta_\circ F)_* \left(\frac{d}{dt}\right) = \dot{x}(t_0)\frac{\partial}{\partial x} + \dot{y}(t_0)\frac{\partial}{\partial y} + \dot{z}(t_0)\frac{\partial}{\partial z}$$

where

$$\dot{x}(t_0) = \left(\frac{dx}{dt}\right)_{t_0} = \frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial x}{\partial v}\frac{dv}{dt}.$$

Evaluated at $\left(x_0, y_{o,} z_0\right)$ and $t = t_0$.substituting and collecting terms, we have

$$(\theta_\circ F)\left(\frac{d}{dt}\right) = \frac{du}{dt}\left(\frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial}{\partial z}\right) + \frac{\partial v}{\partial t}\left(\frac{\partial x}{\partial v}\frac{\partial}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial}{\partial z}\right)$$

$$= \quad \frac{du}{dt}\theta_*\left(\frac{\partial}{\partial u}\right) + \frac{\partial v}{\partial t}\theta_*\left(\frac{\partial}{\partial v}\right) = \dot{u}(t_0)(X_u)_0 + \dot{v}(t_0)(X_v)_0.$$

If we let $u = (t - t_0) + u_{0\,,} v = v_0$, we obtain just $(X_u)_0 = \theta_*\left(\frac{\partial}{\partial u}\right)$ and analogously $(X_v)_0$ are tangent to the parameter curve $u = u_0$, $v = (t - t_0) + v_0$. The coordinate frame vector, are tangent to the coordinate curves.

1.10 Vector Fields

Definition.1.10.1 A vector field is a map that associates a vector $\mathrm{X}(g) \in T_g G$ to each point $g \in$ G.

In a given map we can choose the coordinate basis and write the components as functions of coordinates, i.e.

X = $X^a(x)\partial_a$

Definition 1.10.2 Given two vector fields X and Y, the Lie bracket is a vector field given by:

$$[X,Y][F] = X\big[Y[F]\big] - Y\big[X[F]\big] = [XY-YX][F]$$

This is a reflection of the fact that derivatives on manifolds are not directly straight forward. The lie bracket allows extending the action to vector fields. The lie bracket is thus sometimes called a lie derivative, $\mathcal{L}_x Y = [Y, X]$.

This is not any more truly a directional derivative as it was functions: it depends not only on X at the point. To see this observe that for any function $F: G \to K$ we can define a new vector field $X' = FX$. Assume that f (g) = 1, so that$X'|_g = \mathrm{X}|_g$. Then one could expect that at g also the derivatives coincide, but actually we have.

$$\mathcal{L}_{x'} Y = \mathrm{F}\mathcal{L}_{\mathrm{x}}\mathrm{Y} - \mathrm{Y[F]X}.$$

And the second term doesn't vanish in general.

Definition 1.10.3 Vector fields X of class C^r on M is a function assigning to each point P of M a vector $X_P \in T_P(M)$ whose components in the frames of any local coordinates U, φ are functions of class C^r on the domain U of the coordinates. Unless otherwise noted we will use vector field to mean C^∞ vector field.

Example1.10.4. If we consider$\mathcal{M} = R^3 - \{0\}$, then the gravitational field of an object of unit mass at o is a C^∞ vector field whose components $\propto^1, \propto^2, \propto^3$ relative to the basis

$$\frac{\delta}{\delta x^1} = E_1, \quad \frac{\delta}{\delta x^2} = E_2\ , \frac{\delta}{\delta x^3} = E_3,$$

are $\propto^i = -\frac{x^i}{r^3},\ i = 1\,,2\,,3$,with r: such that.

$r = ((x^1)^2 + (x^2)^2 + (x^3)^2)^{\frac{1}{2}}$

Definition1.10.5 Let $F: M \to N$ a vector field Y on M such that for each $q \in M$ and $P \in F^{-1}(q) \subset N$ we have $F_*(X_p) = Y_q$, then we say that the vector fields X on Y are F-related and we write, briefly,

$Y = F_*(X)$

[We do not require *F* to be on to: if $F^{-1}(q)$ is empty, then the condition is vacuously satisfied].

Theorem 1.10.6 If $\mathrm{F}: \mathrm{M} \to \mathrm{N}$ is a diffeomorphism, then each vector field X on N is F-related to a uniquely determined vector field Y on M.

Proof: Since *F* is diffeomorphism, it has inverse$G: N \to M$, and at each point *P*, we have $F_*: T_p(N) \to T_{\varphi(P)}(M)$ is an isomorphism onto G_* an inverse. Thus given aC^∞_ vector field X on N, then at each point q on *M.* The vector $Y_q = F_*(X_{G(q)})$ is uniquely determined. It then remains to check that Y is a C^∞_ vector field. This is immediate if we introduce local coordinates and apply theorem to the component function.

1.11Invariant

Definition 1.11.1 If $F: M \to N$ is a difteomorphism and X is a C^∞ vector field on *M* such that $F_*(X) = X$. that is, X is F-related to it- self, then X is said to be invariant with respect to *F, or F*- invariant.

Remark 1.11.2 Before prove the theorem let us define two diffeomorphism to Lie group called left translation L_a and right translation R_a by and defined by.

$L_a: \mathrm{G} \to \mathrm{G}, L_a g = ag. \quad R_a: \mathrm{G} \to \mathrm{G}, R_a g = ga.$

Theorem1.11.3 Let G be a Lie group and $T_e(G)$ the tangent space at the identity , then each $X_e \in T_e(G)$ determine uniquely a C^∞ vector field of X on G which is invariant under left translations. In particular, G is parallelizable.

Proof: To each $g \in$ G there corresponding exactly one left translation $L_g$taking e to$g$. Therefore if it exists, X is uniquely determined by the formula: $X_g = L_{g*}(X_e)$. Except for differentiability, this formula does define a left invariant vector field for$a \in G$, we have.

$$L_{g*}(X_g) = L_{a*} o L_{g*}(X_e) = L_{ag*}(X_e) = X_{ag}$$

We must show that, so determined isC^∞. Let U, φ be a coordinate neighborhood of e such that $\varphi(e) = (0, \dots, \dots, \dots, 0)$ and let V be a neighborhood of e satisfying V⊂U. Let$g$, $h \in$ V with co-ordinate $x = (x^1, x^2, \dots, x^n)$ and $y = (y^1, y^2, \dots, y^n)$, respectively, and let $z = (z^1, \dots, z^n)$ be the coordinates of the product $g\,h$.Then, $Z^i = F^i(x, y), i = 1,2, \dots, n$ *are* C^∞functions on

$\varphi(V) \times \varphi(V)$.

If we write $X_e = \sum_{i=1}^{n} \gamma^{\,i} E_{ie}$, $\gamma^1, \dots, \gamma^n$ real numbers, then the formula above for X_g becomes

$$X_g = L_{g*}(X_e) = \sum \gamma^i \left(\frac{\partial f^i}{\partial y^i}\right)_{(x,0)} E_{ig}$$

Since in local coordinate L_g is given by $Z^i = f^i(x, y), i = 1,2, \dots, n$ with the coordinates *x* of g fixed. It follows on V the components of X_g in the coordinates frames are C^∞functions of the local coordinates. However, for any $a \in G$ the open set a V is diffeomorphic image by L_a of V. More over X, as just shown, is L_g-invariant so that for every $g = ah \in aV$ we have

$X_g = L_{a*}(X_h)$. It follows that X on aV is L_g-related to X on V and therefore *X* is C^∞ on aV since X is C^∞ in a neighborhood to each element of G, it is C^∞ on G.

Corollary1.11.4 Let G_1 and G_2 be Lie groups and $F: G_1 \to G_2$ homomorphism. Then to each left- Invariant vector field X on G_1 there uniquely determined left- invariant vector field Y on G_2 which is f-related to *X*

Proof: By the theorem (1.11.3), *X* is determined by X_{e1} , it's value at the identity e_1 of G_1 . Let $e_2 = F(e_1)$ be the identity of G_2 and let *Y* be the uniquely determined left- invariant vector field on G_2 such that Y_{e_2}=$F_*(X_{e_1})$. That Y should have this value at e_2 is surly necessary condition for *Y* to be *F*-related to *X*, and it remains only to see whether this vector field *Y* satisfies. $F_*(X_{e1}) = Y_F(g)$ for every$g \in G_1$. If so Y is indeed *F*-related (and uniquely determined). We write the mapping *F* as a composition.

$F = L_{F(g)}\, ofoL_g$. Using $F(x) = F(g)F(g^{-1}x)$, and note that since both X on Y are left-invariant by assumption. This gives

F (X_e)= $L_{F(g)_*} oF_* oL_{(g)^*}(X_g)$

$F(X_e) = L_{F(g)} oF_*(X_e) = L_{F(g)} Y_e$

Therefore Y meets all conditions and the corollary is true.

1.12Action of Groups

Important groups' action is the following actions of G on itself. Left action: $\mathrm{L}_g: \mathrm{G} \to \mathrm{G}$ is defined by $L_g(h) = gh$

Right action: $\mathrm{R}_g: \mathrm{G} \to \mathrm{G}$ is defined by $\mathrm{R}_g(h) = h\, g^{-1}$

Adjoint action: $\mathrm{Ad}_g: \mathrm{G} \to \mathrm{G}$ is defined by $\mathrm{Ad}_g(h) = ghg^{-1}$

Easily sees that left and right actions are transitive; in fact, each of them is simply transitive. It is also easy to see that left and right actions are commute and that

$\mathrm{Ad}_g = \mathrm{L}_g \mathrm{R}_g$.

Each of these actions also defines the action of G on the spaces of functions, vector fields, forms, etc. On G.

Definition 1.12.1 A vector field $v \in Vect(G)$is left- invariant if $g.v = v$ for every, $g \in$ G, and right -invariant if $v.\ g = v$, for every $g \in$ G. A vector field is called bi-invariant if it is both left- and right-invariant.

Definition1.12.2 (Left invariant vector field)

By using left translations to transport around tangent vectors on G. put g = T_e G, The tangent space to G at the neutral element e∈ G. For X ∈ g and $g \in$ G define.

$$L_X(g) = T_e\lambda_g.X \in T_gG.$$

Definition 1.12.3 Let G is a Lie group. A vector field $\xi \in \mathfrak{x}(G)$ is called left-invariant if and only if$(\lambda_g) * \xi = \xi$ for all $g \in$ G. The space of left-invariant vector fields is denoted by $\mathfrak{x}_L(G)$.

Definition1.12.4 (Right invariant vector fields)

We have used left translations to trivialize the tangent bundle of a Lie group G in (propel) in same way; one can consider the right trivialization. $TG \to G \times g$ Defined by

$\xi_g \to (g, T_g\rho^{g^{-1}}.\xi)$. The inverse of this map is denoted by $(g, X) \to R_X(g)$, and R_X is called the right - invariant vector field generated by$X \in g$. In general, a vector field $\xi \in \mathfrak{X}(G)$ is called right invariant if $(\rho^g)^*\xi = \xi$ for all$g \in$ G. The space of right invariant vector fields is denoted by$\mathfrak{X}_R(G)$. As in (porpl) one shows that

$\xi = \xi(e)$and $X \to R_X$. Are inverse bijections between g and$\mathfrak{X}_R(G)$.

Proposition1.12.5 Let G be a Lie group, then, the vector space of all left – invariant vector fields on G is isomorphic(as a vector space) to T_1G .

Proof: Since X is left invariant the following diagram commutes.

So that $X(a) = (dLa)_1(X(1))$ for all $a \in G$. We denote that $\Gamma(TG)^G$ the set of all left invariant vector fields on G. Define a map $\emptyset: \Gamma(TG)^G \to T_1G$ *by* $\emptyset: X \to X(1)$. Then, $\emptyset$ is linear and injective since if $X, Y \in \Gamma(TG)^G$ and $\emptyset(X) = \emptyset(Y)$

$$X(g) = d\,L_g\big(X(1)\big) = dL_g\,\big(X(1)\big) = Y(g),\ for\ each\ g \in G .$$

Now, $\emptyset$ is also surjective, for $v \in T_1G$ define $X_v \in \Gamma(TG)^G$ by

$X_v(a) = (dL_a)_1(v)$ for $a \in$ G. We claim that X_v is a left invariant vector field. Now $X_v: G \to TG$ is a C^∞ map of manifolds since if $f \in C^\infty$ G then for $a \in G$.

$$(X_v(f))(a) = (dL_a(v))f = v(f \circ L_a).$$

Now, if $x \in G$ we have

$$(f \circ L_a)(x) = (f \circ m)(a, x).$$

Which is a smooth map of *a, x* (here, m is the multiplication map on G).Thus $v(f \circ L_a)$ is smooth and hence so is X_v.

We now show X_v is left invariant. For $a_,\ g \in$ G we have

$$(dL_a)(X_v(a)) = dL_a((dL_a)1(v)) = d(L_g \circ La)(v) = d(L_{ga})(v) = X_v(ga) = X_v(L_a(a))$$

So that X_v is left-invariant. Therefore $\emptyset$ is onto and $\Gamma(\mathrm{TG})^{\mathrm{G}} \cong T_1G$.

We now give T_1G a Lie algebra structure by identifying if with $\Gamma(TG)^G$ with the lie bracket of vector fields. But, we need to show at [,] is in fact a binary operation on $\Gamma(TG)^G$, recall if $\mathrm{f}: \mathrm{M} \to \mathrm{N}$ is a smooth map of manifolds and X, Y are f-related if $d(X(x)) = Y(f(x))$ for every $x \in M$. It is fact from manifolds theory that is X, Y and X', Y' are F-related, then so are $[X, Y]$ and $[X', Y']$, but, left invariant vector fields are L_a related for all $a \in G$ by definition. This justifies.

Proposition 1.12.6 The Lie bracket of two left vector fields is a left invariant vector field. Thus we can regard T_1G as Lie algebra.

Definition 1.12.7 Let G be a group, the Lie algebra g of G is T_1G with the Lie bracket induced by its identification with $\Gamma(TG)^G$.

Example1.12.8 Consider the Lie group $GL\,(n\,,R)$. We have $T_eGL\,(n\,,R)\;=\; M_n(R)$, the set of all $n \times n$ matrices. For any A, B ∈ M_n(R), the Lie bracket is the commutator for; that is

[A, B] = AB – BA.

Proof: To prove this, we compute X_A, the left invariant vector field associated with the matrix $\mathrm{A} \in T_eGL(n, \mathrm{R})$.Now, on M_n(R), we have global coordinate maps given by

x_{ij}(A) = $A_{ij,}$ the ij^{th} entry of the matrix B. So, for g =GL (n, R),

$\left(X_\mathrm{A}(x_{ij})\right)(g) = X_\mathrm{A}(\mathrm{e})\left(x_{ij}\mathrm{o}L_g\right)$. Also, if $h \in \mathrm{GL}(n\,,\mathrm{R})$, then

$\left(x_{ij}\mathrm{o}L_g\right)(h) = x_{ij}(gh) = \sum_k g_{ik}h_{kj} = \sum_k g_{ik}x_{kj}(h)$.

Which, implies that $x_{ij}{}^\circ\mathrm{L_g} = \sum_k g_{ik}x_{kj}$

Now if $f \in \mathrm{C}^\infty(\mathrm{GL}(n\,,\mathrm{R}))$, $X_\mathrm{A}(\mathrm{e})\mathrm{f} = \frac{\mathrm{d}}{\mathrm{dt}}|_{\mathrm{t=o}}\, f(1+tA)$ So that

$X_\mathrm{A}(\mathrm{e})x_{ij} = \frac{\mathrm{d}}{\mathrm{dt}}|_{\mathrm{t=0}}\, x_{ij}\, f(1+tA) = \mathrm{A_{ij}}$.

Putting these remarks together, we see that:

$\mathrm{X}_A\left(x_{ij}\mathrm{o}L_g\right) = \sum_k g_{ik}\;A_{kj}$ =$\sum_k x_{ik}\,(g)\,A_{ki}$

We are now in a position to calculate the lie bracket of the left invariant vector fields associated with elements of $\mathrm{M}_n(\mathrm{R})$:

$([X_A, X_B](e)_{ij}) = [X_A, X_B](e)x_{ij}$ = $X_A X_B$ ($x_{ij)}$ − $X_B X_A$ ($x_{ij)}$ = $(X_A(\sum_k B_{kj} x_{ik})$

$$-X_B(\sum_k A_{kj} x_{ik}) = (\sum_{k,L} B_{kj} x_{il} A_{lk} - A_{kj x_{il}} B_{lk})(e)$$

$$= \sum_{k,l} B_{kj} \delta_{il} A_{lk} - A_{kj,} \delta_{il} B_{lk}$$

$$= \sum_K A_{ik} B_{kj} - \sum_k B_{ik} A_{kj} = (AB - BA)_{ij}$$

So, [A, B] = AB − BA. [3]

1.13 Lie Group Homomorphism

Definition1.13.1 Let G and H is Lie groups. A map $\rho: G \to H$ is Lie group homomorphism if:

1. ρ Is a C^∞ map of manifold and

2. ρ Is a group homomorphism.

Furthermore, we say ρ is a Lie group is isomorphism if it's a group isomorphism.

If g and 𝔥 are Lie algebras, a Lie algebra homomorphism $\tau: g \to \mathfrak{h}$ is a map such that:

1. τ is linear.

2. $\tau([X, Y]) = [\tau(X), \tau(Y)\ )\ ]$ for all $X, Y \in g$

Now, suppose V is an *n*-dimensional vector space over *R*. we define

$GL(V) = \{A: V \to V/A$ a linear isomorphism$\}$.

Since, $V \cong R^n, GL(V) \cong GL(n, R)$.

1.14 The Lie Algebra of A Lie Group

For a Lie group G, left invariant vector fields $\xi, \eta \in \mathfrak{X}_{\mathrm{L}}(\mathrm{G})$ and an element $g \in \mathrm{G}$ we obtain.

$$\lambda_g^*[\xi, \eta] = [\lambda_g^*\xi, \lambda_g^*\eta] = [\xi, \eta]$$

So, $[\xi, \eta]$ is left invariant too. Appling this to L_X and L_Y

For $X, Y \in \mathrm{g} = \mathrm{T_e G}$ we see that $[L_X, L_Y]$ is left invariant.

Detaining$[\mathrm{X}, \mathrm{Y}] \in \mathrm{g}$ as $[L_X, L_Y]$ (e), that,

$$[L_X, L_Y] = L_{[X,Y]}.$$

Proposition1.14. 1 If $X, Y \in \mathrm{g}$, so their Lie bracket $[X, Y]$ is left invariant.

Proof: We need to show that $[X, Y]$ is left invariant if X and Y are left invariant. We first notice

$$Y(\mathrm{f} \circ L_a)(b) = Y_b(f \circ L_a) = (dL_a)_b(Y_b)\mathrm{f} = Y_{ab}\mathrm{f} = (Y\mathrm{f})(L_a b) = (Y\mathrm{f}) \circ L_a(b).$$

For any smooth function $\mathrm{f} \in \mathrm{C}^\infty(\mathrm{G})$. Thus

$$Y_{ab}\mathrm{f}(Y\mathrm{f}) = (dL_a)_b(X_b)(Y\mathrm{f}) = (X_b)((Y\mathrm{f}) \circ L_a) = X_b Y(\mathrm{f} \circ L_a)$$

Similarly:

$$Y_{ab}X\mathrm{f} = Y_b X(\mathrm{f} \circ L_a).$$

Thus

$$dL_a([X, Y]_b)\mathrm{f} = X_b Y(\mathrm{f} \circ L_a) - Y_b X(\mathrm{f} \circ L_a) = X_{ab}(Y\,\mathrm{f}) - Y_{ab}X\,\mathrm{f} = [X, Y]_{ab}(\mathrm{f}).$$

Definition 1.14.2 Let G be a group. The Lie algebras of G is the tangent space *g = T_e G* together with the map $[,]: g \times g \to g$ defined by

$$[X, Y] = [L_X, L_Y](e)$$

Remark 1.14.3 From corresponding properties of the bracket of vector fields, it follows immediately that bracket $[\,,]: g \times g \to g$ is:

1. Bi linear: $[a\,X, Y] \;=\; a\,[X, Y]$ and $[X_1 + X_2, Y] = [X_1, Y] + [X_2, Y]$

2. Skew symmetric: $[X\,, Y] = -[Y\,, X]$

3. satisfies the Jacobi identity: that is

$$\left[X\,, [Y\,, Z]\right] = \left[[X\,, Y], Z\right] + \left[Y\,, [X\,, Z]\right]$$

In general, one defines Lie algebra as a real vector space together with a Lie bracket having those three properties.

Example 1.14.4 Let us consider the fundamental example G = GL (n, R). As a manifold, G is an open subset in the vector space M n (R), so in particular, g = M_n (R) as a vector space. Consider the matrices

A, B, C ∈ M_n (R) we have A (B + t C) = AB + t A C, so left translation by A is a linear map. In particular, this implies that for $\mathrm{A} \in \mathrm{GL}(n\,, \mathrm{R})$ and C ∈ $\mathrm{M}_n(R) = T_e GL\,(n\,, R)$ we obtain

L_C(A) = AC. Viewed as a function $L(n, R) \to \mathrm{M}_n(\mathrm{R})$, the left invariant vector field L_C is therefore given by right multiplication by C and thus extents to all of *M_n(R)*, now viewing vector fields on open subset of R^m as function with values in R^m , the Lie bracket is given by $[\xi\,, \eta](x) = \mathrm{D}\eta(x)\big(\xi(x)\big) - \mathrm{D}\xi(x)\big(\eta(x)\big)$.Since, right multiplication by *a* fixed matrix is a linear map, we conclude that $\mathrm{D}(L_{C'})(\mathrm{e})(\mathrm{C}) = \mathrm{C}C'$ for C ,C' ∈ $M_n(R)$. Hence we obtain $[\,\mathrm{C}, \mathrm{C}'] = [\mathrm{L}_{\mathrm{c}'}, \mathrm{L}_{\mathrm{c}'}\;](\mathrm{e}) = \mathrm{CC}' - \mathrm{C}'\mathrm{C}$, And rth Lie bracket on *M_n (R)* is given by commutate of matrices.

Lemma 1.14.5 *Let* $F : M \to N$ be smooth map, and let $\xi_i \in \mathfrak{X}(M)$ and $\eta_i \in \mathfrak{X}(N)$ be vector fields for i = 1, 2. If $\xi_i \sim_{f_*} \eta_i$ for i = 1, 2, then $f_*[\xi_1, \xi_2] \sim_f [\eta_1, \eta_2]$.

Proof: For a smooth map $\propto : N \to R$ we have $(Tf \circ \xi). \propto = \xi.(\propto \circ f)$ by definition of the tangent map. Hence $\xi \sim f\eta$ is equivalent to

$\xi.(\propto \circ f) = (\eta.\propto) \circ f$ For all $\propto \in C^\infty(N, R)$.Now assuming that $\xi_i \sim_f \eta_i$ for

$i = 1, 2$ We compute

$\xi_1.(\xi_2.(\propto \circ f)) = \xi_1.((\eta_2.\propto) \circ f) = (\eta_1.(\eta_2.\propto)) \circ f$, from definition of Lie bracket, then

$[\xi_1, \xi_2].(\propto \circ f) = ([\eta_1, \eta_2].\propto) \circ f$. And thus, $[\xi_1, \xi_2] \sim_f [\eta_1, \eta_2]$.

Definition1.14.6 **A** Lie algebra homomorphism between two Lie algebras A and B (over the same field) is a linear map that preserves the Lie bracket. i.e. a map

$$f : \begin{bmatrix} A \to B \\ a \to f(a) \end{bmatrix}, \quad . \; f([a, b]) = [f(a), f(b)]$$

An invertible Lie algebra homomorphism is a Lie algebra isomorphism.

Proposition 1.14.7 Let G and H be Lie groups with Lie algebras g and $\mathfrak{h}$.

1. If $\varphi: G \to H$ is a smooth homomorphism then $\varphi' = T_e\varphi: g \to \mathfrak{h}$ is a homomorphism of Lie algebras, i.e. $\varphi'([X, Y]) = [\varphi'(X), \varphi'(Y)]$ for all X, Y ∈ g.

2. If G is cumulative, then the Lie bracket on g is identically zero.

Proof: 1.The equation $\varphi(gh) = \varphi(g)\varphi(h)$ can be interpreted as

$\varphi \, o\lambda_g = \lambda_{\varphi(g)} \circ \varphi$.

Differentiating this equation in $e \in G$ we obtain

$T_g\varphi \circ T_e\lambda_{\varphi(g)} = T_e T_{\varphi(g)} \circ \varphi'$ Inserting $X \in T_eG = g$. we get,

$T_g\varphi . L_X(g) = L_{\varphi'(X)}(\varphi(g))$, and hence the vector fields $L_X \in \mathfrak{X}(G)$ and $L_{\varphi'(X)} \in \mathfrak{X}(H)$ are φ-related for eachX ∈ g. From the lemma, we conclude that for X, Y ∈ g we get $T_\varphi \circ [L_X, L_Y] = [L_{\varphi'}(X), L_{\varphi'}(Y)] \circ \varphi$. Evaluated in e ∈ G thus gives

$\varphi'[X , Y] = [\varphi'(X), \varphi'(Y)]$.

If G is commutative, then $(gh)^{-1} = h^{-1}g^{-1} = g^{-1}h^{-1}$ so inversion map $v : G \to G$ is a group homomorphism. Hence by part (1), v': g → g is a Lie algebra homomorphism.

By part (2) of lemma $v' = id$ and we obtain

$-[X, Y] = v'([X, Y]) = [v'(X), v'(Y)] = [-X , -Y] = [X, Y]$. Thus $[X, Y] = 0$ for all, X, Y∈ g.

Proposition 1.14.8 Let G a Lie group with Lie algebra and inversion $v: G \to G$ then we have

$R_X = v^*(L_{-X})$ for all $X \in$ g .

For X, Y ∈ g, we have $[R_X, R_Y] = R_{-[X,Y]}$. For all, $X, Y \in$ g, we have $[L_X, R_Y] = 0$.

1.15. Exponential Map

Given a Lie group and its Lie algebra g, we would like to construct an exponential map from g → G , which will help to give some information about the structure of g.

Proposition 1.15.1 Let G be a Lie group with Lie algebra g. Then, for each ∈ g , there exists a map $\gamma_X : R \to G$ satisfying: $\gamma_X(o) = I_G$.

$\frac{d}{dt}|_{t=0}\gamma_X(t) = X$, And $\gamma_X(s + t) = \gamma_X(s)\gamma_X(t)$

Proof: Consider the Lie algebra map $\tau: R \to g$ defined by $\tau : t \to tX$ for all $t \in$ g . Now R is connected and simply connected, so by proposition. There exists unique Lie group map $\gamma_X: R \to G$ such that $(d\gamma_X)_0 = \tau$, which is to say $\frac{d}{dt}|_{t=0}\gamma_X(t) = X$.

This leads to the following definition.

Definition1.15.2 Let G be a Lie group with Lie algebra, define the exponential map.

$exp : g \to G$ by $exp(X) = \gamma_X(1)$.

Lemma1.15.3 Let G be a Lie group with Lie algebra g and $\tilde{X} \in$ g. write $\tilde{X}$ for left invariant vector field on g with $\tilde{X}(1) = X$. then, $\varphi_t(a) = a\gamma_X(t)$.

Is the flow of $\tilde{X}$. In particular, $\tilde{X}$ is complete; i.e. the flow exists for all $t \in R$

Proof**:** for $a \in$ G , we have

$\frac{d}{dt}\ |_{t=s} a\gamma_X(t) = (dL_a)_{\gamma_{X(s)}} \left(\frac{d}{dt}\ |_{t=s}\ \gamma_X(t)\right)$

$= \quad (dL_a)_{\gamma_{X(s)}} \left(\frac{d}{dt}|_{t=0}\, \gamma_X(t+s)\right)$

$= \quad (dL_a)_{\gamma X(s)} \left(\frac{d}{dt}|_{t=0}\, \gamma_X(s)\gamma_X(t)\right) = (dL_a)_{\gamma_{X(s)}} \left(\frac{d}{dt}|_{t=0}\, L_{\gamma_{X(s)}}(\gamma_X(t)\right) = (L_{a\gamma_{X(s)}})_1 \left(\frac{d}{dt}|_{t=0}\ \gamma_X\ (t)\right) = \left(dL_{a\gamma x(s)}\right)_1(X) = \tilde{X}\left(a\gamma_x(s)\right),$

$(since\ \tilde{X}\ is\ left\ invariant\ )$

So, a $\gamma_x(t)$ is the flow of $\tilde{X}$ and exist for all.

Lemma1.15.4 The exponential map is C^∞ map.

P**roof:** Consider the vector field *V* on *G*×*g* given by

V (a, X) = (d L_a (X), 0).

Then, $V \in\ C^\infty(G\ , g)$ and the claim is that the flow of *V* is given by

$\psi_t(g, X) = (g\ \gamma_X(t), X)$.To prove this claim, consider the following:

$\frac{d}{dt}|_{t=0}(g\ \gamma_X(t), X) = \left(dL_{a\gamma x(s)}(X), 0\right) = V(g\ \gamma_X(s)\ , X).$

From which we can conclude that γ_X depends smoothly on *X*.

Now, we note that the map $\phi\ :R \times G \times g$ defined by

$\phi(t_{,}a, X) =\ (a\ \gamma_X(t), X)$ is smooth. Thus it $\pi_1 : G \times g\ \rightarrow G$ is projection on the first factor, $(\pi_1 \circ)(I_G, X) =\ \gamma_X(1) = exp(X)\ is\ C^\infty$.

Lemma1.15.5 For all $X\ \in$ g and for all $t \in R$, $\gamma_{tX}(1) =\ \gamma_X(t)$.

Proof: The intent is to prove that for t ∈ R . $\gamma tX(ts)$= $\gamma_X(ts)$. Now, $S \rightarrow \gamma_{tX}(s)$ is the integral curve of the left invariant vector field tX through I_G. But, $\tilde{X} = t\tilde{X}$, so if we prove that, $\gamma_X(ts)$ is an integral curve of $t\tilde{X}$ through I_G, by uniqueness the lemma will be established. To prove this, first let $\sigma(s) =\ \gamma_X(ts)$, then $\sigma(0) =\ \gamma_X(0) = I_G$. We also have

$\frac{d}{ds}\sigma(s) = \frac{d}{ds}\, \gamma_X(ts) = d\,\frac{d}{du}|_{u=ts}\, \gamma X(u)$

= $t\tilde{X}(\, \gamma_X(ts)) = t\tilde{X}(\sigma(s))$. So, σ(s) is also an integral curve of $t\tilde{X}$ through I_G . Thus, $\gamma_{tX}(s) = \ \gamma_X(ts)$, and in particular, when s = 1 we have $\gamma_{tX}(1) = \ \gamma_X(t)$.

Now, we will prove the nice fact about exponential map.

Proposition1.15.6 Let G be a Lie group and g it's Lie algebra. Identity both $T_\circ$ g and I_1G with g.Then $(d\ exp)_0 : Tog \to I_1 G$ is the identity map.

P**roof:** By using definition1.8.1and lemma1.15.3 we have

$(d\ exp)_0(X) = \frac{d}{dt}|_{t=0} exp(0 + X) = \frac{d}{dt}|_{t=0}\ \gamma_{tX}(1)$ $= \frac{d}{dt}|_{t=0}\ \gamma_X(t) = X.$

Corollary1.15.7 For all $t_1, t_2, \in\ R_1$,

1. $(exp\ (t_1 + t_2)\ X)\ = expt_1 X\ *\ exp\ t_2 X$

2. $exp(-tX)$ = $(exp\ (t\ X)^{-1}$

1.16 Representation of Lie Group

Definition 1.16.1 An action of a Lie group G and a manifold M is an assignment to each $g \in$ G a diffeomorphism $\rho(g) \in Diff\ M$ such that

ρ(1) = ρ(g) ρ(h) and such that the map

$G \times M \to M : (g\ , m)\ \to \rho(g).m$ is smooth map.

Example1.16.2 The group *GL (n, R)* (and thus any it's Lie subgroup) acts on R^n

The group $O(n, R)$ acts on the sphere $S^{n-1} \subset R^n$.

The group $U(n)$ acts on the sphere $S^{2n-1} \subset C^n$.

Definition1.16.3 Let G be a Lie group and V a vector space. A representation of a Lie group is a map $\rho : G \to GL(V)$ of Lie groups.

For a Lie group G, consider the action of G on itself by conjugation: for each $g \in$ G we have adiffeomorphism $C_g : G \to G$ given by: $C_g(a) = g\, ag^{-1}$.

Notice that $C_g(1) = 1$ and we have invertible linear map $\left(dC_g\right)_1 : g \to g$

Now $C_{g_1 g_2} = C_{g_1} o C_{g_2}$ for all $g_1, g_2 \in G$ and hence $\left(dC_{g_1}\right)_1 \left(dC_{g_2}\right)_1 = \left(dC_{g_1 g_2}\right)_1$

Definition1.16.4 The Ad joint representation of a Lie group G is the representation $Ad : G \to GL(\mathrm{g})$ defined by: $Ad(g) = (dc_g)_1$

The Adjoin representation of a Lie algebra g is representation

$ad: \mathrm{g} \to \mathrm{gl(g)} = Hom(\mathrm{g}, \mathrm{g})$ defined by $ad(X) = (d\,Ad)_1(X)$.

Proposition 1.16.5 Suppose G is a Lie group, then for all $t \in R$, $g \in G$ and $X \in$ g we have $g\, exp\,(tX) g^{-1} = exp(tAd(g)(X))$ and

$Ad(exp(tX) = ex\,p\big(t\; ad(X)\big)$.

Example1.16.6 We compute that Ad and ad as maps when GL (n, R), recall that for any A, $g \in$ G we have the conjugation map $C_g(A) = gA\, g^{-1}$. Note that conjugation is linear. Thus for $X \in$ g we have

$$Ad(g)(X) = \left(dC_g\right)_1(X) = \frac{d}{dt}|_{t=0} C_g(exp(tX)\,)$$

$$= \frac{d}{dt}|_{t=0}\; g\; exp(tX)\; g^{-1}$$

$$= g\left(\frac{d}{dt}|_{t=0}\; exp(tx)\right) g^{-1} = gX\;\; g^{-1}$$

Also, for $X, Y \in$ g

$$ad(X)Y = \frac{d}{dt}|_{t=0} Ad\ exp(tX)Y = \frac{d}{dt}|_{t=0}\ exp(tX)Y\ \ exp(-tX)$$

=

$$\left(\frac{d}{dt}|_{t=0}\ exp(tX)\right) Y\ exp(-0X) + exp(0X)Y \left(\frac{d}{dt}|_{t=0}\ exp(-tX)\right)$$

= *XY + Y (-X) = XY – YX = [X, Y].*

the commutates of the matrices, X, *Y.*

Theorem 1.16.7 Let G be a Lie group, then for any *X,* $Y \in$ g,

ad(X)Y = [X , Y].

Proof: First note that

$$ad(X)Y = \frac{d}{dt}|_{t=0} Ad\ (exp\ (tX))Y = \frac{d}{dt}|_{t=0}\ d\left(C_{exptX}\right)_1(Y) \qquad (1.1)$$

Also, recall that we have show

$C_g(a) = g\ ag^{-1} = \left(R_{g^{-1}} \circ L_g\right)(\ a).\ \left(dL_g\right)_1(Y) = \tilde{Y}(g)$, where $\tilde{Y}$ is Left invariant vector field with

$$\tilde{Y}(1) = Y \qquad (1.2)$$

The flow $\emptyset_t^{\tilde{Y}}$ of $\tilde{Y}$ is given by

$$\emptyset_t^{\tilde{Y}}(\ a) = \ a(exptX) = \ R_{exptX}(\ a) \qquad (1.3)$$

$$[\tilde{X}\ ,\tilde{Y}](\ a) = \frac{d}{dt}|_{t=0}\ d(\emptyset_t^{\tilde{X}})(\tilde{Y}(\emptyset_t^{\tilde{X}}(\ a)) \qquad (1.4)$$

$$(exptX)^{-1} = exp(-tX) \qquad (1.5)$$

We now put from (1.1) – (1.5) together

$ad(X)Y = \frac{d}{dt}|_{t=0}\, dR_{exp(-tX)}(dL_{exp\,tX}Y)$ =
$\frac{d}{dt}|_{t=0}\, dR_{exp(-tX)}(\tilde{Y}(exptX))$

= $\frac{d}{dt}|_{t=0}\, d(\emptyset_t^{\tilde{X}})(\tilde{Y}\left(\emptyset_t^{\tilde{X}}(1)\right) = [\tilde{X}, \tilde{Y}](1.1), (\,by\ (1.4).$

1.17 Operation on Representations

1.17.1 Sub Representations and Quotient

Definition1.17.1 Let V is a representation of G (respectively g). A sub representation is a vector subspace $w \subset V$ stable under the action: $\rho(g)$ W ⊂W

For all g ∈ G (respectively), $\rho(x)$ W⊂W, for all x∈ g.

If G is a connected Lie group with Lie algebra g, then $W \subset V$ is a sub representation for G if and only if, it is a sub representation for g. If $W \subset V$ is a sub representation, then the quotient space V/W has a canonical structure of a representation, it will be called factor representation, or quotient representation.

Definition1.17.2. (Invariants)

Let V be a representation of a Lie group G. A vector $v \in V$ is called invariant if:

ρ(g)v = v ∀ g ∈ G. The subspace of invariant vector in V is denoted byV^G.

Similarly, let V be a representation of a Lie algebra g. A vector $v \in V$ is called invariant if $\rho(x)$ v = o∀ x ∈ g. The subspace of invariant vectors in V is denoted by V^{g} .

If G is connected with g then for any V of G we have $V^G = V^{\mathrm{g}}$

Chapter Two

Symmetries

2.1 Introduction

In order to understand symmetries of differential equations, it is helpful to consider symmetries of simpler objects. Roughly speaking, symmetry of a geometrical -object is a transformation whose action leaves the object apparently unchanged. For instance, consider the result of rotating an equilateral triangle anticlockwise about its centre. After rotation of $2\pi/3$, the triangle looks the same as it did before the rotation, so this transformation is symmetry. Rotations of $4\pi/3$ and 2π are also symmetries of the equilateral triangle. In fact, rotating by 2π is equivalent to doing nothing, because each point its mapped to itself. The transformation mapping each point to itself is a symmetry of any geometrical object: it is called the trivial symmetry.

On the other hand, consider a circle; any rotation by angle ε may be represented in Cartesian coordinates as the mapping

$$Ƒ_{\varepsilon}\,(x, y) \rightarrow (\,x\cos\varepsilon - y\sin\varepsilon, x\sin\varepsilon + y\cos\varepsilon)$$

In this case ε is continues parameter, similar the set of reflection of circle can be represented by the mapping

$$Ƒ_{p} : (x, y) \rightarrow (\,-x, y).$$

Followed by a rotation $Ƒ_{\varepsilon}$ therefore in the case of a circle we are examining a symmetric group which is not discrete; these kinds of symmetries are known as Lie symmetries, and they form a Lie group which will be explored in the next section.

Symmetries are commonly used to classify geometrical objects. There are certain constraints of on symmetries of geometrical objects. Each, symmetry has a unique inverse, which is itself is symmetry. The combined action of the symmetry and its inverse upon the object leaves of coordinates $q(t)$ and $q'(t)$ are solutions of the same set of equations. This explains the view of symmetry transformation as mapping between different solutions of the equations of motion.

The mapping between different solutions of the equation of motion motivates an alternative view of symmetry transformations, different form the passive view. They are considered as transformations between different time evolutions of the system, describe in the same coordinate frame. This is called an active transformation. It is transformation between different physical stations, rather than a transformation between two different descriptions of the same situation.

2.2. Importance of the Symmetries

Symmetries are for many reasons a highly important subject to study in physics. Symmetries of the fundamental laws of nature tells as something basic about the nature that in many cases can be viewed as even more basic than the laws themselves. A well-known example is the space-time symmetries of the special theory of relativity. The Lorentz transformations were first detected as symmetries of Maxwell's equations, but Einstein realized that they are more fundamental than being symmetries of equations. After him the relativistic symmetries of space-time have been the guiding principle for formulation of all (new) fundamental laws of nature.

In the similar way, when extending the laws of nature in to new realms, in particular in elementary particle physics, the study of observed symmetries have often been used as a tool in the construction of new theoretical models.

When systems become complex and a detailed description becomes difficult, the symmetries may still shine through the complexities as a simplifying principle. In the study of the condensed matter physics the identification of important symmetries is often used as a guiding principle to obtain a correct description o the observed phenomena.

In the present case where we focus on the description of mechanical systems we have notice the possible use of symmetries as generates new solutions from old solutions of the equations of the motion. There is another important effect of symmetries that we will focus on, the connection between symmetries and constant of motion. Loosely speaking, to any (continuous) symmetry there is associated a conserved quantity.

The study of constants of the motion is important for the following reasons.

These constants may tell us something important about time evolution of the system even if we are not able to solve the full problem given by equations of motion.

The presence of constants of motion may simplify the problem since they effectively reduce the number of variables of the system.

In symmetry, a transformation is symmetry if it satisfies the following

1-the transformation preserves the structure.

2-the transformation is a diffeomorphism.

3-the transformation maps is object to itself.

2.3. Local One –Parameter Point Transformations

To begin local one-parameter point transformation, we consider the following equation

$$\bar{x} = G(x, \epsilon) \tag{2.1}$$

be a family of one-parameter $\epsilon \in R$ invertible transformations, of points

$x = (x^1, \dots, x^n) \in R^N$ into points $\bar{x} = (\bar{x}^1, \dots, \bar{x}^n) \in R^N$. These are known as one-parameter transformations, and subject to the conditions

$$\bar{x}\Big|_{\epsilon=0} = x \; this\; is, \;\; G(x,\varepsilon)\Big|_{\epsilon=0} = x \qquad (2.2)$$

Equation (2.1) using Taylor expansion in the some, neighborhood $\epsilon = 0$, we get

$$x + \epsilon_i\left(\frac{\partial G}{\partial \epsilon_i}\Big|_{\epsilon=0}\right) + O(\epsilon^2) \qquad (2.3)$$

Putting

$$\xi(x) = \frac{\partial G}{\partial \epsilon}\Big|_{\epsilon=0} \qquad (2.4)$$

reduces the expansion to

$$\bar{x} = x + \epsilon\xi(x) + O(\epsilon^2) \qquad (2.5)$$

The expression

$$\bar{x} = x + \epsilon\xi(x), \qquad (2.6)$$

is called a local one-parameter point transformation and the component of $\epsilon\xi(x)$ are the infinitesimals of (2.1).[34]

2.4. Local One-Parameter Point Transformation Groups

The set G of transformations

$$\bar{x}_{\epsilon_i} = x + \epsilon_i\left(\frac{\partial G}{\partial \epsilon_i}\Big|_{\epsilon_{i=0}}\right) + \frac{\epsilon_i^2}{2}\left(\frac{\partial^2 G}{\partial \epsilon_i^2}\Big|_{\epsilon_{i=0}}\right) + \cdots, i = 1,2,3,\ldots, (2.7)$$

becomes a grou$p\; only\; when\; truncated\; at\;\; O(\epsilon^2)$

2.4.1 The Group Generator

The local one-parameter point transformation in equation (2.6) we can rewritten in the form

$$\bar{x} = x + \epsilon\xi(x).\nabla x, \qquad (2.8)$$

$$so \;\; \bar{x} = (1 + \epsilon\xi(x).\nabla)x, \;\; an\; operator,$$

$$G = \xi(x).\nabla \qquad (2.9)$$

these implies that

$$\bar{x} = (1 + \epsilon G)x \qquad (2.10)$$

An operator (2.9) has the expanded form

$$G = \sum_{k=1}^{N} \xi^k \frac{\partial}{\partial x^k}, \qquad [10],[34] \qquad (2.11)$$

2.4.2. Prolongations Formulas

The prolongation is happens when the function F(x, y) does not only depend on point x alone, but also on the derivatives. When this case is happened then we have use the prolonged form of the operator G.

When *N =2*, with $x^1 = x \; and \; x^2 = y$ reduces (2.11) to

$$G = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}, \qquad (2.12)$$

In determining the prolongations, it is convenient to use the operator of total differentiation

$$D = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \cdots, \qquad (2.13)$$

$$where \quad y' = \frac{dy}{dx}, \qquad y'' = \frac{d^2y}{dx^2}, \ldots$$

$the\ deerivatives\ of\ the\ transformation\ point\ is\ \ \bar{y}' = \frac{d\bar{y}}{d\bar{x}}$ since

$$\bar{x} = x + \epsilon\boldsymbol{\xi} \quad \text{and} \quad \bar{y} = y + \epsilon\eta \qquad (2.14)$$

Then

$$\bar{y}' = \frac{dy+\epsilon d\eta}{dx+\epsilon d\xi}. \qquad (2.15)$$

So that

$$\bar{y}' = \frac{\boldsymbol{dy/dx+\epsilon d\eta/dx}}{\boldsymbol{dx/dx+\epsilon d\xi/dx}}.$$

We can introduce the operator *D*:

$$\bar{y}' = \frac{y' + \epsilon D(\eta)}{1 + \epsilon D(\xi)}. \qquad (2.16)$$

from the (2.16)implies that

$$\bar{y}' = y' + \epsilon\big(D(\eta) - y'D(\xi)\big), \qquad (2.17)$$

or

$$\bar{y}' = y' + \epsilon\zeta^1, \qquad (2.18)$$

with

$$\zeta^1 = D(\eta) - y'D(\boldsymbol{\xi})$$

It expands into

$$\zeta^1 = \eta_x + (\eta_y - \xi_x)y' - y'^2\xi_{y}. \qquad (2.19)$$

The first prolongation of G is

$$G^{|1|} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \zeta^1 \frac{\partial}{\partial y'}\,. \qquad (2.20)$$

For the second prolongation, we have

$$\bar{y}'' = \frac{y'' + \epsilon D(\zeta^1)}{1+\epsilon D(\xi)} \approx$$
$$y'' + \epsilon\zeta^2, \qquad (2.21)$$

with $\zeta^2 = D(\zeta^1) - y'' D(\boldsymbol{\xi})$ *this expands into*

$$\zeta^2 = \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - y'^3\xi_{yy}$$

$$+(\eta_y - 2\xi_x - 3y'\xi_y)y'' \qquad .$$

(2.22)

the second prolongation of G is

$$G^{|2|} =$$
$$\xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \zeta^1\frac{\partial}{\partial y'} + \zeta^2\frac{\partial}{\partial y''}. \qquad (2.23)$$

Most applications involve up to second order derivatives. For this reasonable we pause here third order. [10].

2.5. Invariant Functions in $\boldsymbol{R^2}$

A function F(x, y) is an invariant of group of transformations (2.11) if for each point (x, y) it is constant along the trajectory determined by the totality of transformed points ($\bar{x}$, $\bar{y}$):

$$F(\bar{x},\bar{y}) = F(x,y). \qquad (2,$$
24)

This requires that

$$GF = 0. \tag{2. 25}$$

Leading to the characteristic system $\frac{dx}{d\xi} = \frac{dy}{d\eta}$.

2.5.1 Multi-Dimensional Cases

In the cases of multi-dimensional, we recast the generator (2.11) as

$$G = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}. \tag{2.26}$$

considered the $kth - order$ partial differential equation

$F \quad (x, \quad u, \quad u_{(1)}, u_{(2)}, \dots, u_{(k)}) where\ x = (x_1, \dots, x_n), u_{(1)} = \frac{\partial u}{\partial x}$. (2.27)

By definition of symmetry, the transformations (2.1) form a symmetry group G of the system (2.27) if the function $\bar{u} = \bar{u}(\bar{x})$ satisfies the initial conditions

$$\bar{x}^i|_{\epsilon=0} = x^i,\ \ \bar{u}^i|_{\epsilon=0} = u^\alpha. whenever\ u = u(x) satisfies (2.27)$$

The transformed derivatives $\bar{u}_{(1)}, \dots, \bar{u}_{(k)}$ are found from (2.3) by using the formula of change variables in the derivatives, $D_i = D_i(f^j)\bar{D}_j$.

Here:

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \left(\frac{\partial}{\partial u^\alpha}\right) + u_{ij}^\alpha \left(\frac{\partial}{\partial x_j^\alpha} + \cdots\right). \tag{2.28}$$

is the total derivative operator with respect to x^i and D_i is given in terms of the transformed variables. The transformations (2.11) together with the transformations on $\bar{u}_{(1)}$ form a group, $G^{|1|}$, which is the first prolonged group which acts in the space $(x, u, u_{(1)})$.

Likewise, we obtain the prolonged group$G^{|2|}$, and so on up to $G^{|k|}$.

The infinitesimal transformations of the prolonged groups are:

$$\bar{u}_i^\alpha \approx u_i^\alpha + \alpha\zeta_i^\alpha\big(x, u, u_{(1)}\big),$$

$$\bar{u}_{ij}^\alpha \approx u_{ij}^\alpha + \alpha\zeta_{ij}^\alpha\big(x, u, u_{(1)}, u_{(2)}\big),$$

$$\vdots$$

$$\bar{u}_{i_1\dots i_k}^\alpha \approx u_{i_1\dots i_k}^\alpha + \alpha\zeta_{i_1\dots i_k}^\alpha\big(x, u, u_{(1)}, \dots, u_{(k)}\big), \qquad (2.29)$$

The functions $\zeta_i^\alpha\big(x, u, u_{(1)}\big), \zeta_{ij}^\alpha\big(x, u, u_{(1)}, u_{(2)}\big) and \zeta_{i_1\dots i_k}^\alpha\big(x, u, u_{(1)}, \dots, u_{(k)}\big)$ are given, recursively, by the prolongation formulas:

$$\zeta_i^\alpha = D_i(\eta^\alpha) - u_j^\alpha\big(\xi^j\big),$$

$$\zeta_{ij}^\alpha = D_j(\zeta_i^\alpha) - u_{il} D_j(\xi^l)$$

$$\vdots$$

$$\zeta_{i_1\dots i_k}^\alpha = D_{i_k}\big(\zeta_{i_1\dots i_{k-1}}^\alpha\big) - u_{l i_1\dots i_{k-1}} D_{ik}\big(\xi^i\big). \qquad (2.30)$$

The generators of the prolonged groups are:

$$G^{|1|} = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha(x, u, u_{(1)})\frac{\partial}{\partial u_1^\alpha}$$

$$\vdots$$

$$G^{|k|} = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha\big(x, u, u_{(1)}\big)\frac{\partial}{\partial u_1^\alpha}$$

$$+ \cdots + \zeta_{i_1\dots i_k}^\alpha\big(x, u, \dots, u_{(k)}\big)\frac{\partial}{\partial u_{i_1\dots i_k}^\alpha} \qquad (2.31)$$

Definition 2.5.1 A differential function *F(x, u …,*$u_{(p)}), p \geq 0$, is *a pth-*order differential invariant of a group G if *F(x, u, …,* $u_{(p)}) = F(\bar{x}, \bar{u}, \dots, u_{(p)})$,

I.e. if F is invariant under the prolonged group $G^{|p|}$, where for $p = 0$,

$u_{(0)} = 0 \; and \; G^{|0|} = G$.

Theorem2.5.2 A differential function *F(x, u,…,*$u_{(p)}), p \geq 0$, is *a* p^{th}-order differential invariant of a group G if

$$G^{|p|}F = 0, \tag{2.32}$$

2.6 Invariant Functions in R^n

A function F(x) is an invariant of the group of transformations (2.11) if for each point x it is constant along the trajectory determined by the totality of transformed points $\bar{x}$,

$$F(\bar{x}) = F(x) \tag{2.33}$$

This implies that, GF= 0. Leading to the characteristic system

$$\frac{dx^1}{d\xi^1} = \cdots = \frac{dx^N}{d\xi^N}. \tag{2.34}$$

2.6.1 Determining Equations

Substitution of equations (2.31) in to

$$G^{|k|}E^{\sigma}\left(x, u, u_{(1),} \dots, u_{(k)}\right)|_{(1)=0} = 0, \sigma = 1, \dots, \widetilde{m},$$

Gives rise to

$$E^{\sigma}\left(\bar{x}, \bar{u}, \bar{u}_{(1),} \dots, \bar{u}_{(k)}\right) \approx E^{\sigma}\left(x, u, u_{(1),} \dots, u_{(k)}\right) + \alpha(G^{|k|}E^{\sigma}) \quad (2.35)$$

Thus, we have

$$G^{|k|}E^{\sigma}\left(x, u, u_{(1),} \dots, u_{(k)}\right) = 0 \qquad (2.36)$$

Then, the equations (2.36) are called the determining equations. They are written as

$$G^{|k|}E^{\sigma}\left(x, u, u_{(1),} \dots, u_{(k)}\right)|_{(1)=0} = 0, \sigma = 1, \dots, \widetilde{m}, \qquad (2.37)$$

where $|_{(1)}$ means evaluated on the surface (2.27).

The determining equations are linear homogenous partial differential equations of order k for the unknown functions$\xi^{i}(x, u)$, $\eta^{\alpha}(x, u)$. These are consequences of the prolongation formula (2.30). Equation (2.37) also involves the derivatives $u_{(1)}, \dots, u_{(k)}$ some of which are eliminated by the system (2.27). We then equate the coefficients of the remaining unconstrained partial derivatives of u to zero. In general (2.27) decomposes into an over determined system of equations, that is, there are more equations than the $n + m$ unknowns ξ^i, η^α.

Since the determining equations are linear homogenous, their solutions form a vector space.

2.6.2 Lie Algebras

There is another important property of the determining equations, that is if the generators

$$G_1 = \xi_1^i(x,u)\frac{\partial}{\partial x^i} + \eta_1^i(x,u)\frac{\partial}{\partial u^\alpha}$$

$$G_2 = \xi_2^i(x,u)\frac{\partial}{\partial x^i} + \eta_2^i(x,u)\frac{\partial}{\partial u^\alpha}$$

Satisfy the determining equations, so do their commutates

$$[G_1, G_2] = G_1G_2 - G_2G_1 = (G_1(\xi_2^i) - G_2(\xi_1^i))\ \frac{\partial}{\partial x^i} + (G_1(\eta_2^i) - G_2(\eta_1^i)\frac{\partial}{\partial u^\alpha}.$$

This obeys the properties of bilinearity, skew-symmetry and Jacobi's identity.

2.7 Lie Equations

A given the infinitesimal transformations $\bar{x}^i = x^i + \epsilon\xi^i(x), \bar{u}^i = u^\alpha + \epsilon\eta^\alpha(x)$ or its symbol G, the corresponding one-parameter group G is obtained by solution of the Lie equations

$$\frac{d\bar{x}^i}{d\epsilon} = \xi^i(\bar{x}, \bar{u}), \frac{d\bar{u}^\alpha}{d\epsilon} = \eta^\alpha(\bar{x}, \bar{u}),$$

with initial conditions

$$\bar{x}^i|_{\epsilon=0} = x^i, \ \ \bar{u}^i|_{\epsilon=0} = u^\alpha \qquad (2.38)$$

2.8 Canonical Parameter

For the expression, $\varphi(\epsilon_1, \epsilon_2) = \epsilon_1 + \epsilon_2$, the canonical parameter exists whenever φ exists.

Theorem2.8.1 for any$\varphi(a, b)$, there exists the canonical parameter

$$\tilde{a} = \int_0^a \frac{da'}{A(a')}, \text{where} \ \ A(a) = \frac{\varphi(a,b)}{b}|_{b=0}.$$

This system, with a as the canonical parameter, transforms form-invariantly in variables t, x, y, u, w, p, μ under

$$\bar{t} = texp\left[\int_0^a \frac{da'}{\bar{\mu}F'(\mu)}\right], \quad \bar{x} = xexp\left[\int_0^a \frac{da'}{\bar{\mu}F'(\mu)}\right], \bar{y} = yexp\left[\int_0^a \frac{da'}{\bar{\mu}F'(\mu)}\right],$$

$$\bar{z} = zexp\left[\int_0^a \frac{da'}{\bar{\mu}F'(\mu)}\right], \quad \bar{u} = uexp\left[\int_0^a \frac{da'}{\bar{\mu}F'(\mu)}\right], \bar{v} = vexp\left[\int_0^a \frac{da'}{\bar{\mu}F'(\mu)}\right],$$

$$\bar{w} = wexp\left[\int_0^a \frac{da'}{\bar{\mu}F'(\mu)}\right], \quad \bar{p} = pexp\left[\int_0^a \frac{da'}{\bar{\mu}F'(\mu)}\right], \qquad F(\bar{\mu}) = a + F(\mu),$$

Where

$$F(\mu) = \frac{1}{\bar{\mu}F'(\mu)}. \quad [36] \qquad (2.39)$$

2.8.1 Canonical Variables

Every one-parameter group of transformations $(\bar{x} = f(x, y, \epsilon), \bar{y} = g(x, y, \epsilon)$

reduces to a group of translations $\bar{t} = t + \epsilon, u = u$

with the generator $X = \frac{\partial}{\partial t}$. By a change of variables

$$t = t(x, y), \qquad u = u(x, y).$$

The variables t, u are called canonical variabls.

2.9 Generate Point Symmetries

2.9.1 One Dependent and Two Independent Variables.

Considering the equation:

$$u_t = u_{xx}, \qquad (2.40)$$

In order to generating point symmetries for equation (2.40) .We first consider a change of variables from t, x and u to t^*, x^* and u^* involving an infinitesimal parameter ϵ. A Taylor's expansion in $\epsilon = 0$ is

$$\left.\begin{aligned} \bar{t} &= t + \epsilon T(t, x, u) \\ \bar{x} &= x + \epsilon \xi(t, x, u) \\ \bar{u} &= u + \epsilon \zeta(t, x, u) \end{aligned}\right\} \qquad (2.41)$$

Differentiating (2.41) with respect to ϵ

$$\left.\begin{aligned} \frac{\partial \bar{t}}{\partial \epsilon}|_{\epsilon=0} &= T(t,x,u) \\ \frac{\partial \bar{x}}{\partial \epsilon}|_{\epsilon=0} &= \xi(t,x,u) \\ \frac{\partial \bar{u}}{\partial \epsilon}|_{\epsilon=0} &= \zeta(t,x,u) \end{aligned}\right\} \qquad (2.42)$$

The tangent vector field (2.42) is associated with an operator

$$G = T\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \zeta\frac{\partial}{\partial u}, \qquad (2.43)$$

This operator is called a symmetry generator. This leads to the invariance condition

$$G^{|2|}[F(t,x,u_t,u_x,u_{tx},u_{tt},u_{xx}]|_{\{F(t,x,u_t,u_x,u_{tx},u_{tt},u_{xx})=0\}} = 0, (2.44)$$

where $G^{|2|}$ *is the second prolongation of* G. it is obtained from:

$$G^{|2|} = G + \zeta_t^1\frac{\partial}{\partial u_t} + \zeta_x^1\frac{\partial}{\partial u_x} + \zeta_{tt}^2\frac{\partial}{\partial u_{tt}} + \zeta_{tx}^2\frac{\partial}{\partial u_{tx}} + \zeta_{xx}^2\frac{\partial}{\partial u_{xx}}, (2.45)$$

where

$$\zeta_t^1 = \frac{\partial g}{\partial t} + u\frac{\partial f}{\partial t} + \left[f - \frac{\partial T}{\partial t}\right]u_t - \frac{\partial \xi}{\partial x}u_x,$$

$$\zeta_x^1 = \frac{\partial g}{\partial t} + u\frac{\partial f}{\partial x} + \left[f - \frac{\partial \xi}{\partial x}\right]u_x - \frac{\partial T}{\partial t}u_t,$$

$$\zeta_{tt}^2 = \frac{\partial^2 g}{\partial t^2} + u\frac{\partial^2 f}{\partial t^2} + \left[2\frac{\partial f}{\partial t} - \frac{\partial^2 T}{\partial t^2}\right]u_t - \frac{\partial^2 \xi}{\partial t^2}u_x + \left[f - 2\frac{\partial T}{\partial t}\right]u_{tt} - 2\frac{\partial \xi}{\partial t}u_{tx},$$

$$\zeta_{xx}^2 = \frac{\partial^2 g}{\partial x^2} + u\frac{\partial^2 f}{\partial x^2} + \left[2\frac{\partial f}{\partial x} - \frac{\partial^2 \xi}{\partial x^2}\right]u_x - \frac{\partial^2 T}{\partial x^2}u_t + \left[f - 2\frac{\partial T}{\partial x}\right]u_{xx} - 2\frac{\partial T}{\partial x}u_{tx},$$

and

$$\zeta_{tx}^2 = \frac{\partial^2 g}{\partial t \partial x} + u\frac{\partial^2 f}{\partial t \partial x} + \left[2\frac{\partial f}{\partial x} - \frac{\partial^2 T}{\partial t \partial x}\right]u_t + \left[2\frac{\partial f}{\partial t} - \frac{\partial^2 \xi}{\partial t \partial x}\right]u_x - \left[f - \frac{\partial T}{\partial t} - \frac{\partial \xi}{\partial x}\right]u_{tx} - \frac{\partial T}{\partial t}u_{tt} - \frac{\partial \xi}{\partial t}u_{xx},$$

2.9.2 One Dependent and Three Independent Variables

In order to generate point symmetries for equation $u_t = u_{xx} + u_{yy,}$we first consider a change of variables from $t, x, y\ and\ u\ to\ t^*, x^*, y^*\ and\ u^*$ involve an infinitesimal parameter ϵ. use Taylor's series in $\epsilon\ near\ \epsilon = 0\ yilds$

$$\left.\begin{aligned} \bar{t} &= t + \epsilon T(t,x,y,u) \\ \bar{x} &= x + \epsilon \xi(t,x,y,u) \\ \bar{y} &= y + \epsilon \varphi(t,x,y,u) \\ \bar{u} &= u + \epsilon \zeta(t,x,y,u) \end{aligned}\right\}, \qquad (2.46)$$

$$\left.\begin{aligned} \frac{\partial \bar{t}}{\partial \epsilon}|_{\epsilon=0} &= T(t,x,y,u) \\ \frac{\partial \bar{x}}{\partial \epsilon}|_{\epsilon=0} &= \xi(t,x,y,u) \\ \frac{\partial \bar{y}}{\partial \epsilon}|_{\epsilon=0} &= \varphi(t,x,y,u) \\ \frac{\partial \bar{u}}{\partial \epsilon}|_{\epsilon=0} &= \zeta(t,x,y,u) \end{aligned}\right\}, \qquad (2.47)$$

The tangent vector field (2.47) is associated with an operator:

$$G = T\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \varphi\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial u}, \qquad (2.48)$$

Called a symmetry generator .this in turn leads to the invariance condition

$$G^{|2|}\left[F(t,x,u_t,u_x,u_y,u_{tx},u_{ty,}\,u_{tt},u_{xx},u_{xy,}u_{yy}\right]|_{\{F(t,x,u_t,u_x,u_y,u_{tx},u_{ty,}\,u_{tt},u_{xx},u_{xy},u_{yy})=0\}}$$

$$= 0, \quad (2.49)$$

where $G^{|2|}$ *is the second prolongation of* G. it is obtained from the formulas:

$$\zeta_t^1 = \frac{\partial g}{\partial t} + u\frac{\partial f}{\partial t} + \left[f - \frac{\partial T}{\partial t}\right]u_t - \frac{\partial \xi}{\partial x}u_x - u_y\frac{\partial \varphi}{\partial t},$$

$$\zeta_x^1 = \frac{\partial g}{\partial t} + u\frac{\partial f}{\partial x} + \left[f - \frac{\partial \xi}{\partial x}\right]u_x - \frac{\partial T}{\partial t}u_t - u_y\frac{\partial \varphi}{\partial t},$$

$$\zeta_y^1 = \frac{\partial g}{\partial t} + u\frac{\partial f}{\partial y} + \left[f - \frac{\partial \varphi}{\partial y}\right]u_y - \frac{\partial \varphi}{\partial t}u_t - u_x\frac{\partial \xi}{\partial y} - u_t\frac{\partial T}{\partial y}$$

$$\zeta_{tt}^2 = \frac{\partial^2 g}{\partial t^2} + u\frac{\partial^2 f}{\partial t^2} + \left[2\frac{\partial f}{\partial t} - \frac{\partial^2 T}{\partial t^2}\right]u_t - \frac{\partial^2 \xi}{\partial t^2}u_x - \frac{\partial^2 \xi}{\partial t^2}u_y + \left[f - 2\frac{\partial T}{\partial t}\right]u_{tt} - 2\frac{\partial \xi}{\partial t}u_{tx} - 2\frac{\partial \varphi}{\partial t}u_{yt},$$

$$\zeta_{xx}^2 = \frac{\partial^2 g}{\partial x^2} + u\frac{\partial^2 f}{\partial x^2} + \left[2\frac{\partial f}{\partial x} - \frac{\partial^2 \xi}{\partial x^2}\right]u_x - \frac{\partial^2 \varphi}{\partial x^2}u_y - \frac{\partial^2 T}{\partial x^2}u_t + \left[f - 2\frac{\partial \xi}{\partial x}\right]u_{xx}$$

$$-2\frac{\partial \varphi}{\partial x}u_{xy} - 2\frac{\partial T}{\partial x}u_{tx},$$

$$\zeta^2_{tx} = \frac{\partial^2 g}{\partial t\partial x} + u\frac{\partial^2 f}{\partial t\partial x} + \left[\frac{\partial f}{\partial x} - \frac{\partial^2 T}{\partial t\partial x}\right]u_t + \left[2\frac{\partial f}{\partial t} - \frac{\partial^2 \xi}{\partial t\partial x}\right]u_x - u_y\frac{\partial^2 \varphi}{\partial t\partial x}$$

$$-\left[2f - 2\frac{\partial T}{\partial t} - \frac{\partial T}{\partial x}\right]u_{tx} - \left[\frac{\partial \xi}{\partial t} - \frac{\partial \xi}{\partial x}\right]u_{xx} - \left[2\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x}\right]u_{xy},$$

$$\zeta^2_{ty} = \frac{\partial^2 g}{\partial t\partial y} + u\frac{\partial^2 f}{\partial t\partial y} + \left[2\frac{\partial f}{\partial y} - \frac{\partial^2 \varphi}{\partial t\partial y}\right]u_y - \frac{\partial^2 \xi}{\partial t\partial y}u_x - \frac{\partial^2 T}{\partial y^2}u_t -$$
$$\left[2\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial y}\right]u_{yy} \quad -2\left[\frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial y}\right]u_{xy} + \left[2f - 2\frac{\partial T}{\partial t} - \frac{\partial T}{\partial y}\right]u_{yt},$$

$$\zeta^2_{yy} = \frac{\partial^2 g}{\partial y^2} + u\frac{\partial^2 f}{\partial y^2} + \left[2\frac{\partial f}{\partial y} - \frac{\partial^2 \varphi}{\partial y^2}\right]u_y - \frac{\partial^2 \xi}{\partial y^2}u_x - \frac{\partial^2 T}{\partial y^2}u_t$$
$$+ \left[f - 2\frac{\partial \varphi}{\partial y}\right]u_{yy} \quad - 2\left[\frac{\partial \xi}{\partial y}\right]u_{xy}$$
$$+ \left[2f - 2\frac{\partial T}{\partial y}\right]u_{yt}, \quad (2.50)$$

2.9.3 One Dependent and n Independent Variables

The local one-parameter point transformations

$$\left.\begin{aligned} x = X_i(x,u,\epsilon) = x_i + \epsilon\xi(x,u) + O(\epsilon^2) \\ u = U(x,u,\epsilon) = u + \epsilon\eta(x,u) + O(\epsilon^2 \end{aligned}\right\}, i = 1, \ldots, n. \qquad (2.51)$$

Acting on (x,u)-space has generator

$X =$
$$\xi_i(x,u)\frac{\partial}{\partial x_i} + \eta(x,u)\frac{\partial}{\partial u}, \qquad (2.52)$$

The kth extended infinitesimals are given by

$$\xi(x,u), \eta(x,u), \eta^{(1)}(x,u,\partial u), \dots, \eta^{(n)}(x,u,\partial u,\dots,\partial u^n), \qquad (2.53)$$

And the corresponding kth extended generator is [34]

$$X^{(k)} = X_i\frac{\partial}{\partial x_i} + \eta\frac{\partial}{\partial u} + \zeta_i^1\frac{\partial}{\partial u_i} + \cdots + \zeta_i^k\frac{\partial}{\partial u_{i_1\dots i_i}}, i = 1,\dots,n \qquad (2.54)$$

2.9.4 *m* Dependent and *n* Independent Variables

We consider the case of n independent variables $x = (x^1,\dots,x^n)$ and m dependent variables $u(x) = u^1(x),\dots,u^n(x)$. with partial derivatives denoted by $u_i^\mu = \frac{\partial u^\mu}{\partial x^i}$. the notation

$$\partial^p u \equiv \partial^1 u = u_1^1(x) \dots u_n^1(x) \dots . u_1^m(x) \dots u_n^m(x)$$

Denotes the set of all first order partial derivatives

$$\partial^p u = \left\{u^p_{i_1\dots i_p} | \mu = 1 \dots m: i_1 \dots i_p = 1 \dots n\right\}$$
$$= \left\{\frac{\partial^p u^\mu(x)}{\partial x^{i_1}\dots\partial x^{i_p}} | \mu = 1 \dots m: i_1 \dots i_p = 1 \dots n\right\}$$

Denotes the set of all partial derivatives of order p. point transformations of the form

$$\left.\begin{aligned}\bar{x} &= f(x,u)\\ \bar{u} &= g(x,u)\end{aligned}\right\}, \qquad (2.55)$$

Acting on the *n+m* dimensional space *(x, y)* has as its p^{th} extended transformation

$$\left.\begin{aligned}(\bar{x})^i &= f^i(x,u)\\ (\bar{u}^\mu) &= g^\mu(x,u)\\ \left(\bar{u}_i^\mu\right) &= h_i^\mu(x,u,\partial u)\\ &\vdots\\ \left(\bar{u}_{i_1\dots i_p}^\mu\right) &= h_{i_1\dots i_p}^\mu(x,u,\partial u,\dots\partial^p u)\end{aligned}\right\}, \qquad (2.56)$$

with $i = i_1, \dots, i_p,\ \mu = 1, \dots, m;\ \frac{\partial(\bar{u}^\mu)}{\partial(\bar{x})^i}$. The transformed components of the first order derivatives are determined by

$$\begin{bmatrix}(\bar{u})_1^\mu\\ (\bar{u})_2^\mu\\ \vdots\\ (\bar{u})_n^\mu\end{bmatrix} = \begin{bmatrix}h_1^\mu\\ h_2^\mu\\ \vdots\\ h_n^\mu\end{bmatrix} = A^{-1}\begin{bmatrix}D_1 g^\mu\\ D_2 g^\mu\\ \vdots\\ D_n g^\mu\end{bmatrix}$$

where A^{-1} is the inverse of the matrix

$$A = \begin{bmatrix}D_1 f^1 & \cdots & D_1 f^n\\ \vdots & \ddots & \vdots\\ D_n f^1 & \cdots & D_n f^n\end{bmatrix}$$

in term of the total derivatives operator

$$D_i = \frac{\partial}{\partial x^i} + u_i^\mu \frac{\partial}{\partial u^\mu} + u_{ii_1}^\mu \frac{\partial}{\partial u_{i_1}^\mu} + \cdots.\ i = 1, \dots, n.$$

The transformed components of the higher-order derivatives are determined by

$$\begin{bmatrix}(\bar{u})^{\mu}_{i_1\ldots i_p\,1}\\(\bar{u})^{\mu}_{i_1\ldots i_p\,2}\\\vdots\\(\bar{u})^{\mu}_{i_1\ldots i_p\,n}\end{bmatrix}=\begin{bmatrix}h^{\mu}_{i_1\ldots i_p\,1}\\h^{\mu}_{i_1\ldots i_p\,2}\\\vdots\\h^{\mu}_{i_1\ldots i_p\,n}\end{bmatrix}=A^{-1}\begin{bmatrix}D_1h^{\mu}_{i_1\ldots i_p\,1}\\D_2h^{\mu}_{i_1\ldots i_p\,2}\\\vdots\\D_nh^{\mu}_{i_1\ldots i_p\,n}\end{bmatrix}$$

The situation where the point transformation (2.55) is a one-parameter group of transformation given by

$$\left.\begin{aligned}x^i&=f^i(x,u,\epsilon)=x^i+\epsilon\xi^i(x,u)+O(\epsilon^2),i=1,\ldots,n\\u^\mu&=g^\mu(x,u,\epsilon)=u^\mu+\epsilon\eta^\mu(x,u)+O(\epsilon^2),\mu=1,\ldots,m\end{aligned}\right\}.\quad(2.57)$$

Will have the corresponding generator given by

$$X=\xi^i(x,u)\frac{\partial}{\partial x^i}+\eta^\mu(x,u)\frac{\partial}{\partial u^\mu},\qquad(2.58)$$

2.10 Lie Group of the Heat Equation

We consider the symmetry analysis and Lie symmetries of one – dimensional and two dimensional heat equations. Also we find the invariant solutions of certain symmetry generators of heat equations [34].

2.10.1 One-Dimensional Heat Equation

Consider the heat equation given by

$$u_{xx}-u_t=0\qquad(2.59)$$

Let x and t two independent variables, and u a dependent variable.

The total derivatives are given by

$$D_x = \frac{\partial}{\partial x} + u_x u_t \frac{\partial}{\partial u}\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{xt}\frac{\partial}{\partial u_t} + \cdots$$

$$D_t = \frac{\partial}{\partial t} + +u_{tx}\frac{\partial}{\partial u_x} + u_{tt}\frac{\partial}{\partial u_t} + \cdots$$

The infinitesimal generator is given by

$$X = T(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}, \qquad (2.60)$$

The second prolongation of X is define by

$$X^{|2|} = X + \zeta_x^1\frac{\partial}{\partial u_x} + \zeta_t^1\frac{\partial}{\partial u_t} + \zeta_{xx}^2\frac{\partial}{\partial u_{xx}} + \zeta_{xt}^2\frac{\partial}{\partial u_{xt}} + \zeta_{tt}^2\frac{\partial}{\partial u_{tt}} \quad , \ (2.61)$$

The coefficients $\zeta_x^1, \zeta_t^1, \zeta_{xx}^2, \zeta_{xt}^2 and\ \zeta_{tt}^2$ are define by

$$\zeta_x^1 = D_x(\eta) - u_x D_x(\xi) - u_t D_x(T) = g_x + f_x u + u_x(f - \xi_x) - u_t T_x$$

$$\zeta_t^1 = D_t(\eta) - u_x D_t(\xi) - u_t D_t(T) = g_t + f_t u + u_t(f - T_t) - u_x \xi_t$$

$$\zeta_{xx}^2 = D_x(\zeta_x^1) - u_{xx} D_{xt}(\xi) - u_t D_x(T)$$

$$= g_{xx} + u f_{xx} + u_x(2f_x - \xi_{xx}) - u_t T_{xx} + u_{xx}(f - 2\xi_x) - 2u_{xt}T_x,$$

$$\zeta_{xt}^2 = D_t(\zeta_x^1) - u_{xx} D_t(\xi) - u_{xt} D_x(T)$$

$$= g_{xt} + f_{xt}u + u_x(2f_t - \xi_{xt}) - u_t(f_x - T_{xt}) - \xi_t u_{xx} + (f - \xi_x - T_t)u_{xt}$$

$-T_x u_{tt},$

$$\zeta_{tt}^2 = D_t(\zeta_t^1) - u_{xt}D_t(\xi) - u_{tt}D_t(T)$$

$$= g_{tt} + f_{tt}u + u_x(\xi_{tt}) - u_t(2f_t - T_{tt}) - 2u_{xt}\xi_t + u_{tt}(f - 2T_t).$$

2.10.2 Symmetries of One Dimensional Heat Equation

The determining equation is obtained from invariance condition

$$\left(T(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} + \zeta_x^1\frac{\partial}{\partial u_x} + \zeta_t^1\frac{\partial}{\partial u_t} + \zeta_{xx}^2\frac{\partial}{\partial u_{xx}} + \zeta_{xt}^2\frac{\partial}{\partial u_{xt}} + \zeta_{tt}^2\frac{\partial}{\partial u_{tt}}\right)(u_t - u_{tt})|_{u_{t=u_{tt}}} = 0$$

where

$$(\zeta_t^1 - \zeta_{tt}^2)|_{u_{t=u_{tt}}} = 0 \qquad (2.62)$$

After substituting $\zeta_{t,}^1$ ζ_{xx}^2 $and\ u_t = u_{xx}$ in equations (2.59), we get

$$g_t + f_t u + u_t(f - T_t) - u_x\xi_t - [g_{xx} + uf_{xx} + u_x(2f_x - \xi_{xx}) - u_tT_{xx} + u_t(f - 2\xi_x) - 2u_{xt}T_x] = 0, \qquad (2.63)$$

Separate coefficients in (2.60) having the following monomials

$$C: \quad g_t - g_{xx} = 0 \qquad (2.64)$$

$$u: \quad f_t - f_{xx} = 0 \qquad (2.65)$$

$$u_t : \quad T_t - T_{xx} - 2\xi_x = 0 \qquad (2.66)$$

$$u_x : \quad \xi_t - \xi_{xx} + 2f_x = \quad o \qquad (2.67)$$

$$u_{xt} : \quad T_x \qquad = \qquad 0 \qquad (2.68)$$

Integrating equation (2.68) with respect to x we get

$$T = \alpha(t) \qquad (2.69)$$

Substituting T in to (2.66) and integrating with respect to x we obtain

$$\xi = \frac{1}{2}\alpha_t x + b(t) \qquad (2.70)$$

Differentiating (2.65) with respect to t we get

$$\xi_t = \frac{1}{2}\alpha_{tt} x + b_t \qquad (2.71)$$

Substituting $\xi_t$in (2.67)and integrating with respect to x yields

$$f = -\frac{1}{8}\alpha_{tt}x^2 - \frac{1}{2}b_t x + c(x) \qquad (2.72)$$

Substituting f in (2.65) we obtain

$$-\frac{1}{8}\alpha_{ttt}x^t - \frac{1}{2}b_{tt}x + c_t + \frac{1}{4}\alpha_{tt} = o \qquad (2.73)$$

Splitting equation (2.68) with respect to the powers x we get

$$x^2 : \alpha_{ttt} = o$$

$$x^1 : b_{tt} = 0$$

$$x^o : c_t + \frac{1}{4}\alpha_{tt} = 0$$

And integrating three equations with respect to t respectively yields

$$\left.\begin{array}{c} \alpha(t) = \dfrac{A_1}{2}t^2 + A_2 t + A_3 \\ b(t) = A_4 t + A_5 \\ c(t) = -\dfrac{1}{4}A_1 t + A_6 \end{array}\right\} \quad (2.74)$$

The infinitesimals:

$$\left.\begin{array}{c} T = \alpha_1 t^2 + 2\alpha_2 t + \alpha_3 \\ \xi = \alpha_1 tx + 2\alpha_2 x + \alpha_4 t + \alpha_5 \\ f = -\dfrac{1}{4}\alpha_1 x^2 - \dfrac{1}{2}\alpha_4 x - \alpha_1 t + \alpha_6 \end{array}\right\} \quad (2.75)$$

2.10.3 Symmetries

The equation

$$X = T\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + (fu + g)\frac{\partial}{\partial u} \quad (2.76)$$

The corresponding symmetries are given by

$$X_1 = \frac{\partial}{\partial x}, \tag{2.77}$$

$$X_2 = \frac{\partial}{\partial t}, \tag{2.78}$$

$$X_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}, \tag{2.79}$$

$$X_4 = xt\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} - \left(\frac{1}{4}x^2 + \frac{1}{2}u\right)u\frac{\partial}{\partial u}, \tag{2.80}$$

$$X_5 = t\frac{\partial}{\partial t} - \frac{1}{2}xt\frac{\partial}{\partial u}, \tag{2.81}$$

$$X_6 = u\frac{\partial}{\partial u}, \tag{2.82}$$

$$X_\infty = g\frac{\partial}{\partial u}, \tag{2.83}$$

2.10.4 Invariant Solutions

Useful tools of the symmetries groups that conserve the set of all solutions in the differential equations admitting these groups. That is, the symmetries transformations simply permute those integrals curves among themselves. Such integral curves are termed invariant solutions.

Theorem 2.10.1 A function F(x, y) is called an invariant of the group G if and only if it solving the following first-order linear differential equations

$$XF = \xi(x,y)\frac{\partial F}{\partial x} + \eta(x,y)\frac{\partial F}{\partial y}, \qquad (2.84)$$

$$X(t,x,u)\frac{\partial u}{\partial x} + T(t,x,u)\frac{\partial u}{\partial t} = \eta(t,x,u), \qquad (2.85)$$

is the general partial differential equation of invariant surface, with the following characteristic equations

$$\frac{dx}{X(t,x,u)} = \frac{dt}{T(t,x,u)} = \frac{du}{\eta(t,x,u)}, \qquad (2.86)$$

For the symmetry equation (s_5) characteristic equation is

$$\frac{xdx}{2t} + \frac{du}{u} = 0. \qquad (2.87)$$

Integrating (2.79) we obtain

$$u = \beta(t)e^{-\frac{x^2}{4t}} \qquad (2.88)$$

Differentiating (2.88) with respect to t and twice with respect to x respectively we obtain

$$u_t = \beta'(t)e^{-\frac{x^2}{4t}} + \beta(t)e^{-\frac{x^2}{4t}}\frac{x^2}{4t^2} \qquad (2.89)$$

$$u_{xx} = \frac{x^2}{4t^2}e^{-\frac{x^2}{4t}}\beta(t) - \frac{1}{2t}e^{-\frac{x^2}{4t}}\beta(t) \qquad (2.90)$$

Substituting u_t and u_{xx} in $(2.59) yields$

$$\beta'(t) + \frac{1}{2}\beta(t) = o \qquad (2.91)$$

Hence the solution is

$$u = \frac{k}{\sqrt{l}} e^{-\frac{x^2}{4t}} \qquad (2.92)$$

Finding the invariant solution the symmetry equation (s_4) is used since it contains (x, t, u) then the invariance condition becomes

$$xtu_x + t^2 u_t = -\left(\frac{1}{4}x^2 + \frac{1}{2}t\right)u \qquad (2.93)$$

The corresponding characteristic equations are given by

$$\frac{dx}{xt} = \frac{dt}{t^2} = \frac{du}{\left(\frac{1}{4}x^2 + \frac{1}{2}t\right)u} \qquad (2.94)$$

By separation of variables and integration, the solution of the characteristic equations yields two invariants of $X_4, \boldsymbol{\zeta} = \frac{\boldsymbol{x}}{\boldsymbol{t}}$ and $v = \sqrt{t} e^{x^2/4t} u$.

Then the solution of the invariant surface equation (2.92) is given by the invariant form $\sqrt{t} e^{x^2/4t} u = \emptyset\left(\frac{x}{t}\right)$ solving for the following equations is obtain

$$u = \theta(x, t) = \frac{1}{\sqrt{t}} e^{x^2/4t} \emptyset(\boldsymbol{\zeta}) \qquad (2.95)$$

Finding the solution:

$$u_t = \frac{x^2 u}{4t^2} - \frac{u}{2t} - \emptyset\left(\frac{x}{t}\right)\left(\frac{xt}{t^2}\right) \qquad (2.96)$$

$$u_{xx} = \ddot{\emptyset}\left(\frac{x}{t}\right)\frac{u}{t^2} - \dot{\emptyset}\left(\frac{x}{t}\right)\left(\frac{xu}{t^2}\right) + \left(\frac{x^2}{4t^2} - \frac{1}{2t}\right)u \qquad (2.97)$$

Substituting $u_t and\ u_{xx}\ in$ (2.59) we obtain

$$\ddot{\emptyset}\left(\frac{x}{t}\right)\frac{u}{t^2} = 0 \qquad (2.98)$$

Hence

$$\ddot{\emptyset} = 0 \qquad (2.99)$$

Solving the differential equation obtained

$$u = \frac{1}{\sqrt{t}}[C_1 + C_2]e^{-\frac{x^2}{4t}} \qquad (2.100)$$

2.11 Two –Dimensional Heat Equation

Consider two dimensional heat equations

$$u_t - u_{xx} - u_{yy} = 0 \qquad (2.101)$$

The dependent variable is u and independent variables are t, x, and y. the infinitesimal generator is given by

$$X = T(t,x,y,u)\frac{\partial}{\partial t} + \xi(t,x,y,u)\frac{\partial}{\partial x} + \varphi(t,x,y,u)\frac{\partial}{\partial y} + \zeta(t,x,y,u)\frac{\partial}{\partial u} \qquad (2.102)$$

The second prolongation of X is

$$X^{[2]} = X + \left(\zeta_t^1\frac{\partial}{\partial u_t} + \zeta_x^1\frac{\partial}{\partial u_x} + \zeta_y^1\frac{\partial}{\partial u_y} + \zeta_{tt}^2\frac{\partial}{\partial u_{tt}} + \zeta_{tx}^2\frac{\partial}{\partial u_{tx}} + \zeta_{ty}^2\frac{\partial}{\partial u_{ty}} + \zeta_{xx}^2\frac{\partial}{\partial u_{xx}} + \zeta_{xy}^2\frac{\partial}{\partial u_{xy}} + \zeta_{yy}^2\frac{\partial}{\partial u_{yy}}\right)$$

With invariance condition

$$X^{[2]}(u_t - u_{xx} - u_{yy})|_{(u_t - u_{xx} = u_{yy})} = 0 \qquad (2.103)$$

That yields

$$\zeta_t^1 - \zeta_{xx}^2 - \zeta_{yy}^2|_{(u_t - u_{xx} = u_{yy})} = o \qquad (2.104)$$

where $\zeta_t^1, \zeta_{xx}^2 and\ \zeta_{yy}^2\ are\ substituted\ in\ (2.104)$

$$\zeta_t^1 = \frac{\partial g}{\partial t} + u\frac{\partial f}{\partial t} + \left[f - \frac{\partial T}{\partial t}\right]u_t - \frac{\partial \xi}{\partial x}u_x - u_y\frac{\partial \varphi}{\partial t},$$

$$\zeta_x^1 = \frac{\partial g}{\partial t} + u\frac{\partial f}{\partial x} + \left[f - \frac{\partial \xi}{\partial x}\right]u_x - \frac{\partial T}{\partial t}u_t - u_y\frac{\partial \varphi}{\partial t},$$

$$\zeta_y^1 = \frac{\partial g}{\partial t} + u\frac{\partial f}{\partial y} + \left[f - \frac{\partial \varphi}{\partial y}\right]u_y - \frac{\partial \varphi}{\partial t}u_t - u_x\frac{\partial \xi}{\partial y} - u_t\frac{\partial T}{\partial y}$$

$$\zeta_{tt}^2 = \frac{\partial^2 g}{\partial t^2} + u\frac{\partial^2 f}{\partial t^2} + \left[2\frac{\partial f}{\partial t} - \frac{\partial^2 T}{\partial t^2}\right]u_t - \frac{\partial^2 \xi}{\partial t^2}u_x - \frac{\partial^2 \xi}{\partial t^2}u_y + \left[f - 2\frac{\partial T}{\partial t}\right]u_{tt} - 2\frac{\partial \xi}{\partial t}u_{tx} - 2\frac{\partial \varphi}{\partial t}u_{yt},$$

$$\zeta_{xx}^2 = \frac{\partial^2 g}{\partial x^2} + u\frac{\partial^2 f}{\partial x^2} + \left[2\frac{\partial f}{\partial x} - \frac{\partial^2 \xi}{\partial x^2}\right]u_x - \frac{\partial^2 \varphi}{\partial x^2}u_y - \frac{\partial^2 T}{\partial x^2}u_t + \left[f - 2\frac{\partial \xi}{\partial x}\right]u_{xx} - 2\frac{\partial \varphi}{\partial x}u_{xy} - 2\frac{\partial T}{\partial x}u_{tx},$$

$$\zeta_{tx}^2 = \frac{\partial^2 g}{\partial t \partial x} + u\frac{\partial^2 f}{\partial t \partial x} + \left[\frac{\partial f}{\partial x} - \frac{\partial^2 T}{\partial t \partial x}\right]u_t + \left[2\frac{\partial f}{\partial t} - \frac{\partial^2 \xi}{\partial t \partial x}\right]u_x - u_y\frac{\partial^2 \varphi}{\partial t \partial x} - \left[2f - 2\frac{\partial T}{\partial t} - \frac{\partial T}{\partial x}\right]u_{tx} - \left[\frac{\partial \xi}{\partial t} - \frac{\partial \xi}{\partial x}\right]u_{xx} - \left[2\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x}\right]u_{xy},$$

$$\zeta_{ty}^2 = \frac{\partial^2 g}{\partial t \partial y} + u\frac{\partial^2 f}{\partial t \partial y} + \left[2\frac{\partial f}{\partial y} - \frac{\partial^2 \varphi}{\partial t \partial y}\right]u_y - \frac{\partial^2 \xi}{\partial t \partial y}u_x - \frac{\partial^2 T}{\partial y^2}u_t$$
$$- \left[2\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial y}\right]u_{yy} - 2\left[\frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial y}\right]u_{xy}$$
$$+ \left[2f - 2\frac{\partial T}{\partial t} - \frac{\partial T}{\partial y}\right]u_{yt},$$

$$\zeta_{yy}^2 = \frac{\partial^2 g}{\partial y^2} + u\frac{\partial^2 f}{\partial y^2} + \left[2\frac{\partial f}{\partial y} - \frac{\partial^2 \varphi}{\partial y^2}\right]u_y - \frac{\partial^2 \xi}{\partial y^2}u_x - \frac{\partial^2 T}{\partial y^2}u_t + \left[f - 2\frac{\partial \varphi}{\partial y}\right]u_{yy} \quad - 2\left[\frac{\partial \xi}{\partial y}\right]u_{xy} + \left[2f - 2\frac{\partial T}{\partial y}\right]u_{yt}, \quad (2.105)$$

2.11.1 Determining Equation of Two Dimensional Heat Equation

Considering the equation

$$\zeta_t^1 = \zeta_{xx}^2 + \zeta_{yy}^2 = 0 \qquad (2.106)$$

implies that

$$\frac{\partial g}{\partial t} + u\frac{\partial f}{\partial t} + \left[f - \frac{\partial T}{\partial t}\right]u_t - \frac{\partial \xi}{\partial x}u_x - u_y\frac{\partial \varphi}{\partial t}$$

$$= \frac{\partial^2 g}{\partial x^2} + u\frac{\partial^2 f}{\partial x^2} + \left[2\frac{\partial f}{\partial x} - \frac{\partial^2 \xi}{\partial x^2}\right]u_x - \frac{\partial^2 \varphi}{\partial x^2}u_y - \frac{\partial^2 T}{\partial x^2}u_t + \left[f - 2\frac{\partial \xi}{\partial x}\right]u_{xx}$$

$$-2\frac{\partial \varphi}{\partial x}u_{xy} - 2\frac{\partial T}{\partial x}u_{tx} + \frac{\partial^2 g}{\partial y^2} + u\frac{\partial^2 f}{\partial y^2} + \left[2\frac{\partial f}{\partial y} - \frac{\partial^2 \varphi}{\partial y^2}\right]u_y - \frac{\partial^2 \xi}{\partial y^2}u_x$$
$$- \frac{\partial^2 T}{\partial y^2}u_t$$

$$+ \left[f - 2\frac{\partial \varphi}{\partial y}\right]u_{yy} \quad - 2\left[\frac{\partial \xi}{\partial y}\right]u_{xy} + \left[2f - 2\frac{\partial T}{\partial y}\right]u_{yt} \quad (2.107)$$

Compare coefficient of constant and u yields

$$C \quad : \quad g_t - g_{xx} - g_{yy} = 0 \tag{2.108}$$

$$u \quad : \quad f_t - f_{xx} - f_{yy} = 0 \tag{2.109}$$

$$u_t \quad : \quad T_t - T_{xx} - T_{yy} - 2\varphi_y = 0 \tag{2.110}$$

$$u_x : \quad \xi_t - \xi_{xx} - \xi_{yy} + 2f_x = 0 \tag{2.111}$$

$$u_y : \quad \varphi_t - \varphi_{xx} - \varphi_{yy} + 2f_y = 0 \tag{2.112}$$

$$u_{xx} : \quad \xi_x - \varphi_y = 0 \tag{2.113}$$

$$u_{xy} : \quad \varphi_x + \xi_y = 0 \tag{2.114}$$

$$u_{xt} \quad : \quad T_x = 0 \tag{2.115}$$

$$u_{yt} : \quad T_y = 0 \tag{2.116}$$

Integrating (2.115) and (2.116) with respect to x and y respectively getting

$$T = \alpha(t) \tag{2.117}$$

Substituting T in (2.110) and integrate with respect to y we obtain

$$\varphi = \frac{1}{2}\alpha_t y + b(t, x) \tag{2.118}$$

Differentiating
φ *twice with respect to x and y respectively we obtain*

$$\varphi_{xx} = b_{xx}(t,x), \ \ \varphi_{yy} = 0 \qquad (2.119)$$

Differentiating φ *with respect to t we get*

$$\varphi_t = \frac{1}{2}\alpha_{tt}y + b_t(t,x) \qquad (2.120)$$

Differentiating φ *with respect to y* and substituting in (2.113) and integrate with respect to x obtained

$$\xi = \frac{1}{2}\alpha_t x + c(t,y) \qquad (2.121)$$

Differentiating **ξ** with respect to t and twice with respect x and y respectively we obtains the following equations

$$\xi_t = \frac{1}{2}\alpha_{tt}x + c_t(t,y) \qquad (2.122)$$

$$\xi_{xx} = 0 \ \ , \xi_{yy} = c_{yy}(t,y) \qquad (2.123)$$

Differentiating (2.114) with respect to x and y respectively obtained the following equations

$$\varphi_{xx} = 0 = b_{xx}, \xi_{yy} = 0 = c_{yy} \qquad (2.124)$$

Integrating (2.124) with respect to x and y respectively we obtained

$$b = A_1x + A_2 \qquad (2.125)$$

$$c = A_3y + A_4 \qquad (2.126)$$

Substituting $\xi_t, \xi_{xx}\ , \xi_{yy}\ in\ (2.111) and\ \varphi_t, \varphi_{xx},\ \ \varphi_{yy}\ in\ (2.112) and$

Integrating with respect to x and y respectively we obtains

$$f = -\frac{1}{8}\alpha_{tt}x^2 - \frac{1}{8}\alpha_{tt}y^2 - \frac{1}{2}c_t(t,y)x - \frac{1}{2}b_t(t,x)y + d(t,x)$$

$$+e(t,y) \qquad (2.127)$$

Differentiating f with respect to t and twice with respect to x and y respectively we obtain

$$f_t = -\frac{1}{8}\alpha_{ttt}x^2 - \frac{1}{8}\alpha_{ttt}y^2 - \frac{1}{2}c_{tt}(t,y)x - \frac{1}{2}b_{tt}(t,x)y + d_t(t,x)$$
$$+ e_t(t,y), (2.128)$$

$$f_{xx} = -\frac{1}{4}\alpha_{tt} + d_{xx}(t,x),\ \ f_{yy} = -\frac{1}{4}\alpha_{tt} + c_{yy}(t,y) \qquad (2.129)$$

Substituting $f_t, f_{xx},\ \ f_{yy}$ in (2.109) yields

$$-\frac{1}{8}\alpha_{ttt}x^2 - \frac{1}{8}\alpha_{ttt}y^2 - \frac{1}{2}c_{tt}(t,y)x - \frac{1}{2}b_{tt}(t,x)y + d_t(t,x) + e_t(t,y)$$

$$+\frac{1}{2}\alpha_{tt} - c_{yy}(t,y) - d_{xx}(t,x) = 0 \qquad (2.130)$$

Splitting (2.130) we obtains minimal equations

:
$$\alpha_{ttt}(t) = 0 \qquad (2.131)$$

:
$$d_{xx}(t,x) = 0 \qquad (2.132)$$

: $e_{yy}(t,y) = 0$ (2.133)

: $d_t(t,x) = 0$ (2.134)

: $e_t(t,y) = 0$ (2.135)

Integrating (2.134), (2.135) and 2.133) with respect to t, x and y respectively yields

$$\alpha(t) = \frac{1}{2}A_5t^2 + A_6t + A_7 \quad (2.136)$$

$$d(x,t) = A_8x + A_9 \quad (2.137)$$

$$e(t,y) = A_{10}y + A_{11} \quad (2.138)$$

The infinitesimals:

$$T = \beta_1t^2 + 2\beta_2t + \beta_3 \quad (2.139)$$

$$\xi = \beta_1tx + 2\beta_2x + \beta_4y + \beta_5 \quad (2.140)$$

$$\varphi = \beta_1ty + 2\beta_2y + \beta_4x + \beta_6 \quad (2.141)$$

$$f = -\frac{1}{4}\beta_1(x^2 + y^2) + \beta_7 x + \beta_8 y + \beta_9, \tag{2.142}$$

Symmetries:

$$X_1 = t\,\frac{\partial}{\partial t}, \tag{2.143}$$

$$X_2 = \frac{\partial}{\partial x}, \tag{2.144}$$

$$X_3 = \frac{\partial}{\partial y}, \tag{2.145}$$

$$X_4 = u\frac{\partial}{\partial u}, \tag{2.146}$$

$$X_5 = xu\frac{\partial}{\partial u}, \tag{2.147}$$

$$X_6 = yu\frac{\partial}{\partial u}, \tag{2.148}$$

$$X_7 = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \tag{2.149}$$

$$X_8 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \tag{2.150}$$

$$X_9 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + ty\frac{\partial}{\partial y} - \frac{1}{4}(x^2 + y^2)u\frac{\partial}{\partial u}, \tag{2.151}$$

$$X_\infty = g\frac{\partial}{\partial u}, \tag{2.152}$$

2.11.2 Invariant Solution

For $X_3 = \frac{\partial}{\partial y}$, the characteristic condition is given by

$$X_3 I = \frac{\partial I}{\partial y} \qquad (2.153)$$

The characteristic equation is

$$\frac{dy}{1} = \frac{dx}{0} = \frac{du}{0} = \frac{dt}{0} \qquad (2.154)$$

This implies that $dy = 0$, thus the invariant solution $u = \emptyset(x)\ or\ u = \emptyset(t)$ for $u = \emptyset(t)$, we substitute this in the original equation $u_t = u_{xx} + u_{yy}$

to get $\emptyset'(t) = 0\ thus\ \emptyset(t) = k$ similarly for $u = \emptyset(x)\ we\ have$

$u_{xx} = \emptyset''(x)$ =0 which implies $\emptyset(x) = C_1 x + C_2$ then the invariant solution becomes

$$u = C_1 x + C_2 \qquad (2.155)$$

For $X_2 = \frac{\partial}{\partial x}$

Similarly we obtain the invariant solution

$$u = A_1 y + A_2 \qquad (2.156)$$

For

$$X_7 = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y},$$

The invariant condition is

$$X_7 I = y\frac{\partial I}{\partial x} + x\frac{\partial I}{\partial y}, \qquad (2.157)$$

The characteristic equation is given by

$$\frac{dx}{y} = \frac{dy}{x} = \frac{du}{0} = \frac{dt}{0} \qquad (2.158)$$

then

$$C = y^2 - x^2 \qquad (2.159)$$

The first invariant, $\Psi_1 = y^2 - x^2$ and the second invariant. , $\Psi_1 = \emptyset(t)$ The invariant solution is $u = \emptyset(t)(y^2 - x^2)$ substituting this solution in the original equation obtain

$$\emptyset'(t)(y^2 - x^2) = 0, \emptyset'(t) = 0, \emptyset(t) = C \qquad (2.160)$$

Thus invariant solution

$$u = C(\, y^2 - x^2) \qquad (2.161)$$

For

$$X_9 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + ty\frac{\partial}{\partial y} - \frac{1}{4}(x^2 + y^2)u\frac{\partial}{\partial u}$$

The invariance condition is given by

$$X_9 I = t^2 \frac{\partial I}{\partial t} + tx \frac{\partial I}{\partial x} + ty \frac{\partial I}{\partial y} - \frac{1}{4}(x^2 + y^2)u \frac{\partial I}{\partial u} \qquad (2.162)$$

The characteristic equation is given by equation

$$\frac{dx}{t^2} = \frac{dy}{tx} = \frac{du}{ty} = -\frac{du}{\frac{1}{4}(x^2 + y^2)u} \qquad (2.163)$$

From the equation (2.163) we get

$$\frac{dt}{t^2} = \frac{dx}{tx} \qquad (2.164)$$

Integrating the equation (2.164) leads to

$$\frac{x}{t} = \varphi_1 \qquad (2.165)$$

From the (2.163) we have

$$\frac{dt}{t^2} + \frac{du}{\frac{1}{4}(x^2 + y^2)u} = 0 \qquad (2.166)$$

Integrating (2.166) obtain

$$u = F\left(\frac{x}{t}\right) e^{\frac{(x^2+y^2)}{4t}} \qquad (2.167)$$

Differentiating (1.167) with respect to t and twice with respect to x and y respectively we obtain

$$u_t = -F \frac{x}{t^2} e^{\frac{(x^2+y^2)}{4t}} - F \frac{x}{t^2} e^{\frac{(x^2+y^2)}{4t}} \frac{(x^2+y^2)}{4t^2} \qquad (2.168)$$

$$u_{xx} = F'' \frac{1}{t^2} e^{\frac{(x^2+y^2)}{4t}} + F' \frac{x}{t^2} e^{\frac{(x^2+y^2)}{4t}} + F e^{\frac{(x^2+y^2)}{4t}} \frac{1}{2t} \qquad (2.169)$$

$$u_{yy} = Fe^{\frac{(x^2+y^2)}{4t}}\left(\frac{1}{2t}\right) + Fe^{\frac{(x^2+y^2)}{4t}}\frac{y^2}{4t^2} \qquad (2.170)$$

Substituting $u_t, u_{xx}\ and\ u_{yy}$ in (2.101) yields

$$F''\frac{1}{t^2} + F'\frac{2x}{t^2} + F\left(\frac{1}{t} + \frac{x^2}{4t^2} + \frac{y^2}{4t^2} + e^{\frac{(x^2+y^2)}{4t}}\right) = 0 \qquad (2.171)$$

The second order differential equation (2.170) reduces to

$$F''\alpha + F'\beta + F\lambda = 0 \qquad (2.172)$$

From characteristic equation (2.163)

$$\frac{dx}{tx} = \frac{dy}{ty} \qquad (2.173)$$

Integrating obtain

$$\frac{x}{y} = \varphi_2 \qquad (2.174)$$

From (2.162)

$$\frac{dy}{ty} + \frac{du}{\frac{1}{4}(x^2+y^2)u} = 0 \qquad (2.175)$$

Integrating (2.175) obtain

$$u = R\left(\frac{x}{y}\right)e^{-\left(\frac{2x^2 lny+y^2}{8t}\right)} \qquad (2.176)$$

Differentiating (2.176) with respect to t and twice with respect to x and y respectively yields

$$u_t = Re^{-\left(\frac{2x^2lny+y^2}{8t}\right)}\left(\frac{2x^2lny+y^2}{8t^2}\right),$$

$$u_{xx} = R''\frac{1}{y^2}e^{-\left(\frac{2x^2lny+y^2}{8t}\right)} - R'\left(\frac{1}{y}e^{-\left(\frac{2x^2lny+y^2}{8t}\right)}\frac{xlny}{ty}\right)$$

$$-R\left(e^{-\left(\frac{2x^2lny+y^2}{8t}\right)}\frac{lny}{2t} + e^{-\left(\frac{2x^2lny+y^2}{8t}\right)}\frac{x^2ln^2y}{4t^2}\right),$$

$$u_{yy} = R''\left(\frac{x^2}{y^4}e^{-\left(\frac{2x^2lny+y^2}{8t}\right)}\right)$$
$$+ R'\left(\frac{2x}{y^3}e^{-\left(\frac{2x^2lny+y^2}{8t}\right)} + \frac{x}{y^2}e^{-\left(\frac{2x^2lny+y^2}{8t}\right)}\frac{x^2+y^2}{2t}\right)$$

$$+R\left(e^{-\left(\frac{2x^2lny+y^2}{8t}\right)}\left(\frac{x^2+y^2}{4ty}\right)^2 + e^{-\left(\frac{2x^2lny+y^2}{8t}\right)}\frac{1}{4t}\frac{x^2+y^2}{y^2}\right),$$

Substituting u_t, u_{xx} and u_{yy} in (2.101) we obtain

$$R''\left(\frac{1}{y^2} - \frac{x^2}{y^2}\right) - R'\left(\frac{xlny}{ty^2} - \frac{2x}{y^3} - \frac{x(x^2+y^2)}{2ty^2}\right)$$

$$-R\left(\frac{2x^2lny+y^2}{8t} + \frac{lny}{2t} + \frac{x^2ln^2y}{4t^2} - \left(\frac{x^2+y^2}{4ty}\right)^2 - \frac{x+y^2}{4ty^2}\right)$$
$$= 0, (2.177)$$

This reduces to

$$R''\delta - R'\kappa - R\gamma = 0, \qquad (2.178)$$

From [36] for two –dimension Lie algebra spanned by $X, Y has\ invariant$

$r = \sqrt{x^2 + y^2}\ and\ v = u^{-\boldsymbol{\kappa}t}$. Looking for invariant solutions in the form $v = \emptyset(r)$, whence $u = \emptyset(r)e^{\boldsymbol{\kappa}t}$, substituting in (2.101) and multiplying by r

The results equation becomes

$$r\emptyset'' + \emptyset' - \boldsymbol{\kappa}r\emptyset = 0, \qquad (2.179)$$

Letting $\boldsymbol{\kappa} < 0$. then setting $\boldsymbol{\kappa} = -\boldsymbol{\alpha}^2\ and\ \bar{r} = \alpha r$ the equation becomes Bessel function $J_0(\bar{r})$ of order zero:

$$\bar{r}\emptyset'' + \emptyset' + \bar{r}\emptyset = 0, \qquad (2.180)$$

Where $\emptyset = J_0(\bar{r})$ and the invariant solution is given by

$$u = J_0(\alpha r)e^{-\alpha^2 t} \qquad (2.181)$$

Also we can obtain the solution of (2.172) and (2.178) similar to (2.180)

Chapter Three

Lagrangian and Hamiltonian System

3.1 Introduction

In this chapter, we introduce Lagrangian and Hamiltonian systems that contain symmetry rules on the vector space by using classical variational calculus, the Euler-Lagrange equations and Hamilton's equations are obtained. We dividing this chapter in to analytical and geometrical formulation with the continuous system of N degrees of freedom, state as Noether's theorems, also we use symplectic form to obtained a symmetry of Hamiltonian system.Mechanics has two main system Lagrangian and Hamiltonian mechanical system [37].

In Lagrangian mechanical system is based on variational principles and it's generalized directly to the general relativistic context. In Hamiltonian mechanics is based directly on the energy concept and it's closely connected to quantum mechanics.

3.2 Analytical Formulation

This chapter we introduce the basic motions of the Lagrangian and Hamiltonian formalisms. First we starts with a system with N degrees of freedom, state as $Noether's\ theorems$.

3.2.1 Lagrangian Formalism

Consider the action associated to a discrete system with N degrees of freedom

$$S(q_i) = \int_{t_1}^{t_2} dt\mathcal{L}(q_i, \dot{q_\iota}, t), i = 1, \dots, N \; . \qquad (3.1)$$

Where $\mathcal{L}(q_i, \dot{q_\iota}, t)$ is Lagrangian of system,{ q_i } are gerneralized coordinates and $\dot{q_\iota} = {dq_i}/_{dt}$ the generalized velocities .In order to get the Euler- Lagrange equations of the motion we need small variations of the generalized coordinates q_i keeping the extremes fixed

$$q_i' = q_i + \delta q_i \;, \qquad \delta q_i(t_1) = \delta q_i(t_2) = 0, \qquad (3.2)$$

Applying the first order Taylor expansion of $Lagrangian$ we obtain

$$L(q_i + \delta q_i, \dot{q_\iota} + \delta \dot{q_i}, \mathrm{t}) = L(q_i, \dot{q_\iota}, t) + \frac{\partial L}{\partial q_\iota}\delta q_i + \frac{\partial L}{\partial \dot{q_\iota}}\delta \dot{q_\iota}$$

$$= L(q_i, \dot{q_\iota}, t) + \delta L, \qquad (3.3)$$

Now we can obtain variation and the differentiation operators:

$$\delta q_i(t) = q_i'(t) - q_i(t) \rightarrow \frac{d}{dt}(\delta q_i) = \dot{q_\iota}'(t) - \dot{q_\iota}(t) = \delta \dot{q_\iota}(t), \; (3.4)$$

Thus we get to the following expression for δL

$$\delta L = \frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q_\iota}}\delta \dot{q_\iota} = \frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q_\iota}}\frac{d}{dt}(\delta q_i) = \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_\iota}}\right)\right)\delta q_i + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_\iota}}\delta q_i\right). \qquad (3.5)$$

Now we apply the Stationary Action principle to get the equation of motion for the physical paths, the action must be maximum, or minimum or an inflexion point. This translates mathematically in to

$$\delta S = \delta \int_{t_1}^{t_2} dtL(q_i, \dot{q_\iota}, t) = \int_{t_1}^{t_2} dt\delta L(q_i, \dot{q_\iota}, t) = 0, \qquad (3.6)$$

Expanding δL we obtain:

$$\delta S = \int_{t1}^{t2} dt \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\iota}\right) \right) \delta q_i + \int_{t1}^{t2} dt \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\iota} \delta q_i\right) = 0, \quad (3.7)$$

Since $\delta q_i\,(t_1) = \delta q_i\,(t_2) = 0$, then this implies that the second integral vanishes:

i.e. $$\int_{t1}^{t2} dt \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\iota} \delta q_i\right) = \left[\frac{\partial L}{\partial \dot{q}_\iota} \delta q_i\right]_{t1}^{t2} = 0, \quad (3.8)$$

Therefore we see that

$$\delta S = \int_{t1}^{t2} dt \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\iota}\right) \right) \delta q_i = 0, \quad (3.9)$$

Then we can obtain the following equations:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\iota}\right) = 0, \quad (3.10)$$

$\forall q_i$. These Equations are called the Euler – Lagrange equations of motion.

Now from the eq. (3.8) we can also conclude an important side of Lagrangians that they are not unique defined:

$$L(q_i\,, \dot{q}_\iota\,, \mathrm{t})\ and\ \tilde{L}(q_i\,, \dot{q}_\iota\,, t\,) = L(q_\iota\,, \dot{q}_\iota, t) + \frac{dF(q_\iota, t)}{dt}, \quad (3.11)$$

generate the same equations of motion. We have an anagram way to directly change that by substituting a function of the form $\frac{dF(q_i,t)}{dt}$ to the

Lagrangian, doesn't alter the equation of motion. Use (3.10) to $\frac{dF(q_i,t)}{dt}$ we obtain:

$$\frac{\partial}{\partial q_i}\left(\frac{dF(q_i,t)}{dt}\right)-\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_\iota}\left(\frac{dF(q_i,t)}{dt}\right)\right)=0, \qquad (3.12)$$

Next we will present $Noether's\,Theorem$ which is one of most important theorem of analytical mechanics:

3.2.2 $\boldsymbol{Noether's\ Theorem}$

In this part we deal with a conserved quantity associated with every symmetry of Lagrangian of a system.

We consider a transformation of a coordinate system

$$q_i \quad \rightarrow \; q_i' = q_i \; + \delta q_i \;\; , \qquad (3.13)$$

Thus the variation of the Lagrangian can be written as the exact differential of the function F: $L(q_i' \,, \dot{q}_i' \,, t) = L(q_i \,, \dot{q}_\iota, t) + \frac{dF(q_i \,,\dot{q}_i \,,\; t)}{dt}$ $\; this\; impiies\; \delta L = \frac{dF(q_i \,,\dot{q}_i \,,\; t)}{dt}$,

(3.14)

In here we allow F to depend on , $\dot{q}_\iota$ (that was not the case of (3 .12)). On the other hand, we can write δLas:

$$\delta L=\left(\frac{\partial L}{\partial q_i}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\iota}\right)\right)\delta q_i+\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\iota}\delta q_i\right)=\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\iota}\delta q_i\right), \qquad (3.15)$$

To obtain the last equality we considered the equations of motion. Suppose we write ($\delta q_i = \epsilon f_i$) , then an infinitesimal variation is form

$$q_i' = q_i \; + \delta q_i \;\; = q_i \; + \epsilon f_i \; , \qquad (3.16)$$

where $\epsilon\; acostant$, and $f_i\; a\; smooth$, in the limit $\epsilon \rightarrow 0$ we get

$$\lim_{\epsilon\to 0} q_i' = q_i \implies \lim_{\epsilon\to 0} \delta L = 0, \qquad (3.17)$$

Thus, F be of the form $F = \epsilon\tilde{F}$, and

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\iota}(\epsilon f_i)\right) = \epsilon \frac{d\tilde{F}(q_i\,,\ \dot{q}_i\,,\ \ t)}{dt}\ , \qquad (3.18)$$

Integrating with respect to t, we get

$$\left(\frac{\partial L}{\partial \dot{q}_\iota}(f_i)\right) = \tilde{F}(q_i\,, \dot{q}_i\,,\ \ t) + C\ , \qquad (3\ .19)$$

where, C is integration constant. Now we can introduced, that the conserved quantity associated to an infinitesimal symmetry [37] as:

$$C = \frac{\partial L}{\partial \dot{q}_\iota} f_i - \tilde{F}(q_i\,, \dot{q}_i\,,\ \ t) \qquad (3\ .20)$$

3.3 Hamiltonian Formalism

The Hamiltonian functional of a physical system is define as

$$H(q_i\ , p_i\ , t) \equiv\ p_i \dot{q}_\iota - L\ , \qquad (3.21)$$

where p_i the canonical conjugated momentum and defined as

$$p_i = \frac{\partial L}{\partial \dot{q}_\iota}\,, \qquad (3\ .22)$$

If the Euler- Lagrange equations (3. 10) are satisfied then:

$$\dot{p}_i = \frac{\partial L}{\partial q_i} \qquad (3.23)$$

Now we are obtain the Hamiltonian equations of motion by applying the principle of the stationary action:

$$\delta S = \int_{t1}^{t2} dt\, \delta L = \int_{t1}^{t2} dt \delta(p_i \dot{q}_i - H) = \int_{t1}^{t2} dt(\delta p_i \dot{q}_i + p_i \delta \dot{q}_i - \frac{\partial H}{\partial q_i}\delta q_i - \frac{\partial H}{\partial p_i}\delta p_i) = \int_{t1}^{t2} dt(\delta p_i \dot{q}_i + \frac{d}{dt}(p_i \delta q_i) - \dot{p}_i \delta q_i - \frac{\partial H}{\partial q_i}\delta q_i - \frac{\partial H}{\partial p_i}\delta p_i)$$

$$= \int_{t1}^{t2} dt \left(\delta p_i \left[\dot{q}_i - \frac{\partial H}{\partial p_i}\right] + \delta q_i \left[-\dot{p}_i - \frac{\partial H}{\partial q_i}\right]\right) + \int_{t1}^{t2} dt\,(p_i \delta q_i)$$

$$= \int_{t1}^{t2} dt(\delta p_i \left[\dot{q}_i - \frac{\partial H}{\partial p_i}\right] + \delta q_i \left[-\dot{p}_i - \frac{\partial H}{\partial q_i}\right]) = 0, \qquad (3.24)$$

From theses equations, for arbitrary, $\delta p_i and\ \delta q_i$, the Hamiltonian equations of motion are easily given as:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad (3.25)$$

If the Hamiltonian displays clear time dependence, it can be simply related to the time dependence of the Lagrangian

$$\frac{dH}{dt} = \frac{d}{dt}(p_i \dot{q}_i - L) = \frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{dH}{dt} = \dot{q}_i \dot{p}_i - \dot{p}_i \dot{q}_i + \frac{\partial H}{\partial t}$$

$$= \dot{p}_i \dot{q}_i + p_i \ddot{q}_i - \frac{\partial L}{\partial q_i}\dot{q}_i - \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i - \frac{\partial L}{\partial t}, \qquad (3.26)$$

Therefore we can obtain the following relation in partial derivatives:

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}, \qquad (3.27)$$

3.4 Continuous Systems

The last paragraph we have considered discrete systems characterized by(N time) degrees of freedom. Now we are considering the systems depends on an infinite number of degrees of freedom ($N \rightarrow \infty$) and replace discrete coordinates q_i by the continuous field that is defined for every point in space and they can vary with time

$$q_i(t) \rightarrow \emptyset(x,t) \equiv \emptyset(x^\mu) = \emptyset(x), \quad (3.28)$$

In here we have spatial dependence besides the time dependence, the following replacement is can justified:

$$\dot{q}_\iota(t) \rightarrow (\partial_t \emptyset(x), \partial_k \emptyset(x)) = \partial_\mu \emptyset(x). \quad (3.29)$$

Remembering that we have introduced the compact relativistic notion and we have supposed that the partial derivatives of the fields are Lorentz or (Poincare) covariant quantity of form $\partial_\mu \emptyset(x)$, with

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = (\partial_t, \nabla) \equiv (\partial_t, \partial_k). \quad (3.30)$$

Now we considering the following important define of contra variant quantity

$$\partial^\mu = g^{\mu v} \partial_v = (\partial_t, -\nabla) \equiv (\partial_t, -\partial_k), \quad (3.31)$$

Then we are only interested in Lagrangians that are invariant under space-time translations besides the Lorentz translations (Poincare groups). Therefore they are not depend explicitly on x^μ. The most general Lagrangian that exhibits all the properties, we have written described as:

$$L = \int_V d^3x \mathcal{L}\left(\emptyset_i(x), \partial_\mu \emptyset_i(x)\right), \quad (3.32)$$

where $\mathcal{L}$ is the Lagrangian density and we can written Lagrangian density as a functional of M (with M finite) fields$\{\emptyset_i(x)\}_i^M$. Thus, the action can be written as an integral of the Lagragian density

$$S = \int_{t_1}^{t_2} dtL = \int_{t_1}^{t_2} dt \int_V d^3x\mathcal{L}x\mathcal{L}\left(\emptyset_i(x), \partial_\mu \emptyset_i(x)\right)$$

$$= \int_{x_1}^{x_2} d^4x\, \mathcal{L}\left(\emptyset_i(x), \partial_\mu \emptyset_i(x)\right), \qquad (3.33)$$

Just like in the discrete case.

3.5 Euler–Lagrange Equations

In order to obtain the Euler- Lagrange equations of motion we considering the variations of the fields, keeping the extremes fixed

$$\emptyset'(x) = \emptyset_i(x) + \delta\emptyset_i(x): \;\; \delta\emptyset_i(x_1) = \delta\emptyset_i(x_2) = 0, \qquad (3\,.34)$$

Under these variations, we can define the following:

$$\delta\left(\emptyset_i(x), \partial_\mu \emptyset_i(x)\right) = \mathcal{L}((\emptyset_i(x) + \delta\emptyset_i(x), \partial_\mu \emptyset_i(x) + \delta\left[\partial_\mu \emptyset_i(x)\right])$$

$$-\mathcal{L}\left(\emptyset_i(x), \partial_\mu \emptyset_i(x)\right), \qquad (3.35)$$

Thus, we can obtain the following operations:

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\emptyset_i(x)}\delta\emptyset_i(x) + \frac{\partial\mathcal{L}}{\partial\left[\partial_\mu \emptyset_i(x)\right]}\delta\left[\partial_\mu \emptyset_i(x)\right]$$

$$= \frac{\partial\mathcal{L}}{\partial\emptyset_i(x)}\delta\emptyset_i(x) + \frac{\partial\mathcal{L}}{\partial\left[\partial_\mu \emptyset_i(x)\right]}\partial_\mu[\delta\emptyset_i(x)]$$

$$= \left(\frac{\partial \mathcal{L}}{\partial \emptyset_i(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial\left[\partial_\mu \emptyset_i(x)\right]}\right) \delta \emptyset_i(x) + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\left[\partial_\mu \emptyset_i(x)\right]} \delta \emptyset_i(x)\right), (3.36)$$

Now, we obtain the Euler- Lagrange equations for continuous systems by applying the principle of stationary action as follows:

$$\delta S = \int_{x_1}^{x_2} \left(\frac{\partial \mathcal{L}}{\partial \emptyset_i(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial\left[\partial_\mu \emptyset_i(x)\right]}\right) \delta \emptyset_i(x) + \int_{x_1}^{x_2} \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\left[\partial_\mu \emptyset_i(x)\right]} \delta \emptyset_i(x)\right)$$

$$= \int_{x_1}^{x_2} \left(\frac{\partial \mathcal{L}}{\partial \emptyset_i(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial\left[\partial_\mu \emptyset_i(x)\right]}\right) \delta \emptyset_i(x), \qquad (3\ .37)$$

Thus, for arbitrary $\delta \emptyset_i(x)$, the equation becomes

$$\frac{\partial \mathcal{L}}{\partial \emptyset_i(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial\left[\partial_\mu \emptyset_i(x)\right]} = 0, \qquad (\ 3.38)$$

For all $\emptyset_i$, $i = 1, \dots, M.$ $We\ can\ also$ see that from the right side of the (3.37) with condition $\delta \emptyset_i(x_1) = \delta \emptyset_i(x_2) = 0$, we found that

$$\int_{x_1}^{x_2} \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\left[\partial_\mu \emptyset_i(x)\right]} \delta \emptyset_i(x)\right) = 0. \qquad (\ 3.39)$$

Thus, if we considering an arbitrary functional of the form $b^{\mu}(\emptyset_i(x))$, then

$$\delta \int_{x_1}^{x_2} \partial_\mu b^{\mu}(\emptyset_i(x)) = \int_{x_1}^{x_2} \partial_\mu \left(\frac{\partial b^{\mu}}{\partial \emptyset_i(x)} \delta[\emptyset_i(x)]\right) = 0, \qquad (3.40)$$

the conclude is that a Lagrangian density is not defined uniquely. Similar to the discrete case, we can add a functional of the form $\partial_\mu b^{\mu}(\emptyset_i(x))$ without changing the equations of the motion. There fore

$$\mathcal{L}\left(\emptyset_i(x), \partial_\mu \emptyset_i(x)\right) \textit{ and } \mathcal{L}\left(\emptyset_i(x), \partial_\mu \emptyset_i(x)\right) + \partial_\mu b^{\mu}\left(\emptyset_i(x)\right), \qquad (3\ .41)$$

give back the same equations of motion.

3.6 Hamiltonian Formalism

The Hamiltonian density is define as

$$\varkappa\left(\pi_i(x), \emptyset_i(x), \nabla\emptyset_i(x)\right) \equiv \dot{\emptyset}_i(x)\pi_i(x) - \mathcal{L}. \qquad (3.42)$$

where $\dot{\emptyset}_i = \partial_t \emptyset_i(x)$ *and* $\pi_i(x)$ *is the canonical momentum associated* to the field $\emptyset_i(x)$:

$$\pi_i(x) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\emptyset}_i(x)}. \qquad (3.43)$$

The action can be written in terms of the Hamiltonian density as

$$S \equiv \int_{x_1}^{x_2} d^4x\mathcal{L} = \int_{x_1}^{x_2} d^4x\big(\dot{\emptyset}_i(x)\pi_i(x) - \varkappa\big). \tag{3.44}$$

Now applying the principle of the stationary action to get

$$\delta S = \int_{x_1}^{x_2} d^4x \left[\delta\pi_i\left\{\dot{\emptyset}_i - \frac{\partial\varkappa}{\partial\pi_i}\right\} - \delta\emptyset_i\left\{\dot{\pi}_\iota + \frac{\partial\varkappa}{\partial\emptyset_i} - \partial_k\frac{\partial\varkappa}{\partial(\partial_k\emptyset_i)}\right\}\right] = 0. \tag{3.45}$$

For any arbitrary $\delta\pi_i\ and\ \delta\emptyset_i$. thus equations of motion simply write as

$$\dot{\emptyset}_i(x) = \frac{\partial\varkappa}{\partial\pi_i(x)}, \qquad \dot{\pi}_\iota = -\frac{\partial\varkappa}{\partial\emptyset_i(x)} + \partial_k\frac{\partial\varkappa}{\partial(\partial_k\emptyset_i)}. \tag{3.46}$$

where $\partial_k\ are\ spatial\ derivatives\ (k = 1, 2, 3)$.

3.7 $Noether's$ Theorem (The General Formulation)

Consider the variation of the shape of the field without changing the space-time coordinates which defined as

$$\delta\emptyset_i(x) \equiv \emptyset'_i(x) - \emptyset_i(x). \tag{3.47}$$

We can also define another type of variation which is closely related, a local variation. It is defined as the difference between the fields evaluated in the same space-time point but in two different coordinates systems:

$$\bar{\delta}\emptyset_i(x) \equiv \emptyset'_i(x') - \emptyset_i(x). \tag{3.48}$$

Now consider a continuous space time translation which define as following

$$x^\mu \rightarrow x'^\mu = x^\mu + \Delta x^\mu . \tag{3.49}$$

Which be proper orthochronous Lorentz transformation or space-time transformation. At first order in Δx, $\bar{\delta}\emptyset_i(x)$ write as:

$$\bar{\delta}\emptyset_i(x) \equiv \emptyset'_i(x) - \emptyset_i(x) = \emptyset'_i(x + \Delta x) - \emptyset_i(x)$$

$$\approx \emptyset'_i(x) + \left(\partial_\mu \emptyset'_i(x)\right) \Delta x^\mu - \emptyset_i(x)$$

$$\approx \emptyset'_i(x) + \left(\partial_\mu \emptyset_i(x)\right) \Delta x^\mu - \emptyset_i(x)$$

$$= \delta\emptyset_i(x) + \left(\partial_\mu \emptyset_i(x)\right) \Delta x^\mu. \qquad (3.50)$$

Therefore, we have found the following relation between $\delta\emptyset_i(x)\ and\ \bar{\delta}\emptyset_i(x)$ for an infinitesimal transformation of the type (3.49):

$$\bar{\delta}\emptyset_i(x) = \delta\emptyset_i(x) + \left(\partial_\mu \emptyset_i(x)\right) \Delta x^\mu. \qquad (3.51)$$

If $\emptyset'_i(x') = \emptyset_i(x)$ ($which\ is\ in\ general\ the\ case$ for scalar field; it is also the case for spinor fields under space-time translations) then

$$\delta\emptyset_i(x) = -\left(\partial_\mu \emptyset_i(x)\right) \Delta x^\mu. \qquad (3.52)$$

Thus, in this case, an equivalent way of making a transformation of the type (3.49).Which acts on the coordinates, is by making an opposite transformation on the field:

$$\emptyset_i(x) \to \theta'_i(x) = \emptyset_i(x - \Delta x) \qquad (3.53)$$

Let us now deduce the Lagrangian transforms under these type of variations.

In order to keep the notation short, we shall introduce the following short-hand notations:

$$\mathcal{L}(x) \equiv \boldsymbol{\mathcal{L}}\big(\emptyset_{\boldsymbol{i}}(\boldsymbol{x}), \boldsymbol{\partial}_{\boldsymbol{\mu}}\emptyset_{\boldsymbol{i}}(\boldsymbol{x})\big), \qquad \boldsymbol{\mathcal{L}}'(x) \equiv \mathcal{L}\big(\emptyset_{\boldsymbol{i}}{}'(\boldsymbol{x}), \boldsymbol{\partial}_{\boldsymbol{\mu}}\emptyset_{\boldsymbol{i}}{}'(\boldsymbol{x})\big),$$

$$\mathcal{L}'(\boldsymbol{x}') \equiv \boldsymbol{\mathcal{L}}\left(\emptyset_{\boldsymbol{i}}'(\boldsymbol{x}'), \boldsymbol{\partial}_{\boldsymbol{\mu}}'\emptyset_{\boldsymbol{i}}{}'(\boldsymbol{x}')\right),\ b^{\mu}(x) \equiv \boldsymbol{b}^{\boldsymbol{\mu}}\big(\emptyset(\boldsymbol{x})\big). \quad (3.54)$$

Where $\partial_{\mu}' \equiv \frac{\partial}{\partial x'^{\mu}}$. keeping only terms up to O(Δx)*we can calculate* $\bar{\delta}\boldsymbol{\mathcal{L}}$(x) under(3.49)

$$\begin{aligned}\bar{\delta}\boldsymbol{\mathcal{L}} &= \boldsymbol{\mathcal{L}}'(\boldsymbol{x}') - \boldsymbol{\mathcal{L}}(\boldsymbol{x}) \\ &= \boldsymbol{\mathcal{L}}\big(\emptyset_{\boldsymbol{i}}(\boldsymbol{x}) + \bar{\boldsymbol{\delta}}\emptyset_{\boldsymbol{i}}(\boldsymbol{x}), \boldsymbol{\partial}_{\boldsymbol{\mu}}\emptyset_{\boldsymbol{i}}(\boldsymbol{x}) + \bar{\boldsymbol{\delta}}\big[\boldsymbol{\partial}_{\boldsymbol{\mu}}\emptyset_{\boldsymbol{i}}(\boldsymbol{x})\big]\big) - \boldsymbol{\mathcal{L}}(\boldsymbol{x})\end{aligned}$$

$$\approx \boldsymbol{\mathcal{L}}(x) + \frac{\partial\mathcal{L}(x)}{\partial\emptyset_i(x)}\bar{\delta}\emptyset_i + \frac{\partial\mathcal{L}(x)}{\partial\big[\partial_{\mu}\emptyset_i(x)\big]}\bar{\delta}\big[\partial_{\mu}\emptyset_i(x)\big] - \boldsymbol{\mathcal{L}}(\boldsymbol{x})$$

$$\approx \delta\boldsymbol{\mathcal{L}}(\boldsymbol{x}) + \big(\boldsymbol{\partial}_{\boldsymbol{\mu}}\boldsymbol{\mathcal{L}}(\boldsymbol{x})\big)\Delta\boldsymbol{x}^{\boldsymbol{\mu}}\ . \qquad (3.55)$$

Now we have introduce the following notation

$$\boldsymbol{\partial}_{\boldsymbol{\mu}}\boldsymbol{\mathcal{L}}(\boldsymbol{x}) \equiv \frac{\partial\mathcal{L}(x)}{\partial\emptyset_i(x)}\partial_{\mu}\emptyset_i(x) + \frac{\partial\mathcal{L}(x)}{\partial\big[\partial_{\mu}\emptyset_i(x)\big]}\partial_{\mu}\partial_{\mathcal{V}}\emptyset_i(x). \qquad (3.56)$$

Also we have used the following approximation:

$$\bar{\delta}\big[\partial_{\mu}\emptyset_i(x)\big] \approx \boldsymbol{\partial}_{\boldsymbol{\mu}}\big[\bar{\boldsymbol{\delta}}\emptyset_{\boldsymbol{i}}(\boldsymbol{x})\big] - \big(\partial_{\mathcal{V}}\emptyset_i(x)\big)\frac{\partial\Delta x^{\mathcal{V}}}{\partial x^{\mu}} = \boldsymbol{\partial}_{\boldsymbol{\mu}}\big[\bar{\boldsymbol{\delta}}\emptyset_{\boldsymbol{i}}(\boldsymbol{x})\big]. \ (3.57)$$

For the last equality we have used that for a Lorentz (Poincare) transformation

$\partial_{\mu}\Delta x^{\mathcal{V}} = 0$. Thus the new variation operator $\bar{\delta}$ also commutes with the derivation operator when restricting ourselves to Lorentz

(Poincare) continuous transformations. There for we have obtained an expression similar to the (3. 51) for$\bar{\delta}$:

$$\bar{\delta}\boldsymbol{\mathcal{L}}(x) = \delta\boldsymbol{\mathcal{L}} + \left(\partial_\mu \boldsymbol{\mathcal{L}}(\boldsymbol{x})\right)\Delta \boldsymbol{x}^{\boldsymbol{\mu}} \,. \tag{3.58}$$

Now considering the transformation of the system, a transformation that leaves the equations of the motion invariant is the symmetry of the system. Under such symmetry the system S will mostly transform as $S \to S' with\, S'$ is given by

$$S' = \int_\Omega d^4x' \boldsymbol{\mathcal{L}}'(\boldsymbol{x}') = \int_\Omega d^4x \boldsymbol{\mathcal{L}}(\boldsymbol{x}) + \int_\Omega d^4 \partial_\mu b^\mu(x)$$

$$= S + \int_\Omega d^4 \partial_\mu b^\mu(x). \tag{3.59}$$

Where $\delta S' = \delta S(thus\ generating\ the\ same\ equations\ of\ motion)$, to introducing the Jacobian matrix we have

$$\int_\Omega \left|\frac{\partial x'}{\partial x}\right| d^4\, x\boldsymbol{\mathcal{L}}'(\boldsymbol{x}') = \int_\Omega d^4\, x\boldsymbol{\mathcal{L}}(\boldsymbol{x}) + \int_{\boldsymbol{\Omega}} \boldsymbol{d^4 \partial_\mu b^\mu}\,(x). \tag{3.60}$$

This must hold for all space-time volume **Ω**, therefore:

$$\left|\frac{\partial x'}{\partial x}\right| \boldsymbol{\mathcal{L}}'(\boldsymbol{x}') = \boldsymbol{\mathcal{L}}(\boldsymbol{x}) + \boldsymbol{\partial_\mu b^\mu}(\boldsymbol{x}). \tag{3.61}$$

The determinant of the Jacobian matrix is equal to 1for a proper orthochronous Lorentz transformation. Thus

$$\bar{\delta}\boldsymbol{\mathcal{L}}(x) - \partial_\mu b^\mu(x) = 0. \tag{3.62}$$

Substituting the (3.54) in to (3.58) to introduces

$$\delta\boldsymbol{\mathcal{L}}(x) + \partial_\mu[\boldsymbol{\mathcal{L}}(\boldsymbol{x})\Delta x^\mu - b^\mu(x)] = 0. \tag{3.63}$$

Put in the explicit form of $\delta\boldsymbol{\mathcal{L}}$ from (3.36) in the last expression (3.63)

We introduce

$$\left(\frac{\partial\mathcal{L}(x)}{\partial\emptyset_i(x)} - \partial_\mu \frac{\partial\mathcal{L}(x)}{\partial\left[\partial_\mu\emptyset_i(x)\right]}\right)\delta\emptyset_i(x) + \partial_\mu\left(\frac{\partial\mathcal{L}(x)}{\partial\left[\partial_\mu\emptyset_i(x)\right]}\delta\emptyset_i(x)\right)$$

$$+\partial_\mu[\boldsymbol{\mathcal{L}}(\boldsymbol{x})\Delta x^\mu - b^\mu(x)] = 0. \qquad (3.64)$$

Using the Euler-Lagrange equations of motion we finally obtain the conservation laws as;

$$\partial_\mu j^\mu(x) = 0,$$

$$where\ j^\mu(x) = \frac{\partial\mathcal{L}(x)}{\partial\left[\partial_\mu\emptyset_i(x)\right]}\delta\emptyset_i(x) + [\boldsymbol{\mathcal{L}}(\boldsymbol{x})\Delta x^\mu - b^\mu(x)]. \qquad (3.65)$$

Where $j^\mu(x)$ is the conserved Noether's current. Our result is hold for continuous space-time transformation of the type

$$x^\mu \to x'^\mu = x^\mu + \Delta x^\mu.$$

And for transformations that only implies afield variation without modifying the space-time configuration. In this last case we would simply set $\Delta x^\mu = 0$ in (3.65). And we defined a conserved charge Q associated to conserved current $j^\mu(x)$ as:

$$Q = \int d^3x\, j^0. \quad , \qquad \frac{dQ}{dt} = \int d^3x \partial_0 j^0 = -\int d^3\, x\Delta j = 0. (3.66)$$

3.8 Geometrical Formulation

3.8.1 Lagrangian Mechanics

Lagrangian mechanics is more fundamental, since it is based on variational principles and it is what generalizes most directly to the general relativistic context.

The Lagrangian formulation of mechanics is based on variational principles behind the fundamental laws of force balance as given by Newton's laws F= ma. If we choose a configuration space Q with coordinate $q^i, i = 1, \dots, n,$ that describe the configuration of the system under study, then when introduces the Lagrangian [15], [16],[19] $L(q_{i,}\, \dot{q}_i, t)$, which is shorthand notation for $L(q^1, \dots, q^n, \dot{q}^1, \dots, \dot{q}^1, \dots, \dot{q}^n, t)$, where, L is the kinetic minus the potential energy of the system, and one takes $\dot{q}^i = dq^i/dt,$ to be the system velocity.

The variational principle of Hamilton states

$$\delta \int_a^b L(q^i, \dot{q}^i, t)dt = 0 \qquad (3.67)$$

In this principle, we choose curves $q^i(t)$ joining two fixed point in Q over fixed time interval$[a, b]$and calculate the integral regarded as a function of this curve. Hamilton's principle states that this function has a critical point at a solution within the space of curves. Now we let δq^i be a variation, that is, the derivate of a family of curves with respect to a parameter, then by the chain rule, (3.67) is equivalent to

$$\sum_{i=0}^{n} \int_a^b \left(\frac{\partial L}{\partial q^i} \delta q^i + \frac{\partial L}{\partial \dot{q}^i} \delta \dot{q}^i \right) dt = 0. \quad [30] \qquad (3.68)$$

$\forall\ variations\ \delta q^i$.

Using equality of mixed partials, we find that,

$$\delta\dot{q}^i = \frac{d}{dt}\delta q^i.$$

Using this integrating the second term of (3.68) by part, and employing the boundary conditions $\delta q^i = o \; at \; t = a \; and \; b$ equation (3.68) becomes [32]

$$\sum_{i=0}^{n}\int_a^b \left[\frac{\partial L}{\partial q^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right)\right]\delta q^i dt = 0. \qquad (3.69)$$

Where $\delta q^i$is arbitrary, and equations (3.69) is equivalent to the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial q^i} - \frac{\partial L}{\partial q^i} = 0, i = 1, \dots, n. \qquad (3.70)$$

For a system of N particles moving in Euclideen 3-space, we choose the configuration space to be $Q = R^{3N} = R^3 \times \dots \times R^3 (N \; times)$, and L has the form of kinetic minus potential energy:

$$L(q_i, \dot{q}_i, t) = \frac{1}{2}\sum_{i=1}^{N} m_i \|\dot{q}_i\|^2 - V(\dot{q}_i). \qquad (3\,.71)$$

Where one can write points in Q as $q_i, \dots, q_N \; where \; q_i \in R^3$.in this case the Euler-Lagrange equations (3.70) reduce to Newton's second law

$$\frac{d}{dt}(m_i \dot{q}_i) = -\frac{\partial V}{\partial q_i}, i = 1, \dots, N\,. \qquad (3.72)$$

There is $F = ma$ for the motion of particles in the potential V.Also in Lagrangian mechanics, we can identifies a configuration space Q with coordinates $(q^1 , \dots , q^n)$ and the forms of velocity phase space TQ, where TQ called the tangent bundle of Q. Coordinates on TQ are denoted by

$$(q^1 , \dots , q^n, \dot{q}^1, \dots , \dot{q}^1, \dots , \dot{q}^n)$$

and the Lagrangian is regarded as a function $L{:}\, TQ \to R$. This stage, interesting links with geometries are possible. If $g_{ij}(q)$is a given metric tensor or mass matrix and we consider the kinetic energy Lagrangian

$$L(q^i, \dot{q}^i) = \frac{1}{2}\sum_{i,j=1}^{n} g_{ij}(q)\dot{q}^i\,\dot{q}^i. \qquad (3\,.73)$$

Then the Euler- Lagrange equations are equivalent to the equations of geodesic motion, as can be verified next lemma.

Lemma 3.8.1.1 Geodesic equations: considering Lagrange's equations of motion for the Lagrangian

$$L(q, \dot{q}) = \frac{1}{2}\sum_{i,k=1}^{n} g_{i\,k}(q)\dot{q}_i\,\dot{q}_k\,. \qquad (3.74)$$

Are given in local coordinates by the system of ordinary differential equations

$$\ddot{q}_i + \sum_{j,k=1}^{n} \Gamma^i_{jk}\,\dot{q}_j\dot{q}_k = 0, \qquad (3.75)$$

Where the quantities Γ^i_{jk} are known as the Christoffel symbols and for

$i, j, k = 1, \dots n$, are given by

$$\Gamma^i_{j,k} = \sum_{i=1}^{n} \frac{1}{2} g^{il} \left(\frac{\partial g_{lj}}{\partial q_k} + \frac{\partial g_{kl}}{\partial q_j} - \frac{\partial g_{jk}}{\partial q_l} \right). \qquad (3.76)$$

Proof: We complete the proof in the steps as follows:

Step1: for the Lagrangian $L = L(q, \dot{q})$ defined in the statement of the lemma, using the product and chain rules, we find that

$$\frac{\partial L}{\partial q_j} = \frac{1}{2} \sum_{i,k=1}^{n} \frac{\partial g_{ik}}{\partial q_j} \dot{q}_j \dot{q}_k$$

And

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{1}{2} \sum_{k=1}^{n} g_{i\,k}\, \dot{q}_k + \frac{1}{2} \sum_{i=1}^{n} g_{i\,j}\, \dot{q}_i = \frac{1}{2} \sum_{k=1}^{n} g_{j\,k}\, \dot{q}_k + \frac{1}{2} \sum_{k=1}^{n} g_{kj}\, \dot{q}_k$$
$$= \sum_{k=1}^{n} g_{j\,k}\, \dot{q}_k. \qquad (3.77)$$

where in the last step we utilized the symmetry of g_{jk}. Using this last expression we see that:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j} \right) = \sum_{k=1}^{n} \frac{d}{dt} \left(g_{j\,k} \dot{q}_k \right) = \sum_{k=1}^{n} \frac{d g_{j\,k}}{dt} \dot{q}_k + \sum_{k=1}^{n} \left(g_{j\,k} \ddot{q}_k \right)$$

$$= \sum_{k,l=1}^{n} \frac{\partial g_{jk}}{\partial q_l} \dot{q}_l \dot{q}_k + \sum_{k=1}^{n} \left(g_{j\,k} \ddot{q}_k \right). \qquad (3.78)$$

Substituting the expressions above into Lagrange's equations of motion, using that the summation indices i, k *can be relabelled and that* g *is symmetric*, we find

$$\sum_{j,k=1}^{n}\left(g_{j\,k}\ddot{q}_k\right)+\sum_{k,l=1}^{n}\left(\frac{\partial g_{jk}}{\partial q_l}-\frac{1}{2}\frac{\partial g_{kl}}{\partial q_j}\right)\dot{q}_k\dot{q}_l=0\ \text{ for each } j = 1,\dots,n\,, \tag{3.79}$$

Step2: we multiply the last equations from step1 by $g^{ij}$and sum over the index j this gives

$$\sum_{j,k=1}^{n}\left(g^{ij}g_{j\,k}\ddot{q}_k\right)+\sum_{k,l=1}^{n}g^{ij}\left(\frac{\partial g_{jk}}{\partial q_l}-\frac{1}{2}\frac{\partial g_{kl}}{\partial q_j}\right)\dot{q}_k\dot{q}_l=0$$

$$\equiv \ddot{q}_i+\sum_{k,l=1}^{n}g^{ij}\left(\frac{\partial g_{jk}}{\partial q_l}-\frac{1}{2}\frac{\partial g_{kl}}{\partial q_j}\right)\dot{q}_k\dot{q}_l=0$$

$$\equiv \ddot{q}_i+\sum_{j,k,l=1}^{n}g^{il}\left(\frac{\partial g_{lj}}{\partial q_l}-\frac{1}{2}\frac{\partial g_{jk}}{\partial q_l}\right)\dot{q}_j\dot{q}_k=0\,. \tag{3.80}$$

where the last step we relabeled summation indices.

Step3: by using the symmetry of g and performing some further relabeling of summation indices, we see that

$$\sum_{j,k,l=1}^{n}g^{il}\left(\frac{\partial g_{lj}}{\partial q_k}-\frac{1}{2}\frac{\partial g_{jk}}{\partial q_l}\right)\dot{q}_j\dot{q}_k = \sum_{j,k,l=1}^{n}\frac{1}{2}g^{il}\left(\frac{\partial g_{lj}}{\partial q_k}+\frac{\partial g_{lj}}{\partial q_k}-\frac{\partial g_{jk}}{\partial q_l}\right)\dot{q}_j\dot{q}_k$$

$$=\sum_{j,k,l=1}^{n}\frac{1}{2}g^{il}\left(\frac{\partial g_{lj}}{\partial q_k}+\frac{\partial g_{kl}}{\partial q_j}-\frac{\partial g_{jk}}{\partial q_l}\right)\dot{q}_j\dot{q}_k=\sum_{j,k=1}^{n}\Gamma^{i}_{ij}\,\dot{q}_j\dot{q}_k\,. \tag{3.81}$$

where the, Γ^i_{ij} are those given in the statement of the lemma.

Conservation laws that are a result of symmetry in a mechanical context can then be applied to yield interesting geometric facts. For instance, theorem about geodesics on surfaces of revolution can be readily proved this way.

The Lagrangian formalism can be extended to the infinite-dimensional case. One view is to replace the q^i by field $\varphi^1, \dots, \varphi^m$ that are, for example, functions of spatial points x^i and time t. Then L is a function of $\varphi^1, \dots, \varphi^m, \dot{\varphi}^1, \dots, \dot{\varphi}^m$ and the spatial derivatives of the fields.

3.8.2 Holonomic Constraints

Definition3.8.2 (Holonomic constraints) for a system of particles with positions given by $r_i(t)$ for$i = 1, \dots, N$, constraints that can be expressed in the form

$$g(r_1, \dots, r_N, t) = 0$$

are said to be $holonomic$. They only involve the configuration coordinates.

We will only consider systems for which the constraints are $holonomic$ system with constraints that are non- $holonomic$ are: gas molecules in a container, or sphere rolling on a rough surface without slipping.

Let us suppose that for the N particles there are m $holonomic$ constraints given by

$$g_k(r_1, \dots, r_N, t) = 0,$$

For, $k = 1, \dots, m$. The positions $r_i(t)$ of all N particles are determined by 3N coordinates. However due to the constraints, the positions $r_i(t)$are not all independent. In principle, we can use the m holonomic constraints to eliminate m of the $3N$ coordinates and we would be left with 3N-m independent coordinates, i.e. the dimension of the configuration space is actually, $3N - m$.

3.8.3 Degrees of Freedom

The dimension of configuration space is called the number of degrees of freedom, thus we can transform from the old coordinates $r_1, \dots, r_N$ to new generalized coordinates $q_1, \dots, q_n$ where

$$n = 3N - m,\ r_1 = r_1(q_1, \dots, q_n, t), \dots ,\ r_N = r_N(q_1, \dots, q_n, t).$$

3.8.4 D' Alembert's Principle

Consider Newton's second law of motion for the i^{th} particles

$$\dot{p}_i = F_i^{ext} + F_i^{con} \quad . \tag{3.82}$$

For $i = 1, \dots, N$ and where F_i^{ext} external force, F_i^{con} constraints force and

$p_i = m_i v_i$ is the linear momentum of the i^{th}particle and $v_i = \dot{r}_i$ velocity

We will restrict ourselves to systems for which the net work of the constraints force is zero i.e. we suppose

$$\sum_{i=1}^{n} F_i^{con} . dr_i = 0. \tag{3.83}$$

for every small charge dr_i, of the configuration of the system (for t fixed).

So here for the i^{th} particles, the constraint force applied is F_i^{con} and suppose it undergoes a small displacement given by the vectors dr_i, since the dot product of two vectors gives the projection of the one vector in the direction of the other , the dot product $F_i^{con}. dr_i$ gives the work done by F_i^{con} in the direction of the displacement of dr_i.

If we combine the assumption that the net work of the constraint forces is zero with Newton's second law, $\dot{p}_i = F_i^{ext} + F_i^{con}$, from the last section , we get

$$\sum_{i=I}^{N} \dot{p}_i . dr_i = \sum_{i=1}^{N} (F_i^{ext} + F_i^{con}) . dr_i$$

$$\Leftrightarrow \sum_{i=1}^{N} \dot{p}_i . dr_i = \sum_{i=1}^{N} F_i^{ext} . dr_i + \sum_{i=1}^{N} F_i^{con} . dr_i$$

$$\Leftrightarrow \quad \sum_{i=1}^{N} \dot{p}_i . dr_i = \sum_{i=1}^{N} F_i^{ext} . dr_i$$

Therefore

$$\sum_{i=1}^{N} (\dot{p}_i - F_i^{ext}) . dr_i = 0. \qquad (3.84)$$

for every small charge dr_i . The equations (3.84) are representing D′Alembert′s principle. In this equations a particles not have forces of constraints only moving by external force

Now consider the transformation to the generalized coordinates

$$r_i = r_i(q_1, \dots, q_n, t),$$

for $i = 1, \dots., N$. If we consider a small increment in the displacements dr_i then the corresponding increment in the work done by the external forces is

$$\sum_{i=1}^{N} F_i^{ext} . dr_i = \sum_{i=1}^{N} F_i^{ext} . \frac{\partial r_i}{\partial q_j} dq_j = \sum_{i=1}^{N} \mathcal{Q}_j \, dq_j . \quad (3.85)$$

In equations (3.85) we have used the chain rule

$$dr_i = \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} dq_j. \qquad (3.86)$$

And for $j = 1, \ldots, n$

$$Q_j = \sum_{i=1}^{N} F_i^{ext} . \frac{\partial r_i}{\partial q_j}. \qquad (3.87)$$

We deal with Q_j as generalized forces.

Now assume the work done by these forces depends on the initial and final configurations only and not on the path between them. In other words we assume there exists a potential function $V = V(q_i, \ldots, q_n)$ such that

$$Q_j = -\frac{\partial V}{\partial q_j}. \qquad (3.88)$$

For, $j = 1, \ldots, n$. Such forces are said to be conservative forces. We defines that total kinetic energy by

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i |v_i|^2. \qquad (3.89)$$

And the Lagrange function or Lagrangian as

$$L = T - V. \qquad (3.90)$$

Theorem 3.8.4.1 (Lagrange's equations) D' Alembert's principle, under the assumption the constraints are holonomic, is equivalent to system of ordinary differential equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0, \qquad (3.91)$$

For, $j = 1, \ldots, n$. These are equations known as Lagrange's equations of motion.

Proof: The change in kinetic energy mediated through the momentum the first term in

D' Alembert's principle due to the increment in the displacements dr_i is given by

$$\sum_{i=I}^{N} \dot{p}_i . dr_i = \sum_{i=1}^{N} m_i \, v_i . dr_i = \sum_{i,j=1}^{N,n} m_i \, v_i . \frac{\partial r_i}{\partial q_j} dq_j. \qquad (3.92)$$

Using product rule we get that

$$\frac{d}{dt}\left(v_i . \frac{\partial r_i}{\partial q_j}\right) \equiv \dot{v}_i . \frac{\partial r_i}{\partial q_j} + v_i \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right) \equiv \dot{v}_i . \frac{\partial r_i}{\partial q_j} + v_i . \frac{\partial v_i}{\partial q_j} \; . \quad (3.93)$$

Also, by differentiating the transformations to generalized coordinates we get

$$v_i = \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} d\dot{q}_j \;\; and \;\; \frac{\partial v_i}{\partial q_j} = \frac{\partial r_i}{\partial q_j}. \qquad (3.94)$$

Using (3.94) we obtain

$$\sum_{i=I}^{N} \dot{p}_i . dr_i = \sum_{j=1}^{n} \left(\sum_{i=1}^{N} m_i v_i . \frac{\partial r_i}{\partial q_j} \right) dq_j$$
$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{N} \left(\frac{d}{dt}\left(m_i v_i . \frac{\partial r_i}{\partial q_j} \right) - m_i v_i . \frac{\partial v_i}{\partial q_j} \right) \right) dq_j$$

$$= \sum_{j=1}^{n}\left(\sum_{i=1}^{N}\left(\frac{d}{dt}\left(m_i v_i . \frac{\partial v_i}{\partial \dot{q}_j}\right) - m_i v_i . \frac{\partial v_i}{\partial q_j}\right)\right) dq_j$$

$$\sum_{j=1}^{n}\left(\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\left(\sum_{i=1}^{N}\frac{1}{2}m_i|v_i|^2\right)\right) - \frac{\partial}{\partial q_j}\left(\sum_{i=1}^{N}\frac{1}{2}m_i|v_i|^2\right)\right) dq_j. \quad (3.95)$$

Hence we see that D′Alembert′s principle is equivalent to

$$\sum_{j=1}^{n}\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j\right) dq_j = 0 \quad (3.96)$$

Since the q_j for, $j = 1, \ldots, n$, where $n = 3N - m$, are all independent, we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j = 0, \quad (3.97)$$

Using the definition for the generalized forces Q_j in term of the potential functions V gives the result.

Remark 3.8.4.2 If the system has forces that are not conservative it my still be possible to find a generalized potential function V such that

$$Q_j = -\frac{\partial V}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_j}\right), \quad (3\ .98)$$

Remark 3.8.4.3 (Non-uniqueness of the Lagrangian).Two Lagrangian's $L_1 and\ L_2$ that differ by the total time derivative of any function of $q = (q_1, \ldots, q_n)^T$ and t generate the same equations of motion. In fact if

$$L_2(q, \dot{q}, t) = L_1(q, \dot{q}, t) + \frac{d}{dt}(f(q, t)), \quad (3.99)$$

Then for $j = 1, \ldots, n$ direct calculation reveals that

$$\frac{d}{dt}\left(\frac{\partial L_2}{\partial \dot{q}_j}\right) - \frac{\partial L_2}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial L_1}{\partial \dot{q}_j}\right) - \frac{\partial L_1}{\partial q_j}, \qquad (3\,.100)$$

3.8.5 Constraints

Consider a Lagrangian $L(q, \dot{q}, t)$ for a system, suppose we realize the system has some constraints (so q_j are not all independent). Suppose we have m holonomic constraints of the form

$$G_k(q_1, \dots, q_n, t) = 0, \qquad (3\,.101)$$

For $j = 1, \dots, m < n$. we can now see the method of Lagrange multipliers with Hamilton's principle to deduce the equations of motion are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = \sum_{k=1}^{m} \lambda_k(t) \frac{\partial G_k}{\partial q_j}, \qquad (3.102)$$

$$G_k(q_1, \dots, q_n, t) = 0,$$

For, $j = 1, \dots, n \; and \; k = 1, \dots, m$. We call the quantities on the right above (3.102)

$$\sum_{k=1}^{m} \lambda_k(t) \frac{\partial G_k}{\partial q_j},$$

the generalized forces of constraints.

3.9 Hamiltonian Mechanics

We consider mechanical system that are holonomic and conservative (or for which the applied forces have a generalized potential). For such a system we can construct a Lagrangian $L(q, \dot{q}, t)$, where $q = (q_1, \dots, q_n)^T$ which is the differences of the total kinetic T and potential V energies. These mechanical system evolve according to the n Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0,$$

for $j = 1, \dots, n$. These are each second order ordinary differential equations and so the system is determined for all time once $2n$ initial conditions $(q(t_0), \dot{q}(t_0))$ are specified (or n conditions at two different times). The state of the system is represented by a point $q = (q_1, \dots, q_n)^T$ in configuration space.

3.9.1 ***Generalized Momenta***

We define the generalized momenta for a Lagrangian mechanical system for $j = 1, \dots, n$ to be

$$p_j = \frac{\partial L}{\partial \dot{q}_j}, \qquad (3.103)$$

where $p_j = p_j(q, \dot{q}, t)$ in general, we have $q = (q_1, \dots, q_n)^T$ and $\dot{q} = (\dot{q}_1, \dots, \dot{q}_n)^T$.

In term of generalized momenta , Lagrange's equation become

$$\dot{p}_j = \frac{\partial L}{\partial q_j}, for\ j = 1, \dots, n. \qquad (3\ .104)$$

Further, in principle, we can solve the relations above which define the generalized momenta, to find functional expressions for the $\dot{q}_j$ in term of $q_i,\ p_i\ and\ t$, i.e. we can solve the relations defining the generalized momenta to find $\dot{q}_j = \dot{q}_j(q, p, t) where$

$$q = (q_1, \dots, q_n)^T \; and \; p = (p_1, \dots, p_n)^T$$

3.9 .2 Hamiltonian Function

We define the Hamiltonian function as the Lagendre transform of the Lagrangian function, i.e. we define it as

$$H(q, p, t) = \dot{q}.p - L(q, \dot{q}, t), \qquad (3.105)$$

where, $q = (q_1, \dots, q_n)^T \; and \; p = (p_1, \dots, p_n)^T$ and we suppose $\dot{q} = \dot{q}(q, p, t)$.

In this definition we used the notion for the dot product

$$\dot{q}.p = \sum_{j=1}^{n} \dot{q}_j \, p_j, \qquad (3.106)$$

From the Lgrange's equations of motion we can deduce Hamilton's equations of motion, using the definitions for the generalized momenta and Hamiltonian,

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \; and \; H = \sum_{j=1}^{n} \dot{q}_j \, p_j - L, \qquad (3.107)$$

Theorem3.9.3 (Hamilton's equations of motion)

Lagrange's equations of motion imply Hamilton's canonical equations, for$i = 1, \dots n$ we have,

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad (3.108)$$

These consist of $2n$ first equations of motion.

Proof: Using the definition of the Hamiltonian in terms of the Lagrangian and with

$\dot{q}_j = \dot{q}_j(q, p, t)$ for $j = 1, \dots, n$ we use,

(1) The chain and product rules;

(2) That $\partial p_j/\partial p_i = 0$ if $i \neq j$,while $\partial p_j/\partial p_i = 1$ for, $i = j$, and

(3) The definition of generalized momenta. Directly computing using this sequence of results gets

$$\frac{\partial H}{\partial p_i} = \sum_{j=1}^{n} \frac{\partial \dot{q}_j}{\partial p_i} p_j - \sum_{j=1}^{n} \dot{q}_j \frac{\partial p_j}{\partial p_i} - \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial p_i}$$
$$= \sum_{j=1}^{n} \frac{\partial \dot{q}_j}{\partial p_i} p_i + \dot{q}_i - \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial p_i}$$

$$= \sum_{j=1}^{n} \frac{\partial \dot{q}_j}{\partial p_i} p_i + \dot{q}_i - \sum_{j=1}^{n} p_j \frac{\partial \dot{q}_j}{\partial p_i} = \dot{q}_i \,, \qquad (3.109)$$

Again we using the definition of the Hamiltonian as above, and we using (1), (2), (3) and the Lagrange's equations of motion in the form, $\dot{p}_j = \partial L/\partial q_j$. Directly computing using this sequence of results yields

$$\frac{\partial H}{\partial q_i} = \sum_{j=1}^{n} \frac{\partial \dot{q}_j}{\partial q_i} p_j - \sum_{j=1}^{n} \frac{\partial L}{\partial q_j} \frac{\partial q_j}{\partial q_i} - \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial q_i}$$
$$= \sum_{j=1}^{n} \frac{\partial \dot{q}_j}{\partial q_i} p_j - \frac{\partial L}{\partial q_i} - \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial q_i}$$

$$= \sum_{j=1}^{n} \frac{\partial \dot{q}_j}{\partial q_i} p_j - \frac{\partial L}{\partial q_i} - \sum_{j=1}^{n} p_j \frac{\partial \dot{q}_j}{\partial q_i} = -\frac{\partial L}{\partial q_i} = -\dot{p}_i, \qquad (3.110)$$

For $i = 1, \dots, n$. Collecting these relations together, we see that Lagrange's equations of motion, implies Hamilton's canonical equations as shown.

We have two useful observations:

First, if the Lagrangian $L = L(q, \dot{q})$ is independent of explicit t, then when we solve the equations that define the generalized momenta we find $\dot{q} = \dot{q}(\,q, p)$. Hence we see that

$$H = \dot{q}(q,p).p - L\big(q, \dot{q}(q,p)\big), \tag{3.111}$$

i.e. the Hamiltonian $H = H(q,p)$is also independent of t explicitly.

Second, in general, using the chain rule and Hamilton's equations we see that

$$\begin{aligned}\frac{dH}{dt} &= \sum_{i=1}^{n} \frac{\partial H}{\partial q_i}\dot{q}_i + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial t} \\ &= \sum_{i=1}^{n} \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\left(-\frac{\partial H}{\partial q_i}\right) + \frac{\partial H}{\partial t} \\ &= \frac{\partial H}{\partial t},\end{aligned} \tag{3.112}$$

Hence we have

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}, \tag{3.113}$$

Hence if H does not explicitly depend on t then

$$H \text{ is a } \begin{cases} constant\ of\ motion, \\ conserved\ quantity, \\ integral\ of\ the\ motion. \end{cases}$$

Hence the absence of explicit t dependence in the Hamiltonian H could serve as more general definition of a conservative system, thought in general H may not be the total energy. However for simple mechanical systems for which the kinetic energy $T = T(q, \dot{q})$ is a homogeneous quadratic function in $\dot{q}$, and the potential $V = V(q)$, then the Hamiltonian H will be the total energy. To see suppose

$$T = \sum_{i,j=1}^{n} c_{ij}\,(q)\dot{q}_i\dot{q}_j, \tag{3.114}$$

so that a homogeneous quadric function in $\dot{q}_i$; to be $c_{ij} = c_{ji}$. Then we have

$$\frac{\partial T}{\partial \dot{q}_k} = \sum_{j=1}^{n} c_{kj}(q)\,\dot{q}_j + \sum_{i=1}^{n} c_{ik}(q)\,\dot{q}_i = 2\sum_{i=1}^{n} c_{ik}\,(q)\dot{q}_i, \tag{115}$$

this implies that

$$\sum_{i,j=1}^{n} \dot{q}_k \frac{\partial T}{\partial \dot{q}_k} = 2T. \tag{3.116}$$

Thus the Hamiltonian $H = 2T - (T - V) = T + V$ is the total energy.

3.10 A Symplectic Form

We begin with a foundational assumption of Hamiltonian mechanics:[21]

First assumption: for any particle, the set of all its possible positions is an n-dimensional $(n \leq 3)$ smooth manifold, $C\ in\, \mathcal{R}^3$, called the configuration space. The manifold that we are interested in, however, is the cotangent bundle of C

$$T^*C \coloneqq \{(r, p^T) \in \mathcal{R}^3 \times (\mathcal{R}^3)^* : r \in \ C and\, p^T \in (T_r\, C)^*\}, \qquad (3.117)$$

Where r is a column vector, p^T is a row vector, and $(T_r\, C)^*$ is the dual space of the tangent space of C at r. in this assumption T^*C is itself a smooth 2n-dimensional manifold, and from now on we shall label such manifold as M. By definition of the tangent space, $T_r\, C$ is the set of all possible momentum p a particles traveling through r can take, and since we are working in finite dimensions, $(T_r\, C)^* \cong T_r\, C$, so we can identify each element of $(T_r\, C)^*$ as the transpose p^T of a momentum column vector p, from now on we shall write p for the row vectors of $(T_r\, C)^*$.

In case we want to incorporate N particles into our system, we may just take the cotangent bundle of a Cartesian product of configuration spaces, such that our manifold M has coordinates $(r_1, \dots, r_N, p_1, \dots, p_N)$ where $r_i\ and\, p_i$ represents the positions and momentum of the ithparticles for $i = 1, \dots, N$. For many systems, such as a free particle, a planetary system and ideal harmonic oscillator systems, we will have $M = \mathcal{R}^{3N} \times (\mathcal{R}^{3N})^*\ or\, M = \mathcal{R} \times \mathcal{R}^*$ for particle constrained to move in one dimension.

Second assumption: we shall make is that the net force F on the particle depends only on the particle's position and momentum. This covers many physicals forces, such that gravitational, electromagnetic, fractional, and spring forces. Newton second law states that

$$F\big(r(t), p(t)\big) = \frac{dp}{dt}, wher\ momentum\, p(t) := m\frac{dr}{dt}\ , (3.118)$$

The existence and uniqueness theorem for first order ordinary differential equations tells us that if we specify r and p at a given time, then we know that particle's position and momentum at an interval around that time. Thus every point (r, p) on our manifold M should define a path through Mthat corresponds to the trajectory of the particle.

Know we need away to find this trajectory, and we can do so given a symplectic form ω and an infinitely differentiable function $H\ on\ M$.

Definition3.10.1.A symplectic form ω is a closed, nondegenerate 2-form on a manifold M. in other words the following two conditions are satisfied.

$(i) d\omega = 0$. (Closedness)

(ii)For all $u \in M$ and nonzero $v \in T_u M$, there exists $w \in T_u M$ such that $\omega_u(v, w) \neq 0$.

(nondegenerate).

As it turns out, every cotangent bundle of a smooth manifold has a canonical symplectic form

$$\omega = \sum dp_i \wedge dr_i \, , \qquad (3.119)$$

or equivalently $\omega = dp \wedge dr$ in more condensed notation. A manifold M with a symplectic form ω is called a symplectic manifold (M, ω). When that manifold is the cotangent bundle of configuration space, (M, ω) is called phase space. For a more complete discussion of the cotangent bundle and it's symplectic structure. For a more details, see [38].

As for an infinitely differentiable function, we can assign to each point of M a total energy , and assume this function is infinitely differentiable (it is all realistic cases). We call this function the Hamiltonian and label it $H: M \to R$. The important fact that is ω and H determine a vector field on M a long whole particle will typically flow.

Lemma3.1o.2 A given a 2-form ω on an n-dimensional smooth manifold M, there exist a family of anti-symmetric matrices $\{A_q\}_{q \in M}$ such that for all, $q \in M \; and \; u, v \in T_q M$, we have

$$\omega_q(u, v) = u^T A_q v. \qquad (3.120)$$

Proof: Since ω is a 2-form, for every $q \in M, \omega_q$ is bilinear and anti-symmetric and can be represented as $\omega_q(u, v) = \sum_{i<j} \alpha_{ij} \, (p) dr_i \wedge dx_j$, (3.121)

where α_{ij} are real valued function on M. then define the matrices $A_q$such that

$$A_q = \{\alpha_{ij}(q)\} \quad \text{for } \; i < j \, , \qquad (3.122)$$

And the remaining entries determined by requiring A_q to be antisymmetric. Then the expression $u^T A_q v$ also define a 2-form that agrees with ω for all the coefficient functions, and so

$$\omega_q(u, v) = u^T A_q v.$$

Theorem 3.10.3 given an infinitely differentiable function H on smooth manifold M with symplectic form ω, there exists a unique vector field X_H satisfying

$$dH(.) = -\,\omega\,(\mathrm{X_H}\,,.), \tag{3.123}$$

Proof: Let A_q denote the matrices from lemma 3.10.1.since ω is symplectic, and therefore nondegenerate, A_q is invertible for all q. We define the vector field X_H as

$$X_H(q) = A_q^{-1}(\nabla H(q))^T, \tag{3.124}$$

Where $\nabla H(q)$is the row vector gradient of H at q. We show that X_H fulfills the desired condition, using the fact that the inverse of an antisymmetric matrix is antisymmetric:

$$\nabla H.(A_q^{-1})^T = X_H^T - \nabla H.A_q^{-1} = X_H^T - \nabla H$$
$$= X_H^T A_q. \tag{3.125}$$

Since dH is the one form that multiplies ∇H to a vector in the tangent space, this proves that X_H satisfies

$$dH(.) = -\,\omega\,(\mathrm{X_H}\,,.).$$

We need to prove X_H is unique. Suppose there exists an another vector field Y_H that satisfies the above property. Then by linearity of ω

$-\omega(X_H - Y_H, .) = -\omega(X_H, .) + \omega(Y_H, .) = dH(.) - dH(.) = 0,$ (3.126)

But by nondegenerate ω, this implies that

$$X_H - Y_H = 0, \qquad (3.127)$$

so X_H is unique.

The significance of the vector field $X_H$and the symplectic form ω can be seen through the example of particle under the influence of conservative forces. This meen that the net force can be written as

$$F = -\nabla U, \qquad (3.128)$$

Where $U(r)$ is real-valued potential energy function depending only the position of the particle.

Corolary3.10.4 A given a Hamiltonian system (M, ω, H) and Hamiltonian vector field X_H, we have

$$X_H H = 0, \qquad (3.129)$$

Proof: the proof is immediately from the definitions.

$X_H H = dH(X_H) = -\omega(X_H - Y_H) = 0$

where $\omega(X_H - Y_H) = 0$ since ω is antisymmetric.

Chapter Four

Symmetries of Lagrangian Systems

4.1 Introduction

The quantum will be based on our chapter and we deal with the concepts of the Classical mechanics. An important concept is that the equations of motion of the Classical Mechanics which based on a variational principle, that along a path describing classical motion the action integral (Hamiltonian Principle of Least Action).[5], [7]

The results of variational calculus derived allow us to formulate the Hamiltonian principle of Least Action of Classical Mechanics and study its equivalence to the Newtonian equations of motion. We used Gauge symmetry to obtained transformation of Lagrangian. Euler operator is importance and we obtained continuous symmetries and conservation laws of Noether's theorem Lorentz invariance is importance, also obtained the reduction of controlled Lagrangian system with symmetry.

Theorem 4.1 Hamiltonian Principle of Least Action

The trajectories $\vec{q}(t)$ of systems of particles described through the Newtonian equations of motion

$$\frac{d}{dt}\left(m_j\dot{q}_j\right) + \frac{\partial U}{\partial q_j} = 0; j = 1,2, \dots, M. \qquad (4.1)$$

are extremals of the functional, so the action integral is

$$S\left(\vec{q}(t)\right) = \int_{t_0}^{t_1} dt L(\vec{q}(t), \dot{q}(t), t). \qquad (4.2)$$

where $L(\vec{q}(t), \dot{\vec{q}}(t), t)$ is the Lagrangian :

$$L\left(\vec{q}(t), \dot{\vec{q}}(t), t\right) = \sum_{j=1}^{M} \frac{1}{2} m_j \dot{q}_j^2 - U(q_1, q_2, \dots, q_M). \quad (4.3)$$

in these equations we consider only velocity-independent potentials.

Proof : to prove the Hamiltonian Principle of Least Action we inspect the Euler- Lagrange conditions associated with the action integral defined through (4.2) , (4.3). The condition lead to

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0 \;\; \rightarrow \;\; -\frac{\partial U}{\partial q_j} - \frac{d}{dt}\left(m_j \dot{q}_j\right) = 0, (4.4)$$

which are obviously equivalent to the Newtonian equations of motion.

4.2 Particle Moving in An Electromagnetic Field

We consider the Newtonian equations of motion for a single particle of charge q with a trajectory $\vec{r}(t) = \left(x_1(t),\ x_2(t), x_3(t)\right)$ moving in electromagnetic field described through the electrical and magnetic field components $\vec{E}(\,r, t)\ and\ \vec{B}(\vec{r}, t)$, respectively. The equations of motion for such a particle are

$$\frac{d}{dt}\left(m\dot{\vec{r}}\right) = \vec{F}(\vec{r}, t);\ \vec{F}(\vec{r}, t) = q\vec{E}(\vec{r}, t) + \frac{q}{c}\vec{v} \times \vec{B}(\vec{r}, t), (4.5)$$

where $\frac{d\vec{r}}{dt} = \vec{v}$ and $\vec{F}(\vec{r}, t)$ is Lorentz force. The fields $\vec{E}(\,r, t)\ and\ \vec{B}(\vec{r}, t)$obey the Maxwell equations

$$\left.\begin{array}{c} \nabla \times \vec{E}(\,\vec{r}, t) + \dfrac{1}{c}\dfrac{\partial}{\partial t}\vec{B}(\vec{r}, t) = 0 \\ \nabla . \vec{B}(\vec{r}, t) = 0 \\ \nabla \times \vec{B}(\vec{r}, t) - \dfrac{1}{c}\dfrac{\partial}{\partial t}\vec{E}(\vec{r}, t) = \dfrac{4\pi\vec{J}}{c} \\ \nabla . \vec{E}(\vec{r}, t) = 4\pi\rho \end{array}\right\}. \qquad (4.6)$$

Where $\rho(\vec{r},t)$ describes the charge density present in the field and $\vec{j}(\vec{r},t)$ describes the charge current density. Equations $\nabla.\vec{B}(\vec{r},t) = 0 \; and \; \nabla \times \vec{B}(\vec{r},t) - \frac{1}{c}\frac{\partial}{\partial t}\vec{E}(\vec{r},t) = \frac{4\pi\vec{j}}{c}$ can be satisfied implicitly if we represent the fields through a scalar potential $V(\vec{r},t)$ and a vector potential $\vec{A}(\vec{r},t)$ as follows

$$\left.\begin{aligned} \vec{B}(\vec{r},t) &= \nabla \times \vec{A}(\vec{r},t) \\ \vec{E}(\vec{r},t) &= -\nabla V(\vec{r},t) - \frac{1}{c}\frac{\partial \vec{A}(\vec{r},t)}{\partial t} \end{aligned}\right\}. \quad (4.7)$$

4.3 Gauge Symmetry of the Electromagnetic Field

We will now consider the relationship between fields and potentials (4.7) allows we to transform the potentials without effecting the fields and without effecting the equations of motion (4.5) of particle moving in the field. The transformation which leaves the fields invariant is

$$\left.\begin{aligned} \vec{A}'(\vec{r},t) &= \vec{A}(\vec{r},t) + \nabla K(\vec{r},t) \\ V'(\vec{r},t) &= V(\vec{r},t) - \frac{1}{c}\frac{\partial}{\partial t}K(\vec{r},t) \end{aligned}\right\}. \quad (4.8)$$

4. 4 Lagrangian of Particle Moving in Electromagnetic Field

Now we want to show that the equation of motion (4.5) follows form the Hamiltonian Principle of Least Action, if we assumes for a particle the Lagrangian

$$L(\vec{r},\dot{\vec{r}},t) = \frac{1}{2}m\vec{v}^2 - qV(\vec{r},t) + \frac{q}{c}\vec{A}(\vec{r},t).\vec{v}. \quad (4.9)$$

For this purpose we consider only we component of the equation of motion (4, 5), namely

$\frac{d}{dt}(mv_1) = F_1 =$
$-q\frac{\partial V}{\partial x_1} + \frac{q}{c}[\vec{v} \times \vec{B}]$. (4.10)

We using equation (4.7), i.e., ($B_3 = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}$) so that

$$[\vec{v} \times \vec{B}] = \dot{x}_2 B_3 - \dot{x}_3 B_2 = \dot{x}_2 \left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right) - \dot{x}_3 \left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right). \quad (4.11)$$

This expression allows us to show that (4.10) is equivalent to the Euler-Lagrange condition

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0. \quad (4.12)$$

The second term in equation (4.12) is

$$\frac{\partial L}{\partial x_1} = -q\frac{\partial V}{\partial x_1} + \frac{q}{c}\left(\frac{\partial A_1}{\partial x_1}\dot{x}_1 + \frac{\partial A_2}{\partial x_1}\dot{x}_2 + \frac{\partial A_3}{\partial x_1}\dot{x}_3\right). \quad (4.13)$$

The first term of equation (4.12) is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = \frac{d}{dt}(m\dot{x}_1) + \frac{q}{c}\frac{dA_1}{dt}$$
$$= \frac{d}{dt}(m\dot{x}_1) + \frac{q}{c}\left(\frac{\partial A_1}{\partial x_1}\dot{x}_1 + \frac{\partial A_1}{\partial x_2}\dot{x}_2 + \frac{\partial A_1}{\partial x_3}\dot{x}_3\right). \quad (4.14)$$

The results of (4.13, 4.14) together yields

$$\frac{d}{dt}(m\dot{x}_1) = q\frac{\partial V}{\partial x_1} + \frac{q}{c}0. \quad (4.15)$$

where

$$0 = \frac{\partial A_1}{\partial x_1}\dot{x}_1 + \frac{\partial A_2}{\partial x_1}\dot{x}_2 + \frac{\partial A_3}{\partial x_1}\dot{x}_3 - \frac{\partial A_1}{\partial x_1}\dot{x}_1 - \frac{\partial A_1}{\partial x_2}\dot{x}_2 - \frac{\partial A_1}{\partial x_3}\dot{x}_3$$

$$= \dot{x}_2 \left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right) - \dot{x}_3 \left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right). \qquad (4.16)$$

Which are identical to the term (4.11) in the Newtonian equation of motion. Comparing the equations (4.15), (4.16) with equation (4.10) shows that the Newtonian equations of motion and Euler- Lagrange condition are, in fact, equivalent.

4.5 Symmetry Properties in Lagrangian Mechanics

Symmetry properties play an eminent role in Quantum Mechanics since they reflect the properties of the elementary constituents of physical systems, and since these properties allow we often simplify mathematical descriptions.

We will consider in following two symmetries, gauge symmetry and symmetries with respect to spatial transformations.

The gauge symmetry, encountered above in connection with the transformations (4.8) of electromagnetic potentials, appear in different, surprisingly simple fashion in Lagrangian Mechanics. They are subject of the following theorem.

Theorem 4.2 (Gauge Transformation of Lagrangian)

The equation of motion (Euler- Lagrange conditions) of classical mechanical systems are unaffected by the following transformation of its Lagrangian

$$L'(\vec{q}, \dot{\vec{q}}, t) = L(\vec{q}, \dot{\vec{q}}, t) + \frac{d}{dt}\frac{q}{c}K(\vec{q}, t), \qquad (4.17)$$

This transformation is termed gauge transformation. The factor $\frac{q}{c}$ has been introduced to make this transformation equivalent to gauge transformation (4.8) of electromagnetic potentials. Note that we adds the total time derivative of a function $K(\vec{r}, t)$ to the Lagrangian. This terms is

$$\frac{d}{dt}K(\vec{r}, t) = \frac{\partial K}{\partial x_1}\dot{x}_1 + \frac{\partial K}{\partial x_2}\dot{x}_2 + \frac{\partial K}{\partial x_3}\dot{x}_3 + \frac{\partial K}{\partial t} = (\nabla K).\vec{v} + \frac{\partial K}{\partial t}. \quad (4.18)$$

Proof: to prove this theorem we determine the action integral corresponding to the transformed $Lagrangian$

$$S'[\vec{q}(t)] = \int_{t_0}^{t_1} dtL'(\vec{q},\dot{\vec{q}},t) = \int_{t_0}^{t_1} dtL(\vec{q},\dot{\vec{q}},t) + \frac{q}{c}K(\vec{q},t)|_{t_0}^{t_1}$$

$$= S[\vec{q}(t)] + \frac{q}{c}K(\vec{q},t)|_{t_0}^{t_1}. \qquad (4.19)$$

Since the condition$\delta\vec{q}(t_1) = \delta\vec{q}(t_0) = 0$ holds for the variational function of Lagrangian Mechanics, eq. (4.19) implies that the gauge transformation amounts to adding a constant term to the action integral, i.e., a term not affected by the variational allowed. We can conclude then immediately that any extermal of $S'[\vec{q}(t)]$ is also an extremal of $S[\vec{q}(t)]$. We want to demonstrate now that the transformation (4.17) is, in fact, equivalent to the gauge transformation (4.8) of electromagnetic potentials. For this purpose we can consider the transformation of the single particle $Lagrangian$ (4.9)

$$L'\left(\vec{r},\dot{\vec{r}},t\right) = \frac{1}{2}m\vec{v}^2 - qV(\vec{r},t) + \frac{q}{c}\vec{A}(\vec{r},t).\vec{v} + \frac{q}{c}\frac{d}{t}K(\vec{r},t). \qquad (4.20)$$

Inserting (4.18) into (4.20) and reordering terms yields using (4.8)

$$L'\left(\vec{r},\dot{\vec{r}},t\right) = \frac{1}{2}m\vec{v}^2 - q\left(V(\vec{r},t) - \frac{1}{c}\frac{\partial K}{\partial t}\right) + \frac{q}{c}\left(\vec{A}(\vec{r},t) + \nabla K\right).\vec{v}$$
$$= \frac{1}{2}m\vec{v}^2 - qV'(\vec{r},t) + \frac{q}{c}\vec{A}'(\vec{r},t).\vec{v}. \qquad (4.21)$$

Obviously, the transformation (4.17) corresponds to replacing in the Lagrangian potentials $V(\vec{r},t),\vec{A}(\vec{r},t)$ by gauge transformed potentials $V'(\vec{r},t),\vec{A}'(\vec{r},t)$. We have proven, therefore, the equivalent of (4.17) and (4.8).

We consider now invariance properties connected with coordinate transformations. Such invariance properties are very familiar, for example, in the case of central force fields which are invariant with respect to rotations of coordinates around the center.

The following description of the spatial symmetry is important in two respects, for connection between invariance properties and constant s of the motion, which has an important analogy in Quantum Mechanics, and for the introduction of infinitesimal transformations which will provide a crucial method for the study of symmetry in Quantum Mechanics. The trans-formations we consider are the most simple kind, the reason being that our interest lies in achieving familiarity with the principles of symmetry properties rather than in providing a general tool in the context of the Classical Mechanics. The transformations considered are specified in the following.

4.6 Infinitesimal One-Parameter Coordinate Transformations

A one-parameter coordinate transformation is described through

$$\vec{r}' = \vec{r}'(\vec{r},\epsilon), \vec{r}, \vec{r}' \in R^{\ddagger}, \epsilon \in R. \qquad (4.22)$$

where the origin of ϵ is chosen such that

$$\vec{r}'(\vec{r},0) = \vec{r}. \qquad (4.23)$$

The corresponding infinitesimal transformation is defined for small ϵ through

$$\vec{r}'(\vec{r},\epsilon) = \vec{r} + \epsilon\vec{R}(\vec{r}) + O(\epsilon);\ \vec{R}(\vec{r}) = \frac{\partial \vec{r}'}{\partial \epsilon}|_{\epsilon=0}\,. \quad (4.24)$$

In the following we will denote unit vectors as $\hat{a}$, i.e., for such vectors holds $\hat{a}.\hat{a} = 1$.

Theorem 4.3 ($\mathbf{Noether's\ Theroem}$)

If the Lagrangian $L(\vec{q},\dot{\vec{q}},t)$ is invariant with respect to an infinitesimal transformation

$\vec{q}'_j = \vec{q}_j + \epsilon\vec{Q}_j(\vec{q})$, then $\sum_{j=1}^{M} Q_j \frac{\partial L}{\partial \dot{x}_j}$ is a constant of motion.

We have generalized in this theorem the definition of infinitesimal coordinate transformation to M-dimensional vectors $\vec{q}$.

Proof: In order to prove $Noethr's\ theorem$ we note

$$\left.\begin{aligned} q'_j &= q_j + \epsilon Q_j(\vec{q}) \\ \dot{q}'_j &= \dot{q}_j + \epsilon \sum_{k=1}^{M} \frac{\partial Q_j}{\partial q_k} \dot{q}_k \end{aligned}\right\}. \qquad (4.25)$$

Inserting these infinitesimal changes of q_j and $\dot{q}_j$ into the Lagrangian $L(\vec{q}, \dot{\vec{q}}, t)$ yields after Taylor expansion, neglecting terms of order $O(\epsilon^2)$,

$$L'(\vec{q}, \dot{\vec{q}}, t) = L(\vec{q}, \dot{\vec{q}}, t) + \epsilon \sum_{j=1}^{M} \frac{\partial L}{\partial q_j} Q_j + \epsilon \sum_{j,k=1}^{M} \frac{\partial L}{\partial \dot{q}_j} \frac{\partial Q_j}{\partial q_k} \dot{q}_k, \quad (4.26)$$

Where we used $\frac{d}{dt} Q_j = \sum_{k=1}^{M} \left(\frac{\partial}{\partial q_k} Q_j\right) \dot{q}_k$. Invariance implies $L' = L$, i.e., the second and third term in (4.26) must cancel each other or both vanish. Using the fact, that along the classical path holds the Euler- Lagrange condition $\frac{\partial L}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)$ we can rewrite the sum of the second and third term in (4.26)

$$\sum_{j=1}^{M} \left(Q_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \frac{\partial L}{\partial \dot{q}_j} \frac{d}{dt} Q_j \right) = \frac{d}{dt}\left(\sum_{j=1}^{M} Q_j \frac{\partial L}{\partial \dot{q}_j} \right) = 0, (4.27)$$

From this follows the statement of the theorem.

4.7 Symmetries of Variational Problems

4.7.1 The calculus of Variations

The starting point will be a discussion of some of foundational result, in the calculus of variations. As usual, we work over an open subset of the total space $E = X \times U \cong R^p \times R^q$ coordinatized by independent variables $x = (x^1, x^2, \ldots, x^p)$ and dependent variables

$u = (u^1, u^2, \dots, u^q)$. The associated n^{th} jet space $j^n$is coordinatized by the derivatives $u^{(n)}$ of the dependent variables. Let $\Omega \subset X$ denote a connected open set with smooth boundary

$\partial\Omega$. By an n^{th} order variational problem, we mean the problem of finding the extremals (maxima and /or minima) of a functional

$$\mathcal{L}[u] = \mathcal{L}_\Omega[u] = \int_\Omega L(x, u^{(n)})dx. \qquad (4.28)$$

Over some space of functions $u = f(x), x \in \Omega$.The integrand $L(x, u^{(n)})$, which is smooth differential function on the jet space J^n, is referred to as the Lagrangian of the variational problem (4.28); the horizontal p-form $Ldx = Ldx^1 \wedge \dots \wedge dx^p$ is the Lagrangian form. The precise space of the functions upon which the functional (4.28) is to be extremied will depend on any boundary conditions which may be imposed –e.g., the Dirichlet conditions

$u = 0 \; on \; \partial\boldsymbol{\Omega}$- as well as smoothness requirements. More generally, although this beyond our scope, we may also impose additional constraints- holonomic, non— honomic, integral.

The most basic result in the calculus of variations is the construction of the fundamental differential equations-Euler-Lagrange equations, which must be satisfied by any smooth extremal. The Euler-Lagrange equation constitute the infinite-dimensional version of the basic theorem from calculus that the maxima and minima of smooth function $f(x)$ occur at the point where the gradient vanishes: $\nabla f = 0$. In the functional context, the gradient's role is played by the "variantional derivatives", whose components, in concrete form, are found by applying the fundamental Euler operators.

4.7.2. Euler operator

Let $1 \leq \alpha \leq q$, the differential operator $E = (E_1, \dots, E_q)$, whose components are

$$E_\alpha = \sum_J (-D)_J \frac{\partial}{\partial u_J^\alpha}, \alpha = 1, \dots, q \,. \qquad (4.29)$$

is known as the Euler operator. In (4.29), the sum is over all symmetric multi-indices

$J = (j_1, \dots, j_k), 1 \leq j_v \leq p, and\ (-D)_J = (-1)^k D_J$ denote the correspond-ing signed higher order total derivative.

Theorem.4.4 The smooth extremals $u = f(x)$ of a variational problem with Lagrangian

$L(x, u^{(n)})$ must satisfy the system of Euler- Lagrange equations

$$E_\alpha(L) = \sum_J (-D)_J \frac{\partial L}{\partial u_J^\alpha} = 0, \alpha = 1, \dots, q. \qquad (4.30)$$

Note that, as with the total derivatives, even though the Euler operator (4.29) is defined using an infinite sum, for any given Lagrangian only finitely many summands are needed to compute the corresponding Euler-Lagrange expressions $E(L)$.

Proof: the proof of this theorem relies on the analysis of the variations of the extremal u.

In general, a one-parameter family of functions $u(x, \varepsilon)$ is called a family of variations of a fixed function $u(x) = u(x, 0)$ provided that, outside a compact subset, $K \subset \boldsymbol{\Omega}$, the functions coincide; $u(x, \varepsilon) = u(x)$ for $x \in K \setminus \Omega$. In particular, all the functions in the family satisfy the same boundary conditions as $\dot{u}$.Therefore, if u is, say, a minimum of the variational problem, then, for any family of variations functions $u(x, \varepsilon)$, the scalar function $h(\varepsilon) = \boldsymbol{\mathcal{L}}[\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{\varepsilon})]$,must have a minimum at $\varepsilon = 0$, and so, by elementary calculus, satisfies $h'(0) = 0$. In view of our smoothness assumptions, we can interchange the integration and differentiation to evaluate this derivative:

$$0 = \frac{d}{d\varepsilon}\boldsymbol{\mathcal{L}}[u(x,\varepsilon)]|_{\varepsilon=0} = \int_{\Omega} \left[\sum_{\alpha=1}^{q}\sum_{J}\frac{\partial L}{\partial u_J^{\alpha}}(x, u^{(n)})D_J v^{\alpha}\right] dx. \quad (4.31)$$

where $v(x) = u_{\varepsilon}(x, 0)$.The method now is to integrate (4.31) by parts. The Leibniz rule

$$PD_iQ = -QD_iP + D_i[PQ], i = 1, \dots, p. \quad (4.32)$$

For total derivatives implies, using the Divergence theorem, the general integration by parts formula

$$\int_{\Omega} PD_i\, Qdx = -\int_{\Omega} QD_iPdx + \int_{\Omega} (-1)^{i-1}\, PQdx^1 \wedge \dots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \dots \wedge dx^p, \quad (4.33)$$

which holds for any smooth function $u = f(x)$. Applying (4.33) repeatedly to integral on the right hand side of (4.31), and using the fact that v and its derivatives vanish on $\partial\boldsymbol{\Omega}$, we find

$$0 = \int_{\Omega} \left[\sum_{\alpha=1}^{q}\sum_{J}(-D)_J\left(\frac{\partial L}{\partial u_J^{\alpha}}\right)v^{\alpha}\right] dx = \int_{\Omega} \left[\sum_{\alpha=1}^{q} E_{\alpha}(L)\, v^{\alpha}\right] d$$

$$= \int_{\Omega} [E(L).v]\, dx.$$

Since the resulting integrand must vanish for every smooth function with compact support $v(x)$, the Euler –Lagrange expression $E(L)$ must vanish everywhere in$\boldsymbol{\Omega}$, completing the proof. $Q.E.D$

Let us specialize to the scalar case, when there is one independent and one dependent variable. Here, the Euler-Lagrange equation associated with an n^{th} order Lagrangian

$L(x, u^{(n)})$ is the ordinary differential equation

$$\frac{\partial L}{\partial u} - D_x\left(\frac{\partial L}{\partial u_x}\right) + D_x^2\left(\frac{\partial L}{\partial u_{xx}}\right) - \cdots + (-1)^n D_x^n\left(\frac{\partial L}{\partial u_n}\right) = 0. \quad (4.34)$$

For example, the Euler-Lagrange equation associated with classical Newtonian variational problem $\mathcal{L}[u] = \int \left[\frac{1}{2}u_x^2 - V(u)\right] dx$ (which equals kinetic energy minus potential energy) is second order differential equation $-u_{xx} - V'(u) = 0$ governing motion in a potential force field.

In general, the Euler-Lagrange equation (4.34) associated with an n^{th} order Lagrangian is an ordinary differential of order $2n$ provided the Lagrangian satisfies the classical nondegeneracy

Condition

$$\frac{\partial^2 L}{\partial u_n^2} \neq 0. \quad (4.35)$$

Isolated points at the $nondegeneracy$ condition (4.35) fails constitute singular points of the Euler-Lagrange equation. Note that, in this context, Lagranginans which are affine functions of the highest order derivative, $L(x, u^{(n)}) = A(x, u^{(n-1)})u_n + B(x, u^{(n-1)})$, are degenerate everywhere. However, straightforward integration by parts will reduce such a Lagrangian to a nondegerate one of lower order, and so the exclusion of the such Lagrangians is not essential;

as (4.33) below.

Of particular interest are the null Lagrangians, which, by definition, are Lagrangians whose Euler- Lagrange expression vanishes identically: $E(L) = 0$. The associated variational problem is completely trivial, since $\mathcal{L}[u]$ depends only on the boundary values of u, and hence every function provides an extremal.

Theroem4.5. A differential function $L(x, u^{(n)})$ defines a null Lagranian, $E(L) = 0$, if and only if it is a total divergence, so $L =$DivP$ = D_1P_1 + \cdots + D_pD_p$, for some $p - tuple$ $P = (P_1, \dots, P_p)$ of differential functions.

Proof: clearly, if $L =$DivP, the Divergence Theorem implies that the integral ((4.28) only depends on boundary values of u. therefore, the functional is unaffected by any variations, and so, $E(L) = 0$. Conversely, suppose $L(x, u^{(n)})$ is null Lagrangian. Consider the expression

$$\frac{d}{dt}L(x, tu^{(n)}) = \sum_{\alpha,J} u_J^\alpha \frac{\partial L}{\partial u_J^\alpha}(x, tu^{(n)}).$$

Each term in this formula can be integrated by parts, using (4.32) repeatedly. The net result is, as in proof of theorem 4.2,

$$\frac{d}{dt}L(x, tu^{(n)}) = \sum_{\alpha=1}^{q} u^\alpha E_\alpha(L)(x, tu^{(2n)}) + DivR(t, x, u^{(2n)}). (4.36)$$

For some well-defined $p - tuple$ of differential functions $R = (R_1, \dots, R_p)$ depending on L and its derivatives. Since E(L)=0 by assumption, we can integrate (4.36)with respect to t from t=o to t=1, producing the desired divergence identity,

$$\frac{d}{dt}L(x, u^{(n)}) = L(x, 0) + Div\hat{P} = DivP.$$

Here $\hat{P}(x, u^{(2n)}) = \int_0^1 R(t, x, u^{(2n)})dt$, and $P = P_0 + \hat{P}$, where $P_0(x)$ is any $p - tuple$ such that $divP_0 = L(x, 0)$.

***Remark* 4.6**: The proof of theorem 4..3 assumes that $L\left(x, u^{(n)}\right)$ is defined everywhere on the Jet space J^n.

Corollary 4.7 Two Lagrangians define the same Euler-Lagrange expressions if and only if they differ by a divergence: $\hat{L} = L + DivP$.

Remark 4.8 It is possible for two Lagrangian to give rise to the same Euler-Lagrange equations even though they do not differ by a divergence. For instance, both of the scalar variational problems $\int u_x^2\, dx$ *and* $\int \sqrt{1 + u_x^2}\, dx$ lead to the same Eule-Lagrange equation $u_{xx} = 0$, even though their Euler-Lagrange expressions are not identical. The characterization of such "*inequivalent Lagrangians*" is a problem of importance in the theory of intgrable systems.

Symmetries of Variational Maps that preserve variational problems serve to define variational symmetries. The precise definition is as follows.

Definition4.9 A point transformation g is called a variational symmetry of the functional (4.28) if and only if the transformed functional agrees with original one, which means that for every smooth function f defined on the domain $\mathbf{\Omega}$, with transformed counterpart $\bar{f} = g.f$ defined on $\bar{\Omega}$, we have

$$\int_{\Omega} L\left(x, f^{(n)}(x)\right) dx = \int_{\bar{\Omega}} L\left(\bar{x}, \bar{f}^{(n)}(\bar{x})\right) d\bar{x}. \qquad (4.37)$$

Thus, a transformation group G is a variational symmetry group if and only if the Lagrangian form $L(x, u^{(n)})dx$ is contact- invariant p-form, so

$$(g^{(n)})^*\left[L(\bar{x},\bar{u}^{(n)})d\bar{x}\right] = L\left(x,u^{(n)}\right)dx + \boldsymbol{\theta}, g \in G \,. \qquad (4.38)$$

for some contact form $\boldsymbol{\theta} = \boldsymbol{\theta}_{\boldsymbol{g}}$, possibly depending on the group element g. In particular, if the group transformation g is fiber-preserving, then, $\boldsymbol{\theta} = \mathbf{0}$, and the Lagrangian form is strictly invariant. In local coordinates, the contact invariance condition (4.38) takes the form

$L\left(x,u^{(n)}\right) = L\left(\bar{x},\bar{f}^{(n)}(\bar{x})\right) det$J,when

$$\left(\bar{x},\bar{u}^{(n)}\right) = g^{(n)}.\left(x,u^{(n)}\right). \qquad (4.39)$$

where $J = \left(D_i\chi^j\right)$ is the total Jacobian matrix. Since the Euler-Lagrange equations are correspondingly transformed under an equivalence map, we immediately deduce the following useful result.

Theorem 4.10 Every variational symmetry group of a variational problem is a symmetry group of the associated Euler-Lagrange equations.

Note, though, the converse to theorem 4.6 is not true.

Theorem4.11 A connected transformation group G is a variational symmetry group of the Lagrangian $L\left(x,u^{(n)}\right)$ if and only if the infinitesimal variational symmetry condition

$$v^{(n)}(L) + LDiv\boldsymbol{\xi} = \mathbf{0}, \qquad (4.40)$$

holds for every infinitesimal generator $v \in g$.

Definition 4.12 A vector field v is a divergence symmetry of a variational problem with Lagrangian L if and only if it satisfies

$$v^{(n)}(L) + LDiv\xi = DivB. \qquad (4.41)$$

For some p-tuple of functions $B = (B_1, \dots, B_p)$.

A divergence symmetry is a divergence self-equivalence of the Lagrangian form, so that (4.38) holds the modulo on exact p-form dΞ. The divergence symmetry groups form the most general class of symmetries related to conservation laws. Indeed, Noether′s theorem provides a one-to-one correspondence between generalized divergence symmetries of a variational problem and conservation laws of the associated Euler-Lagrange equations.

4.8 Invariant Variational Problems

As with differential equations, the most general variational problem admitting a given symmetry group can be readily characterized using the differential invariants of the prolonged group action. The key additional requirement is the existence of a suitable contact-invariant p-form, where p is the number of independent variables. the following theorem is a straightforward consequence of the infinitesimal variational symmetry criterion (4.40) and dates back to Lie.

Theorem 4.13 Let G be transformation group, and assume that the n^{th} prolongation of G acts regularly on (an open subset of) J^n. Assume further that there exists a nonzero contact-invariant horizontal p-form $\Omega_0 = L_0(x, u^{(n)})dx$ on J^n. A variational problem admits G as a variational symmetry group if and only if it can be written in the form $\int I\,\Omega_0 = \int I\, L_0 dx$, where I is a differential invariant of G.

In particular, any contact-invariant co-frame $\omega^1, \dots, \omega^p$ provides a contact-invariant p-form

$\Omega_0 = \omega^1 \wedge \ldots \wedge \omega^p$. Hence every G-invariant variational problem has the form

$$\mathcal{L}[u] = \int L(x, u^{(n)}) dx$$

$$= \int F\left(I_1(x, u^{(n)}), \ldots, I_k(x, u^{(n)})\right) \omega^1 \wedge \ldots \wedge \omega^p. \quad (4.42)$$

Where $I_1, \ldots, I_k$ are a complete set of functionally independent n^{th} order differential invariants.

Definition 4.14 For a system of ordinary differential equations

$\Delta(x, u^{(n)}) = 0$, a first integral is a function $P(x, u^{(m)})$ which is constant on solutions.

Theorem 4.15 If v is an infinitesimal variational symmetry with characteristic Q, then the product $QE(L) = D_x P$ is a total derivative, and thus a first integral of the Euler-Lagrange equation $(L) = 0$.

Proof: If v defines a variational symmetry, then according to (4.40)

$$0 = v^{(n)}(L) + LD_x\boldsymbol{\xi} = \boldsymbol{v}_{\boldsymbol{Q}}^{(\boldsymbol{n})}(L) + D_x(L\xi) = \sum_{i=0}^{n} \left(D_x^i Q\right) . \frac{\partial L}{\partial u_i} + D_x(L\xi)$$

Now, applying the basic integration by parts formula (4.32) repeatedly, we find

$$Q.E(L) - D_x P = Q.\left[\sum_{\boldsymbol{i}=\boldsymbol{0}}^{\boldsymbol{n}} (-\boldsymbol{D}_{\boldsymbol{x}})^{\boldsymbol{i}} \left(\frac{\partial L}{\partial u_i}\right)\right] - D_x P = 0$$

for some function P depending on Q, L, and their derivatives. Thus $D_x P$ is a multiple of the Euler-Lagrange equation, which suffices to prove the result.

4.9 Continuous Symmetries and Conservation Laws (Noether′s Theorem)

In many physical systems, the action is invariant under some continuous set of transformations.

In such systems, there exist local and global conservation laws analogous to current and charge conservation in electrodynamics. The analogs of the charges can be used to generate the symmetry transformation, from which they were derived, with the held of Poisson brackets, or after Quantization, with the held of commutators.

4.9.1 Point Mechanics

We consider a simple mechanical system with a generic action

$$\mathcal{A} = \int_{t_a}^{t_b} dt(q(t), \dot{q}(t), t). \tag{4.43}$$

4.9.2 Continuous Symmetries and Conservation Law

Suppose $\mathcal{A}$ is invariant under a continuous set of transformations of the dynamical variables:

$$q(t) \longrightarrow \dot{q}(t) = f\big(q(t), \dot{q}(t)\big). \tag{4.44}$$

Where $f\big(q(t),\dot{q}(t)\big)$ is some functional of $q(t)$. Such transformations are called symmetry transformations. Thereby it is important that the equations of motion are not used when establishing the invariance of the action under (4.44).

If the action subjected successively to two symmetry transformations, the result is again a symmetry transformation. Thus, symmetry transformations form a group called the symmetry group of the system. For an infinitesimal symmetry transformation (4.44), the difference

$$\delta_s q(t) = q'(t) - q(t). \tag{4.45}$$

will be called a symmetry variation. It has the general form

$$\delta_s q(t) = \epsilon\Delta(q(t),\dot{q}(t),t). \tag{4.46}$$

Symmetry variations must not be confused with ordinary variations $q(t)$ used to derive the Euler-Lagrange equations. While the ordinary variations $\delta q(t)$ vanish at initial and final times,

$\delta q(t_b) = \delta q(t_a) = 0$, the symmetry variations $\delta_s q(t)$ are usually nonzero at the ends.

Let us calculate the change of the action under a symmetry variation (4.46). Using the a chain rule of differentiation and an integration by parts, we obtain from (4.43)

$$\delta_s \mathcal{A} = \int_{t_a}^{t_b} dt\left[\frac{\partial L}{\partial q(t)} - \partial_t \frac{\partial L}{\partial \dot{q}(t)}\right]\delta_s q(t) + \frac{\partial L}{\partial \dot{q}(t)}\delta_s q(t)\Big|_{t_a}^{t_b}. \tag{4.47}$$

For orbit $q(t)$ that satisfy the Euler-Lagrange equations, only boundary terms survive, and we are left with

$$\delta_s \mathcal{A} = \epsilon \frac{\partial L}{\partial \dot{q}(t)} \Delta(q, \dot{q}, t)\Big|_{t_a}^{t_b}. \qquad (4.48)$$

Under the symmetry assumption, $\delta_s \mathcal{A}$ vanishes for any orbit $q(t)$, implying that the quantity

$$Q(t) \equiv \frac{\partial L}{\partial \dot{q}(t)} \Delta(q, \dot{q}, t). \qquad (4.49)$$

is the same at times $t = t_a \; and \; t = t_b$. Since t_b is arbitrary, $Q(t)$ is independent of the time t, i.e., it satisfies

$$Q(t) \equiv Q. \qquad (4.50)$$

It is conserved quantity, a constant of motion, and the expression on the right-hand side of (4.50) is called $Noether\ charge$.

The statement can be generalized to transformations $\delta_s q(t)$ for which the action is not directly invariant but its symmetry variation is equal to an arbitrary boundary term:

$$\delta_s \mathcal{A} = \epsilon \Lambda(q, \dot{q}, t)\Big|_{t_a}^{t_b}. \qquad (4.51)$$

In this case,

$$Q(t) = \frac{\partial L}{\partial \dot{q}(t)} \Delta(q, \dot{q}, t) - \Lambda(q, \dot{q}, t). \qquad (4.52)$$

is a conserved $Noether\ charge$.

$It\ is\ also\ possible\ to\ derive$ the constant of motion (4. 52) without invoking the actions, but starting from the $Lagrangian$. For it we evaluate the symmetry variation as follows:

$$\delta_s L \equiv L(q + \delta_s q, \dot{q} + \delta_s \dot{q}) - L(q, \dot{q}) = \left[\frac{\partial L}{\partial q(t)} - \partial_t \frac{\partial L}{\partial \dot{q}(t)}\right] \delta_s q(t) + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}(t)} \delta_s q(t)\right]. \qquad (4.53)$$

On account of the Euler-Lagrange equations, the first term on the right-hand side vanishes as before, and only the last term survives. The assumption of invariance of the action up to a possible surface term in eq. (4.51) is equivalent to assuming that the symmetry variation of the $Lagrangian$ in a total time derivative of some function $\Lambda(q, \dot{q}, t)$:

$$\delta_s L(q, \dot{q}, t) = \epsilon \frac{d}{dt} \Lambda(q, \dot{q}, t). \tag{4.54}$$

Inserting this into the left-hand side of (4.53), we find

$$\epsilon \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}(t)} \Delta(q, \dot{q}, t) - \Lambda(q, \dot{q}, t)\right] = 0. \tag{4.55}$$

Thus, recovering again the conserved $Noether\ charge$ (4.50).

The existence of conserved quantity for every continuous symmetry, is the content of $Noether's\ theorem$ $[1]$.

4.9.3 Alternative Derivation

Let we do a substantial variation in eq. (4.47) explicitly, and change a classical orbit $q_c(t)$, that extremizes the action, by an arbitrary variation $\delta_a q(t)$. If this does not vanish at the boundaries, the action changes by pure boundary term that follows directly from (4.47)

$$\delta_a \mathcal{A} = \frac{\partial L}{\partial \dot{q}(t)} \delta_a q(t) \Big|_{t_a}^{t_b}. \tag{4.56}$$

From this equation we can derive $Noether's\ theorem$ in yet another way. Suppose we subject a classical orbit to a new type of symmetry variation, to be called local symmetry transformations, which generalizes the previous symmetry variations (4.46) by making the parameter ϵ time-dependent:

$$\delta_s^t q(t) = \epsilon(t)\Delta(q(t), \dot{q}(t), t). \tag{4.57}$$

The superscript t of $\delta_s^t q(t)$ indicates the new time dependence in the parameter $\epsilon(t)$. These variations may be considered as a special set of the general variations $\delta_a q(t)$ introduced above. Thus also $\delta_s^t \mathcal{A}$ must be a pure boundary term of the type (4.56). For the subsequent discussion it is useful to introduce the infinitesimally transformed orbit

$$q^\epsilon(t) \equiv q(t) + \delta_s^t q(t) = q(t) + \epsilon(t)\Delta(q(t), \dot{q}(t), t), \tag{4.58}$$

and the associated $Lagrangian$:

$$L^\epsilon = L\big(q^\epsilon(t), \dot{q}^\epsilon(t)\big). \tag{4.59}$$

Using the time-dependent parameter$\epsilon(t)$, the local symmetry variation of the action can be written as

$$\delta_s^t \mathcal{A} = \int_{t_a}^{t_b} dt \left[\frac{\partial L^\epsilon}{\partial \epsilon(t)} - \frac{d}{dt}\frac{\partial L^\epsilon}{\partial \dot{\epsilon}(t)}\right]\epsilon(t) + \frac{d}{dt}\left[\frac{\partial L^\epsilon}{\partial \dot{\epsilon}(t)}\right]\epsilon(t)\,\Big|_{t_a}^{t_b}. \tag{4.60}$$

Along the classical orbit, the action is $extremal$ and satisfies the equation

$$\frac{\delta \mathcal{A}}{\delta \epsilon(t)} = 0, \tag{4.61}$$

which translates for a local action to an Euler-Lagrange type of equation:

$$\frac{\partial L^{\epsilon}}{\partial \epsilon(t)} - \frac{d}{dt}\frac{\partial L^{\epsilon}}{\partial \dot{\epsilon}(t)} = 0. \tag{4.62}$$

This can also be checked explicitly by differentiating (4.59) according to the chain rule of differentiation:

$$\frac{\partial L^{\epsilon}}{\partial \epsilon(t)} = \frac{\partial L^{\epsilon}}{\partial q(t)}\Delta(q(t),\dot{q}(t),t) + \frac{\partial L^{\epsilon}}{\partial \dot{q}(t)}\dot{\Delta}(q(t),\dot{q}(t),t). \tag{4.63}$$

$$\frac{\partial L^{\epsilon}}{\partial \dot{\epsilon}(t)} = \frac{\partial L^{\epsilon}}{\partial \dot{q}(t)}\Delta(q(t),\dot{q}(t),t). \tag{4.64}$$

and inserting on the right-hand side the ordinary Euler-Lagrange equations $\frac{\partial L^{\epsilon}}{\partial \epsilon(t)} - \frac{d}{dt}\frac{\partial L^{\epsilon}}{\partial \dot{\epsilon}(t)} = 0$.

We now invoke the symmetry assumption that the action is a pure surface term under the time-independent transformation (4.57). This implies that

$$\frac{\partial L^{\epsilon}}{\partial \epsilon(t)} = \frac{d}{dt}\Lambda. \tag{4.65}$$

Combining this with (4.62), we derive a conservation law for the charge:

$$Q = \frac{\partial L^{\epsilon}}{\partial \dot{\epsilon}(t)} - \Lambda \tag{4.66}$$

Inserting here Eq. (4.64), we find that this is the same charge as that derived by the previous method.

4.10 Displacement and Energy Conservation

Consider the case that the $Lagrangian$ does not depend explicitly on time, i.e., that

$L(q,\dot{q},t) = L(q,\dot{q})$. Let us perform a time translation on the coordinate frame:

$$t' = t - \epsilon. \qquad (4.67)$$

In the new coordinate frame, the same orbit has the new description

$$\dot{q}(t') = q(t). \qquad (4.68)$$

i.e., the orbit $\dot{q}(t)$ at the translated time t' is precisely the same as the orbit $q(t)$ at the original time t. If we replace the argument of $\dot{q}(t)$ in (4. 68) by t', we describe a time-translated orbit in terms of the original coordinates. This implies the symmetry variation of the form (4.46)

$$\delta_s q(t) = q'(t) - q(t) = q(t'+\epsilon) - q(t) = q(t') + \epsilon\dot{q}(t') - q(t)$$
$$= \epsilon\dot{q}(t). \qquad (4.69)$$

The symmetry variation of the $Lagrangian$ is in general

$$\delta_s L = L\big(q'(t),\dot{q}'(t)\big) - L\big(q(t),\dot{q}(t)\big) = \frac{\partial L}{\partial q}\delta_s q(t) + \frac{\partial L}{\partial \dot{q}}\delta_s \dot{q}(t). \quad (4.70)$$

Inserting $\delta_s q(t)$ from (4 69) we find, without using the Euler-Lagrange equation,

$$\delta_s L = \epsilon\left(\frac{\partial L}{\partial \dot{q}}\dot{q}(t) + \frac{\partial L}{\partial \dot{q}}\ddot{q}(t)\right) =$$
$$\epsilon\frac{d}{dt}L. \qquad (4.71)$$

This has precisely the form of Eq. (4.54) with $\Lambda = L$ as expected, since time-translations are symmetry translations. Here the function Λ in(4.54) happens to coincide with the Lagrangian.

According to Eq. (4.52) we find the $Noether\ charge$

$$Q = \frac{\partial L}{\partial \dot{q}}\dot{q} - L(q,\dot{q}). \tag{4.72}$$

to be a constant of motion. This recognized as the Legendre transform of the Lagrangian which is, of course, the Hamiltonian of the systems.

Let us briefly check how this $Noether\ charge$ is obtained from the alternative formula (4.52). The time-dependent symmetry variation is here

$$\delta_s^t q(t) = \epsilon(t)\dot{q}(t). \tag{4.73}$$

under which the $Lagrangian$ is change by

$$\delta_s^t L = \frac{\partial L}{\partial q}\epsilon\dot{q} + \frac{\partial L}{\partial \dot{q}}(\dot{\epsilon}\dot{q} + \epsilon\ddot{q})$$
$$= \frac{\partial L}{\partial \dot{\epsilon}}\epsilon + \frac{\partial L^{\epsilon}}{\partial \dot{\epsilon}}\dot{\epsilon}. \tag{4.74}$$

with

$$\frac{\partial L}{\partial \dot{\epsilon}} = \frac{\partial L}{\partial \dot{q}}\dot{q}. \tag{4.75}$$

and

$$\frac{\partial L}{\partial \epsilon} = \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\epsilon\ddot{q} = \frac{d}{dt}L. \tag{4.76}$$

This shows that time translations fulfill the symmetry condition (4.65), and that the $Noether$ charge (4.66) coincides with the Hamiltonian found in Eq. (4.52).

4.11 Momentum and Angular Momentum

While the conservation law of energy follows from the symmetry of action under time translations, conservation laws of momentum and angular momentum are found if the action is invariant under translations and rotations.

Consider a Lagrangian of a point particle in euclideen space

$$L = L\big(x^i(t), \dot{x}^i(t), t\big). \qquad (4.77)$$

In contrast to the previous discussion of time translation invariance, which was applicable to systems with arbitrary Lagrange coordinates $q(t)$, we denote coordinates here by x^i to emphasize that we now consider Cartesian coordinates. If the Lagrangian does depend only on the velocities $\dot{x}^i$ and not on the coordinates x^i themselves, the system is translationally invariant. If it depends, in addition, only on$\dot{x}^2 = \dot{x}^i\dot{x}^i$, it is also rotationally invariant.

The simplest example is the Lagrangian of a point particle of mass m in euclideen space:

$$L = \frac{m}{2}\dot{x}^2 \,. \qquad (4.78)$$

It exhibits both invariances, leading to conserved Noether charge of momentum and angular momentum, as we now demonstrate.

4.11.1 Translational Invariance in Space

Under a spatial translation, the coordinates x^i change to

$$x'^i = x^i + \epsilon^i. \qquad (4.79)$$

where ϵ^i are small numbers, the infinitesimal translations of a particles path are $[compare\ with\ (4.46)]$

$$\delta_s x^i(t) = \epsilon^i. \qquad (4.80)$$

Under these, the Lagrangian changes by

$$\delta_s L = L\left(x'^i(t), \dot{x}'^i(t), t\right) - L\left(x^i(t), \dot{x}^i(t), t\right) = \frac{\partial L}{\partial x^i}\delta_s x^i = \frac{\partial L}{\partial x^i}\epsilon^i$$
$$= 0. \qquad (4.81)$$

By assumption, the Lagrangian is independent of x^i, so that the right-hand side vanishes. This has to be compared with the symmetry variation of the Lagrangian around the classical orbit, calculate via the chain rule, and using the Euler-Lagrange equation:

$$\delta_s L = \left(\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}\right)\delta_s x^i + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}^i}\delta_s x^i\right] = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}^i}\right]\epsilon^i. \qquad (4.82)$$

This has the form (4.48), from which we extract a conserved Noether charge (4. 49) for each coordinate x^i:

$$p^i = \frac{\partial L}{\partial \dot{x}^i}. \qquad (4\ .83)$$

These are simply the canonical momenta of the system.

4.11.2 Rotational Invariance

Under rotations, the coordinates x^i change to

$$x'^i = R^i_j x^j. \tag{4.84}$$

where R^i_j is an orthogonal 3×3-matrix. Infinitesimally, this can be written as

$$R^i_j = \delta^i_j - \omega_k \epsilon_{kij}. \tag{4.85}$$

Where ω is an infinitesimal rotation vector, the corresponding rotation of particle paths is

$$\delta_s x^i(t) = x'^i(t) - x^i(t) = -\omega^k \epsilon_{kij} x^j(\tau). \tag{4.86}$$

It is useful to introduce the anti-symmetric infinitesimal rotation tensor

$$\omega_{ij} = \omega_k \epsilon_{kij} \, . \tag{4.87}$$

In terms of which

$$\delta_s x^i = -\omega_{ij} x^j. \tag{4.88}$$

Then we can write the change of the Lagrangian under $\delta_s x^i$,

$$\delta_s L = L\left(x'^i(t), \dot{x}'^i(t), t\right) - L\left(x^i(t), \dot{x}^i(t), t\right)$$
$$= \frac{\partial L}{\partial x^i}\delta_s x^i + \frac{\partial L}{\partial \dot{x}^i}\delta_s \dot{x}^i. \qquad (4.89)$$

as

$$\delta_s L = -\left(\frac{\partial L}{\partial x^i}\delta_s x^j + \frac{\partial L}{\partial \dot{x}^i}\delta_s \dot{x}^j\right)\omega_{ij} = 0. \qquad (4.90)$$

If the Lagrangian depends only on the rotational variations $x^2, \dot{x}^2, x.\dot{x}$, and on powers thereof, the right-hand side vanishes on account of the anti-symmetric of ω_{ij}. This ensures the rotational symmetry.

Now we calculate once more the symmetry variation of Lagrangian via the chain rule and find using the Euler-Lagrange equations,

$$\delta_s L = \left(\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}\right)\delta_s x^i + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}^i}\delta_s x^i\right] = -\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}^i}x^j\right]\omega_{ij}$$

$$= \frac{1}{2}\frac{d}{dt}\left[x_i \frac{\partial \mathbf{L}}{\partial \dot{\mathbf{x}}^{\mathbf{j}}} - (i \leftrightarrow j)\right]\omega_{ij}. \qquad (4.91)$$

The right-hand side yields the conserved Noether charge of type (4. 49), one for each anti-symmetric pair i, j:

$$L^{ij} = x^i \frac{\partial L}{\partial \dot{x}^j} - x^j \frac{\partial L}{\partial \dot{x}^i} \equiv x^i p^j - x^j p^i\,. \qquad (4.92)$$

These are anti-symmetric components of angular momentum.

Had we worked with the original vector form of the rotation angle ω^k, we would have found the angular momentum in the more common form:

$$L_k = \frac{1}{2}\epsilon_{kij}L^{ij} = (x \times p)^k. \qquad (4.93)$$

The quantum-mechanical operators associated with these, after replacing $p^i \to -i\,\partial/\partial x^i$ have the well-known commutations rules

$$\left[\hat{L}_i,\ \hat{L}_j\right] = i\epsilon_{ijk}\hat{L}_k. \qquad (4.94)$$

In the tensor notation (4.92), these become

$$[\hat{L}_{ij}, \hat{L}_{kl}] = -i(\delta_{ik}\hat{L}_{jl} - \delta_{jl}\hat{L}_{jk} + \delta_{jl}\hat{L}_{ik} - \delta_{jk}\hat{L}_{il}). \qquad (4.95)$$

4.11.3 Center-of-Mass Theorem

Consider now the transformations corresponding to a uniform motion of the coordinate system. We shall study the behavior of a set of free massive point particles in euclidean space described by the Lagrangian

$$L(\dot{x}^i) = \sum_n \frac{m_n}{2}\dot{x}_n^2 . \qquad (4.96)$$

Under Galilei transformations, the spatial coordinates and the time are changed to

$$\dot{x}^i(t) = x^i(t) - v^i t, \qquad t' = t \ . \qquad (4.97)$$

where v^i is the relative velocity along the i^{th} axis. The infinitesimal symmetry variations are

$$\delta_s x^i(t) = \dot{x}^i(t) - x^i(t) = -v^i t. \qquad (4.98)$$

which change Lagrangian by

$$\delta_s L = L(x^i - v^i t, \dot{x}^i - v^i) - L(x^i, \dot{x}^i). \qquad (4.99)$$

Inserting the explicit form (4.96), we find

$$\delta_s L = \sum_n \frac{m_n}{2}\left[\left(\dot{x}_n^i - v^i\right)^2 - \left(\dot{x}_n^i\right)^2\right]. \qquad (4.100)$$

This can be written as a total time derivative:

$$\delta_s L = \frac{d}{dt}\Lambda = \frac{d}{dt}\sum_n m_n\left[-\dot{x}_n^i\, v^i + \frac{v^2}{2}t\right]. \qquad (4.101)$$

to prove that Galilei transformations are symmetry transformations in the Noether sense. By assumption, the velocities v^i in (4.97) are infinitesimal, so that the second term can be ignored.

By calculating $\delta_s L$ once more via the chain rule with help of the Euler-Lagrange equations, and by equating the result with (4.101), we find the conserved Noether charge

$$Q = \sum_n \frac{\partial L}{\partial \dot{x}^i}\delta_s x^i - \Lambda = \left(-\sum_n m_n\, \dot{x}_n^i t + \sum_n m_n\, x_n^i\right) v^i. \qquad (4.102)$$

Since the direction of the velocity v^i is arbitrary, each component is separately a constant of motion:

$$N^i = -\sum_n m_n \dot{x}^i\, t + \sum_n m_n\, x_n^i = constant. \, (4.103)$$

This is the well-known $center\ of\ mass\ theorem$[26]. Indeed, introducing the center of mass theorem coordinates

$$x_{CM}^i \equiv \frac{\sum_n m_n\, x_n^i}{\sum_n m_n}. \qquad (4.104)$$

and the associated velocities

$$v_{CM}^i \equiv \frac{\sum_n m_n\, \dot{x}_n^i}{\sum_n m_n}. \qquad (4.105)$$

the conserved charge (4.103) can be written as

$$N^i = \sum_n m_n\left(-v_{CM}^i t + x_{CM}^i\right). \qquad (4.106)$$

The time-independence of N^i implies that the center-of-mass moves with uniform velocity according to the law

$$x_{CM}^i(t) = x_{0CM}^i + v_{CM}^i t. \qquad (4.107)$$

where

$$x_{0CM}^i = \frac{N^i}{\sum_n m_n}. \qquad (4.108)$$

is the position of the center of mass at $t = 0$.

In the non relativistic physics, the center of mass theorem is a consequence of momentum conservation since momentum = $mass \times velocity$, but this is no longer true in relativistic physics.

4.11.4 Conservation Laws Resulting From Lorentz Invariance

In relativistic physics, particle orbits are described by function in space-time

$$x^\mu(\tau). \qquad (4.109)$$

where τ is an arbitrary Lorentz-invariance parameter. The action is an integral over some Lagrangian:

$$\mathcal{A} = \int d\tau L(x^\mu(\tau), \dot{x}^\mu(\tau), \tau). \qquad (4.110)$$

where $\dot{x}^\mu(\tau)$ is the derivative with respect to the parameter τ. If the Lagrangian depends only on the invariant scalar products $x^\mu x_\mu, x^\mu \dot{x}_\mu, \dot{x}^\mu \dot{x}_\mu$, then it is invariant under the Lorentz transformation

$$x^\mu \to \dot{x}^\mu = \Lambda^\mu_v x^v. \qquad (4.111)$$

where Λ^μ_v *is a* 4×4 matrix satisfying

$$\Lambda_g \Lambda^T = g. \qquad (4.112)$$

with the Minkowski metric

$$g_{\mu v} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}. \qquad (4.113)$$

For a free massive point particle in space-time, the Lagrangian is

$$L(\dot{x}(\tau)) = -Mc\sqrt{g_{\mu v}\dot{x}^\mu \dot{x}^v}\,. \qquad (4.114)$$

It is reparametrization invariant under $\tau \to f(\tau)$, with an arbitrary function $f(\tau)$. Under translations

$$\delta_s x^\mu(\tau) = x^\mu(\tau) - \epsilon^\mu(\tau). \qquad (4.115)$$

The Lagrangian is obviously invariant, satisfying $\delta_s \boldsymbol{\mathcal{L}} = \mathbf{0}$. calculating this variation once more via the chain rule with the help of the Euler-Lagrange equations, we find

$$0 = \int_{\tau_\mu}^{\tau_v} d\tau \left(\frac{\partial L}{\partial x^\mu}\delta_s x^\mu + \frac{\partial L}{\partial \dot{x}^\mu}\delta_s \dot{x}^\mu\right) = -\epsilon^\mu \int_{\tau_\mu}^{\tau_v} d\tau \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right). (116)$$

From this we obtain the Noether charges

$$p_\mu = -\frac{\partial L}{\partial \dot{x}^\mu} = Mc\frac{\dot{x}_\mu(\tau)}{\sqrt{g_{\mu v}\dot{x}^\mu \dot{x}^v}} = Mcu^\mu. \quad (4.117)$$

which satisfy the conservation law

$$\frac{d}{d\tau}p_\mu(t) = 0. \quad (4.118)$$

They are the conserved four-momenta of a free relativistic particle. The quantity

$$u^\mu \equiv \frac{\dot{x}^\mu}{\sqrt{g_{\mu v}\dot{x}^\mu \dot{x}^v}}. \quad (4.119)$$

is the dimensionless relativistic four velocity of the particle. It has the property $x^\mu x_\mu = 1$

and it is reparametrization invariant. By choosing for τ the physical time $t = x^0/c$ we can express u^μ in terms of the physical velocities $v^i = dx^i/dt$ as

$$u^\mu = \gamma(1, v^i/c), with\, \gamma \equiv \sqrt{1 - v^2/c^2}. \quad (4.120)$$

Note the minus sign in the definition (4.117) of the canonical momentum with respect to the non-relativistic case. It is necessary to write Eq.(4.117) covariantly. The derivative with respect to $\dot{x}^\mu$ transforms like a covariant vector with a subscript μ, whereas the physical momenta are p^μ.

For small Lorentz transformations near the identity we write

$$\Lambda^\mu_v = \delta^\mu_v + \omega^\mu_v\,. \quad (4.121)$$

where

$$\omega^\mu_v = g^{\mu\lambda}\omega_{\lambda v}. \quad (4.122)$$

is an arbitrary infinitesimal anti-symmetric matrix. An infinitesimal Lorentz transformation of the particle path is

$$\delta_s x^\mu(\tau) = \dot{x}^\mu(\tau) - x^\mu(\tau) = \omega^\mu_v x^v(\tau). \qquad (4.123)$$

Under it, the symmetry variation of a Lorentz- invariant Lagrangian vanishes:

$$\delta_s L = \left(\frac{\partial L}{\partial x^\mu} x^v + \frac{\partial L}{\partial \dot{x}^\mu} \dot{x}^v\right) \omega^\mu_v = 0. \qquad (4.124)$$

This has to be compared with the symmetry variation of the Lagrangian calculated via the chain rule with the help of the Euler-Lagrange equation

$$\delta_s L = \left(\frac{\partial L}{\partial x^\mu} - \frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^\mu}\right) \delta_s x^\mu + \frac{d}{d\tau}\left[\frac{\partial L}{\partial \dot{x}^\mu}\delta_s x^\mu\right] = \frac{d}{d\tau}\left[\frac{\partial L}{\partial \dot{x}^\mu}\delta_s \dot{x}^v\right]\omega^\mu_v$$

$$= \frac{1}{2}\omega^\mu_v \frac{d}{d\tau}\left(x^\mu \frac{\partial L}{\partial \dot{x}_v} - x^v \frac{\partial L}{\partial \dot{x}_\mu}\right). \qquad (4.125)$$

By equating this with (4.124), we obtain the conserved rotational Noether charges$[containing\ again\ a\ minus\ sign\ as\ in (4.117)]$

$$L^{\mu v} = -x^\mu \frac{\partial L}{\partial \dot{x}_v} + x^v \frac{\partial L}{\partial \dot{x}_\mu} = x^\mu p^v - x^v p^\mu. \qquad (4.126)$$

They are four-dimensional generalizations of the angular momenta (4.92). The quantum-mechanical operators

$$\hat{L}^{\mu v} = i(x^\mu \partial^v - x^v \partial^\mu). \qquad (4.127)$$

obtained after replacement $p^{\mu} \to i\partial/\partial x_{\mu}$ satisfy the four-dimensional space-time generalization of the commutation relations (4.95)

$$[\hat{L}^{\mu v}, \hat{L}^{k\lambda}] = i[g^{\mu k}\hat{L}^{v\lambda} - g^{\mu\lambda}\hat{L}^{vk} + g^{v\lambda}\hat{L}^{\mu k} - g^{vk}\hat{L}^{\mu\lambda}] \,. (4.128)$$

The quantities L^{ij}coincide with the earlier-introduced angular momenta (4.92). The conserved components

$$L^{0i} == x^0 p^i - x^i p^0 = M_i. \qquad (4.129)$$

yield the relativistic generalization of the center-of-mass theorem (4.103)

$$M_i = const. \qquad (4.130)$$

4.12 Generating the Symmetry Transformations

The relation between invariance's and conservation laws has second aspect. With help of Poisson brackets, the charges associated with continuous symmetry transformations can be used to generate the symmetry transformation from which they were derived. Explicitly see, [13]

$$\delta_s \hat{x} = -i\epsilon[\hat{Q}, \hat{x}(t)]. \qquad (4.131)$$

The charge (4.72) is by definition the Hamiltonian

$$Q = H.$$

whose operator version generates infinitesimal time displacements by the Heisenberg equation of motion:

$$\dot{\hat{x}}(t) = -i\big[\hat{H}, \hat{x}(t)\big]. \qquad (4.132)$$

This equation is obviously the same as (4.131)

To quantize the system canonically, we may assume the Lagrangian to have the standard form

$$L(x, \dot{x}) = \frac{M}{2}\dot{x}^2 - V(x). \qquad (4.133)$$

So that the Hamiltonian operator becomes, with the canonical momentum, $p \equiv \dot{x}$:

$$\hat{H} = \frac{\hat{p}^2}{2M} + V(\hat{x}). \qquad (4.134)$$

Equation (4.132) is then a direct consequence of the canonical equal-time commutation rules

$$[\hat{p}(t), \hat{x}(t)] = -i, \qquad [\hat{p}(t), \hat{p}(t)] = 0, \qquad [\hat{x}(t), \hat{x}(t)] = 0. \quad (4.135)$$

the charges (4.83), derived from translational symmetry, are another famous example. After quantization, the commutation rule (4.131) with (4.80) becomes

$$\epsilon^j = i\epsilon^i\big[\hat{p}^i(t), \hat{x}^j(t)\big]. \qquad (4.136)$$

This coincides with one of the canonical commutation relations (here it for time-independent momenta, since the system is translationally invariant).

The relativistic charges (4.117) of space-time generate translations via

$$\delta_s \hat{x}^\mu = \epsilon^\mu = -i\epsilon^v [\hat{p}_v, \hat{x}^\mu(\tau)]. \qquad (4.137)$$

Similarly we find that the quantized versions of the conserved charge L_i in Eq. (4.93) generate infinitesimal rotations:

$$\delta_s \hat{x}^j = -\omega^i \epsilon_{ijk}\, \hat{x}^k(t) = i\omega^i \left[\hat{L}_i, \hat{x}^j(t)\right]. \qquad (4.138)$$

whereas the quantized conserved charges N^i of Eq. (4. 103) generate infinitesimal Galilei transformations, and that the charges M_i of Eq. (4.129) generate pure rotational

Lorentz transformations:

$$\left.\begin{aligned} \delta_s \hat{x}^j &= \epsilon_i \hat{x}^0 = i\epsilon_i \left[M_i, \hat{x}^j\ \right] \\ \delta_s \hat{x}^0 &= \epsilon_i \hat{x}^i = i\epsilon_i [M_i, \hat{x}^0\ ] \end{aligned}\right\}. \qquad (4.139)$$

Since the quantized charges generate the rotational symmetry transformations, they form a representation of the generators of the symmetry group. When commuted with each other, they obey the same commutation rules as the generators of the symmetry group. The charges (4.93) associated with relations, for example, have commutation rules

$$\left[\hat{L}_i, \hat{L}_j\right] = \epsilon_{ijk}\, \hat{L}_j. \qquad (4.140)$$

which are the same as those between the 3×3 generators of the three-dimensional rotations $(L_i)_{jk} = -i\epsilon_{ijk}$.

The quantized charges of the generators (4.126) of the Lorentz group satisfy the commutation rules (4 128) of the 4×4 generators (4.127)

$$\left[\hat{L}^{\mu v}, \hat{L}^{\mu\lambda}\right] = -ig^{\mu\mu}\hat{L}^{v\lambda}. \qquad (4.141)$$

This follows directly from the canonical commutation rules (4. 137)

4.13Reductoin of Controlled Lagrangian System with Symmetry

We use the theory of controlled Lagrangian systems to include systems with symmetry and Lagrangian reduction theory.

4.13.1The Configuration Manifold and Bundle Map

The configuration manifold for the mechanical systems under consideration is denoted Q. We assume that the dimension of Q is $n\ and\ use\ (q^1, \dots, q^n)$ as coordinates on Q. The second order tangent bundle is denotes $T^{(2)}Q$ and consist of second derivatives of the curves in Q. Let G be a Lie group which acts (on the left) on Q free, and properly so that

$$\pi_G(Q): Q \to Q/G$$

becomes a principal bundle. The tangent lift action of G on TQ is free and proper and

$$\tau/G : TQ \to TQ/G$$

becomes a principal bundle. When M is a manifold on which G acts, we let $[m]_G$ denote the equivalence class of $m \in M$ in the quotient space M/G. Even though we do not explicitly specify the manifold M in this part, it will clear in the context. See, [19]

The Euler-Lagrange operator $\varepsilon\boldsymbol{\mathcal{L}}$ assigns to a Lagrangian $L: TQ \to \boldsymbol{\mathcal{R}}$, a bundle map

$\varepsilon\boldsymbol{\mathcal{L}}(\boldsymbol{L})\colon T^{(2)}Q \to T^*Q$ which be written in local coordinates (employing the summation convention) as

$$\varepsilon\boldsymbol{\mathcal{L}}(\boldsymbol{L})_i(q,\dot{q},\ddot{q})dq^i = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}(q,\dot{q}) - \frac{\partial L}{\partial q^i}(q,\dot{q})\right)dq^i. \quad (4.142)$$

In which it is understood that one regards the first term on the right hand side as a function on the second –order tangent bundle $T^{(2)}Q$ by formally applying the chain rule and then replacing everywhere $dq/dt\ by\ \dot{q}$ and $d\dot{q}/dt\ \ by\ \ddot{q}$.

4.13.2 Controlled Lagrangian System

We define a controlled $Lagrangian\,(CL)$ system is a triple (L,F,W) where the function $L\colon TQ \to \boldsymbol{\mathcal{R}}$ is the $Lagrangian$, the fiber-preserving map $F\colon TQ \to T^*Q$ is an external force and W, called the control bundle, is a Sub bundle of T^*Q, representing the actuation directions.

When we choose a specific feedback control map $u\colon TQ \to W$, we call the triple (L,F,u) a closed-loop $Lagrangian$ system. The equation of motion of the closed-loop system (L,F,u) is given by

$$\varepsilon\boldsymbol{\mathcal{L}}(\boldsymbol{L})_i(q,\dot{q},\ddot{q}) = F(q,\dot{q}) + u(q,\dot{q}). \quad (4.143)$$

$A\ CL\ system\ \ (L,F,W) is\ called\ simple\ if\ the\ Lagrangian\ L\ has\ the\ form\ of\ kineti$

$potential\ energy:$ $L(q,\dot{q}) = \frac{1}{2}m(q)(\dot{q},\dot{q}) - V(q)$. We will use the acronym SCL for "simple controlled $Lagrangian$".

4.13.3. Euler-Lagrange Matching Conditions and $\mathbf{CL-Equivalent}$

Given the two simple CL systems (L_1, F_1, W_1) and (L_2, F_2, W_2), the Euler-Lagrange matching conditions are

ELM_1: $W_1 = m_1 m_2^{-1}(W_2)$,

ELM_2: $\mathrm{Im}[(\varepsilon\boldsymbol{\mathcal{L}}(L_1) - F_1) - m_1 m_2^{-1}(\varepsilon\boldsymbol{\mathcal{L}}(L_2) - F_2] \subset W_1$,

where m_i is the mass tensor of L_i and Im means the pointwise image of the map in

bracket.

We say that the two simple $CL\ system$ (L_1, F_1, W_1) and (L_2, F_2, W_2) are $CL - equivalent\ if$

$\mathrm{ELM} - 1$ and $\mathrm{ELM} - 2$ hold.

The following theorem explains the main property of the CL-equivalence relation.

Theorem 4.16 Suppose two simple controlled $Lagrangian\ systems$ $(L_i, F_i, W_i), i = 1, 2$ are CL-equivalent. Then, for an arbitrary control law given for one system, there exist a control law for the other system such that the two closed-loop systems produce the same equations of motion. The explicit relation between the two feedback control laws $u_i, i = 1, 2$ is given by

$$u_1 = (\varepsilon\boldsymbol{\mathcal{L}}(L_1) - F_1) - m_1 m_2^{-1}(\varepsilon\boldsymbol{\mathcal{L}}(L_2) - F_2) + m_1 m_2^{-1} u_2\,. \quad (4.144)$$

$where\ m_i\ is\ the\ mass\ tensor\ of\ L_i$.

4.14 Reduction of Controlled Lagrangian Systems with Symmetry

Consider on the work on the Lagrangian reduction, we develop the reduction theory of controlled Lagrangian systems with symmetry and the reduced CL systems.

4.14.1 Reduction of CL-Systems with Symmetries

We defined the CL-system in 4.13.2. Here, we define G-invariant CL systems on TQ and reduced CL systems on TQ/G where G is a Lie group acting on Q. [29]

Definition 4.17 Let G be a Lie group acting on Q. A G-invariant controlled $Lagrangian$

(G-CL) system is a CL system, (L, F, W), where L is a G-invariant $Lagrangian$, F is G- equivariant force map and W is a G-invariant sub bundle of T^*Q.

Definition 4.18 A reduced controlled Lagrangian (RCL) system is a triple $(l, f, U)$where

$l: TQ/G \to \mathcal{R}$ is a smooth function called the $reduced\ Lagrangian$, the fiber-preserving map f: $TQ/G \to T^*Q/G$ is called the reduced force map, and U, called the reduced control bundle, is a sub bundle of T^*Q/G. A feedback control for the RCL system is a (fiber-preserving) map of $TQ/G\ \ into\ U$.

Suppose that we are given a G-CL system (L, F, W). the G-invariance of L induces a reduced $Lagrangian\ l\ on\ TQ/G\ \ satisfying$

$$l\circ\tau/G = L. \qquad (4.145)$$

The G $-$ equivariance of F induces a reduced force Map $[F]_G : TQ/G \to T^*Q/G\ \ satisfying$

$$[F]_G \circ \tau/G = \pi/G\circ F. \qquad (4.146)$$

This leads to the following definition:

Definition 4.19 The RCL system of a G-CL system (L, F, W) is a triple ($l, [F]_G, W/G$) where l is the reduced Lagrangian satisfying (4.145), and $[F]_G$ is the reduced force satisfying (4.146).

Propposition4.20 Given a RCL system (l, f, U) $on\ TQ/G$, there exists a unique G-CL system (L, F, W) on TQ whose RCL system is (l, f, U).

Proof: define L by (4.145), define a force map F on TQ as follows: for $v_q, w_q \in T_qQ$,

$$\langle F(v_q), w_q\rangle = \langle f^\circ \tau/G(v_q), \tau/G(w_q)\rangle. \qquad (4.147)$$

We can check the G–equivariance of F. We also can check that relation (4.147) defines a unique fiber-preserving map $F\ of\ TQ\ to\ T^*Q$. Let $W := \tau^{-1}/G(U)$. By construction, (L, F, W) is the unique G-CL system whose RCL system is (l, f, U).

By proposition 4.14.1 we can, without loss of generality, write an arbitrary RCL system in the form of the RCL system of a G-CL system.

Given a G-CL system,(L, F, W), the G-invariant of L implies the G-equivalence of the map

$\varepsilon\boldsymbol{\mathcal{L}}(L)$: $T^{(2)}Q \to T^*Q$, which induces a quotient map

$$\mathcal{R}\varepsilon\boldsymbol{\mathcal{L}}(l) := [\varepsilon\boldsymbol{\mathcal{L}}(L)]_G : T^{(2)}Q \to T^*Q/G,$$

which depends only on the reduced $Lagrangian\ l\ on\ TQ/G$ induced from L .The operator $\mathcal{R}\varepsilon\boldsymbol{\mathcal{L}}$ is called the $reduced\ Euler - Lagrange\ operator$. The equation of motion of a RCL

$(l, [F]_G, W/G)$ with a choice of control $[u]_G : TQ/G \to W/G$ is given by

$$\mathcal{R}\varepsilon\boldsymbol{\mathcal{L}}(l)([q, \dot{q}, \ddot{q}]_G) = [F]_G([q, \dot{q}]_G) + [u]_G([q, \dot{q}]_G).$$

To write computable equations of $\mathcal{R}\varepsilon\boldsymbol{\mathcal{L}}$, we have to choose a principle connection on the principle bundle $Q \to Q/G$ to make the following identifications:

$$TQ/G = T(Q/G) \oplus \tilde{g}, T^{(2)}Q/G \to T^{(2)}(Q/G) \times_{Q/G} 2\tilde{g},$$

$$T^*Q/G = T^*(Q/G) \oplus \tilde{g}^*$$

where $\tilde{g}$ is the adjoint bundle Ad(Q), $\tilde{g}^*$ is the coadjiont $b$undle Ad*(Q), $2\tilde{g} := \tilde{g} \oplus \tilde{g}$, and $\oplus$ is the Whitney sum (see [31]. With these identifications, $\mathcal{R}\varepsilon\boldsymbol{\mathcal{L}}(l)$ induces the Lagrange- Poincare operator

$$\mathcal{L}P(l) : T^{(2)}(Q/G) \times_{Q/G} 2\tilde{g} \to T^*(Q/G) \oplus \tilde{g}^*. \qquad (4.148)$$

Hence, the reduced Euler-Lagrange operator, $\mathcal{R}\varepsilon\boldsymbol{\mathcal{L}}$ may be replaced by the Lagrange-Poincare' operator $\mathcal{L}P$ in the following as long as we chooses a connection on $Q \to Q/G$ $[30, 31]$.

We study the relation between trajectories of G-CL systems and trajectories of RCL systems. Let (L, F, W) be a G-CL system and $(l, [F]_G, W/G)$ be its RCL system. Choose an arbitrary G-equivalent feedback control law $u : TQ \to W\ for\ (L, F, W)$. The control u induces a reduced map $[u]_G : TQ/G \to T^* Q/G$. If $(q(t), \dot{q}(t)) \in TQ$ is a trajectory of the closed-loop system (L, F, u), then $\tau/G(q(t), \dot{q}(t)) \in TQ/G$ is the trajectory of the closed-loop system

$(\ l, [F]_G, [u]_G)$.

4.15 Reduced CL-Equivalence

Definition4.21 We define the reduced simple controlled $Lagrangian$ $(RSCL)$ system is the reduced CL system $(\ l, [F]_G, W/G)$ of a G-invariant simple CL system (L, F, W). If the G-invariant simple $Lagrangian\ L\ is\ given\ by$ $L(q, \dot{q}) = \frac{1}{2} m_q(\dot{q}, \dot{q}) - V(q)$, then its reduced $Lagrangian\ L$ is denoted by

$$l([q, q]_G) = \frac{1}{2}[m]_G([q, \dot{q}]_G, [q, \dot{q}]_G) - [V]_G([q]_G)$$

Where $[m]_G \in \Gamma(Q/G,\ T^*Q/G \oplus T^*Q/G)$ are the reduced mass tensor induced from the G-invariance of the mass tensor $m \in \Gamma(T^*Q/G \oplus T^*Q/G)$ and $[V]_G \to R$ is the reduced potential energy, see [30], [31], [32].

We now define an equivalence relation among RCL systems on TQ/G.

Definition4.22 two RSCL systems ($l_i, [F_i]_G, W_i/G$), $i = 1, 2\ are\ said\ to\ be$ reduced-CL-equivalent (RCL-equivalent) if the following reduced Euler-Lagrange matching conditions hold:

$\mathrm{RELM}-1:\ W_1/G =\ [m_1]_G[m_2]_G^{-1}$,

$$\mathrm{RELM}-2: \mathrm{Im}[R\varepsilon\mathcal{L}(l_1) - [F_1]_G - [m_1]_G[m_2]_G^{-1}(R\varepsilon\mathcal{L}(l_2) - [F_2]_G] \in W_1/G$$

where $[m_i]_G$ *is reduced mass tensor of* l_i, $i = 1,2$.

The following proposition explains the relationship between the CL-equivalence relation between G-SCL's and the RCL-equivalence relation among RSCL's.

Proposition 4.23 Two G-SCL systems are CL-equivalent if and only if their associated RSCL systems are RCL-equivalent.

Proof: Let (L, F, W) be a G-SCL system, and $(l, [F]_G, W/G)$ be its associated RSCL system. Then the proposition follows from the G-invariance of W and the following relations:

$$\boldsymbol{\mathcal{R}\mathcal{E}\mathcal{L}(l)} \circ \tau^{(2)}/\boldsymbol{G} = \boldsymbol{\pi}/\boldsymbol{G} \circ \boldsymbol{\mathcal{E}\mathcal{L}}(L), [F]_G \circ \tau/G = \boldsymbol{\pi}/\boldsymbol{G} \circ F.$$

where $\tau^{(2)}/\boldsymbol{G} : \boldsymbol{T}^{(2)}\boldsymbol{Q} \to \boldsymbol{T}^{(2)}\boldsymbol{Q}/\boldsymbol{G}$ is G quotient map. ■

Hence, one can check the RCL-equivalence of two RSCL's in two way:

1: is to directly check it.

2: is to check CL-equivalence of their associated unreduced G-SCL's.

Theorem.24. Suppose that two RSCL systems $(l_i, [F_i]_G, W_i/G), i = 1,2$ are RCL-equivalent.

Then, for an arbitrary control law for one system, there exist a control law for the other system such that the two closed-loop RSCL systems produce the same equations of motion. The explicit relation between the two feedback control laws $[u_i]_G, i = 1,2$ is given by

$$[u_1]_G = \mathcal{REL}(\boldsymbol{l_1}) - [F_1]_G - [m_1]_G[m_2]_G^{-1}(\mathcal{REL}(\boldsymbol{l_2}) - [F_2]_G) +$$

$$[m_1]_G[m_2]_G^{-1}[u_2]_G.$$

(4.149)

$where\ m_i\ is\ mass\ tensor$ of L_i , $i = 1, 2$.

Proof: Let $[u_i]_G$ be a feedback control for ($l_i, [F_i]_G, W_i/G$), $i = 1, 2$. Let (L_i, F_i, W_i) be the unreduced G-SCL system of ($l_i, [F_i]_G, W_i/G$), $i = 1, 2$. By proposition 4.23 the two G-SCL are CL- equivalent. By 4.10, the two closed-loop G-SCL systems (L_i, F_i, W_i), $i = 1, 2$ produce

($l_i, [F_i]_G, W_i/G$), $i = 1, 2$the same equations of motion when $u_1\ and\ u_2$ satisfy (4.144). Hence, the two closed-loop RSCL systems ($l_i, [F_i]_G, W_i/G$), $i = 1, 2$ produce the same equations of motion when $[u_1]_G\ and\ [u_2]_G$ satisfy (4.149) because each term in (4.144) is G-equivariant. In addition, notice that for any choice of $[u_i]_G$, we can choose the other $[u_j]_G$such that (4.149) holds.

4.16 Euler-Poincare' Matching

Here we briefly sketch the proof that the set of Euler-Poincare' matching conditions in [30] is a special case of the reduced Euler-Lagrange matching conditions. This set of matching conditions can handle such examples as a spacecraft with a rotor and underwater vehicles with internal rotors.

Let $Q = G \times H$ be the configuration space where G is a Lie group acting trivially on H, and H is an Abelian Lie group. We choose the trivial connection on $Q \to H$ to write down the Lagrange-Poincare' equation on $TQ/G \simeq \mathrm{g} \times TH$ with the Lie algebra g of the Lie group G. we use $\eta = (\eta^\alpha)$ as coordinates on g and $(\theta\ , \dot{\theta}) = (\theta^a\ , \dot{\theta}^a)$as coordinates on *TH.* The Lagrange-Poincare' operator $\mathcal{LP}$ with respect to trivial connection is given by

$$\mathcal{LP}(l) = \begin{pmatrix} \frac{d}{dt}\frac{\partial l}{\partial \eta^\alpha} - c^\beta_{\alpha\gamma}\eta^\gamma \frac{\partial l}{\partial \eta^\beta} \\ \frac{d}{dt}\frac{\partial l}{\partial \dot{\theta}^\alpha} - \frac{\partial l}{\partial \theta^\alpha} \end{pmatrix}. \qquad (4.150)$$

for any reduced $Lagrangian\ l = l(\eta^\alpha, \dot{\theta}^\alpha,\ \theta^\alpha)$, where $c^\beta_{\alpha\gamma}$ are the structure coefficients of the Lie algebra g. See [31] for derivation of (4.150).

Let $(l, 0, T^*H)$ be the given RCL system with the reduced Lagrangian,

$$l(\eta^\alpha, \dot{\theta}^\alpha) = \frac{1}{2} g_{\alpha\beta}\eta^\alpha\eta^\beta + g_{\alpha a}\eta^\alpha\dot{\theta}^a + \frac{1}{2} g_{ab}\dot{\theta}^a\dot{\theta}^b,$$

where $g_{\alpha\beta}, g_{\alpha a}, g_{ab}$ are constant function on TQ/G. Notice that this Lagrangian cyclic in the Abelian variables θ^α and the controls act only on the cyclic variables. Let $(l_{\tau,\sigma,\rho}, 0, T^*H)$ be an another RCL system with the reduced Lagrangian of the following form:

$$l_{\tau,\sigma,\rho} = l(\eta^\alpha, \dot{\theta}^a + \tau^a_\alpha\eta^\alpha) + \tfrac{1}{2}\sigma_{ab}\tau^a_\alpha\tau^b_\beta\eta^\alpha\eta^\beta \qquad (4.151)$$

$$+\frac{1}{2}(\rho_{ab} - g_{ab})(\dot{\theta}^a + g^{ac}g_{c\alpha}\eta^\alpha + \tau^a_\alpha\eta^\alpha)(\dot{\theta}^b + g^{bc}g_{c\beta}\eta^\beta + \tau^b_\beta\eta^\beta),$$

which exactly the equation (11) in [30].

Now we have the following Euler-Poincare' matching conditions:

$\mathrm{EP}-1:\ \tau^a_\alpha = -\sigma^{ab}g_{b\alpha},$

$\mathrm{EP}-2: \sigma^{ab} + \rho^{ab} = g^{ab}.$

Then, we can show that the two assumptions of $\mathrm{EP}-1$ and $\mathrm{EP}-2$ imply the RCL-equivalence of the two reduced CL systems $(l, 0, T^*H)\ and\ (l_{\tau,\sigma,\rho}, 0, T^*H)$. by theorem4.19.1, we can equivalently work with the second system see also [30, 31].

Chapter Five

Symmetries of Hamiltonian systems

5.1 Introduction

In this part we discuss the relation between one parameter continuous symmetries of dynamics, defined on physical grounds, and conservation laws. In the Hamiltonian formulation, such symmetries of the dynamics in general leave the Hamiltonian invariant only up to a total derivative, $dG(q)/dt$. And we study the infinitesimal symmetries, Newtonoid vector fields, infinitesimal Noether symmetries and conservation laws of Hamiltonian systems. Finally we classify the symmetry groups of an autonomous Hamiltonian system with degrees of freedom. With the exception of the harmonic oscillator or a free particle where the dimension is 15, we obtain all dimensions between 1 and 7. For each system in the classification we examine integrability.

5.2 Infinitesimal Transformation of the Hamiltonian

A one-parameter continuous group of (possibly time dependent) transformations of the dynamical variables which leave their time evolution invariant and therefore leave the Lagrangian invariant up to total derivative, $dG(q)/dt$, induces the following infinitesimal transformations of Hamiltonian

$$\delta H = H(q',p',t) - H(q,p,t) = \varepsilon\frac{\partial F}{\partial t} + \varepsilon\frac{dG(q)}{dt}. \,(5.1)$$

where F is the (possibly time dependent) generator of the corresponding canonical transformation $q,p \to q',p'$.

In this more general case, the Hamiltonian formulation of Noether theorem gives the following conservation law

$$\frac{dQ}{dt} \equiv \frac{d}{dt}(F+G) = 0. \qquad (5.2)$$

Hence, in order to get a conserved quantity we have to add the function G to the generator F of the canonical transformations.

Since the addition of a total derivative to the Lagrangian does not change the dynamics of the Lagrangian variables, q, $\dot{q}$, it leaves invariant all the observables $F(q, \dot{q})$ and has therefore the meaning of a gauge transformation(this point of view is, with different Hamiltonian version of Noether theorem see [28]).

In terms of the canonical variables the addition of the total derivative implies the following transformation of the canonical variables

$$q_i \to q_i, p_i \to p_i - \frac{\partial G}{\partial q_i}. \qquad (5.3)$$

which changes the relation between the conjugate momentum p_i and the time derivative $\dot{q}_i$ of position equation (5.3) states that G is the canonical generator of such gauge transformation. It is worthwhile to recall that for a particle in a magnetic field x and $\dot{x}$ are observable (gauge invariant) quantities, but $p_i \equiv \dot{x}_i + (e/c)A_i$ is not.

In conclusion for one- parameter continuous groups of transformations, which leave the dynamics invariant, but leave the Lagrangian or the Hamiltonian invariant only up to a total derivative, the conservation laws displays a sort of anomaly, the conserved quantity being the sum f the generator of the corresponding canonical transformation plus the generator G of the gauge transformation(5.3).

5.3 Symmetries of the Dynamics and Transformation of the Hamiltonian

The symmetry of the dynamics we mean a transformation of the dynamical variables such that their equations of motion are invariant. In the Lagrangian formulation, the dynamical variables are the Lagrangian coordinates q_i and their time derivatives q_i and a transformation,[23]

$$q_i \rightarrow {q'}_i(q,t), \qquad \dot{q}_i \rightarrow \dot{q}'_i(q,\dot{q},t). \quad (5.4)$$

is a symmetry of the dynamics if it leaves the equations of motion $\ddot{q}_i = F_i(q,\dot{q},t)$ invariant, i.e.

$$\ddot{q}'_i = F_i(q',\dot{q}',t). \qquad (5.5)$$

Then, we have a complete characterization of the symmetries of the dynamics in terms of invariance properties of the Lagrangian [39].

Proposition 5.1 The invariance of the Lagrange equations under a (possible time dependent) transformation of the Lagrangian variables $q_i \rightarrow {q'}_i$, $\dot{q}_i \rightarrow \dot{q}'_i$ is equivalent to the invariance of the Lagrangian up to a total derivative

$$L'(q',\dot{q}',t) = L(q',\dot{q}',t) - \frac{dG(q)}{dt}. \qquad (5.6)$$

Since the Lagrangian transforms covariantly under a change of the Lagrangian coordinates, namely

$$L'(q',\dot{q}',t) = L(q,\dot{q},t). \qquad (5.7)$$

Eq. (5.3) (and therefore the symmetry of the dynamics) is equivalent to

$$L(q', \dot{q}', t) = L(q, \dot{q}, t) + \frac{dG(q)}{dt}. \qquad (5.8)$$

The next step is to characterize the transformation properties of the Hamiltonian under a transformation of the Lagrangian variables which leave the Lagrangian invariant up to a total derivative.

Proposition 5.2. A transformation of Lagrangian Variables

$$q_i \to q'_i(q, t), \; \dot{q}_i \to \dot{q}'_i(q, \dot{q}, t). \qquad (5.9)$$

Such the Lagrangian is invariant up a total derivative, eq. (5.6) defines a canonical transformation of the canonical variables

$$q_i \to q'_i, \qquad p_i \to p'_i. \qquad (5.10)$$

such that

$$H'(q', p', t) = H(q', p', t) - \frac{dG}{dt}. \qquad (5.11)$$

Proof: In fact, we have (sum over repeated indices being understood)

$$H(q', p', t) = \dot{q}'_i \, p'_i - L(q', \dot{q}', t) = \dot{q}'_i \, p'_i - L(q, \dot{q}, t) - \frac{dG(q)}{dt}$$

$$= \dot{q}'_i \, p'_i - L'(q', \dot{q}', t) - \frac{dG(q)}{dt} = H'((q', p', t) + \frac{dG(q)}{dt},$$

where we have eq. (5.8) and eq, (5.7), for one-parameter continuous groups of canonical transformations, the infinitesimal variations of the canonical variables q, p are of the form

$$\delta q_i = \varepsilon\{q_i, F\} = \varepsilon \frac{\partial F(q,p,t)}{\partial p_i},$$
$$\delta p_i = \varepsilon\{p_i, F\} = -\varepsilon \frac{\partial F(q,p,t)}{\partial q_i}. \quad (5.12)$$

Where, $F(q,p,t)$ is the generator of the canonical transformation and $\{\,,\}$ denotes the poisson bracket.

Clearly, a one-parameter group of symmetries of the dynamics is non-trivial, provided $\delta q_i \neq 0$. Then, we have following theorem:

Theorem 5.3 Noether theorem (Hamiltonian form).To each one-parameter group of (non-trivial) symmetries of the dynamics, so that in the Hamiltonian transformation the Hamiltonian is invariant up to a total derivative, eq.(5.11) their corresponds the following constant of motion

$$Q = F + G\,. \quad (5.13)$$

F is a canonical generator of the symmetry transformations, eq. (5.12).

Proof: The first step is to derive the Hamiltonian analog of eq. (5.8), i.e. we must relate

$$H'((q',p',t) \text{ to } H(q,p,t).$$

Contrary to the Lagrangian case, for time dependent transformations

$H'((q',p',t) \;\neq\; H(q,p,t);$

actually one has (see [39])

$$H'((q',p',t) = H(q,p,t) + \frac{\partial \mathcal{F}}{\partial t}. \qquad (5.14)$$

where $\mathcal{F}$ is a generating function of the canonical transformation (5.12). Then, the expression of $\mathcal{F}$ to first order in ε, in eq. (5.14)

$$H'((q',p',t) = H(q,p,t) + \varepsilon \frac{\partial F}{\partial t}. \qquad (5.15)$$

And eq. (5.11) is equivalent to

$$\delta H = H'((q',p',t) - H(q,p,t) = \varepsilon \frac{\partial F}{\partial t} + \varepsilon \frac{\partial G}{\partial t}. \qquad (5.16)$$

Now, on one side, we have

$$\delta H = \varepsilon \left(\frac{\partial H}{\partial q_i} \frac{\partial F}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial F}{\partial q_i} \right) = \varepsilon \{H, F\} = -\varepsilon \left(\frac{dF}{dt} - \frac{\partial F}{\partial t} \right). \quad (5.17)$$

and, on the other side, by eq. (5.16) we have

$$\delta H = \varepsilon \frac{\partial F}{\partial t} + \varepsilon \frac{\partial G}{\partial t}.$$

Hence, it follows that

$$-\frac{dF}{dt} = \frac{dG(q)}{dt}.$$

i.e. $Q = F + G$ is a constant of motion.

The above theorem allows to derive the constant of motion associated to one- parameter group of symmetries of the dynamics, without recourse to the Lagrangian formulation; we have to check only the transformation properties of the Hamiltonian, leaving open its invariance up to a total derivative, eq. (5.11).

The conclusion is the canonical generator F of, a symmetry of the dynamics need not be a constant of motion. The point is that the invariance of the dynamics requires the invariance of the Hamiltonian only up to a total derivative $dG(q)/dt$, with G the generator of a gauge transformation, and in general, only the sum $F+G$ is constant of motion(anomaly).

On the other hand, the standard treatment of the Hamiltonian version of Noether theorem).

, identifies the symmetries as those which leave the Hamiltonian invariant.

5.4 Symmetry of Hamiltonian Systems of Spaces

A Hamiltonian space [40] is a pair (M, H) where M is a differentiable, n-dimensional manifold and H is a function on T^*M with the properties:

1. $H: (x,p) \in T^*M \rightarrow \mathrm{H(x,p)} \in \mathrm{R}$ is differentiable on T^*M and continue on the null section of the projection $\tau: T^*M \rightarrow M$.
2. The Hamiltonian of H with respect to p_i is $n$ondegenerate

$$g^{ij} = \frac{\partial^2 H}{\partial p_i \partial p_j}, \qquad rank\|g^{ij}(x,p)\| = n \;\; on \;\; T^*M \setminus \{0\}. \;\; (5.18)$$

3. The tensor field $g^{ij}(x,p)$ has constant signature on $T^*M \setminus \{0\}$.
   The triple (T^*M, ω, H) is called a Hamiltonian system.
   The Hamiltonian H on T^*M induces a pseudo-Riemannin metric g_{ij} with $g_{ij}g^{jk} = \delta_i^k \; and \; g^{jk}$ given by (5.18) on VT^*M(an integrable vertical distribution). It induces a unique adopted tangent structure denoted

$$\mathcal{J}_H = g_{ij} \otimes \frac{\partial}{\partial p_j}.$$

A $\mathcal{J}$ -regular vector field induced by the regular Hamiltonian H has the form

$$p_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial x^i} + \chi_i \frac{\partial}{\partial p_i}.$$

There exists a unique Hamiltonian vector field $p_H \in \chi(T^*M\ )$ which is a $\mathcal{J}$ -regular vector field such that $i_{p_H}\omega = -dH$, given by

$$p_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial}{\partial p_i}. \qquad (5.19)$$

The symmetric nonlinear connection

$$\mathcal{N} = -\boldsymbol{\mathcal{L}}_{p_H}\mathcal{J}_H,$$

has the coefficients [40], [41]

$$\mathcal{N}_{ij} = \frac{1}{2}\left(\{g_{ij}, H\} - \left(g_{ik}\frac{\partial^2 H}{\partial p_k \partial x^j} + g_{jk}\frac{\partial^2 H}{\partial p_k \partial x^i}\right)\right). \qquad (5.20)$$

where the Poisson bracket is

$$\{g_{ij}, H\} = \frac{\partial g_{ij}}{\partial p_k}\frac{\partial H}{\partial x^k} - \frac{\partial H}{\partial x^k}\frac{\partial g_{ij}}{\partial x^k},$$

it is called the canonical nonlinear connection of the Hamilton space (M, H), which is a metric nonlinear connection, that is $\nabla g = 0\ see[42]$. In this case, the coefficients of the Jacobi endomorphism have the form

$$\mathcal{R}_{jk} = \frac{\partial^2 H}{\partial p_i \partial x^j}\mathcal{N}_{ik} + \frac{\partial^2 H}{\partial p_i \partial x^k}\mathcal{N}_{ji} + \mathcal{N}_{ji}\mathcal{N}_{ik}g^{li} + \frac{\partial^2 H}{\partial x^j \partial x^k} + p_H(\mathcal{N}_{jk}). \quad (5.21)$$

and the action of the dynamical covariant derivative on the Berwald basis is given by

$$\nabla \frac{\delta}{\delta x^j} = h\left[p_H, \frac{\delta}{\delta x^j}\right] = -\left(\frac{\partial^2 H}{\partial p_i \partial x^j} + \mathcal{N}_{jk}g^{ki}\right)\frac{\delta}{\delta x^i}. \quad (5.22)$$

$$\nabla \frac{\partial}{\partial p_j} = v\left[p_H, \frac{\partial}{\partial p_j}\right] = \left(\frac{\partial^2 H}{\partial p_j \partial x^i} + \mathcal{N}_{ik}g^{kj}\right)\frac{\partial}{\partial p_i}. \quad (5.23)$$

In the following we study the symmetries of Hamiltonian systems [43] on the cotangent bundle using the Hamiltonian vector field and the adapted tangent structure.

Definition 5.4 A vector field $X \in \chi(T^*M)$ is an infinitesimal symmetry of Hamiltonian vector field if $[p_H, X] = 0$.

If we consider $X = X^i(x,p)\frac{\partial}{\partial x^i} + Y_i(x,p)\frac{\partial}{\partial p_i}$ then an infinitesimal symmetry is given by equations

$$X\left(\frac{\partial H}{\partial p_i}\right) = p_H(X^i), \qquad X\left(\frac{\partial H}{\partial x^i}\right) + p_H(Y^i) = 0,$$

and the first relation leads to

$$Y_k = g_{ki}\left(p_H(X^i) - X^j\frac{\partial^2 H}{\partial p_i \partial x^j}\right).$$

Definition 5.5 A vector field $\tilde{Z} \in \chi(M)$ is said to be a natural infinitesimal symmetry if its complete lift to T^*M ia an infinitesimal symmetry, that is $[p_H, \tilde{Z}^{C*}] = 0$.

We know that, for $\tilde{Z} = \tilde{Z}^i(x)\frac{\partial}{\partial x^i}$ the complete lift on T^*M is given by

$$\tilde{Z}^{C*} = \tilde{Z}^i(x)\frac{\partial}{\partial x^i} + p_i\frac{\partial \tilde{Z}^j}{\partial x^i}\frac{\partial}{\partial p_i},$$

and a natural infinitesimal symmetry is characterized by the equations

$$\tilde{Z}^{C*}\left(\frac{\partial H}{\partial p_k}\right) = \frac{\partial H}{\partial p_i}\frac{\partial \tilde{Z}^k}{\partial x^i},$$

$$\tilde{Z}^{C*}\left(\frac{\partial H}{\partial x^k}\right) = \frac{\partial H}{\partial x^i}\frac{\partial \tilde{Z}^k}{\partial x^k} - p_i\frac{\partial H}{\partial p_i}\frac{\partial^2 \tilde{Z}^j}{\partial x^k \partial x^i}.$$

Next, we introduce the Newtonoid vector field on T^*M (see [45]for tangent bundle case) which help us to find the canonical nonlinear connection induced by a regular Hamiltonian.

Definition5.6. A vector field $X \in \chi(T^*M)$ is called Newtonoid vector field if $\mathcal{J}_H[p_H, X] = 0$.

In local coordinates we obtain

$$g_{ij}\left(X\left(\frac{\partial H}{\partial p_i}\right) - p_H(X^i)\right)\frac{\partial}{\partial p_i} = 0,$$

and using that $rank\left\|g^{ij}(x,p)\right\| = n$ it result the equation

$$X\left(\frac{\partial H}{\partial p_i}\right) = p_H(X^i),$$

which leads to the expression of a Newtonoid vector field

$$X = X^i \frac{\partial}{\partial x^i} + g_{ki}\left(p_H\left(X^i\right) - X^j \frac{\partial^2 H}{\partial p_i \partial x^j}\right)\frac{\partial}{\partial p_k}.$$

We remark that X is an infinitesimal symmetry if and only if it is Newtonoid vector field and satisfies the equation

$$X\left(\frac{\partial H}{\partial p_i}\right) + p_H\left(Y^i\right) = 0.$$

The set of Newtonoid vector fields is given by

$$\mathfrak{X}_{p_H} = Ker\left(\mathcal{J}_H \circ \mathcal{L}_{p_H}\right) = Im\left(Id + \mathcal{J}_H \circ \mathcal{L}_{p_H}\right).$$

In the following, we will use the dynamical covariant derivative and Jacobi endomorphism in order to find the invariant equations of Newtonoid vector field and infinitesimal symmetries. Let p_H be the Hamiltonian vector field, $\boldsymbol{\mathcal{N}}$ an arbitrary nonlinear connection with induced v, h projectors and ∇ the induced dynamical covariant derivative. We set:

Proposition 5.7 A vector field $X \in \chi(T^*M)$ is a Newtonoid vector field if and only if

$$v(X) = \mathcal{J}_H(\nabla X). \tag{5.24}$$

Proof: We have the relation

$h \circ \mathcal{L}_p \circ \boldsymbol{\mathcal{J}} = -\boldsymbol{h}$, $\boldsymbol{\mathcal{J}} \circ \mathcal{L}_p \circ \boldsymbol{v} = -\boldsymbol{v}$, implies that $\mathcal{J}_H \circ \nabla = \mathcal{J}_H \circ \mathcal{L}_{p_H} + v$ and it results $\mathcal{J}_H[p_H, X] = 0$ if and only if $v(X) = \mathcal{J}_H(\nabla X)$.

Proposition 5.8 A vector field $X \in \chi(T^*M)$ is an infinitesimal symmetry if and only if X is a Newtonoid vector field and satisfies the equation

$$\nabla(\mathcal{J}_H \nabla X) + \Phi(X) = 0. \qquad (5.25)$$

Proof: A vector field $X \in \chi(T^*M)$ is an infinitesimal symmetry if and only if $h[p_H, X] = 0$ and

$v[p_H, X] = 0$. Composing by $\mathcal{J}_H$ we obtain $\mathcal{J}_H h[p_H, X] = \mathcal{J}_H[p_H, X] = 0$ which means that X is a Newtonoid vector field. Also, $v[p_H, X] = v[p_H, vX] + v[p_H, hX] = \nabla(vX) + \Phi(X) = \nabla\big(\mathcal{J}_H(\nabla X)\big) + \Phi(X)$, which ends the proof.

For $f \in C^\infty(T^*M)$ and $X \in \chi(T^*M)$ we define the product

$$f * X = \big(Id + \mathcal{J}_H \circ \mathcal{L}_{p_H}\big)(fX) = fX + f\mathcal{J}_H[p_H, X] + p_H(f)\mathcal{J}_H H,$$

and the result that a vector field X is a Newtonoid if and only if

$$X = X^i(x,p) * \frac{\partial}{\partial p_i},$$

Also,if $X \in \mathfrak{X}_{p_H}$ then $f * X = fX + p_H(f)\mathcal{J}_H H$ (see[46]for the case of tangent bundle). Next theorem proves that the canonical nonlinear connection induced by a regular Hamiltonian can by determine by symmetries.

Theorem 5.9 Let us consider the Hamiltonian vector, field p_H an arbitrary nonlinear connection $\mathcal{N}$ and ∇ the dynamical covariant derivative. The following conditions are equivalent:

1) ∇ *restricts to* $\nabla: \mathfrak{X}_{p_H} \to \mathfrak{X}_{p_H}$ satisfies theLeibnitz rule with respect to the * product.
2) $\nabla \mathcal{J}_H = 0$.

3) $\mathcal{L}_{p_H}\mathcal{J}_H + \mathcal{N} = 0.$

4) $\mathcal{N}_{ij} = \frac{1}{2}\left(\{g_{ij}, H\} - \left(g_{ik}\frac{\partial^2 H}{\partial p_k \partial x^i} + g_{jk}\frac{\partial^2 H}{\partial p_k \partial x^i}\right)\right).$

Proof: For $(2. \Longrightarrow 1.)$ let us consider $X \in \mathfrak{X}_{p_H}$ and using (5.24) we get $vX = \mathcal{J}_H(\nabla X)$.

Applying ∇ to both sides, we obtain $\nabla(vX) = \nabla(\mathcal{J}_H \nabla X)$ which yields

$$(\nabla \mathrm{v})\mathrm{X} + \mathrm{v}(\nabla \mathrm{X}) = (\nabla \mathcal{J}_H)(\nabla X) + \mathcal{J}_H \nabla(\nabla X).$$

Using the relations $\nabla v = 0, \nabla \mathcal{J}_H = 0$ it results $\mathrm{v}(\nabla \mathrm{X}) = \mathcal{J}_H \nabla(\nabla X)$ which implies

$\nabla X \in \mathfrak{X}_{p_H}. For\ X \in \mathfrak{X}_{p_H}$ we obtain,

$$\nabla(f * X) = \nabla(fX + p_H(f)\mathcal{J}_H X)$$
$$= p_H(f)X + f\nabla X + p^2{}_H(f)\mathcal{J}_H X + p_H(f)\nabla(\mathcal{J}_H X),$$

$$\nabla f * X + f * \nabla X = p_H(f)X + f\nabla X + p^2{}_H(f)\mathcal{J}_H X + p_H(f)\mathcal{J}_H(\nabla \mathrm{X}).$$

But $\nabla(\mathcal{J}_H X) = (\nabla \mathcal{J}_H)X + \mathcal{J}_H(\nabla \mathrm{X})$ and from $\nabla \mathcal{J}_H = 0$ it results $\nabla(\mathcal{J}_H X) = \mathcal{J}_H(\nabla \mathrm{X})$ which leads to $\nabla(f * X) = \nabla f * X + f * \nabla X$.

For $(1. \Longrightarrow 2.)$ we prove that $\nabla \mathcal{J}_H$ vanishes on the set $\mathfrak{X}_{p_H} \cup \chi^v(T^*M \setminus \{0\})$ which is a set of generators for $\chi(T^*M \setminus \{0\})$. For X$\in \chi^v(T^*M \setminus \{0\})$ we have $\mathcal{J}_H X = 0$ and $\mathcal{J}_H(\nabla \mathrm{X}) = 0$ which leads to $\nabla \mathcal{J}_H(X) = \nabla(\mathcal{J}_H X) - \mathcal{J}_H(\nabla \mathrm{X}) = 0$. Next, if $X \in \mathfrak{X}_{p_H}$ then from $\nabla(f * X) = \nabla f * X + f * \nabla X = \nabla f * X + f * \nabla X$ it results

$p_H(f)\nabla(\mathcal{J}_H X) = p_H(f)\mathcal{J}_H(\nabla \mathrm{X})$, which implies $p_H(f)(\nabla \mathcal{J}_H)X = 0$ for arbitrary function $f \in C^\infty(T^*M \setminus \{0\})$ and arbitrary vector field $X \in \mathfrak{X}_{p_H}$.Therefore $\nabla \mathcal{J}_H = 0$ which ends the proof. The equivalence of 2) , 3) , 4) results from

$$\nabla \mathcal{J} = \mathcal{L}_p \mathcal{J} + h - v,$$
$$\nabla \mathcal{J} = \left(p(t_{ij}) + t_{kj}\frac{\partial \xi^k}{\partial x^i} - t_{ik}\frac{\partial \chi_j}{\partial p_k} + 2\mathcal{N}_{ij}\right)dx^i \otimes \frac{\partial}{\partial p_j}.$$

Given an adapted tangent structure $\mathcal{J}$ and a $\mathcal{J}$-regular vector field p, then the compatibility condition $\nabla\mathcal{J} = 0$fix the canonical nonlinear connection with h, v projectors

$h = \frac{1}{2}\left(Id - \mathcal{L}_p\mathcal{J}\right),\ v = \frac{1}{2}(Id + \mathcal{L}_p\mathcal{J}).$

Considering the canonical nonlinear connection ($\mathcal{N} = -\mathcal{L}_{p_H}\mathcal{J}_H$), we get the following results.

Proposition 5.10 A vector field $X \in \chi(T^*M)$ is a infinitesimal symmetry if and only if

$$\nabla^2\mathcal{J}_H X + \Phi(X) = 0. \qquad (5.26)$$

which locally yields

$$\nabla^2 g_{ij} X^i + \mathcal{R}_{ij} X^i = 0. \qquad (5.27)$$

Proof: If $\mathcal{N}$ is the canonical nonlinear connection, then $\nabla\mathcal{J}_H = 0$ and using

$\nabla\mathcal{J}_H = \nabla\circ\mathcal{J}_H - \mathcal{J}_H\circ\nabla$ from (5.25) it is results (5.26). Also, we obtain that the local components the vertical vector field (5.26) are (5.27).

Definition 5.11 i) An infinitesimal Noether symmetry of the Hamiltonian H is a vector

$X \in \chi(T^*M)$ such that

$$\mathcal{L}_X\omega = 0, \qquad \mathcal{L}_X H = 0$$

ii) A vector field $\tilde{X} \in \chi(T^*M)$ is said to be an invariant vector field for the Hamiltonian H if $\tilde{X}^{C*}(H) = 0$.

iii) A function $f \in C^\infty(M)$ is a constant of motion (or a conservation Law) for the Hamiltonian H if $\mathcal{L}_X f = 0$.

Proposition 5.12 Every infinitesimal Noether s$ymmetry$ is an infinitesimal symmetry.

Proof: From the symplectic equation $i_{p_H}\omega = -dH$, applying the Lie derivative in both sides, it results

$$\mathcal{L}_X\left(i_{p_H}\omega\right) = -\mathcal{L}_X dH = -d\mathcal{L}_X H = 0.$$

Also, from the formula $i_{\{X,\, p_H\}} = \mathcal{L}_X \circ i_{p_H} - i_{p_H} \circ \mathcal{L}_X$ we obtain

$$\mathcal{L}_X\left(i_{p_H}\omega\right) = i_{\{X,\, p_H\}}\omega + i_{p_H}\mathcal{L}_X\omega = i_{\{X,\, p_H\}}\omega,$$

which leads to $i_{\{X,\, p_H\}}\omega = 0$ and we get $\{X,\ p_H\} = 0$. ■

Proposition 5.13 If $\tilde{X}$ is a vector field on M such that $\mathcal{L}_{\tilde{X}^{C*}}\theta$ is closed and $d\left(\tilde{X}^{C*}H\right) = 0$, then $\tilde{X}$ is a natural infinitesimal symmetry.

Proof: We have

$$i_{\{\tilde{X}^{C*},\, p_H\}}\omega = \mathcal{L}_{\tilde{X}^{C*}}(i_{p_H}\omega) - i_{p_H}(\mathcal{L}_{\tilde{X}^{C*}}\omega) = -\mathcal{L}_{\tilde{X}^{C*}}dH - i_{p_H}(\mathcal{L}_{\tilde{X}^{C*}}d\theta)$$

$$= -d\mathcal{L}_{\tilde{X}^{C*}}dH - i_{p_H}d(\mathcal{L}_{\tilde{X}^{C*}}\theta) = -d\left(\tilde{X}^{C*}H\right) = 0,$$

because $d(\mathcal{L}_{\tilde{X}^{C*}}\theta) = 0$.

Proposition 5.14 The Hamiltonian vector field p_H is an infinitesimal Noether symmetry.

Proof: Using the skew symmetry of the symplectic 2-form ω it results

$$0 = i_{p_H}\omega(p_H) = -dH(p_H) = p_H(H) = \mathcal{L}_{p_H}H.$$

Also, from $d\omega = 0$ we get

$$\mathcal{L}_{p_H}\omega = di_{p_H}\omega + i_{p_H}d\omega = -d(dH) = 0.$$

■

Since Lie and exterior derivatives commute, we obtain for an infinitesimal Noether symmetry

$$d\mathcal{L}_X\theta = \mathcal{L}_X d\theta = \mathcal{L}_X\omega = 0.$$

It results that the 1-form $\mathcal{L}_X\theta$ is a closed 1-form and consequently $\mathcal{L}_{p_H}\theta$ is closed.

Definition 5.15 An infinitesimal Noether symmetry $X \in \chi(T^*M)$ is said to be on exact infinitesimal Noether symmetry if the 1-form $\mathcal{L}_X\theta$ is exact.

The next result, prove that there is a one to one correspondence between the exact infinitesimal Noether symmetry and conservation laws. Also, if X is an exact infinitesimal Noether symmetry, then there is a function $f \in C^\infty(M)$ such that $\mathcal{L}_X\theta = df$.

Theorem 5.16 If X is an exact infinitesimal Noether symmetry, then $f - \theta(X)$ is a conservation law for the Hamiltonian H. conversely, if $f \in C^\infty(M)$ is a conservation law for H, then $X \in \chi(T^*M \setminus \{0\})$ the unique solution of the equation $i_X\omega = -df$ is an exact infinitesimal Noether symmetry.

Proof: We have $p_H\big(f - \theta(X)\big) = d\big(f - \theta(X)\big)(p_H) = \big(\mathcal{L}_X\theta - di_X(\theta)\big)(p_H) = i_Xd\theta(p_H) = i_X\omega(\omega) = i_{p_H}\omega(X) = dH(X) = 0$, and it results that $f - \theta(X)$ is a conservation law for the dynamics associated to the regular Hamiltonian H. Conversely, if X is the solution of the equation $i_X\omega = -df$ then $\mathcal{L}_X\theta = i_Xd\theta + di_X\theta = -df + i_Xd\theta$ is an exact 1-form. Consequently, $0 = d\mathcal{L}_X\theta = \mathcal{L}_Xd\theta = \mathcal{L}_X\omega$. Also, f is a conservation law, and we have

$$0 = p_H(f) = df(p_H) = -i_X\omega(p_H) = i_{p_H}\omega(X) = -dH(X) = -X(H).$$

Therefore, we obtain $\mathcal{L}_XH = 0 \; and \; X$ is an exact infinitesimal Noether symmetry.

Theorem 5.17 If $\tilde{X} \in \chi(M)$ is an invariant vector field for the Hamiltonian H then its complete lift $\tilde{X}^{C*}$ is an exact infinitesimal $Noether$ symmetry and consequently $\tilde{X}$ is an natural infinitesimal symmetry. Moreover, the function $\theta(\tilde{X})$ is a conservation law for the Hamiltonian H.

Proof: We have that $\mathcal{L}_{\tilde{X}^{C*}}H = \tilde{X}^{C*}(H) = 0$. Next we prove that $\mathcal{L}_{\tilde{X}^{C*}}\theta = 0$ using the computation in local coordinates.

$$(\mathcal{L}_{\tilde{X}^{C*}}\theta)\left(\frac{\partial}{\partial x^i}\right) = \tilde{X}^{C*}\left(\theta\left(\frac{\partial}{\partial x^i}\right)\right) - \theta\left[\tilde{X}^{C*}, \frac{\partial}{\partial x^i}\right]$$

$$= \tilde{X}^{C*}(p_i) - \theta\left(-\frac{\partial \tilde{X}^j}{\partial x^i}\frac{\partial}{\partial x^j} + \frac{\partial^2 \tilde{X}^j}{\partial x^k \partial x^i} p_j \frac{\partial}{\partial p_k}\right)$$

$$= -\frac{\partial \tilde{X}^j}{\partial x^i} p_j + \frac{\partial \tilde{X}^j}{\partial x^i} p_j = 0,$$

$$(\mathcal{L}_{\tilde{X}^{C*}}\theta)\left(\frac{\partial}{\partial p_i}\right) = \tilde{X}^{C*}\left(\theta\left(\frac{\partial}{\partial p_i}\right)\right) - \theta\left[\tilde{X}^{C*}, \frac{\partial}{\partial p_i}\right] = -\theta\left(\frac{\partial \tilde{X}^i}{\partial x^j}\frac{\partial}{\partial p_i}\right) = 0.$$

It results that $0 = d\mathcal{L}_{\tilde{X}^{C*}}\theta = \mathcal{L}_{\tilde{X}^{C*}} d\theta = \mathcal{L}_{\tilde{X}^{C*}}\omega$ and $\tilde{X}^{C*}$ is an exact infinitesimal $Noether$ symmetry. Using proposition 5.4and 5.6 we have that $\tilde{X}^{C*}$ is an infinitesimal symmetry and consequently, $\tilde{X}$ is a natural infinitesimal symmetry. Moreover, for $f = 0$ it results that

$$\theta(\tilde{X}^{C*}) = p_i \tilde{X}^i,$$

is a conservation law for the Hamiltonian H.

5.5 Classification of the Symmetries of Hamiltonian Systems with Degrees of Freedom

We consider the motion of a particle of unit mass in the plane (q_1, q_2) under the influence of a potential of the form $V(q_1, q_2)$. We will assume that the Hamiltonian is time independent. This is not really a restriction because a time-dependent n- dimensional system is equivalent to a time-independent (n+1)- dimensional system by regarding the time variable as the new coordinate. For the most part we assume that the system is two-dimensional with Hamiltonian

$$H(q_1, q_2, p_1, p_2) = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + V(q_1, q_2). \qquad (5.28)$$

The real valued function $V(q_1, q_2)$ is assumed to be smooth on some open, connected subset of R^2. Hamilton's equations, in Newtonian form, become

$$\left.\begin{aligned} \ddot{q}_1 &= -\frac{\partial V}{\partial q_1} \\ \ddot{q}_2 &= -\frac{\partial V}{\partial q_2} \end{aligned}\right\} . \qquad (5.29)$$

We search for point symmetries of the system (5.29). That is, we search for the infinitesimal transformations of the form

$$\left.\begin{aligned} t' &= t + \epsilon T(t, q_1, q_2) + O(\epsilon^2) \\ \acute{q}_1 &= q_1 + \epsilon Q_1(t, q_1, q_2) + O(\epsilon^2) \\ \acute{q}_2 &= q_2 + \epsilon Q_2(t, q_1, q_2) + O(\epsilon^2) \end{aligned}\right\}. \qquad (5.30)$$

The function $V(q_1, q_2)$ such that the system (5.29) admit such transformations are completely classified. Therefore, in the following analysis we determine the functions V, T, Q_1 *and* Q_2.

Equations (5.29) admit Lie transformations of the form (5.30) if and only if

$$\left.\begin{aligned} \Gamma^{(2)}\{\ddot{q}_1 + V_{q_1}\} &= 0 \\ \Gamma^{(2)}\{\ddot{q}_2 + V_{q_2}\} &= 0 \end{aligned}\right\}. \qquad (5.31)$$

where $\Gamma^{(2)}$ is the second prolongation of

$$\Gamma = T\frac{\partial}{\partial t} + Q_1\frac{\partial}{\partial q_1} + Q_2\frac{\partial}{\partial q_2}. \qquad (5.32)$$

Equations (5.31) give two identities of the form:

$$\left.\begin{aligned} E_1(t, q_1, q_2, \dot{q}_1, \dot{q}_2\,) = 0 \\ E_2(t, q_1, q_2, \dot{q}_1, \dot{q}_2\,) = 0 \end{aligned}\right\}. \qquad (5.33)$$

Where, we have used that $\ddot{q}_1 = -\frac{\partial V}{\partial q_1}$ and $\ddot{q}_2 = -\frac{\partial V}{\partial q_2}$. The functions E_1 and E_2 are explicit polynomials in $\dot{q}_1$ and $\dot{q}_2$. We impose the condition that equation (5.33) are identities in five variables $t, q_1, q_2, \dot{q}_1$ and $\dot{q}_2$ which are regarded as independent. These two identities enable the infinitesimal transformations to be derived and ultimately impose restrictions on the functional forms of V, T, Q_1 and Q_2.

After some straightforward a calculations we can show, that the generators necessarily have the following form:

$$\left.\begin{aligned} T &= a(t) + b_1(t)q_1 + b_2(t)q_2 \\ Q_1 &= \acute{b}_1(t)q_1^2 + \acute{b}_2(t)q_1q_2 + c_{11}(t)q_1 + c_{12}(t)q_2 + d_1(t) \\ Q_2 &= \acute{b}_1(t)q_1q_2 + \acute{b}_2(t)q_2^2 + c_{21}(t)q_1 + c_{22}(t)q_2 + d_2(t) \end{aligned}\right\}. \qquad (5.34)$$

In this section we classify the symmetry groups of the system according to the form of the generators. Here is a preview of the various cases and the potentials that appear.

Case1. $b_1 \neq 0, b_2 \neq 0.$

In this case the potential is of the form

$$V = \frac{\lambda}{2}(q_1^2 + q_2^2) + \lambda_1 q_1 + \lambda_2 q_2.$$

The symmetry group has maximum dimension. It is a 15-parameter group of transformations isomorphic to $sl(4, R)$.

Case2. $b_1 = b_2 = 0.$

In other words, T is a function of time only. We consider two possibilities according to $a'' \neq 0$ or $a'' = 0$.

First Sub-case: $a'' \neq 0$

2a

$$V = \frac{\lambda}{2}(q_1^2 + q_2^2) + \frac{\mu}{(q_1 + kq_2)^2}$$

In this case we obtain a 6-parameter group.

2b

$$V = \frac{\lambda}{2}(q_1^2 + q_2^2) + \frac{\mu}{(q_2)^2}$$

This is special case of previous one with $k = 0$.

2c

$$V = \frac{\lambda}{2}(q_1^2 + q_2^2) + \frac{1}{q_2^2}\Phi(\xi),$$

where $\xi = \frac{q_1}{q_2}$. For Φ arbitrary we end up with a 3-parameter symmetry group. For special types of Φ we obtain a 4-parameter group.

Second sub-case $a'' = 0$

This is the case where T is a linear function of time,

2d

$$V = q_2^2 + \lambda_1 q_1 q_2 + \Phi(\xi),$$

Where $\xi = q_1 - \lambda q_2$. For Φ arbitrary we end up with a 4-parameter symmetry group. For Φ quadratic we obtain a 7-parameter group and setting $\lambda_1 = 0$, for some special form of Φ (exponential, logarithmic, n^{th} power) results in a 5-parameter group of symmetries.

2e

$$V = \lambda_1 q_2^2 + \Phi(q_1)$$

The case $\lambda_1 = 0$ is the case of a separable potential with one variable missing. We will comment on this case separately. If, $\lambda_1 \neq 0$, we end up with a 4-parameter group.

2f

The dimensions of the symmetry groups in this case are all equal to 2except for the last three systems where the dimension is 3. Specifically, we obtain the following list of potentials:

1.

$$V = q_2^N \Phi\left(\frac{q_1}{q_2}\right),$$

2.

$$V = \lambda_1 \log q_2 + \Phi\left(\frac{q_1}{q_2}\right)$$

3.

$$V = e^{\mu q_1}\, \Phi(q_2)$$

4.

$$V = e^{\mu q_1}\, \Phi(q_1 - \lambda q_2)$$

5-10

$$V = \lambda_1 f_1(q_1) + \lambda_2 f_2(q_2)$$

where $f_1(q_1) = q_1^n$, $\log q_{1,}$ $e^{\mu q_1}$ and $f_2(q_2) = q_2^m$, $\log q_2$, $e^{\mu q_2}$

11.

$$V = \Phi(q_1^2 + q_2^2)$$

12.

$$V = \lambda(q_1^2 + q_2^2)^n, n \neq -1, 0, 1$$

13.

$$V = \lambda \log(q_1^2 + q_2^2)$$

14.

$$V = \lambda sin^{-1} \frac{q_1^2 - q_2^2}{q_1^2 + q_2^2}$$

Finally, we note that the potential $V(q_1, q_2) = q_1^k$ has a 15-parameter group of symmetries for $k = 0, 1$, a 7-parameter group for $k = 2$, a 6-parameter group for $k = -2$, and a 5-parametergroup of symmetries otherwise. These dimensions generalize for arbitrary $n \geq 2$ to

$(n + 2)^2 - 1, n^2 + 3, n^2 + 2, n^2 + 1$ respectively.

In this sections we will not consider the general case of motion in R^n, however for the case $n = 3$, the classification of symmetry groups in progress; see[47]. In this case, we end up again with a maximal dimension of $(n + 2)^2 - 1 = 24$ for the harmonic oscillator or a free particle, but the dimensions of the other groups in the classification vary from 1 to 12 $(n^2 + 3)$. We did not obtain the dimension 8 and any dimension between 13 and 23. In section11 we will also consider the simplest system, with one degree of freedom, mainly to illustrate the procedure we use for the two-dimensional case.

Of course the ultimate goal in classical mechanics is to integrate explicitly the equations of motion. Such systems are called integrable.

Theorem of Liouville 5.18: this theorem is the key result which in the 2-dimensional case translates as follows:

Consider a Hamiltonian system with two degrees of freedom. If in addition to the Hamiltonian H there is a second integral of motion I, independent of H, then the system is integrable, i.e. in principle we can solve the equations by quadratures.

Now, we shill point out that all the systems which do not appear in our classification will have $\frac{\partial}{\partial t}$ as a single symmetry. We can construct a number of systems possessing, only one symmetry, for example, we can take

$$V = q_1^2 + 2q_2^2 + q_1^k q_2. \qquad (5.35)$$

with $k > 1$.

As was mention there are integrable systems which possess only one symmetry, This situation is also, investigate [48], [49]. We would like to point out another example:

$$V = 4q_1^2 + q_2^2 + \frac{r_1}{q_1} + \frac{r_2}{q_2^2}. \qquad (5.36)$$

In this system the associated Hamilton-Jacobi equation is separable in Cartesian coordinates. We should point out that integrable systems have symmetries other than point symmetries.

5.5.1 Systems with One Degree of Freedom

Before attacking the two-dimensional case, we classify the symmetries for a one-dimensional system, just to illustrate the techniques we use on the two-dimensional case. We consider a

Hamiltonian of the form

$$H = \frac{1}{2}p^2 + V(q). \qquad (5.37)$$

The equation of motion of the particle is

$$\ddot{q} = -\frac{\partial V}{\partial q} \, . \qquad (5.38)$$

We search for symmetries for equation (5.38) of form

$$t' = t + \epsilon T(t,q) + O(\epsilon^2)$$

$$q' = q + \epsilon Q(t,q) + O(\epsilon^2). \qquad (5.39)$$

Equation (5.38) admits symmetries of the form (5.39) if and only if

$$\Gamma^{(2)}\{\ddot{q} + V_q\} = 0 \, . \qquad (5.40)$$

where $\Gamma^{(2)}$ is the second prolongation of

$$\Gamma = T\frac{\partial}{\partial t} + Q\frac{\partial}{\partial q} \, . \qquad (5.41)$$

The definition of the second prolongation if the following: first we define the first prolongation

$$\Gamma^{(1)} = \Gamma + \left[-T_q\dot{q}^2 + \left(Q_q - T_t\right)\dot{q} + Q_t\right]\frac{\partial}{\partial \dot{q}} . \, (5.42)$$

The second prolongation of Γ is an extension of $\Gamma^{(1)}$ given by

$$\Gamma^{(2)} = \Gamma^{(1)} + \left[\left(Q_q - 2T_t - 3T_q\dot{q}\right)\ddot{q} - T_{qq}\dot{q}^3 + \left(Q_{qq} - 2T_{tq}\right)\dot{q}^2 + \left(2Q_{tq} - T_{tt}\right)\dot{q} + Q_{tt}\right]\frac{\partial}{\partial \ddot{q}} \, . \qquad (5.43)$$

Equation (5.40) becomes an identity of the form

$$E(t, q, \dot{q}) = 0. \qquad (5.44)$$

using the fact that $\ddot{q} = -V_q$.

The coefficient of $\dot{q}^3$ in (5.44) gives $T_{qq} = 0$. similarly, the coefficient of $\dot{q}^2$ gives $Q_{qq} = 2T_{tq}$.

Therefore,

$$\left.\begin{aligned} T &= a(t) + b(t)q \\ Q &= b'(t)q^2 + c(t)q + d(t) \end{aligned}\right\}. \qquad (5.45)$$

Using equations (5.45), identity (5.44) becomes

$$\left[3b\frac{\partial V}{\partial q} + 3qb" - a" + 2c'\right]\dot{q} + (q^2b' + qc + d)\frac{\partial^2 V}{\partial q^2} + (2a' - c)\frac{\partial V}{\partial q} + q^2b''' + qc'' + d'' = 0. \qquad (5.46)$$

The coefficient of $\dot{q}$ in (5.46) gives

$$3b\frac{\partial V}{\partial q} + 3qb"\text{-}a" + 2c' = 0\,. \qquad (5.47)$$

We split the analysis into two exclusive cases:

Case1. $b \neq 0$.

Case2. $b = 0$

Case1. $b \neq 0$.

From equation (5.47) we obtain

$$V = \frac{\lambda_1}{2}q^2 + \lambda_2 q + \lambda_3\,. \qquad (5.48)$$

We easily calculate that the algebra of symmetries has dimension 8. It is a simple Lie algebra isomorphic to $sl(3, R)$.

Case2. . $b = 0$

From (5.47) we have

$$c = \frac{1}{2}a' + c_1$$

and equation(5.46) becomes

$$[q(a' + 2c_1) + 2d)]\frac{\partial^2 V}{\partial q^2} + (3a' - 2c_1)\frac{\partial V}{\partial q} + a'''q + 2d'' = 0. \quad (5.49)$$

Form equation (5.49) we deduce that V satisfies an O.D.E. of the form

$$(\lambda_1 q + \lambda_2)V_{qq} + \lambda_3 V_q = \lambda_4 q + \lambda_5 \,. \qquad (5.50)$$

In order to solve equation (5.50) we consider the following five possibilities:

1. $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$. In this case we get $a(t) = constant, c_1 = 0, and\ d(t) = 0$.

Therefore V arbitrary we have $T = c_2$, $Q = 0$. In other words the symmetry group is trivial (one dimension).

2. $\lambda_1 = \lambda_3 = 0$, $\lambda_2 \neq 0$. In this case, V is quadratic, a case already examined.

3. $\lambda_1 = 0$, $\lambda_2 \neq 0$, $\lambda_3 \neq 0$. From (5.50) we get $V = \lambda e^{\mu q}$, where we have ignored linear terms. We obtain a 2-parameter group of symmetries with $T = -2c_1 t + c_2$ and $Q = \frac{4c_1}{\mu}$.

4. $\lambda_1 \neq 0$, $\lambda_3 = 0$. From (5.50) we obtain $V = q\log q$. the symmetry group here is also trivial.

5. $\lambda_1 \neq 0, \lambda_3 \neq 0$. Without loss of generality we take $\lambda_2 = 0$ in (5.50) and we obtain either

$V = \lambda q^n,\ n \neq 0{,}1\ 2$, or $V = \lambda \log q$.

First: If $V = \lambda q^n$ we substitute in to (5.49) to get, $a(t) = c_2 t^2 + 2c_3 t + c_4, d(t) = c_1 = 0,$

if $n = -2, and\ a(t) = \frac{2(2-n)}{n+2} c_1 t + c_2, d(t) = 0\ if\ n \neq -2.$ to summarize, we have for $V = \frac{\lambda}{q^2}$,

$$T = c_2 t^2 + 2c_3 t + c_4$$

$$Q = (c_2 t + c_3) q. \qquad (5.51),$$

and for $V = \lambda q^n, n \neq -2, 0, 1, 2$ the generators have the following form:

$$T = \frac{2(2-n)}{n+2} c_1 t + c_2$$

$$Q = \frac{4c_1}{n+2} q. \qquad (5.52)$$

In other words, we obtain either a two-parameter or a three parameter group of symmetries.

Second: When $V = \lambda \log q$ we get $a(t) = c_1 t + c_2$ and $d(t) = 0.$ therefore,

$$T = 2c_1 t + c_2$$

$$Q = 2c_1 q. \qquad (5.53)$$

This is a two parameter group of symmetries.

To summarize the results in the case of one degree of freedom we obtain a maximal dimension of 8 for the harmonic oscillator or a free particle, but the dimensions in the other groups in the classification vary from 1 to 3. We do not obtain any dimension between 4 and 7.

5.5.2 Systems with Two Degrees of Freedom

We return now to the case of two-degree of freedom. The analysis is analogous to the one used in the case of one-degree of freedom. We substitute the form T, Q_1, Q_2 in (5.34) into equations (5.33).

The coefficient of $\dot{q}_1$ in equations (5.33) $[E_2 = 0]$ gives

$$2\left(\frac{\partial V}{\partial q_2} b_1 + \frac{\partial^2 b_1}{\partial t^2} q_2 + \frac{\partial c_{21}}{\partial t}\right) = 0. \tag{5.54}$$

On the other hand, the coefficient of $\dot{q}_2$ in equation (5.33) $[E_1 = 0]$ implies

$$2\left(\frac{\partial V}{\partial q_1} b_2 + \frac{\partial^2 b_2}{\partial t^2} q_1 + \frac{\partial c_{12}}{\partial t}\right) = 0. \tag{5.55}$$

Similarly, the coefficient of $\dot{q}_1$ in equation (5.33) $[E_1 = 0]$ gives

$$3b_1 \frac{\partial V}{\partial q_1} + b_2 \frac{\partial V}{\partial q_2} + 3q_1 \frac{\partial^2 b_1}{\partial t^2} + q_2 \frac{\partial^2 b_2}{\partial t^2} - \frac{\partial^2 a}{\partial t^2} + 2\frac{\partial c_{11}}{\partial t} = 0. \tag{5.56}$$

while the coefficient of $\dot{q}_2$ in equation (5.33) $[E_2 = 0]$ implies

$$3b_2 \frac{\partial V}{\partial q_2} + b_1 \frac{\partial V}{\partial q_1} + 3q_2 \frac{\partial^2 b_2}{\partial t^2} + q_1 \frac{\partial^2 b_1}{\partial t^2} - \frac{\partial^2 a}{\partial t^2} + 2\frac{\partial c_{22}}{\partial t} = 0. \tag{5.57}$$

If $b_1(t) \neq 0 \; and \; b_2(t) \neq 0$, then from equations (5.54) and (5. 55) we deduce that V is quadratic in q_1 and q_2. We also note that if $b_1(t) = 0\,, b_2(t) \neq 0, (\, or \; b_1(t) \neq 0\,, b_2(t) = 0)$,

then from equations (5.55) and (5.55) V has again a quadratic form. We therefore split the analysis into two exclusive cases:

Case1. $b_1(t) \neq 0, b_2(t) \neq 0$

Case2. $b_1(t) = 0, b_2(t) = 0$

Case1.

From equations (5.54) and (5.55) we deduce that

$$V = \lambda_1 q_1^2 + \lambda_2 q_2^2 + \lambda_3 q_1 + \lambda_4 q_2 + \lambda_5. \qquad (5.58)$$

Now, substitute (5.58) into (5.33) the coefficients of $q_1 \dot{q}_1$ in E_1 and $q_2 \dot{q}_1$ in E_2 give respectively:

$$\left.\begin{array}{l} 3(b_1'' + 2\lambda_1 b_1) = 0 \\ 3(b_1'' + 2\lambda_2 b_1) = 0 \end{array}\right\}. \qquad (5.59)$$

Hence, it follows that, $\lambda_1 = \lambda_2 = 0$ or $\lambda_1 = \lambda_2 \neq 0$.

In the case $\lambda_1 = \lambda_2 = 0$, V is linear. We shall present the symmetries for the case, but in the remaining part of the analysis we shall ignore linear terms in the form of V. Adding a constant to equations (5.29) has no effect on the symmetry groups.

$$V = \lambda_1 q_1 + \lambda_2 q_2 + \lambda_3$$

Sub-case 1a

Note that we have taken $\lambda_1 = \lambda_2 = 0$ in (5.58) and then renamed the constants. Without presenting any calculations, we state that the system (5.29) with V linear has the following 15-symmetries:

$$\Gamma_1 = \frac{\partial}{\partial t}$$

$$\Gamma_2 = \frac{\partial}{\partial q_1}$$

$$\Gamma_3 = \frac{\partial}{\partial q_2}$$

$$\Gamma_4 = q_1 \frac{\partial}{\partial t} - \left(\frac{\lambda_1^2}{4} t^3 + \frac{3}{2}\lambda_1 t q_1\right)\frac{\partial}{\partial q_1} - \left(\frac{\lambda_1 \lambda_2}{4} t^3 + \frac{1}{2}\lambda_1 t q_2 + \lambda_2 t q_1\right)\frac{\partial}{\partial q_2}$$

$$\Gamma_5 = t\frac{\partial}{\partial t} + \left(\frac{1}{2}q_1 - \frac{3}{4}\lambda_1 t^2\right)\frac{\partial}{\partial q_1} + \left(\frac{1}{2}q_2 - \frac{3}{4}\lambda_2 t^2\right)\frac{\partial}{\partial q_2}$$

$$\Gamma_6 = t^2\frac{\partial}{\partial t} + \left(q_1 t - \frac{\lambda_1}{2}t^3\right)\frac{\partial}{\partial q_1} + \left(q_2 t - \frac{\lambda_2}{2}t^3\right)\frac{\partial}{\partial q_2}$$

$$\Gamma_7 = q_2\frac{\partial}{\partial t} - \left(\frac{\lambda_1\lambda_2}{4}t^3 + \lambda_1 t q_2 + \frac{1}{2}\lambda_2 t q_1\right)\frac{\partial}{\partial q_1} - \left(\frac{\lambda_2^2}{4}t^3 + \frac{3}{2}\lambda_2 t q_2\right)\frac{\partial}{\partial q_2}$$

$$\Gamma_8 = (2tq_2 + \lambda_2 t^3)\frac{\partial}{\partial t} + \left(2q_1 q_2 + \lambda_2 t^2 q_1 - \lambda_1 t^2 q_2 - \frac{\lambda_1\lambda_2}{2}t^4\right)\frac{\partial}{\partial q_1} + \left(2q_2^2 - \frac{1}{2}\lambda_2^2 t^4\right)\frac{\partial}{\partial q_2}$$

$$\Gamma_9 = (2tq_1 + \lambda_1 t^3)\frac{\partial}{\partial t} + \left(2q_1^2 - \frac{1}{2}\lambda_1^2 t^4\right)\frac{\partial}{\partial q_1} + \left(2q_1 q_2 + \lambda_1 t^2 q_2 - \lambda_2 t^2 q_1 - \frac{\lambda_1\lambda_2}{2}t^4\right)\frac{\partial}{\partial q_2}$$

$$\Gamma_{10} = \left(q_2 + \frac{1}{2}\lambda_2 t^2\right)\frac{\partial}{\partial q_1}$$

$$\Gamma_{11} = \left(q_1 + \frac{1}{2}\lambda_1 t^2\right)\frac{\partial}{\partial q_2}$$

$$\Gamma_{12} = t\frac{\partial}{\partial q_1}$$

$$\Gamma_{13} = t\frac{\partial}{\partial q_2}$$

$$\Gamma_{14} = \left(q_1 + \frac{1}{2}\lambda_1 t^2\right)\frac{\partial}{\partial q_1}$$

$$\Gamma_{15} = \left(q_2 + \frac{1}{2}\lambda_2 t^2\right)\frac{\partial}{\partial q_2}.$$

Remark5.19: this system is of course integrable. The second integral is $I = \frac{1}{2}p_1^2 + \lambda_1 q_1$ or

$I = \frac{1}{2}p_2^2 + \lambda_2 q_2$. It also has constants of motion linear in the momenta, for example

$I = -\lambda_2 p_1 + \lambda_1 p_2$.

Sub-case 1b

We will choose λ=1. We substitute the form of V into equations(5.33) and equate coefficients. In $E_{1,}\ \dot{q}_1 q_1 = 0$ implies

$$\frac{d^2 b_1}{dt^2} + b_1 = 0.$$

In $E_{1,}\ \dot{q}_1 q_2 = 0$ implies

$$\frac{d^2 b_2}{dt^2} + b_2 = 0.$$

In $E_{1,}\ \ \dot{q}_2 = 0$ implies

$$\frac{dc_{12}}{dt} = 0,$$

therefore c_{12} is constant. Similarly by examining the coefficient of $\dot{q}_1$ in E_2 we see that c_{12} is also constant.

In $E_{1,}\ \ \dot{q}_1 = 0$ and in $E_{2,}\ \ \dot{q}_2 = 0$, imply that

$$\frac{d^2 a}{dt^2} - 2\frac{dc_{11}}{dt} = 0,$$

$$\frac{d^2 a}{dt^2} - 2\frac{dc_{22}}{dt} = 0.$$

Similarly, using the coefficient of q_1= 0 in E_1 and q_2=0 in E_2 , we obtain

$$\frac{d^2 c_{11}}{dt^2} + 2\frac{da}{dt} = 0,$$

and

$$\frac{d^2c_{22}}{dt^2}+2\frac{da}{dt}=0.$$

Finally, and $E_1 \ = 0$ and $E_2 \ = 0$ imply that the function $d_1(t)$ and $d_2(t)$ are solutions of the equation

$$\frac{d^2x}{dt^2}+x=0.$$

Therefore the form of the generators in this case is the following:

$$T=k_1+k_2cos2t+k_3sin2t+(k_4cost+k_5sint)q_1+(k_6cost+k_7sint)q_2$$

$$Q_1=(-k_4sint+k_5cost)q_1^2+(-k_6sint+k_7cost)q_1q_2+(-k_2sin2t+k_3cos2t+c_{11})q_1+c_{12}q_2+k_8cost+k_9sint$$

$$Q_2=(-k_4sint+k_5cost)q_1q_2+(-k_6sint+k_7cost)q_2^2+c_{21}q_1+(-k_2sin2t+k_3cos2t+c_{22})q_2+k_{10}cost+k_{11}sint.$$

We note that the system (5.29) with $V=\frac{1}{2}(q_1^2+q_2^2)$ admit a 15-parameter qroup of transformations isomorphic to $sl(4,\mathrm{R})$.

Remark 5.20: This system is the 2-dimensional isotropic oscillator. A second integral is

$$I_1=\frac{1}{2}p_1^2+\frac{1}{2}q_1^2 \text{ or } I_2=\frac{1}{2}p_2^2+\frac{1}{2}q_2^2.$$

We also have constants of motion linear in the in the momenta, for example $I_3=q_2p_1-q_1p_2$.

Remark 5.21: cases $1a \; and \; 1b$ give the most general form of Hamiltonian for which the second invariant is linear in the momenta.

Case 2

We use the identities $E_1 \ = 0$ and $E_2 \ = 0$ in (5.33) to obtain:

$$E_1 \colon \dot{q}_2 = 0 \Leftrightarrow \; c_{12}' = 0$$

$$E_2 \colon \dot{q}_1 = 0 \Leftrightarrow \; c_{21}' = 0$$

$$E_1 \colon \dot{q}_1 = 0 \; \Leftrightarrow 2c_{11}' - a" = 0$$

$$E_2 \colon \dot{q}_2 = 0 \; \Leftrightarrow c_{22}' - a" = 0. \qquad (5.6o)$$

Therefore, $c_{12} = c_{1,}\, c_{21} = c_{2,}\, c_{11} = \frac{1}{2}a' + c_{3,}\, and \;\, c_{22} = \frac{1}{2}a' + c_{4,}$. Here and elsewhere the c_i are constants. Using these results, equations (5.33) take the form $q_1 q_2$

$$(a'q_2 + 2c_2q_1 + 2c_4q_2 + 2d_2)V_{q_1q_2} + (a'q_1 + 2c_3q_1 + 2c_1q_2 + 2d_1)V_{q_1q_1}$$
$$+(3a' - 2c_3)V_{q_1} - 2c_1V_{q_2} + a'''q_1 + 2d_1'' = 0. \qquad (5.61)$$

and

$$(a'q_1 + 2c_3q_1 + 2c_1q_2 + 2d_1)V_{q_1q_2} + (a'q_2 + 2c_2q_1 + 2c_4q_2 + 2d_2)V_{q_2q_2} +(3a' - 2c_4)V_{q_2} - 2c_2V_{q_1} + a'''q_2 + 2d_2'' = 0. \qquad (5.62)$$

We differentiate equations (5.61) and (5.62) with respect to t to get respectively:

$$(a''q_2 + 2d_2')V_{q_1q_2} + (a''q_1 + 2d_1')V_{q_1q_1} + 3a''^{V_{q_1}} + a''''q_1 + 2d_1''' = 0, \qquad (5.63)$$

$$\left(a''q_1 + 2d_1'\right)V_{q_1q_2} + (a''q_2 + 2d_2')V_{q_2q_2} + 3a''^{V_{q_2}} + a''''q_2 + 2d_2''' = 0. \qquad (5.64)$$

We now split the analysis into two parts:

$a'' \neq 0$ or $a'' = 0$.

Non-linear *T* ($a'' \neq 0$)

We divide equations (5.63) and (5.64) by a'' and then differentiate with respect to t to get respectively:

$$2\left(\frac{d_2'}{a''}\right)' V_{q_1q_2} + 2\left(\frac{d_1'}{a''}\right)' V_{q_1q_1} + \left(\frac{a''''}{a''}\right)' q_1 + 2\left(\frac{d_1'''}{a''}\right)' = 0, \qquad (5.65)$$

$$2\left(\frac{d_1'}{a''}\right)' V_{q_1q_2} + 2\left(\frac{d_2'}{a''}\right)' V_{q_2q_2} + \left(\frac{a''''}{a''}\right)' q_2 + 2\left(\frac{d_2'''}{a''}\right)' = 0, \qquad (5.66)$$

From equations (5.65) and (5.66) we deduce that the function $V(q_1, q_2)$ satisfies two partial differential equations of the form

$$\lambda_1 V_{q_1q_2} + \lambda_2 V_{q_1q_1} + \lambda_3 q_1 + \lambda_4 = 0, \qquad (5.67)$$

$$\lambda_2 V_{q_1q_2} + \lambda_1 V_{q_2q_2} + \lambda_3 q_2 + \lambda_5 = 0. \qquad (5.68)$$

In order to solve equations (5.67) and (5.68) we consider the following cases:

- $\lambda_1 \neq 0, \lambda_2 \neq o$
- $\lambda_1 = 0, \lambda_2 \neq o$, (or $\lambda_1 \neq 0, \lambda_2 = o$)
- $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = o$

In the following three sub-cases we determine the form of V from equations (5.67) and (5.68). The corresponding generators may be obtained with the employment of equations (5.61) and (5.66).

Sub-case 2a

In this case, V takes the form

$$V = \frac{\lambda}{2}(q_1^2 + q_2^2) + \frac{\mu}{(q_1 + kq_2)^2}. \qquad (5.69)$$

with the corresponding generators:

$$T = a(t)$$

$Q_1 = \frac{1}{2}a'(t)q_1 + k^2c_4q_1 - kc_4q_2 + kq_2(t)$ (5.70)

$Q_2 = \frac{1}{2}a'(t)q_2 - kc_4q_1 + c_4q_2 - q_2(t)$

where $a'' + 4\lambda a = c_8$ and d_2 is a solution of $d_2'' + \lambda d_2 = 0$.

(A 6-parameter group).

Sub-case 2b

Setting $\lambda_1 = 0$ in (5.67) and (5.68) we get the forms of the group generators:

$$V = k_1q_1^3 + k_2q_1^2 + k_3q_1q_2^2 + k_4q_1q_2 + k_5q_1 + \Phi(q_2). \quad (5.71)$$

Using equations (5.63) and (5.64) we get $\lambda_1 = \lambda_3 = \lambda_4 = 0 \ and \ \Phi = k_2q_2^2 + k_6q_2 + k_7 + \frac{\mu}{q_2^2}$.

Therefore, ignoring the linear terms, V takes the form

$$V = \frac{\lambda}{2}(q_1^2 + q_2^2) + \frac{\mu}{q_2^2}. \quad (5.72)$$

Finally, using equations (5.61) and (5.62) we get the forms of the group generators:

$T = a(t)$

$Q_1 = c_3q_1 + \frac{1}{2}a'q_1 + d_1(t)$ (5.73)

$Q_2 = \frac{1}{2}a'(t)q_2$

where $a'' + 4\lambda a = c_8$ and $d_1'' + \lambda d_1 = 0$ (A 6-parameter group).

The case $\lambda = 0$ is equivalent to the system $V(q_1, q_2) = \frac{1}{q_1^2}$ with generators:

$T = c_1 t^2 + 2c_2 t + c_3$

$Q_1 = (c_1 t + c_2) q_1$ (5.74)

$Q_2 = (c_1 t + c_2 + c_4) q_2 + c_5 t + c_6$

We choose the following basis for the Lie algebra of symmetries:

$$X_1 = \frac{\partial}{\partial t}$$

$$X_2 = 2t\frac{\partial}{\partial t} + q_1\frac{\partial}{\partial q_1} + q_2\frac{\partial}{\partial q_2}$$

$$X_3 = t^2\frac{\partial}{\partial t} + q_1 t\frac{\partial}{\partial q_1} + q_2 t\frac{\partial}{\partial q_2}$$

$$X_4 = q_2\frac{\partial}{\partial q_2}$$

$$X_5 = t\frac{\partial}{\partial q_2}$$

$$X_6 = \frac{\partial}{\partial q_2},$$

with (non-zero) bracket relations

$$[X_1, X_2] = 2X_1$$

$$[X_1, X_3] = X_2$$

$$[X_1, X_5] = X_6$$

$$[X_2, X_3] = 2X_3$$

$$[X_2, X_5] = X_5$$

$$[X_2, X_6] = -X_6$$

$$[X_3, X_6] = -X_5$$

$$[X_4, X_5] = -X_5$$

$$[X_4, X_6] = -X_6.$$

This algebra is not semi-simple since the ideal generated by X_5, X_6 is abetain. It is not solvable either because, $L^{(1)} = \{X_1, X_2, X_3, X_5, X_6\}$, and $L^{(2)} = L^{(1)}$.

Remark5.22: it is clear that sub-case 2b is a special case of sub-case 2a by setting k=0.

Sub-case 2c

Since all the coefficients of the terms in equations (5.65) and (5.66) vanish, the function $a(t)$, $d_1(t)$ and $d_2(t)$ may be determined. Form equations (5.63) and (5.64) we deduce that

$$V = \frac{\lambda}{2}(q_1^2 + q_2^2) + \frac{\mu}{q_2^2}\Phi(\xi). \qquad (5.76)$$

where $\xi = \frac{q_1}{q_2}$. We now use equations (5.61) and (5.62) to determine the forms of the generators. Without presenting any more calculations we state the following results:

1.Φ arbitrary:

$$T = a(t)$$

$$Q_1 = \frac{1}{2}a' q_1 \qquad (5.77)$$

$$Q_2 = \frac{1}{2}a'(t)q_2$$

where $a'' + 4\lambda a = c_8$ (A 3-parameter group).

For λ=0 the Lie algebra has a basis given by

$$X_1 = \frac{\partial}{\partial t}$$

$$X_2 = 2t\frac{\partial}{\partial t} + q_1\frac{\partial}{\partial q_1} + q_2\frac{\partial}{\partial q_2}$$

$$X_3 = t^2\frac{\partial}{\partial t} + q_1 t\frac{\partial}{\partial q_1} + q_2 t\frac{\partial}{\partial q_2}$$

with bracket relations of a simple Lie algebra:

$$[X_1, X_2] = 2X_1$$

$$[X_1, X_3] = X_2$$

$$[X_2, X_3] = 2X_3$$

$\mathbf{2.}\ \Phi = \frac{\boldsymbol{\mu}}{\boldsymbol{\xi}^2} = \boldsymbol{\mu}\frac{\boldsymbol{q_2^2}}{\boldsymbol{q_1^2}}$:

It follows from (5.76)

$$V = \frac{\lambda}{2}(q_1^2 + q_2^2) + \frac{\mu}{q_1^2}. \qquad (5.78)$$

This potential already appeared in 2b.

3. $\Phi = \boldsymbol{\mu} = \boldsymbol{constant}$:

It follows from (5.76) that

$$V = \frac{\lambda}{2}(q_1^2 + q_2^2) + \frac{\mu}{q_2^2}. \qquad (5.79)$$

This system is not different from the previous one.

4. $\Phi = \dfrac{\boldsymbol{\mu}}{\boldsymbol{\xi^2 + 1}} \boldsymbol{e^{c\tan^{-1}\xi}}$:

It follows form (5.76) that

$$V = \frac{\lambda}{2}(q_1^2 + q_2^2) + \frac{\mu}{q_1^2 + q_2^2} \boldsymbol{e}^{\boldsymbol{c\tan^{-1}}\frac{q_1}{q_2}}. \qquad (5.80)$$

The generators take the form:

$T = a(t)$

$$Q_1 = \frac{1}{4}cc_1q_1 + c_2q_2 + \frac{1}{2}a'(t)q_1 \quad . \qquad (5.81)$$

$$Q_2 = \frac{1}{4}cc_1q_2 - c_1q_1 + \frac{1}{2}a'(t)q_2$$

where $a'' + 4\lambda a = c_8$ (A 4-parameter group).

The Lie algebra for this system is a direct sum of an $sl(2, \mathbf{R})$ and a one dimensional Lie algebra. It has a basis consisting of the vectors

$$X_1 = \frac{\partial}{\partial t}$$

$$X_2 = 2t\frac{\partial}{\partial t} + q_1\frac{\partial}{\partial q_1} + q_2\frac{\partial}{\partial q_2}$$

$$X_3 = t^2\frac{\partial}{\partial t} + tq_1\frac{\partial}{\partial q_1} + tq_2\frac{\partial}{\partial q_2}$$

$$X_4 = \left(\frac{1}{4}cq_1 + q_2\right)\frac{\partial}{\partial q_1} + \left(\frac{1}{4}cq_2 - q_1\right)\frac{\partial}{\partial q_2},$$

with bracket relations

$$[X_1, X_2] = 2X_1$$

$$[X_1, X_3] = X_2$$

$$[X_2, X_3] = 2X_3$$

$$[X_i, X_4] = 0, \quad i = 1, 2, 3.$$

This example generalizes in n dimension to Lie algebra which is a direct sum $sl(2, \mathrm{R}) \oplus so(n, \mathrm{R})$.

Remark5.23: in polar coordinates this system is

$$V = \frac{\lambda}{2}(r^2) + \frac{\mu}{r^2}\boldsymbol{e^{c\theta}}.$$

It is integrable, by taking $B(\theta) = \mu e^{c\theta}$ *and* $i = \frac{1}{2}l^2 + B(\theta)$, *where* $l = q_1 p_{2-} p_1 q_2$.

Taking $\Phi(\xi) = \frac{r_1}{\xi^2} + r_2$ we end up with the system

$$V = \frac{\lambda}{2}(q_1^2 + q_2^2) + \frac{r_1}{q_1^2} + \frac{r_2}{q_2^2}.$$

The associated Hamilton-Jacobi equation is integrable in Cartesian and polar coordinates. This system is an example of a system with closed trajectories under the influence of Non-central field [49]. For $\lambda = 1$, the generators take the form:

$$T = c_1 + c_2 cos2t + c_3 sin2t$$

$$Q_1 = (-c_2 sin2t + c_3 cos2t)q_1$$

$$Q_2 = (-c_2 sin2t + c_3 cos2t)q_2.$$

They form a 3-dimensional Lie algebra

$$T = \frac{\partial}{\partial t}$$

$$X_2 = cos2t\frac{\partial}{\partial t} - q_1 sin2t\frac{\partial}{\partial q_1} - \mathrm{q}_2\mathrm{sin2t}\frac{\partial}{\partial q_2}$$

$$X_3 = sin2t\frac{\partial}{\partial t} + q_1 cos2t\frac{\partial}{\partial q_1} + \mathrm{q_2cos2t}\frac{\partial}{\partial q_2},$$

with bracket relations

$$[X_1, X_2] = -2X_1$$

$$[X_1, X_3] = \ 2X_2. \qquad (5.82)$$

$$[X_2, X_3] = 2X_1$$

In other words, it is a simple Lie algebra of type A_1 isomorphic to $so(3, \mathrm{R})$. For $\lambda = 0$ we obtain a Lie algebra isomorphic to $sl(2, \mathrm{R})$. In [50] the most general form of a differential equation invariant under the action of the generators of $sl(2, \mathrm{R})$ is determined.

We note that the similar system

$$V = \frac{\lambda}{2}(4q_1^2 + q_2^2) + \frac{r_1}{q_1^2} + \frac{r_2}{q_2^2}. \qquad (5.83)$$

has $\frac{\partial}{\partial t}$ as the only symmetry .

In general, the system $V = \frac{\lambda}{2}(q_1^2 + q_2^2) + \frac{1}{q_2^2}\Phi(\frac{q_1}{q_2})$ is integrable. Changing to polar coordinates we put $q_1 = rcos\theta, q_2 = rsin\theta$ to find that

$$V = \frac{\lambda}{2}r^2 + \frac{\Phi(cot\theta)}{r^2 sin^2\theta} = \frac{\lambda}{2}r^2 + \frac{B(\theta)}{r^2}.$$

Letting $l = q_1p_2 - p_1q_2$, the second integral is $I = \frac{1}{2}l^2 + B(\theta)$.

The system $V = \frac{\lambda}{2}(q_1^2 + q_2^2) + \frac{\mu}{(q_1 + kq_2)^2}$ is integrable. It is special case of (5.76) with

$$\Phi(\xi) = \frac{\mu}{(\xi + k)^2}.$$

T linear $\quad a'' = 0$

In this case $a(t)$ is a linear function of time.

From equations (5.63) and (5.64) we deduce that the function $V(q_1, q_2)$ satisfies two partial differential equations of form

$$\lambda_1 V_{q_1 q_1} + \lambda_2 V_{q_1 q_2} + \lambda_3 = 0. \qquad (5.84)$$

$$\lambda_1 V_{q_1 q_2} + \lambda_1 V_{q_2 q_2} + \lambda_3 q_2 + \lambda_4 = 0. \qquad (5.85)$$

In order to solve these equations, we consider the following cases:

- $\lambda_1 \neq 0\,, \lambda_2 \neq o$
- $\lambda_1 = 0\,, \lambda_2 \neq o, (\text{or } \lambda_1 \neq 0\,, \lambda_2 = o)$
- $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = o$

Without giving any more details, using equations (5.61) and (5.62), we are led to the following results:

Sub-case 2d

$$V = \lambda_1 q_2^2 + \lambda_2 q_1 q_2 + \Phi(\xi)$$

where $\xi = q_1 - \lambda q_2$. We get various forms of the generators depending on the form of function Φ.

1. $\Phi = \lambda_3 \xi^2$:

That is, V is quadratic of the form

$$V = \lambda_1 q_2^2 + \lambda_2 q_1 q_2 + \lambda_3 q_1^2. \qquad (5.86)$$

We have followed the common practice of renaming the constants. The corresponding generators are:

$$T = c_6$$

$$Q_1 = c_1 q_2 + c_3 q_1 + d_1(\text{t}). \qquad (5.87)$$

$$Q_2 = c_1 q_1 + \left(2\left(\frac{\lambda_1}{\lambda_2} - \frac{\lambda_3}{\lambda_2}\right) c_1 + c_3\right) q_2 + d_2(\text{t})$$

where $d_1(\text{t})\ and\ d_2(\text{t})$ satisfy the O.D.E.'s $d_1'' + 2\lambda_3 d_1(\text{t}) + \lambda_2 d_2(\text{t}) = 0$, and $d_2'' + 2\lambda_1 d_2(\text{t}) + \lambda_2 d_1(\text{t}) = 0$. (A 7-parmeter group).

If $\lambda_2 = o$ then, $\lambda_1 \neq \lambda_3$, $Q_1 = c_3 q_1 + d_1(\mathrm{t}), Q_2 = c_4 q_2 + d_2(\mathrm{t})$ and $d_1(\mathrm{t}), d_2(\mathrm{t})$ satisfy the same O.D.E.'s with $\lambda_2 = o$.

We describe explicitly the Lie algebra for the potential

$V(q_1, q_2) = -\frac{1}{2} q_1^2 + \frac{1}{2} q_2^2$.

The Lie algebra is 7-dimensional with generators:

$$X_1 = \frac{\partial}{\partial t}$$

$$X_2 = q_1 \frac{\partial}{\partial q_1}$$

$$X_3 = e^t \frac{\partial}{\partial q_1}$$

$$X_4 = e^{-t} \frac{\partial}{\partial q_1}$$

$$X_5 = q_2 \frac{\partial}{\partial q_2}$$

$$X_6 = cost \frac{\partial}{\partial q_2}$$

$$X_7 = sint \frac{\partial}{\partial q_2}.$$

with (non-zero) bracket relations

$$[X_1, X_3] = X_3$$

$$[X_1, X_4] = -X_4$$

$$[X_1, X_6] = -X_7$$

$$[X_2, X_3] = -X_3$$

$$[X_1, X_7] = X_6$$

$$[X_1, X_4] = -X_4$$

$$[X_5, X_6] = -X_6$$

$$[X_5, X_7] = -X_7.$$

This Lie algebra L is solvable with $L^{(1)} = [L, L] = \{X_3, X_4, X_6, X_7\}$ and $L^{(2)} = \{0\}$.

Remark5.24: the system with Hamiltonian

$$H = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + \lambda_1 q_1^2 + \lambda_2 q_1 q_2 + \lambda_3 q_2^2. \qquad (5.88)$$

is integrable. We can actually rotate the Hamiltonian to a separable one, obtain the second integral and then rotate black to obtain the invariant in the original coordinates. So, we set

$$q_1 = cos\theta Q_x + sin\theta Q_y$$

$$q_2 = -sin\theta Q_x + cos\theta Q_y$$

$$p_1 = cos\theta p_x + sin\theta p_y$$

$$p_2 = -sin\theta p_x + cos\theta p_y.$$

The Hamiltonian H will be transformed to new Hamiltonian which is a function of Q_x, Q_y, p_x and p_y. The coefficient of $Q_x Q_y$ in the rotated Hamiltonian is

$$(\lambda_1 - \lambda_3) sin 2\theta + \lambda_2 cos 2\theta.$$

If $\lambda_1 = \lambda_3$, we choose $\theta = \frac{\pi}{4}$. If $\lambda_1 \neq \lambda_3$, then we choose θ to satisfy

$$tan 2\theta = \frac{\lambda_2}{\lambda_3 - \lambda_1}.$$

Therefore, in the new coordinates the Hamiltonian is separable of the form

$$\frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \mu_1 Q_x^2 + +\mu_2 Q_y^2.$$

We may choose the second integral to be $p_x^2 + \mu_1 Q_x^2$. The second integral for the original system is

$$I = (cos\theta p_1 - sin\theta p_2)^2 + \mu_1(cos\theta q_1 - sin\theta q_2)^2.$$

<u>2. Φ **arbitrary**</u>:

In this case, V has the form

$$V = \lambda_2\left(q_1 q_2 + \frac{1-\lambda^2}{2\lambda} q_2^2\right) + \Phi(q_1 - \lambda q_2). \qquad (5.89)$$

The corresponding generators are:

$$T = c_6$$

$$Q_1 = \lambda c_2 q_1 + c_2 q_2 + \lambda d_2(\mathrm{t}). \qquad (5.90)$$

$$Q_2 = c_2 q_1 + \frac{1}{\lambda} c_2 q_2 + d_2(\mathrm{t})$$

where $d_2(\mathrm{t})$ satisfies the O.D.E. $d_2'' + \frac{\lambda_2}{\lambda} d_2(t) = 0$. (A 4- parameter group).

Remark5.25: Assume λ=1. The system with Hamiltonian

$$H = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + \lambda_2 q_1 q_2 + \Phi(q_1 - q_2). \qquad (5.91)$$

$is\ integrable$. We can actually transform the Hamiltonian to a separable one, obtain the second integral and then rotate back to obtain the invariant in the original coordinates. So, we set

$$q_1 = \frac{1}{\sqrt{2}}Q_x + \frac{1}{\sqrt{2}}Q_y$$

$$q_2 = -\frac{1}{\sqrt{2}}Q_x + \frac{1}{\sqrt{2}}Q_y$$

$$p_1 = \frac{1}{\sqrt{2}}p_x + \frac{1}{\sqrt{2}}p_y$$

$$p_2 = -\frac{1}{\sqrt{2}}p_x + \frac{1}{\sqrt{2}}p_y.$$

The Hamiltonian H will be transformed to a new Hamiltonian which is a function of Q_x, Q_y, p_x

and p_y. Therefore, in the new coordinates the Hamiltonian is separable of the form

$$\frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \Phi(\sqrt{2}Q_x) + \frac{\lambda_2}{2}(Q_y^2 - Q_x^2).$$

We may choose the second integral to be $\frac{1}{2}p_x^2 + \Phi(\sqrt{2}Q_x) - \frac{\lambda_2}{2}Q_x^2$. The second integral for the original system is

$$I = \frac{1}{4}(p_1 - p_2)^2 + \Phi(q_1 - q_2) - \frac{\lambda_2}{4}(q_1 - q_2)^2.$$

3. $\Phi = \boldsymbol{\lambda_3\xi^n}, \boldsymbol{n \neq 0, 1, 2}$:

In this case, V has the form

$$V = \lambda_2\left(q_1q_2 + \frac{1-\lambda^2}{2\lambda}q_2^2\right) + \lambda_3(q_1 - \lambda q_2)^n. \quad (5.92)$$

with $n \neq 0, 1, 2$.

The generators are:

$$T = 2c_5t + c_6$$

$$Q_1 = c_2(\lambda q_1 + q_2) - \frac{4}{\mathrm{n}-2}\mathrm{c}_5q_1 + \lambda d_2(\mathrm{t}). \quad (5.93)$$

$$Q_2 = c_2\left(q_1 + \frac{1}{\lambda}q_2\right) - \frac{4}{\mathrm{n}-2}\mathrm{c}_5q_2 + \lambda d_2(\mathrm{t})$$

where $d_2(\mathrm{t})$ satisfies the O.D.E. $d_2'' + \frac{\lambda_2}{\lambda} d_2(t) = 0$.

If $\lambda_2 = 0$ then $d_2(\mathrm{t}) = c_3 + c_4 t$ and we end-up with a 5-parameter group. Note that for $n = -2$ we are in subcase 2a with a 6-parameter group.

If $\lambda_2 \neq 0$, then we set $c_5 = 0$ and we end up with a 4-parameter group (the same as Φ arbitrary).

For example, if $V(q_1, q_2) = -q_1 q_2 + (q_1 - q_2)^3$ then the Lie algebra is generated by

$$X_1 = \frac{\partial}{\partial t}$$

$$X_2 = (q_1 + q_2)\left(\frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2}\right)$$

$$X_3 = cost\left(\frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2}\right)$$

$$X_4 = sint\left(\frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2}\right)$$

with (nonzero) bracket relations

$$[X_1, X_3] = -X_4$$

$$[X_1, X_4] = X_3$$

$$[X_2, X_3] = -2X_3$$

$$[X_2, X_4] = -2X_4.$$

This Lie algebra L is solvable with $L^{(1)} = [L, L] = [X_3, X_4]\ and\ L^{(2)} = \{0\}$.

On the other hand, the potential $V(q_1, q_2) = (q_1 - q_2)^3$ gives a five dimensional Lie algebra.

This Lie algebra is isomorphic with the symmetry Lie algebra for the potential $V(q_1, q_2) = {q_1}^3$ which we examine later.

$4. \Phi = \boldsymbol{\lambda_3 e^{\mu\xi}}$:

In this case, V has the form

$$V = \lambda_2\left(q_1q_2 + \frac{1-\lambda^2}{2\lambda}q_2^2\right) + \lambda_3\boldsymbol{e^{\mu(q_1-\lambda q_2)}}\,. \qquad (5.94)$$

The generators are:

$$T = 2c_5t + c_6$$

$$Q_1 = c_2(\lambda q_1 + q_2) - \frac{4}{\mu}\mathrm{c}_5 + \lambda d_2(\mathrm{t}). \qquad (5.95)$$

$$Q_2 = c_2\left(q_1 + \frac{1}{\lambda}q_2\right) + \lambda d_2(\mathrm{t})$$

where $d_2(\mathrm{t})$ satisfies the O.D.E. $d_2'' + \frac{\lambda_2}{\lambda}d_2(t) = 0$.

If $\lambda_2 = 0$ then $d_2(\mathrm{t}) = c_3 + c_4t$ and we end-up with a 5-parameter group.

If $\lambda_2 \neq 0$ then, we set $c_5 = 0$ and we end up with a 4-parameter group (the same as Φ arbitrary).

The case $\lambda = \mu = 1\ and\ \lambda_2 = 0$ is the Toda Lattice, a well-known integrable system [51], [52]. We will calculate the Lie algebra of symmetries for the potential of the Toda Lattice $V(q_1, q_2) = c^{q_1-q_2}$. We obtain a five dimensional Lie algebra with generators

$$X_1 = \frac{\partial}{\partial t}$$

$$X_2 = 2t\frac{\partial}{\partial t} - 4\frac{\partial}{\partial q_1}$$

$$X_3 = (q_1 + q_2)\left(\frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2}\right)$$

$$X_4 = \left(\frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2}\right)$$

$$X_5 = t\left(\frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2}\right),$$

with (nonzero) bracket relations

$$[X_1, X_2] = 2X_1$$

$$[X_1, X_5] = X_4$$

$$[X_2, X_3] = -4X_4$$

$$[X_2, X_5] = 2X_5$$

$$[X_3, X_4] = -2X_4$$

$$[X_3, X_5] = -2X_5$$

This Lie algebra L is solvable with $L^{(1)} = [L, L] = [X_1, X_4, X_5\ ]\ , L^{(2)} = \{X_4\}\ and\ L^{(3)} = \{0\}$.

For the case $\lambda_2 \neq 0$ we obtain a Lie algebra which is identical with the one in Φ arbitrary.

5. $\Phi = \boldsymbol{\lambda_3}\mathbf{log}\boldsymbol{\xi}$ **:**

Setting $\lambda_2 = 0$, V taken the form

$$V = \lambda_3 \log(q_1 - \lambda q_2). \tag{5.96}$$

The generators are:

$$T = 2c_5 t + c_6$$

$$Q_1 = \lambda c_2 q_1 + c_2 q_2 + 2\mathrm{c}_5 q_1 + \lambda d_2(\mathrm{t}). \tag{5.97}$$

$$Q_2 = c_2 q_1 + \tfrac{1}{\lambda} c_2 q_2 + 2\mathrm{c}_5 q_2 + d_2(\mathrm{t})$$

where $d_2(\mathrm{t}) = c_3 t + c_4$ (A 5-parameter group)

If $\lambda_2 \neq 0$, the result again is the same as Φ arbitrary.

Sub-case 2e $\boldsymbol{V = \lambda_1 q_2^2 + \Phi(q_1)}$.

We obtain various forms of the generators depending on the form of the function Φ.

1. Φ arbitrary :

This case, the generators has the form:

$$
\begin{aligned}
T &= c_6 \\
Q_1 &= 0. \\
Q_2 &= c_4 q_2 + d_2(\mathrm{t})
\end{aligned}
\tag{5.98}
$$

where $d_2(\mathrm{t})$ satisfies the O.D.E. $d_2'' + 2\lambda_1 d_2(t) = 0$. (A 4-parameter group).

2. $\boldsymbol{\Phi = \lambda_2 q_1^n}$**,** $\boldsymbol{n \neq -2, o, 1, 2}$**:**

If $\lambda_1 = 0$, then

$$
\begin{aligned}
T &= 2c_5 t + c_6 \\
Q_1 &= \frac{4}{2-\mathrm{n}} \mathrm{c}_5 q_1. \\
Q_2 &= c_4 q_2 + \mathrm{c}_1 t + c_2
\end{aligned}
\tag{5.99}
$$

(A 5-parameter group).

If $\lambda_1 \neq 0$, we set $\mathrm{c}_5 = 0$. We end up with a 4-parameter group. It is the same as in Φ arbitrary.

We will calculate explicitly the Lie algebra for the potential $V(q_1, q_2) = {q_1}^3$. for a basis we choose the following five vector fields:

$$X_1 = \frac{\partial}{\partial t}$$

$$X_2 = t\frac{\partial}{\partial t} - 2q_1\frac{\partial}{\partial q_1}$$

$$X_3 = q_2 \frac{\partial}{\partial q_2}$$

$$X_4 = t \frac{\partial}{\partial q_2}$$

$$X_5 = \frac{\partial}{\partial q_2},$$

with (nonzero)bracket relations

$$[X_1, X_2] = X_1$$

$$[X_1, X_4] = X_5$$

$$[X_2, X_4] = X_4$$

$$[X_3, X_4] = -X_4$$

$$[X_3, X_5] = X_5$$

This Lie algebra L is solvable with $L^{(1)} = [L, L] = [X_1, X_4, X_5\ ]$, $L^{(2)} = \{X_5\}\ and\ L^{(3)} = \{0\}$.

On the other hand, for the potential $V(q_1, q_2) = \frac{1}{2} q_2^2 + q_1^3$ we obtain a 4-parameter group with a basis

$$X_1 = \frac{\partial}{\partial t}$$

$$X_2 = q_2 \frac{\partial}{\partial q_2}$$

$$X_3 = cost \frac{\partial}{\partial q_2}$$

$$X_4 = sint \frac{\partial}{\partial q_2}$$

This Lie algebra L is also solvable with $L^{(1)} = [L, L] = [X_3, X_4\ ]\ and\ L^{(2)} = \{0\}$. It is isomorphic with Lie algebra of symmetries of potential (5.92) which we already examined.

3. $\Phi = \boldsymbol{\lambda_2 e^{\mu q_1}}$:

If $\lambda_1 = 0$, then

$$T = 2c_5 t + c_6$$

$$Q_1 = \frac{-4}{\mu} c_5. \qquad (5.100)$$

$$Q_2 = c_4 q_2 + d_2(t)$$

where d_2(t) satisfies the O.D.E. $d_2'' + 2\lambda_1 d_2(t) = 0$. (A 5-parameter group).

If $\lambda_1 \neq 0$, we set $c_5 = 0$. It is the same as in Φ arbitrary.

4. $\Phi = \boldsymbol{\lambda_2 \log q_1}$:

We set $\lambda_1 = 0$. Then $V = \lambda_2 \log q_1$ and the generators have the form:

$$T = 2c_5 t + c_6$$

$$Q_1 = 2c_5 q_1 \qquad (5.101)$$

$$Q_2 = c_4 q_2 + c_7 t + c_8,$$

(a 5-parameter group).

Remark5.26: the potentials that appear in his case are clearly integrable, being separable potentials. At this point we have completed the analysis of a separable potential with one variable missing. The potential $\frac{1}{q_1^2}$ was considered in subcase 2b. Potentials q_1^n for $n = o, 1$ are covered by case1. The potential q_1^2 was considered in subcase 2d. The potential $f(q_1)$ falls under subcae 2e.

Sub-case 2f:

Equations (5.63) and (5.64) are satisfied ($d_1(t) = constant$, $d_2(t) = constant$). From equations (5.61) and (5.62) we obtain the following results:

$$V = q_2^N \Phi\left(\frac{q_1}{q_2}\right)$$

$$T = \frac{1}{2}c_3(2-N)t + c_6$$

$$Q_1 = c_3 q_1 \qquad (5.102)$$

$$Q_2 = c_3 q_2$$

a 2-parameter group of transformations. The Lie algebra in this case is the two- dimensional non-abelian Lie algebra with bracket $[X_1, X_2] = \frac{1}{2}(2-N)X_1$ if $N \neq 2$ and an abelian 2-dimensinal Lie algebra if $N = 2$. We should mention that for certain choice of Φ we may obtain a larger symmetry group, e.g. for $\Phi(\boldsymbol{x}) = \boldsymbol{x^N}$, but generically the Lie algebra is 2-dimensional. Some values of N will also give different results. For example, N = -2 falls under sub-case 2c.

Remark5.27: in general, this system is not integrable. However, there are some integrable examples. We mention the Holt potentials [53], [54], [55]

$$V(q_1, q_2) = \frac{1}{2}q_2^{-\frac{2}{3}}(cq_2^2 + q_1^2). \qquad (5.103)$$

where $c = \frac{3}{4}, c = \frac{9}{2}$ and $c = 12$.

Also, the Fokos_ Lagerström potential [56].

$$V(q_1, q_2) = \frac{1}{(q_1^2 - q_2^2)^{-\frac{2}{3}}}. \qquad (5.104)$$

Case 2f includes $\mathrm{Henon-Heiles}$ type potential of the form, $cq_2^3 + q_1^2 q_2$. They are $integrable$ for the following values of c: $c = \frac{1}{3}, c = 2$ and $c = \frac{16}{3}$ [57], [58], [59].

Finally we mention the potential

$$V(q_1, q_2) = \frac{q_1}{q_2}. \qquad (5.105)$$

It was shown by Hietarinta [60] that the second integral for this potential is a transcendental functions i.e. the solutions of the equation

1. $y'' + \left(\frac{1}{4}x^2 - a\right)y = 0.$ (5.106)

2 $V = \lambda_1 \boldsymbol{log} q_2 + \Phi\left(\frac{q_1}{q_2}\right)$

$$T = c_3 t + c_6$$

$$Q_1 = c_3 q_1 \qquad (5.107)$$

$$Q_2 = c_3 q_2$$

a 2-parameter group of transformations. The Lie algebra in this case the 2-dimensional non-abelian Lie algebra with bracket $[X_1, X_2\ ] = X_1$.

3. $V = e^{\mu q_1}\Phi(q_2)$

$$T = \frac{1}{2}c_5 t + c_6$$

$$Q_1 = \frac{-c_5}{\mu} \qquad (5.108)$$

$$Q_2 = 0$$

a 2-parameter group of transformations.

Remark5.28: Taking $\Phi(q_2) = e^{-\mu q_2}$ we obtain again the Toda lattice. However, we already have seen that this system has a 5-parameter group of transformations. Generically, the symmetry group is a 2-dimensional. For example, taking $V(q_1, q_2) = e^{q_1}q_2^3$ gives a two-dimensional non-abelian algebra with basis $X_1 = \frac{\partial}{\partial t}$ and $X_1 = t\frac{\partial}{\partial t} - 2\frac{\partial}{\partial q_1}$.

4. $V = e^{\mu q_1}\Phi(q_1 - \lambda q_2)$

$$T = -\frac{1}{2}\lambda\mu c_8 t + c_6$$

$$Q_1 = \lambda c_8 \qquad (5.109)$$

$$Q_2 = c_8$$

a 2-parameter group of transformations. The Lie algebra is again the 2-dimensional non-abelian Lie algebra with bracket $[X_1, X_2] = \frac{1}{2}\lambda\mu X_1$.

Remark5.29: we should mention that because of symmetry we do not list potentials of the form $V((q_1, q_2) = e^{\mu q_2}\Phi(q_2 - \lambda q_1)$. We can also replace $q_1 - \lambda q_2$ with $\alpha q_1 + \beta q_{2.}$ Taking $\mu = 1, \alpha = 1$ and $\beta = -2$ we obtain the potential $V((q_1, q_2) = e^{q_1 - q_2} + e^{q_2}$. This is a generalized Toda Lattice associated with a Lie algebra of type B_2, first considered by $Bogoyavlensky$ in [61]

5. $V = \lambda_1 q_1^n + \lambda_2 q_2^m$

$$T = \frac{1}{2}c_5 t + c_6$$

$$Q_1 = \frac{c_5}{2-n} q_1 \qquad (5.110)$$

$$Q_2 = \frac{c_5}{2-m} q_2,$$

a 2-parameter group of transformations. Here, $n \neq 0, 1, 2$ and $m \neq 0,1,2$ and n, m not both equal to -2. The Lie algebra in this case is the 2-dimensional non-abelian Lie algebra with bracket $[X_1, X_2] = \frac{1}{2}X_1$. The symmetry Lie algebra for the potentials 6-10satisfies precisely same bracket relation.

6. $V = \lambda_1 q_1^n + \lambda_2 \log q_2, n \neq 0, 1, 2$

$$T = \frac{1}{2}c_5 t + c_6$$

$$Q_1 = \frac{c_5}{2-n} q_1 \qquad (5.111)$$

$$Q_2 = \frac{c_5}{2} q_2,$$

7. $$V = \lambda_1 q_1^n + \lambda_2 e^{\mu q_2}$$

$$T = \frac{1}{2} c_5 t + c_6$$

$$Q_1 = \frac{c_5}{2-n} q_1 \qquad (5.112)$$

$$Q_2 = -\frac{c_5}{\mu},$$

8. $$V = \lambda_1 \log q_1 + \lambda_2 \log q_2 \,, n \neq 0, 1, 2$$

$$T = \frac{1}{2} c_5 t + c_6$$

$$Q_1 = \frac{c_5}{2} q_1 \qquad (5.113)$$

$$Q_2 = \frac{c_5}{2} q_2,$$

9. $$V = \lambda_1 \log q_1 + \lambda_2 e^{\mu q_2}$$

$$T = \frac{1}{2} c_5 t + c_6$$

$$Q_1 = \frac{c_5}{2} q_1 \qquad (5.114)$$

$$Q_2 = -\frac{c_5}{\mu},$$

10. $$V = \lambda_1 e^{\mu_1 q_1} + \lambda_2 e^{\mu_2 q_2}$$

$$T = \frac{1}{2} c_5 t + c_6$$

$$Q_1 = -\frac{c_5}{\mu_1} \qquad (5.115)$$

$$Q_2 = -\frac{c_5}{\mu_2},$$

11. $V = \Phi(q_1^2 + q_2^2)$

$T = c_6$

$Q_1 = c_1 q_2$ (5.116)

$Q_2 = -c_1 q_1,$

a 2-parameter group of transformations. This is a unit mass 2-dimensional space moving in a central field, i.e. a potential which is a function of r only. The function $q_1 p_2 - q_2 p_1$ is a second integral. Note that in this case the Lie algebra is abelian.

12. $V = \lambda(q_1^2 + q_2^2)^n, \qquad n \neq -1, 0, 1$

$T = \frac{1}{2} c_5 t + c_6$

$Q_1 = c_1 q_2 - \frac{2}{n-1} c_5 q_1$ (5.117)

$$Q_2 = -c_1 q_1 - \frac{2}{n-1} c_5 q_2,$$

a 3-parameter group of transformations. This Lie algebra is 3-dimensional with only non-zero bracket $[X_1, X_2] = 2X_1$. The case $n = -\frac{1}{2}$ is a Kepler problem. For $n = -1$ the Lie algebra is 4-dimensional, it falls under sub-case 2c. See (5.80) with λ= c =0 and $\mu = 1$.

Remark5.30: This potential is special case of system 1 with $N = 2n$. Taking $n = 2$, we have a system of the form a $q_1^4 + bq_1^2 q_2^2 + cq_2^4$. In general (for $a, b\ c\ non-zero$) this system has a 2-dimensional group of symmetries unless $b = 2a = 2c$. Generically the potential $V(q_1, q_2) = aq_1^4 + bq_1^2 q_2^2 + cq_2^4$ is not integrable, but for certain values of the parameters it becomes integrable. That is the case when $b = 6a = 6c$ or $a = 16c, b = 12c$ or $b = 6a$, $c = 8a$. [53], [62], [63].

13. $V = \lambda \log(q_1^2 + q_2^2)$

$$T = 2c_5 t + c_6$$

$$Q_1 = c_1 q_2 + 2c_5 q_1 \qquad (5.118)$$

$$Q_2 = -c_1 q_1 + 2c_5 q_2,$$

a 3-parameter group of transformations. The Lie algebra, which is the same as in the pervious case, may not seem interesting, but in n-dimensions it is a direct sum of a 2-dimensional Lie algebra with $so(n, R)$.

14. $V = \lambda \sin^{-1} \frac{q_1^2 - q_2^2}{q_1^2 + q_2^2}$

$$T = 2c_5 t + c_6$$

$$Q_1 = c_1 q_2 + 2c_5 q_1 \qquad (5.119)$$

$$Q_2 = -c_1 q_1 + 2c_5 q_2,$$

a 3-parameter group of transformations. This potential can be written in the form $\lambda_1 + \lambda_2 \theta$, in polar coordinates. In other words, it is a linear function of θ.

5.5.3. Generalizations Systems

We consider a Hamiltonian with n degrees of freedom, in n-dimensions

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V(q_1, q_2, \dots, q_n),$$

and the associated Lagrange-Newton equations

$$\ddot{q}_i + V_{q_i} = 0\ , i = 1,2, \dots, n. \qquad (5.120)$$

As in the case of two degrees of freedom we seek point symmetries of equations (5.120). We consider the equations

$$\Gamma^{(2)}\{\ddot{q}_i + V_{q_i}\} = 0\ , \quad i = 1, 2, \dots, n\ , \qquad (5.121)$$

where $\Gamma^{(2)}$ is the second prolongation of

$$\Gamma = T\frac{\partial}{\partial t} + \sum_{i=1}^{n} Q_i \frac{\partial}{\partial q_i}\ . \qquad (5.122)$$

Equations (5.121) give n identities of the form:

$$E_i(\ t, q_1, q_2, \dots, q_n, \dot{q}_{1,}\dot{q}_2, \dots, \dot{q}_n) = 0\ ., \quad i = 1, 2, \dots, n. \qquad (5.123)$$

where, we have used that $\ddot{q}_i = \frac{\partial \mathrm{V}}{\partial \mathrm{q_i}}$. The functions E_i are explicit polynomials in $\dot{q}_{1,}\dot{q}_2, \dots, \dot{q}_n$.

We impose the condition that equations (5.123) are identities in the variables t, q_i, $\dot{q}_i$ which are regarded as independent.

Again, the functions T and Q_i must be of the form

$$T = a(t) + \sum_{i=1}^{n} b_i(t)\, q_i. \qquad (5.124)$$

$$Q_i = \sum_{k=1}^{n} b_k'(t)\, q_i q_k + \sum_{k=1}^{n} c_{ik}(t)\, q_k + d_i(t), \qquad i = 1,2,\dots,n$$

We substitute (5.124) into (5.123). By considering the coefficient of $\dot{q}_k$ in E_j we obtain the following n^2 equations:

For $j \neq k$,

$$b_k''(t) q_j + c_{jk}'(t) + \frac{\partial V}{\partial q_j} b_k = 0. \qquad (5.125)$$

For $j = k$,

$$2c_{jj}'(t) - a''(t) + 3b_j''(t) q_j + 3\frac{\partial V}{\partial q_j} b_j + \sum_{i \neq j} \left(b_i''(t) q_i + \frac{\partial V}{\partial q_i} b_i(t) \right) = 0\,. \qquad (5.126)$$

It follows from equations (5.125) that V is quadratic of the form

$$V = \sum_{i=1}^{n} \lambda_i\, q_i^2 + \sum_{i=1}^{n} \mu_i\, q_i\,. \qquad (5.127)$$

unless $b_i(t) = 0$ for $i = 1,2,\dots,n$.

Substituting (5.127) into (5.126) we obtain

$b_i'' + 2\lambda_i b_i = 0, \qquad i = 2,3,\dots.,n$

and

$b_i'' + 2\lambda_1 b_i = 0, \qquad i = 2,3,\dots.,n.$

Therefore for non-zero $b_i(t)$, we necessarily have

$\lambda_1 = \lambda_2 = \lambda_3 = \cdots = \lambda_n$.

Hence, V is of the form

$$V = \frac{\lambda}{2}\sum_{i=1}^{n} q_i^2,$$

where, the linear terms are ignored.

We can easily deduce the form of the generators:

$a(t)$, is a solution of a second order equation of the form $a'' + 4\lambda a = c$. (3-prameters).

$b_i(t)$, is a solution of $b_i'' + \lambda b_i = 0.\ (\ 2n - \text{parameters})$

$d_i(t)$, is a solution of $d_i'' + \lambda d_i = 0.\ (\ 2n - \text{parameters})$

$c_{ij}(t)$, are constant for $i \neq j$ and $c_{kk} = ct + c_k - 2\lambda \int a(t)dt.\ (n^2 - \text{parameters})$

Therefore, the dimension of the symmetry algebra is $3 + 2n + 2n + n^2 = (n+2)^2 - 1$.

When $\lambda = 0$, the potential energy is zero and we have a free particle moving in R^n. In this case the generators take the following simple form:

$$\left.\begin{aligned} a(t) &= c_1 + c_2 t + ct^2 \\ b_i(t) &= \alpha_1 + \beta_1 t \\ d_i(t) &= \gamma_1 + \delta_1 t \\ c_{ii} &= \epsilon_i + ct \\ c_{ij} &= k_{ij},\ i \neq j \end{aligned}\right\}, \tag{5.128}$$

The dimension is again $(n+2)^2 - 1$.

This dimension is in agreement with the results in[64], where upper bounds for dimension of the symmetry groups are obtained.

In case 2, $b_i(t) = 0$ for $i = 1, 2, .., n$ and equation (5.126) and (5.127) imply that

$$c_{jk}(t) = \ \ k_{ij} for\ j \neq k$$

$$c_{jj}(t) = \ \ \ \frac{1}{2}a'(t) + c_{jj,}$$

where c_{jk} are, constants. Equations (5.123) now become

$$\frac{1}{2}a'''(t)q_i + d_i''(t) + \sum_{k=1}^{n} \frac{\partial^2 V}{\partial q_i \partial q_k}\gamma_k + \frac{3}{2}a'(t)\frac{\partial V}{\partial q_i} - \sum_{k=1}^{n} c_{ik}\frac{\partial V}{\partial q_k} = 0. \quad (5.129)$$

for $i = 1, 2, .., n$, where

$$\gamma_k = \frac{1}{2}q_k a'(t) + \sum_{j=1}^{n} c_{kj}q_j + d_k(t). \quad (5.130)$$

Chapter Six

Applications of Symmetries of Lagrangian and Hamiltonian Systems

6.1 Particle in Rotationally Invariant Potential

Consider the kinetic and potential energies given by

$$T = \frac{1}{2}m\dot{\vec{r}}^2, \quad V = V(r), \qquad (6.1)$$

which give the following Lagrangian in Cartesian coordinates

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V\left(\sqrt{x^2 + y^2 + z^2}\right), \quad (6.2)$$

and in polar coordinates

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 sin^2\theta\dot{\emptyset}^2\right) - V(r), \quad (6.3)$$

The system is clearly symmetric under all rotations about the origin (the center of the potential), but we note that expressed in Cartesian coordinates there is no cyclic coordinate corresponding to these symmetries. In polar coordinates there is a cyclic coordinate, Ø. The corresponding conserved quantity is the conjugate momentum

$$p_\emptyset = \frac{\partial L}{\partial \dot{\emptyset}} = mr^2 sin^2\theta\dot{\emptyset}, \qquad (6.4)$$

and the physical interpretation of $p_\emptyset$ is as the z-component of the angular momentum

$$\left(m\vec{r}\times\dot{\vec{r}}\right)_z = m(x\dot{y}-y\dot{x}) = mr^2 sin^2\theta\dot{\emptyset}, \qquad (6.5)$$

Clearly also the other components of the angular momentum are conserved, but there is no cyclic coordinate corresponding to these components.

We use the expression derived to find the conserved quantities associated with the rotational symmetry. First we note that an infinitesimal rotation can be expressed in the form

$$\vec{r} \rightarrow \vec{r}\,' = \vec{r} + \delta\vec{\alpha}\times\vec{r}, \qquad (6.6)$$

or

$$\delta\vec{r} = \delta\vec{\alpha}\times\vec{r}\,, \qquad (6.7)$$

where the direction of the vector $\delta\vec{\alpha}$ specifies the direction of the axis of rotation and the absolute value $\delta\alpha$ specifies the angle of rotation.

We can explicitly verify that to first order in $\delta\vec{\alpha}$ the transformation (6.6) leaves $\vec{r}^2$ unchanged. Since the velocity $\dot{\vec{r}}$ transforms in the same way (by time derivative of (6.6)) also, $\dot{\vec{r}}^2$ is invariant transformation. Consequently, the Lagrangian is invariant under the infinitesimal rotations (6.6), which is therefore (as we already should know) a symmetry transformation of the system.

By use of the expression $K = \sum_k \frac{\partial L}{\partial \dot{q}_k}\delta q_k$ we find the following expression for the conserved quantity associates with the symmetry transformation,

$$K = \sum_{k=1}^{3}\frac{\partial L}{\partial \dot{x}_k}\delta x_k = m\dot{\vec{r}}.\delta r = m\left(r\times\dot{\vec{r}}\right).\delta\vec{\emptyset}, \qquad (6.8)$$

Since this quantity is conserved for arbitrary, $\delta\vec{\emptyset}$, we conclude that the vector quantity

$$\vec{l} = m\vec{r} \times \dot{\vec{r}}, \qquad (6.9)$$

is conserved. This demonstrates that the general expression we found for a constant of motion reproduces, as expected, the angular momentum as a constant of motion when particle moves in a rotationally invariant potential.

6.2 Conservation of the Total Linear and Angular Momentum

We consider a system of N particles interacting together with potential forces depending on the distances of the particles. This is a Hamiltonian system with total energy

$$H(p,q) = \frac{1}{2}\sum_{i=1}^{N}\frac{1}{m_i}p_i^T p_i + \sum_{i=2}^{N}\sum_{j=1}^{i-1} V_{ij}\ \left(\|q_i - q_j\|\right), \quad (6.10)$$

The $q_i, p_i \in R^3$ represent the position and momentum of the ith particle of mass m_i and $V_{ij}(r)(i > j)$ is the interaction potential between ith and jth particle. The equations of motion read

$$\dot{q}_i = \frac{1}{m_i}p_i, \qquad \dot{p}_i = \sum_{j=1}^{N} v_{ij}(q_i - q_j)\ , \qquad (6.11)$$

Here, for, $i > j$, we have $v_{ij} = v_{ji} = V'_{ij}(r_{ij})/r_{ij}\ \ with\, r_{ij} = \|q_i - q_j\|$. the conservation of the total linear and angular momentum

$$P = \sum_{i=1}^{N} p_i,\ \ L = \sum_{i=1}^{N} q_i \times p_i, \qquad (6.12)$$

is a consequence of the symmetry relation $v_{ij} = v_{ji}$.

6.3 Space-Time Translations

Consider the following infinitesimal space-time translation:

$$x^{\mu} \rightarrow x'^{\mu} = x^{\mu} - \epsilon^{\mu}, \qquad (6.13)$$

with ϵ^{μ} real constants. For scalar or spinor fields we have $\emptyset_i(x) = \emptyset_i'(x')$ thus $\bar{\delta}\emptyset_i(x) = 0$. under this type of transformation our Lagrangians remain unchanged so $\mathcal{L}'(x') = \boldsymbol{\mathcal{L}(x)}$, therefore, by taking $\bar{\delta}\boldsymbol{\mathcal{L}}(\boldsymbol{x}) - \partial_{\mu}b^{\mu}(x) = 0$ we conclude that $\partial_{\mu}b^{\mu}(x) = 0$. we can thus, eliminate the b^{μ} term from (3.65) i.e. $\partial_{\mu}j^{\mu}(x) = 0$. $where\ j^{\mu}(x) = \frac{\partial\mathcal{L}(x)}{\partial[\partial_{\mu}\emptyset_i(x)]}\delta\emptyset_i(x) + [\boldsymbol{\mathcal{L}}(\boldsymbol{x})\Delta x^{\mu} - b^{\mu}(x)]$. and the conserved current is simply given by

$$j^{\mu} = \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\emptyset_i)}\partial_v\emptyset_i\epsilon^v - \boldsymbol{\mathcal{L}\epsilon^{\mu}} = \left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\emptyset_i)}\partial^v\emptyset_i - \boldsymbol{\mathcal{L}g^{\mu v}}\right)\epsilon_v, (6.14)$$

The conservation law $\partial_{\mu}j^{\mu} = 0$ holds for any arbitrary constants ϵ_v, therefore, we actually have four conserved currents:

$$\partial_{\mu} \ = 0, \quad T^{\mu v} = \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\emptyset_i)}\partial^v\emptyset_i - \boldsymbol{\mathcal{L}g^{\mu v}}, \qquad (\mathbf{6}.15)$$

with $T^{\mu v}$ the four- momentum tensor. The conserved Noether charges are given by

$$p^v = \int d^3\,xT^{0v} = \int d^3\,x\left(\frac{\partial\mathcal{L}}{\partial\dot{\emptyset}_i}\partial^v\emptyset_i - g^{0v}\mathcal{L}\right)$$

$$= \int d^3\,x(\pi_i\partial^v\emptyset_i - g^{0v}\mathcal{L})$$

$$= \int d^3 x\left(\pi_i \dot{\emptyset}_\iota - g^{00}\mathcal{L} - \pi_i \nabla \emptyset_i\right) = \int d^3 x(\mathcal{H}, \mathcal{P}) = (H, p), \qquad (6.16)$$

where we have used $\partial^\mu \equiv g^{\mu v}\partial_v = (\partial_t, -\nabla) \equiv (\partial_t, -\partial_k)$. As we can see, the conserved charges are the Hamiltonian and three-momentum operators.

6.4 Conservative Force and Hamiltonian Vector Field

If the forces on a particle are conservative, the Hamiltonian is

$$H = \frac{p^2}{2m} + U(r), \qquad (6.17)$$

Where the first term is the kinetic energy and the second the potential energy. We use the canonical symplectic form $\omega = dp \wedge dr$. let us work out explicitly what X_H is. We denote

$$X_H = \sum_{i=1}^{3}\left(a_i(r,p)\frac{\partial}{\partial r_i} + b_i(r,p)\frac{\partial}{\partial p_i}\right) \coloneqq a(r,p)\frac{\partial}{\partial r} + b(r,p)\frac{\partial}{\partial p}, \quad (6.18)$$

Then

$$-\omega(X_H, .) = dr \wedge dp\left(a(r,p)\frac{\partial}{\partial r} + b(r,p)\frac{\partial}{\partial p}, .\right) = a(r,p)dp - b(r,p)dr, \qquad (6.19)$$

We then calculate dH as

$$dH = \frac{\partial H}{\partial r}dr + \frac{\partial H}{\partial p}dp = \nabla U dr + \frac{p}{m}dp, \qquad (6.20)$$

Setting the coefficient functions of dH and $-\omega(X_H,.)$ equal to each other yields

$$dH = \frac{p}{m}\frac{\partial}{\partial r} - \nabla U \frac{\partial}{\partial p}, \qquad (6.21)$$

Now let us suppose that $\gamma'(t) = X_H(\gamma(t))$. Likewise if $\gamma(t)$ satisfies $\gamma'^{(t)} = X_H(\gamma(t))$, then $\gamma(t)$ solves the equations of motion. This demonstrates that the symplectic form encodes the equations of motion into our Hamiltonian system, and that the vector field X_H points along physical trajectories. When dealing with non-conservative forces, such as magnetic forces, the symplectic form is not always the canonical one and Hamiltonian function does not always correspond to total energy. Ultimately the goal is to choose $\omega \; and \; H$ so as to best encode into our system the equations of motion given by Newton's second law.

Despite the ambiguity in choice of $\omega \; and \; H$, the vector field X_H always points along the direction of constant H, i.e. X_H is tangent to the level sets of H. We take advantage of the partial differentiation notation for X_H to define

$$X_H H = a(r,p)\frac{\partial H}{\partial r} + b(r,p)\frac{\partial H}{\partial p} = dH(X_H), \qquad (6.22)$$

6.5 Simple Harmonic Motion

Consider a particle of mass m moving in a one dimensional Hookeian force field $-kx$, where k is a constant the potential function $V = V(x)$ corresponding to this force field satisfies

$$-\frac{\partial V}{\partial x} = -kx \; \Leftrightarrow V(x) - (0) = \int_0^x k\xi d\xi \; \Leftrightarrow V(x) = \frac{1}{2}kx^2. \; (6.23)$$

The Lagrangian $L = T - V$ is thus given by

$$L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2, \qquad (6.24)$$

From Hamilton's principle the equations of motion are given by Lagrange's equations. Here, taking the generalized coordinate to be $q = x$, the single Lagrange equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0.$$

Using the form for the Lagrangian above we find that

$$\frac{\partial L}{\partial x} = -kx\,, \qquad and\ \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \qquad (6.25)$$

And so Lagrange's equation of motion becomes

$$m\ddot{x} + kx = 0, \qquad (6\ 26)$$

6.6 Kepler Problem

The kepler problem for a mass m moving in an inverse-square central force field with characteristic coefficient μ. The Lagrangian $L = T = V$ is

$$L(r, \dot{r}, \theta, \dot{\theta}) = \frac{1}{2}m\left(\dot{r}^2 + r^2\,\dot{\theta}^2\right) + \frac{\mu m}{r}, \qquad (6.27)$$

Hence the generalized momenta are

$$p_r = \frac{\partial L}{\partial \dot r} = m\dot r \;\; and \;\; p_\theta = \frac{\partial L}{\partial \dot\theta} = mr^2\dot\theta \,, \tag{6.28}$$

These imply $\dot r = p_r/m \quad and \ \dot\theta = p_\theta/mr^2$ and so the Hamiltonian is given by

$$H(r,\theta,p_r,p_\theta) = \dot r\, p_r + \dot\theta p_\theta - L(r,\dot r,\theta,\dot\theta)$$

$$= \frac{1}{m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \left(\frac{1}{2}m\left(\frac{p_r^2}{m^2} + r^2\frac{p_\theta^2}{m^2r^4}\right) + \frac{\mu m}{r}\right)$$
$$= \frac{1}{2}\left(\frac{p_r^2}{m} + r^2\frac{p_\theta^2}{mr^2}\right) - \frac{\mu m}{r}, \tag{6.29}$$

which in this case is also the total energy. Hamilton's equations of motion are

$$\dot r = \partial H/\partial p_r \;\Leftrightarrow\; \dot r = p_r/m$$

$$\dot\theta = \partial H/\partial p_\theta \;\Leftrightarrow\; \dot\theta = p_\theta/m\,r^2$$

$$\dot p_r = -\partial H/\partial r \;\Leftrightarrow\; \dot p_r = p_\theta^2/mr^3 - \mu m/r^2$$

$$\dot p_\theta = -\,\partial H/\partial\theta \;\Leftrightarrow\; \dot p_\theta = 0.$$

Note: for $\dot p_\theta = 0$, we have that p_θ is constant for the motion. This property corresponds to the conservation of angular momentum.

6.7 An Optimal Control Problem

Consider the following distributional system in $\mathbb{R}^2$(driftless control affine system):

$$\left.\begin{aligned} \dot x^1 &= u^1 + u^2x^1 \\ \dot x^2 &= u^2 \end{aligned}\right\}, \tag{6.30}$$

Let x_0 and x_1 be two points in $\mathbb{R}^2$. An optimal control problem consists of finding the trajectories of our control system which connect x_0 and x_1 and minimizing the Lagrangian

$$\min \int_0^T L\big(u(x)\big)dt, \qquad L(u) = \frac{1}{2}((u^1)^2 + (u^2)^2), x(0) = x_0, x(T) = x_1, \tag{6.31}$$

where $\dot{x}^i = \frac{dx^i}{dt}$ and u^1, u^2 are control variables. Using the Pontryagin Maximum principle, we find the Hamiltonian function on the cotangent bundle $T^*\mathbb{R}^2$ in the form

$$H(x,p,u) = p_i\dot{x}^i - L = p_1(u^1 + u^2x^1) + p_2u^2 - \frac{1}{2}((u^1)^2 + (u^2)^2), \tag{6.32}$$

with the condition $\frac{\partial H}{\partial u} = 0$, which leads to $u^1 = p_{1,}\ u^2 = p_1x^1 + p_2$. we obtain

$$\begin{aligned} H(x,p) &= \frac{1}{2}(p_1^2(1 + (x^1)^2) + 2p_1p_2x^1 + p_2^2) \\ &= \frac{1}{2}(p_1^2 + (p_1x^1 + p_2)^2), \end{aligned} \tag{6.33}$$

and it result

$$\frac{\partial H}{\partial p_1} = p_1(1 + (x^1)^2) + p_2x^1, \qquad \frac{\partial H}{\partial p_2} = p_1x^1 + p_2,$$

$$\frac{\partial H}{\partial x^1} = p_1^2 x^1 + p_1 p_2, \qquad \frac{\partial H}{\partial x^2} = 0.$$

The Hamilton's equations lead to the following system of differential equations

$$\begin{cases} \dot{x}^1 = p_1(1 + (x^1)^2) + p_2 x^1 \\ \dot{x}^2 = p_1 x^1 + p_2 \\ \dot{p}_1 = -p_1(p_1 x^1 + p_2) \\ \dot{p}_2 = 0 \ \Rightarrow p_2 = ct \end{cases} \qquad (6.34)$$

The Hessian matrix of H with respect to p is

$$g^{ij}(x) = \frac{\partial^2 H}{\partial p_i \partial p_j} = \begin{pmatrix} 1 + (x^1)^2 & x^1 \\ x^1 & 1 \end{pmatrix}, i, j = 1,2 \qquad (6.35)$$

and it result that H is regular $\left(rank\left\|g^{ij}(x,p)\right\| = 2\right)$ and its inverse matrix has the form

$$g_{ij}(x) = \begin{pmatrix} 1 & -x^1 \\ -x^1 & 1 + (x^1)^2 \end{pmatrix}, \qquad (6.36)$$

The adapted tangent structure is given by

$$\mathcal{J}_H = dx^1 \otimes \frac{\partial}{\partial p_1} - x^1 dx^1 \otimes \frac{\partial}{\partial p_2} - x^1 dx^2 \otimes \frac{\partial}{\partial p_1} + (1 + (x^1)^2) dx^2 \otimes \frac{\partial}{\partial p_2}, \qquad (6.37)$$

The $\mathcal{J}_H$ regular vector field is the Hamiltonian vector field from the equation (5.19) i.e.

$$p_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial}{\partial p_i}$$

So that

$$p_H = (p_1(1 + (x^1)^2) + p_2 x^1)\frac{\partial}{\partial x^1} + p_1(p_1 x^1 + p_2)\frac{\partial}{\partial x^2} - (p_1^2 x^1 + p_1 p_2)\frac{\partial}{\partial p_1}, \qquad (6.37)$$

and from proposition(5.8) is a infinitesimal Noether symmetry for the dynamics induced by the regular Hamiltonian H . Moreover, if $X = p_2 \frac{\partial}{\partial x^2}$ then $[p_H,\ X] = 0$ and it result that X is an infinitesimal symmetry for the Hamiltonian vector field p_H.

The local coefficients of the canonical nonlinear connection (5.20) have the following form

$$N_{11} = -(p_1 x^1 + p_2),$$

$$N_{22} = -x^1(1 + (x^1)^2) + p_2 x^1$$

$$N_{12} = N_{21} = x^1(p_1 x^1 + p_2)$$

By straightforward computation we obtain that

$$p_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial x^i} + \frac{\partial H}{\partial p_i} N_{ij}\frac{\partial}{\partial p_j} = \frac{\partial H}{\partial p_i}\frac{\delta}{\delta x^i}\ , \qquad (6.38)$$

And it results that the Hamiltonian vector field is a horizontal $\mathcal{J}_{H}-$ regular vector field. In this case we obtain that the Jacobi endomorphism (5.21) is given by

$$\mathcal{R}_{ij} = R_{kij}\frac{\partial H}{\partial p_k}\,, \qquad (6.39)$$

where R_{kij} are the local coefficients from

$$R_{kij} = \frac{\delta N_{jk}}{\delta x^i} - \frac{\delta N_{ik}}{\delta x^i}$$

of the curvature of the canonical nonlinear connection with nonzero components

$$R_{121} = 2p_1x^1 + p_2 = -R_{211}$$

$$R_{212} = p_1 + 2p_1(x^1)^2 + p_2x^1 = -R_{122}. \qquad (6.40)$$

Also, $\nabla p_H = D_{p_H}p_H = 0$ and it result that the integral curves of horizontal Hamiltonian vector field p_H are geodesics of the Berwald Linear connection.

6.8 Straight Line R Is Connection (Geodesics) Between Two Points.

We consider application of the rules of the variational calculus to prove the well-known result that a straight line I R is the shortest connection(geodesics) between two points (x_1, y_1) and (x_2, y_2). Let us assume that the two points are connected by the path y(x), y$(x_1) = y_1$, $y(x_2)$= y_2. The length of such path can be determined starting from the fact that the incremental length ds in going from point $\big(x, y(x)\big) to \big(x + ds, y(x + ds)\big) is$

$$ds = \sqrt{(dx)^2 + \left(\frac{dy}{dx}dx\right)^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,, \quad (6.41)$$

The total path Length is then given by the integral

$$s = \int_{x_o}^{x_1} dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,. \qquad (6.42)$$

S is a functional of $y(x)$ of the type $S[\vec{q}(t)] = \int_{t_0}^{t_1} dt L(\vec{q}(t), \dot{q}(t), t)$ with

$$L\left(y(x), \frac{dy}{dx}\right) = \sqrt{1 + (dy/dx)^2}. \qquad (6.43)$$

The shortest path is an external of $s[y(x)]$ which must, according to the theorems above, obey the Euler-Lagrange condition reads

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + (y')^2}}\right) = 0\,, \qquad (6.64)$$

From this follows $y'/\sqrt{1 + (y')^2} = constant$ and hence, $y' = conatant$. this in turn yields $y(x) = ax + b$. the constants a and b are readily identified through the condition y$(x_1) = y_1$ and y$(x_2) = y_2$. One obtains

$$y(x) = \frac{y_1 - y_2}{x_1 - x_2}(x - x_2) + y_2, \qquad (6.65).$$

6.9 Infinitesimal Transformations

An advantage of infinitesimal transformations is that they can be stated in very simple manner [11], [20]. In case of a translation transformation in the direction $\hat{e}$ nothing new is gained. However, we like to provide the transformation as

$$\vec{r}' = \vec{r} + \epsilon\hat{e}\,, \qquad (6.66)$$

A non-trivial example is furnished by the infinitesimal rotation around axis $\hat{e}$

$$\vec{r}' = \vec{r} + \epsilon\hat{e} \times \vec{r}\,, \qquad (6.67)$$

I would Like to derive this transformation in a somewhat complicated, but nevertheless instructive way considering rotations around the $x_3 - axis$. In this case the transformation can be written in matrix form

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} cos\epsilon & -sin\epsilon & 0 \\ sin\epsilon & cos\epsilon & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad (6.68)$$

In case of small ϵ this transformation can be written neglecting terms $O(\epsilon^2)$ using case $cos\epsilon = 1 + O(\epsilon^2),\ sin\epsilon = 1 + O(\epsilon^2)$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 & -\epsilon & 0 \\ \epsilon & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + O(\epsilon^2), (6.69)$$

One can readily verify that in case $\hat{e} = \hat{e}_3(\hat{e}_j$ denoting the unit vector in the direction of the $x_j - axis$) (6.67) read

$$\vec{r}' = \vec{r} - x_2\hat{e}_1 + x_1\hat{e}_2\,, \qquad (6.70)$$

which is equivalent to (6.69).

any time, a classical mechanical system is invariant with respect to a coordinate transformation a constant of motion exists, i.e., a quantity $C(\vec{r},\dot{\vec{r}})$ which is constant along the classical path of the system. We have used there the notation corresponding to single particle motion, however, the property holds for any system.

The property has been shown to hold in a more general context, namely for fields rather than only for particle motion, by Noether. We consider here only the 'particle version' of the theorem. Before the starting on this theorem we will comment on what is meant by the statement that a classical mechanical system is invariant under a coordinate transformation. In the context of Lagrangian mechanics this implies that such transformation leaves the Lagrangian of the system unchanged (the theorem (4.3)).

6.10 The Boussinesq Equation

The Boussinesq equation is not the Euler-Lagrange equation for any variational problem. However, replacing u by u_{xx}, we form the "potential Boussinesq equation" [46]

$$u_{xxtt} + \frac{1}{2} D_x^2(u_{xx}^2) + u_{xxxxxx} = 0, \qquad (6.71)$$

which is the Euler-Lagrange equation for variational problem

$$\mathcal{L}[u] = \int\int \left[\frac{1}{2} u_{xt}^2 + \frac{1}{6} u_{xx}^3 - \frac{1}{2} u_{xxx}^2\right] dx \wedge dt, \qquad (6.72)$$

The symmetry group of the potential form (6.71) is spanned by the translation and scaling vector fields

$$\mathrm{v}_1 = \partial_x, \qquad \mathrm{v}_2 = \partial_t, \qquad \mathrm{v}_3 = x\partial_x + 2t\partial_t, \qquad (6.73)$$

And the two infinite families of vector fields

$$\mathrm{v}_f = f(t)\partial_u, \ \ \mathrm{v}_h = h(t)x\partial_u, \qquad (6.74)$$

Where $f(t)$ and $h(t)$ are arbitrary functions of t; the corresponding group action

$u \mapsto u + f(t) + h(t)x$ indicates the ambiguity in our choice of potential. (Compare with the symmetry group of the usual form of the Boussinesq equation.) The most general varational symmetry is found by substituting a general symmetry vector field

$\boldsymbol{v} = \boldsymbol{c_1 v_1} + \boldsymbol{c_2 v_2} + c_3 v_3 + \boldsymbol{v}_f + \boldsymbol{v}_h$ into the infinitesimal criterion ($\boldsymbol{v}^{(n)}(L) + LDiv\xi = 0$), which requires that

$$\frac{1}{4}c_3(-3u_{xt}^2 + u_{xx}^3 - 3u_{xxx}^2) + h'(t)u_{xt} = 0, \qquad (6.75)$$

Therefore $c_3 = 0$ and h is constant, hence the two translations, and the group $u \mapsto u + cx + f(t)$, with c constant, are variational, whereas the scaling and the more general fields $\boldsymbol{v}_h, h$ not constant, are not ordinary variational symmetries, but do define divergence symmetries, in the following sense:

A vector field v is divergence symmetry of a variational problem with Lagrangian L if and only if it satisfies

$$\boldsymbol{v}^{(n)}(L) + LDiv\xi = DivB, \qquad (6.76)$$

For, some p- tuple of functions $B = \left(B_{1,} \dots, B_p\right)$.

A divergence symmetry is a divergence self- equivalence of the Lagrangian form, so that (4.38) holds modulo an exact p- form $d\Xi$. The divergence symmetry groups form the most general class of symmetries related to conservation laws.

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