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Vector-Valued Gabor Frames and Deformation of Gabor Systems with Gabor Orthonormal Bases

اطارات جابور قيمة - المتجه والتشوه لانظمة جابور مع الاساس المنتظم المتعامد لجابور

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in Mathematics

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Dedication

To My Family.

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Abstract

The description of the spectra tiling properties and Gabor orthonormal bases generated by the unit cubes and of the exponential for the n-cube are characterized. In addition the uniformity of non-uniform Gabor bases, atomic characterizations of modulation spaces through Gabor representations with Weyl-Heisenberg frames on Hilbert space, slanted matrices and Banach frames are clearly improved. We obtain the density, stability, generated characteristic function and Hamiltonian deformations of Gabor frames. We find estimates for vector –valued Gabor frames of Hermite functions plus periodic subsets of the real line. The Gabor frame sets for subspace with totally positive functions and deformation of Gabor systems are considered.

الخلاصة

قمنا بتشخيص الطيف وخصائص السطح المقرمد وأساس جابور المنتظم المتعامد المولد مكعبات الوحدة والأسية للمكعب- n . اضافة تم التحسن الواضح لانتظامية أساس جابور غير المنتظم والتشخيصات الذرية للفضاءات المعدلة خلال تمثيلات جابور مع اطارات ويل-هايسنبيرج على فضاء هلبرت والمصفوفات المائلة وإطارات باناخ . تم الحصول على الكثافة والاستقرارية والدالة المميزة المولدة وتشوهات هاملتوينيان لإطارات جابور. قمنا بإيجاد تقديرات لإجل اطارات جابور قيمة-المتجه لدوال هيرمايت زائداً الفئات الجزئية الدورية للخط الحقيقي . تم الاعتبار لفئات اطار جابور للفضاء الجزئي مع الدوال الموجبة الكلية وتشوه انظمة جابور .

Introduction

Let $Q = [0,1)^d$ denote the unit cube in d-dimensional Euclidean space $\mathbb{R}^d$ and let T be a discrete subset of $\mathbb{R}^d$. We relate the spectra of sets Ω to tiling in Fourier space. We develop such a relation for a large class of sets and apply it to geometrically characterize all spectra for the n-cube. There have been extensive studies on non-uniform Gabor bases and frames. But interestingly there have not been a single example of a compactly supported orthonormal Gabor basis in which either the frequency set or the translation set is non-uniform. Nor has there been an example in which the modulus of the generating function is not a characteristic function of a set. We show that in the one dimension and if we assume that the generating function g(x) of an orthonormal Gabor basis is supported on an interval, then both the frequency and the translation sets of the Gabor basis must be lattices.

Given $s \in R$ and $1 \leq p, q \leq \infty$ the modulation space $M^s_{p,q}(R^m)$ can be described as follows, using the Gauss-function $g_0, g_0(x) \coloneqq \exp(-x^2)$, $M^s_{p,q}(R^m) \coloneqq \left\{\sigma | \sigma \in S', g_0 * \sigma \in L^p(R^m) \text{ and } \left[\sigma | M^s_{p,q} \coloneqq \left[\int_{R^m} \|M_t g_0 * \sigma\|^q_p (1+|t|)^{sp}\right]^{\frac{1}{q}} < \infty\right]\right\}$. (Writing $M_t, M_t f(x) \coloneqq \exp(ix.t) f(x)$, $t, x \in R^m$) for the modulation operator. Among these spaces one has the classical potential spaces $\mathcal{L}^2_s(R^m) = M^s_{2,2}(R^m)$ and the remarkable Segal algebra $S_0(R^m) = M^0_{1,1}(R^m)$. A Gabor system is a set of time-frequency shifts $S(g, \Lambda) = \left\{e^{2\pi i b x} g(x-a)\right\}_{(a,b)\in\Lambda}$ of a function $g \in L^2(\mathbb{R}^d)$. We show that if a finite union of Gabor systems $\bigcup_{k=1}^{r} S(g_k, \Lambda_k)$ forms a frame for $L^2(\mathbb{R}^d)$ then the lower and upper Beurling densities of $\Lambda = \bigcup_{k=1}^{r} \Lambda_k$ satisfy $D^-(\Lambda) \geq 1$ and $D^+(\Lambda) < \infty$. We study the stability of Gabor frames with arbitrary sampling points in the time-frequency plane, in several aspects. We prove that a Gabor frame generated by a window function in the Segal algebra $S_0(\mathbb{R}^d)$ remains a frame even if (possibly) all the sampling points undergo an arbitrary perturbation, as long as this is uniformly small. We give explicit stability bounds when the window function is nice enough, showing that the allowed perturbation depends only on the lower frame bound of the original family and some qualitative parameters of the window under consideration.

A Weyl-Heisenberg frame $\{E_{mb}T_{na}g\}_{m,n\in Z} = \left\{e^{2\pi i m b(\cdot)} g(\cdot - na)\right\}_{m,n\in Z}$ for $L^2(R)$ allows every function $f \in L^2(R)$ to be written as an infinite linear combination of translated and modulated versions of the fixed function $g \in L^2(R)$. We find sufficient conditions for $\{E_{mb}T_{na}g\}_{m,n\in Z}$ to be a frame for $\overline{\text{span}}\{E_{mb}T_{na}g\}_{m,n\in Z}$, which, in general, might just be a subspace of $L^2(R)$. Even our condition for $\{E_{mb}T_{na}g\}_{m,n\in Z}$ to be a frame for $L^2(R)$ is significantly weaker than the previous known conditions. We derive frame bound estimates for vector-valued Gabor systems with window functions belonging to Schwartz space. We provide estimates for windows composed of Hermite functions. The well-known density theorem for one-dimensional Gabor systems of the form $\{e^{2\pi imbx}g(x - na)\}_{m,n\in Z}$, where $g \in L^2(R)$, states that a necessary and sufficient condition for the existence of such a system whose linear span is dense in $L^2(R)$, or which forms a frame for $L^2(R)$, is that the density condition $a\,b \leq 1$ is satisfied. We study the analogous problem for Gabor systems for which the window function g vanishes outside a periodic set $S \subset R$ which is a Z-shift invariant. We obtain measure-theoretic conditions that are necessary and sufficient for the existence of a window g such that the linear span of the corresponding Gabor system is dense in $L^2(S)$.

We investigate the characterization problem which asks for a classification of all the triples (a, b, c) such that the Gabor system $\{e^{i2m\pi bt}\chi_{[na,c+na)}: m, n \in Z\}$ is a frame for $L^2(R)$. With the help of a set-valued mapping defined on certain union of intervals, we are able to provide a complete solution for the case of ab being a rational number. Let g be a totally positive function of finite type, i.e., $\hat{g}(\xi) = \prod_{v=1}^{M}(1 + 2\pi i\delta_v\xi)^{-1}$ for $\delta_v \in R$ and $M \geq 2$. Then the set $\{e^{2\pi i\beta lt}g(t - \alpha k): k, l \in \mathbb{Z}\}$ is a frame for $L^2(\mathbb{R})$, if and only if $\alpha\beta < 1$. This result is a first positive contribution to a conjecture of I. Daubechies from 1990. So far the complete characterization of lattice parameters α, β that generate a frame has been known for only six window functions g.

We consider the problem in determining the countable sets Λ in the time-frequency plane such that the Gabor system generated by the time-frequency shifts of the window $\chi_{[0,1]^d}$ associated with Λ forms a Gabor orthonormal basis for $L^2(\mathbb{R}^d)$. We show that, if this is the case, the translates by elements Λ of the unit cube in $\mathbb{R}^{2d}$ must tile the time-frequency space $\mathbb{R}^{2d}$. By studying the possible structure of such tiling sets, we completely classify all such admissible sets Λ of time-frequency shifts when $d = 1, 2$.

We present a rare combination of abstract results on the spectral properties of slanted matrices and some of their very specific applications to frame theory and sampling problems. We show that for a large class of slanted matrices boundedness below of the corresponding operator in p for some p implies boundedness below in p for all p. Gabor frames can advantageously be redefined using the Heisenberg–Weyl operators familiar from harmonic analysis and quantum mechanics. Not only does this redefinition allow us to recover in a very simple way known results of symplectic covariance, but it immediately leads to the consideration of a general deformation scheme by Hamiltonian isotopies (i.e. arbitrary paths of non-linear symplectic mappings passing through the identity). We introduce a new notion for the deformation of Gabor systems. Such deformations are in general nonlinear and, in particular, include the standard jitter error and linear deformations of phase space. With this new notion we show a strong deformation result for Gabor frames and Gabor Riesz sequences that covers the known perturbation and deformation results.

We investigate vector-valued Gabor frames (sometimes called Gabor superframes) based on Hermite functions H_n. Let $h = (H_0, H_1, \ldots, H_n)$ be the vector of the first $n+1$ Hermite functions. We give a complete characterization of all lattices $\Lambda \subseteq \mathbb{R}^2$ such that the Gabor system $\{e^{2\pi i\lambda 2t}h(t - \lambda_1): \lambda = (\lambda_1, \lambda_2) \in \Lambda\}$ is a frame for $L^2(\mathbb{R}, \mathbb{C}^{n+1})$. We obtain sufficient conditions for a single Hermite function to generate a Gabor frame and a new estimate for the lower frame bound. We investigate Gabor frame sets in a periodic subset $\mathbb{S}$ of $\mathbb{R}$. We characterize tight Gabor sets in $\mathbb{S}$, and obtain some necessary/sufficient conditions for a measurable subset of $\mathbb{S}$ to be a Gabor frame set in $\mathbb{S}$. The notion of vector-valued frame (also called superframe) was first introduced by Balan of multiplexing. It has significant applications in mobile communication, satellite communication, and computer area network. For vector-valued Gabor analysis, existent literatures mostly focus on $L^2(\mathbb{R}, \mathbb{C}^L)$ instead of its subspace. Let $a > 0$, and S be an $a\mathbb{Z}$-periodic measurable set in $\mathbb{R}$ (i.e. $S + a\mathbb{Z} = S$). We addresse Gabor frames in $L^2(S, \mathbb{C}^L)$ with rational time–frequency product. They can model vector-valued signals to appear periodically but intermittently. And the projections of Gabor frames in $L^2(\mathbb{R}, \mathbb{C}^L)$onto $L^2(S, \mathbb{C}^L)$ cannot cover all Gabor frames in $L^2(S, \mathbb{C}^L)$ if $S \neq \mathbb{R}$.

The Contents

Chapter 1
Spectral and Orthonormal Bases with Uniformity

We show that the exponentials $e_t(x) := \exp\ (i2\pi tx), t \in T$ form an othonormal basis for $L^2\ (Q)$ if and only if the translates $Q + t, t \in T$ form a tiling of $\mathbb{R}^d$. We study the behavior of the orthonormal bases for n-cube. We show that the Gabor basis must be the "trivial" one in the sense that $|g(x)| = c_{\chi\Omega}(x)$ for some fundamental interval of the translation set. We also give examples showing that compactly supported non-uniform orthonormal Gabor bases exist in higher dimensions.

Section (1.1): Tiling Properties of the Unit Cube

For $Q := [0, 1)^d$ denote the unit cube in d-dimensional Euclidean space $\mathbb{R}^d$ .Let T be a discrete subset of $\mathbb{R}^d$. We say T is a tiling set for Q, if each $x \in \mathbb{R}^d$ can written uniquely as $x = q + t$, with $q \in Q$ and $t \in T$ .We say T is a spectrum for Q, if the exponentials

$$e_t(x) := e^{i2\pi tx},\ \ t \in T$$

form an orthonormal basis for $L^2(Q)$. Here juxtaposition tx of vectors t, x in $\mathbb{R}^d$ denote the usual inner product $tx = t_1x_1 + \ldots + t_dx_d$ in $\mathbb{R}^d$ and $L^2(Q)$ is equipped with usual inner productive ,vis.,

$$\langle f, g\rangle := \int_Q f\bar{g}\, dm,$$

Where m denotes Lebesgue measure.

Theorem (1.1.1)[1]: Let T be a subset of $\mathbb{R}^d$. Then T is a spectrum for the unit cube Q if and only if T is a tiling set for the unit cube Q.

Remark (1.1.2)[1]: As we shall discuss below there exists highly counter-intuitive cube-tiling's in $\mathbb{R}^d$ for sufficiently large d. Those tiling's can be much more complicated than lattice tiling's. Theorem (1.1.1) is clear if T is a lattice .The point of Theorem (1.1.1) is that the result still holds even if the restrictive lattice assumption is dropped.

Sets whose translates tile $\mathbb{R}^d$ and the corresponding tiling sets have been investigated intensively, see [3], [11], [12]. Even the one-dimensional case $d = 1$ is non-trivial. The study of sets whose L^2-space admits orthogonal bases of exponentials was begun in [4]. Several have appeared, see [6], [15], [13]. It was conjectured in [4] that a set admits a tiling set if and only if it admits a spectrum, i.e, the corresponding L^2-space admits an orthogonal basis of exponentials.

Cube tiling's have a long history beginning with a conjecture due to Minkowski: in every lattice tiling have of $\mathbb{R}^d$ by translates of Q some cubes must share a complete $(d - 1)$-dimensional face. Minkowski's conjecture was show in [5], see [18]. Keller [8] while working on Minkowski's conjecture made the stronger conjecture that one could omit the lattice assumption in Minkowski's conjecture. Using [19] and [2] it was shown in [10] that there are cube tiling's in dimensions $d \geq 10$ not satisfying Keller's conjecture.

The study of the possible spectra for the unit cube was initiated in [7], where Theorem (1.1.1) was conjectured. Theorem (1.1.1) was show in [7] if $d \leq 3$ and for any d if T is periodic. The terminology spectrum for Q originates in a problem about the existence of certain commuting self-adjoint partial differential operators. We say that two self-adjoint operators commute if their spectral measures commute, see [17] for to the theory of unbounded self-adjoint operators. The following result was show in [4] under a mild regularity condition on the boundary; the regularity condition was removed in [14].

Theorem (1.1.3)[1]: Let Ω be a connected open subset of $\mathbb{R}^d$ with finite Lebesque measure, there exists a set T so that the exponentials $e_t,\ t \in T$ form an orthogonal basis for $L^2(\Omega)$ if and only if there exists commuting self-adjoint operators $H = (H_1, \ldots\ldots, H_d)$ so that each H_j is defined on ${C_c}^{\infty}(\Omega)$ and

$$H_j f = \frac{1}{i2\pi}\frac{\partial f}{\partial x_j} \tag{1}$$

For any $f \in {C_c}^{\infty}(\Omega)$ and any $j = 1, \ldots\ldots, d$.

If $e_t,\ t \in T$ is an orthogonal basis for $L^2(\Omega)$ then a commuting tuple $H = (H_1, \ldots\ldots, H_d)$ of self-adjoint operators satisfying (1) is uniquely determined $H_j e_t = t_j e_t, t \in T$ then. Conversely, if $H = (H_1, \ldots\ldots, H_d)$ is a commuting tuple of self-adjoint operators satisfying (1) then the joint spectrum $\sigma(H)$ is discrete and each $t \in \sigma(H)$ is a simple eigen-value corresponding to the eigen-vector e_t, in particular, $e_t, t \in \sigma(H)$ is an orthogonal basis for $L^2(\Omega)$.

We show that any tiling set is a spectrum, the converse is showed. Key ideas in both proofs are that if (g_n) is an orthonormal family in $L^2(Q)$ and $f \in L^2(Q)$ then we have equality in Bessel's inequality

$$\sum |\langle f, g_n\rangle|^2 \leq \|f\|^2$$

if and only if f is in the closed linear span of (g_n), and a sliding lemma (Lemma (1.1.6)) showing that we may translate certain parts of a spectrum or tiling set while preserving the spectral respectively the tiling set property. We show some elementary properties of spectra and tiling sets. For $t \in \mathbb{R}^d$ let $Q + t\colon \{q + t\colon q \in Q\}$ denote the translate of Q by vector . We say (Q, T) is non-overlapping if the cubes $Q + t$ and cubes $Q + t'$ are disjoint for any $t, t' \in T$. Note, T is a tiling set for Q if and only if (Q, T) is non-overlapping and $\mathbb{R}^d = Q_T \coloneqq U_{t\in T}(Q + T)$. We say (Q, T) is orthogonal, if the exponentials $e_t, t \in T$ are orthogonal in $L^2(Q)$. A set T is a spectrum for Q if and only if (Q, T) is orthogonal and

$$\sum_{t\in T} |\langle e_n, e_t\rangle|^2 = 1$$

For all $n \in \mathbb{Z}^d$. Let $\mathbb{N}$ denote the positive integers $\{1,2,3, \ldots\}$ and let $\mathbb{Z}$ denote the set of all integers$\{\ldots, -1,0,1,2, \ldots\}$.

We received a preprint [9] by Lagarias, Reed and Wang proving our main result .Compared to [9] the proof that any spectrum is a tiling set uses completely different techniques, the proof that any tiling set is spectrum is a similar to the proof in [9] in that both proof's makes use of Keller's Theorem (Theorem (1.1.9)) and an argument involving an inequality becoming equality. We wish to thank Lagarias for the preprint and useful remarks. Robert S. Strichartz helped us clarify the exposition.

The basis property is equivalent to the statement that the sum

$$\sum_{t\in T}|\langle e_x, e_t\rangle|^2 = 1 \qquad (2)$$

For all $x \in \mathbb{R}^d$ .it is easy to see that if T has the basis property then the cubes $Q+t$, $t \in T$ are non-overlapping. We show by a geometric argument that if the basis property holds and the tiling property does not hold then the sum in (2) is strictly less than one. Conversely, if T has the tiling property then the exponentials e_t, $t \in T$ are orthogonal by Keller's Theorem. Plancerel's Theorem now implies that the sum in (2) is one. The geometric argument is based Lemma (1.1.6), An analogous lemma was used by Perron [16] in his proof Keller's Theorem.

We begin by proving a simple result characterizing orthogonal subsets of $\mathbb{R}^d$. There is a corresponding (non-trivial) result for tiling's, stated as Theorem (1.1.9) below.

Lemma (1.1.4)[1]: (Spectral version of Keller's theorem). Let T be a discrete subset of $\mathbb{R}^d$. The pair (Q,T) is orthogonal if and only if given any pair $t,t' \in T$, with $t \neq t'$, there exists $j \in \{1,\dots,d\}$ so that $|t_j - t_j'| \in \mathbb{N}$.

Proof. For $t,t' \in \mathbb{R}^d$ we have

$$\langle e_t, e_{t'}\rangle = \prod_{j=1}^{d} \emptyset(t_j - t_j') \qquad (3)$$

Where for $x \in \mathbb{R}$

$$\emptyset(x) := \begin{cases} 1, & \text{if } x = 0; \\ \dfrac{e^{i2\pi x}-1}{i2\pi x}, & \text{if } x \neq 0. \end{cases} \qquad (4)$$

The lemma is now immediate.

We can now state showing that there is a connection between spectra and tiling sets for the unit cube.

Corollary (1.1.5)[1]: Let T be a subset of $\mathbb{R}^d$. if (Q,T) is orthogonal, then (Q,T) is non-overlapping.

A key technical lemma needed for the proofs of both implications in the main result is the following lemma. The lemma shows that a certain part of a spectrum (respectively tiling set) can be translated independently of it's complement without destroying the spectral (respectively tiling set) property. The tiling set part of The lemma if taken from [16].

Lemma (1.1.6)[1]: Let T be a discrete subset of $\mathbb{R}^d$, fix $a,b \in \mathbb{R}$ .Let $c := (b,0,\dots,0) \in \mathbb{R}^d$ And for $t \in T$, let:

$$\alpha_{T,a,b}(t) := \begin{cases} t, & \text{if } t_1 - a \in \mathbb{Z}; \\ t+c, & \text{if } t_1 - a \notin \mathbb{Z}. \end{cases}$$

We have the following conclusions: (a) if T is a spectrum for Q, So is $\alpha_{T,a,b}(T)$. (b) if T is a tiling set for Q, so is $\alpha_{T,a,b}(T)$.

Proof. Suppose T is a spectrum for Q. The orthogonally of $(Q, \alpha_{T,a,b}(T))$ is an easy consequence of Lemma (1.1.4). Let $A_{T,a,b}\, e_t := e_{\alpha_{T,a,b}}(t)$ for $t \in T$. To simplify the notation we will write A_b in place of $A_{T,a,b}$. By orthogonality and linearity A_b extends to an isometry mapping $L^2(Q)$ into itself. We must show that the range $A_b L^2(Q)$ is all of $L^2(Q)$. Let K_+ be the subspace of $L^2(Q)$ spanned by the exponentials e_t, $t \in T$ with

$t_1 - a \in \mathbb{Z}$ and let K_- be the subspace of $L^2(Q)$ spanned by the exponentials e_t , $t \in T$ With $t_1 - a \notin \mathbb{Z}$. Then $A_b f = f$ for all $f \in K_+$, So $A_b K_+ = K_+$ .Since A_b preserves orthogonality, $A_b K_- \subseteq K_-$. We must Show $A_b K_- = K_-$. Since $b \in \mathbb{R}$ is arbitrary, we also have that the map A_{-b} is an isometry mapping K_- into itself. By construction $A_b f = e_c f$ and $A_{-b} f = \overline{e_c} f$ for all $f \in K_-$. it follows that $K_- = A_b A_{-b} K_- \subseteq A_b K_- \subseteq K_-$. Hence $A_b K_- = K_-$ as desired. The proof that $\alpha_{T,a,b}(T)$ is a tiling set provided T is, follows from The last part of the proof of the Theorem (1.1.9) bellow.

For $n' \in \mathbb{Z}^{d-1}$ let $\ell_{n'}$ be the line in $\mathbb{R}^d$ given by $\{(x, n'),\ x \in \mathbb{R}\}$ .The idea of our proof that any spectrum for Q must be a tiling set for Q is as follows .Suppose T is a spectrum but not a tiling set, but not a tiling set. Fix $n' \in \mathbb{Z}^{d-1}$ and pick a $t \in T$ (if any) so that $Q + t$ interseets the line $\ell_{n'}$ applying Lemma (1.1.6) we can insure that $t_1 \in \mathbb{Z}$. Repeating this for each $n' \in \mathbb{Z}^{d-1}$ we can ensure $t_1 \in \mathbb{Z}$ for any $t_1 \in T^{\text{new}}$. Considering each of the remaining coordinate directions we end up with T^{new} being a subset of $\mathbb{Z}^d$. (The meaning of T^{new} changes with each application of lemma (1.1.6)). By Lemma (1.1.8) T^{new} is not a tiling set for Q since T was not a tiling set, so T^{new} is a proper subset of $\mathbb{Z}^d$, contradicting the basis property. The difficulty with this outline is that after we apply Lemma (1.1.6) an infinite number of times the basis property may not hold. In fact, associated to each application of Lemma (1.1.6) is an isometric isomorphism A_{b_n}. Without restrictions on the sequence (b_n) the infinite product $\prod_{n=1}^{\infty} A_{b_n}$ need not be convergent (e.g., with respect to the weak operator topology). Even if the infinite product $\prod_{n=1}^{\infty} A_{b_n}$ is convergent, the limit may be a non- surjective isometry.

If we use Lemma (1.1.6) to put a large finite part of T into $\mathbb{Z}^d$ then we can use decay properties of the Fourier transform of the characteristic functions of the cube Q to contradict (2).

The following lemma shows that sums of the Fourier transform of the characteristic function of the cube Q over certain discrete sets has uniform decay properties.

Lemma (1.1.7)[1]: Let ϕ be given by there exists a constant $C > 0$ so that

$$\sum_{t \in T_N} \prod_{j=1}^{d} |\phi(t_j)|^2 \leq \frac{C}{N}$$

For any $N > 1$, whenever $T \subset \mathbb{R}^d$ is a spectrum for the unit cube Q. Here T_N is the set of $t \in T$ for which$|t_j| > j$, for at least one$j$. Note, the constant C is uniform over all spectra T for the unit cube Q and all $N > 1$.

Proof. Let T be a spectrum for Q. for any partition $P = \{I.II.III.IV\}$ of $\{1, \dots\dots, d\}$, let $T_{N,P}$ denote the set of $t \in T_N$ for so that $t_j > N$ for $j \in I$; $t_j < -N$ for $j \in II$; $0 \leq t_j \leq N$ for $j \in III$ and $-N \leq t_j < 0$ for $j \in IV$ .Note $T_{N,P}$ is empty unless $I \cup II$ is non-empty. For $x \in \mathbb{R}$ let $\psi(x) = 1$, if $-1 < x < 1$ and let $\psi(x) = x^{-2}$ if $|x| \geq 1$. Then for $t \in T_{N,P}$,

$$\prod_{j=1}^{d} |\phi(t_j)|^2 \leq \prod_{j=1}^{d} \psi(s_j) \tag{5}$$

For any $s = (s_1, \dots\dots, s_d)$ in the cube $X_{t,P}$ given by $t_j - 1 \le s_j < t_j$ if $j \in I \cup III$, and $t_j \le s_j < t_j + 1$ if $j \in II \cup IV$. It follows from (5) and disjointness (Lemma (1.1.4)) of the cubes $X_{t,P}$, $t \in T_{N,P}$ that

$$\sum_{t\in T_{N,P}} \prod_{j=1}^{d} \left|\emptyset(t_j)\right|^2 \le \sum_{t\in T_{N,P}} \int_{X_{t,P}} \prod_{j=1}^{d} \psi(s_j)\, ds \le \int_{Y_{N,P}} \prod_{j=1}^{d} \psi(s_j)\, ds,$$

Where $Y_{N,P}$ is the set of $y \in \mathbb{R}^d$ for which $N - 1 < y_j$ for $j \in I$, $y_j < -N + 1$ for $j \in II$, $-1 < y_j < N$ for $j \in III$, and $-N < y_j < 1$ for $j \in IV$. By definition of ψ we have

$$\int_{Y_{N,P}} \prod_{j=1}^{d} \psi(s_j)\, ds \le 3^{d-n} \frac{1}{(N-1)^n},$$

where $n > 0$ is the cardinality of $I \cup II$; since the number of possible partitions $P = \{I, II, III, IV\}$ only depends on the dimension d of $\mathbb{R}^d$, the proof is complete.

The following lemma shows that if T is a spectrum but not a tiling set for Q then the set constructed in Lemma (1.1.6); is also not a tiling set for Q . It is needed because the inverse of the transformation in Lemma (1.1.6) is not of the same form.

Lemma (1.1.8)[1]: If T is a spectrum for Q but not a tiling set for Q , then $\alpha_{T,a,b}(T)$ is not a tiling set for Q.

Proof. Suppose T is a spectrum for Q but not a tiling set for Q. Let $g \notin Q_T$. Let $\ell \coloneqq \{(x, g_2, g_3, \dots\dots, g_d)\}$. If $r, s \in T$ are so that $Q + r$ and $Q + S$ intersect ℓ then it follows from Lemma (1.1.4) that $s_1 - r_1$ is an integer, since $|s_j - t_j| < 1$ for $j \neq 1$ because $Q + r$ and $Q + s$ intersect ℓ. So either $t_1 - a \in \mathbb{Z}$ for all $t \in T$ so that $Q + t$ intersect ℓ or $t_1 - a \notin \mathbb{Z}$ for all $t \in T$ so that $Q + t$ interests ℓ. In the first case $g \notin Q_{\alpha_{T,a,b}}(T)$ in the second case $g + c \notin Q_{\alpha_{T,a,b}}(T)$.

Suppose T is a spectrum for the unit cube Q. By Corollary (1.1.5); the pair (Q, T) is non-overlapping. We must show that the union $Q_T = U_{t\in T}(Q + t)$ is all of $\mathbb{R}^d$. To get a contradiction suppose $g \notin Q_T$. Let N be so large that $g \in (-N + 2, N - 2)^d$.

Let $T(N) \coloneqq T \cap (-N - 1, N + 1)^d$ Let $n'_1 \coloneqq (-N, -N, \dots\dots, -N) \in \mathbb{Z}^{d-1}$. Pick $t \in T(N)$ so that $Q + t$ intersects $\ell_{n'_1}$ (if such a t exists). Use Lemma (1.1.6); with $a = 0$ and $b = b_1 \coloneqq t_1 - \lfloor t_1 \rfloor$ to conclude $T_1 \coloneqq \alpha_{T,a,b}(T)$ has the basis property, it follows from Lemma (1.1.4); that $t_1 \in \mathbb{Z}$ for any $t \in T_1$ so that $Q + t$ intersects $\ell_{n'_1}$.

Let $n'_2 \coloneqq (-N, -N, \dots, -N + 1) \in \mathbb{Z}^{d-1}$. Pick $t \in T_1(N)$ so that $Q + t$ intersects $\ell_{n'_2}$ (if such a t exists). Use Lemma (1.1.6) with $a = 0$ and $b = b_2 \coloneqq t_1 - \lfloor t_1 \rfloor$ if $b_1 + t_1 - \lfloor t_1 \rfloor \ge 1$ and $b = b_2 {:=} t_1 - \lfloor t_1 \rfloor - 1$ if $b_1 + t_1 - \lfloor t_1 \rfloor < 1$ to conclude $T_2 \coloneqq \alpha_{T_1,a,b}(T_1)$ has the basis property. It follows from Lemma (1.1.4); that $t_1 \in \mathbb{Z}$ for any $t \in T_2$, so that $Q + t$ intersects $\ell_{n'_2}$. Note we did not move any of the cubes in T_1 with $-N - 1 < t_j \le -N$, for $j = 2, \dots., d$.

Continuing in this manner, we end up with T' having the basis property so that $t_1 \in \mathbb{Z}$ for any $t \in T'$ with $-N - 1 < t_j < N + 1$ for $j = 2, \dots., d$. Note $-1 < \sum_1^n b_j < 1$ for any n. So if at some stage $t \in T_n$ is derived from $t^{\text{original}} \in T$, Then we have ${t_1}^{\text{original}} - 1 < t_1 < {t_1}^{\text{original}} + 1$ Repeating this process for each of the other coordinate directions we end up with T^{new} so that $T^{\text{new}}(N - 1)$ is a subset of the integer lattice $\mathbb{Z}^d$, any $t \in T^{\text{new}}(N - 1)$ is obtained from some $t^{\text{original}} \in T(N)$, and any

$t^{\text{original}} \in T(N-1)$ is translated onto some $T^{\text{new}}(N)$. In short, we did not move any point in T very much. By Lemma (1.1.8); it follows that $(-N,N)^d \backslash Q_{T^{\text{new}}}$ is non-empty, hence there exists $g^{\text{new}} \in \mathbb{Z}^d$, so that $g^{\text{new}} \in (-N,N)^d \backslash T^{\text{new}}$.Replacing T^{new} by T^{new} by $T^{\text{new}} - g^{\text{new}}$, is necessary, and applying the process described above we may assume $g^{\text{new}} = 0$. To simplify the notation let $T = T^{new}$. We have

$$1 = \sum_{t \in T} |\langle e_t, e_0 \rangle|^2 = \sum_{t \in T(N)} |\langle e_t, e_0 \rangle|^2 \sum_{t \in T_N} |\langle e_t, e_0 \rangle|^2.$$

The first sum $= 0$ since $T(N) \subset \mathbb{Z}^d$ and $0 \notin T(N)$, the second sum is < 1 for N sufficiently large by Lemma (1.1.7); this contradiction completes the proof.

The following result (due to [8]) shows that any tiling set for the cube is orthogonal. It is a key step in our proof that any tiling set for the cube must be a spectrum for the cube and should be compared with Lemma (1.1.4) above. The proof is essentially taken from [16].

Theorem (1.1.9)[1]: (Keller's Theorem). If T is a tilling set for Q, Then given any pair $t, t' \in T$, with $t \neq t'$, there exist a $j \in \{1, \dots \dots, d\}$ so that $|t_j - t'_j| \in \mathbb{N}$.

Proof. Let T be a tiling set for Q. Suppose $t, t' \in T$. The proof is by induction on the number of j's for which$|t_j - t'_j| \geq 1$. Suppose that $|t_j - t'_j| < 1$ for all but one $\in \{1, \dots \dots, d\}$. Let j_0 be the exceptional j, then $|t_{j_0}, t'_{j_0}| \geq 1$. Fix $x_j, j \neq j_0$ so that the line $\ell_{j_0} \coloneqq \{(x_1, \dots \dots, x_j) : x_{j_0} \in \mathbb{R}\}$ passes through both of the cubes $Q + t$ and $Q + t'$. Considering the cubes $Q + t, t \in T$ that intersect ℓ_{j_0} it is immediate that $|t_{j_0}, t'_{j_0}| \in \mathbb{N}$.

For the inductive step, suppose $|t_j - t'_j| < 1$ for k values of j and $|t_j, t'_j| \geq 1$, for the remaining $d-k$ values of j implies $|t_{j_0} - t'_{j_0}| \in \mathbb{N}$ for some j_0. Let $t, t' \in T$ be so that $|t - t'_j| < 1$ for $k-1$ values of j and $|t - t'_j| \geq 1$ for the remaining $d - k + 1$ values of j. interchanging the coordinate axes, if necessary, we may assume

$$|t_j - t'_j| \geq 1, \quad \text{for } j = 1, \dots, d-k+1$$
$$|t - t'_j| < 1, \quad \text{for } j = d-k+2, \dots, d.$$

If $t_1 - t'_1$ is an integer, then there we are done. Assume $t_1 - t'_1 \notin \mathbb{Z}$. Let $c \coloneqq (t_1 - t'_1, 0, \dots, 0)$, and for $\tilde{t} \in T$ let

$$s(\tilde{t}\,) \coloneqq \begin{cases} \tilde{t} - c, & \text{if } \tilde{t}_1 - t_1 \in \mathbb{Z} \\ \tilde{t}, & \text{if } \tilde{t}_1 - t_1 \notin \mathbb{Z}. \end{cases}$$

In particular, $s(t) = t - c$ and $s(t') = t'$. We claim the set $S \coloneqq \{s(\tilde{t}\,) : \tilde{t} \in T\}$ is a tiling set for Q. Assuming, for a moment, that the claim is valid, we can easily complete the proof. In fact, $|s(t)_1 - s(t')_1| = 0$ and $|s(t)_j - s(t')_j| < 1$ for $j = d-k+2, \dots, d$, so by the inductive hypothesis one of the numbers $t_j - t'_j = s(t)_j - s(t')_j$, $j = 2, \dots, d-k+1$ is a non-zero integer.

It remains to show that S is a tiling set for Q. We must show that Q_s is non-overlapping and that $\mathbb{R}^d \subset Q_s$. First we dispense with the non-overlapping part. Let a, a' be distinct points in T. Suppose x is a point in the intersection$(Q + s(a)) \cap (Q + s(a'))$, then $x - s(a), x - s(a') \in Q$ In particular, $0 \leq x_j - a_j < 1$ and $0 \leq x_j - a'_j < 1$ for $j = 2, \dots, d$. It follows that $|a_j - a'_j|$ for $j = 2, \dots, d$, so first paragraph of the proof shows that $|a_j - a_j{}'| \in \mathbb{N}$, hence either $a_1 - t_1, a'_1 - t_1 \in \mathbb{Z}$ or $a_1 - t_1, a'_1 - t_1 \notin \mathbb{Z}$. In both cases we get a contradiction to the non-overlapping property of Q_T. In

fact, if $a_1 - t_1, a' - t_1 \in \mathbb{Z}$, then $(Q + s(a)) \cap (Q + s(a')) = ((Q + a) \cap (Q + a')) - c = \emptyset$.
If $a_1 - t_1, a'_1 - t_1 \notin \mathbb{Z}$, then $Q + s(a)) \cap (Q + s(a')) = ((Q + a) \cap (Q + a')) = \emptyset$.

Let $x \in \mathbb{R}^d$ be an arbitrary point, then $x \in Q_T$. If $x \in (Q + a)$ for some $a \in T$ with $a_1 - t_1 \notin \mathbb{Z}$, then there is nothing to show, Assume $x \in (Q + a)$ for some $a \in T$ with $a_1 - t_1 \in \mathbb{Z}$ .The point $x + c$ is in $Q + b$ for some $b \in T$. First we show that

$|a_1 - b_1| \in \mathbb{N}$. Since $x \in Q + a$ and $x + c \in Q + b$ we have

$$0 \le x_j - a_j < 1, \quad 0 \le x_j - b_j + c_j < 1, \tag{6}$$

for $= 1, \dots, d$, so using $c_j = 0$, for $j = 2, \dots, d$, it follows that $|a_j - b_j| < 1$, for $j = 2, \dots, d$; an application for the first paragraph of the proof yields the desired result that $|a_1 - b_1| \in \mathbb{N}$. Using $a_1 - t_1 \in \mathbb{Z}$ we conclude $b_1 - t_1 \in \mathbb{Z}$; so using the second half of (6) and the definition of $s(b)$ we have $x \in Q + s(b)$; as needed.

Corollary (1.1.10)[1]: If T is a tiling set for Q, then (Q, T) is orthogonal.

Proof. This is a direct consequence of Keller's Theorem and Lemma (1.1.4).

It is now easy to complete the proof that any tiling set for the unit cube Q must be a spectrum for Q.

Suppose T is a tiling set for Q. By Keller's theorem $\{e_t : t \in T\}$ is an orthogonal set of unit vectors in $L^2(Q)$, so by Bessel's inequality

$$\sum_{t \in T} |\langle e_s, e_t \rangle|^2 \le 1 \tag{7}$$

for any $s \in \mathbb{R}^d$. Not that $\langle e_s, e_t \rangle$ is the Fourier transform of the characteristic function of the cube Q at the points $s - t$. For any $r \in \mathbb{R}^d$ we have

$$I = \int_{\mathbb{R}^d} \left|\langle e_y, e_0 \rangle\right|^2 dy = \int_{Q+r} \sum_{t \in T} |\langle e_x, e_t \rangle|^2 dx \le \int_{Q+r} 1 dy = 1,$$

Where we used Plancherel's Theorem, the tiling property, and Bessel's inequality (7); it follows that

$$\sum_{t \in T} |< e_s, e_t >|^2 = 1 \tag{8}$$

for almost every s in $Q + r$, and since r is arbitrary, for almost every s in $\mathbb{R}^d$. Hence for almost every $s \in \mathbb{R}^d$ the exponential e_s is in the closed span of the $e_t, t \in T$. This completes the proof.

Section (1.2): Exponentials for the n-Cube

A compact set Ω in $\mathbb{R}^n$ of positive lebesgue measure is a spectral set if there is some set of exponentials

$$\mathfrak{B}_A := \{e^{2\pi i \langle \lambda, x \rangle} : \lambda \in A\}, \tag{9}$$

which when restricted to Ω gives an orthogonal basis for $L^2(\Omega)$, with respect to the inner product

$$\langle f, g \rangle_\Omega := \int_\Omega \overline{f(x)}\, g(x) dx. \tag{10}$$

Any set A that gives such an orthogonal basis is called a spectrum for Ω. Only very special sets Ω in $\mathbb{R}^n$ are spectral sets. However, when a spectrum exists, it can be viewed as a generalization of Fourier series, because for the n-cube $\Omega = [0,1]^n$, the spectrum $A = \mathbb{Z}^n$ gives the standard Fourier basis of $L^2([0,1]^n)$.

We relate the spectra of sets Ω to tiling in Fourier space. we develop such a relation for a large class of sets and apply it to geometrically characterize all spectra for the n-cube $\Omega = [0,1]^n$.

Theorem (1.2.1)[20]: The following conditions on a set A in $\mathbb{R}^n$ are equivalent.

(i) The set $\mathfrak{B}_A \coloneqq \{e^{2\pi i\langle\lambda,x\rangle} : \lambda \in A\}$ when restricted to $[0,1]^n$ is an orthonormal basis of $L^2([0,1]^n)$.

(ii)The collection of sets $\{\lambda + [0,1]^n : \lambda \in A\}$ is a tiling of $\mathbb{R}^n$ by translates of unit cubes.

Proof. The set $D = [0,1]^n$ is a tight orthogonal packing region for $\Omega = [0,1]^n$. To see this, note that lemma (1.2.20) implies that D is an orthogonal packing region for Ω, and since each of Ω and D has measure 1,it is tight.

(i)⇒(ii). By hypothesis, $\mathfrak{B}_A$ is an orthogonal set in $L^2([0,1]^n)$. We showed above that D is a tight orthogonal packing region for Ω. Now Theorem (1.2.15). Applies to conclude that $A + D$ is a tiling of $\mathbb{R}^n$.

(ii)⇒(i). By hypothesis, $A + D$ is a cube tiling , so by Proposition (1-2-19), $\mathfrak{B}_A$ is an orthogonal set in $L^2([0,1]^n)$. Clearly, $m(\Omega)m(D) = 1$, and since $A + D$ is a cube tiling, it is a fortiori a cube packing. So by Theorem (1.2.14), A is a spectrum.

This result was conjectured by Jorgensen and Pedersen [26], who showed it in dimensions $n \le 3$. We note that in high dimensions , there are many "exotic" cube tiling. There are aperiodic cube tiling's in all dimensions $n \ge 3$, while in dimensions $n \ge 10$ there are cube tilings in which no two cubes share a commone $(n-1)$-face, see Lagarias and and shor [29].

In theorem (1.2.1), the n-cube $[0,1]^n$ appears in both conditions (i) and (ii), but in functorially different context, the n-cube in (i) lies in the space domain $\mathbb{R}^n$ while the $n - cube$ in (ii) lies in the Fourier domain $(\mathbb{R}^n)^*$, so they transform differently under linear change of variables. Thus theorem (1.2.1) is equivalent to the following result.

Theorem (1.2.2)[20]: For any invertible linear transformation $A \in GL(n,\mathbb{R})$ the following condition are equivalent.

(i) $A \subset \mathbb{R}^n$ Is a spectrum for $\Omega_A \coloneqq A([0,1]^n)$.

(ii)The collection of sets $\{\lambda + D_A : \lambda \in A\}$ is a tiling of $\mathbb{R}^n$, where $D_A = (A^T)^{-1}([0,1]^n)$.

The main result gives a necessary and sufficient condition for a general set A to be a spectrum of Ω in terms of a tiling of $\mathbb{R}^n$ by $A + D$, where D is a specified auxiliary set in Fourier space. This result applies whenever a suitable auxiliary set D exists. We show that this is the case when Ω is an n-cube, with D also begin an n-cube, and obtain Theorem (1.2.1).

Spectral sets were originally studied by Fuglede [21], who related them, to the problem of finding commuting self-adjoint extension in $L^2[\Omega]$ of the set of differential operators $-i\frac{\partial}{\partial x_1}, \dots, -i\frac{\partial}{\partial x_n}$ defined on the common dense domain $C_c^\infty(\Omega)$. Our definition of spectrum differs from his by a factor of 2π. Fuglede showed that for sufficiently nice connected open regions Ω, each spectrum A of Ω (in our sense) has $2\pi A$ as a joint spectrum of a set of commuting self-adjoint extensions of

$-i\frac{\partial}{\partial x_1}, \dots, -i\frac{\partial}{\partial x_n}$ and conversely. He also showed that only very special sets Ω are spectral sets. In particular , Fuglede [21, p 120] made the following conjecture.
A set Ω in $\mathbb{R}^n$ is a spectral set if and only if it tiles $\mathbb{R}^n$ by translations.

Much work on spectral sets is due to Jorgenson and Pedersen (see [24]-[26], [33], and [34]), with additional work by Lagarias and Wang [30].

The spectral set conjecture concerns tiling's by Ω in the space domain, Theorem (1.2.2), describes spectra Λ for the n-cube in terms of tiling in the Fourier domain by an auxiliary set D. in general there does not seem to be any simple relation between sets of translations T used to tile Ω in the space domain and the set of spectra Λ for Ω (see [25], [30], and [34]). We indicate a relation between the spectral set conjecture and tiling's in the Fourier domain – this is discussed.

Theorem (1.2.2), also implies a result concerning sampling and interpolation of certain classes of entire functions. Given a compact set Ω of nonzero Lebesgue measure, let $B_2(\Omega)$ denote the set of band-limited functions on Ω,which are those entire functions $f: \mathbb{C}^n \to \mathbb{C}$ whose restriction to $\mathbb{R}^n$ is the Fourier transform of an L^2-function with compact support contained in Ω. A countable set Λ is a set of sampling for $B_2(\Omega)$ if there exist, $A, B > 0$ such that for all $f \in B_2(\Omega)$,

$$A\|f\|^2 \le \sum_{\lambda\in\Lambda} |f(\lambda)|^2 \le B\|f\|^2. \tag{11}$$

A set of sampling is always a set of uniqueness for $B_2(\Omega)$, where a set Λ is a set of uniqueness if for each set of complex values $\{c_\lambda: \lambda \in \Lambda\}$ with $\sum |c_\lambda|^2 < \infty$ there is at most one function $f \in B_2(\Omega)$ with

$$f(\lambda) = c_\lambda, \quad \text{for each } \lambda \in \Lambda. \tag{12}$$

A set Λ is a set of interpolation for $B_2(\Omega)$ if for each such set $\{c_\lambda: \lambda \in \Lambda\}$ there is at least one function $f \in B_2(\Omega)$ such that (12) holds. It is clear that a spectrum Λ of a spectral set Ω is both a set of sampling and a set of interpolation for $B_2(\Omega)$, so Theorem (1.2.2) immediately yields the following theorem.

Theorem (1.2.3)[20]: Given a linear transformation A in $GL(n, \mathbb{R})$, set $\Omega_A = A([0,1]^n)$ and $D_A = (A^T)^{-1}([0,1]^n)$. If $\Lambda + D_A$ is a tiling of $\mathbb{R}^n$, then Λ both a set of sampling and a set of interpolation for $B_2(\Omega_A)$.

Here the set Λ has density exactly equal to the Nyquist rate $|\det(A)|$, as is required by results of Landau (see [31], [32]) for sets of sampling and interpolation.

We apply Theorem (1.2.1) to show that in dimensions $n = 1$ and $n = 2$ any orthogonal set of exponentials in $L^2([0,1]^n)$ can be completed to a basis of exponentials of $L^2([0,1]^n)$but that this is not always the case in dimensions $n \ge 3$.

We conclude with two remarks concerning the relation of spectral sets and tiling's. First, in comparison with other spectral sets, the n-cube $[0,1]^n$ has an enormous variety of spectra Λ. It seems likely that a "generic" spectral set has a unique spectrum, up to translations. Second, the tiling result applies to more general sets Ω than linearly transformed n-cubed $\Omega_A = A([0,1]^n)$; we give the one-dimensional example $\Omega = [0,1] \cup [2,3]$.

After completing a preprint in early 1998, we learned that A. losevich and S. Pedersen [23] simultaneously and independently obtained a proof of Theorem (1.2.1) by a different approach M. Kolountzakis and [28] has showed Conjecture (1.2.12) below, building on the approach.

Natation (1.2.4)[20]: For $x \in \mathbb{R}^n$, let $\|x\|$ denote the Education length of x. We let

$$B(x,T) \coloneqq \{y: \|y-x\| \leq T\}$$

denote the ball of radius T centered at x. The Lebesgue measure of a set Ω in $\mathbb{R}^n$ is denoted $m(\Omega)$. The Fourier transform $\hat{f}(u)$ is normalized by

$$\hat{f}(u) \coloneqq \int_{\mathbb{R}^n} e^{-2\pi i(u,x)} f(x)dx.$$

We let

$$e_\lambda(x) \coloneqq e^{2\pi i(\lambda,x)}, \quad \text{for } x \in \mathbb{R}^n. \tag{13}$$

Some (see [21], [26]) define $e_\lambda(x)$ differently , without the factor 2π in the exponent .

We consider packing's and tiling's in $\mathbb{R}^n$ by compact sets Ω of the following kind.

Definition (1.2.5)[20]: A compact set Ω in $\mathbb{R}^n$ is a regular region if it has positive Lebesgue measure $m(\Omega) > 0$, is the closure of its interior Ω^o, and has a boundary $\partial\Omega = \Omega/\Omega^o$ of measure zero.

Definition (1.2.6)[20]: If Ω is a regular region, then a discrete set A is a packing set for Ω if the sets $\{\Omega + \lambda : \lambda \in A\}$ have disjoint interiors. It is a tiling set if, in addition, the union of the sets $\{\Omega + \lambda : \lambda \in A\}$ covers $\mathbb{R}^n$. In these cases we say $A + \Omega$ is a packing or tiling of $\mathbb{R}^n$ by Ω, respectively.

To a vector λ in $\mathbb{R}^n$, we associate the exponential function.

$$e_\lambda(x) \coloneqq e^{2\pi i\langle \lambda ,x\rangle}, \quad \text{for } x \in \mathbb{R}^n. \tag{14}$$

Given a discrete set A in $\mathbb{R}^n$, we set

$$B_A \coloneqq \{e_\lambda(x) : \lambda \in A\}. \tag{15}$$

Now suppose that B_A restricted to a regular region Ω gives an orthogonal set of exponentials in $L^2(\Omega)$. We derive conditions that the points of A must satisfy. Let

$$x_\Omega(x) = \begin{cases} 1, & \text{for } x \in \Omega, \\ 0, & \text{for } x \notin \Omega \end{cases} \tag{16}$$

be the characteristic function of Ω, and consider its Fourier transform

$$\hat{x}_\Omega(u) = \int_{\mathbb{R}^n} e^{-2\pi i\langle u,x\rangle} x_\Omega(x)dx, \quad u \in \mathbb{R}^n. \tag{17}$$

Since Ω is compact, the function $\hat{x}_\Omega(u)$ is an entire function of $u \in \mathbb{C}^n$. We denote the set of real zeros of $\hat{x}_\Omega(u)$ by

$$Z(\Omega) \coloneqq \{u \in \mathbb{R}^n : \hat{x}_\Omega(u) = 0\}. \tag{18}$$

Lemma (1.2.7)[20]: If Ω is a regular region in $\mathbb{R}^n$, then a set A gives an orthogonal set of exponentials B_A in $L^2(\Omega)$ if and only if

$$A - A \subseteq Z(\Omega) \cup \{0\}. \tag{19}$$

Proof. For distinct $\lambda , \mu \in A$ we have

$$\hat{x}_\Omega(\lambda - \mu) = \int_{R^n} e^{-2\pi i\langle \lambda-\mu,x\rangle} x_\Omega(x)dx$$

$$= \int_{\Omega} e^{-2\pi i\langle\lambda,x\rangle} e^{2\pi i\langle\mu,x\rangle} dx = \langle e_\lambda, e_\mu\rangle_\Omega. \quad (20)$$

If (19) holds, then $\langle e_\lambda, e_\mu\rangle_\Omega = 0$, and conversely.

This lemma implies that the points of A have the property of being "well-spaced" in the sense of being uniformly discrete; that is, there is some positive R such that any two points are no closer than R. Indeed, since $\hat{x}_\Omega(0) = m(\Omega) > 0$, the continuity of $\hat{x}_\Omega(u)$ implies that there is some ball $B(0,R)$ around zero that includes no point of $Z(\Omega)$ hence, $\|\lambda - \mu\| \geq \mathbb{R}$ for all $\lambda, \mu \in A, \lambda \neq \mu$.

Definition (1.2.8)[20]: Let Ω be a regular region in $\mathbb{R}^n$. A regular region D is said to be an orthogonal packing region for Ω if

$$(D^o - D^o) \cap Z(\Omega) = \emptyset. \quad (21)$$

Lemma (1.2.9)[20]: Let Ω be a regular region in $\mathbb{R}^n$, and let D be an orthogonal packing region for Ω. If a set A gives an orthogonal set of exponentials B_A in $L^2(\Omega)$, then A is a packing set for D.

Proof. If $\lambda \neq \mu \in A$, then Lemma (1.2.7) gives $\lambda - \mu \in Z(\Omega)$. By definition of an orthogonal packing region we have $D^O \cap (D^O + u) = \emptyset$, for all $u \in Z(\Omega)$; hence

$$D^O \cap (D^O + \lambda - \mu) = \emptyset,$$

is required.

As indicated above, each regular region Ω has an orthogonal packing region D given by a ball $B(0,T)$ for small enough T. The larger we can take D, the stronger the restrictions imposed on A.

Lemma (1.2.10)[20]: If Ω is a spectral set and if D is an orthogonal packing region for Ω, then

$$m(D)m(\Omega) \leq 1. \quad (22)$$

Proof. Let A be a spectrum for Ω. Then A is a set of sampling for $B_2(\Omega)$, so the density results of Landau [31] (see also Gröchenig and Razafinjatovo [22]) give

$$d(A) = \lim_{n\to\infty} \inf \frac{1}{(2T)^n} \# (A \cap [-T,T]^n) \geq m(\Omega). \quad (23)$$

Now $A + D$ is a packing of $\mathbb{R}^n$, hence if $R = \text{diam}(D)$, we have

$$\frac{m(D)}{(2T)^n} \# (A \cap [-T,T]^n) = \frac{1}{(2T)^n} m(\{\cup_\lambda(\lambda + D): \lambda \in A \cap [-T,T]^n\})$$
$$\leq \frac{m([-T+R,T+R]^n)}{(2T)^n} = \left(1 + \frac{R}{2T}\right)^n. \quad (24)$$

Letting $T \to \infty$ and taking the inferior limit yields

$$m(D)d(A) \leq 1, \quad (25)$$

and now (23); Yields (22).

We give a self-contained proof of Lemma (1.2.10). The inequality of Lemma (1.2.10) Does not hold for general sets Ω. In fact the set $\Omega = [0,1] \cup [2, 2+\theta]$ for suitable irrational θ has a Fourier transform $\hat{x}_\Omega(\xi)$ that has no real zeros; so$Z(\Omega) = \emptyset$, and any regular region D, of arbitrarily large measure, is an orthogonal packing region for Ω.

In view of Lemma (1.2.10) we introduce the following terminology.

Definition (1.2.11)[20]: An orthogonal packing region D for a regular region Ω is tight if

$$m(D) = \frac{1}{m(\Omega)}. \tag{26}$$

This definition transforms in the Fourier domain under linear transformations: If D is a tight orthogonal packing region for a regular region Ω, then for any $A \in GL(n,\mathbb{R})$ the set $(A^T)^{-1}(D)$ is a tight orthogonal packing region for $A(\Omega)$.

There are many spectral sets that have tight orthogonal packing regions. We show that $D = [0,1]^n$ is a tight orthogonal packing region for $\Omega = [0,1]^n$. Another Example in $\mathbb{R}^1$ is the region

$$\Omega = [0,1] \cup [2,3]. \tag{27}$$

We can take

$$D = \left[0, \frac{1}{4}\right] \cup \left[\frac{1}{2}, \frac{3}{4}\right]. \tag{28}$$

Indeed, $\chi_\Omega(x)$ is the convolution of $\chi_{[0,1]}(x)$ with the sum of two delta functions $(\delta_0 + \delta_2)$. Thus

$$\hat{\chi}_\Omega(x) = \left(1 + e^{-4\pi i x}\right)\hat{\chi}_{[0,1]}(x). \tag{29}$$

From this it is easy to check that the zero set is given by

$$z(\Omega) = (\mathbb{Z}\backslash\{0\}) \cup \left(\frac{1}{4} + \mathbb{Z}\right) \cup \left(-\frac{1}{4} + \mathbb{Z}\right), \tag{30}$$

that D is an orthogonal packing region for Ω, and, since $m(D) = 1/2 = 1/m(\Omega)$, that D is tight. A spectrum for Ω is $A = \mathbb{Z} \cup \left(\mathbb{Z} + (1/4)\right)$.

Lemma (1.2.12)[20]: Together with the spectral set conjecture leads us to propose the following.

Conjecture (1.2.13)[20]: If Ω tiles $\mathbb{R}^n$ by translations and if D is an orthogonal packing region for Ω, then

$$m(\Omega)m(D) \le 1. \tag{31}$$

This conjecture has now been proved by Kolountzakis [28, theorem 7].

A main result is the following criterion that relates spectra to tilings in the Fourier domain.

Theorem (1.2.14)[20]: Let Ω be a regular region in $\mathbb{R}^n$, and let A be such that the of exponentials $\mathfrak{B}_A$ is orthogonal for $L^2(\Omega)$, suppose that D is a regular region with

$$m(\Omega)m(D) = 1 \tag{32}$$

Such that $A + D$ is a packing of $\mathbb{R}^n$. Then A is a spectrum for Ω if and only if $A + D$ is a tiling of $\mathbb{R}^n$.

Proof. suppose first that A is a spectrum for Ω. pick a "bump function" $\gamma(x) \in C_c^\infty(\Omega)$, and set

$$\gamma_t(x) = e^{-2\pi i\langle t,x\rangle}\gamma(x), \quad \text{for } t \in \mathbb{R}^n.$$

By hypothesis $\mathfrak{B}_A = \{e_\lambda(x) \colon \lambda \in A\}$ is orthogonal and complete for $L^2(\Omega)$. Thus, on Ω, we have

$$\gamma_t(x) \sim \sum_{\lambda \in A} \frac{\langle e^{2\pi i\langle\lambda,x\rangle}, \gamma_t(x)\rangle_\Omega}{\|e_\lambda\|_2^2} e^{2\pi i\langle\lambda,x\rangle}, \tag{33}$$

with coefficients

$$\frac{\langle e^{2\pi i\langle\lambda,x\rangle}, \gamma_t(x)\rangle_\Omega}{\|e_\lambda\|_2^2} = \frac{1}{m(\Omega)} \int_\Omega e^{-2\pi i\langle\lambda,x\rangle}\gamma_t(x)dx = \frac{1}{m(\Omega)} \int_{\mathbb{R}^n} e^{-2\pi i\langle\lambda+t,x\rangle}\gamma(x)dx$$

$$= \frac{1}{m(\Omega)} \hat{\gamma}(\lambda + t), \tag{34}$$

where $m(\Omega)$ is the Lebesgue measure of Ω. The rapid decrease of $\hat{\gamma}$ with increasing radius $\|x\|$ and the well-spaced property of A show that the right side of (33) converges absolutely and uniformly on $\mathbb{R}^n$. Since $\gamma_t(x)$ is continuous, we have

$$\gamma_t(x) = \frac{1}{m(\Omega)} \sum_{\lambda \in A} \hat{\gamma}(\lambda + t) e^{2\pi i \langle \lambda, x \rangle}, \qquad \text{for all } x \in \Omega. \tag{35}$$

This yields, for all $t \in \mathbb{R}^n$, that

$$\gamma(x) = e^{2\pi i \langle t, x \rangle} \gamma_t(x) = \frac{1}{m(\Omega)} \sum_{\lambda \in A} \hat{\gamma}(\lambda + t) e^{2\pi i \langle \lambda + t, x \rangle}, \qquad \text{for all } x \in \Omega. \tag{36}$$

The series on the right side of (36), converges absolutely and uniformly for all $x \in \mathbb{R}^n$, and t in any fixed compact subset of $\mathbb{R}^n$, but it is only guaranteed to agree with $\gamma(x)$ for $x \in \Omega$.

We now integrate both sides of (36) in t over all $t \in D$ to obtain

$$m(D)\gamma(x) = \gamma(x) \int_{\mathbb{R}^n} X_D(t) dt = \frac{1}{m(\Omega)} \sum_{\lambda \in A} \int_D \hat{\gamma}(\lambda + t) e^{2\pi i \langle \lambda + t, x \rangle} \, dt$$

$$= \frac{1}{m(\Omega)} \int_{A+D} \hat{\gamma}(u) e^{2\pi i \langle u, x \rangle} du, \qquad \text{for all } x \in \Omega. \tag{37}$$

In the last step we used the fact that the translates $\lambda + D$ overlap on sets of measure zero, because $A + D$ is packing of $\mathbb{R}^n$. Since $m(D) = 1/m(\Omega)$, (37) yields

$$\gamma(x) = \int_{\mathbb{R}^n} h(u) \, \hat{y}(u) e^{2\pi i \langle u, x \rangle} du, \qquad \text{for all } x \in \Omega. \tag{38}$$

where

$$h(u) = \begin{cases} 1, & \text{if } u \in A + D, \\ 0, & \text{otherewise.} \end{cases}$$

Define $k \in L^2(\mathbb{R}^n)$ by $\hat{k} = h\hat{y}$, so (38) asserts that $\gamma(x) = k(x)$ for almost all $x \in \Omega$. Plancherel's theorem on $L^2(\mathbb{R}^n)$ applied to k, together with (38), gives

$$\|\hat{\gamma}\|_2^2 \geq \|h\hat{y}\|_2^2 = \|k\|_2^2 \geq \int_\Omega |k(x)|^2 \, dx = \int_\Omega |\gamma(x)|^2 dx = \|\gamma\|_2^2. \tag{39}$$

Since plancherel's theorem also gives $\|\hat{\gamma}\|_2^2 = \|\gamma\|_2^2$, we must have

$$\|\hat{\gamma}\|_2^2 = \|h\hat{\gamma}\|_2^2. \tag{40}$$

We next show that this equality implies that $h(u) = 1$ almost everywhere on $\mathbb{R}^n$. To do this we show that $\hat{\gamma}(u) \neq 0$ in $\mathbb{R}^n$. Since γ has compact support, the paley-wiener theorem states that $\hat{\gamma}(u)$ is the restriction to $\mathbb{R}^n$ of an entire function on $\mathbb{C}^n$ that satisfies an exponential growth condition at infinity, see stein and Weiss [36, Theorem 4.9]. Thus $\hat{\gamma}(u)$ is real analytic on $\mathbb{R}^n$ and is not identically zero, hence

$$Z \coloneqq \{u \in \mathbb{R}^n : \hat{y}(u) = 0\}$$

has Lebesgue measure. Together with (40) this yields

$$h(u) = 1, \quad \text{a.e in } \mathbb{R}^n. \tag{41}$$

Thus, $A + D$ covers all $\mathbb{R}^n$ except a set of measure zero.

Finally we show that $A + D$ covers all of $\mathbb{R}^n$. By the well-spaced property of A and the compactness of D, the set $A + D$ is locally the union of finitely many translates of D ; hence $A + D$ is closed. Thus, the complement of $A + D$ is an open set. But the complement of $A + D$ has zero Lebesgue measure, hence, it is empty, so $A + D$ is a tiling of $\mathbb{R}^n$.

Suppose $A + D$ tiles $\mathbb{R}^n$. by hypothesis, $\mathfrak{B}_A$ is an orthogonal set in $L^2(\Omega)$, and to show that A is a spectrum it remains to show that it is complete in $L^2(\Omega)$. Let S be the closed span of $\mathfrak{B}_A$ in $L^2(\Omega)$.We show that $C_c^\infty(\Omega)$ is contained in S. Since $C_c^\infty(\Omega)$ is dense in $L^2(\Omega)$, this implies $S = L^2(\Omega)$.

For each $\gamma \in C_c^\infty(\Omega)$, set

$$\gamma_t(x) = e^{-2\pi i\langle t,x\rangle}\gamma(x), \quad \text{for } t \in \mathbb{R}^n.$$

Since the elements of $\mathfrak{B}_A$ are orthogonal, Bessel's inequality gives

$$\|\gamma_t\|^2 \geq \sum_{\lambda\in A} \frac{|\langle e_\lambda, \gamma_t\rangle|^2}{\|e_\lambda\|^2} = \frac{1}{m(\Omega)} \sum_{\lambda\in A} |\hat{\gamma}(\lambda + t)|^2, \tag{42}$$

where the last series converges uniformly on compact sets by the rapid decay of $\hat{\gamma}$ at infinity. Integrating this inequality over $t \in D$ yields

$$\int_D \|\gamma_t\|^2 dt \geq \frac{1}{m(\Omega)} \int_D \sum_{\lambda\in A} |\hat{\gamma}(\lambda + t)|^2 \, dt.$$

Since $\|\gamma_t\| = \|\gamma\|$ for all t and since $A + S$ is a tiling, we obtain $m(D)\|y\|^2 \geq \|\hat{y}\|^2/m(\Omega)$. But $m(D) = 1/m(\Omega)$ and $\|\gamma\|^2 = \|\hat{\gamma}\|^2$, so equality must hold in (42). For almost all t:

$$\|\gamma\|^2 = \sum_{\lambda\in A} \frac{|\langle e_\lambda, \gamma_t\rangle|^2}{\|e_\lambda\|_2{}^2}. \tag{43}$$

Now the right side of (43) converges uniformly on compact sets, so (43). holds for all $t = 0$. Hence

$$\|\gamma\|^2 = \sum_{\lambda\in A} \frac{|\langle e_\lambda, \gamma_t\rangle|^2}{\|e_\lambda\|_2^2},$$

and so $\gamma \in S$.

At first glance, the first half of this proof of Theorem (1.2.13) appears too good to be true because it only uses functions $\gamma_t(x)$ supported on a fixed subset of Ω .But the relevant fact is the that sup $\widehat{\gamma_t}$ is dense in Fourier space $\mathbb{R}^n$.

The proof of Theorem (1.2.13) yields a direct proof of Lemma (1.2.10). If D is an orthogonal packing set, then (37), holds for it, hence $m(D)m(\Omega)\gamma(x)$ agrees with $k(x)$ on Ω, and hence

$$m(D)m(\Omega)\|\gamma\|_2 \leq \|K\|_2 \leq \|\gamma\|_2,$$

which shows that (22) holds.

The following result is an immediate corollary of Theorem (1.2.35) [1] , which we state as a theorem for emphasis.

Theorem (1.2.15)[20]: Let Ω be a regular region in $\mathbb{R}^n$, suppose that D is a tight orthogonal packing region for Ω , then $A + D$ is a tiling of $\mathbb{R}^n$.

Proof. the assumption that D is tight orthogonal packing region guarantees that $A + D$ is a packing for all spectra A, so theorem (1.2.14), applies.

Theorem (1.2.15) sheds some light on Fugled's conjecture that every spectral set Ω tiles $\mathbb{R}^n$.

Definitions (1.2.16)[20]: A pair of regular regions $(\Omega,\hat{\Omega})$ is a tight dual pair if each is a tight orthogonal packing region for the other.

We show that $([0,1]^n,[0,1]^n)$ is a tight dual pair of regions; it follows that if $A\in GL(n,\mathbb{R})$, then $(A([0,1]^n),(A^T)^{-1}([0,1]^n))$ is also a tight dual pair of regions.
The sets $\left([0,1]\cup[2,3],\left[0,\frac{1}{4}\right]\cup\left[\frac{1}{2},\frac{3}{4}\right]\right)$ are a tight dual pair in $\mathbb{R}^1$.

If $(\Omega,\hat{\Omega})$ is a tight dual pair, then Theorem (1.2.14) states that if one of $(\Omega,\hat{\Omega})$ is a spectral set, say, Ω, then the other set $\hat{\Omega}$ tiles $\mathbb{R}^n$. if $\hat{\Omega}$ were also a spectral set (as the spectral set conjecture implies), then Theorem (1.2.14) would show that Ω tiles $\mathbb{R}^n$. This raises the question whether the current evidence in favor of Fuglede's conjecture is mainly based on sets Ω that appear in a tight dual pair $(\Omega,\hat{\Omega})$. We can only say that there are many nontrivial examples of tight dual pairs.

We formulate two conjectures.

Conjecture (1.2.17)[20]: (Spectral set duality conjecture). If $(\Omega,\hat{\Omega})$a tight dual pair of regular regions is and if Ω is a spectral set, then $\hat{\Omega}$ is also a spectral set.

In this case Theorem (1.2.15) would imply that both Ω and $\hat{\Omega}$ tile $\mathbb{R}^n$. The corresponding tiling analogue of this conjecture is as follows.

Conjecture (1.2.18)[20]: (weak spectral set conjecture). If $(\Omega,\hat{\Omega})$ is a tight dual pair of regular regions and if one of them tiles $\mathbb{R}^n$, then so does the other, and both Ω and $\hat{\Omega}$ are spectral sets.

We prove Theorem (1.2.1). We use the following basic result of Keller [27], which gives a necessary condition for a set A to give a cube tiling.

Proposition (1.2.19)[20]: (Keller's criterion). If $A+[0,1]^n$ is a tiling of $\mathbb{R}^n$, then each $\lambda,\mu\in A$ has

$$\lambda_i-\mu_i\in\mathbb{Z}\setminus\{0\}\quad\text{for some } i\,,1\le i\le n. \tag{44}$$

Proof. This result was proved by Keller [27] in 1930. A detailed proof appears in Perron [35, Satz 9].

The following lemma shows that Keller's necessary condition for a cube tiling is the same orthogonality of exponentials in the set A.

Lemma (1.2.20)[20]: $\mathfrak{B}_A\coloneqq\{e^{2\pi i\langle\lambda,x\rangle}:\lambda\in A\}$ gives a set of orthogonal functions in $L^2([0,1]^n)$ if and only if ,for any distinct $\lambda,\mu\in A$,

$$\lambda_j-\mu_j\in\mathbb{Z}\setminus\{0\},\quad\text{for some } j\,,1\le j\le n. \tag{45}$$

Proof. For $\Omega=[0,1]^n$ and $u\in\mathbb{R}^n$,

$$\hat{x}_\Omega(u)=\int_{[0,1]^n}e^{-2\pi i\langle u,x\rangle}dx=\prod_{j=1}^{n}h_0(u_j),$$

where $h_0(\omega)\coloneqq\left(1-e^{-2\pi i\omega}\right)/(-2\pi i\omega)\,,\omega\in\mathbb{R}$, and $h_0(0)\coloneqq 1$. Note that $h_0(\omega)\coloneqq 0$. If and only if $\omega\in\mathbb{Z}\setminus\{0\}$. Hence, $\hat{x}_\Omega(u)=0$ if and only if $u_j\in\mathbb{Z}\setminus\{0\}$ for some $j,1\le j\le n$. The lemma now follows immediately from Lemma (1.2.7).

This appendix determines in which dimensions n every orthogonal set of exponentials on the n-cube can be extended to an orthogonal basis of $L^2([0,1]^n)$.

Theorem (1.2.21)[20]: In dimensions $n=1$ and $n=2$,any orthogonal set of exponentials can be completed to an orthogonal basis of exponentials of $L^2([0,1]^n)$. In dimensions $n\ge 3$, this is not always the case.

Proof. we say that a cube packing $\Gamma + [0,1]^n$ is orthogonal if for distinct $\gamma, \mu \in \Gamma$,

$$\gamma_j - \mu_j \in \mathbb{Z} \setminus \{0\}, \quad \text{for some } j, 1 \le j \le n. \tag{46}$$

Now Proposition (1.2.19) (Keller's criterion) and Lemma (1.2.20), together imply that an orthogonal set of exponentials $\{e^{2\pi i \langle \gamma x \rangle}: \gamma \in \Gamma\}$ in $L^2([0,1]^n)$. Corresponds to an orthogonal cube packing using Γ. By Theorem (1.2.1), the question of whether an orthogonal set of exponentials in $L^2([0,1]^n)$ can be extended to an orthogonal basis of exponentials of $L^2([0,1]^n)$ is equivalent to asking whether the associated orthogonal cube packing in $\mathbb{R}^n$ can be completed to a cube tiling by adding extra cubes.

Using the known structure of one- and two-dimensional cube tilings , it is straight-forward to check that a completion of any orthogonal cube packing is always possible. (Two-dimensional cube tiling's always partition into either all horizontal rows of cubes or all vertical columns of cubes).

To show that extendibility is not always possible in dimension 3, consider the set of four cubes $\{v^{(i)} + [0,1]^3: 1 \le i \le 4\}$ in $\mathbb{R}^3$, given by

$$v^{(1)} = \left(-1,0,-\tfrac{1}{2}\right),$$
$$v^{(2)} = \left(-\tfrac{1}{2},-1,0\right),$$
$$v^{(3)} = \left(0,-\tfrac{1}{2},-1\right),$$
$$v^{(4)} = \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right).$$

The orthogonality condition (46) is easily verified. The cubes corresponding to $v^{(1)}$ through $v^{(3)}$ contain $(0,0,0)$ on their boundary and create a corner $(0,0,0)$. Any cube tiling that extended $\{v^i + [0,1]^3 : 1 \le i \le 3\}$ would have to fill this corner by including the cube $[0,1]^3$. However, $[0,1]^3$ has nonempty interior in common with $v^4 + [0,1]^3$.

This construction easily generalizes to $\mathbb{R}^n$ for $n \ge 3$.

Section (1.3): Non-Uniform Gabor Bases

For $\mathcal{F}$ and $\mathcal{T}$ be two discrete subsets $\mathbb{R}^d$, and let $\mathrm{g}(x) \in L^2(\mathbb{R}^d)$. The Gabor system (also known as the *Weyl-Heisenberg system*) with respect to $\mathcal{F}, \mathcal{T}$ and g is following family of functions in $L^2(\mathbb{R}^d)$:

$$G(\mathcal{F},\mathcal{T},\mathrm{g}) := \{e^{2\pi i \lambda x}\mathrm{g}(x-p) \mid \lambda \in \mathcal{F}, p \in \mathcal{T}\}. \tag{47}$$

Such a family was first introduce by Gabor [49] in 1946 for signal processing, and is still widely used today. We call $G(\mathcal{F},\mathcal{T},\mathrm{g})$ an (*orthonormal*) Gabor basis if it is an orthonormal basis for $L^2(\mathbb{R}^d)$, and a *Gabor frame* if it is a frame for $L^2(\mathbb{R}^d)$. Gabor bases and frames have been extensively studied. Apart from their important applications in digital signal processing, they are significant mathematical entities on their own. They are an integral part of time-frequency analysis, and are closely related to the study of wavelets and spectral sets.

Most of the study of Gabor bases have focused on "*uniform*" sets $\mathcal{F}$ and $\mathcal{T}$, i.e. they are taken to be lattices. For full rank lattices $\mathcal{F} = A(\mathbb{Z}^d)$ and $\mathcal{T} = B(\mathbb{Z}^d)$, a Gabor basis $G(\mathcal{F},\mathcal{T},\mathrm{g})$ must satisfy $|\det(AB)| = 1$, see [64] for $d = 1$ and [63] for arbitrary d. Conversely, if $|\det(AB)| = 1$ then there exists a function $\mathrm{g}(x) \in L^2(\mathbb{R}^d)$ such that $G(\mathcal{F},\mathcal{T},\mathrm{g})$ is an orthonormal Gabor basis, see [52]. The function g is not necessarily compactly supported. A compactly supported g can be found if $\mathcal{F}$ and $\mathcal{T}^*$(the dual lattice of $\mathcal{T}$) are commensurable. Also known is the Balian-Low theorem, which states that if $G(\mathcal{F},\mathcal{T},\mathrm{g})$ is an orthonormal Gabor basis and g is compactly supported then g cannot be very smooth, see [38].

The study of non-uniform or irregular Gabor bases and frames, i.e. those without the lattice condition on $\mathcal{F}$ or $\mathcal{T}$, has gained considerable interest (see, [42]). It is known that if $G(\mathcal{F},\mathcal{T},\mathrm{g})$ is a Riesz basis then both $\mathcal{F}$ and $\mathcal{T}$ must be uniformly discrete, i.e there exists an $\varepsilon > 0$ such that they are ε-separated. The density result by Ramanthan and Steger [63] was actually established in a much more general setting. In a Gabor basis $G(\mathcal{F},\mathcal{T},\mathrm{g})$ the sets $\mathcal{F}$ and $\mathcal{T}$ satisfy the density condition $D(\mathcal{F})D(\mathcal{T}) = 1$ where $D(\cdot)$ is the Beurling density. For a set $\mathcal{J}$ in $\mathbb{R}^d$ the upper and lower Beurling density of $\mathcal{J}$ respectively are defined as

$$D^+(\mathcal{J}) = \limsup_{r\to\infty} \sup_{x\in\mathbb{R}^d} \frac{|\mathcal{J} \cap (x + [0,r]^d|}{|r^d|},$$

$$D^-(\mathcal{J}) = \liminf_{r\to\infty} \inf_{x\in\mathbb{R}^d} \frac{|\mathcal{J} \cap (x + [0,r]^d|}{|r^d|}.$$

If $D^+(\mathcal{J}) = D^-(\mathcal{J})$ then $D(\mathcal{J}) = D^+(\mathcal{J}) = D^-(\mathcal{J})$ is the Beurling density of $\mathcal{J}$. But oddly, despite the many studies on non-uniform Gabor bases none contained a single example of an orthonormal Gabor basis that is non-uniform in the sense that either $\mathcal{F}$ or $\mathcal{T}$ is nonperiodic. We have not seen an example given in the Gabor literature in which an orthonormal Gabor basis has non-lattice $\mathcal{F}$ and $\mathcal{T}$. Another observation is that there is not a single example of a compactly supported orthonormal Gabor basis in which the generating function $\mathrm{g}(x)$ does not satisfy $|\mathrm{g}(x)| = (1/\sqrt{\mu(\Omega)})x_\Omega(x))$ for some bounded set Ω. These observations Lead to the following questions: Are there any non-uniform orthonormal Gabor bases, and are there compactly supported orthonormal Gabor bases $G(\mathcal{F},\mathcal{T},\mathrm{g})$ in which $|\mathrm{g}(x)| \neq (1/\sqrt{\mu(\Omega)})x_\Omega(x))$?

As we shall demonstrate, there are indeed non-uniform orthonormal Gabor bases in dimension $d \geq 2$. In the one dimension there exists orthonormal Gabor bases $G(\mathcal{F},\mathcal{T},\mathrm{g})$ in which neither $\mathcal{F}$ nor $\mathcal{T}$ is a lattice. These results follow rather easily from the work on spectral sets. If g is compactly supported we establish:

We assume that supp(g) is an interval, then the main theorem of ours below states that the only such orthonormal Gabor bases are the ‘‘*trivial*‘‘ bases.

We shall use $e(x)$ to denote $e^{2\pi ix}$.

Lemma (1.3.1)[37]: Let $f(x)$ be a compactly supported function in $L^1(\mathbb{R})$, and $\mathcal{T} \subset \mathbb{R}$ be a discrete set with $D^+(\mathcal{T}) < \infty$. Suppose that $\sum_{p\in\mathcal{T}} f(x-p) = c$ for all $x \in \mathbb{R}$. Then $\mathcal{T}$ is a union of (possibly translated) lattices

$$\mathcal{T} = \bigcup_{j=1}^{N} (L_j\mathbb{Z} + b_j)$$

for some real $L_j \neq 0$, and $b_j, 1 \leq j \leq N$.

Proof. See [55].

Theorem (1.3.2)[37]: Let $\mathrm{g}(x) \in L^2(\mathbb{R})$ be compactly supported and let $\mathcal{F},\mathcal{T}$ be subset of $\mathbb{R}$. Suppose that $G(\mathcal{F},\mathcal{T},\mathrm{g})$ is an orthonormal Gabor basis and $\mathcal{F}$ is periodic. Then $\mathcal{T}$ must be periodic.

Proof. $\mathcal{F}$ is periodic so we may write $\mathcal{F} = L\mathbb{Z} + \mathcal{A}$ for some $L \neq 0$ and finite set $\mathcal{A} \subset \mathbb{R}$. Without loss of generality we assume that $L = 1$ and $|A| = m$.

Since we have an orthonormal Gabor basis $G(\mathcal{F}, \mathcal{T}, g)$ applying Parseval's identity to the function $\phi_t(x) = X_{[t,t+1]}(x)$ yields

$$\|\phi_t\|^2 = \sum_{p\in\mathcal{T}}\sum_{\lambda\in\mathcal{F}} |\langle\phi_t(x), e(\lambda, x)\mathrm{g}(x-p)\rangle|^2 = \sum_{p\in\mathcal{T}}\sum_{\lambda\in\mathcal{F}}\left|\int_t^{t+1} e(-\lambda x)\overline{g(x-p)}\,dx\right|^2$$

$$= \sum_{p\in\mathcal{T}}\sum_{a\in A}\sum_{n\in\mathbb{Z}}\left|\int_t^{t+1} e(-(n+a)x)\overline{g(x-p)}\,dx\right|^2.$$

Observe that $\{e((n+a)x): n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2([t, t+1])$ for any t. Therefore another application of parseval's identity yields

$$\sum_{n\in\mathbb{Z}}\left|\int_t^{t+1} e(-(n+a)x)\overline{g(x-p)}dx\right|^2 = \int_t^{t+1} |\mathrm{g}(x-p)|^2 dx.$$

Thus

$$\|\phi_t\|^2 = \sum_{p\in\mathcal{T}}\sum_{a\in A}\sum_{n\in\mathbb{Z}}\left|\int_t^{t+1} e(-(n+a)x)\overline{\mathrm{g}(x-p)}\,dx\right|^2 = \sum_{p\in\mathcal{T}}\sum_{a\in A}\int_t^{t+1} |\mathrm{g}(x-p)|^2 dx$$

$$= |\mathcal{A}|\sum_{p\in\mathcal{T}}\int_t^{t+1} |\mathrm{g}(x-p)|^2 dx$$

Now set $(t) = \int_t^{t+1}|\mathrm{g}(x)|^2 dx$. Then $\int_t^{t+1}|\mathrm{g}(x-p)|^2 dx = f(t-p)$. We have $\|\phi_t\|^2 = 1$ for all t. So

$$\sum_{p\in\mathcal{T}}\int_t^{t+1} |\mathrm{g}(x-p)|^2 dx = \sum_{p\in\mathcal{T}} f(t-p) = |\mathcal{A}|^{-1}. \qquad (48)$$

As g is compactly supported, so must be $f(t)$, and $\int_{\mathbb{R}} f(t)dt < \infty$, By Lemma (1.3.1) $\mathcal{T}$ must be a union of (possibly translated) lattices,

$$\mathcal{T} = \bigcup_{j=1}^{k}(L_j\mathbb{Z} + b_i)$$

for some real $L_j \neq 0$ and b_j, we claim that all $L_i/L_j \in \mathbb{Q}$, if not, say L_1/L_2 is irrational, then a theorem of kronecker (see [43]) states that $L_1\mathbb{Z} - L_2\mathbb{Z}$ is dense in $\mathbb{R}$. Hence there exist $P_1 \in L_1\mathbb{Z} + b_1$ and $p_2 \in L_2\mathbb{Z} + b_2$ such that $p_1 - p_2$ can become arbitrarily small. This contradicts the fact that $\mathrm{g}(x - p_1)$ and $\mathrm{g}(x - p_2)$ are orthogonal in $L^2(\mathbb{R})$. Hence all L_j must be commensurable. So $\mathcal{T}$ is periodic, showing the theorem.

We next show the following theorem.

Lemma (1.3.3)[37]: Let $G(\mathcal{F}, \mathcal{T}, \mathrm{g})$ be an orthonormal Gabor basis. Then $D(\mathcal{F})$ and $D(\mathcal{T})$ both exist and $D(\mathcal{F})D(\mathcal{T}) = 1$.

Proof. It was shown in [26] that $D(\mathcal{F}\times\mathcal{T})$ exists and is equal to 1. But it is easy to show that $D^-(\mathcal{F}\times\mathcal{T})=D^-(\mathcal{F})D^-(\mathcal{T})$ and $D^+(\mathcal{F}\times\mathcal{T})=D^+(\mathcal{F})D^+(\mathcal{T})$. this lemma follows immediately.

Lemma (1.3.4)[37]: Let $\mathcal{F}$ be a uniformly discrete subset of $\mathbb{R}$ with $D^+(\mathcal{F})\le 1$. suppose that for some sequence $c_\lambda\colon \lambda\in\mathcal{F}\in\ell^2(\mathcal{F})$ we have

$$\sum_{\lambda\in\mathcal{F}} c_\lambda\, e(\lambda x)=0$$

in $L^2([a,b])$ with $b-a>1$ then $c_\lambda=0$ for all $\lambda\in\mathcal{F}$.

Proof. Assume $c_{\lambda_0}\neq 0$ for some $\lambda_0\in\mathcal{F}$. Let $\mathcal{F}_0=\mathcal{F}\setminus\{\lambda_0\}$. then

$$e(\lambda_0 x)=\sum_{\lambda\in} b_\lambda e(\lambda x),$$

where $b_\lambda=-c_\lambda/c_{\lambda_0}$ in $L^2([a,b])$. A theorem of Young (see [67, p.129]) states that $\{e(\lambda x)\colon \lambda\in\mathcal{F}_0\}$ is complete in $L^2([a,b])$.

Now Theorem 2.4 of seip [66] states that $\mathcal{F}$ can be extended to $\mathcal{F}'$ so that $\{e(\lambda x)\colon \lambda\in\mathcal{F}'\}$ is a Riesz basis for $L^2([a,b])$. Therefore

$$\left\|\sum_{\lambda\in\mathcal{F}} c_\lambda\, e(\lambda x)\right\|^2\ge B\sum_{\lambda\in\mathcal{F}}|c_\lambda|^2$$

for some $B>0$ this is a contradiction.

Lemma (1.3.5)[37]: Let $\mathcal{F}$ be a uniformly discrete subset of $\mathbb{R}$ such that $D^-(\mathcal{F})>0$. let $\{c_\lambda\colon \lambda\in\mathcal{F}\}$ be a sequence in $\ell^2(\mathcal{F})$. Then $f(x)\coloneqq\sum_{\lambda\in\mathcal{F}} c_\lambda\, e(\lambda x)\in L^2([a,b])$ for any interval $[a,b]$, and $\|f\|^2\le C\sum_{\lambda\in\mathcal{F}}|c_\lambda|^2$, where C depends only on $\mathcal{F}$ and $b-a$.

Proof. The fact that $\mathcal{F}$ is uniformly discrete and $D^-(\mathcal{F})>0$. Implies that there exists a sufficiently small $\delta>0$ such that $\{e(\lambda x)\colon \lambda\in\mathcal{F}\}$ is a frame for $L^2([s,s+\delta])$ for any s. Let $\|\cdot\|_\Omega$ denote the L^2-norm for $L^2(\Omega)$. It follows from [46] that the sum converges in $L^2([s,s+\delta])$ with

$$\|f\|^2_{[s,s+\delta]}\le B\sum_{\lambda\in\mathcal{F}}|c_\lambda|^2, \tag{49}$$

where B is the upper frame bound for the frame. Subdividing $[a,b]$ into $k\le(b-a)/\delta$ intervals of length δ or less yields

$$\|f\|^2_{[a,b]}\le KB\sum_{\lambda\in\mathcal{F}}|c_\lambda|^2. \tag{50}$$

We remark that if the condition $D^-(\mathcal{F})>0$ is dropped in the above lemma then $f(x)\coloneqq\sum_{\lambda\in\mathcal{F}} c_\lambda\, e(\lambda x)\in L^2([a,b])$ still holds, see [67, section 4.3]. It is unclear whether (50) also holds. The weaker result is sufficient for our purpose.

Lemma (1.3.6)[37]: Let $G(\mathcal{F},\mathcal{T},\mathrm{g})$ be an orthonormal Gabor basis for $L^2(\mathbb{R})$ with $\mathrm{supp}(\mathrm{g})=[0,a]$. Suppose that $D(\mathcal{F})=1$ and let $\mathcal{T}=\{y_n\colon n\in\mathbb{Z}\}$ with $y_n<y_{n+1}$. Then $y_{n+1}-y_n\le 1$ for all $n\in\mathbb{Z}$.

Proof. Assume the lemma is false. Then without loss of generality we may assume $y_1-y_0>1$ and $y_0=0$. We shall derive a contradiction.

Clearly $a\ge y_1$, for if not then any function $h(x)\in L^2(\mathbb{R})$, with $\mathrm{supp}(h)\subseteq[a,y_1]$ will be orthogonal to all functions in $G(\mathcal{F},\mathcal{T},\mathrm{g})$, a contradiction. We now choose $\varepsilon>0$, such that $y_1-\varepsilon>1$ and $y_{-1}+\varepsilon<0$. Let $f(x)$ be any L^2 function supported in $[a-\varepsilon,a]$, Then

$$f(x)=\sum_{n\in\mathbb{Z}}\sum_{\lambda\in\mathcal{F}} c_{n,\lambda} e(\lambda x)\mathrm{g}(x-y_n), \quad (51)$$

where $c_{n,\lambda}=\int_{a-\varepsilon}^{a} f(t)e(-\lambda t)\overline{\mathrm{g}(t-y_n)}dt$. By the choice of ε the coefficients $c_{n,\lambda}=0$ for all but finitely many n, in particular $c_{n,\lambda}=0$ for all $n<0$. So

$$f(x)=\sum_{n=0}^{N}\sum_{\lambda\in\mathcal{F}} c_{n,\lambda} e(\lambda x)\mathrm{g}(x-y_n)$$

$$=\sum_{\lambda\in\mathcal{F}} c_{0,\lambda} e(\lambda x)\mathrm{g}(x)+\sum_{n=1}^{N}\sum_{\lambda\in\mathcal{F}} c_{n,\lambda} e(\lambda x)\mathrm{g}(x-y_n). \quad (52)$$

For $n>1$ we have $y_n\ge y_1>1$, so $\mathrm{g}(x-y_n)=0$ for $x\in[0,y_1-\varepsilon]$.

We now restrict $f(x)$ to $x\in[0,y_1-\varepsilon]$ as a function in $L^2([0,y_1-\varepsilon])$, which is 0. Note that $\mathcal{F}$ is uniformly discrete and $D^-(\mathcal{F})>0$, because $G(\mathcal{F},\mathcal{T},\mathrm{g})$ is an orthonormal basis. By Lemma (1.3.5) each sum $\sum_{\lambda\in\mathcal{F}} c_{n,\lambda}e(\lambda x)$ converges in $L^2([0,y_1-\varepsilon])$ for each n. Note that $\mathrm{g}(x-y_n)=0$ on $[0,y_1-\varepsilon]$ for all $n>0$, so (52) yields

$$0=f(x)=\sum_{\lambda\in\mathcal{F}} c_{0,\lambda}e(\lambda x)\,\mathrm{g}(x)$$

on $[0,y_1-\varepsilon]$. But $\mathrm{g}(x)\neq 0$, on $[0,y_1-\varepsilon]$ .Thus

$$\sum_{\lambda\in\mathcal{F}} c_{0,\lambda}e(\lambda x)=0$$

on $[0,y_1-\varepsilon]$, it follows from Lemma (1.3.4) that $c_{0,\lambda}=0$, for all $\lambda\in\mathcal{F}$. However, $c_{0,\lambda}=\langle f(x),e(\lambda x)\mathrm{g}(x)\rangle=0$ implies that $f(x)\overline{\mathrm{g}(x)}$ is orthogonal to the set of functions $\{e(\lambda x):\lambda\in\mathcal{F}\}$ for all functions $f(x)$ with $\mathrm{supp}(f)\subseteq[a-\varepsilon,a]$. This is a contradiction.

For any $\mathcal{F}\in\mathbb{R}$ the upper asymptotic density $D_u(\mathcal{F})$ is defined as

$$D_u(\mathcal{F}):=\limsup_{r\to\infty}\frac{|\mathcal{F}\cap[-r\,,r]|}{2r}.$$

It is easy to see that $D^-(\mathcal{F})\le D_u(\mathcal{F})\le D^+(\mathcal{F})$. In particular if $D(\mathcal{F})$ exists then $D_u(\mathcal{F})=D(\mathcal{F})$. The following lemma is a key to proving Theorem (1.3.8) it was proved in [57] in a much stronger form. We include the weak form here for completeness.

Lemma (1.3.7)[37]: Let $\mathcal{F}$ be a subset of $\mathbb{R}$ with $D(\mathcal{F})=1$.Suppose that $D_u(\mathcal{F}-\mathcal{F})\le 1$. Then $\mathcal{F}-\mathcal{F}=\mathbb{Z}$.

Proof. Without loss of generality we assume that $0\in\mathcal{F}$. Hence $\mathcal{F}-\mathcal{F}\supseteq\mathcal{F}$. Clearly this means $D_u(\mathcal{F}-\mathcal{F})\ge D(\mathcal{F})=1$. This yields $D_u(\mathcal{F}-\mathcal{F})=1$. We show $\mathcal{F}-\mathcal{F}$ is a group.

Denote $G=\mathcal{F}-\mathcal{F}$. For any $a\in\mathcal{F}$ observe that $\mathcal{F}-a$ has Beurling density $D(\mathcal{F}-a)=1$. But $\mathcal{F}-a\subseteq G$ and $D_u(G)=1$. This implies that $G=(\mathcal{F}-a)\cup\varepsilon_a$ with $D_u(\varepsilon_a)=0$ .Similarly $G=(\mathcal{F}-b)\cup\varepsilon_b$ for any $b\in\mathcal{F}$ with $D_u(\varepsilon_b)=0$. Denote $\mathcal{F}_{a\,,b}=(\mathcal{F}-a)\cap(\mathcal{F}-b)$. We therefore must have $D_u(\mathcal{F}_{a\,,b})=1$, since $G\setminus\mathcal{F}_{a\,,b}=\varepsilon_a\cup\varepsilon_b$ has Beurling density 0. It follows that $D_u(\mathcal{F}_{a\,,b}+b)=1$. In other words, $\mathcal{F}\cap(\mathcal{F}-a+b)$ has upper asymptotic density 1.

Now take any $a_1, a_2, b_1, b_2 \in \mathcal{F}$. The above yields that both $\mathcal{F} \cap (\mathcal{F} - a_1 + b_1)$ and $\mathcal{F} \cap (\mathcal{F} - a_2 + b_2)$ have upper asymptotic density 1. But both are subsets of $\mathcal{F}$, which itself has upper asymptotic density 1. Therefore the two sets must intersect, or the upper asymptotic density of 1. Therefore the two sets must intersect ,or the upper asymptotic density of $\mathcal{F}$ would have to be at least 2. This means there exist $c_1, c_2 \in \mathcal{F}$, such that $c_1 - a_1 + b_1 = c_2 - a_1 + b_1$, which gives us $(b_2 - a_2) - (b_1 - a_1) = c_1 - c_2) \in G$. Since $a_1, a_2, b_1, b_2 \in \mathcal{F}$ are arbitrary we conclude that G is closed under subtraction. Hence G is group.

The only discrete subgroups of $\mathbb{R}$ with bounded densities are cyclic groups. So $G = L\mathbb{Z}$ for some $\in \mathbb{R}$. But the density of G is 1. Hence $G = \mathcal{F} - \mathcal{F} = \mathbb{Z}$.

Theorem (1.3.8)[37]: Let $g(x) \in L^2(\mathbb{R})$ such that supp(g) is an interval, and let $\mathcal{F}, \mathcal{T}$ be subset of $\mathbb{R}$. Suppose that $G(\mathcal{F}, \mathcal{T}, g)$ is an orthonormal Gabor basis. Then both $\mathcal{F}$ and $\mathcal{T}$ must be (possibly translated) lattices. In other words there exist real numbers $a > 0$ and b_1, b_2 such that $\mathcal{F} = a\mathbb{Z} + b_1$ and $\mathcal{T} = a^{-1}\mathbb{Z} + b_2$. Furthermore, $|g(x)| = \sqrt{a}x_{\Omega}(x)$ where Ω is an interval of length a^{-1}.

Proof. Without loss of generality we may assume that $D(\mathcal{F}) = D(\mathcal{T}) = 1$ by lemma (1.3.3), $0 \in \mathcal{F}, 0 \in \mathcal{T}$ and that $supp(g) = [0, a]$. Since $\mathcal{T}$ is uniformly discrete we may write it as $\mathcal{T} = \{y_n : n \in \mathbb{Z}\}$ in ascending order.

Claim (1.3.9)[37]: $\mathcal{F} = \mathbb{Z}$ And if $0 < a - (y_k - y_n) \leq 1$, then $a - (y_k - y_n) = 1$.

Proof. For $0 < b := a - (y_k - y_n) \leq 1$ let $h(x) = g(x - y_n)\overline{g(x - y_k)}$. Then $h(x)$ is supported on an interval of length $b \leq 1$. The orthogonality of Gabor basis yields $\hat{h}(\xi) = 0$. for all $\xi \in \mathcal{F} - \mathcal{F}, \xi \neq 0$. (we would even have $\hat{h}(0) = 0$ if $k \neq n$.)

But note that by the Paley-Wiener theorem $\hat{h}$ is an entire function of exponential type restricted to the real's, and such functions can note have "too many" zeros. In fact, it follows from Theorem 8.4.16 of Boas [40] that the set of zeros of $\hat{h}$ has an upper asymptotic density at most b, i.e. $D^+(\mathcal{F} - \mathcal{F}) \leq b \leq 1$. It follows from $D(f) = 1$ and Lemma (1.3.7) that $b = 1$ and $\mathcal{F} - \mathcal{F} = \mathbb{Z}$. Since $0 \in \mathcal{F}$ we get $\mathcal{F} \subseteq \mathbb{Z}$.

Now suppose that $\mathcal{F} \neq \mathbb{Z}$. Then we may replace $\mathcal{F}$ by $\mathbb{Z}$ and the orthonormality is again satisfied with the new Gabor system $G(\mathbb{Z}, \mathcal{T}, g)$. This contradicts the fact that $G(\mathcal{F}, \mathcal{T}, g)$ is a basis. Therefore $\mathcal{F} = \mathbb{Z}$. The claim is proved.

We next show $\mathcal{T} = \mathbb{Z}$ by showing that for any n we have $y_n - y_{n-1} = 1$. Assume that this is not true, then there exists an n such that $y_n - y_{n-1} < 1$. We choose k to be the largest index such that $0 \leq y_k - y_n < a$. Since each $y_{j+1} - y_j \leq 1$. By lemma (1.3.4), $0 < a - (y_k - y_n) \leq 1$. It follows from the claim that $a - (y_k - y_n) = 1$. But we now have $0 < a - (y_k - y_{n-1}) < 1$ Because $y_n - y_{n-1} < 1$. This contradicts claim (1.3.9). Hence $y_n - y_{n-1} = 1$, for all n and $\mathcal{T} = \mathbb{Z}$.

Finally we show $a = 1$ and $|g(x)| = X_{[0,1]}(x)$. it is easy to see that $a \geq 1$, which in fact follows from the claim. Assume that $a > 1$. Then there exists an $n \neq 0$ such that $0 < b := a - n \leq 1$. (the claim actually gives $b = 1$, but we don't need it). Now the function $h(x) = g(x)\overline{g(x - n)}$ is supported on $[n, a]$ and is orthogonal to $e(\lambda x)$ for all $\lambda \in \mathcal{F} = \mathbb{Z}$. This is impossible unless $h(x) = 0$, which is not the case since g is supported on $[0, a]$. Hence $a = 1$. So $|g|^2(x)$ is orthogonal to $e(\lambda x)$ for all $\lambda \in \mathbb{Z} \setminus \{0\}$. This forces $|g|^2(x) = c$. the orthonormality of the Gabor basis now yields $c = 1$, and $|g| = X_{[0,1]}$.

The study of orthonormal Gabor bases is actually closely related to the study of spectral sets and their spectra, a link that has not been exploited in the Gabor. A measurable set Ω in $\mathbb{R}^d$ with positive and finite measure is called a spectral set if there exists an $\mathcal{F} \subset \mathbb{R}^d$ such that set of exponentials $\{e(\lambda.x): \lambda \in \mathcal{F}\}$ is an orthogonal basis for $L^2(\Omega)$. In this case $\mathcal{F}$ is called a spectrum of Ω. A spectral set Ω may have more than one spectrum. Spectral sets have been studied rather extensively. The major unsolved problem concerning spectral sets is the following conjecture of Fuglede [48].

Let Ω be a set in $\mathbb{R}^d$ with positive and finite Lebesgue measure. Then Ω is spectral set if and only if Ω tiles $\mathbb{R}^d$ by translation.

Here by Ω tiles we mean there exists a $\mathcal{T} \subset \mathbb{R}^d$ such that $\Omega + \mathcal{T}$ is a measure-disjoint covering of $\mathbb{R}^d$, i.e $\sum_{p\in\mathcal{T}} x_\Omega\,(x-p) = 1$, for almost all $x \in \mathbb{R}^d$. the set $\mathcal{T}$ is called a tiling set for Ω. The spectral set conjecture remains open in either direction, even in dimension one and for sets that are unions of unit intervals. Furthermore, there appears to be a one-to-one correspondence between spectra of spectral set and its tiling. We give several examples, based on the study of spectral sets. First we establish:

Lemma (1.3.10)[37]: let $\Omega \subset \mathbb{R}^d$ with $0 < \mu(\Omega) < \infty$. Suppose that Ω is spectral set with a spectrum $\mathcal{F}$ and it tiles $\mathbb{R}^d$ by the tiling set $\mathcal{T}$. Let g(x) be any function with $|\mathrm{g}| = (1/\sqrt{\mu(\Omega)})x_\Omega$. Then $G(\mathcal{F},\mathcal{T},\mathrm{g})$ is an orthonormal Gabor basis for $L^2(\mathbb{R}^d)$.

Proof. The proof is rather standard, and we shall give a quick sketch here. The orthonormality is clear. Take any $e(\lambda_1.x)\mathrm{g}(x-p_1)$ and $e(\lambda_2.x)\mathrm{g}(x-p_2)$ in $\boldsymbol{G}(\mathcal{F},\mathcal{T},\mathrm{g})$. If $p_1 \neq p_2$ then $\mathrm{g}(x-p_1)$ and $\mathrm{g}(x-p_2)$ have disjoint support as a result of the tiling property. So the two functions are orthogonal. If $p_1 = p_2$ then $\lambda_1 \neq \lambda_2$. Hence $\langle e(\lambda_1.x), e(\lambda_2.x)\rangle = 0$ by the spectral set property. Therefore

$$\langle e(\lambda_1.x)\mathrm{g}(x-p_1), e(\lambda_2.x)\mathrm{g}(x-p_1)\rangle = \langle e(\lambda_1.x), e(\lambda_2.x)\rangle = 0.$$

To see the completeness observe that the set of functions $\{e(\lambda.x)\mathrm{g}(x-p): \lambda \in \mathcal{F}\}$ is complete in $L^2(\Omega + p)$ becouse $\mathcal{F}$ is a spectrum for $\Omega + p$ and $|\mathrm{g}(x-p)|$ is a nonzero constant on $\Omega + p$. Now every $f(x) \in L^2(\mathbb{R}^d)$ can be expressed as $f(x) = \sum_{p\in\mathcal{T}} f_p(x)$ where $f_p(x) \coloneqq f(x)x_\Omega\,(x-p)$ as a result of the tiling property. But $f_p(x) \in L^2(\Omega + p)$. Standard argument now implies $G(\mathcal{F},\mathcal{T},\mathrm{g})$ is complete in $L^2(\mathbb{R})$, proving the lemma.

We shall refer to an orthonormal Gabor basis obtained in such a way as standard orthonormal Gabor basis. Standard Gabor bases nevertheless yield nontrivial examples of non-uniform Gabor bases.

Example (1.3.11)[37]: In dimension $d \geq 2$ there exist compactly supported orthonormal Gabor bases $G(\mathcal{F},\mathcal{T},\mathrm{g})$ in which both $\mathcal{F}$ and $\mathcal{T}$ are nonperiodic.

Let $\Omega = [0,1]^d$ be the unit d-cube. it is well known that there are nonperiodic tilings using the unit cube. In fact, for $d \geq 3$ there are completely aperiodic cube tilings, see [59]. (For $d = 2$ the tilings must be half periodic in the sense that it must be periodic either in the horizontal or in the vertical direction) one simple nonperiodic tiling set for the cube $[0,1]^2$ in the two dimension is

$$\mathcal{T} = \{(n, m + e^n): n\,, m \in \mathbb{Z}\},$$

which is obtained from the standard lattice tiling by shifting the nth column by e^n. Let $\mathcal{T}$ be any nonperiodic tiling of Ω and set $\mathcal{F} = \mathcal{T}$. Now a theorem of lagarias et al. [58] (and independently by Iosevich and Pedersen [53]) states that $\mathcal{F}$ must also be a spectrum for Ω. Therefore for $\mathrm{g}(x) = x_{\Omega}(x)$ the Gabor system $G(\mathcal{F}, \mathcal{T}, \mathrm{g})$ is an orthonormal Gabor basis. By assumption neither $\mathcal{F}$ nor $\mathcal{T}$ is uniform.

Example (1.3.12)[37]: In the one dimension there exist compactly supported orthonormal Gabor bases $G(\mathcal{F}, \mathcal{T}, \mathrm{g})$ in which neither $\mathcal{F}$ nor $\mathcal{T}$ is a lattice.

Let $\Omega = [0,1] \cup [2,3]$. Again We know that this is a spectral set with spectrum $\mathcal{F} = \mathbb{Z} + \{0, \frac{1}{4}\}$, see, e.g, [61]. Ω tiles with the tiling set $\mathcal{T} = 4\mathbb{Z} + \{0,1\}$. Now for $\mathrm{g}(x) = (1/\sqrt{2})x_{\Omega}(x)$ the Gabor system $G(\mathcal{F}, \mathcal{T}, \mathrm{g})$ is an orthonormal Gabor basis. However, neither $\mathcal{F}$ nor $\mathcal{T}$ is a lattice.

Example (1.3.13)[37]: In an orthonormal Gabor basis $G(\mathcal{F}, \mathcal{T}, \mathrm{g})$ having one of $\mathcal{F}$ or $\mathcal{T}$ being a lattice does not imply the other must be, even in the one dimension.

Let $\Omega = [0,1] \cup [3,4]$. Again we know that this is a spectral set with two distinct spectra: $\mathcal{F}_1 = \mathbb{Z} + \left[0, \frac{1}{6}\right]$ and $\mathcal{F}_2 = \frac{1}{2}\mathbb{Z}$. Ω also has two distinct tiling sets $\mathcal{T}_1 = 6\mathbb{Z} + [0,1,2]$ and $\mathcal{T}_2 = 2\mathbb{Z}$. Let $\mathrm{g}(x) = (1/\sqrt{2})\, x_{\Omega}(x)$. We have an orthonormal Gabor basis $G(\mathcal{F}, \mathcal{T}, \mathrm{g})$ by taking $\mathcal{F} = \mathcal{F}_1$ and $\mathcal{T} = \mathcal{T}_2$, or by taking $\mathcal{F} = \mathcal{F}_2$ and $\mathcal{T} = \mathcal{T}_1$. In either case, one is a lattice and the other is not.

We conclude with the following conjecture on orthonormal Gabor bases $G(\mathcal{F}, \mathcal{T}, \mathrm{g})$.

Conjecture (1.3.14)[37]: Let $\mathrm{g}(x) \in L^2(\mathbb{R}^d)$ be compactly supported. Let $\mathcal{F}$ and $\mathcal{T}$ be discrete sub-set of $\mathbb{R}^d$. Suppose that $G(\mathcal{F}, \mathcal{T}, \mathrm{g})$ is an orthonormal Gabor basis. Then $G(\mathcal{F}, \mathcal{T}, \mathrm{g})$ must be standard.in the other words, there exists a spectral set Ω in $\mathbb{R}^d$ such that

(a) $\mathcal{F}$ is a spectrum of Ω.

(b) $\mathcal{T}$ is a tiling of Ω.

(c) $|g(x)| = (1/\sqrt{\mu(\Omega)})x_{\Omega}(x)$.

Chapter 2
Atomic Characterizations and Arbitrary Sampling Points

We show that for the spaces an atomic characterization similar to known characterization of Besov spaces can be given (with dilation being replaced by modulation). The main theorem is the following: Given $s \in R$ and some $g_0 \neq 0$, $g_0 \in M_{1,1}^{|s|}(R^m)$(e.g., $g \in S(R^m)$ or $g \in L^1$ with compactly supported Fourier transform). Extend the work of Ramanathan and Steger. We show the conjecture that no collection $\bigcup_{k=1}^{r}\{g_k(x - a)\}_{a\in\Gamma_k}$ of pure translates can form a frame for $L^2(\mathbb{R}^d)$. For the perturbation of window functions we show that a Gabor frame generated by any window function with arbitrary sampling points remains a frame when the window function has a small perturbation in $S_0(\mathbb{R}^d)$ sense. We also study the stability of dual frames, which is useful in practice. We give some general results and explain consequences to Gabor frames.

Section (2.1): Modulation Spaces Through Gabor-Type Representations

Theorem (2.1.1)[68]: There exist $\alpha_0 > 0$ and $\beta_0 > 0$ such that, for $\alpha \le \alpha_0$ and $\beta \le \beta_0$, there exists $C = C(\alpha, \beta) > 0$ with the following property: $f \in M^s{}_{p,q}(R^m)$ if and only if $f = \sum_{n,k} a_{n,k} M_{\beta_n} L_{\alpha k}\, \mathrm{g}$, for some double sequence of coefficients satisfying

$$\left[\sum_n \left(\sum_k |a_{n,k}|^p\right)^{q/p} (1 + |n|)^{sq}\right]^{1/q} \le C\|f \mid M^s{}_{p,q}(R^m)\|.$$

The convergence is in the sense of tempered distributions, and in the norm sense for $p, q < \infty$.

The investigation of modulation spaces has been suggested, starting at first from the simple idea of replacing the dyadic partition used in the characterization of Besov-spaces by an equidistant one. It turned out that the characterization used in the abstract (here it would have been sufficient to assume that $\mathrm{g}_0 \neq 0$ is any Schwartz function or $\mathrm{g}_0 \in L^1$ with compactly supported Fourier transform, cf.[78], [79], kap.5.2) is more elegant and admits shorter proof s of some of the basic properties of these spaces, such the invariance of some of these spaces under the Fourier transform.

The most interesting among these spaces are $L^2(R^m) = M_{2,2}^0(R^m)$ and $S_0(R^m) = M_{1,1}^0(R^m)$. For $p = q = 2, s = 0, m = 1$ and g_0 being the Gauss function, our results may considered as Gabor representation for f (cf. [82]), however, with an estimate on the coefficients in ℓ^2. This is in contrast to the classical situation (where the von Neumann lattice with $\alpha\beta = 2\pi$ is chosen; in that case the operators L_x and M_t involved commute, but unbounded coefficients may arise). For $p = q = 1, s = 0$ one obtains an improved atomic characterization for the segal algebra $S_0(G)$ and for $p = q = \infty$ of its dual space (cf.[76], [75],[81]). Atypical feature of our approach is the considerable freedom in the choice of g_0 as “basic” function.

We shall have the following results:

Corollary (2.1.2)[68]: Given $f \in S(R^m), s \in \mathbb{R}$ and $\mathrm{g}_0 \in S(R^m)$, $\mathrm{g}_0 \neq 0$, there exists $\alpha, \beta > 0$ (depending only on g_0) such that $f \in \mathcal{L}_S{}^2(R^m)$ (the Bessel potential space, cf. [75]) if and only if

$$f = \sum_{n,k} a_{n,k}\, M_{\beta_n}\, L_{\alpha_k}\, g_0$$

For a double sequence satisfying $\left[\sum_n \sum_k \left|a_{n,k}\right|^2 (1+|n|)^{2s}\right]^{1/2} < \infty$.

Corollary (2.1.3)[68]: Given $f \in L^1(R^m)$ and $g_0 \in S_0(R^m)$, $g_0 \neq 0$, there exists $\alpha, \beta > 0$ (depending only on g_0) such that $f \in S_0(R^m)$ if and only if

$$f = \sum_{n,k} a_{n,k} M_{\beta_n} L_{\alpha k}\, g_0$$

for a double sequence satisfying $\sum_n \sum_k \left|a_{n,k}\right| < \infty$

As mentioned already in [81] the special choice $g_0 =$ Gauss function implies a number of properties for $S_0(R^m)$ (which are stated for general Ica. groups in [75 ,76], using a fairly different approach).

We want to summarize here a few facts concerning modulation spaces which may be defined as inverse images under the Fourier transform of certain Wiener-type spaces (for basic facts cf. [77] and [80]). In order to describe these spaces we need the following conventions: We shall need the weighted L^q-spaces $L_s^q(R^m)$, given by

$$L_s^q := \left\{ f \middle| \left\|f | L_s^q(R^m)\right\| := \left(\int_{R^m} |f(x)|^q (q+|x|)^{sq} \right)^{1/q} < \infty \right\}$$

which are Banach spaces with respect to their natural norms for $1 \leq q \leq \infty$. Because $\omega = \omega_s : x \to (1+|x|)^s, s > 0,$ satisfies $\omega(xy) \leq C\omega(x)\omega(y)$ for all $x, y \in R^m$ it is a weight function on R^m in the sense of Reiter [82], and L_s^q is invariant under translation, given by $L_x f(z) := f(z-x)$. it also follows that L_s^1 is a Banach convolution algebra (called Beurling algebra, cf. [81]) for $s > 0$ and one has $L_{|s|}^1 * L_s^q \subseteq L_s^q$ for $1 \leq q \leq \infty$, together with the corresponding norm estimate. $C^0(R^m)$ Denotes the space of continuous, complex-valued functions vanishing at infinity, endowed with the sup-norm $\| \quad \|_\infty$, and $M(R^m)$ denotes the space of bounded, regular measures on R^m, which is considered as the dual space to $C^0(R^m)$. We denote by $\mathcal{F}L^p$ the image of L^p (considered as a subspace of $S'(R^m)$) under the Fourier transform and assume that it is endowed with its natural norm, i.e, $\left\|\hat{f}\right\|_{\mathcal{F}L^p} := \|\mathcal{F}\|_{L^p}$. It is now clear from the basic properties of the Fourier transform that $\mathcal{F}L^p$ is a translation invariant Banach space of tempered distributions which is a point wise Banach module over $\mathcal{F}L^1(R^m)$. Consequently it is possible to define the Wiener-type spaces $W(\mathcal{F}L^p, L_s^q)$ (as by [77]) as follows: let $k \in D(R^m)$ be any nonzero Window-function (one should think of a positive plateau-like function, satisfying $k(z) \cong 1$ on a compact set Q, and define the control function as follows:

$$k(f,k)(t) := \|(L_t k) f\|_{FL^p} := \left\|M_t \hat{k} * \hat{f}\right\|_{L^p} \qquad \text{for } t \in R^m.$$

Then

$$W\left(\mathcal{F}L^p, L_s^q\right) := \left\{ f \in S'(R^m) \middle| f \in (\mathcal{F}L^p)_{loc}, K(f,k) \in L_s^q(R^m) \right\},$$

Endowed with its natural norm $\left\|f | W(\mathcal{F}L^p, L_s^q)\right\| := \left\|K(f,k) \middle| L_s^q\right\|$.

Using this definition it is easy to verify that these Wiener-type spaces are translation invariant, but also invariant under multiplication with characters. One has the following estimates for the operator norms of these operators:

$$\left\|L_x f|W\left(\mathcal{F}L^p, L_s^q\right)\right\| \leq (1+|x|)^s \left\|f|W\left(\mathcal{F}L^p, L_s^q\right)\right\| \text{ For all } x \in R^m,$$

And

$$\left\|M_t f|W\left(\mathcal{F}L^p, L_s^q\right)\right\| = \left\|f|W\left(\mathcal{F}L^p, L_s^q\right)\right\| \text{ For all } t \in R^m.$$

An essential tool for the discrete way of describing these spaces (this is the original definition of these spaces) is based on the existence of suitable partitions of unity. Since we do need the most general description in our situation we can stick to the following (restricted) definition of a bounded uniform partition of unity of size $\delta > 0$ (for short a δ-BUPU) in $\mathcal{F}L^1$:

Definition (2.1.4)[68]: Given $\delta > 0$ any bounded family in the Banach space $\Psi = (\psi_n)_{n\in Z^m}$ in $\mathcal{F}L^1(R^m)$ is called a δ-BUPU) in $\mathcal{F}L^1$ if the following properties hold:

(BP1) There is a lattice δZ^m in R^m (for some positive δ) such that supp $\psi_n \subseteq B(\delta_n, \delta)$ (the ball around δ_n with radius δ).

(BP2) $\sum_{n\in Z^m} \psi_n(x) \equiv 1$.

Using BUPUs one can give the following discrete characterization: $f \in W(\mathcal{F}L^p, L_s^q)$ if and only if, for some BUPU, one has

$$\left[\sum_{n\in Z^m} \|f\psi_n\|^q{}_{\mathcal{F}L^p}(1+|n|)^{sq}\right]^{1/q} < \infty$$

(and this expression gives an equivalent norm, cf.[77]).

It is a consequence of this description that any window function k as described above (even any Schwartz function or any $k \in W(\mathcal{F}L^1, L_s^1)$ defines the same space $W(\mathcal{F}L^p, L_s^q)$ and gives an equivalent norm, cf. [79].

The modulation space $M_{p,q}^s(R^m)$ can now defined as inverse images of the spaces $W(\mathcal{F}L^p, L_s^q)$ under the Fourier transform. The invariance properties of Wiener-type spaces are easily translated into invariance properties of modulation spaces. Thus one has isometric translation invariance and the following estimates for the multiplication Operators M_t: $\left\|M_t f|M^s{}_{p,q}\right\| \leq (1+|t|)^s \left\|f|M^s{}_{p,q}\right\|$. In particular, the spaces $M_{1,1}^{|s|}(R^m)$ are character invariant Segal algebras, isometrically translation invariant spaces in $L^1(R^m)$, complete with respect to their own norm, hence Banach ideals in $L^1(R^m)$(cf. [82; chapter 6 & 2.2] for details about Segal algebras). Since Fourier transformation is very well compatible with duality it is clear from the general results on decomposition spaces (of which Wiener-type spaces are a special case, cf. [80], [79], [78]) that modulation spaces show the natural behavior with respect to duality, i.e, one has $(M_{p,q}^s(R^n))' = M_{p',q'}^{-s}(R^n)$, for $1 \leq p, q < \infty$.

We want to show the atomic characterization of modulation spaces indicated in the abstract. Apparently we have to verify two partial results, one on synthesis (i.e, that expressions in the atomic characterizations are actually convergent to elements in $M_{p,q}^s(R^n)$ and, on the other hand , the decomposition result. We shall show the last mentioned first . Because $M_{1,1}^{|s|}(R^m)$ is a Segal algebra we shall write S for this space throughout the proof (fixing s).

A) Let $g \in S$, $g \neq 0$ be given, and $\in M^s_{p,q}(R^m)$. We shall show decomposition result first with respect to functions $g \in S$ with the additional property that supp $\hat{g} \subseteq k$, some compact subset of R^m. Since it is possible to replace g by $M_{\beta_n} g$, if necessary, we may assume that there exists $\delta > 0$ such that $\hat{g}(t) \neq 0$ for $|t| \leq \delta$. Applying Wiener's theorem on the inversion of the Fourier transform (cf.[82; chapter 1 & 3.6]) we find that there exists $h \in L^1(R^m)$ such that $\hat{h}(t)\hat{g}(t) = 1$ for all $|t| \leq \delta$. Without loss of generality we may assume that $\hat{h}$ has compact support, e.g., $\hat{h}(t) = 0$ for $|t| \geq 2\delta$. Now let $\Phi = (\varphi_n)_{n\in Z^m}$ be any bounded, uniform spectral decomposition of unity of Size $\leq \delta$, i.e a family given as inverse image under the Fourier transform of a δ-BUPU. Consequently we have $\sigma = \sum_{n\epsilon Z^m} \Psi_n * \sigma$ for any $\sigma \in s'(R^m)$ (for example), where $\Psi_n = M_{\beta_n}\Psi_0$ for some Ψ_0 with supp $\widehat{\psi_0} \subseteq B(0,\delta) =: Q$. It is our main to start with this spectral decomposition, (at the moment we only have convergence in the week toplogy, but part C) will show that one has norm convergence for $1 \leq p, q < \infty$). Since $\Psi_n = M_{\beta_n}(\Psi_0 * g * h)$ we can write

$$f = \sum_{n\in Z^m} \left(M_{\beta_n}(\Psi_0 * g * h)\right) * f =: \sum_{n\in Z^m} M_{\beta_n}(f_n * g)$$

With $f_n := (\psi_0 * h) * M_{-\beta_n} f$. For later use let us fix the following constants:

(i) For any compact set $Q \subseteq R^m$ there exists $C_Q > 0$ such that

$$\left\| f|M|_{s_{p,q}} \right\| \leq C_Q \|f\|_p$$

And (for later use)

$$\|f|W(C^0, L^p)\| \leq C_Q \|f\|_p \text{ For all } f \in L^p(R^m) \text{ with supp} \hat{f} \subseteq Q$$

(cf. [77], Theorem 5 for a proof of the last statement).

(ii) Using the discrete version of the norm on Wiener-type spaces (cf.[9]) we know that there exists $C_\Phi > 0$ such that

$$\sum_{n\in Z^m} \|f_n\|_p^q (1 + |\beta_n|)^{sq} \leq \sum_{n\in Z^m} \|h_n\|_1^q \left\|M_{\beta_n}\psi_0 * f\right\|_p^q (1 + |\beta_n|)^{sq}$$
$$\leq C_\emptyset^p \|h\|_1^q \left\|f|M^s_{p,q}(R^m)\right\|^q.$$

We apply a variant of Shannon's principle which will allow us to replace, given the $f_n's$ the convolution $f_n * g$ by a discrete sum of translates. Thus let us assume that f belongs to $L^P(R^m)$ and supp$\hat{f} \subseteq Q$.

We proceed as follows, having a look on the Fourier transform side and using the notation ⧢ for the (translation bounded) Radon measure given as $⧢ := \sum_{k\in Z^m} \delta_k$ and $⧢_p := \sum_{k\in Z^m} \delta_{pk}$. Since supp $\hat{g} \subseteq k$ (compact) there exists $\rho > 0$ such that $\hat{f}\hat{g} = [\sum_{k\in Z^m}(L_{pk}\hat{f})]\hat{g} = (⧢_\rho * \hat{f})\hat{g}$, or going back to the functions and using Poisson's formula, telling us that ⧢ is invariant under the Fourier transform

$$f * g = \left(\left(p^{-m} ⧢_{1/p}\right) f\right) * g = p^{-m} \sum_{k\in Z^m} f(k/p) L_{k/p}\, g.$$

Applying the same argument to each f_n we have the following estimate for the sequence of coefficients $a_{n,k} := p^{-m} f_n(k/p)$:

$$\left(\sum_{k\in Z^m} \left|a_{n,k}\right|^p\right)^{1/p} \leq C_p \|f_n|W(C^0, L^p)\| \leq C_\rho C_Q \|f_n\|_p \quad \text{for all } n \in Z^m.$$

Which gives together with the previous estimates, the required sum ability properties for the double sequence ($a_{n,k}$).

B) Let us now consider the case of an arbitrary element g_1 in S. Since the Segal algebra $S \subseteq S_0(R^m)$ is continuously embedded into Wiener's algebra $W(R^m) = W(C^0, L^1)$ (cf. [76]), hence into the Segal algebra $W(L^P, L^1)$ for $1 \leq p < \infty$, it is possible to approximate g_1 in the norm of $W(L^P, L^1)$ by elements g with compactly supported Fourier transform. In order to have the right constants (appropriate a priori estimates) let us note that we have the following facts at our disposition:

(iii) The family $\rho^{-m} ш_{1/\rho}$ is uniformly bounded in the space $W(M, L^\infty)$.

(iv) There is a universal constant $C_p > 0$ (depending only on the norms used) such that the following estimates hold true (cf.[77], [80]):

$$\| g \|_1 \leq C_p \| g |W(L^p, L^1)\| \text{ for all } g \in W(L^p, L^1),$$

$$\left\|f_\mu \middle| W(M, L^p)\right\| \leq C_p \|f|W(C^0, L^p)\| \, \|\mu|W(M, L^p)\|$$
$$\text{for } f \in W(C^0, L^p), \mu \in W(M, L^p),$$

$$\|v * g \|_p \leq C_p \|v|W(M, L^p)\| \, \big\|| g | \, |W(L^p, L^1)|\big\|$$
$$\text{for } v \in W(M, L^p), g \in W(L^p, L^1).$$

(v) Combining these estimates (with (i) above), we find some constant ${C^1}_Q$ (only dependent on the common support of $\hat{f}$ and p) such that

$$\left\|\left(\rho^{-m} \, ш_{1/p}\right) f|W(M, L^p)\right\| \leq C_Q^1 \|f\|_p \quad \text{if supp} \hat{f} \subseteq Q.$$

Writing for brevity, $D_\rho f$ for $\left(\rho^{-m} ш_{1/\rho}\right) f$ (discrete version of f), we obtain the following estimate in L_p (still assuming supp $\hat{f} \subseteq Q$ and ρ chosen depending on the support of $\hat{g}$ as a bove):

$$\|f * g_1 - D_P f * g_1\|_p$$
$$\leq \|f * (g_1 - g)\|_p + \left\|f * g - D_\rho f * g\right\|_p + \left\|D_\rho f * (g - g_1)\right\|_p$$
$$\leq \|f\|_p \| g - g_1\|_1 + 0 + C_p \left\|D_p f|W(M, L^p)\right\| \| g - g_1|W(L^p, L^1)\|$$
$$\leq \|f\|_p C_p \| g - g_1|W(L^p, L^1)\| \left(1 + {C^1}_Q\right).$$

Having this estimate (which does not depend on the support of $\hat{g}$) it is clear that we can choose g such that

$$\| g - g_1|W(L^p, L^1)\| \leq (2C_p\left(+{C^1}_Q\right) C_Q C_\phi \|h\|_1)^{-1},$$

hence

$$\left\|f * g_1 - D_\rho f * g_1 \right\|_p \leq (2C_Q C_\emptyset \|h\|_1)^{-1} \|f\|_p.$$

This estimate, being valid for each $f = f_n$, we obtain, summing over n:

$$\left\|f - \sum_{n \in Z^m} M_{\beta_n}(D_\Psi f_n * g) \middle| M_{p,q}^s\right\| = \left\|\sum_{n \in Z^m} M_{\beta_n}((f_n - D_\Psi f_n) * g) \middle| M_{p,q}^s\right\|$$
$$\leq \sum_{n \in Z^m} \left\|M_{\beta_n}\right| M_{p,q}^s\| \, \| (f_n - D_\Phi f_n) * g |M_{p,q}^s\| \qquad \text{by (i)}$$
$$\leq \sum_{n \in Z^m} (1 + |\beta_n|)^{sq} C_Q \|(f_n - D_\Psi f_n) * g \|_p$$
$$\leq C_Q \sum_{n \in Z^m} (1 + |\beta_n|)^{sq} \|f_n\|_p \left(2C_Q C_\Phi \|h\|_1\right)^{-1} \qquad (\text{by (ii)})$$

$$\leq \left(2C_\Phi C_Q\|h\|_1\right)^{-1} C_Q C_\Phi \|h\|_1 \|f|M^s_{p,q}\| = 1/2 \,.\, \|f|M^s_{p,q}\|.$$

We have thus found a linear mapping $T_\Psi\colon f \longrightarrow \sum_{n\in Z^m} M_{\beta_n}(D_\Psi f_n * \text{ g })$, such that Id-$T_\Psi$ is a contraction on $M^s_{p,q}(R^m)$ for $1 \leq p, q < \infty$. consequently T_Ψ is invertible on $M^s_{p,q}$ and we have

$$f = T_\Psi\left({T_\Psi}^{-1} f\right) = T_\Psi\left(\sum_{i=o}^{\infty}(Id - T_\Psi)^i(f)\right) \coloneqq T_\Psi(h), \text{ with } h \in M^s_{p,q}(R^m).$$

Since $\|h|M^s_{p,q}\| \leq C\|f|M^s_{p,q}\|$ we have

$$f = T_\Psi(h) = \textstyle\sum_{n,k} a_{n,k} M_{\beta_n} L_{\alpha k} \text{ g},$$

with

$$\left[\sum_n (\sum_k |a_{n,k}|^p)^{q/p}(1 + |\beta_n|)^{sq}\right]^{1/q} \leq C_2\|h|M^s_{p,q}\| \leq C_2 C\|f|M^s_{p,q}\|,$$

and the proof is complete in this case.

C) We have now to discuss the synthesis problem, i.e, given any element in $S = M^{|s|}_{1,1}(R^n)$. And a double sequence $(a_{n,k})$ satisfying the sum ability condition stated above, the corresponding Gabor sum defines an element of $M^s_{p,q}(R^m)$.

Given now g $\in M^{|s|}_{1,1}(R^n)$ we start splitting it by means of a uniform spectral decomposition (as used a bove), i.e, we write

$$\text{g} = \sum_{j\in Z^m} Q_j * \text{ g } = \sum_{j\in Z^m} M_{\beta_j} \text{g}_j, \quad \text{with } \text{g}_j \coloneqq \varphi_0 * M_{-\beta_j} \text{g}_j,$$

and

$$\sum_{j\in Z^m} \|\text{ g }_j\|_1 (1 + |j|)^{|s|} \leq C_1 \left\|\text{ g } |M^{|s|}_{1,1}\right\| < \infty.$$

For later use let us note that the $\hat{g}_j{}'s$ have common compact support Q. Consequently they belong to any Segal algebra (cf. [82; chapter 6 & 2.2]), in particular to the Segal algebra $W(L^1, L^p)$. Moreover, there is a constant $C_2 < 0$ such that $\|g_j W(L^p, L^1)\| \leq C_2\|g_j\|_1$ for all $j \in z^m$ (cf. [77, Theorem 5] for an alternative proof of this assertion). Calculating within $S'(R^m)$ we obtain, using the identity $L_x M_t = M_t L_x e^{ix.t}$ for all $x, t \in R^m$:

$$\sum_{n,k} a_{n,k} M_{\beta_n} L_{\alpha k} \text{ g} = \sum_{n,k} a_{n,k} M_{\beta_n} L_{\alpha k}\left(\sum_j M_{\beta_j} \text{g}_j\right)$$

$$= \sum_j M_{\beta_j}\left(\sum_{n,k} {a^j}_{n,k} L_{\alpha k} M_{\beta_n} g_j\right)$$

With $a^j_{n,k} = \exp(i\alpha k.\beta(n - j))$, hence $|a^j_{n,k}| = |a_{n,k}|$ for all $j, n, k \in Z^m$. Rewriting the sum (in order to introduce some notations) one has

$$h \coloneqq \sum_{n,k} a_{n,k} L_{\alpha k} M_{\beta_n} \text{ g } =: \sum_{j,n} M_{\beta_j} h_{j,n} \coloneqq \sum_j M_{\beta_j} h_j,$$

$$h_{j,n} := \sum_k a_{n,k}^j L_{\alpha k} M_{\beta_n} g_j = \left(\sum_k a_{n,k}^j \delta_{\alpha k}\right) * M_{\beta_j g_j} \in W(M, L^p) * W(L^p, L^1) \subseteq L^P$$

and (cf. [77, Theorem 3]) the estimate

$$\|h_{j,n}\|_p \leq C_3 \left\| \sum_k a_{n,k}^j \delta_{\alpha k} \middle| W(M, L^p) \right\| \| g_j | W(L^p, L^1) \|$$
$$\leq C_4 (\sum_k |a_{n,k}|^p)^{1/p} \| g_j \|_1$$

Now $\operatorname{supp} \hat{h}_{j,n} \subseteq \beta_n + \operatorname{supp} \hat{g}_j \subseteq \beta_n + Q$ for all $n, j \in Z^m$. In order to get an estimate for the sum over the $n's$ we observe next that the family $(\beta_n + Q)_{n \in Z^m}$ constitutes an admissible uniform covering of R^m (this f means essentially that is a covering of uniformly bounded height, cf. [80, cor. 2.6]) and consequently (this another characterization of wiener-type spaces) there exists $C_5 > 0$ such that

$$\|h_j | M_{p,q}^s\| = \left\| \sum_n h_{j,n} \,|M_{p,q}^s \right\| = \left\| \sum_n \hat{h}_{j,n} | W(\mathcal{F}L^p, L_s^q) \right\|$$
$$\leq C_5 \left(\sum_n \|h_{j,n}\|_p^q (1 + |\beta_n|)^{sq} \right)^{1/p}$$
$$\leq C_6 \left[\sum_n \left(\sum_k |a_{n,k}|^p \right)^{q/p} (1 + |n|)^{sq} \right]^{1/q} \| g_j \|_1.$$

We can now carry out the last step, i.e, summation over the $j's$ which yield immediately the required estimate, completing the proof.

$$\|h | M_{p,q}^s(R^m)\| \leq \sum_j \|M_{\beta_j} h_j\| \leq \sum_j \| |M_{\beta_j}| \| \, \|h_j | M_{p,q}^s\|$$
$$\leq C_6 \left[\sum_n \left(\sum_k |a_{n,k}|^p \right)^{q/p} (1 + |n|)^{sq} \right]^{1/q} \sum_j C_7 (1 + |j|)^{|s|} \| g_j \|_1$$
$$\leq C_8 \left[\sum_n \left(\sum_k |a_{n,k}|^p \right)^{q/p} (1 + |n|)^{sq} \right]^{1/q} \| g \,| M_{1,1}^{|s|} \|$$

Besides the Corollaries stated already one has among others, the following useful consequence.

Corollary (2.1.5)[68]: (cf. [81]). The Banach space $M_{1,1}^{|s|}(R^n)$ (with $s \geq 0$) is smallest among all Banach spaces satisfying the following conditions:

(a) it is continuously embedded into $S'(R^m)$,
(b) it has non-trivial intersection with $S(R^m)$,
(c) it is isometrically translation invariant,
(d) it satisfies $\|M_t\| = O(1 + |s|)$ for $t \longrightarrow \infty$.

It is also possible to use the atomic characterization of the spaces $\left\| g \,| M_{1,1}^{|s|} \right\| (R^m)$ in order to give an alternative proof showing the freedom in choice of the function g_0

used in the definition of $M^{s}_{p,q}(R^m)$ in the abstract (Note that for all results presented so far we could have worked with a function $g_0 = \hat{k}$, with $k \in D(R^m)$). That we could have used any non-zero window function $k \in W(\mathcal{F}L^1, L^1_s)$ (which is equivalent to the use of $\hat{k} = g_0$ in $\left\| g \,| M^{|s|}_{1,1} \right\|(R^m)$, thus in particular any Schwartz function $g_0 \in S(R^m)$, so especially the Gauss function) in our definition can be shown as follows: if the integral expression involving σ is finite for a given element $g_0 \in \left\| g \,| M^{|s|}_{1,1} \right\|(R^m)$, (e.g, with supp $\widehat{g_0}$ compact) it is easily verified that it is also finite for g_0 replaced by $M_t L_x g_0$, for any $t, x \in R^m$, and in the estimate only an additional factor $(1 + |t|)^s$ arises. Now inserting any $g_1 \in M^{|s|}{}_{1,1}(R^m)$, written in the atomic way based on g_0, it is clear that the integral expression (involving g_1 now instead of g_0) is finite as well. Thus any non-zero element $g_0 \in \left\| g \,| M^{|s|}_{1,1} \right\|(R^m)$, gives another equivalent norm. This was showed using different methods already in [79].

Section (2.2): Gabor Frames and Density

For each $a, b \in R^d$, let T_a and M_b denote the translation and modulation operators on $L^2(R^d)$ defined by

$$T_a g(x) = g(x - a) \quad \text{and} \quad M_b g(x) = e^{2\pi i b x} g(x),$$

Where $bx = b_1x_1 + \cdots + b_dx_d$. A time- frequency shift is a composition of modulation and translation; i.e, it has the form

$$M_b T_a g\,(x) = e^{2\pi i b x} g(x - a).$$

If $\Gamma \subset R^d$, then the collection of translates of g along Γ is defined to be

$$T(g, \Gamma) = \{T_a g\}_{a \in \Gamma}$$

If $A \subset R^{2d}$, then the collection of time-frequency shifts of g along A is defined to be

$$S(g, A) = \{M_b T_a g\}_{(a,b) \in A}.$$

We refer to $S(g, A)$ as the Gabor system generated by g and A.

Gabor system which form frames for $L^2(R^d)$ have a wide variety of applications. One important problem is therefore to determine sufficient conditions on g and A which imply that $S(g, A)$ is a frame. In the case that $d = 1$ and A is a regular lattice of the form $aZ \times bZ$, sufficient conditins for $S(g, A)$ to form a frame for $L^2(R)$ were found by Daubechies [87]. A generalization of this result requiring weaker assumptions, which also applies when $S(g, A)$ only forms a frame for its closed span instead of all of $L^2(R)$, was obtained recently in [85].

We are concerned with the connection between density properties of A and frame properties of $S(g, A)$, and the analogous problem for systems $T(g,\Gamma)$ of pure translates. For the case of Gabor systems, there is rich literature on this subject, especially when A is rectangular lattice $A = aZ^d \times bZ^d$. We briefly review here some of the main results connecting the density of A to properties of $S(g, A)$, and refer to [84] for a more thorough historical discussion and a review of properties of Gabor systems. In addition that we discuss explicitly below, some relevant related articles include [92, 93, 94,98].

For simplicity, consider the one-dimensional setting $d = 1$ and a rectangular lattice $A = aZ \times bZ$. In this case, Rieffel showed (as a corollary of results on C^* algebres) that $S(g, A)$ is incomplete in $L^2(R)$ if $ab > 1$ [97]. The algebraic structure of the lattice is crucial to this result, as the proof follows from computing the coupling constant of the von Neumann algebra generated by the Operators $\{M_{mb} T_{na}\}_{m,n \in Z}$. for the case that $ab > 1$ is rational, Daubechies provided a constructive proof of the incompleteness of $S(\mathrm{g}, A)$ through the use of Zak transform, which is again an algebraic tool highly dependent on the lattice structure of $A = aZ \times bZ$ [87]. Ramanathan and Steger introduced a technique that applies to Countable, nonlattice set A that are uniformly separated; i.e, there is a minimum distance δ between elements of A [96]. It is possible to define an upper Beurling density $D^{+}(A)$ and lower Beurling density $D^{-}(A)$ for such sets (the precise definition of density, along with other fundamental concepts used) for example, for the lattice $A = aZ \times bZ$ these two densities coincide and equal $1/(ab)$; hence this lattice is said to have uniform Beurling density $D(A) = 1/(ab)$. Ramanathan and Steger showed for arbitrary uniformly separated sets A that if $D^{-}(A) < 1$ then $S(\mathrm{g}, A)$ is not a frame. Thus, in the case that $A = aZ \times bZ$, this can be viewed as a weak version of the Rieffel incompleteness result. On the other hand, Ramanathan/Steger result applies to a far broader class of time-frequency translates than does the Rieffel result. Ramanathan and Steger were able to recapture by their techniques the full Rieffel incompleteness result in the case that $A = aZ \times bZ$. In light of the above discussion, Ramanathan and Steger therefore conjectured that $S(\mathrm{g}, A)$ must be incomplete whenever A is a uniformly separated set satisfying $D^{-}(A) < 1$. Walnut and Heil showed that this conjecture is false by constructing for each $\varepsilon > 0$ a function $\mathrm{g} \in L^2(R)$ and a nonlattice $A \subset R^2$ such that $S(\mathrm{g}, A)$ is compete in $L^2(R)$ yet A has uniform Beurling density $D(A) < \varepsilon$ [84]. Hense the algebraic structure of A is in fact critical for the Rieffel incompleteness result.

We extended and apply the Ramanathan/Steger density results. The extension is to higher dimensions, to multiple generating functions, and to completely arbitrary sets of time-frequency shifts. To state our results, for each $k = 1, \dots, r$ let g_k be an element of $L^2(R^d)$ and let $A_k = \{(a_{k,i}, b_{k,i})\}_{i \in I_k}$ be a sequence of points in R^{2d}. Unless specified otherwise, we place no restrictions on the sequences A_k. For example, the index set I_k may be countable or uncountable, and repetitions of points in A_k are allowed. For simplicity, we will write $A_k \subset R^{2d}$, although we always mean that A_k is a sequence of points from R^{2d} and not merely a subset of R^{2d}. Define an index set $I = \{(i,k): i \in I_k, k = 1, \dots, r\}$ and sequence $A = \{(a_{k,i}\, , b_{k,i})\}_{(i,k)\in I} = \{(a_{k,i}, b_{k,i})\}_{i \in I_k}$, $k = 1, \dots\dots, r$, i.e, A is the sequence obtained by amalgamating $A_1, \dots\dots, A_r$. For simplicity, we write $A = \bigcup_{k=1}^{r} A_k$, and say that A is the disjoint union of $A_1, \dots\dots, A_r$. The Gabor system generated by $\mathrm{g}_1, \dots\dots, \mathrm{g}_r$ and $A_1, \dots\dots, A_r$ is then $\bigcup_{k=1}^{r} S(\mathrm{g}_k, A_k)$, the disjoint union of the Gabor systems $S(\mathrm{g}_k, A_k)$ with this notation, our first main result is the following.

We remark that the conclusion in part (a) of Theorem (2.2.10) that A has finite upper Beurling density is equivalent to the statement that A, and hence each A_k, is relatively uniformly separated, i.e, is a finite union of uniformly separated sequences. The proof of Theorem (2.2.10) is given. the result of Ramanathan and strger in [96] corresponds to the special case of Theorem (2.2.10)(b) with $d = 1$ and $k = 1$ and with the added assumption that A is uniformly separated and satisfies $D^{+}(A) < \infty$.

One useful feature of Gabor frames $\bigcup_k S(\mathrm{g}_k\, , A_k)$ generated by functions g_k that are well localized in both time and frequency is that if a function f is expanded in this frame, then a perturbation of f that is well localized in both time and frequency will have a local effect on the frame coefficients. By comparison, a frame of the

$\bigcup_k T(g_k, \Gamma_k)$ consisting solely of translates of finitely many function g_k would have the desirable property that perturbation localized solely in time have localized effects on the frame coefficients. In this regard, Olson and Zalik showed that there do not exist any Riesz bases for $L^2(R)$ generated by translates of a single function [95], and Christensen conjectured that there are no frames for $L^2(R)$ of this form [86]. Since $T(g, \Gamma) = S(g, \Gamma \times \{0\})$, systemes of translates can be considered to be special cases of Gabor systems. We show that Theorem (2.2.10) implies that there are no frames for $L^2(R^d)$ of the form $\bigcup_k T(g_k, \Gamma_k)$. We give a direct proof of the following refinement of this statement.

For $x \in R^d$ and $h > 0$ we let $Q_h(x)$ denote the cube centered at x with side lengths h:

$$Q_h(x) = \prod_{j=1}^{d} [x_j - h/2 \, , x_j + h/2].$$

In particular, $\{Q_h(hn)\}_{n \in Z^d}$ is disjoint cover of R^d. To distinguish between cubes in R^d and those in R^{2d}, we write $Q_h(x, y) = Q_h(x) \times Q_h(y)$ for a cube in R^{2d}.

The Lebesgue measure of $E \subset R^d$ is denoted by $|E|$. In particular, the volume of the cube $Q_h(x)$ is $|\, Q_h(x)| = h^d$. The number of points in $E \subset R^d$ is denoted by $\#E$.

The L^2-inner product is $\langle f, g \rangle = \int f(x)\, \overline{g(x)}\, dx$. The short-time Fourier transform of $f \in L^2(R^d)$ against $g \in L^2(R^d)$ is

$$S_g f(a, b) = \langle f, M_b T_a g \rangle$$

We have $S_g f \in L^2(R^{2d}) \cap C_0(R^{2d})$, with $\|S_g f\|_2 = \|f\|_2 \|g\|_2$.

Given a closed subspace $V \subset L^2(R^d)$, we let P_V denote the orthogonal projection onto V. Then for any $f \in L^2(R^d)$,

$$\operatorname{dist} fV = \|f - P_V f\|_2 = \inf_{u \in V} \|f - u\|_2.$$

A family of elements $\{f_i\}_{i \in I}$ is a frame for a Hilbert space H if there exist constants $A, B > 0$ such that

$$\forall f \in H, \qquad A\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B\|f\|^2. \tag{1}$$

The numbers A, B are called frame bounds. The frame operator $Sf = \sum_i \langle f, f_i \rangle f_i$ is a bounded, invertible, and positive mapping of H onto itself. This provides the frame decomposition

$$f = S^{-1} S f = \sum_{i \in I} \langle f, f_i \rangle \tilde{f}_i \, , \qquad \forall f \in H, \tag{2}$$

Where $\tilde{f}_i = S^{-1} f_i$. The family $\{\tilde{f}_i\}$ is also a frame for H, called the dual frame of $\{f_i\}$, and has frame bounds B^{-1}, A^{-1}. The utility of frames, as compared to sets of functions that are merely complete in $L^2(R^d)$, often lies in the stable reconstruction formula (2).

Riesz bases are special cases of frames, and can be characterized as those frames which are biorthogonal to their dual frames, i.e, such that $\langle f_i, \tilde{f}_j \rangle = \delta_{ij}$.

An arbitrary family $\{f_i\}$ which satisfies the first inequality in (1) (and which may or may not satisfy the second inequality) is said to possess a lower frame bound. Likewise, a family $\{f_i\}$ which satisfies at least the second inequality in (1) is said to possess an upper frame bound. Such a family is also called a Bessel sequence.

Additional information on frames can be found in [87,90].

We now give several definitions related to the "density" of an arbitrary sequence $\Gamma = \{\gamma_i\}_{i\in I}$ of points of R^d. The index set may be countable or uncountable, and since Γ is regarded as a sequence, repetitions of elements of Γ are allowed.

Definition (2.2.1)[83]: Let $\Gamma = \{\gamma_i\}_{i\in I} \subset R^d$.

(a) Γ is δ-uniformly separated if $\delta = \inf_{i\neq j}|\gamma_i - \gamma_j| > 0$. The number δ is the separation constant.

(b) Γ is relatively uniformly separated if it is a finite union of uniformly separated sequences Γ_k. More precisely, this means that I can be partitioned into disjoint sets $I_1, \dots, I_r$ such that each sequence $\Gamma_k = \{\gamma_i\}_{i\in I_k}$ is δ_k-uniformly separated for some $\delta_k > 0$.

Definition (2.2.2)[83]: let $\Gamma = \{\gamma_i\}_{i\in I} \subset R^d$. For each $h > 0$, let $v^+(h)$ and $v^-(h)$ denote the largest and smallest numbers of points of Γ that lie in any $Q_h(x)$:

$$v^+(h) = \max_{x\in R^d} \#\big(\Gamma \cap Q_h(x)\big) \quad \text{and} \quad v^-(h) = \min_{x\in R^d} \#\big(\Gamma \cap Q_h(x)\big).$$

We have $0 \le v^-(h) \le v^+(h) \le \infty$ for each h. The upper and lower Beurling densities of Γ are then

$$D^+(\Gamma) = \limsup_{h\to\infty} \frac{v^+(h)}{h^d} \quad \text{and} \quad D^-(\Gamma) = \liminf_{h\to\infty} \frac{v^-(h)}{h^d}.$$

We have $0 \le D^-(\Gamma) \le D^+(\Gamma) \le \infty$. If $D^+(\Gamma) = D^-(\Gamma)$, then Γ is said to have uniform Beurling density $D(\Gamma) = D^+(\Gamma) = D^-(\Gamma)$.

Note that if Γ is the disjoint union of $\Gamma_1, \dots, \Gamma_r$, then we always have $\#\big(\Gamma \cap Q_h(x)\big) = \sum_{k=1}^{r} \#\big(\Gamma_k \cap Q_h(x)\big)$,
and therefore

$$\sum_{k=1}^{r} D^-(\Gamma_k) \le D^-(\Gamma) \le D^+(\Gamma) \le \sum_{k=1}^{r} D^+(\Gamma_k). \qquad (3)$$

Some or all of the inequalities in (3) may be strict. For example, if Γ_1 is the set of negative integers, Γ_2 is the positive integers, and $\Gamma = \Gamma_1 \cup \Gamma_2$, then $D^-(\Gamma_1) = D^-(\Gamma_2) = 0, D^-(\Gamma) = D^+(\Gamma) = 1$, and $D^+(\Gamma_1) = D^+(\Gamma_2) = 1$.

The following lemma provides some equivalent ways to view the meaning of finite upper Burling density.

Lemma (2.2.3)[83]: Let $\Gamma = \{\gamma_i\}_{i\in I}$ be any sequence of points in R^d. Then the following statements are equivalent.

(a) $D^+(\Gamma) < \infty$.

(b) Γ is relatively uniformly separated.

(c) For some (and therefore every) $h > 0$, there is an integer $N_h > 0$ such that each cube $Q_h(hn)$ contains at most N_h points of Γ. that is,

$$N_h = \sup_{n\in Z^d} \#\big(\Gamma \cap Q_h(hn)\big) < \infty.$$

Proof. (a)⇒(c). If $D^+(\Gamma) < \infty$ then $v^+(h)/h^d < \infty$ for some h.

(c)⇒(b). Assume that there is an $h > 0$ such that each cube $Q_h(hn)$ contains at most N_h elements of Γ. Let $e_1, \dots, e_{2^d}$ denote the vertices of the unit cube $[0,1]^d$, and define

$$Z_k = (2z)^d + e_k \quad \text{and} \quad B_k = \cup_{n\in Z_k} Q_h(hn).$$

Then R^d is the disjoint union of the 2^d Sets B_k. If $m, n \in Z_k$ with $m \neq n$, then

$$\operatorname{dist}(Q_h(hm), Q_h(hn)) = \inf\{|x-y| : x \in Q_h(hm), y \in Q_h(hn)\} \geq h.$$

Further, each cube $Q_h(hn)$contains at most N_h elements of Γ, so the sequences $\{\gamma_i : \gamma_i \in B_k\}$ can be split into N_h uniformly separated sequences. Hence the entire sequence Γ can be split into $2^d N_h$ uniformly separated sequences.

(b)⇒(a). Assume that Γ is relatively uniformly separated. Then we can partition I into sets $I_1, \dots, I_r$ in such a way that each sequence $\Gamma_k = \{\gamma_i\}_{i \in I_k}$ is δ_k-uniformly separated. Let $\delta = \min\{\delta_1/2, \dots, \delta_r/2\}$. Then any cube $Q_\delta(x)$ contains at most one element of Γ_k, and therefore contains at most r elements of Γ. Therefore, if h is any positive number then $Q_{h\delta}(x)$ can contain at most $r(h+1)^d$ elements of Γ. Hence $v^+(h\delta) \leq r(h+1)^d$ for each h, so

$$D^+(\Gamma) \leq \limsup_{h\to\infty} \frac{r(h+1)^d}{(h\delta)^d} = \frac{r}{\delta^d} < \infty.$$

We will show Theorem (2.2.10), we consider part (a) and part (b) of the theorem separately. In particular, we begin by considering the special case of Theorem (2.2.10)(a) when $r = 1$.

Theorem (2.2.4)[83]: Choose a nonzero $g \in L^2(R^d)$ and a sequence $A \subset R^{2d}$. If $S(g, A)$ possesses an upper frame bound, then A is relatively uniformly separated.

Proof. Assume that A is not relatively uniformly separated. Choose any $f \in L^2(R^d)$ with $\|f\|_2 = 1$, and note that

$$|\langle M_q T_p f, M_b T_a g\rangle| = |\langle f, M_{b-q} T_{a-p} g\rangle| = |S_g f(a-p, b-q)|.$$

Since $S_g f$ is nonzero and continuous on R^{2d}, it must be bounded away from zero on some cube, say,

$$\mu = \inf_{(x,y)\in Q_h(c,d)} |S_g f(x,y)| > 0.$$

Now choose any $N > 0$. Then, by Lemma (2.2.3) applied to A, there exists some cube $Q_h(p,q)$ which contains at least N elements of A. However, if $(a,b) \in Q_h(p,q)$, then $(a-p+c, b-q+d) \in Q_h(c,d)$, so

$$\sum_{(a,b)\in A\cap Q_h(p,q)} |\langle M_{q-d} T_{p-c} f, M_b T_a g\rangle|^2$$
$$= \sum_{(a,b)\in A\cap Q_h(p,q)} |S_g f(a-p+c, b-q+d|^2 \geq N\mu^2.$$

Since $\|M_{q-d} T_{p-c} f\|_2 = 1$, it follows that $S(g, A)$ cannot possess an upper frame bound.

The insight provided by [96] is that Gabor frames possess a certain Homogeneous Approximation property, or HAP. This is stated below in our general context as Theorem (2.2.7). The proof given by Ramanathan and Steger relied on weak convergence of translations of A. Grochenig and Razafinjatovo proved an analogue of the HAP for frames of translates in the space of bandlimited functions [89] . Their proof was considerably shorter than the method of [96], but required a restriction on the frame generators. In Gabor systemes , this restriction stems from the fact that the local maximal function of the short-time Fourier transform $S_g f$ is not necessarily square-integrable. We will provide a simple proof of the HAP which imposes no restriction on the generators.

Notation (2.2.5)[83]: Let $g_1, \dots, g_r \in L^2(R^d)$ and $A_1, \dots, A_r \subset R^{2d}$be such that $\bigcup_k S(g_k, A_k)$ is a frame for $L^2(R^d)$, with frame bounds A, B let

$$\{\tilde{g}_{k,a,b}\}_{(a,b)\in A_k, k=1,\dots,r}$$

denote the dual frame of $\bigcup_k S(g_k, A_k)$, in general, this dual frame need not consist of translates and modulates of some finite set of functions.

Given $h > 0$ and $(p, q) \in R^{2d}$, let $W(h, p, q)$ denote the following subspace of $L^2(R^d)$:

$$W(h, p, q) = \text{span}\{\tilde{g}_{k,a,b} : (a, b) \in Q_h(p, q) \cap A_k, k = , \dots, r\}. \quad (4)$$

This space is finite-dimensional because, by Theorem (2.2.10)(a), each A_k is relatively uniformly separated.

Lemma (2.2.6)[83]: Set $\varphi(x) = e^{-(\pi/2)x^2}$, and let $h > 0$ be fixed. Then there exists a constant k such that for each $f \in L^2(R^d)$ and each $(p, q) \in R^{2d}$,

$$\left|\langle\varphi, M_q T_p f\rangle\right|^2 \leq k \iint_{Q_h(p,q)} \left|\langle\varphi, M_y T_x f\rangle\right|^2 dx\, dy.$$

Proof. The Bargmann transform

$$Bf(x + iy) = e^{(\pi/2)(x^2+y^2)} e^{\pi i x y} \langle M_y T_{-x} f, \varphi\rangle$$

maps $L^2(R^d)$ into the space of entire functions on C^{2d} [88, p.40]. Hence, by [91, Theorem 2.2.3] there exists a constant C, independent of f, such that

$$|Bf(0)| \leq C \iint_{Q_h(0,0)} |Bf(z)| dz. \quad (5)$$

Applying (5) to the function $M_q T_p f$ therefore yields

$$\begin{aligned}\left|\langle\varphi, M_q T_p f\rangle\right|^2 &= \left|B(M_q T_p f)(0)\right|^2 \\ &\leq C^2 \left(\iint_{Q_{h(0,0)}} \left|e^{\frac{\pi}{2}(x^2+y^2)} \langle M_y T_{-x}(M_q T_p f), \varphi\rangle\right| dxdy\right)^2 \\ &\leq C^2 \left(\iint_{Q_{h(0,0)}} e^{\pi(x^2+y^2)} dxdy\right) \\ &\quad \times \left(\iint_{Q_{h(0,0)}} \left|\langle M_{q+y} T_{p-x} f, \varphi\rangle\right|^2 dxdy\right) \\ &= K \iint_{Q_{h(p,q)}} \left|\langle M_y T_x f, \varphi\rangle\right|^2 dxdy\end{aligned}$$

We can now state the HAP. The simple proof follows by observing that the HAP for time-frequency shifts of a single function implies the HAP for all functions.

Theorem (2.2.7)[83]: (Homogeneous Approximation property). Let $g_1, \dots, g_r \in L^2(R^d)$ and $A_1, \dots, A_r \subset R^{2d}$, be such that $\bigcup_{k=1}^r S(g_k, A_k)$ is a frame for $L^2(R^d)$. Then for each $f \in L^2(R^d)$,

$$\forall\, \varepsilon > 0, \quad \exists\, R > 0, \quad \forall (p, q) \in R^{2d}, \quad \text{dist}\left(M_q T_p f, W(R, p, q)\right) < \varepsilon. \quad (6)$$

Proof. By Theorem (2.2.10)(a). the assumption that $\bigcup_k S(g_k, A_k)$ is a frame implies that each A_k is relatively uniformly separated. By dividing each A_k into subsequences that are uniformly separated, we may assume without loss of generality that each A_k is δ_k-uniformly separated. Define $\delta = \min\{\delta_1/2, \dots, \delta_r/2\}$.

Let $\mathcal{H}$ be the subset of $L^2(R)$ consisting of all functions f for which (6) holds. It is easy to see that $\mathcal{H}$ is closed under finite linear combinations and L^2-limits. It therefore suffices to show that, for the Gaussian function $\varphi(x) = e^{-(\pi/2)x^2}$, all time-frequency shifts $M_t T_s \varphi$ belong to $\mathcal{H}$, for then $\mathcal{H} = L^2(R^d)$ and the result follows.

Therefore, fix any$(s, t) \in R^{2d}$, and consider any $(p, q) \in R^{2d}$. The function $M_q T_p(M_t T_s \varphi)$ has the frame expansion

$$M_qT_p(M_tT_s\varphi)=\sum_{k=1}^{r}\sum_{(a,b)\in A_k}\langle M_qT_p(M_tT_s\varphi),M_bT_a\mathrm{g}_k\rangle\tilde{\mathrm{g}}_{k,a,b}$$

By definition of distance and the fact that $\{\tilde{\mathrm{g}}_{k,a,b}\}$ is itself a frame with upper frame bound A^{-1}, we have

$$\operatorname{dist}\left(M_qT_p(M_tT_s\varphi),W(R,p,q)\right)^2$$

$$\leq\left\|M_qT_p(M_tT_s\varphi)-\sum_{k=1}^{r}\sum_{(a,b)\in Q_R(p,q)\cap A_k}\langle M_qT_p(M_tT_s\varphi),M_bT_a\mathrm{g}_k\rangle\tilde{\mathrm{g}}_{k,a,b}\right\|_2^2$$

$$=\left\|\sum_{k=1}^{r}\sum_{(a,b)\in A_k\setminus Q_R(p,q)}\langle M_qT_p(M_tT_s\varphi),M_bT_a\mathrm{g}_k\rangle\tilde{\mathrm{g}}_{k,a,b}\right\|_2^2$$

$$\leq A^{-1}\sum_{k=1}^{r}\sum_{(a,b)\in A_k\setminus Q_R(p,q)}\left|\langle M_qT_p(M_tT_s\varphi),M_bT_a\mathrm{g}_k\rangle\right|^2. \quad (7)$$

By Lemma (2.2.6) there exists a constant k such that

$$\left|\langle M_qT_p(M_tT_s\varphi),M_bT_a\mathrm{g}_k\rangle\right|^2=\left|\langle\varphi,M_{b-q-t}T_{a-p-s}\,\mathrm{g}_k\rangle\right|^2$$

$$\leq K\iint_{Q_{\delta(a-p-s,b-q-t)}}\left|\langle\varphi,M_yT_x\mathrm{g}_k\rangle\right|^2dx\,dy$$

$$=K\iint_{Q_{\delta(p+s-a,q+t-b)}}\left|S_\varphi\mathrm{g}_k(x,y)\right|^2dx\,dy. \quad (8)$$

Where $S_\varphi\mathrm{g}_k$ is the short-time Fourier transform of g_k a gainst φ. Combining (7) and (8) with the fact that A_k is δ-separated, we conclude that

$$\operatorname{dist}\left(M_qT_p(M_tT_s\varphi),W(R,p,q)\right)^2$$

$$\leq A^{-1}K\sum_{k=1}^{r}\sum_{(a,b)\in A_k\setminus Q_{R(p,q)}}\iint_{Q_{\delta(p+s-a,q+t-b)}}\left|S_\varphi\mathrm{g}_k(x,y)\right|^2dx\,dy$$

$$\leq A^{-1}K\sum_{k=1}^{r}\iint_{R^{2d}\setminus Q_{R-\delta(s,t)}}\left|S_\varphi\mathrm{g}_k(x,y)\right|^2dxdy. \quad (9)$$

Since each $S_\varphi\mathrm{g}_k\in L^2(R^{2d})$, the last quantity in (9) can be made arbitrarily small, independently of (p,q), by taking R large enough.

Corollary (2.2.8)[83]: (strong HAP). Let $\mathrm{g}_1,\dots,\mathrm{g}_r\in L^2(R^d)$ and $A_1,\dots,A_r\subset R^{2d}$ be such that $\bigcup_{k=1}^{r}S(\mathrm{g}_k,A_k)$ is a frame for $L^2(R^d)$. Then for each $f\in L^2(R^d)$ and each >0 , there exists a constant $R>0$, such that

$$\forall(p,q)\in R^{2d},\quad\forall h>0,\ \forall(x,y)\in Q_h(p,q),\ \operatorname{dist}\left(M_yT_xf,W(h+R,p,q)\right)<\varepsilon.$$

Proof. Simply note that if $(x,y)\in Q_h(p,q)$, then $W(R,x,y)\subset W(h+R,p,q)$, and therefore dist $M_yT_xfW(h+R,p,q)\leq$ dist $M_yT_xfW(R,x,y)$.

We now use the Homogeneous Approximation property to show the following comparison between the density of a Gabor frame and the density of Gabor Riesz basis. The double-projection idea of [96] is an important ingredient.

Theorem (2.2.9)[83] (Comparison Theorem). Let $g_1, \dots, g_r \in L^2(R^d)$ and $A_1, \dots, A_r \subset R^{2d}$ be such that $\bigcup_{k=1}^{r} S(g_k, A_k)$ is a frame for $L^2(R^d)$. Let $\phi_1, \dots, \phi_s \in L^2(R^d)$ and $\Delta_1, \dots, \Delta_s \subset R^{2d}$ be such $\bigcup_{k=1}^{r} S(\phi_k, \Delta_k)$ is a Riesz basis for $L^2(R^d)$. Let A be the disjoint union of $A_1, \dots, A_r$ and let Δ be the disjoint union of $\Delta_1, \dots, \Delta_s$, then

$$D^-(\Delta) \le D^-(A) \quad \text{and} \quad D^+(\Delta) \le D^+(A).$$

Proof. We use the notation defined in Notation (2.2.5). Additionally, we denote the dual frame of $\bigcup_{k=1}^{r} S(\phi_k, \Delta_k)$ by $\{\tilde{\phi}_{k,a,b}\}_{(a,b)\in\Delta_k, k=1,\dots,s}$, and, in analogy to the subspaces $W(h,p,q)$ defined in (4), we set

$V(h,p,q) = \operatorname{span}\{M_b T_a \phi_k : (a,b) \in Q_h(p,q) \cap \Delta_k, k = 1, \dots, s\}$

By Theorem (2.2.10)(a), each Δ_k is relatively uniformly separated, and hence $V(h,p,q)$ is finite-dimensional. Since the elements of any frame are uniformly bounded in norm, we can find a constant C such that $\|\tilde{\phi}_{k,a,b}\| \le C$ for all k, a, and b.

Choose now any $\varepsilon > 0$. Then, by Corollary (2.2.8) applied to the frame $\bigcup_{k=1} S(g_k, A_k)$ and the function $f = \phi_k$, there exists $R_k > 0$ such that

$$\forall\, h > 0, \quad \forall (p,q) \in R^{2d}, \quad \forall (x,y) \in Q_h(p,q),$$
$$\operatorname{dist}\left(M_y T_x f, W(h + R_k, p, q)\right) < \frac{\varepsilon}{C}. \tag{10}$$

Let $= \max\{R_1, \dots, R_s\}$, then (10) holds for each K when R_k, is replaced by R.

Now let $h > 0$ and $(p,q) \in R^{2d}$ be fixed. For simplicity, let us denote the orthogonal projections onto $V(h,p,q)$ and $W(h+R,p,q)$ by $P_V = P_{V(h,p,q)}$ and $P_W = P_{W(h+R,p,q)}$. Define $T: V(h,p,q) \to V(h,p,q)$ by

$$T = P_{V(h,p,q)} P_{W(h+R,p,q)} = P_V P_W.$$

Then, by the biorthogonality of $\bigcup_k S(\emptyset_k, \Delta_k)$ and $\{\tilde{\phi}_{k,a,b}\}$, the trace of T can be computed as

$$\operatorname{tr}(T) = \sum_{k=1}^{s} \sum_{(a,b)\in Q_h(p,q)\cap\Delta_k} \langle T(M_b T_a \phi_k), \tilde{\phi}_{k,a,b} \rangle.$$

However, for $(a,b) \in Q_h(p,q) \cap \Delta_k$, we have

$$T\langle (M_b T_a \phi_k), \tilde{\phi}_{k,a,b} \rangle = \langle P_W M_b T_a \phi_k, P_V \tilde{\phi}_{k,a,b} \rangle$$
$$= \langle M_b T_a \phi_k, P_V \tilde{\phi}_{k,a,b} \rangle + \langle (P_W - 1) M_b T_a \phi_k, P_V \tilde{\phi}_{k,a,b} \rangle.$$

Now $P_V M_b T_a \phi_k = M_b T_a \phi_k$ since $M_b T_a \phi_k \in V(h,p,q)$. Therefore,

$$\langle M_b T_a \phi_k, P_V \tilde{\phi}_{k,a,b} \rangle = \langle P_V M_b T_a \phi_k, \tilde{\phi}_{k,a,b} \rangle = \langle M_b T_a \phi_k, \tilde{\phi}_{k,a,b} \rangle = 1,$$

the last equality following from biorthogonality. Further, by (10) we have

$$\left|\langle (P_W - 1) M_b T_a \phi_k, P_V \tilde{\phi}_{k,a,b} \rangle\right| \le \left\|(P_W - 1) M_b T_a \phi_k, P_V \tilde{\phi}_{k,a,b}\right\|_2 \left\|P_V \tilde{\phi}_{k,a,b}\right\|_2 \le \frac{\varepsilon}{2} \cdot C$$
$$= \varepsilon.$$

Hence, since Δ is disjoint union of $\Delta_1, \dots, \Delta_s$,

$$\operatorname{tr}(A) \ge \sum_{k=1}^{s} \sum_{(a,b)\in Q_h(p,q)\cap\Delta_k} (1 - \varepsilon) = (1 - \varepsilon) \sum_{k=1}^{s} \#(Q_h(p,q) \cap \Delta_k)$$
$$= (1 - \varepsilon)\#(Q_h(p,q) \cap \Delta). \tag{11}$$

On ther other hand, all eigenvalues of T satisfy $|\lambda| \leq \|T\| \leq 1$. Hence,

$$\operatorname{tr}(T) \leq \operatorname{rank}(T) \leq \dim\big(W(h+R,P,q\,)\big) \leq \sum_{k=1}^{s} \#(Q_{h+R}(p,q) \cap A_k)$$

$$= \#(Q_{h+R}(p,q) \cap A). \qquad (12)$$

Therefore, by combining (11) and (12), we see that for each $h > 0$ and each $(p,q) \in R^{2d}$,

$$(1-\varepsilon)\#(Q_h(p,q) \cap \Delta) \leq \#(Q_{h+R}(p,q) \cap A).$$

As a consequence,

$$(1-\varepsilon)\frac{\#(Q_h(p,q)\cap\Delta)}{h^{2d}} \leq \frac{\#(Q_{h+R}(p,q)\cap A)}{(h+R)^{2d}}\frac{(h+R)^{2d}}{h^{2d}}.$$

It follows that

$$(1-\varepsilon)D^-(\Delta) \leq D^-(A) \quad \text{and} \quad (1-\varepsilon)D^+(\Delta) \leq D^+(A),$$

and since ε is arbitrary, the theorem is proved.

The proof of part (b) of Theorem (2.2.10) is now immediate.

Theorem (2.2.10)[83]: For each $k = 1, \dots\dots, r$, choose a nonzero function $\mathrm{g}_k \in L^2(R^d)$ and an arbitrary sequence$A_k \subset R^{2d}$. Let A be the disjoint union of $A_1, \dots\dots, A_r$.

(a) If $\bigcup_{k=1}^{r} S(\mathrm{g}_k, A_k)$ possesses an upper frame bound for $L^2(R^d)$, then $D^+(A) < \infty$.

(b) If $\bigcup_{k=1}^{r} S(\mathrm{g}_k, A_k)$ is a frame for $L^2(R^d)$, then $D^-(A) \geq 1$.

Proof. (a) Suppose that $U_k\, S(\mathrm{g}_k\,, A_k)$ possesses an upper frame bound. then, by Theorem (2.2.4), each A_k is relatively uniformly separated. Hence A is a finite union of relatively uniformly separated sequences, therefore is itself relatively uniformly separated, and hence has finite upper Beurling density.

(b) Define $\phi = \chi_{Q_1(0)}$ and $\Delta = Z^d$. Then $S(\phi, \Delta)$ is an orthonormal basis for $L^2(R^d)$. Therefore, Theorem (2.2.9) implies that $D^-(A) \geq D^-(\Delta) = 1$.

We remark that we cannot replace the conclusion $D^-(A) \geq 1$ of Theorem (2.2.10)(b) by the stronger statement that $\sum_{k=1}^{r} D^-(A_k) \geq 1$. For example, consider again the orthonormal basis $S(\emptyset, \Delta)$ defined by $\emptyset = \chi_{Q_1(0)}$ and $\Delta = Z^d$. We do have $D^- \geq 1$. However, if we define $\Delta_1 = \{n = (n_1, \dots\dots, n_d) \in Z^d : n_1 \geq 0\}$ and $\Delta_2 = \{n = (n_1, \dots, n_d) \in Z^d : n_1 < 0\}$, then $S(\emptyset, \Delta_1) \cup S(\emptyset, \Delta_2)$ is an orthonormal basis for $L^2(R^d)$, yet $D^-(\Delta_1) = D^-(\Delta_2) = 0$.

We conclude with the following consequence of the comparison theorem for Gabor Riesz bases.

Corollary (2.2.11)[83]: Assume that $\phi_1, \dots, \phi_s \in L^2(R^d)$ and $\Delta_1, \dots\dots, \Delta_s \subset R^{2d}$ are such that $\bigcup_{k=1}^{s} S(\phi_K, \Delta_K)$ is a Riesz basis for $L^2(R^d)$. Let Δ be the disjoint union of $\Delta_1, \dots\dots, \Delta_s$, then $D^+(\Delta) = D^-(\Delta) = 1$; i.e, Δ has uniform Beurling density $D(\Delta) = 1$.

Proof. Let $\mathrm{g} = \chi_{Q_1(0)}$ and $A = Z^d$. Then $S(\mathrm{g}, A)$ is an orthonormal basis, and hence a frame, for $L^2(R^d)$. Therefore, Theorem (2.2.9) applied to this frame and to the Riesz Basis $\bigcup_k S(\emptyset_K, \Delta_K)$ implies that $D^-(\Delta) \leq D^-(A) = 1$ and $D^+(\Delta) \leq D^+(A) = 1$. By symmetry, we also have $1 = D^-(A) \leq D^-(\Delta)$ and $1 = D^+(A) \leq D^+(\Delta)$.

We will prove Theorem (2.2.12).

First, however we observe that Theorem (2.2.10) already implies that there are no frames consisting of translates of finitely many functions. to see this, assume that $\mathrm{g}_1, \dots, \mathrm{g}_r \in L^2(R^d)$ and $\Gamma_1, \dots \Gamma_r \subset R^d$ were such that $\bigcup_k T(\mathrm{g}_k, \Gamma_k)$ was a frame for $L^2(R^d)$. Considering that $(\mathrm{g}_k, \Gamma_k \times \{0\}) = T(\mathrm{g}_k, \Gamma_k)$, we see that Theorem (2.2.10)(b) implies that $D^{-}(\Gamma \times \{0\}) \geq 1$, where Γ is the disjoint union of $\Gamma_1, \dots, \Gamma_r$. However, this is certainly a contradiction, since $D^{-}(\Gamma \times \{0\}) = 0$.

Indeed, Theorem (2.2.10)(b) implies that whenever $\bigcup_k S(g_K, \Delta_K)$ is a frame, the disjoint union A cannot contain arbitrarily large gaps , since if for each radius h there existed a point $(x, y) \in R^{2d}$ such that $Q_h(x, y)$ contained no points of A, then we would have $D^{-}(\mathrm{A}) = 0$ and therefore could not have a frame. Thus, the collection A of time-frequency translates must be "spread" throughout the entire time-frequency plane R^{2d}. For example, A could not be restricted to a banded set like $R^d \times Q_1(0)$, or a single "quadrant" in R^{2d}.

We now give the proof of Theorem (2.2.12).

Theorem (2.2.12)[83]: For each $k = 1, \dots \dots, r$, choose a nonzero function $g_k \in L^2(R^d)$ and an arbitrary sequence $\Gamma_k \subset R^d$. Let Γ be the disjoint union of $\Gamma_k, \dots \dots, \Gamma_r$).

(a) If $\bigcup_{k=1}^{r} T(g_k, \Gamma_k)$ possesses an upper frame bound for $L^2(R^d)$, then $D^{+}(\Gamma) < \infty$.

(b) If $\bigcup_{k=1}^{r} T(g_k, \Gamma_k)$ possesses an lower frame bound for $L^2(R^d)$, then $D^{+}(\Gamma) = \infty$.

Proof. (a) Assume that $\bigcup_k T(g_k, \Gamma_k)$ possessed an upper frame bound. Then by Theorem (2.2.10)(a), We have $D^{+}(\Gamma \times \{0\}) < \infty$. This implies that $\Gamma \times \{0\}$ is relatively uniformely separated as a subset of R^{2d}, Hence Γ is relatively uniformly separated as a subset of R^d, and therefore $D^{+}(\Gamma) < \infty$.

(b) We will show the contrapositive statement. Assume that $D^{+}(\Gamma) < \infty$. then Γ, and therefore each Γ_k, is relatively uniformely separated. Hence for each k we can write Γ_k as the union of subsequences Δ_{kj} for $j = 1, \dots, s_k$, each of which is δ_{kj}-Separated. Define $\delta = \min\{\delta_{kj}/2\}$. Then fix any $h < \delta$, and define $Q = Q_h(0)$. Note that the cubes $\{Q + a\}_{a \in \Delta_{kj}}$ are disjoint, and define

$$B_{kj} = \bigcup_{a \in \Delta_{kj}} (Q + a).$$

Then ,

$$\sum_{k=1}^{r} \sum_{a \in \Gamma_k} |\langle x_Q, T_a g_k \rangle|^2 = \sum_{k=1}^{r} \sum_{j=1}^{s_k} \sum_{a \in \Delta_{kj}} |\langle x_Q, x_Q T_a g_k \rangle|^2 \leq \sum_{k=1}^{r} \sum_{j=1}^{s_k} \sum_{a \in \Delta_{kj}} \|x_Q\|_2^2 \|x_Q T_a g_k\|_2^2$$

$$= \|x_Q\|_2^2 \sum_{K=1}^{r} \sum_{J=1}^{s_k} \int_{B_{kj}} |g(x)|^2 \, dx.$$

However, for each fixed k and j, the function $x_{B_{kj}}(x) |g_k(x)|^2$ converges to zero point wise a.e as $h \to 0$, and is dominated by the integrable function $|g_k(x)|^2$. it therefore follows from the lebesgue Dominated Convergence Theorem that $\lim_{h \to 0} \int_{B_{kj}} |g_k(x)|^2 \, dx = 0$. Hence $\bigcup_k T(g_k, \Gamma_k)$ cannot possess a lower frome bound.

Section (2.3): Stability of Gabor Frames

Given a Gabor frames $\{\pi(\gamma) g : \gamma \in \Gamma\}$, also known as a Weyl-Heisenbrg frame, where $g \in L^2(R^d)$, Γ is a sequence in R^{2d}, and

$$\pi(x, y) g(z) = g(z - x) e^{2\pi i \langle z, y \rangle}, \qquad (x, y) \in R^{2d},$$

one is asked if it remains a frame when the window function and /or sampling points have some small perturbation. Since the frame property of this system is equivalent to the question whether it is possible to recover a signal f from L^2 in a stable way from the sampling values of the short-time Fourier transform over Γ, i.e, from the values

$\langle f, \pi(x,y)\mathrm{g}\rangle$, with $(x,y) \in \Gamma$ one also talks about the stability of sampling points, respectively the jitter error problem for the Gabor transform. Although the continuous dependence of the representation coefficients as a function of the precise position of the sampling points has been described in considerable generality as early as 1989 in [111] (Theorem 6.5), this result concerns the coefficients obtained by specific iterative reconstruction methods, which in turn are only guaranteed to converge under the assumption of sufficiently high sampling density. The stability of Gabor frames having the canonical duals in mind has been given by Favier and Zalik [109] and subsequently by [103,104,106,109,126,127,128].Most of them are concerning 1-dimension or Γ being a separable lattice (or product lattice), i.e, $\Gamma = \{(na, mb): n, m \in Z^d\}$ for some $a, b > 0$.

In [114], Feichtinger and Kaiblinger proved that if $\{\pi(Ln)\mathrm{g}: n \in Z^{2d}\}$ is a frame for $L^2(R^d)$, where L is a $2d \times 2d$ matrix and $\mathrm{g} \in S_0(R^d)$, then $\{\pi(\tilde{L}n)\mathrm{g}: n \in Z^{2d}\}$ is also a frame provided $\|L - \tilde{L}\|$ is small enough. These results are valid despite the fact that the corresponding frame operators are not close to each other in the operator norm. We study similar perturbations, but for arbitrary sets of sampling points in the time-frequency plane. Specifically, let $\{\pi(a_n, b_n)\mathrm{g}: n \in Z\}$ be a frame. We ask whether it remains a frame when (a_n, b_n) is replaced by $(Pa_n, P^{-T}b_n)$. here P is a $d \times d$ matrix, P^T is the transpose of P and $P^{-T} = (P^T)^{-1}$. We show that the answer is positive if g satisfies some decay condition and $\|I - P\|$ is small enough. Note that for Γ being a lattice, perturbations of this type preserve the size of the lattice. Typical cases are ordinary dilations (e.g stretching in the time direction and corresponding compression in the frequency domain), or more general symplectic transformations acting on phase space. Since for the regular Gabor the lattices obtained from standard product lattices in this way are called symplectic lattices we call these perturbations symplectic perturbations (cf. [121], p.280).

For the ℓ^∞ perturbation of sampling points, it was shown in [103,104,109,126, 127] that if $\{\pi(na, mb)\mathrm{g}: n, m \in Z\}$ is a frame for $L^2(R)$, g satisfies some decay and smoothness conditions, and $|\lambda_n - na|$ and $|\mu_m - mb|$ are small enough, then $\{\pi(\lambda_n, \mu_m)g: n, m \in Z\}$ is also a frame for $L^2(R)$. In [128], this result is generalized to arbitrary sampling points while the window function is required to stisfy $x^k\mathrm{g}^{(\ell)}(x) \in L^2(R)$ for all $0 \le k, \ell \le 2$, which implies that $\mathrm{g}(x), \mathrm{g}'(x), x\mathrm{g}(x), x\mathrm{g}'(x) \in S_0(R^d)$ (see [116, Lemma 3.11]). We show that the window function belonging to $S_0(R^d)$ is enough to ensure the stability. Specifically, if $\mathrm{g} \in S_0(R^d)$ and $\{\pi(\gamma_n)\mathrm{g}: n \in Z\}$ is a frame for $L^2(R^d)$, then $\{\pi(\gamma_n')\mathrm{g}: n \in Z\}$ is also a frame whenever $\sup_n \|\gamma_n - \gamma_n'\|_\infty$ is small enough. Moreover, we give an explicit stability bound when $X^\alpha\mathrm{g}, D^\alpha\mathrm{g} \in S_0(R^d)$ for all $\alpha \in I_d$ with $|\alpha| = 1$. Note that the sampling points are arbitrary.

On the perturbation of window functions, most of known results [103,109,126] are stated with a decay condition on window functions. Perturbations are allowed only in point wise sense, i.e, $|h(x) - \mathrm{g}(x)| \le \varepsilon|\mathrm{g}(x)|$ or $|\tilde{h}(\omega) - \tilde{\mathrm{g}}(\omega)| \le \varepsilon|\tilde{g}(\omega)|$, where h is the perturbed window function. Here we give some stability conditions with global norms. In particular, if $\{\pi(\gamma)\mathrm{g}: \gamma \in \Gamma\}$ is a frame, then $\{\pi(\gamma)h: \gamma \in \Gamma\}$ is also a frame provided $\|g - h\|_{s_0}$ is small enough. Note that the window function and sampling points are arbitrary.

The stability of dual frames is also needed in practice. However, as far as we know, there are few of results on this topic (except for the typical statements on frame bounds). We show that if two frames are close to each other , so are their dual frames in the same sense.

Notation (2.3.1)[99]: $|E|$ denotes the Lebesgue measure of a measurable set E. A sequence $\Gamma \subset R^d$ is called δ-uniformely discrete for some $\delta > 0$ if $\|\gamma - \gamma'\|_\infty \geq \delta$ for any $\gamma, \gamma' \in \Gamma$ with $\gamma \neq \gamma'$. Γ is called relatively uniformly discrete if it is a finite union of uniformly discrete sequences. For any $a > 0$ and $n \in Z^d$, let $E_{n,a} = na + [0, a]^d$. define the Wiener algebra as

$$W(R^d) = \left\{f: f \text{ is continuous and } \sum_{n\in Z^d} \left\|f . X_{E_{n,a}}\right\|_\infty < \infty\right\}.$$

It can be proved that $W(R^d)$ is Banach space with the norm

$$\|f\|_{W,a} = \sum_{n\in Z^d} \left\|f . X_{E_{n,a}}\right\|_\infty < \infty$$

and different choices of a give of a give equivalent norm [106, p.187].

For any $d \times d$ matrix P, denote its norm by $\|P\| = \sup_{\|x\|_2=1} \|p_x\|_2$.

We use the following set of multi-indices: $I_d \coloneqq \{(i_1, \dots, i_d): i_k = 0 \text{ or } 1\}$, $\alpha = (\alpha_1, \dots, \alpha_d), |\alpha| = \alpha_1 + \cdots + \alpha_d$,

$$x^\alpha = x_1^{\alpha_1}, \dots, x_d^{\alpha_d}, \left(X^\beta f\right)(x) = X^\beta f(x) = x_1^{\beta_1}, \dots, x_d^{\beta_d} f(x),$$

and $D_x^\alpha f(x) = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1}, \dots, \partial_{x_d}^{\alpha_d}} f(x)$ stands for classical partial derivatives. We simply write $D^\alpha f$ if there is no confusion.

The Segal algebra $S_0(R^d)$, also known as Feichtinger's algebra, is defined by

$$S_0(R^d) = \left\{f \in L^2(R^d): \|f\|_{S_0} = \left\|V_{g_0} f\right\|_1 < \infty\right\},$$

Where g_0 is the Gaussian function, i.e , $g_0(x) = e^{-\pi\|x\|_2{}^2}$. $S_0(R^d)$ Coincides with the modulation space $M_{1,1}^0$ as discussed in [110,112].

Note that $\left(V_g f\right)(x, y) = e^{-2\pi i\langle x,y\rangle} \left(V_{\tilde{g}} \tilde{f}\right)(y, -x)$. Consequently $f \in S_0(R^d)$ if and only if $\tilde{f} \in S_0(R^d)$. We give some sufficient conditions (see for example [117]) for a function to be in $S_0(R^d)$, and we see [118] for further properties.

Proposition (2.3.2)[99]: Any of the following conditions implies $f \in S_0(R^d)$:

(i) ([118]) $f \in L^1(R^d)$ and band limited, i.e, $\hat{f}$ is compactly supported.

(ii) ([120]) $f_{\omega_s}, \hat{f}_{\omega_s} \in L^2(R^d)$, where $\omega_s(x) = (1 + \|x\|_2^2)^{s/2}$, for some $s > d$.

Recall that a family of functions $\{\varphi_k : k \in Z\}$ in $L^2(R^d)$ is a frame for $L^2(R^d)$ if there are two positive numbers A and B such that for any $f \in L^2(R^d)$,

$$A\|f\|^2 \leq \sum_{k\in z} |\langle f, \varphi_k\rangle|^2 \leq B\|f\|^2.$$

A And B are called the lower and upper frame bounds, respectively . We see [106, 130] for an overview on frames and Riesz bases.

We study the Symplectic perturbation of sampling points. We consider two cases. the first concerns sets of the form $\Gamma = \left\{\left(a_n, b_{n,m}\right): n, m \in Z\right\}$, the second the more general situation $\Gamma = \{(a_n, b_n): n \in Z\}$. First, we recall a result by Christensen, Deng and Heil.

Proposition (2.3.3)[99]: [107, Theorem 1.1]. Let $g \in L^2(R^d)/\{0\}$ and Γ be a sequence in R^{2d}.

(i) If $\{\pi(\gamma)g: \gamma \in \Gamma\}$ has an upper frame bound, then Γ is relatively uniformly discrete.

(ii) If $\{\pi(\gamma)\, g : \gamma \in \Gamma\}$ is a frame for $L^2(R^d)$, then Γ has a lower Beurling density no less than 1.

The following is a fundamental result in the study of the stability of frames.

Proposition (2.3.4)[99]: [102, Theorem 1]. Let $\{f_k: k \in Z\}$ be a frame for some Hilbert space $\mathcal{H}$ with bounds A and B. if $\{g_k: k \in Z\} \subset \mathcal{H}$ is such that $\{f_k - g_k: k \in Z\}$ is a Bassel sequence with upper bound $M < A$, then $\{g_k: k \in Z\}$ is a frame for $\mathcal{H}$ with bounds $\left(\sqrt{A} - \sqrt{M}\right)^2$ and $\left(\sqrt{B} - \sqrt{M}\right)^2$.

The following lemma is a generalization of Wirtinger's inequality [122, p.184] for multivariate functions, which can be showed by induction (see [116, lemma 3.14] for a proof).

Lemma (2.3.5)[99]: Suppose that E is a cube in R^d with side length δ and $D^\alpha f \in L^2(E)$ for any $\alpha \in I_d$. Then for any $y \in E$,

$$\|f - f(y)\|_{L^2(E)} \le \sum_{\alpha \in I_d \setminus \{0\}} \left(\frac{2\delta}{\pi}\right)^{|\alpha|} \|D^\alpha f\|_{L^2(E)}.$$

The 1-dimensional version of the following lemma essentially appeared in [100, lemma 42]. here we give a multi-dimensional version.

Lemma (2.3.6)[99]: Suppose that a and δ are positive constants. Then for any δ-uniformely discrete sequence $\{b_n: n \in Z\} \subset R^d$, $\left\{e^{2\pi i\langle \cdot, b_n\rangle}: n \in Z\right\}$ is a Bessel sequence for $L^2(E)$ with upper bound $\delta^{-d}(1 + a\delta)^{2d}$ for any cube $E \subset R^d$ with side length a.

Proof. First, we consider the case of $E = [-a/2\,, a/2]^d$. Divide $[-\delta/2\,, \delta/2\,]^d$ into 2^d small cubes U_k, $1 \le k \le 2^d$, such that $|U_k| = (\delta/2)^d$ and 0 is an end point of each U_k.

For any $f \in L^2$ with support in $[-a/2\,, a/2]^d$, the Fourier transform takes the form

$$\hat{f}(y) = \int_{[-a/2, a/2]^d} f(x) e^{-2\pi i\langle x, y\rangle} dx.$$

Then we have

$$\left\|D^\alpha \hat{f}\right\|_2 = \left\|(-2\pi i)^{|\alpha|} X^\alpha f\right\|_{L^2(E)} \le (\pi a)^{|\alpha|} \|f\|_{L^2(E)} = (\pi a)^{|\alpha|} \left\|\tilde{f}\right\|_2.$$

It follows from lemma (2.3.5) that

$$\sum_{k=1}^{2^d} \sum_{n \in Z} \int_{U_k} \left|\hat{f}(y + b_n) - \hat{f}(b_n)\right|^2 dy \le \sum_{k=1}^{2^d} \sum_{n \in Z} \left(\sum_{\alpha \in I_d \setminus \{0\}} \left(\frac{\delta}{\pi}\right)^{|\alpha|} \left\|D^\alpha \hat{f}(.+b_n)\right\|_{L^2(U_k)} \right)^2$$

$$\le \left(\sum_{\alpha \in I_d \setminus \{0\}} \left(\sum_{k=1}^{2^d} \sum_{n \in Z} \left(\frac{\delta}{\pi}\right)^{2|\alpha|} \left\|D^\alpha \hat{f}(\cdot + b_n)\right\|_{L^2(U_k)}^2 \right)^{1/2} \right)^2$$

$$= \left(\sum_{\alpha \in I_d \setminus \{0\}} \left(\sum_{n \in Z} \left(\frac{\delta}{\pi}\right)^{2|\alpha|} \int_{[-\delta/2, \delta/2]^d} \left|(D^\alpha \hat{f})(y + b_n)\right|^2 dy \right)^{1/2} \right)^2$$

$$\leq \left(\sum_{\alpha \in I_d \setminus \{0\}} \left(\frac{\delta}{\pi}\right)^{|\alpha|} \left\| D^\alpha \hat{f} \right\|_2 \right)^2 \leq \left(\sum_{\alpha \in I_d \setminus \{0\}} \left(\frac{\delta}{\pi}\right)^{|\alpha|} (\pi a)^{|\alpha|} \left\| \hat{f} \right\|_2 \right)^2$$
$$= ((a\delta + 1)^d - 1)^2 \left\| \hat{f} \right\|_2^2.$$

Note that

$$\sum_{k=1}^{2^d} \sum_{n \in Z} \int_{U_k} \left| \hat{f}(y + b_n) \right|^2 dy = \sum_{n \in Z} \int_{[-\delta/2, \delta/2]^d} \left| \hat{f}(y + b_n) \right|^2 dy \leq \left\| \hat{f} \right\|_2^2.$$

We have

$$\sum_{n \in Z} \left| \hat{f}(b_n) \right|^2 = \frac{1}{\delta^d} \sum_{k=1}^{2^d} \sum_{n \in Z} \int_{U_k} \left| \hat{f}(b_n) \right|^2 dy \leq \frac{1}{\delta^d} (1 + a\delta)^{2d} \left\| \hat{f} \right\|_2^2.$$

Since $\left\| \hat{f} \right\|_2 = \|f\|_{L^2(E)}, \{e^{2\pi i \langle \cdot, b_n \rangle} : n \in Z\}$ is a Bessel sequence for $L^2([-a/2, a/2]^2)$ with upper bound $\delta^{-d}(1 + a\delta)^{2d}$.

For the general case, by a change of variable of the form $x \to x + x_0$, where x_0 is the center point of E, we can show that $\left\{e^{2\pi i \langle \cdot, b_n \rangle} : n \in Z\right\}$ is a Bessel sequence for $L^2(E)$ with the same upper bound.

The following lemma gives an estimate for the upper frame bound of $\{\pi(a_n, b_{m,n})g : n, m \in Z\}$ when $g \in W(R^d)$.

Lemma (2.3.7)[99]: let $a, \delta > 0$ be constants. Suppose that $\{a_n : n \in Z\} \subset R^d$ is a-uniformly discrete and for any $n \in Z$,$\{b_{n,m} : m \in Z\}$ is δ-uniformly discrete. Then $\{\pi(a_n, b_{m,n})g : n, m \in Z\}$ is a Bessel sequence for $L^2(R^d)$ for any $g \in W(R^d)$, with upper bound

$$M = (2/\delta)^d \, (1 + a\delta)^{2d} \|g\|_{W,a}{}^2.$$

Proof. For any $k \in Z^d$, let $E_{k,a} = ka + [0, a)^d$ and

$$q_{k,n} = \left\| g(\cdot - a_n) \cdot \chi_{E_{k,a}} \right\|_\infty + 2^{-(|k_1| + \cdots + |k_d|)\eta}, \; n \in Z,$$

Where η is appositive constant. We have

$$\sum_{k \in Z^k} q_{k,n} = \sum_{k \in Z^k} \left\| g(\cdot - a_n) \chi_{E_{k,a}} \right\|_\infty + \sum_{k \in Z^k} 2^{-(|k_1| + \cdots + |k_d|)\eta}$$
$$= \sum_{k \in Z^k} \left\| g \cdot \chi_{E_{k,a}} - a_n \right\|_\infty + 3^d \eta$$
$$\leq \sum_{k \in Z} \sum_{\substack{k' \in Z^d \\ \left| E_{k',\alpha} \cap (E_{k,a} - a_n) \right| > 0}} \left\| g \cdot \chi_{E_{k',a}} \right\|_\infty + 3^d \eta$$
$$= \sum_{k' \in Z^d} \sum_{\substack{k \in Z^d \\ \left| E_{k',\alpha} \cap (E_{k,a} - a_n) \right| > 0}} \left\| g \cdot \chi_{E_{k',a}} \right\|_\infty + 3^d \eta$$
$$\leq \sum_{k' \in Z^d} 2^d \left\| g \cdot \chi_{E_{k',a}} \right\|_\infty + 3^d \eta = 2^d \|g\|_{W,a} + 3^d \eta.$$

It follows by Lemma (2.3.6) that

$$\sum_{n,m\in Z}\left|\langle f,\pi(a_n,b_{n,m})g\rangle\right|^2=\sum_{n,m\in Z}\left|\int_{R^d}f(x)\overline{g(x-a_n)}e^{-2\pi i\langle x,b_{n,m}\rangle}dx\right|^2$$

$$=\sum_{n,m\in Z}\left|\sum_{k\in Z^d}\int_{E_{k,a}}f(x)\overline{g(x-a_n)}e^{-2\pi i\langle x,b_{n,m}\rangle}dx\right|^2$$

$$\leq\sum_{n,m\in Z}\sum_{k\in Z^d}q_{k,n}\sum_{k\in Z^d}\frac{1}{q_{k,n}}\left|\int_{E_{k,a}}f(x)\overline{g(x-a_n)}e^{-2\pi i\langle x,b_{n,m}\rangle}dx\right|^2$$

$$\leq\sum_{n,m\in Z}\left(2^d\|g\|_{W,a}+3^d\eta\right)\sum_{k\in Z^d}\frac{1}{q_{k,n}}\left|\int_{E_{k,a}}f(x)\overline{g(x-a_n)}e^{-2\pi i\langle x,b_{n,m}\rangle}dx\right|^2$$

$$\leq\sum_{n\in Z}\frac{1}{\delta^d}(1+a\delta)^{2d}\left(2^d\|g\|_{W,a}+3^d\eta\right)\sum_{k\in Z^d}\frac{1}{q_{k,n}}\int_{E_{k,a}}|f(x)g(x-a_n)|^2dx$$

$$=\frac{1}{\delta^d}(1+a\delta)^{2d}\left(2^d\|g\|_{W,a}+3^d\eta\right)\sum_{k\in Z^d}\int_{E_{k,a}}|f(x)|^2\sum_{n\in Z}|g(x-a_n)|^2\frac{1}{q_{k,n}}dx$$

$$\leq\frac{1}{\delta^d}(1+a\delta)^{2d}\left(2^d\|g\|_{W,a}+3^d\eta\right)\sum_{k\in Z^d}\int_{E_{k,a}}|f(x)|^2\sum_{n\in Z}|g(x-a_n)|\,dx$$

$$=\frac{1}{\delta^d}(1+a\delta)^{2d}\left(2^d\|g\|_{W,a}+3^d\eta\right)\int_{E_{k,a}}|f(x)|^2\sum_{n\in Z}|g(x-a_n)|\,dx$$

$$=\frac{1}{\delta^d}(1+a\delta)^{2d}\left(2^d\|g\|_{W,a}+3^d\eta\right)\int_{R^d}|f(x)|^2\sum_{n\in Z}|g(x-a_n)|\,dx. \qquad (13)$$

Since $\{a_n: n\in Z\}\subset R^d$ is a-uniformly discrete,

$$\#(E_{k,a}\cap\{x-a_n: n\in Z)\leqq 1,\qquad \forall k\in Z^d, x\in R^d.$$

Thus

$$\sum_{n\in Z}|\mathrm{g}(x-a_n)|\leqq\sum_{k\in Z^d}\left\|\mathrm{g}.\chi_{E_{k,a}}\right\|_\infty=\|\mathrm{g}\|_{W,a},\qquad\forall x.$$

By (13) , this implies that

$$\sum_{n,m\in Z}\left|\langle f,\pi(a_n,b_{n,m})\mathrm{g}\rangle\right|^2\leq\frac{1}{\delta^d}(1+a\delta)^{2d}\left(2^d\|\mathrm{g}\|_{W,a}+3^d\eta\right)\|\mathrm{g}\|_{W,a}\|f\|_2^2.$$

Now the conclusion follows by letting $\eta\to 0$.

Lemma (2.3.8)[99]: Suppose that $\mathrm{g}\in W(R^d)$ and P is a $d\times d$ matrix. Let $(D_pg)(x)=|\det P|^{1/2}\cdot\mathrm{g}(px)$. Then we have

$$\lim_{\|I-P\|\to 0}\left\|\mathrm{g}-D_p\mathrm{g}\right\|_{W,a}=0,\ \forall \mathrm{g}\in W(R^d), a>0.$$

Proof. By [114, Lemma 2.1], $\lim_{\|I-p\|\to 0}\|g-g(p^{-1})\|_{W,a}=0$. Since $\|I-p\|\to 0$ is equivalent to $\|1-P^{-1}\|\to 0$, the conclusion follows.

With the results above, we can consider Symplectic perturbations of Gabor frames for the case that the sampling points are on Parallel lines.

Theorem (2.3.9)[99]: Suppose that $\{\pi(a_n,b_{n,m})g : n,m\in Z\}$ is a frame for $L^2(R^d)$, $\{a_n : n\in Z\}$ is relatively uniformly discrete, and $g\in W(R^d)$. Then there is some $\varepsilon>0$. Such that for any matrix P satisfying $\|I-P\|<\varepsilon$, $\{\pi(P_{a_n},P^{-T}b_{n,m})_g : n,m\in Z)\}$ is a frame for $L^2(R^d)$.

Proof. Fix some $\delta>0$. by proposition (2.3.3), Γ is relatively uniformly discrete. Hence there is some $N>0$ such that

$$\#\left(\{(a_n,b_{n,m}) : n,m\in Z\}\cap([0,\delta]^{2d}+x)\right)\leqq N,\qquad \forall x\in R^{2d}.$$

Therefore,

$$\#\left(\{b_{n,m} : m\in Z\}\cap([0,\delta]^{d}+y)\right)\leqq N,\qquad \forall y\in R^{d}, n\in Z.$$

Consequently, we can split $\{b_{n,m} : m\in Z\}$ into at most $r'=(2N)^d$ δ-uniformly discrete subsequences $A_{n,\ell}$, $1\le \ell\le r'$. Similarly, there is some $a>0$ such that we can split $\{a_n : n\in Z\}$ into r a-uniformly discrete subsequences Γ_k, $1\le k\le r$. By Lemma (2.3.7), we have

$$\sum_{n,m\in Z}\left|\langle f,\pi(a_n,b_{n,m})(g-D_P g)\rangle\right|^2 = \sum_{k=1}^{r}\sum_{\ell=1}^{r'}\sum_{\substack{a_n\in\Gamma_k\\ b_{n,m}\in A_{n,\ell}}}\left|\langle f,\pi(a_n,b_{n,m})(g-D_P g)\rangle\right|^2$$
$$\le r'r\left(\frac{2}{\delta}\right)^d(1+a\delta)^{2d}\|g-D_P g\|_{W,a}^2\|f\|_2^2.$$

By Lemma (2.3.8) we can make $\|g-g_P\|_{W,a}$, arbitrary small by choosing $\|I-P\|$ small enough. Hence there is some $\varepsilon>0$ such that

$$\sum_{N,M\in Z}\left|\langle f,\pi(a_n,b_{n,m})(g-D_P g\rangle)\right|^2\leqq \Delta_\varepsilon\|f\|_2^2<A\|f\|_2^2,$$

Whenever $\|I-P\|\leqq\varepsilon$, it follows from Proposition (2.3.5) that $\{\pi(a_n,b_{n,m})D_{P,g} : n,m\in Z\}$ is a frame for $L^2(R^d)$, i.e, there are some constants $A,B>0$ such that

$$A\|f\|_2^2\leqq\sum_{n,m\in Z}|\det P|\cdot\left|\int_{R^d} f(x)\cdot\overline{g(Px-P_{a_n})}e^{-2\pi i\langle x,b_{n,m}\rangle}dx\right|^2\leqq B\|f\|_2^2,\ \forall f.$$

By a change of variable of the form $x\to P^{-1}x$, the conclusion follows.

Remark (2.3.10)[99]: By Proposition (2.3.3) $\{(a_n,b_{m,n}) : n,m\in Z\}$ has to be relatively uniformly discrete for $\{\pi(a_n,b_{m,n})g : n,m\in Z\}$ to be a frame, and so is $\{b_{m,n} : m\in Z\}$ for any $n\in Z$. However, $\{a_n : n\in Z\}$ might not be so. The following is a counter example.

For simplicity, we take $d=1$. Let $\delta>0$ and $\{r_n : n\in Z\}$ be the set of all rational numbers. Define

$$a_n=r_n\delta,\quad b_{n,m}=\begin{cases} m\delta, & r_n\in Z,\\ 2^{2n}(2m+1)\delta, & r_n\notin Z, n\geqq 0,\\ 2^{1-2n}(2m+1)\delta, & r_n\notin Z, n<0.\end{cases}$$

It is easy to see that both $\{(a_n, b_{m,n}): n, m \in Z, r_n \in Z\}$ and $\{(a_n, b_{m,n}): n, m \in Z, r_n \notin Z\}$ are δ-uniformely discrete. Hence $\{(a_n, b_{n,m}): n, m \in Z\}$ is relatively uniformly discrete. Since it contains the lattice $Z^2\delta$, we see from [123,125] that $\{(a_n, b_{m,n})\, g : n, m \in Z\}$ is a frame for $L^2(R)$ if g is the Gaussian and $\delta < 1$. Obviously, $\{a_n: n \in Z\}$ is not relatively uniformly discrete.

Remark (2.3.11)[99]: Also, the continuity of g is necessary for this theorem for example, if $g(x) = \chi_{[0,1]^d}(x)$, then $\{\pi(n, m)g: n, m \in Z^d\}$ is an orthonormal basis for $L^2(R^d)$. But for any $p > 1$, $\{\pi(pn, m/p)g: n, m \in Z^d\}$ is not complete in $L^2(R)$, and so cannot form a frame. On the other hand, it is easy to check that if for a continuous function g satisfying

$$|\, g\,(x)| \leq C \prod_{K=1}^{d} (1 + |\chi_k|)^{-1-\varepsilon}$$

For some $C, \varepsilon > 0$, then $g \in W(R^d)$. Consequently, we can apply Theorem (2.3.9) to Gabor frames generated by g. However, the conclusion is not true if $\varepsilon = 0$.

For example, $h(x) = \prod_{k=1}^{d} \frac{\sin \pi x_k}{\pi \chi_k}$ is continuous and $|h(x)| \leq \prod_{k=1}^{d}(1 + |\chi_k|)^{-1}$. Since $\tilde{h} = \chi_{[-1/2,1/2]^d}$, it is easy to see that $\{\pi(n, m)h: n, m \in Z^d\}$ is an orthonormal basis for $L^2(R^d)$ for any $P > 1$.

We study the stability of Gabor frames with arbitrary sampling points.

Proposition (2.3.12)[99]: ([116, Lemma 3.2.15]). For any $f, g \in S_0(R^d)$, we have $V_g f \in W(R^{2d})$ and

$$\|V_{g_0} g\|_{w,\delta} = \|V_g g_0\|_{w,\delta} \leqq 2^{5d/2}\left(2 + \frac{1}{\delta}\right)^{2d} \|V_{g_0} g_0\|_{w,1} \|g\|_{s_0},$$

$$\|V_g f\|_{w,\delta} \leqq 2^{3d}\left(2 + \frac{1}{\delta}\right)^{2d} \|V_{g_0} g_0\|_{w,1} \|f\|_{s_0} \|g\|_{s_0}.$$

Lemma (2.3.13)[99]: ([116, Lemma 3.3]). Suppose that $g \in S_0(R^d)$. Then $\{\pi(\gamma)\, g : \gamma \in \Gamma\}$ is a Bessel sequence for $L^2(R^d)$ with upper bound

$$M(\, g\, ; \delta) := 2^d \left(2 + \frac{1}{\delta}\right)^{2d} \|V_{g_0}\, g_0\|_{w,1} \|\, g\, \|_{S_0}{}^2,$$

for any δ-uniformely discrete sequence $\Gamma \subset R^{2d}$.

Lemma (2.3.14)[99]: If $\{\pi(a_n, b_n)\, g : n \in Z\}$ has an upper (lower) frame bound, then so does $\{\pi(a_n + x_0, b_n + y_0)\, g : n \in Z\}$ with the same bound for any $x_0, y_0 \in R^d$.

Proof. This is a consequence of the fact that

$$\langle f, \pi(a_n + x_0, b_n + y_0)\, g \,\rangle = e^{-2\pi i \langle x_0, b_n + y_0\rangle} \langle f(\cdot + x_0) e^{-2\pi i \langle \cdot\,, y_0\rangle}, \pi(a_n, b_n)\, g \,\rangle..$$

We are now ready to give a symplectic perturbation result for Gabor frames with arbitrary sampling points.

Theorem (2.3.15)[99]: Suppose that $g \in S_0(R^d)$ and $\{\pi(a_n, b_n)\, g : n \in Z\}$ is a frame for $L^2(R^d)$. Then there is some $\varepsilon > 0$ such that for any matrix P satisfying $\|1 - P\| < \varepsilon$, $\{\pi(Pa_n, P^{-T} b_n)\, g : n \in Z\}$ is a frame for $L^2(R^d)$.

Proof. As in the proof of Theorem (2.3.9) we need only to show that $\{\pi(a_n, b_n) D_P g: n \in Z\}$ is a frame $L^2(R^d)$ if $\|1 - P\|$ is small enough.

By Proposition (2.3.3), $\{(a_n, b_n : n \in Z)\}$ is relatively uniformly discrete. Hence there is some $r, \delta > 0$ such that we can splite $\{(a_n, b_n : n \in Z)\}$ into r δ-uniformely discrete sequences Γ_k. For any $f \in L^2(R^d)$, we have

$$\big(V_{g} f\big)(x,y) = \langle f, \pi(x,y)\,g\rangle = \frac{1}{\|g_0\|_2^2}\langle V_{g_0} f, V_{g_0}\pi(x,y) g\rangle$$

$$= \frac{1}{\|g_0\|_2^2}\iint_{R^{2d}} (V_{g_0} f)(t,\omega)\overline{(V_{g_0}\,g)(t-x,\omega-y)}e^{2\pi i\langle x,\omega-y\rangle}dtd\omega. \quad (14)$$

Similarly,

$$\left(V_{D_P g} f\right)(x,y) = \frac{1}{\|g_0\|_2^2}\iint_{R^{2d}}(V_{D_P g_0} f)(t,\omega)\overline{(V_{D_P g_0} D_P\,g)(t-x,\omega-y)}e^{2\pi i\langle x,\omega-y\rangle}dtd\omega.$$

Hence

$$\left|\big(V_g f\big)(a_n,b_n) - (V_{D_P g} f)(a_n,b_n)\right|$$

$$= \frac{1}{\|g_0\|_2^2}\left|\iint_{R^{2d}} (V_{g_0} f)(t,\omega)\overline{(V_{g_0}\,g)(t-a_n,\omega-b_n)}e^{2\pi i\langle a_n,\omega-b_n\rangle}dtd\omega\right.$$

$$\left. - \iint_{R^{2d}} (V_{D_P g_0} f)(t,\omega)\overline{(V_{D_P g_0} D_P\,g)(t-a_n,\omega-b_n)}e^{2\pi i\langle x,\omega-y\rangle}dtd\omega\right|$$

$$\leqq \frac{1}{\|g_0\|_2^2}\left(\iint_{R^{2d}}\left|(V_{g_0} f)(t,\omega) - (V_{D_P g_0} f)(t,\omega)\right|\right.$$

$$\cdot\left|(V_{g_0}\,g)(t-a_n,\omega-b_n)\right|dtd\omega + \iint_{R^{2d}}\left|(V_{D_P g_0} f)(t,\omega)\right|$$

$$\left.\cdot\left|(V_{g_0}\,g)(t-a_n,\omega-b_n) - (V_{D_P g_0} D_P\,g)(t-a_n,\omega-b_n)\right|dtd\omega\right)$$

$$= \frac{1}{\|g_0\|_2^2}\iint_{R^{2d}}\left|(V_{(g_0 - D_p g_0)} f)(t,\omega)\right|\cdot\left|(V_{g_0}\,g)(t-a_n,\omega-b_n)\right|dtd\omega$$

$$+ \frac{1}{\|g_0\|_2^2}\iint_{R^{2d}}\left|(V_{D_P g_0} f)(t,\omega)\right|\cdot\left|\big(V_g g - V_{D_P g_0} D_P g\big)(t-a_n,\omega-b_n)\right|dtd\omega$$

$$\leqq \frac{1}{\|g_0\|_2^2}\left(\iint_{R^{2d}}\left|(V_{(g_0 - D_p g_0)} f)(t,\omega)\right|^2\cdot\left|(V_{g_0} g)(t-a_n,\omega-b_n)dtd\omega\right|\right)^{1/2}$$

$$\cdot\left(\iint_{R^{2d}}\left|(V_{g_0} g)(t-a_n,\omega-b_n)\right|dtd\omega\right)^{1/2}$$

$$+ \frac{1}{\|g_0\|_2^2}\left(\iint_{R^{2d}}\left|(V_{D_P g_0} f)(t,\omega)\right|^2\right.$$

$$\cdot \left|\left(V_{g}g - V_{D_P g_0} D_P g\right)(t-a_n, \omega - b_n)\right| dt d\omega\right)^{1/2}$$

$$\cdot \left(\iint_{R^{2d}} \left|\left(V_{g}g - V_{D_P g_0} D_P g\right)(t-a_n, \omega - b_n)\right| dt d\omega \right)^{1/2}$$

$$= \frac{\|g\|_{S_0}^{1/2}}{\|g_0\|_2^2} \left(\iint_{R^{2d}} \left|(V_{(g_0 - D_p g_0)} f)(t,\omega)\right|^2 \cdot \left|(V_{g_0}g)(t-a_n, \omega - b_n) dt d\omega\right| \right)^{1/2}$$

$$+ C_p \left(\iint_{R^{2d}} \left|(V_{D_P g_0} f)(t,\omega)\right|^2 \right.$$

$$\left. \cdot \left|(V_{g_0}g - V_{D_P g_0} D_P g)(t-a_n, \omega - b_n)\right| dt d\omega\right)^{1/2},$$

Where $C_p = \|g_0\|_2^{-2} \cdot \left\|V_{g_0}g - V_{D_P g_0} D_P g\right\|_1^{1/2}$. Note that for any $t, \omega \in R^d$, $(t,\omega) - \Gamma_k$. Hence

$$\left(\sum_{n \in Z} \left|(V_g f)(a_n, b_n) - (V_{D_P g} f)(a_n, b_n)\right|^2 \right)^{1/2}$$

$$\leqq \frac{\|g\|_{S_0}^{1/2}}{\|g_0\|_2^2} \left(\iint_{R^{2d}} \left|\left(V_{(g_0 - D_P g_0)} f\right)(t,\omega)\right|^2 \right.$$

$$\left. \cdot \sum_{n \in Z} \left|(V_{g_0} g)(t-a_n, \omega - b_n)\right| dt d\omega \right)^{1/2} + C_p \left(\iint_{R^{2d}} \left|\left(V_{D_P g_0} f\right)(t,\omega)\right|^2 \right.$$

$$\left. \cdot \sum_{n \in Z} \left|(V_{g_0} g - V_{D_P g_0} D_p g)(t-a_n, \omega - b_n)\right| dt d\omega \right)^{1/2}$$

$$= \frac{\|g\|_{S_0}^{1/2}}{\|g_0\|_2^2} \left| \iint_{R^{2d}} \left|(V_{(g_0 - D_P g_0)} f(t,\omega)\right|^2 \right.$$

$$\left. \cdot \sum_{\substack{1 \le k \le r \\ (a_n, b_n) \in \Gamma_k}} \left|(V_{g_0} g)(t-a_n, \omega - b_n)\right| dt d\omega \right|^{1/2} + C_p \left| \iint_{R^{2d}} \left|\left(V_{D_P g_0} f\right)(t,\omega)\right|^2 \right.$$

$$\left. \cdot \sum_{\substack{1 \le k \le r \\ (a_n, b_n) \in \Gamma_k}} \left|(V_{g_0} g - V_{D_P g_0} D_p g)(t-a_n, \omega - b_n)\right| dt d\omega \right|^{1/2}$$

$$\leqq \frac{r^{1/2}}{\|g_0\|_2^2} \left(\left\|V_{g_0} g\right\|_{W,\delta}^{1/2} \|g\|_{S_0}^{1/2} \|g_0 - D_P g_0\|_2 \right.$$

$$+\left\|V_{g_0}g-V_{D_Pg_0}D_pg\right\|_{W,\delta}^{1/2}\left\|V_{g_0}g-V_{D_Pg_0}D_pg\right\|_1^{1/2}\|g_0\|_2\Big)\|f\|. \qquad (15)$$

Note that

$$\left(V_{D_Pg_0}D_pg\right)(x,y)=\int_{R^d}|\det P|_g(pt)\overline{g_0(Pt-Px)}e^{-2\pi i\langle t,y\rangle}dt$$

$$=\int_{R^d}g(t)\overline{g_0(t-p_x)}e^{-2\pi i\langle t,P^{-T}y\rangle}dt=\left(V_{g_0g}\right)(P_x,P^{-T}y),$$

By replacing diag $[P,P^{-T}]$ for P in Lemma (2.3.8), we have

$$\lim_{\|I-P\|\to 0}\left\|V_{g_0g}-V_{D_Pg_0}D_Pg\right\|_{w,\delta}=\lim_{\|I-P\|\to 0}\left\|V_{g_0g}-V_{g_0g}(P,P^{-T}\right\|_{w,\delta}=0.$$

Consequently,

$$\lim_{\|I-P\|\to 0}\left\|V_{g_0g}-V_{D_Pg_0}D_Pg\right\|_1=0.$$

Hence we can find some constants $\varepsilon>0$ and $0<\Delta<A$ such that for any matrix P with$\|I-P\|\le\varepsilon$,

$$\sum_{n\in Z}\left|\left(V_gf\right)(a_n,b_n)-(V_{D_Pg}f)(a_n,b_n)\right|^2\le\Delta\|f\|_2^2.$$

By proposition (2.3.4), $\{\pi(a_n,b_n)D_Pg: n\in Z\}$ is also a frame for $L^2(R^d)$.

We study the perturbation of sampling points in ℓ^∞ sense. The main difference between our result and those in [103,104,109,126,127] is that arbitrary sampling points are considered here. By [16, Lemma 3.11], the assumption on the window function is weaker than that of [128, Theorem 3.2]

Theorem (2.3.16)[99]: Suppose that $g\in S_0(R^d)$ and $\{\pi(\gamma_n)g: n\in Z\}$ is a frame for $L^2(R^d)$ with bounds A and B. Then there is some $\varepsilon>0$, such that for any $\gamma_n'\in R^{2d}$ satisfying $\|\gamma_n-\gamma_n'\|_\infty<\varepsilon, n\in Z$, $\{\pi(\gamma_n')g: n\in Z\}$ is a frame for $L^2(R^2)$.

If $D^\alpha g, X^\alpha g\in S_0, \forall|\alpha|=1$, and $\{\gamma_n: n\in Z\}$ is a union of r δ-uniformly discrete sequences, then ε can be determined by $0<\varepsilon<\delta/3$ and

$$M_\varepsilon:=\varepsilon^2r2^{3d}\left(2+\frac{3}{\delta}\right)^{2d}\left\|V_{g_0}g_0\right\|_{W,1}.$$

$$\left(\sum_{|\alpha|=1}\|D^\alpha g\|_{S_0}+(2\pi)^{1/2}\sum_{|\alpha|=1}\|X^\alpha g\|_{S_0}\right)^2<A.$$

In this case, $\left(A^{1/2}-M_\varepsilon^{1/2}\right)^2$ and $\left(B^{1/2}+M_\varepsilon^{1/2}\right)^2$ are frame bounds for $\{\pi(\gamma_n')g: n\in Z\}$.

Proof. Put $\gamma_n=(a_n,b_n)$ and$\gamma_n'=(a_n',b_n')$. For any $f\in L^2(R^d)$, we see from (14) that

$$\left|(V_gf)(a_n,b_n)e^{2\pi i\langle a_n',b_n\rangle}-(V_gf)(a_n',b_n)e^{2\pi i\langle a_n',b_n\rangle}\right|^2$$

$$=\left|(V_gf)(a_n,b_n),(V_gf)(a_n',b_n)\right|^2$$

$$=\frac{1}{\|g_0\|_2^4}\left|\iint_{R^{2d}}(V_{g_0}f)(t,\omega)\overline{(V_{g_0}g)(t-a_n,\omega-b_n)}e^{2\pi i\langle a_n,\omega-b_n\rangle}\right.$$

$$\left.-\overline{(V_{g_0}g)(t-a_n',\omega-b_n)}e^{2\pi i\langle a_n',\omega-b_n\rangle})dtd\omega\right|^2$$

$$\leqq \frac{1}{\|g_0\|_2^4}\left|\iint_{R^{2d}}\left|(V_{g_0}f)(t,\omega)\right|\cdot\left|(V_{g_0}g)(t-a_n,\omega-b_n)\,e^{2\pi i\langle a_n,\omega-b_n\rangle}\right.\right.$$

$$\left.\left.-(V_{g_0}g)(t-a_n',\omega-b_n)e^{2\pi i-a_n',\omega-b_n}\right|dtd\omega\right|^2$$

$$=2^d\left|\iint_{R^{2d}}\left|(V_{g_0}f)(t,\omega)\right|\cdot\left|(V_{g_0}g)(t-a_n,\omega-b_n)\,e^{2\pi i\langle t-a_n,\omega-b_n\rangle}\right.\right.$$

$$\left.\left.-(V_{g_0}g)(t-a_n',\omega-b_n)e^{2\pi i\langle t-a_n',\omega-b_n\rangle}\right|dtd\omega\right|^2$$

$$=2^d\left|\iint_{R^{2d}}\left|\left(V_{g_0}f\right)(t,\omega)\right|\right.$$

$$\left.\cdot\left|\left(V_g g_0\right)(a_n-t,b_n-\omega)-\left(V_g g_0\right)(a_n'-t,b_n-\omega)\right|dtd\omega\right|^2$$

$$\leqq 2^d\iint_{R^{2d}}\left|\left(V_{g_0}f\right)(t,\omega)\ \right|^2$$

$$\cdot\left|\left(V_g g_0\right)(a_n-t,b_n-\omega)-\left(V_g g_0\right)(a_n'-t,b_n-\omega)dtd\omega\right|$$

$$\cdot\int_{R^{2d}}\left|\left(V_g g_0\right)(a_n-t,b_n-\omega)-\left(V_g g_0\right)(a_n'-t,b_n-\omega)\right|dtd\omega$$

$$\leqq 2^d\Delta_{\varepsilon,1}\left(V_g g_0\right)\iint_{R^{2d}}\left|\left(V_{g_0}f\right)(t,\omega)\right|^2$$

$$\cdot\left|\left(V_g g_0\right)(a_n-t,b_n-\omega)-\left(V_g g_0\right)(a_n'-t,b_n-\omega)\right|dtd\omega \qquad (16)$$

Where $\Delta_{\varepsilon,1}\left(V_g g_0\right)=\sup_{\|x\|_\infty}\leq\left\|\left(V_g g_0\right)(\cdot+x,\cdot)-(V_g g_0)\right\|_1$

By proposition (2.3.3), $\{\gamma_n : n\in Z\}$ is the union of finitely many uniformly discrete sequences. Therefore, we can find some constant $\delta>0$ and r disjoint subsets $A_\ell\subset Z$ such that $\bigcup_{\ell=1}^{r}A_\ell=Z$ and $\{\gamma_n : n\in A_\ell\}$ is δ-uniformely discrete, $1\leq\ell\leq r$.

Assume that $\|\gamma_n{}'-\gamma_n\|_\infty\leq\varepsilon\leq\delta/3$. Then $\{\gamma_n'=(a_n',b_n') : n\in A_\ell\}$ is $\delta/3$-uniformely discrete. Hence for any $t,\omega\in R^d$ and $m\in Z^{2d}$,

$$\#\left(\{(t-a_n,\omega-b_n) : n\in A_\ell\}\cap(m\delta/3+[0,\delta/3)^{2d})\right)\leq 1,$$

$$\#\left(\{(t-a_n',\omega-b_n) : n\in A_\ell\}\cap(m\delta/3+[0,\delta/3)^{2d})\right)\leq 1,\qquad 1\leq\ell\leq r.$$

It follows that

$$\sum_{n\in Z}\left|\left(V_g f\right)(a_n,b_n)e^{2\pi i\langle a_n',b_n\rangle}-\left(V_g f\right)(a_n',b_n)e^{2\pi i\langle a_n',b_n\rangle}\right|^2$$

$$\le 2^d \Delta_{\varepsilon,1}(V_g g_0) \iint_{R^{2d}} \left|(V_{g_0} f)(t,\omega)\right|^2$$
$$\cdot \sum_{n\in Z} \left|(V_g g_0)(a_n - t, b_n - \omega) - (V_g g_0)(a_n' - t, b_n - \omega)\right| dt d\omega$$
$$= 2^d \Delta_{\varepsilon,1}(V_g g_0) \iint_{R^{2d}} \left|(V_{g_0} f)(t,\omega)\right|^2$$
$$\cdot \sum_{\ell=1}^{r} \sum_{n\in A_\ell} \left|(V_g g_0)(a_n - t, b_n - \omega) - (V_g g_0)(a_n' - t, b_n - \omega)\right| dt d\omega$$
$$\le 2^{d/2} \Delta_{\varepsilon,1}(V_g g_0) \cdot 2r \left\|V_g g_0\right\|_{W,\delta/3} \|f\|_2^2$$

Similarly we can show that

$$\sum_{n\in Z} \left|(V_g f)(a_n', b_n) e^{2\pi i\langle a_n', b_n\rangle} - (V_g f)(a_n', b_n') e^{2\pi i\langle a_n', b_n'\rangle}\right|^2$$
$$= \sum_{n\in Z} 2^d \left| \iint_{R^{2d}} (V_{g_0} f)(t,\omega)((V_{g_0} g)(t - a_n', \omega - b_n) e^{2\pi i\langle a_n', \omega\rangle} \right.$$
$$\left. - (V_{g_0} g)(t - a_n', \omega -, b_n') e^{2\pi i\langle a_n', \omega\rangle} dt d\omega \right|^2$$
$$\le \sum_{n\in Z} 2^d \Delta_{\varepsilon,2}(V_{g_0} g) \iint_{R^{2d}} \left|(V_{g_0} f)(t,\omega)\right|^2$$
$$\cdot \left|(V_{g_0} g)(t - a_n', \omega - b_n) - (t - a_n', \omega -, b_n')\right| dt d\omega$$
$$\le 2^{d/2} \Delta_{\varepsilon,2}(V_{g_0} g) \cdot 2r \left\|V_{g_0} g\right\|_{W,\delta/3} \|f\|_2^2,$$

Where $\Delta_{\varepsilon,2}(V_{g_0} g) = \sup_{\|y\|_\infty} \le \varepsilon \left\|(V_{g_0} g)(\cdot,\cdot + y) - V_{g_0} g\right\|_1$. By the triangle inequality,

$$\sum_{n\in Z} \left|(V_g f)(a_n, b_n) e^{2\pi i\langle a_n', b_n\rangle} - (V_g f)(a_n', b_n') e^{2\pi i\langle a_n', b_n'\rangle}\right|^2$$
$$\le 2^{d/2} \cdot 2r \left\|V_{g_0} g\right\|_{W,\delta/3} \left(\Delta_{\varepsilon,1}(V_g g_0)^{1/2} + \Delta_{\varepsilon,2}\left(V_{g_0} g\right)^{1/2}\right)^2 \|f\|_2^2$$
$$\coloneqq M(\varepsilon) \|f\|_2^2. \tag{17}$$

Since $g \in S_0$ implies $V_g g_0$, $V_{g_0} g \in L^1(R^{2d})$, we can choose some $\varepsilon > 0$ such that $M(\varepsilon)$ is less than the lower frame bound of $\{\pi(\gamma_n) g : n \in Z\}$. By Proposition (2.3.4) $\{\pi(\gamma_n') g : n \in Z\}$, is a frame for $L^2(R^d)$ whenever $\|\gamma_n - \gamma_n'\|_\infty < \varepsilon, \forall n \in Z$.

Next we assume that $D^\alpha g, X^\alpha g \in S_0, \forall\ |\alpha| = 1$. We have

$$\iint_{R^{2d}} \left|(V_g g_0)(t+x,\omega) - (V_g g_0)(t,\omega)\right| dt d\omega = \iint_{R^{2d}} \left| \int_0^1 \frac{\partial}{\partial s} (V_g g_0)(t+sx,\omega) ds \right| dt d\omega$$

$$= \iint_{R^{2d}} \left| \int_0^1 \sum_{|\alpha|=1} x^\alpha \big(V_{D^\alpha g} g_0\big)(t+sx,\omega) ds \right| dt d\omega$$

$$\leq \sum_{|\alpha|=1} |x^\alpha| \int_0^1 \iint_{R^{2d}} \left|(V_{D^\alpha g} g_0)(t+sx,\omega)\right| dt d\omega ds = \sum_{|\alpha|=1} |x^\alpha| \cdot \|D^\alpha g\|_{S_0}.$$

Hence

$$\Delta_{\varepsilon,1}\big(V_g g_0\big) = \sup_{\|x\|_\infty \leq \varepsilon} \iint_{R^{2d}} \left|(V_g g_0)(t+x,\omega) - \big(V_g g_0\big)(t,\omega)\right| dt d\omega$$

$$\leq \varepsilon \sum_{|\alpha|=1} \|D^\alpha g\|_{S_0}. \tag{18}$$

As in the first part, we assume that $\{\gamma_n : n \in Z\}$ is a union of r δ-uniformely discrete sequences $\{\gamma_n : n \in A_\ell\}, 1 \leq \ell \leq r$ and $\|\gamma_n - \gamma_n'\|_\infty \leq \varepsilon < \delta/3$. The sequence $\{(a_n - t + s(a_n' - a_n), b_n - \omega) : n \in Z\}$ is a union of r $\delta/3$ uniformly discrete sequences for any $(t,\omega) \in R^{2d}$ and $0 < s < 1$. It follows that

$$\sum_{n\in Z} \left|(V_g g_0)(a_n - t, b_n - \omega) - (V_g g_0)(a_n' - t, b_n - \omega)\right|$$

$$= \sum_{n\in Z} \left| \int_0^1 \frac{\partial}{\partial s} \big(V_g g_0\big)(a_n - t + s(a_n' - a_n), b_n - \omega) ds \right|$$

$$\leq \varepsilon \sum_{n\in Z} \sum_{|\alpha|=1} \int_0^1 \left|\big(V_{D^\alpha g} g_0\big)(a_n - t + s(a_n' - a_n), b_n - \omega)\right| ds$$

$$= \varepsilon \sum_{|\alpha|=1} \int_0^1 \sum_{\ell=1}^{r} \sum_{n\in A_\ell} \left|\big(V_{D^\alpha g} g_0\big)(a_n - t + s(a_n' - a_n), b_n - \omega)\right| ds$$

$$\leq \varepsilon \sum_{|\alpha|=1} \int_0^1 r \left\|V_{D^\alpha g} g_0\right\|_{W,\delta/3} ds$$

$$\leq \varepsilon r 2^{5d/2} \left(2 + \frac{3}{\delta}\right)^{2d} \left\|V_{g_0} g_0\right\|_{W,1} \sum_{|\alpha|=1} \|D^\alpha g\|_{S_0} \tag{19}$$

where we used Proposition (2.3.12) in the last step. Putting (16), (18) and (19) together, we get

$$\sum_{n\in Z} \left|(V_g f)(a_n, b_n) e^{2\pi i \langle a_n', b_n\rangle} - (V_g f)(a_n', b_n) e^{2\pi i \langle a_n', b_n\rangle}\right|^2$$

$$\leq \varepsilon^2 r 2^{3d} \left(2 + \frac{3}{\delta}\right)^{2d} \left\|V_{g_0} g_0\right\|_{W,1} \left(\sum_{|\alpha|=1} \|D^\alpha g\|_{S_0}\right)^2 \|f\|_2^2.$$

Similarly we can show that

$$\sum_{n\in Z}\left|(V_g f)(a_n', b_n)e^{2\pi i\langle a_n', b_n\rangle} - (V_g f)(a_n', b_n')e^{2\pi i\langle a_n', b_n'\rangle}\right|^2$$
$$\le \varepsilon^2 r 2^{3d}\left(2+\frac{3}{\delta}\right)^{2d}\left\|V_{g_0}g_0\right\|_{W,1}\left(2\pi\sum_{|\alpha|=1}\|X^\alpha g\|_{s_0}\right)^2\|f\|_2^2.$$

Hence

$$\sum_{n\in Z}\left|(V_g f)(a_n, b_n)e^{2\pi i< a_n', b_n>} - (V_g f)(a_n{}', b_n{}')e^{2\pi i< a_n', b_n'>}\right|^2$$
$$\le \varepsilon^2 r 2^{3d}\left(2+\frac{3}{\delta}\right)^{2d}\left\|V_{g_0}g_0\right\|_{W,1}\left(\sum_{|\alpha|=1}\|D^\alpha g\|_{s_0} + 2\pi\sum_{|\alpha|=1}\|X^\alpha g\|_{s_0}\right)^2\|f\|_2{}^2.$$

Now the conclusion follows from Proposition (2.3.4).

Remark (2.3.17)[99]: The above theorem says that Gabor frames generated by functions in S_0 with arbitrary sampling points are stable with respect to small perturbations of the sampling points.

The following theorem shows that (regular and) irregular Gabor frames are stable with respect to small changes of the window function in the S_0 norm sense.

Theorem (2.3.18)[99]: Suppose that $\{\pi(\gamma)g:\gamma\in\Gamma\}$ is a frame for $L^2(R^d)$ for some sequence $\Gamma\subset R^{2d}$. Then there is some $\varepsilon>0$ such that $\{\pi(\gamma)h:\gamma\in\Gamma\}$ is a frame for $L^2(R^d)$ provided $\|g-h\|_{S_0}<\varepsilon$.

If $\Gamma=\{(a_n, b_{n,m}):n,m\in Z\}$ and $\{a_n:n\in Z\}$ is relatively discrete, then the condition $\|g-h\|_{S_0}<\varepsilon$ can be replaced by the weaker condition$\|g-h\|_{W,1}<\varepsilon$.

Proof. By Proposition (2.3.3) there is some $\delta>0$ such that we can split Γ into r δ-uniformly discrete subsequences $\Gamma_{\ell,}$ $1\le\ell\le r$. By lemma (2.3.13) $\{\pi(\gamma)(g-h):\gamma\in\Gamma\}$ is a Bessel sequence with upper bound

$$B_\delta := 2^d r\left(1+\tfrac{1}{\delta}\right)^{2d}\left\|V_{g_0}g_0\right\|_{W,1}\cdot\|g-h\|_{S_0}^2.$$

By Proposition (2.3.4) $\{\pi(\gamma)h:\gamma\in\Gamma\}$ is a frame for $L^2(R^2)$ whenever $\|g-h\|_{S_0}$ is small enough. This showed the first part.

The second part follows by Lemma (2.3.7) and Proposition (2.3.4).

Remark (2.3.19)[99]: It is also possible perturb the window function and sampling points simultaneously. By Theorems (2.3.16) and (2.3.18), if $g\in S_0(R^d)$ and $\{\pi(\gamma_n)g:n\in \mathrm{Z}\}$ is a frame for $L^2(R^d)$, then we can find some $\varepsilon>0$ such that $\{\pi(\gamma_n{}')h:n\in \mathrm{Z}\}$ is also a frame for $L^2(R^d)$ provided $\|g-h\|_{S_0}<\varepsilon$ and $\|\gamma_n'-\gamma_n\|_\infty<\varepsilon$.

We study the stability of dual frames. Suppose that $\{g_n:n\in Z\}$ is a frame for some Hilbert space $\mathcal{H}$. Define the frame operator S as

$$Sf=\sum_{n\in Z}\langle f, g_n\rangle g_n \qquad \forall f\in\mathcal{H}.$$

The (canonical) dual frame $\{\widetilde{g_n}:n\in Z\}$, is defined by $\widetilde{g_n}:S^{-1}g_n$.

The following theorem shows that if two frames are close, so are their duals.

Theorem (2.3.20)[99]: Let $\{g_n : n \in Z\}$ and $\{\tilde{g}_n : n \in Z\}$, $\{h_n : n \in Z\}$ and $\{\tilde{h}_n : n \in Z\}$ be two pairs of dual frames for some Hilbert space $\mathcal{H}$. Denote the frame bounds of $\{g_n : n \in Z\}$ and $\{h_n : n \in Z\}$ by (A_1, B_1) and (A_2, B_2), respectively.

(i) If $\{g_n - h_n : n \in Z\}$ is a Bessel sequence with upper bound δ, then $\{\tilde{g}_n - \widetilde{h}_n : n \in Z\}$ is a Bessel sequence with upper bound $\delta\left(\frac{A_1+B_1+B_1^{1/2}B_2^{1/2}}{A_1A_2}\right)^2$.

(ii) If

$$\left|\sum_{n\in Z}|\langle f, g_n\rangle|^2 - \sum_{n\in Z}|\langle f, h_n\rangle|^2\right| \le \delta\|f\|^2, \qquad \forall f \in \mathcal{H},$$

Then

$$\left|\sum_{n\in Z}|\langle f, \widetilde{g_n}\rangle|^2 - \sum_{n\in Z}\left|\langle f, \widetilde{h_n}\rangle\right|^2\right| \le \frac{\delta}{A_1A_2}\|f\|^2, \qquad \forall f \in \mathcal{H}.$$

Proof. First, we show (i). Put

$$Sf = \sum_{n\in Z}\langle f, g_n\rangle g_n \quad \text{and} \quad Tf = \sum_{n\in Z}\langle f, h_n\rangle h_n.$$

Then S and T are self-adjoint, $\widetilde{g_n} = S^{-1}g_n, \widetilde{h_n} = T^{-1}h_n, A_1I \le S \le B_1I$ and $A_2I \le T \le B_2I$. For any $f \in \mathcal{H}$, we have

$$\begin{aligned}\|Sf - Tf\| &= \left\|\sum_{n\in Z}(\langle f, g_n\rangle g_n - \langle f, h_n\rangle h_n)\right\| \\ &\le \left\|\sum_{n\in Z}\langle f, g_n - h_n\rangle g_n\right\| + \left\|\sum_{n\in Z}\langle f, h_n\rangle(g_n - h_n)\right\| \\ &\le B_1^{1/2}\left(\sum_{n\in Z}|\langle f, g_n - h_N\rangle|^2\right)^{1/2} + \delta^{1/2}\left(\sum_{n\in Z}|\langle f, h_n\rangle|^2\right)^{1/2} \\ &\le \delta^{1/2}\left(B_1^{1/2} + B_2^{1/2}\right)\|f\|.\end{aligned}$$

Hence

$$\|S - T\| \le \delta^{1/2}\left(B_1^{1/2} + B_2^{1/2}\right).$$

Therefore,

$$\begin{aligned}\|S^{-1} - T^{-1}\| = \|T^{-1}(T - S)S^{-1}\| &\le \|T^{-1}\| \cdot \|T - S\| \cdot \|S^{-1}\| \\ &\le \frac{1}{A_1A_2}\delta^{1/2}\left(B_1^{1/2} + B_2^{1/2}\right).\end{aligned}$$

Consequently,

$$\begin{aligned}\sum_{n\in Z}|\langle f, (S^{-1} - T^{-1})g_n\rangle|^2 &= \sum_{n\in Z}|\langle (S^{-1} - T^{-1})f, g_n\rangle|^2 \le B_1\|(S^{-1} - T^{-1})f\|^2 \\ &\le \frac{B_1}{A_1^2A_2^2}\delta\left(B_1^{1/2} + B_2^{1/2}\right)\|f\|^2.\end{aligned}$$

On the other hand,

$$\sum_{n\in Z}|\langle f, T^{-1}(g_n - h_n)\rangle|^2 = \sum_{n\in Z}|\langle T^{-1}f, g_n - h_n\rangle|^2 \le \delta\|T^{-1}f\|^2 \le \frac{\delta}{A_2^2}\|f\|^2.$$

Hence,

$$\sum_{n\in Z}\left|\langle f,\widetilde{g_n}-\widetilde{h_n}\rangle\right|^2=\sum_{n\in Z}|\langle f,S^{-1}g_n-T^{-1}h_n\rangle|^2$$
$$=\sum_{n\in Z}|\langle f,(S^{-1}-T^{-1})g_n\rangle+\langle f,T^{-1}(g_n-h_n)\rangle|^2$$
$$\leq\delta\left(\frac{1}{A_2}+\frac{B_1^{1/2}}{A_1A_2}\left(B_1^{1/2}+B_2^{1/2}\right)\right)^2\|f\|^2$$
$$=\delta\left(\frac{A_1+B_1+B_1^{1/2}B_2^{1/2}}{A_1A_2}\right)^2\|f\|^2.$$
Next we prove (ii). Since both S and T are self-adjoint, we have
$$\|S-T\|=\sup_{\|f\|=1}|\langle(S-T)f,f\rangle|=\sup_{\|f\|=1}|\langle Sf,f\rangle-\langle Tf,f\rangle|$$
$$=\sup_{\|f\|=1}\left|\sum_{n\in Z}|\langle f,g_n\rangle|^2-\sum_{n\in Z}|\langle f,h_n\rangle|^2\right|\leq\delta.$$
Therefore,
$$\|S^{-1}-T^{-1}\|\leq\|T^{-1}\|\cdot\|T-S\|\cdot\|S^{-1}\|\leq\frac{1}{A_1A_2}\delta.$$
Since $\widetilde{g_n}=S^{-1}g_n$, we have
$$\sum_{n\in Z}|\langle f,\widetilde{g_n}\rangle|^2=\sum_{n\in Z}|\langle f,S^{-1}g_n\rangle|^2=\sum_{n\in Z}|\langle S^{-1}f,g_n\rangle|^2=\langle SS^{-1}f,S^{-1}f\rangle=\langle f,S^{-1}f\rangle.$$
Similarly,
$$\sum_{n\in Z}\left|\langle f,\widetilde{h_n}\rangle\right|^2=\langle f,T^{-1}f\rangle.$$
It follows that
$$\left|\sum_{n\in Z}|\langle f,\widetilde{g_n}\rangle|^2-\sum_{n\in Z}\left|\langle f,\widetilde{h_n}\rangle\right|^2\right|=|\langle f,(S^{-1}-T^{-1})f\rangle|\leq\|S^{-1}-T^{-1}\|\cdot\|f\|^2$$
$$\leq\frac{\delta}{A_1A_2}\|f\|^2.$$
Example (2.3.21)[99]: Stability of the dual of a Gabor frame $\{\pi(\gamma_n)g : n\in Z\}$. Let the hypotheses be as in Theorem (2.3.16), $\gamma_n=(a_n,b_n),\gamma_n'=(a_n',b_n')$, and A and B be the lower and upper frame bounds for $\{\pi(\gamma_n)g : n\in Z\}$, respectively. Suppose that $\|\gamma_n-\gamma_n'\|_\infty\leq\varepsilon$. We have
$$\left|\left(\sum_{n\in Z}\left|(V_gf)(\gamma_n)\right|^2\right)^{1/2}-\left(\sum_{n\in Z}\left|(V_gf)(\gamma_n')\right|^2\right)^{1/2}\right|$$
$$\leq\left(\sum_{n\in Z}\left|\left(V_gf\right)(a_n,b_n)e^{2\pi i\langle a_n',b_n\rangle}-\left(V_gf\right)(a_n',b_n')e^{2\pi i\langle a_n',b_n'\rangle}\right|^2\right)^{1/2}$$
$$\leq M(\varepsilon)^{1/2}\|f\|_2,$$

Where $M(\varepsilon)$ is defined as in (17) on the other hand, since $\{\pi(\gamma_n')g: n \in Z\}$ has upper frame bound $\left(B^{1/2} + M(\varepsilon)^{1/2}\right)^{1/2}$, we also have

$$\left(\sum_{n\in Z}\left|(V_g f)(\gamma_n)\right|^2\right)^{1/2} + \left(\sum_{n\in Z}\left|(V_g f)(\gamma_n')\right|^2\right)^{1/2} \le \left(2B^{1/2} + M(\varepsilon)^{1/2}\right)\|f\|_2.$$

It follows that

$$\left|\sum_{n\in Z}\left|(V_g f)(\gamma_n)\right|^2 - \sum_{n\in Z}\left|(V_g f)(\gamma_n')\right|^2\right| \le \left(2B^{1/2}M(\varepsilon)^{1/2} + M(\varepsilon)\|f\|_2^2\right).$$

Since $M(\varepsilon)$ tends to zero as ε does, the Gabor frames $\{\pi(\gamma_n)g: n \in Z\}$ and $\{\pi(\gamma_n{}')g: n \in Z\}$ are close. By Theorem (2.3.20), the dual frames are also close in the same sense. Specifically, let $\{\widetilde{g_n}: n \in Z\}$ and $\left\{\widetilde{h_n}: n \in Z\right\}$ be the dual frames of $\{\pi(\gamma_n)g: n \in Z\}$ and $\{\pi(\gamma_n')g: n \in Z\}$, respectively. Then

$$\left|\sum_{n\in Z}|\langle f, \widetilde{g_n}\rangle|^2 - \sum_{n\in Z}\left|\langle f, \widetilde{h_n}\rangle\right|^2\right| \le \frac{2B^{1/2}M(\varepsilon)^{1/2} + M(\varepsilon)}{A(A^{1/2} - M(\varepsilon)^{1/2})^2}\|f\|_2^2.$$

However, the upper bound for the Bessel sequence $\{\pi(\gamma_n)g - \pi(\gamma_n')g : n \in Z\}$ might not tend to zero. Specifically, we have the following .

Proposition (2.3.22)[99]: Let $\{\pi(a_n, b_n)g : n \in Z\}$ be a frame for $L^2(R^d)$.

(i) If $g \in S_0(R^d)$, then

$$\lim_{\|\{a_n - a_n'\}\|_\infty \to 0} \sup_{\|f\|_2 = 1} \sum_{n\in Z} |\langle f, \pi(a_n, b_n)g - \pi(a_n', b_n)g\rangle|^2 = 0.$$

(ii)For any $\varepsilon > 0$, let $y_\varepsilon = (\varepsilon, \dots, \varepsilon) \in R^d$. Then

$$\lim_{\varepsilon\to 0} \sup_{\|f\|_2 = 1} \sup_{n\in Z} |\langle f, \pi(a_n, b_n)g - \pi(a_n, b_n + y_\varepsilon)g\rangle|^2 \ge \|g\|_2^2.$$

Remark (2.3.23)[99]: The second part of the above proposition shows that the upper bound of the Bessel sequence $\{\pi(a_n, b_n)g - \pi(a_n, b_n')g: n \in Z\}$ might not tend to zero even if $\sum_{n\in Z}|b_n - b_n'|^p$ tends to zero for some $p > 0$.

Proof. We see from (16) that (i) holds since $g \in S_0(R^d)$ is equivalent to $V_g g_0 \in L^1(R^{2d})$.

In order to show (ii) put $\Gamma = \{(a_n, b_n) : n \in Z\}$ and $Q_r(x) = \{x': \|x' - x\|_\infty \le r\}$. By proposition (2.3.3), for a sufficiently large r, we have

$$\#\left(\Gamma \bigcap Q_r(x, y)\right) \ge 1, \qquad \forall x, y \in R^d. \tag{20}$$

For any $0 < \varepsilon < 1/(8rd)$, let $x_\varepsilon = (1/(2d\varepsilon), \dots, 1/(2d\varepsilon)) \in R^d$. By (20), there is some $(a_m, b_m) \in \Gamma \cap Q_r(x_\varepsilon, 0)$. Therefore, $\|a_m - x_\varepsilon\|_\infty \le r$, It follows that

$$Q_{1/(4d\varepsilon)}(x_\varepsilon - a_m) \supset Q_{2r}(x_\varepsilon - a_m) \supset Q_r(0).$$

Let

$$f_m(x) = g(x - a_m)e^{2\pi i\langle x, b_m\rangle}\left(1 - e^{2\pi i\langle x, y_\varepsilon\rangle}\right)x_{Q_{1/(4d\varepsilon)}}(x_\varepsilon).$$

Then we have

$$\|f_m\|_2^2 = \iint_{Q_{1/(4d\varepsilon)(x_\varepsilon)}} |g(x - a_m)|^2\left|1 - e^{2\pi i\langle x, y_\varepsilon\rangle}\right|^2 dx$$

$$\leq 4 \iint_{Q_{1/(4d\varepsilon)(x_\varepsilon)}} |g(x-a_m)|^2 dx. \qquad (21)$$

On the other hand, for any $x = (x_1, \dots, x_d) \in Q_{1/(4d\varepsilon)}(x_\varepsilon)$, we have $|x_k - 1/(2d\varepsilon)| \leq 1/(4d\varepsilon)$. It follows that $|x_1 + \cdots + x_d - 1/(2\varepsilon)| \leq 1/(4\varepsilon)$ and thus $|\langle x, y_\varepsilon\rangle - 1/2| \leq 1/4$. Therefore,

$$\left|1 - e^{2\pi i\langle x,y_\varepsilon\rangle}\right|^2 - 4\sin^2(\pi\langle x, y_\varepsilon\rangle) \geq 2, \qquad \forall x \in Q_{1/(4d\varepsilon)}(x_\varepsilon)$$

It follows that

$$\frac{1}{\|f_m\|_2^2} \sup_{n\in Z} |\langle f_m, \pi(a_n, b_n)g - \pi(a_n, b_n + y_\varepsilon)g\rangle|^2$$

$$\geq \frac{1}{\|f_m\|_2^2} |\langle f_m, \pi(a_m, b_m)g - \pi(a_m, b_m + y_\varepsilon)g\rangle|^2$$

$$= \frac{1}{\|f_m\|_2^2} \left| \iint_{Q_{1/(4d\varepsilon)(x_\varepsilon)}} f_m(x)\overline{g(x-a_m)} e^{-2\pi i\langle x,b_m\rangle}(1 - e^{2\pi i\langle x,y_\varepsilon\rangle}) dx \right|^2$$

$$= \frac{1}{\|f_m\|_2^2} \left| \iint_{Q_{1/(4d\varepsilon)(x_\varepsilon)}} |g(x-a_m)|^2 \left|e^{2\pi i\langle x,y_\varepsilon\rangle} - 1\right|^2 dx \right|^2$$

$$\geq \frac{4}{\|f_m\|_2^2} \left| \iint_{Q_{1/(4d\varepsilon)(x_\varepsilon)}} |g(x-a_m)|^2 dx \right|^2 \geq \iint_{Q_{1/(4d\varepsilon)(x_\varepsilon)}} |g(x-a_m)|^2 dx$$

$$= \iint_{Q_{1/(4d\varepsilon)(x_\varepsilon - a_m)}} |g(x)|^2 dx \geq \iint_{Q_{r(0)}} |g(x)|^2 dx.$$

(by (21)). Hence

$$\lim_{\varepsilon\to 0} \sup_{\|f\|_2>0} \frac{1}{\|f_m\|_2^2} \sup_{n\in Z} |\langle f_m, \pi(a_n, b_n)g - \pi(a_n, b_n + y_\varepsilon)g\rangle|^2 \geq \iint_{Q_{r(0)}} |g(x)|^2\, dx.$$

By letting $r \to +\infty$, we get the conclusion.

Chapter 3
Weyl-Heisenberg Frames and Simultaneous Estimates with Density Results

We show that the results shed new light on the classical results concerning frames for $L^2(R)$, showing for instance that the condition $G(x) := \sum_{n\in Z} |g(x - na)|^2 > A > 0$ is not necessary for $\{E_{mb}T_{na}g\}_{m,n\in Z}$ to be a frame for $\text{span}\{E_{mb}T_{na}g\}_{m,n\in Z}$. Inspired by Benedetto and Li, where the relationship between the zero-set of the function G and frame properties of the set of functions $\{g(\cdot - n)\}_{n\in Z}$ is analyzed. We established sampling theorem for the simply connected Heisenberg group, which is translated to a family of frame bound estimates by a direct integral decomposition. We show that if this density condition holds, there exists, in fact, a measurable set $E \subset R$ with the property that the Gabor system associated with the same parameters a, b and the window$=\chi_E$, forms a tight frame for $L^2(S)$.

Section (3.1): Subspaces of $L^2(R)$

For $\mathcal{H}$ denote a separable Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ linear in the first entry. Let I denote a countable index set.

We say that $\{g_i\}_{i\in I} \subseteq \mathcal{H}$ is a frame (for $\mathcal{H}$) if there exist constants $A, B > 0$ such that

$\|f\|^2 \le \sum_{i\in I} |\langle f, g_i\rangle|^2 \le B\|f\|^2, \forall f \in \mathcal{H}.$

In particular a frame for $\mathcal{H}$ is complete, i.e ,$\overline{\text{span}}\{g_i\}_{i\in I} = \mathcal{H}$. In case $\{g_i\}_{i\in I}$ is not complete, $\{g_i\}_{i\in I}$ is a frame for sequence. The numbers A, B that appear in the definition of a frame are called frame bounds.

Orthonormal bases and, more generally, Riesz bases are frames. Recall that $\{g_i\}_{i\in I}$ is a Riesz basis for $\mathcal{H}$ if $\overline{\text{span}}\{g_i\}_{i\in I} = \mathcal{H}$ and

$$\exists A, B > 0 : \quad A\sum_{i\in I}|c_i|^2 \le \left\|\sum_{i\in I} c_i g_i\right\|^2 \le B\sum_{i\in I}|c_i|^2, \quad \forall\{c_i\}_{i\in I} \in \ell^2(I).$$

If $\{g_i\}_{i\in I}$ is a Riesz basis for $\overline{\text{span}}\{g_i\}_{i\in I}$, we say that $\{g_i\}_{i\in I}$ is a Riesz sequence.

The present deals with frames having a special structure: all elements are translated and/or modulated versions of a single function. Let $L^2(R)$ denote the Hilbert space of functions on the real line which are square integrable with respect to the Lebesgue measure. First, define the following operators on functions $f \in L^2(R)$:

Translation by $a \in R$: $(T_a f)(x) = f(x - a), \ x \in R.$

Modulation by $b \in R$: $(E_b g)(x) = e^{2\pi i b x} f(x), \ x \in R.$

A frame for $L^2(R)$ of the form $\{E_{mb}T_{na}g\}_{m,n\in Z}$ is called a Weyl-Heisenberg frame (or Gabor frame). For a collection of different concerning those frames see [136].

Sufficient conditions for $\{E_{mb}T_{na}g\}_{m,n\in Z}$ to be a frame $L^2(R)$ has been known for about 10 years. The basic insight was provided by Daubechies [134]. A slight improvement was proved in [137].

Theorem (3.1.1)[131]: Let $g \in L^2(R)$ and suppose that

$$\exists A, B > 0: \quad A \le \sum_{n\in Z} |g(x - na)|^2 \le B \quad \text{for } a.e\ x \in R, \tag{1}$$

$$\lim_{b\to 0} \sum_{k\ne 0} \left\| \sum_{n\in Z} T_{na} g T_{na+\frac{k}{b}} \bar{g} \right\|_\infty = 0. \tag{2}$$

Then there exists $b_0 > 0$ such that $\{E_{mb}T_{na}g\}_{m,n\in Z}$ is a Weyl-Heisenberg frame for $L^2(R)$ for all $\in]0; b_0[$.

The proof of Theorem (3.1.1) is based on the following identity, valid for all continuous functions f with compact support whenever g satisfies (1):

$$\sum_{m,n\in Z} |\langle f, E_{mb}T_{na}g\rangle|^2 = \frac{1}{b}\int |f(x)|^2 G(x)dx$$
$$+\frac{1}{b}\sum_{k\neq 0}\int \overline{f(x)}f(x-k/b)\sum_{n\in Z} g(x-na)\overline{g(x-na-k/b)}dx. \quad (3)$$

An estimate of the second term in (3) now shows that $\{E_{mb}T_{na}g\}_{m,n\in Z}$ is actually a frame for all values of b for which

$$\sum_{K\neq 0}\left\|\sum_{n\in Z} T_{na}gT_{na+\frac{k}{b}}\bar{g}\right\|_\infty < A. \quad (4)$$

A more recent result can be found in [135]: in Theorem 2.3 it is proved that if (1) is satisfied and there exists a constant $D < A$ such that

$$\sum_{k\neq 0}\sum_{n\in Z}\left|g(x-na)g\left(x-na-\frac{k}{b}\right)\right| \le D \text{ for } a.e.\, x\in R, \quad (5)$$

Then $\{E_{mb}T_{na}g\}_{m,n\in Z}$ is a frame for $L^2(R)$ with bounds $\frac{A-D}{b}, \frac{B+D}{b}$. The reader should observe that [135] does not provide us with a generalization of the results in [134], [137] in a strict sense: there are cases where (5) is satisfied but (4) is not, and vice versa. The main point is that other conditions for $\{E_{mb}T_{na}g\}_{m,n\in Z}$ to be a frame can be derived from (5); cf. Theorem 2.4 in [135].

Define the Fourier transform $\mathcal{F}(f) = \hat{f}$ of $f \in L^1(R)$ by

$$\hat{f}(y) = \int f(x)e^{-2\pi iyx}dx$$

As usual we extend the Fourier transform to an isometry from $L^2(R)$ onto $L^2(R)$. We denote the inverse Fourier transformation of $g \in L^2(R)$ by $\mathcal{F}^{-1}g$ or $\hat{g}$. It is important to observe the following comutator relations, valid for all $a \in R$:

$$\mathcal{F}T_a = E_{-a}\mathcal{F}, \qquad \mathcal{F}E_a = T_a\mathcal{F}.$$

We need a result from [133]. The basic insight was provided by Benedetto and Li [132], who treated the case $a = 1$.

Theorem (3.1.2)[131]: Let $g \in L^2(R)$, then $\{T_{na}g\}_{n\in Z}$ is a frame sequence with bounds A, B if and only if

$$0 < aA \le \sum_{n\in Z}\left|\hat{g}\left(\frac{x+n}{a}\right)\right|^2 \le aB \;\; for\; a.e\; x\; for\; which \sum_{n\in Z}\left|\hat{g}\left(\frac{x+n}{a}\right)\right|^2 \neq 0.$$

In that case $\{T_{na}g\}_{n\in Z}$ is a Riesz sequence if and only if the set of x for which $\sum_{n\in Z}\left|\hat{g}(\frac{x+n}{a})\right|^2 = 0$ has measure zero.

Theorem (3.1.2) leads immediately to an equivalent condition to (1). Define the function G and its kernel N_G by

$$G: R \to [0, \infty],\; G(x) := \sum_{n\in Z}|g(x-na)|^2,$$
$$N_G = \{x \in R | G(x) = 0\}.$$

Corollary (3.1.3)[131]: $\left\{E_{\frac{n}{a}} g\right\}_{n\in Z}$ is a frame sequence with bounds A, B if and only if

$$0<\frac{A}{a}\leq\sum_{n\in Z}|g(x-na)|^2\leq\frac{B}{a}\ \ for\ a.e\ x\in R-N_G.$$

In that case $\left\{E_{\frac{n}{a}} g\right\}_{n\in Z}$ is a Riesz sequence iff N_G has measure zero.

Proof. The inequality

$$0<\frac{A}{a}\leq\sum_{n\in Z}|g(x-na)|^2\leq\frac{B}{a}\ \ for\ a.e\ x\in R-N_G.$$

holds if and only if

$$0<\frac{A}{a}\leq\sum_{n\in Z}|g([x-n]a)|^2\leq\frac{B}{a}\ \ for\ a.e\ x\in R-N_G. \quad (6)$$

By Theorem (3.1.2), (6) is equivalent to $\{T_{na}\hat{g}\}_{n\in Z}$ being a frame sequence with bounds A, B. Appling the Fourier transformation this is equivalent to $\left\{E_{\frac{n}{a}} g\right\}_{n\in Z}$ being a frame sequence with bounds A, B.

From now on we concentrated on Wey-Heisenberg frames $\{E_{mb}T_{na}g\}_{m,n\in Z}$. The first result gives a sufficient condition for $\{E_{mb}T_{na}g\}_{m,n\in Z}$ to a frame sequence. It can be considered as a "subspace version" of a result by Ron and shen; cf. [138]. The condition for $\{E_{mb}T_{na}g\}_{m,n\in Z}$ to be a frame for $L^2(R)$ is significantly weaker than the conditions mentioned.

Let $L^2(R-N_G)$ denote the set of functions in $L^2(R)$ that vanishes at N_G.

Theorem (3.1.4)[131]: let $g\in L^2(R)$ $a,b>0$ and suppose that

$$A:=\inf_{x\in[0,a]\to N_G}\left[\sum_{n\in Z}|g(x-na)|^2-\sum_{k\neq 0}\left|\sum_{n\in Z}g(x-na)\overline{(g\left(x-na-\frac{k}{b}\right)}\right|\right]>0, \quad (7)$$

$$B:=\sup_{x\in[0,a]}\sum_{k\in Z}\left|\sum_{n\in Z}g(x-na)\overline{(g\left(x-na-\frac{k}{b}\right)}\right|<\infty. \quad (8)$$

Then $\{E_{mb}T_{na}g\}_{m,n\in Z}$ is a frame for $L^2(R-N_G)$ with bounds $\frac{A}{b},\frac{B}{b}$.

Proof. First, observe that $\overline{\text{span}}\{E_{mb}T_{na}g\}_{m,n\in Z}\subseteq L^2(R-N_G)$. Now consider a function $f\in L^2(R-N_G)$ which is bounded and has support in a compact set. The Heil-Walnut argument (3) is valid under the assumption (8) and it gives that

$$\sum_{m,n\in Z}|\langle f,E_{mb}T_{na}g\rangle|^2=\frac{1}{b}\int|f(x)|^2\sum_{n\in Z}|g(x-na)|^2dx$$
$$+\frac{1}{b}\sum_{k\neq 0}\int\overline{f(x)}f(x-k/b)\sum_{n\in Z}g(x-na)\overline{g(x-na-k/b)}dx. \quad (3)$$

We want to estimate the second term above. For $k\in Z$, define

$$H_k(x):\sum_{n\in Z}T_{na}g(x)\,\overline{T_{na+k/b}g(x)}.$$

First, observe that

$$\sum_{k\neq 0}\left|T_{-k/b}H_k(x)\right| = \sum_{k\neq 0}\left|T_{-k/b}\sum_{n\in Z}T_{na}g(x)\overline{T_{na+k/b}g(x)}\right|$$
$$= \sum_{k\neq 0}\left|\sum_{n\in Z}T_{na-k/b}g(x)\overline{T_{na}g(x)}\right| = \sum_{k\neq 0}\left|\sum_{n\in Z}T_{na+k/b}g(x)\overline{T_{na}g(x)}\right|$$
$$= \sum_{k\neq 0}\left|\sum_{n\in Z}\overline{T_{na+k/b}g(x)}\,T_{na}g(x)\right| = \sum_{k\neq 0}|H_x(x)|.$$

Now, by a slight modification of the argument in [135], Theorem 2.3,

$$\left|\sum_{k\neq 0}\overline{f(x)}\,f(x-k/b)\sum_{n\in Z}g(x-na)\overline{g(x-na-k/b)}dx\right|$$
$$\leq \sum_{k\neq 0}\int|f(x)|\cdot\left|T_{k/b}f(x)\right|\cdot|H_k(x)|dx$$
$$= \sum_{k\neq 0}\int|f(x)|\sqrt{|H_k(x)|}\cdot\left|T_{K/B}f(x)\right|\sqrt{|H_k(x)|}\,dx$$
$$\leq \sum_{k\neq 0}\left(\int|f(x)|^2|H_k(x)|dx\right)^{1/2}\left(\int\left|T_{k/b}f(x)\right|^2|H_k(x)|dx\right)^{1/2}$$
$$\leq \left(\sum_{k\neq 0}|f(x)|^2|H_k(x)|dx\right)^{1/2}\cdot\left(\int\left|T_{k/b}f(x)\right|^2|H_k(x)|dx\right)^{1/2}$$
$$= \left(\int|f(x)|^2\sum_{k\neq 0}|H_k(x)|dx\right)^{1/2}\cdot\left(\int|f(x)|^2\sum_{k\neq 0}\left|T_{-k/b}H_k(x)\right|^2dx\right)^{1/2}$$
$$= \int|f(x)|^2\sum_{k\neq 0}|H_k(x)|dx.$$

Note that $\sum_{k\neq 0}|H_k(x)| = \sum_{k\neq 0}\left|\sum_{n\in Z}T_{na}g(x)\,\overline{T_{na+k/b}g(x)}\right|$ is a periodic function with periodic a. By (3) and the assumption (7) we now have

$$\sum_{m,n\in Z}|\langle f,E_{mb}T_{na}g\rangle|^2$$
$$\geq \frac{1}{b}\int|f(x)|^2\left[\sum_{n\in Z}|g(x-na)|^2\right.$$
$$\left.-\sum_{k\neq 0}\left|\sum_{n\in Z}g(x-na)\,\overline{g\left(x-na-\frac{k}{b}\right)}\right|\right]dx \geq \frac{A}{b}\,\|f\|^2.$$

Similarly , by (3) and (8),

$$\sum_{m,n\in Z} |\langle f, E_{mb}T_{na}g\rangle|^2$$
$$\leq \frac{1}{b}\int |f(x)|^2 \left[\sum_{n\in Z}|g(x-na)|^2 + \sum_{k\neq 0}\left|\sum_{n\in Z} g(x-na)\,\overline{g\left(x-na-\frac{k}{b}\right)}\right|\right] dx$$
$$= \frac{1}{b}\int |f(x)|^2 \sum_{k\in Z}\left|\sum_{n\in Z} g(x-na)\,\overline{g\left(x-na-\frac{k}{b}\right)}\right| \leq \frac{B}{b}\|f\|^2.$$

Since those two estimates holds on a dense subset of $L^2(R-N_G)$, they hold on $L^2(R-N_G)$. Thus $\{E_{mb}T_{na}g\}_{m,n\in Z}$ is a frame for $L^2(R-N_G)$ with desired bounds.

The advantage of Theorem (3.1.4) compared to the results is that we compare the functions $\sum_{n\in Z}|g(x-na)|^2$ and $\sum_{k\neq 0}|H_k(x)|$ point wise rather than assuming that the supremum of $\sum_{k\neq 0}|H_k(x)|$ is smaller than the infimum of $\sum_{n\in Z}|g(x-na)|^2$. It is easy to give concrete examples where Theorem (3.1.4) shows That $\{E_{mb}T_{na}g\}_{m,n\in Z}$ is a frame for $L^2(R)$ but where the conditions are not satisfied:

Example (3.1.5)[131]: Let $a=b=1$ and define

$$g(x) = \begin{cases} 1+x & \text{if } x\in[0,1[, \\ \frac{1}{2}x & \text{if } x\in[1,2[, \\ 0 & \text{otherwise.} \end{cases}$$

For $x\in[0,1[$ we have

$$G(x) = \sum_{n\in Z}|g(x-n)|^2 = g(x)^2 + g(x+1)^2 = \frac{5}{4}(x+1)^2$$

and

$$\sum_{k\neq 0}\left|\sum_{n\in Z} g(x-a)\overline{g(x-n-k)}\right| = (1+x)^2,$$

so by Theorem (3.1.4) $\{E_{mb}T_{na}g\}_{m,n\in Z}$ is a frame for $L^2(R)$ with bounds $A=\frac{1}{4}, B=9$

But $\inf_{x\in R} G(x) = \frac{5}{4}$ and

$$\sum_{k\neq 0}\left\|\sum_{n\in Z} T_{na}gT_{n+k}\overline{g}\right\|_\infty = 4,$$

so the condition (4) is not satisfied. (5) is not satisfied either.

Remark (3.1.6)[131]: It is well known that G being bounded below is a necessary condition for $\{E_{mb}T_{na}g\}_{m,n\in Z}$ to be a frame for $L^2(R)$; cf. [134]. Theorem (3.1.4) shows that this condition is not necessary for $\{E_{mb}T_{na}g\}_{m,n\in Z}$ to be a frame sequence. However, it is implicit in (7) that G has to be bounded below on $R-N_G$ in order for Theorem (3.1.4) to work, and any easy modification of the proof in [134] shows that this is actually a necessary condition for $\{E_{mb}T_{na}g\}_{m,n\in Z}$ to be a frame for $L^2(R-N_G)$. We shall later give examples of frame sequences for which G is not bounded below on $R-N_G$.

In case g has support in an interval of length $\frac{1}{b}$ an equivalent condition for $\{E_{mb}T_{na}g\}_{m,n\in Z}$ to be a frame sequence can be given. First, observe that by (3) this condition on g implies that for all continuous functions f with compact support, we have

$\sum_{m,n\in Z}\left|\langle f, E_{mb}T_{nag}\rangle\right|^2 = \frac{1}{b}\int|f(x)|^2\, G(x)dx.$

It is not hard to show that this actually holds for all $\in L^2(R)$; cf. [137].

Corollary (3.1.7)[131]: Suppose that $g \in L^2(R)$ has compact support in an interval I of length $|I| \le 1/b$ then $\{E_{mb}T_{na}g\}_{m,n\in Z}$ is a frame sequence with bounds A, B if and only if

$$0 < bA \le \sum_{n\in Z}|g(x-na)|^2 \le bB, \ \ for\ a.e\ x \in R - N_G.$$

In that case $\{E_{mb}T_{na}g\}_{m,n\in Z}$ is actually a frame for $L^2(R - N_G)$.

Proof. Suppose that g has support in an interval I of length $|I| \le \frac{1}{b}$. if $0 < bA \le G(x) \le bB$ for a.e. $x \in R - N_G$, it follows from Theorem (3.1.4) that $\{E_{mb}T_{na}g\}_{m,n\in Z}$ is a frame sequence with the desired bounds. Now suppose that $\{E_{mb}T_{na}g\}_{m,n\in Z}$ is a frame sequence with bounds A, B. then, for every interval $I$of length $|I| \le \frac{1}{b}$ and every function $f \in L^2(I)$,

$$\sum_{m,n}\left|\langle f, \{E_{mb}T_{na}g\}_{m,n\in Z}\rangle\right|^2 = \frac{1}{b}\int_R |f(x)|^2 G(x)dx \le B\|f\|^2.$$

But this is clearly equivalent to

$$G(x) = \sum_{n\in Z}|g(x-na)|^2 \le Bb\ a.e.$$

To prove the lower bound for G we proceed by way of contradiction. Suppose that for some $\epsilon > 0$ we have $0 \le G(x) \le (1-\epsilon)Ab$ on a set of positive measure. in this case there is a set Δ of positive measure and supported in an interval of length $\le \frac{1}{b}$ so that $0 < G(x) \le (1-\epsilon)Ab$ on Δ. Then, for any function $f \in L^2(R)$ supported on Δ, we have

$$\sum_{m,n}|\langle f, E_{mb}T_{na}g\rangle|^2 = \frac{1}{b}\int_R |f(x)|^2 G(x)dx$$

$$\le \frac{(1-\epsilon)Ab}{b}\int_R |f(x)|^2 dx = (1-\epsilon)A\|f\|^2.$$

Since $G(x) > 0$ on Δ, there is a $k \in Z$ so that $x_\Delta T_{ka}g$ is not the zero function. With $\Delta' := \Delta \cap supp(T_{ka}g\ )$ we have

$$f := \chi_{\Delta'}T_{ka}g \in \overline{\text{span}}\{E_{mb}T_{ka}g\ \}_{m\in Z} \subseteq \overline{\text{span}}\{E_{mb}T_{na}g\}_{m,n\in Z},$$

so the above calculation shows that the lower bound for $\{E_{mb}T_{na}g\}_{m,n\in Z}$ is at most $(1-\epsilon)$A, which is a contradiction is a frame. Thus

$$G(x) \ge bA \text{ for } a.e. x \in R - N_G.$$

In case the condition in Corollary (3.1.6) is satisfied, it follows from Theorem (3.1.4) that $\{E_{mb}T_{na}g\}_{m,n\in Z}$ is a frame for $L^2(R - N_G$.

For functions g with the property that the translates $T_{ka}g, n \in Z$, have disjoint support we can give an equivalent condition for $\{E_{mb}T_{na}g\}_{m,n\in Z}$ to be a frame sequence. Define the function

$$\hat{G}(x): R \to [0, \infty], \quad \hat{G}(x) = \sum_{m\in Z} \left|g(x + \frac{m}{b})\right|^2.$$

Proposition (3.1.8)[131]: Let $\in L^2(R)$, $a, b > 0$ and suppose that

$$\operatorname{supp}(g) \cap \operatorname{supp}(T_{na}g) = \emptyset, \quad \forall n \in Z - \{0\}. \tag{9}$$

Then $\{E_{mb}T_{na}g\}_{m,n\in Z}$ is a frame sequence with bounds A, B if and only if there exist $A, B > 0$ such that

$$bA \le \sum_{m\in Z} \left|g\left(x + \frac{m}{b}\right)\right|^2 \le bB \;\; for\; a.e.\; x \in R - N_{\tilde{G}}.$$

In that case $\{E_{mb}T_{na}g\}_{m,n\in Z}$ is a Riesz sequence iff $N_{\tilde{G}}$ has measure zero.

Proof. Because of the support condition (9), it is clear that $\{E_{mb}g\}_{m\in Z}$ is a frame sequence iff $\{E_{mb}T_{na}g\}_{m,n\in Z}$ is a frame sequence, in which case the sequences have the same frame bounds. But by Corollary (3.1.3) $\{E_{mb}g\}_{m\in Z}$ is a frame sequence with bounds A, B iff

$$bA \le \sum_{m\in Z} \left|g\left(x + \frac{m}{b}\right)\right|^2 \le bB \;\; for\; a.e. x \in R - N_{\tilde{G}}.$$

Also, $\{E_{mb}T_{na}g\}_{m,n\in Z}$ is a Riesz sequence iff $\{E_{mb}g\}_{m\in Z}$ is a Riesz sequence, which, by Corollary (3.1.3), is the case iff $N_{\tilde{G}}$ has measure zero.

We are now ready to show that G being bounded below on $R - N_G$ (by a positive number) is not a necessary condition for $\{E_{mb}T_{ka}g\}_{m,n\in Z}$ to be a frame sequence.

Example (3.1.9)[131]: Let $a, b > 0$ and suppose that $\frac{1}{ab} \notin N$. Chose $\epsilon > 0$ such that

$$[0\,, \epsilon] + na \cap \left[\frac{1}{b}\,, \frac{1}{b} + \epsilon\right] = \emptyset, \quad \forall n \in Z.$$

This implies that $\epsilon < \min\left(a\,, \frac{1}{b}\right)$ define

$$g(x) := \begin{cases} x & \text{if } x \in [0, \epsilon], \\ \sqrt{1 - \left(x - \frac{1}{b}\right)^2} & \text{if } x \in \left[\frac{1}{b}\,, \frac{1}{b} + \epsilon\right], \\ 0 & \text{otherwise.} \end{cases}$$

Then the condition (9) in Proposition (3.1.7) is satisfied. Also, for $x \in [0, \epsilon]$

$$\tilde{G}(x) = \sum_{m\in Z} \left|g(x + \frac{m}{b})\right|^2 = g(x)^2 + g(x+1)^2 = 1$$

And for $x \in \left]\epsilon\,, \frac{1}{b}\right]$, we have $\tilde{G}(x) = 0$. Thus, by proposition (3.1.7) $\{E_{mb}T_{ka}g\}_{m,n\in Z}$ is a frame sequence. But for $x \in [0, \epsilon]$,

$$G(x) = \sum_{n\in Z} |g(x - na)|^2 = x^2.$$

Thus G is not bounded below by a positive number on $R - N_G$. By the remark after Theorem (3.1.4) this implies that $\overline{\operatorname{span}}\{\{E_{mb}T_{ka}g\}_{m,n\in Z}\} \neq L^2(R - N_G)$.

For $ab > 1$ it is even possible to construct an orthonormal sequence having all the features of the above example. For example, let $a = 2$, $b = 1$ and

$$g(x) := \begin{cases} x & \text{if } x \in [0,1], \\ \sqrt{2x - x^2} & \text{if } x \in]1,2], \\ 0 & \text{otherwise.} \end{cases}$$

Since

$$\sum_{m \in Z \in} \left| g(x + \frac{m}{b}) \right|^2 = 1, \quad \forall x,$$

it follows by Proposition (3.1.7) that $\{E_{mb}T_{ka}g\}_{m,n \in Z}$ is a Riesz sequence with bounds $A = B = 1$, which implies that $\{E_{mb}T_{ka}g\}_{m,n \in Z}$ is an orthonormal sequence. But $G(x) = \sum_{x \in Z} |g(x - na)|^2$ is not bounded below on $R - N_G$.

G being bounded above is still a necessary condition for $\{E_{mb}T_{ka}g\}_{m,n \in Z}$ to be a frame sequence (repeat the argument in Corollary (3.1.6)) $\tilde{G}$ also has to be bounded above:

Proposition (3.1.10)[131]: If $\{E_{mb}T_{ka}g\}_{m,n \in Z}$ is a frame sequence with upper bound B, then

$$\sum_{m \in Z} \left| G(x + \frac{m}{b}) \right|^2 \leq B \quad a.e.$$

Proof. If $\{E_{mb}T_{ka}g\}_{m,n \in Z}$ is a frame sequence, then $\{\mathcal{F}^{-1}E_{mb}T_{na}g\}_{m,n \in Z} = \{T_{mb}E_{na}\check{g}\}_{m,n \in Z}$ is a frame sequence with the same bounds. In particular the sequence $\{T_{mb}\check{g}\}_{m,n \in Z}$ has the upper frame bound B. By Theorem (3.1.2) (or, more precisely, the proof of in [133]) it follows that

$$\sum_{m \in Z} \left| G(x + \frac{m}{b}) \right|^2 \leq B \text{ for } a.e.x.$$

It follows that $\sum_{m \in Z} \left| g(x + \frac{m}{b} \right|^2 \leq B \;\; a.e.$

As well as Weyl-Heisenberg frames, wavelet frames play a very important role in applications. The theory for the two types of frames was developed at the same time, with the main contribution due to Daubechies. Several results for Weyl-Heisenberg frames have counterparts for wavelet frames. For example, Theorem 5.1.6 in [137] gives sufficient conditions for $\left\{ \frac{1}{a^{n/2}} g(\frac{x}{a^n} - mb) \right\}_{m,n \in Z}$ to be a frame based on a calculation similar to (3).

Also our results for Weyl-Heisenberg frames have counterparts for wavelet frames. The ideas in the proof of Theorem (3.1.4) can be used to modify [137], Theorem 5.1.6 which leads to the following:

Theorem (3.1.11)[131]: let $a > 1$, $b > 0$ and $g \in L^2(R)$ be given. let

$$N := \left\{ \gamma \in [1, a] \middle| \sum_{n \in Z} |\hat{g}(a^n \gamma)|^2 = 0 \right\}$$

and suppose that

$$A := \inf_{|\gamma| \in [1,a] \to N} \left[\sum_{n \in Z} |\hat{g}(a^n \gamma)|^2 - \sum_{K \neq 0} \sum_{n \in Z} |\hat{g}(a^n \gamma) \hat{g}(a^n \gamma + k/b)| \right] > 0,$$

$$B := \sup_{|\gamma|\in[0,a]} \sum_{k,n\in Z} |\hat{g}(a^n\gamma)\hat{g}(a^n\gamma + k/b)| < \infty.$$

Then $\left\{\frac{1}{a^{n/2}}\ g\left(\frac{x}{a^n} - mb\right)\right\}_{m,n\in Z}$ is a frame sequence with bounds $\frac{A}{b}, \frac{B}{b}$.

Section (3.2): Vector-Valued Gabor Frames of Hermite Functions

We derive frame estimates for vector-valued Gabor systems. We consider the space $L^2(\mathbb{R};\mathbb{C}^d)$ of vector–valued signals. Elements of this space can also be understood as vectors $f = (f_1, \dots, f_d)$, with $f_i \in L^2(\mathbb{R};\mathbb{C}^d)$, which amounts to identifying $L^2(\mathbb{R};\mathbb{C}^d)$ with the d-fold direct sum of $L^2(\mathbb{R})$. Gabor systems in this space are obtained by picking a window function $f \in L^2(\mathbb{R};\mathbb{C}^d)$, and applying translations and modulations. Here the translation and modulation operators are given as

$$\left(T_y f\right)(x) = f(x - y), \quad \left(M_\xi f\right)(x) = e^{2\pi i\xi x} f(x) \ \ (y, \xi \in R)$$

For any $\gamma = (\gamma_1, \gamma_2) \in \mathbb{R}^2$, we denote the associated time-frequency shift of a function $f \in L^2(\mathbb{R};\mathbb{C}^d)$ by

$$f_\gamma = T_{\gamma_1} M_{\gamma_2} f.$$

Now, given a lattice $\Gamma \subset \mathbb{R}^2$ and $f \in L^2(\mathbb{R};\mathbb{C}^d)$, the resulting Gabor system $G(f, \Gamma)$ is given by

$$G(f, \Gamma) = (f_\gamma)_{\gamma\in\Gamma}.$$

Recall that a family $(\eta_i)_{i\in I}$ of vectors in a Hilbert space $\mathcal{H}$ is called a frame if it satisfies

$$A\|\varphi\|^2 \leq \sum_{i\in I} |\langle\varphi, \eta_i\rangle|^2 \leq B\|\varphi\|^2, \tag{10}$$

for all $\varphi \in \mathcal{H}$, with constants $0 < A \leq B$. These constants are called frame bounds; in the following frame constants are generally not assumed to be optimal. A Gabor system $G(f, \Gamma)$ that is also a frame is called a Gabor frame.

For case $d = 1$, Gabor frames have been studied extensively; see e.g. [154]. For a treatment of the case $d > 1$, in somewhat different terminology, see [142].

The problem of constructing a Gabor frame in dimension d contains that of simultaneously constructing d Gabor frames: if $G(f, \Gamma)$ is a frame with bounds A, B, then $G(f_i, \Gamma)$ is a frame with frame bounds A, B for each component f_i of f. More generally, whenever $\tilde{d} < d$ and $\tilde{f} = (f_1, \dots, f_{\tilde{d}})$, the system $G(\tilde{f}, \Gamma)$ is a frame of $L^2(\mathbb{R};\mathbb{C}^d)$, again with frame bounds A and B.

But the converse need not be true: the definition of the scalar product in $L^2(\mathbb{R};\mathbb{C}^d)$ entails that

$$\langle g, f_\lambda\rangle = \sum_{i=1}^{d} \langle g_i, f_i|_\lambda\rangle,$$

where we used $f_i|_\lambda$ to denote the action of a time-frequency shift on f_i. Hence, cancellation may prevent the higher-dimensional system from being a frame even when all components $f_1, \dots, f_d$ generate a frame. In fact, one can easily see that an obvious necessary requirement for f to generate a frame is linear independence of its entries $f_1, \dots, f_d$.

Observe, however, that at least the upper frame bound for f can be estimated in terms of the upper frame bounds for the f_i: if $B, B_1, \ldots, B_d$ are optimal upper frame bounds for $f, f_1, \ldots, f_d$, respectively, then the Cauchy-Schwartz inequality entails

$$B \leq d \sum_{i=1}^{d} B_i. \tag{11}$$

Probably the most-studied window function for the case $d = 1$ has been the Gaussian, $g(x) = \pi^{-1/4} e^{-x^2/2}$. This is partly due to historical reasons: Gabor suggested using Gaussian window [151], and the characterization of densities for Gabor frames with Gaussian window took more than 30 years to be fully clarified [156,158]. The choice of this window function is motivated by the way Gabor systems are employed : they are designed to measure time-frequency content in a signal. By the Heisenberg uncertainty relation a Gaussian window has optimal time-frequency concentration, and thus can be expected to yield a good time-frequency resolution. Moreover, for the Gaussian window powerful tools from complex analysis can be employed to study sampling [156,158], which adds to its theoretical appeal.

We intend to derive frame estimates for window functions f consisting of the first $d + 1$ Hermits functions. For $h \in \mathbb{N}_0$, we define the n^{th} Hermits function h_n by

$$h_n(x) = \frac{(-1)^n}{\sqrt{2^2 n! \sqrt{\pi}}} e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2}, \tag{12}$$

where the normalization factor ensures $\|h_n\|_2 = 1$, the above defined Gaussian equals h_0, whence the problem considered here can be viewed as a generalization of Gabor's original question.

The vector-valued windows that we are interested in are given by

$$h^d = (h_0, \ldots, h_d) \in L^2(\mathbb{R}; \mathbb{C}^{d+1}).$$

We intend to give frame conditions and estimates for $G\left(h^d, \mathcal{M}(\mathbb{Z}^2)\right)$, where $\mathcal{M}$ denotes a real-valued invertible 2×2-matrix, in terms of a matrix norm defined by

$$\|\mathcal{M}\| = \sup\{\|\mathcal{M}_Z\|_2 : \|z\|_\infty \leq 1/2\}. \tag{13}$$

Here $\|z\|_P$ denotes the usual ℓ^p-norm on $\mathbb{R}^2$. This choice of matrix norm may seem somewhat peculiar, and in fact the theorems below can be formulated with respect to any other norm on matrix space. As will become clear below, the use of (13) emphasizes the close connection to sampling estimates on the Heisenberg group.

The chief purpose of Theorem (3.2.1) is to allow a better understanding and formulation of Theorem (3.2.6). Theorem (3.2.1) is of independent interest. Even though we expect it somehow to be part of Gabor analysis, we are not aware of any previous source for this result; not even for $d = 1$.

Hence the tightness of the frame estimate, which is the quotient of the two frame bounds, approaches 1 as $\|\mathcal{M}\| \to 0$, with speed proportional to $\|\mathcal{M}\|^2 / C_f^2$.

We observe that Theorem (3.2.1) holds for the supremum C_f^* of all possible constants, and that this choice provides the sharpest possible statement. Then the main result is the following estimate:

It uses a sampling estimate for the Paly-Wiener space $PW(\mathbb{H})$ established in Führ and Gröchenig [150]. The space was introduced by Pesenson [157], using a particular differential operator on $\mathbb{H}$, the so-called sub-Laplacian. Hermite functions enter in the spectral decomposition of this operator, and it is this connection that will allow to relate the sampling theorem to frame estimates for Hermite functions.

The connection between frames and sampling theory is not exactly new, in fact it is at the base of frame theory, which originated from nonharmonic Fourier series and their connections to irregular sampling over the reals, see [144]. For the sake of explicitness, assume we are given a sequence $A = (\lambda_k)_{k\in\mathbb{Z}}$ of sampling points in $\mathbb{R}$. We are looking for criteria that allow to reconstruct a Paley-Wiener function, i.e. $f \in L^2(\mathbb{R})$ whose Fourier transform has support in the unit interval $[-0.5\,,0.5]$, in a stable manner from its restriction to Λ. Noting that

$$f(\lambda_k) = \int_{-0.5}^{0.5} \hat{f}(\xi)e^{2\pi i\lambda_k\xi}d\xi = \langle \hat{f}, e_{\lambda_k}\rangle,$$

we find the following two equivalent conditions, with identical constants A and B in both cases:

(a) The sequence Λ fulfills the sampling estimate

$$A\|f\|_2^2 \le \sum_{k\in\mathbb{Z}} |f(\lambda_k)\,|^2 \le B\|f\|_2^2,$$

for all Paley-Wiener functions.

(b) The sequence $(e_{\lambda_k})_{k\in\mathbb{Z}}$ fulfills the frame estimate

$$A\|F\|_2^2 \le \sum_{k\in\mathbb{Z}} \left|\langle F, e_{\lambda_k}\rangle\right|^2 \le B\|F\|_2^2,$$

for all $F \in L^2([-0.5\,,0.5])$.

This equivalence can be used in two ways: For instance, observing that the choice $\lambda_k = k(k \in \mathbb{Z})$ results in the Fourier orthonormal basis $(e_{\lambda_k})_{k\in\mathbb{Z}}$ of $L^2([-0.5, 0.5])$, the implication (b) ⇒ (a) leads to Shannon's sampling theorem. Conversely, for irregular sampling sets, condition (a) can often be checked using tools from complex analysis, and then (a) ⇒ (b) results in frame estimates for irregularly spaced exponentials.

We use a similar approach for the Heisenberg group $\mathbb{H}$: this time, previously established sampling estimates for Paley-Wiener functions on $\mathbb{H}$ will allow to derive frame estimates for Hermite functions, by an analogue of the implication (a) ⇒ (b). for this purpose we will need to work out the connections between $PW(\mathbb{H})$ and the Hermite functions. But first let us show Theorem (3.2.1).

Theorem (3.2.1)[140]: Let $f = (f_1\,, \dots, f_d) \in L^2(\mathbb{R}\,, \mathbb{C}^d)$ be given, with $f_i \in S(\mathbb{R})$, and $\langle f_i, f_j\rangle = \delta_{ij}$ for all $1 \le i, j \le d$. Then there exists a constants $0 < C_{\boldsymbol{f}} \le 1$ such that for all matrices $\mathcal{M}$ with $\|\mathcal{M}\| < C_{\boldsymbol{f}}$, the system $G\big(\boldsymbol{f}, \mathcal{M}(\mathbb{Z}^2)\big)$ is a frame with frame constants $\frac{1}{|\det(\mathcal{M})|}\left(1 \pm \frac{\|\mathcal{M}\|}{C_f}\right)^2$.

Proof. Let $V_f\colon L^2(\mathbb{R}\,; \mathbb{C}^d) \to L^2(\mathbb{R}^2)$ denote the short-time Fourier transform,

$$V_{fg}(x, \xi) = \big(g, T_x M_\xi f\big).$$

The orthogonally relations for the short-time Fourier transform [154, Theorem 3.2.1] and the pairwise orthogonallity of the components of $\boldsymbol{f}$ imply that V_f is an isometry. Hence its image $\mathcal{H}_f$ is a closed subspace of $L^2(\mathbb{R}^2)$. As outlined in the previous, the isometry property of V_f implies that a frame estimate for $G\big(\boldsymbol{f}, \mathcal{M}(\mathbb{Z}^2)\big)$ is the same as sampling estimate for $\mathcal{H}_f$, with sampling set $\mathcal{M}(\mathbb{Z}^2)$.

We intend to utilize the techniques from Fuhr and Grouching [150] for this purpose, hence we need oscillation estimates. We will need oscillation estimates. We define

$$\operatorname{osc}_r(f)(x) = \sup_{|x-y|<r} |f(x) - f(y)|.$$

Our first aim is to show

$$\|\operatorname{osc}_r(F)\|_2 \leq \frac{r}{C_f} \|F\|_2, \ \forall F \in \mathcal{H}_{\boldsymbol{f}}, \tag{14}$$

for a suitable constant $C_{\boldsymbol{f}}$.

For this purpose we first observe that the projection $P_{\boldsymbol{f}}$ onto $\mathcal{H}_{\boldsymbol{f}}$ is obtained by twisted convolution [154]:

$$\big(P_{\boldsymbol{f}}G\big)(x,\xi) = (\ G\#F)(x,\xi) = \int_{R^2} G(x',\xi')F(x-x',\xi-\xi')e^{\pi i\left(x\xi'-x'\xi\right)}dx'd\xi'$$

where we let $F = V_f \boldsymbol{f}$.

Hence $G = G\#F$ for $G \in \mathcal{H}_{\boldsymbol{f}}$, and therefore

$$\operatorname{osc}_r(G)(x,\xi)$$

$$= \sup_{|(x\,,\xi)-(x'',\xi'')|<\,r} \left| \int_{R^2} G(x',\xi')(F\,(x-x',\xi-\xi')F(x''-x',\xi'' - \xi'))e^{\pi i(x\xi'-x'\xi)}dx'd\xi' \right|$$

$$\leq \int_{R^2} |G(x',\xi')| \sup_{|(x\,,\xi)-(x'',\xi'')|<\,r} |F(x-x',\xi-\xi') - F(x''-x',\xi''-\xi')| dx'd\xi'$$

$$= \int_{R^2} |G(x',\xi')| \operatorname{osc}_r(F)(x-x',\xi-\xi')dx'd\xi' = |G| * \operatorname{osc}_r(F),$$

where convolution is taken with reference to the group structure on R^2. Hence

$$\|\operatorname{osc_r}(G)\|_2 \leq \|G\|_2 \|\operatorname{osc}_r(F)\|_1. \tag{16}$$

In the following estimates we let $B_r(x,\xi) \subset R^2$ denote the Euclidean of radius r centered at (x,ξ). The second factor in (15) can be estimated by

$$\|\mathrm{osc}_r(F)\|_1 = \int_{R^2} \sup_{(x',\xi')\in B_r(x,\xi)} |F(x,\xi) - F(x',\xi')|dxd\xi$$
$$\leq \int_{\mathbb{R}^2} r \sup_{(x',\xi')\in B_r(x,\xi)} \left(\left|\frac{\partial F}{\partial x}(x,\xi)\right| + \left|\frac{\partial F}{\partial \xi}(x,\xi)\right|\right) dxd\xi$$
$$\leq r \int_{\mathbb{R}^2} \sum_{|\alpha|\leq 1} \|D^\alpha F\|_{\infty,B_r(x,\xi)} dxd\xi \leq r \int_{\mathbb{R}^2} C_{B_r} \sum_{1\leq|\alpha|\leq 4} \|D^\alpha F\|_{1,B_r(x,\xi)} dxd\xi$$
$$= r \sum_{1\leq|\alpha|\leq 4} |B_r| C_{B_r} \|D^\alpha F\|_1$$

where we used the mean value theorem for the first inequality, and the Sobolev embedding theorem for the third, applied to both $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial \xi}$. Here C_{B_r} denotes the norm of the embedding $W^{3,1}(B_r) \hookrightarrow C(B_r)$ [141,Theorem 5.4]. Clearly, $|B_r| = r^2|B_1|$; on the other hand, a dilation argument establishes for $r < 1$ that $C_{B_r} \leq r^{-2} C_{B_1}$. Hence

$$\|\mathrm{osc}_r(F)\|_1 \leq r|B_1|C_{B_1} \sum_{|\alpha|\leq 4} \|D^\alpha F\|_1,$$

And thus (15) implies (14), with $C_f = \frac{1}{|B_1|C_{B_1}\sum_{|\alpha|\leq 4}\|D^\alpha F\|_1}$.

Now by letting $k = \mathcal{M}([-0.5\,,0.5]^2)$, we obtain $\mathbb{R}^2 = \bigcup_{\gamma\in\mathcal{M}(\mathbb{Z}^2)} \gamma + k$ as a disjoint union, and K has Lebesgue measure $|\det(\mathcal{M})|$. Moreover, by definition of the norm, $K \subset B_r$, for $r = \|\mathcal{M}\|$. Now by [150, Theorem 3.5], the oscillation estimate (14) results in the sampling estimate

$$\frac{1}{|\det(\mathcal{M})|}\left(1 - \frac{r}{C_f}\right)^2 \|F\|_2^2 \leq \sum_{\gamma\in\mathcal{M}(\mathbb{Z}^2)} |F(\gamma)|^2 \leq \frac{1}{|\det(\mathcal{M})|}\left(1 + \frac{r}{C_f}\right)^2 \|F\|_2^2$$

Which is Theorem (3.2.1).

The (simply connected) Heisenberg group is defined as $\mathbb{H} = \mathbb{R}^3$, with group law

$$(p,q,t)(p',q',t') = (p + p', q + q', t + t' + (pq' - p'q)/2)$$

For the following facts concerning $\mathbb{H}$, see [148]. $\mathbb{H}$ is a step-two nilpotent Lie group, with center $Z(\mathbb{H}) = \{0\} \times \{0\} \times \mathbb{R}$. $\mathbb{H}$ is unimodular, with two-sided invariant measure on $\mathbb{H}$ given by the usual Lebesgue measure of $\mathbb{R}^3$.

Given $\lambda \in \mathbb{R}^* = \mathbb{R}\backslash\{0\}$, the Schrodinger representation ρ_λ of $\mathbb{H}$ acts on $L^2(\mathbb{R})$ via

$$\rho_\lambda(p,q,t) = e^{-2\pi i\lambda(t-pq/2)} T_{\lambda p} M_q.$$

This is an irreducible unitary representation of $\mathbb{H}$. The family of Schrödinger representations provides the basis for the Plancherel transform of the group, a tool that is of key importance.

Before we describe this transform in more detail, let us quickly recall the basics of Hilbert-Schmidt operators: The space of Hilbert-Schmidt operators on a Hilbert space $\mathcal{H}$ is given by all bounded linear operators such that

$$\|T\|_{HS}^2 = \sum_{i\in I} \|T_{\eta_i}\|^2.$$

Is finite; here $(\eta_i)_{i\in I}$ denotes an arbitrary orthonormal basis of $\mathcal{H}$. it is well-known that the norm is independent of the choice of basis, and defines a Hilbert space structure on the Hilbert-Schmidt operators with the Scalar product

$$\langle S,T\rangle = \text{trace}(T^*S) = \sum_{i\in I}\langle S_{\eta_i}, T_{\eta_i}\rangle.$$

Now, given $f \in L^1 \cap L^2(\mathbb{H})$, one defines

$$\rho_\lambda(f) = \int_{\mathbb{H}} \rho_\lambda(x) f(x) dx,$$

Understood in the weak operator sense. This is just the canonical extension of the representation ρ_λ to the convolution algebra $L^1(\mathbb{H})$. The mapping $f \to (\rho_\lambda(f))_{\lambda\in\mathbb{R}^*}$ is called the group Fourier transform. This nomenclature is justified by the observation that the Euclidean Fourier transform is obtained by integration against the characters of the additive group. Moreover, it turns out that for $f \in L^1 \cap L^2(\mathbb{H})$, the group Fourier transform is in fact a family of Hilbert-Schmidt operators, and we have the *Parseval relation*

$$\|f\|_2^2 = \int_{\mathbb{R}} \|\rho_\lambda(f)\|_{HS}^2 \, |\lambda| d\lambda.$$

The Parseval relation allows to extend the Fourier transform to $L^2(\mathbb{H})$, yielding the *Plancherel transform,* a unitary map

$$L^2(\mathbb{H}) \to \int_{\mathbb{R}}^{\oplus} HS(L^2(\mathbb{R}))|\lambda| d\lambda,$$

where the right hand side denotes the direct integral of Hilbert-Schmidt Spaces. We denote the Plancherel transform of $f \in L^2(\mathbb{H})$ as $\hat{f} = \left(\hat{f}(\lambda)\right)_{\lambda\in\mathbb{R}^*}$. This map will play the same role as the Euclidean Fourier transform in the discussion.

The Plancherel transform its algebraic properties, providing a decomposition of various operators and representations acting on $L^2(\mathbb{H})$ and $x, y \in \mathbb{H}$, then

$$\rho_\lambda(L_x f) = \rho_\lambda(x) \circ \rho_\lambda(f),$$

which provides the decomposition of the left regular representation L into a direct integral,

$L \backsimeq \int_{\mathbb{R}^*} \rho_\lambda \otimes 1 |\lambda| d\lambda.$

Similarly, the right regular representation, acting by $R_x(f)(y) = f(yx)$, decomposes by the formula

$$\rho_\lambda(R_x f) = \rho_\lambda(f) \circ \rho_\lambda(x)^*.$$

The decompositions extend to commuting operators: For any bounded operator T commuting with L, there exists a measurable field $(\hat{T}_\lambda)_{\lambda\in\mathbb{R}^*}$ of bounded operators on $L^2(\mathbb{R})$ satisfying $\rho_\lambda(Tf) = \rho_\lambda(f) \circ \hat{T}_\lambda$, or in direct integeral notation

$$T \backsimeq \int_{\mathbb{R}^*} \text{Id} \otimes \hat{T}_\lambda |\lambda| d\lambda.$$

We outline the definition of Paley-Wiener space on $\mathbb{H}$ and its relation to Hermite functions. The central role of Hermite function in the decomposition of the sub-Laplacian has been observed previously, e.g. in Galler [152]; the results presented below can be found also in Thangavelu [160]. For background material on the Heisenberg group, confer Folland [146].

We define a let-invariant differential operator P on $\mathbb{H}$ by

$$(pf)(p,q,t) = \lim_{h\to 0} \frac{f\big((p,q,t)(h,0,0)\big) - f(p,q,t)}{h}, \qquad (16)$$

corresponding to the subgroup $\mathbb{R}\times\{0\}\times\{0\}$, and Q is a left-invariant operator associated to $\{0\}\times\mathbb{R}\times\{0\}$, in the same manner. P, Q are viewed as elements of the Lie algebra $\mathfrak{h}$ of $\mathbb{H}$; we have $[P,Q] = T$, the infinitesimal generator of the group center. This observation exhibits $\mathfrak{h}$ as a stratified Lie algebra,

$$\mathfrak{h} = V_1 + V_2$$

with $V_1 = \text{span}(P,Q)$, and $V_2 = \mathbb{R}\cdot T = [V_1, V_2]$.

Of particular interest for analysis on these groups is the *sup-Laplacian*; as the name suggests, it can be viewed as a replacement for the Laplacian over $\mathbb{R}^n$. For the Heisenberg group, this operator is defined by

$$\mathcal{L} = -P^2 - \frac{Q^2}{4\pi^2}$$

The normalization of Q is chosen for the sake of convenience. $\mathcal{L}$ is a left-invariant positive unbounded operator on $L^2(\mathbb{H})$. We denote its spectral measure by $\pi_{\mathcal{L}}$. The Paley-Wiener space on $\mathbb{H}$ is then defined as

$$PW(\mathbb{H}) = \pi_{\mathcal{L}}([0,1])(L^2(\mathbb{H}))$$

Note that ,up to normalization, this definition is completely analogous to the definition of band limited functions on $\mathbb{R}$, since the Euclidean Fourier transform can also be read as the spectral decomposition of the Laplacian. The projection $\Pi_{\mathcal{L}}([0,1])$ is left – invariant, and is therefore decomposition by Hermit functions. We use the notation $D_a\boldsymbol{f}(\mathrm{x}) = |a|^{-1/2}\boldsymbol{f}(|a|^{-1}x)$. As with translation and modulation operators, we use the same symbol for operators acting on scalar – and on vector – valued functions.

Lemma (3.2.2)[140]:

(a) $(h_n)_{n\in\mathbb{N}_0}$ is an orthonormal basis of $L^2(\mathbb{R})$.

(b) The system $(h_n)_{n\in\mathbb{N}_0}$ is an Eigen basis of the Hermit operator

(c) $Hf(x) = x^2 f(x) - f''(x)$,

$$Hh_n = (2n+1)h_n. \qquad (17)$$

(d) For every real $a \neq 0$, the dilated system $(h_{n,a})_{n\in\mathbb{N}_0}$, defined by

$$h_{n,a}(x) = \left(D_{|a|^{1/2}}h_n\right)(x) = |a|^{-1/4}h_n(|a|^{-1/2}x)\ ,$$

is an eigenbasis of the scaled Hermite operator $H_a f(x) = x^2 f(x) - a^2 f''(x)$

$$H_a h_{n,a} = |a|(2n+1)h_{n,a} \qquad (18)$$

(e) The sub-Laplacian decomposes into a direct integral of scaled Hermite operators:

$$\rho_\lambda(\mathcal{L}f) = \rho_\lambda(f)\circ H_\lambda,$$

for all $f \in C_c^\infty(\mathbb{H})$.

Proof. For part (a) confer [147, Corollary 6.2, Theorem 6.14]. Parts(b) and (c) follow from part (a). Part (d) is established by formal calculation from (16) and the analogous formula for Q, using the decomposition of the right regular representation.

Parts (b) through (d) contain the ingredients of the direct integral decomposition of (ℍ). For the precise formulation of this result and its proof, the tensor product notation for Hilbert-Schmidt operators will be useful. Given vectors η, φ in a Hilbert space $\mathcal{H}$, we let

$$\eta \otimes \varphi : z \mapsto \langle z, \varphi \rangle \eta,$$

which is a rank-one operator on $\mathcal{H}$. Note that the notation is only conjugate linear in φ. The Hilbert-Schmidt scalar product of two elementary tensors is

$$\langle \eta \otimes \varphi, \eta' \otimes \varphi' \rangle HS = \langle \eta\ , \eta' \rangle \mathcal{H} \langle \varphi', \varphi \rangle \mathcal{H}.$$

Moreover, for any pair S, T of bounded operators, $S \circ (\eta \otimes \varphi) \circ T = (S\eta) \otimes (T^* \varphi)$.

Now, given any orthonormal basis $(\varphi_i)_{i \in I}$ of $\mathcal{H}$, every Hilbert-Schmidt operator T has a unique decomposition

$$T = \sum_{i \in I} \eta_i \otimes \varphi_i.$$

Hence, if $S = \sum_{i \in I} \psi_i \otimes \varphi_i$ is another Hilbert-Schmidt Operator, we obtain for the scalar product

$$\langle T, S \rangle = \sum_{i \in I} \langle \eta_i, \psi_i \rangle. \tag{19}$$

Observe in the formulation of the following proposition that $\widehat{P}_\lambda$ involves the first $d(\lambda) + 1$ Hermit functions. For $|\lambda| > 1$, we have $d(\lambda) = -1$, and thus $\widehat{P}_\lambda = 0$.

Proposition (3.2.3)[140]: Letting

$d(\lambda) = \left\lfloor \frac{1}{2|\lambda|} - \frac{1}{2} \right\rfloor$

and

$$\widehat{P}_\lambda = \sum_{n=0,\dots,d(\lambda)} h_{n.\lambda} \otimes h_{n.\lambda},$$

the projection onto Paley-Wiener space is given by

$$(\Pi_{\mathcal{L}}([0,1](f))^\wedge(\lambda) = \hat{f}(\lambda) \circ \widehat{P}_\lambda, \quad \forall f \in L^2(\mathbb{H}). \tag{12}$$

The operator field $(\widehat{P}_\lambda)_{\lambda \in \mathbb{R}^*}$ is the Plancherel transform of a function $p \in L^2(\mathbb{H})$ whence $\Pi_{\mathcal{L}}([0,1](f) = f * p$.

Proof. We apply the above considerations to the case $\mathcal{H} = L^2(\mathbb{H})$ and its orthonormal basis $(h_{n,\lambda})_{n \in \mathbb{N}_0}$. Hence each Hilbert-Schmidt operator T on $L^2(\mathbb{R})$ has a decomposition

$$T = \sum_{n \in \mathbb{N}_0} \eta_n \otimes h_{n,\lambda}, \tag{21}$$

and we obtain from (18) that

$$T \circ H_\lambda = \sum_{n \in \mathbb{N}_0} |\lambda|(2n+1) \eta_n \otimes h_{n,\lambda}\ .$$

This shows that the map $T \longmapsto \eta_n \otimes h_{n,\lambda}$ can be understood as a projection onto an eigenspace of the operator $\mapsto T \circ H_\lambda$, with associated eigenvalue $|\lambda|(2\lambda + 1)$. By definition of Paley-Wiener space, only eigenvalues ≤ 1 are admitted, which shows that the definition of $\widehat{P}_\lambda$ indeed yields (20).

For the second statement, we compute the norm of the operator field in the direct integral space. First observe that $\widehat{P}_\lambda = 0$ for $|\lambda| > 1$. Moreover, the squared Hilbert-

Schmidt norm of a projection equals its rank, whence $\left\|\hat{P}_\lambda\right\|_{HS}^2 = d(\lambda) + 1 < \frac{1}{2|\lambda|+1}$ and thus

$$\int_{\mathbb{R}^*} \left\|\hat{P}_\lambda\right\|_{HS}^2 |\lambda| d\lambda < \int_{-1}^{1} \left(\frac{1}{2|\lambda|} + 1\right) |\lambda| d\lambda = 2.$$

Hence $\left(\hat{P}_\lambda\right)_\lambda$ has a preimage p under the Plancherel transform. Finally, (20) and the convolution Theorem [149, Theorem 4.18] provide that $Pf = f * p$.

The motivation for considering $PW(\mathbb{H})$ is the existence of sampling estimates for this space. The formulation of the sampling theorem requires some additional notation. We fix a quasi-norm $|\cdot|: \mathbb{H} \longrightarrow \mathbb{R}_0^+$ by

$$|(p, q, t)| = (p^2 + q^2 + |t|)^{1/2},$$

and write B_t to denote the unit ball around 0. A discrete subset $\Gamma \subset \mathbb{H}$ is called aquasi-lattice if there exists a relatively compact set $k \subset \mathbb{H}$ such that $\mathbb{H} = \bigcup_{\gamma\in\Gamma} \gamma k$, as a disjoint union. Such a set k is called complement of Γ.

Theorem (3.2.4)[140]: [150, Theorem 5.11] There exists a constant $0 < \boldsymbol{C}_{\mathbb{H}} \leq 1$ with the following property: For all quasi-lattices Γ possessing a complement k contained in a ball of radius $r < \boldsymbol{C}_{\mathbb{H}}$ and all $f \in PW(\mathbb{H})$,

$$\frac{1}{|k|}(1 - r/\boldsymbol{C}_{\mathbb{H}})^2 \|f\|_2^2 \leq \sum_{\gamma\in\Gamma} |f(\gamma)|^2 \leq \frac{1}{|k|}(1 - r/\boldsymbol{C}_{\mathbb{H}})^2 \|f\|_2^2. \tag{22}$$

We stress that the constant $\boldsymbol{C}_{\mathbb{H}}$ is the same as the identically denoted constant from Theorem (3.2.6).

We will now derive Theorem (3.2.6), basically by explicit calculation. The following Lemma can be seen as analog of (1)⇔(2). A version of this result was obtained in [149, Proposition 6.11].

Lemma (3.2.5)[140]: Suppose that $\Gamma' \subset \mathbb{H}$ is of the form $\Gamma' = \Gamma \times \alpha\mathbb{Z}$,with $\Gamma \subset \mathbb{R}^2$ symmetric and $\alpha > 0$. Consider the following statements:

(a) For all $f \in PW(\mathbb{H})$,

$$A\|f\|_2^2 \leq \sum_{\gamma\in\Gamma'} |f(\gamma)|^2 \leq B\|f\|_2^2. \tag{23}$$

(b) For all $d \in \mathbb{N}_0$, and for almost all λ with $|\lambda| < \frac{1}{2\lambda+1}$, the system $\mathcal{G}(h^d, |\lambda|^{1/2}\Gamma)$ is a frame of $L^2(\mathbb{R}; \mathbb{C}^{d+1})$ with frame bounds $\alpha|\lambda|^{-1}A$ and $\alpha|\lambda|^{-1}B$.

Then (a)⇒(b), and if $\alpha < 1/2$, (b)⇒(a).

Moreover, if $\Gamma = \mathcal{M}(\mathbb{Z}^2)$, for a suitable invertible matrix, the frame estimates in (b) are valid for all $\lambda < \frac{1}{2d+1}$.

Proof. For the proof of (a)⇒(b), let $f \in PW(\mathbb{H})$ be given. Then we can write

$$\hat{f}(\lambda) = \sum_{i=0}^{d(\lambda)} \varphi_{i,\lambda} \otimes h_{i,\lambda}, \tag{24}$$

for suitable functions $\varphi_{i,\lambda} \in L^2(\mathbb{R})$. In the following, we also use the notations

$$\Phi_\lambda = (\varphi_{0,\lambda}, \dots, \varphi_{d(\lambda),\lambda})$$

and

$$\boldsymbol{h}_\lambda = (h_{0,\lambda}, \dots, h_{d(\lambda),\lambda}).$$

Let $E \subset \mathbb{R}^*$ be a Borel set contained in an interval I of length $1/\alpha$, and consider g with $\hat{g} = \hat{f} \cdot \mathbf{1}_E$. Observing that $g \in PW(\mathbb{H})$, we can compute the ℓ^2-norm of its restriction to Γ' as follows:

$$\sum_{\gamma\in\Gamma'} |g(\gamma)|^2 = \sum_{\gamma\in\Gamma'} \left|\langle g, L_\gamma p\rangle\right|^2 = \sum_{\gamma\in\Gamma'} \left|\langle \hat{g}, (L_\gamma p)^\wedge\rangle\right|^2 = \sum_{\gamma\in\Gamma'} \left|\int_E \langle \hat{f}(\lambda), \rho_\lambda(\gamma)\hat{p}(\gamma)\rangle |\lambda| d\lambda\right|^2$$

$$= \sum_{(l,k)\in\Gamma, n\in\mathbb{Z}} \left|\int_E \langle \hat{f}(\lambda), \rho_\lambda(l,k,0)\hat{p}(\gamma)\rangle e^{2\pi i\lambda\alpha n} |\lambda| d\lambda\right|^2.$$

Applying the Parseval formula for the interval, we thus obtain

$$\sum_{\gamma\in\Gamma'} |g(\gamma)|^2 = \alpha^{-1} \sum_{(l,k)\in\Gamma} \int_E \left|\langle \hat{f}(\lambda), \rho_\lambda(l,k,0)\hat{p}(\gamma)\rangle\right|^2 |\lambda|^2 d\lambda$$

$$= \alpha^{-1} \int_E \sum_{(l,k)\in\Gamma} \left|\langle \Phi_\lambda, T_{\lambda l} M_k |\lambda|^{1/2} h_\lambda\rangle e^{\pi i\lambda lk}\right|^2 |\lambda| d\lambda$$

$$= \alpha^{-1} \int_E \sum_{(l,k)\in\Gamma} \left|\langle \Phi_\lambda, T_{|\lambda| l} M_k |\lambda|^{1/2} h_\lambda\rangle\right|^2 |\lambda| d\lambda \tag{25}$$

where we used (19) to express the Hilbert-Schmidt scalar products as scalar products as scalar products of vector-valued functions, as symmetry of Γ to replace λ by $|\lambda|$.

On the other hand, by the Plancherel formula, we find

$$\|g\|_2^2 = \int_E \left\|\hat{f}(\lambda)\right\|_{HS}^2 |\lambda| d\lambda = \int_E \|\Phi_\lambda\|^2_{L^2(\mathbb{R},\mathbb{C}^{d(\lambda)})} |\lambda| d\lambda$$

Hence the lower sampling estimate yields

$$A \int_E \|\Phi_\lambda\|^2_{L^2(\mathbb{R},\mathbb{C}^{d(\lambda)})} |\lambda| d\lambda \leq \alpha^{-1} \int_E \sum_{(l,k)\in\Gamma} \left|\langle \Phi_\lambda, T_{|\lambda| l} M_k |\lambda|^{1/2} h_\lambda\rangle\right|^2 |\lambda| d\lambda \tag{26}$$

Since this inequality holds true for all Borel sets E of diameter at most $1/\alpha$, it has to hold point wise a.e for the integerands, i.e, after shifiting constants:

$$\alpha|\lambda|^{-1} A \|\Phi_\lambda\|^2_{L^2(\mathbb{R},\mathbb{C}^{d(\lambda)})} \leq \sum_{(l,k)\in\Gamma} \left|\langle \Phi_\lambda, T_{|\lambda| l} M_k h_\lambda\rangle\right|^2 \text{ (a.e.}\lambda). \tag{27}$$

This is already quite close to the desired lower frame estimate, except that it holds on a set of $\lambda's$ which may depend on the choice of f (or equivalently, on the field $(\Phi_\lambda)_{\lambda\in\mathbb{R}^*}$).

The next step is to establish (27) for all $f \in PW(\mathbb{H})$ and all λ in a set with complement of measure zero, independent of f. For this purpose we pick a sequence $(f_n)_{n\in\mathbb{N}}$ with dense span in $(\mathbb{H})$, and obtain a set $\Omega \subset [-1,1]$ with complement of measure zero such that (27) holds for all $\lambda \in \Omega$ and all f in the $(\mathbb{Q} + i\mathbb{Q})$-span of $(f_n)_{n\in\mathbb{N}}$. But then the $(\mathbb{Q} + i\mathbb{Q})$- span of $(\hat{f}(\lambda))_{n\in\mathbb{N}}$ is dense in $HS(L^2(\mathbb{R})) \circ \hat{P}_\lambda$, for all λ in a Borel set $\Omega' \subset [-1,1]$ with complement of measure zero. Hence for all $\lambda \in \Omega \cap \Omega'$, the frame estimate holds on a dense subset of $HS(L^2(\mathbb{R})) \circ \hat{P}_\lambda$, which is sufficient.

Thus we have finally established (27) for almost all $\lambda \in [-1,1]$, and all $\Phi_\lambda \in L^2(\mathbb{R}, \mathbb{C}^{d(\lambda)})$. The same argument applies to show the upper estimate with constant $\alpha|\lambda|^{-1}B$.

Now using $\boldsymbol{h}_\lambda = D_{|\lambda|^{1/2}} \boldsymbol{h}^{d(\lambda)}$ and the relations

$$M_\xi D_b = D_b M_{b\xi}, \qquad T_x D_b = D_b T_{b^{-1}x}$$

we find that

$$T_{|\lambda| l} M_k \boldsymbol{h}_\lambda = D_{|\lambda|^{1/2}} \left(T_{|\lambda|^{1/2} l} M_{\lambda^{1/2} k} h^{d(\lambda)} \right).$$

Since the image of a frame under a unitary map is a frame with identical constants, we finally obtain that $(T_{|\lambda|^{1/2} l} M_{\lambda^{1/2} k} h^{d(\lambda)})_{(l,k) \in \Gamma}$ is a frame, for almost all $|\lambda| < 1$. Now part (b) follows from $d \leq d(\lambda)$, for $\lambda < \frac{1}{2d+1}$.

For the converse direction observe that by assumption on α. all Plancherel transforms of elements of $PW(\mathbb{H})$ are supported in an interval of length $1/\alpha$. Hence (25) holds for all $g \in PW(\mathbb{H})$, where this time $E = [-1,1]$, and the field $(\Phi_\lambda)_\lambda$ corresponds to the Plancherel transform of g. But then (b)⟹(a) is immediate.

The proof that the "almost everywhere" contained in the statement can be omitted for lattices relies on semi-continunity properties of the frame bounds.

For any unit vector $f \in L^2(\mathbb{R}, \mathbb{C}^d)$, consider the function

$$\Theta_f : \left]0, \frac{1}{2d+1}\right[\ni \lambda \longmapsto \sum_{(l,k) \in |\lambda|^{1/2} \mathcal{M}(\mathbb{Z}^2)} \left| \langle \boldsymbol{f}, T_l M_k |\lambda|^{1/2} h^d \rangle \right|^2.$$

We compute

$$\begin{aligned} \Theta_f(\lambda) &= |\lambda| \sum_{(l,k) \in |\lambda|^{1/2} \mathcal{M}(\mathbb{Z}^2)} \left| \langle \boldsymbol{f}, T_l M_k h^d \rangle \right|^2 = |\lambda| \sum_{(l,k) \in |\lambda|^{1/2} \mathcal{M}(\mathbb{Z}^2)} \left| \sum_{i=0}^{d} \langle f_i, T_l M_k h_i \rangle \right|^2 \\ &= |\lambda| \sum_{(l,k)} \sum_{i,j} \langle f_i, T_l M_k h_i \rangle \overline{\langle f_\iota, T_l M_k h_\iota \rangle} = |\lambda| \sum_{i,j} \sum_{(l,k)} \langle f_i, \langle f_j, T_l M_k h_i \rangle T_l M_k h_i \rangle \\ &= |\lambda| \sum_{i,j} \langle f_i, \sum_{(l,k)} \langle f_j, T_l M_k h_i \rangle T_l M_k h_i \rangle = |\lambda| \sum_{i,j} \langle f_i, S_{h_i, h_j; |\lambda|^{1/2} \mathcal{M}} f_j \rangle. \end{aligned}$$

Here we used the linear operator $S_{g_1, g_2; \mathcal{N}}$ associated to functions g_1, g_2 and an invertible matrix $\mathcal{N}$, defined by

$$S_{g_1, g_2; \mathcal{N}}(f) = \sum_{(l,k) \in \mathcal{N}(\mathbb{Z}^2)} \langle f, T_l M_k g_1 \rangle \, T_l M_k g_2.$$

[145, Theorem 3.6] states that $S: M^1(\mathbb{R}) \times M^1(\mathbb{R}) \times GL(2, \mathbb{R}) \longrightarrow \mathcal{B}(L^2(\mathbb{R}))$ is continuous, where the right-hand side denotes the space of bounded operators endowed with the norm topology, and $M^1(\mathbb{R})$ is the Feichtinger algebra; see e.g. [154] for a definition and basic properties. Now the continuous inclusion $\mathcal{S}(\mathbb{R}) \subset M^1(\mathbb{R})$ entails that the map $\lambda \mapsto \langle f_i, S_{h_i, h_j; |\lambda|^{1/2} \mathcal{M}} f_j \rangle$ is continuous, for all $0 \leq i, j \leq d$, and then Θ_f is continuous.

Next consider the map associating to each λ the optimal upper frame bound, given by

$$B_{opt} : \left]0, \frac{1}{2d+1}\right[\in \mapsto \sup_{\|f\|=1} \Theta_f(\lambda).$$

The supremum is always finite: By [154, Corollary 6.2.3], the frame operator of one-dimensional window in the Schwartz class is always bounded. Hence the upper frame bound also exists in the vector valued case, by (11).

As the supremum of a family of continuous functions, B_{opt} is lower semi-continuous, and then $\lambda \longmapsto B_{opt}(\lambda)|\lambda|^{-1/2}$ is lower semi-continuous as well. We already know that the latter map is bounded from above by αB on a subset of $\left]0, \frac{1}{2d+1}\right[$ with complement of measure zero. This subset is dense, hence lower semi-continuity implies $B_{opt}(\lambda)|\lambda|^{-1/2} \leq \alpha B$ on the whole interval.

The analogous reasoning, replacing lower by upper semi-continuity, applies to the lower frame bound, and we are done.

Theorem (3.2.6)[140]: There exists a constant $C_{\mathbb{H}} \leq 1$ such that for all $d \in \mathbb{N}_0$

$$C^*_{h^d} \geq \frac{C_{\mathbb{H}}}{\sqrt{2d+1}}. \tag{28}$$

Proof. Fix $d \in \mathbb{N}_0$. Suppose that $\mathcal{M}$ is given with $\|\mathcal{M}\| < C_{\mathbb{H}}/\sqrt{2d+1}$. let $K = \mathcal{M}([-0.5\ ,0.5)^2)$, then K is a complement of $\mathcal{M}(\mathbb{Z}^2)$ in $\mathbb{R}^2$, contained in a ball of radius $r_0 = \|\mathcal{M}\|$, and with measure $|\det(\mathcal{M})|$. Moreover, by choosing $\alpha > 0$ small enough, the set $K' = K \times [-\alpha/2\,,\alpha/2)$ is contained in a ball of radius $r_0 + \epsilon < C_{\mathbb{H}}/\sqrt{2d+1}$ (with respect to the quasi-norm on $\mathbb{H}$). In addition, if we define $\Gamma' = \mathcal{M}(\mathbb{Z}^2) \times \alpha\mathbb{Z}$, as in Lemma (3.2.5), then K' is a complement of Γ':for any $(p,q,t) \in \mathbb{H}$, there exist unique $(p_1,q_1) \in \Gamma$ and $(p_2,q_2) \in K$ with $(p_1+p_2, q_1+q_2) = (p,q)$, and finally unique $l \in \mathbb{Z}$ and $s \in [-\alpha/2\,,\alpha/2)$ with $s + \alpha l = t - (p_1q_2 - p_2q_1)/2$. But these choices imply $(p_1,q_1,\alpha l)(p_2,q_2,s) = (p,q,t)$.

We will apply Theorem (3.2.4) to dilated copies of Γ'. We define dilations $\delta_a = \mathbb{H} \longrightarrow \mathbb{H}$ for $a > 0$ by letting $\delta_a(p,q,t) = (ap, aq, a^2t)$. It is easy to check that δ_a is a group automorphism fulfilling $|\delta_a(p,q,t)\,| = a|(p,q,t)|$. Hence $\delta_a(\Gamma')$ is a quasi-lattice, with complement $\delta_a(k)$ contained in a ball of radius $(r_0+\epsilon)$. Hence, for any $a < \mathbb{C}_{\mathbb{H}}\,/(r_0+\epsilon)$, the sampling theorem provides the estimate

$$\frac{1}{a^4|\det(\mathcal{M})|\alpha}\left(1 - \frac{a(r_0+\epsilon)}{\mathbb{C}_{\mathbb{H}}}\right)^2 \|f\|_2^2 \leq \sum_{\gamma\in\delta_a(\Gamma')} |f(\gamma)|^2$$
$$\leq \frac{1}{a^4|\det(\mathcal{M})|\alpha}\left(1 - \frac{a(r_0+\epsilon)}{\mathbb{C}_{\mathbb{H}}}\right)^2 \|f\|_2^2.$$

An application of Lemma (3.2.5) then yields, for all $\lambda < \frac{1}{2n+1}$, that $\mathcal{G}\left(h^d, a|\lambda|^{1/2}\mathcal{M}(\mathbb{Z}^2)\right)$ is a frame with bounds $\frac{1}{a^2|\lambda||\det(\mathcal{M})|}\left(1 \mp \frac{a(r_0+\epsilon)}{C_{\mathbb{H}}}\right)^2$. Letting $a^2|\lambda| = 1$, provides a lower frame bound for $\mathcal{G}\left(h^d, \mathcal{M}(\mathbb{Z}^2)\right)$ given by

$$\sup\left\{\frac{1}{|\det(\mathcal{M})|}\left(1 - \frac{a(r_0+\epsilon)}{C_{\mathbb{H}}}\right)^2 : \sqrt{2d+1} \leq a < C_{\mathbb{H}}/(r_0+\epsilon)\right\}.$$

Observe that the restriction $a \geq \sqrt{2d+1}$ is imposed by $|\lambda| \leq \frac{1}{2d+1}$. By monotonicity, the supremum is $\frac{1}{|\det(\mathcal{M})|}\left(1 - \frac{\sqrt{2d+1}(r_0+\epsilon)}{C_{\mathbb{H}}}\right)^2$. Sending ϵ to zero provides the lower frame bound estimate necessary to establish Theorem (3.2.6) the upper estimate is obtained in the same fashion.

We have employed various ideas that are frequently used in frame theory and the discretization of integral transforms: we have already mentioned the connection between sampling and discretization. The proof of Theorem (3.2.1). Relied on the analysis of the reproducing kernel of the image space of short time Fourier transform, and in this respect it has many precedents in the literature: As a recent influential sources discussing sampling in reproducing kernel Hilbert spaces see [159]. The oscillation approach used to prove Theorem (3.2.1) is an adaptation of techniques employed, e.g, by Grochenig in [153]. Another idea that is often used is the direct connection between Gabor frames and a particular representation of the Heisenberg group $\mathbb{H}$, or ratherble: of a suitable quotient of $\mathbb{H}$.

There are however some aspects that set the results and arguments presented. For instance, we study the asymptotic behavior of the frame bounds, as the density increases to infinity. We are not aware of any previous source for this type of results, not even for the scalar case. The main difference however is in terms of technique: unlike most sources in Gabor analysis, we employ a representation of the simply connected Heisenberg group $\mathbb{H}$, and not of the reduced Heisenberge $\mathbb{H}_r$, which is the quotient of $\mathbb{H}$ by a central discrete subgroup.

It can be argued that $\mathbb{H}_r$ has a more intuitive connection to Gabor analysis. The proof of Theorem (3.2.6) shows that in working with $\mathbb{H}$ one needs to deal with a fair amount of additional technical details, that one avoids by considering $\mathbb{H}_r$. The benefit of this approach lies in the fact that a single sampling estimate, namely (22), gives rise to a whole family of Gabor frame estimates, namely (28), valid for all $d \geq 0$.

The main result provides rather intuitive asymptotic estimates for Gabor frame bounds. A major drawback of these estimates is that they involve an unknown constants. An estimate for $C_{\mathbb{H}}$ is given in Führ and Gröchenig [150], involving operator norms for differential operators on $PW(\mathbb{H})$ as well as a Sobolev constant for the unit ball $\mathbb{H}$; the argument is very similar to the estimate of the constant C_f in the proof of Theorem (3.2.1). While rough estimates for the norms of differential operators are obtainable from the Plancherel transform, which decomposes the differential operators as well as $PW(\mathbb{H})$, we have not managed to obtain an explicit estimate for the Sobolev constant for $\mathbb{H}$. in any case, we stress that the constant $C_{\mathbb{H}}$ in the sampling theorem is the same as in Theorem (3.2.6); this was the chief motivation for the picking the matrix norm (13).

For single Hermite functions, the results obtained here compare in an interesting way with recent results due to Gröchenig and Lyubarskii. Using complex analysis methods, they obtained the following statement [155, Theorem 3.1]:

Theorem (3.2.7)[140]: If $|\det \mathcal{M}| < (d+1)^{-1}$, then $\mathcal{G}(h_d, \mathcal{M})$ is a frame for $L^2(\mathbb{R})$.

For the isotropic case, i.e, $\mathcal{M} = a.\,\mathrm{Id}$, this results provides a criterion that is very close to our Theorem (3.2.6): Any a below a threshold $\sim n^{-1/2}$ guarantees a frame. In the general case however, Theorem (3.2.7) is much more widely applicable: At the same time $|\det \mathcal{M}|$ can be made arbitrarily small and $\|\mathcal{M}\|$ arbitrarily large.

On the other hand, Theorem (3.2.7) does not provide frame bound estimates, and it only applies to the scalar –valued case.

Let us finally comment on possible generalizations. The first possible extension consist in replacing $\mathbb{R}$ by $\mathbb{R}^2$, i.e, studying vector-valued Gabor frames in $L^2(\mathbb{R}^n;\mathbb{C}^d)$. One now considers the $2n+1$-dimensional Heisenberg group $\mathbb{H}_n$. This is a stratified Lie group, possessing a sub-Laplacian, Paley-Wiener space and, finally, a sampling Theorem [150]. As for the one – dimensional case, the spectral decomposition of the sub-Laplacian involves Hermite functions, and an adaptation of arguments for $\mathbb{H}$ should be a straightforward task.

A second, more interesting but also more challenging type of generalization concerns the sampling sets, which could also be irregular. There already exists an irregular sampling theorem for $\mathbb{H}$, however, in the transfer of the associated sampling estimates to Gabor frame estimates, we are crucially relying on the lattice structure of the sampling set. The key result is the continuity statement [145, Theorem 3.6], and the proof of this result makes full use of Gabor theory developed for lattices.

As a result, we can currently only show statements of the following form: For all $d \in \mathbb{N}_0$ and all uniformly discrete and uniformly dense sets $\Gamma \subset \mathbb{R}^2$ there exists a range $(0, a_d)$ of dilation parameters such that $\mathcal{G}(h^d, a\Gamma)$ is a frame, for almost all $a \in (0, a_d)$, including an estimate of the frame bounds. The threshold a_d is of the order $d^{-1/2}$.

Section (3.3): Gabor Systems Associated with Periodic Subsets of the Real Line

The theory of frames was first introduced in 1952 by Duffin and Schaeffer ([167]; see also [179]) dealing with nonharmonic Fourier series. It came back into the limelight in recent years with the apparition of a large number of dealing with specific applications of frames, mostly to wavelets and Gabor systems. Let us briefly recall some important definitions and results of the theory of frames. If $\mathcal{H}$ is an infinite-dimensional separable Hilbert space, with inner product $\langle .,. \rangle$, and $\mathcal{N}$ is a countable index set, we say that a collection $X = \{x_n\}_{n\in\mathcal{N}}$ in $\mathcal{H}$ is a frame for its closed linear span $\mathcal{M}$ if there exist constants $A, B > 0$, called the frame bounds , such that

$$A\|x\|^2 \le \sum_{n\in\mathcal{N}} |\langle x, x_n\rangle|^2 \le B\|x\|^2, \quad x \in \mathcal{M}. \tag{29}$$

A frame X is said to be tight (resp. *a Parseval tight frame*) if $A = B$ (resp. $A = B = 1$) in (29). We call the collection X *Bessel*, with constant B, if the second inequality in (29) holds for all $x \in \mathcal{M}$. the collection $X = \{x_n\}_{n\in\mathcal{N}}$ is called a Riesz family or Riesz sequence with constants $C, D,$ if the inequalities

$$C\sum_{n\in\mathcal{N}} |a_n|^2 \le \left\|\sum_{n\in\mathcal{N}} a_n x_n\right\|^2 \le D\sum_{n\in\mathcal{N}} |a_n|^2,$$

Hold for all (finitely supported) sequences $\{a_n\}$ of complex numbers. If the linear span of a Riesz family X is dense in $\mathcal{H}$, we say that X forms a Riesz basis. If we can choose $C = D = 1$, then X is an orthonormal family and an orthonormal basis if its closed linear span is $\mathcal{H}$. Let $\ell^2(\mathcal{N})$ denote the space of complex-valued square-summable sequences indexed by $\mathcal{N}$. If X is a Bessel collection, the analysis operator or frame transform associated with $X, T_X: \mathcal{M} \to \ell^2(\mathcal{N})$, is defined by

$$T_X(x) = \{\langle x, x_n\rangle\}_{n\in\mathcal{N}}, \quad x \in \mathcal{M}. \tag{30}$$

while its adjoint, the synthesis operator, $T_X^*: \ell^2(\mathcal{N}) \to \mathcal{M}$, is given by

$$T_X^*(\{c_n\}_{n\in\mathcal{N}}) = \sum_{n\in\mathcal{N}} c_n x_n, \qquad \{c_n\}_{n\in\mathcal{N}} \in \ell^2(\mathcal{N}). \tag{31}$$

The *frame operator* S is defined by $S = T_X^* T_X : \mathcal{M} \to \mathcal{M}$. More explicitly

$$S_x = \sum_{n \in \mathcal{N}} \langle x, x_n \rangle \, x_n \,, \quad x \in \mathcal{M}. \tag{32}$$

If X is a frame for $\mathcal{M}$, then $S: \mathcal{M} \to \mathcal{M}$ is a bounded, positive and invertible operator with a bounded inverse. The collection $\{S^{-1} x_n\}_{n \in \mathcal{N}}$ is called the *standard dual frame* of the frame X and we have the reconstruction formula

$$x = \sum_{n \in \mathcal{N}} \langle x, x_n \rangle \, S^{-1} x_n = \sum_{n \in \mathcal{N}} \langle x, S^{-1} x_n \rangle \, x_n \,, \quad x \in \mathcal{M}.$$

We will let $\mathcal{H} = L^2(\mathbb{R})$ and consider expansions in terms of one-dimensional Gabor (also called Weyl-Heisenberg) systems of the form $G = \left\{ e^{2\pi i m b x} g(x - na) \right\}_{m,n \in \mathbb{Z}}$, where $a, b > 0$ two real parameters and g are is a function in $L^2(\mathbb{R})$ called the window function. Such systems have been studied quite extensively, mostly when the expansions are considered on the whole space $L^2(\mathbb{R})$ (see [174,168,169,176,177]), but also (as in [163,171,172,173]).

In the one-dimensional case, the well-known density Theorem for Gabor systems states that a necessary and sufficient condition for the existence of a Gabor system $\boldsymbol{G}$ as above whose linear span is dense in $\mathcal{H} = L^2(\mathbb{R})$ is that $ab \leq 1$. If this last condition holds, there exists a function $g \in L^2(\mathbb{R})$ such that the associated system $\boldsymbol{G}$ forms a tight frame for $L^2(\mathbb{R})$. In fact, it is not difficult to show that $g = \chi_{[0,a)}$ will do the trick. The necessary (and harder) part of this result was first obtained by Daubechies [166] in the rational case (i.e. $ab \in \mathbb{Q}$) and is generally attributed to Baggett [162] and Rieffel [178] in the irrational one. (See [177])

We study related problems for subspaces of $L^2(\mathbb{R})$ of the form $L^2(S) = \{f \in L^2(\mathbb{R}), f = 0 \text{ a.e. on } \mathbb{R} \backslash S\}$, where S is a measurable subset of $\mathbb{R}$ which is $a\mathbb{Z}$-shift invariant, i.e. S has the property that it is invariant under the transformation $x \longmapsto x + a$. If g it is vanishes $a.e.$ Outside of S, it is clear that the closed linear space generated by the corresponding system $\boldsymbol{G}$ will be a subspace of $L^2(S)$. One can then ask for conditions on S depending on a, b for the existence of a system $\boldsymbol{G}$ whose linear span is dense in $L^2(S)$. If this condition holds, one can then ask if there exist such collections $\boldsymbol{G}$ forming a (tight) frame, Riesz basis, etc. for $L^2(S)$. This framework can model a situation where a signal is known to appear periodically but intermittently and one would try to perform a Gabor analysis of the signal in the most efficient way possible while still preserving all the features of the observed data. One could think of the signal as existing for all time t and do the analysis in the usual way but clearly, if the signal is only emitted for very short periods of time, this might not be the optimal way to proceed. Since the correct density condition is $ab \leq 1$ in the case where $S = \mathbb{R}$, one would assume that if S is "smaller" than $\mathbb{R}$, a corresponding smaller density condition might result. One might guess that the correct density condition should be that $b|S \cap [0,1)| \leq 1$, where $| \, . \, |$ denotes the Lebesgue measure. In fact, that condition was proved to be necessary in [172]. As we will show, it turns out to be the right density condition in the irrational case, but not in the rational one. We will show that, if $ab = \frac{p}{q}$, where p and q are two positive integers with $\gcd(p, q) = 1$, the correct density condition is that $\sum_{k=0}^{p-1} \chi_S \left(. + \frac{k}{b} \right) \leq q$ a.e on $\mathbb{R}$. One of the main results, is that, in both cases, if the appropriate density condition is satisfied, we can construct a window g of the form $g = X_E$, where E is a measurable subset of $\mathbb{R}$ with finite measure, such that the corresponding system $\boldsymbol{G}$ actually forms a tight frame for $L^2(S)$. In fact, we will show that the possibility of constructing a Gabor subspace frame of this form for $L^2(S)$ is equivalent to being able to solve a certain tiling problem related to the set S and the density condition is exactly what is needed for the tiling problem to have a solution. We note that the idea of using a window which is the characteristic function of a measurable set was also used by Han and Wang [175] to show the existence of Gabor frames (where the parameters a, b are replaced by invertible matrices) in higher dimensions for the space $L^2(\mathbb{R}^n)$.

We consider the rational case, Given a measurable subset S of the real line, invariant by a $\mathbb{Z}$-translations, and a window $g \in L^2(S)$, we provide a necessary and sufficient condition for the linear span of the linear span of the system $\{e^{2\pi imbx}g(x-na)\}_{m,n\in\mathbb{Z}}$ to be dense in $L^2(S)$ under the assumption that the product ab is a rational number p/q where $\gcd(p,q)=1$ (Theorem (3.3.7)) this condition involves the rank of $aq \times p$ matrix-valued function $\mathcal{G}$ built using the Zak transform of g and implies the density condition for the rational case mentioned earlier. Using an iterative construction is using finitely many steps (in fact, q steps), we show that if this density condition is satisfied, then there exists a measurable set $E \subset \mathbb{R}$ with $|E| < \infty$, such that the Gabor system associated with the window $g = \chi_E$ actually forms a tight frame for $L^2(S)$ (Theorem (3.3.12)), we give a proof of the fact that the condition $b|S\cap[0,a)| \leq 1$ is necessary in order for Gabor system as above to form a frame for $L^2(S)$, whether $a\, b$ is rational or not, and that, if such a frame exists, it will form a Riesz basis if and only if $b|S\cap[0,a)| = 1$ (Theorem (3.3.17)). Finally, we show, for the irrational case. That if the density condition $b|S\cap[0,a)| \leq 1$ hols, one can again construct a window function of the form $g = \chi_E$ such that the associated system forms a tight frame for $L^2(S)$ (Theorem (3.3.19)). The construction of E is done using a similar iterative procedure as for the rational case, but requiring now an infinite number of steps.

We will consider Gabor systems of the form

$$\boldsymbol{G} = \{e^{2\pi imbx}g(x-na)\}_{m,n\in\mathbb{Z}},$$

where $a, b \in \mathbb{Q}$ and g is a window vanishing a.e outside of a set S which is a $\mathbb{Z}$-shift invariant. The Zak transform will be one of the main tools used, which is not unusual when dealing with Gabor systems in the rational case (see [166,174]).

For E be a measurable set in $\mathbb{R}$ with nonzero Lebesgue measure (which will be denoted by $|E|$). We identify $L^2(E)$ with $\{ f \in L^2(\mathbb{R}): f = 0 \text{ a.e on } \mathbb{R}\backslash E \}$. For $x, \omega \in \mathbb{R}$ and $g \in L^2(\mathbb{R})$, we denote by T_x and E_ω, the translation and modulation operators defined respectively by

$$(T_x g)(t) = g(t-x) \quad \text{and} \quad (E_\omega g)(t) = e^{2\pi i\omega t}g(t), \quad t \in \mathbb{R}$$

For a fixed $\alpha > 0$, we define the Zak transform $\mathcal{Z}_\alpha: L^2(\mathbb{R}) \to L^2_{\text{loc}}(\mathbb{R}^2)$ to be the mapping

$$(\mathcal{Z}_\alpha f)(t,v) = \sum_{k\in\mathbb{Z}} f(t+k\alpha)e^{2\pi ikv}, \quad f \in L^2(\mathbb{R}),$$

defined for a.e $(t,v) \in \mathbb{R}^2$. It is easy to check that

$$(\mathcal{Z}_\alpha f)(t+k\alpha, v+l) = e^{-2\pi ikv}(\mathcal{Z}_\alpha f)(t,v) \tag{33}$$

for $k, l \in \mathbb{Z}$ and a.e. $(t, v) \in \mathbb{R}^2$. See [174, Chapter 8], for further information on the Zak transform. The following lemma, although quite elementary, will play an important role in our analysis of the rational case. With a slight abuse of language, we will call the operator that maps a function $f \in L^2(\mathbb{R})$ to the restriction of its Zak transform $\mathcal{Z}_\alpha f$ to a subset of $\mathbb{R}^2$, the restriction of the Zak transform $\mathcal{Z}_\alpha$ to that subset.

Lemma (3.3.1)[161]: let S be a measurable subset of $\mathbb{R}$ which is a $\mathbb{Z}$-shift invariant and define $S_0 = S \cap [0, \alpha)$ then,

(i) The restriction of $\mathcal{Z}_\alpha$ to the set $[0, \alpha) \times [0,1)$ is a unitary operator from $L^2(\mathbb{R})$ onto $L^2([0, \alpha) \times [0,1))$.

(ii) The image of $L^2(S)$ under the restriction of $\mathcal{Z}_\alpha$ to the set $[0, \alpha) \times [0,1)$ is the subspace $L^2(S_0 \times [0,1))$.

Proof. The first statement is a well-known property of the Zak transform. To show the second one, note first that, from the definition of $\mathcal{Z}_\alpha$, if $f \in L^2(S)$, then $\mathcal{Z}_\alpha f = 0$ a.e on $(\mathbb{R} \backslash S) \times \mathbb{R}$. Hence, when we restrict the Zak transform to $[0, \alpha) \times [0,1)$, we deduce that $\mathcal{Z}_\alpha f \in L^2\big(S_0 \times [0,1)\big)$.Conversely, given an arbitrary function $F(t, v) \in L^2\big(S_0 \times [0,1)\big)$, we have, for any $k \in \mathbb{Z}$,

$$(\mathcal{Z}_\alpha^{-1} F)(t + k\alpha) = \int_0^1 F(t, v) e^{-2\pi i k v} dv = 0, \quad \text{for a.e } t \in [0,1) \backslash S_0,$$

which shows that $\mathcal{Z}_\alpha^{-1} F \in L^2(S)$. The mapping $\mathcal{Z}_\alpha : L^2(S) \to L^2(S_0 \times [0,1))$ is thus subjective which shows our claim.

Definition (3.3.2)[161]: If $j \in \mathbb{Z}$, we denote by τ_j the translation operator acting on the finite group $\mathbb{Z}_p$ identified with the set $\{0,1, \dots, p-1\}$ and defined by

$$\tau_j(k) = k - j \bmod(p), \quad k = 0, \dots, p-1.$$

If $A \subset \{0,1, \dots, p-1\}$, we let $\tau_j(A) = \{\tau_j(k), k \in A\}$.

The following lemma is well-known.

Lemma (3.3.3)[161]: Let $p_1, p_2 \in \mathbb{N}$ satisfy $\gcd(p_1, p_2) = 1$. Then, to every $j \in \mathbb{Z}$, there correspond a unique $k \in \mathbb{Z}$ and a unique $r \in \{0,1, \dots, p_1 - 1\}$ such that

$$j = kp_1 + rp_2. \tag{34}$$

Lemma (3.3.4)[161]: Let $a, b > 0$ satisfy $ab = \frac{p}{q}$ with $p, q \in \mathbb{N}$ and $\gcd(p, q) = 1$ and let S be a measurable subset of $\mathbb{R}$ with nonzero measure, and with S being a $\mathbb{Z}$-shift invariant. Define the function

$$h(t) := \sum_{k=0}^{p-1} \chi_S\left(t + \frac{k}{b}\right), \qquad t \in \mathbb{R}$$

and, given $t \in \mathbb{R}$, define the set

$$Ks(t) = \left\{k \in \{0,1, \dots, p-1\}, \chi_S\left(t + \frac{k}{b}\right) = 1\right\}$$

Then, the function h is $\frac{1}{bq}$-periodic and, furthermore, if $j \in \mathbb{Z}$, we have the identity

$$K_S\left(t + \frac{j}{bq}\right) = \tau_{k_0}\big(K_S(t)\big),$$

where k_0 is the unique integer satisfying $\frac{j}{q} = k_0 + \frac{rp}{q}$ with $r \in \{0,1, \dots, q-1\}$.

Proof. Letting $S_0 = S \cap [0, a)$, we have $\chi_S = \sum_{n \in \mathbb{N}} \chi_{S_0}(\cdot + na)$. Thus, using Lemma (3.3.3) with $p_1 = p$ and $p_2 = q$, we have

$$h=\sum_{k=0}^{p-1}\sum_{n\in\mathbb{Z}}\chi_{S_0}\left(\cdot+\frac{k}{b}+\frac{np}{bq}\right)=\sum_{k=0}^{p-1}\sum_{n\in\mathbb{Z}}\chi_{S_0}\left(\cdot+\frac{1}{bq}(kq+np)\right)$$
$$=\sum_{l\in\mathbb{Z}}\chi_{S_0}\left(\cdot+\frac{l}{bq}\right) \quad (35)$$
and this last expression is clearly $\frac{1}{bq}$-periodic, which shows the first part of the claim. Next, note that, for a.e $t\in\mathbb{R}$, the mapping $K\mapsto\chi_S\left(t+\frac{k}{b}\right)$ is p-periodic, since
$$\chi_S\left(t+\frac{K+P}{B}\right)=\chi_S\left(t+\frac{k}{b}+\frac{p}{bq}q\right)=\chi_S\left(t+\frac{k}{b}+aq\right)=\chi_S\left(t+\frac{k}{b}\right).$$
If $\frac{j}{q}=k_0+\frac{rp}{q}$ as above, we have
$$\chi_S\left(t+\frac{j}{bq}+\frac{k}{b}\right)=\chi_S\left(t+\frac{k_0}{b}+\frac{rp}{bq}+\frac{k}{b}\right)=\chi_S\left(t+\frac{k+k_0}{b}\right).$$
Thus, $k\in K_S\left(t+\frac{j}{bq}\right)$ if and only if $\chi_S\left(t+\frac{k+k_0}{b}\right)=1$, which, using the periodicity property just mentioned, is equivalent to the fact that $k\in\tau_{k_0}\big(K_S(t)\big)$.

The analysis in the case where the product ab is rational, will depend in an essential way on properties of a matrix-valued function associated with the window function g and defined using the Zak transform. We denote by $\mathcal{M}_{q,p}$ the space of matrices with complex entries of size $q\times p$. A function taking values in $\mathcal{M}_{q,p}$ is said to be measurable if each of the corresponding entries is measurable.

Definition (3.3.5)[161]: Let $a,b>0$ satisfy $ab=\frac{p}{q}$ with $p,q\in\mathbb{N}$ and $\gcd(p,q)=1$. Given a function g in $L^2(\mathbb{R})$, we associate with the matrix-valued function $\mathcal{G}:\mathbb{R}^2\to\mathcal{M}_{q,p}$ defined by
$$\mathcal{G}(t,v)_{r,k}=\left(\mathcal{Z}_{aq}g\right)\left(t+\frac{k}{b}-ra,v\right),\quad 0\le r\le q-1, 0\le k\le p-1, \quad (36)$$
for a.e. $(t,v)\in\mathbb{R}^2$.

The matrix-valued function $\mathcal{G}$ is related to the so-called Zibulski-Zeevi matrix [180] and has similar properties, but the definition given here is more convenient for our purposes. We will need the following lemma.

Lemma (3.3.6)[161]: Let $a,b>0$ satisfy $ab=\frac{p}{q}$ with $p,q\in\mathbb{N}$ and $\gcd(p,q)=1$ and let S be a measurable subset of $\mathbb{R}$ which is a $\mathbb{Z}$-shift invariant. Given $g\in L^2(S)$, let, for a.e. $(t,v)\in\mathbb{R}^2,\mathcal{G}(t,v)$ be the matrix-valued function defined in (36) and let the matrix $p(t,v)\in\mathcal{M}_{p,p}$ denote the orthogonal projection onto the kernel of $\mathcal{G}(t,v)$. Then, $p(\cdot,\cdot)$ is measurable. Furthermore, the integer-valued function $(t,v)\mapsto\operatorname{rank}(\mathcal{G}(t,v))$ is measurable, $\frac{1}{bq}$-periodic with respect to variable t and satisfies the inequality
$$\operatorname{rank}\big(\mathcal{G}(t,v)\big)\le\sum_{k=0}^{p-1}\chi_S\left(t+\frac{k}{b}\right). \quad (37)$$
Proof. Note first that, for a.e $(v,t)\in\mathbb{R}^2$,
$$p(t,v)=\lim_{n\to\infty}\exp\big(-n(\mathcal{G}^*\mathcal{G})(t,v)\big),$$

by an easy application of the spectral theorem for self-adjoint matrices (see also [166, p.978]). Since $\mathcal{G}(\cdot,\cdot)$ is measurable, the measurability of $p(\cdot,\cdot)$, follows immediately. Using the facts that the sum of the rank of $\mathcal{G}(t,v)$ and the dimension of the kernel of $\mathcal{G}(t,v)$ is equal to p and that the dimension of a subspace of $\mathbb{C}^p$ is the trace of the orthogonal projection onto that subspace, it follows that the rank of $\mathcal{G}(t,v)$ is equal to $p-$ trace $(p(t,v))$ and is thus also measurable. Given any $j\in\mathbb{Z}$, we can write, using Lemma (3.3.3), $\frac{j}{q}=k_0+mp+r_0\frac{p}{q}$ uniquely with $m\in\mathbb{Z}$, $k_0\in\{0,1,\dots,p-1\}$ and $r_0\in\{0,1,\dots,q-1\}$. If $k_1,k_2\in\{0,1,\dots,p-1\}$, we have

$$(\mathcal{G}^*\mathcal{G})(t,v)_{k_1,k_2}=\sum_{r=0}^{q-1}\overline{\mathcal{G}(t,v)}_{r,k_1}\mathcal{G}(t,v)_{r,k_2}$$
$$=\sum_{r=0}^{q-1}\overline{(\mathcal{Z}_{aq}g)\left(t+\frac{k_1}{b}-ra\,,v\right)}(\mathcal{Z}_{aq}g)\left(t+\frac{k_2}{b}-ra,v\right).$$

Hence,

$$(\mathcal{G}^*\mathcal{G})\left(t+\frac{j}{bq},v\right)_{k_1,k_2}$$
$$=\sum_{r=0}^{q-1}\overline{(\mathcal{Z}_{aq}g)\left(t+\frac{j}{bq}+\frac{k_1}{b}-ra,v\right)}(\mathcal{Z}_{aq}g)\left(t+\frac{j}{bq}+\frac{k_2}{b}-ra,v\right)$$
$$=\sum_{r=0}^{q-1}\left\{\overline{(\mathcal{Z}_{aq}g)\left(t+\frac{k_0+k_1}{b}+\frac{mp}{b}-(r-r_0)a,v\right)}\right.$$
$$\left.\times(\mathcal{Z}_{aq}g)\left(t+\frac{k_0+k_2}{b}+\frac{mp}{b}-(r-r_0)a,v\right)\right\}.$$

Using Eq. (33) and the fact that $\frac{mp}{b}=maq$, this expression simplifies to

$$\sum_{r=0}^{q-1}\overline{(\mathcal{Z}_{aq}g)\left(t+\frac{k_0+k_1}{b}-(r-r_0)a,v\right)}(\mathcal{Z}_{aq}g)\left(t+\frac{k_0+k_2}{b}-(r-r_0)a,v\right)$$

or

$$\sum_{r=0}^{r_0-1}\left\{\overline{(\mathcal{Z}_{aq}g)\left(t+\frac{k_0+k_1}{b}-(r-r_0+q)a+qa,v\right)}\right.$$
$$\left.\times(\mathcal{Z}_{aq}g)\left(t+\tfrac{k_0+k_2}{b}-(r-r_0+q)a+qa,v\right)\right\}$$
$$+\sum_{r=r_0}^{q-1}\overline{(\mathcal{Z}_{aq}g)\left(t+\frac{k_0+k_1}{b}-(r-r_0)a,v\right)}(\mathcal{Z}_{aq}g)\left(t+\frac{k_0+k_2}{b}-(r-r_0)a,v\right).$$

Using the Eq. (33), we can rewrite this last expression as

$$\sum_{r=r_0}^{q-1}\overline{(\mathcal{Z}_{aq}g)\left(t+\frac{k_0+k_1}{b}-ra,v\right)}(\mathcal{Z}_{aq}g)\left(t+\frac{k_0+k_2}{b}-ra,v\right).$$

Using the fact that

$$(\mathcal{Z}_{aq}g)\left(t+\frac{k_0+k_1}{b}-ra,v\right)=e^{-2\pi iv}(\mathcal{Z}_{aq}g)\left(t+\frac{k_0+k_1-p}{b}-ra,v\right)$$

it follows thus that, for $k_1,k_2\in\{0,1,\dots\dots,p-1\}$, the entry $(\mathcal{G}^*\mathcal{G})\left(t+\frac{j}{bq},v\right)_{k_1,k_2}$ must be equal to

$$\begin{cases}(\mathcal{G}^*\mathcal{G})(t,v)_{k_1+k_0,\,k_2+k_0}, & \text{if } k_1+k_0<p \text{ and } k_2+k_0<p,\\ e^{-2\pi iv}(\mathcal{G}^*\mathcal{G})(t,v)_{k_1+k_0,\,k_2+k_0-p}, & \text{if } k_1+k_0<p \text{ and } k_2+k_0\ge p,\\ e^{-2\pi iv}(\mathcal{G}^*\mathcal{G})(t,v)_{k_1+k_0-p,\,k_2+k_0}, & \text{if } k_1+k_0\ge p \text{ and } k_2+k_0<p,\\ (\mathcal{G}^*\mathcal{G})(t,v)_{k_1+k_0-p,\,k_2+k_0-p}, & \text{if } k_1+k_0\ge p \text{ and } k_2+k_0\ge p.\end{cases}$$

If $\xi=(\xi_0,\dots,\xi_{p-1})^t\in\mathbb{C}^p$, define $U\xi=\eta=(\eta_0,\dots,\eta_{P-1})^t$, where

$$\eta_i=\begin{cases}e^{-2\pi iv}\xi_{i-k_0+p}, & \text{if } 0\le i\le k_0-1,\\ \xi_{i-k_0}, & \text{if } k_0\le i\le p-1.\end{cases}$$

Then, U is a $p\times p$ unitary matrix and

$$\langle(\mathcal{G}^*\mathcal{G})\left(t+\frac{1}{bq},v\right)\xi,\xi\rangle=\langle(\mathcal{G}^*\mathcal{G})(t,v)\eta,\eta\rangle=\langle(\mathcal{G}^*\mathcal{G})(t,v)U\xi,U\xi\rangle$$
$$=\langle U^*(\mathcal{G}^*\mathcal{G})(t,v)U\xi,\xi\rangle,$$

which shows that $(\mathcal{G}^*\mathcal{G})\left(t+\frac{j}{bq},v\right)=U^*(\mathcal{G}^*\mathcal{G})(t,v)U$ and thus $(\mathcal{G}^*\mathcal{G})\left(t+\frac{j}{bq},v\right)$ and $(\mathcal{G}^*\mathcal{G})(t,v)$ must have the same rank. Since the rank of the any matrix A is the same as that of A^*A, it follows that rank $\left(\mathcal{G}(t,v)\right)$ is $\frac{1}{bq}$-periodic with respect to variable t. Finally, it follows from Lemma (3.3.1) that $(\mathcal{Z}_{aq}g)\left(t+\frac{k}{b}-ra,v\right)=0$ if $\chi_S\left(t+\frac{k}{b}\right)=0$ so that a column of $(\mathcal{G})(t,v)$ corresponding to an index k such that $\chi_S\left(t+\frac{k}{b}\right)=0$ must be identically zero. The rank of $(\mathcal{G})(t,v)$ is then at most equal to the numbers of the other columns which is $\sum_{k=0}^{p-1}\chi_S\left(t+\frac{k}{b}\right)$.this shows the lemma.

The following result provides a characterization for the completeness of the span of Gabor system in $L^2(S)$ in terms of the matrix-valued function $\mathcal{G}$ associated with the window.

Theorem (3.3.7)[161]: Let $a,b>0$ satisfy $ab=\frac{p}{q}$ with $p,q\in\mathbb{N}$ and $\gcd(p,q)=1$. Let S be a measurable set in $\mathbb{R}$ with nonzero measure and suppose that S is a $\mathbb{Z}$-shift invariant. Assume that $g\in L^2(S)$ and let $\mathcal{G}(t,v)$ denote the $q\times p$ matrix-valued function defined by (36).then the following are equivalent:

(a) The linear span of the collection $\{E_{mb}T_{na}g: m,n\in\mathbb{Z}\}$ is dense in $L^2(S)$.

(b) $\text{Rank}\left(\mathcal{G}(t,v)\right)=\sum_{k=0}^{p-1}\chi s\left(t+\frac{k}{b}\right)$ for a.e. $(t,v)\in\left[0,\frac{1}{bq}\right)\times(0,1)$.

(c) $\text{Rank}\left(\mathcal{G}(t,v)\right)=\sum_{k=0}^{p-1}\chi s\left(t+\frac{k}{b}\right)$ for a.e. $(t,v)\in\mathbb{R}^2$.

Proof. The equivalence of (b) and (c) follows from the fact that the functions on either side of the equality in (c) are $\frac{1}{bq}$-periodic with respect to the first variable t, by Lemmas (3.3.4) and (3.3.6), and are also clearly 1-periodic with respect to the second variable v. Define $g_r(\cdot)=g(\cdot-ra)$ for $r=0,1,\dots,q-1$. Then, the linear span of the collection

$\{E_{mb}T_{naq}g_r : m,n \in \mathbb{Z}\}$ is dense in $L^2(S)$ if and only if the span of $\{E_{mb}T_{naq}g_r : 0 \le r \le q-1, m,n \in \mathbb{Z}\}$ is dense in $L^2(S)$. It is easy to check that

$$\left(\mathcal{Z}_{aq}E_{mb}T_{naq}g_r\right)(t,v) = e^{2\pi imbt}e^{2\pi inv}(\mathcal{Z}_{aq}g)(t-ra,v)$$

for $m,n \in \mathbb{Z}$ and $0 \le r \le q-1$. Applying Lemma (3.3.1), we have, for $f \in L^2(\mathbb{R})$, $m,n \in \mathbb{Z}$ and $0 \le r \le q-1$

$$\langle f, E_{mb}T_{naq}g_r\rangle = \int_0^{aq}\int_0^1 (\mathcal{Z}_{aq}f)(t,v)\overline{(\mathcal{Z}_{aq}g)(t-ra,v)}e^{-2\pi imbt}e^{-2\pi inv}dvdt$$

$$= \int_0^{\frac{1}{b}}\int_0^1 \sum_{k=0}^{p-1}(\mathcal{Z}_{aq}f)\left(t+\frac{k}{b},v\right)\overline{(\mathcal{Z}_{aq}g)(t+\frac{k}{b}-ra,v)} \times e^{-2\pi imbt}e^{-2\pi inv}dvdt. \tag{38}$$

If (c) holds, let $f \in L^2(S)$ satisfy that $\langle f, E_{mb}T_{naq}g_r\rangle = 0$ for all $m,n \in \mathbb{Z}$ and $0 \le r \le q-1$. We need to show that $f = 0$. for fixed $(t,v) \in \mathbb{R}^2$, let $F(t,v) = \left(F_0(t,v),\dots,F_{p-1}(t,v)\right)^t \in \mathbb{C}^p$ be defined by $F_i(t,v) = \overline{\left(\mathcal{Z}_{aq}f\right)}\left(t+\frac{i}{b},v\right)$ for $i = 0,\dots,p-1$. By (38), we have

$$\mathcal{G}(t,v)F(t,v) = 0, \quad \text{for a.e. } (t,v) \in \left[0,\frac{1}{b}\right) \times [0,1). \tag{39}$$

If B is a subset of $\{0,1,\dots,p-1\}$, we define

$$I_B = \left\{t \in \left[0,\frac{1}{b}\right), \sum_{k=0}^{p-1}\chi_S\left(t+\frac{k}{b}\right) = \sum_{k\in B}\chi_S\left(t+\frac{k}{b}\right) = \operatorname{card}(B)\right\}.$$

Then, each set I_B is measurable and the collection $\{I_B\}$, where B runs over all subsets of $\{0,1,\dots,p-1\}$, forms a partition of the interval $\left[0,\frac{1}{b}\right)$. If $B = \emptyset$ and $t \in I_\emptyset$, we have $\sum_{k=0}^{p-1}\chi_S\left(t+\frac{k}{b}\right) = 0$ and thus $F(t,v) = 0$ by Lemma (3.3.1), if $B \ne 0$ and $t \in I_B$, we have $\mathcal{G}(t,v)_{r,k} = F_k(t,v) = 0$ if $k \notin B$, by Lemma (3.3.1), since f and $g_r \in L^2(S)$, for each $r = 0,\dots,q-1$. Using our assumption, the sub matrix of $\mathcal{G}(t,v)$ of size $q \times \operatorname{card}(B)$, obtained by removing from $\mathcal{G}(t,v)$ all the columns with an index not in B has thus a rank equal to $\operatorname{card}(B)$, Since the entries corresponding to the removed columns are all zero, and Eq. (39) then implies that $F_k(t,v) = 0$, for $k \in B$. Hence, $F(t,v) = 0$, for $t \in I_B$. Therefore, $F(t,v) = 0$ for $t \in \left[0,\frac{1}{b}\right)$, which shows that $\left(\mathcal{Z}_{aq}f\right)(t,v) = 0$ for $t \in \left[0,\frac{p}{b}\right) = [0,aq)$ and thus that $f = 0$, using Lemma (3.3.1) again.

Conversely, if (c), or equivalently, (b) fails, then, taking into account inequality (37), we deduce the existence of a subset B of $\{0,1,\dots.,p-1\}$ such that

$$\operatorname{Rank}\left(\mathcal{G}(t,v)\right) < \sum_{k=0}^{p-1}\chi_S\left(t+\frac{k}{b}\right), \quad \text{for a.e. } (t,v) \in H, \tag{40}$$

where H is a measurable subset of $I_B \times [0,1)$ with nonzero measure. Let $e_0, e_1, \dots, e_{p-1}$ denote the standard orthonormal basis of $\mathbb{C}^p$ and let $p(t,v): \mathbb{C}^p \to \mathbb{C}^p$ denote the orthogonal projection onto the kernel of $\mathcal{G}(t,v)$. Then, $\mathcal{P}(\cdot,\cdot)$ is measurable by Lemma

(3.3.6). We claim that there exists $k_0 \in B$ such that the vector-valued function $p(t,v)e_{k_0} \neq 0$ on a subset of $I_B \times [0,1)$ having positive measure. Indeed if this were not the case, letting $E = \text{span}\{e_k : k \in B\}$, it would follow that for a.e. $(t,v) \in I_B \times [0,1), p(t,v)x = 0$, for all $x \in E$ or equivalently, that $E \oplus \ker(\mathcal{G}(t,v))$ is a direct sum. Since, in that case,

$$p \geq \dim\big(E \oplus \ker(\mathcal{G}(t,v))\big) = \text{card}(E) + \Big(p - \text{rank}\big(\mathcal{G}(t,v)\big)\Big),$$

this would imply that $\text{rank}(\mathcal{G}(t,v)) \geq \text{card}(E)$ and thus, using the definition of I_B, that

$$\text{rank}\big(\mathcal{G}(t,v)\big) = \sum_{k=0}^{p-1} \chi_S\left(t + \frac{k}{b}\right) \quad \text{for a.e.}\,(t,v) \in I_B \times [0,1).$$

This would contradict (40). With $k_0 \in B$ as a above, we define

$$F(t,v) = \begin{cases} p(t,v)e_{k_0}, & \text{if}\,(t,v) \in I_B \times [0,1), \\ 0, & \text{if}\,(t,v) \in \left(\left[0,\frac{1}{b}\right) \backslash I_B\right) \times [0,1). \end{cases}$$

By construction, we have $\|F(t,v)\|_{\mathbb{C}^p} \leq 1$, so that each component of F is square-integrable on $\left[0,\frac{1}{b}\right) \times [0,1)$ and $F = (F_0, \dots, F_{p-1})^t$ satisfies Eq. (39). Furthermore, if $l \in \{0,1,\dots,p-1\}\backslash B$ and$(t,v) \in I_B \times [0,1)$, we have that $\mathcal{G}(t,v)e_l = 0$ and thus

$$\langle \mathcal{P}(t,v)e_{k_0}, e_l\rangle = \langle e_{k_0}, \mathcal{P}(t,v)e_l\rangle = \langle e_{k_0}, e_l\rangle = 0.$$

This shows that, if $(t,v) \in \left[0,\frac{1}{b}\right) \times [0,1)$, we must have $F_k(t,v) = 0$ whenever $\chi_S\left(t + \frac{k}{b}\right) = 0$. Defining $f \in L^2(\mathbb{R})$ by

$$\left(\mathcal{Z}_{aq}f\right)\left(t + \frac{k}{b}, v\right) = F_k(t,v), \;\; (t,v) \in \left[0,\frac{1}{b}\right) \times [0,1), k = 0,1,\dots.,p-1,$$

we have $f \neq 0$ and $\left\{\mathcal{Z}_{aq}f \neq 0\right\} \subset S_0 \times [0,1)$. Hence, f belong to $L^2(S)$ by Lemma (3.3.1), and, furthermore, using (39), f is orthogonal to the Collection $\left\{E_{mb}T_{nag} : m,n \in \mathbb{Z}\right\}$. Hence, (a) fails which completes the proof.

In the case, $S = \mathbb{R}$, the density condition in the theorem just proved reduces to $\text{rank}(\mathcal{G}) = p$ a.e, a condition which also must hold for the Zibulski-Zeevi matrix ([166,180]; see also [173]). As in the case $S = \mathbb{R}$, this result has an important consequence.

Corollary (3.3.8)[161]: Let $a,b > 0$ satisfy $ab = \frac{p}{q}$ with $p,q \in \mathbb{N}$ and $\gcd(p,q) = 1$ and let S be an $\mathbb{Z}$-shift invariant, measurable subset of $\mathbb{R}$. If there exists a function $g \in L^2(S)$ such that the linear span of the collection $\left\{E_{mb}T_{nag} : m,n \in \mathbb{Z}\right\}$ is dense in $L^2(S)$, then, necessarily

$$\sum_{k=0}^{p-1} \chi_S\left(t + \frac{k}{b}\right) \leq q \quad \text{for a.e. } t \in \mathbb{R}. \tag{41}$$

Proof. By the previous theorem, we have, for a.e. $t \in \mathbb{R}$,

$$\sum_{k=0}^{p-1} \chi_S\left(t + \frac{k}{b}\right) = \text{Rank}\big(\mathcal{G}(t,v)\big) \leq q,$$

Since $\mathcal{G}(t,v) \in \mathcal{M}_{q,p}$.

We show that in the rational case $(ab \in Q)$, condition (41) is sufficient to ensure the existence of function $g \in L^2(S)$ such that the collection $\{E_{mb}T_{nag}: m, n \in \mathbb{Z}\}$ not only has a dense linear span in $L^2(S)$,but forms a tight frame for $L^2(S)$.In fact, we will see that this can be done with g of the form $g = \chi_E$, where E is a subset of S such that $\chi_S = \sum_{n\in\mathbb{Z}} \chi_E(\cdot - na)$. On the other hand, if E is a set satisfying the previous identity and $g = \chi_E$, it is clear that $\{E_{mb}T_{nag}: m, n \in \mathbb{Z}\}$ forms a tight frame for $L^2(S)$ if and only if $\{E_{mb}T_{nag}: m, n \in \mathbb{Z}\}$ is a tight frame for $L^2(E)$. The next Lemma translates this last requirement into geometrical terms. We first need the following definition.

Definition (3.3.9)[161]: Given $a > 0$, two measurable subsets of $\mathbb{R}$, E_1 and E_2, are said to be a $\mathbb{Z}$-congruent if there exist measurable partitions $\{E_{1,k}\}_{k\in\mathbb{Z}}$ of $\{E_{2,k}\}_{k\in\mathbb{Z}}$ of E_2 such that $E_{2,k} = E_{1,k} + ka$, modulo a set of zero measure, for all $k \in \mathbb{Z}$. It is easy to check that E_1 and E_2 are a $\mathbb{Z}$-translation congruent if and only if the identity

$$\sum_{k\in\mathbb{Z}} \chi_{E_1}(\cdot - ak) = \sum_{k\in\mathbb{Z}} \chi_{E_2}(\cdot - ak)$$

Holds a.e. on $\mathbb{R}$.

Lemma (3.3.10)[161]: Let $b > 0$, and E be a measurable subset of $\mathbb{R}$. Then, the following conditions are equivalent:

(a) $\{E_{mb}\chi_E: m \in \mathbb{Z}\}$ is a frame for $L^2(E)$.

(b) The linear span of the collection $\{E_{mb}\chi_E: m \in \mathbb{Z}\}$ is dense in $L^2(E)$.

(c) E is $\frac{1}{b}\mathbb{Z}$-congruent to a subset of $\left[0, \frac{1}{b}\right)$.

(d) $\sum_{k\in\mathbb{Z}} \chi_E\left(\cdot + \frac{k}{b}\right) \le 1$ a.e on $\mathbb{R}$.

In addition, $\{E_{mb}\chi_E: m \in \mathbb{Z}\}$ is a tight frame in $L^2(E)$ with frame bound $\frac{1}{b}$ if any of the above conditions holds.

Proof. If is clear that (a) implies (b). To show that (b) implies (c), assume that (b) holds and define $E_1 = E \cap [l/b, (l + I/b)$ for $\in \mathbb{Z}$. then $\{E_1: l \in \mathbb{Z}\}$ is a partition of E and $\cup_{l\in\mathbb{Z}}(E_1 - l/b) \subset \left[0, \frac{1}{b}\right)$. So it suffices to show that $|(E_1 - l/b) \cap (E_k - k/b)| = 0$ for $l \ne k, l, k \in \mathbb{Z}$. If this were not the case, there would exist $l_0, k_0 \in \mathbb{Z}$ with $l_0 \ne k_0$ such that $F \coloneqq (E_{l_0} - l_0/b) \cap (E_{k_0} - k_0/b)$ has positive measure. Define $f \in L^2(E)$ by

$$f = \begin{cases} 1 & \text{on } F + l_0/b\,; \\ -1 & \text{on } F + k_0/b; \\ 0 & \text{on } E \setminus [(F + l_0/b) \cup (F + k_0/b]. \end{cases}$$

Then, for $m \in \mathbb{Z}$,

$$\int_E f(x)e^{-2\pi imbx}dx = \int_{F+l_0/b} [f(x) + f(x + (k_0 - l_0)/b]e^{-2\pi imbx}dx = 0$$

contradicting the fact that the linear span of the collection $\{E_{mb}\chi_E: m \in \mathbb{Z}\}$ is dense in $L^2(E)$.

The equivalence of (c) and (d) is clear. To finish the proof, we show that (c) implies that $\{E_{mb}\chi_E: m \in \mathbb{Z}\}$ is a tight frame in $L^2(E)$ with frame bound $\frac{1}{b}$ and thus also statement (a). Suppose $\{E_l: l \in \mathbb{Z}\}$ is a partition of E such that $\{E_l - l/b : l \in \mathbb{Z}\}$ is a partition of some subset of $\left[0, \frac{1}{b}\right)$. then for $f \in L^2(E)$, we have

$$\langle f, E_{mb}\chi_E\rangle = \sum_{l\in\mathbb{Z}} \int_{E_l - l/b} f(x - l/b)e^{-2\pi imbx}dx$$

$$= \int_{\left[0,\frac{1}{b}\right)} \sum_{l\in\mathbb{Z}} f(x - l/b)\chi_{E_1 - l/b}(x)e^{-2\pi imbx}dx$$

and, consequently, using Parsval's formula,

$$\sum_{m\in\mathbb{Z}} |\langle f, E_{mb}\chi_E\rangle|^2 = \frac{1}{b} \int_{\left[0,\frac{1}{b}\right)} \left|\sum_{l\in\mathbb{Z}} f(x - l/b)\chi_{E_1 - l/b}(x)\right|^2 dx$$

$$= \frac{1}{b}\sum_{l\in\mathbb{Z}} \int_{E_l - l/b} |f(x - l/b)|^2\, dx = \frac{1}{b}\int_E |f(x)|^2 dx,$$

which completes the proof.

In connection with the previous lemma, we mention the following particular case of a well-known result about spectral pairs due to Fuglede [170].

Proposition (3.3.11)[161]: Let $b > 0$, and E be a measurable subset of $\mathbb{R}$. Then, the following conditions are equivalent:

(a) $\left\{\sqrt{b}E_{mb}\chi_E: m \in \mathbb{Z}\right\}$ is an orthonormal basis for $L^2(E)$.

(b) $\sum_{k\in\mathbb{Z}} \chi_E\left(\cdot + \frac{k}{b}\right) = 1,$ a. e on $\mathbb{R}$.

Theorem (3.3.12)[161]: let $a, b > 0$ satisfy $ab = \frac{p}{q}$ with $p, q \in \mathbb{N}$ and $\gcd(p, q) = 1$. Let S be a measurable subset of $\mathbb{R}$ which is a $\mathbb{Z}$-shift invariant. Then, the following are equivalent.

(i) There exists a measurable set E in $\mathbb{R}$ which is a $\mathbb{Z}$-congruent to $S\cap[0,1)$ such that $\{E_{mb}T_{na}\chi_E: m, n \in \mathbb{Z}\}$ is a tight frame for $L^2(S)$.

(ii)we have the inequality

$$\sum_{k=0}^{p-1} \chi_S\left(\cdot + \frac{k}{b}\right) \le q \quad \text{a. e on } \mathbb{R}. \tag{42}$$

Proof. The necessity of condition (42) is a direct consequence of Corollary (3.3.8) it can also be obtained by the following, more direct, observation. The facts that E is a $\mathbb{Z}$-congruent to $S \cap [0, a)$ and that $\{E_{mb}T_{na}\chi_E: m, n \in \mathbb{Z}\}$ is a tight frame for $L^2(S)$ are equivalent, by Lemma (3.3.10), to

$$\sum_{k=0} \chi_E(\cdot + ka) = \chi_S \quad \text{and} \quad \sum_{k\in\mathbb{Z}} \chi_E(\cdot + k/b) \le 1 \quad \text{a.e on } \mathbb{R},$$

respectively. Therefore, we have

$$\sum_{k=0}^{p-1}\chi_S\left(\cdot+\frac{k}{b}\right)=\sum_{k=0}^{p-1}\sum_{l\in\mathbb{Z}}\chi_E\left(\cdot+\frac{lp}{qb}+\frac{k}{b}\right)=\sum_{j\in\mathbb{Z}}\chi_E\left(\cdot+\frac{j}{qb}\right)=\sum_{r=0}^{q-1}\sum_{k\in\mathbb{Z}}\chi_E\left(\cdot+\frac{r}{qb}+\frac{k}{b}\right)$$
$$\leq q.$$
Let $S_0=S\cap[0,a)$. Clearly, $\{S_0+na : n\in\mathbb{Z}\}$ is a partition of S. to show the sufficiency part of the statement, we need to show, according to Lemma (3.3.10), that there exists a measurable set E in $\mathbb{R}$ such that E is a $\mathbb{Z}$-congruent to S_0 and at the same time is $\frac{1}{b}\mathbb{Z}$-congruent to a subset of $[0,1/b)$. If $ab\leq 1$, we can take $E=S_0$ Since $S_0\subset[a,0)\subset[0,{}^{1}/_{b})$ .We can also reduce the proof of the construction of E to the case $b=1$. Indeed, if b is arbitrary, we can define $\check{S}=bS$. then $\check{S}$ is $ab\mathbb{Z}$-shift invariant, and $\{\check{S}_0+nab: n\in\mathbb{Z}\}$ is a partition of $\check{S}$, where $\check{S}_0=\check{S}\cap[0,ab)$. Furthermore, $\check{S}$ satisfies (42) with b replaced by 1. So, if we can construct a measurable set $\check{E}$ such that $\check{E}$ is $ab\mathbb{Z}$-congruent to $\check{S}_0$, and with $\check{E}$ being $\mathbb{Z}$-congruent to a subset of $[0,1)$, we can then define $E=\frac{1}{b}\check{E}$. We can easily check that E satisfies all of our requirements. We may thus assume, without loss of generality, that $b=1$ and $a>1$. We have $a=\frac{p}{q}$, and, using (35) with $b=1$ and (42), it follows that
$$\sum_{k=0}^{p-1}\chi_S(\cdot+k)=\sum_{k\in\mathbb{Z}}\chi_{S_0}\left(\cdot+\frac{k}{q}\right)\leq q. \tag{43}$$
Note that, since
$$|S_0|=\int_{\left[0,\frac{1}{q}\right)}\sum_{k\in\mathbb{Z}}\chi_{S_0}(x+k/q)dx,$$
inequality (43) implies that $|S_0|\leq 1$ as well as the fact that $|S_0|=1$ if and only if
$$\sum_{k=0}^{p-1}\chi_S(\cdot+k)=q \quad \text{a.e. on } \mathbb{R}. \tag{44}$$
We will divide the proof into two cases: $|S_0|=1$ and $|S_0|<1$.

Case 1: $|S_0|=1$.

In this case, identity (44) holds. Let $g_0=\sum_{k\in\mathbb{Z}}\chi_{S_0}(\cdot+k)$. If $g_0\leq 1$ a.e. on $\mathbb{R}$, then $E=S_0$ is as desired. Otherwise, we follow the following inductive procedure to construct E. Let $H_0=\{g_0>0\}$, where $\{g_0>0\}$ denotes the set $\{t\in\mathbb{R} : g_0(t)>0\}$. (We will use similar notations for the sets H_k defined below). Let T_0 be a measurable subset of S_0 such that
$$\sum_{k\in\mathbb{Z}}\chi_{T_0}(\cdot+k)=\chi_{H_0}.$$
If the sets S_l, T_l, H_l and the function g_l have already been constructed for all indices l with $0\leq l\leq j-1<q-1$, we define $S_j=S_{j-1}\backslash T_{j-1}$,
$$g_j=\sum_{k\in\mathbb{Z}}\chi_{S_j}(\cdot+aj+k),$$

and $H_j = \{g_j > 0, g_0 = g_1 = \cdots = g_{j-1} = 0\}$. We then choose a measurable set T_j contained in S_j and such that

$$\sum_{k\in\mathbb{Z}} \chi_{T_j}(\cdot + aj + k) = \chi_{H_j}.$$

We follow this procedure until the index j above reaches $q-1$ and then stop. We then define $E = \bigcup_{i=0}^{q-1}(T_i - ia)$. Note that the set $H_i, i = 0, \dots, q-1$, are mutually disjoint. We have thus

$$\sum_{k\in\mathbb{Z}} \chi_E(\cdot + k) = \sum_{i=0}^{q-1}\sum_{k\in\mathbb{Z}} \chi_{T_i}(\cdot + ia + k) = \sum_{i=0}^{q-1} \chi_{H_i} \le 1 \qquad \text{a.e. on } \mathbb{R}.$$

So, by Lemma (3.3.10), the collection $\{e^{2\pi imt}\chi_E(t) : m \in \mathbb{Z}\}$ is a tight frame for $L^2(E)$. Since the sets $T_i, i = 0, \dots, q-1$ are disjoint subsets of $S_0 \subset [0, a)$, in order to show that E is a $\mathbb{Z}$-congruent to S_0, we only need to show that $\sum_{i=0}^{q-1}|T_i| = |S_0| = 1$. We have

$$\sum_{i=0}^{q-1}|T_i| = \sum_{i=0}^{q-1} \int_{[0,1)} \sum_{k\in\mathbb{Z}} \chi_{T_i}(t + ia + k)dt = \int_{[0,1)} \sum_{i=0}^{q-1} \chi_{H_i(t)dt}$$

and since $\sum_{i=0}^{q-1} \chi_{H_i(t)} \le 1$ a.e. on $\mathbb{R}$, it suffices to show that $\sum_{i=0}^{q-1} \chi_{H_i} = 1$ a.e. on $[0,1)$. We will argue by contradiction. Suppose that there exists a measurable set $F \subset [0,1)$ with nonzero measure such that $g_i = 0$ on F for all indices i with $0 \le i \le q-1$. If $q = 1$, then $\sum_{k\in\mathbb{Z}} \chi_{S_0}(\cdot + k) = g_0 = 0$ on F, contradicting the fact that $\sum_{k\in\mathbb{Z}} \chi_{S_0}(\cdot + k) = 1$ a.e. on $\mathbb{R}$ (which follows from (43) and (44)). If $q > 1$, we have

$$g_0 = \sum_{k\in\mathbb{Z}} \chi_{S_0}(\cdot + k) = 0 \quad \text{on } F$$

and, since $\chi_{S_l} = \chi_{S_0} - \sum_{i=o}^{l-1} \chi_{T_i}$, the fact that $g_l = 0$ on F for $1 \le l \le q-1$ is equivalent to

$$\sum_{k\in\mathbb{Z}} \chi_{S_0}(\cdot + la + k) = \sum_{i=0}^{l-1}\sum_{k\in\mathbb{Z}} \chi_{T_i}(\cdot + la + k) = \sum_{i=0}^{l-1} \chi_{H_i}(\cdot + (l-i)a)$$

on F for $l = 1, 2, \dots, q-1$. Also observing that

$$\sum_{k=0}^{p-1} \chi_S(\cdot + k) = \sum_{r=0}^{q-1}\sum_{k=0}^{p-1}\sum_{l\in\mathbb{Z}} \chi_{S_0}\left(\cdot + \frac{(lq+r)p}{q} + k\right) = \sum_{r=0}^{q-1}\sum_{k\in\mathbb{Z}} \chi_{S_0}(\cdot + ra + k),$$

We have

$$\sum_{k=0}^{p-1} \chi_S(\cdot + k) = \sum_{r=1}^{q-1}\sum_{i=0}^{r-1} \chi_{H_i}(\cdot + (r-i)a) = \sum_{l=1}^{q-1}\sum_{i=0}^{q-l-1} \chi_{H_i}(\cdot + la) \le q-1 < q$$

on F, which contradicts (44) .

Case 2: $|S_0| < 1$.

For $\boldsymbol{\epsilon} = (\epsilon_0, \epsilon_1, \dots, \epsilon_{p-1}) \in \{0,1\}^p$, define

$$A(\boldsymbol{\epsilon}) = \left\{t \in [0, 1/q) : \chi_{S_0}\left(t + \frac{k}{q}\right) = \epsilon_k \text{ for } 0 \le k \le p-1\right\}.$$

If $\sum_{k=0}^{p-1} \epsilon_k = m < q$, we choose $m - q$ indices $k_i, i = 1, \dots, q-m$, such that $\epsilon_{k_1} = \epsilon_{k_2} = \cdots = \epsilon_{k_{q-m}} = 0$, which can be done since $p > q$, and we define

$$B(\epsilon)=\left(\bigcup_{l=1}^{q-m}\left(A(\epsilon)+\frac{k_l}{q}\right)\right)\bigcup\left(\bigcup_{\epsilon_k\neq 0}\left(A(\epsilon)+\frac{k}{q}\right)\right).$$

If $\sum_{k=0}^{p-1}\epsilon_k=q$, define

$$B(\epsilon)=\bigcup_{\epsilon_k\neq 0}\left(A(\epsilon)+\frac{k}{q}\right).$$

We then let

$$\check{S}_0=\bigcup_{\epsilon\in\{0,1\}^P}B(\epsilon),\quad\text{and}\quad \check{S}=\bigcup_{n\in\mathbb{Z}}(\check{S}_0+na).$$

Note that, by construction, $S_0\subset\check{S}_0\subset[0,a)$. Furthermore, we have

$$\sum_{k\in\mathbb{Z}}\chi_{\check{S}_0}\left(\cdot+\frac{k}{q}\right)=\sum_{k=0}^{p-1}\chi_{\check{S}}(\cdot+k)=q\quad\text{a.e. on }\mathbb{R},$$

Which implies, as before that $|\check{S}_0|=1$. Using Case 1, with S and S_0 replaced with $\check{S}$ and $\check{S}_0$, respectively, we can construct a measurable set $\tilde{E}$ which is a $\mathbb{Z}$-congruent to $\check{S}\bigcap[0,a)$ and such that $\left\{e^{2\pi imt}\chi_{\tilde{E}}(t): m\in\mathbb{Z}\right\}$ is a tight frame for $L^2(\tilde{E})$. The collection $\left\{e^{2\pi imt}\chi_{\tilde{E}}(t-na): m,n\in\mathbb{Z}\right\}$ is thus a tight frame for $L^2(\tilde{S})$ and the set $E:=\tilde{E}\cap S$ satisfies all of our requirements.

Corollary (3.3.13)[161]: Let $a,b>0$ satisfy $ab=\frac{P}{q}$ with $p,q\in\mathbb{N}$ and $\gcd(p,q)=1$ and let S be an a $\mathbb{Z}$-shift invariant, measurable subset of $\mathbb{R}$ and define the set $S_0=S\cap[0,a)$ then, the following are equivalent:

(a) There exists a function $g\in L^2(S)$ such that the linear span of the collection $\{E_{mb}T_{na}\chi_E: m,n\in\mathbb{Z}\}$ is dense in $L^2(S)$.

(b) There exists a measurable set E in $\mathbb{R}$ which is a $\mathbb{Z}$-congruent to $S\cap[0,a)$ such that $\{E_{mb}T_{na}\chi_E: m,n\in\mathbb{Z}\}$ is a tight frame for $L^2(S)$.

(c) $\sum_{k=0}^{p-1}\chi_S\left(\cdot+\frac{k}{b}\right)\leq q$ a.e on $\mathbb{R}$.

In particular, if any of the conditions above holds, then we must have the inequality

$$b|S_0|\leq 1. \tag{45}$$

Proof. The equivalence between statements (a), (b), and (c) follows immediately from Corollary (3.3.8), and Theorem (3.3.12). If (c) holds, then we must have

$$\int_0^{1/b}\sum_{k=0}^{p-1}\chi_S\left(t+\frac{k}{b}\right)dt=\int_0^{p/b}\chi_S(t)dt=\int_0^{a\,q}\chi_S(t)dt=q|S_0|\ \leq\frac{q}{b},$$

which yields inequality (45).

It was showed in [172] that the existence of a function $g\in L^2(S)$ such that the linear span of $\{E_{mb}T_{na}\mathrm{g}: m,n\in\mathbb{Z}\}$ is dense in $L^2(S)$ implies inequality (45), even without the restriction that $ab\in Q$.in the rational case, it is clear that condition (42) is always stronger than condition (45) when $ab=p/q>1$. (Note that both conditions are always clearly satisfied when $ab=p/q\leq 1$). For example, if we define, for $0<\epsilon<\min\left(\frac{1}{bq},\frac{1}{p}\right)$, the set

$$S=\bigcup_{l\in\mathbb{Z}}\left\{\bigcup_{k=0}^{p-1}\left[\frac{k}{b},\frac{k}{b}+\epsilon\right)+la\right\},$$

we have $b|S_0| = \epsilon p \le 1$, but condition (42) clearly fails when $p > q$. However, in the irrational case, condition (45) turns out to be necessary and sufficient for the existence of a function $g \in L^2(S)$ such that the linear span of the collection $\{E_{mb}T_{na}g: m, n \in \mathbb{Z}\}$ is dense (or, forms a tight frame) in $L^2(S)$.

Remark (3.3.14)[161]: In the rational case, if we consider two Gabor systems associated with a fixed set S and with the same parameter a but with a different parameter b, say b_1, and b_2, the fact that density condition (42) holds for the pair (a, b_1) does not imply that it also holds for the pair (a, b_2) when $b_2 < b_1$. For example, if we choose $a = 1$, $b_1 = 5/2$, $b_2 = 2$ and

$$S = \bigcup_{n\in\mathbb{Z}} ([n, 1/10 + n) \cup [5/10 + n, 6/10 + n)),$$

condition (42) holds for the pair $(a, b_1) = (1, 5/2)$ since

$$\sum_{k=0}^{4} \chi_S\left(\cdot + \frac{2k}{5}\right) = 1 \le 2,$$

While it does not for the pair $(a, b_2) = (1,2)$ in view of the fact that

$$\sum_{k=0}^{1} \chi_S\left(\cdot + \frac{k}{2}\right) = 2 > 1$$

on the interval $\left(0, \frac{1}{10}\right)$.

We provide a simple proof that the condition $b|S \cap [a, 0)| \le 1$ is necessary, in both the rational and irrational cases, for the existence of a window such that the associated Gabor system with parameters a, b forms a frame for $L^2(S)$. a although the necessity of this condition was obtained earlier in [172, Corollary 2.4] ,the proof given there, based on methods of operator algebras, is less transparent. Furthermore, we show that, if such a system exists, it will be a Riesz basis for $L^2(S)$ if and only if equality occurs in the condition above. We first need the following lemmas. The first one of these is well-known [165, Proposition 2.1].

Lemma (3.3.15)[161]: If $g \in L^\infty(\mathbb{R})$ is compactly supported, there is a constant C such that

$$\sum_{m,n\in\mathbb{Z}} |\langle f, E_{mb}T_{na}\rangle|^2 \le C\|f\|_2^2, \quad f \in L^2(\mathbb{R}).$$

The second lemma deals with a version of the Walnut representation which was showed in [174, Proposition 7.1.1] under the assumption that the collections $\{E_{mb}T_{na}\, g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}\, \gamma\}_{m,n\in\mathbb{Z}}$ are both Bessel sequences. As we will show here, these conditions on g and γ are not necessary.

Lemma (3.3.16)[161]: Let $a, b > 0$, and $g, \gamma \in L^2(\mathbb{R})$. Then, for $f, h \in L^\infty(\mathbb{R})$ with bounded support,

$$\sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}\, g\rangle\langle E_{mb}T_{na}\, \gamma, h\rangle = \frac{1}{b}\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} G_n(x) f(x - n/b)\,\overline{h(x)}dx, \qquad (46)$$

Where $G_n = \sum_{k\in\mathbb{Z}} \overline{g\left(\cdot - \frac{n}{b} - ka\right)}\gamma(\cdot - ka)$ and both series in (46) converge absolutely.

Proof. By a simple computation, we have

$$\langle f, E_{mb}T_{na}g\rangle = \int\limits_{\left[0,\frac{1}{b}\right)} \sum_{l\in\mathbb{Z}} (T_{na}\overline{g}f)(x - l/b)e^{-2\pi imbx}dx,$$

$$\langle h, E_{mb}T_{na}\gamma\rangle = \int\limits_{\left[0,\frac{1}{b}\right)} \sum_{l\in\mathbb{Z}} (T_{na}\overline{\gamma}h)(x - l/b)e^{-2\pi imbx}dx.$$

Also, observing that the sum of the series in both integrals above define functions in $L^2\left(\left[0,\frac{1}{b}\right)\right)$ since both f and h have bounded support, we have, by Parseval's formula and the fact that both $f$and h are bounded support,

$$\sum_{m\in\mathbb{Z}} \langle f, E_{mb}T_{na}g\rangle\langle E_{mb}T_{na}\gamma, h\rangle$$

$$= \frac{1}{b}\int\limits_{\left[0,\frac{1}{b}\right)} \left[\sum_{l_1\in\mathbb{Z}} (T_{na}\overline{g}f)(x - l_1/b)\right]\left[\sum_{l_2\in\mathbb{Z}} (T_{na}\gamma\bar{h})(x - l_2/b)\right]dx$$

$$= \frac{1}{b}\int\limits_{\mathbb{R}} \left(\sum_{l\in\mathbb{Z}} (T_{na}\overline{g}f)(x - l/b)\right)T_{na}\gamma(x)\overline{h(x)}dx$$

$$= \frac{1}{b}\int\limits_{\mathbb{R}} \sum_{l\in\mathbb{Z}} \overline{g(x - na - l/b)}\gamma(x - na)f(x - l/b)\overline{h(x)}dx.$$

Identity (46) follows immediately. The boundedness of the support of both f and h shows that The series on the right-hand side of (46) converges absolutely since it is actually a finite sum. The absolute convergence of the series on the left-hand side of (46) is obtained by the following argument. By Lemma (3.3.15), both $\{E_{mb}T_{na}f : m, n \in \mathbb{Z}\}$ and $\{E_{mb}T_{na}h : m, n \in \mathbb{Z}\}$ are Bessel sequences in $L^2(\mathbb{R})$. Noting that

$$\langle f, E_{mb}T_{na}g\rangle = e^{-2\pi imnab}\langle E_{-mb}T_{-na}f, g\rangle, \quad m, n \in \mathbb{Z},$$

and

$$\langle h, E_{mb}T_{na}\gamma\rangle = e^{-2\pi imnab}\langle E_{-mb}T_{-na}h, \gamma\rangle, \quad m, n \in \mathbb{Z},$$

it follows that both $\{\langle f, E_{mb}T_{na}g\rangle\}_{m,n\in\mathbb{Z}}$ and $\{\langle h, E_{mb}T_{na}\gamma\rangle\}_{m,n\in\mathbb{Z}}$ are in $l^2(\mathbb{Z}^2)$. The series on the left-hand side of (46) thus converges absolutely using the Cauchy-Schwarz inequality, showing our claim.

Note that part (a) in the following theorem follows from [172, Corollary 3.3.6] under the weaker assumption that the corresponding system is complete in $L^2(S)$. However, as mentioned earlier, that result was obtained by more abstract methods of operator algebras and we prefer to give here a more direct proof of this result (which is needed to show part (b) in any case) under the assumption that the system forms a frame. (See [164] for a similar proof in the case $S = \mathbb{R}$).

Theorem (3.3.17)[161]: Let $a, b > 0$, let S be an $a\mathbb{Z}$-shift invariant measurable set in $\mathbb{R}$ with nonzero measure, and suppose that $\{E_{mb}T_{na}h : m, n \in \mathbb{Z}\}$ is a frame for $L^2(S)$. Then,

(a) $b|S_0| \leq 1, where\ S_0 = S \cap [0, a)$.

(b) $\{E_{mb}T_{na}g : m,n \in \mathbb{Z}\}$ is a Riesz basis for $L^2(S)$ if and only if $b|S_0| = 1$.

Proof. We denote by S the frame operator:

$$Sf = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}g\rangle E_{mb}T_{na}g, \quad f \in L^2(S)$$

and $\gamma^0 = S^{-1}g$. Then

$$f = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}g\rangle E_{mb}T_{na}\gamma^0, \qquad f \in L^2(S). \tag{47}$$

Define $G_n = \sum_{k\in\mathbb{Z}} \overline{g(\cdot - n/b - ka)}\,\gamma^0(\cdot - ka)$ for each $n \in \mathbb{Z}$. Suppose $f, h \in L^\infty(\mathbb{R})$ both vanish outside the set $S_0 \cap I$, where I is an interval of length I/b. Note that $T_{la}f$ and $T_{la}h$ both belong to $L^2(S)$ $whenever$ $l \in \mathbb{Z}$. It follows from identity (47) and Lemma (3.3.16) that

$$\langle f, h\rangle = \langle T_{la}f, T_{la}h\rangle = \sum_{m,n\in\mathbb{Z}} \langle T_{la}f, E_{mb}T_{na}g\rangle\langle E_{mb}T_{na}\gamma^0, T_{la}h\rangle$$

$$= \frac{1}{b}\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} G_n(x) f(x - la - n/b)\overline{h(x - la)}dx$$

$$= \frac{1}{b}\int_{\mathbb{R}} G_0(x) f(x - la)\overline{h(x - la)}dx = \frac{1}{b}\int_{S_0\cap I} G_0(x + la) f(x)\overline{h(x)}dx.$$

Since f and g are arbitrary functions in $L^\infty(\mathbb{R})$ vanishing outside the set $S_0 \cap I$, it follows that $G_0(\cdot + la) = b$ a.e on $S_0 \cap I$ and thus also on S_0, since I is an arbitrary interval of length I/b. Hence, $G_0 = b$ a.e on S and, since the functions $T_{na}\,\gamma^0, n \in \mathbb{Z}$, all belong to $L^2(S), G_0$ vanishes outside of S. Hence, we conclude that $G_0 = b_{\chi_S}$. This implies, in particular, that

$$b|S_0| = \int_{[0,a)} b_{\chi_S}(x)dx = \int_{[0,a)} G_0(x)\,dx = \int_{[0,a)}\sum_{k\in\mathbb{Z}} \overline{g(x - ka)}\gamma^0(x - ka)dx$$

$$= \int_{\mathbb{R}} \overline{g(x)}\,\gamma^0(x)dx = \langle S^{-1}g, g\rangle = \langle S^{-1/2}g, S^{-1/2}g\rangle = \left\|S^{-1/2}g\right\|^2.$$

it follows that

$$\left\|S^{-1/2}g\right\|^2 = b|S_0|. \tag{48}$$

Since $\{E_{mb}T_{na}g : m,n \in \mathbb{Z}\}$ is a frame for $L^2(S)$. It is well-known that the collection $\{S^{-1}E_{mb}T_{na}g : m,n \in \mathbb{Z}\}$ which can also be written as $\left\{E_{mb}T_{na}S^{-1/2}g : m,n \in \mathbb{Z}\right\}$ is a Parseval tight frame for $L^2(S)$. This implies that $\left\|S^{-1/2}g\right\|^2 \le 1$, which together with (48) shows that $b|S_0| \le 1$, and shows (a). the collection $\{E_{mb}T_{na}g : m,n \in \mathbb{Z}\}$ forms a Riesz basis for $L^2(S)$, if and only if the parseval tight frame $\left\{E_{mb}T_{na}S^{-1/2}g : m,n \in \mathbb{Z}\right\}$ is an orthonormal basis for $L^2(S)$, which is equivalent to $\left\|S^{-1/2}g\right\|^2 = 1$ or to $b|S_0| = 1$, using (48). This shows (b) and completes the proof.

We show that, in the irrational case, if the condition $b|S \cap [0,a)| \le 1$ holds, we can construct a measurable set $E \subset S$ whose a $\mathbb{Z}$-translates tile S and such that the

Gabor system with window $g = \chi_E$ and parameters a, b forms a tight frame for $L^2(S)$. We will first need the following Lemma.

Lemma (3.3.18)[161]: Let a be an irrational number and suppose that E is a measurable subset of $\mathbb{R}$ wich is both a $\mathbb{Z}$ and $a\,\mathbb{Z}$ - shift invariant. Then $E = \mathbb{R}$ or $\emptyset$ up to a set of zero measure.

Proof. Since χ_E is 1- periodic, we can express it as a Fourier series

$$\chi_E(x) = \sum_{k\in\mathbb{Z}} c_k e^{2\pi ikx}, \quad x \in \mathbb{R},$$

for some sequence $\{c_k\} \in \ell^2(\mathbb{Z})$ where the series converges locally in L^2. Since χ_E is also a-periodic, we have

$$\chi_E(x) = \chi_E(x+a) = \sum_{k\in\mathbb{Z}} c_k e^{2\pi ika} e^{2\pi ikx},$$

and the uniqueness of the Fourier coefficients implies that $c_k\left(1 - e^{2\pi ika}\right) = 0$ for all integers k. Since a is irrational, this is equivalent to $c_k = 0$ for all $k \in \mathbb{Z}\backslash\{0\}$, i.e. to $\chi_E = c_0$ a.e. on $\mathbb{R}$ from which the claim follows.

Theorem (3.3.19)[161]: Let $a, b > 0$ be such that $a\,b \notin Q$. Let S be an a $\mathbb{Z}$-shift invariant, measurable subset of $\mathbb{R}$ with nonzero measure and satisfying $b|S_0| \le 1$, where $S_0 = S \cap [0, a)$. Then, there exists a measurable set E in $\mathbb{R}$ which is a $\mathbb{Z}$-congruent to S_0 and such that the collection $\{E_{mb}T_{na}\chi_E : m, n \in \mathbb{Z}\}$ is a tight frame for $L^2(S)$.

Proof. By the same argument as that of Theorem(3.3.12) we can assume that $b = 1, a > 1$ without loss of generality, and we only need to show the existence of measurable subset E of $\mathbb{R}$ which is a $\mathbb{Z}$-congruent to S_0 and such that the collection $\left\{e^{2\pi imx}\chi_E(x) \colon m \in \mathbb{Z}\right\}$ is a tight frame for $L^2(E)$. Let $\mathbb{Z}^+ = \{0,1,2,3, \dots\}$ and define a bijection $\sigma : \mathbb{Z}^+ \longrightarrow \mathbb{Z}$ by

$$\sigma(0) = 0, \quad \sigma(2k-1) = k, \quad \sigma(2k) = -k \quad \text{for } k > 0.$$

Let $g_0 = \sum_{k\in\mathbb{Z}} \chi_{S_0}(\cdot + k)$. If $g_0 \le 1$, a.e. on $\mathbb{R}$, then $E = S_0$ is desired. Otherwise, we proceed with the following inductive procedure to construct E, analogous to the construction given in Theorem (3.3.12) (and with similar notation). Let $H_0 = \{g_0 > 0\}$ and let T_0 be a measurable subset of S_0 such that

$$\sum_{k\in\mathbb{Z}} \chi_{T_0}(\cdot + k) = \chi_{H_0}.$$

If the sets S_l, T_l, H_l and the function g_1 have already been constructed for all indices l with $0 \le l \le j-1$, we define $S_j = S_{j-1}\backslash T_{j-1}$,

$$g_j = \sum_{k\in\mathbb{Z}} \chi_{S_j}(\cdot + a\sigma(j) + k),$$

and $H_j = \left\{g_j > 0, g_0 = g_1 = \dots = g_{j-1} = 0\right\}$. We then choose a measurable set T_j contained in S_j and such that

$$\sum_{k\in\mathbb{Z}} \chi_{T_j}(\cdot + a\sigma(j) + k) = \chi_{H_j}.$$

Define $E = \bigcup_{j\in\mathbb{Z}^+}\left(T_j - a\sigma(j)\right)$. Note that the sets $H_j, j \in \mathbb{Z}^+$, are mutually disjoint. We have

$$\sum_{k\in\mathbb{Z}}\chi_E(\cdot+k)=\sum_{j\in\mathbb{Z}^+}\sum_{k\in\mathbb{Z}}\chi_{T_j}(\cdot+a\sigma(j)+k)=\sum_{j\in\mathbb{Z}^+}\chi_{H_j}\le 1 \quad \text{a.e. on } \mathbb{R}.$$

So, by Lemma (3.3.10) the collection $\{E_{mb}\chi_E: m,n\in\mathbb{Z}\}$ is a tight frame for $L^2(E)$. Since $T_j-a\sigma(j)\subset S_0-a\sigma(j)$ for $j\in\mathbb{Z}^+$, and the sets $S_0-a\sigma(j), j\in\mathbb{Z}^+$, are mutually disjoint, so are the sets $T_j-a\sigma(j),\ j\in\mathbb{Z}^+$. Hence, in order to show that E is a $\mathbb{Z}$-congruent to S_0, we only need to show that $\left|S_0\backslash(\cup_{j\in\mathbb{Z}^+}T_j)\right|=0$. Write $Q=S_0\backslash(\cup_{j\in\mathbb{Z}^+}T_j)$. Since, for $j\ge 1$,$S_j=S_0\setminus(\cup_{i=0}^{j-1}T_i)$, we have $Q\subset S_j$, for all $j\ge 0$, and thus

$$\left\{\sum_{k\in\mathbb{Z}}\chi_Q(\cdot+k)>0\right\}\subset\left\{\sum_{k\in\mathbb{Z}}\chi_{S_j}(\cdot+k)>0\right\}=\{g_j(\cdot-a\sigma(j))>0\}$$

$\subset\cup_{m=0}^{\infty}\big(H_m+a\sigma(j)\big)$.

It follows, using the disjointness the sets $H_m, m\ge 0$, that

$$\left\{\sum_{k\in\mathbb{Z}}\chi_Q(\cdot+k)>0\right\}\subset\bigcap_{j=0}^{\infty}\bigcup_{m=0}^{\infty}\big(H_m+a\sigma(j)\big)=\bigcap_{l\in\mathbb{Z}}\bigcup_{m=0}^{\infty}(H_m+al)$$

$$=\bigcap_{l\in\mathbb{Z}}\left(\bigcup_{m=0}^{\infty}\{\chi_{H_m}(\cdot-al)=1\}\right)=\bigcap_{l\in\mathbb{Z}}\left(\left\{\sum_{m\ge0}\chi_{H_m}(\cdot-al)=1\right\}\right).$$

Hence,

$$\left\{\sum_{k\in\mathbb{Z}}\chi_Q(\cdot+k)>0\right\}\subset\bigcap_{l\in\mathbb{Z}}\left\{\sum_{m\ge0}\sum_{k\in\mathbb{Z}}\chi_{T_m}(\cdot+a\sigma(m)-al+k)=1\right\}\coloneqq\tilde{Q}.$$

We will now show that $|Q|=0$ by contradiction. Suppose that $|Q|>0$. Then,

$$\left|\bigcup_{m=0}^{\infty}(T_m+a\sigma(m))\right|=\sum_{m=0}^{\infty}|T_m+a\sigma(m)|=\sum_{m=0}^{\infty}|T_m|<|S_0|\le 1 \tag{49}$$

due to the disjointness of the sets $T_m, m\ge 0$. It is obvious that

$$\tilde{Q}\subset\left\{\sum_{m\ge0}\sum_{k\in\mathbb{Z}}\chi_{T_m}(\cdot+a\sigma(m)+k)=1\right\}=\left\{\sum_{k\in\mathbb{Z}}\chi_{\cup_{m=0}^{\infty}T_{m-a\sigma(m)}}(\cdot+k)=1\right\}$$

Since the sets $T_{m-a\sigma(m)}, m\ge 0$, are disjoint. Since $\tilde{Q}$ is $\mathbb{Z}$-periodic and

$$\left|\tilde{Q}\cap[0,1)\right|=\int_{[0,1)}\chi_{\tilde{Q}}(t)\,dt\le\int_{[0,1)}\sum_{k\in\mathbb{Z}}\chi_{\cup_{m=0}^{\infty}T_m-a\sigma(m)}(\cdot+k)\,dt<1,$$

using (49), it follows that $\tilde{Q}\ne\mathbb{R}$ modulo a set of zero measure. However, $\tilde{Q}$ is both a $\mathbb{Z}$ and $\mathbb{Z}$-periodic and Lemma (3.3.18) shows that $\left|\tilde{Q}\right|=0$. Therefore, we conclude that $\left|\left\{\sum_{k\in\mathbb{Z}}\chi_Q(\cdot+k)>0\right\}\right|=0$ and thus

$$|\mathbb{Q}|=\int_{[0,1)}\sum_{k\in\mathbb{Z}}\chi_{\mathbb{Q}}(t+k)dt\ =0.$$

Which is a contradiction. The proof is completed.

Corollary (3.3.20)[161]: let $a, b > 0$ be such that $ab \neq \mathbb{Q}$. Let S be an a $\mathbb{Z}$-shift invariant, measurable subset of $\mathbb{R}$ and define the set $S_0 = S \cap [0, a)$. Then, the following are equivalent:

(a) There exists a function $g \in L^2(S)$ such that the linear span of the collection $\{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}$ is dense in $L^2(S)$.

(b) There exists a measurable set E in $\mathbb{R}$ which is a $\mathbb{Z}$-congruent to $S \cap [0, a)$ such that $\{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}$ is a tight frame for $L^2(S)$.

(c) $b\,|S_0| \leq 1$.

Proof. As we mentioned earlier, the fact that (a) implies (c) is a result from [172]. The fact that (b) follows from (c) is the content of Theorem (3.3.19) and, clearly (b) implies (a).

Chapter 4
Gabor Frames and Unit Cubes

We show that for the irrational case, we show that the classification problem can be completely settled if the union of some intervals obtained from the set-valued mapping becomes stabilized after finitely many times of iterations, which we conjecture is always true. We provide an uncountable class of functions. As a byproduct of the proof method we derive new sampling theorems in shift-invariant spaces and obtain the correct Nyquist rate. An inductive procedure for constructing such sets Λ in dimension $d \geq 3$ is also given. An interesting and surprising consequence of the results is the existence, for $d \geq 2$, of discrete sets Λ with $G(\chi_{[0,1]^d}, \Lambda)$ forming a Gabor orthonormal basis but with the associated "time"-translates of the window $\chi[0,1]^d$ having significant overlaps.

Section (4.1): Characteristic Function Generates a Gabor Frame

Frames introduced by Duffin and Shaffer in [191] have recently received great attention due to their wide range of applications in both mathematics and engineering science. Gabor frames form a special kind of frames for $L^2(\mathbb{R})$ whose elements are generated by time-frequency shifts of a single window-function or atom. More specifically, let $g \in L^2(\mathbb{R})$ and $a, b \in \mathbb{R}^+$, we use (g, a, b) to denote the Gabor family or system $\{M_{mb}T_{nag} : m, n \in \mathbb{Z}\}$ generated by g where $T_x\, g(t) = g(t-x)$ is the translation unitary operator and $M_\xi\, g(t) = e^{2\pi i \xi t} g(t)$ is the modulation unitary operator. The composition $M_\xi\, T_x$ is called the time-frequency shift operator. We say that (g, a, b) is a Gabor frame for $L^2(\mathbb{R})$ if there exist two constants $C_1, C_2 > 0$ such that

$$C_1\|f\|^2 \leq \sum_{m,n\in\mathbb{Z}} \left|\langle f, M_{mb}T_{nag}\rangle\right|^2 \leq C_2\|f\|^2 \qquad (1)$$

Holds for every $f \in L^2(\mathbb{R})$. See [182,188,190,192,193,195,196,197,204] for some background materials and recent development in Gabor analysis. In order to have a Gabor frame for $L^2(\mathbb{R})$, one important restriction is the density condition which states that if (g, a, b) forms a Gabor frame, then $ab \leq 1$ (cf. [187,201]). Although it is a key condition for the Gabor frame, the density condition is still far from providing an answer to the fundamental question of characterizing classes of functions which can be served as a window-function for a Gabor frame. This is generally believed to a quite difficult problem. In fact, the problem is only solved completely for few functions (cf. [194]). For an excellent survey on this topic see to C. Hail's [196]. In particular we mention that Gabor originally proposed the Gaussian function $g(t) = e^{-t^2}$ as a window with respect the unit time-frequency lattice $\mathbb{Z} \times \mathbb{Z}$ $(i.e. a = b = 1)$. However, though it is a complete system for $L^2(\mathbb{R})$, the Gaussian Gabor system $(g, 1, 1)$ is not a frame. it was conjectured by Daubechies and Grossmann [187,189], and later proved by Lyubarskii, Seip and Wellston [200,202,203] that $\left(e^{-t^2}, a, b\right)$ is a frame if and only if $ab < 1$. Another seemingly natural case is the class of characteristic functions. It was shown by Casazza and kalton [184] that characterizing such sets E that $(\chi_E, 1,1)$ is a frame is equivalent to solving an old open problem of Littlewoods in complex analysis. A further special case of the above problem, Which we will refer to as the abc-problem, asks for a classification of all $a, b, c \in \mathbb{R}^+$ such that $\left(\chi_{[0,c)}, a, b\right)$ is a Gabor frame. This problem was initiated by Janssen [198,199], Casazza and Lammers [185]. Although classification has been obtained for some special cases, this problem appears to be very difficult in general. In many cases it is associated with an extremely complicated set-Janssen's tie [199]. We raise a conjecture concerning the image of certain set-valued mapping on certain union of intervals. We conjecture that the said image will always become stabilized after finitely many times of iterations. The main results (Theorems (4.1.18) and (4.1.3)) indicate that if the conjecture is true, then the abc-problem can be completely solved.

To show how delicate the abc-problem is, we list two results due to Janssen, Casazza and Lammers.

Proposition (4.1.1)[181]: (See [198,199]). Assume $a < 1 < c$.

(i) If a is not rational and $1 < c < 2$, then $(\chi_{[0,c)}, a, 1)$ is a Gabor frame.

(ii) If $a = p/q$ is rational, $gcd(p, q) = 1$ and $2 - 1/q < c < 2$, then $\left(\chi_{[0,c)}, a, 1\right)$ is not a Gabor frame.

(iii) If $a > 3/4$, $c = L - 1 + L(1 - a)$ with integer $L \geq 3$, then $\left(\chi_{[0,c)}, a, 1\right)$ is not a Gabor frame.

(iv) If d is the greatest integer $\leq c$ and $|c - d - 1/2| < 1/2 - a$, then $\left(\chi_{[0,c)}, a, 1\right)$ is a Gabor frame.

Proposition (4.1.2)[181]: (See [185]).

(i) The Gabor system $(\chi_{[0,c)}, 1, c)$ is a Gabor frame if and only if $c = 1$.

(ii) If $2 \leq c \in \mathbb{N}$, then for all $a > 0$, $\left(\chi_{[0,c)}, a, 1\right)$ is not a Gabor frame.

(iii) If $a \leq c < 1$, then $\left(\chi_{[0,c)}, a, 1\right)$ is a Gabor frame, if $c < a < 1$, then $\left(\chi_{[0,c)}, a, 1\right)$ is not a Gabor frame.

We will state all our results for general $a, b, c \in \mathbb{R}^+$ instead of adopting the practice of both Janssen, Casazza and Lammers by letting $b = 1$. From (1) and (3) of Proposition (4.1.2) above, we see that the case of $ab = 1$ and the case of $bc < 1$ are completely solved, so throughout we will always assume that $0 < 1/b \leq c$. We will use M to denote exclusively the largest natural number less than or equal to bc. We will use d exclusively to denote $c - M(1/b)$, thus $0 \leq d < 1/b$ always holds. Almost all the arguments revolve around the behaviors of the following two sets $A = \bigcup_{n\in\mathbb{Z}}[na, na + 1/b)$. We will also use letters A and B for the above mentioned sets exclusively.

We will use the new approach to recover or generalize some of the results obtained in [198,199] and [185]. the main purpose there is to exhibit the new techniques and summarize certain known or generalizable existing results in a suitable way in anticipation for the new results. We will not attempt a detailed comparison of the results stated in the first half with the ones found [198,199] or [185], usually in deferent forms with $b = 1$. It suffices to say, even when these results are not explicitly stated in [198,199] or [185], they are most likely obtainable using the techniques developed. We will recover (2) of Proposition (4.1.1), in a lemma and generalize it to the following result, which is already established by Janssen in [199].

On the other hand, the following (together with Lemma (4.1.5) and Proposition (4.1.11) for the case of $M = 1$) provides us with a complete solution to the abc-problem for the case of $a = \frac{1}{q}(1/b)$ with $q \in \mathbb{N} + 1$, which is also know to Janssen.

We will give proofs of the sufficiency parts of the following two theorems. The first one is a complete solution to the abc-problem for the case of $M = 1$. We point out that this case is already completely solved by Janssen in [199], and his results can be quickly recovered from the theorem below.

Here $\mathcal{M}_1$ and $\mathcal{M}_2$ are set-valued mappings defined by $\mathcal{M}_1(G) = G \cup (A \cap G - 1/b)$ and $\mathcal{M}_2(G) = G \cup (A \cap G + 1/b)$ for any $G \in \mathbb{R}$, where $A = \bigcup_{n\in\mathbb{Z}}[na, na + d)$ and $B = \bigcup_{n\in\mathbb{Z}}[na + d, na + 1/b)$ as we defined earlier. Since it will be made clear that the case of $bc \in \mathbb{N}$ is trivial, what is left to consider is the case of $bc > 2$. This is treated in the next theorem.

The necessity parts of both of these theorems , especially that of Theorem (4.1.18), entail complicated constructions of functions. We will devote most to this task. In fact, instead of proving the necessity part of Theorem (4.1.18), we will show the following slightly stronger result.

Theorem (4.1.3)[181]: Let $0 < a < 1/b < c = M(1/b) + d$ for some $0 < d < 1/b$ and $M \in \mathbb{N} + 1$. If there is an a-periodic proper subset G of $\mathbb{R}$ which contains E and satisfies $\mathcal{M}(G) = G$, then $\left(\chi_{[0,c)}, a, b\right)$ is not a Gabor frame for $L^2(\mathbb{R})$.

We will also show quickly that in the case of $ab \in \mathbb{Q}^+$, it is true that $\mathcal{M}^{S_0+1}(E) = \mathcal{M}^{S_0}(E)$ for some $S_0 \in \mathbb{N}$, thus immediately obtain a complete solution for that case. This naturally leads to our conjecture that when ab is irrational the same still holds.

Due to the nature of the proofs, we will divide them between. Deals with the proofs of Proposition (4.1.9), Theorem (4.1.10), and the proofs of the sufficiency parts of Theorems (4.1.19) and (4.1.18). The necessity part of Theorem (4.1.19) requires some technical details, it will be given together with much more complicated constructions involved in the proof of Theorem (4.1.3), which serves as a proof of the necessity part of Theorem (4.1.18). We then briefly discuss the case of $M > 1$ in comparison with Janssen's counterpart in [199]. In particular, with the help of one of our main results (Theorem (4.1.3)) we negatively settle one conjecture raised by Janssen in [199].

For any fixed triple (a, b, c) satisfying $0 < a < 1/b \le c$, we will henceforth use the letters M, d, A, B exclusively to denote numbers or sets as specified in the previous. Also, for any $n \in \mathbb{Z}$, any bounded and compactly supported function $f \in L^2(\mathbb{R})$ and any $g \in L^2(\mathbb{R})$, we define

$$H_n(t) = \sum_k f(t - k/b)\overline{g(t - na - k/b)}, \tag{2}$$

for each $n \in \mathbb{Z}$. Though H_n also depends both on f and g, for notational simplicity we will avoid indexing H_n by f and g since it will always be clear from the context which f and g are associated with the specific H_n in question. By definition, H_n is $1/b$-periodic and $H_n \in L^2[0, a/b)$. We omit the proof of the following lemma, which is almost identical to the well-known proof of the WH-identity in [182], where it is credited to Heil and Walnut [197]. The same proof can also be found in [188].

Lemma (4.1.4)[181]: Let $g \in L^2(\mathbb{R})$ be such that $\sum_n |g(t - na)|^2 \le C$ for some $C > 0$, and let H_n be defined as in (2) for any bounded and compactly
Supported functions f. then

$$\sum_{m,n\in\mathbb{Z}} \left|\langle f, E_{mb}T_{na}g\rangle\right|^2 = \frac{1}{b}\sum_{n}\int_{0}^{1/b} |H_n(t)|^2 dt.$$

We only consider the case of $g(t) = \chi_{[0,c)}(t)$. In order to examine whether g generates a frame, we need the following observations. First of all, since $g(t) = \chi_{[0,c)}(t)$, we certainly have $\sum_n |g(t-na)|^2 \le C$ for some $C > 0$. Thus Lemma (4.1.4). Applies. Moreover, by the CC-condition due to Casazza and Christensen [182], there is a $C_2 > 0$, such that for any $f \in L^2(\mathbb{R})$,

$$\sum_{m,n\in\mathbb{Z}} \left|\langle f, E_{mb}T_{na}g\rangle\right|^2 \le C_2\|f\|_2^2.$$

Therefore, in order to check whether (g, a, b) is a Gabor frame, we only need to check whether there exists some $C_1 > 0$ such that

$$C_1\|f\|_2^2 \le \sum_{m,n\in\mathbb{Z}} \left|\langle f, E_{mb}T_{na}g\rangle\right|^2$$

for all $f \in L^2(\mathbb{R})$, however, since bounded and compactly supported functions f are dense in $L^2(\mathbb{R})$, it follows from a density argument and Lemma (4.1.4) that we only need to check if there is a $C_1 > 0$ such that

$$C_1\|f\|_2^2 \le \frac{1}{b}\sum_{n}\int_{0}^{1/b} |H_n(t)|^2 dt \tag{3}$$

for all such functions. Secondly, since $g(t) = \chi_{[0,c)}(t)$, for any bounded and compactly supported$f$, if we define $G_n(t) = f(t)\, g(t-na) = f(t)\chi_{[na,na+c)}(t)$ For each $n \in \mathbb{Z}$, then clearly $H_n(t) = \sum_k G_n(t - k/b)$. Simple computation shows that whenever $0 < a < 1/b \le c$ holds, we have:

$$H_n(t) = \begin{cases} F_1(t), & t \in [na, na+d), \\ F_2(t), & t \in [na+d, na+1/b), \end{cases}$$

where

$$F_1(t) = \sum_{j=0}^{M} f\big(t + j(1/b)\big), \qquad F_2(t) = \sum_{j=0}^{M-1} f\big(t + j(1/b)\big).$$

Therefore, if we let L_1, L_2 be the smallest natural numbers greater than or equal to $\frac{d}{a}$ and $\frac{1/b}{a}$ respectively, then for any bounded and compactly supported functions $f \in L^2(\mathbb{R})$, we have

$$\int_A |F_1(t)|^2 dt + \int_B |F_2(t)|^2 dt \le \sum_n \int_0^{1/b} |H_n(t)|^2 dt$$

$$\le L_1\int_A |F_1(t)|^2 dt + L_2\int_B |F_2(t)|^2 dt\,. \tag{4}$$

From now on we will also use L_1, L_2 exclusively to denote such numbers. Let us now show how Lemma (4.1.4) and, specifically, inequality (4) can be used in some simpler cases.

Lemma (4.1.5)[181]: Let $0 < a < 1/b \le c = 1/b + d$ for some $0 \le d \le 1/b$. If $1/b - d \ge a$, then $(\chi_{[0,c)}, a, b)$ is a Gabor frame.

Proof. In this case we have $M = 1$. So $F_2(t) = f(t)$. Note that the condition $1/b - d \ge a$, implies that $B = \mathbb{R}$. Hence

$$\sum_n \int_0^{1/b} |H_n(t)|^2 dt \ge \int_{\mathbb{R}} |F_2(t)|^2 dt = \int_{\mathbb{R}} |f(t)|^2 dt.$$

Note that Lemma (4.1.5) certainly includes the case of $d = 0$ and $c = 1/b$. Next lemma is actually (ii) of Proposition (4.1.2), a result by Casazza and Lammers. It deals with all the other situations of $d = 0$.

Lemma (4.1.6)[181]: Let $0 < a < 1/b < c = M(1/b)$ for some $M \in \mathbb{N} + 1$, then $(\chi_{[0,c)}, a, b)$ is not a Gabor frame.

Proof. In this case we have $d = 0$, So $A = \phi$ and $H_n(t) = F_2(t)$. Similarly, $1/b > a$ implies $B = \mathbb{R}$. Hence for any bounded and compactly supported $f \in L^2(\mathbb{R})$, we have

$$\sum_n \int_0^{1/b} |H_n(t)|^2 dt \le L_2 \int_{\mathbb{R}} F_2(t) dt.$$

To show that $(\chi_{[0,c)}, a, b)$ is not a Gabor frame, we only need to construct a sequence of functions $f_j \in L^2(\mathbb{R})$ with $\|f_j\|$ approaching infinity while the corresponding $\sum_n \int_0^{1/b} |H_n(t)|^2 dt$ stays bounded. In fact, for each $j \in \mathbb{N}$ we define $f_j(t)$ supported on the interval $[0, j/b)$ by letting $f_j(t) = e^{i\frac{k}{M}2\pi t}$ for each $t \in [(k-1)/b, k/b)$ with $k \in \{1,2,\dots,j\}$. Straightforward calculation shows that $\|f_j\|^2 = j/b$, but correspondingly we always have

$$\sum_n \int_0^{1/b} |H_n(t)|^2 dt \le 2L_2 M^3 (1/b).$$

After Lemma (4.1.6), we do not need to worry about the case of $bc \in \mathbb{N}$ anymore. Hence we will assume $d > 0$ throughout the rest of the article. The following result recovers and generalizes (iv) Proposition (4.1.1). We point out that it has already been established by Janssen in this generality in [199].

Lemma (4.1.7)[181]: Let $0 < a < 1/b < c = M(1/b) + d$ for some $M \in \mathbb{N}$ and $0 < d < 1/b$. If $\min\{d, 1/b - d\} \ge a$, then $(\chi_{[0,c)}, a, b)$ is a Gabor frame.

Proof. As before, $\min\{d, 1/b - d\} \ge a$, implies that $A = B = \mathbb{R}$.Thus, by (4).

$$\sum_n \int_0^{1/b} |H_n(t)|^2 dt \ge \int_{\mathbb{R}} |F_1(t)|^2 dt + \int_{\mathbb{R}} |F_2(t)|^2 dt \ge \int_{\mathbb{R}} \frac{1}{2} |F_1(t) - F_2(t)|^2 dt$$

$$= \int_{\mathbb{R}} \frac{1}{2} \left| f\left(t + \frac{M}{b}\right) \right|^2 dt = \frac{1}{2} \int_{\mathbb{R}} |f(t)|^2 dt.$$

Note that because of Lemma (4.1.7), we now only need to consider the case of $c = M(1/b) + d$ with $M \in \mathbb{N}, 0 < d < 1/b$ and $a > \min\{d, 1/b - d\}$ for the next lemma, we need some more terminology. We will call a measurable $E \subset \mathbb{R}$ a-translation periodic or a-translation invariant if $E + na = E$ holds for all $n \in \mathbb{Z}$.

Lemma (4.1.8)[181]: Let $0 < a < 1/b < c = M(1/b) + d$ for some $M \in \mathbb{N}$ and $0 < d < 1/b$. Then $\mathbb{R} \backslash \left(\cup_{k\in\mathbb{Z}}(A + k/b) \cap (\cup_{k\in\mathbb{Z}}(B + k/b))\right)$ has positive measure if and only $a = \frac{p}{q}(1/b)$ with $p, q \in \mathbb{N}$, satisfying $\gcd(p, q) = 1$ and $\frac{a}{p} > \min\{1/b - d, d\}$.

Proof. We show the sufficiency part first: Since $a = \frac{p}{q}(1/b)$ with $\gcd(p, q) = 1$, we have that both $\cup_{k\in\mathbb{Z}}(A + k/b)$ and $\cup_{k\in\mathbb{Z}}(B + k/b)$ are $\frac{1}{qb}$-translation periodic consisting of all $\frac{1}{qb}$-translates of intervals $[0, d)$ or $[d, 1/b)$, respectively. It follows that both sets are actually $\frac{a}{p}$-translation periodic consisting of all $\frac{a}{p}$-translates of intervals $[0, d)$ or $[0, d)$ or $[d, 1/b)$, respectively. Hence the condition $\frac{a}{p} > \min\{1/b - d, d\}$ is equivalent to the condition that at least one of the above two sets is complemented in $\mathbb{R}$ by a set of positive (indeed, infinite) measure, which in turn is equivalent to the desired conclusion.

The argument above implies that we only need to show the necessity part for $a = r(1/b)$ with irrational r. In this case, the set $\left\{\left[\frac{k}{b}\right] : k \in \{0\} \cup \mathbb{N}\right\}$ is dense in the interval$[0, a)$, where $[x] = x \bmod(a)$. Since A, B are both a-periodic sets containing intervals, it follows that $\cup_{k\in\mathbb{Z}}(A + k/b) = \mathbb{R}, \cup_{k\in\mathbb{Z}}(B + k/b) = \mathbb{R}$.

Now we are ready to prove Proposition (4.1.9) and Theorem (4.1.10).

Proposition (4.1.9)[181]: Let $0 < a < 1/b < c = M(1/b) + d$ for some $M \in \mathbb{N} + 1$ and $0 < d < 1/b$. If $a = \frac{p}{q}(1/b)$ with $p, q \in \mathbb{N}$ such that $0 < p < q$, $\gcd(p, q) = 1$ and $\frac{a}{p} > \min\{1/b - d\,, d\}$, then $\left(\chi_{[0,c)}, a, 1\right)$ is a not a Gabor frame.

Proof. According to Lemma (4.1.8), we have either $\mathbb{R} \backslash \cup_{k\in\mathbb{Z}}\left(A + \frac{k}{b}\right)$ or $\mathbb{R} \backslash \cup_{k\in\mathbb{Z}}\left(B + \frac{k}{b}\right)$ has positive measure. Without loss of generality, we assume that the former does, and denote $G = \mathbb{R} \backslash \cup_{k\in\mathbb{Z}}\left(A + \frac{k}{b}\right)$. As noted before, this set is $\frac{1}{qb}\left(\text{or } \frac{a}{p}\right)$-translation invariant. Similar to the proof of Lemma (4.1.6), we only need to define a sequence of functions $f_j \in L^2(\mathbb{R})$ supported on set G with $\|f_j\|$ approaching infinity while the corresponding $\sum_n \int_0^{1/b} |H_n(t)|^2 dt$ stays bounded.

In fact, for each $j \in \mathbb{N}$, we can define $f_j(t)$ supported on $[0, j(\,1/b)) \cap G$ by letting $f_j(t) = e^{i\frac{k}{M}2\pi t}$ for each $t \in [(k-1)\,1/b\,, k(1/b)) \cap G$ with $k \in \{1,2, \dots, j\}$. Then $\|f_j\|^2 = j \cdot \mu(G \cap [0, 1/b))$. On the other hand, for any bounded and compactly supported f defined on G, (4) implies

$$\sum_n \int_0^{1/b} |H_n(t)|^2 dt \le L_2 \int_B |F_2(t)|^2 dt.$$

Similarly we can get an estimate for the corresponding H_n. If we use μ to denote the Lebesgue measure, we have

$$\sum_n \int_0^{1/b} |H_n(t)|^2 dt \le 2L_2M^3 \cdot \mu\big(G \cap [0, 1/b)\big).$$

Theorem (4.1.10)[181]: Let $0 < a < 1/b < c = M(1/b) + d$ for some $M \in \mathbb{N} + 1$, and $0 < d < 1/b$. If $a = \frac{1}{q}(1/b)$ with $q \in \mathbb{N} + 1$, then $(\chi_{[0,\square)}, a, b)$ is a Gabor frame if and only if $a \le \min\{1/b - d, d\}$.

Proof. The sufficiency part follows from Lemma (4.1.7). For the necessity part, we assume to the contrary that $a > \min\{1/b - d, d\}$. Thus Proposition (4.1.9) (the special case of $p = 1$) implies that $(\chi_{[0,c)}, a, b)$ is not a Gabor frame.

Extending Theorem (4.1.10) to the case of $M = 1$ is quick. Note that by Lemma (4.1.5), we only need to look at the case of $a > 1/b - d$.

Proposition (4.1.11)[181]: Let $0 < a < 1/b + d < c = 1/b + d$ for $0 < d < 1/b$. If $a = \frac{p}{q}(1/b)$ with $p, q \in \mathbb{N}$ such that $\gcd(p, q) = 1, \frac{a}{p} > 1/b - d$, then $(\chi_{[0,c)}, a, b)$ is not a Gabor frame.

Proof. In this case, $\mathbb{R} \setminus (\bigcup_{k \in \mathbb{Z}} (B + k/b))$ has positive measure. The rest is the same as the proof of Proposition (4.1.7).

In order to show the necessity parts of Theorems (4.1.19) and (4.1.18), we need to develop a key lemma which already appears in its embryonic form in the proof of Lemma (4.1.7). We will treat the sufficient parts of both.

Proposition (4.1.12)[181]: Let $H_n(t)$ be defined as in (2) for $g(t) = \chi_{[0,c)}(t)$ and any bounded and compactly supported f. Suppose for some $N, L \in \mathbb{N}$ with $N < L$ and some $D_N, D_L \subset \mathbb{R}$, there are $\alpha_N, \alpha_L > 0$ such that

$$\sum_n \int_0^{1/b} |H_n(t)|^2 dt \ge \alpha_L \int_{D_L} \left|\sum_{j=0}^{L-1} f(t + j/b)\right|^2 dt,$$

$$\sum_n \int_0^{1/b} |H_n(t)|^2 dt \ge \alpha_N \int_{D_N} \left|\sum_{j=0}^{N-1} f(t + j/b)\right|^2 dt$$

holds for any bounded and compactly supported f and its corresponding H_n.

Then for any subset D of $\mathbb{R}$ satisfying either

(a) $D \subset D_L, D + \frac{L-N}{b} \subset D_N$ or

(b) $D - \frac{N}{b} \subset D_L, D - \frac{N}{b} \subset D_N$,

there is some $\alpha_D > 0$, such that

$$\sum_n \int_0^{1/b} |H_n(t)|^2 dt \ge \alpha_D \int_D \left|\sum_{j=0}^{L-N-1} f(t + j/b)\right|^2 dt$$

holds for any bounded and compactly supported f and its corresponding H_n.

Proof. Denote $I = \sum_n \int_0^{1/b} |H_n(t)|^2 dt$. For any D satisfying (a), we have

$$\int_D \left|\sum_{j=0}^{L-N-1} f(t + j/b)\right|^2 dt = \int_D \left|\sum_{j=0}^{L-1} f(t + j/b) - \sum_{j=L-N}^{L-1} f(t + j/b)\right|^2 dt$$

$$\leq 2\int_D \left|\sum_{j=0}^{L-1} f(t+j/b)\right|^2 dt + 2\int_D \left|\sum_{j=L-N}^{L-1} f(t+j/b)\right|^2 dt$$
$$\leq 2\int_D \left|\sum_{j=0}^{L-1} f(t+j/b)\right|^2 dt + 2\int_{D+\frac{L-N}{b}} \left|\sum_{j=0}^{N-1} f(t+j/b)\right|^2 dt$$
$$\leq \frac{2}{\alpha_{D_L}} I + \frac{2}{\alpha_{D_N}} I = 2\frac{\alpha_{D_N}+\alpha_{D_L}}{\alpha_{D_N}\times\alpha_{D_L}} I.$$

We may take α_D to be $\frac{\alpha_{D_N}\times\alpha_{D_L}}{2\alpha_{D_N}+2\alpha_{D_L}}$. The other case is similar.

Now we apply Proposition (4.1.12) to the cases which we are interested in. First in order, the case of $M = 1$.

Lemma (4.1.13)[181]: Let $0 < a < 1/b < c = 1/b + d$ with some $0 < d < 1/b$. Suppose for some set F and $\alpha_F > 0$, $\sum_n \int_0^{1/b} |H_n(t)|^2\, dt \geq \alpha_F \int_F |f(t)|^2\, dt$ holds for any bounded and compactly supported f and its corresponding H_n. Then for any $D \subset A$ such that $D + 1/b \subset F$, or any $D \subset A + 1/b$ such that $D \subset F + 1/b$, there exists some $\alpha_D > 0$ such that $\sum_n \int_0^{1/b} |H_n(t)|^2 dt \geq \alpha_D \int_D |f(t)|^2\, dt$ holds for any bounded and compactly supported f and its corresponding H_n.

Proof. Denote $I = \sum_n \int_0^{1/b} |H_n(t)|^2\, dt$. Our assumption can be simply written as $I \geq \alpha_F \int_F |f(t)|^2 dt$. Since $M = 1$, it follows from (4) that

$$I \geq \int_A \left|f(t) + f\big(t + (1/b)\big)\right|^2 dt.$$

Now apply Proposition (4.1.12)(a) with $L = 2$ and $N = 1, D_L = A$ and $D_N = F$, the conclusion follows. The other case is similar.

Lemma (4.1.14)[181]: Let $0 < a < 1/b < c = 1/b + d$ for some $0 < d < 1/b$. If F is any of the three sets $A \cap B + 1/b, A \cap (B - 1/b), B$ or any union of any such sets, then there exists a constant $\alpha_F > 0$ (dependent on F) such that

$\sum_n \int_0^{1/b} |H_n(t)|^2 dt \geq \alpha_F \int_F |f(t)|^2\, dt$

holds for any bounded and compactly supported f and its corresponding H_n.

Proof. We observe that by (4), when $M = 1$, we have

$$\sum_n \int_0^{1/b} |H_n(t)|^2\, dt \geq \int_B |f(t)|^2\, dt.$$

Rest of the conclusion follows readily from Lemma (4.1.13) above.

Part of the following lemma recovers (i) of Proposition (4.1.1) a result by Janssen.

Lemma (4.1.15)[181]: Let $0 < a < 1/b < c = 1/b + d$ for some $0 < d < 1/b$. Let $F = (A \cap B + 1/b) \cup (B - 1/b)) \cup B$. If $\cup_{k=0}^{N}(F - k(1/b)) = \mathbb{R}$ for some $N \in \mathbb{N} \cup \{0\}$, then $(\chi_{[0,c)}, a, b)$ is a Gabor frame. In particular, if $a = r(1/b)$ for some irrational number r, then $(\chi_{[0,c)}, a, b)$ is a Gabor frame.

Proof. We first note that $0 < a < 1/b$ implies $A \cup B = \mathbb{R}$. By starting with the conclusion in Lemma (4.1.14) and applying Lemma (4.1.13) repeatedly , we will obtain that for any $n \in \mathbb{N} \cup \{0\}$, there is some $\alpha_D > 0$ such that

$$\sum_n \int_0^{1/b} |H_n(t)|^2 \, dt \geq \alpha_D \int_D |f(t)|^2 \, dt,$$

where $D = \bigcup_{k=0}^{n} (F - k(1/b))$. Consequently, the first conclusion is established when $n = N$. The second conclusion is proved using the same density argument employed in the proof of the necessity part of Lemma (4.1.8).

We turn our attention to the case of $M > 1$, we have

Lemma (4.1.16)[181]: Let $0 < a < 1/b < c = M(1/b) + d$ for some $0 < d < 1/b$ and $M \in \mathbb{N} + 1$. Let $E = \big(A \cap B + M(1/b)\big) \cup \big(A \cap (B - 1/b)\big)$. Then there is an $\alpha_E > 0$ such that

$$\sum_n \int_0^{1/b} |H_n(t)|^2 dt \geq \alpha_E \int_E |f(t)|^2 \, dt$$

holds for any bounded and compactly supported and its corresponding H_n.

Proof. This follows immediately from proposition (4.1.12). With $L = M + 1$ and $N = M$ and $D_L = A, D_N = B$.

Clearly the sufficiency part of Theorem (4.1.18) follows from the last and the next lemma. Please refer for the definitions of the sets A, B, E and the set-valued mappings $\mathcal{M}$ and p_j with $j \in \{1,2,3,4,5\}$

Lemma (4.1.17)[181]: Let $0 < a < 1/b < c = M(1/b) + d$ for some $0 < d < 1/b$ and $M \in \mathbb{N} + 1$. Suppose that for a measureable set G, there is some $\alpha_G > 0$, such that

$$\sum_n \int_0^{1/b} |H_n(t)|^2 dt \geq \alpha_G \int_G |f(t)|^2 \, dt \qquad (5)$$

holds for any bounded and compactly supported f and its corresponding H_n. Then there exists some $\alpha_{\mathcal{M}(G)} > 0$ such that

$$\sum_n \int_0^{1/b} |H_n(t)|^2 dt \geq \alpha_{\mathcal{M}(G)} \int_{\mathcal{M}(G)} |f(t)|^2 \, dt$$

also holds for any bounded and compactly supported f and its corresponding H_n.

Proof. Observe that by the definition of the set-valued mapping $\mathcal{M}$, it is enough to show that under the assumption of the lemma, for each $j \in \{1,2,3,4,5\}$, there exists some $\alpha_{p_j(G)} > 0$ such that

$$\sum_n \int_0^{1/b} |H_n(t)|^2 dt \geq \alpha_{p_j(G)} \int_{p_j(G)} |f(t)|^2 \, dt$$

also holds for any bounded and compactly supported f and its corresponding H_n.

(i) $p_1(G)$-*case*. Note that by (4), we have

$$\sum_n \int_0^{1/b} |H_n(t)|^2 dt \geq \int_A \left|\sum_{j=0}^{M} f(t+j/b)\right|^2 dt. \qquad (6)$$

Now apply Proposition (4.1.12)(a) to (6) and (5), with $L = M+1, N = 1, D_L = A$ and $D_N = G$, we obtain some $\alpha > 0$ such that

$$\sum_n \int_0^{1/b} |H_n(t)|^2 dt \geq \alpha \int_{A\cap(G-M(1/b))} \left|\sum_{j=0}^{M-1} f(t+j/b)\right|^2 dt. \qquad (7)$$

Now apply proposition (4.1.12)(a) to (7) and (5) again, with $L = M$ and $N = 1, D_L = A \cap \big(G - M(1/b)\big)$ and $D_N = G$, we obtain some $\alpha' > 0$ such that

$$\sum_n \int_0^{1/b} |H_n(t)|^2 dt \geq \alpha' \int_{A\cap(G-M(1/b))\cap(G-(M-1)(1/b))} \left|\sum_{j=0}^{M-1} f(t+j/b)\right|^2 dt. \quad (8)$$

Repeating this process $M-1$ times we get the result.

(ii)$p_2(G)$-*case*. Note that by (4), we also have

$$\sum_n \int_0^{1/b} |H_n(t)|^2 dt \geq \int_B \left|\sum_{j=0}^{M-1} f(t+j/b)\right|^2 dt. \qquad (9)$$

Now apply Proposition (4.1.12)(a) to (9) and (5), with $L = M, N = 1, D_L = B$ and $D_N = G$, we have for some $\beta > 0$,

$$\sum_n \int_0^{1/b} |H_n(t)|^2 dt \geq \beta \int_{B\cap(G-(M-1)(1/b))} \left|\sum_{j=0}^{M-2} f(t+j/b)\right|^2 dt. \qquad (10)$$

The rest is similar to the first case.

(iii)$p_3(G)$-*case*. This is the consequence of applying Proposition (4.1.12)(a) to (6) and (7), with $L = M+1, N = M, D_L = A$ and $D_N = A \cap \big(G - M(1/b)\big)$.

(iv)$p_4(G)$-*case*. This is the consequence of applying Proposition (4.1.12)(a) to (9) and (10), with $L = M, N = M-1, D_L = B$ and $D_N = B \cap \big(G - (M-1)(1/b)\big)$.

(v) $p_5(G)$-case. This is the consequence of applying Proposition (4.1.12)(a) to (9) and (8) with $L = M, N = M-1, D_L = B$ and $D_N = A \cap \big(G - M(1/b)\big) \cap \big(G - (M-1)(1/b)\big)$.

Theorem (4.1.18)[181]: Let $0 < a < 1/b < c = M(1/b) + d$ for some $0 < d < 1/b$ and $M \in \mathbb{N}+1$. If $\mathcal{M}^{S_0+1}(E) = \mathcal{M}^{S_0}(E)$ for some $S_0 \in \mathbb{N}$, then $\big(\chi_{[0,c)}, a, b\big)$ is a Gabor frame for $L^2(\mathbb{R})$ if and only if $\mathcal{M}^{S_0}(E) = \mathbb{R}$.

Here $\mathcal{M}$ is a set-valued mapping defined in turn by several other set-valued mapping p_j for $j \in \{1,2,3,4,5\}$ such that for any $G \in \mathbb{R}$

$$\mathcal{M}(G) = G \cup p_1(G) \cup p_2(G) \cup p_3(G) \cup p_4(G) \cup p_5(G)$$

and

$$p_1(G) = A \cap (\cap_{k=1}^{M}(G - k(1/b)).$$
$$p_2(G) = B \cap (\cap_{k=1}^{M-1}(G - k(1/b)).$$
$$p_3(G) = \big(G - (M+1)(1/b)\big) \cap (A - 1/b) \cap A.$$

$$p_4(G) = \big(G - M(1/b)\big) \cap (B - 1/b) \cap B.$$
$$p_5(G) = (G - M(1/b)) \cap (G - (M+1)(1/b)) \cap (A - 1/b) \cap B,$$
where A, B are unions of intervals specified earlier and $E = (A \cap B + M(1/b)) \cup (A \cap (B - 1/b))$.

Proof. Note that because of Lemma (4.1.15) we only need to consider the case of $a = \frac{p}{q}(1/b)$ with $p, q \in \mathbb{N}$ and $\gcd(p, q) = 1$. Assume that $\mathcal{M}_1^p\left(\mathcal{M}_2^p(B)\right) = \mathbb{R}$. Recall that by definition, for any measurable set $G, \mathcal{M}_1(G) = G \cup (A \cap (G - 1/b))$ and $\mathcal{M}_2(G) = G \cup (A \cap G + 1/b)$. Observe that $A \cup B = \mathbb{R}$ always holds whenever $a < 1/b$ and in the case of $M = 1$ we also have $\sum_n \int_0^{1/b} |H_n(t)|^2\, dt \geq \int_B |f(t)|^2\, dt$. Now apply Lemma (4.1.13) repeatedly $2p$ times starting with set B, the conclusion follows.

This follows immediately from the conclusion of Lemma (4.1.16), by applying Lemma (4.1.17)[181] s_0-times.

We deal with the easier case of Theorem (4.1.19), first.

Theorem (4.1.19)[181]: Let $0 < a < 1/b < c = 1/b + d$ for some $0 < d < 1/b$.

(i) If $a = r(1/b)$ for some irrational r, then $\left(\chi_{[0,c)}, a, b\right)$ is a Gabor frame .

(ii) If $a = \frac{p}{q}(1/b)$ with $p, q \in \mathbb{N}$ and $\gcd(p, q) = 1$, then $\left(\chi_{[0,c)}, a, b\right)$ is a Gabor frame if and only if $\mathcal{M}_1^p\left(\mathcal{M}_2^p(B)\right) = \mathbb{R}$.

Proof. Let $H = \mathbb{R} \backslash \mathcal{M}_1^P(\mathcal{M}_2^P(B))$. Assume that H has positive measure, we are going to prove that $(\chi_{[0,c)}, a, b)$ is not a Gabor frame. We make a few observations about $\mathcal{M}_1$ *and* $\mathcal{M}_2$ first. Note that for any set G containing B, we must have $\mathcal{M}_1 \circ \mathcal{M}_2(G) = \mathcal{M}_2 \circ \mathcal{M}_1(G)$. Indeed, by using the fact that $A \cup B = \mathbb{R}$, We see that $\mathcal{M}_1(G) = G \cup (G - 1/b)$ whenever $B \subset G$. Thus for such G we have
$$\mathcal{M}_1 \circ \mathcal{M}_2(G) = G \cup (A \cap G + 1/b) \cup \big(A \cap (G - 1/b)\big) = \mathcal{M}_2 \circ \mathcal{M}_1(G).$$

Note also that when D is an a-translation periodic set of positive measure, if $a = \frac{P}{q}(1/b)$ with integers $0 < p < q$ and gcd $(p, q) = 1$, then $D - l(1/b) = D - K(1/b)$ whenever $k - l = p_m$ for some integer m. Thus the definition of $\mathcal{M}_1$ says right away that $\mathcal{M}_1^k(G) = \mathcal{M}_1^p(G)$ for any $k \geq p$ and any a-translation periodic set $G \supset B$. Simple computation also shows that $\mathcal{M}_2^k(G)$ is the union of the set G with the sets of the form
$$\left(A + \tfrac{1}{b}\right) \cap \left(A + \tfrac{2}{b}\right) \cap \ldots \cap \left(A + \tfrac{j}{b}\right) \cap \left(G + \tfrac{j}{b}\right)$$
for $j \in \{1,2,3, \ldots, k\}$. By similar argument, we also have $\mathcal{M}_2^k(G) = \mathcal{M}_2^p(G)$ for any $k \geq p$ and any a-translation periodic set $G \supset B$. These facts imply that the set $K = \mathcal{M}_1^p\left(\mathcal{M}_2^p(B)\right)$ is invariant under both maps $\mathcal{M}_1(G) = G \cup (G - 1/b)$ and $\mathcal{M}_2(G) = G \cup (A \cap G + 1/b)$. In particular, it says that
$$k - \tfrac{1}{b} \subseteq K.$$
When $k - \frac{1}{b} = k$, we can employ the same techniques as those found in the proof of both Lemma (4.1.5), and Proposition (4.1.9) we omit the proof to avoid repetition.

Now assume that $k - \frac{1}{b}$ is properly contained in k. It follows that $H + \frac{1}{b}$ is also properly contained in H. Let $J = H \setminus \left(H + \frac{1}{b}\right)$. Then J is a set of positive measure no bigger than $\frac{1}{b}$ and H is the disjoint union of the sets $J+\frac{j}{b}$ with $j \in \{0\} \cup \mathbb{N}$. We will construct a sequence of functions $f_j \in L^2(\mathbb{R})$ with $\|f_j\|$ approaching infinity and the corresponding $\sum_n \int_0^{1/b} |H_n(t)|^2\, dt$ staying bounded. In fact, we define $f_j(t)$ to be supported on the set H so that $f_j(t) = (-1)^k$ for $t \in J + \frac{k}{b}$ with $k \in \{0,1,2,\dots,j\}$. Thus $\|f_j\|^2 = (j+1)\mu(J)$. Because H is disjoint from K, H is also disjoint from B since K contains B. Therefore

$$\sum_n \int_0^{1/b} |H_n(t)|^2\, dt = \int_A \left| f(t) + f\left(t + \frac{1}{b}\right)\right|^2 dt. \tag{11}$$

For any $t \in A$, we have either $t \in A \cap K$ or $t \in A \backslash K$. if $t \in A \cap K$, then by definition of f_j we have $f_j(t) = 0$. Since K is invariant under $\mathcal{M}_2$, we have $t + \frac{1}{b} \in A \cap k + \frac{1}{b} \subset k$. Hence by definition of f_j we have $f_j\left(t + \frac{1}{b}\right) = 0$. On the other hand, if $t \in A\backslash K$, then $f_j(t) = 0$ only when $t \in J + \frac{k}{b}$ for some integer $k \ge j+1$. In this case, by definition of f_j, we always have $f_j\left(t+\frac{1}{b}\right) = 0.$ Lastly, if $t \in A\backslash k$ and $f_j(t) \neq 0$, then $f_j\left(t + \frac{1}{b}\right) \neq 0$ unless $t \in J + \frac{j}{b}$. Using these observations, we have that

$$\sum_n \int_0^{1/b} |H_n(t)|^2\, dt \le L_{1\mu(J)}.$$

Hence $(\chi_{[0,c)}, a, b)$ cannot be a Gabor frame.

It is clear that the necessity part of Theorem (4.1.18) follows readily from Theorem (4.1.3) before embarking on the proof of Theorem (4.1.3). As in the assumption of Theorem (4.1.3), we let G be an a-periodic set satisfying the condition that $E \subset G, \mu(\mathbb{R}\backslash G) > 0$ and $\mathcal{M}(G) = G$. Thought we use $\bar{F}$ to denote $\mathbb{R}\backslash\ F$ for any $F \subset \mathbb{R}$, we will also use H specifically to denote $\bar{G}$. Thus H is also a-periodic and hence $\mu(H \cap [0,a)) > 0$. We also denote $H' = H \cap [0, 1/b)$. Since $1/b > a, \mu(H') > 0$, also holds. Recall that $\mathcal{M}$ is a set-valued mapping defined in turn by several other set-valued mapping mapping p_j for $j \in \{1,2,3,4,5\}$ such that for any $G \subset \mathbb{R}, \mathcal{M}(G) = G \cup p_1(G) \cup p_2(G) \cup p_3(G) \cup p_4(G) \cup p_5(G)$. For the definition of the mappings p_j in terms of the sets A, B and the number M, reader may consult. Note that $G = \mathcal{M}(G)$ implies that $\bar{G} = \bar{G} \cap \overline{p_1(G)} \cap \overline{p_2(G)} \cap \overline{p_3(G)} \cap \overline{p_4(G)} \cap \overline{p_5(G)}$. We will use H_j to denote $\overline{p_j(G)}$ for each $j \in \{1,2,3,4,5\}$. Thus in our simplified notation, we have $H = H \cap H_1 \cap H_2 \cap H_3 \cap H_4 \cap H_5$. Before we begin our construction, we need to establish the following lemmas, which will be used extensively in our proof.

Lemma (4.1.20)[181]: Let G be an a-periodic set satisfying the condition that $E \subset G$, $\mu(\mathbb{R}\backslash G) > 0$ and $\mathcal{M}(G) = G$. Let $H = \mathbb{R}\ \backslash G$. Then the following are true:

(a) If $t \in H$, then either $t - M(1/b) \in A\ \backslash B$ or $t - M(1/b) \in B \backslash A$.

(b) If $t \in A \cap H$, then $t + 1\backslash b \in A\ \backslash B$.

(c) If $t \in A \cap H$, then there exists $k \in \{1,2,3,\dots,M\}$ such that $t + k(1\backslash b) \in H$.

(d) If $t \in B \cap H$, then there exists $k \in \{1,2,3,\dots,M-1\}$ such that $t + k(1\backslash b) \in H$.

(e) If $t \in A \cap H$, then $t + l(1/b) \in H$ for $l = M+1$.

(f) If $t \in B \cap H$ and $t + 1/b \in B$, then $t + l(1/b) \in H$ for $l = M$. Furthermore, in this case $t \in B/A$ always holds.

(g) If $t \in B \cap H$ and $t + 1/b \in A$, then there is an $l \in \{M, M+1\}$ such that $t + l(1/b) \in H$. Furthermore, in the case of $t + M(1/b) \in H$, $t \in B \backslash A$ holds. In the case of $t + (M+1)(1/b) \in H$, $t + 1 \backslash b \in A \backslash B$ holds.

Proof. Since $E = (A \cap B + M(1/b)) \cup (A \cap (B - 1/b)) \subset G$, we have $H \subset \bar{E}$. This can also be expressed as

$$H \subset (\bar{A} \cup \bar{B}) + M(1/b), \qquad H \subset \bar{A} \cup (\bar{B} - 1/b).$$

Items (a) and (b) are then easily derived from the above observation.

On the other hand, $H = H \cap H_1 \cap H_2 \cap H_3 \cap H_4 \cap H_5$ implies that for $j \in \{1,2,3,4,5\}$, we have $H \subset H_j$. This can also be expressed in the following way:

$$H \subset \bar{A} \cup \left(\bigcup_{k=1}^{M} \left(H - K(1/b) \right) \right),$$

$$H \subset \bar{B} \cup \left(\bigcup_{k=1}^{M-1} \left(H - K(1/b) \right) \right),$$

$$H \subset (\bar{A} - 1/b) \cup \bar{A} \cup (H - (M+1)(1/b),$$

$$H \subset (\bar{B} - 1/b) \cup \bar{B} \cup \left(H - M(1/b) \right),$$

$$H \subset (\bar{A} - 1/b) \cup \bar{B} \cup (H - (M+1)(1/b)) \cup (H - M(1/b)).$$

With the help of already established items (a) and (b), items (c)-(g) can be immediately derived from the above conditions concerning these sets.

According to the observation made after Lemma (4.1.4), in order to show that $(\chi_{[0,c)}, a, b)$ is not a Gabor frame, it suffices to construct a sequence of functions $f_{\square}$ with $\| f_j \|$ approaching infinity while the corresponding $\sum_n \int_0^{1/b} |H_n(t)|^2 dt$ stays bounded. We need to define the notion of translation congruence. Suppose E and F are measurable sets, s is a positive real number. We say that E and F are s-translation congruent, if there are measurable partitions $\{E_n \mid n \in \mathbb{Z}\}$ and $\{F_n | n \in \mathbb{Z}\}$ such that $E_n + ns = F_n$ for each $n \in \mathbb{Z}$.

We will construct functions f_j in such a way that for each $j \in \mathbb{N}$, f_j is supported on H, and the range of f_j is $\{-1,0,1\}$. More specifically, for each $j \in \mathbb{N}$, we will make sure that $f_j(t) = 1$ holds whenever $t \in H'$. Also, for each $j \in \mathbb{N}$ and each $t \in H'$, we will find $j+2$ natural numbers $0 = m_0 < m_1 < \cdots < m_{j+1}$ dependent on t, such that $f_j(s) \neq 0$ (namely, $f_j(s) \in \{-1,1\}$) if and only if $s = t + m_l(1/b)$ for some $l \in \{0,1,\dots,j+1\}$. We will also make sure that $f_j(t) = 0$ for any $t \in \mathbb{R}$ and any $j \in \mathbb{N}$, whenever $t + m(1/b) \notin H'$ for any $m \in \mathbb{Z}$. We will define those functions inductively. Once f_j is defined for some $j \in \mathbb{N}$, we define $f_{j+1}(t) = f_j(t)$ for each t satisfying $f_j(t) \in \{-1,1\}$. Then we will construct some set which is $1/b$-translation congruent to H' and on which f_j vanishes, and define $f_{j+1}(t) = 1$ or -1 for each t in such a set. Lastly, we define $f_{j+1}(t) = 0$ for any other $\in \mathbb{R}$. Note that H' is contained in $[0, 1/b)$ with positive measure, we will make sure that each f_j is measurable and the above mentioned conditions hold in a measurable fashion, thus guaranteeing that for each $j \in \mathbb{N}$, $\|f_j\|^2 = (2+j)\mu(H')$.

Because of inequality (4) and the specific forms taken by F_1 and F_2 as discussed, we see that the following two equations are crucial to our construction:

$$\sum_{j=0}^{M} f\big(t + j(1/b)\big) = 0, \tag{12}$$

$$\sum_{j=0}^{M-1} f\big(t + j(1/b)\big) = 0. \tag{13}$$

More importantly, we will construct f_j in such a way that the set of points in A (respectively B) where Eq. (12) (respectively Eq (13)) does not hold for f_j will always have measure no bigger than $2(M+2)\mu(H')$. This will be achieved if we make sure that, in a measurable fashion, for each $t \in H'$, the only possible points where Eqs. (4) and (5) are not satisfied by f_j are contained in the union of sets $\{t-(M+1)(1/b),\ldots,t-(1/b)\}$ and $\{t+m_{j+1}(1/b)-M(1/b), t+m_{j+1}(1/b)-(M-1)(1/b),\ldots,t+m_{j+1}(1/b)\}$ where $m_{j+1} \in \mathbb{N}$ is the largest natural number, dependent on t, for which $f_j\left(t+m_{j+1}(1/b)\right) \in \{-1,1\}$. This will then lead to the estimate that for each f_j constructed, we have

$$\int_A |F_1(t)|^2 dt \le 2(M+2)^3\mu(H'), \qquad \int_B |F_1(t)|^2 dt \le 2(M+2)^3\mu(H').$$

Hence such a sequence f_j will serve the purpose of showing that $(\chi_{[0,c)}, a, b)$ is not a Gabor frame. The construction of each f_j is based on a partition of certain subset of H with positive measure. We deal with the existence of such a partition for H' in the next lemma.

Lemma (4.1.21)[181]: Let $H' = H \cap [0, 1/b)$.then there is a partition of H' into finitely many subsets $H'_{v,k_1,\square_1}$ with $v \in \{1,2,3\}, k_1 \in \{1,2,\ldots.,M\}$ and $l_1 \in \{M, M+1\}$ such that $0 < l_1 - k_1 \le M$. The partition satisfies the following conditions:

(a) For any $k_1, l_1, H'_{1,k_1,l_1} \subset A\backslash B$, $H'_{2,k_1,l_1} \subset A \cap B$, $H'_{3,k_1,l_1} \subset B\backslash A$.

(b) For any $\{v, k_1, l_1\}$, and any $t \in H'_{v,k_1,l_1}, t + k_1(1/b) \in H$ and $t + l_1(1/b) \in H$.

Proof. Under the assumption $a < 1/b$, we always have $A \cup B = \mathbb{R}$. We may partition H' into disjoint union of sets $H' \cap (A\backslash B), H' \cap (A \cap B)$ and $H' \cap (B\backslash A)$. According to Lemma (4.1.20), for each $t \in H' \cap (A\backslash B)$, there is a $k \in \{1,2,3,\ldots,M\}$, such that $t + k(1\backslash b) \in H$. Also $t + l\left(\frac{1}{b}\right) \in H$ for $l = M + 1$. Likewise, for each $t \in H' \cap (A \cap B)$, because of item (a) of Lemma (4.1.20), we know that $t + M\left(\frac{1}{b}\right) \notin H$, So Lemma (4.1.20) Leads to the conclusion that there is a $k \in \{1,2,3,\ldots,M-1\}$, such that $t + k\left(\frac{1}{b}\right) \in H$ and also $t + l\left(\frac{1}{b}\right) \in H$ For $l = M + 1$. The last case is more involved. For each $t \in H' \cap (B\backslash A)$, though there is always a $k \in \{1,2,3,\ldots,M-1\}$ such that $t + k\left(\frac{1}{b}\right) \in H$, it is both possible for $l = M$ or $l = M + 1$ to satisfy $t + l\left(\frac{1}{b}\right) \in H$. The restrictions are stated in items (f) and (g) of Lemma (4.1.20). Yet for the purpose of obtaining a partition of H', if we allow empty sets in the midst, we may ignore such nuance. The above argument is enough to guarantee the existence of such a partition.

The construction of f_1 is based on such a partition for the set H'. For any $t \in H'$ we define $f_1(t) = 1$. For any $t \in H'_{v,k_1,l_1} + k_1\left(\frac{1}{b}\right)$, we define $f_1(t) = -1$. For any $t \in H'_{v,k_1,l_1} + l_1\left(\frac{1}{b}\right)$, we define $f_1(t) = 1$. For any other $t \in \mathbb{R}$, we define $f_1(t) = 0$. According to Lemma (4.1.21), $0 < l_1 - k_1 \leq M$ always holds. So f_1 is well defined. Now we use $\widetilde{H}'$ to denote the set of points where f_1 takes non-zero value. Also we define

$$\mathrm{H}'' = \bigcup_{v,K_1,l_1}\left(H'_{v,K_1,l_1} + k_1\left(\frac{1}{b}\right)\right), \quad H''' = \bigcup_{v,K_1,l_1}\left(H'_{v,K_1,l_1} + l_1\left(\frac{1}{b}\right)\right).$$

Cleary, both H'' and H''' are subsets of $\widetilde{H}'$ and they are both $1/b$ –translation congruent to H'. Hence $\mu(H'') = \mu(H''') = \mu(H')$. In fact, $\widetilde{H}'$ is the disjoint union of $H',\ H''$ and H'''. Therefore $\mu(\widetilde{H}') = 3\mu(H')$ and evidently $\|f_1\|^2 = 3\mu(H')$. Note also that the support of f_1 is contained in

$$\widehat{H}' = \bigcup_{v,k_1,l_1}\left(\bigcup_{m=0}^{l_1} H'_{v,K_1,l_1} + m(1/b)\right).$$

Since $l_1 \leq M+1$ holds for each $l_1 \in \{M, M+1\}$, $\widehat{H}'$ has measure no bigger than $(M+2)\mu(H')$. Also note that for each $t \in H'$ and each $m \in \mathbb{Z}$, $t + m(1/b) \in \widehat{H}'$ if and only if $t + m\left(\frac{1}{b}\right) \in \left\{t, t+1/b, \dots, t + l_1\left(\frac{1}{b}\right)\right\}$. Thus, for any $t \in H'$, the only possible points where Eqs. (4) and (5) are not satisfied by f_1 are contained in the union of sets $\{t - M(1/b), \dots, t - 1/b\}$ and $\left\{t + l_1\left(\frac{1}{b}\right) - M(1/\mathrm{b}), t + l_1(1/\mathrm{b}) - (M-1)(1/\mathrm{b}), \dots, t + l_1(1/\mathrm{b})\right\}$. Note that the later set is contained in $\widehat{H}'$, so the measure of the points in A (respectively B) where Eq. (3) (respectively Eq. (4)) does not hold for f_1 is no bigger that $M_\mu(H') + (M+2)\mu(H') < 2(M+2)\mu(H')$. Hence there is nothing else to check at this stage. Nevertheless, we make the following observation concerning the points which satisfy Eqs. (12) and (13).

Lemma (4.1.22)[181]: For any $m \in \{0,1\}, v \in \{1,2,3\}, k_1 \in \{1,2,3,\dots,M\}$ and $l_1 \in \{M, M+1\}$ and any $t \in H'_{v,K_1,l_1} + m\left(\frac{1}{b}\right)$, the following is true:

(a) f_1 Satisfies Eq. (12) whenever $t \in A$.

(b) f_1 Satisfies Eq. (13) whenever $t \in B$.

Proof. When $v = 1$, according to item (b) of Lemma (4.1.20), for any $m \in \{0,1\}$ and any k_1, l_1 ,if $t \in H'_{1,K_1,l_1} + m(1/b)$, then $t \in A \backslash B$. Thus we only need to check Eq. (12) which can be easily done due to the simple construction of f_1. Likewise, when $v = 2$. according to item (g) of lemma (4.1.20), we need to check both Eqs. (12) and (13) for $t \in H'_{2,K_1,l_1}$ but only Eq. (12) for $t \in H'_{2,K_1,l_1} + 1/b$. Note also that just as in the case of $v = 1$, when $v = 2,$ we always have $l_1 = M+1$. But unlike the case of $v = 1$, when $v = 2, k_1 \leq M-1.$ Likewise, when $v = 3$, we always have $l_1 = M$. Thought in the case of $v = 3$ there are more subcases, it is still quite straight forward to show that for any $m \in \{0,1\}$ and k_1, l_1, any $t \in H'_{3,K_1,l_1} + m(1/b)$, Eq. (12) (respectively Eq. (13)) is satisfied by f_1 whenever $t \in A$ (respectively $t \in B$). We omit the repetitive computations.

Note that as mentioned in the above proof, $l_1 = M$ holds whenever $v = 3$, whereas when $v \in \{1,2\}, l_1 = M + 1$ is always true. Once f_2 is constructed, it will become evident that for $t \in H'_{v,k_1,l_1} + m(1/b)$ with $v \in \{1,2\}, m \in \{0,1\}$ and any k_1, l_1, Eq. (12) (respectively (12)) is satisfied by f_2 whenever $t \in A$ (respectively $t \in B$). As for $t \in H'_{3,k_1,l_1} + m(1/b)$ with $m \in \{0,1\}$, though the same is also true, it takes a bit of proof. In fact, once f_2 is defined, we will show that the construction of f_2 guarantees $f_2\big(t + (M+1)(1/b)\big) = 0$ for any $t \in H'_{3,k_1,l_1}$ with any k_1, l_1. This fact then will lead quickly to the same desired conclusion.

For the purpose of defining f_2, we will first obtain a partition H''. Instead of using indexing triples as in the partition of H', we will use indexing quadruples $\{v, k_1, k_2, l_2\}$ with $v \in \{1,2,3\}, k_1, k_2 \in \{1,2,\dots,M\}$ and $l_2 \in \{M, M+1\}$. The reason for this choice of index will become apparent once f_2 is constructed. Let us first see how the partition is done for H''.

Lemma (4.1.23)[181]: Let H'' be defined as above. Then there is a partition of H'' into finitely many subsets H''_{v,K_1,k_2,l_2} with $v \in \{1,2,3\}, K_1, k_2 \in \{1,2,\dots,M\}$ and $l_2 \in \{M, M+1\}$ such that $0 < l_2 - k_2 \le M$. The partition satisfies the following conditions:

(a) For any $k_1, k_2, l_2, H''_{1,K_1,k_2,l_2} \subset A \backslash B$, $H''_{2,K_1,k_2,l_2} \subset A \cap B$, $H''_{3,K_1,k_2,l_2} \subset B \backslash A$.

(b) For any v, k_1, k_2, l_2 and any $t \in H''_{v,K_1,k_2,l_2}$, $t - k_1(1/b) \in H'$.

(c) For any v, k_1, k_2, l_2, and any $t \in H''_{v,K_1,k_2,l_2}$, $t + k_2(1/b) \in H'''$.

(d) For any v, k_1, k_2, l_2,and any $t \in H''_{v,K_1,k_2,l_2}$, $t + l_2(1/b) \in H$.

Proof. The argument is quite similar to that of the proof of Lemma (4.1.21). Again, Since our main concern is to obtain a partition of H'', to avoid unnecessary complications, we allow that some of the sets indexed by $\{v, K_1, k_2, l_2\}$ may be empty. Clearly, from the discussion of $\widetilde{H}'$ above, for each $t \in H''$, there are unique natural numbers K_1 and k_2 such that $t - k_1(1/b) \in H'$ and $t + k_2(1/b) \in H'''$. On the other hand, Lemma (4.1.20) also guarantees that for each $t \in H''$, there is a $l_2 \in \{M, M+1\}$, such that $\square + l_2(1/b) \in H$. Now the only thing left to show is the following: For each $t \in H''$, let K_1 and K_2 be the unique natural numbers K_1 and K_2 such that $t - K_1(1/b) \in H'$ and $t + K_2\left(\frac{1}{b}\right) \in H'''$, let $l_2 \in \{M, M+1\}$ as guaranteed by Lemma (4.1.20) with $t + l_2(1/b) \in H$, then it is always true that $k_1, k_2 \in \{1, \dots, M\}$ and $0 < k_1 - k_2 \le M$.

Indeed, if we consider the point $t' = t - k_1(1/b) \in H'$, and suppose $t' \in H'_{v,k'_1,l'_1}$, for some $k'_1 \in \{1, \dots, M\}$ and $l'_1 \in \{M, M+1\}$, then we must have $k_1 = k'_1$ and $k_2 = l'_1 - k'_1$. Thus $0 < k_1 \le M$. According to Lemma (4.1.21), $0 < l'_1 - k'_1 \le M$ always holds, Thus we always have $0 < k_2 \le M$. Since $l_2 \in \{M, M+1\}$, $l_2 - k_2 \le M$ clearly holds . Lastly, in order to show $l_2 - k_2 > 0$, we only need to make sure that $k_2 = l_2 = M$, never could happen. Observe that if $k_2 = l'_1 - k'_1 = M$ holds, it must be true that $l'_1 = M + 1$ and $k'_1 = 1$. Let us list all possible cases where this happens, and show that in each case $l_2 = M + 1$ must be true. For $t \in H''$, since $k_1 = 1$, we have must have $t' = t - 1/b \in H'$, thus either $t' \in A$ or $t' \in B \backslash \square$. Yet $t' \in A \cap H'$ implies $t' + 1/b = t \in A \backslash B$ which implies $l_2 = M + 1$.Whereas when $t' \in B \backslash A$, we see that $l'_1 = M + 1$ only when $t' + 1/b = t \in A$, in which case actually $t \in A \backslash B$, hence also $l_2 = M + 1$. Thus $0 < l_2 - k_2 \le M$ always hold.

We note in passing that, as indicated by the above lemma, our choice of the letter k_2 in the definition of the partition of H'' is actually appropriate since k_2 indeed represents one of the k guaranteed by item (c) and (d) of Lemma (4.1.20) for each $t \in H''$.

Now for any $t \in H'' + l_2(1/b)$, we define $f_2(t) = -1$. For any other $t \in \mathbb{R}$, the definition of f_2 agrees with that of f_1. As we have just proved, $k_2 < l_2$ holds for all non-empty sets in the partition of H'', So f_2 is will defined. Similarly, we use $\widetilde{H}''$ to denote the set of points where f_2 takes non-zero value. Clearly, $\widetilde{H}''$ is the disjoint union of H', H'', H''' and the set

$$H^{\{4\}} = \bigcup_{v,k_1,k_2,l_2} \left(H''_{v,k_1,k_2,l_2} + l_2(1/b)\right).$$

Also note that, from the proof of the last Lemma, for any $t \in H''$ with corresponding numbers k_1, k_2 and l_2, the number k_2 can always be expressed uniquely as $k_2 = l_1' - k_1'$ where l_1' and k_1' are the numbers corresponding to the unique point $t' = t - k_1(1/b) \in H'$. Since $H'' = \cup_{v,k_1,l_1}(H'_{v,k_1,l_1} + k_1(1/b))$, it then follows that the set $H''' = \cup_{v,k_1,l_1}(H'_{v,k_1,l_1} + l_1(1/b))$ can be written as

$$H''' = \bigcup_{v,k_1,l_1} \left(H''_{v,k_1,k_2,l_2} + k_2(1/b)\right).$$

Evidently, H', H'', H''' and $H^{\{4\}}$ are mutually $1/b$-translation congruent, therefore $\mu(\widetilde{H}'') = 4\mu(H')$. Thus $\|f_2\|_2^2 = 4\mu(H')$. Now we look at the support of f_2 more closely. According to the partition of H' and H'', the support of f_2 is contained in the disjoint union of the following two sets:

$$\bigcup_{v,k_1,l_1} \bigcup_{m=0}^{k_1+1} (H'_{v,k_1,l_1} + m(1/b)), \qquad \bigcup_{v,k_1,K_2,l_2} \bigcup_{m=2}^{l_2} (H''_{v,k_1,k_2,l_2} + m(1/b)).$$

Similarly, we denote the union of the above two sets as $\widehat{H}''$. Note that the second set above has measure less than $(M+1)\mu(H')$, while the first set above can also be written as

$$\bigcup_{v,k_1,k_2,l_2} \bigcup_{m=-k_1}^{1} H''_{v,k_1,k_2,l_2} + m(1/b).$$

Now it should be clear that once the next lemma is established, due to the fact that $l_2 \le M+1$ always holds, it follows immediately that all possible points where Eqs.(4) and (5) are not satisfied by f_2 are contained in the union of sets $\cup_{m=1}^{M+1} H' - m(1/b)$ and $\cup_{v,k_1,k_2,l_2} \cup_{m=2}^{l_2} H''_{v,k_1,k_2,l_2} + m(1/b)$, thus it has measure no bigger that $2(M+2)\mu(H')$.

Lemma (4.1.24)[181]: Let H'' and its partition be defined as above. For any $t \in H''_{v,k_1,k_2,l_2} + m(1/b)$ With $m \in \{-k_1, \dots, 0, 1\}$ and $v \in \{1,2,3\}$, $k_1, k_2 \in \{1, \dots, M\}$ and $l_2 \in \{M, M+1\}$, Eq. (12) is satisfied by f_2 whenever $t \in A$ and Eq. (13) is satisfied by f_2 whenever $t \in B$.

Proof. We show the case of $m = -k_1$ and $m = -k_1 + 1$ first. For this purpose, note that according to the definition of H' and H'', we have $t \in H''_{v,k_1,k_2,l_2} - k_1\left(\frac{1}{b}\right)$ if and only if $t \in H'_{v',k_1,l'_1}$ for some $v' \in \{1,2,3\}$ and $l'_1 \in \{M, M+1\}$. Likewise, $t \in H''_{v,k_1,k_2,l_2} - k_1\left(\frac{1}{b}\right) + 1/b$ if and only if $t \in H'_{v',k_1,l'_1} + 1/b$ for some $v' \in \{1,2,3\}$ and $l'_1 \in \{M, M+1\}$. Recall that it is already checked in Lemma (4.1.22) that those equations are satisfied by f_1 for such t .Since f_1 and f_2 are identical on the union of the following two sets:

$$\bigcup_{v,k_1,l_1} \bigcup_{m=0}^{k_1} H'_{v,k_1,l_1} + m(1/b), \qquad \bigcup_{v,k_1,k_2,l_2} \bigcup_{m=1}^{l_2-1} H''_{v,k_1,k_2,l_2} + m(1/b).$$

it should be clear that if $t \in H'_{v',k_1,l'_1}$ for some $v' \in \{1,2,3\}$ and $l'_1 \in \{M, M+1\}$, f_2 is also satisfies Eq. (3) (respectively (4)) whenever $t \in A$ (respectively $t \in B$). For the case of $m = -k_1 + 1$, the only non-trivial case that needs detailed discussion is when $k_1 = 1$. Now if $t \in A \cap H'_{v',k_1,l'_1} + 1/b$ and $k_1 = 1$, the corresponding l'_1 for $t' = t - 1/b \in H'_{v',k_1,l'_1}$ is always $M + 1$. Thus the same conclusion is also trivially true. This leaves us with the case of $t \in (B\backslash A) \cap H'_{v',k_1,l'_1} + 1/b$ and $k_1 = 1$, which we discuss in the following.

Note that the assumption $k_1 = 1$ means that for such $t \in (B\backslash A) \cap H'_{v',k_1,l'_1} + 1/b$ we must have $t \in H''$ and $t' = t - 1/b \in H'$. There are three possibilities . First of all, it is possible that $t \in A \cap B$. In this case the corresponding l'_1 for $t' = t - 1/b$ guaranteed by Lemma (4.1.20) must be M (according to the item (g) of Lemma (4.1.20) in this case l'_1 cannot be $M + 1$). Note that in this case we only need to make sure that $t + M(1/b) \notin H^{\{4\}}$, thus it is enough to make sure that $t + M(1/b) \notin H$.

Indeed, this is guaranteed item (a) of Lemma (4.1.20). Secondly ,it is possible that $t \in B\backslash A$. In this case the corresponding l'_1 for such $t' = t - 1/b$ guaranteed by Lemma (4.1.20) must also be M. In this case Eq. (12) is satisfied by f_2 for t vacuously. Whereas Eq. (13) is trivially satisfied by f_2 for such t.

Thirdly, it is possible that $t \in A\backslash B$. In this case the corresponding l'_1 for $t' = t - 1/b$ guaranteed by Lemma (4.1.20) could either be M or $M + 1$. When $l'_1 = M + 1$, it is trivial for the same reason as when $t \in A \cap H'_{v',k_1,l'_1} + 1/b$. We look at the subcase of $l'_1 = M$. In this subcase, again we only need to make sure that $t + M(1/b) \notin H^{\{4\}}$. Indeed, since $t \in A\backslash B$ according to item (c) of Lemma (4.1.20), the number l_2 corresponding to such t is $M + 1$. This means that $t + (M+1)(1/b) \in H^{\{4\}}$. Thus $t + M\left(\frac{1}{b}\right) \notin H$.

After the case of $m = -k_1$and $m = -k_1 + 1$ is done, we see that when $k_1 = 1$ we have already finished checking. Otherwise, $k_1 > 1$. We note that when $m \in \{-k_1 + 2, \dots, -1\}$, the checking is trivially done. When $m \in \{0,1\}$, the checking is exactly the same as in Lemm (4.1.22).

Now we proceed to construct f_3 based on a partition of H'''.

Lemma (4.1.25)[181]: Let H''' be defined as above. Then there is a partition of H''' into finitely many subsets H'''_{v,k_2,k_3,l_3} with $v \in \{1,2,3\}, k_2, k_3 \in \{1,2,\dots,M\}$ and $l_3 \in \{M, M+1\}$such that $0 < l_3 - k_3 \le M$. The partition satisfies the following conditions:

(a) For any k_2, k_3, l_3 $H'''_{1,k_2,k_3,l_3} \subset A\backslash B$, $H'''_{2,k_2,k_3,l_3} \subset A \cap B$, $H'''_{3,k_2,k_3,l_3} \subset B\backslash A$.

(b) For any v, k_2, k_3, l_3 and any $t \in H'''_{v,k_2,k_3,l_3}$, $t - k_2(1/b) \in H''$.

(c) For any v, k_2, k_3, l_3 and any $t \in H'''_{v,k_2,k_3,l_3}, t + k_3(1/b) \in H^{\{4\}}$.

(d) For any v, k_2, k_3, l_3 and any $t \in H'''_{v,k_2,k_3,l_3}, t + l_3(1/b) \in H$.

Proof. The argument is quite similar to that of Lemma (4.1.23), we omit it.

Now for any $t \in H''' + l_3(1/b)$, we define $f_3(t) = 1$. For any other $t \in \mathbb{R}$, the definition of f_3 agrees with that of f_2. As we have just proved, $k_3 < l_3$ holds for all non-empty sets in the partition of H''', so f_3 is well defined. Likewise we define

$$H^{\{5\}} = \bigcup_{v,k_2,k_3,l_3} \left(H'''_{v,k_2,k_3,l_3} + l_3(1/b)\right).$$

Likewise, for any $t \in H'''$ with corresponding numbers k_2, k_3 and l_3, the number k_3 can always be expressed uniquely as $k_3 = l''_2 - k''_2$ where l''_2 and k''_2 are the unique numbers corresponding to the unique point $t'' = t - k_2(1/b) \in H''$. Since $H''' = \bigcup_{v,k_2,k_3,l_3} \left(H''_{v,k_1,k_2,l_2} + k_2(1/b)\right)$, it then f0llows that the set

$$H^{\{4\}} = \bigcup_{v,k_1,k_2,l_2} \left(H''_{v,k_1,k_2,l_2} + l_2(1/b)\right)$$

can be written as $H^{\{4\}} = \bigcup_{v,k_2,k_3,l_3} \left(H''_{v,k_2,k_3,l_3} + k_3(1/b)\right)$.

Similarly we get $\|f_3\|_2^2 = 5\mu(H')$ .Now we look at the support of f_3 more closely. According to the partition of H''', the support of f_3 is contained in the union of the set $\widehat{H}'$ and the following set:

$$\bigcup_{v,k_2,k_3,l_3} \bigcup_{m=2}^{l_3} H'''_{v,k_2,k_3,l_3} + m(1/b).$$

Again we denote the union of the above mentioned two sets as $\widehat{H}'''$. Thus once we prove that for any point $t \in \widehat{H}'$, Eq. (3) (respectively Eq. (4)) is satisfied whenever $t \in A$ (respectively B), it will follow immediately that the set of points in A (respectively B), where f_3 does not satisfy Eq. (3) (respectively Eq. (4)) is contained in the union of $\bigcup_{m=1}^{M+1} H' - m(1/b)$ and $\bigcup_{v,k_2,k_3,l_3} \bigcup_{m=2}^{l_3} H'''_{v,k_2,k_3,l_3} + m(1/b)$, thus it has measure no bigger that $2(M+2)\mu(H')$. Yet if we take Lemma (4.1.24) into consideration, we see that in order to show that for any point $t \in \widehat{H}'$, Eq. (3) (respectively Eq. (4)) is satisfied whenever $t \in A$ (respectively B), it is more than enough to prove the following Lemm:

Lemma (4.1.26)[181]: Let H''' and its partition be defined as above. For any $t \in H'''_{v,k_2,k_3,l_3} + m(1/b)$ with $m \in \{-k_2, \ldots, 0, 1\}$ and $v \in \{1,2,3\}, k_2, k_3 \in \{1, \ldots, M\}$ and $l_3 \in \{M, M+1\}$, Eq. (12) is satisfied by f_3 whenever $t \in A$ and Eq. (13) is satisfied by f_3 whenever $t \in B$.

Proof. The proof is similar to that of Lemma (4.1.24) we omit it.

Lemma (4.1.27)[181]: Let $H^{\{j\}}$ be as defined. Then there is a partition of $H^{\{j\}}$ into finitely many subsets $H^{\{j\}}_{v,k_{j-1},k_j,l_j}$ with $v \in \{1,2,3\}, k_{j-1}, k_j \in \{1,2,\ldots,M\}$ and $l_j \in \{M, M+1\}$ such that $0 < l_j - k_j \le M$. The partition satisfies the following conditions:

(a) For any $k_{j-1}, k_j, l_j, H^{\{j\}}_{1,k_{j-1},k_j,l_j} \subset A \backslash B$, $H^{\{j\}}_{2,k_{j-1},k_j,l_j} \subset A \cap B$, $H^{\{j\}}_{3,k_{j-1},k_j,l_j} \subset B \backslash A$.

(b) For any v, k_{j-1}, k_j, l_j, and any $t \in H^{\{j\}}_{v,k_{j-1},k_j,l_j}, t - k_{j-1}(1/b) \in H^{\{j-1\}}$.

(c) For any v, k_{j-1}, k_j, l_j, and any $t \in H^{\{j\}}_{v,k_{j-1},k_j,l_j}$, $t + k_j(1/b) \in H^{\{j+1\}}$.

(d) For any v, k_{j-1}, k_j, l_j, and any $t \in H^{\{j\}}_{v,k_{j-1},k_j,l_j}$, $t + l_j(1/b) \in H$.

Proof. The argument is quite similar to that of Lemma (4.1.25), we omit it.

Now for any $t \in H^{[j]} + l_3(1/b)$, we define $f_j(t) = (-1)^{j+1}$. For any other $t \in \mathbb{R}$, the definition of f_j agrees with that of f_{j-1}. At this stage, it is already established that $k_j < l_j$ holds for all non-empty sets in the partition of $H^{[j]}$, so f_j is well defined. Likewise we define

$$H^{\{j+2\}} = \bigcup_{v,k_{j-1},k_j,l_j} \left(H^{\{j\}}_{v,k_{j-1},k_j,l_j} + l_j(1/b)\right).$$

Likewise, for any $t \in H^j$ with the unique corresponding numbers k_{j-1}, k_j and l_j, the number k_j can always be expressed uniquely as $k_j = l^{(j-1)}_{j-1} - k^{(j-1)}_{j-1}$ where $l^{(j-1)}_{j-1}$ and $k^{(j-1)}_{j-1}$ are the unique numbers corresponding to the unique point $t^{(j-1)} = t - k_{j-1}(1/b) \in H^{j-1}$. Since

$$H^{\{j\}} = \bigcup_{v,k_{j-2},k_{j-1},l_{j-1}} \left(H^{\{j-1\}}_{v,k_{j-1},k_{j-1},l_{j-1}} + k_{j-1}(1/b)\right),$$

it then follows that $H^{\{j+1\}} = \bigcup_{v,k_{j-2},k_{j-1},l_{j-1}} \left(H^{\{j-1\}}_{v,k_{j-2},k_{\square-1},l_{j-1}} + l_{j-1}(1/b)\right)$ can be written as

$$H^{\{j+1\}} = \bigcup_{v,k_{j-1},k_j,l_j} \left(H^{\{j\}}_{v,k_{j-1},k_j,l_j} + k_j(1/b)\right).$$

We use $\widetilde{H^{\{j\}}}$ to denote the sets of points where f_j takes non-zero value. Then clearly $\widetilde{H^{\{j\}}}$ is a disjoint union of mutually $1/b$-translation congruent sets $H', \dots, H^{\{j+2\}}$. Thus $\mu\left(\widetilde{H^{\{j\}}}\right) = (j+2)\mu(H')$ and $\|f_j\|_2^2 = (j+2)\mu(H')$. Now we look at the support of f_j more closely. The support f_j is contained in the union of the sets $\widehat{H^{\{j-2\}}}$ and the following set:

$$\bigcup_{v,k_{j-1},k_j,l_j} \bigcup_{m=2}^{l_j} H^{\{j\}}_{v,k_{j-1},k_j,l_j} + m(1/b).$$

Again we denote the union of the above mentioned two sets as $\widehat{H^{\{j\}}}$. Note that the latter set has measure no greater than $(M+2)\mu(H')$. Thus once we prove that for any point $t \in \widehat{H^{\{j-2\}}}$, Eq. (3) (respectively Eq. (4)) is satisfied whenever $t \in A$ (respectively B), it will follow immediately that the set of points in A (respectively B) where f_j does not satisfy Eq. (3) (respectively Eq. (4)) is contained in the union of $\bigcup_{m=1}^{M+1} H' - m(1/b)$ and $\bigcup_{v,k_{j-1},k_j,l_j} \bigcup_{m=2}^{l_j} H^{\{j\}}_{,k_{j-1},k_j,l_j} + m(1/b)$, thus it has measure no bigger that $2(M+2)\mu(H')$.

Yet since we proceed inductively, we see that in order to prove that for any point $t \in \widehat{H^{\{j-2\}}}$, Eq. (3) (respectively Eq. (4)) is satisfied whenever $t \in A$ (respectively B), it is more than enough to prove the following lemma:

Lemma (4.1.28)[181]: Let $H^{\{j\}}$ and its partition be as defined. For any $v \in \{1,2,3\}, k_{j-1}, k_j \in \{1, \dots, M\}, l_j \in \{M, M+1\}$ and $m \in \left\{-k_{j-1}, \dots, 0, 1\right\}$ if $t \in H^{\{J\}}_{v,k_{j-1},k_j,l_j} + m(1/b)$, then Eq. (12) is satisfied by f_j whenever $t \in A$ and Eq. (13) is satisfied by f_j whenever $t \in B$.

Proof. The proof is similar to that Lemma (4.1.24) we omit it.

This concludes the proof of Theorem (4.1.3).

Theorem (4.1.29)[181]: Let $0 < a < 1/b < c = M(1/b) + d$ for some $0 < d < 1/b$ and $M \in \mathbb{N} + 1$. If $ab \in \mathbb{Q}$, then there is some $s_0 \in \mathbb{N}$, such that $\mathcal{M}^{s_0+1}(E) = \mathcal{M}^{s_0}(E)$ for some $s_0 \in \mathbb{N}$.

Proof. For the definitions of the sets A, B, E and the set-valued mapping $\mathcal{M}$. In the case of $ab \in \mathbb{Q}$, we may assume that $ab = \frac{q}{p}$ with $p, q \in \mathbb{N}$ and $\gcd(p, q) = 1$. Since A and B are a- translation invariant sets, so there are at most p different elements in the collection of sets $\{A + k(1/b) | k \in \mathbb{Z}\}$. Likewise, there are at most p different elements in the collection of sets $\{B + k(1/b) | k \in \mathbb{Z}\}$. Thus there are at most $2^{2^{2p}}$ elements in the collection $\{\mathcal{M}^l(E) | l \in \mathbb{N}\}$. Hence the conclusion follows.

Conjecture (4.1.30)[181]: Let $0 < a < 1/b < c = M(1/) + d$ for some $0 < d < 1/b$ and $M \in \mathbb{N} + 1$. Then there is some $s_0 \in \mathbb{N}$, such that $\mathcal{M}^{s_0+1}(E) = \mathcal{M}^{s_0}(E)$.

Note that the case $ab \in \mathbb{Q}$ is settled in Theorem (4.1.29). On the other hand, when $c = M(1/b) + d$ for some $M \in \mathbb{N} + 1$ and $0 < d < 1/b$, if $\max\{d, 1/b - d\} \geq a$, then we have either $A = \mathbb{R}$ or $B = \mathbb{R}$. In this case the conjecture is also trivially proven affirmatively. Thus we need to know whether the same holds when $ab \in \mathbb{R} \setminus \mathbb{Q}$, $c = M(1/b) + d$ for some $M \in \mathbb{N} + 1$ and $0 < d < 1/b$, while $a > \max\{d, 1/b - d\}$.

Lastly, we briefly compare our results and those of Janssen's in [199]. As we have already mentioned, the special case of $a = \frac{1}{q}(1/b)$ with $q \in N$, the special case of $M = 1$ and the special case of $a \leq \min\{d, 1/b - d\}$ (as discussed in this article in Theorems (4.1.10), (4.1.19), and Lemma (4.1.7) respectively) are already completely solved by Janssen in [199]. Also, using Theorem (4.1.19), it is quite easy to recover Janssen's more specific conditions in the case of ab being rational.

In the case of $M > 1$ with ab being rational, Janssen developed in [199] an algorithm to determine whether $(\chi_{[0,c)}, a, 1/b)$ is a Gabor frame, while our Theorem (4.1.18), together with Theorem (4.1.29), offers both a complete solution and a concrete criteria in terms of some set-valued mappings.

As we mentioned before, Theorem (4.1.18), leads to a complete solution whenever $\max\{d, 1/b - d\} \geq a$ since in this case we have either $A = \mathbb{R}$ or $B = \mathbb{R}$. Janssen's Theorem 3.3.4.4 in [199] states that when $M > 1$, $c = M(1/b) + d$ with $0 < d < 1/b$, whenever ab is irrational and $\max\{d, 1/b - d\} \geq a$, then $(\chi_{[0,c)}, a, 1/b)$ is always a Gabor frame. This result can be quickly recaptured by Theorem (4.1.18) since the set-valued mapping $\mathcal{M}$ is significantly simplified with either A or B being $\mathbb{R}$, and we only need to look at the mappings p_3 or p_4 to see that the density argument used in the proof of Lemma (4.1.8) applies readily here to quickly get the same conclusion.

For the difficult case of $a \geq \max\{d, 1/b - d\}$, Janssen's Theorem 3.3.5.2 in [199] (which generalizes Theorem 3.3.5.1) states that when $M > 1, d = I(1/b - a)$ with $I \in \{3,4,\dots\}$ satisfying $1/b - a \leq \frac{1}{I+1}$ and $I \equiv 0 \mod (M+1)$, then $(\chi_{[0,c)}, a, 1/b)$ is not a Gabor frame. This result can also be quickly recaptured by Theorem (4.1.3). Indeed, if we assume $I = k(M+1)$ for some natural number k, it is easy to check that the union of $A \cap B + M(1/b)$ and $\bigcup_{j=0}^{k-1}\big((B - 1/b) \cap A - j(M+1)(1/b)\big)$, which is clearly a proper subset of $\mathbb{R}$, is invariant under $\mathcal{M}$.

It is thus quite baffling that we are not able to recover, except in the case of $M = 2$, Janssen's last theorem in [199]. Dealing with the difficult case of $a \geq max\{d, 1/b - d\}$, Theorem 3.3.5.3 states that whenever $M > 1$ and ab is irrational, if $d = I(1/b - a)$ with $I \in \{1,2,3,\dots\}$ satisfying $1/b - a \leq \frac{1}{I+1}$ and $I \not\equiv 0 \mod (M+1)$, then $(\chi_{[0,c)}, a, 1/b)$ is a Gabor frame. The main reason for the failure, it appears, is that the sequence of sets generated by E under iterations of $\mathcal{M}$ does not seem to have regular enough features we would like it to have in order for us to use such density argument as in the proof of Lemma (4.1.8). Thus it is all the more intriguing that Janssen is able to prove the theorem using virtually the same simple density fact as the one in our Lemma (4.1.8).

However, with the help of Theorem (4.1.3), examples can be found which negatively settles Janssen's conjecture in [199] that when ab is irrational, the only cases of $(\chi_{[0,c)}, a, 1/b)$ not being a Gabor frame is described in Theorem 3.3.5.2 in [199].

Example (4.1.31)[181]: Suppose $a = \frac{\sqrt{6}}{3}$, $b = 1, p = 8169, q = 10005$. Let $d = p - (q-1)a, M = p - 1$ and $c = M + d$. Then $0 < \max\{d, 1 - d\} < a < 1/b = 1{,}4(1-a) < d < 5(1-a)$, and $(\chi_{[0,c)}, a, b)$ is not a Gabor frame.

Proof. First we check that the following holds:

$$\frac{p-5}{q-6} < \frac{p}{q} < a < \frac{p-4}{q-5} < \frac{p+1}{q+1} < \frac{p-2}{q-3} < \frac{p-1}{q-2} < \frac{p}{q-1} < 1.$$

Using the above inequalities, we can that hen establish the facts that $0 < d < a, 0 < 1 - d < a$ and $4(1-a) < d < 5(1-a)$. Also useful is the fact that $a - d < 1 - a$ with $a = \frac{\sqrt{6}}{3}, b = 1$, clearly we have $a < 1/b$. We may also substitute $1/b$ with 1 in the definitions of A, B, E and the mapping $\mathcal{M}$ and p_j for $j \in \{1,2,3,4,5\}$.

We then calculate to get $A \cap (B-1) = \bigcup_{n\in\mathbb{Z}}[na + a + d - 1, na + d)$ and $A \cap B = (\bigcup_{n\in\mathbb{Z}}[na + a, na + 1)$. It is also quick to establish the facts that $(B-1) \cap (A-1) \cap A = A \cap (B-1) \cap B = \emptyset$.

From $M = P - 1$ and $p = (q-1)a + d$, it then follows that $A \cap B + M = A \cap B + d - 1 = \bigcup_{n\in\mathbb{Z}}[na + a + d - 1, na + d) = A \cap (B-1)$.Therefore $E = A \cap (B-1) = A \cap B + M$. Finally, we need to check $\mathcal{M}(E) = E$ by proving $p_j(E) = \emptyset$, for each $j \in \{1,2,3,4,5\}$. Observe that $E \cap (E+1) = \emptyset$. The definitions of p_j then lead to the conclusion that $p_j(E) = \emptyset$ whenever $j \in \{1,2,5\}$. On the other hand, the identity $E - M = A \cap B$ leads directly to the conclusion that

$$p_3(E) = (B-1) \cap (A-1) \cap A = \emptyset.$$

$$p_4(E) = (E - M) \cap (B-1) \cap B = A \cap (B-1) \cap B = \emptyset.$$

Section (4.2): Totally Positive Functions

The fundamental problem of Gabor analysis is to determine triples (g,α,β) consisting of an L^2-function g and lattice parameters $\alpha,\beta>0$, such that the set of functions $\mathcal{G}(g,\alpha,\beta)=\{e^{2\pi i\beta lt}g(t-\alpha k): k,l\in\mathbb{Z}\}$ constitutes a frame for $L^2(\mathbb{R})$. Thus the fundamental problems is to determine the set (the frame set)

$$\mathcal{F}(g)=\{(\alpha,\beta)\in\mathbb{R}_+^2: \mathcal{G}(g,\alpha,\beta)) \text{ is a frame}\}. \quad (15)$$

It is stunning how little is known about the nature of the set $\mathcal{F}(g)$, even after twenty years of Gabor analysis. The famous Janssen tie [225] shows that the set $\mathcal{F}(g)$ can be arbitrarily complicated, even for a "simple" function such as the characteristic function $g=\chi_I$ of an interval.

If g is in the Feichtinger algebra M^1, then the set $\mathcal{F}(g)$ is open in $\mathbb{R}_+^2$ [218].Furthermore, if $g\in M^1$, then $\mathcal{F}(g)$ contains a neighborhood U of 0 in $\mathbb{R}_+^2$. Much effort has been spent to improve the analytic estimates and make this neighborhood as large as possible [210,214,231]. The fundamental density theorem asserts that $\mathcal{F}(g)$ is always a subset of $\{(\alpha,\beta)\in\mathbb{R}_+^2: \alpha\beta\le 1\}$ [215,219,222]. If $g\in M^1$, then a subtle version of the uncertainty principle, the so-called Balian-Low theorem, states that $\mathcal{F}(g)\subseteq\{(\alpha,\beta): \alpha\beta<1\}$ [209,213]. This means that $\{(\alpha,\beta): \alpha\beta\le 1\}$ is the maximal set that can occur as a frame set $\mathcal{F}(g)$.

Until now, the catalogue of windows g for which $\mathcal{F}(g)$ is completely known, consists of the following functions: if g is either the Gaussian $g(t)=e^{-\pi t^2}$, the hyperbolic secant $g(t)=(e^t+e^{-t})^{-1}$, the exponential function $e^{-|t|}$, then $\mathcal{F}(g)=\{(\alpha,\beta)\in\mathbb{R}_+^2: \alpha\beta<1\}$; if g is the one-sided exponential function $g(t)=e^{-t}\chi_{\mathbb{R}^+}(t)$, then $\mathcal{F}(g)=\{(\alpha,\beta)\in\mathbb{R}_+^2: \alpha\beta\le 1\}$, in addition, the dilates of these functions and their Fourier transform, $g(t)=(1+2\pi it)^{-1}$ and $g(t)=(1+4\pi^2t^2)^{-1}$, also have the same frame set. The case of the Gaussian was solved independently by Lyubarski [230] and Seip [236] in 1990 with methods from complex analysis in response to a conjecture by Daubechies and Grossman [216]; the case of the hyperbolic secant can be reduced to the Gaussian with a trick of Janssen and Strohmer [227], the case of the exponential functions is due to Janssen [224,226]. We note that in all these cases the necessary density condition $\alpha\beta<1$ (or $\alpha\beta\le 1$) is also sufficient for $\mathcal{G}(g,\alpha,\beta)$ to generate a frame.

The example of the Gaussian lead Daubechies to conjecture that $\mathcal{F}(g)=\{(\alpha,\beta)\in\mathbb{R}_+^2: \alpha\beta<1\}$ whenever g is a positive function in L^1 with positive Fourier transform in L^1 [214,p.981]. This conjecture was disproved in [223].

Surprisingly, no alternatives to Daubechie's conjecture have been formulated so far. We deal with a modification of Daubechie's conjecture and prove that the frame set of a large class of functions is indeed the maximal set $\mathcal{F}(g)=\{(\alpha,\beta)\in\mathbb{R}_+^2: \alpha\beta<1\}$.

This breakthrough is possible by combining ideas from Gabor analysis, approximation theory and spline theory, and sampling theory. The main observation is that all functions above-The Gaussian, the hyperbolic secant, and the exponential functions-are totally positive functions. This means that for every two sets of increasing real numbers $x_1<x_2<\cdots<x_N$ and $y_1<y_2<\cdots<y_N$, $N\in\mathbb{N}$, the determinant of the matrix $[g(x_j-y_k)]_{1\le j,k\le N}$ is non-negative.

Indeed, for a large class of totally positive functions to be defined in (29) we will determine the set $\mathcal{F}(g)$ completely. We have the following:

Theorem (4.2.1)[205]: Assume that $g \in L^2(\mathbb{R})$ is a totally positive function of finite type ≥ 2. Then $\mathcal{F}(g) = \{(\alpha, \beta) \in \mathbb{R}_+^2 : \alpha\beta < 1\}$. In other words, $\mathcal{G}(g, \alpha, \beta)$ is a frame, if and only if $\alpha\beta < 1$.

This theorem increases the number of functions with known frame set from six to uncountable. We will see later that the totally positive functions of finite type can be parameterized by a countable number of real parameters, see (29). Among the examples of totally positive functions of finite type are the two-sided exponential $e^{-|t|}$ (already known), the truncated power functions $g(t) = e^{-t} t^r \chi_{\mathbb{R}_+}$ for $r \in \mathbb{N}$, the function $g(t) = (e^{-at} - e^{-bt})\chi_{\mathbb{R}^+}(t)$ for $a, b > 0$, or the asymmetric exponential $g(t) = e^{at}\chi_{\mathbb{R}^+}(-t) + e^{-bt}\chi_{\mathbb{R}^+}(t)$, and the convolutions of totally positive functions of finite type. In addition the class of g such that $\mathcal{F}(g) = \{(\alpha, \beta) \in \mathbb{R}_+^2 : \alpha\beta < 1\}$ is invariant with respect to dilation, time-frequency shifts, and the Fourier transform. Since $\mathcal{G}(g, \alpha, \beta) = \mathcal{G}(\hat{g}, \beta, \alpha)$, we obtain a complete description of the frame set of the Fourier transforms of totally positive functions. For instance, if $g(t) = (1 + 4\pi^2 t^2)^{-n}$ for $n \in \mathbb{N}$, then $\mathcal{F}(g) = \{(\alpha, \beta) \in \mathbb{R}_+^2 : \alpha\beta < 1\}$.

To compare with Daubechies' original conjecture, we note that every totally positive and even functions possesses a positive Fourier transform. Theorem (4.2.1) yields a large class of functions for which Daubechies' conjecture is indeed true. Furthermore, Theorem (4.2.1) suggests the modified conjecture that the frame set of every continuous totally positive function is $\mathcal{F}(g) = \{(\alpha, \beta) \in \mathbb{R}_+^2 : \alpha\beta < 1\}$.

The main tool is a generalization of the total positivity to infinite matrices. We will show that an infinite matrix of the form $[g(x_j - y_k)]_{j,k \in \mathbb{Z}}$ possesses a left-inverse, when g is totally positive and some natural conditions hold for the sequences (x_j) and (y_k) (Theorem (4.2.8)).

The analysis of Gabor frames and the ideas developed in the proof of Theorem (4.2.1). Lead to a surprising progress on another open problem, namely (nonuniform) sampling in shift-invariant spaces. Fix a generator $g \in L^2(\mathbb{R})$, a step-size $h > 0$, and consider the subspace of $L^2(\mathbb{R})$ defined by

$$V_h(g) = \left\{ f \in L^2(\mathbb{R}) : f = \sum_{k \in \mathbb{Z}} c_k g(. - kh) \right\}.$$

For the case $h = 1$ we write $V(g)$, for short. We assume that translates $g(\cdot - k)$, $k \in \mathbb{Z}$, from a Riesz basis for $V(g)$ so that $\|f\|_2 \asymp \|c\|_2$. Shift-invariant spaces are used as an attractive substitute of bandlimited functions in signal processing to model "almost" bandlimited functions. See the survey [208] for sampling in shift-invariant spaces. An important problem that is related to the analog-digital conversion in signal processing is the derivation of sampling theorems for the space $V(g)$. We say that a set of sampling points x_j, ordered linearly as $x_j < x_{j+1}$, is a set of sampling for $V_h(g)$, if there exist constants $A, B > 0$, such that

$$A\|f\|_2^2 \leq \sum_{j \in \mathbb{Z}} \left| f(x_j) \right|^2 \leq B\|f\|_2^2 \qquad \text{for all } f \in V_h(g).$$

As in the case of Gabor frames there are many qualitative sampling theorems for shift-invariant spaces. Typical results require high oversampling rates. They state that there exists a $\delta > 0$ depending on g such that every set with maximum gap

$\sup_j(x_{j+1} - x_j) = \delta$ is a set of sampling for $V(g)$ [206,208, 239]. In most cases δ is either not specified or too small to be of practical use. The expected result is that for $V(g)$ there exists a Nyquist rate and that $\delta < 1$ is sufficient. And yet, the only generators for which the sharp result is known are the B-splines $b_n = \chi_{[0,1]} * \ldots * \chi_{[0,1]}$ $(n + 1 - \text{times})$. If the maximum gap $\sup_j(x_{j+1} - x_j) = \delta$ satisfies $\delta < 1$, then $\{x_j\}$ is a set of sampling for $V(b_n)$[207]. (This optimal result can also be proved for a generalization of splines, the so-called "ripplets" [211]). It has been an open problem to identify further classes of shift-invariant spaces for which the optimal sampling results hold.

Here we will prove a similar result for totally positive generators.

Theorem (4.2.2)[205]: Let g be a totally positive function of finite type and $x = \{x_j\}$ be a set with maximum gap $\sup_j(x_{j+1} - x_j) = \delta$. If $\delta < 1$, then x is a set of sampling for $V(g)$.

The theorem will be a corollary of a much more general sampling theorem.

We discuss the tool box for the Gabor frame problem. We will review some known characterizations of Gabor frames and derive some new criteria that are more suitable. We then recall the main statements about totally positive functions and prove the main technical theorem about the existence of a left-inverse of the pre-Gramian matrix. We study Gabor frames and discuss some open problems that are raised by our new results. We show the sampling theorem.

There are many results about the structure of Gabor frames and numerous characterizations of Gabor frames. In principle, one has to check that one of the equivalent conditions for a set $\mathcal{G}(g, \alpha, \beta)$ to be a frame is satisfied. This task almost always difficult because it amounts to showing the invertibility of an operator or a family of operators on an infinite-dimensional Hilbert space.

We summarize the most important characterizations of Gabor frames. These are valid in arbitrary dimension d and for rectangular lattices $\alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$. We will use the notation $M_\xi f = e^{2\pi i \xi} f$ and $T_y f = f(\cdot - y), \xi, t \in \mathbb{R}^d$, such that

$$\mathcal{G}(g, \alpha, \beta) = \{M_{l\beta} T_{K\alpha} g : k, l \in \mathbb{Z}^d\}.$$

Then $\mathcal{G}(g, \alpha, \beta)$ is a frame of $L^2(\mathbb{R}^d)$, if there exist constants $A, B > 0$, such that

$$A\|f\|_2^2 \le \sum_{k,l\in\mathbb{Z}} \left|\langle f, M_{l\beta} T_{K\alpha} g\rangle\right|^2 \le B\,\|f\|_2^2 \qquad \text{for all } f \in L^2(\mathbb{R}^d).$$

If only the right-hand inequality is satisfied, then $\mathcal{G}(g, \alpha, \beta)$ is called a Bessel sequence.

Following the fundamental work of Ron and Shen [231] on shift-invariant systems and Gabor frames, we define two families of infinite matrices associated to a given window function $g \in L^2(\mathbb{R}^d)$ and two lattice parameters $\alpha, \beta > 0$. The pre-Gramian matrix $P = P(x)$ is defined by the entries

$$P(x)_{jk} = g\left(x + j\alpha - \frac{k}{\beta}\right), \qquad j, k \in \mathbb{Z}^d. \tag{16}$$

The Ron-shen matrix is $G(x) = P(x) * P(x)$ with the entries

$$G(x)_{kl} = \sum_{j\in\mathbb{Z}^d} g\left(x + j\alpha - \frac{l}{\beta}\right) \bar{g}\left(x + j\alpha - \frac{k}{\beta}\right), \quad k, l \in \mathbb{Z}^d. \tag{17}$$

Theorem (4.2.3)[205]: (characterizations of Gabor frames) Let $g \in L^2(\mathbb{R}^d)$ and $\alpha, \beta > 0$. Then the following conditions are equivalent:

(i) The set $\mathcal{G}(g, \alpha, \beta)$ is a frame for $L^2(\mathbb{R}^d)$.

(ii)There exist $A, B > 0$ such that the spectrum of almost every Ron-Shen matrix $G(x)$ is contained in the interval $[A, B]$:

$$\sigma\big(G(x)\big) \subseteq [A, B] \qquad a.a. x \in \mathbb{R}^d.$$

(iii)There exist $A, B > 0$ such that

$$A\|c\|_2^2 \leq \sum_{j\in\mathbb{Z}^d} \left| \sum_{k\in\mathbb{Z}^d} c_k g\left(x + j\alpha - \frac{k}{\beta}\right) \right|^2 \leq B\,\|c\|_2^2 \;\; a.a. x \in \mathbb{R}^d, c \in \ell^2(\mathbb{Z}^d). \qquad (18)$$

(iv)There exists a so-called dual window γ, such that $\mathcal{G}(\gamma, \alpha, \beta)$ is a Bessel sequence and γ satisfies the biorthogonality condition

$$\langle \gamma, M_{l/\alpha} T_{k/\beta} g \rangle = (\alpha\beta)^d \delta_{k,0} \delta_{l,0} \qquad \forall k, l \in \mathbb{Z}^d. \qquad (19)$$

Lemma (4.2.4)[205]: Let $g \in L^2(\mathbb{R}^d)$ and $\alpha, \beta > 0$. Then the following conditions are equivalent:

(i) The set $\mathcal{G}(g, \alpha, \beta)$ is a frame for $L^2(\mathbb{R}^d)$.

(ii)The set of the pre-Gramians $\{P(x)\}$ is uniformly bounded on $\ell^2(\mathbb{Z}^d)$, and possess a uniformly bounded set of left-inverses, i.e, there exist matrices $\Gamma(x)$, $x \in \mathbb{R}^d$, such that

$$\Gamma(x)P(x) = I \qquad a.a. x \in \mathbb{R}^d, \qquad (20)$$

$$\|\Gamma(x)\|_{op} \leq C \qquad a.a. x \in \mathbb{R}^d. \qquad (21)$$

In this case, the function γ defined by $\gamma(x + \alpha j) = \beta^d \bar{\Gamma}_{0,j}(x)$, where $x \in [0, \alpha)^d$ and $j \in \mathbb{Z}^d$, or equivalently

$$\gamma(x) = \beta^d \sum_{j\in\mathbb{Z}^d} \bar{\Gamma}_{0,j}(x) x_{[0,\alpha)^d}\big(x - \alpha_j\big), \qquad x \in \mathbb{R}^d, \qquad (22)$$

Satisfies the biorthogonality condition (19).

Proof. (i)⇒(v). If $\mathcal{G}(g, \alpha, \beta)$ is a frame, then by Theorem (4.2.3)(ii)

$$\sum_{j\in\mathbb{Z}^d} \left| \sum_{k\in\mathbb{Z}^d} c_k g\left(x + j\alpha - \frac{k}{\beta}\right) \right|^2 = \langle P(x)c, P(x)c \rangle = \langle G(x)c, c \rangle \asymp \|c\|_2^2,$$

with bounds independent of x. Consequently $G(x)$ is bounded and invertible on $\ell^2(\mathbb{Z}^d)$. Therefore the operators $P(x)$ are uniformly bounded on $\ell^2(\mathbb{Z}^d)$,and we can define $\Gamma(x) = G(x)^{-1} P^*(x)$. Then

$$\Gamma(x)P(x) = \Big(\big(P^*(x)P(x)\big)^{-1} P^*(x)\Big) P(x) = Id,$$

and

$$\|\Gamma(x)\|_{op} \leq \|G(x)^{-1}\|_{op} \|P(x)\|_{op} \leq A^{-1} B^{1/2}.$$

(v)⇒(ii). Conversely, if $P(x)$ possesses a bounded left-inverse $\Gamma(x)$, then

$$\|c\|_2^2 = \|\Gamma(x)P(x)c\|_2^2 \leq \|\Gamma(x)\|_{op}^2 \, \|P(x)c\|_{op}^2 \leq C^2 \langle G(x)c, c \rangle \leq C^2 \|P(x)\|_{op}^2 \|c\|_2^2,$$

and this implies condition (ii) of Theorem (4.2.3).

We next verify that γ as defined in (22) satisfies the biorthogonality condition (19):

$$\langle\gamma, M_{l/\alpha}T_{k/\beta}g\rangle = \int_{\mathbb{R}^d} \gamma(x)\bar{g}(x-k/\beta)e^{-2\pi i l\cdot x/\alpha}dx$$

$$= \int_{[0,\alpha]^d} \sum_{j\in\mathbb{Z}^d} \gamma(x+\alpha j)\bar{g}(x+\alpha j-k/\beta)e^{-2\pi i l\cdot x/\alpha}dx$$

$$= \beta^d \int_{[0,\alpha]^d} \sum_{j\in\mathbb{Z}^d} \bar{\Gamma}_{0,j}(x)\bar{g}(x+\alpha j-k/\beta)e^{-2\pi i l\cdot x/\alpha}dx$$

$$= \beta^d \int_{[0,\alpha]^d} \delta_{k,0}e^{-2\pi i l\cdot x/\alpha}dx = (\alpha\beta)^d\delta_{k,0}\delta_{l,0}. \quad (23)$$

The function γ in (22) is a dual window of g, as defined in condition (iv) of Theorem (4.2.3), provided that $\mathcal{G}(g,\alpha,\beta)$ is a Bessel sequence. The following result gives a sufficient condition.

Lemma (4.2.5)[205]: Assume that there exists a (Lévesque measurable) vector-valued function $\sigma(x)$ from $\mathbb{R}^d$ to $\ell^2(\mathbb{Z}^d)$ with period , such that

$$\sum_{j\in\mathbb{Z}^d} \sigma_j(x)\,\bar{g}\left(x+\alpha j-\frac{k}{\beta}\right) = \delta_{k,0} \qquad a.a.\ \ x\in\mathbb{R}^d. \quad (24)$$

If $\sum_{j\in\mathbb{Z}^d}\sup_{x\in[0,\alpha]^d}\left|\sigma_j(x)\right| < \infty$, then $\mathcal{G}(g,\alpha,\beta)$ is a frame. Moreover, with

$$\gamma(x) = \beta^d \sum_{j\in\mathbb{Z}^d} \sigma_j(x)\chi_{[0,\alpha)^d}(x-\alpha_j), \qquad x\in\mathbb{R}^d,$$

the set $\mathcal{G}(\gamma,\alpha,\beta)$ is a dual frame of $\mathcal{G}(g,\alpha,\beta)$.

Proof. The computation in (23) shows that γ satisfies the biorthogonality condition (19). The additional assumption implies that

$$\sum_{k\in\mathbb{Z}^d} \sup_{x\in[0,\alpha]^d} |\gamma(x+\alpha k)| < \infty.$$

Consequently, γ is in the amalgam space $W(\ell^1)$. this property guarantees that $\mathcal{G}(\gamma,\alpha,\beta)$ is a Bessel system [238]. Thus condition (iv) of Theorem (4.2.3) is satisfied, and $\mathcal{G}(g,\alpha,\beta)$ is a frame. The biorthogonality condition (19) implies that $\mathcal{G}(\gamma,\alpha,\beta)$ is a dual frame of $\mathcal{G}(g,\alpha,\beta)$.

The notation of totally positive functions was introduced in 1947 by I. J. Schoenberg [232]. A non-constant measurable function $g:\mathbb{R}\longrightarrow\mathbb{R}$ is said to be totally positive, if it satisfies the following condition: For every two sets of increasing real numbers

$$x_1 < x_2 < \cdots < x_N, \qquad y_1 < y_2 < \cdots < y_N, \qquad N\in\mathbb{N}, \quad (25)$$

We have the inequality

$$D = \det[g(x_j-y_k)]_{1\le j,k\le n} \ge 0 \quad (26)$$

Schoenberg [28] connected the total positivity to factorization of the (two-sided) Laplace transform of g

$$\mathcal{L}[g](s) = \int_{-\infty}^{\infty} e^{-st} g(t)dt =: \frac{1}{\Phi(s)}.$$

Theorem (4.2.6)[205]: (Schoenberg [27]). The function $g: \mathbb{R} \to \mathbb{R}$ is totally positive, if and only if its two-sided Laplace transform exists in a strip $S = \{s \in \mathbb{C}: \alpha < Re\ s < \beta\}$, and

$$\Phi(s) = \frac{1}{\mathcal{L}[g](s)} = Ce^{-\gamma s^2 + \delta s} \prod_{v=1}^{\infty} (1 + \delta_v s) e^{-\delta_v s}, \tag{27}$$

with real parameters $C, \gamma, \delta, \delta_v$ satisfying

$$C > 0, \quad \gamma \geq 0, \quad 0 < \gamma + \sum_{v=1}^{\infty} \delta_v^2 < \infty. \tag{28}$$

A comprehensive study of total positivity is given of Karlin [228]. It is Known that, if g is totally positive and integrable, then g decays exponentially (see [233,p.340]). We restrict our attention to the class of totally positive functions $g \in L^1(\mathbb{R})$ with the factorization

$$\Phi(s) = \frac{1}{\mathcal{L}[g](s)} = Ce^{\delta s} \prod_{v=1}^{M} (1 + \delta_v s), \tag{29}$$

for $M \in \mathbb{N}$ and real δ_v. This means that the denominator of $\mathcal{L}[g]$ has only finitely many roots. Equivalently, the Fourier transform of g can be extended to a meromorphic function with a finite number of poles on the imaginary axis and no other poles. As noted in [234,p.247], the exponential factor can be omitted, as it corresponds to a simple shift of g. in the following we will call a totally positive function satisfying (29) totally positive of finite type and refer to M as the type of g.

Schoenberg and Whitney [234] gave a complete characterization of the case when the determinate D in (26) satisfies $D > 0$.

Theorem (4.2.7)[205]: ([234]). Let $g \in L^1(\mathbb{R})$ be a totally-positive function of finite type. Furthermore, let m be the number of positive δ_v and n be the number of negative δ_v in (29), and $m + n \geq 2$. For a set of points in (25), the determinant $D = \det[g(x_j - y_k)]_{j,k=1,\ldots,N}$ is strictly positive, if and only if

$$x_{j-m} < y_j < x_{j+n} \qquad \text{for } 1 \leq j \leq N. \tag{30}$$

Here, we use convention that $x_j = -\infty$, if $j < 1$, and $x_j = \infty$, if $j > N$.

The conditions in (30) are nowadays called the Schoenberg-Whitney conditions for g. They have been used extensively in the analysis of spline interpolation by Schoenberg and others (see [235]). They will be crucial for our construction of a left inverse of the pre-Gramian matrix in (16).

As a generalization of the pre-Gramian in (16), we consider bi-infinite matrices of the form

$$P = [g(x_j - y_j)]_{j,k \in \mathbb{Z}}, \tag{31}$$

where each sequence $X = (x_j)_{j \in \mathbb{Z}}$ and $Y = (y_k)_{k \in \mathbb{Z}} \subseteq \mathbb{R}$ is strictly increasing. Moreover, the sequence $(x_j)_{j \in \mathbb{Z}}$ is supposed to be denser in the sense of the following condition :

$$(C_r)\begin{cases}\text{(a) every interval } (y_k, y_{k+1}) \text{ contains at least one point } x_j;\\ \text{(b) there is an } r\in\mathbb{N} \text{ such that } |(y_k, y_{k+r})\cap X| \geq r+1 \text{ for all } k.\end{cases}$$

The main tool for the study of Gabor frames and sampling theorems will be the following technical result. It can be interpreted as a suitable extension of total positivity to infinite matrices.

Theorem (4.2.8)[205]: let $g \in L^1(\mathbb{R})$ be totally-positive function of finite type. Let m be the number of positive δ_v , n be the number of negative δ_v in (29), and $M = m + n \geq 1$. Assume that the sequences $(x_j)_{j\in\mathbb{Z}}$ and $(y_k)_{k\in\mathbb{Z}} \subset \mathbb{R}$ satisfy condition (C_r).

Then the matrix $P = [g(x_j - y_j)]_{j,k\in\mathbb{Z}}$ defines a bounded operator on $\ell_2(\mathbb{Z})$. It has an algebraic left- inverse $\Gamma = [\gamma_{k,j}]_{k,j\in\mathbb{Z}}$, and

$$\gamma_{k,j} = 0, \quad \text{if } x_j < y_{k-rm} \text{ or } x_j > y_{k+rn}. \tag{32}$$

Proof. We construct a left-inverse Γ with the desired properties by defining each row of Γ separately. It suffices to consider the row with index $k = 0$, as the construction of all other rows is done in the same way. The goal of the first three steps is to select a finite subset of $x_j's$ and $y_k's$ that satisfy the Schoenberg-Whitney conditions. (our choice of indices is not unique and not symmetric in m and n, it minimizes the number of case distinctions.)

Step 1: Column selection. First, consider the case $m, n > 0$ and set $N := (m + n - 1)(r + 1)$, if $n > 1$, and $N = m(r + 1) + 1$, if $n = 1$. We define an $N \times N$ submatrix P_0 of P in the following way. As columns of P_0, we select columns of P between

$$k_1 = -(r+1)m + 1 \quad \text{and} \quad k_2 = k_1 + N - 1.$$

Hence $k_2 = (r + 1)(n - 1)$ for $n > 1$, and $k_2 = 1$ for $n = 1$ for later purposes, note that $k_1 \leq -m < 0 < n \leq k_2$ .Therefore, the column with index $k = 0$ has at least m columns to its left and n columns to its right.

Step 2: Selection of a Square matrix. Assumption (C_r) and our definition of N imply that the interval $I = \left(y_{k_1+m-1}, y_{k_2-n+1}\right)$ contains at least N points x_j. More precisely, for $n > 1$ we write

$$\left(y_{k_1+m-1}, y_{k_2-n+1}\right) = \left(y_{-rm}, y_{r(n-1)}\right) = \cup_{v=-m}^{n-2}(y_{rv}, y_{r(v+1)})$$

and find at least $r + 1$ points x_j in each subinterval $(y_{rv}, y_{r(v+1)})$ with $-m \leq v \leq n - 2$. This amounts to at least $(m + n - 1)(r + 1) = N$ points in I. If $n = 1$, we have $\left(y_{k_1+m-1}, y_{k_2-n+1}\right) = (y_{-rm}, y_1)$ and find $m(r + 1)$ points x_j in (y_{-rm}, y_0) plus at least one additional point in (y_0, y_1).

We let

$$j_1 := \min\{j: x_j > y_{k_1+m-1}\}, \quad j_2 := \max\{j: x_j < y_{k_2-n+1}\}. \tag{33}$$

We have just shown that the set

$$X_0 = \{x_j: j_1 \leq j \leq j_2\} \subset \left(y_{k_1+m-1}, y_{k_2-n+1}\right)$$

contains at least N elements. We choose a subset

$$X_0' = \{\xi_1 < \cdots < \xi_N\} \subset X_0,$$

that contains precisely N elements and satisfies

$$(y_k, y_{k+1}) \cap X_0' \neq \emptyset \quad \text{for} \quad k_1 + m - 1 \leq k \leq k_2 - n.$$

That is, we choose one point x_j in each interval (y_k, y_{k+1}), with $k_1 + m - 1 \leq k \leq k_2 - n$, and an additional $n + m - 1$ points $x_j \in \left(y_{k_1+m-1}, y_{k_2-n+1}\right)$. Note that

$$y_{k_1+m-1} < \xi_1 = min\, X_0' < y_{k_1+m} < y_{k_2-n} < max\, X_0' = \xi_N < y_{k_2-n+1} \tag{20}$$

Now set

$$\eta_k = y_{k_1+k-1}, \qquad 1 \le k \le N,$$

and define the matrix

$P_0 = \left(g(\xi_j - \eta_k)\right)_{j,k=1,\dots,N}$.

Then P_0 is a quadratic $N \times N$-submatrix of P.

Step 3: Verification of the Schoenberg-Whitney conditions. We next show that P_0 is invertible by checking the Schoenberg-Whitney conditions. First, by (34), we have

$$\xi_1 = \min X_0' < y_{k_1+m} = \eta_{m+1}.$$

By the construction of X_0', this inequality progresses from left to right, i.e,

$$\xi_j < y_{k_1+m-1+j} = \eta_{j+m} \qquad \text{for } 1 \le j \le N-m.$$

Likewise, we also have

$$\eta_{N-n} = y_{k_2-n} < \max X_0' = \xi_N,$$

and this inequality progresses from right to left, i.e,

$$\eta_j < \xi_{j+n} \text{ for } \quad 1 \le j \le N-n.$$

Therefore, the Schoenberg-Whitney conditions (30) are satisfied, and Theorem (4.2.7) implies that $\det P_0 > 0$.

Step 4: Linear dependence of the remaining columns of *P*. We now make some important observations.

Choose indices k_0 and $s \in \mathbb{Z}$ with $k_0 < k_1$ and $m < s \le N$, and consider the new set $\{\eta_k' : k = 1, \dots, N\}$ consisting of the points

$$y_{k_0} < y_{k_1} < \cdots < y_{k_{1+s-2}} < y_{k_1+s} < \cdots < y_{k_2}$$

and the corresponding $N \times N$-matrix $P_0' = \left(g(\xi_j - \eta_k')\right)_{j,k=1,\dots,N}$. This matrix is obtained from P_0 by adding the column $\left(g(\xi_j - y_{k_0})\right)_{1\le j\le N}$ as its first column and deleting the column $\left(g(\xi_j - \eta_s)\right)_{1\le j\le N}$. Then η_m appears in the $m+1$-st column of P_0'. By (34), we see that

$$\eta_{m+1}' = \eta_m = y_{k_1+m-1} < \xi_1.$$

Consequently, the Schoenberg-Whitney conditions are violated and therefore $\det P_0' = 0$ by Theorem (4.2.9). Since this holds for all $s > m$, the vector $\left(g(\xi_j - y_{k_0})\right)_{1\le j\le N}$ must be in the linear span of the first m columns of P_0, namely $\left(g(\xi_j - y_k)\right)_{1\le j\le N}$ for $k = k_1 \dots, k_1+m-1$.

Likewise, choose $k_3 > k_2, 1 \le s \le N-n$, and consider the new set $\{\eta_k'' : k = 1, \dots, N\}$ consisting of the points

$$y_{k_1} < \cdots < y_{k_{1+s-2}} < y_{k_1+s} < \cdots < y_{k_2} < y_{k_3}$$

and the corresponding $N \times N$-matrix $P_0'' = \left(g(\xi_j - \eta_k'')\right)_{j,k=1,\dots,N.}$ This matrix is obtained from P_0 by adding the column $\left(g(\xi_j - y_{k_3})\right)_{1\le j\le N}$ as its last $(= N-\text{th})$ column and deleting the column $\left(g(\xi_j - \eta_s)\right)_{1\le j\le N}$. Then η_{N-n+1} appears in the $n+1-$ st column of P_0'', counted from right to left, and

$$\eta''_{N-n} = \eta_{N-n+1} = y_{k_1+N-n} = y_{k_2-n+1} > \xi_N$$

by (34). Again the Schoenberg-Whitney conditions are violated and therefore $\det P_0'' = 0$. We conclude that the vector $\left(g\left(\xi_j - y_{k_3}\right)\right)_{1\le j\le N}$ must lie in the linear span of the last n columns of P_0.

Step 5: Construction of the left-inverse. Recall that $k_1 = -m(r+1)+1$ and that $\eta_{(r+1)m} = y_{k_1+(r+1)m-1} = y_0$. let C^T denote the $(r+1)_{m-th}$ row vector of P_0^{-1}. By definition of the inverse, we have $\sum_{j=1}^{N} c_j g\left(\xi_j - \eta_k\right) = \delta_{k,(r+1)m}$ or equivalently, for $k_1 \le k \le k_2$,

$$\sum_{j=1}^{N} c_j g\left(\xi_j - y_k\right) = \delta_{k,0}. \tag{35}$$

Let us now consider the other columns with $k < k_1$ or $k > k_2$. Since $k_1 \le -m < 0 < n \le k_2$ and every vector $\left(g\left(\xi_j - y_k\right)\right)_{1\le j\le N}$ lies either in the span of the first m columns of P_0 (for $k < k_1$) or in the span of the last n columns of P_0 (for $k > k_2$), we obtain that

$$\sum_{j=1}^{N} c_j g\left(\xi_j - y_k\right) = 0.$$

Therefore, the identity (35) holds for all $k \in \mathbb{Z}$.

Next we fill the vector c with zeros and define the infinite vector γ_0 by

$$\gamma_{0,j} = \begin{cases} c_{j'} & \text{if } x_j = \xi_{j'} \in X_0' \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\sum_{j\in\mathbb{Z}} \gamma_{0,j} g\left(x_j - y_k\right) = \sum_{j=1}^{N} c_j g\left(\xi_j - y_k\right) = \delta_{k,0}.$$

Thus γ_0 is a row of the left-inverse of P. By construction, γ_0 has most N non-zero entries. In particular, if $x_j < y_{k_1+m-1} = y_{-rm}$, then we have $j < j_1$ and thus $\gamma_{0,j} = 0$. Similarly, if $x_j > y_{k_2-n+1} = y_{r(n-1)}$ for $n > 1$, and $x_j > y_{k_2-n+1} = y_1$ for $n = 1$, then we have $j > j_2$ and $\gamma_{0,j} = 0$. This gives the support properties of the entries $\gamma_{0,k}$ of row $k = 0$ of the left-inverse Γ.

Step 6: the other rows of Γ. The construction of the $k-$th row of Γ is similar. We choose the columns between $k_1 = k-(r+1)m+1$ and $k_2 = k_1 + N - 1$ of P, and, accordingly, we choose suitable rows between the indices $j_1 = \min\{j : x_j > y_{k_1+m-1}\}$ and $j_2 = \max\{j : x_j < y_{k_2-n+1}\}$. Then the column of P containing y_k has at least m columns to its left and n columns to its right. We then proceed to select $\xi_j's$ and define an $N \times N$-matrix P_k and verify that $\det P_k > 0$. The $k-$th row of Γ is obtained by padding the appropriate row (with row-index (r+1)m) of P_K^{-1} with zeros. By this construction one obtains a vector $\gamma_k = (\gamma_{k,j})_{j\in\mathbb{Z}}$, for which

$\sum_{j\in\mathbb{Z}} \gamma_{kj} g\left(x_j - y_l\right) = \delta_{k,l}$

holds. Furthermore, $\gamma_{k,j} = 0$ when $x_j < y_{k-rm}$ and when $x_j > y_{k+rn}$.

Step 7: the remaining cases. The cases where $m = 0$ or $n = 0$ are simple adaptations of the above steps. For $m = 0, n \ge 2$ we choose $N = (n-1)(r+1)$. The indices for the sub matrix P_k that occurs in the construction of the $k-$th row of Γ are $k_1 =$

$k, k_2 = k + N - 1, j_1 = j_1(k) = j_1 = \max\{j: x_j < y_{k_1}\}$ and $j_2 = j_2(k) = \max\{j: x_j < y_{k_2}\}$. Now we proceed as before.

We note that Step 4 simplifies a bit. For $m = 0$ the function g is supported on $(-\infty, 0)$ by [233,p.339]. Consequently, if $k_0 < k_1$ and $j \geq j_1$, then $x_j - y_{k_0} > 0$ and $g(x_j - y_{k_0}) = 0$.

Thus the column vectors of P to the left of the sub matrix P_0 are identically zero and no further proof is needed for linear dependence. The case $n = 0$ is similar. It can be reduced to the previous case by a reflection $x \to -x$, which interchanges the role of m and n.

The special case of $m = 0$ and $n = 1$ can be solved by taking $N = 2, k_1 = k, k_2 = k + 1, j_1 = \max\{j: x_j < y_k\}, j_2 = \min\{j: x_j > y_k\}$, and the 2×2-matrix

$$P_k = \begin{pmatrix} g(x_{j_1} - y_k) & g(x_{j_1} - y_{k+1}) \\ g(x_{j_2} - y_k) & g(x_{j_2} - y_{k+1}) \end{pmatrix}.$$

In this simple case the size of the matrix is independent of the parameter r occurring in condition C_r.

We show the main result about Gabor frames. Recall that a totally positive function is said to be of finite type, if its two-sided Laplace transform factors as $\mathcal{L}[g](s)^{-1} = Ce^{\delta s} \prod_{v=1}^{M} (1 + \delta_v s)$ with real numbers δ, δ_v.

By taking a Fourier transform, we obtain the following corollary.

Corollary (4.2.9)[205]: If $h(\tau) = C \prod_{v=1}^{M} (1 + 2\pi i \delta_v \tau)^{-1}$ for $M \geq 2$, then $\mathcal{F}(h) = \{(\alpha, \beta) \in \mathbb{R}^2: \alpha\beta < 1\}$ and $\mathcal{G}(h, \alpha, \beta)$ possesses a band limited dual window θ with sup $\hat{\theta} \subseteq \left[-\frac{rm}{\alpha} - \beta, \frac{rn}{\alpha} + \beta\right]$.

Theorem (4.2.10)[205]: Assume that g is a totally positive function of finite type and $M = m + n \geq 2$, where m is the number of positive zeros and n the number of negative zeros of $1/\mathcal{L}(g)$. Then $\mathcal{F}(g) = \{(\alpha, \beta) \in \mathbb{R}^2: \alpha\beta < 1\}$. The Gabor frame $\mathcal{G}(g, \alpha, \beta)$ possesses a piecewise continuous dual window γ with compact support in $\left[-\frac{rm}{\beta} - \alpha, \frac{rn}{\beta} + \alpha\right]$, where $r := \left\lfloor \frac{1}{1 - \alpha\beta} \right\rfloor$.

Proof. Since $g \in L^1(\mathbb{R})$ is totally positive, it decays exponentially, and since $M \geq 2$, its Fourier transform $\hat{g}(\xi) = Ce^{2\pi i \delta \xi} \prod_{v=1}^{M} (1 + 2\pi i \delta_v \tau)^{-1}$ decays at least like $|\hat{g}(\xi)| \leq \tilde{C}(1 + \xi^2)^{-1}$. In particular, g is continuous. As a consequence, the assumptions of the Balian-Low theorem are satisfied [209, 219] satisfied [209, 219] and $\mathcal{F}(g) \subseteq \{(\alpha, \beta) \in \mathbb{R}_+^2: \alpha\beta \leq 1\}$.

To show that $\mathcal{G}(g, \alpha, \beta)$ is a frame for $\alpha\beta < 1$, we will construct a family of uniformly bounded left-inverses for the pre- Gramians $P(x)$ of g and then use Lemma (4.2.4).

Fix $x \in [0, \alpha]$ and consider the sequences $x_j = x + \alpha j$ and $y_k = k/\beta, j, k \in \mathbb{Z}$. We first check condition (C_r). By our assumption, we have $\alpha < 1/\beta$ and every interval $(k/\beta, (k+1)/\beta)$ contains at least one point $x + \alpha j$. Every interval $(k/\beta, (k+r)/\beta)$, with $r \in \mathbb{N}$, contains at least $r + 1$ points $x + \alpha j$, if $r/\beta > (r+1)\alpha$, i.e, we have

$$r > \frac{\alpha\beta}{1 - \alpha\beta}, \quad \text{or equivalently} \quad r \geq \left\lfloor \frac{1}{1 - \alpha\beta} \right\rfloor.$$

Consequently, condition (C_r) is satisfied with $r = \left\lfloor \frac{1}{1-\alpha\beta} \right\rfloor$. By Theorem (4.2.8), each pre-Gramian $P(x)$ with entries $g(x+\alpha j - k/\beta)$ Possesses a left-inverse $\Gamma(x)$. To apply Lemma (4.2.5), we need to show that $\Gamma(x), x \in [0,\alpha]$, is a uniformly bounded set of operators on $\ell^2(\mathbb{Z})$.

Let $P_0(x)$ be the $N \times N$-squar sub matrix constructed in Steps 1 and 2. The column indices k_1 and k_2 depend only on the type of g, but not on x. the row indices$j_1 = j_1(x) = \min\{j : x_j > y_{k_1+m-1}\} = \min\{j: x + \alpha j > (k_1 + m - 1)/\beta\}$ and $j_2 = j_2(x)$ are locally constant in x. Likewise the indices that determine which rows $j, j_1 < j < j_2$ of $P(x)$ are contained in $P_0(x)$ are locally constant. Consequently, for every $x \in [0,\alpha)$ there is a neighborhood U_x such that indices used for $P_0(y)_{jk} = g(y + \xi_j - k/\beta)$ do not depend on $y \in U_x$. Since g is continuous, $P_0(y)$ is continuous on U_x, and since det $P_0(x) > 0$ there exists a neighborhood $V_x \subseteq U_x$, such that det $P_0(y) \geq \det P_0(x)/2$ for $y \in V_x$.

We now cover $[0,\alpha]$ with finitely many neighborhoods V_{x_q} and obtain that $\det P_0(y) \geq \min_q \det P_0(x_q)/2 = \delta > 0$ for all $y \in [0,\alpha]$. Since each entry of the inverse matrix $P_0(y)^{-1}$ can be calculated by Cramer's rule, these entries must be bounded by C det $P_0(y)^{-1} \leq C\delta^{-1}$ with a constant C depending only on $\|g\|_\infty$ and the dimension N of $P_0(y)$.

By construction (step 5), the zero-th row $\gamma(x) = (\gamma_0(x), j(x))_{j\in\mathbb{Z}}$ of the left-inverse $\Gamma(x)$ contains at most $N \leq (r+1)M$ non-zero entries, namely those of the row of $P_0(x)^{-1}$ corresponding to $y_0 = 0$. We have thus constructed vector-valued functions $x \to \gamma(x)$ from $[0,\alpha] \to \ell^\infty(\mathbb{Z})$ with the following properties:

(i) $\gamma(x)$ is piecewise continuous,

(ii)card $(\operatorname{supp} \gamma(x)) \leq (r+1)M$, where according to (223).

$$\operatorname{supp} \gamma(x) = \{j: \gamma_{0,j}(x) \not\equiv 0\} \subseteq \left\{j: \frac{-rm}{\beta} \leq x + \alpha j \leq \frac{rn}{\beta}\right\} \subseteq \left\{j : \frac{-rm}{\alpha\beta} - 1 \leq j \leq \frac{rn}{\alpha\beta}\right\},$$

(iii)and $\sup_{x\in[0,\alpha]} \|\gamma(x)\|_\infty = C < \infty$.

Consequently, the dual window $\gamma(x) = \beta \sum_{j\in\mathbb{Z}^d} \overline{\gamma}_{0,j}(x)\chi_{[0,\alpha]}(x - \alpha j)$ corresponding to $\Gamma(x)$ by Lemma (4.2.5), has compact support on the interval $\left[-\frac{rm}{\beta} - \alpha, \frac{rn}{\beta} + \alpha\right]$, is piecewise continuous, and bounded. In particular, it satisfies the Bessel property (see [238] or [219, Cor. 6.2.3]) . We have constructed a dual window for $\mathcal{G}(g,\alpha,\beta)$ satisfying Bessel property. By Theorem (4.2.3) and Lemma (4.2.4). $\mathcal{G}(g,\alpha,\beta)$ is a Gabor frame.

In the proof of Theorem (4.2.10) we have constructed a compactly supported dual window γ for $\mathcal{G}(g,\alpha,\beta)$. This construction is explicit and can be realized numerically, because it requires only the inversion of finite matrices. To determine the values $\gamma(x + j\alpha)$, one has to solve the linear $N \times N$ system $P_0(x)\gamma(x) = e$ for a vector e of the standard basis of $\mathbb{R}^N$.

We observe that the canonical dual window (provided by standard frame theory) has better smoothness properties. The regularity theory for the Gabor frame operator implies that the canonical dual window γ^0 decays exponentially and its Fourier transform $\widehat{\gamma^0}(\xi)$ decays like $O(|\xi|^{-M})$, where M is the type of g. See [217,219,220,237].

Theorem (4.2.10) raises many new questions. Theorem (4.2.10) suggests the natural conjecture that frame set of every totally positive continuous function g in $L^1(\mathbb{R})$ is $\mathcal{F}(g) = \{(\alpha,\beta) \in \mathbb{R}_+^2 : \alpha\beta < 1\}$. Our proof is tailored to totally positive functions of finite type, most likely the proof of the conjecture will require a different method.

In a larger context one may speculate about the set $\mathcal{M}$ of functions such that the frame set is exactly $\mathcal{F}(g) = \{(\alpha,\beta) \in \mathbb{R}_+^2 : \alpha\beta < 1\}$. In other words, when is the necessary density condition $\alpha\beta < 1$ also sufficient for $\mathcal{G}(g,\alpha,\beta)$ to be a frame? the invariance properties of Gabor frames imply that the class $\mathcal{M}$ must be invariant under time-frequency shifts, dilations, involution, and the Fourier transform. Furthermore, if $\mathcal{F}(g) = \{(\alpha,\beta) \in \mathbb{R}^2 : \alpha\beta < 1\}$, then both g and $\hat{g}$ must have infinite support.

A general method for constructing functions in $\mathcal{M}$ can be exracted from [227]. We write $\hat{c}(\xi) = \sum_{k\in\mathbb{Z}} c_k\, e^{2\pi i k\xi}$ for the Fourier series of a sequence (c_k) and then define, for a given function $g_0 \in L^2(\mathbb{R})$,

$$C_{g_0} = \left\{ f \in L^2(\mathbb{R}) : f = \sum_{k,l\in\mathbb{Z}} c_k d_l T_k M_l\, g_0, c,d \in \ell^1(\mathbb{Z}), \inf_{\xi}\left(\left|\hat{c}(\xi)\hat{d}(\xi)\right|\right) > 0 \right\}.$$

Lemma (4.2.11)[205]: Let g_0 be a totally positive function of finite type $M \geq 2$. If $g \in C_{g_0}$, then the frame set of g is $\mathcal{F}(g) = \{(\alpha,\beta) \in \mathbb{R}_+^2 : \alpha\beta < 1\}$. This trick was used by Janssen and Strohmer in [227]. They showed that the hyperbolic secant $g_1(t) = (e^t + e^{-t})^{-1}$ belongs to the set C_φ for the Gaussian window $\varphi(t) = e^{-\pi t^2}$ and then concluded that $\mathcal{F}(g_1) = \{(\alpha,\beta) \in \mathbb{R}_+^2 : \alpha\beta < 1\}$. The general argument is identical.

Each class C_g is completely determined by the zeros of the Zak transform of g. Let $Z_g(x,\xi) = \sum_{k\in\mathbb{Z}} g(t-k)e^{2\pi i k\xi}$ be the Zak transform of g. Since $Z\big(T_k M_{lg}\big)(x,\xi) = e^{2\pi i(lx-k\xi)} Z_g(x,\xi)$, every $g \in C_{g_0}$ has a Zak transform of the form

$$Z_g(x,\xi) = \hat{c}(-\xi)\hat{d}(x) Z_{g_0}(x,\xi).$$

The definition of C_{g_0} implies that Z_g and Z_{g_0} have the same zero set. If g_0 and h are totally positive functions, then the zero sets of Z_{g_0} and Zh are different in general, therefore Lemma (4.2.11), leads to distinct sets C_{g_0}. The zeros of the Zak transform seems to be some kind of invariant for the Gabor frame problem, but their deeper significance is still mysterious.

We exploit the connection between Gabor frames and sampling theorems and show new sharp sampling theorems for shift-invariant spaces. Originally shift-invariant spaces were used as a substitute for band limited functions and were defined as the span of integer translates of a given function g. We refer to the survey [208] for the theory of sampling in shift-invariant spaces. We will deal with a slightly more general class of spaces that are generated by arbitrary shifts.

Let $Y = (y_k)_{k\in\mathbb{Z}}$ be a strictly increasing sequence and consider the quasi shift-invariant space

$$V_Y(g) = \left\{ f \in L^2(\mathbb{R}) : f = \sum_{k\in\mathbb{Z}} c_k g(\cdot - y_k) \right\}.$$

We require that the sequence $Y=(y_k)$ of shift parameters satisfies the conditions

$$0<q_Y=\inf_k(y_{k+1}-y_k)\le\sup_k(y_{k+1}-y_k)=Q_Y<\infty. \qquad (36)$$

Such sequences are called quasi-uniform or uniformly discrete. The numbers $Q_Y,q_y>0$ are the mesh-norm and the separation distance of Y.

For the norm equivalence $\|f\|_2\asymp\|c\|_2$ for $f\in V_Y(g)$ we need that $\{g(\cdot-y_k\colon y_k\in Y\}$ is a Riesz basis for $V_Y(g)$.

Lemma (4.2.12)[205]: Let g be an arbitrary totally positive function.

(i) If $Y=h\mathbb{Z}$ with $h>0$, then $\{g(\cdot-hk),k\in\mathbb{Z}\}$ is a Riesz basis for $V_Y(g)$.

(ii)If Y is quasi-uniform, then $\{g(\cdot-hk)\colon k\in\mathbb{Z}\}$ is a Riesz basis for $V_Y(g)$.

Proof. (i) Since $\hat{g}$ is continuous, does not have any real zeros, and $\hat{g}(\xi)$ decays at least like $C/|\xi|$, every periodization of $|\hat{g}|^2$ is bounded above and below. This property is equivalent to the Riesz basis property, e.g.[212,Thm. 7.2.3].

Of course, (i) also follows from (ii).

(ii) For the general case we use Zygmund's inequality [240,Th. 9.1]: if I is an interval of length $|I|>\frac{1+\delta}{q_Y}$, then

$$\int_I\left|\sum_k c_k e^{2\pi i y_k\xi}\right|^2 d\xi\ge A_\delta|I|\|c\|_2^2$$

for a constant depending only on $\delta>0$.

If $f=\sum_k c_k g(\cdot-y_k)$, then

$$\|f\|_2^2=\left\|\hat{f}\right\|_2^2=\int_{\mathbb{R}}\left|\sum_k c_k e^{-2\pi i y_k\tau}\right|^2|\hat{g}(\tau)|^2d\tau\ge\inf_{\tau\in I}|\hat{g}(\tau)|^2\int_I\left|\sum_k c_k e^{-2\pi i y_k\tau}\right|^2 d\xi$$
$$\ge C|I|A_\delta\|c\|_2^2.$$

Here $\inf_{\tau\in I}|\hat{g}(\tau)|^2>0$, because $\hat{g}$ does not have any real zeros by Theorem (4.2.6).

We are interested to derive sampling theorems for generalized shift-invariant spaces that are generated by a totally positive function g. Our goal is to construct strictly increasing sequences $X=(x_j)$ that yield a sampling inequality

$$A\|f\|_2^2\le\sum_{j\in\mathbb{Z}}\left|f(x_j)\right|^2\le B\|f\|_2^2 \quad \text{for all } f\in V_Y(g) \qquad (37)$$

for some constants $A,B>0$ independent of f. Following Landau [229], a set $X\subset\mathbb{R}$ that satisfies the norm equivalence (37) is called a set of (stable) sampling for $V_Y(g)$. Except for band limited functions and B-spline generators only qualitative results are known about sets of sampling in shif-invariant spaces.

We first give an equivalent condition for sets of sampling in $V_Y(g)$. As in Lemma (4.2.4) we obtain the following characterization of sets of sampling in $V_Y(g)$.

Lemma (4.2.13)[205]: Let $g\in L^2(\mathbb{R})$, and let $Y=(y_k)_{\square\in\mathbb{Z}}\subset\mathbb{R}$ be a strictly increasing sequence. Then a set $\{x_j\}\subset\mathbb{R}$ is a set of sampling for $V_Y(g)$, if and only if the pre-Gramian P with entries $p_{jk}=g(x_j-y_k)$ possesses a left-inverse Γ that is bounded on $\ell^2(\mathbb{Z})$. The case of uniform sampling in shift-invariant spaces is completely settled by the results.

Corollary (4.2.14)[205]: Let g be a totally positive function of finite type $M \geq 2$, and $Y = h\mathbb{Z}$. If $\alpha < h$ and $x \in \mathbb{R}$ is arbitrary, then the set $x + \alpha\mathbb{Z}$ is a set of sampling for $V_Y(g)$. More precisely, there exist positive constants A, B independent of x, such that
$A\|f\|_2^2 \leq \sum_{j\in\mathbb{Z}}|f(x+\alpha j)|^2 \leq B\|f\|_2^2$ for all $f \in V_Y(g)$.

Proof. We showed that Theorem (4.2.10) by verifying the equivalent condition of Theorem (4.2.3) namely (18) stating that

$$A\|c\|_2^2 \leq \sum_{j\in\mathbb{Z}^d}\left|\sum_{k\in\mathbb{Z}^d} c_k g(x + j\alpha - hk)\right|^2 \leq B\|c\|_2^2 \quad \text{for all } x \in \mathbb{R}, c \in \ell^2(\mathbb{Z}). \quad (38)$$

Since $f \in V_Y(g)$ is of the form $f = \sum_k c_k g(\cdot - hk)$ and $\|f\|_2 \asymp \|c\|_2$ by the Lemma (4.2.12), the inequalities (38) are equivalent to the sampling inequality $\|f\|_2^2 \asymp \sum_{j\in\mathbb{Z}}|f(x+\alpha j)|^2$, and the constants are independent of x by Theorem (4.2.3) see [290].

Our methods yield more general sampling theorems. On the one hand, we study non-uniform sampling sets, and on the other hand, we may treat quasi shift –invariant spaces. The auxiliary characterization of Lemma (4.2.13), gives a hint of how to proceed. If the sequences (x_j) and (y_k) satisfy condition (C_r) for some $r > 0$, then by Theorem (4.2.8), the pre-Gramian matrix P possesses an algebraic left-inverse. To obtain a sampling theorem, we need to impose additions on (x_j) and (y_k), so that this left-inverse is bounded on ℓ^2.

To vrify the boundedness of a matrix, we will apply the following lemma which is a direct consequence of Schur's test, see, e.g, [219, Lemma 6.2.1].

Lemma (4.2.15)[205]: Assume that $A = (a_{jk})_{j,k\in\mathbb{Z}}$ is a matrix with bounded entries $|a_{jk}| \leq C$ for$j, k \in \mathbb{Z}$. Furthermore, assume that there exists a strictly increasing sequence $(j_k)_{k\in\mathbb{Z}}$ of row indices $j_k \in \mathbb{Z}$ and $N \in \mathbb{N}$ such that $a_{jk} = 0$ for $|j - j_k| \geq N$. Then

$$\|A\|_{\ell^2\to\ell^2} \leq (2N-1)C. \quad (39)$$

Proof. The conditions give

$$K_2 := \sup_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}|a_{jk}| = \sup_{k\in\mathbb{Z}}\sum_{j=j_k-N+1}^{j_k+N-1}|a_{jk}| \leq (2N-1)C.$$

For the estimate of the column sums, we define the set

$$N_j = \{k \in \mathbb{Z}: a_{jk} \neq 0\} \subseteq \{k \in \mathbb{Z} : |j - j_k| < N\} \quad \text{for } j \in \mathbb{Z}.$$

Since (j_k) is strictly increasing, N_j has at most $(2N-1)$ elements, and this gives

$$K_1 := \sup_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}|a_{jk}| = \sup_{j\in\mathbb{Z}}\sum_{k\in N_j}|a_{jk}| \leq (2N-1)C.$$

The assertion now follows from Schur's test.

We give a sufficient condition for a set X to be a set of sampling for $V_Y(g)$.

Theorem (4.2.16)[205]: Let g be a totally positive function of finite type $M \geq 2$. Let $Y = (y_k)_{k\in\mathbb{Z}} \subseteq \mathbb{R}$ be an increasing quasi-uniform sequence with parameters q_Y, Q_Y defined in (36). Moreover, let $(x_j)_{j\in\mathbb{Z}} \subset \mathbb{R}$ be a strictly increasing sequence, which satisfies the following conditions:

$$(C_r(\epsilon))\begin{cases}\text{There exist } r \in \mathbb{N}, \epsilon \in (0, q_Y/2) \text{ and a quasi} - \text{uniform subsequence} \\ X' \subseteq X, \text{ such that} \\ \text{(a) every interval } (y_k + \epsilon, y_{k+1} - \epsilon) \text{ contains at least one point} \\ x_j \in X'; \\ \text{(b) for every } k \in \mathbb{Z}, \text{ we have } |(y_k + \epsilon, y_{k+1} - \epsilon) \cap X'| \geq r + 1.\end{cases}$$

Then X is a set of sampling for $V_Y(g)$.

Proof. Step 1. First, we construct a left-inverse of the pre-Gramian P as in the proof of Theorem (4.2.8) with a small modification.

We consider only the case $m > 1, n > 1$. For the construction of the row with the index k of the left-inverse Γ, we choose the size $N \in \mathbb{N}$ for a square sub matrix P_k of P as in Step 1 and the column indices $k_1 = k - r(m+1) + 1$ and $k_2 = k_1 + N - 1$ as in Step 6.

To incorporate condition $C_r(\epsilon)$, we modify the selection of the row indices in Step 2 as follows. Assumption $C_r(\epsilon)$ and our definition of N imply that the interval $I = \left(y_{k_1+m-1} + \epsilon, y_{k_2-n+1} - \epsilon\right)$ contains at least N points $x_j \in X'$, where X' is the quasi-uniform subset of X in condition $C_r(\epsilon)$. Define

$$j_1 := \min\{j : x_j \in X', x_j \geq y_{k_1+m-1} + \epsilon\}, \;\; j_2 := \max\{j : x_j \in X', x_j \leq y_{k_2-n+1} - \epsilon\},$$

Then the set

$$X_k = \{x_j \in X' : j_1 \leq j \leq j_2\} \subset \left(y_{k_1+m-1} + \epsilon, y_{k_2-n+1} - \epsilon\right)$$

has at least N elements. We now choose one point $x_j \in (y_l + \epsilon, y_{l+1} - \epsilon) \cap X'$ for each $k_1 + m - 1 \leq l \leq k_2 - n$ and an additional $n + m - 1$ points $x_j \in \left(y_{k_1+m-1} + \epsilon, y_{k_2-n+1} - \epsilon\right) \cap X'$ and obtain a subset

$$X'_k = \{\xi_1 < \cdots < \xi_N\} \subseteq X_k \subset X',$$

containing precisely N elements. As before, we set

$$\eta_1 = y_{k_1+l-1}, \quad 1 \leq l \leq N,$$

and define the quadratic sub matrix P_k of P by

$$P_k = \left(g(\xi_j - \eta_l)\right)_{j,l=1,\ldots,N}.$$

The modified construction leads to a stronger version of the Schoenberg-Whitney conditions, namely

$$\xi_j + \epsilon \leq \eta_{j+m} \text{ for } 1 \leq j \leq N - m, \quad \eta_j + \epsilon \leq \xi_{j+n} \text{ for } 1 \leq j \leq N - n. \quad (40)$$

Step 4 remains unchanged, and the column of P with $k < k_1$ or $k > k_2$ are linearly dependent on the columns of P_k.

Hence P_k is invertible and, by padding the $(r+1)m -$ th row of P_k^{-1} with zeros, we obtain the row $\left(\gamma_{k,j}\right)_{j\in\mathbb{Z}}$ with the row index k of the left-inverse Γ.

Step 2. We show that this left inverse Γ defines a bounded operator on $\ell^2(\mathbb{Z})$.

By construction the $j -$ th row $(\gamma_{k,j})$ of Γ has at most N non-zero entries between $k_1 = k - (r+1)m + 1$ and $k_1 + N - 1$. To apply Lemma (4.2.15), we need to show that the entries of Γ are uniformly bounded, or equivalently, that the entries of P_k^{-1} are bounded with a bound that does not depend on k.

We set up a compactness argument similar to the proof of Theorem (4.2.8), we begin with the simple observation that

$$g(\xi_j - \eta_l) = g\left((\xi_j - \eta_1) - (\eta_l - \eta_1)\right), \;\; j, l = 1, \ldots, N.$$

Let S be the N-dimensional simplex

$$S = \{\tau = (\tau_1, \dots, \tau_N) \in \mathbb{R}^N : 0 \le \tau_1 \le \dots \le \tau_N \le (N-1)Q_Y\}. \qquad (41)$$

Although the finite sequences $(\xi_j)_{1\le j\le N}$ and $(\eta_j)_{1\le l\le N}$depend on the row index k (and we should write $\xi_j^{(k)}$ and $\eta_l^{(k)}$ to make the dependence explicit), we always have

$$0 < \xi_1 - \eta_1 < \xi_N - \eta_1 < \eta_N - \eta_1 \le (N-1)Q_Y.$$

Consequently,

$$(\xi_1 - \eta_1, \dots, \xi_N - \eta_1) \in S, \text{ and } (0, \eta_2 - \eta_1, \dots, \eta_N - \eta_1) \in S.$$

Let $q := \min\{q_{X'}, q_Y\} > 0$ be the minimum of the separation distances of the quasi-uniform sets X' and Y and let

$$S_q = \{\tau = (\tau_1, \dots, \tau_N) \in S : \tau_{j+1} - \tau_j \ge q \text{ for } 1 \le j \le N-1\}.$$

Then S_q is compact and

$$(\xi_1 - \eta_1, \dots, \xi_N - \eta_1) \in S_q \text{ and } (0, \eta_2 - \eta_1, \dots, \eta_N - \eta_1) \in S_q.$$

Finally, we define the compact set

$$K = \{(\tau, \theta) \in S_q \times S_q : \tau_j + \epsilon \le \theta_{j+m} \text{ for } 1 \le j \le N-m, \\ \theta_j + \epsilon \le \tau_{j+n} \text{ for } 1 \le j \le N-n\}.$$

The assumption $C_r(\in)$ implies that

$$((\xi_1 - \eta_1, \dots, \xi_N - \eta_1), (0, \eta_2 - \eta_1, \dots, \eta_N - \eta_1)) \in K.$$

Clearly, the Schoenberg-Whitney conditions are satisfied for every point $(\tau, \theta) \in K$ and therefore every $N \times N$-matrix $\left(g(\tau_j - \theta_l)\right)$ has positive determinant. Since the determinant depends continuously on (τ, θ) and K is compact, we conclude that

$$\inf_{(\tau,\theta)\in K} \det\left(g(\tau_j - \theta_l)\right) = \delta > 0.$$

This construction implies that $\det P_k \ge \delta > 0$ for every k. As in the proof of Theorem (4.2.8) we use Cramer's rule and conclude that all entries of P_k^{-1} are bounded by $(N-1)!\,\delta^{-1}\|g\|_\infty^{N-1}$.

The assumption of the modified Schur test are satisfied, and Lemma (4.2.15) yields that the matrix Γ is bounded as an operator on $\ell^2(\mathbb{Z})$. Finally, Lemma (4.2.13) implies that X is a set of sampling for $V_Y(g)$.

Corollary (4.2.17)[205]: Assume that g is totally positive of finite order $M \ge 2$ and $Y = h\mathbb{Z}$. Let $\alpha = \sup_{j\in\mathbb{Z}}(x_{j+1} - x_j)$ be the maximum $g$ap between consecutive sampling points. If $\alpha < h$, then (x_j) is a set of sampling for $V_Y(g)$.

Proof. The assumption of Theorem (4.2.16) is verified with $\epsilon = h - \alpha$.

Section (4.3): Gabor Orthonormal Bases

For g be a non-zero function in $L^2(\mathbb{R}^d)$ and let Λ be a discrete countable set on $\mathbb{R}^{2d}$, where we identify $\mathbb{R}^{2d}$ to the time-frequency plane by writing $(t, \lambda) \in \Lambda$ with $t, \lambda \in \mathbb{R}^{2d}$. the Gabor system associated with the window g consists of the set of translates and modulates of g:

$$\mathcal{G}(g, \Lambda) = \{e^{2\pi i\langle\lambda, x\rangle} g(x - t) : (t, \lambda) \in \Lambda\}. \qquad (42)$$

Such systems were first introduced by Gabor [248] who used them for applications in the theory of telecommunication, but there has been a more recent interest in using Gabor system to expand functions both from a theoretical and applied perspective. The branch of Fourier analysis dealing with Gabor systems is usually referred to as Gabor, or time-frequency, analysis. Gröchenig's monograph [244] provide an excellent and detailed exposition on this subject.

Recall that the Gabor system is a $frame$ for $L^2(\mathbb{R}^d)$ if there exists constants $A, B > 0$ such that

$$A\|f\|^2 \le \sum_{(t,\lambda)\in\Lambda} \left|\langle f, e^{2\pi i\langle\lambda,.\rangle} g(\cdot - t)\rangle\right|^2 \le B\|f\|^2, \quad f \in L^2(\mathbb{R}^d). \tag{43}$$

It is called an orthonormal basis for $L^2(\mathbb{R}^d)$ if it is complete and the elements of the systems (42) are mutually orthogonal in $L^2(\mathbb{R}^d)$ and have norm 1, or, equivalently, $\|g\| = 1$ and $A = B = 1$ in (43). One of the fundamental problems in Gabor analysis is to classify the windows g and time-frequency sets Λ with the property that the associated Gabor system $\mathcal{G}(g, \Lambda)$ forms a (Gabor) frame or an orthonormal basis for $L^2(\mathbb{R}^d)$. This is of course a very difficult problem and only partial results are known. For example, the complete characterization of time-frequency sets Λ for which (42) is a frame for $L^2(\mathbb{R}^d)$ was only done when $g = e^{-\pi x^2}$, the Gaussian window. Lyubarskii, and Seip and Wallsten [253,258] showed that $\mathcal{G}\left(e^{-\pi x^2}, \Lambda\right)$ is a Gabor frame if and only if the lower Beurling density of Λ is strictly greater than 1. If we assume that Λ is a lattice of the form $a\mathbb{Z} \times b\mathbb{Z}$, then it is well known that $ab \le 1$ is a necessary condition for (42) to form a frame for $L^2(\mathbb{R}^d)$. Gröchenig and Stöcker [246] showed that for totally positive functions, (42) is a frame if and only if $ab < 1$. If we consider $g = \chi_{[0,c)}$, the characteristic function of an interval, the associated characterization problem is known as the abc-problem in Gabor analysis. By rescaling, one may assume that $c = 1$. In that case, the famous Janssen tie showed that the structure of the set couples (a, b) yielding a frame is very complicated [250,249]. A complete solution of the abc-problem was recently obtained by Dai and Sun [243].

We focus our attention on Gabor system of the form (42) which yield orthonormal bases for $L^2(\mathbb{R}^d)$. Perhaps the most natural and simplest example of Gabor orthonormal basis is the system $\mathcal{G}\left(\chi_{[0,c)^d}, \mathbb{Z}^d \times \mathbb{Z}^d\right)$. The orthonormality property for this system easily follows from that facts that the Euclidean space $\mathbb{R}^d$ can be partitioned by the $\mathbb{Z}^d$-translates of the hypercube $[0,1]^d$ and that the exponentials $e^{2\pi\square\langle n,x\rangle}$ form an orthonormal basis for the space of square-integrable functions supported on any of these translated hypercubes. A direct generalization of this observation is the following:

Proposition (4.3.1)[241]: Let $|g| = |K|^{-1/2}\chi_K$, where $|\cdot|$ denotes the Lebesgue measure, and $K \subset \mathbb{R}^d$ is measurable with finite Lebesgue measure. Suppose that

(i) The translates of K by the discrete set $\mathcal{J}$ are pairwise a.e disjoint and cover $\mathbb{R}^d$ up to a set of zero measure.

(ii) For each $t \in \mathcal{J}$, the set of exponentials $\left\{e^{2\pi i\langle\lambda,x\rangle}: \lambda \in A_t\right\}$ is an orthonormal basis for $L^2(K)$.

Let

$$\Lambda = \bigcup_{t\in\mathcal{J}} \{t\} \times \Lambda_t. \tag{44}$$

Then $\mathcal{G}(g, \Lambda)$ is a Gabor orthonormal basis for $L^2(\mathbb{R}^d)$.

Although its proof is straightforward and will be omitted (see also [255]), this proposition gives us a flexible way of constructing large families of Gabor orthonormal basis. The first condition above means that K is a translational tile (with J called an associated tiling set) and the second one that $L^2(K)$ admits an orthonormal basis of exponentials. If this last conditions holds, K is called a spectral set (and each Λ_t is an associated spectrum). The connection between translational tiles and spectral sets is quite mysterious. They were in fact conjectured to be the same class of sets by Fuglede [244], but that statement was later disproved by Tao [259] and the exact relationship between the two classes remains unclear.

For the fixed window $g_d = \chi_{[0,1]^d}$, we call a countable set $\Lambda \subset \mathbb{R}^{2d}$ standard if it is of the form (44). Motivated by the complete solution to the abc-problem, our main objective is to characterize the discrete sets Λ (not necessarily lattices) with the property that the Gabor system $\mathcal{G}(g_d, \Lambda)$ is a Gabor orthonormal basis. First, by generalizing the notion of orthogonal packing region in the work of Lagarias, Reeds and Wang [256] to the setting of Gabor systems, we deduce a general criterion for $\mathcal{G}(g_d, A)$ to be a Gabor orthonormal basis.

Theorem (4.3.2)[241]: $\mathcal{G}(g_d, \Lambda)$ is a Gabor orthonormal basis if and only if $\mathcal{G}(g_d, \Lambda)$ is an orthogonal set and the translates of $[0,1]^d$ by the elements of Λ tile $\mathbb{R}^{2d}$.

This criterion offers a very simple solution to our problem in the one-dimensional case.

However, such a simple characterization cases to exist in higher dimensions. We will introduce an inductive procedure which allows us to construct a Gabor orthonormal basis with window g_d from a Gabor orthonormal basis with window $g_n, n < d$. This procedure can be used to produce many non- standard Gabor orthonormal basis and we call a set Λ obtained through this procedure pseudo-standard. Assuming a mild condition on a low-dimensional time-frequency space, we show that $\mathcal{G}(g_d, \Lambda)$ are essentially pseudo-standard (See Theorem (4.3.14))

Although we do not have a complete description of the sets Λ yielding Gabor orthonormal bases with window g_d in dimension $d \geq 3$, we managed to obtain a complete characterization of those discrete sets $\Lambda \subset \mathbb{R}^4$ such that $\mathcal{G}(g_2, \Lambda)$ form an orthonormal basis for $L^2(\mathbb{R}^2)$.

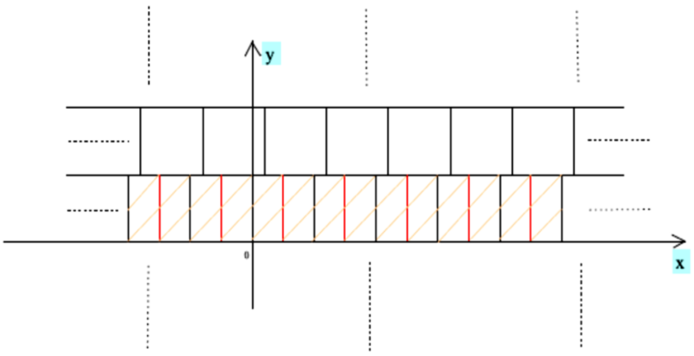

Figure (1)[241]: This figure illustrates the time-domain of Λ in the first situation of Theorem (4.3.22). We basically partition $\mathbb{R}^2$ by horizontal strips. Some strips, like $\mathbb{R} \times [0,1]$ with $n = 0$, have overlapping structure. This corresponds to the first union of Λ. Some strips, like $\mathbb{R} \times [1,2]$ with $n = 1$, have tiling structures. This corresponds to the second union of Λ.

We provide some preliminaries notations and prove Theorem (4.3.2). We show Theorem (4.3.6) and introduce the pseudo-standard time-frequency set. We focus on dimension 2 and prove Theorem (4.3.22).

We explore the relationship between Gabor orthonormal bases and tiling's in the time-frequency space. This theory will be an extension of spectral-tile duality in [256] to the setting of Gabor analysis. Denote by $|K|$ the Lebesgue measure of a set K. We say that a closed set T is a region if $|\partial T| = 0$ and $\overline{T^\circ} = T$. A bounded region T is called a translational tile if we can find a countable set $\mathcal{J}$ such that

(i) $|(T+t) \cap (T+t')| = 0, \quad t, t' \in \mathcal{J}, t \neq t'$, and

(ii)$\bigcup_{t\in\mathcal{J}}(T+t) = \mathbb{R}^d$.

In that case, $\mathcal{J}$ is called a tiling set for T and $T+\mathcal{J}$ a tiling of $\mathbb{R}^d$. We will say that $T+\mathcal{J}$ is a packing of $\mathbb{R}^n$ if (i) above is satisfied. We can generalize the notion of tiling and packing to measures and functions. Given a positive Borel measure μ and $f \in L^1(\mathbb{R}^n)$ with $f \geq 0$, the convolution of f and μ is defined to be

$$f * \mu(x) = \int f(x-y)d\mu(y), \qquad x \in \mathbb{R}^n,$$

(where a Borel measurable function is chosen in the equivalence class of f to define the integral above). We say that $f+\mu$ is a tiling (resp. packing) of $\mathbb{R}^d$ if $f * \mu = 1$ (resp. $f * \mu \leq 1$) almost everywhere with respect to the Lebesgue measure. It is clear that if $f = \chi_T$ and $\mu = \delta_{\mathcal{J}}$ where $\delta_{\mathcal{J}} = \sum_{t\in\mathcal{J}} \delta_t$, then $f * \mu = 1$ is equivalent to $T+\mathcal{J}$ being a tiling.

First we start with the following Theorem which gives us a very useful criterion to decide if a packing is actually a tiling. in fact, special cases of this Theorem were showed by many in different settings (see e.g [256,Theorem 3.1], [252, Lemma 3.1] and [254]), but the following version is the most general one.

Theorem (4.3.3)[241]: Suppose that $F, G \in L^1(\mathbb{R}^n)$ are two functions with $F, G \geq 0$ and $\int_{\mathbb{R}^n} F(x)dx = \int_{\mathbb{R}^n} G(x)dx = 1$. Suppose that μ is a positive Borel measure on $\mathbb{R}^n$ such that

$$F * \mu \leq 1 \quad \text{ and } \quad G * \mu \leq 1.$$

Then, $F * \mu = 1$ if and only if $G * \mu = 1$.

Proof. By symmetry, it suffices to show one side of the equivalence.Assuming that $F * \mu = 1$, we have

$$1 = F * \mu \Rightarrow 1 = 1 * G = G * F * \mu = F * G * \mu.$$

Letting $H = G * \mu$ we have $0 \leq H \leq 1$ and $H * F = 1$. We now show that $H = 1$. indeed letting A be the set $\{x \in \mathbb{R}^n, H(x) < 1\}$ and $B = \mathbb{R}^n \backslash A$, we have

$$(H * F)(x) = \int_{\mathbb{R}^n} H(y)F(x-y)dy = \int_A H(y)F(x-y)dy + \int_B H(y)F(x-y)dy$$

Now, if $|A| > 0$, we have

$$\int_{\mathbb{R}^n}\int_A F(x-y)dydx = |A| > 0$$

and there exists thus a set E with positive measure such that

$$\int_A F(x-y)dy > 0, \qquad x \in E.$$

If $x \in E$, we have

$$\int_A H(y)F(x-y)dy + \int_B H(y)F(x-y)dy < \int_A F(x-y)dy + \int_B F(x-y)dy$$
$$= (1 * F)(x) = 1.$$

This contradicts to the that $H * F = 1$ almost everywhere. Hence, $|A| = 0$ and $H = 1$ follows.

Let, $g \in L^2(\mathbb{R}^d)$. We define the short time Fourier transform of f with respect to the window g be

$$V_g f(t,v) = \int_{\mathbb{R}^{2d}} f(x)\overline{g(x-t)}e^{-2\pi i\langle v,x\rangle}dx.$$

Let $\mathcal{G}(g,\Lambda)$ be a Gabor orthonormal basis. Since translating Λ be an element of $\mathbb{R}^{2d}$ does not affect the orthonormality nor the completeness of the given system, there is no loss of generality in assuming that $(0,0) \in \Lambda$. We say that a region D $(\subset \mathbb{R}^{2d})$ is an orthogonal packing region for g if

$$(D^\circ - D^\circ) \cap \mathcal{Z}(V_g g) = \emptyset.$$

Here $\mathcal{Z}(V_g g) = \{(t,v): V_g g(t,v) = 0\}$.

Lemma (4.3.4)[241]: Suppose that $\mathcal{G}(g,\Lambda)$ is a mutually orthogonal set of $L^2(\mathbb{R}^d)$. Let D be any orthogonal packing region for g. Then $\Lambda - \Lambda \subset \mathcal{Z}(V_g g) \cup \{0\}$ and $A + D$ is a packing of $\mathbb{R}^{2d}$. Suppose furthermore that $\mathcal{G}(g,\Lambda)$ is a Gabor orthonormal basis. Then $|D| \le 1$.

Proof. Let $(t,\lambda),(t',\lambda') \in \Lambda$ be two distinct points in Λ. Then

$$\int g(x-t')\overline{g(x-t)}\, e^{-2\pi i(\lambda-\lambda')x}dx = 0,$$

Or equivalently, after the change of variable $y = x - t'$,

$$\int g(x)\overline{g(x-(t-t'))}\, e^{-2\pi i(\lambda-\lambda')x}dx = 0\,.$$

Hence, $V_g g(t-t',\lambda-\lambda') = 0$ and $(t,\lambda)-(t',\lambda') \in \mathcal{Z}(V_g g)$. This means that $(t,\lambda)-(t',\lambda') \notin D^0 - D^0$. Therefore, the intersection of the sets $(t,\lambda)+D$ and $(t',\lambda')+D$ has zero Lebesgue measure.

Suppose now that $\mathcal{G}(g,A)$ is a Gabor orthonormal basis. Denote by R the diameter of D. By the packing property of $A + D$,

$$|D| \cdot \frac{\#(A \cap [-T,T]^{2d})}{(2T)^{2d}} = \frac{1}{(2T)^{2d}}\left|\bigcup_{\lambda\in\Lambda\cap[-T,T]^{2d}}(D+\lambda)\right| \le \frac{1}{(2T)^{2d}}\left|[-T-R,T+R]^{2d}\right|$$
$$= \left(1+\frac{R}{2d}\right)^{2d}.$$

Taking limit $T \to \infty$ and using the fact that Beurling density of Λ is 1 ([257]), we have $|D| \le 1$.

We say that an orthogonal packing region D for g is tight if we have furthermore $|D| = 1$. We now apply Theorem (4.3.3) to the Gabor orthonormal basis problem.

Theorem (4.3.5)[241]: Suppose that $\mathcal{G}(g,\Lambda)$ is an orthonormal set in $L^2(\mathbb{R}^d)$ and that D is a tight orthogonal packing region for g. Then $\mathcal{G}(g,\Lambda)$ ia a Gabor orthonormal basis for $L^2(\mathbb{R}^d)$ if and only if $\Lambda + D$ is a tiling of $\mathbb{R}^{2d}$.

Proof. Let $F = \chi_D$ and $G = \left|V_g f\right|^2/\|f\|_2^2$. Then $\int_{\mathbb{R}^{2d}} F = 1$ and $\int_{\mathbb{R}^{2d}} G = \|g\|_2^2 = 1$. Now, as D is an orthogonal packing region for g, we have in particular
$\sum_{\lambda\in A} \chi_D(x-\lambda) \le 1$.
This shows that

$$\delta_\Lambda * F = \delta_\Lambda * \chi_D \le 1.$$

Moreover, $\Lambda + D$ is a tilling of $\mathbb{R}^{2d}$ if and only if $\delta_A * \chi_D = 1$. On the other hand, (g,Λ) being a mutually orthogonal set, Bessel's inequality yields

$$\sum_{(t,\lambda)\in\Lambda} \left| \int_{\mathbb{R}^d} f(x)\overline{g(x-t)} e^{-2\pi i\langle\lambda,x\rangle} dx \right|^2 \le \|f\|^2, \qquad f \in L^2(\mathbb{R}^d),$$

or, replacing f by $f(x-\tau)e^{2\pi i v x}$ with $(\tau, v) \in \mathbb{R}^{2d}$,

$$\sum_{(t,\lambda)\in\Lambda} \left|V_g f(\tau - t, v-\lambda)\right|^2 \le \|f\|^2, \qquad f \in L^2(\mathbb{R}^d).$$

Hence,

$$\delta_\Lambda * G = \delta_\Lambda * \frac{\left|V_g f\right|^2}{\|f\|^2} \le 1$$

with equality if and only if the Gabor orthonormal system is in fact a basis. The conclusion follows then from Theorem (4.3.3).

Theorem (4.3.6)[241]: In dimension $d = 1$, the system $\mathcal{G}(g_1, \Lambda)$ is a Gabor orthonormal basis if and only if Λ is standard.

Proof. Let $g_d = \chi_{[0,1]^d}$. Using Theorem (4.3.5), we just need to show that $[0,1]^{2d}$ is a tight orthogonal packing region for g_d.

We first consider the case $d = 1$. For $g_1 = \chi_{[0,1]}$, a direct computation shows that

$$V_{g_1} g_1(t,v) = \begin{cases} 0, & |t| \ge 1; \\ \dfrac{1}{2\pi i v}\left(e^{2\pi i v} - e^{2\pi i v}\right), & 0 \le t \le 1; \\ \dfrac{1}{2\pi i v}\left(1 - e^{2\pi i v(t+1)}\right), & -1 \le t \le 0. \end{cases} \tag{45}$$

The zero set of $V_{g_1} g_1$ is therefore given by

$$Z\left(V_{g_1} g_1\right) = \{(t,v) : |t| \ge 1\} \cup \left\{(t,v) : v(1-|t|) \in \mathbb{Z}\backslash\{0\}\right\}. \tag{46}$$

Hence, $(0,1)^2 - (0,1)^2 = (-1,1)^2$ does not intersect the zero set and therefore $[0,1]^2$ is a tight orthogonal packing region for g_1.

We now consider the case $d \ge 2$. As we can decompose g_d as $\chi_{[0,1]}(x_1) \ldots \chi_{[0,1]}(x_d)$, we have

$V_{g_d} g_d(t,v) = V_{g_1} g_1(t_1,v_1) \ldots V_{g_1} g_1(t_d, v_d)$ where $t = (t_1, \ldots, t_d)$ and $v = (v_1, \ldots, v_d)$.

The zero set $V_{g_d} g_d$ is therefore given by

$$Z(V_{g_d}g_d) = \{(t,v): |t|_{\max} \geq 1\} \cup \left(\bigcup_{i=1}^{d} \{(t,v) : v_i(1-|t_i|) \in \mathbb{Z}\backslash\{0\}\} \right) \quad (47)$$

where $|t|_{\max} = \max\{t_1, \dots, t_d\}$. It follows that $[0,1]^{2d}$ is a tight orthogonal packing region for g_d.

The following example will not be used in later discussion, but it demonstrates the usefulness of the theory for windows other than the unit cube.

Example (4.3.7)[241]: Let $g(x) = \frac{2}{e^{2x}+e^{-2x}}$ be the hyperbolic secant function. It can be shown ([251]; see also [247]) that

$$V_g g(t,v) = \frac{\pi \sin(\pi v t) e^{-\pi i v t}}{\sinh(2t) \sinh(\pi^2 v/2)}$$

and the zero set is given by

$$Z(V_g g) = \{(t,v): tv \in \mathbb{Z}\backslash\{0\}\}.$$

Hence, $[0,1]^2$ is a tight orthogonal packing region for g. Note that the zero set does not contain any point on the $x-$axis and $y-$axis. There is no tiling set Λ for $[0,1]^2$ such that $\Lambda - \Lambda \subset Z(V_g g) \cup \{0\}$ (see also Proposition (4.3.9)) and thus there is no Gabor orthonormal basis using the hyperbolic secant as a window. This can be viewed as a particular case of a version of the Balian-Low Theorem valid for irregular Gabor frames which was recently obtained in [242] and which state that Gabor orthonormal bases cannot exist if the window function is in the modulation space $M^1(\mathbb{R}^d)$.

Using Lemma (4.3.4), Theorem (4.3.2) may be restated in the following way:

Theorem (4.3.8)[241]: $\mathcal{G}(\chi_{[0,1]^d}, \Lambda)$ is a Gabor orthonormal basis if and only if the inclusion $\Lambda - \Lambda \subset Z(V_g g) \cup \{0\}$ holds and $\Lambda + [0,1]^2$, is a tiling.

In view of the previous result, the possible translational tiling's of the unit cube on $\mathbb{R}^{2d}$ play a fundamental role in the solution of our problem. A characterization for these is not available in arbitrary $2d$ dimension but it is easily obtained when $d = 1$. We prove this result here for completeness but it should be well known.

Proposition (4.3.9)[241]: Suppose that $\chi_{[0,1]^2+\mathcal{J}}$ is a tilling of $\mathbb{R}^2$ with $(0,0) \in \mathcal{J}$. Then $\mathcal{J}$ is of either of the following two form:

$$\mathcal{J} = \bigcup_{k\in\mathbb{Z}} (\mathbb{Z} + a_k) \times \{k\} \quad \text{or} \quad \mathcal{J} = \bigcup_{k\in\mathbb{Z}} \{k\} \times (\mathbb{Z} + a_k) \quad (48)$$

where a_k are any real numbers in $[0,1)$ for $k \neq 0$ and $a_0 = 0$.

Proof. By Keller's criterion for square tiling's (see e.g [256, Proposition 4.1]), for any (t_1, t_2) and (t_1', t_2') in $\mathcal{J}$, $t_i - t_i' \in \mathbb{Z}\backslash\{0\}$ for some $i = 1,2$. Taking $(t_1', t_2') = (0,0)$, we obtain that, for any $(t_1, t_2) \in \mathcal{J}\backslash\{(0,0)\}$, one of t_1 or t_2 belongs to $\mathbb{Z}\backslash\{0\}$. If $\mathcal{J} \subset \mathbb{Z}$, we must have $\mathcal{J} = \mathbb{Z}$ for $\chi_{[0,1]^2+\mathcal{J}}$ to be tiling of $\mathbb{R}^2$ and $\mathbb{Z}$ can be written as either of the sets in (48) by taking $a_k = 0$ for all k. Suppose that there exists $(s_1, s_2) \in \mathcal{J}$ such s_1 is not an integer and $s_2 \in \mathbb{Z}$. If $(t_1, t_2) \in \mathcal{J}$ and $t_2 \notin \mathbb{Z}$, then both t_1 and $t_1 - s_1$ must be integers which would imply that s_1 is an integer, contrary to our assumption. Hence, $(s_1, s_2) \in \mathcal{J}$ implies $s_2 \in \mathbb{Z}$ and we can write

$$\mathcal{J} = \bigcup_{k\in\mathbb{Z}} \mathcal{J}_k \times \{k\}.$$

for some discrete set $\mathcal{J}_k \subset \mathbb{R}$. For $\chi_{[0,1]^2} + \mathcal{J}$ to be a tiling of $\mathbb{R}^2$, the set $\mathcal{J}_k$ must be of the form $\mathcal{J}_k = \mathbb{Z} + a_k$. In the case $\mathcal{J}$ can be expressed as one of the sets in the first collection appearing in (48).

Similarly, if there exists $(s_1, s_2) \in \mathcal{J}$ such that s_2 is not an integer and $s_1 \in \mathbb{Z}, \mathcal{J}$ can be expressed as one of the sets in the second collection appearing in (48) this completes the proof.

We say that the Gabor orthonormal basis $\mathcal{G}(\chi_{[0,1]^d,} \Lambda)$ is standard if

$$\Lambda = \bigcup_{t \in \mathcal{J}} \{t\} \times \Lambda_t,$$

where $\mathcal{J} + [0,1]^d$ tiles $\mathbb{R}^d$ and Λ_t is a spectrum for $[0,1]^d$. (Note that, by the result in [256], $\Lambda_t + [0,1]^d$ must then be a tiling of $\mathbb{R}^d$ for every $t \in \mathcal{J}$).

The following result settles the one-dimensional case.

Theorem (4.3.10)[241]: $\mathcal{G}(\chi_{[0,1]^d,} \Lambda)$ is a Gabor orthonormal basis if and only if Λ is standard.

Proof. We just need to show that Λ being standard is a necessary condition for $\mathcal{G}(\chi_{[0,1]^d,} \Lambda)$ to be a Gabor orthonormal basis. We can also assume, for simplicity, that$(0,0) \in \Lambda$. By Proposition (4.3.9), if $\mathcal{G}(\chi_{[0,1]^d,} \Lambda)$ is a Gabor orthonormal basis, then $\Lambda - \Lambda \subset Z(V_g g) \cup \{0\}$ and $\Lambda + [0,1]^2$ Must be a tiling of $\mathbb{R}^2$. By Proposition (4.3.9), Λ must be of either one of the forms in (48). Note that Λ is standard in the second case. In order to deal with the first case, suppose that

$$\Lambda = \bigcup_{k \in \mathbb{Z}} (\mathbb{Z} + a_k) \times \{k\}, \quad \text{with } a_k \in [0,1), \; k \neq 0, \; a_0 = 0.$$

We now show that this is impossible unless $a_k = 0$ for all k (which reduces to the case $\Lambda = \mathbb{Z}^2$, which is standard). We can assume, without loss of generality, that $a_k \neq 0$ for some $k > 0$ with k being the smallest such index. If $a_k \neq 0$ for some k, then both (a_k, k) and $(0, k-1)$ are in Λ. The orthogonality of the Gabor system then implies that $(a_k, 1) \in Z(V_g g)$. Using (46), we deduce that $1. \; (1 - |a_k|) \in \mathbb{Z} \backslash \{0\}$. That means a_k must be an integer, which is a contradiction. Hence, the first case is impossible unless $a_k = 0$ for all k and the proof is completed.

A description of all time-frequency sets Λ for which $\mathcal{G}(\chi_{[0,1]^d,} \Lambda)$ is a Gabor orthonormal basis however become vastly more complicated when $d \geq 2$. In particular, as we will see, the standard structure cannot cover all possible cases. Consider integers $m, n > 0$ such that $m + n = d$. For convenience and to be consistent with our previous notation, we will write the Cartesian product of the two-frequency spaces $\mathbb{R}^{2m}$ and $\mathbb{R}^{2n}$ in the non-standard form

$$\mathbb{R}^{2d} = \mathbb{R}^{2m} \times \mathbb{R}^{2n} = \{(s, t, \lambda, v), (s, \lambda) \in \mathbb{R}^{2m}, (t, v) \in \mathbb{R}^{2n}\}.$$

We will also denote by Π_1 the projection operator from $\mathbb{R}^{2d}$ to $\mathbb{R}^{2m}$ defined by

$$\prod\nolimits_1 \big((s, t, \lambda, v)\big) = (s, \lambda), \quad (s, t, \lambda, v) \in \mathbb{R}^{2d} = \mathbb{R}^{2m} \times \mathbb{R}^{2n}. \tag{49}$$

To simplify the notation, we also define $g_k = \chi_{[0,1]^k}$ for any $k \geq 1$. We now build a new family of time-frequency sets on $\mathbb{R}^{2d}$ as follows. Suppose that $\mathcal{G}(\chi_{[0,1]^m,} \Lambda_1)$ is a Gabor orthonormal basis for $L^2(\mathbb{R}^m)$ and that we associate with each $(s, \lambda) \in \Lambda_1$, a discrete set $\Lambda_{(s,\lambda)}$ in $\mathbb{R}^{2n}$ such that $\mathcal{G}(\chi_{[0,1]^n,} \Lambda_{(s,\lambda)})$ is a Gabor orthonormal basis of $L^2(\mathbb{R}^n)$. We then define

$$\Lambda = \bigcup_{(s,\lambda)\in\Lambda_1} \{(s,t,\lambda,v),(t,v)\in\Lambda_{(s,\lambda)}\}. \quad (50)$$

We say that a Gabor system $\mathcal{G}(\chi_{[0,1]^d,}\Lambda)$ with Λ as in (50) is pseudo-standard.

Proposition (4.3.11)[241]: Every pseudo-standard Gabor system $\mathcal{G}(\chi_{[0,1]^d,}\Lambda)$ is a Gabor orthonormal basis $L^2(\mathbb{R}^d)$.

Proof. If $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$, we have $g_d(x,y) = g_m(x)g_n(y)$ (for $m+n=d$). This yields immediately that

$$V_{g_d}g_d(s,t,\lambda,v) = V_{g_m}g_m(s,\lambda)V_{g_n}g_n(t,v), \quad (s,\lambda)\in\mathbb{R}^{2m},(t,v)\in\mathbb{R}^{2n}. \quad (51)$$

Suppose that $\rho=(s,t,\lambda,v)$ and $\rho'=(s',t',\lambda',v')$ are distinct elements of Λ. If $(s,\lambda)=(s',t')$, then (t,v) and (t',v') are distinct elements of $\Lambda_{(s,\lambda)}$ and we have thus

$$(t'-t,v'-v)\in Z\big(V_{g_n}g_n\big)$$

which implies that $Z\big(V_{g_d}g_d\big)(\rho'-\rho)=0$. On the other hand , if $(s,\lambda)\neq(s',\lambda')$, we have then

$$(s'-s,\lambda'-\lambda)\in Z\big(V_{g_m}g_m\big)$$

Which implies again that $Z\big(V_{g_d}g_d\big)(\rho'-\rho)=0$. This proves the orthonormality of the system $\mathcal{G}(\chi_{[0,1]^d,}\Lambda)$. This proposition can now be proved by invoking Theorem (4.3.8) if we can show that $\Lambda+[0,1]^{2d}$ is a tiling of $\mathbb{R}^{2d}$. To prove this, we note that $\Lambda_1+[0,1]^{2d}$ is a tiling of the subspace $\mathbb{R}^{2m}$ by Theorem (4.3.8) and that, similarly, for each $(t,\lambda)\in\Lambda_{(t,\lambda)}+[0,1]^{2n}$ is a tiling of $\mathbb{R}^{2n}$. This easily implies the required tiling property and concludes the proof.

Example (4.3.12)[241]: Consider the two-dimensional case $d=2$. Let

$$\Lambda_1=\bigcup_{m\in\mathbb{Z}}\{m\}\times(\mathbb{Z}+\mu_m), \quad \mu_m\in[0,1).$$

Associate with each $\gamma=(m,j+\mu_m)\in\Lambda_1$ the set

$$\Lambda_\gamma=\bigcup_{n\in\mathbb{Z}}\{n+s_{m,j}\}\times\big(\mathbb{Z}+v_{n,m,j}\big), \quad s_{m,j}\in\mathbb{R},v_{n,m,j}\in[0,1).$$

Then,

$$\Lambda\coloneqq\{\big(m,n+s_{m,j},j+\mu_m,k+v_{n,m,j}\big):m,n\,j,k\in\mathbb{Z}\}$$

(written in the form of $(t_1,t_2,\lambda_1,\lambda_2)$ where (t_1,t_2) are the translations and (λ_1,λ_2) the frequencies) has the pseudo-standard structure. Note that the parameters $\delta_{m,j}$ can be chosen so that the set Λ is not standard as the set

$$\{\big(m,n+s_{m,j}\big),m,n,j\in\mathbb{Z}\}+[0,1]^2$$

will not tile $\mathbb{R}^2$ in general. For example, for $m=n=0$, we could let $s_{0,0}=0$ and the numbers $s_{0,j}$ could be chosen as distinct numbers in the interval $[0,1)$. The square $[0,1]^2$ would then overlap with infinitely many of its translates appearing as part of the Gabor system.

Using a similar procedure to higher dimension, we can produce many non-standard Gabor orthonormal bases with window $\chi_{[0,1]^d}$. However, the pseudo-standard structure still cannot cover all possible cases of time-frequency sets. A time-frequency set could be a mixture of pseudo-standard and standard structure. For example consider the set

$$\Lambda = \bigcup_{n\in\mathbb{Z}\setminus\{1\}} \{(m + t_{n,k}, n, j + \mu_{k,m,n,} k + v_n): j,k \in \mathbb{Z}\} \cup \{(m,1)\} \times \Lambda_m,$$

where $A_m + [0,1]^2$ tiles $\mathbb{R}^2$. This set consists of two parts. The first part is a subset of a set having the pseudo-standard structure while the second part is a subset of a set having the standard one. Moreover, the translates of the unit square associated with the first part are disjoint with those associated with the second part, showing that $\mathcal{G}(\chi_{[0,1]^2,} \Lambda)$ is a mutually orthogonal set. Since Λ is Clearly a tiling of $\mathbb{R}^4$, Theorem (4.3.8) shows that $\mathcal{G}(\chi_{[0,1]^2,} \Lambda)$ is a Gabor orthonormal basis. We will classify all possible sets $\Lambda \subseteq \mathbb{R}^4$ with the property that $\mathcal{G}(\chi_{[0,1]^2,} \Lambda)$ is a Gabor orthonormal basis for $L^2(\mathbb{R}^2)$. We have

Proposition (4.3.13)[241]: Let $d = m + n$ and suppose that $(\chi_{[0,1]^d,} \Lambda)$ is a Gabor orthonormal basis for $L^2(\mathbb{R}^d)$. If $(s_0, \lambda_0) \in \mathbb{R}^{2m}$, consider the translate of the unit hypercube in $\mathbb{R}^{2m}$, $C = (s_0, \lambda_0) + [0,1)^{2m}$, and define

$$\Lambda(C) := \{(t, v) \in \mathbb{R}^{2n}: (s, t, \lambda, v) \in \Lambda \text{ and } (s, \lambda) \in C\}.$$

Then $(\chi_{[0,1]^n,} \Lambda(C))$ is a Gabor orthonormal basis for $L^2(\mathbb{R}^{2n})$.

Proof. We first show that the system $(\chi_{[0,1]^n,} \Lambda(C))$ is orthogonal. Let (t, v) and (t', v') be distinct elements of $\Lambda(C)$. There exist (s, λ) and (s', λ') in $\mathbb{R}^{2m}$ such that (s, t, λ, v) and (s', t', λ', v') both belong to Λ. Using the mutual orthogonality of the system $(\chi_{[0,1]^d,} \Lambda)$ together with (51), we have

$$V_{g_m} g_m(s - s', \lambda - \lambda') = 0 \text{ or } V_{g_n} g_n(t - t', v - v') = 0.$$

Note that, as both (s, λ) and (s', λ') belong to C, we have $|s - s'|_{\max} < 1$ and $|\lambda - \lambda'|_{\max} < 1$. In particular, $V_{g_m} g_m(s - s', \lambda - \lambda') \neq 0$ and the orthogonallity of the system $(\chi_{[0,1]^n,} \Lambda(C))$ follows.

If $(s, \lambda) \in \Pi_1(\Lambda)$ (as defined in (49)), let

$$\Lambda_{(s,\lambda)} = \{(t, v): (s, t, \lambda, v) \in \Lambda\}.$$

Let $f_1 \in L^2(\mathbb{R}^m), f_2 \in L^2(\mathbb{R}^n)$ and $(s_0, \lambda_0) \in \mathbb{R}^{2m}$. Applying Parseval's identity to the function

$$f(x, y) = e^{2\pi i \lambda_0 x} f_1(x - s_0) f_2(y), \qquad x \in \mathbb{R}^m, y \in \mathbb{R}^n,$$

we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^m} |f_1(x)|^2 dx \int_{\mathbb{R}^n} |f_2(y)|^2 dy \\
&\qquad = \sum_{(s,\lambda)\in\Pi_1(\Lambda)} \sum_{(t,v)\in\Lambda_{(s,\lambda)}} \left|V_{g_m} f_1(s - s_0, \lambda - \lambda_0)\right|^2 \left|V_{g_n} f_2(t, v)\right|^2 \\
&\qquad = \sum_{(s,\lambda)\in\Pi_1(\Lambda)} \sum_{(t,v)\in\Lambda_{(s,\lambda)}} \left|V_{f_1} g_m(s_0 - s, \lambda_0 - \lambda)\right|^2 \left|V_{g_n} f_2(t, v)\right|^2
\end{aligned}$$

Defining

$$w(s, \lambda) = \|f_2\|_2^{-2} \sum_{(t,v)\in\Lambda_{(s,\lambda)}} \left|V_{g_n} f_2(t, v)\right|^2 \text{ and } \mu = \sum_{(s,\lambda)\in\Pi_1(\Lambda)} w(s, \lambda)\delta_{(s,\lambda)}$$

for $f_2 \neq 0$, the above identity can be written as

$$\int_{\mathbb{R}^m} |f_1(x)|^2 dx = \sum_{(s,\lambda)\in\Pi_1(\Lambda)} w(s,\lambda)\left|V_{f_1}g_m(s_0-s,\lambda_0-\lambda)\right|^2$$
$$= \left(\mu * \left|V_{f_1 g_m}\right|^2\right)(s_0,\lambda_0).$$

On the other hand, letting $\check{\chi}_{[0,1)^{2m}}(s,\lambda) = \chi_{[0,1)^{2m}}(-s,-\lambda)$ and defining C and $\Lambda(C)$ as above, we have also

$$\left(\mu * \check{\chi}_{[0,1)^m}\right)(s_0,\lambda_0) = \sum_{(s,\lambda)\in\Pi_1(\Lambda)} w(s,\lambda)\chi_{[0,1)^{2m}}(s-s_0,\lambda-\lambda_0)$$
$$= \sum_{(s,\lambda)\in\Pi_1(\Lambda)\cap C} w(s,\lambda) = \|f_2\|_2^{-2} \sum_{(t,v)\in\Lambda(C)} \left|V_{g_n}f_2(t,v)\right|^2 \leq 1,$$

Where the last inequality results from the orthogonality of the system $\left(\chi_{[0,1)^n,}\Lambda(C)\right)$ proved earlier. Since (s_0,λ_0) is arbitrary in $\mathbb{R}^{2m}$ and

$$\int_{\mathbb{R}^{2m}} \left|V_{f_1}g_m(s,\lambda)\right|^2 ds d\lambda = \|f_1\|_2^2,$$

Theorem (4.3.3) can be used to deduce that $\mu * \check{\chi}_{[0,1)^m} = 1$. This shows that

$$\sum_{(t,v)\in\Lambda(C)} \left|V_{g_n}f_2(t,v)\right|^2 = \|f_2\|^2, \quad f_2 \in L^2(\mathbb{R}^n),$$

and thus that the system $\left(\chi_{[0,1)^n,}\Lambda(C)\right)$ is complete, proving our claim.

Theorem (4.3.14)[241]: Let $d = m+n$ and let $\Pi_1: \mathbb{R}^{2d} \to \mathbb{R}^{2m}$ be defined by (49). Suppose that $(\chi_{[0,1]^d,}\Lambda)$ is a Gabor orthonormal basis and that $\Pi_1(\Lambda) + [0,1]^{2m}$ tiles $\mathbb{R}^{2m}$. Then Λ has the pseudo-standard structure.

Proof. Let $\mathcal{J} = \Pi_1(\Lambda)$ and, for any $(s,\lambda) \in \mathcal{J}$, define

$$\Lambda_{(s,\lambda)} = \{(t,v): (s,t,v,\lambda) \in \Lambda\}.$$

If $(s_0,\lambda_0) \in \mathcal{J}$, let $C = (s_0,\lambda_0) + [0,1)^{2m}$, and

$$\Lambda(C) \coloneqq \{(t,v) \in \mathbb{R}^{2n}: (s,t,v,\lambda) \in \Lambda \text{ and } (s,\lambda) \in C\}.$$

Proposition (4.3.13) shows that the system $\left(\chi_{[0,1)^n,}\Lambda(C)\right)$ forms a Gabor orthonormal basis. By assumption $\mathcal{J} + [0,1)^{2m}$ tiles $\mathbb{R}^{2m}$. Hence, $(s_0,\lambda_0) + [0,1)^{2m}$ contains exactly one point in $\mathcal{J}$, i.e. (s_0,λ_0), and we have

$$\Lambda(C) = \{(t,v): (s_0,t,\lambda_0,v) \in \Lambda\} = \Lambda_{(s_0,\lambda_0)}.$$

Therefore, we can write Λ as

$$\Lambda = \bigcup_{(s_0,\lambda_0)\in\mathcal{J}} \{(s_0,\lambda_0)\} \times \Lambda_{(s_0,\lambda_0)}.$$

Our proof will be complete if we can show that $\mathcal{J}$ is a Gabor orthonormal basis of $L^2(\mathbb{R}^m)$.

As $\mathcal{J}$ is a tiling set, by Proposition (4.3.8) it suffices to show that the inclusion $\mathcal{J} - \mathcal{J} \subset Z(V_{g_m}g_m) \cup \{0\}$ holds. Let (s,λ) and (s',λ') be distinct points in $\mathcal{J}$. As $\Lambda_{(s,\lambda)} + [0,1)^{2n}$ tiles $\mathbb{R}^{2n}$, so does $\Lambda_{(s,\lambda)} + [-1,0)^{2n}$, and we can find $(t,v) \in \Lambda_{(s,\lambda)}$ such that $0 \in (t,v) + [-1,0)^{2n}$, or, equivalently, with $(t,v) \in [0,1)^{2n}$. Similarly, we can fined $(t',v') \in \Lambda_{(s',\lambda')}$ such that $(t',v') \in [0,1)^{2n}$. Using the fact that $\left(\chi_{[0,1)^n,}\Lambda(C)\right)$ is a Gabor orthonormal basis of $L^2(\mathbb{R}^{2d})$, we have

$$(s,t,\lambda,v)-(s',t',\lambda',v')\in Z(V_{g_d}g_d).$$

or, equivalently

$$V_{g_m}g_m(s-s',\lambda-\lambda')=0 \ \text{ or } \ V_{g_n}g_n(t-t',v-v')=0.$$

Note that, since $|t-t'|<1$ and $|v-v'|<1, V_{g_n}g_n(t-t',v-v')\neq 0$. Hence $(s,\lambda)-(s',\lambda')\in Z(V_{g_m}g_m)$ as claimed.

The goal will be to classify all possible Gabor orthonormal basis generated by the unit square on $\mathbb{R}^2$.

Given a fixed Gabor orthonormal basis $\mathcal{G}(\chi_{[01]^2,}\Lambda)$ and a set $A\subset\mathbb{R}^2$, we define the sets

$$\Gamma(A)=\{(\lambda_1,\lambda_2)\in\mathbb{R}^2 : (t_1,t_2,\lambda_1,\lambda_2)\in\Lambda,(t_1,t_2)\in A\}$$

and, for any $(\lambda_1,\lambda_2)\in\mathbb{R}^2$ and any set $B\subset\mathbb{R}^2$, we let

$$T_A(\lambda_1,\lambda_2)=\{(t_1,t_2)\in\mathbb{R}^2 : (t_1,t_2,\lambda_1,\lambda_2)\in\Lambda,(t_1,t_2)\in A\}$$

and

$$T_A(B)=\{(t_1,t_2)\in\mathbb{R}^2 : (t_1,t_2,\lambda_1,\lambda_2)\in\Lambda,(t_1,t_2)\in A,(t_1,t_2)\in B\}.$$

In particular, the set $T_A(\Gamma(A))$ collects all the couples $(t_1,t_2)\in A$ such that $(t_1,t_2,\lambda_1,\lambda_2)\in\Lambda$ for some $(\lambda_1,\lambda_2)\in\mathbb{R}^2$.

We say that a square is half-open if it is a translate of one of the sets

$$[0,1)^2,(0,1]^2,[0,1)\times(0,1] \text{ or } (0,1]\times[0,1).$$

Two measurable subsets of $\mathbb{R}^d$ will be called essentially disjoint if their intersection has zero Lebesgue measure. In the derivation below, we will make use of the identity

$$V_{g_2}g_2(t_1,t_2,\lambda_1,\lambda_2)=V_{g_1}g_1(t_1,\lambda_1)V_{g_1}g_1(t_2,\lambda_2)\,,\quad (t_1,t_2,\lambda_1,\lambda_2)\in\mathbb{R}^4,$$

which implies, in particular, that

$$V_{g_2}g_2(t_1,t_2,\lambda_1,\lambda_2)\Leftrightarrow V_{g_1}g_1(t_1,\lambda_1)=0 \ \text{ or } \ V_{g_1}g_1(t_2,\lambda_2)=0.$$

Moreover, using (47), the zero set of $V_{g_2}g_2$ is given by

$$Z(V_{g_2}g_2)=\{(t,\lambda): |t|_{\max}\geq 1\}\cup\left(\cup_{i=1}^{2}\{(t,v): \lambda_i(1-|t_i|)\in\mathbb{Z}\backslash\{0\})\}\right). \quad (52)$$

This implies that if $|t|_{\max}<1$ and $(t,\lambda)\in Z(V_{g_2}g_2)$, then, there exists $i\in\{1,2\}$ and for some integer $m\neq 0$ such that

$$|\lambda_i|=\frac{|m|}{1-|t_i|}\geq 1.$$

with a strict inequality if $t_i\neq 0$. These properties will be used throughout.

Lemma (4.3.16)[241]: Let $\mathcal{G}(\chi_{[01]^2,}\Lambda)$ be a Gabor orthonormal basis for $L^2(\mathbb{R}^2)$ and let C be a half-open square. Then,

(i) $\Gamma(C)+[0,1]^2$ is a packing of $\mathbb{R}^2$.

(ii)If $(\lambda_1,\lambda_2)\in\Gamma(C)$, then $T_C(\lambda_1,\lambda_2)$ consists of one point.

Proof. (i) let (λ_1,λ_2) and (λ_1',λ_2') be distinct elements of $\Gamma(C)$. By definition, we can find (t_1,t_2) and (t_1',t_2') in C such that $(t_1,t_2,\lambda_1,\lambda_2),(t_1',t_2',\lambda_1',\lambda_2')\in\Lambda$. We then have

$$0=V_{g_1}g_1(t_1-t_1',\lambda_1-\lambda_1')V_{g_1}g_1(t_2-t_2',\lambda_2-\lambda_2')$$

If, without loss of generality, the first factor on the right-hand side of the previous equality vanishes, the fact that $|t_1-t_1'|<1$ shows the existence of an integer $k>0$, such that

$$|\lambda_1-\lambda_1'|=k/(1-|t_1-t_1'|)\geq 1.$$

Hence, the cubes $(\lambda_1,\lambda_2)+[0,1]^2$ and $(\lambda_1',\lambda_2')+[0,1]^2$ are essentially disjoint.

(ii) Suppose that $T_C(\lambda_1,\lambda_2)$ contains two distinct points (t_1,t_2) and (t_1',t_2'). Then,

$$0 = V_{g_1}g_1(t_1 - t_1', 0)\, V_{g_1}g_1(t_2 - t_2', 0).$$

As $V_{g_1}g_1(t, 0) \neq 0$ for any t with $|t| < 1$, we must have $|t_1 - t_1'| \geq 1$ or $|t_2 - t_2'| \geq 1$, contradicting the fact that both (t_1, t_2) and (t_1', t_2') belong to C.

We will denote by ∂A the boundary of a set A. The next result will be useful.

Lemma (4.3.17)[241]: Under the hypotheses of the previous Lemma, consider an element $\lambda = (\lambda_1, \lambda_2)$ of $\Gamma(C)$ and let $T_C(\lambda) = \{(t_1, t_2)\}$. Then for any $x \in \partial(\lambda + [0,1]^2)$, we can find $\lambda_x = (\lambda_{1,x}, \lambda_{2,x}) \in \Gamma(C)$ such that $\in \partial(\lambda_x + [0,1]^2)$. Moreover, for any such λ_x, letting $T_C(\lambda_x) = \{t_x\}$, where $t_x = (t_{1,x}, t_{2,x})$, we can fined $i_0 \in \{1,2\}$ such that $t_{i_0,x} = t_{i_0}$ and $\lambda_{i_0,x} = \lambda_{i_0} + 1$ or $\lambda_{i_0} - 1$.

Proof. We can write $x = (\lambda_1 + \epsilon_1, \lambda_2 + \epsilon_2)$, where $0 \leq \epsilon_i \leq 1$, $i = 1,2$ and $\epsilon_i \in \{0,1\}$ for at least one index i. Let $a = (a_1, a_2) \in \mathbb{R}^2$ with $0 < a_i < 1$, for $i = 1,2$ and consider the point $(t_a, x) \coloneqq (t_1 + a_1, t_2 + a_2, \lambda_1 + \epsilon_1, \lambda_2 + \epsilon_2)$ in $\mathbb{R}^4$. Since $\Lambda + [0,1]^4$ is a tiling on $\mathbb{R}^4$ and the point (t_a, x) is a point on the boundary of $(t, \lambda) + [01]^4$, we can find some point $(t_{x,a}, \lambda_{x,a}) \in \Lambda \backslash \{(t, \lambda)\}$ such that $(t_a, x) \in (t_{x,a}, \lambda_{x,a}) + [01]^4$. Let $t_{x,a} = (t_1', t_2')$ and $\lambda_{x,a} = (\lambda_1', \lambda_2')$. We have

$$\begin{cases} -a_i \leq t_i - t_i' \leq 1 - a_i, \\ -\epsilon_i \leq \lambda_i - \lambda_i' \leq 1 - \epsilon_i, \end{cases} \quad i = 1,2 \tag{53}$$

Using the orthogonality of the system $\mathcal{G}(\chi_{[01]^2}, \Lambda)$, we can find $i_0 \in \{1,2\}$ such that $V_{g_1}g_1(t_{i_0} - t_{i_0}', \lambda_{i_0} - \lambda_{i_0}') = 0$.Note that $t_{i_0} - t_{i_0}' \neq 0$ would imply that $|\lambda_{i_0} - \lambda_{i_0}'| > 1$ which impossible from (53). Hence, $t_{i_0} - t_{i_0}'$ and $\lambda_{i_0} - \lambda_{i_0}' \neq 0$.

Moreover, as $V_{g_1}g_1(0, v) \neq 0$ if $|v| < 1$, $V_{g_1}g_1(t_{i_0} - t_{i_0}', \lambda_{i_0} - \lambda_{i_0}') = 0$ can only occurs if $|\lambda_{i_0} - \lambda_{i_0}'| = 1$. This shows also that $\epsilon_{i_0} \in \{0,1\}$ in that case. This proves the last statement of our claim and the fact that $x \in \partial(\lambda_{x,a} + [0,1]^2)$. the proof will be complete if we can show that $\lambda_{x,a} \in \Gamma(C)$ for some choice of a.

For simplicity, we consider the half -open square to be $C = [b_1, b_1 + 1) \times [b_2, b_2 + 1)$. Our assertion will be true if the point $t_{x,a} = (t_1', t_2')$ Constructed above satisfies the inequalities $b_i \leq t_i' < b_i + 1$ for $i = 1,2$. As $t_{i_0} = t_{i_0}'$, the inequalities clearly hold for $i = i_0$. Suppose that the other index j falls out of the range, say $t_j' < b_j$ (The case $t_j' \geq b_j + 1$ is similar). We consider $(t_{a'}, x)$ with $a_j' = t_j' + 1 - t_j + \delta$ for some small $\delta > 0$. Note that, by (53), we have $t_i + a_i - 1 \leq t_i' \leq t_i + a_i$ for $i = 1,2$, and, in particular,

$$a_j' = t_j' + 1 - t_j + \delta \geq a_j + \delta > 0.$$

We have also $a_j' < 1$. Indeed, the inequality $t_j' - t_j + 1 + \delta \geq 1$ would imply that $t_j' + 1 + \delta \geq 1 + t_j$. This is not possible, as $b_j \leq t_j < b_j + 1$, so $1 + t_j \geq b_j + 1$. But $t_j' < b_j$, so $t_j' + 1 < b_j + 1$, so for δ small,

$$t_j' + 1 + \delta < b_j + 1 \leq 1 + t_j$$

which yields a contradiction.

Using the previous argument with a' replacing a, we guarantee the existence of t_j'' such that $t_j' + \delta = t_j + a_j' - 1 \leq t_j'' \leq t_j + a_j' = t_j' + 1 + \delta$ and the a ssociated point $(t_{a'}, \lambda_{x,a'}) = (t_1'', t_2'', \lambda_1'', \lambda_2'')$ in Λ with the property that $x \in \partial(\lambda_{x,a'} + [0,1]^2)$ for some index i_0' such that $|\lambda_{i_0'} - \lambda_{i_0'}''| = 1$, $t_{i_0'} = t_{i_0'}''$ and $\epsilon_{i_0'} \in \{0,1\}$. We claim that $t_j'' = t_j' + 1$.

Now, $(t_1', t_2', \lambda_1', \lambda_2')$ and $(t_1'', t_2'', \lambda_1'', \lambda_2'')$ are in Λ. The mutual orthogonality property implies that $V_{g_1} g_1(t_i' - t_i'', \lambda_i' - \lambda_i'') = 0$ for some $i = 1,2$.

Suppose that x is not of the corner points of $\lambda + [0,1]^2$. In that case, the index i such that $\epsilon_i \in \{0,1\}$ is unique and it follows that $i_0 = i_0'$. This implies in particular, that $t_{i_0}' = t_{i_0}''$ (as $t_{i_0}' = t_{i_0} = t_{i_0'} = t_{i_0'}'' = t_{i_0}''$). Furthermore, the second set of inequalities in (53) show that $\lambda_{i_0}' = \lambda_{i_0}'' = \lambda_{i_0} - 1$ if $\epsilon_{i_0} = 0$ and $\lambda_{i_0}' = \lambda_{i_0}'' = \lambda_{i_0} + 1$ if $\epsilon_{i_0} = 1$. We have thus $\lambda_{i_0}' = \lambda_{i_0}''$ in both cases. We have thus

$$V_{g_1} g_1\left(t_{i_0}' - t_{i_0}'', \lambda_{i_0}' - \lambda_{i_0}''\right) = V_{g_1} g_1(0,0) = 1.$$

Therefore, the other index j must satisfy $V_{g_1} g_1\left(t_j' - t_j'', \lambda_j' - \lambda_j''\right) = 0$. This inequalities

$$-\epsilon_j \le \lambda_j - \lambda_j' \le 1 - \epsilon_j \quad \text{and} \quad -\epsilon_j \le \lambda_j - \lambda_j'' \le 1 - \epsilon_j$$

yield $-1 \le \lambda_j' - \lambda_j'' \le 1$. However, $\delta \le t_j'' - t_j' \le 1 + \delta$. The $V_{g_1} g_1$ would not be zero unless $t_j'' \ge t_j' + 1 (\ge b_j)$. Hence, $t_j' + 1 \le t_j'' \le t_j' + 1 + \delta$. This forces that $t_j'' = t_j' + 1$. This completes the proof for non-corner points. If x is the corner point, as the square constructed for the non-corner will certainly cover the corner point. therefore, the proof is completed.

With the help of the previous two lemmas, the following tiling result for $\Gamma(C)$ follows immediately.

Corollary (4.3.18)[241]: Let C be a half-open square. Then $\Gamma(C) + [0,1]^2$ is a tilling of $\mathbb{R}^2$.

Proof. It suffices to show the following statements: suppose that $\mathcal{J} + [0,1]^2$ is non-empty packing of $\mathbb{R}^2$. If, for any $x \in \partial(t + [0,1]^2)$ where $t \in \mathcal{J}$, we can fined $t_x \in \mathcal{J}$ with $t_x \ne t$ such that $x \in \partial(t_x + [0,1]^2)$, then $\mathcal{J} + [0,1]^2$ is a tiling of $\mathbb{R}^2$. Indeed, by Lemma (4.3.16)(i) and Lemma (4.3.17) $\Gamma(C) + [0,1]^2$ is a packing of $\mathbb{R}^2$ and satisfies the stated property. It is thus a tiling of $\mathbb{R}^2$.

To prove the previous statements, we note that as $\mathcal{J} + [0,1]^2$ is packing, it is closed set. Suppose that $\mathcal{J} + [0,1]^2$ satisfies the property above and that $\mathbb{R}^d \backslash \mathcal{J} + [0,1]^2 \ne \emptyset$. Let $x \in \partial(\mathcal{J} + [0,1]^2)$ and assume that $x \in t + [0,1]^2$. We can then find $t_x \in \mathcal{J}$ with $t_x \ne t$ such that $x \in \partial(t_x + [0,1]^2)$. Note that if x were not a corner point of either $t + [0,1]^2$ or $t_x + [0,1]^2$, then x would be in the interior of $\mathcal{J} + [0,1]^2$. Hence, x must be a corner point of $t + [0,1]^2$ or $t_x + [0,1]^2$. As the set of all corner points of the squares in $\mathcal{J} + [0,1]^2$ is countable, the Lebesgue measure of the open set $\mathbb{R}^d \backslash (\mathcal{J} + [0,1]^2)$ is zero and $\mathbb{R}^d \backslash (\mathcal{J} + [0,1]^2)$ is thus empty, proving our claim.

Lemma (4.3.19)[241]: Let C be a half-open square and suppose that $(\lambda_1, \lambda_2) \in \Gamma(C)$ with $T_C(\lambda_1, \lambda_2) = \{(t_1, t_2)\}$. Then all the sets $T_C(\lambda_1', \lambda_2')$ with $(\lambda_1', \lambda_2') \in \Gamma(C)$ are either of the form $\{(t_1, t_2 + s)\}$ or $\{(t_1 + s, t_2)\}$ for some real s with $|s| < 1$ depending on (λ_1, λ_2).

Proof. We first make the following remark. If $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in \Gamma(C)$ are such that the two squares $(\alpha_1, \alpha_2) + [0,1]^2$ and $(\beta_1, \beta_2) + [0,1]^2$ intersect each other and also both intersect a third square $(\gamma_1, \gamma_2) + [0,1]^2$ with $(\gamma_1, \gamma_2) \in \Gamma(C)$, then, letting $T_C(\gamma_1, \gamma_2) = (r_1, r_2)$, we have

$$T_C(\alpha_1, \alpha_2) = \{(r_1 + a, r_2)\} \quad \text{and} \quad T_C(\beta_1, \beta_2) = \{(r_1 + b, r_2)\}$$

or

$$T_C(\alpha_1, \alpha_2) = \{(r_1, r_2 + a)\} \quad \text{and} \quad T_C(\beta_1, \beta_2) = \{(r_1, r_2 + b)\},$$

for Some real a, b. Indeed, using Lemma (4.3.17), we have $T_C(\alpha_1, \alpha_2) = \{(r_1 + a, r_2)\}$ or $\{(r_1, r_2 + a)\}$ and $T_C(\beta_1, \beta_2) = \{(r_1 + b, r_2)$ or $(r_1, r_2 + b)\}$. Suppose, for example, that $T_C(\alpha_1, \alpha_2) = \{(r_1 + a, r_2)\}$ and $T_C(\beta_1, \beta_2) = (r_1, r_2 + b)$. Since the two squares intersect each other, we must have $|\alpha_1 - \beta_1| \leq 1$ and $|\alpha_2 - \beta_2| \leq 1$. The orthogonality property also implies that either $(a, \alpha_1 - \beta_1)$ or $(-b, \alpha_2 - \beta_2)$ is in the zero set of $V_{g_1} g_1$. But since we have $|a|, |b| < 1$, this would imply that $|\alpha_1 - \beta_1| > 1$ or $|\alpha_1 - \beta_2| > 1$, which cannot happen. As $\Gamma(C) + [0,1]^2$ is a tiling of $\mathbb{R}^2$, for any square $(\sigma_1, \sigma_2) + [0,1]^2$ intersecting the square $(\lambda_1, \lambda_2) + [0,1]^2$ and with $(\sigma_1, \sigma_2) \in \Gamma(C)$, we can find another square $(\delta_1, \delta_2) + [0,1]^2$, with $(\delta_1, \delta_2) \in \Gamma(C)$ and with $(\delta_1, \delta_2) + [0,1]^2$ intersecting both squares $(\sigma_1, \sigma_2) + [0,1]^2$ and $(\lambda_1, \lambda_2) + [0,1]^2$. By the previous remark, the conclusion of the lemma holds for all the squares that neighbour the square $(\lambda_1, \lambda_2) + [0,1]^2$. Replacing this original square by one of the neighbouring squares and continuing this process, we obtain the conclusion of the lemma for all the squares in the tiling $\Gamma(C) + [0,1]^2$ by an induction argument. This proves our claim.

Suppose that the system $\mathcal{G}(\chi_{[0,1]^2}, \Lambda)$ gives rise to a non-standard Gabor orthonormal basis of $L^2(\mathbb{R}^2)$. Then, some of the squares will have overlaps and, without loss of generality, we can assume that

$$|[0,1]^2 \cap [0,1]^2 + (t_1, t_2)| > 0$$

for some (t_1, t_2) in the translation component of Λ.

Lemma (4.3.20)[241]: If $(0,0,0,0) \in \Lambda$, then the sets $T_{[0,1)^2}(\lambda_1, \lambda_2)$ where $(\lambda_1, \lambda_2) \in \Gamma([0,1]^2)$ are either all of the form $\{(t, 0)\}$ or all ω of the form $\{(0, t)\}$ with some t (depending on (λ_1, λ_2) with $|t| < 1$. In the first case, if there exists some $(\lambda_1, \lambda_2) \in \Gamma([0,1]^2)$ with $T_{[0,1]^2}(\lambda_1, \lambda_2) = (t, 0)$ and $t \neq 0$, then

$$\Gamma([0,1]^2 = \bigcup_{k \in \mathbb{Z}} (\mathbb{Z} + \mu_{k,0}) \times \{k\} \tag{54}$$

for some $0 \leq \mu_{k,0} \leq 1$. Moreover, we can find $0 \leq t_k \leq 1$ such that

$$T_{[0,1)^2}\left((\mathbb{Z} + \mu_{k,0}) \times \{k\}\right) = \{(t_k, 0)\}, \ \ k \in \mathbb{Z}, \tag{55}$$

and

$$\Lambda \cap ([0,1)^2 \times \mathbb{R}^2) = \{(t_k, 0, j + \mu_{k,0}, k) : j, k \in \mathbb{Z}\}. \tag{56}$$

(In the second case, $\Gamma([0,1)^2 = \bigcup_{k \in \mathbb{Z}} \{k\} \times (\mathbb{Z} + \mu_{k,0})$ and $T_{[0,1)^2}(\{k\} \times (\mathbb{Z} + \mu_{k,0})) = \{(0, t_k)\}, \Lambda \cap ([0,1)^2 \times \mathbb{R}^2) = \{(0, t_k, k, j + \mu_{k,0} : j, k \in \mathbb{Z})\}$.

Proof. If $\lambda = (0,0)$, we have $T_{[0,1)^2}(\lambda) = \{(0,0)\}$ as $(0,0,0,0) \in \Lambda$. By Lemma (4.3.19), any $(\lambda_1, \lambda_2) \in \Gamma([0,1)^2$ with the square $(\lambda_1, \lambda_2) + [0,1)^2$ intersecting $[0,1]^2$ on the λ_1, λ_2-plane satisfies $T_{[0,1)^2}(\lambda_1, \lambda_2) = \{(t, 0)\}$ or $T_{[0,1)^2}(\lambda_1, \lambda_2)\{(0, t)\}$ with $|t| < 1$. Without loss of generality, we assume that the first case holds. As $\Gamma([0,1]^2) + [0,1]^2$ is a tiling of $\mathbb{R}^2$, for any square $C = (\lambda_1, \lambda_2) + [0,1]^2$, with $(\lambda_1, \lambda_2) \in \Gamma([0,1]^2)$, we can find squares $C_i = (\lambda_{1,i}, \lambda_{2,i}) + [0,1]^2$ for $i = 0, \ldots, k$ with $(\lambda_{1,i}, \lambda_{2,i}) \in \Gamma([0,1]^2)$ and such that $C_0 = [0,1]^2, C_k = C$, and with C_i and C_{i+1} touching each other for all $i = 0, \ldots, k-1$.

We have $T_{[0,1)^2}(\lambda_{1,1}, \lambda_{2,1}) = \{(t_1, 0)\}$ for some number t_1 with$|t_1| < 1$. Since C_2 and C_0 both intersect C_1, $T_{[0,1)^2}(\lambda_{1,2}, \lambda_{2,2}) = \{(t_2, 0)\}$ by Lemma (4.3.19) again.

Inductively, we have $T_{[0,1)^2}(\lambda_{1,i}, \lambda_{2,i}) = \{(t_i, 0)\}$, $i = 1, \dots, k$, which proves the first part.

Consider the case where, for any $(\lambda_1, \lambda_2) \in \Gamma([0,1]^2)$, there exists a number $t = t(\lambda_1, \lambda_2)$ such that $T_{[0,1)^2}(\lambda_1, \lambda_2) = \{(t, 0)\}$ and assume that $t(\lambda_1, \lambda_2) \neq 0$ for at least one couple $(\lambda_1, \lambda_2) \in \Gamma([0,1]^2)$. Suppose that $\Gamma([0,1]^2)$ is not of the form in (54). By Corollary (4.3.18) and Proposition (4.3.9), we must have $\Gamma([0,1]^2) = \bigcup_{k\in\mathbb{Z}}\{k\} \times (\mathbb{Z} + a_k)$ with $0 \leq a_k \leq 1$ and at least one $a_k \neq 0$. Consider the distinct points

$$(t, 0, k, a_k + j) \quad \text{and} \quad (t', 0, k-1, a_{k-1} + j), \quad \text{both in } \Lambda.$$

We must have that either $(t - t', 1) \in Z(V_{g_1}g_1)$ or $(0, a_k - a_{k-1}) \in Z(V_{g_1}g_1)$. however, since $|a_k - a_{k-1}| < 1$, the second case is impossible. This means that $(t - t', 1) \in Z(V_{g_1}g_1)$ which is possible only if $t = t'$. Therefore the fact $(t, 0, k, a_k + j) \in \Lambda$ implies that $t = t_j$ for some real t_j. We know show by induction on $|j|$ that $t_j = 0$, for all $j \in \mathbb{Z}$. The case $j = 0$ is clear as $(0,0,0,0) \in \Lambda$ by assumption. if our claim is true for all $|j| \leq J$ Where $J \geq 0$, chose $k \in \mathbb{Z}$ such that $a_{k+1} \neq 0$ and $a_k = 0$ if such k exists. Suppose first that $j > 0$. there exist thus $t \in [0,1)$ such that

$$(t_{j+1}, 0, k, j+1) \quad \text{and} \quad (0,0,k+1, a_{k+1} + j) \quad \text{both belong to } \Lambda.$$

This implies that either $(t, -1) \in Z(V_{g_1}g_1)$ or $(0, a_{k+1} - 1) \in Z(V_{g_1}g_1)$. This last case is impossible and the first one is only possible if $t = 0$, showing that $t_{j+1} = 0$. Similarly by considering the points

$$(t_{j-1}, 0, k+1, a_{k+1} + j - 1) \quad \text{and} \quad (0,0,k,j) \quad \text{which both belong to } \Lambda.$$

we can conclude that $t_{j-1} = 0$ for $j < 0$. If k as above does not exist, there exists chose $k' \in \mathbb{Z}$ such that $a_{k'-1} \neq 0$ and $a_{k'} = 0$. By considering the points

$$(t_{j+1}, 0, k', j-1) \quad \text{and} \quad (0,0,k'-1, a_{k'-1} + j) \quad \text{if } j > 0.$$

and the points

$$(t_{j-1}, 0, k'-1, a_{k'-1} + j - 1) \quad \text{and} \quad (0,0,k',j) \quad \text{if } j < 0$$

which all belong to Λ, we conclude that $t_j = 0$ if $|j| = J + 1$. This proves (54).

If we are in the first case, i.e

$$\Gamma([0,1]^2) = \bigcup_{k\in\mathbb{Z}} (\mathbb{Z} + \mu_{k,0}) \times \{k\},$$

let m, m' be distinct integers. We have then

$$T_{[0,1)^2}(m + \mu_{n,0}, n) = \{(t_m, 0)\} \quad \text{and} \quad T_{[0,1)^2}(m' + \mu_{n,0}, n) = \{(t_{m'}, 0)\}$$

which implies that $V_{g_1}g_1(t_m - t_{m'}, m - m') = 0$ or $V_{g_1}g_1(0,0) = 0$. The second case is clearly impossible while the first one is possible only when $t_m = t'_m$. This shows (55) and (56) follows immediately from (45) and (55).

Note that Lemma (4.3.20) implies that $\Gamma([0,1]^2) = \Gamma(\{(x, 0): 0 \leq x < 1\})$ and $\Gamma([0,1]^2) = \emptyset$ if $(0,0,0,0) \in \Lambda$.

Lemma (4.3.21)[241]: Under the assumptions of Lemma (4.3.20), suppose that there exists $(\lambda_1, \lambda_2) \in \Gamma([0,1]^2)$ with $T_{[0,1)^2}(\lambda_1, \lambda_2) = (t, 0)$ and $t \neq 0$. Then we can find numbers t_k with $0 \leq t_k < 1$ and $\mu_{k,0}, k, m \in \mathbb{Z}$, with $0 \leq \mu_{k,m} < 1$, such that

$$\Lambda \cap (\mathbb{R} \times [0,1] \times \mathbb{R}^2) = \{(m + t_k, 0, j + \mu_{k,m,} k): j, k, m \in \mathbb{Z}\}$$

Proof. By the result of Lemma (4.3.20), we have the identities (55) and (56). Let $T = \{t_k, k \in \mathbb{Z}\} \subset [0,1)$ where $t_k, k \in \mathbb{Z}$, are the numbers appearing in (55). let $s_1, s_1 \in T$

with $s_1 < s_2$. Consider the half-open squares $C = (s_1, 0) + [0,1)^2$ and $C' = (s_1, 0) + ((0,1] \times [0,1))$. Then we know that $\Gamma(C) + [0,1]^2$ and $\Gamma(C') + [0,1]^2$ both tile $\mathbb{R}^2$. Let $P_0 = \{(s_1, y): 0 \le y < 1\}$ and $P_1 = \{(s_1 + 1, y): 0 \le y < 1\}$. Note that $\Gamma(P_0) = \Gamma(\{(s_1, 0)\})$. Moreover,

$$\Gamma(C) = \Gamma(P_0) \cup \Gamma(C \backslash P_0), \Gamma(C') = \Gamma(C' \backslash P_1) \cup \Gamma(P_1)$$

and since $C \backslash P_0 = C' \backslash P_1$, $\Gamma(P_0) = \Gamma(P_1)$. We have

$$T_{C'}(\Gamma(P_1)) \subset \{(s_1 + 1, y), 0 \le y < 1\}$$

but since $(s_2, 0) \in C'$, we must have $T_{C'}(\Gamma(P_1)) = (s_1 + 1,0)$ by Lemma (4.3.19). Since

$$\Gamma(P_0) = \{(j + \mu_{k,0}, k): j,k \in \mathbb{Z}, t_k = s_1\}$$

and $\pi_2(\Gamma(P_0)) = \pi_2(\Gamma(P_1))$, where π_2 is the projection to the second coordinate, we have

$$\Gamma(\{(1 + s_1, 0)\}) = \Gamma(P_1) = \{(j + \mu_{k,1}, k): j,k \in \mathbb{Z}, t_k = s_1\}.$$

for some constants $\mu_{k,1}$ with $0 \le \mu_{k,1} < 1$ using Proposition (4.3.9). Applying this argument to $s_1 = 0$ and $s_2 = t$, we obtain that

$$\Lambda \cap (\{1\} \times [0,1) \times \mathbb{R}^2) = \{(j + \mu_{k,1}, k): j,k \in \mathbb{Z}, t_k = 0\}.$$

Similar arguments applied to $s_1 = s$ and $s_2 = 1$ show that, for any $s \in T$, we have

$$\Lambda \cap (\{s + 1\} \times [0,1) \times \mathbb{R}^2) = \{(j + \mu_{k,1}, k): j,k \in \mathbb{Z}, t_k = s\}.$$

and that $\Lambda \cap (\{s + 1\} \times [0,1) \times \mathbb{R}^2)$ is empty if $s \in [0,1) \backslash T$. The same idea can also be used to show the existence of constants $\mu_{k,-1}$ with $0 \le \mu_{k,1} < 1$ such that

$$\Lambda \cap (\{s - 1\} \times [0,1) \times \mathbb{R}^2) = \begin{cases} \{(j + \mu_{k,-1}, k): j,k \in \mathbb{Z}, t_k = s\}, & s \in T, \\ \emptyset, & s \in [0,1) \backslash T. \end{cases}$$

and, more generally using induction, that, for any $m \in \mathbb{Z}$, we can find constants $\mu_{k,m}$ with $0 \le \mu_{k,m} < 1$ such that

$$\Lambda \cap (\{s + m\} \times [0,1) \times \mathbb{R}^2) = \begin{cases} \{(j + \mu_{k,m}, k): j,k \in \mathbb{Z}, t_k = s\}, & s \in T, \\ \emptyset, & s \in [0,1) \backslash T. \end{cases}$$

This proves our claim.

We can now complete the proof of the main result which gives a characterization for the subsets Λ of $\mathbb{R}^4$ with the property that the associated set of time-frequency shifts applied to the window $\chi_{[0,1]^2}$ yields an orthonormal basis for $L^2(\mathbb{R}^2)$.

Theorem (4.3.22)[241]: $\mathcal{G}(\chi_{[0,1]^2}, \Lambda)$ is a Gabor orthonormal basis for $L^2(\mathbb{R}^2)$ if and only if we can partition $\mathbb{Z}$ into $\mathcal{J}$ and $\mathcal{J}'$ such that either

$\Lambda = \cup_{n \in \mathcal{J}} \{(m + t_{n,k,} n, j + \mu_{k,m,n}, K + v_n): m,j,k \in \mathbb{Z}\} \cup \cup_{m \in \mathbb{Z}} \cup_{n \in \mathcal{J}'} \{(m + t_n, n)\} \times \Lambda_{m,n}$.

or

$\Lambda = \cup_{m \in \mathcal{J}} \{(m, n + t_{m,j}, j + v_m, K + \mu_{j,m,n}): n,j,k \in \mathbb{Z}\} \cup \cup_{n \in \mathbb{Z}} \cup_{m \in \mathcal{J}'} \{(m, n + t_m)\} \times \Lambda_{m,n}$.

where $\Lambda_{m,n} + [0,1]^2$ tile $\mathbb{R}^2$ and $t_{n,k}, \mu_{k,m,n}$ and v_n are real numbers in [0,1) as a function of m, n or K.

Proof. It follows from Lemma (4.3.19), that either all $T_{[0,1)^2}(\lambda_1, \lambda_2), (\lambda_1, \lambda_2) \in \Gamma([0,1]^2)$ are either of the form $\{(t, 0)\}$ or all are of the form $\{(0, t)\}$ with some $t \neq 0$. In the first case, we deduce from Lemma (4.3.21) that

$$\Lambda \cap (\mathbb{R} \times [0,1) \times \mathbb{R}^2) = \{(m + t_k, 0, j + \mu_{k,m}, k): j,k,m \in \mathbb{Z}\}$$

for certain numbers t_k and $\mu_{k,m}$ in the interval [0,1). We now show that Λ will be of the first of the two possible forms given in the Theorem. (Similarly, the second form follows from the second case of Lemma (4.3.21)).

Letting $C=[0,1)^2$ and $C'=[0,1)\times(0,1]$, we note that both $\Gamma(C)+[0,1]^2$ and $\Gamma(C')+[0,1]^2$ tile $\mathbb{R}^2$ but $\Gamma((0,1)^2)$ is empty. Hence, $\Gamma(C')=\Gamma(\{(x,1):0\le x\le 1\})$. It means that any set $T_{C'}(\lambda_1,\lambda_2)$ with $(\lambda_1,\lambda_2)\in\Gamma(C')$ is of the form $\{(t,1)\}$ for some $t=t(\lambda_1,\lambda_2)$ with $0\le t<1$. We now have two possible cases : either the cardinality of $T_{C'}(\Gamma(C'))$ is larger than one or equal to one. In the first case, we can find two distinct elements of $T_{C'}(\Gamma(C'))$ and we can then replicate the proof of Lemma (4.3.21), to obtain that

$$\Lambda\cap(\mathbb{R}\times[1,2)\times\mathbb{R}^2)=\{(m+t_k,1,j+\mu_{k,m,1},k):j,k\in\mathbb{Z}\}.$$

In the other case, $T_{C'}(\Gamma(C'))=\{(t_1,1)\}$ for some t_1 with $0\le t_1<1$. If we translate C' horizontally and use the same argument as in the proof of Lemma (4.3.21), we see that

$$\Lambda\cap(\mathbb{R}\times[1,2)\times\mathbb{R}^2)=\{(m+t_1,1)\}\times\Lambda_{m,1},$$

where $A_{m,1}$ is a spectrum for the unit square $[0,1]^2$. This last property is equivalent to $A_{m,1}+[0,1]^2$ being a tiling of $\mathbb{R}^2$ by the result in [256].

We can them prove the Theorem inductively by translating the square C' in the vertical direction using integer steps.

Corollary (4.3.23)[491]: Suppose that $F_r,G_r\in L^1(\mathbb{R}^n)$ are two functions with $F_r,G_r\ge 0$ and $\int_{\mathbb{R}^n}\sum_r F_r(x_r)dx_r=\int_{\mathbb{R}^n}\sum_r G_r(x_r)dx_r=1$. Suppose that μ is a positive Borel measure on $\mathbb{R}^n$ such that

$$\sum_r F_r*\mu\le 1 \text{ and } \sum_r G_r*\mu\le 1.$$

Then, $\sum_r F_r*\mu=1$ if and only if $\sum_r G_r*\mu=1$.

Proof. By symmetry, it suffices to show one side of the equivalence. Assuming that $\sum_r F_r*\mu=1$, we have

$$1=\sum_r F_r*\mu\Rightarrow 1=\sum_r 1*G_r=\sum_r G_r*F_r*\mu=\sum_r F_r*G_r*\mu.$$

Letting $H=\sum_r G_r*\mu$ we have $0\le H\le 1$ and $\sum_r H*F_r=1$. We now show that $H=1$. indeed letting Λ_r be the set $\{x_r\in\mathbb{R}^n,H(x_r)<1\}$ and $A_r+\epsilon=\mathbb{R}^n\setminus\Lambda_r$, we have

$$\sum_r (H*F_r)(x_r)=\int_{\mathbb{R}^n}\sum_r H(y_r)F_r(x_r-y_r)dy_r$$
$$=\int_{A_r}\sum_r H(y_r)F_r(x_r-y_r)dy_r+\int_{A_r+\epsilon}\sum_r H(y_r)F_r(x_r-y_r)dy_r$$

Now, if $|\Lambda_r|>0$,we have

$$\int_{\mathbb{R}^n}\int_{A_r}\sum_r F_r(x_r-y_r)dy_rdx_r=\sum_r |A_r|>0$$

and there exists thus a set E with positive measure such that

$$\int_{A_r}\sum_r F_r(x_r-y_r)dy_r>0,\qquad x_r\in E.$$

If $x_r\in E$, we have

$$\int_{A_r}\sum_r H(y_r)F_r(x_r-y_r)dy_r+\int_{A_r+\epsilon}\sum_r H(y_r)F_r(x_r-y_r)dy_r$$
$$<\int_{A_r}\sum_r F_r(x_r-y_r)dy_r+\int_{A_r+\epsilon}\sum_r F_r(x_r-y_r)dy_r$$
$$=\sum_r (1*F_r)(x_r)=1$$

This contradicts to the that $\sum_r H*F_r=1$ almost everywhere. Hence, $|\Lambda_r|=0$ and $H=1$ follows.

Corollary (4.3.24)[491]: Suppose that $\mathcal{G}(\sum_r g_r,\Lambda_r)$ is a mutually orthogonal set of $L^2(\mathbb{R}^{3+\epsilon})$. Let D be any orthogonal packing region for $\sum_r g_r$. Then $\Lambda_r-\Lambda_r\subset\sum_r \mathcal{Z}(V_{g_r}g_r)\cup\{0\}$ and Λ_r+D is a packing of $\mathbb{R}^{2(3+\epsilon)}$. Suppose furthermore that $\mathcal{G}(\sum_r g_r,\Lambda_r)$ is a Gabor orthonormal basis. Then $|D|\le 1$.

Proof. Let $(t_{r-2},\lambda_{r-2}^2),(t_{r-2}+\epsilon,\lambda_{r-2}^2+\epsilon)\in\Lambda_r$ be two distinct points in Λ_r. Then

$$\int\sum_r g_r\,(x_r-(t_{r-2}+\epsilon))\overline{g_r(x_r-t_{r-2})}\,e^{-2\pi i(-\epsilon)x_r}dx_r=0,$$

Or equivalently, after the change of variable $y_r=x_r-t_{r-2}+\epsilon$,

$$\int\sum_r g_r\,(x_r)\overline{g_r(x_r+\epsilon)}\,e^{2\pi i(\epsilon)x_r}dx_r=0.$$

Hence, $\sum_r V_{g_r}g_r(-\epsilon,-\epsilon)=0$ and $(t_{r-2},\lambda_{r-2}^2)-(t_{r-2}+\epsilon,\lambda_{r-2}^2+\epsilon)\in\sum_r \mathcal{Z}(V_{g_r}g_r)$. This means that $(t_{r-2},\lambda_{r-2}^2)-(t_{r-2}+\epsilon,\lambda_{r-2}^2+\epsilon)\notin D^0-D^0$. Therefore, the intersection of the sets $(t_{r-2},\lambda_{r-2}^2)+D$ and $(t_{r-2}+\epsilon,\lambda_{r-2}^2+\epsilon)+D$ has zero Lebesgue measure.

Suppose now that $\mathcal{G}(\sum_r g_r,\Lambda_r)$ is a Gabor orthonormal basis. Denote by R the diameter of D. By the packing property of Λ_r+D,

$$\sum_r |D|.\frac{\#(\Lambda_r\cap[-T,T]^{2(3+\epsilon)})}{(2T)^{2(3+\epsilon)}}=\frac{1}{(2T)^{2(3+\epsilon)}}\sum_r\left|\bigcup_{\lambda_{r-2}^2\in\Lambda_r\cap[-T,T]^{2(3+\epsilon)}}(D+\lambda_{r-2}^2)\right|$$
$$\le\frac{1}{(2T)^{2(3+\epsilon)}}\left|[-T-R,T+R]^{2(3+\epsilon)}\right|=\left(1+\frac{R}{2(3+\epsilon)}\right)^{2(3+\epsilon)}.$$

Taking limit $T\to\infty$ and using the fact that Beurling density of Λ_r is 1 ([257], we have $|D|\le 1$.

Corollary (4.3.25)[491]: Suppose that $\mathcal{G}(\sum_r g_r,\Lambda_r)$ is an orthonormal set in $L^2(\mathbb{R}^{3+\epsilon})$ and that D is a tight orthogonal packing region for $\sum_r g_r$. Then $\mathcal{G}(\sum_r g_r,\Lambda_r)$ ia a Gabor orthonormal basis for $L^2(\mathbb{R}^{3+\epsilon})$ if and only if Λ_r+D is a tiling of $\mathbb{R}^{2(3+\epsilon)}$.

Proof. Let $F_r=\chi_D$ and $\sum_r G_r=\sum_r \left|V_{g_r}f_r\right|^2/\|f_r\|_2^2$. Then $\int_{\mathbb{R}^{2(3+\epsilon)}}\sum_r F_r=1$ and $\int_{\mathbb{R}^{2(3+\epsilon)}}\sum_r G_r=\sum_r \|g_r\|_2^2=1$. Now, as D is an orthogonal packing region for $\sum_r g_r$, we have in particular

$$\sum_{\lambda_{r-2}^2\in\Lambda_r}\chi_D(x_r-\lambda_{r-2}^2)\le 1.$$

This shows that

$$\sum_r \delta_{v_{r-2}^2} * F_r = \sum_r \delta_{\Lambda_r} * \chi_D \leq 1.$$

Moreover, $\Lambda_r + D$ is a tilling of $\mathbb{R}^{2(3+\epsilon)}$ if and only if $\sum_r \delta_{\Lambda_r} * \chi_D = 1$. On the other hand, $(\sum_r g_r, \Lambda_r)$ being a mutually orthogonal set, Bessel's inequality yields

$$\sum_{(t_{r-2},\lambda_{r-2}^2)\in\Lambda_r} \left| \int_{\mathbb{R}^{2(3+\epsilon)}} \sum_r f_r(x_r)\overline{g_r(x_r - t_{r-2})} e^{-2\pi i \langle v_{r-2}^2, x_r\rangle} dx_r \right|^2 \leq \sum_r \|f_r\|^2 ,$$

$$f_r \in L^2(\mathbb{R}^{3+\epsilon}) ,$$

or, replacing f_r by $f_r(x_r - \tau)e^{2\pi i v_{r-2}^2 x_r}$ with $(\tau, v_{r-2}^2) \in \mathbb{R}^{2(3+\epsilon)}$,

$$\sum_{(t_{r-2},\lambda_{r-2}^2)\in\Lambda_r} \sum_r \left|V_{g_r} f_r\,(\tau - t_{r-2}, v_{r-2}^2 - \lambda_{r-2}^2)\right|^2 \leq \sum_r \|f_r\|^2 , \quad f_r \in L^2(\mathbb{R}^{3+\epsilon}).$$

Hence,

$$\sum_r \delta_{\Lambda_r} * G_r = \sum_r \delta_{\Lambda_r} * \frac{\left|V_{g_r} f_r\right|^2}{\|f_r\|^2} \leq 1$$

with equality if and only if the Gabor orthonormal system is in fact a basis. The conclusion follows then from Corollary (4.3.23).

Corollary (4.3.26)[491]: $\mathcal{G}(\sum_r (g_r)_{2+\epsilon}, \Lambda_r)$ is a Gabor orthonormal basis if and only if $\mathcal{G}(\sum_r (g_r)_{2+\epsilon}, \Lambda_r)$ is an orthogonal set and the translates of $[0,1]^{2+\epsilon}$ by the elements of Λ_r tile $\mathbb{R}^{2(2+\epsilon)}$.

Proof. Let $\sum_r (g_r)_{3+\epsilon} = \chi_{[0,1]^{3+\epsilon}}$. Using Corollary (4.3.25), we just need to show that $[0,1]^{2(3+\epsilon)}$ is a tight orthogonal packing region for $\sum_r (g_r)_{3+\epsilon}$.

We first consider the case $\epsilon = -2$. For $\sum_r (g_r)_1 = \chi_{[0,1]}$, a direct computation shows that

$$\sum_r V_{(g_r)_1}(g_r)_1(t_{r-2}, v_{r-2}^2)$$

$$= \begin{cases} 0 , & |t_{r-2}| \geq 1 \\ \sum_r \frac{1}{2\pi i v_{r-2}^2}\left(e^{2\pi i v_{r-2}^2} - e^{2\pi i v_{r-2}^2}\right) , & 0 \leq t_{r-2} \leq 1 ; \\ \sum_r \frac{1}{2\pi i v_{r-2}^2}\left(1 - e^{2\pi i v_{r-2}^2 (t_{r-2}+1)}\right) , & -1 \leq t_{r-2} \leq 0. \end{cases} \tag{57}$$

The zero set of $V_{(g_r)_1}(g_r)_1$ is therefore given by

$$Z\left(V_{(g_r)_1}(g_r)_1\right)$$
$$= \{(t_{r-2}, v_{r-2}^2) : |t_{r-2}| \geq 1\}$$
$$\cup \left\{(t_{r-2}, v_{r-2}^2) : v_{r-2}^2(1 - |t_{r-2}|) \in \mathbb{Z}\backslash\{0\}\right\} \tag{58}$$

Hence, $(0,1)^2 - (0,1)^2 = (-1,1)^2$ does not intersect the zero set and therefore $[0,1]^2$ is a tight orthogonal packing region for $(g_r)_1$.

We now consider the case $\epsilon \geq 0$. As we can decompose $(g_r)_{2+\epsilon}$ as $\chi_{[0,1]}((x_r)_1) \ldots \chi_{[0,1]}((x_r)_{2+\epsilon})$, we have $\sum_r \left(V_{(g_r)_{2+\epsilon}}(g_r)_{2+\epsilon}(t_{r-2}, v_{r-2}^2)\right) = \sum_r V_{(g_r)_1}(g_r)_1(t_r, v_r) \cdots V_{(g_r)_1}(g_r)_1(t_{r+1+\epsilon}, v_{r+1+\epsilon})$ where $t_{r-2} = (t_r, \ldots, t_{r+1+\epsilon})$ and $v_{r-2}^2 = (v_r, \ldots, v_{r+1+\epsilon})$. The zero set $V_{(g_r)_{2+\epsilon}}(g_r)_{2+\epsilon}$ is therefore given by

$$\sum_r Z\big(V_{(g_r)_1}(g_r)_1\big)$$
$$= \sum_r \{(t_{r-2}, v_{r-2}^2) : |t_{r-2}|_{\max} \geq 1\}$$
$$\cup \left(\bigcup_{i=1}^{2+\epsilon} \{(t_{r-2}, v_{r-2}^2) : v_{r+i-1}(1 - |t_{r+i-1}|) \in \mathbb{Z}\backslash\{0\}\} \right) \quad (59)$$

Where $|t_{r-2}|_{\max} = \max\{t_r, \cdots, t_{r+1+\epsilon}\}$. It follows that $[0,1]^{2(2+\epsilon)}$ is a tight orthogonal packing region for $(g_r)_{2+\epsilon}$.

Corollary (4.3.27)[491]: Suppose that $\chi_{[0,1]^2+\mathcal{J}_r}$ is a tilling of $\mathbb{R}^2$ with $(0,0) \in \mathcal{J}_r$. Then $\mathcal{J}_r$ is of either of the following two form:

$$\mathcal{J}_r = \bigcup_{k\in\mathbb{Z}} (\mathbb{Z} + a_k^r) \times \{k\} \text{ or } \mathcal{J}_r = \bigcup_{k\in\mathbb{Z}} \{k\} \times (\mathbb{Z} + a_k^r) \quad (60)$$

Where a_k^r are any real numbers in $[0,1)$ for $k \neq 0$ and $a_0^r = 0$.

Proof. By Keller's criterion for square tiling's (see e.g [256]), for any (t_r, t_{r+1}) and $(t_r + \epsilon, t_{r+1} + \epsilon)$ in $\mathcal{J}_r$, $t_{r+i-1}, t_{r+i-1} + \epsilon \in \mathbb{Z}\backslash\{0\}$ for some $i = 1,2$. Taking $(t_r + \epsilon, t_{r+1} + \epsilon) = (0,0)$, we obtain that, for any $(t_r, t_{r+1}) \in \mathcal{J}_r\backslash\{(0,0)\}$, one of t_r or t_{r+1} belongs to $\mathbb{Z}\backslash\{0\}$. If $\subset \mathbb{Z}$, we must have $\mathcal{J}_r = \mathbb{Z}$ for $\chi_{[0,1]^2+\mathcal{J}_r}$ to be tiling of $\mathbb{R}^2$ and $\mathbb{Z}$ can be written as either of the sets in (7) by taking $a_k^r = 0$ for all k. Suppose that there exists $(s_r, s_{r+1}) \in \mathcal{J}_r$ such s_r is not an integer and $s_{r+1} \in \mathbb{Z}$. if $(t_r, t_{r+1}) \in \mathcal{J}_r$ and $t_{r+1} \notin \mathbb{Z}$, then both t_r and $t_r - s_r$ must be integers which would imply that s_r is an integer, contrary to our assumption. Hence, $(s_r, s_{r+1}) \in \mathcal{J}_r$ implies $s_{r+1} \in \mathbb{Z}$ and we write

$$\mathcal{J}_r = \bigcup_{k\in\mathbb{Z}} (\mathcal{J}_r)_k \times \{k\}.$$

For some discrete set $(\mathcal{J}_r)_k \subset \mathbb{R}$. For $\chi_{[0,1]^2} + \mathcal{J}_r$ to be a tiling of $\mathbb{R}^2$, the set $(\mathcal{J}_r)_k$ must be of the form $(\mathcal{J}_r)_k = \mathbb{Z} + a_k^r$. In the case $\mathcal{J}_r$ can be expressed as one of the sets in the first collection appearing in (60).

Similarly, if there exists $(s_r, s_{r+1}) \in \mathcal{J}_r$ such that s_{r+1} is not an integer and $s_r \in \mathbb{Z}, \mathcal{J}_r$ can be expressed as one of the sets in the second collection appearing in (60) this completes the proof.

Corollary (4.3.28)[491]: $\mathcal{G}(\chi_{[0,1]^{2+\epsilon}}, \Lambda_r)$ is a Gabor orthonormal basis if and only if Λ_r is standard.

Proof. We just need to show that Λ_r being standard is a necessary condition for $\mathcal{G}(\chi_{[0,1]^{2+\epsilon}}, \Lambda_r)$ to be a Gabor orthonormal basis. We can also assume, for simplicity, that $(0,0) \in \Lambda_r$. By Proposition 3.1, if $\mathcal{G}(\chi_{[0,1]^{2+\epsilon}}, \Lambda_r)$ is a Gabor orthonormal basis, then $\Lambda_r - \Lambda_r \subset Z\big(V_{g_r} g_r\big) \cup \{0\}$ and $\Lambda_r + [0,1]^2$. must be a tiling of $\mathbb{R}^2$. By Corollary (4.3.27), A must be of either one of the forms in (60). Note that Λ_r is standard in the second case. In order to deal with the first case, suppose that

$$\Lambda_r = \cup_{k\in\mathbb{Z}} (\mathbb{Z} + a_k^r) \times \{k\}, \text{ with } a_k^r \in [0,1)\ , k \neq 0\ , a_0^r = 0.$$

We now show that this is impossible unless $a_k^r = 0$ for all k (which reduces to the case $\Lambda_r = \mathbb{Z}^2$, which is standard). we can assume, without loss of generality, that $a_k^r \neq 0$ for some $k > 0$ with k being the smallest such index. If $a_k^r \neq 0$ for some$k$, then both (a_k^r, k) and $(0, k-1)$ are in Λ_r. The orthogonality of the Gabor system then implies that $(a_k^r, 1) \in Z\big(V_{g_r} g_r\big)$. Using (58), we deduce that $1 \cdot (1 - |a_k^r|) \in \mathbb{Z}\backslash\{0\}$. That means a_k^r must be an integer, which is a contradiction.

Hence, the first case is impossible unless $a_k^r = 0$ for all k and the proof is completed.

Corollary (4.3.29)[491]: Every pseudo-standard Gabor system $\mathcal{G}(\chi_{[0,1]^{2+\epsilon}}, \Lambda_r)$ is a Gabor orthonormal basis $L^2(\mathbb{R}^{2+\epsilon})$.

Proof. If $x_r \in \mathbb{R}^{n-\epsilon}$ and $y_r \in \mathbb{R}^n$, we have $(g_r)_{2+\epsilon}(x_r, y_r) = (g_r)_{n-\epsilon}(x_r)(g_r)_n(y_r)$ (for $n - 2(1+\epsilon) = 0$).

This yields immediately that

$$V_{(g_r)_{2+\epsilon}}(g_r)_{2+\epsilon}(s_{r-2}, t_{r-2}, \lambda_{r-2}^2, v_{r-2}^2)$$
$$= V_{(g_r)_{n-\epsilon}}(g_r)_{n-\epsilon}(s_{r-2}, \lambda_{r-2}^2) V_{(g_r)_n}(g_r)_n(t_{r-2}, v_{r-2}^2) \quad ,$$
$$(s_{r-2}, \lambda_{r-2}^2) \in \mathbb{R}^{2(n-\epsilon)}, (t_{r-2}, v_{r-2}^2) \in \mathbb{R}^{2n} \tag{61}$$

Suppose that $\rho = (s_{r-2}, t_{r-2}, \lambda_{r-2}^2, v_{r-2}^2)$ and $\rho' = (s_{r-2}+\epsilon, t_{r-2}+\epsilon, \lambda_{r-2}^2 + \epsilon, v_{r-2}^2 + \epsilon)$ are distinct elements of Λ_r. If $(s_{r-2}, \lambda_{r-2}^2) = (s_{r-2}+\epsilon, t_{r-2}+\epsilon)$, then (t_{r-2}^2, v_{r-2}^2) and $(t_{r-2}+\epsilon, v_{r-2}^2+\epsilon)$ are distinct elements of $(\Lambda_r)_{(s_{r-2}, \lambda_{r-2}^2)}$ and we have thus

$$(\epsilon, \epsilon) \in Z\big(V_{(g_r)_n}(g_r)_n\big)$$

Which implies that $Z\big(V_{(g_r)_{2+\epsilon}}(g_r)_{2+\epsilon}\big)(\rho' - \rho) = 0$. This proves the orthonormality of the system $\mathcal{G}(\chi_{[0,1]^{2+\epsilon}}, \Lambda_r)$. this proposition can now be proved by invoking Theorem (4.3.8) if we can show that $\Lambda_r + [0,1]^{2(2+\epsilon)}$ is a tiling of $\mathbb{R}^{2(2+\epsilon)}$.to prove this, we note that $(\Lambda_r)_1 + [0,1]^{2(2+\epsilon)}$ is a tiling of the subspace $\mathbb{R}^{2(n-\epsilon)}$ by Theorem (4.3.8) and that, similarly, for each $(t_{r-2}, \lambda_{r-2}^2) \in (\Lambda_r)_{(t_{r-2}, \lambda_{r-2}^2)} + [0,1]^{2n}$ is a tiling of $\mathbb{R}^{2n}$. This easily implies the required tiling property and concludes the proof.

Corollary (4.3.30)[491]: Let $n - 2(1+\epsilon) = 0$ and suppose that $\big(\chi_{[0,1]^{2+\epsilon}}, \Lambda_r\big)$ is a Gabor orthonormal basis for $L^2(\mathbb{R}^d)$. if $(s_{r-1}, \lambda_{r-1}^2) \in \mathbb{R}^{2(n-\epsilon)}$, consider the translate of the unit hypercube in $\mathbb{R}^{\boldsymbol{2m}}$, $C = (s_{r-1}, \lambda_{r-1}^2) + [0,1]^{2(n-\epsilon)}$, and define

$\Lambda_r(C) \coloneqq \{(t_{r-2}, v_{r-2}^2) \in \mathbb{R}^{\boldsymbol{2n}} : (s_{r-2}, t_{r-2}, \lambda_{r-2}^2, v_{r-2}^2) \in \Lambda_r$ and $(s_{r-2}, \lambda_{r-2}^2,) \in C\}$.

Then $(\chi_{[0,1]^n}, \Lambda_r(C))$ is a Gabor orthonormal basis for $L^2(\mathbb{R}^{2n})$.

Proof. We first show that the system $\big(\chi_{[0,1]^n}, \Lambda_r(C)\big)$ is orthogonal.Let (t_{r-2}, v_{r-2}^2) and $(t_{r-2}+\epsilon, v_{r-2}^2+\epsilon)$ be distinct elements of $\Lambda_r(C)$. There exist $(s_{r-2}, \lambda_{r-2}^2)$ and $s_{r-2}+\epsilon, \lambda_{r-2}^2+\epsilon)$ in $\mathbb{R}^{2m}$ such that $(s_{r-2}, t_{r-2}, \lambda_{r-2}^2, v_{r-2}^2)$ and $(s_{r-2}+\epsilon, t_{r-2}+\epsilon, \lambda_{r-2}^2 + \epsilon, v_{r-2}^2+\epsilon)$ both belong to Λ_r. Using the mutual orthogonality of the system $\big(\chi_{[0,1]^{2+\epsilon}}, \Lambda_r\big)$ together with (61), we have

$$\sum_r V_{(g_r)_{n-\epsilon}}(g_r)_{n-\epsilon}(-\epsilon, -\epsilon) = 0 \text{ or } \sum_r V_{(g_r)_n}(g_r)_n(-\epsilon, -\epsilon) = 0.$$

Note that, as both $(s_{r-2}, \lambda_{r-2}^2)$ and $s_{r-2}+\epsilon, \lambda_{r-2}^2+\epsilon)$ belong to C, we have $|\epsilon|_{\max} < 1$ and $|\epsilon|_{\max} < 1$. In particular, $\sum_r V_{(g_r)_{n-\epsilon}}(g_r)_{n-\epsilon}(-\epsilon, -\epsilon) \neq 0$ and the orthogonallity of the system $\big(\chi_{[0,1]^n}, \Lambda_r(C)\big)$ follows.

If $(s_{r-2}, \lambda_{r-2}^2) \in \Pi_1(\Lambda_r)$ (as defined in (49)), let

$$\sum_r (\Lambda_r)_{(s_{r-2},\lambda_{r-2}^2)} = \sum_r \{(t_{r-2}, v_{r-2}^2) : (s_{r-2}, t_{r-2}, \lambda_{r-2}^2, v_{r-2}^2) \in \Lambda_r\}.$$

Let $(f_r)_1 \in \mathrm{L}^2(\mathbb{R}^\mathrm{m})$, $(f_r)_2 \in \mathrm{L}^2(\mathbb{R}^\mathrm{n})$ and $(s_{r-1}, \lambda_{r-1}^2) \in \mathbb{R}^{2\mathrm{m}}$. Applying Parseval's identity to the function

$$\sum_r f_r(x_r, y_r) = \sum_r e^{2\pi i \lambda_{r-1}^2 x_r}(f_r)_1(x_r - s_{r-1})(f_r)_2(y_r), \qquad x_r \in \mathbb{R}^{n-\epsilon},$$

$$y_r \in \mathbb{R}^n,$$

we obtain that

$$\int_{\mathbb{R}^{n-\epsilon}} \sum_r (f_r)_1 |x_r|^2 dx_r \int_{\mathbb{R}^{n-\epsilon}} \sum_r (f_r)_2 |y_r|^2 dy_r$$

$$= \sum_{(s_{r-2},\lambda_{r-2}^2)\in\Pi_1(\Lambda_r)} \sum_{(t_{r-2},v_{r-2}^2)\in(\Lambda_r)_{(s_{r-2},\lambda_{r-2}^2)}} \sum_r \left|V_{(g_r)_{n-\epsilon}}\right|(f_r)_1(s_{r-2} - s_{r-1}, \lambda_{r-2}^2 - \lambda_{r-1}^2)^2 \left|V_{(g_r)_n}(f_r)_2(t_{r-2}, v_{r-2}^2)\right|^2$$

$$= \sum_{(s_{r-2},\lambda_{r-2}^2)\in\Pi_1(\Lambda_r)} \sum_{(t_{r-2},v_{r-2}^2)\in(\Lambda_r)_{(s_{r-2},\lambda_{r-2}^2)}} \sum_r \left|V_{(f_r)_1(g_r)_{n-\epsilon}}\right|(f_r)_1(s_{r-1} - s_{r-2}, \lambda_{r-1}^2 - \lambda_{r-2}^2)^2 \left|V_{(g_r)_n}(f_r)_2(t_{r-2}, v_{r-2}^2)\right|^2$$

Defining

$$\sum_r \mathcal{W}(s_{r-2}, \lambda_{r-2}^2) = \sum_r \|(f_r)_2\|_2^{-2} \sum_{(t_{r-2},v_{r-2}^2)\in(\Lambda_r)_{(s_{r-2},\lambda_{r-2}^2)}} \left|V_{(g_r)_n}(f_r)_2(t_{r-2}, v_{r-2}^2)\right|^2$$

and

$$\mu = \sum_{(s_{r-2},\lambda_{r-2}^2)\in\Pi_1(\Lambda_r)} \mathcal{W}(s_{r-2}, \lambda_{r-2}^2)\delta_{(s_{r-2},\lambda_{r-2}^2)}$$

For $(f_r)_2 \neq 0$, the above identity can be written as

$$\int_{\mathbb{R}^{n-\epsilon}} \sum_r (f_r)_1 |x_r|^2 dx_r$$

$$= \sum_{(s_{r-2},\lambda_{r-2}^2)\in\Pi_1(\Lambda_r)} \sum_r \mathcal{W}(s_{r-2}, \lambda_{r-2}^2) \left|V_{(f_r)_1(g_r)_{n-\epsilon}}\right|(s_{r-1} - s_{r-2}, \lambda_{r-1}^2 - \lambda_{r-2}^2)^2 = \sum_r \left(\mu * \left|V_{(f_r)_1(g_r)_{n-\epsilon}}\right|^2\right)(s_{r-1}, \lambda_{r-1}^2).$$

On the other hand, letting $\check{\chi}_{[0,1)^{2(n-\epsilon)}}(s_{r-2}, \lambda_{r-2}^2) = \chi_{[0,1)^{2(n-\epsilon)}}(-s_{r-2}, -\lambda_{r-2}^2)$ and defining C and $\Lambda_r(C)$ as above, we have also

$$\sum_r \left(\mu * \check{\chi}_{[0,1)^{n-\epsilon}}\right)(s_{r-1},\lambda_{r-1}^2)$$
$$= \sum_{(s_{r-2},\lambda_{r-2}^2)\in\Pi_1(\Lambda_r)} \mathcal{W}(s_{r-2},\lambda_{r-2}^2)\chi_{[0,1)^{2(n-\epsilon)}}(s_{r-2}-s_{r-1},\lambda_{r-2}^2-\lambda_{r-1}^2)$$
$$= \sum_{(s_{r-2},\lambda_{r-2}^2)\in\Pi_1(\Lambda_r)\cap C} \mathcal{W}(s_{r-2},\lambda_{r-2}^2)$$
$$= \sum_r \|(f_r)_2\|_2^{-2} \sum_{(t_{r-2},v_{r-2}^2)\in\Lambda_r(C)} \left|V_{(g_r)_n}(f_r)_2(t_{r-2},v_{r-2}^2)\right|^2 \le 1,$$

Where the last inequality results from the orthogonality of the system $\left(\chi_{[0,1)^n},\Lambda_r(C)\right)$ proved earlier. Since $(s_{r-1},\lambda_{r-1}^2)$ is arbitrary in $\mathbb{R}^{2(n-\epsilon)}$ and

$$\int_{\mathbb{R}^{2(n-\epsilon)}} \sum_r \left|V_{(f_r)_1}(g_r)_{n-\epsilon}(s_{r-2},\lambda_{r-2}^2)\right|^2 ds_{r-2}d\lambda_{r-2}^2 = \sum_r \|(f_r)_2\|_2^2,$$

Corollary (4.3.23) can be used to deduce that $\mu * \check{\chi}_{[0,1)^{n-\epsilon}} = 1$. This shows that

$$\sum_{(t_{r-2},v_{r-2}^2)\in\Lambda_r(C)} \sum_r \left|V_{(g_r)_n}(f_r)_2(t_{r-2},v_{r-2}^2)\right|^2 = \sum_r \|(f_r)_2\|^2, \qquad (f_r)_2 \in L^2(\mathbb{R}^n),$$

and thus that the system $\left(\chi_{[0,1)^n},\Lambda_r(C)\right)$ is complete, proving our claim.

Corollary (4.3.31)[491]: Let $n-2(1+\epsilon)=0$ and let $\Pi_1:\mathbb{R}^{2(2+\epsilon)}\to\mathbb{R}^{2(n-\epsilon)}$ be defined by (3.2). Suppose that $(\chi_{[0,1]^{2+\epsilon}},\Lambda_r)$ is a Gabor orthonormal basis and that $\Pi_1(\Lambda_r)+[0,1]^{2(n-\epsilon)}$ tiles $\mathbb{R}^{2(n-\epsilon)}$. Then Λ_r has the pseudo-standard structure.

Proof. Let $\mathcal{J}_r = \Pi_1(\Lambda_r)$ and, for any $(s_{r-2},\lambda_{r-2}^2)\in\mathcal{J}_r$, define

$$(\Lambda_r)_{(s_{r-2},\lambda_{r-2}^2)} = \{(t_{r-2},v_{r-2}^2):(s_{r-2},t_{r-2},v_{r-2}^2,\lambda_{r-2}^2)\in\Lambda_r\}.$$

If $(s_{r-1},\lambda_{r-1}^2)\in\mathcal{J}_r$, let $C=(s_{r-1},\lambda_{r-1}^2)+[0,1)^{2(n-\epsilon)}$, and

$$\Lambda_r(C) \coloneqq \{(t_{r-2},v_{r-2}^2)\in\mathbb{R}^{2n}:(s_{r-2},t_{r-2},v_{r-2}^2,\lambda_{r-2}^2)\in\Lambda_r \text{ and } (s_{r-2},\lambda_{r-2}^2)\in C\}.$$

Corollary (4.3.30) shows that the system $\left(\chi_{[0,1)^n},\Lambda_r(C)\right)$ forms a Gabor orthonormal basis. By assumption $\mathcal{J}_r+[0,1)^{2(n-\epsilon)}$ tiles $\mathbb{R}^{2(n-\epsilon)}$. Hence, $(s_{r-1},\lambda_{r-1}^2)+[0,1)^{2(n-\epsilon)}$ contains exactly one point in $\mathcal{J}_r, i.e. (s_{r-1},\lambda_{r-1}^2)$, and we have

$$\Lambda_r(C) = \{(t_{r-2},v_{r-2}^2):(s_{r-1},\lambda_{r-1}^2,v_{r-2}^2)\in\Lambda_r\} = (\Lambda_r)_{(s_{r-1},\lambda_{r-1}^2)}.$$

Therefore, we can write Λ_r as

$$\Lambda_r = \bigcup_{(s_{r-1},\lambda_{r-1}^2)\in\mathcal{J}_r} \{(s_{r-1},\lambda_{r-1}^2)\}\times(\Lambda_r)_{(s_{r-1},\lambda_{r-1}^2)}.$$

Our proof will be complete if we can show that $\mathcal{J}_r$ is a Gabor orthonormal basis of $L^2(\mathbb{R}^{n-\epsilon})$.

As $\mathcal{J}_r$ is a tiling set, by Proposition (4.3.8) it suffices to show that the inclusion $\mathcal{J}_r-\mathcal{J}_r\subset Z(V_{(g_r)_{n-\epsilon}}(g_r)_{n-\epsilon})\cup\{0\}$ holds. Let $(s_{r-2},\lambda_{r-2}^2)$ and $(s_{r-2}+\epsilon,\lambda_{r-2}^2+\epsilon)$ be distinct points in $\mathcal{J}_r$. As $(\Lambda_r)_{(s_{r-2},\lambda_{r-2}^2)}+[0,1)^{2n}$ tiles $\mathbb{R}^{2n}$, so does $(\Lambda_r)_{(s_{r-2},\lambda_{r-2}^2)}+[-1,0)^{2n}$, and we can find $(t_{r-2},v_{r-2}^2)\in(\Lambda_r)_{(s_{r-2},\lambda_{r-2}^2)}$ such that $0\in(t_{r-2},v_{r-2}^2)+[-1,0)^{2n}$,or, equivalently, with $(t_{r-2},v_{r-2}^2)\in[0,1)^{2n}$.Similarly, we can fined $(t_{r-2}+\epsilon,v_{r-2}^2+\epsilon)\in(\Lambda_r)_{(s_{r-2}+\epsilon,\lambda_{r-2}^2+\epsilon)}$ such that $(t_{r-2}+\epsilon,v_{r-2}^2+\epsilon)\in[0,1)^{2n}$.Using the fact that $\left(\chi_{[0,1)^n},\Lambda_r(C)\right)$ is a Gabor orthonormal basis of $L^2(\mathbb{R}^{2(2+\epsilon)})$, we have

$$(s_{r-2}, t_{r-2}, \lambda_{r-2}^2, v_{r-2}^2) - (s_{r-2}+\epsilon, t_{r-2}+\epsilon, \lambda_{r-2}^2+\epsilon, v_{r-2}^2+\epsilon) \in Z(V_{(g_r)_{2+\epsilon}}(g_r)_{2+\epsilon}).$$

Or, equivalently $\sum_r V_{(g_r)_{n-\epsilon}}(g_r)_{n-\epsilon}(-\epsilon,-\epsilon) = 0$ or $\sum_r V_{(g_r)_n}(g_r)_n(-\epsilon,-\epsilon) = 0$.

Note that, since $|\epsilon| < 1$, $\sum_r V_{(g_r)_n}(g_r)_n(-\epsilon,-\epsilon) \neq 0$. Hence $(s_{r-2}, \lambda_{r-2}^2) - (s_{r-2}+\epsilon, \lambda_{r-2}^2+\epsilon) \in Z(V_{(g_r)_{n-\epsilon}}(g_r)_{n-\epsilon})$ as claimed.

Corollary (4.3.32)[491]: Let $\mathcal{G}(\chi_{[0,1]^2,}\Lambda_r)$ be a Gabor orthonormal basis for $L^2(\mathbb{R}^2)$ and let C be a half-open square. Then,

(i) $\Gamma(C) + [0,1]^2$ is a packing of $\mathbb{R}^2$.

(ii) if $(\lambda_r^2, \lambda_{r+1}^2) \in \Gamma(C)$, then $T_C(\lambda_r^2, \lambda_{r+1}^2)$ consists of one point.

Proof. (i) let $(\lambda_r^2+\epsilon, \lambda_{r+1}^2+\epsilon)$ and $(\lambda_r^2+\epsilon, \lambda_{r+1}^2+\epsilon)$ be distinct elements of $\Gamma(C)$. by definition, we can find (t_r, t_{r+1}) and $(t_r+\epsilon, t_{r+1}+\epsilon)$ in C such that $(t_r, t_{r+1}, \lambda_r^2, \lambda_{r+1}^2), (t_r+\epsilon, t_{r+1}+\epsilon, \lambda_r^2+\epsilon, \lambda_{r+1}^2+\epsilon) \in \Lambda_r$. We then have

$$0 = \sum_r V_{(g_r)_1}(g_r)_1(-\epsilon,-\epsilon)\, V_{(g_r)_1}(g_r)_1(-\epsilon,-\epsilon)$$

If, without loss of generality, the first factor on the right-hand side of the previous equality vanishes, the fact that $|\epsilon| < 1$ shows the existence of an integer $k > 0$, such that

$$|\epsilon| = k/(1-|\epsilon|) \geq 1.$$

Hence, the cubes $(\lambda_r^2, \lambda_{r+1}^2) + [0,1]^2$ and $(\lambda_r^2+\epsilon, \lambda_{r+1}^2+\epsilon) + [0,1]^2$ are essentially disjoint.

(ii) Suppose that $T_C(\lambda_r^2, \lambda_{r+1}^2)$ contains two distinct points (t_r, t_{r+1}) and $(t_r+\epsilon, t_{r+1}+\epsilon)$. Then,

$$0 = \sum_r V_{(g_r)_1}(g_r)_1(-\epsilon, 0)\, V_{(g_r)_1}(g_r)_1(-\epsilon, 0).$$

As $\sum_r V_{(g_r)_1}(g_r)_1(t_{r-2}, 0) \neq 0$ for any t_{r-2} with $|t_{r-2}| < 1$, we must have $|\epsilon| \geq 1$, contradicting the fact that both (t_r, t_{r+1}) and $(t_r+\epsilon, t_{r+1}+\epsilon)$ belong to C.

Corollary (4.3.33)[491]: Under the hypotheses of the previous Lemma, consider an element $\lambda_{r-2}^2 = (\lambda_r^2, \lambda_{r+1}^2)$ of $\Gamma(C)$ and let $T_C(\lambda_{r-2}^2) = \{(t_r, t_{r+1})\}$. Then for any $x_r \in \partial(\lambda_{r-2}^2 + [0,1]^2)$. Moreover, for any such $(\lambda_{r-2}^2)_{x_r} = \left((\lambda_{r-2}^2)_{1,x_r}, (\lambda_{r-2}^2)_{2,x_r}\right) \in \Gamma(C)$ such that $x_r \in \partial\left((\lambda_{r-2}^2)_{x_r} + [0,1]^2\right)$. Moreover, for any such $(\lambda_{r-2}^2)_{x_r}$, letting $T_C\left((\lambda_{r-2}^2)_{x_r}\right) = \left\{(t_{r-2})_{x_r}\right\}$, where $(t_{r-2})_{x_r} = \left((t_{r-2})_{1,x_r}, (t_{r-2})_{2,x_r}\right)$, we can fined $i_0 \in \{1,2\}$ such that $(t_{r-2})_{i_0,x_r} = (t_{r-2})_{i_0}$ and $(\lambda_{r-2}^2)_{i_0,x_r} = (\lambda_{r-2}^2)_{i_0} + 1$ or $(\lambda_{r-2}^2)_{i_0} - 1$.

Proof. We can write $= (\lambda_r^2+\epsilon, \lambda_{r+1}^2+\epsilon)$, where $0 \leq \epsilon_i \leq 1$, $i = 1,2$ and $\epsilon_i \in \{0,1\}$ for at least one index i. Let $a = (a_1, a_2) \in \mathbb{R}^2$ with $0 < a_i < 1$, $i = 1,2$ and consider the point $((t_{r-2})_a, x_r) := (t_r + a_1, t_{r+1} + a_2)(\lambda_r^2+\epsilon), \lambda_{r+1}+\epsilon)$ in $\mathbb{R}^2$. Since $\Lambda_r + [0,1]^4$ is a tiling on $\mathbb{R}^4$ and the point $((t_{r-2})_a, x_r)$ is a point on the boundary of $((t_{r-2})_a, x_r) = \left((t_{r-2})_{x_r,a}, (\lambda_{r-2}^2)_{x_r,a}\right) + [0,1]^4$. Let $(t_{r-2})_{x_r,a} = (t_r+\epsilon, t_{r+1}+\epsilon)$ and $(\lambda_{r-2}^2)_{x_r,a} = (\lambda_r^2+\epsilon, \lambda_{r+1}+\epsilon)$. We have

$$\begin{cases} -a_i \leq \epsilon \leq 1 - a_i, \\ -\epsilon_i \leq -\epsilon \leq 1 - \epsilon_i, \end{cases} \quad i = 1,2. \tag{62}$$

Using the orthogonality of the system $\left(\chi_{[0,1]^2}, \Lambda_r\right)$, we can find $i_0 \in \{1,2\}$ such that $\sum_r V_{(g_r)_1}(g_r)_1(-\epsilon,-\epsilon) = 0$. Note that $\epsilon \neq 0$ would imply that $|\epsilon| > 1$ which impossible from (62). Hence, $\epsilon \neq 0$.
Moreover, as $\sum_r V_{(g_r)_1}(g_r)_1(0, v_{r-2}^2) \neq 0$ if $|v_{r-2}^2| < 1$, $\sum_r V_{(g_r)_1}(g_r)_1(-\epsilon,-\epsilon) = 0$ can only occurs if $|\epsilon| = 1$. This shows also that $\epsilon_{i_0} \in \{0,1\}$ in that case. this proves the last statement of our claim and the fact that $x_r \in \partial(\lambda_{x_r,a} + [0,1]^2)$. The proof will be complete if we can show that $(\lambda_{r-2}^2)_{x_r,a} \in \Gamma(C)$ for some choice of a.

For simplicity, we consider the half -open square to be $C = [b_1, b_1 + 1) \times [b_2, b_2 + 1)$. Our assertion will be true if the point $\epsilon = 0$ constructed above satisfies the inequalities $b_i \leq (t_{r-2})_i + \epsilon < b_i + 1$ for $i = 1,2$. As $(t_{r-2})_{i_0} = (t_{r-2})_{i_0} + \epsilon$, the inequalities clearly hold for $i = i_0$. Suppose that the other index j falls out of the range, say $(t_{r-2})_j + \epsilon < b_j$ (the case $(t_{r-2})_j + \epsilon \geq b_j + 1$ is similar). We consider $((t_{r-2})_{a'}, x_r)$ with $a_j' = 1 + \epsilon + \delta$ for some small $\delta > 0$. Note that, by (62), we have $(t_{r-2})_i + a_i - 1 \leq (t_{r-2})_i + \epsilon \leq (t_{r-2})_i + \epsilon + a_i$ for $i = 1,2$, and, in particular

$$a_j' = 1 + \epsilon + \delta \geq a_j + \delta > 0.$$

We have $a_j' < 1$ indeed, the inequality $\epsilon + \delta \geq 0$. This is not possible, as $b_j \leq (t_{r-2})_j < b_j + 1$, so $1 + (t_{r-2})_j \geq b_j + 1$. But $(t_{r-2})_j + \epsilon < b_j$, so $(t_{r-2})_j + \epsilon + 1 < b_j + 1$, so for δ small,

$$(t_{r-2})_j + \epsilon + \delta < b_j \leq (t_{r-2})_j$$

Which yields a contradiction.

Using the previous argument with a' replacing , we guarantee the existence of $(t_{r-2})_j + 2\epsilon$ such that $(t_{r-2})_j + \epsilon + \delta = (t_{r-2})_j + a_j' - 1 \leq (t_{r-2})_j + 2\epsilon \leq (t_{r-2})_j + a_j' = (t_{r-2})_j + \epsilon + 1 + \delta$ and the a ssociated point $\left((t_{r-2})_{a'}, (\lambda_{r-2}^2)_{x_r,a'}\right) = (t_r + 2\epsilon, t_{r+1} + 2\epsilon, \lambda_r^2 + 2\epsilon, \lambda_{r+1}^2 + 2\epsilon)$ in Λ_r with the property that $x_r \in \partial((\lambda_{r-2}^2)_{x_r,a'} + [0,1]^2)$ for some index i_0' such that $\left|(\lambda_{r-2}^2)_{i_0'}, (\lambda_{r-2}^2)_{i_0'} + 2\epsilon\right| = 1$, $(t_{r-2})_{i_0'} = (t_{r-2})_{i_0'} + 2\epsilon$ and $\epsilon_{i_0'} \in \{0,1\}$. We claim that $\epsilon = 1$. Now, $(t_r + \epsilon, t_{r+1} + \epsilon, \lambda_r^2 + \epsilon, \lambda_{r+1}^2 + \epsilon)$ and $(t_r + 2\epsilon, t_{r+1} + 2\epsilon, \lambda_r^2 + 2\epsilon, \lambda_{r+1}^2 + 2\epsilon)$ are in Λ_r. The mutual orthogonality property implies that $\sum_r V_{(g_r)_1}(g_r)_1(-\epsilon,-\epsilon) = 0$ for some $i = 1,2$.

Suppose that x_r is not of the corner points of $\lambda_{r-2}^2 + [0,1]^2$. in that case, the index i such that $\epsilon_i \in \{0,1\}$ is unique and it follows that $i_0 = i_0'$. this implies in particular, that $\epsilon = 0$ (as $(t_{r-2})_{i_0} + \epsilon = (t_{r-2})_{i_0} = (t_{r-2})_{i_0'} = (t_{r-2})_{i_0'} + 2\epsilon = (t_{r-2})_{i_0} + 2\epsilon)$. Furthermore, the second set of inequalities in (12) show that $(\lambda_{r-2}^2)_{i_0} + \epsilon = (\lambda_{r-2}^2)_{i_0} + 2\epsilon = (\lambda_{r-2}^2)_{i_0} - 1$ if $\epsilon_{i_0} = 0$ and $(\lambda_{r-2}^2)_{i_0} + \epsilon = (\lambda_{r-2}^2)_{i_0} + 2\epsilon = (\lambda_{r-2}^2)_{i_0} + 1$ if $\epsilon_{i_0} = 1$. We have thus $\epsilon = 0$ in both cases. We have thus

$$V_{(g_r)_1}(g_r)_1(-\epsilon,-\epsilon) = V_{(g_r)_1}(g_r)_1(0,0) = 1.$$

Therefore, the other index j must satisfy $\sum_r V_{(g_r)_1}(g_r)_1(-\epsilon,-\epsilon) = 0$. This inequalities

$$\epsilon_j - 1 \leq -\epsilon \leq \epsilon_j \quad \text{and } \epsilon_j - 1 \leq 2\epsilon \leq \epsilon_j \text{ yield } -1 \leq \epsilon \leq 1.$$

However, $\delta \leq \epsilon \leq 1 + \delta$. The $V_{(g_r)_1}(g_r)_1$ would not be zero unless $(t_{r-2})_j + 2\epsilon \geq (t_{r-2})_j + \epsilon + 1 (\geq b_j)$. Hence, $(t_{r-2})_j + \epsilon + 1 \leq (t_{r-2})_j + 2\epsilon \leq (t_{r-2})_j + \epsilon + 1 + \delta$. This forces that $(t_{r-2})_j + 2\epsilon = (t_{r-2})_j + \epsilon + 1$. This completes the proof for non-corner points. if x_r is the corner point, as the square constructed for the non-corner will certainly cover the corner point. therefore, the proof is completed.

Corollary (4.3.34)[491]: Let C be a half-open square. Then $\Gamma(C) + [0,1]^2$ is a tilling of $\mathbb{R}^2$.

Proof. It suffices to prove the following statements: suppose that $\mathcal{J}_r + [0,1]^2$ is non-empty packing of $\mathbb{R}^2$. If, for any $x_r \in \partial(t_{r-2} + [0,1]^2)$ where $t_{r-2} \in \mathcal{J}_r$, we can fined $(t_{r-2})_{x_r} \in \mathcal{J}_r$ with $(t_{r-2})_{x_r} \neq t_{r-2}$ such that $x_r \in \partial((t_{r-2})_{x_r} + [0,1]^2)$, then $\mathcal{J}_r + [0,1]^2$ is a tiling of $\mathbb{R}^2$. Indeed, by Corollary (4.3.32)(i) and Corollary (4.3.33), $\Gamma(C) + [0,1]^2$ is a packing of $\mathbb{R}^2$ and satisfies the stated property. It is thus a tiling of $\mathbb{R}^2$.

To prove the previous statements, we note that as $\mathcal{J}_r + [0,1]^2$ is packing, it is closed set. Suppose that $\mathcal{J}_r + [0,1]^2$ satisfies the property above and that $\mathbb{R}^{2+\epsilon}\backslash(\mathcal{J}_r + [0,1]^2 \neq \emptyset)$. Let $x_r \in \partial(\mathcal{J}_r + [0,1]^2)$ and assume that $x_r \in t_{r-2} + [0,1]^2$. We can then find $(t_{r-2})_{x_r} \in \mathcal{J}_r$ with $(t_{r-2})_{x_r} \neq t_{r-2}$ such that $x_r \in \partial((t_{r-2})_{x_r} + [0,1]^2)$. Note that if x_r were not a corner point of either $t_{r-2} + [0,1]^2$ or $(t_{r-2})_{x_r} + [0,1]^2$, then x_r would be in the interior of $\mathcal{J}_r + [0,1]^2$. Hence, x_r must be a corner point of $t_{r-2} + [0,1]^2$ or $(t_{r-2})_{x_r} + [0,1]^2$. As the set of all corner points of the squares in $\mathcal{J}_r + [0,1]^2$ is countable, the Lebesgue measure of the open set $\mathbb{R}^{2+\epsilon}\backslash(\mathcal{J}_r + [0,1]^2)$ is zero and $\mathbb{R}^{2+\epsilon}\backslash(\mathcal{J}_r + [0,1]^2)$ is thus empty, proving our claim.

Corollary (4.3.35)[491]: Let C be a half-open square and suppose that $(\lambda_r^2, \lambda_{r+1}^2) \in \Gamma(C)$ with $T_C(\lambda_r^2, \lambda_{r+1}^2) = \{(t_r, t_{r+1})\}$. Then all the sets $T_C(\lambda_r^2 + \epsilon, \lambda_{r+1}^2 + \epsilon)$ with $(\lambda_r^2 + \epsilon, \lambda_{r+1}^2 + \epsilon) \in \Gamma(C)$ are either of the form $\{(t_r, t_{r+1} + s_{r-2})\}$ or $\{(t_r + s_{r-2}, t_{r+1})\}$ for some real s_{r-2} with $|s_{r-2}| < 1$ depending on $(\lambda_r^2, \lambda_{r+1}^2)$.

Proof. We first make the following remark. if $(\alpha_r, \alpha_{r+1}), (\beta_r, \beta_{r+1}) \in \Gamma(C)$ are such that the two squares $(\alpha_r, \alpha_{r+1}) + [0,1]^2$ and $(\beta_r, \beta_{r+1}) + [0,1]^2$ intersect each other and also both intersect a third square $(\gamma_r, \gamma_{r+1}) + [0,1]^2$ with $(\gamma_r, \gamma_{r+1}) \in \Gamma(C)$, then, letting $T_C(\gamma_r, \gamma_{r+1}) = (m_r, m_{r+1})$, we have

$$T_C(\alpha_r, \alpha_{r+1}) = \{(m_r + a, m_{r+1})\} \quad \text{and } T_C(\beta_r, \beta_{r+1}) = \{(m_r + b, m_{r+1})\}$$

or

$$T_C(\alpha_r, \alpha_{r+1}) = \{(m_r + a, m_{r+1})\} \quad \text{and } T_C(\beta_r, \beta_{r+1}) = \{(m_r, m_{r+1} + b)\} ,$$

for Some real a, b. Indeed, using Corollary (4.3.33), we have $T_C(\alpha_r, \alpha_{r+1}) = \{(m_r + a, m_{r+1})\}$ or $\{(m_r, m_{r+1} + a)\}$ and $T_C(\beta_r, \beta_{r+1}) = \{(m_r + b, m_{r+1})$ or $(m_r, m_{r+1} + b)\}$. Suppose, for example, that $T_C(\alpha_r, \alpha_{r+1}) = \{(m_r + a, m_{r+1})\}$ and $T_C(\beta_r, \beta_{r+1}) = (m_r, m_{r+1} + b)$. Since the two squares intersect each other, we must have $|\alpha_r - \beta_r| \leq 1$ and $|\alpha_{r+1} - \beta_{r+1}| \leq 1$. The orthogonality property also implies that either $(a, \alpha_r - \beta_r)$ or $(-b, \alpha_{r+1} - \beta_{r+1})$ is in the zero set of $V_{(g_r)_1}(g_r)_1$. But since we have $|a|, |b| < 1$, this would imply that $|\alpha_r - \beta_r| > 1$ or $|\alpha_r - \beta_{r+1}| > 1$, which cannot happen. As $\Gamma(C) + [0,1]^2$ is a tiling of $\mathbb{R}^2$, for any square $(\sigma_r, \sigma_{r+1}) + [0,1]^2$ intersecting the square $(\lambda_r^2, \lambda_{r+1}^2) + [0,1]^2$ and with $(\sigma_r, \sigma_{r+1}) + [0,1]^2$, we can find another square $(\delta_r, \delta_{r+1}) + [0,1]^2$, with $(\delta_r, \delta_{r+1}) \in \Gamma(C)$ and with $(\delta_r, \delta_{r+1}) + [0,1]^2$ intersecting both squares $(\sigma_r, \sigma_{r+1}) + [0,1]^2$ and $(\lambda_r^2, \lambda_{r+1}^2) + [0,1]^2$. by the previous remark, the conclusion of the lemma holds for all the squares that neighbor the square $(\lambda_r^2, \lambda_{r+1}^2) + [0,1]^2$. Replacing this original square by one of the neighbouring squares and continuing this process, we obtain the conclusion of the lemma for all the squares in the tiling $\Gamma(C)+[0,1]^2$ by an induction argument. This proves our claim.

Corollary (4.3.36)[491]: If $(0,0,0,0)\in\Lambda_r$, then the sets $T_{[0,1]^2}(\lambda_r^2,\lambda_{r+1}^2)$ where $(\lambda_r^2,\lambda_{r+1}^2)\in\Gamma([0,1]^2)$ are either all of the form $\{(t_{r-2},0)\}$ or all ω of the form $\{(0,t_{r-2})\}$ with some t_{r-2} (depending on $(\lambda_r^2,\lambda_{r+1}^2)$ with $|t_{r-2}|<1$. in the first case, if there exists some $(\lambda_r^2,\lambda_{r+1}^2)\in\Gamma([0,1]^2)$ with $T_{[0,1]^2}(\lambda_r^2,\lambda_{r+1}^2)=(t_{r-2},0)$ and $\neq 0$, then

$$\Gamma([0,1]^2=\bigcup_{k\in\mathbb{Z}}(\mathbb{Z}+\mu_{k,0})\times\{k\} \qquad (63)$$

for some $0\leq\mu_{k,0}\leq 1$. Moreover, we can find $0\leq(t_{r-2})_k\leq 1$ such that

$$T_{[0,1)^2}\left((\mathbb{Z}+\mu_{k,0})\times\{k\}\right)=\{((t_{r-2})_k,0)\}\ ,\qquad k\in\mathbb{Z} \qquad (64)$$

and

$$\Lambda_r\cap([0,1)^2\times\mathbb{R}^2)=\{((t_{r-2})_k,0,j+\mu_{k,0},k):j,k\in\mathbb{Z}\} \qquad (65)$$

In the second case, $\Gamma([0,1)^2=\bigcup_{k\in\mathbb{Z}}\{k\}\times(\mathbb{Z}+\mu_{k,0})$ and $T_{[0,1)^2}(\{k\}\times(\mathbb{Z}+\mu_{k,0}))=\{(0,(t_{r-2})_k)\},\Lambda_r\cap([0,1)^2\times\mathbb{R}^2)=\left\{\left((0,(t_{r-2})_k,k,j+\mu_{k,0}):j,k\in\mathbb{Z}\right)\right\}$.

Proof. If $(\lambda_r^2,\lambda_{r+1}^2)=(0,0)$ we have $T_{[0,1)^2}(\lambda_{r-2}^2)=\{(0,0)\}$ as $(0,0,0,0)\in\Lambda_r$. By Corollary (4.3.35), any $(\lambda_r^2,\lambda_{r+1}^2)\in\Gamma([0,1)^2$ with the square $(\lambda_r^2,\lambda_{r+1}^2)\in[0,1)^2$ intersecting $[0,1]^2$ on the $\lambda_r^2,\lambda_{r+1}^2$-plane satisfies $T_{[0,1)^2}(\lambda_r^2,\lambda_{r+1}^2)=\{(t_{r-2},0)\}$ or $T_{[0,1)^2}(\lambda_r^2,\lambda_{r+1}^2)\{(t_{r-2},0)\}$ with $|t_{r-2}|<1$. Without loss of generality, we assume that the first case holds. As $\Gamma([0,1]^2)+[0,1]^2$ is a tiling of $\mathbb{R}^2$, for any square $C=(\lambda_r^2,\lambda_{r+1}^2)+[0,1]^2$, with $(\lambda_r^2,\lambda_{r+1}^2)\in\Gamma([0,1]^2)$, we can find squares $C_i=\left((\lambda_{r-2}^2)_{1,i},(\lambda_{r-2}^2)_{2,i}\right)+[0,1]^2$ for $i=0,\dots,k$ with $\left((\lambda_{r-2}^2)_{1,i},(\lambda_{r-2}^2)_{2,i}\right)\in\Gamma([0,1]^2)$ and such that $C_0=[0,1]^2,C_k=C$,and with C_i and C_{i+1} touching each other for all $i=0,\dots,k-1$.

We have $T_{[0,1)^2}\left((\lambda_{r-2}^2)_{1,1},(\lambda_{r-2}^2)_{2,1}\right)=\{(t_r,0)\}$ for some number t_r with $|t_r|<1$. Since C_2 and C_0 both intersect C_1, $T_{[0,1)^2}\left((\lambda_{r-2}^2)_{1,1},(\lambda_{r-2}^2)_{2,1}\right)=\{(t_r,0)\}$ by Corollary (4.3.35) again. inductively, we have $T_{[0,1)^2}\left((\lambda_{r-2}^2)_{1,i},(\lambda_{r-2}^2)_{2,i}\right)=\{(t_{r+i-1},0)\},\ i=1,\dots,k$, which proves the first part.

Consider the case where, for any $(\lambda_r^2,\lambda_{r+1}^2)\in\Gamma([0,1]^2)$, there exists a number $t_{r-2}=t_{r-2}(\lambda_r^2,\lambda_{r+1}^2)$ such that $T_{[0,1)^2}(\lambda_r^2,\lambda_{r+1}^2)=\{(t_{r-2},0)\}$ and assume that $t_{r-2}(\lambda_r^2,\lambda_{r+1}^2)\neq 0$ for at least one couple $(\lambda_r^2,\lambda_{r+1}^2)\in\Gamma([0,1]^2)$. Suppose that $\Gamma([0,1]^2)$ is not of the form in (63). By Corollary (4.3.34) and Corollary (4.3.27), we must have $\Gamma([0,1]^2)=\bigcup_{k\in\mathbb{Z}}\{k\}\times(\mathbb{Z}+a_k^r)$ with $0\leq a_k^r\leq 1$ and at least one $a_k^r\neq 0$. Consider the distinct points

$$(t_{r-2},0,k,a_k^r+j)\text{ and }(t_{r-2}+\epsilon,0,k-1,a_{k-1}^r+j),\text{ both in }\Lambda_r.$$

We must have that either $(-\epsilon,1)\in Z\left(V_{(g_r)_1}(g_r)_1\right)$ or $(0,a_k^r-a_{k-1}^r)\in Z\left(V_{(g_r)_1}(g_r)_1\right)$. however, since $|a_k^r-a_{k-1}^r|<1$, the second case is impossible. this means that $(-\epsilon,1)\in Z\left(V_{(g_r)_1}(g_r)_1\right)$ which is possible only if $t_{r-2}=t_{r-2}+\epsilon$. Therefore the fact $(t_{r-2},0,k,a_k^r+j)\in\Lambda_r$ implies that $t_{r-2}=(t_{r-2})_j$ for some real $(t_{r-2})_j$. we know prove by induction on $|j|$ that $(t_{r-2})_j=0$, for all $|j|\leq J$ Where $J\geq$

0, chose $k \in \mathbb{Z}$ such that $a^r_{k+1} \neq 0$ and $a^r_k = 0$ if such k exists. Suppose first that $j > 0$. There exist thus $t_{r-2} \in [0,1)$ such that

$$\left((t_{r-2})_{j+1}, 0, k, j+1\right) \text{ and } (0,0,k+1, a^r_{k+1} + j) \text{ both belong to } \Lambda_r.$$

This implies that either $(t_{r-2}, -1) \in Z\left(V_{(g_r)_1}(g_r)_1\right)$ or $(0, a^r_{k+1} - 1) \in Z\left(V_{(g_r)_1}(g_r)_1\right)$. This last case is impossible and the first one is only possible if $t_{r-2} = 0$, showing that $(t_{r-2})_{j+1} = 0$. Similarly by considering the points

$$\left((t_{r-2})_{j-1}, 0, k+1, a^r_{k+1} + j - 1\right) \text{ and } (0,0,k,j) \text{ which both belong to } \Lambda_r.$$

We can conclude that $(t_{r-2})_{j-1} = 0$ for $j < 0$. If k as above does not exist, there exists chose $k' \in \mathbb{Z}$ such that $a^r_{k'-1} \neq 0$ and $a^r_{k'} = 0$. By considering the points

$$\left((t_{r-2})_{j+1}, 0, k', j-1\right) \text{ and } \left(0,0,k'-1, a^r_{k'-1} + j\right) \text{ if } \quad j > 0.$$

and the points

$$\left((t_{r-2})_{j-1}, 0, k'-1, a^r_{k'-1} + j - 1\right) \text{ and } (0,0,k',j) \text{ if } j < 0$$

Which all belong to , we conclude that $(t_{r-2})_j = 0$ if $|j| = J + 1$. This proves (63).

If we are in the first case, i.e

$$\Gamma([0,1]^2) = \bigcup_{k \in \mathbb{Z}} \left(\mathbb{Z} + \mu_{k,0}\right) \times \{k\},$$

let $n - \epsilon, n' - \epsilon$ be distinct integers. We have then

$$T_{[0,1)^2}\left(n - \epsilon + \mu_{n,0}, 0\right) = \{(t_{r+n-\epsilon-1}, 0)\} \text{ and } T_{[0,1)^2}\left(n' - \epsilon + \mu_{n,0}, n\right) = \{(t_{r+n'-\epsilon-1}, 0)\}$$

Which implies that $\sum_r V_{(g_r)_1}(g_r)_1(t_{r+n-\epsilon-1} - t_{r+n'-\epsilon-1}, n - n') = 0$ or $\sum_r V_{(g_r)_1}(g_r)_1(0,0) = 0$. This second case is clearly impossible while the first one is possible only when $t_{r+n-\epsilon-1} = t'_{r+n-\epsilon-1}$. This shows (64) and (65) follows immediately from (63) and (64).

Corollary (4.3.37)[491]: Under the assumptions of Corollary (4.3.36), suppose that there exists $(\lambda^2_r, \lambda^2_{r+1}) \in \Gamma([0,1]^2)$ with $T_{[0,1)^2}(\lambda^2_r, \lambda^2_{r+1}) = (t_{r-2}, 0)$ and $t_{r-2} \neq 0$. Then we can find numbers t_{r+k-1} with $0 \leq t_{r+k-1} < 1$ and $\mu_{k,0}, k, n - \epsilon \in \mathbb{Z}$, with $0 \leq \mu_{k,n-\epsilon} < 1$, such that

$$\Lambda_r \cap (\mathbb{R} \times [0,1] \times \mathbb{R}^2) = \left\{\left(n - \epsilon + t_{r+k-1}, 0, j + \mu_{k,n-\epsilon,} k\right) : j, k, n - \epsilon \in \mathbb{Z}\right\}$$

Proof. By the result of Corollary (4.3.36), we have the identities (64) and (65). Let $T = \{t_{r+k-1}, k \in \mathbb{Z}\} \subset [0,1)$ where $t_{r+k-1}, k \in \mathbb{Z}$, are the numbers appearing in (64). Let $s_r, s_{r+1} \in T$ with $s_r < s_{r+1}$. Consider the half-open squares $C = (s_r, 0) + [0,1)^2$ and $C' = (s_r, 0) + \left((0,1] \times [0,1)\right)$. then we know that $\Gamma(C) + [0,1]^2$ and $\Gamma(C') + [0,1]^2$ both tile $\mathbb{R}^2$. Let $P_{r-1} = \{(s_r, y_r) : 0 \leq y_r < 1\}$ and $P_r = \{(s_r + 1, y_r) : 0 \leq y_r < 1\}$. Note that $\Gamma(P_{r-1}) = \Gamma(\{(s_r, 0)\})$. Moreover,

$$\Gamma(C) = \Gamma(P_{r-1}) \cup \Gamma(C \backslash P_{r-1}), \Gamma(C') = \Gamma(C' \backslash P_r) \cup \Gamma(P_r)$$

and since $C \backslash P_{r-1} = C' \backslash P_r$, $\Gamma(P_{r-1}) = \Gamma(P_r)$. We have

$$T_{C'}\left(\Gamma(P_r)\right) \subset \{(s_r + 1, y_r), 0 \leq y_r < 1\}$$

but since $(s_2, 0) \in C'$, we must have $T_{C'}(\Gamma(P_r)) \subset \{(s_r + 1, 1)\}$ by Corollary (4.3.35). Since

$$\Gamma(P_{r-1}) = \left\{\left(j + \mu_{k,0}, k\right) : \; j, k \in \mathbb{Z}, t_{r+k-1} = s_r\right\}$$

and $\pi_2(\Gamma(P_{r-1})) = \pi_2(\Gamma(P_r))$, where π_2 is the projection to the second coordinate, we have

$$\Gamma(\{(1+s_r,0)\}) = \Gamma(P_r) = \{(j+\mu_{k,1},k): j,k\in\mathbb{Z}, t_{r+k-1}=s_r\}.$$

For some constants $\mu_{k,1}$ with $0\le\mu_{k,1}<1$ using Corollary (4.3.27). Applying this argument to $s_r=0$ and $s_{r+1}=t_{r-2}$, we obtain that

$$\Lambda_r\cap(\{1\}\times[0,1)\times\mathbb{R}^2) = \{(j+\mu_{k,1},k): j,k\in\mathbb{Z}, t_{r+k-1}=0\}.$$

Similar arguments applied to $s_r=s_{r-2}$ and $s_{r+1}=1$ show that, for any $s_{r-2}\in T$,we have

$$\Lambda_r\cap(\{s_{r-2}+1\}\times[0,1)\times\mathbb{R}^2) = \{(j+\mu_{k,1},k): j,k\in\mathbb{Z}, t_{r+k-1}=s_{r-2}\}.$$

And that $\Lambda_r\cap(\{s_{r-2}+1\}\times[0,1)\times\mathbb{R}^2)$ is empty if $s_{r-2}\in[0,1)\backslash T$. The same idea can also be used to show the existence of constants $\mu_{k,-1}$ with $0\le\mu_{k,1}<1$ such that

$$\Lambda_r\cap(\{s_{r-2}-1\}\times[0,1)\times\mathbb{R}^2) = \begin{cases}\{(j+\mu_{k,-1},k):\ j,k\in\mathbb{Z}, t_{r+k-1}=s_{r-2}\}, & s_{r-2}\in T\\ \emptyset, & s_{r-2}\in[0,1)\backslash T.\end{cases}$$

And, more generally using induction, that, for any $n-\epsilon\in\mathbb{Z}$, we can find constants $\mu_{k,n-\epsilon}$ with $0\le\mu_{k,1}<1$ such that

$$\Lambda_r\cap(\{s_{r-2}+1\}\times[0,1)\times\mathbb{R}^2) = \begin{cases}\{(j+\mu_{k,n-\epsilon},k):\ j,k\in\mathbb{Z}, t_{r+k-1}=s_{r-2}\}, & s_{r-2}\in T\\ \emptyset, & s_{r-2}\in[0,1)\backslash T.\end{cases}$$

This proves our claim.

Corollary (4.3.38)[491]: $\mathcal{G}(\chi_{[0,1]^2},\Lambda_r)$ is a Gabor orthonormal basis for $\boldsymbol{L^2}(\mathbb{R}^2)$ if and only if we can partition $\mathbb{Z}$ into $\mathcal{J}_r$ and $\mathcal{J}'_r$ such that either

$$\Lambda_r = \bigcup_{n\in\mathcal{J}_r}\{(n-\epsilon+(t_{r-2})_{n,k,}n, j+\mu_{k,n-\epsilon,n}, K+v^2_{r+n-1}): n-\epsilon,j,k\in\mathbb{Z}\} \cup \bigcup_{n-\epsilon\in\mathbb{Z}}\bigcup_{n\in\mathcal{J}'_r}\{(n-\epsilon+(t_{r-2})_n,n)\}\times(\Lambda_r)_{n-\epsilon,n}.$$

$$\Lambda_r = \bigcup_{n-\epsilon\in\mathcal{J}_r}\{(n-\epsilon, n+(t_{r-2})_{n-\epsilon,j,}, j+v^2_{r+n-\epsilon-1}, K+\mu_{j,n-\epsilon,n}): n,j,k\in\mathbb{Z}\}\cup\bigcup_{n\in\mathbb{Z}}\bigcup_{n-\epsilon\in\mathcal{J}'_r}\{(n-\epsilon,n+(t_{r-2})_{n-\epsilon})\}\times(\Lambda_r)_{n-\epsilon,n}.$$

Where $(\Lambda_r)_{n-\epsilon,n}+[0,1]^2$ tile $\mathbb{R}^2$ and $(t_{r-2})_{n,k},\mu_{k,n-\epsilon,n}$ and v^2_{r+n-1} are real numbers in $[0,1)$ as a function of $n-\epsilon,n$ or K.

Proof. It follows from Corollary (4.3.35), that either all $T_{[0,1)^2}(\lambda_r^2,\lambda_{r+1}^2),(\lambda_r^2,\lambda_{r+1}^2)\in\Gamma([0,1]^2)$ are either of the form $\{(t_{r-2},0)\}$ or all are the form $\{(0,t_{r-2})\}$ with some $t_{r-2}\ne 0$. In the first case, we deduce from Corollary (4.3.37) that

$$\Lambda_r\cap(\{s_{r-2}+1\}\times[0,1)\times\mathbb{R}^2) = \{(n-\epsilon+t_{r+k-1},0,j+\mu_{k,n-\epsilon},k): j,k,n-\epsilon\in\mathbb{Z}\}$$

For certain numbers t_{r+k-1} and $\mu_{k,n-\epsilon}$ in the interval $[0,1)$. We now show that Λ_r will be of the first of the two possible forms given in the theorem. (Similarly, the second form follows from the second case of Corollary (4.3.37)).

Letting $C=[0,1)^2$ and $C'=[0,1)\times(0,1]$, we note that both $\Gamma(C)+[0,1]^2$ and $\Gamma(C')+[0,1]^2$, tile $\mathbb{R}^2$ but $\Gamma((0,1)^2)$ is empty. Hence, $\Gamma(C')=\Gamma(\{(x_r,1): 0\le x_r\le$

1}). It means that any set $T_{C'}(\lambda_r^2, \lambda_{r+1}^2)$ with $(\lambda_r^2, \lambda_{r+1}^2) \in \Gamma(C')$ is of the form $\{(t_{r-2}, 1)\}$ for some $t_{r-2} = t_{r-2}(\lambda_r^2, \lambda_{r+1}^2)$ with $0 \leq t_{r-2} < 1$. We now have two possible cases : either the cardinality of $T_{C'}\big(\Gamma(C')\big)$ is larger than one or equal to one. In the first case, we can find two distinct elements of $T_{C'}\big(\Gamma(C')\big)$ and we can then replicate the proof of Corollary (4.3.37), to obtain that

$$\Lambda_r \cap (\mathbb{R} \times [1,2) \times \mathbb{R}^2) = \left\{\left(n - \epsilon + t_{r+k-1}, 1, j + \mu_{k,n-\epsilon,1}, k\right) : j, k \in \mathbb{Z}\right\}.$$

In the other case, $T_{C'}\big(\Gamma(C')\big) = \{(t_r, 1)\}$ for some t_r with $0 \leq t_r < 1$. If we translate C' horizontally and use the same argument as in the proof of Corollary (4.3.37), we see that

$$\Lambda_r \cap (\mathbb{R} \times [1,2) \times \mathbb{R}^2) = \{(n - \epsilon + t_r, 1)\} \times (\Lambda_r)_{n-\epsilon,1}$$

Where $(\Lambda_r)_{n-\epsilon,1}$ is a spectrum for the unit square $[0,1]^2$. This last property is equivalent to $(\Lambda_r)_{n-\epsilon,1} + [0,1]^2$ being a tiling of $\mathbb{R}^2$.

We can them prove the theorem inductively by translating the square C' in the vertical direction using integer steps.

Chapter 5
Slanted Matrices and Hamiltonian Deformation

We establish result to enrich our understanding of Banach frames and obtain new results for irregular sampling problems. We also present a version of a non-commutative Wiener's lemma for slanted matrices. We study in some detail an associated weak notion of Hamiltonian deformation of Gabor frames, using ideas from semiclassical physics involving coherent states and Gaussian approximations. We will thereafter discuss possible applications and extensions, which can be viewed as the very first steps towards a general deformation theory for Gabor frames. The deformation theorem requires a new characterization of Gabor frames and Gabor Riesz sequences. It is in the style of Beurling's characterization of sets of sampling for bandlimited functions and extends significantly the known characterization of Gabor frames "without inequalities" from lattices to non-uniform sets.

Section (5.1): Banach Frames and Sampling

We study certain properties of so-called slanted matrices, which occur naturally in different fields of pure and applied analysis. A matrix is slanted if it has a decay property such that the coefficients vanish away from a diagonal, which is not necessarily the main diagonal; ideally, non-zero coefficients of such a matrix are contained between two parallel slanted lines. Potential applications of the theory of slanted matrices range through wavelet theory and signal processing [277,278,279,284,293], frame and sampling theory [261,262, 263,268,269,290], differential equations [273,274,276], and even topology of manifolds [303]. Here we especially emphasize the use of slanted matrices in frame theory and related fields.

We begin with a few explicit examples illustrating the appearance of slanted matrices. The standard case of banded matrices is a particular case of slanted banded matrices. Below are less trivial examples and the first of them concerns sampling in shift invariant spaces.

Example (5.1.1)[260]: It is well known that the Paley-Wiener space $PW_{1/2} = \{f \in L^2(\mathbb{R}^2) : \operatorname{supp} \hat{f} \subset [-1/2, 1/2]\}$ can also be described as

$$PW_{1/2} = \left\{f \ \in L^2(\mathbb{R}^2) : f = \sum_{k\in\mathbb{Z}} c_k \phi(\cdot - k), c \in \ell^2(\mathbb{Z})\right\}, \qquad (1)$$

where $\phi(x) = \frac{\sin \pi(x-k)}{\pi(x-k)}$ and the series converges in $L^2(\mathbb{R})$ (see [263]). Because of this equivalent description of $PW_{1/2}$, the problem of reconstructing a function $f \in PW_{1/2}$ from the sequence of its integer samples, $\{f(i)\}_{i\in\mathbb{Z}}$, is equivalent to finding the coefficients $c \in \ell^2$ such that $\{f(i)\} = Ac$ where $A = (a_{i,j})$ is the with entries $a_{i,j} = \emptyset(i-j)$. It is immediate, however, that $A = I$ is the identity matrix and, therefore,

$$f = \sum_{k\in\mathbb{Z}} f(k)\phi(\cdot - k).$$

If, instead we sample a function $f \in PW_{1/2}$ on $\frac{1}{2}\mathbb{Z}$, then we obtain the equation $\left\{f(\frac{i}{2})\right\} = Ac$. In this case, the sampling matrix A is defined by $a_{i,j} = \phi\left(\frac{i}{2} - j\right)$ and is no longer diagonal- it has constant values on slanted lines with slopes $1/2$, for instance,

$a_{2j,j} = 1$. If $\phi = \frac{\sin \pi(x-k)}{\pi(x-k)}$ in (1) is replaced by a function ψ supported on $\left[-\frac{M}{2}, \frac{M}{2}\right]$, then the matrix $A = (a_{i,j})$ is zero outside the slanted band $|j - i/2| \le M$. Clearly, this matrix is not banded in the classical sense. If we move to the realm of irregular sampling [263], the sampling matrix will be given by $a_{i,j} = \phi(\chi_i - j)$, where $x_i, i \in \mathbb{Z}$, are the sampling points. In this case, we no longer have constant values on slanted lines, but the slanted structure is still preserved if we have the same number of sampling points per period. An important fact [263] is that any function can be reconstructed from its samples at $x_i, i \in \mathbb{Z}$, if and only if the sampling matrix is bounded below and above. The main emphasis is to study this particular property of abstract slanted matrices.

The next example deals with frames in Hilbert spaces. Meaningful extension of the notion of frames to Banach spaces is a non-trivial problem which provided some inspiration for our abstract results. In the example below we only give a brief introduction and defer precise definitions.

Example (5.1.2)[260]: Let $\mathcal{H}$ be a separable Hilbert space. A sequence $\varphi_n \in \mathcal{H}, n \in \mathbb{Z}$, is a frame for $\mathcal{H}$ if for some $0 < a \le b < \infty$

$$a\|f\|^2 \le \sum_{n\in\mathbb{Z}} |\langle f, \varphi_n\rangle|^2 \le b\|f\|^2 \qquad (2)$$

for all $f \in \mathcal{H}$. The operator$T: \mathcal{H} \to \ell^2, Tf = \{\langle f, \varphi_n\rangle\}_{n\in\mathbb{Z}}, f \in \mathcal{H}$, is called an analysis operator. It is an easy exercise to show that a sequence $\varphi_n \in \mathcal{H}$ is a frame for $\mathcal{H}$ if and only if its analysis operator has a left inverse. The a djoint of the analysis operator, $T^*: \ell^2 \to \mathcal{H}$, is given by $T^*c = \sum_{n\in\mathbb{Z}} c_n\varphi_n, c = (c_n) \in \ell^2$. The frame operator is $T^*T: \mathcal{H} \to \mathcal{H}, T^*Tf = \sum_{n\in\mathbb{Z}}\langle f, \varphi_n\rangle \varphi_n, f \in \mathcal{H}$.

Traditionally (see [280,283,290]), the frame properties are studied via the spectral properties of the frame operator. We show that some work can be done already at the level of the analysis operator. This makes extensions to Banach spaces easier since the analysis operator is more amenable to such. Connection with slanted matrices is readily illustrated if we consider a frame in $\ell^2(\mathbb{Z})$ which consists of two copies of an orthonormal basis. The matrix of the analysis operator with respect to that basis looks like

$$\begin{matrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0. \end{matrix}$$

Clearly, the slant of the matrix may serve as a natural measure of redundancy of a frame. The following example illustrates the role of slanted matrices in wavelet theory.

Example (5.1.3)[260]: In signal processing and communication, a sequence s (a) discrete signal) is often split into finite set compressed sequences $\{s_1, \dots, s_r\}$ from which the original sequence s can be reconstructed or approximated. The compression is often performed with filter banks [284,293] using the cascade algorithm. One way to introduce filters, in the simplest case, is to use the two-scale equation of the multiresolution analysis (MRA):

$$\varphi(x) = \sum_{n\in\mathbb{Z}} a_n \varphi(2x - n),$$

Where $\varphi \in L^2(\mathbb{R})$ is the so-called $scaling\ function$. The filter coefficients$a_n, n \in \mathbb{Z}$, in the above equation are the Fourier coefficients of the $-pass\ filter\ m_0 \in L^2(\mathbb{R}),\ = \mathbb{R}/\mathbb{Z}$, which is a periodic function given by

$$m_0(\xi) = \sum_{n\in\mathbb{Z}} a_n e^{2\pi i n\xi}, \qquad \xi \in \mathbb{R}.$$

It is clear that the two-scale equation has the following equivalent form in the Fourier domain:

$$\hat{\varphi}(2\xi) = m_0(\xi)\hat{\varphi}(\xi),\ \xi \in \mathbb{R}\,.$$

An important role in the MRA theory is played the $periodization\ \sigma_\varphi \in L^\infty(\mathbb{T})$ of the scaling function φ, which is defined by

$$\sigma_\varphi(\xi) = [\hat{\varphi}, \hat{\varphi}](\xi) = \sum_{n\in\mathbb{Z}} |\hat{\varphi}(\xi + n)|^2$$

It is a standard fact (see, e.g., [266, Lemma 2.11]) that this periodization satisfies

$$\sigma_\varphi(\xi/2) = |m_0(\xi)|^2\sigma_\varphi(\xi) + |m_0(\xi + 1/2)|^2\sigma_\varphi(\xi + 1/2).$$

In fact, σ_φ is the Perron-Frobenius eigen-vector of the transfer operator R_{m_0} which acts on different spaces of periodic functions via

$$\left(R_{m_0}f\right)(\xi) = |m_0(\xi)|^2 f(\xi) + |m_0(\xi + 1/2)|^2 f(\xi + 1/2) \qquad (3)$$

In [278] there is a detailed account of the relation between the spectral properties of the transfer operator on different function spaces and the properties of the corresponding MRA filters, scaling functions, and wavelets. Here we will just recall that the convergence rate of the above mentioned cascade algorithm is controlled by the second biggest eigenvalue of R_{m_0}. The reason we use the transfer operator as an example is because of its matrix with respect to the Fourier basis in $L^2(\mathbb{T})$. Following [278, Section 3.2], we let

$$c_n = \sum_{k\in\mathbb{Z}} \overline{a}_k a_{n+k}.$$

Then the Fourier coefficients of $R_{m_0}f$ and f are related via

$$\left(R_{m_0}f\right)_n = \sum_{k\in\mathbb{Z}} c_{2n-k} f_k,$$

and, hence, this is, indeed, a slanted matrix. In particular, if

$$m_0(\xi) = a_0 + a_1 e^{2\pi i\xi} + a_2 e^{2\pi i 2\xi} + a_3 e^{2\pi i 3\xi},$$

this matrix looks like

$$\begin{matrix} c_3 & c_2 & c_1 & c_0 & c_{-1} & c_{-2} & c_{-3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_3 & c_2 & c_1 & c_0 & c_{-1} & c_{-2} & c_{-3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_3 & c_2 & c_1 & c_0 & c_{-1} & c_{-2} & c_{-3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_3 & c_2 & c_1 & c_0 & c_{-1} & c_{-2} & c_{-3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_3 & c_2 & c_1 & c_0 & c_{-1} & c_{-2} & c_{-3}. \end{matrix}$$

Due to the special Laurent-type structure of this matrix there has been a lot of result on the spectral properties of such matrices (see [278]). Since we are interested in more general slanted matrices, we cannot use most of those results. Observe also, that this matrix has the "opposite" slant compared to the matrices in the previous examples. In fact, as shown in Lemma (5.1.10) below such matrices cannot be bounded below and, therefore, are less relevant.

We devoted to abstract results. We give precise definitions of different classes of slanted matrices and study some of their basic properties, we state and show one of our main Theorems. Specifically, slanted matrices with some decay, viewed as operators on

$\ell^P\left(\mathbb{Z}^d,(X_n)\right)$ spaces (where X_n is a Banach space), are either universally bounded below for all $p \in [1,\infty]$, or do not have this property for any $p \in [1,\infty]$. We use this Theorem to obtain a version of Wiener's Tauberian lemma and a result on subspace complementation in Banach spaces. We devoted to some applications of the results. Specifically, the reconstruction formula for Hilbert frames are extended to Banach frames under certain localization conditions related to slanted matrices. Gabor systems having this localization property are then presented as an example. Exhibits an application of slanted matrices to sampling theory.

We prefer to give a straightforward definition of slanted matrices in the relatively simple case that arises in applications presented, mainly in connection with sampling theory. For that reason, we restrict our attention to the group $\mathbb{Z}^d, d \in \mathbb{N}$, and leave the case of more general locally compact Abelian groups for future research in the spirit of [271,272,275]. We believe, also, that some of the results below may be extended to matrices indexed by discrete metric spaces.

For each $n \in \mathbb{Z}^d$ we let X_n and Y_n be (complex) Banach spaces and $\mathfrak{L}^P = \ell^p\left(\mathbb{Z}^d,(X_n)\right)$ be the Banach space of sequences $x = (x_n)_{n\in\mathbb{Z}^d}$, $x_n \in X_n$, with the norm $\|x\|_p = \left(\sum_{n\in\mathbb{Z}^d}\|x_n\|_{X_n}^p\right)^{1/p}$ when $p \in [1,\infty)$ and $\|x\|_\infty = \sup_{n\in\mathbb{Z}^d}\|x_n\|_{X_n}$. By $c_0 = c_0\left(\mathbb{Z}^d,(X_n)\right)$ we denote the subspace of $\mathfrak{L}^\infty$of sequences vanishing at infinity, that is $\lim_{|n|\to\infty}\|x_n\| = 0$, where $|n| = \max_{1\le k\le d}|n_k|$, $n = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d$. We will use this multi-index notation throughout. Note that when $X_n = \mathbb{C}$ for all n, then $\mathfrak{L}^P$ is the standard space of complex-valued sequences $\ell^p(\mathbb{Z}^d)$.

Let $a_{mn}: X_n \to Y_m$ be bounded linear operators. The symbol $\mathbb{A}$ will denote the operator matrix $(a_{mn}), m, n \in \mathbb{Z}^d$. We are interested only in those matrices that give rise to bounded linear operators that map $\mathfrak{L}^P$ into $\mathfrak{L}^P$ for all $p \in [1,\infty]$ and c_0 into c_0. We let $\|A\|_p$ be the operator norm of $\mathbb{A}$ in $\mathfrak{L}^p = \ell^p\left(\mathbb{Z}^d,(Y_n)\right)$ and $\|\mathbb{A}\|_{sup} = \sup_{m,n\in\mathbb{Z}^d}\|a_{mn}\|$. If $X_n, Y_n, n \in \mathbb{Z}^d$, are separable Hilbert spaces, we denote by $\mathbb{A}^\star = (a_{mn}^\star)$ the matrix defined by $a_{mn}^\star = (a_{mn})^*$, where $(a_{mn})^*: Y_n \to X_m$ are the (Hilbert) adjoints of the operators a_{nm}. Clearly, $(\mathbb{A}^\star)^\star = \mathbb{A}$.

To define certain of operator matrices we use the following types of weight functions.

Definition (5.1.4)[260]: A weight is a function $\omega: \mathbb{Z}^d \to [1,\infty)$. A weight is sub multiplicative if

$$\omega(m+n) \le C\omega(m)\omega(n), \quad \text{for some } C > 0.$$

A weight is a GRS-weight if it satisfies the Gel'fand-Raĭkov-Šilov conditiona [286]

$$\lim_{m\to\infty}\omega(mn)^{\frac{1}{m}} = 1, \qquad n \in \mathbb{Z}^d.$$

A weight is balanced if

$$\sup_{n\in\mathbb{Z}^d}\frac{\omega(kn)}{\omega(n)} < \infty, \qquad k \in \mathbb{N}.$$

Finally, an admissible weight is an even sub multiplicative weight.

Example (5.1.5)[260]: A typical weight on $\mathbb{Z}^d$ is given by

$$\omega(n) = e^{a|n|^b}(1+|n|)^s, \qquad a, b, s \ge 0.$$

This weight is admissible when $b \in [0,1]$, is a GRS-weight when $b \in [0,1)$ and is balanced When$b = 0$.

We fix a slant$\alpha \neq 0$. To simplify the notation we use $\beta = \alpha^{-1}$ and $K = \lceil |\beta| \rceil^d$- the dth power of the smallest integer number bigger than or equal to $|\beta|$. By x_S we denote the characteristic function of a set S.

Definition (5.1.6)[260]: For $\alpha \neq 0$ and $j \in \mathbb{Z}^d$ the matrix $A_j = A_j^\alpha = \left(a_{mn}^{(j)}\right)$, $m, n \in \mathbb{Z}^d$, defined by

$$a_{mn}^{(j)} = a_{mn} \prod_{k=1}^{d} \chi_{[j_k, j_k+1)}(\alpha m_k - n_k)$$

is called the $jth\ \alpha\text{-}slant$ of $\mathbb{A}$.

Observe that for every $m \in \mathbb{Z}^d$ there is at most one $n \in \mathbb{Z}^d$ such that $a_{mn}^{(j)} \neq 0$ and at most k different numbers $\ell \in \mathbb{Z}^d$ such that $a_{mn}^{(j)} \neq 0$. Hence, we have $\|A_j\|_P \leq K\|A_j\|_{\sup}$ for any $p \in [1, \infty]$. This allows us to define different classes of matrices with decaying α-slants independently of $p \in [1, \infty]$.

Definition (5.1.7)[260]: We consider the following several classes of matrices.

(i) For some fixed $M \in \mathbb{N}, \mathcal{F}_\alpha^M$ will denote the class of matrices $\mathbb{A}$ that satisfy $\mathbb{A} = \sum_{|j| \leq M-1} A_j$. Observe that for $\mathbb{A} \in \mathcal{F}_\alpha^M$ we have $a_{mn} = 0$ as soon as $|n - \alpha m| > M - 1$. The class $\mathcal{F}_\alpha = \bigcup_{M \in \mathbb{N}} \mathcal{F}_\alpha^M$ consists of operators with finitely many α-slants.

(ii) The class Σ_α^ω of matrices with ω-summable α-slantes consists of matrices $\mathbb{A}$ such that $\|\mathbb{A}\|_{\Sigma_\alpha^\omega} = k \sum_{j \in \mathbb{Z}^d} \|A_j\|_{\sup} \omega(j) < \infty$, where ω is a weight. We have $\Sigma_\alpha^\omega \subset \Sigma_\alpha^1 = \Sigma_\alpha$ -the class of matrices with (unweighted) summable α-slantes.

(iii) The class $\mathcal{E}_\alpha$ of matrices with exponential decay of α-slantes is defined as a subclass of matrices $\mathbb{A}$ from Σ_α such that for some $C \in \mathbb{R}$ and $\tau \in (0,1)$ we have $\|A_j\|_{\Sigma_\alpha} \leq C\tau^{|j|}$.

For $\mathbb{A} \in \Sigma_\alpha^\omega$, we denote by $\mathbb{A}_M \in \mathcal{F}_\alpha^M, M \in \mathbb{N}$, the truncation of $\mathbb{A}$, i.e., the matrix defined by $a_{mn}^M = a_{mn}$ when $|n - \alpha m| \leq M - 1$ and $a_{mn}^M = 0$ otherwise. Equivalently, $\mathbb{A}_M = \sum_{|j| \leq M-1} A_j$ where $A_j, j \in \mathbb{Z}^d$, is the jth α-slant of $\mathbb{A}$. By definition of Σ_α^ω, the operators $\mathbb{A}_M$ converge to $\mathbb{A}$ in the norm $\|\cdot\|_{\Sigma_\alpha^\omega}$.

Here we present some basic properties of slanted matrices that are useful for the reminder.

Lemma (5.1.8)[260]: For some $p \in [1, \infty]$ we consider two operators $\mathbb{A}: \ell^p(\mathbb{Z}^d, (Y_n)) \to \ell^p(\mathbb{Z}^d, (Z_n))$ and $\mathbb{B} : \ell^p(\mathbb{Z}^d, (X_n)) \to \ell^p(\mathbb{Z}^d, (Y_n))$ and let ω be a sub multiplicative balanced weight.

(i) If $\mathbb{A} \in \mathcal{F}_\alpha(\Sigma_\alpha^\omega, or\ \varepsilon_\alpha)$ and $\mathbb{B} \in \mathcal{F}_{\tilde{\alpha}}(\Sigma_{\tilde{\alpha}}^\omega, or\ \varepsilon_{\tilde{\alpha}})$ then we have $\mathbb{AB} \in \mathcal{F}_{\alpha\tilde{\alpha}}(\Sigma_{\alpha\tilde{\alpha}}^\omega, or\ \varepsilon_{\alpha\tilde{\alpha}})$.

If, moreover, $Y_n, Z_n, n \in \mathbb{Z}^d$, are Hilbert spaces, then we have

$$A^\star: \ell^p(\mathbb{Z}^d, (Z_n)) \to \ell^p(\mathbb{Z}^d, (Y_n))$$

and

(ii) $\mathbb{A}$ is inverrible if and only if $A^\star$ is invertible;

(iii) If $\mathbb{A} \in \mathcal{F}_\alpha(\Sigma_\alpha^\omega, or\ \varepsilon_\alpha)$ then $A^\star \in \mathcal{F}_{\alpha^{-1}}\left(\Sigma_{\alpha^{-1}}^\omega, or\ \varepsilon_{\alpha^{-1}}\right)$.

Proof. The last two properties are easily verified by direct computation. For the first one, let $\mathbb{D} = (d_{m,n}) = \mathbb{AB} = (a_{m,n})(b_{m,n})$, and let $\lceil a \rceil = (\lceil a_1 \rceil, \dots, \lceil a_d \rceil) \in \mathbb{Z}^d$, where $a \in \mathbb{R}^d$ and $\lceil a_k \rceil$ is, as before, the smallest integer greater than or equal to $a_k, k = 1, \dots, d$. We have that

$$\begin{aligned}\|d_{m,\lceil \alpha\tilde{\alpha}m\rceil+j}\| &\le \sum_{k\in\mathbb{Z}^d} \|a_{m,k}\| \|b_{k,\lceil \alpha\tilde{\alpha}m\rceil+j}\| \\ &= \sum_{k\in\mathbb{Z}^d} \|a_{m,\lceil \alpha m\rceil+k-\lceil \alpha m\rceil}\| \|b_{k,\lceil \tilde{\alpha}k\rceil+\lceil \alpha\tilde{\alpha}m\rceil+j-\lceil \tilde{\alpha}k\rceil}\| \\ &\le \sum_{k\in\mathbb{Z}^d} r(k-\lceil \alpha m\rceil)s(\lceil \alpha\tilde{\alpha}m\rceil+j-\lceil \tilde{\alpha}k\rceil) \\ &= \sum_{k\in\mathbb{Z}^d} r(k)s(\lceil \alpha\tilde{\alpha}m\rceil+j-\lceil \tilde{\alpha}k+\tilde{\alpha}\lceil \alpha m\rceil\rceil),\end{aligned}$$

Where $r(j) = \|A_j\|_{\sup}$ and $s(j) = \|B_j\|_{\sup}$. For $a, b \in \mathbb{R}$ we have

$$\lceil a\rceil + \lceil b\rceil - 1 \le \lceil a+b\rceil \le \lceil a\rceil + \lceil b\rceil;$$
$$\lceil |a|b\rceil \le \lceil |a|\lceil b\rceil\rceil \le \lceil |a|b\rceil + \lceil |a|\rceil.$$

Hence,

$$\|d_{m,\lceil \alpha\tilde{\alpha}m\rceil+j}\| \le \textstyle\sum_{k\in\mathbb{Z}^d} r(k)\, s(j-\lceil \tilde{\alpha}k\rceil + l),$$

where $l = l(\alpha, \tilde{\alpha}, m, k) \in \mathbb{Z}^d$ is such that $|l| \le \lceil |\tilde{\alpha}|\rceil + 1$.

If $\mathbb{A} \in \mathcal{F}_\alpha$ and $\mathbb{B} \in \mathcal{F}_{\tilde{\alpha}}$, the last inequality immediately implies $\mathbb{D} = \mathbb{AB} \in \mathcal{F}_{\alpha\tilde{\alpha}}$.

If $\mathbb{A} \in \Sigma_\alpha^\omega$ and $\mathbb{B} \in \Sigma_{\tilde{\alpha}}^\omega$, we use the fact that the weight ω is sub multiplicative and balanced to obtain

$$\begin{aligned}\sum_{j\in\mathbb{Z}^d} \sup_{m\in\mathbb{Z}^d} \|d_{m,\lceil \alpha\tilde{\alpha}m\rceil+j}\|\omega(j) &\le \sum_{j,k\in\mathbb{Z}^d} r(k)s(j-\lceil \tilde{\alpha}k\rceil + l)\omega(j) \\ &\le Const \cdot \sum_{j,k\in\mathbb{Z}^d} r(k)s(j-\lceil \tilde{\alpha}k\rceil)\omega(j-\lceil \tilde{\alpha}k\rceil)\frac{\omega(\lceil \tilde{\alpha}k\rceil)}{\omega(k)} \\ &\le Const \cdot \|\mathbb{A}\|_{\Sigma_\alpha^\omega}\|\mathbb{B}\|_{\Sigma_{\tilde{\alpha}}^\omega}.\end{aligned}$$

The case $\mathbb{A} \in \varepsilon_\alpha$ and $\mathbb{B} \in \varepsilon_{\tilde{\alpha}}$ can be treated in a similar way. Since we will not use this result, we omit the proof.

The property of (left, right) invertility of operator matrices in certain operator algebras has been studied extensively by many authors (see [271,287, 295]). The main focus, however, is on a weaker property of boundedness below (or uniform injectivity). As we show matrices with this property play a crucial role in certain applications.

Definitions (5.1.9)[260]: We say that the matrix $\mathbb{A}$ is bounded below in $\mathfrak{L}^p$ or, shorter, $p - bb$, if

$$\|\mathbb{A}x\|_p \ge \wp_p\|x\|_p, \quad \text{for some } \wp_p > 0 \text{ and all } x \in \mathfrak{L}^p. \tag{4}$$

Befor we state the main result, we note an important spectral property of slanted matrices given by the following lemma due to Pfander [297] (see also [298]). We include the proof for completeness and since the matrices considered here are more general.

Lemma (5.1.10)[260]: Assume that $X_n = Y_n, n \in \mathbb{Z}^d$, and that all these spaces are finite-dimensional. If $\mathbb{A} \in \Sigma_\alpha$, for some $\alpha > 1$, then 0 is an approximate eigenvalue of

$\mathbb{A}: \mathfrak{L}^p \to \mathfrak{L}^p, p \in [1, \infty]$. Equivalently, for any $\epsilon > 0$ there exists $x \in \mathfrak{L}^p$ such that $\|x\|_p = 1$ *and* $\|\mathbb{A}x\|_p \leq \epsilon$.

Proof. Let $\mathbb{A} \in \sum_\alpha$. for $\epsilon > 0$ choose M so large that $\|\mathbb{A} - \mathbb{A}_M\|_{\Sigma_\alpha} \leq \epsilon$. Since $\alpha > 1$, there exists N_0 such that $N = \lceil \alpha N_0 \rceil \geq N_0 + 1$. Let $\mathbb{A}_M^N$ be a matrix with an (i, j)-entry coinciding with that of the truncation matrix $\mathbb{A}_M$ if $|i| \leq M + N$, $|j| \leq M + N$, and equal to 0 otherwise. We have $\mathbb{A}_M x_M^N = \mathbb{A}_M^N x_M^N$ for every $x_M^N \in \mathfrak{L}^p$ such that $x_M^N(i) = 0$ for $|i| > M + N$. By assumption, the subspace $\mathcal{X}_M^N$ of such vectors is finite-dimensional and, by construction, it is invariant with respect to $\mathbb{A}_M^N$. Observe that we chose N so large that the restruction of $\mathbb{A}_M^N$ to $\mathcal{X}_M^N$ cannot be invertible because its matrix has a zero "row" Hence, for $\mathbb{A}_M^N$, we can find a vector $\mathcal{X}_M^N \in \mathcal{X}_M^N$ such that $\|x_M^N\| = 1$ and $\mathbb{A}_M x_M^N = \mathbb{A}_M^N x_M^N = 0$. Thus, for any given $\epsilon > 0$, we can find $x_M^N \in \chi$ such that $\|x_M^N\| = 1$, and $\|\mathbb{A}x_M^N\|_p = \|\mathbb{A}x_M^N - \mathbb{A}_M x_M^N\|_p \leq \epsilon$.

We note that without the assumption in Lemma (5.1.10) that $X_n = Y_n$ the lemma may fail. The following Theorem presents our central theoretical result. We observe that it has not been showed before even in the classical case of the slant $\alpha = 1$.

The proof of the Theorem is preceded by several technical lemmas and observations below. We begin with a lemma that provides some insight into the intuition behind the proof. We should also mention that our approach is somewhat similar Sjostrand's proof of a non-commutative Wiener's lemma [299]. We will discuss Wiener-type lemmas in more detail.

Let $\omega^N: \mathbb{R}^d \to \mathbb{R}, N > 1$, be a family of window functions such that $0 \leq \omega^N \leq 1, \omega^N(k) = 0$ for all $|k| \geq N$, and $\omega^N(k) = 1$. By ω_n^N we will denote the translates of ω^N, i.e., $\omega_n^N(t) = \omega^N(t - n)$, and $W_n^N: \ell^p(\mathbb{Z}^d, X) \to \ell^p(\mathbb{Z}^d, X)$ will be the multiplication operator

$$W_n^N x(k) = \omega_n^N(k)x(k), \qquad x \in \mathfrak{L}^p, n \in \mathbb{R}^d.$$

Let $x \in \ell^p(\mathbb{Z}^d, X), p \in [1, \infty]$, and define

$$|||x|||_p^p := \sum_{n \in \mathbb{Z}^d} \sum_{j \in \mathbb{Z}^d} \|W_n^N x(j)\|_p^p = \sum_{n \in \mathbb{Z}^d} \sum_{|j-n|<N} \|W_n^N x(j)\|_p^p, \quad P \in [1, \infty),$$

$$|||x|||_\infty := \sup_n \|W_n^N x\|_\infty.$$

Lemma (5.1.11)[260]: For any $p \in [1, \infty]$, the norms $\|\cdot\|_p$ and $|||\cdot|||_p$ are equivalent norms on $\mathfrak{L}^p$, and we have

$$\|x\|_p \leq |||x|||_p \leq (2N)^{d/p} \|x\|_p, \quad p \in [1, \infty),$$

and

$$\|x\|_\infty = |||x|||_\infty.$$

Proof. For $p = \infty$ the equality is obvious. For $p \in [1, \infty)$, the left-hand side inequality follows from the fact $\|x(n)\|^p \leq \sum_{|j-n| \leq N} \|W_n^N x(j)\|_p^p$, and by summing over n. For the right-hand side inequality we simply note that

$$\sum_{n \in \mathbb{Z}^d} \sum_{|j-n|<N} \|\omega_n^N(j)x(j)\|_P^p \leq \sum_{n \in \mathbb{Z}^d} \sum_{|j|<N} \|x(j+n)\|_p^p \leq (2N)^d \|x\|_p^p.$$

The above equivalence of norms will supply us with the crucial inequality in the proof of the Theorem. The opposite inequality is due to the following observation.

Remark (5.1.12)[260]: We shall make use of the following obvious relation between the norms in finite-dimensional spaces. For every x in a d-dimensional Euclidean space we have

$$\|x\|_p \geq \|x\|_\infty \geq d^{-\frac{1}{p}}\|x\|_p \quad \text{for any } p \in [1, \infty). \tag{5}$$

At this point we choose our window functions to be the family of Cesaro means $\psi^N: \mathbb{R}^d \to \mathbb{R}, N > 1$, defined by

$$\psi^N(k) = \begin{cases} \left(1 - \frac{|k|}{N}\right), & |k| < N; \\ 0, & \text{otherwise.} \end{cases}$$

Observe that their translates $\psi_n^N(k) = \psi^N(k - n), n \in \mathbb{R}^d$, satisfy

$$\psi_{\alpha n}^{\alpha N}(k) = \psi_n^N(\alpha^{-1}k) \tag{6}$$

for any $\alpha > 0$. Again, by $\psi_n^N: \mathfrak{L}^{\mathrm{p}} \to \mathfrak{L}^{\mathrm{p}}, N > 1$, we will denote the operator of multiplication

$$\psi_n^N x(k) = \psi_n^N(k)x(k), \quad x \in \mathfrak{L}^p, n \in \mathbb{R}^d.$$

The following lemma presents yet another estimate crucial for our proof. To simplify the notation we let $\beta = \alpha^{-1}$.

Lemma (5.1.13)[260]: The following estimate holds for any $q \in [1, \infty]$, any $\mathbb{A} \in \Sigma_\alpha = \Sigma_\alpha^1$ and all of its truncations $\mathbb{A}_M \in \mathcal{F}_\alpha^M, M \in \mathbb{N}$.

$$\left\|\mathbb{A}_M\psi_n^N - \psi_{\beta n}^{\beta N}\mathbb{A}_M\right\|_q \leq \frac{(2M)^{d+1}}{2N}\|\mathbb{A}\|_{\sup} =: \aleph/2. \tag{7}$$

Proof. Define $J_k = \{i \in \mathbb{Z}^d : |i - \alpha k| \leq M - 1\}$. Using (6), we have

$$\left|\psi_n^N(i) - \psi_{\beta n}^{\beta N}(k)\right| \leq \frac{M-1}{N}, \quad \text{for } |i - \alpha k| \leq M - 1.$$

Observe that for any $y \in \mathfrak{L}^{\mathrm{q}}$ we have

$$(\mathbb{A}_M\psi_n^N y)(k) = \sum_{i \in J_k} \alpha_{ki}\, \psi_n^N(i)y(i), \quad \left(\psi_{\beta n}^{\beta N}\mathbb{A}_M y\right)(k) = \psi_{\beta n}^{\beta N}(k) \sum_{i \in J_k} a_{ki}\, y(i).$$

Now the following easy computation shows that (7) is true for $q \in [1, \infty)$:

$$\left\|\left(\mathbb{A}_M\psi_n^N - \psi_{\beta n}^{\beta N}\mathbb{A}_M\right)y\right\|_q = \left(\sum_{k \in \mathbb{Z}^d}\left\|\sum_{i \in J_k} a_{ki}\,(\psi_n^N(i) - \psi_{\beta n}^{\beta N}(k)y(i)\right\|^q\right)^{\frac{1}{q}}$$

$$\leq \frac{M}{N}\|\mathbb{A}\|_{\sup}\left(\sum_{k \in \mathbb{Z}^d}\left(\sum_{i \in j_k}\|y(i)\|\right)^q\right)^{\frac{1}{q}} \leq \frac{(2M)^{d+1}}{2N}\|\mathbb{A}\|_{\sup}\|y\|_q.$$

An obvious modification yields it in the case $q = \infty$.

Observe that for $\mathbb{A}_M \in \mathcal{F}_\alpha^M$ the commutator studied in the above lemma satisfies

$$\left(\psi_{\beta n}^{\beta N}\mathbb{A}_M - \mathbb{A}_M\psi_n^N\right)x = \left(\psi_{\beta n}^{\beta N}\mathbb{A}_M - \mathbb{A}_M\psi_n^N\right)P_n^{N+M}x, \tag{8}$$

Where $\beta = \alpha^{-1}$, $P_n^L x(k) = x(k)$ if $|k - n| \leq L$, and $P_n^L x(k) = 0$ otherwise, where $L > 1$. Also observe that for any $p \in [1, \infty]$ and any $L > 1$, we have that

$$\|P_n^L x\|_p \leq 2\|\psi_n^{2L}x\|_p. \tag{9}$$

Combining the above facts we obtain the following estimate.

Lemma (5.1.14)[260]: Let $\mathbb{A} \in \sum_\alpha$, be $p - bb$ for some $p \in [1, \infty]$. As usual, let $\mathbb{A}_M \in \mathcal{F}_\alpha^M$ be the truncations of $\mathbb{A}$ and $\beta = \alpha^{-1}$. Then for all $n \in \mathbb{Z}^n, N > 1$, and all $M \in \mathbb{N}$ with $\gamma_p = \wp_p - \|\mathbb{A} - \mathbb{A}_M\|_p > 0$, we have

$$\|\psi_n^N x\|_p \le \gamma_p^{-1}\left(\left\|\psi_{\beta n}^{\beta N}\mathbb{A}_M x\right\|_p + \aleph\left\|\psi_n^{2(N+M)}x\right\|_p\right). \tag{10}$$

Proof. Observe that

$$\|\psi_n^N x\|_p \le \wp_p^{-1}\big(\|(\mathbb{A}_M\psi_n^N)x\|_p + \|\mathbb{A}-\mathbb{A}_M\|_p\|\psi_n^N x\|_p\big).$$

Hence, using (7), (8), and (9), we get

$$\begin{aligned}\big(1-\wp_p^{-1}\|\mathbb{A}-\mathbb{A}_M\|_p\big)\,\|\psi_n^N x\|_p &\le \wp_p^{-1}\|(\mathbb{A}_M\,\psi_n^N)x\|_p\\ &\le \wp_p^{-1}\left(\left\|\psi_{\beta n}^{\beta N}\mathbb{A}_M x\right\|_p + \left\|\psi_{\beta n}^{\beta N}\mathbb{A}_M - \mathbb{A}_M\psi_n^N)x\right\|_p\right)\\ &\le \wp_p^{-1}\left(\left\|\psi_{\beta n}^{\beta N}\mathbb{A}_M x\right\|_p + \left\|\psi_{\beta n}^{\beta N}\mathbb{A}_M - \mathbb{A}_M\psi_n^N)P_n^{N+M}x\right\|_p\right)\\ &\le \wp_p^{-1}\left(\left\|\psi_{\beta n}^{\beta N}\mathbb{A}_M x\right\|_p + \frac{\aleph}{2}\|P_n^{N+M}x\|_p\right)\\ &\le \wp_p^{-1}\left(\left\|\psi_{\beta n}^{\beta N}\mathbb{A}_M x\right\|_p + \aleph\left\|\psi_n^{2(N+M)}x\right\|_p\right),\end{aligned}$$

which yields the desired inequality.

By iterating (10) $j-1$ times we get

Lemma (5.1.15)[260]: Let $\mathbb{A}\in\sum_\alpha$, be $p-bb$ some $p\in[1,\infty]$. Let $\mathbb{A}_M\in\mathcal{F}_\alpha^M$ be $the\ truncations$ of $\mathbb{A}$ and $\beta=\alpha^{-1}$. Then for all $n\in\mathbb{Z}^n, N>1$ and $M\in\mathbb{N}$ with $\gamma_p = \wp_p - \|\mathbb{A}-\mathbb{A}_M\|_p > 0$, we have

$$\|\psi_n^N x\|_p \le \gamma_p^{-1}\frac{1-\left(\aleph\gamma_p^{-1}\right)^j}{1-\left(\aleph\gamma_p^{-1}\right)}\left\|\psi_{\beta n}^{\beta Z_j}\mathbb{A}_M x\right\|_p + (\aleph\gamma_P^{-1})^j\left\|\psi_n^{Z_{j+1}}x\right\|_p, \tag{11}$$

$where\ Z_j = 2^{j-1}N + \big(2^j-2\big)M,$ for $j\ge 1$.

To simplify the use of (11) we let

$$a_{j,p} := \gamma_p^{-1}\frac{1-\left(\aleph\gamma_p^{-1}\right)^j}{1-\left(\aleph\gamma_p^{-1}\right)} = \frac{1-\left(\dfrac{(2M)^{d+1}\|\mathbb{A}\|_{\sup}}{(\wp_p-\|\mathbb{A}-\mathbb{A}_M\|_p)N}\right)^j}{\wp_p - \|\mathbb{A}-\mathbb{A}_M\|_p - \dfrac{(2M)^{d+1}}{N}\|\mathbb{A}\|_{\sup}} \tag{12}$$

and

$$b_{j,p} := \left(\aleph\gamma_p^{-1}\right)^j = \frac{\left((2M)^{d+1}\|\mathbb{A}\|_{\sup}\right)^j}{\left(\wp_p - \|\mathbb{A}-\mathbb{A}_M\|_p\right)^j N^j}. \tag{13}$$

Now we complete the proof of the main result.

Theorem (5.1.16)[260]: Let $s>(d+1)^2$ and $\omega=(1+|j|)^s$. Then $\mathbb{A}\in\Sigma_\alpha^\omega$ is $p-bb$ for some $p\in[1,\infty]$ if and only if $\mathbb{A}$ is q-bb for all $q\in[1,\infty]$.

Proof. The remainder of the proof will be presented in two major steps. In the first step, we will assume that $\mathbb{A}\in\Sigma_\alpha^\omega$ is ∞-bb and show that this implies that $\mathbb{A}$ is p-bb for all $p\in[1,\infty)$. In the second step we will do the "opposite" that is, assume that $\mathbb{A}\in\Sigma_\alpha^\omega$ is p-bb for some $p\in[1,\infty)$ and show that this implies that $\mathbb{A}$ ∞-bb. This would obviously be enough to complete the proof.

Step 1. Assume that $\mathbb{A}$ is ∞-bb. Using Hölder's inequality and (11), we get for large values of $M\in\mathbb{N}$

$$\|\psi_n^N x\|_\infty^p \le 2^{p-1}a_{j,\infty}^p\left\|\psi_{\beta n}^{\beta Z_j}\mathbb{A}_M x\right\|_\infty^p + 2^{p-1}b_{j,\infty}^p\left\|\psi_n^{Z_{j+1}}x\right\|_\infty^p.$$

Using (5), we get

$$(2N)^{-d}\|\psi_n^N x\|_p^p \le 2^{p-1}a_{j,\infty}^p \left\|\psi_{\beta n}^{\beta Z_j}\mathbb{A}_M x\right\|_p^p + 2^{p-1}b_{j,\infty}^p \left\|\psi_n^{Z_{j+1}}x\right\|_p^p.$$

Summing over n and using Lemma (5.1.11), we get

$$\begin{aligned}\|x\|_p^p &\le (2N)^d 2^{p-1}a_{j,\infty}^p\left(2Z_j\right)^d\|\mathbb{A}_M x\|_p^p + (2N)^d 2^{p-1}b_{j,\infty}^p\left(2Z_{j+1}\right)^d\|x\|_p^p \\ &\le N^d 2^{d+p-1}a_{j,\infty}^p Z_j^d\left(\|\mathbb{A}x\|_p^p + \|\mathbb{A}-\mathbb{A}_M\|_p^p\|x\|_p^p\right) \\ &+N^d 2^{d+p-1}b_{j,\infty}^p Z_{j+1}^d\|x\|_p^p. \qquad (14)\end{aligned}$$

At this point we use the assumption $\mathbb{A}\in\Sigma_\alpha^{(1+|j|)^s}$ to get

$$\begin{aligned}\|\mathbb{A}-\mathbb{A}_M\|_p &\le \left\|\sum_{|j|\ge M} A_j\right\|_p \le K\sum_{|j|\ge M}\left\|A_j\right\|_{\sup}(1+|j|)^s(1+|j|)^{-s} \\ &\le \|A\|_{\Sigma_\alpha^{(1+|j|)^s}}\sup_{|j|\ge M}(1+|j|)^{-s} \le \|\mathbb{A}\|_{\Sigma_\alpha^{(1+|j|)^s}}M^{-s}.\end{aligned}$$

Plugging the above estimate into (14) we obtain

$$\begin{aligned}\|x\|_p^p &\le 2^{2d+p-1}N^d a_{j,\infty}^p Z_j^d\|\mathbb{A}x\|_p^p \\ &+ 2^{2d+p-1}N^d\left(a_{j,\infty}^p Z_j^d\|\mathbb{A}\|_{\Sigma_\alpha^{(1+|j|)^s}}^p M^{-sp} + b_{j,\infty}^p Z_{j+1}^d\right)\|x\|_p^p \\ &= 2^{2d+p-1}a_{j,\infty}^p N^d Z_j^d\|\mathbb{A}x\|_p^p + \widetilde{\aleph}\|x\|_p^p. \qquad (15)\end{aligned}$$

Hence, to complete Step 1 it suffices to show that one can choose $j, M\in\mathbb{N}$ and $N>1$ so that $\widetilde{\aleph}<1$.

We put $N=M^{\delta(d+1)}$ for some $\delta>1$. From (7),(12),(13), and the definition of Z_j in lemma (5.1.11), we get $\aleph=\mathcal{O}\left(M^{(1-\delta)(d+1)}, b_{j,\infty}\right)=\mathcal{O}\left(M^{(1-\delta)(d+1)j}\right)$, $a_{j,\infty}=\mathcal{O}(1)$, and $Z_j=\mathcal{O}\left(M^{\delta(d+1)}\right)$ as $M\to\infty$. Hence,

$$\widetilde{\aleph}\le C_1 M^{\delta(d+1)^2-sp} + C_2 M^{(1-\delta)(d+1)jp+\delta(d+1)^2},$$

where the constants C_1 and C_2 depend on $\mathbb{A}, s, j$, and p but do not depend on M. Since $s>(d+1)^2$, we can choose $\delta\in\left(1,\frac{sp}{(d+1)^2}\right)$ and $j>\frac{\delta(d+1)}{p(\delta-1)}$ then, clearly, $\widetilde{\aleph}=\mathcal{O}(1)$ as $M\to\infty$.

Step 2. Now assume that $\mathbb{A}$ is p-bb, for some $p\in[1,\infty)$. Using (5) and (11), we get

$$\|\psi_n^N\|_\infty \le a_{j,p}\left(2Z_j\right)^{d/p}\left\|\psi_{\beta_n}^{\beta Z_j}\mathbb{A}_M x\right\|_\infty + b_{j,p}\left(2Z_{j+1}\right)^{d/p}\left\|\psi_n^{Z_{j+1}}x\right\|_\infty.$$

As in Step 1, we have $\|\mathbb{A}-\mathbb{A}_M\|_p\le\|A\|_{\Sigma_\alpha^{(1+|j|)^s}}M^{-s}$. Using this estimate and Lemma (5.1.11), we obtain

$$\|x\|_\infty \le a_{j,p}\left(2Z_j\right)^{d/p}\|\mathbb{A}x\|_\infty + 2^{d/p}\left(a_{j,p}Z_j^{d/p}\|A\|_{\Sigma_\alpha^{(1+|j|)^s}}M^{-s} + b_{j,p}Z_{j+1}^{d/p}\right)\|x\|_\infty.$$

Again, as in the previous step, if we choose $\delta\in\left(1,\frac{sp}{(d+1)^2}\right), N=M^{\delta(d+1)}$, and $j>\frac{\delta(d+1)}{p(\delta-1)}$, we get

$$a_{j,p}Z_j^{d/p}\|A\|_{\Sigma_\alpha^{(1+|j|)^s}}M^{-s} + b_{j,p}Z_{j+1}^{d/p} = \mathcal{O}(1)$$

as $M\to\infty$ and the proof is complete.

Careful examination of (15) yields the following result.

Corollary (5.1.17)[260]: let $s>(d+1)^2, \omega=(1+|j|)^s$, and $\mathbb{A}\in\Sigma_\alpha^\omega$ be p-bb for some $p\in[1,\infty]$. Then there exists $\wp>0$ such that for all $q\in[1,\infty]$

$$\|\mathbb{A}x\|_q \geq \wp\|x\|_q, \quad \text{for all } x \in \mathfrak{L}^q.$$

As we have seen in the proof above, the group structure of the index set $\mathbb{Z}^d$ has not been used.Thus, it is natural to conjecture that a similar result holds for matrices indexed by much more general (discrete) metric spaces. We do not pursue this extension. Instead, we show the result for a class of matrices that define operators of $bounded\ flow$.

Definition (5.1.18)[260]: A matrix $\mathbb{A}$ is said to have bounded dispersion if there exists $M \in \mathbb{N}$ such that for every $m \in \mathbb{Z}^d$ there exists $n_m \in \mathbb{Z}^d$ for which $a_{mn} = 0$ as soon as $|n - n_m| > M$. A matrix $\mathbb{A}$ is said to have bounded accumulation if $\mathbb{A}^\star$ has bounded dispersion. Finally, $\mathbb{A}$ is a bounded flow matrix if it has both bounded dispersion and bounded accumulation.

Corollary (5.1.19)[260]: Assume that $\mathbb{A}$ has bounded flow and is p-bb for some $p \in [1, \infty]$. Then $\mathbb{A}$ is $q - bb$ for all $q \in [1, \infty]$.

Proof. In lieu of the proof it is enough to make the following two observations. First, if a matrix is bounded below then any matrix obtained from the original one by permuting its rows (or columns) is also bounded below with the same bound. Second, if a matrix is bounded below then any matrix obtained from the original one by inserting any number of rows consisting entirely of 0 entries is also bounded below with the same bound. Using these observations we can use row permutations and insertions of zero rows to obtain a slanted matrix in $\mathcal{F}_\alpha^M$ for some $\alpha \in \mathbb{R}, |\alpha| > 0$.

Theorem (5.1.20)[260]: Let $X_n, Y_n, n \in \mathbb{Z}^d$, be Hilbert spaces and ω be an admissible balanced GRS-weight. If $\mathbb{A} \in \Sigma_\alpha^\omega$ is invertible for some $p \in [1, \infty]$, then $\mathbb{A}$ is invertible for all $q \in [1, \infty]$ and $\mathbb{A}^{-1} \in \Sigma_{\alpha^{-1}}^\omega$. Moreover, if $\mathbb{A} \in \mathcal{E}_\alpha$, then we also have $\mathbb{A}^{-1} \in \mathcal{E}_{\alpha^{-1}}$.

Proof. First, we observe that $\mathbb{A}^{-1} = (\mathbb{A}^\star\mathbb{A})^{-1}\mathbb{A}^\star$. Second, since Lemma (5.1.8) implies $\mathbb{A}^\star\mathbb{A} \in \Sigma_1^\omega$ (or ε_1), [272, Theorem 2] guarantees that $(\mathbb{A}^\star\mathbb{A})^{-1} \in \Sigma_1^\omega$(or ε_1). Finally, applying Lemma (5.1.8) once again we get the desired results.

Theorem (5.1.21)[260]: Let $X_n = \mathcal{H}_X$ and $Y_n = \mathcal{H}_Y$ be the same Hilbert (or Euclidean) spaces for all $n \in \mathbb{Z}^d$ and $\mathbb{A} \in \Sigma_\alpha^\omega$ where $\omega(j) = (1 + |j|)^s, s > (d + 1)^2$. Let also $p \in [1, \infty]$.

(i) If $\mathbb{A}$ is p-bb, then $\mathbb{A}$ is left invertible for all $q \in [1, \infty]$ and a left inverse is given by $\mathbb{A}^\# = (\mathbb{A}^\star\mathbb{A})^{-1}\mathbb{A}^\star \in \Sigma_{\alpha^{-1}}^\omega$.

(ii)If $\mathbb{A}^\star$ is p-bb, then $\mathbb{A}$ is right invertible for all $q \in [1, \infty]$ and a right inverse is given by $\mathbb{A}^b = \mathbb{A}^\star(\mathbb{A}\mathbb{A}^\star)^{-1} \in \Sigma_{\alpha^{-1}}^\omega$.

Proof. Since (i) and (ii) are equivalent, we prove only (i). Theorem (5.1.16) implies that $\|\mathbb{A}x\|_2 \geq \wp_2\|x\|_2$ for some $\wp_2 > 0$ and all $x \in \mathfrak{L}^2$. Under the specified conditions the Banach spaces $\ell^2(\mathbb{Z}^d, (\mathrm{X}_n))$ and $\ell^2(\mathbb{Z}^d, (Y_n))$ are, however, Hilbert spaces and $\mathbb{A}^\star$ defines the Hilbert a djoint of $\mathbb{A}$. Since $\langle\mathbb{A}^\star\mathbb{A}x, x\rangle = \langle\mathbb{A}x, \mathbb{A}x\rangle \geq \wp_2\langle x, x\rangle$, we have that the operator $\mathbb{A}^\star\mathbb{A}$ is invertible in $\mathfrak{L}^2$. It remains to argue as in (5.1.20) and apply Lemma (5.1.8) and [272, Theorem 2 and Corolary 3].

Corollary (5.1.22)[260]: If $\mathbb{A}$ is as in Theorem (5.1.21)(i) then, for any $q \in [1, \infty]$, Im $\mathbb{A}$ is a subspace of $\mathfrak{L}^q$ that can be complemented.

We will address several fundamental questions. Given a sampling set for some $p \in [1, \infty]$ can we deduce that this set is a set of sampling for all p? Under which condition is a p-frame for some $p \in [1, \infty]$ also a Banach frame for all p ? These and a few other questions are discussed and an answer in terms of slanted matrices is presented.

The first part concerns Banach frames and the second one concerns sampling theory.

The notion of a frame in a separable Hilbert space has already become classical. The pioneering work [282] explicitly introducing it was published in 1952. It analoues in Banach spaces, however, are non-trivial (see [265,268,269, 280,285,290]). We show that in the case of certain localized frames the simplest possible extension of the definition remains meaningful.

Definition (5.1.23)[260]: Let $\mathcal{H}$ be a separable Hilbert space. A sequence $\varphi_n \in \mathcal{H}, n \in \mathbb{Z}^d$, is a frame for $\mathcal{H}$ if for some $0 < a \le b < \infty$

$$a\|f\|^2 \le \sum_{n\in\mathbb{Z}^d} |\langle f, \varphi_n\rangle|^2 \le b\|f\|^2 \qquad (16)$$

for all $f \in \mathcal{H}$.

The operator $T: \mathcal{H} \to \ell^2,\ f = \{\langle f, \varphi_n\rangle\}_{n\in\mathbb{Z}^d}, f \in \mathcal{H}$, is called an analysis operator. It is an easy exercise to show that a sequence $\varphi_n \in \mathcal{H}$ is a frame for $\mathcal{H}$ for if and only if its analysis operator has a left inverse. The adjoint of the analysis operator $T^*: \ell^2 \to \mathcal{H}$, is given by $T^*c = \sum_{n\in\mathbb{Z}^d} c_n\varphi_n$, $c = (c_n) \in \ell^2$. The frame operator is $T^*T: \mathcal{H} \to \mathcal{H}$, $T^*Tf = \sum_{n\in\mathbb{Z}^d}\langle f, \varphi_n\rangle\varphi_n, f \in \mathcal{H}$. A gain, a sequence $\varphi_n \in \mathcal{H}$ is a frame for $\mathcal{H}$ if and only if its frame operator is invertible. The canonical dual frame $\tilde{\varphi}_n \in \mathcal{H}$ is then $\tilde{\varphi}_n = (T^*T)^{-1}\varphi_n$ and the (canonical) synthesis operator is $T^\#: \ell^2 \to H$, $T^\# = (T^*T)^{-1}T^*$, so that

$$f = T^\#Tf = \sum_{n\in\mathbb{Z}^d}\langle f, \varphi_n\rangle\tilde{\varphi}_n = \sum_{n\in\mathbb{Z}^d}\langle f, \tilde{\varphi}_n\rangle\varphi_n$$

for all $f \in \mathcal{H}$.

Generalizing the notion of frames to Banach spaces requires some care. In general Banach spaces one cannot use just the equivalence of norms similar to (16). The above construction breaks down because, in this case, the analysis operator ends up being bounded below and not necessarily left invertible. As a result a "frame decomposition" remains possible but "frame reconstruction" no longer makes sense. Theorem (5.1.21)(i) indicates, however, that often this obstruction does not exist. The idea is to make the previous statement precise. To simplify the exposition we remain in the realm of Banach spaces $\ell^p(\mathbb{Z}^d, \mathcal{H})$ and use other chains of spaces such as the one in [290] only implicitly.

Definition (5.1.24)[260]: A sequence $\varphi^n = (\varphi^n_m)_{m\in\mathbb{Z}^d} \in \ell^1(\mathbb{Z}^d, \mathcal{H}), n \in \mathbb{Z}^d$, is a p-frame (for $\ell^p(\mathbb{Z}^d, \mathcal{H})$) for some $p \in [1, \infty)$ if

$$a\|f\|^p \le \sum_{n\in\mathbb{Z}^d}\left|\sum_{m\in\mathbb{Z}^d}\langle f_m, \varphi^n_m\rangle\right|^p \le b\|f\|^p \qquad (17)$$

for some $0 < a \le b < \infty$ and all $f = (f_m)_{m\in\mathbb{Z}^d} \in \ell^p(\mathbb{Z}^d, \mathcal{H})$.if

$$a\|f\| \le \sup_{n\in\mathbb{Z}^d}\left|\sum_{m\in\mathbb{Z}^d}\langle f_m, \varphi^n_m\rangle\right| \le b\|f\| \qquad (18)$$

for some $0 < a \le b < \infty$ and all $f = (f_m)_{m\in\mathbb{Z}^d} \in \ell^\infty(\mathbb{Z}^d, \mathcal{H})$, then the sequence φ^n is called an ∞-frame. It is called a 0-frame if (18) holds for all $f \in c_0(\mathbb{Z}^d, \mathcal{H})$.

The definition of p-frame above is consistent with the standard one found [265,281,288]. For example, if $\mathcal{H} = \mathbb{C}$ we obtain the standard definition of p-frames for $\ell^p(\mathbb{Z}^d)$.

The operator $T_\varphi = T: \ell^p(\mathbb{Z}^d, \mathcal{H}) \to \ell^p(\mathbb{Z}^d) = \ell^p(\mathbb{Z}^d, \mathbb{C})$, given by

$$Tf = \langle f, \varphi_n \rangle := \left\{ \sum_{m\in\mathbb{Z}^d} \langle f_m, \varphi_m^n \rangle \right\}_{n\in\mathbb{Z}^d}, \quad f \in \ell^p(\mathbb{Z}^d, \mathcal{H}),$$

is called a p-analysis operator, $p \in [1, \infty]$. The 0-analysis operator is defined in the same way for $f \in c_0(\mathbb{Z}^d, \mathcal{H})$.

Definition (5.1.25)[260]: A p-frame φ^n with the p-analysis operator T, $p \in \{0\} \cup [1, \infty]$, is (s, α)-localized for some $s > 1$ and $\alpha \neq 0$, if there exists an isomorphism $J: \ell^\infty(\mathbb{Z}^d, \mathcal{H}) \to \ell^\infty(\mathbb{Z}^d, \mathcal{H})$ which leaves invariant c_0 and all $\ell^q(\mathbb{Z}^d, \mathcal{H}), q \in [1, \infty)$, and such that

$$TJ_{|\ell^p} \in \Sigma_\alpha^\omega,$$

where $\omega(n) = (1 + |n|)^s, n \in \mathbb{Z}^d$, see Remark (5.1.11).

As a direct corollary of Theorem (5.1.21) and the above definition we obtain the following result.

Theorem (5.1.26)[260]: Let $\varphi^n, n \in \mathbb{Z}^d$, be an (s, α)-localized p-frame for some $p \in \{0\} \cup [1, \infty]$ with $s > (d + 1)^2$. Then

(i) The q-analysis operator T is well defined and left invertible for all $q \in \{0\} \cup [1, \infty]$, and the q-synthesis operator $T^\# = (T^*T)^{-1}T^*$ is also well defined for all $q \in \{0\} \cup [1, \infty]$.

(ii) The sequence $\varphi^n, n \in \mathbb{Z}^d$, and its dual sequence $\tilde{\varphi}^n = (T^*T)^{-1}\varphi^n, n \in \mathbb{Z}^d$, are both (s, α)-localized q-frames for all $q \in \{0\} \cup [1, \infty]$.

(iii) *In* c_0 and $\ell^q(\mathbb{Z}^d, \mathcal{H}), q \in [1, \infty)$, we have the reconstruction formula

$$f = T^\# Tf = \sum_{n\in\mathbb{Z}^d} \langle f, \varphi_n \rangle \tilde{\varphi}_n = \sum_{n\in\mathbb{Z}^d} \langle f, \tilde{\varphi}_n \rangle \varphi_n.$$

For $f \in \ell^\infty(\mathbb{Z}^d, \mathcal{H})$ the reconstruction formula remains valid provided the convergence is understood in the weak*-topology.

Theorem (5.1.26)(iii) shows that an (s, α)-localized p-frame is a Banach frame for c_0 and all $\ell^q(\mathbb{Z}^d, \mathcal{H}), q \in [1, \infty]$, in the sense of the following definition.

Definition (5.1.27)[260]: (see [289, Definition 13.6.1]) . A countable sequence $\{x_n\}_{x_n\in J} \subset X'$ in the dual of a Banach space X is a Banach frame for X if there exist an associated sequence space $X_d(J)$, a constant $C \ge 1$, and a bounded operator $R: X_d \to X$ such that for all $f \in X$

$$\frac{1}{C}\|f\|_X \le \|\langle f, x_n \rangle\|_{X_d} \le C\|f\|_X,$$

$$R(\langle f, x_n \rangle_{j\in J}) = f.$$

Example (5.1.28)[260]: Following [289], let $g \in S \subset C^\infty(\mathbb{R}^d)$ be a non-zero window function in the Schwartz class $\mathcal{S}$, and V_g be the short time Fourier transform

$$\left(V_g f\right)(x,\omega) = \int_{\mathbb{R}^{2d}} f(t)\overline{g(t-x)}e^{-2\pi i t\cdot\omega}dt\,, \quad x,\omega \in \mathbb{R}^d.$$

let M^p,$1 \le p \le \infty$, be the modulation spaces of tempered distributions with the norms

$$\|f\|_{M^p} = \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\left|V_g f(x,\omega)\right|^p dx\right)d\omega\right)^{1/p}, \qquad 1 \le p < \infty,$$

$$\|f\|_{M^\infty} = \left\|V_g f\right\|_\infty.$$

It is known that these modulation spaces do not depend on the choice of $g \in S$ and are isomorphic to $\ell^p(\mathbb{Z}^{2d})$, with isomorphism's provided by the Wilson bases.

Let $g \in M^1$ be a window such that Gabor system

$$\mathcal{G}(g,a,b) = \left\{g_{k,n}(x) = e^{-2\pi i(x-ak).bn}g(x-ak),\ \ k,n \in \mathbb{Z}^d,\ \ x \in \mathbb{R}^d\right\}$$

is a tight Banach frame for all M^p, $1 \le p \le \infty$. By this we mean that the p-analysis operator $T_{\mathcal{G}}: M^p \to \ell^p, T_{\mathcal{G}}f = \left\{\langle f, g_{k,n}\rangle\right\}$, is left invertible and the frame operator $T_{\mathcal{G}}^* T_{\mathcal{G}}$ is a scalar multiple of the identity operator for all $p \in [1,\infty]$. Assume that a sequence $\Phi = \left\{\phi_{i,j}\right\}_{i,j\in\mathbb{Z}^d}$ of distributions in M^∞ is such that $\left\{\varphi^{(i,j)} = T_{\mathcal{G}}\emptyset_{i,j}\right\}, (i,j) \in \mathbb{Z}^{2d}$, is an (s,α)-localized p-frame for some $p \in \{0\} \cup [1,\infty]$, and $s > (d+1)^2$. Since, by Definition (5.1.24), $\left\{\varphi^{(i,j)} = T_{\mathcal{G}}\emptyset_{i,j}\right\}$ must be in $\ell^1(\mathbb{Z}^{2d},\mathbb{C})$, then by [289, Corollary 12.2.8] $\Phi \subset M^1$. Moreover, by Theorem (5.1.26) we have that $\left\{\varphi^{(i,j)}\right\}$ is an (s,α)-localized q-frame for all $q \in \{0\} \cup [1,\infty]$, and a Banch frame. Finally, since $\mathcal{G}$ is a tight Banach frame for all $M^q, q \in [1,\infty]$, we have that

$$\langle f, \phi_{i,j}\rangle = Const\langle T_{\mathcal{G}}^* T_{\mathcal{G}} f, \phi_{i,j}\rangle = Const\langle T_{\mathcal{G}} f, T_{\mathcal{G}}\phi_{i,j}\rangle, \text{for all } f \in M^q,$$

and, hence, the frame operator

$$f \mapsto T_{\mathcal{G}} f \mapsto \left\{\langle T_{\mathcal{G}} f, T_{\mathcal{G}}\phi_{i,j}\rangle\right\} \mapsto \left\{\langle f, \phi_{i,j}\rangle\right\}: M^q \to \ell^q(\mathbb{Z}^{2d},\mathbb{C})$$

is left invrtible and, therefore, Φ is a Banach frame for all $M^q, q \in [1,\infty]$.

Example (5.1.29)[260]: Here we would like to highlight the role of the slant α in the previous example. Using the same notation as above, let Φ be the frame consisting of two copies of the frame $\mathcal{G}$. then (renumbering Φ if needed) it is easy to see that matrix $\left(\langle \phi_{i,j}\ , g_{k,n}\rangle\right)_{(i,j),(k,n)\in\mathbb{Z}^{2d}}$ is $\frac{1}{2}$-slanted. Hence, the slant α serves as a measure of relative redundancy of Φ with respect to $\mathcal{G}$ and a measure of absolute redundancy of Φ if $\mathcal{G}$ is a basis.

We apply the previous results to handle certain problems in sampling theory. Theorem (5.1.32) below was the principal motivation for us to show Theorem (5.1.16).

The sampling/reconstruction problem includes devising efficient methods for representing a signal (function) in terms of a discrete (finite or countable) set of its samples (values) and reconstructing the original signal from its samples. We assume that the signal is a function f that belongs to space

$$V^p(\Phi) = \left\{\sum_{k\in\mathbb{Z}^d} c_k\varphi_k\right\},$$

where $c = (c_k) \in \ell^p(\mathbb{Z}^d)$ when $p \in [1,\infty], c \in c_0$ when $p = 0$, and $\Phi = \{\varphi_k\}_{k\in\mathbb{Z}^d} \subset L^p(\mathbb{R}^d)$ is a countable collection of continuous functions. To avoid convergence issues in the definition of $V^p(\Phi)$, we assume that the functions in Φ satisfy the condition

$$m_p\|c\|_{\ell^p} \le \left\|\sum_{k\in\mathbb{Z}^d} c_k\varphi_k\right\|_{L^p} \le M_p\|c\|_{\ell^p}, \qquad \text{for all } c \in \ell^p, \tag{19}$$

for some $m_p, M_p > 0$ independent of c. This is a p-Riesz basis condition for $p \in [1,\infty) \cup \{0\}$[260]. We assume that the functions in Φ belong to a Wiener-amalgam space W^1_ω defined as follows.

Definitions (5.1.30)[260]: A measurable function φ belongs to W^1_ω for a certain weight, if it satisfies

$$\|\varphi\|_{W^1_\omega} = \left(\sum_{k\in\mathbb{Z}^d} \omega(k)\cdot \operatorname{ess\,sup}\{|\varphi|(x+k): x \in [0,1]^d\}\right) < \infty. \tag{20}$$

When a function φ in W^1_ω is continuous we write $\varphi \in W^1_{0,\omega}$. In many applications $V^p(\Phi)$ is a shift invariant space, that is, $\varphi_k(x) = \varphi(x-k), k \in \mathbb{Z}^d$, for some $\varphi \in W^1_\omega$.

Sampling is assumed to be performed by a countable collection of finite complex Borel measures $\mu = \{\mu_j\}_{j\in\mathbb{Z}^d} \subset \mathcal{M}(\mathbb{R}^d)$. A μ-sample is a sequence $f(\mu) = \int f\, d\mu_j, j \in \mathbb{Z}^d$. If $f(\mu) \in \ell^p$ and $\|f(\mu)\|_{\ell^p} \le C\|f\|_{L^p}$ for all $f \in V^p(\Phi)$, we say that μ is a (Φ,p)-sampler. If a sampler μ is a collection of Dirac measures then it is called a (Φ,p)-ideal sampler. Other-wise, it is a (Φ,p)-average sampler.

We determine when a sampler μ is stable. That is, when f is uniquely determined by its μ-sample and a small perturbation of the sampler results in a small perturbation of $f \in V^p(\Phi)$. The above condition can be formulated as follows [263].

Definition (5.1.31)[260]: A sampler μ is stable on $V^p(\Phi)$ (in other words, μ is a stable(Φ,p)-sampler) if the bi-infinite matrix $\mathbb{A}^\Phi_\mu$ defined by

$$\left(\mathbb{A}^\Phi_\mu c\right)(j) = \sum_{k\in\mathbb{Z}^d} \int c_k\varphi_k d\mu_j,\ c \in \ell^p(\mathbb{Z}),$$

defines a bounded sampling operator $\mathbb{A}^\Phi_\mu: \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ which is bounded below In ℓ^p (or p-bb).

We assume that the generator Φ and the sampler μ are such that the operator $\mathbb{A}^\Phi_\mu$ is a bounded on c_0 and all $\ell^p, p \in [1,\infty]$; we say that such sampling system (Φ,μ) is sparse. This situation happens, for example, when the generator Φ has sufficient decay at ∞ and the sampler is separated. The following Theorem is a direct corollary of Theorem (5.1.21) and the above definitions.

Theorem (5.1.32)[260]: Assume that $\omega(n) = (1+|n|)^n, n \in \mathbb{Z}^d, s > (d+1)^2, \Phi$ satisfies (19), for all $q \in \{0\} \cup [1,\infty]$ and μ is (Φ,p)-sampler for every $p \in [1,\infty]$. Assume also that the sampling operator $\mathbb{A}^\Phi_\mu$ is p-bb for some $p \in \{0\} \cup [1,\infty]$ and $\mathbb{A}^\Phi_\mu \in \Sigma^\omega_\alpha$ for some $\alpha \ne 0$. Then μ is a stable sampler on $V^q(\Phi)$ for every $q \in \{0\} \cup [1,\infty]$.

Below we study the case of ideal sampling in shift invariant spaces in greater detail and obtain specific examples of the above Theorem. From now on we assume that $\varphi_k(x) = \varphi_k(x-k), k \in \mathbb{Z}^d$, for some $\varphi \in C \cap W^1_\omega =: W^1_{0,\omega}$.

Definition (5.1.33)[260]: If $\mu = (\mu_i)$ is a stable ideal sampler on $V^p(\Phi)$ and the measures μ_i are supported on $\{x_j\}, j \in \mathbb{Z}^d$, then set $X = \{x_j, j \in \mathbb{Z}^d\}$ is called a (stable) set of sampling on $V^p(\Phi)$. A set of sampling $X \subset \mathbb{R}^d$ is separated if

$$\inf_{j \neq k \in \mathbb{Z}^d} |x_j - x_k| = \delta > 0.$$

A set of sampling $X \subset \mathbb{R}^d$ is homogeneous if

$$\#\{X \cap [n, n+1)\} = M$$

is constant for every $n \in \mathbb{Z}^d$.

We are interested in the homogeneous sets of sampling because of the following result.

Lemma (5.1.34)[260]: Let $\varphi \in W_{0,\omega}^1, \Phi = \{\varphi(\cdot - k)\}$, and $\mu \in \ell^\infty(\mathbb{Z}^d, \mathcal{M}(\mathbb{R}^d))$ be an ideal sampler with a separated homogenous sampling set X. Then the sampling operator $\mathbb{A}_\mu^\Phi$ belongs to Σ_α^ω for $\alpha = M^{-1}$.

Proof. Follows by direct computation.

The following lemma shows that we can restrict our attention to homogeneous sets of sampling without any loss of generality. The intuition behind this result is that we can count each measurement at a point in X not once but finitely many times and still obtain unique and stable reconstructions.

Lemma (5.1.35)[260]: Let $\mathbb{A}$ be an infinite matrix that defines a bounded operator on $\ell^p, p \in [1, \infty]$, and $\mathbb{A}$ be a (bounded) operator on ℓ^p obtained from $\mathbb{A}$ by duplicating each row at most M times. Then $\mathbb{A}$ is p-bb if and only if $\widetilde{\mathbb{A}}$ is p-bb.

Proof. The proof for $p < \infty$ follows from the inequalities

$$\|\mathbb{A}x\|_p^p \leq \|\widetilde{\mathbb{A}}x\|_p^p \leq (M+1)\|\mathbb{A}x\|_p^p, \qquad x \in \ell^p.$$

For $p = \infty$, we have $\|\mathbb{A}x\|_\infty = \|\widetilde{\mathbb{A}}x\|_\infty$, $x \in \ell^\infty$.

As a direct corollary of Theorems (5.1.21), (5.1.32), Lemmas (5.1.34), (5.1.35), and Remark (5.1.10) we obtain the following Theorem.

Theorem (5.1.36)[260]: Let $\omega(n) = (1 + |n|)^s, n \in \mathbb{Z}^d, s > (d+1)^2, \varphi \in W_{0,\omega}^1$, and

$$a_p \|f\|_{L^p} \leq \|\{f(x_j)\}\|_{\ell^p} \leq b_p \|f\|_{L^p}, \qquad \text{for all } f \in V^p(\Phi),$$

for some $p \in [1, \infty] \cup \{0\}$ and a separated set $X = \{x_j, j \in \mathbb{Z}^d\}$. Then X is a stable set of sampling on $V^q(\Phi)$ for all $q \in [1, \infty] \cup \{0\}$.

Now we can show a Beurling-Landau type Theorem [1-4] for shift-invariant spaces generated by piecewise differentiable functions.

Theorem (5.1.37)[260]: Let Φ be a sequence generated by the translates of a piecewise differentiable function $\varphi \in W_{0,\omega}^1$ such that

$a\|c\|_\infty \leq \|\sum_{k \in \mathbb{Z}} c_k \varphi_k\|_\infty \leq b\|c\|_\infty$ and $\|\sum_{k \in \mathbb{Z}} c_k \varphi_k'\|_\infty \leq b'\|c\|_\infty$.

for all $c \in c_0(\mathbb{Z}^d)$. Then every $X = \{x_j\}$ that satisfies $\gamma(X) = \sup(x_{j+1} - x_j) < 2a/b'$ is a set of sampling for $V^p(\Phi)$ for all $p \in \{0\} \cup [1, \infty]$.

Proof. We show the result for everywhere differentiable functions φ and omit the obvious generalization.

Let $f \in V^0(\Phi)$ be such that $f' = \sum_{k \in \mathbb{Z}^d} c_k \varphi_k'$, where the series has finitely many non-zero terms. The set of such functions is dense in $V^0(\Phi)$ and if we show that for all such f

$$\|\{f(x_j)\}\|_\infty = \sup_{j \in \mathbb{Z}^d} |f(x_j)| \geq \wp_\infty \|c\|_\infty,$$

the result would follow immediately from Theorem (5.1.36).

Let $x^* \in \mathbb{R}$ be such that $\|f\|_\infty = |f(x^*)|$. There exists $j \in J$ such that $|x_j - x^*| \leq \frac{1}{2}\gamma(X)$. Using the fundamental Theorem of calculus, we get

$$|f(x_j)| = \left|\int_{x_j}^{x^*} f'(t)dt - f(x^*)\right| \geq \|f\|_\infty - \left|\int_{x_j}^{x^*} c_k\varphi_k'(t)dt\right|$$

$$\geq \|f\|_\infty - \left|\int_{x_j}^{x^*} \left\|\sum_{k\in\mathbb{Z}^d} c_k\varphi_k'\right\|_\infty dt\right| \geq \left(a - \frac{1}{2}b'\gamma(X)\right)\|c\|_\infty.$$

Since $(X) < \frac{2a}{b'}$, we have $\wp_\infty > a - \frac{1}{2}b' \cdot \frac{2a}{b'} = 0$.

Corollary (5.1.38)[260]: Let Φ be a sequence generated by the translates of a piecewise twice differentiable function $\varphi \in W_{0,\omega}^1$ such that

$$a\|c\|_\infty \leq \left\|\sum_{k\in\mathbb{Z}} c_k\varphi_k\right\|_\infty \leq b\|c\|_\infty \text{ and } \left\|\sum_{k\in\mathbb{Z}} c_k\varphi_k''\right\|_\infty \leq b''\|c\|_\infty$$

for all $c \in c_0(\mathbb{Z})$. Then every $X = \{x_j\}$ that satisfies $\gamma(X) = \sup(x_{j+1} - x_j) < \sqrt{\frac{8a}{b''}}$ is a set of sampling for $V^p(\Phi)$ for all $p \in \{0\} \cup [1,\infty]$.

Proof. Using the same notation as in the proof of Theorem, we see that $f'(x^*) = 0$ and, therefore

$$|f(x_j)| = \left|\int_{x_j}^{x^*} f'(t)dt - f(x^*)\right| \geq \|f\|_\infty - \left|\int_{x_j}^{x^*}\int_t^{x^*} f''(u)dudt\right|$$

$$\geq \|f\|_\infty - \frac{b''}{2}|x^* - x_j|^2\|c\|_\infty \geq \left(a - \frac{1}{8}b''\gamma^2(X)\right)\|c\|_\infty.$$

At this point the statement easily follows.

In the next two examples we apply the above Theorem and its corollary to spaces generated by B-splines $\beta_1 = \chi_{[0,1]} * \chi_{[0,1]}$ and $\beta_2 = \chi_{[0,1]} * \chi_{[0,1]} * \chi_{[0,1]}$.

Example (5.1.39)[260]: Let $\varphi = \beta_1$. This function satisfies the conditions of Theorem (5.1.37) with $a = 1$ and $b' = 2$. Hence, if $\gamma(X) < 1$, we have that X is a set of sampling for $V^0(\varphi)$ with the lower bound $1 - \gamma(X)$. Using the estimates in the proof of Theorem (5.1.16) one can obtain explicit lower bounds for any $V^p(\varphi)$, $p \in [1,\infty]$ and a universal bound for all $p \in [1,\infty]$.

Example (5.1.40)[260]: Let $\varphi = \beta_2$. This function satisfies the conditions of Corallary (5.1.38) with $a = \frac{1}{2}$ and $b'' = 4$. Hence, if $\gamma(X) < 1$, we have that X is a set of sampling for $V^0(\varphi)$ with the lower bound $\frac{1}{2}(1 - \gamma^2(X))$. Again, Using the estimates in the proof of Theorem (5.1.16) one can obtain explicit lower bounds for any $V^p(\varphi)$, $p \in [1,\infty]$ and a universal bound for all $p \in [1,\infty]$.

Slanted matrices have also been studied in wavelet theory and signal processing (see Bratteli and Jorgensen [278], [277,279,284,293],). They also occur in k-theory of operator algebras and its applications to topology of manifolds [303]. the technique may be applied to these situations as well. Finally, the results may be useful in the study of differential equations with unbounded operator coefficients similar to the ones described in [273,274,246].

Section (5.2): Hamiltonian Deformations of Gabor Frames

The theory of Gabor frames (or Weyl-Heisenberg frames as they are also called) is a rich and expanding topic of applied harmonic analysis. It has numerous applications in time-frequency analysis, signal theory, and mathematical physics. We show a systematic study of the symplectic transformation properties of Gabor frames, both in the linear and nonlinear cases. Strangely enough, the use of symplectic techniques in the theory of Gabor frames is often ignored; one example (among many others) being Casazza's seminal [311] on modern tools for Weyl-Heisenberg frame theory, where the word "symplectic" does not appear a single time. This is of course very unfortunate: it is a thumb-rule in mathematics and physics that when symmetries are present in a theory their use always leads to new insights in the mechanisms underlying that theory. To name just one single example, the study of fractional Fourier transforms belongs to the area of symplectic analysis and geometry; remarking this would avoid to many unnecessary efforts and complicated calculations. On the positive side, there are however (a few) exceptions to this refusal to include symplectic techniques in applied harmonic analysis: for instance, in Gröchenig's treatise [330] the metaplectic representation is used to study various symmetries in time frequency analysis, and the recent by Pfander et al. [354] elaborates on earlier work [333] by Han and Wang, where symplectic transformations are exploited to study various properties of Gabor frames.

We consider deformations of Gabor systems using Hamiltonian isotopies . A Hamiltonian isotopy is a curve $(f_t)_{0\leq t\leq 1}$ of diffeomorphisms of phase space $\mathbb{R}^{2n}$ starting at the identity, and such that there exists a (usually time-dependent) Hamiltonian function H such that (generalized) phase flow $(f_t^H)_t$ determined by the Hamilton equations

$$\dot{x} = \partial_p H(x,p,t), \qquad \dot{p} = -\partial_x H(x,p,t) \tag{21}$$

consists of the mappings f_t for $0 \leq t \leq 1$. In particular Hamiltonian isotopies consist of symplectomorphisms (or canonical transformations, as they are called in physics). Given a Gabor system $\mathcal{G}(\phi,\Lambda)$ with window (or atom) ϕ and lattice Λ we want to find a working definition of the deformation of $\mathcal{G}(\phi,\Lambda)$ by a Hamiltonian isotopy $(f_t)_{0\leq t\leq 1}$. While it is clear that the deformed lattice should be the image $\Lambda_t = f_t(\Lambda)$ of the original lattice Λ, it is less clear what the deformation $\phi_t = f_t(\phi)$ of the window ϕ should be. A clue is however given by the linear case: assume that the mappings f_t are linear, i.e. symplectic matrices S_t; assume in addition that there exists an infinitesimal symplectic transformation X such that $S_t = e^{tX}$ for $0 \leq t \leq 1$. Then $(S_t)_t$ is the flow determined by the Hamiltonian function

$$H(x,p) = -\tfrac{1}{2}(x,p)^T JX(x,p) \tag{22}$$

where J is the standard symplectic matrix. it is well-known that in this case there exists a one-parameter group of unitary operators $\left(\hat{S}_t\right)_t$ satisfying the operator Schrödinger equation

$$i\hbar \frac{d}{dt}\hat{S}_t = H(x,-i\hbar\partial_x)\hat{S}_t$$

where the formally self-adjoint operator $H(x,-i\hbar\partial_x)$ is obtained by replacing formally p with $-i\hbar\partial_x$ in (22); the matrices S_t and the operators $\hat{S}_t$ correspond to each other by the metaplectic representation of the symplectic group. This suggests that we define the deformation of the initial window ϕ by $\phi_t = \hat{S}_t\phi_t$. It turns out that this definition is satisfactory, because it allows to recover, setting $t = 1$, known results on the image of Gabor frames by linear symplectic transformations. This example is thus a good guideline; however one encounters difficulties as soon as one want to extend it to more general situations. While it is "reasonably" easy to see what one should do when the Hamiltonian isotopy consists of an arbitrary path of symplectic matrices, it is not clear at all what a "good" definition should be in the general nonlinear case: this is discussed, where we suggest that a natural choice would be to extend the linear case by requiring that ϕ_t should be the solution of the Schrödinger equation

$$i\hbar\frac{d}{dt}\phi_t = \widehat{H}\phi_t$$

associated with the Hamiltonian function H determined by the equality $(f_t)_{0\le t\le 1} = (f_t^H)_{0\le t\le 1}$; the Hamiltonian operator $\widehat{H}$ would then be associated with the function H by using, for instance, the Weyl correspondence. Since the method seems to be difficult to study theoretically and to implement numerically, we propose what we call a notation of weak deformation, where the exact definition of the transformation $\phi \mapsto \phi_t$ of the window ϕ is replaced with a correspondence used in semiclassical mechanics, and which consists in propagating the "center" of a sufficiently sharply peaked initial window ϕ (for instance a coherent state, or a more general Gaussian) along the Hamiltonian trajectory. This definition coincides with the definition already given in the linear case, and has the advantage of being easily computable using the method of symplectic integers since all what is needed is the knowledge of the phase flow determined by a certain Hamiltonian function. Finally we discuss possible extensions of our method.

We notice that the notion of general deformations of Gabor frames is an ongoing topic in Gabor analysis; see for instance the recent contribution by Gröchenig et al. [332], also Feichtinger and Kaiblinger [320] where lattice deformations are studied.

The generic point of the phase space $\mathbb{R}^{2n} \equiv \mathbb{R}^n \times \mathbb{R}^n$ is denoted by $\mathcal{Z} = (x, p)$ where we have set $x = (x_1, \ldots, x_n)$, $p = (p_1, \ldots, p_n)$. The scalar product of two vectors, say p and x, is denoted by $p \cdot x$ or simply px. When matrix calculations are performed, $\mathcal{Z}, x, p$ are viewed as column vectors.

We will write $d\mathcal{Z} = dxdp$ where $dx = dx_1 \ldots dx_n$ and $dp = dp_1 \ldots dp_n$. The scalar product on $L^2(\mathbb{R}^2)$ is defined by

$$(\psi|\phi) = \int_{\mathbb{R}^n} \psi(x)\overline{\phi(x)}dx$$

and the associated norm by $\|\cdot\|$. The Schwartz space of rapidly decreasing functions is denoted by $S(\mathbb{R}^n)$ and its dual (the space of tempered distributions) by $S'(\mathbb{R}^n)$.

We review the basics of the modern theory of Hamiltonian mechanics from the symplectic point of view; for details see [306,315,342,355]; we are following here the elementary accounts we have given in [325,327].

We will equip $\mathbb{R}^{2n}$ with the standard symplectic structure

$$\sigma(\mathcal{Z}, \mathcal{Z}') = p \cdot x' - p' \cdot x;$$

in matrix notation $\sigma(\mathcal{Z}, \mathcal{Z}') = (\mathcal{Z}')^T J_{\mathcal{Z}}$ where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (0 and I are here the $n \times n$ zero and identity matrices. The symplectic group of $\mathbb{R}^{2n}$ is denoted by $S_{P(n)}$; it consists of all linear outomorphisms of $\mathbb{R}^{2n}$ such $\sigma(Sz, Sz') = \sigma(z, z')$ for all $z, z' \in \mathbb{R}^{2n}$. Working in the canonical basis $S_{P(n)}$ is identified with the group of all real $2n \times 2n$ matrices S such that $S^T J S = J$ (or, equivalently, $SJS^T = J$). A diffeomorphism $f: \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is called a symplectomorphism if the Jacobin matrix $Df(z)$ is symplectic at every $z \in \mathbb{R}^{2n}$:

$$Df(z)^T J Df(z) = Df(z) J Df(z)^T = J. \quad (23)$$

(Symplectomorphisms are often called "canonical transformations" in physics.) the symplectomorphisms of $\mathbb{R}^{2n}$ form a subgroup $\text{Symp}(n)$ of the group $\text{Diff}(\mathbb{R}^{2n})$ of all diffeomorphisms of $\mathbb{R}^{2n}$ (this follows from formula (23) above, using the chain rule). Of course $\text{Sp}(n)$ is a subgroub of $\text{Symp}(n)$.

Let $H \in C^\infty(\mathbb{R}^{2n} \times \mathbb{R})$ be real-valued; we will call H a Hamiltonian fuction. The associated Hamilton equations with initial data z' at time t' are

$$\dot{z}(t) = J\partial_z H(z(t), t), \quad z(t') = z' \quad (24)$$

(cf.Eqs. (21)). A ssuming existence and uniqueness of the solution for every choice of (z', t') the time-dependent flow $\left(f_{t,t'}^H\right)$ is the family of mapping which associates to every initial z' the value $z(t) = f_{t,t'}^H(z')$ of the solution of (24). The importance of symplectic geometry in Hamiltonian mechanics comes from the following result:

Proposition (5.2.1)[304]: Each diffeomorphism $f_{t,t'}^H$ is a symplectomorphism of $\mathbb{R}^{2n}$: $f_{t,t'}^H \in \text{Symp}(n)$. Equivalently

$$[Df_t^H(z)]^T J Df_t^H(z) = Df_t^H(z) J [Df_t^H(z)]^T = J \quad (25)$$

where $Df_t^H(z)$ is the Jacobian matrix of f_t^H calculated at z.

Proof. See for instance [306,325,327].

It is common practice to write $f_t^H = f_{t,0}^H$. Obviously

$$f_{t,t'}^H = f_{t,0}^H \left(f_{t',0}^H\right)^{-1} = f_t^H \left(f_{t'}^H\right)^{-1} \quad (26)$$

and the $f_{t,t'}^H$ satisfy the groupoid property

$$f_{t,t'}^H f_{t',t''}^H = f_{t,t''}^H, \quad f_{t,t}^H = I_d \quad (27)$$

for all t, t' and t''. Notice that it follows in particular that $\left(f_{t,t'}^H\right)^{-1} = f_{t,t'}^H$.

A remarkable fact is that composition and inversion of Hamiltonian flows also yield Hamiltonian flows:

Proposition (5.2.2)[304]: Let (f_t^H) and $\left(f_t^k\right)$ be the phase flows determined by two Hamiltonian functions $H = H(z, t)$ and $K = K(z, t)$. We have

$$f_t^H f_t^k = f_t^{H\#k} \quad with\ H\#K(z,t) = H(z,t) + K((f_t^H)^{-1}(z), t). \quad (28)$$

$$(f_t^H)^{-1} = f_t^{\bar{H}} \quad with\ \bar{H}(z,t) = -H(f_t^H(z), t). \quad (29)$$

Proof. It is based on the transformation properties of the Hamiltonian fields $X_H = J\partial_z H$ under diffeomorphisms; see [325,342,355] for detailed proofs.

We notice that even if H and K are time-independent Hamiltonians the functions $H\#K$ and $\bar{H}$ are generically time-dependent.

We will call a symplectomorphism f such that $f = f_t^H$ for some Hamiltonian function H and time $t = 1$ a Hamiltonian symplectomorphism. The choice of time $t = 1$ in this definition is of course arbitrary, and can be replaced with any other choice $t = a$ noting that we have $f = f_t^{H_a}$ where $H_a(z, t) = aH(z, at)$.

Hamiltonian symplectomorphisms form a subgroup $\mathrm{Ham}(n)$ of the group $\mathrm{Symp}(n)$ of all symplectomorphisms; it is in fact a normal subgroup of $\mathrm{Symp}(n)$ as follows from the conjugation formula

$$g^{-1}f_t^H g = f_t^{H\circ g} \tag{30}$$

valid for every symplectomorphisms g of $\mathbb{R}^{2n}$ (see [342,325,327]). This formula is often expressed in Hamiltonian mechanics by saying that "Hamilton's equations are covariant under canonical transformations". That $\mathrm{Ham}(n)$ is a group follows from the two formulas (28) and (29) in Proposition (5.2.2) above.

The following result is, in spite of its simplicity, a deep statement about the structure of the group $\mathrm{Ham}(n)$. It says that every continuous path of Hamiltonian transformations passing through the identity is itself the phase flow determined by a certain Hamiltonian function.

Proposition (5.2.3)[304]: Let $(f_t)_t$ be a smooth one-parameter family of Hamiltonian transformations such that $f_0 = I_d$. There exists a Hamiltonian function $H = H(z,t)$ such that $f_t = f_t^H$. More precisely, $(f_t)_t$ is the phase flow determined by the Hamiltonian function

$$H(z,t) = -\int_0^1 z^T J\left(\dot{f}_t \circ f_t^{-1}\right)(\lambda z)d\lambda \tag{31}$$

Where $\dot{f}_t = df_t/dt$.

Proof. See Banyaga [308]; Wang [361] gives an elementary proof of formula (31).

We will call a smooth path (f_t) in $\mathrm{Ham}(n)$ joining the identity to some element $f \in \mathrm{Ham}(n)$ Hamiltonian isotopy. Proposition (5.2.3) above says that every Hamiltonian isotopy is a Hamiltonian flow restricted to sometime interval.

Consider in particular the case of the symplectic $\mathrm{Sp}(n)$. We claim that every path in $\mathrm{Sp}(n)$ joining an element $S \in \mathrm{Sp}(n)$ to the identity is a Hamiltonian isotopy. Since $\mathrm{Sp}(n)$ is connected there exists a C^1 path $t \mapsto S_t, 0 \le t \le 1$ (in fact infinitely many) joining the identity to S in $\mathrm{Sp}(n)$. in view of Proposition (5.2.3) above there exists a Hamiltonian function H such that $S_t = f_t^H$. The following result gives an explicit description of that Hamiltonian without using formula (31):

Proposition (5.2.4)[304]: Let $t \mapsto S_t, 0 \le t \le 1$, be a Hamiltonian isotopy in $\mathrm{Sp}(n)$. There exists a Hamiltonian function $H = H(z,t)$ such that S_t is the phase flow determinant by the Hamilton equations $\dot{z} = J\partial_z H$. Writing

$$S_t = \begin{pmatrix} A_t & B_t \\ C_t & D_t \end{pmatrix} \tag{32}$$

The Hamiltonian function is the quadratic form

$$H = \frac{1}{2}\left(\dot{D}_t C_t^T - \dot{C}_t D_t^T\right)x^2 + \left(\dot{D}_t A_t^T - \dot{C}_t B_t^T\right)p \cdot x + \frac{1}{2}\left(\dot{B}_t A_t^T - \dot{A}_t B_t^T\right)P^2 \tag{33}$$

where $\dot{A}_t = dA_t/dt$, etc.

Proof. The matrices S_t being symplectic we have $S_t^T J S_t = J$ Differentiating both sides of this equality with respect to t we get $\dot{S}_t^T J S_t + S_t^T J \dot{S}_t = 0$ or, equivalently,

$$J\dot{S}_t S_t^{-1} = -(S_t^T)^{-1}\dot{S}_t^T J = \left(J\dot{S}_t S_t^{-1}\right)^T$$

This equality can be rewritten $J\dot{S}_t S_t^{-1} = \left(J\dot{S}_t S_t^{-1}\right)^T$ hence the matrix $J\dot{S}_t S_t^{-1}$ is symmetric. Set $J\dot{S}_t S_t^{-1} = M_t (= M_t^T)$; then

$$\dot{S}_t = X_t S_t, \quad X_t = -JM_t \tag{34}$$

(these relations reduce to $S_t = e^{tx}$ when M_t is time-independent: see (47) below). Define now

$$H(z,t) = -\frac{1}{2} z^T (JX_t) z; \tag{35}$$

using (34) one verifies that the phase flow determined by H consists precisely of the symplectic matrices S_t and that H is given by formula (33).

Symplectic integrators are designed for the numerical solution of Hamilton's equations; they are algorithms which preserve the symplectic character of Hamiltonian flows. The literature on the topic is immense; a well-cited is channel and Scovel [312]. Among many recent contributions, a highlight is [344] by Kang Feng and Mengzhao Q in; also see the comprehensive by Xue-Shen Liu et al. [364], and Marsden's online lecture notes [351, Chapter 9].

Let (f_t^H) be a Hamiltonian flow; let us first assume that H is time-independent so that we have the one-parameter group property $f_t^T f_{t'}^T = f_{t+t'}^H$. Choose an initial value z_0 at time $t = 0$. A mapping f_{Δ_t} on $\mathbb{R}^{2n}$ is an algorithm with time step-size Δ_t for (f_t^H) if we have

$$f_{\Delta_t}^H(z) = f_{\Delta_t}(z) + O(\Delta t^k);$$

the number k (usually an integer ≥ 1) is called the order of the algorithm. In the theory of Hamiltonian systems one requires that f_{Δ_t} be a symplectomorphism; f_{Δ_t} is then called a symplectic integrator. One of the basic properties one is interested in is convergence: setting $\Delta_t = t/N$ (N an integer) when do we have $\lim_{N\to\infty} \left(f_{t/N}\right)^N(z) = f_t^H(z)$? One important requirement is stability, i.e. $\left(f_{t/N}\right)^N(z)$ must remain close to z for small t (Chorin et al. [313]).

Here are two elementary examples of symplectic integrators. We assume that the Hamiltonian H has the physical form

$$H(x,p) = U(p) + V(x).$$

(i) First order algorithm, one defines $(x_{k+1}, p_{k+1}) = f_{\Delta_t}(x_k, p_k)$ by

$$x_{k+1} = x_k + \partial_p U(p_k - \partial_x V(x_k)\Delta t)\Delta t$$
$$p_{k+1} = p_k - \partial_x V(x_k)\Delta t.$$

(ii) Second order algorithm Setting

$$x'_k = x_k + \frac{1}{2}\partial_p U(p_k)$$

We take

$$x_{k+1} = x_k + \frac{1}{2}\partial_p U(p_k)$$
$$p_{k+1} = p_k - \partial_x V(x'_k)\Delta t.$$

One can show, using Proposition (5.2.3) that both schemes are not only symplectic, but also Hamiltonian (Wang [361]). For instance, for the first order algorithm described above, we have $f_{\Delta t} = f_{\Delta t}^k$ where k is the now time-dependent Hamiltonian

$$K(x,p,t) = U(p) + V(x - \partial_p U(p)t). \tag{36}$$

When the Hamiltonian H is itself time-dependent its flow does no longer enjoy the group property $f_t^H f_{t'}^H = f_{t+t'}^H$, so one has to redefine the notion of algorithm in some way. This can be done by considering the time-dependent flow $\left(f_{t+t'}^H\right)$ defined by

(26) : $f_{t+t'}^{H} = f_t^H\left(f_{t'}^{H}\right)^{-1}$. One then uses the following trick: define the suspended flow $\widetilde{(f_t^H)}$ by the formula

$$\widetilde{f_t^H}(z',t') = \left(f_{t+t'}^{H}(z'), t+t'\right); \qquad (37)$$

One verifies that the mappings $\widetilde{f_t^H}: \mathbb{R}^{2n} \times \mathbb{R} \to \mathbb{R}^{2n} \times \mathbb{R}$ (the "extended phase space") satisfy the one-parameter group law $\widetilde{f_t^H}\widetilde{f_{t'}^H} = \widetilde{f_{t+t'}^H}$ and one may then define a notion of algorithm approximating $\widetilde{f_t^H}$(see Struckmeier [359] for a detailed study of the extended phase space approach to Hamiltonian dynamics). For details see [313] by Chorin et al. where a general Lie-Trotter is developed.

Gabor frames are a generalization of the usual notion of basis; see for instance Grochenig [330], Feichtinger and Grochenig [319], Balan et al. [307], Heil [337], Casazza [311] for a detailed treatment of this topic. In what follows we give a slightly modified version of the usual definition, better adapted to the study of symplectic symmetries.

let ϕ be an a non-zero square integrable function (hereafter called window) on $\mathbb{R}^n$, and a lattice Λ in $\mathbb{R}^{2n}$, i.e a discrete subset of $\mathbb{R}^{2n}$. Observe that we do not require that Λ be regular (i.e a subgroup of $\mathbb{R}^{2n}$). The associated $\hbar - Gabor$ system is the set of square-integrable functions

$$G(\phi,\Lambda) = \left\{\hat{T}^h(z)\phi : z \in \Lambda\right\}$$

where $\hat{T}^h(z) = e^{-i\sigma(\hat{z},z)/\hbar}$ is the Heisenberg operator. The action of this operator is explicitly given by the formula

$$\hat{T}^{\hbar}(z_0)\phi(x) = e^{\frac{i(p_0 x - p_0 x_0/2)/\hbar}{\hbar}}\phi(x-x_0) \qquad (38)$$

(see e.g [325,327,348]) .We will call the Gabor system $\mathcal{G}(g,\Lambda)$ a $\hbar$-frame for $L^2(\mathbb{R}^2)$, if there exist constants $a,b > 0$ (the frame bounds) such that

$$a\|\psi\|^2 \le \sum_{z_0 \in \Lambda} \left|(\psi|\hat{T}^h(z_0)\phi)\right|^2 \le b\|\psi\|^2 \qquad (39)$$

for every square integrable function ψ on $\mathbb{R}^n$. When $a = b$ the $\hbar$-frame $\mathcal{G}(g,\Lambda)$ is said to be tight.

For the choice $\hbar = 1/2\pi$ the notion of $\hbar$-Gabor frame coincides with the usual notion of Gabor frame as found in the literature. In fact, in this case, writing $\hat{T}(z) = \hat{T}^{1/2\pi}(z)$ and $p = \omega$, we have

$$\left|(\psi|\hat{T}\quad(z)\phi)\right| = |(\psi|\tau(z)\phi)|$$

where $\mathcal{T}(z)$ is the time-frequency shift operator defined by

$$\mathcal{T}(z_0)\phi(x) = e^{2\pi i\omega_0 x}\phi(x-x_0)$$

for $z_0 = (x_0,\omega_0)$. The two following elementary results can be used to toggle between both definitions:

Proposition (5.2.5)[304]: Let $D^{\hbar} = \begin{pmatrix} 1 & 0 \\ 0 & 2\pi hI \end{pmatrix}$. The system $\mathcal{G}(\phi,\Lambda)$ is a Gabor frame if and only if $\mathcal{G}(\phi, D^{\hbar}\Lambda)$ is a h-Gabor frame.

Proof. We have $\hat{T}^{\hbar}(x_0, 2\pi h p_0) = \hat{T}(x_0,p_0)$ where $\hat{T}(x_0,p_0) = \hat{T}^{1/2\pi}(x_0,p_0)$.by definition $\mathcal{G}(\phi,\Lambda)$ is a Gabor frame if and only if

$$a\|\psi\|^2 \le \sum_{z_0 \in \Lambda}\left|\left(\psi|\hat{T}(z_0)\phi\right)\right|^2 \le b\|\psi\|^2$$

for every $\psi \in L^2(\mathbb{R}^n)$ that is

$$a\|\psi\|^2 \leq \sum_{(x_0,p_0)\in\Lambda}\left|\left(\psi\middle|\hat{T}(x_0,p_0)\phi\right)\right|^2 \leq b\|\psi\|^2;$$

this inequality is equivalent to

$$a\|\psi\|^2 \leq \sum_{(x_0,p_0)\in\Lambda}\left|\left(\psi\middle|\hat{T}^{\hbar}(x_0,2\pi\hbar p_0)\phi\right)\right|^2 \leq b\|\psi\|^2$$

that is to

$$a\|\psi\|^2 \leq \sum_{(x_0,(2\pi\hbar\,)^{-1}p_0)\in\Lambda}\left|\left(\psi\middle|\hat{T}^{\hbar}(x_0,p_0)\phi\right)\right|^2 \leq b\|\psi\|^2$$

hence the result since $(x_0,(2\pi\hbar\,)^{-1}p_0) \in \Lambda$ is equivalent to the condition $(x_0,p_0) \in D^{\hbar}\Lambda$.

We can also rescale simultaneously the lattice and the window (which amounts to a "change of Planck's constant"):

Proposition (5.2.6)[304]: Let $\mathcal{G}(\phi,\Lambda)$ be a Gabor system, and set

$$\phi^{\hbar}(x) = (2\pi\hbar\,)^{-n/2}\phi\left(x/\sqrt{2\pi\hbar}\right). \quad (40)$$

Then $\mathcal{G}(\phi,\Lambda)$ is a frame if and only if $\mathcal{G}\left(\phi^{\hbar},\sqrt{2\pi\hbar}\Lambda\right)$ is a $\hbar$-frame.

Proof. We have $\phi^{\hbar} = \widehat{M}_{1/\sqrt{2\pi\hbar}I,0}\phi$ where $\widehat{M}_{1/\sqrt{2\pi\hbar}I,0} \in \mathrm{Mp}(n)$ has projection

$$M_{1/\sqrt{2\pi\hbar}} = \begin{pmatrix} (2\pi\hbar)^{1/2}I & 0 \\ 0 & (2\pi\hbar)^{-1/2}I \end{pmatrix}$$

On $\mathrm{Sp}(n)$. the Gabor system $\mathcal{G}\left(\phi^{\hbar},\sqrt{2\pi\hbar}\Lambda\right)$ is a $\hbar$-frame if and only

$$a\|\psi\|^2 \leq \sum_{z_0\in\sqrt{2\pi\hbar}\Lambda}\left|(\psi|\hat{T}(z_0)\widehat{M}_{1/\sqrt{2\pi\hbar}I,0}\phi)\right|^2 \leq b\|\psi\|^2$$

for every $\psi \in L^2(\mathbb{R}^n)$, that is, taking the symplectic covariance formula (41) into account, if and only if

$$a\|\psi\|^2 \leq \sum_{z_0\in\sqrt{2\pi\hbar}\Lambda}\left|\left(\widehat{M}_{\sqrt{2\pi\hbar}I,0}\psi\middle|\hat{T}\left((2\pi\hbar)^{-1/2}x_0,(2\pi\hbar)^{-1/2}p_0\right)\phi\right)\right|^2 \leq b\|\psi\|^2.$$

But this is inequality is equivalent to

$$a\|\psi\|^2 \leq \sum_{z_0\in D^{\hbar}\Lambda}\left|\left(\psi\middle|\hat{T}(z_0)\phi\right)\right|^2 \leq b\|\psi\|^2.$$

And one concludes using Proposition (5.2.5).

Gabor frames behave well under symplectic transformations of the lattice (or, equivalently, under metaplectic transformations of the window). Formula (41) bellow will play a fundamental role in our transformations, and metaplectic operators. Let $\hat{S} \in Mp(n)$ have projection $\pi^{\hbar}(\hat{S}) = S \in \mathrm{Sp}(n)$. Then

$$\hat{T}^{\hbar}(z)\hat{S} = \hat{S}\hat{T}^{\hbar}(S^{-1}z) \quad (41)$$

(see e.g. [325,327,348]); one easy way to derive this intertwining relation is to show it separately for each generator $\hat{J},\widehat{M}_{L,m},\widehat{V}_P$ of the metaplectic group described in formula (81), (82), (83). We remark the time-frequency shift operators do not satisfy any simple analogue of property (41). As a consequence, the covariance properties we will study below do not appear in any "abvious" way when using the standard tools of Gabor analysis.

The following result is well-known, and appears in many places (see e.g. Grochenig [330], Pfander et al. [354], Luo [349]). Our proof is somewhat simpler since it exploits the symplectic covariance property of the Heisenberge-Weyl operators, which we explain now.

Proposition (5.2.7)[304]: Let $\phi \in L^2(\mathbb{R}^n)$ (or $\phi \in S(\mathbb{R}^n)$). A Gabor system $\mathcal{G}(\phi,\Lambda)$ is a $\hbar$-frame if and only if $\mathcal{G}\left(\hat{S}_{\phi},S\Lambda\right)$ is a $\hbar$-frame; when this is the case both frames have the same bounds. In particular, $\mathcal{G}(\phi,\Lambda)$ is a tight $\hbar$-frame if and only if $\mathcal{G}\left(\hat{S}_{\phi},S\Lambda\right)$ is.

Proof. Using formula (41) intertwining metaplectic and Heisenberg-Weyl operators we have

$$\sum_{z\in S\Lambda}\left|\left(\psi\middle|\hat{T}^{\hbar}(z)\hat{S}_{\phi}\right)\right|^{2} = \sum_{z\in S\Lambda}\left|\left(\psi\middle|\hat{S}\hat{T}^{\hbar}(S^{-1}z)\phi\right)\right|^{2}$$

$$= \sum_{z\in\Lambda}\left|\left(\hat{S}^{-1}\psi\middle|\hat{T}^{\hbar}(z)\phi\right)\right|^{2}$$

and hence, since $\mathcal{G}(\phi,\Lambda)$ is a $\hbar$-frame,

$$a\left\|\hat{S}^{-1}\psi\right\|^{2} \le \sum_{z\in S\Lambda}\left|\left(\psi\middle|\hat{T}^{\hbar}(z)\hat{S}_{\phi}\right)\right|^{2} \le b\left\|\hat{S}^{-1}\psi\right\|^{2}.$$

The result follows since $\left\|\hat{S}^{-1}\psi\right\| = \|\psi\|$ because metaplectic operatoes are unitary; the case $\phi\in S(\mathbb{R}^{n})$ is similar since metaplectic operators are linear automorphisms of $S(\mathbb{R}^{n})$.

The problem of constructing Gabor frames $\mathcal{G}(\phi,\Lambda)$ in $L^{2}(\mathbb{R}^{n})$ with an arbitrary window ϕ and lattice Λ is difficult and has been tackled by many (see [331], also [354]).Very little is known about the existence of frames in the general case. We however have the following characterization of Gaussian frames which extends a classical result of Lyubarskii [320] and Seip and Wallsten[357]:

Proposition (5.2.8)[304]: Let $\phi_{0}^{\hbar}(x) = (\pi\hbar)^{-n/4}e^{-|x|^{2}/2\hbar}$ (the standard centered Gaussian) and $\Lambda_{\alpha\beta} = \alpha\mathbb{Z}^{n}\times\beta\mathbb{Z}^{n}$ with $\alpha = (\alpha_{1},\dots,\alpha_{n})$ and $\beta = (\beta_{1},\dots,\beta_{n})$. Then $\mathcal{G}\left(\phi_{0}^{\hbar},\Lambda_{\alpha\beta}\right)$ is a frame if and only if $\alpha_{j}\beta_{j} < 2\pi\hbar$ for $1\le j\le n$.

Proof. Bourouihiya [309] showes this for $\hbar = 1/2\pi$; the result for arbitrary $\hbar > 0$ follows using Proposition (5.2.6).

It turns out that using the result above one can construct infinitely many symplectic Gaussian frames using the theory of metaplectic operators:

Proposition (5.2.9)[304]: Let $\phi_{0}^{\hbar}$ be the standard Gaussian. The Gabor system $\mathcal{G}\left(\phi_{0}^{\hbar},\Lambda_{\alpha\beta}\right)$ is a frame if and only if $\mathcal{G}\left(\hat{S}\phi_{0}^{\hbar},S\Lambda_{\alpha\beta}\right)$ is a frame (with same bounds) for every $\hat{S}\in\mathrm{Mp}(n)$. Writing S in block-matrix form $\begin{pmatrix}A & B\\ C & D\end{pmatrix}$ the window $\hat{S}\phi_{0}^{\hbar}$ is the Gaussian

$$\hat{S}\phi_{0}^{\hbar}(x) = \left(\frac{1}{\pi\hbar}\right)^{n/4}(\det X)^{1/4}e^{-\frac{1}{2\hbar}(X+iY)x\cdot x} \tag{42}$$

where

$$X = -(CA^{T}+DB^{T})(AA^{T}+BB^{T})^{-1} \tag{43}$$

$$Y = (AA^{T}+BB^{T})^{-1} \tag{44}$$

are symmetric matrices, and $X > 0$.

Proof. That $\mathcal{G}\left(\phi_{0}^{\hbar},\Lambda_{\alpha\beta}\right)$ is a frame if and only if $\mathcal{G}\left(\hat{S}\phi_{0}^{\hbar},S\Lambda_{\alpha\beta}\right)$ is a frame follows from Proposition (5.2.7). To calculate $\hat{S}\phi_{0}^{\hbar}$

Let us choose $\hbar/2\pi$ and consider the rotations

$$S_{t} = \begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix} \tag{25}$$

(we assume $n = 1$). The matrices S_{t} form a one-parameter subgroup of the symplectic group Sp(1). To (S_{t}) corresponds a unique one-parameter subgroup $(\hat{S}_{t})$ of the metaplectic group Mp(1) such that $S_{t} = \pi^{1/2\pi}(\hat{S}_{t})$. $(\hat{S}_{t\phi})$ is explicitely given for $t\neq k\pi$ (k integer) by

$$\hat{S}_{t}\phi(x) = i^{m(t)}\left(\frac{1}{2\pi i|\sin t|}\right)^{1/2}\int_{-\infty}^{\infty}e^{2\pi iW(x,x',t)}\phi(x')dx'$$

where $m(t)$ is an integer (the “Maslov index”) and

$$W(x,x',t)=\frac{1}{2\sin t}\left((x^2+x'^2)\cos t-2xx'\right).$$

The metaplectic operators $\hat{S}_t$ are the “fractional Fourier transforms” familiar from time-frequency analysis (see e.g Almedia [305], Namias [352]). The argumentation above clearly shows that the study of these fractional Fourier transforms belong to the area of symplectic and metaplectic analysis and geometry.

Applying Proposition (5.2.7) we recover without any calculation the results of Kaiser [343, Theorem 1 and Corollary 2] about rotations of Gabor frames; in the notation:

Corollary (5.2.10)[304]: Let $\mathcal{G}(\phi,\Lambda)$ be a frame; then $\mathcal{G}(\hat{S}_t\phi,S_t\Lambda)$ is a frame for every $t\in\mathbb{R}$.

Notice that fractional Fourier transforms (and their higher-dimensional generalizations) are closely related to the theory of the quantum mechanical harmonic oscillator: the metaplectic operators $\hat{S}_t$ are solutions of the operator Schrödinger equation

$$i\hbar\frac{d}{dt}\hat{S}_t=\frac{1}{2}\left(-\hbar^2\frac{d^2}{dx^2}+x^2\right)\hat{S}_t.$$

The symplectic covariance property of Gabor frames studied above can be interpreted as a first result on Hamiltonian deformations of frames because, as we will see, every symplectic matrix is the value of the flow (at some time t) of Hamiltonian function which is a homogeneous quadratic polynomial (with time-depending coefficients) in the variables x_j,p_k. We will in fact extend this result to deformations by affine flows corresponding to the case where the Hamiltonian is an arbitrary quadratic function of these coordinates.

The first example (the fractional Fourier transform) can be interpreted as a statement about continuous deformations of Gabor frames. For instance, assume that $S_t=e^{tx},X$ in the Lie algebra $\mathfrak{sp}(n)$ of the symplectic goup $\mathrm{Sp}(n)$ (it is the algebra of all $2n\times 2n$ matrices X such that $XJ+JX^T=0$; when $n=1$ this condition reduces to $\mathrm{Tr}X=0$; see e.g. [322,325]). The family (S_t) can be identified with the flow determined by the Hamilton equations $\dot{z}=J\partial_zH$ where

$$H(z)=-\frac{1}{2}z^T(JX)z \tag{46}$$

is a quadratic polynomial in the variables x_j,p_k (cf. formula (35)). That flow satisfies the matrix differential equation

$$\frac{d}{dt}S_t=XS_t. \tag{47}$$

We now make the following fundamental observation: in view of the unique lifting property of covering spaces, to the path of symplectic matrices $t\longmapsto S_t, 0\le t\le 1$, corresponds a unique path $t\longmapsto \hat{S}_t, 0\le t\le 1$, of metaplectic operators such that $\hat{S}_0=I_d$ and $\hat{S}_1=\hat{S}$ it can be shown that this path satisfies the operator Schrödinger equation

$$i\hbar\frac{d}{dt}\hat{S}_t=\widehat{H}\hat{S}_t \tag{48}$$

Where $\widehat{H}$ is the Weyl quantization of function H (for a detailed discussion of the correspondence between symplectic and metaplectic pathes see de Gosson [325,327], Leray [347]; it is also hinted at in at Folland [322]). Collecting these facts, one sees that

$\mathcal{G}(\hat{S}\phi_0^\hbar, S\Lambda_{\alpha\beta})$ is obtained from the initial Gabor frame $\mathcal{G}(\phi_0^\hbar, \Lambda_{\alpha\beta})$ by a smooth deformation

$$t \mapsto \mathcal{G}(\hat{S}_t\phi_0^\hbar, S_t\Lambda_{\alpha\beta}), \quad 0 \le t \le 1. \tag{49}$$

Let S be an arbitrary element of the symplectic group $\mathrm{Sp}(n)$. Such an element can in general no longer be written as an exponential e^X, $X \in \mathfrak{sp}(n)$, so we cannot define an isotopy joining I_d to S by the formula $S_t = e^{tX}$. However, in view of Proposition (5.2.4), such an isotopy $t \mapsto S_t$ exists (but it does not satisfy the group property $S_tS_{t'} = S_{t+t'}$ as in the case $S_t = e^{tx}$). Exactly as above , to this isotopy corresponds a path $t \mapsto \hat{S}_t$ of metaplectic operators such that $\hat{S}_0 = I_d$ and $\hat{S}_1 = \hat{S}$, and this path again satisfies a Schrödinger equation (48) where the explicit form of the Hamiltonian function is given by formula (33) in Proposition (5.2.4). Thus, it makes sense to consider smooth deformations (49) for arbitrary symplectic isotopies. This situation will be generalized to the nonlinear case later.

A particular simple example of transformation is that of the translations $T(z_0): z \mapsto z + z_0$ in $\mathbb{R}^{2n}$. On the operator level they correspond to the Heisenberg – Weyl operators $\hat{T}^\hbar(z_0)$. This correspondence is very easy to the understand in terms of "quantization": for fixed z_0 consider the Hamiltonian function

$$H(z) = \sigma(z, z_0) = p \cdot x_0 - p_0 \cdot x$$

The associated Hamilton equations are just $\dot{x} = x_0, \dot{p} = p_0$ whose solutions are $x(t) = x(0) + tx_0$ and $p(t) = p(0) + tp_0$, that is $z(t) = T(tz_0)z(0)$. Let now

$$\widehat{H} = \sigma(\hat{z}, z_0) = (-i\hbar\partial_x) \cdot x_0 - p_0 \cdot x$$

be the "quantization" of H, and consider the Schrödinger equation

$$i\hbar\partial_t\phi = \sigma(\hat{z}, z_0)\phi$$

Its solution is given by

$$\phi(x,t) = e^{-t\sigma(\hat{z},z_0)/\hbar}\phi(x,0) = \hat{T}^\hbar(tz_0)\phi(x,0)$$

(the second equality can be verified by a direct calculation, or using the Campbell-Hausdorff formula [322,325,327,348]).

Translations act in a particularly simple way on Gabor frames; writing $T(z_1)A = A + z_1$ we have:

Proposition (5.2.11)[304]: Let $z_0, z_1 \in \mathbb{R}^{2n}$. A Gabor system $\mathcal{G}(\phi, \Lambda)$ is a $\hbar$-frame if and only if $\mathcal{G}(\hat{T}^\hbar(z_0)\phi, T(z_1)\Lambda)$ is a $\hbar$-frame; the frame bounds are in this case same for all values of z_0, z_1.

Proof. We will need the following well-known [322,325,327,348] properties of the Heisenberge –Weyl operators:

$$\hat{T}^\hbar(z)\hat{T}^\hbar(z') = e^{i\sigma(z,z')/\hbar}\hat{T}^\hbar(z)\hat{T}^\hbar(z') \tag{50}$$

$$\hat{T}^\hbar(z+z') = e^{-i\sigma(z,z')/\hbar}\hat{T}^\hbar(z)\hat{T}^\hbar(z'). \tag{51}$$

Assume first $z_1 = 0$ and let us show that $\mathcal{G}(\hat{T}^\hbar(z_0)\phi, \Lambda)$ is a $\hbar$-frame if and only if $\mathcal{G}(\phi, \Lambda)$ is. We have, using formula (50) and the unitary of $\hat{T}^\hbar(z_0)$,

$$\sum_{z\in\Lambda} \left|(\psi|\hat{T}^\hbar(z)\hat{T}^\hbar(z_0)\phi)\right|^2 = \sum_{z\in\Lambda}\left|(\psi|e^{i\sigma(z,z')/\hbar}\hat{T}^\hbar(z_0)\hat{T}^\hbar(z)\phi)\right|$$

$$= \sum_{z\in\Lambda}\left|(\psi|\hat{T}^\hbar(z_0)\hat{T}^\hbar(z)\phi)\right| = \sum_{z\in\Lambda}\left|(\hat{T}^\hbar(-z_0)\psi|\hat{T}^\hbar(z)\phi)\right|;$$

it follows that the inequality

$$a\|\psi\|^2 \leq \sum_{z\in\Lambda} \left|(\psi|\widehat{T}^\hbar(z)\widehat{T}^\hbar(z_0)\phi)\right|^2 \leq b\|\psi\|^2$$

is equivalently to

$$a\|\psi\|^2 \leq \sum_{z\in\Lambda} \left|(\psi|\widehat{T}^\hbar(z)\phi)\right|^2 \leq b\|\psi\|^2$$

hence our claim in the case $z_1 = 0$. We next assume that $z_0 = 0$; we have, using this time formula (51),

$$\sum_{z\in T(z_1)\Lambda} \left|(\psi|\widehat{T}^\hbar(z)\phi)\right|^2 = \sum_{z\in\Lambda} \left|(\psi|\widehat{T}^\hbar(z+z_1)\phi)\right|^2 = \sum_{z\in\Lambda} \left|(\psi|\widehat{T}^\hbar\big(z_1)\widehat{T}^\hbar(z)\big)\phi)\right|^2$$

$$= \sum_{z\in\Lambda} \left|\widehat{T}^\hbar(-z_1)\psi|\widehat{T}^\hbar(z)\phi)\right|^2$$

and one conclude as in the case $z_1 = 0$. The case of arbitrary z_0, z_1 immediately follows.

Identifying the group of translations with $\mathbb{R}^{2n}$ the inhomogeneous (or affine) symplectic group $\mathrm{ISp}(n)$ is the semi-direct product $\mathrm{Sp}(n) \times \mathbb{R}^{2n}$ (see [310, 322,325]); the group law is given by

$$(S,z)(S',z) = (SS', z + Sz').$$

Using the conjugation relation (cf. (41))

$$S^{-1}T(z_0)S = T(S^{-1}z_0) \tag{52}$$

one checks that $\mathrm{ISp}(n)$ is isomorphic to the group of all affine transformations of $\mathbb{R}^{2n}$ of the type $ST(z_0)$ (or $T(z_0)S$) where $S \in \mathrm{Sp}(n)$.

The group $\mathrm{ISp}(n)$ appears in a natural way when one considers Hamiltonians of the type

$$H(z,t) = \frac{1}{2}M(t)z \cdot z + m(t) \cdot z \tag{53}$$

where $M(t)$ is symmetric and $m(t)$ is a vector. In fact, he phase flow determined by the Hamilton equation's for (53) consist of elements of $\mathrm{ISp}(n)$. Assume for instance that the coefficients M and m are time-independent; the solution of Hamilton's equations $\dot{z} = JM_z + J_m$ is

$$z_t = e^{tJM}z_0 + (JM)^{-1}(e^{tJM} - I)JM \tag{54}$$

provided that $\det M \neq 0$. When $\det M = 0$ the solution (54) is still formally valid and depends on the nilpotency degree of $X = JM$. Since $X = JM \in \mathfrak{sp}(n)$ we have $s_t = e^{tx} \in Sp(n)$; setting $\xi_t = X^{-1}(e^{tx} - I)u$ the flow (f_t^H) is thus given by

$$f_t^H = T(\xi_t)S_t \in ISp(n).$$

The metaplectic group $\mathrm{Mp}(n)$ is a unitary representation of the double cover $\mathrm{Sp}_2(n)$ of $\mathrm{Sp}(n)$. There is an analogue when $\mathrm{Sp}(n)$ is replaced with $\mathrm{ISp}(n)$: it is the Weyl-metaplectic group $\mathrm{WMp}(n)$, which consists of all products $\widehat{T}(z_0)\hat{S}$; notice that formula (41), which we can rewrite

$$\hat{S}^{-1}\widehat{T}^h(z)\hat{S} = \widehat{T}^h(S^{-1}z) \tag{55}$$

is the operator version of formula (52).

We now turn to the central topic, which is to propose and study "reasonable" definitions of the notion of deformation of a Gabor frame by a Hamiltonian isotopy. We begin by briefly recalling the notion of Weyl quantization.

Let H be a Hamiltonian which we assume to be well-behaved at infinity ; more specifically we impose for fixed t, the condition

$$H(\cdot,t) \in C^{\infty}(\mathbb{R}^{2m}) \cap S'(\mathbb{R}^{2m}).$$

We will call such a Hamiltonian function admissible. We denote by $\widehat{H} = \mathrm{Op}(H)$ the pseudo-differential operator on $\mathbb{R}^n$ associated to H by the Weyl rule. Formally, for $\psi \in S(\mathbb{R}^n)$,

$$\widehat{H}\psi(x) = \left(\frac{1}{2\pi h}\right)^n \int\int e^{ip(x-y)/h} H\left(\frac{1}{2}(x+y), p, t\right)\psi(x)dpdy;$$

more rigorously (that is avoiding convergence problems in the integeral above)

$$\widehat{H}\psi(x) = \left(\frac{1}{2\pi h}\right)^n \int H_\sigma(z)\widehat{T}^h(z_0)\psi(x)dz_0$$

where H_σ is the symplectic Fourier of H and $\widehat{T}^h(z_0)$ is the Heisenberg-Weyl operator defined by formula (38). An essential observation is that the operator $\widehat{H}$ is (formally) self-adjoint (because a Hamiltonition is a real function). We refer to the standard literature on pseudo-differential calculus for details (see [322,325,327,353,358,362]; a nice review accessible to non-specialists is given by Littlejohn in [348]. Our choice of this particular type of quantization-among all others available on the market – is not arbitrary; it is due to the fact that the Weyl rule is the only [362] quantization procedure which is symplectically covariant in the following sense: let $\hat{S}$ be an arbitrary element of the metaplectic group $\mathrm{Mp}(n)$; if $\hat{S}$ has projection $S \in \mathrm{Sp}(n)$ then

$$\mathrm{Op}(H \circ S) = \hat{S}\mathrm{Op}(H)\hat{S}^{-1}. \tag{56}$$

This property, which easily follows from the intertwining relation (41) for Heisenberg-Weyl operators, is essential in our context, since our aim is precisely to show how symplectic convariance properties provide a powerful tool for the study of transformations of Gabor frames.

It is usually to consider the Schrödinger equation associated with an admissible Hamiltonian function H: It is the linear partial differential equation

$$ih\frac{\partial\psi}{\partial t} = \widehat{H}\psi, \quad \psi(\cdot,0) = \psi_0 \tag{57}$$

where the initial function is usually chosen in the Schwartz space $S(\mathbb{R}^n)$. Every solution ψ can be written

$$\psi(x,t) = \widehat{U}_t\psi_0(x)$$

and $\widehat{U}_t$ is called the evolution operator (or “propagator”) for the Schrödinger equation (57). An essential property is that the $\widehat{U}_t$ are unitary operators on $L^2(\mathbb{R}^n)$. To see this, set $u(t) = \left(\widehat{U}_t\psi \middle| \widehat{U}_t\psi\right)$ where ψ is in the domain of $\widehat{H}$ (for instance $\psi \in S(\mathbb{R}^n)$); differentiating with respect to t and using the product rule we have

$$ih\dot{u}(t) = \left(\widehat{H}\widehat{U}_t\psi \middle| \widehat{U}_t\psi\right) - \left(\widehat{U}_t\psi \middle| \widehat{H}\widehat{U}_t\psi\right) = 0$$

since $\widehat{H}$ is (formally) self- adjoint; it follows that $\left(\widehat{U}_t\psi \middle| \widehat{U}_t\psi\right) = (\psi|\psi)$ hence $\widehat{U}_t$ is unitary as claimed.

We now turn to the description of the problem. Let $f \in \mathrm{Ham}(n)$ and $(f_t)_{0\le t\le 1}$ be a Hamiltonian isotopy joining the identity to f; in view of Proposition (5.2.3) there exists a Hamiltonian function H such that $f_t = f_t^H$ for $0 \le t \le 1$. we want to study the deformation of a h-Gabor frame $\mathcal{G}(\phi,\Lambda)$ by $(f_t)_{0\le t\le 1}$; that is we want to define a deformation

$$\mathcal{G}(\phi, A) \xrightarrow{f_t} \mathcal{G}(\widehat{U}_t\phi, f_t\Lambda); \tag{58}$$

here $\widehat{U}_t$ is an (unknown) operator associated in some (yet unknown) way with f_t. We will proceed by analogy with the case $f_t = S_t \in \mathrm{Sp}(n)$ where we defined the deformation by

$$\mathcal{G}(\phi, \Lambda) \xrightarrow{S_t} \mathcal{G}(\hat{S}_t\phi, S_t\Lambda); \tag{59}$$

where $\hat{S}_t \in \mathrm{Mp}(n), S_t = \pi^h(\hat{S}_t)$. This suggests that we require that:

(iii) The operator $\widehat{U}_t$ should be unitary in $L^2(\mathbb{R}^n)$;

(iv) The deformation (58) should reduce to (59) when the isotopy $(f_t)_{0\leq t\leq 1}$ lies than $\mathrm{Sp}(n)$.

The following property of the metaplectic representation gives us a clue. Let (S_t) be a Hamiltonian isotopy in $\mathrm{Sp}(n) \subset \mathrm{Ham}(n)$. We have seen in Proposition (5.2.4) that there exists a Hamiltonian function

$$H(z,t) = \frac{1}{2}M(t)z\cdot z$$

with associated phase flow precisely (S_t). Consider now the Schrödinger equation

$$ih\frac{\partial\psi}{\partial t} = \widehat{H}\psi, \ \ \psi(\cdot,0) = \psi_0$$

where $\widehat{H}$ is the Weyl quantization of H (real that $\widehat{H}$ is a formally self-adjoint operator). It is well-known [325,327,322] that $\psi = \hat{S}_t\psi_0$ where $(\hat{S}_t)$ is the unique path in $Mp(n)$ passing through the identity and covering (S_t). This suggests that we should choose $(\widehat{U}_t)_t$ in the following way: let H be the Hamiltonian function determined by the Hamiltonian isotopy (f_t): $f_t = f_t^H$. the quantize H into an operator $\widehat{H}$ using the Weyl correspondence, and let $\widehat{U}_t$ be the solution of Schrödinger's equation

$$ih\frac{d}{dt}\widehat{U}_t = \widehat{H}\widehat{U}_t, \qquad \widehat{U}_0 = I_d. \tag{60}$$

While definition (59) of a Hamiltonian deformation of a Gabor system is "reasonable", its practical implementation is difficult because it requires the solution of a Schrödinger equation. We will therefore try to find a weaker, more tractable definition of the correspondence (58), which is easier to implement numerically.

The "weak Hamiltonian deformation" scheme method we are going to use is the so-called Gasussian wave packet method which comes from semiclassical mechanics and is widely used in chemistary; it is due to Heller and his collaborators (Heller [338,339], Davis and Heller [314]) and Littlejohn [348]. (for a rather up to date discussion of various Gaussian wave packet methods see Heller [340].) for fixed z_0 we set $z_t = f_t^H(z_0)$ and define the new Hamilton function

$$H_{z_0}(z,t) = (\partial_z, H)(z_t,t)(z-z_t) + \frac{1}{2}D_z^2H(z_t,t)(z-z_t)^2; \tag{61}$$

it is the Taylor series of H at z_t with terms of order 0 and > 2 suppressed. The corresponding Hamilton equations are

$$\dot{z} = J\partial_z H(z_t,t) + JD_z^2H(z_t,t)(z-z_t). \tag{62}$$

We make the following obvious but essential observation: in view of the uniqueness theorem for the solution of (32) with initial value z_0 is the same as that of the Hamiltonian system

$$\dot{z}(t) = (\partial_z, H)(z(t),t) \tag{63}$$

with $z(0)=z_0$. Denoting by $\left(f_t^{H_{z_0}}\right)$ the Hamiltonian flow determined by H_{z_0} we thus have $f_t^H(z_0)=f_t^{H_{z_0}}(z_0)$. More generally, the flows $\left(f_t^{H_{z_0}}\right)$ and (f_t^H) are related by a simple formula involving the "linearized flow" (S_t):

Proposition (5.2.12)[304]: The solutions of Hamilton's equations (62) and (63) are related by the formula

$$z(t)=z_t+S_t(z(0)-z_0) \tag{64}$$

where $z_t=f_t^H(z_0)$, $z_t=f_t^H(z)$ and (S_t) is the phase flow determined by the quadratic time-dependent Hamiltonian

$$H^0(z,t)=\frac{1}{2}D_z^2H(z_t,t)\,z\cdot z \tag{65}$$

Equivalently,

$$f_t^H(z)=T[z_t-S_t(z_0)]\,S_t\big(z(0)\big) \tag{66}$$

where $T(\cdot)$ is the translation operator.

Proof. let us set $u=z-z_t$. we have, taking (62) into account,

$$\dot{u}+\dot{z}_t=J\partial_zH(z(t),t)+JD_z^2H(z_t,t)u$$

that is, since $\dot{z}_t=J\partial_zH(z_t,t)$,

$$\dot{u}=JD_z^2H(z_t,t)u$$

It follows that $u(t)=S_t(u(0))$ and hence

$$z(t)=f_t^H(z_0)+S_tu(0)=z_t-S_t(z_0)+S_t\big(z(0)\big)$$

Which is precisely (64).

Remark (5.2.13)[304]: the function $t\mapsto S_t(z)=Df_t^H(z)$ satisfies the "variational equation"

$$\frac{d}{dt}S_t(z)=JD_z^2H(f_t^H(z),t)S_t(z),\qquad S_0(z)=I \tag{67}$$

(this relation can be used to show that $S_t(z)$ is symplectic [325,327]; it thus gives a simple proof of the fact that Hamiltonian phase flows consist of symplectomorphisms [325,327]).

The thawed Gaussian approximation (TGA) (also sometimes called the nearby orbit method) consists in making the following Ansatz:

The approximate solution to Schrödinger's equation

$$ih\frac{\partial\psi}{\partial t}=\widehat{H}\psi,\;\;\psi(\cdot\,,0)=\phi_{z_0}^h$$

where

$$\phi_{z_0}^h=\widehat{T}^h(z_0)\phi_0^h \tag{68}$$

is the standard coherent state centered at z_0 is given by the formula

$$\tilde{\psi}(x,t)=\;e^{\frac{i}{h}\gamma(t,z_0)}\widehat{T}^h(z_t)\,\tilde{S}_t(z_0)\widehat{T}^h(z_0)^{-1}\phi_{z_0}^h \tag{69}$$

where the phase $\gamma(t,z_0)$ is the symmetrized action

$$\gamma(t,z_0)=\int_0^1\left(\frac{1}{2}\sigma(z_{t'},\dot{z}_{t'})-H(z_{t'},t')\right)dt' \tag{70}$$

calculated along the Hamiltonian trajectory leading from z_0 at time $t_0=0$ to z_t at t. One shows that under suitable conditions on the Hamiltonian H the approximate solution satisfies, for $|t|\le T$, an estimate of the type

$$\left\|\psi(\cdot,t)-\tilde{\psi}(\cdot,t)\right\| \leq C(z_0,T)\sqrt{h}|t| \tag{71}$$

where $C(z_0,T)$ is a positive constant depending only on the initial point z_0 and the time interval $[-T,T]$ (Hagedorn [335,336], Nazaikiinskii et al. [353]).

We consider a Gaussian Gabor system $\mathcal{G}(\phi_0^h,\Lambda)$; applying the nearby orbit method to ϕ_0^h yields the approximation

$$\phi_0^h = e^{\frac{i}{h}\gamma(t,0)}\hat{T}^h(z_t)\hat{S}_t\phi_0^h \tag{72}$$

where we have set $\hat{S}_t = \hat{S}_t(0)$. Let us consider the Gabor system $\mathcal{G}(\phi_0^h,\Lambda)$ where $A_t = f_0^h(\Lambda)$.

Proposition (5.2.14)[304]: The Gabor system $\mathcal{G}(\phi_0^h,\Lambda)$ is a Gabor h-frame if and only if $\mathcal{G}(\phi_0^h,\Lambda)$ is a Gabor h-frame; when this is the case both frames have the same bounds.

Proof. Writing

$$I_t(\psi) = \sum_{z\in\Lambda_t}\left|\left(\psi\middle|\hat{T}^h(z_t)\phi_0^h\right)\right|^2$$

we set out to show that the inequality

$$a\|\psi\|^2 \leq I_t(\psi) \leq b\|\psi\|^2 \tag{73}$$

(for all $\psi\in L^2(\mathbb{R}^2)$) holds for every t if and only if it holds for $t=0$ (for all $\psi\in L^2(\mathbb{R}^2)$). In view of definition (72) we have

$$I_t(\psi) = \sum_{z\in\Lambda_t}\left|\left(\psi\middle|\hat{T}^h(z)\hat{T}^h(z_t)\hat{S}_t\phi_0^h\right)\right|^2;$$

the commutation formula (50) yields

$$\hat{T}^h(z)\hat{T}^h(z_t) = e^{i\sigma(z,z_t)/h}\hat{T}^h(z_t)\hat{T}^h(z)$$

and hence

$$\begin{aligned} I_t(\psi) &= \sum_{z\in\Lambda_t}\left|\left(\psi\middle|\hat{T}^h(z_t)\hat{T}^h(z)\hat{S}_t\phi_0^h\right)\right|^2 \\ &= \sum_{z\in\Lambda}\left|\left(\psi\middle|\hat{T}^h(z_t)\hat{T}^h(f_t^H(z)\hat{S}_t\phi_0^h\right)\right|^2. \end{aligned}$$

Since $\hat{T}^h(z_t)$ is unitary the inequality (73) is thus equivalent to

$$a\|\psi\|^2 \leq \sum_{z\in\Lambda}\left|\left(\psi\middle|\hat{T}^h(f_t^H(z)\hat{S}_t\phi_0^h\right)\right|^2 \leq b\|\psi\|^2. \tag{74}$$

In view of formula (64) we have, since $S_t z_0 = 0$ because $z_0 = 0$,

$$f_t^H(z) = S_t z_0 + f_t^H(0) = S_t z + z_t$$

hence the inequality (74) can be written

$$a\|\psi\|^2 \leq \sum_{z\in\Lambda}\left|\left(\psi\middle|\hat{T}^h(S_t z + z_t)\hat{S}_t\phi_0^h\right)\right|^2 \leq b\|\psi\|^2. \tag{75}$$

In view of the product formula (51) for Heisenberg-Weyl operators we have

$$\hat{T}^h(S_t z + z_t) = e^{i\sigma(S_t z,z_t)/2h}\hat{T}^h(z_t)\hat{T}^h(S_t z)$$

so that (75) becomes

$$a\|\psi\|^2 \leq \sum_{z\in\Lambda}\left|\left(\psi\middle|\hat{T}^h(z_t)\hat{T}^h(S_t z)\hat{S}_t\phi_0^h\right)\right|^2 \leq b\|\psi\|^2; \tag{76}$$

the unitarity of $\hat{T}^h(z_t)$ implies that (76) is equivalent to

$$a\|\psi\|^2 \leq \sum_{z\in\Lambda} \left|\left(\psi \middle| \hat{T}^h(S_t z)\hat{S}_t\phi_0^h\right)\right|^2 \leq b\|\psi\|^2. \tag{77}$$

Using the symplectic covariance formula (51) we have

$$\hat{T}^h(S_t z)\hat{S}_t\phi_0^h = \hat{S}_t\hat{T}^h(z)$$

so that the inequality (77) can be written

$$a\|\psi\|^2 \leq \sum_{z\in\Lambda} \left|\left(\psi \middle| \hat{S}_t\hat{T}^h(z)\phi_0^h\right)\right|^2 \leq b\|\psi\|^2;$$

since $\hat{S}_t$ is unitary, this is equivalent to

$$a\|\psi\|^2 \leq \sum_{z\in\Lambda} \left|\left(\psi \middle| \hat{T}^h(z)\phi_0^h\right)\right|^2 \leq b\|\psi\|^2.$$

The Proposition follows.

The fact that we assumed that the window is the centered coherent state ϕ_0^h is not essential. For instance, proposition (25) shows that the result remains valid if we replace ϕ_0^h with a coherent state having arbitrary center, for instance $\phi_{z_0}^h = \hat{T}^h(z_0)\phi_0^h$. More generally:

Corollary (5.2.15)[304]: Let $\mathcal{G}(\phi, \Lambda)$ be a Gabor system where the window ϕ is the Gaussian

$$\phi_M^h(x) = \left(\frac{\det \operatorname{Im} M}{(\pi h)^n}\right)^{1/4} e^{\frac{i}{2\pi}M_{x.x}} \tag{78}$$

where $M = M^T, \operatorname{Im}M > 0$. then $\mathcal{G}(\phi_t^h, \Lambda_t)$ is a Gabor h-frame if and only if it is the case for $\mathcal{G}(\phi, \Lambda)$.

Proof. It follows from the properties of the action of the metaplectic group on Gaussians that there exists $\hat{S} \in \operatorname{Mp}(n)$ such that $\phi_M^h = \hat{S}\phi_0^h$. Let $S = \pi^h(\hat{S})$ be the projection on $\operatorname{Sp}(n)$ of $\hat{S}$; the Gabor system $\mathcal{G}(\phi_M^h, \Lambda)$ is a h-frame if and only if $\mathcal{G}(\hat{S}^{-1}\phi_M^h, S^{-1}\Lambda) = \mathcal{G}(\phi_0^h, S^{-1}\Lambda)$ is a h-frame in view of Proposition (5.2.7) .the result now follows Proposition (5.2.14).

We finally remark that the fact that we have been using Gaussian windows (coherent states and their generalizations) is a matter of pure convenience. in fact, that definition of weak Hamiltonian deformations of a Gabor frame as given above is valid for arbitrary windows $\phi \in S(\mathbb{R}^n)$ $\left(\text{or } \phi \in L^2(\mathbb{R}^n)\right)$. it suffices for this to replace the defining formula (72) with

$$\phi_t = e^{\frac{1}{h}\gamma(t,0)}\hat{T}^h(z_t)\phi_0^h\hat{S}_t\phi. \tag{79}$$

One can prove that if ϕ is sufficiently concentrated around the origin, then ϕ_t is again a good semiclassical approximation to the true solution of Schrödinger's equation. This question is related to the uncertainty principle, see [323,324,329]. However, when one wants to the initial window to belong to more sophisticated functional spaces than $S(\mathbb{R}^n)$ or $L^2(\mathbb{R}^n)$ one might be confronted to technical difficulties if one wants to prove that the deformed window (79) belongs to the same space. However, there is a very important case where this difficulty does not appear, namely if we assume that the initial window ϕ belongs to Feichtinger's algebra $S_0(\mathbb{R}^n)$. Since our definition of weak transformations of Gabor frames only makes use of phase space translations $\hat{T}^h(z)$ and of metaplectic operators it follows that $\phi_t \in S_0(\mathbb{R}^n)$ if and only if $\phi \in S_0(\mathbb{R}^n)$ (see de Gosson [326]). This is due to the fact that the Feichtinger's algebra is the smallest

Banach algebra invariant under these operations, and is thus preserved under the semiclassical propagation scheme used here. It is unknown whether this property is conserved under passage to the general definition (59) b, that is

$$\mathcal{G}(\phi, \Lambda) \xrightarrow{f_t} \mathcal{G}(\widehat{U}_t\phi, f_t\Lambda) \tag{80}$$

where $\widehat{U}_t$ is the solution of the Schrödinger equation associated with the Hamiltonian operator corresponding to the Hamiltonian operator corresponding to the Hamiltonian isotopy $(f_t)_{0\leq t\leq 1}$: one does not know at the time of writing if the solution to Schrödinger equations with initial data in $S_0(\mathbb{R}^n)$ also is in $S_0(\mathbb{R}^n)$ for arbitrary Hamiltonians. The same difficulty appears when one considers other more general functions spaces (e.g modulation spaces).

We shortly discuss some future issues that will be studied in forthcoming; the list is of course far from being exhaustive, since these "first steps" of a general theory of Hamiltonians of Gabor frames will hopefully become a marathon!

We briefly indicate here how the weak Hamiltonian deformation method could be practically and numerically implemented; we will come back to this important practical issue in a forthcoming where experimental results will be given. The main observation is that a weak deformation of a Gabor frame consists of two objects: a Hamiltonian flow and a family of operators approximating the quantized version of that flow (semi classical propagator). First, the action of the Hamiltonian isotopy on the Gabor lattice can be computed (to an arbitrary degree of precision) using the symplectic algorithms reviewed; a host of numerical implementations can be found, see for instance the already mentioned works [312,344, 364]. The corresponding deformation of the window should not be more difficult to compute numerically, since the essence of the method consists in replacing the "true" quantum propagation with a linearized operator, expressed in terms of translations and metaplectic operators as in formula (69), which says that (up to an unessential phase factor) the propagated coherent state is an expression of the type

$$\widehat{T}^h(z_t)\hat{S}_t(z_0)\widehat{T}^h(z_0)^{-1}\phi^h_{z_0} = \widehat{T}^h(z_t)\hat{S}_t(z_0)\phi^h_{z_0}.$$

Numerically, this term can be calculated using the symplectic algorithm to evaluate $\widehat{T}^h(z_t) = \widehat{T}^h(f^H_t(z_0))$ and then calculate $\hat{S}_t(z_0)$ by numerical (or explicit) methods for generating metaplectic operators given. Of course, precise error bounds have to be proven, but this should not be particularly difficult, these approximation theories being well-established parts of the toolbox of numerical analysis.

Since our definition of weak deformations was motivated by semi classical considerations one could perhaps consider refinements of this method using the asymptotic expansions of Hagedorn [335,336] and his followers; this could then lead to "higher order" weak deformations, depending on the number of terms that are retained. The scheme we have been exposing is a standard and robust method; its advantage is its simplicity. We will discuss other interesting possibilities. For instance, in [338,339] Heller proposes a particular simple semi classical approach which he calls the "frozen Gaussian approximation" (FGA). It is obtained by surrounding the Hamiltonian trajectories by a fixed ("frozen") Gaussian function (for instance ϕ^h_0) and neglecting its "squeezing" by metaplectic operators used in the TGA. Although this method seems to be rather crud, it yields astoundingly accurate numerical results applied to super positions of infinitely many Gaussians; thus it inherently has a clear relationship with frame expansions. A more sophisticated procedure would be the use of the Kluk-Herman (HK) approximate propagator, which has been widely discussed in the chemical literature (Herman [341] shows that the evolution associated with the HK propagator is unitary, and Swart and Rousse [360] put the method on a firm mathematical footing by relating it with the theory of Fourier integral operators; in [334] Grossmann and Herman discuss questions of terminology relating to the FGA and the HK propagator). Also see Heller [340] and Kay [345] where the FGA and the respective merits of various semi classical approximation methods are discussed .

Still, there remains the question of the general definition (80) where the exact quantum propagator is used. It would indeed be more intellectually (and also probably practically!) satisfying to study this definition in detail. We preferred to consider a weaker version because it is relatively easy to implement numerically using symplectic integrators. The general case (80) is challenging, but probably not out of reach . from a theoretical point of view, it amounts to construct an extension of the metaplectic representation in the non-linear case; that such a representation indeed exists has been shown with Hily [328] (a caveat: one sometimes finds in the physical literature a claim following which such an extension could not be constructed, a famous theorem of Greenwood and Van Hove being invoked to sustain this claim. This is merely a misunderstanding of this theorem, which only says that there is no way to extend the metaplectic representation so that the Dirac correspondence between Poisson brackets and commutators is preserved). There remains the problem of how one could prove that there is no way to extend the metaplectic representation so that the Dirac correspondence between Poisson brackets and commutators is preserved). There remains the problem of how one could show that the deformation scheme (80) preserves the frame property; a possible approach could consist in using a time-slicing (as one does for symplectic integrators); this would possible also lead to some insight on whether the Feichtinger algebra is preserved by general quantum evolution. This is an open question which is being actively investigated.

Section (5.3): Gabor Systems

The question of robustness of a basis or frame is a fundamental problem in functional analysis and in many concrete applications. It has its historical origin in the work of Paley and Wienter (see, e.g, [405]) who studied the perturbation of Fourier bases and was subsequently investigated in complex analysis and harmonic analysis. Particularly fruitful was the study of the robustness of structured function systems, such as reproducing kernels, sets of sampling in a space of analytic functions, wavelets, or Gabor systems. We take a new look at the stability of Gabor frames and Gabor Riesz sequences with respect to general deformations of phase space.

To be explicit, let us denote the time-frequency shift of a function $g \in L^2(\mathbb{R}^2)$ along $z = (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d \simeq \mathbb{R}^{2d}$ by

$$\pi(z)g(t) = e^{2\pi i \xi t} g(t - x).$$

For a fixed non-zero function $g \in L^2(\mathbb{R}^2)$, usually called a “window function”, and $\Lambda \subseteq \mathbb{R}^2$, a Gabor system is a structured function system of the form

$$\mathcal{G}(b, \Lambda) = \{\pi(\lambda)g := e^{2\pi i \xi t} g(\cdot - x) : \lambda = (x, \xi) \in \Lambda\}.$$

The index set Λ is a discrete subset of the phase space $\mathbb{R}^{2d}$ and λ indicates the localization of time-frequency shift $\pi(\lambda)g$ in phase space.

The Gabor system $\mathcal{G}(g,\Lambda)$ is called a frame (a Gabor frame), if

$$A\|f\|_2^2 \leq \sum_{\lambda\in\Lambda} |\langle f,\pi(\lambda)g\rangle|^2 \leq B\|f\|_2^2, \qquad f\in L^2(\mathbb{R}^2),$$

for some constants $0 < A \leq B < \infty$. In this case every function $f \in L^2(\mathbb{R}^2)$ possesses an expansion $f = \sum_\lambda c_\lambda \pi(\lambda) g$, for some coefficient sequence $c \in \ell^2(\Lambda)$ such that $\|f\|_2 \asymp \|c\|_2$. The Gabor system $\mathcal{G}(g,\Lambda)$ is called a Riesz sequence (or Riesz basis for its span), if $\|\sum_\lambda c_\lambda \pi(\lambda) g\|_2 \asymp \|c\|_2$ for all $c \in \ell^2(\Lambda)$.

For meaningful statements about Gabor frame it is usually assumed that

$$\int_{\mathbb{R}^{2d}} |\langle g,\pi(z)g\rangle| dz < \infty.$$

This condition describes the modulation space $M^1(\mathbb{R}^d)$, also known as the Feichtinger algebra. Every Schwartz function satisfies this condition.

We study the stability of the spanning properties of $\mathcal{G}(g,\Lambda)$ with respect to a set $\Lambda \subseteq \mathbb{R}^{2d}$. If Λ' is "close enough" to Λ, then we expect $\mathcal{G}(g,\Lambda')$ to possess the same spanning properties. We distinguish perturbations and deformations. Whereas a perturbation is local and Λ' is obtained by slightly moving every $\lambda \in \Lambda$, a deformation is a global transformation of $\mathbb{R}^{2d}$. the existing literature is rich in perturbation results, but not much is known about deformations of Gabor frames.

(a) *Perturbation or jitter error*: The jitter describes small point wise perturbations of Λ. For every Gabor $\mathcal{G}(g,\Lambda)$ with $g \in M^1(\mathbb{R}^d)$ there exists a maximal jitter $\epsilon > 0$ with the following property: if $\sup_{\lambda\in\Lambda}\inf_{\lambda'\in\Lambda}|\lambda-\lambda'| < \epsilon$ and $\sup_{\lambda'\in\Lambda'}\inf_{\lambda\in\Lambda}|\lambda-\lambda'| < \epsilon$, then $\mathcal{G}(g,\Lambda')$ is also a frame. See [384, 388] for a general result in coorbit theory, see [386], and Christensen's book on frames [375].

Conceptually the jitter error is easy to understand, because the frame operator is continuous in the operator norm with respect to the jitter error. The proof techniques go back to Paley and Wiener and amount to norm estimates for the frame operator. See [375] and [405] for a modern exposition.

(b) *Linear deformations:* The fundamental deformation result is due to Feichtinger and Kaiblinger [385]. Let $g \in M^1(\mathbb{R}^d), \Lambda \subseteq \mathbb{R}^{2d}$ be a lattice, and assume that $\mathcal{G}(g,\Lambda)$ is a frame for $L^2(\mathbb{R}^d)$. Then there exists $\epsilon > 0$ with the following property: if A is a $2d \times 2d$-matrix with $\|A - I\| < \epsilon$ (in some given matrix norm), then $\mathcal{G}(g,A\Lambda)$ is again a frame .only recently, this result was generalized to non-uniform Gabor frames [368]. The proof for the of a lattice [385] was based oduality theory of Gabor frames, the proof for non-uniform Gabor frames in [368] relies on the stability under chirps of the Sjostrand symbol class for pseudo differential operators, but this technique does not seem to adapt to nonlinear deformations. Compared to perturbations, (linear) deformations of Gabor frames are much more difficult to understand, because the frame operator no longer depends (norm-) continuously on Λ and a deformation may change the density of Λ (which may affect significantly the spanning properties of $\mathcal{G}(g,\Lambda)$).

Perhaps the main difficulty is to find a suitable notion for deformations that preserves Gabor frames. Except for linear deformations and some preliminary observations in [377,380] this question is simply unexplored. We introduce a general concept of deformation, which we call *Lipschitz deformations*. Lipchitz deformations include both the jitter error and linear deformations as a special case. The precise definition is somewhat technical and will be given. For simplicity we formulate a representative special case.

Theorem (5.3.1)[356]: Let $g \in M^1(\mathbb{R}^d)$, and $\Lambda \subseteq \mathbb{R}^{2d}$. Let $T_n: \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ for $n \in \mathbb{N}$ be a sequence of differentiable maps with Jacobian DT_n. Assume that

$$\sup_{z\in\mathbb{R}^{2d}} |DT_n(z) - I| \longrightarrow 0 \qquad as\, n \to \infty, \tag{81}$$

Then the following holds.

(a) If $\mathcal{G}(g, \Lambda)$ is a frame, then $\mathcal{G}(g, T_n(\Lambda))$ is a frame for all sufficiently large n.

(b) If $\mathcal{G}(g, \Lambda)$ is a Riesz sequence, then $\mathcal{G}(g, T_n(\Lambda))$ is a Riesz sequence for all sufficiently large n.

We would like to emphasize that Theorem (5.3.1) is quite general. It deals with non-uniform Gabor frames (not just lattices) under nonlinear deformations.in particular, Theorem (5.3.1) implies the main results of [385,368]. The counterpart for deformations of Gabor Riesz sequences (item (b)) is new even for linear deformations.

Condition (81) roughly states that the mutual distances between the points of Λ are preserved locally under the deformation T_n. Our main insight was that the frame property of a deformed Gabor system $\mathcal{G}(g, T_n(\Lambda))$ does not depend so much on the position or velocity of the sequences $(T_n(\lambda))_{n\in\mathbb{N}}$ for $\lambda \in \Lambda$, but on the relative distances $|T_n(\lambda) - T_n(\lambda')|$ for $\lambda, \lambda' \in \Lambda$. For an illustration see Example (5.3.26).

As an application of Theorem (5.3.1), we derive a non-uniform Balian-Low Theorem (BLT). For this, we recall that the lower Beurling density of a set $\Lambda \subseteq \mathbb{R}^{2d}$ is given by

$$D^-(\Lambda) = \lim_{R\to\infty} \min_{z\in\mathbb{R}^{2d}} \frac{\#\Lambda \cap B_R(z)}{\mathrm{vol}(B_R(0))},$$

and likewise the upper Beurling density $D^+(\Lambda)$ (where the minimum is replaced by a supremum). The fundamental density Theorem of Raman than and Steger [396] asserts that if $\mathcal{G}(g, \Lambda)$ is a frame then $D^-(\Lambda) \geq 1$. Analogously, if $\mathcal{G}(g, \Lambda)$ is a Riesz sequence, then $D^+(\Lambda) \leq 1$ [370]. The so-called Balian-low Theorem (BLT) is a stronger version of the density Theorem and asserts that for "nice" window s g the inequalities in the Balian -Low Theorem is a consequence of [385]. A Balian-Low Theorem for non-uniform Gabor frames was open for a long time and was proved only recently by Ascensi, Feichtinger, and Kaiblinger [368]. The corresponding statement for Gabor Riesz sequences was open and is settled here as an application of our deformation Theorem. See Heil's detailed survey [392] of the numerous contributions to the density Theorem for Gabor frames after [396] and to [378] for the Balian-low Theorem.

As an immediate consequence of Theorem (5.3.1) we obtain the following version of the Balian-Low therem for non-uniform Gabor systems.

Corollary (5.3.2)[356]: (Non-uniform Balian-Low Theorem). Assume that $g \in M^1(\mathbb{R}^d)$.

(a) If $\mathcal{G}(g, \Lambda)$ is a frame for $L^2(\mathbb{R}^d)$, then $D^-(\Lambda) > 1$.

(b) If $\mathcal{G}(g, \Lambda)$ is a Riesz sequence in $L^2(\mathbb{R}^d)$, then $D^+(\Lambda) < 1$.

Proof. We only prove the new statement (b), part (a) is similar [368]. Assume $\mathcal{G}(g, \Lambda)$ is a Riesz sequence but the $D^+(\Lambda) = 1$. Let $\alpha_n > 1$ such that $\lim_{n\to\infty} \alpha_n = 1$ and set $T_n z = \alpha_n z$. Then the sequence T_n satisfies condition (81). on the one hand, we have $D^+(\alpha_n\Lambda) = \alpha_n^{2d} > 1$, and on the other hand, Theorem (5.3.1) implies that $\mathcal{G}(g, \alpha_n\Lambda)$ is a Riesz sequence for n large enough. This is a contradiction to the density Theorem, and thus the assumption $D^+(\Lambda) = 1$ cannot hold.

The proof of Theorem (5.3.1) does not come easily and technical. It combines methods from the theory of localized frames [387,390], the stability of operators on ℓ^p-spaces [366,401] and weak limit techniques in the style of Beurling [373]. We say that $\Gamma \subseteq \mathbb{R}^{2d}$ is a weak limit of translates of $\Lambda \subseteq \mathbb{R}^{2d}$, if there exists a sequence $(z_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^{2d}$, such that $\Lambda + z_n \to \Gamma$ uniformly on compact sets. For the precise definition and more details on weak limits.

We will prove the following characterization of non-uniform Gabor frames "without inequalities".

Theorem (5.3.3)[356]: Assume that $g \in M^1(\mathbb{R}^d)$ and $\Lambda \subseteq \mathbb{R}^{2d}$. Then $\mathcal{G}(g,\Lambda)$ is a frame for $L^2(\mathbb{R}^d)$, if and only if for every weak limit Γ of Λ the map $f \to (\langle f, \pi(\gamma)g\rangle)_{\gamma\in\Gamma}$ is one-to-one on $\left(M^1(\mathbb{R}^d)\right)^*$.

The full statement with five equivalent conditions characterizing a non-uniform Gabor frame will be given, Theorem (5.3.14). An analogous characterization of Gabor Riesz sequences with weak limits is stated in Theorem (5.3.17).

For the special case when Λ is a lattice, the above characterization of Gabor frames without inequalities was already proved in [391]. In the lattice case, the Gabor system $\mathcal{G}(g,\Lambda)$ possesses additional invariance properties that facilitate the application of methods from operator algebras. The generalization of [391] to non-uniform Gabor systems was rather surprising for us and demands completely different methods.

To make Theorem (5.3.3) more plausible, we make the analogy with Beurling's results on balayage in Paley-Wiener space. Beurling [373] characterized the stability of sampling in the Paley-Wiener space of bandlimited functions $\{f \in L^2(\mathbb{R}^d) : \operatorname{supp} \hat{s} \subseteq S\}$ for a compact spectrum $S \subseteq \mathbb{R}^d$ in terms of sets of uniqueness for this space. It is well-known that the frame property of a Gabor system $\mathcal{G}(g,\Lambda)$ is equivalent to a sampling Theorem for an associated transform. Precisely, let $z \in \mathbb{R}^{2d} \to V_g f(z) = \langle f, \pi(z)g\rangle$ be the short-time Fourier transform, for fixed non-zero $g \in M^1(\mathbb{R}^d)$ and $f \in \left(M^1(\mathbb{R}^d)\right)^*$. Then $\mathcal{G}(g,\Lambda)$ is a frame, if and only if Λ is a set of sampling for the short-time Fourier transform on $(M^1)^*$. In this light, Theorem (5.3.3) is the precise analog of Beurling's Theorem for bandlimited functions.

One may therefore try to adapt Beurling's methods to Gabor frames and the sampling of short-time Fourier transforms. Beurling's ideas have been used for many sampling problems in complex analysis following the pioneering work of Seip on the Fock space [397,398] and the Bergman space [399], see also [372] for a survey. A remarkable fact in Theorem (5.3.3) is the absence of a complex structure (except when g is a Gaussian). This explains why we have to use the machinery of localized frames and the stability of operators in the proof. We mention that Beurling's ideas have been transferred to a few other contexts outside complex analysis, such as sampling Theorems with spherical harmonics in the sphere [393], or, more generally, with eigenvectors of the Laplace operator in Riemannin manifolds [395].

We collect the main definitions from time-frequency analysis. We discuss time-frequency molecules and their ℓ^p-stability. We devoted to the details of Beurling's notation of weak convergence of sets. We state and prove the full characterization of no-uniform Gabor frames and Riesz sequences without inequalities. We introduce the general concept of a Lipschitz deformation of a set and prove the main properties. We state and show the main result, the general deformation result.

Let, $|x| := (|x_1|^2 + \cdots + |x_d|^2)^{1/2}$ denotes the Euclidean norm, and $B_r(x)$ denotes the Euclidean ball. Given two functions $f, g : X \to [0, \infty)$, we say that $f \lesssim g$ if there exists a constant $C > 0$ such that $f(x) \leq Cg(x)$, for all $x \in X$. We say that $f \asymp g$ if $f \lesssim g$ and $g \lesssim f$.

A set $\Lambda \subseteq \mathbb{R}^d$ is called relatively separated if

$$\mathrm{rel}(\Lambda) := \sup\{\#(\Lambda \cap B_1(x)) : x \in \mathbb{R}^d\} < \infty. \tag{82}$$

It is called separated if

$$\mathrm{sep}(\Lambda) := \inf\{|\lambda - \lambda'| : \lambda \neq \lambda' \in \Lambda\} > 0. \tag{83}$$

We say that Λ is δ-separated if $\mathrm{sep}(\Lambda) \geq \delta$. A separated set is relatively separated and

$$\mathrm{rel}(\Lambda) \lesssim \mathrm{sep}(\Lambda)^{-1}, \qquad \Lambda \subseteq \mathbb{R}^d. \tag{84}$$

Relatively separated sets are finite unions of separated sets.

The hole of a set $\Lambda \subseteq \mathbb{R}^d$ is define as

$$\rho(\Lambda) := \sup_{x \in \mathbb{R}^d} \inf_{\lambda \in \Lambda} |x - \lambda|. \tag{85}$$

A sequence Λ is called relatively dense if $\rho(\Lambda) < \infty$. Equivalently, Λ is relatively dense if there exists $R > 0$ such that

$$\mathbb{R}^d = \bigcup_{\lambda \in \Lambda} B_R(\lambda).$$

In terms of the Beurling defined, a set Λ is relatively separated if and if $D^+(\Lambda) < \infty$ and it is relatively dense if and only if $D^-(\Lambda) > 0$.

The amalgam space $W(L^\infty, L^1)(\mathbb{R}^d)$ consists of all functions $f \in L^\infty(\mathbb{R}^d)$ such that

$$\|f\|_{W(L^\infty, L^1)} := \int_{\mathbb{R}^d} \|f\|_{L^\infty(B_1(x))} dx \; x \in \mathbb{R}^d \asymp \sum_{k \in \mathbb{Z}^d} \|f\|_{L^\infty([0,1]^d + k)} < \infty.$$

The space $C_0(\mathbb{R}^d)$ consists of all continuous functions $f : \mathbb{R}^d \to \mathbb{C}$ such that $\lim_{x \to \infty} f(x) = 0$, consequently the (closed) subspace of $W(L^\infty, L^1)(\mathbb{R}^d)$ consisting of continuous functions is $(C_0, L^1)(\mathbb{R}^d)$. This space will be used as a convenient collection of test functions.

We will repeated use the following sampling inequality: Assume that $f \in W(C_0, L^1)(\mathbb{R}^d)$ and $\Lambda \subseteq \mathbb{R}^d$ is relatively separated, then

$$\sum_{\lambda \in \Lambda} |f(\lambda)| \lesssim \mathrm{rel}(\Lambda) \|f\|_{W(L^\infty, L^1)}. \tag{86}$$

The dual space of $W(C_0, L^1)(\mathbb{R}^d)$ will be denoted $W(L^\infty, L^1)(\mathbb{R}^d)$. It consists of all the complex-valued Borel measures $\mu : \beta(\mathbb{R}^d) \to \mathbb{C}$ such that

$$\|\mu\|_{W(\mathcal{M}, L^\infty)} := \sup_{x \in \mathbb{R}^d} \|\mu\|_{B_1(x)} = \sup_{x \in \mathbb{R}^d} |\mu|(B_1(x)) < \infty.$$

For the general theory of Wiener amalgam spaces see [382].

The time-frequency shifts of a function $f : \mathbb{R}^d \to \mathbb{C}$ are

$$\pi(z) f(t) := e^{2\pi i \xi t} f(t - x), \quad z = (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d, t \in \mathbb{R}^d.$$

These operators satisfy the commutation relations

$$\pi(x,\xi)\pi(x',\xi') = e^{-2\pi i\xi' x}\,\pi(x+x',\xi+\xi'),\ \ (x,\xi),(x',\xi') \in \mathbb{R}^d \times \mathbb{R}^d. \quad (87)$$

Given a non-zero Schwartz function $g \in S(\mathbb{R}^d)$, the short-time Fourier transform of a distribution $g \in S'(\mathbb{R}^d)$ with respect to the window g is defined as

$$V_g f(z) \coloneqq \langle f, \pi(z) g\rangle, \qquad z \in \mathbb{R}^{2d}. \quad (88)$$

For $\|g\|_2 = 1$ the short-time Fourier transform is an isometry:

$$\|V_g f\|_{L^2(\mathbb{R}^{2d})} = \|f\|_{L^2(\mathbb{R}^{2d})}, \qquad f \in L^2(\mathbb{R}^{2d}). \quad (89)$$

The commutation rule (87) implies the covariance property of the short-time Fourier transform:

$$V_g(\pi(x,\xi)f)(x',\xi') = e^{-2\pi i x(\xi'-\xi)} V_g f(x'-x,\xi'-\xi),\ \ (x,\xi),(x',\xi') \in \mathbb{R}^d \times \mathbb{R}^d.$$

In particular,

$$|V_g \pi(z) f| = |V_g f(\cdot - z)|, \qquad z \in \mathbb{R}^{2d}. \quad (90)$$

We then define the modulation spaces as follows: fix a non-zero $g \in S(\mathbb{R}^d)$ and let

$$M^P(\mathbb{R}^d) \coloneqq \{f \in S'(\mathbb{R}^d) : V_g f \in L^2(\mathbb{R}^{2d})\},\ \ 1 \le p \le \infty, \quad (91)$$

endowed with the norm $\|f\|_{M^p} \coloneqq \|V_g f\|_{L^p}$. Different choice of non-zero windows $g \in S(\mathbb{R}^d)$ yield the same space with equivalent norms, see [383]. We note that for $g \in M^1(\mathbb{R}^d)$ and $f \in M^p(\mathbb{R}^d), 1 \le p \le \infty,$ the short-time Fourier transform $V_g f$ is a continuous function, we may therefore argue safely with the point wise values of $V_g f$.

The space $M^1(\mathbb{R}^d)$, known as the Feichtinger algebra plays a central role. it can also be characterized as

$$M^0(\mathbb{R}^d) = \{f \in M^\infty(\mathbb{R}^d) : V_g f \in L^1(\mathbb{R}^{2d})\}.$$

The modulation space $M^0(\mathbb{R}^d)$ is defined as the closure of the Schwartz-class with respect to the norm $\|\cdot\|_{M^\infty}$. Then $M^0(\mathbb{R}^d)$ is a closed subspace of $M^\infty(\mathbb{R}^d)$ and can also be characterized as

$$M^0(\mathbb{R}^d) = \{f \in M^\infty(\mathbb{R}^d) : V_g f \in C_0(\mathbb{R}^{2d})\}.$$

The duality of modulation spaces is similar to sequence spaces; we have $M^0(\mathbb{R}^d)^* = M^\infty(\mathbb{R}^d)$ with respect to the duality $\langle f, h\rangle \coloneqq \langle V_g f, V_g h\rangle$.

We consider a fixed function $g \in M^1(\mathbb{R}^d)$ and will be mostly concerned with $M^1(\mathbb{R}^d)$, its dual space $M^\infty(\mathbb{R}^d)$, and $M^2(\mathbb{R}^d) = L^2(\mathbb{R}^d)$. The weak* topology in $M^\infty(\mathbb{R}^d)$ will be denoted by $\sigma(M^\infty, M^1)$ and the weak* topology on $M^1(\mathbb{R}^d)$ by $\sigma(M^1, M^0)$, Hence, a sequence $\{f_k : k \ge 1\} \subseteq M^\infty(\mathbb{R}^d)$ convergence to $f \in M^\infty(\mathbb{R}^d)$ in $\sigma(M^\infty, M^1)$ if and only if for every $h \in M^1(\mathbb{R}^d)$: $\langle f_k, h\rangle \to \langle f, h\rangle$.

We mention the following facts that will be used repeatedly (see for example [384, Theorem 4.1] and [389, proposition 12.1.11])

Lemma (5.3.4)[356]: Let $g \in M^1(\mathbb{R}^d)$ be nonzero. Then the following hold true.

(a) If $f \in M^1(\mathbb{R}^d)$, then $V_g f \in W(C_0, L^1)(\mathbb{R}^{2d})$.

(b) Let $\{f_k : k \ge 1\} \subseteq M^\infty(\mathbb{R}^d)$ be a bounded sequence and $f \in M^\infty(\mathbb{R}^d)$. Then $f_k \to f$ in $\sigma(M^\infty, M^1)$ if and only if $V_g f_k \to V_g f$ uniformly on compacts sets.

(c) Let $\{f_k : k \ge 1\} \subseteq M^1(\mathbb{R}^d)$ be a bounded sequence and $f \in M^1(\mathbb{R}^d)$. Then $f_k \to f$ in $\sigma(M^1, M^0)$ if and only if $V_g f_k \to V_g f$ uniformly on compacts sets.

In particular, if $f_n \to f$ in $\sigma(M^\infty, M^1)$ and $z_n \to z \in \mathbb{R}^{2d}$, then $V_g f_n(z_n) \to V_g f(z)$.

Given $g \in M^1(\mathbb{R}^d)$ and a relatively separated Set $\Lambda \subseteq \mathbb{R}^{2d}$, consider the analysis operator and the synthesis operator that are formally defined as

$$C_{g,\Lambda} f := (\langle f, \pi(\lambda) g\rangle)_{\lambda\in\Lambda}, \qquad f \in M^\infty(\mathbb{R}^d),$$

$$C^*_{g,\Lambda} c := \sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda) g\,, \quad c \in \ell^\infty(\Lambda).$$

These maps are bounded between M^p and ℓ^p spaces [389, Cor.12.1.12] with estimates

$$\left\|C_{g,\Lambda} f\right\|_{\ell^p} \lesssim \mathrm{rel}(\Lambda) \|g\|_{M^1} \|f\|_{M^p},$$

$$\left\|C^*_{g,\Lambda} c\right\|_{M^p} \lesssim \mathrm{rel}(\Lambda) \|g\|_{M^1} \|c\|_{\ell^p}.$$

The implicit constants in the last estimates are of $p \in [1, \infty]$.

For $z = (x, \xi) \in \mathbb{R}^{2d}$, the twisted shift is the operator $\kappa(z): \ell^\infty(\Lambda) \to \ell^\infty(\Lambda + z)$ given by

$$(\kappa(z)c)_{\lambda+z} := e^{-2\pi i x\lambda_2} c_\lambda, \quad \lambda = (\lambda_1, \lambda_2) \in \Lambda.$$

As a consequence of the commutation relations (87), the analysis and synthesis operators satisfy the covariance property

$$\pi(z) C^*_{g,\Lambda} = C^*_{g,\Lambda+z} \kappa(z) \text{ and } e^{2\pi i x\xi} C_{g,\Lambda} \pi(-z) = e^{-2\pi i x\xi} \kappa(-z) C_{g,\Lambda+z} \qquad (92)$$

for $z = (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$.

A Gabor system $\mathcal{G}(g, \Lambda)$is a frame if and only if $C_{g,\Lambda}: L^2(\mathbb{R}^d) \to \ell^2(\Lambda)$ is bounded below, and $\mathcal{G}(g, \Lambda)$ is a Riesz sequence if and only if $C^*_{g,\Lambda}: \ell^2(\Lambda) \to L^2(\mathbb{R}^d)$ is bounded below. As the following lemma shows, each of these conditions implies a restriction of the geometry of the set Λ.

Lemma (5.3.5)[356]: Let $g \in L^2(\mathbb{R}^d)$ and let $\Lambda \subseteq \mathbb{R}^{2d}$ be a set. Then the following holds.

(a) If $\mathcal{G}(g, \Lambda)$ is a frame, then Λ is relatively separated and relatively dense.

(b) If $\mathcal{G}(g, \Lambda)$ is a Riesz sequence, then Λ is separated.

Proof. For part (a) see for example [374, Theorem 1.1]. for part (b), suppose that Λ is not separated. Then there exist two sequences $\{\lambda_n : n \geq 1\} \subseteq \Lambda$ with $\lambda_n \neq \gamma_n$ such $|\lambda_n - \gamma_n| \to 0$. Hence we derive the following contradiction: $\sqrt{2} = \left\|\delta_{\lambda_n} - \delta_{\gamma_n}\right\|_{\ell^2(\mathbb{R}^d)} \asymp \|\pi(\lambda_n) g - \pi(\gamma_n) g\|_{L^2(\mathbb{R}^d)} \to 0$.

We extend the previous terminology to other values of $p \in [1, \infty]$. We say that $\mathcal{G}(g, \Lambda)$ is a p-fame for $M^p(\mathbb{R}^d)$ if $C_{g,\Lambda}: M^p(\mathbb{R}^d) \to \ell^p(\Lambda)$ is bounded below, and that $\mathcal{G}(g, \Lambda)$ is a p −Riesz sequence within $M^p(\mathbb{R}^d)$ if $C^*_{g,\Lambda}: \ell^p(\Lambda) \to M^p(\mathbb{R}^d)$ is bounded below. Since boundedness below and left invertibility are different properties outside the context of Hilbert spaces, there are other reasonable definitions of frames and Riesz sequences for $\mathcal{M}^p$. This is largely immaterial for Gabor frames with $g \in \mathcal{M}^1$, since the theory of localized frames asserts that when such a system is a frame for L^2, then it is a frame for all $\mathcal{M}^p$ and moreover the operator $C_{g,\Lambda}: \mathcal{M}^p \to \ell^p$ is a left invertible [390,387,369,370]. Similar statements apply to Riesz sequences.

We say that $\{f_\lambda : \lambda \in \Lambda\} \subseteq \mathrm{L}^2(\mathbb{R}^d)$ is a set of time-frequency molecules, if $\Lambda \subseteq \mathbb{R}^{2d}$ is a relatively separated set and there exists a non-zero $g \in \mathcal{M}^1(\mathbb{R}^d)$ and an envelope function $\Phi \in W(L^\infty, L^1)(\mathbb{R}^{2d})$ such that

$$\left|V_g f_\lambda(z)\right| \leq \Phi(z - \lambda), \qquad \text{a.e. } z \in \mathbb{R}^d, \lambda \in \Lambda. \qquad (93)$$

If (93) holds for some $g \in \mathcal{M}^1(\mathbb{R}^d)$, then it holds for all $g \in \mathcal{M}^1(\mathbb{R}^d)$ (with an envelope depending on g).

Theorem (5.3.6)[356]: Let $\{f_\lambda: \lambda \in \Lambda\}$ be a set of time-frequency molecules. Then the following holds.

(a) Assume that

$$\|f\|_{M^p} \asymp \|(\langle f, f_\lambda\rangle)_{\lambda\in\Lambda}\|_p, \qquad \forall f \in \mathcal{M}^p(\mathbb{R}^d), \tag{94}$$

Holds for some $1 \le p \le \infty$. Then (94) holds for all $1 \le p \le \infty$. in other words, if $\{f_\lambda: \lambda \in \Lambda\}$ is a p-frame for $\mathcal{M}^p(\mathbb{R}^d)$ for some $p \in [1, \infty]$, then it is a p-frame for $\mathcal{M}^p(\mathbb{R}^d)$ for all $p \in [1, \infty]$.

(b) Assume That

$$\left\|\sum_{\lambda\in A} c_\lambda f_\lambda\right\|_{M^p} \asymp \|c\|_p, \qquad c \in \ell^p(\Lambda), \tag{95}$$

Holds for some $1 \le p \le \infty$. Then (95) holds for all $1 \le p \le \infty$.

The result is similar in spirit to other results [402,366,400, 404, 403], but none of these is directly applicable to our setting. We postpone the proof of Theorem (5.3.6), so as not to interrupt the natural flow. the proof elaborates on Sjöstrand's Wiener-type lemma [401].

As a special case of Theorem (5.3.6). We record the following corollary.

Corollary (5.3.7)[356]: Let $g \in M^1(\mathbb{R}^d)$ and let $\Lambda \subseteq \mathbb{R}^{2d}$ be a relatively separated set. Then the following holds.

(a) If $\mathcal{G}(g, \Lambda)$ is a p-frame for $M^p(\mathbb{R}^d)$ for some $p \in [1, \infty]$, then it is a p-frame for $M^p(\mathbb{R}^d)$ for all $p \in [1, \infty]$.

(b) If $\mathcal{G}(g, \Lambda)$ is a p-Riesz sequence in $M^p(\mathbb{R}^d)$ for some $p \in [1, \infty]$, then it is a p-Riesz sequence in $M^p(\mathbb{R}^d)$ for all $p \in [1, \infty]$.

The space $M^1(\mathbb{R}^d)$ is the largest space of windows for which the corollary holds. Under a stronger condition on g, statement (a) was already derived in [366], the general case was left open.

The Hausdorff distance between two sets $X, Y \subseteq \mathbb{R}^d$ is defined as

$$d_H(X, Y) \coloneqq \inf\{\epsilon > 0: X \subseteq Y + B_\epsilon(0), Y \subseteq X + B_\epsilon(0)\}.$$

Note that $d_H(X, Y) = 0$ if and only if $\bar{X} = \bar{Y}$.

Let $\Lambda \subseteq \mathbb{R}^{2d}$ be a set. A sequence $\{\Lambda_n: n \ge 1\}$ of subsets of $\mathbb{R}^d$ convergence weakly to Λ, in short $\Lambda_n \xrightarrow{\omega} \Lambda$, if

$$d_H\left(\left(\Lambda_n \cap \bar{B}_R(z)\right) \cup \partial\bar{B}_R(z)\right), \left(\Lambda \cap \bar{B}_R(z) \cup \partial\bar{B}_R(z)\right) \to 0, \ \ \forall\, z \in \mathbb{R}^d, R > 0. \tag{96}$$

(to understand the role of the boundary of the ball in the definition, consider the following example in dimension $d = 1$: $\Lambda_n \coloneqq \{1 + 1/n\}, \Lambda \coloneqq \{1\}$ and $B_R(z) = [0,1]$).

The following lemma provides an alternative description of weak convergence.

Lemma (5.3.8)[356]: Let $\Lambda \subseteq \mathbb{R}^{2d}$ and $\Lambda_n \subseteq \mathbb{R}^d, n \ge 1$ be sets. Then $\Lambda_n \xrightarrow{\omega} \Lambda$ if and only if for every $R > 0$ and $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for $n \ge n_0$

$$\Lambda \cap B_R(0) \subseteq \Lambda_n + B_\epsilon(0) \quad \text{and} \quad \Lambda_n \cap B_R(0) \subseteq \Lambda_n + B_\epsilon(0)$$

The following consequence of Lemma (5.3.8) is often useful to identity weak limits.

Lemma (5.3.9)[356]: let $\Lambda_n \xrightarrow{\omega} \Lambda$ and $\Gamma_n \xrightarrow{\omega} \Gamma$. Suppose that for every $R > 0$ and $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$

$$\Lambda_n \cap B_R(0) \subseteq \Gamma_n + B_\epsilon(0).$$

Then $\bar{\Lambda} \subseteq \bar{\Gamma}$.

The notion of weak convergence will be a technical tool in the proofs of deformation results.

We explain how the weak convergence of sets can be understood by the convergence of some associated measures. First we note the following semi continuity property, that follows directly from Lemma (5.3.19).

Lemma (5.3.10)[356]: Let $\{\mu_n : n \geq 1\} \subseteq W(\mathcal{M}, L^\infty)(\mathbb{R}^d)$ be a sequence of measures of measures that converges to a measure $\mu \in W(\mathcal{M}, L^\infty)(\mathbb{R}^d)$ in the $\sigma\big(W(\mathcal{M}, L^\infty), W(C_0, L^1)\big)$ topology. Suppose that $\text{supp}(\mu_n) \subseteq \Lambda_n$ and that $\Lambda_n \xrightarrow{\omega} \Lambda$. Then $\text{supp}(\mu_n) \subseteq \bar{\Lambda}$.

The example $\mu_n = \frac{1}{n}\delta, \mu = 0$ shows that in Lemma (5.3.10) the inclusions cannot in general be improved to equalities. Such improvement is however possible for certain classes of measures. A Borel measure μ is called natural-valued if for all Borel sets E the value $\mu(E)$ is a non-negative integer or infinity. For these measures the following holds.

Lemma (5.3.11)[356]: Let $\{\mu_n : n \geq 1\} \subset W(\mathcal{M}, L^\infty)(\mathbb{R}^d)$ be a sequence of natural-valued measures that converges to a measure $\mu \in W(\mathcal{M}, L^\infty)(\mathbb{R}^d)$ in the $\sigma\big(W(\mathcal{M}, L^\infty), W(C_0, L^1)\big)$ topology. Then $\text{supp}(\mu_n) \xrightarrow{\omega} \text{supp}(\mu)$.

The proof of Lemma (5.3.11) is elementary and therefore we skip it. Lemma (5.3.11) is useful to deduce properties of weak convergence of sets from properties of convergence of measures, as we now show. For a set $\Lambda \subseteq \mathbb{R}^{2d}$, let us consider the natural-valued measure

$$Ш_A := \sum_{\lambda \in \Lambda} \delta_\lambda. \tag{97}$$

One can readily verify that Λ is relatively separated if and only if $Ш_\Lambda \in W(\mathcal{M}, L^\infty)(\mathbb{R}^d)$ and moreover,

$$\|Ш_\Lambda\|_{W(\mathcal{M}, L^\infty)} \asymp \text{rel}(\Lambda). \tag{98}$$

For sequence of sets $\{\Lambda_n : n \geq 1\}$ with uniform separation, i.e,

$$\inf_n \text{sep}(\Lambda_n) = \inf\{|\lambda - \lambda'| : \lambda \neq \lambda', \lambda, \lambda' \in \Lambda_n, n \geq 1\} > 0,$$

the convergence $\Lambda_n \xrightarrow{\omega} \Lambda$ is equivalent to convergence $Ш_{\Lambda_n} \to Ш_\Lambda$ in $\sigma\big(W(\mathcal{M}, L^\infty), W(C_0, L^1)\big)$. For sequences without uniform separation the situation is slightly more technical because of possible multiplicities in the limit set.

Lemma (5.3.12)[356]: Let $\{\Lambda_n : n \geq 1\}$ be a sequence of relatively separated sets in $\mathbb{R}^d$. Then the following hold.

(a) If $Ш_{\Lambda_n} \to \mu$ in $\sigma\big(W(\mathcal{M}, L^\infty), W(C_0, L^1)\big)$ for some measure $\mu \in W(\mathcal{M}, L^\infty)$, then $\sup_n \text{rel}(\Lambda_n) < \infty$ and $\Lambda_n \xrightarrow{\omega} \Lambda := \text{supp}(\mu)$.

(b) If $\limsup_n \text{rel}(\Lambda_n) < \infty$, then there exists a subsequence $\{\Lambda_n : k \geq 1\}$ that converges weakly to a relatively separated set.

(c) If $\limsup_n \text{rel}(\Lambda_n) < \infty$, and $A_n \xrightarrow{\omega}$ for some set $\Lambda \subseteq \mathbb{R}^d$, then Λ is relatively separated (and is particular closed).

The lemma follows easily from lemma (5.3.11), (98) and the weak*-compactness of the ball of $W(\mathcal{M}, L^\infty)$, and hence we do not prove it. We remark that the limiting measure μ in the lemma is not necessarily $ш_A$. For example, if $d = 1$ and $\Lambda_n \coloneqq \{0, 1/n, 1, 1 + 1/n, 1 - 1/n\}$, then $ш_{\Lambda_n} \to 2\delta_0 + 3\delta_1$. The measure μ in (a) can be shown to be natural-valued, and therefore we can interpret it as representing a set with multiplicities.

The following lemma provides a version of (98) for linear combinations of point measures.

Lemma (5.3.13)[356]: Let $\Lambda \subseteq \mathbb{R}^d$ be a relatively separated set and consider a measure

$$\mu \coloneqq \sum_{\lambda\in\Lambda} c_\lambda\, \delta_\lambda$$

with coefficients $c_\lambda \in \mathbb{C}$. Then

$$\|\mu\| = |\mu|(\mathbb{R}^d) = \|c\|_1,$$
$$\|c\|_\infty \le \|c\|_{W(\mathcal{M},L^\infty)} \lesssim \mathrm{rel}(\Lambda)\|c\|_\infty.$$

Proof. The identity $|\mu|(\mathbb{R}^d) = \|c\|_1$ is elementary. The estimate for $\|\mu\|_{W(\mathcal{M},L^\infty)}$ follows from the fact that, for all $\lambda \in \Lambda$, $|c_\lambda|\delta_\lambda \le \|c\|_\infty ш_\Lambda$, where $ш_\Lambda$ is defined by (17).

As a first step towards the main results, we characterize frames and Riesz bases in terms of uniqueness properties for certain limit sequences. The corresponding results for lattices have been derived by different methods in [391]. For the proofs we combine Theorem (5.3.6) with Beurling's methods [373, pp. 351-365].

For a relatively separated set $\Lambda \subseteq \mathbb{R}^{2d}$, let $W(\Lambda)$ be the set of weak limits of the translated sets $\Lambda + z, z \in \mathbb{R}^{2d}$, i.e., $\Gamma \in W(\Lambda)$ if there exists a sequence $\{z_n : n \in \mathbb{N}\}$ such that $\Lambda + z_n \xrightarrow{\omega} \Gamma$. It is easy to see that then Γ is always relatively separated. When Λ is a lattice, i.e, $\Lambda = \Lambda\mathbb{Z}^{2d}$ for an invertible real-valued $2d \times 2d$-matrix Λ, then $W(\Lambda)$ consists only of translates of Λ.

We use repeated the following special case of Lemma (5.3.12)(b,c): given a relatively separated set $\Lambda \subseteq \mathbb{R}^{2d}$ and any sequence of points $\{z_n : n \ge 1\} \subseteq \mathbb{R}^{2d}$, there is a subsequence $\{z_{n_k} : k \ge 1\}$ and relatively separated set $\Gamma \subseteq \mathbb{R}^{2d}$ such $\Lambda + z_{n_k} \xrightarrow{\omega} \Gamma$.

We characterize the frame property of Gabor systems in terms of the sets in (Λ).

Theorem (5.3.14)[356]: Assume that $g \in M^1(\mathbb{R}^d)$ and that $\Lambda \subseteq \mathbb{R}^{2d}$ is relatively separated. Then the following are equivalent.

(i) $\mathcal{G}(g, \Lambda)$ is a frame for $L^2(\mathbb{R}^d)$.

(ii) $\mathcal{G}(g, \Lambda)$ is a $\mathcal{P}$-frame for $M^p(\mathbb{R}^d)$ for some $p \in [1, \infty]$ (for all $p \in [1, \infty]$).

(iii) $\mathcal{G}(g, \Lambda)$ is an ∞-frame for $M^\infty(\mathbb{R}^d)$.

(iv) $C^*_{g,\Lambda}$ is surjective from $\ell^1(\Lambda)$ onto $M^1(\mathbb{R}^d)$.

(v) $C_{g,\Gamma}$ is bounded below on $M^\infty(\mathbb{R}^d)$ for every weak limit $\Gamma \in W(\Lambda)$.

(vi) $C_{g,\Gamma}$ is one-to-one on $M^\infty(\mathbb{R}^d)$ for every weak limit $\Gamma \in W(\Lambda)$.

Proof. The equivalence of (i), (ii) and (iii) follows immediately from Corollary (5.3.7).

We will use several times the following version of the closed range the Orem [376, p.166]: Let $T: X \to Y$ be bounded operator between two Banach spaces X and Y. Then T is onto Y, if and only if $T^*: Y^* \to X^*$ is one-to-one on Y^* and has closed range in X^*, if and only if T^* is bounded below.

Condition (iii) and (iv) are equivalent by applying the closed range Theorem to the synthesis operator $C^*_{g,\Lambda}$ on $\ell^1(\Lambda)$.

For the remaining equivalences we adapt Burling's methods.

(iv) ⇒ (v). Consider a convergent sequence of translates $\Lambda - z_n \xrightarrow{\omega} \Gamma$. Since $C^*_{g,\Lambda}$ maps $\ell^1(\Lambda)$ onto $M^1(\mathbb{R}^d)$, because of (92) and the open mapping Theorem, the synthesis operators $C^*_{g,\Lambda-z_n}$ are also onto $M^1(\mathbb{R}^d)$ with bounds on preimages independent of n. Thus for every $f \in M^1(\mathbb{R}^d)$ there exist sequences $\{c^n_\lambda\}_{\lambda\in\Lambda-z_n}$ with $\|c^n\|_1 \lesssim 1$ such that

$$f = \sum_{\lambda\in\Lambda-z_n} c^n_\lambda \pi(\lambda)\, g,$$

with convergence in $M^1(\mathbb{R}^d)$.

Consider the measures $\mu_n \coloneqq \sum_{\lambda\in\Lambda-z_n} c^n_\lambda \delta_\lambda$. Note that $\|\mu_n\| = \|c^n\|_1 \lesssim 1$. By passing to a subsequence we may assume that $\mu_n \to \mu$ in $\sigma(\mathcal{M}, C_0)$ for some measure $\mu \in \mathcal{M}(\mathbb{R}^{2d})$.

By assumption $\operatorname{supp}(\mu_n) \subseteq \Lambda - z_n$, $\Lambda - z_n \xrightarrow{w} \Gamma$, and Γ is relatively separated and thus closed. It follows from Lemma (5.3.10) that $\operatorname{supp}(\mu_n) \subseteq \Gamma$. Hence,

$$\mu = \sum_{\lambda\in\Gamma} c_\lambda \delta_\lambda$$

for some sequence c. In addition, $\|c\|_1 = \|\mu\| \le \liminf_n \|\mu_n\| \lesssim 1$. Let $f' \coloneqq \sum_{\lambda\in\Gamma} c_\lambda \pi(\lambda) g$. This is well-defined in $M^1(\mathbb{R}^d)$, because $c \in \ell^1(\Gamma)$. Let $\in \mathbb{R}^{2d}$. Since by Lemma (5.3.4), $V_g\pi(z)g \in W(C_0, L^1)(\mathbb{R}^{2d}) \subseteq C_0(\mathbb{R}^{2d})$ we can compute

$$\langle f, \pi(z)g\rangle = \sum_{\lambda\in\Lambda-z_n} c^n_\lambda \overline{V_g\pi(z)g(\lambda)}$$

$$= \int_{\mathbb{R}^{2d}} \overline{V_g\pi(z)g}\, d\mu_n \to \int_{\mathbb{R}^{2d}} \overline{V_g\pi(z)g}\, d\mu = \langle f', \pi(z)g\rangle.$$

(Here, the interchange of summation and integration is justified because c and c^n are summable). Hence $f = f'$ and thus $C^*_{g,\Gamma} : \ell^1(\Gamma) \to M^1(\mathbb{R}^d)$ is surjective. By duality $C_{g,\Gamma}$ is one-to-one from $M^\infty(\mathbb{R}^d)$ to $\ell^\infty(\Gamma)$ and has closed range, whence $C_{g,\Gamma}$ is bounded below on $M^\infty(\infty)$.

(v) ⟹ (vi) is clear.

(vi) ⟹ (iii). Suppose $\mathcal{G}(g,\Lambda)$ is not an ∞-frame for $M^\infty(\mathbb{R}^d)$. then there exists a sequence of functions $\{f_n : n \ge 1\} \subset M^\infty(\mathbb{R}^d)$ such that $\|V_g f_n\|_\infty = 1$ and $\sup_{\lambda\in\Lambda} |V_g f_n(\lambda)| \to 0$. Let $z_n \in \mathbb{R}^{2d}$ be such that $|V_g f_n(z_n)| \ge 1/2$ and consider $h_n \coloneqq \pi(-z_n) f_n$. By passing to a subsequence we may assume that $h_n \to h$ in $\sigma(M^\infty, M^1)$ for some $h \in M^\infty(\mathbb{R}^d)$, and that $\Lambda - z_n \xrightarrow{\omega} \Gamma$ for some relatively separated Γ. Since $|V_g h_n(0)| = |V_g f_n(z_n)| \ge 1/2$ by (90), it follows from Lemma (5.3.4)(b) that $h \ne 0$. Given $\gamma \in \Gamma$, there exists a sequence $\{\lambda_n : n \ge 1\} \subseteq \Lambda$ such that $\lambda_n - z_n \to \gamma$. Since, by Lemma (5.3.4), $V_g h_n \to V_g h$ uniformly on compact sets, we can use (90) to obtain that

$$|V_g h_n(\gamma)| = \lim_n |V_g f_n(\lambda_n - z_n)| = \lim_n |V_g f_n(\lambda_n)| = 0.$$

As $\gamma \in \Gamma$ is arbitrary, this contradicts (vi).

Theorem (5.3.14) seems to be purely qualitative, it can be used to derive quantities estimates for Gabor frames. We fix a non-zero window g in $M^1(\mathbb{R}^d)$ and assume that $\|g\|_2 = 1$. We measure the modulation space norms with respect to this window by $\|f\|_{M^p} = \|V_g f\|_p$ and observe that the isometry property of the short-time Fourier transform extends to $M^\infty(\mathbb{R}^d)$ as follows: if $\in M^\infty(\mathbb{R}^d)$ and $h \in M^1(\mathbb{R}^d)$, then

$$\langle f, h\rangle = \int_{\mathbb{R}^{2d}} V_g f(z)\overline{V_g h(z)} dz = \langle V_g f, V_g h\rangle. \tag{99}$$

For $\delta > 0$, we define the M^1-modulus of continuity of g as

$$\omega_\delta(g)_{M^1} = \sup_{\substack{z,w\in\mathbb{R}^{2d} \\ |z-w|\le\delta}} \|\pi(z)g - \pi(w)g\|_{M^1}$$
$$= \sup_{\substack{z,w\in\mathbb{R}^{2d} \\ |z-w|\le\delta}} \left\|V_g(\pi(z)g - \pi(w)g)\right\|_{L^1}. \tag{100}$$

It is easy to verify that $\lim_{\delta\to 0+\omega_\delta} (g)_{M^1} = 0$, because time-frequency shifts are continuous on $M^1(\mathbb{R}^d)$.

Then we deduce the following quantitative condition for Gabor frames from Theorem (5.3.14).

Corollary (5.3.15)[356]: For $g \in M^1(\mathbb{R}^d)$ with $\|g\|_2 = 1$ choose $\delta > 0$ so that $\omega_\delta(g)_{M^1} < 1$.

If $\Lambda \subseteq \mathbb{R}^{2d}$ is relatively separated and $\rho(A) \le \delta$, Then $\mathcal{G}(g,\Lambda)$ is a frame for $L^2(\mathbb{R}^d)$.

Proof. We argue by contradiction and assume that $\mathcal{G}(g,\Lambda)$ is not a frame. By condition (vi) of Theorem (5.3.14) there exists a weak limit $\Gamma \in W(\Lambda)$ and non-zero $f \in M^\infty(\mathbb{R}^d)$, such that $V_g f\big|_\Gamma = 0$. Since $\rho(\Lambda) \le \delta$, we also have $\rho(\Gamma) \le \delta$. By normalizing, we may assume that $\|f\|_{M^\infty} = \|V_g f\|_\infty = 1$. For $0 < \epsilon < 1 - \omega_\delta(g)_{M^1}$ we find $z \in \mathbb{R}^{2d}$ such that $|V_g f(z)| = |\langle f, \pi(z)g\rangle| > 1 - \epsilon$. By lemma (5.3.12), Γ is relatively separated and, in particular, closed. Since $\rho(\Gamma) \le \delta$ there is a $\gamma \in \Gamma$ such that $|z - \gamma| \le \delta$. Consequently, since $V_g f\big|_\Gamma = 0$, we find that

$$\begin{aligned} 1 - \epsilon &< |\langle f, \pi(z)g\rangle - \langle f, \pi(\gamma)g\rangle| = |\langle f, \pi(z)g - \pi(\gamma)g\rangle| \\ &= |\langle V_g f, V_g(\pi(z)g - \pi(\gamma)g)\rangle| \\ &\le \|V_g f\|_\infty \|V_g(\pi(z)g - \pi(\gamma)g)\|_1 \\ &= \|f\|_{M^\infty}\|\pi(z)g - \pi(\gamma)g\|_1 \\ &\le \omega_\delta(g)_{M^1}. \end{aligned}$$

Since we have chosen $1 - \epsilon > \omega_\delta(g)_{M^1}$, we have arrived at a contradiction. Thus $\mathcal{G}(g,\Lambda)$ is a frame.

This Theorem is analogous to Beurling's famouse sampling Theorem for multiveariate bandlimited functions [371]. The proof is in the style of [394].

We now derive analogouse results for Riesz sequences.

Lemma (5.3.16)[356]: Let $g \in M^1(\mathbb{R}^d)$, $g \ne 0$, and let $\{\Lambda_n : n \ge 1\}$ be a sequence of uniformly separated subsets of $\mathbb{R}^{2d}$, i.e,

$$\inf_n \operatorname{sep}(\Lambda_n) = \delta > 0. \tag{101}$$

For every $n \in \mathbb{N}$, let $c^n \in \ell^\infty(\Lambda_n)$ be such that $\|c^n\|_\infty = 1$ and suppose that
$$\sum_{\lambda\in\Lambda_n} c_\lambda^n \pi(\lambda)g \to 0 \quad in\ M^\infty(\mathbb{R}^d), \qquad as\ n \to \infty.$$
Then there exist a subsequence $(n_k) \subset \mathbb{N}$, points $\lambda_{n_k} \in \Lambda_{n_k}$, a separated set $\Gamma \subseteq \mathbb{R}^{2d}$, and a non-zero sequence $c \in \ell^\infty(\Gamma)$ such that
$$\Lambda_{n_k} - \lambda_{n_k} \xrightarrow{\omega} \Gamma, \quad \text{as } k \to \infty$$
$$\text{and} \qquad \sum_{\lambda\in\Gamma} c_\lambda \pi(\lambda) g = 0.$$
Proof. Combining the hypothesis (111) and observation (84), we also have the uniform relative
$$\sup_n \operatorname{rel}(\Lambda_n) < \infty. \tag{102}$$
Since $\|c^n\|_\infty = 1$ for every $n \geq 1$, we may choose $\lambda_n \in \Lambda_n$ be such that $|c_\lambda^n| \geq 1/2$. Let $\theta_{\lambda,n} \in \mathbb{C}$ such that
$$\theta_{\lambda,n}\pi(\lambda - \lambda_n) = \pi(-\lambda_n)\pi(\lambda),$$
and consider the measures $\mu_n \coloneqq \sum_{\lambda\in A_n} \theta_{\lambda,n} c_\lambda^n \delta_{\lambda-\lambda_n}$. Then by Lemma (5.3.13), $\|\mu_n\|_{W(\mathcal{M},L^\infty)} \lesssim \operatorname{rel}(\Lambda_n - \lambda_n)\|c^n\|_\infty = \operatorname{rel}(\Lambda_n)\|c^n\|_\infty \lesssim 1$. Using (102) Lemma (5.3.12) we may pass to a subsequence such that (i) $\Lambda_n - \lambda_n \xrightarrow{\omega} \Gamma$ for some relatively separated set $\Gamma \subseteq \mathbb{R}^{2d}$ and (ii) $\mu_n \to \mu$ in $\sigma\big(W(\mathcal{M}, L^\infty), W(C_0, L^1)\big)(\mathbb{R}^{2d})$ for some measure $\mu \in W(\mathcal{M}, L^\infty)(\mathbb{R}^{2d})$. The uniform separation condition in (101) implies that Γ is also separated.

Since $\operatorname{supp}(\mu_n) \subseteq \Lambda_n - \lambda_n$ it follows from Lemma (5.3.10), that $\operatorname{supp}(\mu) \subseteq \bar{\Gamma} = \Gamma$. Hence,
$$\mu = \sum_{\lambda\in\Gamma} c_\lambda\, \delta_\lambda,$$
for some sequence of complex numbers, and, by Lemma (5.3.13), $\|c\|_\infty \leq \|\mu\|_{W(\mathcal{M},L^\infty)} < \infty$.

From (101) it follows that for all $n \in \mathbb{N}, B_{\delta/2}(\lambda_n) \cap A_n = \{\lambda_n\}$. Let $\varphi \in C(\mathbb{R}^{2d})$ be real-valued, supported on $B_{\delta/2}(0)$ and such that $(0) = 1$. Then
$$\left|\int_{\mathbb{R}^{2d}} \varphi d\mu\right| = \lim_n \left|\int_{\mathbb{R}^{2d}} \varphi d\mu_n\right| = \lim_n \left|c_{\lambda_n}^n\right| \geq 1/2.$$
Hence $\mu \neq 0$ and therefore $c \neq 0$.

Finally, we show that the short-time Fourier transforms of $\sum_\lambda c_\lambda \pi(\lambda) g$ is zero. Let $z \in \mathbb{R}^{2d}$ be arbitrary and recall that by Lemma (5.3.4) $V_g\pi(z)g \in W(C_0, L^1)(\mathbb{R}^{2d})$. Now we estimate
$$\left|\langle \sum_{\lambda\in\Gamma} c_\lambda \pi(\lambda) g, \pi(z)\, g\rangle\right| = \left|\sum_{\lambda\in\Gamma} c_\lambda \overline{V_g\pi(z)}\, g(\lambda)\right|$$
$$= \left|\int_{\mathbb{R}^{2d}} \overline{V_g\pi(z) g} d\mu\right| = \lim_n \left|\int_{\mathbb{R}^{2d}} \overline{V_g\pi(z) g} d\mu_n\right|$$

$$= \lim_n \left| \langle \sum_{\lambda\in\Lambda_n} \theta_{\lambda,n} c_\lambda^n \pi(\lambda-\lambda_n)g, \pi(z)g\rangle \right|$$

$$\leq \lim_n \left\| \sum_{\lambda\in\Lambda_n} \theta_{\lambda,n} c_\lambda^n \pi(\lambda-\lambda_n)g \right\|_{M^\infty} \|g\|_{M^1}$$

$$= \lim_n \left\| \pi(-\lambda_n) \sum_{\lambda\in\Lambda_n} c_\lambda^n \pi(\lambda)g \right\|_{M^\infty} \|g\|_{M^1}$$

$$= \lim_n \left\| \sum_{\lambda\in\Lambda_n} c_\lambda^n \pi(\lambda)g \right\|_{M^\infty} \|g\|_{M^1} = 0.$$

We have shown that $V_g(\sum_{\lambda\in\Gamma} c_\lambda \pi(\lambda)g) \equiv 0$ and thus $\sum_{\lambda\in\Gamma} c_\lambda \pi(\lambda)g \equiv 0$, as desired.

Theorem (5.3.17)[356]: Assume that $g \in M^1(\mathbb{R}^d)$ and $\Lambda \subseteq (\mathbb{R}^{2d})$ is separated separated. Then the following are equivalent.

(i) $\mathcal{G}(g,\Lambda)$ is a Riesz sequence in $L^2\mathbb{R}^d$.

(ii) $\mathcal{G}(g,\Lambda)$ is a p-Riesz sequence in $M^p(\mathbb{R}^d)$ for some $p \in [1,\infty]$ (for all $p \in [1,\infty]$).

(iii) $\mathcal{G}(g,\Lambda)$ is a ∞-Riesz sequence in $M^\infty(\mathbb{R}^d)$, i.e, $C_{g,\Lambda}^* : \ell^\infty(\Gamma) \to M^\infty(\mathbb{R}^d)$ is bounded below.

(iv) $C_{g,\Lambda}: M^1 \to \ell^1(\Gamma)$ is surjective.

(v) $C_{g,\Gamma}^* : \ell^\infty(\Gamma) \to M^\infty(\mathbb{R}^d)$ is bounded below for every weak limit $\Gamma \in \mathrm{W}(\Lambda)$.

Proof. The equivalence of (i), (ii) and (iii) follows from Corollary (5.3.7)(b), and the equivalence of (iii) and (iv) follows by duality.

(iv) $\Rightarrow$ **(v)**. Assume (iv) and consider a sequence $\Lambda - z_n \xrightarrow{\omega} \Gamma$. Let $\lambda \in \Gamma$ be arbitrary and let $\{\lambda_n : n \in \mathbb{N}\} \subseteq \Lambda$ be a sequence such that $\lambda_n - z_n \to \lambda$. By the open map Theorem, every sequence $c \in \ell^1(\Lambda)$ with $\|c\|_1 = 1$ has a preimage $c = C_{g,\Lambda}(f)$ with $\|f\|_{M^1} \lesssim 1$. With the covariance property (92) we deduce that there exist $f_n \in M^1(\mathbb{R}^d)$, such that $c = C_{g,A-z_n}(f_n)$ and $\|f_n\|_{M^1} \lesssim 1$.

In particular, for each $n \in \mathbb{N}$ there exists an interpolating function $h_n \in M^1(\mathbb{R}^d)$ such that $\|V_g h_n\|_1 \lesssim 1, V_g h_n(\lambda_n - z_n) = 1$ and $V_g h_n \equiv 0$ on $\Lambda - z_n \backslash \{\lambda_n - z_n\}$. By passing to a subsequence we may assume that $h_n \to h$ in $\sigma(M^1, M^0)$. It follows that $\|h\|_{M^1} \lesssim 1$. Since $V_g h_n \to V_g h$ uniformly on compact sets by Lemma (5.3.4), we obtain that

$$V_g h(\lambda) = \lim_n V_g h_n(\lambda_n - z_n) = 1.$$

Similarly, given $\gamma \in \Gamma \backslash \{\lambda\}$, there exists a sequence $\{\gamma_n : n \in \mathbb{N}\} \subseteq \Lambda$ such that $\gamma_n - z_n \to \gamma$. Since $\lambda \neq \gamma$, for $n \gg 0$ we have that $\gamma_n \neq \lambda_n$ and consequently $V_g h(\gamma_n - z_n) = 0$. It follows that $V_g h(\lambda) = 0$.

Hence, we have shown that for each $\lambda \in \Gamma$ there exists an interpolating function $h_\lambda \in M^1(\mathbb{R}^d)$ such that $\|h_\lambda\|_{M^1} \lesssim 1, V_g h_\lambda(\lambda) = 1$ and $V_g h_\lambda \equiv 0$ on $\Gamma \backslash \{\lambda\}$. Given an arbitrary sequence $c \in \ell^1(\Gamma)$ we consider

$$f := \sum_{\lambda \in \Gamma} c_\lambda h_\lambda.$$

It follows that $f \in M^1(\mathbb{R}^d)$ and that $C_{g,\Gamma} f = c$. Hence, $C_{g,\Gamma}$ is onto $\ell^1(\Gamma)$, and therefore $C_{g,\Gamma}^*$ is bounded below.

(v) ⇒ (vi) is clear.

(vi) ⇒ (iii). Suppose that (iii) does not hold. Then there exists a sequence $\{c^n : n \in \mathbb{N}\} \subseteq \ell^\infty(\Lambda)$ such that $\|c^n\|_\infty = 1$ and

$$\left\| \sum_{\lambda \in \Lambda} c_\lambda^n \pi(\lambda) g \right\|_{M^\infty} \to 0, \quad \text{as } n \to \infty.$$

We now apply Lemma (5.3.16), with $\Lambda_n := \Lambda$ and obtain a set $\Gamma \in W(\Lambda)$ and a non-zero sequence $c \in \ell^\infty(\Gamma)$ such that $\sum_{\lambda \in \Gamma} c_\lambda \pi(\lambda) g = C_{g,\Gamma}^*(c) = 0$. This contradicts (vi).

The characterizations of Theorem (5.3.14) suggest that Gabor frames are invariant under "weak deformations" of Λ. One might expect that if $\mathcal{G}(g, \Lambda)$ is a frame and Λ' is close to Λ in the weak sense, then $\mathcal{G}(g, \Lambda')$ is also a frame. This view is too simplistic. Just choose $\Lambda_n = \Lambda \cap B_n(0)$, then $\Lambda_n \xrightarrow{\omega} \Lambda$, but Λ_n is a finite set and thus $\mathcal{G}(g, \Lambda_n)$ is never a frame. For a deformation result we need to introduce a finer notion of convergence.

Let $\Lambda \subseteq \mathbb{R}^d$ be a (countable) set. We consider a sequence $\{\Lambda_n : n \geq 1\}$ of subsets of $\mathbb{R}^d$ produced in the following way. For each $n \geq 1$, let $\tau_n : \Lambda \to \mathbb{R}^d$ be a map and let $\Lambda_n := \tau_n(\Lambda) = \{\tau_n(\lambda) : \lambda \in \Lambda\}$. We assume that $\tau_n(\lambda) \to \lambda$, as $n \to \infty$, for all $\lambda \in \Lambda$. The sequence of sets $\{\Lambda_n : n \geq 1\}$ together with the maps $\{\tau_n : n \geq 1\}$ is called a deformation of Λ. We think of each sequence of points $\{\tau_n(\lambda) : n \geq 1\}$ as a (discrete) path moving towards the end point λ.

We will often say that $\{\Lambda_n : n \geq 1\}$ of Λ is called Lipschitz, denote by $\Lambda_n \xrightarrow{Lip} \Lambda$, if the following two conditions hold:

(L1) Given $R > 0$,

$$\sup_{\substack{\lambda, \lambda' \in \Lambda \\ |\lambda - \lambda'|_\infty \leq R}} \left| \left(\tau_n(\lambda) - \tau_n(\lambda')\right) - (\lambda - \lambda') \right| \to 0, \quad \text{as } n \longrightarrow \infty.$$

(L2) Given $R > 0$, there exist $R' > 0$ and $n_0 \in \mathbb{N}$ such that if $|\tau_n(\lambda) - \tau_n(\lambda')| \leq R$ for some $n > n_0$ and some $\lambda, \lambda' \in \Lambda$, then $|\lambda - \lambda'| \leq R'$.

Condition (L1) means that $\tau_n(\lambda) - \tau_n(\lambda') \to \lambda - \lambda'$ uniformly in $|\lambda - \lambda'|$. In particular, by fixing λ', we see that Lipschitz convergence implies the weak convergence $\Lambda_n \xrightarrow{\omega} \Lambda$. Furthermore, if $\{\Lambda_n : n \geq 1\}$ is Lipschitz convergent to Λ, then so is every subsequence $\{\Lambda_{n_k} : k \geq 1\}$.

Example (5.3.18)[356]: Jitter error: Let $\Lambda \subseteq \mathbb{R}^d$ be relatively separated and let $\{\Lambda_n : n \geq 1\}$ be a deformation of Λ. If $\sup_\lambda |\tau_n(\lambda) - \lambda| \to 0$, as $n \to \infty$, then $\Lambda_n \xrightarrow{\text{Lip}} \Lambda$.

Example (5.3.19)[356]: Linear deformations: Let $\Lambda = A\mathbb{Z}^{2d} \subseteq \mathbb{R}^{2d}$, with Λ an invertible $2d \times 2d$ matrix, $\Lambda_n = \Lambda_n \mathbb{Z}^{2d}$ for a sequence of invertible $2d \times 2d$- matrices and assume that $\lim \mathrm{A}_n = \Lambda$. Then $\Lambda_n \xrightarrow{\text{Lip}} \Lambda$. in this case conditions (L1) and (L2) are easily checked by taking $\tau_n = A_n A^{-1}$.

The third class of examples contains differentiable, nonlinear deformations.

Lemma (5.3.20)[356]: let $p \in (d, \infty]$. For each $n \in \mathbb{N}$, let $T_n = (T_n^1, \dots, T_n^d): \mathbb{R}^d \to \mathbb{R}^d$ be a map such that each coordinate function $T_n^k: \mathbb{R}^d \to \mathbb{R}$ is continuous, locally integrable and has a weak derivative in $L^p_{\text{loc}}(\mathbb{R}^d)$. Assume that

$$T_n(0) = 0,$$
$$|DT_n - I| \to 0 \quad \text{in } L^d(\mathbb{R}^d).$$

(Here, DT_n is the Jacobian matrix consisting of the partial derivatives of T_n and the second condition means that each entry of the matrix $DT_n - I$ tends to 0 in L^p) .

Let $\Lambda \subseteq \mathbb{R}^d$ be a relatively separated set and consider the deformation $\Lambda_n := T_n(\Lambda)$ (i.e. $\tau_n := T_n|_\Lambda$). Then Λ_n is Lipschitz convergent to Λ.

Proof. Let $\alpha := 1 - \frac{d}{p} \in (0,1]$. We use the following Sobolev embedding known as Morrey's inequality (see for example [381, Chapter 4, Theorem 3]). If $f: \mathbb{R}^d \to \mathbb{R}$ is locally integrable and possesses a weak derivative in $L^p(\mathbb{R}^d)$, then f is α-Holder continuous (after being redefine on a set of measure zero). If $x, y \in \mathbb{R}^d$, then

$$|f(x) - f(y)| \lesssim \|\nabla f\|_{L^p(\mathbb{R}^d)} |x - y|^\alpha, \qquad x, y \in \mathbb{R}^d.$$

Applying Morrey's inequality to each coordinate function of $T_n - I$ we obtain that there is a constant $C > 0$ such that

$$|(T_n x - T_n y) - (x - y)| = |(T_n - I)x - (T_n - I)y| \le C\|DT_n - I\|_{L^p(\mathbb{R}^d)} |x - y|^\alpha, \qquad x, y \in \mathbb{R}^d.$$

Let $\epsilon_n = C\|DT_n - I\|_{L^p(\mathbb{R}^d)}$, where $\|DT_n - I\|_{L^p(\mathbb{R}^d)}$ is the L^p-norm of $|DT_n(\cdot) - I|$. Then $\epsilon_n \to 0$ by assumption and

$$|(T_n x - T_n y) - (x - y)| \le \epsilon_n |x - y|^\alpha, \qquad x, y \in \mathbb{R}^d. \tag{103}$$

Choose $x = \lambda$ and $y = 0$, then $T_n(\lambda) \to \lambda$ for all $\lambda \in \Lambda$ (since $T_n(0) = 0$). Hence Λ_n is a deformation of Λ.

If $\lambda, \lambda' \in \Lambda$ and $|\lambda - \lambda'| \le R$, then (103) implies that

$$|(T_n\lambda - T_n\lambda') - (\lambda - \lambda')| \le \epsilon_n \mathbb{R}^\alpha.$$

Thus condition (L1) is satisfied.

For condition (L2), choose n_0 such that $\epsilon_n \le 1/2$ for $n \ge n_0$, $\lambda, \lambda' \in \Lambda$ then by (103) we obtain

$$|(T_n\lambda - T_n\lambda') - (\lambda - \lambda')| \le 1/2\, |\lambda - \lambda'|^\alpha \le 1/2\, |(\lambda - \lambda')|.$$

This implies that

$$|\lambda - \lambda'| \le 2|(T_n\lambda - T_n\lambda')|, \quad \text{for all } n \ge n_0. \tag{104}$$

Since $|T_n\lambda - T_n\lambda'| \le R$, we conclude that $|\lambda - \lambda'| \le 2R$, and we may actually choose $R' = \max(1, 2R)$ in condition (L2).

Lemma (5.3.21)[356]: Let $\{\Lambda_n : n \ge 1\}$ be a deformation of a relatively separated set $\Lambda \subseteq \mathbb{R}^d$. Then the following hold.

(a) If Λ_n is Lipschitz convergent to Λ and $\text{sep}(\Lambda) > 0$, then $\liminf_n \text{sep}(\Lambda_n) > 0$.

(b) If Λ_n is Lipchitz convergent to Λ and then $\liminf_n \text{sep}(\Lambda_n) < \infty$.

(c) If Λ_n is Lipchitz convergent to Λ and $\rho(\Lambda) < \infty$, then $\limsup_n \rho(A_n) < \infty$.

Proof. (a) By assumption $\delta := \text{sep}(\Lambda) > 0$. Using (L_2), let $n_0 \in \mathbb{N}$ and $R' > 0$ be such that if $|\tau_n(\lambda) - \tau_n(\lambda')| \le \delta/2$ for some $\lambda, \lambda' \in \Lambda$ and $n > n_0$, then $|\lambda - \lambda'| \le R'$. By (L_1), choose $n_1 \ge n_0$ such that for $n \ge n_1$

$$\sup_{|\lambda-\lambda'|\leq R'} \left|\left(\tau_n(\lambda) - \tau_n(\lambda')\right) - (\lambda - \lambda')\right| < \delta/2.$$

Claim (5.3.22)[356]: $\mathrm{sep}(\Lambda_n) \geq \delta/2$ for $n \geq n_1$.

If the claim is not true, then for some $n \geq n_0$ there exist two distinct points $\lambda, \lambda' \in \Lambda$ such that $|\tau_n(\lambda) - \tau_n(\lambda')| \leq \delta/2$. Then $|\lambda - \lambda'| \leq R'$ and consequently

$$|\lambda - \lambda'| \leq |\tau_n(\lambda) - \tau_n(\lambda') - (\lambda - \lambda')| + |\tau_n(\lambda) - \tau_n(\lambda')| < \delta,$$

contradicting the fact that Λ is δ-separated.

(b) Since Λ is relatively separated we can split it into finitely many separated sets $\Lambda = \Lambda^1 \cup \ldots \cup \Lambda^L$ with $\mathrm{sep}(\Lambda^k) > 0$.

Consider the sets defined by restricting the deformation τ_n to each Λ^k

$$\Lambda_n^K \coloneqq \{\tau_n(\lambda): \lambda \in \Lambda^k\}.$$

As proved above in (a), there exists $n_0 \in \mathbb{N}$ and $\delta > 0$ such that $\mathrm{sep}(\Lambda_n^K) \geq \delta$ for all $n \geq n_0$ and $1 \leq k \leq L$. Therefore, using (84),

$$\mathrm{rel}(\Lambda_n) \leq \sum_{k=1}^{L} \mathrm{rel}(\Lambda_n^K) \lesssim L\delta^{-d}, \quad n \geq n_0.$$

and the conclusion follows.

(c) By (b) we may assume that each Λ_n is relatively separated. Assume that $\rho(\Lambda) < \infty$. Then there exists $r > 0$ such that every cube $Q_r(z) \coloneqq z + [-r, r]^d$ Intersects. By (L1), There is $n_0 \in \mathbb{N}$ such that for $n \geq n_0$,

$$\sup_{\substack{\lambda,\lambda' \in \Lambda \\ |\lambda-\lambda'|_\infty \leq 6r}} \left|\left(\tau_n(\lambda) - \tau_n(\lambda')\right) - (\lambda - \lambda')\right|_\infty \leq r. \quad (105)$$

Let $R \coloneqq 8r$ and $n \geq n_0$. We will show that every cube $Q_R(z)$ intersects Λ_n. This will give a uniform upper bound for $\rho(\Lambda_n)$. Suppose on the contrary that some cube $Q_R(z)$ does not meet Λ_n and consider a larger radius $R' \geq R$ such that Λ_n intersects the boundary but not the interior of $Q_{R'}(z)$. (this is possible because Λ_n is relatively separated and therefore closed) . Hence, there exists $\lambda \in \Lambda$ such that $|\tau_n(\lambda) - z|_\infty = R'$. Let us write

$$\left(z - \tau_n(\lambda)\right)_k = \delta_k c_k, \qquad k = 1, \ldots, d.$$
$$\delta_k \in \{-1,1\}, \qquad k = 1, \ldots, d,$$
$$0 \leq c_k \leq R', \qquad k = 1, \ldots, d,$$

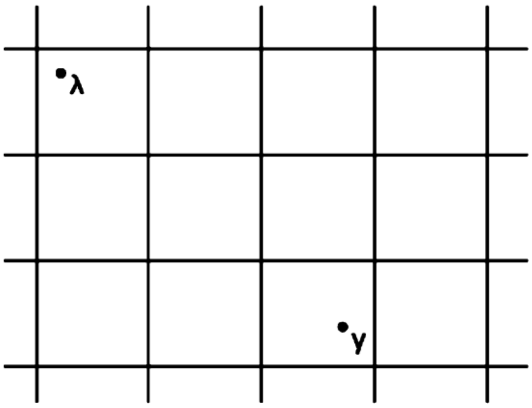

Fig. (1)[356]: The selection of the point γ satisfying (106) and (107).

and $c_k = R'$ for some k. We now argue that we can select a point $\gamma \in \Lambda$ such that

$$(\lambda - \gamma)_k = -\delta_k c_k', \quad k = 1, \ldots, d, \quad (106)$$

with coordinates

$$2r \leq c'_k \leq 6r, \qquad k = 1, \dots, d. \tag{107}$$

Using the fact that Λ intersects each of the cubes $\{Q_r(2rj): j \in \mathbb{Z}^d\}$, we first select an index $j \in \mathbb{Z}^d$ such that $\lambda \in Q_r(2rj)$. Second, we define a new index $j' \in \mathbb{Z}^d$ by $j'_k = j_k + 2\delta_k$ for $k = 1, \dots, d$. We finally select a point $\in \Lambda \cap Q_r(2rj')$. This guarantees that (106) and (107) hold true. See Fig. 1.

Since by (106) and (107) $|\lambda - \gamma|_\infty \leq 6r$, we can use (105) to obtain

$$\left(\tau_n(\lambda) - \tau_n(\gamma)\right)_k = -\delta_k c''_k, \qquad k = 1, \dots, d \,,$$

with coordinates

$$r \leq c''_k \ \leq 7r, \qquad k = 1, \dots, d.$$

We write $\left(z - \tau_n(\gamma)\right)_k = \left(z - \tau_n(\lambda)\right)_k + \left(\tau_n(\lambda) - \tau_n(\gamma)\right)_k = \delta_k(c_k - c''_k)$ and note that $-7r \leq c_k - c''_k \leq R' - r$. Hence,

$$|z - \tau_n(\gamma)|_\infty \leq \max\{R' - r, 7r\} = R' - r,$$

since $7r = R - r \leq R' - r$. This shown that $Q_{R'-r}(z)$ intersects Λ_n, contradicting the choice of R'.

The following lemma relates Lipchitz convergence to the weak-limit techniques.

Lemma (5.3.23)[356]: Let $\Lambda \subseteq R^d$ be relatively separated and let $\{\Lambda_n : n \geq 1\}$ be a Lipschitz deformation of Λ. Then the following holds.

(a) Let $\Gamma \subseteq \mathbb{R}^d$ and $\{\lambda_n : n \geq 1\} \subseteq \Lambda$ be some sequence in Λ. If $\Lambda_n - \tau_n(\lambda_n) \xrightarrow{\omega} \Gamma$, then $\Gamma \in W(\Lambda)$.

(b) Suppose that Λ is relatively dense and $\{z_n : n \geq 1\} \subseteq \mathbb{R}^d$ is an arbitrary sequence if $\Lambda_n - z_n \xrightarrow{\omega} \Gamma$, then $\Gamma \in W(\Lambda)$.

Proof. (a) We first note that Γ is relatively separated indeed, Lemma (5.3.21) says that

$$\limsup_{n\to\infty} \operatorname{rel}\left(\Lambda_n - \tau_n(\lambda_n)\right) = \limsup_{n\to\infty} \operatorname{rel}(\Lambda_n) < \infty,$$

and Lemma (5.3.12)(c) implies that Γ is relatively separated (and in particular closed).

By extracting a subsequence, we assume that $\Lambda - \lambda_n \xrightarrow{\omega} \Gamma'$ for some relatively separated set $\Gamma' \in W(\Lambda)$. We will show that $\Gamma' = \Gamma$ and consequently $\Gamma \in W(\Lambda)$.

Let $R > 0$ and $0 < \epsilon \leq 1$ be given. By (L1), there exists $n_0 \in \mathbb{N}$ such that

$$\lambda, \lambda' \in \Lambda, |\lambda - \lambda'| \leq R, n \geq n_0 \Longrightarrow \left|\left(\tau_n(\lambda) - \tau_n(\lambda')\right) - (\lambda - \lambda')\right| \leq \varepsilon. \tag{108}$$

If $n \geq n_0$ and $z \in (\Lambda - \lambda_n) \cap B_R(0)$, then there exists $\lambda \in \Lambda$ such that $z = \lambda - \lambda_n$ and $|\lambda - \lambda_n| \leq R$. Consequently (108) implies that

$$\left|\left(\tau_n(\lambda) - \tau_n(\lambda_n)\right) - z\right| = \left|\left(\tau_n(\lambda) - \tau_n(\lambda_n)\right) - (\lambda - \lambda_n)\right| \leq \epsilon.$$

This shows that

$$(\Lambda - \lambda_n) \cap B_R(0) \subseteq \left(\Lambda_n - \tau_n(\lambda_n)\right) + B_\varepsilon(0) \quad \text{for } n \geq n_0. \tag{109}$$

Since $\Lambda - \lambda_n \xrightarrow{\omega} \Gamma'$ and $\Lambda_n - \tau_n(\lambda_n) \xrightarrow{\omega} \Gamma$, it follows from (109) and Lemma (5.3.9) that $\Gamma' \subseteq \bar{\Gamma} = \Gamma$.

For the reverse inclusion, let again $R > 0$ and $0 < \varepsilon \leq 1$. Let $\mathrm{R}' > 0$ and $n_0 \in \mathbb{N}$ be the numbers associated with R in(L2). Using (L1), choose $n_1 \geq n_0$ such that

$$\lambda, \lambda' \in \Lambda, |\lambda - \lambda'| \leq \mathrm{R}', n \geq n_1 \ \Longrightarrow \left|\left(\tau_n(\lambda) - \tau_n(\lambda')\right) - (\lambda - \lambda')\right| \leq \varepsilon. \tag{110}$$

If $n \geq n_1$ and $z \in \left(\Lambda_n - \tau_n(\lambda)\right) \cap B_R(0)$, then $z = \tau_n(\lambda) - \tau_n(\lambda)$ For some $\lambda \in \Lambda$ and $|\tau_n(\lambda) - \tau_n(\lambda_n)| \leq R$. Condition (L2) now implies that $|\lambda - \lambda_n| \leq \mathrm{R}'$ and therefore, using (110) with $\lambda' = \lambda_n$, we get

$$|z - (\lambda - \lambda_n)| = |\tau_n(\lambda) - \tau_n(\lambda_n)) - (\lambda - \lambda_n)| \leq \varepsilon.$$

Hence we have proved that

$$z \in (\Lambda_n - \tau_n(\lambda)) \cap B_R(0) \subseteq (\Lambda - \lambda_n) + B_\varepsilon(0), \quad \text{for } n \geq n_1.$$

Since $\Lambda_n - \tau_n(\lambda_n) \xrightarrow{\omega} \Gamma$ and $\Lambda - \lambda_n \xrightarrow{\omega} \Gamma'$, Lemma (5.3.9) implies that $\Gamma \subseteq \overline{\Gamma}' \in W(\Lambda)$, as desired.

(b) since $\rho(\Lambda) < \infty$, Lemma (5.3.21)(c), implies that $\limsup_n \rho(\Lambda_n) < \infty$. By omitting finitely many n, there exists $L > 0$ such that $\Lambda_n + B_L(0) = \mathbb{R}^d$ for all $n \in \mathbb{N}$. This implies the existence of a sequence $\{\lambda_n \geq 1\} \subseteq \Lambda$ such that $|z - \tau_n(\lambda_n)| \leq L$. By passing to a subsequence we may assume that $z - \tau_n(\lambda_n) \to z_0$ for some $z_0 \in \mathbb{R}^d$.

Since $\Lambda_n - z_n \xrightarrow{\omega} \Gamma$ and $z_n - \tau_n(\lambda_n) \to z_0$, it follows that $\Lambda_n - \tau_n(\lambda_n) \xrightarrow{\omega} \Gamma + z_0$. By (a), we deduce that $\Gamma + z_0 \in W(\Lambda)$, as desired.

We now show the main results on the deformation of Gabor systems. The proofs combine the characterization of non-uniform Gabor frames and Riesz sequences without inequalities and the fine details of Lipschitz convergence.

First we formulate the stability of Gabor frames under a class of nonlinear deformations.

Theorem (5.3.1)(a) now follows by combining Theorem (5.3.24) and Lemma (5.3.20). Note that in Theorem (5.3.1) we may assume without loss of generality that $T_n(0) = 0$, because the deformation problem is invariant under translations.

Theorem (5.3.24)[356]: Let $g \in M^1(\mathbb{R}^d)$, $\Lambda \subseteq \mathbb{R}^{2d}$ and assume that $\mathcal{G}(g,\Lambda)$ is a frame for $L^2(\mathbb{R}^d)$. If Λ_n is Lipchitz convergent to Λ, then $\mathcal{G}(g,\Lambda_n)$ is a frame for all sufficiently large n.

Proof. Suppose that $\mathcal{G}(g,\Lambda_n)$ is a frame. According to Lemma (5.3.5*), Λ is relatively separated and relatively dense. Now suppose that the conclusion does not hold. By passing to a subsequence we may assume that $\mathcal{G}(g,\Lambda_n)$ fails to be a frame for all $n \in \mathbb{N}$.

By Theorem (5.3.14) every $\mathcal{G}(g,\Lambda_n)$ also fails to an ∞-frame for $M^\infty(\mathbb{R}^d)$. It follows that for every $n \in \mathbb{N}$ there exist $f_n \in M^\infty(\mathbb{R}^d)$ such that $\|V_g f_n\|_\infty = 1$ and

$$\|C_{g,\Lambda_n}(f_n)\|_{\ell^\infty(\Lambda_n)} = \sup_{\lambda\in\Lambda} |V_g f_n(\tau_n(\lambda))| \to 0. \quad \text{as } n \to \infty.$$

For each $n \in \mathbb{N}$, let $z_n \in \mathbb{R}^{2d}$ be such that $|V_g f_n(z_n)| \geq 1/2$ and let us consider $h_n := \pi(-z_n) f_n$. By passing to a subsequence we may assume that $h_n \to h$ in $\sigma(M^\infty, M^1)$ for some function $h \in M^\infty$. Since $|V_g h_n(0)| = |V_g f_n(z_n)| \geq 1/2$, it follows that $|V_g h_n(0)| \geq 1/2$ and the weak*-limit h is not zero.

In addition, by Lemma (5.3.21)

$$\limsup_{n\to\infty} \operatorname{rel}(\Lambda_n - z_n) = \limsup_{n\to\infty} \operatorname{rel}(\Lambda_n) < \infty.$$

Hence, using Lemma (5.3.12) and passing to a further subsequence, we may assume that $\Lambda_n - z_n(\lambda_n) \xrightarrow{\omega} \Gamma$, for some relatively separated set $\Gamma \subseteq \mathbb{R}^{2d}$. Since A is relatively dense, Lemma (5.3.23) guarantees that $\Gamma \in W(\Lambda)$.

Let $\gamma \in \Gamma$ be arbitrary. Since $\Lambda_n - z_n(\lambda_n) \xrightarrow{\omega} \Gamma$, there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq \Lambda$ such that $\tau_n(\lambda_n) - z_n \to \gamma$. By Lemma (5.3.4), the fact $h_n \to h$ in $\sigma(M^\infty, M^1)$ implies that $V_g h_n \to V_g h$ uniformly on compact sets. Consequently, by (90),

$$|V_g h(\gamma)| = \lim_n |V_g h_n(\tau_n(\lambda_n) - z_n)| = \lim_n |V_g h_n(\tau_n(\lambda_n))| = 0.$$

Hence, $h \not\equiv 0$ and $V_g h \equiv 0$ on $\Gamma \in W(\Lambda)$. According to Theorem (5.3.14)(vi), $\mathcal{G}(g,\Lambda)$ is not a frame, thus contradicting the initial assumption.

The corresponding deformation result for Gabor Riesz sequences reads as follows.

Theorem (5.3.25)[356]: Let $g \in M^1(\mathbb{R}^d)$, $\Lambda \subseteq \mathbb{R}^{2d}$, and assume that $\mathcal{G}(g,\Lambda)$ is a Riesz sequences reads in $L^2(\mathbb{R}^d)$.

If Λ_n is Lipchitz convergent to, then $\mathcal{G}(g,\Lambda_n)$ is a Riesz sequence for all sufficiently large n.

Proof. Assume that $\mathcal{G}(g,\Lambda)$ is a Riesz sequence, Lemma (5.3.5*) implies that Λ is separated. With Lemma (5.3.21) we may extract a subsequence such that each Λ_n is separated with a uniform separation constants, i.e.,

$$\inf_{n\geq 1} \operatorname{sep}(\Lambda_n) > 0. \tag{111}$$

we argue by contradiction and assume that the conclusion does not hold. By passing to a further subsequence, we may assume that $\mathcal{G}(g,\Lambda_n)$ fails to be a Riesz sequence for all $n \in \mathbb{N}$. As a consequence of Theorem (5.3.17)(iii), there exist sequences $c^n \in \ell^\infty(\Lambda_n)$ such that $\|c^n\|_\infty = 1$ and

$$\left\|C^*_{g,A_n}(c^n)\right\|_{M^\infty} = \left\|\sum_{\lambda\in\Lambda} c^n_{\tau_n(\lambda_n)}\pi\big(\tau_n(\lambda_n)\big)\,g\right\|_{M^\infty} \to 0,\ \text{ as } n \to \infty. \tag{112}$$

Thus g, Λ_n, c^n satisfy the assumptions of Lemma (5.3.16) the conclusion of Lemma (5.3.16) yields a subsequence (n_k), a separated set $\Gamma \subseteq \mathbb{R}^{2d}$, a non-zero sequence $c \in \ell^\infty(\Gamma)$, and a sequence of points $\{\lambda_{n_k} : k \geq 1\} \subseteq \Lambda$ such that

$$\Lambda_{n_k} - \tau_{n_k}\big(\lambda_{n_k}\big) \xrightarrow{\omega} \Gamma$$

and

$$\sum_{\gamma\in\Gamma} c_\gamma \pi(\gamma) g = C^*_{g,\Gamma}(c) = 0.$$

By Lemma (5.3.23), we conclude that $\Gamma \in W(\Lambda)$. According to condition (vi) of Theorem (5.3.17), $\mathcal{G}(g,\Lambda)$ is not a Riesz sequence, which is a contradiction.

Example (5.3.26)[356]: From [368] or from Theorem (5.3.24) we know that if $g \in M^1(\mathbb{R}^d)$ and $\mathcal{G}(g,\Lambda)$ is a frame, then $\mathcal{G}(g,(1+1/n)\Lambda)$ is also a frame for sufficiently large n. For every n we now construct a deformation of the form $\tau_n(\lambda) := \alpha_{\lambda,n}\lambda$ where $\alpha_{\lambda,n}$ is either 1 or $(1+1/n)$ with roughly half of the multipliers equal to 1. Since only a subset of A is moved, we would think that this deformation is "smaller" than the full dilation $\lambda \to \left(1+\frac{1}{n}\right)\lambda$, and thus it should preserve the spanning properties of the corresponding Gabor system.

Surprisingly, this is completely false. We now indicate how the coefficients $\alpha_{\lambda,n}$ need to be chosen. Let $\mathbb{R}^{2d} = \bigcup_{l=0}^{\infty} B_l$ be a partition of $\mathbb{R}^{2d}$ into the annuli

$$B_l = \left\{z \in \mathbb{R}^{2d} : \left(1+\frac{1}{n}\right)^l \leq |z| < \left(1+\frac{1}{n}\right)^{l+1}\right\}, \quad l \geq 1,$$

$$B_0 := \left\{z \in \mathbb{R}^{2d} : |z| < \left(1+\frac{1}{n}\right)\right\}$$

and define

$$\alpha_{\lambda,n} = \begin{cases} 1 & \text{if } \lambda \in B_{2l}, \\ 1 + \dfrac{1}{n} & \text{if } \lambda \in B_{2l+1}. \end{cases}$$

Since $\left(1 + \frac{1}{n}\right) B_{2l+1} = B_{2l+2}$, the deformed set $\Lambda_n = \tau_n(\Lambda) = \{\alpha_{\lambda,n}\lambda: \lambda \in \Lambda\}$ is contained in $\bigcup_{l=0}^{\infty} B_{2l}$ and thus contains arbitrarily large holes. So $\rho(\Lambda_n) = \infty$ and $D^-(\Lambda_n) = 0$. Consequently the corresponding Gabor system $\mathcal{G}(g, \Lambda_n)$ cannot be frame. See Fig. 2 for a plot of this deformation in dimension 1.

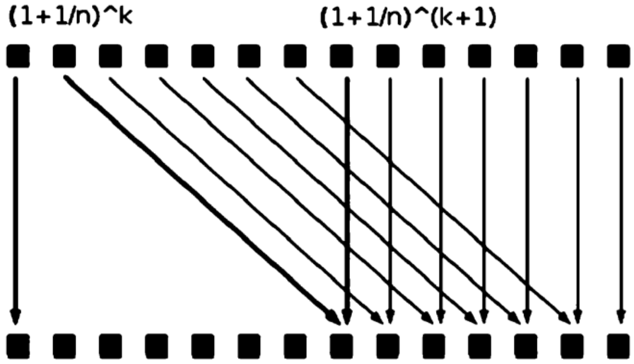

Fig. (2)[356]: A deformation "dominated" by the dilation $\lambda \to (1 + 1/n)\lambda$.

Chapter 6
Vector-Valued Gabor Frames with Hermits Functions

We show that the main tools are growth estimates for the Weierstrass σ-function, a new type of interpolation problem for entire functions on the Bargmann–Fock space, and structural results about vector-valued Gabor frames. We characterize those sets $\mathbb{S}$ admitting tight Gabor sets, and obtain an explicit construction of a class of tight Gabor sets in such $\mathbb{S}$ for the case that the product of time-frequency shift parameters is a rational number. By introducing a suitable Zak transform matrix, we characterize completeness and frame condition of Gabor systems, obtain a necessary and sufficient condition on Gabor duals of type I (resp. II) for a general Gabor frame, and establish a parametrization expression of Gabor duals of type I (resp. II). All conclusions are closely related to corresponding Zak transform matrices. This allows to easily realize these conclusions by designing the corresponding matrix-valued functions. An example theorem is also presented.

Section (6.1): Gabor Super Frames

Given a function $g \in L^2(\mathbb{R})$ and a lattice $\Lambda \subset \mathbb{R}^2$, we study the frame property of the set $\{e^{2\pi i\lambda_2 t} g(t-\lambda_1): \lambda = (\lambda_1, \lambda_2) \in \Lambda\}$. Precisely, write $\pi_\lambda g = e^{2\pi i\lambda_2 t} g(t-\lambda_1)$ for the time-frequency shift by $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$. Then we call the set $\mathcal{G}(g, \Lambda) = \{\pi_\lambda g: \lambda \in \Lambda\}$ a Gabor frame or Weyl-Heisenberg frame, whenever there exist constants $A, B > 0$ such that, for all $f \in L^2(\mathbb{R})$,

$$A\|f\|^2_{L^2(\mathbb{R})} \le \sum_{\lambda\in\Lambda} \left|\langle f, \pi_\lambda g\rangle_{L^2(\mathbb{R})}\right|^2 \le B\|f\|^2_{L^2(\mathbb{R})}. \tag{1}$$

Gabor frames originate in quantum mechanics through J. von Neumann and in information theory through Gabor [418] and nowadays have many application in signal processing. A large body of results describes the structure of Gabor frames and provides sufficient conditions for $\mathcal{G}(g, \Lambda)$ to form a (Gabor) frame, see [412, 420].

We will also study vector-valued Gabor frames in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}, \mathbb{C}^n)$ of all vector-valued functions $f(t) = (f_1(t), \dots, f_n(t))$ with inner product

$$\langle f, g\rangle = \sum_{j=1}^{n} \int_{-\infty}^{\infty} f_j(t)\overline{g_j(t)}dt = \sum_{j=1}^{n} \langle f_j, g_j\rangle. \tag{2}$$

The *time-frequency shifts* $\pi_z, z = (x, \xi)$ act coordinate –wise by

$$\pi_z f(t) = e^{2i\pi\xi t} f(t-x). \tag{3}$$

The vector-valued Gabor system $\mathcal{G}(g, \Lambda) = \{\pi_z g : \lambda \in \Lambda\}$ is a frame for $L^2(\mathbb{R}, \mathbb{C}^n)$, if there exist constant $A, B > 0$ such that

$$A\|f\|^2_{L^2(\mathbb{R},\mathbb{C}^n)} \le \sum_{\lambda\in\Lambda} |\langle f, \pi_z g\rangle|^2 \le B\|f\|^2_{L^2(\mathbb{R},\mathbb{C}^n)}, \quad \forall f \in L^2(\mathbb{R}, \mathbb{C}^n). \tag{4}$$

If only the inequality on the right-hand side is satisfied, then the Gabor system $\mathcal{G}(g, \Lambda)$ is called a Bessel sequence.

Vector-valued Gabor frames were introduced under the name "super frames" in signal processing by Balan [409] in the context of multiplexing and studied in [408,424] for their own sake.The idea of multiplexing is to encode n independent signals (functions) $f_j \in L^2(\mathbb{R}), j = 1, \dots, n$, as a single sequence that captures the time-frequency information of each f_j. Fixing suitable windows $g_j \in L^2(\mathbb{R})$ and using vector-valued notation $f = (f_1, \dots, f_n)$, one then considers the sequence of numbers $\langle f, \pi_z g\rangle = \sum_{j=1}^{n} \langle f_j, \pi_\lambda g_j\rangle$ for $\lambda \in \Lambda$, i.e., the inner product in $L^2(\mathbb{R}, \mathbb{C}^n)$. These numbers then measure the time-frequency content of the whole f at the point λ in the time- frequency plane.

Now one requires that f is completely determined by these inner products and that there exists a stable reconstruction. This requirement leads to the definition of a vector-valued Gabor frame (4).

The general problem of characterizing all lattices Λ for which $\mathcal{G}(g, \Lambda)$ is a frame seems to be extremely difficult. In fact, this problem is solved only for three classes of basis functions, namely for the Gaussians $H_0(t) = e^{-at^2}, a > 0$, in [432,435], for hyperbolic secant $g(t) = (\cosh at)^{-1}$ in [431]. For the one-sided exponential function $g(t) = e^{-a|t|}\chi_{[0,\infty)}(t)$ [429]. For the Gaussian, our understanding is based on the connection between the frame property of $\mathcal{G}(g, \Lambda)$ and a classical interpolation problem in the Bargmann-Fock space of entire functions. The case of the hyperbolic secant is reduced to the case of the Gaussian.

For other choices of the g, where the connections for $\mathcal{G}(g, a\mathbb{Z} \times b\mathbb{Z})$ to form a frame become very different and are often rather intriguing [430]. Even less is known about Gabor super frames. Necessary density conditions were studied in [408], sufficient density conditions can be derived from coorbit theory [414] and from the sampling theory on the Heisenberg group [417]. In particular, these results imply that for any sufficiently dense lattice Λ and a mild condition on the vector g the Gabor system $\mathcal{G}(g, \Lambda)$ is a Gabor super frame.

We study the frame property of $\mathcal{G}(h, \Lambda)$ in the case when h is the vector of the first $n+1$ Hermite function $h = (H_0, \dots, H_n)$. In the scalar-valued case, the study of Gabor frames with the Gaussian window H_0 is natural, because the Gaussian minimizes the uncertainty principle. Likewise, for vector-valued frames, the study of Gabor super frames with the Hermit window h is natural, because the first Hermite functions are the unique orthonormal set $\{f_0, \dots, f_n\}$ in $L^2(\mathbb{R})$ of size $n+1$ satisfying the normalizations $\int_{-\infty}^{\infty} t\left|f_j(t)\right|^2 dt = 0$ and $\int_{-\infty}^{\infty} \xi\left|\hat{f}_j(\xi)\right|^2 d\xi = 0, j = 0, \dots, n$ such that the uncertainty

$$\sum_{j=0}^{n}\left(\int_{-\infty}^{\infty} t^2\left|f_j(t)\right|^2 dt + \int_{-\infty}^{\infty} \xi^2\left|\hat{f}_j(\xi)\right|^2 d\xi\right) \tag{5}$$

is minimized. Another motivation comes again from signal processing where Hermite functions are used, see [425].

The case of the Hermite vector window has been already investigated by employing techniques related to sampling in the space of band limited functions on the Heisenberg group. He proved the following result.

Theorem (6.1.1)[406]: let $h = (H_0, \dots, H_n)$. There exists a constant $C > 0$ with the following property: If the diameter of the (smallest) fundamental domain of Λ is less that $C(n+1)^{-1/2}$, then $\mathcal{G}(h, \Lambda)$ is a frame for $L^2(\mathbb{R}, \mathbb{C}^{n+1})$.

Unfortunately nothing can be said about the constant C within such an approach. Fuhr uses the so-called "oscillation method" which is used for existence results, but in general does not yield sharp results.

We give a complete characterization of vector-valued frames with Hermite functions. If the lattice is given as $\Lambda = A\mathbb{Z}^2$ for some invertible, real-valued 2×2-matrix A, let $s(\Lambda) = |\det A|$ be the area of the fundamental domain of Λ. The main result can be formulated as follows.

By specializing to the n-th coordinate of the h, we obtain a condition for scalar-valued Gabor frames with Hermite functions. The following result was already announced in [423].

Proposition (6.1.2)[406]: If $s(\Lambda) < \frac{1}{n+1}$, then $\mathcal{G}(H_n, \Lambda)$ is a frame for $L^2(\mathbb{R})$.

For $n = 0$, the case of the Gaussian, we recover (the lattice part of) the results in [432,435]. Our method of proof yields several new results about the dual window. On the one hand, we construct a dual window for $\mathcal{G}(H_n, \Lambda)$ with Gaussian decay in time and frequency; on the other hand, we derive a new estimate for the lower frame bound of $\mathcal{G}(H_0, \Lambda)$. Furthermore, we discuss an example for $n = 1$ which suggests that the sufficient conditions of Proposition (6.1.2) are sharp for all n.

In order to show these results we combine the techniques of Gabor analysis and complex-analytic methods. The (now) classical structural results related to the scalar Gabor frame systems can be formulated for the vector case as well. They lead to an interpolation problem in the Fock space of entire functions.

This problem is not "purely holomorphic": the values of linear combinations of functions from the Fock space and their derivatives are prescribed in the lattice points; however, the coefficients of such combinations are anti-holomorphic polynomials. The classical methods of complex analysis cannot be applied for problems of this kind. Fortunately, in our particular case, i.e., for the lattices of sufficiently small density, one may use the well-developed machinery of elliptic functions.

We collect the necessary facts related to vector-valued frames, in particular, the frame operator and the vector-valued version of the structural theorems: Janssen's representation and the Wexler-Raz biorthogonality criteria .the complex analytic tools, including the Fock spaces and the Weierstrass σ-function ,are presented. We contain the proofs of Theorem (6.1.12) and Proposition (6.1.2) we also show a result (Proposition (6.1.14)) indicating that the density condition in Proposition (6.1.2) might be sharp. The rest contains estimates of dual windows for the Hermitian frames and of the lower frame bound.

To derive the criterion for Gabor super frames from Hermit functions Theorem (6.1.12), we need some general results about Gabor super frames, such as Janssen's representation of the frame operator and the Wexler-Raz identities. Although these are among the fundamentals results about Gabor frames, they have not yet been formulated for Gabor super frames. For convenience and later reference we formulate them explicitly for Gabor frames on $\mathbb{R}^d$ (instead of $\mathbb{R}$). Since the vector-valued case is a simple consequence of the well-known scalar-valued case.

Let $\Lambda = A\mathbb{Z}^{2d}$ be a lattice in $\mathbb{R}^{2d}$, where A is a non-singular real $2d \times 2d$-matrix. Let $s(\Lambda) = |\det A|$ be the volume of a fundamental domain of Λ. The adjoint lattice is defined by the commutant property as

$$\Lambda^\circ = \{\mu \in \mathbb{R}^{2d} : \pi(\lambda)\pi(\mu) = \pi(\mu)\pi(\lambda) \text{ for all } \lambda \in \Lambda\}. \tag{6}$$

If $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$, then $\Lambda^\circ = \beta^{-1}\mathbb{Z}^d \times \alpha^{-1}\mathbb{Z}^d$. For general $\Lambda = A\mathbb{Z}^{2d} \subseteq \mathbb{R}^{2d}$

$$\Lambda^\circ = \mathcal{J}(A^T)^{-1}\mathbb{Z}^{2d}, \tag{7}$$

where A^T is the transpose of A and $\mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (consisting of $d \times d$ blocks) is the matrix defining the standard symplectic form [415].

Furthermore, from (7) we see that
$$s(\Lambda^\circ) = s(\Lambda)^{-1}. \quad (8)$$
The density of Λ is defined as $d(\Lambda) = s(\Lambda)^{-1}$, so that $d(\Lambda)$ coincides with the usual notions of density.

Given two vector-valued functions (windows) $\gamma = (\gamma_j)_1^n, g = (g_j)_1^n \in L^2(\mathbb{R}^d, \mathbb{C}^n)$, the associated vector-valued Gabor frame-type operator is defined to be
$$Sf = S_{g,\gamma}^{\Lambda} f = \sum_{\lambda \in \Lambda} \langle f, \pi_\lambda g \rangle \pi_\lambda \gamma. \quad (9)$$
For arbitrary $g, \gamma \in L^2(\mathbb{R}^d, \mathbb{C}^n)$ the right-hand side in (9) defines a continuous operator from $S(\mathbb{R}^d, \mathbb{C}^n)$ to $S'(\mathbb{R}^d, \mathbb{C}^n)$ with weak*-convergence of the sum.

Under slight conditions on the component functions g_j and γ_j the series in (9) converges in $L^2(\mathbb{R}^d, \mathbb{C}^n)$-norm .Recall that a function g on $\mathbb{R}^d$ belongs to the Feichtinger a lgebra $M^1(\mathbb{R}^d)$, if
$$\|g\|_{M^1} := \int_{\mathbb{R}^{2d}} |\langle g, \pi_z \varphi \rangle| dz < \infty, \quad (10)$$
where $\varphi(t) = 2^{d/4} e^{-\pi t^2}$ is the L^2-normalized Gaussian. This condition is met if, for example, both g and its Fourier transform $\hat{g}$, decay sufficiently fast, see [420, Ch,7], and also [419] for discussion and the proofs. The convergence of (9) follows from the following lemma taken from [420] by looking at each component separately (the statement is not stated as explicitly as we want it, but follows from combining Propositions 11.1.4, 12.1.11, and 12.2.1).

Lemma (6.1.3)[406]: If $g \in M^1(\mathbb{R}^d)$, then
$$\sum_{\lambda \in \Lambda} |\langle f, \pi_\lambda g \rangle|^2 \le n(\Lambda) \|g\|_{M^1}^2 \|f\|_2^2 \quad (11)$$
and
$$\left\| \sum_{\lambda \in \Lambda} C_\lambda \pi_\lambda g \right\|_2 \le n(\Lambda)^{1/2} \|g\|_{M_1} \|c\|_2, \quad (12)$$
where $n(\Lambda) = c\max_{k \in \mathbb{Z}^{2d}}$ card $(\Lambda \cap (K + [0,1]^{2d}))$ for some absolute constant $c > 0$.

As a consequence, if $g = (g_j)_1^n \in M^1(\mathbb{R}^d)$ for $j = 1, .., n$, then the sequence $\mathcal{G}(g, \Lambda)$ is a Bassel sequence in $L^2(\mathbb{R}^d, \mathbb{C}^n)$ for every lattice.

As in the scalar case, the frame operator can be represented as a sum of time-frequency shifts over the adjoint lattice.

In the scalar case $n = 1$, this is a theorem of Rieffel [434] and Janssen [428], the vector-valued version follows easily from the scalar case. Given a Gabor (super) frame $\mathcal{G}(g, \Lambda)$ with $g \in L^2(\mathbb{R}^d, \mathbb{C}^n)$, we say that $\gamma \in L^2(\mathbb{R}^d, \mathbb{C}^n)$ is a dual window (with respect to the lattice Λ) if
$$S_{g,\gamma}^{\Lambda} = S_{\gamma,g}^{\Lambda} = I. \quad (13)$$

Dual windows always exist, a special choice is given by the canonical dual window $\gamma^\circ \coloneqq \left(S_{g,g}^{\Lambda}\right)^{-1} g$; clearly the invertibility of $S_{g,g}^{\Lambda}$ is equivalent to the form property (4).

From Janssen's representation we obtain a criterium for the system $\mathcal{G}(g,\Lambda)$ to form a frame in $L^2(\mathbb{R}^d,\mathbb{C}^n)$ with dual window γ, that is a vector analog of the Wexler-Raz condition.

As in the scalar case we obtain Balan's necessary density condition [408] by adjusting an argument of Janssen [427].

Proposition (6.1.4)[406]: (Density theorem). If $\mathcal{G}(g,\Lambda)$ is a frame for $L^2(\mathbb{R}^d,\mathbb{C}^n)$, then $s(\Lambda) \le n^{-1}$.

Proof. Let $\gamma^\circ = S_{g,g}^{-1} g$ be the canonical dual window. Then $f = \sum_{\lambda\in\Lambda}\langle f,\pi_\mu\gamma^\circ\rangle\,\pi_\lambda g$ holds for every $f \in L^2(\mathbb{R}^d,\mathbb{C}^n)$. If f has also the representation $f = \sum_{\lambda\in\Lambda} c_\lambda \pi_\lambda g$ then by [413]

$$\sum_{\lambda\in\Lambda}\left|\langle f,\pi_\mu\gamma^\circ\rangle\right|^2 \le \sum_{\lambda\in\Lambda}|c_\lambda|^2.$$

we apply this argument to the trivial expansion $g = 1\cdot g + \sum_{\lambda\neq 0} 0\cdot\pi_\lambda g$. thus we obtain

$$|\langle g,\gamma^\circ\rangle|^2 \le \sum_{\lambda\in\Lambda}|\langle g\,,\pi_\lambda\gamma^\circ\rangle|^2 \le 1.$$

The Wexler-Raz identities (56) yield

$$\langle g\,,\gamma^\circ\rangle = \sum_{j=1}^{n}\langle g_j,\gamma_j\rangle = ns(\Lambda).$$

and the result follows.

The Wexler-Raz relations also yield an estimate for the lower frame bound that deserves to be better known and is stated here for the scalar case.

Corollary (6.1.5)[406]: Assume that $\mathcal{G}(g,\Lambda)$ is a frame for $L^2(\mathbb{R}^d)$ with some dual $\gamma \in M^1(\mathbb{R}^d)$. Then the optimal lower frame bound $A_{\text{opt}} = \left\|S_{g,g}^{-1}\right\|_{L^2\to L^2}^{-1}$ satisfies

$$A_{\text{opt}} \ge n(\Lambda)^{-1}\|\gamma\|_{M^1}^{-2}. \tag{14}$$

Proof. Since $f = S_{g,\gamma}^{\Lambda} f = \sum_{\lambda\in\Lambda}\langle f,\pi_\lambda g\,\rangle\,\pi_\lambda\gamma$. Lemma (6.1.3) implies that

$$\|f\|_2^2 \le n(\Lambda)\|\gamma\|_{M^1}^2\sum_{\lambda\in\Lambda}|\langle f,\pi_\lambda g\,\rangle|^2,$$

whence the estimate for the lower frame bound follows.

The following duality result is not needed, but is included for completeness. It answers a question of our engineering colleague G.Matz, see [425].

The complex analytic techniques we use are concentrated around the Fock space of entire functions and precise estimates for the Weierstrass σ-function. These topics are closely related: roughly speaking σ-function deliver examples of functions of "maximal possible growth" in the Fock space.

We recall the basic properties of the Fock space, as discussed and proved, for instance, in [416,420].

The Fock $\mathcal{F}$ is the Hilbert space of all entire functions such that

$$\|F\|_{\mathcal{F}}^2 = \int_{\mathbb{C}} |F(z)|^2 e^{-\pi|z|^2} dm_z < \infty, \qquad (15)$$

where dm is the planaer Lebesgue measure. The natural inner product in $\mathcal{F}$ is denoted by $\langle .,. \rangle_{\mathcal{F}}$.

Below we list the properties of the Fock space, which will be used in the sequel.

(a) Each point evaluation is a bounded linear functional in $\mathcal{F}$, and the corresponding reproducing kernel is the function $w \mapsto e^{\pi \bar{Z} w}$, precisely,

$$F(z) = \langle F(w), e^{\pi \bar{z} w} \rangle_{\mathcal{F}} \qquad \forall F \in \mathcal{F}. \qquad (16)$$

(b) A function in $\mathcal{F}$ grows at most like the Gaussian (see e.g. [433]), more precisely,

$$|F(z)| = o(1) e^{\frac{\pi}{2}|z|^2}, \ \ as\, z \to \infty, \ \ F \in \mathcal{F}. \qquad (17)$$

(c) The collection of monomials

$$e_n(z) = \left(\frac{\pi^n}{n!}\right)^{1/2} z^n, \ \ n = 0,1, \dots \qquad (18)$$

forms an orthonormal basis in $\mathcal{F}$.

(d) Define the Bargmann transform of a function $f \in L^2(\mathbb{R}^d)$ by

$$f \longmapsto \mathcal{B}f(z) = F(z) = 2^{1/4} e^{-\pi z^2/2} \int_{\mathbb{R}} f(t) e^{-\pi t^2} e^{2\pi t z} dt. \qquad (19)$$

Proposition (6.1.6)[406]: The Bargmann transform is a unitary mapping between $L^2(\mathbb{R})$ and $\mathcal{F}$.

(e) The Hermite functions are defined by

$$H_n(t) = c_n e^{\pi t^2} \frac{d^n}{dt^n} e^{-2\pi t^2}, \qquad n = 0,1,2, \dots \qquad (20)$$

where the coefficients c_n are chosen in order to have $\|H_n\|_2 = 1$. It is a classical result that the set of Hermite functions $\{H_n\}_{n=0}^{\infty}$ forms an orthonormal basis in $L^2(\mathbb{R}^d)$.

Their image under the Bargmann transform is $\{e_n\}_{n=0}^{\infty}$-the natural orthonormal basis in $\mathcal{F}$:

$$\mathcal{B}H_n(z) = e_n(z), \quad n = 0,1, \dots. \qquad (21)$$

(f) In what follows we identify $\mathbb{C}$ and $\mathbb{R}^2$. In particular for each $\zeta = \xi + i\eta \in \mathbb{C}$ we write $\pi_\zeta = \pi_{(\xi,\eta)}$. Define the shift $\beta_\zeta : \mathcal{F} \to \mathcal{F}$ in the Fock space by

$$\beta_\zeta \mathrm{F(z)} = e^{i\pi\xi\eta} e^{-\pi|\zeta|^2/2} e^{\pi\xi z} F(z - \bar{\zeta}). \qquad (22)$$

Then β_ζ is unitary on $\mathcal{F}$, and the Bargmann transform intertwines the Fock space shift and the time-frequency shift:

$$\beta_\zeta \mathcal{B} = \mathcal{B}\pi_\zeta. \qquad (23)$$

With these properties one can easily obtain the inner product of a function $\mathcal{F} \in L^2(\mathbb{R}^d)$ with the time-frequency shifts of a Hermite function as in [411]. This quantity is the short-time Fourier transform with respect to a Hermite function.

Proposition (6.1.7)[406]: Let $f \in L^2(\mathbb{R})$ and $F(z) = \mathcal{B}f(z)$. Then, for all $\zeta \in \mathbb{C}$,

$$\langle f, \pi_\xi H_n \rangle_{L^2(\mathbb{R}^d)} = \frac{1}{\sqrt{H^n n!}} e^{-i\pi\xi\eta} e^{-\frac{\pi|\zeta|^2}{2}} \sum_{k=0}^{n} \binom{n}{k} (-\pi\zeta)^k F^{(n-k)}(\bar{\zeta}). \qquad (24)$$

Proof. Using the intertwining property (23), we have

$\langle f, \pi_\xi H_n \rangle_{L^2(\mathbb{R})} = \langle F, \beta_\xi \mathcal{B} H_n \rangle_{\mathcal{F}}$

$$= \left(\frac{\pi^n}{n!}\right)^{1/2} e^{-i\pi\xi\eta} e^{-\frac{\pi|\zeta|^2}{2}} \langle F(z), e^{\pi\zeta z}(z-\bar{\zeta})^n\rangle$$
$$= \left(\frac{\pi^n}{n!}\right)^{1/2} e^{-i\pi\xi\eta} e^{-\frac{\pi|\zeta|^2}{2}} \sum_{k=0}^{n} \binom{n}{k} (-\zeta)^k \langle F(z), z^{n-k} e^{\pi\zeta z}\rangle$$
$$= \left(\frac{\pi^n}{n!}\right)^{1/2} e^{-i\pi\xi\eta} e^{-\frac{\pi|\zeta|^2}{2}} \sum_{k=0}^{n} \binom{n}{k} (-\zeta)^k \pi^{-n+k} \frac{d^{n-k}}{d\bar{\zeta}^{n-k}} \langle F(z), e^{\pi\zeta z}\rangle.$$

It remains to apply relation (16).

Finally we give a description of the space M^1 in terms of the Bargmann transform. Since $|\langle f, \pi_{\bar{z}} H_0\rangle| = |\mathcal{B}f(z)| e^{-\frac{|z|^2}{2}}$ (e.g., [420]or [416]), we obtain the following reformulation of condition (10).

Proposition (6.1.8)[406]: A function $f \in L^2(\mathbb{R})$ belongs to M^1, if and only if its Bargmann transform $F(z) = \mathcal{B}f$ satisfies

$$\|f\|_{M^1} = \int_{\mathbb{C}} |f(z)| e^{-\pi|z|^2/2} dm_z < \infty. \tag{25}$$

We collect definitions and known facts about the Weierstrass functions. They will be used.

Given two numbers $\omega_1, \omega_2 \in \mathbb{C}$ such that

$$\Im(\omega_2/\omega_1) > 0. \tag{26}$$

we consider the lattice $\Lambda = \{m_1\omega_1 + m_2\omega_2 : m_1, m_2 \in \mathbb{Z}\} \subset \mathbb{C}$. The numbers $\omega_1, \omega_2 \in \mathbb{C}$ are called periods of the lattice Λ. A lattice on $\mathbb{R}^2$ written in the form $\Lambda' = A\mathbb{Z}^2 \subset \mathbb{R}^2$ with $\det A > 0$ coincides with Λ after the natural identification of $\mathbb{R}^2$ and $\mathbb{C}$. in this case ω_1, ω_2 correspond to the columns of A.

The parallelogram $\Pi_\Lambda = A[0,1)^2$ with basis on ω_1 and ω_2 is a fundamental domain for Λ (also called the period parallelogram). Its area can be expressed through the periods as

$$s(\Lambda) = \text{Area}(\Pi_\Lambda) = \det A = \Im(\bar{\omega}_1\omega_2) = -\frac{i}{2}(\bar{\omega}_1\omega_2 - \bar{\omega}_2\omega_1). \tag{27}$$

Next consider the Weierstrass sigma-function (σ-function):

$$\sigma(z) = z \prod_{\omega\in\Lambda'} \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{z^2}{2\omega^2}}, \tag{28}$$

where $\Lambda' = \Lambda\backslash\{0\}$. The lattice Λ plays a special role for the Weierstrass σ-function: σ has simple zeros precisely on Λ and no other zeros.

The main property of the Weierstrass σ-function is its quasi-periodicity. This means that there exist $\eta_1, \eta_2 \in \mathbb{C}$, such that

$$\sigma(z + \omega_k) = -\sigma(z) e^{\eta_k z + \frac{1}{2}\eta_k\omega_k}, \quad z \in \mathbb{C}, k = 1,2. \tag{29}$$

See, e.g., [407] for detailed proofs. Furthermore, the constants η_k satisfy the Legendre relation

$$\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i. \tag{30}$$

In order to understand the growth properties of the Weierstrass σ-function in dependence of the lattice, we follow an elegant argument of Hayman [426]. He realized that, after a proper normalization and growth compensation, the absolute value of σ

becomes a double periodic function. We include this arguments for the sake of completeness.
Proposition (6.1.9)[406]: Set

$$\alpha(\Lambda) = \frac{i\pi}{\overline{\omega}_1\omega_2 - \overline{\omega}_2\omega_1} = \frac{\pi}{2S(\Lambda)}, \quad a(\Lambda) = \frac{\eta_2\overline{\omega}_1 - \eta_2\overline{\omega}_2}{2\omega_1\overline{\omega}_2 - \omega_2\overline{\omega}_1}. \tag{31}$$

and

$$\sigma_\Lambda(z) = \sigma(z)e^{a(\Lambda)z^2}. \tag{32}$$

Then the function $|\sigma_\Lambda(z)|e^{-\alpha(\Lambda)|z|^2}$ is periodic with periods ω_1 and ω_2.
Proof. It follows from (29) that

$$\left|\sigma(z+\omega_k)e^{a(\Lambda)(z+\omega_k)^2}\right|e^{-\alpha(\Lambda)|z+\omega_k|^2} = |\sigma(z)|e^{a(\Lambda)|z|^2}e^{\Re A_k(z)}, \ \ k = 1,2,$$

where

$$A_k(z) = z(\eta_k + 2a\omega_k - 2\alpha\overline{\omega}_k) + \left(\frac{1}{2}\eta_k\omega_k + a\omega_k^2 - \alpha|\omega_k|^2\right), \ \ k = 1,2. \tag{33}$$

An explicit calculation, based on relations (31) and (30), shows that

$$\eta_k + 2a\omega_k - 2\alpha\overline{\omega}_k = 0, \qquad k = 1,2.$$

and also

$$\Re\left(\frac{1}{2}\eta_k\omega_k + a\omega_k^2 - \alpha|\omega_k|^2\right) = 0, \quad k = 1,2.$$

Proposition (6.1.9) now follows.
The periodicity implies a groth estimate for the modified sigma-function σ_A.
Proposition (6.1.10)[406]: Set

$$c(\Lambda) = \sup_{z\in\Pi_\Lambda} |\sigma_\Lambda(z)|e^{-\alpha(\Lambda)|z|^2}. \tag{34}$$

Then

$$|\sigma_\Lambda(z)| \le c(\Lambda)e^{\frac{\pi}{2s(\Lambda)}|z|^2}, \quad \forall z \in \mathbb{C}. \tag{35}$$

For each $\epsilon > 0$ we have

$$|\sigma_\Lambda(z)| \asymp e^{\alpha(\Lambda)|z|^2}, \quad \text{whenever } \operatorname{dist}(z,\Lambda) > \epsilon. \tag{36}$$

Proof. The function $|\sigma_\Lambda(z)|e^{-\alpha(\Lambda)|z|^2}$ is bounded in Π_Λ and has its only zeros at the vertices of Π_Λ. Hence it is bounded away from 0 on every compact subset of Π_Λ that does not contain its vertices. Relation (36) follows now from the periodicity.

We show Theorem (6.1.12) and thus give a complete characterization of Gabor superframes with Hermite functions.

We divide proof into several steps. First we will use the matrix form of the Wexler-Raz biorthogonality relations (56) and translate these relations to an interpolation problem on Fock space.

Let, as earlier, $h = (H_j)_{j=0}^n$ be the vector-valued window consisting of the first $n+1$ Hermite functions, For each $\gamma = (\gamma_j)_{j=0}^n \in L^2(\mathbb{R}, \mathbb{C}^{n+1})$ denote

$$G_j = \mathcal{B}_{\gamma_j}, \quad j = 0,1,\dots,n. \tag{37}$$

Taking $g = h$ in the biorthogonality relations (56) and using Proposition (6.1.7). We can rewrite these relations as an interpolation problem for the function $G_j \in \mathcal{F}$:

$$\langle\gamma_j, \pi_\mu H_l\rangle_{L^2(\mathbb{R})} = e^{-i\pi\Im\mu\Re\mu}e^{-\pi|\mu|^2/2}\frac{1}{\sqrt{\pi^l l!}}\sum_{k=0}^{l}\binom{l}{k}(-\pi\mu)^k G_j^{(l-k)}(\bar{\mu})$$

$$= s(\Lambda)\delta_{\mu,0}\delta_{j,l}, \quad j,l = 0,1,\dots,n; \mu \in \Lambda^\circ. \tag{38}$$

The $\mathcal{G}(h,\Lambda)$ is a frame for $L^2(\mathbb{R},\mathbb{C}^{n+1})$, if and only if there exist functions $G_j \in \mathcal{F}$ satisfying (38) and if $\mathcal{G}(\gamma,\Lambda)$ is a Bessel sequence in $L^2(\mathbb{R},\mathbb{C}^{n+1})$.

This interpolation problem can be rewritten in a simpler way. Indeed, for each $\mu \in \Lambda^\circ\backslash\{0\}$ and $j \in \{0,1,\dots,n\}$, the system (38) is a triangle linear system in the variables $\left\{G_j^{(m)}(\bar{\mu})\right\}_{m=0}^{n}$ with non-zero diagonal coefficients and zero right-hand side. Clearly, it has just zero solutions. Thus if $G = (\mathrm{G_j})_{j=0}^{n}$ satisfies (38), then $G_j^{(m)}(\bar{\mu}) = 0, j,m = 0,1,\dots,n, \mu \in \Lambda^\circ\backslash\{0\}$. For $\mu = 0$, the relations (38) take the form

$$\frac{1}{\sqrt{\pi^l l!}} G_j^{(l)}(0) = s(\Lambda)\delta_{j,l}, \quad j,l = 0,1,\dots,n.$$

By adjusting the normalization of the G_j, we obtain the following statement.

Proposition (6.1.11)[406]: The Gabor system $\mathcal{G}(h,\Lambda)$ is a Gabor superframe for $L^2(\mathbb{R},\mathbb{C}^{n+1})$, if and only if there exist $n+1$ functions $\gamma_j \in L^2(\mathbb{R}^d)$ such that $\mathcal{G}(\gamma_j,\Lambda)$ is a Bessel sequence and $G_j = \mathcal{B}_{\gamma_j}, j = 0,\dots,n$, satisfy the (Hermite) interpolation problem

$$G_j^{(\ell)}(\bar{\mu}) = \delta_{\mu,0}\delta_{j,\ell}, \quad \text{for } \mu \in \Lambda^\circ, j,\ell = 0,\dots,n. \tag{39}$$

Theorem (6.1.12)[406]: Let $h = (H_0,\dots,H_n)$ be the vector of the first $n+1$ Hermit functions. Then $\mathcal{G}(h,\Lambda)$ is a frame for $L^2(\mathbb{R},\mathbb{C}^{n+1})$, if and only if

$$s(\Lambda) < \frac{1}{n+1}. \tag{40}$$

Proof. Sufficiency of (40). We use the Weierstrass functions described. Construct the Weierstrass σ-function for the lattice $\overline{\Lambda^\circ}$, as given by (6).

Let the construct $\alpha(\overline{\Lambda^\circ}), a(\overline{\Lambda^\circ})$ be defined by (31) and the function $\sigma_{(\overline{A^0})}$ be defined by (32), where the σ-function is constructed for the lattice $(\overline{\Lambda^\circ})$, consider the function

$$S(z) = \left(\sigma_{(\overline{\Lambda^\circ})}(z)\right)^{n+1} = \sigma(z)^{n+1} e^{a(\overline{\Lambda^\circ})(n+1)z^2}. \tag{41}$$

The zero set of S is $(\overline{\Lambda^{0\circ}})$, and the multiplicity of each zero is precisely $n+1$. In our case $(\overline{\Lambda^\circ}) = \frac{\pi}{2s(\Lambda^\circ)} = \frac{\pi}{2}s(\Lambda)$, and the growth estimates (35) and (36) can be rewritten as

$$|S(s)| \le e^{\frac{\pi}{2}(n+1)S(\Lambda)|s|^2}, \qquad \text{for all } s \in \mathbb{C}, \tag{42}$$

$$|S(s)| \asymp e^{\frac{\pi}{2}(n+1)S(\Lambda)|s|^2}, \qquad \text{if } \operatorname{dist}(s,\overline{A^0}) > \epsilon. \tag{43}$$

Now assume that $s(\Lambda) < (n+1)^{-1}$. Then the functions

$$S_m(z) = \frac{1}{z^{n+1-m}} S(z), \qquad m = 0,1,\dots,n \tag{44}$$

belong to $\mathcal{F}$. they have zero of order $n+1$ at each $\mu \in \overline{\Lambda^\circ}\backslash\{0\}$, and also

$$S_m^{(l)}(0) = 0, \text{ for } 0 \le l \le m-1, \text{ and } S_m^{(m)}(0) \ne 0.$$

The solutions G_j to the interpolation problem (39) can be now found in the form

$$G_j = \sum_{m=j}^{n} c_{m,j} S_m, \qquad j = 0,1,\dots,n. \tag{45}$$

These functions have a zero of multiplicity $n+1$ at all points from $\overline{\Lambda^\circ}\backslash\{0\}$, while, for each j, the condition $G_j^{(l)}=\delta_{j,l}$ leads one to a triangular system of linear equations with respect to the coefficients $\{c_{m,j}\}_{m=j}^{n}$ with non-zero diagonal entries. Clearly this system has a (unique) Solution.

It remains to ensure that each system $\mathcal{G}(\gamma_j,\Lambda)$ is a Bessel sequence. Each G_j inherits its growth from the functions S_j and from S, thus (42) and Proposition (6.1.8) imply that $\gamma_i\in M^1(\mathbb{R})$ For $j=0,\dots,n$. Now apply Lemma (6.1.3).

Necessity. Let now $s(\Lambda)\geq(n+1)^{-1}$. The growth estimate (43) implies that

$$|S(z)|\geq \text{Const}\, e^{\frac{\pi}{2}|z|^2},\quad \text{whenever } \operatorname{dist}(z,\overline{\Lambda^\circ})>\epsilon. \tag{46}$$

Now assume that $\mathcal{G}(h,\Lambda)$ is a frame for $L^2(\mathbb{R},\mathbb{C}^{n+1})$. Then there exists a system of functions $G_0,G_1,\dots,G_n\in\mathcal{F}$ which satisfy the interpolation problem (39). The function G_n has zeros of multiplicity $n+1$ at $\overline{\Lambda^\circ}\backslash\{0\}$ and a zero of multiplicity n at the origin. Therefore

$$\Phi(\mathcal{Z})=\frac{zG_n(z)}{S(z)}$$

is an entire function. Estimates (46) and (17) yield

$$|\Phi(z)|=o(|z|),\ \ z\to\infty,\ \ \operatorname{dist}(z,\Lambda^\circ)>\epsilon.$$

By the maximum principle, the restriction $\operatorname{dist}(z,\Lambda^\circ)>\epsilon$ can be removed, and thus Φ is a bounded entire function. Liouville's theorem implies that Φ is constant:

$$G_n(z)=C\frac{S(z)}{z}.$$

But $z^{-1}S(z)\notin\mathcal{F}$, this follows again from (46). Therefore $C=0$ and hence G_n is identically contradiction to (39). this completes the proof of necessity [410].

Next we consider scalar-valued Gabor frames with Hermite functions.

We remark that if for some $g\in L^2(\mathbb{R},\mathbb{C}^{n+1})$ and lattice $\Lambda\subset\mathbb{R}^2$ the system $\mathcal{G}(g,\Lambda)$ is a vector-valued frame in $L^2(\mathbb{R},\mathbb{C}^{n+1})$, then trivially, for each $j=0,1,\dots,n$, the Gabor system $\mathcal{G}(g_j,\Lambda)=\{\pi_\lambda g_j,\lambda\in\Lambda\}$ is a frame for $L^2(\mathbb{R})$. More generally, for each $c\in\mathbb{C}^{n+1}, c\neq 0$, the system $\mathcal{G}(h,\Lambda)=\{\pi_\lambda g_j,\lambda\in\Lambda\}$ with $h=\sum_{j=0}^{n}c_jg_j$ also is a frame in $L^2(\mathbb{R})$. By applying this observation to the window $g=(H_0,\dots,H_n)$ we obtain Proposition (6.1.2), which gives a condition for the one dimensional system with a Hermite window to be a frame for $L^2(\mathbb{R})$. Actually a slightly more general result holds true.

Proposition (6.1.13)[406]: Let $n\in\mathbb{Z}, n\geq 0$ and $h=\sum_{k=0}^{n}c_kH_k, \sum_0^n|c_k|\neq 0$. If $s(\Lambda)<(n+1)^{-1}$, then $\mathcal{G}(h,\Lambda)$ is a frame for $L^2(\mathbb{R})$.

Amazingly enough the sufficient density of Proposition (6.1.13) might be sharp, as is suggested by following counter-example.

Proposition (6.1.14)[406]: If $\Lambda=a\mathbb{Z}\times b\mathbb{Z}$ and $ab=1/2$, then $\mathcal{G}(H_{2n+1},\Lambda)$ is not a Gabor frame for $L^2(\mathbb{R})$ for all integers $n\geq 0$.

Proof. We use a Zak transform argument. For $a>0$ the Zak transform is defined as

$$Z_af(x,\xi)=\sum_{k=-\infty}^{\infty}f(x-ak)e^{2i\pi ak\xi}. \tag{47}$$

We refer the reader to [420, Sect.8.3] for a detailed discussion of its properties. In particular it is a unitary mapping between $L^2(\mathbb{R})$ and $L^2(Q_a)$- the space of square-integrable functions in the rectangle $Q_a = [0, a) \times [0, 1/a)$. For the case $ab = 1/2$ the Gabor frame operator is unitarily equivalent to a multiplication operator on $L^2(Q_a)$ by maens of the Zak transform. Precisely,

$$Z_a S_{H_n,H_n} f(x,\xi) = \left(|Z_a H_n(x,\xi)|^2 + \left|Z_a H_n(x - \tfrac{a}{2}, \xi)\right|^2\right) Z_a f(x,\xi), \; f \in L^2(\mathbb{R}).$$

Consequently $\mathcal{G}(H_n, \Lambda)$ is a frame for $L^2(\mathbb{R})$ if and only if

$$0 < \inf_{Q_a}\left\{\left(|Z_a H_n(x,\xi)|^2 + \left|Z_a H_n\left(x - \frac{a}{2}\right), \xi\right|^2\right)\right\}$$
$$\leq \sup_{Q_a}\left\{\left(|Z_a H_n(x,\xi)|^2 + \left|Z_a H_n\left(x - \frac{a}{2}\right), \xi\right|^2\right)\right\} < \infty,$$

see relation (8.21)in [420].

Clearly, $Z_a H_n(x,\xi)$ is a continuous function Since H_{2n+1} is an odd function, its Zak transform satisfies $Z_a H_{n+1}(0,0) = Z_a H_{n+1}\left(\frac{a}{2}, 0\right) = 0$. This contradicts the above criterium, therefore $\mathcal{G}(H_{2n+1}, \Lambda)$ cannot be a frame.

In [423, Prop. 3.3] we showed that $\mathcal{G}(H_n, \Lambda)$ is a frame, if and only if there exists a $\gamma_n \in L^2(\mathbb{R})$, such that $\mathcal{G}(\gamma_n, \Lambda)$ is a Bessel system and the Bargmann transform $G_n = \mathcal{B}_{\gamma_n}$ is in $\mathcal{F}$ and satisfies the interpolation problem

$$\sum_{k=0}^{n} \binom{n}{k} (-\pi\mu)^k G_n^{(n-k)}(\bar{\mu}) = \delta_{\mu,0} \quad \forall \mu \in \Lambda^\circ. \tag{48}$$

Note that (48) coincides with (38) for $j = l = n$ and that for $s(\Lambda) < 1/(n+1)$ the function

$$G_n(z) = S_n(z) = cz^{-1}\sigma\frac{n+1}{\Lambda^\circ}(z) \tag{49}$$

is a solution of (48) in $\mathcal{F}$.

If $s(A) \geq 1/(n+1)$, then we do not know whether (48) has any solution in $\mathcal{F}$. the difficulty is that this interpolation problem is not entirely holomorphic, and the standard complex variable methods do not seem sufficient to investigate this problem. In the light of Proposition (6.1.14) it is conceivable that the sufficient condition $s(\Lambda) < 1/(n+1)$ in Proposition (6.1.13) is also necessary.

We assume that $s(\Lambda) < \frac{1}{n+1}$, and we estimate the L^2 and M^1-norm of the dual window for the frame $\mathcal{G}(H_n, \Lambda)$. Thought the estimates are simple, they seem to be new even for the Gaussian case $n = 0$.

Lemma (6.1.15)[406]: Assume that $(n+1)S(\Lambda) < 1$ and let γ_n be the dual windows with Bargmann transform G_n defined in (49). Setting $= \frac{1-(n+1)S(\Lambda)}{3-(n+1)S(\Lambda)}$, then

$$|\gamma_n(t)| + |\hat{\gamma}_n(t)| \leq Ce^{-\pi k t^2}. \tag{50}$$

Furthermore,

$$\|\gamma\|_{M_1} \leq 4c(\overline{\Lambda^\circ})^{n+1}\frac{1}{1-(n+1)S(\Lambda)}, \tag{51}$$

and

$$\|\gamma_n\|^2_{L^2(\mathbb{R})} \asymp \log \frac{1}{1-(n+1)S(\Lambda)}. \tag{52}$$

where $c(\Lambda)$ is defined in (34).

Proof. The estimates involve a routine calculation with Gausians and are a consequence of the growth estimate (35) of Proposition (6.1.10). Set $\rho = 1-(n+1)s(\Lambda)$.

For the decay property, we use the inversion formula for the short-time Fourier transform (e.g., [420,Prop. 3.2.3]) which states

$$\gamma_n(t) = \int_{\mathbb{C}} \langle \gamma_n, \pi_z H_0 \rangle \pi_z H_0(t) dm_z$$

with absolute convergence of the integral for each $t \in \mathbb{R}$. Since by (35)

$$|\langle \gamma_n, \pi_{\bar{z}} H_0 \rangle| = |\mathcal{B}\gamma_n(z)| e^{-\pi |z|^2/2} \leq C e^{-\pi |z|^2/2}$$

and since $\pi_{x+i\xi} H_0(t) = e^{2\pi i \xi t} H_0(t-x)$, we obtain that

$$|\gamma_n(t)| \leq C \int_{\mathbb{R}} e^{-\pi\rho\xi^2/2} d\xi \int_{\mathbb{R}} e^{-\pi\rho x^2/2} e^{-\pi\rho(t-x)^2} dx.$$

The convolution of the two Gaussians in this integral is a multiple of the Gaussian $e^{-\pi\frac{\rho}{2+\rho}t^2}$ by the semigroup property og Gaussians with respect to convolution [420, Lemma 4.4.5].

The M^1-norm of γ_n is readily estimated by

$$\begin{aligned}\|\gamma\|_{M^1} &= \int_{\mathbb{C}} |G_n(z)| e^{-\pi |z|^2/2} dm_z \\ &\leq c(\overline{\Lambda^\circ})^{n+1} \int_{\mathbb{C}} \frac{1}{|z|} e^{-\pi\rho |z|^2/2} dm_z \\ &= 2\pi(\overline{\Lambda^\circ})^{n+1} \int_0^\infty e^{-\pi\rho r/2} dr = 4c(\overline{\Lambda^\circ})^{n+1} + \frac{1}{\rho}.\end{aligned}$$

For the estimate of $\|\gamma_n\|_2^2$ we have similarly

$$\|\gamma_n\|^2_{L^2(\mathbb{R})} = \|G\|^2_{\mathcal{F}} = \int_{\mathbb{C}} |z|^{-2} |\sigma_{\overline{\Lambda^\circ}}(z)|^{2(n+1)} e^{-\pi|z|^2} dm_z.$$

When $|z| \leq 1$ the integrated is bounded uniformly, so this part does not bring an essential contribution into the whole norm. It follows now from (36) that

$$\|\gamma_n\|^2_{L^2(\mathbb{R})} \asymp \int_{|z|>1} |z|^{-2} e^{-\pi\rho|z|^2} dm_z \asymp \int_1^\infty \frac{1}{r} e^{-\pi\rho r^2} dr \asymp -\log(1-(n+1)s(\Lambda)).$$

This is the desired inequality.

Finally let us briefly describe how the lower frame bound of the Gabor frame $\mathcal{G}(H_0, \Lambda)$ with Gaussian window behaves, when the lattice approaches the critical size $s(\Lambda) = 1$. Since the constants in Lemma (6.1.15) depend on (the excentricity of) the lattice Λ, we fix a lattice Λ with size $s(\Lambda) = 1$ and study the behavior of lower frame bound of $\mathcal{G}(H_0, q\Lambda)$, as q tends to 1. As long as $q < 1, \mathcal{G}(H_0, q\Lambda)$ is aframe by the classical results in [432,435],however, when $q = 1$, then $\mathcal{G}(H_0, q\Lambda)$ cannot be a frame by the Balian-Low theorem [410]. The upper frame bound can be controlled uniformly with Proposition (6.1.3), therefore the lower frame bound must A converge to 0 as $q \to 1$.

Proposition (6.1.16)[406]: Assume that $s(\Lambda) = 1$ and $q < 1$. Let A_q denote the optimal lower frame bound of $\mathcal{G}(H_0, q\Lambda)$. Then

$$A_q \geq c(1 - q^2)^2 = c(1 - s(q\Lambda))^2$$

for some constant independent of q.

Proof. Let γ_0 be the dual window defined by (49). Corollary (6.1.19) of the Wexler-Raz relations and Lemma (6.1.15) imply that

$$A_q \geq (n(q\Lambda)\|\gamma_0\|_{M^1}^2)^{-1} \geq \left(16n(q\Lambda)c\left(\overline{(q\Lambda)^\circ}\right)^2\right)^{-1}\left(1 - s(q\Lambda)\right)^2$$

Thus we need to show that the two constants n and c are bounded, as long as q is bounded away from $0, q \geq 1/2$, say. Clearly the constant $(q\Lambda)$, which measures the maximal number of lattice points in a unite cube (Lemma (6.1.3)), can be bounded uniformly. As for c, which is the supremum of the σ-function over the period parallelogram (34), we note that $(q\Lambda)^\circ = q^{-1}\Lambda^\circ$ and that $\sigma_{q^{-1}\Lambda}(z) = q^{-1}\sigma_\Lambda(qz)$ by (28), (30) and (31). Consequently, $\sup_{1/2 \leq q \leq 1} c\left(q^{-1}\overline{\Lambda^\circ}\right) < \infty$, and we are done.

We show how the structural results about Gabor super frames can be derived from the corresponding well-known results for scalar Gabor frames.

Proposition (6.1.17)[406]: (Janssen representation for Gabor super frames) Assume that $\gamma, g \in M^1(\mathbb{R}^d, \mathbb{C}^n)$. Then the frame-type operator $S_{\gamma,g}^\Lambda$ can be written as

$$Sf = S_{\gamma,g}^\Lambda f = \sum_{\mu \in \Lambda^\circ} \Gamma(\mu)\pi_\mu f, \tag{53}$$

where $\Gamma(\mu)$ is the $n \times n$ matrix with entries

$$\Gamma(\mu)_{kl} = s(\Lambda)^{-1}\langle \gamma_k, \pi_\mu g_l \rangle \tag{54}$$

and the sum convergence in the operator norm on $L^2(\mathbb{R}^d, \mathbb{C}^n)$.

Proof. We look at the l–th component of Sf. Using definition (9) it can be written explicitly as a sum of scalar frame-type operators

$$(Sf)_l = \sum_{\lambda \in \Lambda} \langle f, \pi_\lambda g \rangle \pi_\lambda \gamma_l = \sum_{j=1}^{n} \sum_{\lambda \in \Lambda} \langle f_j, \pi_\lambda g_j \rangle \pi_\lambda \gamma_l = \sum_{j=1}^{n} S_{g_j,\gamma_l}^\Lambda f_j.$$

If all g_j, γ_j are in M^1, then the well-known scalar version of Janssen's representation (53) can be applied to each of the frame-type operators occurring above sum and we obtain that

$$(Sf)_l = s(\Lambda)^{-1} \sum_{\mu \in \Lambda^\circ} \sum_{j=1}^{n} \langle \gamma_l, \pi_\mu\, g_j \rangle\, \pi_\mu f_j = \sum_{\mu \in \Lambda^\circ} \left(\Gamma(\mu)\pi_\mu f\right)_l, \tag{55}$$

as was to be shown.

Proposition (6.1.18)[406]: (Wexler-Raz biorthogonality). Assume that both $\mathcal{G}(g, \Lambda)$ and $\mathcal{G}(\gamma, \Lambda)$ are Bessel sequences in $L^2(\mathbb{R}^d, \mathbb{C}^n)$. Then $\mathcal{G}(g, \Lambda)$ is a vector-valued frame in $L^2(\mathbb{R}^d, \mathbb{C}^n)$ with dual window γ if and only if

$$\frac{1}{S(\Lambda)}\langle\gamma_l,\pi_\mu g_j\rangle=\delta_{0,\mu}\delta_{l,j}\quad\text{for }\mu\in\Lambda^\circ,j,l=1,2,\dots,n. \tag{56}$$

Proof. We remark first that time-frequency shifts on a lattice are linearly independent in the following sense: if $c=(c_\mu)_{\mu\in\Lambda^\circ}\in\ell^\infty(\Lambda^\circ)$ and $\sum_{\mu\in\Lambda^\circ}c_\mu\pi(\mu)=0$ as an operator from M^1 to $(M^1)^*$, then $c_\mu=0$ for all $\mu\in\Lambda^\circ$. See, e.g., [421,434]. If $S_{g,\gamma}=\text{I}$, then by Janssen's representation

$$f_l=(Sf)_\ell=s(\Lambda)^{-1}\sum_{\mu\in\Lambda^\circ}\sum_{j=1}^{n}\langle\gamma_\ell,\pi_\mu\, g_j\rangle\pi_\mu f_j,$$

and so linear independence forces $s(\Lambda)^{-1}\langle\gamma_\ell,\pi_\mu\, g_j\,\rangle=\delta_{\mu,0}\delta_{j,\ell}$ or in shart notation $s(\Lambda)^{-1}\Gamma(\mu)=\delta_{\mu,0}\text{I}$.

Conversely, if the biorthogonality condition $s(\Lambda)^{-1}\Gamma(\mu)=\delta_{\mu,0}\text{I}$ holds, then obviously $S_{g,\gamma}=1$.

Theorem (6.1.19)[406]: (Janssen-Ron-Shen duality). The Gabor system $\mathcal{G}(g,\Lambda)$ is a frame for $L^2(\mathbb{R}^d,\mathbb{C}^n)$, if and only if the union of Gabor systems $\bigcup_{j=1}^n\mathcal{G}(g_j,\Lambda^\circ)$ is a Riesz sequence for $L^2(\mathbb{R}^d)$.

Proof. Assume first that $\mathcal{G}(g,\Lambda)$ is a frame for $L^2(\mathbb{R}^d,\mathbb{C}^n)$. then the canonical dual is given by $\gamma^\circ=S^{-1}g$, and by general frame theory γ° satisfies $S_{g,\gamma^\circ}=1$. Furthermore, $\mathcal{G}(\gamma^\circ,\Lambda)$ is again a frame for $L^2(\mathbb{R}^d,\mathbb{C}^n)$, in particular it is a Bessel sequence. If

$$f=\sum_{j=1}^{n}\sum_{\mu\in\Lambda^\circ}c_{j,\mu}\pi_\mu\, g_j \tag{57}$$

For some sequence $(c_{j,\mu})\in\ell^2(\{1,\dots,n\}\times\Lambda^\circ)$, Then by the Bessel property of $\mathcal{G}(\text{g}_j,\Lambda^\circ)$ for each, we obtain that

$$\|f\|_2^2=\left\|\sum_{j=1}^{n}\sum_{\mu\in\Lambda^\circ}c_{j,\mu}\pi_\mu\, g_j\right\|\le B\sum_{j=1}^{n}\sum_{\mu\in\Lambda^\circ}\left|c_{j,\mu}\right|_2^2=B\|c\|_2^2.$$

For the converse inequality we use the Wexler-Raz relations. If $f\in L^2(\mathbb{R}^d)$ is given by (57), then the coefficients are uniquely determined by

$$c_{j,\mu}=\langle f,\pi_\mu\gamma_j\rangle.$$

Again, since $\mathcal{G}(\gamma_j,\Lambda^\circ)$ possesses the Bessel property, we find that $\|c\|_2^2\le A'\|f\|_2^2$. Altogether we have shown that the set $\bigcup_{j=1}^n\mathcal{G}(g_j,\Lambda^\circ)$ is a Riesz sequence in $L^2(\mathbb{R}^d)$.

Conversely, assume that $\bigcup_{j=1}^n\mathcal{G}(g_j,\Lambda^\circ)$ is a Riesz sequence that generates a sub-space $\mathcal{K}\subseteq L^2(\mathbb{R}^d)$. Then there exists a biorthogonal basis $\{e_{j,\mu}\}$ contained in $\mathcal{K}$. By the invariance of the Gabor systems $\mathcal{G}(g_j,\Lambda^\circ)$, we find that the biorthogonal basis must be of the form $e_{j,\mu}=\pi_\mu\gamma_j$ for some $\gamma_j\in L^2(\mathbb{R}^d)$. By the general properties of Riesz bases, $\mathcal{G}(g_j,\Lambda^\circ)$ is a Bessel sequence for each j, and, after some rescaling, the biorthogonality states that $s(\Lambda)^{-1}\langle\pi_{\mu'}\gamma_j,\pi_\mu g_\ell\rangle=\delta_{\mu,\mu'}\delta_{j,\ell}$. According to the Wexler-Raz relations, this implies that $\mathcal{G}(g,\Lambda)$ is a frame for $L^2(\mathbb{R}^d,\mathbb{C}^n)$.

Section (6.2): Gabor Frame Sets for Subspace

We denote by $\mathbb{Z}, \mathbb{N}, \mathbb{Q}$ and $L^2(\mathbb{R})$ the set of integers, the set of natural numbers, the set of rational numbers and the Hilbert space of all squares-integrable functions on $\mathbb{R}$, respectively. Unless otherwise specified, relations between two measurable sets in $\mathbb{R}$ such as equality, disjointness or inclusion, are always understood up to a set of Lebesgue measure zero, and similarly, equality or inequality between functions is always understood in the "almost everywhere" sense. Given any $M \in \mathbb{N}$, write $\mathbb{N}_M = \{0,1, \dots, M-1\}$. For $c > 0$ two measurable subsets $\mathbb{S}_1$ and $\mathbb{S}_2$ of $\mathbb{R}$ are said to be $c\mathbb{Z}$-congruent if there exist measurable partitions $\{\mathbb{S}_{1,k}\}_{k\in\mathbb{Z}}$ of $\mathbb{S}_1$ and $\{\mathbb{S}_{2,k}\}_{k\in\mathbb{Z}}$ of $\mathbb{S}_2$ such that $\mathbb{S}_{2,k} = \mathbb{S}_{1,k} + ck$ for all $k \in \mathbb{Z}$. For a measurable set F in $\mathbb{R}$, we denote by $\mu(F)$ its Lebesgue measure.

Let $\mathcal{H}$ be a separable Hilbert space, and let I be a countable index set. A sequence $\{h_n\}_{n\in I}$ in $\mathcal{H}$ is called a frame for $\mathcal{H}$ if there exist constants A and B with $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \sum_{n\in I} |\langle f, h_n\rangle|^2 \le B\|f\|^2, \quad f \in \mathcal{H}. \tag{58}$$

The maximum constant A and the minimal constant B for the inequalities (58) to hold are called the upper frame bound and the lower frame bound respectively. The sequence $\{h_n\}_{n\in I}$ is called a Bessel sequence in $\mathcal{H}$ if only the right-hand side inequality in (58) holds. See [437,442] and [456] for the fundamentals of frame theory. Given $a, b > 0$. Define the modulation operator $E_{mb}, m \in \mathbb{Z}$, and translation operator $T_{na}, n \in \mathbb{Z}$, by

$$E_{mb}f(\cdot) := e^{2\pi imb} f(\cdot) \text{ and } T_{na}f(\cdot) := f(\cdot - na), \quad f \in L^2(\mathbb{R}),$$

respectively. For an arbitrary $g \in L^2(\mathbb{R})$, we denote by $\mathcal{G}(g, a, b)$ the Gabor system associated with g and the time-frequency shift parameters a, b:

$$\mathcal{G}(g, a, b) := \{E_{m,b}T_{na}g\}_{m,n\in\mathbb{Z}}.$$

A Measurable set $\mathbb{S}$ in $\mathbb{R}$ with positive measure is said to be $a\mathbb{Z}$-periodic if $\mathbb{S} + na = \mathbb{S}$ for $n \in \mathbb{Z}$. We restrict ourselves to the closed subspace $L^2(\mathbb{S})$ of $L^2(\mathbb{R})$:

$$L^2(\mathbb{S}) := \{f \in L^2(\mathbb{R}) : f = 0 \text{ on } \mathbb{R}\backslash\mathbb{S}\},$$

where $\mathbb{S}$ is an $a\mathbb{Z}$-periodic set in $\mathbb{R}$. Clearly, it is a Hilbert space with the inner product in $L^2(\mathbb{R})$. We deal with the Gabor systems of the form $\mathcal{G}(\chi_F, a, b)$ where F a measurable subset of is $\mathbb{S}$, and χ_F denotes the characteristic function of F. A measurable set F in $\mathbb{S}$ is called a Gabor frame set (tight Gabor set) in $\mathbb{S}$ if $\mathcal{G}(\chi_F, a, b)$ is a frame (tight frame) for $L^2(\mathbb{S})$, and called a Gabor Bessel set in $\mathbb{S}$ if $\mathcal{G}(\chi_F, a, b)$ is a Bessel sequence in $L^2(\mathbb{S})$.

During the last 20 years Gabor systems have been extensively studied in $L^2(\mathbb{R})$ (see [443,444,445,451]), and also in the setting of subspaces [438,446,447,449,454,455]. We concern Gabor analysis in $L^2(\mathbb{S})$, where $\mathbb{S}$ is an $a\mathbb{Z}$-periodic set in $\mathbb{R}$. Such a scenario can be used to model a signal that appears periodically but intermittently. Although classical Gabor analysis tools in $L^2(\mathbb{R})$ can be adjusted to treat such a scenario by padding with zeros outside the set $\mathbb{S}$, Gabor systems that fit exactly such a scenario might have been more efficient. The following proposition shows that only periodic $\mathbb{S}$ in $\mathbb{R}$ are suitable for Gabor analysis.

Proposition (6.2.1)[436]: Given $a, b > 0$ and a measurable set $\mathbb{S}$ in $\mathbb{R}$ with positive measure. Assume that there exists $g \in L^2(\mathbb{R})$ such that $\mathcal{G}(g, a, b)$ is complete in $L^2(\mathbb{S})$. Then $= \bigcup_{n\in\mathbb{Z}}(\text{supp}(g) + na)$.

Significant achievement has been made on wavelet sets and frame wavelet sets (see [437,440,441,450,452,453]). In recent years, the study of Gabor frame sets in $\mathbb{R}$ attracted many mathematicians. In [439], Casazza and Kalton proved that the characterization of Gabor frame sets in $\mathbb{R}$ is equivalent to solving an old open problem of Littlewoods in complex analysis. Janssen in [444] obtained many interesting results about Gabor frame sets in $\mathbb{R}$ of the form $(0, c_0)$. With the help of set-valued mapping defined on certain union of intervals, Gu and Han in [448] provided a complete solution to Gabor frame sets in $\mathbb{R}$ of the form $(0, c_0)$ for the case of ab being a rational number. For the irrational case,some related results were also obtained. Given $, b > 0$. For Gabor analysis in $a\mathbb{Z}$-periodic se, Gabardo and [449] provided some conditions on $\mathbb{S}$ for the existence of a complete Gabor system in $L^2(\mathbb{S})$ and the existence of tight Gabor sets. We devoted to Gabor frame sets (especially tight Gabor sets) in $\mathbb{S}$. The results are also new even if $\mathbb{S} = \mathbb{R}$.

We characterize Gabor Bessel sets in $\mathbb{R}$. Focuses on characterization of Gabor frame sets (especially on tight Gabor sets) in periodic setting. In particular, given $a, b > 0$ and an $a\mathbb{Z}$-periodic Set $\mathbb{S}$ in $\mathbb{R}$. Some results related to Gabor frame sets are given, and a necessary and sufficient condition on a subset of $\mathbb{S}$ being a tight Gabor set is established. We discuss Gabor frame sets with rational ab. We characterize $\mathbb{S}$ admitting tight Gabor Sets, and obtain an explicit construction of a class of tight Gabor sets in such in such $\mathbb{S}$. For irrational, we derive a necessary and sufficient condition on $\mathbb{S}$ admitting tight Gabor sets.

We will characterize Gabor Bessel sets in $\mathbb{R}$. We begin with a proposition on set decomposition .Given a measurable set F in $\mathbb{R}$ and a constant $c > 0$. Define the function τ_c of $\mathbb{R}$ into $\mathbb{N}\cup\{\infty\}$ by

$$\tau_c(x) := \text{card}(\{y \in F : y = x + cj \text{ for some } j \in \mathbb{Z}\}), \quad x \in \mathbb{R}. \tag{59}$$

and

$$F(c, k) := \{x \in F : \tau_c(x) = k\}, \quad k \in \mathbb{N} \cup \{\infty\}. \tag{60}$$

Here card (E) denotes the cardinality of E for a set E.

Proposition (6.2.2)[436]: Given a Lebesgue measurable set F in $\mathbb{R}$ and a constant $c > 0$. Define τ_c and $F(c, k)$ as in (59) and (60) for $k \in \mathbb{N}\cup\{\infty\}$ respectively. Then

(i) τ_c is c-periodic.

(ii) $F(c, k) \cap F(c, k') + cj) = \emptyset$ for $j \in \mathbb{Z}$ and $k, k' \in \mathbb{N}\cup\{\infty\}$ with $k \neq k'$, and $F = (\bigcup_{k\in\mathbb{N}} F(c, k)) \cup F(c, \infty)$.

(iii) For each $k \in \mathbb{N}\cup\{\infty\}$, $F(c, k)$ is measurable, and $F(c, k)$ is a disjoint union of k measurable subsets $F^{(j)}(c, k), j \in \mathbb{N}_k$, such that all $F^{(j)}(c, k)$ are $c\mathbb{Z}$-congruent to the same subset of $[0, c)$ where $\mathbb{N}_\infty = \mathbb{N}$.

(iv) $\sum_{n\in\mathbb{Z}} \mu(F(c, k) \cap (F + cn))\, k\mu(F(c, k))$ for $k \in \mathbb{N}$.

(v) $F(c, \infty) = \emptyset$ when $\mu(F) < \infty$.

Proof. By the definitions of τ_c and (c, k), we have the conclusion (i) and $F = (\bigcup_{K\in\mathbb{N}} F(c, k)) \cup F(c, \infty)$. The conclusion (iii) follows from Lemma 1 in [441], and the equality in (v) is borrowed from Remark 1 in [440]. To show (ii), we only need to show that

$$F(c,k) \cap (F(c,k')+cj) = \emptyset$$

for $j \in \mathbb{Z}$ and $k,k' \in \mathbb{N}\cup\{\infty\}$ with $k \neq k'$. We will establish the above result by indirect proof. Suppose there exists a set E with $\mu(E) > 0$ such that $E \subset F(c,k)\cap(F(c,k')+cj)$ for some $j \in \mathbb{Z}$ and $k,k' \in \mathbb{N}\cup\{\infty\}$ with $k \neq k'$. Then $\tau_c(\cdot) = k$ and $\tau_c(\cdot - cj) = k'$ on E by the definition of τ_c, which implies that $k = k'$ by (i). This is a contradiction. Now we turn to prove (iv). Given $k \in \mathbb{N}$. By (ii) and (iii),

$$\begin{aligned} F(c,k)\cap(F+cn) &= \bigcup_{l\in\mathbb{N}\cup\{\infty\}} \big(F(c,k)\cap(F(c,l)+cn)\big) \\ &= F(c,k)\cap(F(c,k)+cn) \\ &= \bigcup_{j=0}^{k-1}\Big(F(c,k)\cap\big(F^{(j)}(c,k)+cn\big)\Big) \end{aligned}$$

for $n \in \mathbb{Z}$. Also note that all $F^{(j)}(c,k), j \in \mathbb{N}_k$, are $c\mathbb{Z}$-congruent to the same subsets of $[0,c)$ by (iii). It follows that

$$\begin{aligned} \sum_{n\in\mathbb{Z}} \mu\big(F(c,k)\cap(F(c,k)+cn)\big) &= \sum_{n\in\mathbb{Z}}\sum_{j=0}^{k-1} \mu\Big(F(c,k)\cap\big(F^{j}(c,k)+cn\big)\Big) \\ &= \sum_{j=0}^{k-1} \mu\left(\bigcup_{n\in\mathbb{Z}}\Big(F(c,k)\cap\big(F^{j}(c,k)+cn\big)\Big)\right) \quad (61) \end{aligned}$$

Again by (iii), $F^{l}(c,k)$ is $c\mathbb{Z}$-congruent to $F^{j}(c,k)$ for each $(l,j) \in \mathbb{N}_k \times \mathbb{N}_k$. It follows that

$$F^{l}(c,k) \subset \bigcup_{n\in\mathbb{Z}}\Big(\big(F^{j}(c,k)+cn\big)\Big),$$

and consequently,

$$F(c,k) = \bigcup_{l=0}^{k-1} F^{l}(c,k) \subset \bigcup_{n\in\mathbb{Z}}\big(F^{j}(c,k)+cn\big).$$

which together with (61) leads to the conclusion (iv).

Lemma (6.2.3)[436]: Let $b > 0$ and Γ be a subset in $\mathbb{R}$ with positive measure such that $\Gamma = \Gamma\left(\frac{1}{b},1\right)$. For an arbitrary function $f \in L^2(\mathbb{R})$, write

$$\tilde{f}(\cdot) = \sum_{\gamma\in\mathbb{Z}} f\left(\cdot + \frac{\gamma}{b}\right)\chi_r\left(\cdot + \frac{\gamma}{b}\right). \quad (62)$$

Then $\tilde{f} \in L^2\left[0,\frac{1}{b}\right)$,

$$\tilde{f}(\cdot) = b\sum_{\gamma\in\mathbb{Z}} \langle f, E_{mb}\chi_r\rangle e^{2\pi imb}. \quad (63)$$

in $L^2\left[0,\frac{1}{b}\right)$, and the right-hand side series in (62) converges absolutely a.e on $\mathbb{R}$.

Proof. Since Γ is $\frac{1}{b}\mathbb{Z}$-congruent to a subset of $\left[0,\frac{1}{b}\right)$, we have

$$\left|\tilde{f}(\cdot)\right|^2 = \sum_{\gamma\in\mathbb{Z}} \left|f\left(\cdot + \frac{\gamma}{b}\right)\chi_r\left(\cdot + \frac{\gamma}{b}\right)\right|^2$$

on $\mathbb{R}$. It follows that

$$\left|\tilde{f}(\cdot)\right|^2 dt = \sum_{\gamma\in\mathbb{Z}} \int_{\left[0,\frac{1}{b}\right)} \left|f\left(t+\frac{\gamma}{b}\right)\chi_r\left(t+\frac{\gamma}{b}\right)\right|^2 dt$$

$$= \int_{\Gamma} \left|\tilde{f}(t)\right|^2 dt \le \|f\|^2 < \infty.$$

Then $\tilde{f} \in L^2\left[0,\frac{1}{b}\right)$, and Consequently,

$$\int_{\left[0,\frac{1}{b}\right)} \sum_{\gamma\in\mathbb{Z}} \left|f\left(t+\frac{\gamma}{b}\right)\chi_r\left(t+\frac{\gamma}{b}\right)\right| dt$$

$$\le \frac{1}{\sqrt{b}}\left[\int_{\left[0,\frac{1}{b}\right)} \left(\sum_{\gamma\in\mathbb{Z}} \left|f\left(t+\frac{\gamma}{b}\right)\chi_r\left(t+\frac{\gamma}{b}\right)\right|\right)^2 dt\right]^{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{b}}\|\tilde{f}\| < \infty, \qquad (64)$$

where we have used the fact that Γ is $\frac{1}{b}\mathbb{Z}$-congruent to a subset of $\left[0,\frac{1}{b}\right)$. Therefore,

$$\int_{\left[0,\frac{1}{b}\right)} \tilde{f}(t)e^{-2\pi imbt}dt = \sum_{\gamma\in\mathbb{Z}} \int_{\left[0,\frac{1}{b}\right)} f\left(t+\frac{\gamma}{b}\right)\chi_r\left(t+\frac{\gamma}{b}\right)e^{-2\pi imbt}dt = \langle f, E_{mb}\chi_r\rangle.$$

for $m \in \mathbb{Z}$. Also observing that $\left\{\sqrt{b}e^{2\pi imb}\right\}_{m\in\mathbb{Z}}$ is an orthonormal basis for $\tilde{f} \in L^2\left[0,\frac{1}{\mathrm{b}}\right)$, we obtain the equation (63). Form (64) and $\frac{1}{b}$-periodicity of $\tilde{f}$, it follows that the right-hand side series in (63) converges absolutely a.e. on $\mathbb{R}$.

Lemma (6.2.4)[436]: Given $b > 0$ and a measurable set F in $\mathbb{R}$ with positive measure. Then $\{E_{mb}\chi_F\}_{m\in\mathbb{Z}}$ is a tight frame for $L^2(F)$ if and only if $F = F\left(\frac{1}{b},1\right)$. In this case, $\{E_{mb}\chi_F\}_{m\in\mathbb{Z}}$ is a tight frame for $L^2(F)$ with frame bound $\frac{1}{b}$.

Proof. The conclusions follow from Lemma 2.10 in [449].

Lemma (6.2.5)[436]: Given $b > 0$ and a measurable set F in $\mathbb{R}$ with finite positive measure. Then $\{E_{mb}\chi_F\}_{m\in\mathbb{Z}}$ is a Bessel sequence in $L^2(F)$ if and only if there exists $K \in \mathbb{N}$ such that $F\left(\frac{1}{b},1\right) = \emptyset$ for $k > K$.

Proof. We show the necessity by indirect proof. Suppose $\{E_{mb}\chi_F\}_{m\in\mathbb{Z}}$ is Bessel sequence in $L^2(F)$ with Bessel bound B, and $\mu\left(F\left(\frac{1}{b},k_B\right)\right) > 0$ for some $k_B > bB$. By Proposition (6.2.2), $F\left(\frac{1}{b},k_B\right)$ can be represented as a disjoint union of k_B measurable subsets $F^{(j)}\left(\frac{1}{b},k_B\right), j \in \mathbb{N}_{k_B}$, such that all $F^{(j)}\left(\frac{1}{b},k_B\right)$ are $\frac{1}{b}\mathbb{Z}$-congruent to the same subset of $\left[0,\frac{1}{b}\right)$. Define $f := \chi_{F\left(\frac{1}{b},k_B\right)}$. Then

$$\sum_{m\in\mathbb{Z}}\langle f, E_{mb}\chi_F\rangle E_{mb}\chi_F(\cdot)$$

$$= \sum_{m\in\mathbb{Z}}\sum_{j=0}^{k_B-1}\langle \chi_{F^{(j)}\left(\frac{1}{b},k_B\right)}, E_{mb}\chi_F\rangle\, e^{2\pi imb} f(\cdot)$$

$$= f(\cdot)\sum_{j=0}^{k_B-1}\sum_{m\in\mathbb{Z}}\langle \chi_{F^{(j)}\left(\frac{1}{b},k_B\right)}, E_{mb}\chi_{F^{(j)}\left(\frac{1}{b},k_B\right)}\rangle\, e^{2\pi imb}. \quad (65)$$

on $F\left(\frac{1}{b},k_B\right)$. Applying Lemma (6.2.3) to $\chi_{F^{(j)}\left(\frac{1}{b},k_B\right)}$ and $F^{(j)}\left(\frac{1}{b},k_B\right)$, we have

$$\sum_{m\in\mathbb{Z}}\langle \chi_{F^{(j)}\left(\frac{1}{b},k_B\right)}, E_{mb}\chi_{F^{(j)}\left(\frac{1}{b},k_B\right)}\rangle\, e^{2\pi imb} = \frac{1}{b}\sum_{\gamma\in\mathbb{Z}}\chi_{F^{(j)}\left(\frac{1}{b},k_B\right)+\frac{\gamma}{b}}(\cdot)$$

on $\mathbb{R}$. This together with (65) leads to

$$\sum_{m\in\mathbb{Z}}\langle f, E_{mb}\chi_F\rangle E_{mb}\chi_F(\cdot) = \frac{K_B}{b}f(\cdot)$$

on $\left(\frac{1}{b},k_B\right)$, where we have also used the facts that $F\left(\frac{1}{b},k_B\right)$ can be represent as a disjoint union of k_B measurable subsets $F^{(j)}\left(\frac{1}{b},k_B\right), j\in\mathbb{N}_{K_B}$, and that all $F^{(j)}\left(\frac{1}{b},k_B\right)$ are $\frac{1}{b}\mathbb{Z}$-congruent to the same subset of $\left[0,\frac{1}{b}\right)$. Therefore,

$$\sum_{m\in\mathbb{Z}}|\langle f, E_{mb}\chi_F\rangle|^2 = \frac{k_B}{b}\|f\|^2 > B\|f\|^2,$$

which is a contradiction as $\{E_{mb}\chi_F\}_{m\in\mathbb{Z}}$ is a Bessel sequence in $L^2(F)$ with Bessel bound B.

Now we prove the sufficiency. For an arbitrary function $f\in L^2(F)$,

$$\sum_{m\in\mathbb{Z}}|\langle f, E_{mb}\chi_F\rangle|^2 = \sum_{m\in\mathbb{Z}}\left|\sum_{k=1}^{k}\langle f, E_{mb}\chi_{F\left(\frac{1}{b},k\right)}\rangle\right|^2$$

$$= \sum_{m\in\mathbb{Z}}\left|\sum_{k=1}^{k}\langle f\chi_{F\left(\frac{1}{b},k\right)}, E_{mb}\chi_{F\left(\frac{1}{b},k\right)}\rangle\right|^2.$$

By Proposition (6.2.2), for each $k=1,\dots,k, F\left(\frac{1}{b},k\right)$ can be represented as a disjoint union of k measurable subsets $F^{(j)}\left(\frac{1}{b},k\right), j\in\mathbb{N}_k$, such that all $F^{(j)}\left(\frac{1}{b},k\right)$ are $\frac{1}{b}\mathbb{Z}$-congruent to the same subset of $\left[0,\frac{1}{b}\right)$. It follows that

$$\langle f\chi_{F\left(\frac{1}{b},k\right)}, E_{mb}\chi_{F\left(\frac{1}{b},k\right)}\rangle = \sum_{j=0}^{k-1}\langle f\chi_{F^{(j)}\left(\frac{1}{b},k\right)}, E_{mb}\chi_{F^{(j)}\left(\frac{1}{b},k\right)}\rangle$$

for $m\in\mathbb{Z}$ and $1\le k\le K$. Therefore,

$$\sum_{m\in\mathbb{Z}}|\langle f, E_{mb}\chi_F\rangle|^2 = \sum_{m\in\mathbb{Z}}\left|\sum_{k=1}^{k}\sum_{j=0}^{k-1}\langle f\chi_{F^{(j)}\left(\frac{1}{b},k\right)}, E_{mb}\chi_{F\left(\frac{1}{b},k\right)}\rangle\right|^2$$

$$\leq \frac{K(K+1)}{2}\sum_{k=1}^{k}\sum_{j=0}^{k-1}\sum_{m\in\mathbb{Z}}\left|\langle f\chi_{F^{(j)}\left(\frac{1}{b},k\right)}, E_{mb}\chi_{F^{(j)}\left(\frac{1}{b},k\right)}\rangle\right|^2. \quad (66)$$

Applying Lemma (6.2.4) $F^{(j)}\left(\frac{1}{b},k\right)$ leads to

$$\sum_{m\in\mathbb{Z}}\left|\langle f\chi_{F^{(j)}\left(\frac{1}{b},k\right)}, E_{mb}\chi_{F\left(\frac{1}{b},k\right)}\rangle\right|^2 = \frac{1}{b}\int_{F^{(j)}\left(\frac{1}{b},k\right)} |f(t)|^2 dt,$$

which together with (66) implies that

$$\sum_{m\in\mathbb{Z}}|\langle f, E_{mb}\chi_F\rangle|^2 \leq \frac{K(k+1)}{2b}\sum_{k=1}^{k}\sum_{j=0}^{k-1}\int_{F^{(j)}\left(\frac{1}{b},k\right)} |f(t)|^2 dt = \frac{K(k+1)}{2b}\|f\|^2.$$

Then $\{E_{mb}\chi_F\}_{m\in\mathbb{Z}}$ is a Bessel sequence in $L^2(f)$, as the function f in the above inequality is chosen arbitrarily.

Theorem (6.2.6)[436]: Given $a, b > 0$ and a measurable set F in $\mathbb{R}$ with finite positive measure. Then F is a Gabor Bessel set in $\mathbb{R}$ if and only if there exists $k \in \mathbb{N}$ such that $F\left(\frac{1}{b},k\right) = F(a,k) = \emptyset$ for $k > K$.

Proof. Necessity. Suppose F is a Gabor Bessel set in $\mathbb{R}$ with Bessel bound B.Then

$$\sum_{n\in\mathbb{Z}}\chi_F(\cdot - na) = \sum_{n\in\mathbb{Z}}|\chi_F(\cdot - na)|^2 \leq bB$$

on $\mathbb{R}$ by Proposition 8.3.2 in [437], Also note that $\{E_{mb}\chi_F\}_{m\in\mathbb{Z}}$ is a Bessel sequence in $L^2(F)$. By Lemma (6.2.5), there exists $L \in \mathbb{N}$ such that $F(a,k) = \emptyset$ for $k > K$, when $K > \max\{b, B, L\}$.

Sufficiency. Suppose $K \in \mathbb{N}$ is such that $F\left(\frac{1}{b},k\right) = F(a,k) = \emptyset$ for $k > K$. Write $F_n = F + na$ for $n \in \mathbb{Z}$. Then $F_n\left(\frac{1}{b},k\right) = F\left(\frac{1}{b},k\right) = \emptyset$ for $k > K$, and

$$\sum_{m\in\mathbb{Z}}|\langle f, E_{mb}T_{na}\chi_F\rangle|^2 \leq \sum_{n\in\mathbb{Z}}\left|\langle f\chi_{F_n}, E_{mb}\chi_{F_n}\rangle\right|^2$$

for $f \in L^2(\mathbb{R})$ and $n \in \mathbb{Z}$. So, by the proof of sufficiency in Lemma (6.2.5)

$$\sum_{m\in\mathbb{Z}}|\langle f, E_{mb}T_{na}\chi_F\rangle|^2 \leq \frac{K(K+1)}{2b}\|fT_{na}\chi_F\|^2$$

for $f \in L^2(\mathbb{R})$ and $n \in \mathbb{Z}$. It follows that

$$\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}|\langle f, E_{mb}T_{na}\chi_F\rangle|^2 \leq \frac{K(K+1)}{2b}\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} |f(t)|^2\chi_F(t-na)dt$$

$$= \frac{K(K+1)}{2b}\int_{\mathbb{R}} |f(t)|^2\sum_{n\in\mathbb{Z}}\chi_F(t-na)dt$$

for $f \in L^2(\mathbb{R})$. Also observing that $\sum_{n\in\mathbb{Z}}\chi_F(t-na) \leq K$ due to $F(a,k) = \emptyset$ for $k > K$

$$\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}|\langle f, E_{mb}T_{na}\chi_F\rangle|^2 \leq \frac{K^2(K+1)}{2b}\|f\|^2$$

for $f \in L^2(\mathbb{R})$. That F is a Gabor Bessel set in $\mathbb{R}$.

We will focus on Gabor frame sets, especially on tight Gabor sets in $\mathbb{S}$, where $a, b > 0$, and $\mathbb{S}$ is an a$\mathbb{Z}$-periodic set in $\mathbb{R}$. We first need to establish two Lemmas.

Lemma (6.2.7)[436]: Given $b > 0$ and two measurable sets F_1 and F_2 in $\mathbb{R}$ with positive measure. Assume that

$$F_1 = F_1\left(\frac{1}{b}, 1\right), \quad F_1 \cap \left(F_2 + \frac{K}{b}\right) = \emptyset \quad \text{for } k \in \mathbb{Z}.$$

Then

$$\sum_{m\in\mathbb{Z}} \langle f, E_{mb}\chi_{F_1}\rangle E_{mb}\chi_{F_1}(\cdot) = 0 \tag{67}$$

Holds on $\mathbb{R}$ for any $f \in L^2(\mathbb{R})$.

Proof. Applying Lemma (6.2.3) to F_1 and f, we have

$$\sum_{m\in\mathbb{Z}} \langle f, E_{mb}\chi_{F_1}\rangle e^{2\pi imb} = \frac{1}{b}\sum_{\gamma\in\mathbb{Z}} f\left(\cdot + \frac{\gamma}{b}\right)\chi_{F_1}\left(\cdot + \frac{\gamma}{b}\right).$$

It follows that

$$\sum_{m\in\mathbb{Z}} \langle f, E_{mb}\chi_{F_1}\rangle E_{mb}\chi_{F_2}(\cdot) = \frac{1}{b}\sum_{\gamma\in\mathbb{Z}} f\left(\cdot + \frac{\gamma}{b}\right)\chi_{(F_1 - \frac{\gamma}{b})\cap F_2}(\cdot),$$

which, together with $F_1 \cap \left(F_2 + \frac{k}{b}\right) = \emptyset$ for $k \in \mathbb{Z}$, shows the conclusion (67).

Lemma (6.2.8)[436]: Given $a, b > 0$ and an a$\mathbb{Z}$-periodic set $\mathbb{S}$ in $\mathbb{R}$. Then, for an arbitrary measurable subset F of $\mathbb{S}$ with finite positive measure, F is a Gabor Bessel set in $\mathbb{S}$ if and only if there exists $k \in \mathbb{N}$ such that $F\left(\frac{1}{b}, k\right) = F(a, k) = \emptyset$ for $k > K$.

Proof. Since $E_{mb}T_{na}\chi_F \in L^2(\mathbb{S})$ for $m, n \in \mathbb{Z}$, F is a Gabor Bessel set in $\mathbb{S}$ if and only if it is a Gabor Bessel set in $\mathbb{R}$. the lemma therefore follows by Theorem (6.2.6).

By Proposition (6.2.1) and Lemma (6.2.8), we obtain the following necessary condition for a set to be a Gabor frame set in $\mathbb{S}$.

Theorem (6.2.9)[436]: Given $a, b > 0$ and an a$\mathbb{Z}$-periodic set $\mathbb{S}$ in $\mathbb{R}$. Assume that F is a Gabor frame set in $\mathbb{S}$. Then

(i) $\bigcup_{n\in\mathbb{Z}}\left(F\left(\frac{1}{b}, 1\right) + na\right) = \mathbb{S}$; and

(ii)There exists $K \in \mathbb{N}$ such that $F\left(\frac{1}{b}, k\right) = F(a, k) = \emptyset$ for $k > K$.

The following theorem gives a sufficient condition for a set to be a Gabor frame set in $\mathbb{S}$.

Theorem (6.2.10)[436]: Given $a, b > 0$ and an a$\mathbb{Z}$-periodic set $\mathbb{S}$ in $\mathbb{R}$. Assume that F is a measurable subset of $\mathbb{S}$ with finite positive measure, and that F satisfies the following conditions:

(i) $\bigcup_{n\in\mathbb{Z}}\left(F\left(\frac{1}{b}, 1\right) + na\right) = \mathbb{S}$; and

(ii) There exists $K \in \mathbb{N}$ such that $F\left(\frac{1}{b}, 1\right) = F(a, k) = \emptyset$ for $k > K$.

Then F is a Gabor frame set in $\mathbb{S}$.

Proof. By Lemma (6.2.8), F is a Gabor Bessel set in $\mathbb{S}$, and the function

$$H_F f := \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \langle f, E_{mb}T_{na}\chi_F\rangle E_{mb}T_{na}\chi_F \tag{68}$$

converges for any $f \in L^2(\mathbb{R})$. Then it suffices to prove

$$\langle H_F f, f\rangle \geq \frac{1}{b}\|f\|^2, \quad f \in L^2(\mathbb{S}). \tag{69}$$

When $K = 1$, we have that $F = F\left(\frac{1}{b}, 1\right)$, and consequently,

$$\langle H_F f, f\rangle = \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} |\langle T_{-na} f\chi_F, E_{mb}T_{na}\chi_F\rangle|^2 = \frac{1}{b}\sum_{n\in\mathbb{Z}}\int_{F+na} |f(t)|^2 dt$$

for $f \in L^2(\mathbb{S})$ by Lemma (6.2.4).This, together with the condition (i), establishes (69).

When $K > 1$, by Proposition (6.2.4), for $f \in L^2(\mathbb{S})$, $H_F f$ can be be rewritten as

$$H_F f = \sum_{n\in\mathbb{Z}}\sum_{k=1}^{k}\sum_{l=1}^{k}\sum_{j=0}^{k-1}\sum_{m\in\mathbb{Z}} \langle f, E_{mb}T_{na}\chi_{F^{(j)}\left(\frac{1}{b},K\right)}\rangle E_{mb}T_{na}\chi_{F^{(j)}\left(\frac{1}{b},K\right)}$$

where $F\left(\frac{1}{b}, 1\right) \cap \left(F\left(\frac{1}{b}, l\right) + \frac{1}{b}\right) = \emptyset$ for $l' \in \mathbb{Z}$ and $k, l \in \{1, \dots, k\}$ with $k \neq l$, $F\left(\frac{1}{b}, k\right) = \bigcup_{j=0}^{k-1} F^{(j)}\left(\frac{1}{b}, k\right), j \in \mathbb{N}_k$, are $\frac{1}{b}\mathbb{Z}$-congruent to the same subset of $\left(0, \frac{1}{b}\right)$. Then, by Lemma (6.2.7),

$$\begin{aligned} H_F f &= \sum_{n\in\mathbb{Z}}\sum_{k=1}^{k}\sum_{m\in\mathbb{Z}} \langle f, E_{mb}T_{na}\chi_{F\left(\frac{1}{b},k\right)}\rangle E_{mb}T_{na}\chi_{F\left(\frac{1}{b},k\right)} \\ &= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \langle f, E_{mb}T_{na}\chi_{F\left(\frac{1}{b},1\right)}\rangle E_{mb}T_{na}\chi_{F\left(\frac{1}{b},1\right)} \\ &\quad + \sum_{n\in\mathbb{Z}}\sum_{k=2}^{k}\sum_{m\in\mathbb{Z}} \langle f, E_{mb}T_{na}\chi_{F\left(\frac{1}{b},k\right)}\rangle E_{mb}T_{na}\chi_{F\left(\frac{1}{b},k\right)} \end{aligned} \tag{70}$$

for $f \in L^2(\mathbb{S})$. It follows that

$$\begin{aligned} \langle H_F f, f\rangle &\geq \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \left|\langle f, E_{mb}T_{na}\chi_{F\left(\frac{1}{b},1\right)}\rangle\right|^2 \\ &= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \left|\langle T_{-na} f\chi_{F\left(\frac{1}{b},1\right)}, E_{mb}\chi_{F\left(\frac{1}{b},1\right)}\rangle\right|^2 \end{aligned} \tag{71}$$

for $f \in L^2(\mathbb{S})$. By Lemma (6.2.4) and the condition (i).

$$\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \left|\langle T_{-na} f\chi_{F\left(\frac{1}{b},1\right)}, E_{mb}\chi_{F\left(\frac{1}{b},1\right)}\rangle\right|^2 = \frac{1}{b}\sum_{n\in\mathbb{Z}}\int_{F\left(\frac{1}{b},1\right)+na} |f(t)|^2 dt \geq \frac{1}{b}\|f\|^2$$

for $f \in L^2(\mathbb{S})$, which together with (71) proves (69).

Theorem (6.2.11)[436]: Given $a, b > 0$ and an $a\mathbb{Z}$-periodic set $\mathbb{S}$ in $\mathbb{R}$. Assume that F is a measurable subset of $\mathbb{S}$, and that

$$F = F(a, 1), \text{and } F\left(\frac{1}{b}, k\right) \neq \emptyset \quad \text{for some } k > 1.$$

Then F is not a Gabor frame set in $\mathbb{S}$.

Proof. By Proposition (6.2.2), there exist $F^{(j)}\left(\frac{1}{b}, k\right), j \in \mathbb{N}_k$, such that

$$F\left(\frac{1}{b}, k\right) = \bigcup_{j\in\mathbb{N}_k} F^{(j)}\left(\frac{1}{b}, k\right),$$

that $F^{(j)}\left(\frac{1}{b},k\right), j \in \mathbb{N}_k$, are mutually disjoint, and they are all $\frac{1}{b}\mathbb{Z}$-congruent to the same subset of $\left(0,\frac{1}{b}\right)$. Define $f(\cdot) := \chi_{F^{(0)}\left(\frac{1}{b},k\right)}(\cdot) - \chi_{F^{(0)}\left(\frac{1}{b},k\right)}(\cdot)$ then $0 \neq f \in L^2(\mathbb{S})$ and

$$\langle f, E_{mb}T_{na}\chi_F\rangle = \int_{F+na} \chi_{F^{(0)}\left(\frac{1}{b},k\right)}(t)e^{-2\pi imbt}dt - \int_{F+na} \chi_{F^{(1)}\left(\frac{1}{b},k\right)}(t)e^{-2\pi imbt}dt$$

for $m,n \in \mathbb{Z}$. Also observing that $F = F(a,1)$, we have

$$F^{(j)}\left(\frac{1}{b},k\right) \cap F = F^{(j)}\left(\frac{1}{b},k\right),\ F^{(j)}\left(\frac{1}{b},k\right) \cap (F + an) = \emptyset \text{ when } 0 \neq n \in \mathbb{Z}$$

for $j = 0,1$. It follows that

$$\sum_{m,n\in\mathbb{Z}} |\langle f, E_{mb}T_{na}\chi_F\rangle|^2 = \sum_{m\in\mathbb{Z}} \left| \int_{F^{(0)}\left(\frac{1}{b},k\right)} e^{-2\pi imbt}dt - \int_{F^{(1)}\left(\frac{1}{b},k\right)} e^{-2\pi imbt}dt \right|^2 = 0,$$

where we have used the fact that $F^{(0)}\left(\frac{1}{b},k\right)$ and $F^{(0)}\left(\frac{1}{b},k\right)$ are both $\frac{1}{b}\mathbb{Z}$-congruent to the same subset of $\left[0,\frac{1}{b}\right)$. Therefore, F is not Gabor frame set in $\mathbb{S}$.

Theorem (6.2.12)[436]: Let $a,b > 0$ and $\mathbb{S}$ be an a$\mathbb{Z}$-periodic set in $\mathbb{R}$. For an arbitrary measurable subset F of $\mathbb{S}$ with finite positive measure, the following statements are equivalent:

(i) F is a tight Gabor set in $\mathbb{S}$.

(ii)$\bigcup_{n\in\mathbb{Z}}(F + na) = \mathbb{S}$ and $F = F\left(\frac{1}{b},1\right) = F(a,k)$ for Some $k \in \mathbb{N}$.

(iii)There exist k mutually disjoint measurable sets $F_0, F_1, \dots, F_{k-1}$ in $\mathbb{S}$ such that $F = \bigcup_{\gamma=0}^{k-1} F_\gamma$ for some $k \in \mathbb{N}, F = F\left(\frac{1}{b},1\right)$, and each F_γ is a$\mathbb{Z}$-congruent to $[0,a) \cap \mathbb{S}$.

Furthermore, the number k in (ii) and (iii) are the same, and, when one of (ii) and (iii) is satisfied, F is a tight Gabor set in $\mathbb{S}$ with frame bound $\frac{k}{b}$.

Proof.

(i)⇒(ii) Suppose F is a tight Gabor set in $\mathbb{S}$ with frame bound A. By Proposition (6.2.1), we have $\bigcup_{n\in\mathbb{Z}}(F + na) = \mathbb{S}$. Suppose $\mu\left(F\left(\frac{1}{b}\ ,k_0\right)\right) > 0$ for some $k_0 > 1$. Then, by Proposition (6.2.2), $F\left(\frac{1}{b}\ ,k_0\right)$ can be represented as a disjoint union of k_0 measurable subsets $F^{(j)}\left(\frac{1}{b},k_0\right), j \in \mathbb{N}_{k_0}$, such that all $F^{(j)}\left(\frac{1}{b},k_0\right)$ are $\frac{1}{b}\mathbb{Z}$-congruent to the same subset of $\left[0,\frac{1}{b}\right)$. Take $f_0 = \chi_{F^{(0)}\left(\frac{1}{b},k_0\right)}$ and $f_1 = \chi_{F^{(1)}\left(\frac{1}{b},k_0\right)}$. Then $f_0, f_1 \in L^2(\mathbb{S})$ and $\langle f_0, f_1\rangle = 0$ since $F^{(0)}\left(\frac{1}{b},k_0\right) \cap F^{(1)}\left(\frac{1}{b},k_0\right) = \emptyset$. Define

$$H_F f := \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \langle f, E_{mb}T_{na}\chi_F\rangle E_{mb}T_{na}\chi_F$$

for $f \in L^2(\mathbb{S})$. Again observing that F is a tight Gabor set in $\mathbb{S}$ with frame bound $A > 0$ leads to

$$\langle H_F f_0, f_1\rangle = A\langle f_0, f_1\rangle = 0 \tag{72}$$

On the other hand,

$$\langle H_F f_0, f_1\rangle = \sum_{m,n\in\mathbb{Z}} \langle f_0, E_{mb}T_{na}\chi_F\rangle\langle E_{mb}T_{na}\chi_F, f_1\rangle$$
$$= \sum_{m,n\in\mathbb{Z}} \left(\int_{F^{(0)}\left(\frac{1}{b},k_0\right)\cap(F+na)} e^{-2\pi imbt}dt \right)$$
$$\times \left(\int_{F^{(1)}\left(\frac{1}{b},k_0\right)\cap(F+na)} e^{2\pi imbt}dt \right).$$

For each $n \in \mathbb{Z}$, $F^{(j)}\left(\frac{1}{b}, k_0\right) \cap F(F + an)$ is $\frac{1}{b}\mathbb{Z}$-congruent to a subset, denoted by $G_n^{(j)}$, of $\left[0, \frac{1}{b}\right)$ for $j = 0,1$. It follows that

$$\langle H_F f_0, f_1\rangle = \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \left(\int_{G_n^{(0)}} e^{-2\pi ibmt}dt \right)\left(\int_{G_n^{(1)}} e^{2\pi ibmt}dt \right)$$
$$= \frac{1}{b}\sum_{n\in\mathbb{Z}} \int_{G_n^{(0)}\cap G_n^{(1)}} dt \geq \frac{1}{b}\mu\left(G_0^{(0)} \cap G_0^{(1)}\right).$$

Note that $G_0^{(0)} = G_0^{(1)}$ and $\mu\left(G_0^{(0)}\right) = \mu\left(F^{(0)}\left(\frac{1}{b}, k_0\right)\right) > 0$ by Proposition (6.2.2). This leads to

$$\langle H_F f_0, f_1\rangle > 0,$$

Which contradicts (72), and therefore $F = F\left(\frac{1}{b}, 1\right)$. It, together with Lemma (6.2.4), yields that

$$\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} |\langle f, E_{mb}T_{na}\chi_F\rangle|^2 = \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} |\langle T_{-na}f\chi_F, E_{mb}\chi_F\rangle|^2$$
$$= \frac{1}{b}\sum_{n\in\mathbb{Z}} \int_{F+na} |f(t)|^2 dt \tag{73}$$

for $f \in L^2(\mathbb{S})$.Next we show that $F = F(a,k)$ for some $k \in \mathbb{N}$ by indirect proof. Since $\mu(F) < \infty, F(a,\infty) = \emptyset$ by Proposition (6.2.2). Suppose $\mu\big(F(a,k_1)\big) > 0$ and $\mu\big(F(a,k_2)\big) > 0$ for some $k_1, k_2 \in \mathbb{N}$ with $k_1 \neq k_2$. Let $f_1 = \chi_{F(a,k_1)}$, $f_2 = \chi_{F(a,k_2)}$. Applying (73) to functions f_1 and f_2, and using Proposition (6.2.2),we obtain that

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} |\langle f_1, E_{mb}T_{na}\chi_F\rangle|^2 = \frac{k_1}{b}\|f_1\|^2,$$
$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} |\langle f_2, E_{mb}T_{na}\chi_F\rangle|^2 = \frac{k_2}{b}\|f_2\|^2,$$

Which contradicts the fact that F is a tight Gabor set in $\mathbb{S}$.

(ii)⇒(iii) Suppose that the condition (ii) is satisfied. By Proposition (6.2.2), there exist mutually disjoint measurable subsets $F^{\gamma}(a,k), \gamma \in \mathbb{N}_k$, such that $F =$

$F(a,k)=\bigcup_{\gamma=0}^{k-1}F^{\gamma}(a,k)$, and all $F^{\gamma}(a,k)$ are a$\mathbb{Z}$-congruent to the same subset S of $[0,a)$. Then

$$\bigcup_{n\in\mathbb{Z}}(S+na)=\bigcup_{n\in\mathbb{Z}}\left(F^{(\gamma)}(a,k)+na\right)=\bigcup_{n\in\mathbb{Z}}(F+na)=\mathbb{S}$$

For each $\gamma\in\mathbb{N}_k$, which implies that $[0,a)\cap\mathbb{S}=[0,a)\cap(\bigcup_{n\in\mathbb{Z}}(S+na))=S$ Since $S\subset[0,a)$. The conclusion (iii) therefore follows.

(iii)⇒(i) Suppose that the condition (iii) is satisfied. Since $=F\left(\frac{1}{b},1\right)$, the equation (73) holds. It follows that

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}|\langle f,E_{mb}T_{na}\chi_F\rangle|^2=\frac{1}{b}\sum_{\gamma=0}^{k-1}\sum_{n\in\mathbb{Z}}\int_{F_\gamma+na}|f(t)|^2dt$$

For $f\in L^2(\mathbb{S})$ since F is the disjoint union of $F_\gamma,\gamma\in\mathbb{N}_k$. Note that each F_γ is a$\mathbb{Z}$-congruent to $[0,a)\cap\mathbb{S}$, and that $\{([0,a)\cap\mathbb{S}+na:n\in\mathbb{Z}\}$ is also a partition of $\mathbb{S}$. We therefore have

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}|\langle f,E_{mb}T_{na}\chi_F\rangle|^2=\frac{K}{b}\|f\|^2$$

For $f\in L^2(\mathbb{S})$. We then finish the proof as (ii) holds for the same K as in (iii).

By Theorem (6.2.12), we have the following corollary.

Corollary (6.2.13)[436]: Given $a,b>0$ and an a$\mathbb{Z}$-periodic set $\mathbb{S}$ in $\mathbb{R}$. If F is a tight Gabor set in $\mathbb{S}$ with frame bound A, then $A=\frac{k}{b}$ for some $k=1$.

As an immediate consequence of Theorem (6.2.12) and Corollary (6.2.13), We can obtain a necessary condition of the existence of tight Gabor sets in $\mathbb{S}$, which can be obtained from Theorem 3.3 in [449] when $k=1$.

Theorem (6.2.14)[436]: Given $k\in\mathbb{N},a,b>0$ and an a$\mathbb{Z}$-periodic set in $\mathbb{S}$ in $\mathbb{R}$. If there exists a tight Gabor set in $\mathbb{S}$ with frame bound $\frac{k}{b}$, then

$$bk_\mu([0,a)\cap\mathbb{S})\le 1. \qquad (74)$$

Given positive number a,b such that ab is a rational number. We devoted to characterization of a$\mathbb{Z}$-periodic set $\mathbb{S}$ which admits tight Gabor sets, and to the construction of a class of tight Gabor sets in such a$\mathbb{Z}$-periodic $\mathbb{S}$. We first need to establish some auxiliary lemmas.

Lemma (6.2.15)[436]: Let $a,b>0$ satisfy $ab=\frac{p}{q}$ with $p,q\in\mathbb{N}$ and $\gcd(p,q)=1$. Then $\left\{\left[0,\frac{1}{bq}\right)+\frac{l}{b}+na:(l,n)\in\mathbb{N}_p\times\mathbb{Z}\right\}$ and $\left\{\left[0,\frac{1}{bq}\right)+\frac{1}{b}-ra+maq:(l,r,m)\in\mathbb{N}_p\times\mathbb{N}_q\times\mathbb{Z}\right\}$ are both partitions of $\mathbb{R}$.

Proof. Note that $\left\{\left[0,\frac{1}{bq}\right)+\frac{j}{b}:j\in\mathbb{Z}\right\}$ is a partition of $\mathbb{R}$. By Lemma 2.3 in [449], to every $j\in\mathbb{Z}$ there corresponds a unique $(l,n)\in\mathbb{N}_q\times\mathbb{Z}$ such that $j=lq+np$, from which we can obtain a unique $(l,r,m)\in\mathbb{N}_q\times\mathbb{N}_q\times\mathbb{Z}$ such that $j=lq+(mq-r)p$ the lemma therefore follows.

Lemma (6.2.16)[436]: Let $a,b>0$ satisfy $ab=\frac{p}{q}$ with $p,q\in\mathbb{N}$ and $\gcd(p,q)=1$, and let $\mathbb{S}$ be an a$\mathbb{Z}$-periodic set in $\mathbb{R}$. For $B\subset\mathbb{N}_p$, define

$$I_B := \left\{ t \in \left[0, \frac{1}{bq}\right) : \sum_{l \in \mathbb{N}_p} \chi_{\mathbb{S}}\left(t + \frac{l}{b}\right) = \sum_{l \in \mathbb{N}_p} \chi_{\mathbb{S}}\left(t + \frac{l}{b}\right) = \operatorname{card}(B) \right\} \quad (75)$$

if $B \neq \emptyset$, and

$$I_\emptyset := \left\{ t \in \left[0, \frac{1}{bq}\right) : \sum_{l \in \mathbb{N}_p} \chi_{\mathbb{S}}\left(t + \frac{l}{b}\right) = 0 \right\}. \quad (76)$$

Then

$$\mu([0, a) \cap \mathbb{S}) = \sum_{B \subset \mathbb{N}_p} \operatorname{card}(B) \mu(I_B).$$

Proof. Write $\mathbb{S}_0 = [0, a) \cap \mathbb{S}$. By Lemma (6.2.15),

$$\mu(\mathbb{S}_0) = \chi_{\mathbb{S}_0}(t)dt = \int_{\left[0, \frac{1}{bq}\right)} \sum_{l \in \mathbb{N}_p} \sum_{n \in \mathbb{Z}} \chi_{\mathbb{S}_0}\left(t + \frac{l}{b} + na\right) dt.$$

Note that $\{\mathbb{S}_0 + na : n \in \mathbb{Z}\}$ is a partition of $\mathbb{S}$ and $\{I_B : B \subset \mathbb{N}_p\}$ is a partition of $\left[0, \frac{1}{bq}\right)$. So,

$$\mu(\mathbb{S}_0) = \sum_{B \subset \mathbb{N}_p} \int_{I_B} \sum_{l \in \mathbb{N}_p} \chi_{\mathbb{S}}\left(t + \frac{1}{b}\right) dt = \sum_{B \subset \mathbb{N}_p} \operatorname{card}(B) \mu(I_B).$$

Lemma (6.2.17)[436]: Given $k \in \mathbb{N}, a, b > 0$ and an $a\mathbb{Z}$-periodic set $\mathbb{S}$ in $\mathbb{R}$. Let $ab = \frac{p}{q}$ with $p, q \in \mathbb{N}$ and $\gcd(p, q) = 1$, and let I_B be defined as in Lemma (6.2.16) for $B \subset \mathbb{N}_p$, Assume that, for $\emptyset \neq B \subset \mathbb{N}_p$ with $I_B \neq \emptyset$ and $(\gamma, l) \in \mathbb{N}_k \times B, m_{B,\gamma,l} \in \mathbb{Z}$ and $\tau_{B,\gamma}$ are mappings from B into $\mathbb{N}_q$ satisfying $\tau_{B,\gamma}(l) \neq \tau_{B,\gamma'}(l')$ for $(\gamma, l) \neq (\gamma', l')$. Define

$$F_\gamma := \bigcup_{\substack{\emptyset \neq B \subset \mathbb{N}_p \\ I_B \neq \emptyset}} \bigcup_{l \in B} \left(I_B + \frac{1}{b} - \tau_{B,\gamma}(l) a + m_{B,\gamma,l} aq \right). \quad (77)$$

for $\gamma \in \mathbb{N}_k$ =, and

$$F := \bigcup_{\gamma \in \mathbb{N}_k} F_\gamma. \quad (78)$$

Then F is a tight Gabor set in $\mathbb{S}$ with frame bound $\frac{k}{b}$ =.

Proof. By Theorem (6.2.12), it suffices to show that $F = \left(\frac{1}{b}, 1\right), F_\gamma, \gamma \in \mathbb{N}_k$, are mutually disjoint, and each F_γ is a$\mathbb{Z}$-congruent to $[0, a) \cap \mathbb{S}$.

First we show that $F = \left(\frac{1}{b}, 1\right)$. For this purpose, we only need to prove that $k = 0$ whenever $k \in \mathbb{Z}$ satisfies that

$$E + \frac{1}{b} - \tau_{B,\gamma}(l) a + m_{B,l,\gamma} aq = E' + \frac{l'}{b} - \tau_{B',\gamma'}(l') a + m_{B',l',\gamma'} aq + \frac{k}{b} \quad (79)$$

for some $\gamma, \gamma' \in \mathbb{N}_k, B, B' \subset \mathbb{N}_p, l \in B, l' \in B'$, and $\emptyset \neq E \subset I_B, \emptyset \neq E' \subset I_{B'}$.

Suppose (79) holds, then $E' = E + \frac{j}{bq}$, where $j = (l - l')q - \left(\tau_{B,\gamma}(l) - \tau_{B',\gamma'}(l')\right)p + \left(m_{B,l,\gamma} - m_{B',l',\gamma'}\right)pq - kq \in \mathbb{Z}$. Note that $E, E' \subset \left[0, \frac{1}{bq}\right)$. It follows that $j = 0$, and thus $E = E'$, which in turn implies that $B = B'$ since $E = E' \subset I_B \cap I_{B'}$ and $I_B \cap I_{B'} = \emptyset$ if $B \neq B'$. So the equation $j = 0$ can be rewritten as

$$(l - l')q - \left(\tau_{B,\gamma}(l) - \tau_{B,\gamma'}(l)\right)p + \left(m_{B,l,\gamma} - m_{B,l',\gamma'}\right)pq - kq = 0. \quad (80)$$

It follows that $q | \left(\tau_{B,\gamma}(l) - \tau_{B,\gamma'}(l)\right)$, and thus $\tau_{B,\gamma}(l) = \tau_{B,\gamma'}(l')$ since $\tau_{B,\gamma}(l)$, $\tau_{B,\gamma'}(l') \in \mathbb{N}_q$. Therefore, $(\gamma, l) = (\gamma', l')$, which implies that $k = 0$ by (80). From the above argument, we also obtain that $F_\gamma, \gamma \in \mathbb{N}_k$, are mutually disjoint.

Fix a $\gamma \in \mathbb{N}_k$. Similarly, we can show that F_γ is a$\mathbb{Z}$-congruent to a subset of $[0, a)$. Also observing that $F_\gamma \subset \mathbb{S}$, we have F_γ is a$\mathbb{Z}$-congruent to a subset of $[0, a) \cap \mathbb{S}$. By the above argument, $I_B + \frac{1}{b} - \tau_{B,\gamma}(l)a + m_{B,l,\gamma}aq, \emptyset \neq B \subset \mathbb{N}_p$ with $I_B \neq \emptyset, l \in B$, are mutually disjoint. It follows that

$$\mu(F_\gamma) = \sum_{B \subset \mathbb{N}_p} \operatorname{card}(B)\mu(I_B) = \mu([0, a) \cap \mathbb{S})$$

by Lemma (6.2.16), therefore, F_γ is a$\mathbb{Z}$-congruent to $[0, a) \cap \mathbb{S}$.

Remark (6.2.18)[436]: Under the hypothesis of Lemma (6.2.17) the sets of the form (78) cannot run over all tight Gabor sets in $\mathbb{S}$ with frame bound $\frac{k}{b}$. Indeed we can construct other tight Gabor sets in $\mathbb{S}$ with frame bound $\frac{k}{b}$ by cut-and-paste operations to a set of the form (78), which are not of the form (78). Suppose F is a set of the form (78). Arbitrarily fix $\gamma_0 \in \mathbb{N}_k, \emptyset \neq B_0 \subseteq \mathbb{N}_k, \emptyset \neq B_0 \subseteq \mathbb{N}_p$ with $I_{B_0} \neq \emptyset$ and $l_0 \in B_0$, fix $0 \neq m \in \mathbb{Z}$ and decompose $I_{B_0} + \frac{l_0}{b} - \tau_{B_0,\gamma_0}(l_0)a + m_{B_0,\gamma_0,l_0}aq$ as

$$I_{B_0} + \frac{l_0}{b} - \tau_{B_0,\gamma_0}(l_0)a + m_{B_0,\gamma_0,l_0}aq = S_1 \cap S_2 \neq \emptyset \text{ with } S_1, S_2 \neq \emptyset \text{ and } S_1 \cap S_2 = \emptyset.$$

Define

$$\tilde{F}_{\gamma_0} := \left(F_{\gamma_0} \backslash S_2\right) \cup (S_2 + maq),$$

and

$$\tilde{F} := \left(\bigcup_{\gamma_0 \neq \gamma \in \mathbb{N}_k} F_\gamma\right) \cup \tilde{F}_{\gamma_0}.$$

Then, by the same procedure as in Lemma (6.2.17). We can show that $\tilde{F}$ is a tight Gabor set in $\mathbb{S}$ with frame bound $\frac{k}{b}$. Also as $\tilde{F}_{\gamma_0}$ is not of the form (77), $\tilde{F}$ is not of the form (78) too.

The following theorem provides us with a necessary and sufficient condition for the existence of tight Gabor sets with frame bound $\frac{k}{b}$.

Theorem (6.2.19)[436]: Given $k \in \mathbb{N}$. Let $a, b > 0$ satisfy $ab = \frac{p}{q}$ with $p, q \in \mathbb{N}$ and $\gcd(p, q) = 1$, and let $\mathbb{S}$ be an a$\mathbb{Z}$-periodic set in $\mathbb{R}$. Then there exists a tight Gabor set in $\mathbb{S}$ with frame bound $\frac{k}{b}$ if and only if

$$k\sum_{l\in\mathbb{N}_p}\chi_{\mathbb{S}}\left(\cdot+\frac{1}{b}\right)\le q. \tag{81}$$

on $\left[0,\frac{1}{bq}\right)$.

Proof. Necessity. Suppose F is a tight Gabor set in $\mathbb{S}$ with frame bound $\frac{k}{b}$, then, by Theorem (6.2.12), there exist k mutually disjoint sets $F_\gamma, \gamma\in\mathbb{N}_k$, such that $F=\bigcup_{\gamma\in\mathbb{N}_k}F_\gamma$, $F=F\left(\frac{1}{b},1\right)$, and each F_γ is a$\mathbb{Z}$-congruent to $[0,a)\cap\mathbb{S}$. Also observing that $\{([0,a)\cap\mathbb{S})+na: n\in\mathbb{N}\}$ forms a partition of $\mathbb{S}$ by a$\mathbb{Z}$-periodicity of $\mathbb{S}$, we conclude that $\{F_\gamma+na: n\in\mathbb{N}\}$ is a partition of $\mathbb{S}$ for each $\gamma\in\mathbb{N}_k$. Then

$$\sum_{l\in\mathbb{N}_p}\chi_{\mathbb{S}}\left(\cdot+\frac{1}{b}\right)=\sum_{l\in\mathbb{N}_p}\sum_{n\in\mathbb{Z}}\chi_{F_\gamma}\left(\cdot+na+\frac{1}{b}\right)=\sum_{l\in\mathbb{N}_p}\sum_{n\in\mathbb{Z}}\chi_{F_\gamma}\left(\cdot+\frac{np+lq}{bq}\right)$$

on $\left[0,\frac{1}{bq}\right)$ for each $\gamma\in\mathbb{N}_k$. Taking sum over γ to both sides of the above equality, we obtain that

$$k\sum_{l\in\mathbb{N}_p}\chi_{\mathbb{S}}\left(\cdot+\frac{1}{b}\right)=\sum_{l\in\mathbb{N}_p}\sum_{n\in\mathbb{Z}}\chi_{F}\left(\cdot+\frac{np+lq}{bq}\right) \tag{82}$$

on $\left[0,\frac{1}{bq}\right)$, where we have used the facts that $F=F=\bigcup_{\gamma\in\mathbb{N}_k}F_\gamma$ and that $F_\gamma,\gamma\in\mathbb{N}_k$, are mutually disjoint. By Lemma 2.3 in [449], every $j\in\mathbb{Z}$ corresponds a unique $(l,n)\in\mathbb{N}_p\times\mathbb{Z}$ such that $j=lq+np$, from which we rewrite (81) as

$$k\sum_{l\in\mathbb{N}_p}\chi_{\mathbb{S}}\left(\cdot+\frac{1}{b}\right)=\sum_{n\in\mathbb{Z}}\chi_F\left(\cdot+\frac{n}{bq}\right)=\sum_{l\in\mathbb{N}_p}\sum_{n\in\mathbb{Z}}\chi_F\left(\cdot+\frac{l}{bq}+\frac{n}{b}\right)$$

on $\left[0,\frac{1}{bq}\right)$. Note that $\sum_{n\in\mathbb{Z}}\chi_F\left(\cdot+\frac{n}{b}\right)\le 1$ on $\mathbb{R}$ due to the fact that $F=F\left(\frac{1}{b},1\right)$. This inequality (81) therefore follows.

Sufficiency. By the definition of I_B in Lemma (6.2.16) and the fact $\{I_B: B\subset\mathbb{N}_p\}$ is a partition of $\left[0,\frac{1}{bq}\right)$, (80) holds on $\left[0,\frac{1}{bq}\right)$ if and only if

$$k\,\mathrm{card}(B)\le q \tag{83}$$

for $\emptyset\ne B\subset\mathbb{N}_p$ with $I_B\ne\emptyset$. Then, by Lemma (6.2.17) we only need to show that the sets of the form (78) are well-defined if (83) holds for $\emptyset\ne B\subset\mathbb{N}_p$ with $I_B\ne\emptyset$. Note that the inequality (83) is equivalent to $\mathrm{card}(\mathbb{N}_k\times B)\le q$ for $\emptyset\ne B\subset\mathbb{N}_p$ with $I_B\ne\emptyset$. So there exist k mappings $\tau_{B,\gamma}$ from B into $\mathbb{N}_q$, $\gamma\in\mathbb{N}_k$, such that they meet the requirement in Lemma (6.2.17) for each $\tau_{B,\gamma}(l)$, fix an integer $m_{B,\gamma,l}$. Then we can define a set of the form (78).

By Lemma (6.2.17) and Theorem (6.2.19), we have the following theorem, which also gives a class of tight Gabor sets in $\mathbb{S}$.

Theorem (6.2.20)[436]: Given $a,b>0$ and an $a\mathbb{Z}$-periodic set $\mathbb{S}$ in $\mathbb{R}$. Let $ab=\frac{p}{q}$ with $p,q\in\mathbb{N}$ and $\gcd(p,q)=1$. Then there exists a tight Gabor set in $\mathbb{S}$ if and only if

$$\sum_{l\in\mathbb{N}_p}\chi_{\mathbb{S}}\left(\cdot+\frac{1}{b}\right)\le q \tag{84}$$

on $\left[0,\frac{1}{bq}\right)$. In this case, define k_0 by

$$k_0 := \max\left\{k \in \mathbb{N}: k\sum_{l\in\mathbb{N}_p}\chi_{\mathbb{S}}\left(\cdot+\frac{1}{b}\right) \le q \text{ on } [0,\frac{1}{bq})\right\}.$$

Then every set G_k of the form

$$G_k = \bigcup_{\gamma\in\mathbb{N}_k}\bigcup_{\substack{\emptyset\neq B\subset\mathbb{N}_p\\ I_B\neq\emptyset}}\bigcup_{l\in B}\left(I_B+\frac{1}{b}-\tau_{B,\gamma}(l)a+m_{B,\gamma,l}aq\right)$$

is a tight Gabor set in $\mathbb{S}$ with frame bound $\frac{k}{b}$ for each $k \in \mathbb{N}_{k_0}+1$, where I_B are defined as in Lemma (6.2.16), $\tau_{B,\gamma}$ are mappings from B into $\mathbb{N}_q$ satisfying $\tau_{B,\gamma}(l) \neq \tau_{B,\gamma'}(l')$ and $m_{B,\gamma,l} \in \mathbb{Z}$ for $(\gamma,l),(\gamma',l') \in \mathbb{N}_k \times B$ with $(\gamma,l) \neq (\gamma',l')$ and $\emptyset \neq B \subset \mathbb{N}_p$ with $I_B \neq \emptyset$.

We begin with the following example. Given $k \in \mathbb{N}$ and $a,b > 0$ satisfying $\frac{1}{k} < ab = \frac{p}{q}$ with $p,q \in \mathbb{N}$ and $\gcd(p,q) = 1$. Let $0 < \epsilon \le \frac{1}{kbp}$, let $\Gamma = \bigcup_{l\in\mathbb{N}_p}\left[\frac{1}{b},\frac{1}{b}+\epsilon\right)$, and let $\mathbb{S} = \bigcup_{l\in\mathbb{N}_p}[\Gamma+na)$. Then

$$[0,a)\cap\mathbb{S} = \bigcup_{l\in\mathbb{N}_q}\big((\Gamma\cap(la,(la+1)a)\big)-la).$$

It follows that

$$\mu([0,a)\cap\mathbb{S}) \le \sum_{l\in\mathbb{N}_q}(\mu(\Gamma\cap(la,(la+1)a)-la) = \sum_{l\in\mathbb{N}_q}\mu\big(\Gamma\cap(la,(la+1)a)\big)$$
$$= \mu(\Gamma) = \epsilon p,$$

which implies that

$$kb\mu([0,a)\cap\mathbb{S}) \le 1.$$

However, it is obvious that $\sum_{l\in\mathbb{N}_q}\chi_{\mathbb{S}}\left(\cdot+\frac{1}{b}\right) = kp > q$ on $(0,\epsilon)$. Then by Theorems (6.2.14) and (6.2.19), this examples shows that, when ab is a rational number, the inequality (44) is a necessary but not a sufficient condition for the existence of tight Gabor set in $\mathbb{S}$ with frame bound $\frac{k}{b}$. Next we will show that, when ab is an irrational number, (44) is also sufficient for the existence of tight Gabor set in $\mathbb{S}$. However, it is open how to obtain an explicit expression of general tight Gabor sets in $\mathbb{S}$ for this case.

By a careful observation of Theorem 4.2 in [449] and Theorem (6.2.14), we have the following Lemma.

Lemma (6.2.21)[436]: Given $a,b > 0$ with $ab \notin Q$, and an $a\mathbb{Z}$-periodic set $\mathbb{S}$ in $\mathbb{R}$. There exists a tight Gabor set in $\mathbb{S}$ with frame bound $\frac{1}{b}$ if and only if

$$b\mu([0,a)\cap\mathbb{S}) \le 1.$$

Theorem (6.2.22)[436]: Given $k \in \mathbb{N}, a,b > 0$ with $ab \notin Q$, and an $a\mathbb{Z}$-periodic set $\mathbb{S}$ in $\mathbb{R}$. There exists a tight Gabor set in $\mathbb{S}$ with frame bound $\frac{k}{b}$ if and only if

$$bk\mu([0,a)\cap\mathbb{S}) \le 1. \tag{85}$$

Proof. By Theorem (6.2.14) and Lemma (6.2.21), we only need to show the sufficiency for the case that $k > 1$. Suppose that (85) is satisfied. Note that

$$[0,ka)\cap\mathbb{S}=\bigcup_{\gamma\in\mathbb{N}_k}\big(([0,a)\cap\mathbb{S})+\gamma a\big).$$

It follows that

$$\mu([0,ka)\cap\mathbb{S})=k\mu([0,a)\cap\mathbb{S}),\tag{86}$$

and thus, $b\mu([0,ka)\cap\mathbb{S})\leq 1$. Note that $\mathbb{S}$ is also $ka\mathbb{Z}$-periodic. So, by Lemma (6.2.21), there exists $F\subset\mathbb{S}$ such that $\{E_{mb}T_{nka}\,\chi_F\}_{m,n\in\mathbb{Z}}$ is a tight frame for $L^2(\mathbb{S})$ with frame bound $\frac{1}{b}$, which, by Theorem (6.2.12), implies that $F=F\left(\frac{1}{b},1\right)$ and F is $ka\mathbb{Z}$-congruent to $[0,ka)\cap\mathbb{S}$. Put

$$F_\gamma=\left(\bigcup_{l\in\mathbb{Z}}([0,a)+(\gamma+lk)a)\right)\cap F$$

for $\gamma\in\mathbb{N}_k$. Then $F=\bigcup_{\gamma\in\mathbb{N}_k}F_\gamma$, and $F_\gamma,\gamma\in\mathbb{N}_k$, are mutually disjoint. So it suffices to show that F_γ is $a\mathbb{Z}$-congruent to $[0,a)\cap\mathbb{S}$ for each $\gamma\in\mathbb{N}_k$ by Theorem (6.2.12). By $a\mathbb{Z}$-periodicity of $\mathbb{S}$ and the fact that $F_\gamma\subset\mathbb{S}$ for $\gamma\in\mathbb{N}_k$,

$$F_\gamma=\{\textstyle\bigcup_{l\in\mathbb{Z}}[(([0,a)\cap\mathbb{S})+\gamma a)+lka]\}\cap F$$

for $\gamma\in\mathbb{N}_k$. It follows that each F_γ with $\gamma\in\mathbb{N}_k$ is $ka\mathbb{Z}$-congruent to a subset of $([0,a)\cap\mathbb{S})+\gamma a)$, and thus $a\mathbb{Z}$-congruent to a subset of $[0,a)\cap\mathbb{S}$. So, we only need to show that $\mu(F_\gamma)=\mu([0,a)\cap\mathbb{S})$ for each $\gamma\in\mathbb{N}_k$ by indirect proof. Suppose $\mu(F_{\gamma_0})<\mu([0,a)\cap\mathbb{S})$ for some $\gamma_0\in\mathbb{N}_k$. Also observing that $\{F_\gamma:\gamma\in\mathbb{N}_k\}$ is a partition of F, we have

$\mu(F)=\sum_{\gamma\in\mathbb{N}_k}\mu(F_\gamma)<K\mu([0,a)\cap\mathbb{S})$.

It together with (86) follows that

$$\mu(F)<\mu([0,ka)\cap\mathbb{S}),$$

which contradicts the fact that F is $ka\mathbb{Z}$-congruent to $[0,ka)\cap\mathbb{S}$.

Section (6.3): Periodic Subsets of the Real Line

For $\mathcal{H}$ be a separable Hilbert space. An at most countable sequence $\{h_i\}_{i\in\mathrm{I}}$ in $\mathcal{H}$ is a called a frame for $\mathcal{H}$ if there exist $0<A\leq B<\infty$ such that

$$A\|f\|^2\leq\sum_{i\in\mathrm{I}}|\langle f,h_i\rangle|^2\leq B\|f\|^2.\tag{87}$$

for $f\in\mathcal{H}$. Where A,B are called frame bounds; it is called a tight frame (Parseval frame) if $A=B(A=B=1)$ in (87); and a Bessel sequence in $\mathcal{H}$ if the right-hand side inequality in (87) holda. A frame for $\mathcal{H}$ is called a Riesz basis if it ceases to be a frame whenever any one of its elements is removed. Given two Bessel sequences $\{g_i\}_{i\in\mathrm{I}}$ and $\{h_i\}_{i\in\mathrm{I}}$ in $\mathcal{H}$, define the operator $S_{h,g}:\mathcal{H}\to\mathcal{H}$ by

$$S_{h,g}f=\sum_{i\in\mathrm{I}}\langle f,h_i\rangle g_i\tag{88}$$

for $f\in\mathcal{H}$. Then $S_{h,g}$ is bounded operator on $\mathcal{H}$. Let $\{g_i\}_{i\in\mathrm{I}}$ be a frame $\{h_i\}_{i\in\mathrm{I}}$ is called a dual of $\{g_i\}_{i\in\mathrm{I}}$ if $S_{h,g}=I$ on $\mathcal{H}$, where I denotes the identity operator. It is well-known that, for two Bessel sequences $\{g_i\}_{i\in\mathrm{I}}$ and $\{h_i\}_{i\in\mathrm{I}}$ in $\mathcal{H}$, whenever $S_{h,g}=I$ on $\mathcal{H}$, they are both frames for $\mathcal{H}$ and are duals of each other. If $g_i=h_i$ in (88) and $\{g_i\}_{i\in\mathrm{I}}$ is a frame for $\mathcal{H}$ with frame bounds A and B, it is also well-known that $S_{g,g}$ is bounded and invertible, that $\{S_{g,g}^{-1}h_i\}_{i\in\mathrm{I}}$ is also a frame for $\mathcal{H}$ with frame bounds B^{-1} and A^{-1}, and a dual of $\{g_i\}_{i\in I}$, he which is the so-called canonical dual. The fundamentals of frames can be found in [466,467,477,488].

Given a positive integer L, let $L^2(\mathbb{R},\mathbb{C}^L)$ be the vector-valued Hilbert space $L^2(\mathbb{R},\mathbb{C}^L)$ endowed with the inner product defined by

$$\langle f, h\rangle = \sum_{l=1}^{L} \int_{\mathbb{R}} f_l(x)\overline{h_l(x)}dx \textit{ for } f = (f_1, f_2, \ldots, f_L), h = (h_1, h_2, \ldots, h_L) \in L^2(\mathbb{R},\mathbb{C}^L).$$

Obviously, it is exactly the direct sum Hilbert space $\bigoplus_{l=1}^{L} L^2(\mathbb{R})$. *In what follows, for* $f \in L^2(\mathbb{R},\mathbb{C}^L)$ *and* $1 \le l \le L$, *we always denote by* f_l *its* l*-th component. For* $a, b > 0$ *and* $g \in L^2(\mathbb{R},\mathbb{C}^L)$, *we define Gabor system* $G(g,a,b)$ *by*

$$G(\mathbf{g}, a, b) = \{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}. \tag{89}$$

where

$$E_{mb}T_{na}g = \left(e^{2\pi imb}g_1(\cdot - na), e^{2\pi imb}g_2(\cdot - na), \ldots, e^{2\pi imb}g_1(\cdot - na)\right).$$

We also call it vector-valued Gabor system since L is not necessarily 1. When $L = 1$, it is the usual Gabor system in $L^2(\mathbb{R})$ and called scalar-valued Gabor system in contrast to a general L. A set S in $\mathbb{R}$ with positive measure is said to be $a\mathbb{Z}$-periodic if $S + na = S$ for $n \in \mathbb{Z}$. For such S, we denote by $L^2(\mathbb{R}\,,\mathbb{C}^L)$ the closed subspace of $L^2(\mathbb{R},\mathbb{C}^L)$ of the form

$$L^2(\mathbb{R},\mathbb{C}^L) = \{f \in L^2(\mathbb{R},\mathbb{C}^L) : f = 0 \text{ on } \mathbb{R}\backslash S\}.$$

The addresses Gabor analysis on $L^2(S,\mathbb{C}^L)$.

Vector-valued frame is also called Super frame. It was introduced in [461] under the setting of general Hilbert spaces by Balan in the context of "multiplexing". Which has been widely used in mobile communication network, satellite communication network and computer area network. In recent years, vector-valued wavelet and Gabor frames in $L^2(\mathbb{R},\mathbb{C}^L)$ have interested some mathematicians and engineering specialist (see [459,460,462,468,469,470,478, 479,481,482]), and in [486,487] vector-valued analysis also occurred as a technical tool in the study of ordinary frames. Let us first recall some related works.

Führ in [470] derived frame bound estimates for vector-valued Gabor system in $L^2(\mathbb{R},\mathbb{C}^L)$ with window functions belonging to Schwartz space, and obtained estimates for the window $\boldsymbol{h} = (h_0, h_1, \ldots, h_L) \in L^2(\mathbb{R},\mathbb{C}^{L+1})$ composed of the first $L+1$ Hermite functions. Grochenig and Lyubarskii in [478] characterized all lattices $\Lambda \subset \mathbb{R}^2$ such that Gabor system $\{E_{\lambda_2}T_{\lambda_1}\boldsymbol{h} : \lambda = (\lambda_1, \lambda_2) \in \Lambda\}$ is a frame for $L^2(\mathbb{R},\mathbb{C}^L)$. Abreu [458] gave a simple proof of this characterization. It has the advantage of also characterizing all lattices $\Lambda \subset \mathbb{R}^2$ such that $\{E_{\lambda_2}T_{\lambda_1}\boldsymbol{h} : \lambda = (\lambda_1, \lambda_2) \in \Lambda\}$ is a Riesz sequence in $L^2(\mathbb{R},\mathbb{C}^L)$. Also observe that Brekke and Seip in [464] characterized sets generating multi-window Gabor frames (resp. Riesz sequences) with Hermite functions. For general vector-valued Gabor systems, necessary density conditions were studied in [460] by Balan. For vector-valued Gabor systems with rational time-frequency lattices, a sufficient and necessary density condition was obtained in [482] by Li and Han. And a Zak transform matrix method was developed in [484] by Li and Zhou. There authors characterized complete vector-valued Gabor systems and Gabor frames, and obtained a parameterization of all its Gabor duals of a general vector-valued Gabor frame.

The theory of subspace Gabor frames includes two aspects. One is to ask whether $G(g,a,b)$a frame for its closed linear span for is given $g \in L^2(\mathbb{R})$ and $a,b > 0$, and [463,465,471,472,473,489] belong to this. The other is, given $a,b > 0$ and an $a\mathbb{Z}$-periodic set S in $\mathbb{R}$, to fined g such that $G(\boldsymbol{g},a,b)$ is a frame for $L^2(S)$. See [474,475,476,480,485]. Gabor analysis on $L^2(S)$ interests us because of the following reasons:

- From the perspective of application .Gabor systems on $L^2(S)$ can model a situation where a signal is known to appear periodically but intermittently, and one would try to perform Gabor analysis for the signal in the most efficient way possible while still preserving all the features of the observed data. Although one can think of the signal as existing for all time and do the analysis in the usual way, this is not optimal way to proceed if the signal is only emitted for very short periods of time.
- From the perspective of theory. The $a\mathbb{Z}$-periodicity of S is a natural requirement since one can show that S must be $a\mathbb{Z}$-periodicity if $L^2(S)$ admits a complete Gabor system. The projections of Gabor frames in $L^2(\mathbb{R})$ onto $L^2(S)$ onto $L^2(S)$ cannot cover all Gabor frames $L^2(S)$. Indeed, let $ab \leq 1$, and S be an $a\mathbb{Z}$-periodic measurable subset of $\mathbb{R}$ with positive measure. It is easy to check that, if $G(g,a,b)$ is a frame for $L^2(\mathbb{R})$, then its projection $G(g\chi_S,a,b)$ onto $L^2(S)$ is a frame for $L^2(S)$, where χ_S is the characteristic function of S. However, when $ab > 1$ and $S \neq \mathbb{R}$, $G(g,a,b)$ cannot be a frame in $L^2(\mathbb{R})$ for any $g \in L^2(\mathbb{R})$, while it is possible that there exists some g such $G(g,a,b)$ is a frame for $L^2(S)$. In addition, Theorems 2.7, 2.12, 3.3, 4.2 and Corollaries 2.13, 4.3 in [475] show that there exist significant differences between Gabor analysis on $L^2(S)$ and one on $L^2(\mathbb{R})$.

For rational ab and $g \in L^2(\mathbb{R},\mathbb{C}^L)$, Li and Zhang in [483] investigated the Gabor system $G(\boldsymbol{g},a,b)$ of the form (89). Using a suitable Zak transform matrix, they obtained a characterization for $G(\boldsymbol{g},a,b)$ to be a frame, Riesz basis, and orthonormal basis for its closed linear span. They also characterized the uniqueness of two types of Gabor duals, and using it they extended the classical Balian-Low Theorem. The classical Ron-Shen dual principle was pointed out to be invalid under this setting. For general ab and $a\mathbb{Z}$-periodic set S in $\mathbb{R}$, a densiry Theorem for Gabor systems in $L^2(\mathbb{R},\mathbb{C}^L)$ was presented in [474, Theorem 1.5] by Gabardo et al. However, nothing is known about general vector-valued Gabor systems in $L^2(\mathbb{R},\mathbb{C}^L)$ except for [474, Theorem 1.5]. Motivated by the above works, we consider Gabor systems in $L^2(S,\mathbb{C}^L)$. By introduction of a suitable Zak transform matrix, we investigate completeness, frame conditions of Gabor systems, and two types of Gabor duals for a general Gabor frame. We will work under the following assumptions:

Assumption (6.3.1)[457]: $a,b > 0$, and $ab = \frac{p}{q}$ with p and q being relatively prime positive integers.

Assumption (6.3.2)[457]: L is a positive integer.

Assumption (6.3.3)[457]: S is an $a\mathbb{Z}$-periodic set in $\mathbb{R}$.

Focuses on completeness and frame characterization. We devoted to dual characterization and expression. For a general frame $G(\boldsymbol{g},\mathrm{a},\mathrm{b})$ in $L^2(S,\mathbb{C}^L)$. We characterize its Gabor duals of type I and II. Obtain an explicit expression of its canonical Gabor dual. And establish a parameterization expression of all its Gabor duals of type I and a class of its Gabor duals of type II. We present an example Theorem for all previous Theorems. This allows us to easily construct Gabor frames and their Gabor duals by designing corresponding Zak transform matrices.

We denote by $\mathbb{N}$ the set of positive integers, by $\mathcal{M}_{s,t}$ the set of all $s\times t$ complex matrices for $s,t\in\mathbb{N}$, by $\mathcal{A}^*$ its conjugate transpose for $\mathcal{A}\in\mathcal{M}_{s,t}$, by I the identity operator and by I_t the $t\times t$ identity matrix when want to specify its size, by χ_F the characteristic function of F for a set F, and by $\mathbb{N}_M$ the set $\mathbb{N}_M=\{0,1,2,\dots,M-1\}$ for $M\in\mathbb{N}$. For $\mathcal{A}\in\mathcal{M}_{s,t}$, and $\emptyset\neq\Gamma\subset\mathbb{N}_s$, $\emptyset\neq\Omega\subset\mathbb{N}_t$, define $\mathcal{A}_{\Gamma,\Omega}$ as the sub matrix of $\mathcal{A}$ with row indices in Γ and column indices in Ω. In particular, we write $\mathcal{A}_{\Gamma,\Omega}=\mathcal{A}_\Gamma$ if $\Gamma=\Omega$. Given a subspace V of Euclidean Space $\mathbb{C}^\mu$, we denote by $\mathcal{P}_V$ the orthogonal projection operator from $\mathbb{C}^\mu$ onto V. And we denote by $\mathcal{M}(\boldsymbol{g},a,b)$ the closed linear subspace of $L^2(\mathbb{R},\mathbb{C}^L)$ generated by $G(\boldsymbol{g},a,b)$ for $g\in L^2(\mathbb{R},\mathbb{C}^L)$ and $a,b>0$. Let $G(\boldsymbol{g},a,b)$ and $G(\boldsymbol{h},a,b)$ be both Bessel sequences in $L^2(\mathbb{R},\mathbb{C}^L)$. Define $S_{h,\boldsymbol{g}}\colon L^2(\mathbb{R},\mathbb{C}^L)\to L^2(\mathbb{R},\mathbb{C}^L)$ by

$$S_{h,\boldsymbol{g}}f=\sum_{m,n\in\mathcal{W}\mathbb{Z}}\langle f,E_{m,b}T_{na}h\rangle E_{m,b}T_{na}g.$$

We also make the following conventions: relations between two measurable sets in $\mathbb{R}$ such as equality, disjointness or inclusion, are always understood up to a set of measure zero. And similarly, equality between two functions is always understood up to a set of measure zero, and similarly, equality or inequality between two functions is always understood in the “almost-everywhere” sense.

Let $\mathcal{A}\in\mathcal{M}_{s,t}$ which we consider as a linear mapping from $\mathbb{C}^t$ into $\mathbb{C}^s$. Define the mapping $\tilde{\mathcal{A}}\colon\ker(\mathcal{A})^\perp\to\mathrm{range}(\mathcal{A})$ by $\tilde{\mathcal{A}}x=\mathcal{A}x$ for $x\in\left(\ker(\mathcal{A})\right)^\perp$. Then $\tilde{\mathcal{A}}$ is a bijection, and thus it has an inverse $(\tilde{\mathcal{A}})^{-1}$. We extend $(\tilde{\mathcal{A}})^{-1}$ to an operator $\mathcal{A}^\dagger\colon\mathbb{C}^s\to\mathbb{C}^t$ by defining

$$\mathcal{A}^\dagger(y+z)=(\tilde{\mathcal{A}})^{-1}y \text{ for } y\in\mathrm{range}(\mathcal{A}) \text{ and } z\in\left(\mathrm{range}(\mathcal{A})\right)^\perp.$$

The operator $\mathcal{A}^\dagger$ is called the pseudo-inverse of $\mathcal{A}$.

Definition (6.3.4)[457]: Let $G(\boldsymbol{g},a,b)$ be a frame for $\mathcal{M}(\boldsymbol{g},a,b)$.

(i) If $\boldsymbol{h}\in\mathcal{M}(\boldsymbol{g},a,b)$ is such that $G(\boldsymbol{h},a,b)$ is a Bessel sequence, and that

$$S_{h,\boldsymbol{g}}f=I \text{ on } \mathcal{M}(\boldsymbol{g},a,b),$$

then $G(\boldsymbol{g},a,b)$ is called a Gabor dual of type I for $G(\boldsymbol{g},a,b)$.

(ii) If $h\in L^2(\mathbb{R},\mathbb{C}^L)$ (not necessarily in $\mathcal{M}(\boldsymbol{g},a,b)$ is such that $G(\boldsymbol{g},a,b)$ is a Bessel sequence,

$$\left\{\{\langle f,E_{m,b}T_{na}h\rangle\}_{m,n\in\mathbb{Z}}:\boldsymbol{f}\in L^2(\mathbb{R},\mathbb{C}^L)\right\}\subset\left\{\{\langle f,E_{m,b}T_{na}\boldsymbol{g}\rangle\}_{m,n\in\mathbb{Z}}\boldsymbol{f}\in L^2(\mathbb{R},\mathbb{C}^L)\right\}$$

and

$$S_{h,\boldsymbol{g}}=I \text{ on } \mathcal{M}(\boldsymbol{g},a,b),$$

then $G(\boldsymbol{h},a,b)$ is called a Gabor dual of type II for $G(\boldsymbol{g},a,b)$.

Definition (6.3.4) is a direct generalization of Definition 2.1 in [472]. Recall that the Gabor dual of type I corresponds to the usual Gabor dual where the dual window belongs to $\mathcal{M}(\boldsymbol{g}, a, b)$.while the dual window is not required to be in $\mathcal{M}(\boldsymbol{g}, a, b)$ for the Gabor dual of type II, instead the range of the analysis operator of the dual frame is required to be contained in that of the original frame.

Given a measurable set F in $\mathbb{R}$, a countable collection $\{F_i: i \in \mathrm{I}\}$ of measurable sets is called a partition of F sets is called a partition of F if

$$\bigcup_{i\in\mathrm{I}} F_i = F, \quad \text{and} \quad F_i \cap F_{i'} = \emptyset \text{ for } i \neq i' \in \mathrm{I}.$$

Given $\alpha > 0$, two measurable sets F_1 and F_2 are said to be $\alpha\mathbb{Z}$-congruent if there exists a partition $\{F_{1,k}: k \in \mathbb{Z}\}$ of F_1 such that $\{F_{1,k} + k: k \in \mathbb{Z}\}$ is a partition of F_2. In particular, only finitely many $F_{1,k}$ among $\{F_{1,k}: k \in \mathbb{Z}\}$ are nonempty, and the others are empty if both F_1 and F_2 are bounded in addition. Let a, b and L satisfy Assumptions (6.3.1) and (6.3.2). Define Zak transform $\mathcal{Z}_{aq}: L^2(\mathbb{R}) \to L^2_{\mathrm{loc}}(\mathbb{R}^2)$ by

$$\mathcal{Z}_{aq} f(t, v) = \sum_{k\in\mathbb{Z}} f(t + aqk) e^{2\pi ikv} \text{ for } f \in L^2(\mathbb{R}) \text{ and a.e.} (t, v) \in \mathbb{R}^2$$

and vector-valued Zak transform $\mathcal{Z}_{aq}: L^2(\mathbb{R}, \mathbb{C}^L) \to L^2_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{C}^L)$ by

$$\mathcal{Z}_{aq} f(t, v) = \left(\mathcal{Z}_{aq} f_1(t, v), \mathcal{Z}_{aq} f_1(t, v), \ldots, \mathcal{Z}_{aq} f_L(t, v)\right)$$
$$\text{for } f \in L^2(\mathbb{R}, \mathbb{C}^L) \text{ and a.e } (t, v) \in \mathbb{R}^2.$$

It is easy to check that $\mathcal{Z}_{aq}$ has the quasi-periodicity:

$$\mathcal{Z}_{aq} f(t + kaq, v + n) = e^{-2\pi ikv} \mathcal{Z}_{aq} f(t, v)$$
$$\text{for } f \in L^2(\mathbb{R}, \mathbb{C}^L), (k, n) \in \mathbb{Z}^2 \text{ and a.e } (t, v) \in \mathbb{R}^2. \tag{90}$$

By [475, Lemma 2.1], the restrictions of $\mathcal{Z}_{aq}$ and $\boldsymbol{Z}_{aq}$ to $[0, aq) \times [0,1)$ are respectively unitary operators from $L^2([0, aq) \times [0,1))$ and from $L^2(\mathbb{R}, \mathbb{C}^L)$ onto $L^2([0, aq) \times [0,1)\mathbb{C}^L)$.

For $f \in L^2(\mathbb{R}, \mathbb{C}^L)$, define the $\mathbb{C}^{Lp}$-valued function $\mathcal{F}(t, v)$ on $\mathbb{R}^2$ by

$$\mathcal{F}(t, v) = \begin{pmatrix} \mathcal{F}_1(t, v) \\ \mathcal{F}_2(t, v) \\ \vdots \\ \mathcal{F}_L(t, v) \end{pmatrix}, \tag{91}$$

where

$$\mathcal{F}_l(t, v) = \left(\mathcal{Z}_{aq} f_l \left(t + \frac{k}{b}, v\right)\right)_{k\in\mathbb{N}_p} \tag{92}$$

for each $1 \leq l \leq L$ and a.e. $(t, v) \in \mathbb{R}^2$.

Definition (6.3.5)[457]: Let a, b, L satisfy Assumptions (6.3.1) and (6.3.2). Given $g \in L^2(\mathbb{R}, \mathbb{C}^L)$, we associate it with the matrix-valued function $G: \mathbb{R}^2 \to \mathcal{M}_{q,L_p}$ Defined by

$$G(t, v) = \left(G_1(t, v), G_2(t, v), \ldots, G_L(t, v)\right) \tag{93}$$

for a.e. $(t, v) \in \mathbb{R}^2$, where $G_1(t, v)$ is the matrix-valued function from $\mathbb{R}^2$ into $\mathcal{M}_{q,p}$ given by

$$(G(t, v))_{r,k} = \mathcal{Z}_{aq} g_l \left(t + \frac{k}{b} - ra, v\right) \text{ for } (r, k) \in \mathbb{N}_q \times \mathbb{N}_p.$$

Observe that, even if $L = 1$, $G(t, v)$ is different from Zibulski-Zeevi matrix in [470], but it is more direct and convenient for our purpose. Similarly, for $\boldsymbol{h}, \boldsymbol{w}$ and $f \in L^2(\mathbb{R}, \mathbb{C}^L)$, we associate it with $H(t, v), W(t, v)$ and $F(t, v)$ as in (93) respectively.

Write

$$\Delta = \bigcup_{k\in\mathbb{N}_p} \bigcup_{r\in\mathbb{N}_q} \left(\left[0, \frac{1}{bq}\right) + \frac{k}{b} - ra\right).$$

It is easy to check that Δ is $aq\mathbb{Z}$-congruent to $[0, aq)$. So, by quasi-periodicity and unitarity of $\mathcal{Z}_{aq}$, an arbitrary $q \times L_p$ matrix-valued function defined on $\left[0, \frac{1}{bq}\right) \times [0,1)$ with $L^2\left(\left[0, \frac{1}{bq}\right) \times [0,1)\right)$ entries determines a unique $g \in L^2(\mathbb{R}, \mathbb{C}^L)$ via (93) restricted to $\left[0, \frac{1}{bq}\right) \times [0,1)$.

Let a, b, L and S satisfy Assumptions (6.3.1)-(6.3.2). Now we check how a $G(t, v)$ defined on $\left[0, \frac{1}{bq}\right) \times [0,1)$ determines a g in $L^2(S, \mathbb{C}^L)$. For $E \subset \mathbb{N}_p$, define

$$S_E = \left\{t \in \left[0, \frac{1}{bq}\right) : t + \frac{k}{b} \in S \text{ for } k \in E, t + \frac{k}{b} \notin S \text{ for } k \in \mathbb{N}_p \backslash E\right\}.$$

Then $\{S_E : E \subset \mathbb{N}_p\}$ is a partition of $\left[0, \frac{1}{bq}\right)$, and the set

$$\bigcup_{r\in\mathbb{N}_q} \bigcup_{E\subseteq\mathbb{N}_p} \bigcup_{k\in\mathbb{N}_{p\backslash E}} \left(S_E - ra + \frac{k}{b}\right)$$

is $aq\mathbb{Z}$-congruent to $[0, aq)\backslash S$ by Lemma 2.5 in [476]. Moreover,

$$\{\boldsymbol{Z}_{aq}\boldsymbol{f}|_{[0,aq)\times[0,1)} : \boldsymbol{f} \in L^2(S, \mathbb{C}^L)\}$$
$$= \{\boldsymbol{F} : \boldsymbol{F} \in L^2([0, aq) \times [0,1), \mathbb{C}^L), \boldsymbol{F} = 0 \text{ on } ([0, aq)\backslash S) \times [0,1)\}$$

by Lemma 2.1 in [475]. So, by quasi-periodicity and unitary of Z_{aq}, for an arbitrary $q \times L_p$ matrix–valued function $G(t, v)$ defined on $\left[0, \frac{1}{bq}\right) \times [0,1)$ with $L^2([0, aq) \times [0,1))$ entries, whenever all its kth columns vanish on $S_E \times [0,1)$ for $k \in \mathbb{N}_p\backslash E$ with $\boldsymbol{E} \nsubseteq \mathbb{N}_p$, it determines a unique $g \in L^2(S, \mathbb{C}^L)$ via (93) restricted to $\left[0, \frac{1}{bq}\right), [0,1)$.

In what follows, we always write

$$B(t) = \left\{k \in \mathbb{N}_p : t + \frac{k}{b} \in S\right\}, \tag{94}$$

$$D(t) = \operatorname{diag}\left(\chi_{B(t)}(0), \chi_{B(t)}(1), \dots, \chi_{B(t)}(p-1)\right) \tag{95}$$

and

$$\widetilde{D}(t) = \operatorname{diag}\big(D(t), D(t), \dots, D(t)\big) \text{ (with } L \text{ blocks)} \tag{96}$$

for $t \in \mathbb{R}$.

Let a, b, L and S satisfy Assumptions (6.3.1)-(6.3.3). Focuses on the completeness and frame characterization of a Gabor system in $L^2(S, \mathbb{C}^L)$, We begin with some lemmas.

The following three Lemmas are borrowed from Theorem 2.1 and Theorem 2.2 in [484] respectively:

Lemma (6.3.6)[457]: For $\boldsymbol{g} \in L^2(S, \mathbb{C}^L)$, $G(\boldsymbol{g}, a, b)$ is a frame for $\mathcal{M}(\boldsymbol{g}, a, b)$ with frame bounds A and B if and only if $bA(G(t,v)G^*(t,v)) \leq \left(G(t,v)G^*(t,v)\right)^2 \leq bB\left(G(t,v)G^*(t,v)\right)$ for a.e $(t,v) \in \left[0, \frac{1}{bq}\right) \times [0,1)$.

Lemma (6.3.7)[457]: Let $G(\boldsymbol{g}, a, b)$ be a Bessel sequence in $L^2(S, \mathbb{C}^L)$. Then

$$\operatorname{rank}\left(G(t,v)\right) = q \text{ for a.e } (t,v) \in \left[0, \frac{1}{bq}\right) \times [0,1)$$

if and only if $G(\boldsymbol{g}, a, b)$ has Riesz property, i.e., $c = 0$ is a unique solution to $\sum_{m,n\in\mathbb{Z}} c_{m,n} E_{mb} T_{na} \boldsymbol{g} = 0$ in $l^2(\mathbb{Z}^2)$.

Lemma (6.3.8)[457]: For every $j \in \mathbb{Z}$, there exists a $\mathbb{Z}$-periodic $L_P \times L_P$ unitary matrix-valued measurable function $U_j(v)$ such that

$$G^*\left(t + \frac{j}{bq}, v\right) H\left(t + \frac{j}{bq}, v\right) = U_j^*(v) G^*(t,v) H(t,v) U_j(v)$$

for $\boldsymbol{g}, \boldsymbol{h} \in L^2(S, \mathbb{C}^L)$ and a.e. $(t,v) \in \mathbb{R}^2$.

Lemma (6.3.9)[457]: $\boldsymbol{g} \in L^2(S, \mathbb{C}^L)$. Then $\mathcal{P}_{\ker(G(t,v))}(t,v), \operatorname{rank}\left(G(t,v)\right)$ are both measurable, $\operatorname{rank}\left(G(t,v)\right)$ is $\frac{1}{bq}\mathbb{Z}$-periodic with respect to t and satisfies

$$\operatorname{rank}\left(G(t,v)\right) \leq L \sum_{k=0}^{p-1} \chi_S\left(t + \frac{k}{b}\right). \tag{97}$$

Proof. Using an argument similar to Lemma 2.6 in [475], we can prove the measurability of $\mathcal{P}_{\ker(G(t,v))}(t,v)$ and $\operatorname{rank}\left(G(t,v)\right)$. By Lemma (6.3.8), $\operatorname{rank}\left(G(t,v)\right)$ is $\frac{1}{bq}\mathbb{Z}$-periodic with respect to t. Also observe that, for each $k \in \bigcup_{l=1}^{L}\left(\mathbb{N}_p + (l-1)p\right)$, we must have $k \in \bigcup_{l=1}^{L}(B(t) + (l-1)p)$ whenever the k-th column of $G(t,v)$ is a nonzero vector. This implies that $\operatorname{rank}\left(G(t,v)\right)$ is at most $L\sum_{k=0}^{p-1}\chi_S\left(t + \frac{k}{b}\right)$, the cardinality of $\bigcup_{l=1}^{L}(B(t) + (l-1)p)$. The proof is completed.

Theorem (6.3.10)[457]: Let $\boldsymbol{g} \in L^2(S, \mathbb{C}^L)$. Then the following are equivalent:

(i) $\mathcal{M}(\boldsymbol{g}, a, b) = L^2(S, \mathbb{C}^L)$;

(ii) $\operatorname{rank}\left(G(t,v)\right) = \sum_{k=0}^{p-1}\chi_S\left(t + \frac{k}{b}\right)$ for a.e. $(t,v) \in \left[0, \frac{1}{bq}\right) \times [0,1)$;

(iii) $\operatorname{range}\left(G^*(t,v)\right) = \operatorname{range}\left(\widetilde{D}(t)\right)$ for a.e. $(t,v) \in \left[0, \frac{1}{bq}\right) \times [0,1)$;

(iv) $\mathcal{P}_{\operatorname{range}(G^*(t,v))} = \widetilde{D}(t)$ for a.e. $(t,v) \in \left[0, \frac{1}{bq}\right) \times [0,1)$.

Proof. By an argument similar to Theorem 2.7 in [475], we can prove the equivalence between (i) and (ii). Next we prove the equivalence between (ii) and (iii), and the equivalence between (iii) and (iv) to finish the proof. Write $T = \bigcup_{l=1}^{L}\left(\mathbb{N}_p \backslash B(t) + (l-1)p\right)$. For almost every $(t,v) \in \left[0, \frac{1}{bq}\right) \times [0,1)$, by the definition of $G(t,v)$ we deduce that the kth row of $G^*(t,v)$ is a zero vector if $k \in T$. This implies that

$$\operatorname{range}\left(G^*(t,v)\right) \subset \left\{ x \in \mathbb{C}^{Lp}: x = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{L_{p-1}} \end{pmatrix}, x_k = 0 \text{ for } k \in T \right\} = \operatorname{range}\left(\widetilde{D}(t)\right)$$

and thus (iii) holds if and only if

$$\operatorname{rank}\big(G^*(t,v)\big) = \operatorname{rank}\left(\widetilde{D}(t)\right) \text{ for a.e. } (t,v) \in \left[0,\frac{1}{bq}\right) \times [0,1).$$

Since $\operatorname{rank}\left(\widetilde{D}(t)\right) = L\sum_{k=0}^{p-1}\chi_S\left(t+\frac{k}{b}\right)$, (ii) is equivalent to (98). The equivalence between (ii) and (iii) therefore follows. Observe that $\widetilde{D}(t)$ is an orthogonal projection for each $t \in \left[0,\frac{1}{bq}\right)$. It follows that (iii) is equivalent to (iv) [483, Theorem 3.1].

Theorem (6.3.11)[457]: Let $\boldsymbol{g} \in L^2(S,\mathbb{C}^L)$. Then $G(\boldsymbol{g},a,b)$ is a frame for $L^2(S,\mathbb{C}^L)$ with frame bounds A and B if and only if

$$bA\widetilde{D}(t) \le G^*(t,v)G(t,v) \le bB\widetilde{D}(t) \text{ for a.e} (t,v) \in \left[0,\frac{1}{bq}\right) \times [0,1). \tag{99}$$

Proof. It is easy to check that

$$\left\|\widetilde{D}(t)x\right\|^2 = \langle D(t)x,x\rangle \text{ and } G(t,v)\widetilde{D}(t) = G(t,v) \tag{100}$$

for $x \in \mathbb{C}^{Lp}$ and a.e. $(t,v) \in \mathbb{R}^2$. By Lemma (6.3.6) and Theorem (6.3.10), $G(\boldsymbol{g},a,b)$ is a frame for $L^2(S,\mathbb{C}^L)$ with frame bounds A and B if and only if

$$bAG(t,v)G^*(t,v) \le \big(G(t,v)G^*(t,v)\big)^2 \le bB\big(G(t,v)G^*(t,v)\big), \tag{101}$$

$$\operatorname{range}\big(G^*(t,v)\big) = \operatorname{range}\left(\widetilde{D}(t)\right) \tag{102}$$

for a.e. $(t,v) \in \left[0,\frac{1}{bq}\right) \times [0,1)$. Observe that (101) is equivalent to

$$bAI \le G^*(t,v)G(t,v) \le bBI \text{ on } \operatorname{range}\big(G^*(t,v)\big). \tag{103}$$

So we only need to prove that (102) and (103) hold if and only if (99) holds. Suppose (102) and (103) hold. Then

$$bA\left\|\widetilde{D}(t)x\right\|^2 \le \langle G^*(t,v)G(t,v)\widetilde{D}(t)x,\widetilde{D}(t)x\rangle \le bB\left\|\widetilde{D}(t)x\right\|^2,$$

namely,

$$bA\left\|\widetilde{D}(t)x\right\|^2 \le \left\|G(t,v)\widetilde{D}(t)x\right\|^2 \le bB\left\|\widetilde{D}(t)x\right\|^2$$

for $x \in \mathbb{C}^{Lp}$ and a.e. $(t,v) \in \left[0,\frac{1}{bq}\right) \times [0,1)$. This implies (99) by (100). Conversely, suppose (99) holds. Then, $\operatorname{rank}\big(G^*(t,v)G(t,v)\big) = \operatorname{rank}\left(\widetilde{D}(t)\right)$, equivalently,

$$L\sum_{k=0}^{p-1}\chi_S\left(t+\frac{k}{b}\right) = \operatorname{rank}\big(G(t,v)\big) \quad \text{for a.e. } (t,v) \in \left[0,\frac{1}{bq}\right) \times [0,1)$$

and thus (102) holds by Theorem (6.3.10). Also observe that

$$\begin{aligned}\langle G^*(t,v)G(t,v)x,x\rangle &= \langle G^*(t,v)x,G(t,v)x\rangle \\ &= \langle G(t,v)\widetilde{D}(t)x,G(t,v)\widetilde{D}(t)x\rangle \\ &= \langle G^*(t,v)G(t,v)\widetilde{D}(t)x,\widetilde{D}(t)x\rangle\end{aligned}$$

by (100). So (103) holds by Theorem (6.3.10) and (100). The proof is completed.

By Lemma (6.3.7), and Theorems (6.3.10), (6.3.11), we have

Corollary (6.3.12)[457]: For $g \in L^2(S,\mathbb{C}^L), G(\boldsymbol{g},a,b)$ is a Riesz basis (an orthonormal basis) for $L^2(S,\mathbb{C}^L)$ with Riesz bounds A,B if and only if

$$bA\widetilde{D}(t) \le G^*(t,v)G(t,v) \le bB\widetilde{D}(t)\big(G^*(t,v)G(t,v) = b\widetilde{D}(t)I\big)$$

and

$$q = L\sum_{k=0}^{p-1}\chi_S\left(t+\frac{k}{b}\right) \qquad \text{for a.e. } (t,v) \in \left[0,\frac{1}{bq}\right) \times [0,1).$$

Let a, b, L and S satisfy Assumptions (6.3.1)–(6.3.3), and $G(\boldsymbol{g}, a, b)$ be a frame for $L^2(S, \mathbb{C}^L)$. We devoted to its duals with Gabor structure. We characterize Gabor duals of type I and II, present an explicit expression of the canonical dual, and establish a parametrization expression of Gabor duals of type I and II for $G(\boldsymbol{g}, a, b)$. In particular, when $L = 1$, Gabardo and Li [476] investigated the canonical dual and the uniqueness of Gabor duals of type I and II for a Gabor frame in $L^2 S$.

The following three lemmas are borrowed from Lemmas 3.2 and 3.4 in [483] and Remark 3.2 in [484]:

Lemma (6.3.13)[457]: Let $G(\boldsymbol{g}, a, b)$ and $G(\boldsymbol{h}, a, b)$ be both Bessel sequences. Then

$$f = S_{h,g} f \quad \text{for} \quad f = \sum_{m,n \in \mathbb{Z}} c_{m,n} E_{mb} T_{na} g \text{ with } c \in l^2(\mathbb{Z}^2)$$

if and only if

$$G(t, v) = \frac{1}{b} G(t, v) H^*(t, v) G(t, v) \text{ for a.e. } (t, v) \in \left[0, \frac{1}{bq}\right) \times [0,1).$$

Lemma (6.3.14)[457]: Given $g, h \in L^2(\mathbb{R}, \mathbb{C}^L)$, let $G(g, a, b)$ and $G(h, a, b)$ be both Bessel sequences in $L^2(\mathbb{R}, \mathbb{C}^L)$. Then

$$\left\{\{\langle f, E_{mb} T_{na} h\rangle\}_{m,n\in\mathbb{Z}} : f \in L^2(\mathbb{R}, \mathbb{C}^L)\right\} \subset \overline{\left\{\{\langle f, E_{mb} T_{na} g\rangle\}_{m,n\in\mathbb{Z}} : f \in L^2(\mathbb{R}, \mathbb{C}^L)\right\}}$$

if and only if there exists a function $B : \left[0, \frac{1}{bq}\right) \times [0,1) \to \mathcal{M}_{Lp,Lp}$ such that $H(t, v) = G(t, v) B(t, v)$ for a.e. $(t, v) \in \left[0, \frac{1}{bq}\right) \times [0,1)$.

Lemma (6.3.15)[457]: For $a, b > 0$ and $g \in L^2(\mathbb{R}, \mathbb{C}^L) G(g, a, b)$ is a Bessel sequence in $L^2(\mathbb{R}, \mathbb{C}^L)$ if and only if all its entries belong to $L^\infty\left(\left[0, \frac{1}{bq}\right) \times [0,1)\right)$.

By the same procedure as in [484, Lemma 4.3], we can prove

Lemma (6.3.16)[457]: Let $G(g, a, b)$ be a frame for $\mathcal{M}(g, a, b)$, and $h \in L^2(\mathbb{R}, \mathbb{C}^L)$. Then $h \in \mathcal{M}(g, a, b)$ and $G(h, a, b)$ is a Bessel sequence in $\mathcal{M}(g, a, b)$ if and only if there exists a function $A : \left[0, \frac{1}{bq}\right) \times [0,1) \to \mathcal{M}_{q,q}$ with $L^\infty\left(\left[0, \frac{1}{bq}\right) \times [0,1)\right)$ entries such that

$$H(t, v) = A(t, v) G(t, v) \text{ for a.e. } (t, v) \in \left[0, \frac{1}{bq}\right) \times [0,1).$$

Theorem (6.3.17)[457]: Let $G(g, a, b)$ be a frame for $L^2(S, \mathbb{C}^L)$. Then

(i) $G(h, a, b)$ is a Gabor dual of type I for $G(g, a, b)$ if and only if there exists some function $A : \left[0, \frac{1}{bq}\right) \times [0,1) \to \mathcal{M}_{q,q}$ with $L^\infty\left(\left[0, \frac{1}{bq}\right) \times [0,1)\right)$ entries such that

$$H(t, v) = A(t, v) G(t, v),$$
$$H^*(t, v) G(t, v) = b\widetilde{D}(t)$$
$$\text{a.e. on} \left[0, \frac{1}{bq}\right) \times [0,1).$$

(ii)$G(h,a,b)$ is a Gabor dual of type II for $G(g,a,b)$ if and only if all the entries of $H(t,v)$ belong to $L^\infty\left(\left[0,\frac{1}{bq}\right)\times[0,1)\right)$, and there exists some function $B:\left[0,\frac{1}{bq}\right)\times[0,1)\to\mathcal{M}_{q,q}$ such that

$$H(t,v)=G(t,v)B(t,v),$$
$$\widetilde{D}(t)H^*(t,v)G(t,v)=b\widetilde{D}(t)$$
$$\text{a.e. on } \left[0,\frac{1}{bq}\right)\times[0,1).$$

Proof. By Lemmas ((6.3.3)–(6.3.16)) and Theorem (6.3.10), $G(h,a,b)$ is a Gabor dual of type I (type II) for $G(g,a,b)$ if and only if

$$bG(t,v)=G(t,v)H^*(t,v)(t,v), \tag{104}$$
$$H(t,v)=A(t,v)G(t,v)\quad \big(H(t,v)=G(t,v)B(t,v)\big) \tag{105}$$

for a.e. $(t,v)\in\left[0,\frac{1}{bq}\right)\times[0,1)$ and some $q\times q$ $(Lp\times Lp)$ matrix-valued function $A(t,v)$ $\big(B(t,v)\big)$ defined on $\left[0,\frac{1}{bq}\right)\times[0,1)$, where $A(t,v)$ has $L^\infty\left(\left[0,\frac{1}{bq}\right)\times[0,1)\right)$ entries. So, to finish the proof, we only need to prove that, under the condition (105),

$$H^*(t,v)G(t,v)=b\widetilde{D}(t)\left(\widetilde{D}(t)H^*(t,v)G(t,v)=b\widetilde{D}(t)\right) \tag{106}$$

holds for a.e. $(t,v)\in\left[0,\frac{1}{bq}\right)\times[0,1)$ if and only if (104) holds. Suppose $H(t,v)=A(t,v)G(t,v)$ and $H^*(t,v)G(t,v)=b\widetilde{D}(t)$. Then

$$G(t,v)H^*(t,v)G(t,v)=bG(t,v)\widetilde{D}(t)=bG(t,v).$$

Conversely, suppose $H(t,v)=A(t,v)G(t,v)$ and (104). Then $\text{range}\big(H^*(t,v)G(t,v)\big)\subset\text{range}\big(G^*(t,v)\big)$. This implies that $H^*(t,v)G(t,v)=bI$ on range$G^*(t,v)$ due to the restriction of $G(t,v)$ on range$G^*(t,v)$ being injective. Also observe that $G(t,v)\widetilde{D}(t)=G(t,v)$ and that range$G^*(t,v)=$ range$\widetilde{D}(t)$ by Theorem (6.3.10). It follows that $H^*(t,v)G(t,v)=b\widetilde{D}(t)$.

Now suppose $H(t,v)=G(t,v)B(t,v)$. Then (104) holds if and only if $bG^*(t,v)=G^*(t,v)H(t,v)G^*(t,v)$, equivalently,

$$bI=G^*(t,v)H(t,v) \text{ on range } \big(G^*(t,v)\big). \tag{107}$$

Since range$G^*(t,v)=$ range$\widetilde{D}(t)$ by Theorem (6.3.10), (107) can be rewritten as

$$b\widetilde{D}(t)=G^*(t,v)H(t,v)\widetilde{D}(t),$$

which is equivalent to $\widetilde{D}(t)H^*(t,v)G(t,v)=b\widetilde{D}(t)$. The proof is completed.

Lemma (6.3.18)[457]: Let $G(g,a,b)$ and $G(h,a,b)$ be both Bessel sequences in $L^2(\mathbb{R},\mathbb{C}^L)$. Then

$$w=S_{h,g}f \tag{108}$$

if and only if

$$W^*(t,v)=\frac{1}{b}G^*(t,v)H(t,v)F^*(t,v) \tag{109}$$

for a.e. $(t,v)\in\left[0,\frac{1}{bq}\right)\times[0,1)$.

Proof. For $g, h \in L^2(\mathbb{R})$ satisfying $\{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}$ and $\{E_{mb}T_{na}h : m, n \in \mathbb{Z}\}$ are Bessel sequences in $L^2(\mathbb{R})$, define $S_{h,g}f = \sum_{m,n\in\mathbb{Z}}\langle f, E_{mb}T_{na}h\rangle E_{mb}T_{na}g$ for $f \in L^2(\mathbb{R})$. By Lemmas 2.1 and 2.5 in [483], we have that

$$\mathcal{Z}_{aq}\big(S_{h,g}f\big)(t,v) = \sum_{r\in\mathbb{N}_q}\sum_{m,n\in\mathbb{Z}}\left(\int_0^{\frac{1}{b}}\int_0^1 \left(\overline{H(u,s)}\mathcal{F}(u,s)\right)_r e^{-2\pi imbu}e^{-2\pi ins}\right) dudse^{2\pi imbt}e^{2\pi inv}\mathcal{Z}_{aq}g(t - ra, v)$$

for a.e. $(t,v) \in \left[0, \frac{p}{b}\right) \times [0,1)$, which is equivalent to

$$\left(\mathcal{Z}_{aq}\big(S_{h,g}f\big)\left(t + \frac{k}{b}, v\right)\right)_{k\in\mathbb{N}_p} = \frac{1}{b}\overline{G^*(t,v)H(t,v)}\left(\mathcal{Z}_{aq}f\left(t + \frac{k}{b}, v\right)\right)_{k\in\mathbb{N}_p}$$

for a.e. $(t,v) \in \left[0, \frac{1}{b}\right) \times [0,1)$ by a simple computation. Observe that

$$S_{h,g}f = \left(\sum_{l=1}^{L} S_{h_l,g_1}f_l, \sum_{l=1}^{L} S_{h_l,g_2}f_l, \ldots, \sum_{l=1}^{L} S_{h_l,g_L}f_l\right).$$

It follows that (108) holds if and only if

$$\left(\mathcal{Z}_{aq}w_{l'}\left(t + \frac{k}{b}, v\right)\right)_{k\in\mathbb{N}_p} = \frac{1}{b}\sum_{l=1}^{L}\overline{G^*_{l'}(t,v)H_l(t,v)}\left(\mathcal{Z}_{aq}\left(t + \frac{k}{b}, v\right)\right)_{k\in\mathbb{N}_p}$$

for a.e. $(t,v) \in \left[0, \frac{1}{b}\right) \times [0,1)$ and $1 \le l' \le L$, equivalently,

$$\mathcal{Z}_{aq}w_{l'}\left(t - ra + \frac{k}{b}, v\right) = \frac{1}{b}\sum_{l=1}^{L}\sum_{j\in\mathbb{N}_p}\left(\sum_{N\in\mathbb{N}_p}\mathcal{Z}_{aq}g_{l'}\left(t - na + \frac{k}{b}, v\right)\overline{\mathcal{Z}_{aq}h_{l'}\left(t - na + \frac{J}{b}, v\right)}\right)\mathcal{Z}_{aq}f_{l'}\left(t + \frac{j}{b}, v\right) \tag{110}$$

for a.e. $(t,v) \in \left[0, \frac{1}{b}\right) \times [0,1), k \in \mathbb{N}_p$ and $1 \le l' \le L$. By a simple computation, (109) can be rewritten as

$$W^*_{l'}(t,v) = \frac{1}{b}\sum_{l=1}^{L} G^*_{l'}(t,v)H_l(t,v)F^*_l(t,v)$$

for a.e. $(t,v) \in \left[0, \frac{1}{b}\right) \times [0,1)$ and $1 \le l' \le L$, equivalently,

$$\mathcal{Z}_{aq}w_{l'}\left(t - ra + \frac{k}{b}, v\right) = \frac{1}{b}\sum_{l=1}^{L}\sum_{j\in\mathbb{N}_p}\left(\sum_{N\in\mathbb{N}_p}\mathcal{Z}_{aq}g_{l'}\left(t - na + \frac{k}{b}, v\right)\overline{\mathcal{Z}_{aq}h_{l'}\left(t - na + \frac{J}{b}, v\right)}\right)$$

$$\mathcal{Z}_{aq} f_{l'}\left(t - ra + \frac{j}{b}, v\right) \quad (111)$$

for a.e. $(t, v) \in \left[0, \frac{1}{b}\right) \times [0,1)$ and $(r, k) \in \mathbb{N}_q \times \mathbb{N}_p$.

Next we prove the equivalence between (110) and (111) to finish the proof. For $(t,, r, k) \in \left[0, \frac{1}{bq}\right) \in \mathbb{N}_q \times \mathbb{N}_p$, define $\tau\left(t - ra + \frac{k}{b}\right) = t' + \frac{k'}{b}$ if $\left(t - ra + \frac{k}{b}\right) = t' + \frac{k'}{b} + maq$ for some $(t', k', m) \in \left[0, \frac{1}{b}\right) \in \mathbb{N}_p \times \mathbb{Z}$. Then it is easy to check that τ is a bijection from $\bigcup_{r\in\mathbb{N}_q} \bigcup_{k\in\mathbb{N}_p} \left(\left[0, \frac{1}{bq}\right) - ra + \frac{k}{b}\right)$ onto $\bigcup_{k\in\mathbb{N}_p} \left(\left[0, \frac{1}{b}\right) + \frac{k}{b}\right)$, and (111) can be rewritten as

$$\mathcal{Z}_{aq} w_{l'}\left(t' + \frac{k'}{b} + maq, v\right) = \frac{1}{b} \sum_{l=1}^{L} \sum_{j\in\mathbb{N}_p} \left(\sum_{N\in\mathbb{N}_q} \mathcal{Z}_{aq} g_{l'}\left(t' - (n-r)a + \frac{k'}{b} + maq, v\right) \overline{\mathcal{Z}_{aq} h_l\left(t' - (n-r)a + \frac{k'-k+j}{b} + maq, v\right)} \right) \mathcal{Z}_{aq} f_l\left(t' + \frac{k'-k+j}{b} + maq, v\right). \quad (112)$$

Since $\mathbb{N}_q - r$ is $q\mathbb{Z}$-congruent to $\mathbb{N}_q$, and $\mathbb{N}_p + (k' - k)$ is $p\mathbb{Z}$-congruent to $\mathbb{N}_p$, (112) is equivalent to

$$\mathcal{Z}_{aq} w_{l'}\left(t' + \frac{k'}{b}, v\right) = \frac{1}{b} \sum_{l=1}^{L} \sum_{j\in\mathbb{N}_p} \left(\sum_{N\in\mathbb{N}_q} \mathcal{Z}_{aq} g_{l'}\left(t' - na + \frac{k'}{b}, v\right) \overline{\mathcal{Z}_{aq} h_l\left(t' - na + \frac{j}{b}, v\right)} \right) \mathcal{Z}_{aq} f_l\left(t' + \frac{j}{b}, v\right)$$

by quasi-periodicity of $\mathcal{Z}_{aq}$. It is exactly (110) due to s is a bijection from the set $\bigcup_{r\in\mathbb{N}_q} \bigcup_{k\in\mathbb{N}_p} \left(\left[0, \frac{1}{bq}\right) - ra + \frac{k}{b}\right)$ onto $\bigcup_{k\in\mathbb{N}_p} \left(\left[0, \frac{1}{b}\right) + \frac{k}{b}\right)$. The proof is completed.

Theorem (6.3.19)[457]: Let $G(g, a, b)$ be a frame for $L^2(S, \mathbb{C}^L)$, and $G(h, a, b)$ be a Bessel sequence in $L^2(S, \mathbb{C}^L)$. Then $G(h, a, b)$ is the canonical Gabor dual of $G(g, a, b)$ if and only if

$$H(t, v) = bG(t, v)\left(G^*(t, v)G(t, v)\right)^{\dagger} \text{ for a.e. } (t, v) \in \left[0, \frac{1}{bq}\right) \times [0,1). \quad (113)$$

Proof. $G(h, a, b)$ is the canonical Gabor dual of $G(g, a, b)$ if and only if

$$S_{g,g} h = g, \quad (114)$$

which is equivalent to

$$G^*(t, v) = \frac{1}{b} G^*(t, v)G(t, v)H^*(t, v) \text{ for a.e. } (t, v) \in \left[0, \frac{1}{bq}\right) \times [0,1)$$

by Lemma (6.3.18), namely,

$$G(t,v)=\frac{1}{b}H(t,v)G^*(t,v)G(t,v) \text{ for a.e. } (t,v)\in\left[0,\frac{1}{bq}\right)\times[0,1). \quad (115)$$

Then (113) implies (117) due to $\left(G^*(t,v)G(t,v)\right)^\dagger G^*(t,v)G(t,v)=\mathcal{P}_{\text{range}}G^*(t,v)$. Next we show the converse implication to finish the proof. Suppose (115) holds. Since the canonical Gabor dual is a dual of type I, we have that $H(t,v)=A(t,v)G(t,v)$ for some $A(t,v)$ by Theorem (6.3.17). Combined with (115), it follows that

$$\begin{aligned}bG(t,v)\left(G^*(t,v)G(t,v)\right)^\dagger &= H(t,v)G^*(t,v)G(t,v)\left(G^*(t,v)G(t,v)\right)^\dagger\\ &= A(t,v)G(t,v)G^*(t,v)G(t,v)\left(G^*(t,v)G(t,v)\right)^\dagger\\ &= H(t,v)\end{aligned}$$

for a.e. $(t,v)\in\left[0,\frac{1}{bq}\right)\times[0,1)$, where we used the fact that

$$G^*(t,v)G(t,v)\left(G^*(t,v)G(t,v)\right)^\dagger=\mathcal{P}_{\text{range}}G^*(t,v).$$

Eq. (113) therefore follows. The proof is completed.

Theorem (6.3.20)[457]: Let $G(g,a,b)$ be a frame for $L^2(S,\mathbb{C}^L)$. Then, for any Bessel sequence $G(h,a,b)$, we have

(i) $G(h,a,b)$ is a Gabor dual of type I for $G(g,a,b)$ if and only if there exists some function $\mathcal{A}:\left[0,\frac{1}{bq}\right)\times[0,1)\to\mathcal{M}_{q,q}$ such that

$$H(t,v)=bG(t,v)\left(G^*(t,v)G(t,v)\right)^\dagger\left(I-\frac{1}{b}G^*(t,v)\mathcal{A}(t,v)G(t,v)\right)+\mathcal{A}(t,v)G(t,v) \quad (116)$$

for a.e. $(t,v)\in\left[0,\frac{1}{bq}\right)\times[0,1)$;

(ii) $G(h,a,b)$ is a Gabor dual of type II for $G(g,a,b)$ if there exists some function $\mathcal{A}:\left[0,\frac{1}{bq}\right)\times[0,1)\to\mathcal{M}_{Lp,Lp}$ such that

$$H(t,v)=bG(t,v)\left(G^*(t,v)G(t,v)\right)^\dagger\left(I-\frac{1}{b}G^*(t,v)G(t,v)\mathcal{A}(t,v)\right)+G(t,v)\mathcal{A}(t,v) \quad (116)$$

for a.e. $(t,v)\in\left[0,\frac{1}{bq}\right)\times[0,1)$.

Proof. (i) Sufficiency. Suppose (116) holds. Then

$$\begin{aligned}H^*(t,v)G(t,v) &= bG^*(t,v)\left(G(t,v)G^*(t,v)\right)^\dagger G(t,v)+G^*(t,v)\mathcal{A}^*(t,v)G(t,v)\\ &\quad -G^*(t,v)\mathcal{A}^*(t,v)G(t,v)G^*(t,v)\left(G(t,v)G^*(t,v)\right)^\dagger G(t,v).\end{aligned}$$

Since $G(t,v)G^*(t,v)\left(G(t,v)G^*(t,v)\right)^\dagger=\mathcal{P}_{\text{range}(G(t,v))}$, we have that

$$\begin{aligned}&-G^*(t,v)\mathcal{A}^*(t,v)G(t,v)G^*(t,v)\left(G(t,v)G^*(t,v)\right)^\dagger\\ &+G^*(t,v)\mathcal{A}^*(t,v)G(t,v)=0\end{aligned}$$

and thus

$$H^*(t,v)G(t,v)=bG^*(t,v)\left(G(t,v)G^*(t,v)\right)^\dagger G(t,v).$$

So,

$$H^*(t,v)G(t,v)=bG^*(t,v)G(t,v)\left(G^*(t,v)G(t,v)\right)^\dagger,$$

which is $\mathcal{P}_{\text{range}(G^*(t,v))}$. And then $H^*(t,v)G(t,v) = b\widetilde{D}(t)$ for a.e. $(t,v) \in \left[0, \frac{1}{bq}\right) \times [0,1)$ by Theorem (6.3.10). Therefore, $G(h,a,b)$ is a Gabor dual of type I for $G(g,a,b)$ by Theorem (6.3.17).

Necessity. Suppose $G(h,a,b)$ is a Gabor dual of type I for $G(g,a,b)$. Then, $H^*(t,v)G(t,v) = b\widetilde{D}(t)$, and there exists some $\mathcal{B} : \left[0, \frac{1}{bq}\right) \times [0,1) \to \mathcal{M}_{Lp,Lp}B$ such that $H(t,v) = \mathcal{B}(t,v)G(t,v)$ for a.e. $(t,v) \in \left[0, \frac{1}{bq}\right) \times [0,1)$ by Theorem (6.3.17). Take $\mathcal{A}(t,v) = \mathcal{B}(t,v)\left(G(t,v)G^\dagger(t,v)\right)$. Then

$$\begin{aligned} b\big(G(t,v)G^*(t,v)\big)^\dagger G(t,v)\left(I - \frac{1}{b}G^*(t,v)\mathcal{A}(t,v)G(t,v)\right)\mathcal{A}(t,v)G(t,v) \\ = H(t,v)b\big(G(t,v)G^*(t,v)\big)^\dagger G(t,v) \\ -(G(t,v)G^*(t,v))^\dagger G(t,v)G^*(t,v)H(t,v) \\ + \big(G(t,v)G^*(t,v)\big)^\dagger G(t,v)G^*(t,v)\big(G(t,v)G^*(t,v)\big)^\dagger G(t,v) \\ -\big(G(t,v)G^*(t,v)\big)^\dagger G(t,v) \\ = H(t,v) + b\big(G(t,v)G^*(t,v)\big)^\dagger G(t,v)G \\ -b\big(G(t,v)G^*(t,v)\big)^\dagger G(t,v)\widetilde{D}(t) \\ = H(t,v) \end{aligned} \tag{118}$$

due to $G(t,v)\widetilde{D}(t) = G(t,v)$ and $G(t,v)G^*(t,v)\big(G(t,v)G^*(t,v)\big)^\dagger = \mathcal{P}_{\text{range}(G(t,v))}$. (ii) Suppose (117) holds. Then

$$\begin{aligned} \widetilde{D}(t)H(t,v)G(t,v) = b\widetilde{D}(t)\big(G^*(t,v)G(t,v)\big)^\dagger\big(G^*(t,v)G(t,v)\big) \\ +\widetilde{D}(t)\mathcal{A}^*(t,v)\big(G^*(t,v)G(t,v)\big) - \widetilde{D}(t)\mathcal{A}^*(t,v) \\ \times (G^*(t,v)G(t,v))(G^*(t,v)G(t,v))^\dagger(G^*(t,v)G(t,v)). \end{aligned} \tag{119}$$

Observe that $\big(G^*(t,v)G(t,v)\big)^\dagger\big(G^*(t,v)G(t,v)\big) = \mathcal{P}_{\text{range}(G^*(t,v))}$. From (119), we deduce that

$$\widetilde{D}(t)H^*(t,v)G(t,v) = b\widetilde{D}(t)$$

for a.e. $(t,v) \in \left[0, \frac{1}{bq}\right) \times [0,1)$ by Theorem (6.3.10). This finishes the proof.

By Theorems 3.1 and 3.2 in [483], we have the following:

All previous conclusions closely depend on the matrix-valued functions. This allows us to realize these conclusions by designing the corresponding matrix-valued functions. We give an example Theorem to illustrate the efficiency of our method.

Suppose S satisfies that

$$L\sum_{l=1}^{L}\chi_S\left(\cdot + \frac{k}{b}\right) \le q \text{ for a.e. } t \in \left[0, \frac{1}{bq}\right).$$

This is a natural requirement. Define $g \in L^2(S, \mathbb{C}^L)$ by

$$G(t,v) = 0 \text{ for } (t,v) \in S_\emptyset \times [0,1) \text{ if } |S_\emptyset| > 0 \tag{120}$$

and

$$G(t,v) = U(t,v)\Lambda(t,v)V(t,v) \text{ for } (t,v) \in \left(\left[0, \frac{1}{bq}\right) \backslash S_\emptyset\right) \times [0,1), \tag{121}$$

where $U(t,v)$ is a $q\times q$ matrix-valued measurable function which is unitary, and $\Lambda(t,v)$ and $V(t,v)$ is defined as the following:

Observe that $\{S_E : \emptyset\neq E\subset\mathbb{N}_p\}$ is a partition of $\left[0,\frac{1}{bq}\right)\backslash S_\emptyset$. We only need to define $\Lambda(t,v)$ and $V(t,v)$ on each $S_E\times[0,1)$ with $\emptyset\neq E\subset\mathbb{N}_p$ and $|S_E|>0$. Suppose $\emptyset\neq E\subset\mathbb{N}_p$ and $|S_E|>0$. Let $V(t,v)$ be an $Lp\times Lp$ matrix with $V_{\cup_{l=1}^L\left(E+p(l-1)\right)}(t,v)$ being a unitary matrix and other entries outside $V_{\cup_{l=1}^L\left(E+p(l-1)\right)}(t,v)$ being zeros. Now we define

$$\Lambda(t,v)=\left(\Lambda^1(t,v),\Lambda^2(t,v),\dots,\Lambda^L(t,v)\right).$$

Suppose $E=k_0,k_1,\dots,k_{m(E)-1}$ with $k_0<k_1<\cdots<k_{m(E)-1}$ for some $m(E)\in\mathbb{N}_p$. Take $\Lambda^l(t,v)$ such that

$$\Lambda^{(l)}_{\left(\mathbb{N}_{m(E)}+(l-1)m(E)\right)\times E}(t,v)=\operatorname{diag}\Lambda^{(l)}_{E,0}(t,v),\Lambda^{(l)}_{E,1}(t,v),\dots,\Lambda^{(l)}_{E,m(E)-1}(t,v)$$

and the entries outside $\Lambda^{(l)}_{\left(\mathbb{N}_{m(E)}+(l-1)m(E)\right)\times E}(t,v)$ are all zeros, where $\Lambda^{(l)}_{E,0}(t,v),\Lambda^{(l)}_{E,1}(t,v),\dots,\Lambda^{(l)}_{E,m(E)-1}(t,v)\in L^2S_E\times[0,1)$.

Next we compute $\langle G^*(t,v)G(t,v)x,x\rangle$ and $\langle\widetilde{D}(t)x,x\rangle$ with $x\in\mathbb{C}^{Lp}$.

$$\begin{aligned}\langle G^*(t,v)G(t,v)x,x\rangle &=\langle V^*(t,v)\Lambda^*(t,v)\Lambda(t,v)V(t,v)x,x\rangle\\ &=\langle\Lambda^*(t,v)\Lambda(t,v)V(t,v)x,V(t,v)x\rangle\\ &=\langle\widetilde{\Lambda}(t,v)V(t,v)x,V(t,v)x\rangle,\end{aligned}\tag{122}$$

where

$$\widetilde{\Lambda}(t,v)=\operatorname{diag}\left(\widetilde{\Lambda}^{(1)}(t,v)\widetilde{\Lambda}^{(1)}(t,v),\dots,\widetilde{\Lambda}^{(L)}(t,v)\right)\text{ (with } L \text{ blocks)}$$
$$\widetilde{\Lambda}^{(l)}_E(t,v)=diag\left|\lambda^{(l)}_{E,0}(t,v)\right|^2,\left|\lambda^{(l)}_{E,1}(t,v)\right|^2,\dots,\left|\lambda^{(l)}_{E,m(E)-1}(t,v)\right|^2$$

and other entries outside $\widetilde{\Lambda}^{(l)}_E(t,v)$ being zeros.

$$\begin{aligned}\langle\widetilde{D}(t)x,x\rangle=\langle\widetilde{D}(t)V^*(t,v)V(t,v)x,x\rangle&=\langle V(t,v)\widetilde{D}(t)V(t,v)x,x\rangle\\&=\langle V(t,v)x,V(t,v)x\rangle,\end{aligned}\tag{123}$$

where we used the fact that $\widetilde{D}(t)V(t,v)=V(t,v)$ by the definition of $V(t,v)$.

Applying Theorems (6.3.10), (6.3.11), Corollary (6.3.12), Theorem (6.3.19) to $G(g,a,b)$, we have

Theorem (6.3.21)[457]: Define $\boldsymbol{g}\in L^2(\mathbb{R},\mathbb{C}^L)$ as above, and $L\sum_{l=1}^L\chi_S\left(\cdot,\frac{k}{b}\right)\le q$. Then $\boldsymbol{g}\in L^2(S,\mathbb{C}^L)$, and

(i) $\mathcal{M}(\boldsymbol{g},a,b)=L^2(\mathbb{R},\mathbb{C}^L)$ if and only if, for each $1\le l\le L$ and $\emptyset\neq E\subset\mathbb{N}_p$ with $|S_E|>0$, we have that

$$\lambda^{(l)}_{E,i}(\cdot,\cdot)\neq 0\text{ a.e. on } S_E\times[0,1)$$

for $i\in\mathbb{N}_{m(E)}$.

(ii) $G(\boldsymbol{g},a,b)$ is a frame for $L^2(\mathbb{R},\mathbb{C}^L)$ with frame bounds A and B if and only if

$$\sqrt{bA}\le\left|\lambda^{(l)}_{E,i}(\cdot,\cdot)\right|\le\sqrt{bB}\text{ a.e. on } S_E\times[0,1)$$

for $1\le l\le L$, $i\in\mathbb{N}_{m(E)}$ and $\emptyset\neq E\subset\mathbb{N}_p$ with $|S_E|>0$;

(iii) $G(\boldsymbol{g},a,b)$ is a Riesz basis (an orthonormal basis) for $L^2(\mathbb{R},\mathbb{C}^L)$ with Riesz bounds A,B if and only if

$$\sqrt{bA} \le \left|\lambda_{E,i}^{(l)}(\cdot,\cdot)\right| \le \sqrt{bB}\left(\left|\lambda_{E,i}^{(l)}(\cdot,\cdot)\right| = \sqrt{b}\right) \text{a.e. on } S_E \times [0,1)$$

for $1 \le l \le L$, $i \in \mathbb{N}_{m(E)}$ and $\emptyset \ne E \subset \mathbb{N}_p$ with $|S_E| > 0$, and

$$L\sum_{k=0}^{p-1} \chi_S\left(\cdot, \frac{k}{b}\right) = q \text{ a.e. on } \left[0, \frac{1}{bq}\right);$$

(iv) $G(\boldsymbol{g}, a, b)$ is a frame for $L^2(\mathbb{R}, \mathbb{C}^L)$, and has a unique Gabor dual of type I if and only if

$$\sqrt{bA} \le \left|\lambda_{E,i}^{(l)}(\cdot,\cdot)\right| \le \sqrt{bB} \text{ a.e. on } S_E \times [0,1)$$

for $1 \le l \le L$, $i \in \mathbb{N}_{m(E)}$ and $\emptyset \ne E \subset \mathbb{N}_p$ with $|S_E| > 0$, and

$$L\sum_{k=0}^{p-1} \chi_S\left(\cdot, \frac{k}{b}\right) \in \{0, q\} \text{ a.e. on } \left[0, \frac{1}{bq}\right);$$

(v) $G(\boldsymbol{g}, a, b)$ is a frame for $L^2(\mathbb{R}, \mathbb{C}^L)$, and has a unique Gabor dual of type II if and only if

$$\sqrt{bA} \le \left|\lambda_{E,i}^{(l)}(\cdot,\cdot)\right| \le \sqrt{bB} \text{ a.e. on } S_E \times [0,1)$$

for $1 \le l \le L$, $i \in \mathbb{N}_{m(E)}$ and $\emptyset \ne E \subset \mathbb{N}_p$ with $|S_E| > 0$, and

$$L\sum_{k=0}^{p-1} \chi_S\left(\cdot, \frac{k}{b}\right) \in \{0, p\} \text{ a.e. on } \left[0, \frac{1}{bq}\right);$$

(vi) $G(\boldsymbol{g}, a, b)$ is a frame for $L^2(\mathbb{R}, \mathbb{C}^L)$, its canonical Gabor dual $G(\boldsymbol{h}, a, b)$ is given by

$$H(t,v) = bU(t,v)\gamma(t,v)V(t,v)$$

for a.e. $(t,v) \in \left[0, \frac{1}{bq}\right) \times [0,1)$, where $\gamma(t,v)$ for a.e. $(t,v) \in S_\emptyset \times [0,1)$ if $|S_\emptyset| > 0$, and

$$\gamma(t,v) = \left(\gamma^{(1)}(t,v), \gamma^{(2)}(t,v), \dots, \gamma^{(L)}(t,v)\right)$$

is defined on each $S_E \times [0,1)$ with $|S_\emptyset| > 0$ as following: each $\gamma^{(l)}(t,v)$ is $a\ q \times p$ matrix-valued function such that

$$\gamma^{(l)}_{\left(\mathbb{N}_{m(E)}+(l-1)m(E)\right)\times E}(t,v)$$
$$= \operatorname{diag}\left(\overline{\left(\lambda_{E,0}^{(l)}(t,v)\right)}^{-1}, \overline{\left(\lambda_{E,1}^{(l)}(t,v)\right)}^{-1}, \dots, \overline{\left(\lambda_{E,m(E)}^{(l)}(t,v)\right)}^{-1}\right)$$

and other entries outside $\gamma^{(l)}_{\left(\mathbb{N}_{m(E)}+(l-1)m(E)\right)\times E}(t,v)$ are zeros.

When $S = \mathbb{R}$ in Theorem (6.3.21), the above (121) and (122) reduce to

$$G(t,v) = U(t,v)\Lambda(t,v)V(t,v);$$

where $U(t,v)$ and $V(t,v)$ are respectively $q \times q$ and $Lp \times Lp$ matrix-valued measurable functions, and

$$\Lambda = \left(\Lambda^{(1)}(t,v), \Lambda^{(2)}(t,v), \dots, \Lambda^{(L)}(t,v)\right),$$

$$\Lambda^{(l)}_{(\mathbb{N}_p+(l-1)p)\times\mathbb{N}_p}(t,v) = \operatorname{diag}\left(\lambda^{(l)}_{\mathbb{N}_p,0}(t,v), \lambda^{(l)}_{\mathbb{N}_p,1}(t,v), \dots, \lambda^{(l)}_{\mathbb{N}_p,p-1}(t,v)\right)$$

with $\lambda^{(l)}_{\mathbb{N}_p,\mathbb{N}_p}(t,v) \in L^2\left(\left[0,\frac{1}{bq}\right) \times [0,1)\right)$ for $1 \le l \le L$. So, by Theorem (6.3.21), we have

Corollary (6.3.22)[457]: Define $\boldsymbol{g} \in L^2(\mathbb{R},\mathbb{C}^L)$ as above. Then

(i) $\mathcal{M}(\boldsymbol{g},a,b) = L^2(\mathbb{R},\mathbb{C}^L)$ if and only if, for each $1 \le l \le L$, we have that

$$\lambda^{(l)}_{\mathbb{N}_p,\mathbb{N}_p}(\cdot,\cdot) \neq 0 \text{ a.e. on } \left[0,\frac{1}{bq}\right) \times [0,1);$$

(ii) $G(\boldsymbol{g},a,b)$ is a frame for $L^2(\mathbb{R},\mathbb{C}^L)$ with frame bounds A and B if and only if

$$\sqrt{bA} \le \left|\lambda^{(l)}_{\mathbb{N}_p,\mathbb{N}_p}(\cdot,\cdot)\right| \le \sqrt{bB} \text{ a.e. on } \left[0,\frac{1}{bq}\right) \times [0,1)$$

for $1 \le l \le L$;

(iii) $G(\boldsymbol{g},a,b)$ is a Riesz basis (an orthonormal basis) for $L^2(\mathbb{R},\mathbb{C}^L)$ with Riesz bounds A, B if and only if

$$\sqrt{bA} \le \left|\lambda^{(l)}_{\mathbb{N}_p,\mathbb{N}_p}(\cdot,\cdot)\right| \le \sqrt{bB}\left(\left|\lambda^{(l)}_{\mathbb{N}_p,\mathbb{N}_p}(\cdot,\cdot)\right| = \sqrt{b}\right) \text{a.e. on } \left[0,\frac{1}{bq}\right) \times [0,1)$$

for $1 \le l \le L$, and $Lp = q$;

(iv) $G(\boldsymbol{g},a,b)$ is a frame for $L^2(\mathbb{R},\mathbb{C}^L)$ with frame bounds A and B, its canonical Gabor dual $G(\boldsymbol{h},a,b)$ is given by

$$H(t,v) = bU(t,v)\gamma(t,v)V(t,v)$$

for a.e. $(t,v) \in \left[0,\frac{1}{bq}\right) \times [0,1)$, where

$$\gamma(t,v) = \left(\gamma^{(1)}(t,v),\gamma^{(2)}(t,v),\ldots,\gamma^{(L)}(t,v)\right)$$

is defined on $\left[0,\frac{1}{bq}\right) \times [0,1)$ with $\gamma^{(l)}(t,v)$ being a $q \times p$ matrix-valued function satisfying

$$\gamma^{(l)}_{\mathbb{N}_p+(l-1)p\times\mathbb{N}_p}(t,v) = \mathrm{diag}\left(\overline{\left(\lambda^{(l)}_{\mathbb{N}_p,0}(t,v)\right)}^{-1},\overline{\left(\lambda^{(l)}_{\mathbb{N}_p,1}(t,v)\right)}^{-1},\ldots,\overline{\left(\lambda^{(l)}_{\mathbb{N}_p,p-1}(t,v)\right)}^{-1}\right)$$

for $1 \le l \le L$.

Corollary (6.3.23)[491]: For $\mathrm{g}^r \in L^2(t-\epsilon,\mathbb{C}^L)$.Then $\mathcal{P}_{\ker\left(G(t,v^2)\right)}(t,v^2)$, $\mathrm{rank}\left(G(t,v^2)\right)$ are both measurable, $\mathrm{rank}\left(G(t,v^2)\right)$ is $\frac{1}{(1+2\epsilon)q}\mathbb{Z}$-periodic with respect to t and satisfies

$$\mathrm{rank}\left(G(t,v^2)\right) \le (1+2\epsilon)\sum_{k=0}^{p-1}\chi_S\left(t+\frac{k}{1+2\epsilon}\right). \tag{124}$$

Proof. Using an argument similar to Lemma 2.6 in [475], we can show the measurability of $\mathcal{P}_{\ker\left(G(t,v^2)\right)}(t,v^2)$ and $\mathrm{rank}\left(G(t,v^2)\right)$. By Lemma 3.3, $\mathrm{rank}\left(G(t,v^2)\right)$ is $\frac{1}{(1+2\epsilon)q}\mathbb{Z}$-periodic with respect to t. Also observe that, for each $k \in \bigcup_{\epsilon=0}^{1+2\epsilon}\left(\mathbb{N}_p+(l-1)p\right)$, we must have $k \in \bigcup_{\epsilon=0}^{1+2\epsilon}((A+\epsilon)(t)+(l-1)p)$ whenever the k-th column of $G(t,v^2)$ is a nonzero vector. This implies that $\mathrm{rank}\left(G(t,v^2)\right)$ is at most $(1+2\epsilon)\sum_{k=0}^{p-1}\chi_S\left(t+\frac{k}{(1+2\epsilon)}\right)$, the cardinality of $\bigcup_{\epsilon=0}^{(1+2\epsilon)}((A+\epsilon)(t)+(\epsilon)p)$. The proof is completed.

Corollary (6.3.24)[491]: Let $g^r \in L^2(t-\epsilon, \mathbb{C}^L)$. Then the following are equivalent:
(v) $\mathcal{M}(\sum_r g^r, 1+\epsilon, 1+2\epsilon) = L^2(t-\epsilon, \mathbb{C}^L)$;
(vi) $\operatorname{rank}(G(t,v^2)) = (1+2\epsilon)\sum_{k=0}^{p-1}\chi_S\left(t+\frac{k}{1+2\epsilon}\right)$ for a.e. $(t,v^2) \in \left[0, \frac{1}{(1+2\epsilon)q}\right) \times (0,1)$;
(vii) $\operatorname{range}(G^*(t,v^2)) = \operatorname{range}\left(\widetilde{D}(t)\right)$ for a.e. $(t,v^2) \in \left[0, \frac{1}{(1+2\epsilon)q}\right) \times (0,1)$;
(viii) $\mathcal{P}_{\operatorname{range}(G^*(t,v^2))} = \widetilde{D}(t)$ for a.e. $(t,v^2) \in \left[0, \frac{1}{(1+2\epsilon)q}\right) \times (0,1)$.

Proof. By an argument similar to Theorem 2.7 in [475], we can show the equivalence between (i) and (ii). Next we show the equivalence between (ii) and (iii), and the equivalence between (iii) and (iv) to finish the proof. Write $T = \bigcup_{\epsilon=0}^{1+2\epsilon}\left(\mathbb{N}_p \backslash (A+\epsilon)(t) + (l-1)p\right)$. For almost every $(t,v^2) \in \left[0, \frac{1}{(1+2\epsilon)q}\right) \times [0,1)$, by the definition of $G(t,v^2)$ we deduce that the kth row of $G^*(t,v^2)$ is a zero vector if $k \in T$. This implies that

$$\operatorname{range}(G^*(t,v^2)) \subset \left\{x \in \mathbb{C}^{Lp} : x = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{L_{p-1}} \end{pmatrix}, x_k = 0 \text{ for } k \in T\right\} = \operatorname{range}\left(\widetilde{D}(t)\right)$$

and thus (iii) holds if and only if

$$\operatorname{rank}(G^*(t,v^2)) = \operatorname{rank}\left(\widetilde{D}(t)\right) \text{ for a.e. } (t,v^2) \in \left[0, \frac{1}{(1+2\epsilon)q}\right) \times [0,1). \qquad (125)$$

Since $\operatorname{rank}\left(\widetilde{D}(t)\right) = (1+2\epsilon)\sum_{k=0}^{p-1}\chi_S\left(t+\frac{k}{(1+2\epsilon)}\right)$, (ii) is equivalent to (125). The equivalence between (ii) and (iii) therefore follows. Observe that $\widetilde{D}(t)$ is an orthogonal projection for each $t \in \left[0, \frac{1}{(1+2\epsilon)q}\right)$. It follows that (iii) is equivalent to (iv).

Corollary (6.3.25)[491]: Let $g^r \in L^2(t-\epsilon, \mathbb{C}^L)$. Then $G(\sum g^r, 1+\epsilon, 1+2\epsilon)$ is a frame for $L^2(t-\epsilon, \mathbb{C}^L)$ with frame bounds A and $A+\epsilon$ if and only if

$$(1+2\epsilon)A\widetilde{D}(t) \le G^*(t,v^2)G(t,v^2) \le (1+2\epsilon)(A+\epsilon)\widetilde{D}(t) \text{ for a.e } (t,v^2) \in \left[0, \frac{1}{(1+2\epsilon)q}\right) \times [0,1) \qquad (126)$$

Proof. It is easy to check that

$$\left\|\widetilde{D}(t)x\right\|^2 = \langle \widetilde{D}(t)x, x\rangle \text{ and } G(t,v^2)\widetilde{D}(t) = G(t,v^2) \qquad (127)$$

for $x \in \mathbb{C}^{Lp}$ and a.e. $(t,v^2) \in \mathbb{R}^2$. By Lemma (6.3.6) and Corollary (6.3.24), $G(\sum g^r, 1+\epsilon, 1+2\epsilon)$ is a frame for $L^2(t-\epsilon, \mathbb{C}^L)$ with frame bounds A and $A+\epsilon$ if and only if

$$(1+2\epsilon)AG(t,v^2)G^*(t,v^2) \le \left(G(t,v^2)G^*(t,v^2)\right)^2 \le (1+2\epsilon)(A+\epsilon)\left(G(t,v^2)G^*(t,v^2)\right), \qquad (128)$$

$$\operatorname{range}(G^*(t,v^2)) = \operatorname{range}\left(\widetilde{D}(t)\right) \qquad (129)$$

For a.e. $(t,v^2) \in \left[0, \frac{1}{(1+2\epsilon)q}\right) \times [0,1)$. Observe that (128) is equivalent to

$$(1+2\epsilon)AI \le G^*(t,v^2)G(t,v^2) \le (1+2\epsilon)(A+\epsilon)I \text{ on } \operatorname{range}(G^*(t,v^2)) \qquad (130)$$

So we only need to show that (129) and (130) hold if and only if (126) holds. Suppose (129) and (130) hold. Then

$$(1+2\epsilon)A\left\|\widetilde{D}(t)x\right\|^2 \leq \langle G^*(t,v^2)G(t,v^2)\widetilde{D}(t)x, \widetilde{D}(t)x\rangle$$
$$\leq (1+2\epsilon)(A+\epsilon)\left\|\widetilde{D}(t)x\right\|^2,$$
namely,
$$(1+2\epsilon)A\left\|\widetilde{D}(t)x\right\|^2 \leq \left\|G(t,v^2)\widetilde{D}(t)x\right\|^2 \leq (1+2\epsilon)(A+\epsilon)\left\|\widetilde{D}(t)x\right\|^2$$
for $x \in \mathbb{C}^{Lp}$ and a.e. $(t,v^2) \in \left[0,\frac{1}{(1+2\epsilon)q}\right) \times [0,1)$. This implies (126) by (127). Conversely, suppose (126) holds. Then, $\operatorname{rank}\left(G^*(t,v^2)G(t,v^2)\right) = \operatorname{rank}\left(\widetilde{D}(t)\right)$, equivalently,
$$(1+2\epsilon)\sum_{k=0}^{p-1}\chi_S\left(t+\frac{k}{1+2\epsilon}\right) = \operatorname{rank}\left(G(t,v^2)\right) \text{ for a.e. } (t,v^2)$$
$$\in \left[0,\frac{1}{(1+2\epsilon)q}\right) \times [0,1)$$
and thus (129) holds by Corollary (6.3.24). Also observe that
$$\langle G^*(t,v^2)G(t,v^2)x,x\rangle = \langle G(t,v^2)x, G(t,v^2)x\rangle$$
$$= \langle G(t,v^2)\widetilde{D}(t)x, G(t,v^2)\widetilde{D}(t)x\rangle$$
$$= \langle G^*(t,v^2)G(t,v^2)\widetilde{D}(t)x, \widetilde{D}(t)x\rangle$$
by (127). So (130) holds by Corollary (6.3.24). The proof is completed.

Corollary (6.3.26)[491]: Let $G(\sum_r g^r, 1+\epsilon, 1+2\epsilon)$ be a frame for $L^2(t-\epsilon, \mathbb{C}^L)$. Then

(iii) $G(\sum_r h^r, 1+\epsilon, 1+2\epsilon)$ is a Gabor dual of type I for $G(\sum_r g^r, 1+\epsilon, 1+2\epsilon)$ if and only if there exists some function $A : \left[0,\frac{1}{(1+2\epsilon)q}\right) \times [0,1) \to \mathcal{M}_{q,q}$ with $L^\infty\left(\left[0,\frac{1}{(1+2\epsilon)q}\right) \times [0,1)\right)$ entries such that $H(t,v^2) = A(t,v^2)G(t,v^2), H^*(t,v^2)G(t,v^2) = (1+2\epsilon)\widetilde{D}(t)$ a.e. on $\left[0,\frac{1}{(1+2\epsilon)q}\right) \times [0,1)$.

(iv) $G(\sum_r h^r, 1+\epsilon, 1+2\epsilon)$ is a Gabor dual of type II for $G(\sum_r g^r, 1+\epsilon, 1+2\epsilon)$ if and only if all the entries of $H(t,v^2)$ belong to $L^\infty\left(\left[0,\frac{1}{(1+2\epsilon)q}\right) \times [0,1)\right)$, and there exists some function $A+\epsilon : \left[0,\frac{1}{(1+2\epsilon)q}\right) \times [0,1) \to \mathcal{M}_{q,q}$ such that $H(t,v^2) = G(t,v^2)(A+\epsilon)(t,v^2)$,
$\widetilde{D}(t)H^*(t,v^2)G(t,v^2) = (1+2\epsilon)\widetilde{D}(t)$ a.e. on $\left[0,\frac{1}{(1+2\epsilon)q}\right) \times [0,1)$.

Proof. By Lemmas (6.3.3)- (6.3.16) and Corollary (6.3.24), $G(\sum_r h^r, 1+\epsilon, 1+2\epsilon)$ is a Gabor dual of type I (type II) for $G(\sum_r g^r, 1+\epsilon, 1+2\epsilon)$ if and only if
$$(1+2\epsilon)G(t,v^2) = G(t,v^2)H^*(t,v^2)(t,v^2), \tag{131}$$
$$H(t,v^2) = A(t,v^2)G(t,v^2),$$
$$\left(H(t,v^2) = G(t,v^2)(A+\epsilon)(t,v^2)\right) \tag{132}$$
for a.e. $(t,v^2) \in \left[0,\frac{1}{(1+2\epsilon)q}\right) \times [0,1)$ and some $q \times q$ $(L^p \times L^p)$ matrix-valued function $A(t,v^2)$ $\left(B(t,v^2)\right)$ defined on $\left[0,\frac{1}{(1+2\epsilon)q}\right) \times [0,1)$, where $A(t,v^2)$ has

$L^\infty\left(\left[0,\frac{1}{(1+2\epsilon)q}\right)\times[0,1)\right)$ entries. So, to finish the proof, we only need to show that, under the condition (132),

$$H^*(t,v^2)G(t,v^2)=(1+2\epsilon)\widetilde{D}(t)\left(\widetilde{D}(t)H^*(t,v^2)G(t,v^2)=(1+2\epsilon)\widetilde{D}(t)\right) \quad (133)$$

holds for a.e. $(t,v^2)\in\left[0,\frac{1}{(1+2\epsilon)q}\right)\times[0,1)$ if and only if (131) holds. Suppose $H(t,v^2)=A(t,v^2)G(t,v^2)$ and $H^*(t,v^2)G(t,v^2)=(1+2\epsilon)\widetilde{D}(t)$.Then

$$G(t,v^2)H^*(t,v^2)G(t,v^2)=(1+2\epsilon)G(t,v^2)\widetilde{D}(t)=(1+2\epsilon)G(t,v^2).$$

Conversely, suppose $H(t,v^2)=A(t,v^2)G(t,v^2)$ and (131). Then $\operatorname{range}\left(H^*(t,v^2)G(t,v^2)\right)\subset\operatorname{range}\left(G^*(t,v^2)\right)$.This implies that $H^*(t,v^2)G(t,v^2)=(1+2\epsilon)I$ on range$G^*(t,v^2)$ due to the restriction of $G(t,v^2)$ on range $G^*(t,v^2)$ being injective. Also observe that $G(t,v^2)\widetilde{D}(t)=G(t,v^2)$ and that range$G^*(t,v^2)=$ range$\widetilde{D}(t)$ by Corollary (6.3.24). It follows that $H^*(t,v^2)G(t,v^2)=(1+2\epsilon)\widetilde{D}(t)$.

Now suppose $H(t,v^2)=G(t,v^2)(A+\epsilon)(t,v^2)$. Then (131) holds if and only if $(1+2\epsilon)G^*(t,v^2)=G^*(t,v^2)H(t,v^2)G^*(t,v^2)$, equivalently,

$$(1+2\epsilon)I=G^*(t,v^2)H(t,v^2)\text{ on range }\left(G^*(t,v^2)\right). \quad (134)$$

Since range $G^*(t,v^2)=$ range $\widetilde{D}(t)$ by Corollary (6.3.24), can be rewritten as

$$(1+2\epsilon)\widetilde{D}(t)=G^*(t,v^2)H(t,v^2)\widetilde{D}(t),$$

Which is equivalent to $\widetilde{D}(t)H^*(t,v^2)G(t,v^2)=(1+2\epsilon)\widetilde{D}(t)$. The proof is completed.

Corollary (6.3.27)[491]: Let $G(\sum_r g^r,1+\epsilon,1+2\epsilon)$ and $G(\sum_r h^r,1+\epsilon,1+2\epsilon)$ be both Bessel sequences in $L^2(\mathbb{R},\mathbb{C}^L)$. Then

$$\sum_r w^r=\sum_r \mathcal{S}_{h^r,g^r}f^r \quad (135)$$

if and only if

$$W^*(t,v^2)=\frac{1}{1+2\epsilon}G^*(t,v^2)H(t,v^2)F^*(t,v^2) \quad (136)$$

for a.e. $(t,v^2)\in\left[0,\frac{1}{(1+2\epsilon)q}\right)\times[0,1)$.

Proof. For $g^r,h^r\in L^2(\mathbb{R})$ satisfying $\left\{E_{m(1+2\epsilon)}T_{(m+\epsilon)(1+\epsilon)}g^r : m,(m+\epsilon)\in\mathbb{Z}\right\}$ and $\left\{E_{m(1+2\epsilon)}T_{(m+\epsilon)(1+\epsilon)}h^r : m,(m+\epsilon)\in\mathbb{Z}\right\}$ are Bessel sequences in $L^2(\mathbb{R})$, define $\sum_r \mathcal{S}_{h^r,g^r}f^r=\sum_{m,n\in\mathbb{Z}}\sum_r \langle f^r,E_{m(1+2\epsilon)}T_{(m+\epsilon)(1+\epsilon)}h^r\rangle E_{m(1+2\epsilon)}T_{(m+\epsilon)(1+\epsilon)}g^r$ for $f^r\in L^2(\mathbb{R})$. By Lemmas 2.1 and 2.5 in [483], we have that

$$\sum_{r} \mathcal{Z}_{(1+\epsilon)q}\left(\mathcal{S}_{h^r,\mathrm{g}^r} f^r\right)(t,v^2)$$

$$= \sum_{r\in\mathbb{N}_q} \sum_{m,(m+\epsilon)\in\mathbb{Z}} \left(\int_0^{\frac{1}{1+2\epsilon}} \int_0^1 \left(\overline{H(u^2,s)}\mathcal{F}(u^2,s)\right)_r e^{-2\pi i m(1+2\epsilon)u^2} e^{-2\pi i(m+\epsilon)(t-\epsilon)} du^2 d(t - \epsilon) \right) e^{2\pi i m(1+2\epsilon)t} e^{2\pi i(m+\epsilon)v^2} \mathcal{Z}_{(1+\epsilon)q}\mathrm{g}^r(t-r(1+\epsilon),v^2)$$

for a.e. $(t,v^2)\in\left[0,\frac{p}{(1+2\epsilon)}\right)\times[0,1)$, which is equivalent to

$$\sum_{r}\left(\mathcal{Z}_{(1+\epsilon)q}\left(\mathcal{S}_{h^r,\mathrm{g}^r} f^r\right)\left(t+\frac{k}{1+2\epsilon},v^2\right)\right)_{k\in\mathbb{N}_p} = \frac{1}{1+2\epsilon}\overline{G^*(t,v^2)H(t,v^2)}\sum_{r}\left(\mathcal{Z}_{(1+\epsilon)q} f^r\left(t+\frac{k}{1+2\epsilon},v^2\right)\right)_{k\in\mathbb{N}_p}$$

for a.e. $(t,v^2)\in\left[0,\frac{1}{1+2\epsilon}\right)\times[0,1)$ by a simple computation. Observe that

$$\sum_{r}\mathcal{S}_{h^r,\mathrm{g}^r} f^r = \left(\sum_{\epsilon=0}^{(1+2\epsilon)}\sum_{r}\mathcal{S}_{{h^r}_{(1+\epsilon)},{\mathrm{g}^r}_1} f^r_{1+\epsilon}, \sum_{\epsilon=0}^{(1+2\epsilon)}\sum_{r}\mathcal{S}_{{h^r}_{(1+\epsilon)},{\mathrm{g}^r}_1} {f^r}_{(1+\epsilon)}, \dots, \sum_{\epsilon=0}^{(1+2\epsilon)}\sum_{r}\mathcal{S}_{{h^r}_{1+\epsilon},\mathrm{g}^r_1} f^r_{1+\epsilon}\right).$$

It follows that (135) holds if and only if

$$\sum_{r}\left(\mathcal{Z}_{(1+\epsilon)q} w^r_{(1+\epsilon)'}\left(t+\frac{k}{1+2\epsilon},v^2\right)\right)_{k\in\mathbb{N}_p} = \frac{1}{1+2\epsilon}\sum_{\epsilon=0}^{(1+2\epsilon)}\sum_{r}\overline{G^*_{(1+\epsilon)'}(t,v^2)H_{(1+\epsilon)}(t,v^2)}\left(\mathcal{Z}_{(1+\epsilon)q} f^r\left(t+\frac{k}{1+2\epsilon},v^2\right)\right)_{k\in\mathbb{N}_p}$$

for a.e. $(t,v^2)\in\left[0,\frac{1}{1+2\epsilon}\right)\times[0,1)$ and $1\le(l+\epsilon)'\le l+2\epsilon$, equivalently,

$$\sum_{r}\mathcal{Z}_{(1+\epsilon)q} w^r_{l'}\left(t-r(1+\epsilon)+\frac{k}{1+2\epsilon},v^2\right)$$

$$= \frac{1}{1+2\epsilon} \sum_{\epsilon=0}^{1+2\epsilon} \sum_{j\in\mathbb{N}_p} \sum_{r} \left(\sum_{n\in\mathbb{N}_p} \mathcal{Z}_{(1+\epsilon)q} \mathrm{g}^{r}_{(1+\epsilon)'} \left(t-(m+\epsilon)(1+\epsilon) + \frac{k}{1+2\epsilon}, v^2\right) \overline{\mathcal{Z}_{(1+\epsilon)q} h^{r}{}_{(1+\epsilon)'} \left(t-(m+\epsilon)+1+\epsilon+\frac{J}{1+2\epsilon}, v^2\right)} \right) \times \mathcal{Z}_{(1+\epsilon)q} f^{r}{}_{(1+\epsilon)} \left(t+\frac{j}{1+2\epsilon}, v^2\right) \qquad (137)$$

For a.e. $(t, v^2) \in \left[0, \frac{1}{1+2\epsilon}\right) \times [0,1), k \in \mathbb{N}_p$ and $1 \le l+\epsilon \le l+2\epsilon$. By a simple computation, can be rewritten as

$$W^{*}_{(1+\epsilon)'}(t, v^2) = \frac{1}{1+2\epsilon} \sum_{\epsilon=0}^{(1+2\epsilon)} G^{*}_{(1+\epsilon)'}(t, v^2) H_{(1+\epsilon)}(t, v^2) F^{*}_{(1+\epsilon)}(t, v^2)$$

for a.e. $(t, v^2) \in \left[0, \frac{1}{(1+2\epsilon)q}\right) \times [0,1)$ and $1 \le (1+\epsilon)' \le (l+2\epsilon)$, equivalently,

$$\sum_{r} \mathcal{Z}_{(1+\epsilon)q} w^{r}{}_{(1+\epsilon)'} \left(t-r(1+\epsilon)+\frac{k}{1+2\epsilon}, v^2\right) = \frac{1}{1+2\epsilon} \sum_{\epsilon=0}^{l+2\epsilon} \sum_{j\in\mathbb{N}_p} \sum_{r} \left(\sum_{N\in\mathbb{N}_p} \mathcal{Z}_{(1+\epsilon)q} \mathrm{g}^{r}{}_{(1+\epsilon)'} \left(t-(m+\epsilon)(1+\epsilon) + \frac{k}{1+2\epsilon}, v^2\right) \overline{\mathcal{Z}_{(1+\epsilon)q} h^{r}{}_{(1+\epsilon)} \left(t-(m+\epsilon)(1+\epsilon)+\frac{J}{1+2\epsilon}, v^2\right)} \right) \times \sum_{r} \mathcal{Z}_{(1+\epsilon)q} f^{r}_{(1+\epsilon)} \left(t-r(1+\epsilon)+\frac{j}{1+2\epsilon}, v^2\right), \qquad (138)$$

for a .e. $(t, v^2) \in \left[0, \frac{1}{(1+2\epsilon)q}\right) \times [0,1)$ and $(r, k) \in \mathbb{N}_q \times \mathbb{N}_p$.

Next we show the equivalence between (137) and (138) to finish the proof.

For$(t,, r, k) \in \left[0, \frac{1}{(1+2\epsilon)q}\right) \in \mathbb{N}_q \times \mathbb{N}_p$, define $\tau\left(t-r(1+\epsilon)+\frac{k}{1+2\epsilon}\right) = t' + \frac{k'}{1+2\epsilon}$ if $\left(t-r(1+\epsilon)+\frac{k}{1+2\epsilon}\right) = t' + \frac{k'}{1+2\epsilon} + m(1+\epsilon)q$ for some $(t', k', m) \in \left[0, \frac{1}{1+2\epsilon}\right) \times \mathbb{N}_p \times \mathbb{Z}$. Then it is easy to check that τ is a bijection from $\bigcup_{r\in\mathbb{N}_q} \bigcup_{k\in\mathbb{N}_p} \left(\left[0, \frac{1}{(1+2\epsilon)q}\right) - r(1+\epsilon) + \frac{k}{1+2\epsilon}\right)$ onto $\bigcup_{k\in\mathbb{N}_p} \left(\left[0, \frac{1}{1+2\epsilon}\right) + \frac{k}{1+2\epsilon}\right)$, and (138) can be rewritten as

$$\sum_{r} \mathcal{Z}_{(1+\epsilon)q} w^{r}{}_{(1+\epsilon)'} \left(t' + \frac{k'}{1+2\epsilon} + m(1+\epsilon)q, v^2\right)$$

$$= \frac{1}{1+2\epsilon} \sum_{\epsilon=0}^{(1+2\epsilon)} \sum_{j\in\mathbb{N}_p} \sum_{r} \left(\sum_{n\in\mathbb{N}_q} \mathcal{Z}_{(1+\epsilon)q} \mathrm{g}^r{}_{(1+\epsilon)'} \left(t' - \big((m+\epsilon) - r\big)(1+\epsilon) + \frac{k'}{1+2\epsilon} \right.\right.$$

$$\left. + m(1+\epsilon)q, v^2 \right)$$

$$\left. \times \overline{\mathcal{Z}_{(1+\epsilon)q} h^r{}_{(1+\epsilon)} \left(t' - ((m+\epsilon) - r)(1+\epsilon) + \frac{k'-k+j}{1+2\epsilon} + m(1+\epsilon)q, v^2 \right)} \right)$$

$$\times \mathcal{Z}_{(1+\epsilon)q} f^r{}_{h^r{}_{(1+\epsilon)}} \left(t' + \frac{k'-k+j}{1+2\epsilon} + m(1+\epsilon)q, v^2 \right). \qquad (139)$$

Since $\mathbb{N}_q - r$ is $q\mathbb{Z}$-congruent to $\mathbb{N}_q$, and $\mathbb{N}_p + (k' - k)$ is $p\mathbb{Z}$-congruent to $\mathbb{N}_p$, (139) is equivalent to

$$\mathcal{Z}_{(1+\epsilon)q} w^r{}_{(1+\epsilon)'} \left(t' + \frac{k'}{1+2\epsilon}, v^2 \right)$$

$$= \frac{1}{1+2\epsilon} \sum_{\epsilon=0}^{(1+2\epsilon)} \sum_{j\in\mathbb{N}_p} \left(\sum_{n\in\mathbb{N}_q} \mathcal{Z}_{(1+\epsilon)q} \mathrm{g}^r{}_{(1+\epsilon)'} \left(t' - (m+\epsilon)(1+\epsilon) \right.\right.$$

$$\left.\left. + \frac{k'}{1+2\epsilon}, v^2 \right) \overline{\mathcal{Z}_{(1+\epsilon)q} h^r{}_{(1+\epsilon)} \left(t' - (m+\epsilon)(1+\epsilon) + \frac{j}{1+2\epsilon}, v^2 \right)} \right) \mathcal{Z}_{(1+\epsilon)q} f^r{}_{(1+\epsilon)} \left(t' \right.$$

$$\left. + \frac{j}{1+2\epsilon}, v^2 \right)$$

by quasi-periodicity of $\mathcal{Z}_{(1+\epsilon)q}$. It is exactly (137) due to τ is a bijection from the set $\bigcup_{r\in\mathbb{N}_q} \bigcup_{k\in\mathbb{N}_p} \left(\left[0, \frac{1}{(1+2\epsilon)q}\right) - r(1+\epsilon) + \frac{k}{1+2\epsilon} \right)$ onto $\bigcup_{k\in\mathbb{N}_p} \left(\left[0, \frac{1}{1+2\epsilon}\right) + \frac{k}{1+2\epsilon} \right)$. The proof is completed.

Corollary (6.3.28)[491]: Let $G(\sum_r \mathrm{g}^r, 1+\epsilon, 1+2\epsilon)$ be a frame for $L^2(S, \mathbb{C}^L)$, and $G(\sum_r h^r, 1+\epsilon, 1+2\epsilon)$ be a Bessel sequence in $L^2(t-\epsilon, \mathbb{C}^L)$. Then $G(\sum_r h^r, 1+\epsilon, 1+2\epsilon)$ is the canonical Gabor dual of $G(\sum_r \mathrm{g}^r, 1+\epsilon, 1+2\epsilon)$ if and only if

$$H(t, v^2) = (1+2\epsilon) G(t, v^2) \big(G^*(t, v^2) G(t, v^2) \big)^{\dagger} \text{ for a.e. } (t, v^2) \in \left[0, \frac{1}{(1+2\epsilon)q}\right) \times [0,1). \qquad (140)$$

Proof. $G(\sum_r h^r, 1+\epsilon, 1+2\epsilon)$ is the canonical Gabor dual of $G(\sum_r \mathrm{g}^r, 1+\epsilon, 1+2\epsilon)$ if and only if

$$\sum_r \mathcal{S}_{\mathrm{g}^r, \mathrm{g}^r} \boldsymbol{h}^r = \sum_r \mathrm{g}^r, \qquad (141)$$

Which is equivalent to

$$G^*(t, v^2) = \frac{1}{1+2\epsilon} G^*(t, v^2) G(t, v^2) H^*(t, v^2) \text{ for a.e. } (t, v^2) \in \left[0, \frac{1}{(1+2\epsilon)q}\right) \times [0,1)$$

by Corollary (6.3.27), namely,

$$G(t,v^2)=\frac{1}{1+2\epsilon}H(t,v^2)G^*(t,v^2)G(t,v^2) \text{ for a.e. } (t,v^2) \in \left[0,\frac{1}{(1+2\epsilon)q}\right)\times[0,1). \quad (142)$$

Then (140) implies (142) due to $\left(G^*(t,v^2)G(t,v^2)\right)^\dagger G^*(t,v^2)G(t,v^2)=\mathcal{P}_{\text{range}}G^*(t,v^2)$.

Next we show the converse implication to finish the proof. Suppose (142) holds. Since the canonical Gabor dual is a dual of type I, we have that $H(t,v^2)=A(t,v^2)G(t,v^2)$ for some $A(t,v^2)$ by Corollary (6.3.26). Combined with (142), it follows that

$$\begin{aligned}(1+2\epsilon)G(t,v^2)\left(G^*(t,v^2)G(t,v^2)\right)^\dagger &= H(t,v^2)G^*(t,v^2)G(t,v^2)\left(G^*(t,v^2)G(t,v^2)\right)^\dagger\\ &= A(t,v^2)G(t,v^2)G^*(t,v^2)G(t,v^2)\left(G^*(t,v^2)G(t,v^2)\right)^\dagger = H(t,v^2)\end{aligned}$$

for a.e. $(t,v^2)\in\left[0,\frac{1}{(1+2\epsilon)q}\right)\times[0,1)$, where we used the fact that

$$G^*(t,v^2)G(t,v^2)\left(G^*(t,v^2)G(t,v^2)\right)^\dagger=\mathcal{P}_{\text{range}}G^*(t,v^2).$$

Eq. (140) therefore follows. The proof is completed.

Corollary (6.3.29)[491]: Let $G(\sum_r g^r, 1+\epsilon, 1+2\epsilon)$ be a frame for $L^2(t-\epsilon,\mathbb{C}^L)$. Then, for any Bessel sequence $G(\sum_r h^r, 1+\epsilon, 1+2\epsilon)$, we have

(iii) $G(\sum_r h^r, 1+\epsilon, 1+2\epsilon)$ is a Gabor dual of type I for $G(\sum_r g^r, 1+\epsilon, 1+2\epsilon)$ if and only if there exists some function $\mathcal{A}:\left[0,\frac{1}{(1+2\epsilon)q}\right)\times[0,1)\to\mathcal{M}_{q,q}$ such that

$$H(t,v^2)=(1+2\epsilon)(G(t,v^2)G^*(t,v^2)G(t,v^2))^\dagger G(t,v^2)\left(I-\frac{1}{1+2\epsilon}G^*(t,v^2)\mathcal{A}(t,v^2)G(t,v^2)\right)+\mathcal{A}(t,v^2)G(t,v^2) \quad (143)$$

for a.e. $(t,v^2)\in\left[0,\frac{1}{(1+2\epsilon)q}\right)\times[0,1)$;

(iv) $G(\sum_r h^r, 1+\epsilon, 1+2\epsilon)$ is a Gabor dual of type II for $G(\sum_r g^r, 1+\epsilon, 1+2\epsilon)$ if there exists some function $\mathcal{A}:\left[0,\frac{1}{(1+2\epsilon)q}\right)\times[0,1)\to\mathcal{M}_{q,q}$ such that

$$H(t,v^2)=(1+2\epsilon)G(t,v^2)\left(G^*(t,v^2)G(t,v^2)\right)^\dagger\left(I-\frac{1}{1+2\epsilon}G^*(t,v^2)G(t,v^2)\mathcal{A}(t,v^2)\right)+G(t,v^2)\mathcal{A}(t,v^2) \quad (144)$$

for a.e. $(t,v^2)\in\left[0,\frac{1}{(1+2\epsilon)q}\right)\times[0,1)$.

Proof. (i) Sufficiency. Suppose (143) holds. Then

$$\begin{aligned}H^*(t,v^2)G(t,v^2) &= (1+2\epsilon)G^*(t,v^2)\left(G(t,v^2)G^*(t,v^2)\right)^\dagger G(t,v^2)\\ &+G^*(t,v^2)\mathcal{A}^*(t,v^2)G(t,v^2)\\ &-G^*(t,v^2)\mathcal{A}^*(t,v^2)G(t,v^2)G^*(t,v^2)\left(G(t,v^2)G^*(t,v^2)\right)^\dagger G(t,v^2).\end{aligned}$$

Since $G(t,v^2)G^*(t,v^2)\left(G(t,v^2)G^*(t,v^2)\right)^\dagger=\mathcal{P}_{\text{range}(G(t,v))}$, we have that

$$-G^*(t,v^2)\mathcal{A}^*(t,v^2)G(t,v^2)G^*(t,v^2)\big(G(t,v^2)G^*(t,v^2)\big)^\dagger + G^*(t,v^2)\mathcal{A}^*(t,v^2)G(t,v^2) = 0,$$

and thus

$$H^*(t,v^2)G(t,v^2) = (1+2\epsilon)G^*(t,v^2)\big(G(t,v^2)G^*(t,v^2)\big)^\dagger G(t,v^2).$$

So

$$H^*(t,v^2)G(t,v^2) = (1+2\epsilon)G^*(t,v^2)G(t,v^2)\big(G^*(t,v^2)G(t,v^2)\big)^\dagger,$$

which is $\mathcal{P}_{\mathrm{range}\left(G^*(t,v^2)\right)}$. And then $H^*(t,v^2)G(t,v^2) = (1+2\epsilon)\widetilde{D}(t)$ for a.e. $(t,v^2) \in \left[0,\frac{1}{(1+2\epsilon)q}\right) \times [0,1)$ by Corollary (6.3.24). Therefore, $G(\sum_r h^r, 1+\epsilon, 1+2\epsilon)$ is a Gabor dual of type I for $G(\sum_r \mathrm{g}^r, 1+\epsilon, 1+2\epsilon)$ by Corollary (6.3.26).

Necessity, Suppose $G(\sum_r h^r, 1+\epsilon, 1+2\epsilon)$ is a Gabor dual of type I for $G(\sum_r \mathrm{g}^r, 1+\epsilon, 1+2\epsilon)$. Then, $H^*(t,v^2)G(t,v^2) = (1+2\epsilon)\widetilde{D}(t)$, and there exists some $\mathcal{B} : \left[0,\frac{1}{(1+2\epsilon)q}\right) \times [0,1) \to \mathcal{M}_{Lp,Lp}$ such that $H(t,v^2) = \mathcal{B}(t,v^2)G(t,v^2)$ for a.e. $(t,v^2) \in \left[0,\frac{1}{(1+2\epsilon)q}\right) \times [0,1)$ by Corollary (6.3.26). Take $\mathcal{A}(t,v^2) = \mathcal{B}(t,v^2) - (G(t,v^2)G^*(t,v^2))^\dagger$. Then

$$\begin{aligned}
(1+2\epsilon)\big(G(t,v^2)G^*(t,v^2)\big)^\dagger G(t,v^2)\left(I - \frac{1}{1+2\epsilon}G^*(t,v^2)\mathcal{A}(t,v^2)G(t,v^2)\right)& \\
+ \mathcal{A}(t,v^2)G(t,v^2)& \\
= H(t,v^2) + (1+2\epsilon)\big(G(t,v^2)G^*(t,v^2)\big)^\dagger G(t,v^2)& \\
- (G(t,v^2)G^*(t,v^2))^\dagger G(t,v^2)G^*(t,v^2)H(t,v^2)& \\
+ \big(G(t,v^2)G^*(t,v^2)\big)^\dagger G(t,v^2)G^*(t,v^2)\big(G(t,v^2)G^*(t,v^2)\big)^\dagger G(t,v^2)& \\
- \big(G(t,v^2)G^*(t,v^2)\big)^\dagger G(t,v^2)& \\
= H(t,v^2) + (1+2\epsilon)\big(G(t,v^2)G^*(t,v^2)\big)^\dagger G(t,v^2)G - (1& \\
+ 2\epsilon)\big(G(t,v^2)G^*(t,v^2)\big)^\dagger G(t,v^2)\widetilde{D}(t)& \\
= H(t,v^2)&
\end{aligned} \tag{145}$$

Due to $G(t,v^2)\widetilde{D}(t) = G(t,v^2)$ and $(G(t,v^2)G^*(t,v^2))\big(G(t,v^2)G^*(t,v^2)\big)^\dagger = \mathcal{P}_{\mathrm{range}\left(G(t,v^2)\right)}$. (ii) Suppose (144) holds. Then

$$\begin{aligned}
\widetilde{D}(t)H^*(t,v^2)G(t,v^2) &= (1+2\epsilon)\widetilde{D}(t)\big(G^*(t,v^2)G(t,v^2)\big)^\dagger\big(G^*(t,v^2)G(t,v^2)\big) \\
&+\widetilde{D}(t)\mathcal{A}^*(t,v^2)\big(G^*(t,v^2)G(t,v^2)\big) - \widetilde{D}(t)\mathcal{A}^*(t,v^2) \\
&\times (G^*(t,v^2)G(t,v^2))(G^*(t,v^2)G(t,v^2))^\dagger(G^*(t,v^2)G(t,v^2)).
\end{aligned} \tag{146}$$

Observe that $\big(G^*(t,v^2)G(t,v^2)\big)^\dagger\big(G^*(t,v^2)G(t,v^2)\big) = \mathcal{P}_{\mathrm{range}\left(G^*(t,v^2)\right)}$. From (146), we deduce that

$$\widetilde{D}(t)H^*(t,v^2)G(t,v^2) = (1+2\epsilon)\widetilde{D}(t)$$

for a.e. $(t,v^2) \in \left[0,\frac{1}{(1+2\epsilon)q}\right) \times [0,1)$ by Theorem (124). This finishes the proof.

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