بسم الله الرحمن الرحيم

Sudan University of Science &Technology

College of Graduate Studies

Spatial Evolution in the Interaction Model and Schrodinger Equation in a Curved Space-Time

التطور الإحداثي في النموذج التفاعلي و معادلة شوردينجر في الفراغ الزمكاني المنحني

A thesis submitted in fulfillment for the requirements

of PhD degree in physics

BY:

Abdelkarim Gism Alla Khogli Mohamed

Supervisor:

Prof: Mubarak Dirar Abdalla

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الآيـة

بسم الله الرحمن الرحيم

اقْرَأْ بِاسْمِ رَبِّكَ الَّذِي خَلَقَ (1) خَلَقَ الْإِنْسَانَ مِنْ عَلَقٍ (2) اقْرَأْ وَرَبُّكَ الْأَكْرَمُ (3) الَّذِي عَلَّمَ بِالْقَلَمِ (4) عَلَّمَ الْإِنْسَانَ مَا لَمْ يَعْلَمْ (5)

صدق الله العظيم

سورة العلق من الآية (1-5)

الإهداء

اهدي هذا البحث إلي أمي فاطمة مصطفي قسم الباري

والي روح (أبي قسم الله خوجلي وأخي عوض) عليهم رحمة الله والي زوجتي عنايات بشير وبناتي براءة وبلسم وبيان

والي إخواني (صلاح وخوجلي وسوسن وشجن وبسمه)

والي خالتي مدينة محمد الحسن والي خالي الفاضل مصطفي والي جميع اخوالي و خالاتي و ابنا اخواني و اخواتي والي أخواني طلحه محمد احمد و محمد صالح بشير وخالد البدوي و محمد عبدالواحد وغسان ومحمد ومجتبي ومصعب وغسي احمد عجيل و يس محمد عثمان والشفيع عبدالله وربيع بشير وعوض بشير ومحمد عبدالله(ود المناقل) والي كل الزملاء

الشكر

الشكر أولاً وأخيرً إلي الله سبحانه وتعالي ثم الي البروفسور مبارك درار عبدالله الذي أشرف علي هذا البحث

والي الوالدة العزيزة فاطمة مصطفي قسم الباري

والي زوجتي عنايات بشير

والي أخواني صلاح و خوجلي وأخواتي سوسن وشجن وبسمه

والي جدة بناتي مدينة و طلحه محمد احمد حداد و خالي الفاضل مصطفي قسم الباري الي كل من وقف بجانبي لانجاز هذا البحث

ABSTRACT

This study is concerned with derivation of quantum relation in a curved space –time . The importance of this study emerges from the use of energy and momentum in many applications. This study use mathematical derivation trend. The research problem is related to the lack of suitable consistent relations for quantum eigen equations in a curved space , beside lack of relations for the special quantum evolution in the interaction model. Therefore this research aims to use the expression of time and distance in a curved space time to find a useful expression of energy and momentum Eigen equations in a curved space .These relation are used to derive the corresponding relation in the Euclidean space . The corresponding Energy –momentum relation for both curved and Euclidian space gives a relation between energy and momentum typical to that obtained from the energy and momentum Eigen equation . The expression of mass in a curved space similar to that of the generalized relativity is also found.

Using generalized special relativity a useful expression of the perturbed momentum is found . This expression is used to describe the behavior of the quantum system in the interaction model . It is found that the spatial evolution of the Schrodinger equation in the interaction model is similar to that of time evolution , where the time differential is replaced by the space one, and the Hamiltonian by the momentum operator . The same holds for the unitary operator , where the time integral is replaced by the space one and the Hamiltonian with the momentum operator .

The unitary operator in the Heisenberg picture and the spatial evolution of the quantum system was found by using simple mathematics and the ordinary laws of differentiation and integration .This expression describes successfully the spatial evolution of the quantum operator . The metrics in the curved space is found to be related to the Lorentz transformation coefficient .

المستخلص

تناولت هذه الدراسة اشتقاق علاقات كمية في الزمكان المنحني . وتنبع أهمية الدراسة من الاستخدامات المتعددة للطاقة وكمية التحرك . واستخدمت الدراسة المنهج الاستنباطي الرياضي.

تتمثل مشكلة الدراسة في عدم وجود صيغ مقبولة للمعادلات الذاتية الكمية في الفراغ المنحني وعدم وجود صيغ للتغير الإحداثي الكمي في النموذج التفاعلي . لذا هدفت هذه الدراسة لاستخدام علاقات الزمن و المسافة في الفراغ المنحني لإيجاد علاقات مفيدة لمعادلات الطاقة و الاندفاع في الفراغ المنحني . هذه العلاقات استخدمت لاستنباط العلاقات المناظرة في الفراغ الإقليدي . وقد أعطت علاقات الطاقة و الاندفاع المناظرة لكل من الفراغ المنحني و الإقليدي علاقات مطابقة لتلك التي تم الحصول عليها من معادلات الطاقة و الاندفاع الذاتية وقد تطابقت صيغة الكتلة في الفراغ المنحني مع صيغتها في النسبية الخاصة المعممة.

تم الحصول ايضاً علي علاقات مفيدة للاندفاع في حالة الاضطراب باستخدام نظرية النسبية الخاصة المعممة . هذه الصيغة استخدمت لوصف سلوك المنظومة الكمية في النموذج التفاعلي . ووجد أن التغير الإحداثي لمعادلة شوردينجر في النموذج التفاعلي مشابه لذلك الذي للتغير الزمني , حيث استبدلنا التفاضل الزمني بالإحداثي و الهاملتونيان بمؤثر الاندفاع . ونفس الوضع ينطبق علي مؤثر الوحدة حيث استبدل التكامل الزمني بالإحداثي و الهاملتوني بمؤثر الاندفاع .

تم الحصول ايضاً علي مؤثر الوحدة في نموذج هيزنبرج و التغير الإحداثي للمنظومة الكمية باستخدام رياضيات مبسطة مع العلاقات العادية التفاضلية و التكاملية . هذه الصيغة تصف بنجاح التغير الإحداثي للمؤثر الكمي . ووجد أن المقياس في الفراغ المحدب له علاقة بمعامل تحويل لورينس .

Contents

Chapter one

Introduction

(1.1) History Of Quantum Mechanics (Q.M)

Quantum theory is the theory that is concerned with atomic world based on the radical theoretical proposals that were not based on accepted classical physics . The Quantum mechanics was created between 1900 and 1925 . At the end of 1900 , Max . Planck presented a new form of the black body radiation spectral distribution Law , based on a revolutionary hypothesis . He postulated that the energy of an oscillator of given frequency cannot take arbitrary values between zero and infinity , but can only take on the discrete values . However , it was not long before the quantum concept was used to explain other phenomena . Indeed , in 1905 , A . Einstein was able to interpret the photoelectric effect by introducing the idea of photons , or ligh t quanta . And in 1907 he used plank formula for the average energy of an oscillator to derive the law of Dulongand Patitconcerning the specific heat of solids . subsequently N. Bohr in 1913 was able to invoke the idea of quantisation of atomic energy levels to explin the existence of line spectra . [1,2,3]

In 1924 , L . debrolie mode a great unifying , but speculative , hypothesis ,that just as radiation has particle –like properties , electrons and other material particles possess wave-like properties . The particle properties of electromagnetic waves are also demonstrated in the Compton Effect and using the momentum and energy .

In 1924 de Broglie suggests particles can behave as waves in 1925. Thus the atomic world particles have dual particle - wave nature Werner Heisenberg introduces Matrix Mechanics to consider observable quantities . He used matrices , which were not that familiar at the time to describe the dual nature of atomic particles .[4,5]

Erwin Schrödinger proposes wave mechanics , he used waves , which is more familiar to scientists at the time . Heisenberg’s and Schrödinger’s formulations were competing Eventually, Schrödinger showed they were equivalent; different descriptions which produced the same predictions . Later on the so called interaction picture is introduced to simplify solving problems of quantum systems .

Applied quantum mechanics is widely in spectrum , For example spectral techniques are used in mineral exploration as well as identifying Chemical compounds . laser is widely used in telecommunication , computer , and medicine .These techniques were based on quantum physics . Atoms are building block of matter . Atoms themselves consist of elementary particles like electrons, protons and neutrons . The behavior of atoms described by quantum lows . The one which is biased the classical Newtonian energy – momentum relationship is known as Schrodinger equation [6]. That who riles on relativistic energy –momentum relation is Known as Klien – Gordan and Dirac relativistic quantum equations [7]. Fortunately a quantum law successfully describes a wade veriety of phenomena , like atomic spectra of isolated atoms beside the spectra of some solids like semi-conductor (SC). However quantum mechanics (qm) suffer from noticeable setbacks .For instant there is no quantum gravity low that moreover the unification of force under the umbrella of quantum low is so difficult within the framework of conventional quantum lows [8]. Also the superconductivity behavior for high temperature superconductors (HTSC) can hot be described easily and fully by the existing models [9]. This forces many researchers to construct new models that modify quantum lows to cure some of these defects [10,11]. These attempts encourages to purpose quantum model that can help in finding quantum gravity equation . This model was an expression of the wave function in a curved space to find energy and momentum Eigen equation in a curved space .

Quantum laws are used to describe the behavior of atoms and elementary particles . According to the time evolution there are three versions . The first one is the Schrodinger picture in which the time evolution is described by the wave function . The second one is the Heisenberg picture in which the time evolution is described by the operator . The third representation is the so called interaction representation in which the time evolution of the system is described by the wave vector and the operator which is the interaction Hamiltonian instead of the total Hamiltonian [3,12.13]

These versions succeeded in describing the time evolution but says nothing about the spatial evaluation of the quantum system . This motivated some authors to propose some models to cure this defect [14,15,16] .In one of them the ordinary Schrodinger equation is developed to describe the behavior of the system using the momentum operator [17]. In another approach the system by using a perturbed momentum [18,19] . Different attempts were also be made to make quantum laws more flexible in describing the quantum system [20,21]. This encourages to construct a new model to help in describing the spatial evolution of the quantum system .

Quantum systems are described by operator and wave function . In the Schrodinger representation the evolution of the quantum system is described by the wave function . In the Heisenberg picture the evolution is described by operator [22.2]. The change from the Schrodinger to the Heisenberg picture time evolution is done by using mathematical transformation . Schrödinger picture is needed for the probability distribution ,while the Heisenberg picture time evolution is needed for the quantum average of the physical quantity [3,14]. This transformation is different from the Lorentz transformation which aims to find the effect of motion and fields on the physical quantities[13,16]. The quantum system is described by these transformation services successfully . However the spatial evolution of the quantum system is not fully supported .Some attempts have been made to derive Heisenberg spatial evolution of the quantum system [23, 24,25] but it needs to be done by another simple approach.

(1.2) Research problem

The research problem stems from the lack of useful expression for quantum eigen equations in a curved space . Also the spatial evolution of the quantum system in the interaction model dos not recognized .

(1.3) Literature review

A seminal paper published and one by M . Dirar recognises the effect of friction by using Maxwell's equation [26] . Another attempts' were made by M . Mamoun based on harmonic oscillator expression for energy is also mode to derive new Schrödinger equation[27] .

Different attempts were made to modify Schrodinger equation [28, 29, 30]. Some of them work in a curved space –time [31, 32, 33] . Some of these attempts tries to describe quantum gravity [34, 35, 36] . While some are concerned with bulk matter [37.38,39].

The paper published by Nilesh P.BARDE,(2015) derive Schrodinger equation from wave –mechanics , Schrodinger time independent equation , classical Hamilton –Jacobi equation [29]

The concept of time dependent Schrodinger equation (TDSE) is mostly complex for advanced learners . It is shown that TDSE may be derived using wave mechanics ,time independent equation , classical &Hamilton –Jacobi's equation . similar attempts have bee done earlier by some researches . However , this work provides a comprehensive , lucid and well derived derivation , derived using various approaches , which would make this article unique .

Another work was done to derive Schrodinger equation .In the work of PRANAB RUDRA SARMA ,(2010), he derive Schrodinger equation from Hamilton –Jacobi equation using uncertainty principle [30] .In deriving Schrodinger's wave equation the momentum and energy of a particle are taken to be operators acting on a wave function .Here one shows that the wave equation can be directly derived from the classical Hamilton –Jacobi equation , if a basic uncertainty is assumed to be present in the momentum .In this derivation one dose not have to assume the momentum and energy to be operators .

The unitary operator in a curved space time was also tackled by some works. The work done by C.FRONSDAL , determine unitary operator in a curved space .He discuss only the case of constant curvature . Then operators of angular and linear momentum exist , and we show that the interesting irreducible unitary representations of the group of motions reduce very simply to those of the inhomogeneous Lorentz group in limit of zero curvature[40].

In the paper of L.C.W. Jeronimus , (2012).Elementary particles can be indentified with the unitary irreducible representations(UIR,S) of the isometry group of a given space time . These (UIR,S)are labeled by the eigenvalues of asimir operators of the isometry group and hence they represent invariant physical properties of the elementary particles . These properties therefore depend entirely on the space time background of the particle . To compare these labels for different space time background , one can use the method of contraction[41] . D.Aresnovic ,(2014) derived Lagrangian formulation of quantum mechanical from Schrodinger equation . It is developed and illustrated on eigenbasis of the Hamiltonian and in the coordinate representation . The Lagrangian formulation of physically plausible quantum system results in a wall defined second order equation on a real vector space . The Klein –Gordon equation for areal field in shown to be the Lagrangian form of the corresponding Schrodinger equation[42] .

Schrodinger and Dirac equation are also derived using new approach .In the work done by Spyros Efthimiades ,he derived the Schrodinger and Dirac equation from basic principles[43]. First we determine that each eigenfunction of a bound particle is a specific superposition of plane wave states that fulfills the averaged energy relation . The Schrodinger equation is derived to be the condition the particle eigen function must satisfy , at each space -time point in order to fulfill the averaged energy relation . The same approach is applied to derive Dirac equation involving electromagnetic potentials .Effectively , the Schrodinger and Dirac equation are space –time versions of the respective averaged energy relations .

In the work of Steven Carlip , (2019) , gravity is asymptotically safe , operators will exhibit anomalous Scaling at the ultraviolet fixed point in a way that makes the theory effectively two –dimensional .A number of independent lines of evidence , based on different approaches to quantization , indicate a similar short –distance dimensional reduction . The physical question of what one means by "dimension" in quantum space time , and the possible mechanisms that could explain the universality is shown in terms of curved space quantization[44] .

Quantum gravity models are discussed by Giampiero Esposito , (2011). The various commenting theories , e.g. string theory and loop quantum gravity , have still to be checked against observations .Classical and quantum foundations are necessary to study field –theory approaches to quantum gravity .The passage from old to new unification in quantum field –theory ,needs canonical quantum gravity , the use of functional integrals , the properties of gravitational intentions , the use of spectral zeta-functions in the quantum theory of the universe , Hawking radiation ,some theoretical achievements and some key experimental issues [45].

A work was done by some anthers to derive Schrodinger equation using the concept of amplitudes .K .Young and S. S. Tong , (2014) ,uses principle of least action , is used to drive Schrodinger equation . The advantages of using the action S over Newton's laws are explained , through examples using generalized coordinates and bringing in the concepts of the lagrangian and the Hamiltonian . Quantum amplitudes along a path are simply summed over all possible paths leads to Schrodinger equation . The geodesic equation in general relativity is also quickly sketched as a simple extension of the least action principle [46].

Maxwell F. Parsons , (2017), uses Hubbard model to study exotic phases of matter using strong correlations in quantum many –body systems. Quantum gas microscopy affords the opportunity to study these correlations .Here one report site–resolved observations of antiferromagnetic correlations in a two-dimensional , Hubbard –region optical lattice and demonstrate the ability to measure the spin –correlation function over any distance .One measure the in –situ distributions of the particle density and magnetic correlations , extract thermodynamic quantities from comparisons to theory ,and observe statistically significant correlations over three lattice sites . The temperatures that reached approach the limits of available numerical simulations . The direct access to many–body physics at the single –particle level demonstrated by the results will improve understanding of how the interplay of motion and magnetism gives rise to new states of matter[37].

(1.4) Aim Of The Work

The aim of this work is to use momentum perturbation to find the spatial evolution in the interaction model and Schrodinger equation in a curved space-time . This is achieved by finding the momentum eigen equation in the interaction picture and obtaining unitary operator. The Schrodinger equation in a curved space is found from the form of the wave function and the coordinate in a curved space.

(1.5) Thesis Lay Out :

The thesis consists of 4 chapters . Chapter one and two are devoted for introduction and theoretical background . While chapter three and four are concerned with contribution and results beside discussion and conclusion .

Chapter 2
Theoretical Background

(2.1) Introduction:

According to the Laws of quantum mechanics the state of motion of a particle is specified by the wave function . The fundamental question is to predict how the state of motion will evolve in time .

In quantum mechanics the equation of motion is the time-dependent Schrödinger equation . the time-dependent Schrödinger equation determines the wave function at any other time .

(2.2) Planck Discovery :

In 1900 Planck discovered a formula for black body radiation that is in complete agreement with experiment at all wave lengths. [47]

Planck 's analysis led to the curve shown is figure blow [48].

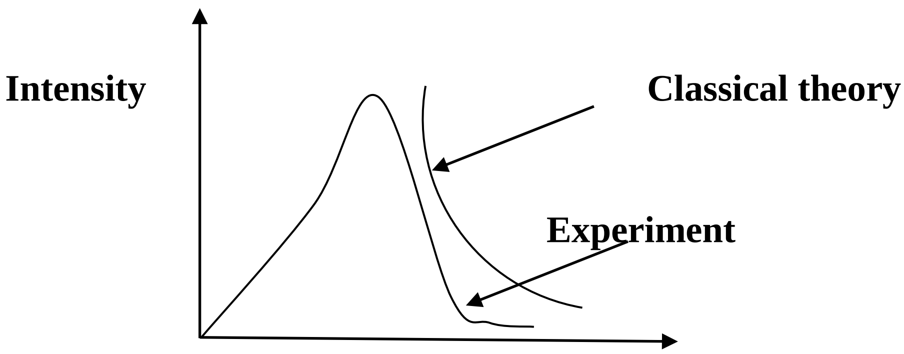

Figure (2.2.1)

Figure (2.2.1) comparison of the experimental results with the curve predicted by the Rayleigh Jeans classical model for the distribution of the blackbody radiation [49, 50].

As Known the concept of quantization developed by Planck in 1900 the quantization model assumes that the energy of light wave is present in bundles of energy called photons , hence, the energy is said to be quantized . Any quantity that appears in discrete bundles in side to be quantized , just as charge and other properties are quantized.

According to Planck's theory , the energy of photon is proportional to frequency of electromagnetic wave .

$$E = hf \qquad (2.2.1)$$

Where E is the energy , f is frequency and h is plank constant .

It important to not that this theory retains some features of the particle theory of the light .

Planck mode two bold and controversial assumption concerning the nature of the oscillating molecules at the surface of the blackbody.

The molecules can have only discrete units of energy E_n

$$E_n = nhf \qquad (2.2.2)$$

There $n = 1 , 2 ,3 \dots..$

Where n is a positive integer called a quantum number and f is frequency of the vibration of the molecules . Because the energy of molecules can have only discrete values given by equation (2.2.1) the energy is quantized , each discrete energy value represents a different quantum state .

The molecules emit or absorb energy in discrete plackets called photons.

The key points in Planck theory is radical a assumption of quantized energy state . This development the birth of quantum theory when Planck presented his theory , most scientists (include Planck) did not consider the quantum concept to be realistic . Hence , Planck and other continued to search for a more rationale explanation of blackbody radiation . However , subsequent development showed that a theory based on the quantum concept (rather than on classical concept) had to be used to explain many other phenomena at atomic [51,52].

(2.3) The Schrodinger equation :

The Schrodinger equation is the key equation of quantum mechanics.

This second partial differential equation determines the spatial shape and the temporal evolvement of wave function in a given potential and for a given boundary conditions .

The one dimensional Schrodinger equation is used when the particle of interest is confined to one spatial dimension nature of many semiconductor hetero structures, the Schrodinger equation is sufficient for most applications [53].

To derive the one – dimensional Schrodinger equation , one starts with the total energy equation (i,e) the sum of kinetic and potential energy $V(x)$.

$$\frac{P^2}{2m} + V(x) = E \tag{2.3.1}$$

Multiply both side by the wave function ψ gives

$$E\psi = \frac{P^2}{2m}\psi + V\psi \tag{2.3.2}$$

Substitution of the dynamical variables by their quantum mechanical operators which act on the wave function $\psi(x,t)$, this yields the one-dimensional time – dependent Schrodinger equation .

The operators were found by using the wave function of a free particle which is given by

$$\psi = Ae^{i(kx-\omega t)} = Ae^{\frac{i}{\hbar}(Px-Et)}$$

Where

$$P = \frac{h}{\lambda} = \hbar k \tag{2.3.3}$$

$$E = hf = \hbar\omega \tag{2.3.4}$$

k and ω are the wave number and angular frequency respectively. Differentiating ψ w.r.t to t and x gives

$$i\hbar\frac{\partial\psi}{\partial t} = E\psi$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi = \frac{P^2}{2m}\psi \tag{2.3.5}$$

Thus substituting (2.3.5) in (2.3.2) gives Schrodinger equation.

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x,t) + V(x)\psi(x,t) = -\frac{\hbar}{i}\frac{d}{dt}\psi(x,t) = E\psi(x,t) \tag{2.3.6}$$

The left side of this equation can be written by using the Hamiltonian or total energy operator .

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x) \tag{2.3.7}$$

By using the notation of the Hamiltonian operator the time-dependent Schrodinger equation can be written as

$$\hat{H}\psi(x,t) = -\frac{\hbar}{i}\frac{\partial}{\partial t}\psi(x,t) \tag{2.3.8}$$

Since the Schrodinger equation is partial differential equation , the product method can be used to separate the equation in to spatial and temporal parts .

$$\psi(x,t) = X(x)T(t) \tag{2.3.9}$$

Where $X(x)$ depends only on (x) and $T(t)$ depend only on (t) .

Insertion of equation (2.3.9) into the Schrodinger equation one gets .

$$\frac{1}{X_{(x)}}\hat{H}X(x) = -\frac{\hbar}{iT_{(t)}}\frac{\partial}{\partial t}T(t)$$

$$\frac{1}{X_{(x)}}\hat{H}X(x) = \frac{i\hbar}{T_{(t)}}\frac{\partial}{\partial t}T(t) \tag{2.3.10}$$

The right side of this equation depends on (t) only , while the left side depends on (x) because (x) and (t) are completely intendment variables.

The equation can be true if the both sides are constant .

$$\frac{i\hbar}{T_{(t)}}\frac{\partial T(t)}{\partial t} = constant$$

$$T(t) = e^{-\frac{i}{\hbar}Et} \tag{2.3.11}$$

Substitution this result into equation (2.3.9) yields to the time- dependent wave function .

$$\psi(x,t) = X(x)e^{-\frac{i}{\hbar}Et} \tag{2.3.12}$$

If E is real , then the wave function has amplitude $X(x)$ and has a phase $e^{-\frac{i}{\hbar}Et}$.

One now consider the case where the potential V , is not a function of time and where , according to classical mechanics , energy is conserved.

If V is time independent one can apply the standard separation of variables ,technique to the Schrödinger equation an in(2.3.9) , for the time independent .

One get

$$-\frac{ħ^2}{2m}\frac{\partial^2}{\partial x^2}X(x) + V(x)X(x) = EX(x) \qquad (2.3.13)$$

The solution of equation (2.3.13) depends on the particular form of $V(x)$. The above equation is Known as the one – dimensional time –independent Schrödinger equation .In the special case of free particle , the origin of potential energy can be chosen so that $V(x) = 0$ and asolution to (2.3.13) is then

$$X(x) = Ae^{ikx} \qquad (2.3.14)$$

Where $K = (2mE/ħ^2)^{1/2}$ and A is a constant . Thus the wave function has the form ; $(\omega = E/ħ)$:

$$\psi = Ae^{i(kx-\omega t)} \qquad (2.3.15)$$

In the case of any closed system , therefore , we can obtain solutions to the time- dependent Schrödinger equation corresponding to a given value of the energy of the system by solving the appropriate time-independent phase factor (2.3.11).

Provided the energy of the system is Known and remains constant (and it is only this case which one shall be considering for the moment) the phase factor , T ,has no physical significance . In particular , one that the probability distribution , $|\psi|^2$, is now indentical to $|X(x)|^2$, so that the normalization condition becomes [54]

$$\int_{-\infty}^{\infty} |X(x)|^2 = 1 \qquad (2.3.16)$$

Dirac Notation :

The physical state of the system represented in quantum mechanic by elements of liner space ,Hilbert space , these elements are called state vector . one can represent the stat vectors in different basics by mean of function expansion. This is analogous to specifying a Euclidean vector by components in various coordinate system .

The manning of a vector is , independent of the coordinate system chooses represents its components .

To free state vector from coordinate meaning Dirac introduced what was to become an invaluable notation in quantum mechanics has denoted the state vector ψ by what he called the Ket vector $|\psi\rangle$, its complex conjugate ψ^* by a bra $\langle\psi|$ and the scalar (inner) product $(\emptyset,\psi)$ by the a bra – Ket $\langle\emptyset|\psi\rangle$:

$$\psi \quad \rightarrow \quad |\psi\rangle \quad , \tag{2.3.17}$$

$$\psi^* \quad \rightarrow \quad \langle\psi| \quad , \tag{2.3.18}$$

$$(\emptyset,\psi) \quad \rightarrow \quad \langle\emptyset|\psi\rangle \quad . \tag{2.3.19}$$

When a Ket (or bra) is multiplied by a complex number , one get also a Ket (or bra).

In the wave mechanics one deal with wave functions $\psi(x,t)$, but in the more general formalism of quantum mechanics one deal with Ket vector $|\psi\rangle$. for every Ket there exists a unique bra and vice versa . So that Ket represents the system completely ,and hence $|\psi\rangle$ means knowing all than on classical concepts had to be used to explain phenomena ate atomic level .[38]

(2.4) Klein – Gordon equation :

Schrodinger equation can not describes very fast particles which moves with speed near to that of light [1] . This is since Schrodinger equation is derived from classical Newtonian energy . To describe fast particles one must use Einstein energy – momentum relation [55 ,56].

$$E^2 = P^2C^2 + {m_0}^2C^4 \tag{2.4.1}$$

Multiply both side by ψ yields

$$E^2\psi = P^2C^2\psi + {m_0}^2C^4\psi \tag{2.4.2}$$

Using equation (2.3.3) yields

$$-\hbar^2\frac{\partial^2\psi}{\partial t^2} = E^2\psi \qquad -\hbar^2\nabla^2\psi = P^2\psi \tag{2.4.3}$$

A direct insertion of (2.4.3) in (2.4.2) gives

$$-\hbar^2 \frac{\partial^2\psi}{\partial t^2} = -C^2\hbar^2\nabla^2\psi + {m_0}^2C^4\psi \qquad (2.4.4)$$

Which is the Klein –Gordon equation .

(2.5) Dirac equation :

Klein – Gordon equation describes successfully spin less particles , but it can not describe particles having spins [57,58]. This motivates to construct linear energy –momentum relation in form

$$E = c\alpha.P + \beta m_0C^2 \qquad (2.5.1)$$

Multiply both sides by ψ to get

$$E\psi = c\alpha.P\psi + \beta m_0C^2\psi \qquad (2.5.2)$$

Using equation (2.3.3) , yields

$$i\hbar\frac{\partial\psi}{\partial t} = E\psi \qquad \frac{\hbar}{i}\nabla\psi = P\psi \qquad (2.5.3)$$

Inserting (2.5.3) in (2.5.2) to get

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{\hbar}{i}\alpha.\nabla\psi + \beta m_0C^2\psi \qquad (2.5.4)$$

Which is the Dirac equation .

(2.6) Harmonic Oscillator :

The potential energy of the harmonic oscillator is given by [59]

$$V = \frac{1}{2}kx^2 \qquad (2.6.1)$$

Thus time independent Schrödinger equation takes the form

$$-\frac{\hbar^2}{2m}\acute{\acute{u}} + \frac{1}{2}kx^2u = Eu \qquad (2.6.2)$$

To simplify this equation , define y and α to satisfy

$$y = \alpha\, x \quad , \quad \left(\frac{mk}{\hbar}\right)^{\frac{1}{4}} = \alpha \qquad (2.6.3)$$

Thus inserting (2.6.3) in (2.6.2) yields

$$\acute{\acute{u}} + (\lambda - y^2)u = 0 \qquad (2.6.4)$$

Where

$$\lambda = \frac{2mE}{\hbar^2\alpha^2} = \frac{2E}{\hbar\omega} \qquad (2.6.5)$$

And

$$k = m\omega^2 \qquad (2.6.6)$$

For classical oscillator . Now try a solution

$$u = He^{-\frac{1}{2}y^2}$$

$$\acute{u} = [\acute{H} - yH]e^{-\frac{1}{2}y^2}$$

$$\acute{\acute{u}} = \left[\acute{\acute{H}} - H - y.\acute{H}\right]e^{-\frac{1}{2}y^2} - y[\acute{H} - yH]e^{-\frac{1}{2}y^2}$$

$$= \left[\acute{\acute{H}} - 2\acute{y} - y^2.H - H\right]e^{-\frac{1}{2}y^2} \qquad (2.6.7)$$

A direct substitution of (2.6.7) in (2.6.4) gives

$$\acute{\acute{H}} = 2y\acute{H} + (\lambda - 1)H = 0 \qquad (2.6.8)$$

Consider now H to be in the form

$$H = \sum a_s\, y^s, \quad \acute{H} = \sum s a_s\, y^{s-1} \quad ,\acute{\acute{H}} = \sum s(s-1)a_s\, y^{s-2} \qquad (2.6.9)$$

Inserting (2.6.9) in (2.6.8) gives

$$\sum s(s-1)a_s\, y^{s-2} + \sum[\lambda - 1 - 2s]a_s\, y^s = 0 \qquad (2.6.10)$$

Thus

$$\sum(s+2)(s+1)a_s + 2y^s + \sum[\lambda - 1 - 2s]a_s\, y^s = 0 \qquad (2.6.11)$$

Equating the coefficients of y^s gives

$$(s+2)(s+1)a_{s+2} + [\lambda - 1 - 2s]a_s = 0$$

hence

$$a_{s+2} = \frac{[2s+1-\lambda]}{(s+1)(s+2)}a_s \qquad (2.6.12)$$

For the wave function to be finite , the thus polynomial must have finite terms . the lest term is n .

Thus

$$H = \sum a_s y^s \qquad , s = n \tag{2.6.13}$$

It follows that

$$a_n \neq 0 \qquad a_{n+1} = 0 \quad , \quad a_{n+2} = 0 \tag{2.6.14}$$

Substitute $(s = n)$ in (2.6.12) to get

$$a_{n+2} = \frac{[2n+1-\lambda]}{(n+1)(n+2)} a_n \tag{2.6.15}$$

In view of equation (2.6.14) , one gets

$$0 = \frac{[2n+1-\lambda]}{(n+1)(n+2)} a_n \tag{2.6.16}$$

This requires

$$2n + 1 - \lambda = 0$$

There fore

$$\lambda = 2n + 1 \tag{2.6.17}$$

Bearing in mind equation (2.6.5)

$$2E = (2n + 1)\hbar\omega \tag{2.6.18}$$

Thus the energy of the harmonic oscillator is given by

$$E = (n + \tfrac{1}{2})\hbar\omega \tag{2.6.19}$$

(2.7) Hamiltonian formalizm 0f quantum laws :

(2.7.1) Lagrange's and Hamilton's equation :

In this sec on one , considers tow reformulation of Newtonian mechanics , the Lagrangian and the Hamiltonian formalism .

The first is naturally associated with configuration space , extended by time . while the latter is the natural description for working in phase space .

Lagrange developed his approach in 1764 in study of the libration of the moon but it is best thought of as general method of treating dynamics in terms of generalized coordinates for configuration space . It so transcends its origin that the Lagranian is considered the fundamental object which describes a quantum field theory [60] .

(2.7.2) Lagrangian for unconstrained systems :

For a collection of particles with conservative forces described by potential we have in inertial Cartesian coordinates [61]

$$m\ddot{x}_i = F_i \tag{2.7.1}$$

The left hand side of this equation is determined by Kinetic energy function as the time derivative of the momentum $P_i = \frac{\partial T}{\partial x_i}$,while the right hand side is derivative of the potential energy ,$-\frac{\partial U}{\partial x_i}$ .As T is independent of $x_i$and U is independent of x_i in these coordinates .we can write both sides in terms of the Lagrangian . $L = T - U$ which is then a function of both the coordinates and there velocities . Thus we have established .

$$\frac{d}{dt}\frac{\partial L}{\partial x_i} - \frac{\partial L}{\partial x_i} = 0 \tag{2.7.2}$$

Which once we generalize it to arbitrary coordinates , will be known as Lagrange's equation .

We assume we have a set of generalized coordinates $\{q_j\}$ which parameterize all of coordinate space , so that each point may be described by the $\{q_i\}$ or by the $\{x_i\}, i , j \in [1, N]$ and thus each set may be thought of as a function of the other and time .

$$q_{j=}q_j(x_1, x_2, \ \dots \ x_{N\ ,}t\ ) \qquad x_i = x_i(q_1\ , q_2, \dots \ q_N, t)\ . \tag{2.7.3}$$

We my consider L as a function of the generalized coordinates $q_j$and $\dot{q}_j$, and ask whether the same expression in these coordinates .

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \tag{2.7.4}$$

Also vanishes . The chain rule tells us

$$\frac{\partial L}{\partial \dot{x}_j} = \sum \frac{\partial L}{\partial q_k}\frac{\partial q_k}{\partial \dot{x}_j} + \sum \frac{\partial L}{\partial \dot{q}_k}\frac{\partial \dot{q}_k}{\partial \dot{x}_j} \qquad (2.7.5)$$

The first term vanishes because q_k depends only on the coordinates x_k and t , but not on the $\dot{x}_k$

$$\dot{q}_j = \sum_i \frac{\partial q_j}{\partial x_i} + \frac{\partial q_j}{\partial t}, \qquad (2.7.6)$$

$$\frac{\partial \dot{q}_j}{\partial \dot{x}_i} = \frac{\partial q_j}{\partial x_i}$$

Using this in (2.7.5)

$$\frac{\partial L}{\partial \dot{x}_i} = \sum_j \frac{\partial L}{\partial \dot{q}_j} + \frac{\partial q_j}{\partial x_i} \qquad (2.7.7)$$

Lagrange's equation involves the time derivative of this .For any function $f(x,t)$ of extended contended configuration space , this total time derivative is

$$\frac{df}{dt} = \sum \frac{\partial f}{\partial x_j}\dot{x}_j + \frac{\partial f}{\partial t} \qquad (2.7.8)$$

Using Leibnitz ,rule on (2.7.7) and using (2.7.8) in the second term we find λ

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = \sum_j (\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j})\frac{\partial q_j}{\partial x_i} + \sum \frac{\partial L}{\partial \dot{q}_j}(\sum \frac{\partial^2 q_j}{\partial x_i \partial x_k}\dot{x}_k + \frac{\partial^2 q_j}{\partial x_i \partial t}) \qquad (2.7.9)$$

On the other hand the chain rule also tells us

$$\frac{\partial L}{\partial x_i} = \sum \frac{\partial L}{\partial q_j}\frac{\partial q_j}{\partial x_i} + \sum \frac{\partial L}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial x_i}$$

Where the last term does not necessarily vanish , as $\dot{q}_j$ in general depends on both the coordinates and velocities . In fact from (2.7.6)

$$\frac{\partial \dot{q}_j}{\partial x_i} = \sum \frac{\partial^2 q_j}{\partial x_i \partial x_k}\dot{x}_k + \frac{\partial^2 q_j}{\partial x_i \partial t}$$

$$\frac{\partial L}{\partial x_i} = \sum_j \frac{\partial L}{\partial q_j}\frac{\partial q_j}{\partial x_i} + \sum_j \frac{\partial L}{\partial q_j}(\sum_k \frac{\partial^2 q_j}{\partial x_i \partial x_k}\dot{x}_k + \frac{\partial^2 q_j}{\partial x_i \partial t} \qquad (2.7.10)$$

Lagrange's equation in Cartesian coordinates says (2.7.9)and (2.7.10) are equal and in subtracting them the second terms cancels so [61]

$$0 = \sum_j \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j}\right) \frac{\partial q_j}{\partial x_i} \qquad (2.7.11)$$

The matrix $\frac{\partial q_j}{\partial x_i}$ is nonsingular as its inverse ,so have derived Lagrange's equation in generalized coordinates

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \qquad (2.7.12)$$

Thus we see that Lagrange's equation are form invariant under changes of the generalized coordinates used to describe the configuration of the system .

(2.7.3) Lagrangian for constrained system :

We now to generalize our discussion to include constraint . At the same time we will also consider possibly neoconservative forces .As we mentioned in section , we often have a system with internal forces whose effect is better understood than the forces them selves , with which we may not be concerned .

We will assume the constraints are holonomic , expressible as k real function $\phi_\alpha(\vec{r}_1, \dots . \vec{r}_n, t) = 0$, which are somehow enforced by constraint forces $\vec{F}_i^c$ on the particles { i } . there may also be other forces , which we will call $\vec{F}_i^D$ and will treat as having a dynamical effect . These are given by known function of the configuration and time , possibly but not necessarily in terms of a potential .

We will assume that the constraint forces in general satisfy this restriction that no net virtual work is done by the forces of constraint for any possible virtual displacement . Newton's law tells us that

$$\vec{P}_i = F_i = \vec{F}_i^c + \vec{F}_i^D . \qquad (2.7.13)$$

We can multiply by an arbitrary virtual displacement

$$\sum_i \left(F_i^D \ \vec{P}_i\right). \delta\vec{r}_i = -\sum_i F_i^c . \delta\vec{r}_i = 0 \qquad (2.7.14)$$

Where the first equality would be true even if $\delta\vec{r}_i$ did not satisfy the constraints but the second requires $\delta\vec{r}_i$ to be an allowed virtual displacement

$$\sum_i \left(F_i^D - \vec{P}_i\right). \delta\vec{r}_i = 0 \qquad (2.7.15\ )$$

Which is known as D,Alembert's principle .

Then $\frac{\partial \vec{r}_i}{\partial q_j}$ is no longer an invertable or even square , matrix , but we still have

$$\Delta \vec{r}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \vec{r}_i}{\partial t} \quad . \tag{2.7.16}$$

But fore a virtual displacement $\Delta t = 0$ we have

$$\delta \vec{r}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j \quad . \tag{2.7.17}$$

Differentiating (2.7.16) we not that

$$\frac{\partial \vec{V}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j}, \tag{2.7.18}$$

$$\frac{\partial \vec{V}_i}{\partial \dot{q}_j} = \sum_k \frac{\partial^2 \vec{r}_i}{\partial q_j \partial q_k} \dot{q}_k + \frac{\partial^2 \vec{r}_i}{\partial q_i \partial t} = \frac{d}{dt} \frac{\partial \vec{r}_i}{\partial q_j} \ , \tag{2.7.19}$$

Where the lest equality comes form applying (2.7.18) with coordinates q_j rather than x_j to $f = \frac{\partial r_i}{\partial q_j}$

The first term in the equation (2.7.16) stating D,Alembert's principle is

$$\sum_i \vec{F}_i \delta \vec{r}_i = \sum_j \sum_i \vec{F}_i \frac{\partial \vec{r}}{\partial q_j} \delta q_i = \sum_j Q_j \delta q_i \tag{2.7.20}$$

The generalized force Q_j has the same form as in the unconstrained case .

The second term of (2.7.19) involves

$$\sum_i \vec{F}_i \delta \vec{r}_i = \sum_i \frac{dP_i}{dt} \frac{\partial \vec{r}_i}{\partial q_j} \ \delta q_j$$

$$= \sum_j \frac{d}{dt} (\sum_i \vec{P}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}) \ \delta q_j - \sum_{i\ j} P_i . (\frac{d}{dt} \frac{\partial \vec{r}_i}{\partial q_j}) \ \delta q_j$$

$$= \sum_j \frac{d}{dt} (\sum_i \vec{P}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j}) \ \delta q_j - \sum_{i\ j} P_i . \frac{\partial \vec{v}_i}{\partial q_j} \ \delta q_j$$

$$= \sum_j \frac{d}{dt} (\sum_i m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j}) \ \delta q_j - \sum_i m_i v_i . \frac{\partial \vec{v}_i}{\partial q_j} \ \delta q_j$$

$$= \sum[\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j}]\delta q_j \tag{2.7.21}$$

Where we used (2.7.18) and (2.7.19) to get the third line . Plugging in the expressions we have found for the two terms in D,Alembert's principle .

$$\sum_j \left[\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - Q_j \right] \delta q_j = 0 \tag{2.7.22}$$

We assumed we had a holonomic system and the g' s were all independent , so this equation holds for arbitrary virtual displacements δq_j , and therefore

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - Q_j = 0 \tag{2.7.23}$$

Now let us restrict ourselves to forces given by a potential , with

$\vec{F}_i - \nabla_\mathrm{i} U(\{\vec{r}\}, t)$, or

$$Q_j = -\sum_j \frac{\partial \vec{r}_i}{\partial q_j} . \vec{\nabla}_\mathrm{i} U = -\frac{\partial U(\{q\},t)}{\partial q_j}\bigg|_t \tag{2.7.24}$$

Notice that Q_j depends only on the value of U on the constrained surface also , U is independent of the q_i ' s , so

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} + \frac{\partial U}{\partial q_j} = 0 = \frac{d}{dt}\frac{\partial (T-U)}{\partial \dot{q}_j} - \frac{\partial (T-U)}{\partial q_j}, \tag{2.7.25}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \tag{2.7.26}$$

This is LaGrange's equation , which we have now derived in the more general context of constrained system .

(2.7.4) Hamilton's Equation :

We have written the Lagrangian as function of q_i , $\dot{q}_i$ and t , so it is a function of $N + N + 1$ variables . For a free particle we can write the Kinetic energy either as $\frac{1}{2}m\dot{x}^2$ or as $P^2/_{2m}$. more generally , we can reexpress the dynamics in terms of the $2N + 1$ variables q_k , p_k *and* t.

The motion of the system sweeps out a path in the space $(q , \dot{q} , t)$ or path in (q , p , t) . Along this line , the variation of L is[62]

$$dL = \sum_k \left(\frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k + \frac{\partial L}{\partial q_k} dq_k \right) + \frac{\partial L}{\partial t} dt$$

$$= \sum_k P_k d\dot{q}_k + \dot{P}_k dq_k + \frac{\partial L}{\partial t} dt \qquad (2.7.27)$$

Where for first term we used the definition of the generalized momentum and in the second we have used the equation of motion $\dot{P}_k = \frac{\partial L}{\partial q_k}$ .Then examining the change in the Hamiltonian $H = \sum_k P_k \dot{q}_k - L$ along this actual motion .

$$dH = \sum_k (P_k d\dot{q}_k + \dot{q}_k dP_k) - dL$$

$$= \sum_k \dot{q}_k d - \dot{p}_k dP_k) - \frac{\partial L}{\partial t} dt \, . \qquad (2.7.28)$$

If we think of $\dot{q}_k$ and H as functions of $q \ and \ P$, and think of H as function of $q \ , P, and \ t$ we see that the physical motion obeys .

$$\dot{q}_k = \left.\frac{\partial H}{\partial p_k}\right|_{q,t} \ , \dot{p}_k = -\left.\frac{\partial H}{\partial q_k}\right|_{p,t} , \qquad \left.\frac{\partial H}{\partial t}\right|_{q,p} = -\left.\frac{\partial L}{\partial t}\right|_{q,\dot{q}} \qquad (2.7.29)$$

The first two constitute Hamilton , equations of motion , which are first order equations for the motion of the point representing the system in phase space .

Let's work out a simple example , the one dimension harmonic oscillator .

Here the kinetic energy is $T = \frac{1}{2} m\dot{x}^2$, the potential energy $U = \frac{1}{2} kx^2$, so $L = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2$,the only generalized momentum is $P = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$, and the Hamiltonian is

$$H = P\dot{x} - L = {P^2}/{m} - \left({P^2}/{2m} - \frac{1}{2} kx^2\right) = {P^2}/{2m} + \frac{1}{2} kx^2 \qquad (2.7.30)$$

Not this is just the sum of the kinetic and potential energies , or the total energy

Hamilton's equation give

$$\dot{x} = \left.\frac{\partial H}{\partial p}\right|_x = \frac{P}{m} \quad , \quad \dot{P} = -\left.\frac{\partial H}{\partial x}\right|_p = -Kx = F \quad . \qquad (2.7.31)$$

These two equations verify the usual connection of the momentum velocity and give Newton's second law .

The identification of H with the total energy is more general than our particular example . If T is purely quadratic in velocities , we can write $T = \frac{1}{2}\sum_{ij} m_{ij}\dot{q}_i\dot{q}_j$ in terms of symmetric mass independent of velocities

$$L = \frac{1}{2}\sum_{ij} m_{ij}\dot{q}_i\dot{q}_j - U(q) \qquad (2.7.32)$$

$$P_k = \frac{\partial L}{\partial \dot{q}_k} = \sum_i m_{ki}\dot{q}_i \qquad (2.7.33)$$

Which as matrix equation in a n- dimensional space is $P = M.\dot{q}$ Assuming M is invertible , we also have $\dot{q} = M^{-1}.P$, so

$$H = P^T\dot{q} - L$$

$$= P^T M^{-1}.P - (\frac{1}{2}\dot{q}^T M\dot{q} - U(q)\ )$$

$$= P^T M^{-1}.P - \frac{1}{2}P^T\ M^{-1}.M.M^{-1}.P + U(q)$$

$$= \frac{1}{2}P^T M^{-1}.P + U(q) = T + U \qquad (2.7.34)$$

So we see that the Hamiltonian is indeed the total energy under these circumstances .

New let's assume L is regular , so

$$\lambda : TQ \xrightarrow{\sim} \quad x\underline{c}\ \dot{T}Q$$

$$(q, \dot{q}) \rightarrow (q, P) \qquad (2.7.35)$$

This lets us have the best of both worlds : we can identify treat q_i , P_i , L , H etc , all as function on x(or TQ) . Thus writing

$$\dot{q}_i(function\ on\ TQ)$$

For the function

$$\dot{q}_i\ o\ \lambda^{-1}(function\ on\ \ x)$$

In particular

$P_i = \frac{\partial L}{\partial q_i}$ (Euler – LaGrange equation)

Which is really a function on TQ , will be treated as a function on x . Now let's calculate

$$dL = \frac{\partial L}{\partial \dot{q}_i} dq_i + \frac{\partial L}{\partial q_i} d\dot{q}_i$$

$$= \dot{P}_i \, dq_i + P_i d\dot{q}_i \qquad (2.7.36)$$

While

$$dH = d(P_i \dot{q}_i - L)$$

$$= \dot{q}_i dP_i + P_i d\dot{q}_i - L$$

$$= \dot{q}_i dP_i + P_i d\dot{q}_i - (\dot{P}_i dq_i + P_i d\dot{q}_i) \qquad (2.7.37)$$

So

$$dH = \dot{q}_i dP_i - \dot{P}_i dq_i) \qquad (2.7.38)$$

Assume the Lagrangian $L = TQ \rightarrow \mathcal{R}$ is regular, so

$$\lambda : TQ \rightrightarrows \quad x\underline{c} \;\; T^*Q$$

$$q, \dot{q}) \rightarrow (q, P)$$

Is a diffeomorphism . This lets us regard both L and the Hamiltonian $H = P_i \dot{q}_i - L$ as function on the phase space x .

And use $(q_i, \dot{q}_i)$als local coordinates on x . As we've seen this gives us

$$dL = \dot{P}_i \, dq_i + P_i d\dot{q}_i$$

$$dH = \dot{P}_i dP_i - \dot{P}_i dq_i)$$

But we can also work out dH directly , this time using local coordinates (q_i, P_i) to get

$$dH = \frac{\partial H}{\partial P_i} dP_i - \frac{\partial H}{\partial q_i} dq_i) \qquad (2.7.39)$$

Since dP_i, dq_i form a basic of 1 – forms , we conclude .

$$\dot{q}_i = \frac{\partial H}{\partial P_i} \qquad P_i = \frac{\partial H}{\partial q_i} \qquad (2.7.40)$$

These are Hamilton's Equation .

(2.7.5) Lagrangian formalism of Schrödinger equation :

The Lagrangian equation for a quantum system described by the wave function ψ is given by

$$\frac{\partial L}{\partial \psi^*} - \frac{\partial}{\partial t}\left(\frac{\partial}{\partial t \partial \psi^*}\right) - \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x \partial \psi^*}\right) = 0 \qquad (2.7.41)$$

Where ψ^* acts as a field variable . For Schrödinger equation consider the Lagrangian

$$L = -i\hbar\dot{\psi}^* - \frac{\hbar^2}{2m}\partial_x\psi^*\partial_x\psi - V\psi^*\psi$$

$$L = -i\hbar\dot{\psi}^*\psi - \frac{\hbar^2}{2m}\partial_x\psi^*\partial_x\psi - V\psi\psi^* \qquad (2.7.42)$$

Differentiating L w.r.t to ψ^* , $\dot{\psi}^*$ gives

$$\frac{\partial L}{\partial \psi^*} = -V\psi$$

$$\frac{\partial}{\partial \partial_t \psi^*} = \frac{\partial L}{\partial \dot{\psi}^*} = -i\hbar\psi \qquad (2.7.43)$$

Thus

$$\frac{\partial L}{\partial t}\left(\frac{\partial L}{\partial \partial_x \psi^*}\right) = -i\hbar\frac{\partial \psi}{\partial t}$$

Also

$$\frac{\partial L}{\partial \partial_x \psi^*} = -\frac{\hbar^2}{2m}\partial_x\psi$$

Hence

$$\partial_x\left(\frac{\partial L}{\partial \partial_x \psi^*}\right) = -\frac{\hbar^2}{2m}\partial^2{}_x\psi \qquad (2.7.44)$$

A direct insertion of (2.7.43) and (2.7.44) in(2.7.41) gives

$$-V\psi + i\hbar\frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2m}\partial_x^2\psi = 0$$

Thus

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\partial_x^2\psi + V\psi \tag{2.7.45}$$

In 3-dimension

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi \tag{2.7.46}$$

Which is the ordinary

(2.8) Principle of least action :

The principle of least action is based on the action integral [62]

$$I = \int_a^b L\, dx_i = 0$$

$$\delta I = \int_a^b \delta L\, \delta x_i = 0 \tag{2.8.1}$$

But the Lagrangian depends on q, $\partial_i q$ and x_i hence :

$$L = L(\, q, \partial_i q\ , x_i) \tag{2.8.2}$$

Thus using calculus of variation

$$\delta L = \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \partial_i q}\partial_i q$$

$$= \frac{\partial L}{\partial q}\delta q + \partial_i\left[\frac{\partial L}{\partial \partial_i q}\partial_i q\right] - \partial_i\left[\frac{\partial L}{\partial \partial_i q}\right]\delta q$$

$$= \left[\frac{\partial L}{\partial q} - \partial_i\left[\frac{\partial L}{\partial \partial_i q}\right]\right]\delta q - \partial_i\left[\frac{\partial L}{\partial \partial_i q}\partial_i q\right] \tag{2.8.3}$$

Inserting (2.8.3) in (2.8.1) gives

$$= \int\left[\frac{\partial L}{\partial q} - \partial_i\left[\frac{\partial L}{\partial \partial_i q}\right]\right]\delta q - \frac{\partial L}{\partial \partial_i q}\delta q(b) + \frac{\partial L}{\partial \partial_i q}\delta q(b)$$

$$= \int\left[\frac{\partial L}{\partial q} - \partial_i\left[\frac{\partial L}{\partial \partial_i q}\right]\right]\delta q dx_i = 0 \tag{2.8.4}$$

Thus

$$\frac{\partial L}{\partial q} - \partial_i\left[\frac{\partial L}{\partial \partial_i q}\right] = 0 \tag{2.8.5}$$

Choosing the field variables to be

$$q = \psi^* \tag{2.8.6}$$

One gets

$$\frac{\partial L}{\partial \psi^*} - \partial_t \left(\frac{\partial L}{\partial \partial_t \psi^*}\right) - \frac{\partial}{x}\left(\frac{\partial L}{\partial \partial_x \psi^*}\right) = 0 \tag{2.8.6}$$

Using the same Lagrangian in section (2.8.5) and the same steps , one gets again Schrödinger equation in the form

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi \tag{2.8.7}$$

(2.9) Gravity ,Curved Space ,and Newtonian Limit :

The space – time interval in a curved space –time is given by [63,64,65]

$$c^2 d\tau^2 = -g_{\mu\gamma} dX^\mu dX^\gamma \tag{2.9.1}$$

Where c , τ , $g_{\mu\gamma}$ X^μ stands for speed of light , proper time , metric and coordinates .

However in .Eucleadian space it takes the form

$$c^2 d\tau^2 = -\zeta_{\mu\gamma} dy^\alpha dy^\beta \tag{2.9.2}$$

Where y depends on x, i.e

$$y^\alpha = y^\alpha(X^\mu) \tag{2.9.3}$$

Thus according to the laws of partial differential equations

$$dy^\alpha = \frac{\partial y^\alpha}{\partial X^\mu}\, dX^\mu \tag{2.9.4}$$

Equation (2.9.3) can be rewritten explicitly in the form

$$y^\alpha(X^\mu) = y^\alpha(X^1, X^2, . . . , X^\mu,) \tag{2.9.5}$$

Where

$$dy^\beta = \frac{\partial y^\beta}{\partial X^\gamma}\, dX^\gamma \tag{2.9.6}$$

Inserting (2.9.4) and (2.9.6)in(2.9.2) gives

$$c^2 d\tau^2 = -\zeta_{\alpha\beta} \frac{\partial y^\alpha}{\partial X^\mu} dX^\mu \frac{\partial y^\beta}{\partial X^\gamma} \; dX^\gamma \tag{2.9.7}$$

Using equation (2.9.1) and(2.9.7) , yields

$$-g_{\mu\gamma} dX^\mu dX^\gamma = -\zeta_{\alpha\beta} \frac{\partial y^\alpha}{\partial X^\mu} \frac{\partial y^\beta}{\partial X^\gamma} \; dX^\mu dX^\gamma \tag{2.9.8}$$

Thus

$$g_{\mu\gamma} = \zeta_{\alpha\beta} \frac{\partial y^\alpha}{\partial X^\mu} \frac{\partial y^\beta}{\partial X^\gamma} \tag{2.9.9}$$

Consider now a freely falling particle and elevator under the action of a gravitational field .

Thus for elevator no acceleration exists . Hence

$$a = \frac{d^2 y^\alpha}{d\tau^2} = 0 \tag{2.9.10}$$

Therefore

$$a = \frac{d^2 y^\alpha}{d\tau^2} = \frac{d}{d\tau} [\frac{dy^\alpha}{d\tau}] = 0 \tag{2.9.11}$$

Inserting x in equation (2.9.11) gives

$$\frac{d}{d\tau} \left[\frac{\partial y^\alpha}{\partial X^\mu} \frac{dX^\mu}{d\tau} \right] = 0 \tag{2.9.12}$$

$$\frac{d}{d\tau} \left[\frac{\partial y^\alpha}{\partial X^\mu} \right] \frac{dX^\mu}{d\tau} + \frac{\partial y^\alpha}{\partial X^\mu} \frac{d^2 X^\mu}{d\tau^2} = 0 \tag{2.9.13}$$

Now let f be defined

$$f = \frac{\partial y^\alpha}{\partial X^\mu} \tag{2.9.14}$$

Where f also depends on x , i.e

$$f = f(X^1, X^2, . \; . \; . , X^\mu, \ldots) \tag{2.9.15}$$

Thus its total differentiation becomes

$$df = \frac{\partial f}{\partial X^\gamma} dX^\gamma \tag{2.9.16}$$

Hence

$$\frac{df}{d\tau} = \frac{\partial f}{\partial X^{\gamma}}\frac{dX^{\gamma}}{d\tau} \tag{2.9.17}$$

Thus equation (2.9.13) becomes

$$\left[\frac{df}{d\tau}\right]\frac{dX^{\mu}}{d\tau} + f\,\frac{d^2X^{\mu}}{d\tau^2} = 0$$

In view of (2.9.17) one gets

$$\left[\frac{\partial f}{\partial X^{\gamma}}\frac{dX^{\gamma}}{d\tau}\right]\frac{dX^{\mu}}{d\tau} + \frac{\partial y^{\alpha}}{\partial X^{\mu}}\frac{d^2X^{\mu}}{d\tau^2} = 0 \tag{2.9.18}$$

Hence

$$\frac{\partial}{\partial X^{\gamma}}\left[\frac{\partial y^{\alpha}}{\partial X^{\mu}}\right]\frac{dX^{\gamma}}{d\tau}\frac{dX^{\mu}}{d\tau} + \frac{\partial y^{\alpha}}{\partial X^{\mu}}\frac{d^2X^{\mu}}{d\tau^2} = 0 \tag{2.9.19}$$

Therefore

$$\left[\frac{\partial^2 y^{\alpha}}{\partial X^{\gamma}\partial X^{\mu}}\right]\frac{dX^{\gamma}}{d\tau}\frac{dX^{\mu}}{d\tau} + \frac{\partial y^{\alpha}}{\partial X^{\mu}}\frac{d^2X^{\mu}}{d\tau^2} = 0 \tag{2.9.20}$$

Multiply both sides of (2.9.20) by

$$\frac{\partial X^{\lambda}}{\partial y^{\alpha}} \tag{2.9.21}$$

Then using the fact that

$$\frac{\partial y^{\alpha}}{\partial X^{\mu}}\frac{\partial X^{\lambda}}{\partial y^{\alpha}} = \delta^{\lambda}_{\mu} = \begin{cases} 1, & \lambda = \mu \\ 0, & \lambda \neq \mu \end{cases} \tag{2.9.22}$$

Equation (2.9.19) becomes

$$\frac{dX^{\gamma}}{d\tau}\frac{dX^{\mu}}{d\tau}\left[\frac{\partial X^{\lambda}}{\partial y^{\alpha}}\frac{\partial^2 y^{\alpha}}{\partial X^{\gamma}\partial X^{\mu}}\right] + \left[\frac{\partial y^{\alpha}}{\partial X^{\mu}}\frac{\partial X^{\lambda}}{\partial y^{\alpha}}\right]\frac{d^2X^{\mu}}{d\tau^2} = 0 \tag{2.9.23}$$

$$\frac{dX^{\gamma}}{d\tau}\frac{dX^{\mu}}{d\tau}\frac{\partial X^{\lambda}}{\partial y^{\alpha}}\frac{\partial^2 y^{\alpha}}{\partial X^{\gamma}\partial X^{\mu}} + \delta^{\lambda}_{\mu}\frac{d^2X^{\mu}}{d\tau^2} = 0 \tag{2.9.24}$$

But

$$\delta^{\lambda}_{\mu}\frac{d^2X^{\mu}}{d\tau^2} = \frac{d^2X^{\lambda}}{d\tau^2} \tag{2.9.25}$$

There fore equation (2.9.24) reduces to

$$\Gamma^{\lambda}_{\mu\gamma}\frac{dX^{\gamma}}{d\tau}\frac{dX^{\mu}}{d\tau} + \frac{d^2X^{\lambda}}{d\tau^2} = 0 \tag{2.9.26}$$

Where

$$\Gamma^{\lambda}_{\mu\gamma} = \frac{\partial X^{\lambda}}{\partial y^{\alpha}} \frac{\partial^2 y^{\alpha}}{\partial X^{\gamma} \partial X^{\mu}} \qquad (2.9.27)$$

$$\Gamma^{\lambda}_{\mu\gamma} = \frac{g^{\upsilon\lambda}}{2} [-g_{\mu\gamma,\upsilon} + g_{\mu\upsilon;\gamma} + g_{\upsilon\gamma,\mu}]$$

$$g_{\mu\gamma,\upsilon} = \frac{\partial g_{\mu\gamma}}{\partial X^{\upsilon}} \qquad (2.9.28)$$

The Riemann Crystofal symbol satisfies

$$\Gamma^{\lambda}_{\mu\gamma} = \Gamma^{\lambda}_{\gamma\mu} \qquad (2.9.29)$$

$$\Gamma^{\lambda}_{\mu\gamma} = 0 \qquad g_{\lambda\mu} = constant \qquad (2.9.30)$$

(2.9.1)Newtonian Limit :

In the Newtonian Limit the velocities

$$\frac{dx}{dt} \quad , \quad \frac{dy}{dt} \quad , \quad \frac{dz}{dt} \qquad (2.9.31)$$

Are very small compared to light speed , where

$$x^0 = ct \qquad x^1 = x$$

$$x^2 = y \qquad x^3 = z \qquad (2.9.32)$$

Therefore

$$\Gamma^{\lambda}_{\mu\gamma} \frac{dx^{\mu}}{dt} \frac{dx^{\gamma}}{dt} = c^2 \Gamma^{\lambda}_{00} (\frac{dt}{dt})^2 + \Gamma^{\lambda}_{11} (\frac{dx}{dt})^2 + \Gamma^{\lambda}_{22} (\frac{dy}{dt})^2 + \Gamma^{\lambda}_{33} (\frac{dz}{dt})^2 \qquad (2.9.33)$$

But since the velocities are small thus one can assume that

$$\frac{dx}{c^2 dt} = 0 \quad , \quad \frac{dy}{c^2 dt} = 0 \quad , \quad \frac{dz}{c^2 dt} = 0 \qquad (2.9.34)$$

Fore $(\lambda = \mu)$equation (2.9.26) becomes

$$\left[\frac{dt}{d\tau}\right]^2 \left[\frac{d^2 X^{\mu}}{dt^2} + c^2 \Gamma^{\lambda}_{00} (\frac{dt}{dt})^2\right] = 0 \qquad (2.9.35)$$

For a particle moving a long the x axis

$$\mu = 1 \quad , \quad X^{\mu} = X^1 = x \qquad (2.9.36)$$

Where

$$\frac{d^2X^\mu}{d\tau^2} = \frac{d^2X^1}{d\tau^2} = \frac{d^2x}{d\tau^2} = \frac{d}{dt}\left[\frac{dx}{dt}\right]\left[\frac{dt}{d\tau}\right]$$

$$= \frac{d}{dt}\left[\frac{dx}{d\tau}\right]\left[\frac{dt}{d\tau}\right] = \frac{d}{dt}\left[\frac{dx}{dt}\frac{dt}{d\tau}\right]\frac{dt}{d\tau} \qquad (2.9.37)$$

$$\frac{d^2X^\mu}{d\tau^2} = \frac{d^2x}{d\tau^2} = \frac{d^2x}{dt^2}\left(\frac{dt}{d\tau}\right)^2 \qquad (2.9.38)$$

thus

$$\frac{d^2x}{dt^2}\left(\frac{dt}{d\tau}\right)^2 + c^2\Gamma_{00}^{\lambda}\left(\frac{dt}{d\tau}\right)^2 = 0 \qquad (2.9.39)$$

Therefore

$$\frac{d^2x}{dt^2} = -c^2\Gamma_{00}^{\lambda} \qquad (2.9.40)$$

When one considers only time and x components

$$\Gamma_{\alpha\beta}^{\lambda} = -\frac{1}{2}g^{\lambda\gamma}\frac{\partial g_{\alpha\beta}}{\partial X^\gamma} \qquad (2.9.41)$$

$$\alpha = 0 \quad , \quad \beta = 0 \qquad (2.9.42)$$

Thus

$$\Gamma_{00}^{\lambda} = -\frac{1}{2}g^{\lambda\gamma}\frac{\partial g_{00}}{\partial X^\gamma} \qquad (2.9.43)$$

For very week field , the metric becomes

$$g_{\lambda\gamma} = y_{\lambda\gamma} + h_{\lambda\gamma} \quad \rightarrow \quad g_{\lambda\gamma} = \zeta_{\lambda\gamma} + h_{\lambda\gamma} \qquad (2.9.44)$$

Where h is very small and much Less than one , and

$$\zeta_{\lambda\gamma} = \mp 1 \qquad (2.9.45)$$

And

$$h_{\mu\gamma} \ll 1 \qquad (2.9.46)$$

$$x^1 = x \quad , \; \mu = 1 \qquad \gamma = 1 \qquad (2.9.47)$$

Thus equation (2.9.43) reads

$$\Gamma_{00}^{\lambda} = \Gamma_{00}^{1} = -\frac{1}{2}g^{11}\frac{\partial g_{00}}{\partial X^1} = -\frac{1}{2}g^{11}\frac{\partial g_{00}}{\partial x} \qquad (2.9.48)$$

For

$$\mu = 0 \quad , \; \gamma = 0 \tag{2.9.49}$$

Equation(2.9.44) becomes

$$g_{00} = \zeta_{00} + \; h_{00} = \zeta_{00} + \; h_{00} \tag{2.9.50}$$

$$\Gamma_{00}^{\lambda} = \Gamma_{00}^{1} = -\frac{1}{2} g^{11} \frac{\partial(\zeta_{00} + h_{00})}{\partial x} = -\frac{1}{2} g^{11} \frac{\partial h_{00}}{\partial x} \tag{2.9.51}$$

Similarly the spatial component can be wirillen as

$$g^{\lambda\gamma} = y^{\lambda\gamma} + h^{\lambda\gamma} \tag{2.9.52}$$

Where

$$\lambda = 1 \quad , \gamma = 1 \tag{2.9.53}$$

Again , one can write

$$g^{11} = \zeta^{11} + h^{11} \tag{2.9.54}$$

Where

$$h^{11} \ll 1 \tag{2.9.55}$$

Thus equation (2.9.51)reads

$$\Gamma_{00}^{\lambda} = -\frac{1}{2}(\zeta^{11} + h^{11}) \frac{\partial h_{00}}{\partial x} = -\frac{1}{2} \zeta^{11} \frac{\partial h_{00}}{\partial x} - \frac{1}{2} h^{11} \frac{\partial h_{00}}{\partial x}) \tag{2.9.56}$$

Since

$$h^{11} \ll 1 \qquad h_{00} \ll 1 \tag{2.9.57}$$

It f0llows that

$$\Gamma_{00}^{\lambda} = -\frac{1}{2} \zeta^{11} \frac{\partial h_{00}}{\partial x} \tag{2.9.58}$$

But

$$\zeta^{11} = 1 \tag{2.9.59}$$

Hence

$$\Gamma_{00}^{\lambda} = -\frac{1}{2} \frac{\partial h_{00}}{\partial x} \tag{2.9.60}$$

Thus equation (2.9.40) reads

$$\frac{d^2x}{dt^2} = c^2\frac{\partial h_{00}}{\partial x} \quad (2.9.61)$$

In 3-dimension

$$\frac{d^2x}{dt^2} = \frac{c^2}{2}\nabla h_{00} \quad (2.9.62)$$

On the other hand Poisson equation takes the form

$$\frac{d^2x}{dt^2} = -\nabla\emptyset \quad (2.9.63)$$

This equation comes from the Newton second law

$$m\frac{d^2x}{dt^2} = F = -\nabla V = -\nabla m\emptyset = -m\nabla\emptyset \quad (2.9.64)$$

Thus comparing equations   and

$$h_{00} = -\frac{2\emptyset}{c^2} \quad (2.9.65)$$

$$\zeta_{00} = -1 \quad (2.9.66)$$

Thus from (2.9.50) one gets

$$g_{00} = \zeta_{00} + h_{00} = -(1 + \frac{2\emptyset}{c^2}) \quad (2.9.67)$$

(2.10) Einstein Gravitational Field Equation :

According to Poisson equation [66,67]

$$\nabla^2\emptyset = 4\pi G\rho \quad (2.10.1)$$

Where ρ is the matter density . Thus from equation (2.9.67) , one get

$$\nabla^2 g_{00} = -\frac{c^2}{2}\nabla\emptyset \quad (2.10.2)$$

Rearranging

$$\nabla^2 g_{00} = -\frac{8\pi G\rho}{c^2} \quad (2.10.3)$$

This equation can be written in a tonsorial form for time –time component to be in the form

$$G_{00} = -\frac{8\pi G}{c^2}T_{00} \quad (2.10.4)$$

More generally , for all components , one gets

$$G_{\mu\gamma} = -\frac{8\pi G}{c^2} T_{\mu\gamma} \quad (2.10.5)$$

Where G , represents the geometrical tensor and T, the matter tensor .

Differentiating w .r .t x^μ gives

$$G_{\mu\gamma;\mu} = -\frac{8\pi G}{c^2} T_{\mu\gamma;\mu} \quad (2.10.6)$$

Since the energy momentum tensor is conserved thus

$$T_{\mu\gamma;\mu} = 0 \quad (2.10.7)$$

thus

$$G_{\mu\gamma;\mu} = 0 \quad (2.10.8)$$

This can be satisfied when

$$G_{\mu\gamma} = R_{\mu\gamma} - \frac{1}{2} g_{\mu\gamma} R \quad (2.10.9)$$

Thus according to equation (2.10.5) Einstein equation

takes the form

$$R_{\mu\gamma} - \frac{1}{2} g_{\mu\gamma} R = -\frac{8\pi G}{c^2} T_{\mu\gamma} \quad (2.10.10)$$

(2.11)Potential Dependent Frictional Schrödinger Equation:

This work is done by treating particles as harmonic oscillator one obtains the friction energy related to the momentum . The energy and the corresponding Newtonian operator is found . This results in a new Schrodinger equation accounting for the effect of friction . This new equation shows that the energy and mass are quantized , if one treat particles as strings the radioactive decay law and collision probability is also derived.[68,69,70]

(2.11.1)Schrodinger equation for frictional medium :

According to Plank and de Broglie hypothesis the quantum quanta are treated as wave packets .

According to string theory matter building blocks are treated as vibrating string . Motivated by this hypothesis , the energy dissipated by friction can be derived consider now a frictional force F_f in terms of mass m , relaxation time τ and velocity v to be [71,72]

$$E_F = \frac{mv}{\tau} \tag{2.11.1}$$

Considering matter building blocks as strings the speed is given by

$$v = v_0 e^{i\omega t} \tag{2.11.2}$$

Thus , the displacement is given by :

$$x = \int v\, dt = v_0 \int e^{i\omega t}\, \mathrm{dt} = \frac{v_0}{i\omega} e^{i\omega t} = \frac{v}{i\omega} \tag{2.11.3}$$

The total dissipative energy E_f is given by :

$$E_F = \int E_f . dx = \frac{m}{\mathrm{i}\omega\tau} \int v\, dv = \frac{mv^2}{2i\omega\tau} = \frac{imv^2}{2i^2\omega\tau} = \frac{-imv^2}{2\omega\tau} = -\frac{i}{\omega\tau}\left(\frac{1}{2}mv^2\right) = \frac{-i}{\omega\tau}\left(\frac{P^2}{2m}\right) \tag{2.11.4}$$

But the total energy can be expressed in terms of the kinetic and potential energy V in the form

$$E = K + V = \frac{P^2}{2m} + \mathrm{V} \tag{2.11.5}$$

Thus according to equation (2.11.5)and equation (2.11.4) E_f is given by

$$E_f = \frac{-i}{\omega\tau}(E - V) \tag{2.11.6}$$

But using plank hypothesis the energy E is given by :

$$E = \hbar\omega \tag{2.11.7}$$

In view of equations(2.11.6)and (2.11.7) the frictional energy is given by

$$E_F = \frac{-i\hbar}{\hbar\omega\tau}(E - V) = \frac{i\hbar}{\tau \mathrm{E}}(E - V)$$

$$E_F = \frac{i\hbar}{\tau}\left(\frac{V}{E} - 1\right) \tag{2.11.8}$$

Thus the Hamiltonian classical relation for a particle in frictional medium is given by

$$E = H = \frac{P^2}{2m} + \mathrm{V} + \frac{i\hbar}{\tau}\left(\frac{V}{E} - 1\right) = \frac{P^2}{2m} + \mathrm{V} + \frac{i\hbar}{\tau}\left(\frac{V-E}{E}\right) \tag{2.11.9}$$

Therefore

$$E^2 = \left(\frac{P^2}{2m} + \mathrm{V}\right)\mathrm{E} + \frac{i\hbar}{\tau}(V - E) \tag{2.11.10}$$

To find the Schrodinger equation corresponding to this relation , one multiplies both sides of equation (2.11.10) by ψ to get :

$$E^2\psi = \left(\frac{P^2}{2m} + \mathrm{V}\right)\mathrm{E}\psi + \frac{i\mathrm{c}}{\tau}(V - E)\psi \tag{2.11.11}$$

Considering the wave function

$$\psi = Ae^{\frac{i}{\hbar}(Px-Et)} \tag{2.11.12}$$

Hence

$$E\psi = i\hbar\frac{\partial\psi}{\partial t} \tag{2.11.13}$$

$$-\hbar^2\frac{\partial^2\psi}{\partial t^2} = E^2\psi \tag{2.11.14}$$

Similarly differentiating the wave function respect to x yields

$$\frac{\partial\psi}{\partial x} = \frac{i}{\hbar}P\psi$$

$$\frac{\hbar}{i}\frac{\partial\psi}{\partial x} = P\psi$$

$$-\hbar^2\frac{\partial^2\psi}{\partial x^2} = -\hbar^2\nabla^2\psi = P^2\psi \tag{2.11.15}$$

Thus inserting equations (2.11.13), (2.11.14)and (2.11.15)into equation (2.11.11) yields

$$-\hbar^2\frac{\partial^2\psi}{\partial x^2} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)i\hbar\frac{\partial\psi}{\partial t} + \frac{i\hbar}{\tau}\left(-i\hbar\frac{\partial\psi}{\partial t} + V\psi\right)$$

$$-\hbar^2\frac{\partial^2\psi}{\partial x^2} = i\hbar\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{\tau}\frac{\partial\psi}{\partial t} + \frac{i\hbar}{\tau}V\psi \tag{2.11.16}$$

(2.11.2)Harmonic oscillator solution :

To see how friction force consider the solution of equation(2.11.12)in the form

$$\psi = e^{-i\frac{E}{\hbar}t}u(v) = f(t)u(v) = fu$$

$$\frac{\partial\psi}{\partial t} = -i\frac{E}{\hbar}fu$$

$$\frac{\partial^2\psi}{\partial t^2} = \frac{i^2E^2}{\hbar^2}fu = -\frac{E^2}{\hbar^2}fu \tag{2.11.17}$$

A direct substitution in equation(2.11.16) gives

$$E^2fu = i\hbar\left(-\frac{\hbar^2}{2m}\nabla^2u + Vu\right)f\left(\frac{-iE}{\hbar}\right) - i\frac{E\hbar^2}{\hbar\tau}fu + i\frac{\hbar}{\tau}Vfu \tag{2.11.18}$$

Dividing both sides of equation (2.11.18)by f yields

$$E^2u = +E\left(-\frac{\hbar^2}{2m}\nabla^2u + Vu\right) - i\frac{E\hbar^2}{\hbar\tau}u + i\frac{\hbar}{\tau}Vu \tag{2.11.19}$$

Dividing both sides of equation (2.11.19)by +E yields

$$\left(E + \frac{i\hbar}{\tau}\right)u = -\frac{\hbar^2}{2m}\nabla^2u + V\left(1 + \frac{i\hbar}{\tau E}\right)u$$

$$-\frac{\hbar^2}{2m}\nabla^2u + c_1Vu = E_1u \tag{2.11.20}$$

Where

$$c_1 = 1 + \frac{i\hbar}{\tau E}$$

$$E_1 = E + \frac{i\hbar}{\tau} \tag{2.11.21}$$

For harmonic oscillator on finds

$$V = \frac{1}{2}kx^2 \tag{2.11.22}$$

Thus substituting this expression in equation(2.11.20) gives

$$-\frac{\hbar^2}{2m}\nabla^2u + c_1\frac{1}{2}kx^2 = E_1u \tag{2.11.23}$$

Let now

$$k_0 = c_1 k \quad (2.11.24)$$

Therefore equation (3.2.23) become

$$-\frac{\hbar^2}{2m}\nabla^2 u + \frac{1}{2}k_0 x^2 = E_1 u \quad (2.11.25)$$

Thus substituting equation(2.11.21)into equation(2.11.25) gives

$$E_1 = \mathrm{E} + \frac{i\hbar}{\tau} = \left(\mathrm{n} + \frac{1}{2}\right)\hbar\omega \quad (2.11.26)$$

$$\mathrm{E} = \left(\mathrm{n} + \frac{1}{2}\right)\hbar\omega - \frac{i\hbar}{\tau} \quad (2.11.27)$$

The frequency is given according to equation (3.2.24)and equation (2.11.21)to be

$$k_0 = m\omega^2$$

$$c_1 k = \left(1 + \frac{i\hbar}{\tau \mathrm{E}}\right) k = m\omega^2$$

$$\left(\mathrm{E} + \frac{i\hbar}{\tau}\right) k = m\omega^2 E \quad (2.11.28)$$

Thus

$$\mathrm{E} = \left(\frac{m\omega^2}{k} - 1\right)^{-1} \frac{i\hbar}{\tau} \quad (2.11.29)$$

From (2.11.27) and(2.11.29)

$$0 = \frac{m\omega^2}{k} + \left(\mathrm{n} + \frac{1}{2}\right)\hbar\omega$$

$$m = \left(1 + \frac{i}{\tau\left(\mathrm{n} + \frac{1}{2}\right)}\right)\frac{k}{\omega^2} \quad (2.11.30)$$

Thus from equation (2.11.30) one find the mass is quantized

(2.11.3) radioactive decay low and collision probability :

Consider equation (2.11.16) for constant potential V_0

Using the separation of variables let the wave function ψ be in the form

$$\psi(r,t) = f(t)u(r) = fu \tag{2.11.31}$$

A direct substitution of equation (2.11.31)in equation (2.11.16)gives

$$-\hbar^2 u\frac{\partial^2 f}{\partial t^2} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_0\right)u\left(i\hbar\frac{\partial f}{\partial t}\right) + \frac{i\hbar}{\tau}\frac{\partial f}{\partial t}V_0 uf + \frac{\hbar^2}{\tau}u\frac{\partial f}{\partial t}$$

Thus

$$\left(-\hbar^2\frac{\partial^2 f}{\partial t^2} - \frac{i\hbar}{\tau}V_0 f - \frac{\hbar^2}{\tau}\frac{\partial f}{\partial t}\right)u = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_0\right)u\left(i\hbar\frac{\partial f}{\partial t}\right) \tag{2.11.32}$$

Divide both sides of equation (2.11.32)by fu to get

$$\left(i\hbar\frac{\partial f}{\partial t}\right)^{-1}\left(-\hbar^2\frac{\partial^2 f}{\partial t^2} - \frac{iV_0\hbar}{\tau}f - \frac{\hbar^2}{\tau}\frac{\partial f}{\partial t}\right) = \frac{1}{u}\left(-\frac{\hbar^2}{2m}\nabla^2 + V_0\right)u = E_0 \tag{2.11.33}$$

Taking the time part of equation(2.11.33)only gives

$$-\hbar^2\frac{\partial^2 f}{\partial t^2} - \frac{iV_0\hbar}{\tau}f - \frac{\hbar^2}{\tau}\frac{\partial f}{\partial t} = i\hbar E_0\frac{\partial f}{\partial t} \tag{2.11.34}$$

Consider the case when the potential vanishes

$$V_0 = 0 \tag{2.11.35}$$

Hence

$$-\hbar^2\frac{\partial^2 f}{\partial t^2} - \frac{\hbar^2}{\tau}\frac{\partial f}{\partial t} = i\hbar E_0\frac{\partial \mathrm{f}}{\partial t} \tag{2.11.36}$$

Consider now a solution

$$\mathrm{f} = Ae^{-\frac{i}{\hbar}Et}$$

$$\frac{\partial \mathrm{f}}{\partial t} = -\frac{i}{\hbar}Et$$

$$\frac{\partial^2 f}{\partial t^2} = +\frac{i^2}{\hbar^2}E^2 f = -\frac{E^2}{\hbar^2}f \tag{2.11.37}$$

Inserting equation (2.11.37) in equation (2.11.36) yields

$$E^2 f + \frac{i\hbar}{\tau}Ef = i\hbar E_0\left(-\frac{i}{\hbar}Ef\right) \tag{2.11.38}$$

Dividing both sides of equation (2.11.38)by gives

$$E^2 + \frac{i\hbar}{\tau}E = E_0 E \tag{2.11.39}$$

Rearranging both sides of equation (2.11.39) gives

$$E^2 = \left(E_0 - \frac{i\hbar}{\tau}\right)E \tag{2.11.40}$$

Dividing both sides of equation (2.11.40)by E gives

$$E = \left(E_0 - \frac{i\hbar}{\tau}\right) \tag{2.11.41}$$

Inserting equation (2.11.41) in equation (2.11.37)gives

$$\mathrm{f} = Ae^{\frac{-i}{\hbar}\left(E_0 - \frac{i\hbar}{\tau}\right)t} = Ae^{-\frac{t}{\tau}}e^{-\frac{i}{\hbar}E_0 t}$$

Hence

$$f = Ae^{-\frac{t}{\tau}}e^{-\frac{i}{\hbar}E_0 t} \tag{2.11.42}$$

Since the probability and number of particles are given by

$$n = |f|^2 = f\,\bar{f} = A^2 e^{\frac{-2t}{\tau}} \tag{2.11.43}$$

Equation (3.2.43) is the ordinary radioactive decay low with

$$\lambda = \frac{2}{\tau} \quad , \quad n_0 = A^2 \tag{2.11.44}$$

i.e.

$$n = n_0 e^{-\lambda t} \tag{2.11.45}$$

This expression also gives collision probability P with

$$p = n \quad , \quad p_0 = A^2$$

$$\tau_0 = {}^{\tau}/_{2} \tag{2.11.46}$$

To get

$$p = p_0 e^{\frac{-t}{\tau_0}} \tag{2.11.47}$$

Equation (2.11.47) is the ordinary collision probability relation

(2.12)Time Dependent Schrödinger Equation for Two Level Systems to Find Traverse Relaxation Time :

This model was made by in this work Schrodinger equation in energy space for two level system was used to find transverse relaxation time . By suggesting sine and cosine beside complex solutions a useful expression for transverse relaxation time was found . When electric interaction dominate , i.e. for dielectric materials the transverse relaxation time depends on the electric dipole moment . However the magnetic materials having magnetic spin and magnetic dipole moment , it depends on the internal filed as well as spin quantum number .[71,73]

$$sin\omega\tau = 1 \tag{2.12.1}$$

$$\omega\tau = \frac{\pi}{2}$$

$$\therefore \tau = \frac{\pi}{2\omega} = \frac{\pi}{4\pi f} = \frac{\pi}{4f} \tag{2.12.2}$$

(2.13) Quantum and Generalized Special Relativistic Model for Electron Charge Quantization :

This research was done by the explanation of electron self-energy and charge quantization. In this work one quantizes electron and elementary particles charges on the basis of electromagnetic Hamiltonian in a curved space –time at vacuum stage of the universe , using quantum spin angular momentum an Klein –Gordon equation beside generalized special relatively . Electron charge is found to be quantized and the electron self-energy is finite . the radius of the electron is also found .[74]

In this case according to generalized special relativity model the electron mass is given by

$$m = \left(1 - \frac{2\emptyset_g}{c^2}\right) \mathrm{m}_0 \tag{2.13.1}$$

Assume for simplicity

$$m = 10^{13}\,\mathrm{m}_0 = 10^{13} \times 9 \times 10^{-31} = 9 \times 10^{-18} kg \tag{2.13.2}$$

Thus

$$\mathrm{r}_0 = \frac{\hbar}{2\mathrm{mc}} = \frac{\mathrm{h}}{4\pi\mathrm{mc}} = \frac{6.63\times10^{-34}}{2\pi 9\times10^{-18}\times3\times10^{8}}$$

$$r_0 = 1.954 \times 10^{-26} m \tag{2.13.3}$$

Which is quite reasonable as far as nucleus or proton radius for very light atoms are

$$r_b = 10^{-14} m \qquad r_p = 10^{-16} m \tag{2.13.4}$$

(2.14)Classical Newtonian Model For Destruction of Superconductors by Magnetic Field :

Newton second law is used to describe the destruction of super conductivity for type 1& type 2.The electron is assumed to be affected by external electric and magnetic field as well as the internal magnetic field. The conductivity and resistance depends on the internal as well as external magnetic field . For type 1 the superconducting state is destroyed when the external magnetic field exceeds the maximum internal field . For type 2 the superconductivity is destroyed partially in the region where the local maximum field is the lowest , and enters completely when the external field exceeds the maximum local internal field .[75]

Thus the maximum produced atomic fields is

$$b_{am} = \frac{\mu i_m}{2r} \tag{2.14.1}$$

Where the maximum current produced is

$$i = \frac{-e z_0 \omega_m}{2\pi} \tag{2.14.2}$$

Where

$$\omega_r = \frac{1}{r}\sqrt{\frac{2E_b}{m}} \tag{2.14.3}$$

Thus the internal field attains maximum value

$$b_{im} = \sum b_{am} \tag{2.14.4}$$

When B exceeds this maximum value in the region of lowest B_i , i.e

$$B > B_{im} \tag{2.14.5}$$

The resistivity will no longer vanishes , where

$$\rho = \frac{\mathrm{B} - \mathrm{B}_{\mathrm{imL}}}{\mathrm{ne}} > 0 \tag{2.14.6}$$

(2.15)Energy Quantization of Electrons for Spherically Symmetric Atoms and Nano Particles According to Schrodinger Equation :

Schrodinger equation for spherical atoms and nano particles was used to describe the behavior of electrons and phonons by treating them as strings oscillating thermally and under the action of external force .The solution shows that for thermally excited phonons and electrons the energy and frequency are quantized . For electrons excited by external force the energy and frequency are also quantized . The energy in both cases resembles the zero point energy for harmonic oscillator of the quantum system .The solution also describes free as well as bounded electrons . The results obtained agree with previous models and observation [76].

$$E_n = \pm \frac{2n\pi\hbar^2}{m^2 d^2} \tag{2.15.1}$$

When

$$\omega_0 = 0$$

$$E = \frac{1}{2}\hbar\omega \tag{2.15.2}$$

But when $\omega = 0$

$$E = \frac{1}{2}\hbar\omega_0 \tag{2.15.3}$$

This represents thermal photons with minimum energy which represents rest mass energy . In view of equation (2.15.1) and (2.15.3) the phonon energy is quantized .

(2.16) Time Independent Generalized Special Relativity Quantum Equation and Travelling Wave Solution:

The effects of fields on physical systems is recognized using the generalized special relativity (GSR) . A new quantum equation Dirac equation consisting of a potential term is derived .[77]

(2.16.1)Potential dependent Dirac quantum equation :

According to GSR model the linear energy is given by

$$E = g_{00}^{1/2}\beta m_0 c^2 + g_{00}^{-1/2} c\alpha.p = \beta m_0 c^2\left(1+\frac{V}{m_0c^2}\right) + c\left(1-\frac{V}{m_0c^2}\right)\alpha.p \qquad (2.16.1)$$

Where

$$g_{00}^{1/2} = 1+\frac{V}{m_0c^2} \quad , \quad g_{00}^{-1/2} = 1-\frac{V}{m_0c^2}$$

Multiplying both side of equation (3.7.1) byψ gives

$$E\psi = c\left(1-\frac{V}{E}\right)\alpha.p\psi + \left(1+\frac{V}{E}\right)\beta m_0c^2\psi$$

$$E^2\psi = cE\left(1-\frac{V}{E}\right)\alpha.p\psi + E\left(1+\frac{V}{E}\right)\beta m_0c^2\psi$$

$$\therefore E^2\psi = c(E-V)\alpha.p\psi + (E+\mathrm{V})\beta m_0c^2\psi$$

$$E^2\psi = c\alpha.pE\psi - cV\alpha.p\psi + \beta m_0c^2E\psi + \beta m_0c^2\mathrm{V}\psi \qquad (2.16.2)$$

Where

$$E \to \hat{\mathrm{H}} = i\hbar\frac{\partial}{\partial t} \; and \quad p = \hat{p} = \frac{\hbar}{i}\vec{\nabla} \qquad (2.16.3)$$

From equation (2.16.2) and (2.16.3)

$$-\hbar^2\frac{\partial^2\psi}{\partial t^2} = c\hbar^2\alpha.\vec{\nabla}\left(\frac{\partial\psi}{\partial t}\right) - cV\alpha\frac{\hbar}{i}\vec{\nabla}\psi + i\hbar\beta m_0c^2\left(\frac{\partial\psi}{\partial t}\right) + \beta m_0c^2V\psi \qquad (2.16.4)$$

From (2.16.4) , by suggesting a solution

$$\psi = u(\underline{r})e^{-i\omega_0 t} = ue^{-i\omega_0 t} = ue^{\frac{iE}{\hbar}t}$$

$$\frac{\partial\psi}{\partial t} = -i\omega_0\psi \quad , \quad \frac{\partial^2\psi}{\partial t^2} = -\omega_0{}^2\psi \qquad (2.16.5)$$

A direct substitution of (2.16.5) in (2.16.4) gives

$$\hbar^2\omega_0{}^2\psi = c\hbar^2\omega_0\alpha.\nabla\psi + ic\hbar\mathrm{V}\alpha\nabla\psi - \beta m_0c^2\hbar\omega_0\psi + \beta m_0c^2V\psi \qquad (2.16.6)$$

,

Where

$\mathrm{E} = \hbar\omega_0 \quad , \; E_0 = m_0 c^2$

$$E^2\psi = c\hbar E\alpha\,\nabla\psi + ic\hbar V\alpha\,\nabla\psi - \beta E_0 E\psi + \beta E_0 \mathrm{V}\psi \tag{2.16.7}$$

$$e^{-i\omega_0 t}(E^2 u) = e^{-i\omega_0 t}\big(c\hbar E\alpha\,\nabla\mathrm{u} + ic\hbar V\alpha\,\nabla\mathrm{u}\big) + e^{-i\omega_0 t}(\beta\mathrm{E} + \beta\mathrm{V})E_0 u$$
, (2.16.8)

The time decaying exponential term can be cancelled on both sides to get

$$(E^2 - \beta(\mathrm{E} + \mathrm{V})E_0)u = c\hbar(E + iV)\,\alpha\nabla\mathrm{u} \tag{2.16.9}$$

This can be written as

$$c_1 u - c_2 V u = c_3\,\alpha\nabla\mathrm{u} + ic_4 V\,\alpha\nabla\mathrm{u} \tag{2.16.10}$$

Where

$$c_1 = E^2 - \beta\mathrm{E}E_0\,,\; c_2 = +\beta E_0\,, c_3 = c\hbar E \quad , c_4 = \mathrm{c}\hbar \tag{2.16.11}$$

(2.16.2) Travelling wave solution

$$\psi = Ae^{i(kr-\omega t)} \tag{2.16.12}$$

But

$$\psi = e^{-i\omega t} \qquad \mathrm{u} = e^{-i\omega t}\mathrm{u}\big(\underline{r}\big) \tag{2.16.13}$$

$$u = Ae^{ikr} \qquad , \nabla u = iku \tag{2.16.14}$$

$$c_1 u - c_2 V u = ik[\alpha](c_3 + ic_4 V)\,\mathrm{u} \tag{2.16.15}$$

Equating coefficients of u and vu yields

$$c_1 = ik.\alpha c_3 \qquad , k = \frac{-ic_1}{c_3\alpha\cos\theta} - ik_0$$

From equation (2.16.11)

$$k = \frac{-i(\mathrm{E}-\beta E_0)}{c_3 h\alpha\cos\theta} - ik_0 \tag{2.16.16}$$

$$c_2 = -i^2 k.\alpha c_4 = k.\alpha c_4 \tag{2.16.17}$$

$$k.\alpha = k\alpha\cos\theta = \frac{c_2}{c_4}$$

$$k = \frac{c_2}{c_4 \alpha \, cos\theta}$$

$$k = \frac{\beta E_0}{c\hbar\alpha \, cos\theta} = k_1$$

The first expression for k in equation (2.16.12) where $k = -ik_0$ gives

$$\psi = Ae^{k_0 r} e^{-i\omega t} \tag{2.16.19}$$

The second expression for k in equation (2.16.12)where $k \rightarrow k_1$ gives

$$\psi = Ae^{i(k_1 r - \omega t)} = u(\underline{r})e^{-i\omega t}$$

$$u(\underline{r}) = u = Ae^{ik_1 r} \tag{2.16.20}$$

Consider the outer most shell where electrons occupy this shell when the radius of the atom is a. in this case

$$|u(a)|^2 = 1$$

$$|u(a)| = 1$$

$$cosk_1 a + isink_1 a = 1 \tag{2.16.21}$$

Thus

$$cosk_1 a = 1 \quad , \quad sink_1 a = 0 \qquad k_1 a = 2\pi n$$

There fore

$$k_1 = \frac{2\pi n}{a} \tag{2.16.22}$$

Thus the momentum is given by

$$\text{p} = \hbar k_1 = \frac{hn}{a} \tag{2.16.23}$$

hence the energy takes the form

$$E^2 = c^2 p^2 + m_0{}^2 c^4 \tag{2.16.24}$$

$$E^2 = \frac{c^2 n^2 h^2}{a^2} + m_0{}^2 c^4 \tag{2.16.25}$$

The liner energy is given by

$$E = c\alpha . p + \beta m_0 c^2$$

$$E = \frac{c\alpha hn}{a} + \beta m_0 c^2 \tag{2.16.26}$$

It is very interesting to not that the velocity is given by

$$V_0 = \lambda_0 f = \frac{\omega}{k_0} = \frac{\omega c_3 \alpha \cos\theta}{c_1} \tag{2.16.27}$$

Becomes infinite when

$$c_1 = E(E - \beta E_0) = 0$$

$$E = \beta E_0 \tag{2.16.28}$$

Where equation (2.16.27) gives

$$V_0 = 0 \tag{2.16.29}$$

In this case equation (2.16.16)gives

$$k_0 = 0 \tag{2.16.30}$$

Thus equation (2.16.19)become in the form

$$\psi = Ae^{i\omega t} \tag{2.16.31}$$

This represents a stationary oscillating wave .Fortunately equations (2.16.29) and (2.16.31) describe the behavior of biophotons which are stationary waves that spread themselves simultaneously through the surrounding media.

(2.17) Describing a Bell and Breathe Solitons by Using Harmonic Oscillator soliton :

Schrodinger harmonic oscillator equation in the momentum space in friction medium (Harmonic Oscillator Soliton model)was used to describe properties of two types of solitons , permanent and time dependent . First one a bell soliton has a permanent profile , while other one is breathers have an internal dynamic , even so ,their shape oscillates in time .[78]

$$\alpha = \frac{E_0}{2c_0} = \frac{E_0\hbar\omega + V_0 + \gamma_0 kT - \frac{i\hbar}{\tau}}{2c_0} = \beta_0 - \frac{i\hbar}{\tau m\hbar^2\omega_0^2}$$

$$\alpha = \beta_0 - \frac{i\hbar}{\tau m c^2\hbar^2\omega_0} = \beta_0 - \frac{i\lambda_0}{\tau m c^2\hbar^2(2\pi f_0\lambda_0)}$$

$$\alpha = \beta_0 - \frac{i\mathrm{k}_0}{\pi\tau \mathrm{c}^2 \mathrm{m}^2 v_0} = \beta_0 - i\gamma x_0 \qquad (2.17.1)$$

Where one assumes $\mathrm{mc}^2 = \hbar\omega_0$

$$\gamma = (\pi\tau \mathrm{c}^2 \mathrm{m}^2 v_0)^{-1} \qquad (2.17.2)$$

The solioton in the momentum space is given by

$$\psi(p,t) = \psi(p)f(t) = Ae^{-i\omega t}e^{i\gamma p^2 x_0}e^{-\beta p^2}$$

$$\psi(p,t) = Ae^{-\beta p^2}e^{i(\gamma p^2 x_0 - \omega t)}e^{i\gamma p^2 x_0} \qquad (2.17.3)$$

(2.18)Relativistic Hamiltonian Formalism in Quantum Field Theory and Micro- Noncausality :

An attempt is made to extend Heisenberg –Pauli 'stheory of quantized fields in a relativistically invariant way . the transformation theory of Dirac is used as a basis for that purpose . It is assumed that a mass variable is canonically conjugate to an invariant –time variable , being a common time to all fields. Considering that the field function in the usual quantum field theory are those expressed in terms of a mass representation , we transform the field functions into those in an invariant –time representation . It is shown that in the new representation a relativistic Hamiltonian formalism of quantum field theory can be obtained and the configuration space method in nonrelativistic case can be generalized .The new formalism is applied to the bound states problem . It shown that ,fore the interaction between two particles in bound states , the condition of micro-noncausality plays an important role . As simple examples , the bound states of deuteron –like and hydrogen –like systems are discussed . For simplicity the new formalism is developedfor charged spin zero fields , but the extensions to other cases is obvious [79].

$$S_1(R,\omega) \sim \left(\frac{R}{|v|}\right)^{\frac{1}{2}} sin\big(|v| \log R + \grave{A}\big), \qquad (2.18.1)$$

$$S_2(R,\omega) \sim \left(\frac{R}{|v|}\right)^{\frac{1}{2}} sos\big(|v| \log R + \grave{A}\big), \qquad (2.18.2)$$

The boundary conditions

$$[S_1(R,0)G](0) = 0 \qquad (2.18.3)$$

$$G(R,\mathfrak{R}) = A(\mathfrak{R})S_2(R,\omega) - B(\mathfrak{R})S_1(R,\omega), \qquad (2.18.4)$$

One fields

$$A(\Re) = \left(\frac{2\Re}{isinh\pi|v|}\right)^{\frac{1}{2}} sin\left[\beta(v) + \grave{A} - |v|\log\frac{\Re}{2i}\right], \tag{2.18.5}$$

$$B(\Re) = \left(\frac{2\Re}{isinh\pi|v|}\right)^{\frac{1}{2}} sos\left[\beta(v) + \grave{A} - |v|\log\frac{\Re}{2i}\right], \tag{2.18.6}$$

$$\mathcal{P} = 2\frac{|\Re Q|^2}{|v|} \tag{2.18.7}$$

(2.19) Solitary wave solutions for nonlinear fractional Schrödinger equation in Gaussian nonlocal media:

This article is devoted to the study of nonlinear fractional Schrodinger equation with a Gaussian nonlocal response . We firstly prove the existence of solitary wave solution by using the variational method and Mountain Pass theorem . Numerical simulations are presented to verify the findings of the existence theorem .And we also investigate the impacts of Gaussian nonlocal response and fractional –order derivatives on the solitary waves , which enable us to perform control experiments for the development of rogue waves in quantum mechanics and optics [80].

$$\Gamma_n = \iint_{R^2d} K_0(|x-y|)|u_n(y)|^P|u_n(x)|^{P-2}u_n(x)\varphi(x)dxdy - \iint_{R^2d} K_0(|x-y|)|u_0(y)|^P|u_0(x)|^{P-2}u_0(x)\varphi(x)dxdy - \iint_{R^2d} K_0(|x-y|)|u_n(y)-u_0(y)|^P|u_n(x)-u_0(x)|^{P-2}(u_n(x)-u_0(x))\varphi(x)dxdy \tag{2.19.1}$$

By a variant of Brezis – Lib's Lemma , we know that $\Gamma_n \to 0$ as $n \to \infty$. Thus we have $\langle \grave{J}\,(u_n - u_0), \varphi\rangle \to 0$ as $n \to \infty$. for a fixed $\lambda \geq 1$, setting $\varphi = u_n - u_0$

$$\|\varphi_n - u_0\|^2 \leq \int_{Rd}\left(\left|(-\Delta)^{\alpha/2}(u_n - u_0)\right|^2 + \lambda|u_n - u_0|^2\right)dx$$

$\langle \grave{J}\,(u_n - u_0), u_n - u_0\rangle + \beta\iint_{R^2d} K_0(|x-y|)|u_n(y)-u_0(y)|^P|u_n(x) - u_0(x)|^P dxdy \leq \langle \grave{J}\,(u_n - u_0), u_n - u_0\rangle + C\left\|(-\Delta)^{\alpha/2}(u_n - u_0)\right\|_{L^2}^{2v} \|u_n - u_0)\|_{L^2}^{2(1-v)} \to 0$, (2.19.2)

As $n \to 0$ thanks to $\langle \grave{J}\,(u_n - u_0), u_n - u_0\rangle \to$and $\|u_n - u_0)\|_{L^2} \to 0$

(2.20) Summary and Critique :

Different attempts were made to modify Schrodinger equation to use it to describe bulk mater[81, 82,83].

Some of them are used describe super conductivity or super fluidity [84,85].

Some modifications are used to describe the early universe to solve some cosmological problems [86,87].

Chapter 3

Methods Theoretical Derivation of Interaction , Schrodinger and Heisenberg Picture Spatial Evolution In a Curved Space- Time

(3.1) Introduction :

The spatial evolution of the system in the interaction picture is deduced , beside the derivation of Schrodinger equation in a curved space time . The physical meaning of the metric is also exhibited here.

(3.2)Interaction picture:

The conventional wave function ψ is related to that of interaction picture ψ_I according to the relation

$$\psi = e^{-iH_0 t}\psi_I$$

$$|\psi\rangle = e^{-iH_0 t}|\psi\rangle_I \tag{3.2.1}$$

ψ_I is the wave function in the interaction picture . The Schrodinger equation is given by

$$\frac{id\psi}{dt} = \hat{H}\psi \tag{3.2.2}$$

By redefining the wave function , one needs the Schrodinger equation to be in terms of H_I only , where H_I represent the interaction Hamiltonian . Inserting (3.2.1) in (3.2.2) the L.H.s gives

$$\frac{ide^{-iH_0 t}\psi_I}{dt} = e^{-iH_0 t}(H_0\,\psi_I + \frac{id\psi_I}{dt}) \tag{3.2.3}$$

but the action of the Hamiltonian on ψ gives

$$\hat{H}\psi = (\frac{-\hbar^2\nabla^2}{2m} + V)\psi$$

$$= (\frac{-\hbar^2\nabla^2}{2m} + V)e^{-iH_0 t}\psi_I \tag{3.2.4}$$

Since the expectation value is the same in all representations , it follows that

$$\langle\psi|_I H_I|\psi\rangle_I = \langle\psi|H|\psi\rangle \tag{3.2.5}$$

In view of equation (3.2.1) the r.h.s of (3.2.5) is given by

$$\langle\psi|H|\psi\rangle = \langle\psi|_I e^{-iH_0 t} H e^{-iH_0 t}|\psi\rangle_I \tag{3.2.6}$$

Comparing (3.2.5) and (3.2.6) yields

$$H_I = e^{-iH_0 t}\hat{H}e^{-iH_0 t} = e^{iH_0 t}(H_0 + H_i)e^{-iH_0 t}$$

$$= e^{iH_0 t}H_0 e^{-iH_0 t} + e^{iH_0 t}H_i e^{-iH_0 t} = e^{iH_0 t}e^{-iH_0 t}H_0 + e^{iH_0 t}H_i e^{-iH_0 t}$$

$$= H_0 + e^{iH_0 t}H_i e^{-iH_0 t} \tag{3.2.7}$$

Where

$$\hat{H} = g_{00}H_0 \tag{3.2.8}$$

Thus form (3.2.6) and (3.2.7)

$$\langle\psi|\hat{H}|\psi\rangle = \langle\psi|_I H_0|\psi\rangle_I + \langle\psi|_I e^{iH_0 t}H_i e^{-iH_0 t}|\psi\rangle_I \tag{3.2.9}$$

In interaction picture , $\hat{H}_0$ should give no contribution . thus

$$\langle\psi|_I H_0|\psi\rangle_I = \langle\psi_I|\hat{H}_0|\psi_I\rangle = 0 \tag{3.2.10}$$

$$\langle\psi|\hat{H}|\psi\rangle = \langle\psi_I|e^{iH_0 t}H_i e^{-iH_0 t}|\psi_I\rangle \tag{3.2.11}$$

Comparing (3.2.11) and (3.2.5) one gets

$$\langle\psi_I|H_I|\psi_I\rangle = \langle\psi_I|e^{iH_0 t}H_i e^{-iH_0 t}|\psi_I\rangle$$

$$H_I = e^{iH_0 t}H_i e^{-iH_0 t}$$

(3.2.1)The Interaction picture for spatial Evolution :

In Schrödinger picture

$$i\hbar\frac{d|\psi\rangle}{dx} = \hat{P}|\psi\rangle \tag{3.2.12}$$

Define the interaction state to be

$$|\psi\rangle_I = U(x)|\psi\rangle = e^{i\hat{P}_0 x}|\psi\rangle$$

$$|\psi\rangle = e^{-i\hat{P}_0 x}|\psi\rangle_I \tag{3.2.13}$$

Since the expectation values in Schrodinger and interaction pictures are equal it follows that

$$\langle\psi|_I \, P_I \, |\psi\rangle_\mathrm{I} = \langle\psi| \, P \, |\psi\rangle \tag{3.2.14}$$

$$\langle\psi|_I \, P_I \, |\psi\rangle_\mathrm{I} = \langle\psi|_I \, \mathrm{e}^{\mathrm{i}p_0\mathrm{x}} \, P \, \mathrm{e}^{-\mathrm{i}p_0\mathrm{x}}|\psi\rangle_\mathrm{I} \tag{3.2.15}$$

If one assumes that

$$P = P_0 + \mathrm{P}_i \tag{3.2.16}$$

Thus

$$\langle\psi|_I P_I|\psi\rangle_I = \langle\psi|_I \, e^{iP_0x} \, P_0 \, e^{-iP_0x}|\psi\rangle_I + \langle\psi|_I \, e^{iP_0x} \, P_i \, e^{-iP_0x}|\psi\rangle_I$$

$$= \langle\psi|_I \, P_0 \, |\psi\rangle_I + \langle\psi|_I \, e^{iP_0x} \, P_i \, e^{-iP_0x}|\psi\rangle_I \tag{3.2.17}$$

Assuming that

$$\langle\psi|_I \, P_0 \, |\psi\rangle_\mathrm{I} = 0 \tag{3.2.18}$$

$$\langle\psi|_I P_I|\psi\rangle_I = \langle\psi|_I \, e^{iP_0x} \, P_i \, e^{-iP_0x}|\psi\rangle_I \tag{3.2.19}$$

Thus

$$P_\mathrm{I} = e^{iP_0x} \, P_i e^{-iP_0x} \tag{3.2.20}$$

This relation can also be fond by using equation (3.2.12),(3.2.13) and (3.2.16) to get

$$i\hbar \frac{d e^{-\mathrm{iP}x}|\psi\rangle_I}{dt} = (\; P_0 + \; P_\mathrm{i})e^{-iP_0x}|\psi\rangle_I \tag{3.2.21}$$

For $\quad \hbar = 1$

$$i(-i \; P_0 e^{-iP_0x}|\psi\rangle_I) + i\mathrm{e}^{-iP_0\mathrm{x}}\frac{d|\psi\rangle_I}{dt} = \; P_0 e^{-\mathrm{iP}x}|\psi\rangle_I + \; P_i e^{-iP_0x}|\psi\rangle_I$$

$$P_0 e^{-iP_0x}|\psi\rangle_I + i \, e^{-iP_0x}\frac{d|\psi\rangle_I}{dt} = \; P_0 e^{-iP_0x}|\psi\rangle_I + \; P_i e^{-iP_0x}|\psi\rangle_I$$

$$i \, e^{-iP_0x}\frac{d|\psi\rangle_I}{dt} = \; P_i e^{-iP_0x}|\psi\rangle_I \tag{3.2.22}$$

Multiplying both sides by e^{iP_0x} gives

$$i\frac{d|\psi\rangle_I}{dt} = \; e^{iP_0x}P_i e^{-iP_0x}|\psi\rangle_I \tag{3.2.23}$$

But the formal expression for wave function evolution in the interaction picture is

$$i\frac{d|\psi\rangle_I}{dx} = P_I|\psi\rangle_I \qquad (3.2.24)$$

Thus comparing equations (3.2.23) and (3.2.24) gives

$$P_{\mathrm{I}} = \mathrm{e}^{\mathrm{i}P_0\mathrm{x}}\,P_i\mathrm{e}^{-\mathrm{i}P_0\mathrm{x}} \qquad (3.2.25)$$

$$i\hbar\frac{d}{dx}|\psi\rangle_I = P_I|\psi\rangle_I \qquad (3.2.26)$$

Let now the eigen vector be in the form

$$|\psi\rangle_I = \mathrm{U}|\psi\rangle_0 \qquad (3.2.27)$$

Where at $t = t_0$

$$P_I = 0 \qquad\qquad |\psi\rangle_I = |\psi\rangle_0 \qquad (3.2.28)$$

Thus form (3.2.24)

$$\mathrm{U} = \mathrm{U}(t_0) = \mathrm{I}$$

Hence equations (3.2.26) and (3.2.28) gives

$$i\hbar\frac{d|\psi\rangle_0}{dt} = 0 \qquad (3.2.29)$$

Thus

$$|\psi\rangle_0 = constant$$

i.e it is independent of x . A direct substitution of (3.2.27) in (3.2.26) gives

$$i\hbar\frac{d\,U|\psi\rangle_0}{dx} = P_I U|\psi_0\rangle \qquad (3.2.30)$$

Thus

$$i\hbar\frac{d\,U}{dx} = P_I U$$

$$\int_{x_0}^{x_1} d\,U = \frac{1}{i\hbar}\int_{x_0}^{x_1} P_I U dt$$

$$U(x_1) - U(x_0) = \frac{1}{i\hbar}\int_{x_0}^{x_1} P_I U dt \qquad (3.2.31)$$

For $t \approx t_0$, one can write the U of the integrand to be

$$U(x_1) = U(x_0) - \frac{i}{\hbar}\int_{t_0}^{t_1} P_I \mathrm{U}(t_0)\mathrm{dx}'$$

$$U(x_1) = I - \frac{i}{\hbar}\int_{x_0}^{x_1} P_I \mathrm{dx}' \qquad (3.2.32)$$

Consider now the next point where

$x_2 > x_1$ $\qquad$ $x_2 \approx x_1$

$$\int_{x_1}^{x_2} \mathrm{dU} = -\frac{i}{\hbar}\int_{x_1}^{x_2} P_I \mathrm{Udx}''$$

$$U(x_2) - U(x_1) = -\frac{i}{\hbar}\int_{x_1}^{x_2} P_I \mathrm{Udx}''$$

$$U(x_2) = U(x_1) - \frac{i}{\hbar}\int_{x_1}^{x_2} P_I(\mathrm{x}'')\left[I - \frac{i}{\hbar}\int_{x_0}^{x_1} P_I(\mathrm{x}')\mathrm{dx}'\right]\mathrm{dx}''$$

$$= U(x_1) + (-i)\int_{x_1}^{x_2} P_I(\mathrm{x}'')\mathrm{dx}'' + \frac{(-i)^2}{\hbar^2}\int_{x_2}^{x_1}\int_{x_0}^{x_1} P_I(\mathrm{x}'')\, P_I(\mathrm{x}')\mathrm{dx}'\mathrm{dx}''$$
(3.2.33)

(3.3)Energy –momentum relation and Eiegen equations in a curved space time :

The energy within the framework of the GSR and SR are given by

$$g_{00}E^2 = g_{xx}P^2C^2 + g_{00}{m_0}^2C^4 \,,\; E_0^2 = P_0^2C^2 + {m_0}^2C^4 \qquad (3.3.1)$$

Where is the ordinary SR energy .

Thus the GSR energy E is given by

$$E = g_{00}^{-1/2}E_0 \qquad (3.3.2)$$

The wave function in the curved space is thus

$$\psi = Ae^{\left(\frac{i}{\hbar}\right)(Px_c - Et_c)} \qquad (3.3.3)$$

Energy Eigen equation and time independent Schrodinger equation in the Euclidean space takes the form

$$i\hbar\frac{\partial\psi}{\partial t} = E_0\psi \qquad (3.3.4)$$

Also the momentum Eigen equation in the Euclidian space is given by

$$\frac{\hbar}{i}\nabla\,\psi = \frac{\hbar}{i}\frac{\partial\psi}{\partial x} = \mathrm{P}_0\psi \tag{3.3.5}$$

In a curved space GSR Wave function for free particle is given by

$$\psi = Ae^{\left(\frac{i}{\hbar}\right)(\sqrt{g_{xx}}Px-\sqrt{g_{00}}Et)} \tag{3.3.6}$$

Where $dt_c = \sqrt{g_{00}}dt \qquad dx_c = \sqrt{g_{xx}}dx$

Schrodinger equation in the curved space , where the time is denoted by t_c , can read

$$i\hbar\frac{\partial\psi}{\partial t_c} = i\hbar\frac{\partial\psi}{\partial\sqrt{g_{00}}dt} = \frac{i\hbar}{\sqrt{g_{00}}}\left[\frac{i\hbar}{\sqrt{g_{00}}}\frac{\partial\psi}{\partial t} = \frac{-i}{\hbar}\sqrt{g_{00}}E\psi\right] = \frac{\sqrt{g_{00}}}{\sqrt{g_{00}}}E\psi = E\psi \tag{3.3.7}$$

Thus

$$i\hbar\frac{\partial\psi}{\partial t_c} = E\psi \tag{3.3.8}$$

But form (3.3.3)

$$i\hbar\frac{\partial\psi}{\partial t_c} = E\psi \tag{3.3.9}$$

This is completely consistent with equation(3.3.8) . conversely from (3.3.6), (3.3.9) and (3.3.2)

$$i\hbar\frac{\partial\psi}{\partial t_c} = i\hbar\frac{\partial\psi}{\sqrt{g_{00}}\partial t} = \mathrm{E}\psi$$

$$i\hbar\frac{\partial\psi}{\partial t} = \sqrt{g_{00}}\mathrm{E}\psi = E_0\psi$$

Thus

$$\sqrt{g_{00}}\mathrm{E} = E_0 \tag{3.3.10}$$

which agrees with equation (3.3.4)and (3.3.2)

The momentum Eigen equation for the momentum in Euclidean space takes the form

$$\frac{\hbar}{i}\frac{\partial\psi}{\partial x} = P_0\psi \tag{3.3.11}$$

In curved space , the momentum Eigen equation becomes

$$\frac{\hbar}{i}\frac{\partial\psi}{\partial x_c} = P\psi$$

With

$$dx_c = \sqrt{g_{xx}}dx \qquad (3.3.12)$$

Thus

$$\frac{\hbar}{i}\frac{\partial\psi}{\sqrt{g_{xx}}\partial x} = P\psi \qquad (3.3.13)$$

Thus , one can write

$$\frac{\hbar}{i}\frac{\partial}{\partial x}\psi = \sqrt{g_{xx}}P\psi \qquad (3.3.14)$$

Comparing this relation with (3.3.11)

$$P_0 = \sqrt{g_{xx}}P \qquad (3.3.15a)$$

Where

$$P_0\psi = \sqrt{g_{xx}}P\psi \qquad (3.3.15b)$$

Thus equation (3.3.14) and (3.3.15a) gives

$$\frac{\hbar}{i}\frac{\partial}{\partial x}\psi = P_0\psi \qquad (3.3.15c)$$

This is the ordinary momentum Eigen equation in the Euclidian space.

The velocity in a curved space is define to be

$$\mathrm{v} = \frac{\mathrm{d}x_c}{\mathrm{d}t_c}$$

$$\mathrm{v} = \frac{\sqrt{g_{xx}}}{\sqrt{g_{00}}}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\sqrt{g_{xx}}}{\sqrt{g_{00}}}\mathrm{v}_0 \qquad (3.3.16)$$

But the momentum in curved space an Euclidean

$$P = mv$$

$$P_0 = m_0 v_0 \qquad (3.3.17)$$

Using equation (3.3.15)

$$P = (g_{xx})^{-1/2} P_0$$

Thus

$$mv = (g_{xx})^{-1} m_0 v_0$$

$$m = \frac{\sqrt{g_{00}}}{\left(\sqrt{g_{xx}}\right)^2} m_0 = \frac{\sqrt{g_{00}}}{g_{xx}} m_0 \qquad (3.3.18)$$

Since in driving GSR , one assumes that

$$g_{xx} = 1 \qquad (3.3.19)$$

It follows that

$$m = \sqrt{g_{00}} m_0 \qquad (3.3.20)$$

But the mass in GSR is given by

$$m = \frac{g_{00} m_0}{\sqrt{g_{00} - v^2/c^2}} \qquad (3.3.21)$$

For mass at rest

$$v = 0$$

$$\mathrm{m} = \frac{\mathrm{g}_{00}\mathrm{m}_0}{\sqrt{\mathrm{g}_{00}}} = \sqrt{\mathrm{g}_{00}}\,\mathrm{m}_0 \qquad (3.3.22)$$

This relation is consistent with equation (3.3.20)

To find the expression , which relates E to P in a curved space ---time on uses the relation

$$c^2 dt^2 = c^2 g_{00} dt_0^2 - g_{xx} dx^2$$

$$\gamma^{-1} = \left(\frac{d\tau}{dt}\right) = \left[g_{00} - g_{xx}\frac{v_0^2}{c^2}\right]^{\frac{1}{2}} \qquad (3.3.23)$$

Thus

$$E = mc^2 = g_{00}\gamma \mathrm{m}_0 = \frac{g_{00}\mathrm{m}_0 c^2}{\sqrt{g_{00} - g_{xx}\frac{v_0^2}{c^2}}} \qquad (3.3.24)$$

But from(3.3.16)

$$g_{xx} v_0^2 = g_{00} v^2$$

$$E = \frac{g_{00}\mathrm{m}_0 c^2 E}{\sqrt{g_{00}E^2 - g_{xx}P^2c^2}}$$

$$g_{00}E^2 - g_{00}P^2c^2 = g_{00}^2 m_0^2 c^4$$

$$g_{00}E^2 = g_{00}P^2c^2 + g_{00}^2 m_0^2 c^4 \qquad (3.3.25)$$

Setting

$$E_0^2 = g_{00}E^2 \quad , \; P_0^2 = g_{00}P^2 \qquad , \quad \widetilde{\mathrm{m}}_0 = g_{00}\mathrm{m}_0 \qquad (3.3.26)$$

One gets

$$E_0^2 = P_0^2 c^2 + \widetilde{\mathrm{m}}_0^2 c^4 \qquad (3.3.27)$$

However , when one replaces v_0 by v in equation (3.3.23), one gets

$$\gamma^{-1} = \left[g_{00} - g_{xx}\frac{v_0^2}{c^2}\right]^{\frac{1}{2}} \qquad (3.3.28)$$

As a result , energy becomes

$$E = \frac{g_{00}\mathrm{m}_0 c^2}{\sqrt{g_{00}E^2 - g_{xx}v^2/c^2}} = \frac{g_{00}\mathrm{m}_0 c^2}{\sqrt{\frac{g_{00}m^2C^4 - g_{xx}m^2\,v^2C^2}{m^2C^4}}}$$

$$= \frac{g_{00}\mathrm{m}_0 c^2}{\sqrt{\frac{g_{00}E^2 - g_{xx}P^2\,C^2}{E^2}}}$$

$$E = \frac{g_{00}\mathrm{m}_0 c^2 E}{\sqrt{g_{00}E^2 - g_{xx}P^2c^2}}$$

$$g_{00}E^2 - g_{xx}P^2c^2 = g_{00}^2 m_0^2 c^4$$

$$g_{00}E^2 = g_{xx}P^2c^2 + g_{00}^2 m_0^2 c^4 \qquad (3.3.29)$$

By setting

$$E_0^2 = g_{00}E^2 \qquad , P_0^2 = g_{xx}P^2$$

$$\mathrm{E}_0 = g_{00}^{\frac{1}{2}}E \qquad , \qquad \mathrm{P}_0 = \sqrt{g_{xx}}\,P \qquad (3.3.30)$$

This relation agrees with (3.3.10) and (3.3.15a) one gets

$$E_0^2 = P_0^2 c^2 + \widetilde{m}_0^2 c^4 \qquad (3.3.31)$$

Where

$$\tilde{\mathrm{m}}_0 = g_{00}\mathrm{m}_0 \tag{3.3.32}$$

The energy of harmonic oscillator in a curved space can be found from the Schrodinger equation in a curved space which is given by

$$i\hbar\frac{\partial\psi}{\partial t_c} = E\psi \qquad\qquad i\hbar\frac{\partial\psi}{\partial\sqrt{g_{00}}\,t} = E\psi$$

$$i\hbar\frac{\partial\psi}{\partial t} = \sqrt{g_{00}}\,E\psi = E_0\psi \tag{3.3.33}$$

For harmonic oscillator in Euclidean space

$$E_0 = \left(n + \frac{1}{2}\right)\hbar\omega \tag{3.3.34}$$

$$(1+x)^n \approx 1 + nx$$

For $x < 1$

$$\frac{2\emptyset}{c^2} = x \tag{3.3.35}$$

$$(g_{00})^{-\frac{1}{2}} = \left(1 - \left(-\frac{1}{2}\right)\frac{2\emptyset}{c^2}\right)$$

$$(g_{00})^{-\frac{1}{2}} = \left(1 + \frac{\emptyset}{c^2}\right) \tag{3.3.36}$$

This approximation is justifiable since $\frac{2\emptyset}{mc^2} < 1 \quad 2\emptyset < mc^2$

Which means that the total energy is the greater than potential energy . Thus equation (3.3.30) gives

$$E = E_0(g_{00})^{-\frac{1}{2}} = E_0\left(1 + \frac{\emptyset}{c^2}\right) \tag{3.3.37}$$

$$E = \left(n + \frac{1}{2}\right)\left(1 + \frac{\emptyset}{c^2}\right)\hbar\omega$$

$$E = \left(n + \frac{1}{2}\right)\hbar\omega + \frac{\emptyset}{c^2}\left(n + \frac{1}{2}\right)\hbar\omega \tag{3.3.38}$$

$$E = E_0\left(1 + \frac{\emptyset}{c^2}\right) = E_0\left(1 + \frac{\mathrm{m}_0\emptyset}{\mathrm{m}_0 c^2}\right) = E_0\left(1 + \frac{\mathrm{V}_0}{\mathrm{E}_0}\right) = \mathrm{E}_0 + \mathrm{V}_0 \tag{3.3.39}$$

(3.4) Time evolution of quantum system within the frame work of generalized special relativity :

The energy in generalized special relativity is given by

$$E = mc^2 = \frac{g_{00}\mathrm{m}_0 c^2}{\sqrt{g_{00} - v^2/c^2}} = \frac{g_{00}\mathrm{E}_0}{\sqrt{g_{00} - v^2/c^2}} \tag{3.4.1}$$

For very small velocity compared to the speed of light

$v \ll c$

Thus

$$E = g_{00}^{\frac{1}{2}}\mathrm{E}_0 = g_{00}^{\frac{1}{2}}\mathrm{E}_0 \tag{3.4.2}$$

Using the fact that ($v < E, m\varphi < c^2 m, \varphi < c^2$)

$$g_{00}^{\frac{1}{2}} = (1 + 2\,{}^{\varphi}/_{c^2})^{\frac{1}{2}}$$

$$E = \left(1 + \frac{\emptyset}{c^2}\right) E_0 = \left(1 + \frac{m_0\emptyset}{m_0 c^2}\right) E_0$$

$$\mathrm{E} = \mathrm{E}_0 + \frac{\mathrm{vE}_0}{\mathrm{E}_0} = \mathrm{E}_0 + \mathrm{v} \tag{3.4.3}$$

Thus the corresponding Hamiltonian is given by

$$\widehat{H} = \widehat{H}_0 + \widehat{H}_i \tag{3.4.4}$$

Where $\widehat{H}_0$ standing for the unperturbed Hamiltonian , while $\widehat{H}_i$ represents the interaction Hamiltonian which causes perturbation .

To explain equation (3.2.10)and to simplify treatment ,it is convenient to modify Schrodinger equation . This modification requires the time evolution of the wave equation to be in terms of the interaction Hamiltonian instead of the total Hamiltonian . This requires

$$|\psi\rangle = \mathrm{e}^{\frac{-i}{\hbar}\hat{\mathrm{H}}_0\mathrm{t}}|\psi\rangle_I$$

$$|\psi\rangle_I = \mathrm{e}^{\frac{i}{\hbar}\hat{\mathrm{H}}_0\mathrm{t}}|\psi\rangle$$

$$|\psi\rangle = |\psi\rangle_I \mathrm{e}^{\frac{i\hat{\mathrm{H}}_0\mathrm{t}}{\hbar}} \tag{3.4.5}$$

$$i\hbar\frac{d|\psi\rangle}{dt} = \hat{\mathrm{H}}|\psi\rangle \tag{3.4.6}$$

$$\hat{\mathrm{H}} = \hat{\mathrm{H}}_0 + \hat{\mathrm{H}}_i \tag{3.4.7}$$

$$i\hbar \frac{d|\psi\rangle}{dt} = \hat{\mathrm{H}}|\psi\rangle$$

$$i\hbar \frac{d}{dt}\left[\mathrm{e}^{\frac{-i\hat{\mathrm{H}}_0\mathrm{t}}{\hbar}}|\psi\rangle_I\right] = \hat{\mathrm{H}}|\psi\rangle \tag{3.4.8}$$

$$i\hbar\left[\mathrm{e}^{\frac{-i\hat{\mathrm{H}}_0\mathrm{t}}{\hbar}}\left(\frac{-i\hat{\mathrm{H}}_0}{\hbar}\right)|\psi\rangle_I + \frac{d|\psi\rangle_I}{dt}\right] = \hat{\mathrm{H}}|\psi\rangle$$

$$=\hat{\mathrm{H}}_0\mathrm{e}^{\frac{-i\hat{\mathrm{H}}_0\mathrm{t}}{\hbar}}|\psi\rangle_I + \hat{\mathrm{H}}_0\mathrm{e}^{\frac{-i\hat{\mathrm{H}}_0\mathrm{t}}{\hbar}}|\psi\rangle_I + i\hbar\mathrm{e}^{\frac{-i\hat{\mathrm{H}}_0\mathrm{t}}{\hbar}}\frac{d|\psi\rangle_I}{dt}$$

$$= \left(\hat{\mathrm{H}}_0 + \hat{\mathrm{H}}_i\right)\mathrm{e}^{\frac{-i\hat{\mathrm{H}}_0\mathrm{t}}{\hbar}}|\psi\rangle_I$$

$$= \hat{\mathrm{H}}_0\mathrm{e}^{\frac{-i\hat{\mathrm{H}}_0\mathrm{t}}{\hbar}}|\psi\rangle_I + i\hbar\mathrm{e}^{\frac{-i\hat{\mathrm{H}}_0\mathrm{t}}{\hbar}}\frac{d|\psi\rangle_I}{dt} = \hat{\mathrm{H}}_0\mathrm{e}^{\frac{-i\hat{\mathrm{H}}_0\mathrm{t}}{\hbar}}|\psi\rangle_I + \hat{\mathrm{H}}_i\mathrm{e}^{\frac{-i\hat{\mathrm{H}}_0\mathrm{t}}{\hbar}}|\psi\rangle_I \tag{3.4.9}$$

Cancelling terms and multiplying both side by $\mathrm{e}^{\frac{i\hat{\mathrm{H}}_0\mathrm{t}}{\hbar}}$ gives

$$i\hbar\frac{d|\psi\rangle_I}{dt} = \mathrm{e}^{\frac{i\hat{\mathrm{H}}_0\mathrm{t}}{\hbar}}\hat{\mathrm{H}}_i\mathrm{e}^{\frac{-i\hat{\mathrm{H}}_0\mathrm{t}}{\hbar}}|\psi\rangle_I \tag{3.4.10}$$

To simplify this equation it is convenient to define operator

$$\hat{\mathrm{H}}_I = \mathrm{e}^{\frac{i\hat{\mathrm{H}}_0\mathrm{t}}{\hbar}}\hat{\mathrm{H}}_i\mathrm{e}^{\frac{-i\hat{\mathrm{H}}_0\mathrm{t}}{\hbar}} \tag{3.4.11}$$

Inserting equation(3.4.10) in equation (3.4.11) yields

$$i\hbar\frac{d|\psi\rangle_I}{dt} = \hat{\mathrm{H}}_I|\psi\rangle_I \tag{3.4.12}$$

Which is ordinary Schrodinger equation in the interaction representation .

This equation can also be derived by bearing in mind that the expect value is the same in Schrödinger and interaction picture . i.e

$$\langle\psi_I|H|\psi_I\rangle = \langle\psi|_I H_I|\psi\rangle_I \tag{3.4.13}$$

In view of equation (3.4.5) , (3.4.7) and (3.4.11) one gets

$$\langle\psi_I|\hat{\mathrm{H}}|\psi_I\rangle = |\psi\rangle_I\mathrm{e}^{\frac{-i\hat{\mathrm{H}}_0\mathrm{t}}{\hbar}}\left(\hat{\mathrm{H}}_0 + \hat{\mathrm{H}}_i\right)\mathrm{e}^{\frac{-i\hat{\mathrm{H}}_0\mathrm{t}}{\hbar}}\langle\psi|_I$$

$$|\psi\rangle_I\mathrm{e}^{\frac{-i\hat{\mathrm{H}}_0\mathrm{t}}{\hbar}}\hat{\mathrm{H}}_0\mathrm{e}^{\frac{-i\hat{\mathrm{H}}_0\mathrm{t}}{\hbar}}\langle\psi|_I + |\psi\rangle_I\mathrm{e}^{\frac{-i\hat{\mathrm{H}}_0\mathrm{t}}{\hbar}}\hat{\mathrm{H}}_i\mathrm{e}^{\frac{-i\hat{\mathrm{H}}_0\mathrm{t}}{\hbar}}\langle\psi|_I$$

$$\langle\psi_I|\hat{\mathrm{H}}_0|\psi_I\rangle + \langle\psi|_I\hat{\mathrm{H}}_I|\psi\rangle_I \tag{3.4.14}$$

This means that for equation (3.4.14) and (3.4.13) to be typical to each other , the expectation value in Schrodinger picture . This can be satisfied only when

$$\hat{\mathrm{H}}_0|\psi\rangle_I = 0$$

Thus

$$\langle\psi_I|\hat{\mathrm{H}}_0|\psi_I\rangle = 0 \tag{3.4.15}$$

Which conforms with equation (3.2.10)

This consistent with the fact that in the interaction picture the original Hamiltonian is absorbed in the wave vector and disappear as an energy operator according to this transformation .

$$|\psi\rangle \to |\psi\rangle_I = \mathrm{e}^{\frac{i\hat{\mathrm{H}}_0\mathrm{t}}{\hbar}}|\psi\rangle$$

$$\hat{\mathrm{H}} = \hat{\mathrm{H}}_0 + \hat{\mathrm{H}}_i \to H_I = \mathrm{e}^{\frac{i\hat{\mathrm{H}}_0\mathrm{t}}{\hbar}}\hat{\mathrm{H}}_i\mathrm{e}^{\frac{-i\hat{\mathrm{H}}_0\mathrm{t}}{\hbar}} \tag{3.4.16}$$

This is equivalent to make

$$\hat{\mathrm{H}}_0 \to 0 \tag{3.4.17}$$

Thus it is quite natural to have

$$\hat{\mathrm{H}}_0 \to 0 \Longrightarrow \hat{\mathrm{H}}_0|\psi\rangle_I = 0 \tag{3.4.18}$$

This explains equation (3.2.10)

(3.5) Momentum perturbation equation in the interaction picture:

The momentum operator is related to the spatial differential change according to the relation

$$\hat{P} = \frac{\hbar}{i}\vec{\nabla} \tag{3.5.1}$$

In one dimension

$$\hat{P} = \frac{\hbar}{i}\frac{\partial}{\partial x} \tag{3.5.2}$$

To see how the momentum operator look like in a curved space time , one uses the expression for x and t in a curved space time for velocity , i . e .

$$v=\frac{dx_c}{dt_c}=\frac{\sqrt{g_{xx}}dx}{\sqrt{g_{00}}dt} \qquad (3.5.3)$$

Where

$$dx_c=\sqrt{g_{xx}}dx \quad , \quad dt_c=\sqrt{g_{00}}dt \qquad (3.5.4)$$

But in schwardzcheid solution and special relativity

$$g_{xx}=g_{00}^{-1} \qquad (3.5.5)$$

Where

$$dt=\sqrt{g_{00}}dt_0 \qquad , \qquad dx=\sqrt{g_{xx}}dx_0$$

$$g_{xx}=\gamma^2=\left(1-{v^2}/{c^2}\right) \qquad (3.5.6)$$

Thus equation (3.5.3) and (3.5.5) gives

$$v=\frac{1}{g_{00}}\frac{dx}{dt}=\frac{1}{g_{00}}v_0 \qquad (3.5.7)$$

Which is the expression for the velocity in a curved space –time .

The momentum in curved space –time takes the form

$$P=mv \qquad (3.5.8)$$

Where the mass is given by

$$m=\frac{g_{00}\mathrm{m}_0}{\sqrt{g_{00}-v_i^2/c^2}} \qquad (3.5.9)$$

Inserting equation (3.5.7) in (3.5.9) yields

$$P=mv=\frac{g_{00}\mathrm{m}_0 v_0}{g_{00}\sqrt{g_{00}-v_i^2/c^2}}=P_0\left(g_{00}-v_i^2/c^2\right)^{-\frac{1}{2}} \qquad (3.5.10)$$

Where the momentum in Euclidean free space is given by

$$P_0=m_0\mathrm{v}_0 \qquad (3.5.11)$$

Bearing in mind that for weak fined

$$g_{00} = \left(1 + \frac{2\emptyset}{c^2} - v_i^2/c^2\right)$$

The momentum is given by

$$P = P_0\left(1 + \frac{2\emptyset}{c^2} - v_i^2/c^2\right)^{-\frac{1}{2}}$$

This expression relates momentum in a curved space –time to that in Euclidean space .

Since the potential is less than the total energy

$\mathrm{v}_0 < E_0$

$m_0\emptyset < m_0c^2$

Therefore

$$\frac{\emptyset}{c^2} < 1 \tag{3.5.12}$$

Similarly the kinetic energy is also less than the total energy . Hence

$\frac{1}{2}m_0v_i^2 < m_0c^2$

$$v_i^2 < c^2 \tag{3.5.13}$$

As a result

$$\left(1 - \frac{2\emptyset}{c^2} - v_i^2/c^2\right)^{-\frac{1}{2}} = \left(1 - (-\frac{1}{2})\frac{2\emptyset}{c^2} - \frac{1}{2}\frac{v_i^2}{c^2}\right)$$

$$= \left(1 + \frac{\left(m_0\emptyset - \frac{1}{2}m_0v_i^2\right)}{m_0c^2}\right)$$

$$= \left(1 + \frac{(T_0 - \mathrm{v}_0)}{E_0}\right) = \left(1 + \frac{L_0}{E_0}\right) \tag{3.5.14}$$

Where the free space Lagrangian is defined to be

$$L_0 = T_0 - \mathrm{v}_0 \tag{3.5.15}$$

Hence , the curved space operator can be written as sum of perturbed and non – perturbed momentum in form

$$\hat{P} = \hat{P}_0\left(1 + \frac{L_0}{E_0}\right) = \hat{P}_0 + \hat{P}_i \tag{3.5.16}$$

Where the perturbed momentum is given by

$$P_i = \frac{P_0 L_0}{E_0} \tag{3.5.17}$$

To explain equation (3.2.18), the Schrodinger Hamiltonian is related to the interaction one according to the relation

$$|\psi\rangle = \mathrm{e}^{\frac{iP_0 x}{\hbar}} |\psi\rangle_I$$

$$\langle\psi| = \langle\psi|_I \, \mathrm{e}^{\frac{-iP_0 x}{\hbar}} \tag{3.5.18}$$

The spatial evaluation of the system is related to momentum operator according to the relation

$$\frac{\hbar}{i}\frac{d}{dx}|\psi\rangle = \hat{P}|\psi\rangle \tag{3.5.19}$$

In view of equation (3.5.18) and equation (3.5.19)

$$\frac{\hbar}{i}\frac{d}{dx}\mathrm{e}^{\frac{i\hat{P}_0 x}{\hbar}}|\psi\rangle_I = \frac{\hbar}{i}\left[\frac{i}{\hbar}\hat{P}_0 \mathrm{e}^{\frac{i\hat{P}_0 x}{\hbar}}|\psi\rangle_I + \mathrm{e}^{\frac{i\hat{P}_0 x}{\hbar}}\frac{d}{dx}|\psi\rangle_I\right]$$

$$= \left(\hat{P}_0 + \hat{P}_\mathrm{i}\right)\mathrm{e}^{\frac{i\hat{P}_0 x}{\hbar}}|\psi\rangle_I \hat{P}_0 \mathrm{e}^{\frac{i\hat{P}_0 x}{\hbar}}|\psi\rangle_I + \frac{\hbar}{i}\mathrm{e}^{\frac{i\hat{P}_0 x}{\hbar}}\frac{d}{dx}|\psi\rangle_I$$

$$= P_0 \mathrm{e}^{\frac{i\hat{P}_0 x}{\hbar}}|\psi\rangle_I + P_\mathrm{i}\mathrm{e}^{\frac{i\hat{P}_0 x}{\hbar}}|\psi\rangle_I$$

Multiply both sides by $\mathrm{e}^{\frac{-i\hat{P}_0 x}{\hbar}}$ one gets

$$\frac{\hbar}{i}\frac{\partial}{\partial x}|\psi\rangle_I = \mathrm{e}^{\frac{-i\hat{P}_0 x}{\hbar}} P_\mathrm{i} \mathrm{e}^{\frac{i\hat{P}_0 x}{\hbar}}|\psi\rangle_I$$

$$\frac{\hbar}{i}\frac{\partial}{\partial x}|\psi\rangle_I = \hat{P}_\mathrm{I}|\psi\rangle_I \tag{3.5.20}$$

Where

$$P_\mathrm{I} = \mathrm{e}^{\frac{-i\hat{P}_0 x}{\hbar}} P_\mathrm{i} \mathrm{e}^{\frac{i\hat{P}_0 x}{\hbar}} \tag{3.5.21}$$

The mathematical form of equation (3.5.21) can also be found using the fact that the expectation values are the same in all representations .

Thus

$$\langle\psi|\hat{P}|\psi\rangle = \langle\psi|\hat{P}_\mathrm{I}|\psi\rangle_I \tag{3.5.22}$$

With the aid of equation (3.5.16) (3.5.21) and (3.5.22) one gets

$$\langle\psi|_I\, e^{\frac{-iP_0x}{\hbar}}\left(\hat{P}_0+\hat{P}_\mathrm{i}\right)e^{\frac{i\hat{P}_0x}{\hbar}}\,|\psi\rangle_I$$

$$=\langle\psi|_I\, e^{\frac{-iP_0x}{\hbar}}\hat{P}_0 e^{\frac{i\hat{P}_0x}{\hbar}}\,|\psi\rangle_I+\langle\psi|_I\, e^{\frac{-iP_0x}{\hbar}}\hat{P}_\mathrm{i} e^{\frac{i\hat{P}_0x}{\hbar}}\,|\psi\rangle_I$$

$$=\langle\psi|_I\,\hat{P}_0\,|\psi\rangle_I+\langle\psi|_I\,\hat{P}_\mathrm{I}\,|\psi\rangle_I \qquad (3.5.23)$$

Equation (3.5.23) should be typical to (3.5.22) this requires

$$\langle\psi|_I\,\hat{P}_0\,|\psi\rangle_I=0 \qquad (3.5.24)$$

One can prove this by bearing that in the interaction picture

$$|\psi\rangle\to|\psi\rangle_I=e^{\frac{-i\hat{P}_0x}{\hbar}}|\psi\rangle$$

$$\hat{P}=\hat{P}_0+\hat{P}_\mathrm{i}\to\hat{P}_\mathrm{I}=e^{\frac{-iP_0x}{\hbar}}P_\mathrm{i}e^{\frac{i\hat{P}_0x}{\hbar}} \qquad (3.5.25)$$

In view of equation (3.5.20) and (3.5.16) it is clear that $\hat{P}_0$ gives no contribution to the equation of motion . Thus as if

$$\hat{P}_\mathrm{i}\to 0 \qquad (3.5.26)$$

Thus

$$P_0|\psi\rangle_I=0|\psi\rangle_I=0 \qquad (3.5.27)$$

Hence is equations(3.5.23) becomes

$$\langle\psi|\hat{P}|\psi\rangle=\langle\psi|_I\,\hat{P}_\mathrm{I}\,|\psi\rangle_I \qquad (3.5.28)$$

Which is typical to equation (3.5.22).

(3.6) Spatial evolution of unitary operator :

The spatial evolution of the wave function in the wave vector space takes the form

$$\frac{\hbar}{i}\frac{d}{dx}|\psi\rangle_I=\hat{P}_\mathrm{I}\,|\psi\rangle_I \qquad (3.6.1)$$

The unitary operator $\mathfrak{U}$ can be defined to be

$$|\psi\rangle_I=\hat{\mathfrak{U}}\,|\psi\rangle_0 \qquad (3.6.2)$$

Where the stationary wave vector is defined to satisfy

$x = x_0 = 0$

$P_{\mathrm{I}} = 0$ (3.6.3)

$|\psi(x)\rangle_I = |\psi\rangle_I = |\psi(x = 0)\rangle_I = |\psi\rangle_0$ (3.6.4)

$|\psi(t = 0)\rangle_I = |\psi\rangle_0 = \widehat{\mathfrak{U}}\,(0)\,|\psi_0\rangle$ (3.6.5)

Hence

$\mathfrak{U}_0 = \mathfrak{U}(0) = \mathrm{I}$ (3.6.6)

But since at

$x = x_0 = 0$

$P_{\mathrm{I}} = 0$ (3.6.7)

It is follow that

$\frac{\hbar}{i}\,\frac{d|\psi_0\rangle}{dx} = \frac{\hbar}{i}\,\frac{d|\psi\rangle_0}{dx} = 0\,|\psi\rangle_0$ (3.6.8)

Thus

$|\psi\rangle_0 = constant$ (3.6.9)

Inserting (3.6.2) and(3.6.9) in (3.6.1) gives

$\frac{\hbar}{i}\,\frac{d}{dx}\mathfrak{U}(x)|\psi_0\rangle = P_I\mathfrak{U}(x)|\psi_0\rangle$ (3.6.10)

Therefore

$\frac{\hbar}{i}\,\frac{d}{dx}\mathfrak{U} = P_I\mathfrak{U}$ (3.6.11)

Thus using iterated integral method and approximation , the zeroth , first ,second orders of $\mathfrak{U}$ are given by

$\int_{x_0}^{x_1} d\mathfrak{U} = \frac{i}{\hbar}\int_{x_0}^{x_1} P_I\mathfrak{U}dx$

Where

$\mathfrak{U}(x_1) - \mathfrak{U}(x_0) = \frac{i}{\hbar}\int_{x_0}^{x_1} P_I(x)\,\mathfrak{U}(x)\,dx$ (3.6.12)

When

$$x_1 > x_0 \quad , \quad x_1 \approx x_0 \tag{3.6.13}$$

$$\mathfrak{U}(x_1) = \mathfrak{U}(x_0) + \frac{i}{\hbar}\int_{x_0}^{x_1} P_I(x_0)\,\mathfrak{U}(x_0)\,dx_0 = \mathfrak{U}(x_0) + I_0 \tag{3.6.14}$$

Similarly

$$\int_{x_1}^{x_2} d\mathfrak{U} = \frac{i}{\hbar}\int_{x_1}^{x_2} P_I\mathfrak{U}dx \tag{3.6.15}$$

$$\mathfrak{U}(x_2) - \mathfrak{U}(x_1) = \frac{i}{\hbar}\int_{x_1}^{x_2} P_I(x)\,\mathfrak{U}(x)\,dx \tag{4.6.16}$$

When

$$x_2 > x_1 \quad , \quad x_2 \approx x_1 \tag{3.6.17}$$

$$\mathfrak{U}(x_2) = \mathfrak{U}(x_1) + \frac{i}{\hbar}\int_{x_1}^{x_2} P_I(x_1)\,\mathfrak{U}(x_1)\,dx_1$$

$$\mathfrak{U}(x_2) = \mathfrak{U}(x_1) + \frac{i}{\hbar}\int_{x_1}^{x_2} P_I(x_1)\,[\mathfrak{U}(x_0) + I_0]\,dx_1$$

$$\mathfrak{U}(x_2) = \mathfrak{U}(x_0) + I_0 + \frac{i}{\hbar}\int_{x_1}^{x_2} P_I(x_1)[\mathfrak{U}(x_0) + I_0]\,dx_1 \tag{3.6.18}$$

(3.7) Heisenberg Picture :

In Heisenberg picture the time evolution of the system is described by operators instead of the wave function (wf) . Thus one needs changing Schrödinger equation

$$i\hbar\frac{d\psi(t)}{dx} = H\,\psi(t)$$

Or

$$i\hbar\frac{d|\psi\rangle}{dt} = H|\psi\rangle \tag{3.7.1}$$

Where

$$\hat{\mathrm{H}} = \hat{\mathrm{H}}_0 + \hat{\mathrm{H}}_i \qquad\qquad \psi(t) = \psi(\underline{\mathrm{r}}\,, t) \tag{3.7.2}$$

To shift time dependence of ψ to be that of the operator Ô by defining

$$|\psi(t)\rangle = \mathfrak{U}(t)|\psi(0)\rangle = \mathfrak{U}|\psi_0\rangle = \mathfrak{U}|\psi\rangle_0 = \mathfrak{U}|\psi\rangle_H$$

$$\psi(t) = \psi_s \qquad\qquad \psi_0 = \psi_H \qquad (3.7.3)$$

S= Schrödinger H= Heisenberg

Thus the wave function in the Schrödinger picture and Heisenberg picture are related according to the relation

$$|\psi\rangle_s = \mathfrak{U}(t)|\psi\rangle_H = \mathfrak{U}(t)|\psi\rangle_0$$

$$\psi_s = \mathfrak{U}(t)\psi_H = \mathfrak{U}(t)\psi_0 \qquad (3.7.4)$$

Two methods can be used to find Heisenberg equation of motion . In the simplest one , one uses the fact that the expectation values in all representation take the same form . Thus the expectation of the operator $\hat{\mathrm{O}}$ are equal , i.e

$$\langle\psi|_H \,\hat{\mathrm{O}}_{\mathrm{H}}\, |\psi\rangle_{\mathrm{H}} = \langle\psi|_s \mathrm{O}_{\mathrm{s}}\, |\psi\rangle_{\mathrm{s}}$$

$$\langle\psi|_H \,\mathfrak{U}^{-1} O_{\mathrm{s}}\, \mathrm{U}|\psi\rangle_{\mathrm{H}} \qquad (3.7.5)$$

$$\hat{\mathrm{O}}_{\mathrm{H}} = \mathfrak{U}^{-1} O_{\mathrm{s}}\, \mathfrak{U} \qquad (3.7.6)$$

To find U , one must solve (3.7.1) to get

$$\frac{d|\psi\rangle}{|\psi\rangle} = \frac{\hat{H}}{i\hbar} dt$$

$$\int \frac{d|\psi\rangle}{|\psi\rangle} = \int \frac{\hat{H}}{i\hbar} dt \qquad (3.7.7)$$

$$ln|\psi\rangle = \frac{-i}{\hbar}\int \hat{H}\, dt + c_0$$

$$|\psi\rangle = \mathrm{c}_1 \mathrm{e}^{\frac{-i}{\hbar}\int \hat{H}\, dt} = \mathrm{e}^{\frac{-i}{\hbar}\int \hat{H}\, dt}\mathrm{c}_1 \qquad (3.7.8)$$

But at $t = 0$

$$|\psi(t)\rangle = |\psi(0)\rangle = |\psi\rangle_0 = c_1$$

$$|\psi\rangle = \mathrm{e}^{\frac{-i}{\hbar}\int \hat{\mathrm{H}}\, dt}|\psi\rangle_0 \qquad (3.7.9)$$

From (4.7.3) it follows that

$$\mathfrak{U}(t) = \mathrm{e}^{\frac{-i}{\hbar}\int \hat{\mathrm{H}}\, dt} \qquad (3.7.10)$$

from (3.7.1) and (3.7.3)

$$i\hbar \frac{d\mathfrak{U}}{dt} |\psi\rangle_0 = \hat{\mathrm{H}}\, \mathfrak{U}|\psi\rangle_0 \tag{3.7.11}$$

Since the wave vector is time independent it follows that

$$i\hbar \frac{d\mathfrak{U}}{dt} = \hat{\mathrm{H}}\, \mathfrak{U}$$

$$i\hbar \int \mathfrak{U} = \int \hat{\mathrm{H}}\, dt$$

$$\int \frac{d\mathfrak{U}}{U} = \int \frac{\hat{\mathrm{H}}}{i\hbar}\, dt \tag{3.7.12}$$

$$\ln \mathfrak{U} = \frac{1}{i\hbar} \int \hat{\mathrm{H}}\, dt \tag{3.7.13}$$

To find the second term consider again the action of the integral on ψ_0 , to get

$$\int \hat{\mathrm{H}}\, dt|\psi_0\rangle = \int i\hbar \frac{d}{\mathrm{dt}} dt|\psi_0\rangle$$

$$= i\hbar \int \frac{d|\psi_0\rangle}{\mathrm{dt}} dt \quad = i\hbar|\psi_0\rangle \tag{3.7.15}$$

Assume that

$$\int \hat{\mathrm{H}}\, dt = \hat{\mathrm{H}} \int \mathrm{dt} = \hat{\mathrm{H}}\,\mathrm{t} \tag{3.7.16}$$

Thus

$$\int \hat{\mathrm{H}}\, dt|\psi_0\rangle = (\hat{\mathrm{H}}t)|\psi_0\rangle = \left(i\hbar \frac{d}{\mathrm{dt}} t\right) |\psi_0\rangle = i\hbar|\psi_0\rangle \tag{3.7.17}$$

Thus the assumption (3.7.16) is true .

Hence

$$\ln \mathfrak{U} = \frac{\hat{\mathrm{H}}t}{i\hbar} + C_0$$

$$\mathfrak{U}(t) = \mathfrak{U} = \mathrm{e}^{C_0} \mathrm{e}^{\frac{-i\hat{\mathrm{H}}t}{\hbar}} \tag{3.7.18}$$

To find C_0, one can use relation (3.7.3), Where at t=0

$$|\psi(0)\rangle = \mathfrak{U}(0)|\psi(0)\rangle$$

Thus

$$\mathfrak{U}\,(0) = I \tag{3.7.19}$$

According to equation (3.7.18)

$$\mathfrak{U}(0) = \mathfrak{U} = e^{C_0} \tag{3.7.20}$$

There fore

$$\mathfrak{U}(t) = e^{\frac{-i\hat{H}\,t}{\hbar}} \tag{3.7.21}$$

In this work plank constant ħ is assumed to be unity to get

$$\mathfrak{U}_H(t) = \mathfrak{U}(t) = e^{\frac{-i\hat{H}\,t}{\hbar}} = e^{-iH\,t} \tag{3.7.22}$$

Where the subscript H is used to differentiate it from that in the interaction picture

In view of equations (3.7.6) and (3.7.21)

$$\hat{O}_H = e^{\frac{i\hat{H}\,t}{\hbar}}\hat{O}_s e^{\frac{-i\hat{H}\,t}{\hbar}}$$

Differentiating both sides with respect to time yields

$$\frac{d\hat{O}_H}{dt} = \frac{-i}{\hbar}\hat{H}e^{\frac{i\hat{H}\,t}{\hbar}}\hat{O}_s e^{\frac{-i\hat{H}\,t}{\hbar}} + e^{\frac{i\hat{H}\,t}{\hbar}}\frac{\partial\hat{O}_s}{\partial t}e^{\frac{-i\hat{H}\,t}{\hbar}} - \frac{-i}{\hbar}e^{\frac{i\hat{H}\,t}{\hbar}}\hat{O}_s e^{\frac{-i\hat{H}\,t}{\hbar}}$$

$$= \frac{i}{\hbar}\hat{H}\hat{O}_H - \frac{i}{\hbar}\hat{O}_H\hat{H} + \left(\frac{\partial\hat{O}_s}{\partial t}\right)_H = \frac{i}{\hbar}\left[\hat{H}\,,\hat{O}_H\right] + \left(\frac{\partial\hat{O}_s}{\partial t}\right)_H \tag{3.7.23}$$

Comparing this equation (3.7.9)with equation (3.7.3), yields

$$\mathfrak{U} = e^{\frac{-i}{\hbar}\int\hat{H}\,t} = e^{\frac{-i}{\hbar}g}$$

$$\mathfrak{U}^{-1} = e^{\frac{-i}{\hbar}g} \tag{3.7.24}$$

Where

$$g = \int\frac{dg}{dt}dt = \int Hdt$$

$$\frac{dg}{dt} = H \tag{3.7.25}$$

This equation (3.7.25) together with equation (3.7.6) can be used to obtain the time evolution equation of the Heisenberg operator .

Therefore , ones get:

$$\frac{\mathrm{d}o_{\mathrm{H}}}{\mathrm{dt}}=\frac{\mathrm{d}}{\mathrm{dt}}[\mathfrak{U}^{-1}O_{\mathrm{s}}\,\mathfrak{U}]=\frac{\mathrm{d}}{\mathrm{dt}}\left[\mathrm{e}^{\frac{i}{\hbar}g}\mathrm{oe}^{\frac{-i}{\hbar}g}\right]$$

$$=\frac{i}{\hbar}\frac{dg}{dt}\mathrm{e}^{\frac{i}{\hbar}g}\mathrm{Oe}^{\frac{-i}{\hbar}g}+g\mathrm{e}^{\frac{i}{\hbar}g}\frac{do}{dt}\mathrm{e}^{\frac{-i}{\hbar}g}-\frac{i}{\hbar}\mathrm{e}^{\frac{i}{\hbar}g}\mathrm{Oe}^{\frac{-i}{\hbar}g}\frac{dg}{dt}$$

$$\frac{i}{\hbar}[H\;O_H\;,O_HH]+(\frac{\partial O_s}{\partial t})_H$$

$$\frac{dO_H}{dt}=\frac{i}{\hbar}[H,O_H]+(\frac{dO}{dt})_H \tag{3.7.26}$$

On other

$$\left(\int\hat{H}dt\right)|\psi_0\rangle=\int i\hbar\frac{d}{\mathrm{dt}}dt|\psi\rangle_0=i\hbar\int d|\psi\rangle_0=i\hbar|\psi\rangle_0 \tag{3.7.27}$$

But the same result (3.7.27) can be found if one proposes that

$$\hat{H}=i\hbar\frac{d}{\mathrm{dt}}$$

To be out of the integration sign to get

$$\int\hat{H}dt|\psi\rangle_0=\hat{H}\int dt|\psi\rangle_0=\;i\hbar\frac{d}{\mathrm{dt}}\int dt|\psi_0\rangle=\int dt|\psi\rangle_0$$

$$i\hbar\frac{d}{\mathrm{dt}}t|\psi\rangle_0=i\hbar t|\psi\rangle_0 \tag{3.7.28}$$

Thus comparing(3.7.27)and (3.7.28) yields

$$\int\hat{H}dt=\hat{H}\int dt=\;i\hbar$$

$$\hat{H}=\;i\hbar \tag{3.7.29}$$

Thus from (3.7.9), (3.7.24)and(3.7.29)

$$\mathfrak{U}_0=\mathfrak{U}(t=0)=\mathrm{e}^{\frac{-i}{\hbar}\int H\,dt}=\mathrm{e}^0=\mathrm{I} \tag{3.7.30}$$

(3.7.1)New Derivation of Heisenberg special Evolution :

In this work an new tread based on the unitary operator which is found using some simple mathematical techniques besed on the ordinary differentiation and integration is used . The quantum average of the momentum operator in the Schrodinger and Heisenberg picture is also used . Acting on the wave function by momentum operator

$$i\hbar\frac{d}{\mathrm{dt}}|\psi\rangle = \hat{P}|\psi\rangle \tag{3.7.31}$$

The unitary operator is defined by

$$|\psi(x)\rangle = |\psi(x)\rangle_s = \mathfrak{U}(x)|\psi(0)\rangle = \mathfrak{U}(x)|\psi\rangle_0$$

$$= \mathfrak{U}|\psi\rangle_0 = \mathfrak{U}|\psi\rangle_H \tag{3.7.32}$$

The quantum average of the operator O is equal in the Schrodinger and Heisenberg picture .Therefore

$$\langle\psi|_H O_\mathrm{H} |\psi\rangle_\mathrm{H} = \langle\psi|_s \mathrm{O_s} |\psi\rangle_\mathrm{s}$$

$$\langle\psi|_H O_\mathrm{H} |\psi\rangle_\mathrm{H} = \langle\psi|_H \mathfrak{U}^{-1}\mathrm{O_s}\, \mathfrak{U}|\psi\rangle_\mathrm{H} \tag{3.7.33}$$

Hence the operator in the Heisenberg picture is given by

$$O_\mathrm{H} = \mathfrak{U}^{-1}\mathrm{O_s}\,\mathfrak{U} \tag{3.7.34}$$

$$\int \frac{d}{|\psi\rangle}|\psi\rangle = \frac{i}{\hbar}\int P\,dx$$

$$Ln|\psi\rangle = \frac{i}{\hbar}\int P\,dx + C_2$$

$$|\psi\rangle = \mathrm{e}^{\frac{i}{\hbar}\int P dx}C_2 \tag{3.7.35}$$

But at

$$x = 0 \qquad |\psi(x)\rangle = |\psi(0)\rangle = C_2 \tag{3.7.36}$$

Thus equation (3.7.35)and (3.7.32) give

$$|\psi\rangle = \mathrm{e}^{\frac{i}{\hbar}\int P(x)dx}|\psi\rangle_0 = \mathrm{e}^{\frac{i}{\hbar}f(x)}|\psi\rangle_0 = \mathrm{e}^{\frac{i}{\hbar}f(x)}|\psi\rangle_\mathrm{H} \tag{3.7.37}$$

Where

$$f(x) = \int \frac{df}{dx}dx = \int P(x)dx$$

Thus

$$P = \frac{df}{dx} \tag{3.7.38}$$

Hence equation(3.7.2) and (3.7.9) can be compared to get

$$\mathfrak{U} = e^{\frac{i}{\hbar}\int P(x)dx} = e^{\frac{i}{\hbar}f(x)} = e^{\frac{i}{\hbar}f} \tag{3.7.39}$$

Thus equation (4) is given by

$$O_H = e^{-\frac{i}{\hbar}f} O e^{\frac{i}{\hbar}f}$$

The spatial evolution of the operator is therefore given by :

$$\frac{dO_H}{dx} = -\frac{i}{\hbar}\frac{df}{dx} e^{-\frac{i}{\hbar}f} O e^{\frac{i}{\hbar}f} + e^{-\frac{i}{\hbar}f}\frac{dO}{dx} e^{\frac{i}{\hbar}f} = \frac{i}{\hbar} e^{-\frac{i}{\hbar}f} O e^{\frac{i}{\hbar}f}\frac{df}{dx}$$

$$= -\frac{i}{\hbar}[PO_H - O_H P] + \left(\frac{dO}{dx}\right)_H$$

$$\frac{\hbar}{i}\frac{dO_H}{dx} = [O_H\,, P] + \left(\frac{dO}{dx}\right)_H \tag{3.7.41}$$

On the other hand

$$\int P\,dx|\psi\rangle_0 = \int \frac{\hbar}{i}\frac{d}{dx} dx\,|\psi\rangle_0$$

$$\int P\,dx|\psi\rangle_0 = \int dx\,|\psi\rangle_0 = \frac{\hbar}{i}|\psi\rangle_0 \tag{3.7.42}$$

The same result can be obtained by suggesting that

$$P = \frac{\hbar}{i}\frac{d}{dx}$$

To be out of integration to get

$$\int P\,dx|\psi\rangle_0 = P\int dx\,|\psi\rangle_0 = \frac{\hbar}{i}\frac{d}{dx}(x)|\psi\rangle_0 = \frac{\hbar}{i}|\psi\rangle_0$$

Thus

$$\int P\,dx = P\int dx = \frac{\hbar}{i} \tag{3.7.43}$$

Form (4.7.25) , when

$$|\psi\rangle = |\psi\rangle x = 0$$

$$\mathfrak{U}(x=0) = \mathfrak{U}_0 = e^{\frac{i}{\hbar}\int P dx} = e^0 = I \tag{3.7.44}$$

(3.7.2)Special and General Relativistic Meaning of the Matrix:

The proper length or proper time I a relativistic space- time takes the general form

$$c^2 d\tau^2 = -g_{\mu v} dx^{\mu} dx^{v} = -g'_{\mu v} dx^{\mu'} dx^{v'} \tag{3.7.45}$$

Consider two space points in the frame s and s' that measured simultaneously . In this case

$$dx^0 = 0 \qquad , \qquad dx^{0'} = 0$$

Therefore

$$g_{xx} dx^2 = -g'_{xx} dx'^2 \tag{3.7.46}$$

An observer at s observe a rod which is at rest in s' moving with constant speed v . So

$$g_{xx} = 1 \qquad , g'_{xx} = 1 - {v^2}/{c^2} \tag{3.7.47}$$

To get the ordinary length contraction relation

$$dx = \sqrt{1 - {v^2}/{c^2}}\, dx' = \gamma dx' \tag{3.7.48}$$

When s' moves with speed of light

$$dt' = 0 \qquad \dot{x}' = c^2 \tag{3.7.49}$$

Thus from (4.7.45)

$$-c^2 dt^2 - g_{xx} dx^2 = -g'_{xx} dx'^2 \tag{3.7.49}$$

$$-c^2 - v^2 g_{xx} = -c^2 g'_{xx}$$

Thus since

$$g_{xx} = -1$$

$$g'_{xx} = 1 - {v^2}/{c^2} \tag{3.7.50}$$

Consider also a clock at rest at a certain point in s' . In this case one must use the time metric relation to get

$$dx^0 = \sqrt{g'_{00}}dx^{0'}$$

$$dt = \left(1 - {}^{v^2}/_{c^2}\right)^{-\frac{1}{2}} dt' \quad (3.7.51)$$

$$= \left(1 - {}^{v^2}/_{c^2}\right)^{-\frac{1}{2}} dt_0$$

It is very interesting to note that

$$g_{xx} = g_{xx}^{-1} \quad (3.7.52)$$

This conforms to Schwarzschild solution . thus in a curved space time $(Curved \equiv C)$ which is equivalent to an accelerated frame s' with respect to an observer which is at rest in s

$$dx_c = \sqrt{g'_{xx}}dx' \quad (3.7.53)$$

Chapter 4

Results , Discussion and Conclusion

(4.1) Introduction :

In this chapter consist of the results , discussion and conclusion .

(4.2) Results:

For the first time the spatial evolution of the quantum system in the interaction picture has bee obtained in the form [see(3.2.26)

$$i\hbar \frac{d}{dx} |\psi\rangle_I = P_I |\psi\rangle_I \tag{4.2.1}$$

The viability and the reality of this equation comes from the fact that its form resembles the ordinary form momentum eigen equation . It also of resembles the time evolution when replacing time with coordinate and Hamiltonian with momentum operator. The unitary operator is given by equation (3.2.33)

$$U(x_2) = U(x_1) - \frac{i}{\hbar}\int_{x_1}^{x_2} P_I(\mathrm{x}'') \left[I - \frac{i}{\hbar}\int_{x_0}^{x_1} P_I(\mathrm{x}')\mathrm{dx}' \right] \mathrm{dx}''$$

$$= U(x_1) + (\frac{-i}{\hbar}) \int_{x_1}^{x_2} P_I(\mathrm{x}'')\mathrm{dx}'' + \frac{(-i)^2}{\hbar^2} \int_{x_2}^{x_1} \int_{x_0}^{x_1} P_I(\mathrm{x}'')\, P_I(\mathrm{x}')\mathrm{dx}'\mathrm{dx}'' \tag{4.2.2}$$

Which describes its spatial evolution .Its viability comes from the fact that is resembles the time dependent one with the time replaced coordinate and Hamiltonian with the momentum operator . The energy relation in a curved space time is given by equation (3.3.9)

$$i\hbar \frac{\partial \psi}{\partial t_c} = E\psi \tag{4.2.3}$$

From which one can deduce the relation between energy in a curved space and Eucleadian space in the form

$$E = (g_{00})^{-\frac{1}{2}} E_0 \tag{4.2.4}$$

Which conforms with the fact that for static mass

$$m = (g_{00})^{-\frac{1}{2}} m_0 \tag{4.2.5}$$

$$E = mc^2 = (g_{00})^{-\frac{1}{2}} m_0 c^2 = (g_{00})^{-\frac{1}{2}} E_0 \tag{4.2.6}$$

The same hold for momentum .

The momentum in a curved space time is given according to equation (3.3.12) by

$$\frac{ħ}{i} \frac{\partial \psi}{\partial x_c} = P\psi$$

With

$$dx_c = \sqrt{g_{xx}} dx \tag{4.2.7}$$

When compare it with that of Eucleadian space (4.3.15c)

$$\frac{ħ}{i} \frac{\partial \psi}{\partial x} = P_0 \psi \tag{4.2.8}$$

It gives

$$P = (g_{00})^{-\frac{1}{2}} P_0 \tag{4.2.9}$$

Using the proper time relation [see(3.3.23)]

$$c^2 dt^2 = c^2 g_{00} dt_0^2 + g_{xx} dx^2$$

$$\gamma^{-1} = \left(\frac{d\tau}{dt}\right) = \left[g_{00} - g_{xx} \frac{v_0^2}{c^2}\right]^{\frac{1}{2}} \tag{4.2.10}$$

One gets

$$E_0^2 = P_0^2 c^2 + \tilde{m}_0^2 c^4 \tag{4.2.11}$$

Which resembles the ordinary energy –momentum relation with the mass given by equation (3.3.32)to be

$$m = \sqrt{g_{00}} m_0 \tag{4.2.12}$$

Which resembles that of GSR for static mass . The perturbed momentum is given by

$$\hat{P} = \hat{P}_0 \left(1 + \frac{L_0}{E_0}\right) = \hat{P}_0 + \hat{P}_i \tag{4.2.13}$$

This means that the field changes momentum through the lagrangian which consists of a potential term . This effect can not be recognized by the ordinary model.

The comparison of the proper interval shows that the metric space components related to Lorentz coefficient γ through the relation

$$g'_{xx} = 1 - {v^2}/{c^2} = \gamma^{-2} \tag{4.2.14}$$

(4.3) Discussion :

The wave function in a curved time-space is given by equations (4.3.1) and (4.3.6) . Using this expression together with definition of time in a curved space –time in equation (4.3.10) , the curved space-Eigen equation take a form typical to Euclidian space ,this is very apparent when comparing equation (4.3.5) and(4.3.7) . It is very striking to note that the energy Eigen equation in curved space-time [equation (4.3.9)] can be used to derive the energy Eigen equation in the Euclidean space . the same hold for momentum Eigen equation in curved space in equation (4.3.12) can be used to derive that Euclidian space in equation (4.3.15.c).Fortunately , the relation between the momentum in a curved space and momentum in equation (4.3.30) is typical to that obtained for Eigen equation in equation (4.3.10) and (4.3.15.a) .

Appling energy Eigen equation for harmonic oscillator in a curved space-time shows that the energy in the curved space-time is equivalent to the existence of additional potential term typical to that Newton . This means that a particle in a curved space behavior is typically to behavior of particle moving in a potential .

The energy expression (4.4.1) in a curved space –time within the frame work of GSR is utilized to get a useful expression for the Hamiltonian (4.4.4). here one assumes that the velocity is less than the speed of light . Both of Schrodinger equation and expectation Values of the Hamiltonian in Schrodinger and interaction picture(see (4.4.11)&(4.4.16)). These tow expression are typicall each other only when the unperturbed Hamiltonian gives no contribution to the energy in the interaction picture . This is in agreement with the fact that the Hamiltonian in the interaction picture is only that wich causes perturbation .

Spatial evolution of the quantum system in the interaction picture is also derived using the expression of the momentum in a curved space-time within the frame work of the GSR. Here one assumes that the velocity is less than the speed of light and the potential is also less than the rest mass energy . The perturbed momentum is found to be proportional to the Lagrangin of the system thus also to the perturbation energy (see(4.5.3) . Fortunately this new expression resembles that of the Hamiltonian , where the time differential is replaced by the space one and the Hamiltonian is replaced by the momentum (see (4.5.20) . The expression of the momentum using the Schrodinger equation is typical to the one found by equating the expectation values in the interaction picture and Schrodinger picture as shown in equations (4.5.25)& (4.5.21) . This requires that the unperturbed momentum to give no contribution in the interaction picture as shown in equation (4.5.24) . Finally the spatial evolution of the unitary operator is derived using the momentum operator . It is very interesting to note that this spatial evolution resembles that of time but here one replaces time integral by spatial one , and the Hamiltonian by the momentum operator.

Using the quantum average of Hamiltonian of the quantum system in equation(4.7.5) and the definition of the unitary operator in (4.7.6)one finds the expression of the unitary operator in the Heisenberg time evolution equation in (4.7.26). Using the quantum average of momentum operator of quantum system in the Schrodinger and Heisenberg picture in equation (4.7.33)beside integration and differentiation technique ,one finds the functional form of the unitary operator defined in equation(4.7.32) . Then the Heisenberg spatial evolution is found in equation (4.7.41). The proper length in a curved space is written in equation(4.7.46). comparing this expression with the corresponding Special relativity expressions , one finds that the spatial and time metric is related to the Lorentz transformation factor as shown in equations (4.7.47)& (4.7.51). They satisfy a Schwarzschild relation.

(4.4) Conclusion :

The energy and momentum Eigen equation in a curved space can be used to derive that the Euclidean space using the energy –momentum relation analogous in Euclidian and curved space . Its the expression of mass in a curved space was similar to that of GSR.

A useful expression of the spatial evolution of the quantum system in the interaction picture was derived . This expression was found to be typical to the Hamiltonian one when one replaces the time differential with the spatial one . Another expression of the spatial evolution of the unitary operator was also found to be typical to that of the Hamiltonian one . Here one replaces the time integral with the spatial one , and the Hamiltonian with the momentum .

The spatial evolution of the quantum system was found using unitary operator and simple mathematics based on the ordinary differentiation and integration . The metrics in the curved space time was found to be related to the Lorentz transformation factor.

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