Chapter (1)

Differentiable Manifolds

We discuss some fundamental concepts, preliminary notions and some fundamental results that is required for the development of the subject in the dissertation. Specifically, the notions of manifold, function, and vector field, and the concept of differentiability (smoothness).

The notion of a differentiable manifold is necessary for extending the methods of differential calculus to spaces more general than $\mathbb{R}^n$.

Differentiable manifold is a type of manifold that is locally similar enough to a Euclidean space to allow one to do calculus. Any manifold can be described by a collection of charts.

1.1: Topological Manifolds

A topological space is called separable if it contains a countable dense subset, i.e., there exist sequences $\{C_i\}_{i=1}^{\infty}$ of elements of the space such that every non-empty open subset of the space contains at least one element of the sequence.

Let M be a separable topological space. We assume that M satisfies the Hausdorff separation axiom which states that any two different points in M can be separated by disjoint open sets.

An open chart on M is a pair (U, φ) where U is an open subset of M and φ is a homeomorphism of U onto an open subset of $\mathbb{R}^n$, where $\mathbb{R}^n$ is an n-dimensional Euclidean space.

1.1.1. Definition. Let X be an arbitrary set. A local parametrization of X is an injective mapping $\varphi: \Omega \to X$ from an open subset Ω of $\mathbb{R}^n$ onto a subset of X.

The inverse $\varphi^{-1}: \varphi(\Omega) \to \Omega$ of such a parametrization is called a chart because through φ^{-1} the regions $\mathrm{im}\varphi \subset X$ is "charted" on $U \subset \mathbb{R}^n$ just as region of the earth is charted on a topographic or a political map.

φ^{-1} is also called a local coordinate system because through φ^{-1} each point $p \in \mathrm{im}\varphi$ corresponds to an n-tuple of real numbers, the coordinates of p.

1.1.2. Definition. Let M be a topological space. A covering of M is a collection of open subsets of M whose union is M. A covering $\{U_\alpha\}_{\alpha\in A}$ of M is said to be locally finite if each $p \in M$ has a neighborhood (an open subset of M containing p) which intersects only finitely many of the sets U_α .

A Hausdorff space M is called paracompact if for each covering $\{U_\alpha\}_{\alpha\in A}$ of M there exists a locally finite covering $\{V_\beta\}_{\beta\in B}$ which is a refinement of $\{U_\alpha\}_{\alpha\in A}$ (That is, each V_β is contained in some U_α). It is known that a locally compact Hausdorff space which has a countable base is paracompact.

1.1.3. Definition. An m-dimensional manifold in $\mathbb{R}^n$ is a nonempty $S \subset \mathbb{R}^n$ satisfying the following property for each point $p \in S$. There exists an open neighborhood $W \subset \mathbb{R}^n$ of p an m-dimensional embedded parameterized manifold $\sigma: U \to \mathbb{R}^n$ with image $\sigma(U) = S \cap W$.

1.1.4. Definition. A topological space M is locally Euclidean of dimension n if every point p in M has a neighborhood U such that there is homeomorphism φ from U onto an open subset of $\mathbb{R}^n$. We call the pair $(U, \varphi: U \to \mathbb{R}^n)$ a chart, U a coordinate neighborhood or a coordinate open set, and φ a coordinate map or a coordinate neighborhood system on U. We say that a chart (U, φ) is a centered at $p \in U$ if $\varphi(p) = 0$.

1.1.5. Definition. Suppose M is a topological space. We say that M is a topological manifold of dimension n or topological n-manifold if it has the following properties:

(i) M is a Hausdorff space,
(ii) M is locally Euclidean of dimension n, and
(iii) M is second-countable.

1.1.1: Some Examples of Topological Manifolds

(a) The Euclidean space $\mathbb{R}^n$ is covered by a single chart $(\mathbb{R}^n, I_{\mathbb{R}^n})$, where $I_{\mathbb{R}^n}: \mathbb{R}^n \to \mathbb{R}^n$ is the identity map. It is the prime example of a topological manifold. Every open subset of $\mathbb{R}^n$ is also a topological manifold, with chart $(U,\ I_U)$.

Recall that the Hausdorff condition and second countability are "hereditary properties" that is, they are inherited by subspace: a subspace of a Hausdorff space is a Hausdorff and a subspace of a second countable space is second countable. So any subspace of $\mathbb{R}^n$ is automatically Hausdorff and second countable.

(**b**) (Spheres). Let S^n denote the (unit) n-sphere which is the set of unit length vectors in $\mathbb{R}^n$:

$$S^n = \{x \in \mathbb{R}^{n+1}: |x| = 1\}$$

It is Hausdorff and second countable because it is a subspace of $\mathbb{R}^n$.

To show that it is locally Euclidean, for each index $i = 1, \dots, n+1$, let U_i^+denote the subset of S^n where the i-th coordinate is positive

$$U_i^+ = \left\{(x^1, \dots, x^{n+1}) \in S^n: x^i > 0\right\}$$

Similarly, U_i^- is the set where $x^i < 0$. For each i, define maps $\varphi_i^\pm: U_i^\pm \to \mathbb{R}^n$ by

$$\varphi_i^\pm(x^1, \dots, x^{n+1}) = (x^1, \dots, \widehat{x^i}, \dots, x^{n+1})$$

where the hat over x^i indicates that x^i is omitted.

Each $\varphi_i^\pm$ is evidently a continuous map, being the restriction to S^n of a linear map on $\mathbb{R}^{n+1}$.

It is homeomorphism onto it is the unit ball $B^n \subset \mathbb{R}^n$, because it has a continuous inverse given by

$$\varphi^{\pm^{-1}}(u^1, \dots, u^n) = (u^1, \dots, u^{i-1}, \pm\sqrt{1 - |u|^2}, u^i, \dots, u^n).$$

Since every point in S^{n+1} is in the domain of one these $2n$+2 charts, S^n is locally Euclidean of dimension n and is thus a topological n-manifold.

(**c**) The simplest examples of manifolds not homeomorphic to open subset of Euclidean space are the circle S^1 and the 2-sphere S^2, which may be defined to be all points of $\mathbb{E}^2$, or $\mathbb{E}^3$, respectively, which are at unit distance from a fixed point 0.

These are to be taken with the subspace topology so that (i) and (iii) are immediate.

To see that they are locally Euclidean we introduce coordinate axes with 0 as origin in the corresponding ambient Euclidean space. Thus in the case of S^2 we identify $\mathbb{E}^3$ or $\mathbb{R}^3$, and S^2 becomes the unit sphere centered at the origin. At each point p of S^2 we have a tangent plane and a unit normal vector N_p . There will be a coordinate axis which is not perpendicular to N_p and some neighborhood U of p on S^2 will then project in a continuous and one-to-one fashion onto an open set $U^{'}$ of the coordinate plane perpendicular to the axis.

1.1.2: Coordinate Charts

Let M be topological n-manifolds. A coordinate chart (or just a chart) on M is a pair (U, φ), where U is an open subset of M and $\varphi : U \to \widetilde{U}$ is a homeomorphism from U to an open subset $\widetilde{U} = \varphi(U) \subset \mathbb{R}^n$.

If the addition $\widetilde{U}$ is an open ball in $\mathbb{R}^n$, then U is called coordinate ball.

The definition of a topological manifold implies that each point $p \in M$ is contained in the domain of some chart (U, φ).

If $\varphi(p) = 0$ we say that a chart is centered at p. Given a chart (U, φ) we call the set U a coordinate domain, or a coordinate neighborhood of each of its points.

The map φ is called a (local) coordinate map, and the component functions of φ are called local coordinate on U.

1.1.3: Compatible Charts

Suppose $(U, \varphi: U \to \mathbb{R}^n)$ and $(V, \Psi: V \to \mathbb{R}^n)$ are two charts of a topological manifold. Since $U \cap V$ is open in U and $\varphi: U \to \mathbb{R}^n$ is a homeomorphism onto an open subset of $\mathbb{R}^n$, the image $\varphi(U \cap V)$ will also be an open subset of $\mathbb{R}^n$.

1.1.6. Definition. Two charts $(U,\ \varphi: U \to \mathbb{R}^n)$, $(V,\ \ \Psi: V \to \mathbb{R}^n)$ of topological manifold are C^∞-compatible if the two maps

$$\varphi \circ \psi^{-1}: \psi(U \cap V) \to \varphi(U \cap V),$$

$$\psi \circ \varphi^{-1}: \varphi(U \cap V) \to \psi(U \cap V)$$

are C^∞.

These two maps are called the transition functions between the charts.

If $U \cap V$ is nonempty, then the two charts are automatically C^∞-compatible.

1.1.7. Definition. A C^∞-atlas or simply an atlas on a locally Euclidean space M is a collection $\mathfrak{A} = \{(U_\alpha, \varphi_\alpha)\}$ of pairwise C^∞-compatible charts that cover M, i.e., such that $M = \bigcup_\alpha U_\alpha$.

We say that a chart (V, ψ) is a compatible with an atlas $\{(U_\alpha, \varphi_\alpha)\}$ if it is compatible with all charts $(U_\alpha, \varphi_\alpha)$ of the atlas.

1.1.8. Lemma. Let $\{(U_\alpha, \varphi_\alpha)\}$ is an atlas on a locally Euclidean space. If two charts (V, ψ) and (W, σ) are both compatible with the atlas $\{(U_\alpha, \varphi_\alpha)\}$ then they are compatible with each other.

Proof. Let $p \in V \cap W$. We need to show that $\sigma \circ \psi^{-1}$ is C^∞ at $\psi(p)$.

Since $\{(U_\alpha, \varphi_\alpha)\}$ is an atlas for M, $p \in U_\alpha$ for some α.Then p is in the triple intersection $V \cap W \cap U_\alpha$.

Now

$$\sigma \circ \psi^{-1} = (\sigma \circ \varphi_\alpha^{-1}) \circ (\varphi_\alpha \circ \psi^{-1}) \text{ is } C^\infty \text{ on } \psi(V \cap W \cap U_\alpha)$$

Hence at $\psi(p)$. Since p was arbitrary point of $(V \cap W)$, this proves that $\sigma \circ \psi^{-1}$ is C^∞ on $\psi(V \cap W)$.

Similarly $\psi \circ \sigma^{-1}$ is C^∞ on $\sigma(V \cap W)$. ■

1.1.9. Definition. A differentiable structure on a topological manifold M is a family $\mathfrak{A} = \{U_\alpha, \varphi_\alpha\}$ of coordinate neighborhoods such that

(1) The U_α cover M,
(2) For any α, β the neighborhoods U_α, φ_α and U_β, φ_β are C^∞-compatible.
(3) Any coordinate neighborhood V, ψ compatible with every U_α, $\varphi_\alpha \in \mathfrak{A}$ is itself in $\mathfrak{A}$.

1.1.10. Definition. A smooth or C^∞-manifold is a topological manifold M together with a maximal atlas.

The maximal atlas is also called a differentiable structure on M.

Statement. In the C^∞-atlas all Jacobians $\det\left|\partial x^i / \partial y^k\right|$ are nonzero.

Proof. Since $\vec{x}(\vec{y})$ and $\vec{y}(\vec{x})$ are both C^∞ we have $\det\left|\partial x^i / \partial y^k\right| \neq 0$.

1.1.11. Definition. Oriented atlas is such that all $\det\left|\partial x^i / \partial y^k\right| > 0$.

Oriented manifold $\coloneqq$ there exists an oriented atlas.

1.1.12. Definition. Let M be a smooth manifold of dimension n, a function $f: M \to \mathbb{R}$ is said to be C^∞ or smooth at a point p in M if there is a chart (U, φ) about p in M such that $f \circ \varphi^{-1}$, a function defined of the open subset $\varphi(U)$ of $\mathbb{R}^n$ is C^∞ at $\varphi(p)$.

The function f is said to be C^∞ on M if it is C^∞ at every point of M.

1.1.4: Examples of Smooth Manifolds

(**i**) (Euclidean Space) .The Euclidean space $\mathbb{R}^n$ is a smooth manifold with a single chart $(\mathbb{R}^n, r^1, \dots, r^n)$, where $r^1, \dots, r^n$ are the standard coordinates on $\mathbb{R}$.

(**ii**) (Open subset of a manifold). Any open subset V of a manifold M is also a manifold. If $\{(U_\alpha, \varphi_\alpha)\}$ is an atlas for M, then $\left\{(U_\alpha \cap V, \varphi_\alpha|_{U_\alpha \cap V}\right\}$ is an atlas for V, where $\varphi_\alpha|_{U_\alpha \cap V}: U_\alpha \cap V \to \mathbb{R}^n$ denotes the restriction of φ_α to the subset $U_\alpha \cap V$.

(**iii**) (Manifolds of dimension zero). In a manifold of dimension zero, every singleton subset in homeomorphic to $\mathbb{R}^0$ and so is open. Thus, a zero dimensional manifold is a discrete set. By second countability, this discrete set must be countable.

1.1.13. Lemma. Let $S \subset \mathbb{R}^n$ be non-empty. Then S is a m-dimensional manifold if and only if it satisfies the following condition for each $p \in S$:

There exist an open neighborhood $W \subset \mathbb{R}^n$ of p, such that $S \cap W$ is the graph of a smooth function h, where $n - m$ of the variables $x_1, \dots, x_n$ are considered as functions of the remaining m variables.

1.1.14. Definition. A neighborhood of a point p_0 in M is any subset of M containing all p whose coordinate point $x(p)$ in some coordinate system satisfy $\|x(p) - x(p_o)\| < \epsilon$, for some $\epsilon > 0$. A subset of M is open if it contains a neighborhood of each of its points, and closed if its complement is open.

1.1.15. Definition. If M and N are manifolds, the Cartesian product

$$M \times N = \{(p, q): p \in M, q \in N\}$$

becomes a manifold with the coordinate systems $(x^1, \dots, x^n, y^1, \dots, y^m)$ where $(x^1, \dots, x^n)$ is a coordinate system of M, $(y^1, \dots, y^m)$ for N.

1.1.16. Definition. Let N and M be manifolds. And let $f: N \to M$ be a continuous mapping. A mapping f is called a homeomorphism between N and M if f is continuous and has a continuous inverse $f^{-1}: M \to N$.

In this case the manifolds N and M are said to be homeomorphic.

1.1.17. Definitions. Given two manifolds N and M, a mapping $f: N \to M$ is said to be C^∞ or smooth if for every $p \in N$ there exist charts (U, φ) of p and (V, ψ) of $f(p)$ with $f(U) \subset V$ such that $\psi \circ f \circ \varphi^{-1}: \varphi(U) \to \psi(V)$ is a C^∞-mapping (from $\mathbb{R}^n$ to $\mathbb{R}^m$).

Let $u_1, \dots, u_n$ be a local coordinate system on U_α and $y_1, \dots, y_m$ a local coordinate system on V_β.

If φ is a differentiable map of M into N, then locally φ can be expressed by a set of differentiable functions:

$$y_1 = y_1(u_1, \dots, u_n), \dots, y_m = y_m(u_1, \dots, u_n).$$

By a differentiable map of a closed interval $[a, b]$ into a manifold M, we mean the restriction of a differentiable map of an open interval $I \supset [a, b]$ into M.

By a differentiable curve in a manifold M, we shall mean a differentiable mapping of a closed interval $[a, b]$ of $\mathbb{R}$ into M.

We shall now define a tangent vector (or simply a vector) at a point p of M. Let $\mathcal{F}(p)$ be the algebra of differentiable functions in a neighborhood of a point p.

Let $\gamma(t)$ $(a \le t \le b)$ be a differentiable curve in M with $\gamma(t_0) = p$. The vector tangent to the curve $\gamma(t)$ at p is a map $X: \mathcal{F}(p) \to \mathbb{R}$ defined by

$$Xf = \left. \frac{df\big(\gamma(t)\big)}{dt} \right|_{t=t_0}.$$

In other words, Xf is the derivative of f in the direction of the curve $\gamma(t)$ at $t = t_0$. The vector X satisfies the following two conditions:

(1) X is a linear map of $\mathcal{F}(p)$ into $\mathbb{R}$. i.e.,

$X\,(af + bg) = aX\,(f) + bX\,(g)$ for $f, g \in \mathcal{F}\,(p)$, $a, b \in \mathbb{R}$;

(2) X is a derivation: $X\,(fg) = (Xf)\, g(p) + f(p)\, Xg$ for $f, g \in \mathcal{F}(p)$.

The set of maps X of $\mathcal{F}(p)$ into $\mathbb{R}$ satisfying these two conditions forms a real vector space.

Let $u_1, \dots, u_n$ be a local coordinate system defined in a coordinate neighborhood U of p. For each i, $(\partial/\partial u_i)_p$ is a map from $\mathcal{F}(p)$ into $\mathbb{R}$ satisfying conditions (1) and (2) above. We shall show that the set of vectors at p forms the vector space with basis $(\partial/\partial u_1)_p, \dots, (\partial/\partial u_n)_p$.

Given a curve $\gamma(t)$ with $\gamma(t_0) = p$, let $u_i = \gamma_i(t)$ be its equations in terms of $u_1, \dots, u_n$. Then we have

$$\left(\frac{df\big(\gamma(t)\big)}{dt}\right)_{t_0} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial u_i}\right)_p \left(\frac{d\gamma_i(t)}{dt}\right)_{t_0}.$$

Thus every vector at p is a linear combination of $(\partial/\partial u_1)_p, \dots, (\partial/\partial u_n)_p$.

Conversely, for a given linear combination $X = \sum_i a_i (\partial/\partial u_i)_p$, let us consider the curve defined by

$$u_i = u_i(p) + a_i t, \quad i = 1, \dots, n.$$

Then X is the vector tangent to this curve at $t = 0$.

To prove the linear independence, we assume that $\sum_i a_i (\partial/\partial u_i)_p = 0$.

Then we have

$$0 = \sum a_i \left(\frac{\partial u_j}{\partial u_i}\right)_p = a_j, \quad j = 1, \dots, n.$$

Consequently, we obtain the following

1.1.17. Proposition. Let M be an n-dimensional manifold and $p \in M$. If $u_1, \dots, u_n$ is a local coordinate system on a coordinate neighborhood containing p, then the set of vectors at p tangent to M is an n-dimensional vector space or $\mathbb{R}$ with basis $(\partial/\partial u_1)_p, \dots, (\partial/\partial u_n)_p$.

A vector field X on a differentiable manifold M is a function that assigns to each point $p \in M$ a tangent vector $X_p \in T_pM$ to M at p. If f is a differentiable function on M, then Xf is a function on M such that $(Xf)(p) = X_pf$. A vector field X is called differentiable if Xf is differentiable for every differentiable function f on M.

In terms of a local coordinate system $u_1, \dots, u_n$, a differentiable vector field X can be expressed by

$$X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial u_i},$$

where the coefficients X^i are differentiable functions.

Let $\mathfrak{X}(M)$ be the set of all differentiable vector fields on M. Then $\mathfrak{X}(M)$ is a real vector space under the natural addition and scalar multiplication.

Given two vector fields X, Y on M, we define the bracket $[X, Y]$ as a map from the ring of functions on M into itself by

$$[X, Y]f = X(Yf) - Y(Xf). \tag{1.1.1}$$

Then $[X, Y]$ is again a vector field on M. In terms of a local coordinate system $u_1, \dots, u_n$, we write

$$X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial u_i} \quad , \quad Y = \sum_{j=1}^{n} X^j \frac{\partial}{\partial u_j}$$

Then we have

$$[X, Y]f = \sum_{j,k=1}^{n} \left\{ X^k \frac{\partial Y^j}{\partial u_k} - Y^k \frac{\partial X^j}{\partial u_k} \right\} \frac{\partial}{\partial u_j}. \tag{1.1.2}$$

With respect to this bracket operation, $\mathfrak{X}(M)$ becomes Lie algebra over $\mathbb{R}$ (of infinite dimension). In particular, we have the Jacobi identity:

$$\big[[X, Y], Z\big] + \big[[Y, Z], X\big] + \big[[Z, X], Y\big] = 0 \tag{1.1.3}$$

for $X, Y, Z \in \mathfrak{X}(M)$.

We may regard $\mathfrak{X}(M)$ as a module over the ring $\mathcal{F}(M)$ of differentiable functions on M as follows:

If $f \in \mathcal{F}(M)$ and $X \in \mathfrak{X}(M)$, then we define fX by $(fX)_p = f(p)X_p$ we have

$$[fX, gY] = fg[X, Y] - f(Xg)Y - g(Yf)X$$

for $f, g \in \mathcal{F}(M)$ and $X, Y \in \mathfrak{X}(M)$.

1.1.19. Definition. A C^∞ mapping $F: M \to N$ between C^∞ manifolds is a diffeomorphism if it is a homeomorphism and F^{-1} is C^∞. M and N are diffeomorphic if there exists a diffeomorphism $F: M \to N$.

1.1.20. Definition. Let $F: N \to M$ be a map and h a function on M. The pull-back of h by F denoted by $F^* h$, is the composite function $h \circ F$.

A function f on M is C^∞ on a chart (U, φ) if and only if its Pull-back $(\varphi^{-1})^* f$ by φ^{-1} is C^∞ on the subset $\varphi(U)$ of Euclidean space.

1.1.21. Definition. A complex manifold M is a Hausdorff second countable topological space X, with an atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha): \alpha \in A\}$ the coordinate functions φ_α take values in $\mathbb{C}^n$ and so all the overlap maps are holomorphic. The number n

is called the complex dimension of M and one writes $dim_{\mathbb{C}} M = n$.

A maximal set of charts is now called a complex structure.

1.1.22. Definition. A topological space M is called an m-dimensional manifold-with boundary $\partial M \subset M$, if the following conditions hold:

1. M is a Hausdorff space,

2. For any point $p \in M$ there exist a neighborhood U of p, which is homeomorphic to an open subset $V \subset \mathcal{H}^M$ and

3. M has a countable basis of open sets.

1.1.23. Definition. The stereographic projection from the North Pole $N = (0, \ldots, 0, 1)$ (resp., South Pole $S = (0, \ldots, 0, -1)$) of the sphere

$$S^n = \left\{ (x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} \left(x^i\right)^2 = 1 \right\}$$

Onto the equatorial plane $x^{n+1} = 0$ is the map sending $p \in S^n \setminus \{N\}$(resp., $p \in S^n \setminus \{S\}$) to the point where the straight line through N (resp., S) and p intersects the plane $x^{n+1} = 0$.

The inverse of the stereographic projection is the map from $x^{n+1} = 0$ to $S^n \setminus \{N\}$ (resp., $p \in S^n \setminus \{S\}$) sending the point q in the plane $x^{n+1} = 0$ to the point where the straight line through q and N (resp., S) intersects S^n.

1.1.24. Definition. Let $\Phi: M \to N$ be a C^∞ map. A point $p \in M$ is said to be critical point of Φ if the differential $\Phi_*: T_p M \to T_{\Phi(p)} N$ is not surjective (or onto).

A point $q \in N$ is said to be a critical value of Φ if the set $\Phi^{-1}(q)$ contains a critical point of Φ. A point of N which is not a critical value is called a regular value of Φ.

Let $f \in C^\infty M$. A point $p \in M$ is called a critical point of f if $f_{*p} = 0$. If we choose a coordinate system $(U, x^1, \dots, x^n)$ around $p \in M$, this means that

$$\frac{\partial f}{\partial x^1}(p) = \cdots = \frac{\partial f}{\partial x^n}(p) = 0.$$

The real number $f(p)$ is then called a critical value of f. A critical point is called non-degenerate if the matrix

$$\left(\frac{\partial^2 f}{\partial x^i \partial x^j}(p) \right)$$

is non-singular. Non-degeneracy does not depend on the choice of coordinate system.

1.1.25. Definition. A frame on an m-dimensional manifold M is an ordered set of vector fields $\mathcal{V} = \{V_1, \dots, V_m\}$ having the property that they form a basis for the tangent space $T_p M$ at each point $p \in M$.

1.1.26. Definition. A subset S of $\mathbb{R}^n$ is said to have measure zero if for every $\varepsilon > 0$, there is a covering of S by a countable number of open cubes $C_1, C_2, \dots$, such that the Euclidean volume $\sum_{i=1}^{\infty} v(C_i) < \varepsilon$.

A subset S of a differentiable n-manifold M has measure zero if there exist a countable family $(U_1, \varphi_1), (U_2, \varphi_2), \dots$, of charts in the differentiable structure of M such that $\varphi_i(U_i \cap S)$ has measure zero in $\mathbb{R}^n$ for every $i = 1,2, \dots$.

1.1.27. Theorem (Sard's Theorem). Let M and N be two manifolds of dimension m and n, respectively. If the $\Phi: M \to N$ is a differentiable map. Then the set of critical value of Φ has measure zero.

1.1.28. Theorem (Inverse Map Theorem). Let $f = (f^1, \dots, f^n): U \to \mathbb{R}^n$ be a C^∞ map defined on an open subset $U \subseteq \mathbb{R}^n$. Given a point $x_0 \in U$, assume

$$\frac{\partial(f^1, \dots, f^n)}{\partial(x^1, \dots, x^n)}(x_0) \neq 0.$$

Then there exists an open neighborhood $V \subseteq U$ of x_0 such that:

(i) $f(V)$ is an open subset of $\mathbb{R}^n$;
(ii) $f: V \to f(V)$ is one-to-one;
(iii) $f^{-1}: f(V) \to V$ is C^∞.

1.1.29. Theorem (Implicit Map Theorem). Denote the coordinates on $\mathbb{R}^n \times \mathbb{R}^m$ by $(x^1, \dots, x^n, y^1, \dots, y^m)$. Let $U \subseteq \mathbb{R}^n \times \mathbb{R}^m$ be an open subset, and let

$$f = (f^1, \dots, f^m): U \to \mathbb{R}^m$$

be a smooth map. Given a point $(x_0, y_0) \in U$, assume:

(i) $f(x_0, y_0) = 0$;

(ii) $\frac{\partial(f^1, \dots, f^m)}{\partial(y^1, \dots, y^m)}(x_0, y_0) \neq 0$.

Then, there exist an open neighborhood V of x_0 in $\mathbb{R}^n$ and an open neighborhood W of y_0 in $\mathbb{R}^m$ such that $V \times W \subset U$, and there exists a unique C^∞ map $g: V \to \mathbb{R}^m$, such that for each $(x, y) \in V \times W$:

$$f(x, y) = 0 \text{ if and only if } y = g(x).$$

1.1.5: Further Examples of Manifolds

A hemispherical cap (including the equator) or a right circular cylinder (including the circles at the ends) is typical examples of manifolds with boundary.

Except for the equator, or the end circles, they are 2-manifolds and these boundary sets are themselves manifolds of dimension one less. In fact, they are homeomorphism to S^1 or $S^1 \cup S^1$ in these two cases. An even simpler example is the upper half-plane H^2, or more generally H^n, where we shall mean by H^n the subspace of $\mathbb{R}^n$ defined by

$$H^n = \{(x^1, \dots, x^n) \in \mathbb{R}^n : x^n \geq 0\}.$$

Every point $p \in H^n$ has a neighborhood U which is homeomorphism to an open subset U' of $\mathbb{R}^n$ except the set of points $(x^1, \dots, x^{n-1}, 0)$, which obviously forms a subspace homeomorphic $\mathbb{R}^{n-1}$, called the boundary of H^n and denoted by ∂H^n.

We shall define a manifold with boundary to be a Hausdorff space M with a countable basis of open sets which has the property that each $p \in M$ is contained in an open set U with a homeomorphism φ to either (a) an open set U' of H^n-∂H^n or (b) to an open set U' of H^n with $\varphi(p) \in \partial H^n$, that is, a boundary point of H^n.

It can be shown (as consequence of invariance of domain) that $p \in M$ is in one class or the other but not both; those p of the first type are called interior points of M and those p mapped onto the boundary of H^n by one, and hence by all, homeomorphisms of their neighborhoods into H^n are called boundary points.

The collection of boundary points is then denoted by ∂M and is called the boundary of M. It is a manifold of dimension $n - 1$.

1.1.30. Definition (Tangent Spaces). Let $M \subset \mathbb{R}^k$ be a smooth m-dimensional manifold and fix a point $p \in M$. A vector $v \in \mathbb{R}^k$ is called a tangent vector of M at p if there is a smooth curve $\gamma: \mathbb{R} \to M$ such that

$$\gamma(0) = p, \quad \dot{\gamma}(0) = v.$$

The set $T_pM := \{\dot{\gamma}(0): \gamma: \mathbb{R} \to M \text{ is smooth}, \gamma(0) = p\}$ of tangent vectors of M at p is called the tangent space of M at p. We also write $T_p(M)$ instead of T_pM.

We denote the union of tangent spaces $T_p(M)$, for $p \in M$, by $T(M)$.

A tangent vector field to M is a continuous function $F: M \to T(M)$ such that $F(p) \in T_p(M)$ for each $p \in M$.

1.1.31. Definition. Let M be a smooth m-dimensional manifold, and let T_pM be the tangent space at some $p \in M$. The cotangent space T_p^*M is defined as the dual vector space of T_pM, i.e.,

$$T_p^*M := \left(T_pM\right)^*.$$

In other words $\omega_p \in T_p^*M$ if and only if $\omega_p : T_pM \to \mathbb{R}$ is a linear map. We denote the action of ω_p on a vector $X_p \in T_pM$ by

$$\omega_p\left(X_p\right) = \langle \omega_p, X_p \rangle$$

Since ω_p is a linear map we have

$$\langle \omega_p, X_p + \lambda Y_p \rangle = \omega_p\left(X_p + \lambda Y_p\right) = \omega_p\left(X_p\right) + \lambda \omega_p\left(Y_p\right) = \langle \omega_p, X_p \rangle + \lambda \langle \omega_p, Y_p \rangle$$

and we may also take, in effect by definition,

$$\langle \omega_p + \lambda \eta_p, X_p \rangle = \left(\omega_p + \lambda \eta_p\right)\left(X_p\right) = \omega_p\left(X_p\right) + \lambda \eta_p\left(X_p\right)$$

$$= \langle \omega_p, X_p \rangle + \lambda \langle \eta_p, X_p \rangle.$$

Thus $\langle\ ,\ \rangle$ is a linear in each of its entries.

For each $f \in \mathcal{F}(M)$, the total differential df of f is defined by

$$\langle (df)_p, X \rangle = Xf$$

for $X \in T_p(M)$. If $u_1, \ldots, u_n$ is a local coordinate system in M, then the total differentials $(du_1)_p, \ldots, (du_n)_p$ form a basis of $T_p^*(M)$. In fact, they form the dual basis of the basis $(\partial/\partial u_1)_p, \ldots, (\partial/\partial u_n)_p$ of $T_p(M)$.

1.1.32. Definition (The Maximal Rank Condition). Let $F: M \to N$ be a smooth mapping from an m-dimensional manifold M to an n-dimensional manifold N.

The rank of F at a point $x \in M$ is the rank of the $n \times m$ Jacobian matrix $\left({\partial F^i}/{\partial x^j}\right)$ at x, where $y = F(x)$ expressed in any convenient local coordinates near x. The mapping F is of maximal rank on a subset $S \subset M$ if for each $x \in S$ the rank of F is as large as possible (i.e., the minimum of m and n).

The definition of the rank of F at x does not depend on the particular local coordinates chosen on M or on N. For example, the rank of $F(x, y) = xy$ on $\mathbb{R}^2$ is 1 at all points except the origin (0, 0) since its Jacobian matrix $\left(F_x, F_y\right) = (y, x)$ is nonzero except at $x = y = 0$. (Here, subscripts denote derivatives, so $F_x = \partial F/\partial x$, etc.)

1.1.33. Theorem. Let $F: M \to N$ be of maximal rank at $x_0 \in M$. Then there local coordinates $x = (x^1, \dots, x^m)$ near x_0, and $y = (y^1, \dots, y^n)$ near $y_0 = F(x_0)$ such that in these coordinates F has the simple form $y = (x^1, \dots, x^m, 0, \dots, 0)$, if $n > m$, or $y = (x^1, \dots, x^n)$, if $n \leq m$.

1.1.34. Definition. A q-dimensional distribution on a manifold M is a mapping $\underline{D}$ defined on M which assigns to each point p in M a q-dimensional linear subspace $\underline{D}_p$ of $T_p M$.

A q-dimensional distribution $\underline{D}$ is called differentiable if there are q differentiable vector fields on a neighborhood of p which, for each point Q in this neighborhood, form a basis of $\underline{D}_Q$.

The set of these q vector fields is called a local basis of the distribution $\underline{D}$. A vector field X belongs to the distribution $\underline{D}$, and we write $X \in \underline{D}$ if, for any point $p \in M, X_p \in \underline{D}_p$.

A distribution $\underline{D}$ is said to be involutive if, for all differentiable vector fields X, Y belonging to $\underline{D}$, we have $[X, Y] \in \underline{D}$. By a distribution we shall always mean a differentiable distribution.

1.1.35. Definition (Submanifolds). Let M be a C^∞ manifold. A manifold N is a submanifolds of M if there is a one-to-one C^∞-map $i: N \to M$ such that di is one-to-one at every point. We call i an imbedding and say that N is imbedded in M by i.

1.1.36: The Differential of a Map

Let $F: N \to M$ be a C^∞ map between two manifolds. At each point $p \in N$, the map F induces a linear map of tangent spaces, called its differential at p, $F_*: T_p N \to T_{F(p)} M$ as follows.

If $X_p \in T_p N$, then $F_*(X_p)$ is the tangent vector in $T_{F(p)} M$ defined by

$$\left(F_*(X_p)\right) f = X_p(f \circ F) \in \mathbb{R} \text{ for } f \in C^\infty_{F(p)}(M)$$

here f is a germ at $F(p)$, represented by a C^∞ function in a neighborhood of $F(p)$.

In terms of local coordinates,

$$F_*(X_p) = F_*\left(\sum_{i=1}^{m} X^i \frac{\partial}{\partial u^i}\right) = \sum_{j=1}^{n}\left(\sum_{i=1}^{m} X^i \frac{\partial F^j}{\partial u^i}\right)\frac{\partial}{\partial y^j}.$$

Consequently, the differential F_* defines a linear map from T_pN to $T_{F(p)}M$, whose local coordinate matrix expression is just the $n \times m$ Jacobian matrix $\left(\partial F^j/\partial u^i\right)$ of F at p.

Let $F: N \to M$ and $G: M \to P$ be smooth maps of manifolds, and $p \in N$. The differential of F at p and G at $F\,(p)$ are linear maps

$$T_p(N) \xrightarrow{F_{*},p} T_{F(p)}M \xrightarrow{G_{*},F(p)} T_{G(F(p))}P.$$

1.1.37. Theorem (The Chain Rule). If $F: N \to M$ and $G: M \to P$ are smooth maps of manifolds and $p \in N$, then $G \circ F$ is smooth at p, and

$$(G \circ F)_{*,p} = G_*\left(F_p\right) \circ F_{*,p}.$$

Proof. Let $X_p \in T_pN$ and let f be a smooth function at $G\big(F(p)\big)$ in p. Then

$$\big((G \circ F)_* X_p\big)f = X_p(f \circ G \circ F)$$

and

$$\Big((G_* \circ F_*)X_p\Big)f = \Big(G_*\big(F_* X_p\big)\Big)f = \big(F_* X_p\big)(f \circ G) = X_p(f \circ G \circ F).$$

Remark: The differential of the identity $\mathbb{I}_M: M \to M$ at any point p in M is the identity map $\mathbb{I}_{T_pM}: T_pM \to T_pM$, because $\big((\mathbb{I}_M)_* X_p\big)f = X_p(f \circ \mathbb{I}_M) = X_p f$, for any $X_p \in T_pM$ and $f \in C_p^\infty(M)$.

1.1.38. Corollary. If $F: N \to M$ is a diffeomorphism of manifolds and $p \in N$, then $F_*: T_pN \to T_{F(p)}M$ is an isomorphism of vector spaces.

Proof. To say that F is a diffeomorphism means that it has a differentiable inverse $G: M \to N$ such that $G \circ F = \mathbb{I}_N$ and $F \circ G = \mathbb{I}_M$.

By the chain rule,

$$(G \circ F)_* = G_* \circ F_* = (\mathbb{I}_N)_* = \mathbb{I}_{T_pN},$$

$$(F \circ G)_* = F_* \circ G_* = (\mathbb{I}_M)_* = \mathbb{I}_{T_{F(p)}M}.$$

Hence, F_* and G_* are isomorphism. ■

1.1.39. Corollary (Invariance of Dimension). If an open set $U \subset \mathbb{R}^n$ is diffeomorphic to an open set $V \subset \mathbb{R}^m$, then $n = m$.

Proof. Let $F: U \to V$ be a diffeomorphism and let $p \in U$. By Corollary 1.1.38, $F_{*,p}: T_p U \to T_{F(p)} V$ is an isomorphism of vector spaces. Since there are vector space isomorphisms $T_p U \simeq \mathbb{R}^n$ and $T_{F(p)} \simeq \mathbb{R}^m$, we must have that $n = m$. ■

1.1.40. Differential Forms

Differential forms are generalizations of real valued functions on a manifold.

Instead of assigning to each point of the manifold a number, a differential p-form assigns to each point a p-covector on its tangent space.

For $p = 0$, differential 0-form is a scalar function $f(x): M^n \to \mathbb{R}$. And for $p = 1$, differential 1-form is a convector field ω written in the form

$$\omega = \sum \omega_i(x) dx^i.$$

For $p = n$, differential n-form = object of integration:

$$\Omega = f(x) dx^1 \wedge \ldots \wedge dx^n, \qquad \text{(locally)}$$

such that for $x = x(y)$ we have

$$\Omega = f\big(x(y)\big) dx^1 \wedge \ldots \wedge dx^n, dx^i = \frac{\partial x^i}{\partial y^j} dy^j, \qquad \text{(summation in } j\text{)}.$$

1.1.41. Definition. Differential k-form in M^n is a smooth quantity Ω which locally in every chart of atlas $(x^1, \ldots, x^n)$ can be written in the form

$$\Omega = \textstyle\sum_I f_I(x) dx^{i_1} \wedge \ldots \wedge dx^{i_k} = f_I dx^I,$$

$dx^I = dx^{i_1} \wedge \ldots \wedge dx^{i_k}$, where $I = (i_1 < i_2 \ldots < i_k)$, and for $x = x(y)$ we have

$$\Omega = f_I\big(x(y)\big) dx^i = g_J(y) dy^J, dy^J = dy^{j_1} \wedge \ldots \wedge dy^{j_k}, \; j_1 < j_2 < \cdots < j_k.$$

The symbols dx^i form an associative algebra under multiplication $\wedge$ (Exterior Product), such that

$$dx^i \wedge dx^j = -dx^j \wedge dx^i \quad \text{(Bilinear, associative)}$$

If $\omega = \sum_I \omega_I dx^I$ is a p-form defined on an open subset of $\mathbb{R}^n$, we define a $(p+1)$ - form called exterior derivative of ω as

$$d\omega := \sum_I d\omega_I \wedge dx^I.$$

1.1.42. Definition. A (smooth) k-dimensional vector bundle is a pair of smooth manifolds E (the total space) and M (the base), together with a surjective map $\pi: E \to M$ (the projection), satisfying the following conditions:

(i) Each set $E_p := \pi^{-1}(p)$ (called the fiber of E over p) is endowed with the structure of a vector space.

(ii) For each $p \in M$, there exist a neighborhood U of p and a diffeomorphism $\varphi: \pi^{-1}(U) \to U \times \mathbb{R}^k$, called a local trivialization of E, such that the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U) & \overset{\varphi}{\to} & U \times \mathbb{R}^k \\ \pi \downarrow & & \downarrow \pi_1 \\ U & = & U \end{array}$$

(where π_1 is the projection onto the first factor).

(iii)The restriction of φ to each fiber,$\varphi: E_p \to \{p\} \times \mathbb{R}^k$, is a linear isomorphism.

1.1.43. Definition. If $\pi: E \to M$ is a vector bundle over M, a section of E is a map $F: M \to E$ such that $\pi \circ F = Id_M$, or equivalent, $F(p) \in E_p$ for all p.

It is said to be a smooth section if it is smooth as a map between manifolds.

1.2: Riemannian Manifolds

1.2.1. Definition. A Riemannian metric on a smooth manifold M is a tensor field g of type (0, 2) that is symmetric $\big(i.e., \ g(X,Y) = g(Y,X)\big)$ and positive definite $(\text{i.e.}, g(X,X) > 0 \text{ if } X \neq 0)$. A Riemannian metric thus determines an inner product on each tangent space T_pM, which is typically written

$$\langle X, Y \rangle := g(X,Y) \text{ for } X, Y \in T_pM.$$

A manifold M with Riemannian metric g is called a Riemannian manifold. If the metric is not positive definite it is called a pseudo Riemannian manifold.

The notion of derivatives can be generalized to manifolds with the concept of a connection.

1.2.2. Definition. An affine connection on a manifold M is a rule ∇ which assigns to each $X \in \mathfrak{X}(M)$ a linear mapping ∇_X of the vector space $\mathfrak{X}(M)$ into itself satisfying the following two conditions:

(1) $\nabla_{fX+gY} \quad = f\nabla_X + g\nabla_Y$;

(2) $\nabla_X(fY) \ = f\nabla_X Y + (Xf)Y$

for f , $g \in \mathcal{T}(M)$, $X, Y \in \mathfrak{X}(M)$. The operator ∇_X is called the covariant differentiation with respect to X.

We denote by $\mathfrak{X}(M)$ the set of all differentiable vector fields on M. And denote by $\mathcal{T}(M)$ the set of differentiable real-valued functions on M.

We define the covariant differentiation of a function f with respect to X by

$$\nabla_X f = Xf.$$

Thus for any tensor field S of type (0, s) or (1, s) we define the covariant derivative $\nabla_X S$ of S with respect to X by

$$(\nabla_X S)(X_1, \dots, X_s) = \nabla_X\big(S(X_1, \dots, X_s)\big) - \sum_{i=1}^{s}\{S(X_1, \dots, \nabla_X X_i, \dots, X_s)\}, \quad (1.2.*)$$

for any $X_i \in \mathfrak{X}(M), i = 1, \dots, s$. In a similar way we can define the covariant of a tensor field of type (r, s), but for our purpose (1.2.*) is sufficient.

The tensor field S is said to be parallel with respect to the affine connection ∇ if we have

$$\nabla_X S = 0 \text{ , for any } X \in \mathfrak{X}(M).$$

The torsion tensor T of an affine connection ∇ is a tensor field T of type (1, 2) defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

for any $X, Y \in \mathfrak{X}(M)$, where $[X, Y]$ is the Lie bracket of vector fields X and Y defined by

$$[X, Y](f) = X(Yf) - Y(Xf),$$

for any $f \in \mathcal{T}(M)$. A torsion-free connection is an affine connection with vanishing torsion tensor field.

1.2.3. Lemma. A connection ∇ is local: if U is an open subset of M, $X|_U \equiv 0 \Rightarrow \nabla_X Y|_U \equiv 0$ and $(\nabla_Y X \equiv 0)$ for all $Y \in \mathfrak{X}(M)$.

Proof. Let $p \in U$ and choose $f: M \to \mathbb{R}$ such that $f \equiv 0$ on a neighborhood of p and $f \equiv 1$ outside U. Then $X|_U = 0$ implies that $fX = X$, and hence

$$\begin{aligned}
(\nabla_X Y)(p) &= \left(\nabla_{fX} Y\right)(p) \\
&= f(p)(\nabla_X Y)(p) \\
&= 0, \\
(\nabla_Y X)(p) &= \left(\nabla_Y (fX)\right)(p) \\
&= (Yf)(p)X(p) + f(p)\nabla_Y X(p) \\
&= 0.
\end{aligned}$$

Since p is an arbitrary point of U, $(\nabla_X Y)|_U \equiv 0$ and $(\nabla_Y X)(p) = 0$. ■

1.2.4. Definition. Suppose ∇ is an affine connection on M and f is a diffeomorphism of M. A new affine connection ∇' can be defined on M by

$$\nabla'_X Y = f_*^{-1}\left(\nabla_{f_* X} f_* Y\right),$$

for $X, Y \in \mathfrak{X}(M)$.

The affine connection ∇ is called invariant under f if

$$\nabla = \nabla'$$

In this case f is called an affine transformation of M. Similarly we can define an affine transformation of one manifold onto another.

1.2.5. Theorem. On a Riemannian manifold M there exist one and only one affine connection satisfying the following two conditions:

(1) The torsion tensor T vanishes, i.e., $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = 0$;
(2) g is parallel, i.e., $\nabla_X g = 0$.

Proof. Existence: given vector fields X and Y on M, we define $\nabla_X Y$ by setting

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) + g([X,Y],Z) + g([Z,X],Y)$$
$$+g(X,[Z,Y]) \qquad\qquad (*)$$

for any vector field Z on M. Then the mapping $(X,Y) \to \nabla_X Y$ defines an affine connection on M. From the definition of $\nabla_X Y$, we have $T(X,Y) = 0$ and

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$

which shows that $\nabla_X g = 0$, that is ∇ is a metric connection on M.

Uniqueness: By a straight forward computation, we can see that,

If $\nabla_X Y$ satisfies $\nabla_X g = 0$ and $T(X,Y) = 0$, then it satisfies the equation which defines $\nabla_X Y$. ■

The connection ∇ given by $(*)$ is called the Riemannian connection (sometimes called the Levi Civita Connection).

Putting $X = \partial/\partial x^j$, $Y = \partial/\partial x^i$ and $Z = \partial/\partial x^k$ in $(*)$, the components Γ^i_{jk} of the Riemannian connection with respect to a local coordinate system $\{x^1, \dots, x^n\}$ are given by

$$\sum_1 g_{1k}\Gamma^1_{ji} = i\left(\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

1.2.6. Isometric Immersion:

Let M and $\bar{M}$ be a Riemannian manifolds with Riemannian metric g and $\bar{g}$ respectively. A mapping $f: M \to \bar{M}$ is called isometric at a point x of M if $g(X,Y) = \bar{g}(f_*X, f_*Y)$ for all $X, Y \in T_x(M)$. In this case, f_* is injective at x, because $f_*X = 0$ implies $X = 0$. A mapping f which is isometric at every point of M is thus an immersion, which we call an isometric immersion. If moreover, f is one-to-one, then it is called an isometric imbedding of M onto $\bar{M}$.

In this case, the differential of the isometry f commutes with the parallel translation.

An immersion f (or an embedding) of a Riemannian manifold (M, g) into another Riemannian manifold $(\widetilde{M}, \tilde{g})$ is said to be isometric if it satisfies the condition $f^*\tilde{g} = g$. In this case, M is called a Riemannian submanifold (or simply a submanifold) of $\widetilde{M}$.

1.2.7. Definition. Let (M, g) and $(\widetilde{M}, \tilde{g})$ be Riemannian manifolds. A map $\varphi: (M, g) \rightarrow (\widetilde{M}, \tilde{g})$ is said to be conformal if there exists a function $\lambda: M \rightarrow \mathbb{R}$ such that $e^{\lambda(p)}. g_p(X_p, Y_p) = \tilde{g}_{\varphi(p)}\left(d\varphi_p(X_p), d\varphi_p(Y_p)\right)$, for all $X, Y \in C^\infty(TM)$ and $p \in M$. The positive real valued function e^λ is called the formal factor of φ. A conformal map with $\lambda \equiv 0$ i.e., $e^\lambda \equiv 1$ is said to be isometric. An isometric diffeomorphism is called an isometry.

1.2.8. Definition. Let $(\bar{M}, \bar{g})$ be a Riemannian manifold and M be a submanifold.

For a point $p \in M$, we define the normal space N_pM of M at p by

$$N_pM = \left\{X \in T_p\bar{M}: \bar{g}_p(X, Y) = 0 \text{ for all } Y \in T_pM\right\}.$$

For all $p \in M$ we have the orthogonal decomposition

$$T_p\bar{M} = T_pM \oplus N_pM.$$

The normal bundle of M in $\bar{M}$ is defined by

$$NM = \left\{(p, X): p \in M, X \in N_pM\right\}.$$

1.2.9 The Levi-Civita Connection (The Riemannian Connection)

In the Euclidean space $\mathbb{R}^3$ there are at least two ways to define a line segment.

(a) A line segment is the shortest path connecting two given points.
(b) A line segments is a smooth path $\gamma: [0,1] \rightarrow \mathbb{R}^3$ satisfying

$$\ddot{\gamma}(t) = 0 \qquad (1.2.1)$$

Since we have not said anything about calculus of variations which deals precisely with problems of type (a), we will use the second interpretation as our starting point. We will soon see however that both points of view yield the same conclusion.

Let us first reformulate (1.2.1). As we know, the tangent bundle of $\mathbb{R}^3$ is equipped with a natural trivialization, and as such, it has a natural trivial connection ∇^0 defined by

$$\nabla_i^0 \partial_i = 0 \quad \forall\, i, j \text{ , where } \partial_i := \partial_{x^i},\ \nabla_i := \nabla_{\partial_i}$$

i.e., all the Christoffel symbols vanish. Moreover, if g_0 denote the Euclidean metric, then

$$(\nabla_i^0 g_0)(\partial_j, \partial_k) = \nabla_i^0 \delta_{jk} - g_0(\nabla_i^0 \partial_j, \partial_k) - g_0(\partial_j, \nabla_i^0 \partial_k) = 0,$$

i.e., the connection is compatible with the metric. Condition (1.2.1) can be rephrased as

$$\nabla_{\dot{\gamma}(t)}^0 \dot{\gamma}(t) = 0, \tag{1.2.2}$$

So that problem of defining “Lines” in a Riemann manifold reduces to choosing a “natural” connection on the tangent bundle.

We would like this connection to be compatible with metric, but even so, there are infinitely many connections to choose from. The following fundamental result will solve this dilemma.

1.2.10. Proposition. Consider a Riemann manifold (M, g). Then there exist a unique symmetric connection ∇ on TM compatible with the metric g i.e.,

$$T(\nabla) = 0, \quad \nabla g = 0.$$

The connection ∇ is usually called the Levi-Civita connection associated to the metric g.

1.2.11. Definition. The connection ∇ is symmetric if

$$\nabla_X Y - \nabla_Y X = [X, Y] \tag{1.2.3}$$

for all vector fields $X, Y \in \mathfrak{X}(M)$.

1.2.12. Lemma. Let E_i be a local moving frame such that $[E_i, E_j] = 0,\ 1 \leq i,\ j \leq n$ (for instance $E_i = \partial_i$ could be a coordinate frame). Then ∇ is symmetric if and only if

$$\Gamma_{ij}^k = \Gamma_{ji}^k, 1 \leq i,\ j, k \leq n$$

Proof. $\nabla_{E_i} E_j - \nabla_{E_j} E_i = \left(\Gamma_{ij}^k - \Gamma_{ji}^k\right) E_k$. ■

1.2.13. Definition. The connection ∇ is compatible with the metric g if

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \tag{1.2.4}$$

for any three vector fields $X, Y, Z \in \mathfrak{X}(M)$.

1.2.14. Lemma. Let E_i be a local moving frame. Then ∇ is compatible metric g if and only if

$$E_k g_{ij} = \Gamma_{ki}^{\ell} g_{j\ell} + \Gamma_{kj}^{\ell} g_{i\ell}, \qquad 1 \leq i, j, k, \ell \leq n$$

Proof. $E_k\langle E_i, E_j\rangle - \langle \nabla_{E_k} E_i, E_j\rangle - \langle E_i, \nabla_{E_k} E_j\rangle = E_k g_{ij} - \Gamma_{ki}^{\ell} g_{j\ell} - \Gamma_{kj}^{\ell} g_{i\ell}$. ■

1.2.15. Definition. A geodesic of a Riemann manifold (M, g) is a smooth path

$$\gamma: (a, b) \to M$$

satisfying

$$\nabla_{\dot\gamma(t)} \dot\gamma(t) = 0 \tag{1.2.5}$$

where ∇ is the Levi-Civita connection.

Using local coordinates $(x^1, \dots, x^n)$ with respect to which the Christoffel symbols are $\left(\Gamma_{ij}^k\right)$, and the path γ is described by $\gamma(t) = \left(x^1(t), \dots, x^n(t)\right)$, we can rewrite to geodesic equation as a second order, nonlinear system of ordinary differential equations.

1.2.16. Theorem. The operator ∇_X has the following properties:

(i) ∇_X is derivation of the mixed tensor algebra T;

(ii) ∇_X preserves type of tensor;

(iii) ∇_X commutes with contractions.

On a manifold M with an affine connection, we put

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \tag{1.2.6}$$

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}, \tag{1.2.7}$$

where X, Y are vector fields on M. Note that

$T(X,Y) = -T(Y,X)$ and $R(X,Y) = -R(Y,X)$.

A Riemannian manifold M or a Riemannian metric g on M is said to be complete if the metric function d is complete, that is, all Cauchy sequences converge. It is well known that the following conditions on M are equivalent:

(1) M is complete;
(2) Every bounded subset of M with respect to d is relatively complete;
(3) All geodesic arc can be extended in two directions indefinitely with respect to the arc length.

1.2.17. Theorem. (Hopf-Rinow Theorem) Let $(M, \langle\, , \rangle)$ be a connected Riemannian manifold, and $p_0 \in M$. The following are equivalent:

a. exp_{p_0} is defined in $T_{p_0}M$;
b. Closed bounded subset of M are compact;
c. (M, d) is a complete metric space;
d. $(M, \langle\, , \rangle)$ is (geodesically) complete;
e. There is a sequence of compact sets $K_n \subset K_{n+1} \subset M, \bigcup_n K_n = M$ such that if $p_n \notin K_n \ \forall_n \Longrightarrow \lim_{n\to+\infty} d(p_0, q_0) = +\infty$.

In addition, any of these is equivalent to the following:

f. $\forall\, p, q \in M$, there is a minimizing geodesic joining p and q.

1.2.18. Corollary. M compact $\Longrightarrow$ M is complete $\forall \langle\, , \rangle$.

1.2.19. Theorem (T. Levi- Civita (1929)) if M is a Riemannian manifold, then there exists a unique connection (called the Levi- Civita connection) for which

$$\nabla_X Y = \nabla_Y X + [X, Y] \tag{1.2.8}$$

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, \tag{1.2.9}$$

for all differentiable vector fields $X, Y, Z \in \mathfrak{X}(M)$.

Proof. Since at each point the inner product is a non-degenerate bilinear form, to calculate $\nabla_X Y$ it suffices to calculate $\langle \nabla_X Y, Z\rangle$ for any $X, Y, Z \in \mathfrak{X}(M)$.

To this end we have, using (1.2.8), (1.2.9) alternatively,

$$\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle - \langle Y, \nabla_X Z\rangle$$

$$= X\langle Y,Z\rangle - \langle Y,\nabla_Z X\rangle - \langle Y,[X,Z]\rangle$$

$$= X\langle Y,Z\rangle - Z\langle Y,X\rangle + \langle \nabla_Z Y, X\rangle - \langle Y,[X,Z]\rangle$$

$$= X\langle Y,Z\rangle - Z\langle Y,X\rangle + \langle \nabla_Y Z, X\rangle + \langle [Z,Y],X\rangle - \langle Y,[X,Z]\rangle$$
$$= X\langle Y,Z\rangle - Z\langle Y,X\rangle + Y\langle Z,X\rangle - \langle Z,\nabla_Y X\rangle + \langle [Z,Y],X\rangle - \langle Y,[X,Z]\rangle$$

$$= X\langle Y,Z\rangle - Z\langle Y,X\rangle + Y\langle Z,X\rangle - \langle Z,\nabla_X Y\rangle \ - \langle\ Z,[Y,X]\rangle +$$
$$\langle [Z,Y],X\rangle - \langle Y,[X,Z]\rangle$$

Therefore, we have

$$\langle \nabla_X Y, Z\rangle = \tfrac{1}{2}\left\{\begin{matrix} X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle - \langle X,[Y,Z]\rangle - \langle Y,[X,Z]\rangle \\ - \langle Z,[Y,X]\rangle \end{matrix}\right\} \qquad (1.2.10)$$

That is, if we are given the restrictions (1.2.4), (1.2.5), then we have the explicit calculation of $\langle \nabla_X Y, Z\rangle$-thus $\nabla_X Y$ is uniquely determined.

To establish the existence of ∇, one takes (1.2.10) to define ∇ and verifies directly that ∇ indeed defines a connection satisfying (1.2.8), (1.2.9).■

1.2.20. Definition. The Riemann-Christoffel curvature tensor is the map

$$R: \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \text{ for } X,Y,Z \in \mathfrak{X}(M).$$

1.2.21. Proposition. The curvature tensor R satisfies the following symmetries:

1. $R(X,Y)Z = -R(Y,X)Z$.

2. $R(X,Y)Z = R(Y,Z)X + R(Z,X)Y = 0$.

3. $\langle R(X,Y)Z,W\rangle = -\langle R(X,Y)W,Z\rangle$.

4. $\langle R(X,Y)Z,W\rangle = \langle R(Z,W)X,Y\rangle$.

1.2.22. Definition. A connection is said to be flat if its curvature vanishes identically.

1.2.23 Sectional curvature

Sectional curvature is the generalization (to any Riemannian manifold) of the classical notion of the Gaussian curvature of a surface in $\mathbb{R}^3$. All we need know for this research is that if M has constant sectional curvature c then M is locally isometric to a domain in one of the following "models":

Euclidean space ($c = 0$), spherical space ($c > 0$), or hyperbolic (i.e., Lobachevski) space ($c < 0$).

If $c = 0$ one also says that M is flat.

The analogous notion in a Kählerian manifold is constant holomorphic sectional curvature c; the corresponding "model spaces" are C^n (if $c = 0$), CP^n (if $c > 0$), and the unit disc in C^n with "Bergman kernel metric" if $c < 0$.

Let $(M,\ g)$ be a Riemannian manifold. We shall give a geometric interpretation of the Riemannian curvature tensor of M.

Let p be a point of M. For each 2-dimensional subspace π of the tangent space T_pM, let $S_\pi \subset M$ be the surface in M that is the image under $\exp_p$ of π, or rather the image of the part π where $\exp_p$ is a diffeomorphism, $S_\pi = \exp_p\left(\pi \cap \varepsilon_p\right)$.

We give S_π the induced metric so that $S_\pi \subset M$ is a Riemannian embedding. Note that the tangent space of S_π is

$$T_pS_\pi = T_p\exp_p\left(\pi \cap \varepsilon_p\right) = \left(\exp_p\right)_* T_0\pi = T_0\pi = \pi \subset T_pM.$$

1.2.24. Definition. Sectional curvature at $p \in M$ is the function that to any tangent plane $\pi \subset T_pM$ to the manifold associates the Gaussian curvature $K(\pi) = K(S_\pi)_p$ at p of the Riemannian surface S_π.

1.2.25. Proposition. The sectional curvature of the tangent plane $\pi \subset T_pM$ is

$$K(\pi) = -\frac{Rm(X,Y;Y,X)}{|X|^2|Y|^2 - \langle X,Y\rangle^2}$$

where X, Y is any basis for π.

If the sectional curvature is equal to a constant c for all plane sections, M is called a space of constant curvature or a real-space-form.

1.2.26. Corollary. If M has constant sectional curvature $c \in \mathbb{R}$, then

$$R(X,Y;Z,W) = c(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle).$$

1.3: Symplectic Manifolds

A symplectic form on a manifold M is a closed non-degenerate 2-form Ω. Here non-degenerate means that for all $p \in M$ if there is a vector field $X \in T_pM$ such that $\Omega(X,Y) = 0$ for all $Y \in T_pM$, then $X = 0$.

1.3.1. Definition. A symplectic manifold is a pair (M,Ω), a manifold and a symplectic form on it.

For algebraic reasons symplectic manifolds must have even dimensions. In symplectic manifolds, there is a kind of special submanifolds, called isotropic submanifolds were the symplectic form restricts to zero. The most important case of the isotropic submanifolds is that of Lagrangian submanifolds, which is defined as follows.

1.3.2. Definition. A Lagrangian submanifold is an isotropic submanifold of maximal dimensions, namely half the dimension of the ambient symplectic manifold.

1.4: Kähler Manifolds

A complex analytic manifold id defined as a space covered with complex coordinate charts in such a way that the coordinates undergo holomorphic coordinate transformations in the common region of the charts. A Riemann surface, C^n and its projective space CP^{n-1} are simple examples.

Any n-dimensional complex manifold Z is also a real (analytic) manifold K of dimension $2n$ with $z^r = x^r + iy^r$ as complex coordinates for x^r, y^r as real coordinates $(r = 1, \dots, n)$. The tangent space $T_z(Z)$ of C^n can be endowed with a canonical automorphism $\mathrm{J}: u \to iu$ such that J defines an almost complex structure $(\mathrm{J}^2 = -I)$ on K. J is said to be a complex structure if and only if its torsion tensor $N \equiv 0$, i.e.,

$$N(X,Y) \equiv [\mathrm{J}X, \mathrm{J}Y] - [X,Y] - \mathrm{J}[X,\mathrm{J}Y] - \mathrm{J}[\mathrm{J}X,Y] = 0 \qquad (1.4.1)$$

for every X, Y of K.

1.4.1. Definition. An almost complex structure on a differentiable manifold M is a differentiable map $\mathtt{J}: TM \to TM$, such that:

(i) $\mathtt{J}$ maps linearly $T_p(M)$ into $T_p(M)$ for all $p \in M$;
(ii) $\mathtt{J}^2 = -\mathtt{I}$ on each $T_p(M)$, where $\mathtt{I}$ stands for the identity map.

1.4.2. Definition. A pair $(M, \mathtt{J})$ is called an almost complex manifold if $\mathtt{J}$ is an almost complex structure on M.

1.4.3. Definition. Let $(M, \mathtt{J})$ be a complex manifold, $\dim_{\mathbb{C}}(M) = n$, and g is Riemannian metric. Then g on a complex manifold M is called Hermitian if $g(\mathtt{J}X, \mathtt{J}Y) = g(X, Y)$, i.e., g and $\mathtt{J}$ are compatible, for any X, Y in $T(M)$.

K is a Kähler manifold if it is Hermitian and $\nabla\mathtt{J} = 0$, where ∇ is an affine connection on K. The two form ω of a Hermitian K is defined by

$$\omega(\mathrm{X}, \mathrm{Y}) = \mathrm{g}(\mathrm{X}, \mathrm{JY}).$$

1.4.4. Remark. Since $\mathtt{J}^2 = -\mathtt{I}d$, it is equivalent to $g(\mathtt{J}X, Y) = -g(X, \mathtt{J}Y)$.

The form $\omega(X, Y) := g(X, \mathtt{J}Y)$ is a skew-symmetric.

1.4.5. Definition. The differential form ω is called the Hermitian form $(M, \mathtt{J}, g)$.

1.4.6. Definition. A complex Hermitian manifold is called Kähler if $d\omega = 0$, that is, ω is closed.

Example: C^n, with the Hermitian metric

$$ds^2 = \sum_r dz_r dz^{-r}$$

is Kähler.

We say that $(M, \mathtt{J}, g)$ is an almost Hermitian manifold if tangent bundle of M has an almost complex structure $\mathtt{J}$ (i.e. $\mathtt{J}^2 = -\mathtt{I}$) and a Riemannian metric g such that

$$g(X, Y) = g(\mathtt{J}X, \mathtt{J}Y)$$

for $X, Y \in \Gamma(TM)$. Here $\Gamma(TM)$ denotes the lie algebra of vector fields on M.

The manifold M is orientable and even dimensional $2m$.

1.4.7 Some Important Classes of Almost Hermitian Manifold (AH-Manifold).

An AH-manifold with J integrable is called a Hermitian manifold (H-manifold).

The fundamental 2-form Ω an AH-manifold is defined by

$$\Omega(X,Y) = g(X, \mathrm{J}Y), \text{ for any } X,Y \in \Gamma(TM).$$

An AH-manifold is called an almost Kähler manifold (AK-manifold) if Ω is closed. Using the Levi-Civita connection ∇ of an AH-manifold (M, J, g), we define a nearly Kähler manifold (NK-manifold) and a quasi-Kähler manifold (QK-manifold) by the conditions:

$$(NK): (\nabla_X \mathrm{J})Y + (\nabla_Y \mathrm{J})X = 0$$

$$(QK): (\nabla_X \mathrm{J})Y + (\nabla_{\mathrm{J}X} Y)\mathrm{J}Y = 0$$

for all $X,Y \in \Gamma(TM)$.

An AH-manifold is Kähler if and only if $\nabla \mathrm{J} = 0$. An AH-manifold is Hermitian if and only if:

$$(H) \qquad (\nabla_X \mathrm{J})Y - (\nabla_{\mathrm{J}X} \mathrm{J})\mathrm{J}Y = 0.$$

We have the strict inclusion relations

$$K \subset AK \subset QK \subset AH \qquad K \subset NK \subset QK \qquad K \subset H \subset AH.$$

The Kähler manifolds form a very interesting class of manifolds in differential geometry. It is well known that the geometrical and topological structure of these manifolds is very rich.

The nearly Kähler manifolds, which are not Kähler, are perhaps the most interesting nonintegrable almost Hermitian manifolds.

Now we define the class of locally conformal Kähler manifolds ($l.\ c.\ K$ manifolds) which has been developed mainly in the last 20 years.

AH-manifolds is called $l.\ c.\ K$-manifold (M, J, g) if each of points has a neighborhood U on which the restriction of the metric g is conformal with a Kähler metric g' on U, such that $g|_U = exp(f_U)g'$ where f is a differentiable function on U.

Equivalently, a H-manifold is $l.\ c.\ K$-manifold if and only if there exists on M a closed 1-form ω, called the lee form, such that $d\Omega = \omega \wedge \Omega$.

A $l.\ c.\ K$-manifold with parallel lee form is called a generalized Hopf manifold ($g.$ H -manifold).

We shall suppose $\omega \neq 0$ everywhere. For a $g.$ H -manifold the lee 1-form ω has constant length:

let $|\omega| = \frac{c}{2}$. We denote $u = \frac{\omega}{2}$. let U be the vector field defined by $g(U, X) = u(X)$ for all X tangent to M and $V = \mathrm{J}U$.

We say that an AH-manifold M is a para-Kähler manifold or a F-space if the curvature tensor R satisfies

$$R(X, Y, Z, W) = R(X, Y, \mathrm{J}Z, \mathrm{J}W) \tag{1.4.2}$$

for all $X, Y, Z, W \in \Gamma(TM)$.

1.4.8. Remark. Kähler manifolds are the main object of complex algebraic geometry (algebraic geometry over $\mathbb{C}$).

1.5: Contact Manifold

1.5.1 Definition. A contact manifold M is an odd dimensional manifold with a 1-form ω such that $\omega\wedge(d\omega)^n \neq 0$, where $\dim M = 2n + 1$ and the exponent denotes the n^{ih} exterior power. We call ω a contact form of M.

Assume now that (M, ω) is a given contact manifold of dimension $2n + 1$. Then ω defines a $2n$-dimensional vector bundle over M, where the fiber at each point $p \in M$ is given by

$$\xi_p = \ker \omega_{\mathrm{p}}.$$

Since $\omega \wedge (d\omega)^n$ defines a volume form on M, we see that

$$\Omega := d\omega$$

is a called non-degenerate 2-form $\xi \oplus \xi$ and hence it defines a symplectic product of ξ such that $\left(\xi, \Omega|_{\xi \oplus \xi}\right)$ becomes a symplectic vector bundle. A consequence of this fact is that there exists an almost complex bundle structure

$$\widetilde{\mathrm{J}}: \xi \mapsto \xi$$

compatible with $d\omega$. i.e., a bundle endomorphism satisfying:

(1) $\widetilde{\mathrm{J}}^2 = -Id_{\xi}$,

(2) $d\omega(\widetilde{\mathrm{J}}X, \widetilde{\mathrm{J}}Y) = d\omega(X, Y)$ for all $X, Y \in \xi$,

(3) $d\omega(X, \widetilde{\mathrm{J}}X) > 0$ for $X \in \xi \backslash 0$.

1.6: Sasakian Manifold

1.6.1. Definition. Let S be a $(2n+1)$-dimensional manifold equipped with a structure (φ, ξ, η, g) such that:

(1) φ is a (1, 1) tensor field,

(2) ξ is a vector field,

(3) η is a field of a 1-form,

(4) g is a Riemannian metric.

Assume, in addition, that for any vector fields X and Y on S, (φ, ξ, η, g) satisfy the following algebraic conditions:

(1) $\varphi^2 X = -X + \eta(X)\xi$,

(2) $\eta(\xi) = 1$,

(3) $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$,

(4) $g(\xi, X) = \eta(X)$,

and the following differential conditions

(5) $N_{\varphi} + d\eta \otimes \xi = 0$, where

$$N_{\varphi}(X, Y) = [\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]$$

is the Nijenhuis tensor for φ.

(6) $d\eta(X, Y) = g(\varphi X, Y)$.

Then S is called a Sasakian manifold.

1.6.2. Example. A standard example of a Sasakian manifold is the odd dimensional sphere

$$S^{2k+1} = \left\{C^{k+1} \ni (z^1, \dots, z^{k+1}) : |z^1|^2 + \cdots + \left|z^{k+1}\right|^2 = 1\right\} \subset C^{k+1},$$

viewed as a submanifold of C^{k+1}. Let J be standard complex structure on C^{k+1}, $\tilde{g}$ the standard flat metric on $C^{k+1} \equiv R^{2k+2}$, and n be the unit normal to the sphere. The vector field ξ on S^{2k+1} is defined by $\xi = -\mathrm{J}n$. If X a tangent vector to the sphere then $\mathrm{J}X$ uniquely decomposes onto the part parallel to n and the part tangent to the sphere. Denote this decomposition by

$$\mathrm{J}X = \eta(X)n + \varphi(X).$$

This defines the 1-form η and the tensor field φ on S^{2k+1}. Denoting the restriction of $\tilde{g}$ to S^{2k+1} by g we obtain (φ, ξ, η, g) structure on S^{2k+1}. It is a matter of checking that this structure equips S^{2k+1} with a structure of a sasakian- Einstein manifold.

This construction is, in a certain sense, a Sasakian counterpart of the Fubini-study Kähler structure on CP^k.

Chapter (2)

The Theory of Submanifolds

We study problems in submanifold theory since the invention of calculus and it was started with curvature of plane curves. For a surface in Euclidean 3-space one has the two important quantities, namely, the mean curvature and the Gauss curvature. The mean curvature is an extrinsic invariant which measures the surface tension of the surface arisen from the ambient space.

2.1: Submanifolds of the Euclidean Space

We consider some aspects of the extrinsic geometry of a submanifold M of dimension m of the Euclidean space $\mathbb{R}^{m+n}$, with $n \geq 1$. When $n = 1$, M is a hypersurface and has locally a well defined unit normal vector field which can be extend to the whole hypersurface if it is orientable. This normal vector field defines a map, called the Gauss map, from M to the unit hypersphere in $\mathbb{R}^{m+1}$. A great deal of the extrinsic geometry of the hypersurface M can be derived from the Gauss map and its derivative map, the shape operator. The shape operator is, at each point of M, a self adjoint operator from the tangent space of M at p to itself. Its eigenvalues are called the principal curvatures and when they all are equal at a given point p, the point p is called an umbilic point. A hypersurface is totally umbilic if all its points are umbilic points.

At each point p of a submanifold M of codimension $n > 1$, there is an n-dimensional space of normal vectors to M at p. One can define a shape operator on M along a fixed unit normal vector field on M.

A hypersurface in $\mathbb{R}^{m+1}$ is a codimension one submanifold. The notion of a submanifold of an abstract smooth manifold will now be defined.

In fact, there exist two different notions of submanifolds, ‘embedded submanifold’ and ‘immersed submanifold’.

The examples of surfaces in $\mathbb{R}^3$-the sphere and the torus-are special cases of the general notion of a submanifold. Naïvely, given a smooth manifold M, a submanifold $N \subset M$ should be a subset which is also a smooth manifold in its own right. There are also several methods of describing submanifolds, either implicitly by the vanishing of some smooth functions, or parametrically by some local parametrization.

2.1.1. Definition. A submanifold of a manifold M is a pair (N, φ), where φ is a differentiable map from a manifold N into M such that, for each point $p \in N$, $(\varphi_*)_p$ is injective. In this case, φ is called an immersion. If, furthermore, φ is also injective, (N, φ) is called imbedded submanifold of M and φ is an imbedding.

Facts: (1) Recall the subspace topology. Let X be a topological space and let $S \subset X$ be any subset, then the subspace or relative topology on S (induced by the topology on X) is defined as follows. A subset $U \subset S$ is an open if there exist an open set $V \subset X$ such that $U = V \cap S$. In this case S is called a (topological) subspace of X.

(2)The rank of a linear transformation $L: V \to W$ between finite dimensional vector space is the dimension of the image $L(V)$ as a subspace of W, while the rank of a matrix A is the dimension of its column space. If L is represented by a matrix A relative to a basis for V and a basis for W, then the rank of L is the same as the rank of A, because the image $L(V)$ is simply the column space of A.

2.1.2. Definition. Let M be a smooth manifold. A submanifold of M is a subset $N \subset M$, together with a smooth, one-to-one map $\varphi: \widetilde{N} \to N \subset M$ satisfying the maximal rank condition everywhere, where the parameter space $\widetilde{N}$ is some other manifold and $N = \varphi(\widetilde{N})$ is the image of φ. In particular, the dimension of N is the same as that of $\widetilde{N}$, and does not exceed the dimension of M.

The map φ is often called an immersion, and serves to define a parametrization of the submanifold N. Often such a submanifold is referred to as an immersed-submanifold, to emphasize the difference between this definition and other notions of submanifold.

2.1.3. Definition. A regular submanifold N of a manifold M is a submanifold parametrized by $\varphi: \widetilde{N} \to M$ with the property that for each x in N there exist arbitrarily small open neighborhoods U of x in M such that $\varphi^{-1}[U \cap N]$ is a connected open subset of $\widetilde{N}$.

2.1.4. Lemma. A n-dimensional submanifold $N \subset M$ is regular if and only if for each $x_0 \in N$ there exist local coordinates $x = (x^1, \dots, x^m)$ defined on a neighborhood U of x_0 such that

$$N \cap U = \{x : x^{n+1} = \cdots = x^m = 0\}.$$

Such a coordinate chart is called a flat coordinate chart on M. Note that, in view of this lemma, for regular submanifolds $N \subset M$ we can dispense with the parametrizing manifold $\widetilde{N}$ and just treat N as a manifold in its own right.

Namely the flat local coordinates $x = (x^1, \dots, x^m)$ on $U \subset M$ induce local coordinate, namely $\tilde{x} = (x^1, \dots, x^n)$, on $U \cap N$. The parametrization thereby is replaced by the natural inclusion $N \subset M$.

2.1.5. Theorem. Let M be a smooth m-dimensional manifold, and $F: M \to \mathbb{R}^n, n \leq m$, be a smooth map. If F is of maximal rank on the subset $N = \{x: F(x) = 0\}$, then N is a regular, $(m - n)$- dimensional submanifold of M.

2.1.6. Definition. A subset $N \subset M$ is called a smooth embedded n-dimensional submanifold in M if for every $p \in N$, there exists a chart (U, φ) for M, with $p \in U$, such that

$$\varphi(U \cap N) = \varphi(U) \cap (\mathbb{R}^n \times \{0\}) = \{x \in \varphi(U): x_{n+1} = \cdots = x_m = 0\}.$$

The codimension of N is defined as: $\operatorname{codim} N = \dim M - \dim N$.

If $F: N \to M$ is a differentiable map, we define the rank of F at $p \in N$ to be the rank of the linear map $F_*: T_p N \to T_{F(p)} M$; it is of course just the rank of the matrix of partial derivatives of F in any coordinate chart, or the dimension of $\operatorname{Im} F_* \subset T_{F(p)} M$.

If F has the same rank k at every point it has constant rank, and write $\operatorname{rank} F = k$.

2.1.7. Definition. A differentiable mapping $F: N \to M$ is called an immersion if $\operatorname{rank} F = \dim N$ at all points of N.

Similarly a differentiable mapping $F: N \to M$ is called a submersion if

$$\operatorname{rank} F = \dim M.$$

One special kind of immersion is particularly important. A (smooth) embedding is an injective immersion $F: N \to M$ that is also a topological embedding, i.e., a homeomorphism onto its image $F(N) \subset M$ in the subspace topology.

2.1.8. Implicit Submanifolds

Instead of defining a surface S in $\mathbb{R}^3$ parametrically, an alternative method is to define it implicitly by the vanishing of a smooth function:

$$S = \{F(x, y, z) = 0\}.$$

Let $f: M \to N$ be a smooth maps between manifolds. We say that f is a local diffeomorphism at p if f maps neighborhood of p diffeomorphically onto a neighborhood of $f(p)$.

2.1.9 Theorem (The inverse function theorem).Suppose that $f: M \to N$ is a smooth map whose derivative df_p at the point p is an isomorphism. Then f is a local diffeomorphism at p.

2.1.10. Definition. A point $p \in M^m$ is called a regular value of a smooth map $f: M^m \to L^l, m \geq l$, if for any point $x \in f^{-1}(p)$ the differential (tangent map) $Df(x)$ has rank l.

2.1.11 Remark Let M^m be a smooth manifold and N^n a submanifold equipped with the induced topology. The restriction of the chart (U_p, φ_p) to N^n provide a chart on N^n.Thus N is a topological manifold and the induced charts $\left(U(p) \cap N^n, \varphi_p\big|_{(U(p))\cap N^n}\right)$, $p \in N^n$ provide a smooth atlas for N.

2.1.12. Theorem (Implicit function theorem). Let M^m and N^n be a smooth manifold with $m \geq n$, and let q be a regular value of a smooth map $f: M^m \to N^n$. Then the set $f^{-1}(q)$ is a smooth submanifold of M^m.

2.1.13. Embeded Submanifolds

Smooth submanifolds are modeled locally on the standard embedding of $\mathbb{R}^k$ into $\mathbb{R}^n$, identifying $\mathbb{R}^k$ with the subspace

$$\{(x^1, \ldots, x^k, x^{k+1}, \ldots, x^n): x^{k+1} = \cdots = x^n = 0\}$$

of $\mathbb{R}^n$. Somewhat more generally, if U is an open subset of $\mathbb{R}^n$, a k-slice, of U is any subset of the form

$$S = \{(x^1, \dots, x^k, x^{k+1}, \dots, x^n) \in U : x^{k+1} = c^{k+1}, \dots, x^n = c^n\}$$

for some constant $c^{k+1}, \dots, c^n$.

Clearly any k-slice is homeomorphic to an open subset of $\mathbb{R}^k$.

Let M be a smooth n-manifold, and let (U, φ) be a smooth chart on M. We say a subset $S \subset U$ is a k-slice of U if $\varphi(S)$ is a k-slice of $\varphi(U)$. A subset $N \subset M$ is called an embedded submanifold of dimension k (or an embedded k-submanifold or a regular submanifold) of M if for each point $p \in N$ there exists a chart (U, φ) for M such that $p \in U$ and $U \cap N$ is a slice of U. In this situation, we call the chart (U, φ) a slice chart for N in M, and the corresponding coordinates$(x^1, \dots, x^n)$ are called slice coordinates.

The difference $n - k$ is called the co-dimension of N in M.

2.1.14. Lemma. Let M be a smooth manifold and N a subset of M. Suppose every point $p \in N$ has a neighborhood $U \subset M$ such that $U \cap N$ is an embedded submanifold of U. Then N is an embedded submanifold of M.

The proposition below explains the reason for the name "embedded submanifold".

2.1.15. Proposition. Let $N \subset M$ be an embedded k-dimensional submanifold of M. With the subspace topology, N is a topological manifold of dimension k, and it has a unique smooth structure such that the inclusion map $N \hookrightarrow M$ is a smooth embedding.

2.1.16. Proposition (Smoothness of inverse map). Suppose M and N are smooth manifolds of the same dimension, and $F: M \to N$ is a homeomorphism that is also a smooth immersion. Then F^{-1} is smooth, so F is a diffeomorphism.

2.1.17. Proposition. Let S be a regular submanifold of N and $\mathcal{U} = \{(U, \varphi)\}$a collection of compatible adapted charts of N that over S.Then $\{(U \cap S, \varphi_S)\}$ is an atlas for S. Therefore, a regular submanifold is itself a manifold. If N has dimension n and S is locally defined by the vanishing of $n - k$ coordinate, then $\dim S = k$.

2.1.18. Level Sets of a Function

A level set of a map $F: N \to M$ is a subset

$$F^{-1}(\{c\}) = \{p \in N: F(p) = c\}$$

for some $c \in M$.The usual notation for a level set is $F^{-1}(c)$, rather than the more correct $F^{-1}(\{c\})$. The value $c \in M$ is called the level of the level set $F^{-1}(c)$.

If $F: N \to \mathbb{R}^m$, then

$$Z(F) := F^{-1}(0)$$

is the zero set of F. Recall that c is a regular value of F if and only if either c is not in the image of F or at every point $p \in F^{-1}(c)$, the differential $F_{*,p}: T_pN \to T_{F(p)}M$ is surjective. The inverse image $F^{-1}(c)$ of a regular value is called a regular level set. If the zero set $F^{-1}(0)$ is a regular level set of $F: N \to \mathbb{R}^m$, it is called a regular zero set.

2.1.19. Theorem (Regular Level set theorem). Let $F: N \to M$ be a C^∞ map of manifolds, with $\dim N = n$ and $\dim M = m$. Then a nonempty regular level set $F^{-1}(c)$, where $c \in M$, is a regular submanifold of N of dimension equal to $n - m$.

2.1.20. Definition. A subset $N \subset M$ is called an immersed submanifold if N is a smooth n-dimensional manifold. And the mapping $i: N \to M$ is an immersion.

2.1.21. Theorem. Let $f: N \to M$ be a constant rank mapping with $rk(f) = k$. Then for each $q \in f(N)$, the level set $S = f^{-1}(q)$ is an embedded submanifold in N with co-dimension equal to k.

2.1.22. Definition. Let M be a n-dimensional manifold, S a subset of M. A point $p \in S$ is called a regular point of S if p has an open neighborhood U in M that lies in the domain of some coordinate system $x^1, \dots, x^n$ on M with the property that the points of S in U are precisely those points in U whose coordinate satisfy $x^{m+1} = 0, \dots, x^n = 0$ for some m. This m is called the dimension of S at p.

Otherwise p is called a singular point of S. S is called a m-dimensional (regular) submanifold of M if every point of S in regular of the same dimensions m.

2.1.23. Examples of Regular Submanifolds

(1) (Hypersurface). Show that the solution set S of $x^3 + y^3 + z^3 = 1$ in $\mathbb{R}^3$ is a manifold of dimension 2.

Solution: Let $f(x, y, z) = x^3 + y^3 + z^3$. Then

$$S = f^{-1}(1).$$

Since

$$\frac{\partial f}{\partial x} = 3x^2, \quad \frac{\partial f}{\partial y} = 3y^2, \quad \text{and } \frac{\partial f}{\partial z} = 3z^2,$$

the only critical point of f is (0, 0, 0), which is not in S.Thus, 1 is a regular value of $f: \mathbb{R}^3 \to \mathbb{R}$.

By the regular level set theorem (Theorem 2.1.19) S is a regular submanifold of $\mathbb{R}^3$ of dimension 2. So S is a manifold. Proposition (2.1.17).

(2) (Solution set of two polynomial equations). Decide whether the subset S of $\mathbb{R}^3$ defined by the two equations

$$x^3 + y^3 + z^3 = 1$$

$$x + y + z = 0$$

is a regular submanifold of $\mathbb{R}^3$.

Solution. Define $F: \mathbb{R}^3 \to \mathbb{R}^2$ by

$$(u, v) = F(x, y, z) = (x^3 + y^3 + z^3, x + y + z).$$

Then S in the level set $F^{-1}(1, 0)$. The Jacobian matrix of F is

$$\mathrm{J}(F) = \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \end{bmatrix} = \begin{bmatrix} 3x^2 & 3y^2 & 3z^2 \\ 1 & 1 & 1 \end{bmatrix},$$

where $u_x = \frac{\partial u}{\partial x}$ and so forth. The critical points of F are the points $(x, y, z)$where the matrix $\mathrm{J}(F)$ has rank < 2. That is precisely where all 2×2 minors of $\mathrm{J}(F)$ are zero:

$$\begin{vmatrix} 3x^2 & 3y^2 \\ 1 & 1 \end{vmatrix} = 0, \qquad \begin{vmatrix} 3x^2 & 3z^2 \\ 1 & 1 \end{vmatrix} = 0 \qquad (*)$$

(The third condition

$$\begin{vmatrix} 3y^2 & 3z^2 \\ 1 & 1 \end{vmatrix} = 0$$

is a consequence of these two.) Solving $(*)$, we get $y = \pm x,\ z = \pm x$.

Since $x + y + z = 0$ on S, this implies that $(x, y, z) = (0, 0, 0)$. Since $(0, 0, 0)$ does not satisfy the first equation $x^3 + y^3 + z^3 = 1$, there are no critical points of F on S. Therefore, S is a regular level set. By the regular level set theorem, S is a regular submanifold of $\mathbb{R}^3$ of dimension 1.

2.1.24. Theorem. If M is an r-dimensional regular submanifold of $\mathbb{R}^n$ then for every $p \in M$ there exist at least one r-dimensional coordinate plane P such that linear projection $P \to \mathbb{R}^n$ restricts to a coordinate system for M defined in a neighborhood of p.

2.1.25: First Fundamental Form of a Submanifold

The important geometrical characteristic of a submanifold is its first fundamental form. Let γ be a curve in a submanifold. Then parameters $u^1, \dots, u^n$ are certain functions of parameters t:

$$u^i = u^i(t), \qquad i = 1, \dots, n.$$

The position vector of γ has the form:

$$r = r\big(u^1(t), \dots, u^n(t)\big) = \tilde{r}(t).$$

A general rule given us an expression for the arc-length of γ:

$$s = \int_{t_1}^{t_2} \sqrt{\left|\tilde{r}_t'\right|^2}\, dt.$$

Differentiating $\tilde{r}(t)$ as a composite function we get

$$\tilde{r}_t' = \sum_{i=1}^{n} r_{u^i}\, \frac{du^i}{dt}$$

Therefore

$$|\tilde{r}_t'|^2 = \sum_{i,j=1}^{n} (r_{u^i} r_{u^j}) \frac{du^i}{dt} \frac{du^j}{dt}$$

The coefficients of the first fundamental form are the functions

$$g_{ij} = (r_{u^i} r_{u^j}).$$

The form

$$ds^2 = \sum_{i,j=1}^{n} g_{ij}\ du^i\ du^j$$

is called the fundamental form.

We say this form is induced by the metric of ambient Euclidean space. The length of any curve in the submanifold is the integral

$$s = \int ds = \int \sqrt{\sum_{i,j=1}^{n} g_{ij}\ du^i\ du^j}\ .$$

For two given directions

$$a = r_{u^i}\, du^i, \ \ b = r_{u^j}\, \delta u^j,$$

we get their scalar product as

$$(ab) = |a||b| \cos\varphi = \sum g_{ij}\ du^i\ \delta u^j$$

Since

$$|a| = \sqrt{\sum g_{ij}\ du^i\ du^i}, \qquad |b| = \sqrt{\sum g_{ij}\ \delta u^i\ \delta u^j},$$

we can evaluate the cosine of the angle between a and b by the formula

$$\cos\varphi = \frac{\sum g_{ij}\ du^i\ \delta u^j}{\sqrt{\sum g_{ij}\ du^i\ du^j}\ \sqrt{g_{ij}\ \delta u^i\ \delta u^j}}$$

A property of the manifold is intrinsic if it depends only on the metric. All intrinsic properties form the intrinsic geometry of the manifold.

By the first fundamental form we define the volume of a submanifold, which is a notion analogous to the area of a surface. Consider an infinitely small n-dimensional curvilinear parallelepiped, which is built on coordinate curves as its edges. Its volume dV is approximately equals the volume of a straight line parallelepiped:

$$d_1 r = r_{u^1}\, du^1, \dots, d_n r = r_{u^n}\, du^n.$$

Given n-vectors $a, b, \dots, c$ in n-dimensional Euclidean space E^n, the volume of a parallelepiped built on them equals a mixed product, that is, the determinant of an n-matrix formed with the coordinate of vectors as rows:

$$V(a.b.\dots,c) = \begin{vmatrix} a_1 & a_2 \dots a_n \\ b_1 & b_2 \dots b_n \\ \dots & \dots \dots \dots \\ c_1 & c_2 \cdots c_n \end{vmatrix}$$

Evidently

$$V^2(a.b.\dots,c) = \begin{vmatrix} a_1 & a_2 \cdots a_n \\ b_1 & b_2 \cdots b_n \\ \dots & \dots \dots \dots \\ c_1 & c_2 \cdots c_n \end{vmatrix} \begin{vmatrix} a_1 & a_2 \cdots a_n \\ b_1 & b_2 \cdots b_n \\ \dots & \dots \dots \dots \\ c_1 & c_2 \cdots c_n \end{vmatrix} = \begin{vmatrix} a^2 & (ab) \cdots (ac) \\ (ab) & b^2 \quad \cdots \\ \dots & \dots \quad \dots \\ (ac) & \dots \dots c^2 \end{vmatrix}$$

Apply the latter formula to $d_1 r, \dots, d_n r$, which lie in tangent space $T_p M$. If we denote $\det\|g_{ij}\|$ by g, then

$$dV = \sqrt{g}\, du^1 \dots du^n.$$

Hence, the volume of the submanifold M is equal to

$$V = \int dV = \int_M \sqrt{g}\, du^1 \dots du^n.$$

2.2. Submanifolds of a Riemannian Manifold

We develop the basic concepts of the theory of Riemannian submanifolds, and then to use these concepts to derive a quantitative interpretation of the curvature tensor.

Let $i: M \to \widetilde{M}$ be an immersion of an n-dimensional manifold M into an m-dimensional Riemannian manifold $(\widetilde{M}, \tilde{g})$. Denote by $g = i^* \tilde{g}$ the induced Riemannian metric on M.

Thus, i become an isometric immersion and M is also a Riemannian manifold with the Riemannian metric $g(X, Y) = \tilde{g}(X, Y)$ for any vector fields X, Y in M.

The Riemannian metric g on M is called the induced metric on M. In local components, $g_{ij} = g_{AB} B_j^B B_i^A$ with $g = g_{ij}\, dx^j\, dx^i$ and $\tilde{g} = g_{BA} du^B du^A$.

If a vector field ξ_p of $\widetilde{M}$ at a point $p \in M$ satisfies

$$\tilde{g}(X_p, \xi_p) = 0 \tag{2.2.1}$$

For any vector X_p of M at p, then ξ_p is called a normal vector of M in $\widetilde{M}$ at p.

A unit normal vector field of M in $\widetilde{M}$ is sometimes called a normal section on M, or a normal direction on M.

By $T^\perp M$, we denote the vector bundle of all normal vectors of M in $\widetilde{M}$. Then, the tangent bundle of $\widetilde{M}$, restricted to M, is the direct sum of the tangent bundle TM of M and the normal bundle $T^\perp M$ of M in $\widetilde{M}$, i.e.,

$$T\widetilde{M}\big|_M = TM \oplus T^\perp M. \tag{2.2.2}$$

We note that if the submanifold M is of codimension one in $\widetilde{M}$ and they are both orientable, we can always choose a normal section ξ on M, i.e.,

$$g(X, \xi) = 0, \qquad g(\xi, \xi) = 1, \tag{2.2.3}$$

where X is any arbitrary vector field on M.

We denote by $\widetilde{\nabla}$ the Riemannian connection on $\widetilde{M}$ with respect to its Riemannian metric $\tilde{g}$. The Riemannian connection $\widetilde{\nabla}$ has no torsion, that is,

$$\widetilde{\nabla}_{\tilde{X}} \tilde{Y} - \widetilde{\nabla}_{\tilde{Y}} \tilde{X} - [\tilde{X}, \tilde{Y}] = 0, \tag{2.2.4}$$

and is metric, that is,

$$\widetilde{\nabla}_{\tilde{X}}\left(\tilde{g}(\tilde{Y},\tilde{Z})\right) = \tilde{g}(\widetilde{\nabla}_{\tilde{X}}\tilde{Y},\tilde{Z}) + \tilde{g}(\tilde{Y},\nabla_{\tilde{X}}\tilde{Z}) \tag{2.2.5}$$

where $\tilde{X}, \tilde{Y}$ and $\tilde{Z}$ are arbitrary vector fields on $\widetilde{M}$.

2.2.1. Definition. (The Second Fundamental Form) We define the second fundamental form of the submanifold M (or immersion i) to be the map (or bilinear form)

$$h : \Gamma(TM) \times \Gamma(TM) \to \Gamma(T^{\perp}M)$$

given by:

$$h(X,Y) = \widetilde{\nabla}_X Y - \nabla_X Y$$
$$= \left(\widetilde{\nabla}_X Y\right)^{\perp}.$$

where X, Y are extended arbitrarily of $\widetilde{M}$.

If $h = 0$ identically, then submanifold M is said to be totally geodesic, where $\Gamma(T^{\perp}M)$ is the set of the differentiable vector fields on normal bundle of M.

Totally geodesic submanifolds are simplest submanifolds.

2.2.2. Lemma. The second fundamental form is

(a) independent of the extensions of X and Y,

(b) bilinear over $C^{\infty}(M)$; and

(c) symmetric in X and Y.

Proof. First we show that the symmetry of h follows from the symmetry of the connection $\widetilde{\nabla}$.

Let X and Y be extended arbitrarily to M. Then

$$h(X,Y) - h(Y,X) = \left(\widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X\right)^{\perp} = [X,Y]^{\perp}.$$

Since X and Y are tangent to M at all points of M, so their Lie bracket. Therefore $[X,Y]^{\perp} = 0$, so h is symmetric.

Because $\widetilde{\nabla}_X Y\big|_p$ depends only in X_p, it is clear that $h(X,Y)$ is independent of the extension chosen for X, and that $h(X,Y)$ is linear over $C^\infty(M)$ in X. By symmetry, the same is true for Y. ■

2.2.3. Proposition. Let ∇ and $\widetilde{\nabla}$ be the respective Levi-Civita connections of M and $\widetilde{M}$. Then we have

$$\nabla_X Y = \left(\widetilde{\nabla}_X Y\right)^T$$

where X, Y are arbitrarily extensions to vector field on $\widetilde{M}$.

2.2.4. Definition. Let M be an n-dimensional submanifold of an m-dimensional Riemannian manifold $(\widetilde{M}, \tilde{g})$. By h, we denote the second fundamental form of M in $\widetilde{M}$. $H = \frac{1}{n} trace(h)$ is called the mean curvature vector of M in $\widetilde{M}$. If $H = 0$, the submanifold is called minimal.

Let $i: M \to \widetilde{M}$ be an immersion of an n-dimensional manifold M into an m-dimensional Riemannian manifold $\widetilde{M}$ with the Riemannian metric $\tilde{g}$. Denote by $g = i^*\tilde{g}$ the induced metric on M. Equipped with g, i becomes an isometric immersion. We shall identify X with its image $i_* X$ for any $X \in T(M)$.

If X, Y are vector fields tangent to M, we put

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y) \qquad (2.2.6),$$

where $\nabla_X Y$ and $h(X,Y)$ are the tangential and the normal components of $\widetilde{\nabla}_X Y$, respectively.

Formula (2.2.6) is called the Gauss formula.

2.2.5. Proposition. ∇ is the Riemannian connection of the induced metric $g = i^*\tilde{g}$ on M and $h(X,Y)$ is a normal vector field over M which is symmetric and bilinear in X and Y.

Proof. Replacing X and Y be αX and βY, respectively, α, β being functions on M, we have

$$\widetilde{\nabla}_{\alpha X}(\beta Y) = \alpha\left\{(X\beta)Y + \beta\widetilde{\nabla}_X Y\right\}$$

$$= \{\alpha(X\beta)Y + \alpha\beta\nabla_X Y\} + \alpha\beta h(X,Y),$$

from which we find

$$\nabla_{\alpha X}(\beta Y) = \alpha(X\beta)Y + \alpha\beta\nabla_X Y \tag{2.2.7}$$

$$h(\alpha X, \beta Y) = \alpha\beta h(X, Y) \tag{2.2.8}$$

Equation (2.2.7) shows that ∇ defines an affine connection on M and equation (2.2.8) shows that h is bilinear in X and Y since additivity is trivial.

Since the Riemannian connection $\tilde{\nabla}$ has no torsion, we have

$$\begin{aligned} 0 &= \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y] \\ &= \nabla_X Y + h(X, Y) - \nabla_Y X - h(X, Y) - [X, Y], \end{aligned}$$

from which by comparing the tangential and normal part, we obtain

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

and

$$h(X, Y) = h(Y, X).$$

These equations show that ∇ has no torsion and h is a symmetric bilinear map.

Since the metric $\tilde{g}$ is parallel, we can easily see that

$$\begin{aligned} \nabla_X g(Y, Z) = \tilde{\nabla}_X \tilde{g}(Y, Z) &= \tilde{g}\big(\tilde{\nabla}_X Y, Z\big) + \tilde{g}(Y, \tilde{\nabla}_X Z) \\ &= \tilde{g}(\nabla_X Y + h(X, Y), Z) + \tilde{g}(Y, \nabla_X Z + h(X, Z)) \\ &= \tilde{g}(\nabla_X Y, Z) + \tilde{g}(Y, \nabla_X Z) \\ &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \end{aligned}$$

for any vector fields X, Y, Z tangent to M. This shows that ∇ is the Riemannian connection of the induced metric g on M. ■

We call h the second fundamental form of the submanifold M (or of the immersion i). Let ξ be a normal vector field on M and X be a tangent vector field on M.

We decompose $\tilde{\nabla}_X \xi$ as

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{2.2.9}$$

where $-A_{\xi}X$ and $D_X\xi$ are the tangential and normal components of $\widetilde{\nabla}_X\xi$ respectively. We can easily see that $A_{\xi}X$ and $D_X\xi$ are both differentiable vector fields on M and normal bundle on M, respectively.

Moreover, (2.2.8) implies that $h(X_p, Y_p)$ depends only on $X_p, Y_p \in T_pM$, not on their extensions X, Y. Formula (2.2.9) is called the Weingarten formula.

2.2.6. Proposition. Let M be a submanifold of a Riemannian manifold $(\widetilde{M}, \tilde{g})$.

Then

(a) $A_{\xi}(X)$ is bilinear in vector fields ξ and X. Hence, $A_{\xi}(X)$ at a point $p \in M$ depends only on vector ξ_p and X_p. And

(b)For each normal vector field ξ of M and tangent vectors X, Y of M, we have

$$g(A_{\xi}X, Y) = \tilde{g}(h(X,Y), \xi) \tag{2.2.10}$$

Proof. Let α and β be any two functions on M. Then, we have

$$\widetilde{\nabla}_{\alpha X}(\beta\xi) = \alpha\widetilde{\nabla}_X(\beta\xi) = \alpha\{(X\beta)\xi + \beta\widetilde{\nabla}_X\xi\} \tag{2.2.11}$$

$$-A_{\beta\xi}(\alpha X) + D_{\alpha X}(\beta\xi) = \alpha(X\beta)\xi - \alpha\beta A_{\xi}X + \alpha\beta D_X\xi.$$

This implies that

$$A_{\beta\xi}(\alpha X) = \alpha\beta A_{\xi}X \tag{2.2.12}$$

and

$$D_{\alpha X}(\beta\xi) = \alpha(X\beta)\xi + \alpha\beta D_X\xi. \tag{2.2.13}$$

Thus, $A_{\xi}X$ is bilinear in ξ and X, since additivity is trivial. This proves (a). To prove (b), we notice that for any arbitrary vector field Y tangent to M, we have

$$\begin{aligned} 0 &= \tilde{g}(\widetilde{\nabla}_X Y, \xi) + \tilde{g}(Y, \widetilde{\nabla}_X\xi) \\ &= \tilde{g}(\nabla_X Y, \xi) + \tilde{g}(h(X,Y), \xi) - g\left(Y, A_{\xi}(X)\right) + \tilde{g}(Y, D_X\xi) \\ &= \tilde{g}(h(X,Y), \xi) - g(Y, A_{\xi}X). \end{aligned}$$

This shows (b). ■

Let $T^{\perp}(M)$ denote the normal bundle of the immersion $i: M \to \widetilde{M}$. From (2.2.11) we find

$$D_{\alpha X}(\beta\xi) = \alpha(X\beta)\xi + \alpha\beta D_X\xi.$$

Moreover, it is easy to verify that

$$D_{X+Y} = D_X + D_Y. \tag{2.2.14}$$

Equations (2.2.13) and (2.2.14) justified that D is a connection on the normal bundle $T^{\perp}(M)$.

In fact, we have the following

2.2.7. Proposition. D or $(\nabla^{\perp})$ is a metric connection in the normal bundle $T^{\perp}(M)$ of M in $\widetilde{M}$ with respect to induced metric on $T^{\perp}(M)$.

Proof. From (2.2.13) we see that D defines an affine connection on the normal bundle $T^{\perp}M$.

Moreover, for any two normal vector fields ξ and η in $T^{\perp}M$, we have

$$\widetilde{\nabla}_X\xi = -A_{\xi}X + D_X\xi; \quad \widetilde{\nabla}_X\eta = -A_{\eta}X + D_X\eta.$$

Hence, we get

$$\begin{aligned}\tilde{g}(D_X\xi,\eta) + \tilde{g}(\xi,D_X\eta) &= \tilde{g}\big(\widetilde{\nabla}_X\xi,\eta\big) + \tilde{g}(\xi,\widetilde{\nabla}_X\eta)\\ &= \widetilde{\nabla}_X\tilde{g}(\xi,\eta) = D_X\tilde{g}(\xi,\eta).\end{aligned}$$

This shows that D is a metric connection. ■

A normal vector field ξ on M is said to be parallel in the normal bundle, or simply parallel, if we have $\nabla^{\perp}\xi = 0$ identically.

2.3. Equations of Gauss, Codazzi and Ricci

Let $f: (M, g) \to \big(\widetilde{M}, \tilde{g}\big)$ be an immersion isometric. Denote by ∇ and $\widetilde{\nabla}$ the metric connections of M and $\widetilde{M}$, respectively.

For vector fields X and Y tangent to M, the tangential component of $\widetilde{\nabla}_X Y$ is equal to $\nabla_X Y$.

Let: $\quad h = \widetilde{\nabla}_X Y - \nabla_X Y \quad$ (2.3.1)

The h is a normal bundle valued symmetric (0, 2) tensor field on M which is called the second fundamental form of the submanifold. Formula (2.3.1) is known as the Gauss formula.

For a normal vector ξ at the point $x \in M$. We put

$$g\big(A_\xi X\,,Y\big) = \tilde{g}(h(X,Y),\xi). \tag{2.3.2}$$

Then A_ξ is a symmetric linear transformation on the tangent space $T_x M$ of M at x, which is called the shape operator (or the Weingarten map) in the direction of ξ. The eigenvalues of A_ξ are called the principle curvatures in the direction of ξ.

The metric connection on the normal bundle $T^\perp M$ induced from the metric connection of $\widetilde{M}$ is called the normal connection of M.

Now, Let M be a submanifold of a Riemannian manifold $\big(\widetilde{M},\tilde{g}\big)$, and h and A_ξ denote the second fundamental form and shape operator of M, respectively. The covariant derivative of h and A_ξ is, respectively, defined by

$$\big(\widetilde{\nabla}_X h\big)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z)$$

and

$$(\nabla_X A)_\xi Y = \nabla_X\big(A_\xi Y\big) - A_{D_X\xi} Y - A_\xi \nabla_X Y$$

For any vector fields X, Y tangent to M and any vector field ξ normal to M. If $\nabla_X h = 0$ for all X, then the second fundamental form of M is said to be parallel, which is equivalent to $\nabla_X A = 0$.

By direct calculations, we get the relation

$$g\big((\nabla_X h)(Y,Z),\xi\big) = g\big((\nabla_X A)_\xi Y,Z\big)$$

Let D denote covariant differentiation with respect to the normal connection. For a tangent vector field X and a normal vector ξ on M, we have

$$\widetilde{\nabla}_X\,\xi = -A_\xi X + D_X \xi \tag{2.3.3}$$

where $-A_\xi X$ is the tangential component of $\widetilde{\nabla}_X \xi$. (2.3.3) is known as the Weingarten formula, named after the 1861 J. Weingarten (1836-1910).

Let R and $\tilde{R}$ and R^D denote the Riemannian curvature tensors of ∇ , $\widetilde{\nabla}$ and D, respectively. Then the integrability condition for (2.3.1) implies

$$\tilde{R}(X\,,Y)Z = R(X\,,Y)Z + A_{h(X\,,Z)}Y - A_{h(Y\,,Z)}X + \big(\widetilde{\nabla}_X h\big)(Y,Z) - \big(\widetilde{\nabla}_Y h\big)(X\,,Z) \qquad (2.3.4)$$

For tangent vector fields X, Y, Z of M, where $\widetilde{\nabla}$ is the covariant differentiation with respective to the connection in $TM \oplus T^{\perp}M$. The tangential normal components of (2.3.4) yield the following equation of Gauss:

$$\langle R(X,Y)Z\,,W\rangle = \langle \tilde{R}(X,Y)Z,W\rangle + \langle h(X,W),h(Y,Z)\rangle - \langle h(X,Z),h(Y,W)\rangle \qquad (2.3.5)$$

and the equation of Codazzi:

$$(R(X,Y)Z)^{\perp} = \big(\widetilde{\nabla}_X h\big)(Y,Z) - \big(\widetilde{\nabla}_Y h\big)(X,Z)\,, \qquad (2.3.6)$$

where X, Y, Z and W are tangent vector of M, $(R(X,Y)Z)^{\perp}$ is the normal component of $R(X,Y)Z$ and $\langle\ \ ,\ \ \rangle$ is inner product.

If the Codazzi equation vanishes identically, then submanifold M is said to be curvature invariant submanifold.

Similary, for normal vector field ξ and η the relation

$$\langle \tilde{R}(X,Y)\xi\,,\eta\rangle = \langle R^D(X,Y)\xi\,,\eta\rangle - \langle \big[\mathrm{A}_{\xi}\,,\mathrm{A}_{\eta}\big]X,Y\rangle \qquad (2.3.7)$$

holds, which is called the equation of Ricci.

Equations (2.3.1), (2.3.3), (2.3.5), (2.3.6) and (2.3.7) are called the fundamental equations of the isometric immersion $f: M \to \widetilde{M}$.

As a special case, suppose the ambient space $\bar{M}$ is a Riemannian manifold of constant sectional curvature c. Then the equations of Gauss, Codazzi and Ricci reduce respectively to

$$\langle R(X,Y)Z,W\rangle = \langle h(X,W),h(Y,Z)\rangle - \langle h(X,Z),h(Y,W)\rangle + c\{\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle\}. \qquad (2.3.8)$$

$$\big(\widetilde{\nabla}_X h\big)(Y,Z) = \big(\widetilde{\nabla}_Y h\big)(X,Z), \qquad (2.3.9)$$

$$\langle R^D(X,Y)\xi\,,\eta\rangle = \langle [\mathrm{A}_\xi, \mathrm{A}_\eta]X, Y\rangle\,. \tag{2.3.10}$$

These formulas at that time already published by G. Mainardi, (1800-1879) in [Mainardi 1856].

The fundamental importance of formulas was fully recognized by O. Bonnet (1819-1892) in [Bonnet 1867]. The equations of Gauss and Codazzi for general submanifolds were first given by A. Voss in 1880. The equation (2.3.10) of Ricci was first given by G. Ricci (1853-1925) in 1888.

2.4. Proof the Gauss, Codazzi and Ricci Equations

2.4.1. Proof of Gauss Equation

$$\langle R(X,Y)Z,W\rangle = \langle \tilde{R}(X,Y)Z,W\rangle + \langle h(X,W),h(Y,Z)\rangle - \langle h(X,Z),h(Y,W)\rangle$$

Proof. Let M be an n-dimensional submanifold of an m-dimensional Riemannian manifold $\widetilde{M}$.

Let $\tilde{R}$ denote the curvature tensor of Riemannian manifold $\widetilde{M}$. Then for any vector X, Y, Z tangent to M, we have

$$\tilde{R}(X,Y)Z = \widetilde{\nabla}_X\widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y\widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]}Z$$

Applying the Gauss formula (2.4.1.1), we find that

$$\begin{aligned}\tilde{R}(X,Y)Z &= \widetilde{\nabla}_X\big(\nabla_Y Z + h(Y,Z)\big) - \widetilde{\nabla}_Y\big(\nabla_X Z + h(X,Z)\big)\\ &\qquad - \Big(\nabla_{[X,Y]}Z + h([X,Y],Z)\Big).\\ &= \nabla_X\nabla_Y Z + h(X,\nabla_Y Z) + \widetilde{\nabla}_X h(Y,Z) - \nabla_Y\nabla_X Z\\ &\qquad - h(Y,\nabla_X Z) - \widetilde{\nabla}_Y h(X,Z) - \nabla_{[X,Y]}Z - h([X,Y],Z).\\ &= R(X,Y)Z + h(X,\nabla_Y Z) - h(Y,\nabla_X Z)\\ &\qquad - h([X,Y],Z) + \widetilde{\nabla}_X h(Y,Z) - \widetilde{\nabla}_Y h(X,Z),\end{aligned}$$

where R denotes the curvature tensor of the submanifold M. Let $\xi_1,\ldots,\xi_{m-n}$ be orthonormal normal vector fields of M and let h^{x} be the corresponding second fundamental forms, that is,

$$\mathrm{h}(\mathrm{X},\mathrm{Y}) = \mathrm{h}^{\mathrm{x}}(\mathrm{X},\mathrm{Y})\xi_{\mathrm{x}},$$

where h is the second fundamental form of the submanifold M.

By using Weingarten formula (2.4.1.4) we obtain

$$\tilde{R}(X,Y)Z = R(X,Y)Z - \mathrm{A}_{h(Y,Z)}X + \mathrm{A}_{h(X,Z)}Y + (\nabla_X^{\perp}h)(Y,Z) - (\nabla_Y^{\perp}h)(X,Z)$$

The tangential component of this equation gives

$$k(\langle Y,Z\rangle X - \langle X,Z\rangle Y) = \left(\tilde{R}(X,Y)Z\right)^{\mathrm{T}} = R(X,Y)Z - \mathrm{A}_{h(Y,Z)}X + \mathrm{A}_{h(X,Z)}Y,$$

And the normal component gives

$$0 = \left(\tilde{R}(X,Y)Z\right)^{\perp} = (\nabla_X^{\perp}h)(Y,Z) - (\nabla_Y^{\perp}h)(X,Z).$$

If W is another vector field, then Gauss equation can be rewritten as:

$$k(\langle Y,Z\rangle\langle X,W\rangle - \langle X,Z\rangle\langle Y,W\rangle) = \langle R(X,Y)Z,W\rangle - \langle h(Y,Z),h(X,W)\rangle + \langle h(X,Z),h(Y,W)\rangle.$$

Therefore, we get that:

$$\langle \tilde{R}(X,Y)Z,W\rangle = \langle R(X,Y)Z,W\rangle + \langle h(X,Z),h(Y,W)\rangle - \langle h(Y,Z),h(X,W)\rangle. \blacksquare$$

As a special case, suppose the ambient space $\bar{M}$ is a Riemannian manifold of constant sectional curvature k . Then the equation of Gauss becomes:

$$\langle R(X,Y)Z,W\rangle = \langle h(X,W),h(Y,Z)\rangle - \langle h(X,Z),h(Y,W)\rangle + k(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle).$$

2.4.2. Proof of Codazzi Equation

Since: $\left(\widetilde{\nabla}_X h\right)(Y,Z) = \left(\widetilde{\nabla}_Y h\right)(X,Z) \Rightarrow$

$$(R(X,Y)Z)^{\perp} = \left(\widetilde{\nabla}_X h\right)(Y,Z) - \left(\widetilde{\nabla}_Y h\right)(X,Z) \Rightarrow$$

$$0 = \left(\widetilde{\nabla}_X h\right)(Y,Z) - \left(\widetilde{\nabla}_Y h\right)(X,Z). \blacksquare$$

2.4.3. Proof of Ricci Equation

Since:

$$\langle \tilde{R}(X,Y)\xi,\ \eta\rangle = \langle R^D(X,Y)\xi\ ,\ \ \eta\rangle - \langle [\mathrm{A}_\xi,\ \mathrm{A}_\eta]X,\ Y\rangle,$$

where R^D denotes the curvature tensor of the normal connection $\nabla^\perp$ on the normal bundle $T^\perp M$.

Appling the Gauss and Weingarten formula, we find that

$$\begin{aligned}
\tilde{R}(X,Y)\xi &= \tilde{\nabla}_X\tilde{\nabla}_Y\xi - \tilde{\nabla}_Y\tilde{\nabla}_X\xi - \tilde{\nabla}_{[X,Y]}\xi \\
&= \tilde{\nabla}_X\left(-\mathrm{A}_\xi Y + \nabla^\perp_Y\xi\right) - \tilde{\nabla}_Y\left(-\mathrm{A}_\xi X + \nabla^\perp_X\xi\right) \\
&\quad +\mathrm{A}_\xi[X,Y] - \nabla^\perp_{[X,Y]}\xi \\
&= -\nabla_X\left(\mathrm{A}_\xi Y\right) - h\left(X,\mathrm{A}_\xi Y\right) - \mathrm{A}_{\nabla^\perp_Y\xi}X \\
&\quad +\nabla^\perp_X\nabla^\perp_Y\xi + \nabla_Y\left(\mathrm{A}_\xi X\right) + h\left(Y,\mathrm{A}_\xi X\right) \\
&\quad +\mathrm{A}_{\nabla^\perp_X\xi}Y - \nabla^\perp_Y\nabla^\perp_X\xi + \mathrm{A}_\xi\nabla_X Y - \mathrm{A}_\xi\nabla_Y X \\
&\quad -\nabla^\perp_{[X,Y]}\xi \\
&= (\nabla_Y A)_\xi X - (\nabla_X A)_\xi Y + R^\perp(X,Y)\xi \\
&\quad +h\left(A_\xi X,Y\right) - h\left(X,A_\xi Y\right).
\end{aligned}$$

Here, $R^\perp(X,Y)\xi = \nabla^\perp_X\nabla^\perp_Y\xi - \nabla^\perp_Y\nabla^\perp_X\xi - \nabla^\perp_{[X,Y]}\xi$ is the curvature tensor of the normal bundle with respect to the normal connection $\nabla^\perp$, the so-called normal curvature tensor of M.

The normal part gives the so called Ricci equation. Namely:

$$0 = (\tilde{R}(X,Y)\xi)^\perp = R^\perp(X,Y)\xi + h\left(A_\xi X,Y\right) - h\left(X,A_\xi Y\right).$$

If η is another normal vector field of M, the Ricci equation can be written as

$$\langle R^\perp(X,Y)\xi\ ,\eta\rangle = \langle [\mathrm{A}_\xi,\mathrm{A}_\eta]X,Y\rangle$$

where $[\mathrm{A}_\xi,\mathrm{A}_\eta] = \mathrm{A}_\xi\mathrm{A}_\eta - \mathrm{A}_\eta\mathrm{A}_\xi$.

If $R^\perp = 0$, then the normal connection of the submanifold M is said to be <u>flat</u>.

When $\left(\tilde{R}(X,Y)\xi\right)^{\perp}=0$, the normal connection of the submanifold M is flat if and only if the second fundamental form is commutative, i.e.

$$[A_{\xi},A_{\eta}]=0$$

for all ξ,η. If the ambient space $\bar{M}$ is real space form, then $\left(\tilde{R}(X,Y)\xi\right)^{\perp}=0$ and hence the normal connection of M is flat if and only if the second fundamental form is commutative.

On the other hand, if the ambient space $\bar{M}$ is a space of constant curvature c, then we have

$$\tilde{R}(X,Y)Z=c\{g(Y,Z)X-g(X,Z)Y\}$$

for any vector fields X, Y and Z on $\bar{M}$.

2.5. Fundamental Theorems of Submanifolds

2.5.1. Existence Theorem: Let $(M,\ g)$ be a simply-connected Riemannian n-manifold and suppose there is a given m-dimensional Riemannian vector bundle $v(M)$ over M with curvature tensor R^{D} and a $v(M)$-valued symmetric (0, 2) tensor h on M. For a cross section ξ of $v(M)$, define A_{ξ} by $g\left(A_{\xi}X,Y\right)=\langle h(X,Y),\xi\rangle$ where $\langle\ ,\ \rangle$ is the fiber metric of $v(M)$. If they satisfy (2.3.8), (2.3.9) and (2.3.10), then M can be isometrically immersed in an $(n+m)$-dimensional complete, simply connected Riemannian manifold $R^{n+m}(c)$ of constant curvature c in such way that $v(M)$ is the normal bundle and h is the second fundamental form.

2.5.2 Uniqueness Theorem: Let $f,\ f^{'}:M\to R^{m}(c)$ be to isometric immersions of a Riemannian n-manifold M into a complete, simply connected Riemannian m-manifold of constant curvature c with normal bundles v and $v^{'}$ equipped with their canonical bundle metrics, connections and second fundamental forms, respectively. Suppose that there is an isometry $\varphi\colon M\to M$ such that φ can be covered by a bundle map $\bar{\varphi}\colon v\to v^{'}$ which preserves the bundle metrics, the connections and the metrics, second fundamental form. Then there is an isometry Φ of $R^{m}(c)$ such that $\quad \Phi\circ f=f^{'}\circ\varphi$.

2.6. Theorem. Suppose M^{n} is a submanifold of $\mathbb{R}^{n}$ and is given the induce Riemannian metric $\langle\,,\rangle$. Then the curvature R of M is given by the formula:

$$\langle R(x,y),z,w\rangle = h(x,w)\cdot h(y,z) - h(x,z)\cdot h(y,w)$$

for $x,y,z,w \in T_pM$; the dot product on the right hand denoting the usual dot product in $\mathbb{R}^N$.

Proof. If D is the Levi-Civita connection on $\mathbb{R}^N$, since Euclidean space has zero curvature,

$$D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z = 0 \text{ for } X,Y,Z \in \mathfrak{X}(M^n).$$

Hence if $W \in \mathfrak{X}(M)$,

$$\begin{aligned}
0 &= \left(D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z\right)\cdot W \\
&= X(D_Y Z\cdot W) - (D_Y Z)\cdot(D_X W) - Y(D_X Z\cdot W) + \\
&\qquad (D_X Z)\cdot(D_Y W) - \left(D_{[X,Y]}Z\right)\cdot W \\
&= X\langle\nabla_Y Z, W\rangle - \langle\nabla_Y Z, \nabla_X W\rangle - h(Y,Z)\cdot h(X,W) - Y\langle\nabla_X Z, W\rangle \\
&\qquad + \langle\nabla_X Z, \nabla_Y W\rangle + h(X,Z)\cdot h(Y,W) - \langle\nabla_{[X,Y]}Z, W\rangle \\
&= \langle\nabla_X\nabla_Y Z, W\rangle - h(Y,Z)\cdot h(X,W) - \langle\nabla_Y\nabla_X Z, W\rangle + \\
&\qquad h(X,Z)\cdot h(Y,W) - \langle\nabla_{[X,Y]}Z, W\rangle \\
&= \langle R(X,Y)Z, W\rangle - h(Y,Z)\cdot h(X,W) + h(X,Z)\cdot h(Y,W).
\end{aligned}$$

as desired. ■

Chapter (3)

Some Special Submanifolds

Let M be an n-dimensional Riemannian submanifold of a Riemannian manifold $\widetilde{M}$. A point $x \in M$ is called a geodesic point if the second fundamental form h vanishes at x. The submanifolds is said to be totally geodesic if every point of M is a geodesic point. A Riemannian submanifold M is a totally geodesic submanifold of $\widetilde{M}$ if and only if every geodesic of M is a geodesic of $\widetilde{M}$.

Let M be a submanifold of $\widetilde{M}$ and let $e_1, \dots, e_n$ be an orthonormal basis of $T_x M$. Then the mean curvature vector $\vec{H}$ at x is defined by

$$\vec{H} = \frac{1}{n} \sum_{j=1}^{n} h(e_j, e_j).$$

The length of $\vec{H}$ is called the mean curvature which is denoted by H. M is called a minimal submanifold of $\widetilde{M}$ if the mean curvature vector field vanishes identically.

A point $x \in M$ is called an umbilical point if $h = g \otimes \vec{H}$ at x, that is, the shape operator A_ξ is proportional to the identity transformation for all $\xi \in T_x^\perp M$.

The submanifold is said to be totally umbilical if every point of the submanifold is an umbilical point.

A submanifold N of a Riemannian manifold M is called parallel submanifold if the second fundamental form h is parallel, that is, $\overline{\nabla} h = 0$, identically.

3.1 The First Variational Formula.

First of all we need the following algebraic result.

3.1.1 Lemma. Let $A(t) = \left(\left(G_{ij}(t)\right)\right)$; $t \in \mathbb{I}$ be a smooth family of $m \times m$ matrices such that $A(0) = I$ (the identity matrix). Then

$$\frac{d}{dt} det\big(A(t)\big)\Big|_{t=0} = trace\big(A'(0)\big).$$

Let $\widetilde{M}$ be a Riemannian manifold and let $f: M \to \widetilde{M}$ be an immersion where M is a compact oriented manifold with boundary ∂M.

3.1.2 Definition. By a smooth variation of f we mean a C^{∞}- mapping $F: \mathrm{I} \times M \to \widetilde{M}$, where $\mathrm{I} = (-1,1)$, such that

(a) Each map $f_t = F(t,.): M \to \widetilde{M}$ is an immersion.

(b) $f_0 = f$.

(c) $f_t|\partial M = f|\partial M$ for all $t \in \mathrm{I}$.

Let $\frac{\partial}{\partial t}$ denote the canonical vector field along the I factor in $\mathrm{I} \times M$ and set $E = F_* \frac{\partial}{\partial t}\Big|_{t=0}$. E is called the deformation vector field of the map f and it is considered as a section of $T(M) \oplus N(M)$. Finally, let A(t) be the volume of M at time t, i.e., let dV_t be the volume element of the metric induced by f_t and set $A(t) = \int_M dV_t$ then we have

3.1.3 Theorem (The First Variational Formula):

$$\frac{dA}{dt}\Big|_{t=0} = -\int \langle H, E \rangle dV_0$$

3.2. Minimal Submanifold in Euclidean Space

Let M be a Riemannian manifold of dimension m. Consider the Laplace operator $\Delta: C^{\infty}M \to C^{\infty}M$. For $f \in C^{\infty}(M)$ choose a local orthonormal frame field $\{e_1, \dots, e_m\}$ in M. Then

$$\Delta f = e_i e_i(f) - \left(\nabla_{e_i} e_i\right) f \tag{3.2.1}$$

Around each point p, there are local coordinate $(x^1, \dots, x^m)$, where the Riemannian metric on M can be written as $ds^2 = g_{ij}\, dx^i dx^j$. If we denote

$$\left(g^{ij}\right) = (g_{ij})^{-1} \quad \text{and} \quad g = \det\left(g_{ij}\right), \text{then}$$

$$\Delta f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ij} \frac{\partial f}{\partial x^j}\right) \tag{3.2.2}$$

In general, for any differential form with values in a vector bundle we can define exterior differential operator d and codifferential operator δ and the Hodge Laplace operator $d\delta + \delta d$.

The minus sign of the Hodge-Laplace operator acting on a smooth function f, a cross-section of the trivial bundle $M \times R$ is just the ordinary Laplace operator

$$\Delta f = -\delta df \qquad (3.2.3)$$

Any $f \in C^{\infty}M$ satisfying $\Delta f = 0$ is called a harmonic function.

We have the Hopf maximum principle for harmonic functions: any harmonic function on a Riemannian manifold has to be a constant, if it attains the local maximum in an interior point.

3.2.1. Proposition (Xin 2003**).** Let $\psi: M \to \mathbb{R}^n$ be an isometric immersion with the mean curvature vector H, then

$$\Delta\psi = mH \qquad (3.2.4)$$

where $\Delta\psi = (\Delta\psi^1, \dots, \Delta\psi^n)$

Proof. Note the fact $X(\psi) = \psi_* X \cong X$ for any $X \in TM$. Let $\{e_i\}$ be a local orthonormal frame field. Then

$$\begin{aligned} \Delta\psi &= e_i\big(e_i(\psi)\big) - \big(\nabla_{e_i} e_i\big)(\psi) \\ &= \bar{\nabla}_{e_i}\bar{\nabla}_{e_i}\psi - \big(\nabla_{e_i} e_i\big)(\psi) \\ &= \bar{\nabla}_{e_i} e_i - \nabla_{e_i} e_i \\ &= \big(\bar{\nabla}_{e_i} e_i\big)^N = mH. \blacksquare \end{aligned}$$

3.2.2. Corollary. An isometric immersion $\psi: M \to \mathbb{R}^n$ is a minimal immersion if and only if each component of ψ is a harmonic function on M.

3.2.3. Remark. In this case the equation (3.2.4) reduces to $\Delta\psi = 0$. However, this is not a linear equation, since the induced metric would change when the immersion Ψ changes, and so does the operator Δ**.**

From Corollary (3.2.2) and the Hopf maximum principle we have immediately:

3.2.4. Corollary. There is no compact minimal submanifold in Euclidean space. From Corollary 3.2.4, it is natural to ask the question whether there exists a bounded but complete minimal submanifold is Euclidean space. This is the well

Known Calabi-Yan problem, which has been answered positively a few years ago by Nadirashivili.

From the first variational formula:

$$\frac{d}{dt}\mathrm{v}ol(f_t M)|_{t=0} = -\int_M \langle nH, V\rangle\, d\mathrm{v}ol$$

where $V = \left.\frac{df_t}{dt}\right|_{t=0}$ to be the variational vector field along f.

We know that $H = 0$ is the Euler-Langrangians equations for the volume functional of immersed submanifolds in an ambient manifold. What is the equation look like? Let us see the simplest situation. In $\mathbb{R}^{n+1}$ a minimal graph M is defined by

$$x^{n+1} = f(x^1, \dots, x^n).$$

We denote $f_i = \frac{\partial f}{\partial x^i}$. The induced metric on M is $ds^2 = g_{ij}\, dx^i\, dx^j$, where

$$g_{ij} = \delta_{ij} + f_i f_j.$$

Denote $w = \sqrt{1 + \sum_i f_i^2}$. We have $g^{ij} = \delta_{ij} - \frac{1}{w^2} f_i f_j$. The unit normal vector to M is

$$v = \frac{1}{w}(f_1, \dots, f_n, -1).$$

It is obvious that

$$\overline{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \frac{\partial}{\partial x^i}\left(0, \dots, 0,1,0, \dots, \frac{\partial f}{\partial x^j}\right) = \left(0, \dots, f_{ij}\right)$$

and

$$\langle B_{\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}}, v\rangle = \langle \overline{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, v\rangle = -\frac{1}{w} f_{ij}$$

From $H = 0$ it follows that $g^{ij} f_{ij} = 0$. Thus; we obtain the minimal hypersurface equation

$$\left(1+\sum f_i^2\right)f_{ij} - f_i f_j f_{ij} = 0 \tag{3.2.5}$$

which is equivalent to

$$\frac{\partial}{\partial x^i}\left(\frac{1}{w}\frac{\partial f}{\partial x^i}\right) = 0 \tag{3.2.6}$$

When $n = 2$ in (3.2.5) reduces to

$$\left(1+f_y^2\right)f_{xx} - 2f_x f_y f_{xy} + (1+f_x^2)f_{yy} = 0 \tag{3.2.7}$$

where we denote $x = x^1$, $y = x^2$. It is a nonlinear elliptic PDE. On a minimal submanifold in $\mathbb{R}^n$ there is another important equation. In fact, we have

3.2.5. Proposition (Xin 2003**).** Let M be an oriented hypersurface with constant mean curvature in $\mathbb{R}^{n+1}$ and with second fundamental form B. Let v the unit normal vector to M. Then for any fixed vector $a \in \mathbb{R}^{n+1}$,

$$\Delta\langle a, v\rangle + |B|^2\langle a, v\rangle = 0 \tag{3.2.8}$$

Proof. Choose a local orthonormal frame field $\{e_i\}$ with $\nabla_{e_j} e_i = 0$, at the considered point. Then

$$\begin{aligned}
\Delta\langle a, v\rangle &= \nabla_{e_i}\nabla_{e_i}\langle a, v\rangle \\
&= \nabla_{e_i}\langle a, \bar{\nabla}_{e_i} v\rangle \\
&= \langle a, \bar{\nabla}_{e_i}\bar{\nabla}_{e_i} v\rangle \\
&= \langle a\,, \bar{\nabla}_{e_i}(\nabla_{e_i} v - A^v(e_i))\rangle \\
&= -\langle a\,, \bar{\nabla}_{e_i} A^v(e_i)\rangle \\
&= -\langle a\,, \nabla_{e_i} A^v(e_i) + (\bar{\nabla}_{e_i} A^v(e_i))^N\rangle
\end{aligned}$$

Noting that the ambient Euclidean space has vanishing curvature and the unit normal vector field v is parallel in the normal bundle,

$$\begin{aligned}
\nabla_{e_i} A^v(e_i) &= \nabla_{e_i}\langle B_{e_i e_j}\,, v\rangle\, e_i \\
&= \nabla_{e_i}\langle \bar{\nabla}_{e_i e_j}\,, v\rangle\, e_j \\
&= \left(\langle \bar{\nabla}_{e_i}\bar{\nabla}_{e_i e_j}\,, v\rangle + \langle \bar{\nabla}_{e_j e_i}\,, \bar{\nabla}_{e_i} v\rangle\right) e_j
\end{aligned}$$

$$= \left(\langle \bar{\nabla}_{e_j} \bar{\nabla}_{e_i} e_i \,, v \rangle + \langle \bar{\nabla}_{e_j} e_i \,, \left(\bar{\nabla}_{e_i} v \right)^T \rangle \right) e_j$$

$$= \left(\langle \bar{\nabla}_{e_j} \left(\nabla_{e_i} e_i + B_{e_i e_i} \right), v \rangle + \langle \nabla_{e_j} e_i \,, \left(\bar{\nabla}_{e_i} v \right)^T \rangle \right) e_j$$

$$= \langle B_{e_j} \nabla_{e_i} e_i \,, v \rangle + \langle n \nabla_{e_j} H \,, v \rangle = 0$$

Therefore,

$$\Delta \langle a \,, v \rangle = - \langle a \,, \left(\bar{\nabla}_{e_i} A^v (e_i) \right)^N \rangle$$

$$= - \langle a \,, B_{e_i} A^v (e_i) \rangle = - \langle a \,, v \rangle |B|^2 . \blacksquare$$

3.3. Minimal Submanifold in the Euclidean Sphere

Let $\bar{M} \subset \mathbb{R}^n$ be an embedded submanifold, and for any $p \in \bar{M}$ and $X \in T_p(\mathbb{R}^n)$. Let X^T denote the orthogonal projection of X onto $T_p(\bar{M})$. Suppose now that $\Psi: M \to \bar{M} \subset \mathbb{R}^n$ is an immersion with mean curvature vector fields K in $\bar{M}$ and K^* in $\mathbb{R}^n$. Then

$$K = (K^*)^T = (\Delta \Psi)^T \tag{3.3.1}$$

3.3.1. Proposition (Xin 2003**)**. Let M be a Riemannian m-manifold and let $\Psi: M \to S^n \subset \mathbb{R}^n$ be an isometric immersion. Then Ψ is a minimal immersion S^n if and only if

$$\Delta \Psi = -m \Psi \tag{3.3.2}$$

Proof. By equation (3.3.1) above we see that Ψ is minimal if and only if for all $p \in M$, $\Delta \Psi(p)$ is parallel to the normal to S^n at $\Psi(p)$, i.e., if and only if $\Delta \Psi = \lambda \Psi$ for some $\lambda \in C^\infty(M)$.

However, from the lemma says that: Let $A(t) = \left(G_{ij}(t) \right)$, $t \in I$ be a smooth family of $M \times m$ matrices such that $A(0) = Identity$.

Then $\frac{\mathrm{d}}{\mathrm{dt}} \det\left(\mathrm{A(t)}\right)\Big|_{\mathrm{t=0}} = \mathrm{trace}\left(\mathrm{A^t(0)}\right)$. And the condition that $|\Psi|^2 = 1$ we see that if $\Delta \Psi = \lambda \Psi$, then

$$0 = \tfrac{1}{2} \Delta |\Psi|^2 = \langle \Psi \,, \Delta \Psi \rangle + |\nabla \Psi|^2 = \lambda |\Psi|^2 + |\nabla \Psi|^2 = \lambda + |\nabla \Psi|^2 \,.$$

Hence, $\lambda = -|\nabla\Psi|^2 = -\sum_k \langle \sum_k \Psi \ , \sum_k \Psi \rangle = -\sum_k |\Psi_* \sum_k \quad |^2 = -m$, and the proposition is proved. ■

Thus we see that the minimal immersions of a differentiable manifold M into S^n are just those immersions whose coordinate functions in the ambient Euclidean space are eigenfunctions of the Laplace-Beltrami operator in the induced metric with eigen$value = -dim(M)$.

Moreover, we have the following useful fact.

For each $r > 0$ let,

$$S^n(r) = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_k x_k^2 = r^2\}.$$

3.3.2 Proposition (Takahashi 1966**).** Let M a Riemannian m-manifold and $\Psi: M \to \mathbb{R}^{n+1}$ an isometric immersion such that

$$\Delta\Psi = -\lambda\Psi$$

for some constant $\lambda \neq 0$**.**Then

i. $\lambda > 0$

ii. $\Psi(M) \subset S^n(r)$, where $S^n(r)$ is a hypersphere of $\mathbb{R}^{n+1}$ centered at the origin 0 and $r^2 = \frac{m}{\lambda}$,

iii. The immersion $\Psi: M \to S^n(r)$ is minimal.

Proof. From Proposition (3.3.1) we have that $\Delta\Psi = -\lambda\Psi = K$, and therefore at any point $p \in M$ the vector $\Psi(p)$ is normal to the immersion. Hence, for any tangent vector field X on M we have

$$X.\langle \Psi , \Psi \rangle = 2\langle X.\Psi , \Psi \rangle = 2(\Psi_* X , \Psi)(= 2\langle X , \Psi \rangle) = 0$$

And it follows that $|\Psi|^2 = constant \overset{\text{def}}{=} r^2$.Then, as above, we have

$$0 = \frac{1}{2}\Delta|\Psi|^2 = \langle \Psi , \Delta\Psi \rangle + |\nabla\Psi|^2 = -\lambda r^2 + m \text{ , and so } \lambda = \frac{m}{r^2} > 0.$$

The minimality of Ψ follows immediately from equation (3.3.1). ■

3.3.3. Corollary. Let G/H be a Riemannian homogeneous space where G is compact Lie group and where the isotropy representation of H (on the tangent space at the point $e.H \in G/H$) is irreducible.

Let $E_\lambda = \{\varphi \in C^\infty(G/H) : \Delta\varphi = -\lambda\varphi\}$ be a non-trivial eigenvalue space of the Laplace-Beltrami operator, and introduce on E_λ an inner product invariant under the natural action $\varphi \longmapsto g_*\varphi = \varphi \circ g$ of G on E_λ. Choose an orthonormal basis $\varphi_1, \ldots, \varphi_N$ for E_λ in this inner product. Then, for an appropriate real number $\alpha \neq 0$ the mapping $\Psi = (\alpha\varphi_1, \ldots, \alpha\varphi_N)$ is an isometric minimal immersion $\Psi: G/H \to S^{N-1}(r)$ for some $r > 0$.

3.4. Totally Geodesic Submanifolds

The notion of totally geodesic submanifolds was introduced in 1901 by J. Hadamard (1865-1963). Hadamard defined (Totally) geodesic submanifolds of a Riemannian manifold as submanifolds such that each geodesic of them is a geodesic of the ambient space. This condition is equivalent to the vanishing on the second fundamental form on the submanifolds. One dimensional totally geodesic submanifolds are nothing but geodesics. Totally geodesic submanifolds are the simplest and the most fundamental submanifolds of Riemannian manifolds.

Totally geodesic submanifold of a Euclidean space is affine subspace and totally geodesic submanifolds of a Riemannian sphere are the greatest spheres. It is much more difficult to classify totally geodesic submanifolds of a Riemannian manifold in general.

The notion of totally geodesic submanifolds is a higher dimensional generalization of geodesics. But, those are very few in general situation. Note that geodesics are critical points of the arc-length functional.

The simplest example gives the plane $\mathbb{E}^2 \subset \mathbb{E}^3$. Geodesics in $\mathbb{E}^2$ are the straight lines only. At the same time they are geodesics in ambient space $\mathbb{E}^3$. So, the plane is a totally geodesic submanifold in $\mathbb{E}^3$.

We are going to state the criterion for the surface to be totally geodesic.

Let τ be a unit tangent vector to the curve in n-dimensional submanifold M. Decompose its curvature vector $\widetilde{\nabla}_\tau\tau$ in m-dimensional Riemannain manifold $\widetilde{M}$ into tangent to and normal to M components:

$$\widetilde{\nabla}_\tau \tau = \left(\widetilde{\nabla}_\tau \tau\right)^T + \left(\widetilde{\nabla}_\tau \tau\right)^N. \tag{3.4.1}$$

From what was proved above. $\left(\widetilde{\nabla}_\tau \tau\right)^T = \nabla_\tau \tau$. Hence

$$\widetilde{\nabla}_\tau \tau = \nabla_\tau \tau + \left(\widetilde{\nabla}_\tau \tau\right)^N. \tag{3.4.2}$$

If γ is geodesic in M, then $\nabla_\tau \tau = 0$. Therefore γ is geodesic in $\widetilde{M}$ if and only if $\left(\widetilde{\nabla}_\tau \tau\right)^N = 0$. We have $\widetilde{\nabla}_\tau \tau = \left(\widetilde{\nabla}_{u_i} a^i u_i\right)^N a^j = a^i a^j \left(\widetilde{\nabla}_{r_j} r_i\right)^N = a^i a^j L_{ij}^\alpha \xi_\alpha$.

The normal ξ_α is linearly independent. Hence $a^i a^j L_{ij}^\alpha = 0$ for all $\alpha = 1, \dots, p$. If this supposed for any τ, then $L_{ij}^\alpha = 0$. So we conclude the following:

3.4.1 Theorem A submanifold M is totally geodesic if and only if its second fundamental forms are identically zero.

3.4.2 Theorem (Cartan's Theorem) Let M be a Riemannian n-manifold with $n \geq 3$. For a vector V in the tangent space T_pM at $p \in M$ denoted by γ_V the geodesic through p whose tangent vector at p is V.

Denoted by $\mathrm{R_V(t)}$ the (1, 3)-tensor on T_pM obtained by the parallel translation of the curvature tensor at $\gamma_V(t)$ along the geodesic γ_V**.** Also define (1, 2)-tensor $r_v(t)$ on T_pM by

$$r_v(t)(x, y) = R_V(t)(v, x)y, \; x, y \in T_pM.$$

The following result of È. Cartan provides necessary and sufficient condition for the existence of totally geodesic submanifolds in Riemannian manifolds in general.

Let V be a subspace of the tangent space T_pM of a Riemannian manifold M at a point p. Then the following three conditions are equivalent.

(1) There is a totally geodesic submanifold of M through p whose tangent space at p is V.

(2) There is a positive number ϵ such that for any unit vector $v \in V$ any $t \in (-\epsilon, \epsilon)$, $R_v(t)(x, y)z \in V$ for any $x, y, z \in V$.

(3) There is a positive number ϵ such that for any unit vector $v \in V$ and any $t \in (-\epsilon, \epsilon)$, $r_v(t)(x, y) \in V$ any $x, y \in V$.

3.4.3. Proposition [O'Neill 1983, p.104]. For a submanifold M of a Riemannian manifold $\widetilde{M}$, the following assertions are equivalent:

a. M is totally geodesic in $\widetilde{M}$;

b. every geodesic of M is a geodesic of $\widetilde{M}$;

c. the geodesic γ_v of $\widetilde{M}$ with initial velocity $v \in T_pM$ is contained in M for small time (and hence is a geodesic in M).

Proof. Since h is symmetric, M is totally geodesic in $\widetilde{M}$ if and only if $h(X,X) = 0$ for all $X \in \Gamma(TM)$ if and only if $h(v,v) = 0$ for all $v \in TM$.

Gauss's formula (2.2.6) says that this is the case if and only if $\widetilde{\nabla}_X X = \nabla_X X$ for all $X \in \Gamma(TM)$**.** If this equation is true, plainly every geodesic in M will be a geodesic in $\widetilde{M}$. Conversely, assume every geodesic in M is a geodesic in $\widetilde{M}$.

Given $0 \neq v \in T_pM$, let γ_v be the geodesic in M with $\gamma_v'(0) = v$. Extend γ_v' to a smooth vector field $X \in \Gamma(TM)$ defined on a neighborhood of p. Since γ_v is also a geodesic of $\widetilde{M}$, we have

$$\widetilde{\nabla}_X X = 0 = \nabla_X X.$$

This proves the equivalence between (a) and (b). Next, if (b) holds, then the uniqueness of geodesics for given initial conditions says that all geodesics of $\widetilde{M}$ initially tangent to M come from geodesics of M, which implies (c). Finally, $\nabla_X X$ is the tangential component of $\widetilde{\nabla}_X X$ for X $\in \Gamma(TM)$, so a geodesic of $\widetilde{M}$ which is contained in M is a geodesic of M, which finishes the proof of the equivalence between (b) and (c). Note that the geodesic γ_v as in (c) is entirely contained in M, if M is complete. ∎

3.4.4 Corollary. A connected complete totally geodesic submanifold M of a Riemannian manifold $\widetilde{M}$ is completely characterized by T_pM for any given $p \in M$.

Proof. In fact, it follows from the Hopf-Rinow theorem and Proposition (3.4.3) that $M = \widetilde{\exp}_p(T_pM)$, where $\widetilde{\exp}$ denotes the exponential map of $\widetilde{M}$.

3.4.5 Proposition. (Totally geodesic submanifolds of space forms) The connected complete totally geodesic submanifolds of:

a. $\mathbb{R}^n$ are the affine subspaces;

b. S^n are the great subspheres, namely, intersections of S^n with linear subspaces of $\mathbb{R}^{n+1}$;

c. $\mathbb{R}H^n$ are the intersections of hyperboloid model with linear subspaces of $\mathbb{R}^{1,n}$.

Proof. (a) Affine subspaces are clearly totally geodesic in $\mathbb{R}^n$. Since a totally geodesic submanifold is completely determined by its tangent space at a point, there can be no other examples. (b)

Great circles of the subsphere are great circles of S^n, so this is a totally geodesic submanifold.

The rest follows as in (a). The proof of (c) is similar. ■

3.4.6 Theorem (cf.[Klingenberg 1995,1.10.15, p.95]). Let $f: (\bar{M}, \bar{g}) \to (\bar{M}, \bar{g})$ be an isometry of the Riemannian manifold $(\bar{M}, \bar{g})$. Then every connected component M of the fixed point set $\{y \in \bar{M}: f(y) = y\}$, with the induced Riemannian metric is a totally geodesic submanifold.

3.4.7 Examples:

(a) A geodesic $\gamma: \mathbb{R} \to M$ can be viewed as a totally geodesic submanifold of dimension one.

(b) Consider the standard sphere

$$S^n := \{(x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1}; x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}.$$

For $1 \leq k < n$ the k–sphere

$$S^k = \{(x_1, x_2, \dots, x_{n+1}) \in S^n; x_{k+1} = \cdots = x_{n+1} = 0\}$$

is a totally geodesic submanifold of S^n. It is the fixed point set of the isometry

$f: S^n \to S^n$:

$$f(x_1, x_2, \dots, x_{n+1}) = (x_1, x_2, \dots, x_k, -x_{k+1}, \dots, -x_{n+1}).$$

One can see immediately that any complex k–dimensional totally geodesic submanifold of S^n is of this form up to an isometry of the sphere.

3.5. Totally Geodesic Submanifolds of Symmetric Spaces

The class of Riemannian manifolds with parallel Riemannian curvature tensor, that is, $\nabla R = 0$ was first introduced independently by P. A. Shirokov (1895–1944) in 1925 and by Levy 1926.

An isometry S of a Riemannian manifold M is called involutive if its iterate $S^2 = S \circ S$ is the identity map.

A Riemannian manifold M is called a symmetric space, if for each point $p \in M$ there exists an involutive isometry s_p of M such that p is an isolated fixed point of s_p. The s_p is called the (point) symmetry of M at the point p.

Denote by G_M, or simply G, the closure of the group of isometrics generated by $\{s_p : p \in M\}$ in the compact open topology. Then G is a Lie group which acts say at 0, is compact and $M = G/H$.

Every complete totally geodesic submanifold of a symmetric space is a symmetric space.

For a symmetric space M, the dimension of a maximal flat totally geodesic submanifold of M is a well defined natural number which is called the rank of M, denoted by $rk(M)$.

3.6. Totally Umbilical Submanifolds

Let N be an n-dimensional submanifold of an m-dimensional Riemannian manifold M with metric g. ∇ and $\widetilde{\nabla}$ be the covariant differentiations on N and M, respectively. Then the second fundamental form h of the immersion is defined by the equation

$$h(X,Y) = \widetilde{\nabla}_X Y - \nabla_X Y \qquad (3.6.1),$$

where X and Y are vector fields tangent to N. The submanifold N is said to be totally umbilical if

$$h(X,Y) = g(X,Y)H, \qquad (3.6.2)$$

for all vector fields X, Y tangent to N, where

$$H = \frac{1}{n} trace\ h \qquad (3.6.3)$$

is the mean curvature vector of N in M.

The length of H is called the mean curvature of N in M. A totally umbilical submanifold with vanishing mean curvature is a totally geodesic submanifold.

A totally umbilical submanifold with nonzero parallel mean curvature is called an extrinsic sphere. Totally geodesic submanifolds are the simplest submanifolds in Riemannian manifolds. It corresponds to linear subspaces of Euclidean spaces.

Despite its simplicity, totally geodesic submanifolds in rank one symmetric space are not classified until 1963.Totally geodesic submanifolds in other symmetric spaces has been studied rather extensively in the last few years.

Extrinsic spheres can be regarded as the extrinsic analogous of the ordinary n-spheres in Euclidean spaces. It can be considered as the simplest example submanifolds next to totally geodesic submanifolds. The class of totally umbilical submanifolds includes the class of all totally geodesic submanifolds and extrinsic spheres. The class of totally umbilical submanifolds with constant mean curvature lies between these two classes. Give a Riemannian or Kähler manifold M, it is fundamental and interesting to know the relationship between those classes and find some fundamental properties of them.

3.6.1. Proposition. A totally umbilic submanifold of dimension at least two in a space form is an extrinsic sphere.

Proof. Differentiate (3.6.2) with respect to $Z \in \Gamma(TM)$ and use $\nabla g = 0$ to get

$$(\nabla_Z^{\perp} h)(X, Y) = g(X, Y)\nabla_Z^{\perp} H.$$

Now the Codazzi equation (2.5.6) says that

$$g(X, Y)\nabla_Z^{\perp} H = g(Z, Y)\nabla_X^{\perp} H.$$

Since $\dim M \geq 2$, we can choose $Y \perp Z$ and $X = Y$ to deduce $\nabla_Z^{\perp} H = 0$. Since Z is arbitrary, H is parallel. ■

3.6.2. Propostion. A totally umbilical submanifold M in a real space form $\widetilde{M}$ of constant curvature c is also of constant curvature.

Proof. Since M is a totally umbilical submanifold of $\widetilde{M}$ of constant curvature c, by using equations (3.6.2) and (2.3.8), we have

$$g(R(X,Y)Z,W) = c\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}$$
$$+g(H,H)\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}$$
$$= \{c + g(H,H)\}\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W\}$$

This shows that the submanifold M is of constant curvature $c + \|H\|^2$ for $n > 2$.

If $n = 2$, $\|H\| =$ constant follows from the equation of Codazzi. This proves the proposition. ■

3.6.3. Proposition (Chen 1973, p. 50). A totally umbilical submanifold M in a space form $\mathbb{R}^m(c)$ is either totally geodesic in $\mathbb{R}^m(c)$ or contained in a hypersphere of an $(n+1)$-dimensional totally geodesic subspace of $\mathbb{R}^m(c)$.

3.7. Totally Umbilical Submanifolds in Conformally Flat Spaces

Let N be an n-dimensional submanifold of an m-dimensional Riemannian manifold M with metric g. Let h be the second fundamental form of N in M. Then h is normal-bundle-valued symmetric 2-form on N. Let ξ be a normal vector field on N, we write

$$\widetilde{\nabla}_X \xi = -A_\xi(X) + D_X \xi, \tag{3.7.1}$$

where $-A_\xi(X)$ and $D_X\xi$ denote the tangential and normal component of $\widetilde{\nabla}_X \xi$, respectively. Then we have

$$g\big(A_\xi(X), Y\big) = g(h(X,Y), \xi) \tag{3.7.2}$$

A normal vector field ξ on N is said to be parallel if $D_X\xi = 0$ for all tangent vector fields X. Let R, $\tilde{R}$ and R^N be the curvature tensors associated with ∇, $\widetilde{\nabla}$ and D. For example, R^D is given by

$$R^D(X,\mathrm{Y}) = D_X D_Y - D_Y D_X - D_{[X,Y]}$$

for X, Y tangent to N.

For vector fields X, Y, Z, W tangent to N and vector fields ξ, η normal to N, the equation of Gauss and Ricci are then given respectively by

$$\tilde{R}(X,Y;Z,W) = R(X,Y;Z,W) + g\big(h(X,Z), h(Y,W)\big) -$$
$$g\big(h(X,W), h(Y,Z)\big), \tag{3.7.3}$$

$$\tilde{R}(X,Y;\xi,\eta) = R^N(X,Y;\xi,\eta) - g\left(\left[A_\xi, A_\eta\right](X), Y\right), \tag{3.7.4}$$

where $R(X,Y;Z,W) = g(R(X,Y)Z,W)$, ..., etc.

For the second fundamental form h, we define the covariant derivative in $TN \oplus T^\perp N$, denote by $\tilde{\nabla}_X h$, to be

$$\left(\tilde{\nabla}_X h\right)(Y,Z) = D_X(h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \tag{3.7.5}$$

The equation of Codazzi is given by

$$\left(\tilde{R}(X,Y)Z\right)^\perp = \left(\tilde{\nabla}_X h\right)(Y,Z) - \left(\tilde{\nabla}_Y h\right)(X,Z), \tag{3.7.6}$$

where $\perp$denote the normal component.

Let X and Y be two orthonormal vectors tangent to N. The sectional curvature $K(X \wedge Y)$, of plane section $X \wedge$ Y spanned by X, Y is given by

$$K(X \wedge Y) = R(X,Y;Y,X) \tag{3.7.7}$$

Let $E_1, \ldots, E_n$ be an orthonormal frame tangent to N. Then

$$S(X,Y) = \sum_{i=1}^{n} R(E_i, X; Y, E_i) \tag{3.7.8}$$

defines a global tensor field S of type (0, 2), called the Ricci tensor of N. Moreover, from the tensor field S, we can define a global scalar field

$$\tau = \sum_{i=1}^{n} S(E_i, E_i) \tag{3.7.9}$$

This scalar field is called the scalar curvature of N. N is called an Einstein space if the Ricci tensor of N is proportional to the metric tensor. And N is said to be locally Symmetric, if its curvature tensor is parallel, that is $\nabla R = 0$. It is well-known that every irreducible locally symmetric space is Einsteinian. A Riemannian manifold is called a real-space-form if it has constant sectional curvature.

For a Riemannian manifold N of dimension n, we define a tensor field of type (0, 2) by

$$L(X,Y) = \frac{-1}{n-2} S(X,Y) + \frac{\tau}{2(n-1)(n-2)} g(X,Y) \tag{3.7.10}$$

Then the Weyl conformal curvature tensor C is defined by

$$C(X,Y)Z = R(X,Y)Z + L(Y,Z)X - L(X,Z)Y + g(Y,Z)L^*X -$$

$$g(X,Z)L^*Y \tag{3.7.11}$$

where L^* is the (2,5)-tensor associated with L, that is,

$$g(L^*X,Y) = L(X,Y) \tag{3.7.12}$$

A Riemannian manifold of dimension ≥ 4 is conformally flat if and only if its Weyl conformal curvature tensor vanishes. Moreover, the Weyl conformal curvature tensor vanishes identically for Riemannian manifolds of dimension ≤ 3.

The follows Result is well-Known.

3.7.1. Proposition (Schouten 1954**).** Every totally umbilical submanifold of dimension > 3 in a conformally flat space is conformally flat.

For the totally geodesic submanifold in conformally flat spaces, we have the following characterization.

3.7.2. Proposition (Chen, Vanhecke and Verstraelen 1978**).** A minimal submanifold N of a conformally flat space M is a totally geodesic submanifold if and only if

$$L(X,Y) = \tilde{L}(X,Y) \tag{3.7.13}$$

for all X, Y tangent to N, where $\tilde{L}$ denotes the corresponding quantity for M.

The following result is well-known which in fact follows easily from equation (3.6.1).

3.7.3. Proposition (Chen 1979**).** Submanifold N in a Riemannian manifold is totally geodesic if and only if every geodesic in N is a geodesic in M.

One dimensional totally geodesic submanifolds are geodesic, they exist extensively. In fact, for every point p in a Riemannian manifold M, and every tangent vector T at p, there always exists a geodesic through p with T as its velocity vector at p.

However, totally geodesic submanifolds and totally umbilical submanifolds of higher dimensions do not necessarily exist in general. In fact we have the following two results of E. Cartan and J. A. Schouten:

3.7.4. Theorem (Carten 1946**).** Let M be an m-dimensional Riemannian manifold. If there exist an integer n; $2 \leq n \leq m-1$, such that for any $p \in M$ and any linear n-subspace V of T_pM, there always exist a totally geodesic submanifold N through p with $T_pN = V$, then M is real-space-form.

3.7.5. Theorem (Schouten 1954**).** Let M be an m-dimensional Riemannian manifold. If there exist an integer n, $3 \leq n \leq m-1$, such that for any $p \in M$ and any linear n-subspace V of T_pM, there always exist a totally umbilical submanifold N through p with $T_pN = V$, then M is a conformally flat space.

Moreover we have

3.7.6. Theorem (Miyazawa and Chūman 1972**).** Every totally umbilical submanifold of dimension ≥ 4 in a locally symmetric space is either totally geodesic or conformally flat.

Theorem 3.7.6 was generalized to totally umbilical submanifolds in conformally recurrent space by Z. Olszak.

3.8. Totally Umbilical Submanifolds in Kähler Manifolds

Let N be a Kähler manifold with complex structure J, Kähler metric g. Then we have $\nabla \mathrm{J} = 0$, thus we obtain

$$R(\mathrm{J}X, \mathrm{J}Y) = R(X, Y), \tag{3.8.1}$$

$$R(X, Y)\mathrm{J}Z = \mathrm{J}R(X, Y)Z \tag{3.8.2}$$

Therefore, the sectional curvature of M determined by any orthonormal vector X and Y satisfies

$$K(X \wedge \mathrm{Y}) = K(\mathrm{J}X \wedge \mathrm{J}\mathrm{Y}), \tag{3.8.3}$$

$$K(X \wedge \mathrm{J}\mathrm{Y}) = K(\mathrm{J}X \wedge \mathrm{Y}) \tag{3.8.4}$$

For a unit vector X the holomorphic sectional curvature $H(X)$ determined by X is defined by

$$H(X) = K(X \wedge \mathrm{J}X).$$

A Kähler manifold is called a complex-space-form if it has constant holomorphic sectional curvature c. It is known that the curvature tensor of such Kähler manifold satisfies

$$R(X,Y)Z = \frac{c}{4}\left\{\begin{matrix} g(Y,Z)X - g(X,Z)Y + g(\mathrm{J}Y,Z)\mathrm{J}X - g(\mathrm{J}X,Z)\mathrm{J}Y \\ +2g(X,\mathrm{J}Y)\mathrm{J}Z \end{matrix}\right\} \quad (3.8.5)$$

Let N be an n-dimensional Kähler manifold. We introduce a tensor field $\boldsymbol{\Gamma}$ of type (0, 2) by

$$\Gamma(X,Y) = -\frac{1}{n+4}S(X,Y) + \frac{\tau}{2(n+2)(n+4)}g(X,Y) \qquad (3.8.6)$$

Then the Bochner curvature tensor B is defined by

$$\begin{aligned} B(X,Y;Z,W) &= R(X,Y;Z,W) + \Gamma(X,W)g(Y,Z) - \Gamma(X,Z)g(Y,W) \\ &+\Gamma(Y,Z)g(X,W) - \Gamma(Y,W)g(X,Z) + \Gamma(\mathrm{J}X,W)g(\mathrm{J}Y,Z) \\ &-\Gamma(\mathrm{J}X,Z)g(\mathrm{J}Y,W) + \Gamma(\mathrm{J}Y,Z)g(\mathrm{J}X,W) - \Gamma(\mathrm{J}Y,W)g(\mathrm{J}X,Z) \\ &-2\Gamma(\mathrm{J}X,Y)g(\mathrm{J}Z,W) - 2\Gamma(\mathrm{J}Z,W)g(\mathrm{J}X,Y) \qquad (3.8.7) \end{aligned}$$

A Kähler manifold is called Bochner-flat if its Bochner curvature tensor vanishes.

Let M be a Kähler manifold with complex structure J. A submanifold N of M is said to be holomorphic if the complex structure J carries tangent spaces of N into tangent spaces of N. It is called totally real if J carries the tangent spaces of N into normal spaces $T^{\perp}N$ of N.

3.9. Extrinsic Spheres.

Let N be a n–dimensional submanifold of a Riemannian manifold M with $\langle\ ,\ \rangle$ as its first fundamental form. Let $\tilde{\nabla}$ be the Riemannian connection on M and let ∇ be the induced connection on N. Then the second fundamental form h of the immersion is given by

$$h(X,Y) = \tilde{\nabla}_X Y - \nabla_Y X$$

where X and Y are vector fields tangent to N. It is well known that h is a normal-bundle valued symmetric 2–form on N. The submanifold N is said to be totally umbilical in M if there exists a normal vector field H, called the mean curvature vector field, such that

$$h(X,Y) = \langle X,Y\rangle H.$$

When h vanishes identically N is called a totally geodesic submanifold of M. A totally umbilical submanifold with parallel non−zero mean curvature vector (this is $DH = 0$ and $H \neq 0$ where D denotes the normal connection of N in M) is called an extrinsic sphere. A circle is by definition a 1-dimensional extrinsic sphere.

The following theorem of Nomizu-Yano gives a characterization of extrinsic spheres by circles.

3.9.1. Theorem [Nomizu and Yano 1974]. Let N be an n-dimensional $(n \geq 2$ submanifold of a Riemannian manifold M. If, for some $r>0$, every circle of radius r in N is a circle in M; then N is an extrinsic sphere in M. Conversely, if N is an extrinsic sphere in M then every circle in N is a circle in M.

As for geodesics, circles always exist at every point x and every direction T in a Riemannian manifold.

3.9.2. Theorem (Chen 1979**)**. Let N be an n-dimensional submanifold in a Riemannian manifold M. Then N is an extrinsic sphere in M if and only if M admits an $(n+1)$-dimensional totally geodesic submanifold N' such that N lies in N' as an extrinsic hypersphere.

Thus the maximal dimension of extrinsic spheres is less than the maximal dimension of totally geodesic submanifold in a Riemannian manifold by at least one.

If the ambient space M is locally symmetric, we have the following sharper result.

3.9.3. Theorem (Chen 1979**).** Let N be an n-dimensional submanifold in a locally symmetric space $\widetilde{M}$.Then N is an extrinsic sphere in M if and only if N is a real-space-form and N is an extrinsic hypersphere in an $(n+1)$-dimensional real-space-form N' which is imbedded in M as a totally geodesic submanifold.

It is known that every real-space-form admit extrinsic hyperspheres.

A submanifold N in a Kähler manifold M is called a purely real submanifold if

$$TN \cap \mathrm{J}(TN) = \{0\}.$$

For extrinsic spheres in Kähler manifolds, we have the following.

3.9.4. Theorem (Chen 1979**).** If N is an extrinsic sphere of a totally geodesic submanifold N' of constant curvature in a Kähler manifold M, then either N is purely real or N' is flat.

Combining theorems (3.9.3) and (3.9.4) we have immediately the following.

3.9.5. Theorem. If N is an n-dimensional extrinsic sphere in a Hermitian locally symmetric space M, then either N an extrinsic sphere in a flat totally geodesic submanifold or $\dim N \leq \frac{1}{2} \dim M$.

A complete extrinsic sphere in a Euclidean space is nothing but an ordinary sphere. It is natural to ask when an extrinsic sphere is isometric to an ordinary sphere.

3.9.6. Corollary (Chen 1979**)**. Every complete, extrinsic sphere in a compact symmetric space is isometric to an ordinary sphere.

This corollary follows from theorem (3.9.3) simply because every hypersphere in real projective spaces (or spheres) are isometric to an ordinary sphere. For extrinsic sphere in Kähler manifolds with flat normal connection we have the following.

3.9.7. Theorem (Chen 1976**).** A complete, simply- connected even- dimensional extrinsic sphere N in a Kähler manifold M is isometric to an ordinary sphere if its normal connection is flat (that is, $R^N \equiv 0$.)

For extrinsic spheres with flat normal connection in Hermitian symmetric spaces we have:

3.9.8. Theorem (Chen 1977**).** If N is a complete, n-dimensional, simply- connected extrinsic sphere with flat normal connection in a Hermitian locally symmetric space M. Then $\dim N \leq \operatorname{rank} M$ and N is a isometric to an ordinary n-sphere.

Now, we shall give the proof of the following.

3.9.9. Theorem (Chen 1979**).** Every totally umbilical hypersurface N of an Einstein space M is either totally geodesic or an extrinsic sphere.

Proof. Let N is a totally umbilical hypersurface of an Einstein space M. Then we have

$$h(X,Y) = g(X,Y).H, \tag{3.9.1}$$

where X and Y are vector fields tangent to N. By (3.7.5) we see that

$$(\tilde{\nabla}_X h)(Y,Z) = g(Y,Z).D_X H; \tag{3.9.2}$$

Therefore equation (3.7.6) of codazzi reduces to

$$\left(\tilde{R}(X,Y)Z\right)^{\perp} = g(Y,Z).D_X H - g(X,Z).D_Y H. \tag{3.9.3}$$

Let $E_1, \ldots, E_n$ be an orthonormal basis of $T_x N$, $x \in N$. Then we have

$$\left(\tilde{R}(E_1,E_i)E_i\right)^{\perp} = D_{E_1} H. \tag{3.9.4}$$

Consequently, we get

$$\tilde{R}(E_1,E_i;E_i,H) = \tfrac{1}{2} E_1(\alpha^2), \tag{3.9.5}$$

where $\alpha^2 = g(H,H)$. Thus the Ricci tensor $\tilde{S}$ of M satisfies

$$\tilde{S}(E_1,H) = \tfrac{n-1}{2} E_1(\alpha^2). \tag{3.9.6}$$

On other hand, since M is Einsteinian, $\tilde{S}(E_1,H) = 0$. Thus $E_1(\alpha^2) = 0$.

Since E_1can be chosen as any unit vector tangent to N, the mean curvature α is constant.

If $\alpha = 0$, N is totally geodesic. If $\alpha \neq 0$, then by the assumption on codimension the mean curvature vector H is parallel. Thus N is an extrinsic sphere. This proves the theorem.

3.10. Parallel Submanifolds

The first fundamental form, that is, the metric tensor, of a submanifold of a Riemannian submanifold is automatically parallel, thus, $\nabla g \equiv 0$ with respect to the Riemannian connection ∇ on the tangent bundle TM.

3.10.1. Definition. A submanifold M of a Riemannian submanifold $\widetilde{M}$ is said to be parallel, if the second fundamental form h of M is parallel, that is

$\overline{\nabla} h \equiv 0$ with respect to the connection $\overline{\nabla}$ on $TM \oplus T^{\perp}M$.

3.10.2. Proposition. A submanifold M in $\widetilde{M}$ is parallel if and only if the parallel transport of the second fundamental form with respect to $\nabla^{\perp}$ along curve in M is equal to the second fundamental form acting on the parallel transport of two tangent vectors to M along the same curve.

Proof. Let $p \in M$ and $\gamma: I \subset \mathbb{R} \to M$ a curve in M with $\gamma(t_0) = p$. Consider two vector fields $U, V \in \mathfrak{X}(\gamma)$ so that $U_p = u$ and $V_p = v$, and $\nabla_{\gamma'} U = \nabla_{\gamma'} V = 0$.

Assume that M is parallel, i.e. $\overline{\nabla} h = 0$. Because the parallel transport defines a unique vector field it is sufficient to prove that $\nabla^{\perp}_{\gamma'} h(U, V) = 0$. In fact,

$$\nabla^{\perp}_{\gamma'} h(U, V) = h\big(\nabla_{\gamma'} U, V\big) + h\big(U, \nabla_{\gamma'} V\big) = 0$$

Conversely, let us assume that $h(u, v)^{*\perp} = h(u^*, v^*)$. Then

$$(\overline{\nabla} h)(\gamma'(t_0), u, v)\Big|_p = \Big(\nabla^{\perp}_{\gamma'} h(U, V) - h\big(\nabla_{\gamma'} U, V\big) - h\big(U, \nabla_{\gamma'} V\big)\Big)\Big|_p$$

$$= \nabla^{\perp}_{\gamma'} h(U, V)\Big|_p = 0.$$

Because p, u, v and γ can be chosen arbitrary this implies $\overline{\nabla} h = 0$. ∎

The first result on parallel submanifolds was given by V. F. Kagan in 1948 who showed that the class of parallel surfaces in E^3 consists of open parts of plans, round spheres, and circular cylinders $S^1 \times E^1$. U. Simon and A. Weinstein (1969) determined parallel hypersurfaces of Euclidean $(n + 1)$- space.

A general classification theorem of parallel submanifolds in Euclidean space was obtained by D. Ferus in 1974.

An affine subspace of E^m or a symmetric R-space $M \subset E^m$, which is minimally embedded in a hypersphere of E^m as described in [Takeuchi-Kobayashi 1965] is parallel submanifold of E^m.

D. Ferus (1974) proved that essentially these submanifolds exhaust all parallel submanifolds of E^m in the following sense: A complete full parallel submanifold of the Euclidean m-space E^m is congruent to

(1) $M = E^{m_0} \times M_1 \times \ldots \times M_s \subset E^{m_0} \times E^{m_1} \times \ldots \times E^{m_s} = E^m, s \geq 0$, or to

(2) $M = M_1 \times \ldots \times M_s \subset E^{m_1} \times \ldots \times E^{m_s}, s \geq 1$, where each $M_i \subset E^{m_i}$ is

an irreducible symmetric R-space.

3.10.3. Definition. For an n-dimensional submanifold $f: M \to E^m$, for each point $x \in M$ and each unit tangent vector X at x, the vector $f_*(X)$and the normal space $T_x^\perp$ determine an $(m-n+1)$ dimensional subspace $E(x, X)$of E^m.The intersection $f(M)$ and $E(x, X)$ defines a curve γ in a neighborhood of $f(x)$,which is called the normal section of f at x in the direction X. A point p on a plane curve is called a vertex if its curvature function $\kappa(s)$ has a critical point at p.

Parallel submanifolds of E^m are characterized by the following simple geometric property: normal sections of M at each point $x \in M$ are plane curves with x as one of its vertices.

3.10.4. Definition. A submanifold $f: M \to E^m$ is said to be extrinsic symmetric if, for each $x \in M$, there is an isometry φ of M into itself such that $\varphi(x) = x$ and $f \circ \varphi = \sigma_x \circ f$ where σ_x denotes the reflection at the normal space $T_x^\perp M$ at x that is the motion of E^m which fixes the space through $f(x)$ normal to $f_*(T_x M)$ and reflects $f(x) + f_*(T_x M)$ at $f(x)$.

3.10.5. Definition. The submanifold $f: M \to E^m$ is said to be extrinsic locally symmetric, if each point $x \in M$ has a neighborhood U and an isometry φ of U into itself, such that $\varphi(x) = x$ and $f \circ \varphi = \sigma_x \circ f$ on U.

In other words, a submanifold M of E^m is extrinsic locally symmetric if each point $x \in M$ has a neighborhood which is invariant under the reflection of $E^m$with respect to the normal space at x.

D. Ferus (1980) proved that extrinsic locally symmetric submanifolds of E^m have parallel second fundamental form and vice versa.

A canonical connection on a Riemannian manifold (M, g) is defined as any metric connection ∇^c on M such that the difference tensor $\widehat{D}$ between ∇^c and the Levi-Civita connection ∇ is ∇^c-parallel.

3.10.6. Definition. An embedded submanifold M of E^m is said to be an extrinsic homogeneous submanifold with constant principal curvature if, for any given $x, y \in M$ and a given piecewise differentiable curve γ from x to y, there exists an isometry φ of E^m satisfying

(1) $\varphi(M) = M$,

(2) $\varphi(x) = y$, and

(3) $\varphi * x|T_x^{\perp}M: T_x^{\perp}M \to T_x^{\perp}M$ coincides with $\widehat{D}$-parallel transport along γ.

C. Olmos and C. Sanchez (1991) extended Ferus' result and obtained the following:

Let M be a connected compact Riemannian submanifold fully in E^m, and let h be its second fundamental form. Then the following three statements are equivalent:

i. M admits a canonical connection ∇^c such that $\nabla^c h = 0$,
ii. M is an extrinsic homogeneous submanifold with constant principal curvature,
iii. M is an orbit of an s-representation, that is, of an isotropy representation of a semi simple Riemannian symmetric space.

3.10.7. Definition. Regarding the unit $(m-1)$-sphere S^{m-1} as an ordinary hypersphere of E^m, a submanifold $M \subset S^{m-1}$ is parallel if and only if $M \subset S^{m-1} \subset E^m$ is a parallel submanifold of E^m.

Consequently, Ferus' result implies that M is a parallel submanifold of S^{m-1} if and only if M is obtained by submanifold of type (2).

3.11. Examples

The minimal surface equation (3.2.7) is a nonlinear partial differential equation. It is hard to solve. Besides the linear functions, what are its solutions? As early as 1776 J. L. Meunier obtained two nonlinear solutions to the equation firstly. Their graphs are catenoid and helicoid.

The catenoid is defined by

$$x = \cosh^{-1}\sqrt{x^2 + y^2}, \qquad (3.11.1)$$

Take a catenary in X-Z coordinates plane. Letting it rotating about Z-axis gives the catenoid.

Furthermore, we have the following result.

3.11.1. Proposition. Any minimal surface which is also a surface of revolution in $\mathbb{R}^3$ is a catenoid or a plane up to a rigid motion in $\mathbb{R}^3$.

The catenoid is a complete surface whose Gauss curvature is

$$K = \frac{1}{(x^2+y^2)^2} \tag{3.11.2}$$

and the total curvature

$$\int_M KdM = -4\pi \tag{3.11.3}$$

The helicoid is defined by

$$z = \tan^{-1}\frac{x}{y} \tag{3.11.4}$$

Let a line in X-axis screw about Z-axis. The result surface is a helicoid.

The helicoid is also a complete surface with the Gauss curvature

$$K = -\frac{1}{(1+x^2+y^2)^2}$$

Its total curvature is infinite.

3.11.2. Proposition. Up to a rigid motion a ruled minimal surface in $\mathbb{R}^3$ has to be a helicoid or a plane.

Consider a special solution to (3.2.7) of the type

$$f(x,y) = g(x) + h(y)$$

By direct computation we obtain

$$f(x,y) = \frac{1}{a}\log\frac{\cos ax}{\sin ay} \tag{3.11.5}$$

Its graphs is called Scheck's surface which we obtained in 1835.

We now give some examples of minimal submanifolds in the sphere.

Let $\Psi: M \to S^n \subset \mathbb{R}^{n+1}$ and $\Psi': M' \to S^{n'} \subset \mathbb{R}^{n'+1}$ be minimal immersions. For any constants c and c'

$$c\Psi \oplus c'\Psi': M \times M' \to \mathbb{R}^{n+n'+2}$$

is also an isometric immersion of the product manifold $M \times M'$ to $\mathbb{R}^{n+n'+2}$.

If we choose c and c' with $c^2 + c'^2 = 1$, then the image of $M \times M'$ under $c\Psi \oplus c'\Psi'$ lies in the spheres $S^{n+n'+1}$.We know that induced metric on M under $c\Psi$ is $c^2 ds^2$, where ds^2 is the original metric on M. Then, the Laplacian on M with respect to the metric $c^2 ds^2$ in $\frac{1}{c^2}\Delta_m$.

By Proposition (3.3.1)

$$\frac{1}{c^2}\Delta_m(c\Psi) = \frac{1}{c^2}c\Delta_M\Psi = -\frac{m}{c^2}(c\Psi),$$

so dose for $c'\Psi'$ and

$$\Delta_{M\times M'}(c\Psi \oplus c'\Psi') = \frac{m}{c^2}c\Psi \oplus \frac{-m'}{c'^2}c'\Psi'.$$

If c and c' also satisfy

$$\frac{m}{c^2} = \frac{m}{c'^2},$$

then by Proposition (3.3.1) we obtain a minimal immersion

$$c\Psi \oplus c'\Psi' : M \times M' \to S^{n+n'+1}.$$

In particular, $M = S^n$ and $M' = S^{n'}$ we have the Clifford minimal hypersurface

$$S^n\left(\sqrt{\frac{n}{n+n'}}\right) \times S^{n'}\left(\sqrt{\frac{n'}{n+n'}}\right) \to S^{n+n'+1} \qquad (3.11.6)$$

A unit normal vector to the Clifford minimal hypersurface is

$$v = -c'\Psi + c\Psi,$$

because it is orthogonal to $d\Psi, d\Psi'$, and to $c\Psi + c'\Psi'$.

Hence,its second fundamental form is:

$$-\langle dx, dv\rangle = cc'(\langle d\Psi, d\Psi\rangle - \langle d\Psi', d\Psi'\rangle)$$

On the other hand, its induced metric is

$$ds^2 = c^2\langle d\Psi, d\Psi\rangle + c'^2\langle d\Psi', d\Psi'\rangle.$$

Noting that

$$c = \sqrt{\frac{n}{n+n'}} \quad \text{and} \quad c' = \sqrt{\frac{n'}{n+n'}}$$

it has principal curvature $\sqrt{\frac{n'}{n}}$ with the multiplicity n and $-\sqrt{\frac{n}{n'}}$ with the multiplicity n'. Therefore, the sum of squares of the principal curvature is $n + n'$, which is the squared norm $|B|^2$ of the second fundamental form B for the Clifford hypersurface (3.11.6). We thus have

$$|B|^2 = n + n' \tag{3.11.7}$$

Let us consider another minimal submanifold in the sphere. Let

$$P(d) = \{\text{homogeneous polynomials of degree in } \mathbb{R}^{n+1}\}$$

and

$$H(d) = \{f \in P(d) : \Delta f = 0\}.$$

Then

$$SH(d) = \left\{f|_{S^n(c)}; f \in H(d)\right\}$$

denotes the spherical harmonic functions of degree d.

3.11.3. Lemma. If $f \in SH(d)$, then

$$\Delta_{S^n(c)} f = \frac{-d(n+d-1)}{c^2} f \tag{3.11.8}$$

Proof. Let $\{e_i, \dots, e_n\}$ be an orthonormal frame field in $S^n(c), v = \frac{x}{c}$ be the unit normal vector field along S^n. Then

$$\begin{aligned} \Delta_{S^n(c)} f &= e_i e_i(f) - \left(\nabla_{e_i} e_i\right)(f) \\ &= e_i e_i(f) + vv(f) - (\overline{\nabla}_v v(f)) + B_{e_i e_i}(f) - vv(f) + \overline{\nabla}_v v(f) \qquad (3.11.9) \\ &= \Delta_{\mathbb{R}^{n+1}} f + nH(f) - vv(f) \\ &= -\frac{n}{c} v(f) - vv(f), \end{aligned}$$

where $H = \frac{1}{c} v$ is the mean curvature vector of $S^n(c)$ in $\mathbb{R}^{n+1}$.

Since f is a homogeneous function,

$$vf(x)|_{S^n(c)} = \frac{1}{c}\frac{\partial}{\partial t} f(tx)\bigg|_{t=1} = \frac{df}{c} \tag{3.11.10}$$

And

$$vvf(x)|_{S^n(c)} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} f(tx)\bigg|_{t=1} = \frac{d(d-1)f(x)}{c^2} \tag{3.11.11}$$

Substituting (3.11.10) and (3.11.11) into (3.11.9) gives (3.11.8) and the proof is completed. ■

Proposition 3.3.2 and Lemma 3.11.3 enable us to define minimal immersions by using homogeneous spherical harmonic functions

$$S^n\left(\sqrt{\frac{d(n+d-1)}{n}}\right) \to S^N(1),$$

where $N + 1 = \dim SH(d)$. In the case of $n = 2$ and $d = 2$ we have $S^2(\sqrt{3}) \to S^4$, which can be realized by the map

$$\Psi(x, y, z) = \left(\frac{1}{\sqrt{3}}xy, \frac{1}{\sqrt{3}}xz, \frac{1}{\sqrt{3}}yz, \frac{1}{2\sqrt{3}}(x^2 - y^2), \frac{1}{6}(x^2 + y^2 - 2x^2)\right) \tag{3.11.12}$$

where $x^2 + y^2 + z^2 = 3$. It is called the veronese surface which is an imbedding of the real projective plane of curvature $\frac{1}{3}$ into S^4.

The Clifford minimal hypersurface and the veronese surface are important minimal submanifolds in the sphere.

3.12. CR-Submanifolds of a Kähler Manifold

Let $\widetilde{M}$ be a Kähler manifold with complex structure J, N a Riemannian manifold isometrically immersed in $\widetilde{M}$, and $\mathcal{D}_x$ the maximal holomorphic subspace of the tangent space $T_x N$ of N. If the dimension of $\mathcal{D}_x$ is the same for all x in N, $\mathcal{D}_x$ gives a holomorphic distribution $\mathcal{D}$ on N.

Recently, A. Bejancu introduced the notion of a CR-submanifold of $\widetilde{M}$ as follows. A submanifold N in a Kähler manifold $\widetilde{M}$ is called a CR-submanifold if there exist on N a differentiable holomorphic distribution $\mathcal{D}$ such that its orthogonal complement $\mathcal{D}^{\perp}$ is a totally real distribution, i.e. $\mathrm{J}\mathcal{D}^{\perp} \subseteq T_x^{\perp}N$.

Let $\widetilde{M}$ be a complex m-dimensional Kähler manifold with complex structure J, and N a real n-dimensional Riemannian manifold isometrically immersed in $\widetilde{M}$. We denote by $\langle\, ,\rangle$ the metric tensor of $\widetilde{M}$ as well as that induced on N. Let ∇ and $\widetilde{\nabla}$ be the covariant differentiations on N and $\widetilde{M}$, respectively. Then the Gauss and Weingarten formulas for N are given respectively by

$$\widetilde{\nabla}_{\mathrm{X}}\mathrm{Y} = \nabla_{\mathrm{X}}\mathrm{Y} + \sigma(\mathrm{X}, \mathrm{Y}) \tag{3.12.1}$$

$$\widetilde{\nabla}_{\mathrm{X}}\xi = -\mathrm{A}_{\xi}\mathrm{X} + \mathrm{D}_{\mathrm{X}}\xi \tag{3.12.2}$$

For any vector fields X, Y tangent to N and any vector field ξ normal to N, where σ denotes the second fundamental form, and D the linear connection, called the normal connection, induced in the normal bundle $T^{\perp}N$.The second fundamental tensor A_{ξ} is related to σ by

$$\langle A_{\xi}X, Y\rangle = \langle \sigma(X, Y), \xi\rangle \tag{3.12.3}$$

For any vector field X tangent to N, we put

$$\mathrm{J}X = PX + FX, \tag{3.12.4}$$

where PX and FX are the tangential and normal components of $\mathrm{J}X$, respectively. Then P is an endomorphism of the tangent bundle TN, and F is a normal bundle valued 1-form on TN. For any vector field ξ normal to, we put

$$\mathrm{J}\xi = t\xi + f\xi, \tag{3.12.5}$$

where $t\xi$ and $f\xi$ are the tangential and normal components of $\mathrm{J}\xi$, respectively. Then f is an endomorphism of the normal bundle $T^{\perp}N$ and t is a tangent bundle valued 1-form on $T^{\perp}N$.

A Kähler manifold N is called a complex space form if it is of constant holomorphic sectional curvature. We denote by $\widetilde{M}(c)\left(or\ \widetilde{M}^m(c)\right)$ a complex m-dimensional complex space form of constant holomorphic sectional curvature. Then the curvature tensor $\widetilde{R}$ of $\widetilde{M}(c)$ is given by

$$\tilde{R}(X,Y)Z = \tfrac{c}{4}\begin{Bmatrix}\langle Y,Z\rangle X - \langle X,Z\rangle Y + \langle \mathrm{J}Y,Z\rangle \mathrm{J}X - \langle \mathrm{J}X,Z\rangle \mathrm{J} \\ +2\langle X,\mathrm{J}Y\rangle \mathrm{J}Z\end{Bmatrix} \tag{3.12.6}$$

for any vector fields X, Y and Z tangent to $\widetilde{M}(c)$. We denote the curvature tensors associated with ∇ and D by and $R^{\perp}$ respectively.

For the second fundamental form σ, we define the covariant differentiation $\overline{\nabla}$ with respect to the connection in $(TN) \oplus (T^{\perp}N)$ by

$$(\overline{\nabla}_X\sigma)(Y,Z) = D_X(\sigma(Y,Z)) - \sigma(\nabla_X Y,Z) - \sigma(Y,\nabla_{\mathrm{X}} Z) \tag{3.12.7}$$

for any vector fields X,Y and Z tangent to N. The equations of Gauss, Codazzi, and Ricci are then given respectively by

$$R(X,Y;Z,W) = \tilde{R}(X,Y;Z,W) + \langle \sigma(X,W),\sigma(Y,Z)\rangle - \langle \sigma(X,Z),\sigma(Y,W)\rangle \tag{3.12.8}$$

$$\left(\tilde{R}(X,Y)Z\right)^{\perp} = (\overline{\nabla}_X\sigma)(Y,Z) - (\overline{\nabla}_Y\sigma)(X,Z) \tag{3.12.9}$$

$$\tilde{R}(\mathrm{X},Y;\xi,\eta) = R^{\perp}(X,Y;\xi,\eta) - \langle \left[A_{\xi},A_{\eta}\right]X,Y\rangle \tag{3.12.10}$$

where $\tilde{R}(X,Y;Z,W) = \langle R(X,Y)Z,W\rangle$ ...etc, X, Y, Z, W are tangent to N, ξ and η are normal to N, and $\perp$ in (3.12.9) denotes the normal component.

3.12.1. Definition. A submanifold N of a Kähler manifold $\widetilde{M}$ is called a CR-submanifold if there is a differentiable distribution $\mathfrak{D}: x \to \mathfrak{D}_x \subseteq T_xN$ on N satisfying the following conditions:

i. $\mathfrak{D}$ is holomorphic, i.e.$\mathrm{J}\mathfrak{D}_x = \mathfrak{D}_x$ for each $x \in N$, and

ii. the complementary orthogonal distribution $\mathfrak{D}^{\perp}: x \to \mathfrak{D}_x^{\perp} \subseteq T_xN$ is totally real,

i.e. $\mathrm{J}\mathfrak{D}_x^{\perp} \subset T_x^{\perp}N$ for each $x \in N$.

If $\dim \mathfrak{D}_x^{\perp} = 0$ (respectively, $\dim \mathfrak{D}_x = 0$), then the CR-submanifold N is a holomorphic submanifold (respectively, totally real submanifold).

If $\dim \mathfrak{D}_x^{\perp} = \dim T_x^{\perp}$, then the CR-submanifold is an anti-holomorphic submanifold (or generic submanifold). A CR-submanifold is called a proper CR-submanifold if it is neither holomorphic nor totally real.

We shall always denote by h the complex dimension of $\mathfrak{D}_x$ and by p the real dimension of $\mathfrak{D}_x^{\perp}$, i.e. $h = \dim_C \mathfrak{D}_x$ and $p = \dim_R \mathfrak{D}_x^{\perp}$.

We denote by ν the complementary orthogonal subbundle of $\mathrm{J}\mathfrak{D}^{\perp}$ in $T^{\perp}N$. Hence we have

$$T^{\perp}N = \mathrm{J}\mathfrak{D}^{\perp} \oplus \nu, \quad \mathrm{J}\mathfrak{D}^{\perp} \perp \nu \tag{3.12.11}$$

Let $\widetilde{M}$ be a Kähler manifold. Then we have

$$\tilde{\nabla}\mathrm{J} = 0$$

If N is a CR-submanifold of $\widetilde{M}$, then (3.12.1) and (3.12.2) give

$$\mathrm{J}\nabla_U Z + \mathrm{J}\sigma(U, Z) = -A_{\mathrm{J}z}U + D_U \mathrm{J}Z \tag{3.12.12}$$

for U tangent to N and Z in $\mathfrak{D}^{\perp}$.

3.12.2. Lemma (Chen 1981**).** Let N be a CR–submanifold of a Kähler manifold $\widetilde{M}$. Then we have

$$\langle \nabla_U Z, X\rangle = \langle \mathrm{J}A_{\mathrm{J}Z}U, X\rangle, \tag{3.12.13}$$

$$A_{\mathrm{J}Z}W \quad = A_{\mathrm{J}W}Z, \tag{3.12.14}$$

$$A_{\mathrm{J}\xi}X \quad = -A_{\xi}\mathrm{J}X, \tag{3.12.15}$$

for U tangent to N, X in $\mathfrak{D}$, Z and W in $\mathfrak{D}^{\perp}$, and ξ in ν.

Proof. (3.12.13) and (3.12.14) follow immediately from (3.12.12). And (3.12.15) follows from that fact that $\langle \sigma(\mathrm{J}X, Y), \xi\rangle = \langle \tilde{\nabla}_Y \mathrm{J}X, \xi\rangle = \langle \mathrm{J}\sigma(X, Y), \xi\rangle$.

3.12.3. Lemma. Let N be a CR-submanifold of a Kähler manifold $\widetilde{M}$. Then for any Z, W in $\mathfrak{D}^{\perp}$ we have

$$D_W \mathrm{J}Z - D_Z \mathrm{J}W \in \mathrm{J}\mathfrak{D}^{\perp} \tag{3.12.16}$$

Proof. For any ξ in ν and Z, W in $\mathfrak{D}^{\perp}$ we have

$$\langle A_{\mathrm{J}\xi}Z, W\rangle = -\langle \tilde{\nabla}_Z \mathrm{J}Z, W\rangle = \langle D_Z\xi, \mathrm{J}W\rangle = -\langle \xi, D_Z \mathrm{J}W\rangle$$

Thus we obtain

$$\langle \xi, D_W \mathrm{J} Z - D_Z \mathrm{J} W\rangle = \langle A_{\mathrm{J}\xi} Z, W\rangle - \langle A_{\mathrm{J}\xi} W, Z\rangle = 0$$

Since this true for all ξ in ν, (3.12.16) holds.

From Lemma (3.12.2) it follows that we have

$$\mathrm{J}[Z, W] = \mathrm{J}(\nabla_Z W - \nabla_W Z) = D_Z \mathrm{J} W - D_W \mathrm{J} Z.$$

Thus by Lemma (3.12.3) we obtain

3.12.4. Lemma (Chen 1981**).** The totally real distribution $\mathfrak{D}^{\perp}$ of a CR-submanifold in a Kähler manifold is integrable.

3.12.5. Lemma. Let N be a CR-submanifold of a Kähler manifold $\widetilde{M}$. Then $\mathfrak{D}$ is integrable if and only if

$$\langle \sigma(X, \mathrm{J}Y), \mathrm{J}Z\rangle = \langle \sigma(\mathrm{J}X, Y), \mathrm{J}Z\rangle$$

for any vectors X, Y in $\mathfrak{D}$, and Z in $\mathfrak{D}^{\perp}$.

3.12.6. Lemma. For a submanifold N in a Kähler manifd $\widetilde{M}$, the leaf $N^{\perp}$ of $\mathfrak{D}^{\perp}$ is totally geodesic in N if and only if

$$\langle \sigma(\mathfrak{D}, \mathfrak{D}^{\perp}), \mathrm{J}\mathfrak{D}^{\perp}\rangle = 0 \tag{3.12.17}$$

The following Lemma can be obtained easily from Lemma (3.12.3.4).

3.12.7. Lemma. If (3.12.17) holds and $\mathfrak{D}$ is integrable, then for any X in $\mathfrak{D}$ and ξ in $\mathrm{J}\mathfrak{D}^{\perp}$, we have

$$A_{\xi} \mathrm{J} X = -\mathrm{J} A_{\xi} X \tag{3.12.18}$$

Let P, F, t and f be the endomorphism and vector-valued 1-forms defined by (3.12.15) and (3.12.16). Put

$$(\bar{\nabla}_U P)V = \nabla_U(PV) - P\nabla_U V, \tag{3.12.19}$$

$$(\bar{\nabla}_U F)V = D_U(FV) - F(\nabla_U V), \tag{3.12.20}$$

$$(\bar{\nabla}_U t)\xi = \nabla_U(t\xi) - tD_U\xi, \tag{3.12.21}$$

$$(\bar{\nabla}_U f)\xi = D_U(f\xi) - fD_U\xi \tag{3.12.22}$$

for U, V tangent to N, and ξ normal to N. Then the endomorphism P (respectively, endomorphism f, 1-forms F or t) is parallel if

$$\bar{\nabla}P = 0 \text{ (respectively, } \bar{\nabla}f = 0, \quad \bar{\nabla}F = 0, \quad \text{or} \quad \bar{\nabla}t = 0).$$

From (3.12.12), (3.12.13) and (3.12.15) we obtain

$$(\bar{\nabla}_U P)V = t\sigma(U,V) + A_{FV}U \tag{3.12.23}$$

3.12.1. CR-Products in Kähler Manifolds

According to Lemma (3.12.4), every CR-submanifold N of a Kähler manifold is foliated by totally real submanifolds.

3.12.1.1. Definition. A CR-submanifold N of a Kähler manifold $\widetilde{M}$ is called a CR-product if it is locally a Riemannian product of a holomorphic submanifold $N^{\perp}$ and a totally real submanifold $N^{\perp}$ on $\widetilde{M}$.

First we give the following characterization of CR-products.

3.12.1.2. Theorem (Chen 1981**).** A CR-submanifold of a Kähler manifold $\widetilde{M}$ is CR-products if and only if P is parallel, i.e.,

$$\bar{\nabla}P = 0.$$

Proof. If P is parallel, (3.12.23) gives

$$t\sigma(U,V) = -A_{FV}U \tag{3.12.1.1}$$

for any vectors U,V tangent to N. In particular, if $X \in \mathfrak{D}$ then $FX = 0$. Hence (3.12.1.1) implies $t\sigma(U,X) = 0$, i.e.,

$$A_{JZ}X = 0, \tag{3.12.1.2}$$

for any Z in $\mathfrak{D}^{\perp}$ and X in $\mathfrak{D}$.Thus by Lemma (3.12.15) and Lemma (3.12.16) we know that $\mathfrak{D}$ is integrable and the leaf $N^{\perp}$of $\mathfrak{D}^{\perp}$ is totally geodesic in N. Let $N^{\perp}$ be a leaf of $\mathfrak{D}$. For any X, Y in $\mathfrak{D}$, and Z in $\mathfrak{D}^{\perp}$, (3.12.1) and Lemma (3.12.2) give

$$0 = \langle A_{JZ}Y, X\rangle = \langle JA_{JZ}Y, JX\rangle = \langle \nabla_Y Z, JX\rangle = -\langle Z, \nabla_Y JX\rangle.$$

From this we may conclude that $N^{\perp}$ is totally geodesic in N, and N is a CR–product in $\widetilde{M}$.

Conversely, if N is a CR-product, then $\nabla_U Y \in \mathfrak{D}$ for any Y in $\mathfrak{D}$ and U tangent to N. Thus by (3.12.2.1) and (3.12.2.2), we may obtain

$$\mathrm{J}\sigma(U,Y) = \sigma(U,\mathrm{J}Y).$$

From this, together with (3.12.1) and (3.12.2) we may prove that

$$(\bar{\nabla}_U P)Y = 0.$$

Similarly, from $\nabla_U Z \in \mathfrak{D}^{\perp}$ for any Z in $\mathfrak{D}^{\perp}$ and U tangent to N, we may also prove that $(\bar{\nabla}_U P)Z = 0$.

From the proof of Theorem (3.12.1.2) we have the following.

3.12.1.2. Lemma. A CR-submanifold N in a kähler manifold $\widetilde{M}$ is a CR-product if and only if

$$A_{\mathrm{J}\mathfrak{D}^{\perp}}\mathfrak{D} = 0$$

Remark. In Bejancu-Kon-Yano proved that if N is an anti-holomorphic submanifold and $\bar{\nabla}P = 0$, then N is a CR-product.

3.12.1.3. Lemma (Bejancu 1986**).** Let N be a CR-product of a Kähler manifold $\widetilde{M}$.Then for any unit vectors X in $\mathfrak{D}$ and Z in $\mathfrak{D}^{\perp}$ we have

$$\widetilde{H}_B(X,Z) = 2\|\sigma(X,Z)\|^2,$$

where $\widetilde{H}_B(X,Z) = \tilde{R}(X,\mathrm{J}X;\mathrm{J}Z,Z)$ is the holomorphic bisectional curvature of

$X \wedge \mathrm{Z}$.

Proof: Let N be a CR-product in $\widetilde{M}$.Then we have (3.12.1.2) for any Z in $\mathfrak{D}^{\perp}$ and X in $\mathfrak{D}$. Thus by equation (3.12.9) of Codazzi we obtain

$$\tilde{R}(X,\mathrm{J}X;Z,\mathrm{J}Z) = \langle D_X\sigma(\mathrm{J}X,Z) - D_{\mathrm{J}X}\sigma(X,Z),\mathrm{J}Z\rangle,$$

where we have used the fact that N^T is totally geodesic in N. Since

$\langle\sigma(\mathfrak{D},\mathfrak{D}^{\perp}),\mathrm{J}\mathfrak{D}^{\perp}\rangle = 0$, (3.12.1.2) and (3.12.1.3) imply

$$\tilde{R}(X,\mathrm{J}X;Z,\mathrm{J}Z) = \langle\sigma(X,Z),D_{\mathrm{J}X}\mathrm{J}Z\rangle - \langle\sigma(\mathrm{J}X,Z),D_X\mathrm{J}Z\rangle$$
$$= \langle\sigma(X,Z),\mathrm{J}\tilde{\nabla}_{JX}Z\rangle - \langle\sigma(\mathrm{J}X,Z),\mathrm{J}\tilde{\nabla}_X Z\rangle$$

$$= \langle \sigma(X, Z), \mathrm{J}\sigma(\mathrm{J}X, Z)\rangle - \langle \sigma(\mathrm{J}X, Z), \mathrm{J}\sigma(X, Z)\rangle \tag{3.12.1.4}$$

Thus by (3.12.1.2) and Lemma (3.12.7) we obtain the lemma.

3.12.1.4. Theorem (Chen 1981). Let $\widetilde{M}$ be a kähler manifold with negative holomorphic bisectional curvature. Then every CR-product in $\widetilde{M}$ is either a holomorphic submanifold or a totally real submanifold. In particular, there exist no proper CR-product in any complex hyperbolic space $\widetilde{M}^m(c)(c < 0)$.

3.13. Totally Real Submanifolds

Let M be an n-dimensional Riemannian manifold and $\widetilde{M}$ be a Kähler manifold of dimension $2(n+p), p \geq 0$. Let $\widetilde{\mathrm{J}}$ be the almost complex structure of $\widetilde{M}$ and let g (resp. $\tilde{g}$) be the Riemannian metric of M $\left(\text{resp.} \widetilde{M}\right)$. We call M an totally real submanifold of $\widetilde{M}$ if M admits an isometric immersion into $\widetilde{M}$ such that for all x, $\widetilde{\mathrm{J}}\left(T_x(M)\right) \subset v_x(M)$, where $T_x(M)$ denotes the tangent space of M at x and v_x the normal space at x. By a plane section we mean a 2-dimensional linear subspace of a tangent space. A plane section τ is called holomorphic (respectively, anti -holomorphic) if $\widetilde{\mathrm{J}}\tau = \tau$ $\left(\text{respectively, if } \widetilde{\mathrm{J}}\tau \text{ is perpendicular to } \tau\right)$. A totally geodesic submanifold $P_n(\mathbb{R})$ in $P_n(₵)$, $S^1 \times S^1$ in $P_2(₵)$ and an immersion $P_n(₵) \to P_{n(n+2)}(₵)$ defined by $[z_i] \to [z_i \bar{z}_j]$ give typical examples of totally real submanifolds.

3.13.1. Proposition (Chen and Ogiue 1974**).** Let M be a submanifold immersed in an almost Hermition manifold $\widetilde{M}$. Then M is a totally real submanifold of $\widetilde{M}$ if and only if every plane section of M is antiholomorphic.

Proof. Let X be an arbitrary vector in $T_x(M)$, and let $e_1 = X, e_2, \dots, e_n$ be a basis of $T_x(M)$.We denote by τ_{ij} the plane section spanned by e_i and e_j.

Assume that every plane section is antiholomorphic. Then $\widetilde{\mathrm{J}}\tau_{1j}$ are perpendicular to τ_{1j} for $j = 2, \dots, n$. Therefore $\widetilde{\mathrm{J}}X$ is perpendicular to $e_1, e_2, \dots, e_n$ so that $\widetilde{\mathrm{J}}X \in v_x$. this implies that M is a totally real submanifold of M. The converse is clear. ■

Chapter (4)

Submanifolds of Finite Type

4.1. Introduction

The study of submanifolds of finite type began in the late 1970's through B. Y. Chen's attempts to find the best possible estimate of the total mean curvature of a compact submanifold of a space and to find a notion of "degree" for submanifolds of Euclidean space. Similar to minimal submanifolds, submanifolds of finite type are characterized by a variational minimal principle in a natural way.

The main objects of studies in algebraic geometry are algebraic varieties. Because an algebraic variety is defined by using algebraic equations, one can define the degree of an algebraic variety by its algebraic structure.

On the other hand, according to Nash's imbedding theorem, every Riemannian manifold can be realized as a Riemannian submanifold in some Euclidean space with sufficiently high codimension. However, one lacks the notion of the degree for Riemannian submanifolds in Euclidean spaces.

The family of submanifolds of finite type is huge, which contains many important families of submanifolds, including minimal submanifolds of Euclidean space, minimal submanifolds of hypersphere, parallel submanifolds as well as equivariant immersed compact homogeneous submanifolds.

On one hand, the notion of finite type submanifolds provides a very natural way to apply spectral geometry to study submanifolds. One the other hand, one can also apply the theory of finite type submanifolds to investigate the spectral geometry of submanifolds.

4.2. Order and Type of Submanifolds

As mentioned in introduction, one lacks the concept of degree for a submanifold of a Euclidean m-space $\mathbb{E}^m$. However, one can use the induced Riemannian structure on a submanifold M of $\mathbb{E}^m$ to introduce a pair of well- defined numbers p and q associated with the submanifold M.

Here p is a natural number and q is either a natural number $\geq p$ or $+\infty$. We call the pair $[p, q]$ the order of submanifold M; more precisely, p is the lower order and q the upper order of the submanifold. The submanifold is said to be of finite type if its upper order is finite and it is of infinite type if its upper order is $+\infty$.

The order of a submanifold is defined as follows.

Let (M, g) be a compact Riemannian n-manifold with Riemannian connection ∇. And let $\Delta = -\text{trace } \nabla^2$ denote the Laplacian operator of (M, g) acting as an elliptic differential operator on $C^\infty(M)$, the space of all smooth functions on M. It is well known that the eigenvalues of Δ form a discrete infinite sequence:

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots \nearrow \infty. \tag{4.2.1}$$

Let $V_k = \{f \in C^\infty(M): \Delta f = \lambda_k f\}$ be the eigenspace of Δ with eigenvalue λ_k. Then V_k is finite dimensional. We define an inner product (,) on $C^\infty(M)$ by

$$(f, h) = \int_M fh\, dV \tag{4.2.2}$$

where dV is the volume element of (M, g) and for $f, h \in C^\infty(M)$, then the decomposition $\sum_{k=0}^{\infty} V_k$ is orthogonal with respect to this structure. Moreover $\sum_{k=0}^{\infty} V_k$ is dense in $C^\infty(M)$ (in L^2 -sense).

Since M is closed, V_0 is one-dimensional and it consists only of constant functions.

If we denote by $\widehat{\bigoplus} V_k$ the completion of $\sum V_k$, we have

$$C^\infty(M) = \widehat{\bigoplus}_k V_k. \tag{4.2.3}$$

For each function $f \in C^\infty(M)$, let f_t denote the projection of f onto the subspace V_t $(t = 0,1,2, \dots)$. Then we have the following spectral decomposition:

$$f = \sum_{t=0}^{\infty} f_t\,, \left(\text{in } L^2\text{– sense}\right). \tag{4.2.4}$$

Because V_0 is 1-dimensional, for any non-constant function $f \in C^\infty(M)$, there is a positive integer $p \geq 1$ such that $f_p \neq 0$ and

$$f - f_0 = \sum_{t \geq 1} f_t, \tag{2.4.5}$$

where $f_0 \in V_0$ is a constant. If there are infinite many f_t's which are nonzero, we put $q = +\infty$.

Otherwise, there is an integer q, $q \geq p$ such that $f_q \neq 0$ and

$$f - f_0 = \sum_{t=p}^{q} f_t \tag{4.2.6}$$

If we allow q to be $+\infty$, we have the decomposition (4.2.6) in general. The set

$$T(f) = \{t \in N_0 : f_t \neq 0\} \tag{4.2.7}$$

is called the order of f. The smallest element in $T(f)$ is called the lower order of f, denoted by $l.o.(f)$, and the supremum of $T(f)$ is called the upper order of f, denoted by $u.o.(f)$. A function f in $C^\infty(M)$ is said to be of finite if $T(f)$ is a finite set, i.e., if its spectral decomposition contains only finitely many non-zero terms. Otherwise f is said to be of infinite type. f is said to be of k-type if $T(f)$ contains exactly k elements.

For an isometric immersion $x: M \to \mathbb{E}^m$ of a compact Riemannian manifold M into a Euclidean m-space, we put

$$x = (x_1, \dots, x_m), \tag{4.2.8}$$

where x_A is the A-th Euclidean coordinate function of M in $\mathbb{E}^m$. For each x_A, we have

$$x_A - (x_A)_0 = \sum_{t=p_A}^{q_A} (x_A)_t \, , A = 1, \dots, m. \tag{4.2.9}$$

For each isometric immersion $x: M \to \mathbb{E}^m$, we put

$$p = \inf_{A}\{p_A\}, \qquad q = \sup_{A}\{q_A\} \tag{4.2.10}$$

where A ranges among all $A = 1,2, \dots, m$ such that $x_A - (x_A)_0 \neq 0$. It is easy to see that p and q are well-defined geometric invariants such that p is a positive integer and q is either $+\infty$ or an integer $\geq p$. By using the (4.2.8), (4.2.9) and (4.2.10) we have the following spectral decomposition of x (in vector form);

$$x = x_0 + \sum_{t=p}^{q} x_t \ . \tag{4.2.11}$$

An immersion x is sometime said to be of mono-order (bi-order, tri-order…) if there are only 1 (2, 3…) of x_t which is (are) non-zero. If $p = q$, we just say that x is order p. We define $T(x)$ by

$$T(x) = \{t \in N_0 : x_t \neq 0\}.$$

The immersion x or the submanifold M is said to be of k-type if $T(x)$ contains exactly k elements. Similarly we can define the lower order and the upper order of the immersion.

The immersion x is said to be of finite type if its upper order q is finite; and the immersion is said to be of infinite type if its upper order is $+\infty$.

For an isometric immersion $x: M \to \mathbb{E}^m$ of a compact Riemannian manifold M into $\mathbb{E}^m$, the constant vector x_0 in (4.2.11) is exactly the center of mass of M in $\mathbb{E}^m$.

Given two $\mathbb{E}^m$-valued functions v, w on M, we define the inner product v, w by

$$(v, w) = \int_M \langle v, w \rangle dV \tag{4.2.12}$$

where $\langle v, w \rangle$ denotes the Euclidean inner product of v, w.

For an isometric immersion $x: M \to \mathbb{E}^m$ of a compact Riemannian manifold M into $\mathbb{E}^m$, the components of the spectral decomposition (4.2.11) are mutually orthogonal, i.e.

$$(x_t, x_s) = 0, \ t \neq s \tag{4.2.13}$$

One cannot make the spectral decomposition of a function on a noncompact Riemannian manifold in general. However, it remains possible to define the notion of a function of finite type and the related notions of order… etc for those functions. For example, a function f is said to be of finite type if it is a finite sum of eigenfunctions of the Laplacian. More precisely, a function is of finite type if it can be written as a finite sum:

$$f = \sum_{i=1}^{k} f_i$$

where $f_1, \ldots, f_k$ are (non zero) eigenfunctions of the Laplacian with eigenvalues $\lambda_{t_1}, \ldots, \lambda_{t_k}$ are assumed to be mutually distinct.

We define the order $T(f)$ of f to be the set $\{t_1, \ldots, t_k\}$; Moreover,

$$u.o.(f) = max\, T(f) \text{ and } l.o.(f) = min\, T(f).$$

Similarly, for an isometric immersion $x: M \to \mathbb{E}^m$ of a non-compact manifold M into $\mathbb{E}^m$, the immersion (or the submanifold) is said to be of finite type if it admits a finite spectral decomposition

$$x = \sum_{t=p}^{q} x_t\,, \quad \Delta x_t = \lambda_t x_t \tag{4.2.14}$$

for some natural numbers p and q. Otherwise, the immersion is said to be of infinite type. When M is compact, the compact x_0 in spectral decomposition (4.2.11) is a constant vector.

However, when M is noncompact, the component x_0 is not necessary a constant vector. A finite type immersion $x: M \to \mathbb{E}^m$ is said to be null if the component x_0 in its spectral decomposition (4.2.14) is non-constant.

For instance, a null 2-type immersion is an immersion with the following simple spectral decomposition:

$$x = x_0 + x_p, \quad \Delta x = \lambda_p x_p, \tag{4.2.15}$$

for some non-constant vector functions x_0, x_p, where λ_p is a non-zero eigenvalue of the Laplacian of M.

It is easy to see that if two isometric immersions x and y of a Riemannian manifold M into a Euclidean space is congruent, then they have the same order, i.e.,$T(x) = T(y)$, hence they also have the same type number.

For finite type isometric immersions of a Riemannian manifold, we have the following general result.

4.2.1. Theorem (Chen 1996**)**. Let $x: M \to \mathbb{E}^m$ be a k- type isometric immersion of order $T(x) = \{t_1, \ldots, t_k\}$ and $L: \mathbb{E}^m \to \mathbb{E}^N$ be a linear map. If the composition $L \circ x: M \to \mathbb{E}^N$ is isometric, then $L \circ x$ is of finite type.

Moreover, the type number of $L \circ x$ is at most k and the order $T(L \circ x)$ is a subset of the order $T(x)$ of x.

This result follows simply from the fact that a linear combination of some linear combinations of eigenfunctions of the Laplacian is also a linear combination of eigenfunctions of the Laplacian. For example, let $\gamma: S^1(1) \to \mathbb{E}^4$ be a 2-type isometric immersion of order $\{1,3\}$ given by

$$\gamma(s) = \left(\tfrac{1}{\sqrt{2}} \sin s\,, \tfrac{1}{\sqrt{2}} \cos s\,, \tfrac{1}{3\sqrt{2}} \sin 3s\,, \tfrac{1}{3\sqrt{2}} \cos 3s\right)^T, \tag{4.2.16}$$

which is regarded as a column vector and let L be the linear map from $\mathbb{E}^4$ into $\mathbb{E}^3$ defined by

$$L = \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \tag{4.2.17}$$

Then the composition $L \circ \gamma: S^1(1) \to \mathbb{E}^3$ is an isometric immersion given by

$$L \circ \gamma(s) = \left(\sin s\,, -\tfrac{1}{2} \cos s + \tfrac{1}{6} \cos 3s\,, -\tfrac{1}{2} \sin s + \tfrac{1}{6} \sin 3s\right)^T$$

which is also a 2-type curve of order $\{1,3\}$ lying fully in $\mathbb{E}^3$ but not fully in $\mathbb{E}^4$.

4.2.1. Remark. Theorem 4.2.1 implies, in particular, that if $x: M \to \mathbb{E}^m$ is an isometric immersion and $\bar{x} = L \circ x: M \to \mathbb{E}^m \subset \mathbb{E}^{\bar{m}}$ where $L: \mathbb{E}^m \subset \mathbb{E}^{\bar{m}}$ is inclusion, then $T(x) = T(\bar{x})$.

4.2.2. Remark. For an isometric immersion $x: M \to \mathbb{E}^m$ of a Riemannian n-manifold into $\mathbb{E}^m$, one has the following formula of E-Beltrami:

$$\Delta x = -nH \tag{4.2.18}$$

where H denotes the mean curvature vector of the immersion.

4.2.3. Remark. Let $x: M \to \mathbb{E}^m$ be an isometric immersion a Riemannian n-manifold. Then the Laplacian operator Δ of M gives rise to a differentiable map

$$L: M \to \mathbb{E}^m \tag{4.2.19}$$

which is called the Laplace map. The image $L(M)$ of the Laplace map is called the Laplace image and the transformation $\mathfrak{T}: M \to L(M)$ from M onto the Laplace image via the Laplace operator is called the Laplace transformation of the immersion $x: M \to \mathbb{E}^m$.

The Laplace map of a submanifold in a Euclidean space is closely related with the submanifold via Beltrami's formula (4.2.18).

4.3. Finite Type Submanifolds

We recall some basic definitions, results and formulas.

Let $x: M \to \mathbb{E}^m$ be an isometric immersion of a (connected) Riemannian manifold M into the Euclidean m-space $\mathbb{E}^m$. Denote by Δ the Laplace operator of M. The immersion x is said to be of finite type if the position vector field of M in $\mathbb{E}^m$, also denoted by x, can be expressed as finite sum of $\mathbb{E}^m$-valued eigenfunctions of the Laplace operator, i.e., if x can be expressed as

$$x = c + x_1 + x_2 + \cdots + x_k \tag{4.3.1}$$

where c is a constant vector in $\mathbb{E}^m$ and $x_1, \ldots, x_k$ are nonconstant $\mathbb{E}^m$-valued maps satisfying

$$\Delta x_i = \lambda_i x_i, i = 1,2, \ldots, k \tag{4.3.2}$$

The composition (4.3.1) is called the spectral decomposition or the spectral resolution of the immersion x. In particular, if all of the eigenvalues $\lambda_1, \ldots, \lambda_k$ associated with the spectral decomposition are mutually different, then the immersion x (or the submanifold M) is said to be of k-type. In particular, if one of $\lambda_1, \ldots, \lambda_k$ is zero, then the immersion is said to be of null k-type.

Clearly, every submanifold of null k-type is non-compact. A submanifold is said to be of infinite type if it is not of finite type. In terms of finite type submanifolds, a result of states that a submanifolds of $\mathbb{E}^m$ is of 1-type if and only if it is

either a minimal submanifold of $\mathbb{E}^m$ or a minimal submanifold of a hypersphere of $\mathbb{E}^m$.

For a sphere isometric immersion $x: M \to {S_c}^{m-1} \subset \mathbb{E}^m$, the immersion is called Mass-symmetric in ${S_c}^{m-1}$ if the center of gravity of M in $\mathbb{E}^m$ coincides with the center c of the hyperspheres ${S_c}^{m-1}$ in $\mathbb{E}^m$.

4.4. Minimal Polynomial of Finite Type Submanifolds

For a finite type submanifold M satisfying (4.3.1) and (4.3.2), the polynomial P defined by

$$P(t) = \prod_{i=1}^{k} (t - \lambda_i), \tag{4.4.1}$$

satisfies $P(\Delta)(x-c)=0$. This polynomial P is called the minimal polynomial M. For n-dimensional submanifold M of a Euclidean space, the mean curvature vector H satisfies Beltrami's formula:

$$\Delta x=-nH \tag{4.4.2}$$

It follows from (4.4.2) that the minimal polynomial Q also satisfies

$$Q(\Delta)H=0.$$

Conversely, if M is compact and if there exists a constant vector c and a non-trivial polynomial Q such that $Q(\Delta)(x-c)=0$(or $Q(\Delta)H=0$), then M are always of finite type. This characterization of finite type submanifold via the minimal polynomial plays an important role in the study of finite type submanifolds.

When M is non-compact, the existence of a non-trivial polynomial Q satisfying $Q(\Delta)H=0$ does not guarantee M to be finite. On the other hand, if either M is one dimensional or Q is a polynomial of degree k with exactly k distinct roots, then the existence of the polynomial Q satisfying $Q(\Delta)(x-c)=0$ for some constant vector c does guarantee that M is finite type.

4.4.1. Proposition (Chen 1984**).** Let M be a compact submanifold of $\mathbb{E}^m$. Then M is of 1-type if and only if M is a minimal submanifold of a hypersphere of $\mathbb{E}^m$.

For finite type submanifolds, we have the following characterization theorem.

4.4.2. Theorem (Chen 1984**).** Let M be a compact submanifold of $\mathbb{E}^m$. Then M is of finite type if and only if there is a polynomial $Q(t)\neq 0$ such that

$Q(\Delta)H=0$, where H is the mean curvature vector of M in $\mathbb{E}^m$.

4.5. A Basic Formula for ΔH

The following basic formula of ΔH derived in plays important role in the study of submanifolds of low type as well as in the study of biharmonic submanifolds:

$$\Delta H=\Delta^D H+\sum_{i=1}^{n} h(e_i,A_H e_i)+2trace(A_{DH})+\tfrac{n}{2}\,grad\langle H,H\rangle \tag{4.5.1}$$

where $\Delta^D$is the Laplacian operator associated with the normal connection D, h is the second fundamental form, and $\{e_1,\dots,e_n\}$ is a local orthonormal frame of

M. In particular, if M is a hypersurface of a Euclidean space $\mathbb{E}^{n+1}$, then formula (4.5.1) reduces to

$$\Delta H = (\Delta\alpha + \alpha\|h\|^2)\xi + 2trace(A_{DH}) + \tfrac{n}{2}grad\langle H, H\rangle, \tag{4.5.2}$$

where α is the mean curvature and ξ is a unit normal vector of M in $\mathbb{E}^{n+1}$.

Similar formulas hold as well if the ambient space is pseudo-Euclidean.

4.6. δ-Invariants and Ideal Immersions

Let M be a Riemannian n-manifold. Denote by $K(H)$ the sectional curvature of a plane section $\pi \subset T_pM, p \in M$. For any orthonormal basis $e_1, \dots, e_n$ of T_pM the scalar curvature τ at p is

$$\tau(p) = \sum_{i>j} K(e_i \wedge e_j)$$

Let L be a r-subspace of T_pM with $r \geq 2$ and let $\{e_1, \dots, e_r\}$ be an orthonormal-basis of L. The scalar curvature $\tau(L)$ of L is defined by

$$\tau(L) = \sum_{\alpha<\beta} K(e_\alpha \wedge e_\beta), 1 \leq \alpha, \beta \leq r. \tag{4.6.1}$$

For given integers $n \geq 3, k \geq 1$, we denoted by $S(n, k)$ the finite set consisting of k-tuples $(n_1, \dots, n_k)$ of integers satisfying $2 \leq n_1, \dots, n_k < n$ and $\sum_{i=1}^{k} n_i \leq n$.

Put $S(n) = \cup_{k\geq 1} S(n, k)$. For each k-tuple $(n_1, \dots, n_k) \in S(n)$, B. Y. Chen introduced in 1990s the Riemannian invariant $\delta(n_1, \dots, n_k)$ by

$$\delta(n_1, \dots, n_k)(p) = \tau(p) - inf\{\tau(L_1) + \cdots + \tau(L_k)\}, \tag{4.6.2}$$

$p \in M$, where $L_1, \dots, L_k$ run over all k mutually orthogonal submanifold of $\mathrm{T_p}M$ such that $\dim L_j = n_j, j = 1, \dots, k$. For an n-dimensional submanifold of $\mathbb{E}^m$ and for a k-tuple $(n_1, \dots, n_k) \in S(n)$, B. Y. Chen proved the following general sharp inequality:

$$\delta(n_1, \dots, n_k) \leq \frac{n^2(n+k-1\sum n_j)}{2(n+k-\sum n_j)}|H|^2, \tag{4.6.3}$$

where $|H|^2 = \langle H, H\rangle$ denotes the squared mean curvature of M.

A submanifold M of $\mathbb{E}^m$ is called $\delta(n_1, \dots, n_k)$ ideal if it satisfies the equality case of (4.6.3) identically. Roughly speaking, ideal submanifolds are submanifolds which receive the least possible tension from its ambient space.

4.7. Proper and ϵ-Super Biharmonic Submanifolds

An immersed submanifold M of a Riemannian manifold $\widetilde{M}$ is said to be properly immersed if the immersion is a proper map, i.e., the preimage of each compact set in $\widetilde{M}$ is compact in M.

A hypersurface of a Euclidean space is called weakly convex if it has nonnegative principle curvatures. Also, a hypersphere of an $(n+1)$-sphere is called isoparametric if it has constant principle curvatures. The total mean curvature of a submanifold M in a Riemannian manifold is defined to be

$$\int |H|^2 dv.$$

Let M be a submanifold of a Riemannian manifold with inner product $\langle\, , \rangle$. Then M is called ϵ–superbiharmonic if it satisfies

$$\langle \Delta H, H \rangle \geq (\epsilon - 1)|\nabla H|^2, \qquad (4.7.1)$$

where $\epsilon \in [0,1]$ is constant.

For a complete Riemannian manifold (N, h) and $\alpha \geq 0$, if the sectional curvature K^N of N satisfies

$$K^N \geq -L(1 + distN(.,q_0)^2)^{\frac{\alpha}{2}} \qquad (4.7.2)$$

For some $L > 0$ and $q_0 \in N$, then we call that K^N has a polynomial growth bound of order α from below.

4.8. Spherical Hypersurfaces of Finite Type

4.8.1. Finite Type Spherical Hypersurfaces

In contrast to Euclidean hypersurfaces, there do exist many 1–type and 2–typespherical hypersurfaces. B. Y. Chen proved in that every compact hypersurfaces of a hypersphere $S^{n+1} \subset \mathbb{E}^{n+2}$, not a small hypersphere, is mass-symmetric and of 2-type if and only if it has non-zero constant mean curvature and constant scalar curvature.

Consequently, every isoparametric hypersurfaces of a hypersphere is either of 1-type or mass-symmetric and 2-type. Since there exist non-minimal isoparametric hypersurfaces in hyperspheres, there do exist 2-type hypersurfaces of hyperspheres.

4.8.2. Finite Type Hypersurfaces of a Euclidean Space

The class of submanifolds of finite type is very large. For example, minimal submanifolds of hyperspheres are of 1-type. Isoparametric hypersurfaces of a hypersphere are either of 1-type or of 2-type. Moreover, parallel submanifolds and compact homogeneous Riemannian manifolds, equivariantly immersed in a Euclidean space, are of finite type.

On the other hand, very few hypersurfaces of finite type are known. For example, no surfaces of finite type in a Euclidean 3-space are known, other than minimal surfaces, circular cylinders and spheres.

In views of these facts, B. Y. Chen's proposed more than a decade ago the following basic problem in the theory of finite type.

Problem 1. Classify finite type hypersurfaces of a Euclidean space. In particular, classify finite type surfaces of a Euclidean 3-space. For compact finite type surfaces in a Euclidean 3-space, B. Y. Chen

Conjecture 1: The only compact finite type surfaces of a Euclidean 3-space are the spheres.

For closed 2-type hypersurfaces of a Euclidean space we have the following result proved by the B. Y. Chen and H. S. Lue.

4.8.2.1. Theorem (Chen and Lue 1988**).**

(1) Every compact 2-type hypersurface of a Euclidean space has non-constant mean curvature.

(2) If M is a 2-type hypersurface of a Euclidean space with constant mean curvature, then M is of null 2-type.

Also we have the following result on 2-type hypersurfaces.

4.8.2.2. Theorem (Chen 1991**).** Null 2-type hypersurfaces and open portions of hyperspheres are the only hypersurfaces of a Euclidean space with have non zero constant mean curvature and constant scalar curvature.

For null 2-type surfaces, we have the following:

4.8.2.3. Theorem (Chen 1988**).** A surface M in a Euclidean 3-space is of null 2-type if and only if M is an open portion of a circular cylinder.

Theorem 4.8.2.3 implies that null 2-type surfaces in $\mathbb{E}^3$ have nonzero constant mean curvature and constant scalar curvature.

4.9: 2- Type Submanifolds in $\mathbb{E}^m$

For an n-dimensional submanifold M in $\mathbb{E}^m$, we denote by $h, A, H, \nabla,$ and D, the second fundamental form, the Weingarten map, the mean curvature vector, the Riemannian curvature and the normal connection of the submanifold M, respectively. A submanifold M is said to have parallel mean curvature vector if $DH = 0$ identically. For a hypersurface M, the parallelism of mean curvature vector equivalent to the constancy of mean curvature $\alpha = \|H\|$.

If the submanifold M is closed (i.e., M is compact and without boundary), then every eigenvalue λ_t of Δ is ≥ 0 and the only harmonic functions on M are constant functions. In this case, the constant vector c in the spectral decomposition (4.3.1) is nothing but the center of mass of M in $\mathbb{E}^m$. A submanifold M of a hypersphere S^{m-1} in $\mathbb{E}^m$ is said to be mass-symmetric if the center of mass of M in $\mathbb{E}^m$ is center of the hypersphere S^{m-1} in $\mathbb{E}^m$. We study 2- type submanifolds in Euclidean space with parallel mean curvature vector.

Since 2-type submanifolds are the "simplest submanifolds" next to minimal submanifolds, 2-type submanifolds, in particular mass-symmetric spherical 2-type submanifolds, deserve special attention.

Mass-symmetric spherical 2-type submanifolds have some special properties.

For instances, every mass-symmetric spherical 2-type submanifolds has constant mean curvature (which is completely determined by its order and such a submanifold is pointwise orthogonal).

4.9.1. Theorem. Let M be a 2-type submanifold. If M has parallel mean curvature vector, then one of the following two cases occurs,

(i) M is spherical;

(ii) M is of null 2-type.

In particular, if M is closed, then M is spherical and mass-symmetric.

Proof. Let X, Y be two vector fields tangent to M. Then, for any fixed vector a in $\mathbb{E}^m$, we have

$$XY\langle H,a\rangle = \langle D_Y D_X H, a\rangle - \langle \nabla_Y(A_H X), a\rangle - \langle A_{D_X H} Y, a\rangle - \langle h(Y, A_H X), a\rangle, \quad (4.9.1)$$

where $\langle \, , \rangle$ denotes the inner product of $\mathbb{E}^m$. Let $e_1,\ldots,e_n$ be an orthonormal local frame field tangent to M. Then equation (4.9.1) implies

$$\Delta H = \Delta^D H + \sum\left\{h(e_i, A_H e_i) + A_{D_{e_i}H} e_i + \left(\nabla_{e_i} A_H\right) e_i\right\} \quad (4.9.2)$$

where

$$\Delta^D H = \sum\left\{D_{\nabla_{e_i} e_i} H - D_{e_i} D_{e_i} H\right\} \quad (4.9.3)$$

is the Laplacian of H with respect to the normal connection D. Regard ∇A_H and A_{DH} as (1, 2)-tensor M and we set

$$\overline{\nabla} A_H = \nabla A_H + A_{DH}$$

Then we have

$$\overline{\nabla} A_H = \sum\left\{\left(\nabla_{e_i} A_H\right) e_i + A_{D_{e_i}H} e_i\right\} \quad (4.9.4)$$

Let $e_{n+1}, \ldots, e_m$ be an orthonormal normal basis of M such that e_{n+1} is parallel to H. Then we have

$$\sum h(e_i, A_H e_i) = \|A_{n+1}\|^2 H + a(H) \quad (4.9.5)$$

where

$$A_r = A_{e_r}, \quad \|A_{n+1}\|^2 = tr(A_{n+1})^2,$$

and

$$a(H) = \sum_{r=n+2}^{m}\left(tr(A_H A_r)\right) e_r \quad (4.9.6)$$

is called the allied mean curvature vector of M in $\mathbb{E}^m$. Combining (4.9.2), (4.9.4), (4.9.5) and (4.9.6), we have the following useful formula:

$$\Delta H = \Delta^D H + \|A_{n+1}\|^2 H + a(H) + tr(\overline{\nabla} A_H). \tag{4.9.7}$$

Moreover, we also have the following

$$tr(\overline{\nabla} A_H) = \left(\tfrac{n}{2}\right) grad\, \alpha^2 + 2trA_{DH}, \;\; \alpha^2 = \langle H, H \rangle. \tag{4.9.8}$$

Therefore, if $DH = 0$, then we have

$$\Delta^D H = tr(\overline{\nabla}_{AH}) = 0$$

which implies

$$\Delta H = \|A_{n+1}\|^2 H + a(H). \tag{4.9.9}$$

Now, assume that M is of 2-type in $\mathbb{E}^m$. Then the position vector x of M in $\mathbb{E}^m$ has the following spectral decomposition:

$$x - c = x_p + x_q, \; \Delta x_p = \lambda_p x_p, \quad \Delta x_q = \lambda_q x_q \tag{4.9.10}$$

For (4.9.10) we have

$$\Delta^2 x = (\lambda_p + \lambda_q)\Delta x - \lambda_p \lambda_q (x - c). \tag{4.9.11}$$

On the other hand, we also have

$$\Delta x = -nH \tag{4.9.12}$$

Therefore, by using (4.9.9), (4.9.11), (4.9.12), we obtain

$$\|A_{n+1}\|^2 H + a(H) = (\lambda_p + \lambda_q)H + (\lambda_p \lambda_q / n)(x - c). \tag{4.9.13}$$

From (4.9.13) we have either $\lambda_p \lambda_q = 0$ or $x - c$ is normal to M at every point in M. If $\lambda_p \lambda_q = 0$, then M is of null 2-type. If $x - c$ is normal to M, then $\langle x - c, x-c$ is positive constant. In this case, M is contained in a hypersphere $Sm-1$ centered at c. In particular, if M is closed, then because λ_p and λ_q are positive, M cannot be null. Moreover, in this case, because c is the center of mass of M in $\mathbb{E}^m$, M is mass-symmetric in S^{m-1}. ■

4.9.2. Corollary. Every 2-type closed hypersurface of $\mathbb{E}^{n+1}$ has nonconstant mean curvature.

For surfaces in $S^3(r)$, we have the following classification theorem.

4.9.3. Theorem (Chen 1984**).** Let M be a compact, mass-symmetric surface of $S^3(r)$ in $\mathbb{R}^4$. Then M is of 2-type if and only if M is the product of two plane circles of different radii, that is, $M = S^1(a) \times S^1(b), a \neq b$.

For the proof, see [B. Y. Chen, Total Mean Curvature and Submanifolds of Finite Type, P. 279].

4.9.4. Theorem. Let M be a closed 2-type surface in $\mathbb{E}^m$. Then M has parallel mean curvature vector if and only if M is the product of two plane circles with different radii.

Proof. If M has mean curvature vectors, then M is one the following surfaces (cf. [B.Y.Chen, Geometry of submanifolds, P.106]):

(1) a minimal surface of $\mathbb{E}^m$,

(2) a minimal surface of a hypersphere of $\mathbb{E}^m$,

(3) a surface in a 3-dimensional linear subspace $\mathbb{E}^3$ or

(4) a surface in a 3-sphere S^3 in a 4-dimensional linear subspace.

For the first two cases, M is of 1-type which contradicts to the hypothesis. If M lies in a 3-dimensional linear subspace, then, by Theorem (4.9.1), M is a 2-sphere which is of 1-type again.

If M lies in a 3-sphere, then by parallelism of H, we see that the mean curvature is constant and so M is mass-symmetric. Consequently, according to theorem (4.9.3), we know that M is the product surface of two plane circles with different radii. ■

4.9.5. Spherical 2-type hypersurfaces:

Problem 2. Study and classify 2-type hypersurfaces in a hypersphere of $\mathbb{E}^{n+2}$. In particular, classify 3-dimensional 2-type of a hypersphere S^4 in $\mathbb{E}^5$.

4.10. Biharmonic Submanifolds

let $x: M \to \mathbb{E}^m$ be an isometric immersion.As we mentioned in preliminaries, the position vector of M in $\mathbb{E}^m$ satisfies

$$\Delta x = -nH \tag{4.10.1}$$

Formula (4.10.1) implies that the immersion is minimal if and only if the immersion is harmonic, that is $\Delta x = 0$. An isometric immersion $x: M \to \mathbb{E}^m$ is called biharmonic if we have

$$\Delta^2 x = 0, \text{ that is, } \ \Delta H = 0 \tag{4.10.2}$$

It is obvious that minimal immersions are biharmonic.

Problem 3. Other than minimal submanifolds of $\mathbb{E}^m$, which submanifolds of $\mathbb{E}^m$ are biharmonic?

From (4.10.1) and (4.10.2) it follows that there are no compact biharmonic submanifolds of positive dimension in $\mathbb{E}^3$. Moreover, by using the formula

$$\Delta H = (\Delta\alpha + \alpha\|h\|^2\xi + 2tr\ (A_{DH})) + \frac{n}{g}\, grad\langle H, H\rangle,$$

B. Y. Chen's proved in 1985 that every biharmonic surface in $\mathbb{E}^3$ is in fact a minimal surface.

Conjecture 2. The only biharmonic submanifolds in Euclidean Spaces are the minimal ones.

4.11. Submanifolds Theory and Parallel Transport.

The most symmetric of all Riemannian manifolds $(M,\ g)$ are the real space forms, i.e., the manifolds with contant sectional curvature $K = c$.Their (0, 4) - Riemann Christoffel curvature tensor R is given by $R = \frac{c}{2} g \wedge g$, ($\wedge$ denoting the Nomizu Kulkarni product of (0, 2)-tensor), and they were characterized by Riemann, Helmholtz and Lie as the Riemannian spaces which satisfy the axiom of free mobility. The class of the real space forms can be obtained by applying projective transformations to the locally flat spaces, i.e., to the manifolds $(M,\ g)$ for which $K = 0$, or equivalently, for which $R = 0$. The Riemann-Christoffel curvature tensor according to Schouten essentially measures the change of direction when a vector $v \in T_p M$ is parallelly transported all around infinitesimal coordinate parallelograms to a vector $v^* \in T_p M$.

The locally flat spaces are characterized by the fact that such parallel transport leaves v invariant, i.e., such that $v^* = v$ for all such coordinate parallelograms cornered at p.

In the 1920ties, Cartan introduced the locally symmetric spaces, i.e. the Riemannian manifolds (M, g) for which R is parallel,$\nabla R = 0$, where ∇ denotes the Levi-Civita connection of the metric. As shown by Cartan, the locally symmetric spaces are the Riemannian manifolds for which locally all geodesic reflections or symmetries σ_p in all points p of M actually are isometries, and, as shown by Levy, they can also be characterized as the Riemannian manifolds for which the sectional curvature $K(p, \pi)$ remains invariant under parallel transport along any curve in M, i.e. for which $K(p^*, \pi^*) = K(p, \pi)$,where $\pi^* \subset T_{p^*}M$ is the plane obtained by moving π parallelly form p to p^* along any curve γ joining p and p^*.The study of the locally symmetric spaces was independently started by P. A. Shirokov.

Every locally symmetric space satisfies $R \cdot R = 0$, whereby the first R stands for the curvature operator of $(M,\ g)$, i.e. for tangent vector fields X and Y one has $R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$, which acts as a derivation on the second R which stands for the Riemann-Christoffel curvature tensor. The converse however does not hold in general.

The Riemannian manifolds for which $R \cdot R = 0$ are called semi-symmetric spaces and were classified by Z. Szabȯ. They can be characterized by the geometric property that, up to second order, $K(p, \pi^*) = K(p, \pi)$, whereby π is any tangent 2-plane to M at p and π^* is the tangent 2-plane to M at p obtained by parallelly transporting π all around any infinitesimal coordinate parallelogram cornered at p. For short, their sectional curvatures are invariant under parallel transport around infinitesimal coordinate parallelograms.

In the 1980ties, R. Deszcz introduced the pseudo-symmetric spaces as follows. Let $Q(g, R) \equiv \wedge_g \cdot R$ be the Tachibana tensor of a Riemannian manifold $(M,\ g)$, i.e. the (0, 6)-tensor $\wedge_g \cdot R$ where the metrical endomorphism $\left(X \wedge_g Y\right): TM \to TM$ given by $\left(X \wedge_g Y\right)Z = g(Y, Z)X - g(X, Z)Y$ acts as a derivation on the (0, 4)-curvature tensor R,then M is said to be pseudo symmetric if the (0, 6)-tensors $R \cdot R$ and $Q(g,\ R)$ are proportional, i.e. if $R \cdot R = LQ(g, R)$ for some scalar valued function $L: M \to \mathbb{R}$. This function is called the double sectional curvature or the sectional curvature of Deszcz.

The class of pseudo symmetric manifolds can be obtained by applying projective transformations to the semi symmetric manifolds, i.e. to the manifolds for which $L = 0$.

In analogy with the above intrinsic symmetries of Riemannian manifolds concerning their Riemann-Christoffel curvature tensor R, table 1 list the corresponding extrinsic symmetries of submanifolds concerning their second fundamental form.

TABLE.1. Comparison between intrinsic and extrinsic symmetries

Intrinsic		**Extrinsic**	
Flat space	$R = 0$	Totally geodesic	$h = 0$
Space form	$R = \frac{c}{2} g \wedge g$	Totally umbilical	$h = gH$
Locally symmetric	$\nabla R = 0$	Parallel	$\bar{\nabla} h = 0$
Semi-symmetric	$R \cdot R = 0$	Semi-parallel	$\bar{R} \cdot h = 0$
Pseudo-symmetric	$R \cdot R = LQ(g, R)$	Pseudo-parallel	$\bar{R} \cdot h = LQ(g, h)$

Let (M^n, g) and $(\widetilde{M}^{n+m}, \tilde{g})$ be two Riemannian manifolds with dimension n and $n + m$, and with respective Levi-Civita connections ∇ and $\widetilde{\nabla}$. Assume that $(M,\, g)$ is isometrically immersed in $(\widetilde{M}, \tilde{g})$.We can decompose the covariant derivative $\widetilde{\nabla}$ of two tangent vector fields X and Y to M, i.e. $X, Y \in \mathfrak{X}(M)$, into its tangential and normal part as follows,

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{4.11.1}$$

where $h(X,\, Y)$ is normal to M and h is called the second fundamental form.

Equation (4.11.1) is known as the Gauss formula. A submanifold is called totally geodesic if $h = 0$.

We can further define the normal connection $\nabla^{\perp}$ through the decomposition of the tangent vector field $X \in \mathfrak{X}(M)$ into its tangential and normal parts,

$$\widetilde{\nabla}_X \xi = -A_\xi X + \nabla^{\perp}_X \xi, \tag{4.11.2}$$

where A_ξ is called the shape operator with respect to ξ. The shape operator is related to the second fundamental form by $g(A_\xi X, Y) = \tilde{g}(h(X, Y), \xi)$.

Equation (4.10.2) is known as the Weingarten formula. A point p of M is called umbilic if $h(u,v) = g(u,v)H$, for all $u, v \in T_pM$. H is called the mean curvature vector field, or the Bompiani vector field, of the submanifold M in $\widetilde{M}$.

The Bortolotti-Van Waerden connection $\overline{\nabla}$ of M in $\widetilde{M}$ acting on h gives,

$$(\overline{\nabla}h)(X,Y,Z) := \nabla_X^{\perp}\big(h(Y,Z)\big) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z),$$

with $X, Y, Z \in \mathfrak{X}(M)$.

4.11.1. A Geometrical Interpretation of Semi-Parallel Submanifolds

We will give some geometrical interpretations of totally geodesic and parallel submanifolds in terms of parallel transport. Further, we will present a new geometrical interpretation of semi-parallel submanifolds. Every semi-parallel submanifold satisfies condition $R^{\perp}(X,Y)H = 0$. However, the converse is not true in general. We call spaces which satisfy this condition H-semi-parallel because for these spaces the mean curvature vector H is invariant under parallel transport around infinitesimal coordinate parallelograms.

Let $\gamma: I \subseteq \mathbb{R} \to M$ be a curve in a Riemannian manifold M. Choose points $p = \gamma(t_0)$ and $p^* = \gamma(t_0^*)$ on the curve and a vector $v \in T_pM$. Let V be the unique vector field along γ such that

$$V(t_0) = v, \quad \nabla_{\gamma'} V = 0,$$

where ∇ is the Levi-Civita connection of $(M,\ g)$. Then we call $v^* = V(t_0^*)$ the parallel transport of v from p to p^* along the curve γ with respect to the connection ∇.

If M is immersed in another Riemannian manifold $\widetilde{M}$, we can also transport the vector v parallel along γ in $\widetilde{M}$ with respect to the Levi-Civita connection $\widetilde{\nabla}$ of $\big(\widetilde{M}, \tilde{g}\big)$.

The following result can be proven straightforwardly.

4.11.1.1. Proposition. A submanifold M in $\widetilde{M}$ is totally geodesic if and only if the parallel transports of tangent vectors to M with respect to the connections ∇ on M and $\widetilde{\nabla}$ on $\widetilde{M}$ are the same.

Given a curve γ in M and two vectors $u, v \in T_p M$, with $\gamma(t_0) = p$, we have the vector $h(u, v)$ in the normal space of M at the point $p, T_p^{\perp} M$. At the point $\gamma(t_0^*) = p^*$, we can consider two normal vectors. First, the parallel translate of $h(u, v)$ by $\nabla^{\perp}$, which we denote by $h(u, v)^{*\perp}$, and secondly, the vector $h(u^*, v^*)$ obtained after first parallelly translating u and v by ∇, and then applying h.

Chapter (5)

Chen's Inequality and Some Invariants

We study submanifolds satisfying Chen's equality in an Euclidean space and the conditions $R \cdot P = 0$ and $P \cdot P = 0$ in an Euclidean space. Also we study pseudo symmetry type hypersurface in the Euclidean space $\mathbb{E}^{n+1}$ satisfying B. Y. Chen's equality.

5.1. Introduction

One of the basic problems in submanifold theory is to find simple relationships between the extrinsic and intrinsic invariants of a submanifold. B. Y. Chen established inequalities in this respect, called Chen inequalities. The main extrinsic invariants include the classical curvature invariants namely the scalar curvature and the Ricci curvature; and the well Known modern curvature invariant namely Chen invariant.

In 1993 Chen obtained an interesting basic inequality for submanifolds in a real space form involving the squared mean curvature and the Chen invariant and found several of its applications. This inequality is now well known as Chen's inequality; and in the equality case it is known as Chen's equality.

Now we give the following general Lemmas for later use.

5.1.1. Lemma [Chen 1993]. Let $a_1, \dots, a_n, c$ be $n + 1 (n \geq 2)$ real numbers such that

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + c\right) \qquad (5.1.1)$$

Then $2a_1a_2 \geq c$, with equality holding if and only if $a_1 + a_2 = a_3 = \cdots = a_n$.

Proof. If $n = 2$ there is nothing to prove. So we may assume $n > 2$. Because (5.1.1) can be written as

$$(n-2)a_n^2 - 2\left(\sum_{i=1}^{n-1} a_i\right)a_n + (n-1)\left(c + \sum_{i=1}^{n-1} a_i^2\right) - \left(\sum_{i=1}^{n-1} a_i\right)^2 = 0 \qquad (5.1.2)$$

and a_n is real number, we have

$$\left(\sum_{i=1}^{n-1} a_i\right)^2 = (n-2)\left(\sum_{i=1}^{n-1} a_i^2 + c + e_{n-1}\right), \qquad (5.1.3)$$

for some real number $e_{n-1} \geq 0$.

If $n = 3$, this implies $2a_1a_2 = c + e_2 \geq c$. From (5.1.1) with $n = 3$, it follows that $2a_1a_2 = c$ if and only if $a_3 = a_1 + a_2$.

If $n > 3$, by continuing the same process $(n-2)$ times, we obtain

$$\left(\sum_{i=1}^{k} a_i\right)^2 = (k-1)\left(\sum_{i=1}^{k} a_i^2 + c + e_{n-1} + \cdots + e_k\right), k = 2, \dots, n-1, \tag{5.1.4}$$

for some non-negative numbers $e_2, \dots, e_{n-1}$. In particular, we obtain

$$2a_1a_2 = c + e_{n-1} + \cdots + e_2 \geq c.$$

If $2a_1a_2 = c$, then $e_{n-1} = \cdots = e_2 = 0$. Hence (5.1.4) yields

$$\left(\sum_{i=1}^{k} a_i\right)^2 = (k-1)\left(\sum_{i=1}^{k} a_i^2 + c\right), \quad k = 2, \dots, n. \tag{5.1.4–k}$$

From (5.1.4-k), we may obtain $(k-2)a_k = a_1 + \cdots + a_{k-1}$. Therefore, $a_3 = \cdots = a_n = a_1 + a_2$. ■

Let M be an immersed n-dimensional submanifold of an m-dimensional Riemannian manifold $R^m(c)$ of constant sectional curvature c. Denote by h the second fundamental form of the immersion. Then the mean curvature vector H of the immersion is given by $H = \frac{1}{n}\, trace\ h$.

We choose a local field of orthonormal frames $e_1, \dots, e_n, e_{n+1}, \dots, e_m$ in $R^m(c)$

such that, restricted to M, the vectors $e_1, \dots, e_n$ are tangent to M and hence $e_{n+1}, \dots, e_m$ are normal to M.

Let $\{h_{ij}^r\}$, $i, j = 1, \dots, n; r = n+1, \dots, m$, be the coefficients of the second fundamental form h with respect to $\mathrm{e}_1, \dots, \mathrm{e}_\mathrm{n}, \mathrm{e}_{\mathrm{n}+1}, \dots, \mathrm{e}_\mathrm{m}$. Then we have

$$h_{ij}^r = \langle h(e_i, e_j), e_r \rangle = \langle A_{e_r} e_i, e_j \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product.

Denote by R the Riemannian curvature tensor of M. Then the equation of Gauss is given by

$$R(X,Y;Z,W) = c(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle) + \langle h(X,W), h(Y,Z)\rangle - \langle h(X,Z), h(Y,W)\rangle \qquad (5.1.5)$$

Denote by $K(\pi)$ the sectional curvature of M of the plane section $\pi \subset T_pM$, $p \in M$. For any orthonormal basis of tangent space T_pM, the scalar curvature τ at p is defined by

$$\tau = \sum_{i,j=1}^{n} K(e_i \wedge e_j), \qquad (5.1.6)$$

where $e_1, \dots, e_n$ are orthonormal vectors tangent to M.

5.1.2. Lemma [Chen 1993]. Let M be an n-dimensional $(n \geq 2)$ submanifold of a Rie-mannian manifold $R^m(c)$ of constant sectional curvature c. Then

$$\inf K \geq \frac{1}{2}\left\{\tau - \frac{n^2(n-2)}{n-1}|H^2| - (n+1)(n-2)c\right\}. \qquad (5.1.7)$$

Equality holds if and only if, with respect to suitable orthonormal frame fields e_1, ..., e_n, e_{n+1}, ..., e_m, the shape operators $A_r = A_{e_r}$, $r = n+1, \dots, m$, of M in $R^m(c)$ take the following forms:

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & \cdots & 0 \\ 0 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \mu \end{pmatrix}, a + b = \mu, \qquad (5.1.8)$$

$$A_r = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 & \cdots & 0 \\ h_{12}^r & -h_{11}^r & 0 & \dots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \mu \end{pmatrix}, r = n+2, \dots, m. \qquad (5.1.9)$$

Proof. From equation (5.1.5) of Gauss we have

$$\tau = n^2|H|^2 - |h|^2 + n(n-1)c. \qquad (5.1.10)$$

Let $\delta = \tau - \frac{n^2(n-2)}{n-1}|H|^2 - (n+1)(n-2)c. \qquad (5.1.11)$

Then (5.1.10) and (5.1.11) yield

$$n^2|H|^2 = (n-1)|h|^2 + (n-1)(\delta - 2c). \tag{5.1.12}$$

Let $\pi \subset T_p M$ be a plane section. If we choose an orthonormal frame $e_1, \dots, e_m$ such that $\pi = \text{span}\{e_1, e_2\}$ and e_{n+1} is in the direction of the mean curvature vector H, then (5.1.12) gives

$$\left(\sum_{i=1}^n h_{ii}^{n+1}\right)^2 =$$
$$(n-1)\left\{\sum_{i=1}^n \left(h_{ii}^{n+1}\right)^2 + \sum_{i\neq j}\left(h_{ij}^{n+1}\right)^2 + \sum_{r=n+2}^m \sum_{i,j=1}^n \left(h_{ij}^r\right)^2 + \delta - 2c\right\}. \tag{5.1.13}$$

By applying Lemma 1.5.1 we obtain

$$2\, h_{11}^{n+1}\, h_{22}^{n+1} \geq \sum_{i\neq j}\left(h_{ij}^{n+1}\right)^2 + \sum_{r=n+2}^m \sum_{i,j=1}^n \left(h_{ij}^r\right)^2 + \delta - 2c. \tag{5.1.14}$$

From this we get

$$K(\pi) \geq \sum_{r=n+1}^m \sum_{j>2} \left\{\left(h_{1j}^r\right)^2 + \left(h_{2j}^r\right)^2\right\} + \frac{1}{2}\sum_{i\neq j>2}\left(h_{ij}^{n+1}\right)^2 + \frac{1}{2}\sum_{r=n+2}^m \sum_{i,j>2}\left(h_{ij}^r\right)^2$$
$$+ \frac{1}{2}\sum_{r=n+2}^m (h_{11}^r + h_{22}^r)^2 + \frac{\delta}{2} \geq \frac{\delta}{2}. \tag{5.1.15}$$

Combining (5.1.11) and (5.1.15), we get (5.1.7).

If the equality signs of (5.1.7) holds, then the inequalities in (5.1.14) and (5.1.15) become equalities. Thus, we have

$$h_{1j}^{n+1} = 0, j > 2; \quad h_{ij}^{n+1} = 0, i \neq j > 2;$$

$$h_{1j}^r = h_{2j}^r = h_{ij}^r = 0, r = n+2, \dots, m;\ i, j = 3, \dots, n;$$

$$h_{11}^{n+2} + h_{22}^{n+2} = \cdots = h_{11}^m + h_{22}^m = 0.$$

Furthermore, we may choose e_1, e_2 such that $h_{12}^{n+1} = 0$. Moreover, by applying Lemma (5.1.1), we also have

$$h_{11}^{n+1} + h_{22}^{n+1} = h_{33}^{n+1} = \cdots = h_{nn}^{n+1}.$$

Therefore, with respect to a suitable frame field, the shape operators of M take the forms (5.1.8) and (5.1.9).

The converse of this can be verified by straight forward computation. ■

Let $(M,g), n \geq 3$, be a connected Riemannian manifold of class C^∞.We denote by ∇, R, C, S and τ the Levi-Civita connection, the Riemann-Christoffel curvature tensor, the Weyl conformal curvature tensor, the Ricci tensor and scalar curvature of (M, g), respectively. The Ricci operator S is defined by

$$g(SX,Y) = S(X,Y),$$

where $X, Y \in \chi(M), \chi(M)$ being Lie algebra of vector fields on M. We next define endomorphism $X \wedge Y$, $R(X,Y)$ and $C(X,Y)Z$ of $\chi(M)$ by

$$(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y \,, \tag{5.1.16}$$

$$R(X,Y,Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z, \tag{5.1.17}$$

$$C(X,Y)Z = R(X,Y,Z) - \frac{1}{n-2}\left(X \wedge SY + SX \wedge Y - \frac{\tau}{n-1} X \wedge Y\right)Z, \tag{5.1.18}$$

The Riemann Christoffel curvature tensor R and the Weyl curvature tensor C of (M, g) are defined by

$$R(X,Y;Z,W) = g(R(X,Y)Z,W), \tag{5.1.19}$$

$$C(X,Y,Z,W) = g(C(X,Y)Z,W), \tag{5.1.20}$$

respectively, where $W \in \chi(M)$.

For a $(0, k)$-tensor field $T, k \geq 0$, on (M, g) we define the tensors $R \cdot T$ and $Q(g,T)$ by

$$(R(X,Y) \cdot T)(X_1, \dots, X_k) = -T(R(X,Y)X_1, X_2, \dots, X_k) \; - \cdots -$$

$$T(X_1, \dots, X_{k-1}, R(X,Y)X_k), \tag{5.1.21}$$

$$Q(g,T)(X_1, \dots, X_k; X, Y) = (X \wedge Y)T(X_1, \dots, X_k) - T\big((X \wedge Y)X_1, X_2, \dots, X_k\big)$$

$$- \cdots - T(X_1, \dots, X_{k-1}, (X \wedge Y)X_k), \tag{5.1.22}$$

respectively.

If the tensors $R \cdot R$ and $Q(g,R)$ are linearly dependent then M is called pseudo-symmetric. This is equivalent to

$$R \cdot R = L_R Q(g,R), \tag{5.1.23}$$

holding on the set $U_R = \{x: Q(g,R) \neq 0\ at\ x\}$, where L_R is some function on U_R. If $R \cdot R = 0$ then M is called semi-symmetric.

If the tensors $R \cdot S$ and $Q(g,S)$ are linearly dependent, then M is called Ricci-Pseudo-symmetric. This equivalent to

$$R \cdot S = L_S Q(g,S) \tag{5.1.24}$$

holding on the set $U_S = \left\{x: S \neq \frac{\tau}{n} g\ at\ x\right\}$, where L_S is some function on U_S.

Every pseudo-symmetric manifold is Ricci pseudo-symmetric but the converse statement is not true. If $R \cdot S = 0$, then M is called Ricci-semisymmetric.

If the tensors $R \cdot C$ and $Q(g,C)$ are linearly dependent then M is called Weyl-pseudosymmetric. This is equivalent to

$$R \cdot C = L_C Q(g,C) \tag{5.1.25}$$

holding on the set $U_C = \{x: C \neq 0\ at\ x\}$. Every pseudosymmetric manifold is Weyl pseudosymmetric but the converse statement is not true. If $R \cdot C = 0$ then M is called Weyl-semisymmetric.

The manifold is a manifold with pseudosymmetric Weyl tensor field if and only if

$$C \cdot C = L_C Q(g,C) \tag{5.1.26}$$

holds on the set U_C, where L_C is some function on U_C.The tensor $C \cdot C$ is defined in the same way as the tensor $R \cdot R$.

5.2. Submanifolds Satisfying Chen's Inequality

Let M be an n-dimensional submanifold of an $(n+m)$-dimensional Euclidean space $\mathbb{E}^{n+m}$. The Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y) \text{ and } \widetilde{\nabla}_X \xi = -A_\xi X + \nabla_X^\perp \xi$$

for all $X, Y \in TM$ and $\xi \in T^\perp M$, where $\widetilde{\nabla}$, ∇ and $\nabla^\perp$ are respectively the Riemannian, induced Riemannian and induced normal connections in $\widetilde{M}$, M and the normal bundle $T^\perp M$ of M respectively, and h is the second fundamental form related to the shape operator A by

$$\langle h(X,Y),\xi\rangle = \langle A_\xi X, Y\rangle.$$

The equation of Gauss is given by

$$R(X,Y,Z,W) = \langle h(X,W), h(Y,Z)\rangle - \langle h(X,Z), h(Y,W)\rangle \tag{5.2.1}$$

for all $X,Y,Z,W \in TM$, where R is the curvature tensor of M.

The mean curvature vector H is given by $H = \frac{1}{n} trace(h)$. The submanifold M is totally geodesic in $\mathbb{E}^{m+n}$ if $h = 0$, and minimal if $H = 0$.

Let $\{e_1, \dots, e_n\}$ be an orthonormal tangent frame field on M. For the plane section $e_i \wedge e_j$ of the tangent bundle TM spanned by the vectors e_i and e_j $(i \neq j)$ the scalar curvature of M is defined by $\tau = \sum_{i,j=1}^{n} K(e_i \wedge e_j)$ where K denotes the sectional curvature of M. Consider the real function $\inf K$ on M^n defined for every $x \in M$ by

$$(\inf K)(x) := \inf\{K(\pi) \colon \pi \text{ is a plane in } T_x M^n\}.$$

Note that since the set of plane at a certain point is compact, this infimum is actually a minimum.

5.2.1. Lemma [Chen 1993]. Let $M, n \geq 2$, be any submanifold of $\mathbb{E}^{n+m}$. Then

$$\inf K \geq \frac{1}{2}\left\{\tau - \frac{n^2(n-2)}{n-1}|H|^2\right\}. \tag{5.2.2}$$

Equality holds in (5.2.2) at a point x if and only if with respect to suitable local orthonormal frames $e_1, \dots, e_n \in T_x M^n$, the Weingarten maps A_t with respect to the normal sections $\xi_t = e_{n+t}$, $t = 1, \dots, p$ are given by

$$A_1 = \begin{bmatrix} a & 0 & 0 & 0 \dots 0 \\ 0 & b & 0 & 0 \dots 0 \\ 0 & 0 & \mu & 0 \dots 0 \\ 0 & 0 & 0 & \mu \dots 0 \\ \vdots & \vdots & \vdots & \vdots \ddots \vdots \\ 0 & 0 & 0 & 0 \dots \mu \end{bmatrix}$$

$$A_t = \begin{bmatrix} c_t & d_t & 0 \dots 0 \\ d_t & -c_t & 0 \dots 0 \\ 0 & 0 & 0 \dots 0 \\ \vdots & \vdots & \vdots \ddots \vdots \\ 0 & 0 & 0 \dots 0 \end{bmatrix}, \quad (t > 0), \tag{5.2.3}$$

where $\mu = a + b$ for any such frame, $\inf K(x)$is attained by the plane $e_1 \wedge e_2$.

The relation (5.2.2) is called Chen's inequality.

Remark. For dimension $n = 2$, the Chen's equality (5.2.2) is always true (Trivially satisfied).

We denote shortly $K_{ij} = K(e_i \wedge e_j)$. Let M be an n-dimensional ($n \geq 3$) submanifold of an Euclidean space $\mathbb{E}^{n+m}$ satisfying Chen's equality. Then, from Lemma 5.2.1 we immediately have the following

$$K_{12} = ab - \sum_{r=1}^{m}(c_r^2 + d_r^2), \qquad (5.2.4)$$

$$K_{1j} = a\mu, \qquad (5.2.5)$$

$$K_{2j} = b\mu, \qquad (5.2.6)$$

$$K_{ij} = \mu^2, \qquad (5.2.7)$$

$$S(e_1, e_1) = K_{12} + (n-2)a\mu, \qquad (5.2.8)$$

$$S(e_2, e_2) = K_{12} + (n-2)b\mu, \qquad (5.2.9)$$

$$S(e_i, e_i) = (n-2)\mu^2, \qquad (5.2.10)$$

and

$$S(e_i, e_j) = 0 \text{ if } i \neq j \qquad (5.2.11).$$

where $i, j > 2$. Furthermore, $R(e_i, e_j)e_k = 0$ if i, j and k are mutually different. From now on we assume that M^n is hypersurface in $\mathbb{E}^{n+1}$.

5.2.2. Corollary. Let M be a hypersurface of $\mathbb{E}^{n+1}$, $n \geq 3$, satisfying Chen's equality then

$$K_{12} = ab,\ K_{1j} = a\mu,\ K_{2j} = b\mu,\ K_{ij} = \mu^2, \qquad (5.2.12)$$

where $i, j > 2$. Furthermore, $R(e_i, e_j)e_k = 0$ if i, j and k are mutually different.

5.2.3. Theorem [Dillen, F., Petrovic, M., and Verstraelen, L. 1997]. Let M^n, $n \geq 3$, be a submanifold of $\mathbb{E}^m$ satisfying Chen's equality. Then M^n is semisymmetric if and only if M^n is a minimal submanifold (in which case M^n is

$(n-2)$-ruled) or M^n is a round hypercone in some totally geodesic space $\mathbb{E}^{n+1}$ of $\mathbb{E}^m$.

Now our aim is to give an extension of Theorem (5.2.3) for case M is a pseudo-symmetric hypersurface in the Euclidean space $\mathbb{E}^{n+1}$.

5.2.4. Lemma. Let $M, n \geq 3$, be a hypersurface of $\mathbb{E}^{n+1}$ satisfying Chen's equality. Then

$$(R(e_1, e_3) \cdot R)(e_2, e_3)e_1 = a\mu b^2 e_2, \tag{5.2.13}$$

$$(R(e_2, e_3) \cdot R)(e_1, e_3)e_2 = b\mu a^2 e_1. \tag{5.2.14}$$

Proof. Using (5.1.21) we have

$$\begin{aligned}(R(e_1, e_3) \cdot R)(e_2, e_3)e_1 = R(e_1, e_3)(R(e_2, e_3)e_1) \\ -R(R(e_1, e_3)e_2, e_3)e_1 \\ -R(e_2, R(e_1, e_3)e_3)e_1 \\ -R(e_2, e_3)(R(e_1, e_3)e_1)\end{aligned} \tag{5.2.15}$$

and

$$\begin{aligned}(R(e_2, e_3) \cdot R)(e_1, e_3)e_2 = R(e_2, e_3)(R(e_1, e_3)e_2) \\ -R(R(e_2, e_3)e_1, e_3)e_2 \\ -R(e_1, R(e_2, e_3)e_3)e_2 \\ -R(e_1, e_3)(R(e_2, e_3)e_2).\end{aligned} \tag{5.2.16}$$

Since

$$R(e_i, e_j)e_k = (A_\xi e_i \wedge A_\xi e_j)e_k$$

then using (5.2.12) one can get

$$\begin{aligned} R(e_1, e_3)e_1 = -K_{13}e_1, \qquad R(e_1, e_3)e_3 = K_{13}e_1 \\ R(e_2, e_1)e_1 = K_{12}e_2, \qquad R(e_2, e_1)e_2 = -K_{12}e_1 \\ R(e_2, e_3)e_2 = -K_{23}e_2, \qquad R(e_2, e_3)e_3 = K_{23}e_2. \end{aligned} \tag{5.2.17}$$

Therefore substituting (5.2.17), (5.2.12) into (5.2.15) and (5.2.16) respectively we get the result.

5.2.5. Lemma. Let $M, n \geq 3$, be a hypersurface of $\mathbb{E}^{n+1}$ satisfying Chen's equality. Then

$$Q(g,R)(e_2,e_3,e_1;e_1,e_3) = b^2 e_2 \tag{5.2.18}$$

$$Q(g,R)(e_1,e_3,e_2;e_2,e_3) = a^2 e_1 \tag{5.2.19}$$

Proof. Using the relation (5.1.22) we obtain

$$Q(g,R)(e_2,e_3,e_1;e_1,e_3) = (e_1 \wedge e_3)R(e_2,e_3)e_1 - R\big((e_1 \wedge e_3)e_2,e_3\big)e_1$$
$$-R(e_2,(e_1 \wedge e_3)e_3) - R(e_2,e_3)\big((e_1 \wedge e_3)e_1\big). \tag{5.2.20}$$

and

$$Q(g,R)(e_1,e_3,e_2;e_2,e_3) = (e_2 \wedge e_3)R(e_1,e_3)e_2 - R\big((e_2 \wedge e_3)e_1,e_3\big)e_2$$
$$-R(e_1,(e_2 \wedge e_3)e_3)e_2 - R(e_1,e_3)\big((e_2 \wedge e_3)e_2\big). \tag{5.2.21}$$

So substituting respectively (5.2.17) and (5.2.12) into (5.2.20) and (5.2.21) we obtain (5.2.18)-(5.2.19). ■

5.2.6. Theorem [Özgür, C. and Arslan, K., 2002]. Let $M, n \geq 3$, be a hyersurface of $\mathbb{E}^{n+1}$ satisfying Chen's equality. Then M is pseudosymmetric if and only if

(i) $M = \mathbb{E}^n$, or

(ii) M is a round hypercone in $\mathbb{E}^{n+1}$, or

(iii) M is a minimal hypersurface in $\mathbb{E}^{n+1}$(in which case M is $(n-2)$-ruled), or

(iv) The shape operator of M in $\mathbb{E}^{n+1}$ is of the form

$$A_\xi = \begin{bmatrix} a & 0 & 0 & 0 \cdots 0 \\ 0 & a & 0 & 0 \cdots 0 \\ 0 & 0 & 2a & 0 \cdots 0 \\ 0 & 0 & 0 & 2a \cdots 0 \\ \vdots & \vdots & \vdots & \vdots \ddots \vdots \\ 0 & 0 & 0 & \cdots\ 2a \end{bmatrix} \tag{5.2.22}$$

Proof. Let M be a pseudosymmetric hypersurface in $\mathbb{E}^{n+1}$.Then by definition one can write

$$\big(R(e_1,e_3)\cdot R(e_2,e_3)\big)e_1 = L_R Q(g,R)(e_2,e_3,e_1;e_1,e_3) \tag{5.2.23}$$

and

$$(R(e_2,e_3)\cdot R)(e_1,e_3)e_2 = L_R Q(g,R)(e_1,e_3,e_2;e_2,e_3). \tag{5.2.24}$$

Since M satisfies B.Y. Chen equality then lemma 5.2.4 and lemma 5.2.5 the equations (5.2.23) and (5.2.24) turns, respectively, into

$$(a\mu - L_R)b^2 = 0 \tag{5.2.25}$$

and

$$(b\mu - L_R)a^2 = 0 \tag{5.2.26}$$

i) Firstly, suppose that M is semisymmetric, i.e., M is trivially pseudosymmetric

then $L_R = 0$.

So the equations (5.2.25) and (5.2.26) can be written as the following

$$ab\mu = 0.$$

Now suppose $a = 0$, $b \neq 0$ then $\mu = b$ and M is a round hypercone in $\mathbb{E}^{n+1}$.

If $a \neq 0$, $b = 0$ then $\mu = a$ and similarly M is a round hypercone in $\mathbb{E}^{n+1}$.

If $\mu = 0$, then M is minimal. If $a = 0$, $b = 0$ then $\mu = 0$ so $M = \mathbb{E}^n$.

ii) Secondly, suppose M is not semisymmetric, i.e., $R \cdot R \neq 0$. For the subcases $a = b = 0$, $a = 0$, $b \neq 0$ or $a \neq 0$, $b = 0$ we get $R \cdot R = 0$ which contradicts the facts that $R \cdot R \neq 0$. Therefore the only remaining possible subcase is a $\neq 0$, $b \neq 0$. So by the use of (5.2.25) and (5.2.26) we have $(a-b)\mu = 0$. Since $\mu = a + b \neq 0$ then $a = b$ and by Lemma 5.2.1 the shape operator of M is of the forms (5.2.22). This completes the proof of the theorem.

5.3. Projective Curvature Tensor of Submanifolds Satisfying Chen's Equality.

We consider projectively semi-symmetric submanifolds satisfying Chen's equality in an Euclidean space. We also consider submanifolds satisfying the condition $P \cdot P = 0$.

The projective curvature tensor P of an n-dimensional Riemannian manifold (M, g) is defined by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y]. \tag{5.3.1}$$

It is well-known that if the condition $R \cdot P = 0$ holds on M, then M is said to be projectively semi-symmetric.

So from (5.2.4)-(5.2.10) we have the following corollary:

5.3.1. Corollary. Let M be an n-dimensional $(n \geq 3)$ submanifold in an Euclidean space satisfying Chen's equality, then

$$P_{122} = \frac{n-2}{n-1}(K_{12} - b\mu)e_1, \tag{5.3.2}$$

$$P_{133} = \mu\left(a - \frac{n-2}{n-1}\mu\right)e_1, \tag{5.3.3}$$

$$P_{131} = \frac{1}{n-1}(K_{12} - a\mu)e_3, \tag{5.3.4}$$

$$P_{233} = \mu\left(b - \frac{n-2}{n-1}\mu\right)e_2, \tag{5.3.5}$$

$$P_{211} = \frac{n-2}{n-1}(K_{12} - a\mu)e_2 \tag{5.2.6}$$

$$P_{232} = \frac{1}{n-1}(K_{12} - b\mu)e_3, \tag{5.3.7}$$

and

$$P_{ijk} = 0 \text{ if } i, j, k \text{ are mutually different.} \tag{5.3.8}$$

5.3.2. Theorem. Let M be an n-dimensional $(n \geq 3)$ submanifold of an Euclidean space $\mathbb{E}^{n+m}$ satisfying Chen's equality. If M is projectively semi-symmetric then

(i) M is totally geodesic, or

(ii) M is minimal, or

(iii) M is a round hypercone in some totally geodesic subspace $\mathbb{E}^{n+1}$ of $\mathbb{E}^{n+m}$, or

(iv) $\inf K = 0$, or

(v) $a = b$, in this case if $n = 3$ then M is totally geodesic, if $n = 4$ then M is a pseudosymmetric hypersurface of $\mathbb{E}^5$ which has a shape operator of the form

$$A_1 = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 2a & 0 \\ 0 & 0 & 0 & 2a \end{bmatrix}, \tag{5.3.9}$$

or

(vi) M is a submanifold in some totally geodesic subspace $\mathbb{E}^{n+m-1}$ which has shape operators of the form (5.2.3).

Proof. Assume that the condition $R \cdot P = 0$ holds on M. Then, we can write

$$\begin{aligned}(R(e_1, e_3).P)(e_2, e_3, e_1) &= R(e_1, e_3)P(e_2, e_3)e_1 \\ &\quad -P(R(e_1, e_3)e_2, e_3)e_1 - P(e_2, R(e_1, e_3)e_3)e_1 \\ &\quad -P(e_2, e_3)R(e_1, e_3)e_1 = 0 \end{aligned} \tag{5.3.10}$$

and

$$\begin{aligned}(R(e_2, e_3).P)(e_1, e_3, e_2) &= R(e_2, e_3)P(e_1, e_3)e_2 \\ &\quad -P(R(e_2, e_3)e_1, e_3)e_2 - P(e_1, R(e_2, e_3)e_3)e_2 \\ &\quad -P(e_1, e_3)R(e_2, e_3)e_2 = 0. \end{aligned} \tag{5.3.11}$$

Then, using (5.2.4)-(5.2.7) and (5.3.2)-(5.3.8), we get

$$a\mu[b\mu - (n-2)ab + (n-2)\sum_{r=1}^{m}(c_r^2 + d_r^2)] = 0 \tag{5.3.12}$$

and

$$b\mu[a\mu - (n-2)ab + (n-2)\sum_{r=1}^{m}(c_r^2 + d_r^2)] = 0 \tag{5.3.13}$$

Case I. If M is totally geodesic, the condition $R \cdot P = 0$ holds trivially.

Case II. If $\mu = 0$, then M is minimal.

Case III. If $\mu \neq 0$ and $a = 0$ then $\mu = b$.Hence, from (5.3.13), we get

$(n-2)\sum_{r=1}^{m}(c_r^2 + d_r^2) = 0$. This gives us $c_r = d_r = 0$. So, M is around hyper-cone in some totally geodesic subspace $\mathbb{E}^{\mathrm{n+1}}$ of $\mathbb{E}^{\mathrm{n+m}}$.

Case IV. If $\mu \neq 0$ and $b = 0$, then we obtain again the same result in case III.
Case V. a, b, $\mu \neq 0$, then from (5.3.12) and (5.3.13) we obtain $a = b$, or $\mu = 0$, or $K_{12} = 0$.

If $\mu = 0$, then M is minimal. If $K_{12} = 0$, then $\inf K = 0$. Assume that $a = b$. Then, from (5.3.13) we have

$$(4-n)a^2 + (n-2)\sum_{r=1}^{m}(c_r^2 + d_r^2) = 0.$$

In this case, if $n = 3$, then $c_r = d_r = 0$. Hence, M is totally geodesic. If $n = 4$, then $c_r = d_r = 0$, so by theorem (5.2.6), M is a pseudosymmetric hypersurface in some totally geodesic subspace $\mathbb{E}^{\mathrm{n+1}}$ of $\mathbb{E}^{\mathrm{n+m}}$, which has a shape operator of the form (5.3.9).

Case VI. If $a = b = 0$, then M is a submanifold in some totally geodesic sub-space $\mathbb{E}^{n+m-1}$, which has a shape operators of the form (5.3.9). This completes the proof of the theorem. ■

5.4. Chen Invariant or δ-Invariant.

5.4.1. Definition of Chen Invariant

Let M be a Riemannian manifold of dimension m and let $\{e_1, e_2, \dots, e_m\}$ be an orthonormal basis of the tangent space T_pM at any point $p \in M$. Then the scalar curvature τ at $p \in M$ is given by

$$\tau = \sum_{1 \le i < j \le m} K\left(e_i \wedge e_j\right).$$

For any point $p \in M$, we denote

$$(\inf K)(p) = \inf\left\{K(\pi) : \pi \subset T_pM, \dim \pi = 2\right\}.$$

where $K(\pi)$ denotes the sectional curvature of M associated with a plane section $\pi \subset T_p M$ at $p \in M$.

The Chen first invariant δ_M at any point $p \in M$ is defined as

$$\delta_M(p) = \tau(p) - (\inf K)(p).$$

Let L be a subspace of $T_p M$ of dimension $r \geq 2$ and let $\{e_1, \dots, e_m\}$ an orthonormal basis of L. The scalar curvature $\tau(L)$ of the r-plane section L is defined by

$$\tau(L) = \sum_{1 \leq \alpha < \beta \leq r} K(e_\alpha \wedge e_\beta).$$

Given an orthonormal basis $\{e_1, \dots, e_m\}$ of the tangent space $T_p M$, we simply denote by $\tau_{1\dots r}$ the scalar curvature of the r-plane section spanned by $e_1, \dots, e_r$. The scalar curvature $\tau(p)$ of M at p is nothing but the scalar curvature of the tangent space of M at p, and if L is a 2-plane section, $\tau(L)$ is nothing but the sectional curvature $K(L)$ of L.

Geometrically, $\tau(L)$ is nothing but the scalar curvature of the image $\exp_p(L)$ of L at p under the exponential map at p.

For an integer $k \geq 0$ denote by $S(n,k)$ the finite set consisting of unordered k-tuples $(n_1, \dots, n_k)$ of integer ≥ 2 satisfying $n_1 < n$ and $n_1 + \cdots + n_k \leq n$. Denote by $S(n)$ the set of unordered k-tuples with $k \geq 0$ for a fixed n. For each k-tuple $(n_1, \dots, n_k) \in S(n)$ the Riemannian invariants $\delta(n_1, \dots, n_k)(p)$ is defined to be

$$\delta(n_1, \dots, n_k)(p) = \tau(p) - \inf\{\tau(L_1) + \cdots + \tau(L_k)\}, \qquad (5.4.1.)$$

where $L_1, \dots, L_k$ run over all k mutually orthogonal subspaces of $T_p M$ such that $\dim L_j = n_j, j = 1, \dots, k$.We note that the Chen invariant with $k = 0$ is nothing but the scalar curvature τ.

Similarly, we have also defined $\hat{\delta}(n_1, \dots, n_k)(p)$ by

$$\hat{\delta}(n_1, \dots, n_k)(p) = \tau(p) - \sup\{\tau(L_1) + \cdots + \tau(L_k)\}, \qquad (5.4.2)$$

where $L_1, \dots, L_k$ run over all k mutually orthogonal subspaces of $T_p M$ such that $\dim L_j = n_j, j = 1, \dots, k$.

Obviously, one has

$$\delta(n_1, \dots, n_k) \geq \hat{\delta}(n_1, \dots, n_k), \tag{5.4.3}$$

for any k-tuple $(n_1, n_2, \dots, n_k) \in S(n)$.

For simplicity, a Riemannian manifold M is called an $S(n_1, \dots, n_k)$-space if it satisfies $\delta(n_1, \dots, n_k) = \hat{\delta}(n_1, \dots, n_k)$ identically. It follows from (5.4.1) and (5.4.2) that a Riemannian n-manifold is an $S(n_1, \dots, n_k)$-space if and only if $\tau(L_1) + \cdots + \tau(L_k)$ is independent of the choice of k mutually orthogonal subspace $L_1, \dots, L_k$ which satisfy $\dim L_j = n_j$, $j = 1, \dots, k$.

Let $\{e_1, \dots, e_n\}$ be any orthonormal basis of T_pM. Then we denote the scalar curvature of the j-space spanned by $e_{i_1}, \dots, e_{i_j}$ by $\tau_{i_1 \dots i_j}$.

Let $\#S(n)$ denote the cardinal number of $S(n)$. Then $\#S(n)$ increases quite rapidly with n. For instance, for

$n = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \dots, 20, \dots, 50, \dots, 100, \dots, 200, \dots,$

$\#S\ (n)$ are given respectively by

1, 2, 4, 6, 10, 14, 21, 29, 41, 54, 76, …, 626,…, 204225,…, 190569291,…,3972999029387,….

In general, the cardinal number $\#S\ (n)$ is equal to $p(n) - 1$, where $p\ (n)$ denotes the partition function.

The asymptotic behavior of $\#S(n)$ is given by

$$\#S(n) \approx \frac{1}{4n\sqrt{3}} \exp\left[\pi\sqrt{\frac{2n}{3}}\right] \ as\ n \to \infty.$$

For a submanifold M of a real space form $\bar{M}(c)$, Chen has given a basic inequality in terms of the intrinsic invariant δ_M and the squared mean curvature of the immersion as

$$\delta_M \leq \frac{m^2(m-2)}{2(m-1)} \|H\|^2 + \frac{1}{2}(m+1)(m-2)c. \tag{5.4.4}$$

The inequality (5.4.4) also holds well in case M is an anti-invariant submanifold of complex space form $\bar{M}(c)$.

Note:

$$\delta(n_1, \dots, n_k)(p) = \tau(p) - \inf\{\tau(L_1) + \dots + \tau(L_k)\}$$

where $L_1, \dots, L_k$ run over all k mutually orthogonal subspaces of $T_p M$ with $\dim L_j = n_j, \quad j = 1, \dots, k$. In particular we have:

$\delta(\emptyset) = \tau \ (k = 0, \text{the trivial } \delta\text{– invariant}),$

$\delta(2) = \tau - \inf K$, where K is the sectional curvature,

$\delta(n-1)(p) = \max \mathrm{Ric}(p).$

The non-trivial $(k > 0)$ δ-invariants are very different in nature from the "classical" scalar and Ricci curvatures; simply due to the fact that both scalar and Ricci curvatures are the "total sum" of sectional curvatures on a Riemannian manifold.

5.4.2. Relations Between δ-Invariants and Einstein Conformally Flat Manifolds.

A manifold $(M,\ g)$ is called conformally flat, if locally we can write $g = h \cdot g_0$ where $g_0 =$ Euclidean metric, and h, a positive real valued function on M.

A well-known theorem of Weyl is that: if $\dim M \geq 4$, then $(M,\ g)$ is conformally flat if and only if $\mathrm{C} = 0$. Where

$$\mathrm{C} = \mathrm{R} - \frac{1}{\mathrm{n} - 2}\mathcal{R}ic + \frac{sc}{(n-1)(n-2)}\,\mathrm{I}$$

(Here, $n = \dim M$, and R is Riemannian curvature structure defined by

$$R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$$

and I is (1, 3) tensor field given by

$$\mathrm{I}(X,Y)Z = \langle X,Z\rangle Y - \langle Y,Z\rangle X$$

and $\mathcal{R}ic$ is Ricci curvature structure defined by

$$\mathcal{R}ic(X,Y)Z = \{Ric(X,Z)Y - Ric(Y,Z)X + \langle X,Z\rangle Ric_0 Y - \langle Y,Z\rangle Ric_0 X\}$$

where Ric_0 the corresponding linear transformation defined by

$$\langle Ric_0 X, Y\rangle = Ric(X, Y)$$

and $sc = scalar\ curvature = trace\ Ric_0$).

By using the notion of δ-invariant, we have the following simple characterization of Einstein spaces which generalizes the well-known characterization of Einstein 4-manifolds given by I. M. Senger and J. A. Thorpe.

5.4.2.1. Theorem. Let M be a Riemannian $2r$-manifold. Then M is an Einstein space if and only if we have

$$\tau(L) = \tau(L^{\perp}) \tag{5.4.5}$$

for any r-plane section $L \subset T_p M, p \in M$, where $L^{\perp}$ denotes the orthogonal of L in $T_p M$.

Proof. Let L be an arbitrary r-plane section at p. Choose an orthonormal basis $\{e_1, \dots, e_{2r}\}$ at p such that L is spanned by $e_1, \dots, e_r$.

If M is an Einstein space, then the Ricci curvatures of M satisfy

$$Ric(e_1) + \cdots + Ric(e_r) = Ric(e_{r+1}) + \cdots + Ric(e_{2r}) \tag{5.4.6}$$

equation (5.4.6) yields $\tau(L) = \tau(L^{\perp})$.

Conversely, suppose that we have $\tau(L) = \tau(L^{\perp})$ for any r-plane section

$L \subset T_p M$. Then we have

$$\tau_{12\dots r} - \tau_{2\dots r+1} = \tau_{r+1\dots 2r} - \tau_{1r+2\dots 2r}. \tag{5.4.7}$$

Equation (5.4.7) implies

$$K_{12} + \cdots + K_{1r} - \left(K_{(r+1)2} + \cdots + K_{(r+1)r}\right) =$$

$$K_{(r+1)(r+2)} + \cdots + K_{(r+1)(2r)} - \left(K_{1(r+2)} + \cdots + K_{1(2r)}\right),$$

which yields $Ric(e_1) = Ric(e_{r+1})$. Since $r > 1$, this implies that M is Einstein.■

We first prove the following generalization of a result of Kulkarni.

5.4.2.2. Theorem. Let M^n be a Riemannian manifold with $n \geq 4$, and let s be any integer satisfying $2 < 2s \leq n$. Then M is conformally flat if and only if for any orthonormal set $\{e_1, \dots, e_{2s}\}$ of vectors one has

$$\tau_{1\dots s} + \tau_{s+1\dots 2s} = \tau_{1\dots s-1\, s+1} + \tau_{s\, s+2\dots 2s} \qquad (5.4.8)$$

Proof. For $s = 2$ this is Kulkarni's result. For completeness, we include a proof of this case.

If M is conformally flat, Weyl's conformal curvature tensor vanishes. Thus

$$K_{ij} = \frac{1}{n-2}\left(Ric(e_i) + Ric\left(e_j\right)\right) - \frac{\tau}{(n-1)(n-2)},\ i \neq j \qquad (5.4.9)$$

from which we conclude that

$$K_{ij} + K_{k\ell} = K_{ik} + K_{j\ell}, \text{ for distinct } i, j, k, \ell \qquad (5.4.10)$$

Conversely, if (5.4.10) holds for distinct i, j, k, ℓ, then by fixing i, j, k in (5.4.10) and summing up over all remaining ℓ, one obtains

$$(n-2)K_{ij} + Ric(e_k) = (n-2)K_{ik} + Ric\left(e_j\right) \qquad (5.4.11)$$

fixing i, j in (5.4.11) and summing up over all remaining k, we obtain (5.4.9), which implies the vanishing of Weyl's conformal curvature tensor. Therefore, M is conformal flat.

Now we prove the theorem for $s > 2$.First we prove that a conformally flat space satisfies (5.4.8). So suppose that M^n is conformally flat and let s be any integer satisfying $2 < 2s \leq n$.

From Kulkarni's result we know that $K_{is} + K_{ks+1} = K_{is+1} + K_{ks}$ for any $i < s$ and $k > s + 1$.

Therefore we have that

$$\tau_{1\dots s} + \tau_{s+1\dots 2s} = \tau_{1\dots s-1} + \sum_{i=1}^{s-1}(K_{is} + K_{s+1\, s+1+i}) + \tau_{s+2\dots 2s}$$

$$= \tau_{1\dots s-1} + \sum_{i=1}^{s-1}(K_{is+1} + K_{ss+1+i}) + \tau_{s+2\dots 2s}$$

$$= \tau_{1\dots s-1\, s+1} + \tau_{sj+2..2s}.$$

Next we prove (5.4.8) implies conformal flatness. For this we use (5.4.8) twice to obtain

$$0 = (\tau_{1\ldots s} + \tau_{s+1\ldots 2s}) - (\tau_{1\ldots s-1\,s+1} + \tau_{ss+2\ldots 2s})$$
$$-\begin{pmatrix} (\tau_{1\ldots s-2s+2s} + \tau_{s+1\,s-1\,s+3\ldots 2s}) \\ -(\tau_{1\ldots s-2\,s+2\,s+1} + \tau_{ss-1\,s+3\ldots 2s}) \end{pmatrix} \qquad (5.4.12)$$

It is clear that K_{ik} does not occur in (5.4.2.8) unless both i and k belong to the set $\{s-1, s, s+1, s+2\}$. Taking this into consideration, (5.4.12) becomes

$$0 = 2\big((K_{s-1\,s} + K_{s+1\,s+2}) - (K_{s-1\,s+1} + K_{ss+2})\big),$$

and Kulkarni's result implies that M is conformally flat. ■

In general, the δ-invariants $\delta(n_1, \ldots, n_k)$ are independent invariants. However, theorem (5.4.5) implies that, for a $2r$-dimensional manifold, we have the following relations

$$2\delta(r) - \delta(r,r) = 2\hat{\delta}(r) - \hat{\delta}(r,r). \qquad (5.4.13)$$

For any k-tuple $(n_1, \ldots, n_k) \in S(n)$, let us put

$$\Delta(n_1, \ldots, n_k) = \frac{\delta(n_1,\ldots,n_k)}{c(n_1,\ldots,n_k)}, \qquad (5.4.14)$$

Since a Riemannian n-manifold with $n \geq 3$ satisfies inequality

$$\Delta(2) > \Delta(\emptyset) = \tau$$

if and only if $\inf K < {}^{\tau}/_{(n-1)^2}$.

Thus, a Riemannian n-manifold ($n \geq 3$) with vanishing scalar curvature satisfies

$$\Delta_0(2) > \Delta_0(\emptyset)$$

automatically, unless M is flat.

For compact homogeneous Einstein Kähler manifold, we also have the following relationship between the δ-invariants and scalar curvature.

5.4.2.3. Proposition. Let M be a compact homogeneous Einstein Kähler manifold with positive scalar curvature. Then, for each $(n_1, \ldots, n_k) \in S(n)$, we have

$$\Delta(n_1, \dots, n_k) \leq \left(2 - \frac{2}{n}\right)\Delta(\emptyset),$$

where n denotes the real dimension of M.

5.4.2.1. Lemma. For a given integer j with $2 \leq j \leq n-2$, if M is an $S(j)$-space, then it is an $S(j+1)$-space.

Proof. For simplicity, we start with a special case $j = 3$. If M is an $S(3)$-space, then τ_{123} is a number, say c, which is independent of the choice of the 3-plane. In particular, from the definition of scalar curvature of a j-plane, we have

$$\tau_{234} = \tau_{1234} - K_{12} - K_{13} - K_{14} = c, \tag{5.4.15}$$

where K_{ij} denotes the sectional curvature of the 2-plane spanned by e_i, e_j.

On the other hand, since M is an $S(3)$-space, we have

$$K_{12} + K_{13} + K_{23} = c, \tag{5.4.16}$$

$$K_{13} + K_{14} + K_{34} = c, \tag{5.4.17}$$

$$K_{12} + K_{14} + K_{24} = c. \tag{5.4.18}$$

Summing up (5.4.16)–(5.4.18) we obtain

$$\tau_{1234} + K_{12} + K_{13} + K_{14} = 3c \tag{5.4.19}$$

Combining (5.4.15) and (5.4.19), yields $\tau_{1234} = 2c$. Since the orthonormal basis can be chosen arbitrarily, this implies that M is as $S(4)$-space.

In general, if M is an $S(j)$-space ,then

$$\tau_{2\dots j+1} = \tau_{1\dots j+1} - K_{12} - \dots - K_{1j+1} = c \tag{5.4.20}$$

On the other hand, similar to (5.4.19) we also have

$$(j-2)\tau_{1\dots j+1} + K_{12} + \dots + K_{1j+1} = jc \tag{5.4.21}$$

Combining (5.4.20) and (5.4.21) yields $(j-1)\tau_{1\dots j+1} = (j+1)c$.

This implies M is an $S(j+1)$-space. ■

The $S(n_1, \dots, n_k)$-spaces are completely by the following two propositions.

5.4.2.4. Proposition. Let M be a Riemannian n-manifold with $n > 2$.Then

(1) For any integer j with $2 \leq j \leq n-2$, M is an $S(j)$-space if and only if M is a Riemannian space form.

(2) M is an $S(n-1)$-space if and only if M is an Einstein space.

5.4.2.5. Proposition. Let M be a Riemannian n-manifold such that n is not a prime and k an integer ≥ 2.Then

(1) If M is an $S(n_1, \dots, n_k)$-space, then M is a Riemannian space form unless $n_1 = \cdots = n_k$ and $n_1 + \cdots + n_k = n$, and

(2) M is an $S(n_1, \dots n_k)$-space with $n_1 = \cdots = n_k$ and $n_1 + \cdots + n_k = n$ if and only if M is a conformally flat space.

5.5. Fundamental Inequalities Involving δ-Invariants

Let M be an n-dimensional submanifold of a Riemannian m-manifold $\widetilde{M}^m$. We choose a local field of orthonormal frame $e_1, \dots e_n, e_{n+1}, \dots, e_m$ in $\widetilde{M}^m$ such that, restricted to M, the vectors $e_1, \dots, e_n$ are tangent to M and hence $e_{n+1}, \dots, e_m$ are normal to M. Let $K(e_i \wedge e_j)$ and $\widetilde{K}(e_i \wedge e_j)$ denote respectively the sectional curvatures of M and $\widetilde{M}^m$ of the plane section spanned by e_i and e_j. The Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{5.5.1}$$

$$\widetilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{5.5.2}$$

for any vector fields X, Y tangent to M and vector field ξ normal to M, where h denotes the second fundamental form, D the normal connection and A the shape operator of the submanifold.

Let $\{h_{ij}^r\}, i, j = 1, \dots, n; r = n+1, \dots, m$, denote the coefficients of the second fundamental form h with respect to $e_1, \dots, e_n, e_{n+1}, \dots, e_m$. Then we have

$$h_{ij}^r = \langle h(e_i, e_j), e_r \rangle = \langle A_{e_r} e_i, e_j \rangle, \tag{5.5.3}$$

where $\langle\ ,\ \rangle$ denotes the inner product. The mean curvature vector $\vec{H}$ is defined by

$$\vec{H} = \frac{1}{n} trace\ h = \frac{1}{n} \sum_{i=1}^n h(e_i, e_i), \tag{5.5.4}$$

where $\{e_1, \dots, e_n\}$ is a local orthonormal frame of the tangent bundle TM of M.

The squared mean curvature is then given by

$$H^2 = \langle \vec{H}, \vec{H} \rangle. \tag{5.5.5}$$

A submanifold M is called minimal in $\widetilde{M}^m$ if its mean curvature vector vanishes identically.

Denote by R and $\tilde{R}$ the Riemann curvature tensors of M and $\widetilde{M}^m$, respectively. Then the equations of Gauss and Codazzi are given respectively by

$$R(X,Y;Z,W) = \tilde{R}(X,Y;Z,W) + \langle h(X,W), h(Y,Z) \rangle$$
$$-\langle h(X,Z), h(Y,W) \rangle, \tag{5.5.6}$$

$$\left(\tilde{R}(X,Y)Z\right)^{\perp} = (\overline{\nabla}_X h)(Y,Z) - (\overline{\nabla}_Y h)(X,Z), \tag{5.5.7}$$

where X, Y, Z, W are tangent to M and ∇h is defined by

$$(\nabla h)(X,Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{5.5.8}$$

for vectors X, Y, Z, W tangent to M.

A submanifold M is called a parallel submanifold if we have $\overline{\nabla} h = 0$ identically.

For each $(n_1, \dots, n_k) \in S(n)$, let $c(n_1, \dots, n_k)$ and $b(n_1, \dots, n_k)$ denote the constants given by

$$c(n_1, \dots, n_k) = \frac{n^2\left(n+k-1-\sum n_j\right)}{2\left(n+k-\sum n_j\right)} \tag{5.5.9}$$

$$b(n_1, \dots, n_k) = \frac{1}{2}\left(n(n-1)\right) - \sum_{j=1}^{k} n_j\left(n_j - 1\right) \tag{5.5.10}$$

For any isometric from a Riemannain submanifold into another Riemannian manifold, we have the following general optimal inequality.

5.5.1. Theorem (Chen, 2005). Let $\varphi: M \to \widetilde{M}$ be an isometric immersion of a Riemannian n-manifold into a Riemannian m-manifold. Then for each point $p \in M$ and each k-tuple $(n_1, \dots, n_k) \in S(n)$, we have the following inequality:

$$\delta(n_1, \dots, n_k)(p) \leq c(n_1, \dots, n_k) H^2(p) + b(n_1, \dots, n_k) \max \widetilde{K}(p) \tag{5.5.11}$$

where $\max \widetilde{K}(p)$ denotes the maximum of the sectional curvature function of $\widetilde{M}^m$ restricted to 2-plane sections of tangent space T_pM of M at p.

The equality case of inequality (5.5.11) holds at $p \in M$ if and only if the following conditions hold:

(1) there exists an orthonormal basis $e_1, \dots, e_m$ at p, such that the shape operator of M in $\widetilde{M}^m$ at p take the following form:

$$A_{e_r} = \begin{pmatrix} \begin{matrix} A_1^r \dots 0 \\ \vdots \ \ddots \ \vdots \\ 0 \dots A_k^r \end{matrix} & 0 \\ 0 & \mu_r I \end{pmatrix}, r = n+1, \dots, m, \tag{5.5.12}$$

where I is an identity matrix and A_j^r is a symmetric $n_j \times n_j$ submatrix such that

$$trace\ (A_1^r) = \cdots = trace\ (A_k^r) = \mu_r \tag{5.5.13}$$

(2**)** For any k mutual orthogonal subspaces $L_1, \dots, L_k$ of T_pM which satisfy

$$\delta(n_1, \dots, n_k) = \tau - \sum_{j=1}^{k} \tau(L_j)$$

at p we have $\widetilde{K}\left(e_{\alpha_i}, e_{\alpha_j}\right) = \max \widetilde{K}(p)$ for any $\alpha_i \in \Gamma_i, \alpha_j \in \Gamma_j$ with $0 \le i \ne j \le k$, where

$$\Gamma_0 = \{1, \dots, n_1\},$$

$$\dots$$

$$\Gamma_{k-1} = \{n_1 + \cdots + n_{k-1} + 1, \dots, n_1 + \cdots + n_k\},$$

$$\Gamma_k = \{n_1 + \cdots + n_k + 1, \dots, n\}.$$

REFERENCES:

[1] A. Bejancu, Geometry of CR−Submanifolds, D.Reidel Publishing Company, 1986.

[2] A. Bejancu, CR-submanifolds of a Kähler manifold, Proc. Amer. Math. Soc. 69 (1978), 135−142.

[3] A. Kriegl, Analysis on Manifolds, Vienna, 2018.

[4] B. Y. Chen, Geometry of Submanifolds, Marcel Dekker, INC. New York 1973.

[5] B. Y. Chen, Geometry of Submanifolds and Its Applications, Science University of Tokyo, 1981.

[6] B. Y. Chen, Total mean curvature and submanifolds of finite type, World Scientific, 1984.

[7] B. Y. Chen, Total mean curvature and submanifolds of finite type, World Scientic, 2015.

[8] B. Y. Chen, Riemannian Submanifolds, in Handbook of Differential Geometry, North Holland Volume 1 (2000), 187−418.

[9] B. Y. Chen, 'Totally umbilical submanifolds of quaternion space forms', J. Austral. Math. Soc. 26 (1978), 154−162.

[10] B. Y. Chen, 'Totally umbilical submanifolds', Soochow Journal of Math. 5 (1979), 9−37.

[11] B. Y. Chen, 'Totally umbilical submanifolds of Kähler manifolds', Archiv der Mathematik 36 (1981), 83−91.

[12] B. Y. Chen, Totally Umbilical Submanifolds of Cayley Plane, Soochow Journal of Mathematical & Natural Sciences, Vol. 3, 1977.

[13] B. Y. Chen, Submanifolds with parallel mean curvature vector in Riemannian and indefinite space forms, 2000 Mathematics Subject Classification. Primary: 53A05; Secondary 53C40, 53C42.

[14] B. Y. Chen, A general optimal inequality for arbitrary Riemannian submanifolds, Journal of Inequalities in Pure and Applied Mathematics, 2000 Victoria University.

[15] B. Y. Chen, 'On submanifolds of finite type', Soochow J. Math. 9 (1983)

[16] B. Y. Chen, Finite type submanifolds and generalizations, University of Rome, Rome, 1985.

[17] B. Y. Chen, '2–type submanifolds and their applications', Chiness J. Math. 14 (1986), 1−14.

[18] B. Y. Chen, 'Submanifolds of finite type in hyperbolic spaces', Chiness J. Math. 20 (1992), 5−21.

[19] B. Y. Chen, Submanifolds of finite type and applications, Pro. Geometry and Topology Research center (Proc.-3rd International workshop of Topology and Geometry), Teagu, 3 (1993), 1−48.

[20] B. Y. Chen, A report on submanifolds of finite type, Soochow J. Math., 22 (1996), 117−337.

[21] B. Y. Chen, Surfaces of finite type in Euclidean 3-space, Bull. Sco. Math. Belg. Ser. B, 39 (1987), 243−254.

[22] B. Y. Chen, Null 2-type surfaces in Euclidean space, Algebra, Analysis and Geometry, 1988, 1−18 (A special issue dedicated to Profs. K. S. Shih and C. J. Hsu).

[23] B. Y. Chen, Null 2-type surfaces in $\mathbb{E}^3$ are circular cylinders, Kodai Math. J., 11(1988), 295-299.

[24] B. Y. Chen, finite type submanifolds in pseudo-Euclidean spaces and applications, Kodai Math. J., Vol. 8, (1985), 358−374.

[25] B. Y. Chen, on the total curvature of immersed manifolds, IV: Spectrum and total mean curvature, Bull. Inst. Math. Acad. Sinica, Vol. 7, (1979), 301–311.

[26] B. Y. Chen, on the total curvature of immersed manifolds, VI: Submanifolds of finite type and their applications, Bull. Inst. Math. Acad. Sinica, Vol. 11, (1983), 309−328.

[27] B. Y. Chen, Some Open Problems and Conjectures on Submanifolds of Finite Type, Soochow Journal of Mathematics-Volume 17, No. 2, pp. 169−188, 1991.

[28] B. Y. Chen, Some Open Problems and Conjectures on Submanifolds of Finite Type: Recent Development, Tamkang Journal of Mathematics, Vol. 45, N. 1, (2014) 87−108.

[29] B. Y. Chen, CR-Submanifolds and δ-Invariants, Springe Science + Business Media Singapore, 2016.

[30] B. Y. Chen, 'CR−submanifolds of a Kähler manifold I', J. Differential Geometry 16 (1981), 305−323.

[31] B. Y. Chen, 'CR−submanifolds of a Kähler manifold II', J. Differential Geometry 16 (1981), 493−509.

[32] B. Y. Chen, δ−Invariants, Inequalities of Submanifolds and Their Applications, Topics in Diff. Geom., Editura Academiei Române, 2008, pp. 29−155.

[33] B. Y. Chen, Some Pinching and Classification Theorems for Minimal Submanifolds, Birkhäuser Verlag, Basel, Arch. Math., Vol. 60, 568−578 (1993).

[34] B. Y. Chen, A Riemannian Invariant for Submanifolds in Space Forms and Its Applications, In Geometry and topology of Submanifolds, V1,World Scientific, Singapore, 58-81, 1994.

[35] B. Y. Chen, Mean curvature and shape operator of isometric immersions in real space forms, Glasgow Math. J. 38 (1996), 87−97.

[36] B. Y. Chen and K. Ogiue, 'On totally real submanifolds', Trans amer. Math. Soc. 193 (1974), 257−266.

[37] B. Y. Chen, C. S. Houh and H. S. Lue, Totally Real Submanifolds, J. Differential Geometry, 12 (1977) 473−480.

[38] B. Y. Chen and P. Piccinni, Submanifolds with finite type Gauss map, Bull. Austral. Math. Soc., Vol. 35 (1987) 161−186.

[39] B. Y. Chen and T. Nagano, totally geodesic submanifolds of symmetric spaces I, Duke Math. J., Vol. 44, (1977), 745−755.

[40] B. Y. Chen and T. Nagano, totally geodesic submanifolds of symmetric spaces-II, Duke Math. J., Vol. 45, (1978), 405−425.

[41] B. Y. Chen, L. Vanhecke and L. Verstraelen, Totally Geodesic Submanifolds, Simon Stevin, A Quarterly Journal of Pure and Applied Mathematics, Vol. 52 (1978).

[42] B. Y. Chen, J. M. Morvan and T. Nore, Energy, Tension and Finite Type maps, Kodai Math. J., Vol. 9, (1986), 406−418.

[43] B. Y. Chen and K. Yano, minimal submanifolds of a higher dimensional sphere, Tensor, 22 (1971), 369−393.

[44] B. Y. Chen and L. Verstraelen, Differential geometry of Laplace transformations, Geometry and Topology of Submanifolds, IV (1992), 51−73.

[45] B. Y. Chen and L. Verstraelen, Laplace transformation of submanifolds, PADGE, Brussel-Leuven, 1995.

[46] B. Y. Chen and L. Verstraelen, Differential geometry of Laplace transformations of submanifolds, Differential geometry in honor of Radu Rosca, Kath. Univ. Leuven, (1991), 51−72.

[47] B. Y. Chen and H. S. Lue, some 2-type submanifolds and applications, Annales de la faculté des sciences de Toulouse 5^{e} série, tome 9, n^{0} 1(1988), p. 121−131.

[48] B. Y. Chen and M. Petrovic, On Spectral Decomposition of Immersions of Finite Type, Bull. Austral. Math. Soc., Vol. 44, (1991), (117−129).

[49] B. O'Neill, Semi−Riemannian Geometry, Academic Press Inc., New York, 1983.

[50] C. Özgür and U. C. De, On Some Classes of Submanifolds Satisfying Chen's Equality in an Euclidean Space, Italian Journal of pure and Applied Mathematics-N.28-2011 (109−116).

[51] C. Özgür and M. M. Tripathi, 'On submaifolds satisfying Chen's equality in a real space form', The Arabian Journal for Science and Engineering, Vol. 33, N. 2A, 2008.

[52] C. Özgür and K. Arslan, on some class of hypersurfaces in $\mathbb{E}^{n+1}$ satisfying Chen’s equality, Turk J Math, 26 (2002), 283−293.

[53] D. V. Lindt and P. Verheyen, Extrisic Spheres in Positively Curved Kähler Manifolds, Soochow Journal of Mathematics, Vol. 9, 1983.

[54] D. Fetcu, Submanifolds with parallel Mean Curvature and Biharmonic Submanifolds in Riemannian Manifolds, IMAR, Buchararest, 2017.

[55] F. Defever, J. M. Morvan, I. Van de Woestije, L. Verstraelen and G. Zafindratafa, Geometry and topology of submanifolds IX, World Scientific Publishing Co Pte Ltd, 1999.

[56] F. Dillen, J. fastenakels, S. Haesen, J. Van der Veken and L. Verstraelen, Submanifolds Theory and Parallel Transport, Kragujevac Journal of Mathematics, Vol. 37(1) (2013), Pages 33−43.

[57] F. Dillen, M. Petrovic and L. Verstraelen, Einstein, conformally flat and semi-symmetric submanifolds satisfying Chen's equality, Israel J. Math., 100 (1997), 163—169.

[58] F. Manfio and F. Vitório, Minimal Immersions of Riemannian Manifolds in Products of Space Forms, Journal of mathematical Analysis and Applications, 424 (2015) 260−268.

[59] H. Blaine Lawson, JR., Lectures on Minimal Submanifolds, Volume 1.

[60] I. Chavel, Riemannian geometry, Cambridge University Press 1994, 2006.

[61] John M. Lee, Introduction to Smooth Manifolds, Version 3.0, 2000.

[62] John M. Lee, Riemannian Manifolds: An Introduction to Curvature, Springer−Verlag New York, Inc., 1997.

[63] J. M. Lee, Manifolds and Differential Geometry, American Mathematical Society, Vol. 107, 2009.

[64] J. S. kim, Y. M. Song and M. M. Tripathi, B. Y. Chen inequalities for submanifolds in generalized complex space forms, Bull. Korean Math. Soc. 40 (2003), No. 3, pp. 411—423.

[65] J. F. Nash, The imbedding problem for Riemannian manifolds, Ann. of Math. 63 (1956), 20—63. MR 0075639.

[66] J. S. Pak and W. H. Sohn, Some curvature conditions of n-dimensional CR-submanifolds of $(n-1)$ CR-dimensional in a complex projective space, Bull. Korean Math. Soc. 38 (2001), No. 3, PP. 575-

[67] K. Yano, Notes on Submanifolds in a Riemannian Manifold, Kodai Math Sem. Rep. 21 (1969), 496–509.

[68] K. Yano and M. Kon, Structures on Manifolds, World Scientific, 1985.

[69] K. Yano and M. Kon, CR -submanifolds of a complex projective space, J. Diff. Geom.,16 (1981) 431-444.

[70] K. L. Duggal, Lorentzian Geometry of CR Submanifolds, Kluwer Academic Publishers, Acta Applicandae Mathematicae 17: 171–193, 1989.

[71] L. Cao and H. Li, Variational problems in geometry of submanifolds, Proceeding of the Eleventh International workshop on Diff. Geom. 11 (2007) 41-71

[72] M. A. Bashir, On the three-dimensional CR-submanifolds of the six dimensional sphere, Internat. J. Math. & Math. Sci., Vol. 14, No. 4 (1991) 675-678.

[73] M. A. Bashir, CR-hypersurfaces of a complex projective space, Internat. J. Math. & Math. Sci., Vol. 17, No. 3, (1994) 613-616.

[74] M. A. Bashir, Some totally umbilical CR-submanifolds of a Kähler manifold, Math. Chronicle, Vol. 20, (1991), 67-73.

[75] M. A. Bashir, On CR-submanifolds of the six-dimensional sphere, Internat. J. Math. & Math. Sci., Vol. 18, No. 1 (1995) 201-203.

[76] M. A. Bashir, Totally real surfaces of the six-dimensional sphere, Glasgow Math. J. 33 (1991) 83-87.

[77] M. Atceken, Ümit Y., S. Dirik, Submanifolds of a Riemannian Manifold, In Tech open, 2017.

[78] Mukut Mani Tripathi, Certain basic Inequalities for Submanifolds, Proceedings of the Tenth International Workshop on Diff. Geom. 10 (2006), 99–145.

[79] M. Morohashi, Certain Properties of a Submanifold in a Sphere, Hokkaido Math. Jour., 2 (1973) 40–54.

[80] M. Okumura, Submanifolds of codimension 2 with certain properties, J. Diff. Geometry 4 (1970), 457−467.

[81] M. Barros, Spherical finite type submanifolds, Lecture Notes in Math., 1410 (1988), 60−70.

[82] Noel J. Hicks, Notes on Differential Geometry, Van Nostrand Reinhold Company, 1965.

[83] N. Ejiri, Totally real submanifolds in a 6-sphere, Proceeding of The American Math. Soc. Vol. 83, No. 4, 1981.

[84] Nirmala Prakash, Differential Geometry, Tata McGraw-Hill Publishing Company Limited, 1981.

[85] N. Nadirashivili, Hadamard's and Calabi-Yan's conjectures on negative curved and minimal surfaces, Invent. Math. 126(1996), 457-465.

[86] O. J. Garay and L. Verstraelen, on submanifolds of finite Chen type and some related topics, Preprint series, Dept. Math. Kath. Univ. Leuven, 4 (1992), 5−28.

[87] P. Alegre, A. Carriazo, Y. H. Kim, and D. W. Yoon, B. Y. Chen's inequality for submanifolds of generalized space forms, Indian J. Pure Appl. Math., 38(3): 185−201, (2007).

[88] P. Zhang, L. Zhang, and W. Song, Chen's inequalities for submanifolds of a Riemannian manifold of quasi-constant curvature with a semi-symmetric connection, Taiwanese J. Math., Vol. 18, No. 6, pp. 1841−1862, 2014.

[89] P. Zhang, X. Pan, and L. Zhang, Inequalities for submanifolds of Riemannian manifold of nearly quasi-constant curvature with a semi-symmetric non-metric connection, REVISTA DE LA UNION MATEMATICA ARGENTINA, Vol. 56, No. 2, 2015, 1−19.

[90] R. S. Gupta and I. Ahmad, B. Y. Chen's inequality and its application to slant immersions into Kenmotsu manifolds, KYUNGPOOK Math. J. 44 (2004), 101−110.

[91] R. S. Palais and C. L. Terng, Critical Point Theory and Submanifold Geometry, Springer-Verlag, 1988.

[92] R. S. Gupta, B. Y. Chen's inequalities for bi-slant submanifolds in cosymplectic space forms, Sarajevo Journal of Mathematics, Vol.9 (21) (2013), 117−128.

[93] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Volume I, Interscience, New York, 1963.

[94] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Volume II, Interscience, New York, 1969.

[95] S. S. Chern, Minimal Submanifolds in a Riemannian Manifold, University of Kansas Press, 1968.

[96] S. Helgason, Differential Geometry, Lie Groups, and Symmetric Spaces, Academic Press, New York 1978.

[97] S. Helgason, Differential geometry and symmetric spaces, Academic Press Inc., New York and London 1962.

[98] S. Ianus, Submanifolds of Almost Hermitian Manifolds, Riv. Mat. Univ. Parma (5) 3 (1994), 123−142.

[99] S. Ishikawa, Classification problems of finite submanifolds and biharmonic submanifolds, Doctoral Thesis, Kyushu University, 1999.

[100] S. Klein, Totally geodesic submanifolds in Riemannian symmetric space, arXiv: 0810.4413 V1 [math. DG] 2008.

[101] S. Sular, Chen inequalities for submanifolds of generalized space forms with a semi-symmetric metric connection, Hacettepe J. of Math. and Stat., Vol. 45(3) (2016), 811−825.

[102] T. Nagai and H. Kôjyô, Notes on Totally Submanifolds in Constant Curvature Spaces, Hokkaido Math. J., 11 (1982), 328−336.

[103] T. Yamada, Submanifolds of codimension greater than 1 with certain properties, to appear.

[104] Tobias H. Colding and William P. Minicozzi II, Minimal Submanifolds, Bull. of the London Mathematical Society, 2005.

[105] T. Takahashi, Minimal immersions of Riemannian manifolds, J. Math. Sco. Japan, Vol. 18, (1966), 380–385.

[106] W. M. Boothby, An Introduction to Differentiable Manifolds and Riemannian Geometry, Academic Press, New York, 2003.

[107] W. M. Boothby, An Introduction to Differentiable Manifolds and Riemannian Geometry, Academic Press, Inc, 1975.

[108] Wijsman, Robert A. 3. Differentiable Manifolds, Tangent Spaces, and Vector fields, Invariant Measures on Groups and their use in Statistics, 38–52, Institute of Mathematical Statistics, Harward, CA, 1990.

[109] W. P. A. Klingenberg, Riemannian Geometry, Walter de Gruyter & Co., 1995.

[110] W. S. Massey, Algebraic topology: An introduction. Harcourt, New York,1967.

[111] Yano K. and Ishihara S., ‘Submanifolds with parallel mean curvature’, J. Defferential Geometry 6 (1971), 95–118.

[112] Y. Xin, Minimal submanifolds and related topics, World Scientific Publishing Co. Pte. Ltd., 2003.

[113] Y. Katsurada and T. Nagai, On Some Properties of Submanifold with Constant Mean Curvature in a Riemannian Space, J. Fac. Sci. Hokkaido Univ. Ser. 1. Vol. 20 (1968), 79–89.

[114] Y. Katsurada, T. Nagai and H. Kôjyô, On Submanifolds with Constant Mean Curvature in a Riemannian Manifold; Publ. Study Group of Geom., 9 (1975), 1–129.