Sudan University of Science and Technology
College of Graduate Studies

Asymptotic of Eigenvalues of Pseudo-Differential Operators and Coulson-Type Integral Formulas with Common Hypercylic Functions

مقاربة القيم الذاتية للمؤثرات شبه التفاضلية وصيغ تكامل نوع-كولسون مع الدوال الدورية المفرطة العامة

A Thesis Submitted in Fulfillment of the Requirements for the Degree of Ph.D in Mathematics

By
Afnan Ali Hasan Alrabiaa

Supervisor
Prof. Dr. Shawgy Hussein AbdAlla

2019

Dedication

To my Family.

Acknowledgements

I would like to thank with all sincerity Allah, and my family for their supports throughout my study. Many thanks are due to my thesis guide, Prof. Dr. Shawgy Hussein AbdAlla Sudan University of Science and Technology.

Abstract

The growth of frequently hypercyclic functions and common hypercyclic vectors for differential and translation operators and for the conjugate class of a hypercyclic operator and a Sot-Dense path of chaotic operators are studied. The spectral properties of the Cauchy process and asymptotic estimate of eigenvalues of pseudo-differential operators on half-line and interval with eigenvalues and refined semiclassical asymptotics for fractional Laplace operators, trace estimates and two-term estimates for unimodal Levy and relativistic stable processes are determined. We also classify the sum of powers of the Laplacian eigenvalues and normalized incidence energy of graphs with Coulson-type integral formulas for the general Laplacian-energy-like invariant of graphs.

الخلاصة

قمنا بدراسة النمو للدوال الدورية المفرطة والمتجهات الدورية المفرطة العامة لأجل المؤثرات التفاضلية والانسحابية ولأجل العائلة المرافقة للمؤثر الدوري المفرط والمسار الكثيف القوى للمؤثر التبولوجي للمؤثرات الفوضوية. ثم تحديد الخصائص الطيفية لعملية كوشي والتقدير التقاربي للقيم الذاتية للمؤثرات شبه التفاضلية على نصف-الخط والفترة مع القيم الذاتية والمقاربات شبه التقليدية المحسنة لأجل مؤثرات لابلاس الكسرية وتقديرات الأثر وتقديرثنائية-الحد لأجل عمليات ليفي أحادية الواسطة واستقراية النسبوية. أيضاً تم تصنيف جمع القوى للقيم الزائدية للابلسيان والطاقة الحادثة الناظمة للبيانات مع صيغ تكامل نوع-كولسون لأجل لامتغيرمثل-طاقة اللابلسيان العامة للبيانات.

Introduction

We investigate the conjugate indicator diagram or, equivalently, the indicator function of (frequently) hypercyclic functions of exponential type for differential operators. Given a separable, infinite dimensional Hilbert space, it was shown that there is a path of chaotic operators, which is dense in the operator algebra with the strong operator topology, and along which every operator has the exact same dense G_δ set of hypercyclic vectors.

We study the spectral properties of the transition semigroup of the killed one-dimensional Cauchy process on the half-line $(0, \infty)$ and the interval $(-1, 1)$. This process is related to the square root of one-dimensional Laplacian $A = -\sqrt{-(d^2/dx^2)}$ with a Dirichlet exterior condition, and to a mixed Steklov problem in the half-plane. For the half-line, an explicit formula for generalized eigenfunctions ψ_λ of A is derived, and then used to construct a spectral representation of A. Two-term Weyl-type asymptotic law for the eigenvalues of the one-dimensional fractional Laplace operator $(-\Delta)\alpha/2$ $(\alpha \in (0, 2))$ in the interval $(-1, 1)$ is given: the n-th eigenvalue is equal to $\left(\frac{n\pi}{2} - \frac{(2-\alpha)\pi}{8}\right)^\alpha + O\left(\frac{1}{n}\right)$. We consider the fractional Laplacian on a domain and investigate the asymptotic behavior of its eigenvalues.

Let G be a simple graph with n vertices and m edges. Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of the adjacency matrix of G, and let $\mu_1, \mu_2, \cdots, \mu_n$ be the eigenvalues of Laplacian matrix of G. The energy of G is defined as $E(G) = \sum_{i=1}^{n} |\lambda_i|$. For a graph G and a real $\alpha \neq 0$, we study the graph invariant $s_\alpha(G)$ – the sum of the αth power of the non-zero Laplacian eigenvalues of G. The cases $\alpha \neq 2, \frac{1}{2}$ and -1 have appeared in different problems.

Many have obtained interesting results on the existence of a dense G_δ set of common hypercyclic vectors for a path of operators. We show that on a separable infinite dimensional Hilbert space, there is a path of chaotic operators that is dense in the operator algebra with the strong operator topology. Let $H(\mathbb{C})$ be the set of entire functions endowed with the topology of local uniform convergence. Fix a sequence of non-zero complex numbers (λ_n), $|\lambda_n| \to +\infty$, which satisfies the following property:for every $M > 0$ there exists a subsequence (μ_n) of (λ_n) such that (i) $|\mu_{n+1}| - |\mu_n| > M$ for every $n = 1, 2, \ldots$ and (ii) $\sum_{n=1}^{+\infty} \frac{1}{|\mu_n|} = +\infty$.

We give two-term small-time approximation for the trace of the Dirichlet heat kernel of bounded smooth domain for unimodal L´evy processes satisfying the weak scaling conditions. We show a two-term Weyl-type asymptotic law, with error term $O\left(\frac{1}{n}\right)$, for the eigenvalues of the operator $\psi(-\Delta)$ in an interval, with zero exterior condition, for complete Bernstein functions ψ such that $\xi\psi'(\xi)$ converges to infinity as $\xi \to \infty$.

The energy of a graph G is the sum of the singular values of its adjacency matrix. It is a graph invariant used in mathematical chemistry. The normalized incidence energy of the graph G, denoted by *NIE(G)*, is defined as the sum of the singular values of its normalized incidence matrix. Let G be a simple graph. Its energy is defined as $E(G) = \sum_{k=1}^{n} |\lambda_k|$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of G. A well-known result on the energy of graphs is the Coulson integral formula which gives a relationship between the energy and the characteristic polynomial of graphs. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ be the Laplacian eigenvalues of G. The general Laplacian-energy-like invariant of G, denoted by $LEL_\alpha(G)$, is defined as $\sum_{\mu_k \neq 0} \mu_k^\alpha$ when $\mu_1 \neq 0$, and 0 when $\mu_1 = 0$, where α is a real number. Let G be a graph of order n and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ the eigenvalues of G. The energy of G is defined as $E(G) = \sum_{k=1}^{n} |\lambda_k|$. A well-known result on the energy of graphs is the Coulson integral formula which gives a relationship between the energy and the characteristic polynomial of graphs. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ be the Laplacian eigenvalues of G. The general Laplacian-energy-like invariant of G, denoted by $LEL_\alpha(G)$, is defined as $\sum_{\mu_k \neq 0} \mu_k^\alpha$ when $\mu_1 \neq 0$, and 0 when $\mu_1 = 0$, where α is a real number. We give some Coulson-type integral formulas for the general Laplacian-energy-like invariant of graphs in the case that α is a rational number.

The Contents

Chapter 1
Growth of Hypercyclic Functions and Common Hypercyclic Vectors

We give growth conditions of the functions on particular rays or sectors. The research extends known results in several respects. We show that the conjugate set of any hypercyclic operator on a separable, infinite dimensional Banach space always contains a path of operators which is dense with the strong operator topology, and yet the set of common hypercyclic vectors for the entire path is a dense G_δ set. As a corollary, the hypercyclic operators on such a Banach space form a connected subset of the operator algebra with the strong operator topology.

Section (1.1): Differential Operators

A continuous operator $T : X \to X$, with X a topological vector space, is called hyper-cyclic if there exists a vector $x \in X$ sucht that the orbit $\{T^n x : n \in \mathbb{N}\}$ is dense in X. Such a vector x is said to be a hypercyclic vector. By $\mathcal{HC}(T, X)$, we denote the set of all hypercyclic vectors for T (on X). The operator is called frequently hypercyclic if there exists some $x \in X$ such that for every non-empty open set $U \subset X$ the set $\{n : T^n x \in U\}$ has positive lower density. The vector x is called a frequently hypercyclic vector in this case and the set of all these vectors shall be denoted by $\mathcal{FHC}(T, X)$ in the following. We recall that the lower density of a discrete set $\Lambda \subset \mathbb{C}$ is defined by

$$\liminf_{r\to\infty} \frac{\#\{\lambda \in \Lambda : |\lambda| \le r\}}{r} =: \underline{\text{dens}}(\Lambda).$$

We are only concerned with spaces consisting of holomorphic functions and therefore the hypercyclic vectors are called hypercyclic functions.

In [10], G. Godefroy and J. H. Shapiro show that for every non-constant entire func-tion $\varphi(z) = \sum_{n=0}^{\infty} c_n z^n$ of exponential type, the induced differential operator

$$\varphi(D): H(\mathbb{C}) \to H(\mathbb{C}), f \mapsto \sum_{n=0}^{\infty} c_n f^{(n)}, \quad (1)$$

where $H(\mathbb{C})$ is endowed with the usual topology of locally uniform convergence, is hypercyclic. This results also applies for the case of frequent hypercyclicity as it is shown in [7]. Actually, in both [10] and [7], the outlined results are given for the case of $H(\mathbb{C}^N)$. The possible rate of growth of the corresponding (frequently) hypercyclic functions is widely investigated (cf. [4], [5], [7], [8], [9], [11]). It turns out that the level set

$$C_\varphi := \{z : |\varphi(z)| = 1\} \quad (2)$$

plays a cruical role. Under certain additional assumptions, it is known that for $\tau_\varphi := \text{dist}(0, C_\varphi)$ there are functions of exponential type τ_φ that belong to $\mathcal{HC}(\varphi(D), H(\mathbb{C}))$, while every function of exponential type less than τ_φ cannot belong to $\mathcal{HC}(\varphi(D), H(\mathbb{C}))$ (cf. [4]). It is also known that for every $\varepsilon > 0$ there are functions in $\mathcal{FHC}(\varphi(D), H(\mathbb{C}))$ that are of exponential type less or equal than $\tau_\varphi + \varepsilon$ (cf. [7]). For the case of the translation operator $f \mapsto f(\cdot + 1)$, which is the differential operator induced by the exponential function, and the ordinary differentiation operator D, growth conditions are achieved in [9], [8], [11] and [5].

All investigations in this direction have in common that the rate of growth is measured with respect to the maximum modulus $M_f(r) := \max_{|z|=r} |f(z)|$ or L^p-Averages $M_{f,p}(r) := (1/2\pi \int_0^{2\pi} |f(re^{it})|^p dt)^{1/p}$, where $p \in [1, \infty)$. We extend some of these results by considering growth conditions with respect to rays emanating from the origin.
For the sake of completeness, we recall that an entire function f is said to be of exponential type τ if

$$\limsup_{r\to\infty} \frac{\log M_f(r)}{r} =: \tau(f) = \tau,$$

where we set $\log(0) := -\infty$, and f is said to be a function of exponential type when the above lim sup is not equal to $+\infty$. The indicator function of an entire function of exponential type is defined by

$$h_f(\Theta) := \limsup_{r\to\infty} \frac{\log|f(re^{i\Theta})|}{r}, \qquad \Theta \in [-\pi, \pi].$$

It is known that h_f is determined by the support function of a certain compact and convex set $K(f) \subset \mathbb{C}$, to be more specific, for $z = re^{i\Theta}$ we have

$$rh_f(\Theta) = H_{K(f)}(z) := \sup\{Re(zu) : u \in K(f)\}$$

(cf. [3]). The set $K(f)$ is called the conjugate indicator diagram of f. Note that for $f \equiv 0$, we have $K(f) = \emptyset$. We give necessary and sufficient conditions for the location and the size of the conjugate indicator diagram of (frequently) hypercyclic functions for differential operators $\varphi(D)$. According to the above relations, this yields information about the growth on particular rays or sectors in terms of the indicator function. Since

$$\max_{\Theta\in[-\pi,\pi]} h_f(\Theta) = \max_{u\in K(f)} |u| = \tau(f),$$

this also includes information about the possible exponential type. In particular, f is of exponential type zero if and only of $K(f) = \{0\}$.

We abbreviate the exponential function $z \mapsto e^{\alpha z}$ by e_α, for α some complex number. For $\alpha = \tau e^{i\psi}$ the indicator function of e_α is given by

$$h_{e_\alpha}(\Theta) = \tau \cos(\Theta + \psi)$$

and the conjugate indicator diagram is the singleton $\{\alpha\}$.
With [6] it follows that for an entire function f of exponential type we have $K(f) = \{\alpha\}$ if and only if there is some entire function f_0 of exponential type zero with $f = f_0 e_\alpha$. In that sense, functions which have singleton conjugate indicator diagram are close to the corresponding exponential function. In particular, the indicator functions of f and e_α coincide, which implies that f decreases exponentially in the half plane $|arg(z) + \psi| > \pi/2$ if $\alpha \neq 0$.

The first result shows that the conjugate indicator diagram of hypercyclic functions for differential operators are not restricted with respect to their size and shape.

Let $\Omega \subset \mathbb{C}$ be a domain and K a compact subset of Ω. A cycle Γ in $\Omega \setminus K$, is called a Cauchy cycle for K in Ω if $\mathrm{ind}_\Gamma(u) = 1$ for every $u \in K$ and $\mathrm{ind}_\Gamma(w) = 0$ for every $w \in \mathbb{C} \setminus \Omega$. The existence of such a cycle is always guaranteed and, moreover, the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\xi)}{\xi - z} d\xi$$

is valid for every $z \in K$ (see [17]). By $|\Gamma|$ we denote the trace of and $\text{len}(\Gamma) := \int_a^b |\Gamma(t)|\, dt$ is the length of Γ. In the following, K always has a simply connected complement. In this case, may be chosen as a simple closed path. For a given compact and convex set $K \subset C$, we denote by $\text{Exp}(K)$ the space of all entire functions f of exponential type that satisfy $K(f) \subset K$. This space naturally of analytic functionals (cf. [14], [15], [3]). The differential operators are mainly considered on $Exp(K)$, which is very convenient as we will see.
For a function f of exponential type, $\mathcal{B}f(z) := \sum_{n=0}^{\infty} f^{(n)}(0)/z^{n+1}$ is called the Borel transform of f. The Borel transform is a holomorphic function on some neighbourhood of infinity that vanishes at infinity. It is known that the conjugate indicator diagram $K(f)$ is the smallest convex, compact set such that $\mathcal{B}f$ admits an analytic continuation to $\mathbb{C}\setminus K(f)$ and that the inverse of the Borel transform is given by

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma} \mathcal{B}f(\xi)e^{\xi z}d\xi$$

where is a Cauchy cycle for $K(f)$ in $\mathbb{C}$ (cf. [6], [3]). This integral formula is known as the Pólya representation.
Finally, we make use of the following notions: $\mathbb{C}_\infty$ is the extended complex plane $\mathbb{C} \cup \{\infty\}, \mathbb{D} := \{z : |z| < 1\}$ and $\mathbb{T} := \{z : |z| = 1\}$. If $A \subset \mathbb{C}$, then $A^{-1} := \{z : 1/z \in A\}$, where as usual $1/0 := \infty, \overline{A}$ is the closure of A and $\text{conv}(A)$ is the convex hull of A. For an open set $\Omega \subset \mathbb{C}_\infty$ the space of functions holomorphic on Ω and vanishing at ∞ (if $\infty \in \Omega$) endowed with the topology of uniform convergence on compact subsets is denoted by $H(\Omega)$. Recall that a function f is said to be holomorphic at infinity if $f(1/z)$ is holomorphic at the origin.
For the proof of the next proposition, see [15].
Proposition (1.1.1)[1]: Let $K \subset \mathbb{C}$ be a compact and convex set.
(i) For every $n \in \mathbb{N}$,

$$\|f\|_{K,n} := \sup_{z\in\mathbb{C}} |f(z)|\, e^{-H_K(z)-\frac{1}{n}|z|}$$

defines a norm $\|\cdot\|_{K,n}$ on $\text{Exp}(K)$ and the space $\text{Exp}(K)$, endowed with the topology induced by the sequence $\{\|\cdot\|_{K,n} : n \in \mathbb{N}\}$, is a Fréchet space.
(ii) The Borel transfrom

$$\mathcal{B} = \mathcal{B}_K : \text{Exp}(K) \to H(\mathbb{C}_\infty \setminus K), \ f \mapsto \mathcal{B}f|_{\mathbb{C}_\infty \setminus K}$$

is an isomorphism.
By differentiation of the parameter integral, the Polya representation yields

$$f^{(n)}(z) = \frac{1}{2\pi i}\int_{\Gamma} \mathcal{B}f(\xi)\xi^n e^{\xi z}d\xi.$$

Inspired by this formula, we introduce a class of operators on $\text{Exp}(K)$ by replacing ξ^n in the above integral by a function holomorphic on some neighborhood of K. We define $H(K)$ to be the space of germs of holomorphic functions on K, where $K \subset \mathbb{C}$ is some compact set. In order to simplify the notation, an element of $H(K)$ shall always be identified with some of its representatives φ which is defined on an open neighbourhood Ω_φ of K. In case that K is convex we always assume Ω_φ to be simply connected (actually we may suppose Ω_φ to be even convex).
Now, for a fixed compact and convex set $K \subset \mathbb{C}$ and a germ $\varphi \in H(K)$, we define

$$\varphi(D)f(z) := \frac{1}{2\pi i}\int_{\Gamma} \mathcal{B}f(\xi)\varphi(\xi)e^{\xi z}\,d\xi \qquad (3)$$

where is a Cauchy cycle for K in Ω_{φ}. Obviously, this definition is independent of the particular choice of Γ. If φ extends to an entire function $\varphi(z) = \sum_{n=0}^{\infty} c_n z^n$, the interchange of integration and summation immediately yields

$$\sum_{n=0}^{\infty} c_n f^{(n)}(z) = \frac{1}{2\pi i}\int_{\Gamma} \mathcal{B}f(\xi)\varphi(\xi)e^{\xi z}\,d\xi\,.$$

We see that the operators $\varphi(D)$ from (3) are a natural extension of the differential operators in (1).

Proposition (1.1.2)[1]: Let K be a compact, convex set in $\mathbb{C}$ and $\varphi \in H(K)$. Then $\varphi(D)$ defined by (3) is a continuous operator on $\mathrm{Exp}(K)$.

Proof: For a given positive integer n, we choose Γ such that $|\Gamma| \subset \frac{1}{n}\overline{\mathbb{D}} + K$. Then $H_{conv(|\Gamma|)} \leq H_{K+\frac{1}{n}\overline{\mathbb{D}}}$ and that means $(Re(\xi z) - H_K(z) - n^{-1}|z|) \leq 0$ for all $\xi \in |\Gamma|$ and all $z \in \mathbb{C}$. Consequently, $\left|e^{\xi z - H_K(z) - \frac{1}{n}|z|}\right| \leq 1$ for all $z \in \mathbb{C}$ and all $\xi \in |\Gamma|$. As $\mathcal{B}$: $\mathrm{Exp}(K) \to H(\mathbb{C}_{\infty}\setminus K)$ is an isomorphism and $|\Gamma|$ is compact in $\mathbb{C}\setminus K$, there is an $m \in \mathbb{N}$ and a constant $C > 0$ such that $\sup\{|\mathcal{B}f(\xi)| : \xi \in |\Gamma|\} \leq C\|f\|_{K,m}$. With $M := \frac{1}{2\pi}\int_{\Gamma}|\varphi(\xi)|d\xi$, we now obtain

$$\|\varphi(D)f\|_{K,n} = \sup_{z\in\mathbb{C}}\left|\frac{1}{2\pi i}\int_{\Gamma}\varphi(\xi)\mathcal{B}f(\xi)e^{\xi z}\,d\xi\right| e^{-H_K(z)-\frac{1}{n}|z|}$$
$$\leq \sup_{z\in\mathbb{C}}\frac{1}{2\pi}\int_{\Gamma}|\varphi(\xi)||\mathcal{B}f(\xi)|\left|e^{\xi z - H_K(z)-\frac{1}{n}|z|}\right| d\xi \leq MC\|f\|_{K,m}.$$

This proves that $\varphi(D)$ is a self-mapping on $\mathrm{Exp}(K)$ and the continuity of this operator.

Proposition (1.1.3)[1]: Let $K \subset \mathbb{C}$ be a compact and convex set.

(i) For any $\alpha \in K$, the set $\{Pe_{\alpha} : P \text{ polynomial }\}$ is dense in $\mathrm{Exp}(K)$.

(ii) If A is an infinite subset of K, then $\mathrm{span}\{e_{\alpha} : \alpha \in A\}$ is dense in $\mathrm{Exp}(K)$.

Proof: Let Σ denote the space of all polynomials. In a first case we assume that $0 \in K$. For a function $f \in Exp(K)$ we have that $\widetilde{\mathcal{B}}f := (1/\cdot)\,\mathcal{B}f\,(1/\cdot) \in H(\mathbb{C}_{\infty}\setminus K^{-1})$. Since $\mathcal{B} : Exp(K) \to H(\mathbb{C}_{\infty}\setminus K)$ is an isomorphism, one verifies that $\widetilde{\mathcal{B}} : Exp(K) \to H(\mathbb{C}_{\infty}\setminus K^{-1})$ is also an isomorphism. Now, Σ is dense in $H(\mathbb{C}_{\infty}\setminus K^{-1})$ by Runge's theorem and observing that $\widetilde{\mathcal{B}}^{-1}(\Sigma) = \Sigma$ this shows that Σ is dense in $Exp(K)$.

Let K be an arbitrary compact and convex set. By means of [6] it follows that for every entire function f of exponential type and $\alpha \in \mathbb{C}$ we have $K(fe_{-\alpha}) = K(f) - \{\alpha\}$. Thus, if $g = f/e_{\alpha}$ for an $f \in Exp(K)$ and $\alpha \in K$,

$$||f||_{K,n} = \sup_{z\in\mathbb{C}} |g(z)||e^{\alpha z}|e^{-H_K(z)-\frac{1}{n}|z|}$$
$$= \sup_{z\in\mathbb{C}} |g(z)|e^{-H_K(z)-H_{-\{\alpha\}}(z)-\frac{1}{n}|z|}$$
$$= \sup_{z\in\mathbb{C}} |g(z)|e^{-H_{K-\{\alpha\}}(z)-\frac{1}{n}|z|}$$
$$= ||g||_{K-\{\alpha\},n}$$

which shows that $f \longmapsto f/e_{\alpha}$ is an isometric isomorphism from $Exp(K)$ to $Exp(K - \{\alpha\})$.

With the first part, this implies (i).
Without loss of generality, we may assume $0 \notin A$. It is easily seen that $\mathcal{B}e_\alpha = 1/(\cdot - \alpha)$ and thus $\mathcal{B}(span\{e_\alpha : \alpha \in A\}) = span\{1/(\cdot - \alpha) : \alpha \in A\}$. Since A has an accumulation point in K, a variant of Runge's theorem (see. [13]) yields that $span\{1/(\cdot - \alpha) : \alpha \in A\}$ is dense in $H(\mathbb{C}_\infty \setminus K)$. According to the fact that $\mathcal{B} : Exp(K) \to H(\mathbb{C}_\infty \setminus K)$ is an isomorphism, this shows (ii).
A germ $\varphi \in H(K)$ is said to be zero-free if there exists a representative φ which is zero-free on some open neighbourhood of K. In this case, we always assume that Ω_φ is so small that φ is zero-free on Ω_φ and thus $1/\varphi \in H(\Omega_\varphi)$.
Proposition (1.1.4)[1]: Let $K \subset \mathbb{C}$ be a compact, convex set and φ, ψ in $H(K)$. Then we have $\varphi(D)\psi(D) = \varphi\psi(D)$. In particular, if φ is zero-free, then

$$\varphi(D)\,(1/\varphi)(D) = (1/\varphi)(D)\,\varphi(D) = \mathrm{id}_{Exp(K)}$$

and hence $\varphi(D)$ is invertible with $\varphi(D)^{-1} = (1/\varphi)(D)$.
Proposition (1.1.4) is an immediate consequence of
Lemma (1.1.5)[1]: Let K be a compact, convex set in $\mathbb{C}$, $f \in Exp(K)$ and $\varphi \in H(K)$. Then for all $h \in H(\Omega_\varphi)$, we have

$$\int_\Gamma \mathcal{B}f(\xi)\,\varphi(\xi)\,h(\xi)\,d\xi = \int_\Gamma \mathcal{B}(\varphi(D)f)(\xi)\,h(\xi)\,d\xi$$

Where Γ is a Cauchy cycle for K in Ω_φ.
Proof: Considering Runge's theorem, one verifies that, for fixed $\alpha \in \mathbb{C}, \{Pe_\alpha : P \text{ polynomial}\}$ is dense in $H(\Omega)$ whenever Ω is a simply connected open subset of $\mathbb{C}$. According to the fact that $Pe_\alpha \in Exp(K)$ for $\alpha \in K$, this shows that $Exp(K)$ is densely embedded in $H(\Omega)$ for every non-empty, compact and convex $K \subset \mathbb{C}$. Thus, as a consequence of Proposition (1.1.3) (ii), $E := span\{e_\alpha : \alpha \in \mathbb{C}\}$ is dense in $H(\Omega_\varphi)$ since Ω_φ is simply connected.
We consider the functional

$$\langle \Lambda, h\rangle := \int_\Gamma (\mathcal{B}f(\xi)\,\varphi(\xi) - \mathcal{B}(\varphi(D)f)(\xi))\,h(\xi)\,d\xi$$

on $H(\Omega_\varphi)$. With the Polya representation for $(D)f$, the following holds:

$$\frac{1}{2\pi i}\int_\Gamma \mathcal{B}(\varphi(D)f)(\xi)e^{\xi\alpha}\,d\xi = \varphi(D)f(\alpha) = \frac{1}{2\pi i}\int_\Gamma \mathcal{B}f(\xi)\varphi(\xi)e^{\xi\alpha}\,d\xi\,.$$

Hence $\langle \Lambda, e_\alpha\rangle = 0$ for all $\alpha \in \mathbb{C}$ and consequently $\Lambda|_E = 0$. As E is dense in $H(\Omega_\varphi)$, we have $\Lambda = 0$.
Proposition (1.1.6)[1]: Let $K \subset \mathbb{C}$ be a compact, compact set. Then the set of all $f \in Exp(K)$ with $K(f) = K$ is residual in $Exp(K)$.
Proof: Let $M \subset H(\mathbb{C}_\infty \setminus K)$ be the set of functions that are exactly holomorphic in $\mathbb{C}_\infty \setminus K$, that means, for every $w \in \mathbb{C} \setminus K$ the radius of convergence of the Taylor series with center w equals $\mathrm{dist}(w, K)$. Due to a result of V. Nestoridis (see [16]), M is a dense G_δ-set in $H(\mathbb{C}_\infty \setminus K)$. Since $\mathcal{B}^{-1}(M) \subset \{f \in Exp(K) : K(f) = K\}$ and $\mathcal{B} : Exp(K) \to H(\mathbb{C}_\infty \setminus K)$ is an isomorphism, we obtain the assertion.
Theorem (1.1.7)[1]: Let K be a convex, compact subset of $\mathbb{C}$ and $\varphi \in H(K)$ non-constant. Then $\mathcal{HC}(\varphi(D), \mathrm{Exp}(K)) \neq \emptyset$ if and only if $\varphi(K) \cap \mathbb{T} \neq \emptyset$. Further, if

$\mathcal{HC}(\varphi(D), \text{Exp}(K)) \neq \emptyset$, then the set of all $f \in \mathcal{HC}(\varphi(D), \text{Exp}(K))$ with $K(f) = K$ is residual in Exp(K) in the sense of Baire categories.

Before giving the proof, some auxiliary results for $\text{Exp}(K)$ and $\varphi(D)$ are established.

Proof: Firstly, assume that $\varphi(K) \subset \mathbb{D}$. Let be a Cauchy cycle for K in Ω_φ being so close to K that $|\varphi| < \delta < 1$ on . Then, considering Proposition (1.1.4), for any $f \in Exp(K)$ we have

$$|\varphi(D)^n f(0)| = \left|\frac{1}{2\pi i}\int_\Gamma \mathcal{B}f(\xi)\varphi^n(\xi)d\xi\right| \leq \frac{\delta^n}{2\pi}\int_\Gamma |\mathcal{B}f(\xi)|\, d\xi \to 0$$

as n tends to infinity. Consequently, $\varphi(D)$ cannot be hypercyclic on $Exp(K)$. If $\varphi(K) \subset \mathbb{C}\setminus\overline{\mathbb{D}}$, then $\varphi(D)$ is zero-free, as an element of $H(K)$, and thus, by Proposition (1.1.4), it is invertible on $Exp(K)$ with $\varphi(D)^{-1} = (1/\varphi)(D)$. Now, since $(1/\varphi)(K) \subset \mathbb{D}$, we have $\mathcal{HC}((1/\varphi)(D), Exp(K)) = \emptyset$, and this is equivalent to $\mathcal{HC}(\varphi(D), Exp(K)) = \emptyset$ (see [18]).

Let us now assume that $\varphi(K) \cap T \neq \emptyset$ and K to have non-emtpy interior. Taking into account that φ is non-constant, we have that $\varphi(K)$ has non-empty interior and thus, $\text{span}\{e_\alpha : \alpha \in K, |\varphi(\alpha)| > 1\}$ and $\text{span}\{e_\alpha : \alpha \in K, |\varphi(\alpha)| < 1\}$ are dense in $Exp(K)$ by Proposition (1.1.3) (i). Observing that $\varphi(D)e_\alpha = \varphi(\alpha)e_\alpha$, the Godefroy-Shapiro Criterion (see [18]) yields $\mathcal{HC}(\varphi(D), Exp(K)) \neq \emptyset$. In order to prove the hypercyclicity for the case that K has empty interior, we show that $\varphi(D)$ is transitive on $Exp(K)$ (i.e. for every pair of non-empty open sets $U, V \subset Exp(K)$ there exists a positive integer k such that $T^k(U) \cap V \neq \emptyset$), which is equivalent to the hypercyclicity of $\varphi(D)$ on $Exp(K)$ (cf. [18]).

For every positive integer n we find some convex, compact set $K \subset L \subset \Omega$ that has non-empty interior such that $H_L(z) + 1/(n+1)|z| < H_K(z) + 1/n|z|$ implying $\|\cdot\|_{L,n+1} < \|\cdot\|_{K,n}$. Consequently, for given non-empty of open sets $U, V \subset Exp(K)$, we may assume the existence open sets $\tilde{U}, \tilde{V} \subset \text{Exp}(L)$ with $U = \tilde{U} \cap Exp(K)$ and $V = \tilde{V} \cap Exp(K)$. By the above, $\varphi(D)$ is hypercyclic and hence transitive on $Exp(K)$. This implies the existence of a positive integer k such that $\tilde{U} \cap T^{-k}(\tilde{V})$ is a non-empty open set in $Exp(L)$. The denseness of $Exp(K)$ in $Exp(L)$, which is for instance a consequence of Proposition (1.1.3) (i), yields $Exp(K) \cap \tilde{U} \cap T^{-k}(\tilde{V}) = U \cap T^{-k}(\tilde{V}) \neq \emptyset$ so that $T^k(U) \cap V \neq \emptyset$.

Since $Exp(K)$ is a Frechet space, $\mathcal{HC}(\varphi(D), Exp(K)) \neq \emptyset$ implies that $\mathcal{HC}(\varphi(D), Exp(K))$ is a dense G_δ-set in $Exp(K)$ (see. [12]). Due to Proposition (1.1.6), we obtain that $\{f \in Exp(K) : K(f) = K\} \cap \mathcal{HC}(\varphi(D), Exp(K))$ is residual in $Exp(K)$.

As an easy consequence of Theorem (1.1.7) we obtain.

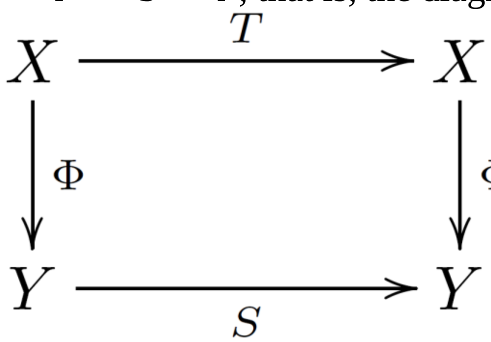
Theorem (1.1.8)[1]: Let φ be a non-constant entire function of exponential type. Then for every compact and convex set $K \subset \mathbb{C}$ that intersects C_φ there exists an $f \in \mathcal{HC}(\varphi(D), H(\mathbb{C}))$ that is of exponential type with $K(f) = K$.

Theorem (1.1.8) implies that for every $\alpha \in C_\varphi$ there exists some f_0 of exponential type zero such that $f = f_0 e_\alpha \in \mathcal{HC}(\varphi(D), H(\mathbb{C}))$. Consequently, in the case that C_φ intersects the origin, there is a function $f \in \mathcal{HC}(\varphi(D), H(\mathbb{C}))$ that is of exponential type zero. For the translation operator $e_1(D)$, a much stronger result is due to S. M. Duyos-Ruiz. She proved that functions $f \in \mathcal{HC}(e_1(D), H(\mathbb{C}))$ can have arbitrary slow tranzendental rate of growth, that is, for every $q : [0,\infty) \to [1,\infty)$ such that $q(r) \to \infty$ as r tends to infinity, there are functions $f \in \mathcal{HC}(e_1(D), H(\mathbb{C}))$ such that $M_f(r) = O(r^{q(r)})$ (cf. [9]). In [8], this result is extended the Hilbert spaces consisting of entire functions of small growth.

We will introduce a transform that quasi-conjugates differential operators and which enables us to extend the result of S. M. Duyos-Ruiz to the whole class of differential operators in the following sense.

Proof: As mentioned in the proof of Lemma (1.1.5), $Exp(K)$ is densely embedded in $H(\mathbb{C})$ for every non-empty, compact and convex set $K \subset \mathbb{C}$. Now, if φ is an entire function of exponential type, we obtain $\mathcal{HC}(\varphi(D), Exp(K)) \subset \mathcal{HC}(\varphi(D), H(\mathbb{C}))$ and see that Theorem (1.1.8) is an immediate consequence of Theorem (1.1.7).

Let $T : X \to X$ and $S : Y \to Y$ be two continuous operators acting on topological vector spaces $,Y$. A very useful tool to link the dynamics of such operators is to show that they are (quasi-) conjugated. That means, find a continuous mapping $\Phi : X \to Y$ having dense range and such that $\Phi \circ T = S \circ \Phi$, that is, the diagram



commutes. Then S is said to be quasi-conjugated to T (by Φ). If Φ is bijective and Φ^{-1} is continuous, then T and S are said to be conjugated.

Proposition (1.1.9)[1]: If S is quasi-conjugated to T by Φ, then $\Phi(\mathcal{HC}(T,X)) \subset \mathcal{HC}(S,Y)$ and $\Phi(\mathcal{FHC}(T,X)) \subset \mathcal{FHC}(S,Y)$.

This result is immediately deduced from the definition of quasi-conjugacy (cf. [12]). We introduce a transform that quasi-conjugates the operators.

Let $K \subset \mathbb{C}$ be a compact and convex set and $\varphi \in H(K)$. As in the definition of the operators $\varphi(D)$, our starting point is the Polya representation. For $f \in Exp(K)$,
we set

$$\Phi_\varphi f(z) := \frac{1}{2\pi i}\int_\Gamma \mathcal{B}f(\xi)e^{\varphi(\xi)z}\, d\xi \qquad (4)$$

where is a Cauchy cycle for K in Φ_φ. It is clear that this definition is independent of the particular choice of Γ.

Lemma (1.1.10)[1]: Let $K \subset \mathbb{C}$ be a compact, convex set and (K_n) a sequence of compact, convex supersets of K such that $K_n^\circ \supset K_{n+1}$ and $\bigcap_{n\in N} K_n = K$. Then $Exp(K) = \bigcap_{n\in N} Exp(K_n)$ in algebraic and topological sense.

Proof: The equality in algebraic sense is clear. That the spaces also coincide in topological sense is an immediate consequenc of the observation that for a given $\in N$, we have $H_{K_n,j} \leq H_{K,l}$ for a suitable choice of $n, j \in N$.

Proposition (1.1.11)[1]: Let K be a compact, convex subset of $\mathbb{C}$ and $\varphi \in H(K)$ non-constant.

Then, for each $f \in Exp(K)$, the function $\Phi_\varphi f$ defined by (4) is an entire function of exponential type with $K\big(\Phi_\varphi f\big) \subset \operatorname{conv}(\varphi(K(f)))$. Further

$$\Phi_\varphi : Exp(K) \to Exp(\operatorname{conv}(\varphi(K)))$$

is a continuous operator that has dense range.

Proof: One immediately verifies that $\Phi_\varphi f$ is an entire function. We fix some positive integer n and choose Γ such that $\varphi(|\Gamma|)$ is contained in $\operatorname{conv}(\varphi(K(f))) + \frac{1}{n}\overline{\mathbb{D}}$.

Then

$$H_{conv(\varphi(|\Gamma|))}(z) \leq H_{\operatorname{conv}(\varphi(K(f)))+\frac{1}{n}\overline{\mathbb{D}}}(z) = H_{\operatorname{conv}(\varphi(K(f)))}(z) + \frac{1}{n}|z|$$

and thus

$$\phi(\xi)z - H(z) - |z|$$

$$\begin{aligned} ||\Phi_\varphi f||_{\operatorname{conv}(\varphi(K(f))),n} &= \sup_{z\in\mathbb{C}} \left| \frac{1}{2\pi i} \int_\Gamma \mathcal{B}f(\xi) e^{\varphi(\xi)z}\, d\xi \right| e^{-H_{\operatorname{conv}(\varphi(K(f)))}(z) - \frac{1}{n}|z|} \\ &\leq \frac{\operatorname{len}(\Gamma)}{2\pi} \sup_{\xi\in|\Gamma|} |\mathcal{B}f(\xi)|\, e^{H_{conv(\varphi(|\Gamma|))}(z)} e^{-H_{\operatorname{conv}(\varphi(K(f)))}(z) - \frac{1}{n}|z|} \qquad (5) \\ &\leq \frac{\operatorname{len}(\Gamma)}{2\pi} \sup_{\xi\in|\Gamma|} |\mathcal{B}f(\xi)| < \infty. \end{aligned}$$

As n was arbitrary, this yields that $K(\Phi_\varphi f)$ is contained in $\operatorname{conv}(\varphi(K(f)))$, which in particular implies that $\Phi_\varphi f$ is of exponential type and $\Phi f \in Exp(\operatorname{conv}(\varphi(K)))$.

We proceed with the second assertion. Taking into account that for some $C < \infty$ and $m \in \mathbb{N}$ we have $\sup_{\xi\in|\Gamma|} |\mathcal{B}f(\xi)| \leq C\, ||f||_{K,m}$ due to the fact that $\mathcal{B} : Exp(K) \to H(\mathbb{C}_\infty \setminus K)$ is an isomorphism, the continuity of Φ_φ follows from (5) when $K(f)$ is replaced by K. It remains to show that $\Phi_\varphi(Exp(K))$ is dense in $Exp(conv(\varphi(K)))$. Therefore, let $K_1, K_2, \ldots$ be a sequence of compact, convex sets in Ω_φ such that $K_n^\circ \supset K_{n+1}$ and the intersection of all these sets is equal to K. As already noted above, the Borel transform of e_α is given by $\xi \mapsto 1/(\xi - \alpha)$. Inserting this in (4), the Cauchy integral formula yields $\Phi_\varphi(e_\alpha) = e_{\varphi}(\alpha)$ for all α in some K_n. Consequently, for arbitrary $n \in \mathbb{N}$

$$\Phi_\varphi(span\{e_\alpha : \alpha \in K_n\}) = span\{e_\varphi(\alpha) : \alpha \in K_n\} \subset Exp(conv(\varphi(K_n)))$$

which implies that $\Phi_\varphi : Exp(K_n) \to Exp(conv(\varphi(K_n))$ has dense range according to Proposition (1.1.3)(ii) and the fact that φ is non-constant. Since $Exp(K)$ is dense in $Exp(K_n)$, we obtain that $\Phi_\varphi(Exp(K))$ lies densely in $Exp(conv(\varphi(K_n)))$. Furthermore, we have

$$\bigcap_{n\in\mathbb{N}} conv(\varphi(K_n)) = conv(\varphi(K))$$

and hence

$$\bigcap_{n\in\mathbb{N}} Exp(conv(\varphi(K_n))) = Exp(conv(\varphi(K)))$$

in algebraic and topological sense by Lemma (1.1.10). It is now obvious that $\Phi_\varphi(Exp(K))$ is dense in $Exp(conv(\varphi(K)))$.

In the formulation of Theorem (1.1.11), it is necessary to form the convex hull in the image space $Exp(conv(\varphi(K)))$, since $Exp(K)$ is only defined for convex sets K. However, we show that the Borel transform of $\Phi_\varphi f$ actually admits an analytic continuation beyond $\mathbb{C}_\infty \backslash conv(\varphi(K))$. For that purpose, we have to introduce a further notation: For a compact set $K \subset \mathbb{C}$, the polynomially convex hull K is defined as the union of K with the bounded components of its complement. Let $K \subset \mathbb{C}$ be a compact, convex set,
$f \in Exp(K)$ and $\varphi \in H(K)$. For $w \in \mathbb{C}\backslash\widehat{\varphi(K)}$ we set

$$H\varphi(w) := \frac{1}{2\pi i}\int_\Gamma \frac{\mathcal{B}f(\xi)}{w - \varphi(\xi)}\, d\xi$$

With Γ a Cauchy cycle for $K \in \Omega_\varphi$ being so near to K that $\varphi(|\Gamma|)$ is contained in a simply connected, compact set $L \supset \widehat{\varphi(K)}$ such that $w \in \mathbb{C} \backslash L$. This definition is independent of the particular choice of Γ. Since $\varphi(|\Gamma|)$ can be arbitrarily near to $\varphi(K)$, we obtain a function $H_\varphi| \in H(\mathbb{C}_\infty \backslash \widehat{\varphi(K)})$.

Proposition (1.1.12)[1]: The function $H_\varphi \in H(\mathbb{C}_\infty \backslash \widehat{\varphi(K)})$ defines an analytic continuation of $\mathcal{B}(\Phi_\varphi) \in H(\mathbb{C}_\infty \setminus conv(\varphi(K)))$.

Proof: Let Γ_0 be a Cauchy cycle for $conv(\varphi(K))$ in $\mathbb{C}$. Then we can chose a Cauchy cycle for K in Ω_φ being so near to K that $\operatorname{ind}_{\Gamma_0}(\varphi(u)) = 1$ for all $u \in |\Gamma|$. Then

$$\frac{1}{2\pi i}\int_{\Gamma_0} H_\varphi(w) e^{wz}\, dw = \frac{1}{2\pi i}\int_\Gamma \mathcal{B}f(\xi) \frac{1}{2\pi i}\int_{\Gamma_0} \frac{e^{wz}}{w - \varphi(\xi)}\, dw\, d\xi$$

$$= \frac{1}{2\pi i}\int_\Gamma \mathcal{B}f(\xi) e^{\varphi(\xi) z}\, d\xi = \Phi_\varphi f(z)$$

by the Cauchy integral formula. Considering that $\mathcal{B}_{conv(\varphi(K))}$ is an isomorphism, we can conclude $H_\varphi|_{\mathbb{C}_\infty \backslash conv(\varphi(K))} = \mathcal{B}(\Phi_\varphi)|_{\mathbb{C}_\infty \backslash conv(\varphi(K))}$.

Now, let f be an entire function of exponential type and $\varphi \in H(K(f))$. Interchanging integration and differentiation yields

$$(\Phi_\varphi f)^{(n)}(z) = \frac{1}{2\pi i}\int_\Gamma \mathcal{B}f(\xi)\varphi^n(\xi) e^{\varphi(\xi) z}\, d\xi \tag{6}$$

which implies that the Taylor expansion of $\Phi_\varphi f$ at the origin is given by

$$\Phi_\varphi f\,(z) = \sum_{n=0}^{\infty} \frac{\varphi(D)^n f(0)}{n!} z^n. \tag{7}$$

Further, in accordance with our conventions, if $\varphi \in H(K(f))$ is zero-free, Φ_φ is a simply connected domain that contains no zeros of φ. These conditions ensure the existence of a logarithm function $\log \varphi \in H(\Phi_\varphi)$ for φ. Then for each non-negative integer n, we have

$$e_1(D)^n \Phi_{\log \varphi} f(0) = \Phi_{\log\varphi} f\,(n) = \frac{1}{2\pi i}\int_\Gamma \mathcal{B}f(\xi) e^{n \log \varphi(\xi)}\, d\xi$$

$$= \frac{1}{2\pi i}\int_\Gamma \mathcal{B}f(\xi)\varphi^n(\xi) d\xi = \varphi(D)^n f\,(0). \tag{8}$$

We extend (6) and (8) by showing that Φ_φ commutes with differential operators on $Exp(K)$. For that purpose, we have to introduce another terminology: A germ $\varphi \in H(K)$ is said to be biholomorphic, if Ω_φ can be choosen so that $\varphi : \Omega_\varphi \to \varphi(\Omega_\varphi)$ is biholomorphic. In this case, we always assume Ω_φ to be so small that the above property is ensured.

Proposition (1.1.13)[1]: Let K be a compact, convex subset of $\mathbb{C}$ and let $\varphi \in H(K)$.
(i) $D: Exp(conv(\varphi(K))) \to Exp(conv(\varphi(K)))$ is quasi conjugated to $\varphi(D): Exp(K) \to Exp(K)$ by Φ_φ;
(ii) If φ is zero-free then $e_1(D): Exp(conv(\log\varphi(K))) \to Exp(conv(\log\varphi(K)))$ is quasi conjugated to $\varphi(D)$: $Exp(K) \to Exp(K)$ by $\Phi_{\log\varphi}$;
(iii) If C is a compact, convex subset of $\mathbb{C}$ and $\psi \in H(C)$ is biholomorphic and satisfies $\psi(C) \supset \varphi(K)$ then $\psi(D): Exp(conv(\psi^{-1}\circ\varphi(K))) \to Exp(conv(\psi^{-1}\circ\varphi(K)))$ is quasi conjugated to $\varphi(D)$: $Exp(K) \to Exp(K) by\ \Phi_{\psi^{-1}\circ\varphi}$.

Proof: Let $f \in Exp(K)$. In order to see (i), consider the Taylor expansion for $\Phi_\varphi f$ in (7) and observe that

$$D\left(\sum_{n=0}^{\infty}\frac{\varphi(D)^n f(0)}{n!}z^n\right) = \sum_{n=0}^{\infty}\frac{\varphi(D)^{n+1}f(0)}{n!}z^n$$

Considering Lemma (1.1.5), we obtain

$$e_1(D)^n\Phi_{\log\varphi}f(z) = \frac{1}{2\pi i}\int_\Gamma \mathcal{B}f(\xi)e^{(z+n)\log\varphi(\xi)}d\xi = \frac{1}{2\pi i}\int_\Gamma \mathcal{B}f(\xi)\varphi^n(\xi)e^{z\log\varphi(\xi)}d\xi$$
$$= \frac{1}{2\pi i}\int_\Gamma \mathcal{B}\varphi(D)^n f(\xi)e^{z\log\varphi(\xi)}d\xi = \Phi_{\log\varphi}\varphi(D)^n f(z).$$

With $n = 1$, this is the assertion in (ii).

In order to show (iii), we consider an arbitrary $z \in \mathbb{C}\setminus C$ and choose a Cauchy cycle Γ_1 for K in Ω_φ such that $\varphi(|\Gamma_1|) \subset \Omega_{\psi^{-1}}$ and $\psi^{-1}\circ\varphi(|\Gamma_1|)$ is contained in some compact set $L \subset \Omega_\psi$ with $z \in \mathbb{C}\setminus L$. Further, let Γ_2 be a Cauchy cycle for L in Ω_ψ. Then, according to Proposition (1.1.12) see [8], we have

$$\psi(D)(\Phi_{\psi^{-1}\circ\varphi}f)(z) = \frac{1}{2\pi i}\int_{\Gamma_2}\mathcal{B}(\Phi_{\psi^{-1}\circ\varphi}f)(w)e^{wz}dw$$
$$= \frac{1}{2\pi i}\int_{\Gamma_2}\frac{1}{2\pi i}\int_{\Gamma_1}\frac{\mathcal{B}f(\xi)}{w-\psi^{-1}\circ\varphi(\xi)}d\xi\, e^{wz}\,dw$$
$$= \frac{1}{2\pi i}\int_{\Gamma_2}\mathcal{B}f(\xi)\frac{1}{2\pi i}\int_{\Gamma_1}\frac{\psi(w)e^{wz}}{w-\psi^{-1}\circ\varphi(\xi)}\,dw\,d\xi$$
$$= \frac{1}{2\pi i}\int_{\Gamma_1}\mathcal{B}f(\xi)\varphi(\xi)e^{\psi^{-1}\circ\varphi(\xi)z}\,d\xi = \Phi_{\psi^{-1}\circ\varphi}(\varphi(D)f)(z).$$

Theorem (1.1.14)[1]: (Duyos-Ruiz - Chan and Shapiro). For every admissible comparison function a and every $\alpha \in \mathbb{C}\setminus\{0\}$, there is an $f \in \mathcal{HC}(e_1(D), Exp(\{0\}))$ such that $M_f(r) = O(a(r))$.

By means of the transfrom Φ_φ, we show that this result extends to the operators $\varphi(D)$ as follows:

Lemma (1.1.15)[1]: Let $K \subset \mathbb{C}$ be a compact, convex set, $\varphi \in H(K)$ and $\alpha \in \mathbb{C}$. Then for every $f \in Exp(K)$, we have $\varphi(D)f = e_\alpha\,\varphi_\alpha(D)(f/e_\alpha)$ where $\varphi_\alpha := \varphi(\cdot+\alpha)$.

Proof: For $\lambda \in K$, we have $\varphi(D)e_\lambda = \varphi(\lambda)e_\lambda$ and hence

$$\varphi(D)e_\lambda = e_\alpha\varphi(\lambda)e_{\lambda-\alpha} = e_\alpha\varphi(\lambda-\alpha+\alpha)e_{\lambda-\alpha},$$

which shows the assertion for $f = e_\lambda, \lambda \in K$. Since φ is holomorphic in a neighbourhood of K, we can assume that K has non-empty interior. Then $\mathrm{span}\{e_\lambda : \lambda \in K\}$ is dense in

$Exp(K)$ by Proposition (1.1.3) (ii). Further, as outlined in the proof of Theorem (1.1.3), $f \longmapsto f/e_\alpha$ is an isometric isomorphism from $Exp(K)$ to $Exp(K-\{\alpha\})$ and we can conclude that the above equality extends to all $f \in Exp(K)$.

Theorem (1.1.16)[1]: Let $K \subset \mathbb{C}$ be a compact, convex set and $\varphi \in H(K)$ non-constant. Then, for every $\alpha \in K$ such that $|\varphi(\alpha)| = 1, \varphi'(\alpha) \neq 0$ and every admissible comparison function a, there is some $f_0 \in Exp(\{0\})$ that satisfies $M_{f_0}(r) = O(a(r))$ and such that $f = f_0 e_\alpha \in \mathcal{HC}(\varphi(D), Exp(K))$.

Proof: Let $a(z) = \sum_{n=0}^{\infty} a_n z^n$ be an admissible comparison function.

Without loss of generality $a \in Exp(\{0\})$. Due to Lemma (1.1.15) we can assume that $\alpha = 0$ and thus we only have to show the existence of some $f \in \mathcal{HC}(\varphi(D), Exp(\{0\}))$ with $M_f(r) = O(a(r)), r > 0$. We define $b(z) := \sum_{n=0}^{\infty} b_n z^n$ with $b_n := a_n/n!$ which is again an admissible comparison function. Now, as outlined above, the results in [8] yield a function $g \in E^2(b) \cap \mathcal{HC}(e_1(D), Exp(\{0\}))$. By the definition of E^2 and (b_n),

$$\sum_{n=0}^{\infty} \frac{|g^{(n)}(0)|^2}{(n!\, b_n)^2} = \sum_{n=0}^{\infty} \frac{|g^{(n)}(0)|^2}{a_n^2} < \infty.$$

This implies that $G(z) := \sum_{n=0}^{\infty} |g^{(n)}(0)|\, z^n \in E^2(a)$ and hence, as again outlined above, $M_G(r) = O(a(r))$.

According to the condition $\varphi'(0) \neq 0$, we have that φ is biholomorphic as an element of $H(\{0\})$. We can assume that $\varphi(0) = 1$, otherwise, replace e_1 by $\varphi(0)e_1$ in what follows and notice that $g \in \mathcal{HC}(\varphi(0)e_1(D), Exp(\{0\}))$ (see [18]). Then $f := \Phi_{\varphi^{-1}\circ e_1}\, g \in \mathcal{HC}(\varphi(D), Exp(\{0\}))$ due to Proposition (1.1.9) and Proposition (1.1.13). We find some small $\delta > 0$ and $0 < c < \infty$ such that $|\varphi^{-1}(e_1(\xi))| \leq c|\xi|$ for all $|\xi| < \delta$. We fix an $r > 0$ with $1/r \leq \delta$ and such that for $\Gamma_r : [0, 2\pi) \to \mathbb{C}, t \longmapsto r^{-1}e^{it}$ we have $e^1(|\Gamma_r|) \subset \Omega_{\phi^{-1}}$. Now, considering that $Bg(\xi) = \sum_{n=0}^{\infty} g^{(n)}(0)/\xi^{n+1}$ on every compact subset of $\mathbb{C}\backslash\{0\}$, we have

$$\begin{aligned} M_f(r) &\leq \max_{|z|=r} \left| \frac{1}{2\pi i} \int_{\Gamma_r} \mathcal{B}g(\xi) e^{(\varphi^{-1}\circ e_1)(\xi) z}\, d\xi \right| \\ &\leq \max_{|z|=r} \sum_{n=0}^{\infty} |g^{(n)}(0)| \left| \frac{1}{2\pi i} \int_{\Gamma_r} \frac{e^{(\varphi^{-1}\circ e_1)(\xi) z}}{\xi^{n+1}}\, d\xi \right| \\ &\leq \sum_{n=0}^{\infty} |g^{(n)}(0)|\, r^n e^{\frac{c}{r}r} \\ &= e^c\, G(r). \end{aligned}$$

Thus, $M_f(r) = O(M_G(r)) = O(a(r))$ and this completes the proof.

Theorem (1.1.17)[1]: Let φ be a non-constant entire function of exponential type and let $\alpha \in C_\varphi$ be so that $\varphi'(\alpha) \neq 0$. Then for every $q : [0, \infty) \to [1, \infty)$ such that $q(r) \to \infty$ as r tends to infinity, there is an entire function f_0 with $M_{f_0}(r) = O(r^{q(r)})$ and so that $f_0 e_\alpha \in \mathcal{HC}(\varphi(D), H(\mathbb{C}))$.

The above results fail to hold in the case of frequent hypercyclicity. Here, some expansion of the conjugate indicator diagram is required.

Proof: According to the proof of Theorem (1.1.8), we have the inclusion $\mathcal{HC}(\varphi(D), Exp(K)) \subset \mathcal{HC}(\varphi(D), H(\mathbb{C}))$ provided that $\varphi(D)$ extends to a continuous operator on $H(\mathbb{C})$. Now, the assertion of Theorem (1.1.17) follows from the observation that for each $q(r) : [0,\infty) \to [1,\infty)$ with $q(r) \to \infty$ as $r \to \infty$, there exists an admissible comparison function a such that $a(r) = O(r^{q(r)})$ and the application of Theorem (1.1.16).

We apply Φ_φ to extend known results for frequently hypercyclic functions for $e_1(D)$ to the whole class of differential operator $\varphi(D)$ on $Exp(K)$ as well as on $H(\mathbb{C})$.
[2], proves the following

Theorem (1.1.18)[1]: If $K \subset \mathbb{C}$ is a compact, convex set that contains two distinct points of the imaginary axis, then $\mathcal{FHC}(e_1(D), Exp(K)) \neq \emptyset$.

We can conclude that it is sufficient to require that $e_1(K) \cap \mathbb{T}$ contains a continuum in order to have $\mathcal{FHC}(e_1(D), Exp(K)) \neq \emptyset$. Similarily, this result holds in the general situation:

Theorem (1.1.19)[1]: Let $K \subset \mathbb{C}$ be a compact, convex set and $\varphi \in H(K)$ non-constant such that $\varphi(K) \cap \mathbb{T}$ contains a continuum. Then we have $\mathcal{FHC}(\varphi(D), Exp(K))) \neq \emptyset$.

Proof: Our assumptions ensure the existence of a compact, convex set $\widetilde{K} \subset K$ such that $\varphi(\widetilde{K})$ contains some continuum of $\mathbb{T}$ and φ is biholomorphic as an element of $H(\widetilde{K})$. We choose suitable real numbers $a < b$ so that $e^{[ia,ib]} \subset \varphi(\widetilde{K})$. The preceding result yields an $f \in \mathcal{FHC}(e_1(D), Exp([ia, ib]))$, and, by Proposition (1.1.9) and Proposition (1.1.13), we have

$$\Phi_{\varphi^{-1}\circ e_1} f \in \mathcal{FHC}(\varphi(D), Exp(\widetilde{K})) \subset \mathcal{FHC}(\varphi(D), Exp(K)).$$

The next result shows that the assumption in Theorem (1.1.19) are sharp.

Theorem (1.1.20)[1]: Let λ be a complex number and $\varphi \in H(\{\lambda\})$. Then we have $\mathcal{FHC}(\varphi(D), Exp(\{\lambda\})) = \emptyset$.

Proof: If there exists some $f \in \mathcal{FHC}(\varphi(D), Exp(\{\lambda\})$, then, by Proposition (1.1.9) and Proposition (1.1.13),

$$\Phi_\varphi f \in \mathcal{FHC}(D, Exp(\{\varphi(\lambda)\})) \subset \mathcal{FHC}(D, H(\mathbb{C})),$$

contradicting Theorem (1.1.21)(ii).

Theorem (1.1.21)(ii) is stronger than the previous result since it excludes frequent hypercyclicity with respect to the weaker topology of $H(\mathbb{C})$. Unfortunately, the transform Φ_φ does not carry over (frequent) hypercyclicity with respect to this topology. Thus, some extra argument is required to show Theorem (1.1.21)(ii).

Theorem (1.1.21)[1]: Let φ be a non-constant entire function of exponential type.

(i) If $K \subset \mathbb{C}$ is a compact and convex set such that the intersection of K and C_φ contains a continuum, then there is a function $f \in \mathcal{FHC}(\varphi(D), H(\mathbb{C}))$ that is of exponential type and so that $K(f) \subset K$.

(ii) There is no function $f \in \mathcal{FHC}(\varphi(D), H(\mathbb{C}))$ that is of exponential type and so that $K(f)$ is a singleton.

In particular, the second part of the above result states that, in contrast to the case of hypercyclicity, a function f of exponential type zero is never frequently hypercyclic for any differential operator $\varphi(D)$ (on $H(\mathbb{C})$).

Proof: The first part is an immediate consequence of Theorem (1.1.19) since $\mathcal{FHC}(\varphi(D), Exp(K)) \subset \mathcal{FHC}(\varphi(D), H(\mathbb{C}))$. Thus, there is only (ii) left to prove.

We suppose there is some entire function f of exponential type such that $K(f) = \{\lambda\}, \lambda \in \mathbb{C}$ and $f \in \mathcal{FHC}(\varphi(D), H(\mathbb{C}))$. Then necessarily, $|\varphi(\lambda)| \geq 1$ because otherwise $\varphi(D)^n f(0) \to 0$ as $n \to \infty$ as it turns out from the proof of Theorem (1.1.7), and φ is non-constant. Hence in some sufficiently small and simply connected, open neighbourhood Ω of λ, the function $\tilde{\varphi} := \varphi/\varphi(\lambda)$ is zero-free, which implies the existence of a logarithm function $\log\tilde{\varphi}$ for $\tilde{\varphi}$ on Ω with $\log\tilde{\varphi}\,(\lambda) = 0$. We set $h := \Phi_{\log\tilde{\varphi}} f$. Then $K(h) = \{0\}$ by Proposition (1.1.11) and, according to (8) applied to $\tilde{\varphi}$, we have

$$h(n) = \frac{1}{\varphi(\lambda)}\varphi(D)^n f(0) \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{9}$$

Let S be the sector $\{z : |arg(z)| \le \frac{\pi}{5}\} \setminus \{0\}$. By the Casorati-Weierstrass theorem, we can choose $\alpha \in \mathbb{C}$ such that $\varphi(\alpha)$ is close enough to $\pi\varphi(\lambda)$ to ensure that

$$\frac{\varphi(\alpha)}{\varphi(\lambda)} S \subset \left\{z : |arg(z) - \pi| \le \frac{\pi}{4}\right\} \tag{10}$$

and $\varphi(\alpha) \neq 0$. Now, according to the continuity of $\varphi(D)$ on $H(\mathbb{C})$, for every $\varepsilon > 0$, there are some $r > 0$ and $\delta > 0$ such that for all $g \in H(\mathbb{C})$ that satisfy

$$\sup_{z \in r\overline{\mathbb{D}}} |g(z) - e_\alpha(z)| < \delta, \tag{11}$$

we have

$$|\varphi(D)g(0) - \varphi(D)e_\alpha(0)| = |\varphi(D)g(0) - \varphi(\alpha)| < \varepsilon.$$

We assume that $\delta, \varepsilon > 0$ are so small that, whenever g satisfies (11), we have

$$g(0) \in S \text{ and } \varphi(D)g(0) \in \varphi(\alpha)\, S. \tag{12}$$

Our assumption implies the existence of some sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers with dens$((n_k)_{k\in\mathbb{N}} > 0$ and such that $\sup_{z \in r\overline{\mathbb{D}}} |\varphi(D)^{n_k} f(z) - e_\alpha(z)| < \delta$ for all $k \in \mathbb{N}$. The interpolating property of h in (9) combined with (12) yields

$$h(n_k) \in \frac{1}{\varphi(\lambda)^{n_k}} S \text{ and } h(n_k + 1) \in \frac{\varphi(\alpha)}{\varphi(\lambda)^{n_k+1}} S \text{ for all } k \in \mathbb{N}. \tag{13}$$

Condition (10) implies that the factor $\frac{\varphi(\alpha)}{\varphi(\lambda)}$ rotates S by an angle larger than $\frac{\pi}{2}$. Hence, from (13), it follows that for each $k \in \mathbb{N}$ either $Re(h)$ or $Im(h)$ has a sign change in $[n_k, n_k + 1]$. The intermediate value theorem yields a sequence $(w_k)_{k\in\mathbb{N}}$ of positive numbers with $w_k \in (n_k, n_k + 1)$ and

$$Re(h(w_k))\, Im(h(w_k)) = 0 \text{ for all } k \in \mathbb{N}. \tag{14}$$

Assuming that the Taylor series of h is given by $\sum_{v=0}^{\infty} \frac{h_v}{v!} z^v$, we set $h_1(z) := \sum_{v=0}^{\infty} \frac{\mathrm{Re}(h_v)}{v!} z^v$ and $h_2(z) := \sum_{v=0}^{\infty} \frac{\mathrm{Im}(h_v)}{v!} z^v$. The functions h_1, h_2 are of exponential type zero due to the of fact that h is exponential type and thus $h_1 h_2$ is a function of exponential type zero.
Since $\mathrm{Re}(h(x)) = h_1(x)$ and $\mathrm{Im}(h(x)) = h_2(x)$ for every real x, we obtain $h_1 h_2(w_k) = 0$ for all $k \in \mathbb{N}$ by (14). Taking into account that $(w_k)_{k\in\mathbb{N}}$ has obviously the same lower density as $(n_k)_{k\in\mathbb{N}}$, we have that $h_1 h_2$ is a function of exponential type zero having zeros of positive lower density which is impossible unless it is constantly zero (cf. [6]).

Section (1.2): Conjugate Class of a Hypercyclic Operator

Let X be a separable, infinite dimensional Banach space over the scalar field $\mathbb{C}$ or $\mathbb{R}$, and let $B(X)$ denote the algebra of bounded linear operators $T : X \to X$. An operator T in $B(X)$ is hypercyclic if there is a vector x in X for which its orbit, $Orb(T, x) = \{T^n x : n \ge 0\}$, is dense in X. Such a vector x is called a hypercyclic vector for . An operator T in $B(X)$

is hypercyclic if and only if the set of hypercyclic vectors for T , denoted by $\mathcal{HC}(T)$, is a dense G_δ set; see Kitai [42]. For a countable family $\mathcal{F}$ of hypercyclic operators, a direct application of the Baire Category Theorem implies that the set $\bigcap_{T\in\mathcal{F}} \mathcal{HC}(T)$ of common hypercyclic vectors is also a dense G_δ set. However, for the situation when $\mathcal{F}$ is an uncountable family of hypercyclic operators, we cannot apply this Baire Category Theorem argument to show the set $\bigcap_{T\in\mathcal{F}} \mathcal{HC}(T)$ of common hypercyclic vectors is a dense G_δ set, or is even nonempty. This observation has prompted research on the existence of common hypercyclic vectors for uncountable families of hypercyclic operators. Bayart and Matheron [24], Chan and Sanders [32], and Costakis and Sambarino [38] have separately developed different sufficient conditions for an uncountable family of operators to have a dense G_δ set of common hypercyclic vectors. Other results on common hypercyclic vectors include the work of Abakumov and Gordon [20], Aron, Bès, León, and Peris [22], Bayart [23], Bayart and Grivaux [26], Conejero, Müller, and Peris [37], and León and Müller [43]. In much of the above work on common hypercyclic vectors, the uncountable family of operators maintains some sort of continuity within the family. This brings us to the definition of a path of operators. A family of operators $\{F_t \in B(X): t \in I\}$, where I is an interval of real numbers, is a path of operators if the map $F : I \longrightarrow (B(X), \|\cdot\|)$, defined by $F(t) = F_t$, is continuous with respect to the usual topology on the interval I and the operator norm topology on $B(X)$. If the interval $I = [a,b]$, then the path $\{F_t \in B(X): t \in I\}$ is a path of operators between F_a and F_b. For any path, a vector x in X is called a common hypercyclic vector for the path if $x \in \bigcap_{t\in I} \mathcal{HC}(F_t)$.

We examine common hypercyclic vectors for a family of operators which consists of the conjugates of a single hypercyclic operator. Let $\mathcal{S}(T) = \{L^{-1}T L: L \text{ invertible}\}$ be the conjugate set of the operator.

The conjugate set $\mathcal{S}(T)$ is also often referred to as the similarity orbit of T. A standard similarity argument shows that an operator T in $B(X)$ is hypercyclic if and only if each operator in the conjugate set $\mathcal{S}(T)$ is hypercyclic. From this observation, one can ask whether the set $\bigcap_{A\in\mathcal{S}(T)} \mathcal{HC}(A)$ of common hypercyclic vectors for the entire conjugate set $\mathcal{S}(T)$ of a hypercyclic operator T is a dense G_δ set. In Proposition (1.2.1) below, we show this set of common hypercyclic vectors has only two possibilities. If every nonzero vector in X is a hypercyclic vector for T, then the set $\bigcap_{A\in\mathcal{S}(T)} \mathcal{HC}(A)$ of common hypercyclic vectors for the conjugate set $\mathcal{S}(T)$ contains every nonzero vector also. Otherwise, the set $\bigcap_{A\in\mathcal{S}(T)} \mathcal{HC}(A)$ of common hypercyclic vectors for the conjugate set $\mathcal{S}(T)$ is empty.

Not only does the conjugate set $\mathcal{S}(T)$ of a hypercyclic operator T consist entirely of hypercyclic operators, those hypercyclic operators are dense in $B(X)$ with respect to the strong operator topology, or SOT. This result was proved by Bès and Chan [28] by applying a fundamental property of the strong operator topology established by Hadwin, Nordgren, Radjavi, and Rosenthal [41]. As we have mentioned above, if $\mathcal{HC}(T) \neq X \setminus \{0\}$, then the set $\bigcap_{A\in\mathcal{S}(T)} \mathcal{HC}(A)$ of common hypercyclic vectors for the conjugate set must be empty. We show the conjugate set $\mathcal{S}(T)$ must contain a path $\{F_t \in B(X): t \in [1,\infty)\}$ of operators which is SOT-dense in $B(X)$, and yet the set $\bigcap_{t\in[1,\infty)} \mathcal{HC}(F_t)$ of common hypercyclic vectors for the whole path is a dense G_δ set; see Theorem (1.2.4) below. As a corollary, we show the hypercyclic operators in $B(X)$ form an SOT-connected subset of $B(X)$; see Corollary (1.2.10) below. Also using Theorem (1.2.4), we show that for any nonzero vector g in X, the set $\{T \in B(X): g \in \mathcal{HC}(T)\}$ is SOT-dense, as well as SOT-connected in $B(X)$; see Corollary (1.2.9) below.

A hypercyclic operator clearly has orbits which exhibit wild behavior. It may also possess orbits with simple behavior. A vector x in X is a periodic point of the operator T if $T^n x = x$ for some positive integer n. An operator T in $B(X)$ is called chaotic if it is hypercyclic and the set of periodic points for T is dense in X. Recently, Chan and Sanders [34] showed that every separable, infinite dimensional Hilbert space H over the scalar field $\mathbb{C}$ admits a path of chaotic operators which is SOT-dense in $B(H)$, and yet each operator along the path shares the exact same set of hypercyclic vectors. However, Bonet, Martínez-Giménez, and Peris [30] provided examples of separable, infinite dimensional Banach spaces which fail to support even a single chaotic operator. Hence, the techniques in [34] do not work for an arbitrary separable, infinite dimensional Banach space. For this general setting, even though we are not able to show that there is an SOT-dense path of hypercyclic operators, each of which has the exact same set of hypercyclic vectors, Theorem (1.2.4) below exhibits such a path with a dense G_δ set of common hypercyclic vectors. In the case where the Banach space does support a chaotic operator, using Theorem (1.2.4), we show there does exist a path of chaotic operators which is SOT-dense in $B(X)$, and for which the set of common hypercyclic vectors for the whole path is a dense G_δ set. Furthermore, the chaotic operators in $B(X)$ form a connected subset of $B(X)$; see Corollary (1.2.11) below.

As stated, an operator T in $B(X)$ is hypercyclic if and only if every operator in the conjugate set $\mathcal{S}(T) = \{L^{-1}T L: L \text{ invertible}\}$ is hypercyclic. In fact, one can easily verify that

$$x \in \mathcal{HC}(L^{-1}T L) \quad \text{if and only if} \quad Lx \in \mathcal{HC}(T). \qquad (15)$$

We show that the set of common hypercyclic vectors for the conjugate set $\mathcal{S}(T)$ of an operator T has only two possibilities, either the set of all nonzero vectors or the empty set.

Proposition (1.2.1)[19]: Let T be an operator in $B(X)$.

(i) If $\mathcal{HC}(T) = X \setminus \{0\}$, then the set $\bigcap_{A\in\mathcal{S}(T)} \mathcal{HC}(A)$ of common hypercyclic vectors for the conjugate set $\mathcal{S}(T)$ is also $X \setminus \{0\}$.

(ii) If $\mathcal{HC}(T) \neq X \setminus \{0\}$, then the set $\bigcap_{A\in\mathcal{S}(T)} \mathcal{HC}(A)$ of common hypercyclic vectors for the conjugate set $\mathcal{S}(T)$ is empty.

Proof: Part (i) follows directly from the statement given in (15). For part (ii), let y be any nonzero vector in X which fails to be a hypercyclic vector for the operator . For any nonzero vector x in X, there exists an invertible operator L such that $Lx = y$. For instance, if x and y are linearly independent, we may take $Lx = y$ and $Ly = x$ and $L = I$ on a closed subspace complementary to the finite dimensional subspace spanned by x and y. If $y = \alpha x$ for some nonzero scalar α, then let $L = \alpha I$ on X. Since $Lx = y \notin \mathcal{HC}(T)$, by (15), we have $x \notin \mathcal{HC}(L^{-1}T L)$. Therefore, $\bigcap_{A\in\mathcal{S}(T)} \mathcal{HC}(A) = \emptyset$.

Read [45] provided an example of an operator T on ℓ^1 for which every nonzero vector is a hypercyclic vector. Thus, it is possible for the set of common hypercyclic vectors for a conjugate set to be nonempty. On the other hand, every separable, infinite dimensional Banach space X admits a hypercyclic operator T for which $\mathcal{HC}(T) \neq X \setminus\{0\}$; see the hypercyclic operator constructed by Ansari in [21] or by Bernal in [27]. Since the conjugate set $\mathcal{S}(T)$ of this particular hypercyclic operator fails to have a single hypercyclic vector in common, it follows trivially that the set of all hypercyclic operators in $B(X)$ fails to have a single hypercyclic vector in common.

The conjugate set $\mathcal{S}(T)$ of a hypercyclic operator T is SOT-dense in the operator algebra $B(X)$. However, in many cases, this SOT-dense set fails to have a single common hypercyclic vector. On the positive side, it does contain a path of operators which is SOT-dense in $B(X)$, and for which the set of common hypercyclic vectors for the whole path is a dense G_δ set. For this, we need two technical results.

Lemma (1.2.2)[19]: Let $x_1, x_2, \dots, x_k$ be k linearly independent vectors in X, and let

$$d = \min_{1\le j\le k} dist\big(x_j, span\{x_i : i \neq j\}\big) .$$

There exists $a\ \delta > 0$ such that whenever $y_1, y_2, \dots, y_k$ are k vectors in X satisfying $\|x_j - y_j\| < \delta$ for each integer j with $1 \le j \le k$, we have

$$\min_{1\le j\le k} dist\big(y_j, span\{y_i : i \neq j\}\big) \ge \frac{d}{2}.$$

Proof: Since all norms are equivalent on the finite dimensional space span$\{x_1, x_2, \dots, x_k\}$, there is a constant $C > 0$ such that

$$\sum_{i=1}^{k} |\alpha_i| \le C \left\| \sum_{i=1}^{k} \alpha_i x_i \right\| \tag{16}$$

for any scalars $\alpha_1, \alpha_2, \dots, \alpha_k$. Choose $a\ \delta > 0$ such that

$$(1 - C\delta) \ge \frac{1}{2} . \tag{17}$$

Let $y_1, y_2, \dots, y_k$ be any k vectors in X satisfying $\|x_j - y_j\| < \delta$ for $1 \le j \le k$. For any integer j with $1 \le j \le k$ and for any scalars $\alpha_1, \alpha_2, \dots, \alpha_{j-1}, \alpha_{j+1}, \dots, \alpha_k$, we have

$$\left\| y_j - \sum_{i\neq j} \alpha_i y_i \right\| = \left\| \left(x_j - \sum_{i\neq j} \alpha_i x_i \right) + (y_j - x_j) + \sum_{i\neq j} \alpha_i (x_i - y_i) \right\|$$

$$\ge \left\| x_j - \sum_{i\neq j} \alpha_i x_i \right\| - \|x_j - y_j\| - \sum_{i\neq j} |\alpha_i| \|x_i - y_i\|$$

$$> \left\| x_j - \sum_{i\neq j} \alpha_i x_i \right\| - \delta \left(1 + \sum_{i\neq j} |\alpha_i| \right) \ge \left\| x_j - \sum_{i\neq j} \alpha_i x_i \right\| - \delta C \left\| x_j - \sum_{i\neq j} \alpha_i x_i \right\| , \text{by (16)}$$

$$\ge (1 - C\delta) d \ge \frac{d}{2} , \text{by (17)}.$$

Thus, our result follows.

The second result involves the union of a finite linearly independent set with the tail end of an orbit generated by a hypercyclic vector.

Proposition (1.2.3)[19]: Let $T \in B(X)$ be a hypercyclic operator. If $g \in \mathcal{HC}(T)$ and $x_1, x_2, \dots, x_k$ are k linearly independent vectors in X, then there is an integer $N \ge 0$ such that the set $\{x_1, x_2, \dots, x_k\} \cup \{T^n g : n \ge N\}$ is linearly independent.

Proof: By way of contradiction, we suppose that no such integer N exists; that is, the set $\{x_1, x_2, \dots, x_k\} \cup \{T^n g : n \ge N\}$ is linearly dependent for each integer $N \ge 0$. If we take $N = 1$, then by the linear independence of the vectors $x_1, x_2, \dots, x_k$ and the linear independence of the orbit of a hypercyclic vector (see, for example Bourdon [31]), we obtain a nonzero polynomial p_1 for which $p_1(T)g \in span\{x_1, x_2, \dots, x_k\}$. For similar reasons, by taking $N = 1 + deg\, p_1$, we obtain a nonzero polynomial p_2 with $deg\, p_2 > deg\, p_1$ and $p_2(T)g \in span\{x_1, x_2, \dots, x_k\}$. After $k + 1$ steps, we obtain nonzero polynomial p_{k+1} with $deg\, p_{k+1} > deg\, p_k$ and $p_{k+1}(T)g \in span\{x_1, x_2, \dots, x_k\}$. Since the $k + 1$ vectors $p_1(T)g, p_2(T)g, \dots, p_{k+1}(T)g$ lie in the subspace $span\{x_1, x_2, \dots, x_k\}$, which is k-dimensional, it follows that they must be linearly dependent. However, this contradicts the fact that g is a hypercyclic vector.

We are now ready to prove that every conjugate set of a hypercyclic operator contains a path of operators which is SOT-dense in $B(X)$.

Theorem (1.2.4)[19]: Let T be a hypercyclic operator in $B(X)$. The conjugate set $\mathcal{S}(T) = \{L^{-1}T\,L: L \text{ invertible}\}$ contains a path $\{F_t \in B(X): t \in [1, \infty)\}$ of operators which is SOT-dense in $B(X)$, and for which the set $\bigcap_{t\in[1,\infty)} \mathcal{HC}(F_t)$ of common hypercyclic vectors for the whole path is a dense G_δ set.

Proof: We begin with an outline of the construction of the desired path of hypercyclic operators. The path must contain a hypercyclic operator in every nonempty SOT-basic open set $\mathcal{O}$ in $B(X)$, which is of the form

$$\mathcal{O} = \{A \in B(X): \|Ax_l - Bx_l\| < \epsilon \text{ for } 1 \le l \le k\},$$

where $B \in B(X), \epsilon > 0$, and $x_l \in X$. The vectors x_l and Bx_l provide a starting point of our construction of an invertible operator L so that $L^{-1}T\,L$ is in $\mathcal{O}$ and it can be joined to the given hypercyclic operator T with a path having a dense G_δ set of common hypercyclic vectors. For that, we may assume that the vectors x_l are linearly independent and use Proposition (1.2.3) to choose appropriate powers of T on a hypercyclic vector $g \in \mathcal{HC}(T)$ that can approximate x_l and Bx_l. Then we use Lemma (1.2.2) to control the norms of L and L^{-1} so that the terms $L^{-1}T\,L(x_l) - Bx_l$ qualify $L^{-1}TL$ to be in the set $\mathcal{O}$.

Furthermore, to create the desired path we first note that we can trivially write T as $I^{-1}T\,I$, where I is the identity, and so we have to join I with L with an appropriate path of invertible operators. The operator L takes the form of the sum of the identity and a finite rank operator K whose range is the linear span of carefully chosen powers $T^m\, g$. The path will then be in the form of $I + t\,K$, where t in $[0, 1]$ is the parameter for the path. However, in order to carefully select vectors $T^m\, g$ to make our argument work, we need to have good estimations on their distances from each other and separate them in terms of linear functionals.

To this end, let $g \in \mathcal{HC}(T)$. Let $\mathcal{E}$ be the collection of all sets E of the form

$$E = \{T^{m_1}\, g, T^{m_2}\, g, \dots, T^{m_{2k}}\, g, T^N\, g, T^{N+1}\, g, \dots, T^{N+2k-1}\, g\} \qquad (18)$$

where N, k are integers with $N \ge 0$ and $k \ge 1$ and $m_1, m_2, \dots, m_{2k}$ are distinct integers with each $m_j \ge N + 2k$. Note that the collection $\mathcal{E}$ is countable. For the set E given in (18) and for each integer j with $1 \le j \le 2k$, define

$$d_{j,E} = dist\left(T^{m_j}g, span\,(E \setminus \{T^{m_j}\, g\})\right) \text{ and } D_{j,E} = dist\left(T^{N+\,j-1}\, g, span\left(E \setminus \{T^{N+\,j-1}\, g\}\right)\right).$$

Then define Δ_E by

$$\Delta_E = min\left\{d_{1,E}\,, d_{2,E}\,, \dots, d_{2k,E}\,, D_1, E\,, D_{2,E}\,, \dots, D_{2k,E}\,\right\}.$$

Since the orbit of the hypercyclic vector g must be linearly independent, each set $E \in \mathcal{E}$ is linearly independent, and so $\Delta_E > 0$.

Claim (1.2.5)[19]: For the set $E \in \mathcal{E}$ given in (18), there are $2k$ linear functionals $\lambda_{1,E}, \lambda_{2,E}, \dots, \lambda_{2k,E}$ in the dual space X^* such that for any integers i,j with $1 \le i,j \le 2k$, we have $\|\lambda_{j,E}\| \le \frac{2}{\Delta_E}$ and

$$\lambda_{j,E}\,(T^{m_i}g) = \lambda_{j,E}\left(T^{N+i-1}g\right) = \begin{cases} 1, & if\ i = j, \\ 0, & if\ i \neq j. \end{cases}$$

Proof: As a corollary of the Hahn–Banach Theorem, there exist linear functionals $\varphi_{1,E}, \dots, \varphi_{2k,E}$ and $\psi_{1,E}, \dots, \psi_{2k,E}$ in the dual space X^* such that for integers i,j with $1 \le i,j \le 2k$, we have $\varphi_{j,E}(T^{m_j}g) = 1$ and $\varphi_{j,E}(x) = 0$ for all $x \in span(E \setminus \{T^{m_j}g\})$, and $\psi_{j,E}\,(T^{N+j-1}g) = 1$ and $\psi_{j,E}(x) = 0$ for all $x \in span(E \setminus \{T^{N+j-1}g\})$. Furthermore, $\|\varphi_{j,E}\| = \frac{1}{d_{j,E}} \le \frac{1}{\Delta_E}$ and $\|\psi_{j,E}\| = \frac{1}{D_{j,E}} \le \frac{1}{\Delta_E}$. Letting $\lambda_{j,E} = \varphi_{j,E} + \psi_{j,E}$ for each integer j with $1 \le j \le 2k$ completes the proof of Claim (1.2.5).

We now use Claim (1.2.5) to form a countable collection of invertible operators in $B(X)$. For the set $E \in \mathcal{E}$ given in (18), define the operator $L_E \colon X \longrightarrow X$ by

$$L_E(x) = x + \sum_{j=1}^{2k} \lambda_{j,E}(x)\left(T^{N+j-1}g - T^{m_j}g\right) . \tag{19}$$

To see that the operator L_E is invertible, define the operator $A_E : X \longrightarrow X$ by

$$A_E(x) = x + \sum_{i=1}^{2k} \lambda_{i,E}(x)\left(T^{m_i}g - T^{N+i-1}g\right) .$$

For any $x \in X$, observe that

$$L_E A_E\,(x) = L_E(x) + \sum_{i=1}^{2k} \lambda_{i,E}(x) L_E\left(T^{m_i}g - T^{N+i-1}g\right) .$$

By Claim (1.2.5), for any integers i,j with $1 \le i,j \le 2k$, we have $\lambda_{j,E}(T^{m_i}g - T^{N+i-1}g) = 0$, and so by (19),

$$L_E\left(T^{m_i}g - T^{N+i-1}g\right) = T^{m_i}g - T^{N+i-1}g.$$

Thus,

$$\begin{aligned} L_E A_E\,(x) &= L_E(x) + \sum_{i=1}^{2k} \lambda_{i,E}(x) L_E\, T^{m_i}g - T^{N+i-1}g \\ &= x + \sum_{j=1}^{2k} \lambda_{j,E}(x)\left(T^{N+j-1}g - T^{m_j}g\right) \\ &+ \sum_{i=1}^{2k} \lambda_{i,E}(x)\left(T^{m_i}g - T^{N+i-1}g\right) = x. \end{aligned}$$

Likewise, $A_E L_E(x) = x$ for any $x \in X$. Therefore, the operator L_E is invertible and $L_E^{-1} = A_E$. Moreover, by definitions of L_E , L_E^{-1} and by Claim (1.2.5), both operators L_E , L_E^{-1} satisfy the inequality

$$\|L_E^{-1}\|, \|L_E\| \le 1 + \sum_{j=1}^{2k} \|\lambda_{j,E}\| \|T^{N+j-1}g - T^{m_j}g\|$$
$$\le 1 + \frac{2}{\Delta_E}\sum_{j=1}^{2k} \|T^{N+j-1}g - T^{m_j}g\|. \qquad (20)$$
Using the countable collection $\{L_E : E \in \mathcal{E}\}$ of invertible operators, we generate a countable SOT-dense subset of $\mathcal{S}(T)$.

Claim (1.2.6)[19]: The countable collection $\{L_E^{-1}\, T\, L_E : E \in \mathcal{E}\}$ is SOT-dense in $B(X)$.

Proof: Let $\mathcal{U}$ be a nonempty SOT-open set in $B(X)$. Then there exists an operator $B \in B(X)$, an $\epsilon > 0$, and nonzero vectors $x_1, x_2, \dots, x_k$ in X such that
$$\{A \in B(X) : \|Ax_l - Bx_l\| < \epsilon \text{ for } 1 \le l \le k\} \subseteq \mathcal{U}.$$
Without loss of generality, we may assume the set $\{x_1, x_2, \dots, x_k\}$ is linearly independent. By Proposition (1.2.3), there is an integer $N \ge 0$ such that the set $\{x_1, x_2, \dots, x_k\} \cup \{T^n g : n \ge N\}$ is linearly independent. Since $g \in \mathcal{HC}(T)$, we can choose k distinct integers $m_2, m_4, \dots, m_{2k}$ satisfying
$$m_{2l} \ge N + 2k \quad \text{and} \quad \|T^{m_{2l}}g - Bx_l\| < \frac{\epsilon}{2} \quad \text{for } 1 \le l \le k. \qquad (21)$$
Consider the linearly independent set
$$\tilde{E} = \{x_1, T^{m_2} g, x_2, T^{m_4} g, \dots, x_k, T^{m_{2k}} g, T^N g, T^{N+1} g, \dots, T^{N+2k-1} g\},$$
and define
$$\tilde{d}_{2l-1} = dist\left(x_l, span\left(\tilde{E} \setminus \{x_l\}\right)\right) \text{ for } 1 \le l \le k,$$
$$\tilde{d}_{2l} = dist\left(T^{m_{2l}}g, span\left(\tilde{E} \setminus \{T^{m_{2l}}g\}\right)\right) \text{ for } 1 \le l \le k,$$
$$\widetilde{D}_j = dist\left(T^{N+j-1}g, span\left(\tilde{E} \setminus \{T^{N+j-1}g\}\right)\right) \text{ for } 1 \le j \le 2k,$$
$$\widetilde{\Delta} = min\{\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_{2k}, \widetilde{D}_1, \widetilde{D}_2, \dots, \widetilde{D}_{2k}\}.$$
Set
$$M = k + \sum_{l=1}^{k} \|T^{N+2l-1}g - T^{m_{2l}}g\| + \sum_{l=1}^{k} \|T^{N+2l-2}g - x_l\|. \qquad (22)$$
Since the set $\tilde{E}$ is linearly independent, by Lemma (1.2.2), there is a $\delta > 0$ such that whenever $m_1, m_3, \dots, m_{2k-1}$ are k distinct integers with
$$E = \{T^{m_1}g, T^{m_2}g, \dots, T^{m_{2k-1}}g, T^{m_{2k}}g, T^N g, \dots, T^{N+2k-1}g\} \in \mathcal{E}$$
And
$$\|T^{m_{2l-1}}g - x_l\| < \delta \quad \text{for } 1 \le l \le k, \qquad (23)$$
We get
$$\Delta_E \ge \frac{\widetilde{\Delta}}{2}. \qquad (24)$$
We may further assume that δ satisfies
$$\delta < min\left\{1, \frac{\epsilon}{2\|T\|\left(1 + \frac{4}{\Delta}M\right)^2}\right\}. \qquad (25)$$

Note that there exists such an $E \in \mathcal{E}$ because g is a hypercyclic vector for T. Moreover, for any such $E \in \mathcal{E}$ and for any integer l with $1 \le l \le k$, we have

$$\begin{aligned} &\|L_E^{-1} T\, L_E(x_l) - B x_l\| \\ &\qquad \le \|L_E^{-1}\, T\, L_E\,(x_l) - L_E^{-1} T\, L_E\, T^{m_{2l-1}} g\| \\ &\qquad + \|L_E^{-1}\, T\, L_E\, T^{m_{2l-1}} g \;-\; B x_l\| \;. \end{aligned} \tag{26}$$

To estimate the first summand on the right-hand side of (26), note that

$$\begin{aligned} \|L_E^{-1}\, T\, L_E(x_l) - L_E^{-1}\, T\, L_E(T^{m_{2l-1}} g)\| &\le \|L_E^{-1}\|\|T\|\|L_E\|\|x_l - T^{m_{2l-1}} g\| \\ &< \|L_E^{-1}\|\|T\|\|L_E\|\delta, \quad \text{by (23)} \\ &\le \left(1 + \frac{2}{\Delta_E} \sum_{j=1}^{2k} \left\|T^{N+j-1} g - T^{m_j} g\right\|\right)^2 \|T\|\delta, \quad \text{by (20)} \\ &\le \left(1 + \frac{4}{\tilde{\tilde{\Delta}}} \sum_{j=1}^{2k} \left\|T^{N+j-1} g - T^{m_j} g\right\|\right)^2 \|T\|\delta, \quad \text{by (24). (27)} \end{aligned}$$

We now estimate the above summation

$$\begin{aligned} \sum_{j=1}^{2k} \left\|T^{N+j-1} g - T^{m_j} g\right\| &= \sum_{l=1}^{k} \left\|T^{N+2l-1} g - T^{m_{2l}} g\right\| + \sum_{l=1}^{k} \left\|T^{N+2l-2} g - T^{m_{2l-1}} g\right\| \\ &\le \sum_{l=1}^{k} \left\|T^{N+2l-1} g - T^{m_{2l}} g\right\| + \sum_{l=1}^{k} \left\|T^{N+2l-2} g - x_l\right\| + \sum_{l=1}^{k} \|x_l - T^{m_{2l-1}} g\| \\ &< k + \sum_{l=1}^{k} \left\|T^{N+2l-1} g - T^{m_{2l}} g\right\| + \sum_{l=1}^{k} \left\|T^{N+2l-2} g - x_l\right\|, \text{by (23), (25)} \\ &= M, \quad \text{by (22)}. \end{aligned}$$

Combining inequality (27) with the above inequality gives us

$$\|L_E^{-1}\, T\, L_E(x_l) - L_E^{-1}\, T\, L_E\,(T^{m_{2l-1}} g)\| < \left(1 + \frac{4}{\Delta} M\right)^2 \|T\|\delta < \frac{\epsilon}{2}, \text{by (25)}. \tag{28}$$

To estimate the second summand on the right-hand side of (26), observe that for each integer i with $1 \le i \le 2k$, we have

$$\begin{aligned} L_E\,(T^{m_i}\, g) &= T^{m_i} g + \sum_{j=1}^{2k} \lambda_{j,E}(T^{m_i} g)\left(T^{N+j-1} g - T^{m_j} g\right), \text{ by (19)} \\ &= T^{m_i} g + \left(T^{N+i-1} g - T^{m_i} g\right), \text{by Claim} \\ &1 = T^{N+i-1} g. \end{aligned} \tag{29}$$

Thus,

$$\begin{aligned} \|L_E^{-1}\, T\, L_E\, T^{m_{2l-1}} g - B x_l\| &= \left\|L_E^{-1}\, T\, T^{N+2l-2} g - B x_l\right\|, \text{by (29)} \\ &= \left\|L_E^{-1}\, T^{N+2l-1} g - B x_l\right\| = \|T^{m_{2l}} g - B x_l\|, \text{by (29)} \\ &< \frac{\epsilon}{2}, \quad \text{by (21)}. \end{aligned} \tag{30}$$

Combining inequality (26) with inequalities (28) and (30) yields $\|L_E^{-1}\, T\, L_E(x_l) - B x_l\| < \epsilon$, and so $L_E^{-1}\, T\, L_E \in \mathcal{U}$ which completes the proof of Claim (1.2.6).

We now construct a path of operators between T and $L_E^{-1}\, T\, L_E$ which lies entirely within the conjugate set $\mathcal{S}(T)$ of the operator T.

Claim (1.2.7)[19]: For each $E \in \mathcal{E}$, there is a path of operators between T and $L_E^{-1}\, T\, L_E$ contained in the conjugate set $\mathcal{S}(T)$ for which the set of common hypercyclic vectors for the whole path is a dense G_δ set.

Proof: Let E be the set in $\mathcal{E}$ given in (18). For each $t \in [0,1]$, define an operator $L_{t,E} : X \longrightarrow X$ by

$$L_{t,E}(x) = x + \sum_{j=1}^{2k} t\lambda_{j,E}(x)\left(T^{N+j-1}g - T^{m_j}g\right) .$$

Using computations similar to those before Claim (1.2.5), each operator $L_{t,E}$ is invertible, and its inverse $L_t^{-1}, E : X \longrightarrow X$ is given by

$$L_{t,E}^{-1}(x) = x + \sum_{i=1}^{2k} t\lambda_{i,E}(x)\left(T^{m_i}g - T^{N+i-1}g\right) .$$

Consider the path of operators $\{G_t \in B(X) : t \in [0,1]\}$, where $G_t = L_{t,E}^{-1}\, T\, L_{t,E}$. Clearly this path of operators is between T and $L_E^{-1}\, T\, L_E$ and lies entirely in the conjugate set $\mathcal{S}(T)$. To show $\bigcap_{t\in[0,1]} \mathcal{HC}(G_t)$ is a dense G_δ set, first note that by Corollary (1.2.3) in [32], the set $\bigcap_{t\in[0,1]} \mathcal{HC}(G_t)$ is a G_δ set, and so it suffices to show this set is also dense. We do this by proving {span $Orb(T, g)$}\{0} is contained inside the set $\bigcap_{t\in[0,1]} \mathcal{HC}(G_t)$. To begin, note that given any nonzero polynomial p and any $t \in [0,1]$, we have $L_{t,E}p(T)g \neq 0$ because $p(T)g \neq 0$ by the linear independence of the orbit of g, and because the operator $L_{t,E}$ is invertible. Furthermore,

$$L_{t,E}p(T)g = p(T)g + \sum_{j=1}^{2k} t\lambda_{j,E}p(T)g\left(T^{N+j-1}g - T^{m_j}g\right) \in \{span\ Orb(T, g)\}\{0\} \subseteq \mathcal{HC}(T),$$

because every nonzero vector from the linear span of a dense orbit is a hypercyclic vector; see Bourdon [31] and Bès [29]. Therefore, by statement (15), we get $p(T)g \in \mathcal{HC}(L_{t,E}^{-1}T\, L_{t,E}) = \mathcal{HC}(G_t)$. Hence, $\{span\ Orb(T, g)\}\backslash\{0\} \subseteq \bigcap_{t\in[0,1]} \mathcal{HC}(G_t)$, and this concludes the proof of Claim (1.2.7).

We construct the desired SOT-dense path of operators in the conjugate set $\mathcal{S}(T)$. Let $\{E_n : n \geq 1\}$ be an enumeration of the countable set $\mathcal{E}$. By Claim (1.2.7), if for each integer $n \geq 1$, let $G_{t,n} = L_{2t,E_n}^{-1}\, T\, L_{2t,E_n}$ for $t \in [0, 1/2]$ and $G_{t,n} = L_{2-2t,E_n}^{-1}\, T\, L_{2-2t,E_n}$ for each $t \in [1/2, 1]$, then $\{G_{t,n} \in B(X) : t \in [0,1]\}$ is a path of operators in the conjugate set $\mathcal{S}(T)$ such that $G_{0,n} = G_{1,n} = T$ and $L_{E_n}^{-1}\, T\, L_{E_n} = G_{1/2,n} \in \{G_{t,n} \in B(X) : t \in [0,1]\}$, and in addition, the set $\bigcap_{t\in[0,1]} \mathcal{HC}(G_{t,n})$ is a dense G_δ set. For each $t \in [n, n+1]$, let $F_t = G_{t-n,n}$. Then $\{F_t \in B(X) : t \in [1,\infty)\}$ is a path of operators in the conjugate set $\mathcal{S}(T)$ which is SOT-dense by Claim (1.2.6), and for which the set $\bigcap_{t\in[1,\infty)} \mathcal{HC}(F_t) = \bigcap_{n=1}^{\infty} \bigcap_{t\in[0,1]} \mathcal{HC}(G_{t,n})$ is a dense G_δ set.

The SOT-dense path $\{F_t \in B(X) : t \in [1,\infty)\}$ in the previous theorem consists of operators of the form $L^{-1}T\, L$, which share many properties that each other has; in fact, any properties preserved by similarity. For instance, if one of them is chaotic, then every operator in the whole path is chaotic. If one of them has a nontrivial kernel, then every one in the whole path has one. If one of them is surjective, then every one is. If one of them has a nontrivial invariant subspace, then every one has one. If every nonzero vector is a hypercyclic vector for one single operator in the path, then the same holds true for every operator in the path. If one of them has a hypercyclic subspace, which is an infinite dimensional closed subspace consisting, except the zero vector, of hypercyclic vector of the operator, then every one in the path has such a subspace. We study more common properties that a path of operators may share.

Theorem (1.2.4) has several interesting corollaries. First, let us examine the linear structure within the dense G_δ set of common hypercyclic vectors for the path of operators given within the proof of Theorem (1.2.4). For each set $E \in \mathcal{E}$, consider the path of operators $\{G_t \in B(X): t \in [0,1]\}$ between the operators T and $L_E^{-1}\, T\, L_E$ given in the proof of Claim (1.2.7). To show the set $\bigcap_{t\in[0,1]} \mathcal{HC}(G_t)$ of common hypercyclic is dense, we prove that if g is a hypercyclic vector for , then $\{span\ Orb(T, g)\}\backslash\{0\}$ is contained inside the set $\bigcap_{t\in[0,1]} \mathcal{HC}(G_t)$. Furthermore, these paths of operators are the building blocks for the desired SOT-dense path of operators. Thus, the set of common hypercyclic vectors for the path of operators constructed within the proof of Theorem (1.2.4) contains some natural linear structure.

Corollary (1.2.8)[19]: Let T be a hypercyclic operator in $B(X)$, and let $g \in \mathcal{HC}(T)$. There exists a path $\{F_t \in B(X): t \in [1,\infty)\}$ of operators, contained entirely in the conjugate set $\mathcal{S}(T)$, which is SOT-dense in $B(X)$, and for which $\{span\ Orb(T, g)\}\backslash\{0\}$ is contained within the dense G_δ set $\bigcap_{t\in[1,\infty)} \mathcal{HC}(F_t)$ of common hypercyclic vectors.

Since the orbit of a hypercyclic vector is linearly independent, the set of common hypercyclic vectors for the path of operators given in Corollary (1.2.8) contains an infinite dimensional linear manifold for which every nonzero vector is a common hypercyclic vector. However, the linear manifold given in Corollary (1.2.8) is not closed. Corollary 3.5 of Sanders [46] provides a natural sufficient condition for the set of common hypercyclic vectors for a path of operators to contain a closed, infinite dimensional subspace of which every nonzero vector is a common hypercyclic vector.

The existence of a path of hypercyclic operators that is SOT-dense in $B(X)$ gives us information about the connectedness of the hypercyclic operators in $B(X)$. Recall that if Y and Z are subsets of a topological space X satisfying $\subseteq Z \subseteq \bar{Y}$, and if Y is connected, then Z is also connected; see Munkres [44]. A path of operators in $B(X)$ is SOT-connected, and so any set of operators in $B(X)$, which contains an SOT-dense path of operators, is also SOT-connected. From this⟶ topological argument and Corollary (1.2.8), we get the next result.

Corollary (1.2.9)[19]: Let g be any nonzero vector in a separable, infinite dimensional Banach space X. Then the set $\mathcal{A} = \{T \in B(X): g \in \mathcal{HC}(T)\}$ is SOT-dense and SOT-connected in $B(X)$. Furthermore, its set of common hypercyclic vectors is $\bigcap_{T\in\mathcal{A}} \mathcal{HC}(T) = (span\{g\}) \setminus \{0\}$.

Proof: For the first part of the proof, it suffices to show there is an operator T in $B(X)$ with $g \in \mathcal{HC}(T)$. By Corollary (1.2.8), it follows that the set $\{T \in B(X): g \in \mathcal{HC}(T)\}$ contains a path of operators which is SOT-dense in $B(X)$, and consequently SOT-connected by the topological argument above. To this end, let T_0 be a hypercyclic operator with

$\mathcal{HC}(T_0) \neq X \setminus \{0\}$; see Ansari in [21] or by Bernal in [27]. Let $g_0 \in \mathcal{HC}(T_0)$. Choose an invertible map $L : X \longrightarrow X$ such that $Lg = g_0$, and set $T = L^{-1}T_0 L$. Since $Lg = g_0 \in \mathcal{HC}(T_0)$, by statement (15) we get $g \in \mathcal{HC}(L^{-1}T_0 L) = \mathcal{HC}(T)$.

For the second part, observe that $(span\{g\}) \setminus \{0\} \subseteq \bigcap_{T\in\mathcal{A}} \mathcal{HC}(T)$ because $g \in \bigcap_{T\in\mathcal{A}} \mathcal{HC}(T)$. To establish the reverse set inequality, let $h_0 \notin \mathcal{HC}(T_0)$ with $h_0 \neq 0$. For any $h \notin span\{g\}$, the sets $\{g, h\}$ and $\{g_0, h_0\}$ are each linearly independent, and so there is an invertible map $L_0 : X \longrightarrow X$ with $L_0\ g = g_0$ and $L_0 h = h_0$. Again by statement (15), this implies $g \in \mathcal{HC}(L_0^{-1}\, T_0\, L_0)$ and $h \notin \mathcal{HC}(L_0^{-1}\, T_0\, L_0)$. Thus, $h \notin \bigcap_{T\in\mathcal{A}} \mathcal{HC}(T)$.

In many of the known cases, the set of common hypercyclic vectors for an uncountable family of hypercyclic operators is either empty or a dense G_δ set. Corollary (1.2.9) provides an example of a set of common hypercyclic vectors which is a G_δ set that fails to be dense.

When H is a separable, infinite dimensional Hilbert space over the scalar field $\mathbb{C}$, the invertible operators are path connected; see Douglas [40]. Thus, the conjugate set of a hypercyclic operator is both SOT-dense and SOT-connected in $B(H)$. By the topological argument given before Corollary (1.2.9), the hypercyclic operators in $B(H)$ then form an SOT-connected subset of $B(H)$; see [34]. For the Banach space version of the result, we can combine Theorem (1.2.4) and the topological argument given before Corollary (1.2.9).

Corollary (1.2.10)[19]: Let X be a separable, infinite dimensional Banach space. The set of all hypercyclic operators is SOT-connected in $B(X)$.

The argument used in Corollary (1.2.9) can be used to show certain well-known classes of hypercyclic operators are SOTconnected in $B(X)$. For example, from the definition of a chaotic operator, one can easily see that an operator is chaotic if and only if each operator in its conjugate set is chaotic. Using the same argument as with Corollary (1.2.10), we get the following result.

Corollary (1.2.11)[19]: Let X be a separable, infinite dimensional Banach space which admits a chaotic operator. The set of all chaotic operators is SOT-connected in $B(X)$.

For another example, an operator T in $B(X)$ satisfies the Hypercyclicity Criterion if and only if each operator in its conjugate set satisfies the criterion. Moreover, every separable, infinite dimensional Banach space admits an operator which satisfies the Hypercyclicity Criterion; see the hypercyclic operator constructed by Ansari in [21] or by Bernal in [27]. Thus, the collection of all hypercyclic operators in $B(X)$ which satisfies the Hypercyclicity Criterion is SOT-connected in $B(X)$. De la Rosa and Read [39] provided an example of a Banach space which admits a hypercyclic operator that fails to satisfy the Hypercyclicity Criterion. Using techniques inspired by De la Rosa and Read, Bayart and Matheron [25] showed some common Banach spaces, including the sequence Hilbert space ℓ^2, also admit such hypercyclic operators. Since a hypercyclic operator fails to satisfy the Hypercyclicity Criterion if and only if each operator in its conjugate set fails to satisfy the criterion, we get that whenever a Banach space X admits a hypercyclic operator that fails to satisfy the Hypercyclicity Criterion, then the collection of all such operators is SOT-connected in $B(X)$. Again, by a similar argument, if the Banach space X admits an operator with no nontrivial, closed, invariant subset, then the collection of all such operators is SOTconnected in $B(X)$. Recently, Chan and Seceleanu [35],[36] provided classes of operators for each of which having one orbit with a nonzero limit point imply the operator be hypercyclic. An operator has this property if and only if each operator in the conjugate set also has this property. By our topological argument, the collection of all operators having this property is an SOT-connected subset of $B(X)$.

We discuss some natural questions which arise from the results in the previous. To begin, Proposition (1.2.1) states that the set of common hypercyclic vectors for the entire conjugate set is either all nonzero vectors or the empty set. In the Hilbert space setting, the unitary orbit, $\mathcal{U}(T) = \{U^{-1}T\,U : U \text{ unitary}\}$, of an operator T is a well-studied subset of the conjugate set $\mathcal{S}(T)$. Obviously, the unitary orbit $\mathcal{U}(T)$ is strictly smaller than the conjugate set $\mathcal{S}(T)$. Hence, in view of Proposition (1.2.1), one may ask whether the set $\bigcap_{A\in\mathcal{U}(T)} \mathcal{HC}(A)$ of common hypercyclic vectors for the unitary orbit $\mathcal{U}(T)$ is always a dense G_δ set if T is hypercyclic.

As it turns out, the answer is still negative, unless we have the trivial case that every nonzero vector is a hypercyclic vector for T and hence the set of common hypercyclic vectors is $H \setminus \{0\}$. Otherwise, we have a unit vector y that is not a hypercyclic vector for . Extend the singleton set $\{y\}$ to an orthonormal basis of H. For any unit vector x, extend the singleton set $\{x\}$ to an orthonormal basis of H. Let $V : H \longrightarrow H$ be a unitary operator taking the second orthonormal basis one-to-one and onto the first orthonormal basis with $V x = y$. Since $y \notin \mathcal{HC}(T)$, by statement (15), we get $x \notin \mathcal{HC}(V^{-1}T\,V)$. From this, we can conclude $\bigcap_{A\in\mathcal{U}(T)} \mathcal{HC}(T) = \emptyset$.

The unitary orbit $\mathcal{U}(T)$ cannot be SOT-dense in $B(H)$ because every operator in the unitary orbit $\mathcal{U}(T)$ has the same norm as the operator . Along that line, a question one may ask is whether we can have a path of operators in the unitary orbit $\mathcal{U}(T)$ of a hypercyclic operator T that is SOT-dense in $\|T\| \cdot Sph(H)$, where $Sph(H)$ denotes the unit sphere of H. This may appear to have a positive answer. However, the answer is negative because the unitary orbit $\mathcal{U}(T)$ is not necessarily SOT-dense in $\|T\| \cdot Sph(H)$. One can easily construct the following counterexample in the sequence space $\ell^2(\mathbb{Z}) = \{\sum_{-\infty}^{\infty} a_n e_n : \sum |a_n|^2 < \infty\}$, where $\{e_n : n \in \mathbb{Z}\}$ is the canonical orthonormal basis of $\ell^2(\mathbb{Z})$. $Let\ T : \ell^2(\mathbb{Z}) \longrightarrow \ell^2(\mathbb{Z})$ be the bilateral weighted backward shift on the sequence space defined by

$$T\left(\sum_{n=-\infty}^{\infty} a_n e_n\right) = \sum_{n=-\infty}^{-1} \frac{1}{2} a_n e_{n-1} + \sum_{n=0}^{\infty} 2a_n e_{n-1}.$$

The above formula defines a hypercyclic shift T due to a result of Salas [47]. Let $A : \ell^2(\mathbb{Z}) \longrightarrow \ell^2(\mathbb{Z})$ be defined by

$$A\left(\sum_{n=-\infty}^{\infty} a_n e_n\right) = 2a_0 e_0.$$

Clearly $\|T\| = \|A\| = 2$. If we let

$$\mathcal{O} = \left\{S \in B\,\ell^2(\mathbb{Z}) : \|Se_1 - Ae_1\| < \frac{1}{4}\right\},$$

be an SOT-open set containing A, then one can easily show that no operator in the unitary orbit $\mathcal{U}(T)$ is in $\mathcal{O}$. In fact, $\|T f\| \ge \|f\|/2$ for every vector f in $\ell^2(\mathbb{Z})$, and so $\|U^{-1}TUf\| \ge \|f\|/2$ for any unitary operator U. Hence we have

$$\|U^{-1}TUe_1 - Ae_1\| \ge \frac{1}{2},$$

and so $U^{-1}T\,U \notin \mathcal{O}$.

We now switch our focus to weak hypercyclicity in the setting of a separable, infinite dimensional Banach space X. An operator T in $B(X)$ is weakly hypercyclic if there is a vector x in X for which its orbit $Orb(T, x)$ is dense in X with respect to the weak topology. Any such vector x is called a weakly hypercyclic vector for T, and we use $\mathcal{WHC}(T)$ to denote the set of all weakly hypercyclic vectors for the operator T. By a similarity argument, an operator is weakly hypercyclic but not hypercyclic if and only if the same is true for each operator in the conjugate set; see Chan and Sanders [33] or Shkarin [48] for the existence of such operators. Bès and Chan [28] showed that the conjugate set of a weakly hypercyclic operator is SOT-dense in the operator algebra $B(X)$.

Chapter 2
Spectral Properties with Eigenvalues and Refined Semiclassical Asymptotics

We show that explicit formulas for the transition density of the killed Cauchy process on the half-line (or the heat kernel of A in $(0,\infty)$), and for the distribution of the first exit time from the half-line follow. The formula for ψ_λ is also used to construct approximations to eigenfunctions of A in the interval. For the eigenvalues λ_n of A in the interval the asymptotic formula $\lambda_n = n\pi/2 - \pi/8 + O(1/n)$ is derived, and all eigenvalues λ_n are shown to be simple. Efficient numerical methods of estimation of eigenvalues λ_n are applied to obtain lower and upper numerical bounds for the first few eigenvalues up to the ninth decimal point. Simplicity of eigenvalues is proved for $\alpha \in [1,2)$. L^2 and L^∞ properties of eigenfunctions are studied. We also give precise numerical bounds for the first few eigenvalues. Extending methods from semi-classical analysis we are able to show a two-term formula for the sum of eigenvalues with the leading (Weyl) term given by the volume and the subleading term by the surface area. Our result is valid under very weak assumptions on the regularity of the boundary.

Section (2.1): Cauchy Process on Half-Line and Interval

Let (X_t), with $t \leq 0$, be the one-dimensional Cauchy process, that is, a one-Dimensional symmetric α-stable process for $\alpha = 1$. Let us consider the Cauchy process killed upon first exit time from D for $D = (0,\infty)$ and $D = (-1,1)$. The purpose is to study the spectral properties of the transition semigroup of this killed process, defined by

$$P_t^D f(x) = E_x(f(X_t); X_s \in D for\ all\ s \in [0,t]),\ \ f \in L^P(D)$$

and it's infinitesimal generator$\mathcal{A}_D$, which is the operator $-\sqrt{-(d^2/dx^2)}$ with a Dirichlet exterior condition (on D^c). The key problem is the description of eigenfunctions and eigenvalues of $\mathcal{A}_D$and P_t^D . The study of the spectral theoretic properties of the semigroups of killed symmetric α-stable processes has been the subject of many in recent years; see, for example, [50]–[52], [64]–[66], [69], [70]. We show the continuation of the work of Banuelos and Kulczycki [50].

The identification of the spectral problem for P_t^D and the so-called mixed Steklov problem in two dimensions, a method developed in [50], is applied for the case of the half-line $D = (0,\infty)$. Instead of searching for a function f satisfying

$$P_t^D f(x) = e^{-\lambda t} f(x) \text{ for } x \in D,$$

and $f(x) = 0$ for$x \in D^c$, we solve the equivalent mixed Steklov problem

$$\Delta u(x,y) = 0, x \in \mathbb{R}, y > 0, \tag{1}$$

$$\frac{\partial}{\partial x} u(x,0) = -\lambda u(x,0), x \in D, \tag{2}$$

$$u(x,0) = 0, \quad x \neq D, \mathbb{R} \tag{3}$$

Where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplace operator in $\mathbb{R}^2$. The relation between f and u is given here by $u(x,y) = E_x f(X_y)$. in this way a nonlocal spectral problem for the pseudodifferential operator on $\mathbb{R}$ (or its semigroup (P_t^D) on a domain D) is transformed into a Local one for a harmonic function of two variables, with spectral parameter in the Boundary conditions. From the point of view of stochastic processes, this Corresponds to the identification of the jump-type process (X_t) with the trace left on the horizontal axis by the two-dimensional Brownian motion. Similar or related methods were also applied, for example, by DeBlassie and Mendez-H´ ernandez [68]–[70], [85], and the idea can be traced back to the work of Spitzer [94]; see also [86].

When $D = (0, \infty)$, the spectrum of $\mathcal{A}_D$ is equal to $(-\infty, 0]$ and is of continuous type, and so there are no eigenfunctions of $\mathcal{A}_D$ in $L^P(D)$ (this follows easily from scaling properties of $\mathcal{A}_D$; see also Theorem (2.1.5) below). It turns out, however, that, for all$\lambda > 0$, there exist continuous generalized eigenfunctions $\psi_\lambda \in L^\infty(D)$. We have $P_t^D \psi_\lambda = e^{-\lambda t}\psi_\lambda$. Using the identification described, an explicit formula for$\psi_\lambda$is derived; see (26) and (27).

There are no earlier works concerning the spectral problem. $P_t^D f(x) = e^{-\lambda t} f(x)$ For $x \in D$ and $f(x) = 0$ for $x \in D^c$ for the half-line$D = (0, \infty)$, or the equivalent problem (1)–(3). However, there is an extensive literature concerning the related sloshing problem in the half-plane, that is, the problem given by (1), (2) and the Neumann condition

$$\frac{\partial}{\partial y} u(x, 0) = 0, x \notin D,$$

In place of the Dirichlet one (3). The sloshing problem is one of the fundamental problems in the theory of linear water waves; see, for example, [74]. The explicit solution of the sloshing problem in the half-plane for $D = (0, \infty)$ was first obtained by Friedrichs and Lewy in 1947 [75]; see also [62], [78], [81]. Both methods and results are closely related to their counterparts for the sloshing problem in the half-plane.

Certain holomorphic functions play an important role in the derivation ofψ_λ, and one of these functions is studied. In particular, the Fourier–Laplace transform of ψ_λ is derived; See (43). The formula for ψ_λ is of the form $\psi_\lambda(x) = \sin(\lambda x + \pi/8) - r(\lambda x)$, where r is the Laplace transform of a positive integrable function. We obtain estimates of the function r.

It is proved that ψ_λ yield a generalized eigenfunction expansion of $\mathcal{A}_D$ for $D = (0, \infty)$ in the sense of [76]; see, for example, [87], [93]. The transformation $\Pi f = \langle f, \psi_\lambda \rangle$ is an isometric (up to a constant) mapping of $L^2(D)$ onto $L^2(0, \infty)$ which diagonalizes$\mathcal{A}_D$, with $\Pi, \mathcal{A}_D = \lambda \mathcal{A}_D f$; see Theorem (2.1.5).

The spectral decomposition and enable us to derive an explicit formula for the kernel function $P_t^D(x, y)$ of P_t^D that is, the transition density of the Cauchy process killed on exiting $D = (0, \infty)$ (or the heat kernel for $\sqrt{-d^2/\ dx^2}$ —-with Dirichlet exterior condition onD^c); see Theorem (2.1.6). This extends the results of [58], [59], [63], where two-sided Estimates for $P_t^D(x, y)$ were obtained. As a Corollary, we obtain a new proof of the result by Darling [67], the explicit formula for the density of the distribution Of the first exit time from $(0, \infty)$; see Theorem (2.1.7). This can be rephrased in terms of the two dimensional Brownian motion; see Corollary (2.1.8); namely, we obtain a formula for the distribution of some local time of two-dimensional Brownian motion at some entrance time.

The spectral problem for the interval $D = (-1,1)$. We remark that due to translation invariance and scaling property of (X_t), the results for $(-1,1)$ extend easily to any open interval. It is well known that there is an infinite Sequence of continuous eigenfunction $\varphi_n \in D$ such that $\mathcal{A}_D \varphi_n = -\lambda_n \varphi_n$ on D and $\varphi_n \equiv 0$ on D^c, where $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \to \infty$. each φ_n is either symmetric or antisymmetric. The study of the properties of φ_n and $\lambda_n$dates back of Blumenthal and Getoor [54], where the Weyl-type asymptotic law was proved for a class of Markov processes in Domains.

In [54] it was proved that $\lambda_n/n \longrightarrow \pi/2$ as $n \to\infty$. Over the last few years, there have been an increasing amount of research related to this topic; see, [50], [51], [65], [66], [69], [70], [83], [84]. In [50] it was shown that $\lambda_n \leq n\pi/2$. The best known estimates for general λ_n, namely$/4 \leq \lambda_n \leq n\pi/2$, were proved in [65], where subordinate Brownian motions in bounded domains are studied. The simplicity of eigenvalues was studied in [50], where λ_2 andλ_3 are proved to be simple (simplicity of $\lambda_1$is standard), and in [84], where all eigenvalues are proved to have at most double multiplicity.
All these results are improved below.

Approximations $\hat{\varphi}_n$ to eigenfunctions φ_n are constructed by interpolating the translated eigenfunction for the half-lines $\psi_\lambda(1 + x)$ and $\psi_\lambda(1\text{-}x)$ with $\lambda = n\pi/2 - \pi/8$. It is then shown that $\mathcal{A}_D\hat{\varphi}_n$ is nearly equal to$-\lambda\hat{\varphi}_n$. We show that

$$\left|\lambda_n - \left(\frac{n\pi}{2} - \frac{\pi}{8}\right)\right| \leq \frac{1}{n}, n \geq 1.$$

and that the eigenvalues λ_n are simple; see Theorem (2.1.11). Finally, various properties of φ_n, see Corollary (2.1.13)– Corollary (2.1.16).

An application of numerical methods for estimation of eigenvalues to our problem is described. To get the upper bounds we use the Rayleigh–Ritz method for the Green operator, and for the lower bounds the Weinstein–Aronszajn method of intermediate problems is applied for (1)–(3). The numerical bounds of an approximate 10-digit accuracy are given by formula (88).

We use purely analytic arguments. In fact, the Cauchy process and related probabilistic notions are only used to give a concise definition of the killed semigroup (P_t^D).

We begin with a brief introduction to the Cauchy process (X_t) and its relation to the Steklov problem. We only collect the properties used in what follows; for a more detailed exposition see to [50] or [57], [64], [82]. For an introduction to more general Markov processes, see, [55], [72], [92]. Basic facts concerning the Fourier transform, the Hilbert transform and Paley–Wiener theorems are recalled.
The one-dimensional Cauchy process (X_t) is the symmetric 1-stable process, that is, the L´evy process with one-dimensional distributions

$$P_x(X_t) = p_t(y - x)dy = \frac{1}{\pi}\frac{t}{t^2 + (y - x)^2}dy.$$

Here P_x corresponds to the process starting at $x \in \mathbb{R}$; we denote by **E***x* the expectation with respect toP_x. Clearly, the P_x-distributions of $(X_t + a)$ and (bX_t) are equal to P_{x+a} distribution of $(X_t + a)$ and (bX_t)-distribution of (X_{bt}), respectively; these are the translation invariance and scaling property mentioned. The transition semigroup of (X_t) is defined by

$$P_t f(x) = E_x f(X_t) = \mathrm{f} * P_t(x), f \in L^P(\mathbb{R}), p \in [1, \infty], t > 0,$$

and $P_0 f(x) = f(x)$. This is a contraction semigroup on each $L^P(\mathbb{R})$, with $p \in [1, \infty]$, strongly continuous if$p \in [1, \infty)$, and when f is continuous and bounded, then $P_t f$ converges to f locally uniformly as $t \searrow 0$. The infinitesimal generator $\mathcal{A}$of (P_t) acting on $L^2(\mathbb{R})$ is the square root of the second derivative operator. For a smooth function f with compact support we have

$$\mathcal{A}f(x) = -\sqrt{-\frac{d^2}{dx^2}}f(x) = \frac{1}{\pi}PV\int_{-\infty}^{\infty}\frac{f(y)-f(x)}{(y-x)^2}dy$$

Where the integral is the Cauchy principal value (PV).

D always denotes the interval $(-1,1)$ or the half-line $(0,\infty)$. The time of the first exit from D is defined by $\tau_D = \inf\{t > 0 : X_t \notin D\}$, and the semigroup of the process (X_t) killed at timeτ_D is given by

$$P_t^D f(x) = E_x(f(X_t); X_s \in D$$

For all $s \in [0,t]) = E_x(f(X_t); t < \tau_D)$, where $t \geq 0$. This is again a well-defined contraction semigroup on every $L^P(D)$ space, with $p \in [1,\infty]$, strongly continuous if$p \in [1,\infty)$. If f is continuous and bounded in $\mathbb{R}$ and vanishes in $(-\infty,0]$, then$P_t^D f$ converges to f locally uniformly as t 0. The semigroup (P_t^D) admits a jointly continuous kernel function$P_t^D(x,y)$ $(t > 0, x, y \in D)$; clearly, $P_t^D f(x,y) \leq p_t(y - x) \leq 1/\pi t$. By $\mathcal{A}_D$ we denote the infinitesimal generator of (P_t^D) acting on$L^2(D)$. Since this is a Friedrichs extentsion on $L^2(D)$ of A restricted to the class of smooth functions supported in a compact subset of D, we sometimes say that $\mathcal{A}_D$is the square root of Laplacian with Dirichlet exterior conditions (on D^c).

Let us describe in more detail the connection between the spectral problem for the semigroup (P_t^D) and the mixed Steklov problem (1)–(3), established in [50]. The main idea is to consider the harmonic extension $u(x,y)$ of a function f to the upper half-plane $x \in \mathbb{R}$ and $y > 0$. Let $f \in L^P(\mathbb{R})$for some $p \in [1,\infty]$,and define

$$u(x,y) = P_y f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y}{y^2+(z-x)^2}f(z)dz \quad 1$$

Then u is harmonic in the upper half-plane $\mathbb{R} \times (0,\infty)$, and if $p \in [1,\infty)$, then $u(\cdot,y)$ converges to f in $L^P(\mathbb{R})$ as y 0. Conversely, for $p \in (1,\infty)$, if $u(x,y)$ is harmonic in the upper halfplane and the $L^P(\mathbb{R})$ norms of $u(\cdot,y)$ are bounded for $y > 0$, then $u(\cdot,y)$ converges in $L^P(\mathbb{R})$ to some f when y 0, and $u(x,y) = P_y f(x)$. By the definition,

$$\frac{\partial}{\partial y}u(x,0) = \lim_{y\searrow 0}\frac{P_y f(x)-f(y)}{y}$$

Point wise for all $x \in \mathbb{R}$. When f is in the domain of$\mathcal{A}$, then the above limit exists in $L^2(\mathbb{R})$and it is equal to $\mathcal{A}f$.

The motivation to study the mixed Steklov problem (1)–(3) comes from the following simple extension of [50] to the case of unbounded domains. A partial converse is given in the proof of Theorem (2.1.3).

Proposition (2.1.1)[49]: Let $D = (0,\infty)$ and $\lambda > 0$. Suppose that f: $\mathbb{R} \to \mathbb{R}$ *is* continuous and bounded, $f(x) = 0$ for $x \geq 0$, and $u(x,y) = P_y f(x)$. If $P_t^D f(x) = e^{-\lambda t} f(x)$ for all $x \in D$ and $t > 0$, then u satisfies (1)–(3).

Proof: Formulas (1) and (3) hold true by the definition of u. Since $P_y \; P_t^D f(x) = e^{-\lambda t} f(x)$, we have

$$\frac{u(x,y) - u(x,0)}{y} = \frac{P_y f(x) - f(x)}{y} = \frac{e^{-\lambda t}f(x)-1}{y}f(x) - \frac{P_y f P_y(x) - P_t^D f(x)}{y}.$$

As $y \searrow 0$, the first summand converges to $-\lambda f(x)$. the second one is estimated using formula (101) (see also [50]). If $0 < y < x$, then we have

$$\left|\frac{P_y f(x) - P_t^D f(x)}{y}\right| \le \int_0^\infty \frac{P_y(z-x) - P_t^D(x,z)}{y} |f(z)| dz$$

$$\le \frac{\|f\|_\infty}{\pi} \int_0^\infty \min\left(\frac{1}{x^2}, \frac{y}{x^2 z}, \frac{y}{x z^2}\right) dz = \frac{y(2+\log(x\backslash y))\|f\|_\infty}{\pi x^2}.$$

and this tends to 0 as $y \searrow 0$. Therefore, (2) is also satisfied.

The Fourier transform of a (complex-valued) function $f \in L^1(\mathbb{R})$ is given by $\hat{f}(x) = \int f(t) e^{-itx} dt$; this can be continuously extended to $L^P(\mathbb{R})$ whenever $1 \le p \le 2$. For $p \in (1, \infty)$, the Hilbert transform of $f \in L^P(\mathbb{R})$, denoted by Hf, satisfies $(Hf)\hat{}(t) = -\hat{f}t)(isignt)$. This is a bounded linear operator on $L^P(\mathbb{R})$, and for almost all t, $Hf(t) = 1$

$$Hf(t) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{f(s)}{t-s} ds \tag{4}$$

If f is Holder continuous, then the above formula holds for all $t \in \mathbb{R}$ and Hf is continuous; see, [95].

Let $C_+\{z \in \boldsymbol{C}: Im\, z > 0\}$ and $\bar{C}_+ = \{z \in \boldsymbol{C} : Im\, z \ge 0\}$; in a similar manner C_- and are defined. Let $1 < p < \infty$. If F is in the (complex) Hardy space $H^P(C_+)$, that is, F is holomorphic in C_+ and the $L^P(\mathbb{R})$ norms of $F(\cdot + i\varepsilon)$ are bounded in $\varepsilon > 0$, then, we find that $F(\cdot + i\varepsilon)$ converges in $L^P(\mathbb{R})$ to some $f \in L^P(\mathbb{R})$, which is said to be the boundary limit of F. In this case

$$Im\, f = H(Ref) \text{and } Ref = -H(Im\, f). \tag{5}$$

We also have

$$H\hat{f}(t) = -Hf(-t), \quad \text{where } \hat{f}(t) = f(-t). \tag{6}$$

The following version of the Paley–Wiener theorem is important in what follows; see, for example, [71]. For $p \in (1, \infty)$, a function $f \in L^P(\mathbb{R})$ is a boundary limit of some function $F \in H^P(C_+)$ if and only if vanishes in $(-\infty, 0)$. in this case

$$F(z) = \frac{1}{2\pi} \int_0^\infty \hat{f}(x) e^{-izx} dx, \qquad z \in C_+. \tag{7}$$

We use small letters to denote functions of the real variable and capital letters for functions on the upper half-plane C_+. Real-valued functions are denoted by Greek letters, whereas Latin letters are used for complex-valued functions.

We study the eigenproblem (1)–(3) for the half-line $D = (0, \infty)$ using methods that were earlier applied to the sloshing problem with a semiinfinite dock; see [62], [75], [78]. The solution u is given as the imaginary part of a holomorphic function F of a complex variable $z = x + iy$, where $x \in \mathbb{R}$ and $y \ge 0$. Such a function is automatically harmonic, and hence (1) is satisfied. Using the Cauchy–Riemann equations, we may restate (2) and (3) in the following equivalent form:

$$Im(iF'(x) + \lambda F(x)) = 0, \quad x > 0, \tag{8}$$

$$Im\, F(x) = 0, x \le 0. \tag{9}$$

Observe that for all $\vartheta \in \mathbb{R}$ and $t < 0$, the bounded holomorphic functions $F(z) = e^{i\lambda z + i\vartheta}$ and $F(z) = e^{i\lambda z - iarctan\, t}$ satisfy (8), and for all $t > 0$ the bounded holomorphic function

$$F(z) = e^{i\lambda z + i\vartheta}$$

Satisfies (9). This suggests searching for a solution of the form

$$F(z) = e^{i\lambda z+i\vartheta} - \int_{-\infty}^{0} \varrho(t)e^{i\lambda z-iarctan\, t}\, dt, \quad Re\, z \geq\ 0, Im\, z \geq\ 0, \qquad (10)$$

$$F(z) = \int_{0}^{\infty} \varrho(t)e^{i\lambda z}\, dt \qquad Rez \leq 0, Imz \geq 0, \qquad (11)$$

where is an unknown real function, say in some $L^P(\mathbb{R})$, with $p \in (1,2]$, and $\vartheta \in \mathbb{R}$. The values of F given by (10) and (11) must agree when Re $z = 0$ and $Im\ z \geq 0$, that is,

$$\int_{-\infty}^{\infty} e^{i\chi(t)}\varrho(t)e^{it\lambda y}dt = e^{-i\lambda y+i\vartheta}, y > 0.$$

Where $\chi(t)\ =\ \arctan t- =\ \arctan(max(-t,0))$. Replacing λy by -s yields that

$$\int_{-\infty}^{\infty} e^{i\chi(t)}\varrho(t)e^{it\lambda y}dt = e^{-i\lambda y+i\vartheta}\ \ s <\ 0. \qquad (12)$$

The right-hand side is the Fourier transform of $g(t) = (e^{e\vartheta}/2\pi)(1/(1+it))$. Therefore, formula (12) is equivalent to the condition the function $a(t)\ = e^{i\chi(t)}\varrho(t) - g(t)$ satisfies

$$\hat{a}(s) = 0 \text{ for } s < 0. \qquad (13)$$

Note that both and ϱ are in$L^P(\mathbb{R})$, so that $\hat{a}$ is well defined and $\hat{a} \in L^P(\mathbb{R})$. The foregoing remarks can be summarized as follows: any real function $\varrho \in L^P(\mathbb{R})$ satisfying (13) yields a solution to the problem (8)–(9).

By the Paley–Wiener theorem, (13) is satisfied if and only if a is the boundary limit of a unique function A in the Hardy space $H^P(C_+)$ in the upper half-plane $C_+ = \{z \in \boldsymbol{C} : Im\ z\ > 0\}$. such a function A can be derived as follows. (Formula (19)), a function B holomorphic in C_+ and continuous on $\bar{C}_+$ is defined, such that $i\chi(t) - B(t) \in \mathbb{R}$ for all$t \in \mathbb{R}$. The function

$$e^{-B(t)}a(t) = e^{i\chi(t)-B(t)} - e^{-B(t)}g(t)$$

Is therefore the boundary limit of $e^{-B(z)}A(z)$. Note that $e^{i\chi(t)-B(t)}$ is real. The function $g(t) = (e^{i\vartheta}/2\pi)(1/(1+it))$ is the boundary limit of a meromorphic function $G(z) = (e^{i\vartheta}/2\pi)(1+iz))$. The function G has a simple pole at i, so that$G(z)(e^{iB(z)-B(t)})$ is holomorphic inC_+. It follows that

$$e^{-B(t)}a(t) + g(t)\left(e^{-B(t)} - e^{-B(i)}\right) = e^{i\chi(t)-B(t)}\varrho(t) - e^{-B(i)}g(t) \qquad (14)$$

Is a boundary limit of$\hat{A}(z) = e^{-B(z)}A(z) + G(z)\left(e^{-B(z)} - e^{-B(i)}\right)$, $z \in C_+$. Since G and A are in $H^P(C_+)$, and $\left|e^{-B(i)}\right|$ is bounded (see (109)), we must have $\hat{A} \in H^P(C_+)$.

Let $\hat{G}(z)$ = $(e^{-i\vartheta}/2\pi)$ (1/(1 - iz)). Note that, by (14), the boundary limit of the function $\tilde{A}(z) - \overline{e^{-B(i)}}\tilde{G}(z)$ (belonging to $H^P(C_+)$) is equal to

$$e^{i\chi(t)-B(t)}\varrho(t) - e^{-B(i)}g(t) - \overline{e^{-B(i)}g(t)}, \qquad (15)$$

Which is real for all$t \in \mathbb{R}$. The real part of the boundary limit of and $H^P(C_+)$ function is the negative of the Hilbert transform of its imaginary part. Therefore, the function defined by (15).

Is the Hilbert transform of the constant 0, and so it is identically 0. It follows that

$$e^{i\chi(t)-B(t)}\varrho(t) = e^{-B(i)}g(t) + \overline{e^{-B(i)}g(t)} = 2\mathrm{Re}\left(e^{-B(i)}g(t)\right), t \in \mathbb{R}. \quad (16)$$

Also, $\tilde{A}(z) - \overline{e^{-B(i)}\,\tilde{G}(z)}$) has a boundary limit 0, and so it is identically zero in C_+. Hence, for $z \in C_+$ we have

$$A(z) = e^{-B(i)}\left(\tilde{A}(z) - G(z)\left(e^{B(z)} - e^{-B(i)}\right)\right) = \frac{e^{-i\vartheta}}{2\pi}\frac{e^{B(z)-\overline{B(i)}}}{1-iz} - \frac{e^{i\vartheta}}{2\pi}\frac{1-e^{B(z)-B(i)}}{1+iz}$$

Since $\left|e^{B(z)}\right|$ is bounded by a constant multiple of $1 + |z|\sqrt{1+|z|}$(see (109)), it follows that A defined by the above formula is in $H^P(C_+)$ for any $p \in (2,\infty)$, and given by (16) is in $L^P(\mathbb{R})$; later we show that in fact $\varrho \in L^P(\mathbb{R})$ for $p \in (1,\infty)$ if $\vartheta = \pi/8$. Also, the boundary limit of A is the function a defined in (13) (this can be verified, for example, by a direct calculation), so that indeed is a solution to (13).

We now come to the construction of the function B. We want it to be holomorphic in C_+ and continuous in C_+, and $i\chi(t) - B(t)$ is to be real for all $t \in \mathbb{R}$. Therefore

$$Im\, B(t) = \chi(t) = arctan(t_-), \quad t \in \mathbb{R}. \quad (17)$$

Clearly B is not in $H^P(C_+)$, so that $ReB(t)$ cannot be expressed directly as the Hilbert transform of $ImB'(t) = \chi'(t)$. We can, however, apply the Hilbert transform to$ImB(t) = \chi(t)$, which is an $L^2(\mathbb{R})$ function. It follows that

$$ReB'(t) = -H(Im\, B')(t) = \frac{1}{\pi}PV\int_{-\infty}^{0}\frac{1}{(t-s)(1+s^2)}ds, \quad t \in \mathbb{R},$$

The integral on the right-hand side being the Cauchy principal value for $t < 0$. This equation is studied. It follows that up to an additive constant, which we choose to be zero, we have $\mathrm{Re}B(t) = \eta(t)$, where η is given by (102). By (103) and (107), for all $t \in \mathbb{R}$,

$$B(t) = i\chi(t) + \eta(t) = i \arctan(t_-) + \log\sqrt[4]{1+t^2} - \frac{1}{\pi}\int_0^t \frac{\log|s|}{1+s^2}ds$$

$$= Iarctan(t_-) + \frac{1}{\pi}\int_{-\infty}^{0}\frac{\log|t-s|}{1+s^2}ds. \quad (18)$$

this formula is easily extended to complex arguments, whenever$Im\, z \geq 0$, and we Have

$$B(z) = \frac{1}{\pi}\int_{-\infty}^{0}\frac{\log|z-s|}{1+s^2}ds \quad (19)$$

provided that the continuous branch of log is chosen on the upper half-plane $\overline{C_+}$ (that is, the principal branch with$\log s = \log|s| + i\pi\, for\ s < 0$). We emphasize that (18) and (19) agree for $z = t < 0$.

For the explicit formula for, the function $B(i)$ needs to be computed. By (111) and (112),

$$B(i) = \frac{1}{\pi}\int_0^\infty \frac{\log(i+s)}{1+s^2}ds = \frac{1}{\pi}\int_0^\infty \frac{\log(i+s)}{1+s^2}ds$$

$$= \frac{1}{2\pi}\int_0^\infty \frac{\log|1+s^2|}{1+s^2}ds + \frac{i}{\pi}\int_0^\infty \frac{\pi/2}{1+s^2}ds = \frac{\log 2}{2} + \frac{i\pi}{8}. \quad (20)$$

Now (16) yields that

$$\varrho(t) = 2e^{B(t)-i\chi(t)}\mathrm{Re}\big(e^{-B(i)}g(t)\big) = 2e^{\eta(t)}\mathrm{Re}\left(\frac{e^{i(\vartheta-\pi/8)}}{2\pi\sqrt{2}}\frac{1}{1+it}\right)$$
$$= \frac{\sqrt{2}}{2\pi}e^{\eta(t)}\frac{\cos(\vartheta-\pi/8)+t\sin(\vartheta-\pi/8)}{1+t^2}, t \in \mathbb{R}.$$

Since $\vartheta \in \mathbb{R}$ is arbitrary, we conclude that there are two linearly independent solutions for ϱ, corresponding to $\vartheta = \pi/8$ and $\vartheta = 5\pi/8$, respectively:

$$\varrho(t) = \frac{\sqrt{2}}{2\pi}\frac{1}{1+t^2}e^{\eta(t)} \text{ and } \tilde{\varrho}(t) = \frac{\sqrt{2}}{2\pi}\frac{1}{1+t^2}e^{\eta(t)}$$

the solution to the problem (8)–(9) corresponding to $\vartheta = \pi/8$ and as above is therefore given by

$$F(z) = e^{i\lambda z+i\pi/8} - \frac{\sqrt{2}}{2\pi}\int_{-\infty}^{0}\frac{1}{1+t^2}e^{\eta(t)}e^{t\lambda z - i\arctan t}dt, Re\ z \geq 0, Imz \geq 0 \qquad (21)$$

$$F(z) = \frac{\sqrt{2}}{2\pi}e^{\eta(t)}\int_{-\infty}^{0}\frac{1}{1+t^2}e^{\eta(t)}e^{t\lambda z}dt,\ Rez \leq 0, Imz \geq 0. \qquad (22)$$

By (106), we have $\varrho \in L^1(\mathbb{R})$, and so F is bounded and continuous. Furthermore, it can be easily verified that the solution corresponding to $\vartheta = 5\pi/8$ and ˜ is given by $F'(z)/\lambda$. Since $\tilde{\varrho}$ decays at infinity as $|t|^{-1/2}$, it follows that $F'(z)$ has a singularity of order $|z|^{-1/2}$ at zero and it is not bounded near 0. For that reason, in what follows we only study the solution $F(z)$ given by (21) and (22).

Since $e^{-i\arctan t} = (1-it)/\sqrt{1+t^2}$ and $e^{\eta(t)} = e^{-\eta(-t)}\sqrt{1+t^2}$√1 + t2 (see (104)), we can rewrite (21) as

$$F(z) = e^{t\lambda z+\vartheta-\pi/8} - \frac{\sqrt{2}}{2\pi}\int_{0}^{\infty}\frac{1+it}{1+t^2}e^{\eta(t)}e^{-t\lambda z}dt,\ \ Re\ z \geq 0, Imz \geq 0 \qquad (23)$$

Therefore, we have proved the following theorem.

Theorem (2.1.2)[49]: The bounded solution of (1)–(3) for $D = (0,\infty)$ is given by $u(x,y) = e^{\lambda y}\sin\left(\lambda x + \frac{\pi}{8}\right)$

$$\frac{\sqrt{2}}{2\pi}\int_{0}^{\infty}\frac{\cos(t\lambda y)-t\sin(t\lambda y)}{1+t^2}exp\left(-\frac{1}{\pi}\int_{0}^{\infty}\frac{log(i+s)}{1+s^2}ds\right)e^{t\lambda zx}dt, \qquad (24)$$

For $x \geq 0$ and $y \geq 0$, and

$$u(x,y) = \frac{\sqrt{2}}{2\pi}\int_{0}^{\infty}\frac{t\sin(t\lambda y)}{1+t^2}exp\left(-\frac{1}{\pi}\int_{0}^{\infty}\frac{log(i+s)}{1+s^2}ds\right)e^{t\lambda zx}dt, \qquad (25)$$

For $x \leq 0$ and $y \geq 0$.

We stated below, follows from Theorem (2.1.2) and a partial converse to Proposition (2.1.1).

Theorem (2.1.3)[49]: Let $D = (0,\infty)$. For $\lambda > 0$, the function (see Figure (1) [49])

$$\psi_\lambda(x)\sin\left(\lambda x + \frac{\pi}{8}\right) - r_\lambda(x), x > 0, \qquad (26)$$

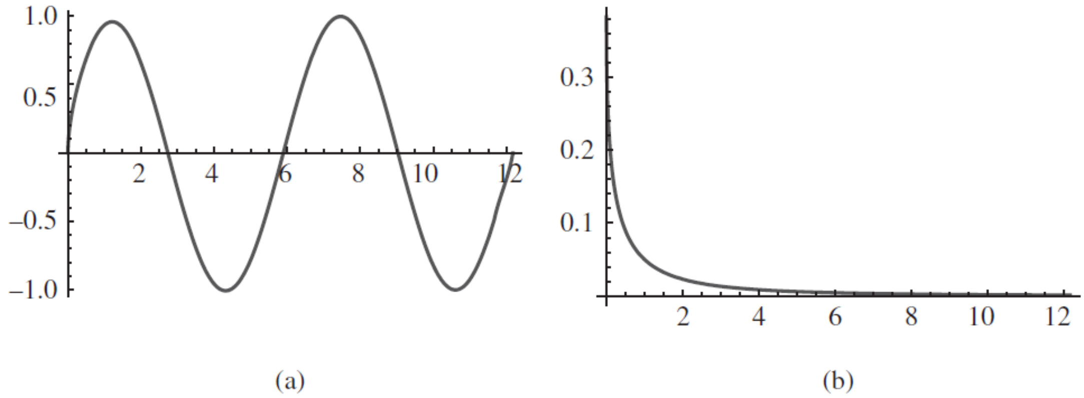


Figure (1)[49]: (a) Graph of ψ_1 and (b) graph of the remainder term $r(x) = sin\left(x + \frac{\pi}{8}\right) - \psi_1(x)$.wherer

$$r_\lambda(x) = r(\lambda x)\frac{\sqrt{2}}{2\pi}\int_0^\infty \frac{t}{1+t^2} exp\left(-\frac{1}{\pi}\int_0^\infty \frac{log(i+s)}{1+s^2}ds\right)e^{-t\lambda x}dt \qquad (27)$$

is the eigenfunction of the semigroup (P_t^D) acting on $C(D)$ corresponding to eigenvalue λ.

Proof. With the notation of Theorem (2.1.2), we have$\psi_\lambda(x) = u(x,0)$; we extend ψ_λ to be 0 on $(-\infty, 0]$. since u is harmonic and bounded in the upper half-plane, we have$P_y\psi_\lambda(x) = u(x,y)(y > 0, x \in \mathbb{R})$. Since u satisfies (2), for all$x > 0$, $(1/y)\left(P_y\psi_\lambda(x) - \psi_\lambda(x)\right)$converges to $-\psi_\lambda(x)$ as $y\searrow 0$. We will now prove (formula (33)) that this convergence is dominated by an appropriate function.

Below we assume that $\lambda > 0, x > 0$ and$0 < y < 1/\lambda$. By formula (24), we have

$$\frac{P_y\psi_\lambda(x) - e^{t\lambda y}\psi_\lambda(x)}{y} = \frac{\sqrt{2}}{2\pi}\int_0^\infty \frac{cos(t\lambda y) - sin(t\lambda y) - e^{-\lambda y}t}{(1+t^2)y}e^{-\eta(t)}e^{-t\lambda x}dt, \qquad (28)$$

since $|1-\cos z| \le z^2/2, |z - \sin z| \le z^3/3, |1 - z - e^{-z}| \le z^2/2$ and $\lambda_y < 1$, we have $\left|t\cos(t\lambda y) - \sin(t\lambda y) - e^{-\lambda y}t\right| \le \lambda^2 t\left(\frac{t^2}{2} + \frac{t^2\lambda y}{3} + \frac{1}{2}\right)y^2 < \lambda^2 t(1+t^2)y^2$.

Using also $e^{-\eta(t)} \le e^{C/\pi}(1+t^2)^{-1/4}$ and then$(1+t^2) \ge t^2$, , we obtain

$$\left|\int_0^{1/\lambda y} \frac{\cos(t\lambda y) - \sin(t\lambda y) - e^{-\lambda y}t}{(1+t^2)y}e^{-\eta(t)}e^{-t\lambda x}dt,\right|$$

$$\le e^{C/\pi}\lambda^2 y\int_0^{1/\lambda y}\frac{t(1+t^2)}{(1+t^2)^{5/4}}e^{-t\lambda x}dt \le e^{C/\pi}\lambda^2 y\int_0^{1/\lambda y}\sqrt{t}e^{-t\lambda x}dt. \qquad (29)$$

In Furthermore,

$$\int_0^{1/\lambda y}\sqrt{t}e^{-t\lambda x}dt \le (\lambda y)^{-3/4}\int_0^\infty t^{1/4}e^{-t\lambda x}dt \le \frac{\Gamma(3/4)}{(\lambda^2 x y)^{3/4}} \qquad (30)$$

In a similar manner, by using$\left|\cos(t\lambda y) - \sin(t\lambda y) - e^{-\lambda y}t\right| \le t + t\lambda y + t \le 3t$, we obtain

$$\left|\int_{1/\lambda y}^{\infty}\frac{\cos(t\lambda y)-\sin(t\lambda y)-e^{-\lambda y}t}{(1+t^2)y}e^{-\eta(t)}e^{-t\lambda x}dt,\right|$$

$$\le 3e^{C/\pi}\int_{1/\lambda y}^{\infty}\frac{t}{(1+t^2)^{5/4}y}e^{-t\lambda x}dt$$

$$\le \frac{3e^{C/\pi}}{y}\int_{1/\lambda y}^{\infty}t^{-3/2}e^{-t\lambda x}dt, \tag{31}$$

and

$$\int_{1/\lambda y}^{\infty}t^{-3/2}e^{-t\lambda x}dt, \le (\lambda y)^{5/4}\int_{0}^{\infty}t^{-1/4}e^{-t\lambda x}dt \le \frac{\Gamma(3/4)(\lambda y)^{5/4}}{(\lambda x)^{3/4}} \tag{32}$$

Formulas (28)–(32) yield, after simplification, that

$$\left|\frac{P_y\psi_\lambda(x)-e^{-\lambda y}\psi_\lambda(x)}{y}\right| \le \frac{2\sqrt{2}e^{C/\pi}\Gamma(3/4)\lambda^{1/2}y^{1/4}}{\pi x^{3/4}} = \frac{c_1(\lambda)y^{1/4}}{x^{3/4}}. \tag{33}$$

With some constant$c_1(\lambda)$

We are now going to replace P_y by P_y^Din (33). It is proved (using only the definition (26) and (27) ofψ_λ) that $|\psi_\lambda(x)|=|\psi_1(\lambda x)|\le 2\sqrt{\lambda x}$; see (53). This and (101), for $0<y<x$, yield that

$$\left|\frac{P_y\psi_\lambda(x)-e^{-\lambda y}\psi_\lambda(x)}{y}\right| \le \int_0^{\infty}\frac{P_y(z-x)-P_y^D(x,z)}{y}|\psi_\lambda(z)|\,dz$$

$$\le \frac{2\sqrt{\lambda}}{\pi}\int_0^{\infty}min\left(\frac{1}{x^2},\frac{y}{x^2z},\frac{y}{z^2x}\right)\sqrt{z}dz = \frac{8\sqrt{\lambda}y\left(3\sqrt{x}-\sqrt{y}\right)}{3\pi x^2} \le \frac{8\sqrt{\lambda}y}{\pi x^{3/2}}$$

$$\le \frac{8\sqrt{\lambda}y}{\pi x^{3/4}} \tag{34}$$

When $0<x<y$, in a similar manner (35)

$$\left|\frac{P_y\psi_\lambda(x)-P_y^D\psi_\lambda(x)}{y}\right| \le \frac{2\sqrt{\lambda}}{\pi}\int_0^{\infty}min\left(\frac{1}{y^2},\frac{1}{z^2}\right)\sqrt{z}dz = \frac{16\sqrt{\lambda}}{3\pi\sqrt{y}} \le \frac{16\sqrt{\lambda}y^{1/4}}{\pi x^{3/4}} \tag{35}$$

Finally, by (33)–(35), there is a constant $c_2(\lambda)$ such that

$$\left|\frac{P_y^D\psi_\lambda(x)-e^{-\lambda y}\psi_\lambda(x)}{y}\right| \le c_2(\lambda)\frac{y^{1/2}}{x^{3/4}} \tag{36}$$

For any fixed $x>0$ and$t>0$, the one-sided derivative of $e^{\lambda y}P_y^D\psi_\lambda(x)$ with respect to t equals

$$\frac{\partial}{\partial t_+}\left(e^{\lambda y}P_y^D\psi_\lambda(x)\right) = \lim_{y\searrow 0}\frac{e^{\lambda(t+y)}P_{t+y}^D\psi_\lambda(x)-e^{\lambda t}P_y^D\psi_\lambda(x)}{y}$$

$$= \lim_{y\searrow 0} e^{\lambda(t+y)}\int_0^{\infty}P_y^D(x,z)\,\frac{P_y^D\psi_\lambda(z)-e^{\lambda y}\psi_\lambda(z)}{y}dz$$

Since $P_y^D(x,z) \le 1/t$, by (36) we have

$$\left| \int_0^\infty P_y^D(x,z) \, \frac{P_y^D \psi_\lambda(z) - e^{-\lambda y}\psi_\lambda(z)}{y} dz \right| \le \int_0^\infty \frac{c_1(\lambda) y^{1/4}}{z^{3/4}} P_y^D(x,z) dz$$

$$\le c_2(\lambda) y^{1/4} \left(\int_1^\infty P_y^D(x,z)dz + \frac{1}{t}\int_0^1 \frac{1}{z^{3/4}} dz \right) \le \left(1 + \frac{4}{t}\right) c_2(\lambda) y^{1/4}$$

The right-hand side tends to zero as $y \searrow 0$, that

$$\frac{\partial}{\partial t_+}\left(e^{\lambda y} P_y^D \psi_\lambda(x)\right) = 0$$

For all $t > 0$ and$x > 0$. By (36) (with both sides multiplied by$e^{\lambda y}$), this also holds for$t = 0$.

Finally, the function $e^{\lambda t} P_y^D \psi_\lambda(x)$ is continuous with respect to *t* for each *x* > 0 (this follows from the weak continuity of $P_y^D(x,z)dz$ with respect to *t*, which is a consequence of the stochastic continuity of the killed Cauchy process; one can also prove this using the explicit formula for p_i and (101)). It follows that$e^{\lambda t} P_y^D \psi_\lambda(x)$ is constant in $t \ge 0$, and since $P_0^D \psi_\lambda(x) = \psi_\lambda(x)$,this completes the proof.

We study the properties of the function *B*. As an interesting corollary, the Laplace transform of the eigenfunctions $\psi_\lambda$is computed.

The function *B* defined by (19) extends to a holomorphic function on $\boldsymbol{C} \setminus (-\infty, 0]$, satisfying $B(\bar{z}) = \overline{B(z)}$. Therefore *B* is defined on whole $\boldsymbol{C}$, it is holomorphic in $\boldsymbol{C} \setminus (-\infty, 0]$ with a branch cut on $(-\infty, 0]$, and it is continuous on $\bar{C}_+$. The following properties of *B* will play an important role.

When $Im\, z > 0$, we have

$$B(z) + B(-z) = \frac{1}{\pi}\int_{-\infty}^{0} \frac{\log(z-s)}{1+s^2} ds + \frac{1}{\pi}\int_{-\infty}^{0} \frac{\log(-z-s)}{1+s^2} ds$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\log(z+s)}{1+s^2} ds + \frac{1}{\pi}\int_{-\infty}^{0} \frac{\log(z+s) - i\pi}{1+s^2} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log(z-s)}{1+s^2} ds - \frac{i\pi}{2}\ .$$

On the right-hand side, the function$s \to log(z-s)$, holomorphic (and therefore harmonic) inC_-, is integrated against the Poisson kernel of the lower half-plane $p_1(s) = (1/\pi)(1/1+s^2)$. The result is the value of $log(z-s)$ at$s = -i$. It follows that

$$B(z) + B(-z) = \log\big(z - (-i)\big) - \frac{i\pi}{2} = \log(1 - iz).$$

By $B(\bar{z}) = \overline{B(z)}$ we get $B(z)B(-z) = = log(1 + iz)$ whenever $Im\, z < 0$, and so

$$e^{B(z)} = \big(1 - iz\sigma(z)\big)e^{-B(-Z)} \tag{37}$$

where $\sigma(z) = 1$ when $Im\ z > 0$ and $\sigma(z) = -1$ when$Im\ z\ <\ 0$. A similar relation for η was used earlier in (23), see also (104). By continuity of $B(z)$ in C_+ the formula (37) is also valid for $z \in \mathbb{R}$ if we let $\sigma(z) = 1$ for $z < 0$ and $\sigma(z) = -1$ for $z > 0$. For completeness, we let$\sigma(0) = 0$ By (21), (27), the relation between F, ψ_λ and r_λ, and using $B(t) = \eta(t) + iarctan$

$$r(x) = \frac{\sqrt{2}}{2\pi}\int_0^\infty \tau(t)e^{-tx}dt \text{ Where } \tau(t) = Im\frac{e^{B(-t)}}{1+t^2} \tag{38}$$

Note that by the definition, $\tau(t) = 0$ for $t \leq 0$. In what follows, we need the Hilbert transform of τ, which can be computed. The function $e^{B(z)}/(1+z^2)$ is Meromorphic in the upper half-plane with a simple pole at i, so that the function

$$\frac{e^{B(z)}}{1+z^2} - \frac{1}{2}\frac{e^{B(t)}}{1+iz} - \frac{1}{2}\frac{e^{\overline{B(\iota)}}}{1+iz}$$

Is holomorphic inC_+. In fact it is in $H^P(C_+)$ for$p \in (1,\infty)$; see (109). Its boundary limit on $\mathbb{R}$ is equal to

$$\frac{e^{B(t)}}{1+t^2} - \frac{1}{2}\frac{e^{B(t)}}{1+it} - \frac{1}{2}\frac{e^{\overline{B(\iota)}}}{1+it} = \frac{e^{B(t)} - \sqrt{2}\cos(\pi/8) - t\sqrt{2}\sin(\pi/8)}{1+t^2},$$

and the imaginary part of this function is just$\tau(-t)$. Therefore, the Hilbert transform of $\tau(-t)$ is the negative of the real part of the above function. It follows by (6) that, for $t \in \mathbb{R}$,

$$H\tau(t) = Re\ \frac{e^{B(-t)} - \sqrt{2}\,cos(\pi/8) + t\sqrt{2}\,sin(\pi/8)}{1+t^2}. \tag{39}$$

We are now able to compute the Laplace transform $\mathcal{L}\psi_\lambda$ofψ_λ. By a direct computation, we have

$$\int_0^\infty \sin\left(x+\frac{\pi}{8}\right)e^{-tx}dx = \frac{\sqrt{2}\cos(\pi/8) - t\sqrt{2}\sin(\pi/8)}{1+t^2} \quad t > 0. \tag{40}$$

On the other hand, by Fubini's theorem and (38),

$$\int_0^\infty r(x)e^{-tx}\,dx = \frac{\sqrt{2}}{2\pi}\int_0^\infty \frac{e^{B(z)}}{t+s}ds = -\frac{\sqrt{2}}{2}H\tau(-t)\ t \geq 0.$$

By (39) we have

$$\int_0^\infty r(x)e^{-tx}\,dx = \frac{\sqrt{2}}{2}\frac{e^{B(t)}}{1+t^2} + \frac{\cos(\pi/8) - t\sin(\pi/8)}{1+t^2} \quad t \geq 0. \tag{41}$$

In particular,

$$\int_0^\infty r(x)e^{-tx} = \cos\left(\frac{\pi}{8}\right) - \frac{\sqrt{2}}{2}. \tag{42}$$

Formulas (40) and (41) give

$$L\psi_1(t) = \int_0^\infty \psi_1(x)r(x)e^{-tx}\,dx = \frac{\sqrt{2}}{2}\frac{e^{B(t)}}{1+t^2}, \qquad t > 0. \tag{43}$$

By scaling and the uniqueness of the holomorphic continuation, we obtain the following result.

Corollary (2.1.4)[49]: The Laplace transform of ψ_λ is equal to

$$\psi_\lambda(z) = \int_0^\infty \psi_\lambda r(x) e^{-zx}\, dx = \frac{\sqrt{2}}{2} \frac{\lambda e^{B(z/\lambda)}}{\lambda^2 + z^2}, \qquad Rez > 0, \tag{44}$$

where $B(z)$ is given by (19).

We devoted to a detailed analysis of the remainder term r_λ; see (27). Recall that $r_\lambda(x) = r_\lambda(x)$, where

$$r(x) = \frac{\sqrt{2}}{2\pi} \int_0^\infty \frac{t}{(1+t^2)^{5/4}} exp\left(\frac{1}{\pi} \int_0^t \frac{\log s}{1+s^2} ds \right) e^{-tx} dt. \tag{45}$$

since r is the Laplace transform of a positive function, it is completely monotone, that is, all functions$(-1)^n r^{(n)}$ are nonnegative and monotonically decreasing (see, for example, [73]). In most of our estimates we simply use the inequality

$$-C \leq \int_0^t \frac{\log s}{1+s^2} ds \leq 0 \quad \text{for} t > 0$$

and formula (113). The $L^1(\mathbb{R})$ norm of r, however, we have already calculated in (42) as

$$\int_0^\infty r(x) = \cos\left(\frac{\pi}{8}\right) - \frac{\sqrt{2}}{2} \in (0.216{,}0.217). \tag{46}$$

Since $1/(t+s) \leq 1/2\sqrt{ts}$, by Fubini's theorem, it follows that

$$\int_0^\infty (r(x))^2 \leq \frac{1}{2\pi^2} \int_0^\infty \int_0^\infty \frac{t}{(1+t^2)^{5/4}} s \frac{t}{(1+s^2)^{5/4}} \frac{1}{t+s} dtds \leq \frac{1}{4\pi^2} \left(\int_0^\infty \frac{\sqrt{t}}{(1+t^2)^{5/4}} dt \right)^2$$

$$= \frac{\left(\Gamma(3/4)\right)^2}{\pi\left(\Gamma(1/4)\right)^2} < 0.037. \tag{47}$$

In a similar manner,$1/(t+\ s) \geq 1\, 1/\left(\sqrt{1+s^2}\sqrt{1+t^2}\right)$, so that

$$\int_0^\infty r(x)^2 dx \geq \frac{e^{-2c/\pi}}{2\pi^2} \left(\int_0^\infty \frac{\sqrt{t}}{(1+t^2)^{5/4}} dt \right)^2 = \frac{e^{-2c/\pi}}{9\pi^2} > \ 0.012. \tag{48}$$

For $x > 0$, we have

$$r(x) = \frac{\sqrt{2}}{2\pi} \int_0^\infty \frac{\sqrt{t}}{(1+t^2)^{5/4}} exp\left(\int_0^t \frac{\log s}{1+s^2} ds \right) e^{-tx} dt \leq \frac{\sqrt{2}}{2\pi} \int_0^\infty t e^{-tx} dt \leq \frac{\sqrt{2}}{2\pi x^2}. \tag{49}$$

In a similar manner,

$$(-1)^n r^{(n)}(x) \leq \frac{\sqrt{2}}{2\pi} \frac{(n+1)!}{x^{n+2}} \tag{50}$$

also,

$$r(x) \leq r(0) = \sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2} < 0.383, \tag{51}$$

and

$$-r'(x) \le \frac{\sqrt{2}}{2\pi}\int_0^\infty \frac{e^{-tx}}{\sqrt{t}}dt = \frac{1}{\sqrt{2\pi x}} \tag{52}$$

For$x > 0$, it follows that

$$|\psi_1(x)| \le \left|\sin\left(x+\frac{\pi}{8}\right) - \sin\frac{\pi}{8}\right| + |r(x) - r(0)| \le x + \int_0^x |r'(y)|dy \le x + \sqrt{\frac{2x}{\pi}}.$$

Since clearly $|\psi_1(x)| \le |\sin(x + \pi/8)| + |r(x)| \le 2$,we have

$$|\psi_1(x)| \le min\left(x + \sqrt{\frac{2x}{\pi}}, 2\right) \le min(2\sqrt{x}, 2). \tag{53}$$

This property was already used in the proof of Theorem (2.1.3).

We estimate the supremum norm of ψ_λ. We have $r(x) > 0$, so that $\psi_1(x) < 1$ for all $x > 0$. Furthermore, since r is monotonically decreasing, the global minimum of ψ_1 is its first local minimum, say $\psi_1(x_0)$, which is attained at the second zero x_0 of $\psi'_1(x) = \cos(x + \pi/8) - r'(x)(x) > 0$). Since -$r$ is decreasing, by (50), we find that $x0$ is not less than the second zero of$cos(x + \pi/8) + (\sqrt{2}/\pi)x^{-3}$; hence $x_0 > 4.31$. It follows that

$$\|\psi_\lambda\|_\infty \le 1 + r(x_0) = 1 + \frac{\sqrt{2}}{2\pi}\int_0^\infty \frac{te^{-x_0 t}}{(1+t^2)^{5/4}}dt < 1.01; \tag{54}$$

For the last inequality, integrate by parts the left-hand side of formula 3.387(7) in [77]. The estimate (54) is only used in Corollary (2.1.16), where a weaker version of (54) would only result in a larger constant in (87). In fact, for the present constant 3, we only need that $\|\psi_\lambda\|_\infty \le 1.19$, which is easily obtained by (49) and $x_0 \ge 4$.

Let $D = (0, \infty)$. We study the $L^2(D)$ properties of the operators P_t^D. For $f \in C_c(D)$, define

$$\Pi f(x) = \int_0^\infty f(\lambda)\psi_\lambda(x)d\lambda\,, x \in D, \tag{55}$$

Where $\psi_\lambda = sin(\lambda x + \pi/8) - r\lambda(x)$ is given by (26) and (27). Note that

$$F_1(x) = \int_0^\infty f(\lambda)\sin\left(\lambda x + \frac{\pi}{8}\right)d\lambda\,, x \in D,$$

Satisfies $\|F_1\|_2 \le c_1\|f\|_2$. Also, for

$$F_2(x) = \int_0^\infty f(\lambda)r_\lambda(x)d\lambda\,, x \in D$$

we may apply (49) and (51) to obtain

$$\int_0^\infty \left(F_2(x)\right)^2 dx \le \int_0^\infty \int_0^\infty \int_0^\infty |f(\mu)||f(\lambda)|r(\mu x)r(\lambda x)d\mu d\lambda dx$$

$$\le \int_0^\infty \int_0^\infty \left(\int_0^\infty \frac{c_0}{(1+\mu x)^2(1+\lambda x^2)^2}dx\right)|f(\mu)||f(\lambda)|d\mu d\lambda$$

$$\le \int_0^\infty \int_0^\infty \left(\int_0^\infty \frac{c_0}{(1+(\mu+\lambda)x)^2}dx\right)|f(\mu)||f(\lambda)|d\mu d\lambda$$

$$= c_2 \int_0^\infty \int_0^\infty \frac{|f(\mu)||f(\lambda)|}{\mu+\lambda}d\mu d\lambda,$$

Which is bounded by $c_2\pi\|f\|_2^2$ and so by the Hardy-Hilbert inequality,

$$\|F_2\|_2^2 \le c_2\pi\|f\|_2^2.$$

It follows that $\|\Pi f\|_2 = \|F_1 - F_2\|_2 \le c_3\|f\|_2$, and therefore Π can be continuously extended to a unique bounded linear operator on $L^2(D)$.

For $f \in C_c(D)$, we find that $p_t^D(x,y)f(\lambda)\psi_\lambda(y)$ is integrable in $(y,\lambda) \in D \times D$, so that by Theorem (2.1.3),

$$p_t^D \Pi f(x) = \int_0^\infty e^{-\lambda t} f(\lambda)\psi_\lambda(x)d\lambda,\ x \in D. \tag{56}$$

Let $f, g \in C_c(D)$ and we define $f_k(\lambda) = e^{-k\lambda t}f(\lambda)$ and $g_k(\lambda) = e^{-k\lambda t}g(\lambda)$. From (56) it follows that $p_t^D \Pi f_k = \Pi f_{k+1}$ and $p_t^D \Pi_{gk} = \Pi_{gk+1}$. Since the operators p_t^D are self-adjoint, we have

$$\int_0^\infty \Pi f(x)\Pi g(x)dx$$

$$= \int_0^\infty p_t^D \Pi f_{-1}(x)\Pi g(x)dx = \int_0^\infty \Pi f_{-1}(x)p_t^D \Pi g(x)dx = \int_0^\infty \Pi f_{-1}(x)\Pi g_1(x)dx\,.$$

By induction,

$$\int_0^\infty \Pi f(x)\Pi g(x)dx = \int_0^\infty \Pi f_{-k}(x)\Pi g_k(x)dx$$

Suppose that sup$f \subseteq (0,\lambda_0)$ and supp $g \subseteq (\lambda_0,\infty)$. Then we have

$$\int_0^\infty \Pi f(x)\Pi g(x)dx = \int_0^\infty \Pi\left(e^{-k\lambda_0 t}f_{-k}\right)(x)\Pi\left(e^{k\lambda_0 t}g_k\right)(x)dx.$$

Both $e^{-k\lambda_0 t}f_{-k}$ and $e^{k\lambda_0 t}g_k$ tend to zero uniformly as $k \to \infty$, and so $\Pi\left(e^{-k\lambda_0 t}f_{-k}\right)$ and $\Pi\left(e^{-k\lambda_0 t}f_{-k}\right)$ converge to zero in $L^2(D)$. We conclude that Πf and Πg are orthogonal in $L^2(D)$ .By an approximation argument, this is true for any $f, g \in L^2(D)$, provided that $f(\lambda)$ = 0 for $\lambda \ge \lambda_0$ and $(\lambda)\ =\ 0$ for $\lambda \le \lambda_0$.

We define

$$\mu(A)=\int_0^{\infty}\left(\Pi 1_A(x)\right)^2dx, \quad A \subseteq D.$$

Clearly

$$\mu(A)\le c_3\|1_A\|_2^2=c_3|A|, \qquad A \subseteq D.$$

Whenever $A\subseteq(0,\lambda_0)$and $B\subseteq(\lambda_0,\infty)$, we have

$\mu(A\cup B)=\int_0^{\infty}\left(\Pi 1_A(x)\right)^2dx+\int_0^{\infty}\left(\Pi 1_B(x)\right)^2dx+2\int_0^{\infty}\Pi 1_A(x)\Pi 1_B(x)dx=\mu(A)+\mu(B)$.

Finally, when $A=\bigcup_{n=1}^{\infty}A_n$, where $A_1\subseteq A_2\subseteq\cdots$ and $|A|<\infty$, the sequence 1_{A_n} converges in $L^2(D)$ to 1_Aas $n\to\infty$. Hence $\Pi 1_{A_n}$ converges to $\Pi 1_A$ in$L^2(D)$, and so $\mu(A)=\lim_{n\to\infty}\mu(A_n)$.

It follows that μ is an absolutely continuous measure on $(0,\ \infty)$. By an approximation argument, we have

$$\int_0^{\infty}\Pi f(x)\Pi g(x)dx\int_0^{\infty}f(\lambda)g(\lambda)\mu(d\lambda)$$

For any$f,g\in L^2(D)$.

Note that $\psi_\lambda(qx)=\psi_{\lambda/q}(x)$, and therefore $\Pi f_q(x)=q\Pi f(qx)$, where$f_q(x)=f(x/q)$. It follows that $\mu(qA)=q\mu(A)$and so μ must be a multiple of the Lebesgue measure on $(0,\infty)$, say $\mu(A)=c_4|A|$.This result is a version of Plancherel's theorem, where the Fourier transform is replaced by Π:

$$\int_0^{\infty}\Pi f(x)\Pi g(x)dx\int_0^{\infty}f(\lambda)g(\lambda)d\lambda$$

For any $f,g\in L^2(D)$.

The constant c_4 can be determined by considering $f(\lambda)=(1/\sqrt{q})1_{[1,1+q]}(\lambda)$, where $q>0$.

We then have$\|f\|_2=1$. On the other hand,

$$\Pi f(x)=\frac{1}{x\sqrt{q}}=\left(\cos\left(x+\frac{\pi}{8}\right)-\cos\left((1+q)x+\frac{\pi}{8}\right)\right)-\frac{1}{\sqrt{q}}\int_1^{1+q}r(\lambda x)d\lambda.$$

The $L^2(D)$ norm of the first summand converges to $\sqrt{\pi/2}$ as $q\searrow 0$, just as in the case of the Fourier sine transform. The second summand is bounded by $\sqrt{qr}(x)$ and so it converges to zero in$L^2(D)$. It follows that$c_4=\pi/2$. The Plancherel's theorem can be therefore written as

$$\int_0^{\infty}\Pi f(x)\Pi g(x)dx\int_0^{\infty}f(\lambda)g(\lambda)d\lambda \tag{57}$$

In particular, $\sqrt{\pi/2}\,\Pi$ is an isometry on$L^2(D)$. Since$\psi_\lambda(x)=\psi_x(\lambda)$, for $f,g\in C_c(D)$ and therefore for any $f,g\in L^2(D)\big)$ we also have

$$\int_0^\infty \Pi f(x)g(x)dx \int_0^\infty f(\lambda)\Pi g(\lambda)d\lambda,$$

Which combined with (57) yields that $\Pi^2 f = (\pi/2)f$. we collect the above results in the following theorem.

Theorem (2.1.5)[49]: The operator $\sqrt{\pi/2}\,\Pi: L^2(D) \to L^2(D)$ gives a spectral representation of $\mathcal{A}_D$ and the semi group (P_t^D), acting on$L^2(D)$, where $D = (0,\infty)$; that is, for any $f \in L^2(D)$, we have the following:

(a) $\|f\|_2 = \sqrt{\pi/2}\,\|\Pi f\|_2 f$ (Plancherel's theorem);
(b) $\Pi P_t^D f(\lambda) = e^{-\lambda t}\Pi f(\lambda)$;
(c) f is in the domain of $\mathcal{A}_D$ if and only if $\lambda \Pi f(\lambda)$ is square integrable;
(d) $\Pi \mathcal{A}_D f(\lambda) = -\lambda \Pi f(\lambda)$.

Furthermore, $\Pi^2 = (\pi/2)\mathrm{Id}$ (inversion formula).

The aim is to compute an explicit formula of the transition density $P_t^D(x,y)$ of the Cauchy process killed on exiting a half-line $D = (0,\infty)$, or the heat kernel for$\mathcal{A}_D$. Let us note that the transition density of the Brownian motion killed on exiting a half-line (0, ∞) equals

$$\frac{1}{\sqrt{2\pi t}} e^{-|x-y|^2/2t} - \frac{1}{\sqrt{2\pi t}} e^{-|x+y|^2/2t},$$

Which follows from the reflection principle. For the Cauchy process we cannot
Use the reflection principle and the computation of $P_t^D(x,y)$ requires using much more complicated methods.

Theorem (2.1.6)[49]: For D = (0, ∞) and any$g \in L^p(D)$, with p∈ [1, ∞], we have

$$P_t^D g(x) = \int_0^\infty P_t^D(x,y)g(y)dy, \quad t,x > 0, \tag{58}$$

Where

$$P_t^D(x,y) = \frac{1}{\pi}\frac{t}{t^2 + (x-y)^2} - \frac{1}{xy}\int_0^t \frac{f(s/x)f\left(\frac{(t-s)}{y}\right)}{s/x + (t-s)/y} ds, \ \ t,x,y > 0, \tag{59}$$

and

$$f(s) = \frac{1}{\pi}\frac{s}{1+s^2} exp\left(\frac{1}{\pi}\int_0^\infty \frac{\log(s+w)}{1+w^2} dw,\right), \qquad s > 0. \tag{60}$$

For $s > 0$, note that f is positive continuous and bounded. This follows by the fact that $f(s) = (1/\pi)(s/1+s^2)e^{\eta}(s)$ and (105). The function $P_t^D(x,y)$ can be effectively computed by numerical integration. Indeed, by the same arguments we have

$$f(s) = \frac{1}{\pi}\frac{s^{1-arctan/\pi}}{(1+s^2)^{3/4}} exp\left(\frac{i}{2\pi}\left(L_{i_2}(is) - L_{i_2}(-is)\right)\right),$$

Where $L_{i_2}$is the dilogarithm function.

Proof. For $g \in C_c(D)$ we have $\Pi P_t^D g(\lambda) = e^{-\lambda t}\Pi g(\lambda)$ (see (55) and Theorem (2.1.5)). Applying $\Pi^{-1} = (2/\pi)\Pi$ to both sides of this identity yields

$$P_t^D g(x) = \frac{2}{\pi}\int_0^\infty e^{-\lambda t}\Pi g(\lambda)\psi_\lambda(x) = \frac{2}{\pi}\int_0^\infty \int_0^\infty e^{-\lambda t}\Pi g(\lambda)\psi_\lambda(x)\psi_\lambda(y)g(y)dyd\lambda.$$

By the Fubini's theorem, (58) holds with

$$P_t^D(x,y) = \frac{2}{\pi}\int_0^\infty \psi_\lambda(x)\psi_\lambda(y)e^{-\lambda t}d\lambda. \tag{61}$$

By an approximation argument, (58) holds for $g \in L^p(\mathbb{R})$ with any $p \in [1,\infty]$. we will now prove (59).

Suppose first that$x < y$, and let$t = t_1 + t_2 > 0$, with$t_1, t_2 > 0$. By Plancherel's theorem and identities$\psi_\lambda(x) = \psi_\lambda(\lambda)$, and $\overline{\mathcal{L}\psi_y(z)} = \mathcal{L}\psi_y(\bar{z})$ we have

$$P_t^D(x,y) = \frac{\pi}{2}\int_0^\infty \left(\psi_x(\lambda)e^{-t_1\lambda}\right)\left(\psi_y(\lambda)e^{-t_2\lambda}\right)d\lambda$$

$$= \frac{1}{\pi^2}\int_{-\infty}^\infty \mathcal{L}\psi_x(t_1 + is)\,\mathcal{L}\psi_y(t_2 - is)ds = \frac{1}{2\pi i}\int_{t_1-i\infty}^{t_1+i\infty} R(z)dz, \tag{62}$$

Where (see (43))

$$R(z) = \frac{2}{\pi}\mathcal{L}\psi_x(z)\mathcal{L}\psi_y(t-z) = \frac{1}{\pi}\frac{xyexp\left(B(z/x) + B((t-z)/y)\right)}{(x^2+z^2)(y^2 + (t-z)^2)}.$$

Note that R is defined on $\mathbf{C}$ and it is meromorphic in $\boldsymbol{C} \setminus ((-\infty,0] \cup [t,\infty))$ with simple poles at $\pm ix$ and$t \pm iy$. Let $z \in \boldsymbol{C} \setminus [0,t]$. By (37) and the identity$(1 + iz\sigma(z))(1 - iz\sigma(z)) = 1 + (z\sigma(z))^2 = 1 + z^2$, for all$z \in \boldsymbol{C}$, we have

$$R(z) = \frac{1}{\pi}\frac{\left(1 - i(z/x)\sigma(z/x)\right)\left(1 - i\left((t-z/y)\sigma\,(t-z)/y\right)\right)exp\left(-B(-z/x) - B(-\,(t-z)/y)\right)}{xy(1 + z^2/x^2)(1 + (t-z)^2/y^2)}$$

$$= \frac{1}{\pi}\frac{exp\left(-B(-z/x) - B(-\,(t-z)/y)\right)}{xy\left(1 + i(z/x)\sigma(z/x)\right)\left(1 + i(t-z)/y\sigma\left((t-z)/y\right)\right)}$$

Since $\sigma((t-z)/y) = -\sigma(z/x)$ for $z \in \boldsymbol{C} \setminus [0,t]$, it follows that, for $z \in \boldsymbol{C} \setminus [0,t]$,

$$R(z) = \frac{\exp\left(-B(-z/x) - B(-(t-z)y)\right)}{\pi xy(z/x + (t-z)/y)}\left(\frac{z/x}{1 + i(z/x)\sigma(z/x)} + \frac{(t-z)/y}{1 + i\,(t-z)/y\sigma((t-z)/y)}\right).$$

We therefore have $R(z) = R_1(z) + R_2(z)$for z $\boldsymbol{C} \setminus [0,t]$, where, again using (37), we obtain

$$R_1(z) = \frac{\exp\left(-B(-z/x) - B(-(t-z)y)\right)}{\pi xy(z/x + (t-z)/y)}.\frac{z/x}{1 + i(z/x)\sigma(z/x)} \tag{63}$$

and

$$R_1(z) = \frac{\exp\left(-B(z/x) - B(-(t-z)y)\right)}{\pi xy(z/x + (t-z)/y)}.\frac{z/x}{1 + z^2/x^2}, \tag{64}$$

Also, in a similar manner,

$$R_2(z) = \frac{\exp\big(-B(-z/x) + B(-(t-z)y)\big)}{\pi xy(z/x + (t-z)/y)} . \frac{(t-z)\ /y}{1 + (t-z)^2/y^2}, \tag{65}$$

and

$$R_2(z) = \frac{\exp\big(B(z/x) + B(-(t-z)y)\big)}{\pi xy(z/x + (t-z)/y)} . \frac{(t-z)\ /y}{1 + (t-z)^2/y^2} . \frac{1}{1 - i(z/x)\sigma(z/x)} \tag{66}$$

The only zero of $\frac{z}{x} + \frac{t-z}{y} is\ z = \frac{tx}{x-y} < 0$. Hence $R_1(z)$ is holomorphic in the set $\{Rez > 0\}\ [0,t]$(by (63)), bounded in the neighbourhood of $[0,t]$, and it decays as $|z|^{-2}$ at infinity (by (110)). Also, $R_2(z)$ is meromorphic in the set $\{Re\ z < t\}\ [0,t]$ (by (65)) with a simple pole at $\frac{tx}{x-y}$, bounded near$[0,t]$, and it decays as $|z|^{-2}$ at infinity.

For $n = 1,2,\ldots$, let γ be the positively oriented contour consisting of the following:

(i) Two vertical segments $\gamma_1 = \left[t_1 - ni, t_1 - \frac{i}{n}\right]$ $and\ \gamma_5 = \left[t_1 + \frac{i}{n}, t_1 + ni\right]$;

(ii) two horizontal segments $\gamma_2 = \left[t_1 - \frac{i}{n}, -\frac{i}{n}\right]$ $and\ \gamma_4 = \left[\frac{i}{n}, t_1 + \frac{i}{n}\right]$;

(iii) two semicircles $\gamma_3 = \left\{|z| = \frac{1}{n}, Re\ z \le 0\right\}$ $and\ \gamma_6 = \{|z - t_1| = n, Re\ z \le t_1\}$.

Clearly, $\int_{\gamma_2\cup\gamma_4} R_2(z)dz \to \int_0^{t_1} R_2(z)$ Converges to $\int_{t_1-i\infty}^{t_1+i\infty} R_2(z)dz$ as $n \to \infty$. The integrals over γ_3 and γ_6 converge to zero by the properties of R_2. Finally, by (66),

$$\int_{\gamma_2\cup\gamma_4} R_2(z)dz \to \int_0^{t_1} R_2(z) = \frac{\exp\Big(B(s/x) + B\big((t-s)y\big)\Big)}{\pi xy(s/x + (t-s)/y)} . \frac{(t-s)\ /y}{1 + (t-s)^2/y^2}$$

$$\times \left(\frac{1}{1 - i(s/x)} - \frac{1}{1 - i(s/x)}\right) ds = \int_0^{t_1} \frac{2\pi i f(s/x) f((t-s)/y)}{xy(s/x + (t-s)/y)} ds.$$

Therefore, by the residue theorem,

$$\frac{1}{2\pi i} \int_{t_1-i\infty}^{t_1+i\infty} R_2(z)dz = -\int_0^{t_1} \frac{f(s/x) f((t-s)/y)}{xy(s/x + (t-s)/y)} ds + Re\ s\left(R_2, \frac{tx}{x-y}\right). \tag{67}$$

In a similar manner, using (64) and analogous contours γ consisting of two segments of the line Re $z = t_1$, two segments parallel to $[t_1, t]$, and two semi-circles centered at *t* (the small one) and t_1 (the large one), both contained in $\{Re\ z \ge t_1\}$, we obtain that 1

$$\frac{1}{2\pi i} \int_{t_1-i\infty}^{t_1+i\infty} R_1(z)dz = -\int_{t_1}^{t} \frac{f(s/x) f((t-s)/y)}{xy(s/x + (t-s)/y)} ds. \tag{68}$$

Therefore, (62), (67) and (68) yield that

$$P_t^D(x,y) = \int_0^t \frac{f(s/x) f((t-s)/y)}{xy(s/x + (t-s)/y)} ds + Res\left(R_2, \frac{tx}{x-y}\right).$$

For $z = tx/(x-y)$ we have $z/x = t/(x-y) = -(t-z)/y$. Therefore, by (65) we get

$$\text{Res}\left(R_2,\frac{tx}{x-y}\right)=\frac{1}{\pi(y-x)}.\frac{-t/(x-y)}{1+t^2/(x-y)^2}=\frac{1}{\pi}\frac{t}{t^2+(x-y)^2},$$
and (59) follows for $x < y$.

When $x > y$, simply note that $P_t^D(x,y)=P_t^D(y,x)$ (see (61) or, for example, [64]), and that the right-hand side of (59) has the same symmetry property (this follows by a substitutions $s=t-v$). Finally, for $x = y$ simply use the continuity of $P_t^D(x,y)$ and f.

For the next result, we need the following simple observation, similar to the derivation of (39). By (37) we have $Im\, e^{-B(-S)} = -(s/(1+S^2))e^{B(S)}$ for $s > 0$. Hence the function f defined by (60) satisfies

$$f(s)=\frac{1}{\pi}\frac{s}{1+s^2}e^{\eta(s)}.\frac{1}{\pi}\frac{s}{1+s^2}e^{B(s)}=m-\frac{1}{\pi}Im\, e^{-B(-s)},\qquad s > 0.$$

If we extend f by $f(s) = 0$ for $s < 0$, then $f(-s) = (1/\pi)\text{Im}\, e^{-B(s)}$ for all real s. Since $e^{-B(z)}$ is in $H^p(C_+)$ for $p \in (2,\infty)$ (see (109)), the Hilbert transform of f is given by (see (6))

$$Hf(s)= -\frac{1}{\pi}Re\, e^{-B(-s)}=\frac{1}{\pi}e^{-\eta(-s)}, s\in\mathbb{R}.$$

It follows that

$$Hf(-s)=\tfrac{1}{\pi^2}\tfrac{s}{1+s^2}\tfrac{1}{f(s)},\ \ s>0,\ \text{and}\qquad Hf(0)=\tfrac{1}{\pi}\ . \tag{69}$$

The following result has been previously obtained with different methods by Darling [67];see also [53].

Theorem (2.1.7)[49]: (Darling [67]). For $D=(0,\infty)$, we have

$$P_x(\tau_{\boldsymbol{D}}\in dt)=\frac{1}{\pi}\frac{s}{1+s^2}exp\left(\frac{1}{\pi}\int_0^\infty\frac{\log(t/x+w)}{1+w^2}dw\right)dt \tag{70}$$

Using the function f defined in (60), we have $P_x(\tau_{\boldsymbol{D}}\in dt)=(1/t)f(t/x)dt$.

Proof: By Theorem (2.1.6) we have

$$P_x(\tau_{\boldsymbol{D}}>t)=\int_{\boldsymbol{0}}^\infty P_t^D(x,y)dy$$

$$\frac{1}{\pi}\int_0^\infty\frac{t}{t^2+(x-y)^2}dy-\int_0^t\int_0^\infty\frac{1}{xy}\frac{f(s/x)f((t-s)/y)}{(s/x+(t-s)/y)}dyds. \tag{71}$$

By a substitution $w = (t - s)/y$ we obtain

$$\int_0^\infty\frac{f(s/x)f((t-s)/y)}{(s/x+(t-s)/y)}dy=\frac{f(s/x)}{x}\int_0^\infty\frac{f(w)}{w(s/x+w)}dw$$
$$=\frac{f(s/x)}{s}\left(\int_0^\infty\frac{f(w)}{w}dw-\int_0^\infty\frac{f(w)}{s/x+w}d\,w\right).$$

The right-hand side equals $(\pi/s)f(s/x)(-Hf(0)+Hf(-s/x))$. this, (69) and (71) give

$$P_x(\tau_{\boldsymbol{D}} > t) = \frac{1}{\pi}\int_0^{\infty} \frac{t}{t^2 + (x-y)^2}dy - \int_0^t \frac{f(s/x)}{x}ds + \frac{1}{\pi}\int_0^t \frac{x}{x^2+s^2}ds$$

By substitution of $v = x - y$ in the first integral and $v = xt/s$ in the third one, we obtain

$$P_x(\tau_{\boldsymbol{D}} > t) = \frac{1}{\pi}\int_{-\infty}^{x} \frac{t}{t^2+v^2}dy - \int_0^t \frac{f(s/x)}{s}ds + \frac{1}{\pi}\int_x^{\infty} \frac{t}{x^2+v^2}ds = 1 - \int_0^t \frac{f(s/x)}{s}ds.$$

The result follows by differentiation and (60).

Theorem (2.1.7) can be stated in terms of the two-dimensional Brownian motion; namely, we obtain the distribution of some local time of the two-dimensional Brownian motion at some entrance time. For the one-dimensional Brownian motion similar results were widely studied and are usually called Ray–Knight theorems [56], [60], [61], [79] [80], [88], [90].

Corollary (2.1.8)[49]: Let $B_t = \left(B_t^{(1)}, B_t^{(2)}\right)$ be the two-dimensional Brownian motion and let

$$L(t) = \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon}\int_0^t \chi(-\varepsilon,\varepsilon)\left(B_t^{(2)}\right)ds$$

Be the local time of $B_t$on the line $(-\infty,\infty) \times \{0\}$. Let $A = (-\infty, 0] \times \{0\}$ and let$T_A = inf\ \{t \geq 0 : B(t) \in A\}$ be the first entrance time for A. Then, for any x > 0, we have

$$P^{(x,0)}(L(T_A) \in dt) = \frac{1}{\pi}\frac{x}{x^2+t^2} exp\left(\frac{1}{\pi}\int_0^{\infty} \frac{\log(t/x+s)}{1+s^2}ds\right)dt$$

For $(x, y) \in \mathbb{R}^{\mathbf{2}}, y \neq 0\ and\ t \geq 0$ we have

$$P^{(x,y)}(L(T_A) < t) = \frac{1}{\pi}\int_{-\infty}^{0} \frac{|y|}{y^2+(x-u)^2}du + \frac{1}{\pi}\int_{-\infty}^{0} \frac{|y|}{y^2+(x-u)^2}P^{(x,0)}(L(T_A) \leq t).$$

Proof: Let $\eta_t = inf\{s > 0 : L(s) > t\}$ be the inverse of the local time $L(t)$. It is well known (see, for example, [94]) that the one-dimensional Cauchy process can be identified with$B(1)(\eta t)$. With this relation, we have$L(T_A) = \tau(0,\infty)$, where $\tau(0,\infty) = inf\ \{t \geq 0 : X_t \notin (0,\infty)\}$.

This and Theorem (2.1.7) give the first equality. The second equality follows by the harmonicity of $(x,y) \to P\ (x,y)(L(T_A)\ \ t)$ in $\{(x,y) \in \mathbb{R}^{\mathbf{2}} : y > 0\}$ and in $\{(x,y) \in \mathbb{R}^{\mathbf{2}} : y < 0\}$.

The interval $D = (-1,1)$ is studied. Let *n* be a positive integer and$\mu_n = n\pi/2 - \pi/8$. Our goal is to show that μ_n is close toλ_n, the *n*th eigenvalue of the semigroup (P_t^D).

Let q be the function equal to 0 on $(\infty, \frac{1}{3})$ and to 1 on $(\frac{1}{3}, \infty)$, defined by (115). We construct approximations to eigenfunctions of (P_t^D) by combining the eigenfunctions $\psi_{\mu_n}(1+x)$ and $\psi_{\mu_n}(1-x)$ for half-line. For a symmetric eigenfunction, when *n* is odd, let

$$\tilde{\varphi}_n(x) = q(-x)\psi_{\mu_n}(1+x) - q(x)\psi_{\mu_n}(1-x).$$

$$= (-1)^{(n-1)/2}\cos(\mu_n x)\,1_D(x) - q(-x)r_{\mu_n}(1+x) - q(x)r_{\mu_n}(1-x) \quad (72)$$

For an antisymmetric eigenfunction, when n is even, we define

$$\tilde{\varphi}_n(x) = q(-x)\psi_{\mu_n}(1+x) - q(x)\psi_{\mu_n}(1-x)$$

$$= (-1)^{(n-1)/2}\sin(\mu_n x)\,1_D(x) - q(-x)r_{\mu_n}(1+x) - q(x)r_{\mu_n}(1-x) \quad (73)$$

Lemma (2.1.9)[49]: With the above definitions,

$$\|\mathcal{A}_D\tilde{\varphi}_n + \mu_n\tilde{\varphi}_n\|_2 < \sqrt{1.21 + \frac{8.00}{\mu_n} + \frac{13.66}{\mu^2 n}}.\frac{1}{\mu_n}. \quad (74)$$

Proof: Note that we have

$$\tilde{\varphi}_n(x) - \psi_{\mu_n}(1+x) = (1 - q(-x)\psi_{\mu_n}(1+x) - (-1)^n q(x)\psi_{\mu_n}(1-x)$$
$$= -q(x)\left(\psi_{\mu_n}(x+1) + (-1)^n\psi_{\mu_n}(1-x)\right)$$
$$= q(x)\left(\psi_{\mu_n}(x+1) + (-1)^n\psi_{\mu_n}(1-x)\right)$$
$$- \sin\left(\mu_n(1+x) + \frac{\pi}{8}\right)1_{[1,\infty)}(x)$$

Define $h(x) = \sin(\mu_n(1+x) + \pi/8)(\mathbf{1})_{[\mathbf{1},\infty)}(x)$ and $f(x) = r_{\mu_n}(1+x) + (-1)^n r_{\mu_n}(1-x)$, $g(x) = q(x)f(x)$. by (49), (50) and (46), we have

$$M_0 = \sup_{x\in(-1/3,1/3)}|f(x)| \le r\left(\frac{2\mu_n}{3}\right) + r\left(\frac{4\mu_n}{3}\right) \le \frac{45\sqrt{2}}{32\pi\mu_n^2},$$

$$M_1 = \sup_{x\in(-1/3,1/3)}|f'(x)| \le -\mu_n r'\left(\frac{2\mu_n}{3}\right) - \mu_n r'\left(\frac{4\mu_n}{3}\right) \le \frac{243\sqrt{2}}{64\pi\mu_n^2}$$

$$M_2 = \sup_{x\in(-1/3,1/3)}|f''(x)| \le \mu_n^2 r''\left(\frac{2\mu_n}{3}\right) + \mu_n^2 r''\left(\frac{4\mu_n}{3}\right) \le \frac{4131\sqrt{2}}{256\pi\mu_n^2},$$

$$1 = \int_0^\infty |f(x)|\,dx \le \int_0^\infty r_{\mu_n}(1+x)dx + \int_0^1 r_{\mu_n}(1-x)dx = \frac{1}{\mu_n}\int_0^\infty r(y)dy$$
$$= \left(\cos\frac{\pi}{8} - \frac{\sqrt{2}}{2}\right)\frac{1}{\mu_n};$$

The notation here corresponds. By (116) and (117),

$$|\mathcal{A}_D g(z)| < \frac{0.605}{\mu_n^2} + \frac{0.156}{\mu_n}, z \in \left(-1, -\frac{1}{3}\right); \quad (75)$$

$$|\mathcal{A}_D g(z)| < \frac{4.444}{\mu_n^2} + \frac{0.622}{\mu_n}, \quad z \in \left(-\frac{1}{3}, 0\right). \quad (76)$$

Furthermore, $|g(z)| = 0$ for $z \in \left(-1, -\frac{1}{3}\right)$ and

$$|\mu_n g(z)| < \frac{\mu_n}{2}M_0 + \frac{0.317}{\mu_n}, z \in \left(-\frac{1}{3}, 0\right). \quad (77)$$

Finally, for $z < 0$ we have

$$\left|(-\Delta)^{1/2}h(z)\right| = \frac{1}{\pi}\left|\int_0^\infty \frac{\sin(\mu_n(1+x)+\pi/8)}{(x-z)^2}\right|$$

$$\leq \frac{1}{\pi(1-z)^2}\int_1^{1+\pi/\mu_n}\left|\frac{\sin(\mu_n(1+x)+\pi/8)}{(x-z)^2}\right|dx = \frac{1}{\pi\mu_n(1-z)^2},$$

So that

$$\left|(-\Delta)^{1/2}h(z)\right| < \frac{0.180}{\mu_n}, z \in \left(-1,-\frac{1}{3}\right); \tag{78}$$

$$\left|(-\Delta)^{1/2}h(z)\right| < \frac{0.319}{\mu_n}, z \in \left(-\frac{1}{3},0\right). \tag{79}$$

Since, for $z \in (-1,0)$, we have

$$|\mathcal{A}_D\tilde{\varphi}_n(z) + \mu_n\tilde{\varphi}_n(z)| \leq \left|(-\Delta)^{1/2}h(z)\right| + \left|(-\Delta)^{1/2}g(z)\right| + |\mu_n g(z)|$$

The estimates (75)–(79) yield that

$$|\mathcal{A}_D\tilde{\varphi}_n(z) + \mu_n\tilde{\varphi}_n(z)| < \frac{0.605}{\mu_n^2} + \frac{1.258}{\mu_n},\ \ z \in \left(-1,-\frac{1}{3}\right); \tag{80}$$

$$|\mathcal{A}_D\tilde{\varphi}_n(z) + \mu_n\tilde{\varphi}_n(z)| < \frac{4.444}{\mu_n^2} + \frac{0.622}{\mu_n},\ \ z \in \left(-\frac{1}{3},0\right); \tag{81}$$

By symmetry, estimates similar to (80) and (81) hold for$z \in (0,1)$. The estimate (74) follows.

The estimate of the $L^2(D)$ norm of $\tilde{\varphi}_n$ plays an important role in what follows. We have

$$\sqrt{1-\frac{0.52}{\mu_n}} \leq \|\tilde{\varphi}_n\|_2 \leq \sqrt{1+\frac{1.37}{\mu_n}}. \tag{82}$$

indeed, the lower bound follows by (46), (72), (73) and symmetry:

$$\|\tilde{\varphi}_n\|_2^2 \geq \int_{-1}^{1}\left(\sin\left(\mu_n(x+1)+\frac{\pi}{8}\right)\right)^2 dx$$

$$-4\int_{-1}^{1}\left|q(-x)r_{\mu_n}(1+x)\sin\left(\mu_n(x+1)+\frac{\pi}{8}\right)\right|dx$$

$$\geq \left(1+\frac{\sqrt{2}}{4\mu_n}\right) - \frac{4}{\mu_n}\left(\cos\frac{\pi}{8}-\frac{\sqrt{2}}{2}\right)$$

In a similar manner, using also (47),

$$\|\tilde{\varphi}_n\|_2^2 \leq \left(1+\frac{\sqrt{2}}{4\mu_n}\right) - \frac{4}{\mu_n}\left(\cos\frac{\pi}{8}-\frac{\sqrt{2}}{2}\right) + 4\int_{-1}^{1}\left(r\left(\mu_n(1+x)\right)\right)^2 dx$$

$$\leq \left(1+\frac{\sqrt{2}}{4\mu_n}\right) - \frac{4}{\mu_n}\left(\cos\frac{\pi}{8}-\frac{\sqrt{2}}{2}\right) + \frac{4\left(\Gamma(3/4)\right)^2}{\pi\left(\Gamma(1/4)\right)^2\mu_n}.$$

We continue denoting by φ_j the eigenfunctions of (P_t^D), by λ_j $(\lambda_j > 0)$ the corresponding eigenvalues, and by $\tilde{\varphi}_n$ and μ_n the approximations of the previous. Fix

$n \geq 1$. Since $\tilde{\varphi}_n \in L^2(D)$, we have $\tilde{\varphi}_n = \sum_j a_j \varphi_j$ for some a_j. Moreover, $\|\tilde{\varphi}_n\|_2^2 = \sum_j a_j^2$ and $\mathcal{A}_D \tilde{\varphi}_n = -\sum_j a_j \varphi_j$.

Let $\lambda_{k(n)}$ be the eigenvalue nearest to μ_n. Then

$$\|\mathcal{A}_D \tilde{\varphi}_n - \mu_n \tilde{\varphi}_n\|_2^2 = \sum_{j=1}^{\infty} (\lambda_j - \mu_n)^2 a_j^2$$
$$\geq (\lambda_{k(n)} - \mu_n)^2 \sum_{j=1}^{\infty} a_j^2 \geq (\lambda_{k(n)} - \mu_n)^2 \|\tilde{\varphi}_n\|_2^2.$$

By (74) and (82), it follows that

$$|\lambda_{k(n)} - \mu_n| \leq \sqrt{\frac{1.21 + 8.00/\mu_n + 13.66/\mu_n^2}{1 - 0.52/\mu_n}} \cdot \frac{1}{\mu_n} \tag{83}$$

The right-hand side is a decreasing function of n, so that $|\lambda_{k(n)} - \mu_n| < 0.098\pi < \pi/10$ whenever $n \geq 4$. Hence we have the following result.

Lemma (2.1.10)[49]: Each interval $(n\pi/2 - \pi/4, n\pi/2)$, with $n \geq 4$, contains an eigenvalue $\lambda_{k(n)}$.

In particular $\lambda_{k(n)}$ are distinct for $n \geq 4$. We will now prove that there are only three eigenvalues not included in the above lemma. For $t > 0$, we have (see, for example, [52], [82])

$$\sum_{j=1}^{\infty} e^{-\lambda_j t} = \int_D \sum_{j=1}^{\infty} e^{-\lambda_j t} \left(\varphi_j(x)\right)^2 = \int_D P_t^D(x,x) dx \leq \int_D p_t(0)\, dx = \frac{2}{\pi t}.$$

On the other hand,

$$\sum_{j=1}^{\infty} e^{-\lambda_{k(n)} t} \geq \sum_{j=4}^{\infty} e^{-(n\pi/2)t} = \frac{e^{-2\pi t}}{1 - e^{-(\pi/2)t}} \geq \frac{2}{\pi t} - \frac{7}{2}$$

For small $t > 0$. It follows that there are at most three eigenvalues of (P_t^D) other than $\lambda_{k(n)}(4)$. Furthermore, we have $1 < \lambda_1 < 3\pi/8, 2 \leq \lambda_2 \leq \pi$ and $3.4 \geq \lambda_3 \quad 3\pi/2$ by [50]. Therefore, $k(n) = n$ for $n \geq 4$, and also by (83), we see that $\lambda_3 > 3.83$. We have thus proved the following theorem.

Theorem (2.1.11)[49]: We have

$$1 < \lambda_1 < \frac{3\pi}{8}, 2 \leq \lambda_2 \leq \pi, 3.83 < \lambda_3 \leq \frac{3\pi}{2},$$

and

$$\frac{n\pi}{2} - \frac{\pi}{8} - \frac{\pi}{10} < \lambda_n < \frac{n\pi}{2} - \frac{\pi}{8} + \frac{\pi}{10} \quad (n \geq 4).$$

In particular, all eigenvalues of (P_t^D) are simple, $|\lambda_n - \lambda_m| > 0.69$ when $n \neq m$ and $|\lambda_n - \lambda_m| > 3\pi/10$ if, moreover, $n \geq 4$. Furthermore, as $n \to \infty$,

$$\lambda_n = \frac{n\pi}{2} - \frac{\pi}{8} + O\left(\frac{1}{n}\right). \tag{84}$$

More precisely,

$$\left|\lambda_n - \left(\frac{n\pi}{2} - \frac{\pi}{8}\right)\right| \leq \frac{1}{n}, \quad n \geq 1, \tag{85}$$

That is, the constant in $O(1/n)$ notation in (84) is not greater than 1. Indeed, by (83), formula (85) holds for $n \geq 7$, and for $n \leq 6$ one can use the estimates (88). Without referring to numerical calculation of upper and lower bounds, one can use (83) for $n \geq 4$

and estimatesof λ_1, λ_2 and λ_4 of Theorem (2.1.11) to obtain (85) with $1/n$ replaced by $3/2n$.

The approximations $\tilde{\varphi}_n$ to the eigenfunctions φ_n were constructed and it was proved that $\mu_n = n\pi/2 - \pi/8$ is close to λ_n. Now we show that $\tilde{\varphi}_n$ is close to φ_n in $L^2(D)$. Let $n \geq 4$ be fixed. Recall that $\tilde{\varphi}_n = \sum_{j=1} a_j \varphi_j$; with no loss of generality we may assume that$a_n > 0$. For $j \neq n$ we have $|\mu_n - \lambda_j| \geq 3\pi/10$. Therefore,

$$\|\mathcal{A}_D \tilde{\varphi}_n - \mu_n \tilde{\varphi}_n\|_2^2 = \sum\nolimits_{j=1}^{\infty} (\mu_n - \lambda_j)^2 a_j^2 \geq (\mu_n - \lambda_j)^2 a_n^2 + \frac{9\pi^2}{100} \sum_{j \neq n} a_j^2.$$

We denote the left-hand side byM_n^2; the upper bound for M_n is given in (74). We have

$$\|\tilde{\varphi}_n - a_n \tilde{\varphi}_n\|_2^2 = \sum\nolimits_{j \neq n}^{\infty} a_j^2 \leq \frac{100 M_n^2}{9\pi^2}.$$

Therefore,

$$\|\tilde{\varphi}_n - \|\tilde{\varphi}_n\|_2 \varphi_n\|_2 \leq \|\tilde{\varphi}_n - a_n \varphi_n\|_2 + (\|\tilde{\varphi}_n\|_2 - a_n) \leq 2\|\tilde{\varphi}_n - a_n \varphi_n\|_2 \leq \frac{20 M_n}{3\pi}$$

This, together with (82), yields the following result.

Lemma (2.1.12)[49]: Let $n \geq 4$. With the notation of the previous, we have $1 - \frac{0.52}{\mu_n} < \|\tilde{\varphi}_n\|_2^2 < 1 + \frac{1.37}{\mu_n}$, and

$$\|\tilde{\varphi}_n - a_n \varphi_n\|_2 \leq \frac{20}{3\pi} \sqrt{1.21 + \frac{8.00}{\mu_n} + \frac{13.66}{\mu_n^2}} . \frac{1}{\mu_n} .$$

In particular, for $n \geq 4$, by the above result and (82),

$$\left\| \frac{\tilde{\varphi}_n}{\|\tilde{\varphi}_n\|_2} - \varphi_n \right\|_2 \leq \frac{20}{3\pi} \frac{M_n}{\sqrt{1 - 0.52/\mu_n}} < \frac{20}{3\pi} . \frac{\pi}{10} = \frac{2}{3}$$

Since $\tilde{\varphi}_n$ is symmetric or antisymmetric when *n* is odd or even, respectively, we have the alternating type of symmetry of φ_n.

Corollary (2.1.13)[49]: The function $\varphi_n$is symmetric when n is odd, and antisymmetric when n is even.

Proof: For $n \leq 3$ this is a result of [50]. When $n \geq 4$, we find that $\varphi_n$is either symmetric or antisymmetric, and the distance between φ_n and the normed $\tilde{\varphi}_n$ does not exceed 2 3. Therefore $\varphi_n$has the same type of symmetry as $\tilde{\varphi}_n$.

Corollary (2.1.14)[49]: As $n \to \infty$,

$$\left\| \varphi_n - \sin\left(\left(\frac{n\pi}{2} - \frac{\pi}{8} \right) (1 + x) + \frac{\pi}{8} \right) \right\|_2 = o\left(\frac{1}{\sqrt{n}} \right).$$

By a rather standard argument, $\|\varphi_n\|_\infty \leq \sqrt{e\lambda_n/\pi}$; see, for example, [83]. A slight modification gives the following result.

Proposition (2.1.15)[49]: Let $c = \|\tilde{\varphi}_n\|_2$. Then

$$\|\varphi_n\|_\infty \leq \frac{1}{c} \left(\sqrt{\frac{e\lambda_n}{\pi}} . \|c\varphi_n - \tilde{\varphi}_n\|_2 + \|\psi_{\mu_n}\|_\infty \right). \tag{86}$$

Proof: Let $t = 1/2\lambda_n$. Using the Cauchy–Schwarz inequality, Plancherel theorem and inequality $P_t^D(x, y) \leq p_t(x - y)$, we obtain

$$c|\varphi_n(x)| \le e^{\lambda_n t}|P_t^D(c\varphi_n - \tilde{\varphi}_n)(x)|_2 + e^{\lambda_n t}|P_t^D\tilde{\varphi}_n(x)|$$
$$\le \sqrt{e}.\sqrt{\int_{-\infty}^{\infty}(p_t(x-y))^2 dy}\,.\,\|c\varphi_n - \tilde{\varphi}_n\|_2 + \sqrt{e}\|\tilde{\varphi}_n\|_\infty$$
$$= \sqrt{\frac{e}{2\pi}\int_{-\infty}^{\infty} e^{-2t|z|}dz}\,.\,\|c\varphi_n - \tilde{\varphi}_n\|_2 + \sqrt{e}\left\|\psi_{\mu_n}\right\|_\infty$$
$$= \sqrt{\frac{e}{2\pi t}}\|c\varphi_n - \tilde{\varphi}_n\|_2 + \sqrt{e}\left\|\psi_{\mu_n}\right\|_\infty,$$
and the proposition follows.

Corollary (2.1.16)[49]: The functions $\varphi_n(x)$ are uniformly bounded in n≥ 1 and $x \in D$. More precisely, for $n \ge 1$, we have

$$\|\varphi_n\|_\infty < 3. \qquad (87)$$

Indeed, for $n \ge 6$, this follows from (86) when the right-hand side is estimated using Theorem (2.1.11), Lemma (2.1.12) and (54). For $\le$ it is a consequence of

$$\|\varphi_n\|_\infty \le \sqrt{e\lambda_n/\pi} \text{ and } \lambda_n \le n\pi/2.$$

We give numerical estimates for the eigenvalues λ_n of the semigroup (P_t^D) when$D = (-1,1)$. The following estimates hold true; the upper bounds are given in the superscript and the lower bounds in the subscript:

$$\begin{aligned} \lambda_1 &= 1.1577738836977^{92}_{58}, & \lambda_6 &= 9.032852690^{50838}_{48857},\\ \lambda_2 &= 2.75475474221^{695}_{510}, & \lambda_7 &= 10.6022930996^{3854}_{1113},\\ \lambda_3 &= 4.31680106659^{758}_{303}, & \lambda_8 &= 12.1741182627^{6180}_{2585},\\ \lambda_4 &= 5.8921474709^{4751}_{3908}, & \lambda_9 &= 13.744109059^{44402}_{39799},\\ \lambda_5 &= 7.460175739^{41122}_{39764}, & \lambda_{10} &= 15.3155549960^{8382}_{2690}. \end{aligned} \qquad (88)$$

This is the result of numerical computation of the eigenvalues of 900 × 900 matrices using Mathematica 6.01. Different methods are used for the upper and lower bounds, as is described below. For the Green operator and the Green function, see [55]. The explicit formula for the Green function of the interval was first obtained by Riesz [91].

For the upper bounds, we use the Rayleigh–Ritz method, see [96]. Let $G_D$be the Green operator forP_t^D. Then$G_D\varphi_n = (1/\lambda_n)\varphi_n$. The following min–max variational characterization of eigenvalues of G_D is well known; see, for example, [89]:

$$\frac{1}{\lambda_n} = \text{Max}\left\{\min_{f\in E} R(f) : E \; is \; n - \text{dimensional subspace} \; of \; L^2(D)\right\} \qquad (89)$$

Where $R(f)$ is the Rayleigh quotient for G_D, given by

$$R(f) = \frac{\int_{-1}^{1} f(x)G_D f(x)dx}{\|f\|_2^2}.$$

Letf_n, where $n = 1,2,\ldots$, be a complete orthonormal system in $L^2(D)$ and let E_N be the subspace spanned by f_n, where $n = 1,2,\ldots,N$. By replacing $L^2(D)$ by E_N in (89), we clearly obtain the upper bound $\lambda_{n,N}^+$ for λ_n, with $n = 1,2,\ldots,N$. On the other hand, $\left(\lambda_{n,N}^+\right)^{-1}$ is the nth largest eigenvalue of the $N \times N$ matrix A_N of the coefficients $a_{m,n}$ of the operator G_D in the basis $(f_1, f_2, \ldots. f_N)$ (note that $a_{m,n}$ do not depend on N).

The main difficulty is to find a convenient basis f_n for which the Approximations converge sufficiently fast, while the entries of A_N can be Computed explicitly.
For the sake of comparison, recall that analytical computation in [50] gives the upper bound $3\pi/8 \approx 1.178$. Our first attempt to use the Rayleigh–Ritz method for A_D instead of G_D, with $f_n(x) = sin((n\pi/2)(x + 1))$, resulted in relatively poor estimates. For example, for $N = 1000$ the upper bound for the first eigenvalue is $\lambda_1,1000 \approx 1.1579$, accurate up to the third decimal place. A more efficient approach, described below, uses Legendre polynomials we begin with computation the values of the Green operator of the interval $(-1,1)$ on the polynomials $g_n(x) = x^n$. Recall that the Green function of the interval $D = (-1,1)$ for theCauchy process is given by

$$G_D(x,y) = \frac{1}{2\pi}\int_0^{(1-x^2)(1-y^2)/(x-y)^2} \frac{du}{\sqrt{u}\sqrt{u+1}} = \frac{1}{\pi}\log\frac{1-xy+\sqrt{1-x^2}\sqrt{1-y^2}}{|x-y|}$$

Where $x, y \in D$. Integrating by parts gives, after some simplification,

$$G_D g_n(y) = \int_{-1}^{1} G_D(x,y) g_n(x)dx = \frac{1}{\pi}\frac{\sqrt{1-y^2}}{n+1} PV \int_{-1}^{1} \frac{x^{n+1}dx}{\sqrt{1-x^2}(x-y)}$$

$$= \frac{1}{\pi}\frac{\sqrt{1-y^2}}{n+1}\int_{-1}^{1}\frac{(x^{n+1}+y^{n+1})dx}{\sqrt{1-x^2}(x-y)} + \frac{1}{\pi}\frac{\sqrt{1-y^2}y^{n+1}}{n+1} I(y),$$

Where

$$I(y) = PV\int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}(x-y)}.$$

The indefinite integral is given by

$$\frac{1}{\sqrt{1-y^2}}\log\frac{|x-y|\sqrt{1-y^2}}{\sqrt{1-x^2}(x-y)},$$

and there fore $I(y) = 0$. Consequently, we have

$$G_D g_n(y) == \frac{1}{\pi}\frac{\sqrt{1-y^2}}{n+1}\int_{-1}^{1}\frac{(x^{n+1}+y^{n+1})dx}{\sqrt{1-x^2}(x-y)} = \frac{1}{\pi}\frac{\sqrt{1-y^2}}{n+1}\sum_{i=0}^{n} y^{n-i}\int_{-1}^{1}\frac{x^i dx}{\sqrt{1-x^2}}$$

$$= \frac{1}{\pi}\frac{\sqrt{1-y^2}}{n+1}\sum_{j=0}^{[n/2]} y^{n-2j}\frac{\Gamma\left(j+\frac{1}{2}\right)}{\Gamma(j+1)}.$$

Finally, for $m, n = 0,1,2,\ldots$ such that $m + n$ is even, we get

$$G_{m,n} = \int_{-1}^{1} g_m(y) G_D g_n(y)dy$$

$$= \frac{1}{n+1}\sum_{j=0}^{[n/2]} y^{n-2j}\frac{\Gamma(j+1/2)}{\sqrt{\pi}\Gamma(j+1)}\int_{-1}^{1}\sqrt{1-y^2}y^{n+m-2j}dy$$

$$\frac{1}{2(n+1)}\sum_{j=0}^{[n/2]}\frac{\Gamma(j+1/2)}{\Gamma(j+1)}\frac{\Gamma((n+m+1)/2-j)}{\Gamma((n+m)/2+2-j)}.$$

By simple induction, one can prove that in this case

$$G_{m,n}=\begin{cases}\frac{1}{m+n+1}\cdot\frac{\Gamma(m+1/2)}{\Gamma(m/2+1)}\cdot\frac{\Gamma((n+1)/2)}{\Gamma(n/2+1)} & \text{for } m,n \text{ even,}\\ \frac{1}{m+n+2}\cdot\frac{\Gamma(m/2+1)}{\Gamma((m+3)/2)}\cdot\frac{\Gamma(n/2+1)}{\Gamma((n+3)/2)} & \text{for } m,n \text{ odd.}\end{cases} \quad (90)$$

If $m+n$ is odd, then we obviously have$G_{m,n}=0$.

The Legendre polynomials are defined by

$$f_n(x)=\sum_{j=0}^{[n/2]}c_{n},x^{n-2i},$$

Where

$$c_{n,i}=\frac{(-1)^i(2n-2i)!}{2^n i!\,(n-i)!\,(n-2i)!}=\frac{(-1)^i\Gamma(2n-2i-1)}{2^n i!\,\Gamma(n-i+1)!\,(n-2i+1)}, \quad (91)$$

Form the orthogonal basis in$L^2(D)$. Therefore, we have

$$a_{m,n}=\int_{-1}^{1}f_m(y)G_D f_n(y)dy=\sum_{j=0}^{[n/2]}\sum_{j=0}^{[n/2]}c_{n,i}c_{m,j}G_{i,j}, \quad (92)$$

With $c_{n,i}$ and $G_{i,j}$ given by (90) and (91), respectively. The upper bound for $\lambda_n$is$\lambda^+_{n,N}$ where $\left(\lambda^+_{n,N}\right)^{-1}$ is the nth greatest eigenvalue of the $N\times N$ matrix $A_N=\left(a_{m,n}\right)$.

To find the lower bounds to the eigenvalues of the problem (1)–(3) for an interval $D=(-1,1)$, we apply the Weinstein–Aronszajn method of intermediate problems. We use the method described in [74], where the sloshing problem is considered.

The analytic function $sin(z)=(sin\,\xi\, cosh\,\eta, sinh\xi\, cos\,\eta)$, where $z=\xi+i\eta$, transforms the semiinfinite strip $R=\{(\xi,\eta)\in\mathbb{R}^2:-\pi/2\le\xi\le\pi/2,\eta\ge 0\}$ onto the upper half-space $H\{(x,y)\in\mathbb{R}^2\ \ y\ge 0\}$. Let u be a solution to the eigenproblem (1)–(3) with $D=(-1,1)$.Then the image $v(z)=u(\eta(z))$ of the function u under η is a solution to the followingequivalent problem

$$\Delta v(\xi,\eta)=0,-\frac{\pi}{2}\xi<\frac{\pi}{2},\eta>0, \quad (93)$$

$$\frac{\partial y}{\partial x}v(\xi,0)=-\lambda\cos\xi v(\xi,0),\ \ \frac{\pi}{2}\le\xi\le\frac{\pi}{2},\eta=0, \quad (94)$$

$$v\left(-\frac{\pi}{2},\eta\right)=v\left(\frac{\pi}{2},\eta\right)=0,\qquad \eta\ge 0. \quad (95)$$

For $f\in L^2(-\pi/2,\pi/2)$ we denote by Af (not to be confused with $\mathcal{A}f$) the normal derivative of the harmonic function agreeing with f on $(-\pi/2,\pi/2)$ and vanishing on $\{-\pi/2,\pi/2\}\times[0,\infty)$ (this is an analogue of the Dirichlet–Neumann operator). Since $v(\xi,\eta)=sin(k(\xi+\pi/2))e^{-k\eta}$atisfies (93) and (95), the eigenfunctions of A are simply

$$gk(\xi) = \sqrt{\frac{2}{\pi}}\, sin\left(k\left(\xi + \frac{\pi}{2}\right)\right) \;\; and\; A_{gk} \; = \; k_{gk}.$$

We define the operator of multiplication by the function sign $\xi\sqrt{1-\cos\xi}$

$$(Tf)(\xi) = sign\xi\sqrt{1-\cos\xi}\, f(\xi), \;\; f \in L^2\left(-\frac{\pi}{2},\frac{\pi}{2}\right).$$

The problem (93)–(95) can be written in the operator form as

$$(Af)(\xi) = \lambda(1-T^2)f(\xi). \tag{96}$$

Let P_N be the orthogonal projection of $L^2(D)$ onto a linear subspace E_N of $L^2(D)$ spanned by the first N of the linearly dense set of functions $f_1, f_2, \ldots$. Then the eigenvalues $\lambda_{n,N}^-$ of the spectral problem

$$Af(\xi) = \lambda(1 - TP_N T)f \tag{97}$$

Are lower bounds for the eigenvalues of (96) and consequently to the eigenvalues λ_n of the problem (93)–(95). Roughly, this is because

$$\int_{-\pi/2}^{\pi/2} f(x)TP_N Tf(x)dx = \|P_N Tf(x)\|_2^2 \le \|Tf(x)\|_2^2 = \int_{-\pi/2}^{\pi/2} f(x)T^2 f(x)dx,$$

and so the Rayleigh quotient associated with (97) is dominated by the Rayleigh quotient for (96), namely

$$\frac{\int_{-\pi/2}^{\pi/2} f(x)Af(x)dx}{\int_{-\pi/2}^{\pi/2} f(x)(1-TP_N T)f(x)dx} \le \frac{\int_{-\pi/2}^{\pi/2} f(x)Af(x)dx}{\int_{-\pi/2}^{\pi/2} f(x)(1-T^2)f(x)dx}$$

The problem (97) is called the intermediate problem. We shall later choose f_n so that each Tf_n is a linear combination of g_i, the eigenfunctions of A, say

$$Tf_n = \sum_{i=1}^{K} c_i, ig_i, \quad n = 1,2,\ldots,N, \tag{98}$$

where $K \ge N$. Let C be the $N \times K$ matrix with entries $c_{n,i}$ and let B be the $N \times N$ Gram matrix of the functions $f_1, \ldots, f_N$, that is, the matrix with the entries

$$b_{m,n} = \int_{-\pi/2}^{\pi/2} f_m(x) f_n(x)dx.$$

Finally, let D be the $K \times K$ diagonal matrix of the first K eigenvalues $1,2,\ldots,K$ of A. Note that, for each $j > K$, the function g_j is the solution of (97) with an eigenvalue $\lambda = j$ (this is because $T_{gj} = 0$). On the other hand, if f is the linear combination of $g_1, g_2, \ldots, g_K$ with the coefficients $\alpha = (\alpha_1, \ldots, \alpha_K)$, then f satisfies (97) if and only if α is the solution to the $K \times K$ relative matrix Eigenvalue problem,

$$D\alpha = \lambda(I - C^T B^{-1} C)\alpha \tag{99}$$

By arranging the eigenvalues of (99) and eigenvalues $K+1, K+2, \ldots$ in the nondecreasing order, we obtain the sequence of eigenvalues $\lambda_{n,N}^-$ of the intermediate problem (97). As already noted, these are the lower bounds for λ_n. We define

$$f_n(x) = 2\sqrt{1+\cos x}\, g_n(x).$$

It follows that

$$Tf_n(x) = 2\sin x g_n(x) = (-1)^n g_{n-1}(x) + (-1)^{n+1} g_{n+1}(x),$$

Using the convention that $g_0(x) = 0$. Consequently, C is $N \times (N+1)$ matrix of the form

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & (-1)^N & 0 \\ 0 & 0 & 0 & 0 & \cdots & (-1)^N & 0 & (-1)^{N+1} \end{pmatrix}$$

The coefficients of the Gram matrix B can be easily computed, and we have

$$b_{m,n} = \frac{(-1)^{1+(m+n)/2} 32mn}{\pi((m-n)^2-1)((m+n)^2-1)} + 4\delta_{m,n}$$

Whenever $m+n$ is even, and $b_{m,n} = 0$ otherwise. Finally, the solutions of the spectral problem (99) are simply the inverses of the eigenvalues of the matrix $D^{-1}(I - C^T B^{-1} C)$. These numbers turn out to be less than $N+2$, and therefore they form $\lambda_{n,N}^{-}$, where$n = 1,2,\dots,N+1$.

Let $D = (0,\infty)$. Let $p_t(x,A) = P_x(X_t \in A)$ for $A \subseteq \mathbb{R}$, and fix $x > 0$. By the strong Markov property,

$$2P_x(X_t \le 0) = 2P_x(X_t \in D^c) = E_x\big(2p_{t-\tau D}\big(X_{(\tau D)}, D^c\big); \tau_D \le t\big).$$

Since $2p_S(y, D^C) \ge 1$ for $y \le 0$ and $s > 0$, it follows that the right-hand side is bounded below by$P_x(\tau_D \le t)$. Therefore, for $t > 0$ and$x > 0$,

$$P_x(\tau_D \le t) \le \frac{2}{\pi} \int_{-\infty}^{0} \frac{t}{t^2 + (y-x)^2} dy = 1 - \frac{2}{\pi} arctan \frac{x}{t} \le \min\left(1, \frac{t}{x}\right). \quad (100)$$

For$t > 0, x, y \in D = (0,\infty)$, we have (see [50])

$$\frac{p_t(y-x) - p_t^D(x,y)}{t} = \frac{1}{t} E_x\big(p_{t-\tau D}\big(y - X(\tau D)\big); \tau D \le t\big)$$

$$= \frac{1}{\pi t} E_x\left(\frac{t - \tau D}{(t-\tau D)^2 + \big(y - X(\tau D)\big)^2}; \tau D \le t\right)$$

$$\le \frac{1}{\pi y^2} P_x(\tau D \le t) \le min\left(\frac{1}{\pi y^2}, \frac{t}{\pi x y^2}\right).$$

By symmetry, also

$$\frac{p_t(y-x) - p_t^D(x,y)}{t} \le min\left(\frac{1}{\pi x^2}, \frac{t}{\pi x^2 y}\right)$$

Since$p_t(y-x) \le 1/\pi t$ we conclude that

$$0 \le \frac{p_t(y-x) - p_t^D(x,y)}{t} \le \frac{1}{\pi} \le min\left(\frac{1}{t^2}, \frac{1}{x^2}, \frac{1}{y^2}, \frac{t}{x^2 y}, \frac{t}{x y^2}\right), t, x, y > 0. \quad (101)$$

A function η being the generalized Hilbert transform of -arctan t- is sought. More precisely, η is the function satisfying η (0) = 0 and

$$\eta'(t) = \frac{1}{\pi} PV \int_{-\infty}^{0} \frac{1}{(t-s)(1+s^2)} ds, t \in \mathbb{R}, \quad (102)$$

The integral being the Cauchy principal value when$t < 0$. Observe that

$$\int \frac{1}{(t-s)(1+s^2)} ds = \frac{1}{1+t^2} \int \left(\frac{s+t}{1+s^2} + \frac{1}{t-s}\right) ds$$
$$= \frac{1}{1+t^2}\left(t \arctan s + \frac{1}{2}\log(1+s^2) - \log|t-s|\right).$$

Hence we have

$$\eta'(t) = \frac{1}{1+t^2}\left(\frac{t}{2} - \frac{1}{\pi}\log|t|\right)$$

and so

$$\eta(t) = \frac{\log(1+t^2)}{4} - \frac{1}{\pi}\int_0^t \frac{\log|s|}{1+s^2} ds, t \in \mathbb{R}. \tag{103}$$

In particular,

$$\eta(-t) = -\eta(t) + \log\sqrt{1+t^2}, t \in \mathbb{R}. \tag{104}$$

The integrals of $log\,|s|/(1\,+\,s^2)$ over $(0,\infty)$ and over $(-\infty,0)$ are zero (this follows by asubstitution $u = 1/s$); and the maximum and minimum of the integral of $log\,|s|/(1\,+\,s^2)$, equal to the Catalan constant $C \approx 0.916$ and to -C, respectively, are attained at -1 and 1. It follows that

$$\frac{1}{4}\log(1+t^2)\frac{C}{\pi} \le \eta(t) \le \frac{1}{4}\log(1+t^2) + \frac{C}{\pi},\ t \in \mathbb{R}, \tag{105}$$

and in particular,

$$e^{\eta(t)} \sim \sqrt{|t|} \text{as} |t| \ \to \infty. \tag{106}$$

On the other hand, by (102),

$$\eta'(t) = \frac{1}{\pi}\frac{d}{dt}\int_{-\infty}^{0} \frac{\log|t-s|}{1+s^2} ds,$$

and for $t\ =\ 0$,

$$\frac{1}{\pi}\int_{-\infty}^{0} \frac{\log|s|}{1+s^2} ds = \left(\int_0^1 + \int_0^\infty\right)\frac{\log s}{1+s^2} ds = \int_0^1 \frac{\log s}{1+s^2} ds + \int_0^1 \frac{-\log s}{1+s^{-2}}\frac{ds}{s^2} = 0.$$

Therefore,

$$\eta(t) = \frac{1}{\pi}\int_0^t \frac{\log|t-s|}{1+s^2} ds,\ t \in\ \mathbb{R}. \tag{107}$$

A related holomorphic function B plays a major role. It is defined by

$$B(z) = \frac{1}{\pi}\int_{-\infty}^{0} \frac{\log(z-s)}{1+s^2} ds,\ z\ \in\ \boldsymbol{C}. \tag{108}$$

Here we agree that $log(z) = log\,|z| + i\pi/2$ for $z \in (-\infty, 0]$, that is, log (and therefore also B) is continuous on $(-\infty, 0]$ when approached from $\boldsymbol{C}_+$, but not from $\boldsymbol{C}_-$. The function $ReB(z)$ is harmonic in $\boldsymbol{C} \setminus (-\infty, 0]$, continuous in whole **C** and Re $B(t) = \eta(t)$ for $t \in \mathbb{R}$. For $z \in \boldsymbol{C}$, we have

$$\text{Re}B(z) = \frac{1}{\pi}\int_{-\infty}^{0} \frac{\log|z-s|}{1+s^2} ds \le \frac{1}{\pi}\int_{-\infty}^{0} \frac{\log(|z|-s)}{1+s^2} ds = \eta(|z|).$$

and in a similar manner

$$\mathrm{Re}B(z) = \frac{1}{\pi}\int_{-\infty}^{0}\frac{\log|z-s|}{1+s^2}ds \geq \frac{1}{\pi}\int_{-\infty}^{0}\frac{\log|-|z|-s|}{1+s^2}ds = \eta(-|z|),$$

$$\frac{1}{4}\log(1+|z|^2) - \frac{C}{\pi} \leq \mathrm{Re}\,B(z) \leq \frac{1}{4}\log(1+|z|^2) + \frac{C}{\pi}, \mathrm{z} \in \mathbf{C}. \qquad (109)$$

In particular,

$$\left|e^{B(Z)}\right| \sim \sqrt{|z|}\ \mathrm{as}|z| \to \infty. \qquad (110)$$

The function $ReB'(t) = \eta'(t)$ is the Hilbert transform of $(-\,arctan\,t-),'$ and at the same time Re $B(t)$ is the Hilbert transform of $-Im\,B(t)$; hence Im $B(t)$ = (arctan t−). Since Im $B(0) = 0 = \arctan 0 -$, we conclude that$Im\,B'(t)\; = (\;\arctan \mathrm{t}-)'$.
The following auxiliary computations related to the functions η and B. We have

$$\int\frac{\pi/2 - arctan\,s}{1+S^2}ds = \frac{1}{2}\arctan s - (\arctan s)^2,$$

So that.

$$\frac{1}{\pi}\int_{0}^{\infty}\int\frac{\pi/2 - arctan\,s}{1+S^2}ds = ds = \frac{\pi}{8} \qquad (111)$$

By a substitution $s = 1/tan\,t$, we have

$$\int_{0}^{-\infty}\frac{\log(1+s^2)}{1+s^2}ds = -\,2\int_{0}^{\pi/2}\log\sin t\,dt.$$

We have

$$2\int_{0}^{\pi/2}\log\sin t\,dt = \int_{0}^{\pi/2}\log\sin t\,dt + \int_{0}^{\pi/2}\log\cos t\,dt$$

$$= \int_{0}^{\pi/2}\log\sin(2t)\,dt - \frac{\pi\log 2}{2}$$

$$= \frac{1}{2}\int_{0}^{\pi}\log\sin u\,du - \frac{\pi\log 2}{2} = \int_{0}^{\pi/2}\log\sin u\,du - \frac{\pi\log 2}{2}.$$

Therefore,

$$\frac{1}{\pi}\int_{0}^{\infty}\frac{\log(1+s^2)}{1+s^2}ds = \log 2. \qquad (112)$$

Whenever $a > -1$ and$b > (1\;+\;a)/2$, we have by a substitution $1 + t^2 =\; 1/s$ and a formula for the beta integral

$$\int_0^\infty \frac{t^a}{(1+s^2)^b} dt$$
$$= \frac{1}{2}\int_0^1 (1-2)^{(a-1)/2} s^{b-(a+3)/2} ds$$
$$= \frac{\Gamma((a+1)/2)\Gamma(b-(a-1)/2)}{2\Gamma(b)}. \quad (113)$$

Also, by integration by parts and $\Gamma(1/2) = \sqrt{\pi}$,

$$\int_0^\infty \frac{1-e^{tx}}{t^{3/2}} dt = 2x\int_0^\infty \frac{e^{-tx}}{\sqrt{t}} dt = 2\sqrt{\pi x}, x > 0. \quad (114)$$

Estimates for the generator on a piecewise smooth function the following estimate. Define an auxiliary piecewise C^2 function

$$\begin{cases} 0 & for\ x \in \left(-\infty, -\frac{1}{3}\right), \\ \frac{9}{2}\left(x+\frac{1}{3}\right)^2 & for\ x\left(-\frac{1}{3}, 0\right), \\ 1-\frac{9}{2}\left(x-\frac{1}{3}\right)^2 & for\ x \in \left(0, \frac{1}{3}\right), \\ 1 & for\ x \in \left(\frac{1}{3}, \infty\right). \end{cases} \quad (115)$$

Note that$q(x) + q(-x) = 1$. Let f be a piecewise C^2 function on $\mathbb{R}$ and let $g(x) = q(x)f(x)$.

Suppose that g has compact support. We estimate $Ag(x)$ for $x \in (-1,0)$. Choose M_0, M_1 and M_2 so that $|f(x)| \le M_0, |f'(x)| \le M_1\ and\ |f''(x)| \le M_2forx \in (-\frac{1}{3}, \frac{1}{3})$.

Let$\int_0^\infty |f(x)|dx$.. Then

$$|q''(x)| \le M_0|q''(x)| + 2M_1|q'(x)| + M_2|q(x)| \le 9M_0 + 6M_1 + M_2.$$

If$z \in (-\frac{1}{3}, \frac{1}{3})$. , then $g(z) = 0$, and so $Ag(z)$ is estimated (up to the factor $\frac{1}{\pi}$) by

$$\int_{-1/3}^\infty \frac{|g(x)|}{(x-z)^2} dx \le M_0 \int_{-1/3}^{1/3} \frac{|g(x)|}{(x-z)^2} dx + \frac{9}{4}\int_{1/3}^\infty |f(x)|dx \le 3M_0 + \frac{9I}{4};$$

Here we used $q(x)/(x-z)^2 \le \frac{9}{2}$ for $x\left(-\frac{1}{3}, \frac{1}{3}\right)$In the second inequality. For $z \in (-\frac{1}{3}, 0)$ the principal value integral in the definition of A can be estimated by splitting it into two parts. By Taylor's expansion of g, we have sup

$$\left|PV\int_{z-1/3}^{z+1/3} \frac{g(x)-g(z)}{(x-z)^2} dx\right| \le \frac{2}{3}.\frac{1}{2} sup\left\{|g''(x)| : x \in \left(z-\frac{1}{3}, z+\frac{1}{3}\right).\right\}$$
$$\le \frac{1}{3}\ sup\left\{|g''(x)| : x \in \left(z-\frac{1}{3}, z+\frac{1}{3}\right).\right\} \le 3M_0 + 2M_1 + \frac{2M_2}{3}$$

For the second inequality note that $g''(x) = 0$for$x < -\frac{1}{3}$. Furthermore,

$$\left|\left(\int_{-\infty}^{z-1/3} + \int_{z+1/3}^{\infty}\right)\frac{g(x)-g(z)}{(x-z)^2}dx\right|$$
$$\leq |g(z)|\left(\int_{-\infty}^{z-1/3} + \int_{z+1/3}^{\infty}\right)\frac{1}{(x-z)^2}dx + 9\int_{z+1/3}^{\infty}|f(x)|\,dx \leq 6M_0 + 9I$$

We conclude that

$$|\mathcal{A}g(z)| \leq \frac{3M_0 + (9/4)I}{\pi}, z \in \left(-1, \frac{1}{3}\right); \tag{116}$$

$$|\mathcal{A}g(z)| \leq \frac{3M_0 + 2M_1 + (2/3)M_2 + 9I}{\pi}, z \in \left(-\frac{1}{3}, 0\right); \tag{117}$$

Section (2.2): Fractional Laplace Operator in the Interval

Let $D = (-1,1)$ and $\alpha \in (0,2)$. Below we study the asymptotic behavior of the eigenvalues of the spectral problem for the one-dimensional fractional Laplace operator in the interval D:

$$\left(-\frac{d^2}{dx^2}\right)^{\alpha/2} \varphi(x) = \lambda\varphi(x), x \in D, \tag{118}$$

where $\varphi \in L^2(D)$ is extended to $\mathbb{R}$ by 0. It is known that there exists an infinite sequence of eigenvalues $\lambda_n, 0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$, and the corresponding eigenfunctions φ_n form a complete orthonormal set in $L^2(D)$.

By following carefully the proof, one can take e.g. $C = 30\,000$ and $C' = 4000$ in Theorem (2.2.3). Note that the constant in the error term $O(1/n)$ in (140) tends to zero as α approaches 2, and in the limiting case $\alpha = 2$ (not considered below), we have $\lambda_n = (n\pi/2)^2$ without an error term. Theorem (2.2.3) for $\alpha = 1$ (with better numerical constants) was proved in [49]. The proof of Theorem (2.2.3) is modeled after [49], and the idea can be sketched as follows. In [101], an explicit formula for the solution of the spectral problem similar to (118) in half-line $(0,\infty)$ was given: for all $\lambda > 0$ there is an eigenfunction $F_\lambda(x)$ such that $(-d^2/dx^2)^{\alpha/2}F_\lambda(x) = \lambda^\alpha F_\lambda(x)$ for $x \in (0,\infty)$, and $F_\lambda(x) = 0$ for $x \leq 0$. Furthermore, $F_\lambda(x) \approx \sin\left(\lambda x + \frac{(2-\alpha)\pi}{8}\right)$ when λ_x is large enough. The fractional Laplace operator $(-d^2/dx^2)^{\alpha/2}$ is a non-local operator, so the eigenfunctions in half-line are not restrictions of eigenfunctions in the entire real line. When λ is large enough and x is not too close to 0, then $F_\lambda(x)$ behaves nearly as $\sin\left(\lambda x + \frac{(2-\alpha)\pi}{8}\right)$, which is an eigenfunction of $(-d^2/dx^2)^{\alpha/2}$ in $\mathbb{R}$. One may expect a similar approximate localization phenomenon for the solutions of the spectral problem (118) in the interval D: locally near -1 and 1, the eigenfunctions $\varphi_n(x)$ on the interval D are expected to be close to the eigenfunctions in half-lines $(-1,\infty)$ and $(-\infty,1)$ respectively. In other words, for n large enough, and with $\mu_n \approx \lambda_n^{1/\alpha}$, we expect that

$$\varphi_n(x) \approx \begin{cases} C_1 F_{\mu_n}(1+x) \text{ for } x \text{ close to } -1, \\ C_2 F_{\mu_n}(1+x) \text{ for } x \text{ close to } 1, \\ C_3 \sin(\mu_n x + \theta_n) \text{ for } x \in D \text{ away from the boundary,} \end{cases},$$

for some constants C_1, C_2, C_3, θ_n. The above observation is exploited as follows. We define the function $\tilde{\varphi}_n(x)$ to be equal to $F_{\mu_n}(1 + x)$ for x close to $-1, \tilde{\varphi}_n(x) = \pm F_{\mu_n}(1 - x)$ for x close to 1 (the sign depends on n), and so that $\tilde{\varphi}_n(x)$ is approximately equal to $\pm\cos(\mu_n x)$ or $\pm\sin(\mu_n x)$ (again, depending on n) for $x \in D$ away from the boundary. Such a construction is possible when $\mu_n = \frac{n\pi}{2} - \frac{(2-\alpha)\pi}{8}$.

Table (1)[97]:

Comparison of the approximation $\tilde{\lambda}_n = \left(\frac{n\pi}{2} - \frac{(2-\alpha)\pi}{8}\right)^\alpha$ (roman font), and numerical approximations to λ_n obtained using the method of [102] with 5000 × 5000 matrices (slanted font).

α	λ_1		λ_2		λ_3	
0.01	0.998	0.997	1.009	1.009	1.014	1.014
0.1	0.981	0.973	1.091	1.092	1.147	1.148
0.2	0.971	0.957	1.195	1.197	1.319	1.320
0.5	0.991	0.970	1.598	1.601	2.029	2.031
1	1.178	1.158	2.749	2.754	4.316	4.320
1.5	1.611	1.597	5.055	5.059	9.592	9.597
1.8	2.056	2.048	7.500	7.501	15.795	15.801
1.9	2.248	2.243	8.594	8.593	18.710	18.718
1.99	2.444	2.442	9.733	9.729	21.820	21.829

Then we are able to prove that $A\tilde{\varphi}_n(x) \approx \mu_n^\alpha \tilde{\varphi}_n(x)$ for all $x \in D$. This means that $\tilde{\varphi}_n$ is an approximate eigenfunction. Using $L^2(D)$ decomposition of $\tilde{\varphi}_n$ in the orthonormal basis of (true) eigenfunctions φ_k, we can show that μ_n^α must be close to some eigenvalue λ_k . This proves that there is an infinite sequence of eigenvalues satisfying (140). It remains to prove that there are no other igenfunctions. This is achieved using a trace estimate for the semigroup generated by $(-d^2/dx^2)^{\alpha/2}$ on D (with zero exterior condition).

We briefly recall the history of the problem (118) and state it a more formal way. An auxiliary estimate for the fractional Laplace operator is given. The formula from [101] for the eigenfunctions $F_\lambda(x)$ on the half-line is recalled. An approximation $\tilde{\varphi}_n$ to eigenfunctions, and Theorem (2.2.3). Further properties of eigenfunctions and eigenvalues in [49]. Proposition (2.2.6) gives simplicity of the eigenvalues when $\alpha \in [1, 2)$. This result follows relatively easily from the result for $\alpha = 1$ in [49], and monotonicity in α properties from [69]. $L^2(D)$ and $L^\infty(D)$ bounds for the eigenfunctions are given. Numerical estimates of λ_n in terms of eigenvalues of large dense matrices are obtained.

The spectral problem studied has long history. First-term Weyl-type asymptotic law for λ_n was proved by Blumenthal and Getoor in 1959 [54]. The best known general estimate for λ_n is $\frac{1}{2}\left(\frac{n\pi}{2}\right)^\alpha \le \lambda_n \le \left(\frac{n\pi}{2}\right)^\alpha$ due to DeBlassie [69] and Chen and Song [65], also known in a more general setting. The important case of $\alpha = 1$ was studied in detail, see [50], [49]. It is known that $(\lambda_n)^{1/\alpha}$ is continuous and increasing in $\alpha \in (0, 2]$, see [65]–[70]. For a discussion of related results see [50], [49]. Theorem (2.2.3) is of interest in physics, the asymptotic formula (140) (without the information about the order of the error term) was stated, and supported by numerical experiments, in [102].

The values of C and C' given above are rather large, numerical evidence suggests that the error term in formula (140) is rather small also for small n in the full range of $\alpha \in (0, 2)$,. It is an interesting open problem to prove Theorem (2.2.3) with C and C' non-exploding as α approaches 0. This is related to simplicity of eigenvalues λ_n, conjectured to hold for all $\alpha \in (0, 2)$, proved for $\alpha = 1$ in [49], and extended to $\alpha \in [1, 2)$ in Proposition (2.2.6).

By the results of [49] and [101], as well as by Theorem (2.2.3) above, one can conjecture asymptotic law similar to (140) for eigenvalues on an interval for more general operators $A = \psi(-d^2/dx^2)$, studied in [101]. While such a result for each individual complete Bernstein function ψ should present no difficulty (under some reasonable regularity and growth assumptions on ψ), it is an interesting (and much more difficult) problem to obtain estimates uniform also in ψ, for a given class of ψ. One important example here is the family of Klein–Gordon square-root operators $A = \sqrt{m^2 - d^2/dx^2} - m$, with mass m ranging from 0 to ∞. This operator is close to $\sqrt{-d^2/dx^2}$ for small m, but when m is large, it more similar to $-d^2/dx^2$.

To give a formal statement of the spectral problem (118), first we recall the definition of the one-dimensional fractional Laplace operator $A = (-d^2/dx^2)^{\alpha/2}$. It is defined pointwise by the principal value integral, if convergent,

$$Af(x) = c_\alpha \,\mathrm{pv}\int_{-\infty}^{\infty} \frac{f(x) - f(y)}{|x - y|^{1+\alpha}}\,dy, \qquad x \in \mathbb{R}, \tag{119}$$

where

$$c_\alpha = \frac{2^\alpha \Gamma\left(\frac{1+\alpha}{2}\right)}{\sqrt{\pi}\left|\Gamma\left(-\frac{\alpha}{2}\right)\right|} = \frac{\Gamma(1 + \alpha)\sin\frac{\alpha\pi}{2}}{\pi}\ ;$$

$Af(x)$ is convergent if, for example, f is C_2 in a neighborhood of x and bounded on $\mathbb{R}$. Note that

$$\frac{1}{8}\,\alpha(2 - \alpha) \le c_\alpha \le \frac{1}{2}\,\alpha(2 - \alpha). \tag{120}$$

Indeed, for the lower bound simply use $\sin\frac{\alpha\pi}{2} \ge \frac{\pi}{4}\,\alpha(2-\alpha)$ and $(1+\alpha) \ge \frac{1}{2}$, and for the upper bound, we have $\Gamma(1 + \alpha) \le \max(1, \alpha)$ and $\max(1, \alpha)\sin\frac{\alpha\pi}{2} \le \frac{\pi}{2}\,\alpha(2 - \alpha)$. For $f \in C_c^\infty(\mathbb{R})$, the Fourier transform of Af is equal to $|\xi|$ αf $\hat{f}(\xi)$, and A extends to an unbounded self-adjoint operator on $L^2(\mathbb{R})$. We write A_D for the operator A on D with zero exterior condition on $\mathbb{R}\backslash D$. For $f \in C_c^\infty(D)$, $A_D f$ is defined to be the restriction of Af to D. Again, the Friedrich's extension of A_D is an unbounded self-adjoint operator on $L^2(D)$, denoted by the same symbol A_D.

The operator $-A$ is the generator of the one-dimensional symmetric α-stable process X_t, and $-A_D$ is the generator of X_t killed upon leaving the interval D. This probabilistic interpretation is a primary source of our motivation, but will not be exploited in the sequel.

Throughout, C, C', C'' denote generic absolute constants (independent of α), and their values may be different in each displayed equation. We will track the dependence of other constants employed below on α to catch their asymptotic behavior as $\alpha \searrow 0$ and $\alpha \nearrow 2$. For brevity, we denote $\beta = 2 - \alpha$.

Define, as in [49], an auxiliary function (see Fig. 1):

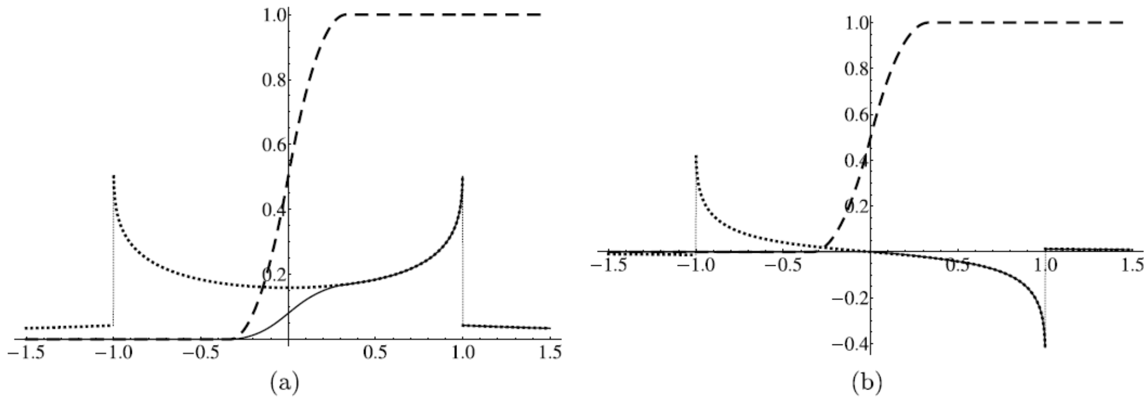


Fig. (1)[97]: Plot of $q(x)$ (dashed line), $f\ (x)$ (dotted line) and $g(x)\ =\ q(x)f\ (x)$ (solid line). With the notation of Lemma (2.2.1), plots correspond to $\alpha\ = \frac{1}{5}$ and (a) $n\ =\ 1$; (b) $n\ =\ 2$.

$$q(x)\ =\ \begin{cases} 0 & \text{for } x\ \in\ \left(-\infty, -\frac{1}{3}\right), \\ \frac{9}{2}\left(x\ + \frac{1}{3}\right)^2 & \text{for } x\ \in\ \left(-\frac{1}{3}\ , 0\right), \\ 1\ - \frac{9}{2}\left(x\ - \frac{1}{3}\right)^2 & \text{for } x\ \in\ \left(0, \frac{1}{3}\right), \\ 1 & \text{for } x\ \in\ \left(\frac{1}{3}\ , \infty\right). \end{cases} \tag{121}$$

Note that q, q' are continuous and bounded on $\mathbb{R}$, and q'' is continuous and bounded on $\mathbb{R} \setminus \left\{-\frac{1}{3}\ , 0, \frac{1}{3}\right\}$. Furthermore, $q(x)\ +\ q(-x)\ =\ 1$. Assume that f is an integrable function on $\mathbb{R}$ such that $f\ , f'$ and f'' exist and are bounded in $\left[-\frac{1}{3}\ , \frac{1}{3}\right]$. We define $g(x)\ =\ q(x)f\ (x)$. Below we estimate Ag on $(-1, 0)$ in a very similar way as in [49].

Let M be the supremum of $\max(|f\ (x)|, |f'\ \ (x)|, |f''(x)|)$ over $x\ \in\ \left[-\frac{1}{3}\ , \frac{1}{3}\right]$, and let $I\ =\ \int_0^{\infty}\ |f\ (x)|\ dx$. Then $g''(x)\ =\ 0$ for $x\ <\ -\frac{1}{3}$ and

$$|g''(x)| \le |f\ (x)q''(x)|\ +\ 2\ |f'\ \ (x)q'\ (x)|\ + |f''(x)q(x)| \le CM,$$
$$x\ \in \left(-\frac{1}{3}, \frac{1}{3}\right) \setminus \{0\}.$$

Suppose first that $x\ \in\ \left(-1, -\frac{1}{3}\right]$. Since g vanishes in $\left(-1, -\frac{1}{3}\right]$, we have

$$c_{\alpha}^{-1}\ |Ag(x)| \le \int_{-\frac{1}{3}}^{\infty}\ \frac{q(y)|f\ (y)|}{|x\ -\ y|^{1+\alpha}} dy$$

$$\le M \int_{-\frac{1}{3}}^{\frac{1}{3}} \frac{q(y)}{\left|y\ + \frac{1}{3}\right|^{1+\alpha}}\ dy\ + \int_{\frac{1}{3}}^{\infty}\ \frac{|f\ (y)|}{|x\ -\ y|^{1+\alpha}}\ dy$$

$$\le \frac{9M}{2} \int_{-\frac{1}{3}}^{\frac{1}{3}} \left|y\ + \frac{1}{3}\right|^{1-\alpha} dy\ + \frac{1}{\left(\frac{2}{3}\right)^{1+\alpha}} \int_{\frac{1}{3}}^{\infty}\ |f\ (y)| dy$$

$$\le \frac{2^{1-\alpha}3^{\alpha}M}{2-\alpha} + \frac{3^{1+\alpha}I}{2^{1+\alpha}}\frac{CM}{\beta} + CI.$$

In the third inequality we used the estimate $q(y)\frac{9}{2}\left(y+\frac{1}{3}\right)^2$ $(y \in \mathbb{R})$. For $x \in \left(-\frac{1}{3}, 0\right)$ the principal value integral in the definition of $Ag(x)$ can be estimated by splitting it into two parts. By Taylor's expansion of g, for $y \in \left(-\infty, \frac{1}{3}\right]$ we have

$$|g(x) - g(y) - (y-x)g'(x)| = \left|\int_x^y g''(z)(x-z)dz\right|$$

$$\le (\operatorname{ess\,sup} |g''(z)|)\left|\int_x^y (x-z)dz\right|$$

$$\frac{CM(x-y)^2}{2}.$$

Hence,

$$\left|\mathrm{pv}\int_{x-\frac{1}{3}}^{x+\frac{1}{3}} \frac{g(x)-g(y)}{|x-y|^{1+\alpha}}dy\right| = \left|\int_{x-\frac{1}{3}}^{x+\frac{1}{3}} \frac{g(x)-g(y)-(x-y)g'(x)}{|x-y|^{1+\alpha}}dy\right|$$

$$\le \frac{CM}{2}\int_{x-\frac{1}{3}}^{x+\frac{1}{3}} \frac{(x-y)^2}{|x-y|^{1+\alpha}}dy = \frac{2^{1-\alpha}CM}{3^{2-\alpha}(2-\alpha)} \le \frac{CM}{2\beta}.$$

Furthermore,

$$\left|\left(\int_{-\infty}^{x-\frac{1}{3}} + \int_{x+\frac{1}{3}}^{\infty}\right)\frac{g(x)-g(y)}{|x-y|^{1+\alpha}}dy\right|$$

$$\le \left(\int_{-\infty}^{x-\frac{1}{3}} + \int_{x+\frac{1}{3}}^{\infty}\right)\frac{1}{|x-y|^{1+\alpha}}dy + 3^{1+\alpha}\int_{x+\frac{1}{3}}^{\infty} |f(y)|\,dy$$

$$\frac{CM}{\alpha} + CI.$$

We conclude that

$$c_{\alpha}^{-1}\,|Ag(x)|\,\frac{CM}{\alpha\beta} + CI, \qquad x \in (-1,0). \tag{122}$$

The main result of [101] is the formula for generalized eigenfunctions for a class of operators on $(0,\infty)$. The case of the fractional Laplace operator is studied in [101]. In particular, the eigenfunction F_{λ} of $A(0,\infty)$ (defined pointwise, or as an operator on $L^{\infty}(0,\infty)$; see [101] for more details) corresponding to the eigenvalue λ^{α} $(\lambda > 0)$ is given by $F_{\lambda}(x) = F(\lambda x) = \sin\left(\lambda x + \frac{\beta\pi}{8}\right) - G(\lambda x)$ (recall that $\beta = 2-\alpha$), where G is a completely monotone function.

G is the Laplace transform of a positive function $\gamma(s)$ $(s > 0)$, given by the formula

$$\gamma(s) = \frac{\sqrt{2\alpha}\sin\left(\frac{\alpha\pi}{2}\right)}{2\pi}\frac{s^{\alpha}}{1+s^{2\alpha}-2s^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)}\exp\left(\frac{1}{\pi}\int_0^{\infty}\frac{1}{1+r^2}log\ \frac{1-r^{\alpha}s^{\alpha}}{1-r^2s^2}dr\right). \tag{123}$$

By [101], for $x > 0$ we have

$$G(x) \leq \sin\left(\frac{\beta\pi}{8}\right) \leq C\beta, \tag{124}$$

and

$$\int_0^\infty G(x)dx = \cos\left(\frac{\beta\pi}{8}\right) - \sqrt{\frac{\alpha}{2}}\, C\beta. \tag{125}$$

Note that the exponent in (123) is negative. Furthermore, for $\alpha \in (0,1]$ we have

$$1 + s^{2\alpha} - 2s^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right) \geq \left(\sin\left(\frac{\alpha\pi}{2}\right)\right)^2 \geq \alpha^2,$$

while for $\alpha \in (1,2)$, the left-hand side is not less than one. Hence, for all $\alpha \in (0,2]$,

$$1 + s^{2\alpha} - 2s^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right) \geq \min(\alpha^2,1) \geq \frac{\alpha^2}{4}.$$

Finally, $\sin\left(\frac{\alpha\pi}{2}\right)\alpha(2-\alpha) = \alpha\beta$. Therefore,

$$\gamma(s) \leq \frac{2\sqrt{2\alpha}\beta}{\alpha\pi}\, s^{\alpha}. \tag{126}$$

By direct integration, we find that for $x > 0$,

$$G(x) = \int_0^\infty e^{-xs}\gamma(s)ds \leq \frac{2\sqrt{2\alpha}\beta\Gamma(1+\alpha)}{\alpha\pi}\, x^{-1-\alpha} \leq \frac{C\beta}{\sqrt{\alpha}}\, x^{-1-\alpha}. \tag{127}$$

Furthermore, $-G'$ and G'' are the Laplace transforms of $s\gamma(s)$ and $s2\gamma(s)$ respectively. Hence, (126) gives

$$-G'(x) \leq \frac{C\beta}{\sqrt{\alpha}}\, x^{-2-\alpha}, G''(x) \leq \frac{C\beta}{\sqrt{\alpha}}\, x^{-3-\alpha} \tag{128}$$

for $x > 0$. For simplicity, we let $F(x) = 0$ and $G(x) = 0$ for $x \leq 0$.

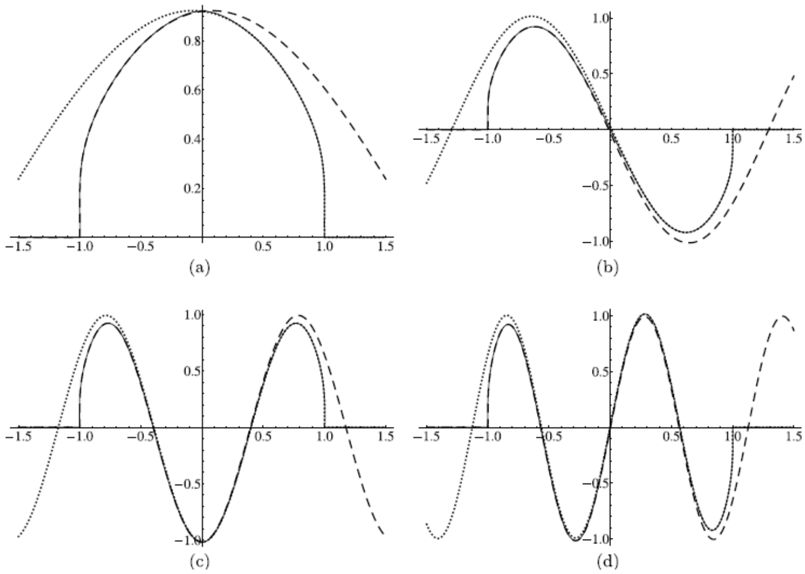


Fig. (2)[97]: Plot of the approximation $\tilde{\varphi}_n(x)$ (solid line), and the shifted eigenfunctions $F_{\mu_n}(1+x)$ (dashed line) and $F_{\mu_n}(1-x)$ (dotted line), for $\alpha = \frac{1}{5}$ and (a) $n = 1$; (b) $n = 2$; (c) $n = 3$; (d) $n = 4$.

Let n be a fixed positive integer and $\mu_n = \frac{n\pi}{2} - \frac{\beta\pi}{8}$. Our goal is to show that μ_n^{α} is close to λ_n. Note that $\mu_n \geq \frac{\pi}{4}$ and $\frac{n\pi}{4}$ μn $\frac{n\pi}{2}$.

We construct approximations $\tilde{\varphi}_n$ to eigenfunctions φ_n by combining shifted eigenfunctions for half-line, $F_{\mu_n}(1+x)$ and $F_{\mu_n}(1-x)$, and using the auxiliary function q given above in (121) to join them in a sufficiently smooth way. We let (see Fig. (2))

$$\tilde{\varphi}_n(x) = q(-x)F_{\mu_n}(1+x) - (-1)^n q(x)F_{\mu_n}(1-x), x \in \mathbb{R}. \qquad (129)$$

Note that $\tilde{\varphi}_n(x) = 0$ for $x \notin (-1,1)$. Suppose that n is odd, $n = 2m+1$. Then $\tilde{\varphi}_n$ is an even function. Furthermore,

$$\sin\left(\mu_n(1-x)+\frac{\beta\pi}{8}\right) = \sin\left(\frac{n\pi}{2}-\frac{\mu_n}{x}\right) = (-1)^m\cos(\mu_n x)$$
$$= \sin\left(\frac{n\pi}{2}+\mu_n x\right) = \sin\left(\mu_n(1+x)+\frac{\beta\pi}{8}\right).$$

Recall that $F_\lambda(x) = \sin\left(\lambda x+\frac{\beta\pi}{8}\right) - G(\lambda x)$. Hence, for $x \in (-1,1)$,

$$\tilde{\varphi}_n(x) = q(-x)F_{\mu_n}(1+x) + q(x)F_{\mu_n}(1-x)$$
$$= (-1)^m\big(q(-x)+q(x)\big)\cos(\mu_n x) + q(-x)G\big(\mu_n(1+x)\big) + q(x)G\big(\mu_n(1-x)\big)$$
$$= (-1)^m\cos(\mu_n x) + q(-x)G\big(\mu_n(1+x)\big) + q(x)G\big(\mu_n(1-x)\big). \qquad (130)$$

In a similar manner, when n is even, n = 2m, then for x ∈ (−1, 1),

$$\tilde{\varphi}_n(x) = (-1)^m\sin(\mu_n x) + q(-x)G\big(\mu_n(1+x)\big) - q(x)G\big(\mu_n(1-x)\big). \qquad (131)$$

It follows that away from the boundary of $D = (-1,1)$, $\tilde{\varphi}_n$ is close to $\pm\cos(\mu_n x)$ or $\pm\sin(\mu_n)x$, and it converges to zero near ±1.

Lemma (2.2.1)[97]: We have

$$\|AD\tilde{\varphi}_n - \mu_n^\alpha\tilde{\varphi}_n\|_2 \le \frac{C\beta}{\sqrt{\alpha}}\frac{1}{n}. \qquad (132)$$

Proof: Note that for all x ∈ R we have (see Fig. (2))

$$\tilde{\varphi}_n(x) - F_{\mu_n}(1+x) = \big(q(-x)-1\big)F_{\mu_n}(1+x) - (-1)^n q(x)F_{\mu_n}(1-x)$$
$$= -q(x)\Big(F_{\mu_n}(1+x) + (-1)^n F_{\mu_n}(1-x)\Big).$$

Observe that

$$\sin\left(\mu_n(1+x)+\frac{\beta\pi}{8}\right) + (-1)^n\sin\left(\mu_n(1-x)+\frac{\beta\pi}{8}\right)$$
$$= \sin\left(\frac{n\pi}{2}+\mu nx\right) + (-1)^n\sin\left(\frac{n\pi}{2}-\mu_n x\right) = 0.$$

Since $F_\lambda(x) = \sin\left(\lambda x+\frac{\beta\pi}{8}\right)1_{(0,\infty)}(x) - G(\lambda x)(x \in \mathbb{R})$, it follows that for all $x \in \mathbb{R}$ we have

$$\tilde{\varphi}_n(x) - F_{\mu_n}(1+x) = q(x)\Big(G\big(\mu_n(1+x)\big) + (-1)^n G\big(\mu_n(1-x)\big)\Big)$$
$$-\sin\left(\mu_n(1+x)+\frac{\beta\pi}{8}\right)1_{[1,\infty)}(x).$$

For $x \in \mathbb{R}$, denote (see Fig. 1)

$$h(x) = \sin\left(\mu_n(1+x)+\frac{\beta\pi}{8}\right)1_{[1,\infty)}(x),$$
$$f(x) = G\big(\mu_n(1+x)\big) + (-1)^n G\big(\mu_n(1-x)\big),$$
$$g(x) = q(x)f(x).$$

It follows that $\tilde{\varphi}_n(x) = F_{\mu_n}(1 + x) + g(x) - h(x)(x \in \mathbb{R})$. For $x \in (-1, 0)$, we have
$AF_{\mu_n}(1 + x) - \mu_n^\alpha F_{\mu_n}(1 + x) = 0$ and $h(x) = 0$. Hence,

$$|A\tilde{\varphi}_n(x) - \mu_n^\alpha \tilde{\varphi}_n(x)| \le |Ag(x)| + |\mu_n^\alpha g(x)| + |Ah(x)|, x \in (-1, 0). \quad (133)$$

We will now estimate each summand on the right-hand side.

Recall that G is completely monotone, so that $G, -G$ and G are positive, convex functions on (0,∞). This fact and estimates (125), (127) and (128) give

$$\sup_{x\in\left[-\frac{1}{3},\frac{1}{3}\right]} |f(x)| \le G\left(\frac{2}{3}\mu_n\right) + G\left(\frac{4}{3}\mu_n\right) \le \frac{C\beta}{\sqrt{\alpha}}\mu_n^{-1-\alpha},$$

$$\sup_{x\in\left[-\frac{1}{3},\frac{1}{3}\right]} |f'(x)| \le -\mu_n G'\left(\frac{2}{3}\mu_n\right) - \mu_n G'\left(\frac{4}{3}\mu_n\right) \le \frac{C\beta}{\sqrt{\alpha}}\mu_n^{-1-\alpha},$$

$$\sup_{x\in\left[-\frac{1}{3},\frac{1}{3}\right]} |f''(x)| \le -\mu_n^2 G''\left(\frac{2}{3}\mu_n\right) - \mu_n^2 G''\left(\frac{4}{3}\mu_n\right) \le \frac{C\beta}{\sqrt{\alpha}}\mu_n^{-1-\alpha},$$

$$\int_0^\infty |f(x)|\,dx \int_0^\infty G_{\mu_n}(1 + x)dx + \int_0^1 G_{\mu_n}(1 - x)dx$$
$$= \frac{1}{\mu_n}\int_0^\infty G(y)dy \le \frac{C\beta}{\mu_n}. \quad (134)$$

By (122) and (120),

$$|Ag(x)| \le \frac{C\beta}{\sqrt{\alpha}}\mu_n^{-1-\alpha} + C\alpha\beta^2\mu_n^{-1}, \qquad x \in (-1, 0). \quad (135)$$

For the second term in (133), we have $|g(x)| = 0$ for $x \in (-1, -\frac{1}{3})$. Furthermore, since $q(x) \le \frac{1}{2}$ for $x < 0$, the estimate (134) gives

$$|\mu_n^\alpha g(x)| = \mu_n^\alpha q(x)|f(x)| \le \left|\frac{\mu_n^\alpha |f(x)|}{2}\right| \le \frac{C\beta}{\sqrt{\alpha}}\mu_n^{-1}, \qquad x \in -1\ 3, 0. \quad (136)$$

Finally, for the third term in (133), we use the following estimate: if u is a decreasing differentiable function such that $\lim_{z\to\infty} u(z) = 0$, then, by integration by parts, for any $a, \vartheta \in \mathbb{R}$ and $\lambda > 0$ we have

$$\left|\int_a^\infty u(z)\sin(\lambda z + \vartheta)\,dz\right| = \left|\frac{1}{\lambda}\int_a^\infty u(z)\cos(\lambda a + \vartheta) - \cos(\lambda z + \vartheta)\,dz\right|$$
$$\le \frac{2}{\lambda}\int_a^\infty |u'(z)|\,dz = \frac{2u(a)}{\lambda}.$$

It follows that for all $x < 0$, we have

$$|Ah(x)| = \left|c_\alpha \int_1^\infty \frac{\sin\left(\mu_n(1 + y) + \frac{\beta\pi}{8}\right)}{|x - y|^{1+\alpha}}\,dy\right| \le \left|\frac{2c_\alpha}{\mu_n|x - 1|^{1+\alpha}}\right|$$
$$\le \alpha\beta\mu_n^{-1}. \quad (137)$$

Estimates (135)–(137) applied to (133) yield that

$$|A\tilde{\varphi}_n n(z) - \mu_n^\alpha \tilde{\varphi}_n(z)| \le \frac{C\beta}{\sqrt{\alpha}}\mu_n^{-1}, \qquad z \in (-1, 0). \quad (138)$$

By symmetry, (138) also holds for z ∈ (0, 1). Formula (132), with $A_D\tilde{\varphi}_n$ understood in the pointwise sense, follows. It remains to prove that $\tilde{\varphi}_n$ is in the domain of A_D. To this end, we will use the notion of the Green operator $G_D = A_D^{-1}$. See [99] for formal definition and properties of GD. Since $A\tilde{\varphi}_n$ is bounded on D, the function $\tilde{\varphi}_n - G_D A\tilde{\varphi}_n$ is a bounded, continuous in D, weakly α-harmonic function in $D = (-1, 1)$ with zero exterior condition. Such a function is necessarily zero (see [57], [100]). It follows that $\tilde{\varphi}_n = G_D A\tilde{\varphi}_n$, and hence $\tilde{\varphi}_n$ is in the $L^\infty(D)$ domain of A_D. Since convergence in $L^\infty(D)$ is stronger than the one in $L^2(D)$, the proof is complete.

Lemma (2.2.2)[97]: We have

$$1 - \frac{C\beta}{n} \le \|\tilde{\varphi}_n\|_2 \le 1 + \frac{C\beta}{n}. \tag{139}$$

In particular, there is an absolute constant K such that $\|\tilde{\varphi}_n\|_2 \ge \frac{1}{2}$ for $n \ge K$.

Proof: By (129), we have (see also (130) and (131))

$$\tilde{\varphi}_n(x) = \sin\left(\mu_n(1 + x) + \frac{\beta\pi}{8}\right) + q(-x)G\big(\mu_n(1 + x)\big) - (-1)^n q(x)G\big(\mu_n(1 - x)\big).$$

Hence,

$$\begin{aligned}
&\big|\|\tilde{\varphi}_n\|_2 - 1\big| \\
&= \left|\int_{-1}^{1} \big(\tilde{\varphi}_n(x)\big)^2 - \frac{1}{2}\, dx\right| \le \left|\int_{-1}^{1} \left(\left(\sin\left(\mu_n(1 + x) + \frac{\beta\pi}{8}\right)\right)^2 - \frac{1}{2}\right) dx\right| \\
&\quad + 2\left|\int_{-1}^{1} q(-x)G\big(\mu_n(1 + x)\big) \right.\\
&\qquad \left. - (-1)^n q(x)G\big(\mu_n(1 - x)\big)(\sin\left(\mu_n(1 + x) + \frac{\pi}{8}\right) dx\right| \\
&\quad + \int_{-1}^{1} \Big(q(-x)G\big(\mu_n(1 + x)\big) - (-1)^n q(x)G\big(\mu_n(1 - x)\big)\Big)^2 dx.
\end{aligned}$$

We estimate each term on the right-hand side. First, by direct integration,

$$\begin{aligned}
\left|\int_{-1}^{1} \left(\left(\sin\left(\mu_n(1 + x) + \frac{\beta\pi}{8}\right)\right)^2 - \frac{1}{2}\right) dx\right| &= \frac{1}{2}\left|\int_{-1}^{1} \cos\left(2\mu_n(1 + x) + \frac{\beta\pi}{4}\right) dx\right| \\
&= \frac{1}{4\mu_n}\left|\sin\left(4\mu_n + \frac{\beta\pi}{4}\right) - \sin\frac{\beta\pi}{4}\right| \quad C\beta\ \mu n .
\end{aligned}$$

By (124) and (125),

$$\int_{-1}^{1} G\big(\mu_n(1 + x)\big)^2 dx \le C\beta \int_{-1}^{\infty} G\big(\mu_n(1 + x)\big) \le \frac{C\beta^2}{\mu_n}.$$

Hence,

$$\begin{aligned}
&\int_{-1}^{1} \Big(q(-x)G\big(\mu_n(1 + x)\big) - (-1)^n q(x)G\big(\mu_n(1 - x)\big)\Big)^2 dx \\
&2\int_{-1}^{1} G\big(\mu_n(1 + x)\big)^2 dx + 2\int_{-1}^{1} G\big(\mu_n(1 - x)\big)^2 dx \le \frac{C\beta^2}{\mu_n}.
\end{aligned}$$

Finally, again by (125),

$$\left|\int_{-1}^{1} q(-x)G(\mu_n(1 + x)\sin\left(\mu_n(1 + x)+\frac{\pi}{8}\right)dx\right| \le \frac{1}{\mu_n}\int_{-1}^{1} G(\mu_n(1 + x)dx \le \frac{C\beta}{\mu_n},$$

and we can replace $q(-x)G\big(\mu_n(1 + x)\big)$ by $q(x)G\big(\mu_n(1 - x)\big)$. Formula (139) follows.

Theorem (2.2.3)[97]: We have

$$\lambda_n = \left(\frac{n\pi}{2} - \frac{(2 - \alpha)\pi}{8}\right)^{\alpha} + O\left(\frac{1}{n}\right). \tag{140}$$

More precisely, there are absolute constants C, C' such that

$$\left|\lambda_n - \left(\frac{n\pi}{2} - \frac{(2 - \alpha)\pi}{8}\right)^{\alpha}\right| \le C\frac{(2 - \alpha)}{\sqrt{\alpha}}\frac{1}{n}$$

for $n \ge (C'/\alpha)^{3/(2\alpha)}$.

The scaling property of the fractional Laplace operator $(-d^2/dx^2)^{\frac{\alpha}{2}}$ and its translation invariance imply that for a similar spectral problem in $D' = (a,b)$, the corresponding eigenvalues λ_n' satisfy $\lambda_n' = \big((b - a)/2\big)^{-\alpha}\lambda_n(D)$. Hence, one easily finds the asymptotic formula for λ_n'.

Proof: Since $\tilde{\varphi}_n \in L^2(D)$, we have $\tilde{\varphi}_n = \sum_j a_j\varphi_j$ for some $a_j \in \mathbb{R}$. Moreover, $\|\tilde{\varphi}_n\|_2^2 = \sum_j a_j^2$ and $A_D\tilde{\varphi}_n = \sum_j \lambda_j a_j\varphi_j$. Let $\lambda_k(n)$ be the eigenvalue nearest to μ_n^{α}. Then

$$\|A_D\tilde{\varphi}_n - \mu_n^{\alpha}\tilde{\varphi}_n\|_2^2 = \sum_{j=1}^{\infty} \left(\lambda_j - \mu_n^{\alpha}\right)^2 a_j^2 \ \left(\lambda_{k(n)} - \mu_n^{\alpha}\right)^2\sum_{j=1}^{\infty} a_j^2$$
$$= \left(\lambda_{k(n)} - \mu_n^{\alpha}\right)^2\|\tilde{\varphi}_n\|_2^2.$$

Let K be the constant defined in Lemma (2.2.2). By (132) and Lemma (2.2.2), it follows that for $n \ge K$,

$$\left|\lambda_{k(n)} - \mu_n^{\alpha}\right| \le \frac{C\beta}{\sqrt{\alpha}}\frac{1}{n}. \tag{141}$$

This will enable us to derive the two-term asymptotic formula for λ_j.

Denote $\varepsilon = \frac{1}{2}\frac{\beta\pi}{8}$. We claim that for each α ∈ (0, 2), there is a positive integer L_α such that $\lambda_{k(n)} \in ((\mu_n - \varepsilon)^{\alpha}, (\mu_n + \varepsilon)^{\alpha})$ for $n \ge L_\alpha$. Namely, we take

$$L_\alpha = \left[\left(\frac{A\beta}{\alpha^{32}\varepsilon}\right)^{1\alpha}\right] = \left[\left(\frac{CA}{\alpha^{32}}\right)^{1\alpha}\right], \tag{142}$$

with the constant A large enough. In particular, we take $A \ge 2^{34}K^2\pi/16$, so that $L_\alpha \ge K$ for all $\alpha \in (0,2)$. By (141) and (142), for $n \ge L_\alpha$ we have

$$\left|\lambda_{k(n)} - \mu_n^{\alpha}\right| \le \frac{C\beta}{\sqrt{\alpha}}\frac{1}{n} \le \frac{C\beta}{\sqrt{\alpha}}\frac{1}{n}\cdot\frac{\alpha^{32}\varepsilon n^{\alpha}}{A\beta} = \frac{C\alpha\varepsilon n^{\alpha-1}}{A}. \tag{143}$$

On the other hand, we have $\frac{n\pi}{8}\ \mu_n - \varepsilon\,\mu_n + \varepsilon \le \frac{n\pi}{2}$. Hence, by the mean value theorem,

$$|(\mu_n \pm \varepsilon)^{\alpha} - \mu_n^{\alpha}| \ge (\alpha\varepsilon\min(\mu_n - \varepsilon)^{\alpha-1}, (\mu_n + \varepsilon)^{\alpha-1})$$
$$\ge \alpha\varepsilon n^{\alpha-1}\min\left(\left(\frac{\pi}{7}\right)^{\alpha-1}, \left(\frac{\pi}{2}\right)^{\alpha-1}\right)C\alpha\varepsilon n^{\alpha-1}. \tag{144}$$

If A is large enough, then (143) and (144) give $\lambda_{k(n)} \in ((\mu_n - \varepsilon)^{\alpha}, (\mu n + \varepsilon)^{\alpha})$.

This proves our claim. The intervals $((\mu_n - \varepsilon)^\alpha, (\mu_n + \varepsilon)^\alpha)$ are mutually disjoint. Thus, $\lambda_{k(n)}$ for $n \geq L_\alpha$ are all distinct. We claim that there are strictly less than L_α eigenvalues not included in the above class. As in [49], the key step will be the trace estimate.

Let J be the set of those $j > 0$ for which $j \neq k(n)$ for all $n \geq L_\alpha$. We need to show that $\#J < L_\alpha$. Denote by $p_t\,(x - y)$ and $p_t^D\,(x, y)$ the heat kernels for A and A_D respectively; we have $\hat{p}_t\,(\xi\,) = \exp(-t\,|\xi'|^\alpha)$. For $t > 0$, we have (see e.g. [52], [82])

$$\sum_{j=1}^{\infty} e^{-\lambda_j t} = \int_D \sum_{j=1}^{\infty} e^{-\lambda_j t} \left(\varphi_j(x)\right)^2 dx = \int_D p_t^D\,(x,x)dx$$
$$\leq \int_D p_t(0)dx = 2p_t(0) = \frac{2}{\pi}\int_0^\infty e^{-ts^\alpha} ds.$$

In the last step, Fourier inversion formula was used. We find that

$$\frac{\pi}{2}\sum_{j\in J} e^{-\lambda_j t} = \frac{\pi}{2}\sum_{j=1}^{\infty} e^{-\lambda_j t} - \frac{\pi}{2}\sum_{n=L_\alpha}^{\infty} e^{-\lambda_{k(n)}t}$$
$$\leq \int_0^\infty e^{-ts^\alpha} ds - \sum_{n=L_\alpha}^{\infty} \frac{\pi}{2} e^{-t(\mu_n+\varepsilon)^\alpha}.$$

The series on the right-hand side is an upper Riemann sum for the integral of e^{-ts^α} over $(\mu L_\alpha + \varepsilon, \infty)$. Hence,

$$\frac{\pi}{2}\sum_{j\in J} e^{-\lambda_j t} \leq \int_0^{\mu_{L_\alpha}+\varepsilon} e^{-ts^\alpha} ds \leq \mu L_\alpha + \varepsilon.$$

As $t \searrow 0$, the left-hand side converges to $(\pi/2)\#J$. It follows that

$$\#J \leq \frac{2}{\pi}\left(\mu_{L_\alpha} + \varepsilon\right) = L_\alpha - \frac{\beta}{4} + \frac{2\varepsilon}{\pi}.$$

Since $\varepsilon < \frac{\beta\pi}{8}$, the right-hand side is less than L_α, and our claim is proved.

By [66], [69], for $j < L_\alpha$ we have $\lambda_j \leq (j\pi/2)^\alpha \leq \left((L_\alpha - 1)\pi/2\right)^\alpha$.on the other hand, $\lambda_{k(n)} \geq (\mu_n - \varepsilon)^\alpha > \left((L_\alpha - 1)\pi/2\right)^\alpha$ for $n \geq L_\alpha$. It follows that J contains $\{1, 2, \ldots, L_\alpha - 1\}$. But since $\#J \leq L_\alpha - 1$, we must have $J = \{1, 2, \ldots, L_\alpha - 1\}$. We conclude that $k(n) = n$ for all $n \geq L_\alpha$. Theorem (2.2.3) follows now from (141).

We study three additional properties of φ_n and λ_n: the $L^2(D)$ estimates of $\varphi_n - \tilde{\varphi}_n$, the $L^\infty(D)$ bound for φ_n, and simplicity of λ_n. This part is modeled after [49].

Proposition (2.2.4)[97]: (Cf. Lemma 3 and Corollary 4 in [49].) We can choose the sign of the eigenfunctions φ_n in such a way that there are constants C, C with the following property: for $n\ (C/\alpha)^{3/(2\alpha)}$,

$$\|\tilde{\varphi}_n - \varphi_n\|_2 \leq \frac{C'(2 - \alpha)}{n} \quad when\ \alpha \geq 1,$$
$$\|\tilde{\varphi}_n - \varphi_n\|_2 \leq \frac{C'(2 - \alpha)}{\alpha^{\frac{3}{2}} n^\alpha} \quad when\ \alpha < 1.$$

In particular, if $\varphi_n^*(x) = (-1)^{(n-1)/2}\cos(\mu_n x)$ for odd n and $\varphi_n^*(x) = (-1)^{n/2}\sin(\mu_n x)$ for even n, then there is a constant C'' such that for $n \geq (C/\alpha)^{3/(2\alpha)}$,

$$\|\varphi_n^* - \varphi_n\|_2 \le \frac{C''(2 - \alpha)}{\sqrt{n}} \quad when\ \alpha \ge \frac{1}{2},$$
$$\|\varphi_n^* - \varphi_n\|_2 \le \frac{C'(2 - \alpha)}{\alpha^{3/2} n^{\alpha}} \quad when\ \alpha < \frac{1}{2}.$$
Proof. Fix $n \ge L_\alpha + 1$ and $\varepsilon = \frac{1}{2}\frac{\beta\pi}{8}$, $\tilde{\varphi}_n = \sum_j a_j\varphi_j$. By changing the sign of φ_n if necessary, we may assume that $a_n > 0$. Recall that L_α was chosen in such a way that $\left|\lambda_j - \mu_n^\alpha\right| \ge C\alpha n^{\alpha-1}$ for $j = n$ (see (144) and following discussion). Hence,
$$\|A_D\tilde{\varphi}_n - \mu_n^\alpha\tilde{\varphi}_n\|_2^2 = \sum_{j=1}^{\infty} \left(\lambda_j - \mu_n^\alpha\right)^2 a_j^2 \ge C(\alpha n^{\alpha-1})^2 \sum_{j\neq n} a_j^2 .$$
By (132), we obtain that
$$\|\tilde{\varphi}_n - a_n\varphi_n\|_2^{2\sim} = \sum_{j\neq n} a_j^2 \le C\left(\frac{\beta}{\sqrt{\alpha}}\frac{1}{n}\right)^2 \frac{1}{(\alpha n^{\alpha-1})^2} = C\left(\frac{\beta}{\alpha^{3/2}n^\alpha}\right)^2 . \quad (145)$$
Since $\|\varphi_n\|_2 = 1$, we have
$$\|\tilde{\varphi}_n - \|\tilde{\varphi}_n\|_2\varphi_n\|_2 \ \|\tilde{\varphi}_n - a_n\varphi_n\|_2 + |a_n - \|\tilde{\varphi}_n\|_2| .$$
Furthermore,
$$|a_n - \|\tilde{\varphi}_n\|_2|^2 \le (\|\tilde{\varphi}_n\|_2 - a_n)(\|\tilde{\varphi}_n\|_2 + a_n)\|\tilde{\varphi}_n\|_2^2 - a_n^2 \ \|\tilde{\varphi}_n - a_n\varphi_n\|_2^2.$$
Hence, by (145),
$$\|\tilde{\varphi}_n - \|\tilde{\varphi}_n\|_2\varphi_n\|_2 \le 2\,\|\tilde{\varphi}_n - a_n\varphi_n\|_2 \le \frac{C\beta}{\alpha^{3/2}n^\alpha} . \quad (146)$$
Finally, by (139) and (146),
$$\|\tilde{\varphi}_n - \varphi_n\|_2 \le \|\tilde{\varphi}_n - \|\tilde{\varphi}_n\|_2\varphi_n\|_2 2 + |\|\tilde{\varphi}_n\|_2 - 1| \le \frac{2C\beta}{\alpha^{3/2}n^\alpha} + \frac{C\beta}{n}.$$
The first part of the proposition is proved. The other part is a simple consequence of the first one and the definition of $\tilde{\varphi}_n$. Indeed, by (130) and (131),
$$\|\tilde{\varphi}_n - \varphi_n^*\|^2 \le \left(\int_{-1}^{1} \left(G\left(\mu_n(1 + x)\right)\right)^2 dx\right)^{1/2} + \left(\int_{-1}^{1} \left(G\left(\mu_n(1 - x)\right)\right)^2 dx\right)^{1/2}$$
$$= \frac{2}{\sqrt{\mu_n}}\left(\int_0^{2\mu_n} \left(G(y)\right)^2 dy\right)^{1/2} \le \frac{C\beta}{\sqrt{n}};$$
the last step follows by (124), (125) and the inequality $\mu_n \ge C_n$.

Proposition (2.2.5)[97]: (Cf. Corollary 5 in [49].) If $\ge \frac{1}{2}$, then the eigenfunctions $\varphi_n(x)$ are bounded uniformly in $n \ge 1$ and $x \in D$.

Proof: Let $P_t^D = \exp(-tA_D)$ $(t > 0)$ be the heat semigroup for $-A_D$ (or the transition semigroup of the symmetric α-stable process in D), and let $p_t^D(x,y)$ be the corresponding heat kernel (or transition density). We have $P_t^D\varphi_n(x) = e^{-t\lambda_n}\varphi_n(x)$ for $x \in D$. It is well known that $p_t^D(x,y) \le p_t(y - x)$, where $p_t(x)$ is the heat kernel for $-A, \hat{p}t(\xi) = \exp(-t|\xi|^\alpha)$; see e.g. [98].

By Cauchy–Schwarz inequality and Plancherel's theorem, we obtain

$$
\begin{aligned}
e^{-\lambda_n t}\ \left|\varphi_{n(x)}\right| &\le |P_t^D\ (\varphi_n - \tilde{\varphi}_n)(x)| + |P_t^D\ \tilde{\varphi}_n(x)| \\
&\le \int_D p_t(x-y)|\varphi_n(y) - \tilde{\varphi}_n(y)|\,dy + \|P_t^D\ \tilde{\varphi}_n\|_\infty \\
&\le \left(\int_{-\infty}^{\infty} \left(p_t(x-y)\right)^2 dy\right)^{1/2} \|\varphi_n - \tilde{\varphi}_n\|_2 + \|\varphi_n\|_\infty \\
&= \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-2t|z|^\alpha}\,dz\right)^{12} \|\varphi_n - \tilde{\varphi}_n\|_2 + \sup_{x\in(0,\infty)} |F(x)| \\
&\le 2\sqrt{\Gamma\,(1 + 1/\alpha)}(2t)^{-1/(2\alpha)}\|\varphi_n - \tilde{\varphi}_n\|_2 + 2.
\end{aligned}
$$

Let $t = 1/\lambda_n$. Then $e^{-\lambda_n t} = 1/e$ and $t^{-1/(2\alpha)} = \lambda_n^{1/(2\alpha)}\,(n\pi 2)^{1/2}$. If $n \ge L_\alpha + 1$ and $\alpha \ge \frac{1}{2}$, then $\|\varphi_n - \tilde{\varphi}_n\|_2 \le C\beta/\sqrt{n}$, and finally $|\varphi_n(x)| \le C$ (for all $n \ge L_\alpha + 1, x \in D$). Since $\varphi_n \in L^\infty(D)$ also for each $n \le L_\alpha$, the proof is complete.

Proposition (2.2.6)[97]: (Cf. Theorem 6 in [49].) If $\alpha \ge 1$, then the eigenvalues λ_n are simple.

Proof. Let us write $\lambda_{n,\alpha}$ for λ_n in this proof. Since $\left(\lambda_{n,\alpha}\right)^{1/\alpha}$ is increasing in α, we have

$$\left(\lambda_{n,\alpha}\right)^{1/\alpha} \le \left(\lambda_{n,2}\right)^{1/2} = \frac{n\pi}{2}.$$

By Theorem 6 in [49], for $n \ge 3$ we have

$$\frac{(n+1)\pi}{2} - \frac{\pi}{8} - \frac{\pi}{10} < \lambda_{n+1,1}\left(\lambda_{n+1,\alpha}\right)^{1/\alpha}.$$

Therefore, $\lambda_{n,\alpha} < \lambda_{n+1,\alpha}$, except perhaps $n = 1$ or $n = 2$. But a similar argument works also for $n = 1$ and $n = 2$, since by [50] we have

$$\left(\lambda_{1,\alpha}\right)^{1/\alpha}\ \left(\lambda_{1,2}\right)^{1/2} = \frac{\pi}{2} < 2 \le \lambda_{2,1}\left(\lambda_{2,\alpha}\right)^{1/\alpha},$$

$$\left(\lambda_{2,\alpha}\right)^{1/\alpha}\left(\lambda_{2,2}\right)^{1/2} = \pi < 3.83\,\lambda_{3,1}\left(\lambda_{3,\alpha}\right)^{1/\alpha}.$$

The proof is complete.

Numerical experiments suggest that φ_n are uniformly bounded also for $\alpha < \frac{1}{2}$. Furthermore, it would be interesting to obtain an upper estimate of $\sup_n \|\varphi_n\|_\infty$, and in particular, to find its behavior when α approaches 0. Finally, better bounds for λ_n may yield simplicity of eigenvalues also when $\alpha < 1$.

For $\alpha = 1$, a satisfactory method (an application of Rayleigh–Ritz and Weinstein–Aronszajn methods) is described in [49]. For general α, even approximation of λ_n is difficult: all known methods converge rather slowly, and thus the computation of eigenvalues of very large matrices is required. A version of finite element method for obtaining a lower bound for λ_n is described. It shares the main drawbacks of many related algorithms: compared to the technique applied in [102], it converges slowly, and it suffers large errors as α approaches 2. On the other hand, the method presented below gives mathematically correct lower bounds, and there is no error estimate for the numerical scheme of [102]. A somewhat similar method for the upper bound for λ_1 is given. It gives satisfactory results for large α, but deteriorates as α gets close to 0.

It should be pointed out that in some cases (e.g. α close to 2 or n large), the bound $\frac{1}{2}\left(\frac{n\pi}{2}\right)^\alpha \le \lambda_n \le \left(\frac{n\pi}{2}\right)^\alpha$ of [65], [69] is sharper than the estimates obtained below, unless extremely large matrices are used. Also, good numerical estimates of λ_n are available for

$\alpha = 1$ due to [49]. By the monotonicity of $(\lambda_n)^{1/\alpha}$ in α, this gives a lower bound for λ_n when $\alpha \in (1,2)$ and an upper bound for $\alpha \in (0,1)$. Finally, a good estimate for λ_1 can be found in [50]. For a comparison of the above, see Table (2).

The method for the lower bound works for the fractional Laplace operator in an arbitrary bounded open set $D \subseteq \mathbb{R}^d$, for any $d \geq 1$; in fact, it can be easily extended to more general pseudo-differential operators (or Lévy processes). We use the following monotonicity property: the eigenvalues λ_n decrease when the kernel of A, i.e. the function $c_{d,\alpha}|x - y|^{-d-\alpha}$, is replaced by a smaller one. This fact is a simple consequence of the Rayleigh– Ritz variational formula. We cover the set D with small cubes I_k, and replace the kernel of A by a smaller kernel, which is constant whenever $x \in I_k$ and $y \in I_l$. The eigenvalues of the integral operator corresponding to the latter kernel can be easily expressed in terms of eigenvalues of a certain matrix.

Fix $\varepsilon > 0$ and let $\{I_k : k \in \mathbb{Z}^d\}$ be the partition of $\mathbb{R}^d$ into cubes $I_k = \prod_{j=1}^{d} [k_j \varepsilon, (k_j + 1)\varepsilon], k \in \mathbb{Z}^d$. Let $K_\varepsilon \subseteq \mathbb{Z}^d$ be the set of those $k \in \mathbb{Z}^d$ for which I_k intersects D, and let D_ε be the interior of $\bigcup_{k \in K_\varepsilon} I_k$. Note that $D \subseteq D_\varepsilon$.

The definition of $A = (-\Delta)^{\alpha/2}$ in higher dimension is similar to (119): for smooth bounded functions we have

$$Af(x) = c_{d,\alpha} \,\mathrm{pv} \int_{\mathbb{R}^d} \frac{f(x) - f(y)}{|x - y|^{d+\alpha}} dy, \qquad x \in \mathbb{R}^d,$$

where $c_{d,\alpha} = 2\alpha\Gamma((d + \alpha)/2) / \left(\pi^{d/2} \left|\Gamma\left(-\frac{\alpha}{2}\right)\right|\right)$. Fractional Laplace operator in D with zero exterior condition, denoted A_D, is defined as in dimension one. Below we denote by λ_n the eigenvalues of A_D. By domain monotonicity of λ_n, the eigenvalues for D are not less than the eigenvalues of its superset D_ε. For notational convenience, we assume that $D = D_\varepsilon$.

The Dirichlet form $E(f,f)$ corresponding to A_D is given by

$$E(f,f) = \frac{c_{d,\alpha}}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\left(f(x) - f(y)\right)^2}{|x - y|^{d+\alpha}} dxdy, f \in L^2(D).$$

As usual, $f \in L^2(D)$ is extended to $\mathbb{R}^d$ so that $f(x) = 0$ for $x \in \mathbb{R}^d \setminus D$. By Rayleigh–Ritz variational principle,

$$\lambda_n = \inf\{sup\{\mathcal{E}_\varepsilon(f,f) : f \in U, \|f\|_2 = 1\} : U < L^2(D), dim \ U = n\}.$$

where $U < L^2(D)$ means that U is a linear subspace of $L^2(D)$. For $k \in \mathbb{Z}^d$, we denote

$$\varrho(k) = \sqrt{\sum_{j=1}^{d} \left(|k_j| + 1\right)^2}.$$

Table (2)[97]: Comparison of bounds and approximations to λ_n. Each cell contains six numbers: lower bound $\lambda_{n,\varepsilon}$ with $\varepsilon = \frac{1}{2500}$, the best lower bound known before, approximation $\left(\frac{n\pi}{2} - \frac{(2-\alpha)\pi}{8}\right)^\alpha$, numerical approximation of [102], upper bound $\lambda^*_{1,\varepsilon}$,the best upper bound known before. The better estimates are underlined.

α	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}
0.01	<u>0.9966</u> 0.9943[1] 0.9976	<u>1.0086</u> 0.5057[2] 1.0086	<u>1.0137</u> 0.5078[2] 1.0138	<u>1.0171</u> 0.5092[2] 1.0172	<u>1.0196</u> 0.5104[2] 1.0198	<u>1.0217</u> 0.5113[2] 1.0218	<u>1.0234</u> 0.5121[2] 1.0235	<u>1.0248</u> 0.5128[2] 1.0250	<u>1.0261</u> 0.5134[2] 1.0263	<u>1.0273</u> 0.5139[2] 1.0274

	0.9966[4] 13.5210 0.9974[1]	1.0087[4] n/a 1.0102[3]	1.0137[4] n/a 1.0148[3]	1.0172[4] n/a 1.0179[3]	1.0197[4] n/a 1.0203[3]	1.0218[4] n/a 1.0223[3]	1.0235[4] n/a 1.0239[3]	1.0250[4] n/a 1.0254[3]	1.0263[4] n/a 1.0266[3]	1.0274[4] n/a 1.0277[3]
0.1	0.9724 0.9513[1] 0.9809 0.9726[4] 1.8351 0.9786[1]	1.0919 0.5606[2] 1.0913 1.0922[4] n/a 1.1067[3]	1.1469 0.5838[2] 1.1477 1.1473[4] n/a 1.1575[3]	1.1863 0.6008[2] 1.1867 1.1868[4] n/a 1.1941[3]	1.2159 0.6144[2] 1.2167 1.2165[4] n/a 1.2226[3]	1.2405 0.6257[2] 1.2412 1.2413[4] n/a 1.2462[3]	1.2611 0.6354[2] 1.2620 1.2620[4] n/a 1.2664[3]	1.2791 0.6440[2] 1.2802 1.2802[4] n/a 1.2840[3]	1.2950 0.6516[2] 1.2962 1.2962[4] n/a 1.2997[3]	1.3094 0.6585[2] 1.3107 1.3107[4] n/a 1.3138[3]
0.2	0.9572 0.9181[1] 0.9712 0.9575[4] 1.2376 0.9675[1]	1.1960 0.6286[2] 1.1948 1.1965[4] n/a 1.2247[3]	1.3182 0.6817[2] 1.3199 1.3191[4] n/a 1.3398[3]	1.4093 0.7221[2] 1.4012[2] 1.4105[4] n/a 1.4258[3]	1.4801 0.7550[2] 1.4819 1.4817[4] n/a 1.4947[3]	1.5402 0.7831[2] 1.5420 1.5421[4] n/a 1.5530[3]	1.5915 0.8076[2] 1.5939 1.5938[4] n/a 1.6036[3]	1.6373 0.8294[2] 1.6399 1.6400[4] n/a 1.6485[3]	1.6780 0.8492[2] 1.6812 1.6811[4] n\a 1.6890[3]	1.7154 0.8673[2] 1.7188 1.7188[4] n\a 1.7260[3]
0.5	0.9692 0.8862[2] 0.9908 0.97014 1.0002 0.9863[1]	1.5991 .08862[2] 1.5977 1.6015[4] n/a 1.6598[3]	2.0247 1.0854[2] 2.0306 2.0288[4] n/a 2.0777[3]	2.3809 1.2533[2] 2.3862 2.3871[4] n/a 2.4274[3]	2.6862 1.4012[2] 2.6954 2.6947[4] n/a 2.7314[3]	2.9618 1.5349[2] 2.9725 2.9728[4] n/a 3.0055[3]	3.2118 1.6579[2] 3.2259 3.2255[4] n/a 3.2562[3]	3.4443 1.7724[2] 3.4608 3.4610[4] n/a 3.4892[2]	3.6608 1.8799[2] 3.6808 3.6805[4] n/a 3.7074[3]	3.8654 1.9816[2] 3.8883 3.8883[4] n/a 3.9136[3]
1	1.1516 1.1577[3] 1.1781 1.1577[4]	2.7343 2.7547[3] 2.7489 2.7545[4]	4.2756 4.3168[3] 4. 3197 4.3164[4]	5.8236 5.8921[3] 5.8905 5.8916[4]	7.3584 7.4601[3] 7.4613 7.4594[4]	8.8919 9.0328[3] 9.0321 9.0319[4]	10.4166 10.6022[3] 10.6029 10.6012[4]	11.9382 12.1741[3] 12.1737 12.1729[4]	13.4528 13.7441[3] 13.7445 13.7427[4]	14.9636 15.3155[3] 15.315 15.3140[4]

Table (2)[97]: (continued)

α	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}
	1.1608 1.1578[3]	n/a 2.7548[3]	n/a 4.3169[3]	n/a 5.8922[3]	n/a 7.4602[3]	n/a 9.0329[3]	n/a 10.6023[3]	n/a 12.1742[3]	n/a 13.7442[3]	n/a 15.3156[3]
1.5	1.5139 1.3293[1] 1.6114 1.5971[4] 1.5989 1.6224[1]	4.7367 4.5721[3] 5.0545 5.0586[4] n/a 5.5684[2]	8.8817 8.9689[3] 9.5970 9.5921[4] n/a 10.2297[2]	13.7668 14.3024[3] 15.0171 15.0154[4] n/a 15.7497[2]	19.2502 20.3762[3] 21.1905 21.1846[4] n/a 22.0108[2]	25.2613 27.1479[3] 28.0344 28.0289[4] n/a 28.9339[2]	31.7334 34.5222[3] 35.4886 35.4800[4] n/a 36.4609[2]	38.6263 42.4772[3] 43.5067 43.4972[4] n/a 44.5467[2]	45.8896 50.9536[3] 52.0514 52.0392[4] n/a 53.1550[2]	53.5266 59.9375[3] 61.0922 61.0786[4] n/a 62.2558[2]
1.8	1.4483 1.6765[1] 2.0555 2.0481[4] 2.0501 2.0777[1]	5.1149 6.1965[3] 7.5003 7.5007[4] n/a 7.8501[2]	10.4447 13.9088[3] 15.8014 15.7948[4] n/a 16.2868[2]	17.2231 243496[3] 26.7233 26.7156[4] n/a 27.3353[2]	25.2907 37.2347[3] 40.1148 40.1012[4] n/a 40.8472[2]	34.5448 52.5393[3] 55.8658 55.8481[4] n/a 56.7138[2]	44.8969 70.1002[3] 73.8905 73.8661[4] n/a 74.8501[2]	56.2813 89.9057[3] 94.1188 94.0884[4] n/a 95.1871[2]	68.6385 111.8432[3] 116.4923 116.4541[4] n/a 117.6664[2]	81.9210 135.9060[3] 140.9605 140.9145[4] n/a 142.2381[2]
1.9	1.0353 1.8273[1] 2.2477 2.2432[4] 2.2455 2.2748[1]	3.7704 6.8573[3] 8.5942 8.5926[4] n/a 8.8021[2]	7.8734 16.0993[3] 18.7177 18.7101[4] n/a 19.0178[2]	13.1989 29.0750[3] 32.4615 32.4503[4] n/a 32.8505[2]	19.6379 45.5221[3] 49.7204 49.7021[4] n/a 50.1962[2]	27.1159 65.4737[3] 70.4157 70.3905[4] n/a 70.9766[2]	35.5691 88.7686[3] 94.4848 94.4503[4] n/a 95.1293[2]	44.9481 115.4333[3] 121.8754 121.8313[4] n/a 122.6024[2]	55.2082 145.3521[3] 152.5433 152.4878[4] n/a 153.3517[2]	66.3127 178.5468[3] 186.4500 186.3822[4] n/a 187.3822[4]
1.99	0.1474 1.9816[1] 2.4441 2.4427[4] 2.4452 2.4563[2]	0.5494 7.5121[3] 9.7330 9.7293[4] n/a 9.7573[2]	1.1671 18.3642[3] 21.8288 21.8200[4] n/a 21.8651[2]	1.9816 34.1070[3] 34.7113 38.6960[4] n/a 38.7595[2]	2.9788 54.5469[3] 60.3666 60.3426[4] n/a 60.4267[2]	4.1482 79.8163[3] 86.7839 86.7495[4] n/a 86.8560[2]	5.4811 109.7856[3] 117.9546 117.9077[4] n/a 118.0385[2]	6.9705 144.5508[3] 153.8713 153.8100[4] n/a 153.9670[2]	8.6101 184.0144[3] 194.5275 194.4500[4] n/a 194.6351[2]	10.3994 22.82517[3] 239.9178 239.8220[4] n/a 240.0373[2]

(i) See [50], [65], [102]. (ii) Combination of [49] with monotonicity in α.

Hence, when $x \in I_k, y \in I_l, k, l \in \mathbb{Z}^d$, we have $|x - y| \leq \varepsilon\varrho(k - l)$. We define

$$\nu_k = \big(\varrho(k)\big)^{-d-\alpha}, \qquad \bar{\nu} = \sum_{k\in\mathbb{Z}^d} \nu_k,$$

and

$$\mathcal{E}_\varepsilon(f,f) = \frac{c_{d,\alpha}\varepsilon^{-d-\alpha}}{2} \sum_{k,l\in\mathbb{Z}^d} \nu_{k-l} \int_{I_k} \int_{I_l} \big(f(x) - f(y)\big)^2 dx\, dy.$$

Clearly, $\mathcal{E}_\varepsilon(f,f) \le \mathcal{E}(f,f)$. By Rayleigh–Ritz variational principle, the eigenvalues λ_n are bounded below by the sequence $\lambda_{n,\varepsilon}$ of eigenvalues of the operator corresponding to the Dirichlet form $\mathcal{E}_\varepsilon$. Here $\lambda_{n,\varepsilon}$ are defined in the usual way,

$$\lambda_{n,\varepsilon} = \inf\{sup\{\mathcal{E}_\varepsilon(f,f): f \in U, \|f\|_2 = 1\}: U < L^2(D), dimU = n\}.$$

We now express $\lambda_{n,\varepsilon}$ as eigenvalues of a matrix. For $f \in L^2(D)$ and $k \in \mathbb{Z}^d$, let $f_k = \varepsilon^{-d} \int_{I_k} f(x)dx$ be the mean value of f on I_k, and define f^* to be equal to f_k on each $I_k, k \in \mathbb{Z}^d$. Hence $f^* \in L^2(D)$ is the orthogonal projection of f onto the space of functions constant on each I_k, and $\int_{I_k} f^*(x)\,dx = \int_{I_k} f(x)dx$. In particular, $\|f\|_2^2 = \|f^*\|_2^2 + \|f - f^*\|_2^2$.Furthermore,

$$\mathcal{E}_\varepsilon(f,f) = \frac{c_{d,\alpha}\varepsilon^{-d-\alpha}}{2} \sum_{k,l\in\mathbb{Z}^d} \nu_{k-l} \int_{I_k} \int_{I_l} \left(\big(f(x)\big)^2 - 2f(x)f(y) + \big(f(y)\big)^2\right) dx\, dy$$

$$= c_{d,\alpha}\varepsilon^{-\alpha}\left(\bar{\nu}\|f\|_2^2 - \varepsilon^d \sum_{k,l\in\mathbb{Z}^d} \nu_{k-l} f_k f_l\right).$$

Since $(f^*)_k = f_k$ for all $k \in \mathbb{Z}^d$, and $\|f\|_2^2 - \|f^*\|_2^2 = \|f - f^*\|_2^2$, we obtain that

$$\mathcal{E}_\varepsilon(f,f) = \mathcal{E}_\varepsilon(f^*,f^*) + c_{d,\alpha}\varepsilon^{-\alpha}\bar{\nu}\|f - f*\|_2^2. \tag{147}$$

This proves that the two orthogonal subspaces, $\{f \in L^2(D) : f^* = 0\}$ and $\{f \in L^2(D): f^* = f\}$, are invariant under the action of the operator corresponding to $\mathcal{E}_\varepsilon$. By (147), the former subspace is in fact the eigenspace corresponding to the eigenvalue $c_{d,\alpha}\varepsilon^{-\alpha}$. The latter one is finite dimensional, and when $f^* = f$, we have

$$f(x) = \sum_{k\in K_\varepsilon} f_k \mathbf{1}_{I_k}(x),$$

$$\mathcal{E}_\varepsilon(f,f) = c_{d,\alpha}\varepsilon^{d-\alpha}\left(\bar{\nu}\sum_{k\in K_\varepsilon} f_k^2 - \sum_{k,l\in K_\varepsilon} \nu_{k-l} f_k f_l\right). \tag{148}$$

The normalized indicator functions of I_k, that is, the functions $\varepsilon^{-d/2}\mathbf{1}_{I_k}, k \in K_\varepsilon$, form an orthonormal basis of the space $\{f \in L^2(D): f^* = f\}$. By (148), in this basis, the action of $\mathcal{E}_\varepsilon$ is given by the following $|K_\varepsilon| \times |K_\varepsilon|$ matrix V : if κ is an enumeration of the elements of K_ε (that is, a bijection between $\{1, 2, \dots, |K_\varepsilon|\}$ and K_ε), then $V_{p,q} = c_{d,\alpha}\varepsilon^{-\alpha}\big(\delta_{p,q}\bar{\nu} - \nu_{\kappa(p)-\kappa(q)}\big)$.

We conclude that the sequence $\lambda_{n,\varepsilon}$ starts with all eigenvalues of the matrix V which are less than $c_{d,\alpha}\varepsilon^{-\alpha}\bar{\nu}$ (there are at most $|K_\varepsilon|$ of them), and then it is a constant sequence $c_{d,\alpha}\varepsilon^{-\alpha}\bar{\nu}$. We have thus proved the following result.

Proposition (2.2.7)[97]: Let $D \subseteq \mathbb{R}^d$ be an open set in $\mathbb{R}^d$, and let $\varepsilon > 0$. Let K_ε be the set of those $k \in \mathbb{Z}^d$ for which $D \cap \prod_{j=1}^d \big[k_j\,\varepsilon, \big(k_j + 1\big)\varepsilon\big]$ is nonempty, and let $\kappa : \{1, 2, \dots, |K_\varepsilon|\} \to K_\varepsilon$ be the enumeration of the elements of K_ε. Finally, for $k \in \mathbb{Z}^d$ let

$$\varrho(k) = \sqrt{\sum_{j=1}^{d} \left(|k_j| + 1\right)^2}, \qquad \text{and}\ \overline{\nu} = \sum_{k\in\mathbb{Z}^d} \left(\varrho(k)\right)^{-d-\alpha}.$$

Define a $|K_\varepsilon| \times |K_\varepsilon|$ matrix V with entries

$$V_{p,q} = -\frac{c_{d,\alpha}}{\varepsilon^\alpha}\varrho\big(\kappa(p) - \kappa(q)\big)^{-d-\alpha}, \qquad p,q = 1,2,\ldots,|K_\varepsilon|, p \neq q;$$

$$V_{p,p} = \frac{c_{d,\alpha}}{\varepsilon_\alpha}\left(\overline{\nu} - d^{-(d+\alpha)/2}\right), \qquad p = 1,2,\ldots,|K_\varepsilon|.$$

If $n \leq |K_\varepsilon|$ and $n\text{-}th$ smallest eigenvalue of V does not exceed $c_{d,\alpha}\varepsilon^{-\alpha}\overline{\nu}$, then let λ_n be this eigenvalue. Otherwise, let $\lambda_{n,\varepsilon} = c_{d,\alpha}\varepsilon^{-\alpha}\overline{\nu}$. Then the eigenvalues λ_n of A_D satisfy $\lambda_n \geq \lambda_{n,\varepsilon}$.

Note that if $\overline{\nu}$ is replaced by a smaller number in the definition of V, the eigenvalues $\lambda_{n,\varepsilon}$ decrease. Hence, when doing numerical computations using Proposition (2.2.7), one should approximate $\overline{\nu}$from below.

In the one-dimensional case, we have $c_{1,\alpha} = c_\alpha$, and $\overline{\nu} = 2\,\zeta(1 + \alpha) - 1$, where ζ is the Riemann zeta function. Consider now $= (-1,1) \subseteq \mathbb{R}$, and $\varepsilon = \frac{2}{N}$. For simplicity, assume that *N* is an even positive integer. Then $K_\varepsilon = \left\{-\frac{N}{2}, -\frac{N}{2} + 1, \ldots, \frac{N}{2} - 1\right\}$, so it is natural to choose $\kappa(p) = p - \frac{N}{2} - 1, p \in \{1,2,\ldots,N\}$. Furthermore, V is a Toeplitz matrix, that is, $V_{p,q} = V_{p-q}$ depends only on $p - q$. In this case we can prove that all eigenvalues of the matrix V are less than $c_\alpha\varepsilon^{-\alpha}\overline{\nu}$. Indeed, the symbol of the Toeplitz matrix V is given by (we omit some technical details here)

$$\sum_{k=-\infty}^{\infty} V_k e^{ikx} = \frac{2c_\alpha}{\varepsilon^\alpha}\left(\zeta(1+\alpha) - \sum_{k=0}^{\infty}\frac{\cos(kx)}{(1+k)^{1+\alpha}}\right)$$

$$= \frac{2c_\alpha}{\varepsilon^\alpha}\left(\zeta(1+\alpha) - \operatorname{Re}\left(\frac{Li_{1+\alpha}\left(e^{ix}\right)}{e^{ix}}\right)\right)$$

$$= \frac{2c_\alpha}{\varepsilon^\alpha}\left(\zeta(1+\alpha) - \frac{1}{1+\alpha}\int_0^\infty \frac{t^\alpha(e^t - \cos x)}{e^{2t} - 2e^t\cos x + 1}dt\right).$$

The right-hand side is easily checked to be symmetric, 2π-periodic and increasing in $x \in [0,\pi]$,and so it attains its global maximum for $x = \pi$. The symbol of V is therefore bounded above by $2c_\alpha\varepsilon^{-\alpha}\big(\zeta(1+\alpha) - Li_{1+\alpha}(-1)\big) = 2^{1-\alpha}c_\alpha\varepsilon^{-\alpha}\zeta(1+\alpha) \leq c_\alpha\varepsilon^{-\alpha}\overline{\nu}$. By a general result, the eigenvalues of *V* are bounded above by the supremum of the symbol. It follows that all *N* eigenvalues

Table (3)[97]: Comparison of estimates of λ_n for a square $(-1,1)^2$. LB and UB mean lower bounds and upper bounds respectively.
Estimates are given in roman font, best numerical estimates known before are typeset in slanted font.
Better estimates are underlined.

α	λ_1 (LB)		λ_1 (UB)	λ_2 (LB)		λ_2 (UB)

0.1	1.0308	0.5230[1]	1.0462[1]	1.0880	0.5415[1]	1.0831[1]
0.2	1.0506	0.5472[1]	1.0946[1]	1.1691	0.5865[1]	1.1731[1]
0.5	1.1587	0.6266[1]	1.2534[1]	1.4908	0.7452[1]	1.4905[1]
1	1.3844	0.7853[1]	1.5708[1]	2.1807	1.1107[1]	2.2215[1]
1.5	1.4135	0.9843[1]	1.9688[1]	2.6029	1.6554[1]	3.3110[1]
1.8	0.9167	1.1271[1]	2.2544[1]	1.8164	2.1033[1]	4.2068[1]
1.9	0.5427	1.1792[1]	2.3585[1]	1.0984	2.2781[1]	4.5563[1]

See [65].

Table (4)[97]: Comparison of estimates of λ_n for a unit disk. LB and UB mean lower bounds and upper bounds respectively. Estimates are given in roman font, best numerical estimates known before are typeset in slanted font. Better estimates are underlined.

α	λ_1 (LB)		λ_1 (UB)		λ_2 (LB)		λ_2 (UB)
0.1	1.0381	1.0157[1]	6.6198	1.0641[1]	1.0953	0.5718[2]	1.1609[2]
0.2	1.0655	1.0396[1]	3.8878	1.1342[1]	1.1849	0.6541[2]	1.3476[2]
0.5	1.1986	1.1618[1]	2.5081	1.3943[1]	1.5404	0.9787[2]	2.1079[2]
1	1.4734	1.5707[1]	2.7588	2.0944[1]	2.3201	1.9158[2]	4.4429[2]
1.5	1.5387	2.3891[1]	4.0668	3.4131[1]	2.8379	3.7502[2]	9.3648[2]
1.8	1.0087	3.2210[1]	5.5014	4.7468[1]	2.0045	5.6114[2]	14.6487[2]
1.9	0.5990	3.5834[1]	6.1369	5.2974[2]	1.2165	6.4182[2]	17.0045[2]

See [50], [65], of V are included in the sequence $\lambda_{n,\varepsilon}$, as desired. Therefore, we have the following specialized version of Proposition (2.2.7) (the case of odd N is very similar).

Proposition (2.2.8)[97]: Let $D = (-1,1), N > 0$ and $\varepsilon = 2/N$. Let V be an $N \times N$ Toeplitz matrix with entries

$$V_{p,q} = -\frac{c_\alpha}{\varepsilon^\alpha}\frac{1}{(|p - q| + 1)^{1+\alpha}}, \qquad p,q = 1,2,\ldots,N, p \neq q;$$

$$V_{p,p} = \frac{2c_\alpha(\zeta(1 + \alpha) - 1)}{\varepsilon^\alpha}, \qquad p = 1,2,\ldots,N.$$

Define $\lambda_{n,\varepsilon}$ to be the n-th smallest eigenvalue of V when $n \leq N$, and $\lambda_{n,\varepsilon} = c_\alpha\varepsilon^{-\alpha}(2\zeta(1+\alpha)-1)$ for $n > N$. Then the eigenvalues λ_n of A_D satisfy $\lambda_n \geq \lambda_{n,\varepsilon}$.

The lower bounds $\lambda_{n,\varepsilon}$ for the interval $D = (-1,1)$ are presented in Table 2 above. In higher dimensions, the complexity of computations increases dramatically. For example, a unit disk $B(0,1)$ or a square $(-1,1)^2$ with $\varepsilon = \frac{1}{25}$ require handling matrices larger than 2000×2000. Some results for these two cases are given in Tables (3) and (4).

In principle, the upper bound is much more difficult. The above approach can be modified to give an upper bound for the first eigenvalue λ_1 whenever the Green function for D can be computed. For the fractional Laplace operator, this is the case when D is a ball. By a scaling property, it is enough to consider $D = B(0,1)$.

Let $G_D(x,y)$ be the Green function of $D, G_D(x,y) = \int_0^\infty p_t^D(x,y)\,dt$, where p_t^D is the heat kernel for A_D (see the proof of Proposition (2.2.5)). The Green function is the kernel of A_D^{-1}. M. Riesz proved that

$$G_D(x,y) = \frac{\Gamma(\frac{d}{2})|x-y|^{\alpha-d}}{2\alpha\pi^{d/2}\left(\Gamma\left(\frac{\alpha}{2}\right)\right)^2}\int_0^{\frac{(1-x^2)(1-y^2)}{|x-y|^2}} \frac{s^{\alpha 2-1}}{(1+s)^{d/2}}\, ds$$

$$= \frac{\Gamma\left(\frac{d}{2}\right)(1-x^2)^{\alpha 2}(1-y^2)^{\alpha 2}}{2^{\alpha}\pi^{d2}\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(1+\frac{\alpha}{2}\right)|x-y|^d}\, 2F_1\left(\frac{\alpha}{2},\frac{d}{2};\, 1+\frac{\alpha}{2};-\frac{(1-x^2)(1-y^2)}{|x-y|^2}\right).$$

The eigenvalues of the Green operator $GD = A^{-1}D$ are λ_n^{-1} . Hence,

$$\frac{1}{\lambda_1} = \sup\left\{\int_D\int_D G_D(x,y)f(x)f(y)dx\,dy: f \in L^2(D), \|f\|_2 = 1\right\}.$$

Since $G_D(x,y)$ is nonnegative, we may restrict the supremum to nonnegative functions only. It follows that whenever $0 \le g(x,y) \le G_D(x,y)$, we have

$$\lambda_1 \le \left(\sup\left\{\int_D\int_D g(x,y)f(x)f(y)dx\,dy: f \in L^2(D), \|f\|_2 = 1\right\}\right)^{-1}.$$

For $k,l \in \mathbb{Z}^d$, let $g_{k,l}$ be the infimum of $G_D(u,v)$ over $u \in I_k$ and $v \in I_l$. When $\in I_k$, $y \in I_l$, we choose $g(x,y) = g_{k,l}$. Hence, by an argument similar to one used for the lower bounds, λ_1 is bounded above by $\lambda_{1,\varepsilon}^*$, the reciprocal of the largest eigenvalue of the matrix U with entries $U_{i,j} = \varepsilon^d g_{\kappa(i),\kappa(j)}$.

The results for $D = (-1,1) \subseteq \mathbb{R}$ and some values of α are given in Table (2). Estimates for the unit disk and the square $(-1,1)^2$ are given in Tables (3) and (4). For the unit disk and $\varepsilon = \frac{1}{25}$, the estimate $\lambda_{1,\varepsilon}^*$ is worse than the one obtained in [50] using analytical methods.

Section (2.3): Fractional Powers of the Laplace Operator

We study the asymptotic behavior of eigenvalues for fractional powers of the Laplacian. The operator $(-\Delta)^s$ with $0 < s < 1$ appears in numerous fields of mathematical physics, mathematical biology and mathematical finance. The key difference between this operator and the usual Laplacian is the non-locality of $(-\Delta)^s$, which allows one to model long-range interactions in applications and leads to challenging mathematical problems.

From a probabilistic point of view, the fractional Laplacian of order s on a domain $\Omega \subset \mathbb{R}^d$ can be defined as the generator of the $2s$-stable process killed upon exiting Ω. A more operator theoretic definition, which we employ here, is in terms of the quadratic form

$$C_{s,d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}\,dxdy = \int_{\mathbb{R}^d}|p|^{2s}|\hat{u}(p)|^2\,dp, \qquad (149)$$

restricted to functions $u \in H^s(\mathbb{R}^d)$ which satisfy $u = 0$ in $\mathbb{R}^d\backslash\bar{\Omega}$. Here $H^s(\mathbb{R}^d)$ is the Sobolev space of order s,

$$\hat{u}(p) = (2\pi)^{-d/2} = \int_{\mathbb{R}^d} e^{i_p\cdot x}u(x)\,dx$$

is the Fourier transform of u and $C_{s,d}$ is an explicit constant given in (153). The identity in (149) is an easy consequence of Plancherel's theorem.

For bounded domains Ω the spectrum of the fractional Laplacian is discrete and we denote its eigenvalues (in increasing order, repeated according to multiplicities) by $\lambda_n^{(s)}$. Two-term asymptotic expansion of the sum of these eigenvalues,

$$\frac{1}{N}\sum_{n=1}^{N} \lambda_n^{(s)} = C_{d,s}^{(1)} |\Omega|^{-2s/d} N^{2s/d} + C_{d,s}^{(2)} |\partial\Omega| \, |\Omega|^{-\frac{(d-1+2s)}{d}} N^{\frac{(2s-1)}{d}} \big(1 + o(1)\big) \quad (150)$$

as $N \to \infty$. Here $|\Omega|$ and $|\partial\Omega|$ denote the d-dimensional measure of Ω and the $(d-1)$-dimensional surface measure of $\partial\Omega$, respectively, and $C_{d,s}^{(1)}$ and $C_{d,s}^{(2)}$ are positive, universal constants, depending only on d and s, for which we shall obtain explicit expressions. Our result is valid for non-smooth domains, requiring only that $\partial\Omega \in C^{1,\alpha}$ for some (arbitrarily small) $\alpha > 0$. It is remarkable that, despite the fact that we are dealing with a non-local operator, both coefficients in (150) have a local form, depending only on Ω and $\partial\Omega$, just like in the case of the Laplacian. This will become clearer from the reformulation given in Theorem (2.3.3) below.

We emphasize that the fractional Laplacian of order s on a domain Ω is different from the Dirichlet Laplacian on Ω raised to the s-th power. For the Dirichlet Laplacian, and hence for its fractional powers, asymptotics analogous to (150) are well known. One of our results is that, while the first terms in (150) coincide for both operators, the second terms do not. This means, in particular, that our result cannot be obtained from the study of the (local) Dirichlet Laplacian, and that our analysis needs to take into account the non-locality inherent in (150). For further results about the relation between the fractional Laplacian on a domain and the fractional power of the Dirichlet Laplacian see [106].

The one-term asymptotics

$$\lambda_N^{(s)} = \frac{(d+2s)}{d} C_{d,s}^{(1)} |\Omega|^{-2s/d} N^{2s/d} \big(1 + o(1)\big),$$

which is a fractional version of Weyl's law, is a classical result of Blumenthal–Getoor [54]. Bañuelos–Kulczycki [52] and Bañuelos–Kulczycki–Siudeja [104] have shown a two-term asymptotic formula for $\sum_{n=1}^{\infty} \exp\left(t\lambda_n^{(s)}\right)$ as $t \to 0$. Note that $\sum_{n=1}^{\infty} \exp\left(-t\lambda_n^{(s)}\right)$ and $N^{-1}\sum_{n=1}^{N} \lambda_n^{(s)}$ correspond to the Abel and Cesàro summation of the sequence $\lambda_n^{(s)}$, respectively. As is well known, asymptotics of Cesàro means imply asymptotics of Abel means, but not vice versa. Hence for $C^{1,\alpha}$ domains we recover and improve upon the result of [52], [104].

This is, actually, a significant improvement since our asymptotics are no longer derived for the infinitely smooth function e^{-tE} of the fractional Laplacian, but, as we shall see shortly, for the Lipschitz function $(\Lambda - E)_+$. Moreover, since we are no longer able to apply the probabilistic machinery available for the partition function, we have to find new and more robust tools. The methods also work for the ordinary Dirichlet Laplacian on a bounded domain, and in [108] we use the techniques developed here to give an elementary and short proof of two-term asymptotics in that case.

Another point in which we go beyond [52], [104] is that we give an expression for the constant $C_{d,s}^{(2)}$ in (150) in terms of a model operator on a half-line instead of a model operator on a half-space. In this way our expression is similar to familiar two-term formulas in semiclassical analysis; see, for instance, [118]. This is possible due to some recent beautiful results of Kwasnicki [112] about a general class of half-line operators.

We find it convenient to prove (150) in an equivalent form, namely,

$$\sum_{1=n}^{\infty}\left(\Lambda-\lambda_n^{(s)}\right)_+ = L_{s,d}^{(1)}|\Omega|\Lambda^{\frac{1+d}{2s}} - L_{s,d}^{(2)}|\partial\Omega|\Lambda^{1+\frac{d-1}{2s}}\big(1+o(1)\big) \tag{151}$$

as $\Lambda \to \infty$. Here $x_+ := \max\{x,0\}$ denotes the positive part of a number x. Note that (151) can be rewritten as

$$\sum_{n=1}^{\infty}\left(1-h^{2s}\lambda_n^{(s)}\right)_+ = L_{s,d}^{(1)}|\Omega|h^{-d} - L_{s,d}^{(2)}|\partial\Omega|h^{-d+1}\big(1+o(1)\big) \tag{152}$$

as $h \to 0+$, and this is the form in which we shall state and prove the main theorem. The small parameter h has the interpretation of Planck's constant and (152) emphasizes the semi-classical nature of the problem.

The approach extends the multiscale analysis to the fractional setting. By this we mean that we localize simultaneously on different length scales according to the distance from the boundary. A main difficulty when dealing with our non-local operator comes from the treatment of the localization error. At this point we have to improve upon previous results from [115], [119]. Another major impass, as compared to the local case, is the analysis of a onedimensional model operator for which an (almost) explicit diagonalization is far from trivial. This is where Kwasnicki's work [112] enters. It requires, however, still substantial work to bring these results into a form which is useful for us.

We assume that the dimension $d \geq 2$. In the one-dimensional case (the fractional Laplacian on an interval) considerably stronger results are known [49], [97]. The powerful methods developed there are, however, intrinsically one-dimensional and seem of little help in the multi-dimensional case. The question raised in [104] of whether an analogue of Ivrii's two-term asymptotics [111] holds for $\lambda_n^{(s)}$ in $d \geq 2$ without Abel or Cesàro averaging remains a challenging open problem.

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded open set. For $h > 0$ and $0 < s < 1$ let

$$H_\Omega = (-h^2\Delta)^s - 1$$

be the self-adjoint operator in $L^2(\Omega)$ generated by the quadratic form

$$(u, H_\Omega u) = \int_{\mathbb{R}^d} (|hp|^{2s} - 1)|\,|\hat{u}(p)|^2\, dp$$

with form domain

$$\mathscr{H}^s(\Omega) = \{u \in H^s(\mathbb{R}^d): u = 0 \; on \; \mathbb{R}^d \setminus \overline{\Omega}\}.$$

For $0 < s < 1$ we have the representation

$$(u, H_\Omega u) = C_{s,d}\, h^{2s} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(x)\, u(y)|^2}{|x\; y|^{d+2s}}\, dxdy \int_\Omega |u(x)|^2\, dx$$

with constant

$$C_{s,d} = 2^{2s-1}\pi^{d/2}\,\frac{\Gamma(d2+\; s)}{|\Gamma(-s)|} > 0. \tag{153}$$

The main results hold without any global geometric conditions on Ω. We only require weak smoothness conditions on the boundary – namely that the boundary belongs to the class $C^{1,\alpha}$ for some $\alpha > 0$. That is, the local charts of $\partial\Omega$ are differentiable and the derivatives are Hölder continuous with exponent α.

We will derive several representations of the constant $L_{s,d}^{(2)}$ in (161). One of these, which emphasizes the semi-classical nature of the problem, leads to a rewriting of (161) as

$$\operatorname{Tr}(H_\Omega)_- = \iint_{T*\Omega} (|p|^{2s} - 1)\frac{dpdx}{(2\pi h)^d} - \iint_{T*\partial\Omega} (|p'|^{2s})\frac{dp'\sigma(x)}{(2\pi h)^{d-1}} + R_h, \qquad (154)$$

where $T * \Omega = \Omega \times \mathbb{R}^d$ and $T * \partial\Omega = \partial\Omega \times \mathbb{R}^{d-1}$ are the cotangent bundles over Ω and $\partial\Omega$, respectively, and where $d\sigma$ is the surface element of $\partial\Omega$. Here $\xi$is a universal (i.e., depending on s, but independent of Ω or d) function, which has the interpretation of an energy shift (the integral of a spectral shift). It is given in terms of a one-dimensional model operator A^+ on the half-line $\mathbb{R}_+$ and its analogue A on the whole line by

$$\xi(\mu) = \mu^{-1}\int_0^\infty \left(a(t,t,\mu) - a^+(t,t,\mu)\right)dt, \qquad \mu > 0;$$

where $a(t,u,\mu)$ and $a^+(t,u,\mu)$ denote the integral kernels of $(A-\mu)_-$ and $(A^+-\mu)_-$, respectively. Another representation, derived, shows that our result is consistent with the result of [52], [104].

We show that $L_{s,d}^{(2)} > 0$. We compare this constant with the one obtained from the corresponding fractional power of the Dirichlet Laplacian.

Proposition (2.3.1)[103]: Let $0 < s < 1$ and assume that the boundary of Ω satisfies $\partial\Omega \in C^{1,\alpha}$ with some $0 < \alpha \le 1$. Let $-\Delta_\Omega$ be the Dirichlet Laplacian on Ω. Then

$$\operatorname{Tr}((-h^2\Delta_\Omega)^s - 1)_- = L_{s,d}^{(1)}|\Omega|h^{-d} - \tilde{L}_{s,d}^{(2)}|\partial\Omega|h^{-d+1} + R_h \qquad (155)$$

with $R_h = o(h^{-d+1})$ as $h \to 0+$. Here $L_{s,d}^{(1)}$ is the same as in (162) and $L_{s,d}^{(2)}$ satisfies

$$L_{s,d}^{(2)} < \tilde{L}_{s,d}^{(2)}. \qquad (156)$$

In other words, the operators H_Ω and $(-h^2\Delta_\Omega)^s - 1$ differ semi-classically to first subleading order.

The proof of Theorem (2.3.3) is divided into three main steps: First, we localize the operator H_Ω into balls, whose size varies depending on the distance to the complement of Ω. Then we can analyze separately the semiclassical limit in the bulk and at the boundary.

The key idea is to choose the localization depending on the distance to the complement of Ω, see [110] and [120]. Let $d(u) = \inf\{|x-u| : x \notin \Omega\}$ denote the distance of $u \in \mathbb{R}^d$ to the complement of Ω. We set

$$l(u) = \frac{1}{2}\left(1 + (d(u)^2 + l_0^2)^{-\frac{1}{2}}\right)^{-1}, \qquad (157)$$

where $0 < l_0 \le 1/2$ is a small parameter depending only on h. Indeed, we will finally choose l_0 proportional to h^β with suitable $0 < \beta < 1$.

We construct real-valued functions $\phi_u \in C_0^\infty(\mathbb{R}^d)$ with support in the ball $B_u = \{x \in \mathbb{R}^d : |x-u| < l(u)\}$.For all $u \in \mathbb{R}^d$ these functions satisfy

$$\|\phi_u\|_\infty \le C, \qquad \left\|\Delta_{\phi_u}\right\|_\infty \le Cl(u)^{-1} \qquad (158)$$

and for all $x \in \mathbb{R}^d$

$$\int_{\mathbb{R}^d} \phi_u^2(x)l(u)^{-d}\,du = 1. \qquad (159)$$

Here and in the following the letter C denotes various positive constants that are independent of u, l_0 and h.

Proposition (2.3.2)[103]: Assume that $\phi \in C_0^1(\Omega)$ is real-valued, supported in a ball of radius $l > 0$ and

$$\|\nabla_\phi\|_\infty \leq C\, l^{-1}. \tag{160}$$

Then for all $h > 0$ the estimates

$$-Cl^{d-2}h^{-d+2} \leq \operatorname{Tr}(\phi H_\Omega \phi)_- - L_{s,d}^{(1)} \int_\Omega \phi^2(x)\, dx\, h^{-d} \leq 0$$

hold with a constant depending only on the constant in (160).

Close to the boundary of Ω, more precisely, if the support of ϕ intersects the boundary, a boundary term of the order h^{-d+1} appears.

Theorem (2.3.3)[103]: Let $0 < s < 1$ and assume that the boundary of Ω satisfies $\partial\Omega \in C^{1,\alpha}$ with some $0 < \alpha \leq 1$. Then

$$\operatorname{Tr}(H_\Omega)_- = L_{s,d}^{(1)} |\Omega| h^{-d} - L_{s,d}^{(2)} |\, |\partial\Omega| h^{-d+1} + R_h \tag{161}$$

with $R_h = o(h^{-d+1})$ as $h \to 0+$. Here

$$L_{s,d}^{(1)} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (|p|^{2s} - 1)_-\, dp \tag{162}$$

and the positive constant $L_{s,d}^{(2)}$ is given in (220).

More precisely, we have the lower bound $R_h \geq -Ch^{-d+1+\epsilon_-}$ for any

$$0 < \epsilon_- < \begin{cases} \dfrac{\alpha}{\alpha+2} & \text{if} \dfrac{1}{2} \leq s < 1, \\ \dfrac{2s\alpha}{\alpha+1+2s} & \text{if } 0 < s < \dfrac{1}{2}. \end{cases}$$

and the upper bound $R_h \leq Ch^{-d+1+\epsilon_-}$ for any

$$0 < \epsilon_+ < \frac{\alpha}{\alpha+2} \qquad \text{if } 1 - \frac{d}{4} \leq s < 1,$$

$$0 < \epsilon_= \leq \frac{\alpha(2s - 1 + d/2)}{\alpha + 2s + d/2} \qquad \text{if } 0 < s < 1 - \frac{d}{4}.$$

We do not claim that our remainder estimates are sharp. They show, however, that our methods are rather explicit and they correctly reflect the intuitive fact that the estimate worsens as the boundary gets rougher. We also mention that for not too small s we (almost) get the same remainder estimate $h^{-d+1+\alpha/(\alpha+2)}$ that our method yields in the local case $s = 1$, see [108].

Proof: In order to apply Proposition (2.3.12) to the operators $\phi_u H_\Omega \phi_u$, we need to estimate $l(u)$ uniformly. Let

$$U(\Omega) = \{u \in \mathbb{R}^d : B_u \cap \partial\Omega \neq \emptyset\}$$

be a small neighborhood of the boundary. For $u \in U(\Omega)$ we have $d(u) \leq l(u)$, which by the definition of $l(u)$ implies

$$l(u) \leq \frac{l_0}{\sqrt{3}}. \tag{163}$$

In view of (158) and (163) we can apply Proposition (2.3.2) and Proposition (2.3.12) to all functions $\phi_u, u \in \mathbb{R}^d$, if l_0 is sufficiently small. Combining these results with Proposition (2.3.15) we get

$$-C \int_{\Omega \setminus U(\Omega)} l(u)^{-2}\, du\, h^{-d+2} \int_{U(\Omega)} \tilde{R}_{bd}(l(u), h) l(u)^{-d}\, du$$
$$\leq \operatorname{Tr}(H_\Omega)_- L_{s,d}^{(1)} \int_{\mathbb{R}^d} \int_\Omega \phi_u^2(x) dx \frac{du}{l(u)^d} h^{-d}$$
$$+ L_{s,d}^{(2)} \int_{\mathbb{R}^d} \int_{\partial\Omega} \phi_u^2(x)\, d\sigma(x) \frac{du}{l(u)^d} h^{-d+1}$$
$$\leq \int_{U(\Omega)} R_{bd}(l(u), h) l(u)^d\, du + C\, h^{-d+2}\, l_0^{-1} R_{\text{loc}}(l_0, h).$$

Now we change the order of integration and in view of (159) we obtain

$$-C \int_{\Omega \setminus U(\Omega)} l(u)^2\, du\, h^{-d+2} - \int_{U(\Omega)} \tilde{R}_{bd}(l(u), h) l(u)^{-d} du \tag{164}$$
$$\leq \operatorname{Tr}(H_\Omega)\, L_{s,d}^{(1)} |\Omega| h^{-d} + L_{s,d}^{(2)} |\partial\Omega| h^{-d+1}$$
$$\leq \int_{U(\Omega)} R_{bd}(l(u), h) l(u)^{-d}\, du + C\, h^{-d+2}\, l_0^{-1} R_{\text{loc}}(l_0, h).$$

It remains to estimate the error terms.

By the definition of l.u/ we have

$$l(u) \geq \frac{1}{4} \min(d(u), 1) \quad \text{and} \quad l(u) \geq \frac{l_0}{4} \tag{165}$$

for all $u \in \mathbb{R}^d$ and $l_0 \leq 1$. For $u \in \Omega \setminus U(\Omega)$, we find $d(u) \geq l(u) \geq l_0/4$. Hence, we can estimate

$$\int_{\Omega \setminus U(\Omega)} l(u)^2\, du \leq C \left(1 + \int_{\{d(u) l_0/4\}} d(u)^2\, du\right) \leq C(1 + \int_{l_0/4}^{\infty} t^{-2} |\partial\Omega_t|\, dt\, ;$$

where $|\partial\Omega_t|$ denotes the surface area of the boundary of $\Omega_t = \{x \in \Omega : d(x) > t\}$. Using the fact that $|\partial\Omega_t|$ is uniformly bounded and that $|\partial\Omega_t| = 0$ for large t, we get

$$\int_{\Omega \setminus U(\Omega)} (\Omega) l(u)^2\, du \leq C\, l_0^{-1}. \tag{166}$$

For $u \in U(\Omega)$ the inequalities (163) and (165) show that $l(u)$ is proportional to l_0. Since $B_u \cap \partial\Omega \neq \emptyset$, we find $d(u) \leq l(u) \leq l_0$ and

$$\int_{U(\Omega)} l(u)^a\, du \leq C\, l_0^a \int_{\{d(u) \leq l_0\}} du \leq C\, l_0^{a+1} \tag{167}$$

for any $a \in \mathbb{R}$.

We insert (166) and (167) into (164) and get (using the fact that $h \leq C^{-1} l_0$)

$$-C \left(l_0^{\delta_2} h^{\delta_2} + l_0^{2\alpha} + l_0^{\alpha+1} h^{-1}\right) \tag{168}$$
$$\leq h^{d-1} \left(\operatorname{Tr}(H_\Omega)\, L_{s,d}^{(1)} |\Omega| h^{-d} + L_{s,d}^{(2)} |\partial\Omega| h^{d+1}\right)$$
$$\leq C \left(l_0^{\delta_1} h^{\delta_1} + l_0^{\alpha+1} h^{-1} + l_0^{-1} h\, R_{\text{loc}}(l_0, h)\right).$$

In order to choose l_0 we need to distinguish several cases. For the lower bound we recall that $0 < \delta_2 < \min\{1, 2s\}$. The stated lower bound on R_h follows with l_0 proportional to h^{β} , where $\beta = (1+\delta_2)/(1+\alpha+\delta_2)$.

For the upper bound we have $0 < \delta_1 < 1$. If $1 - d/4 < s < 1$, then we pick l_0 proportional to h^{β} , where $\beta = (1+ \delta_1)/(1+\alpha+\delta_1)$. If $0 < s \leq 1 - d/4$, then we pick h^{β} , where $\beta = (2s + d/2)/(\alpha + 2s + d/2)$. This completes the proof of Theorem (2.3.3).

We analyze the local asymptotics in the bulk and prove Proposition (2.3.2). We consider the local asymptotics in the case where Ω is replaced by a half-space. We reduce the problem close to the boundary to the analysis of a one-dimensional model operator given on a half-line and give an analogue of Proposition (2.3.12) for a half-space. We show how Proposition (2.3.12) follows from the previous considerations by local straightening of the boundary. We perform the localization and, in particular, prove Proposition (2.3.15). We provide some technical results about the one-dimensional model operator introduced.

We define the positive and negative parts of $x \in \mathbb{R}$ by $x_{\pm} = \max\{0, \pm x\}$. We use a similar notation for the Heaviside function, namely, $x_{\pm}^0 = 1$ if $\pm x \geq 0$ and $x_{\pm}^0 = 0$ if $\dot{}x < 0$. For a self-adjoint operator X, the operators $X^{\pm}$ and $X_{\pm}^0$ are defined similarly via the Spectral Theorem.

Warm-up dealing with the spectral asymptotics in the boundaryless case. Although the estimates in this case are essentially known, we include a proof for the sake of completeness and in order to introduce the methods that will be important later on. We divide the proof of Proposition (2.3.2). The operator

$$H_0 = (-h^2\Omega)^s - 1 \quad \text{in} \quad L^2(\mathbb{R}^d),$$

defined with form domain $H^s(\mathbb{R}^d)$, will appear frequently.

The lower bound is given by a variant of the Berezin–Lieb–Li–Yau inequality, see [105], [113], [114].

Lemma (2.3.4)[103]: For any $\phi^2\, L^2(\mathbb{R}^d)$ and $h > 0$

$$\mathrm{Tr}(\phi H_\Omega \phi)_- \leq L_{s,d}^{(1)} \int_{\mathbb{R}^d} \phi^2(x)\, dx\, h^{-d}.$$

Proof: We apply the variational principle for the sum of the eigenvalues

$$\mathrm{Tr}(\phi H_\Omega \phi) = \inf_{0\leq\gamma\leq 1} \mathrm{Tr}(\phi H_\Omega \phi),$$

where the infimum is taken over all trial density matrices, i.e., over all trace-class operators $0 \leq \gamma \leq 1$ with range belonging to the form domain of H_Ω. We apply this twice and find

$$\mathrm{Tr}(\phi H_\Omega \phi)_- \leq \mathrm{Tr}(\phi H_0 \phi)_- \leq \mathrm{Tr}(\phi (H_0) \phi)\,.$$

Applying the Fourier transform to diagonalize the operator (H_0) yields the bound

$$\mathrm{Tr}(\phi (H_0)_- \phi) = \frac{1}{(2\pi h)^d} \iint \phi(x)^2\, (|p|^{2s} - 1)_-\, dp dx = L_{s,d}^{(1)} \int \phi(x)^2\, dx\, h^{-d}\,,$$

as claimed.

We now assume that - satisfies the conditions of Proposition (2.3.2). In particular, we assume that - has support in Ω. To derive the upper bound, we put $\gamma = \chi_\phi (H_0)_-^0 \chi_\phi$, where χ_ϕ denotes the characteristic function of the support of ϕ. Then

$$\gamma(x,y) = (2\pi h)^{-d} \chi_\phi(x)\chi_\phi(y) \int_{|p|<1} e^{ip.(x-y)/h} dp,$$

and we obtain that

$$-\mathrm{Tr}(\phi H_\Omega \phi)_- \le \mathrm{Tr}(\gamma\phi H_\Omega \phi) = \mathrm{Tr}(\gamma\phi H_O \phi)$$

$$= \int_{|p|<1} \left(\left\| (-h^2\Delta)^{s2} \phi e^{ip.h} \right\|_2^2 - \|\phi\|_2^2 \right) \frac{dp}{(2\pi h)^d} \tag{169}$$

Lemma (2.3.5)[103]: For $\phi \in C_0^\infty(\mathbb{R}^d)$ and $h > 0$ we have

$$\left\| (-h^2\Delta)^{s/2} \phi e^{ip.h} \ \right\|_2^2 = |p|^{2s} \|\phi\|_2^2 + \int \ (\frac{1}{2}\ (|p +\ h|^{2s} + |p -\ h\eta|^{2s})\ \left|\hat{\phi}(\eta)\right|^2 d\eta.$$

Proof: By Plancherel's theorem we get

$$\left\| (-h^2\Delta)^{s/2} \phi e^{ip.h} \right\|_2^2 = (2\pi h)^d \ \iiint \ |\xi|^{2s} \phi(x)\phi(y) e^{i(p\,-\,\xi).(x-\,y)/h}\, dy d\xi dx.$$

Since $\phi \in \ C_0^\infty\ (\mathbb{R}^d)$, we can use the fact that

$$\iint \ \phi(x)\phi(y) e^{i(p-\xi).(x-\,y)/h}\, dxdy$$

$$= \lim_{\delta\to 0+} \ \iint \ e^{-\delta|x\,-\,y|^2}\ \phi(x)\phi(y) e^{i(p-\,\xi).(x-\,y)/h}\, dxdy$$

and since

$$|\xi|^{2s} \iint \ \phi(x)\phi(y) e^{i(p-\xi).(x-\,y)/h}\, dxdy$$

is absolutely integrable as a function of $\xi \in \ \mathbb{R}^d$, we find

$$\left\| (-h^2\Delta)^{s2} \phi e^{ip.h} \ \right\|_2^2 = \lim_{\delta\to 0+} \ \iiint \ e^{\delta|x-y|^2}\ |\xi|^{2s} \phi(x)\phi(y) e^{i(p-\xi).(x-y)/h} \frac{dydxd}{(2\pi h)^d}$$

$$= \lim_{\delta\to 0+} \ \iiint \ e^{\delta|x-y|^2} |\xi|^{2s} (\phi^2(x) +\ \phi^2\ (y)\ \left(\phi(x) - \phi(y)\right)^2$$

$$\times\, e^{i(p-\xi).\frac{(x-y)}{h}} \frac{dxdyd}{2(2\pi h)^d}. \tag{170}$$

By symmetry in x and y the first two terms on the right side give

$$\iiint \ e^{\delta|x-y|^2}\ |\xi|^{2s} \phi^2(x) e^{i(p-\xi).(x-y)h} \frac{dxdyd}{(2\pi h)^d}$$

$$= \left(\frac{\pi}{\delta}\right)^{d2} \iiint \ e^{-|p-\xi|^2}/(4\delta h^2)\ |\xi|^{2s} \phi^2\ (x) \frac{dxd\xi}{(2\pi h)^d}$$

Now we can substitute $|q|^2 = |p-\xi|^2/(4\delta h^2)$ to get

$$\lim_{\delta\to 0+} \ \iiint \ e^{\delta|x-y|^2} |\xi|^{2s} \left(\phi^2(x) + \phi^2(y)\right) e^{i(p-\xi).(x-y)/h} \frac{dxdyd}{2(2\pi h)^d}$$

$$= |p|^{2s} \int \ \phi^2\ (x)\, dx. \tag{171}$$

We are left with calculating the third term on the right side of (170), namely

$$\iiint \ e^{\delta|z|^2} |\xi|^{2s} (\phi(x) - \left(\phi(x +\ z)\right)^2\ e^{i(p-\xi).z/h} \frac{dxdzd}{2(2\pi h)^d}.$$

Again by Plancherel's theorem we see that it equals

$$\iiint \ e^{\delta|z|^2}\ |\xi|^{2s} \left|\hat{\phi}\left(\frac{n}{\eta}\right)\right|^2 \left|1 - e^{-iz.\eta/h}\right|^2 e^{i(p-\xi).z/h} \frac{d\eta dz d\xi}{2(2\pi)^d h^{2d}}\,.$$

We can write

$$\left|1 - e^{-iz.\eta/h}\right|^2 = \ 2 - e^{iz.\eta/h} - e^{-iz.\eta/h}$$

and from the first summand we get

$$\iiint \ e^{\delta|z|^2}|\xi|^{2s}\left|\hat{\phi}\left(\frac{\eta}{h}\right)\right|^2 e^{i(p-\xi).z/h}\frac{d\eta dz d\xi}{(2\pi)^d h^{2d}}$$

$$= \iint \ e{-|q|^2}\left|p+\ 2h\sqrt{\delta}q\right|^{2s}\left|\hat{\phi}\left(\frac{\eta}{h}\right)\right|^2\frac{d\eta dq}{\pi^{d2}h^d}.$$

In the same way we can treat the second and third summand and after taking the limit $\delta \to 0+$ we finally find

$$\lim_{\delta\to 0+} \iiint \ e^{-\delta|x-y|^2}\ |\xi|^{2s}\big(\phi(x)-\phi(y)\big)^2 e^{i(p-\xi).(x-y)h}\frac{dx dy d\xi}{2(2\pi h)^d}$$

$$=\frac{1}{h^d}\int \left(|p|^{2s}-\frac{1}{2}(|p+\eta|^{2s}+\ |p-\eta|^{2s})\right)\left|\hat{\phi}\left(\frac{\eta}{h}\right)\right|^2 d\eta. \qquad (172)$$

Hence, combining (170), (171) and (172) yields the claim.

In view of identity (169) and Lemma (2.3.5) we conclude

$$\mathrm{Tr}(\phi H_0\phi)=(2\pi h)^d\int_{|p|<1}(|p|^{2s}-1)\,dp\,\|\phi\|_2^2+(2\pi h)^{-d}\int_{|p|<1}R_h(p)dp \qquad (173)$$

with

$$R_h(p)=\int\left(\frac{1}{2}(|p+h\eta|^{2s}+|p-h\eta|^{2s})-|p|^{2s}\right)\left|\hat{\phi}(\eta)\right|^2 d\eta.$$

We proceed to estimate $R_h(p)$. Note that for any $a>0$

$$\max_{|t|\le a}\ ((a+\ t)^s+(a-\ t)^s\ )\ =\ 2a^s\,.$$

Taking $a=|p|^2+|\eta|^2$ and $t=\ 2p\,.\eta$, we deduce that

$$\frac{1}{2}(|p+\ \eta|^{2s}+|p|^{2s})\ |p|^{2s}\le\ (|p|^2+|\eta|^2\ )^s\ |p|^{2s}.$$

Next, for $0\ <\ s\ <\ 1$ concavity implies that $(a+\ b)^s\le\ a^s+\ sa^{s-1}b$ for $a,b\ >\ 0$, from which we learn that

$$(|p|^2+|\eta|^2\ )^s-|p|^{2s}\ \le s|p|^{2(s-1)}|\eta|^2.$$

Hence, replacing η with $h\eta$ and using (160), we can estimate

$$R_h(p)\int\ |p|^{-2+2s}|h\eta|^2\left|\hat{\phi}(\eta)\right|^2 d\eta=\ s|p|^{-2+2s}h^2\int\ \left|\nabla_\phi\right|^2 dx\le\ C\,h^2\,|p|^{-2+2s}l^{d-2}.$$

Thus the upper bound follows from (169) and (173).

We prove the analogue of Proposition (2.3.12) in the case where Ω is the half-space $\mathbb{R}^d_+\ =\{(x',x_d)\colon x_d\ >\ 0\}$. We define the operator H^+ on $L^2\,(\mathbb{R}^d_+)$, in the same way as H_Ω, with form domain

$$\mathscr{H}^s\,(\mathbb{R}^d_+)=\left\{v\in H^s(\mathbb{R}^d)\,:\ v=\ 0 \text{ on } \mathbb{R}^d\setminus\overline{\mathbb{R}^d_+}\right\}.$$

We shall prove

We collect some facts about the one-dimensional operator

$$A^+=\left(\frac{d^s}{dt^2}+1\right)^s$$

in $L^2(\mathbb{R}_+)$ with form domain $\mathscr{H}^s\,(\mathbb{R}_+)$, and about the corresponding operator A in $L^2\,(\mathbb{R})$, defined analogously to A^+, but with form domain $H^s\,(\mathbb{R})$.

For $\mu\ >\ 0$ and t; $u\in\ \mathbb{R}_+$, let $e^+(t,u,\mu)$ and $a^+(t,u,\mu)$ be the integral kernels of the operators $(A^+-\mu)^0_-$ and $(A^+-\mu)_-$, respectively. Similarly, we define $a(t,u,\mu)$ via $(A-\mu)$. To simplify notation we abbreviate $a^+(t,\mu)=\ a^+(t,t,\mu)$. We also note that

$a(\mu) = a(t,t,\mu)$ is independent of $t \in \mathbb{R}_+$. The inequality $A^+ \geq 1$ implies that $a^+(t,u,\mu) = e^+(t,u,\mu) = 0$ for $\mu < 1$ and similarly for $a(t,u,\mu)$ and $e(t,u,\mu)$.

The following two results about the integral kernels $e^+(t,\mu)$ and $a^+(t,\mu)$ are rather technical. The first one provides a rough a-priori bound on $e^+(t,u,\mu)$.

Corollary (2.3.6)[103]: For $x = (x', x_d) \in \mathbb{R}^d_+$ and $y = (y', y_d) \in \mathbb{R}^d_+$ the integral kernels of $(H^+)^0_-$ and $(H^+)_-$ are related to those of $(A^+ - \mu)^0_-$ and $(A^+ - \mu)_-$ by

$$(H^+)^0_-(x,y) = \frac{1}{h^d}\int_{\mathbb{R}^{d-1}} |\xi'| e^{i\xi'.(x'-y')/h} \times e^+\left(\frac{x_d|\xi'|}{h}, \frac{y_d|\xi'|}{h}, \frac{1}{|\xi'|^{2s}}\right)\frac{d\xi'}{(2\pi)^{d-1}} \quad (174)$$

and

$$(H^+)(x,y) = \frac{1}{h^d}\int_{\mathbb{R}^{d-1}} |\xi'|^{1+2s} e^{i\xi'.(x'-y')/h} a^+\left(\frac{x_d|\xi'|}{h}, \frac{y_d|\xi'|}{h}, \frac{1}{|\xi'|^{2s}}\right)\frac{d\xi'}{(2\pi)^{d-1}} \quad (175)$$

Proof: Observe that Lemma (2.3.7) and the Spectral Theorem imply that for any bounded, measurable function ϕ on $\mathbb{R}$,

$$\mathcal{U}\phi(H^+ + 1)\mathcal{U}^* = \int^{\oplus}_{\mathbb{R}^{d-1}} (|\xi'|^{2s}A^+)d\xi'.$$

This formula means that for any $f \in L^2(\mathbb{R}^d_+)$,

$$(f, \phi(H^+ + 1)f) = \int_{\mathbb{R}^{d-1}} (\mathcal{U}f)_{\xi'}, \phi(|\xi'|^{2s}A^+)(\mathcal{U}f)_{\xi'} d\xi'.$$

From this, we easily conclude that if $\phi(|\xi'|^{2s}A^+)$ has an integral kernel for all $\xi' \in \mathbb{R}^{d-1}$, then $\phi(H^+ + 1)$ has an integral kernel given by

$$\phi(H^+ + 1)(x,y) = \frac{1}{h^d}\int_{\mathbb{R}^{d-1}} |\xi'| e^{i\xi'.(x'-y')h} \phi(|\xi'|^{2s}A^+)\left(\frac{x_d|\xi'|}{h}, \frac{y_d|\xi'|}{h}, \frac{1}{|\xi'|^{2s}}\right)\frac{d\xi'}{(2\pi)^{d-1}}.$$

The corollary now follows from the fact that for $\phi(E) = (E-1)^0_-$ one has

$$\phi(|\xi'|^{2s}A^+) = (A^+|\xi'|^{2s})^0_-$$

and for $\phi(E) = (E-1)^0_-$ one has

$$\phi(|\xi'|^{2s}A^+) = |\xi'|^{2s}(A^+ |\xi'|^{2s}).$$

We now give

Lemma (2.3.7)[103]: The mapping

$$(\mathcal{U}f)_{\xi}(t) = (2\pi h)^{-(d-1)/2} h^{1/2}|\xi'|^{-1/2}\int_{\mathbb{R}^{d-1}} f(x', |\xi'|^{-1}ht)e^{i\xi'.x'/h}\, dx',$$
$$\xi' \in \mathbb{R}^{d-1}, t > 0,$$

defines a unitary operator from $L^2(\mathbb{R}^d)$ to $\int^{\oplus}_{\mathbb{R}^{d-1}} L^{2(0,\infty)} d\xi'$. Moreover,

$$\mathcal{U}(H^+ + 1)\mathcal{U}^* = \int^{\oplus}_{\mathbb{R}^{d-1}} |\xi'|^{2s}A^+\, d\xi'.$$

Before giving the proof we show how to deduce formulas for spectral projections.

Proof: The fact that $\mathcal{U}$ is unitary follows from Plancherel's theorem together with a dilation. To prove the formula for H^+, let $f \in \mathscr{H}^s(\mathbb{R}^d_+)$, the form domain of H^+, and denote by $\hat{f}$ as before the Fourier transform of f with respect to both x' and x_d. Since $f \in \mathscr{H}^s(\mathbb{R}^d_+)$, its extension to Rd by zero belongs to $\mathscr{H}^s(\mathbb{R}^d_+)$ and we can also extend $(\mathcal{U}f)_{\xi'}$ by zero to $\mathbb{R}$. A short computation shows that

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (\mathcal{U}f)_{\xi'}(t)e^{i\omega t}\, dt = h^{-d/2}|\xi'|^{1/2}\hat{f}(h^{-1}\xi', h^{-1}\,|\xi'|\omega),$$

and thus,

$$\int_{\mathbb{R}^d} |hp|^{2s}\left|\hat{f}\,(p)\right|^2 dp = \int_{\mathbb{R}^{d-1}} \left(\int_{\mathbb{R}} (|hp'|^2 + (hpd)^2)^s \left|\hat{f}(p',pd)\right|^2 dpd\right) dp'$$
$$= h^{-d} \int_{\mathbb{R}^{d-1}} |\xi'|^{1+2s} \left(\int_{\mathbb{R}} (1+\omega^2)^s \left|\hat{f}(h^{-1}\xi', h^{-1}|\xi'|\omega)\right|^2 d\omega\right) d\xi'$$
$$= \int_{\mathbb{R}^{d-1}} |\xi'|^{2s} \left(\int_{\mathbb{R}} (1+\omega^2)^s \left|\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (\mathcal{U}f)_{\xi'}(t)e^{i\omega t} dt\right|^2 d\omega\right) d\xi'.$$

Since $(\mathcal{U}f)_{\xi'}$ vanishes on $(\infty, 0)$, the previous formula can be rewritten as

$$\int_{\mathbb{R}^d} |hp|^{2s}\left|\hat{f}(p)\right|^2 dp = \int_{\mathbb{R}^{d-1}} |\xi'|^{2s}\left\|(A^+)^{1/2}(\mathcal{U}f)_{\xi'}\right\|^2 d\xi'.$$

This is equivalent to

$$\mathcal{U}(H^+ + 1)\mathcal{U} = \int_{\mathbb{R}^{d-1}}^{\oplus} |\xi'|^{2s} A^+ \, d\xi'$$

and concludes the proof.

We state upper and lower bounds on $-\mathrm{Tr}(\phi H^+ \phi)_-$ in terms of the one-dimensional model operators A and A^+, in particular, in terms of the function $K(t)$ given in (219). As explained below, Proposition (2.3.9), will be a direct consequence of the following estimates.

Proposition (2.3.8)[103]: Assume that $\phi \in C_0^\infty(\mathbb{R}^d)$ is supported in a ball of radius $l = 1$ and assume that (160) is satisfied with $l = 1$. Then for any $0 < \delta_2 < \min\{1, 2s\}$ there is a constant C_{δ_2} such that for all $h > 0$ we have

$$-\mathrm{Tr}(\phi H^+ \phi)_- \geq -L_{s,d}^{(1)} \int_{\mathbb{R}_+^d} \phi^2(x) dx\, h^{-d} + \int_{\mathbb{R}_+^d} \phi^2(x) \frac{1}{h} K\left(\frac{x_d}{h}\right) dx\, h^{-d+1}, \tag{176}$$

$$-\mathrm{Tr}(\phi H^+ \phi)_- \geq -L_{s,d}^{(1)} \int_{\mathbb{R}_+^d} \phi^2(x) dx\, h^{-d} + \int_{\mathbb{R}_+^d} \phi^2(x) \frac{1}{h} K\left(\frac{x_d}{h}\right) dx\, h^{-d+1}, + C_{\delta_2}\, h^{-d+1+\delta_2}. \tag{177}$$

Assuming Proposition (2.3.8), we now give

Proposition (2.3.9)[103]: Assume that $\phi \in C_0^1(\mathbb{R}^d)$ is supported in a ball of radius $l > 0$ and assume that (160) is satisfied. Then for $h > 0$ and any $0 < \delta_1 < 1$ and $0 < \delta_2 < \min\{1, 2s\}$ we have

$$-C_{\delta_1,\delta_2}\left(l^{d-1-\delta_1}\, h^{-d+1+\delta_1} + l^{d-1-\delta_2}\, h^{-d+1+\delta_2}\right)$$
$$\leq \operatorname{Tr}(\phi H^+\phi)_- - L_{s,d}^{(1)} \int_{\mathbb{R}_+^d} \phi^2(x)\, dx\, h^{-d}$$
$$+ L_{s,d}^{(2)} \int_{\mathbb{R}^{d-1}} \phi^2\,(x',0)dx'\, h^{-d+1} \leq C_{\delta_1} l^{d-1-\delta_1}\, h^{-d+1+\delta_1}\,.$$

This result depends on a more or less explicit diagonalization of the operator H^+, which is far from obvious. Relying crucially on recent results of Kwasnicki [112] about non-local operators on a half-line.

Proof: To prove the proposition, we may rescale - and hence assume. Proposition (2.3.9) is then an immediate consequence of Proposition (2.3.8) provided we can show that for any $0 < \delta_1 < 1$ there is a C_{δ_1} such that for all $h > 0$

$$\left|\int_{\mathbb{R}_+^d} \phi^2(x)\frac{1}{h}K\left(\frac{x_d}{h}\right)dx - L_{s,d}^{(2)}\int_{\mathbb{R}_+^d} \phi^2(x',0)dx'\right| \leq C_{\delta_1}\, h^{\delta_1} \tag{178}$$

In order to obtain the latter bound, we substitute $x_d = th$ and write, recalling (220),

$$\int_{\mathbb{R}_+^d} \phi^2(x)\frac{1}{h}K\left(\frac{x_d}{h}\right)dx - L_{s,d}^{(2)}\int_{\mathbb{R}_+^d} \phi^2(x',0)dx'$$
$$= \int_0^\infty K(t)\int_{\mathbb{R}^{d-1}} \int_0^{th} \partial_\tau\phi^2(x',\tau)d\tau dx' dt.$$

By Hölder's inequality we can further estimate

$$\left|\int_{\mathbb{R}^{d-1}} \int_0^{th} \partial_\tau\phi^2(x',\tau)d\tau dx'\right|$$
$$\leq \left(\int_0^{th} d\tau\right)^{\delta_1} \left(\int_0^\infty \left|\int_{\mathbb{R}^{d-1}} \partial_\tau\phi^2(x',\tau)d\tau dx'\right|^{(1-\delta_1)^{-1}}\right)^{1-\delta_1} \leq Ct^{\delta_1}\, h^{\delta_1}\,.$$

Since $\int_0^\infty t^{\delta_1}|K(t)|dt < \infty < 1$ by Lemma (2.3.25), we obtain inequality (178).

We shall prove the lower and the upper bound in Proposition (2.3.8), respectively. (2.3.7). Lower bound on $-\operatorname{Tr}(\phi H^+\phi)_-$. To prove (176) we use the fact that

$$-\operatorname{Tr}(\phi H^+\phi)_- \geq -\operatorname{Tr}(\phi(H^+)_-\phi)\,.$$

The lower bound follows from this by integrating the identity

$$(H^+)_-(x,x) = h^{-d}L_{s,d}^{(1)}\, h^{-d}K\left(\frac{x_d}{h}\right), \tag{179}$$

against ϕ^2 . Equation (179) is a consequence of (175). Indeed, by the same argument we learn that

$$(H_0)(x,x) = \frac{1}{(2\pi)^{d-1}}\frac{1}{h^d}\int_{\mathbb{R}^{d-1}} |\xi'|^{1+2s}a(|\xi'|^{2s})d\xi'.$$

On the other hand, by direct diagonalization, we find that

$$(H_0)(x,x) = h^{-d}L_{s,d}^{(1)}.$$

Comparing these two identities with (175), we arrive at (179), thus establishing (176).
To prove (177) we set= $(H^+)_-^0$. Its integral kernel is given by (174) in terms of the kernel $e^+(\cdot\,,\cdot\,,\mu)$ of $(A^+ - \mu)_-^0$. By the variational principle it follows that

$$-\mathrm{Tr}(\phi H^+\phi)_- \le -\mathrm{Tr}(\phi\gamma\phi H^+)$$
$$= \frac{1}{h^{2d}}\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^d} |\xi'|\, e^{i\xi'.\frac{x'-y'}{h}} \times e^+(x_d|\xi'|\,h^{-1}, y_d|\xi'|\,h^{-1}, |\xi'|^{-2s})$$
$$\times (|p|^{2s}-1)e^{ip.(y-x)h}\phi(x)\phi(y)\frac{dpd\xi'dxdy}{(2\pi)^{2d-1}}. \qquad (180)$$

We insert the identity

$$\phi(x)\phi(y) = \frac{1}{2}(\phi^2(x)\phi^2(y) - |\phi(x)\phi(y)|^2),$$

and by a similar argument as in the proof of Lemma (2.3.5) we can use the symmetry in x and y and substitute $q = pd\,/|p'|$ to obtain

$$-\mathrm{Tr}(\phi H^+\phi)_- \le I_h[\phi] - R_h[\phi]$$

with the main term

$$I_h[\phi] = \frac{1}{h^{2d}}\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^d} |\xi'|\, e^{i\xi'.(x'-y')/h} \times e^+(x_d|\xi'|\,h^{-1}, y_d|\xi'|\,h^{-1}, |\xi'|^{-2s})$$
$$\times e^{i(y_p-x_p)|p'|q/h}((q^2+1)^s - |p'|^{-2s})|p'|^{1+2s}\phi^2(x)\frac{dpd\xi'dxdy}{(2\pi)^{2d-1}}$$

and the remainder

$$R_h[\phi] = \frac{1}{h^{2d}}\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^d} |\xi'|\, e^{i\xi'.(x'-y')/h} e^+(x_d|\xi'|\,h^{-1}, y_d|\xi'|\,h^{-1}, |\xi'|^{-2s})$$
$$\times |p|^{2s}e^{ip.(y-x)/h}|\phi\,(x)-\phi(y)|^2\frac{dpd\xi'dxdy}{2(2\pi)^{2d-1}}.$$

Since $\phi \in\ C_0^\infty(\mathbb{R}^d)$, we can perform the y' -integration in $I_h[\phi]$. We use the fact that

$$\int_{\mathbb{R}}\int_0^\infty e^+(x_p, y_p x\mu)((q^2+1)^s - |p'|^{-2s})e^{i(y_p-x_p)q}dy_p dq = -2\pi a^+(x_d, z_d, \mu)$$

and obtain

$$I_h[\phi] = \frac{1}{h^{d+1}}\int_{\mathbb{R}^d_+}\int_0^\infty\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^d} |\xi'|^{2s+2}\, e^+(x_d|\xi'|\,h^{-1}, y_d|\xi'|\,h^{-1}, |\xi'|^{-2s})$$
$$\times ((q^2+1)^s - |p'|^{-2s})e^{i(y_p-x_p)|\xi'|q/h}\phi^2(x)\frac{dqd\xi'dy_d dx}{(2\pi)^d}$$
$$= -\frac{1}{h^d}\int_{\mathbb{R}^d_+}\phi^2(x)\int_{\mathbb{R}^{d-1}} |\xi'|^{2s+2}\, a^+(x_d|\xi'|\,h^{-1}, |\xi'|^{-2s})\frac{d\xi' dx}{(2\pi)^{d-1}}.$$

Using again (179), we find that

$$I_h[\phi] = L^{(1)}_{s,d}\int_{\mathbb{R}^d_+}\phi^2(x)\,dx\,h^{-d} + \int_{\mathbb{R}^d_+}\phi^2(x)\,K\left(\frac{x_d}{h}\right)dx\,h^{-d}. \qquad (181)$$

It remains to study $R_h[\phi]$. We claim that for any $\frac{1}{2} - s < \sigma < \min\left\{\frac{1}{2}, 1-s\right\}$ there is a constant $C_\sigma > 0$ such that

$$|R_h[\phi]| \le C_\sigma\, h^{-d+2s+2\sigma} \qquad (182)$$

for all $h > 0$. This, together with (181) will complete the proof of (177). In order to show (182) we perform the p integration and find that

$R_h[\phi]$
$$= \frac{C}{h^{d-2s}} \int_{\mathbb{R}^d_+} \int_{\mathbb{R}^d_+} \int_{\mathbb{R}^{d-1}} |\xi'| \, e^{i\xi'.(x'-y')/h} \left(\frac{x_d|\xi'|}{h}, \frac{y_d|\xi'|}{h}, \frac{1}{|\xi'|^{-2s}} \right) \frac{|\phi\,(x) - \phi(y)|^2}{|x-y|^{d+2s}} d\xi' dx dy.$$
We insert
$$e^{i\xi'.(x'-y')/h} = \frac{h^{2\sigma}}{|\xi'|^{2\sigma}} (\Delta_{x'})^\sigma e^{i\xi'.(x'-y')/h}$$
and integrate by parts to get
$$R_h[\phi] = \frac{C}{h^{d-2s-2\sigma}} \int_{\mathbb{R}^d_+} \int_{\mathbb{R}^d_+} \int_{\mathbb{R}^{d-1}} |\xi'|^{1-2\sigma} e^{i\xi'.(x'-y')/h} e^+ \left(\frac{x_d|\xi'|}{h}, \frac{y_d|\xi'|}{h}, \frac{1}{|\xi'|^{-2s}} \right) d\xi'$$
$$\times (\Delta_{x'})^\sigma \frac{|\phi\,(x) - \phi(y)|^2}{|x-y|^{d+2s}} dx dy.$$
By Lemma (2.3.22) and the fact that $e^+(t, u, \mu) = 0$ for $\mu \le 1$ we arrive at
$$R_h[\phi] \le \frac{C}{h^{d-2s-2\sigma}} \int_{\mathbb{R}^d_+} \int_{\mathbb{R}^d_+} \int_{\{\xi' \in \mathbb{R}^{d-1} |\xi'| < 1\}} |\xi'|^{2\sigma} d\xi' \left| (\Delta_{x'})^\sigma \frac{|\phi\,(x) - \phi(y)|^2}{|x-y|^{d+2s}} \right| dx dy$$
$$\le \frac{C}{h^{d-2s-2\sigma}} \int_{\mathbb{R}^d_+} \int_{\mathbb{R}^d_+} \left| (\Delta_{x'})^\sigma \frac{|\phi\,(x) - \phi(y)|^2}{|x-y|^{d+2s}} \right| dx dy$$
According to Lemma (2.3.26) this implies (182) and hence completes the proof of (177).

We show Proposition (2.3.12). After having analyzed the half-space, we now show how the case of a general domain follows. We shall transform the operator H_Ω locally to an operator given on the half-space
$$\mathbb{R}^d_+ = \{(y', y_d) \in \mathbb{R}^{d-1} \times \mathbb{R} : y_d > 0\}$$
and we shall quantify the error made by this straightening of the boundary.

Under the conditions of Proposition (2.3.12), let B denote the open ball of radius $l > 0$, containing the support of ϕ. For $x_0 \in B \cap \partial\Omega$ let v_{x_0} be the inner normal unit vector at x_0. We choose a Cartesian coordinate system such that $x_0 = 0$ and $v_{x_0} = (0, \dots, 0, 1)$, and we write $x = (x_0, x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$ for $x \in \mathbb{R}^d$.

For sufficiently small $l > 0$ one can introduce new local coordinates near the boundary. Let D denote the projection of B on the hyperplane given by $x_d = 0$. Since the boundary of Ω is compact and $C^{1,\alpha}$, there is a constant $c > 0$ such that for $0 < l \le c$ we can find a real function $f \in C^{1,\alpha}$ given on D satisfying
$$\partial\Omega \cap B = \{(x_0, x_d) : x' \in D, x_d = f(x')\} \cap B.$$
The choice of coordinates implies $f(0) = 0$ and $\nabla f(0) = 0$. Hence, we can estimate
$$\sup_{x' \in D} |\nabla f(x')| = \sup_{x' \in D} |\nabla f(x') - \nabla f(0)| \le C_f |x'|^\alpha \le C_f l^\alpha.$$
Since the boundary of Ω is compact, we can choose a constant $C > 0$, depending only on Ω, in particular independent of f, such that the following bound holds:
$$\sup_{x' \in D} |\nabla f(x')| \le C l^\alpha. \tag{183}$$
We introduce new local coordinates via the diffeomorphism $\varphi : D \times \mathbb{R} \to \mathbb{R}^d$, given by
$$y_j = \varphi_j(x) = x_j \quad \text{for} \quad j = 1, \dots, d-1$$
and
$$y_d = \varphi_d(x) = x_d f(x').$$

Note that the determinant of the Jacobian matrix of ' equals 1 and that the inverse of ' is given on ran $\varphi = D \times \mathbb{R}$. In particular, we get

$$\varphi(\partial\Omega \cap B) \subset \partial\mathbb{R}^d_+ = \{y \in \mathbb{R}^d : y_d = 0\}.$$

Fix $v \in \mathscr{H}^s(\Omega)$ with support in $\overline{B}$. For $y \in$ ranφ put $\overline{v}(y) = v \circ \varphi^{-1}(y)$ and extend $\overline{v}$ by zero to $\mathbb{R}^d$.

Lemma (2.3.10)[103]: The function $\tilde{v}$ belongs to $\mathscr{H}^s(\mathbb{R}^d_+)$ and there exist positive constants c and C depending only on Ω such that for $0 < l \leq c$ we have

$$\left|\left(\tilde{v}, (-\Delta)^s_{\mathbb{R}^d_+}\tilde{v}\right) - (\tilde{v}, (-\Delta)^s_\Omega\ v)\right| \leq C\, l^\alpha \min\left\{\left(v, (-\Delta)^s_{\mathbb{R}^d_+} v\right), (v, (-\Delta)^s_\Omega v)\right\}.$$

Proof: By definition, $\tilde{v}$ belongs to $\mathscr{H}^s(\mathbb{R}^d)$ and for $y \in \mathscr{H}^s \mathbb{R}^d / \mathbb{R}^d_+$ we find

$$x_d = y_d + f(y') < f(x'),$$

thus $\tilde{v}(y) = v(x) = 0$. Therefore $\tilde{v}$ belongs to $\mathscr{H}^s(\mathbb{R}^d_+)$.

Using the new local coordinates we get

$$(v, (-\Delta)^s_\Omega v) = C_{s,d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|v(x) - v(w)|^2}{|x-w|^{d+2s}} dx dw$$

$$= C_{s,d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|\tilde{v}(y) - \tilde{v}(z)|^2}{|x-w|^{d+2s}} dy dz, \quad (184)$$

where $y = \varphi(x)$ and $z = \varphi(w)$, thus $x = \left(y', y_d + f(y')\right)$ and $w = \left(z', z_d + f(z')\right)$. Let us write

$$\left|\frac{1}{|y-z|^{d+2s}} - \frac{1}{|x-w|^{d+2s}}\right|$$

$$= \frac{1}{|y-z|^{d+2s}} \left|\frac{|y-z|^{d+2s}}{\left[|y'-z'|^2 + \left(y_d + f(y') - z_d - f(z')\right)^2\right]^{d/2+s}}\right|.$$

After multiplying out, the last fraction equals

$$\left(1 + \frac{\left(f(y') - f(z')\right)^2 + 2(y_d - z_d)\left(f(y') - f(z')\right)}{|y-z|^2}\right)$$

and we can employ (183) to estimate

$$\left|\frac{\left(f(y') - f(z')\right)^2 + 2(y_d - z_d)\left(f(y') - f(z')\right)}{|y-z|^2}\right|$$

$$\sup|\nabla f| \frac{|y'-z'|^2}{|y-z|^2} + \sup|\nabla f| \frac{|y'-z'|\,|y_d - z_d|}{|y-z|^2} < Cl^\alpha.$$

Choosing l small enough, we can assume $C\, l^\alpha < 1/2$. Then, combining the foregoing relations, we find

$$\left|\frac{1}{|x-w|^{d+2s}} - \frac{1}{|y-z|^{d+2s}}\right| < C \frac{l^\alpha}{|y-z|^{d+2s}}. \quad (185)$$

From (184) and (185) we conclude

$$\left|\left(\tilde{v}, (-\Delta)^s_{\mathbb{R}^d_+}\tilde{v}\right) - (\tilde{v}, (-\Delta)^s_\Omega\ v)\right|$$

$$\leq C_{s,d} \iint |\tilde{v}(y) - \tilde{v}(z)|^2 \left|\frac{1}{|x-w|^{d+2s}} - \frac{1}{|y-z|^{d+2s}}\right| dy dz$$

$$\leq Cl^{\alpha}\left(\tilde{v},(-\Delta)^{s}_{\mathbb{R}^d_+}\tilde{v}\right).$$

This proves the first claim of the lemma. The second claim follows by interchanging the roles of $(-\Delta)^{s}_{\mathbb{R}^d_+}$ and $(-\Delta)^{s}_{\Omega}$.

On the range of ' we define $\tilde{\phi}=\phi\circ\varphi^{-1}$ and extend the function by zero to $\mathbb{R}^d$ such that $\tilde{\phi}\in C_0^{\infty}(\mathbb{R}^d)$ and $\|\nabla\tilde{\phi}\|\leq C\,l^{-1}$ hold. Using Lemma (2.3.10), we show the following relations.

Lemma (2.3.11)[103]: For $0<l\leq c$ and any $h>0$ the estimate

$$|\mathrm{Tr}(\phi H_{\Omega}\phi)_{-}-\mathrm{Tr}(\phi H^{+}\phi)_{-}|\leq Cl^{d+\alpha}h^{-d} \tag{186}$$

holds. Moreover, we have

$$\int_{\Omega}\phi^2(x)d\sigma(x)=\int_{\mathbb{R}^d_+}\tilde{\phi}^2(y)dy \tag{187}$$

and

$$0\leq\int_{\partial\Omega}\phi^2(x)d\sigma(x)-\int_{\mathbb{R}^d_+}\tilde{\phi}^2(y',0)dy'\leq Cl^{d-1+2\alpha} \tag{188}$$

Proof: The definition of $\tilde{\phi}$ and the fact that the Jacobian of ϕ equals 1 immediately give (187). Using (183), we estimate

$$\int_{\partial\Omega}\phi^2(x)d\sigma(x)=\int_{\mathbb{R}^{d-1}}\tilde{\phi}^2(y',0)\sqrt{1+|\nabla f|^2}dy'\leq\int_{\mathbb{R}^{d-1}}\tilde{\phi}^2(y',0)dy'+Cl^{d-1+2\alpha}.$$

from which (188) follows. To prove (186) we refer to the variational principle once more and note that

$$-\mathrm{Tr}(\phi H_{\Omega}\phi)_{-}=\inf_{0\leq\gamma\leq 1}\mathrm{Tr}(\phi\gamma\phi H_{\Omega}),$$

where we can assume that infimum is taken over trial density matrices supported in $\overline{B}\times\overline{B}$. Fix such a matrix . For y and z from $D\times\mathbb{R}$ set

$$\tilde{\gamma}(y,z)=\gamma\left(\varphi^{-1}(y),\varphi^{-1}(z)\right),$$

so that $0\leq\tilde{\gamma}\leq 1$ and the range of $\tilde{\gamma}$ belongs to the form domain of $\tilde{\phi}H^{+}\tilde{\phi}$. According to Lemma (2.3.10) it follows that

$$\mathrm{Tr}(\phi\gamma\phi H_{\Omega})\geq\mathrm{Tr}\left(\tilde{\phi}\gamma\tilde{\phi}(h^{2s}(1-Cl^{\alpha})(-\Delta)^{s}_{\mathbb{R}^d_+}-1)\right)$$
$$\geq\mathrm{Tr}\left(\tilde{\phi}\left((1-Cl^{\alpha})h^{2s}(-\Delta)^{s}_{\mathbb{R}^d_+}-1\right)\tilde{\phi}\right)_{-}$$

and consequently

$$\mathrm{Tr}(\phi H_{\Omega}\phi)_{-}\geq\mathrm{Tr}\left(\tilde{\phi}\left((1-Cl^{\alpha})h^{2s}(-\Delta)^{s}_{\mathbb{R}^d_+}-1\right)\tilde{\phi}\right)_{-}.$$

Set $\varepsilon=2Cl^{\alpha}$ and assume l to be sufficiently small, so that $0<\varepsilon\leq 1/2$. Then

$$\mathrm{Tr}(\phi H_{\Omega}\phi)_{-}\leq\mathrm{Tr}\left(\tilde{\phi}\left((1-Cl^{\alpha})h^{2s}(-\Delta)^{s}_{\mathbb{R}^d_+}-1\right)\tilde{\phi}\right)_{-}$$

$$\mathrm{Tr}(\phi H_{\Omega}\phi)_{-}\leq\mathrm{Tr}\left(\tilde{\phi}\left((-h^{2s}\Delta-1)^{s}_{\mathbb{R}^d_+}\right)\tilde{\phi}\right)_{-}+\mathrm{Tr}\left(\tilde{\phi}\left((\varepsilon-Cl^{\alpha})h^{2s}(-\Delta)^{s}_{\mathbb{R}^d_+}-\varepsilon\right)\tilde{\phi}\right)_{-}$$
$$\leq\mathrm{Tr}\left(\tilde{\phi}H^{+}\tilde{\phi}\right)_{-}+\varepsilon\,\mathrm{Tr}\left(\tilde{\phi}\left((h^{2s}/2)(-\Delta)^{s}_{\mathbb{R}^d_+}-1\right)\tilde{\phi}\right)_{-}.$$

Using Lemma (2.3.4), we estimate $\mathrm{Tr}\left(\tilde{\phi}\left((h^{2s}/2)(-\Delta)^{s}_{\mathbb{R}^d_+}-1\right)\tilde{\phi}\right)_{-}\leq Cl^{d}h^{-d}$ and it follows that

$$\mathrm{Tr}(\phi H_\Omega \phi)_- \le \mathrm{Tr}\big(\tilde{\phi} H^+ \tilde{\phi}\big)_- + C l^{d+\alpha} h^{-d}$$

Finally, by interchanging the roles of H_Ω and H^+, we get an analogous lower bound and the proof of the lemma is complete.

Proposition (2.3.12)[103]: There is a constant $c > 0$ depending only on Ω such that the following holds. Assume that $\phi \in C_0^1(\mathbb{R}^d)$ is real-valued, supported in a ball of radius $0 < l \le c$ intersecting the boundary of Ω and satisfies (160). Then for all $h > 0$ the estimates

$$-\tilde{R}_{bd}(l,h) \le \mathrm{Tr}(\phi H_\Omega \phi)_- L_{s,d}^{(1)} \int_\Omega \phi^2(x)\,dx\,h^{-d} + L_{s,d}^{(2)} \int_{\partial\Omega} \phi^2(x) d\sigma(x)\,h^{-d+1}$$
$$\le \tilde{R}_{bd}(l,h)$$

hold. Here d denotes the $(d-1)$-dimensional volume element of $\partial\Omega$ and the remainder terms satisfy for any $0 < \delta_1 < 1$ and $0 < \delta_2 < \min\{1, 2s\}$

$$R_{bd}(l,h) \le C_{\delta_1}\left(\frac{l^{d-1-\delta_1}}{h^{d-1\delta_1}} + \frac{l^{d+\alpha}}{h^d}\right),$$
$$\tilde{R}_{bd}(l,h) \le C_{\delta_1,\delta_2}\left(\frac{l^{d-1-\delta_1}}{h^{d-1-\delta_1}} + \frac{l^{d-1-\delta_2}}{h^{d-1-\delta_2}} + \frac{l^{2\alpha+d-1}}{h^{d-1}} + \frac{l^{d+\alpha}}{h^d}\right),$$

with positive constants $C_{\delta_1}, C_{\delta_1,\delta_2}$ depending on $\delta_1, \delta_2, \Omega$ and the constant in (160).

Based on these propositions we can complete the proof of Theorem (2.3.3).

Proof: It suffices to combine Lemma (2.3.11) and Proposition (2.3.9).

We construct the family of localization functions $(\phi_u)_{u\in\mathbb{R}^d}$ and prove Proposition (2.3.15). Fix a real-valued function $\phi \in C_0^\infty(\mathbb{R}^d)$ with support in $\{x \in \mathbb{R}^d : |x| < 1\}$ that satisfies $\|\phi\|^2 = 1$. We recall the definition of the local length scale from (157). For $u, x \in \mathbb{R}^d$ let $J(x,u)$ be the Jacobian of the map $u \mapsto (x-u)/l(u)$. We define

$$\phi_u(x) = \phi\left(\frac{x-u}{l(u)}\right)\sqrt{J(x,u)}\, l(u)^{d/2},$$

such that ϕ_u is supported in the ball $B_u = \{x \in \mathbb{R}^d \ |x-u| < l(u)\}$.

By definition, the function $l(u)$ is smooth and satisfies $0 < l(u) \le 1/2$ and $\|\nabla l\|_\infty \le 1/2$. Therefore, according to [120], the functions ϕ_u satisfy (158) and (159) for all $u \in \mathbb{R}^d$.

To prove the lower bound in Proposition (2.3.15), we follow some ideas from [115]. In particular, we need the following auxiliary results; the first one gives an IMS-type localization formula for the fractional Laplacian.

Lemma (2.3.13)[103]: For the family of functions $(\phi_u)_{u\in\mathbb{R}^d}$ introduced above and for all functions $f \in \mathscr{H}^s(\Omega)$ the identity .

$$(f,(-\Delta)^s f) = \int_{\Omega^*} (\phi_u f, (-\Delta)^s \phi_u f) l(u)^{-d}\,du - (f, Lf)$$

holds with $\Omega^* = \{u \in \mathbb{R}^d \text{ supp}\,\phi_u \cap \Omega \ne \emptyset\}$. The operator L is of the form

$$L = \int_{\Omega^*} L_{\phi_u}\, l(u)^{-d}\,du, \tag{189}$$

where L_{ϕ_u} is a bounded operator with integral kernel

$$L_{\phi_u}(x,y) = C_{s,d}\,\frac{|\phi_u(x) - \phi_u(y)|^2}{|x-y|^{d+2s}}\,\chi_\Omega(x)\chi_\Omega(y).$$

Here χ_Ω denotes the characteristic function of Ω.

Lemma (2.3.13) implies that for any operator with range in $\mathscr{H}^s(\Omega)$

$$\mathrm{Tr}\gamma(-\Delta)^s = \int_{\mathbb{R}^d} \mathrm{Tr}(\gamma\phi_u(-\Delta)^s\phi_u)l(u)^{-d}\,du - \mathrm{Tr}\gamma L. \tag{190}$$

The next result allows to estimate the localization error Tr γL.

Lemma (2.3.14)[103]: For $u \in \mathbb{R}^d$ and $0 < \delta \le 1/2$ we have

$$\mathrm{Tr}\gamma\, L_{\phi_u} \le \mathrm{Tr}\gamma(C\,\delta^{2-2s}(u)^{2s}\chi_\delta\chi_\Omega) + C\|\gamma\|(u)^{2s}\,\delta^{-d+2-2s}r(\delta)$$

with

$$r(\delta) = \begin{cases} 1 & if\ 1-\frac{d}{4} < s < 1, \\ |\ln\delta| & if\ \ 0 < s = 1-\frac{d}{4}, \\ \delta^{d+4s-4} & if\ 0 < s < 1-\frac{d}{4}, \end{cases}$$

where χ_δ denotes the characteristic function of $\{x \in \mathbb{R}^d\ |x - u| < l(u)(1+\delta)\}$.

Proof: By translation and scaling we can assume that $u = 0$ and $l(u) = 1$, and hence we write $\phi_u = \phi$. (This rescaling changes Ω, but the bound we are going to prove is independent of the domain and therefore not affected by this dilation.) We set

$$L_\phi^1(x,y) - \begin{cases} L_\phi(x,y)\chi_\delta(x)\chi_\delta(y) & if|x-y| < \delta, \\ 0 & if|x-y| \ge \delta, \end{cases}$$

$$L_\phi^0(x,y) = L_\phi(x,y) - L_\phi^1(x,y)$$

and

$$\theta(x) = \int L_\phi^1(x,y)dy.$$

By a simple adaption of the arguments of [115] we find that for any $\epsilon > 0$

$$\mathrm{Tr}\gamma L_\phi \le \mathrm{Tr}\gamma(\theta + \varepsilon\chi_0) + \frac{\|\gamma\|}{2\varepsilon}\mathrm{Tr}\big(L_\phi^0\big)^2. \tag{191}$$

It remains to bound θ and $\mathrm{Tr}\big(L_\phi^0\big)^2$.

We begin by estimating θ. By definition, for $|x| \ge 1+\delta$ we have $L_\phi^1(x,y) = 0$ and hence $\theta(x) = 0$, and for $|x| < 1+\delta$ we get

$$\theta(x) = C_{s,d}\int_{\substack{|x-y|<\delta \\ |y|<1+\delta}} \frac{\big(\phi_u(x)-\phi_u(y)\big)^2}{|x-y|^{d+2s}}\,\chi_\Omega(x)\chi_\Omega(y)dy$$

Thus, for all $x \in \mathbb{R}^d$

$$C\|\nabla\phi\|_\infty^2\,\chi_\Omega(x)\int_{|x-y|<\delta} \frac{1}{|x-y|^{d+2s}}dy. \tag{192}$$

Finally, we estimate $\mathrm{Tr}\big(L_\phi^0\big)^2$ .The symmetry $L_\phi^0(x,y)$ implies

$$\mathrm{Tr}\big(L_\phi^0\big)^2 \iint_A \left(\frac{\big(\phi_u(x)-\phi_u(y)\big)^2}{|x-y|^{d+2s}}\right)^2 dxdy,$$

where A denote the set $\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : |x| < \min(|y|,1), |x-y \ge \delta\}$. Set

$$A_1 = \{(x,y) \in\ A : |y| \ge 2\}\ \text{ and } A_2 = \{(x,y) \in\ A : |y| <\ 2\}.$$

Then

$$\mathrm{Tr}\left(L_\phi^0\right)^2 \le \iint_{A_1} \left(\frac{\phi(x)^4}{|x-\ y|^{d+2s}}\right) dxdy + C\|\nabla\phi\|_\infty^4 \iint_{A_2} \frac{1}{|x-\ y|^{2d+4s-4}} dxdy.$$

Observe that the right-hand side is bounded by $C\,\delta^{-d-4s+4}$ for $1-\ d/4 < s < 1$, by $C|\ln\ \delta|$ for $0 < s = 1-d/4$, and by C for $0 < s < 1-\ d/4$. Finally, we choose $\varepsilon = \delta^{2-2s}$ and combining the last estimates with (191) and (192) yields the claimed result.

Proposition (2.3.15)[103]: There is a constant $C > 0$ depending only on s and d such that for all $0 < l_0 \le 1/2$ and all $0 < h \le C^{-1}\, l_0$ the estimates

$$0 \le \mathrm{Tr}(H_\Omega)_- \int_{\mathbb{R}^d} \mathrm{Tr}(\phi_u H_\Omega \phi_u)_- l(u)^{-d}\, du \le C\, h^{-d+2}\, l_0^{-1}\, R_{\mathrm{loc}}(l_0,h)$$

hold with a remainder

$$R_{\mathrm{loc}}(l_0,h) = \begin{cases} 1 & if\ \ 1-\frac{d}{4} < s < 1, \\ \left|\ln\left(\frac{l_0}{h}\right)\right|^{\frac{1}{2}} & if\ 0 < s = 1-\frac{d}{4} \\ \left(\frac{l_0}{h}\right)^{2-2s-d2} & if\ 0 < s < 1-\frac{d}{4} \end{cases}$$

In view of this result, one can analyze the local asymptotics, i.e., the asymptotic behavior of $\mathrm{Tr}(\phi_u H_\Omega \phi_u)$, separately on different parts of Ω. First, we consider the bulk, where the influence of the boundary is not felt.

Proof: We apply Lemma (2.3.14) with a parameter $0 < \delta_u \le 1/2$ to be specified later. For ease of notation we write u instead of δ_u . Identities (189) and (190) and the estimate from Lemma (2.3.14) imply

$$\begin{aligned}\mathrm{Tr}\gamma(-\Delta)^s \ge \int_{\Omega^*} &\mathrm{Tr}\gamma(\phi_u(-\Delta)^s\phi_u - C\,\delta_u^{2-2s}\, l(u)^{-2s}\chi_u\chi_\Omega)\, l(u)^{-d} du \\ &- C\|\gamma\| \int_{\Omega^*} \delta_u^{-d+2-2s}\, r(\delta_u) l(u)^{-d-2s}\, du. \qquad (193)\end{aligned}$$

If the supports of u and $\phi_{u'}$ overlap, we have $|u-u'|\ (3/2)l(u) + l(u')$. It follows that $l(u') - l(u) \le \|\nabla l\|_\infty.\big((3/2)l(u) + l(u')\big)$. Since $\|\nabla l\|_\infty \le 1/2$, we find $l(u')^{-1} \le C\, l(u)$ and $l(u)^{-1} \le Cl(u')^{-1}$. Similarly, we get $l(u) \le C\, l(u')$. We assume now that ıu satisfies

$$\delta_u \le C\delta_{u'} \quad \text{if} \quad |u-u'| \le \frac{3}{2}\left(l(u) + l(u')\right) \qquad (194)$$

Using these locally uniform bounds on $l(u)/\, l(u')$ and $\delta_u = \delta_{u'}$, together with (159), we can deduce the pointwise bound for all $x \in \mathbb{R}^d$

$$\begin{aligned}&\int_{\Omega^*} \delta_u^{2-2s} l(u)^{-2s}\chi_{\mathrm{u}}(x)\chi_\Omega(x)\frac{du}{l(u)^d} \\ &= \int_{\Omega^*} \delta_u^{2-2s} l(u)^{-2s}\chi_{\mathrm{u}}(x)\chi_\Omega(x)\left(\int \phi_{u'}(x)^2\frac{du'}{l(u')^d}\right)\frac{du}{l(u)^d} \\ &\le C\int_{\Omega^*} \phi_{u'}(x)\, \delta_{u'}^{2-2s} l(u')^{-2s}\phi_{u'}(x)\, \frac{du'}{l(u')^d}.\end{aligned}$$

Rewriting the last integral with u as integration variable, in view of (193), we find

$$\mathrm{Tr}\gamma(-\Delta)^s \geq \int_{\Omega^*} \mathrm{Tr}\gamma\left(\int \phi_u\left((-\Delta)^s - \frac{C\delta_u^{2-2s}}{l(u)^{2s}}\right)\phi_u\right)\frac{du}{l(u)^d}$$
$$-C\|\gamma\|\int_{\Omega^*} \delta_u^{-d+2-2s} r(\delta_u)\frac{du}{l(u)^{d+2s}}.$$

By the variational principle it follows that

$$\begin{aligned}\mathrm{Tr}(H_\Omega)_- &= \inf_{0\le\gamma\le 1} \mathrm{Tr}\gamma((-h^2\Delta)^s - 1)\\ &\le \int_{\Omega^*} \mathrm{Tr}\gamma(\phi_u((-h^2\Delta)^s - 1 - C\,h^{2s}\delta_u^{2-2s}l(u)^{-2s})\phi_u)\frac{du}{l(u)^d}\\ &+ Ch^{2s}\int_{\Omega^*} \delta_u^{-d+2-2s} r(\delta_u)\frac{du}{l(u)^{d+2s}}. \qquad (195)\end{aligned}$$

To bound the first term, we use Lemma (2.3.4). For any $u \in \mathbb{R}^d$, let ρ_u be another parameter satisfying $0 < \rho_u \le 1/2$ and estimate

$$\begin{aligned}&\mathrm{Tr}(\phi_u((-h^2\Delta)^s - 1 - C\,h^{2s}\delta_u^{2-2s}l(u)^{-2s})\phi_u)_-\\ &\le \mathrm{Tr}(\phi_u H_\Omega\phi_u)_- + C\mathrm{Tr}(\phi_u(\rho_u h^{2s}(-\Delta)^s - \rho_u - h^{2s}\delta_u^{2-2s}l(u)^{-2s})\phi_u)_-\\ &\le \mathrm{Tr}(\phi_u H_\Omega\phi_u)_- + Cl(u)^d(\rho_u h^{2s})^{-d/(2s)}(\rho_u + h^{2s}\delta_u^{2-2s}l(u)^{-2s})^{1+d/(2s)}.\end{aligned}$$

We pick $\rho_u = h^{2s}\delta_u^{2-2s}l(u)^{-2s}$. By (165) and our assumption that $\delta_u \le 1/2$, we see that

$$\rho_u \le \left(\frac{h}{l_0}\right)^{2s} 2^{6s-2}.$$

We assume now that $h \le C^{-1}\, l_0$ (with a possibly large constant C) in order to guarantee that $\rho_u \le 1/2$. With this choice we find

$$\mathrm{Tr}\left(\int \phi_u\left((-\Delta)^s - 1 - \frac{Ch^{2s}\delta_u^{2-2s}}{l(u)^{2s}}\right)\phi_u\right)_- \le \mathrm{Tr}(\phi_u H_\Omega\phi_u)_- + C\frac{\delta_u^{2-2s}l(u)^{d-2s}}{h^{d-2s}}. \quad (196)$$

Combining (195) and (196), we obtain

$$\mathrm{Tr}(H_\Omega)_- \le \int_{\Omega^*} \mathrm{Tr}(\phi_u H_\Omega\phi_u)_- \frac{du}{l(u)^d} + C\int_{\Omega^*}\left(\frac{\delta_u^{2-2s}}{h^{d-2s}l(u)^{2s}} + \frac{h^{2s}\delta_u^{-d}r(\delta_u)}{l(u)^{d+2s}}\right)du. \quad (197)$$

At this point we choose δ_u in order to minimize the second integrand, which we shall denote by I_u. We pick

$$\delta_u = \begin{cases} h/l(u) & \text{if } 1 - \frac{d}{4} < s < 1,\\ (h/l(u))|\ln\ (l(u)/h)|^{1/(4-4s)} & \text{if } 0 < s = 1 - \frac{d}{4},\\ (hl(u)\,)^{d/(4-4s)} & \text{if } 0 < s < 1 - \frac{d}{4},\end{cases}$$

and note that $\delta_u \le 1/2$ if $h \le C^{-1}l_0$ by (165). Moreover, (194) is an easy consequence of the corresponding estimate for $l(u)/l(u')$. With this choice we arrive at the bounds

$$I_u \le C\begin{cases} h^{-d+2}l(u)^{-2} & \text{if } 1 - \frac{d}{4} < s < 1,\\ h^{-d+2}l(u)^{-2}|\ln\ (l(u)/h)|^{1/2} & \text{if } 0 < s = 1 - \frac{d}{4},\\ h^{-d/2+2s}l(u)^{d/2-2s} & \text{if } 0 < s < 1 - \frac{d}{4},\end{cases}$$

Finally, we integrate with respect to u. The same arguments that lead to (166) and (167) yield

$$\int_{\Omega^*} I_u du \le C \begin{cases} h^{-d+2} l_0^{-1} & if\ \ 1-\frac{d}{4}<s<1, \\ h^{-d+2} l_0^{-1} |\ln\ \ (l(u)/h)|^{1/2} & if\ \ 0<s=1-\frac{d}{4}, \\ h^{-d/2+2s} l l_0^{-d/2-2s+1} & if\ \ 0<s<1-\frac{d}{4}, \end{cases}$$

This completes the proof of the lower bound with the remainder stated in Proposition (2.3.15).

To prove the upper bound, we put

$$\gamma = \int_{\mathbb{R}^d} \mathrm{Tr}(\phi_u H_\Omega \phi_u)_-^0 \phi_u l(u)^{-d} du.$$

Obviously, $\gamma \ge\ 0$ holds and in view of (159) also $\gamma \ge\ 1$. The range of belongs to $\mathscr{H}^s(\Omega)$ and by the variational principle it follows that

$$\mathrm{Tr}(H_\Omega)_- \le \mathrm{Tr}\gamma H_\Omega = -\int_{\mathbb{R}^\mathrm{d}} \mathrm{Tr}(\phi_u H_\Omega \phi_u)_- l(u)^{-d} du.$$

This yields the upper bound and finishes the proof of Proposition (2.3.15).

We study the second term of (161) in more detail. First we derive representation (154).

Proposition (2.3.16)[103]: One has

$$L_{s,d}^{(2)} = \int_{\mathbb{R}^{d-1}} \xi(|p'|^{-2s}) \frac{dp'}{(2\pi)^{d-1}}$$
$$= \frac{|S^{d-2}|}{(2\pi)^{d-1}} \frac{2s}{(d-1)(d-1+\ 2s)} \mathrm{\ Tr}\ \left[\chi A^{\frac{(d-1)}{2s}} \chi - (A^+)^{\frac{(d-1)}{2s}}\right]. \tag{198}$$

Here is the characteristic function of $\mathbb{R}_+$ and

$$\xi(\mu) = \mu^{-1} \int_0^\infty \big(a(\mu) - a^+(t,\mu)\big)\, dt. \tag{199}$$

Proof: The first identity follows immediately from (219) and (220). The second identity follows from the fact that

$$\int_{\mathbb{R}^{d-1}} |p'|^{2s} (E - |p'|^{-2s})_- \frac{dp'}{(2\pi)^{d-1}} = \frac{|S^{d-2}|}{(2\pi)^{d-1}} \frac{2s}{(d-1)(d-1+\ 2s)} E^{-(d-1)/2s}$$

for any E > 0, which by the Spectral Theorem implies that

$$\int_{\mathbb{R}^{d-1}} |p'|^{2s} a^+(t, |p'|^{-2s})\ \frac{dp'}{(2\pi)^{d-1}} = \frac{|S^{d-2}|}{(2\pi)^{d-1}} \frac{2s}{(d-1)(d-1+\ 2s)} (A^+)^{-(d-1)/2s}(t,t)$$

and similarly for A.

Remark (2.3.17)[103]: There is another representation, namely,

$$L_{s,d}^{(2)} = \frac{2s}{d-1+\ 2s} \int_{\mathbb{R}^{d-1}} \xi(|p'|^{-2s})_- \frac{dp'}{(2\pi)^{d-1}} \tag{200}$$

where

$$\xi(\mu) = \int_0^\infty \big(e(\mu) - e^+(t,\mu)\big)\, dt. \tag{201}$$

Here $e(\mu)$ and $e^+(t,\mu)$ are the diagonals of the integral kernels of the spectral projections $(A-\mu)^0_-$ and $(A^+-\mu)^0_-$, respectively. We have not shown that the integral in (201) converges, since we will not use (200). Identity (200) is an easy consequence of (198) and the fact that

$$a(\mu)=\int_0^\mu e(\tau)d\tau, \qquad a^+(t,\mu)=\int_0^\mu e^+(t,\tau)d\tau,$$

which follows by the Spectral Theorem from $(E-\mu)_- = \int_0^\mu (E-\tau)^0_- d\tau$. Representation (200) is natural since in terms of this function the conjectured formula for the number of negative eigenvalues of H_Ω takes the form

$$\iint_{T^*\Omega} (|p'|^{-2s})^0_- \frac{dp\,dx}{(2\pi h)^d} - \iint_{T^*\partial\Omega} \xi(|p'|^{-2s})_- \frac{dp'd\sigma(x)}{(2\pi h)^{d-1}} + o(h^{-d+1})$$

which is the analogue of well-known two-term semi-classical formulas in the local case; see [111], [118]. The function ξ plays the role of a spectral shift. Note that we avoided to write (199) and (201) in terms of a trace. While the integrals on the diagonals converge, we do not expect the operators to be trace class, see [117].

Lemma (2.3.18)[103]: Let B be a non-negative operator with $\ker B=\{0\}$ and let P be an orthogonal projection. Then for any operator monotone function $\phi:(0,\infty)\to\mathbb{R}$,

$$P_\phi(PBP)P \geq P_\phi(B)P. \tag{202}$$

If, in addition, B is positive definite and - is not affine linear, then $P\phi(PBP)P = P\phi(B)P$ implies that the range of P is a reducing subspace of B.

By definition, the range of P is a reducing subspace of a non-negative (possibly unbounded) operator if $(B+\tau)^{-1}\,\mathrm{ran}P \subset \mathrm{ran}P$ for some $\tau>0$. We note that this is equivalent to $(B+\tau)^{-1}$ commuting with P, and we see that the definition is independent of τ since

$$\begin{aligned}(B+\tau')^{-1}P - P(B+\tau')^{-1} \\ = (B+\tau)(B+\tau')^{-1}((B+\tau)^{-1}P - P(B+\tau')^{-1})(B+\tau)(B+\tau')^{-1}.\end{aligned}$$

Hansen [109] has proved Lemma (2.3.18) for bounded B and without the equality statement. It is not clear how to extend his proof to our general case and we provide a different argument.

We recall Löwner's theorem [107] which characterizes operator monotone functions on $(0,\infty)$ by the representation

$$\phi(E) = a + bE + \int_{(0,\infty)} \frac{\tau E-1}{E+\tau}\, d\rho(\tau) \tag{203}$$

with $a\in\mathbb{R}, b\geq 0$ and a finite, positive measure ρ on $[0,\infty)$. Note that the function $\phi(E) = E^s, 0<s<1$, to which we apply this lemma, is operator monotone in view of the representation

$$E^s = \frac{\sin(\pi s)}{\pi}\int_0^\infty \tau^{s-1}\frac{E}{E+\tau}d\tau, \qquad 0<s<1.$$

This is of the form (203) above with $d\rho(\tau) = (\sin(\pi s)\,\pi)(1+\tau^2)^{-1}\tau^s d\tau, a = \int_0^\infty \tau^{-1}d\rho(\tau)$ and $b=0$.

Proof: We first prove that

$$PB^{-1}P \geq P\,(PBP)^{-1}P. \tag{204}$$

Here on the right side, the operator PBP is inverted as an operator on the range of P. By a monotone convergence argument we may assume that B is positive definite. Let f be an arbitrary element in the Hilbert space. For any in the form domain of B we can write

$$(f, PB^{-1}Pf) = (\psi, B\,\psi) + 2\mathrm{Re}(Pf, \psi) + \left\|B^{1/2}\psi - B^{-1/2}Pf\right\|^2 .$$

We apply this to $\psi = P\,(PBP)^{-1}Pf$. Note that belongs to the operator domain of PBP and hence also to the form domain of PBP, which means that $P\psi = \psi$ belongs to the form domain of B. We find

$$(f, PB^{-1}Pf) = (f, P(PBP)^{-1}Pf) + \left\|B^{1/2}P\,(PBP)^{-1}Pf - B^{-1/2}Pf\right\|^2.$$

This proves (204). Moreover, if equality holds in (204) (still assuming that B is positive definite), then $B^{1/2}P(PBP)^{-1}Pf = B^{1/2}Pf$ for all f , that is, $P(PBP)^{-1}Pf = B^{-1}Pf$ for all f , which means that $B^{-1}\,\mathrm{ran}P \subset \mathrm{ran}P$. Thus, $\mathrm{ran}P$ reduces B.

Now assume that ϕ is of the form (203) and rewrite

$$\frac{\tau E - 1}{E + \tau} = \frac{\tau^2 + 1}{E + \tau}.$$

By the Spectral Theorem,

$$P\phi(B)P = aP + bPBP + \int_{[0,\infty)} (\tau P\,(\tau^2 + 1)P\,(B + \tau)^{-1}P)\,d\rho(\tau)\,.$$

Similarly, PBP is a self-adjoint operator in the range of P and by the Spectral Theorem in that space

$$P\phi(PBP)P = aP + bPBP + \int_{[0,\infty)} (\tau P\,(\tau^2 + 1)P\,(PBP + \tau P)^{-1}P)\,d\rho(\tau)$$

Here, as before PBP C P is inverted in the range of P. Thus,

$$P\phi(PBP)P - P\phi(B)P = \int_{[0,\infty)} (P\,(PBP + \tau P)^{-1}P - P\,(B + \tau)^{-1}P)(\tau^2 + 1)d\rho(\tau).$$

By (204) with B replaced by $B + \tau$, the integrand is a non-positive operator for every $\tau \in [0, \infty)$. Thus, $P\phi(PBP)P \geq P\phi(B)P$, as claimed.

This argument shows that $P\phi(PBP)P = P\phi(B)P$ implies

$$P(PBP + \tau P)^{-1}P = P\,(B + \tau)^{-1}P$$

for $\tau \in [0, \infty)$. If ϕ is not affine linear, then the measure ρ is not identically zero and there is $\tau \in [0, \infty)$ with $P(PBP + \tau P)^{-1}P = P\,(B + \tau)^{-1}P$. Now the analysis of equality in (204) (note that $B + \tau$ is positive definite) implies that ranP reduces B.

Here we shall prove

Proposition (2.3.19)[103]: For any $0 < s < 1$ and $d \geq 2$ one has $L^{(2)}_{s,d} > 0$.

Proof: We shall show that for arbitrary non-negative operators B with $\ker B = \{0\}$ and orthogonal projections P,

$$\mathrm{Tr}[PB^{\alpha}P\,(PBP)^{-\alpha}] \geq 0 \text{ for all } \alpha > 0. \tag{205}$$

If B is positive definite, then equality holds iff the range of P is a reducing subspace of B.

We apply this to the second representation in (198) with $B = A$ and $P = \chi$ and note that $A^{+} = \chi A \chi$. Thus (205) implies $L^{(2)}_{s,d} \geq 0$. Since $B \geq 1$ and since the range of P is not a reducing subspace for B (indeed, $(A + \tau)^{-1}$f does not necessarily vanish on $(\infty, 0)$ if f does), we even have $L^{(2)}_{s,d} > 0$, as claimed.

It remains to prove (205). The argument is somewhat different depending on whether $\alpha \leq 1$ or not. In the first case we learn from Lemma (2.3.18) with $\phi(E) = E^{-\alpha}$ that

$$PB^{-\alpha}P \geq (PBP)^{-\alpha}$$

with equality if and only if ran$P$ reduces B. This immediately implies (205) and the equality statement. Now assume that $\alpha > 1$. Then Lemma (2.3.18) with $\phi(E) = E^{-1/\alpha}$ yields

$$PBP \geq (PB^{-\alpha}P)^{-1/\alpha}$$

with equality if and only if ran$P$ reduces $B^{-\alpha}$. Since $E \mapsto E^{-\alpha}$ is strictly monotone decreasing, we obtain again (205) and, using the Spectral Theorem, the equality statement. It is well known that the Dirichlet Laplacian $-\Delta\Omega$ on Ω satisfies

$$\operatorname{Tr}(-h^2\Delta_\Omega - 1) = L_{1,d}^{(1)}|\Omega|h^{-d}\, L_{1,d}^{(2)}|\partial\Omega|h^{-d+1} + o(h^{-d+1}),$$

see, [108] for a proof under the sole assumption that $\partial\Omega \in C^{1,\alpha}$ for some $0 < \alpha \leq 1$. Here

$$L_{1,d}^{(1)} = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} (|p|^2 - 1)\, dp$$

and, by an argument similar to that in our Proposition (2.3.16), one can bring the second constant in the form

$$L_{1,d}^{(2)} = \frac{|S^{d-2}|}{(2\pi)^{d-1}} \frac{2s}{(d-1)(d-1+2s)} \operatorname{Tr}\left[\chi B^{(d-1)/2s}\chi - (B^+)^{(d-1)/2s}\right],$$

where $B = -d^2/dt^2 + 1$ in$L_{1,d}^{(2)}$ and $B^+ = d^2/dt^2 + 1$ with Dirichlet boundary conditions in $L_{1,d}^{(2)}$. A short computation, using the fact that

$$(E^s - 1) = s(1-s)\int_0^1 (E-\tau)_-\tau^{s-2}\, d\tau + s(E - 1),$$

gives

$$\begin{aligned}
&\operatorname{Tr}((-h^2\Delta)^s - 1)_- \\
&\quad = L_{1,d}^{(1)}|\Omega|h^{-d}\, s\left((1-s)\int_0^1 \tau^{\frac{d}{2}+s-1}\, d\tau + 1\right) \\
&\quad - L_{1,d}^{(2)}|\partial\Omega|h^{-d+1}\, s\left((1-s)\int_0^1 \tau^{(d-1)/2+s-1}\, d\tau + 1\right) + o(h^{-d+1}) \\
&\quad = L_{1,d}^{(1)}|\Omega|h^{-d} - \frac{s(d+1)}{d-1+2s} L_{1,d}^{(2)}|\partial\Omega|h^{-d+1} + o(h^{-d+1}),
\end{aligned}$$

that is,

$$\begin{aligned}
\tilde{L}_{1,d}^{(2)} &= \frac{s(d+1)}{d-1+2s} L_{1,d}^{(2)} \\
&= \frac{|S^{d-2}|}{(2\pi)^{d-1}} \frac{2s}{(d-1)(d-1+2s)} \operatorname{Tr}\left[\chi B^{(d-1)/2s}\chi - (B^+)^{(d-1)/2s}\right],
\end{aligned}$$

Since

$$B^{(d-1)/2}(t,t) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{(1+p^2)^{(d-1)/2}}\, dp = A^{-(d-1)/2s}(t,t)\,,$$

we find that

$$\tilde{L}_{1,d}^{(2)} - L_{1,d}^{(2)} = \frac{|S^{d-2}|}{(2\pi)^{d-1}} \frac{2s}{(d-1)(d-1+2s)} \operatorname{Tr}\left[(A^+)^{(d-1)/2s} - (B^+)^{(d-1)/2}\right].$$

We now apply Lemma (2.3.18) with $B = d^2/dt^2 + 1$ in $L^2(\mathbb{R})$, with P being the projection onto $L^2(\mathbb{R}_+)$ and with $\phi(E) = E^s$. Then $P\phi(PBP)P = (B^+)^s$ and $P\,\phi(B)P = A^+$, and therefore (202) yields $(B^+)^s \geq A^+$. Since $E \mapsto E^{(d-1)/2s}$ is strictly monotone and since the operators A^+ and $(B^+)^s$ are not identical, we conclude that

$$\mathrm{Tr}\left[(A^+)^{(d-1)/2s}(B^+)^{(d-1)/2}\right] > 0.$$

This shows that $\tilde{L}_{1,d}^{(2)} - L_{1,d}^{(2)} > 0$ and completes the proof of Proposition (2.3.1).

For the sake of completeness we have

Lemma (2.3.20)[103]: Let $(\lambda_k)_{k\in\mathbb{N}}$ be a non-decreasing sequence of real numbers, and let $A, C > 0, B, D \in \mathbb{R}$ and $1 < a - 1 < b < a$ be related by

$$C = A^{1/a}a(a+1)^{(1+a)/a}, \qquad D = B\big(A(a+1)\big)^{-(1+b)/a}.$$

Then the asymptotic formula

$$\sum_{k=1}^{N} \lambda_k = AN^{a+1} + BN^{b+1} + o(N^{b+1}), \qquad N \to \infty, \tag{206}$$

is equivalent to

$$\sum_{k\in\mathbb{N}} (\Lambda - \lambda_k)_+ = C\Lambda^{(1+a)/a} - D\Lambda^{\frac{(1+b)}{a}} + o(\Lambda^{\frac{(1+b)}{a}}, \ \Lambda \to \infty. \tag{207}$$

Proof: This lemma is a consequence of Hardy, Littlewood and Polya's majorization theorem, which says that for any non-decreasing sequences (a_k) and (b_k)

$$\sum_{k=1}^{N} a_k \leq \sum_{k=1}^{N} b_k \text{ for all } N \in \mathbb{N} \tag{208}$$

is equivalent to

$$\sum_{k=1}^{\infty} (\Lambda - a_k)_+ \geq \sum_{k=1}^{\infty} (\Lambda - b_k)_+ \text{ for all } \Lambda \in \mathbb{R},$$

see, [116]. As usual, we will denote property (208) by $(a_k) \prec (b_k)$.

We fix $\epsilon > 0$ and set $\beta_k^{\pm} = A(a+1)k^a + (B \pm \epsilon)(b+1)k^b$. Note that the assumptions on a and b imply

$$\sum_{k=1}^{N} \beta_k^{\pm} = AN^{a+1}\,(B \pm \epsilon)N^{b+1} + o(N^{b+1}), \qquad N \to \infty \tag{209}$$

and

$$\sum_{k\in\mathbb{N}} \left(\Lambda - \beta_k^{\pm}\right)_+ = \frac{aA}{\big(A(a+1)\big)^{1+1/a}}\Lambda^{(1+a)/a} \tag{210}$$

$$\frac{B \pm \epsilon}{\big(A(a+1)\big)^{(1+b)/a}}\Lambda^{(1+b)/a} + o\big(\Lambda^{(1+b)/a}\big), \qquad \Lambda \to \infty$$

First, we assume that (206) holds. Then, by (206) and (209) there is an $N \in \mathbb{N}$ such that for all $N \geq N_\epsilon$

$$\sum_{k=1}^{N} \beta_k^{-} \leq \sum_{k=1}^{N} \lambda_k \leq \sum_{k=1}^{N} \beta_k^{+}.$$

We put $\alpha_k^+ = \beta_k^+$ for $k \geq N_\epsilon$ and $\alpha_k^+ = \max(\beta_k^+, \lambda_k), \alpha_k^- = \min(\beta_k, \lambda_k)$ for $k < N_\epsilon$. Thus

$$(\alpha_k^-) \prec (\lambda_k) \prec (\alpha_k^+),$$

and therefore

$$\sum_{k\in\mathbb{N}} (\Lambda - \alpha_k^+)_+ \leq \sum_{k\in\mathbb{N}} (\Lambda - \lambda_k)_+ \leq \sum_{k\in\mathbb{N}} (\Lambda - \alpha_k^-)_+ \quad \text{for all } \Lambda \in \mathbb{R}.$$

Since

$$\sum_{k\in\mathbb{N}} \left(\Lambda - \alpha_k^\pm\right)_+ = \sum_{k\in\mathbb{N}} \left(\Lambda - \beta_k^\pm\right)_+ + O(1),$$

the assertion (207) follows from (210). The converse implication is proved similarly.

We outline the calculations that are necessary to complete the analysis of the model operator A^+ introduced. The results depend on the following spectral representation of the operator A^+ found in [112].

Theorem (2.3.21)[103]: For $E > 0$ let

$$\psi(E) = (E + 1)^s - 1$$

and for $\lambda > 0$ put $\gamma_\lambda(\xi) = 0$ if $0 < \xi < 1$ and

$$\gamma_\lambda(\xi) = \frac{1}{\pi} \frac{\lambda\psi'(\lambda^2)\sin(\pi s)(\xi^2 - 1)^s}{\psi(\lambda^2)^2 + (\xi^2 - 1)^s - 2\psi(\lambda^2)(\xi^2 - 1)\cos(\pi s)} \times \exp\left(-\frac{1}{\pi}\int_0^\infty \frac{\xi}{\xi^2 + \zeta^2} \ln \frac{\psi'(\lambda^2)(\lambda^2 - \zeta^2)}{\psi(\lambda^2) - \psi(\zeta^2)} d\zeta\right)$$

if $\xi \geq 1$ Moreover, define a phase-shift

$$\vartheta_\lambda = \frac{1}{\pi}\int_0^\infty \frac{\lambda}{\xi^2 + \zeta^2} \ln \frac{\psi'(\lambda^2)(\lambda^2 - \zeta^2)}{\psi(\lambda^2) - \psi(\zeta^2)} d\zeta \tag{211}$$

and functions

$$F_\lambda(x) = \sin(\lambda x + \vartheta_\lambda) + \int_0^\infty e^{x\xi}\gamma_\lambda(\xi)d\xi, \qquad x > 0. \tag{212}$$

Then

$$\Phi f(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)F_\lambda(x)\, dx$$

defines a unitary operator from $L^2(\mathbb{R}_+)$ to $L^2(\mathbb{R}_+)$.

This operator diagonalizes A^+ in the sense that a function $f \in L^2(\mathbb{R}_+)$ is in the domain of A^+ if and only if $(\lambda^2 + 1)^s\Phi f(\lambda)$ is in $L^2(\mathbb{R}_+)$, and in this case

$$\Phi A^+ f(\lambda) = (\lambda^2 + 1)^s \Phi f(\lambda)$$

According to [112] the Laplace transform of γ_λ is a completely monotone function bounded by one. From (212) it follows that for all $t \geq 0$

$$|F_\lambda(t)| \leq 2. \tag{213}$$

Theorem (2.3.21) states that the functions F_λ are generalized eigenfunctions of the operator A^+. Hence, we can write

$$e^+(t, u, \mu) = \frac{2}{\pi}\int_0^\infty ((\lambda^2 + 1)^s - \mu)_-^0 F_\lambda(t)F_\lambda(u)d\lambda. \tag{214}$$

From (219), (220), and Proposition (2.3.16) it follows that

$$L_{s,d}^{(2)} = \frac{4s}{(d-1+\ 2s)(d\ -1)}\frac{|S^{d-2}|}{(2\pi)^d}$$
$$\times \int_0^\infty \int_0^\infty \left(1-\ 2F_\lambda^2(t)\right)(\lambda^2+1)^{s(d-1)/2}\, d\lambda dt. \quad (215)$$

Lemma (2.3.22)[103]: For any $\mu > 0$ and $t, u \in \mathbb{R}_+$ one has $|e^+(t,u,\mu)| \le C\mu^{1/2s}$. Quantifies that $a^+(t,\mu)$ is close to $a(\mu)$ for large t.

Proof: Lemma (2.3.22) is an immediate consequence of (214). In view of (213) we estimate

$$|e^+(t,u,\mu) \le C \int_0^{(\mu^{1/s}-1)_+^{1/2}} d\lambda \le\ C\mu^{1/(2s)}.:$$

This proves the lemma.

We need the following technical result about ϑ_λ.

Lemma (2.3.23)[103]: The phase-shift ϑ_λ is monotone increasing and twice differentiable in $\lambda > 0$. It satisfies

$$\vartheta_0 =\ 0 \text{ and } \vartheta_\lambda \to \frac{\pi}{4}\,(1\ -\ s)\ as\ \lambda \to \infty$$

The first and second derivatives are bounded and one has, as $\lambda \to \infty$,

$$\frac{d\vartheta_\lambda}{d\lambda} = \frac{d^2\vartheta_\lambda}{d\lambda^2} =\ O\left(\frac{1}{\lambda}\right).$$

Proof: Following [112], we substitute $\zeta = \lambda z$ for $\zeta \in (0,\lambda)$ and $\zeta = \lambda/z$ for $\zeta \in (\lambda,\infty)$ in the definition of ϑ_λ and obtain

$$\vartheta_\lambda = \frac{1}{\pi}\int_0^1 \frac{1}{1-z^2}\ln\left(\frac{1}{z^2}\frac{\psi(\lambda^2)-\psi(\lambda^2 z^2)}{\psi(\lambda^2/z^2)-\psi(\lambda^2)}\right)dz.$$

Note that the function

$$\frac{1}{z^2}\frac{\psi(\lambda^2)-\psi(\lambda^2 z^2)}{\psi(\lambda^2/z^2)-\psi(\lambda^2)} = \frac{1}{z^2}\frac{(1+\lambda^2)^s-(1+\lambda^2 z^2)}{(1+\lambda^2/z^2)^s-(1+\lambda^2)^s}$$

equals 1 for $\lambda =\ 0$ and that for all $z \in (0,1)$ it is increasing in $\lambda\ >\ 0$ and tends to z^{-2-s2} as λ tends to infinity. By Lebesgue's dominated convergence we find $\vartheta_0 =\ 0$ and

$$\lim_{\lambda\to\infty} \vartheta_\lambda = \frac{1}{\pi}\int_0^1 \frac{1}{1-z^2}\ln\ (z^{2s-2})\,dz = \frac{\pi}{4}\,(1-\ s).$$

By (211), we also have

$$\vartheta_\lambda = \frac{1}{\pi}\int_0^\infty b_\lambda(\zeta)\,d\zeta$$

with

$$b_\lambda(\zeta) = \frac{\lambda}{\zeta-\lambda}\ln\left(\frac{s(1+\lambda^2)^{s-1}-(\lambda^2-\zeta^2)}{(\lambda^2+1)^s-(\zeta^2+1)^s}\right).$$

We remark that

$$|\vartheta_\lambda b_\lambda(\zeta)| \le \vartheta_\lambda b_\lambda(\zeta)|_{\lambda=0}\frac{1}{\zeta^2}\ln\left(\frac{s\zeta^2}{(1+\zeta^2)^s-1}\right)$$

for all $\zeta \in (0,\infty)$. Since the last expression is integrable in $\zeta \in (0,\infty)$, it follows that

$$\frac{d\vartheta_\lambda}{d\lambda} = \frac{1}{\pi}\int_0^\infty \vartheta_\lambda b_\lambda(\zeta)\, d\zeta$$

is bounded and, in particular, we obtain

$$\frac{d\vartheta_\lambda}{d\lambda}|_{\lambda=0} = \frac{1}{\pi}\int_0^\infty \frac{1}{\zeta^2}\ln\left(\frac{s\zeta^2}{\left(1+\zeta^2\right)^s - 1}\right) d\zeta. \qquad (216)$$

Similarly, we can show existence and boundedness of the second derivative and decay of the derivatives as $\lambda \to \infty$ by explicit calculations and Lebesgue's dominated convergence.

To simplify notation we put

$$\psi_\lambda(E) = \frac{1 - E/\lambda^2}{1 - \psi(E)/\psi(\lambda^2)}$$

for $E > 0$. Moreover, we write G_λ for the Laplace transform of γ_λ and g_λ for the Laplace transform of G_λ. According to [112] we have

$$g_\lambda(t) = \frac{\lambda\cos\vartheta_\lambda + t\sin\vartheta_\lambda}{\lambda^2 + t^2}\ \lambda^2\sqrt{\frac{\psi'(\lambda^2)}{\psi(\lambda^2)}}\frac{\varphi_\lambda(t)}{\lambda^2 + t^2}, \qquad t > 0; \qquad (217)$$

With

$$\varphi_\lambda(t) = \exp\left(\frac{1}{\pi}\int_0^\infty \frac{t}{t^2 + \zeta^2} ln\ \left(\psi_\lambda(\zeta^2)\right) d\zeta\right).$$

To prove Lemma (2.3.25), we need the following properties of φ_λ.

Lemma (2.3.24)[103]: The function $t \mapsto \varphi_\lambda(t)$ is differentiable in $t > 0$ and its derivative satisfies

$$\varphi_\lambda'(0) = o(1) \text{ as } \lambda \to \infty,$$

$$\varphi_\lambda'(0) = \frac{d\vartheta_\lambda}{d\lambda}|_{\lambda=0} + O(\lambda) \quad \text{as } \lambda \to \infty.$$

Proof: For fixed $\zeta \in (0,\infty)$ the function $\lambda \mapsto \psi_\lambda(\zeta^2)$ is non-increasing in $\lambda > 0$ and tends to 1 as $\lambda \to \infty$. Moreover,

$$\frac{1}{\zeta^2}\ln\ \left(\psi_0(\zeta^2)\right) = \left(\frac{s\zeta^2}{\left(1+\zeta^2\right)^s - 1}\right)$$

is integrable with respect to $\zeta \in (0,\infty)$. Hence we find that

$$\varphi_\lambda'(0) = \frac{1}{\pi}\int_0^\infty \frac{1}{\zeta^2}\ln\ \left(\psi_\lambda(\zeta^2)\right) d\zeta$$

and $\varphi_\lambda'(0) = o(1)$ as $\lambda \to \infty$ by Lebesgue's theorem.

In view of (216)

$$\varphi_\lambda'(0)|_{\lambda=0} = \frac{1}{\pi}\int_0^\infty \frac{1}{\zeta^2}\ln\ \left(\psi_0(\zeta^2)\right) d\zeta = \frac{d\vartheta_\lambda}{d\lambda}|_{\lambda=0}.$$

The second claim now follows from the fact that the derivative of $\lambda \mapsto \varphi_\lambda'(0)$ is bounded.

Lemma (2.3.25)[103]: For any $0 \le \gamma < 1$ there is a constant C such that for all $\mu \ge 1$

$$\int_0^\infty t|a^+(t,\mu)\, a(\mu)|\, dt \le C_\gamma \mu((\ln\mu)^2 + 1). \qquad (218)$$

In particular, the function

$$K(t) = \frac{1}{(2\pi)^{d-1}} \int_{\mathbb{R}^{d-1}} |\xi'|^{1+2s}\big(a(|\xi'|^{-2s}) - a^+(t|\xi'|, |\xi'|^{-2s})\big)d\xi', \qquad t > 0, (219)$$

satisfies for every $0 \le \gamma < 1$

$$\int_0^\infty t|K(t)|\, dt < 1.$$

With this lemma at hand we can now define the constant $L_{s,d}^{(2)}$ which appears in our main theorem by

$$L_{s,d}^{(2)} = \int_0^\infty K(t)\, dt. \qquad (220)$$

(This integral converges by Lemma (2.3.25).) See also (215), we will derive different representations for $L_{s,d}^{(2)}$.

The spectral projections of the operator H^+ on the half-space in terms of those of the operator A^+ on the half-line. Since H^+ commutes with translations parallel to the boundary of $\mathbb{R}_+^d$, it can be written as a direct integral; see, [89] for definitions and properties of direct integrals.

Proof: In view of Theorem (2.3.21) we can write

$$a(\mu) - a^+(t,\mu) = \frac{1}{\pi} \int_0^\infty ((\lambda^2 + 1)^s - \mu)_-\big(1 - 2F_\lambda^2\big)d\lambda$$

and by (212)

$$1 - 2F_\lambda^2(t)^2 = \cos(2\lambda t + 2\vartheta_\lambda)\, 4\sin(\lambda t + \vartheta_\lambda)\, G_\lambda(t) - 2G_\lambda(t)^2 .$$

We get

$$\int_0^\infty t^\gamma |a(\mu) - a^+(t,\mu)|\, dt \le R_1(\mu) + R_2(\mu)$$

with

$$R_1(\mu) = \int_0^\infty t^\gamma \left| \int_0^{(\mu^{1/s}-1)_+^{1/2}} (\mu - (\lambda^2+1)^s) \cos(2\lambda t + 2\vartheta_\lambda)\, d\lambda \right| dt,$$

$$R_2(\mu) = \int_0^\infty t^\gamma \left| \int_0^{(\mu^{1/s}-1)_+^{1/2}} (\mu - (\lambda^2+1)^s)(2\sin(\lambda t + \vartheta_\lambda)\, G_\lambda(t) + G_\lambda(t)^2) d\lambda \right| dt.$$

To estimate $R_1(\mu)$we split the integration in t and integrate over $t \in [0,1]$ first. We assume $0 < \gamma < 1$. The proof for $\gamma = 0$ follows similarly.

We write

$$\cos(2\lambda t + 2\vartheta_\lambda) = \frac{1}{2t}\frac{d}{d\lambda}\sin(2\lambda t + 2\vartheta_\lambda) - \frac{\cos(2\lambda t + 2\vartheta_\lambda)}{t}\frac{d\vartheta_\lambda}{d_\lambda}$$

and insert this identity in the expression for $R_1(\mu)$. After integrating by parts in the λ-integral one can estimate

$$\int_0^1 t^\gamma \left| \int_0^{(\mu^{1/s}-1)_+^{1/2}} (\mu - (\lambda^2+1)^s) \cos(2\lambda t + 2\vartheta_\lambda)\, d\lambda \right| dt \le C\mu((\ln\ \mu)^2 + 1) .$$

To estimate the integral over $t \in [1,1]$, we proceed similarly. We integrate by parts twice and get

$$\int_0^\infty t^\gamma \left| \int_0^{(\mu^{1/s}-1)_+^{1/2}} (\mu - (\lambda^2+1)^s)\cos(2\lambda t + 2\vartheta_\lambda)\, d\lambda \right| dt \le C\mu(\ln\ \mu + 1).$$

We conclude

$$R_1(\mu) \le\ C\mu((\ln\ \mu)^2 +\ 1)$$

and turn to estimating $R_2(\mu)$.

Since G_λ is non-negative and uniformly bounded, we have

$$R_2(\mu) \le C \int_0^{\left(\mu^{\frac{1}{s}}-1\right)_+^{\frac{1}{2}}} (\mu - (\lambda^2+1)^s) \int_0^\infty t^\gamma G_\lambda(t)\, dt d\lambda. \tag{221}$$

By definition,

$$g_\lambda(0) = \int_0^\infty G_\lambda(t)\, dt \quad \text{and} \quad g_\lambda'(0) = \int_0^\infty t\, G_\lambda(t)\, dt.$$

We note that, by (217),

$$g_\lambda(0) = \frac{\cos\vartheta_\lambda}{\lambda} - \sqrt{\frac{\psi'(\lambda^2)}{\psi(\lambda^2)}}$$

and apply Lemma (2.3.23) to estimate

$$\int_0^\infty G_\lambda(t)\, dt \le\ C(\lambda \wedge \lambda^{-1}).$$

Moreover, by (217),

$$g_\lambda'(0) = \frac{\sin\vartheta_\lambda}{\lambda} - \sqrt{\frac{\psi'(\lambda^2)}{\psi(\lambda^2)}}\varphi_\lambda'(0)$$

and we apply Lemma (2.3.23) and Lemma (2.3.24) to estimate

$$\int_0^\infty tG_\lambda(t)\, dt \le\ C(1 \wedge \lambda^{-1}).$$

It follows that

$$\int_0^\infty tG_\lambda(t)\, dt \le\ C(1 \wedge \lambda^{-1}).$$

Thus, by (221), we arrive at

$$R_2(\mu) \le\ C \int_0^{(\mu^{1/s}-1)_+^{1/2}} (\mu - (\lambda^2+1)^s)(1 \wedge \lambda^{-1}) d\lambda \le C\mu(\ln\ \mu + 1).$$

This finishes the first part of the proof of Lemma (2.3.25).

In order to prove the assertion about $K(t)$, we bound

$$\int_0^\infty t^\gamma |K(t)| dt \le \int_{|\xi'|<1} |\xi'|^{1+2s} \int_0^\infty t^\gamma |a^+(t|\xi'|, |\xi'|^{-2s}) - a(|\xi'|^{-2s})| dt d\xi'.$$

Here we also used that, since $a(\mu) = a^+(t,\mu) = 0$ for $\mu \le 1$, we can restrict the integration in the definition of K to $|\xi'| < 1$.On the other hand, from (218) we know that

$$\int_0^\infty t^\gamma \left| a^+\left(t\mu^{-1/2s}, \mu\right) - a(\mu) \right| dt \le\ C_\gamma \mu^{1+(\gamma+1)/(2s)}((\ln\ \mu)^2 +\ 1).$$

Combining these two bounds and using that $\gamma < 1 \leq d - 1$, we obtain the second part of Lemma (2.3.25).
The following technical lemma was needed in the proof of the upper bound near the boundary.
Lemma (2.3.26)[103]: Assume that $\phi \in C_0^1(\mathbb{R}^d)$ is supported in a ball of radius $l = 1$ and that (160) is satisfied with $l = 1$. Then for any $\frac{1}{2}\ 1 - s < \sigma < \min\left\{\frac{1}{2}, 1 - s\right\}$ one has

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left|(-\Delta_{x'})^{\sigma}\frac{|\phi(x)-\phi(y)|^2}{|x-y|^{d+2s}}\right| dxdy \leq C. \tag{222}$$

Proof: For $x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}^d$ and $y = (y', y_d) \in \mathbb{R}^{d-1} \times \mathbb{R}^d$ put

$$F_{x_d,y}(x') = \frac{\left(\phi(x',x_d) - \phi(y',y_d)\right)^2}{(|x'-y'|^2 + (x_d - y_d)^2)^{d/2+s}}.$$

To establish (222) we use the fact that

$$\left|(-\Delta_{x'})^{\sigma}\frac{|\phi(x)-\phi(y)|^2}{|x-y|^{d+2s}}\right| \leq C\int_{\mathbb{R}^{d-1}}\frac{\left|F_{x_d,y}(x') - F_{x_d,y}(z')\right|}{|x'-z'|^{d-1+2\sigma}}dz' \tag{223}$$

and split the integration in $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$ into four parts. First we assume that x and y are in B_1. Then we have to show that

$$\begin{aligned}\int_{B_1}\int_{B_1}\int_{\mathbb{R}^{d-1}}&\frac{\left|F_{x_d,y}(x') - F_{x_d,y}(z')\right|}{|x'-z'|^{d-1+2\sigma}}dz'dxdy \\ &= \int_{B_1}\int_{B_1}\int_{|x'-z'|<|x-y|/2}\frac{\left|F_{x_d,y}(x') - F_{x_d,y}(z')\right|}{|x'-z'|^{d-1+2\sigma}}dz'dxdy \\ &+ \int_{B_1}\int_{B_1}\int_{|x'-z'|<\frac{|x-y|}{2}}\frac{\left|F_{x_d,y}(x') - F_{x_d,y}(z')\right|}{|x'-z'|^{d-1+2\sigma}}dz'dxdy\end{aligned} \tag{224}$$

is bounded from above.

To estimate the first integral over $|x'-z'| < |x-y|/2$, we use the fact that

$$F(z') - F(x') = \sum_{j=1}^{d-1}\frac{(z_j - x_j)}{|x'-z'|}\int_0^{|x'-z'|}\left(\partial_j F\right)\left(x' + t\frac{(z_j - x_j)}{|x'-z'|}\right)dt.$$

For $j = 1, \ldots, d-1$ we have

$$\left(\partial_j F_{x_d,y}\right)(x') = \frac{\left(\phi(x',x_d) - \phi(y)\left(\partial_j\phi(x)\right)\right)}{|x-y|^{d+2s}} - (d+2s)\left(x_j - y_j\right)\frac{\left(\phi(x) - \phi(y)\right)^2}{|x-y|^{d+2s+2}},$$

thus

$$\left(\partial_j F_{x_d,y}\right)(x') \leq C|x-y|^{d+2s}.$$

Hence, we obtain

$$\begin{aligned}&\left|F_{x_d,y}(z') - F_{x_d,y}(x')\right| \\ &\leq C|x'-z'|^{\alpha}\left(\int_0^{|x'-z'|}\left(\left|x' + t\frac{(z_j - x_j)}{|x'-z'|} - y'\right|^2 + (x_d - y_d)^2\right)^{\beta}dt\right)^{1-\alpha}\end{aligned}, \tag{225}$$

with $0 < \alpha < 1$ and $\beta = \left(\frac{d-1}{2} + s\right)/(\alpha - 1)$, by applying Hölder's inequality. Note that

$$\left|x' + t\frac{(z_j - x_j)}{|x' - z'|} - y'\right|^2 + (x_d - y_d)^2 = |x-y|^2 + t^2 + 2t\frac{(x'-y').(z'-x')}{|x'-z'|}$$
$$\geq (|x-y|-t)^2.$$
Inserting this into (225), we get for $|x' - z'| < |x - y|/2$
$$\left|F_{x_d,y}(z') - F_{x_d,y}(x')\right| \leq C|x' - z'|^\alpha \left(\int_0^{|x-y|/2} (|x - y| - t)^{2\beta} dt\right)^{1-\alpha}$$
$$\leq C|x' - z'|^\alpha |x - y|^{(2\beta+1)(1-\alpha)},$$
where $(2\beta+1)(1-\alpha) = -d - 2s + 2 - \alpha$.We conclude that for any $2\sigma < \sigma < 1$ and $\sigma < 1 \leq s$
$$\int_{B_1}\int_{B_1}\int_{|x'-z'|<|x-y|/2} \frac{\left|F_{x_d,y}(x') - F_{x_d,y}(z')\right|}{|x'-z'|^{d-1+2\sigma}} dz'dxdy$$
$$\leq C\int_{B_1}\int_{B_1}\int_{|x'-z'|<\frac{|x-y|}{2}} |x' - z'|^{-d+1-2\sigma+\alpha} dz'|x-y|^{-d-2s+2-\alpha} dxdy$$
$$\leq C. \tag{226}$$
Now we turn to the second integral in (224) over $|x'-z'| \geq \frac{|x-y|}{2}$. Since
$$0 \leq F_{x_d,y}(x') \leq |x-y|^{-d-2s+2-\alpha} \tag{227}$$
and $\sigma < 1 - s$, we have
$$\int_{B_1}\int_{B_1}\int_{|x'-z'|<|x-y|/2} \frac{F_{x_d,y}(x')}{|x'-z'|^{d-1+2\sigma}} dz'dxdy \leq C\int_{B_1}\int_{B_1} \frac{1}{|x-y|^{d-1+2\sigma}} dxdy \leq C. \tag{228}$$
Moreover,
$$\int_{|x'-z'|<|x-y|/2} \frac{F_{x_d,y}(x')}{|x'-z'|^{d-1+2\sigma}} dz'$$
$$\leq C|x-y|^{-d+1-2\sigma+(d-1)/P} \left(\int_{|x'-z'|<|x-y|/2} F^q_{x_d,y}(z')\,dz'\right)^{1/q}$$
with $1/p + 1/q = 1$, by Hölder's inequality. Since $\sigma > 1/2 - s$, we can choose $p > \frac{d-1}{2\sigma}$ and $q > \frac{d-1}{d+2s-2}$. By (227), we have
$$\left(\int_{|x'-z'|<|x-y|/2} F^q_{x_d,y}(z')\,dz'\right)^{1/q}$$
$$\leq \left(\int_{\mathbb{R}^{d-1}} (|z'-y'|^2 + (x_d - y_d)^2)^{-qd(d/2+s-1)} dz'\right)^{1/q}$$
$$\leq C|x_d - y_d|^{-d-2s+2+(d-1)/q}.$$
It follows that
$$\int_{B_1}\int_{B_1}\int_{|x'-z'|<|x-y|/2} \frac{F_{x_d,y}(x')}{|x'-z'|^{d-1+2\sigma}} dz'dxdy$$
$$\leq \int_{B_1}\int_{B_1} |x-y|^{-d-1-2\sigma+(d-1)/p}|x_d - y_d|^{-d-2s+2+(d-1)/q}$$
$$\leq C\int_0^2 t^{-d-2s+2(d-1)/q}\int_0^2 r^{d-2}(r^2+t^2)^{(-d+1-2\sigma)/2+2(d-1)/2p} drdt,$$

where we substituted $t = |x_d - y_d|$ and $r = |x' - y'|$. Since $p > \frac{d-1}{2\sigma}$ and $\sigma < 1 - s$, we find

$$\int_{B_1}\int_{B_1}\int_{|x'-z'|<|x-y|/2} \frac{F_{x_d,y}(z')}{|x'-z'|^{d-1+2\sigma}} dz'dxdy \le C\int_0^2 t^{-d-2s+2\sigma}dt \le C. \quad (229)$$

The estimates (228) and (229) show that

$$\int_{B_1}\int_{B_1}\int_{|x'-z'|<|x-y|/2} \frac{\left|F_{x_d,y}(x') - F_{x_d,y}(z')\right|}{|x'-z'|^{d-1+2\sigma}} dz'dxdy \le C \quad (230)$$

and from (223), (226), and (230) it follows that

$$\int_{B_1}\int_{B_1}\left|(-\Delta_{x'})^{\sigma}\frac{|\phi(x)-\phi(y)|^2}{|x-y|^{d+2s}}\right| dxdy \le C.$$

The proof that the respective integrals over the domains $B_1 \times (\mathbb{R}^d / B_1)$, $(\mathbb{R}^d / B_1) \times B_1$ and $(\mathbb{R}^d / B_1) \times (\mathbb{R}^d / B_1)$ are finite is similar but easier, since $\mathrm{supp}\phi \subset B_1$ and we only have to handle one singularity at a time.

Chapter 3
Laplacian Energy-Like Invariant and Sum of Powers

We investigate a Laplacian energy-like graph invariant $LEL(G) = \sum_{i=1}^{n} \sqrt{\mu_i}$. There is a great deal of analogy between the properties of $E(G)$ and $LEL(G)$. We also establish a few sharp lower and upper bounds of $LEL(G)$. Here we establish some properties for sα with $\alpha \neq 0, 1$. We also discuss the cases $\alpha = 2, \frac{1}{2}$.

Section (3.1): A Laplacian Energy-Like Invariant of a Graph

Let G be a stmple graph with n vertices and m edges. In what follows we write $G(n,m)$ for it. Let A be the symmetric $(0,1)$-adjacency matrix of G and $D = diag(d_1, d_2, \ldots d_n)$ be the diagonal matrix of vertex degrees. The Laplacian matrix of G is $C = D - A$. Let $\lambda_1, \lambda_3 \ldots \lambda_n$be the adjacenry spectrum of G, and let $\mu_1, \mu_2, \ldots \mu_n$ be the Laplacian spectrum of G. The adjacency and Laplacian spectrum obey the following relations

$$\sum_{i=1}^{n} \lambda_i = 0; \quad \sum_{i=1}^{n} \lambda_i^2 = 2m, \tag{1}$$

$$\sum_{i=1}^{n} \mu_i = 2m; \quad \sum_{i=1}^{n} \mu_i^2 = 2m + \sum_{i=1}^{n} d_i^2 \tag{2}$$

Furthermore, if the graph G has p components $(p \geq 1)$, and if the Laplacian eigenvalues are labelled so that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$, then [139]

$$\mu_{n-1} = 0 \quad for\ i = 0, \ldots, p-1 \quad and \quad \mu_{n-p} > 0 \tag{3}$$

Eichinger [141] has shown how the spectrum of C may be used to calculate the radius of gyration of a Gaussian molecule. Mohar [145] argues that, because of its importance in various physical and chemical theories, the spectrum of C is more natural and important than the more widely studied adjacency spectrum. The energy of the graph G is defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i| \tag{4}$$

his quantity, introduced by I. Gutman in 1978 ([125]), has a long known chemical application (see [126]-[128]). For some of the most recent works along these lines sec [132]-[138].

$E(G)$ has the following basic properties

(a) $E(G) \geq 0$; equality is attained if and only if $m = 0$.

(b) If the graph G consists of (disconnected) components G_1 and G_2, then $E(G) = E(G_1) + E(G_2)$.

(c) If one component of the graph G is G_1 and all other components are isolated vertices, then $E(G) = E(G_1)$.

The Laplacian energy of the graph G has recently been defined ([129]) as

$$LE(G) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right| \tag{5}$$

The Laplacian energy $LE(G')$ and the ordinary energy $E(G)$ were found to have a number of analogous properties ([129], [131]), but $LE(G)$ does not possess the basic properties (b), (c) as above.

We conceive a new graph-energy-like quantity, that instead of Eq. (5) would be defined intense of Laplacian eigenvalues, and that hopefully would preserve properties (b), (c). We introduce the auxiliary eigenvalues $p_i,\ i = 1,2,\dots n$, defined via $p_i = \sqrt{\mu_i}$. Then we have,

$$\sum_{i=1}^{n} p_i^2 = 2m = \sum_{i=n}^{n} \lambda_i^2 \tag{6}$$

Definition (3.1.1)[121]: If the Laplacian eigenvalues of $G(n,m)$ are $\mu_1, \mu_2, \dots, \mu_n$, then the Laplucian-energy-like invariant of G, denoted by $LEL(G)$, is equal to $\sum_{i=1}^{n} \sqrt{\mu_i}$, $i.e$

$$LEL(G) = \sum_{i=1}^{n} \rho_i$$

where $\rho_i = \sqrt{\mu_i}$, $i = 1, 2, \dots, n$.

We report some properties of $LEL(G)$ and show that the above definition is well chosen. A few sharp lower and upper bounds of $LEL(G)$ arc established.

We present some properties of $LEL(G)$ which have a great deal of analogy with the properties (a),(b), (c) of $E(G)$.

Proposition (3.1.2)[121]: (a) $LEL(G) \geq 0$; equality is attained if and only if $m = 0$.

(b) If the graph G consists of (disconnected) components G_1 and G_2, then

$$LEL(G) = LEL(G_1) + LEL(G_2).$$

(c) If one component of the graph G is G_1 and all other components are isolated vertices, then $LEL(G) = LEL(G_1)$.

Proposition (3.1.3)[121]: $LEL(G) \leq \sqrt{2m(n-p)}$, p is the number of components of $G(n,m)$. Equality is attained if and only if G is regular of degree 0 or G consists of n_1 copies of complete graphs of order k and $n - kn_1$ isolated vertices.

Proof: Let

$$S = \sum_{i=1}^{n-p} \sum_{j=1}^{n-p} \left(\sqrt{\mu_i} = \sqrt{\mu_j}\right)^2 = 2\sum_{i=1}^{n-p} 2m - 2\left(\sum_{i=1}^{n-p} \sqrt{\mu_i}\right)\left(\sum_{j=1}^{n-p} \sqrt{\mu_j}\right)$$
$$= 4m(n-p) - 2LEL(G)^2$$

Since $S \geq 0$,we have $LEL(G) \leq \sqrt{2m(n-p)}$

The equality is attained if and only if $\sqrt{\mu_i} = \sqrt{\mu_j}$, for all $i,j = 1,2,..,n-p$, and then from above we conclude that G has at most two distinct Laplacian eigenvalues

(i) $\mu_1 = \cdots = \mu_{n-p} = \frac{2m}{n-p} (n \neq p)$;

(ii) $\mu_{n-p+1} = \cdots = \mu_n = 0$.

If $m \neq 0$, then G has exactly two distinct Laplacian eigenvalues. A connected graph has curtly two distinct Laplacian eigenvalues if and only if Its diameter is equal to unity, i. e., if it is a complot graph.

If $n = p$ or $m = 0$, then G is regular of degree 0.

The following lemma will be used in next proposition.

Lemma (3.1.4)[121]: [130] If G has at lease one edge, then $\mu_1 \geq \Delta + 1$ (Δ is the greatest vertex degree in G). For G being a connected graph on $n > 1$ vertices, equality is attained if and ouly if $\Delta = n - 1$.

Proposition (3.1.5)[121]: If G has at least one edge, then

$$LEL(G): \sqrt{\Delta + 1}\sqrt{(n-p-1)(2m\Delta - 1)},$$

Proof: Using the Cauchy-Schwarz inequality

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right), \tag{7}$$

which holds for arbitrary real-valued numbers $a_i, b_i, i = 1,2 \dots, n$, we have

$$\left(\sum_{i=1}^{n-p} \sqrt{\mu_i}\right)^2 \leq (n-p-1)\left(\sum_{i=1}^{n-p} \mu_i\right)$$

(choosing in (7) $a_i = \sqrt{\mu_i}$, and $b_i = 1$).

$$\left(LEL(G) - \sqrt{\mu_i}\right)^2 \leq (n - p - 1)(2m - \mu_1).$$

Thus $LEL(G) \leq \sqrt{\mu_i} + \sqrt{(n-p-1)(2m - \mu_1)}$.

Since $\mu_1 \geq \Delta + 1$ $(m \neq 0)$, where Δ is the greatest vertex degree of G.By direct analysis we verify that the function $f(x) = x + \sqrt{(n-1)(2m - x^2)}$ monotonically decreases in the interval$\left(\frac{\sqrt{2m}}{n}, \sqrt{2m}\right)$, for both $\sqrt{\Delta + 1}$ and $\sqrt{\mu_1}$

belong to this interval, and therefore, We have

$$LEL(G) \leq S\sqrt{\Delta + 1} + \sqrt{(n-p-1)(2m - \Delta - 1)}. \tag{8}$$

Proposition (3.1.6)[121]: $\sqrt{2m} \leq LEL(G) \leq \sqrt{2m}$, the right equality is attained if and only if $G = rK_2 \cup (n - 2r)K_1$, where $0 \leq r \leq \left[\frac{a}{2}\right]$, $[x]$ is the integral part of x, while $LEL(G) = \sqrt{2m} = \sqrt{2}m$ if and only if $G = rK_2 \cup (n-2r)K_1$, $r = 0,1$.

Proof: (i) Let p be the number of components of $G(n, m)$, then

$$LEL(G) = \sum_{i=1}^{n-p} \sqrt{\mu_i}$$

Therefore

$$\left(LEL(G)\right)^2 = \sum_{i=1}^{n-p} \left(\sqrt{\mu_i}\right)^2 + 2\sum_{i \neq j} \sqrt{\mu_i}\sqrt{\mu_j} \geq \sum_{i=1}^{n-p} \mu_i = 2m.$$

The left equality is analysis if and only if $\mu_1 = \ldots = \mu_{n-p} = 0$ or $\mu_1 > 0$ and $\mu_2 = \ldots = \mu_{n-p} = 0$, i.e., if G is regular of degree 0 or $G = K_2 \cup (n-2)K_1$,

(ii) Since $LEL(G) \leq \sqrt{2m(n-p)}$ (Proposition (3.1.3)), and $n - p = m$, where $p \geq 1$, we obtain $LEL(G) \leq \sqrt{2m}$. Note that Proposition (3.1.3) and $n - p = m$ if and only if G is a forest. we obtain that $LEL(G) = \sqrt{2m}$ if and only if $G = rK_2 \cup (n-2r)K_1$, where $0 \leq r \leq \left[\frac{n}{2}\right]$

Combining (i) and (ii), we complete the proof of Proposition (3.1.6).

Now, we study the relation among the iterated line graphs of G.

The line graph of G will be denoted by $L(G)$. The iterated line graphs of G are then defined recursively as $L^2\{G) = L(L(G)), L^3(G) = L(L^2(G)) \dots,$

$L^k(G) = L\left(L^{k-1}(G)\right), \dots$ It is consistent to set $L(G) \equiv L^1(G)$ and $G \equiv L^0(G)$.

The line graph $L(G)$ of a regular graph G is a regular graph. Let n_i and r_i denote the order and degree of $L^1(G)$ respectively, $t = 1, 2, \dots, k$. Then (see [122], [142])

$$n_k = \frac{1}{2} r_{k-1} n_{k-1} \quad and \quad r_k = 2r_{k-1} - 2 \dots \dots$$

Therefore,

$$r_k = 2^k r_0 - 2^{k+1} + 2 \tag{9}$$

and

$$n_k = \frac{n_0}{2^k} \prod_{i=0}^{k-1} r_i = \frac{n_0}{2^k} \prod_{i=0}^{k-1} \left(2^i r_0 - 2^{i+1} + 2\right) \tag{10}$$

Proposition (3.1.7)[121]: Let G be a regular graph of order n_0 and of degree r_0 then

$$LEL\left(L^k(G)\right) = LEL\left(L^{k-1}(G)\right) + \sqrt{2r_{k-1}}(n_k - n_{k-1}). \tag{11}$$

Proof: Let $C_G(\mu)\left(\text{ or } C_{L(G)}(\mu)\right)$ be the Laplacian characteristic polynomial of $G(\text{ or } L(G))$, and let $P_G(\lambda)$ $\left(\text{or } P_{L(G)}(\lambda)\right)$ be the characteristic polynomial of the adjacency matrix of G $(\text{or } L(G))$.

It is well known that

$$P_{L(G)}(\lambda) = (\lambda + 2)^{n_1 - n_0} P_G(\lambda + 2 - r_0)$$

and

$$C_G(\mu) = (-1)^{n_0} P_G(-\mu + r_0) \tag{12}$$

Then by equation (12) we have

$$C_{L(G)}(\mu) = (-1)^{n_1} P_{L(G)}(-\mu + (2r_0 - 2)) \tag{13}$$

Combating (11) and (13), we get $C_{L(G)}(\mu) = (\mu - 2r_0)^{n_1 - n_0} C_G(\mu)$.

Therefore, the Laplacian spectrum of $L(G)$ is

$$\begin{pmatrix} 2r_0 & \mu_1 \mu_2 & \dots & \mu_{n_0} \\ n_1 - n_0 & 1\,1 & \dots & 1 \end{pmatrix}$$

where $\mu_1 \geq \mu_3 \geq \dots \geq \mu_{n_0}$ is the Laplacian spectrum of G.

In an analogous manner as above, we have the Laplacian spectrum of $L^2(G)$

$$\begin{pmatrix} 2r_0 & 2r_0 & \mu_1 \mu_2 & \dots & \mu_{n_0} \\ n_2 - n_1 & n_1 - n_0 & 1\,1 & \dots & 1 \end{pmatrix}$$

then for the Laplacian spectrum of $L^k(G)$

$$\begin{pmatrix} 2r_{k-1} & 2r_{k-2} & \dots & 2r_0 & \mu_1\mu_2 \dots \mu_{n_0} \\ n_k - n_{k-1} & n_{k-1} - n_{k-2} & \dots & n_1 - n_0 & 1\ 1\ \dots\ 1 \end{pmatrix}$$

for all $k = 1,2,\dots$

Therefore $LEL\left(L^k(G)\right) = LEL\left(L^{k-1}(G)\right) + \sqrt{2r_{k-1}}(n_k - n_{k-1})$.

The proof is complete.

Now, we would like to give a pair of non-cospcctral graphs of the same order, having equal LEL-energies. Let $G_1 = K_{1.7}$ be a star of order 8. Then the Laplacian eigenvalues of G_1 arc $\mu_{11} = 8, \mu_{12} = \mu_{13} = \dots = \mu_{17} = 1, \mu_{18} = 0$. Let $G_2 = 2K_2 \cup K_4$ be another graph of order 8. Then the Laplacian it is straightforward to check that $LEL(G_1) = LEL(G_2)$.

We point out the dissimilarities between $E(G), LE(G)$, and $LEL(G)$.

Dissimilarity (3.1.8)[121]: In Proposition (3.1.2), we can see that $LEL(G)$ and $E(G)$ preserve the three elementary properties (a), (b), (c), and also they have the same square sum by equality (6). This is the advantage of $LEL(G)$ over $LE(G)$. Since the ordinary energy $E(G)$ has a long known application in molecular-orbital theory of organic molecules (see [126]-[128]), we preconceive that $LEL(G)$ would also have some chemical application.

Dissimilarity (3.1.9)[121]: If the graph G is regular of degree k, then $LE(G) = E(G)$, while $LEL(G) = \sum_{i=1}^{n} \sqrt{k - \lambda_{n-i+1}}$ differs from $E(G)$. This is an advantage of $LE(G)$ over $LEL(G)$. However. if $k = 0$ or $G = K_4$, then we have $E(G) = LE(G) = LEL(G)$. In addition, for a regular graph G. $LEL(G)$ satisfies Proposition (3.1.7) as above.

There are numerous known results (especially lower and upper bounds) that are obtained by using the relations (1) and that depend on the parameters n and m. Then one could expect analogous results for LEL, obtained by means of the relations (2), that would depend on the parameters n, m and d_i.

We point out a few more (n, m)-type new bounds for $LEL(G)$. Furthermore, we prove that for all simple graphs with n vortices. The complete graph K_n has the maximum $LEL(G)$.

In Proposition (3.1.3) we proved that,

$$LEL(G) \leq \sqrt{2m(n - p)}, \tag{14}$$

We now show that the right-hand side expression in (14) is a decreasing function of the parameter p, then we have,

Theorem (3.1.10)[121]:

For any graph $G, LEL(G) \leq \sqrt{2m(n-1)}$. Equality holds if and only if $G = K_n$,

Proof: We consider the function

$$f(x) = \sqrt{2m(n-x)} \quad 1 \leq x \leq n.$$

Then

$$f(x) = \frac{-m}{\sqrt{2m(n-x)}} \leq 0 \quad 1 \leq x \leq n.$$

Because the upper bound (14) increases with decreasing p, by setting $p = 1$ we obtain the estimate

$$LEL(G) \leq \sqrt{2m(n-1)}, \tag{15}$$

which holds for all graphs G. And combining with the Proposition (3.1.3), we have that the equality holds if and only if $G = K_n$.

Let $f\ (m) = \sqrt{2m(n-1)}\quad 0 \le 2m \le n(n-1)$. Obviously, $f(m)$ is an increasing function of the parameter m, then we have proved,

Theorem (3.1.11)[121]:

Let G be a simple graph of order n, then $LEL(G) \le (n-1)\sqrt{n}$. Equality holds if and only if $G = K_n$, i.e., I.he graph of order n with maximum LEL is K_n.

In Proposition (3.1.5) we proved,

$$LEL(G) \le \sqrt{d+1}\sqrt{(n-p-1)(2m-d_1-1)}. \tag{16}$$

Similar to the proof of inequality (15), we now show that the right-hand side expression in (16) is a decreasing function of the parameter p.Then the following result holds immediately,

Theorem (3.1.12)[121]: If G has at least one edge. then $LEL(G) \le \sqrt{d+1}\sqrt{(n-2)(2m-d_1-1)}$. Equality holds if and only if $G = K_n$.

Proof: We consider the function

$$f(x) = \sqrt{d+1}\sqrt{(n-2)(2m-d_1-1)}\quad 1 \le x \le n.$$

Then

$$f'(x) = \frac{d_1+1-2m}{\sqrt{(n-2)(2m-d_1-1)}}\quad 1 \le x \le n.$$

The derived function $f'(x) \le 0$ if and only if $d_1 + 1 \le 2m$, which holds for any graph G has at least one edge.

Because the upper bound (16) increases with decreasing p. By setting $p = 1$ we obtain the estimate

$$LEL(G) \le \sqrt{d+1}\sqrt{(n-2)(2m-d_1-1)} \tag{17}$$

The inequality (17) is sharp. Equality holds if and only if $G = K_n$.

We now show that the bound (17) is better than (15) indeed,

$$\sqrt{d+1} + \sqrt{(n-2)(2m-d_1-1)}\sqrt{2m(n-1)}$$

holds if and only if

$$(n-2)(2m-d_1-1) \le \left(\sqrt{2m(n-1)}\sqrt{d_1+1}\right)^2$$

i.e.,

$$(2m+)(n-1)(d_1+1) \le \sqrt{2m(n-1)(d_1+1)},$$

which is directly transformed into

$$\left(\sqrt{2m}\sqrt{(n-1)(d_1+1)}\right)^3 \ge 0$$

and holds for any m, n, d_1. The equality holds if and only if $2m = (n-1)(d_1+1) = (n-1)d_1 + (n-1)$. Since $2m = \sum_{i=1}^{n} (n-1)d_1 + (n-1)$, hence $G = K_n$, i.e., the equality holds if and only if $G = K_n$.

We present sonic bounds for $LEL(G)$ which depend on the vertex degrees, and we show that for all connected graphs with n vertices. the star $K_{1,n-1}$ has the minimal $LEL(G)$.

Lemma (3.1.13)[121]: [144] If G is a connected graph on $n > 2$ vertices, then $\mu_2 \ge d_2$

Theorem (3.1.14)[121]: If G is a connected graph on $n > 2$ vertices, then $LEL(G) \geq \sqrt{d_1+1}\sqrt{d_2}$.
Equality is attained if and only if $G = P_3$ (P_a is the path of order n).
Proof: It is easy to see from the Lemma (3.1.4) and Lemma (3.1.13) that $LEL(G) \geq \sqrt{d_1+1}\sqrt{d_2}$, equality is attained if and only if $\mu_1 = d_1 + 1, \mu_2 = d_2,\ \mu_3 =\ldots= \mu_0 = 0$. Since G is a connected graph, so we have $p = 1$, this implies $n = 3$. From Lemma (3.1.4), we have $d_1 = 2, \mu_1 = 3$. Since $\sum_{i=1}^{3} \mu_i, = \sum_{i=1}^{3} d_i$, we have $(d_1 + 1) + d_2 + 0 = d_1 + d_2 + d_3$, thus $d_3 = 1$ and then $d_2 = 1$. Therefore $G = P_3$.

Now, we will give another lower bound of $LEL(G)$ for the connected graphs.

Let $a = (a_1, a_2, \ldots, a_n),\ a_k \geq 0, 1 \geq k \geq n$, then $A_n(a) = \frac{1}{n}\sum_{k=1}^{n} a_k$ is called the algebraic average value of $a_1, a_2, \ldots, a_n$, $G_n(a) = \sqrt[n]{a_1 a_2, \ldots, a_n}$ is called the geometry average value of $a_1, a_2, \ldots, a_n$. It is well known that,
Lemma (3.1.15)[121]: [143]

$$G_n(a) \leq A_n(a) \tag{19}$$

Equality holds if and only if $a_1 = a_2 = \ldots = a_n$.
Let G be a connected graph and let $t(G)$ denote the number of spanning trees contained in G.
Lemma (3.1.16)[121]: [124] Let G be a connected multigraph on n vertices, then $t(G) = \frac{1}{n}\prod_{i=1}^{n-1} \mu_i$.

It is easy to see that if G is connected then $t(G) \geq 1$. Thus we prove the following result.
Theorem (3.1.17)[121]: Let G be a connected simple graph on n vertices, then $LEL(G) \geq \sqrt{n} + (n-2)$. Equality holds if and only if $G = K_{1,n-1}$, i.e., the connected simple graph of order n with minimal LEL is $K_{1,n-1}$.
Proof: Using inequality (19), we have

$$\frac{\sqrt{\mu_2}+\sqrt{\mu_3}+\cdots+\sqrt{\mu_{n-1}}}{n-2} \geq \sqrt[2(n-2)]{\mu_2\mu_3 \ldots \mu_{n-1}}.$$

equality holds if and only if $\mu_1 = \mu_3 = \cdots = \mu_{n-1}$.

It is well known that $\mu_1 \leq n$. So

$$\prod_{i=2}^{n-1} \mu_i > t(G) \geq 1,$$

the first equality holds if and only if $\mu_1 = n$, the second equality holds if and only if G is a tree.
Hence $\sqrt{\mu_2} + \sqrt{\mu_3} + \ldots + \sqrt{\mu_{n-1}} \geq n-2$.
Therefore

$$LEL(G) = \sum_{i=1}^{n} \sqrt{\mu_i} = \sqrt{\mu_1} + \left(\sqrt{\mu_2}+\ldots+\sqrt{\mu_{n-1}}\right) \geq \sqrt{\mu_1} + (n-2),$$

equality holds if and only if $\mu_1 = n$ and $\mu_2 = \mu_3 = \cdots = \mu_{n-1}$ and G is a tree, which implies that $G = K_{1,n-1}$ and $LEL(G) \geq \sqrt{n} + (n-2)$

Now, we will give upper bounds of $LEL(G)$ for the connected graphs.

Definition (3.1.18)[121]: [130] If vector $(a) = (a_1, a_2, \ldots, a_r)$ and $(b) = (b_1, b_2, \ldots, b_s)$ are nonincreasing sequences of real numbers, then (a) majorizes (b) if

$$\sum_{i=1}^{k} a_i \geq \sum_{i=1}^{k} b_i \quad k = 1,2,\ldots,\min\{r,s\}$$

and

$$\sum_{i=1}^{r} a_i \equiv \sum_{i=1}^{s} b_i.$$

we denote it by $b \prec a$.

Definition (3.1.19)[121]: [146] The relation $x \prec\prec y$ means that $x \prec y$ and x is not the rearrangement of y.

Definition (3.1.20)[121]: [143] A real valued function $f(x)$ defined on a convex set D is said to be convex if

$$f(\lambda + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y),$$

for all $0 \leq \lambda \leq 1$ and all $x, y \in D$. If the above inequality is always strict for $0 < \lambda < 1$ and $x \neq y$, then f is called strictly convex. If f is a convex function, then f is called concave.

Lemma (3.1.21)[121]: [146] Let $(x) = (x_1, x_3, \ldots, x_n)$ be majorized by $(y) = (y_1, y_2, \ldots, y_n)$, i.e., $x \prec y$, then for any convex function φ_i, the following inequality holds,

$$\sum_{j=1}^{n} \varphi(x_j) \leq -\sum_{j=1}^{n} \varphi(y_j),$$

Lemma (3.1.22)[121]: [146] Let $x \prec\prec y$ then for any strictly convex function φ, the following inequality holds,

$$\sum_{j=1}^{n} \varphi(x_j) < \sum_{j=1}^{n} \varphi(y_j).$$

For convenience. letting (d) denote the nonincreasing sequence $(d) = (d_1 + 1, d_2, \ldots, d_{n-1}, d_n - 1)$ of vertex degrees and letting (μ) denote the non-increasing sequence$(\mu) = (\mu_1, \mu_2, \ldots, \mu_{n-1}, \mu_n)$ of nonnegative real Laplacian eigenvalues.

Lemma (3.1.23)[121]: [142] Let G be a connected graph on $n \geq 2$ vertices, then (d) is majorized by (μ).

Theorem (3.1.24)[121]: Let G be a connected graph on $n \geq 2$ vertices, then $LEL(G) \leq \sqrt{d_1 + 1} + \sqrt{d_2} + \cdots + \sqrt{d_{n-1}} + \sqrt{d_n - 1}$, where the equality holds if and only if $(d) = (\mu)$.

Proof: Let $\varphi(x) = -\sqrt{x}, x \in (0, +\infty)$, then $\varphi(x)$ is a convex function. Since (d) is majorized by (μ), using Lemma (3.1.21) we have,

$$\left(-\sqrt{d_1 + 1}\right) + \left(-\sqrt{d_2}\right) + \cdots + \left(-\sqrt{d_{n-1}}\right) + \left(-\sqrt{d_n - 1}\right)$$
$$\leq \left(-\sqrt{\mu_1}\right) + \left(-\sqrt{\mu_2}\right) + \cdots + \left(-\sqrt{\mu_{n-1}}\right) + \left(-\sqrt{\mu_n - 1}\right)$$

which is directly transformed into

$$\left(\sqrt{\mu_1}\right)+\left(\sqrt{\mu_2}\right)+\cdots+\left(\sqrt{\mu_{n-1}}\right)+\left(\sqrt{\mu_n-1}\right)$$
$$\leq\left(\sqrt{d_1+1}\right)+\left(\sqrt{d_2}\right)+\cdots+\left(\sqrt{d_{n-1}}\right)+\left(\sqrt{d_n-1}\right)$$

i.e.,

$$LEL(G)\leq\sqrt{d_1+1}+\sqrt{d_2}+\cdots+\sqrt{d_{n-1}}+\sqrt{d_n-1} \qquad (20)$$

And it is easy to see from Lemma (3.1.22) that the equality holds if and only if $(d) = (\mu)$.

Remark (3.1.25)[121]: We show that the bounds (15) and (20) are also incomparable. Let H_1, H_2 be the graphs shown in Fig. (1).

Fig. (1)[121]:

Then for $G = H_1$ the upper bound (15) is better than (20). On the other hand, if $G = H_2$ then the upper bound (20) is better than (15).

We now discuss the case $LEL(G) = \sqrt{d_1+1}+\sqrt{d_2}+\cdots+\sqrt{d_{n-1}}+\sqrt{d_n-1}$.

Lemma (3.1.26)[121]: [140] If an isolated vertex is connected by edges to all the vertices of a graph G of order n, then the Laplacian eigenvalues of the resultant graph are as follows: one of the eigenvalues is $n+1$, the other eigenvalues can be obtained by incrementing the eigenvalues of the old graph G by 1 except the lowest one and 0 as another eigenvalue.

Example (3.1.27)[121]: Let G_1, G_2 be the graphs shown in Fig. (2). The Laplacian spectrum of G_1 is (3, 1, 0). We want to find out the spectrum of G_2.

Fig. (2)[121]:

Applying Lemma (3.1.26), we can easily get the spectrum of G_2 is (4, 4, 2, 0).

Theorem (3.1.28)[121]: Let G be a connected graph on $n\geq 2$ vertices, then $(d) = (\mu)$ if and only if $G = K_{1,n-1}$.

Proof: If $G = K_{1,n-1}$ then (d) = (μ).

Conversely, let $(d) = (\mu)$, we are to show that G is a star. If $(d_1+1, d_2, \cdots, d_{n-1}, d_n-1) = (\mu_1, \mu_2, \cdots, \mu_{n-1}, \mu_n)$, then we have $\mu_1 = d_1+1$.

Since G is a connected graph, using Lemma (3.1.4), we have $d_1 = n-1$.

Let $G' = (V', E')$ be a graph with vertex set $V' = \{v_1, v_2, \dots, v_{n-1}\}$, and edge set $E' \neq \emptyset$, let $a_1 \geq a_3 \geq \cdots \geq a_{n-1} = 0$ be the Laplueian eigenvalues of G', and let $b_1 \geq b_2 \geq \cdots \geq b_{n-1}$ be the non increasing vertex degrees of G'. Let G be a graph obtained from C' by adding a new vertex v_{11}, which is connected by edges to all the vertices of G'.

Applying Lemma (3.1.26), the Laplacian spectrum of G is $(n, a_1 + 1, a_2 + 1, \cdots a_{n-2} + 1, 0)$, which we denoted it by$(\mu)$. If$(\mu) = (d) = (d_1 + 1, d_2, \cdots, d_{n-1}, d_n - 1)$ then $d_1 = n - 1$, $d_2 = b_1 + 1 = n_1 + 1$. Hence $b_1 = a_1$. Since $E' \neq \emptyset$, using Lemma (3.1.4), we have $a_1 \geq b_1 + 1$, a contradiction. Thus $E' = 0$, which implies

According to Theorem (3.1.24) and Theorem (3.1.28) and noting $(d) \prec (\mu)$ we obtain the following theorem.

Theorem (3.1.29)[121]: Let G be a connected graph on $n \geq 2$ vertices, then $LEL(G) = \sqrt{d_1 + 1} + \sqrt{d_2} + \cdots + \sqrt{d_{n-1}} + \sqrt{d_n - 1}$ if and only if $C = K_{1,n-1}$

Now we give another upper bound depended on the vertex degrees.

Theorem (3.1.30)[121]: Let G be a connected graph on $n \geq 2$ vortices, then

$$LEL(G) \leq \frac{1}{2}\left(\sqrt{n-1} + \sqrt{\frac{1}{n-1}}\right)\left(\sum_{i=1}^{n} \sqrt{d_i}\right)$$

Proof: Using Cauchy-Schwarz inequality, we have

$$\left(\sum_{i=1}^{n} \sqrt{\mu_i}\right)^2 \leq n\left(\sum_{i=1}^{n} \mu_i\right)$$

On the other hand, since G is a connected graph on $n \geq 2$ vertices, thus $1 \leq d_i \leq n - 1$ for any integer $1 \leq i \leq n$. Then by Polya-Szegö inequality, we have

$$n\left(\sum_{i=1}^{n} d_i\right) \leq \frac{1}{4}\left(\sqrt{n-1} + \sqrt{\frac{1}{n-1}}\right)^2\left(\sum_{i=1}^{n} \sqrt{d_i}\right)^2$$

Since $n\left(\sum_{i=1}^{n} \mu_i\right) = n\left(\sum_{i=1}^{n} d_i\right)$ hence

$$LEL(G) \leq \frac{1}{2}\left(\sqrt{n-1} + \sqrt{\frac{1}{n-1}}\right)\left(\sum_{i=1}^{n} \sqrt{d_i}\right) \quad (21)$$

This completes the proof of the theorem.

Remark (3.1.31)[121]: Using Cauchv-Schwarz inequality, we have

$$\left(\sum_{i=1}^{n} \sqrt{d_i}\right) \leq \sqrt{n\left(\sum_{i=1}^{n} \sqrt{d_i}\right)} = \sqrt{2mn}.$$

Combining with inequality (21), we obtain

$$LEL(G) \leq \frac{1}{2}\left(\sqrt{n-1} + \sqrt{\frac{1}{n-1}}\right)\sqrt{2mn}. \quad (22)$$

Unfortunately, the bound (22) is not better than the bound (15). In fact, by direct calculation, $\sqrt{2m(n-1)} \le \frac{1}{2}\left(\sqrt{n-1}+\sqrt{\frac{1}{n-1}}\right)\sqrt{2mn}$ if and only if $n^3 - 4n^2 + 8n - 4 \ge 0$,where the inequality always holds for any integer number n.

Section (3.2): Laplacian Eigenvalues of Graphs

Let G be a simple finite undirected graph with vertex set $V(G)$. Let $A(G)$ be the $(0,1)$ adjacency matrix of G and $D(G)$ the diagonal matrix of vertex degrees. Then $L(G) = D(G) - A(G)$ is called the Laplacian matrix of G. It is symmetric, positive semidefinite and singular. The Laplacian eigenvalues of G are the eigenvalues of $L(G)$. Let $\mu_1, \mu_2, \ldots, \mu_n$ be the Laplacian eigenvalues of G arranged in a non-increasing manner, where $n = |V(G)|$. When more than one graph is under discussion, we write $\mu_i(G)$ instead of μ_i. It is known that $\mu_n = 0$ and the multiplicity of 0 is equal to the number of connected components of G. Let α be a non-zero real number. Let G be a graph with n vertices. Let $s_\alpha(G)$ be the sum of the αth power of the non-zero Laplacian eigenvalues of G, i.e.,

$$s_\alpha(G) = \sum_{i=1} \mu_i^\alpha ,$$

where h is the number of non-zero Laplacian eigenvalues of G. The case $\alpha = 1$ is trivial as $s_1(G) = 2m$, where m is the number of edges. Some properties for s_2 were established in [154], where Lazic called it the Laplacian energy of the graph. Recall that the energy of a graph is equal to the sum of the absolute values of its ordinary eigenvalues [127] and that an energy like quantity was proposed and studied in [129] based on the Laplacian eigenvalues. Some properties of $s_{\frac{1}{2}}$ were given in [121]. We also note that for a connected graph G with n vertices, $ns_{-1}(G)$ is equal to its Kirhhoff index or quasi-Wiener index, which found applications in electric circuit, probabilistic theory and chemistry [152], [157].

We establish some properties for s_α, where α is a real number with $\alpha \ne 0, 1$. We also discuss further properties for s_2 and $s_{\frac{1}{2}}$.

Let K_n and P_n be respectively the complete graph and the path on n vertices. Let $K_{a,b}$ be the complete bipartite graph with two partite sets having a and b vertices, respectively. We need some properties of the Laplacian eigenvalues. For more details, see [130], [145]. Let $\bar{G}$ be the complement of the graph G with n vertices. The Laplacian eigenvalues of $\bar{G}$ are $n - \mu_{n-1}(G), n - \mu_{n-2}(G), \ldots, n - \mu_1(G), 0$.

Lemma (3.2.1)[147]: [145]. Let G be a non-complete graph with n vertices. If G^* is obtained from G by adding an edge, then

$$\mu_1(G^*) \ge \mu_1(G) \ge \mu_2(G^*) \ge \mu_2(G) \cdots$$
$$\ge \mu_{n-1}(G^*) \ge \mu_{n-1}(G) \ge \mu_n(G^*) = \mu_n(G) = 0.$$

Lemma (3.2.2)[147]: [149]. Let G be a graph with at least one edge and maximum vertex degree Δ. Then

$$\mu_1 \ge 1 + \Delta$$

with equality for connected graph if and only if $\Delta = n - 1$.

Lemma (3.2.3)[147]: [130]. Let G be a connected graph with diameter d. Then G has at least $d + 1$ distinct Laplacian eigenvalues.

Lemma (3.2.4)[147]: Let G be a graph with n vertices. Then $\mu_1 = \cdots = \mu_{n-1}$ if and only if $G \cong K_n$ or $G \cong K_n$.

Proof: Suppose that $\mu_1 = \cdots = \mu_{n-1}$. If G is connected, then by Lemma (3.2.3), $G \cong K_n$. If G is not connected, then $\mu_{n-1} = 0$ and so all Laplacian eigenvalues are equal to zero, which obviously implies that $G \cong K_n$. Conversely, it is easily seen that $\mu_1 = \cdots = \mu_{n-1}$ if $G \cong K_n$ or $G \cong K_n$.

Lemma (3.2.5)[147]: Let G be a connected graph with $n \geq 2$ vertices. Then $\mu_2 = \cdots = \mu_{n-1}$ and $\mu_1 = 1 + \Delta$ if and only if $G \cong K_n$ or $G \cong K_{1,n-1}$.

Proof: Suppose that $\mu_2 = \cdots = \mu_{n-1}$ and $\mu_1 = 1 + \Delta$. By Lemma (3.2.2), $\Delta = n - 1$. Then $\bar{G}$ has an isolated vertex, say v, and the Laplacian eigenvalues of $\bar{G} - v$ are $n - \mu_{n-1}, \ldots, n - \mu_2, 0$. By Lemma (3.2.4), $\overline{G} - v \cong K_{n-1}$ or $\overline{G} - v \cong \overline{K_{n-1}}$. Thus $G \cong K_n$ or $G \cong K_{1,n-1}$.

Conversely, it is easy to see that $\mu_2 = \cdots = \mu_{n-1}$ and $\mu_1 = 1 + \Delta \; if \; G \cong K_n$ or $G \cong K_{1,n-1}$.

For a graph G, let $Z(G) = \sum_{u \in V(G)} d_u^2$, where d_u stands for the degree of vertex u in G.

Lemma (3.2.6)[147]: [153]. Let G be a connected bipartite graph with n vertices. Then $\mu_1 \geq 2\sqrt{\frac{Z(G)}{n}}$ with equality if and only if G is a regular bipartite graph.

The subdivision graph $S(G)$ of a graph G is obtained by inserting a new vertex (of degree 2) on each edge of G. The ordinary spectrum of a graph G is the spectrum of its adjacency matrix.

Lemma (3.2.7)[147]: [158]. Let G be a bipartite graph with n vertices and m edges. If the non-zero Laplacian eigenvalues of G are$\mu_i, i = 1, \ldots, h$,then the ordinary spectrum of $S(G)$ consists of the numbers $\pm\sqrt{\mu_i}, i = 1, \ldots, h$, and of $n + m - 2h$ zeros.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the ordinary eigenvalues of the graph G, where $n = |V(G)|$. Then the energy of G is defined as [127], [150]

$E(G) = \sum_{i=1}^{n} |\lambda_i|$.

Lemma (3.2.8)[147]: [156], [134]. Let G be a graph with n vertices, $m \geq 1$ edges and q quadrangles. Then

$$E(G) \geq \sqrt{\frac{(2m)^3}{2Z(G) - 2m + 8q}}$$

with equality if and only if G is the vertex-disjoint union of $K_{a_1,b_1}, \ldots, K_{a_r,b_r}$ with $a_1 b_1 = \cdots = a_r b_r$ and $r \geq 1$, and isolated vertices.

It is obvious that for any graph G with n vertices, $s_\alpha(G) \geq 0$ for $\alpha \neq 0$ with equality if and only if $G \cong \overline{K_n}$.

Theorem (3.2.9)[147]: (i) For any non-complete graph G, if G^* is obtained from G by adding an edge, then $s_\alpha(G) < s_\alpha(G^*)$ for $\alpha > 0$ and $s_\alpha(G) > s_\alpha(G^*)$ for $\alpha < 0$.

(ii) For any graph G with n vertices

$s_\alpha(G) \leq (n-1)n^\alpha \quad if \quad \alpha > 0,$

$s_\alpha(G) \geq (n-1)n^\alpha \quad if \quad \alpha < 0$

with either equality if and only if G is the complete graph K_n.

Proof: Note that $\sum_{i=1}^{n-1} \mu_i(G^*) - \sum_{i=1}^{n-1} \mu_i(G) = 2$. By Lemma (3.2.1), the result in (i) follows. Note that $\mu_1(K_n) = \cdots = \mu_{n-1}(K_n) = n$ and $\mu_n(K_n) = 0$. From (i), we have (ii).

Theorem (3.2.10)[147]: Let α be a real number with $\alpha \neq 0, 1$, and let G be a connected graph with $n \geq 3$ vertices, t spanning trees and maximum vertex degree Δ. Then

$$s_\alpha(G) \geq (1+\Delta)^\alpha + (n-2)\left(\frac{tn}{1+\Delta}\right)^{\frac{\alpha}{n-2}} \tag{23}$$

with equality if and only if $G \cong K_n$ or $G \cong K_{1,n-1}$.

Proof: By the matrix-tree theorem (see [145]), $\prod_{i=1}^{n-1} \mu_i = tn$. By the arithmetic–geometric mean inequality

$$s_\alpha(G) = \mu_1^\alpha + \sum_{i=2}^{n-1} \mu_i^\alpha \geq \mu_1^\alpha + (n-2)\left(\prod_{i=2}^{n-1} \mu_i^\alpha\right)^{\frac{1}{n-2}} = \mu_1^\alpha + (n-2)\left(\frac{tn}{\mu_1}\right)^{\frac{\alpha}{n-2}}$$

with equality if and only if $\mu_2 = \cdots = \mu_{n-1}$. Let $(x) = x^\alpha + (n-2)\left(\frac{tn}{x}\right)^{\frac{\alpha}{n-2}}$. By solving $f'(x) = \alpha\left(x^{\alpha-1} - (tn)^{\frac{\alpha}{n-2}} x^{-\frac{\alpha}{n-2}-1}\right) \geq 0$, it may be easily seen that $f(x)$ is increasing for $x \geq (tn)^{\frac{1}{n-1}}$ whether $\alpha > 0$ or $\alpha < 0$. Obviously, $2m \leq n\Delta \leq (n-1)(1+\Delta)$. By Lemma (3.2.2)

$$\mu_1 \geq 1+\Delta \geq \frac{2m}{n-1} = \frac{\sum_{i=1}^{n-1} \mu_i}{n-1} \geq \left(\prod_{i=1}^{n-1} \mu_i\right)^{\frac{1}{n-1}} = (tn)^{\frac{1}{n-1}}$$

and then $s_\alpha(G) \geq f(1+\Delta) = (1+\Delta)^\alpha + (n-2)\left(\frac{tn}{1+\Delta}\right)^{\frac{\alpha}{n-2}}$. Hence (23) follows, and equality holds in (23) if and only if $\mu_2 = \cdots = \mu_{n-1}$ and $\mu_1 = 1 + \Delta$, which, by Lemma (3.2.5), is equivalent to $G \cong K_n$ or $G \cong K_{1,n-1}$.

Theorem (3.2.11)[147]: Let G be a connected graph with $n \geq 3$ vertices, m edges and maximum vertex degree Δ:

(i) If $\alpha < 0$ or $\alpha > 1$, then

$$s_\alpha(G) \geq (1+\Delta)^\alpha + \frac{(2m-1-\Delta)^\alpha}{(n-2)^{\alpha-1}} \tag{24}$$

with equality if and only if $G \cong K_n$ or $G \cong K_{1,n-1}$.

(ii) If $0 < \alpha < 1$, then

$$s_\alpha(G) \leq (1+\Delta)^\alpha + \frac{(2m-1-\Delta)^\alpha}{(n-2)^{\alpha-1}} \tag{25}$$

with equality if and only if $G \cong K_n$ or $G \cong K_{1,n-1}$.

Proof: Observe that for $\alpha \neq 0, 1$ and $x > 0, x^\alpha$ is a strictly convex function if and only if $\alpha < 0$ or $\alpha > 1$.

Suppose that $\alpha < 0$ or $\alpha > 1$.Then

$$\left(\sum_{i=2}^{n-1} \frac{1}{n-2}\mu_i\right)^\alpha \leq \sum_{i=2}^{n-1} \frac{1}{n-2}\mu_i^\alpha,$$

i.e.,

$$\sum_{i=2}^{n-1} \mu_i^{\alpha} \geq \frac{1}{(n-2)^{\alpha-1}} \left(\sum_{i=2}^{n-1} \mu_i \right)^{\alpha}$$

with equality if and only if $\mu_2 = \cdots = \mu_{n-1}$. It follows that

$$s_{\alpha}(G) \geq \mu_1^{\alpha} + \frac{1}{(n-2)^{\alpha-1}} \left(\sum_{i=2}^{n-1} \mu_i \right)^{\alpha} = \mu_1^{\alpha} + \frac{(2m - \mu_1)^{\alpha}}{(n-2)^{\alpha-1}}.$$

Let $(x) = x^{\alpha} + \frac{(2m-x)^{\alpha}}{(n-2)^{\alpha-1}}$. By solving $g'(x) = \alpha \left(x^{\alpha-1} - \frac{(2m-x)^{\alpha-1}}{(n-2)^{\alpha-1}} \right) \geq 0$, it is easily seen that $g(x)$is increasing for $\geq \frac{2m}{n-1}$. Note that $(n-1)(1+\Delta) \geq 2m$. By Lemma (3.2.2),$\mu_1 \geq 1 + \Delta \geq \frac{2m}{n-1}$ and then

$$s_{\alpha}(G) \geq g(1+\Delta) = (1+\Delta)^{\alpha} + \frac{(2m - 1 - \Delta)^{\alpha}}{(n-2)^{\alpha-1}}$$

with equality if and only if $\mu_2 = \cdots = \mu_{n-1}$ and $\mu_1 = 1 + \Delta$. By Lemma (3.2.5), equality holds in (24) if and only if $G \cong K_n$ or $G \cong K_{1,n-1}$.

Now suppose that $0 < \alpha < 1$.Then

$$\left(\sum_{i=2}^{n-1} \frac{1}{n-2} \mu_i \right)^{\alpha} \geq \sum_{i=2}^{n-1} \frac{1}{n-2} \mu_i^{\alpha}$$

with equality if and only if $\mu_2 = \cdots = \mu_{n-1}$, and $g(x)$ is decreasing for $x \geq \frac{2m}{n-1}$. By similar arguments as above, the second part of the theorem follows.

We consider bipartite graphs.

Theorem (3.2.12)[147]: Let α be a real number with $\alpha \neq 0, 1$, and let G be a connected bipartite graph with $n \geq 3$ vertices, t spanning trees. Then

$$s_{\alpha} \geq \left(2\sqrt{\frac{Z(G)}{n}} \right)^{\alpha} + (n-2) \left(\frac{tn}{2\sqrt{\frac{Z(G)}{n}}} \right)^{\frac{\alpha}{n-2}} \tag{26}$$

with equality if and only if $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

Proof: By Lemma (3.2.6), we have $\mu_1 \geq 2\sqrt{\frac{Z(G)}{n}} \geq \frac{4m}{n} \geq \frac{2m}{n-1} \geq (tn)^{\frac{1}{n-1}}$. Thus, by similar arguments as in the proof of Theorem (3.2.10), we have $s_{\alpha} \geq f\left(2\sqrt{\frac{Z(G)}{n}} \right)$, from which (26) follows, and equality holds in (26) if and only if $\mu_2 = \cdots = \mu_{n-1}$ and $\lambda_1 = 2\sqrt{\frac{Z(G)}{n}}$.

Suppose that equality holds in (26). Then G is a regular bipartite graph with at most three distinct Laplacian eigenvalues. Thus, by Lemma (3.2.3), G is a regular bipartite graph with at most diameter 2, i.e., $\cong K_{\frac{n}{2},\frac{n}{2}}$.

Conversely, it is easily seen that $\mu_2 = \cdots = \mu_{n-1}, \lambda_1 = 2\sqrt{\frac{Z(G)}{n}}$, and then (26) is an equality if $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

Theorem (3.2.13)[147]: Let G be a connected bipartite graph with $n \geq 3$ vertices, m edges:

(i) If $\alpha < 0$ or $\alpha > 1$, then

$$s_\alpha(G) \geq \left(2\sqrt{\frac{Z(G)}{n}}\right)^\alpha + \frac{\left(2m - 2\sqrt{\frac{Z(G)}{n}}\right)^\alpha}{(n-2)^{\alpha-1}} \qquad (27)$$

with equality if and only if $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

(ii) If $0 < \alpha < 1$, then

$$s_\alpha(G) \leq \left(2\sqrt{\frac{Z(G)}{n}}\right)^\alpha + \frac{\left(2m - 2\sqrt{\frac{Z(G)}{n}}\right)^\alpha}{(n-2)^{\alpha-1}} \qquad (28)$$

with equality if and only if $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

Proof: By Lemma (3.2.6), we have $\mu_1 \geq 2\sqrt{\frac{Z(G)}{n}} \geq \frac{4m}{n} \geq \frac{2m}{n-1}$. Thus, by similar arguments as in the proof of Theorem (3.2.11), we have $s_\alpha \geq g\left(2\sqrt{\frac{Z(G)}{n}}\right)$ for $\alpha < 0$ or $\alpha > 1$, and then (27) follows. Similarly, $s_\alpha \leq g\left(2\sqrt{\frac{Z(G)}{n}}\right)$ for $0 < \alpha < 1$, and then (28) follows.

Either equality in (27) or (28) holds if and only if $\mu_2 = \cdots = \mu_{n-1}$ and $\lambda_1 = 2\sqrt{\frac{Z(G)}{n}}$, which, by the arguments in the proof of Theorem (3.2.12), is equivalent to $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

Now we consider the special case $\alpha = 2$. Note that $s_2(G)$ is equal to the trace of L^2 where $L = L(G)$, from which it may be shown that [154]

$$s_2(G) = \sum_{u \in V(G)} (d_u^2 + d_u) = Z(G) + 2m,$$

where m is the number of edges of G. Thus, if both the number of vertices and the number of edges are given, then the study of $s_2(G)$ is equivalent to that of $Z(G)$.

Let G be a graph with n vertices and m edges. As restatements of the results in [151], [155] on $Z(G)$, respectively, we have

$$s_2(G) \geq 2m\left(\left\lfloor\frac{2m}{n}\right\rfloor + \left\lceil\frac{2m}{n}\right\rceil + 1\right) - \left\lfloor\frac{2m}{n}\right\rfloor\left\lceil\frac{2m}{n}\right\rceil n$$

with equality if and only if any degree of G is either $\left\lfloor\frac{2m}{n}\right\rfloor$ or $\left\lceil\frac{2m}{n}\right\rceil$, and

$$s_2(G) \leq m\left(\frac{2m}{n-1} + n\right)$$

with equality if and only if G is $K_{1,n-1}$ or K_n.

Let G be a connected graph with $n \geq 2$ vertices. It was proved in [154] that

$$s_2(G) \geq 6n - 8$$

with equality if and only if G is the path P_n. An alternate argument is as follows: By Theorem (3.2.9), if $s_2(G) \geq s_2(T)$ with equality if and only if $= T$, where T is a spanning tree of

G. Note that T has at least two vertices of degree one. By the Cauchy–Schwarz inequality, we have

$$s_2(G) \geq s_2(T) = Z(T) + 2(n-1) \geq 2 + \frac{(2(n-1)-2)^2}{n-2} + 2(n-1) = 6n - 8$$

and then $s_2(G) \geq 6n - 8$ with equality if and only if G is a tree that has exactly two vertices of degree one and all other vertices have equal degrees, i.e., G is the path P_n. Finally, we turn to the special case $= \frac{1}{2}$.

Theorem (3.2.14)[147]: Let G be a bipartite graph with n vertices and $m \geq 1$ edges. Then

$$s_{\frac{1}{2}}(G) \geq 2\frac{\sqrt{2}m}{\sqrt{n+2}} \tag{29}$$

with equality if and only if $G \cong K_2$.

Proof: For $u, v \in V(G), u \sim v$ means that u and v are adjacent in G. It follows that $Z(G) = \sum_{u\sim v} (d_u + d_v) \leq \sum_{u\sim v} n = mn$ with equality if and only if $d_u + d_v = n$ for any edge uv of G, i.e., G is a complete bipartite graph. Note that $S(G)$ possesses $2m$ edges, it is quadrangle-free and $Z(S(G)) = Z(G) + 4m$. By Lemma (3.2.8)

$$E(S(G)) \geq \sqrt{\frac{(2 \cdot 2m)^3}{2Z(S(G)) - 2 \cdot 2m}} = \sqrt{\frac{(4m)^3}{2(Z(G) + 4m) - 4m}}$$
$$\geq \sqrt{\frac{(4m)^3}{2(mn + 4m) - 4m}} = 4\frac{\sqrt{2}m}{\sqrt{n+2}}.$$

By Lemma (3.2.7), we have $s_{\frac{1}{2}}(G) = \frac{1}{2}E(S(G))$ and thus (29) follows. From the proof above, equality in (29) if and only if G and $S(G)$ are both complete bipartite graphs, i.e., $G \cong K_2$. The lower bound in Theorem (3.2.14) is asymptotically best possible. For example, let G be the complete bipartite graph $K_{k,k}$ and then $s_{\frac{1}{2}}(G) = \sqrt{2k} + (2k-2)\sqrt{k}$ and the corresponding lower bound is equal to $= \frac{2k^2}{\sqrt{k+1}}$. Obviously, $\lim_{k\to\infty} \frac{s_{\frac{1}{2}}(G)}{c} = \lim_{k\to\infty} \frac{(2k-2+\sqrt{2})\sqrt{k+1}}{2k\sqrt{k}} = 1.$

Chapter 4
A Sot-Dense Path and Common Hypercyclic Functions

We show that each operator along the path has the exact same dense G_δ set of hypercyclic vectors. The operators having that particular set of hypercyclic vectors form a connected subset of the operator algebra with the strong operator topology. We show that there exists a residual set $G \subset H(\mathbb{C})$ such that for every $f \in G$ and every non-zero complex number a the set $\{f(z + \lambda_n a) : n = 1, 2, \ldots\}$ is dense in $H(\mathbb{C})$. This answers in the affirmative and it also provides an extension of a theorem due to Costakis and Sambarino.

Section (4.1): Chaotic Operators with Same Hypercyclic Vectors

Let H be a separable, infinite dimensional Hilbert space over the complex field $\mathbb{C}$, and let $B(H)$ denote the algebra of all bounded linear operators $T : H \to H$. An operator T in $B(H)$ is hypercyclic if there is a vector x in H for which its orbit, $\mathrm{Orb}(T, x) = \{T^n x : n \geq 0\}$, is dense in H. Such a vector x is called a hypercyclic vector for T, and we use the notation $\mathcal{HC}(T)$ to denote the set of hypercyclic vectors for T. The behavior of the orbit of a hypercyclic vector is wild. On the other hand, even when the operator is hypercyclic, the orbit of a certain vector may be finite. A vector x in H is called a periodic point for the operator T if $T^n x = x$ for some positive integer n. The operator T is chaotic if it is hypercyclic and has a dense set of periodic points. Godefroy and Shapiro [10] showed this definition of chaos is equivalent to the notion of chaos proposed by Devaney [165].

Whenever the operator T is hypercyclic, the set $\mathcal{HC}(T)$ of hypercyclic vectors is a dense G_δ set; see Kitai [42]. It follows from the Baire Category Theorem that if we have a countable collection $\{T^n \in B(H): n > l\}$ of hypercyclic operators, then the set $\bigcap_{n=1}^{\infty} \mathcal{HC}(T^n)$ of their common hypercyclic vectors is still a dense G_δ set. Since this argument fails when the collection of operators is uncountable, it becomes interesting to study their common hypercyclic vectors, especially when the operators in the uncountable collection are in fact related by continuity. This leads to the following definition. A collection of operators $\{F_t \in B(H): t \in 1\}$ is a path of operators if the map $F : I \to (B(H), \| \bullet \|)$, defined on an interval I by $F(t) = F_t$, is continuous with respect to the operator norm topology of $B(H)$ and the usual topology of the interval I. The set $\bigcap_{t \in I} \mathcal{HC}(F_t)$ is referred to as the set of common hypercyclic vectors for the whole path, and any vector in the set is called a common hypercyclic vector. If I is $[a, b]$, then the collection $\{F_t \in B(H): t \in 1\}$ is called a path of operators between F_a and F_b.

Many results on common hypercyclic vectors were obtained. Leon-Saavedra and Müller [43] showed that every operator in the path of rotations $\{e^{i\theta}T : \theta \in [0, 2\pi]\}$ of a single hyper cyclic operator T has the exact same hypercyclic vectors as the operator T itself. Later, Conejero, Müller, and Peris [37] studied common hypercyclic vectors for a semigroup of operators. The first example of a specific class of operators with common hypercyclic vectors is perhaps the unilateral weighted backward shifts. We provide a formal definition here.

Definition (4.1.1)[159]: An operator C in $B(H)$ is called a unilateral weighted back ward shift if there is an orthonormal basis $\{e_0, e_1, e_2, \ldots\}$ and a sequence of nonzero scalars $\{w_j : j \geq 1\}$ in C such that $Be_0 = 0$ and $Be_j = w_j e_{j-1}$ for each integer; $j \geq 1$. It is easy to check that B in $B(H)$ implies that the weight sequence $\{w_j : j \geq 1\}$ is bounded. This class of operators was well studied in Shields [31]. If all the weights satisfy $w_{j\,-\,1}$, then B is simply called the unilateral backward shift. This particular shift B was used to provide the first examples of hypercyclic operators on a Hilbert space, as Rolewicz [28] showed that tB is hypercyclic whenever $t > 1$. Then Abakumov and Gordon [20] showed the path $\{tB : t \in (1, \infty)\}$ indeed has a dense set of common hypercyclic vectors. This result was reobtained by Costakis and Sambarino [38] who introduced a sufficient condition for a path of general operators to have such a dense set. In fact, they also provided many natural examples of paths of operators with common hypercyclic vectors, including a path of unilateral weighted backward shifts. Another sufficient condition was provided by Bayart and Matheron [24] with applications for which Costakis and Sambarino's condition does not apply. A necessary and sufficient condition was provided by Chan and Sanders [32] for a path of operators to have a dense G_δ set of common hypercyclic vectors. They used this condition to prove another sufficient condition that reduces to the well known Hypercyclicity Criterion for the case when the whole path contains exactly one operator. Chan and Sanders used their conditions to reprove the result of Abakumov and Gordon, and further showed that between any two hypercyclic unilateral weighted backward shifts, there is a path of such shifts whose common hypercyclic vectors form a dense G_δ set. For techniques that are totally different from those of Chan and Sanders [32], their study on unilateral weighted backward shifts help motivate some of the ideas here. Common hypercyclic vectors include that of Bayart [23], and Bayart and Grivaux [26] who studied composition operators on spaces of analytic functions, and Costakis [164] who studied Cesaro hypercyclic operators. For nonexistence results, Aron, Bes, Leon, and Peris [22] showed in their Example (4.1.4) that there does not exist a vector in H which is hypercyclic for every hypercyclic operator in $B(H)$. They proved this by showing for any nonzero vector x in H, there is a hypercyclic operator T in $B(H)$ for which $Tx = 0$. Along this line, Chan and Sanders [32] showed there is a path of hypercyclic unilateral weighted backward shifts which fails to have a common hypercyclic vector.

We need to turn our attention to the density of the hypercyclic operators in $B(H)$. Clearly the norm of any hypercyclic operator must be strictly greater than 1, it is still easy to see that they collectively are not dense, with respect to the norm topology, in the complement of the unit ball of $B(H)$; see Chan [163]. As it turns out, they are dense in the whole operator algebra $B(H)$ with a weaker topology called the strong operator topology, abbreviated SOT. This result was first obtained by Chan [163], and was generalized to the Frechet space case by Bes and Chan [161] using a fundamental property of the strong operator topology provided by Hadwin, Nordgren, Radjavi, and Rosenthal [167]. In fact, Bès and Chan showed that if T is a hypercyclic operator, then the set of conjugates $\{ATA^{-1} : A \text{ invertible in } B(H)\}$ is SOT-dense in $B(H)$. Even further, if the hypercyclic operator T is chaotic, this SOT-dense set of conjugates consists entirely of chaotic operators. On the other hand, that conjugate set is path connected be cause the collection of all invertible operators in $B(H)$ is indeed path connected; see Douglas ([40], Corollary 5.30). Hence, any single chaotic operator generates a path connected set of chaotic operators that is SOT-dense in $B(H)$. As a result, it is interesting to see whether we can improve the above results, by raising the following question: Does there exist a path of chaotic operators which is SOT-dense in B(H), and yet has a dense G_δ set of common hypercyclic vectors?

The above question is answered in the positive with a constructive proof in Theorem (4.1.7) below. Even more interesting, we show each operator along this path has the exact same dense G_δ set of hypercyclic vectors. It should be noted that Bonet, Martinez, and Peris [30] have shown that there is a Banach space that fails to admit a chaotic operator, and so Theorem (4.1.7) is purely a Hilbert space result. Since the collection of all hypercyclic operators in $B(H)$ fails to have a common hypercyclic vector, the path that we construct is necessarily a proper subset of all hypercyclic operators. In fact, the path consists entirely of operators that satisfy the Hypercyclicity Criterion due to the result by Bes and Peris [162] that every chaotic operator satisfies the criterion. It should be pointed out here that de la Rosa and Read [39] showed that there is a Banach space which admits a hypercyclic operator that does not satisfy the hypercyclicity Criterion. Inspired by de la Rosa and Read, Bayart and Matheron [25] were able to obtain an analogous result for a Hilbert space. We conclude with a discussion about the SOT-connectedness of the hypercyclic operators in $B(H)$.

We first examine unilateral weighted backward shifts in $B(H)$. In particular, we show that for any given orthonormal basis of H, there is a path of chaotic hypercyclic unilateral weighted backward shifts which is SOT-dense in the set of all unilateral weighted backward shifts on that particular basis and for which all operators along this path have the exact same set of hypercyclic vectors. As a consequence, the common hypercyclic vectors for the whole path is a dense G$ set; see Theorem (4.1.4) below.

Since hypercyclic vectors may form some sort of linear structure, common hypercyclic vectors follow this natural pattern as well.

Definition (4.1.2)[159]: By the term hypercyclic subspace for an operator T, we mean a closed, infinite dimensional subspace consisting entirely, except the zero vector, of hypercyclic vectors for T.

A sufficient condition for the existence of common hypercyclic subspaces was obtained by Bayart [160]. Different sufficient conditions were obtained by Aron, Bes, Leon, and Peris in [22], and by Sanders in [30]. We show the SOT-dense paths of chaotic operators given in Theorem (4.1.4) and Theorem (4.1.7) can be chosen to have common hypercyclic subspaces; see Corollary (4.1.5) and Corollary (4.1.10) below.

Let B be the subset of $B(H)$ consisting of all unilat eral weighted backward shifts of a fixed orthonormal basis $\{e_0, e_1, e_2, ...\}$. We show there is a path of chaotic shifts in B which is SOT-dense in B.

The first examples of hypercyclic operators on a Hilbert space, provided by Rolewicz [28], were unilateral weighted backward shifts. Salas [29] later completely characterized hypercyclic unilateral weighted backward shifts in terms of the weight sequences. His result was originally stated for positive weight sequences. Since a unilateral weighted backward shift with the complex weight sequence $\{w_j : j \geq 1\}$ is unitarily equivalent to one with the positive weight sequence $\{|w_j| : j \geq 1\}$; see Shields ([31], Corollary 1), Salas′ characterization can be stated in term of complex weights: A unilateral weighted backward shift is hypercyclic if and only if its weight sequence satisfies

$$\sup\left\{\prod_{j=1}^{n} |w_j| : n \geq 1\right\} = \infty \tag{1}$$

A more general version of Salas' characterization was established by Grosse Erdmann [166]. Martinez and Peris ([169], Example (4.1.12)) also characterized the hypercyclic unilateral weighted backward shifts which are chaotic in terms of weight sequences: A unilateral weighted backward shift is chaotic if and only if its weight sequence satisfies

$$\sum_{n=1}^{\infty} \frac{1}{\prod_{j=1}^{n} |w_j|^2} < \infty. \tag{2}$$

From Salas' characterization (1), it is easy to see that if one takes the weight sequence of a hypercyclic shift in B, and multiply one of its weights by a nonzero complex scalar, the new resulting shift in B will also be hypercyclic. In fact, we now show they have the exact same set of hypercyclic vectors.

Proposition (4.1.3)[159]: Let $\lambda \in \mathbb{C} \setminus \{0\}$ and let m be a positive integer. If A is a unilateral weighted backward shift in B with weight sequence $\{w_j : j \geq 1\}$, and if B is another shift in B whose weight sequence $\{v_j : j > 1\}$ satisfies $v_j = w_j$ for any positive integer $j \neq m$ and $v_m = \lambda w_m$, then $\mathcal{HC}(A) = \mathcal{HC}(B)$.

Proof: Observe that for any integer $n \geq m$, we have:

$$A^n e_{j+n} = \left(\prod_{i=1}^{n} w_{j+i}\right) e_j, \qquad \text{for } j \geq 0$$

$$B^n e_{j+n} = \left(\prod_{i=1}^{n} w_{j+i}\right) e_j, \qquad \text{for } j \geq m, \text{and}$$

$$B^n e_{j+n} = \lambda\left(\prod_{i=1}^{n} w_{j+i}\right) e_j, \qquad \text{for} 0 \leq j \leq m-1$$

Therefore, for any vector $g \in H$ and integer $n > m$, we have

$$\langle A^n g, e_j\rangle = \langle B^n g, e_j\rangle, \text{ for } j \geq m, \text{and} \tag{3}$$

$$\langle A^n g, e_j\rangle = \lambda^{-1}\langle B^n g, e_j\rangle, \text{ for } 0 \leq j \leq m-1, \text{and} \tag{4}$$

Let $P : H \to H$ be the orthogonal projection onto the subspace span$\{e_j : 0 \leq j \leq m - 1\}$. That is, for any vector $g \in H$, we have $\sum_{j=0}^{m-1} \langle g, e_j\rangle e_j$. Hence, by equations (3) and (4), we have that for any integer $n \geq m$,

$$A^n g = \lambda^{-1} P(B^n g) + (I - P)B^n g, \text{and } B^n g = \lambda P(A^n g) + (I - P)A^n g.$$

It follows that the orbit $\operatorname{Orb}(A, g)$ is dense if and only if the orbit $\operatorname{Orb}(B, g)$ is dense.

Using induction with Proposition (4.1.3), we get that if A and B are two hyper cyclic unilateral weighted backward shifts in $\mathcal{B}$ whose weight sequences differ by only a finite number of members, then the two shifts satisfy $\mathcal{HC}(A) = \mathcal{HC}(B)$. This observation together with Martinez and Peris' necessary and sufficient condition (2) allows us to create a path of chaotic shifts in $\mathcal{B}$ which is SOT-dense in $\mathcal{B}$, and each operator along the path has the exact same set of hypercyclic vectors.

Theorem (4.1.4)[159]: There is a path $\{F_t \in B(H): t \in [1, \infty)\}$ of chaotic unilateral weighted backward shifts in $\mathcal{B}$ which is SOT-dense in $\mathcal{B}$. Moreover, for each $t \in [1, \infty)$, we have $\mathcal{HC}\,(F_t) = \mathcal{HC}(F_l)$; that is, each operator along the path has the exact same dense G$ set of hypercyclic vectors.

Proof: Let B_0 be a chaotic unilateral weighted backward shift in B with weight sequence $\{w_j : j \geq 1\}$ satisfying $w_j \in (\mathbb{Q} + i\mathbb{Q}) \setminus \{0\}$ for each integer $j \geq 1$. Consider the collection A of all weight sequences $w = \{w_j : j \geq 1\}$ satis fying that each weight $w_j \in (\mathbb{Q} + i\mathbb{Q}) \setminus \{0\}$ and $w_j = \tilde{w}_j$ for all but finitely many positive integers j. The collection A is countable, and so let $w^{(n)} = \{w_j^{(n)} : j > 1\}$, where $n = 1,2,3,\ldots$, be an enumeration of A. From the definition of the weight sequences in A, there is a sequence $(k_n)_{n=1}^{\infty}$ of positive integers such that $w^{(n)} = w_j^{(n+1)} = \tilde{w}_j$ for any integer $j \geq k_n + 1$. For integers n, j with $n > 1$ and $1 \leq j \leq kn$, write $w_j^{(n)} = r_j^{(n)} \exp\left(i\theta_j^{(n)}\right)$ where $r_j^{(n)} > 0$ and $0 \leq \theta_j^{(n)} < 2\pi$.

To define the path of operators, for each integer $n \geq 1$ and for each $t \in [0,1]$, let $G_{t,n}$ be the unilateral weighted backward shift in $\mathcal{B}$ whose weight sequence $\{v_j^{(t)} : j > 1\}$ is given by $v_j^{(t)} = \tilde{w}_j$ if $j > k_n + 1$, and

$$v_j^{(t)} = \left[(1 - t)r_j^{(n)} + tr_j^{(n+1)}\right] e^{i\left[(1-t)\theta_j^{(n)} + t\theta_j^{(n+1)}\right]} \quad \text{if} \quad 1 \leq j \leq k_n.$$

For each $t \in [n, n + 1]$, define $F_t = G_{t-n,n}$. Since $G_{1,n} = G_{0,n+1}$ for each integer $n \geq 1$, the map $F : [1, \infty) \to (B(H), \|\cdot\|)$, given by $F(t) = F_t$, is well defined. Moreover, the map $F : [1, \infty) \to (B(H), \|\cdot\|)$ is continuous because the map $1 \mapsto G_{t,n}$ is continuous on $[0,1]$ for each integer $n \geq 1$. Lastly, note that the series $\sum_{n=1}^{\infty} \prod_{j=1}^{n} \left|v_j^{(t)}\right|^{-2}$ converges if and only if the series $\sum_{n=1}^{\infty} \prod_{j=1}^{n} \left|\tilde{w}_j\right|^{-2}$ converges. Thus, from condition (2), we get each operator Ft is chaotic. Therefore, $\{F_t \in B(H): t \in [1, \infty)\}$ is a path of chaotic unilateral weighted backward shifts in B. To show $\mathcal{KC}\,(F_t) = \mathcal{HC}(F_i)$ for each $t \in [1, \infty)$, observe that the weights of each F_t are the same as F_1 except at most a finite number of them. Thus, by Proposition (4.1.3), we get $\mathcal{HC}\,(F_t) = \mathcal{HC}(F_1)$.

To show the path $\{F_t \in B(H): t \in [1, \infty)\}$ is SOT-dense in B, let B be a shift in B with the weight sequence $\{v_j : j \geq 1\}$. Let $f_1, \ldots, f_r$ be r nonzero vectors in H, and let $e > 0$. Consider an SOT-basic open set U given by

$$U = \{A \in B(H): \|\,(A - B)f_k\| < \varepsilon, \text{whenever } 1 \leq k \leq r\}.$$

Let $\gamma_2 = \sup\left\{\left(|\tilde{w}_j| + |v_j|\right)^2 : j \geq 1\right\}$, and choose an integer $N > 1$ such that for each vector f_k, we have

$$\sum_{j=N+1}^{\infty} |\langle f_k, e_j \rangle|^2 < \frac{\varepsilon}{2\gamma_1}$$

Let $\gamma_2 = \max\{\|f_k\|^2 : 1 < k \leq r\}$, and choose $w_1, \ldots, w_N$ $(\mathbb{Q} + i\mathbb{Q}) \setminus \{0\}$ to satisfy

$$|w_j - v_j|^2 < \frac{\varepsilon^2}{2\gamma_2}.$$

Consider the weight sequence $\{w_1, w_2, \ldots, w_N, \tilde{w}_{N+1}, \tilde{w}_{N+2}, \ldots\}$. From the definition of the collection A, there exists an integer $n_0 \geq 1$ such that

$$\left\{w_j^{(n_0)} : j \geq 1\right\} = \{w_1, w_2, \dots, w_N, \widetilde{w}_{N+1}, \widetilde{w}_{N+2}, \dots\}$$

Thus, for each vector f_k, we have

$$\left\|(F_{n_0} - B)f_k\right\|^2 = \sum_{j=1}^{\infty} \left|w_j^{(n_0)} - v_j\right|^2 \left|\langle f_k, e_j\rangle\right|^2$$

$$= \sum_{j=1}^{N} \left|w_j - v_j\right|^2 \left|\langle f_k, e_j\rangle\right|^2 + \sum_{j=N+1}^{\infty} \left|\widetilde{w}_j - v_j\right|^2 \left|\langle f_k, e_j\rangle\right|^2$$

$$< \frac{\varepsilon^2}{2\gamma_2} \sum_{j=1}^{N} \left|\langle f_k, e_j\rangle\right|^2 + \sum_{j=N+1}^{\infty} \left(\left|\widetilde{w}_j\right| + \left|v_j\right|\right)^2 \left|\langle f_k, e_j\rangle\right|^2$$

$$< \frac{\varepsilon^2}{2\gamma_2} \|f_k\|^2 + \gamma_1 \sum_{j=N+1}^{\infty} \left|\langle f_k, e_j\rangle\right|^2 < \frac{\varepsilon^2}{2\gamma_2} + \gamma_1 \frac{\varepsilon^2}{2\gamma_1} = \varepsilon^2,$$

and so, $F_{n_0} \in U$.

In fact, the above proof can be modified to show the following interesting connection with some linear structure of the hypercyclic vectors.

Corollary (4.1.5)[159]: There is a path of chaotic shifts in $\mathcal{B}$ that is SOT-dense in $\mathcal{B}$, and the shifts along the whole path have a common hypercyclic subspace.

To see that, we first note that Leon and Montes [168] completely characterized, in terms of weight sequences, the unilateral weighted backward shifts which possess a hypercyclic subspace. Their characterization, expressed in terms of complex weights, states that a unilateral weighted backward shift has a hyper cyclic subspace if and only if its weight sequence satisfies

$$\sup\left\{\prod_{j=1}^{n} : n \geq 1\right\} = \infty, \text{and } \lim_{n\to\infty}\left(\inf \prod_{j=0}^{n-1} \left|w_{k+j}\right|\right)^{1n} \leq 1.$$

Using this condition, we see that each operator along the path given in the proof of Theorem (4.1.4) may fail to have a hypercyclic subspace. However, if we select the chaotic shift B_0 in the proof of Theorem (4.1.4) to also have a hypercyclic subspace, then each operator F_t along the corresponding path of shifts in B is chaotic and has a hypercyclic subspace. Furthermore, since $\mathcal{HC}\ (F_t)$— $\mathcal{HC}(F_1)$ for each t, the whole path has a common hypercyclic subspace within the dense G_δ set of common hypercyclic vectors. The shift in B with the weight sequence $\left\{\frac{j+1}{j} : j \geq 1\right\}$ is an example of such a chaotic shift with a hypercyclic subspace.

We focused on the collection of all unilateral weighted back ward shifts of a fixed orthonormal basis in the Hilbert space H. However, this collection fails to be SOT-dense in $B(H)$. To construct the path of chaotic operators desired in Theorem (4.1.7), we turn our attention to generalized backward shifts. An operator T in $B(H)$ is a generalized backward shift if the kernel, $\text{Ker}(T)$, of T is one dimensional, and the set $U\{\text{Ker}(T^n) : n \geq 1\}$ is dense in H. Godefroy and Shapiro ([10], Proposition (4.1.10)) showed that if the operator T is a generalized backward shift, then there is a sequence $\{e_j : j \geq 0\}$ of vectors in H for which

$Tx_j = x_{j-1}$ for each integer; $j \geq 1$, and $\text{Ker}(T) = \text{span}\{x_0\}$. For more on general ized backward shifts, see Godefroy and Shapiro [10].

Proposition (4.1.6) below is the major building block of Theorem (4.1.7). By using Proposition (4.1.3) and the fact that the invertible operators on the Hilbert space H are path connected, Proposition (4.1.6) creates a path between a hypercyclic unilateral weighted backward shift and a specific generalized backward shift. Let $\{e_j : j \geq 0\}$ be any orthonormal basis of the Hilbert space H, and let $B_0: H \to H$ be a hypercyclic unilateral weighted backward shift of the basis $\{e_j : j \geq 0\}$ with weight sequence $\{w_j : j \geq 1\}$. Let $\{g_j : j > 0\}$ be a sequence of vectors in H and $\{v_j : j \geq 1\}$ a weight sequence for which there is an integer $N \geq 0$ such that

$$\text{span}\{g_j : 0 \leq j \leq N\} = \text{span}\{e_j : 0 \leq j \leq N\}, \tag{5}$$

and for any integer $N + 1$, we have

$$g_j = e_j \quad \text{and} \quad v_j = w_j. \tag{6}$$

Define an operator $B_1 : H \to H$ by $B_1 g_j = v_j g_{j-1}$ for each integer $j \geq 1$, and $B_1 g_0 = 0$. The operator B_1 is a generalized backward shift with $x_0 = g_0$, and

$$x_j = \left[\prod_{i=1}^{j} v_i\right]^{-1} g_j \text{ for integers } j > 1.$$

Proposition (4.1.6)[159]: There is a path $\{G_t \in B(H): t \in [0,1]\}$ of hypercyclic operators between Bo and Bi such that for each t in $[0,1]$, we have $\mathcal{HC}(G_t) = \mathcal{HC}(B_0)$. Furthermore, if the operator B_0 is chaotic, then this path may be chosen to consistentirely of chaotic operators.

Proof. To begin our proof, first observe we can assume $v_j = w_j$ for each inte ger $j \geq 1$. To see this, note that in the general case where $v_j = w_j$ for each integer $j \geq N + 1$, we can use the argument in the first half of the proof of Theorem (4.1.4) to create a path of hypercyclic operators between B_0 and the unilateral weighted backward shift of the basis $\{e_j : j \geq 0\}$ with weight sequence $\{v_j : j \geq 1\}$ where each operator along the path has the same set of hypercyclic vectors as the oper ator B_0. Moreover, if the operator B_0 is chaotic, then this path can be chosen so each operator along the path is chaotic.

To create the path of operators described in our proposition, let $H_N = \text{span}\{g_j : 0 \leq j \leq N\} = span\{e_j : 0 \leq j \leq N\}$, and let $A : H_N \to H_N$ be the invertible operator satisfying

$$Ae_j = g_j \text{ for } \quad 0 \leq j \leq N. \tag{7}$$

Since the invertible operators on H_N are path connected, see Douglas ([40], Corollary 5.30), there exists a path $\{A_t \in B(H_N) : t \in [0,1]\}$ of invertible operators such that $A_0 = I$ and $A_1 = A$. For each $t \in [0,1]$ and each integer $j > 0$, let

$$g_{t,j} = \begin{cases} A_t e_j & \text{if } 0 \leq j \leq N, \\ e_j & \text{if } j \geq N + 1, \end{cases} \tag{8}$$

and define the operator $G_t : H \to H$ by

$$G_t g_j = \begin{cases} v_j g_{t,j-1} & \text{if } j \geq 1, \\ 0 & \text{if } j = 0. \end{cases}$$

Then $\{G_t \in B(H): t \in [0,1]\}$ is a path of operators between $G_0 = B_0$ and $G_1 = B_1$. To show $\mathcal{HC}(G_t) = \mathcal{HC}(B_0)$ for each $t \in [0,1]$, let $P : H \to H$ be the orthogonal projection onto the closed subspace H_N. For each $f \in [0,1]$, define the operator $St : H \to H$ by $S_t = A_t P + (I — P)$. Since the operator S_t is invertible, it suffices to show $G_t^n = S_t B_0^n$ for each integer $n \geq N + 1$. For such an integer n, we observe that by (5), (6), (8), and the definition of B_0, we get $\mathrm{Ker}(B_0^n) = \mathrm{span}\{g_{t,j} : 0 \leq j \leq n-1\}$ which give us

$$S_t B_0^n g_{t,j} = 0 = G_t^n g_{t,j} \quad \text{for} \quad 0 \leq \mathrm{j} \leq \mathrm{n}-1. \tag{9}$$

From (8) and the definition of S_t, we get $S_t e_j = g_{t,j}$ for each integer $j \geq 0$. Thus, for any integer $j \geq n$, we have

$$S_t B_0^n g_{t,j} = S_t B_0^n e_j \ (by\ (8))$$
$$\prod_{i=0}^{n-1} w_{j—i} S_t e_{j—n} = \prod_{i=0}^{n-1} v_{j-i} g_{t,j-n} = G_t^n\, g_{t,j}.$$

Therefore, $G^n = S_t\, B_0^n$ whenever $n \geq N + 1$.

To complete the proof of our proposition, it remains to show that if the operator B_0 has a dense set of periodic points, then so has each operator G_t. For that we observe that if / is a periodic point of B_0, then we choose an integer $n \geq N + 1$ such that $B_0^n f = f$. Now, the vectors $A_t P f$ and $Pf \in H_N \subseteq \mathrm{Ker}(B_0^n)$, and so

$$B_0^n S_t f = B_0^n (A_t P f + (I - P) f) = B_0^n (Pf + (I - P) f) = B_0^n f = f.$$

It follows that $G_t^n S_t f = S_t B_0^n S_t f = S_t f$. Hence $S_t f$ is a periodic point of G_t if the f is a periodic point of B_0. Since the operator S_t is invertible, it takes the dense set of periodic points of B_0 to a dense set of periodic points of Gt .

Each operator along the path $\{G_t: t \in [0,1]\}$ given in the proof of Proposition (4.1.6) is, in fact, a generalized backward shift. We now use Proposition (4.1.6).

Theorem (4.1.7)[159]: Let H be a separable, infinite dimensional Hilbert space over C. Then there is a path $\{F_t \in B(H): t \in [1,\infty)\}$ of chaotic operators which is SOT-dense in $B(H)$. Furthermore, for each t in $[0,\infty]$, we have $\mathcal{HC}(F_t) = \mathcal{HC}(F_1)$; that is, each operator along the path has the same dense G_δ set of hypercyclic vectors.

Proof: To start, fix an orthonormal basis $\{e_j : j \geq 0\}$ of the Hilbert space H. Let $\mathcal{D}$ be the collection of all nonzero finite rank operators $D \in B(H)$ each of which has an integer $n > 1$ such that $De_j \in \{\sum_{k=0}^{n} a_k e_k : a_k \in \mathbb{Q} + i\mathbb{Q}\}$ whenever $0 \leq j \leq n$, and $De_j = 0$ whenever $j \geq n + 1$. Clearly $\mathcal{D}$ is a countable collection. It is also easy to see that $\mathcal{D}$ is SOT-dense in $B(H)$ because if $T \in B(H)$ and $P_n : H \to H$ is the orthogonal projection onto $span\{e_j : 0 \leq j \leq n\}$, then $P_n T P_n \to T$ in the strong operator topology. Let $\{D_\alpha : \alpha > 1\}$ be an enumeration of the collection $\mathcal{D}$ such that

$$D_\alpha e_j \in \left\{\sum_{k=0}^{\alpha} a_k e_k : a_k \in \mathbb{Q} + i\mathbb{Q}\right\}, \text{whenever } 0 < j < \alpha,$$

and

$$D_\alpha e_j = 0, \qquad \text{whenever} \quad j \geq \alpha + 1. \tag{10}$$

Let $B_0 : H \to H$ be a chaotic unilateral weighted backward shift of the basis $\{e_j : j \geq 0\}$ with weight sequence $\{w_j: j \geq 1\}$. For each D_α and each pair of integers $\beta, \gamma \geq 1$, we define the linear operator $T_{\alpha,\beta,\gamma} : H \to H$ in the following manner:

$$T_{\alpha,\beta,\gamma}e_j = D_\alpha e_j + \frac{1}{\gamma}e_{\alpha+\beta+1+j}, \qquad \text{for } 0 \le j \le \alpha; \tag{11}$$

$$T_{\alpha,\beta,\gamma}e_{\alpha+1} = e_0 \tag{12}$$

$$T_{\alpha,\beta,\gamma}e_{\alpha+1+j} = \frac{1}{\gamma^{j+1}}e_{\alpha+j}, \qquad \text{for } 0 \le j \le \beta-1; \tag{13}$$

$$T_{\alpha,\beta,\gamma}e_{\alpha+\beta+1+j} = -\gamma T_{\alpha,\beta,\gamma}D_\alpha e_j + \gamma e_{j+1}, \qquad \text{for } 0 \le j \le \alpha; \tag{14}$$

$$T_{\alpha,\beta,\gamma}e_{2\alpha+\beta+1} = -\gamma T_{\alpha,\beta,\gamma}D_\alpha e_\alpha; \tag{15}$$

$$T_{\alpha,\beta,\gamma}e_{2\alpha+\beta+2} = -e_{2\alpha+\beta+2}e_{\alpha+\beta}; \tag{16}$$

$$T_{\alpha,\beta,\gamma}e_j = w_j e_{j-1}, \qquad \text{for } j \ge 2\alpha+\beta+3. \tag{17}$$

Equations (14) and (15) define $T_{\alpha,\beta,\gamma}$ because $\mathrm{Ran}(D_\alpha) \subseteq span\{e_k : 0 \le \kappa \le 0\}$. In fact, the operator $T_{\alpha,\beta,\gamma}$ is a compact perturbation of a chaotic unilateral weighted backward shift, and hence a bounded linear operator on H.

Claim (4.1.8)[159]: The set $\{T_{\alpha,\beta,\gamma} : \alpha,\beta,\gamma \ge 1\}$ is $\mathrm{SOT} -$ dense in $B(H)$.

Proof: Let U be a nonempty SOT-open set in $B(H)$. Since $\mathcal{D}$ is SOT-dense in $B(H)$, there is an $\alpha \ge I$, nonzero vectors $f_1,\dots,f_r \in H$, and $\varepsilon > 0$ for which the basic SOT-open set

$$\{A \in B(H) : \|(A - D_\alpha)f_k\| < \varepsilon, \qquad \text{whenever } 1 \le k \le r\} \subseteq U.$$

Let $M_1 = \max\{\|f_k\| : 1 \le k \le r\}$, and let $M_2 = (a+1)^2(\|D_\alpha\| + 2)^2$. Choose an integer $\gamma \ge 2$ such that

$$\frac{(\alpha+1)M_1}{\gamma} < \frac{\varepsilon}{4}, \qquad \text{and} \sum_{j=0}^{\infty} \frac{1}{\gamma^{j+1}} < \frac{\varepsilon}{4M_1} \tag{18}$$

Since $\sup\{|w_j| : j \ge 1\} = \|B_0\| < \infty$, we can then choose an integer $B \ge 1$ such that for each vector f_k,

$$|\langle f_k, e_j\rangle| < \frac{\varepsilon}{4\gamma M_2}, \qquad \text{if } \; j \ge \alpha+\beta+1; \tag{19}$$

$$\left(\sum_{j=\alpha+\beta+1}^{\infty} |w_j|^2 |\langle f_k, e_j\rangle|^2\right) < \frac{\varepsilon}{4}. \tag{20}$$

To show $T_{\alpha,\beta,\gamma} \in \mathcal{U}$, let $T = T_{\alpha,\beta,\gamma}$, and let k be an integer with $1 \le k \le r$ Observe that

$$\|T - D_\alpha\| = \left\|\sum_{j=0}^{\infty} |\langle f_k, e_j\rangle|(T - D_\alpha)e_j\right\|$$

$$\le \sum_{j=0}^{\alpha} |\langle f_k, e_j\rangle| \|(T-D_\alpha)e_j\| + \sum_{j=0}^{\beta+1} |\langle f_k, e_{\alpha+1+j}\rangle| \|(T-D_\alpha)e_{\alpha+1+j}\| \tag{21}$$

$$\le \sum_{j=0}^{\alpha} |\langle f_k, e_{\alpha+\beta+1+j}\rangle| \|(T-D_\alpha)e_{\alpha+\beta+1+j}\| + \left\|\sum_{j=2\alpha+\beta+2}^{\infty} |\langle f_k, e_j\rangle|(T-D_\alpha)e_j\right\|$$

To estimate each of the above four summation, we note that for the first term

$$\sum_{j=0}^{\alpha} |\langle f_k, e_j\rangle| \|(T-D_\alpha)e_j\| = \sum_{j=0}^{\alpha} |\langle f_k, e_j\rangle| \frac{1}{\gamma} \text{ (by(11))}$$
$$\leq \frac{1}{\gamma}\sum_{j=0}^{\alpha} \|f_k\| \tag{22}$$
To estimate the second summation, observe that
$$\sum_{j=0}^{\beta-1} |\langle f_k, e_{\alpha+1+j}\rangle| \|(T-D_\alpha)e_{\alpha+1+j}\| = \sum_{j=0}^{\beta-1} |\langle f_k, e_{\alpha+1+j}\rangle| \frac{1}{\gamma^{j+1}} \text{(by(3.6),(12)(13))}$$
$$\leq \|f_k\| \sum_{j=0}^{\infty} \frac{1}{\gamma^{j+1}} \tag{23}$$
$$= \frac{\varepsilon}{4}.$$
To estimate the second summation, observe that by equalities (10),(16), and (17), we have
$$\left\| \sum_{j=2\alpha+\beta+2}^{\infty} |\langle f_k, e_j\rangle| (T-D_\alpha)e_j \right\|^2$$
$$= \left\| w_{2\alpha+\beta+2} |\langle f_k, e_{2\alpha+\beta+2}\rangle| e_{\alpha+\beta} + \sum_{j=2\alpha+\beta+2}^{\infty} w_j |\langle f_k, e_j\rangle| e_{j-1} \right\|$$
$$= \sum_{j=2\alpha+\beta+2}^{\infty} |w_j|^2 |\langle f_k, e_j\rangle|^2 \tag{24}$$
$$< \frac{\varepsilon^2}{16} \text{(by(20))}.$$
Lastly, to estimate the third summation, we observe that $\mathrm{Ran}(D_\alpha) \subseteq span\{e_k : 0 \leq k \leq \alpha\}$, and so for any vector $g \in H$, we have
$$\|TD_\alpha g\| = \left\| \sum_{j=0}^{\alpha} |D_\alpha g, e_j| T e_j \right\|$$
$$\leq \sum_{j=0}^{\alpha} |\langle D_\alpha g, e_j\rangle| \left\| D_\alpha e_j + \frac{1}{\gamma} e_{\alpha+\beta+1+j} \right\| \text{(by(11))}$$
$$\leq \sum_{j=0}^{\alpha} \|D_\alpha g\| \left(\|D_\alpha g\| + \frac{1}{\gamma} \right) \leq \sum_{j=0}^{\alpha} (\|D_\alpha\|+1)^2 \|g\| = (\alpha+1)(\|D_\alpha\|+1)^2 \|g\|.$$
Thus, for any integer$j$ with $0 \leq j \leq \alpha - 1$
$$\|(T-D_\alpha)e_{\alpha+\beta+1+j}\| = \|-\gamma T D_\alpha e_j + e_{\alpha+1}\| \text{ (by(10),(14))}$$
$$\leq \gamma(\alpha+1)(\|D_\alpha\|+1)^2 + \gamma \leq \gamma(\alpha+1)(\|D_\alpha\|+1)^2.$$
For similar reasons,
$$\|(T-D_\alpha)e_{\alpha+\beta+1}\| \leq \gamma(\alpha+1)(\|D_\alpha\|+1)^2.$$
Therefore,

$$\sum_{j=0}^{\alpha} |\langle f_k, e_{\alpha+\beta+1+j}\rangle| \|(T-D_\alpha)e_{\alpha+\beta+1+j}\| < \sum_{j=0}^{\alpha} \frac{\varepsilon^2}{4M_2\gamma} \|(T-D_\alpha)e_{\alpha+\beta+1+j}\| (\text{by}(19))$$

$$< \frac{\varepsilon^2}{4M_2\gamma}\sum_{j=0}^{\alpha} \gamma(\alpha+1)(\|D_\alpha\|+1)^2 = \frac{\varepsilon^2}{4M_2} M_2 = \frac{\varepsilon^2}{4}. \quad (25)$$

Combining inequalities (22), (23),(24) and (25) with (21) yields

$$\|(T-D_\alpha)f_k\| < \varepsilon,$$

which completes the proof of Claim (4.1.8).

We now use Proposition (4.1.6) to connect the chaotic shift B_0 with the operator $T_{\alpha,\beta,\gamma}$.

Claim (4.1.9)[159]: For each triple of integers $\alpha, \beta, \gamma > 1$, there is a path $\{G_t \in B(H): t \in [0,1]\}$ of chaotic operators between B_0 and $T_{\alpha,\beta,\gamma}$ for which $\mathcal{HC}(Gt) = \mathcal{HC}(B_0)$for each $t \in [0,1]$.

Proof: To prove Claim (4.1.9) using Proposition (4.1.6), we define as proof of Claim (4.1.9). To prove Claim (4.1.9) using Proposition (4.1.6), we define a sequence $\{g_j : j \geq 0\}$ of vectors in H and a weight sequence $\{v_j : j \geq 1\}$ in the following manner. For each integer j with $0 \leq j \leq \alpha$, let 1

$$g_{2j} = D_\alpha e_{\alpha-j} + \frac{1}{\gamma} e_{\alpha+\beta+1+(\alpha-j)}, \quad (26)$$

$$g_{2j+1} = e_{\alpha-j}, \text{and} \quad (27)$$

$$v_{2j} = v_{2j+1} = 1. \quad (28)$$

For each integer j with $0 \leq j \leq \beta - 1$, let

$$g_{2\alpha+2+j} — e_{\alpha+1+j} \quad \text{and} \quad v_{2\alpha+2+j} = \frac{1}{\gamma^{j+1}} \quad (29)$$

and for integers $j \geq 2\alpha + \beta + 2$, let

$$g_j = e_j \quad \text{and} \quad v_j = w_j. \quad (30)$$

Since $\operatorname{Ran}(D_\alpha) \subseteq span\{e_j : 0 \leq j \leq \alpha\}$, we get $span\{g_j : 0 \leq j \leq 2\alpha+\beta+1\} = \operatorname{span}\{e_j : 0 \leq j \leq 2\alpha+\beta+1\}$. To show $T_{\alpha,\beta,\gamma} g_j = v_j g_{j-1}$ for each and $T_{\alpha,\beta,\gamma} g_0 = 0$, observe that

$$T_{\alpha,\beta,\gamma} g_0 = T_{\alpha,\beta,\gamma} D_\alpha e_\alpha + \frac{1}{\gamma} T_{\alpha,\beta,\gamma} e_{\alpha+\beta+1} \text{ (by(26))}$$

$$= T_{\alpha,\beta,\gamma} D_\alpha e_\alpha + \frac{1}{\gamma}\left(-\gamma T_{\alpha,\beta,\gamma} D_\alpha e_\alpha\right) \text{(by (15))}$$

$$= 0, \text{and}$$

$$T_{\alpha,\beta,\gamma} g_1 = T_{\alpha,\beta,\gamma} e_\alpha \text{ (by (27))}$$

$$= D_\alpha e_\alpha + \frac{1}{\gamma} e_{\alpha+\beta+1} \text{(by (11))}$$

$$= v_1 g_0 \text{ (by (26) and (28))}.$$

Using a similar argument, for integers j with

$$T_{\alpha,\beta,\gamma} g_{2j} = T_{\alpha,\beta,\gamma} D_\alpha e_{\alpha-j} + \frac{1}{\gamma} T_{\alpha,\beta,\gamma} e_{\alpha+\beta+1+(\alpha-j)} (by\ (3-22))$$

$$= e_{\alpha-j+1} \text{ (by (14))}$$

$$= v_{2j} g_{2j-1} \text{ (by (27) and (28)), and}$$

$$T_{\alpha,\beta,\gamma} g_{2j+1} = T_{\alpha,\beta,\gamma} e_{\alpha-j} \text{ (by (27))}$$

$$= D_\alpha e_{\alpha-j} + \frac{1}{\gamma} e_{\alpha+\beta+1} \text{ (by (11))}$$

$$= v_{2j+1} g_{2j} \text{ (by (26) and (28)).}$$

Next, note that

$$T_{\alpha,\beta,\gamma} g_{\alpha+2+j} = T_{\alpha,\beta,\gamma} e_{\alpha+1} \text{ (by (29))}$$

$$= \frac{1}{\gamma} e_0 \text{ (by (12))}$$

$$= v_{2\alpha+2} g_{2\alpha+1} \text{(by (27) and (29)),}$$

and for integers j with$1 \leq j \leq \beta - 1$ and for integers j with

$$T_{\alpha,\beta,\gamma} g_{2\alpha+2+j} = T_{\alpha,\beta,\gamma} e_{\alpha+1+j} \text{ (by (29))}$$

$$= \frac{1}{\gamma^{j+1}} e_{\alpha+1} \text{(by (13))}$$

$$= v_{2\alpha+2+j} g_{2\alpha+1+j} \text{(by (29))}$$

Lastly, observe that

$$T_{\alpha,\beta,\gamma} g_{2\alpha+\beta+2} = T_{\alpha,\beta,\gamma} e_{\alpha+\beta+2} \text{(by (30))}$$

$$= w_{\alpha+\beta+2} e_{\alpha+\beta} \text{ (by (16))}$$

$$= v_{\alpha+\beta+2} g_{\alpha+\beta+1} \text{ (by (29) and (30)),}$$

and for integers $j \geq 2\alpha + \beta + 3$,

$$T_{\alpha,\beta,\gamma} g_j = T_{\alpha,\beta,\gamma} e_j \text{(by (30))}$$

$$= w_j e_{j-1} \text{ (by (17))}$$

$$= v_j g_{j-1} \text{ (by (30)),}$$

which completes the proof of Claim (4.1.9).

To construct the desired SOT-dense path of chaotic operators. Let $\{T_{\alpha_k,\beta_k,\gamma_k} : k \geq 1\}$ be an enumeration of the countable set $\{T_{\alpha,\beta,\gamma} \geq 1\}$. By Claim (4.1.9), for each integer $k > 1$, there is a path $\{G_{t,k} \in B(H): t \in [0,1]\}$ of chaotic operators such that $G_{0,k} = G_{1,k} = B_0$ and $T_{\alpha_k,\beta_k,\gamma_k} \in \{G_{t,k} \in B(H): t \in [0,1]\}$, and in addition $\mathcal{HC}(G_{t,k}) = \mathcal{HC}(B_0)$ for each $t \in [0,1]$. For each $t \in [k, k+1]$, let $F_t = G_{t-k,k}$. Then $\{F_t \in B(H): t \in [1,\infty)\}$ is a path of chaotic operators which is SOT-dense in $B(H)$ by Claim (4.1.8), and for which $\mathcal{HC}(F_t) = \mathcal{HC}(B_0) = \mathcal{HC}(F_1)$ for each $t \in [1,\infty)$.

If we choose the chaotic shift B_0 given within the proof of Theorem (4.1.7) to have a hypercyclic subspace, then the corresponding path of operators in the theorem maintains the linear structure.

Corollary (4.1.10)[159]: There is a path of chaotic shifts that is SOT-dense in $B(H)$, and the shifts along the whole path have a common hypercyclic subspace.

Not only does the strong operator topology play an important role in the density of the hypercyclic operators, it also plays a role in the connectedness of those operators. To explain, recall Bes and Chan [161] showed that if an operator T in $B(H)$ is hypercyclic, then its conjugate class $\{ATA^{-1} : A \text{ invertible in } B(H)\}$ is an SOT-dense collection of hypercyclic operators in $B(H)$. That conjugate class is also path connected because the invertible operators in $B(H)$ are path connected; see Douglas ([40], Corollary 5.30). Hence, the conjugate class consisting entirely see Douglas ([40], Corollary 5.30). Hence, the conjugate class consisting entirely of hypercyclic operators is SOT-dense and SOT-cortnected in $B(H)$. On the other of hypercyclic operators is SOT-dense and SOT-cortnected in $B(H)$. On the other hand, we observe that if Υ and Z are two subsets of a topological space X satis fying $\Upsilon \subseteq Z \subseteq \overline{\Upsilon}$ and if Υ is connected, then Z is connected; see Munkres ([44], Theorem 1.4, page 149). This observation and our discussion above lead to the following fact

Proposition (4.1.11)[159]: The set of all hypercyclic operators is SOT-connected in $B(H)$. Now, if the hypercyclic operator that generates the SOT-dense conjugate class is chaotic, then the conjugate class consists entirely of chaotic operator, which by the same discussion as above, implies that the set of all chaotic tors is also SOT-connected. Furthermore, one can easily verify that an operator satisfies the Hypercyclicity Criterion if and only if every operator in gate class does. Hence, the same argument shows the set of operators satisfying the Hypercyclicity Criterion is SOT-connected. Similarly, the set of hypercyclic operators not satisfying the criterion is SOT-connected as well.

With the same topological argument, we see that if we let $\mathcal{G}$ be the dense G_δ set of common hypercyclic vectors in Theorem (4.1.7), then we have the following conclusion.

Corollary (4.1.12)[159]: The set of operators T in $B(H)$ with $\mathcal{G} \subseteq \mathcal{HC}(T)$ is SOT connected.

Likewise, if we let $\mathcal{G}$ be the common hypercyclic subspace in Corollary (4.1.10), then the set of all operators T for which $\mathcal{G} \subseteq \mathcal{HC}(T)$ is also SOT-connected in $B(H)$.

Related to hypercyclicity are the concepts of supercyclicity and cyclicity. An operator T in $B(H)$ is supercyclic if there is a vector x in H for which the set $\{\lambda T^n x : n \geq 0 , \lambda \in \mathbb{C}\}$, consisting of all scalar multiples of vectors from the orbit $\mathrm{Orb}(T, x)$, is dense in H. An operator T is cyclic if there is a vector x in H for which the linear span of the orbit, span $\mathrm{Orb}(T, x)$, is dense in H. Clearly, hypercyclic ity implies supercyclicity, and supercyclicity implies cyclicity. From the above topological argument, the supercyclic operators in $B(H)$ are SOT-connected and SOT-dense. Furthermore, as in Corollary (4.1.12), the set of supercyclic operators T in $B(H)$ having the prescribed dense G_δ set of supercyclic vectors forms an SOT connected subset of $B(H)$. Likewise, the same holds true for the cyclic operators in $B(H)$.

Section (4.2): Translation Operators with Large Gaps

By $H(\mathbb{C})$ we denote the set of entire functions endowed with the topology of local uniform convergence. For a subset A of $H(\mathbb{C}).\bar{A}$ denotes the closure of A with respect to the topology of local duniform convergence. Let X be a topological vector space. A subset G of a X is called G_δ if it can be written as a countable intersection of open sets in X and a subset Y of X is called residual if it contains a $G_\delta$and dense subset of X. The symbol ∞ whenever appears in the present work denotes the complex infinity.

Let $(T_n: X \to X)$ be a sequence of continuous linear operators on a topological vector space X. If $\left(T_n(x)\right)_{n\geq 1}$ is dense in X for some $x \in X$, then x is called hypercyclic for (T_n) and we say that (T_n) is hypercyclic [180], [12]. The symbol $HC(\{T_n\})$ stands for the collection of all hypercyclic vectors for (T_n). In the case where the sequence (T_n) comes from the iterates of a single operator $T: X \to X$, i. e. $T_n := T^n$, then we simply say that T is hypercyclic and x is hypercyclic for T. If $T: X \to X$ is hypercyclic then the symbol $HC(T)$ stands for the collection of all hypercyclic vectors for $T. A$ simple consequence of Baire's category theorem is that for every continuous linear operator T on a separable topological vector space X, if $HC(T)$ is non-empty then it is necessarily (G_δ and) dense. For an account of results on the subject of hypercyclicity see [180], [12], see also [196].

We deal with translation operators. For every $a \in \mathbb{C} \setminus \{0\}$ consider the translation operator $T_a: H(\mathbb{C}) \to H(\mathbb{C})$ defined by

$$T_a(f)(z) = f(z + a), \qquad f \in H(\mathbb{C}).$$

An old result of Birkhoff [184] says that there exist entire functions the integer translates of which dense in the space of all entire functions are endowed with the topology of local uniform convergence. In other words T_1 is hypercyclic. Actually, it is not difficult to see that for every $a \in \mathbb{C} \setminus \{0\}, T_a$ s hypercyclic and hence $HC(T_a)$ is G_δ and dense in $H(\mathbb{C})$. Costakis and Sambarino [38] strengthened Birkhoff's result by showing that the family $\{\{T_a\}\ a \in \mathbb{C} \setminus \{0\}\}$ has a residual set of common hypercyclic vectors i.e., the set $\bigcap_{a\in\mathbb{C}\setminus\{0\}} HC\ (\{T_{na}\})$ is residual in $H(\mathbb{C})$. In particular, it is non-empty. What makes their result nontrivial is the uncountable range of a. At this point, let us mention a relevant observation due to Bayart and Matheron, [180], [24]: suppose X is a Fréchet space and $\{S_{a,n} | a \in A, n \in \mathbb{N}\}$ is a collection of sequences of continuous linear operators on X, labelled by the elements a of a set A. If A is a σ-compact topological space, the maps $\to S_{a,n}$ are SOT-continuous and each sequence $(S_{a,n})_{n\in\mathbb{N}}$ has a dense set of hypercyclic vectors then either $\bigcap_{a\in A} HC(\{S_{a,n}\}) = \emptyset$ or $\bigcap_{a\in A} HC(\{S_{a,n}\})$ is a dense G_δ-set in X. This observation applies to all the collections of operators considered.

Recall that the set $\bigcap_{a\in\mathbb{C}\setminus\{0\}} HC(\{T_{na}\})$ is residual in $H(\mathbb{C})$, [38]. Subsequently, Costakis [188] asked whether, in this result, the sequence (n) can be replaced by more general sequences (λ_n) of non-zero complex numbers. In this direction Costakis [188] showed that, if the sequence (λ_n) satisfies the following condition ($\sum$): for every $M > 0$ there exists a subsequence (μ_n) of (λ_n) such that

(i) $|\mu_{n+1}| - |\mu_n| > M$ for every $n = 1,2, \ldots$ and

(ii) $\sum_{n=1}^{+\infty} \frac{1}{|\mu_n|} = +\infty$

Then the desired conclusion holds if we restrict attention to $a \in C(0,1) \left\{z \in \frac{\mathbb{C}}{|z|} = 1\right\}$, that is the set $\bigcap_{a\in C(0,1)} HC(\{T_{\lambda_n a}\})$ is residul in $H(\mathbb{C})$.

In view of the above, Costakis led to the following question, see Question 1 in [188].

Theorem (4.2.1)[170]: Fix a sequence of non-zero complex numbers $=\Lambda = (\lambda_n)$ that tends to infinity and satisfies the above condition ($\sum$). Ten $\bigcap_{a\in\mathbb{C}\setminus\{0\}} HC(\{T_{\lambda_n a}\})$ is a G_δ and dense subset of $H(\mathbb{C})$.

We mention here that one is forced to impose certain natural restrictions on the sequence (λ_n) is order conclude that the set $\bigcap_{a\in\mathbb{C}\setminus\{0\}} HC(\{T_{\lambda_n a}\})$ is non-empty. Indeed, [191] show that if $\lim \inf_{a\in\mathbb{C}\setminus\{0\}} \frac{|\lambda_{n+1}|}{|\lambda_n|} > 2$ then $\bigcap_{a\in\mathbb{C}\setminus\{0\}} HC(\{T_{\lambda_n a}\}) = \emptyset$. In particular, $\bigcap_{a\in\mathbb{C}\setminus\{0\}} HC(\{T_{e^n a}\}) = \emptyset$ However, for sequences (λ_n) with

$$1 < \lim_n \inf(\{T_{\lambda_n a}\}) = \emptyset,$$

although it is plausible to conjecture that this is the case. In particular, we do not know what happens when $\lambda_n = 2^n$ or $\lambda_n = (3/2)^n$. This work can be seen as a try to understand the nature of this restriction. In any case, it seems a quite difficult problem to fully characterize the sequences (λ_n) for which the conclusion of Theorem (4.2.1) holds.

We stress that Theorem (4.2.1) complements the main result from [203]. [203] showed that the conclusion of Theorem (4.2.1) holds for sequences (λ_n) satisfying another type of condition different from $(\sum\ )$; condition, which we call it $(\sum'\ )$, is also not very restrictive, in the sense that it still allows sequences (λ_n) with "large gaps". We postpone the definition of condition $(\sum'\ )$. We note that although sequences of polynomial type of degree bigger than one, such as $(n^2), (n^3), (n^4 + n^5)$ and so on, clearly do not satisfy condition $(\sum\ )$ they do satisfy $(\sum'\ )$. However, there exist sequences satisfying both conditions $(\sum\ )$ and $(\sum'\ )$.

The main argument uses Baire's A few words about the proof of Theorem (4.2.1). Of course the main argument uses Baire's category theorem, but in order to do so the first and most difficult thing is to construct a suitable two dimensional partition on a given sector of the plane. After, to each point of the partition we assign a suitable closed disk of constant radius so that these disks are pairwise disjoint and their union almost fills the sector.

Having done these steps we are ready for the final argument which involves a standard use of Runge's or Mergelyan's approximation theorem along with Baire's theorem. It is important to say that in our framework one cannot use Ansari's theorem [174], as Costakis and Sambarino did in their proof, since now the sequence (λ_n) lacks the semigroup structure, i.e. $\lambda_n + \lambda_m \neq \lambda_{n+m}$ in general. Actually, this was the reason that led us to seek higher order partitions in order to make things work. Overall, we elaborate on the work of Costakis and Sambarino and we offer a general strategy how to construct two dimensional partitions relevant to the problem. The proof shares certain similarities with the proof in [203] and so we feel that will get a more clear and integrated picture by reading in parallel the present in [203]. However, the methods of constructing the partitions in [203] differentiate drastically. The reason for this, is that always the partition reflects the structure of the sequence (λ_n). The construction of the partition in [203] is very tight and quite delicate and comes from our effort to deal firstly with the most natural sequence which fails condition $(\sum\ )$, namely the sequence (n^2). It is also evident that there is a huge distance between sequences satisfying condition $(\sum\ )$ and the sequences satisfying condition $(\sum'\ )$. It would be desirable to exhibit a condition and a corresponding partition, if any, which imply the main result of the present as well as the main result in [203]. Unfortunately, this is unclear to us.

There are several results concerning either the existence or the nonexistence of common hypercyclic vectors for uncountable families of operators, such as weighted shifts, adjoints of multiplication operators, differentiation and composition operators; see [171], [23]-[181], [182] [185]-[186], [37]-[191], [193], [12], [43], [198], [46], [201], [202], [203].

The proof of Theorem (4.2.1) has several steps and occupies. We compare Theorem (4.2.1) with the main result from [203] and we exhibit examples of sequences which illustrate the main theorem.

We describe the steps for the proof of Theorem (4.2.1). Consider the sectors

$$S_n^k := \left\{a \in \mathbb{C} | \exists\, r \in \left[\frac{1}{n}, n\right] \text{ and } t \in \left[\frac{k}{4}, \frac{k+1}{4}\right] \text{ such that } a = re^{2\pi i t}\right\}$$

For $k = 0,1,2,3$ and $n = 2,3, \ldots$ since

$$\bigcap_{a \in \mathbb{C} \setminus \{0\}} HC(\{T_{\lambda_n a}\}) = \bigcap_{k=0}^{3} \bigcap_{n=2}^{+\infty} \bigcap_{a \in S_n^k} HC(\{T_{\lambda_n a}\})$$

An appeal of Baire's category theorem reduces Theorem (4.2.1) to the following.

Proposition (4.2.2)[170]: Fix a sequence (λ_n) of non-zero complex numbers that tends to infinity which satisfies the above condition $(\sum)$. Fix four real numbers $r_0, R_0, \theta_0, \theta_T$ such that $0 < r_0 < 1 < R_0 < +\infty, 0 \leq \theta_0 < \theta_T \leq 1$, $\theta_T - \theta_0 = \frac{1}{4}$ and consider the sector S defined by

$$S := \{a \in \mathbb{C} |\ there\ exist\ r \in [r_0, R_0] \quad and \quad t \in [\theta_0, \theta_T] \quad such\ that\ \ a = re^{2\pi i t}\}$$

Then $\bigcap_{a \in S} HC(\{T_{\lambda_n a}\})$ is a G_δ and dense subset of $H(\mathbb{C})$.

For the proof of Proposition (4.2.2) we introduce some notation which will be carried. Let (p_j), $j = 1,2, \ldots$ be a dense sequence of $H(\mathbb{C})$, (for instance, all the polynomials in one complex variable with coefficients in $\mathbb{Q} + i\mathbb{Q}$). For every $m, j, s, k \in \mathbb{N}$ we consider the set

$$E(m,j,s,k) := \left\{f \in H(\mathbb{C}) \,\middle|\, \forall a \in S \ \ \exists n \in \mathbb{N},\ \ n \leq m \colon \sup_{|z| \leq k} \left| f(z + \lambda_n a) - p_j(z) < \frac{1}{s} \right| \right\}$$

By Baire's category theorem and the three lemmas stated below, Proposition (4.2.2) readily follows.

Lemma (4.2.3)[170]:

$$\bigcap_{a \in S} HC(\{T_{\lambda_n a}\}) = \bigcap_{j=1}^{+\infty} \bigcap_{s=1}^{+\infty} \bigcap_{k=1}^{+\infty} \bigcup_{m=1}^{+\infty} E(m,j,s,k).$$

Lemma (4.2.4)[170]: For every $m, j, s, k \in \mathbb{N}$ the set $E(m,j,s,k)$ is open in $H(\mathbb{C})$.

For the sequel we fix four positive numbers c_1, c_2, c_3, c_4 such that $c_1 > 1$, $c_2 \in (0,1)$, $c_3 > 1$, $c_4 > 1$, where $c_3 := \frac{c_4}{r_0 c_2}$, $c_1 := 4(c_3 + 1)$. We also consider four positive real numbers $\theta_0, \theta_T, r_0, R_0$ as in Proposition (4.2.2) and a sequence $\Lambda = (\lambda_n)$ of non zero complex numbers which satisfies condition $(\sum)$ and such that $\lambda_n \to \infty$ as $n \to +\infty$. After the definition of the above numbers we fix a subsequence (μ_n) of (λ_n) such that:

$$|\mu_n| > c_1, \ \ |\mu_{n+1}| - |\mu_n| > c_1 \quad \text{for every} \quad n = 1,2, \ldots \ \ and \ \ \sum_{k=1}^{+\infty} \frac{1}{|\mu_k|} = +\infty.$$

We succeed the elementary structure of our construction. The following two steps are based in this first one. For every positive integer m we shall construct a corresponding partition Δ_m of $[\theta_0, \theta_T]$. So, let $m \in \mathbb{N}$ be fixed.

The condition $\sum_{n=1}^{+\infty} \frac{1}{|\mu_n|} = +\infty$ implies that for every positive integer $m = 1, 2, \ldots$ there exists the minimum natural number $m_1(m)$ such that:

$$\sum_{k=m}^{m_1(m)} \frac{1}{|\mu_k|} > c_3 . \frac{1}{|\mu_m|} \tag{31}$$

Clearly $m_1(m) \geq m + 1$ for every $m = 1,2, \ldots$ because $c_3 > 1$. We defined the numbers

$\theta_0^{(m)} := \theta_0, \theta_1^{(m)} := \theta_0^{(m)} + \frac{c_2}{|\mu_m|}, \theta_2^{(m)} := \theta_1^{(m)} + \frac{c_2}{|\mu+1|}, \dots, \theta_{m_1(m)-m+1}^{(m)} := \theta_{m_1(m)-m}^{(m)} + \frac{c_2}{|\mu_{m_1(m)}|}$, or generally:

$$\theta_{n+1}^{(m)} := \theta_n^{(m)} + \frac{c_2}{|\mu_{m+n}|}, \qquad n = 0,1, \dots, m_1(m) - m, \tag{32}$$

Where $m_1(m) - m \geq 1$. Define

$$\sigma_m := \theta_{m_1(m)-m+1}^{(m)} - \theta_0.$$

Now let any positive integer ν with

$$\nu > m_1(m) - m + 1.$$

For such a ν there exists a unique pair $(k, j) \in \mathbb{N}^2$, where $j \in \{0,1, \dots, m_1(m)m\}$, such that:

$$\nu = k(m_1(m) - m + 1) + j.$$

We define

$$\theta_\nu^{(m)} := \theta_j^{(m)} + k\sigma_j^{(m)} + k\sigma_m.$$

It is obvious that $\lim_{\nu\to+\infty} \theta_\nu^{(m)} = +\infty$ and the sequence $\left(\theta_\nu^{(m)}\right)$ is strictly increasing, in respect to ν. So there exists a maximum natural number $\nu_m \in \mathbb{N}$ such that $\theta_{\nu_m}^{(m)} \leq \theta_T$. We set

$$\Delta_m := \left\{\theta_0^{(m)}, \theta_1^{(m)}, \dots, \theta_{\nu_m}^{(m)}\right\}.$$

It holds that $\nu_m \geq m_1(m) - m + 1$ (see Lemma (4.2.5)).

Consider the function $\phi: [\theta_0, \theta_T] \times (0, +\infty) \to \mathbb{C}$ given by

$$\phi(t, r) := re^{2\pi it}, \qquad (t, r) \in [\theta_0, \theta_T].$$

For any given positive integer $m, \phi_r(\Delta_m)$ is a partition of the arc $\phi_r([\theta_0, \theta_T])$, where Δ_m is the partition of the interval $[\theta_0, \theta_T]$ constructed in Step 1. For every $r > 0, m \in \mathbb{N}$ define

$$p_0^{r,m} := \phi_r(\Delta_m)$$

Which we call partition of the arc $\phi_r([\theta_0, \theta_T])$ with height r, density m and order 0.

Consider the partition $p_0^{r_0,1}$ from the previous step, Step 2 and set

$$r_1 := r_0 + \frac{c_2}{|\mu_{m_1(1)}|} \tag{33}$$

After, we consider the partition $p_0^{r_1,m_1(1)+1}$ and we set

$$m_2 := m_1(m_1(1) + 1),$$

$$r_2 := r_1 + \frac{c_2}{|\mu_{m_2}|}.$$

Inductively we define two sequences (r_ν), $\nu = 0,1,2, \dots, (m_\nu)$, $\nu = 2, \dots$. As follows: r_0, r_1, r_2 and m_2 are above, see (33). Suppose that we have constructed the numbers m_ν, r_ν, for some $\nu \geq 2$. Then, taking into account the partition $p_0^{r_\nu,m_\nu+1}$, we set

$$m_{\nu+1} = m_1(m_\nu + 1) \tag{34}$$

And

$$r_{\nu+1} := r_\nu + \frac{c_2}{|\mu_{m_\nu+1}|} \tag{35}$$

For the next step, consider the partition $p_0^{r_\nu+1,m_{\nu+1}+1}$. We show that $\lim_{\nu\to+\infty} r_\nu = +\infty$. Therefore there exists a maximum natural number $\nu_0 \in\setminus\mathbb{N}$ such that $r_{\nu_0} \leq R_0$ because the sequence (r_ν) is strictly increasing. In view of the above, we define

$$p := p_0^{r_0,1} \cup \left(\bigcup_{v=1}^{v0} p_0^{r_v,m_v+1} \right),$$

Which is the desire partition of our sector S.

Lemma (4.2.5)[170]: Let some fixed $m \in \mathbb{N}$. Then

$$\sigma_m = \theta_{m_1(m)-m+1}^{(m)} - \theta_0 < \frac{1}{4}$$

In particular, $v_m \geq m_1(m) - m + 1$.

Proof: By the definition of the numbers $\theta_j^{(m)},\ j = 0,1,\dots,m_1(m)+1$ we have

$$\theta_{m_1(m)-m+1}^{(m)} - \theta_0 = c_2. \sum_{k=m}^{m_1(m)} \frac{1}{|\mu_k|}, \tag{36}$$

And by the definition of the number $m_1(m)$ it follows that

$$\sum_{k=m}^{m_1(m)} \frac{1}{|\mu_m|} \leq c_3. \frac{1}{|\mu_m|} + \frac{1}{|\mu_{m_1(m)}|} < (c_3+1) \frac{1}{|\mu_m|}. \tag{37}$$

Our hypotheses imply $c_1 = 4(c_3+1)$ and $|\mu_m| > c_1 = 4(c_3+1) > 4c_2(c_3+1)$ because $c_2 \in (0,1)$. This gives

$$\frac{c_3+1}{|\mu_m|} < \frac{1}{4c_2}. \tag{38}$$

Thus, (36), (37) and (38) yield $\sigma_m < \frac{1}{4}$ and the proof is complete.

Lemma (4.2.6)[170]: $\lim\limits_{v \to +\infty} r_v = +\infty$.

Proof: Below, let us rewrite the relations that define the numbers $(r_v), v = 0,1,2,\dots$.

$$r_1 = r_0 + \frac{c_2}{|\mu_{m_1(1)}|}, \tag{39}$$

$$r_2 = r_1 + \frac{c_2}{|\mu_{m_2}|}, \tag{40}$$

$$r_{v+1} = r_v + \frac{c_2}{|\mu_{m_v+1}|}, \quad v = 1,2,\dots \tag{41}$$

Where $m_2 := m_1(m_1(1)+1)$. Equalities (39), (40), (41) imply

$$r_v = r_0 + c_2 \sum_{k=1}^{v} \frac{1}{|\mu_{m_k}|} \quad for\ \ v = 1,2,\dots, \quad where\ m_1 = m_1(1) \tag{42}$$

By the definition of $m_1(1), m_2$ we have

$$\sum_{k=1}^{m_1(1)} \frac{1}{|\mu_k|} \leq c_3. \frac{1}{|\mu_1|} + \frac{1}{|\mu_{m_1(1)}|} < (c_3+1) \frac{1}{|\mu_1|}, \tag{43}$$

$$\sum_{k=m_1(1)+1}^{m_2} \frac{1}{|\mu_k|} \leq c_3. \frac{1}{|\mu_{m_1(1)+1}|} + \frac{1}{|\mu_{m_2}|} < (c_3+1). \frac{1}{|\mu_{m_1(1)}|}. \tag{44}$$

Inductively, for every $v \geq 2$ we get

$$\sum_{k=m_v+1}^{m_{v+1}} \frac{1}{|\mu_k|} \le c_3 . \frac{1}{|\mu_{m_v+1}|} + \frac{1}{|\mu_{m_v+1}|} < (c_3+1) \frac{1}{|\mu_{m_v}|}, \tag{45}$$

Because the sequence $(|\mu_n|)$ is strictly increasing. So by (43), (44) and (45) we conclude that

$$\sum_{k=1}^{m_v+1} \frac{1}{|\mu_k|} < (c_3+1) . \sum_{k=0}^{v} \frac{1}{|\mu_{m_k}|}, \tag{46}$$

Where

$$m_0 := 1, \quad m_1 := m_1(1).$$

On the other hand $\sum_{k=1}^{+\infty} \frac{1}{|\mu_k|} = +\infty$ by our assumption. This fact and (46) give us

$$\sum_{k=0}^{+\infty} \frac{1}{|\mu_{m_k}|} = +\infty. \tag{47}$$

Now by (42) and (47) we conclude that $\lim_{v \to +\infty} r_v = +\infty$ and the proof is complete.

Fix the numbers $r_0, R_0, \theta_0, \theta_T, c_1, c_2, c_3, c_4$ which are defined. We fix a subsequence (μ_n) of (λ_n) satisfying the following:

(i) $|\mu_n|, |\mu_{n+1}| - |\mu_n| > c_1$ for $n = 1,2, \dots$

(ii) $\sum_{k=1}^{+\infty} \frac{1}{|\mu_k|} = +\infty$

Finally, on the basis of the above, we consider the partition p constructed.

We construct a certain family of pairwise disjoint disks, based on the previous partition p of the sector S. This family points out how one can use Runge's theorem to conclude the Proposition (4.2.2). Let us describe, very briefly, the highlights of our argument. The main idea is to assign to each point w of the partition p a suitable closed disk $B(w\mu(w), c_4)$ with center $w\mu(w)$ and radius c_4 (the radius will be the same for every member of the family of the disks), where $\mu(w)$ will be chosen from the sequence (μ_n), so that on the one hand the disks $B(w\mu(w), c_4)$, $w \in p$ are pairwise disjoint and on the other hand the union of the disks, $\bigcup_{w \in p} B(w\mu(w), c_4)$ "almost fills" the sector S.

So, let us begin with the desired construction. We set

$$\mathcal{B} := \{z \in \mathbb{C} / |z| \le c_4\}.$$

Let $w \in p$ be a fixed point in p. By the definition of p there exist unique $r' \in \{r_0, r_1, \dots, r_{v_0}\}, m_1 \in \{1, m_1(1), m_2+1, \dots, m_{v_0}+1\}$ such that $w \in p_0^{r',m'}$. By definition, $p_0^{r',m'} = \phi_{r'}(\Delta_{m'})$. So there exists unique $k \in \mathbb{N}. k \ge 1$ and $j \in \{0,1, \dots, m_1(m') - m'\}$ such that $n = k(m_1(m') - m' + 1) + j$; so we define

$$\mu(w) := \mu_{m'+j}.$$

Thus we assign, in a unique way, a term of the sequence (μ_n) to each one from the point of p. Finally we set

$$B_w := B + w\mu(w).$$

The desired family of disks is the following:

$$\mathfrak{D} := \{B\} \cup \{B_w : w \in p\}.$$

Lemma (4.2.7)[170]: We have $B \cap B_w = \emptyset$ for every $w \in p$.

Proof: $c_3 = \frac{c_4}{r_0 c_2} > \frac{c_4}{r_0}$, since $c_2 \in (0,1)$. So $2c_3 > 2\frac{c_4}{r_0}$ and in view of $c_1 4(c_3+1) > 2c_3$ we get

$$c_1 > \frac{2c_4}{r_0} \tag{48}$$

Take $w \in p$. The closed disks B, B_w, are centered at, 0, $w\mu(w)$ respectively and they have the same radius c_4. Hence, we have to show that $|w\mu(w)| > 2c_4$. Since $|w| \geq r_0$, it satisfices to prove that $|\mu(w)| > \frac{2c_4}{r_0}$. Respectively and they have the same radius $\mu(w)$,

$$\mu(w) = \mu \tag{49}$$

For some positive integer $n \in \mathbb{N}$ and from the choice of (μ_n)

$$|\mu_n| > c_1 \quad \text{for every} \quad n \in \mathbb{N}. \tag{50}$$

Now, (48), (49) and (50) imply $|\mu(w)| > 2\frac{c_4}{r_0}$ and this finishes the proof of the lemma.

Lemma (4.2.8)[170]: Let $w_1, w_2 \in p$ such that $|w_1| < |w_2|$. Then $B_{w_1} \cap B_{w_2} = \emptyset$.

Proof: We have

$$m_0 = 1 < m_1(1) + 1$$
$$m_2 = m_1(m_1(1) + 1) > m_1(1)$$

And generally

$$m_{v+1} = m_1(m_v + 1) > m_v \quad for \quad v = 1,2m \ldots, v_0$$

Since $w_1, w_2 \in p$, we have $w_1 \in p_0^{r_{v_1}, m_{v_1}+1}, w_2 \in p_0^{r_{v_2}, m_{v_2}+1}$ for some $v_1, v_2 \in \{0,1,\ldots,v_0\}$ and so $|w_1| = r_{v_1}, |w_2| = r_{v_2}$. Our hypothesis $|w_1| < |w_2| \Leftrightarrow r_{v_1} < r_{v_2}$ and the fact the sequence (r_v) is strictly increasing gives us $v_1 < v_2$. Thus $m_{v_1} + 1 < m_{v_2} + 1$, because the finite sequence $(m_v), v \in \{0,1 \ldots, v_0\}$ is strictly increasing; recall that $m_0 = 1, m_1 = m_1(1)$. By the definition of $\mu(w)$ for $w \in p_0^{r', m'} \subset p$ we get $\mu(w) = \mu_{m'+j}$ for some $j \in \{0,1, \ldots., m_1(m') - m'\}$, so $|\mu_{m'}| \leq |\mu(w)| \leq \left|\mu_{m_1(m')}\right|$, since the sequence $(|\mu_n|)$ is strictly increasing. The fact that $w_1 \in p_0^{r_{v_1}, m_{v_1}+1}$ implies

$$\left|\mu_{m_{v_1}+1}\right| \leq |\mu(w_1)| \leq \left|\mu_{m_1(m_{v_1}+1)}\right| = \left|\mu_{m_{v_1}+1}\right| < \left|\mu_{m_{v_1}+1}\right| \leq \left|\mu_{m_{v_2+1}}\right|,$$

Since $v_1 + 1 \leq v_2$ and the sequence $(|\mu_n|)$ is strictly increasing (48). On the other hand we have $w_2 \in p_0^{r_{v_2}. m_{v_2+1}}$, so

$$\left|\mu_{m_{v_2+1}}\right| \leq |\mu(w_2)| \leq \left|\mu_{m_{v_2}+1}\right|.$$

Hence, the last two inequalities above give

$$|\mu(w_1)| < |\mu(w_2)|,$$

Which iin turn implies

$$|w_2 \mu(w_2)| > |w_1 \mu(w_1)|. \tag{51}$$

By (51) and the hypothesis we get

$$|w_2\mu(w_2) - w_1\mu(w_1)| \geq \big||w_2\mu(w_2)| - |w_1\mu(w_1)|\big| = |w_2\mu(w_2)| - |w_1\mu(w_1)|$$
$$> |w_1||\mu(w_2)| - |w_1||\mu(w_1)| \geq r_0(|\mu(w_2)| - |(w_1)|) > r_0 c_1 > 2c_4,$$

Where the last inequality in the right hand side above follows from $c_1 > \frac{2c_4}{r_0}$ which is already established in Lemma (4.2.7). This shows that $B_{w_1} \cap B_{w_2} = \emptyset$.

Lemma (4.2.9)[170]: Let $w_1, w_2 \in p$ such that $w_1 \neq w_2$ and $|w_1| = |w_2|$. Then $B_{w_1} \cap B_{w_2} = \emptyset$.

Proof: We distinguish two cases:
(i) $|\mu(w_1)| < |\mu(w_2)|$.
In this case, by our hypothesis, we have

$$|w_2\mu(w_2) - w_1\mu(w_1)| \geq \big||w_2\mu(w_2)| - |w_1\mu(w_1)|\big| = |w_1|.\left(\big|\mu(w_2) - |\mu(w_1)|\big|\right) \geq r_0.c_1 > 2c_4.$$

Therefore $B_{w_1} \cap B_{w_2} = \emptyset$.
(ii) $|\mu(w_1)| = |\mu(w_2)|$.
Since $w_1, w_2 \in p$ it follows that $w_1 \in p_0^{r_{v_1}, m_{v_1}+1}, w_2 \in p_0^{r_{v_2}, m_{v_2}+1}$ for some $v_1, v_2 \in \{0,1,\dots,v_0\}$. By the equalities $|w_1| = r_{v_1}, |w_2| = r_{v_2}$ and the hypothesis $|w_1| = |w_2|$ we conclude that $r_{v_1} = r_{v_2}$, which in turn implies $v_1 = v_2$, since the sequence (r_v) is strictly increasing. Setting $v_1 = v_2 = v'$ we get $w_1, w_2 \in p_0^{r_{v'}, m_{v'}+1}$ for some $v' \in \{0,1,\dots,v_0\}$, that is w_1, w_2belong to the same partition of zero order. For simplicity we write $m_{v'} + 1 = m'$. Also set $r_{v'} = r'$. So, $w_1, w_2 \in p_0^{r',m'}$ and the definition of the set $p_0^{r',m'}$gives us $w_1 = r'.e^{2\pi i\theta_{n_1}^{(m')}}, w_2 = r'.e^{2\pi i\theta_{n_2}^{(m')}}$ for some $n_1, n_2 \in \{0,1,\dots,v_{m'}\}$, $n_1 \neq n_2$, since $w_1 \neq w_2$. Without loss of generality suppose that $n_1 < n_2$. Now, there exists a unique pair (k_1, j_1), where $k_1 \in \mathbb{N}, j_1 \in \{0,1,\dots,m_1(m') - m'\}$ and a unique pair (k_2, j_2) where $k_2 \in \mathbb{N}$ and $j_2 \in \{0,1,\dots,m_1(m') - m'\}$ such that

$$n_1 = k_1(m_1(m') - m' + 1) + j_1 \tag{52}$$

and

$$n_2 = k_2(m_1(m') - m' + 1) + j_2 \tag{53}$$

By definition, $\mu(w_1) = \mu_{m'+j_1}$ and $\mu(w_2\,) = \mu_{m'+j_2}$and our hypothesis implies

$$|\mu(w_1)| = |\mu(w_2)| \Leftrightarrow \mu(w_1) = \mu(w_2).$$

So we have $j_1 = j_2 = j_0$ and

$$\theta_{n_1}^{(m')} = \theta_{j_0}^{(m')} + k_1\sigma_{m'},$$
$$\theta_{n_2}^{(m')} = \theta_{j_0}^{(m')} + k_2\sigma_{m'},$$

Thus

$$\theta_{n_2}^{(m')} - \theta_{n_1}^{(m')} = (k_2 - k_1)\sigma_{m'}. \tag{54}$$

By (52), (53) and the fact that $n_1 < n_2$ and $j_1 = j_2$ we have $k_1 < k_2 \Rightarrow k_2 \geq k_1 + 1$. So, in view of (54) we arrive at

$$\theta_{n_2}^{(m')} - \theta_{n_1}^{(m')} \geq \sigma_{m'} > 0. \tag{55}$$

The previous imply the following bound.

$$\begin{aligned}
|w_2\mu(w_2) - w_1\mu(w_1)| &= |\mu(w_1)| \cdot |w_1 - w_2| \geq \mu_{m'}| \cdot |w_1 - w_2| \\
&= |\mu_{m'}| \cdot \left|r' \cdot e^{2\pi i\theta_{n_2}^{(m')}} - r'e^{2\pi i\theta_{n_1}^{(m')}}\right| = r'|\mu_{m'}| \cdot \left|e^{2\pi i\theta_{n_2}^{(m')}} - e^{2\pi i\theta_{n_1}^{(m')}}\right| \\
&= r'|\mu_{m'}| \cdot 2\sin\left(\pi\left(\theta_{n_2}^{(m')} - \theta_{n_1}^{(m')}\right)\right) \\
&\geq r_0 \cdot |\mu_{m'}| \cdot 2\sin\left(\pi\left(\theta_{n_2}^{(m')} - \theta_{n_1}^{(m')}\right)\right)
\end{aligned} \tag{56}$$

Now, consider Jordan’s inequality

$$\sin x > \frac{2}{\pi}x, \qquad x \in \left(0, \frac{\pi}{2}\right).$$

We have

$$0 < \theta_{n_2}^{(m')} - \theta_{n_1}^{(m')} \le \frac{1}{4} \Longrightarrow 0 < \pi\left(\theta_{n_2}^{(m')} - \theta_{n_1}^{(m')}\right) < \frac{\pi}{4}.$$

So, applying Jordan's inequality for

$$x = \pi\left(\theta_{n_2}^{(m')} - \theta_{n_1}^{(m')}\right)$$

We get

$$\sin\left(\pi\left(\theta_{n_2}^{(m')} - \theta_{n_1}^{(m')}\right)\right) > 2\left(\theta_{n_2}^{(m')} - \theta_{n_1}^{(m')}\right) \tag{57}$$

By (55), (56) and (57) it follows that

$$|w_2\mu(w_2) - w_1\mu(w_1)| > 4r_0|\mu_{m'}| \cdot \sigma_{m'} \tag{58}$$

The definition of the number $\sigma_{m'}$ and relation (36) of Lemma (4.2.5) yield

$$\sigma_{m'} = c_2 \cdot \sum_{k=m'}^{m_1(m')} \frac{1}{|\mu_k|}.$$

By this fact, inequality (58) and the definition of the number $m_1(m')$ we get

$$\begin{aligned} |w_2\mu(w_2) - w_1\mu(w_1)| &> 4r_0|\mu_{m'}| \cdot c_2 \sum_{k=m'}^{m_1(m')} \frac{1}{|\mu_k|} \\ &> 4r_0|\mu_{m'}| \cdot c_2 \frac{c_3}{|\mu_{m'}|} = 4r_0c_2c_3. \end{aligned} \tag{59}$$

Recall that $c_3 = \frac{c_4}{r_0c_2}$. So

$$4r_0c_2c_3 = 4r_0c_2 \cdot \frac{c_4}{r_0c_2} = 4c_4 > 2c_4.$$

The last bound along with (59) give $B_{w_1} \cap B_{w_2} = \emptyset$ and the proof of the lemma is complete.
By Lemmas (4.2.7), (4.2.8), (4.2.9) we conclude the following

Corollary (4.2.10)[170]: The family $\mathfrak{D} \coloneqq \{B\} \cup \{B_w : w \in p\}$ consists of pairwise disjoint disks.

Lemma (4.2.11)[170]: For every $j, s, k \in \mathbb{N}$ the set $\bigcup_{m=1}^{+\infty} E(m, j, s, k)$ is dense in $H(\mathbb{C})$.

The proof of Lemma (4.2.3) is in [203]. The proof of Lemma (4.2.4) is similar to that in Lemma 9 of [38] and it is omitted.

We now move on to Lemma (4.2.11).

Proof: Let $j_1, s_1, k_1 \in \mathbb{N}$ be fixed. Our aim is to prove that the set $\bigcup_{m=1}^{+\infty} E(m, j_1, s_1, k_1)$ is dense in $H(\mathbb{C})$. For simplicity we write $p_{j_1} = p$. Fix $g \in H(\mathbb{C})$, a compact set $C \subseteq \mathbb{C}$ and $\varepsilon_0 > 0$. We seek $f \in H(\mathbb{C})$ and a positive integer m_1 such that

$$f \in E(m_1, j_1, s_1, k_1) \tag{60}$$

And

$$\sup_{z \in C} |f(z) - g(z)| < \varepsilon_0. \tag{61}$$

Fix $R_1 > 0$ sufficiently large so that

$$C \cup \{z \in \mathbb{C} | |z| \le k_1\} \subset \{z \in \mathbb{C} | |z| \le R_1\}$$

And then choose $0 < \delta_0 < 1$ such that

$$\text{if } |z| \le R_1 \text{ and } |z - w| < \delta_0, \quad w \in \mathbb{C}, \text{ then } |p(z) - p(w)| < \frac{1}{2s_1}. \tag{62}$$

Define

$$B := \{z \in \mathbb{C} \mid |z| \leq R_1 + \delta_0\},$$
$$c_4 := R_1 + \delta_0, \qquad c_2 := \frac{\delta_0}{2(2R_0\pi + 1)},$$
$$c_3 = \frac{c_4}{r_0 c_2} = \frac{R_1 + \delta_0}{r_0 \frac{\delta_0}{2(2R_0\pi + 1)}} = \frac{2(R_1 + \delta_0)(2R_0\pi + 1)}{r_0\delta_0},$$
$$c_1 = 4(c_3 + 1) = 4 \cdot \left(\frac{2(R_1 + \delta_0)(2R_0\pi + 1)}{r_0\delta_0} + 1\right).$$
After the definition of the above numbers we choose a subsequence (μ_n) of (λ_n) such that
(i) $|\mu_n| > c_1 \quad |\mu_{n+1}| - |\mu_n| > c_1$ for $n = 1,2, \dots$ and
(ii) $\sum_{n=1}^{+\infty} \frac{1}{|\mu_n|} = +\infty$
On the basis of the fixed numbers $r_0, R_0, \theta_0, \theta_T, c_1, c_2, c_3, c_4$ and the choice of the sequence (μ_n) we define the set L as follows:
$$L: B \cup \left(\bigcup_{w \in p} B_w\right)$$
where the partition p and the discs $B_w, w \in p$ are constructed. By Corollary (4.2.10), the family $\mathfrak{D}$ consists of pairwise disjoint disks. Therefore the compact set L has connected complement. This property is needed in order to apply Mergelyan's theorem. We now define the function h on the compact set $L, h: L \to \mathbb{C}$ by
$$h(z) = \begin{cases} g(z), & z = B \\ p\big(z - w\lambda(w)\big), & z \in B_w, \; w \in p. \end{cases}$$
By Mergelya's theorem [199] there exists an entire function f (in fact a polynomial) such that
$$\sup_{z \in L} |f(z) - h(z)| < \min\left\{\frac{1}{2s_1}, \varepsilon_0\right\}. \tag{63}$$
The definition of h and (63) give
$$\sup_{z \in C} |f(z) - g(z)| \leq \sup_{z \in B} |f(z) - g(z)| = \sup_{z \in L} |f(z) - h(z)| < \varepsilon_0,$$
Which implies the desired inequality (61). It remains to show (60).
Let $a \in S$. Then $a = re^{2\pi i\theta}$ for some $r \in [r_0, R_0]$ and $[\theta_0, \theta_T]$. There exists a unique $n_0 \in \{0,1, \dots, v_0 - 1\}$ such that either $r_{n_0} < r < r_{n_0+1}$ or $r_{v_0} \leq r \leq R_0$.
We set
$$r_1 := r_{n_0}, \qquad r_2 :=_{n_0+1} \qquad if \quad r_{n_0} \leq r \leq r_{n_0+1}$$
and
$$r_1 := r_{v_0}, \quad r_2 := R_0 \quad if \quad r_{v_0} \leq r \leq R_0.$$
By the construction of the partition p there exists a unique $m' \in \mathbb{N}$ such that $p_0^{r_1, m'} \subset p$. In addition, there exists unique $\rho \in \{0,1, \dots, v_{m'} - 1\}$ such that either
$$\theta_\rho^{(m')} \leq \theta < \theta_{\rho+1}^{(m')} \quad or \quad \theta_{v_{m'}}^{(m')} \leq \theta \leq \theta_T.$$
Define now
$$\theta_1 := \theta_\rho^{(m')}, \quad \theta_\rho^{(m')} \quad if \quad \theta_\rho^{(m')} \leq \theta < \theta_{\rho+1}^{(m')}$$

and

$$\theta_1 \coloneqq \theta_{v_{m'}}^{(m')}, \quad \theta_2 \coloneqq \theta_T \quad if \quad \theta_{v_{m'}}^{(m')} \le \theta \le \theta_T$$

and then set

$$w_0 \coloneqq r_1 \cdot e^{2\pi i\theta_1} \in p_0^{r_1, m'}.$$

We shall prove now that for every $z \in \mathbb{C}$ with $|z| \le R_1,\ z + a\mu(w_0) \in B_{w_0}$.
Recall that $B_{w_0} \coloneqq B + w_0\mu(w_0) = \overline{D}(w_0\mu(w_0), R_1 + \delta_0)$. It suffices to prove that

$$|z + a\mu(w_0) - w_0\mu(w_0)| < R_1 + \delta_0 \qquad for \quad |z| \le R_1. \tag{64}$$

For $|z| \le R_1$ we have,

$$\begin{aligned} |z + a\mu(w_0) - w_0\mu(w_0)| &\le R_1 + |\mu(w_0)||a - w_0| \\ &= R_1 + |\mu(w_0)| \cdot \left|r \cdot e^{2\pi i\theta} - r_1 e^{2\pi i\theta_1}\right|. \end{aligned} \tag{65}$$

By (64) and (65) it suffices to prove

$$|\mu(w_0)| \cdot \left|re^{2\pi i\theta} - r_1 e^{2\pi i\theta_1}\right| < \delta_0. \tag{66}$$

We now have

$$\begin{aligned} \left|re^{2\pi i\theta_1} - r_1 e^{2\pi i\theta_1}\right| &= \left|re^{2\pi i\theta} + r_1 e^{2\pi i\theta} - r_1 e^{2\pi i\theta_1}\right| \\ &\le \left|re^{2\pi i\theta} - r_1 e^{2\pi i\theta}\right| + \left|r_1 e^{2\pi i\theta_1}\right| \\ &\le |r - r_1| + r_1\left|e^{2\pi i\theta} - e^{2\pi i\theta_1}\right| \\ &\le |r_1 - r_2| + R_0 2\sin\big(\pi(\theta_1 - \theta)\big) \\ &\le (r_2 - r_1) + R_0 2\sin\big(\pi(\theta_2 - \theta_1)\big) \\ &< (r_2 - r_1) + 2R_0\pi(\theta_2 - \theta_1) \\ &\le R_0\pi \cdot \frac{c_2}{|\mu(w_0)|} + \frac{c_2}{|\mu(w_0)|} \\ &= (2R_0\pi + 1) \cdot \frac{\delta_0}{2(2R_0\pi + 1)} . \frac{1}{|\mu(w_0)|} = \frac{\delta_0}{2|\mu(w_0)|} \end{aligned}$$

which implies (66). So we proved that for every $z \in \mathbb{C},\ |z| \le R_1$

$$z + a\mu(w_0) \in B_{w_0}. \tag{67}$$

By the definition of h and (67) we have that for every $z \in \mathbb{C}$ with $|z| \le R_1$

$$\left|f\big(z + a\mu(w_0)\big) - p\left(z + \mu(w_0)\big(re^{2\pi i\theta} - r_1 e^{2\pi i\theta_1}\big)\right)\right| < \frac{1}{2s_1}. \tag{68}$$

Take any $z \in \mathbb{C}$ with $|z| \le R_1$. By (62) and (66)

$$\left|p\left(z + \mu(w_0)\big(re^{2\pi i\theta} - r_1 e^{2\pi i\theta_1}\big)\right) - p(z)\right| < \frac{1}{2s_1} \tag{69}$$

and the triangle inequality gives

$$\begin{aligned} \left|f\big(z + a\mu(w_0)\big) - p(z)\right| &\le \left|f\big(z + a\mu(w_0)\big) - p\left(z + \mu(w_0)\big(re^{2\pi i\theta} - r_1 e^{2\pi i\theta_1}\big)\right)\right| \\ &+ \left|p\left(z + \mu(w_0)\big(re^{2\pi i\theta} - r_1 e^{2\pi i\theta_1}\big)\right) - p(z)\right|. \end{aligned} \tag{70}$$

Using (68), (69), (70) we arrive at

$$\left|f\big(z + a\mu(w_0)\big) - p(z)\right| < \frac{1}{s_1}$$

And since $k_1 \le R_1$ it readily follows that

$$\sup_{|z| \le k_1} \left|f\big(z + a\mu(w_0)\big) - p(z)\right| < \frac{1}{s_1} \tag{71}$$

Set

$$m_1 := \max\{n \in \mathbb{N} | \lambda_n = \mu(w), \quad \text{for some} \quad w \in p\}$$

And observe that the definition of $m_1$is independent from $a \in S$. Thus, by the previous we conclude that for every $a \in S$ there exists some $n \in \mathbb{N}$, $n \leq m_1$ such that

$$\sup_{|z| \leq k_1} |f(z + +a\lambda_n) - p(z)| < \frac{1}{s_1},$$

Where $f \in H(\mathbb{C})$, since f is a polynomial. This completes the proof of the lemma.

By the remark in [188] we have a sample of first examples satisfying condition ($\sum$);

$$\lambda_n = n, \quad \lambda_n = n(\log n)^p \quad for \quad p \leq 1, \quad \lambda_n = n \log n \log \log n.$$

In all the above examples we also have $\left|\frac{\lambda_{n+1}}{\lambda_n}\right| \to 1$ as $n \to +\infty$. However, for sequences (λ_n), such that $\lambda_n \to \infty$ and $\left|\frac{\lambda_{n+1}}{\lambda_n}\right| \to 1$ we have that the conclusion of Theorem (4.2.1) holds by the main result in [203]. It is our aim to show that there exist sequences (λ_n), such that: $\lambda_n \to \infty$, (λ_n) satisfies condition ($\sum$) and the ratio $\left|\frac{\lambda_{n+1}}{\lambda_n}\right|$ does not tend to 1.

Let us see things more specifically. Consider a sequence $\Lambda = (\lambda_n)$ of non-zero complex numbers and define

$$B(\Lambda) := \left\{a \in 0, +\infty | \exists(\mu_n) \subset \Lambda \ with \ a = \limsup_n \left|\frac{\mu_{n+1}}{\mu_n}\right|\right\},$$

$$i(\Lambda) := \inf B(A).$$

Clearly

$$i(\Lambda) \in [0, +\infty]$$

and

If $\lambda_n \to \infty$ then $B(\Lambda) \subset [1, +\infty]$ and $i(\Lambda) \in [1. +\infty]$.

We say that a sequence of non-zero complex numbers $\Lambda = (\lambda_n)$ satisfies condition ($\sum'$) if $i(\Lambda) = 1$. In [203] we established the following result.

If $\Lambda = (\lambda_n)$ is a sequence of non-zero complex numbers such that $\lambda_n \to \infty$ and Λ satisfies condition ($\sum'$), then the conclusion of Theorem (4.2.1) holds. In view of the above result the following question arises naturally.

Below we construct specific examples of sequences $\Lambda = (\lambda_n)$ such that $\lambda_n \to \infty, i(\Lambda) = M$ for any fixed positive number $M > 1$ and Λ ssatidfies ($\sum$).

(i) Firstly, we give affirmative reply to Question 1 of [188].

(ii) Secondly, for certain sequences, we also give a positive answer.

(iii) Thirdly, we exhibit a variety of examples of sequences $\Lambda = (\lambda_n)$ of non-zero complex numbers with $\lambda_n \to \infty$ such that Λ satisfies condition ($\sum$) and it does not satisfy condition ($\sum'$).

The above discussion shows that the problem of deciding whether a sequence $\Lambda = (\lambda_n)$, such that $\lambda_n \to \infty$ and $i(\Lambda) = M$ for some $M > 1$ satisfies the conclusion of Theorem (4.2.1).

Proposition (4.2.12)[170]: For every $M > 1$ there exist a sequence $\Lambda = (\lambda_n)$ such that $\lambda_n \to \infty$, $i(\Lambda) = M$ and condition ($\sum$) holds for Λ. Thus, for every $M > 1$ there exists a sequence of non-zero complex numbers $\Lambda = (\lambda_n)$ such that $\lambda_n \to \infty$ as $n \to +\infty$, $i(\Lambda) = M$ and $\bigcap_{a \in \mathbb{C}\setminus\{0\}} HC(\{T_{\lambda_n a}\})$ is a G_δ and dense subset of $H(\mathbb{C})$.

Proof: Fix a positive number $M_0 > 1$. We shall construct a sequence of non-zero complex numbers $\Lambda = (\lambda_n)$ such that $\lambda_n \to \infty$, $i(\Lambda) = M_0$ and condition ($\sum$) holds for Λ. The sequence Λ will be a strictly increasing sequence of positive numbers such that $\lambda_n \to +\infty \ \ as\ n \to +\infty$.

We construct inductively a countable family $\{\mathfrak{D}_n\}$, $n = 1,2,\dots$ of sets $\mathfrak{D}_n \subset [1,+\infty)$ according to the following rules.

(i) $\mathfrak{D}_1 = \{1\}$.

(ii) $\mathfrak{D}_n = \{a_n + v | v = 0,1,\dots,([a_n]+1)!\}, \quad n = 1,2,\dots$

(iii) $\min \mathfrak{D}_{n+1} = M_0 \max \mathfrak{D}_n$ for each $n = 1,2,\dots,$

Where $a_n = \min \mathfrak{D}_n$ and $[x]$ denotes the integer part of the real number x as usual. Observe that every $n,m \in \mathbb{N}, \quad n \neq m, \mathfrak{D}_n \cap \mathfrak{D}_m = \emptyset$. Set

$$\tilde{\Lambda} = \bigcup_{n=1}^{+\infty} \mathfrak{D}_n.$$

We define the sequence $\Lambda = (\lambda_n)$ to be the enumeration of $\tilde{\Lambda}$ by the natural order. It is obvious that $\lambda_n \neq 0\ \forall n \in \mathbb{N}$, $\lim_{n\to+\infty} \lambda_n = +\infty$, and (λ_n) is a strictly increasing sequence of positive numbers. We prove now the following

Claim (4.2.13)[170]: For every subsequence $\mu = (\mu_n)$ of Λ we have $\lim_{n\to+\infty} \sup \frac{\mu_{n+1}}{\mu_n} \geq M_0$..

Proof: Firstly we prove that for every natural number $m \in \mathbb{N}$, there exists some $N \in \mathbb{N}\ N \geq m$ such that

$$\frac{\mu_{N+1}}{\mu_N} \geq M_0$$

So, take any $m \in \mathbb{N}$ and let m_1 be the unique natural number such chat $\mu_m \in \mathfrak{D}_{m_1}$.

Setting $A_{m_1} \coloneqq \{n \in \mathbb{N} | \mu_n\} \in \mathfrak{D}_{m_1}\}$, it is obvious that $A_{m_1} \neq \emptyset$, since $m \in A_{m_1}$.

We set $m_2 \coloneqq \max A_{m_1}$. Then $\mu_{m_2+1} \notin \mathfrak{D}_{m_1}$and so $\mu_{m_2+1} \geq \min \mathfrak{D}_{m_1+1}$. We have $\mu_{m_2} \leq \max \mathfrak{D}_{m_1}$, thus

$$\frac{\mu_{m_2+1}}{\mu_{m_2}} \geq \frac{\min \mathfrak{D}_{m_1+1}}{\max \mathfrak{D}_{m_1}} = M_0 \quad and \quad m_2 \geq m_1.$$

So we proved that for every $m \in \mathbb{N}$, there exist some $N \geq m$ such that $\frac{\mu_{N+1}}{\mu_N} \geq M_0$.

We incorporate the last fact into an inductive argument and obtain the following.

For $m = 1$ there exists $k_1 \in \mathbb{N},\ k_1 \geq 1$ such that $\frac{\mu_{k_1+1}}{\mu_{k_1}} \geq M_0$. For $m = k_1 + 1$, there exists some $k_2 \geq k_1 + 1$ (especially $k_2 > k_1$) such that $\frac{\mu_{k_2+1}}{\mu_{k_2}} \geq M_0$. Suppose that for some $v \in \mathbb{N}$ we have some $k_v \in \mathbb{N}$ such that $\frac{\mu_{k_2+1}}{\mu_{k_2}} \geq M_0$. Then for $m = k_v + 1$

There exist some $k_{v+1} \geq k_v + 1$ (especially $k_{v+1} > k_v$) such that $\frac{\mu_{k_{v+1}+1}}{\mu_{k_{v+1}}} \geq M_0$.

Therefore we obtain a subsequence $(\mu_{k_v}),\ \ v = 1,2,\dots$ of (μ_n) such that $k_{v+1} > k_v$ for each $v = 1,2,\dots$ and $\frac{\mu_{k_v+1}}{\mu_{k_v}} \geq M_0$.

This gives $\lim_{v\to+\infty} \sup \frac{\mu_{k_v+1}}{\mu_{k_v}} \geq M_0$, which in turn implies

$$\lim_{n\to+\infty}\sup\frac{\mu_{n+1}}{\mu_n}\geq M_0|$$

This completes the proof of Claim (4.2.13).

Claim (4.2.14)[170]: $\lim_{n\to+\infty}\sup\frac{\lambda_{n+1}}{\lambda_n}=M_0$.

Proof: Let $n\in\mathbb{N}$. If $\lambda_n,\lambda_{n+1}\in\mathfrak{D}_m$ for some positive integer m, then by the construction of $\mathfrak{D}_m$ we have

$$\lambda_{n+1}=\lambda_n+1\Rightarrow\frac{\lambda_{n+1}}{\lambda_n}=1+\frac{1}{\lambda_n}\tag{72}$$

If there is no $m\in\mathbb{N}$ such that $\lambda_n,\lambda_{n+1}\in\mathfrak{D}_m$, then this can happen only if $\lambda_n=\max\mathfrak{D}_m$ and $\lambda_{n+1}=\min\mathfrak{D}_{m+1}$ for some $m\in\mathbb{N}$, hence

$$\frac{\lambda_{n+1}}{\lambda_n}=M_0.\tag{73}$$

By (72), (73) and since $\lambda_n\to+\infty$ the conclusion follows. This completes the proof of Claim (4.2.14).

Claims (4.2.13) and (4.2.14) imply that $i(\Lambda)=M_0$.

Claim (4.2.15)[170]: The sequence Λ satisfies condition $(\sum\ )$.

Proof: Fix some natural number $N_0\geq 2$. We will show that there exists a subsequence (μ_n) of Λ such that

(i) $\mu_{n+1}-\mu_n>N_0$ for every $n=1,2,\ldots$ and

(ii) $\sum_{n=1}^{+\infty}\frac{1}{\mu_n}=+\infty$.

Recall that $a_n=\min\mathfrak{D}_n>N_0$ for every $n\geq 2$. Since

$$\lim_{n\to+\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{2}-\log n\right)=\gamma,$$

Where $\gamma\simeq 0,57722156649\ldots$ is the Euler constant, there exists some natural Number $n_0\in\mathbb{N}$ such that

$$-\frac{1}{2}<\sum_{k=1}^{n}\frac{1}{k}-\log n-\gamma<\frac{1}{2}\quad for\quad n\geq n_0>2.$$

Let some $m,n\in\mathbb{N},m>n\geq n_0$. Then we have

$$\begin{aligned}\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{m}&=\sum_{k=n+1}^{m}\frac{1}{k}=\sum_{k=1}^{m}\frac{1}{k}-\sum_{n=1}^{n}\frac{1}{k}\\&=\left(\sum_{k=1}^{m}\frac{1}{k}-\log m-\gamma\right)-\left(\sum_{k=1}^{n}\frac{1}{k}-\log n-\gamma\right)+\log m-\log n>\log\frac{m}{n}-1\\&=\log\frac{m}{n}+\log e^{-1}=\log\left(\frac{m}{n}.e^{-1}\right)\\&=\log\left(\frac{m}{ne}\right)\end{aligned}\tag{74}$$

It is easy to show that $a_2>n$ for $n\geq 2$. Set $n_1:=\max\{n_0,N_0\}+2$. Let now some $n\in\mathbb{N}$ with $n\geq n_1$. Recall that

$$\begin{aligned}\mathfrak{D}_n&=\{a_n,a_n+1,\ldots,a_n+([a_n]+1)!\}\\&=\{a_n+j|j=0,1,\ldots,([a_n]+1)!\}\end{aligned}$$

Setting $N_1:=N_0+1$ we obtain

$$\frac{1}{a_n}+\frac{1}{a_n+N_1}+\frac{1}{a_n+2N_1}+\cdots+\frac{1}{a_n+\frac{([a_n]+1)}{N_1}.N_1}$$
$$>\frac{1}{N_1a_n}+\frac{1}{N_1a_n+N_1}+\frac{1}{N_1a_n+2N_1}+\cdots+\frac{1}{N_1a_n+N_1.\frac{([a_n]+1)!}{N_1}}$$
$$=\frac{1}{N_1}.\sum_{j=0}^{\frac{([a_n]+1)!}{N_1}}\frac{1}{a_n+j}>\frac{1}{N_1}.\sum_{j=0}^{\frac{([a_n]+1)!}{N_1}}\frac{1}{([a_n]+1)+j} \tag{75}$$

We write for simplicity $v=[a_n]+1$. So by (74), (75) we get

$$\sum_{k=0}^{\frac{v!}{N_1}}\frac{1}{a_n+kN_1}>\frac{1}{N_1}.\log\left(\frac{v+\frac{v!}{N_1}}{(v-1)e}\right)>\frac{1}{N_1}.\log\left(\frac{(v-1)!}{N_1e}\right) \tag{76}$$

We will show that

$$\frac{1}{N_1}.\log\left(\frac{(v-1)!}{N_1e}\right)>v$$

For v big enough. It follows that

$$\frac{1}{N_1}.\log\left(\frac{(v-1)!}{N_1e}\right)>v\Leftrightarrow\log\left(\frac{(v-1)!}{N_1e}\right)>N_1v$$
$$\Leftrightarrow(v-1)!>N_1e.e^{N_1v}=N_1.e^{N_1v+1}$$

Let us consider the sequence $\gamma_v=\frac{(v-1)!}{N_1e^{N_1v+1}}$. By the ratio criterion for (γ_v) we have

$$\frac{\gamma_{v+1}}{\gamma_v}=\frac{\frac{v!}{N_1e^{N_1(v+1)+1}}}{\frac{(v-1)!}{N_1e^{N_1v+1}}}=\frac{v!\cdot e^{N_1v+1}}{(v-1)!\cdot e^{N_1(v+1)+1}}=\frac{v}{e^{N_1}}.$$

So $\lim\limits_{v\to+\infty}\left(\frac{\gamma_{v+1}}{\gamma_v}\right)=+\infty$ which implies that there exists some $n_2\geq n_1$ such that $\gamma_n>1$ for $n\geq n_2$ or equivalently

$$\frac{1}{N_1}\cdot\log\left(\frac{(n-1)!}{N_1e}\right)>n,\qquad n\geq n_2. \tag{77}$$

Thus by (76) and (77) we have:

$$\sum_{k=0}^{\frac{v!}{N_1}}\frac{1}{a_n+kN_1}>[a_n]+1\quad for\quad n\geq n_2.$$

Now for $n\geq n_2$ define the set

$$\mathfrak{D}'_n:=\left\{a_n,a_n+N_1,a_n+2N_1,\dots,a_n+\frac{([a_n]+1)!}{N_1}\cdot N_1\right\}$$

And consider the union

$$\mathfrak{D}':=\bigcup_{n\geq n_2}\mathfrak{D}'_n.$$

Let (μ_n) be the sequence we get when we enumerate $\mathfrak{D}'$ by its natural order. Clearly (μ_n) is a subsequence of Λ and satisfies the desired properties (i) and (ii). This completes the proof of Claim (4.2.15) and hence that of Proposition (4.2.12) using Theorem (4.2.1)

Corollary (4.2.16)[170]: There exists a sequence $\Lambda = (\lambda_n)$ of non-zero complex numbers with $\lambda_n \to \infty$ such that Λ satisfies condition $(\sum\ )$ and it does not satisfy condition $(\sum'\ )$.

Proof: Every sequence $\Lambda = (\lambda_n)$ of non-zero complex numbers with $\lambda_n \to \infty$ which satisfies the conclusion of Proposition (4.2.12), clearly does not satisfy $(\sum'\ )$.

We point out that sequences of the form $(n^2), (n^3), (n^4)$ ..., satisfy condition $(\sum'\ )$. but they do not satisfy $(\sum\ )$. To complete the picture we observe that there are sequences with sufficiently slow growth, such as $(n), \left(\sqrt{n}\right), (\log(n+1)), (\log\log(n+1))$, that satisfy both conditions $(\sum\ )$ and $(\sum'\ )$. Hence, neither $(\sum\ )$ nor $(\sum'\ )$ implies the other and, in addition, they have non-empty intersection. This in turn shows that Theorem (4.2.1) does not follow by the main result in [203].

Chapter 5
Trace with Two-Term Trace and Asymptotic Estimate

We show trace estimates for the relativistic α-stable process extending the result of Bañuelos and Kulczycki in the stable case. We extend previous results obtained for the fractional Laplace operator ($\psi(\xi) = \xi^{\alpha/2}$) and for the Klein–Gordon square root operator ($\psi(\xi) = (1 + \xi^2)^{1/2} - 1$). The formula for the eigenvalues in $(-a, a)$is of the form $\lambda_n = \psi(\mu_n^2) + O(\frac{1}{n})$, where μ_n is the solution of $\mu_n = \frac{n\pi}{2a} - \frac{1}{a}\vartheta(\mu_n)$, and $\vartheta(\mu) \in [0, \frac{\pi}{2})$ is given as an integral involving ψ.

Section (5.1): Unimodal Levy Processes

A two-term small-time uniform approximation for the trace of the transition density of the Wiener process killed off bounded R-smooth domain $D \subset \mathbb{R}^d$, i.e. the classical Dirichlet heat kernel, was obtained by van den Berg [216]. The first term of the approximation is proportional to the domain's volume $|D|$ and the second–to the surface measure $|\partial D|$ of the boundary, with explicit coefficient depending on time.

Asymptotic non-uniform expansions of the trace of the heat kernel were given earlier in [212], see [216].

Bañuelos and Kulczycki [52] obtained a uniform two-term approximation for the isotropic α-stable L$\acute{e}$vy processes. The closely related case of the relativistic α-stable L$\acute{e}$vy processes was resolved by Bañuelos, Mijena and Nane [219]. A similar two-term approximation for Lipschitz domains was given for the Wiener process by Brown [209], and for the isotropic α-stable L$\acute{e}$vy processes–by Bañuelos, Kulczycki and Siudeja [104].

Park and Song [213] obtained a two-term small-time approximation of the trace for the relativistic α-stable L$\acute{e}$vy processes on Lipschitz domains, and gave an explicit power expansion of the first term.

We investigate those L$\acute{e}$vy processes X_t in $\mathbb{R}^d$, where $d \geq 2$, which are unimodal and satisfy the so-called weak lower and upper scaling conditions, denoted WLSC and WUSC respectively, of orders strictly between 0 and 2. The isotropic stable and relativistic L$\acute{e}$vy processes are included as special cases but at present the orders of the lower and upper scalings may differ. For bounded R-smooth open sets $D \subset \mathbb{R}^d$ (also called $C^{1,1}$ open sets) our main result gives a two-term small-time approximation of the trace of the corresponding Dirichlet heat kernel. We resolve sums of independent isotropic stable L$\acute{e}$vy processes with different indexes.

We let ψ be the L$\acute{e}$vy-Khintchine exponent and $p_t(x)$ be the transition density of X_t. We consider

$$\tau_D = \{t > 0 : X_t \notin D\},$$

the first time that X_t exits D. For $t > 0$ and $x, y \in \mathbb{R}^d$, we define the heat remainder

$$r_D(t, x, y) = \mathbb{E}^x[\tau_D < t, p_{t-\tau_D}(X(\tau_D) - y)] . \quad (1)$$

The Dirichlet heat kernel for X_t is given by the Hunt formula:

$$p_D(t, x, y) = p_t(y - x) - r_D(t, x, y), \quad (2)$$

and the trace of X_t on D is

$$tr(t, D) = \int p_D(t, x, x)dx , \qquad t > 0. \quad (3)$$

We denote $\mathbb{H} = \{(x_1,\dots,x_d) \in \mathbb{R}^d : x_1 > 0\}$, a half-space, and for $t > 0$ we let

$$C_{\mathbb{H}}(t) = \int_0^\infty r_{\mathbb{H}}(t,(q,0,\cdot\ \cdot\ \cdot,0),(q,0,\cdot\ \cdot\ \cdot,0))dq.$$

For instance, $C_{\mathbb{H}}(t) = ct^{-d/\alpha+1/\alpha}$ for the isotropic α-stable Lévy process [52].

We remark in passing that the trace can also be studied and interpreted within the spectral theory of the corresponding semigroup given by the integral kernel p_D [52].

We note that sharp pointwise estimates of $r_D(t,x,y)$ complementing [207] would be of considerable interest. We also note that two-term approximations of the trace of the heat kernel of general unimodal Lévy processes are open for Lipschitz domains.

A Borel measure on $\mathbb{R}^d$ is called isotropic unimodal, in short: unimodal, if on $\mathbb{R}^d \setminus \{0\}$ it is absolutely continuous with respect to the Lebesgue measure and has a radially nonincreasing, in particular rotationally invariant, or isotropic density function. Recall that Lévy measure is an arbitrary Borel measure concentrated on $\mathbb{R}^d \setminus \{0\}$ and such that

$$\int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu(dx) < \infty.$$

In what follows we assume that ν is a unimodal Lévy measure and define

$$\psi(\xi) = \int_{\mathbb{R}^d} (1 - cos\,\langle \xi, x\rangle)\nu(dx), \quad \xi \in \mathbb{R}^d, \tag{4}$$

the Lévy -Khintchine exponent. It is a radial function, and we often let $\psi(r) = \psi(\xi)$, where $\xi \in \mathbb{R}^d$ and $r = |\xi| \geq 0$. The same convention applies to all radial functions.

The (radially nonincreasing) density function of the unimodal Lévy measure ν will also be denoted by ν, so $\nu(dx) = \nu(x)dx$ and $\nu(x) = \nu(|x|)$. We point out that for $\lambda \geq 1$ and $r \geq 0$, $\psi(\lambda r) \geq \pi^{-2}\psi(r)$ and $\psi(\lambda r) \geq \pi^{-2}\lambda^2\psi(r)$ [206]. More restrictive inequalities of this type define what are called the weak scaling conditions.

We consider the pure-jump Lévy process $X = (X_t, t \geq 0)$ on $\mathbb{R}^d$ [92], in short: X_t, determined by the Lévy-Khintchine formula

$$\mathbb{E}\, e^{i\langle \xi, X_t\rangle} = e^{-t\psi(\xi)} = \int_{\mathbb{R}^d} e^{i\langle \xi, x\rangle} p_t(dx).$$

The process is (isotropic) unimodal, meaning that all its one-dimensional distributions $p_t(dx)$ are (isotropic) unimodal; in fact the unimodality of ν is also necessary for the unimodality of X_t [217]. In what follows we always assume that ψ is unbounded, equivalently that $\nu(\mathbb{R}^d) = \infty$. In other words X_t below is not a compound Poisson process. Clearly, $\psi(0) = 0$ and $\psi(u) > 0$ for $u > 0$. By [207], $p_t(dx)$ have bounded, in fact smooth density functions $p_t(x)$ for all $t > 0$ if and only if the following Hartman-Wintner condition holds,

$$\frac{\lim_{|\xi|\to\infty} \psi(\xi)}{\ln|\xi|} = \infty. \tag{5}$$

Let V be the renewal function of the corresponding ladder-height process of the first coordinate of X_t. Namely we consider $X_t^{(1)}$, the first coordinate process of X_t, its running maximum $M_t := \sup_{0\leq s\leq t} X_s^{(1)}$ and the local time L_t of $M_t - X_t^{(1)}$ at 0 so normalized that its inverse function L_t^{-1} is a standard $1/2$-stable subordinator. The resulting ladder-height process $\eta(t) := X^{(1)}(L_t^{-1})$ is a subordinator with the Laplace exponent

$$\kappa(u) = -\log \mathbb{E}e^{-u\eta(1)} = \exp\left\{\frac{1}{\pi}\int_0^\infty \frac{\log\psi(u\zeta)}{1+\zeta^2}d\zeta\right\}, \qquad u \geq 0,$$

and $V(x)$ is defined as the accumulated potential of η:

$$V(x) = \mathbb{E}\int_0^\infty 1_{[0,x]}(\eta_t)dt, \qquad x \geq 0.$$

For $x < 0$ we let $V(x) = 0$. For instance, if $\psi(\xi) = |\xi|^\alpha$ with $\alpha \in (0,2)$, then $V(x) = x_+^{\frac{\alpha}{2}}$ [215]. Silverstein studied V and V' as g and ψ in [214]. The Laplace transform of V is

$$\int_0^\infty V(x)e^{-ux}dx = \frac{1}{u\kappa(u)}, \qquad u > 0.$$

The function V is continuous and strictly increasing from $[0,\infty)$ onto $[0,\infty)$. We have $\lim_{r\to\infty} V(r) = \infty$. Also, V is subadditive:

$$V(x+y) \leq V(x) + V(y), \qquad x,y \in \mathbb{R}. \tag{6}$$

For a more detailed discussion of V see [205] and [214].

In estimates we can use V and interchangeably because by [207],

$$V(r) \approx \left[\psi\left(\frac{1}{r}\right)\right]^{-\frac{1}{2}}, \quad r > 0. \tag{7}$$

The above means that there is a constant, i.e. a number $C \in (0,\infty)$, such that for all $r > 0$ we have $C^{-1}V(r) \leq [\psi(1/r)]^{-1/2} \leq CV(r)$. In fact in (7) we have $C = C(d)$, meaning that C may be so chosen to depend only on the dimension, see ibid. To give full justice to V, the function is absolutely crucial in the proofs of [205], [207]. By (6),

$$\frac{1}{2}\varepsilon V(r) \leq V(\varepsilon r) \leq V(r), \qquad 0 < \varepsilon \leq 1, \qquad 0 < r < \infty. \tag{8}$$

We shall assume relative power-type behaviors of $\psi(r)$ at infinity.

We say that ψ satisfies the weak lower scaling condition at infinity (WLSC) if there are numbers $\underline{\alpha} > 0$, $\underline{\theta} \in [0,\infty)$ and $\underline{C} \in (0,1]$, such that

$$\psi(\lambda r) \geq \underline{C}\lambda^{\underline{\alpha}}\psi(r) \text{ for } \lambda \geq 1, \qquad r > \underline{\theta}.$$

Put differently and more explicitly, $\psi(r)/r^{\underline{\alpha}}$ is almost increasing on $(\underline{\theta},\infty)$, i.e.

$$\frac{\psi(s)}{s^{\underline{\alpha}}} \geq \underline{C}\frac{\psi(r)}{r^{\underline{\alpha}}}, \qquad \text{if} \quad s \geq r > \underline{\theta}.$$

In short we write $\psi \in \mathrm{WLSC}(\underline{\alpha},\underline{\theta},\underline{C})$, $\psi \in \mathrm{WLSC}(\underline{\alpha},\underline{\theta})$, $\psi \in \mathrm{WLSC}(\underline{\alpha})$ or $\psi \in \mathrm{WLSC}$, depending on how specific we wish to be about the constants. If $\psi \in \mathrm{WLSC}(\underline{\alpha},\underline{\theta})$, then we say that ψ satisfies the global weak lower scaling condition (global WLSC) if $\underline{\theta} = 0$.

If $\underline{\theta} \geq 0$, then we can emphasize this by calling the scaling local at infinity. We always assume that $\psi \not\equiv 0$, therefore in view of $\psi \in \mathrm{WLSC}$ we have the Hartman-Wintner condition (5) satisfied, and so $\mathbb{R}^d \ni x \mapsto p_t(x)$ is smooth for each $t > 0$.

Similarly, the weak upper scaling condition at infinity (WUSC) means that there are numbers $\overline{\alpha} < 2$, $\overline{\theta} \geq 0$ and $\overline{C} \in [1,\infty)$ such that

$$\psi(\lambda r) \leq \overline{C}\lambda^{\overline{\alpha}}\psi(r) \text{ for } \lambda \geq 1, \qquad r > \overline{\theta}.$$

In short, $\psi \in \mathrm{WUSC}(\overline{\alpha},\overline{\theta},\overline{C})$ or $\psi \in \mathrm{WUSC}$. Global WUSC is $\mathrm{WUSC}(\overline{\alpha},0)$, etc.

We call $\underline{\alpha},\underline{\theta},\underline{C},\overline{\alpha},\overline{\theta},\overline{C}$ the scaling characteristics of ψ. As pointed out in [207], by inflating $\underline{C}$ and $\overline{C}$ we can replace $\underline{\theta}$ with $\underline{\theta}/2$ and $\overline{\theta}$ by $\overline{\theta}/2$ in the scalings, therefore we can always

choose the same, arbitrarily small value $\theta = \underline{\theta} = \overline{\theta} > 0$ in both local scalings WLSC and WUSC, if they hold at all. The scalings characterize the so-called common bounds for $p_t(x)$ [206], and so they are natural conditions on ψ in the unimodal setting. See [206] many examples of Lévy-Khintchine exponents which satisfy WLSC or WUSC.
For instance $\psi(\xi) = |\xi|^\alpha$, the Lévy-Khintchine exponent of the isotropic α-stable Lévy process in $\mathbb{R}^d$ with $\alpha \in (0,2)$, satisfies WLSC$(\alpha, 0, 1)$ and WUSC$(\alpha, 0, 1)$. The characteristic exponent $\psi(\xi) = (1 + |\xi|^2)^{\alpha/2} - 1$ of the relativistic α-stable Lévy process with $\alpha \in (0,2)$ satisfies WLSC$(\alpha, 0)$ and WUSC$(\alpha, 1)$. Other examples include $\psi(\xi) = |\xi|^{\alpha_1} + |\xi|^{\alpha_2} \in$ WLSC$(\alpha_1, 0, 1) \cap$ WUSC$(\alpha_2, 0, 1)$, where $0 < \alpha_1 < \alpha_2 < 2$, etc. If $\psi(r)$ is α-regularly varying at infinity and $0 < \alpha < 2$, then $\psi \in$ WLSC$(\underline{\alpha}) \cap$ WUSC$(\overline{\alpha})$, with any $0 < \underline{\alpha} < \alpha < \overline{\alpha} < 2$. The connection of the scalings to the so-called Matuszewska indices of $\psi(r)$ is explained in [206].
If $\psi \in$ WLSC$(\underline{\alpha}, \theta)$, then by (7) (or see [207]) we get the following scaling at 0:

$$V(\varepsilon r) \leq C\varepsilon^{\frac{\underline{\alpha}}{2}} V(r), \qquad 0 < \varepsilon \leq 1, \qquad 0 < r < 1/\theta. \tag{9}$$

Here the range is $0 < r < \infty$ if the lower scaling of ψ is global, in agreement with (9) and the convention $1/0 = \infty$. If $\psi \in$ WUSC$(\overline{\alpha}, \theta)$, then, similarly,

$$V(\varepsilon r) \geq C\varepsilon^{\overline{\alpha}/2} V(r), \qquad 0 < \varepsilon \leq 1, \qquad 0 < r < 1/\theta. \tag{10}$$

We shall need V^{-1}, the inverse function of V on $[0, \infty)$. We let

$$T(t) = V^{-1}\left(\sqrt{t}\right), \qquad t \geq 0. \tag{11}$$

Put differently, $\left[V\left(T(t)\right)\right]^2 = t$. For instance, $T(t) = t^{1/\alpha}$ for the isotropic α-stable Lévy process. The functions V and T allow us to handle intrinsic difficulties which hampered extensions of [216], [52], [219], [213] to general unimodal Lévy processes, namely the lack of explicit formulas and estimates for the involved potential-theoretic objects.
We note that $T(t) < a$ if and only if $t < V^2(a)$, wherever $a, t \geq 0$. The scaling properties of T at zero reflect those of ψ (at infinity) as follows.

Lemma (5.1.1)[204]: If (9) holds, $0 < \varepsilon \leq 1$ and $0 \leq t < V\left(\frac{1}{\theta}\right)^2$, then $T(\varepsilon t) \geq c\varepsilon^{1/\underline{\alpha}} T(t)$.

If (10) holds, $0 < \varepsilon \leq 1$ and $0 \leq t < V\left(\frac{1}{\theta}\right)^2$, then $T(\varepsilon t) \leq c\varepsilon^{1/\overline{\alpha}} T(t)$.

Proof. To prove the first assertion we note that T is increasing. If $0 < t < V\left(\frac{1}{\theta}\right)^2$, and $0 \leq \varepsilon \leq 1$, then $T(t) < 1/\theta$ and $T(\varepsilon t)/T(t) \leq 1$. By (9),

$$\sqrt{\varepsilon} = \frac{V(T(\varepsilon t))}{V(T(t))} \leq C\left(\frac{T(\varepsilon t)}{T(t)}\right)^{\underline{\alpha}/2},$$

as needed. The proof of the second inequality is analogous but uses (10).
By (8) and the proof of Lemma (5.1.1) we always have

$$T(\varepsilon t) \leq c\sqrt{\varepsilon} T(t), \qquad 0 < \varepsilon \leq 1, \qquad 0 < r < \infty. \tag{12}$$

In what follows we always assume that ν is an infinite unimodal Lévy measure on $\mathbb{R}^d$ with $d \geq 2$ and the Lévy-Khintchine exponent defined by (4) satisfies

$$\psi \in \text{WLSC}(\underline{\alpha}, \theta) \cap \text{WUSC}(\overline{\alpha}, \theta),$$

where $0 < \underline{\alpha} \leq \overline{\alpha} < 2$, and $\theta \geq 0$. Many partial results below need less assumptions but for simplicity of presentation.

Definition (5.1.2)[204]: We say that (**H**) holds if for every $r > 0$ there is $H_r \geq 1$ such that

$$V(z) - V(y) \leq H_r V'(x)(z - y) \quad \text{whenever} \quad 0 < x \leq y \leq z \leq 5x \leq 5r.$$

We say that (**H***) holds if $H_\infty := \sup_{r>0} H_r < \infty$.

We may and do chose H_r nondecreasing in r. By [205], (**H**) always holds in our setting because ψ satisfiesWLSC andWUSC. If $\psi \in \mathrm{WLSC}(\underline{\alpha}, 0) \cap \mathrm{WUSC}(\overline{\alpha}, 0)$, then (**H***) even holds.

By [207], there is a $C_1 = C_1(d)$ such that

$$p_t(x) \leq C_1 \frac{t}{|x|^d V^2(|x|)}, \qquad t > 0, \qquad x \in \mathbb{R}^d \setminus \{0\}, \tag{13}$$

hence [206],

$$\nu(x) \leq C_1 \frac{1}{V^2(|x|)|x|^d}, \qquad x \neq 0. \tag{14}$$

Since $\psi \in \mathrm{WLSC}(\underline{\alpha}, \theta)$, by [207] we have

$$p_t(x) \leq cT^{-d}(t), \qquad t < V^2(\theta^{-1}), \qquad x \in \mathbb{R}^d. \tag{15}$$

We now discuss the heat remainder and the heat kernel of open sets $D \subset \mathbb{R}^d$. As usual, $0 \leq r_D(t, x, y) \leq p_t(x - y)$. Indeed, one directly checks that $[0, t) \ni s \mapsto Y_s = p(t - s, X_s, y)$ is a $\mathbb{P}_x$-martingale for each $x, y \in \mathbb{R}^d$. The martingale almost surely converges to 0 as $s \to t$, and we let $Y_t = 0$. By optional stopping, quasileft continuity of X and Fatou's lemma, for every stopping time $T \leq t$ we have $\mathbb{E}_x Y_T \leq \mathbb{E}_x Y_0 = p(t, x, y)$. The inequality $r_D(t, x, y) \leq p_t(x - y)$ follows by taking $T = \tau_D \wedge t$. The next result is a consequence of the strong Markov property of X_t.

Lemma (5.1.3)[204]: Consider open sets $D \subset F \subset \mathbb{R}^d$. For all $t > 0$ and $x, y \in \mathbb{R}^d$,

$$p_F(t, x, y) - p_D(t, x, y) = \mathbb{E}^y\left[\tau_D < t, X(\tau_D) \in F \setminus D;\ p_F\left(t - \tau_D, X(\tau_D), x\right)\right].$$

Proof: We repeat verbatim the proof of [52].

Here is a well-known Ikeda-Watanabe formula for the joint distribution of $X(\tau_D)$ and τ_D, see [211] or [208] for proof.

Lemma (5.1.4)[204]: Let $D \subset \mathbb{R}^d$ be open. For $x \in D, t_2 \geq t_1 \geq 0$ and $A \subset \left(\overline{D}\right)^c$,

$$\mathbb{P}_x(X(\tau_D) \in A, t_1 < \tau_D < t_2) = \int_D \int_{t_1}^{t_2} p_D(s, x, y) ds \int_A \nu(y - z) dz dy.$$

We denote $\delta_D(x) := \mathrm{dist}(x, D^c),\ x \in \mathbb{R}^d$.

Lemma (5.1.5)[204]: We have

$$r_D(t, x, y) \leq CT(t)^{-d}, \tag{16}$$

and

$$r_D(t, x, y) \leq C_1 \frac{t}{V^2\left(\delta_D(x)\right)\delta_D^d(x)}, \qquad x, y \in \mathbb{R}^d. \tag{17}$$

Proof. Since $\psi \in \mathrm{WLSC}(\underline{\alpha}, \theta, \underline{C})$, we have (15), which yields (16). By (1), (13), and symmetry,

$$r_D(t, x, y) = r_D(t, y, x) \leq \mathbb{E}^y\left[\tau_D < t;\ C_1 \frac{t - \tau_D}{V^2(|X(\tau_D) - x|)|X(\tau_D) - x|^d}\right].$$

Since $|X(\tau_D) - x| \leq \delta_D(x)$ and V is increasing, we obtain (17).

Recall that $\mathbb{H}$ is a half-space and $C_{\mathbb{H}}(t)$ is defined immediately before Theorem (5.1.12).

Lemma (5.1.6)[204]: If $T(t) < 1/\theta$, then $C_{\mathbb{H}}(t) \le cT(t)^{-d+1}$.
Proof: Denote $r(t,q) = r_{\mathbb{H}}(t,(q,0,\cdot\ \cdot\ \cdot,0),(q,0,\cdot\ \cdot\ \cdot,0))$. By (17) and (9),

$$\int_{T(t)}^{\infty} r(t,q)dq \le c\int_{T(t)}^{\infty} \frac{V^2(T(t))}{V^2(q)q^d}dq \le c\int_{T(t)}^{\infty} \frac{T(t)^{\underline{\alpha}}}{q^{d+\underline{\alpha}}}dq = cT(t)^{1-d}.$$

Using (16) we get

$$\int_0^{T(t)} r(t,q)dq \le c\int_0^{T(t)} T(t)^{-d}dq = cT(t)^{1-d}.$$

To obtain a lower bound for $C_{\mathbb{H}}(t)$ we shall use the existing heat kernel estimates for geometrically regular domains. Recall that open set $D \subset \mathbb{R}^d$ satisfies the inner (outer) ball condition at scale $R > 0$ if for every $Q \in \partial D$ there is a ball $B(x',R) \subset D$ (a ball $B(x'',R) \subset D^c$) such that $Q \in \partial B(x',R)$ ($Q \in \partial B(x'',R)$, respectively). An open set D is R-smooth if it satisfies both the inner and the outer ball conditions at some scale $R > 0$. We call $B(x',R)$ and $B(x'',R)$ the inner ball and the outer ball, respectively.
In the next lemma we collect a number of results from [207]. For brevity in what follows we sometimes write $T = T(t)$, where $t > 0$ is given.
Lemma (5.1.7)[204]: Let open $D \subset \mathbb{R}^d$ satisfy the outer ball condition at scale $R < 1/\theta$. There is a constant c such that for $T \vee |x-y| < 1/\theta$,

$$p_D(t,x,y) \le c\left(\frac{V(\delta_D(x))}{V(T \wedge R)} \wedge 1\right)\left(\frac{V(\delta_D(y))}{V(T \wedge R)} \wedge 1\right)\left(T^{-d} \wedge \frac{V^2(T)}{|x-y|^d V^2(|x-y|)}\right).$$

Proof. We have (**H**). We note that $\sqrt{t} = V(T)$ and use the second part of [207]. We need to justify that the quotient $\frac{H_R}{J^4(R)}$ is bounded, where H_R is the constant from (**H**) and $J(R) = \inf_{0<r\le R} \nu(B(0,r)^c)V^2(r)$. To this end we observe that H_R is increasing, and $J(R)$ is nonincreasing, hence we get an upper bound for this quotient by replacing R with $\frac{1}{\theta}$. If $\theta = 0$, which we also allow, then by [205] the quotient is bounded as a function of R. By [207] with $r = \frac{1}{2}$, we also have $p_{\frac{t}{2}}(0) \le cT^{-d}(t)$.
Lemma (5.1.8)[204]: We have $C_{\mathbb{H}}(t) \approx T(t)^{-d+1} \approx p_t(0)T(t)\ as\ t \to 0$.
Proof: By Lemma (5.1.7) and (2) there is $\varepsilon > 0$ such that $r(t,q) \ge \frac{1}{2}p_t(0)$ if $V(q) < \varepsilon\sqrt{t}$.
Since $\psi \in$ WUSC, by scaling of V there is $c > 0$ such that for $0 < q \le cT(t)$ the condition is satisfied and we have

$$\int_0^{cT(t)} r(t,q)dq \ge \frac{1}{2}\int_0^{cT(t)} T(t) - ddq = \frac{c}{2}T(t)^{1-d}.$$

By WUSC and WLSC we have $p_t(0) \approx T(t)^{-d}$, see [206].
For $M \ge 0$, the truncated Green function of D is defined as

$$G_D^M(x,y) = \int_0^M p_D(t,x,y)dt, \qquad x,y \in \mathbb{R}^d.$$

The Green function of D is

$$G_D(x,y) = \int_0^\infty p_D(t,x,y)dt = G_D^\infty(x,y).$$

Lemma (5.1.9)[204]: Let open $D \subset \mathbb{R}^d$ satisfy the outer ball condition at scale $R < 1/\theta$, $x, y \in \mathbb{R}^d$ and $|x - y| < 1/\theta$. Let $M = V^2(R)$. Then

$$G_D^M(x,y) \leq c\frac{V\big(\delta_D(y)\big)V\big(\delta_D(x)\big)}{|x - y|^d}, \tag{18}$$

and

$$G_D^M(x,y) \leq c\frac{V\big(\delta_D(y)\big)V(|x - y|)}{|x - y|^d}. \tag{19}$$

Furthermore, if $d > 2$ or WUSC$(\overline{\alpha}, 0)$ holds, then (18) and (19) even hold for $M = V^2(1/\theta)$, including the case of global WLSC ($M = \infty$).

Proof: Assuming $T < R \wedge |x - y|$, by Lemma (5.1.7) we get

$$p_D(t,x,y) \leq cV\big(\delta_D(y)\big)\frac{V\big(T \wedge \delta_D(x)\big)}{V^2(|x - y|)|x - y|^d},$$

hence

$$\begin{aligned}\int_0^{V^2(|x-y|\wedge R)} p_D(t,x,y)dt &\leq c\frac{V\big(\delta_D(x)\big)}{V^2(|x - y|)|x - y|^d}\int_0^{V^2(|x-y|\wedge R)} V(T \wedge \delta_D(x))dt \\ &\leq c\frac{V\big(\delta_D(x)\big)V^2(|x - y| \wedge R)V(|x - y| \wedge \delta_D(x))}{|x - y|^d V^2(|x - z|)} \\ &\leq c\frac{V(\delta_D(x))V(|x - y| \wedge \delta_D(x))}{|x - y|^d}.\end{aligned}$$

This establishes (19) and (18) for small times. Then,

$$\int_{V^2(|x-y|)}^{V^2(R)} p_D(t,x,y)dt \leq cV(\delta_D(x))\int_{V^2(|x-y|)}^{V^2(R)} \frac{T^{-d}(t)}{\sqrt{t}}dt.$$

By WUSC and Lemma (5.1.1),

$$\frac{1}{T(t)} \leq \frac{c\varepsilon^{1/\overline{\alpha}}}{T(\varepsilon t)}.$$

With this in mind we obtain

$$\begin{aligned}\int_{V^2(|x-y|)}^{V^2(R)} \frac{T^{-d}(t)}{\sqrt{t}}dt &\leq c\int_{V^2(|x-y|)}^{\infty} \frac{V^{\frac{2d}{\overline{\alpha}}}(|x - y|)}{t^{\frac{d}{\overline{\alpha}}+\frac{1}{2}}T^d(V^2(|x - y|))}dt \\ &= c\frac{V^{\frac{d}{\overline{\alpha}}}(|x - y|)}{|x - y|^d}[V^2(|x - y|)]^{-d/\overline{\alpha}-1/2+1},\end{aligned}$$

where the integral converges, because $d/\overline{\alpha} + 1/2 > 1$ (recall that $\overline{\alpha} < 2$). We thus get (19). To finish the proof of (18) we note that

$$\int_{V^2(|x-y|)}^{V^2(R)} p_D(t,x,y)dt \leq cV(\delta_D(x))V(\delta_D(y))\int_{V^2(|x-y|)}^{V^2(R)} \frac{T^{-d}(t)}{t}dt,$$

and we proceed as before.

For $M \geq 0$, the truncated Poisson kernel is defined as

$$K_D^M(x,z) = \int_D G_D^M(x,y)\nu(y - z)dy, \qquad x \in D, \qquad z \in D^c.$$

Lemma (5.1.10)[204]: Let open $D \subset \mathbb{R}^d$ satisfy the outer ball condition at scale R. If $\operatorname{diam}(D \cup \{z\}) < 1/\theta$, then

$$K_D^{\frac{V(R^2)}{2}}(x,z) \le \frac{V\big(\delta_D(x)\big)}{V\big(\delta_D(z)\big)} \frac{c}{|x - z|^d}, \qquad x \in D, \qquad z \in D^c.$$

Proof: The previous lemma gives an estimate for $G_D^{V^2(R)}$, and the Lévy measure is controlled by (14). Thus,

$$K_D^{\frac{V(R2)}{2}}(x,z) \le cV\left(\delta_D(x)\right) \int_D \frac{V(|x-y|) \wedge V\big(\delta_D(y)\big)}{|x - y|^d |y - z|^d V^2(|y - z|)} dy.$$

Note that $|x-y| \ge |x-z|/2$ or $|y - z| \ge |x-z|/2$. Furthermore, if $|x-y| \ge |y - z|$, then $|x - y| \ge |x - z|/2$. Therefore, it is enough to verify that

$$I := \int_D \frac{V(\delta_D(y))}{|y - z|^d V^2(|y - z|)} dy \le \frac{C}{V(\delta_D(z))}, \text{and}$$

$$II := \int_{D\cap\{|x-y|<|y-z|\}} \frac{V(|x - y|)}{|x - y|^d V^2(|y - z|)} dy \le \frac{C}{V(\delta_D(z))}.$$

Considering I we note that $\delta_D(y) \le |y - z|$, hence

$$I \le \int_{|y-z|>\delta_D(z)} \frac{|y - z|^{-d}}{V(|y - z|)} dy \le c \int_{\delta_D(z)}^{1/\theta} \frac{dr}{rV(r)}$$

Using the scaling (9) we get

$$I \le \frac{c}{V(\delta_D(z))} \int_{\delta_D(z)}^{\infty} \left(\frac{\delta_D(z)}{r}\right)^{\underline{\alpha}/2} \frac{dr}{r} = \frac{c}{V(\delta_D(z))}.$$

To verify the estimate for II we also use the scaling properties of V. For $y \in D$ we have $|y - z| < 1/\theta$, hence

$$II \le c \int_{|x-y|\le|y-z|} \left(\frac{|x - y|}{|y - z|}\right)^{\frac{\underline{\alpha}}{2}} \frac{dy}{|x - y|^d V(|y - z|)}$$

$$\le \frac{c}{V(\delta_D(z))} \int_0^{|y-z|} \left(\frac{r}{|y - z|}\right)^{\underline{\alpha}/2} \frac{dr}{r} = \frac{c}{V(\delta_D(z))} \frac{2}{\underline{\alpha}}.$$

In the following statement we repeat our standing assumptions; see also the definition of V and that of T in (11).

Theorem (5.1.11)[204]: Let ν be an infinite unimodal Lévy measure on $\mathbb{R}^d$ with $d \ge 2$, and let the Lévy-Khintchine exponent (4) satisfy $\psi \in \mathrm{WLSC}(\underline{\alpha}, \theta) \cap \mathrm{WUSC}(\overline{\alpha}, \theta)$, where $0 < \underline{\alpha} \le \overline{\alpha} < 2$ and $\theta \ge 0$. Let open bounded set $D \subset \mathbb{R}^d$ be R-smooth with $0 < R < 1/\theta$. There is a constant c_θ depending only on ν and θ such that if $0 < t < V^2(\theta^{-1})$, or $T(t) < 1/\theta$, then the trace (3) of the Dirichlet heat kernel (2) satisfies

$$|tr(t,D) - |D|p_t(0) + |\partial D| C_{\mathbb{H}}(t)| \le c_\theta |D| p_t(0) \frac{T(t)^2}{R^2}. \qquad (20)$$

If $\theta = 0$, then (20) holds for all $t > 0$.

Recall that Lemma (5.1.8) asserts that $C_{\mathbb{H}}(t) \approx p_t(0)T(t)$ and $p_t(0) \approx T(t)^{-d}$ as $t \to 0$, so the approximation of the trace in Theorem (5.1.11) is given in terms of powers of $T(t)$.

Theorem (5.1.12)[204]: If bounded open set $D \subset \mathbb{R}^d$ is R-smooth, WLSC and WUSC hold for ψ, and $t \to 0$, then $t_r(t,D)$ equals $p_t(0)|D| - C_{\mathbb{H}}(t)|\partial D|$ plus lower order terms.
Heuristically, if $x \in D$ and $t > 0$ is small, then $r_D(t,x,x)$ is small and so $p_D(t,x,x)$ is close to $p_{\mathbb{R}^d}(t,x,x) = p_t(0)$. Therefore the first approximation to $t_r(t,D)$ is $p_t(0)|D|$.
The second term in Theorem (5.1.12), $C_{\mathbb{H}}(t)|\partial D|$, approximates $\int_D r_D(t,x,x)dx$. As we shall see, $r_D(t,x,x)$ depends primarily on the distance of x from ∂D. It is here that the R-smoothness of D plays a role by allowing for an asymptotic coefficient independent of D, that is $C_{\mathbb{H}}(t)$. In view of the definition of $C_{\mathbb{H}}(t)$, the appearance of $|\partial D|$ in the second term of the approximation of the trace is natural.

Including the relativistic stable Lévy process, explicit expansions of $p_t(0)$ can be given [213]. In more general situations $p_t(0)$, $C_{\mathbb{H}}(t)$ and the bounds for the error terms cannot be entirely explicit but Lemma (5.1.8) and Theorem (5.1.11) below provide a satisfactory formulation.
Technically we only need to estimate $\int_D r_D(t,x,x)dx$ to prove Theorem (5.1.12). In this connection we note that sharp global estimates for $p_D(t,x,y)$ were recently obtained by Bogdan, Grzywny and Ryznar [207], but these estimates do not easily translate into sharp estimates of $r_D(t,x,y)$. Namely, if $p_D(t,x,y)$ is only known to be proportional to $p_t(y-x)$, then essential further work is needed to accurately estimate $r_D(t,x,y)$.

We give a unimodal Lévy processes with scaling, their heat kernel, Green function and Poisson kernel for R-smooth open sets. We show Theorem (5.1.11), a stronger and more detailed variant of Theorem (5.1.12). The most technical step of the proof of Theorem (5.1.11) is given separately.
Proof: The result is a direct consequence of (15), Lemma (5.1.8) and Theorem (5.1.11), where we take $\theta > 0$ so small that $R < 1/\theta$.
In the course of the proof of Theorem (5.1.11), which now follows, we usually write $T = T(t)$. As mentioned in the Introduction,

$$tr(t,D) - |D|p_t(0) = \int_D p_D(t,x,x)dx - \int_D p(t,x,x)dx = -\int_D r_D(t,x,x)dx.$$

We only need to show that

$$\left|\int_D r_D(t,x,x)dx - |\partial D|C_{\mathbb{H}}(t)\right| \le \frac{cT^2}{T^d R^2}. \tag{21}$$

We first consider $T = T(t) \ge R/2$, and we have

$$\int_D r_D(t,x,x) \le \int_D p_t(0)dx \le |D|p_t(0) \le 4|D|p_t(0)\frac{T^2}{R^2}.$$

By Lemma (5.1.6),

$$|\partial D|C_{\mathbb{H}}(t) = |\partial D|\int_0^\infty r_{\mathbb{H}}\big(t,(q,0,\cdots,0),(q,0,\cdots,0)\big)dq \le \frac{c|D|}{R}T^{1-d} \le \frac{c|D|T^{2-d}}{R^2}.$$

By [206], we see that (20) holds trivially in this case.
From now on we assume that $T < R/2$. For $r > 0$ we let $D_r = \{x \in D : \delta_D(x) > r\}$. We have $D = D_{R/2} \cup (D \setminus D_{R/2})$. In analyzing the decomposition we shall often use our assumptions $R < 1/\theta$ and $|x-y| < 1/\theta$, and the heat kernel estimates from Lemma (5.1.7). By Lemma (5.1.5),

$$\int_{D_{R/2}} r_D(t,x,x)dx \le C\left|D_{\frac{R}{2}}\right| \frac{V^2(T)}{V^2\left(\frac{R}{2}\right)R^d} \le C|D|\frac{1}{R^2R^{d-2}} \le C|D|\frac{1}{R^2T^{d-2}}. \quad (22)$$

Thus, the integral gives insignificant contribution to the trace.

To handle the integration near ∂D, we shall estimate the heat remainder of D using the heat remainder of halfspace. Let $x^* \in \partial D$ be such that $|x - x^*| = \delta_D(x)$. Let I and O be the (inner and outer) balls with radii R such that $\partial I \cap \partial O = \{x^*\}$ and $I \subset D \subset O^c$. Let $\mathbb{H}(x)$ denote the halfspace satisfying $I \subset \mathbb{H}(x) \subset O^c$. By domain monotonicity of the heat remainder, and by Lemma (5.1.3),

$$\left|r_D(t,x,x) - r_{\mathbb{H}(x)}(t,x,x)\right| \le r_I(t,x,x) - r_{O^c}(t,x,x)$$
$$= p_{O^c}(t,x,x) - p_I(t,x,x)$$
$$= \mathbb{E}^x[\tau_I < t, X(\tau_I) \in O^c;\ p_{O^c}(t - \tau_I, X(\tau_I), x)].$$

The next result is an analogue of [52].

Lemma (5.1.13)[204]: If $T < R/2$, then

$$\left|\int_{D\setminus D_{R/2}} r_D(t,x,x) - r_{\mathbb{H}(x)}(t,x,x)\,dx\right| \le \frac{c|D|T^2}{R^2T^d}. \quad (23)$$

Proof. This is an analog of [52] and is proved as follows. By the coarea formula and Proposition (5.1.15) we find that the left side of (23) is bounded above by

$$\frac{cT}{RT^d}\int_0^{R/2} |\partial D_q| \left(\frac{T^{d-1}V(T)}{q^{d-1}V(q)} \wedge 1\right)dq.$$

Therefore [52] gives a simplified bound

$$\frac{c|\partial D|}{RT^{d-1}}\int_0^{R/2} \left(\frac{T^{d-1}V(T)}{q^{d-1}V(q)} \wedge 1\right)dq.$$

The integral over $(0,T)$ is clearly bounded by T. To estimate the integral from T to $R/2$ we note that scaling (9) for $q \in [T, R/2)$ yields $V(T) \le C(T/q)^{\underline{\alpha}/2}V(q)$. Also,

$$\int_T^{R/2} q^{1-d-\underline{\alpha}/2}dq \le \int_T^{\infty} q^{1-d-\underline{\alpha}/2}dq < \infty,$$

since $d + \underline{\alpha}/2 > 2$.

Recall that $r(t,q) = r_{\mathbb{H}}(t,(q,0,\cdots,0),(q,0,\cdots,0))$, and $C_{\mathbb{H}}(t) = \int_0^\infty r(t,q)dq$.

Lemma (5.1.14)[204]: If $T < R/2$, then

$$\left|\int_{D\setminus D_{R/2}} r_{\mathbb{H}(x)}(t,x,x)dx - |\partial D|\int_0^{R/2} r(t,q)dq\right| \le \frac{c|D|T^2}{R^2T^d}. \quad (24)$$

Proof: Using the coarea formula we get

$$\int_{D\setminus D_{R/2}} r_{\mathbb{H}(x)}(t,x,x)dx = \int_0^{R/2} |\partial D_q| r(t,q)dq.$$

Hence the left side of the inequality (24) is bounded by

$$\int_0^{R/2} \left||\partial D_q| - |\partial D|\right| r(t,q)dq \le \frac{C|D|}{R^2}\int_0^{R/2} q\, r(t,q)dq,$$

as follows from Corollary 2.14(iii) in [52]. For $q \in (0,T]$ we have $r(t,q) \le p_t(0)$, hence

$$\int_0^T qr(t,q)dq \le c\int_0^T \frac{q}{T^d}dq = cT^{2-d}.$$

For the remaining integration, using (17) and (9), we get

$$\int_T^{\frac{1}{\theta}} qr(t,q)dq \ \le\ c\int_T^{\frac{1}{\theta}} \frac{t}{q^{d-1}V^2(q)}dq \le c\int_T^{\frac{1}{\theta}} \frac{V^2(T)}{q^{d-1}V^2(q)}dq$$

$$\le\ c\int_T^{1/\theta} \left(\frac{T}{q}\right)^{\underline{\alpha}} \frac{dq}{q^{d-1}} \le\ cT^{2-d}\int_1^\infty q^{-d+1-\underline{\alpha}}dq.$$

The last integral converges since $d\ \ge\ 2$ and $\underline{\alpha}\ >\ 0$.

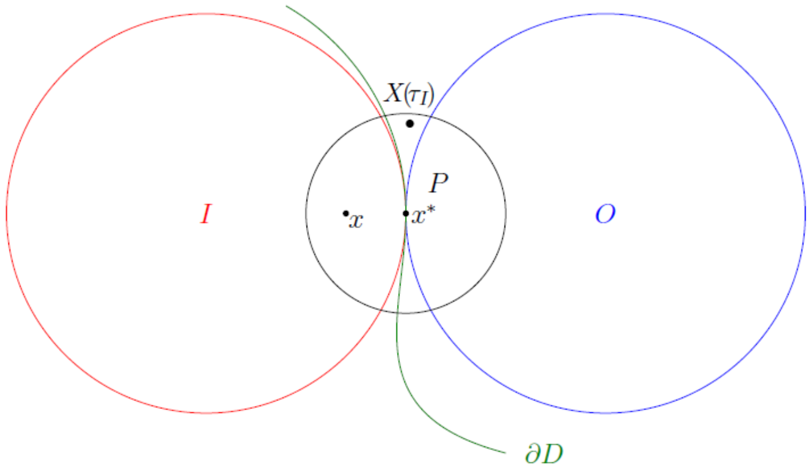


Fig. (1)[204]: Balls $I\ \subset\ D$ (left), $O\ \subset\ D^c$ (right) and P (middle), and "a short jump" to point $X(\tau_I)$. Here $x\ \in\ P$ and $|x|\ =\ \delta_I(x)$.

Thus, for $T\ <\ R/2$ we have by Lemma (5.1.5)

$$|\partial D|\int_{\frac{R}{2}}^\infty r(t,q)dq \le \frac{c|D|}{R}\int_{\frac{R}{2}}^\infty \frac{V^2(T)}{q^dV^2(q)}dq \le \frac{c|D|}{R}\int_{\frac{R}{2}}^\infty \frac{dq}{T^{d-2}q^2} = \frac{CT^2}{R^2T^d},$$

which is a lower order term. By Lemma (5.1.13), Lemma (5.1.14) and (22) we obtain (21).

Proposition (5.1.15)[204]: If $T\ <\ R/2$, then

$$\mathbb{E}^x[\tau_I\ <\ t, X(\tau_I)\ \in\ O^c;\ p_{O^c}(t-\tau_I, X(\tau_I), x)] \le \frac{c}{R}\left(\frac{V(T)}{\delta_D(x)^{d-1}V\big(\delta_D(x)\big)}\wedge T^{1-d}\right).$$

Proof: Let $x^*\ =0, a\ =\ (-R,0,\dots,0), b\ =\ (R,0,\dots,0), I\ =\ B(a,R)$ and $O\ =\ B(b,R)$.

This also means that $x\ =\ (x_0,0,\dots,0)$ with $0\ \le\ x_0\ <\ R/2$, and $\delta_I(x)\ =\ |x|$, see Fig. (1). Recall that $t\ <\ V^2(R/2)$ or equivalently $T\ <\ R/2$. Before we proceed to the heart of the matter we need the following lemma based on spherical integration developed in [210] and later used in [52], [104].

Lemma (5.1.16)[204]: For $s\ <\ R$ we have

$$\int_{(O^c\setminus I)\cap B(0,s)} \frac{dz}{|x-z|^\beta}\frac{V\big(\delta_{O^c}(z)\big)}{V\big(\delta_I(z)\big)} \le c\begin{cases} \dfrac{|x|^{d+1-\beta}}{R} & if\ \beta\ >\ d\ +\ 1, \\ s^{d+1-\beta}/R & if\ \beta\ <\ d\ +\ 1. \end{cases} \tag{25}$$

Proof: First we consider $V\ (x)\ =\ x^{\alpha/2}$ with $\alpha\ \in\ [0,2)$. Let $z\ \in\ A\ =\ (O^c\setminus I)\ \cap\ B(0,s)$.

Note that $|x - z| \geq |x|$. If $|x - z| \leq 2|x|$, then $|z| \leq |x - z| + |x| \leq 3|x|$, which leads to the integral

$$\int_{A\cap\{|x-z|\leq 2|x|\}} \frac{dz}{|x - z|^\beta} \frac{\delta_{O^c}^\alpha(z)}{V(\delta_I(z))} \leq \frac{1}{|x \wedge s|^\beta} \int_{A\cap\{|z|\leq 3(|x|\wedge s)\}} \frac{\delta_{O^c}^\alpha(z)}{\delta_I^\alpha(z)} dz.$$

The last integral is similar to [52]. Using [52] we get the following upper bound

$$\frac{c}{|x \wedge s|^\beta} \int_0^{3(|x|\wedge s)} \frac{r^d}{R} dr = \frac{c(|x| \wedge s)^{d+1-\beta}}{R}.$$

If $|x - z| \geq 2|x|$, then $|x - z| \geq |z|/2$ and $|z| \geq |z - x| - |x| \geq |x|$. By [52],

$$\int_{A\cap\{|x-z|>2|x|\}} \frac{dz}{|x - z|^\beta} \frac{\delta_{O^c}^\alpha(z)}{\delta_I^\alpha(z))} \leq c \int_{A\cap\{s\geq|z|\geq|x|\}} \frac{1}{|z|^\beta} \frac{\delta_{O^c}^\alpha(z)}{\delta_I^\alpha(z)} dz \leq \frac{c}{R} \int_{|x|\wedge s}^{s} r^{d-\beta} dr.$$

If $\beta > d + 1$, then the last integral is bounded by $c|x|^{d+1-\beta}$, while for $\beta < d + 1$ we get the upper bound $cs^{d+1-\beta}$.

This settles (25) for $V(x) = x^{\alpha/2}$ with $\alpha \in [0, 2)$. Note that the form of the right hand side of (25) does not depend on α.

Consider general $\psi \in \mathrm{WUSC}(\overline{\alpha})$ and the corresponding ladder-height function V.

Due to the scaling property (10) we have

$$\frac{V(\delta_{O^c}(z))}{V(\delta_I(z))} \leq c \frac{\delta_{O^c}^{\overline{\alpha}}(z)}{\delta_I^{\overline{\alpha}}(z))}, \qquad \text{if } \delta_{O^c}(z) \geq \delta_I(z).$$

If $\delta_{O^c}(z) \leq \delta_I(z)$, then the fraction is bounded by 1, since V is monotone. Therefore, we can use the previous special case with $\alpha = \overline{\alpha}$ and $\alpha = 0$ to finish the proof.

We return to the core proof of Proposition (5.1.15). In view of Lemma (5.1.4) we want to estimate

$$\begin{aligned}\mathbb{E}^x[\tau_I < t, X(\tau_I) \in O^c; p_{O^c}(t - \tau_I, X(\tau_I), x)] \\ = \int_I \int_0^t p_I(s, x, y) \int_{O^c\setminus I} \nu(y - z) p_{O^c}(t - s, x, z) dz ds dy \\ = I_1 + I_2 + I_3,\end{aligned}$$

which splits the integration into three subregions, as specified and estimated below:

$I_1 : |z| > R/2,$

$I_2 : t/2 < s < t$ and $|x - z| < T$ and $|z| \leq R/2,$

$I_3 : (s < t/2$ or $|x - z| > T)$ and $|z| \leq R/2.$

The setting, especially that of I_2, is illustrated on Fig. (1).

On I_1 we have $|z| > R/2$, hence $|x - z| \geq R/3$, thus by (13)

$$\begin{aligned} I_1 &= \int_I \int_0^t p_I(s, x, y) \int_{|z|>\frac{R}{2}} \nu(y - z) p(t - s, z, x) ds dz dy \\ &\leq \frac{ct}{R^d V^2\left(\frac{R}{3}\right)} \int_I \int_0^t p_I(s, x, y) \int_{p^c} \nu(y - z) ds dz dy \\ &= \frac{ct}{RdV^2\left(\frac{R}{3}\right)} \mathbb{P}^x(\tau_I < t, |X(\tau_I)| > R/2) \leq \frac{cV^2(T)}{R^d V^2(R/2)},\end{aligned}$$

where the last inequality follows from sublinearity (8) of V. Since $T < R/2$, we have

$$\frac{cV^2(T)}{R^dV^2(R/2)} \le \frac{c}{R^d} \le \frac{c}{RT^{d-1}}.$$

Since $|x| < R/2$, by monotonicity of V we get

$$\frac{cV^2(T)}{R^dV^2(R/2)} \le \frac{cV(T)}{R^dV(R/2)} \le \frac{cV(T)}{R|x|^{d-1}V(|x|)}.$$

Here we have $|x| \le |x-z| < T$, and $|z| \le |x - z| + |x| < 2T$. By Lemma (5.1.7), $t/2 < q < T$ and (12),

$$p_I(q,x,y) \le T^{-d}\frac{V(\delta_I(y))}{V(T)}.$$

Let $S = (O^c \setminus I) \cap \{|z| < 2T\}$. We get the following upper bound,

$$\begin{aligned}
I_2 &= \int_I \int_{t/2}^t p_I(q,x,y) \int_S \nu(y-z)p(t-q,z,x)dqdzdy \\
&\le c\int_I T(t)^{-d}\frac{V\big(\delta_I(y)\big)}{V(T)} \int_S \frac{1}{|y-z|^dV^2(|y-z|)} G_{O^c}^{V^2(R/2)}(x,z)dzdy \\
&\le \frac{cT^{-d}}{V(T)} \int_S \int_I \frac{V(\delta_I(z))}{|y-z|^dV(|y-z|)} \frac{G_{O^c}^{V^2(R/2)}(x,z)}{V(\delta_I(z))} dydz,
\end{aligned}$$

where we use $\delta_I(y) \le |y-z|$. Scaling (9) gives

$$I_2 \le \frac{cT^{-d}}{V(T)} \int_S \int_{B^c(z,\delta_I(z))} \frac{\delta_I^{\frac{\alpha}{2}}(z)}{|y-z|^{d+\alpha/2}} \frac{G_{O^c}^{V^2(R/2)}(x,z)}{V(\delta_I(z))} dydz.$$

We then rewrite the inner integral in spherical coordinates, use Green function estimate (18) and $|x| < T$,

$$\begin{aligned}
I_2 &\le \frac{cT^{-d}}{V(T)} \int_1^\infty \frac{\delta_I^{\frac{\alpha}{2}}(z)dr}{r^{1+\frac{\alpha}{2}}} \int_S \frac{V(|x|)V\big(\delta_{O^c}(z)\big)}{|x-z|^dV\big(\delta_I(z)\big)}dz \\
&\le cT^{-d}\int_1^\infty \frac{dr}{r^{1+\frac{\alpha}{2}}} \int_S \frac{V\big(\delta_{O^c}(z)\big)}{|x-z|^dV(\delta_I(z))}dz \\
&= cT^{-d}\int_S \frac{V\big(\delta_{O^c}(z)\big)}{|x-z|^dV(\delta_I(z))}dz.
\end{aligned} \tag{26}$$

Using Lemma (5.1.16) with $\beta = d$ and $s = 2T$ we get

$$I_2 \le \frac{cT^{1-d}}{R}.$$

Since $|x| < T$, we get the desired estimate from Proposition (5.1.15).

Let $S = (O^c \setminus I) \cap \{|z| < R/2\}$. We have $|x-z| > T$ or $s < t/2$. In either case, Lemma (5.1.7) and sublinearity of V implies

$$p_{O^c}(t-s,x,z) \le \left(T^{-d} \wedge \frac{V^2(T)}{|x-z|^dV^2(|x-z|)}\right)\frac{V\big(\delta_{O^c}(z)\big)}{V(T)}.$$

Therefore by Lemma (5.1.10),

$$I_3 \leq \int_I \int_0^{V^2(R/2)} p_I(s,x,y) \int_S \nu(y-z)\left(T^{-d} \wedge \frac{V^2(T)}{|x-z|^d V^2(|x-z|)}\right) \frac{V\big(\delta_{O^c}(z)\big)}{V(T)}\, dzdsdy$$
$$= c\int_S K_I^{V^2(R)}(x,z)\left(T^{-d} \wedge \frac{V^2(T)}{|x-z|^d V^2(|x-z|)}\right) \frac{V\big(\delta_{O^c}(z)\big)}{V(T)}\, dz$$
$$\leq c\int_S \frac{V(|x|)}{V\big(\delta_I(z)\big)} \frac{1}{|x-z|^d}\left(T^{-d} \wedge \frac{V^2(T)}{|x-z|^d V^2(|x-z|)}\right) \frac{V\big(\delta_{O^c}(z)\big)}{V(T)}\, dz.$$

If $|x-z| < T$, then we are satisfied with T^{-d} from the minimum and we note $V(|x|) < V(T)$. We arrive at (26), and finish the proof in the same way as in the previous cases. We are left with the case $|x-z| > T$, and we have

$$I_3 \leq cV(T)\int_S \frac{V(|x|)}{|x-z|^{2d} V^2(|x-z|)} \frac{V\big(\delta_{O^c}(z)\big)}{V(\delta_I(z))}\, dz.$$

Since $\psi \in \mathrm{WLSC}(\underline{\alpha})$, we get

$$I_3 \leq cV(T)\int_S \frac{|x|^{\underline{\alpha}/2}}{|x-z|^{2d+\underline{\alpha}/2} V(|x-z|)} \frac{V\big(\delta_{O^c}(z)\big)}{V(\delta_I(z))}\, dz$$
$$\leq \frac{cV(T)|x|^{\frac{\underline{\alpha}}{2}}}{(T \vee |x|)^{d-1} V(T \vee |x|)} \int_S \frac{V\big(\delta_{O^c}(z)\big)}{|x-z|^{2d+\underline{\alpha}/2} V(\delta_I(z))}\, dz,$$

where the last inequality follows from the monotonicity of V, since $|x-z| \geq |x| \vee T$. Now we use Lemma (5.1.16) with $\beta = d + 1 + \underline{\alpha}/2$, to get

$$I_3 \leq \frac{cV(T)}{(T \vee |x|)^{d-1} V(T \vee |x|) R}.$$

Here the right hand side is comparable with the required upper bound.

Section (5.2): Relativistic Stable Processes

For $m > 0$, an $\mathbb{R}^d$-valued process with independent, stationary increments having the following characteristic function

$$\mathbb{E}e^{i\xi \cdot X_t^{\alpha,m}} = e^{-t\left\{\left(m^{2/\alpha}+|\xi|^2\right)^{\alpha/2}-m\right\}}, \qquad \xi \in \mathbb{R}^d,$$

is called relativistic α-stable process with mass m. We assume that sample paths of $X_t^{\alpha,m}$ are right continuous and have left-hand limits a.s. If we put $m = 0$ we obtain the symmetric rotation invariant α-stable process with the characteristic function $e^{-t|\xi|^\alpha}, \xi \in \mathbb{R}^d$. We refer to this process as isotropic α-stable Lévy process. We keep α, m and $d \geq 2$ fixed and drop α, m in the notation, when it does not lead to confusion. Hence from now on the relativistic α-stable process is denoted by X_t and its counterpart isotropic α-stable Lévy process by $\tilde{X}_t$. We keep this notational convention consistently throughout, e.g., if $p_t(x-y)$ is the transition density of X_t, then $\tilde{p}_t(x-y)$ is the transition density of $\tilde{X}_t$.

In Ryznar [224] Green function estimates of the Schödinger operator with the free Hamiltonian of the form

$$\left(-\Delta + m^{2/\alpha}\right)^{\alpha/2} - m,$$

were investigated, where $m > 0$ and Δ is the Laplace operator acting on $L^2(\mathbb{R}^d)$. Using the estimates in Lemma (5.2.9) below and proof in Bañuelos and Kulczycki (2008) we provide an extension of the asymptotics in [52] to the relativistic α-stable processes for any $0 < \alpha < 2$.

Brownian motion has characteristic function

$$\mathbb{E}^0 e^{i\xi\cdot B_t} = e^{-t|\xi|^2}, \qquad \xi \in \mathbb{R}^d.$$

Let $\beta = \alpha/2$. Ryznar showed that X_t can be represented as a time-changed Brownian motion. Let $T_\beta(t), t > 0$, denote the strictly β-stable subordinator with the following Laplace transform

$$\mathbb{E}^0 e^{-\lambda T_\beta(t)} = e^{-t\lambda_\beta}, \qquad \lambda > 0. \tag{27}$$

Let $\theta_\beta(t,u), u > 0$, denote the density function of $T_\beta(t)$. Then the process $B_{T_\beta}(t)$ is the standard symmetric α-stable process.

Ryznar [224] showed that we can obtain $X_t = B_{T_\beta(t,m)}$, where a subordinator $T_\beta(t,m)$ is a positive infinitely divisible process with stationary increments with probability density function

$$\theta_\beta(t,u,m) = e^{-m^{1/\beta}u+mt}\theta_\beta(t,u), \qquad u > 0.$$

Transition density of $T_\beta(t,m)$ is given by $\theta_\beta(t,u-v,m)$. Hence the transition density of X_t is $p(t,x,y) = p(t,x-y)$ given by

$$p(t,x) = e^{mt}\int_0^\infty \frac{1}{(4\pi u)^{d/2}} e^{-\frac{|x|^2}{4u}} e^{-m^{1/\beta}u}\theta_\beta(t,u)du. \tag{28}$$

Then

$$p(t,x,x) = p(t,0) = e^{mt}\int_0^\infty \frac{1}{(4\pi u)^{d/2}} e^{-m^{1/\beta}u}\theta_\beta(t,u)du.$$

The function $p(t,x)$ is a radially symmetric decreasing and that

$$p(t,x) \le p(t,0) \le e^{mt}\int_0^\infty \frac{1}{(4\pi u)^{d/2}}\theta_\beta(t,u)du = e^{mt}t^{-d/\alpha}\,\frac{\omega_d\Gamma(d/\alpha)}{(2\pi)^d\alpha}, \tag{29}$$

where $\omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the surface area of the unit sphere in $\mathbb{R}^d$. For an open set D in $\mathbb{R}^d$ we define the first exit time from D by $\tau_D = inf\{t \ge 0: X_t \notin D\}$.

We set

$$r_D(t,x,y) = \mathbb{E}^x\left[p(t-\tau_D, X_{\tau_D}, y);\ \tau_D < t\right], \tag{30}$$

And

$$p_D(t,x,y) = p(t,x,y) - r_D(t,x,y), \tag{31}$$

for any $x,y \in \mathbb{R}^d, t > 0$. For a nonnegative Borel function f and $t > 0$,let

$$P_t^D f(x) = \mathbb{E}^x\,[f(X_t): t < \tau_D] = \int_D p_D(t,x,y)f(y)dy,$$

be the semigroup of the killed process acting on $L^2(D)$, see, Ryznar [224].

Let D be a bounded domain (or of finite volume). Then the operator P_t^D maps $L^2(D)$ into $L^\infty(D)$ for every $t > 0$. This follows from (29), (30), and the general theory of heat semigroups as described in [223]. It follows that there exists an orthonormal basis of eigenfunctions $\{\varphi_n: n = 1,2,3,\ldots\}$ for $L^2(D)$ and corresponding eigenvalues $\{\lambda_n: n = 1,2,3,\ldots\}$ of the generator of the semigroup P_t^D satisfying

$\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$, with $\lambda_n \to \infty$ as $n \to \infty$. By definition, the pair $\{\varphi_n, \lambda_n\}$ satisfies

$$P_t^D\varphi_n(x) = e^{-\lambda_n t}\varphi_n(x), \qquad x \in D, t > 0.$$

Under such assumptions we have

$$p_D(t,x,y) = \sum_{n=1}^{\infty} e^{-\lambda_n t}\, \varphi_n(x)\varphi_n(y). \quad (32)$$

We are interested in the behavior of the trace of this semigroup

$$Z_D(t) = \int_D p_D(t,x,x)dx. \quad (33)$$

Because of (32) we can write (33) as

$$Z_D(t) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \int_D \varphi_n^2(x)dx = \sum_{n=1}^{\infty} e^{-\lambda_n t}. \quad (34)$$

We denote d-dimensional volume of D by $|D|$.

The first result is Weyl's asymptotic for the eigenvalues of the relativistic Laplacian.

Definition (5.2.1)[218]: The boundary, ∂D, of an open set D in $\mathbb{R}^d$ is said to be R-smooth if for each point $x_0 \in \partial D$ there are two open balls B_1 and B_2 with radii R such that $B_1 \subset D, B_2 \subset \mathbb{R}^d \backslash (D \cup \partial D)$ and $\partial B_1 \cap \partial B_2 = x_0$.

Remark (5.2.2)[218]: When $m = 0, 0 < \alpha \le 2, C_2(t) = C_4 t^{1/\alpha}/t^{d/\alpha}$. Then the result in Theorem (5.2.12) becomes, for bounded domains with R-smooth boundary,

$$\left| Z_D(t) - \frac{C_1|D|}{t^{d/\alpha}} + \frac{C_4|\partial D|t^{1/\alpha}}{t^{d/\alpha}} \right| \le \frac{C_7|D|t^{2/\alpha}}{R^2 t^{d/\alpha}}, \quad (35)$$

where C_1, C_4 are as in Theorem (5.2.12). This was established by Bañuelos and Kulczycki [52] recently.

The asymptotic for the trace of the heat kernel when $\alpha = 2$ (the case of the Laplacian with Dirichlet boundary condition in a domain of $\mathbb{R}^d$), has been extensively studied. For Brownian motion van den Berg [216], proved that under the R-smoothness condition

$$\left| Z_D(t) - (4\pi t)^{-d/2} \left(|D| - \frac{\sqrt{\pi t}}{2} |\partial D| \right) \right| \le \frac{C_d |D| t^{1-d/2}}{R^2}, t > 0. \quad (36)$$

For domains with C^1 boundaries the result

$$Z_D(t) = (4\pi t)^{-d/2} \left(|D| - \frac{\sqrt{\pi t}}{2} |\partial D| + o\left(t^{1/2}\right) \right), as\ t \to 0, \quad (37)$$

was proved by Brossard and Carmona [222], for Brownian motion.

Let the ball in $\mathbb{R}^d$ with center at x and radius $r, \{y: |y - x| < r\}$, be denoted by $B(x,r)$. We will use $\delta_D(x)$ to denote the Euclidean distance between x and the boundary, ∂D, of D. That is, $\delta_D(x) = dist(x, \partial D)$. Define

$$\psi(\theta) = \int_0^{\infty} e^{-v}\, v^{p-1/2}(\theta + v2)^{p-1/2}\, dv, \qquad \theta \ge 0,$$

where $p = (d + \alpha)/2$. We put $\mathcal{R}(\alpha, d) = \mathcal{A}(-\alpha, d)/\psi(0)$, where $\mathcal{A}(v, d) = (\Gamma((d - v)/2))/(\pi^{d/2} 2^v\, |\Gamma(v/2)|)$. Let $\nu(x), \tilde{\nu}(x)$ be the densities of the Lévy measures of the relativistic α-stable process and the standard α-stable process, respectively. These densities are given by

$$\nu(x) = \frac{\mathcal{R}(\alpha, d)}{|x|^{d+\alpha}}\, e^{-m^{1/\alpha}|x|}\, \psi\left(m^{1/\alpha}|x|\right), \quad (38)$$

and

$$\tilde{v}(x) = \frac{\mathcal{A}(-\alpha, d)}{|x|^{d+\alpha}}\,. \tag{39}$$

We need the following estimate of the transition probabilities of the process X_t which is given in [211]: For any $x, y \in \mathbb{R}^d$ and $t > 0$ there exist constants $c_1 > 0$ and $c_2 > 0$,

$$p(t, x, y) \le c_1 e^{mt} \min\left\{\frac{t}{|x - y|^{d+\alpha}}\, e^{-c_2|x-y|}, t^{-d/\alpha}\right\}. \tag{40}$$

We will also use the fact [221] that if $D \subset \mathbb{R}^d$ is an open bounded set satisfying a uniform outer cone condition, then $P^x(X(\tau_D) \in \partial D) = 0$ for all $x \in D$. For the open bounded set D we will denoted by $G_D(x, y)$ the Green function for the set D equal to,

$$G_D(x, y) = \int_0^\infty p_D(t, x, y)dt, \qquad x, y \in \mathbb{R}^d.$$

For any such D the expectation of the exit time of the processes X_t from D is given by the integral of the Green function over the domain. That is

$$E^x(\tau_D) = \int_D G_D(x, y)dy.$$

Lemma (5.2.3)[218]: Let $D \subset \mathbb{R}^d$ be an open set. For any $x, y \in D$ we have

$$r_D(t, x, y) \le c_1 e^{mt}\left(\frac{t}{\delta_D^{d+\alpha}(x)} e^{-c_2\delta_D(x)} \wedge t^{-d/\alpha}\right).$$

Proof: Using (30) and (40) we have

$$\begin{aligned} r_D(t, x, y) &= E^y\left(p(t - \tau_D, X(\tau_D), x)\ ;\ \tau_D < t\right) \\ &\le c_1 e^{mt} E^y\left(\frac{t}{|x - X(\tau_D)|^{d+\alpha}}\, e^{-c_2|x - X(\tau_D)|} \wedge t^{-d/\alpha}\right) \\ &\le c_1 e^{mt}\left(\frac{t}{\delta_D^{d+\alpha}(x)}\, e^{-c_2\delta_D(x)} \wedge t^{-d/\alpha}\right). \end{aligned}$$

We need the following result for the proof of Proposition (5.2.5).

Lemma (5.2.4)[218]:

$$\lim_{t\to 0} p(t, 0)e^{-mt}t^{d/\alpha} = C_1, \tag{41}$$

Where

$$C_1 = (4\pi)^{d/2}\int_0^\infty u^{-d/2}\theta_\beta(1, u)du = \frac{\omega_d \Gamma\,(d/\alpha)}{(2\pi)^d\alpha}.$$

Proof: By (28) we have

$$p(t, x, x) = p(t, 0) = e^{mt}\int_0^\infty \frac{1}{(4\pi u)^{d/2}}\, e^{-m^{1/\beta}u}\theta_\beta(t, u)du.$$

Now using the scaling of stable subordinator $\theta_\beta(t, u) = t^{-1/\beta}\theta_\beta(1, ut^{-1/\beta})$ and a change of variables we get

$$p(t, 0) = \frac{e^{mt}}{(4\pi)^{d/2}t^{d/\alpha}}\int_0^\infty z^{-d/2}e^{-m^{1/\beta}}t^{1/\beta z}\theta_\beta(1, z)dz = \frac{C_1(t)e^{mt}}{t^{d/\alpha}},$$

then by dominated convergence theorem we obtain

$$\lim_{t\to 0} p(t, 0)e^{-mt}t^{d/\alpha} = \frac{1}{(4\pi)^{d/2}}\int_0^\infty z^{-d/2}\theta_\beta(1, z)dz,$$

and this last integral is equal to the density of α-stable process at time 1 and $x = 0$ which was calculated in [52] to be

$$\frac{\omega_d \Gamma\,(d/\alpha)}{(2\pi)^d \alpha}.$$

Proposition (5.2.5)[218]:

$$\lim_{t\to 0} \quad t^{d/\alpha} e^{-mt} Z_D(t) = C_1|D|, \tag{42}$$

where $C_1 = \frac{\omega_d \Gamma(d/.\alpha)}{(2\pi)^d \alpha}$.

Let $N(\lambda)$ be the number of eigenvalues $\{\lambda_j\}$ which do not exceed λ. It follows from (42) and the classical Tauberian theorem (see [73]) where $L(t) = C_1|D|e^{m/t}$ is our slowly varying function at infinity that

$$\lim_{\lambda\to\infty} \quad \lambda^{-d/\alpha} e^{-m/\lambda} N(\lambda) = \frac{C_1|D|}{\Gamma\,(1 + d/\alpha)}. \tag{43}$$

This is the analogue for relativistic stable process of the celebrated Weyl's asymptotic formula for the eigenvalues of the Laplacian.

We obtain the second term in the asymptotics of $Z_D(t)$ under some additional assumptions on the smoothness of D. The result is inspired by result for trace estimates for stable processes by Bañuelos and Kulczycki [52]. We need the following property of the domain D.

Proof: By (30) we see that

$$\frac{p_D(t,x,x)}{C_1 e^{mt} t^{-d/\alpha}} = \frac{p(t,0)}{C_1 e^{mt} t^{-d/\alpha}} - \frac{r_D(t,x,x)}{C_1 e^{mt} t^{-d/\alpha}}. \tag{44}$$

Since the first term tend to 1 as $t \to 0$ by (41), in order to prove (42), we show that

$$\frac{t^{d/\alpha}}{C_1 e^{mt}} \int_D r_D(t,x,x)dx \to 0, \qquad as\, t \to 0. \tag{45}$$

For $q \geq 0$, we define $D_q = \{x \in D\colon \delta_D(x) \geq q\}$. Then for $0 < t < 1$, consider the subdomain $D_{t^{1/2\alpha}} = \{x \in D\colon \delta_D(x) \geq t^{1/2\alpha}\}$ and its complement $D^C_{t^{1/2\alpha}} = \{x \in D\colon \delta_D(x) < t^{1/2\alpha}\}$. Recalling that $|D| < \infty$, by Lebesgue dominated convergence theorem we get $|D^C_{t^{1/2\alpha}}| \to 0$, as $t \to 0$. Since $p_D(t,x,x) \leq p(t,x,x)$, by (29) we see that

$$\frac{r_D(t,x,x)}{C_1 e^{mt} t^{-d/\alpha}} \leq 1,$$

for all $x \in D$. It follows that

$$\frac{t^{d/\alpha}}{C_1 e^{mt}} \int_{D^C_{t^{12/\alpha}}} r_D(t,x,x)dx \to 0, \qquad as\, t \to 0. \tag{46}$$

On the other hand, by Lemma (5.2.4) in [211] we obtain

$$\begin{aligned}\frac{r_D(t,x,x)}{C_1 e^{mt} t^{-d/\alpha}} &= \frac{\mathbb{E}^x\left[p\left(t - \tau_D, X_{\tau_D}, x\right); t \geq \tau_D\right]}{C_1 e^{mt} t^{-d\alpha}} \\ &\leq c\mathbb{E}^y \min\left\{\frac{t^{1+d/\alpha}}{|x - X(\tau_D)|^{d+\alpha}} e^{-c_2|x-X(\tau_D)|}, 1\right\} \\ &\leq c \min\left\{\frac{t^{1+d/\alpha}}{\delta_D(x)^{d+\alpha}}\, e^{-c_2\delta_D(x)}, 1\right\}.\end{aligned} \tag{47}$$

For $x \in D_{t^{1/2\alpha}}$ and $0 < t < 1$, the right-hand side of (47) is bounded above by $ct^{d/2\alpha+1/2}$ and hence

$$\frac{t^{d/\alpha}}{C_1 e^{mt}} \int_{D_t^{1/2\alpha}} r_D(t,x,x)dx \le ct^{d/2\alpha+1/2}|D|, \tag{48}$$

and this last quantity goes to 0 as $t \to 0$.

For an open set $D \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$, the distribution $P^x(\tau_D < \infty, X(\tau_D) \in \cdot)$ will be called the relativistic α-harmonic measure for D. The following Ikeda–Watanabe formula recovers the relativistic α-harmonic measure for the set D from the Green function.

Proposition (5.2.6)[218]: (See [211].) Assume that D is an open, nonempty, bounded subset of $\mathbb{R}^d$, and A is a Borel set such that $\text{dist}(D,A) > 0$. Then

$$P^x(X(\tau_D) \in A, \tau_D < \infty =) \int_D G_D(x,y) \int_A v(y - z)dzdy, x \in D. \tag{49}$$

We need the following generalization already stated and used in [52].

Proposition (5.2.7)[218]: (See [79], [211].) Assume that D is an open, nonempty, bounded subset of R^d, and A is a Borel set such that $A \subset D^c \backslash \partial D$ and $0 \le t_1 < t_2 < \infty, x \in D$. Then we have

$$P^x(X(\tau_D) \in A, t_1 < \tau_D < t_2) = \int_D \int_{t_1}^{t_2} p_D(s,x,y)ds \int_A v(y-z)dzdy.$$

The following proposition holds for a large class of Lévy processes

Proposition (5.2.8)[218]: (See [52].) Let D and F be open sets in $\mathbb{R}^d$ such that $\subset F$. Then for any $x, y \in \mathbb{R}^d$ we have

$$p_F(t,x,y) - p_D(t,x,y) = E^x(\tau_D < t, X(\tau_D) \in F/D;\ p_F(t-\tau_D, X(\tau_D), y)\ ).$$

Lemma (5.2.9)[218]: (See [224].) Let $D \subset \mathbb{R}^d$ be an open set. For any $x, y \in D$ and $t > 0$ the following estimates hold

$$p_D(t,x,y) \le e^{mt}\tilde{p}_D(t,x,y),$$
$$r_D(t,x,y) \le e^{2mt}\tilde{r}_D(t,x,y). \tag{50}$$

We need the following lemma given by van den Berg in [216].

Lemma (5.2.10)[218]: (See [216].) Let D be an open bounded set in $\mathbb{R}^d$ with R-smooth boundary ∂D and for $0 \le q < R$ denote the area of boundary of ∂D_q by $|\partial D_q|$.Then

$$\left(\frac{R - q}{R}\right)^{d-1} |\partial D| \le \left|\partial D_q\right| \left(\frac{R}{R - q}\right)^{d-1} |\partial D|, \quad 0 \le q < R. \tag{51}$$

Corollary (5.2.11)[218]: (See [52]) Let D be an open bounded set in $\mathbb{R}^d$ with R-smooth boundary. For any $0 < q \le R$ we have

(i) $2^{-d+1}|\partial D| \le \left|\partial D_q\right| \le 2^{d-1}\,|\partial D|$,

(ii) $|\partial D| \le \frac{2^d|D|}{R}$,

(iii) $\left|\partial D_q\right| - |\partial D| \le \frac{2^d dq|\partial D|}{R} \le \frac{2^{2d}dq|D|}{R^2}$.

Theorem (5.2.12)[218]: Let $D \subset \mathbb{R}^d, d \ge 2$, be an open bounded set with R-smooth boundary. Let $|D|$ denote the volume (d-dimensional Lebesgue measure) of D and $|\partial D|$ denote its surface area (($d - 1$)-dimensional Lebesgue measure) of its boundary. Suppose $\alpha \in (0,2)$. Then

$$\left| Z_D(t) - \frac{C_1(t)^{emt}|D|}{t^{d/\alpha}} + C_2(t)|\partial D| \right| \le \frac{C_3 e^{2mt}|D|t^{2/\alpha}}{R^2 t^{d/\alpha}}, t > 0, \tag{52}$$

where

$$C_1(t) = \frac{1}{(4\pi)^{d/2}} \int_0^{\infty} z^{-d/2} e^{-(mt)^{1/\beta} z} \theta_\beta(1, z) dz \to C_1 = \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^d \alpha}, as\ t \to 0,$$

$$C_2(t) = \int_0^{\infty} r_H\big(t, (x_1, 0, \dots, 0), (x_1, 0, \dots, 0)\big)\, dx_1 \le \frac{C_4 e^{2mt} t^{1/\alpha}}{t^{d/\alpha}}, t > 0,$$

$$C_4 = \int_0^{\infty} \tilde{r}_H\,(1, (x_1, 0, \dots, 0), (x_1, 0, \dots, 0)) dx_1,$$

$C_3 = C_3(d, \alpha), H = \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_1 > 0\}$ and r_H is given by (30).

Proof: For the case $t^{1/\alpha} > R/2$ the theorem holds trivially. Indeed, by Eq. (29)

$$Z_D(t) \le \int_D p(t, x, x) dx \le \frac{c_1 e^{mt} |D|}{t^{d/\alpha}} \le \frac{c_1 e^{mt} |D| t^{2/\alpha}}{R^2 t^{d/\alpha}}.$$

By Corollary (5.2.11) and Lemma (5.2.9) we also have

$$C_2(t)|\partial D| \le \frac{C_4 e^{2mt} |\partial D| t^{1/\alpha}}{t^{d/\alpha}} \le \frac{2^d C_4 e^{2mt} |D| t^{1/\alpha}}{R t^{d/\alpha}} \le \frac{2^{d+1} C_4 e^{2mt} |D| t^{2/\alpha}}{R^2 t^{d/\alpha}},$$

$$\frac{C_1(t) e^{mt} |D|}{t^{d/\alpha}} \le \frac{C_1 e^{mt} |D| t^{2/\alpha}}{R^2 t^{d/\alpha}}.$$

Therefore for $t^{1/\alpha} > R/2$ (52) holds. Here and in sequel we consider the case $t^{1/\alpha} \le R/2$. From (31) and the fact that $p(t, x, x) = \frac{C_1(t) e^{mt}}{t^{d/\alpha}}$, we have that

$$Z_D(t) - \frac{C_1(t) e^{mt} |D|}{t^{d/\alpha}} = \int_D p_D(t, x, x) dx - \int_D p(t, x, x) dx$$

$$= -\int_D r_D(t, x, x) dx, \tag{53}$$

where $C_1(t)$ is as stated in the theorem. Therefore we must estimate (53). We break our domain into two pieces, $D_{R/2}$ and its complement $D_{R/2}^C$. We will first consider the contribution of $D_{R/2}$.

Claim (5.2.13)[218]: For $t^{1/\alpha} \le R/2$ we have

$$\int_{D_{R/2}} r_D(t, x, x) dx \le \frac{c e^{2mt} |D| t^{2/\alpha}}{R^2 t^{d/\alpha}}. \tag{54}$$

Proof: By Lemma (5.2.9) we have

$$\int_{D_{R/2}} r_D(t, x, x) dx \le e^{2mt} \int_{D_{R/2}} \tilde{r}_D(t, x, x) dx, \tag{55}$$

and by scaling of the stable density the right-hand side of (55) equals

$$\frac{e^{2mt}}{t^{d/\alpha}} \int_{DR2} \tilde{r}_{D/t^{1/\alpha}} \left(1, \frac{x}{t^{1/\alpha}}, \frac{x}{t^{1/\alpha}}\right) dx. \tag{56}$$

For $x \in D_{R/2}$ we have $\delta_{D/t^{1/\alpha}}(x/t^{1/\alpha}) \ge R/(2t^{1/\alpha})$ 1. By [52], we get

$$\tilde{r}_{D/t^{1/\alpha}} \left(1, \frac{x}{t^{1/\alpha}}, \frac{x}{t^{1/\alpha}}\right) \le \frac{c}{\delta^{d+\alpha}_{D/t^{1/\alpha}}(x/t^{1/\alpha})} \le \frac{c}{\delta^2_{D/t^{1/\alpha}}(x/t^{1/\alpha})} \le \frac{c t^{2/\alpha}}{R^2}.$$

Using the above inequality, we get

$$\int_{D_{R/2}} r_D(t,x,x)dx \le \frac{e^{2mt}}{t^{d/\alpha}} \int_{D_{R/2}} \frac{ct^{2/\alpha}}{R^2}dx \le \frac{ce^{2mt}|D|t^{2/\alpha}}{R^2 t^{d/\alpha}},$$

which proves (54).

Since D has R-smooth boundary, for any point $y \in \partial D$ there are two open balls B_1 and B_2 both of radius R such that $B_1 \subset D, B_2 \subset \mathbb{R}^d \backslash (D \cup \partial D), \partial B_1 \cap \partial B_2 = y$. For any $x \in D_{R/2}$ there exists a unique point $x_* \in \partial D$ such that $\delta_D(x) = |x - x_*|$. Let $B_1 = B(z_1, R), B_2 = B(z_2, R)$ be inner/outer balls for the point x_*. Let $H(x)$ be the half-space containing B_1 such that $\partial H(x)$ contains x_* and is perpendicular to the segment $\overline{z_1 z_2}$.

We will need the following very important proposition in the proof of Theorem (5.2.12). Such a proposition has been proved for the stable process in [52].

Proposition (5.2.14)[218]: Let $D \subset R^d, d \ge 2$, be an open bounded set with R-smooth boundary ∂D. Then for any $x \in D^C_{R/2}$ and $t > 0$ such that $t^{1/\alpha} \le R/2$ we have

$$\left| r_D(t,x,x) - r_{H(x)}(t,x,x) \right| \le \frac{ce^{2mt} t^{1/\alpha}}{R t^{d/\alpha}} \left(\left(\frac{t^{1/\alpha}}{\delta_D(x)} \right)^{d+\alpha/2-1} \wedge 1 \right). \qquad (57)$$

Proof: Exactly as in [52], let $x_* \in \partial D$ be a unique point such that $|x - x_*| = dist(x, \partial D)$ and B_1 and B_2 be balls with radius R such that $B_1 \subset D, B_2 \subset \mathbb{R}^d \backslash (D \cup \partial D), \partial B_1 \cap \partial B_2 = x_*$. Let us also assume that $x_* = 0$ and choose an orthonormal coordinate system $(x_1, x_2, \dots, x_d)$ so that the positive axis $0x_1$ is in the direction of $\overrightarrow{0p}$ where p is the center of the ball B_1. Note that x lies on the interval $0p$ so $x = (|x|, 0, 0, \dots, 0)$. Note also that $B_1 \subset D \subset (\overline{B_2})^c$ and $B_1 \subset H(x) \subset (\overline{B_2})^c$. For any open sets A_1, A_2 such that $A_1 \subset A_2$ we have $r_{A_1}(t,x,y) \ge r_{A_2}(t,x,y)$ so

$$\left| r_D(t,x,x) - r_{H(x)}(t,x,x) \right| \le r_{B_1}(t,x,x) - r_{(\overline{B_2})^c}(t,x,x).$$

So in order to prove the proposition it suffices to show that

$$r_{B_1}(t,x,x) - r_{(\overline{B_2})^c}(t,x,x) \le \frac{ce^{2mt} t^{1/\alpha}}{R t^{d/\alpha}} \left(\left(\frac{t^{1/\alpha}}{\delta_D(x)} \right)^{d+\alpha/2-1} \wedge 1 \right),$$

for any $x = (|x|, 0, \dots, 0), |x| \in (0, R/2]$. Such an estimate was proved for the case $m = 0$ in [52]. In order to complete the proof it is enough to prove that

$$r_{B_1}(t,x,x) - r_{(\overline{B_2})^c}(t,x,x) \le ce^{2mt} \{ \tilde{r}_{B_1}(t,x,x) - \tilde{r}_{(B_2)^c}(t,x,x) \}.$$

To show this given the ball B_2, we set $U = (\overline{B_2})^c$. Now using the generalized Ikeda–Watanabe formula, Proposition (5.2.8) and Lemma (5.2.9) we have

$$\begin{aligned}
r_{B_1}(t,x,x) - r_U(t,x,x) &= E^x\left[t > \tau_{B_1}, X(\tau_{B_1}) \in U \backslash B_1; p_U\left(t - \tau_{B_1}, X(\tau_{B_1}), x\right)\right] \\
&= \int_{B_1} \int_0^t p_{B_1}(s,x,y)ds \int_{U \backslash B_1} v(y-z) p_U(t-s,z,x)dzdy \\
&\le e^{2mt} \int_{B_1} \int_0^t \tilde{p}_{B_1}(s,x,y)ds \int_{U \backslash B_1} \tilde{v}(y-z) \tilde{p}_U(t-s,z,x)dzdy \\
&\le ce^{2mt} E^x\left[t > \tilde{\tau}_{B_1}, \tilde{X}(\tau_{B_1}) \in U \backslash B_1; \tilde{p}_U\left(t - \tilde{\tau}_{B_1}, \tilde{X}(\tilde{\tau}_{B_1}), x\right)\right] \\
&= ce^{2mt} \tilde{r}_{B_1}(t,x,x) - \tilde{r}_U(t,x,x) \le \frac{ce^{2mt} t^{1/\alpha}}{R t^{d/\alpha}} \left(\left(\frac{t^{1/\alpha}}{\delta_D(x)} \right)^{d+\alpha/2-1} \wedge 1 \right).
\end{aligned}$$

The last inequality follows by Proposition (5.2.14) in [52].

Now using this proposition we estimate the contribution from $D\backslash D_{R/2}$ to the integral of $r_D(t,x,x)$ in (53).

Claim (5.2.15)[218]: For $t^{1/\alpha} \le R/2$ we get

$$\left| \int_{D\backslash D_{R/2}} r_D(t,x,x)dx - \int_{D\backslash D_{R/2}} r_{H(x)}(t,x,x)dx \right| \le \frac{ce^{2mt}|D|t^{2\alpha}}{R^2 t^{d\alpha}} . \tag{58}$$

Proof: By Proposition (5.2.14) the left-hand side of (58) is bounded above by

$$\frac{ce^{2mt}t^{1/\alpha}}{Rt^{d/\alpha}} \int_0^{R/2} \left|\partial D_q\right| \left(\left(\frac{t^{1/\alpha}}{q} \right)^{d+\alpha/2-1} \wedge\ 1 \right) dq.$$

By Corollary (5.2.11), (i), the last quantity is smaller than or equal to

$$\frac{ce^{2mt}t^{1/\alpha}|\partial D|}{Rt^{d/\alpha}} \int_0^{R/2} \left(\left(\frac{t^{1/\alpha}}{q} \right)^{d+\alpha/2-1} \wedge\ 1 \right) dq.$$

The integral in the last quantity is bounded by $ct^{1/\alpha}$. To see this observe that since $t^{1/\alpha} \le R/2$ the above integral is equal to

$$\int_0^{t^{1/\alpha}} \left(\left(\frac{t^{1/\alpha}}{q} \right)^{d+\alpha/2-1} \wedge\ 1 \right) dq + \int_{t^{1/\alpha}}^{R/2} \left(\left(\frac{t^{1/\alpha}}{q} \right)^{d+\alpha/2-1} \wedge\ 1 \right) dq$$

$$= \int_0^{t1/\alpha} 1 dq + \int_{t^{1/\alpha}}^{R/2} \left(\frac{t^{1/\alpha}}{q} \right)^{d+\alpha/2-1} dq \le ct^{1/\alpha}.$$

Using this and Corollary (5.2.11), (ii), we get (58).

Recall that $H = \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_1 > 0\}$. For abbreviation let us denote

$$f_H(t,q) = r_H\left(t,(q,0,\dots,0),(q,0,\dots,0)\right), \qquad t,q > 0.$$

of course we have $r_H(x)(t,x,x) = f_H(t,\delta_H(x))$. In the next step we will show that

$$\left| \int_{D\backslash D_{R/2}} r_{H(x)}(t,x,x)dx - |\partial D| \int_0^{R/2} f_H(t,q)dq \right| \le \frac{ce^{2mt}|D|t^{2/\alpha}}{R^2 t^{d/\alpha}} . \tag{59}$$

We have

$$\int_{D\backslash D_{R/2}} r_H(x)(t,x,x)dx = \int_0^{R/2} \left|\partial D_q\right| f_H(t,q)dq.$$

Hence the left-hand side of (59) is bounded above by

$$\int_0^{R/2} |\partial D_q| - |\partial D| f_H(t,q)dq.$$

By Corollary (5.2.11), (iii), this is smaller than

$$\frac{c|D|}{R^2}\int_0^{R/2} qf_H(t,q)dq \le \frac{c|D|e^{2mt}}{R^2}\int_0^{R/2} q\tilde{f}_H(t,q)dq$$
$$= \frac{c|D|e^{2mt}}{R^2}\int_0^{R/2} qt^{-d/\alpha}\tilde{f}_H\left(1,qt^{-1/\alpha}\right)dq$$
$$= \frac{c|D|e^{2mt}}{R^2t^{d/\alpha}}\int_0^{R/2t^{1/\alpha}} qt^{2/\alpha}\tilde{f}_H(1,q)dq$$
$$\le \frac{c|D|e^{2mt}t^{2/\alpha}}{R^2t^{d/\alpha}}\int_0^{\infty} q(q^{-d-\alpha}\wedge 1)dq \le \frac{c|D|e^{2mt}t^{2/\alpha}}{R^2t^{d/\alpha}}.$$

This shows (59). Finally, we have

$$\left||\partial D|\int_0^{R/2} f_H(t,q)dq - |\partial D|\int_0^{\infty} f_H(t,q)dq\right| \le |\partial D|\int_{R/2}^{\infty} f_H(t,q)dq$$
$$\le \frac{c|D|}{R}\int_{R/2}^{\infty} f_H(t,q)dq \text{ by Corollary (5.2.11), (ii)}$$
$$\le \frac{c|D|e^{2mt}}{Rt^{d/\alpha}}\int_{R/2}^{\infty} f_H\left(1,qt^{-1/\alpha}\right)dq = \frac{c|D|e^{2mt}t^{1/\alpha}}{Rt^{d/\alpha}}\int_{R/2t^{1/\alpha}}^{\infty}\tilde{f}_H(1,q)dq.$$

Since $R/2t^{1/\alpha} \ge 1$, so for $q \ge R/2t^{1/\alpha} \ge 1$ we have $\tilde{f}_H(1,q) \le cq^{-d-\alpha} \le cq^{-2}$. Therefore,

$$\int_{R/2t^{1/\alpha}}^{\infty}\tilde{f}_H(1,q)dq \le c\int_{R/2t^{1/\alpha}}^{\infty}\frac{dq}{q^2} \le \frac{ct^{1/\alpha}}{R}.$$

Hence,

$$\left||\partial D|\int_0^{R/2} f_H(t,q)dq - |\partial D|\int_0^{\infty} f_H(t,q)dq\right| \le \frac{c|D|e^{2mt}t^{\frac{2}{\alpha}}}{R^2t^{\frac{d}{\alpha}}}. \quad (60)$$

Note that the constant $C_2(t)$ which appears in the formulation of Theorem (5.2.12) satisfies $C_2(t) = \int_0^{\infty} f_H(t,q)dq$. Now Eqs. (53), (54), (58), (59), (60) give (52).

Section (5.3): Eigenvalues of Pseudo-Differential Operators in an Interval

The fractional Laplace operator $(-\Delta)^{\alpha/2}$was considered in [49] for $\alpha = 1$ and in [97] for general $\alpha \in (0,2)$, while in [236] the case of the Klein–Gordon square-root operator $(-\Delta+1)^{1/2}-1$ was solved (Δ dentotes the second derivative operator, the Laplace operator in dimension one). We extend the above results to operators $\psi(-\Delta)$, where ψis an arbitrary complete Bernstein function such that $\xi\psi'(\xi)$converges to infinity as $\xi \to \infty$.
Let λ_n denote the nondecreasing sequence of eigenvalues of $\psi(-\Delta)$ in an interval $D = (-a,a)$, with zero condition in the complement of D. Furthermore, for $\mu > 0$ define

$$\vartheta_\mu = \frac{1}{\pi}\int_0^{\infty}\frac{\mu}{r^2-\mu^2}\log\frac{\psi'(\mu^2)(\mu^2-r^2)}{\psi(\mu^2)-\psi(r^2)}dr. \quad (61)$$

We note that $\vartheta_\mu \in [0,\frac{\pi}{2})$ and $\frac{d}{d\mu}\vartheta_\mu = O\left(\frac{1}{\mu}\right)$ as $\mu \to \infty$. Finally, let μ_n be a solution of

$$\mu_n = \frac{n\pi}{2a} - \frac{1}{a}\vartheta_{\mu_n}. \quad (62)$$

We remark that the solution is unique for n large enough, and

$$\mu_n = \frac{n\pi}{2a} - \frac{1}{a}\vartheta_{(n\pi)/(2a)} + O(\frac{1}{n}).$$

The following is the main result.

Example (5.3.1)[225]: Let $\psi(\xi) = \xi^{\alpha/2} + \xi^{\beta/2}$, where $0 < \beta < \alpha \le 2$. Then (see Example (5.3.12))

$$\vartheta_\mu = \frac{(2-\alpha)\pi}{8} + O\left(n^{\beta-\alpha}\right), \quad \mu_n = \frac{n\pi}{2a} - \frac{(2-\alpha)\pi}{8a} + O(n^{\beta-\alpha}),$$

and consequently

$$\lambda_n = \left(\frac{n\pi}{2a} - \frac{(2-\alpha)\pi}{8a}\right)^\alpha + \left(\frac{n\pi}{2a} - \frac{(2-\alpha)\pi}{8a}\right)^\beta + O(n^{\beta-1}).$$

Example (5.3.2)[225]: If ψ is regularly varying at infinity with index $\frac{\alpha}{2} \in (0, 1]$, then one has $\lim_{\mu\to\infty} \vartheta_\mu = \frac{(2-\alpha)\pi}{8}$(see(72)). Therefore,

$$\mu_n = \frac{n\pi}{2a} - \frac{(2-\alpha)\pi}{8a} + O(1),$$

and, using Karamata's monotone density theorem, one easily finds that

$$\lambda_n = (1 - \frac{(2-\alpha)\alpha}{4n} + o(\frac{1}{n}))\psi(\left(\frac{n\pi}{2a}\right)^2).$$

` We point out that relatively little is known about λ_n. Most results, including all listed below, cover also higher-dimensionaldomains, but provide significantly less detailed information. Extension of Theorem (5.3.26) for higher-dimensional domains seems out of reach with the present methods.

Best known estimates of λ_n, proved in [65], are given in terms of the corresponding eigenvalues λ_n^Δ of the Laplace operator $-\Delta$, namely

$$C\psi(\lambda_n^\Delta) \le \lambda_n \le \psi(\lambda_n^\Delta);$$

a more direct statement for the case of an interval is given in (69) below. First term of the asymptotic expansion of λ_n, namely $\lambda_n \sim \psi(\lambda_n^\Delta)$, is given in many cases in [54]. This result follows by a Tauberian theorem from the asymptotic expression for the trace $\sum_{n=1}^\infty e^{-t\lambda_n}$ as $t \to 0^+$.

Second term of the asymptotic expansion of the trace has been found in [52], [227] for $(-\Delta)^{\alpha/2}$, in [219], [213]for $(-\Delta + 1)^{\alpha/2} - 1$, and finally in [204]for a rather general class of isotropic Lévy processes with unimodal Lévy measure, satisfying some mild regularity conditions. Tauberian theory is, however, insufficient to obtain a result similar to Theorem(5.3.26)from the two-term expansion of the trace. In this case, as well as for many other local operators, two-term asymptotic formula for the eigenvalues in appropriate domains was derived by V.Ivrii [232], [233], thus resolving the famous Weyl conjecture. The only related result for non-local operators, proved in [103], provides an analogous two-term asymptotic expansion of Cesàro means $\frac{1}{N}\sum_{n=1}^N \lambda_n$ for $(-\Delta)^{\alpha/2}$, using the methods of semi-classical analysis.

The proof of Theorem (5.3.26) is based on the explicit expression for the generalised eigenfunctions of the operator $\psi(-\Delta)$ in the half-line, found in [49] for $(-\Delta)^{1/2}$, and in

[112], [238] for $\psi(-\Delta)$ for a general complete Bernstein function ψ. The asymptotic expression (85)for $(-\Delta)^{\alpha/2}$ simplifies to

$$\lambda_n = \left(\frac{n\pi}{2a} - \frac{(2-\alpha)\pi}{8a}\right)^{\alpha} + O\left(\frac{1}{n}\right) \qquad as \qquad n \to \infty,$$

because $\vartheta_\mu = \frac{(2-\alpha)\pi}{8}$. As mentioned above, this was proved for $\alpha = 1$ in [49], with constant 1 in the asymptotic notation $O\left(\frac{1}{n}\right)$, and for general $\alpha \in (0,2)$ in [97], with a rather big constant in the term $O\left(\frac{1}{n}\right)$.

A very careful estimate of [236] yielded a version of (85) uniform in $a > 0$ for the operator $(-\Delta + 1)^{1/2} - 1$.

We do not pay attention to the constant in the asymptotic term $O(\frac{1}{n})$. All our estimates are, however, explicit, and so it is theoretically possible to trace the dependence of this constant on a and ψ.

We sketch the main idea of the proof. The generalised eigen function of $\psi(-\Delta)$ in the half-line $(0,\infty)$ corresponding to the eigenvalue $\psi(\mu^2)$ is given by an explicit formula $F_\mu(x) = sin(\mu x + \vartheta_\mu) - G_\mu(x)$, where G_μ is the Laplace transform of a certain non-negative measure. We construct approximation $\tilde{\varphi}_n$ to eigenfunctions of $\psi(-\Delta)$ in $(-a,a)$ by interpolating between $F_\mu(a + x)$ near $-a$ and $\pm F_\mu(a - x)$ near a. In order that the sine terms agree, we need to set $\mu = \mu_n$ defined in(62). Due to non-locality of $\psi(-\Delta)$, $\tilde{\varphi}_n$ is not an eigenfunction; we show that the $L^2(D)$ distance of $\psi(-\Delta)\,\tilde{\varphi}_n$ and $\mu_n\,\tilde{\varphi}_n$ does not exceed $O\left(\frac{1}{n}\right)$. This is sufficient to prove that there is some eigenvalue $\lambda_{k(n)}$ within the $O(\frac{1}{n})$ range from $\psi(\mu_n^2)$. Using the assumption that $\xi\psi'(\xi)$ diverges to infinity as $\xi \to \infty$, one easily finds that the numbers $k(n)$ are distinct for sufficiently large n. It remains to estimate the number of eigenvalues λ_j not counted as $\lambda_{k(n)}$: this turn out to follow from an estimate of the trace (Lemma (5.3.25)).

We conjecture that (85) holds for arbitrary complete Bernstein functions, without the moderate growth condition $\lim_{\xi\to\infty} \xi\psi'(\xi) = \infty$. Note that, however, if this growth condition is not satisfied (for example, when $\psi(\xi) = log(1 + \xi)$) and a is large enough, then one cannot expect that the numbers $k(n)$ are distinct. Therefore, an extension of Theorem(5.3.26)to general complete Bernstein function would require a completely different approach. It is also natural to expect that (85) holds for more general functions ψ, for example, for all Bernstein functions ψ satisfying the growth condition. However, no expressions for the generalised eigenfunctions F_μ are known unless ψ is a complete Bernstein function, and so our approach cannot currently be used in this case. Finally, we believe that the semi-classical argument of [103], combined with the results of [112], [238] and for the family of complete Bernstein functions $\psi(\sqrt{a^2 + \xi^2})$, may lead to a two-term asymptotic formula for Cesàro means $\frac{1}{N}\sum_{n=1}^{N} \lambda_n$ of eigenvalues of the operator $\psi(-\Delta)$ in sufficiently smooth domains in $\boldsymbol{R}^d$.

The method described above has been designed in [49] and successfully used in [97] and [236]. The core of the argument remains the same. We showing Theorem (5.3.26) in this generality requires rather non-obvious estimates of $\vartheta_\mu, \frac{d}{d\mu}\vartheta_\mu \; and \; G_\mu(x)$, as well as many other modifications.

All functions considered below are Borel measurable. For $p \in [1,\infty)$ and an open set $D \subseteq \boldsymbol{R}$, the Lebesgue space $L^p(D)$ is the set of functions f on D such that

$\|f\|_{L^p(D)} = \left(\int_D |f(x)|^p dx\right)^{1/p}$ is finite, and $f \in L^\infty(D)$ if and only if the essential supremum $\|f\|_{L^\infty(D)}$ of $|f(x)|$ over $x \in D$ is finite. The space of smooth functions with compact support contained in D is denoted by $C_c^\infty(D)$. By $C_0(D)$ we denote the space of continuous functions in $\boldsymbol{R}$ which are equal to 0 in $\boldsymbol{R}\backslash D$ and which satisfy the condition $\lim_{x\to\pm\infty} f(x) = 0$.

The Fourier transform of a function $f \in L^2(\boldsymbol{R})$ is denoted by $\mathcal{F} f$. If $f \in L^2(\boldsymbol{R}) \cap L^1(\boldsymbol{R})$, then $\mathcal{F}f(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx$. The Laplace transform of a function f is denoted by $\mathcal{L}f$, $\mathcal{L}f(\xi) = \int_0^\infty f(x)e^{-\xi x}dx.$

Symbols x, y, z are used for spatial variables, while ξ, η, μ typically correspond to 'Fourier space' variables.

We write $f(n) = O\big(g(n)\big)$ if $\limsup_{n\to\infty} |f(n)/g(n)| < \infty$, and $f(n) = o\big(g(n)\big)$ if $\lim_{n\to\infty} |f(n)/g(n)| = 0$.

We recall several classical definitions. A function $f(x)$ on $(0,\infty)$ is said to be completely monotone if $(-1)^n f^{(n)}(x) \geq 0$ for all $x > 0$ and $n = 0,1,2,\ldots$. By Bernstein's theorem (see [239]), f is completely monotone if and only if it is the Laplace transform of a (possibly infinite) Radon measure on $[0,\infty)$. If f is nonnegative on $(0,\infty)$ and f' is completely monotone, then f is said to be a Bernstein function. By Bernstein's theorem, Bernstein functions have the representation

$$f(x) = cx + \tilde{c} + \int_{(0,\infty)} (1 - e^{-zx})M(dz) \tag{63}$$

for some $c, \tilde{c} \geq 0$ and a Radon measure M such that $\int_{(0,\infty)} min(z,1)M(dz) < \infty$. The above formula extends to complex x such that $e\ x \geq 0$, and defines a continuous function holomorphic in the region $Re\ x > 0$.

If the measure M in (63) is absolutely continuous with respect to the Lebesgue measure, and the density function is completely monotone, then f is said to be a complete Bernstein function. One easily verifies that in this case

$$f(x) = cx + \tilde{c} + \frac{1}{\pi}\int_{(0,\infty)} \frac{x}{x+z}\frac{m(dz)}{z} \tag{64}$$

for some $c, \tilde{c} \geq 0$ and a Radon measure m such that $\int_{(0,\infty)} min(1/z, 1z^2)m(dz) < \infty$. The above formula defines a holomorphic extension of f in the region $C \setminus (-\infty, 0]$.

Bernstein and complete Bernstein functions appear in a number of different areas of mathematics. For more information on these objects, see [239].

We will need the following technical result.

It is enough to assume that ψ is an increasing, nonnegative function on $[0,\infty)$, which satisfies

$$1 + \psi(\xi + \eta) \leq C(1 + \eta)^\alpha\big(1 + \psi(\xi)\big) \tag{65}$$

for all $\xi \geq \eta \geq 0$ and some $C, \alpha \geq 1$. When ψ is a complete Bernstein function, then (65) holds with $\alpha = 1$ and $C = 1$, because $\psi(\xi + \eta) \leq \psi(\xi) + \psi(\eta) \leq \psi(\xi) + C(1 + \eta)$.
The operator $A = \psi(-\Delta)$ is an unbounded, non-local, self-adjoint operator on $L^2(R)$, defined as follows. The domain $\mathcal{D}(A)$ of A consists of functions $f \in L^2(R)$ such that $\left(1 + \psi(\xi^2)\right)\mathcal{F}f(\xi)$ is square integrable. Clearly, $\mathcal{D}(A)$ contains $C_c^\infty(R)$. For $f \in \mathcal{D}(A)$,
$$\mathcal{F}Af(\xi) = \psi(\xi^2)\mathcal{F}f(\xi).$$
In other words, A is a Fourier multiplier with symbol $\psi(\xi^2)$. This explains the notation $A = \psi(-\Delta)$: the second derivative operator Δ is a Fourier multiplier with symbol $-\xi^2$. Furthermore, by Plancherel's theorem, A is positive-definite.

Let $\mathcal{D}(\mathcal{E})$ denote the space of $f \in L^2(R)$ such that $\left(1 + \psi(\xi^2)\right)^{1/2}\mathcal{F}f(\xi)$ is square integrable. For $f, g \in \mathcal{D}(\mathcal{E})$ the quadratic form $\mathcal{E}(f, g)$ associated to A is defined by
$$\mathcal{E}(f, g) = \frac{1}{2\pi}\int_R \psi(\xi^2)\mathcal{F}f(\xi)\mathcal{F}g(\xi)d\xi.$$
The inner product $\mathcal{E}_1(f, g) = \langle f, g\rangle + \mathcal{E}(f, g)$ makes $\mathcal{D}(\mathcal{E})$ into a Hilbert space. If $f \in \mathcal{D}(A)$, then $\mathcal{E}(f, g) = \langle Af, g\rangle$, and $\mathcal{D}(A)$ is a dense subset of the Hilbert space $\mathcal{D}(\mathcal{E})$.
Let D be an open subset of R. The following definition states that the operator A_D is the Friedrichs extension (or the minimal self-adjoint extension) of the restriction of A to $C_c^\infty(D)$.
Definition (5.3.3)[225]: The domain $\mathcal{D}(\mathcal{E}_D)$ of the form $\mathcal{E}_D$ is the closure of $C_c^\infty(D)$ in the Hilbert space $\mathcal{D}(\mathcal{E})$, and $\mathcal{E}_D(f, g) = \mathcal{E}(f, g)$ for $f, g \in \mathcal{D}(\mathcal{E}_D)$. The operator A_D is associated to the form $\mathcal{E}_D$: $f \in \mathcal{D}(\mathcal{E}_D)$ is in the domain $\mathcal{D}(A_D)$ of A_D if and only if there is a function $A_D f \in L^2(D)$ such that $\mathcal{E}(f, g) = \langle A_D f, g\rangle$ for $g \in \mathcal{D}(\mathcal{E}_D)$ (or, equivalently, for $g \in C_c^\infty(D)$).
The following result is well-known of general Dirichlet forms and generators of Lévy processes, see [231], [92] for more general results in this direction.
Proposition (5.3.4)[225]: (See [236].) If D is a bounded interval, then $f \in \mathcal{D}(\mathcal{E}_D)$ if and only if $f \in \mathcal{D}(\mathcal{E})$ and $f = 0$ almost everywhere in $R \setminus D$.
Proof: By definition, if $f \in \mathcal{D}(\mathcal{E}_D)$, then $f \in \mathcal{D}(\mathcal{E})$ and $f = 0$ almost everywhere in $R \setminus D$. Let $f \in \mathcal{D}(\mathcal{E})$ and $f = 0$ almost everywhere in $R \setminus D$. The result follows from the following claim: there is a sequence $f_n \in C_c^\infty(D)$ such that
$$\mathcal{E}_1(f_n - f, f_n - f) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(1 + \psi(\xi^2)\right)|\mathcal{F}f_n(\xi) - \mathcal{F}f(\xi)|^2 d\xi$$
converges to 0 as $n \to \infty$.
Let $h_n \in C_c^\infty(R^D)$ be an approximation to the identity such that $h_n(x) = nh(nx), h(x) \geq 0, \int_R h(x)dx = 1 \; and \; h(x) = 0$ for $x \notin (-1, 1)$. Note that h_n is zero outside $(-\frac{1}{n}, \frac{1}{n})$.
Let
$$g_n(x) = h_n * f(x), \qquad f_n(x) = g_n((x - b_n)/a_n),$$
where $(x - b_n)/a_n$ maps the $\frac{2}{n}$-neighbourhood of I into I, with $a_n \geq 1, \lim_{n\to\infty} a_n = 1$ and $\lim_{n\to\infty} b_n = 0$. Observe that $f_n \in C_c^\infty(D)$ and
$$\mathcal{F}f_n(\xi) = a_n e^{-ib_n\xi}\mathcal{F}g_n(a_n\xi) = a_n e^{-ib_n\xi}\mathcal{F}h\left(\frac{1}{n}(a_n\xi)\right)\mathcal{F}f(a_n\xi).$$

Since $f, g \in L^1(R), \mathcal{F}f$ and $\mathcal{F}h$ are continuous. Furthermore, $\mathcal{F}h(0) = 1$ and $|\mathcal{F}h(\xi)| \leq 1$ for $\xi \in R$. It follows that $\mathcal{F}f_n$ converges pointwise to $\mathcal{F}f$, and for n large enough

$$|\mathcal{F}f_n(\xi)| \leq 2|\mathcal{F}f(a_n\xi)|$$

for all $\xi \in R$. Hence, if $u(\xi) = \left(1 + \psi(\xi^2)\right)|\mathcal{F}f(\xi)|^2$, then for n large enough,

$$\left(1 + \psi(\xi^2)\right)|\mathcal{F}f_n(\xi) - \mathcal{F}f(\xi)|^2 \leq 2(1 + \psi(\xi^2))(|\mathcal{F}f_n(\xi)|^2 + |\mathcal{F}f(\xi)|^2)$$
$$\leq 4u(a_n\xi) + 2u(\xi)$$

for all ξ. By the assumption, $u(\xi)$ is integrable. Therefore, the family of functions $\left(1 + \psi(\xi^2)\right)|\mathcal{F}f_n(\xi) - \mathcal{F}f(\xi)|^2$ is tight and uniformly integrable. By the Vitali's convergence theorem, $\mathcal{E}_1(f_n - f, f_n - f)$ converges to 0 as $n \to \infty$, as desired.

We remark that the above result in general fails to be true for arbitrary open sets D. It is, in particular, not true when $D = R \setminus \{0\}$ and $\psi(\xi) = \xi^{\alpha/2}$ with $\alpha \in (1, 2]$.

From now on, ψ is a complete Bernstein function. The operator $-A = -\psi(-\Delta)$ generates a strongly continuous semigroup of self-adjoint contractions

$$T(t) = exp(-tA),$$

where $t \geq 0$. Note that $T(0)$ is the identity operator, $T(t)$ is the Fourier multiplier with symbol $exp(-t\psi(\xi^2))$, and

$$\mathcal{D}(A) = \left\{f \in L^2(R): \lim_{t \to 0^+} \frac{T(t)f - f}{t} \textit{ exists in } L^2(R)\right\},$$
$$-Af = \lim_{t \to 0^+} \frac{T(t)f - f}{t}. \tag{66}$$

For $t > 0$, the operator $T(t)$ is a convolution operator: for all $f \in L^2(R)$,

$$T(t)f(x) = \int_{-\infty}^{\infty} f(x - y)T(t; dy), \tag{67}$$

where $T(t; dx)$ is a sub-probability measure with characteristic function $exp(-t\psi(\xi^2))$ and total mass $e^{-t\psi(0)}$. Furthermore,

$$T(t; dx) = e^{-t\psi(\infty)}\,\delta_0(dx) + T(t; x)dx,$$

where $T(t; x) = T(t; -x)$ is a decreasing function of $x > 0$ (see [239]). Hence, $T(t)$ is a Markov operator, and formula (67) defines a contraction on every $L^p(R)$ ($p \in [1, \infty]$), and also on $C_0(R)$. In each of these Banach spaces, the generator of the semigroup $T(t)$ is defined in a similar way as in (66); for example,

$$\mathcal{D}\left(A;\, C_0(R)\right) = \left\{f \in C_0(R): \lim_{t \to 0^+} \frac{T(t)f - f}{t} \textit{ exists in } C_0(R)\right\},$$
$$-Af = \lim_{t \to 0^+} \frac{T(t)f - f}{t}.$$

Since the above definitions of Af are consistent on the intersections of domains with limits in different function spaces: $L^p(R)$ for $p \in [1, \infty]$ or $C_0(R)$, we abuse the notation and use the same symbol $-A$ for the generator of the semigroup $T(t)$ in any of these spaces. Observe that $C_c^\infty(R)$ is contained in $\mathcal{D}(A, L^p(R))$ ($p \in [1, \infty]$) and in $\mathcal{D}(A;\, C_0(R))$, and it is the core of A in each of these Banach spaces except $L^\infty(R)$ (see [226], [234]). Whenever we write $\mathcal{D}(A)$, we mean $\mathcal{D}(A; L^2(R))$.

If $\psi(\xi)$ has the representation given in (64), then for $f \in C_c^\infty(R)$ we have

$$Af(x) = -cf''(x) + \tilde{c}f(x) + pv\int_{-\infty}^{\infty} \left(f(x) - f(y)\right)\nu(x - y)dz, \tag{68}$$

where by the subordination formula,

$$\nu(z) = \frac{1}{2\pi} \int_{(0,\infty)} e^{-|z|\zeta^{1/2}} \frac{m(d\zeta)}{\zeta^{1/2}},$$

and '$pv\int$ ' denotes the Cauchy principal value integral:

$$pv\int_{-\infty}^{\infty} \big(f(x) - f(x+z)\big)\nu(z)dz = \lim_{\varepsilon\to 0^+} \int_{R\setminus(-\varepsilon,\varepsilon)} \big(f(x) - f(x+z)\big)\nu(z)dz;$$

see, [112].

Let D be a (possibly unbounded) interval. The operator $-A_D$ generates a strongly continuous semigroup of operators

$$T_D(t) = exp(-tA_D).$$

The operators $T_D(t)$ are given by

$$T_D(t)f(x) = \int_D f(y)T_D(t;\, x, dy),$$

Where

$$T_D(t;\, x, dy) = e^{-t\psi(\infty)}\, \delta_x(dy) + T_D(t;\, x, y)dy.$$

It is known that $0 \le T_D(t;\, x, y) \le T(t, x - y)$, and we let $T_D(t;\, x, y) = 0$ whenever $x \notin D$ or $y \notin D$. Hence, $T_D(t)$ form a contraction semigroup on each of the spaces $L^p(D)$ ($p \in [1,\infty]$), and if ψ is unbounded, then also on $C_0(D)$ (see [228], [112], [92]). The generator of each of these semigroups is again denoted by $-A_D$, and it acts on an appropriate domain $\mathcal{D}(A_D; L^p)$ or $\mathcal{D}(A_D;\, C_0)$.

Suppose that D is a bounded interval and that $exp(-2t\psi(\xi^2))$ is integrable for some $t > 0$. Then $T_D(t;\, x, y)$ is a Hilbert–Schmidt kernel, and so $T_D(t)$ is a compact operator on $L^2(D)$. Hence, there is a complete orthonormal set of eigenfunctions $\varphi_n \in L^2(D)$ of $T_D(t)$. By strong continuity and the semigroup property, the eigenfunctions do not depend on $t > 0$, and the corresponding eigenvalues have the form $e^{-t\lambda_n}$ for all $t > 0$, where the sequence λ_n is nondecreasing and converges to ∞.

By translation invariance, we may assume that $D = (-a, a)$. By symmetry, $T_D(t;\, x, y) = T_D(t;\, -x, -y)$, and hence the spaces of odd and even $L^2(D)$ functions are invariant under the action of $T_D(t)$. Therefore, we may assume that every φ_n is either an odd or an even function. The ground state eigenvalue λ_1 is positive and simple (unless ψ is constant), and the corresponding ground state eigenfunction has constant sign in D; we choose it to be positive in D. The functions φ_n are also the eigenfunctions of A_D (because $-A_D$ is the generator of the semigroup $T_D(t)$), and λ_n are the corresponding eigenvalues.

No closed-form expression for λ_n and φ_n is available, except when $\psi(\xi) = c\xi + \tilde{c}$. By a general result of [54] (see Theorem (5.3.4) therein), $\lambda_n \sim \psi((\frac{n\pi}{2a})2)$ as $n \to \infty$ (the original statement includes only the case when $\psi(\xi) \sim \xi^{\alpha/2}$ for some $\alpha \in (0,2)$, but it can be easily extended to more general ψ). Best known general estimates of λ_n are found in [65], where it is proved that:

$$\frac{1}{2}\,\psi\left(\left(\frac{n\pi}{2a}\right)^2\right) \le \lambda_n \le \psi\left(\left(\frac{n\pi}{2a}\right)^2\right). \tag{69}$$

Note that the upper bound in (69) follows relatively easily from the operator monotonicity of ψ: the form associated to A_D is bounded above by the form of $\psi(-\Delta_D)$, and the eigenvalues of the latter are equal to $\psi(\left(\frac{n\pi}{2a}\right)^2)$. The proof of the lower bound is more intricate.

The spectrum of A_D for an unbounded interval D is continuous. When $D = R$, then $A_D = A$ takes diagonal form in the Fourier space, and $e^{i\xi x}$ $(\xi \in R)$ are the L^∞ eigenfunctions of A. Similar eigenfunction expansion was obtained for the half-line using an appropriate version of the Wiener–Hopf method in [112], [238]. Due to translation invariance and symmetry, it suffices to consider $D = (0, \infty)$.

Definition (5.3.5)[225]: Suppose that ψ is a non-constant complete Bernstein function such that $\psi(0) = 0$. For $x, \mu > 0$, let

$$F_\mu(x) = sin(\mu x + \vartheta_\mu) - G_\mu(x),$$

where $\vartheta_\mu \in [0, \frac{\pi}{2})$ and G_μ is a completely monotone function on $(0, \infty)$. More precisely,

$$\vartheta_\mu = \frac{1}{\pi}\int_0^\infty \frac{\mu}{r^2 - \mu^2}\, log \frac{\psi'(\mu^2)(\mu^2 - r^2)}{\psi(\mu^2) - \psi(r^2)}\, dr$$

(as in (61)), and G_μ is the Laplace transform of a measure γ_μ,

$$G_\mu(x) = \mathcal{L}\gamma_\mu(x) = \int_{(0,\infty)} e^{-x\xi}\gamma_\mu(d\xi),$$

With

$$\gamma_\mu(d\xi) = \frac{1}{\pi}\lim_{\varepsilon\to 0^+} Im\left(\frac{\mu\psi'(\mu^2)}{\psi(\mu^2) - \psi(-e^{-i\varepsilon}\xi^2)}\right) \times exp\left(-\frac{1}{\pi}\int_0^\infty \frac{\xi}{\xi^2 + r^2}\, log \frac{\psi'(\mu^2)(\mu^2 - r^2)}{\psi(\mu^2) - \psi(r^2)}\, dr\right) d\xi$$

for $\mu, \xi, x > 0$.

Equivalently, $F_\mu(x)$ is defined by its Laplace transform: for $\xi \in C$ with $Re\, \xi > 0$,

$$\mathcal{L}F_\mu(\xi) = \int_0^\infty F_\mu(x) e^{-\xi x} dx = \frac{\mu}{\mu^2 + \xi^2}\, exp\left(\frac{1}{\pi}\int_0^\infty \frac{\xi}{\xi^2 + r^2}\, log \frac{\psi'(\mu^2)(\mu^2 - r^2)}{\psi(\mu^2) - \psi(r^2)}\right) dr,$$

see [238] and [112]. We have the short-hand expressions

$$\mathcal{L}F_\mu(\xi) = \frac{\mu}{\mu^2 + \xi^2} \frac{\psi_\mu^\dagger(\xi)}{\sqrt{\psi_\mu(\mu^2)}},$$

$$\vartheta_\mu = Arg\, \psi_\mu^\dagger(i\mu),$$

$$\gamma_\mu(d\xi) = \frac{1}{\pi}\lim_{\varepsilon\to 0^+} \frac{\mu}{\mu^2 + \xi^2} \frac{Im\, \psi_\mu(-e^{-i\varepsilon}\xi^2)}{\sqrt{\psi_\mu(\mu^2)\psi_\mu^\dagger(\xi)}}\, d\xi,$$

again see [112] and [238]. The expressions for $\gamma_\mu(d\xi)$ given above are slightly different than in [112], [238], so we provide a short justification. By Lemma (5.3.30) and the identity

$$\mathcal{L}G_\mu(\xi) = \frac{\mu\, cos\vartheta_\mu + \xi\, sin\vartheta_\mu}{\mu^2 + \xi^2} - \mathcal{L}F_\mu(\xi),$$

we have

$$\gamma_\mu(d\xi) = \frac{1}{\pi}\lim_{\varepsilon\to 0^+} \ Im\left(\mathcal{L}G_\mu\left(e^{-i\varepsilon}\xi\right)\right)d\xi \ = \ -\frac{1}{\pi}\lim_{\varepsilon\to 0^+} \ Im(\mathcal{L}F_\mu(e^{-i\varepsilon}\xi))d\xi.$$

The expression for $\mathcal{L}F_\mu(\xi)$ and the Wiener–Hopf identity $\psi_\mu^\dagger(\xi)\psi_\mu^\dagger(-\xi) = \psi_\mu(-\xi^2)$ give

$$\gamma_\mu(d\xi) = \frac{1}{\pi}\lim_{\varepsilon\to 0^+} \ Im\left(\frac{\mu}{\mu^2 + e^{-2i\varepsilon}\xi^2} \ \frac{\psi\mu\left(-e^{-2i\varepsilon}\xi^2\right)}{\sqrt{\psi_\mu(\mu^2)\psi_\mu^\dagger(e^{-i\varepsilon}\xi)}}\right)d\xi$$

$$= \frac{1}{\pi}\lim_{\varepsilon\to 0^+} \ \frac{\mu}{\mu^2 \ + \ \xi^2} \ \frac{Im\,\psi_\mu\left(-e^{-i\varepsilon}\xi^2\right)}{\sqrt{\psi_\mu(\mu^2)\psi_\mu^\dagger(\xi)}}\ d\xi,$$

as desired; here we used Lemma (5.3.30) again.

We extend the definition of F_μ and G_μ to R so that $F_\mu(x) \ = \ G_\mu(x) \ = \ 0$ for $x \ \leq \ 0$. The functions F_μ $(\mu \ > \ 0)$ are L^∞ eigenfunctions of A_D and play a similar role for A_D as the Fourier kernel $e^{i\xi x}(\xi \ \in \ R)$ for A. This is formally stated in the following result.

Theorem (5.3.6)[225]: (See [112] and [238].) The functions F_μ are L^∞ eigenfunctions of $A_{(0,\infty)}$; the corresponding eigenvalues are $\psi(\mu^2)$. The operator $A_{(0,\infty)}$ takes a diagonal form under the integral transform with kernel $F_\mu(x)$. More precisely, let

$$\Pi f(\mu) \ = \int_0^\infty \ f(x)F_\mu(x)dx$$

for $f \in L^2((0,\infty)) \ \cap \ L^1((0,\infty))$. Then $\left(\frac{2}{\pi}\right)^{1/2}\Pi$ extends to a unitary mapping on $L^2((0,\infty))$, such that for $f \in L^2((0,\infty))$,

$$f \ \in \mathcal{D}(A_{(0,\infty)}) \Leftrightarrow \ (1 \ + \ \psi(\mu^2))\Pi f(\mu) \ \in \ L^2((0,\infty)),$$

and if $f \ \in \mathcal{D}(A_{(0,\infty)})$, then

$$\Pi\left(A_{(0,\infty)}f\right)(\mu) = \ \psi(\mu^2)\Pi f(\mu), \qquad \Pi(T_D(t)f) \ = \ e^{-t\psi(\mu^2)}\,\Pi f(\mu).$$

We only use the first part of the above result, namely, that F_μ are the $L^\infty((0,\infty))$ eigenfunctions of $A_{(0,\infty)}$. We remark that a similar eigenfunction expansion is available for $D \ = \ R\setminus\{0\}$, see [235], [237], and there are no other known explicit expressions for the eigenfunctions of A_D unless $D \ = \ R$ or $\psi(\xi) \ = \ c\xi \ + \ \tilde{c}$.

Recall that according to (61), Definition (5.3.5) and [112],

$$\vartheta_\mu = \frac{1}{\pi}\int_0^\infty \ \frac{\mu}{s^2 \ - \ \mu^2} \ log\frac{\psi'(\mu^2)(\mu^2 - s^2)}{\psi(\mu^2) - \ \psi(s^2)} \ ds \tag{70}$$

$$= \frac{1}{\pi}\int_0^1 \frac{1}{1 - z^2} \ log \ \frac{\psi(\mu^2) - \psi(\mu^2 z^2)}{z^2\left(\psi(\mu^2/z^2) - \psi(\mu^2)\right)}\ dz. \tag{71}$$

We remark that if ψ is regularly varying at infinity with index $\alpha \in (0,2]$, then, by dominated convergence,

$$\lim_{\mu\to\infty} \ \vartheta_\mu = \frac{1}{\pi}\lim_{\mu\to\infty} \ \int_0^1 \frac{1}{1 \ - z^2}\log \ \frac{1 - \psi(\mu^2 z^2)/\psi(\mu^2)}{z^2(\psi(\mu^2/z^2)/\psi(\mu^2) - \ 1)}\ dz$$

$$= \frac{1}{\pi}\int_0^1 \ \frac{1}{1 \ - z^2} \ log \ \frac{1 - z^\alpha}{z^2(z^{-\alpha} \ - \ 1)}dz = \frac{2-\alpha}{\pi} \ \int_0^1 \frac{-\log \ z}{1 \ - z^2} \ dz$$

$$= \frac{(2-\alpha)\pi}{8}\,, \tag{72}$$

see [112]. By [112], dominated convergence can be used to differentiate the right-hand side of (70) in $\mu > 0$ under the integral sign. This yields

$$\frac{d\vartheta_\mu}{d\mu} = \frac{2}{\pi\mu}\int_0^1 \frac{1}{1-z^2}\left(\frac{\mu^2\psi'(\mu^2)-\mu^2z^2\psi'(\mu^2z^2)}{\psi(\mu^2)-\psi(\mu^2z^2)} - \frac{(\mu^2/z^2)\psi'(\mu^2/z^2)-\mu^2\psi'(\mu^2)}{\psi(\mu^2/z^2)-\psi(\mu^2)}\right)dz \quad (73)$$

for all $\mu > 0$. We prove two properties of ϑ_μ that are needed in the remaining part of the article. First, we find estimates of ϑ_μ that imply that the lower limits of ϑ_μ as $\mu \to 0^+$ or $\mu \to \infty$ do not exceed $\frac{3\pi}{8}$ (Lemma (5.3.10)). Next, a simple estimate of $\frac{d}{d\mu}\vartheta_\mu$ is found (Lemma (5.3.11)).

By [238], we have the following general estimate of ϑ_μ:

$$\left(\inf_{\xi>0} -\frac{\xi\psi''(\xi)}{\psi'(\xi)}\right)\frac{\pi}{4} \le \vartheta_\mu \le \left(\sup_{\xi>0} -\frac{\xi\psi''(\xi)}{\psi'^{(\xi)}}\right)\frac{\pi}{4}$$

for all $\mu > 0$. Furthermore, the supremum in the upper bound is always not greater than 2. If ψ is a Thorin–Bernstein function (see [239]), then one easily checks that the supremum is in fact not greater than 1, and therefore $\vartheta_\mu \le \frac{\pi}{4}$. Below we find more refined bounds for ϑ_μ. By [238],

$$\frac{1}{\pi}\left(\arcsin^2\sqrt{Q} + \arcsin^2\sqrt{\frac{Q}{1-P}} - \arcsin^2\sqrt{\frac{PQ}{1-P}}\right) \le \vartheta_\mu \le \frac{\pi}{2} - arcsin\sqrt{P} \quad (74)$$

With

$$P = \frac{\mu^2\psi'(\mu^2)}{\psi(\mu^2)}, \qquad Q = \frac{-\mu^2\psi''(\mu^2)}{2\psi'(\mu^2)}$$

(note that the factor $\frac{1}{\pi}$ is missing in the lower bound in the original statement). By the same argument as in the proof of the lower bound of [238] (using the lower bound for $\psi_\lambda(\lambda^2\zeta^2)$ and the upper bound for $\psi_\lambda(\lambda^2/\zeta^2)$), one easily shows that, with the same P and Q,

$$\vartheta_\mu \le \frac{\pi}{4} - \frac{1}{\pi}\left(\arcsin^2\sqrt{1-Q} + \arcsin^2\sqrt{1-\frac{Q}{1-P}} - \arcsin^2\sqrt{1-\frac{PQ}{1-P}}\right). \quad (18)$$

One can also verify that this bound is always at least as good as the upper bound of (74), with equality when $P + Q = 1$.

The following technical result states that $P + Q \le 1$. This in fact follows indirectly from the proof of [238] (note that the right-hand side of (75) is not well-defined when $P + Q > 1$), but we choose to give a simple, direct argument.

Lemma (5.3.7)[225]: If ψ is a non-constant complete Bernstein function, then

$$-\frac{\xi\psi''(\xi)}{\psi'(\xi)} \le 2\,\frac{-2\xi\psi'(\xi)}{\psi(\xi)}.$$

Proof: The lemma is equivalent to the inequality

$$-\xi\psi(\xi)\psi''(\xi) \le 2\psi'(\xi)(\psi(\xi)-\xi\psi'(\xi)).$$

Assuming ψ has the representation (64), we need to prove

$$\xi\left(c\xi+\tilde{c}+\frac{1}{\pi}\int_{(0,\infty)}\frac{\xi}{\xi+s}\frac{\mu(ds)}{s}\right)\left(\frac{1}{\pi}\int_{(0,\infty)}\frac{s}{(\xi+s)^3}\frac{\mu(ds)}{s}\right)$$
$$\leq\left(c+\frac{1}{\pi}\int_{(0,\infty)}\frac{s}{(\xi+s)^2}\frac{\mu(ds)}{s}\right)\left(\tilde{c}+\frac{1}{\pi}\int_{(0,\infty)}\frac{\xi^2}{(\xi+s)^2}\frac{\mu(ds)}{s}\right).$$

This follows by simple integration from the following bounds: $0\leq c\tilde{c}$,

$$\xi(c\xi+\tilde{c})\frac{s}{(\xi+s)^3}\leq c\frac{\xi^2}{(\xi+s)^2}+\tilde{c}\frac{s}{(\xi+s)^2},$$

And

$$\xi\left(\frac{\xi}{\xi+s_1}\frac{s_2}{(\xi+s_2)^3}+\frac{\xi}{\xi+s_2}\frac{s_1}{(\xi+s_1)^3}\right)$$
$$\leq\frac{s_1}{(\xi+s_1)^2}\frac{\xi^2}{(\xi+s_2)^2}+\frac{s_2}{(\xi+s_2)^2}\frac{\xi^2}{(\xi+s_1)^2};$$

the last two inequalities are easily proved by direct calculations.

Lemma (5.3.8)[225]: The left-hand side of (74) is decreasing in $P\in[0,1-Q]$. The right-hand side of (75) is increasing in $P\in[0,1-Q]$.

Proof: Let $P=1-\frac{Q}{s+Q}=\frac{s}{s+Q}$, $s\in[0,1-Q]$. Note that P increases with increasing s, and the left-hand side of (74) is equal to

$$\frac{1}{\pi}(\arcsin^2\sqrt{Q}+\arcsin^2\sqrt{s+Q}-\arcsin^2\sqrt{s}).$$

Since $\arcsin^2\sqrt{s}$ is convex, the above expression is increasing in s. In a similar way, with $P=1-\frac{Q}{1-s}=\frac{1-s-Q}{1-s}$, $s\in[0,1-Q]$, the right-hand side of (75) is equal to

$$\frac{\pi}{4}-\frac{1}{\pi}(\arcsin^2\sqrt{1-Q}+\arcsin^2\sqrt{s}-\arcsin^2\sqrt{s+Q}),$$

which is again an increasing function of s, but now P decreases with increasing s. Substituting $P=0$, we obtain immediately the following elegant result.

Corollary (5.3.9)[225]: If ψ is a non-constant complete Bernstein function such that $\psi(0)=0$, then

$$\frac{2}{\pi}\arcsin^2\sqrt{\frac{-\mu^2\psi''(\mu^2)}{2\psi'(\mu^2)}}\leq\vartheta_\mu\leq\frac{\pi}{2}-\frac{2}{\pi}\arcsin^2\sqrt{1+\frac{\mu^2\psi''(\mu^2)}{2\psi'(\mu^2)}}.$$

Lemma (5.3.10)[225]: If ψ is a non-constant complete Bernstein function such that $\psi(0)=0$, then

$$\liminf_{\mu\to0^+}\vartheta_\mu\leq\frac{3\pi}{8}.$$

If ψ is unbounded, then also

$$\liminf_{\mu\to\infty}\vartheta_\mu\leq\frac{3\pi}{8}.$$

Proof: Suppose that $\liminf_{\mu\to0^+}\vartheta_\mu>\frac{3\pi}{8}$. Then there are $\mu_0>0$ and $q\in(0,1)$ such that $\vartheta_\mu\geq\frac{\pi}{2}-\frac{q\pi}{8}$ for $\mu\in(0,\mu_0)$. By Corollary (5.3.9),

$$\arcsin^2 \sqrt{1+\frac{\mu^2\psi''(\mu^2)}{2\psi'(\mu^2)}} \le \frac{q\pi^2}{16}$$

for $\mu \in (0,\mu_0)$, and hence

$$\frac{-\mu^2\psi''(\mu^2)}{\psi'(\mu^2)} \ge 2 - 2\left(sin\,\frac{\pi\sqrt{q}}{4}\right)^2$$

for $\mu \in (0,\mu_0)$. If α denotes the right-hand side, then $\alpha > 1$. By integration (see [235]), we have $\psi'(\mu^2)/\psi'(\mu_0^2) \ge (\mu_0^2/\mu^2)^\alpha$ for all $\mu \in (0,\mu_0)$, which contradicts integrability of ψ' at 0. This proves the first statement of the lemma.

In a similar manner, if $\liminf_{\mu\to\infty} \vartheta_\mu > \frac{3\pi}{8}$, then there are $\mu_0 > 0$ and $q \in (0,1)$ such that $\vartheta_\mu \ge \frac{\pi}{2} - \frac{q\pi}{8}$ for $\mu \in (\mu_0,\infty)$. Again this implies

$$\frac{-\mu^2\psi''(\mu^2)}{\psi'(\mu^2)} \ge 2 - 2\left(sin\,\frac{\pi\sqrt{q}}{4}\right)^2$$

for $\mu \in (\mu_0,\infty)$. If α denotes the right-hand side, then $\alpha > 1$, and by integration, $\psi'(\mu^2)/\psi'(\mu_0^2) \le (\mu_0^2/\mu^2)^\alpha$ for all $\mu \in (\mu_0,\infty)$. This implies integrability of ψ' at ∞.

We conjecture that the above lemma holds with $\frac{3\pi}{8}$ replaced with $\frac{\pi}{4}$. An example of a complete Bernstein function ψ for which the set of partial limits of ϑ_μ as $\mu \to 0^+$ is equal to $[0,\frac{\pi}{2}]$ is given in [238].

Lemma (5.3.11)[225]: If ψ is a non-constant complete Bernstein function such that $\psi(0) = 0$, then for all $\mu > 0$,

$$\left|\frac{d\vartheta_\mu}{d\mu}\right| < \frac{3}{\mu}.$$

Proof: By (73) and the Cauchy's mean value theorem, for some $\xi_z \in (\mu^2z^2,\mu^2)$ and $\xi_{1/z} \in (\mu^2,\mu^2/z^2)$ (where $z \in (0,1)$),

$$\frac{d\vartheta_\mu}{d\mu} = \frac{2}{\pi\mu}\int_0^1 \frac{1}{1-z^2}\left(\frac{\xi_z\psi''(\xi_z)+\psi'(\xi_z)}{\psi'(\xi_z)} - \frac{\xi_{1z}\psi''(\xi_{1z})+\psi'(\xi_{1z})}{\psi'(\xi_{1z})}\right)dz$$

$$= \frac{2}{\pi\mu}\int_0^1 \frac{1}{1-z^2}\left(\frac{\xi_z\psi''(\xi_z)}{\psi(\xi_z)} - \frac{\xi_{1/z}\psi''(\xi_{1/z})}{\psi'(\xi_{1/z})}\right)dz.$$

By (64), $0 \le -\xi\psi''(\xi) \le 2\psi'(\xi)$ and $0 \le \xi^2\psi^{(3)}(\xi) \le 6\psi'(\xi)$. Hence,

$$\xi\frac{d}{d\xi}\left(\frac{\xi\psi''(\xi)}{\psi'(\xi)}\right) = \frac{\xi^2\psi^{(3)}(\xi)}{\psi'(\xi)} - \frac{-\xi\psi''(\xi)}{\psi'(\xi)} - \frac{(-\xi\psi''(\xi))^2}{(\psi'(\xi))^2} \in [-6,6].$$

Furthermore, $\xi_z\psi''(\xi_z)/\psi'(\xi_z) - \xi_{1/z}\psi''(\xi_{1/z})/\psi'(\xi_{1/z}) \in [-2,2]$. It follows that

$$\left|\frac{d\vartheta_\mu}{d\mu}\right| \le \frac{2}{\pi\mu}\int_0^1 \frac{1}{1-z^2}\, min\left(2,\int_{\xi_{1/z}}^{\xi_z}\left|\frac{d}{dr}\left(\frac{r\psi''(r)}{\psi'(r)}\right)\right|dr\right)dz$$

$$\le \frac{2}{\pi\mu}\int_0^1 \frac{1}{1-z^2}\, min\left(2, 6\,log\frac{\xi_{1/z}}{\xi_z}\right)dz.$$

Recall that $\xi_{1/z}/\xi_z \le z^{-4}$. Hence,

$$\left|\frac{d\vartheta_\mu}{d\mu}\right| \le \frac{2}{\pi\mu}\int_0^1 \frac{min(2,-24\, log\, z)}{1-z^2}\, dz.$$

Since $-\,log\, z \le \frac{1}{z} - 1$,we have

$$\left|\frac{d\vartheta_\mu}{d\mu}\right| \le \frac{2}{\pi\mu}\int_0^1 \frac{min\big(2, 24(1z\, -\, 1)\big)}{1-z^2}\, dz = \frac{100\, log\, 5\, -\, 48\, log24}{\pi\mu} < \frac{3}{\mu}.$$

We conjecture that in fact $-\,\frac{1}{\mu} < \frac{d}{d\mu}\vartheta_\mu \le \frac{1}{2\mu}$. We with the following simple example.

Example (5.3.12)[225]: Let $\psi(\xi) = \xi^{\frac{\alpha}{2}} + \xi^{\beta/2}$, where $0 < \beta < \alpha$ By a short calculation

$$\frac{\psi(\mu^2) - \psi(\mu^2 z^2)}{z^2\big(\psi(\mu^2/z^2) - \psi(\mu^2)\big)} = \frac{1}{z^{2-\alpha}}\left(\frac{1-z^\alpha}{1-z^\beta} + \frac{1}{\mu^{\alpha-\beta}}\right)\left(\frac{1-z^\alpha}{1-z^\beta} + \frac{z^{\alpha-\beta}}{\mu^{\alpha-\beta}}\right)^{-1}.$$

If we denote $w = (1-z^\alpha)/(1-z^\beta)$, then

$$\frac{\psi(\mu^2) - \psi(\mu^2 z^2)}{z^2\big(\psi(\mu^2/z^2) - \psi(\mu^2)\big)} = \frac{1}{z^{2-\alpha}}\,\frac{\mu^{\alpha-\beta}w\, + 1}{\mu^{\alpha-\beta}w + z^{\alpha-\beta}} = \frac{1}{z^{2-\alpha}}\left(1 + \frac{1-z^{\alpha-\beta}}{\mu^{\alpha-\beta}w + z^{\alpha-\beta}}\right).$$

As in the last equality of (72), we obtain

$$\vartheta_\mu = \frac{(2\, -\, \alpha)\pi}{8} + \frac{1}{\pi}\int_0^1 \frac{1}{1\, - z^2}\, log\left(1 + \frac{1-z^{\alpha-\beta}}{\mu^{\alpha-\beta}w + z^{\alpha-\beta}}\right) dz.$$

Clearly, the integrand is nonnegative. Since $log(1 + s) \le s, z^{\alpha-\beta} \ge z^2$ and $w \ge 1$,

$$\int_0^1 \frac{1}{1-z^2}\, log\left(1 + \frac{1-z^{\alpha-\beta}}{\mu^{\alpha-\beta}w\, +\, 1}\right) dz \le \int_0^1 \frac{1}{1-z^2}\frac{1-z^{\alpha-\beta}}{\mu^{\alpha-\beta}w\, + z^{\alpha-\beta}}\, dz$$

$$\le \int_0^1 \frac{1}{1-z^2}\frac{1-z^2}{\mu^{\alpha-\beta}} dz = \frac{1}{\mu^{\alpha-\beta}}.$$

Therefore,

$$\frac{(2\, -\, \alpha)\pi}{8} \le \vartheta_\mu \le \frac{(2\, -\, \alpha)\pi}{8} + \frac{1}{\pi\mu^{\alpha-\beta}}.$$

The remaining part of the article we will need the following simple estimate of $\mathcal{L}F_\mu$ and a more refined estimate of G_μ.

Lemma (5.3.13)[225]: If ψ is a non-constant complete Bernstein function such that $\psi(0) = 0$, then for all $\mu > 0$ and ξ such that $Re\, \xi > 0$

$$\left|\mathcal{L}F_\mu(\xi)\right| \le 2\sqrt{2}\frac{\mu}{|\mu^2 + \xi^2|}\sqrt{\frac{\psi'(\mu^2)(\mu^2 - |\xi|^2)}{\psi(\mu^2) - \psi(|\xi|^2)}}.$$

Proof: Recall that $(\mu^2 + \xi^2)\mathcal{L}F_\mu(\xi) = \mu\left(\psi_\mu(\mu^2)\right)^{-1/2}\psi_\mu^\dagger(\xi)$ is a complete Bernstein function of ξ, and hence by [112] and [238],

$$|\mu^2 + \xi^2|\left|\mathcal{L}F_\mu(\xi)\right| \le \sqrt{2}\,(\mu^2 + |\xi|^2)\mathcal{L}F_\mu(|\xi|) \le 2\sqrt{2}\mu\sqrt{\frac{\psi'(\mu^2)(\mu^2 - |\xi|^2)}{\psi(\mu^2) - \psi(|\xi|^2)}}$$

for all ξ such that $Re\, \xi > 0$.

Lemma (5.3.14)[225]: If ψ is a non-constant complete Bernstein function such that $\psi(0) = 0$, then for all $\mu, x > 0$ such that $\mu x \ne 1$,

$$G_\mu(x) \le \frac{1}{\pi x}\frac{\psi(1/x^2)}{\psi(\mu^2)}\sqrt{\frac{\psi'(\mu^2)}{\psi(\mu^2)}}\frac{1-\psi(\mu^2)/\big(\mu^2x^2\psi(1/x^2)\big)}{1-\psi(1/x^2)/\psi(\mu^2)}.$$

In particular, if ψ is unbounded, then

$$\limsup_{\mu\to\infty}\left(\mu\psi(\mu^2)G_\mu(x)\right) \le \frac{\psi(1/x^2)}{\pi x}.$$

Proof: Recall that $\psi_\mu^\dagger(\xi) \ge \psi_\mu^\dagger(0) = \psi_\mu(0) = 1$. Hence,

$$\gamma_\mu(d\xi) \le \frac{1}{\pi\mu\sqrt{\psi_\mu(\mu^2)}}\lim_{\varepsilon\to 0^+} Im\,\psi_\mu(-e^{-i\varepsilon}\xi^2)d\xi.$$

After a substitution $\xi = \sqrt{s}$ it follows that

$$G_\mu(x) = \int_0^\infty e^{-\xi x}\gamma_\mu(d\xi) \le \frac{1}{2\pi\mu\sqrt{\psi_\mu(\mu^2)}}\lim_{\varepsilon\to 0^+}\int_0^\infty \sqrt{s}\,e^{-x\sqrt{s}}\,\frac{Im\,\psi_\mu\big(-e^{-i\varepsilon}s\big)ds}{s}.$$

Since $x\sqrt{s}\,e^{-x\sqrt{s}} \le 2/(1+x^2s)$, we have

$$G_\mu(x) \le \frac{1}{\pi\mu x\sqrt{\psi_\mu(\mu^2)}}\lim_{\varepsilon\to 0^+}\int_0^\infty \frac{1}{1+x^2s}\frac{Im\,\psi_\mu\big(-e^{-i\varepsilon}s\big)ds}{s} \le \frac{\psi_\mu(1/x^2)-1}{\pi\mu x\sqrt{\psi_\mu(\mu^2)}};$$

for the last inequality note that the integral converges to the integral term in the representation (64) for the complete Bernstein function ψ_μ, and we have $\psi_\mu(0) = 1$ (therefore the inequality becomes equality if ψ_μ contains no linear term, that is, if ψ is unbounded). To prove the first statement, it remains to use the definition of ψ_μ. The other statement of the lemma follows from the first one by the inequality $\xi\psi'(\xi) \le \psi(\xi)$.

We implicitly assume that ψ is a non-constant complete Bernstein function such that $\psi(0) = 0$, that is, $\tilde{c} = 0$ in the representation (64) for ψ. By c and ν we denote the constant and the measure in the representation (64) for ψ. Finally, we let $D = (-a,a)$ for some $a > 0$.

Recall that $A = \psi(-d^2/dx^2)$, and for $f \in C_c^\infty(R)$ we have, as in (68),

$$Af(x) = -cf''(x) + pv\int_{-\infty}^\infty (f(x)-f(y))\nu(x-y)dy$$
$$= -cf''(x) + \int_0^\infty (2f(x)-f(x+z)-f(x-z))\nu(z)dz. \qquad (76)$$

We denote the right-hand side by $\mathcal{A}f(x)$ (with a calligraphic letter $\mathcal{A}$) whenever the integral converges and, if $c > 0, f''$ is well-defined. The following estimates of $\mathcal{A}f(x)$ are proved in [236] in the special case $\psi(\xi) = (\xi+1)^{1/2}-1$, but their proofs rely only on the symmetry, unimodality and positivity of the kernel function ν. Note that in [236] the notation $\mathcal{A}_0$ is used for $\mathcal{A}$.

Lemma (5.3.15)[225]: (See [236].) Let $x \in R, b > 0$, and let g have an absolutely continuous derivative in $(x-b,x+b)$. Then

$$|\mathcal{A}g(x)| \le c|g''(x)| + \left(\sup_{y\in(x-b,x+b)}|g''(y)|\right)\int_0^b$$
$$= z^2\nu(z)dz + \int_{R\setminus(x-b,x+b)}(|g(x)|+|g(y)|)\nu(y-x)dy.$$

As in [236], [49], [97],for $b > 0$ we define an auxiliary function:

$$q(x) = \begin{cases} 0 & for\ x \in (-\infty, -b], \\ (1/2)(x/b + 1)^2 & for\ x \in [-b, 0], \\ 1 - (1/2)(x/b - 1)^2 & for\ x \in [0, b], \\ 1 & for\ x \in [b, \infty), \end{cases} \tag{77}$$

Note that q is C^1, q' is absolutely continuous, $0 \le q''(x) \le 1/b^2$ (for $x \in R \setminus \{-b, 0, b\}$), the distributional derivative $q^{(3)}$ is a finite signed measure, and $q(x) + q(-x) = 1$.

Lemma (5.3.16)[225]: (See [236].) Let $b > 0$, let $f \in L^1(R)$, and suppose that the second derivative $f''(x)$ exists for $x \in [-b, b]$ and it is continuous in $[-b, b]$. Define

$$M_{-1} = \int_0^\infty |f(x)| dx, \qquad M_0 = \sup_{x \in [-b,b]} |f(x)|,$$
$$M_1 = \sup_{x \in [-b,b]} |f'(x)|, \qquad M_2 = \sup_{x \in [-b,b]} |f''(x)|.$$

Let $q(x)$ be given by (77), and define $g(x) = q(x)f(x)$. For $x \in (-\infty, 0)$, we have

$$|\mathcal{A}g(x)| \le C(b, \psi)(M_{-1} + M_0 + M_1 + M_2).$$

More precisely, for $x \in (-\infty, -b]$ we have

$$|\mathcal{A}g(x)| \le \frac{M_0}{2b^2} \int_0^{2b} z^2 \nu(z) dz + \nu(2b) M_{-1}$$

and for $x \in (-b, 0)$,

$$|\mathcal{A}g(x)| \le M_2 c + \left(\frac{M_0}{b^2} + \frac{2M_1}{b} + M_2\right) \int_0^b z^2 \nu(z) dz + 2M_0 \int_b^\infty \nu(z) dz + \nu(b) M_{-1}.$$

Recall that $D = (-a, a)$. Following [236], [49], [97], for $n \ge 1$, let μ˜n be the largest solution of

$$a\tilde{\mu}_n + \vartheta_{\tilde{\mu}_n} = \frac{n\pi}{2}, \tag{78}$$

with ϑ_μ defined in (61); this agrees with the definition of μ_n in (62), but we choose to use the notation $\tilde{\mu}_n$, so that all approximations are clearly distinguished from true values by the presence of a tilde. Although we are interested in large n only, note that by Lemma (5.3.10), the equation $a\mu + \vartheta_\mu = \frac{n\pi}{2}$ has a solution for all $n \ge 1$, and every such solution satisfies

$$\frac{(n-1)\pi}{2a} \le \tilde{\mu}_n \le \frac{n\pi}{2a}.$$

We remark that (78) may fail to have a unique solution for $n = 1$ (for example, when $a = 1$ and $\psi(\xi) = \xi/(10^4 + \xi) + \xi/10^7$). Nevertheless, if $n \ge 3$ and $\mu \ge \frac{(n-1)\pi}{2a} = \frac{\pi}{a}$, then, by Lemma (5.3.11),

$$\frac{d}{d\mu}\left(a\mu + \vartheta_\mu\right) > a - \frac{3}{\mu} \ge a - \frac{3a}{\pi} > 0,$$

and so the solution $\tilde{\mu}_n$ is in fact unique.
We let

$$\tilde{\lambda}_n = \psi(\tilde{\mu}_n^2).$$

In order to show that $\tilde{\lambda}_n$ is close to some eigenvalue of A_D, we construct an approximate eigenfunction $\tilde{\varphi}_n$ of A_D, using the eigenfunctions $F_{\tilde{\mu}_n}(a - x), F_{\tilde{\mu}_n}(a + x)$ for the one-sided problems corresponding to $A_{(-\infty,a)}$ and $A_{(-a,\infty)}$. As in [236], [49], [97], we define

$$\tilde{\varphi}_n(x) = q(-x)F_{\tilde{\mu}_n}(a + x) - (-1)^n q(x)F_{\tilde{\mu}_n}(a - x), \tag{79}$$
with the auxiliary function q defined by (77). Here $x \in R$, but we have $\tilde{\varphi}_n(x) = 0$ for $x \notin D$, so that $\tilde{\varphi}'_n$ is equal to zero in the complement of D. Clearly, $\tilde{\varphi}_n$ is continuously differentiable in $D, \tilde{\varphi}'_n$ is absolutely continuous in $D, \tilde{\varphi}''_n$ exists in $D \setminus \{-b, b\}$, and $\tilde{\varphi}''_n$ is locally bounded in D. Note that $\tilde{\lambda}_n$ depends on a and n, while $\tilde{\phi}_n(x)$ depends also on b. We could fix b in order to optimise the constants (in many cases $b = \frac{1}{3}a$ seems to be a reasonable choice), but since we do not track the exact value of the constants, we will simply indicate their dependence on b. Note also that $\tilde{\varphi}_n$ is not normed in $L^2(D)$, its norm is approximately equal to $\sqrt{a}$ (see Lemma (5.3.19)).
The following result is intuitively clear, although its formal proof is rather long and technical.
Lemma (5.3.17)[225]: (See [236].) We have $\tilde{\varphi}_n \in \mathcal{D}(A_D)$ and $A_D\tilde{\varphi}_n(x) = \mathcal{A}\tilde{\varphi}_n(x)$ for almost all $x \in D$.
Proof: For brevity, in this proof we write $\tilde{\mu} = \tilde{\mu}_n$ and $\tilde{\varphi} = \tilde{\varphi}_n$. The domain of A_D is described in Definition (5.3.3): we need to prove that $\tilde{\varphi} \in \mathcal{D}(\mathcal{E})$ and that $\langle \tilde{\varphi}, \mathcal{A}g \rangle = \langle \mathcal{A}\tilde{\varphi}, g \rangle$ for all $g \in C_c^\infty(D)$. We first verify the latter condition.
Note that $\mathcal{A}\tilde{\varphi}(x)$ is well-defined for all $x \in D \setminus \{-b, b\}$, since $\tilde{\varphi}$ is smooth in $D \setminus \{-b, b\}$ and bounded on R. Let $g \in C_c^\infty(D)$. Since $\tilde{\varphi}'$ is absolutely continuous in $(-a, a)$, integration by parts gives
$$\int_{-a}^{a} (-c\tilde{\varphi}''(x))g(x)dx = \int_{-a}^{a} \tilde{\varphi}(x)(-cg''(x))dx.$$
Furthermore, by the definition of $\mathcal{A}$ (see (76)),
$$\int_{-a}^{a} \mathcal{A}\tilde{\varphi}(x)g(x)dx - \int_{-a}^{a} \tilde{\varphi}(x)Ag(x)dx$$
$$= \int_{-a}^{a} \left(\int_{0}^{\infty} \big(g(x+z)\tilde{\varphi}(x) + g(x-z)\tilde{\varphi}(x) - g(x)\tilde{\varphi}(x+z) - g(x)\tilde{\varphi}(x-z)\big)\nu(z)dz\right)dx.$$
We claim that the double integral exists. Then, by Fubini, it is equal to 0, and so $\langle \tilde{\varphi}, \mathcal{A}g \rangle = \langle \mathcal{A}\tilde{\varphi}, g \rangle$, as desired.
Denote the integrand by $I(x,z)\nu(z)$, and let $\varepsilon = \frac{1}{3}\operatorname{dist}(supp\ g, R \setminus D)$, so that supp $g \subseteq (-a + 3\varepsilon, a - 3\varepsilon)$. When $z \geq \varepsilon$, then $|I(x,z)| \leq 4\, \|\tilde{\varphi}\|_{L^\infty(R)}\, \|g\|_{L^\infty(R)}$. Suppose that $z \in (0, \varepsilon)$. If $x \notin (-a + 2\varepsilon, a - 2\varepsilon)$, then $I(x,z) = 0$. Otherwise, by first-order Taylor's expansion of $I(x,z)$ around $z = 0$ (note that $I(x,0) = \frac{\partial}{\partial z} I(x,0) = 0$) with the remainder in the integral form, we obtain that
$$|I(x,z)| \leq \int_{0}^{z} (z-s)\frac{\partial^2}{\partial s^2} \quad I(x,s)ds$$
$$\leq z^2(\, \|\tilde{\varphi}\|_{L^\infty(R)}\, \|g''\|_{L^\infty(R)} + \|\tilde{\varphi}''\|_{L^\infty((-a+\varepsilon, a-\varepsilon))}\, \|g\|_{L^\infty(R)})$$
(recall that $\tilde{\varphi}''$ is bounded in $(-a + \varepsilon, a - \varepsilon)$). We conclude that $|I(x,z)\nu(z)| \leq C_1(\tilde{\varphi}, g)\, min(1, z^2)\nu(z)$, which implies joint integrability of $I(x,z)\nu(z)$. Our claim is proved.

It remains to verify that $\tilde{\varphi} \in \mathcal{D}(\mathcal{E})$, that is, $\left(1 + \psi(\xi^2)\right)|\mathcal{F}\tilde{\varphi}(\xi)|^2$ is integrable. Let $f(x) = q(a - x)F_{\tilde{\mu}}(x)$, so that $\tilde{\varphi}(x) = f(a + x) - (-1)^n f(a - x)$ (see (79)). It suffices to prove integrability of $\left(1 + \psi(\xi^2)\right)|\mathcal{F}f(\xi)|^2$.

Fix $\varepsilon > 0$ and let $\tilde{q}(x) = q(a - x)e^{\varepsilon x}$. Since the distributional derivatives q, q' and q'' are integrable functions, and the third distributional derivative of $q(x)$ is a finite signed measure on R, the function $\tilde{q}(x)$ has the same property. Therefore, $\mathcal{F}q(\xi)$ and $\mathcal{F}q^{(3)}(\xi) = -i\xi^3\mathcal{F}q(\xi)$ are bounded functions, and so $|\mathcal{F}\tilde{q}(\xi)| \le C_2(\varepsilon, a, b)/(1 + |\xi|)^3$. The Fourier transform of $e^{-\varepsilon x}F_{\tilde{\mu}}(x)$ is equal to $\mathcal{L}F_{\tilde{\mu}}(\varepsilon + i\xi)$, and the Fourier transform of $f(x) = q(a - x)F_{\tilde{\mu}}(x) = \tilde{q}(x)e^{-\varepsilon x}F_{\tilde{\mu}}(x)$ is given by the convolution

$$\mathcal{F}f(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}\tilde{q}(\xi - s)\mathcal{L}F_{\tilde{\mu}}(\varepsilon + is)ds.$$

Suppose that $\xi > 0$. To estimate $|\mathcal{F}f(\xi)|$, we write

$$\mathcal{F}f(\xi) = \frac{1}{2\pi}\int_{\xi/2}^{\infty} \mathcal{F}\tilde{q}(\xi - s)\mathcal{L}F_{\tilde{\mu}}(\varepsilon + is)ds + \frac{1}{2\pi}\int_{\xi/2}^{\infty} \mathcal{F}\tilde{q}(s)\mathcal{L}F_{\tilde{\mu}}(\varepsilon + i(\xi - s))ds. \quad (80)$$

By Lemma (5.3.13), we have

$$\left|\mathcal{L}F_{\tilde{\mu}}(\varepsilon + is)\right| \le 2\sqrt{2}\frac{\tilde{\mu}}{|\tilde{\mu}^2 + (\varepsilon + is)^2|}\sqrt{\frac{\psi'(\tilde{\mu}^2)(\tilde{\mu}^2 - |\varepsilon + is|^2)}{\psi(\tilde{\mu}^2) - \psi(|\varepsilon + is|^2)}}$$
$$\le C_3(\varepsilon, \tilde{\mu}, \psi)\ \frac{1}{1 + s}\ \sqrt{\frac{1}{1 + \psi(s^2)}}$$

(for the second inequality observe that the expression under the square root is bounded by a constant when $s \le 2\tilde{\mu}$ and by $\psi'(\tilde{\mu}^2)(1 + s^2)/(\psi(s^2) - \psi(\tilde{\mu}^2))$ when $s > 2\tilde{\mu}$). The right-hand side decreases with $s > 0$. Hence,

$$\left|\int_{\xi/2}^{\infty} \mathcal{F}\tilde{q}(\xi - s)\mathcal{L}F_{\tilde{\mu}}(\varepsilon + is)ds\right| \le \frac{C_3(\varepsilon, \tilde{\mu}, \psi)}{(1 + \xi/2)\left(1 + \psi(\xi^2/4)\right)^{1/2}}\int_{\xi/2}^{\infty} |\mathcal{F}\tilde{q}(\xi - s)|ds$$
$$\le \frac{C_3(\varepsilon, \tilde{\mu}, \psi)C_2(\varepsilon, a, b)}{(1 + \xi/2)\left(1 + \psi(\xi^2/4)\right)^{1/2}}\int_{\xi/2}^{\infty} \frac{1}{(1 + |\xi - s|)^3}\, ds$$
$$\le \frac{8C_3(\varepsilon, \tilde{\mu}, \psi)C_2(\varepsilon, a, b)}{(1 + \xi)\left(1 + \psi(\xi^2)\right)^{1/2}};$$

in the last inequality we used the fact that $4\psi(\xi^2/4) \ge \psi(\xi^2)$ and that the integral is bounded by 1. The estimate of the other integral in (80) is simpler: $|\mathcal{L}F_{\tilde{\mu}}(\varepsilon + is)| \le C_4(\varepsilon, \tilde{\mu})$ for all $s \in R$, and hence

$$\left|\int_{\xi/2}^{\infty} \mathcal{F}\tilde{q}(s)\mathcal{L}F_{\tilde{\mu}}\left(\varepsilon + i(\xi - s)\right)ds\right| \le C_4(\varepsilon, \tilde{\mu})\int_{\xi/2}^{\infty} |\mathcal{F}\tilde{q}(s)|ds \le \frac{C_4(\varepsilon, \tilde{\mu})C_2(\varepsilon, a, b)}{2(1 + \xi/2)^2}.$$

Therefore, for $\xi > 0$,

$$|\mathcal{F}f(\xi)| \le C_5(\varepsilon, a, b, \tilde{\mu})\left(\frac{1}{(1 + |\xi|)\left(1 + \psi(\xi^2)\right)^{1/2}} + \frac{1}{(1 + |\xi|)^2}\right).$$

Since $\mathcal{F}f(-\xi) = \overline{\mathcal{F}f(\xi)}$, the above estimate extends to all $\xi \in R$. We conclude that for all $\xi \in R$,

$$\left(1+\psi(\xi^2)\right)|\mathcal{F}f(\xi)|^2 \le 2\left(C_5(\varepsilon,a,b,\tilde{\mu})\right)^2\left(\frac{1}{(1+|\xi|)^2}+\frac{1+\psi(\xi^2)}{(1+|\xi|)^4}\right),$$

and the right-hand side is integrable because $(1+|\xi|)^{-2}(1+\psi(\xi^2))$ is bounded. Following [236], we introduce the following notation:

$$\nu_0(x)=c+\int_0^x z^2\nu(z)dz, \qquad \nu_\infty(x)=\int_x^\infty \nu(z)dz,$$

$$I_\mu=\int_0^\infty G_\mu(x)dx, \qquad G_{\mu,b}(x)=G_\mu(x-b)+G_\mu(x+b).$$

We recall two fundamental estimates, which were proved in [236] for $\psi(\xi)=(\xi+1)^{1/2}-1$, but their proofs work for general non-constant complete Bernstein functions ψ such that $\psi(0)=0$. One minor change is required in the proof of Lemma (5.3.18): an extra term M_2c appears when Lemma (5.3.16) is applied (as compared to the application of [236] in the proof of [236]). This extra term is absorbed into $M_2\nu_0(b)$. Also, note two typos in the first displayed formula in the original statement of [236]: the norm in the left-hand side should not be squared, and the term $\tilde{\lambda}_n G_{\tilde{\mu}_n,b}(a)$ is missing in the right-hand side. (These typos did not appear in the other displayed formula in the original statement, which was the one used later in the proof of the main result.)

Lemma (5.3.18)[225]: (See [236].) We have

$$\left\|A_D\tilde{\varphi}_n-\tilde{\lambda}_n\tilde{\varphi}_n\right\|_{L^2(D)} \le C(a,b,\psi)\left(\left(1+\tilde{\lambda}_n\right)G_{\tilde{\mu}_n,b}(a)-G'_{\tilde{\mu}_n,b}(a)+G''_{\tilde{\mu}_n,b}(a)+I_{\tilde{\mu}_n}+\frac{1}{\tilde{\mu}_n}\right).$$

More precisely, we have

$$\left\|A_D\tilde{\varphi}_n-\tilde{\lambda}_n\tilde{\varphi}_n\right\|^2_{L^2(D)} \le 2(a-b)\left(\frac{G_{\tilde{\mu}_n,b}(a)\nu_0(2b)}{2b^2}+\nu(2b)I_{\tilde{\mu}_n}+\frac{2\nu(a)}{\tilde{\mu}_n}\right)^2 + 2b\left(\frac{\left(G_{\tilde{\mu}_n,b}(a)-2bG'_{\tilde{\mu}_n,b}(a)+b^2G''_{\tilde{\mu}_n,b}(a)\right)\nu_0(b)}{b^2} + 2G_{\tilde{\mu}_n,b}(a)\nu_\infty(b)+\nu(b)I_{\tilde{\mu}_n}+\frac{\tilde{\lambda}_nG_{\tilde{\mu}_n,b}(a)}{2}+\frac{2\nu(a)}{\tilde{\mu}_n}\right)^2.$$

Lemma (5.3.19)[225]: (See [236].) We have

$$\left|\|\tilde{\varphi}_n\|^2_{L^2(D)}-a\right|\le 8(I_{\tilde{\mu}_n}+1/\tilde{\mu}_n).$$

More precisely,

$$a-\frac{\sin\left(\vartheta_{\tilde{\mu}_n}\right)}{\tilde{\mu}_n}-4I_{\tilde{\mu}_n}\le\|\tilde{\varphi}_n\|^2_{L^2(D)}\le a+\frac{\sin\left(\vartheta_{\tilde{\mu}_n}\right)}{\tilde{\mu}_n}+4I_{\tilde{\mu}_n}(1+\sin\vartheta_{\tilde{\mu}_n}).$$

Lemma (5.3.20)[225]: If ψ is unbounded, then for $n\ge 2$,

$$\left\|A_D\tilde{\varphi}_n-\tilde{\lambda}_n\tilde{\varphi}_n\right\|_{L^2(D)}\le\frac{C(a,b,\psi)}{n}$$

And

$$a-\frac{20a}{n\pi}\le\|\tilde{\varphi}_n\|^2_{L^2(D)}\le a+\frac{36a}{n\pi}.$$

Proof: By [112],

$$I_\mu = \mathcal{L}G_\mu(0) = \frac{\cos\ \vartheta_\mu}{\mu} - \mathcal{L}F_\mu(0^+) = \frac{cos\,\vartheta_\mu}{\mu} - \sqrt{\frac{\psi'(\mu^2)}{\psi(\mu^2)}} \le \frac{1}{\mu}\,. \qquad (81)$$

Furthermore, by complete monotonicity,

$$I_\mu \ge \int_0^x G_\mu(z)dz \ \ge \int_0^x \left(G_\mu(x) - G'_\mu(x)(x-z) + \frac{1}{2}G'_\mu(x)(x-z)^2\right)dz$$
$$= xG_\mu(x) - \frac{1}{2}x^2G'_\mu(x) + \frac{1}{6}x^3G'_\mu(x),$$

so that

$$G_\mu(x) \le \frac{1}{\mu x}\,, \qquad G'_\mu(x) \le \frac{2}{\mu x^2}\,, \qquad G''_\mu(x) \le \frac{6}{\mu x^3}\,.$$

By Lemma (5.3.14), for $\mu \ \ge \ \tilde{\mu}_2$

$$\psi(\mu^2)G_\mu(x) \le \frac{C(\psi,x)}{\mu}\,.$$

Finally, $\tilde{\mu}_n \ \ge \frac{(n-1)\pi}{2a} \ge \frac{n\pi}{4a}$ for $n \ \ge \ 2$. The result follows from Lemmas (5.3.18) and (5.3.19).

Let $\sigma(A_D)$ denote the spectrum of A_D. Recall that the spectrum of A_D is purely discrete, and the eigenvalues of A_D are denoted by λ_n. The following result was given in [236] for $\psi(\xi) \ = \ (\xi \ + \ 1)^{1/2} \ - \ 1$ only, but the proof extends to arbitrary self-adjoint operators A_D that preserve the spaces of even and odd functions.

Lemma (5.3.21)[225]: (See [236].) We have

$$dist\left(\tilde{\lambda}_n, \sigma(A_D)\right) \le \frac{\left\|A_D\tilde{\varphi}_n \ - \ \tilde{\lambda}_n\tilde{\varphi}_n\right\|_{L^2(D)}}{\|\tilde{\varphi}_n\|_{L^2(D)}}\,. \qquad (82)$$

In fact, if A_D^{even} and A_D^{odd} are the restrictions of A_D to the (invariant) subspaces of $L^2(D)$ consisting of even and odd functions, respectively, then (82) holds with $\sigma(A_D)$ replaced by $\sigma(A_D^{even})$ when n is odd, and by $\sigma(A_D^{odd})$ when n is even.

The following result is an immediate consequence of Lemmas (5.3.20) and (5.3.21).

Corollary (5.3.22)[225]: If ψ is unbounded, for all $n \ \ge \ 7$ there is a positive integer $k(n)$ such that

$$\left|\tilde{\lambda}_n \ - \ \lambda_{k(n)}\right| \le \frac{C(a,b,\psi)}{n}\,.$$

Lemma (5.3.23)[225]: Suppose that $\lim\limits_{\xi\to\infty} \ \xi\psi'(\xi) \ = \ \infty$. For n larger than some (integer) constant $C(a,b,\psi)$ the numbers $k(n)$ are distinct. Moreover, for any $\varepsilon \ > \ 0$, for n larger than some (integer) constant $C(a,b,\psi,\varepsilon)$,

$$\psi((\tilde{\mu}_n \ - \ \varepsilon)^2) \ < \ \lambda_{k(n)} \ < \ \psi((\tilde{\mu}_n \ + \ \varepsilon)^2). \qquad (83)$$

Proof: Let $\varepsilon \ \in \ (0,\frac{\pi}{4a})$. For some $\xi_n \in (\tilde{\mu}_n, \tilde{\mu}_n \ + \ \varepsilon)$,

$$\psi((\tilde{\mu}_n \ + \ \varepsilon)^2) \ - \ \psi(\tilde{\mu}_n^2) = 2\varepsilon\xi_n\psi'(\xi_n^2).$$

Since $\xi_n \ \le \frac{n\pi}{2a} \ + \ \varepsilon \ \le \frac{n\pi}{a}$, it follows that

$$\psi((\tilde{\mu}_n \ + \ \varepsilon)^2) - \ \psi(\tilde{\mu}_n^2) \ge \frac{2a\varepsilon\xi_n^2\psi'(\xi_n^2)}{n\pi}\,.$$

Since $\xi_n \geq \frac{(n-1)\pi}{2a}$, we have $\lim_{n\to\infty} \xi_n^2\psi'(\xi_n^2) = \infty$, and so, by Corollary (5.3.22), for n greater than some constant $C(a, b, \psi, \varepsilon)$,

$$\psi((\tilde{\mu}_n + \varepsilon)^2) - \psi(\tilde{\mu}_n^2) > |\tilde{\lambda}_n - \lambda_{k(n)}|.$$

Since ψ is concave,

$$\psi(\tilde{\mu}_n^2) - \psi((\tilde{\mu}_n - \varepsilon)^2) \geq \psi((\tilde{\mu}_n + \varepsilon)^2) - \psi(\tilde{\mu}_n^2).$$

Finally, $\tilde{\lambda}_n = \psi(\tilde{\mu}_n^2)$. This proves (83).
Observe that, by Lemma (5.3.11),

$$a\tilde{\mu}_{n+1} - a\tilde{\mu}_n = \frac{\pi}{2} + \vartheta_{\tilde{\mu}_n} - \vartheta_{\tilde{\mu}_{n+1}} \geq \frac{\pi}{2} - \frac{3}{\tilde{\mu}_n}(\tilde{\mu}_{n+1} - \tilde{\mu}_n)$$
$$\geq \frac{\pi}{2} - \frac{6a}{(n-1)\pi}(\tilde{\mu}_{n+1} - \tilde{\mu}_n),$$

so that $\tilde{\mu}_{n+1} - \tilde{\mu}_n \geq \frac{\pi}{2a}\left(1 + \frac{6}{(n-1)\pi}\right)^{-1} \geq \frac{\pi}{4a}$ for $n \geq 3$. The first statement of the lemma follows hence from (83) with $\varepsilon = \frac{\pi}{8a}$.

Lemma (5.3.24)[225]: Suppose that $\lim_{\xi\to\infty} \xi\psi'(\xi) = \infty$. Then $k(n) \geq n$ for infinitely many n.

Proof: By Lemma (5.3.23),

$$\lambda_{k(n)} \geq \psi\left(\left(\tilde{\mu}_n - \frac{\pi}{16a}\right)^2\right)$$

for n large enough. On the other hand, by (69),

$$\lambda_{n-1} \leq \psi\left(\left(\frac{(n-1)\pi}{2a}\right)^2\right)$$

for all $n \geq 1$. Finally, by Lemma (5.3.10) and Lemma (5.3.11), $\vartheta_{\tilde{\mu}_n} < \frac{3\pi}{8} + \frac{\pi}{16}$ for infinitely many n, and hence

$$\tilde{\mu}_n - \frac{\pi}{16a} = \frac{n\pi}{2a} - \frac{1}{a}\vartheta_{\tilde{\mu}_n} - \frac{\pi}{16a} > \frac{n\pi}{2a} - \left(\frac{3\pi}{8a} + \frac{\pi}{16a}\right) - \frac{\pi}{16a} = \frac{(n-1)\pi}{2a}$$

for infinitely many n.

Recall that the kernel functions of the operators $exp(-tA)$ and $exp(-tA_D)$ are denoted by $T(t;\, x - y)$ and $T_D(t;\, x, y)$, respectively. Furthermore, $0 \leq T_D(t;\, x, y) \leq T(t;\, x - y)$ for all $t > 0$ and $x, y \in D = (-a, a)$, and the Fourier transform of $T(t;\, x)$ is $exp(-t\psi(\xi^2))$. In order to estimate the number of eigenvalues λ_n not counted as $\lambda_{k(n)}$ for n large enough, we use the trace estimate method, applied previously in [49], [97] and [236], see also [52], [82].

Lemma (5.3.25)[225]: Suppose that $\lim_{\xi\to\infty} \xi\psi'(\xi) = \infty$. For n greater than some constant $C(a, b, \psi)$ we have $k(n) = n$.

Proof: Let $\varepsilon = \frac{\pi}{6a}$ and let N be the constant $C(a, b, \psi, \varepsilon)$ in Lemma (5.3.23). Define $J = \{k(n) : n > N\}$ and let $J' = \{j \geq 1 : j \notin J\}$. We claim that it suffices to show that $|J'| \leq N$. Indeed, there is $n_0 > N$ such that $k(n_0) = 1 + max\, J'$, and $k(n)$ is strictly increasing for $n > N$. It follows that $k(n) = k(n0) + n - n_0$ for $n \geq n_0$. If $|J'| \leq N$, then $k(n_0) = |J'| + (n_0 - N) \leq n_0$, so that $k(n) \leq n$ for $n \geq n_0$. Since $k(n) \geq n$ infinitely many times by Lemma (5.3.24), necessarily $k(n) = n$ for $n \geq n_0$, as desired.

Let $t > 0$. By the assumption, $\psi(\xi) \geq \frac{1}{t} \log \xi - C(t)$ for some constant $C(t)$, and therefore $exp(-t\psi(\xi^2))$ is integrable. Therefore, $T(t;\, x)$ is bounded in $x \in R$. In particular, $T_D(t;\, x,\cdot)$ is in $L^2(D)$, and so, by Parseval's identity,

$$\int_{-a}^{a} \int_{-a}^{a} \left(T_D(t;\, x,y)\right)^2 dydx = \int_{-a}^{a} \sum_{n=1}^{\infty} \left(\int_{-a}^{a} T_D(t;\, x,y)\varphi_j\,(y)dy\right)^2 dx$$

$$= \int_{-a}^{a} \sum_{j=1}^{\infty} e^{-2\lambda_j t} \left(\varphi_j\,(x)\right)^2 dx = \sum_{j=1}^{\infty} e^{-2\lambda_j t}.$$

On the other hand, by Plancherel's identity,

$$\int_{-a}^{a} \int_{-a}^{a} \left(T_D(t;\, x,y)\right)^2 dydx \leq 2a \int_{-\infty}^{\infty} \left(T(t;\, x-y)\right)^2 dy = \frac{2a}{\pi} \int_{0}^{\infty} e^{-2t\psi(\xi^2)}\, d\xi.$$

It follows that for all $t > 0$,

$$\sum_{j=1}^{\infty} e^{-\lambda_j t} \leq \frac{2a}{\pi} \int_{0}^{\infty} e^{-t\psi(\xi^2)}\, d\xi. \tag{84}$$

Observe that

$$\sum_{j\in J} e^{-\lambda_j t} = \sum_{n=N}^{\infty} e^{-\lambda_{k(n)} t} \geq \sum_{n=N+1}^{\infty} e^{-\psi((\tilde{\mu}_n+\varepsilon)^2)t} \geq \sum_{n=N}^{\infty} e^{-\psi((n\pi/(2a)+\varepsilon)^2)t}.$$

Denote $\xi_n = n\pi/(2a) + \varepsilon = (n + \frac{1}{3})\pi/(2a)$. Since $e^{-t\psi(z)}$ is concave in $z > 0$,

$$\int_{\xi_n}^{\xi_{n+1}} e^{-t\psi(\xi^2)}\, d\xi \leq \int_{\xi_n}^{\xi_{n+1}} \left(\frac{\xi_{n+1}^2 - \xi^2}{\xi_{n+1}^2 - \xi_n^2}\, e^{-t\psi(\xi_n^2)} + \frac{\xi^2 - \xi_n^2}{\xi_{n+1}^2 - \xi_n^2}\, e^{-t\psi(\xi_{n+1}^2)}\right) d\xi$$

$$= \frac{2\xi_{n+1}^2 - \xi_n\xi_{n+1} - \xi_n^2}{3(\xi_n + \xi_{n+1})}\, e^{-t\psi(\xi_n^2)} + \frac{\xi_{n+1}^2 + \xi_n\xi_{n+1} - 2\xi_n^2}{3(\xi_n + \xi_{n+1})}\, e^{-t\psi(\xi_{n+1}^2)}$$

$$= \frac{\pi}{2a}\left(\frac{3n+3}{6n+5}\, e^{-t\psi(\xi_n^2)} + \frac{3n+2}{6n+5}\, e^{-t\psi(\xi_{n+1}^2)}\right).$$

Hence,

$$\frac{2a}{\pi} \int_{\xi_N}^{\infty} e^{-t\psi(\xi^2)}\, d\xi \leq \sum_{n=N}^{\infty} \left(\frac{3n+3}{6n+5}\, e^{-t\psi(\xi_n^2)} + \frac{3n+2}{6n+5}\, e^{-t\psi(\xi_{n+1}^2)}\right)$$

$$\leq \frac{3N+3}{6N+5}\, e^{-t\psi(\xi_N^2)} + \sum_{n=N+1}^{\infty} e^{-t\psi(\xi_n^2)} \leq \frac{3N+3}{6N+5}\, e^{-t\psi(\xi_N^2)} + \sum_{j\in J} e^{-t\lambda_j}$$

(the second inequality is a consequence of $\frac{3n+2}{6n+5} + \frac{3(n+1)+3}{6(n+1)+5} \leq 1$, while the last one follows from $\lambda_{k(n)} \leq \psi((\tilde{\mu}_n + \varepsilon)^2) \leq \psi(\xi_n^2)$ for $n > N$). By (84),

$$\sum_{j\in J'} e^{-\lambda_j t} \leq \frac{2a}{\pi} \int_{0}^{\infty} e^{-t\psi(\xi^2)}\, d\xi - \sum_{j\in J} e^{-\lambda_j t}$$

$$\leq \frac{2a}{\pi} \int_{0}^{\xi_N} e^{-t\psi(\xi^2)} d\xi + \frac{3N+3}{6N+5}\, e^{-t\psi(\xi_N^2)}.$$

Passing to a limit as $t \to 0^+$, we obtain

$$|J'| \leq \frac{2a}{\pi}\xi_N + \frac{3N+3}{6N+5} = N + \frac{1}{3} + \frac{3N+3}{6N+5} < N+1.$$

This shows that $|J'| \leq N$, as desired.

Theorem (5.3.26)[225]: If ψ is a complete Bernstein function and $\lim_{\xi\to\infty} \xi\psi'(\xi) = \infty$, then

$$\lambda_n = \psi(\mu_n^2) + O\left(\frac{1}{n}\right) \quad as \quad n \to \infty. \tag{85}$$

In many cases, μ_n can be approximated with more explicit expressions, at the price of a weaker estimate of the error term. We provide two examples.

Proof: By Lemma (5.3.25), $k(n) = n$ for n large enough. Hence, by Corollary (5.3.22),

$$\lambda_n = \tilde{\lambda}_n + O(\frac{1}{n}) = \psi(\tilde{\mu}_n^2) + O(\frac{1}{n}).$$

As in [236], [49], [97], the intermediate results in the proof of Theorem (5.3.26) provide some approximation results for the eigenfunctions. The details of the argument differ slightly from that of [236], [49], [97], so we sketch the proofs.

Proposition (5.3.27)[225]: (See [97] and [236].) Suppose that $\lim_{\xi\to\infty} \xi\psi'(\xi) = \infty$. With the appropriate choice of the signs of φ_n and with

$$\beta_n = \|\tilde{\varphi}_n\|_{L^2(D)}$$

we have $\beta_n = \sqrt{a} + O(\frac{1}{n})$ as $n \to \infty$, and

$$\|\tilde{\varphi}_n - \beta_n \varphi_n\|_{L^2(D)} = O\left(\frac{1}{\left(\frac{n\pi}{2a}\right)^2 \psi'\left(\left(\frac{n\pi}{2a}\right)^2\right)}\right) \qquad \text{as } n \to \infty.$$

Proof: By Lemma (5.3.20), indeed $\beta_n = \sqrt{a} + O(\frac{1}{n})$. Let $\alpha_{n,j} = \langle\tilde{\varphi}_n, \varphi_j\rangle_{L^2(D)}$, so that $\tilde{\varphi}_n = \sum_{j=1}^{\infty} \alpha_{n,j}\varphi_j$ in $L^2(D)$. We choose the sign of φ_n so that $\alpha_{n,n} \geq 0$. We have

$$\begin{aligned}\|\tilde{\varphi}_n - \beta_n\varphi_n\|_{L^2(D)} &\leq \left\|\tilde{\varphi}_n - \alpha_{n,n}\varphi_n\right\|_{L^2(D)} + |\alpha_{n,n} - \beta_n| \\ &= \left\|\tilde{\varphi}_n - \alpha_{n,n}\varphi_n\right\|_{L^2(D)} + \left|\left\|\alpha_{n,n}\varphi_n\right\|_{L^2(D)} - \|\tilde{\varphi}_n\|_{L^2(D)}\right| \\ &\leq 2\left\|\tilde{\varphi}_n - \alpha_{n,n}\varphi_n\right\|_{L^2(D)}.\end{aligned}$$

As in the proof of Lemma (5.3.23), for n larger than some constant, if $j \neq n$ and $\varepsilon = \frac{\pi}{8a}$, then

$$\begin{aligned}\left|\lambda_j - \tilde{\lambda}_n\right| &\geq max(\psi((\tilde{\mu}_{n+1} - \varepsilon)^2) - \psi((\tilde{\mu}_n + \varepsilon)^2), \psi((\tilde{\mu}_n - \varepsilon)^2) - \psi((\tilde{\mu}_{n-1} + \varepsilon)^2) \\ &\geq 2\frac{(n-1)\pi}{2a}\psi'\left(\left(\frac{(n+1)\pi}{2a}\right)^2\right)\cdot\left(\frac{\pi}{2a} - 2\varepsilon\right) \geq \frac{1}{C_1}\frac{n\pi}{2a}\psi'(\left(\frac{n\pi}{2a}\right)^2).\end{aligned}$$

Therefore,

$$\left\|\tilde{\varphi}_n - \alpha_{n,n}\varphi_n\right\|_{L^2(D)}^2 = \sum_{j\neq n} \left|\alpha_{n,j}\right|^2 \leq \frac{C_1}{\frac{n\pi}{2a}\psi'\left(\left(\frac{n\pi}{2a}\right)^2\right)} \sum_{j\neq n} \left(\lambda_j - \tilde{\lambda}_n\right)^2\left|\alpha_{n,j}\right|^2$$

$$\leq \frac{C_1}{\frac{n\pi}{2a}\psi'\left(\left(\frac{n\pi}{2a}\right)^2\right)} \left\|A_D\tilde{\varphi}_n - \tilde{\lambda}_n\tilde{\varphi}_n\right\|_{L^2(D)}^2 \leq \frac{C_2(a,b,\psi)}{\left(\frac{n\pi}{2a}\right)^2\psi'\left(\left(\frac{n\pi}{2a}\right)^2\right)},$$

again by Lemma (5.3.20).

Proposition (5.3.28)[225]: (See [97] and [236].) Suppose that $\lim_{\xi\to\infty} \xi\psi'(\xi) = \infty$. With the appropriate choice of the signs of φ_n and with

$$f_n(x) = \begin{cases} (-1)^{(n-1)/2}\dfrac{1}{\sqrt{a}}\, cos(\tilde{\mu}_n x) & when\, n\, is\, odd, \\ (-1)^{n/2}\dfrac{1}{\sqrt{a}}\, sin(\tilde{\mu}_n x) & when\, n\, is\, even, \end{cases}$$

we have

$$\|f_n - \varphi_n\|_{L^2(D)} = O\left(\frac{1}{\sqrt{n}} + \frac{1}{\left(\frac{n\pi}{2a}\right)^2 \psi'\left(\left(\frac{n\pi}{2a}\right)^2\right)}\right) \qquad as\, n \to \infty.$$

Proof: Clearly,

$$\|f_n - \varphi_n\|_{L^2(D)} \le \left\|f_n - \frac{1}{\sqrt{a}}\tilde{\varphi}_n\right\|_{L^2(D)} + \frac{1}{\sqrt{a}}\|\tilde{\varphi}_n - \beta_n\varphi_n\|_{L^2(D)} + \left|\frac{\beta_n}{\sqrt{a}} - 1\right| \|\varphi_n\|_{L^2(D)}.$$

The middle summand is $O(1/(\left(\frac{n\pi}{2a}\right)^2 \psi'(\left(\frac{n\pi}{2a}\right)^2)))$, while the last one is $O(\frac{1}{n})$. Finally, by the definition (79) of $\tilde{\varphi}_n$ and the properties of $q(x)$ and $F_\mu(x)$,

$$\begin{aligned}\left\|\sqrt{a}\, f_n - \tilde{\varphi}_n\right\|^2_{L^2(D)} &= \int_{-a}^{a} \left(q(-x)G_{\tilde{\mu}_n}(a + x) - (-1)^n q(x)G_{\tilde{\mu}_n}(a - x)\right)^2 dx \\ &\le 4\int_0^\infty \left(G_{\tilde{\mu}_n}(s)\right)^2 ds \le 4G_{\tilde{\mu}_n}(0)\int_0^\infty G_{\tilde{\mu}_n}(s)ds = 4G_{\tilde{\mu}_n}(0)\mathcal{L}G_{\tilde{\mu}_n}(0).\end{aligned}$$

Since $G_\mu(0) = cos\,\vartheta_\lambda \le 1$ and $\mathcal{L}G_\mu(0) = I_\mu \le \frac{1}{\mu}$ (see (81)), we have

$$\left\|\sqrt{a}\, f_n - \tilde{\varphi}_n\right\|_{L^2(D)} = O(\frac{1}{\sqrt{n}}).$$

Proposition (5.3.29)[225]: (See [97] and [236].) Suppose that if $\xi_2 > \xi_1 > 1$, then

$$\frac{\psi(\xi_2)}{\psi(\xi_1)} \ge M\left(\frac{\xi_2}{\xi_1}\right)^\varepsilon \tag{86}$$

for some $M, \varepsilon > 0$. Suppose in addition that

$$\liminf_{\xi\to\infty} \xi^{3/4}\psi'(\xi) > 0. \tag{87}$$

Then $\varphi_n(x)$ are bounded uniformly in $n \ge 1$ and $x \in (-a, a)$.

Condition (86) is known under various names, including weak lower scaling condition and subregularity; such a function ψ is also said to have positive lower Matuszewska index. We remark that although (87) does not imply (86), examples of complete Bernstein functions which satisfy (87), but not (86), are rather artificial.

Proof: Observe that $\xi\psi'(\xi)$ diverges to ∞ as $\xi \to \infty$, and therefore main results of the present article apply. Furthermore, by (86), we have $T(t, 0) \le C_1(\psi)\sqrt{\psi^{-1}(1/t)}$ for $t \le 1$, see, for example, [206]. We have

$$\begin{aligned}\|\varphi_n\|_{L^\infty(D)} &= e^{\lambda_n t}\|T_D(t)\varphi_n\|_{L^\infty(D)} \\ &\le e^{\lambda_n t}\left\|T_D(t)\left(\varphi_n - \frac{1}{\beta_n}\tilde{\varphi}_n\right)\right\|_{L^\infty(D)} + e^{\lambda_n t}\frac{1}{\beta_n}\|T_D(t)\varphi_n\|_{L^\infty(D)}.\end{aligned}$$

Since $|\varphi_n(x)| \le 2$, the latter term in the right-hand side does not exceed $\frac{2}{\beta_n} e^{\lambda_n t}$. For the former one, observe that $|T_D(t)f(x)| \le \|T_D(t,x,\cdot)\|_{L^2(D)} \|f\|_{L^2(D)}, T_D(t,x,y) \le T(t,x-y)$, and, by Plancherel's theorem,

$$\|T(t,\cdot)\|^2_{L^2(R)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(e^{-t\psi(\xi^2)}\right)^2 d\xi = T(2t,0).$$

Finally, $T(2t,0) \le C_1(\psi)\sqrt{\psi^{-1}\big(1/(2t)\big)} \le C_1(\psi)\sqrt{\psi^{-}1(1/t)}$ when $t \le 1$. Therefore, with $t = \frac{1}{\lambda n}$,

$$\|\varphi_n\|_{L^\infty(D)} \le \frac{e}{\beta_n}\left(C_1(\psi)\right)^{1/2}\left(\psi^{-1}(\lambda_n)\right)^{1/4} \|\beta_n\varphi_n - \tilde{\varphi}_n\|_{L^2(D)} + \frac{2e}{\beta_n}.$$

In the right-hand side, $\beta_n = O(1), \psi^{-1}(\lambda_n) \le \left(\frac{n\pi}{2a}\right)^2$ (by (69)), and, by Lemma (5.3.20),

$$\|\beta_n\varphi_n - \tilde{\varphi}_n\|_{L^2(D)} = O\left(\frac{1}{\left(\frac{n\pi}{2a}\right)^2 \psi'(\left(\frac{n\pi}{2a}\right)^2)}\right).$$

Lemma (5.3.30)[225]: Let f is a complete Bernstein function with representation (64). Let g be a holomorphic function in $\{w \in C : |Arg\, w| < C_1\}$ (with $0 < C_1 < \frac{\pi}{2}$) such that $g(x)$ is real for $x > 0$, and let h be a continuous function on $(0,\infty)$. Denote

$$G(y) = \sup_{\substack{y/4\le|z|\le 4y \\ |Arg\, z|<C_1}} |g(z)| \qquad H(y) = \sup_{y/4\le x\le 4y} |h(x)|$$

and suppose that

$$G(x)H(x) \le C_2\, min(x^{-1}, x^{-2}), \qquad C_3 = \int_0^\infty (1+y)G(y)H(y)dy < \infty$$

for $x > 0$. Then

$$\int_{(0,\infty)} g(x)h(x)m(dx) = \lim_{\varepsilon\to 0^+} \frac{1}{\pi}\int_0^\infty Im\left(f\left(-e^{-i\varepsilon}x\right)g\left(e^{-i\varepsilon}x\right)\right)h(x)dx \tag{88}$$

$$= \lim_{\varepsilon\to 0^+} \frac{1}{\pi}\int_0^\infty Im\left(f\left(-e^{-i\varepsilon}x\right)\right)g(x)h(x)dx. \tag{89}$$

Following [112], [238], if ψ is a non-constant complete Bernstein function such that $\psi(0) = 0$, and $\mu > 0$, we denote

$$\psi_\mu(\xi) = \frac{1-\xi/\mu^2}{1-\psi(\xi)/\psi(\mu^2)}$$

for $\xi \in C \setminus ((-\infty,0] \cup \{\mu^2\})$, and $\psi_\mu(\mu^2) = \psi(\mu^2)/(\mu^2\psi'(\mu^2))$. We also let

$$\psi^\dagger(\xi) = exp\left(\frac{1}{\pi}\int_0^\infty \frac{\xi}{\xi^2 + r}\, log\, \psi(r^2)dr\right)$$

for $\xi \in C$ with $Re\,\xi > 0$. Then ψ_μ is a complete Bernstein function, $\psi^\dagger$ extends to a complete Bernstein function, and we have the Wiener–Hopf identity $\psi^\dagger(\xi)\psi^\dagger(-\xi) = \psi(-\xi^2)$ for $\xi \in C \setminus R$, see, for example, [112]. Finally, we denote $\psi^\dagger_\mu = \left(\psi_\mu\right)^\dagger$.

In principle we could extend the definition of ψ_μ to general non-constant complete Bernstein functions ψ, so that $\psi_\mu(\xi) = (1-\xi^2/\mu^2)/(1-(\psi(\xi)-\psi(0))/(\psi(\mu^2)-$

$\psi(0))$). All results proved below hold true with this definition. We will typically assume that $\psi(0) = 0$. For brevity, we also denote $\psi(\infty) = \lim_{\xi\to\infty} \psi(\xi) \in [0,\infty]$.

We assume that $\psi(\xi)$ is a non-constant complete Bernstein function which satisfies $\psi(0) = 0$, that is, $\tilde{c} = 0$ in representation (64) for ψ.

Proof: Let $x > 0, 0 < \varepsilon < \frac{1}{2}C_1$ and $y > 0$, and denote for simplicity $\xi = -e^{-i\varepsilon}x$. By the representation (64) of the complete Bernstein function f and Fubini, we have

$$\int_0^\infty Im\big(f(\xi)g(-\xi)\big)h(x)dx$$
$$= c\int_0^\infty Im\big(\xi g(-\xi)\big)h(x)dx + \tilde{c}\int_0^\infty Im\big(g(-\xi)\big)h(x)dx + \frac{1}{\pi}\int_{(0,\infty)}\int_0^\infty Im\frac{\xi g(-\xi)}{\xi+z}\,h(x)dx\frac{m(dz)}{z} \quad (90)$$

We provide estimates for the integrands and find their pointwise limits as $\varepsilon \to 0^+$ in order to apply dominated convergence.

For the first integral in the right-hand side of (90), we simply use $|\xi g(-\xi)h(x)| \le xG(x)H(x)$, integrability of $xG(x)H(x)$ and $Im(\xi g(-\xi)) \to 0$ as $\varepsilon \to 0^+$. By dominated convergence, the limit as $\varepsilon \to 0^+$ of the first integral in the right-hand side of (90) is zero. Similarly, $|g(-\xi)h(\xi)| \le G(x)H(x), G(x)H(x)$ is integrable and $Im(g(-\xi)) \to 0$ as $\varepsilon \to 0^+$, and so also the second integral in the right-hand side of (90) converges to zero as $\varepsilon \to 0^+$.

To estimate the last integral in the right-hand side of (90), we consider separately two cases. When $x \le \frac{y}{2}$ or $x \ge 2y$, we have

$$\left|-\frac{\xi}{\xi+y}\,g(-\xi)\right| \le \frac{1}{|x-y|}\,xG(x) \le \frac{3}{x+y}\,xG(x) \le 3\,min(1,xy^{-1})G(x)$$
$$\le 3\,min(1,y^{-1})(1+x)G(x),$$

so that by dominated convergence,

$$\left(\int_0^{y/2} + \int_{2y}^\infty\right)\left|Im\left(\frac{\xi}{\xi+y}\,g(-\xi)\right)h(x)\right|dx \le 3C_3\min(1,y^{-1}),$$
$$\lim_{\varepsilon\to 0^+}\left(\int_0^{y/2} + \int_{2y}^\infty\right)Im\left(\frac{\xi}{\xi+y}\,g(-\xi)\right)h(x)dx = 0. \quad (91)$$

When $\frac{y}{2} < x < 2y$, we need a more careful estimate. Observe that

$$\frac{\xi g(-\xi)}{\xi+y} = \frac{yg(y) - (-\xi)g(-\xi)}{y-(-\xi)} - \frac{yg(y)}{\xi+y}.$$

The estimate for g and Cauchy's integral formula for g' easily give

$$|g'(z)| \le C_4y^{-1}G(y)$$

in $\{z \in C : |Arg\,z| < \frac{1}{2}C_1, y/2 \le |z| \le 2y\}$, with $C_4 = 4C_1^{-1}$. By the mean value theorem,

$$\left|\frac{yg(y) - (-\xi)g(-\xi)}{y-(-\xi)}\right| \le C_4y^{-1}G(y)$$

when $\frac{y}{2} \leq x \leq 2y$, and therefore, by dominated convergence,

$$\int_{y/2}^{2y} \left| Im\left(\frac{yg(y) - (-\xi)g(-\xi)}{y - (-\xi)}\right) h(x)\right| dx \leq \frac{3}{2} C_4 y^{-1} G(y)H(y) \leq \frac{3}{2} C_2 C_4 \min(1, y^{-1}),$$

$$\lim_{\varepsilon\to 0^+} \int_{y/2}^{2y} Im\left(\frac{yg(y) - (-\xi)g(-\xi)}{y - (-\xi)}\right) h(x)dx = 0. \quad (92)$$

Finally, if $P_t(s)$ and $Q_t(s)$ denote the (classical) Poisson and conjugate Poisson kernels for the half-plane, then

$$Im\left(-\frac{1}{\xi + y}\right) = \pi\, cos(\varepsilon) P_{y\, sin\, \varepsilon}(x - y\, cos\, \varepsilon) + \pi\, sin(\varepsilon) Q_{y\, sin\, \varepsilon}(x - y\, cos\, \varepsilon).$$

Clearly, $P_{y\, sin\, \varepsilon}(x - y\, cos\, \varepsilon)1_{(y/2,2y)}(x)dx$ converges weakly to $\delta_y(x)$, and therefore

$$\int_{y/2}^{2y} \left|\pi\, cos(\varepsilon) P_{y\, sin\, \varepsilon}(x - y \cos \varepsilon) yg(y)h(x)\right| dx \leq \pi yg(y)H(y) \leq C_2 \pi \min(1, y^{-1}),$$

$$\lim_{\varepsilon\to 0^+} \int_{y/2}^{2y} \pi\, cos(\varepsilon) P_{y\, sin\, \varepsilon}(x - y\, cos\, \varepsilon) yg(y)h(x)dx = \pi yg(y)h(y).$$

Furthermore, $|tQ_t(s)| \leq \frac{1}{\pi}$ and $tQ_t(s) \to 0$ as $t \to 0^+$, and hence, by dominated convergence,

$$\int_{y/2}^{2y} \left|\pi\, sin(\varepsilon) Q_{y\, sin\, \varepsilon}(x - y\, cos\, \varepsilon) yg(y)h(x)\right| dx \leq \frac{3}{2}\, yg(y)H(y) \leq \frac{3}{2} C_2 \min(1, y^{-1}),$$

$$\lim_{\varepsilon\to 0^+} \int_{y/2}^{2y} \pi\, sin(\varepsilon) Q_{y\, sin\, \varepsilon}(x - y\, cos\, \varepsilon) yg(y)h(x)dx = 0.$$

We have thus proved that

$$\int_{y2}^{2y} \left| Im\left(-\frac{yg(y)}{\xi + y}\right) h(x)\right| dx \leq C_2\left(\pi + \frac{3}{2}\right) \min(1, y^{-1}),$$

$$\lim_{\varepsilon\to 0^+} \int_{y/2}^{2y} Im\left(-\frac{yg(y)}{\xi + y}\right) h(x)dx = \pi yg(y)h(y). \quad (93)$$

Due to estimates (91), (92) and (93), as well as the integrability condition on m, indeed we could use Fubini in (90). The same estimates allow us to use dominated convergence in the limit as $\varepsilon \to 0^+$. We conclude that

$$\lim_{\varepsilon\to 0^+} \int_0^\infty Im(f(\xi)g(-\xi))h(x)dx = \pi \int_{(0,\infty)} \int_0^\infty g(y)h(y)m(dy).$$

This proves the first equality in (88). The other one follows by replacing the pair $g(z), h(x)$ with 1 and $g(x)h(x)$.

Corollary (5.3.31)[274]: (See [236].) If D is a bounded interval, then $f \in \mathcal{D}(\mathcal{E}_D)$ if and only if $f \in \mathcal{D}(\mathcal{E})$ and $f = 0$ almost everywhere in $R \setminus D$.

Proof. By definition, if $f \in \mathcal{D}(\mathcal{E}_D)$, then $f \in \mathcal{D}(\mathcal{E})$ and $f = 0$ almost everywhere in $R \setminus D$. Let $f \in \mathcal{D}(\mathcal{E})$ and $f = 0$ almost everywhere in $R \setminus D$. The result follows from the following claim: there is a sequence $f_n \in C_c^\infty(D)$ such that

$$\mathcal{E}_1(f_n - f, f_n - f)$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(1 + \psi((\xi+\epsilon)^2)\right)|\mathcal{F}f_n(\xi+\epsilon) - \mathcal{F}f(\xi+\epsilon)|^2 d(\xi+\epsilon)$$
converges to 0 as $n \to \infty$.

Let $h_n \in C_c^\infty(R^D)$ be an approximation to the identity such that $h_n(x) = nh(nx), h(x) \geq 0, \int_R h(x)dx = 1$ and $h(x) = 0$ for $x \notin (-1,1)$. Note that h_n is zero outside $(-\frac{1}{n}, \frac{1}{n})$. Let
$$g_n(x) = h_n * f(x), \qquad f_n(x) = g_n((x - b_n)/a_n),$$
where $(x - b_n)/a_n$ maps the $\frac{2}{n}$-neighbourhood of I into I, with $a_n \geq 1, \lim_{n\to\infty} a_n = 1$ and $\lim_{n\to\infty} b_n = 0$. Observe that $f_n \in C_c^\infty(D)$ and
$$\mathcal{F}f_n(\xi+\epsilon) = a_n e^{-ib_n(\xi+\epsilon)}\mathcal{F}g_n(a_n(\xi+\epsilon))$$
$$= a_n e^{-ib_n(\xi+\epsilon)}\mathcal{F}h\left(\frac{1}{n}\left(a_n(\xi+\epsilon)\right)\right)\mathcal{F}f(a_n(\xi+\epsilon)).$$
Since $f, g \in L^1(R), \mathcal{F}f$ and $\mathcal{F}h$ are continuous. Furthermore, $\mathcal{F}h(0) = 1$ and $|\mathcal{F}h(\xi+\epsilon)| \leq 1$ for $(\xi+\epsilon) \in R$. It follows that $\mathcal{F}f_n$ converges pointwise to $\mathcal{F}f$, and for n large enough
$$|\mathcal{F}f_n(\xi+\epsilon)| \leq 2|\mathcal{F}f(a_n(\xi+\epsilon))|$$
for all $(\xi+\epsilon) \in R$. Hence, if $u(\xi+\epsilon) = \left(1 + \psi((\xi+\epsilon)^2)\right)|\mathcal{F}f(\xi+\epsilon)|^2$, then for n large enough,
$$\left(1 + \psi((\xi+\epsilon)^2)\right)|\mathcal{F}f_n(\xi+\epsilon) - \mathcal{F}f(\xi+\epsilon)|^2$$
$$\leq 2(1 + \psi((\xi+\epsilon)^2))(|\mathcal{F}f_n(\xi+\epsilon)|^2 + |\mathcal{F}f(\xi+\epsilon)|^2)$$
$$\leq 4u(a_n(\xi+\epsilon)) + 2u(\xi+\epsilon)$$
for all $(\xi+\epsilon)$. By the assumption, $u(\xi+\epsilon)$ is integrable. Therefore, the family of functions $\left(1+\psi((\xi+\epsilon)^2)\right)|\mathcal{F}f_n(\xi+\epsilon) - \mathcal{F}f(\xi+\epsilon)|^2$ is tight and uniformly integrable. By the Vitali's convergence theorem, $\mathcal{E}_1(f_n - f, f_n - f)$ converges to 0 as $n \to \infty$, as desired.

Corollary (5.3.32)[274]: If ψ is a non-constant complete Bernstein function, then
$$-\frac{(\xi+\epsilon)\psi''(\xi+\epsilon)}{\psi'(\xi+\epsilon)} \leq 2 - \frac{2(\xi+\epsilon)\psi'(\xi+\epsilon)}{\psi(\xi+\epsilon)}.$$
Proof. The lemma is equivalent to the inequality
$$-(\xi+\epsilon)\psi(\xi+\epsilon)\psi''(\xi+\epsilon) \leq 2\psi'(\xi+\epsilon)(\psi(\xi+\epsilon) - (\xi+\epsilon)\psi'(\xi+\epsilon)).$$
Assuming ψ has the representation (64), we need to prove
$$(\xi+\epsilon)\left(c(\xi+\epsilon) + \tilde{c} + \frac{1}{\pi}\int_{(0,\infty)} \frac{(\xi+\epsilon)}{(\xi+\epsilon+s)}\frac{(\mu+\epsilon)(ds)}{s}\right)\left(\frac{1}{\pi}\int_{(0,\infty)} \frac{1}{(\xi+\epsilon+s)^3}(\mu+\epsilon)(ds)\right)$$
$$\leq \left(c + \frac{1}{\pi}\int_{(0,\infty)} \frac{1}{(\xi+\epsilon+s)^2}(\mu+\epsilon)(ds)\right)\left(\tilde{c} + \frac{1}{\pi}\int_{(0,\infty)} \frac{(\xi+\epsilon)^2}{(\xi+\epsilon+s)^2}\frac{(\mu+\epsilon)(ds)}{s}\right).$$
This follows by simple integration from the following bounds: $0 \leq c\tilde{c}$,
$$(\xi+\epsilon)(c(\xi+\epsilon)+\tilde{c})\frac{s}{(\xi+\epsilon+s)^3} \leq c\frac{(\xi+\epsilon)^2}{(\xi+\epsilon+s)^2} + \tilde{c}\frac{s}{(\xi+\epsilon+s)^2},$$
and

$$(\xi+\epsilon)\left(\frac{(\xi+\epsilon)}{(\xi+\epsilon+s_1)}\frac{s_2}{(\xi+\epsilon+s_2)^3}+\frac{(\xi+\epsilon)}{(\xi+\epsilon+s_2)}\frac{s_1}{(\xi+\epsilon+s_1)^3}\right)$$
$$\leq\frac{s_1}{(\xi+\epsilon+s_1)^2}\frac{(\xi+\epsilon)^2}{(\xi+\epsilon+s_2)^2}+\frac{s_2}{(\xi+\epsilon+s_2)^2}\frac{(\xi+\epsilon)^2}{(\xi+\epsilon+s_1)^2};$$
the last two inequalities are easily proved by direct calculations.

Corollary (5.3.33)[274]: The left-hand side of (74) is decreasing in $P \in [0,1-Q]$. The right-hand side of (75) is increasing in $P \in [0,1-Q]$.

Proof. Let $P=1-\frac{Q}{s+Q}=\frac{s}{s+Q}$, $s \in [0,1-Q]$. Note that P increases with increasing s, and the left-hand side of (74) is equal to
$$\frac{1}{\pi}(\text{arc}\sin^2\sqrt{Q}+\text{arc}\sin^2\sqrt{s+Q}-\text{arc}\sin^2\sqrt{s}).$$
Since $\text{arc}\sin^2\sqrt{s}$ is convex, the above expression is increasing in s. In a similar way, with $P=1-\frac{Q}{1-s}=\frac{1-s-Q}{1-s}$, $s \in [0,1-Q]$, the right-hand side of (75) is equal to
$$\frac{\pi}{4}-\frac{1}{\pi}\left(\text{arc}\sin^2\sqrt{1-Q}+\text{arc}\sin^2\sqrt{s}-\text{arc}\sin^2\sqrt{s+Q}\right),$$
which is again an increasing function of s, but now P decreases with increasing s. Substituting $P=0$, we obtain immediately the following elegant result (see [225]).

Corollary (5.3.34)[274]: If ψ is a non-constant complete Bernstein function such that $\psi(0)=0$, then
$$\liminf_{\mu+\epsilon\to0^+}\vartheta_{\mu+\epsilon}\leq\frac{3\pi}{8}.$$
If ψ is unbounded, then also
$$\liminf_{\mu+\epsilon\to\infty}\vartheta_{\mu+\epsilon}\leq\frac{3\pi}{8}.$$
Proof. Suppose that $\liminf_{\mu+\epsilon\to0^+}\vartheta_{\mu+\epsilon}>\frac{3\pi}{8}$. Then there are $(\mu+\epsilon)_0>0$ and $0<\epsilon<1$ such that $\vartheta_{\mu+\epsilon}\geq\frac{\pi}{2}-\frac{(1-\epsilon)\pi}{8}$ for $(\mu+\epsilon)\in(0,(\mu+\epsilon)_0)$. By Corollary (5.3.9),
$$\text{arc}\sin^2\sqrt{1+\frac{(\mu+\epsilon)^2\psi''((\mu+\epsilon)^2)}{2\psi'((\mu+\epsilon)^2)}}\leq\frac{(1-\epsilon)\pi^2}{16}$$
for $(\mu+\epsilon)\in(0,(\mu+\epsilon)_0)$, and hence
$$\frac{-(\mu+\epsilon)^2\psi''((\mu+\epsilon)^2)}{\psi'((\mu+\epsilon)^2)}\geq2-2\left(\sin\frac{\pi\sqrt{1-\epsilon}}{4}\right)^2$$
for $(\mu+\epsilon)\in(0,(\mu+\epsilon)_0)$. If $(1+\epsilon)$ denotes the right-hand side, then $\epsilon>0$. By integration (see [235]), we have $\psi'((\mu+\epsilon)^2)/\psi'((\mu+\epsilon)_0^2)\geq((\mu+\epsilon)_0^2/(\mu+\epsilon)^2)^{1+\epsilon}$ for all $(\mu+\epsilon)\in(0,(\mu+\epsilon)_0)$, which contradicts integrability of ψ' at 0. This proves the first statement of the corollary.

In a similar manner, if $\liminf_{\mu+\epsilon\to\infty}\vartheta_{\mu+\epsilon}>\frac{3\pi}{8}$, then there are $(\mu+\epsilon)_0>0$ and $0<\epsilon<1$ such that $\vartheta_{\mu+\epsilon}\geq\frac{\pi}{2}\left(\frac{3+\epsilon}{4}\right)$ for $(\mu+\epsilon)\in((\mu+\epsilon)_0,\infty)$. Again this implies

$$\frac{-(\mu+\epsilon)^2\psi''((\mu+\epsilon)^2)}{\psi'((\mu+\epsilon)^2)} \geq 2\left(1-\left(sin\,\frac{\pi\sqrt{1-\epsilon}}{4}\right)^2\right)$$

for $(\mu+\epsilon) \in ((\mu+\epsilon)_0,\infty)$. If $(1+\epsilon)$ denotes the right-hand side, then $\epsilon > 0$, and by integration, $\psi'((\mu+\epsilon)^2)/\psi'((\mu+\epsilon)_0^2) \leq ((\mu+\epsilon)_0^2/(\mu+\epsilon)^2)^{1+\epsilon}$ for all $(\mu+\epsilon) \in ((\mu+\epsilon)_0,\infty)$. This implies integrability of ψ' at ∞.

Corollary (5.3.35)[274]: If ψ is a non-constant complete Bernstein function such that $\psi(0) = 0$, then for all $\mu+\epsilon > 0$,

$$\left|\frac{d\vartheta_{\mu+\epsilon}}{d(\mu+\epsilon)}\right| < \frac{3}{\mu+\epsilon}.$$

Proof. By (73) and the Cauchy's mean value theorem, for some $(\xi+\epsilon)_z \in ((\mu+\epsilon)^2z^2,(\mu+\epsilon)^2)$ and $(\xi+\epsilon)_{\frac{1}{z}} \in ((\mu+\epsilon)^2,(\mu+\epsilon)^2/z^2)$ (where $z \in (0,1)$),

$$\frac{d\vartheta_{\mu+\epsilon}}{d(\mu+\epsilon)} = \frac{2}{\pi(\mu+\epsilon)}\int_0^1 \frac{1}{1-z^2}\left(\frac{(\xi+\epsilon)_z\psi''((\xi+\epsilon)_z)+\psi'((\xi+\epsilon)_z)}{\psi'((\xi+\epsilon)_z)}\right.$$
$$\left.-\frac{(\xi+\epsilon)_{\frac{1}{z}}\psi''\left((\xi+\epsilon)_{\frac{1}{z}}\right)+\psi'\left((\xi+\epsilon)_{\frac{1}{z}}\right)}{\psi'\left((\xi+\epsilon)_{\frac{1}{z}}\right)}\right)dz$$
$$= \frac{2}{\pi(\mu+\epsilon)}\int_0^1 \frac{1}{1-z^2}\left(\frac{(\xi+\epsilon)_z\psi''((\xi+\epsilon)_z)}{\psi((\xi+\epsilon)_z)} - \frac{(\xi+\epsilon)_{\frac{1}{z}}\psi''\left((\xi+\epsilon)_{\frac{1}{z}}\right)}{\psi'\left((\xi+\epsilon)_{\frac{1}{z}}\right)}\right)dz.$$

By (64), $0 \leq -(\xi+\epsilon)\psi''(\xi+\epsilon) \leq 2\psi'(\xi+\epsilon)$ and $0 \leq (\xi+\epsilon)^2\psi^{(3)}(\xi+\epsilon) \leq 6\psi'(\xi+\epsilon)$. Hence,

$$(\xi+\epsilon)\frac{d}{d(\xi+\epsilon)}\left(\frac{(\xi+\epsilon)\psi''(\xi+\epsilon)}{\psi'(\xi+\epsilon)}\right)$$
$$= \frac{(\xi+\epsilon)^2\psi^{(3)}(\xi+\epsilon)}{\psi'(\xi+\epsilon)} - \frac{-(\xi+\epsilon)\psi''(\xi+\epsilon)}{\psi'(\xi+\epsilon)} - \frac{\left(-(\xi+\epsilon)\psi''(\xi+\epsilon)\right)^2}{\left(\psi'(\xi+\epsilon)\right)^2}$$
$$\in [-6,6].$$

Furthermore, $(\xi+\epsilon)_z\psi''((\xi+\epsilon)_z)/\psi'((\xi+\epsilon)_z) - (\xi+\epsilon)_{\frac{1}{z}}\psi''((\xi+\epsilon)_{\frac{1}{z}})/\psi'((\xi+\epsilon)_{\frac{1}{z}}) \in [-2,2]$. It follows that

$$\left|\frac{d\vartheta_{\mu+\epsilon}}{d(\mu+\epsilon)}\right| \leq \frac{2}{\pi(\mu+\epsilon)}\int_0^1 \frac{1}{1-z^2}\,min\left(2,\int_{(\xi+\epsilon)_{\frac{1}{z}}}^{(\xi+\epsilon)_z}\left|\frac{d}{dr}\left(\frac{r\psi''(r)}{\psi'(r)}\right)\right|dr\right)dz$$
$$\leq \frac{2}{\pi(\mu+\epsilon)}\int_0^1 \frac{1}{1-z^2}\,min\left(2,6\,log\frac{(\xi+\epsilon)_{\frac{1}{z}}}{(\xi+\epsilon)_z}\right)dz.$$

Recall that $\frac{(\xi+\epsilon)_{\frac{1}{z}}}{(\xi+\epsilon)_z} \leq z^{-4}$. Hence,

$$\left|\frac{d\vartheta_{\mu+\epsilon}}{d(\mu+\epsilon)}\right| \le \frac{2}{\pi(\mu+\epsilon)} \int_0^1 \frac{min(2, -24\, log\, z)}{1-z^2}\, dz.$$

Since $-\, log\, z \le \frac{1}{z} - 1$,we have

$$\left|\frac{d\vartheta_{\mu+\epsilon}}{d(\mu+\epsilon)}\right| \le \frac{2}{\pi(\mu+\epsilon)} \int_0^1 \frac{min\big(2, 24(1z\, -\, 1)\big)}{1-z^2}\, dz = \frac{100\, log\, 5\, -\, 48\, log 24}{\pi(\mu+\epsilon)}$$
$$< \frac{3}{\mu+\epsilon}.$$

We conjecture that in fact $-\frac{1}{\mu+\epsilon} < \frac{d}{d(\mu+\epsilon)}\vartheta_{\mu+\epsilon} \le \frac{1}{2(\mu+\epsilon)}$. We close this section with the following simple example(see [225]).

Corollary (5.3.36)[274]: If ψ is a non-constant complete Bernstein function such that $\psi(0) = 0$, then for all $\mu+\epsilon > 0$ and $(\xi+\epsilon)$ such that $Re\,(\xi+\epsilon) > 0$

$$\left|\mathcal{L}F_{\mu+\epsilon}(\xi+\epsilon)\right| \le 2\sqrt{2}\frac{\mu+\epsilon}{|(\mu+\epsilon)^2 + (\xi+\epsilon)^2|}\sqrt{\frac{\psi'((\mu+\epsilon)^2)((\mu+\epsilon)^2 - |\xi+\epsilon|^2)}{\psi((\mu+\epsilon)^2) - \psi(|\xi+\epsilon|^2)}}.$$

Proof. Recall that $((\mu+\epsilon)^2 + (\xi+\epsilon)^2)\mathcal{L}F_{\mu+\epsilon}(\xi+\epsilon) = (\mu+\epsilon)\left(\psi_{\mu+\epsilon}((\mu+\epsilon)^2)\right)^{-1/2}\psi^{\dagger}_{\mu+\epsilon}(\xi+\epsilon)$ is a complete Bernstein function of $(\xi+\epsilon)$, and hence by [112] and [238],

$$|(\mu+\epsilon)^2 + (\xi+\epsilon)^2|\left|\mathcal{L}F_{\mu+\epsilon}(\xi+\epsilon)\right| \le \sqrt{2}\,((\mu+\epsilon)^2 + |\xi+\epsilon|^2)\mathcal{L}F_{\mu+\epsilon}(|\xi+\epsilon|)$$
$$\le 2\sqrt{2}(\mu+\epsilon)\sqrt{\frac{\psi'((\mu+\epsilon)^2)((\mu+\epsilon)^2 - |\xi+\epsilon|^2)}{\psi((\mu+\epsilon)^2) - \psi(|\xi+\epsilon|^2)}}$$

for all $(\xi+\epsilon)$ such that $Re\,(\xi+\epsilon) > 0$.

Corollary (5.3.37)[274]: If ψ is a non-constant complete Bernstein function such that $\psi(0) = 0$, then for all $(\mu+\epsilon), x > 0$ such that $(\mu+\epsilon)x \ne 1$,

$$G_{\mu+\epsilon}(x) \le \frac{1}{\pi x}\frac{\psi(1/x^2)}{\psi((\mu+\epsilon)^2)}\sqrt{\frac{\psi'((\mu+\epsilon)^2)}{\psi((\mu+\epsilon)^2)}\;\frac{1\, -\, \psi((\mu+\epsilon)^2)/\big((\mu+\epsilon)^2x^2\psi(1/x^2)\big)}{1\, -\, \psi(1/x^2)/\psi((\mu+\epsilon)^2)}}.$$

In particular, if ψ is unbounded, then

$$\limsup_{\mu+\epsilon\to\infty}\Big((\mu+\epsilon)\psi((\mu+\epsilon)^2)G_{\mu+\epsilon}(x)\Big) \le \frac{\psi\left(\frac{1}{x^2}\right)}{\pi x}.$$

Proof. Recall that $\psi^{\dagger}_{\mu+\epsilon}(\xi+\epsilon) \ge \psi^{\dagger}_{\mu+\epsilon}(0) = \psi_{\mu+\epsilon}(0) = 1$. Hence,

$$\gamma_{\mu+\epsilon}(d(\xi+\epsilon)) \le \frac{1}{\pi(\mu+\epsilon)\sqrt{\psi_{\mu+\epsilon}((\mu+\epsilon)^2)}}\lim_{\epsilon\to 0^+}\; Im\,\psi_{\mu+\epsilon}(-e^{-i\epsilon}(\xi+\epsilon)^2)d(\xi+\epsilon).$$

After a substitution $(\xi+\epsilon) = \sqrt{s}$ it follows that

$$G_{\mu+\epsilon}(x) = \int_0^\infty e^{-(\xi+\epsilon)x}\gamma_{\mu+\epsilon}(d(\xi+\epsilon))$$
$$\le \frac{1}{2\pi(\mu+\epsilon)\sqrt{\psi_{\mu+\epsilon}((\mu+\epsilon)^2)}}\lim_{\epsilon\to 0^+}\int_0^\infty \sqrt{s}\, e^{-x\sqrt{s}}\;\frac{Im\,\psi_{\mu+\epsilon}\big(-e^{-i\epsilon}s\big)ds}{s}.$$

Since $x\sqrt{s}\, e^{-x\sqrt{s}} \leq 2/(1 + x^2 s)$, we have

$$G_{\mu+\epsilon}(x) \leq \frac{1}{\pi(\mu+\epsilon)x\sqrt{\psi_{\mu+\epsilon}((\mu+\epsilon)^2)}} \lim_{\epsilon\to 0^+} \int_0^\infty \frac{1}{1+x^2 s} \frac{Im\, \psi_{\mu+\epsilon}(-e^{-i\epsilon}s)ds}{s}$$
$$\leq \frac{\psi_{\mu+\epsilon}(1/x^2) - 1}{\pi(\mu+\epsilon)x\sqrt{\psi_{\mu+\epsilon}((\mu+\epsilon)^2)}} ;$$

for the last inequality note that the integral converges to the integral term in the representation (64) for the complete Bernstein function $\psi_{\mu+\epsilon}$, and we have $\psi_{\mu+\epsilon}(0) = 1$ (therefore the inequality becomes equality if $\psi_{\mu+\epsilon}$ contains no linear term, that is, if ψ is unbounded). To prove the first statement, it remains to use the definition of $\psi_{\mu+\epsilon}$. The other statement of the lemma follows from the first one by the inequality $(\xi+\epsilon)\psi'(\xi+\epsilon) \leq \psi(\xi+\epsilon)$.

Corollary (5.3.38)[274]: (See [236].) We have $\tilde{\varphi}_n \in \mathcal{D}(A_D)$ and $A_D\tilde{\varphi}_n(x) = \mathcal{A}\tilde{\varphi}_n(x)$ for almost all $x \in D$.

Proof. For brevity, in this proof we write $(\widetilde{\mu+\epsilon}) = (\widetilde{\mu+\epsilon})_n$ and $\tilde{\varphi} = \tilde{\varphi}_n$. The domain of A_D is described in Definition (5.3.3): we need to prove that $\tilde{\varphi} \in \mathcal{D}(\mathcal{E})$ and that $\langle\tilde{\varphi}, \mathcal{A}g\rangle = \langle\mathcal{A}\tilde{\varphi}, g\rangle$ for all $g \in C_c^\infty(D)$. We first verify the latter condition.

Note that $\mathcal{A}\tilde{\varphi}(x)$ is well-defined for all $x \in D \setminus \{-(b-\epsilon), b-\epsilon\}$, since $\tilde{\varphi}$ is smooth in $D \setminus \{-(b-\epsilon), b-\epsilon\}$ and bounded on R. Let $g \in C_c^\infty(D)$. Since $\tilde{\varphi}'$ is absolutely continuous in $(-(a-\epsilon), (a-\epsilon))$, integration by parts gives

$$\int_{-(a-\epsilon)}^{a-\epsilon} (-c\tilde{\varphi}''(x))g(x)dx = \int_{-(a-\epsilon)}^{a-\epsilon} \tilde{\varphi}(x)(-cg''(x))dx.$$

Furthermore, by the definition of $\mathcal{A}$ (see (76)),

$$\int_{-(a-\epsilon)}^{a-\epsilon} \mathcal{A}\tilde{\varphi}(x)g(x)dx - \int_{-(a-\epsilon)}^{a-\epsilon} \tilde{\varphi}(x)Ag(x)dx$$
$$= \int_{-(a-\epsilon)}^{a-\epsilon} \left(\int_0^\infty \big(g(x+z)\tilde{\varphi}(x) + g(x-z)\tilde{\varphi}(x) - g(x)\tilde{\varphi}(x+z) - g(x)\tilde{\varphi}(x-z)\big)\nu(z)dz\right)dx.$$

We claim that the double integral exists. Then, by Fubini, it is equal to 0, and so $\langle\tilde{\varphi}, \mathcal{A}g\rangle = \langle\mathcal{A}\tilde{\varphi}, g\rangle$, as desired.

Denote the integrand by $I(x,z)\nu(z)$, and let $\epsilon = \frac{1}{3}\operatorname{dist}(\operatorname{supp} g, R \setminus D)$, so that supp $g \subseteq (-a+4\epsilon, a-4\epsilon)$. When $z \geq \epsilon$, then $|I(x,z)| \leq 4\,\|\tilde{\varphi}\|_{L^\infty(R)}\,\|g\|_{L^\infty(R)}$. Suppose that $z \in (0,\epsilon)$. If $x \notin (-a+3\epsilon, a-3\epsilon)$, then $I(x,z) = 0$. Otherwise, by first-order Taylor's expansion of $I(x,z)$ around $z = 0$ (note that $I(x,0) = \frac{\partial}{\partial z} I(x,0) = 0$) with the remainder in the integral form, we obtain that

$$|I(x,z)| \leq \int_0^z (z-s)\frac{\partial^2}{\partial s^2} \quad I(x,s)ds$$
$$\leq z^2(\,\|\tilde{\varphi}\|_{L^\infty(R)}\,\|g''\|_{L^\infty(R)} + \|\tilde{\varphi}''\|_{L^\infty((-a+2\epsilon,a-2\epsilon))}\,\|g\|_{L^\infty(R)})$$

(recall that $\tilde{\varphi}''$ is bounded in $(-a+2\epsilon, a-2\epsilon)$). We conclude that $|I(x,z)\nu(z)| \leq C_1(\tilde{\varphi}, g)\, min(1, z^2)\nu(z)$, which implies joint integrability of $I(x,z)\nu(z)$. Our claim is proved.

It remains to verify that $\tilde{\varphi} \in \mathcal{D}(\mathcal{E})$, that is, $\left(1+\psi((\xi+\epsilon)^2)\right)|\mathcal{F}\tilde{\varphi}(\xi+\epsilon)|^2$ is integrable. Let $f(x) = q(a-\epsilon-x)F_{\widetilde{(\mu+\epsilon)}}(x)$, so that $\tilde{\varphi}(x) = f(a-\epsilon+x) - (-1)^n f(a-\epsilon-x)$ (see (79)). It suffices to prove integrability of $\left(1+\psi((\xi+\epsilon)^2)\right)|\mathcal{F}f(\xi+\epsilon)|^2$.

Fix $\epsilon > 0$ and let $\tilde{q}(x) = q(a-\epsilon-x)e^{\epsilon x}$. Since the distributional derivatives q, q' and q'' are integrable functions, and the third distributional derivative of $q(x)$ is a finite signed measure on R, the function $\tilde{q}(x)$ has the same property. Therefore, $\mathcal{F}q(\xi+\epsilon)$ and $\mathcal{F}q^{(3)}(\xi+\epsilon) = -i(\xi+\epsilon)^3\mathcal{F}q(\xi+\epsilon)$ are bounded functions, and so $|\mathcal{F}\tilde{q}(\xi+\epsilon)| \leq \frac{C_2(\epsilon, a-\epsilon, b-\epsilon)}{(1+|\xi+\epsilon|)^3}$. The Fourier transform of $e^{-\epsilon x}F_{\widetilde{(\mu+\epsilon)}}(x)$ is equal to $\mathcal{L}F_{\widetilde{(\mu+\epsilon)}}(\epsilon+i(\xi+\epsilon))$, and the Fourier transform of $f(x) = q(a-\epsilon-x)F_{\widetilde{(\mu+\epsilon)}}(x) = \tilde{q}(x)e^{-\epsilon x}F_{\widetilde{(\mu+\epsilon)}}(x)$ is given by the convolution

$$\mathcal{F}f(\xi+\epsilon) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}\tilde{q}(\xi+\epsilon-s)\mathcal{L}F_{\widetilde{(\mu+\epsilon)}}(\epsilon+is)ds.$$

Suppose that $(\xi+\epsilon) > 0$. To estimate $|\mathcal{F}f(\xi+\epsilon)|$, we write

$$\mathcal{F}f(\xi+\epsilon) = \frac{1}{2\pi}\int_{\frac{\xi+\epsilon}{2}}^{\infty} \mathcal{F}\tilde{q}(\xi+\epsilon-s)\mathcal{L}F_{\widetilde{(\mu+\epsilon)}}(\epsilon+is)ds + \frac{1}{2\pi}\int_{\frac{\xi+\epsilon}{2}}^{\infty} \mathcal{F}\tilde{q}(s)\mathcal{L}F_{\widetilde{(\mu+\epsilon)}}\left(\epsilon+i(\xi+\epsilon-s)\right)ds. \tag{94}$$

By Corollary (5.3.36), we have

$$\left|\mathcal{L}F_{\widetilde{(\mu+\epsilon)}}(\epsilon+is)\right| \leq 2\sqrt{2}\frac{\widetilde{(\mu+\epsilon)}}{\left|\widetilde{(\mu+\epsilon)}^2+(\epsilon+is)^2\right|}\sqrt{\frac{\psi'\left(\widetilde{(\mu+\epsilon)}^2\right)\left(\widetilde{(\mu+\epsilon)}^2-|\epsilon+is|^2\right)}{\psi\left(\widetilde{(\mu+\epsilon)}^2\right)-\psi(|\epsilon+is|^2)}}$$
$$\leq C_3\left(\epsilon, \widetilde{(\mu+\epsilon)}, \psi\right)\left(\frac{1}{1+s}\right)\sqrt{\frac{1}{1+\psi(s^2)}}$$

(for the second inequality observe that the expression under the square root is bounded by a constant when $s \leq 2\widetilde{(\mu+\epsilon)}$ and by $\psi'(\widetilde{(\mu+\epsilon)}^2)(1+s^2)/(\psi(s^2)-\psi(\widetilde{(\mu+\epsilon)}^2))$ when $s > 2\widetilde{(\mu+\epsilon)}$). The right-hand side decreases with $s > 0$. Hence,

$$\left|\int_{\frac{\xi+\epsilon}{2}}^{\infty} \mathcal{F}\tilde{q}(\xi+\epsilon - s)\mathcal{L}F_{\widetilde{(\mu+\epsilon)}}(\epsilon+is)ds\right|$$
$$\le \frac{C_3\left(\epsilon,\widetilde{(\mu+\epsilon)},\psi\right)}{\left(1+\frac{\xi+\epsilon}{2}\right)\left(1+\psi\left(\frac{(\xi+\epsilon)^2}{4}\right)\right)^{\frac{1}{2}}}\int_{\frac{\xi+\epsilon}{2}}^{\infty} |\mathcal{F}\tilde{q}(\xi+\epsilon-s)|ds$$
$$\le \frac{C_3\left(\epsilon,\widetilde{(\mu+\epsilon)},\psi\right)C_2(\epsilon,a-\epsilon,b-\epsilon)}{\left(1+\frac{\xi+\epsilon}{2}\right)\left(1+\psi\left(\frac{(\xi+\epsilon)^2}{4}\right)\right)^{\frac{1}{2}}}\int_{\frac{\xi+\epsilon}{2}}^{\infty} \frac{1}{(1+|\xi+\epsilon-s|)^3}\,ds$$
$$\le \frac{8C_3\left(\epsilon,\widetilde{(\mu+\epsilon)},\psi\right)C_2(\epsilon,a-\epsilon,b-\epsilon)}{(1+\xi+\epsilon)\left(1+\psi((\xi+\epsilon)^2)\right)^{\frac{1}{2}}};$$

in the last inequality we used the fact that $4\psi\left(\frac{(\xi+\epsilon)^2}{4}\right) \ge \psi((\xi+\epsilon)^2)$ and that the integral is bounded by 1. The estimate of the other integral in (94) is simpler: $|\mathcal{L}F_{\widetilde{(\mu+\epsilon)}}(\epsilon + is)| \le C_4(\epsilon,\widetilde{(\mu+\epsilon)})$ for all $s \in R$, and hence

$$\left|\int_{\frac{\xi+\epsilon}{2}}^{\infty} \mathcal{F}\tilde{q}(s)\mathcal{L}F_{\widetilde{(\mu+\epsilon)}}\left(\epsilon+i(\xi+\epsilon-s)\right)ds\right| \le C_4\left(\epsilon,\widetilde{(\mu+\epsilon)}\right)\int_{\frac{\xi+\epsilon}{2}}^{\infty} |\mathcal{F}\tilde{q}(s)|ds$$
$$\le \frac{C_4\left(\epsilon,\widetilde{(\mu+\epsilon)}\right)C_2(\epsilon,a-\epsilon,b-\epsilon)}{2\left(1+\frac{\xi+\epsilon}{2}\right)^2}.$$

Therefore, for $\xi+\epsilon > 0$,

$$|\mathcal{F}f(\xi+\epsilon)| \le C_5\left(\epsilon,a-\epsilon,b-\epsilon,\widetilde{(\mu+\epsilon)}\right)\left(\frac{1}{(1+|\xi+\epsilon|)\left(1+\psi((\xi+\epsilon)^2)\right)^{1/2}} + \frac{1}{(1+|\xi+\epsilon|)^2}\right).$$

Since $\mathcal{F}f(-(\xi+\epsilon)) = \overline{\mathcal{F}f(\xi+\epsilon)}$, the above estimate extends to all $(\xi+\epsilon) \in R$. We conclude that for all $(\xi+\epsilon) \in R$,

$$\left(1+\psi((\xi+\epsilon)^2)\right)|\mathcal{F}f(\xi+\epsilon)|^2 \le 2\left(C_5\left(\epsilon,a-\epsilon,b-\epsilon,\widetilde{(\mu+\epsilon)}\right)\right)^2\left(\frac{1}{(1+|\xi+\epsilon|)^2} + \frac{1+\psi((\xi+\epsilon)^2)}{(1+|\xi+\epsilon|)^4}\right),$$

and the right-hand side is integrable because $(1+|\xi+\epsilon|)^{-2}\left(1 + \psi((\xi+\epsilon)^2)\right)$ is bounded.

Corollary (5.3.39)[274]: If ψ is unbounded, then for $n \ge 2$,

$$\left\|A_D\tilde{\varphi}_n - \widetilde{(\lambda+\epsilon)}_n\tilde{\varphi}_n\right\|_{L^2(D)} \le \frac{C(a-\epsilon,b-\epsilon,\psi)}{n},$$

and

$$(a-\epsilon) - \frac{20(a-\epsilon)}{n\pi} \le \|\tilde{\varphi}_n\|_{L^2(D)}^2 \le (a-\epsilon) + \frac{36(a-\epsilon)}{n\pi}.$$

Proof. By [112],

$$I_{\mu+\epsilon} = \mathcal{L}G_{\mu+\epsilon}(0) = \frac{\cos\ \vartheta_{\mu+\epsilon}}{\mu+\epsilon} - \mathcal{L}F_{\mu+\epsilon}(0^+) = \frac{\cos\vartheta_{\mu+\epsilon}}{\mu+\epsilon} - \sqrt{\frac{\psi'((\mu+\epsilon)^2)}{\psi((\mu+\epsilon)^2)}} \le \frac{1}{\mu+\epsilon}. \quad (95)$$

Furthermore, by complete monotonicity,

$$I_{\mu+\epsilon} \ge \int_0^x G_{\mu+\epsilon}(z)dz \ge \int_0^x \left(G_{\mu+\epsilon}(x) - G'_{\mu+\epsilon}(x)(x-z) + \frac{1}{2}G'_{\mu+\epsilon}(x)(x-z)^2\right)dz$$
$$= xG_{\mu+\epsilon}(x) - \frac{1}{2}x^2G'_{\mu+\epsilon}(x) + \frac{1}{6}x^3G'_{\mu+\epsilon}(x),$$

so that

$$G_{\mu+\epsilon}(x) \le \frac{1}{(\mu+\epsilon)x}, \qquad G'_{\mu+\epsilon}(x) \le \frac{2}{(\mu+\epsilon)x^2}, \qquad G''_{\mu+\epsilon}(x) \le \frac{6}{(\mu+\epsilon)x^3}.$$

By Corollary (5.3.37), for $\mu+\epsilon \ge \widetilde{(\mu+\epsilon)}_2$

$$\psi((\mu+\epsilon)^2)G_{\mu+\epsilon}(x) \le \frac{C(\psi,x)}{\mu+\epsilon}.$$

Finally, $\widetilde{(\mu+\epsilon)}_n \ge \frac{(n-1)\pi}{2(a-\epsilon)} \ge \frac{n\pi}{4(a-\epsilon)}$ for $n \ge 2$. The result follows from Lemmas (5.3.18) and (5.3.19).

Corollary (5.3.40)[274]: Suppose that $\lim_{\xi+\epsilon\to\infty} (\xi+\epsilon)\psi'(\xi+\epsilon) = \infty$. For n larger than some (integer) constant $C(a-\epsilon, b-\epsilon, \psi)$ the numbers $k(n)$ are distinct. Moreover, for any $\epsilon > 0$, for n larger than some (integer) constant $C(a-\epsilon, b-\epsilon, \psi, \epsilon)$,

$$\psi\left(\left(\widetilde{(\mu+\epsilon)}_n - \epsilon\right)^2\right) < (\lambda+\epsilon)_{k(n)} < \psi\left(\left(\widetilde{(\mu+\epsilon)}_n + \epsilon\right)^2\right). \quad (96)$$

Proof. Let $\epsilon \in (0, \frac{\pi}{4(a-\epsilon)})$. For some $(\xi+\epsilon)_n \in (\widetilde{(\mu+\epsilon)}_n, \widetilde{(\mu+\epsilon)}_n + \epsilon)$,

$$\psi\left(\left(\widetilde{(\mu+\epsilon)}_n + \epsilon\right)^2\right) - \psi\left(\widetilde{(\mu+\epsilon)}_n^2\right) = 2\epsilon(\xi+\epsilon)_n\psi'((\xi+\epsilon)_n^2).$$

Since $(\xi+\epsilon)_n \le \frac{n\pi}{2(a-\epsilon)} + \epsilon \le \frac{n\pi}{a-\epsilon}$, it follows that

$$\psi\left(\left(\widetilde{(\mu+\epsilon)}_n + \epsilon\right)^2\right) - \psi\left(\widetilde{(\mu+\epsilon)}_n^2\right) \ge \frac{2(a-\epsilon)\epsilon(\xi+\epsilon)_n^2\psi'((\xi+\epsilon)_n^2)}{n\pi}.$$

Since $(\xi+\epsilon)_n \ge \frac{(n-1)\pi}{2(a-\epsilon)}$, we have $\lim_{n\to\infty} (\xi+\epsilon)_n^2\psi'((\xi+\epsilon)_n^2) = \infty$, and so, by Corollary (5.3.22), for n greater than some constant $C(a-\epsilon, b-\epsilon, \psi, \epsilon)$,

$$\psi(\left(\widetilde{(\mu+\epsilon)}_n + \epsilon\right)^2) - \psi(\widetilde{(\mu+\epsilon)}_n^2) > |\widetilde{(\lambda+\epsilon)}_n - (\lambda+\epsilon)_{k(n)}|.$$

Since ψ is concave,

$$\psi\left(\widetilde{(\mu+\epsilon)}_n^2\right) - \psi\left(\left(\widetilde{(\mu+\epsilon)}_n - \epsilon\right)^2\right) \ge \psi\left(\left(\widetilde{(\mu+\epsilon)}_n + \epsilon\right)^2\right) - \psi\left(\widetilde{(\mu+\epsilon)}_n^2\right).$$

Finally, $\widetilde{(\lambda+\epsilon)}_n = \psi(\widetilde{(\mu+\epsilon)}_n^2)$. This proves (96).

Observe that, by Corollary (5.3.35),

$$(a-\epsilon)\widetilde{(\mu+\epsilon)}_{n+1} - (a-\epsilon)\widetilde{(\mu+\epsilon)}_n = \frac{\pi}{2} + \vartheta_{\widetilde{(\mu+\epsilon)}_n} - \vartheta_{\widetilde{(\mu+\epsilon)}_{n+1}}$$
$$\ge \frac{\pi}{2} - \frac{3}{\widetilde{(\mu+\epsilon)}_n}\left(\widetilde{(\mu+\epsilon)}_{n+1} - \widetilde{(\mu+\epsilon)}_n\right)$$
$$\ge \frac{\pi}{2} - \frac{6(a-\epsilon)}{(n-1)\pi}\left(\widetilde{(\mu+\epsilon)}_{n+1} - \widetilde{(\mu+\epsilon)}_n\right),$$

so that $(\widetilde{\mu+\epsilon})_{n+1} - (\widetilde{\mu+\epsilon})_n \geq \frac{\pi}{2(a-\epsilon)}\left(1+\frac{6}{(n-1)\pi}\right)^{-1} \geq \frac{\pi}{4(a-\epsilon)}$ for $n \geq 3$. The first statement of the lemma follows hence from (96) with $\epsilon = \frac{\pi}{8(a-\epsilon)}$.

Corollary (5.3.41)[274]: Suppose that $\lim_{\xi+\epsilon\to\infty} (\xi+\epsilon)\psi'(\xi+\epsilon) = \infty$. Then $k(n) \geq n$ for infinitely many n.

Proof. By Corollary (5.3.40),

$$(\lambda+\epsilon)_{k(n)} \geq \psi\left(\left((\widetilde{\mu+\epsilon})_n - \frac{\pi}{16(a-\epsilon)}\right)^2\right)$$

for n large enough. On the other hand, by (69),

$$(\lambda+\epsilon)_{n-1} \leq \psi\left(\left(\frac{(n-1)\pi}{2(a-\epsilon)}\right)^2\right)$$

for all $n \geq 1$. Finally, by Corollary (5.3.34) and Corollary (5.3.35), $\vartheta_{(\widetilde{\mu+\epsilon})_n} < \frac{3\pi}{8} + \frac{\pi}{16}$ for infinitely many n, and hence

$$\begin{aligned}(\widetilde{\mu+\epsilon})_n - \frac{\pi}{16(a-\epsilon)} &= \frac{n\pi}{2(a-\epsilon)} - \frac{1}{a-\epsilon}\vartheta_{(\widetilde{\mu+\epsilon})_n} - \frac{\pi}{16(a-\epsilon)} \\ &> \frac{n\pi}{2(a-\epsilon)} - \left(\frac{3\pi}{8(a-\epsilon)} + \frac{\pi}{16(a-\epsilon)}\right) - \frac{\pi}{16(a-\epsilon)} = \frac{(n-1)\pi}{2(a-\epsilon)}\end{aligned}$$

for infinitely many n.

Corollary (5.3.42)[274]: Suppose that $\lim_{\xi+\epsilon\to\infty} (\xi+\epsilon)\psi'(\xi+\epsilon) = \infty$. For n greater than some constant $C(a-\epsilon, b-\epsilon, \psi)$ we have $k(n) = n$.

Proof. Let $\epsilon = \frac{\pi}{6(a-\epsilon)}$ and let N be the constant $C(a-\epsilon, b-\epsilon, \psi, \epsilon)$ in Corollary (5.3.40). Define $J = \{k(n) : n > N\}$ and let $J' = \{j \geq 1 : j \notin J\}$. We claim that it suffices to show that $|J'| \leq N$. Indeed, there is $n_0 > N$ such that $(n_0) = 1 + \max J'$, and $k(n)$ is strictly increasing for $n > N$. It follows that $k(n) = k(n_0) + n - n_0$ for $n \geq n_0$. If $|J'| \leq N$, then $k(n_0) = |J'| + (n_0 - N) \leq n_0$, so that $k(n) \leq n$ for $n \geq n_0$. Since $k(n) \geq n$ infinitely many times by Corollary (5.3.41), necessarily $k(n) = n$ for $n \geq n_0$, as desired.

Let $\epsilon > -1$. By the assumption, $\psi(\xi+\epsilon) \geq \frac{1}{1+\epsilon} \log \xi + \epsilon - C(1+\epsilon)$ for some constant $C(1+\epsilon)$, and therefore $exp(-(1+\epsilon)\psi((\xi+\epsilon)^2))$ is integrable. Therefore, $T(1+\epsilon; x)$ is bounded in $x \in R$. In particular, $T_D(1+\epsilon; x,\cdot)$ is in $L^2(D)$, and so, by Parseval's identity,

$$\begin{aligned}&\int_{-(a-\epsilon)}^{a-\epsilon} \int_{-(a-\epsilon)}^{a-\epsilon} \left(T_D(1+\epsilon;\, x, y)\right)^2 dy dx \\ &= \int_{-(a-\epsilon)}^{a-\epsilon} \sum_{n=1}^{\infty} \left(\int_{-(a-\epsilon)}^{a-\epsilon} T_D(1+\epsilon;\, x, y)\varphi_j(y) dy\right)^2 dx \\ &= \int_{-(a-\epsilon)}^{a-\epsilon} \sum_{j=1}^{\infty} e^{-2(\lambda+\epsilon)_j(1+\epsilon)} \left(\varphi_j(x)\right)^2 dx = \sum_{j=1}^{\infty} e^{-2(\lambda+\epsilon)_j(1+\epsilon)}.\end{aligned}$$

On the other hand, by Plancherel's identity,

$$\int_{-(a-\epsilon)}^{a-\epsilon}\int_{-(a-\epsilon)}^{a-\epsilon}\left(T_D(1+\epsilon;\,x,y)\right)^2 dydx \le 2(a-\epsilon)\int_{-\infty}^{\infty}\left(T(1+\epsilon;\,x-y)\right)^2 dy$$
$$= \frac{2(a-\epsilon)}{\pi}\int_0^{\infty} e^{-2(1+\epsilon)\psi((\xi+\epsilon)^2)}\,d(\xi+\epsilon).$$

It follows that for all $\epsilon > -1$,

$$\sum_{j=1}^{\infty} e^{-(\lambda+\epsilon)_j(1+\epsilon)} \le \frac{2(a-\epsilon)}{\pi}\int_0^{\infty} e^{-(1+\epsilon)\psi((\xi+\epsilon)^2)}\,d(\xi+\epsilon). \tag{97}$$

Observe that

$$\sum_{j\in J} e^{-(\lambda+\epsilon)_j(1+\epsilon)} = \sum_{n=N}^{\infty} e^{-(\lambda+\epsilon)_{k(n)}(1+\epsilon)} \ge \sum_{n=N+1}^{\infty} e^{-\psi\left(\left((\widetilde{\mu+\epsilon})_n+\epsilon\right)^2\right)(1+\epsilon)}$$
$$\ge \sum_{n=N}^{\infty} e^{-\psi\left(\left(\frac{n\pi}{2(a-\epsilon)}+\epsilon\right)^2\right)(1+\epsilon)}.$$

Denote $(\xi+\epsilon)_n = \frac{n\pi}{2(a-\epsilon)} + \epsilon = \frac{\left(n+\frac{1}{3}\right)\pi}{2(a-\epsilon)}$. Since $e^{-(1+\epsilon)\psi(z)}$ is concave in $z > 0$,

$$\int_{(\xi+\epsilon)_n}^{(\xi+\epsilon)_{n+1}} e^{-(1+\epsilon)\psi((\xi+\epsilon)^2)}\,d(\xi+\epsilon)$$
$$\le \int_{(\xi+\epsilon)_n}^{(\xi+\epsilon)_{n+1}} \left(\frac{(\xi+\epsilon)_{n+1}^2-(\xi+\epsilon)^2}{(\xi+\epsilon)_{n+1}^2-(\xi+\epsilon)_n^2}\, e^{-(1+\epsilon)\psi((\xi+\epsilon)_n^2)}\right.$$
$$\left.+\frac{(\xi+\epsilon)^2-(\xi+\epsilon)_n^2}{(\xi+\epsilon)_{n+1}^2-(\xi+\epsilon)_n^2}\, e^{-(1+\epsilon)\psi\left((\xi+\epsilon)_{n+1}^2\right)}\right) d(\xi+\epsilon)$$
$$= \frac{2(\xi+\epsilon)_{n+1}^2-(\xi+\epsilon)_n(\xi+\epsilon)_{n+1}-(\xi+\epsilon)_n^2}{3((\xi+\epsilon)_n+(\xi+\epsilon)_{n+1})}\, e^{-(1+\epsilon)\psi((\xi+\epsilon)_n^2)}$$
$$+\frac{(\xi+\epsilon)_{n+1}^2+(\xi+\epsilon)_n(\xi+\epsilon)_{n+1}-2(\xi+\epsilon)_n^2}{3((\xi+\epsilon)_n+(\xi+\epsilon)_{n+1})}\, e^{-(1+\epsilon)\psi\left((\xi+\epsilon)_{n+1}^2\right)}$$
$$= \frac{\pi}{2(a-\epsilon)}\left(\frac{3n+3}{6n+5}\, e^{-(1+\epsilon)\psi((\xi+\epsilon)_n^2)} + \frac{3n+2}{6n+5}\, e^{-(1+\epsilon)\psi\left((\xi+\epsilon)_{n+1}^2\right)}\right).$$

Hence,

$$\frac{2(a-\epsilon)}{\pi}\int_{(\xi+\epsilon)_N}^{\infty} e^{-(1+\epsilon)\psi((\xi+\epsilon)^2)}\,d(\xi+\epsilon)$$
$$\le \sum_{n=N}^{\infty}\left(\frac{3n+3}{6n+5}\, e^{-(1+\epsilon)\psi((\xi+\epsilon)_n^2)} + \frac{3n+2}{6n+5}\, e^{-(1+\epsilon)\psi\left((\xi+\epsilon)_{n+1}^2\right)}\right)$$
$$\le \frac{3N+3}{6N+5}\, e^{-(1+\epsilon)\psi\left((\xi+\epsilon)_N^2\right)} + \sum_{n=N+1}^{\infty} e^{-(1+\epsilon)\psi((\xi+\epsilon)_n^2)}$$
$$\le \frac{3N+3}{6N+5}\, e^{-(1+\epsilon)\psi\left((\xi+\epsilon)_N^2\right)} + \sum_{j\in J} e^{-(1+\epsilon)(\lambda+\epsilon)_j}$$

(the second inequality is a consequence of $\frac{3n+2}{6n+5}+\frac{3(n+1)+3}{6(n+1)+5} \leq 1$, while the last one follows from $(\lambda+\epsilon)_{k(n)} \leq \psi\left(\left(\widetilde{(\mu+\epsilon)}_n + \epsilon\right)^2\right) \leq \psi((\xi+\epsilon)_n^2)$ for $n > N$). By (97),

$$\sum_{j\in J'} e^{-(\lambda+\epsilon)_j(1+\epsilon)} \leq \frac{2(a-\epsilon)}{\pi}\int_0^{\infty} e^{-(1+\epsilon)\psi((\xi+\epsilon)^2)}\,d(\xi+\epsilon) - \sum_{j\in J} e^{-(\lambda+\epsilon)_j(1+\epsilon)}$$
$$\leq \frac{2(a-\epsilon)}{\pi}\int_0^{(\xi+\epsilon)_N} e^{-(1+\epsilon)\psi((\xi+\epsilon)^2)} d(\xi+\epsilon) + \frac{3N+3}{6N+5}\, e^{-(1+\epsilon)\psi\left((\xi+\epsilon)_N^2\right)}.$$

Passing to a limit as $\epsilon \to -1$, we obtain

$$|J'| \leq \frac{2(a-\epsilon)}{\pi}\,(\xi+\epsilon)_N + \frac{3N+3}{6N+5} = N + \frac{1}{3} + \frac{3N+3}{6N+5} < N+1.$$

This shows that $|J'| \leq N$, as desired.

Corollary (5.3.43)[274]: If ψ is a complete Bernstein function and $\lim_{\xi+\epsilon\to\infty}(\xi+\epsilon)\psi'(\xi+\epsilon) = \infty$, then

$$(\lambda+\epsilon)_n = \psi((\mu+\epsilon)_n^2) + O\left(\frac{1}{n}\right) \quad \text{as} \quad n\to\infty. \tag{98}$$

In many cases, $(\mu+\epsilon)_n$ can be approximated with more explicit expressions, at the price of a weaker estimate of the error term. We provide two examples [225].

Proof. By Corollary (5.3.42), $k(n) = n$ for n large enough. Hence, by Corollary (5.3.22),

$$(\lambda+\epsilon)_n = \widetilde{(\lambda+\epsilon)}_n + O\left(\frac{1}{n}\right) = \psi(\widetilde{(\mu+\epsilon)}_n^2) + O\left(\frac{1}{n}\right).$$

Corollary (5.3.44)[274]: (See [97] and [236].) Suppose that $\lim_{\xi+\epsilon\to\infty}(\xi+\epsilon)\psi'(\xi+\epsilon) = \infty$. With the appropriate choice of the signs of φ_n and with

$$\beta_n = \|\tilde{\varphi}_n\|_{L^2(D)}$$

we have $\beta_n = \sqrt{a-\epsilon} + O\left(\frac{1}{n}\right)$ as $n \to \infty$, and

$$\|\tilde{\varphi}_n - \beta_n\varphi_n\|_{L^2(D)} = O\left(\frac{1}{\left(\frac{n\pi}{2(a-\epsilon)}\right)^2 \psi'\left(\left(\frac{n\pi}{2(a-\epsilon)}\right)^2\right)}\right) \quad \text{as } n\to\infty.$$

Proof. By Corollary (5.3.39), indeed $\beta_n = \sqrt{a-\epsilon} + O\left(\frac{1}{n}\right)$. Let $\alpha_{n,j} = \langle\tilde{\varphi}_n, \varphi_j\rangle_{L^2(D)}$, so that $\tilde{\varphi}_n = \sum_{j=1}^{\infty}\alpha_{n,j}\varphi_j$ in $L^2(D)$. We choose the sign of φ_n so that $\alpha_{n,n} \geq 0$. We have

$$\|\tilde{\varphi}_n - \beta_n\varphi_n\|_{L^2(D)} \leq \left\|\tilde{\varphi}_n - \alpha_{n,n}\varphi_n\right\|_{L^2(D)} + |\alpha_{n,n} - \beta_n|$$
$$= \left\|\tilde{\varphi}_n - \alpha_{n,n}\varphi_n\right\|_{L^2(D)} + \left|\left\|\alpha_{n,n}\varphi_n\right\|_{L^2(D)} - \|\tilde{\varphi}_n\|_{L^2(D)}\right|$$
$$\leq 2\left\|\tilde{\varphi}_n - \alpha_{n,n}\varphi_n\right\|_{L^2(D)}.$$

As in the proof of Corollary (5.3.40), for n larger than some constant, if $j \neq n$ and $= \frac{\pi}{8(a-\epsilon)}$, then

$$
\begin{aligned}
&\left|(\lambda+\epsilon)_j-(\widetilde{\lambda+\epsilon})_n\right| \\
&\qquad \geq \max\Big(\psi\big(\big((\widetilde{\mu+\epsilon})_{n+1}-\epsilon\big)^2\big)-\psi\big(\big((\widetilde{\mu+\epsilon})_n+\epsilon\big)^2\big),\psi\big(\big((\widetilde{\mu+\epsilon})_n-\epsilon\big)^2\big) \\
&\qquad -\psi\big(\big((\widetilde{\mu+\epsilon})_{n-1}+\epsilon\big)^2\big)\Big) \\
&\qquad \geq 2\frac{(n-1)\pi}{2(a-\epsilon)}\,\psi'\left(\left(\frac{(n+1)\pi}{2(a-\epsilon)}\right)^2\right)\cdot\left(\frac{\pi}{2(a-\epsilon)}-2\epsilon\right) \\
&\qquad \geq \frac{1}{C_1}\frac{n\pi}{2(a-\epsilon)}\,\psi'\left(\left(\frac{n\pi}{2(a-\epsilon)}\right)^2\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\left\|\tilde{\varphi}_n-\alpha_{n,n}\varphi_n\right\|_{L^2(D)}^2 &= \sum_{j\neq n}\left|\alpha_{n,j}\right|^2 \\
&\leq \frac{C_1}{\frac{n\pi}{2(a-\epsilon)}\,\psi'\left(\left(\frac{n\pi}{2(a-\epsilon)}\right)^2\right)}\sum_{j\neq n}\left((\lambda+\epsilon)_j-(\widetilde{\lambda+\epsilon})_n\right)^2\left|\alpha_{n,j}\right|^2 \\
&\leq \frac{C_1}{\frac{n\pi}{2(a-\epsilon)}\,\psi'\left(\left(\frac{n\pi}{2(a-\epsilon)}\right)^2\right)}\left\|A_D\tilde{\varphi}_n-(\widetilde{\lambda+\epsilon})_n\tilde{\varphi}_n\right\|_{L^2(D)}^2 \\
&\leq \frac{C_2(a-\epsilon,b-\epsilon,\psi)}{\left(\frac{n\pi}{2(a-\epsilon)}\right)^2\psi'\left(\left(\frac{n\pi}{2(a-\epsilon)}\right)^2\right)},
\end{aligned}
$$

again by Corollary (5.3.39).

Corollary (5.3.45)[274]: (See [97] and [236].) Suppose that $\lim_{\xi+\epsilon\to\infty}(\xi+\epsilon)\psi'(\xi+\epsilon)=\infty$. With the appropriate choice of the signs of φ_n and with

$$
f_n(x)=\begin{cases}(-1)^{(n-1)/2}\dfrac{1}{\sqrt{a-\epsilon}}\cos\big((\widetilde{\mu+\epsilon})_n x\big) & \text{when } n \text{ is odd,}\\[2ex] (-1)^{n/2}\dfrac{1}{\sqrt{a-\epsilon}}\sin\big((\widetilde{\mu+\epsilon})_n x\big) & \text{when } n \text{ is even,}\end{cases}
$$

we have

$$
\|f_n-\varphi_n\|_{L^2(D)}=O\left(\frac{1}{\sqrt{n}}+\frac{1}{\left(\frac{n\pi}{2(a-\epsilon)}\right)^2\psi'\left(\left(\frac{n\pi}{2(a-\epsilon)}\right)^2\right)}\right)\qquad \text{as } n\to\infty.
$$

Proof. Clearly,

$$
\begin{aligned}
&\|f_n-\varphi_n\|_{L^2(D)} \\
&\qquad \leq \left\|f_n-\frac{1}{\sqrt{a-\epsilon}}\tilde{\varphi}_n\right\|_{L^2(D)}+\frac{1}{\sqrt{a-\epsilon}}\|\tilde{\varphi}_n-\beta_n\varphi_n\|_{L^2(D)} \\
&\qquad +\left|\frac{\beta_n}{\sqrt{a-\epsilon}}-1\right|\|\varphi_n\|_{L^2(D)}.
\end{aligned}
$$

The middle summand is $O\left(\frac{1}{\left(\frac{n\pi}{2(a-\epsilon)}\right)^2 \psi'\left(\left(\frac{n\pi}{2(a-\epsilon)}\right)^2\right)}\right)$, while the last one is $O\left(\frac{1}{n}\right)$. Finally, by the definition (79) of $\tilde{\varphi}_n$ and the properties of $q(x)$ and $F_{\mu+\epsilon}(x)$,

$$\left\|\sqrt{a-\epsilon}\, f_n - \tilde{\varphi}_n\right\|^2_{L^2(D)}$$

$$= \int_{-(a-\epsilon)}^{a-\epsilon} \Big(q(-x)G_{\widetilde{(\mu+\epsilon)}_n}(a-\epsilon + x)$$

$$- (-1)^n q(x) G_{\widetilde{(\mu+\epsilon)}_n}(a-\epsilon - x)\Big)^2 dx \le 4\int_0^\infty \left(G_{\widetilde{(\mu+\epsilon)}_n}(s)\right)^2 ds$$

$$\le 4G_{\widetilde{(\mu+\epsilon)}_n}(0)\int_0^\infty G_{\widetilde{(\mu+\epsilon)}_n}(s)ds = 4G_{\widetilde{(\mu+\epsilon)}_n}(0)\mathcal{L}G_{\widetilde{(\mu+\epsilon)}_n}(0).$$

Since $G_{\mu+\epsilon}(0) = \cos\vartheta_{\lambda+\epsilon} \le 1$ and $\mathcal{L}G_{\mu+\epsilon}(0) = I_{\mu+\epsilon} \le \frac{1}{\mu+\epsilon}$ (see (95)), we have

$$\left\|\sqrt{a-\epsilon}\, f_n - \tilde{\varphi}_n\right\|_{L^2(D)} = O\left(\frac{1}{\sqrt{n}}\right).$$

Corollary (5.3.46)[274]: (See [97] and [236].) Suppose that if $(\xi+\epsilon)_2 > (\xi+\epsilon)_1 > 1$, then

$$\frac{\psi((\xi+\epsilon)_2)}{\psi((\xi+\epsilon)_1)} \ge M\left(\frac{(\xi+\epsilon)_2}{(\xi+\epsilon)_1}\right)^\epsilon \tag{99}$$

for some $M, \epsilon > 0$. Suppose in addition that

$$\liminf_{\xi+\epsilon\to\infty} (\xi+\epsilon)^{\frac{3}{4}}\psi'(\xi+\epsilon) > 0. \tag{100}$$

Then $\varphi_n(x)$ are bounded uniformly in $n \ge 1$ and $x \in (-(a-\epsilon), (a-\epsilon))$.

Condition (99) is known under various names, including weak lower scaling condition and subregularity; such a function ψ is also said to have positive lower Matuszewska index. We remark that although (100) does not imply (99), examples of complete Bernstein functions which satisfy (100), but not (99), are rather artificial [225].

Proof. Observe that $(\xi+\epsilon)\psi'(\xi+\epsilon)$ diverges to ∞ as $\xi+\epsilon \to \infty$, and therefore main results of the present article apply. Furthermore, by (99), we have $T(1-\epsilon, 0) \le C_1(\psi)\sqrt{\psi^{-1}\left(\frac{1}{1-\epsilon}\right)}$ for $\epsilon \ge 0$, see, for example, [206]. We have

$$\|\varphi_n\|_{L^\infty(D)} = e^{(\lambda+\epsilon)_n(1-\epsilon)}\|T_D(1-\epsilon)\varphi_n\|_{L^\infty(D)}$$

$$\le e^{(\lambda+\epsilon)_n(1-\epsilon)}\left\|T_D(1-\epsilon)\left(\varphi_n - \frac{1}{\beta_n}\tilde{\varphi}_n\right)\right\|_{L^\infty(D)}$$

$$+ e^{(\lambda+\epsilon)_n(1-\epsilon)}\frac{1}{\beta_n}\|T_D(1-\epsilon)\varphi_n\|_{L^\infty(D)}.$$

Since $|\varphi_n(x)| \le 2$, the latter term in the right-hand side does not exceed $\frac{2}{\beta_n} e^{(\lambda+\epsilon)_n(1-\epsilon)}$. For the former one, observe that $|T_D(1-\epsilon)f(x)| \le \|T_D(1-\epsilon, x,\cdot)\|_{L^2(D)}\|f\|_{L^2(D)}, T_D(1-\epsilon, x, y) \le T(1-\epsilon, x-y)$, and, by Plancherel's theorem,

$$\|T(1-\epsilon,\cdot)\|^2_{L^2(R)} = \frac{1}{2\pi}\int_{-\infty}^\infty \left(e^{-(1-\epsilon)\psi((\xi+\epsilon)^2)}\right)^2 d(\xi+\epsilon) = T(2(1-\epsilon), 0).$$

Finally, $T(2(1-\epsilon),0) \le C_1(\psi)\sqrt{\psi^{-1}\big(1/(2(1-\epsilon))\big)} \le C_1(\psi)\sqrt{\psi^{-1}\left(\frac{1}{1-\epsilon}\right)}$ when $\epsilon \ge 0$. Therefore, with $1-\epsilon = \frac{1}{(\lambda+\epsilon)_n}$,

$$\|\varphi_n\|_{L^\infty(D)} \le \frac{e}{\beta_n}\left(C_1(\psi)\right)^{1/2}\left(\psi^{-1}((\lambda+\epsilon)_n)\right)^{1/4}\|\beta_n\varphi_n - \tilde{\varphi}_n\|_{L^2(D)} + \frac{2e}{\beta_n}.$$

In the right-hand side, $\beta_n = O(1), \psi^{-1}((\lambda+\epsilon)_n) \le \left(\frac{n\pi}{2(a-\epsilon)}\right)^2$ (by (69)), and, by Corollary (5.3.39),

$$\|\beta_n\varphi_n - \tilde{\varphi}_n\|_{L^2(D)} = O\left(\frac{1}{\left(\frac{n\pi}{2(a-\epsilon)}\right)^2 \psi'\left(\left(\frac{n\pi}{2(a-\epsilon)}\right)^2\right)}\right).$$

Corollary (5.3.47)[274]: Let f be a complete Bernstein function with representation (64). Let g be a holomorphic function in $\{w \in C : |Arg\, w| < C_1\}$ (with $0 < C_1 < \frac{\pi}{2}$) such that $g(x)$ is real for $x > 0$, and let h be a continuous function on $(0,\infty)$. Denote

$$G(y) = \sup_{\substack{y/4\le|z|\le 4y \\ |Arg\, z|<C_1}} |g(z)| \qquad H(y) = \sup_{y/4\le x\le 4y} |h(x)|$$

and suppose that

$$G(x)H(x) \le C_2\, min(x^{-1}, x^{-2}), \qquad C_3 = \int_0^\infty (1+y)G(y)H(y)dy < \infty$$

for $x > 0$. Then

$$\int_{(0,\infty)} g(x)h(x)m(dx) = \lim_{\epsilon\to 0^+}\frac{1}{\pi}\int_0^\infty Im\left(f\left(-e^{-i\epsilon}x\right)g\left(e^{-i\epsilon}x\right)\right)h(x)dx \tag{101}$$

$$= \lim_{\epsilon\to 0^+}\frac{1}{\pi}\int_0^\infty Im\left(f\left(-e^{-i\epsilon}x\right)\right)g(x)h(x)dx. \tag{102}$$

Proof. Let $x > 0, 0 < \epsilon < \frac{1}{2}C_1$ and $y > 0$, and denote for simplicity $\xi + \epsilon = -e^{-i\epsilon}x$. By the representation (64) of the complete Bernstein function f and Fubini, we have

$$\int_0^\infty Im\big(f(\xi+\epsilon)g(-(\xi+\epsilon))\big)h(x)dx$$
$$= c\int_0^\infty Im\big((\xi+\epsilon)g(-(\xi+\epsilon))\big)h(x)dx$$
$$+ \tilde{c}\int_0^\infty Im\big(g(-(\xi+\epsilon))\big)h(x)dx$$
$$+ \frac{1}{\pi}\int_{(0,\infty)}\int_0^\infty Im\frac{(\xi+\epsilon)g(-(\xi+\epsilon))}{\xi+\epsilon+z}h(x)dx\frac{m(dz)}{z} \tag{103}$$

(an estimate which allows us to use Fubini is shown below). Our goal is to provide estimates for the integrands and find their pointwise limits as $\epsilon \to 0^+$ in order to apply dominated convergence.

For the first integral in the right-hand side of (103), we simply use $|(\xi+\epsilon)g(-(\xi+\epsilon))h(x)| \le xG(x)H(x)$, integrability of $xG(x)H(x)$ and $Im((\xi+\epsilon)g(-(\xi+\epsilon))) \to 0$

as $\epsilon \to 0^+$. By dominated convergence, the limit as $\epsilon \to 0^+$ of the first integral in the right-hand side of (103) is zero. Similarly, $|g(-(\xi+\epsilon))h(\xi+\epsilon)| \le G(x)H(x), G(x)H(x)$ is integrable and $Im(g(-(\xi+\epsilon))) \to 0$ as $\epsilon \to 0^+$, and so also the second integral in the right-hand side of (103) converges to zero as $\epsilon \to 0^+$ [225].

To estimate the last integral in the right-hand side of (103), we consider separately two cases. When $x \le \frac{y}{2}$ or $x \ge 2y$, we have

$$\left|-\frac{\xi+\epsilon}{\xi+\epsilon+y}\, g(-(\xi+\epsilon))\right| \le \frac{1}{|x-y|}\, xG(x) \le \frac{3}{x+y}\, xG(x)$$
$$\le 3\, min(1, xy^{-1})G(x) \le 3\, min(1, y^{-1})(1+x)G(x),$$

so that by dominated convergence,

$$\left(\int_0^{y/2} + \int_{2y}^{\infty}\right)\left|Im\left(\frac{\xi+\epsilon}{\xi+\epsilon+y}\, g(-(\xi+\epsilon))\right) h(x)\right| dx \le 3C_3 \min(1, y^{-1}),$$

$$\lim_{\epsilon\to 0^+}\left(\int_0^{y/2} + \int_{2y}^{\infty}\right) Im\left(\frac{\xi+\epsilon}{\xi+\epsilon+y}\, g(-(\xi+\epsilon))\right) h(x)dx = 0. \qquad (104)$$

When $\frac{y}{2} < x < 2y$, we need a more careful estimate. Observe that

$$\frac{(\xi+\epsilon)g(-(\xi+\epsilon))}{\xi+\epsilon+y} = \frac{yg(y) - (-(\xi+\epsilon))g(-(\xi+\epsilon))}{y - (-(\xi+\epsilon))} - \frac{yg(y)}{\xi+\epsilon+y}.$$

The estimate for g and Cauchy's integral formula for g' easily give

$$|g'(z)| \le C_4 y^{-1} G(y)$$

in $\{z \in C : |Arg\, z| < \frac{1}{2}C_1, y/2 \le |z| \le 2y\}$, with $C_4 = 4C_1^{-1}$. By the mean value theorem,

$$\left|\frac{yg(y) - (-(\xi+\epsilon))g(-(\xi+\epsilon))}{y - (-(\xi+\epsilon))}\right| \le C_4 y^{-1} G(y)$$

when $\frac{y}{2} \le x \le 2y$, and therefore, by dominated convergence,

$$\int_{y/2}^{2y}\left|Im\left(\frac{yg(y) - (-(\xi+\epsilon))g(-(\xi+\epsilon))}{y - (-(\xi+\epsilon))}\right) h(x)\right| dx \le \frac{3}{2}C_4 y^{-1} G(y)H(y)$$
$$\le \frac{3}{2}C_2C_4 \min(1, y^{-1}),$$

$$\lim_{\epsilon\to 0^+}\int_{y/2}^{2y} Im\left(\frac{yg(y) - (-(\xi+\epsilon))g(-(\xi+\epsilon))}{y - (-(\xi+\epsilon))}\right) h(x)dx = 0. \qquad (105)$$

Finally, if $P_{1+\epsilon}(s)$ and $Q_{1+\epsilon}(s)$ denote the (classical) Poisson and conjugate Poisson kernels for the half-plane, then

$$Im\left(-\frac{1}{\xi+\epsilon+y}\right)$$
$$= \pi\, cos(\epsilon)P_{y\, sin\,\epsilon}(x - y\, cos\,\epsilon) + \pi\, sin(\epsilon)Q_{y\, sin\,\epsilon}(x - y\, cos\,\epsilon).$$

Clearly, $P_{y\, sin\,\epsilon}(x - y\, cos\,\epsilon)1_{(y/2,2y)}(x)dx$ converges weakly to $\delta_y(x)$, and therefore

$$\int_{y/2}^{2y}\left|\pi\, cos(\epsilon)P_{y\, sin\,\epsilon}(x - y\cos\epsilon)yg(y)h(x)\right|dx \le \pi yg(y)H(y) \le C_2\pi \min(1, y^{-1}),$$

$$\lim_{\epsilon\to 0^+} \int_{y/2}^{2y} \pi\, cos(\epsilon) P_{y\, sin\, \epsilon}(x - y\, cos\, \epsilon) y g(y) h(x) dx = \pi y g(y) h(y).$$

Furthermore, $|(1+\epsilon)Q_{1+\epsilon}(s)| \le \frac{1}{\pi}$ and $(1+\epsilon)Q_{1+\epsilon}(s) \to 0$ as $\epsilon \to -1$, and hence, by dominated convergence,

$$\int_{y/2}^{2y} \left|\pi\, sin(\epsilon) Q_{y\, sin\, \epsilon}(x - y\, cos\, \epsilon) y g(y) h(x)\right| dx \le \frac{3}{2}\, y g(y) H(y) \le \frac{3}{2} C_2 \min(1, y^{-1}),$$

$$\lim_{\epsilon\to 0^+} \int_{y/2}^{2y} \pi\, sin(\epsilon) Q_{y\, sin\, \epsilon}(x - y\, cos\, \epsilon) y g(y) h(x) dx = 0.$$

We have thus proved that

$$\int_{y2}^{2y} \left| Im\, \left(-\frac{y g(y)}{\xi + \epsilon + y}\right)\, h(x)\right| dx \le C_2 \left(\pi + \frac{3}{2}\right) \min(1, y^{-1}),$$

$$\lim_{\epsilon\to 0^+} \int_{y/2}^{2y} Im\, \left(-\frac{y g(y)}{\xi + \epsilon + y}\right)\, h(x) dx = \pi y g(y) h(y). \tag{106}$$

Due to estimates (104), (105) and (106), as well as the integrability condition on m, indeed we could use Fubini in (103). The same estimates allow us to use dominated convergence in the limit as $\epsilon \to 0^+$. We conclude that

$$\lim_{\epsilon\to 0^+} \int_0^\infty Im(f(\xi+\epsilon) g(-(\xi+\epsilon))) h(x) dx = \pi \int_{(0,\infty)} \int_0^\infty g(y) h(y) m(dy).$$

This proves the first equality in (101). The other one follows by replacing the pair $g(z), h(x)$ with 1 and $g(x)h(x)$.

Chapter 6
Normalized Incidence Energy and Coulson-Type Integral Formulas

We find some upper and lower bounds and determine the Coulson integral formula for *NIE(G)*. Based on the integral formula, we give a way to compare the normalized incidence energies. We show a relation between normalized incidence energy and Randi´c energy. We give a Coulson-type integral formula for the general Laplacian-energy-like invariant for $\alpha = \frac{1}{p}$ with $p \in \mathbb{Z}^{+}\backslash\{1\}$. This implies integral formulas for the Laplacian-energy-like invariant, the normalized incidence energy and the Laplacian incidence energy of graphs. We further give some Coulson-type integral formulas for the general energy and general Laplacian energy of graphs in the case that α is a rational number. We also show that our formulas hold when α is an irrational number with $0 < |\alpha| < 1$ and do not hold with $|\alpha| > 1$.

Section (6.1): Incidence Energy of a Graph

Let G be a simple graph on n vertices and let $v_1, v_2, \ldots, v_n$ be its vertices. The eigenvalues of G are the eigenvalues of its adjacency matrix $A(G)$ [124]. These eigenvalues, arranged in a non-increasing order, are denoted as $\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)$. Then the energy of the graph G is defined as

$$E(G) = \sum_{k=1}^{n} |\lambda_k(G)|.$$

This concept was proposed by Gutman [125]. Research on graph energy is nowadays very active in mathematical chemistry, as seen from [243], [249]–[251], [253], and [254].
We use d_k^G to denote the degree of a vertex v_k in G, and $D(G)$ to denote the diagonal matrix of order n whose (k,k)-entry is d_k^G . If there is only one graph in question, we simply write d_k and D. The normalized Laplacian matrix of a graph G, denoted by $\hat{L}(G)$ or $\hat{L}$, is defined to be the matrix with entries

$$\hat{L}(i,j) = \begin{cases} 1 & \text{if } i = j \text{ and } d_j \neq 0; \\ -\frac{1}{\sqrt{d_i d_j}} & \text{if } v_i \text{ and } v_j \text{ are adjacent in } G; \\ 0 & otherwise. \end{cases}$$

It is well known that 0 is an eigenvalue of $\hat{L}$ and that the remaining eigenvalues lie in the interval [0,2] (see [245]).

Let A be the adjacency matrix of G. The Randi´c matrix of G, denoted by $\hat{A}(G)$ or $\hat{A}$, is the square matrix of order n whose (i,j)-entry is equal to $\frac{1}{\sqrt{d_i d_j}}$ if v_i and v_j are adjacent in G, and zero otherwise.
Let $D^{-1/2}$ is the matrix with entries

$$D^{-1/2}(i,j) = \begin{cases} D(i,j)^{-1/2} & \text{if } i = j \text{ and } d_j = 0; \\ 0 & \text{otherwise.} \end{cases}$$

Then $\hat{A} = D^{-1/2} A D^{-1/2}$. The Randic matrix appeared in [242], [259].

Let $\widehat{D}$ be the square matrix of order n where (i,j)-entry is equal to 1 if $i = j$ and $d_j \neq 0$, and is equal to 0 otherwise. Then it follows that $\hat{L} = \widehat{D} - \hat{A}$. We unit matrix of order n is denoted by I or I_n. If G has no isolated vertices, then

$$\hat{L} = I - \hat{A}. \tag{1}$$

If M is a real symmetric matrix of order n, we denote the eigenvalues of M (or M-eigenvalues) by $\lambda_1(M), \lambda_2(M), \ldots, \lambda_n(M)$. If G is a graph of order n and M is a real symmetric matrix associated with G, then the M-energy of G is

$$E_M(G) = \sum_{k=1}^{n} \left| \lambda_k(M) - \frac{tr(M)}{n} \right| , \tag{2}$$

where $tr(M)$ is the trace of M (see [246]).

Let G be a graph of order n. The Randi´c energy $RE(G)$ of G, is defined as $RE(G) = \sum_{k=1}^{n} |\lambda_k(\hat{A})|$.

For a graph G of order n with no isolated vertices, Caversetal. [244] introduced the normalized Laplacian energy $E_{\hat{L}}(G)$ of G by (2), i.e.,

$$E_{\hat{L}}(G) = \sum_{k=1}^{n} |\lambda_k(\hat{L}(G)) - 1|.$$

Then it can be verified that $E_{\hat{L}}(G) = RE(G)$ for a graph G with no isolated vertices.

The normalized signless Laplacian matrix of a graph G, denoted by $\hat{L} + (G)$ or $\hat{L}^{+}$, is defined to be the matrix with entries

$$\hat{L}^{+}(i,j) = \begin{cases} 1 & \text{if } i = j \text{ and } d_j \neq 0; \\ \frac{1}{\sqrt{d_i d_j}} & \text{if } v_i \text{ and } v_j \text{ are adjacent in } G; \\ 0 & \text{otherwise.} \end{cases}$$

Then

$\hat{L}^{+} = \widehat{D} + \hat{A}$.

Specially for a graph with no isolated vertices, we have

$$\hat{L}^{+} = I + \hat{A}. \tag{3}$$

Let G be a graph of order n with no isolated vertices. With M taken to be $\hat{L}^{+}$ in (2), the normalized signless Laplacian energy $NSE(G)$ of G is defined as

$$NSE(G) = \sum_{k=1}^{n} |\lambda_k(\hat{L}^{+}(G)) - 1|.$$

By (1) and (3), we have $RE(G) = E_{\hat{L}}(G) = NSE(G)$ if G has no isolated vertices.

The concept of graph energy was extended to any matrix by Nikiforov in the following manner. The singular values of a real $n \times m$ matrix M, denoted by $\sigma_1(M), \sigma_2(M), \ldots, \sigma_n(M)$, are the square roots of the eigenvalues of the square matrix MM^t , where M^t denotes the transpose of M. The energy $\mathcal{E}(M)$ of the matrix M is then defined as the sum of its singular values [257]

$$\mathcal{E}(M) = \sum_{k=1}^{n} \sigma_k(M).$$

Obviously, $E(G) = \mathcal{E}(A(G))$.

For a graph G with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{e_1, e_2, \ldots, e_m\}$, the (vertex-edge) incidence matrix of G, denoted by $I(G)$, is defined to be the $n \times m$ matrix with entries

$$I(G)(i,j) = \begin{cases} 1 & \text{if } v_i \text{ is incident with } e_j; \\ 0 & \text{otherwise.} \end{cases}$$

The incidence matrix should not be confused with the unit matrix of order p, which is denoted by I or I_p. The normalized incidence matrix of G is $\hat{I}(G) = D^{-1/2} I(G)$. Then $\hat{I}(G)\hat{I}(G)^t = \widehat{D} + \hat{A} = \hat{L}^+$.

The energy for the incidence matrix was introduced by [248], [252]. Now we discuss the energy for the normalized incidence matrix.

For a graph G of order n, the normalized incidence energy $NIE(G)$ is defined as $NIE(G) = \mathcal{E}(\hat{I}(G))$. If G is an empty graph, i.e., G contains no edges, then we define $NIE(G) = 0$. Therefore we have $NIE(G) = \sum_{k=1}^{n} \sigma_k(\hat{I}(G)) = \sum_{k=1}^{n} \sqrt{\lambda_k(\hat{L}^+(G))}$.

$E(G)$ has the following basic properties.

(a) $E(G) \geq 0$; equality is attained if and only if G is an empty graph.

(b) If the graph G consists of connected components G_1 and G_2, then $E(G) = E(G_1) + E(G_2)$.

(c) If one connected component of the graph G is G_1 and all other connected components are isolated vertices, then $E(G) = E(G_1)$.

We will discuss some properties of $NIE(G)$ and give some bounds for $NIE(G)$. We will determine the Coulson integral formula for $NIE(G)$. A relation between $NIE(G)$ and Randi´c energy will be shown.

We first present some properties of $NIE(G)$ which are analogous to the properties (a), (b) and (c) of $E(G)$.

(a) $NIE(G) \geq 0$; equality is attained if and only if G is an empty graph.

(b) If the graph G consists of connected components G_1 and G_2, then $NIE(G) = NIE(G_1) + NIE(G_2)$.

(c) If one component of the graph G is G_1 and all other components are isolated vertices, then $NIE(G) = NIE(G_1)$.

If a connected component of the graph G contains at least one edge, then the connected component is said to be nontrivial. We denote by $d(G)$ the diameter of a connected graph G. If G contains only one vertex, then we define $d(G) = 0$. We use K_n for the complete graph on n vertices. For a graph $G, \bar{G}$ denotes the complement of G.

From (1) and (3), if G has no isolated vertices, then

$$\hat{L}^+ + \hat{L} = 2I. \tag{4}$$

From some properties for the normalized Laplacian matrix in [245], we obtain their corresponding results for the normalized signless Laplacian matrix by (4). We state them as follows.

Lemma (6.1.1)[240]: Suppose the n-vertex graph G has no isolated vertices and p connected components. If the eigenvalues of $\hat{L}^+(G)$ are ordered and denoted by $\mu_1^+ \geq \mu_2^+ \geq \cdots \mu_n^+$, then $\mu_1^+ = \cdots = \mu_p^+ = 2$ and $\mu_{p+1}^+ < 2$.

Lemma (6.1.2)[240]: Suppose G is a graph. If μ^+ is an eigenvalues of $\hat{L}^+(G)$, then $\mu^+ \geq 0$.

Lemma (6.1.3)[240]: Suppose the n-vertex connected graph G is not a complete graph. If the eigenvalues of $\hat{L}^+(G)$ are ordered and denoted by $\mu_1^+ \geq \mu_2^+ \geq \cdots \geq \mu_n^+$, then $\mu_2^+ \geq 1$.

Lemma (6.1.4)[240]: Let G be a connected graph with diameter d and s distinct $\hat{L}^+$-eigenvalues. Then $d \le s - 1$.

Thus by Lemma (6.1.1), we have

Lemma (6.1.5)[240]: Suppose the n-vertex graph G has p nontrivial connected components where $p \ge 1$. If the eigenvalues of $\hat{L}^+(G)$ are ordered and denoted by $\mu_1^+ \ge \mu_2^+ \ge \cdots \ge \mu_n^+$, then $\mu_1^+ = \cdots = \mu_p^+ = 2$ and $\mu_{p+1}^+ < 2$.

Lemma (6.1.6)[240]: Let G be a graph with s distinct $\hat{L}^+$-eigenvalues. Then $d \le s - 1$, where $d = max\{d(G^*) : G^*$ is a connected component of $G\}$.

Proof. Suppose G_1 is a connected component of G such that $d(G_1) = d$. Let s_1 be the number of the distinct eigenvalues of $\hat{L}^+(G_1)$. Then $d \le s_1 - 1$ by Lemma (6.1.4). Since $s_1 \le s$, the result follows.

The following result is immediate.

Lemma (6.1.7)[240]: If the connected graph G has exactly two distinct $\hat{L}^+$-eigenvalues, then the diameter of G is 1, i.e., G is complete.

We present an upper bound for $NIE(G)$.

Theorem (6.1.8)[240]: Suppose G has n vertices, p nontrivial connected components, and t isolated vertices where $p \ge 1$. Then

$$NIE(G) \le \sqrt{2}p + \sqrt{(n - t - p)(n - t - 2p)}$$

with equality holding if and only if G is $pK_r \cup \overline{K_t}$ where $r = \frac{n-t}{p}$.

Proof. Suppose the eigenvalues of $\hat{L}^+(G)$ are ordered and denoted by $\mu_1^+ \ge \mu_2^+ \ge \cdots \ge \mu_n^+$. Then $\mu_{n-t+1}^+ = \cdots = \mu_n^+ = 0$ by the fact that G has t isolated vertices. By Lemma (6.1.5), we have $NIE(G) = \sqrt{2}p + \sum_{k=p+1}^{n-t} \sqrt{\mu_k^+}$. By the Cauchy–Schwarz inequality,

$$NIE(G) \le \sqrt{2}p + \sqrt{(n-t-p)\sum_{k=p+1}^{n-t} \mu_k^+} = \sqrt{2}p + \sqrt{(n-t-p)\left(tr\left(\hat{L}^+\right) - 2p\right)}$$

$$= \sqrt{2}p + \sqrt{(n - t - p)(n - t - 2p)}$$

with equality holding if and only if $\mu_1 = \mu_2 = \cdots = \mu_p = 2, \mu_{p+1} = \cdots = \mu_{n-t}$ and $\mu_{n-t+1} = \cdots = \mu_n = 0$.

Suppose $\mu_1 = \mu_2 = \cdots = \mu_p = 2, \mu_{p+1} = \cdots = \mu_{n-t}$ and $\mu_{n-t+1} = \cdots = \mu_n = 0$. Then for every nontrivial connected component $G', \hat{L}^+(G')$ has exactly two distinct eigenvalues. A connected graph has exactly two distinct normalized signless Laplacian eigenvalues if and only if its diameter is equal to unity, i.e., it is a complete graph. Therefore the graph G must consist of connected components that are mutually isomorphic complete graphs (say, of order r) and t isolated vertices. Thus $rp + t = n$ and then $= \frac{n-t}{p}$. Therefore G is $pK_r \cup \overline{K_t}$ where $= \frac{n-t}{p}$.

Corollary (6.1.9)[240]: If the connected graph G has n vertices where $n \ge 2$, then $NIE(G) \le \sqrt{2} + \sqrt{(n - 1)(n - 2)}$ with equality holding if and only if G is the complete graph K_n.

Theorem (6.1.10)[240]: Let G be a connected graph on n vertices where $n \geq 2$. If G is not a complete graph, then $NIE(G) \leq 1+\sqrt{2}+\sqrt{(n-2)(n-3)}$ with equality holding if and only if G is a graph with $\hat{L}^{+}$-eigenvalues $2, 1, \frac{n-3}{n-2}, \ldots, \frac{n-3}{n-2}$ $\left(n-2\text{'s } \frac{n-3}{n-2}\right)$.

Proof: Suppose the eigenvalues of $\hat{L}^{+}(G)$, arranged in a non-increasing order, are $\mu_1^{+}, \mu_2^{+}, \ldots, \mu_n^{+}$. Then $\mu_1^{+} = 2$ from Lemma (6.1.5). By the Cauchy–Schwarz inequality,

$$\sum_{k=3}^{n} \sqrt{\mu_k^{+}} \leq \sqrt{(n-2)\sum_{k=3}^{n} \mu_k^{+}} = \sqrt{(n-2)(n-2-\mu_2^{+})}$$

with equality holding if and only if $\mu_3^{+} = \cdots = \mu_n^{+}$.

Let $f(x) = \sqrt{x} + \sqrt{(n-2)(n-2-x)}$. Then $f'(x) = \frac{1}{2\sqrt{x}} - \frac{\sqrt{n-2}}{2\sqrt{n-2-x}}$ and thus $f(x)$ decreases with $x \geq 1$.

By Lemma (6.1.3), we have

$$NIE(G) = \sqrt{\mu_1^{+}} + \sqrt{\mu_2^{+}} + \sum_{k=3}^{n} \sqrt{\mu_k^{+}} \leq \sqrt{2} + \mu_2^{+} + \sqrt{(n-2)(n-2-\mu_2^{+})}$$
$$\leq \sqrt{2} + 1 + \sqrt{(n-2)(n-3)},$$

with equality holding if and only if $\mu_2^{+} = 1$ and $\mu_3^{+} = \cdots = \mu_n^{+}$. Then the equality holds if and only if the $\hat{L}^{+}$-eigenvalues of G are $2, 1, \frac{n-3}{n-2}, \ldots, \frac{n-3}{n-2}$ $\left(n-2\text{'s } \frac{n-3}{n-2}\right)$.

Next we give some lower bounds for $NIE(G)$.

Theorem (6.1.11)[240]: Let G be a graph on n vertices. Then $NIE(G)\sqrt{n-t}$ where t is the number of isolated vertices in G, and equality holds if and only if G is $\overline{K_n}$ or $K_2 \cup \overline{K_{n-2}}$.

Proof: Let the eigenvalues of $\hat{L}^{+}(G)$ be $\mu_1^{+}, \mu_2^{+}, \ldots, \mu_n^{+}$. Then

$$NIE(G) = \sum_{k=1}^{n} \sqrt{\mu_k^{+}} \geq \sqrt{\sum_{k=1}^{n} \mu_k^{+}} = \sqrt{n-t},$$

and equality holds if and only if G has at most one positive $\hat{L}^{+}$-eigenvalue if and only if G is $\overline{K_n}$ or $K_2 \cup \overline{K_{n-2}}$.

As a corollary, we have

Corollary (6.1.12)[240]: Let G be a graph of order n with no isolated vertices. Then $NIE(G) \geq \sqrt{n}$, with equality if and only if G is K_2.

Let $a_1, a_2, \ldots, a_s$ be positive integers. By Hölder's inequality, we obtain

$$\sum_{i=1}^{s} a_i = \sum_{i=1}^{s} a_i^{2/3} a_i^{1/3} \leq \left(\sum_{i=1}^{s} a_i^2\right)^{1/3} \left(\sum_{i=1}^{s} a_i^{1/2}\right)^{2/3},$$

and hence

$$\sum_{i=1}^{s} a_i^{1/2} \geq \sqrt{\frac{\left(\sum_{i=1}^{s} a_i\right)^3}{\sum_{i=1}^{s} a_i^2}}, \qquad (5)$$

with equality if and only if $a_1 = a_2 = \cdots = a_s$.

Let G be a graph of order n with no isolated vertices. The general Randi´c index $R_\alpha(G)$ is defined as

$$R_\alpha(G) = \sum_{v_i \sim v_j} \left(d_i d_j\right)^\alpha, \tag{6}$$

where the summation is over all (unordered) edges $v_i v_j$ in G, and $\alpha \neq 0$ is a fixed real number. In 1975, Randi´c [258] introduced a topological index R (with $= -\frac{1}{2}$) under the name 'branching index'. In 1998, BollobKas and ErdRos [241] generalized this index by replacing the $\frac{1}{2}$ with any real number $R_\alpha(G)$ (as defined in (6)).

The following result is from [255].

Lemma (6.1.13)[240]: [255]. Let G be a graph of order n with no isolated vertices. Then $R_{-1}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$, with equality if and only if either (i) n is even and G is the disjoint union of $\frac{n}{2}$ paths of length 1, or (ii) n is odd and G is the disjoint union of $\frac{n-3}{2}$ paths of length 1 and one path of length 2.

Now we improve Corollary (6.1.12).

Theorem (6.1.14)[240]: Let G be a graph of order n with no isolated vertices. Then $NIE(G) \geq \sqrt{\frac{2n^3}{4n-1+(-1)^n}}$ with equality if and only if n is even and G is the disjoint union of $\frac{n}{2}$ paths of length 1.

Proof: Let the eigenvalues of $\hat{L}^+(G)$ be $\mu_1^+, \mu_2^+, \ldots, \mu_n^+$. By (5), we see that

$$NIE(G) = \sum_{i=1}^{n} \sqrt{\mu_i^+} \geq \sqrt{\frac{\left(\sum_{i=1}^n \mu_i^+\right)^3}{\sum_{i=1}^n \left(\mu_i^+\right)^2}},$$

with equality if and only if all nonzero $\hat{L}^+$ -eigenvalues are equal, i.e., n is even and G is the disjoint union of $\frac{n}{2}$ paths of length 1.

Since G contains no isolated vertices, it follows that $\sum_{i=1}^n \mu_i^+ = n$ and $\sum_{i=1}^n (\mu_i^+)^2 = tr\left(\left(\hat{L}^+\right)^2\right) = n + 2R_{-1} \leq n + 2\left\lfloor \frac{n}{2} \right\rfloor$ by Lemma (6.1.13).

Therefore $(G) \geq \sqrt{\frac{n^3}{n+2\left\lfloor \frac{n}{2} \right\rfloor}} = \sqrt{\frac{2n^3}{4n-1+(-1)^n}}$, with equality if and only if n is even and G is the disjoint union of $\frac{n}{2}$ paths of length 1.

Corollary (6.1.15)[240]: Let G be a graph of order n with no isolated vertices. Then $(G) \geq \frac{n}{\sqrt{2}}$, with equality if and only if n is even and G is the disjoint union of $\frac{n}{2}$ paths of length 1.

The graph-energy concept has been extended to polynomials in [256]. Let $\psi = \psi(\lambda)$ be a (complex or real) monic polynomial of degree n, written in the form $\psi(\lambda) = \lambda^n + \sum_{k=1}^n a_k \lambda^{n-k}$, and let $z_1, z_2, \ldots, z_n$ be its zeros. Set $\prod^+ = \{z_k : Re\, z_k > 0 \text{ and } 1 \leq k \leq n\}$ and $\prod^- = \{z_k : Re\, z_k < 0 \text{ and } 1 \leq k \leq n\}$. Let s^+ (respectively s^-) be the sum of zeros of ψ in $\prod^+$ (respectively $\prod^-$), counting multiplicities. The energy $E(\psi)$ of the polynomial ψ, is defined as

$$E(\psi) = s^+ - s^-.$$

For a matrix $M, \phi(M,\lambda)$ denotes its characteristic polynomial, i.e., $\phi(M,\lambda) = det(\lambda I - M)$. As early as in 1940, Charles Coulson [247] obtained a formula in which $E(G)$ was expressed in terms of the characteristic polynomial $\phi(A(G),\lambda)$:

$$E(G) = \frac{1}{\pi}\ v.p.\int_{-\infty}^{+\infty} \left[n - ix\frac{\phi'(A(G),ix)}{\phi(A(G),ix)}\right]dx,$$

where n is the order of G. In this formula $v.p.\int_{-\infty}^{+\infty} F(x)$ dx stands for the principle value of the respective integral, i.e., $\lim_{t\to+\infty} \int_{-t}^{t} F(x)\,dx$. This result has been generalized in [256].

Theorem (6.1.16)[240]: [256]. Let f be a polynomial of degree n with leading coefficient 1. Then

$$E(f) = \frac{1}{\pi}\ v.p.\int_{-\infty}^{+\infty} \left[n - ix\,(\frac{f'(ix)}{f(ix)}\right]dx.$$

For the normalized incidence energy of a graph G, we have

Theorem (6.1.17)[240]: Let G be a graph of order n. Then

$$NIE(G) = \frac{1}{2\pi}\ v.p.\int_{-\infty}^{+\infty} \left[2n - ix\,(\frac{f'(ix)}{f(ix)}\right]dx,$$

where $f(x) = \phi(\hat{L}^{+}(G),x^2)$.

Proof: Let $\mu_1^+, \mu_2^+, \dots, \mu_n^+$ be the eigenvalues of $\hat{L}^{+}(G)$. Then the zeros of $f(x)$ are $\pm\sqrt{\mu_1^+}, \pm\sqrt{\mu_2^+}, \dots, \pm\sqrt{\mu_n^+}$. Hence $E(f) = 2\sum_{k=1}^{n} \sqrt{\mu_k^+} = 2NIE(G)$, and thus $NIE(G) = \frac{1}{2}E(f)$. The result follows from Theorem (6.1.16).

Corollary (6.1.18)[240]: Let G be a graph of order n. Then

$$NIE(G) = \frac{1}{\pi}\int_{0}^{+\infty} \left[2n - ix\,\frac{f'(ix)}{f(ix)}\right]\ dx, \qquad (7)$$

where $f(x) = \phi(\hat{L}^{+}(G),x^2)$.

Proof: It follows from the fact that $g(x) = 2n - ix\frac{f'(ix)}{f(ix)}$ is an even function.

The Coulson integral formula gives us a new way to obtain the normalized incidence energy of a graph G. For example, take $G = K_{1,n-1}$, a tree on n vertices with one vertex having degree $n - 1$.

Then $\phi(\hat{L}^{+}(K_{1,n-1}),x) = x(x-1)^{n-2}(x-2)$. Therefore $f(x) = x^2(x^2 - 1)^{n-2}(x^2 - 2)$, and thus $2n - ix\frac{f'(ix)}{f(ix)} = \frac{2n-4}{x^2+1} + \frac{4}{x^2+2}$. So $NIE(K_{1,n-1}) = n - 2 + \sqrt{2}$.

Now we present another way to write the Coulson integral formula and use it to compare the normalized incidence energies.

The characteristic polynomial of $\hat{L}^{+}(G)$ can be written in the coefficient form as $\phi(\hat{L}^{+}(G),\lambda) = \sum_{k=0}^{n} (-1)^k b_k(G)\lambda^{n-k}$, or $\phi(\hat{L}^{+}(G),\lambda) = \sum_{k=0}^{n} (-1)^k b_k\lambda^{n-k}$. We see that $b_0 = 1$. By Lemma (6.1.2), $b_k \geq 0$.

Theorem (6.1.19)[240]: Let G be a graph of order n and let the characteristic polynomial of $\hat{L}^{+}(G)$ be of the form $\sum_{k=0}^{n} (-1)^k b_k\lambda^{n-k}$. Then

$$NIE(G) = \frac{1}{\pi}\int_0^{+\infty} \; ln\left(\sum_{k=0}^{n} \; b_k x^{2k}\right)\frac{dx}{x^2}\ .$$

Proof: Set $x \ = \frac{1}{y}$ in (7), and it follows that

$$NIE(G) = \frac{1}{\pi}\int_0^{+\infty} \left[2n \ - \frac{i}{y}\frac{f'\left(\frac{i}{y}\right)}{f\left(\frac{i}{y}\right)}\right]\frac{dy}{y^2},$$

where $f(x) \ = \phi(\hat{L}^{+}(G), x^2)$.

Integrating by parts and considering

$$u \ = \frac{1}{y} \text{ and } dv \ = \left[\frac{2n}{y} - \frac{i}{y^2} \ \frac{f'\left(\frac{i}{y}\right)}{f\left(\frac{i}{y}\right)}\right] dy,$$

one can obtain that

$$NIE(G) = \frac{1}{\pi}\left(\frac{1}{y}\ ln\left|y^{2n}f\ \left(\frac{i}{y}\right)\right|\right)_0^{+\infty} + \frac{1}{\pi}\ \int_0^{+\infty} \frac{1}{y^2}\ ln\left|y^{2n}f\left(\frac{i}{y}\right)\right| dy$$
$$= \frac{1}{\pi}\int_0^{+\infty} \ ln\left(\sum_{k=0}^{n} \ b_k y^{2k}\right)\frac{dy}{y^2}.$$

We introduce a quasi-order relation here. For two n-vertex graphs G_1 and G_2, if

$$b_k(G_1) \le b_k(G_2) \qquad (8)$$

holds for $0 \le k \le n$, then we write $G_1 \preccurlyeq \ G_2$. Moreover, if at least one of the inequalities in (8) is strict, then we write $G_1 \ \prec \ G_2$.

From Theorem (6.1.19), we have

Corollary (6.1.20)[240]: If $G_1 \preccurlyeq G_2$, then $NIE(G_1) \le NIE(G_2)$. If $G_1 \prec G_2$, then $NIE(G_1) < NIE(G_2)$.

We give a connection between the eigenvalues of Randi´c matrix and $\hat{L}^{+}(G)$ using a method from Zhou and Gutman [158]. The subdivision graph $\tilde{G}$ of a graph G, is obtained by inserting an additional vertex into each edge of G. If G is a graph with n vertices and m edges, then $\tilde{G}$ has $n \ + \ m$ vertices and $2m$ edges.

Theorem (6.1.21)[240]: Let G be a graph with n vertices and m edges, and let $\tilde{G}$ be its subdivision graph. Then $\phi(\hat{A}(\tilde{G}), \lambda) \ = \ 2^{-n}\lambda^{m-n}\phi(\hat{L}^{+}(G), 2\lambda^2)$.

Proof: We have $\phi(\hat{A}(\tilde{G}), \lambda) \ = \ det[\lambda I_{n+m} \ - \ \hat{A}(\tilde{G})] \ = det\begin{pmatrix}\lambda I_n & -M \\ -M^t & \lambda I_m\end{pmatrix}$, where $M \ = \frac{1}{\sqrt{2}}\ D^{-1/2}I(G)$.

It follows that

$$\phi(\hat{A}(\tilde{G}), \lambda) \ = \ \lambda^{m-n} det[\lambda^2 I_n - MM^t\ ] \ = \ \lambda^{m-n} det\left[\lambda^2 I_n \ - \frac{1}{2}\ \hat{I}(G)\hat{I}(G)^t\ \right]$$
$$= \ 2^{-n}\lambda^{m-n} det[2\lambda^2 I_n - \hat{L}^{+}(G)] \ = \ 2^{-n}\lambda^{m-n}\phi(\hat{L}^{+}(G), 2\lambda^2).$$

Corollary (6.1.22)[240]: Let G a graph with n vertices and m edges, $\tilde{G}$ its subdivision graph. If μ_k^{+} are the non-zero eigenvalues of the normalized signless Laplacian matrix of G, then the spectrum of the Randi´c matrix of $\tilde{G}$ consists of the number $\pm\sqrt{\mu_k^{+}/2}, k \ = \ 1, \ldots, h,$ and of $n \ + \ m \ - \ 2h$ zeros.

By Corollary (6.1.22), we have

$$NIE(G) = \sum_{k=1}^{h} \sqrt{u_k^+} = \frac{\sqrt{2}}{2}\left(2\sum_{k=1}^{h} \sqrt{u_k^+/2}\right) = \frac{\sqrt{2}}{2} RE(\tilde{G}).$$

Corollary (6.1.23)[240]: Let G be a graph. Then $NIE(G) = \frac{\sqrt{2}}{2} RE(\tilde{G})$.

Section (6.2): General Laplacian-Energy-Like Invariant of Graphs

All graphs considered are finite and simple. See to Cvetković et al. [124].

Let G be a graph with n vertices and m edges. The eigenvalues of the adjacency matrix $A(G)$ of G are said to be the eigenvalues of G and form the spectrum of G. We denote the eigenvalues of G by $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n$ in non-increasing order. The matrix $L(G) = D(G) - A(G)$ is called the Laplacian matrix of G, where $D(G) = diag(d_1, d_2, \ldots, d_n)$ is the diagonal matrix of vertex degrees of G. It is well known that $L(G)$ is a positive semi-definite symmetric matrix, and moreover 0 is the smallest eigenvalue of $L(G)$. We denote the eigenvalues of $L(G)$ by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$, which are called the Laplacian eigenvalues of G.

The energy of a graph G is defined as $E(G) = \sum_{k=1}^{n} |\lambda_k|$, which is derived from the total π-electron energy [267]. Graph energy has been studied extensively by many mathematicians and chemists, and there have been many results obtained on this invariant of graphs (see [264]). In the theory of graph energy there is an important result called the Coulson integral formula which makes it possible to calculate the energy of a graph without knowing its spectrum. For a graph G, its Coulson integral formula is

$$E(G) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \left[n - \frac{ix\phi_A'(G, ix)}{\phi_A(G, ix)} \right] dx,$$

where $\phi_A(G, x)$ is the characteristic polynomial of $A(G)$ (called the characteristic polynomial of G).

This formula was obtained by Coulson [247], and has many applications in the theory of graph energy (see [264]).

For a graph G, since $\mu_k \geq 0$ for $k = 1, 2, \ldots, n$, it would be trivial to define its Laplacian energy as $\sum_{k=1}^{n} |\mu_k| = \sum_{k=1}^{n} \mu_k = 2m$. Gutman and Zhou [129] defined the Laplacian energy of a graph G as

$$LE(G) = \sum_{k=1}^{n} |\mu_k - \frac{2m}{n}|$$

Later, Liu and Liu [121] introduced the *Laplacian-energy-like* invariant of G, which is similar to the definition of the graph energy, as

$$LEL(G) = \sum_{k=1}^{n} \sqrt{\mu_k}.$$

This invariant has many similar properties as the energy of a graph. For more results on the Laplacian-energy-like invariant, we refer the reader to the references [263], [121], [266], [268].

In [147], Zhou studied the sum of powers of the Laplacian eigenvalues of graphs, which can be regarded as a generalization of the Laplacian-energy-like invariant. Here we call this invariant the general Laplacian-energy-like invariant of graphs.

Definition (6.2.1)[260]: Let G be a graph of order $n, \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$the Laplacian eigenvalues of G and α a real number. The general Laplacian-energy-like invariant of G, denoted by $LEL_\alpha(G)$, is defined as $\sum_{\mu_k \neq 0} \mu_k^\alpha$ when $\mu_1 \neq 0$, and 0 when $\mu_1 \neq 0$.

Obviously, $LEL(G) = LEL_{\frac{1}{2}}(G)$.

We obtain a Coulson-type integral formula for the general Laplacian-energy-like invariant of graphs. We present a Coulson-type integral formula for the general energy of polynomials, which is an extension of the general Laplacian- energy-like invariant of graphs, and show that it implies two known integral formulas for the normalized incidence energy and the Laplacian incidence energy.

We first introduce some basic concepts and results from complex analysis which will be used later. Let D be a bounded domain. The boundary of D is denoted by ∂D.

The following two results in complex analysis are well known (see [262]).

Lemma (6.2.2)[260]: (Cauchy's theorem). Let D be a bounded domain with piecewise smooth boundary. If $f(z)$ is analytic on D, and extends smoothly to ∂D, then

$$\int_{\partial D} f(z)dz = 0.$$

Lemma (6.2.3)[260]: (Cauchy integral formula). Let D be a bounded domain with piecewise smooth boundary. If $f(z)$ is analytic on D, and extends smoothly to ∂D, then

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta, \qquad z \in D.$$

We also need the following simple lemmas. The proofs are omitted here.

Lemma (6.2.4)[260]: Let S_r be the arc $z(\theta) = a_0 + re^{i\theta}, \theta_1 \leq \theta \leq \theta_2$, where $r > 0$ is a real number. If $f(z)$ is a continuous function on the arc S_r for all small r such that

$$\lim_{r \to 0+} \max_{\theta \in [\theta_1, \theta_2]} \left| re^{i\theta} f\left(a_0 + re^{i\theta}\right) - \lambda \right| = 0,$$

then

$$\lim_{r \to 0+} \int_{S_r} f(z)dz = i(\theta_2 - \theta_1)\lambda.$$

Lemma (6.2.5)[260]: Suppose that Γ is a piecewise smooth curve. If$f(z)$is a continuous function on Γ, then $\left|\int_\Gamma f(z)dz\right| \leq \int_\Gamma |f(z)| \cdot |d_z|$. Further, if Γ has length L, and $|f(z)| \leq M$ on Γ, then

$$\left|\int_\Gamma f(z)dz\right| \leq M L.$$

Setting $f(z) = 1$ in Lemmas (6.2.2) and (6.2.3), we get

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{d\zeta}{\zeta - z_0} = \begin{cases} 1, & if\ z_0 \in \operatorname{int}(\partial D); \\ 0, & if\ z_0 \in \operatorname{ext}(\partial D), \end{cases}$$

where $z_0 \in \operatorname{int}(\partial D)$ and $z_0 \in \operatorname{ext}(\partial D)$ mean that z_0 lies in the interior of ∂D and in the exterior of ∂D, respectively.

Let $\phi(z) = \sum_{k=0}^{n} a_k z^{n-k} = a_0 \prod_{k=1}^{n}(z - z_k)$ be a complex polynomial of degree n. By direct computing, we get

$$\frac{z\phi'(z)}{\phi(z)} = \sum_{k=1}^{n} \frac{z}{z-z_k} = n + \sum_{k=1}^{n} \frac{z_k}{z-z_k}$$

That is

$$\frac{z\phi'(z)}{\phi(z)} - n = \sum_{k=1}^{n} \frac{z_k}{z-z_k}.$$

If $z_1, z_2, \dots, z_n \in \text{int}(\partial D)$, then we have

$$\frac{1}{2\pi i}\int_{\partial D}\left(\frac{z\phi'(z)}{\phi(z)} - n\right)dz = \frac{1}{2\pi i}\int_{\partial D}\sum_{k=1}^{n}\frac{z_k}{z-z_k}dz = \sum_{k=1}^{n} z_k\,.$$

Coulson-type integral formula for the general Laplacian-energy-like invariant of graphs

Theorem (6.2.6)[260]: Let G be a graph of order $n, \phi_L(G,x)$ the characteristic polynomial of the Laplacian matrix $L(G)$, and $\alpha = 1/p$ a number with $p \in \mathbb{Z}^+\backslash\{1\}$. Then the general Laplacian-energy-like invariant of G can be given by the following integral formula

$$LEL_\alpha(G) = \frac{1}{\pi}\int_0^{+\infty}\left(\frac{px^p\phi_L'(G,-x^p)}{\phi_L(G,-x^p)} + pn\right)\cdot sin\frac{\pi}{p}dx.$$

Proof: Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ be the roots of $\phi_L(G,x)$. It is well known that if G has $c\ (< n)$ components, then the multiplicity of $\mu_n = 0$ is c, which means that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-c} > \mu_{n-c+1} = \cdots = \mu_n = 0$. Let $\varphi_L(G,z) = \phi_L(G,z^p)$. Then we have

$$\varphi_L(G,z) = (z^p)^c \cdot \prod_{k=1}^{n-c}(z^p - \mu_k) = z^{cp} \cdot \prod_{k=1}^{n-c}\left[\prod_{t=0}^{p-1}(z - \mu_k^{\frac{1}{p}} e^{i\frac{2t\pi}{p}})\right]$$

and

$$\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)} = z\left[\frac{cp}{z} + \sum_{k=1}^{n-c}\left(\sum_{t=0}^{p-1}\frac{1}{z - \mu_k^{\frac{1}{p}} e^{i\frac{2t\pi}{p}}}\right)\right] = cp + \sum_{k=1}^{n-c}\sum_{t=0}^{p-1}\frac{z}{z - \mu_k^{\frac{1}{p}} e^{i\frac{2t\pi}{p}}}$$

$$= cp + \sum_{k=1}^{n-c}\sum_{t=0}^{p-1}\left(\frac{\mu_k^{\frac{1}{p}} e^{i\frac{2t\pi}{p}}}{z - \mu_k^{\frac{1}{p}} e^{i\frac{2t\pi}{p}}} + 1\right) = cp + (n-c)p + \sum_{k=1}^{n-c}\sum_{t=0}^{p-1}\frac{\mu_k^{\frac{1}{p}} e^{i\frac{2t\pi}{p}}}{z - \mu_k^{\frac{1}{p}} e^{i\frac{2t\pi}{p}}}$$

Therefore,

$$\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)} - pn = \sum_{k=1}^{n-c}\sum_{t=0}^{p-1}\frac{\mu_k^{\frac{1}{p}} e^{i\frac{2t\pi}{p}}}{z - \mu_k^{\frac{1}{p}} e^{i\frac{2t\pi}{p}}}$$

Let $\Gamma = \Gamma_R \cup L_1 \cup S_r \cup L_2$ be the positively (i.e., counterclockwise) oriented piecewise smooth Jordan curve (see Fig. (1)), where $R > \max\left\{\mu_1, \mu_1^{\frac{1}{p}}\right\}, 0 < r <$

$\min\left\{\mu_{n-c}^{\frac{1}{p}}, \mu_{n-c}\right\}$, Γ_R is the contour $\left\{z(\theta) = Re^{i\theta}, -\frac{\pi}{p} \le \theta \le \frac{\pi}{p}\right\}$, $L_1$is the line $\left\{z(\theta) = \rho e^{i\theta},\ r \le \rho \le R,\ \theta = \frac{\pi}{p}\right\}$, S_r is the curve $\left\{z(\theta) = re^{i\theta}, -\frac{\pi}{p} \le \theta \le \frac{\pi}{p}\right\}$, and L_2 is the line $\{z(\theta) = \rho e^{i\theta}, r \le \rho \le R, \theta = -\frac{\pi}{p}\}$. Then the points $\mu_1^{\frac{1}{p}}, \mu_2^{\frac{1}{p}}, \dots, \mu_{n-c}^{\frac{1}{p}}$ are all in the interior of the contour Γ, and the points $0, \mu_1^{\frac{1}{p}} e^{i\frac{2t\pi}{p}}, \mu_2^{\frac{1}{p}} e^{i\frac{2t\pi}{p}}, \dots, \mu_{n-c}^{\frac{1}{p}} e^{i\frac{2t\pi}{p}}, t = 1, 2, \dots, p-1$, are all in the exterior of the contour Γ. It follows from Lemmas (6.2.2) and (6.2.3) that

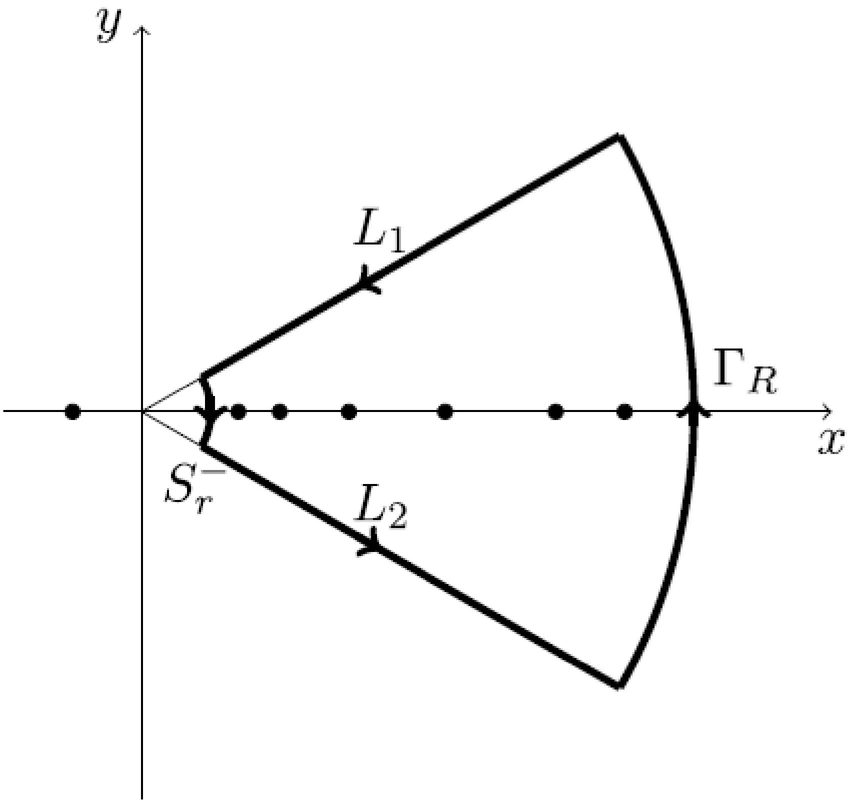


Fig. (1)[260]: The contour Γ in Theorem (6.2.6).

$$\frac{1}{2\pi i}\int_{\Gamma}\left[\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)} - pn\right]dz = \frac{1}{2\pi i}\int_{\Gamma}\sum_{k=1}^{n-c}\sum_{t=0}^{p-1}\frac{\mu_k^{\frac{1}{p}} e^{i\frac{2t\pi}{p}}}{z - \mu_k^{\frac{1}{p}} e^{i\frac{2t\pi}{p}}}dz = \frac{1}{2\pi i}\int_{\Gamma}\sum_{k=1}^{n-c}\frac{\mu_k^{\frac{1}{p}}}{z - \mu_k^{\frac{1}{p}}}dz$$

$$= \sum_{k=1}^{n-c}\frac{1}{2\pi i}\int_{\Gamma}\frac{\mu_k^{\frac{1}{p}}}{z - \mu_k^{\frac{1}{p}}}dz = \sum_{k=1}^{n-c}\mu_k^{\frac{1}{p}} = LEL_{\frac{1}{p}}(G).$$

Since the value of the integral

$$\frac{1}{2\pi i}\int_{\Gamma}\left[\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)} - pn\right]dz$$

is independent of r and R, we obtain that

$$LEL_\alpha(G) = \frac{1}{2\pi i} \lim_{\substack{R\to+\infty \\ r\to 0^+}} \int_{\Gamma} \left[\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)} - pn\right] dz$$

$$= \frac{1}{2\pi i} \lim_{\substack{R\to+\infty \\ r\to 0^+}} \left[\int_{\Gamma_R} \left(\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)} - pn\right) dz + \int_{L_1} \left(\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)} - pn\right) dz \right.$$

$$\left. + \int_{S_r^-} \left(\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)} - pn\right) dz + \int_{L_2} \left(\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)} - pn\right) dz\right]$$

where S_r^- is the same curve as S_r but has clockwise orientation. Suppose that $z = \rho(cos\,\theta + i\,sin\,\theta)$, where $\rho > 0$. Then

$$\left|1 - \frac{\mu_k^{\frac{1}{p}} e^{i\frac{2t\pi}{p}}}{z}\right| = \frac{|z - \mu_k^{\frac{1}{p}} e^{i\frac{2t\pi}{p}}|}{|z|} = \frac{|\rho(cos\,\theta + i\,sin\,\theta) - \mu_k^{\frac{1}{p}}(cos\frac{2t\pi}{p} + i\,sin\frac{2t\pi}{p})|}{\rho}$$

$$= \frac{\sqrt{\rho^2 + \mu_k^{\frac{2}{p}} - 2\cos\left(\theta - \frac{2t\pi}{p}\right)\rho\mu_k^{\frac{1}{p}}}}{\rho} \geq \frac{\sqrt{\rho^2 + \mu_k^{\frac{2}{p}} - 2\rho\mu_k^{\frac{1}{p}}}}{\rho} = \frac{|\rho - \mu_k^{\frac{1}{p}}|}{\rho}$$

Thus,

$$\left|z\left[\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)} - pn\right]\right| \leq \sum_{k=1}^{n-c}\sum_{t=0}^{p-1} \left|\frac{z\mu_k^{\frac{1}{p}} e^{i\frac{2t\pi}{p}}}{z - \mu_k^{\frac{1}{p}} e^{i\frac{2t\pi}{p}}}\right| = \sum_{k=1}^{n-c}\sum_{t=0}^{p-1} \frac{\left|\mu_k^{\frac{1}{p}} e^{i\frac{2t\pi}{p}}\right|}{\left|1 - \frac{\mu_k^{\frac{1}{p}} e^{i\frac{2t\pi}{p}}}{z}\right|} \leq \sum_{k=1}^{n-c}\sum_{t=0}^{p-1} \frac{\rho\mu_k^{\frac{1}{p}}}{\left|\rho - \mu_k^{\frac{1}{p}}\right|}$$

Since

$$\sum_{k=1}^{n-c}\sum_{t=0}^{p-1} \frac{\rho\mu_k^{\frac{1}{p}}}{\left|\rho - \mu_k^{\frac{1}{p}}\right|} \to 0, \quad \text{for } \rho \to 0,$$

by Lemma (6.2.4) we have

$$\int_{S_r^-} \left(\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)} - pn\right) dz \to 0, \qquad for\ \rho \to 0.$$

Suppose that $\omega_{kt} = \mu_k^{\frac{1}{p}} e^{i\frac{2t\pi}{p}}$. Then $|\omega_{k0}| = |\omega_{k1}| = \cdots = |\omega_{k(p-1)}| = \mu_k^{\frac{1}{p}}$. Thus,

$$\left|z\sum_{t=0}^{p-1}\frac{\mu_k^{\frac{1}{p}}e^{i\frac{2t\pi}{p}}}{z-\mu_k^{\frac{1}{p}}e^{i\frac{2t\pi}{p}}}\right|$$

$$=\left|z\sum_{t=0}^{p-1}\frac{\mu_k^{\frac{1}{p}}e^{i\frac{2t\pi}{p}}\left(z^{p-1}+z^{p-2}\mu_k^{\frac{1}{p}}e^{i\frac{2t\pi}{p}}+\cdots+z(\mu_k^{\frac{1}{p}}e^{i\frac{2t\pi}{p}})^{p-2}+(\mu_k^{\frac{1}{p}}e^{i\frac{2t\pi}{p}})^{p-1}\right)}{z^p-\mu_k}\right|$$

$$=\left|z\sum_{t=0}^{p-1}\frac{\omega_{kt}(z^{p-1}+z^{p-2}\omega_{kt}+\cdots+z\omega_{kt}^{p-2}+\omega_{kt}^{p-1})}{z^p-\mu_k}\right|$$

$$=\left|\sum_{t=0}^{p-1}\frac{\omega_{kt}+\frac{\omega_{kt}^2}{z}+\cdots+\frac{\omega_{kt}^{p-1}}{z^{p-2}}+\frac{\omega_{kt}^{p}}{z^{p-1}}}{1-\frac{\mu_k}{z^p}}\right|.$$

Obviously, there exists $N_k > 0$, for $k \in \{1,2,\ldots,n-c\}$, such that $|1-\frac{\mu_k}{z^p}| \geq \frac{1}{2}$ for $|z| > N_k$. For any $\varepsilon > 0$, there exists $M_k > 0$, for $k \in \{1,2,\ldots,n-c\}$, such that

$$\left|\sum_{t=0}^{p-1}\left(\frac{\omega_{kt}^2}{z}+\cdots+\frac{\omega_{kt}^{p-1}}{z^{p-2}}+\frac{\omega_{kt}^{p}}{z^{p-1}}\right)\right|<\frac{\varepsilon}{2n}$$

for $|z| > M_k$. Noting that $\sum_{t=0}^{p-1}\omega_{kt}^r = 0$ unless $r = p$. Therefore, for any $\varepsilon > 0$, there exists $N = \max\{N_1, N_2, \ldots, N_{n-c}, M_1, M_2, \ldots, M_{n-c}\}$ such that

$$\left|z\left[\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)}-pn\right]\right| = \left|\sum_{k=1}^{n-c} z\sum_{t=0}^{p-1}\frac{\mu_k^{\frac{1}{p}}e^{i\frac{2t\pi}{p}}}{z-\mu_k^{\frac{1}{p}}e^{i\frac{2t\pi}{p}}}\right|$$

$$=\left|\sum_{k=1}^{n-c}\sum_{t=0}^{p-1}\frac{\omega_{kt}+\frac{\omega_{kt}^2}{z}+\cdots+\frac{\omega_{kt}^{p-1}}{z^{p-2}}+\frac{\omega_{kt}^{p}}{z^{p-1}}}{1-\frac{\mu_k}{z^p}}\right|$$

$$\leq 2\left|\sum_{k=1}^{n-c}\sum_{t=0}^{p-1}\omega_{kt}\right| + 2\sum_{k=1}^{n-c}\left|\sum_{t=0}^{p-1}\left(\frac{\omega_{kt}^2}{z}+\cdots+\frac{\omega_{kt}^{p-1}}{z^{p-2}}+\frac{\omega_{kt}^{p}}{z^{p-1}}\right)\right|$$

$$< 0 + 2\sum_{k=1}^{n-c}\frac{\varepsilon}{2n} < \varepsilon$$

for $|z| > N$. By Lemma (6.2.5), it can be obtained that for any $\varepsilon > 0$ there exists $N = \max\{N_1, N_2, \ldots, N_{n-c}, M_1, M_2, \ldots, M_{n-c}\}$ such that

$$\int_{\Gamma_R}\left(\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)}-pn\right)dz \leq \frac{2\pi R}{p}\max_{z\in\Gamma_R}\left|\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)}-pn\right| = \frac{2\pi}{p}\max_{z\in\Gamma_R}\left|z\left[\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)}-pn\right]\right|$$

$$< \frac{2\pi}{p}\varepsilon$$

for $|z| > N$. In other words,

$$\int_{\Gamma_R} \left(\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)} - pn\right) dz \to 0, \qquad \text{for} |z| \to +\infty.$$

Consequently, we have

$$LEL_\alpha(G) = \frac{1}{2\pi i} \lim_{\substack{R\to+\infty \\ r\to0^+}} \left[\int_{\Gamma_R} \left(\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)} - pn\right) dz + \int_{L_1} \left(\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)} - pn\right) dz \right.$$
$$\left. + \int_{S_r^-} \left(\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)} - pn\right) dz + \int_{L_2} \left(\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)} - pn\right) dz \right]$$
$$= \frac{1}{2\pi i} \lim_{\substack{R\to+\infty \\ r\to0^+}} \left[\int_{L_1} \left(\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)} - pn\right) dz + \int_{L_2} \left(\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)} - pn\right) dz \right]$$
$$= \frac{1}{2\pi i} \lim_{\substack{R\to+\infty \\ r\to0^+}} \left[\int_R^r \left(\frac{p(\rho e^{i\frac{\pi}{p}})^p \phi_L'(G,(\rho e^{i\frac{\pi}{p}})^p)}{\phi_L(G,(\rho e^{i\frac{\pi}{p}})^p)} - pn\right) d(\rho e^{i\frac{\pi}{p}}) \right.$$
$$\left. + \int_r^R \left(\frac{p(\rho e^{-i\frac{\pi}{p}})^p \phi_L'(G,(\rho e^{-i\frac{\pi}{p}})^p)}{\phi_L(G,(\rho e^{-i\frac{\pi}{p}})^p)} - pn\right) d(\rho e^{-i\frac{\pi}{p}}) \right]$$
$$= \frac{1}{2\pi i} \lim_{\substack{R\to+\infty \\ r\to0^+}} \left[\int_R^r \left(-\frac{p\rho^p \phi_L'(G,-\rho^p)}{\phi_L(G,-\rho^p)} - pn\right) e^{i\frac{\pi}{p}} d\rho \right.$$
$$\left. + \int_r^R \left(-\frac{p\rho^p \phi_L'(G,-\rho^p)}{\phi_L(G,-\rho^p)} - pn\right) e^{-i\frac{\pi}{p}} d\rho \right]$$
$$= \frac{1}{2\pi i} \lim_{\substack{R\to+\infty \\ r\to0^+}} \left[\int_r^R \left(\frac{p\rho^p \phi_L'(G,-\rho^p)}{\phi_L(G,-\rho^p)} - pn\right) (cos\frac{\pi}{p} + isin\frac{\pi}{p}) d\rho \right.$$
$$\left. - \int_r^R \left(\frac{p\rho^p \phi_L'(G,-\rho^p)}{\phi_L(G,-\rho^p)} - pn\right) (cos(-\frac{\pi}{p}) + isin(-\frac{\pi}{p})) d\rho \right]$$
$$= \frac{1}{2\pi i} \lim_{\substack{R\to+\infty \\ r\to0^+}} \int_r^R \left(\frac{p\rho^p \phi_L'(G,-\rho^p)}{\phi_L(G,-\rho^p)} + pn\right) \cdot 2isin\frac{\pi}{p} d\rho$$
$$= \frac{1}{\pi} \int_0^{+\infty} \left(\frac{px^p \phi_L'(G,-x^p)}{\phi_L(G,-x^p)} + pn\right) \cdot sin\frac{\pi}{p} dx.$$

If G has n components, which means that $\mu_1 = \cdots = \mu_n = 0$, then $\phi(G,x) = x_n$. Thus, we have

$$\frac{px^p \phi_L'(G,-x^p)}{\phi_L(G,-x^p)} + pn = \frac{px^p n(-x^p)^{n-1}}{(-x^p)^n} + pn = 0$$

and $LEL_{\frac{1}{p}}(G) = \sum_{k=1}^{n} \mu_k^{\frac{1}{p}} = 0$. Therefore,

$$LEL_\alpha(G) = \frac{1}{\pi}\int_0^{+\infty} \left(\frac{px^p\phi_L'(G,-x^p)}{\phi_L(G,-x^p)} + pn\right) \cdot sin\frac{\pi}{p} dx.$$

This completes the proof.

Clearly, it is easy to obtain the following result from Theorem (6.2.6).

Corollary (6.2.7)[260]: Let G be a graph of order n, and $\phi_L(G,x)$ the characteristic polynomial of the Laplacian matrix $L(G)$. Then the Laplacian-energy-like invariant of G can be given by the following integral formula

$$LEL(G) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \left(\frac{x^2\phi_L'(G,-x^2)}{\phi_L(G,-x^2)} + n\right) dx.$$

Proof: By Theorem (6.2.6), it can be obtained that

$$LEL(G) = LEL_{\frac{1}{2}}(G) = \frac{1}{\pi}\int_0^{+\infty} \left(\frac{2x^2\phi_L'(G,-x^2)}{\phi_L(G,-x^2)} + 2n\right) sin\frac{\pi}{2} dx$$

$$= \frac{2}{\pi}\int_0^{+\infty} \left(\frac{x^2\phi_L'(G,-x^2)}{\phi_L(G,-x^2)} + n\right) dx = \frac{1}{\pi}\int_{-\infty}^{+\infty} \left(\frac{x^2\phi_L'(G,-x^2)}{\phi_L(G,-x^2)} + n\right) dx,$$

which completes the proof.

Let G_1 and G_2 be two graphs of same order. The Coulson–Jacobs formula (see [261], [264]) gives the difference of their energies, that is

$$E(G_1) - E(G_2) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \ln\left|\frac{\phi(G_1,ix)}{\phi(G_2,ix)}\right| dx$$

where $\phi(G,x)$ is the characteristic polynomial of the matrix $A(G)$. Similar to this, we obtain the following theorem on the difference of the general Laplacian-energy-like invariant of two graphs.

Theorem (6.2.8)[260]: Let G_1 and G_2 be two graphs of equal order. Then

$$LEL_{\frac{1}{p}}(G_1) - LEL_{\frac{1}{p}}(G_2) = \frac{1}{\pi}\int_0^{+\infty} \ln\left|\frac{\phi_L(G_1,x^p)}{\phi_L(G_2,x^p)}\right| \cdot sin\frac{\pi}{p} dx, p \in \mathbb{Z}^+\backslash\{1\},$$

where $\phi_L(G,x)$ is the characteristic polynomial of the Laplacian matrix $L(G)$.

Proof. By Theorem (6.2.6), it can be obtained that

$$LEL_{\frac{1}{p}}(G_1) - LEL_{\frac{1}{p}}(G_2) = \frac{1}{\pi}\int_0^{+\infty} \left(\frac{px^p\phi_L'(G_1,-x^p)}{\phi_L(G_1,-x^p)} - \frac{px^p\phi_L'(G_2,-x^p)}{\phi_L(G_2,-x^p)}\right) \cdot sin\frac{\pi}{p} dx$$

$$= -\frac{1}{\pi}\int_0^{+\infty} x\left(\frac{\phi_L'(G_1,-x^p)}{\phi_L(G_1,-x^p)} - \frac{\phi_L'(G_2,-x^p)}{\phi_L(G_2,-x^p)}\right) \cdot sin\frac{\pi}{p} d(-x^p)$$

$$= -\frac{1}{\pi}\int_0^{+\infty} x sin\frac{\pi}{p} d \ln\frac{\phi_L(G_1,-x^p)}{\phi_L(G_2,-x^p)}$$

$$= -\frac{\sin\frac{\pi}{p}}{\pi}\left(x\ln\left|\frac{\phi_L(G_1,-x^p)}{\phi_L(G_2,-x^p)}\right|\Bigg|_0^{+\infty} - \int_0^{+\infty} \ln\left|\frac{\phi_L(G_1,-x^p)}{\phi_L(G_2,-x^p)}\right| dx \right)$$

and

$$\lim_{x\to+\infty}\left(\frac{\phi_L(G_1,-x^p)}{\phi_L(G_2,-x^p)}\right)^x = \lim_{x\to+\infty}\left(1+\frac{\phi_L(G_1,-x^p)-\phi_L(G_2,-x^p)}{\phi_L(G_2,-x^p)}\right)^x$$

$$= \lim_{x\to+\infty}\left(1+\frac{\phi_L(G_1,-x^p)-\phi_L(G_2,-x^p)}{\phi_L(G_2,-x^p)}\right)^{\frac{\phi_L(G_2,-x^p)[\phi_L(G_1,-x^p)-\phi_L(G_2,-x^p)]x}{[\phi_L(G_1,-x^p)-\phi_L(G_2,-x^p)]\phi_L(G_2,-x^p)}}$$

$$= e^0 = 1,$$

since the degree of $[\phi_L(G_1,-x^p)-\phi_L(G_2,-x^p)]x$ is less than the degree of $\phi_L\ (G_2,-x^p)$. Thus,

$$\lim_{x\to+\infty} x\ln\left|\frac{\phi_L(G_1,-x^p)}{\phi_L(G_2,-x^p)}\right| = 0$$

Suppose that

$$\phi_L(G_j,x) = x\prod_{k=1}^{n-1}(x - \mu_k(G_j)),\ \ j = 1,2,$$

where $\mu_1(G_j),\dots,\mu_n(G_j)$ are the Laplacian eigenvalues of G_j ($j = 1,2$). Then

$$\frac{\phi_L(G_1,-x^p)}{\phi_L(G_2,-x^p)} = \prod_{k=1}^{n-1}\frac{-x^p-\mu_k(G_1)}{-x^p-\mu_k(G_2)}$$

Since $\lim_{x\to 0} x\ln x = 0$, we have

$$\lim_{x\to 0} x\ln\left|\frac{\phi_L(G_1,-x^p)}{\phi_L(G_2,-x^p)}\right| = \lim_{x\to 0} x\ln\prod_{k=1}^{n-1}\frac{-x^p-\mu_k(G_1)}{-x^p-\mu_k(G_2)} = 0$$

Therefore,

$$LEL_{\frac{1}{p}}(G_1) - LEL_{\frac{1}{p}}(G_2) = \frac{1}{\pi}\int_0^{+\infty} \ln\left|\frac{\phi_L(G_1,-x^p)}{\phi_L(G_2,-x^p)}\right|\cdot \sin\frac{\pi}{p}dx$$

This completes the proof.

Corollary (6.2.9)[260]: Let G be a simple graph of order n, and $\phi_L(G,x) = \sum_{k=0}^{n} a_k x^{n-k}$. Then

$$LEL_{\frac{1}{p}}(G) = \frac{1}{2\pi}\int_0^{+\infty} x^{-2}\ln\left(\sum_{k=0}^{n}(-1)^k a_k x^{pk}\right)^2\cdot \sin\frac{\pi}{p}dx\,, p\ \in \mathbb{Z}^+\backslash\{1\}.$$

Proof: Noting that $(\overline{K_n},x) = x^n$, by Theorem (6.2.8) we have

$$LEL_{\frac{1}{p}}(G) = \frac{1}{\pi}\int_0^{+\infty} \ln\left|\sum_{k=0}^{n} a_k(-x^{-p})^{-k}\right|\cdot \sin\frac{\pi}{p}dx = \frac{1}{\pi}\int_{+\infty}^{0} \ln\left|\sum_{k=0}^{n} a_k(-x^{-p})^{-k}\right|\cdot \sin\frac{\pi}{p}d(x^{-1})$$

$$= \frac{1}{\pi}\int_0^{+\infty} x^{-2}\ln\left|\sum_{k=0}^{n}(-1)^k a_k x^{pk}\right|\cdot \sin\frac{\pi}{p}dx$$

$$= \frac{1}{2\pi} \int_0^{+\infty} x^{-2} \ln \left(\sum_{k=0}^{n} (-1)^k a_k x^{pk} \right)^2 \cdot sin \frac{\pi}{p} dx$$

Thus, the proof is complete.

We first extend the concept of the general Laplacian-energy-like invariant of graphs to complex polynomials.

Definition (6.2.10)[260]: Let

$$\phi(z) = \sum_{k=0}^{n} a_k z^{n-k} = a_0 \prod_{k=1}^{n} (z - z_k)$$

be a complex polynomial of degree n and α a real number. The general energy of $\phi(z)$, denoted by $E_\alpha(\phi(z))$, is defined as $\sum_{z_k \neq 0} |z_k|^\alpha$ when there exists $i_0 \in \{1, 2, \ldots, n\}$ such that $z_{i_0} \neq 0$, and 0 when $z_1 = \cdots = z_n = 0$.

By an analogous argument in the proof of Theorem (6.2.6), we can obtain the following result on the general energy of polynomials for $\alpha = 1/p$ with $p \in \mathbb{Z}^+ \backslash \{1\}$.

Theorem (6.2.11)[260]: Let $\phi(z)$ be a monic polynomial of degree n, whose roots are all non-negative real numbers, and $\alpha = 1/p$ a number with $p \in \mathbb{Z}^+ \backslash \{1\}$. Then the general energy of $\phi(z)$ can be given by the following integral formula

$$LEL_\alpha(G) = \frac{1}{\pi} \int_0^{+\infty} \left(\frac{px^p \phi'(-x^p)}{\phi(-x^p)} + pn \right) \cdot sin \frac{\pi}{p} dx$$

As an extension of the concept of graph energy, the energy $E(M)$ of a real $n \times m$ matrix M is defined by Nikiforov [257] as the sum of its singular values, which are the square roots of the eigenvalues of the square matrix M^T, where M^T is the transpose of M. Let $\sigma_1(M), \sigma_n(M), \ldots, \sigma_n(M)$ be the singular values of . Then

$$E(M) = \sum_{k=1}^{n} \sigma_k(M).$$

The normalized incidence energy $NIE(G)$ of G, introduced by Cheng and Liu in [240], is the energy of the matrix $\hat{I}(G) = D^{-\frac{1}{2}}(G) I(G)$, where $I(G)$ is the incidence matrix of $G, D^{-\frac{1}{2}}(G)$ is the diagonal matrix with entries $D^{-\frac{1}{2}}(G)(k,k) = 1/\sqrt{d_k}$ if $d_k \neq 0$ and $D^{-\frac{1}{2}}(G)(k,k) = 0$ otherwise. Then

$$NIE(G) = \sum_{k=1}^{n} \sigma_k(\hat{I}(G)) = \sum_{k=1}^{n} \sqrt{\lambda_k(\hat{I}(G)\hat{I}(G)^T)},$$

where $\lambda_k(\hat{I}(G)\hat{I}(G)^T)(k = 1, \ldots, n)$ are the eigenvalues of the matrix $\hat{I}(G)\hat{I}(G)^T$. Obviously, $\hat{I}(G)\hat{I}(G)^T$ is a positive semi-definite matrix.

Cheng and Liu [240] gave an integral formula for the normalized incidence energy of graphs. We find that their result is an immediate consequence of Theorem (6.2.11).

Corollary (6.2.12)[260]: (See Cheng and Liu [240].) Let G be a graph of order n, and $\phi(x)$ the characteristic polynomial of the matrix $\hat{I}(G)\hat{I}(G)^T$. Then the normalized incidence energy of G can be given by the following integral formula

$$NIE(G) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[2n \;-\; ix\frac{f'(ix)}{f(ix)}\right]dx,$$

where $f(x) \;=\; \phi(x^2)$.

Proof: Clearly $\hat{I}(G)\hat{I}(G)^T$ is a positive semi-definite matrix, and all the roots of $\phi(x)$ are nonnegative. Note that

$$f'(ix) \;=\; 2\cdot(ix)\phi'((ix)^2) \;=\; 2ix\phi'(-x^2)$$

and

$$2n - ix\frac{f'(ix)}{f(ix)} = \; 2n - ix\frac{2ix\phi'(-x^2)}{\phi(-x^2)} = \; 2[n + \frac{x^2\phi'(-x^2)}{\phi(-x^2)}].$$

By Theorem (6.2.11), it can be obtained that

$$NIE(G) \; = E_{\frac{1}{2}}(\phi(x)) = \frac{1}{\pi}\int_{0}^{+\infty}\left[\frac{2x^2\phi'(-x^2)}{\phi(-x^2)} + 2n\right]\sin\frac{\pi}{2}dx = \frac{2}{\pi}\int_{0}^{+\infty}\left[\frac{x^2\phi'(-x^2)}{\phi(-x^2)} + n\right]dx$$

$$= \frac{1}{\pi}\int_{-\infty}^{+\infty}\left[\frac{x^2\phi'(-x^2)}{\phi(-x^2)} + n\right]dx = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\left[\frac{x^2\phi'(-x^2)}{\phi(-x^2)} + n\right]dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[2n - ix\frac{f'(ix)}{f(ix)}\right]dx$$

Thus, the proof is complete.

By assigning an arbitrary orientation to the edges of G with vertex set $V(G) = \{v_1, \ldots, v_n\}$, the vertex-arc incidence matrix $S(\vec{G}) \;=\; (s_{ie})$ of $\vec{G}$ is defined as

$$s_{ie} = \begin{cases} 1, & \text{if } v_i \text{ is the head of } e; \\ -1, & \text{if } v_i \text{ is the tail of } e; \\ 0, & \text{otherwise.} \end{cases}$$

The normalized oriented incidence matrix of $\vec{G}$, denoted by $S'(\vec{G})$, is defined as $S'(\vec{G}) \;=\; D^{-\frac{1}{2}} - (G)S(\vec{G})$. The normalized Laplacian matrix of G, denoted by $NL(G) \;=\; (l_{ij})$, is the matrix with entries

$$l_{ij} = \begin{cases} 1, & \text{if } i \;=\; j \text{ and } d_i \neq 0; \\ -\dfrac{1}{\sqrt{d_i d_j}}, & \text{if } v_i \text{ and } v_j \text{ are adjacent in } G; \\ 0, & otherwise. \end{cases}$$

Clearly, $NL(G) \;=\; S'(\vec{G})S'(\vec{G})^T$, where $\vec{G}$ is an arbitrary oriented graph of G. The Laplacian incidence energy $LIE(G)$ of G, introduced by Shi and Wang in [265], is defined as

$$LIE(G) \; = \sum_{k=1}^{n}\sigma_k(S'(\vec{G})) \; = \sum_{k=1}^{n}\sqrt{\lambda_k(NL(G))},$$

where λ_k $(NL(G))$ $(k \;=\; 1, \ldots, n)$ are the eigenvalues of $NL(G)$.

Shi and Wang [265] gave an integral formula for Laplacian incidence energy of graphs. Their result is also an immediate consequence of Theorem (6.2.11).

Corollary (6.2.13)[260]: (See Shi and Wang [265].) Let G be a graph of order n, and $\phi(x)$ the characteristic polynomial of the normalized Laplacian matrix $NL(G)$ of G. Then the Laplacian incidence energy of G can be given by the following integral formula

$$LIE\ (G)\ = \frac{1}{\pi}\int_{-\infty}^{+\infty}\left[n+\frac{x^2\phi'(-x^2)}{\phi(-x^2)}\right]dx$$

We omit the proof of this corollary here.

Section (6.3): General Laplacian-Energy-Like Invariant of Graphs

We only consider simple graphs. See [124].

Let G be a graph of order n. The spectrum of G consists of the eigenvalues $\lambda_1 \geq \lambda_1 \geq \cdots \geq \lambda_n$ of the adjacency matrix $A(G)$ of G, which are called the eigenvalues of G. It is well known that $\lambda_1 = \max\{|\lambda_1|, \cdots, |\lambda_n|\}$ The Laplacian matrix of G is the matrix $L(G) = D(G) - A(G)$, where $D(G) = diag(d_1, d_2, \cdots, d_n)$ is the diagonal matrix of vertex degrees of G. The Laplacian eigenvalues of G are the eigenvalues of $L(G)$, denoted by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. As we all know, $L(G)$ is a positive semi-definite symmetric matrix and $\mu_n =\ 0$.

The *energy* $E(G)$ of G is defined as the sum of the absolute values of the eigenvalues of G, which is an invariant related to total π-electron energy [267]. Many mathematicians and chemists have done lots of work in the field of the theory of graph energy (see [273]). In 1940, Coulson [247] obtained an important integral formula which makes it possible to calculate the energy of a graph without knowing its spectrum. For a graph G on n vertices, its energy is

$$E(G) = \frac{1}{\pi}\int_{-\infty}^{\infty}\left[n - \frac{ix\phi_A'(G.ix)}{\phi_A(G,ix)}\right]dx$$

where $\phi_A(G,x)$ is the characteristic polynomial of $A(G)$ (the *characteristic polynomial* of G). This formula is called the Coulson integral formula, and has many applications in the theory of graph energy (see [273]).

Moreover, Gutman and Zhou [129] defined the Laplacian energy of G as

$$LE(G) = \sum_{k=1}^{n}\left|\mu_k - \frac{2m}{n}\right|,$$

where n and m are the number of vertices and edges of G, respectively. At the same time, Liu and Liu [121] defined the Laplacian-energy-like invariant of G as

$$LEL(G) = \sum_{k=1}^{n}\sqrt{\mu_k}$$

This invariant has many similar properties as the energy of graphs. For results and problems on these two invariants, see [272], [273].

In [147], Zhou studied the sum of powers of the Laplacian eigenvalues of graphs, which can be regarded as a generalization of the Laplacian-energy-like invariant and is called the general Laplacian-energy-like invariant of graphs in [260].

Definition (6.3.1)[269]: Let G be a graph of order $n, \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ the Laplacian eigenvalues of G and α a real number. The general Laplacian-energy-like invariant of G, denoted by $LEL_\alpha(G)$, is defined as $\sum_{\mu_k \neq 0}\mu_k^\alpha$ when $\mu_1 \neq 0$, and 0 when $\mu_1 = 0$ Qiao et al. [260] obtained an integral formula for general Laplacian-energy-like invariant in the case

that $\alpha = 1/p, p \in \mathbb{Z}^+\{1\}$, gave an extension of the general Laplacian-energylike invariant of graphs to complex polynomials and obtained an integral formula for it.
Theorem (6.3.2)[269]: ([260])**.** Let G be a graph of order $n, \phi_L(G, x)$ the characteristic polynomial of the Laplacian matrix $L(G)$ of G, and $\alpha = 1/p$ with $p \in \mathbb{Z}^+\backslash\{1\}$.
Then the general Laplacian-energy-like invariant of G can be given by the following integral formula

$$LEL_\alpha(G) = \frac{p}{\pi}\int_0^{+\infty}\left(\frac{x^p\phi_L'(G, x^p)}{\phi_L(G - x^{pp})}\right).\sin\frac{\pi}{p}dx$$

Definition (6.3.3)[269]: ([260]). Let

$$\phi(z) = \sum_{k=0}^{n} a_k z^{n-k} = a_0 \prod_{k=1}^{n}(z - z_k)$$

Be a complex polynomial of degree n and α a real number. The general energy of $\phi(z)$, denoted by $E_\alpha(\phi(z))$, is defined as $\sum_{z_k \neq 0}|z_k|^\alpha$ when there exist $i_0 \in \{1,2,\cdots,n\}$ that $z_{i_0} \neq 0$, and 0 when $z_1 = \cdots = z_n = 0$
Theorem (6.3.4)[269]: ([260])**.** Let $\phi(z)$ be a monic polynomial of degree n, whose roots are all nonnegative real numbers, and $\alpha = 1/p$ with $p \in \mathbb{Z}^+\backslash\{1\}$. Then the general energy of $\phi(z)$ can be given by the following integral formula

$$E_\alpha(\phi(z)) = \frac{p}{\pi}\int_0^{+\infty}\left(\frac{x^p\phi'(-x^p)}{\phi(-x^p)} + n\right).\sin\frac{\pi}{p}dx$$

The two following concepts are regarded as generalizations of graph energy and Laplacian graph energy, respectively.
Definition (6.3.5)[269]: Let *G* be a graph of order $n, \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ the eigenvalues of *G* and α a real number. The general energy of *G*, denoted by $E_\alpha(G)$, is defined as $\sum_{\lambda_k \neq 0}|\lambda_k|^\alpha$ when $\lambda_1 \neq 0$, and 0 when $\lambda_1 = 0$.
Definition (6.3.6)[269]: Let *G* be a graph, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ the Laplacian eigenvalues of *G* and α a real number. The general Laplacian energy of *G* is defined as

$$LE_\alpha(G) = \sum_{\mu_k \neq \frac{2m}{n}} \left|\mu_k - \frac{2m}{n}\right|^\alpha$$

We obtain some Coulson-type integral formulas for the general Laplacian energy-like invariant of graphs and the general energy of polynomials with $\alpha \in \mathbb{Q}$. We present Coulson-type integral formulas for the general energy and general Laplacian energy of graphs with $\alpha \in \mathbb{Q}$, respectively. We also show that our formulas in Theorem (6.3.10) (i) and (iv), Theorem (6.3.11) (i) and (iv) and Theorem (6.3.12) (i) and (iv) hold when α is an irrational number with $0 < |\alpha| < 1$ and do not hold with $|\alpha| > 1$.

We first introduce some basic concepts and results in complex analysis which will be used later.
Let D be a bounded domain. The boundary of D is denoted by ∂D. We need the following simple lemma. The proofs are omitted here.
Lemma (6.3.7)[269]: Let S_r be the are $z(\theta) = a_0 re^{i\theta}, \theta_1 \leq \theta \leq \theta_2$ where a_0 and $r > 0$ are two real numbers. If $f(z)$ is a continuous function on the arc S_r for all small r such that

$$\lim_{r\to 0^+}\max_{\theta\in|\theta_1,\theta_2|}\left|re^{i\theta}\,f\left(a_0+re^{i\theta}\right)-\lambda\right|=0$$

Then

$$\lim_{r\to 0^+}\int_{S_r} f(z)dz=i(\theta_2-\theta_1)\lambda.$$

Suppose $f(z) = 1$. By Cauchy's Theorem and Integral Formula, we get

$$f(z_0)=\frac{1}{2\pi i}\int_{\partial D}\frac{d\zeta}{\zeta-z_0}\begin{cases}1, & if \quad z_0\in int\,(\partial)\\ 0, & if \quad z_0\in ext\,(\partial D)\end{cases}$$

Where $z_0 \in int\,(\partial D)$ mean that z_0 lies in the interior of ∂D and in the exterior of ∂D, respectively.

Let $\phi(z)=\sum_{k=0}^{n}a_k z^{n-k}=a_0\prod_{k=1}^{n}(z-z_k)$ be a complex polynomial of degree n. By direct computing, we get

$$\frac{z\phi'(z)}{\phi(z)}=\sum_{k=1}^{n}\frac{z}{z-z_k}=n+\sum_{k=1}^{n}\frac{z_k}{z-z_k}$$

That is

$$\frac{z\phi'(z)}{\phi(z)}-n=\sum_{k=1}^{n}\frac{z_k}{z-z_k}$$

If $z_1, z_2, \cdots, z_n \in int\,(\partial D)$, then we have

$$\frac{1}{\pi i}\int_{\partial D}\left(\frac{z\phi'(z)}{\phi(z)}-n\right)dz=\frac{1}{2\pi i}\int_{\partial D}\sum_{k=1}^{n}\frac{z_k}{z-z_k}dz=\sum_{k=1}^{n}z_k$$

Suppose that $\phi(z)=(z-z_1)(z-z_2)\cdots(z-z_n)$. Then

$$\phi\left(\sqrt{z}\right)\phi\left(-\sqrt{-z}\right)=\prod_{k=1}^{n}\left(\sqrt{z}-z_k\right)\left(-\sqrt{z}-z_k\right)$$

$$=\prod_{k=1}^{n}(z-z_k^2)(-1)^n=(-1)^n\prod_{k=1}^{n}(z-z_k^2)$$

Therefore, we have

$$\varphi(z)=(-1)^n\phi\left(\sqrt{z}\right)\phi\left(-\sqrt{z}\right)=\prod_{k=1}^{n}(z-z_k^2).$$

Thus, by Theorem (6.3.4) it is easy to get the following theorem.

Theorem (6.3.8)[269]: Let $\phi(z)$ be a monic polynomial of degree n, whose roots are all non-negative real numbers, and $\alpha = 1$. Then $E_1\left(\phi(z)\right)$ can be given by the following integral formula

$$E_1\left(\phi(z)\right)=\frac{2}{\pi}\int_0^{+\infty}\left(\frac{x^2\varphi'(-x^2)}{\varphi(-x^2)}+n\right)dx$$

Where $\varphi(z)=(-1)^n\phi\left(\sqrt{z}\right)\phi\left(-\sqrt{z}\right)$.

Theorem (6.3.9)[269]: Let G be a graph of order n with $c\ (< n)$ components, $\phi_L(G,x)$ the characteristic polynomial of the Laplacian matrix $L(G)$ of G, and $\alpha \in \mathbb{Q}$. Suppose that

$\varphi_L(G,z) = e^{i(q-1)n\pi}\phi_L\left(G, z^{\frac{1}{q}}e^{-i\frac{2\pi}{q}}\right)\cdots\phi_L\left(G, z^{\frac{1}{q}}e^{-i\frac{2(q-1)\pi}{q}}\right)$. then the general Laplacing – energy - like invariant of G can be given as follows

(i) If $\alpha = \frac{1}{p}, p \in \mathbb{Z}^+\backslash\{1\}$, then

$$LEL_\alpha(G) = \frac{p}{\pi}\int_0^{+\infty}\left(\frac{x^p\phi_L'(G,-x^p)}{\phi_L(G,-x^p)} + n\right).\sin\frac{\pi}{p}dx$$

(ii) if $\alpha = q, q \in \mathbb{Z}^+$, then

$$LELE_\alpha(G) = \frac{2}{\pi}\int_0^{+\infty}\left(\frac{x^2\varphi'(-x^2)}{\varphi(-x^2)} + n\right)dx$$

Where $\varphi(z) = (-1)^n\varphi_L(G,\sqrt{z})\varphi_L(G,-\sqrt{z})$.

(iii) if $\alpha = \frac{q}{p}, p, q \in \mathbb{Z}^+\backslash\{1\}$, then

$$LEL_\alpha(G) = \frac{p}{\pi}\int_0^{+\infty}\left(\frac{x^p\varphi_L'(G,-x^p)}{\varphi_L(G,-x^p)}\right).\sin\frac{\pi}{p}dx$$

(iv) If $\alpha = -\frac{1}{p}, p \in \mathbb{Z}^+\backslash\{1\}$, then

$$LEL_\alpha(G) = \frac{p}{\pi}\int_0^{+\infty}\left(\frac{-x^{-2}\varphi'(-x^{-2})}{\varphi'(-x^{-2})} - c\right)dx,$$

Where $\varphi(z) = (-1)^n\varphi_L(G,\sqrt{z})\varphi_L(G,-\sqrt{z})$

(vi) if $\alpha = -\frac{q}{p}, q \in \mathbb{Z}^+\backslash\{1\}$, then

$$LEL_\alpha(G) = \frac{p}{\pi}\int_0^{+\infty}\left(\frac{-x^{-p}\varphi_L'(G,-x^{-p})}{\varphi_L(G,-x^{-p})}\right).\sin\frac{\pi}{p}dx$$

Proof: (i) this is just the result of Theorem (6.3.2)

Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ be the roots of $\phi_L(G,x)$. Then $\phi_L(G,x) = (x-\mu_1)\cdots(x-\mu_2)$. Therefore, we have

$$\begin{aligned}\varphi_L(G,z) &= e^{i(q-1)n\pi}\phi_L\left(G,z^{\frac{1}{q}}\right)\phi_L\left(G,z^{\frac{1}{q}}e^{-i\frac{2\pi}{q}}\right)\cdots\phi_L\left(G,z^{\frac{1}{q}}e^{-i\frac{2(q-1)\pi}{q}}\right)\\
&= e^{i(q-1)n\pi}\prod_{k=1}^{n}\left(z^{\frac{1}{q}}-\mu_k\right)\left(z^{\frac{1}{q}}e^{-i\frac{2\pi}{q}}-\mu_k\right)\cdots\left(z^{\frac{1}{q}}e^{-i\frac{2(q-1)\pi}{q}}-\mu_k\right)\\
&= e^{i(q-1)n\pi}\prod_{k=1}^{n}\left(z^{\frac{1}{q}}-\mu_k\right)\left(z^{\frac{1}{q}}-\mu_k e^{i\frac{2\pi}{q}}\right)\cdots\left(z^{\frac{1}{q}}\mu_k e^{i\frac{2(q-1)\pi}{q}}\right)e^{-i(q-1)\pi}\\
&= \prod_{k=1}^{n}\left(z-\mu_k^q\right).\end{aligned}$$

By Theorem (6.3.8) we obtain that

$$LEL_\alpha(G) = \sum_{k=1}^{n}\mu_k^q = E_1(\varphi_L(G,z)) = \frac{2}{\pi}\int_0^{+\infty}\left(\frac{x^2\varphi'(-x^2)}{\varphi(-x^2)} + n\right)dx$$

Where $\varphi(z) = (-1)^n\varphi_L(G,\sqrt{z})\varphi_L(G,-\sqrt{z})$

(iii) By Theorem (6.3.4), it is easy to obtain that

$$LEL_\alpha(G)=\sum_{k=1}^{n}\mu_k^{\frac{q}{p}}=\sum_{k=1}^{n}\left(\mu_k^{q}\right)^{\frac{1}{p}}=E_{\frac{1}{p}}\big(\varphi_L(G,z)\big)$$
$$=\frac{p}{\pi}\int_0^{+\infty}\left(\frac{x^p\varphi_L'(G,-x^p)}{\varphi_L(G,-x^p)}+n\right).\sin\frac{\pi}{p}dx$$

(iv) Suppose that $\mu_1\geq\mu_2\geq\cdots\geq\mu_{n-c}>\mu_{n-(c+1)}=\cdots=\mu_n=0$ are the roots of $\phi_L(G,x)$. Thus we can write $\phi_L(G,x)$ as $\phi_L(G,x)=x^c\prod_{k=1}^{n-c}(x-\mu_k)$. Therefore, we obtain that

$$\varphi_L(G,z)=z^{pn}\phi_L\left(G,\frac{1}{z^p}\right)=z^{pn}|.\left(\frac{1}{z^p}\right)^c\prod_{k=1}^{n-c}\left(\frac{1}{z^p}-\mu_k\right)=(-1)^{n-c}\prod_{k=1}^{n-c}(z^p\mu_k-1)$$
$$=(-1)^{n-c}\prod_{k=1}^{n-c}\mu_k\left(z^p-\frac{1}{\mu_k}\right)=(-1)^{n-c}\prod_{k=1}^{n-c}\mu_k\prod_{k=1}^{p-1}\left(z-\mu_k^{-\frac{1}{p}}e^{-i\frac{2t\pi}{p}}\right)$$
$$=(-1)^{n-c}\prod_{k=1}^{n-c}\left[(-1)^{p-1}\prod_{t=0}^{p-1}\left(z\mu_k^{\frac{1}{p}}e^{i\frac{2t\pi}{p}}-1\right)\right]$$

Then

$$\frac{\varphi_L'(G,z)}{\varphi_L(G,z)}=\sum_{k=1}^{n-c}\sum_{t=0}^{p-1}\frac{z}{z-\mu_k^{-\frac{1}{p}}e^{-i\frac{2t\pi}{p}}}=\sum_{k=1}^{n-c}\sum_{t=0}^{p-1}\left(1+\frac{\mu_k^{\frac{1}{p}}e^{-i\frac{2t\pi}{p}}}{z-\mu_k^{-\frac{1}{p}}e^{-i\frac{2t\pi}{p}}}\right)$$
$$=p(n-c)+\sum_{k=1}^{n-c}\sum_{t=0}^{p-1}\frac{\mu_k^{-\frac{1}{p}}e^{-i\frac{2t\pi}{p}}}{z-\mu_k^{-\frac{1}{p}}e^{-i\frac{2t\pi}{p}}}$$

Suppose that $\Gamma=\Gamma_R\cup L_1\cup S_r\cup L_2$ see Figure (1) is a positively (i.e., counter clockwisely) oriented piecewise smooth Jordan curve, where $R>\max\left\{\mu_1,\mu_{n-c}^{-1},\mu_{n-c}^{-\frac{1}{p}}\right\}$, $0<r<\min\left\{\mu_{n-c},\mu_1^{-1},\mu_1^{-\frac{1}{p}}\right\}$, Γ_R is the curve $\left\{z(\theta)=Re^{i\theta},\ -\frac{\pi}{p}\leq\theta\leq\frac{\pi}{p}\right\}$, L_1 is the line $\left\{z(\theta)=pe^{i\theta},r\leq\rho\leq R,\ \theta=\frac{\pi}{p}\right\}$. Then the point $\mu_1^{-\frac{1}{p}},\mu_2^{-\frac{1}{p}},\cdots,\mu_{n-c}^{-\frac{1}{p}}$ are $t=1,2,\dots,p-1$, are all in the exterior of the curve Γ. By Cauchy's Theorem and Integral Formula, we get

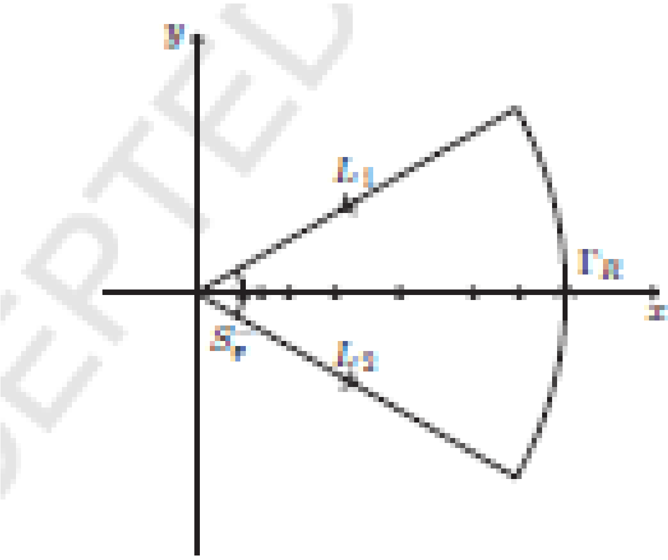


Figure (1): The curve Γ in Corollary (6.3.13) (iv).

$$\frac{1}{2\pi i}\int_{\Gamma}\left[\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)}-p(n-c)\right]dz=\frac{1}{2\pi i}\int_{\Gamma}\sum_{k=1}^{n-c}\sum_{t=0}^{p-1}\frac{\mu_k^{-\frac{1}{p}}e^{-i\frac{2t\pi}{p}}}{z-\mu_k^{-\frac{1}{p}}e^{-i\frac{2t\pi}{p}}}dz$$

$$=\frac{1}{2\pi i}\int_{\Gamma}\sum_{k=1}^{n-c}\frac{\mu_k^{-\frac{1}{p}}}{z-\mu_k^{-\frac{1}{p}}}dz=\sum_{k=1}^{n-c}\frac{1}{2\pi i}\int_{\Gamma}\frac{\mu_k^{-\frac{1}{p}}}{z_k^{-\frac{1}{p}}}dz=\sum_{k=1}^{n-c}\mu_k^{-\frac{1}{p}}=LEL_{-\frac{1}{p}}(G)$$

Since the value of the integral

$$\frac{1}{2\pi i}\int_{\Gamma}\left[\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)}-p(n-c)\right]dz$$

Is independent of the actual values of R and r, it can be gotten that

$$LEL_{\alpha}(G)=\frac{1}{2\pi i}\lim_{\substack{R\to+\infty\\ r\to 0^+}}\int_{\Gamma}\left[\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)}-p(n-c)\right]dz$$

$$=\frac{1}{2\pi i}\lim_{\substack{R\to+\infty\\ r\to 0^+}}\left[\int_{\Gamma_R}\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)}-p(n-c)\right]dz+\int_{\Gamma_1}\left[\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)}-p(n-c)\right]dz$$

$$+\int_{S_r^-}\left[\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)}-p(n-c)\right]dz+\int_{L_2}\left[\left[\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)}-p(n-c)\right]dz\right]$$

When S_r^- is the same curve as S_r but has clockwise orientation

Suppose that $z=\rho(\cos\theta+i\,sin\theta)$, where $\rho>0$. Then

$$\left|1-\frac{\mu_k^{-\frac{1}{p}}e^{-i\frac{2t\pi}{p}}}{z}\right|=\left|\frac{z-\mu_k^{-\frac{1}{p}}e^{-i\frac{2t\pi}{p}}}{|z|}\right|=\frac{\left|\rho(\cos\theta+i\sin\theta)-\mu_k^{-\frac{1}{p}}\left(\cos\frac{2t\pi}{p}-i\sin\frac{2t\pi}{p}\right)\right|}{\rho}$$

$$\geq\frac{\sqrt{\rho^2+\mu_k^{\frac{2}{p}}-2\rho\mu_k^{-\frac{1}{p}}}}{\rho}=\frac{\left|\rho-\mu_k^{-\frac{1}{p}}\right|}{\rho}$$

Thus we have

$$\left|z\left[\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)}-p(n-c)\right]\right|\leq\sum_{k=1}^{n-c}\sum_{t=0}^{p-1}\left|\frac{z\mu_k^{-\frac{1}{p}}e^{-i\frac{2t\pi}{p}}}{z-\mu_k^{-\frac{1}{p}}e^{-i\frac{2t\pi}{p}}}\right|=\sum_{k=1}^{n-c}\sum_{t=0}^{p-1}\frac{\left|\mu_k^{-\frac{1}{p}}e^{-i\frac{2t\pi}{p}}\right|}{\left|1-\frac{\mu_k^{-\frac{1}{p}}e^{-i\frac{2\pi t}{p}}}{z}\right|}$$

$$\leq\sum_{k=1}^{n-c}\sum_{t=0}^{p-1}\frac{\rho\mu_k^{-\frac{1}{p}}}{\left|\rho-\mu_k^{-\frac{1}{p}}\right|}.$$

Obviously,

$$\sum_{k=1}^{n-c}\sum_{t=0}^{p-1}\frac{\rho\mu_k^{\frac{1}{p}}}{\left|\rho-\mu_k^{\frac{1}{p}}\right|}\to 0,\ \ for\ \rho\to 0$$

Then by Lemma (6.3.7) we get that

$$\int_{S_r^-}\left[\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)}-p(n-c)\right]dz\to 0,\quad for\ r\to 0.$$

Suppose that $\omega_{kt}=\mu_k^{-\frac{1}{p}}e^{-i\frac{2t\pi}{p}}$. Then $|\omega_{k1}|=|\omega k1|=\cdots=\left|\omega_{k(p-1)}\right|=\mu_k^{-\frac{1}{p}}$. We have

$$\left|z\sum_{t=0}^{p-1}\frac{\mu_k^{-\frac{1}{p}}e^{-i\frac{2t\pi}{p}}}{z-\mu_k^{-\frac{1}{p}}e^{-i\frac{2t\pi}{p}}}\right|$$

$$=\frac{\left|z\sum_{t=0}^{p-1}\mu_k^{-\frac{1}{p}}e^{-i\frac{2t\pi}{p}}\left(z^{p-1}+z^{p-2}\mu_k^{-\frac{1}{p}}e^{-i\frac{2t\pi}{p}}+\cdots z\left(\mu_k^{-\frac{1}{p}}e^{-i\frac{2t\pi}{p}}\right)^{p-2}+\left(\mu_k^{-\frac{1}{p}}e^{-i\frac{2t\pi}{p}}\right)^{p-1}\right)\right|}{z^p-\mu_k^{-1}}$$

$$=\left|z\sum_{t=0}^{p-1}\frac{\omega_{kt}\left(z^{p-1}+z^{p-2}\omega_{kt}+\cdots+\omega_{kt}^{p-2}+\omega_{kt}^{p-1}\right)}{z^p-\mu_k^{-1}}\right|=\left|\sum_{t=0}^{p-1}\frac{\omega_{kt}+\omega_{kt}^2+\cdots+\frac{\omega_{kt}^{p-1}}{z^{p-2}}+\frac{\omega_{kt}^{p}}{z^{p-1}}}{1-\mu_{z^p}^{-1}}\right|$$

It is easy to get that there exists $N_k>0$, for $k\in\{1,2,\cdots,n-c\}$, such that $\left|1-\frac{\mu_k^{-1}}{z^p}\right|\geq\frac{1}{2}$ for $|z|>N_k$. For any $\varepsilon>0$, there exists $M_k>0$, for $k\in\{1,2,\cdots,n-c\}$, such that

$$\left|\sum_{t=0}^{p-1}\left(\frac{\omega_{kt}^2}{z}+\cdots+\frac{\omega_{kt}^{p-1}}{z^{p-2}}+\frac{\omega_{kt}^{p}}{z^{p-1}}\right)\right|<\frac{\varepsilon}{2n}$$

For $|z|>M_k$. Note that $\sum_{t=0}^{p-1}\omega_{kt}^r=0$ unless $r=p$. Therefore, for any $\varepsilon>0$, there exists $N=\max\{N_1,N_2,\cdots N_{n-c},M_1,M_2,\cdots,M_{n-c}\}$ such that

$$\left|z\left[\frac{\varphi_L' G,z}{\varphi_L(G,z)}-p(n-c)\right]\right| = \left|\sum_{k=1}^{n-c}\sum_{t=0}^{p-1}\frac{\mu_k^{-\frac{1}{p}}e^{-i\frac{2t\pi}{p}}}{z-\mu_k^{-\frac{1}{p}}e^{-i\frac{2t\pi}{p}}}\right|$$

$$= \left|\sum_{k=1}^{n-c}\sum_{t=0}^{p-1}\frac{\omega_{kt}+\frac{\omega_{kt}^2}{z}+\cdots+\frac{\omega_{kt}^{p-1}}{z^{p-2}}+\frac{\omega_{kt}^p}{z^{p-1}}}{1-\mu_k^{-\frac{1}{p}}}\right|$$

$$\leq 2\left|\sum_{k=1}^{n-c}\sum_{t=0}^{p-1}\omega_{kt}\right| + 2\sum_{k=1}^{n-c}\left(\left|\sum_{t=0}^{p-1}\left(\frac{\omega_{kt}^2}{z}+\cdots+\frac{\omega_{kt}^{p-1}}{z^{p-2}}+\frac{\omega_{kt}^p}{z^{p-1}}\right)\right|\right)$$

$$< 0+2\sum_{k=1}^{n-c}\frac{\varepsilon}{2n} < \varepsilon,$$

For $|z| > N$. By stander estimate, we obtain that, for any $\varepsilon > 0$, there exists $N = \max\{N_1, N_2, N_{n-c}, M_1, M_2, \cdots M_{n-c}\}$ such the integral

$$\int_{\Gamma_R}\left[\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)}-p(n-c)\right]dz \leq \frac{2\pi R}{p}\max_{z\in\Gamma_R}\left|\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)}-p(n-c)\right|$$

$$= \frac{2\pi}{p}\max_{z\in\Gamma_R}\left|z\left[\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)}-p(n-c)\right]\right| < \frac{2\pi}{p}\varepsilon,$$

For $|z| > N$. This implies

$$\int_{\Gamma_R}\left[\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)}-p(n-c)\right]dz \to 0, for\ |z| \to +\infty.$$

Therefore, we obtain that

$$LEL_\alpha(G) = \frac{1}{2\pi i}\lim_{\substack{R\to+\infty\\ r\to 0^+}}\left[\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)}-p(n-c)\right]dz + \int_{L_1}\left[\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)}-p(n-c)\right]dz$$

$$+\int_{L_2}\left[\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)}-p(n-c)\right]dz + \int_{L_2}\left[\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)}-p(n-c)\right]dz$$

$$= \frac{1}{2\pi i}\lim_{\substack{R\to+\infty\\ r\to 0^+}}\left[\int_{L_1}\left[\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)}-p(n-c)\right]dz\right.$$

$$\left.+\int_{L_2}\left[\frac{z\varphi_L'(G,z)}{\varphi_L(G,z)}-p(n-c)\right]dz\right]$$

$$= \frac{1}{2\pi i}\lim_{\substack{R\to+\infty\\ r\to 0^+}}\left[pc-\frac{\phi_L'(G,z)}{\phi_L(z^{-p})}pz^{-p}\right]dz + \int_{L_2}\left[pc-\frac{\phi_L'(G,z^{-p})}{\phi_L(,z^{-p})}pz^{-p}\right]dz$$

$$= \frac{1}{2\pi i}\lim_{\substack{R\to+\infty\\ r\to 0^+}}\left[\int_{L_1}\left[pc-\frac{\phi_L'(G,z^{-p})}{\phi_L(G,z^{-p})}pz^{-p}\right]dz\right.$$

$$\left.+\int_{L_2}\left[pc-\frac{\phi_L'(G,z^{-p})}{\phi_L(G,z^{-p})}pz^{-p}\right]dz\right]$$

$$= \frac{1}{2\pi i} \lim_{\substack{R\to+\infty \\ r\to 0^+}} \left[pc - \frac{\phi_L'\left(G, \left(\rho e^{-i\frac{\pi}{p}}\right)^p\right)}{\phi_L\left(G, \left(\rho e^{-i\frac{\pi}{p}}\right)^{-p}\right)} p\left(\rho e^{-i\frac{\pi}{p}}\right) \right] d\left(\rho e^{-i\frac{\pi}{p}}\right)$$

$$+ \int_r^R \left[pc - \frac{\phi_L'\left(G, \left(\rho e^{-i\frac{\pi}{p}}\right)^p\right)}{\phi_L\left(G, \left(\rho e^{-i\frac{\pi}{p}}\right)^{-p}\right)} p\left(\rho e^{-i\frac{\pi}{p}}\right)^{-p} \right] d\left(\rho e^{-i\frac{\pi}{p}}\right)$$

$$= \frac{1}{2\pi i} \lim_{\substack{R\to+\infty \\ r\to 0^+}} \left[pc - \frac{\phi_L'(G, -\rho^{-p})}{\phi_L(G, -\rho^{-p})} (-p)\rho^{-p} \right] e^{i\frac{\pi}{p}} \, d\rho$$

$$+ \int_r^R \left[pc - \frac{\phi_L'\left(G, \left(\rho e^{-i\frac{\pi}{p}}\right)^{-p}\right)}{\phi_L\left(G, \left(\rho e^{-i\frac{\pi}{p}}\right)^{-p}\right)} p\left(\rho e^{-i\frac{\pi}{p}}\right)^{-p} \right] d\left(\rho e^{-i\frac{\pi}{p}}\right)$$

$$= \frac{1}{2\pi i} \lim_{\substack{R\to+\infty \\ r\to 0^+}} \left[\int_R^r pc - \frac{\phi_L'(G, -\rho^{-p})}{\phi_L(G, -\rho^{-p})} (-p)\rho^{-p} \right] e^{i\frac{\pi}{p}} d\rho$$

$$+ \int_r^R \left[pc - \frac{\phi_L'(G, -\rho^{-p})}{\phi_L(G, -\rho^{-p})} (-p)\rho^{-p} \right] e^{-i\frac{\pi}{p}} d\rho$$

$$= \frac{1}{2\pi i} \lim_{\substack{R\to+\infty \\ r\to 0^+}} \left[pc + \frac{\phi_L'(G, -\rho^{-p})}{\phi_L(G, -\rho^{-p})} p\rho^{-p} \right] \left(\cos\frac{\pi}{p} + i\sin\frac{\pi}{p}\right) d\rho$$

$$= -\frac{p}{\pi} \lim_{\substack{R\to+\infty \\ r\to 0^+}} \left(c + \frac{\phi_L'(G, -\rho^{-p})}{\phi_L(G, -\rho^{-p})} \right) . \sin\frac{\pi}{p} d\rho$$

$$= \frac{p}{\pi} \int_0^{+\infty} \left(\frac{-x^{-p}\phi_L'(G, -\rho^p)}{\phi_L(G, -x^{-p})} - c \right) . \sin\frac{\pi}{p} \, dx$$

Note that the formula above also holds for the general energy $E_\alpha(\phi(z))$ of $\phi(z)$ whose roots are all nonnegative (here c is the multiplicity of 0 as root of $\varphi(z)$).

(v) Clearly, we have that

$$LEL_{-q}(G) = E_{-\frac{1}{2}}\big(\varphi(z)\big) = \frac{2}{\pi} \int_0^{+\infty} \left(\frac{-x^{-2}\varphi'(-x^{-2})}{\varphi(-x^{-2})} \right) dx$$

Where $\varphi(z) = (-1)^n \varphi_L\big(G, \sqrt{z}\big)\big(G, -\sqrt{z}\big)$

(vi) it can be easy to get that

$$LEL_{-\frac{q}{p}}(G) = \sum_{k=1}^{n} \mu_k^{-\frac{q}{p}} = \sum_{k=1}^{n} \big(\mu_k^q\big)^{-\frac{1}{p}} = ELE_{-\frac{1}{p}}\left(\varphi_{-\frac{1}{p}}(G, x)\right)$$

$$= \frac{p}{\pi} \int_0^{+\infty} \left(\frac{-x^{-p}\varphi_L'(G, -x^{-p})}{\varphi_L(G, -x^{-p})} - c \right) . \sin\frac{\pi}{p} \, dx$$

The proof is complete

Suppose that $\phi(z)$ is a monic polynomial whose roots are all non-negative real numbers Similar to the proof of the above theorem, we can get the integral formulas for the general energy of $\phi(x)$ as follows.

Theorem (6.3.10)[269]: Let $\phi(z)$ be a monic complex polynomial, $c \in \{0,1,\dots,n-1\}$ the multiplicity of 0 as root of $\phi(z)$ and $\alpha \in \mathbb{Q}$. Suppose that $\varphi(z) = e^{i(q-1)n\pi}\phi\left(z^{\frac{1}{q}}e^{-i\frac{2\pi}{q}}\right)\cdots\phi\left(z^{\frac{1}{q}}e^{-i\frac{2(q-1)\pi}{q}}\right)$. Then the general energy of $\phi(z)$ can be given as follows

(i) if $\alpha = \frac{1}{p}, p \in \mathbb{Z}^+\backslash\{1\}$, then

$$E_\alpha(\phi(z)) = \frac{p}{\pi}\int_0^{+\infty}\left(\frac{x^p\phi'(-x^p)}{\phi(-x^p)}\right).\sin\frac{\pi}{p}dx$$

(ii) if $\alpha = q, q \in \mathbb{Z}^+$, then

$$E_\alpha(\phi(z)) = \frac{2}{\pi}\int_0^{+\infty}\left(\frac{x^2P'(-x^2)}{P(-x^2)} + n\right)dx.$$

Where $P(z) = (-1)^n\varphi(\sqrt{z}\varphi)\varphi(-\sqrt{z})$.

(iii) if $\alpha = \frac{q}{p},, p, q \in \mathbb{Z}^+\backslash\{1\}$, then

$$E_\alpha(\phi(z)) = \frac{p}{\pi}\int_0^{+\infty}\left(\frac{x^p\varphi'(-x^p)}{\varphi(-x^p)} + n\right).\sin\frac{\pi}{p}dx$$

(iv) if $\alpha = -\frac{1}{p}, p \in \mathbb{Z}^+\backslash\{1\}$, then

$$E_\alpha(\phi(z)) = \frac{p}{\pi}\int_0^{+\infty}\left(\frac{-x^{-p}\phi'(-x^{-p})}{\phi(-x^{-p})} - c\right).\sin\frac{\pi}{p}dx$$

(v) if $\alpha = -q, q \in \mathbb{Z}^+$, then

$$E_\alpha(\phi(z)) = \frac{2}{\pi}\int_0^{+\infty}\left(\frac{-x^{-2}P'(-x^{-2})}{P(-x^{-2})} - c\right)dx,$$

Where $P(z) = (-1)^n\varphi_L(G,\sqrt{z})\varphi_L(G,\sqrt{z})$.

(vi) if $\alpha = -\frac{q}{p}, q \in \mathbb{Z}^+\backslash\{1\}$, then

$$E_\alpha(\phi(z)) = \frac{p}{\pi}\int_0^{+\infty}\left(\frac{-x^{-p}\varphi'(-x^{-p})}{\varphi(-x^{-p})} - c\right).\sin\frac{\pi}{p}dx$$

We define a new polynomial $\varphi_A(G,z) = (-1)^n\phi_A(G,\sqrt{z})\phi_A(G,-\sqrt{z})$. Then the roots of $\varphi_A(G,z)$ are $\lambda_1^2, \lambda_2^2, \cdots, \lambda_n^2$. Note that

$$E_\alpha(G) = \sum_{\lambda\neq 0}|\lambda_k|^\alpha = \sum_{\lambda_k\neq 0}\left|\lambda_k^2\right|^{\frac{\alpha}{2}} = E_{\frac{\alpha}{2}}(\varphi_A(G,z))$$

Thus, by Theorems (6.3.4) and (6.3.8) we have the following results.

Theorem (6.3.11)[269]: Let G be a graph of order $n, \phi_A(G,x)$ the characteristic polynomial of the adjacency matrix A(G) of G, and $c \in \{0,1,\cdots n-1\}$ is multiplicity of 0 as root of $\phi_A(G,z)$. Suppose that $\varphi_A(G,z) = (-1)^n\phi_A(G,\sqrt{z})\phi_A(G,-\sqrt{z})$. Then the general energy of G can be given as follows

(i) if $\alpha = \frac{1}{p}, p \in \mathbb{Z}^+$, then

$$E_\alpha(G)=\frac{2p}{\pi}\int_0^{+\infty}\left(\frac{x^{2p}\varphi_A'(G,-x^{2p})}{\varphi_A(G,-x^{2p})}+n\right).\sin\frac{\pi}{2p}dx$$

(ii) if $\alpha=\frac{p}{q}, p,q\in\mathbb{Z}^+$, then

$$E_\alpha(G)=\frac{2p}{\pi}\int_0^{+\infty}\left(\frac{x^{2p}\varphi_A'(G,-x^{2p})}{\varphi_A(G,-x^{2p})}+n\right).\sin\frac{\pi}{2p}dx$$

Where $\varphi_A(z)=e^{i(q-1)n\pi}\varphi_A\left(G,z^{\frac{1}{q}}e^{-i\frac{2\pi}{q}}\right)\cdots\varphi_A\left(G,z^{\frac{1}{q}}e^{-i\frac{2(q-1)x}{q}}\right)$.

(iii) If $\alpha=-\frac{1}{p}, p\in\mathbb{Z}^+$, then

$$E_\alpha(G)=\frac{2p}{\pi}\int_0^{+\infty}\left(\frac{-x^{2p}\varphi_A'(G,-x^{-2p})}{\varphi_A(G,-x^{-2p})}-c\right).\sin\frac{\pi}{2p}dx$$

(iv)If $\alpha=-\frac{q}{p}, p,q\in\mathbb{Z}^+$, then

$$E_\alpha(G)=\frac{2p}{\pi}\int_0^{+\infty}\left(\frac{-x^{2p}\varphi_A'(G,-x^{-2p})}{\varphi_A(G,-x^{-2p})}-c\right).\sin\frac{\pi}{2p}dx$$

Where $\varphi_A(z)=e^{i(q-1)n\pi}\varphi_A\left(G,z^{\frac{1}{p}}\right)\varphi_A\left(G,z^{\frac{1}{q}}e^{-i\frac{2\pi}{q}}\right)\cdots\varphi_A\left(G,z^{\frac{1}{q}}e^{-i\frac{2(q-1)\pi}{q}}\right)$

Let G be a graph of order n and size m. Suppose that $\varphi_L(G,z)=(-1)^n\phi_L\left(G,\sqrt{z}+\frac{2m}{n}\right)\phi_L\left(G,-\sqrt{z}+\frac{2m}{n}\right)$. Then the roots of $\varphi_L(G,z)$ are $\left(\mu_1-\frac{2m}{n}\right)^2,\left(\mu_2-\frac{2m}{n}\right)^2,\cdots,\left(\mu_n-\frac{2m}{n}\right)^2$. Thus get that

$$LE_\alpha(G)=\sum_{\mu_k\neq\frac{2m}{n}}\left|\mu_k-\frac{2m}{n}\right|^\alpha=\sum_{\mu_k\neq\frac{2m}{n}}\left[\left(\mu_k-\frac{2m}{n}\right)^2\right]^{\frac{\alpha}{2}}=E_{\frac{\alpha}{2}}\left(\varphi_L(G,z)\right)$$

By Theorem (6.3.4) and (6.3.8), we can get the following results.

Theorem (6.3.12)[269]: Let G be a graph of order n and size m, $\phi_L(G,x)$ the characteristic polynomial of the Laplacian matrix $L(G)$ of G, and $c\in\{0,1,\dots n-1\}$ the multiplicity of $\frac{2m}{n}$ as roots of $\phi_L(G,z)=(-1)^n\phi_L\left(G,\sqrt{z}+\frac{2m}{n}\right)\phi_L\left(G,-\sqrt{z}+\frac{2m}{n}\right)$.

Then the general Laplacing energy of G can be given as follows

(i) if $\alpha=\frac{1}{p}, p\in\mathbb{Z}^+$, the

$$LE_\alpha(G)=\frac{2p}{\pi}\int_0^{+\infty}\left(\frac{x^{2p}\varphi_L'(G,-x^{2p})}{\varphi_L(G,-x^{2p})}+n\right).\sin\frac{\pi}{2p}dx$$

(ii) If $\alpha=\frac{q}{p}, p,q\in\mathbb{Z}^+$, then

$$LE_\alpha(G)=\frac{2p}{\pi}\int_0^{+\infty}\left(\frac{x^{2p}\varphi_L'(G,-x^{2p})}{\varphi_L(G,-x^{2p})}+n\right).\sin\frac{\pi}{2p}dx$$

Where $\varphi_L(z)=e^{i(q-1)n\pi}\varphi_L\left(G,z^{\frac{1}{q}}\right)\varphi_L\left(G,z^{\frac{1}{q}}e^{-i\frac{2\pi}{q}}\right)\cdots\varphi_L\left(Gz^{\frac{1}{q}}e^{-i\frac{2(q-1)\pi}{q}}\right)$

(iii) If $\alpha=-\frac{1}{p}, p\in p\in\mathbb{Z}^+$, then

$$LE_\alpha(G) = \frac{2p}{\pi}\int_0^{+\infty}\left(\frac{-x^{-2p}\varphi_L'(G,-x^{-2p})}{\varphi_L(G,-x^{-2p})}+n\right).\sin\frac{\pi}{2p}dx$$

(iv) If $\alpha = -\frac{q}{p}, p, q \in \mathbb{Z}^+$, then

$$LE_\alpha(G) = \frac{2p}{\pi}\int_0^{+\infty}\left(\frac{-x^{-2p}\varphi_L'(G,-x^{-2p})}{\varphi_L(G,-x^{-2p})}-c\right).\sin\frac{\pi}{2p}dx$$

Where $\varphi_L(z) = e^{i9(q-1)n\pi}\varphi_L\left(G, z^{\frac{1}{q}}e^{-i\frac{2\pi}{q}}\right)\cdots\varphi_L\left(G, z^{\frac{1}{q}}e^{-i\frac{2(q-1)\pi}{q}}\right)$

It is natural to ask that whether the formulas obtained above hold for the case that α is an irrational number. Now we consider this problem.

Let $0 < \alpha = 1/p < 1$ be an irrational number. Then $1 < p \in \mathbb{R}\backslash\mathbb{Q}$. For a graph G, $LEL_\alpha(G) = \sum_{\mu_k\neq 0}\mu_k^{1/p}$. The integral in Theorem (6.3.9) (i) is

$$\frac{p}{\pi}\int_0^{+\infty}\left(\frac{x^p\phi_L'(G,-x^p)}{\phi_L(G,-x^p)}+n\right).\sin\frac{\pi}{p}dx$$

$$= \frac{p}{\pi}\sin\frac{\pi}{p}\int_0^{+\infty}\left(\frac{x^p}{-x^p-\mu_k}+n\right)dx$$

$$= \frac{p}{\pi}\sin\frac{\pi}{p}\int_0^{+\infty}\sum_{k=1}^{n}\frac{\mu_k}{x^p+\mu_k}dx$$

$$= \sum_{\mu_k\neq 0}\frac{\mu_k p}{\pi}\sin\frac{\pi}{p}\int_0^{+\infty}\frac{1}{x^p+\mu_k}dx$$

By using the software mathematica, we have

$$\int\frac{1}{x^\alpha+b}dx = \frac{x}{b}\,{}_2F_1\left(1,\frac{1}{a};1+\frac{1}{a};\frac{x^a}{b}\right),$$

Where ${}_2F_1(a_0,a_1;b_0;x) = \sum_{k=0}^{\infty}\frac{(a_0)_k(a_1)_k x^k}{(b_0)_k}\frac{x^k}{k!}$ is a hypergeometric function with $a > 1, b > 0$ and $(z)_k = z(z+1)(z+2)\cdots(z+k-1)$ (see |1|)Again, using the software Mathematica, we get

$$\lim_{x\to+\infty} x\;{}_2F_1\left(1,\frac{1}{a};1+\frac{1}{a};\frac{x^a}{b}\right) = b^{\frac{1}{a}}\Gamma\left(1+\frac{1}{a}\right)\Gamma\left(1-\frac{1}{a}\right)$$

and

$$\lim_{x\to 0} x\;{}_2F_1\left(1,\frac{1}{a};1+\frac{1}{a};\frac{x^a}{b}\right) = 0$$

Where $\Gamma(x)$ is the Gamma function, since

$$\Gamma(1-x)\Gamma(x) = \frac{\pi}{\sin\pi x}$$

For $0 < x < 1$ and

$$\Gamma(1+x) = xzG\;(x)$$

We have

$$\frac{p\mu_k}{\pi}siin\frac{\pi}{p}\int_0^{+\infty}\frac{1}{x^p+\mu_k}dx = \frac{p\mu_k}{\pi}\sin\frac{\pi}{p}.\frac{x}{\mu_k}\,{}_2F_1\left(,\frac{1}{p};1+\frac{1}{p};\frac{x^a}{\mu_k}\right)\Bigg|_0^{+\infty}$$

$$= \frac{p}{\pi}\sin\frac{\pi}{p}.\mu_k^{\frac{1}{p}}\Gamma\left(\frac{1}{p}\right)\Gamma\left(1-\frac{1}{p}\right)$$

$$\mu_k^{\frac{1}{p}}\frac{p}{\pi}\sin\frac{\pi}{p}.\frac{1}{p}\Gamma\left(\frac{1}{p}\right)\Gamma\left(1-\frac{1}{p}\right)=\mu_k^{\frac{1}{p}}\frac{1}{\pi}\sin\frac{\pi}{p}.\frac{\pi}{\sin\frac{\pi}{p}}=\mu_k^{\frac{1}{p}}$$

Therefore,

$$\frac{p}{\pi}\int_0^{+\infty}\left(\frac{x^p\phi_L'(G,-x^p)}{\phi_L(G,-x^p)}\right).\sin\frac{\pi}{p}dx=\sum_{\mu_k\neq 0}\frac{\mu_k p}{\pi}\sin\frac{\pi}{p}\int_0^{+\infty}\frac{1}{x^p+\mu_k}dx=\sum_{\mu_k\neq 0}\mu_k^{\frac{1}{p}}$$
$$=LEL_\alpha(G)$$

Let $-1<\alpha=-1/p<0$ be an irrational number. Then $1<p\in\mathbb{R}\backslash\mathbb{Q}$ a graph With c components, $LEL_\alpha(G)=\sum_{\mu_k\neq 0}\mu_k^{-1/p}$. The integral in Theorem (6.3.9) (iv) is

$$\frac{p}{\pi}\int_0^{+\infty}\left(\frac{-x^{-p}\phi_L'(G,-x^{-p})}{\phi_L(G,-x^{-p})}-c\right).\sin\frac{\pi}{p}dx=\frac{p}{\pi}\sin\frac{\pi}{p}\int_0^{+\infty}\left(\frac{-x^{-p}}{-x^{-p}-\mu_k}-c\right)dx$$
$$=\frac{\mathrm{p}}{\pi}\sin\frac{\pi}{p}\int_0^{+\infty}\sum_{k=1}^{n-c}\frac{1}{1+\mu_k^{-1}}dx=\frac{p}{\pi}\sin\frac{\pi}{p}\sum_{k=1}^{n-c}\int_0^{+\infty}\frac{\mu_k^{-1}}{x^p+\mu_k^{-1}}dx=\sum_{k=1}^{n-c}p\mu_k^{-1}$$
$$=LEL_\alpha(G).$$

Different from the case that α is an irrational number with $0<|\alpha|<1$, the integral formulas in Theorem (6.3.9) for the case $|\alpha|>1$ do not hold when α is irrational. Note that

$$\frac{p}{\pi}\int_0^{+\infty}\left(\frac{x^p\phi_L'(G,-x^p)}{\phi_L(G,-x^p)}+n\right).\sin\frac{\pi}{p}dx=\sum_{\mu_k}\frac{\mu_k p}{\pi}\sin\frac{\pi}{p}\int_0^{+\infty}\frac{1}{x^p+\mu_k}dx.$$

Then it follows from that the important integral

$$\int_0^{+\infty}\frac{1}{x^p+a}dx\,,(0<p<1,a>0)$$

Diverges that the integral

$$\frac{p}{\pi}\int_0^{+\infty}\left(\frac{x^p\phi_L'(G,-x^p)}{\phi_L(G,-x^p)}+n\right).\sin\frac{\pi}{p}dx$$

Diverges.

It can also be shown that the formulas in Theorem (6.3.10) (i) and (iv), Theorem (6.3.11) (i) and (iv) and Theorem (6.3.12) (i) and (iv) hold when α is an irrational number with $0<|\alpha|<1$ and do not hold with $|\alpha|>1$.

Corollary (6.3.13)[274]: Let G be a graph of order $n+r-1$ with $c\,(<n+r-1)$ components, $\phi_L(G,x)$ the characteristic polynomial of the Laplacian matrix $L(G)$ of G, and $\alpha^2\in\mathbb{Q}$. Suppose that

$$\varphi_L(G,z_{r-2})=e^{i(q^2-1)(n+r-1)\pi}\phi_L\left(G,z_{r-2}^{\frac{1}{q^2}}e^{-i\frac{2\pi}{q^2}}\right)\cdots\phi_L\left(G,z_{r-2}^{\frac{1}{q^2}}e^{-i\frac{2(q^2-1)\pi}{q^2}}\right).$$

Then the general Laplacing - energy - like invariant of G can be given as follows

(i) If $\alpha=\frac{1}{p},p^2\in\mathbb{Z}^+\backslash\{1\}$, then

$$LEL_{\alpha^2}(G)=\frac{p^2}{\pi}\int_0^{+\infty}\left(\frac{x^{p^2}\phi_L'\left(G,-x^{p^2}\right)}{\phi_L\left(G,-x^{p^2}\right)}+n+r-1\right).\sin\frac{\pi}{p^2}dx$$

(ii) if $\alpha=q,q^2\in\mathbb{Z}^+$, then

$$LELE_{\alpha^2}(G) = \frac{2}{\pi}\int_0^{+\infty}\left(\frac{x^2\varphi'(-x^2)}{\varphi(-x^2)} + n + r - 1\right)dx$$

Where $\varphi(z_{r-2}) = (-1)^{n+r-1}\varphi_L(G,\sqrt{z_{r-2}})\varphi_L(G,-\sqrt{z_{r-2}})$.

(iii) if $\alpha = \frac{q}{p}, p^2, q^2 \in \mathbb{Z}^+\backslash\{1\}$, then

$$LEL_{\alpha^2}(G) = \frac{p^2}{\pi}\int_0^{+\infty}\left(\frac{x^{p^2}\varphi_L'\left(G,-x^{p^2}\right)}{\varphi_L\left(G,-x^{p^2}\right)}\right).\sin\frac{\pi}{p^2}dx$$

(iv) If $\alpha^2 = -\frac{1}{p^2}, p^2 \in \mathbb{Z}^+\backslash\{1\}$, then

$$LEL_{\alpha^2}(G) = \frac{p^2}{\pi}\int_0^{+\infty}\left(\frac{-x^{-2}\varphi'(-x^{-2})}{\varphi'(-x^{-2})} - c\right)dx,$$

Where $\varphi(z_{r-2}) = (-1)^{n+r-1}\varphi_L(G,\sqrt{z_{r-2}})\varphi_L(G,-\sqrt{z_{r-2}})$

(vi) if $\alpha^2 = -\frac{q^2}{p^2}, q^2 \in \mathbb{Z}^+\backslash\{1\}$, then

$$LEL_{\alpha^2}(G) = \frac{p^2}{\pi}\int_0^{+\infty}\left(\frac{-x^{-p^2}\varphi_L'\left(G,-x^{-p^2}\right)}{\varphi_L\left(G,-x^{-p^2}\right)}\right).\sin\frac{\pi}{p^2}dx$$

Proof. (i) This is just the result of Theorem (6.3.2). Let $\mu_r \geq \mu_{r+1} \geq \cdots \geq \mu_{n+r-1} = 0$ be the roots of $\phi_L(G,x)$. Then $\phi_L(G,x) = (x-\mu_r)\cdots(x-\mu_{r+1})$. Therefore, we have

$$\begin{aligned}
&\varphi_L(G,z_{r-2})\\
&= e^{i(q^2-1)(n+r-1)\pi}\phi_L\left(G,z_{r-2}^{\frac{1}{q^2}}\right)\phi_L\left(G,z_{r-2}^{\frac{1}{q^2}}e^{-i\frac{2\pi}{q^2}}\right)\cdots\phi_L\left(G,z_{r-2}^{\frac{1}{q^2}}e^{-i\frac{2(q^2-1)\pi}{q^2}}\right)\\
&= e^{i(q^2-1)(n+r-1)\pi}\prod_{k=r}^{n+r-1}\left(z_{r-2}^{\frac{1}{q^2}} - \mu_k\right)\left(z_{r-2}^{\frac{1}{q^2}}e^{-i\frac{2\pi}{q^2}} - \mu_k\right)\cdots\left(z_{r-2}^{\frac{1}{q^2}}e^{-i\frac{2(q^2-1)\pi}{q^2}} - \mu_k\right)\\
&= e^{i(q^2-1)(n+r-1)\pi}\prod_{k=r}^{n+r-1}\left(z_{r-2}^{\frac{1}{q^2}} - \mu_k\right)\left(z_{r-2}^{\frac{1}{q^2}}\right.\\
&\left.- \mu_k e^{i\frac{2\pi}{q^2}}\right)\cdots\left(z_{r-2}^{\frac{1}{q^2}}\mu_k e^{i\frac{2(q^2-1)\pi}{q^2}}\right)e^{-i(q^2-1)\pi} = \prod_{k=r}^{n+r-1}\left(z_{r-2} - \mu_k^{q^2}\right).
\end{aligned}$$

By Theorem (6.3.8) we obtain that

$$LEL_{\alpha^2}(G) = \sum_{k=r}^{n+r-1}\mu_k^{q^2} = E_1\left(\varphi_L(G,z_{r-2})\right) = \frac{2}{\pi}\int_0^{+\infty}\left(\frac{x^2\varphi'(-x^2)}{\varphi(-x^2)} + n + r - 1\right)dx$$

Where $\varphi(z_{r-2}) = (-1)^{n+r-1}\varphi_L(G,\sqrt{z_{r-2}})\varphi_L(G,-\sqrt{z_{r-2}})$

(iii) By Theorem (6.3.4), it is easy to obtain that

$$\begin{aligned}
LEL_{\alpha^2}(G) &= \sum_{k=r}^{n+r-1}\mu_k^{\frac{q^2}{p^2}} = \sum_{k=r}^{n+r-1}\left(\mu_k^{q^2}\right)^{\frac{1}{p^2}} = E_{\frac{1}{p^2}}\left(\varphi_L(G,z_{r-2})\right)\\
&= \frac{p^2}{\pi}\int_0^{+\infty}\left(\frac{x^{p^2}\varphi_L'\left(G,-x^{p^2}\right)}{\varphi_L\left(G,-x^{p^2}\right)} + n + r - 1\right).\sin\frac{\pi}{p^2}dx
\end{aligned}$$

(iv) Suppose that $\mu_r \geq \mu_{r+1} \geq \cdots \geq \mu_{n-c+r-1} > \mu_{n-c+r-2} = \cdots = \mu_{n+r-1} = 0$ are the roots of $\phi_L(G,x)$. Thus we can write $\phi_L(G,x)$ as $\phi_L(G,x) = x^c \prod_{k=r}^{n-c+r-1}(x-\mu_k)$. Therefore, we obtain that

$$\varphi_L(G,z_{r-2}) = z_{r-2}^{p^2(n+r-1)}\phi_L\left(G,\frac{1}{z_{r-2}^{p^2}}\right) = z_{r-2}^{p^2(n+r-1)}\Big|.\left(\frac{1}{z_{r-2}^{p^2}}\right)^c \prod_{k=r}^{n-c+r-1}\left(\frac{1}{z_{r-2}^{p^2}} - \mu_k\right)$$

$$= (-1)^{n+r-1-c}\prod_{k=r}^{n-c+r-1}\left(z_{r-2}^{p^2}\mu_k - 1\right)$$

$$= (-1)^{n+r-1-c}\prod_{k=r}^{n-c+r-1}\mu_k\left(z_{r-2}^{p^2}\right.$$

$$\left.-\frac{1}{\mu_k}\right)(-1)^{n+r-1-c}\prod_{k=r}^{n-c+r-1}\mu_k\prod_{k=r}^{p^2+r-2}\left(z_{r-2} - \mu_k^{-\frac{1}{p^2}}e^{-i\frac{2t\pi}{p^2}}\right)$$

$$= (-1)^{n+r-1-c}\prod_{k=r}^{n-c+r-1}\left[(-1)^{p^2-1}\prod_{t=0}^{p^2-1}\left(z_{r-2}\mu_k^{\frac{1}{p^2}}e^{i\frac{2t\pi}{p^2}} - 1\right)\right]$$

Then

$$\frac{\varphi_L'(G,z_{r-2})}{\varphi_L(G,z_{r-2})} = \sum_{k=r}^{n-c+r-1}\sum_{t=0}^{p^2-1}\frac{z_{r-2}}{z_{r-2} - \mu_k^{-\frac{1}{p^2}}e^{-i\frac{2t\pi}{p^2}}}$$

$$= \sum_{k=r}^{n-c+r-1}\sum_{t=0}^{p^2-1}\left(1 + \frac{\mu_k^{\frac{1}{p^2}}e^{-i\frac{2t\pi}{p^2}}}{z_{r-2} - \mu_k^{-\frac{1}{p^2}}e^{-i\frac{2t\pi}{p^2}}}\right)$$

$$= p^2(n+r-1-c) + \sum_{k=r}^{n-c+r-1}\sum_{t=0}^{p^2-1}\frac{\mu_k^{-\frac{1}{p^2}}e^{-i\frac{2t\pi}{p^2}}}{z_{r-2} - \mu_k^{-\frac{1}{p^2}}e^{-i\frac{2t\pi}{p^2}}}$$

Suppose that $\Gamma = \Gamma_R \cup L_1 \cup S_r \cup L_2$ see Figure (2) is a positively (i.e., counter clockwisely) oriented piecewise smooth Jordan curve, where $R > \max\left\{\mu_r, \mu_{n-c+r-1}^{-1}, \mu_{n-c+r-1}^{-\frac{1}{p^2}}\right\}$, $0 < r < \min\left\{\mu_{n-c+r-1}, \mu_r^{-1}, \mu_r^{-\frac{1}{p^2}}\right\}$, Γ_R is the curve $\left\{z(\theta_{r-2}) = Re^{i\theta_{r-2}},\ -\frac{\pi}{p^2} \leq \theta_{r-2} \leq \frac{\pi}{p^2}\right\}$, L_1 is the line $\left\{z(\theta_{r-2}) = p^2 e^{i\theta_{r-2}}, r \leq \rho_{r-2} \leq R,\ \theta_{r-2} = \frac{\pi}{p^2}\right\}$. Then the point $\mu_r^{-\frac{1}{p^2}}, \mu_{r+1}^{-\frac{1}{p^2}}, \cdots, \mu_{n-c+r-1}^{-\frac{1}{p^2}}$ are $t = 1,2,\ldots,p^2-1$, are all in the exterior of the curve Γ. By Cauchy's Theorem and Integral Formula, we get

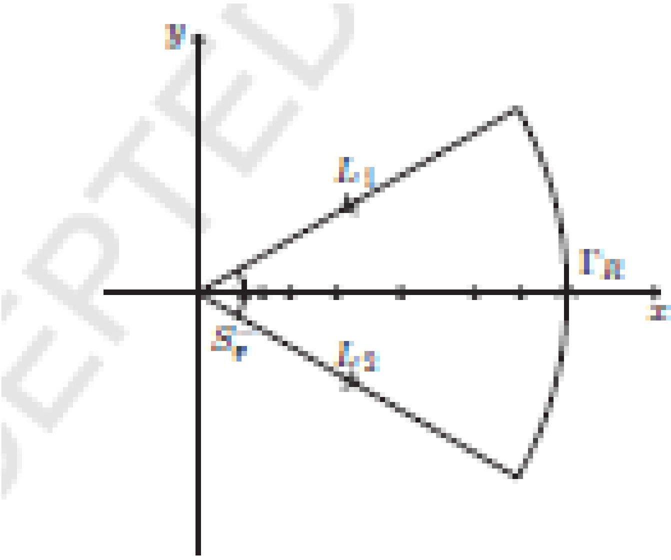


Figure (2): The curve Γ in Corollary (6.3.13) (iv).

$$\frac{1}{2\pi i}\int_\Gamma\left[\frac{z\varphi_L'(G,z_{r-2})}{\varphi_L(G,z_{r-2})}-p^2(n+r-1-c)\right]dz_{r-2}$$

$$=\frac{1}{2\pi i}\int_\Gamma\sum_{k=r}^{n-c+r-1}\sum_{t=0}^{p^2-1}\frac{\mu_k^{-\frac{1}{p^2}}e^{-i\frac{2t\pi}{p^2}}}{z_{r-2}-\mu_k^{-\frac{1}{p^2}}e^{-i\frac{2t\pi}{p^2}}}dz_{r-2}$$

$$=\frac{1}{2\pi i}\int_\Gamma\sum_{k=r}^{n-c+r-1}\frac{\mu_k^{-\frac{1}{p^2}}}{z_{r-2}-\mu_k^{-\frac{1}{p^2}}}dz=\sum_{k=r}^{n-c+r-1}\frac{1}{2\pi i}\int_\Gamma\frac{\mu_k^{-\frac{1}{p^2}}}{z_k^{-\frac{1}{p^2}}}dz_{r-2}$$

$$=\sum_{k=r}^{n+r-1-c}\mu_k^{-\frac{1}{p^2}}=LEL_{-\frac{1}{p^2}}(G)$$

Since the value of the integral

$$\frac{1}{2\pi i}\int_\Gamma\left[\frac{z_{r-2}\varphi_L'(G,z_{r-2})}{\varphi_L(G,z_{r-2})}-p^2(n+r-1-c)\right]dz_{r-2}$$

Is independent of the actual values of R and r, it can be gotten that

$$LEL_{\alpha^2}(G)=\frac{1}{2\pi i}\lim_{\substack{R\to+\infty\\ r\to0^+}}\int_\Gamma\left[\frac{z_{r-2}\varphi_L'(G,z_{r-2})}{\varphi_L(G,z_{r-2})}-p^2(n+r-1-c)\right]dz_{r-2}$$

$$=\frac{1}{2\pi i}\lim_{\substack{R\to+\infty\\ r\to0^+}}\left[\int_{\Gamma_R}\frac{z_{r-2}\varphi_L'(G,z_{r-2})}{\varphi_L(G,z_{r-2})}-p^2(n+r-1-c)\right]dz_{r-2}$$

$$+\int_{\Gamma_1}\left[\frac{z_{r-2}\varphi_L'(G,z_{r-2})}{\varphi_L(G,z_{r-2})}-p^2(n+r-1-c)\right]dz_{r-2}$$

$$+\int_{S_r^-}\left[\frac{z_{r-2}\varphi_L'(G,z_{r-2})}{\varphi_L(G,z_{r-2})}-p^2(n+r-1-c)\right]dz_{r-2}$$

$$+\int_{L_2}\left[\left[\frac{z_{r-2}\varphi_L'(G,z_{r-2})}{\varphi_L(G,z_{r-2})}-p^2(n+r-1-c)\right]dz_{r-2}\right]$$

When S_r^- is the same curve as S_r but has clockwise orientation suppose that $z_{r-2} = \rho_{r-2}(\cos\theta_{r-2} + i\, sin\theta_{r-2})$, where $\rho_{r-2} > 0$. Then

$$\left|1 - \frac{\mu_k^{-\frac{1}{p^2}} e^{-i\frac{2t\pi}{p^2}}}{z_{r-2}}\right| = \left|\frac{z_{r-2} - \mu_k^{-\frac{1}{p^2}} e^{-i\frac{2t\pi}{p^2}}}{|z_{r-2}|}\right|$$

$$= \frac{\left|\rho_{r-2}(\cos\theta_{r-2} + i\sin\theta_{r-2}\,) - \mu_k^{-\frac{1}{p^2}}\left(\cos\frac{2t\pi}{p^2} - i\sin\frac{2t\pi}{p^2}\right)\right|}{\rho_{r-2}}$$

$$\geq \frac{\sqrt{\rho_{r-2}^2 + \mu_k^{\frac{2}{p^2}} - 2\rho_{r-2}\mu_k^{-\frac{1}{p^2}}}}{\rho_{r-2}} = \frac{\left|\rho_{r-2} - \mu_k^{-\frac{1}{p^2}}\right|}{\rho_{r-2}}$$

Thus we have

$$\left|z_{r-2}\left[\frac{z_{r-2}\varphi_L'(G, z_{r-2})}{\varphi_L(G, z_{r-2})} - p^2(n + r - 1 - c)\right]\right| \leq \sum_{k=r}^{n-c+r-1}\sum_{t=0}^{p^2-1}\left|\frac{z_{r-2}\mu_k^{-\frac{1}{p^2}} e^{-i\frac{2t\pi}{p^2}}}{z_{r-2} - \mu_k^{-\frac{1}{p^2}} e^{-i\frac{2t\pi}{p^2}}}\right|$$

$$= \sum_{k=r}^{n-c+r-1}\sum_{t=0}^{p^2-1}\frac{\left|\mu_k^{-\frac{1}{p^2}} e^{-i\frac{2t\pi}{p^2}}\right|}{\left|1 - \frac{\mu_k^{-\frac{1}{p^2}} e^{-i\frac{2\pi t}{p^2}}}{z_{r-2}}\right|} \leq \sum_{k=r}^{n-c+r-1}\sum_{t=0}^{p^2-1}\frac{\rho_{r-2}\mu_k^{-\frac{1}{p^2}}}{\left|\rho_{r-2} - \mu_k^{-\frac{1}{p^2}}\right|}.$$

Obviously,

$$\sum_{k=r}^{n-c+r-1}\sum_{t=0}^{p^2-1}\frac{\rho_{r-2}\mu_k^{\frac{1}{p^2}}}{\left|\rho_{r-2} - \mu_k^{\frac{1}{p^2}}\right|} \to 0, \;\; \text{for } \rho_{r-2} \to 0$$

Then by Lemma (6.3.7) we get that

$$\int_{S_r^-}\left[\frac{z_{r-2}\varphi_L'(G, z_{r-2})}{\varphi_L(G, z_{r-2})} - p^2(n + r - 1 - c)\right] dz_{r-2} \to 0, \quad \text{for } r \to 0.$$

Suppose that $\omega_{kt} = \mu_k^{-\frac{1}{p^2}} e^{-i\frac{2t\pi}{p^2}}$. Then $|\omega_{k_1}| = |\omega_{k_1}| = \cdots = |\omega_{k(p^2-1)}| = \mu_k^{-\frac{1}{p^2}}$. We have

$$\left|z_{r-2}\sum_{t=0}^{p^2-1}\frac{\mu_k^{-\frac{1}{p^2}}e^{-i\frac{2t\pi}{p^2}}}{z_{r-2}-\mu_k^{-\frac{1}{p^2}}e^{-i\frac{2t\pi}{p^2}}}\right|$$

$$=\frac{\left|z_{r-2}\sum_{t=0}^{p^2-1}\mu_k^{-\frac{1}{p^2}}e^{-i\frac{2t\pi}{p^2}}\left(z_{r-2}^{p^2-1}+z_{r-2}^{p^2-2}\mu_k^{-\frac{1}{p^2}}e^{-i\frac{2t\pi}{p^2}}+\cdots z_{r-2}\left(\mu_k^{-\frac{1}{p^2}}e^{-i\frac{2t\pi}{p^2}}\right)^{p^2-2}+\left(\mu_k^{-\frac{1}{p^2}}e^{-i\frac{2t\pi}{p^2}}\right)^{p^2-1}\right)\right|}{z_{r-2}^{p^2}-\mu_k^{-1}}$$

$$=\left|z_{r-2}\sum_{t=0}^{p^2-1}\frac{\omega_{kt}\left(z_{r-2}^{p^2-1}+z_{r-2}^{p^2-2}\omega_{kt}+\cdots+\omega_{kt}^{p^2-2}+\omega_{kt}^{p^2-1}\right)}{z_{r-2}^{p^2}-\mu_k^{-1}}\right|$$

$$=\left|\sum_{t=0}^{p^2-1}\frac{\omega_{kt}+\omega_{kt}^2+\cdots+\frac{\omega_{kt}^{p^2-1}}{z_{r-2}^{p^2-2}}+\frac{\omega_{kt}^{p^2}}{z_{r-2}^{p^2-1}}}{1-\mu^{-1}_{z_{r-2}^{p^2}}}\right|$$

It is easy to get that there exists $N_k>0$, for $k\in\{1,2,\cdots,n+r-1-c\}$, such that $\left|1-\frac{\mu_k^{-1}}{z_{r-2}^{p^2}}\right|\geq\frac{1}{2}$ for $|z_{r-2}|>N_k$. For any $\varepsilon>0$, there exists $M_k>0$, for $k\in\{1,2,\cdots,n+r-1-c\}$, such that

$$\left|\sum_{t=0}^{p^2-1}\left(\frac{\omega_{kt}^2}{z_{r-2}}+\cdots+\frac{\omega_{kt}^{p^2-1}}{z_{r-2}^{p^2-2}}+\frac{\omega_{kt}^{p^2}}{z_{r-2}^{p^2-1}}\right)\right|<\frac{\varepsilon}{2(n+r-1)}$$

For $|z_{r-2}|>M_k$. Note that $\sum_{t=0}^{p^2-1}\omega_{kt}^r=0$ unless $r=p$. Therefore, for any $\varepsilon>0$, there exists $N=\max\{N_r,N_{r+1},\cdots N_{n+r-1-c},M_r,M_{r+1},\cdots,M_{n+r-1-c}\}$ such that

$$\left|z_{r-2}\left[\frac{\varphi_L'G,z_{r-2}}{\varphi_L(G,z_{r-2})}-p^2(n+r-1-c)\right]\right|=\left|\sum_{k=r}^{n-c+r-1}\sum_{t=0}^{p^2-1}\frac{\mu_k^{-\frac{1}{p^2}}e^{-i\frac{2t\pi}{p^2}}}{z_{r-2}-\mu_k^{-\frac{1}{p^2}}e^{-i\frac{2t\pi}{p^2}}}\right|$$

$$=\left|\sum_{k=r}^{n-c+r-1}\sum_{t=0}^{p^2-1}\frac{\omega_{kt}+\frac{\omega_{kt}^2}{z_{r-2}}+\cdots+\frac{\omega_{kt}^{p^2-1}}{z_{r-2}^{p^2-2}}+\frac{\omega_{kt}^{p^2}}{z_{r-2}^{p^2-1}}}{1-\mu_k^{-\frac{1}{p^2}}}\right|$$

$$\leq 2\left|\sum_{k=r}^{n+r-1-c}\sum_{t=0}^{p^2-1}\omega_{kt}\right|+2\sum_{k=1}^{n+r-1-c}\left(\left|\sum_{t=0}^{p^2-1}\left(\frac{\omega_{kt}^2}{z_{r-2}}+\cdots+\frac{\omega_{kt}^{p^2-1}}{z_{r-2}^{p^2-2}}+\frac{\omega_{kt}^{p^2}}{z_{r-2}^{p^2-1}}\right)\right|\right)$$

$$<0+2\sum_{k=1}^{n+r-1-c}\frac{\varepsilon}{2(n+r-1)}<\varepsilon,$$

For $|z_{r-2}|>N$. By standard estimate, we obtain that, for any $\varepsilon>0$, there exists $N=\max\{N_r,N_{r+1},\cdots N_{n+r-1-c},M_r,M_{r+1},\cdots,M_{n+r-1-c}\}$ such the integral

$$\int_{\Gamma_R}\left[\frac{\varphi_L' G, z_{r-2}}{\varphi_L(G,z_{r-2})}-p^2(n+r-1-c)\right]dz_{r-2}$$
$$\leq \frac{2\pi R}{p^2}\max_{z_{r-2}\in\Gamma_R}\left|\frac{z_{r-2}\varphi_L'(G,z_{r-2})}{\varphi_L(G,z_{r-2})}-p^2(n+r-1-c)\right|$$
$$=\frac{2\pi}{p^2}\max_{z_{r-2}\in\Gamma_R}\left|z_{r-2}\left[\frac{z_{r-2}\varphi_L'(G,z_{r-2})}{\varphi_L(G,z_{r-2})}-p^2(n+r-1-c)\right]\right|<\frac{2\pi}{p^2}\varepsilon,$$

For $|z_{r-2}|>N$. This implies

$$\int_{\Gamma_R}\left[\frac{z_{r-2}\varphi_L'(G,z_{r-2})}{\varphi_L(G,z_{r-2})}-p^2(n+r-1-c)\right]dz_{r-2}\to 0, \text{for } |z_{r-2}|\to+\infty.$$

Therefore, we obtain that

$$LEL_{\alpha^2}(G)=\frac{1}{2\pi i}\lim_{\substack{R\to+\infty\\ r\to 0^+}}\left[\frac{z_{r-2}\varphi_L'(G,z_{r-2})}{\varphi_L(G,z_{r-2})}-p^2(n+r-1-c)\right]dz_{r-2}$$
$$+\int_{L_1}\left[\frac{z_{r-2}\varphi_L'(G,z_{r-2})}{\varphi_L(G,z_{r-2})}-p^2(n+r-1-c)\right]dz_{r-2}$$
$$+\int_{L_2}\left[\frac{z_{r-2}\varphi_L'(G,z_{r-2})}{\varphi_L(G,z_{r-2})}-p^2(n+r-1-c)\right]dz_{r-2}$$
$$=\frac{1}{2\pi i}\lim_{\substack{R\to+\infty\\ r\to 0^+}}\left[\int_{L_1}\left[\frac{z_{r-2}\varphi_L'(G,z_{r-2})}{\varphi_L(G,z_{r-2})}-p^2(n+r-1-c)\right]dz_{r-2}\right.$$
$$\left.+\int_{L_2}\left[\frac{z_{r-2}\varphi_L'(G,z_{r-2})}{\varphi_L(G,z_{r-2})}-p^2(n+r-1-c)\right]dz_{r-2}\right]$$
$$=\frac{1}{2\pi i}\lim_{\substack{R\to+\infty\\ r\to 0^+}}\left[p^2c-\frac{\phi_L'(G,z_{r-2})}{\phi_L\left(z_{r-2}^{-p^2}\right)}p^2 z_{r-2}^{-p^2}\right]dz_{r-2}$$
$$+\int_{L_2}\left[p^2c-\frac{\phi_L'\left(G,z_{r-2}^{-p^2}\right)}{\phi_L\left(,z_{r-2}^{-p^2}\right)}p^2 z_{r-2}^{-p^2}\right]dz_{r-2}$$
$$=\frac{1}{2\pi i}\lim_{\substack{R\to+\infty\\ r\to 0^+}}\left[\int_{L_1}\left[p^2c-\frac{\phi_L'\left(G,z_{r-2}^{-p^2}\right)}{\phi_L\left(G,z_{r-2}^{-p^2}\right)}p^2 z_{r-2}^{-p^2}\right]dz_{r-2}\right.$$
$$\left.+\int_{L_2}\left[p^2c-\frac{\phi_L'\left(G,z_{r-2}^{-p^2}\right)}{\phi_L\left(G,z_{r-2}^{-p^2}\right)}p^2 z_{r-2}^{-p^2}\right]dz_{r-2}\right]$$
$$=\frac{1}{2\pi i}\lim_{\substack{R\to+\infty\\ r\to 0^+}}\left[p^2c-\frac{\phi_L'\left(G,\left(\rho_{r-2}e^{-i\frac{\pi}{p^2}}\right)^{p^2}\right)}{\phi_L\left(G,\left(\rho_{r-2}e^{-i\frac{\pi}{p^2}}\right)^{-p^2}\right)}p^2\left(\rho_{r-2}e^{-i\frac{\pi}{p^2}}\right)\right]d\left(\rho_{r-2}e^{-i\frac{\pi}{p^2}}\right)$$
$$+\int_r^R\left[p^2c-\frac{\phi_L'\left(G,\left(\rho_{r-2}e^{-i\frac{\pi}{p^2}}\right)^{p^2}\right)}{\phi_L\left(G,\left(\rho_{r-2}e^{-i\frac{\pi}{p^2}}\right)^{-p^2}\right)}p^2\left(\rho_{r-2}e^{-i\frac{\pi}{p^2}}\right)^{-p^2}\right]d\left(\rho_{r-2}e^{-i\frac{\pi}{p^2}}\right)$$

$$= \frac{1}{2\pi i} \lim_{\substack{R\to+\infty \\ r\to 0^+}} \left[p^2 c - \frac{\phi_L'\left(G, -\rho_{r-2}^{-p^2}\right)}{\phi_L\left(G, -\rho_{r-2}^{-p^2}\right)} (-p^2)\rho_{r-2}^{-p^2} \right] e^{i\frac{\pi}{p^2}}\, d\rho_{r-2}$$

$$+ \int_r^R \left[p^2 c - \frac{\phi_L'\left(G, \left(\rho_{r-2} e^{-i\frac{\pi}{p^2}}\right)^{-p^2}\right)}{\phi_L\left(G, \left(\rho_{r-2} e^{-i\frac{\pi}{p^2}}\right)^{-p^2}\right)} p^2 \left(\rho_{r-2} e^{-i\frac{\pi}{p^2}}\right)^{-p^2} \right] d\left(\rho_{r-2} e^{-i\frac{\pi}{p^2}}\right)$$

$$= \frac{1}{2\pi i} \lim_{\substack{R\to+\infty \\ r\to 0^+}} \left[\int_R^r p^2 c - \frac{\phi_L'\left(G, -\rho_{r-2}^{-p^2}\right)}{\phi_L\left(G, -\rho_{r-2}^{-p^2}\right)} (-p^2)\rho_{r-2}^{-p^2} \right] e^{i\frac{\pi}{p^2}} d\rho_{r-2}$$

$$+ \int_r^R \left[p^2 c - \frac{\phi_L'\left(G, -\rho_{r-2}^{-p^2}\right)}{\phi_L\left(G, -\rho_{r-2}^{-p^2}\right)} (-p^2)\rho_{r-2}^{-p^2} \right] e^{-i\frac{\pi}{p^2}} d\rho_{r-2}$$

$$= \frac{1}{2\pi i} \lim_{\substack{R\to+\infty \\ r\to 0^+}} \left[p^2 c + \frac{\phi_L'\left(G, -\rho_{r-2}^{-p^2}\right)}{\phi_L\left(G, -\rho_{r-2}^{-p^2}\right)} p^2 \rho_{r-2}^{-p^2} \right] \left(\cos\frac{\pi}{p^2} + i \sin\frac{\pi}{p^2}\right) d\rho_{r-2}$$

$$= -\frac{p^2}{\pi} \lim_{\substack{R\to+\infty \\ r\to 0^+}} \left(c + \frac{\phi_L'\left(G, -\rho_{r-2}^{-p^2}\right)}{\phi_L\left(G, -\rho_{r-2}^{-p^2}\right)} \right) . \sin\frac{\pi}{p^2}\, d\rho_{r-2}$$

$$= \frac{p^2}{\pi} \int_0^{+\infty} \left(\frac{-x^{-p^2} \phi_L'\left(G, -\rho_{r-2}^{p^2}\right)}{\phi_L\left(G, -x^{-p^2}\right)} - c \right) . \sin\frac{\pi}{p^2}\ dx$$

Note that the formula above also holds for the general energy $E_{\alpha^2}(\phi(z_{r-2}))$ of $\phi(z_{r-2})$ whose roots are all nonnegative (here c is the multiplicity of 0 as root of $\varphi(z_{r-2})$).
(v) Clearly, we have that

$$LEL_{-q^2}(G) = E_{-\frac{1}{2}}\big(\varphi(z_{r-2})\big) = \frac{2}{\pi} \int_0^{+\infty} \left(\frac{-x^{-2}\varphi'(-x^{-2})}{\varphi(-x^{-2})} \right) dx$$

Where $\varphi(z_{r-2}) = (-1)^{n+r-1} \varphi_L(G, \sqrt{z_{r-2}})(G, -\sqrt{z_{r-2}})$
(vi) it can be easy to get that

$$LEL_{-\frac{q^2}{p^2}}(G) = \sum_{k=1}^{n+r-1} \mu_k^{-\frac{q^2}{p^2}} = \sum_{k=1}^{n+r-1} \left(\mu_k^{q^2}\right)^{-\frac{1}{p^2}} = ELE_{-\frac{1}{p^2}}\left(\varphi_{-\frac{1}{p^2}}(G, x)\right)$$

$$= \frac{p^2}{\pi} \int_0^{+\infty} \left(\frac{-x^{-p^2} \varphi_L'\left(G, -x^{-p^2}\right)}{\varphi_L\left(G, -x^{-p^2}\right)} - c \right) . \sin\frac{\pi}{p^2}\ dx$$

The proof is complete.

List of Symbols

Symbol		Page
inf:	infimum	1
dens:	dense	1
dist:	distante	1
max:	maximum	1
L^p:	Lebesgue space	1
sup:	supremum	2
Re:	Real	2
arg:	argument	2
Exp:	exponential	3
conv:	convex	3
Im:	Imaginary	13
orb:	orbit	13
SOT:	Strong operator topology	14
min:	minimum	15
deg:	degree	16
ℓ^2:	Hilbert space	23
sph:	sphere	23
L^2:	Hilbert space	25
L^∞:	essential Lebesgue space	25
PV:	principle value	28
L^1:	Lebesgue space on the real line	29
H^p:	Hardy space	30
dim:	dimension	70
LB:	lower bound	73
UB:	upper bound	73
loc:	local	80
Tr:	Trace	81
$\oplus$:	Direct sum	84
supp:	support	91
ker:	kernel	97
LEL:	Laplacian-Energy-Like	107
diag:	diagonal	107
LE:	Laplacian Energy	108
WLSC:	Weak lower Scaling conditions	152
WUSC:	Weak upper Scaling conditions	152
a. s:	almost sure	165
RE:	Randi’c energy	216
NSE:	normalized signless energy	216
det:	determinant	222
int:	interior	236

References

[1] H.-P. Beise and J. Müller, Growth of (frequently) hypercyclic functions for differential operators, Studia Math. 207 (2011), 97-115.

[2] H.-P. Beise. Frequently Birkhoff-universal functions. Submitted

[3] C. A. Berenstein und R. Gay. Complex analysis and special topics in harmonic analysis. Springer, New York, 1995.

[4] L. Bernal-Gonz´alez and A. Bonilla. Exponential type of hypercyclic entire functions. Arch. Math., 78: 283-290, 2002.

[5] O. Blasco, A. Bonilla, and K.-G. Große-Erdmann. Rate of growth of frequently hypercyclic functions. Proc. Edinb. Math. Soc., 53: 39-50, 2010.

[6] R. P. Boas, Entire functions. Academic Press, New York, 1954.

[7] A.Bonilla and K.-G. Große-Erdmann. On a theorem of Godefroy and Shapiro. Integral Equations Operator Theory, 56: 151-162, 2006.

[8] K. C. Chan and J. H. Shapiro. The cyclic behaviour of translation operators on Hilbert spaces of entire functions. Indiana Univ. Math. J., 40: 1421-1449, 1991.

[9] S. M. Duyos-Ruiz. On the existence of universal functions. Soviet Math. Dokl., 27: 9-13, 1983.

[10] G. Godefroy and J. H. Shapiro. Operators with dense, invariant cyclic vector manifolds. J. Funct. Anal., 98: 229-269, 1991.

[11] K.-G. Große-Erdmann. On the universal functions of G. R. MacLane. Complex Variables Theory Appl., 15: 193-196, 1990.

[12] K.-G. Große-Erdmann and A. Peris. Linear Chaos. Springer, Berlin, 2011.

[13] D. H. Luecking and L. A. Rubel. Complex analysis, a functional analysis approach. Springer, New York, 1984.

[14] A. Martineau. Sur les fonctionelles analytiques et la transformation de Fourier-Borel. J. Anal. Math., 11: 1-163, 1963.

[15] M. Morimoto. An introduction to Sato's hyperfunctions. Trans. Math. Monogr. Amer. Math. Soc., 1994.

[16] V. Nestoridis. Non extendable holomorphic functions. Math. Proc. Cambridge Philos. Soc., 139: 351-360, 2005.

[17] W. Rudin. Real and complex analysis. Mc-Graw-Hill, New York., 3. edition, 1974.

[18] F. Bayart and E. Matheron. ´ Dynamics fo Linear Operators. Cambridge University Press, Cambridge, 2009.

[19] Kit C. Chan, Rebecca Sanders, Common hypercyclic vectors for the conjugate class of a hypercyclic operator, J. Math. Anal. Appl. 375 (2011) 139–148.

[20] E. Abakumov, J. Gordon, Common hypercyclic vectors for multiplies of backward shift, J. Funct. Anal. 200 (2) (2003) 494–504.

[21] S.I. Ansari, Existence of hypercyclic operators on topological vector spaces, J. Funct. Anal. 148 (1997) 384–390.

[22] R. Aron, J. Bès, F. León, A. Peris, Operators with common hypercyclic subspaces, J. Operator Theory 54 (2) (2005) 251–260.

[23] F. Bayart, Common hypercyclic vectors for composition operators, J. Operator Theory 52 (2004) 353–370.

[24] F. Bayart, E. Matheron, How to get common universal vectors, Indiana Univ. Math. J. 56 (2) (2007) 553–580.

[25] F. Bayart, E. Matheron, Hypercyclic operators failing the hypercyclicity criterion on classical Banach spaces, J. Funct. Anal. 250 (2) (2007) 426–441.
[26] F. Bayart, S. Grivaux, Hypercyclicity and unimodular point spectrum, J. Funct. Anal. 226 (2005) 281–300.
[27] L. Bernal-González, On hypercyclic operators on Banach space, Proc. Amer. Math. Soc. 127 (1999) 1003–1110.
[28] J. Bès, K. Chan, Approximation by chaotic operators and by conjugate classes, J. Math. Anal. Appl. 284 (2003) 206–212.
[29] J. Bès, Invariant manifolds of hypercyclic vectors for the real scalar case, Proc. Amer. Math. Soc. 127 (1999) 1801–1804.
[30] J. Bonet, F. Martínez-Giménez, A. Peris, A Banach space which admits no chaotic operator, Bull. Lond. Math. Soc. 33 (2001) 196–198.
[31] P.S. Bourdon, Invariant manifolds of hypercyclic vectors, Proc. Amer. Math. Soc. 118 (1993) 845–847.
[32] K. Chan, R. Sanders, Two criteria for a path of operators to have common hypercyclic vectors, J. Operator Theory 61 (1) (2009) 191–223.
[33] K. Chan, R. Sanders, A weakly hypercyclic operator that is not norm hypercyclic, J. Operator Theory 52 (1) (2004) 39–59.
[34] K. Chan, R. Sanders, An SOT-dense path of chaotic operators with common hypercyclic vectors, J. Operator Theory, in press.
[35] K. Chan, I. Seceleanu, Orbital limit points and hypercyclicity of operators on analytic function spaces, Math. Proc. R. Ir. Acad. 110A (2010) 99–109.
[36] K. Chan, I. Seceleanu, Hypercyclicity of shifts as a zero-one law of orbital limit points, J. Operator Theory, in press.
[37] J.A. Conejero, V. Müller, A. Peris, Hypercyclic behaviour of operators in a hypercyclic C0-semigroup, J. Funct. Anal. 244 (1) (2007) 342–348.
[38] G. Costakis, M. Sambarino, Genericity of wild holomorphic functions and common hypercyclic vectors, Adv. Math. 182 (2) (2004) 278–306.
[39] M. De la Rosa, C. Read, A hypercyclic operator whose direct sum $T \oplus T$ is not hypercyclic, J. Operator Theory 61 (2) (2009) 369–380.
[40] R. Douglas, Banach Algebra Techniques in Operator Theory, Academic Press, New York, 1972.
[41] D.W. Hadwin, E.A. Nordgren, H. Radjavi, P. Rosenthal, Most similarity orbits are strongly dense, Proc. Amer. Math. Soc. 76 (1979) 250–252.
[42] C. Kitai, Invariant closed sets for linear operators, PhD thesis, Univ. of Toronto, 1982.
[43] F. León-Saavedra, V. Müller, Rotations of hypercyclic and supercyclic operators, Integral Equations Operator Theory 50 (3) (2004) 385–391.
[44] J. Munkres, Topology: A First Course, Prentice Hall, 1975.
[45] C. Read, A solution to the invariant subspace problem, Bull. Lond. Math. Soc. 16 (4) (1984) 337–401.
[46] R. Sanders, Common hypercyclic vectors and the Hypercyclicity Criterion, Integral Equations Operator Theory 65 (2009) 131–149.
[47] H. Salas, Hypercyclic weighted shifts, Trans. Amer. Math. Soc. 347 (3) (1995) 993–1004.

[48] S. Shkarin, Non-sequential weak supercyclicity and hypercyclicity, J. Funct. Anal. 242 (1) (2007) 37–77.
[49] Tadeusz Kulczycki, Mateusz Kwa´snicki, Jacek Malecki and Andrzej Stos, Spectral properties of the Cauchy process on half-line and interval, Proc. London Math. Soc. (3) 101 (2010) 589–622.
[50] R. Banuelos ˜ and T. Kulczycki, ‘The Cauchy process and the Steklov problem’, J. Funct. Anal. 211 (2004) 355–423.
[51] R. Banuelos ˜ and T. Kulczycki, ‘Spectral gap for the Cauchy process on convex symmetric domains’, Comm. Partial Differential Equations 31 (2006) 1841–1878.
[52] R. Banuelos ˜ and T. Kulczycki, ‘Trace estimates for stable processes’, Probab. Theory Related Fields 142 (2009) 313–338.
[53] N. H. Bingham, ‘Maxima of sums of random variables and suprema of stable processes’, Probab. Theory Related Fields 26 (1973) 273–296.
[54] R. M. Blumenthal and R. K. Getoor, ‘The asymptotic distribution of the eigenvalues for a class of Markov operators’, Pacific J. Math. 9 (1959) 399–408.
[55] R. M. Blumenthal and R. K. Getoor, Markov processes and potential theory (Academic Press, Reading, MA, 1968).
[56] R. M. Blumenthal, R. K. Getoor and D. B. Ray, ‘On the distribution of first hits for the symmetric stable processes’, Trans. Amer. Math. Soc. 99 (1961) 540–554.
[57] K. Bogdan and T. Byczkowski, ‘Potential theory for the α-stable Schrödinger operator on bounded Lipschitz domains’, Studia Math. 133 (1999) 53–92.
[58] K. Bogdan and T. Grzywny, ‘Heat kernel of fractional Laplacian in cones’, Colloq. Math. 118 (2010) 365–377.
[59] K. Bogdan, T. Grzywny and M. Ryznar, ‘Heat kernel estimates for the fractional Laplacian’, Preprint, 2009, Ann. Probab., arXiv:0905.2626, to appear.
[60] T. Byczkowski, J. Malecki and M. Ryznar, ‘Bessel potentials, hitting distributions and Green functions’, Trans. Amer. Math. Soc. 361 (2009) 4871–4900.
[61] T. Byczkowski, J. Malecki and M. Ryznar, ‘Hitting half-spaces by Bessel–Brownian diffusions’, Potential Anal., http://www.springerlink.com/content/h346720387j31621/ to appear.
[62] A. Chakrabarti, B. N. Mandal and R. Gayen ‘The dock problem revisited’, Int. J. Math. Math. Sci. 2005 (2005) 3459–3470.
[63] Z. Q. Chen, P. Kim and R. Song, ‘Heat kernel estimates for Dirichlet fractional Laplacian’, J. Eur. Math. Soc., to appear.
[64] Z. Q. Chen and R. Song, ‘Intrinsic ultracontractivity and conditional gauge for symmetric stable processes’, J. Funct. Anal. 150 (1997) 204–239.
[65] Z. Q. Chen and R. Song, ‘Two sided eigenvalue estimates for subordinate Brownian motion in bounded domains’, J. Funct. Anal. 226 (2005) 90–113.
[66] Z. Q. Chen and R. Song, ‘Continuity of eigenvalues of subordinate processes in domains’, Math. Z. 252 (2006) 71–89.
[67] D. A. Darling, ‘The maximum of sums of stable random variables’, Trans. Amer. Math. Soc. 83 (1956) 164–169.
[68] R. D. DeBlassie, ‘The first exit time of a two-dimensional symmetric stable process from a wedge’, Ann. Probab. 18 (1990) 1034–1070.

[69] R. D. DeBlassie, ‘Higher order PDE’s and symmetric stable processes’, Probab. Theory Related Fields 129 (2004) 495–536.
[70] R. D. DeBlassie and P. J. Mendez-Hern ´ andez ´ , ‘α-continuity properties of the symmetric α-stable process’, Trans. Amer. Math. Soc. 359 (2007) 2343–2359.
[71] P. Duren, Theory of Hp spaces (Academic Press, New York, 1970).
[72] E. B. Dynkin, Markov processes, vols. I and II (Springer, Berlin, 1965).
[73] W. Feller, An introduction to probability theory and its applications, vol. 2, 3rd edn (Wiley, New York, 1971).
[74] D. W. Fox and J. R. Kuttler, ‘Sloshing frequencies’, Z. Angew. Math. Phys. 34 (1983) 668–696.
[75] K. O. Friedrichs and H. Lewy, ‘The dock problem’, Comm. Pure Appl. Math. 1 (1948) 135–148.
[76] R. K. Getoor, ‘Markov operators and their associated semi-groups’, Pacific J. Math. 9 (1959) 449–472.
[77] I. S. Gradstein and I. M. Ryzhik, Table of integrals, series, and products, 7th edn (eds A. Jeffrey and D. Zwillinger; Academic Press, New York, 2007).
[78] R. L. Holford, ‘Short surface waves in the presence of a finite dock. I, II’, Math. Proc. Cambridge Philos. Soc. 60 (1964) 957–983, 985–1011.
[79] N. Ikeda and S. Watanabe, ‘On some relations between the harmonic measure and the Lévy measure for a certain class of Markov processes’, Probab. Theory Related Fields 114 (1962) 207–227.
[80] F. B. Knight, ‘Random walks and a sojourn density process of Brownian motion’, Trans. Amer. Math. Soc. 109 (1963) 56–86.
[81] V. Kozlov and N. G. Kuznetsov, ‘The ice-fishing problem: the fundamental sloshing frequency versus geometry of holes’, Math. Methods Appl. Sci. 27 (2004) 289–312.
[82] T. Kulczycki, ‘Intrinsic ultracontractivity for symmetric stable processes’, Bull. Pol. Acad. Sci. Math. 46 (1998) 325–334.
[83] M. Kwa´snicki, ‘Spectral gap estimate for stable processes on arbitrary bounded open sets’, Probab. Math. Statist. 28 (2008) 163–167.
[84] M. Kwa´snicki, ‘Spectral gap estimate for stable processes on arbitrary bounded open sets’, Semigroup Forum 79 (2009) 183–192.
[85] P. J. Mendez-Hern ´ andez ´ , ‘Exit times from cones in $\mathbb{R}^n$ of symmetric stable processes’, Illinois J. Math. 46 (2002) 155–163.
[86] S. A. Molchanov and E. Ostrowski, ‘Symmetric stable processes as traces of degenerate diffusion processes’, Theory Probab. Appl. 14(1) (1969) 128–131.
[87] T. Poerschke, G. Stolz and J. Weidmann, ‘Expansions in generalized eigenfunctions of selfadjoint operators’, Math. Z. 202 (1989) 397–408.
[88] D. B. Ray, ‘Sojourn times of a diffusion process’, Illinois J. Math. 7 (1963) 615–630.
[89] M. Reed and B. Simon, Methods of modern mathematical physics, vol. 4, Analysis of operators (Academic Press, New York, 1978).
[90] D. Revuz and M. Yor, Continuous Martingales and Brownian motion (Springer, New York, 1999).
[91] M. Riesz, ‘Intégrales de Riemann–Liouville et potentiels’, Acta Sci. Math. Szeged (1938).

[92] K. Sato, Lévy processes and infinitely divisible distributions (Cambridge University Press, Cambridge, 1999).
[93] B. Simon, 'Schrödinger semigroups', Bull. Amer. Math. Soc., (N.S.) 7 (1982) 447–526.
[94] F. Spitzer, 'Some theorems concerning 2-dimensional Brownian motion', Trans. Amer. Math. Soc. 87 (1958) 187–197.
[95] E. M. Stein, Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals, Princeton Mathematical Series 43 (Princeton University Press, Princeton, NJ, 1993).
[96] M. M. Vainberg, Variational method and method of monotone operators in the theory of nonlinear equations (Wiley, New York, 1973).
[97] Mateusz Kwasnicki, Eigenvalues of the fractional Laplace operator in the interval, Journal of Functional Analysis 262 (2012) 2379–2402.
[98] K. Bogdan, T. Byczkowski, T. Kulczycki, M. Ryznar, R. Song, Z. Vondracek, Potential Analysis of Stable Processes and Its Extensions, Lecture Notes in Math., vol. 1980, Springer, 2009.
[99] K. Bogdan, T. Kulczycki, M. Kwasnicki, Estimates and structure of ´ α-harmonic functions, Probab. Theory Related Fields 140 (3–4) (2008) 345–381.
[100] E.B. Dynkin, Markov Processes, vols. I and II, Springer-Verlag, Berlin/Götingen/Heidelberg, 1965.
[101] M. Kwasnicki, Spectral analysis of subordinate Brownian motions in half-line, Studia Math. 206 (2011) 211–271.
[102] A. Zoia, A. Rosso, M. Kardar, Fractional Laplacian in bounded domains, Phys. Rev. E 76 (2007) 061121.
[103] R. Frank, L. Geisinger, Refined semiclassical asymptotics for fractional powers of the Laplace operator, J. Reine Angew. Math. 712 (2016) 1–37.
[104] R. Bañuelos, T. Kulczycki and B. Siudeja, On the trace of symmetric stable processes on Lipschitz domains, J. Funct. Anal. 257 (2009), no. 10, 3329–3352.
[105] F. A. Berezin, Covariant and contravariant symbols of operators, Izv. Akad. Nauk SSSR Ser. Mat. 13 (1972), 1134–1167.
[106] Z. Chen and R. Song, Two-sided eigenvalue estimates for subordinate processes in domains, J. Funct. Anal. 226 (2005), no. 1, 90–113.
[107] W. F. Donoghue, Jr., Monotone matrix functions and analytic continuation, Springer-Verlag, Berlin 1974.
[108] R. L. Frank and L. Geisinger, Two-term spectral asymptotics of the Dirichlet Laplacian on a bounded domain, in: Mathematical results in quantum physics. Proceedings of the QMath11 conference, World Scientific, Singapore (2011), 138–147.
[109] F. Hansen, An operator inequality, Math. Ann. 246 (1980), no. 3, 249–250.
[110] L. Hörmander, The analysis of linear partial differential operators. IV: Fourier integral operators, SpringerVerlag, Berlin 1985.
[111] V. J. Ivrii, The second term of the spectral asymptotics for the Laplace–Beltrami operator on manifolds with boundary and for elliptic operators acting in vector bundles, Soviet Math. Dokl. 20 (1980), no. 1, 1300–1302.

[112] M. Kwasnicki ´ , Spectral analysis of subordinate Brownian motions on the half-line, Studia Math. 206 (2011), no. 3, 211–271.
[113] P. Li and S. T. Yau, On the Schrödinger equation and the eigenvalue problem, Comm. Math. Phys. 88 (1983), no. 3, 309–318.
[114] E. H. Lieb, The classical limit of quantum spin systems, Comm. Math. Phys. 31 (1973), 327–340.
[115] E. H. Lieb and H. T. Yau, The stability and instability of relativistic matter, Comm. Math. Phys. 118 (1988), no. 2, 177–213.
[116] A. W. Marshall and I. Olkin, Inequalities: Theory of majorization and its applications, Math. Sci. Eng. 143, Academic Press, New York 1979.
[117] A. Pushnitski, The scattering matrix and the differences of spectral projections, Bull. Lond. Math. Soc. 40 (2008), 227–238.
[118] Y. Safarov and D. Vassiliev, The asymptotic distribution of eigenvalues of partial differential operators, Transl. Math. Monogr. 155, American Mathematical Society, Providence 1996.
[119] J. P. Solovej, T. Ø. Sørensen and W. L. Spitzer, Relativistic Scott correction for atoms and molecules, Comm. Pure Appl. Math. 63 (2010), 39–118.
[120] J. P. Solovej and W. L. Spitzer, A new coherent states approach to semiclassics which gives Scott's correction, Comm. Math. Phys 241 (2003), 383–420.
[121] J. Liu and B. Liu, A Laplacian-energy-like invariant of a graph, Match Commun. Math. Comput. Chem. 59 (2008) 355-372.
[122] F. Buckley, Iterated line graphs, Congr. Numer. 33 (1981) 390-394.
[123] F. Buckley, The size of iterated line graphs, Graph Theory Notes N.Y. 25(1993) 33-36.
[124] D. M. Cvetkovi´c, M. Doob, H. Sachs, Spectra of Graphs – Theory and Application, Academic Pres, New York, 1980.
[125] I. Gutman, The energy of a graph, Ber. Math.-Statist. Sekt. Forschungsz. Graz 103 (1978) 1–22.
[126] I. Gutman, Toal π-electron energy of benzenoid hydroearbons, Toies Curr. Chem 162 (1992) 29-63.
[127] I. Gutman, The energy of a graph: Old and new results, in: (A. Betten, A. Kohnert, R. Laue, A. Wassermann (Eds.), Algebraic Combinatorics and Applications, Springer, Berlin, 2001, pp. 196–211.
[128] I. Gutman, Topology and stability of conjugated hydrocarbons. The dependence of total π-electron energy on molecular topology, J. Serb. Chem. Soc. 70 (2005) 441–456.
[129] I. Gutman, B. Zhou, Laplacian energy of a graph, Lin. Algebra Appl. 414 (2006) 29–37.
[130] R. Merris, Laplacian matrices of graphs: a survey, Lin. Algebra Appl. 197–198 (1994) 143–176.
[131] B. Zhou, I. Gutman, On Laplacian energy of graphs, MATCH Commun. Math. Comput. Chem. 57(2007) 311-320.
[132] G. Indulal, A. Vijayakumar, On a pair of equienergetic graphs, MATOH Commun. Math. Comput. Chem. 55(2006) 83-90.

[133] B. Zhou, Lower bund for energy of quadrangle-free graphs, MATOM Commun. Math. Comput. Chem. 55(2006) 91-94
[134] B. Zhou, I. Gutman J.A de la Paña, J. Rada L. Mendoza, On spectral moments and energy of graphs, MATOM Commun. Math. Comput. Chem. 57(2007) 183-191.
[135] H. S. Ramane, H. B. Walikar, Construction of equienergetic graphs, MATOM Commun. Math. Comput. Chem. 57(2007) 203-210.
[136] Y. Hou, Z. Teng, C. Woo, On the spectral radius, k-deree and the upper bound of energy of a graphs, MATOM Commun. Math. Comput. Chem. 57 (2007) 341-350.
[137] L. Xu, Y. Hou, Equienergetic bipartite graphs, MATOM Commun. Math. Comput. Chem. 57 (2007) 363-370.
[138] I. Gutman, S. Zare Firoozabadi, J. A de la Peña, J. Rada, On the energy of regular graphs, MATOM Commun. Math. Comput. Chem. 57 (2007) 435-442.
[139] I. M. Chakravarti, On a characterization of irreducibility of a nonnegative matrix, Linear Algebra and its Applications, 10 (1975) 103-109.
[140] K. CH. Das, The Laplacian spectrum of a graphs, Computers and Mathematics with Applications, 48(2004) 715-724.
[141] B. Eichinger, Configuration statistics of Gaussian molecules, Macromolecules, 13(1980) 1-11.
[142] Grone R., Eigenvalues and the degree sequences of graphs, [J]. Linear and Multilinear Algebra, 1995, 39: 133-136.
[143] Kuang Jichang, Applied inequalities, Shandong Scienee and Technology Press, 2004 (Chinese).
[144] Li Jiongsheng, Pan Yongliang, A note on the second largest eigenvalue of the Laplacian matrix of a graph, [J]. Linear and Multilinear Algebra, 2000, 48(2): 117-121.
[145] B. Mohar, The Laplacian spectrum of graphs, in: Y. Alavi et al., eds., Graph Theory Combinatorics, and Applications (Wiley, New York, 1991) 871-888.
[146] Wang Boying, Brief discussion on the applications in trigonometry of majorizing inequalities, Bulletin of Mathematics, 1985, 9: 35-37 (Chinese)
[147] Bo Zhou, On sum of powers of the Laplacian eigenvalues of graphs, Linear Algebra and its Applications 429 (2008) 2239–2246.
[148] M. Fiedler, Algebraic connectivity of graphs, Czechoslovak Math. J. 23 (1973) 298–305.
[149] R. Grone, R. Merris, The Laplacian spectrum of a graph II, SIAM J. Discrete Math. 7 (1994) 221–229.
[150] I. Gutman, On graphs whose energy exceeds the number of vertices, Linear Algebra Appl., in press, doi:10.1016/j.laa.2007.09.024.
[151] I. Gutman, Graphs with smallest sum of squares of vertex degrees, Kragujevac J. Math. 25 (2003) 51–54.
[152] I. Gutman, B. Mohar, The quasi-Wiener and the Kirchhoff indices coincide, J. Chem. Inf. Comput. Sci. 36 (1996) 982–985.
[153] Y. Hong, X. Zhang, Sharp upper and lower bounds for the largest eigenvalue of the Laplacian matrices of trees, Discrete Math. 296 (2005) 187–197.
[154] M. Lazic, On the Laplacian energy of a graph, Czechoslovak Math. J. 56 (2006) 1207–1213. ´

[155] J. Li, Y. Pan, De Caen’s inequality and bounds on the largest Laplacian eigenvalues of a graph, Linear Algebra Appl. 328 (2001) 153–160.
[156] J. Rada, A. Tineo, Upper and lower bounds for the energy of bipartite graphs, J. Math. Anal. Appl. 289 (2004) 446–455.
[157] J. Palacios, Foster’s formulas via probability and the Kirchhoff index, Methodol. Comput. Appl. Probab. 6 (2004) 381–387.
[158] B. Zhou, I. Gutman, A connection between ordinary and Laplacian spectra of bipartite graphs,Linear and Multilinear Algebra 56 (2008) 305–310.
[159] K. C. Chan, R. Sanders, An SOT-dense path of chaotic operators with same hypercyclic vectors, J. Operator Theory 66 (2011), 107-124.
[160] F. Bayart, Common hypercyclic subspaces, Integral Equations Operator Theory 53(2005), 467-476.
[161] J. Bes, K. Chan, Denseness of hypercyclic operators on a Frechet space, Houston J. Math. 29(2003), 195-206.
[162] J. Bes, A. Peris, Hereditarily hypercyclic operators, J. Fund. Anal. 167(1999), 94-112.
[163] K.C. Chan, The density of hypercyclic operators on a Hilbert space, J. Operator The ory 47(2002), 131-143.
[164] G. COSTAKIS, Common Cesaro hypercyclic vectors, preprint, 2006.
[165] R.L. DEVANEY, An Introdudion to Chaotic Dynamical Systems, Second Edition, Addison-Wesley, Reading, MA 1989.
[166] K.-G. GROSSE-ERDMANN, Hypercyclic and chaotic weighted shifts, Studia Math. 139(2000), 47-68.
[167] D.W. Hadwin, E.A. Nordgren, H. Radjavi, P. Rosenthal, Most similarity or bits are strongly dense, Proc. Amer. Math. Soc. 76(1979), 250-252.
[168] F. LeOn-Saavedra, A. MONTES-RODRIGUEZ, Spectral theory and hypercyclic sub spaces, Trans. Amer. Math. Soc. 353(2001), 247-267.
[169] F. MartInez-Gimenez, A. Peris, Chaos for backward shift operators, Internat. J. Bifurcation and Chaos 12(2002), 1703-1715.
[170] N. Tsirivas, Common Hypercyclic Functions for Translation Operators with Large Gaps, (2016) 1-26.
[171] E. Abakumov, J. Gordon, Common hypercyclic vectors for multiples of backward shift, J. Funct. Anal. 200 (2003), 494-504.
[172] M. P. Aldred and D. H. Armitage, Harmonic analogues of G. R. Mac Lane’s universal functions II, J. Math. Anal. Appl. 220 (1998), 382-395.
[173] M. P. Aldred and D. H. Armitage, Harmonic analogues of G. R. MacLane’s universal functions, J. London Math. Soc. 57 (1998), 148-156.
[174] S. I. Ansari, Hypercyclic and cyclic vectors, J. Funct. Anal. 128 (1995), 374-383.
[175] D. H. Armitage, Permissible growth rates for Birkhoff type universal harmonic functions, J. Approx. Theory 136 (2005), 230-243.
[176] F. Bayart, Topological and algebraic genericity of divergence and universality, Studia Math. 167 (2005), 161-181.
[177] F. Bayart, Dynamics of holomorphic groups, Semigroup Forum 82 (2011), 229-241.
[178] F. Bayart, G. Costakis, D. Hadjiloucas, Topologically transitive skew-products of operators. Ergodic Theory Dynam. Systems 30 (2010), 33-49.

[179] F. Bayart, S. Grivaux, Frequently hypercyclic operators, Trans. Amer. Math. Soc. 358 (2006), 5083-5117.
[180] F. Bayart and E. Matheron, Dynamics of linear operators, Cambridge Tracts in Math. 179, Cambridge Univ. Press, 2009.
[181] F. Bayart, S. Grivaux, R. Mortini, Common bounded universal functions for composition operators, Illinois J. Math. 52 (2008), 995-1006.
[182] L. Bernal-Gonz´alez, Common hypercyclic functions for multiples of convolution and nonconvolution operators, Proc. Amer. Math. Soc. 137 (2009), 3787-3795.
[183] L. Bernal-Gonz´alez and A. Bonilla and G. Costakis, On the growth of zero-free MacLaneuniversal entire functions, Indag. Math. (N.S.) 23 (2012), 311-317.
[184] G. D. Birkhoff, Démonstration d' un théor`em él`ementaire sur les fonctions enti`eres, C. R. Acad. Sci. Paris, 189 (1929), 473-475.
[185] K. C. Chan, R. Sanders, Common supercyclic vectors for a path of operators, J. Math. Anal. Appl. 337 (2008), 646-658.
[186] K. C. Chan, R. Sanders, Rebecca Common hypercyclic vectors for the unitary orbit of a hypercyclic operator, J. Math. Anal. Appl. 387 (2012), 17-23.
[187] K. C. Chan and J. H. Shapiro, On cyclic behavior of translation operators on Hilbert spaces of entire functions, Indiana Univ. Math. J. 40 (1991) 1421-1449.
[188] G. Costakis, Approximation by translates of entire functions, Complex and harmonic analysis, 213-219 Destech Publ., Inc., Lancaster, PA, 2007.
[189] G. Costakis, Common Ces`aro hypercyclic vectors, Studia Math. 201 (2010), 203-226.
[190] G. Costakis, P. Mavroudis, Common hypercyclic entire functions for multiples of differential operators, Colloq. Math. 111 (2008), 199-203.
[191] G. Costakis, N. Tsirivas and V. Vlachou, Non-existence of common hypercyclic vectors for certain families of translations operators, in preparation.
[192] D. Drasin and E. Saksman, Optimal growth of entire functions frequently hypercyclic for the differentiation operator, J. Funct. Anal. 263 (2012), 3674-3688.
[193] E. Gallardo-Gutierrez, J. R. Partington, Common hypercyclic vectors for families of operators, Proc. Amer. Math. Soc. 136 (2008), 119-126.
[194] M. C. G´omez-Collado, F. Mart´ınez-Giménez, A. Peris and F. Rodenas, Slow growth for universal harmonic functions, J. Inequal. Appl. 2010, Art. ID 253690, 6 pp.
[195] S. M. Duios-Ruis, Universal functions and the structure of the space of entire functions, Dokl. Akad. Nauk SSSR 279 (1984) 792-795 (Russian); Soviet Math. Dokl. 30 (1984) 713- 716 (English translation).
[196] K.-G. Grosse-Erdmann, Universal families and hypercyclic operators, Bull. Amer. Math. Soc. (N.S.) 36 (1999), 345-381.
[197] K.-G. Grosse-Erdmann, Rate of growth of hypercyclic entire functions, Indag. Math. (N.S.) 11 (2000), 561-571.
[198] E. Matheron, Subsemigroups of transitive semigroups, Ergod. Th. Dynam. Sys., to appear.
[199] W. Rudin, Real and Complex Analysis, Mc Graw-Hill, New York, 1966.

[200] S. A. Shkarin, On the growth of D-universal functions. (Russian) Vestnik Moskov. Univ. Ser. I Mat. Mekh. 1993, 80-83 (1994); translation in Moscow Univ. Math. Bull. 48 (1993), 49-51.
[201] S. Shkarin, Universal elements for non-linear operators and their applications, J. Math. Anal. Appl. 348 (2008), 193-210.
[202] S. Shkarin, Remarks on common hypercyclic vectors, J. Funct. Anal. 258 (2010), 132-160.
[203] N. Tsirivas, Existence of common hypercyclic vectors for translation operators, submitted.
[204] K. Bogdan And B. A. Siudeja, Trace Estimates for Unimodal Lévy Processes, arXiv:1503.09126v1 [math.FA] 31 Mar 2015.
[205] K. Bogdan, T. Grzywny, and M. Ryznar. Barriers, exit time and survival probability for unimodal Lévy processes. Probability Theory and Related Fields, pages 1–44, 2014.
[206] K. Bogdan, T. Grzywny, and M. Ryznar. Density and tails of unimodal convolution semigroups. J. Funct. Anal., 266(6):3543–3571, 2014.
[207] K. Bogdan, T. Grzywny, and M. Ryznar. Dirichlet heat kernel for unimodal Lévy processes. Stochastic Process. Appl., 124(11):3612–3650, 2014.
[208] K. Bogdan, J. Rosi´nski, and L. Wojciechowski. Lévy systems and moment formulas for interlaced multiple Poisson integrals. ArXiv e-prints, Nov. 2014.
[209] R. M. Brown. The trace of the heat kernel in Lipschitz domains. Trans. Amer. Math. Soc., 339(2):889–900, 1993.
[210] T. Kulczycki. Properties of Green function of symmetric stable processes. Probab. Math. Statist., 17(2, Acta Univ. Wratislav. No. 2029):339–364, 1997.
[211] T. Kulczycki and B. Siudeja. Intrinsic ultracontractivity of the Feynman-Kac semigroup for relativistic stable processes. Trans. Amer. Math. Soc., 358(11):5025–5057, 2006.
[212] S. Minakshisundaram. Eigenfunctions on Riemannian manifolds. J. Indian Math. Soc. (N.S.), 17:159–165 (1954), 1953.
[213] H. Park and R. Song. Trace estimates for relativistic stable processes. Potential Anal., 41(4):1273–1291, 2014.
[214] M. L. Silverstein. Classification of coharmonic and coinvariant functions for a Lévy process. Ann. Probab., 8(3):539–575, 1980.
[215] R. Song and Z. Vondraˇcek. On suprema of Lévy processes and application in risk theory. Ann. Inst. Henri Poincaré Probab. Stat., 44(5):977–986, 2008.
[216] M. van den Berg. On the asymptotics of the heat equation and bounds on traces associated with the Dirichlet Laplacian. J. Funct. Anal., 71(2):279–293, 1987.
[217] T. Watanabe. The isoperimetric inequality for isotropic unimodal Lévy processes. Z. Wahrsch. Verw. Gebiete, 63(4):487–499, 1983.
[218] Rodrigo Bañuelos, Jebessa B. Mijena, Erkan Nane, Two-term trace estimates for relativistic stable processes, J. Math. Anal. Appl. 410 (2014) 837–846.
[219] R. Bañuelos, Trace asymptotics for stable heat semigroups, Schrödinger Institute Lecture, Vienna, 2009, http://www.math.purdue.edu/~banuelos/Lectures/Vienna2009.pdf.

[220] R. Bañuelos, Heat and Weyl asymptotics for fractional Laplacians, 34th SPA2010 Lecture, Osaka, Japan, 2010, http://www.math.purdue.edu/~banuelos/Lectures/SPA2010.pdf.
[221] K. Bogdan, The boundary Harnack principle for the fractional Laplacian, Studia Math. 123 (1986) 103–122.
[222] J. Brossard, R. Carmona, Can one hear the dimension of a fractal?, Comm. Math. Phys. 104 (1986) 103–122.
[223] E.B. Davies, Heat Kernels and Spectral Theory, Cambridge Univ. Press, Cambridge, 1989.
[224] M. Ryznar, Estimates of Green function for relativistic α-stable process, Potential Anal. 17 (2002) 1–23.
[225] Kamil Kaleta, Mateusz Kwaśnicki, Jacek Małecki, Asymptotic estimate of eigenvalues of pseudo-differential operators in an interval, J. Math. Anal. Appl. 439 (2016) 896–924.
[226] D. Applebaum, Lévy Processes and Stochastic Calculus, Cambridge University Press, Cambridge, 2004.
[227] R. Bañuelos, T. Kulczycki, B. Siudeja, On the traces of symmetric stable processes on Lipschitz domains, J. Funct. Anal. 257 (10) (2009) 3329–3352.
[228] K.L. Chung, Doubly-Feller process with multiplicative functional, in: Seminar on Stochastic Processes, Gainesville, FL, 1985, in: Prog. Probab. Stat., vol. 12, Birkhäuser Boston, Boston, MA, 1986, pp. 63–78.
[229] A.M.J. Davis, Waves in the presence of an infinite dock with gap, J. Inst. Math. Appl. 6 (1970) 141–156.
[230] B. Dyda, A. Kuznetsov, M. Kwaśnicki, Eigenvalues of the fractional Laplace operator in the unit ball, preprint, arXiv:1509. 08533v1, 2015.
[231] M. Fukushima, Y. Osima, M. Takeda, Dirichlet Forms and Symmetric Markov Processes, de Gruyter Stud. Math., vol. 19, de Gruyter, Berlin, Boston, 2010.
[232] V. Ivrii, Precise Spectral Asymptotics for Elliptic Operators Acting in Fiberings over Manifolds with Boundary, Lecture Notes in Math., vol. 1100, Springer-Verlag, Berlin, Heidelberg, 1984.
[233] V. Ivrii, Microlocal Analysis and Precise Spectral Asymptotics, Springer Monogr. Math., Springer-Verlag, Berlin, Heidelberg, 1998.
[234] N. Jacob, Pseudo Differential Operators and Markov Processes, vol. 1, Imperial College Press, London, 2001.
[235] T. Juszczyszyn, M. Kwaśnicki, Hitting times of points for symmetric Lévy processes with completely monotone jumps, Electron. J. Probab. 20 (48) (2015) 1–24.
[236] K. Kaleta, M. Kwaśnicki, J. Małecki, One-dimensional quasi-relativistic particle in the box, Rev. Math. Phys. 25 (8) (2013) 1350014.
[237] M. Kwaśnicki, Spectral theory for one-dimensional symmetric Lévy processes killed upon hitting the origin, Electron. J. Probab. 17 (83) (2012) 1–29.
[238] M. Kwaśnicki, J. Małecki, M. Ryznar, First passage times for subordinate Brownian motions, Stochastic Process. Appl. 123 (2013) 1820–1850.
[239] R. Schilling, R. Song, Z. Vondraček, Bernstein Functions: Theory and Applications, de Gruyter Stud. Math., vol. 37, de Gruyter, Berlin, Boston, 2012.

[240] Bo Cheng, Bolian Liu, The normalized incidence energy of a graph, Linear Algebra and its Applications 438 (2013) 4510–4519.
[241] B. Bollobás, P. Erdös, Graphs of extremal weights, Ars. Combin. 50 (1998) 225–233.
[242] ¸S.B. Bozkurt, A.D. Güngör, I. Gutman, A.S. Çevik, Randi´c matrix and Randi´c energy, MATCH Commun. Math. Comput. Chem. 64 (2010) 239–250.
[243] R. Brualdi, Energy of a graph, in: Notes for AIM Workshop on Spectra of Families of Matrices Described by Graphs, Digraphs, and Sign Patterns, 2006.
[244] M. Cavers, S. Fallat, S. Kirkland, On the normalized Laplacian energy and general Randi´c index R−1 of graphs, Linear Algebra Appl. 433 (2010) 172–190.
[245] F.R.K. Chung, Spectral Graph Theory, American Mathematical Society, Providence, 1997.
[246] V. Consonni, R. Todeschini, New spectral indices for molecule description, MATCH Commun. Math. Comput. Chem. 60 (2008) 3–14.
[247] C.A. Coulson, On the calculation of the energy in unsaturated hydrocarbon molecules, Math. Proc. Cambridge Philos. Soc. 36 (1940) 201–203.
[248] I. Gutman, D. Kiani, M. Mirzakhah, On incidence energy of graphs, MATCH Commun. Math. Comput. Chem. 62 (2009) 573–580.
[249] I. Gutman, D. Kiani, M. Mirzakhah, B. Zhou, On incidence energy of a graph, Linear Algebra Appl. 431 (2009) 1223–1233.
[250] I. Gutman, X. Li, J. Zhang, Graph energy, in: M. Dehmer, F. Emmert-Streib (Eds.), Analysis of Complex Networks. From Biology to Linguistics, Wiley-VCH, Weinheim, 2009, pp. 145–174.
[251] I. Gutman, J.Y. Shao, The energy change of weighted graphs, Linear Algebra Appl. 435 (2011) 2425–2431.
[252] M. Jooyandeh, D. Kiani, M. Mirzakhah, Incidence energy of a graph, MATCH Commun. Math. Comput. Chem. 62 (2009) 561–572.
[253] J.H. Koolen, V. Moulton, Maximal energy graphs, Adv. Appl. Math. 26 (2001) 47–52.
[254] J.H. Koolen, V. Moulton, Maximal energy bipartite graphs, Graphs Combin. 19 (2003) 131–135.
[255] X. Li, Y. Yang, Sharp bounds for the general Randi´c index, MATCH Commun. Math. Comput. Chem. 51 (2004) 155–166.
[256] M. Mateljevi´c, Vladimir Bo˘zin, I. Gutman, Energy of a polynomial and the Coulson integral formula, J. Math. Chem. 48 (2010) 1062–1068.
[257] V. Nikiforov, The energy of graphs and matrices, J. Math. Anal. Appl. 326 (2007) 1472–1475.
[258] M. Randi´c, On characterization of molecular branching, J. Amer. Chem. Soc. 97 (1975) 6609–6615.
[259] J.A. Rodríguez, A spectral approach to the Randi´c index, Linear Algebra Appl. 400 (2005) 339–344.
[260] Lu Qiao, Shenggui Zhang ∗, Bo Ning, Jing Li, Coulson-type integral formulas for the general Laplacian-energy-like invariant of graphs I, J. Math. Anal. Appl. 435 (2016) 1249–1261.

[261] C.A. Coulson, J. Jacobs, Conjugation across a single bond, J. Chem. Soc. (1949) 2805–2812.
[262] T.W. Gamelin, Complex Analysis, Springer-Verlag, New York, 2001.
[263] I. Gutman, B. Zhou, B. Furtula, The Laplacian-energy like invariant is an energy like invariant, MATCH Commun. Math. Comput. Chem. 64 (2010) 85–96.
[264] X. Li, Y. Shi, I. Gutman, Graph Energy, Springer, New York, 2012.
[265] L. Shi, H. Wang, The Laplacian incidence energy of graphs, Linear Algebra Appl. 439 (2013) 4056–4062.
[266] D. Stevanović, Laplacian-like energy of trees, MATCH Commun. Math. Comput. Chem. 61 (2009) 407–417.
[267] K. Yates, Hückel Molecular Orbital Theory, Academic Press, New York, 1978.
[268] B. Zhu, The Laplacian-energy like of graphs, Appl. Math. Lett. 24 (2011) 1604–1607
[269] Lu Qiao, Shenggui Zhang, Jing Li, Coulson-type integral formulas for the general Laplacian-energy-like invariant of graphs II, (2016) 1-17.
[270] G.E. Andrews, R. Askey, R. Roy, Special funtions, Cambridge University Press, Cambridge, 1999.
[271] T.W. Gamelin, Complex Analysis, Springer-Verlag, New York, 2001.
[272] I. Gutman, X. Li (Eds), Energies of Graphs — Theory and Applications, Math. Chem. Monogr., vol.17, ISBN 978-86-6009-033-3, 2016, p.III+290, Kragujevac.
[273] X. Li, Y. Shi, I. Gutman, Graph Energy, Springer, New York, 2012.
[274] Shawgy Hussein and Afnan Ali, Asymptotic of Eigenvalues of Pseudo-Differential Operators and Coulson-Type Integral Formulas with Common Hypercylic Functions, Ph.D. Thesis Sudan University of Science and Technology, Sudan (2019).