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Application of the G'/G –Expansion Method for the Schrödinger Equation

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Abstract

In this paper, we present travelling wave solutions for the Schrödinger equation. The (G'/G)-expansion method is used to determine travelling wave solution of equation. The travelling wave solutions are expressed by the hyperbolic functions, the trigonometric functions and the rational functions. It is shown that the proposed method is direct, effective and can be used for many other nonlinear evolution equations in mathematical physics. We have verified all the obtained solutions with the aid of Maple.

Keywords: (G'/G) –expansion method; Schrödinger equation; travelling wave solutions.

Introduction

During the past decades, the investigation of the exact travelling wave solutions of nonlinear partial differential equations (NLPDEs) plays an important role in the study of complex physical and mechanical phenomena. Several effective methods for obtaining exact solutions of NLPDEs, such as Painleve method (Tabor and Carnevale, 1982). Jacobi elliptic function method (Dai and Zhang, 2006; Hirota, 2004) .The sine-cosine function method (Wazwaz ,2004). The tanh - coth function method (Malfliet and Hereman, 1996). The exp-function method (He and Wu, 2006).

The homogeneous balance method and so on have been presented. Recently (Wang et al, 2008). Proposed the (G'/G) –expansion method to find travelling wave solutions of NLEEs (Bekir, 2008; Aslan, 2009). Applied this method to obtain travelling wave Solutions of some NLEEs. More recently, some authors (Tong and Wang, 2008; Zhang and Wei, 2008). Applied this method to improve and extend al's work, to solve variable coefficient and high dimensional equations expansion problem for using the method to solve nonlinear differential difference equations (Zhou and Chao, 2009). Modified the method to derive travelling wave solutions for solved the equations with the balance numbers which are not positive integers, by this method.

For studying the Vakhnenko equation (Wen et al, 2009). Presented a new function expansion method which can be thought of as the generalization of the (G'/G) –expansion method (Hirota, 2004). Applied this method for solving the combined and the double combined sinh-cosh-Gordon equations.

In this work, we apply the (G'/G) - expansion method to solve the Schrödinger equation.

The Schrödinger equation appears in various areas of applied mathematics, This equation has illuminated sufficiently atomic phenomena and dynamical centerpiece of quantum wave mechanics.

We assume that Erwin Schrödinger generated his equation based on three major principles such as de Broglie's hypothesis of matter wave, the law of conservation of energy and classical plane wave equation.

NLSE has been indicated to manage the evolution of a wave packet in a weakly nonlinear and dispersive medium and has eventuated diverse fields such as nonlinear optics, water waves and plasma physics. Another implementation of this equation is in pattern formation, where it has been used to model some non-equilibrium pattern forming systems.

Especially, this equation is now widely used in the optics field as a good model for optical pulse propagation in nonlinear fibers (Ablowitz et al, 2004; Wazwaz, 2006).

Materials and Method

Let us consider a general non-linear PDE with independent variables $x, y, z \; and \; t$ is of in the form

$$P\,(u, u_x\,, u_y\,, u_t\,, u_{xx}\,, u_{yy}\,, u_{xy}\,,\; u_{txy}\,,\ldots) = 0\;, \qquad (1)$$

where u$(x\,,y\,,t)$ is an unknown function and P is polynomial in u$(x\,,y\,,t)$ and its partial derivatives, and contains higher order derivatives and nonlinear terms.

step 1

We combine the real variables $x\,,y$ and t by a compound variable:

$$u(x,y,t) = u(\xi) := k_1 x + k_2\, y + k_3\, t\,, \qquad (2)$$

the travelling wave transformation (2) converts Eq.(1) into an ordinary differential equation (ODE)for :

$$Q(u\,,u',u'',\ldots)=0 \qquad (3)$$

where Q is a polynomial of u and it's derivatives and the superscripts indicate the ordinary derivatives with respect to ξ.

step 2

According to possibility Eq. (3) can be integrated term by term one or more time yields constant(s) of integration. The integral constant may be zero, for simplicity.

step 3

Suppose the travelling wave solution of Eq. (1) can be expressed as follows:

$$u(\xi)=\sum_{i=0}^{N} a_i \left(\frac{G'(\xi)}{G(\xi)}\right)^i \qquad (4)$$

Where (a_i $(i=0,1,2,\ldots,N)$ are arbitrary constants to be determined later and G = G(ξ)

satisfies the following auxiliary equation:

$$G''(\xi)+\lambda G'(\xi)+\mu G(\xi)=0 \qquad (5)$$

λ and μ are arbitrary constants.

step 5

Substitute Eq.(4) and Eq.(5) into Eq.(3) with the value of N obtained in step 4, we obtain polynomials in $\left(\frac{G'(\xi)}{G(\xi)}\right)^i$ $(i=0,1,2,\ldots,N)$ we collect each coefficient of the resulted polynomials and setting them to zero yields a set of algebraic equation for k_1, k_2, k_3, μ, λ and a_i $(i=0,1,2,\ldots,N)$.

step 6

Suppose that the value of the constants k_1 , k_2, k_3, μ, λ and $a_i\,(i = 0, 1\,,2, \dots, N)$ can be found by solving of eq.(5) is well known to us, obtained in step 5. Since the general of Eq.(5) is well known to us, inserting the values of k_1 , k_2, k_3, μ, λ and $a_i\,(i = 0, 1\,,2, \dots, N)$ into Eq. (4),we obtain more general type new exact travelling wave solution of the nonlinear partial differential equation (1) using the general solution of Eq.(5), we have the following ratios :

$$\frac{G'(\xi)}{G(\xi)} = \begin{cases} \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} . \frac{C_1 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{C_1 \operatorname{sish}\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)} ; \lambda^2 - 4\mu > 0 \\ \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} . \frac{C_1 \sin\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + C_2 \cos\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{C_1 \cos\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + C_2 \sin\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)} ; \lambda^2 - 4\mu < 0 \\ \frac{-\lambda}{2} + \frac{C_2}{C_1 + C_2 \xi} ; \qquad \lambda^2 - 4\mu = 0 \end{cases} \tag{6}$$

Where C_1 and C_2 are arbitrary constants

The exact solution of Schrödinger equation by (G'/G)- expansion method:

We will apply the generalized (G'/G) expansion method to construct the exact solution of the following KdV equation.

Application:

$$u_t + u_{ttt} + 6\, u u_x = 0 \tag{7}$$

Using the transformation u(x , t) = u(ξ) where ξ = $k_1 x + k_2\, t$ the PDE is reduced to an ODE

$$k_1\, u' + k_2^3\, u''' - 6 k_1\, u u' = 0 \tag{8}$$

Where the primes denote the derivative with respect to ξ.

Using the balancing uu' and u''' i.e n + n+ 1 = n+ 3 and thus the substitution Eq.(4) for u(ξ) is written as:

$$u(\xi) = a_0 + a_0 \left(\frac{G'(\xi)}{G(\xi)}\right) + a_2 \left(\frac{G'(\xi)}{G(\xi)}\right)^2 , \quad (9)$$

where $a_0, a_1 \; a_2$, λ and μ are arbitrary constants to be determined later .

Substituting eqs.(9) and (5) into (8) and equating the coefficients of $\left(\frac{G'(\xi)}{G(\xi)}\right)^i \; (i = 0,1,2,\ldots,N)$

to zero, we obtain a system of algebraic equation in a_0, a_1, a_2, λ and μ as following

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^0 : -k_2 a_1 \mu^2 - 6 k_2^3 a_2 \lambda \mu^2 - k_2^3 a_1 \lambda^2 \mu + 6 k_1 a_0 a_1 \mu^2 = 0 ,$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^1 : -16k_2^3 a_2 \mu^2 + 6k_1 a_1^2 \mu + 12k_1 a_0 a_2 \mu - k_2 a_1 \lambda - 14k_2^3 a_2 \lambda^2 \mu - 2k_2 a_2 \mu$$
$$-k_2^3 a_1 \lambda^3 + 6k_1 a_0 a_1 \lambda - 8k_2^3 \lambda \mu = 0 ,$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^2 : -2k_2 2 \lambda - 8k_2^3 a_2 \lambda^3 + 6k_1 a_0 a_1 - k_2 a_1 + 6 k_2 a_1^2 \lambda - 8k_2^3 a_1 \mu$$
$$+6k_1 a_0 a_1 - 2k_2 a_1 + 6k - a_1^2 \lambda - 8k_2^3 a_1 \mu - 52k_2^3 a_2 \lambda \mu$$
$$-7k_2^3 a_1 \lambda^2 + 12k_1 a_0 a_2 \lambda + 18 k_1 a_1 a_2 \mu = 0,$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^3 : 12k_1 a_2^2 \mu - 2k_2 a_2 - 40k_2^3 a_2 \mu + 12k_1 a_0 a_2 + 18k_1 a_1 a_2 \lambda + 6 k_1 a_1^2$$
$$-38k_2^3 a_2 \lambda^2 - 12 k_2^3 a_1 \lambda = 0,$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^4 : -6 k_2^3 a_1 + 18k_1 a_1 a_2 + 12k_2^3 a_2^2 \lambda - 54k_2^3 a_2 \lambda = 0 ,$$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^5 : \ 12k_1 a_2^2 - 24 k_2^3 a_2 = 0 \ . \tag{10}$$

Solving this system by Maple gives:

The first of the solution:

$$\begin{cases} a_0 = \frac{1}{24} \frac{a_1^2 + 32k_2^4\mu + 4k_2^2}{k_2^2} \ , \quad a_1 = 2k_2^2 \lambda \ , \\ a_2 = 2k_2^2 \ , \quad \mu = \mu \ , \quad k_1 = k_1 \ , \quad k_2 = k_2 \ . \end{cases} \tag{11}$$

The second of the solution:

$$\begin{cases} a_0 = \frac{1}{6} + \frac{1}{6} k_2^2 \lambda^2 \ , \quad a_1 = 2k_2^2 \lambda \\ a_2 = 2k_2^2 \ , \ \lambda = \lambda \ , \quad \mu = \mu \ , \ k_1 = k_2 \ . \end{cases} \tag{12}$$

Case 1

When $(\lambda^2 - 4\mu > 0)$, we obtain the hyperbolic solution in the form

$$\frac{G'(\xi)}{G(\xi)} = \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} . \frac{C_1 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right)}{C_1 \operatorname{sish}\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right)} \tag{14}$$

$$u_1(\xi) = \frac{1}{24} \frac{a_1^2 + 32k_2^4\mu + 4k_2^2}{k_2^2} + 2k_2^2 \lambda \left(\frac{G'(\xi)}{G(\xi)}\right) + 2k_2^2 \left(\frac{G'(\xi)}{G(\xi)}\right)^2 \tag{15}$$

$$\{ \ \lambda = 2 \ , \ \mu = 0 \ , k_1 = 1 \ , \ k_2 = 1 \ , C_1 = 0 \ , C_2 \neq 0 \tag{16}$$

$$u_1(\xi) = \frac{-28}{24} + 2coth^2(\xi) \quad (17)$$

We find:

$$\begin{cases} u_1(\xi) = \frac{-28}{24} + 2coth^2(\xi) \\ \xi = x + t \end{cases} \quad (18)$$

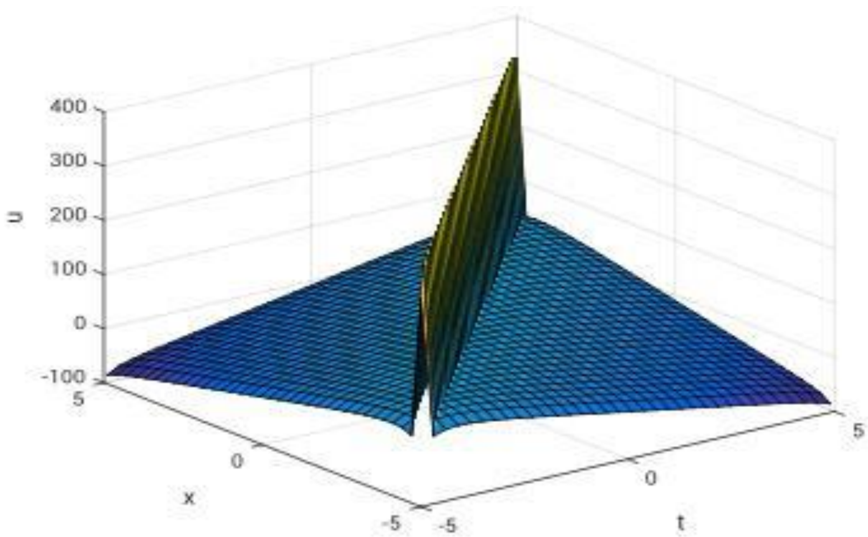

Figure 1: The graph of exact travelling wave solution of eq. (18)

and

$$u_2(\xi) = \frac{1}{6} + \frac{1}{6} k_2^2\lambda^2 + 2k_2^2 \lambda \left(\frac{G'(\xi)}{G(\xi)}\right) + 2k_2^2 \left(\frac{G'(\xi)}{G(\xi)}\right)^2 \quad (19)$$

$$\{\lambda = 2 , \ \mu = 0 \ , k_1 = 2 , \ k_2 = 1 \ , C_1 = 0 , C_2 \neq 0 \quad (20)$$

We fin

$$\begin{cases} u_2(\xi) = \frac{-7}{6} + 2tanh^2(\xi) \\ \xi = x + t \end{cases} \quad (21)$$

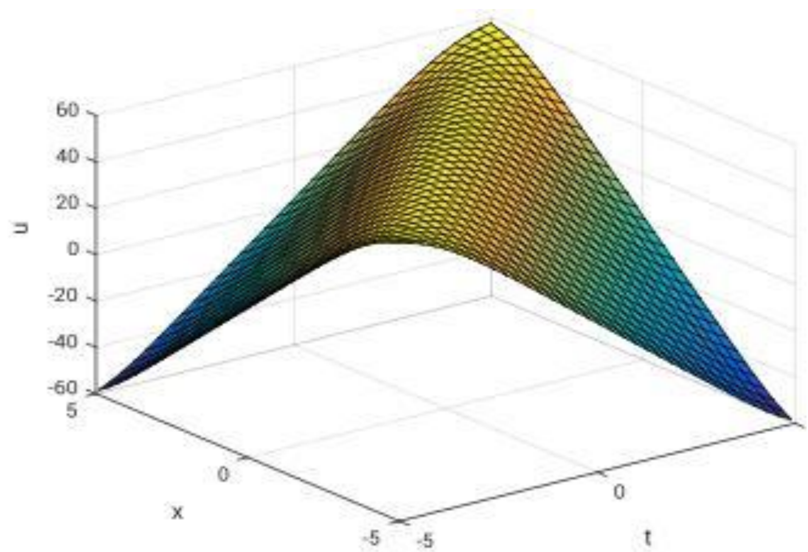

Figure 2: The graph of exact travelling wave solution of eq. (21)

Case 2

When ($\lambda^2 - 4\mu < 0$), we have the trigonometric function solution in the form :

$$\frac{G'(\xi)}{G(\xi)} = \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2} \cdot \frac{C_1 \sin\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + C_2 \cos\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{C_1 \cos\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + C_2 \sin\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)} \quad (22)$$

$$u_3(\xi) = \frac{1}{24}\,\frac{a_1^2 + 32k_2^4\mu + 4\,k_2^2}{k_2^2} + 2k_2^2\,\lambda\left(\frac{G'(\xi)}{G(\xi)}\right) + 2k_2^2\left(\frac{G'(\xi)}{G(\xi)}\right)^2 \quad (23)$$

$$\{\ \lambda = 2\ ,\ \ \mu = 2\ \ ,k_1 = 2\ ,\ k_2 = 2\ ,C_1 \neq 0\ \ ,C_2\ = 0\ , \quad (24)$$

We find:

$$\begin{cases} u_3(\xi) = \ 224\ + 8cot^2(\xi) \\ \xi = 2x + 2t \end{cases} \quad (25)$$

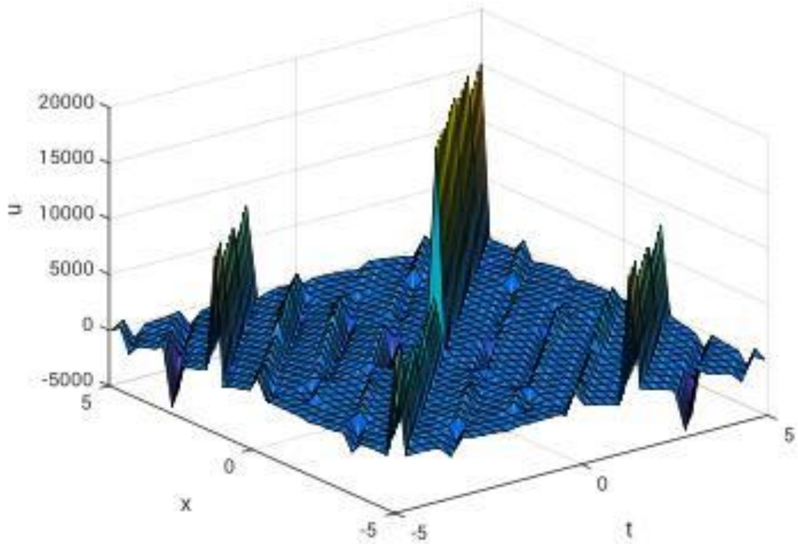

Figure 3: The graph of exact travelling wave solution of eq. (25)

and

$$u_4(\xi) = \frac{1}{6} + \frac{1}{6}\, k_2^2\lambda^2 \; + 2k_2^2\, \lambda \left(\frac{G'(\xi)}{G(\xi)}\right) + 2k_2^2 \left(\frac{G'(\xi)}{G(\xi)}\right)^2 \qquad (26)$$

$$\{\ \lambda = 2\ ,\ \ \mu = 0\ \ ,k_1 = 2\ ,\ k_2 = 1\ \ , C_1\ = 0\ , C_2 \neq 0 \qquad (27)$$

We find:

$$\begin{cases} u_4(\xi) = \dfrac{-31}{6} + 2tanh^2(\xi) \\ \xi = 2x + 2t \end{cases} \qquad (28)$$

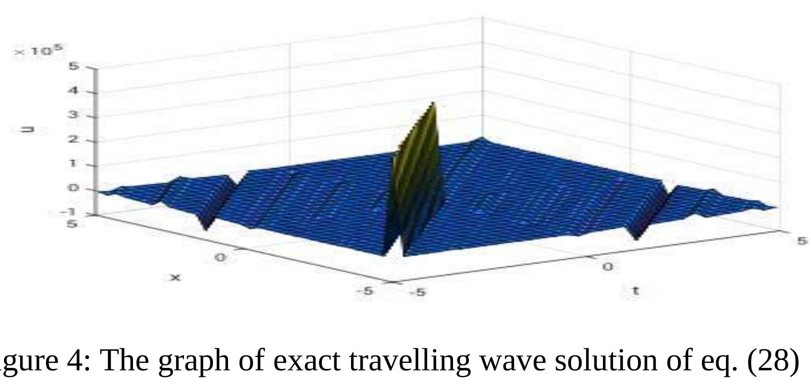

Figure 4: The graph of exact travelling wave solution of eq. (28)

Case 3

When $(\lambda^2 - 4\mu\ ) = 0$, we get the rational function solution in the form:

$$\frac{G'(\xi)}{G(\xi)} = \frac{-\lambda}{2} + \frac{C_2}{C_1 + C_2\xi} \tag{29}$$

$$u_5(\xi) = \frac{1}{24}\,\frac{a_1^2 + 32k_2^4\mu + 4\,k_2^2}{k_2^2} + 2k_2^2\,\lambda\left(\frac{G'(\xi)}{G(\xi)}\right) + 2k_2^2\left(\frac{G'(\xi)}{G(\xi)}\right)^2 \tag{30}$$

$$\{\ \lambda = 2\ ,\ \ \mu = 1\ \ \ , k_1 = 1\ ,\ k_2 = 1\ , C_1 = 1\ \ , C_2\ = 1\ , \tag{31}$$

we find:

$$\begin{cases} u_5(\xi) = \dfrac{1}{6} + \dfrac{2}{(1+\xi)^2} \\ \xi = x + t \end{cases} \tag{32}$$

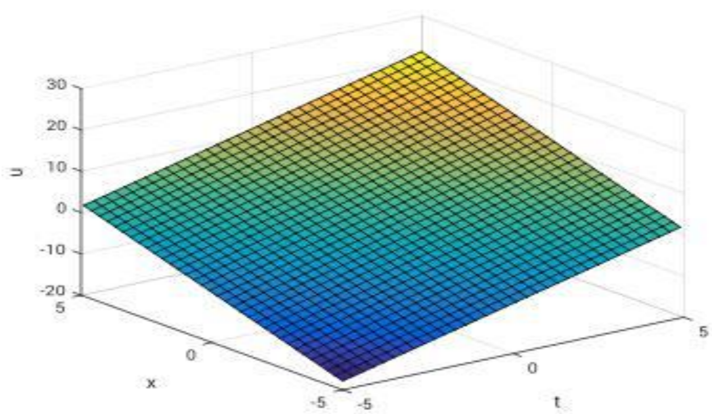

Figure 5: The graph of exact travelling wave solution of eq. (32)

and

$$u_6(\xi) = \frac{1}{6} + \frac{1}{6}\, k_2^2\lambda^2 \; + 2k_2^2\, \lambda \left(\frac{G'(\xi)}{G\,(\xi)}\right) + 2k_2^2 \left(\frac{G'(\xi)}{G\,(\xi)}\right)^2 \qquad (33)$$

$$\{\ \lambda = 2\ ,\ \ \mu = 1\ \ ,k_1 = 1\ ,\ k_2 = 1\ \ ,C_1\ = 1\ ,C_2 = 1 \qquad (34)$$

we find:

$$\begin{cases} u_6(\xi) = \dfrac{29}{6} + \dfrac{2}{(1+\xi)^2} \\ \xi = x + t \end{cases} \qquad (35)$$

Results and Discussion:

It is worth declaring that some of our obtained solutions are in good agreement with already published results which are presented .

Moreover some of obtained exact solutions are described in Figures 1- 5

Beyond this table, we obtain new exact travelling wave solutions $u_1(\xi), u_2(\xi), u_3(\xi), u_4(\xi), u_5(\xi)\ and\ u_6(\xi)$

Which are not being established in the previous literature

Conclusions:

In this paper, abundant l wave solutions of nonlinear partial differential equations have been found via presented (G'/G) –expansion method. We have used this method to the Schrödinger equation. As a result, we obtained plentiful new exact solutions including hyperbolic functions, the trigonometric functions and the rational functions which might have significant impact on future researches. The obtained solutions with free parameters may be important to explain some physical phenomena. It is shown that the performance of this method is productive, effective and well-built mathematical tool for solving nonlinear evolution equations.

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