Chapter 4

Control of Mechanical and Nonholonomic Systems

Analysis and synthesis of control strategies for nonlinear systems with nonholonomic constraints are the subject of extensive research. These systems are typical of mechanical applications such as wheeled mobile robots (rolling constraints), free-space manipulators (conservation of angular momentum) and redundant manipulators subject to a given inverse kinematic control. From the theoretical point of view, the control of nonholonomic systems presents interesting aspects. First, the control problem is a true nonlinear one since a nonlinear nonholonomic system is not linearly controllable. Moreover, controllability in the nonlinear setting - which is strictly related to the nonholonomic nature of the system - does not imply stabilizability by smooth time-invariant feedback. As a consequence, a combination between feedforward (off-line planning) and feedback laws of a more general kind (e.g. discontinuous or periodic time-varying control is necessary.

4.1 Background in Kinematic Nonholonomic Control Systems

We provide a summary of recent developments in control of nonholonomic systems. It is is organized so as to give an introduction to nonholonomic control systems and where they arise in applications, classification of models of nonholonomic control systems, control problem formulations, motion planning results, stabilization results, and current and future research topics.

4.1.1 Nonholonomic Motion Planning

In nonholonomic motion planning one's goal is to use open-loop control to reach a desired point in phase space. Nonholonomic systems, by virtue of the nonintegrable nature of their constraints, are amenable to rather elegant path planning algorithms. The basic situation considered is usually that of kinematic control systems, where the vector fields defining the system velocities do not span the state space, but nonetheless one *can* move from any point of the space to any other. This is, as we have seen, a fundamental property of nonholonomic systems.

We shall consider the class of completely controllable kinematic systems of the form

$$\dot{x} = \sum_{i}^{m} u_i(t) g_i(x), \tag{4.1.1}$$

Where $x \in R^n$ for a suitable class of functions g_i on R^n and a suitable class of function u_i on R^+ .

The ***motion planning problem*** is to find an efficient algorithm that gives for every pair of points p and q an open loop control $t \to u(t) = (u_1(t), \ldots, u_n(t))$ that steers the system from p to q.

4.1.1 Example (Generalized Heisenberg System). We consider first a generalization of the Heisenberg system. The system is the following:

$$\dot{x} = u \tag{4.1.2}$$

$$\dot{Y} = x\,u^T - u\,x^T \tag{4.1.3}$$

where $x, u \in R^n$ and $Y \in so(n), n \geq 2$. Here $so(n)$ is the Lie algebra of $n \times n$ skew-symmetric matrices, and elements of R^n are viewed as column vectors.

In terms of components, the last equation reads

$$\dot{Y}_{ij} = x_i u_j - x_j u_i \,. \tag{4.1.4}$$

The importance of this system is that it is a canonical form for a class of controllable systems of the form $\dot{x} = B(x)u,\ u \in R^n, x \in R^{n(n+1)/2}$.

The class in question is as follows: Let E^0 be the subbundle of the tangent bundle spanned by the control fields, and define the ***first derived algebra*** to be given by $E^1 = E^0 + [E^0, E^0]$. Then this system is a normal form for the controllable systems of this type, where the first derived algebra of control vector fields spans the tangent space $TR^{n(n+1)/2}$ at any point.

That is, Brockett showed that such a system can be transformed to the form (4.1.2)-(4.1.3) up to a suitable order in a neighborhood of a given point such as the origin.

The key to controlling this system is being able to change Y without changing x. Since x is directly controlled, it is easily changed. We present here a method of changing Y using sinusoids, which is motivated by the optimal control problem.

To solve the motion planning problem for this system, the idea is to proceed along loops in x- space, which gradually drives one through Y- space. This is just a reflection of the holonomy in the system. Motivated by the fact that the optimal solution of the Heisenberg system gives a u that consists of sinusoids, we choose the control law

$$u_i = \sum_k a_{ik} \sin kt + \sum_k b_{ik} \cos kt, \qquad k = 1,2, \ldots, \tag{4.1.5}$$

where a_{ik} and b_{ik} are real numbers. Since $\dot{x}_i = u_i$, integration gives

$$x_i = -\sum_k \frac{a_{ik}}{k} \cos kt + \sum_k \frac{b_{ik}}{k} \sin kt + C_i, \tag{4.1.6}$$

where C_i is a constant depending on the initial value of x.

Substituting these equations for $x_i(t)$ and $u_i(t)$ into equation (4.1.4) and integrating yields

$$Y_{ij}(2\pi) = \sum_k \frac{2\pi}{k} \left(b_{ik}\, a_{jk} - b_{jk}\, a_{ik} \right) + Y_{ij}(0), \tag{4.1.7}$$

since all integrals except those of the squares of cosine and sine vanish. Under this input, the $x's$ remain unchanged.

Thus, this gives the following solution to the motion planning problem:

First drive the x to the desired final value; then use the control to drive Y to the desired final value. ♦

Chained Systems. Similar algorithms may also be given for higher order systems. One such class that may be easily handled is the class of chained systems, which are systems of the form

$$\begin{aligned}\dot{\xi}_1 &= v_1,\\ \dot{\xi}_2 &= v_2,\\ \dot{\xi}_3 &= \xi_2 v_1,\\ &\cdots\\ \dot{\xi}_n &= \xi_{n-1} v_1.\end{aligned} \tag{4.1.8}$$

One can show that a large class of kinematic two-input systems may be put into this form. To make this specific, we state the following result and then illustrate the proof of the theorem for the Heisenberg system.

4.1.2 Proposition. *Consider a controllable system*

$$\dot{x} = u_1 g_1(x) + u_2 g_2(x), \tag{4.1.9}$$

where g_1 and g_2 are linearly independent and smooth. Define the distributions

$$\begin{aligned}\Delta 0 &\equiv span\{g_1, g_2, ad g_1\, g_2, \cdots, ad_{g1}^{n-2}\, g_2\},\\ \Delta 1 &\equiv span\{g_2, ad g_1\, g_2, \cdots, ad_{g1}^{n-2}\, g_2\},\\ \Delta 2 &\equiv span\{g_2, ad g_1\, g_2, \cdots, ad_{g1}^{n-3}\, g_2\}.\end{aligned} \tag{4.1.10}$$

If there exists an open set $U \in R^n$ such that $\Delta_0(x) = R^n$ for all $\in U$, Δ_1, and Δ_2 are involutive on U, and there exists a smooth function $h_1 : U \to R$ such that $dh_1 \cdot \Delta_1 = 0$ and $£_{g1}h_1 = 1$, then there exists a local feedback transformation

$$\xi = \emptyset(x), \qquad u = \beta(x)v \tag{4.1.11}$$

such that the transformed system is in the chained form (4.1.8).

4.1.3 Example. Consider now the Heisenberg system

$$\begin{aligned} \dot{x} &= u_1, \\ \dot{y} &= u_2, \\ \dot{z} &= xu_2 - yu_1. \end{aligned} \tag{4.1.12}$$

In this case $\Delta_0 = R^3$, since the system is controllable. Also,

$$\begin{aligned} \Delta 1 &= span\left\{\frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3}, 2\frac{\partial}{\partial x_3}\right\}, \\ \Delta 2 &= span\left\{\frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3}\right\}. \end{aligned} \tag{4.1.13}$$

Now we choose $\xi_1 = h_1 = x_1$. In the following the prescription, we construct a function h_2 such that $dh_2 \cdot \Delta 2 = 0$ and $dh_2.ad_{g1}^{n-2}\ g2 \neq 0$. This means that

$$\frac{\partial h_2}{\partial x_2} + x_1\frac{\partial h_2}{\partial x_3} = 0, \qquad \frac{\partial h_2}{\partial x_3} \neq 0, \tag{4.1.14}$$

which is satisfied by the function

$$h_2 = x_3 - x_1x_2 .$$

Now set

$$\xi_2 = £_{g1}h_2 = -2x_2 \tag{4.1.15}$$

and

$$v_1 = u_1, \qquad v_2 = \left(£_{2g}^2\ h_2\right)u_1 + \left(£_{g2}£_{g1}h_2\right)u_2$$

$$= -2u_2 . \tag{4.1.16}$$

Then

$$\begin{aligned} \dot{\xi}_1 &= \dot{x}_1 = u_1 = v_1, \\ \dot{\xi}_2 &= -2\dot{x}_2 = v_2, \\ \dot{\xi}_3 &= \dot{x}_3 - \dot{x}_1 x_2 - x_1 \dot{x}_2 = -2x_2 u_1 = \xi_2 v_1, \end{aligned} \tag{4.1.17}$$

which puts the system into the chained form desired.

Extended Systems. Aside from such special classes of systems it is possible to use rather general motion planning algorithms. One of the key ideas in this work is to use an ***extended*** system of the form

$$\begin{aligned} \dot{x} = {} & v_1 g_1(x) + \ldots + v_m g_m(x) + v_{m+1} g_{m+1}(x) + \ldots \\ & + v_r g_r(x), \end{aligned} \tag{4.1.18}$$

where $g_{m+1}(x), \ldots, g_r(x)$ are higher-order Lie brackets of the g_i chosen so that $g_1(x), \ldots, g_r(x)$ span all R^n. The idea is then to compute a motion controller for the extended system (which is easy, since we have an independent control vector field for each independent direction in R^n) and then use that to construct one for the original system. [5]

4.2 Stabilization of the Heisenberg System

Here we are considering a discontinuous approach to the stabilization problem for the nonholonomic integrator or Heisenberg system. This is the prototypical example for which smooth feedback fails. The idea is to use the natural algebraic structure of the system together with ideas from sliding mode theory. Another interesting problem for such systems is the problem of tracking.

$$\dot{x} = u, \tag{4.2.1}$$

$$\dot{y} = v, \tag{4.2.2}$$

$$\dot{z} = xv - yu, \tag{4.2.3}$$

The problem of stabilizing this system, even locally, is not a trivial task, since, as can be easily seen, the linearization in the vicinity of the origin gives the noncontrollable system

$$\dot{x} = u,$$

$$\dot{y} = v,$$

$$\dot{z} = 0,$$

The main difficulty is the fact that stabilization of x and y leads to a zero right-hand side of (4.2.3), and therefore, the variable z cannot be steered to zero. That simple observation implies that to stabilize the system one needs to make z converge "faster" than x and y.

We consider the control law

$$u = -\alpha x + \beta y \, sign(z), \tag{4.2.4}$$

$$v = -\alpha y - \beta x \, sign(z), \tag{4.2.5}$$

where α and β are positive constants.

Let us show that there exists a set of initial conditions such that trajectories starting there converge to the origin. To do this, consider a Lyapunov function for the (x, y) −subspace:

$$\mathrm{V} = \frac{1}{2}(\mathrm{x}^2 + \mathrm{y}^2). \tag{4.2.6}$$

The time derivative of V along the trajectories of the system (4.2.3) is negative:

$$\begin{aligned} \dot{V} = -\alpha x^2 + \beta xy \, \mathrm{sign}(z) - \alpha y2 - \beta xy \, \mathrm{sign}(z) &= -\alpha(x^2 + y^2) \\ &= -2\alpha V. \end{aligned} \tag{4.2.7}$$

Therefore, under the control 4.2.4, 4.2.5 the variables x and y are stabilized.

Now let us consider the variable z. Using 4.2.3, 4.2.4, and 4.2.5, we obtain

$$\dot{z} = xv - yu = -\beta(x^2 + y^2)\text{sign}(z)$$
$$= -2\beta V\ \text{sign}(z). \tag{4.2.8}$$

Since V does not depend on z and is a positive function of time, the absolute value of the variable z will decrease and will reach zero in finite time if the inequality

$$2\beta \int_0^\infty V(\tau)d\tau > |z(0)| \tag{4.2.9}$$

holds. If $z(0)$ is such that

$$2\beta \int_0^\infty V(\tau)d\tau = |z(0)|, \tag{4.2.10}$$

then z(t) converges to the origin in infinite time (asymptotically). Otherwise, it converges to some constant nonzero value of the same sign as $z(0)$.
If the above inequality 4.2.9 holds, the system trajectories are directed to the surface $z = 0$, and the variable $z(t)$ is stabilized at the origin in finite time. (The variables x and y, as follows from 4.2.7, always converge to the origin while within that surface.)

This phenomenon is known as ***sliding mode***. The manifold $z = 0$ is a stable integral manifold of the closed-loop system (4.2.1)-(4.2.3), (4.2.4), (4.2.5). Its characteristic feature is reachability in finite time. Using a smooth control, even a control satisfying a local Lipschitz condition (in the vicinity of $\{z = 0\}$) such fast convergence cannot be achieved. On the other hand, within the sliding manifold $\{z = 0\}$ the system behavior is described in accordance with the definition for systems of differential equations with discontinuous right-hand sides.
The version of this definition that we are using is as follows: We consider the system

$$\dot{x} = f(x), \tag{4.2.11}$$

with $f(x)$ a discontinuous function composed of a finite number of functions

$$f(x) \equiv f_k(x) for\ x \in M_k, \tag{4.2.12}$$

where the open regions M_k have piecewise smooth boundaries ∂M_k. Then we define the right-hand side of 4.2.11 within ∂M_k to be

$$\dot{x} = \sum_{k \in I(x)} \mu_k f_k(x). \tag{4.2.13}$$

The sum is taken over the set $I(x)$ of all k such that $x \in \partial M_k$ and the variables μ_k satisfy

$$\sum_{k \in I(x)} \mu_k = 1; \tag{4.2.14}$$

i.e., the right-hand side belongs to the convex closure $\mathrm{co}\{f_k(x) : k \in I(x)\}$ of the vector fields $f_k(x)$ for all $k \in I(x)$. Actually, the definition replaces the differential equation 4.2.11by a differential inclusion

$$\dot{x} \in \mathrm{co}\{f_k(x): k \in I(x)\} \tag{4.2.15}$$

for the points x belonging to the boundaries ∂M_k. If within the convex closure there exists a vector field tangent to all or some of the boundaries, then there is a solution of the differential inclusion belonging to ∂M_kthat corresponds to the sliding mode.

In the above relatively simple case, the definition provides a unique solution and implies that the system on the manifold is

$$\dot{x} = -\alpha x,$$

$$\dot{y} = -\alpha y.$$

From 4.2.7 it follows that

$$V(t) = V(0)e^{2\alpha t} = \frac{1}{2}\left(x^2(0) + y^2(0)\right)e^{2\alpha t}. \tag{4.2.16}$$

Substituting this expression in (4.2.9) and integrating, we find that the condition for the system to be stabilized is

$$\frac{\beta}{2\alpha}[x^2(0) + y^2(0)] \geq |z(0)|. \tag{4.2.17}$$

The inequality

$$\frac{\beta}{2\alpha}[x^2 + y^2] \geq |z|. \tag{4.2.18}$$

defines a parabolic region P in the state space.

The above derivation can be summarized in the following theorem:

Theorem 4.2.1. *If the initial conditions for the system (4.2.1)–(4.2.3) belong to the complement P^c of the region P defined by (4.2.18), then the control (4.2.4), (4.2.5) stabilizes the state.*

If the initial data are such that (4.2.18) is true, i.e., the state is inside the paraboloid, we can use any control law that steers it outside. In fact, any nonzero constant control can be applied. Namely, if $u \equiv u_0 = const, v \equiv v_0 = const,$ then

$$x(t) = u_0 t + x_0,$$
$$y(t) = v_0 t + y_0,$$
$$z(t) = t(x_0 v_0 - y_0 u_0) + z_0.$$

With such $x, y,$ and $z,$ the left-hand side of (4.2.18) is quadratic with respect to time t, while the right-hand side is linear. Hence, as the time increases, the state inevitably will leave P.

A global feedback control law in the form of the feedback (although discontinuous) can be described as follows:

$$(u, v)^T = \begin{cases} (u_0, v_0)^T & if\ (x,y,z)^T \in P, \\ (4.2.4),(4.2.5)^T & if\ (x,y,z)T \in P^c. \end{cases} \tag{4.2.19}$$

Theorem 4.2.2. *The closed system* (4.2.1) (4.2.3), (4.2.19) *is globally asymptotically stable at the origin.*

Global asymptotic stability mans that.

(i) For all initial conditions we have x (t), y(t), z(t) →0.when t→ ∞;

(ii) For all ε >0 there exists$\delta > 0$ such that $x_0^2 + y_0^2 + z_0^2 > \delta^2$ implies

x^2 (t) + y^2 (t) + z^2 (t)$< \varepsilon^2$ for any t ≥ 0.

We have already shown above that (i) is true, and (ii) follows from the fact that outside P and on the surface of the paraboloid ðP the state monotonically approaches the origin in. For initial conditions inside P we have

$$x^2\,(t) + y^2\,(t) + z^2\,(t) = (u_0\, t + x_0)^2 += (v_0\, t + y_0)^2 +[(\, x_0\, v_0 - y_0\, u_0)t + z_0]^2. \tag{4.2.20}$$

The maximum of the expression (4.2.20) is achieved for t = 0 or t_f, where t_f is the first moment of time when the state reaches ðp. This moment is defined by an equating.

$$\frac{\beta}{2\propto}(u_0\, t_f + x_0)^2 + (\, v_0\, t_f + y_0)^2 = [x_0\, v_{.0} - y0\, u_{\,0})\, t_f + z_0] \tag{4.2.21}$$

As can be easily seen from (4.2.21), for fixed u_0, v_0, the solution of this equation t_f tends to zero if x_0 , y_0, z_0 tend simultaneously to zero. That proves (ii).

The parameters$\propto$, β define the size of the paraboloid.

Simulations of the algorithm for two types of initial conditions are shown in figure 4.2.1. The figure shows the trajectories exiting from the set P under constant control and then being driven to the origin under the feedback (4.2.4), (4.2.5).

When $\frac{\beta}{\alpha} \rightarrow \infty$ the parabolic region P is limited to the z-axis. From that point of view, to stabilize the system(4.2.1 – 4.2.3), it is reasonable to increase β as the state approaches the origin (if we decrease α, the convergence of x and y approach the origin:

$$u = -\alpha\, x + \beta\, \frac{y}{x^2 + y^2}\, sign(z), \tag{4.2.22}$$

$$v = -\alpha\, y - \beta\, \frac{x}{x^2 + y^2}\, sign(z), \tag{4.2.22}$$

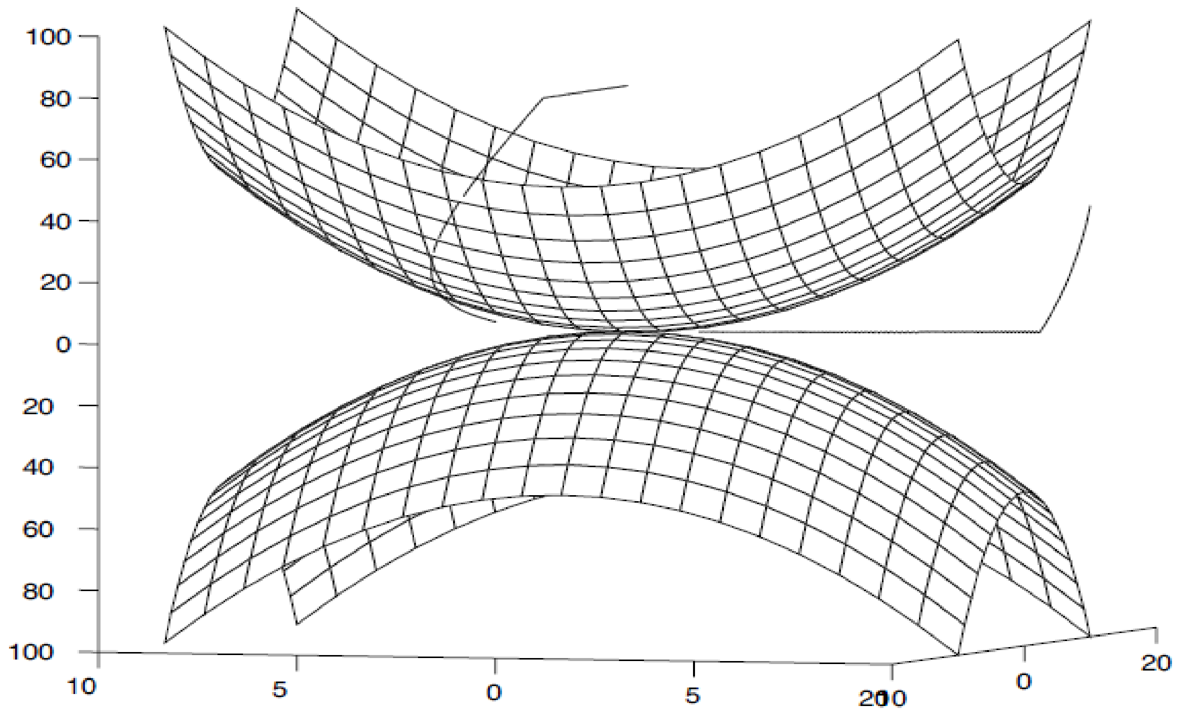

FIGURE 4.2.1. Stabilization of the nonholonomic integrator.

Or even

$$u = \alpha\, x + \beta\, \frac{y}{x^2+y^2}\, z,$$

(4.2.24)

$$v = -\alpha\, y - \beta\, \frac{x}{x^2+y^2}\, z.$$

(4.2.25)

Then from (4.2.3) we have

$$\dot{z} = -\beta \text{ sign } (z),$$

for the controls (4.2.22),(4.2.23), or

$$\dot{z} = -\beta z,$$

respectively, for the controls (4.2.24), (4.2.25).

In both cases, the state converges to the origin form any initial conditions, except the ones belonging to the z–axis. But in contrast to (4.2.4), (4.2.5), the control laws (4.2.22), (4.2.23).and (4.2.24), (4.2.25) are unbounded in a neighborhood of the z–axis (on the axis it is not defined). If the initial conditions belong to this set, again we can apply any nonzero constant control for an arbitrarily small period of time and then switch to (4.2.22) or (4.2.24), (4.2.25).

An ε – stabilizing control (to a neighborhood of the origin) may be obtained by switching $\propto$.

Let $\propto$ be the following function of x and y,

$$\propto = \propto_0 \text{ sing } (x^2 + y^2 - \varepsilon^2), \tag{4.2.26}$$

Where $\propto_0 > 0$, $\beta > 0$ are constants, and let the control be

$$u = -\alpha\, x + \beta\, y\, z, \tag{4.2.27}$$

$$u = -\alpha\, y - \beta\, x\, z. \tag{4.2.28}$$

(one deals with initial data on the z-axis as above.)

Using (4.2.7) we find that from any initial conditions x and y the state reaches an $\varepsilon -$ sphere of the $(x, y) -$ space origin:

$$x^2 + y^2 = \text{const } \varepsilon^2 . \tag{4.2.29}$$

After that, the equation for variable z is

$$\dot{z} = -\beta\, \varepsilon^2\, z. \tag{4.2.30}$$

Therefore, z $\to 0$ *as* t $\to\ \infty$, while the variables x *and* y stay in an ε- vicinity of the origin. Of course, in (4.2.27), (4.2.28) z can be replaced by any function g(z) that guarantees asymptotic stability of the equation

$$\dot{z} = -\beta\, \varepsilon^2\, (z), \tag{4.2.31}$$

For example, (z) = sign(z). [24]

4.3 Stabilization of a Generalized Heisenberg System.

We discuss here the stabilization of the canonical generalization of the Heisenberg system (4.1.2), (4.1.3) by discontinuous feedback. We also demonstrate a rather interesting connection with isospectral flows (flows that preserve eigenvalues). Such flows are fundamental to integrable systems theory.

Lie– Algebraic Generalization. We consider here a system that generalizes (4.1.2) – (4.1.3) and can be described as follows. Let $\mathfrak{g}$ be a Lie algebra with a direct sum decomposition $\mathfrak{g} = \mathrm{m} \oplus \mathfrak{h}$ such that $\mathfrak{h}$ is a Lie subalgebra, $[\mathfrak{h}, \mathrm{m}] \subseteq \mathrm{m}$, and $[\mathrm{m}, \mathrm{m}] = \mathfrak{h}$. We will consider the following system in $\mathfrak{g}$:

$$\dot{x} = u, \tag{4.3.1}$$

$$\dot{Y} = [u, x], \tag{4.3.2}$$

Where $x, u \in \mathrm{m}$, $\mathrm{Y} \in \mathfrak{h}$.

The $\mathfrak{so}\,(n)$ system (4.1.2) – (4.1.3) is of the type (4.3.1) – (4.3.2), as we now show. Let $\mathfrak{h} = \mathfrak{so}\,(n)$ and let $\mathrm{m} = \mathbb{R}^n$. For $x, u \in \mathrm{m}$, define $[x, u] \equiv x\,\mathrm{u}^T - x\,\mathrm{u}^T \in \mathfrak{h}$. Y For $\mathrm{Y} \in \mathfrak{h}, x \in \mathrm{m}$, define $[\mathrm{Y}, x] = -[x, \mathrm{Y}] \equiv \mathrm{Y}\,x$. It is easy to see that the Lie algebra $\mathfrak{g} \equiv \mathrm{m} \otimes \mathfrak{h}$ is isomorphic to $\mathfrak{so}\,(n+1)$: Identify $\mathrm{Y} \in \mathfrak{so}\,(n)$ with the matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & Y \end{pmatrix}$$

And identify $x \in \mathbb{R}^n$ with the matrix

$$\begin{pmatrix} 0 & -x^T \\ x & Y \end{pmatrix}$$

The adjoint action of $\mathfrak{h}$ on m agrees with standard action of $\mathfrak{so}\,(n)$ on $\mathbb{R}^n$, and it is straightforward to check that the desired commutation relations hold.

Our goal is to find a stabilizing control for the system (4.3.1) – (4.3.2).

Since this system fails the necessary condition for the existence of a continuous feedback law, our goal here is to find a discontinuous law.

The General System. Let $\mathfrak{g}$ be a real semi simple Lie algebra with Killing form B : $\mathfrak{g}\times\mathfrak{g}\rightarrow \mathbb{R}$. Assume that $\mathfrak{g}$ has a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathrm{m}$, where $\mathfrak{h}$ is a compactly embedded subalgebra that contains no ideals of $\mathfrak{g}$, and m is the orthogonal complement of $\mathfrak{h}$ relative to B. Then the commutation relations $[\mathfrak{h}, \mathrm{m}] \subseteq \mathrm{m}$ and $[\mathrm{m}, \mathrm{m}] = \mathfrak{h}$ hold, the restriction of B to $\mathfrak{h}$ is negative definite, and the representation of $\mathfrak{h}$ on m is faithful. Note that if $\mathfrak{g}$ is a simple Lie algebra and $\mathfrak{g} = \mathfrak{h}\otimes\mathrm{m}$ is a Cartan decomposition, then our hypotheses are satisfied.

We will consider stabilization of the system (4.3.1) – (4.3.2) in $\mathfrak{g}$, where x , $u \in \mathrm{m}$, $Y \in \mathfrak{h}$ .We may assume without loss of generality that $\mathfrak{g}$ is either of noncompact type or of compact – type (under the given hypotheses, $\mathfrak{g}$ splits into a B- orthogonal direct sum of a compact – type ideal and one of noncom pact type. It is straightforward to show that (4.3.1) – (4.3.2) decouples into systems in each ideal, and stabilization of (4.3.1) – (4.3.2) follows from stabilization of each of the compact and noncompact cases separately.) It follows that the restriction of B to m is positive definite if $\mathfrak{g}$ is of noncompact type and negative definite if $\mathfrak{g}$ is of compact type.

Let

$$\varepsilon = \begin{cases} 1 & \text{if } \mathfrak{g} \text{ is of noncompact type} \\ -1 & \text{if } \mathfrak{g} \text{ is of compact type.} \end{cases} \tag{4.3.3}$$

We will use the inner product on   defined by the Killing form:

$$<x_1 + Y_1, x_2 + Y_2> \equiv \varepsilon B(x_1, x_2) - B(Y_1, Y_2), \tag{4.3.4}$$

For $x_1, x_2 \in \mathrm{m}, Y_1, Y_2 \in \mathfrak{h}$. The corresponding norm will be denoted by $\|\cdot\|$.

For $x \in$ m, let.

$$M(x) = \varepsilon(\mathrm{ad}_x)^2 \mid_{\mathfrak{h}} \tag{4.3.5}$$

If $\mathfrak{g}$ is noncompact , then ad_x is B – symmetric, while if $\mathfrak{g}$ is compact , then ad_x is B – skew - symmetric. In either case, M (x) = ε $(\mathrm{ad}_x)^2$ is a nonnegative symmetric operator $\mathfrak{h}$. Next, for Y $\in \mathfrak{h}$, let

$$N(Y) = (\mathrm{ad}_Y)^2 \mid_{\mathrm{m}} \tag{4.3.6}$$

Since ad_Y is B- skew – symmetric, N (Y) is a nonnegative symmetric operator on m.

We will make frequent use of two identities relating the operators $M(x)$and $N(Y)$. First, the Jacobi identity implies

$$[Y, M(x)Y] = \varepsilon\, [x, N(Y)x] \tag{4.3.7}$$

For all $x \in$ m, $Y \in \mathfrak{h}$. Second, the invariance of the Killing form implies

$$[Y, M(x)Y] = \|[Y, x]\|^2 = < x, N(Y)x> \tag{4.3.8}$$

For all $x \in$ m, $Y \in \mathfrak{h}$.

We also require two estimates arising from$M(x)$ and $N(Y)$. First, we have the inequality

$$tr\ (M(x)) \le \|x\|^2 \tag{4.3.9}$$

for all $x \in$m .Second , there exists a constant 0 $<\eta<$1 such that

$$\mathrm{tr}(Y, (Y)) > \eta = \|Y\|^2 \tag{4.3.10}$$

for all $Y \in \mathfrak{h}$.

Controls. We consider the following controls for system for the system(4.3.1)-(4.3.2):

$$u = -\propto x + \beta[Y, x] + \gamma\, N(Y)\, x, \tag{4.3.11}$$

Where $\propto, B, \gamma : \mathfrak{g} \to \mathbb{R}$ are real – valued functions, with $\propto, \gamma \ge 0$ and $\beta\varepsilon \le 0$.With the control (4.3.11) (and using (4.3.7), the system (4.3.1) – (4.3.2) becomes

$$\dot{x} = -\propto x + \beta[Y, x] + \gamma\, N(Y)\, x, \tag{4.3.12}$$

$$\dot{Y} = -\beta\varepsilon M(x)Y - \gamma\varepsilon[Y, M(x), Y\ ]. \tag{4.3.13}$$

Using (4.3.12) and the skew –symmetry of ad γ, we easily compute

$$\frac{d}{dt}\|x\|^2 = -2\propto \|x\|^2 + 2\gamma < x, N(Y)x >. \tag{4.3.14}$$

Let λ_* denote the largest eigenvalue of $N(Y)$. Then $< x, N(Y)\, x > \ \leq \lambda_* \|x\|^2$ for all $x \in$ m, and thus the right-land side of (4.3.14) is nopositive if $\lambda_* \gamma \geq \propto$. In this case $\|x\|$is nonincreasing, and if $\propto = \gamma = 0$, then $\|x\|$ is constant.

Using (4.3.13), we obtain

$$\frac{d}{dt}\|Y\|^2 = 2\,\beta\varepsilon < Y, M(x)Y >. \tag{4.3.15}$$

Since $\beta\varepsilon \leq 0$ and $M(x)$ is a nonnegative operator, the right hand side of (4.3.15) is nonpositive. Thus $\|Y\|$ is nonincreasing in general, and is constant if $\beta = 0$.

Our (necessarily discontinuous) stabilization algorithm will involve a witching the control (4.3.11) among the following three cases: (i) $\propto > 0, \beta = \gamma = 0$; (ii) $\propto = k\,\lambda_*$, $\gamma = k, and\ \beta = 0$, where, as above, λ_* is the largest eigenvalue of N(Y) and where k is a positive function; (iii) $\propto = \gamma = 0, \beta\varepsilon < 0$. We now discuss the dynamics of the system (4.3.12) – (4.3.13) in each of these cases.

Case I: $\propto > 0$, $\beta = \gamma = 0$;

In this case, the system (4.3.12) – (4.3.13)

$$\dot{x} = -\propto x, \tag{4.3.16}$$

$$\dot{Y} = 0. \tag{4.3.17}$$

Here x is driven to 0 radially while Y remains fixed. If Y was not already 0, implementing (4.3.11) with these parameter values will render the system unstabilizable. Hence this case will be used only if $Y \equiv 0$.

Case II: $\propto = k\,\lambda_*$, $\gamma = k$, $\beta = 0$;

As noted above, $\kappa > 0$. In this case, the control (4.3.11) has the form

$$u = -\kappa\,(\lambda_*\, x - N(Y)\, x), \tag{4.3.18}$$

while the system (4.3.12) – (4.3.13) is

$$\dot{x} = -\kappa\,(\lambda_*\, x - N(Y)\, x), \tag{4.3.19}$$

$$\dot{Y} = -\kappa\varepsilon\,[Y, M(x)\, Y], \tag{4.3.20}$$

In this case, $\|Y\|$ is constant. In addition, (4.3.20) is a Lax equation in Y. It follows that the spectrum of ad_Y is constant. Therefore, the spectrum of the operator N(Y) is constant, as are the dimensions of its eigenspaces. In particular, the eigenvalue λ_*, which occurs in (4.3.19), is constant.

Let $0 \leq \lambda_0 < \lambda_1 < \cdots < \lambda_s = \lambda_*$ denote those eigenvalues of N(Y) that are distinct (thus $\leq \dim m - 1$). Let $x = x_0 + x_1 + \cdots + x_s$ be the unique decomposition of x into the eigenspaces of N(Y), then the differential equation (4.3.19) decouples into the following system of equations in m:

$$\begin{aligned} \dot{x}_0 &= -\kappa(\lambda_* - \lambda_0)\, x_0 \\ \dot{x}_1 &= -\kappa(\lambda_* - \lambda_1)\, x_1 \\ &\vdots \\ \dot{x}_{s-1} &= -\kappa(\lambda_* - \lambda_{s-1})\, x_{s-1}, \\ \dot{x}_s &= 0. \end{aligned} \tag{4.3.21}$$

Since $\kappa(\lambda_* - \lambda_j) > 0$ for $j = 0,1, \ldots, s-1$, it follows that $x_j \to 0$ asymptotically. If we let x_* denote the projection of x onto the λ_*-eigenspace of N(Y), that is, $x_* = x_s$, then noting that $x_* \equiv x_*\,|_{t=0}$ is constant, we conclude that

$$x \longrightarrow x_*$$

asymptotically.

Note that (4.3.19) - (4.3.20) and (4.3.7) imply the following:

$$\dot{Y} = -\kappa[x, N(Y)\, x] = [x, \dot{x}], \tag{4.3.22}$$

Since x converges to a λ_* - eigenvector of N(Y), the right-hand side of (4.3.19) converges to 0 and thus $\dot{x}$ converges to 0. Therefore, (4.3.22) implies that $\dot{Y}$ converges to 0.

Summarizing this case, we have that Y evolves isospectrally (with constant spectrum) and with constant norm and asymptotically vanishing velocity, while x is driven to x_*, the (constant) projection of x onto the λ_* - eigenspace of N(Y).

Case III: $\alpha = \gamma = 0$, $\beta\varepsilon < 0$:

The system (4.3.12) – (4.3.13) for this case is

$$\dot{x} = \beta[Y, x], \tag{4.3.23}$$

$$\dot{Y} = \beta\varepsilon\, M(x)\, Y, \tag{4.3.24}$$

In this case, $\|x\|$ is constant. In addition, (4.3.23) is a Lax equation in x, and thus ad_x has constant spectrum. Therefore, the spectrum of the operator $M(x)$ is constant, as are the dimensions of its eigenspaces. Let $0 \le \mu_0 < \mu_1 < \cdots < \mu_r$ denote those eigenvalues of $M(x)$ that are distinct (thus $r \le \dim \mathfrak{h} - 1$). For $Y \in \mathfrak{h}$, let $Y = Y_o + \cdots + Y_r$ denote the unique decomposition of Y into the eigenspaces of $M(x)$. Then the differential equation (4.3.24) decouples into the following system of equations in $\mathfrak{h}$:

$$\begin{aligned} \dot{Y}_0 &= \beta\varepsilon\mu_0 Y_0, \\ \dot{Y}_1 &= \beta\varepsilon\mu_1 Y_1, \\ &\vdots \\ \dot{Y}_r &= \beta\varepsilon\mu_r Y_r, \end{aligned} \tag{4.3.25}$$

Since $\beta\varepsilon\mu_j < 0$ for $j = 1, \ldots, r$, we have that $Y_j \to 0$ asymptotically. If $\mu_0 < 0$, then tha sme applies to Y_0. Otherwise, if $M(x)$ has $\mu_0 = 0$ as an eigenvalue, then Y_0 remains constant. Thus we have $Y \to 0$ or $Y \to Y_0$ asymptotically, where $Y_0 \equiv Y_0 \,|_{t=0}$ is constant. In either case, if we let Y# denote the projection of Y onto the nullspace of $M(x)$, then noting that $Y_\# \equiv Y_\# \,|_{t=0}$ is constant, we conclude that

$$Y \to Y_\#$$

asymptotically.

Using system (4.3.23) – (4.3.24), we can derive the equation

$$\frac{d}{dt} M(x)^n Y = \beta\varepsilon\,[Y, M(x)^n Y] + \beta\varepsilon\, M(x)^{n+1} Y \tag{4.3.26}$$

for every nonnegative integer *n*. Indeed, the case *n* = 0 is just (4.3.24). Using the induction hypothesis, we have for n > 0,

$$\frac{d}{dt}M(x)^nY = \varepsilon\big[\dot{x},[x,M(x)^{n-1}Y]\big] + \varepsilon[x,[\dot{x},M(x)^{n-1}Y]]$$

$$+ \mathrm{M}(x)(\,\beta\varepsilon\,[\mathrm{Y},M(x)^{n-1}Y] + \beta\varepsilon\,\mathrm{M}(x)^{n}\mathrm{Y}).$$

(4.3.27)

Now,

$$[\dot{x},\big[x,M(x)^{n-1}\big]\big] = \beta[[Y,x],[x,M(x)^{n-1}Y]]$$

(4.3.28)

and

$$[x,\big[\dot{x},M(x)^{n-1}\big]\big] = \beta[x,[[Y,x],M(x)^{n-1}Y]]$$

(4.3.29)

while applying the Jacobi identity repeatedly gives

$$M(x)[Y,M(x)^{n-1}Y] = [Y,M(x)^nY] + \varepsilon\big[[x,Y],[x,M(x)^{n-1}Y]\big]$$
$$+\varepsilon[x,[[x,Y],M(x)^{n-1}Y]]$$

(4.3.30)

Substituting (4.3.28), (4.3.29), and (4.3.30) into (4.3.27) and simplifying gives (4.3.26).

Then from (4.3.26),

$$\tfrac{d}{dt}f\big(M(x)\big)Y = \beta\varepsilon\big[Y,f\big(M(x)\big)Y\big] + \beta\varepsilon f\big(M(x)\big)M(x)Y \qquad (4.3.31)$$

follows immediately for every real analytic function *f*. As an interesting special case of this, let $p(\mu)$ be the minimal polynomial of $M(x)$ and assume that $\mu_0 = 0$ is an eigenvalue of $M(x)$ (so that Y does not converge to 0).

Then $p(\mu) = \mu\, q(\mu)$ for some polynomial q. Taking $f = q$ in (4.3.31) gives

$$\tfrac{d}{dt}q\big(M(x)\big)Y = \beta\varepsilon\big[Y,q\big(M(x)\big)Y\big]. \qquad (4.3.32)$$

It follows the spectrum of $q\,(M(x))\,Y$ remains constant: that is, it evolves isospectrally.

Summarizing this case, we have that x evolves isospectrally with constant norm. Y is driven to $Y_{\#}$, its (constant) projection onto the nullspace of $M(x)$, and $q\,(M(x))\,Y$ evolves with constant spectrum.

Remark.

It is interesting to compare the system of equations of Case III with the double bracket equations discussed. The isospectral flow $\dot{L} = [L, [L, N]], L, N$ lying in a compact algebra, N fixed, was considered. This flow is the gradient flow of $\langle L, N \rangle$ on an adjoint orbit of the corresponding Lie group with respect to the so-called normal metric. Equation (4.3.24) is, on the other hand, of the form $\dot{Y} = \beta \in [X, [X, Y]]$, which is *not* isospectral (although it is coupled to the isospectral equation (4.3.23). Further, as we have seen, we have a different function, $\langle Y, Y \rangle$, decreasing along its flow, which is precisely what is needed.

The Stabilization Algorithm.

We now describe our feedback strategy. As before, λ_* denotes the largest eigenvalue of the operator $N(Y)$, x_* denotes the projection of x onto the λ_*-eigenspace of $N(Y)$, and $Y_\#$ denotes the projection of Y onto the nullspace of $M(x)$. Let $\delta > 0$ be a prescribed error tolerance. In informal pseudocode, the algorithm can be described as follows:

begin

while $\|Y\| \geq \delta$

1. Let $r = \|x\|$. Implement the control (4.3.11) with $\alpha = \lambda_* k$, $\gamma = k$, and $\beta = 0$. Then Y evolves isospectrally with constant norm, while x converges to the constant x_*. If $x_* \neq 0$, then go to step 3.
2. Let z_* denote a fixed λ_*-eigenvector of $N(Y)$ with
$$\|z_*\| = r(1 - 1/\dim m)^{1/2}$$
Let $u = -\alpha\,(x - z_*)$, where $\alpha > 0$. The x converges to z_*while Y remains constant.
3. Implement the control (4.3.11) with $\alpha = \gamma = 0$, $\beta\varepsilon < 0$. Then x evolves isospectrally with constant norm, while Y converges to the constant $Y_\#$.

end while

if $\|x\| \geq \delta$, **then**

4. implement the control (4.3.11) with $\alpha > 0$, $\beta = \gamma = 0$. Then x will converge to 0 radially, while Y remains 0.

end

In Step 1, if α is a constant, then x will converge to x_* in infinite time: if, for example, $\alpha = 1/\|x - x*\|$, then x will converge in infinite time. Similarly, in Step 3, if β is a constant, then Y will converge to $Y_\#$ in infinite time: if, for example, $\beta=1/\|Y - Y\#\|$, then Y will converge in infinite time. To establish the convergence claim made in Step 2, we simply note that in this case $x(t)$ has the form $x(t) = f(t)z_*$, where $f(t)$ is a scalar-valued function satisfying

$$\dot{f} = -\alpha(f - 1), f(0) = 0.$$

(For instance, if $\alpha > 0$ is constant, we have $f(t) = 1 - e^{at}$.) It follows from (4.3.2) that $\dot{Y} = [u, x] = 0$, so that Y is constant, as claimed.

Step 2 is implemented if x converges to 0 in Step 1. One instance where this could happen occurs if the initial value of x is 0, in which case the first implementation of Step 1 is trivial. More generally, the case where the projection of x onto the λ_*-eigenspace of $N(Y)$ is 0 seems to be the natural higher-dimensional analogue of the situation in the Heisenberg system where the initial value starts on the z-axis. As in Steps 1 and 3, Step 2 can also be implemented in finite time.

The λ_*-eigenspace of $N(Y)$ will, in general, have dimension greater than 1 (since the nonzero eigenvalues of the B-skew-symmetric operator ad_Y come in complex conjugate pairs). Thus there is no unique choice of eigenvector z_* in Step 2. Any lexicographic ordering of the eigenvectors relative to a coordinate basis will suffice as a selection scheme. The rationale behind the particular normalization of z_* will be explained below.

We will now show that our algorithm successfully stabilizes the system (4.3.1)-(4.3.2) by showing that each of $\|x\|$ and $\|Y\|$ can be brought to within the prescribed error tolerance, note that as soon as the test condition of the while loop fails, that is, as soon as $\|Y\| < \delta$, then the system will be stabilized whether Step 4 needs to be executed or not. Thus we may assume that the initial value of Y satisfies $\|Y\| \geq \delta$ so that the while loop will be executed at least once. If Y ever converges to 0 in Step 3 because $Y_{\#} = 0$, then the test condition of the while loop will eventually fail. As noted, this is enough to guarantee that the system is stabilizable.

Assume that for every iteration of Step 3 we have $Y_{\#} \neq 0$. We will show that after finitely many iterations of the while loop, the test condition will fail. In other words, the projection of Y onto the nullspace of $M(x)$ is eventually arbitrarily small in norm. in fact, we will show a stronger result, for when this situation occurs, then it turns out that $\|x\|$ is simultaneously brought to within the error tolerance. Thus as soon as the while loop's test condition fails, the test condition of the if- then statement (Step 4) will also fail, and the system will already be stabilized.

Assume first that Step 3 is about to be executed. Since Step 1 and possible Step 2 have already been executed, the initial values $x(0) = x_*$ and $Y(0) = Y_*$ satisfy $N(Y_*) x_* = \lambda_* x_*$. As before, let Y_j denote the projection of Y onto the μ_j-eigenspace of $M(x)$. Recall that.

$$Y_{\#} = Y0 \equiv Y_0(0)$$

throughout Step 3, and that $Y(t) \rightarrow Y_{\#}$ asymptotically. Using the orthogonality of the eigenspace, we compute

$$\begin{aligned} \| Y_{\#} \|^2 &= \|Y_*\|^2 - \textstyle\sum_{j=1}^{r} \|Y_j(0)\|^2 \\ &\leq \| Y_* \|^2 - \frac{1}{\sum_{j=0}^{r} \mu_j} \textstyle\sum_{j=0}^{r} \mu_j \|Y_j(0)\|^2 \end{aligned} \tag{4.3.33}$$

Note that we are using $\mu_0 = 0$. Now using the orthogonality once again, we compute

$$\textstyle\sum_{j=0}^{r} \mu_j \|Y_j(0)\|^2 = \langle Y_* \textstyle\sum_{j=0}^{r} \mu_j Y_j(0) \rangle = \langle Y_* , M(x_*) Y_* \rangle$$

$$= \langle x_* , N(Y_*) x_* \rangle \tag{4.3.34}$$

$$= \lambda_* \| x_* \|^2. \tag{4.3.35}$$

Here we have used (4.3.8) to obtain (4.3.34). In addition, using (4.3.9), we have

$$\textstyle\sum_{j=1}^{r} \mu j \leq tr(M(x_*)) \leq \| x_* \|^2.. \tag{4.3.36}$$

Applying (4.3.35) and (4.3.36) to (4.3.33) yields

$$\| Y_{\#} \|^2 \leq \| Y_* \|^2 - \lambda_*. \quad (4.3.37)$$

Now using (6.3.10), we have

$$\lambda_* \geq \frac{1}{\dim m} \operatorname{tr}(N(Y_*)) \frac{\eta}{\dim m} \| Y_* \|^2$$

(4.3.83)

applying (4.3.38) to (4.3.37) gives our final estimate for Step 3:

$$\| Y_{\#} \|^2 < \left(1 - \frac{\eta}{\dim m}\right) \| Y_* \|^2 \quad (4.3.39)$$

Now assume that Step 3 has already been executed and that Step 1 is about to be executed again. Then the initial values $x(0) = x_{\#}$ and $Y(0) = Y_{\#}$ in Step 1 satisfy $M(x_{\#}) Y_{\#} = 0$. By (4.3.8), this implies $\langle x_{\#}, N(Y_{\#})\, x_{\#} \rangle = 0$. As before, let x_j denote the projection of x into the λj-eigenspace of $N(Y)$. Recall that $x_* = x_s \equiv x_s(0)$ throughout Step 1, and that x(t) $\rightarrow$ x_* asymptotically. Using the orthogonality of the eigenspaces, we compute

$$\| x_* \|^2 = \| x_{\#} \|^2 - \sum_{j=0}^{s-1} \| x_j (0) \|^2$$
$$\leq \| x_* \|^2 - \frac{1}{\sum_{j=0}^{s} (\lambda s - \lambda j)} \sum_{j=0}^{s} (\lambda_s - \lambda_j) \| x_j (0) \|^2. \quad (4.3.40)$$

Using orthogonality again, we compute

$$\sum_{j=0}^{s} (\lambda_s - \lambda_j) \| x_j (0) \|^2 = \lambda_s \| x_{\#} \|^2 - \langle x_{\#}, \sum_{j=0}^{s} \lambda_s \; x_j (0) \rangle$$
$$= \lambda_s \| x_{\#} \|^2 - \langle x_{\#}, N(Y_{\#}) \, x_{\#} \rangle$$
$$= \lambda_s \| x_{\#} \|^2. \quad (4.3.41)$$

Also,

$$\sum_{j=0}^{s} (\lambda_s - \lambda_j) = s \lambda_s - \sum_{j=0}^{s-1} \lambda_j .$$

(4.3.42)

Applying (4.3.41) and (4.3.42) to (4.3.40) gives

$$\| x_* \|^2 \leq \left(1 - \frac{\lambda s}{s\lambda s - \sum_{j=0}^{s-1} \lambda j}\right) \| x_\# \|^2. \tag{4.3.43}$$

Finally,

$$\frac{\lambda_s}{s\, \lambda_s \;-\; \sum_{j=0}^{s-1} \lambda_j} \geq \frac{1}{s} \geq \frac{1}{\dim m} \tag{4.3.44}$$

And applying (4.3.44) to (4.3.43) gives our final estimate for Step 1:

$$\| x_* \| \leq \left(1 - \frac{1}{\dim\, m}\right) \| x_* \|^2. \tag{4.3.45}$$

Now assume that Step 2 is executed because x = 0 (that is, x_* = 0 in Step 1). Rename x_* = z_* ,where z_* is the chosen λ_*-eigenvector. Then the normalization of z_* described in Step 2 immediately implied that (4.3.45) holds as an equality.

Define two sequences of real numbers as follows: Let a_j and b_j denote, respectively, the initial values of $||x||^2$ and $||Y||^2$ prior to the (j +1)st iteration of the while loop, where j = 0, 1, ….. Recall that $||Y||$ remains constant during Steps 1 and 2 and $||Y||$ remains constant during Step 3. Our estimates (4.3.39) and (4.3.45) imply that the sequences $\{aj\}$ and $\{bj\}$ satisfy

$$aj+1 \leq \left(1 - \frac{1}{\dim\, m}\right) aj, \tag{4.3.46}$$

$$bj+1 < \left(1 - \frac{\eta}{\dim\, m}\right) bj, \tag{4.3.47}$$

since

$$0 < 1 - \frac{1}{\dim\, m} < 1 - \frac{\eta}{\dim\, m} < 1. \tag{4.3.48}$$

It follows from (4.3.46)-(4.3.47) that the sequence $\{aj\}$ and $\{bj\}$ each converges to 0. In particular, it is immediate that each of $||x||$and $||Y||$ can be brought to within the prescribed error tolerance $\delta > 0$ in finitely many iterations of the while loop.

In summary, we have proven the following result.

4.3.1 Theorem. The algorithm given in Step 1-4 above globally stabilizes the system (4.3.1)-(4.3.2).

We remark that while we have used the error tolerance δ above to indicate how the stabilization algorithm works in practice, the formal proof of stability follows from letting δ approaches zero. [25]

4.4 Controllability, Accessibility and Stabilizability

We consider a class of nonholonomic dynamic control systems and various control and stabilizability properties. We consider the class of mechanical (Lagrangian) nonholonomic control systems described by the equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = \sum_{j=1}^{m} \lambda_j a_i^j + \sum_{j=1}^{r} b_i^j \, u_j \qquad (4.4.1)$$

$$\sum_{i=1}^{n} a_i^j \dot{q}^i = 0 \qquad j = 1, \dots, m \qquad (4.4.2)$$

These equations are a controlled version of the nonholonomic equations in Lagrange multiplier form. We assume here that we have a Lagrangian on the tangent bundle to an arbitrary configuration space$Q$, given by $L : TQ \to R$. In coordinates $q^i, i = 1, \dots, n$, on Q with induced coordinates $(q^i, \dot{q}^i)$ for the tangent bundle, we have $L(q^i, \dot{q}^i)$. All computations here will be local, however, and for the moment we will assume $Q = R^n$. Here L is taken to be the mechanical Lagrangian

$$L = \frac{1}{2} \sum_{i,j=1}^{n} g_{ij}(q)\, \dot{q}^i \dot{q}^j - V(q). \qquad (4.4.3)$$

Hence equation (4.4.1) takes the explicit form

$$\dot{q}^i \dot{q}^j + \frac{\partial g_{ij}}{\partial q^k} \dot{q}^k \dot{q}^j - \frac{1}{2} \frac{\partial g_{jk}}{\partial q^i} \dot{q}^j \dot{q}^k + \frac{\partial V}{\partial q^k}$$

$$= \sum_{j=1}^{m} \lambda_j a_i^j + \sum_{j=1}^{r} b_i^j \, u_j \qquad (4.4.4)$$

For convenience below we shall sometimes rewrite equation (4.4.4) as

$$g_{ij}\,\ddot{q}^j + f_i(q\,,\dot{q}\,) \;= \sum_{j=1}^{m} \lambda_j\, a_i^j \,+\, \sum_{j=1}^{r} b_i^j\; u_j \tag{4.4.5}$$

All functions are assumed to be smooth.

Stabilization to an Equilibrium Manifold. We now study the problem of stabilization of equations (4.4.1), (4.4.2) to a smooth equilibrium submanifold of M defined by

$$\mathrm{N_e} \,=\, \{(q,\dot{q}) \mid \dot{q} \,=\, 0\,, w(q) \,=\, 0\}\,,$$

where w(q) is a smooth $(n - m)$-vector function. We show that, with appropriate assumptions, there exists a smooth feedback such that the closed loop is locally asymptotically stable to $\mathrm{N_e}$.

The smooth stabilization problem is the problem of giving conditions such that there exists a smooth feedback function $U : M \to R^l$ such that $\mathrm{N_e}$ is locally asymptotically stable. Of course, we are interested not only in demonstrating that such a smooth feedback exists but also in indicating how such an asymptotically stabilizing smooth feedback can be constructed.

We now assume that we have here nonholonomic control systems whose normal form equations satisfy the property that if $r(t)$ and $\dot{r}(t)$ are exponentially decaying functions, then the solution to

$$\dot{s} \,=\, -A(r(t), s)\dot{r}(t)$$

is bounded (all the physical examples of nonholonomic systems, of which we are aware, satisfy this assumption).

Note also that the first and second time derivatives of $w(q)$ are given by

$$\dot{w} = \frac{\partial w(q)}{\partial q}\, C(q)\dot{r}\,,$$

$$\ddot{w} \,=\, \frac{\partial}{\partial q}\Big(\frac{\partial\, w(q)}{\partial\, q}\, C(q)\dot{s}\,\Big)C(q)\dot{r} \,+\, \frac{\partial\, w(q)}{\partial\, q}\, C(q)v\,.$$

Theorem 4.5.4. *Assume that the above solution property holds. Then the nonholonomic control system defined by equations* (4.4.1) and (4.4.2), *is locally asymptotically stabilizable to*

$$\boldsymbol{N_e} = \{(q, \dot{q}) \mid \dot{q} = 0, w(q) = 0\}, \tag{4.5.11}$$

using smooth feedback, if the transversality condition

$$\det\left(\frac{\partial\, w(q)}{\partial\, q}\right)\det\left(\frac{\partial\, w(q)}{\partial\, q}\, C(q)\right) \neq 0 \tag{4.5.12}$$

is satisfied.

Proof. It is sufficient to analyze the system in the normal form (4.5.7), (4.5.8), (4.5.9). By the transversality condition, the change of coordinates from $(r, s, \dot{r})$ to $(w, s, \dot{w})$ is a diffeomorphism.

Let

$$v = -\left(\frac{\partial\, w(q)}{\partial q}\, C(q)\right)^{-1} \left[\left(\frac{\partial\, w(q)}{\partial q}\, C(q)\dot{r}\right) C(q)\, \dot{r} + K_1 \frac{\partial\, w(q)}{\partial q} C(q)\dot{r} + K_2 w(q)\right],$$

where K_1 and K_2 are symmetric positive definite $(n-m) \times (n-m)$ constant matrices. Then obviously,

$$\ddot{w} + K_1\dot{w} + K_2 w = 0$$

is asymptotically stable to the origin so that $(w, \dot{w}) \rightarrow 0$ as $t \rightarrow \infty$. The remaining system variables satisfy equation (4.5.7) of the normal form equations (with $x_2 = s$) and by our assumption on the constraint matrix A, these variables remain bounded for all time. Thus $(q(t), \dot{q}(t)) \rightarrow N_e$ as $t \rightarrow \infty$. [10]