Chapter 3

Foundations of Symplectic Geometry

Symplectic geometry is the study of symplectic manifolds. A symplectic manifold is a differentiable manifold equipped with a non-degenerate skew-symmetric bilinear closed 2-form, the symplectic form ω. A diffeomorphism between two symplectic manifolds which preserves the symplectic form is called a symplectomorphism. Non-degenerate skew-symmetric bilinear forms can only exist on even dimensional vector spaces, so symplectic manifolds necessarily have even dimension. In dimension 2, a symplectic manifold is just a surface endowed with an area form and a symplectomorphism is an area-preserving diffeomorphism.

Hamiltonian geometry is the geometry of symplectic manifolds equipped with a moment map, that is, with a collection of quantities conserved by symmetries.

About two centuries ago, symplectic geometry provided a language for classical mechanics; through its recent fast development, it conquered a rich territory, asserting itself as a central branch of differential geometry and topology. Besides its activity as an independent subject, symplectic geometry is significantly stimulated by important interactions with dynamical systems, global analysis, mathematical physics, low-dimensional topology, representation theory, microlocal analysis, partial differential equations, algebraic geometry, Riemannian geometry, geometric combinatorics, equivariant cohomology, etc.

This chapter covers foundations of symplectic geometry in a modern language. We start by describing symplectic manifolds and examples. Next we study the local description of symplectic manifold where along the way various examples are also given for the purpose of illustration. We end the chapter by some properties of the symplectic manifold like the symmetries and the integrability.

3.1 Symplectic Manifolds

A symplectic manifold is a smooth manifold, M, equipped with a closed nondegenerate differential 2-form,ω, called the sympletic form. The study of symplectic manifolds is called symplectic geometry or symplectic topology. Symplectic manifolds arise naturally in abstract formulations of classical mechanics and analytical mechanics as the cotangent bundles of manifolds. For example, in the Hamiltonian formulation of classical mechanic, which provides one of the major motivations for the field, the set of all possible configurations of a system is modeled as a manifold, and this manifold's cotangent bundle describes the phase space of the system.

Let ω be closed nondegenerate 2-form on a manifold M, that is, for each $p \in M$, the map $\omega_p : T_pM \times T_pM \rightarrow R$ is skew-symmetric bilinear on the tangent space to M at p, and ω_p varies smoothly in p. We say that ω is closed if it satisfies the differential equation dω = 0, where d is the exterior derivative.

Definition 3.1.1 The 2-form ω is **symplectic** if ω *is closed and* ω_p *is symplectic for all* $p \in M$. If ω is symplectic, then dim T_pM = dim M must be even.

Necessarily then M is even dimensional and if dim M = 2n, then $\frac{1}{n!}\omega^n$ is a volume *2n*-form on M. So M is orientable and ω determines in this way an orientation. However, not every orientable, even-dimensional, smooth manifold admits a symplectic structure. If (M,ω) is a compact, symplectic manifold of dimension 2n, then ω defines a real cohomology class a = $[\omega] \in H^2(M, \mathbb{R})$ and the cohomology class $a^n = a \cup \ldots \cup a \in H^{2n}(M,R)$ is represented by $\omega^N = \omega \wedge$

$\ldots \wedge \omega$. So, $a^n \neq 0$ and the symplectic form ω cannot be exact. It follows that if M is an orientable, compact, smooth manifold such that $H^2(M, \mathbb{R}) = \{0\}$, then M admits no symplectic structure. For example, the n-sphere S^n cannot be symplectic for n > 2, as well as the 4-manifold $S^1 \times S^3$.

A smooth map f : (M, ω) → (M′, ω') between symplectic manifolds is called *symplectic* if $f^*\omega' = \omega$. If f is also a diffeomorphism, it is called *symplectomorphism*.

In this way symplectic manifolds form a category. The product of two symplectic manifolds (M_1, ω_1) and (M_2, ω_2) is the symplectic manifold

$$(M_1 \times M_2, \pi_1^* \omega_1 + \pi_2^* \omega),$$

where $\pi_j : M_1 \times M_2 \to M_j$, j = 1, 2, are the projections.

Example 3.1.2

Let M = $\mathbb{R}^{2n}$ with linear coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$. The form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$$

is symplectic as can be easily checked; the set

$$\left\{\left(\frac{\partial}{dx_1}\right)_p, \ldots, \left(\frac{\partial}{dx_n}\right)_p, \left(\frac{\partial}{dy_1}\right)_p, \ldots, \left(\frac{\partial}{dy_n}\right)_p\right\} \qquad (3.1.1)$$

is a symplectic basis of $T_p M$.

Example 3.1.3

Let M = $\mathbb{C}^n$ with linear coordinates $z_1, \ldots, z_n$. The form

$$\omega_0 = \frac{i}{2} \sum_{k=1}^{n} dz_k \wedge d\check{z}_k$$

is symplectic. In fact, this form equal that of previous example under identification

$$\mathbb{C}^n \approx \mathbb{R}^{2n},\ z_k = x_k + \mathrm{i}\ y_k.$$

Example 3.1.4 The basic example of a symplectic manifold is the cotangent bundle T^*M of any smooth n- manifold M with the symplectic 2-form $c = -d\theta$, where θ is the Liouville canonical 1-form on T^*M. Recall the locally ω is given by the formula

$$\omega_{\,locally} = \sum_{i=1}^{n} dq^i \wedge dp_i$$

And compare with 3.1.2.

Proposition 3.1.5 *A smooth manifold M of dimension 2n has an almost complex structure if and only if it has an almost symplectic structure.*

Proof. Let J be an almost complex structure on M. Let g_0 be any Riemannian metric on M and g be the Riemannian metric defined by

$$g(v,w) = g_0(v,w) + g_0(J_v\,, J_w) \qquad (3.1.2)$$

for $v, w \in T_pM\,, p \in M$. Then

$$g(J\,v\,, J\omega) = g_0(J\,v\,, J\omega) + g_0(J^2 v\,, J^2\omega) = g_0(J\,v\,, J\omega) + g_0(-v, -w)$$
$$= g(v,w). \qquad (3.1.3)$$

The smooth 2-form ω defined by

$$g(v,\omega) = g(J\,v\,,\omega)$$

Is non-degenerate, because

$$\omega\,(\,v\,, J\,v) > 0 \ \text{ for } \neq 0.$$

The proof of the converse is clear, where now linear maps are replaced by vector bundle morphisms. Suppose that ω is an almost symplectic structure on M and let again g be any Riemannian metric on M. There exists a smooth bundle automorphism $A : TM \rightarrow TM$ (depending on g) such that $\omega\ (v,\omega)$ = g (Av, ω)

for all $v, \omega \in T_pM, p \in M$. Since ω is non-degenerate and skew-symmetric, A is an automorphism and skew-symmetric (with respect to g). Therefore, - A^2 is positive definite and symmetric (with respect to g). Therefore, it has a unique square root, which means that there is a unique smooth bundle automorphism $B : TM \rightarrow TM$ such that $B^2 = - A^2$. Moreover, B commutes with A. Then, $J = AB^{-1}$ is an almost complex structure on M.

In the proof of the converse statement in Proposition 3.1.5 we have used the easily proved fact that if ω_t, $t \in R$, is a family of symplectic bilinear forms on R^{2n}, and g_t, t ∈ R, is a family of positive definite inner products all depending smoothly on t we end up with a smooth family of corresponding compatible complex structures J_t, t ∈ R, on R^{2n}. This guarantees the smoothness of the almost complex structure J on M.

If (M, ω) is a symplectic manifold, an almost complex structure J on M is called compatible with ω if $g_x(u, v) = -\omega\,(J(u), v)$, for $u, v \in T_x M$, $x \in M$, is a Riemannian metric on M preserved by J. As the proof of Proposition 3.1.5 shows, any symplectic manifold carries compatible almost complex structures. If J_0 and J_1 are two almost complex structures compatible with ω, there exists a smooth family J_t, $0 \leq t \leq 1$, of compatible almost complex structures from J_0 to J_1. We can globalize the arguments to prove that the space $J\,(M, \omega)$ of compatible almost complex structures is contractible. This is important for uniqueness of invariants arising from a compatible almost complex structure.

The tangent bundle of a symplectic manifold (M, ω), or more generally of an almost complex manifold, is the realification of a complex vector bundle. This situation is a particular case of a symplectic vector bundle. A symplectic vector bundle (E, ω) over a smooth manifold M is a real smooth vector bundle $p : E \rightarrow M$ with a symplectic form ω_x on each fiber $E_x = p^{-1}(x)$ which varies smoothly with $x \in M$. In other words, ω is a smooth section of the bundle $E^* \wedge E^*$, where E^* is the dual bundle. Two symplectic vector bundles (E_1, ω_1) and (E_2, ω_2) over M are isomorphic if there exists a smooth vector bundle isomorphism $f : E_1 \to E_2$ such that $f^*(\omega_2) = \omega_1$. [4]

3.2 Local Description of Symplectic Manifolds

Even though we have defined the symplectic structure in analogy to the Riemannian structure, their local behaviour differs drastically. We shall show that in the neighbourhood of any point on a symplectic 2n-manifold (M, ω) there are suitable local coordinates $(q^1, \dots, q^n, p^1, \dots, p^n)$ such that

$$\omega_{\ locally} = \sum_{i=1}^{n} dq^i \wedge dp_i$$

This shows that in symplectic geometry there are no local invariants, in contrast to Riemannian geometry, where there are highly non-trivial local invariants. In other words, the study of symplectic manifolds is of global nature and one expects to use mainly topological methods.

Lemma 3.2.1. (Moser) *Let M and N be two smooth manifolds and* $F : M \times R \to N$ *be a smooth manifolds and F: M × R → N be a smooth map. For every t ∈ R let* $X_t : M \to TN$ *be the smooth vector field along* $F_t = F(.,t)$ *defined by*

$$X_t(p) = \frac{\partial}{\partial s} \ F(p,s) \in T_{F_t(p)}N. \tag{3.2.1}$$

If $(w_t)_{t \in R}$ *is a smooth family of k-forms on N, then*

$$\frac{d}{dt}(F_t^* \omega_t) = F_t^* \frac{d\omega_t}{dt} + i_{X_t} d\omega_t) + d\big(F_t^* i_{X_t} \omega_t\big). \tag{3.2.2}$$

If moreover F_t is a diffeomorphism for every t ∈ R, then

$$\frac{d}{dt}(F_t^* \omega_t) = F_t^*(\frac{d\omega_t}{dt} + i_{X_t} d\omega_t + d i_{X_t} \omega_t). \tag{3.2.3}$$

Note that if F_t is not a diffeomorphism then X_t is not in general a vector field on N. The meaning of the symbol $F_t^* \, i_{X_t} \, \omega_t$ will be clear in the proof.

Proof. (a) First we shall prove the formula in the special case M = N = P × R and

$$F_t = \Psi_t, \text{ where } \Psi_t(x,s) = (x, s+t).$$

Then

$$\omega_t = ds \wedge a(x,s,t)dx^k + b(x,s,t)dx^{k+1},$$

Where

$$a(x,s,t)dx^k = \sum_{i_1<i_2<\cdots<i_k} a_{i_1 i_2 \ldots i_k}(x,s,t)dx^{i1} \wedge dx^{i2} \wedge \ldots \wedge dx^{ik}$$

And similarly for b(x, s, t) dx^{k+1}.

So

$$\Psi_t^* \, \omega_t = ds \wedge a(x, s+t, t)dx^k + b(x, s+t, t)dx^{k+1}$$

and

$$\frac{d}{dt}(\Psi_t^* \, \omega_t) = ds \wedge \frac{\partial a}{\partial s}(x, s+t, t)dx^k + \frac{\partial b}{\partial s}(x, s+t, t)dx^{k+1} + ds \wedge \frac{\partial a}{\partial s}(x, s+t, t)dx^k + \frac{\partial b}{\partial s}(x, s+t, t)dx^{k+1},$$

Obviously,

$$\Psi_t^* \left(\frac{d\omega_t}{dt}\right) = ds \wedge \frac{\partial a}{\partial t}(x, s+t, t)dx^k + \frac{\partial b}{\partial t}(x, s+t, t)dx^{k+1}. \qquad (3.2.4)$$

On the other hand $X_t = \frac{\partial}{\partial s}$. So $i_{X_t} \, \omega_t = a(x, s, t)dx^k$ and

$$d(i_{X_t} \, \omega_t) = \sum_{i_1<i_2<\cdots<i_k} da_{i_1 i_2 \ldots i_k}(x,s,t) \wedge dx^{i1} \wedge dx^{i2} \wedge \ldots \wedge dx^{ik} =$$

$$\sum_{i_1<i_2<\cdots<i_k} \left(\frac{\partial a_{i_1 i_2 \ldots i_k}}{\partial s}(x,s,t)ds \wedge dx^{i1} \wedge dx^{i2} \wedge \ldots \wedge dx^{ik} + \right.$$

$$\sum_{j \notin \{i_1 < i_2 < \cdots < i_k\}} \frac{\partial a_{i_1 i_2 \ldots i_k}}{\partial x^j} \, dx^j \wedge dx^{i1} \wedge dx^{i2} \wedge \ldots \wedge dx^{ik}).$$

We shall write for brevity

$$d(i_{X_t}\, \omega_t) = \frac{\partial a}{\partial s}(x,s,t)ds \wedge dx^k + d_x a(x,s,t)dx^{k+1}.$$

So

$$\Psi_t^*(d(i_{X_t}\, \omega_t))$$

$$= \frac{\partial a}{\partial s}(x,s,t)ds \wedge dx^k$$

$$+ d_x a(x,s,t)dx^{k+1}. \qquad (3.2.5)$$

Using the symbol d_x in the same way, we have

$$d\omega_t = -ds \wedge d_x a(x,s,t)dx^k + \frac{\partial b}{\partial s}(x,s,t)ds \wedge dx^{k+1} + d_x b(x,s,t)dx^{k+2},$$

and thus

$$\Psi_t^*(i_{X_t}\, d\omega_t)$$

$$= -d_x a(x,s+t,t)dx^{k+1}$$

$$+ \frac{\partial b}{\partial s}(x,s+t,t)\, dx^{k+1}. \qquad (3.2.6)$$

Summing up now (3.2.4), (3.2.5) and (3.2.6) we get

$$\Psi_t^*\left(\frac{d\omega_t}{dt}\right) + \Psi_t^*\, d(i_{X_t}\, \omega_t) + \Psi_t^*\, d(i_{X_t}\, d\omega_t) = \frac{d}{dt}\Psi_t^*\omega_t\,.$$

(b) The general case follows from part (a) using the decomposition $F_t = F \circ \Psi_t \circ j$, where $j : M \to M \times R$ is the inclusion j(p) = (p, 0) and Ψ_t is the same as in part (a) Now we have

$$X_t(p) = \frac{\partial}{\partial s} F(p,s) = F_{*(p,t)}(\frac{\partial}{\partial s})_{(p,t)}.$$

If each F_t is a diffeomorphism, then X_t is a vector field on N and $i_{X_t}\, d\omega_t$ is defined.

If not, the term $F_t^*\left(i_{X_t}\, d\omega_t\right)$ has the following meaning. By definition,

$$F_t^*\left(i_{X_t}\,\omega_t\right)_p(v_1, \dots, v_{k-1}) = (\omega_t)_{F_t(p)}\left(X_t(p), (F_t)_{*p}(v_1), \dots, (F_t)_{*p}(v_{k-1})\right) =$$

$$(\omega_t)_{F(p,t)}\ (F_{*(p,t)}(\frac{\partial}{\partial s})_{(p,t)}\ , F_{*(p,t)}(v_1, 0), \dots, F_{*(p,t)}(v_{k-1}, 0) =$$

$$(F^*\omega_t)\ _{(p,t)}\left(\left(\frac{\partial}{\partial s}\right)_{(p,t)}, (v_1, 0), \dots, (v_{k-1}, 0)\right) =$$

$$\left(i_{\partial/\partial s} F^*\omega_t\right)_{(p,t)}\left((v_1, 0), \dots, (v_{k-1}, 0)\right) =$$

$$j^*\Psi_t^*(i_{\partial/\partial s} F^*\omega_t)_p(v_1, \dots, v_{k-1})$$

for $v_1, \dots, v_{k-1} \epsilon\, T_pM$. Therefore, $F_t^*\left(i_{X_t}\,\omega_t\right) = j^*\Psi_t^*(i_{\partial/\partial s} F^*\omega_t)$ and similarly $F_t^*\left(i_{X_t} d\omega_t\right) = j^*\Psi_t^*(i_{\partial/\partial s} d(F^*\omega_t))$. Since j^* does not depend on t, we have

$$\frac{d}{dt}(F_t^*\omega_t) = \ j^*\frac{d}{dt}(\Psi_t^* F^*\omega_t)$$

And applying part (a) to $F^*\omega_t$ we get

$$\frac{d}{dt}(F_t^*\omega_t) = j^*\Psi_t^*\left(\frac{d(F^*\omega_t)}{dt}\right) + j^*\Psi_t^*(i_{\partial/\partial s} d(F^*\omega_t)) + j^*d(\Psi_t^* i_{\partial/\partial s}(F^*\omega_t)) =$$

$$j^*\Psi_t^* F^*\left(\frac{d\omega_t}{dt}\right) + F_t^*\left(i_{X_t} d\omega_t\right) + \ d(F_t^*\left(i_{X_t} d\omega_t\right)) =$$

$$F_t^*\left(\frac{d\omega_t}{dt}\right) + F_t^*\left(i_{X_t} d\omega_t\right) + d(F_t^*\left(i_{X_t} d\omega_t\right)).$$

Corollary 3.2.2. *Let X be a smooth vector field on a smooth manifold M. If ω is a differential form on M, then* $L_X\,\omega = i_X d\,\omega + d i_X\,\omega$.

Proof. If X is complete and $(\emptyset_t)_{t\in R}$ is its flow, we apply Lemma 3.2.1 for $F_t = \emptyset_t$, M = N and $\omega_t = \ \omega$ and we have

$$L_X\, \omega = \frac{d}{dt}_{t=0}\, \emptyset_t^* \omega = i_X \mathrm{d}\, \omega + d i_X\, \omega.$$

If X is not complete, then M has an open covering $\mathcal{U}$ such that for every U ∈ $\mathcal{U}$ there exists some ϵ > 0 and a local flow map $\emptyset$: (- ϵ, ϵ) × $U \to M$ of X. Again we apply Lemma 3.2.1 for $F_t = \emptyset_t$ on U this time to get the desired formula on every U∈ $\mathcal{U}$, hence on M.

We are now in a position to prove the main theorem.

Theorem 3.2.2 (Darboux) *Let ω_0 and ω_1 be two symplectic 2-forms on a smooth 2n-manifold M and p ϵ M. If ω_0(p) = ω_1(p), there exists an open neighbourhood U of p in M and a diffeomorphism $F : U \to F(U) \subset M$, where F(U) is an open neighbourhood of p, such that F(p) = p and $F^*\omega_1 = \omega_0$.*

Proof. Let ω_t = (1-t) ω_0+ t ω_1, 0 ≤ t ≤ 1. Since ω_t(p) = ω_0(p) = ω_1(p), there exists an open neighbourhood U_1 of p diffeomorphic to R^{2n} such that $\omega_t \,|\, U_1$ is symplectic for every 0 ≤ t ≤ 1. By the lemma of Poincare, there exists a 1-form a on U_1 such that $\omega_0 - \omega_1$ = da on U_1 and a (p) = 0. For every 0 ≤ t ≤ 1 there exists a smooth vector field Y_t on U_1 such that $i_{Y_t}\omega_t$ = a. Obviously, Y_t (p) = 0 and the above hold for every - ϵ < t < 1 + ϵ, for some ϵ > 0. Now $\overline{Y}$ = ($\frac{\partial}{\partial s}$, Y_s (p)) is a smooth vector field on (- ϵ, 1 + ϵ) × U_1. If ϕ_t is the flow of $\overline{Y}$, then ϕ_t *(s, x) = (s + t, f_t(s, x)),* for some smooth f_t : (- ϵ, 1 + ϵ) × $U_1 \to M$. Therefore, ϕ_t(0, x) = (t, F_t(x)), where $F_t : U_1 \to F_t(U_1)$ is a diffeomorphism. Since ϕ_t*(0, p) = (0, p),* that is F_t(p) = p, there exists an open neighbourhood U of p such that F_t is defined on U and F_t(U) ⊂ U1 for every 0 ≤ t ≤ 1. Obviously, $\mathrm{Y_t} = \frac{\partial \mathrm{F_t}}{\partial \mathrm{t}}$ and so from Lemma 3.2.1 we have

$$\frac{d}{dt}\,(F_t^*\omega_t) = F_t^*\left(\frac{d\omega_t}{dt} + i_{Y_t} d\omega_t + d i_{Y_t}\omega_t\right) = F_t^*(\,\omega_1 - \omega_0 + 0 + da) = 0.$$

Hence

$F_t^* \omega_t = F_0^* \omega_0 = \omega_0$ for every $0 \leq t \leq 1$, since $F_0 = id$.

Corollary 3.2.3 Let (M,ω) be a symplectic 2n-manifold and p ϵ M. There exists an open neighbourhood U of p and a diffeomorphism $F : U \rightarrow F(U) \subset R^{2n}$ such that

$$\omega\big|_U = F^* \left(\sum_{i=1}^{n} dx^i \wedge dy^i \right).$$

Proof. Let (W, Ψ) be a chart of M with p ϵ W, Ψ (W) = R^{2n} and Ψ(p) = 0. Then the 2-form $\omega_1 = (\Psi^{-1})^* \omega$ on R^{2n} is symplectic. Composing with a linear transformation if necessary, we may assume that $\omega_1(0) = \omega_0(0)$, where ω_0 is the standard symplectic 2-form on R^{2n}. By Darboux's theorem, there exists an open neighbourhood V of 0 in R^{2n} and a diffeomorphism $\phi : V \rightarrow \phi(V)$ with ϕ (0) = 0 and $\phi^* \omega_1 = \omega_0$. It suffices to set now F = $(\Psi^{-1} \circ \phi)^{-1}$.

At this point, we cannot resist the temptation to use Moser's trick in order to prove the following result, also due to J. Moser.

Theorem 3.2.4 (Moser) *Let M be a connected, compact, oriented, smooth n-manifold and ω_0, ω_1 be two representatives of the orientation. If*

$$\int \omega_0 = \int \omega_1$$

there exists a diffeomorphism $f : M \rightarrow M$ such that $f^* \omega_1 = \omega_0$.

Proof. For every *$0 \leq t \leq 1$ the n*-form ω_t = *(1- t)* ω_0 +*t* ω_0 is a representative of the orientation, that is a positive volume element of M. Since

$$\int (\omega_0 - \omega_1) = 0$$

there exists a $(n - 1)$-form a on M such that $\omega_0 - \omega_1 = da$. There exists a unique smooth vector field X_t on M such that $i_{X_t}\,\omega_t = a$. As in the proof of Darboux's theorem, there exists a smooth isotopy $F : M \times [0, 1] \to M$ with $F_0 = id$ and

$$X_t = \frac{\partial F_t}{\partial t},$$

because M is compact. Again from Lemma 3.2.1 we have

$$\frac{d}{dt}(F_t^*\,\omega_t) = F_t^*\,(\omega_1 - \omega_0 - 0 + da) = 0$$

Hence $F_t^*\,\omega_t = \omega_0$ for every $0 \le t \le 1$. □ [22]

3.3 Examples of Symplectic Manifolds

The main object is to show that *locally* all finite-dimensional symplectic manifolds look alike. On the other hand, a *global* examination of symplectic structures is usually made difficult by additional geometric propertics of the manifold. Therefore we restrict our considerations and illustrating examples to the three most frequently encountered types of symplectic manifolds, namely cotangent bundles and the geodiscs manifolds.

3.3.1 The Geometry of the Tangent Bundle

We shall study the geometry of the tangent bundle of a Riemannian manifold. Its structure is useful in Riemannian geometry and when studying mechanical problems within the framework of Newtonian mechanics.

Let M be a smooth n-dimensional manifold and let $p : TM \to M$ be its tangent bundle. There exists a canonical subbundle V of T(TM), which is just $V = \mathrm{Ker}\, p_*$ and is called the vertical subbundle. In other words, for each $u \in TM$ the fiber V_u is the tangent space to the fiber $T_x M$ of TM at u, where $x = p(u)$.

It would be desirable to have a canonical complementary to V subbundle of T(TM). Unfortunately, this is impossible. In order to construct a complementary subbundle, we must use a Riemmanian metric on M. So from now on we assume that M is a Riemannian manifold with metric g. The corresponding Levi-Civita connection ∇ induces the connection map $K : T(TM) \to TM$ which is defined as follows:

Let $u \in T_xM$, $x \in M$. Let W be a normal neighbourhood of x in M, that is $W = \exp_x(U)$, where U is a star-shaped open neighbourhood of $0 \in T_xM$ and $\exp_x\big|_U : U \to W$ is a diffeomorphism. Let $\tau : p^{-1}(W) \to T_xM$ be the smooth map which sends each $v \in T_yM$, $y \in W$ to its parallel translation at x along the unique geodesic in W form y to x. For $w \in T_xM$, we let $R_{-w} : T_xM \to T_xM$ be the translation by the vector $-w$. The connection map $K_u : T_u(TM) \to T_xM$ is defined by

$$K_u(\xi) = (\exp_x \circ R_{-u} \circ \tau)_{*u}(\xi). \tag{3.3.1}$$

Obviously, this is a well-defined linear map. An alternative definition in terms of covariant differentiation is given by the following.

Proposition 3.3.2 *Let $z : (-\epsilon, \epsilon) \to TM, \epsilon > 0$, be a smooth curve such that $z(0) = u$ and $\dot{z}(0) = \xi$. If $\gamma = p \circ z : (-\epsilon, \epsilon) \to M$ and X is the smooth vector field along γ such that $z(t) = (\gamma(t), X(t)) \in T_{\gamma(t)}M$, $|t| < \epsilon$, then $K_u(\xi) = \nabla_{\dot{\gamma}(0)}X$.*

Proof. The definition of the connection map and the chain rule imply that

$$K_u(\xi) = (\exp_x \circ R_{-u} \circ \tau)_{*z(0)}(\dot{z}(0)) = \frac{d}{dt}\Big|_{t=0}(\exp_x \circ R_{-u} \circ \tau \circ z).$$

Since ($\exp_x \circ R_{-u} \circ \tau \circ$ z)(t) $= \exp_x\big(\tau\big(z(t)\big) - u\big), we\ get$

$$K_u(\xi) = (\exp_x)_{*0}\left(\frac{d}{dt}\Big|_{t=0} \tau\big(X(t)\big)\right) = \nabla_{\dot{\gamma}(0)}X \qquad (3.3.2)$$

since $(\exp_x)_{*0}$ is the identity.

The horizontal subbundle H of T(TM) is now the one whose fiber at $u \in TM$ is

$$H_u = Ker K_u.$$

It is evident that horizontal curves in TM, that is smooth curves tangent to H, correspond to parallel vector fields along curves in M. To be more precise, given $u \in T_xM$ and $\gamma : (-\epsilon, \epsilon) \to M$ a smooth curve with $\gamma(0) = x$ and $\dot{\gamma}(0) = u$, let $X(t)$, $|t| < \epsilon$, be the parallel transport of u along γ . Let also $\sigma : (-\epsilon, \epsilon) \to TM$ be the smooth curve $\sigma(t) = (\gamma(t), X(t)$. Then $\sigma(t) = (x, u)$ and if $\xi = \dot{\sigma}(0)$, we have

$$u = \dot{\gamma}(0) = p_{*u}\big(\dot{\sigma}(0)\big) = p_{*u}(\xi).$$

This shows that:

$$p_{*u}(H_u) = T_xM,$$

since

$$K_u(\xi) = \nabla_{\dot{\gamma}(0)}X = 0.$$

Moreover, $p_{*u}\big|H_u : H_u \to T_xM$ is an isomorphism. Indeed, let $\xi \in T_u(TM)$ be such that $p_{*u}(\xi) = 0$. There exists a vertical smooth curve $z : (-\epsilon, \epsilon) \to T_xM \subset TM$ such that $z(0) = u$ and $\dot{z}(0) = \xi$. Thus, $\gamma = p \circ z$ takes the constant value $\gamma(t) = x$ for all $|t| < \epsilon$ and therefore $(i_x \circ \tau)_{*u}(\xi) = \xi$, where $i_x : T_xM \to T\,M$ denotes the inclusion. Since ($\exp_x \circ R_{-u})_{*u}$ is an isomorphism, we conclude that if $\xi \in H_u$, then $\xi = 0$. The above argument also shows that $H_u \cap V_u = \{0\}$ and $H_u\big|V_u : V_u \to T_xM$ is also an isomorphism. Hence $T_u(TM) = H_u \oplus V_u$ and the linear map $j_u : T_u(TM) \to T_xM \oplus T_xM$ given by

$$j_u(\xi) = (p_{*u}(\xi), K_u(\xi))$$

is a isomorphism.

If now $X \in T_x M$,the horizontal lift of X to $u \in TM$ is the unique vector $X^h \in H_u$ such that $p_{*u}\left(X^h\right) = X$. the vertical lift of X is the unique vector $X^v \in V_u$ such that $X^v(\tilde{f}) = X(f)$ for all smooth functions f where $\tilde{f}$ is the smooth function on TM with $\tilde{f}(u) = u(\tilde{f})$.

Using the above decomposition of T(TM) as the Whitney direct sum of two subbundles , we can define a Riemannian metric $\langle , \rangle$on TM such that H and V become orthogonal subbundles and the tangent bundle projection $p: TM \rightarrow M$ becomes a Riemannian submersion. This Riemannian metric is called the Sasaki metric and is defined by:

$$\langle \xi, \xi \rangle_u = g\left(H_{*u}(\xi), p_{*u}(\xi)\right) + g\left(K_u(\xi), K_u(\xi)\right).$$

It is worth to note that the geodesic vector field $G: TM \rightarrow T(TM)$ has a very simple expression under the isomorphidm j_u , $u \in TM$. if$_{\gamma u}$ denotes the geodesic with $\gamma u(0) = x$ and $\dot{\gamma} u(0) = u$,

Then

$$G(u) = \frac{d}{dt}\Big|_{t=0} \gamma \dot{u}(t).$$

Since $\dot{\gamma}_u(t)$ is the parallel transport of u along γ_u, is follows that

$$P_{*u}\left(G(u)\right) = u$$

and

$$K_u\left(G(u)\right) = 0 .$$

Therefore,

$$j_u\left(G(u)\right) = (u, 0).$$

3.3.3 The Manifold of Geodesics

If (M,g) is a Riemannian manifold, then the concept of *length* makes sense for any piecewise smooth (in fact, C^1) curve on M. Then, it is possible to define the structure of a metric space on M, where d(p, q) is the greatest lower bound of the length of all curves joining p and q. Curves on M which locally yield the shortest distance between two points are of great interest. These curves called *geodesics* play an important role and the goal is to study some of their properties.

Let (M, g) be a complete Riemannian n-manifold. A unit speed geodesic $\gamma : R \rightarrow M$ is called periodic of period $\ell > 0$ if $\gamma(t + \ell) = \gamma(t)$ for every $t \in R$ and ℓ is the smallest positive real number with this property. In this case the length of γ is ℓ.

If every geodesic of M is periodic of the same period ℓ, then (M, g) is called a C_ℓ -manifold and its metric a C_ℓ-metric. The geodesic flow of a C_ℓ-manifold is periodic and there exists a smooth free action of S^1 on the unit tangent bundle T^1M of M whose orbit space is smooth 2n-manifold CM. Also the quotient map $q : T^1M \rightarrow CM$ is a principle S^1-bundle. The manifold CM is called the manifold of oriented geodesics of M.

Example 3.3.4 The sphere S^n, $n \geq 2$, equiped with the usual euclidean Riemannian metric is a $C_{2\pi}$-manifold. From the uniuqeness of geodesics follows that the oriented geodesics on S^n are in one-to-one correspondence with the oriented 2-dimensional linear subspaces of R^{n+1}. Therefore, CS^n is diffeomorphic to SO(n+1,R) / SO(2,R) × SO(n-1,R). The same space is the manifold of geodesics of the real projective space RP^n with its standard Riemannian metric which is a C_π-manifold and is doubly covered by S^n.

The manifold of geodesics of any C_ℓ -metric on S^2 can be determined from the honotopy exact sequence

$$\dots \to \pi_1(S^1) \to \pi_1(T^1S^2) \to \pi_1(C\, S^2) \to \{1\}$$

of the fibration $q : T^1S^2 \to C\, S^2$. Recall that T^1S^2 is diffeomorphic to RP^3 and so $\pi_1(T^1S^2) \cong Z_2$. It follows that $\pi_1(C\, S^2)$ is either trivial or isomorphic to Z_2. However, we shall show shortly that the manifold of geodesics carries a symplectic structure and is therefore orientable. Hence $C\, S^2$ is diffeomorphic to S^2.

The manifold of geodesics CM of a C_ℓ- manifold M can be given a natural symplectic structure. Recall that TM has a symplectic structure $\Omega = -dA$, where A is the pullback of the Liouville canonical 1-form on T^*M under the natural bundle isomorphism $L: TM \to T^*M$ defined by the Riemannian metric g. Let $\eta = A|_{T^1M}$. Recall also that $T^1M = H^{-1}(1/2)$, where $H: TM \to R$ is the kinetic energy. Let $(\psi_{t\in R})$ be the smooth flow on T M defined by $\psi_t(u) = e^t u$. Its infinitesimal generator Y is smooth vector field on T M which in local coordinates $(q^1, \dots, q^n, v^1, \dots, v^n)$ on T M is represented as

$$Y|_{locally} = \sum v^k \frac{\partial}{\partial v^k}.$$

Since $dH(Y) = 2H$, it follows that Y is transverse to T^1M. [3]

3.4 Symmetries and Integrability

We will introduce some properties of symplectic manifolds on finite-dimensional integrable dynamical systems related to Hamiltonian G-actions. Within a framework of noncommutative integrability we study integrability of G-invariant systems, collective motions and reduced integrability. We also consider reductions of the Hamiltonian flows restricted to their invariant submanifolds.

3.4.1 Symplectic Group Actions

Let M be a smooth manifold, G a Lie group with Lie algebra g and $\phi : G \times M \rightarrow M$ be a smooth group action. If $X \in g$, the fundamental vector field $\phi_*(X) \in X(M)$ of the action which corresponds to X is the infinitesimal generator of the flow $\phi_X : R \times M \rightarrow M$ defined by $\phi_X(t,p) = \phi(exp(tX),p)$. Note that for $g \in G$ the transformed vector field $(\phi_g)_*(\phi_*(X))$ is the fundamental vector field $\phi_*(Ad_g(X))$,

that is

$$(\phi_g)_{*p}(\phi_*(X)(p)) = \phi_*(Ad_g(X))(\phi_g(p)) \tag{3.4.1}$$

for every $p \in M$. Indeed,

$$\phi_*\left(Ad_g(X)\right)\left(\phi_g(p)\right) = \frac{d}{dt}\Big|_{t=0} \phi^{\phi_g(p)}\left(\exp\left(tAd_g(X)\right)\right) =$$

$$(\phi^{\phi_g(p)})_{*e}\left(\frac{d}{dt}\Big|_{t=0} \exp\left(tAd_g(X)\right)\right) = (\phi^{\phi_g(p)})_{*e}\left(Ad_g(X)\right) =$$

$$\frac{d}{dt}\Big|_{t=0} \phi(g\exp(tX)\,g^{-1}, \phi(g,p) = \frac{d}{dt}\Big|_{t=0}\phi(g\exp(tX), p) =$$

$$\frac{d}{dt}\Big|_{t=0}(\phi^p \circ L_g)(\exp(\mathrm{t\,X})) = \frac{d}{dt}\Big|_{t=0}(\phi_g \circ \phi^p)(\exp(tX)) =$$

$$(\phi_g)_{*p}\left((\phi^p)_{*e}(X)\right) = (\phi_g)_{*p}\left((\phi_*(X)(p)\right). \tag{3.4.2}$$

Lemma 3.4.2 *The linear map* $\phi_* : g \rightarrow X(M)$ *is an anti-homomorphism of Lie algebras, meaning that* $\phi_*([X,Y]) = -[\phi_*(X), \phi_*(Y)]$ *for every* $X, Y \in g$.

Proof. If $p \in M$, then we compute

$$[\phi_*(X), \phi_*(Y)](p) = \frac{d}{dt}\Big|_{t=0} \left(\phi_{\exp(-tX)}\right)_{*\phi_{exp\,(tX)}(p)} \left(\phi_*(Y)\right)(\phi_{exp\,(tX)}(p))) =$$

$$\frac{d}{dt}\Big|_{t=0} \ \phi_*\left(Ad_{\exp(-tX)}(Y)\right)(p) = \phi_*\left(-ad_X(Y)\right)(p) = -\phi_*([X,Y]).$$

Although ϕ_* is an anti-homomorphism of Lie algebras, it follows that $\phi_*(g)$ is a Lie subalgebra of $X(M)$ of finite dimension.

Proposition 3.4.3

Let X_H *be a Hamiltonian vector field with flow* $\emptyset_t$ *on a symplectic manifold M. Then*

$$\frac{d}{dt}(F \circ \emptyset_t) = \{F \circ \emptyset_t, H\} = \{F, H\} \circ \emptyset_t \qquad (3.4.3)$$

For every $F \in C^\infty(M)$

Proof. By the chain rule, for every $p \in M$ we have

$$\frac{d}{dt}(F \circ \emptyset_t)(p) = (dF)(\emptyset_t(p))\, X_H(\emptyset_t(p)) = \{F, H\}(\emptyset_t(p))$$

$$= \{F \circ \emptyset_t, H \circ \emptyset_t\}(p) = \{F \circ \emptyset_t, H\}(p)$$

Since H is a first integral of X_H.

Definition 3.4.4 Let (M, ω) be a symplectic manifold and G a Lie group. A smooth group action $\phi : G \times M \rightarrow M$ is called symplectic if $\phi_g = \phi(g,.) : M \rightarrow M$ is a symplectomorphism for every $g \in G$.

If ϕ is symplectic, then $\phi_*(g) \subset sp(M,\omega)$, and therefore

$$\phi_*([g,g]) \subset [sp(M,\omega), sp(M,\omega)] \subset h(M,\omega),$$

by Proposition 3.4.3. If $H_\phi : g \to H^1_{DR}(M)$ is the linear map defined by $H_\phi(X) = [i_{\phi*(X)}\omega]$, then $X \in Ker\, H_\phi$ if and only if ϕ_* (X) is a Hamiltonian vector field, and $[g,g] \subset Ker\, H_\phi$.

Definition 3.4.5 A symplectic group action ϕ is called Hamiltonian if H_ϕ = 0.

Thus, if $H^1(M;R) = \{0\}$, then every symplectic group action on M is Hamiltonian. In particular, every symplectic group action on a simply connected symplectic manifold is Hamiltonian. Also if the Lie algebra g of G is perfect, meaning that $g = [g,g]$, then every symplectic group action of G is Hamiltonian. This happens for example in the case $G = SO(3,R)$, because $so(3,R)$ is isomorphic to the Lie algebra (R, ×), which is obviously perfect.

If ϕ is a Hamiltonian group action, in general there is no canonical way to choose a Hamiltonian function for $\phi_*(X)$ since adding a constant to a Hamiltonian function yields a new Hamiltonian function. If there is a linear map $\rho : g \to C^\infty(M)$ such that $\rho(X)$ is a Hamiltonian function for $\phi_*(X)$ for every $X \in g$, there is a smooth map $\mu : M \to g^*$ defined by $\mu(p)(X) = \rho(X)(p)$.

Examples 3.4.6 (a) Let M be a smooth manifold, G a Lie group with Lie algebra g and $\phi : G \times M \to M$ a smooth group action. Then, ϕ is covered by a group action $\check{\phi}$ of G on T^*M defined by $\check{\phi}(g,a) = a \circ (\phi_{g^{-1}})_{*\phi_g}(\pi(a))$, where $\pi : T^*M \to M$ is the cotangent bundle projection. Since $\pi \circ \check{\phi}_g = \phi_g \circ \pi$, differentiating we get

$$\pi_{*\check{\phi}_g(a)} \circ (\check{\phi}_g)_{*a} = (\phi_g)_{*\pi(a)} \circ \pi_{*a} \tag{3.4.5}$$

for every $a \in T^*M$ and $g \in G$. The Liouville 1-form θ on T^*M remains invariant under the action of G, because

$$((\check{\phi}_g)^*\theta)_a = \theta_{\check{\phi}_g(a)} \circ (\check{\phi}_g)_{*a} = a \circ \left(\phi_g^{-1}\right)_{*\phi_{g(\pi(a))}} \circ (\phi_g)_{*\pi(a)} \circ \pi_{*a}$$

$$= a \circ \pi_{*a} = \theta_a.$$

Consequently, the action of G on T^*M is symplectic with respect to the canonical symplectic structure $\omega = -d\theta$. Moreover, it is Hamiltonian, because

$$0 = L_{\check{\phi}_*(X)}\theta = i_{\check{\phi}_*(X)}(d\theta) + d(i_{\check{\phi}_*(X)}\theta)$$

and therefore $i_{\check{\phi}_*(X)}\omega = d\left(i_{\check{\phi}_*(X)}\theta\right)$. Here we have a linear map $\rho : g \to C^\infty(T^*M)$ defined by $\rho(X) = i_{\check{\phi}_*(X)}\theta$ and $\mu : T^*M \to g^*$ is given by the formula

$$\mu(a)(X) = \theta_a\left(\widetilde{\phi_*}(X)\right).$$

(b) Let G be a Lie group with Lie algebra g and O be a coadjoint orbit. The symplectic Kirillov 2-form ω^- is Ad^*-invariant and so the natural action of G on O is symplectic. Recall that

$$\omega_v^-\left(X_{g*}(v), Y_{g*}(v)\right) = -v([X,Y]) = (v \circ ad_Y)(X) = -Y_{g*}(v)(X) =$$
$$-X(Y_{g*}(v)) \tag{3.4.6}$$

for every $X, Y \in g$ and $v \in O$, having identified g^{**} with g. If now $\rho_X \in C^\infty(g^*)$ is the (linear) function defined by $\rho_X(v) = -v(X)$, then $d\rho_X(v) = -X$ (again we identify g^{**} with g. It follows that $iX_{g*}\omega^- = d\rho_X$, which shows that the action of G on O is Hamiltonian.

Let $\phi : G \times M \to M$ be a Hamiltonian group action of the Lie group G with Lie algebra g on a connected, symplectic manifold (M, ω). We assume that we have a linear lift $\rho : g \to C^{\infty}(M)$ such that $\phi_*(X) = X_{\rho(X)}$ for every $X \in g$. We shall study the possibility to change ρ to a new lift which is also a Lie algebra homomorphism.

From Proposition 3.4.3 and Lemma 3.4.2 we have

$$X_{\{\rho(X_0),\rho(X_1)\}} = -[X_{\rho(X_0)}, X_{\rho(X_1)}] = \emptyset_*([X_0, X_1]) = X_{\rho([X_0,X_1])},$$

for every $X_0, X_1 \in g$. Since M is connected, there exists $c(X_0, X_1) \in R$ such that

$$\{\rho(X_0), \rho(X_1)\} = \rho([X_0, X_1]) + c(X_0, X_1).$$

Obviously, $c : g \times g \to R$ is a skew-symmetric, bilinear form. Moreover, $\delta c = 0$, from the Jacobi identity and the linearity of ρ. Hence $c \in Z^2(g)$. If $\breve{\rho} : g \to C^{\infty}(M)$ is another linear lift and $\sigma = \breve{\rho} - \rho$, then $\sigma \in g^*$ and

$$\{\breve{\rho}(X_0), \breve{\rho}(X_1)\} = \{\rho(X_0), \rho(X_1)\} = \rho([X_0, X_1]) + c(X_0, X_1)$$
$$= \breve{\rho}([X_0, X_1]) + c(X_0, X_1) - \sigma([X_0, X_1]).$$

Hence, $\breve{c}(X_0, X_1) - c(X_0, X_1) = -\sigma([X_0, X_1]) = (\delta\sigma)(X_0, X_1)$. We conclude that there is a choice of $\breve{\rho}$ such that $\breve{c} = 0$ if and only if $[c] = 0$ in $H^2(g)$. Thus, in case $H^2(g) = \{0\}$, we can always select a linear lift $\rho : g \to C^{\infty}(M)$ which is a Lie algebra homomorphism.

Examples 3.4.7 (a) Let M be a smooth manifold, G a Lie group with Lie algebra g and $\phi : G \times M \to M$ a smooth group action. As we saw in Example 3.4.6 (a), the covering action $\breve{\phi}$ on T^*M is Hamiltonian and $\rho : g \to C^{\infty}(T^*M)$ is given by the formula $\rho(X) = i_{\breve{\emptyset}_*(X)}\theta$, where θ is the invariant Liouville 1-form. Then,

$$c(X_0, X_1) = -d\theta\left(\breve{\emptyset}_*(X_0), \breve{\emptyset}_*(X_1)\right) - \theta(\breve{\emptyset}_*([X_0, X_1])$$
$$= -L_{\breve{\emptyset}_*(X_0)}\rho(X_1) + L_{\breve{\emptyset}_*(X_1)}\rho(X_0) + \theta(\breve{\emptyset}_*([X_0, X_1])$$
$$= -\{\rho(X_1) + \rho(X_0)\} + \{\rho(X_0) + \rho(X_1)\} - 2\theta(\breve{\emptyset}_*([X_0, X_1])$$
$$= 2c(X_0, X_1)$$

and hence c = 0.

(b) If G is a Lie group with Lie algebra g and O is a coadjoint orbit, then $\rho(X)(v) = -v(X)$ for every $X \in g$ and $v \in o \subset g^*$, as we saw in Example 3.4.6 (b).

Therefore, $c = 0$, from the definition of the Kirillov 2-form.

(c) We shall now describe a simple example, where $[c]$ is a non-zero element of $H^2(g)$. Let $G = (R^2, +)$, in which case $g = R^2$ with trivial Lie bracket.

Let $M = R^2$ endowed with the euclidean area 2-form $dx \wedge dy$. Let G act on M by translations.

The action is symplectic and if $X = (a, b) \in g$, then

$$\phi_*(X) = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y}, \tag{3.4.7}$$

which is Hamiltonian with Hamiltonian function $\rho(X)(x, y) = ay - bx$. Then,

$$c\big((a_0, b_0), (a_1, b_1)\big) = a_0 b_1 - a_1 b_0 \tag{3.4.8}$$

and therefore $[c] = c \neq 0$. [1]

3.5 Momentum Maps

We will show how to obtain conserved quantities for Lagrangian and Hamiltonian systems with symmetries. This is done using the concept of a momentum mapping, which is a geometric generalization of the classical linear and angular momentum. This concept is more than a mathematical reformulation of a concept that simply describes the well-known Noether theorem. Rather, it is a rich concept that is ubiquitous in the modern developments of geometric mechanics. It has led to surprising insights into many areas of mechanics and geometry.

Let (M, ω) be a connected, simplistic manifold, G be a Lie group with Lie algebra g and $\emptyset: M \to M$ be a Possion action .

Definition 3.5.1 A momentum map for $\emptyset$ is a smooth map $\mu: M \to g^*$ such that $\rho: g \to C^\infty(M)$ defined by $\rho(X)(p) = \mu(p)(X)$ for $X \in g$ and $p \in M$ satisfies

i. $\emptyset_*(X) = X_\rho(X)$, and
ii. $\{\rho(X), \rho(Y)\} = \rho([X, Y])$ for every $X, Y \in g$.

From the point of view of dynamical system, one person to study momentum maps is the following . If H: M→ R is a G-invariant , smooth function, then μ is constant along the integral curves of the Hamiltonian vector field X_H. Indeed, for every $X \in g$ we have

$$LX_H \, P(X) = \{p(X), H\} = -\{H, p(X)\} = -L\emptyset_*(X)^H = 0. \qquad (3.5.1)$$

Theorem 3.5.2 If G is a connected Lie group , then a momentum map $\mu: M \to g^*$ is $G -$ equariant with respect to the coadjoint action on g^*.

Proof. The momentum map is μ is $g -$ equivariant when $\mu\left(\emptyset_g(p)\right) = \mu(p) \circ Ad_{g^{-1}}$ or equivivalently

$$\rho(X)\big(\emptyset g(P)\big) = \rho(Ad_{g^{-1}}(X))(P) \qquad (3.5.2)$$

For every X∈ g, $g \in G$ and $p \in M$,then this is also true for the element $g_1 g_2$

Recall that since G is connected, if V is any connected ,open neighborhood of the identity $e \in G$ with $V = V^{-1}$ then $G = \bigcup_{n=1}^{\infty} V^n$, where $V^n = V \cdot \ldots \cdot V$, n-times.

It follows that it suffices to prove the above equality for $g = \exp tY$ for $Y \in g$ and $t \in R$. In other words, it suffices to show that,

$$\rho(X)(\emptyset_{\exp(tY)}(P)) = \rho(Ad_{\exp(-tY)}(X))(P) \tag{3.5.3}$$

For every X,Y, $\in$ g, p $\in$ M and t $\in$ R. As this is true for t = 0, we need only show that the two sides have equal derivatives with respect to t. The derivative of the left hand side is:

$$\frac{d}{dt}\rho(X)\emptyset_{\exp(tY)}(P)) = d\rho(X)\emptyset_{\exp(tY)}(P))(\frac{d}{dt}\emptyset_{\exp(tY)}(P)) =$$

$$\omega\left(\emptyset_*(X)\left(\emptyset_{\exp(tY)}(P)\right), \emptyset_*(Y)\left(\emptyset_{\exp(tY)}(P)\right)\right) =$$

$$\omega(\emptyset_{g)*p}\,(\emptyset_*\left(Ad_{\exp(-tY)}(X)\right)(P), \left(\emptyset_{g)*p}\left(\left(Ad_{\exp(-tY)}(Y)\right)(P)\right)\right) =$$

$$\omega(\emptyset_*\left(Ad_{\exp(-tY)}(X)\right)(P), \left(\emptyset_*\left(\left(Ad_{\exp(-tY)}(Y)\right)(P)\right)\right) =$$

$$\omega\left(\emptyset_*\left(Ad_{\exp(-tY)}(X)\right)(P), \emptyset_*(Y)(P)\right). \tag{3.5.4}$$

Since $\left(Ad_{\exp(-tY)}(Y)\right) = Y,$ the action is simplictic. The derivative of the right hand side is

$$\frac{d}{dt}\rho\left(Ad_{\exp(-tY)}(X)\right) = \rho\frac{d}{dt}AD_{\exp(-tY)}(X)(P) =$$

$$\rho\left(ad_{(-Y)}\left(Ad_{\exp(-tY)}(X)\right)\right)(P) = \rho([-Y, AD_{\exp(-tY)}(X)](P) =$$

$$\left\{\rho\left(\left(Ad_{\exp(-tY)}(X)\right)\right), \rho(Y)\right\}(P) = \omega\left(\emptyset_*\left(Ad_{\exp(-tY)}(X)\right)(P), \emptyset_*(Y)(P)\right).$$

$$\tag{3.5.5}$$

In general, for every $X \in g$ and $g \in G$ the smooth function

$$(\emptyset^{*}_{g)}\big(p(X)\big) - \rho\big(AD_{g}^{-1}\big)(X)): M \to R$$

has differential

$$d(\left(\emptyset_{g}\right)^{*}\left(\rho(X)\right) - \rho\left(Ad_{g^{-1}}\right)(X))) = \left(\emptyset_{g}\right)^{*}(d\rho(X)) - d\rho(Ad_{g^{-1}})(X)) =$$

$$(\emptyset_{g})^{*}(\breve{\omega}^{-1})(\emptyset_{*})(X))) - \breve{\omega}^{-1}(\emptyset_{*}(Ad_{g^{-1}}(X)))$$

$$= \breve{\omega}^{-1}((\left(\emptyset_{g^{-1}}\right)_{*}\emptyset_{*}(X) - \emptyset_{*}Ad_{g^{-1}}(X))) = 0,$$

because the group action is simplictic. Since M is connected , it is constant and so we have a function $c{:}G \to g^{*}$ defined by

$$c(g) = (\emptyset_{g})^{*}\big(\rho(X)\big) - \rho\left(Ad_{g^{-1}}(X)\right) = \mu\left(\emptyset_{g}(P)\right) - Ad_{g^{*}}(\mu(P))$$

For any $p \in M$. If now $g_0\ g_1 \in G$ then

$$c(g_0\ g_1) = \mu\left(\emptyset_{g0}\left(\emptyset_{g1}(P)\right)\right) - Ad_{g0}{}^{*}\left(Ad_{g1}{}^{*}\big(\mu(P)\big)\right) =$$

$$\mu\left(\emptyset_{g0}\left(\emptyset_{g1}(P)\right)\right) - Ad_{g0}{}^{*}\left(Ad_{g1}{}^{*}\big(\mu(P)\big)\right)$$

$$= Ad_{g0}{}^{*}\mu\left(\emptyset_{g1}\big((P)\big)\right) - Ad_{g0}{}^{*}\left(Ad_{g1}{}^{*}\big(\mu(P)\big)\right) =$$

$$C(g_0) + Ad_{g0^{*}}(\mu\left(\emptyset_{g1}(P)\right) - Ad_{g1^{*}}(\mu\left(\emptyset_{g1}(P)\right) = c(g0) + Ad_{g0^{*}}\big(c(g1)\big).$$

This means that c is a 1-cocycle with respect to the group co homology of

G^{δ} with coefficients in the G module g^{*}, with respect to the coadjoint action, where G^{δ} denotes G made discrete. If $\acute{\mu}` = \mu + a$. The corresponding cocycle $\acute{c}`$ is given by the formula:

$$\acute{c}`(g) = \mu\left(\emptyset_g(P)\right) + a - Ad_{g^*}\mu(P)) - Ad_{g^*}(a) = (c - \delta a)(g)$$

Where δ denotes the coboundary operator in group co homology. Thus the cohomology class $[c] \in H^1 \ (G^\delta; \ g^*)$ does not depend on the choice of the momentum map but only on the group action. [6]