Chapter 2

Analytical Mechanics

Joseph Louis Lagrange reformulated Newton's Laws in a way that eliminates the need to calculate forces on isolated parts of a mechanical system. Any convenient variables obeying the constraints on a system can be used to describe the motion. If Lagrangian mechanics rather than Newtonian mechanics are used, it is only necessary to consider a single function of the dynamical variables that describe the motion of the entire system. The differential equations governing the motion are obtained directly from this function without any vector force diagrams. Lagrangian mechanics is extremely efficient: There are only as many equations to solve as there are physically significant variables.

Lagrange did not introduce new physical principles to mechanics. The physical concepts are due to Newton and Galileo. But he succeeded in giving a more powerful and sophisticated way to formulate the mathematical equations of classical mechanics, an approach that has spread its influence over physics far beyond the purely mechanical problems.

2.1 Newtonian Mechanics

The mathematical study of the motion of everyday objects and the forces that affect them is called classical mechanics. Classical mechanics is often called Newtonian mechanics because nearly the entire study builds on the work of Isaac Newton.

In newtonian mechanics the state of a mechanical system is described by a finite number of real parameters. The set of all possible positions, of a material point for example, is a finite dimensional smooth manifold M, called the *configuration space*.

A motion of the system is a smooth curve $\gamma: I \to M$, where $I \subset R$ is an open interval. The velocity field of γ is smooth curve $\dot{\gamma}: I \to TM$. The (total space of the) tangent bundle TM of M is called the *phase space*.

According to Newton, the total force is a vector field F that acts on the points of the configuration space. Locally, a motion is a solution of the second order differential equation $F = m\ddot{x}$, where m is the mass. Equivalently, $\dot{\gamma}$ is locally a solution of the first order differential equation

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} v \\ \frac{1}{m} F(x, v, t) \end{pmatrix}. \tag{2.1.1}$$

Consider a system of N particles in $\mathbb{R}^3$ subject to some forces. If x_i denotes the position of the i-th particle then the configuration space is $(\mathbb{R}^3)^N$ and Newton's law of motion is

$$m_i \frac{d^2 x_i}{dt^2} = F_i(x_1, \dots, x_N, \dot{x}_1, \dots, \dot{x}_N, t), \quad 1 \le i \le N \tag{2.1.2}$$

where m_i is the mass and F_i is the force on the i-th particle. Relabeling the variables setting q^{3i}, q^{3i+1} and q^{3i+2} the coordinates of x_i in this order, the configuration space becomes $\mathbb{R}^n$, $n = 3N$, and the equations of motion take the form

$$m_j \frac{d^2 q^j}{dt^2} = F_j(q^1, \dots, q^n, \dot{q}^1, \dots, \dot{q}^n, t), \quad 1 \le j \le n \tag{2.1.3}$$

Suppose that the forces do not depend on time and are conservative. This means that there is a smooth function $V: \mathbb{R}^n \to \mathbb{R}$ such that

$$F_j(q^1, \dots, q^n, \dot{q}^1, \dots, \dot{q}^n, t) = -\frac{\partial V}{\partial q^j}, \quad 1 \le j \le n \tag{2.1.4}$$

For instance, this is the case if N particles interact by gravitational attraction. Rewritting Newton's law as a system of first order ordinary differential equations

$$\frac{dq^j}{dt} = v^j, \; m_j \frac{d^2 v^j}{dt^2} = -\frac{\partial V}{\partial q^j}, 1 \le j \le n \tag{2.1.5}$$

or changing coordinates to $p_j = m_j v^j$ we have

$$\frac{dq^j}{dt} = \frac{1}{m_j}\, p_j \quad , \frac{d^2 p_j}{dt^2} = -\frac{\partial V}{\partial q^j} \quad , 1 \leq j \leq n \tag{2.1.6}$$

The solutions of the above system of ordinary differential equations are the integral curves of the smooth vector field

$$X = \sum_{j=1}^{n} \frac{1}{m_j}\, p_j \frac{\partial}{\partial q^j} - \sum_{j=1}^{n} \frac{\partial V}{\partial q^j} \frac{\partial}{\partial p_j} \tag{2.1.7}$$

Note that the smooth function

$$H(q^1, \dots, q^n, p_1, \dots, p_n) = \sum_{j=1}^{n} \frac{1}{2m_j}\, p_j^2 + V(q^1, \dots, q^n) \tag{2.1.8}$$

Is constant along solution , because

$$dH = \sum_{j=1}^{n} \frac{\partial V}{\partial q^j}\, \partial q^j + \sum_{j=1}^{n} \frac{1}{m_j}\, p_j\, dp_j \tag{2.1.9}$$

and so dH(X) = 0. Actually, H completely determines X in the following sense.

Let

$$\omega = \sum_{j=1}^{n} dq^j \wedge dp_j \,.$$

Then

$$i_X\, \omega = \sum_{j=1}^{n} dq^j\, (X)\, dp_j - \sum_{j=1}^{n} dp_j\, (X) dp^j = \sum_{j=1}^{n} \frac{1}{m_j}\, p_j\, dp_j + \sum_{j=1}^{n} \frac{\partial V}{\partial q^j}\, \partial q^j = dH \tag{2.1.10}$$

The smooth 2-form ω is closed and non-degenerate. The latter means that given any smooth 1-form η the equation $i_Y\, \omega$ = η has a unique solution Y. Indeed, for any smooth vector field Y we have

$$Y = \sum_{j=1}^{n} \omega(Y, \frac{\partial}{\partial p_j}) \frac{\partial}{\partial q^j} - \sum_{j=1}^{n} \omega(Y, \frac{\partial}{\partial q^j}) \frac{\partial}{\partial p_j} \tag{2.1.11}$$

and so $i_Y\, \omega = 0$ if and only if $Y = 0$.

Returning to Newtonian mechanics, we give the following definition.

Definition 2.1.1 An (autonomous) newtonian mechamical system is a triple (*M, g, X*), where M is a smooth manifold, g is a Riemannian metric on M and X is a smooth vector field on *TM* such $\pi_*X = id$. A motion of (*M, g, X*) is a smooth curve $\gamma: I \rightarrow M$ such that $\dot{\gamma}: I \rightarrow TM$ is an integral curve of *X*. The smooth function $T: TM \rightarrow$ R defined by $T(v) = \frac{1}{2}\, g(v, v)$ is called the kinetic energy.

Examples 2.1.2 (a) Let M = $\mathbb{R}$, so that we may identify TM with $\mathbb{R}^2$ and π with the projection onto the first coordinate. If g is the Euclidean Riemannian metric on $\mathbb{R}$ and

$$X = v\frac{\partial}{\partial x} + \frac{1}{m}(-k^2x - \rho v)\frac{\partial}{\partial v}\ ,\ k > 0,\ \rho \geq 0 \qquad (2.1.12)$$

then obviously $\pi_*X = id$ and a motion is a solution of the second order differential equation

$$m\ddot{x} = -k^2x - \rho\dot{x}$$

This mechanical system describes the oscillator.

(b) The geodesic vector field G of a Riemannian *n*-manifold (M, g) defines a newtonian mechanical system. Locally it has the expression

$$G = \sum_{k=1}^{n} v^k \frac{\partial}{\partial x^k} - \sum_{i,j,k=1}^{n} \Gamma_{ij}^k v^i v^j \frac{\partial}{\partial v^k} \qquad (2.1.13)$$

where Γ_{ij}^k are the Christofell symbols.

Often a mechanical system has potential energy. This is a smooth function $V: M \rightarrow$ R . Let gradV be the gradient of V with respect to the Riemannian metric. If for every $v \in TM$ we set $\overline{gradV} = \frac{d}{dt}\left(v + t\ grandV\big(\pi(v)\big)\right), t = 0$, then $\overline{gradV} \in$ *X(TM)* and $\pi_*\ \overline{gradV}$ = 0, since π (*v* + *t* grad*V* (π (*v*))) = π (*v*), for every *t* ∈ R. Locally, if g = (g_{ij}) and (g_{ij})$^{-1}$ = (g^{ij}), then

$$grad\ v = \sum_{i,j=1}^{n} g^{ij} \frac{\partial V}{\partial x^i}\frac{\partial}{\partial x^j} \quad \text{and} \quad \overline{grad\ v} = \sum_{i,j=1}^{n} g^{ij} \frac{\partial V}{\partial x^i}\frac{\partial}{\partial v^j} \qquad (2.1.14)$$

Definition 2.1.3 A newtonian mechanical system with potential energy is a triple (M, g, V), where (M, g) is a Riemannian manifold and $V : M \rightarrow \mathrm{R}$ is a smooth function called the potential energy.

The corresponding vector field on *TM* is *Y* = *G* - $\overline{grad\, v}$, where *G* is the geodesic vector field. The smooth function $E = T + V \circ \pi : TM \rightarrow \mathrm{R}$ is called the mechanical energy. [12]

Proposition 2.1.4 (Conservation of energy) *In a newtonian mechanical system with potential energy* (M, g, V) *the mechanical energy is a constant of motion.*

Proof. We want to show that *Y* (*E*) = 0. We compute locally, where we have

$$E = \frac{1}{2} \sum_{i,j=1}^{n} g_{ij}\, v^i v^j + V \quad and$$

$$Y = \sum_{k=1}^{n} v^k \frac{\partial}{\partial x^k} - \sum_{k=1}^{n} \left(\sum_{i,j=1}^{n} \Gamma_{ij}^k v^i v^j + \sum_{i=1}^{n} \frac{\partial V}{\partial x^i} g^{ik} \right) \frac{\partial}{\partial v^k}.$$

Recall that

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{l=1}^{n} g^{kl} \left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right)$$

We can now compute

$$Y = \sum_{k=1}^{n} v^k \frac{\partial V}{\partial x^k} - \frac{1}{2} \sum_{i,j,k=1}^{n} \frac{\partial g_{i,j}}{\partial x^k} v^i v^j v^k - \sum_{k=1}^{n} \left(\sum_{i,j=1}^{n} \Gamma_{ij}^k v^i v^j + \sum_{i=1}^{n} \frac{\partial V}{\partial x^i} g^{ik} \right) \frac{\partial V}{\partial v^k}$$

$$- \sum_{k=1}^{n} \left(\sum_{i,j=1}^{n} \Gamma_{ij}^k v^i v^j + \sum_{i=1}^{n} \frac{\partial V}{\partial x^i} g^{ik} \right) \left(\sum_{i=1}^{n} g_{ik}\ v^i \right)$$

$$= \sum_{k=1}^{n} v^k \frac{\partial V}{\partial x^k} - \left(\sum_{i=1}^{n} \frac{\partial V}{\partial x^i} g^{ik} \right) \left(\sum_{i=1}^{n} g_{ik}\ v^i \right) + \frac{1}{2} \sum_{i,j,k=1}^{n} \frac{\partial g_{i,j}}{\partial x^k} v^i v^j v^k$$

$$- \sum_{k=1}^{n} (\sum_{r=1}^{n} g_{rk} \ v^r)(\sum_{i,j}^{n} \frac{1}{2} \sum_{l=1}^{n} g^{kl} \left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right) v^i v^j)$$

$$= \frac{1}{2} \sum_{i,j,k=1}^{n} \frac{\partial g_{i,j}}{\partial x^k} v^i v^j v^k - \frac{1}{2} \sum_{i,j,k=1}^{n} \left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right) v^i v^j v^l = 0.$$

2.2 Lagrangian Mechanics

Lagrangian mechanics is a reformulation of classical mechanics, introduced by the Italian-French mathematician Joseph-Louis Lagrange in 1788.

In Lagrangian mechanics, the trajectory of a system of particles is derived by solving the Lagrange equations in one of two forms, either the Lagrange equations of the first kind, which treat constraints explicitly as extra equations, often using Lagrange multipliers; or the Lagrange equations of the second kind, which incorporate the constraints directly by judicious choice of generalized coordinates. In each case, a mathematical function called the Lagrangian is a function of the generalized coordinates, their time derivatives, and time, and contains the information about the dynamics of the system.

Let (M, g, V) be a newtonian mechanical system with potential energy V and let $L : TM \rightarrow R$ be the smooth function $L = T - V \circ \pi$, where T is the kinetic energy and $\pi : TM \rightarrow M$ is the tangent bundle projection.

Most mechanical systems interact with the environment through generalized forces. The basic mechanisms that describe these interactions are Newton's laws and the Lagrange–d'Alembert principle. We describe below very briefly the D'Alembert-Lagrange principle and its relationship with constrained dynamics.

Theorem 2.2.1. (D'Alembert-Lagrange) *A smooth curve* $\gamma : I \rightarrow M$ *is a motion of the mechanical system* (*M, g, V*) *if and only if*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\,(\dot{\gamma}(t))\right) = \frac{\partial L}{\partial x^i}(\dot{\gamma}(t)) \tag{2.2.1}$$

for every $t \in I$ *and* $i = 1, 2..., n$, *where n is the dimension of M.*

Proof. Suppose that in local coordinates we have $= (x^1, x^2, \dots, x^n)$. Recall that γ is a motion of (*M, g, V)* if and only if

$$\ddot{x}^k = -\sum_{i,j}^{n} \Gamma_{ij}^k \dot{x}^i \dot{x}^j - \sum_{l=1}^{n} \frac{\partial V}{\partial x^l}\, g^{lk}.$$

Since

$$L(\dot{\gamma}) = \frac{1}{2}\sum_{i,j=1}^{n} g_{i,j}\, \dot{x}^i \dot{x}^j - V(\gamma),$$

For every i = 1,2 , …, n we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\big(\dot{\gamma}(t)\big)\right) - \frac{\partial L}{\partial x^i}\big(\dot{\gamma}(t)\big)$$

$$= \frac{d}{dt}\left(\sum_{i,j=1}^{n} g_{i,j}\, \dot{x}^j\right) - \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial g_{ml}}{\partial x^i}\dot{x}^m \dot{x}^l + \frac{\partial V}{\partial x^i}\big(\gamma(t)\big) =$$

$$\sum_{j=1}^{n}\sum_{l=1}^{n} \frac{\partial g_{ij}}{\partial x^l}\dot{x}^l \dot{x}^j + \sum_{j=1}^{n} g_{i,j}\, \ddot{x}^j - \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial g_{ml}}{\partial x^i}\dot{x}^m \dot{x}^l + \frac{\partial V}{\partial x^i}\big(\gamma(t)\big) =$$

$$\sum_{m,l=1}^{n}\left(\frac{\partial g_{im}}{\partial x^l} - \frac{1}{2}\frac{\partial g_{ml}}{\partial x^i}\right)\dot{x}^m \dot{x}^l + \sum_{j=1}^{n} g_{i,j}\, \ddot{x}^j \;\; \frac{\partial V}{\partial x^i}\big(\gamma(t)\big). \tag{2.2.2}$$

Taking the image of the vector with these coordinates by $(g_{ij})^{-1} = (g^{ij})$, we see that the equations in the statement of the theorem are equivalent to

$$0 = \sum_{i=1}^{n} g^{ik} (\sum_{m,l=1}^{n} (\frac{\partial g_{im}}{\partial x^l} - \frac{1}{2}\frac{\partial g_{ml}}{\partial x^i}) \dot{x}^m \dot{x}^l) + \sum_{j=1}^{n} g^{ik} g_{ij} \ddot{x}^j \sum_{i=1}^{n} \frac{\partial V}{\partial x^i} g^{ik} =$$

$$\ddot{x}^k + \sum_{m,l}^{n} \sum_{i=1}^{n} g^{ik} (\frac{\partial g_{im}}{\partial x^l} - \frac{1}{2}\frac{\partial g_{ml}}{\partial x^i}) \dot{x}^m \dot{x}^l + \sum_{i=1}^{n} \frac{\partial V}{\partial x^i} g^{ik} =$$

$$\ddot{x}^k + \sum_{m,l=1}^{n} \Gamma^k_{ml} \dot{x}^m \dot{x}^l + \sum_{i=1}^{n} \frac{\partial V}{\partial x^i} g^{ik}.$$

Generalizing we give the following definition.

Definition 2.2.2 An autonomous Lagrangian system is a couple (*M*,*L*), where *M* is a smooth manifold and $L : TM \to \mathbb{R}$ is a smooth function, called the Lagrangian. A Lagrangian motion is a smooth curve $\gamma : I \to M$ which locally satisfies the system of differential equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\big(\dot{\gamma}(t)\big)\right) - \frac{\partial L}{\partial x^i}\big(\dot{\gamma}(t)\big) \tag{2.2.3}$$

for $t \in I$ and $i = 1, 2, ..., n$, where n is the dimension of *M*. These equations are called the Euler-Lagrange equations.

The variational interpretation of the Euler-Lagrange equations is given by the Least Action Principle due to Hamilton.

Theorem 2.2.3. (Least Action Principle) *Let* (*M*,*L*) *be a Lagrangian system. A smooth curve* $\gamma: [a, b] \to M$ *is a Lagrangian motion if and only if for every smooth variation* $\Gamma : (-\epsilon, \epsilon) \times [a, b] \to M$ of γ *of with fixed endpoints, so that* $\Gamma(0, t) = \gamma(t)$ *for* $a \le t \le b$*, we have*

$$\frac{\partial}{\partial s} = \int_a^b L(\frac{\partial \Gamma}{\partial t}(s,t))\, dt = 0, \quad under\ the\ restriction\ s = 0$$

Proof. It suffices to carry out the computations locally. We have

$$\frac{\partial}{\partial s}\int_a^b L(\frac{\partial\Gamma}{\partial t}(s,t))\,dt = \int_a^b \frac{\partial}{\partial s} L(\frac{\partial\Gamma}{\partial t}(s,t))\,dt =$$

$$\int_a^b [\sum_{i=1}^{n} \frac{\partial L}{\partial x^i}\big(\dot\gamma(t)\big)\frac{\partial\Gamma}{\partial s}(0,t) + \sum_{i=1}^{n} \frac{\partial L}{\partial v^i}\big(\dot\gamma(t)\big)\frac{\partial}{\partial s}\ \Big(\frac{\partial\Gamma}{\partial t}(s,t)\Big]\,dt =$$

$$\int_a^b [\sum_{i=1}^{n} \frac{\partial L}{\partial x^i}\big(\dot\gamma(t)\big)\frac{\partial\Gamma}{\partial s}(0,t) + \sum_{i=1}^{n} \frac{\partial L}{\partial v^i}\big(\dot\gamma(t)\big)\frac{\partial}{\partial t}\ \Big(\frac{\partial\Gamma}{\partial s}(0,t)\Big]\,dt =$$

$$\int_a^b [\sum_{i=1}^{n} \frac{\partial L}{\partial x^i}\big(\dot\gamma(t)\big)\frac{\partial\Gamma}{\partial s}(0,t) + \frac{d}{dt}\left(\sum_{i=1}^{n} \frac{\partial L}{\partial v^i}\big(\dot\gamma(t)\big)\ \frac{\partial\Gamma}{\partial s}(0,t)\right)$$

$$-\sum_{i=1}^{n}\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\big(\dot\gamma(t)\big)\right)\frac{\partial\Gamma}{\partial s}(0,t)\,]\,dt =$$

$$\int_a^b [\sum_{i=1}^{n} \frac{\partial L}{\partial x^i}\big(\dot\gamma(t)\big) - \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\big(\dot\gamma(t)\big)\right)\frac{\partial\Gamma}{\partial s}(0,t)\,dt,$$

because

$$\frac{\partial\Gamma}{\partial s}(0,a) = \frac{\partial\Gamma}{\partial s}(0,b) = 0$$

since the variation is with fixed endpoints. This means that

$$\frac{\partial}{\partial s}\int_a^b L(\frac{\partial\Gamma}{\partial t}(s,t))\,dt = 0$$

if and only if

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\big(\dot\gamma(t)\big)\right) = \frac{\partial L}{\partial x^i}\big(\dot\gamma(t)\big)$$

For i = 1, 2, … , n, because $\frac{\partial\Gamma}{\partial s}(0,t)$ can take any value.

Example 2.2.4. Consider a particle of charge e and mass m in $\mathbb{R}^3$ moving under the influence of an electromagnetic field with electrical and magnetic components E and B, respectively. The fields E and B satisfy Maxwell's equations

$$curl E + \frac{1}{c}\frac{\partial B}{\partial t} = 0, \quad div\, B = 0,$$

$$curl B - \frac{1}{c}\frac{\partial E}{\partial t} = 4\pi J, \quad div\, E = 4\pi\rho \tag{2.2.4}$$

where c is the speed of light, ρ is the charge density and J is the charge current density. By the Poincare lemma, there exists a vector potential $A = (A_1, A_2, A_3)$
such that B = curl A. So

$$curl\left(E + \frac{1}{c}\frac{\partial A}{\partial t}\right) = 0$$

and there exists a scalar potential $V : \mathbb{R}^3 \to \mathbb{R}$ such that

$$E = -gradV - \frac{1}{c}\frac{\partial A}{\partial t}$$

the gradient being euclidean.

Suppose the electromagnetic field does not depend on time and let L: $\mathbb{R}^3 \to \mathbb{R}$ be the Lagrangian

$$L(x, v) = \frac{1}{c}\, m < v, v > +e\left(\frac{1}{c} < A(x), v > -V(x)\right).$$

We shall describe only the first for i = 1 of the corresponding Euler-Lagrange equations, the other two being analogous. The right hand side is

$$\frac{\partial L}{\partial x^1} = -e\,\frac{\partial V}{\partial x^1} + \frac{e}{c} < \frac{\partial A}{\partial x^1}, v >.$$

The left hand side is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial x^1}\right) = m\,\dot{v}^1 + \frac{e}{c}\,[v^1\frac{\partial A_1}{\partial x^1} + v^2\frac{\partial A_1}{\partial x^2} + v^3\frac{\partial A_1}{\partial x^3}]$$

It follows that the first of the Euler-Lagrange equations takes the form

$$m\,\dot{v}^1 = -e\,\frac{\partial V}{\partial x^1} + \frac{e}{c}\,[v^2\left(\frac{\partial A_2}{\partial x^1} - \frac{\partial A_1}{\partial x^2}\right) - v^3\left(\frac{\partial A_1}{\partial x^3} - \frac{\partial A_3}{\partial x^1}\right)]$$

(2.2.5)

The right hand side is the first coordinate of the vector $e\left(E + \frac{1}{\mathrm{c}}\,V\;\times B\right)$. Since the other two equations are analogous and have the same form by cyclically permuting indices, we conclude that the Euler-Lagrange equations give Lorentz's equation of motion

$$m\,\frac{d^2x}{dt^2} = e\left(E + \frac{1}{\mathrm{c}}\,V\;\times B\right)$$

We globalize the above situation as follows. Let (M, g) be a pseudo-Riemannian n-manifold, A be a smooth 1-form on M and $V\!: M \to \mathbb{R}$ a smooth function. The Lagrangian

$$L(x,v) = \frac{1}{2}\,mg(v,v) + A_x(v) - V\,(x)$$

generalizes the motion of a charged particle of mass m under the influence of an electromagnetic field. Let $(U, x^1, x^2, \dots,\ x^n)$ be a local system of coordinates on M and $(\pi^{-1}(U), x^1, x^2, \dots,\ x^n, v^1, v^2, \dots,\ v^n)$ be the corresponding local system of coordinates on TM. In local coordinates L is given by the formula

$$L(x^1, x^2, \dots, x^n, v^1, v^2, \dots, v^n) = \frac{1}{2}m\sum_{i,j=1}^{n} g_{i,j}\;v^i v^j + \sum_{i=1}^{n} A_i\;v^i$$

(2.2.6)

where (g_{ij}) is the matrix of the pseudo-Riemannian metric g and $A = \sum_{i=1}^{n} A_i\,dx^i$ on U.

A smooth curve $\gamma\!: I \to M$ is a Lagrange motion if and only if it satisfies the Euler-Lagrange equations. In our case the right hand side of the Euler-Lagrange equations is

$$\frac{\partial L}{\partial x^k}\big(\dot{\gamma}(t)\big) = \frac{1}{2}\,m\sum_{i,j=1}^{n}\frac{\partial g_{ij}}{\partial x^k}\,\frac{dx^i}{dt}\,\frac{dx^j}{dt} + \sum_{i=1}^{n}\frac{\partial A_i}{\partial x^k}\,\frac{dx^i}{dt} - \frac{\partial V}{\partial x^k},$$

and the left hand side

$$\frac{d}{dt}\left(\frac{\partial L}{\partial x^k}\big(\dot{\gamma}(t)\big)\right) = m\sum_{i,j=1}^{n}\frac{\partial g_{ik}}{\partial x^j}\frac{dx^j}{dt}\frac{dx^i}{dt} + m\sum_{i=1}^{n} g_{ik}\frac{d^2x^i}{\partial t^2} + \sum_{i=1}^{n}\frac{\partial A_k}{\partial x^i}\frac{dx^i}{dt}$$

So the Euler-Lagrange equations are equivalent to

$$\sum_{i=1}^{n}\left(\frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i}\right)\frac{dx^i}{dt} - \frac{\partial V}{\partial x^k} = m\sum_{i=1}^{n} g_{ik}\frac{d^2x^i}{\partial t^2} + m\sum_{i,j=1}^{n}\left(\frac{\partial g_{ik}}{\partial x^j} - \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}\right)\frac{dx^i}{dt}\frac{dx^j}{dt}$$

On the other hand

$$dA\left(\dot{\gamma}(t), \frac{\partial}{\partial x^k}\right) = \sum_{i,j=1}^{n}\left(\frac{\partial A_i}{\partial x^j}\, dx^j \wedge dx^i\right)\left(\dot{\gamma}(t), \frac{\partial}{\partial x^k}\right) = \sum_{i=1}^{n}\left(\frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i}\right)\frac{dx^i}{dt}$$

Recall that the covariant derivative formula along γ gives

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \sum_{i=1}^{n}\frac{d^2x^i}{\partial t^2}\frac{\partial}{\partial x^i} + \sum_{i,j,l=1}^{n}\Gamma_{i,j}^{l}\frac{dx^i}{dt}\frac{dx^j}{dt}\frac{\partial}{\partial x^l}$$

And so

$$g\left(m\nabla_{\dot{\gamma}}\dot{\gamma}, \frac{\partial}{\partial x^k}\right) = m\sum_{l=1}^{n} g_{lk}\frac{d^2x^l}{\partial t^2} + m\sum_{i,j,l=1}^{n} g_{lk}\Gamma_{i,j}^{l}\frac{dx^i}{dt}\frac{dx^j}{dt}$$

Since

$$\sum_{l=1}^{n} g_{lk}\Gamma_{i,j}^{l} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

we get

$$\sum_{i,j,l=1}^{n} g_{lk}\Gamma_{i,j}^{l}\frac{dx^i}{dt}\frac{dx^j}{dt} = \sum_{i,j=1}^{n}\left(\frac{\partial g_{ik}}{\partial x^j} - \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}\right)\frac{dx^i}{dt}\frac{dx^j}{dt}$$

(2.2.7)

we conclude now that the Euler – Lagrange equations have the form

$$g\left(m\nabla_{\dot{\gamma}}\dot{\gamma}, \frac{\partial}{\partial x^k}\right) = -d\, A\left(\dot{\gamma}(t), \frac{\partial}{\partial x^k}\right) - g\left(gradV, \frac{\partial}{\partial x^k}\right), \quad k = 1, 2, \dots, n$$

Or independently of local coordinates

$$m\nabla_{\dot{\gamma}}\dot{\gamma} = -grad\, \left(i_{\dot{\gamma}}(dA)\right) - grad\, V,$$

where the gradient is taken with respect to the pseudo-Riemannian metric g.

As in newtonian mechanical systems with potential energy, one can define the notion of mechanical energy for Lagrangian systems also. In order to do this, we shall need to define first the Legendre transformation. So let $L : TM \rightarrow R$ be a Lagrangian and $p \in M, v \in T_pM$. The derivative

$$\left(Lj\middle|T_pM\right) * v : T_v\left(T_pM\right) \cong T_pM \rightarrow R$$

can be considered as an element of the dual tangent space T_p^*M.

Definition 2.2.5 The Legendre transformation of a Lagrangian system (*M*, *L*) is the map $L : TM \rightarrow T^*M$ defined by *L* (*p*,*v*) = $\left(Lj\middle|T_pM\right)_{*v}$. In other words, for every $w \in T_pM$ we have

$$L(p, v)(w) = \frac{d}{dt} L(p, v + tw). \tag{2.2.8}$$

It is worth to note that *L* is not in general a vector bundle morphism, as it may not be linear on fibers. For instance, if $M = \mathbb{R}$ and $L(x, v) = e^v$ (this Lagrangian has no physical meaning), then $L : TR \rightarrow T^*R \cong R^2$ is given by $L(x, v) = (x, e^v)$, which is not linear in the variable *v*.

Example 2.2.6 If $L = \frac{1}{2}g - V$ is the Lagrangian of a newtonian mechanical system with potential energy (*M*, *g*, *V*), then for every $p \in M$ and $v, w \in T_pM$ we have *L* (*p*, *v*) (*w*) = *g* (*v*, *w*). Thus, in this particular case the Legendre transformation $L : TM \rightarrow T^*M$ is the natural isomorphism defined by the Riemannian metric.

Definition 2.2.7 The energy of a Lagrangian system (M, L) is the smooth function $E : TM \to R$ defined by $E(p, v) = L(p, v)(v) - L(p, v)$.

If $(x^1, x^2, \dots, x^n)$ is a system of local coordinates on M with corresponding local coordinates $(x^1, x^2, \dots, x^n, v^1, v^2, \dots, v^n)$ on TM, then

$$E(x^1, x^2, \dots, x^n, v^1, v^2, \dots, v^n) = \sum_{i=1}^{n} \frac{\partial L}{\partial v^i} v^i - L(x^1, x^2, \dots, x^n, v^1, v^2, \dots, v^n). \tag{2.2.9}$$

In the case of a newtonian mechanical system with potential energy (M, g, V) the above definition gives

$$E(p, v) = L(p, v)(v) - L(p, v) = g(v, v) - \frac{1}{2} g(v, v) + V(p)$$

$$= \frac{1}{2} g(v, v) + V(p),$$

which coincides with the previous definition.

Example 2.2.8 We shall compute the Legendre transformation and the energy of the Lagrangian system of example (2.2.4) using the same notation. Considering local coordinates $(x^1, x^2, \dots, x^n, v^1, v^2, \dots, v^n)$ on TM, we have

$$\left(Lj\middle|T_pM\right)(v^1, v^2, \dots, v^n) = \frac{1}{2} m \sum_{i,j=1}^{n} g_{ij} v^i dv^j - \sum_{i=1}^{n} A_i v^i - V(p).$$

Differentiating we get

$$\left(Lj\middle|T_pM\right)_{*v} = m \sum_{i,j=1}^{n} g_{ij} v^i dv^j - \sum_{i=1}^{n} A_i dv^i .$$

We conclude now that

$$L(p, v)(w) = \left(Lj\middle|T_pM\right)_{*v}(w) = mg(v, w) + A_p(w).$$

The energy here is

$$E(p,v) = L(p,v)(v) - L(p,v) = \frac{1}{2}mg(v,v) + V(p), \tag{2.2.11}$$

and so does not depend on the 1-form *A*, which represents the magnetic field. This reflects the fact that the magnetic field does not produce work.

Theorem 2.2.9 (Conservation of energy) *In a Lagrangian system the energy is a constant of motion.*

Proof. Considering local coordinates on *M*, let $\gamma = (x^1, x^2, \dots, x^n)$ be a Lagrangian motion. Then

$$E(\dot{\gamma}(t)) = \sum_{i=1}^{n} \frac{\partial L}{\partial v^i} \frac{dx^i}{dt} - L(\dot{\gamma}(t))$$

and differentiating

$$\begin{aligned}
\frac{d}{dt}\left(E(\dot{\gamma}(t))\right) \\
&= \sum_{i,j=1}^{n} \left(\frac{\partial^2 L}{\partial v^i \partial x^j} \frac{dx^i}{dt} \frac{dx^j}{dt} + \frac{\partial^2 L}{\partial v^i \partial v^j} \frac{dx^i}{dt} \frac{d^2 x^j}{dt^2} \right) \\
&+ \sum_{i=1}^{n} \frac{\partial L}{\partial v^i} \frac{d^2 x^i}{dt^2} \\
&- \sum_{i=1}^{n} \frac{\partial L}{\partial x^i} \frac{dx^i}{dt} \\
&- \sum_{i=1}^{n} \frac{\partial L}{\partial v^i} \frac{d^2 x^i}{dt^2} \\
&= \sum_{i,j=1}^{n} \left(\frac{\partial^2 L}{\partial v^i \partial x^j} \frac{dx^i}{dt} \frac{dx^j}{dt} + \frac{\partial^2 L}{\partial v^i \partial v^j} \frac{dx^i}{dt} \frac{d^2 x^j}{dt^2} \right) - \sum_{i=1}^{n} \frac{\partial L}{\partial x^i} \frac{dx^i}{dt}.
\end{aligned}$$

But from the Euler-Lagrange equations we have

$$\frac{\partial L}{\partial x^i} = \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) = \sum_{j=1}^{n}\left(\frac{\partial^2 L}{\partial v^i \partial x^j}\frac{dx^j}{dt} + \frac{\partial^2 L}{\partial v^i \partial v^j}\frac{d^2 x^j}{dt^2}\right)$$

and so substituting we get

$$\frac{d}{dt}\Big(E\big(\dot{\gamma}\ (t)\big)\Big) = 0.$$

Apart from the energy, one can have constants of motion from symmetries of the Lagrangian.

Theorem 2.2.10 (Noether) *Let* (M,L) *be a Lagrangian system and* X *a complete smooth vector field on* M *with flow* $(\phi_t)t \in R$*. If* $L\ ((\phi_t)_{*p}(v)) = L(v)$ *for every* $v \in T_p M,\ p \in M$ *and* $t \in \mathbb{R}$*, then the smooth function* $f_X : TM \to \mathbb{R}$ *defined by*

$$f_X(v) = \lim_{s \to 0} \frac{L\Big(v + s\, X\big(\pi(v)\big)\Big) - L(v)}{s}$$

is a constant of motion.

Proof. Considering local coordinates, let $\phi_t = (\phi_t^1, \phi_t^1, \dots, \phi_{t)}^1$. Since L is $(\phi_t)_*$ invariant, if $\gamma = (x^1, x^2, \dots, x^n)$ is a Lagrangian motion, differentiating the equation $L((\phi_s)_{*\gamma(t)}(\dot{\gamma}\ (t))) = L(\dot{\gamma}\ (t))$ with respect to s, we have

$$\sum_{i=1}^{n}\frac{\partial L}{\partial x^i}\left(\frac{\partial \emptyset_s^i}{\partial s}\right)_{s=0} + \sum_{i=1}^{n}\frac{\partial L}{\partial v^i}\left(\frac{\partial^2 \emptyset_s^i}{\partial x^j\, \partial s}\right)_{s=0}\frac{dx^j}{dt} = 0.$$

Since $f_X(v)$ is the directional derivative of $L_{\ T_{\pi(v)}M}$ in the direction of $X(\pi(v))$ and

$$X = \sum_{i=1}^{n}\left(\frac{\partial \emptyset_t^i}{\partial t}\right)_{t=0}\frac{\partial}{\partial x^i}$$

we have

$$f_X(\dot{\gamma}\ (t)) = \sum_{i=1}^{n}\frac{\partial L}{\partial v^i}\left(\frac{\partial \emptyset_s^i}{\partial s}\right)_{s=0}$$

Using now the Euler-Lagrange equations we compute

$$f_X(\dot{\gamma}\ (t)) = \sum_{i=1}^{n} \frac{\partial L}{\partial v^i} (\frac{\partial \emptyset_s^i}{\partial s})_{s=0}$$

$$\frac{d}{dt} f_X(\dot{\gamma}\ (t)) = \sum_{i=1}^{n} \frac{d}{dt} (\frac{\partial L}{\partial v^i})(\frac{\partial \emptyset_s^i}{\partial s})_{s=0} + \sum_{i=1}^{n} \frac{\partial L}{\partial v^i} \frac{d}{dt} (\frac{\partial \emptyset_s^i}{\partial s})_{s=0}\ .$$

Examples 2.2.11 (a) Let (*M, g, V)* be a newtonian mechanical system with potential energy and *X* be a complete vector field, which is a symmetry of the system. Then $f_X(v)$ = g (v, $X(\pi(v))$). The restriction to a fiber of the tangent bundle of f_X is linear in this case.

(b) Let *X* be a complete vector field which we assume to be symmetry of the Lagrangian system of Example (2.2.3). For instance, this is the case if the flow of *X* preserves the pseudo-Riemannian metric on *M* and the 1-form *A*. Then the Lagrangian is *X*-invariant and the first integral provided form Noether's theorem is $f_X(v) = mg(v, X) - A(X)$.

Let (*M*,*L*) be a Lagrangian system. Let $(x^1, x^2, \dots, x^n, v^1, v^2, \dots, v^n)$ be a system of local coordinates in *TM* coming from local coordinates $(x^1, x^2, \dots, x^n)$ on *M* and let $(q^1, q^2, \dots, q^n, p_1, p_2, \dots, p_n)$ be the corresponding local coordinates on T^*M, so that $x^j = q^j$, $1 \leq j \leq n$. The local representation of the Legendre transformation is

$$L(x^1, x^2, \dots, x^n, \sum_{j=1}^{n} v^j \frac{\partial}{\partial x^j}) = (x^1, x^2, \dots, x^n, \sum_{j=1}^{n} \frac{\partial L}{\partial v^j} dx^j).$$

The local forms

$$\sum_{j=1}^{n} \frac{\partial L}{\partial v^j} dx^j$$

over all charts on *TM* fit together and give a global smooth 1-form θ_L on *TM*. This may be verified directly. Alternatively, we note that

$$L^*\left(\sum_{i=1}^{n} p_i dq^i\right) = \sum_{i=1}^{n} (p \circ L) d(x^i \circ L) = \sum_{i=1}^{n} \frac{\partial L}{\partial v^i} dx^i.$$

The local 1-forms $\sum_{i=}^{n} p_i dq^i$ on T^*M fit together to a global smooth 1-form θ on T^*M. Actually, θ is precisely the 1-form defined by

$$\theta_a(v) = a(\pi_{*a}(v))$$

for $v \in T_a(T^*M)$ and $a \in T^*M$, where $\pi : T^*M \to M$ is the cotangent bundle projection. Indeed,

$$\theta|_{locally} = \sum_{i=1}^{n} \theta\left(\frac{\partial}{\partial q^i}\right) dq^i + \sum_{i=1}^{n} \theta\left(\frac{\partial}{\partial p_i}\right) dp_i.$$

If now $a = (q^1, q^2, \ldots, q^n, p_1, p_2, \ldots, p_n)$, then $\pi_{*a}\left(\frac{\partial}{\partial p_i}\right) = 0$, and therefore $\theta\left(\frac{\partial}{\partial p_i}\right) = 0$.

Moreover,

$$\theta\left(\frac{\partial}{\partial q_i}\right) = a\left(\pi_{*a}\left(\frac{\partial}{\partial q_i}\right)\right) = p_i.$$

It follows that

$$\theta|_{locally} = \sum_{i=1}^{n} p_i dq^i$$

The smooth 1-form θ is called the *Liouville canonical* 1-*form* on T^*M.

Remark 2.2.12 The 2-form $d\theta_L$ in local coordinates $(x^1, x^2, \ldots, x^n, v^1, v^2, \ldots, v^n)$ on *TM* is given by the formula

$$\theta|_{locally} = \sum_{i,j=1}^{n} \frac{\partial^2 L}{\partial x^j \partial v^i} dx^j \wedge dx^i + \sum_{i,j=1}^{n} \frac{\partial^2 L}{\partial v^j \partial v^i} dv^j \wedge dx^i \tag{2.2.12}$$

It follows that $d\theta_L$ is non-degenerate if and only if the vertical Hessian matrix

$$(\frac{\partial^2 L}{\partial v^j \partial v^i})_{1 \leq i,j \leq n}$$

is everywhere invertible. A Lagrangian system is called non-degenerate if the vertical Hessian of the Lagrangian is everywhere invertible. [5]

2.3 The equations of Hamilton

In Hamiltonian mechanics, a classical physical system is described by a set of canonical coordinates r = (q, p), where each component of the coordinate q_i, p_i is indexed to the frame of reference of the system.

The time evolution of the system is uniquely defined by Hamilton's equations:

$$\frac{dp}{dt} = -\frac{dH}{dq} \;, \frac{dq}{dt} = +\frac{dH}{dp} \tag{2.3.1}$$

where H = H(q, p, t) is the Hamiltonian, which often corresponds to the total energy of the system. For a closed system, it is the sum of the kinetic and potential energy in the system.

A Lagrangian system (M, L) is called hyperregular if the Legendre transformation $L : TM \to T^*M$ is a diffeomorphism. For example a newtonian mechanical system with potential energy and the system of Example (2.2.4) are hyperregular.

Definition 2.3.1. In a hyperregular Lagrangian system as above, the smooth function $H = E \circ L^{-1} : T^*M \to R$, where E is the energy, is called the Hamiltonian function of the system.

Example 2.3.2. Let (*M, g, V)* is a newtonian mechanical system with potential energy. The Legendre transformation gives

$$q^i = x^i, \quad p_i = \frac{\partial L}{\partial v^i} = \sum_{j=1}^{n} g_{ij} \, v^j.$$

The inverse Legendre transformation is given by

$$x^i = q^i, \quad v^i = \sum_{j=1}^{n} g^{ij} \, p_j.$$

So we have

$$E = \frac{1}{2}\sum_{i,j=1}^{n} g_{ij} \, v^i v^j + V(x^1, x^2, \dots, x^n),$$

$$L = \frac{1}{2}\sum_{i,j=1}^{n} g_{ij} \, v^i v^j - V(x^1, x^2, \dots, x^n),$$

And therefore

$$H = \tfrac{1}{2}\textstyle\sum_{j=1}^{n} g^{ij} \, p_i p_j + V(q^1, q^2, \dots, q^n). \qquad (2.3.2)$$

Theorem 2.3.3. (Hamilton) *Let* (*M, L*) *be a hyperregular Lagrangian system on the n-dimensional manifold M. A smooth curve* $\gamma: I \rightarrow M$ *is a Lagrangian motion if and only if the smooth curve* $L \circ \gamma: I \rightarrow T^* M$ *locally solves the system of differential equations*

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}^i = \frac{\partial H}{\partial q_i}, \qquad i = 1, 2, \dots, n.$$

Proof. In the local coordinates $(x^1, x^2, \dots, x^n)$ of a chart on *M* the Legendre transformation is given by the formulas

$$q^i = x^i, \quad p_i = \frac{\partial L}{\partial v^i}, \qquad i = 1, 2, \dots, n,$$

where $(x^1, x^2, \dots, x^n, v^1, v^2, \dots, v^n)$ are the local coordinates of the corresponding chart on *TM*. Inversing,

$$x^i = q^i, \quad v^i = y^i(q^1, q^2, \dots, q^n, p_1, p_2, \dots, p_n), \qquad i = 1, 2, \dots, n$$

for some smooth functions $y^1, y^2, \dots, y^n$. From the definitions of the energy E and the Hamiltonian *H*, we have

$$H = E \circ L^{-1} = \sum_{j=1}^{n} p_j y^j - L(q^1, q^2, \dots, q^n, y^1, y^2, \dots, y^n)$$

and differentiating the chain rule gives

$$\frac{\partial H}{\partial p_i} = y^i + \sum_{j=1}^{n} p_j \frac{\partial y^j}{\partial p_i} - \sum_{j=1}^{n} \frac{\partial L}{\partial v^j} \frac{\partial y^j}{\partial p_i} = y^i$$

$$\frac{\partial H}{\partial q^i} = \sum_{j=1}^{n} p_j \frac{\partial y^j}{\partial q^i} - \frac{\partial L}{\partial x^i} - \sum_{j=1}^{n} \frac{\partial L}{\partial v^j} \frac{\partial y^j}{\partial q^i} = -\frac{\partial L}{\partial x^i}.$$

If $\gamma(t) = (x^1(t), x^2(t), \dots, x^n(t))$ is a smooth curve in local coordinates on *M*, then

$$L\left(\dot{\gamma}(t)\right) = \left(x^1(t), x^2(t), \dots, x^n(t), \frac{\partial L}{\partial v^1}\left(\dot{\gamma}(t)\right), \frac{\partial L}{\partial v^2}\left(\dot{\gamma}(t)\right), \dots, \frac{\partial L}{\partial v^n}\left(\dot{\gamma}(t)\right)\right).$$

Now γ is a Lagrangian motion if and only if

$$\dot{x}^i = v^i$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}(\dot{\gamma})\right) = \frac{\partial L}{\partial x^i}(\dot{\gamma}) \qquad (1)$$

or equivalently

$$\dot{q}^i = \dot{x}^i = v^i = y^i = \frac{\partial H}{\partial p_i} \qquad (2)$$

and

$$\dot{p}_i = \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}(\dot{\gamma})\right) = \frac{\partial L}{\partial x^i}(\dot{\gamma}) = -\frac{\partial H}{\partial q^i}.$$

The equations provided by Theorem 2.3.3 on T^*M are *Hamilton's equations*. The cotangent bundle T^*M is called the phase space of the Lagrangian system (M, L).

Corollary 2.3.4. *The Hamiltonian is constant on solutions of Hamilton's equations.*

Proof. Indeed, if $\gamma(t) = q^1(t), q^2(t), \ldots, q^n(t), p_1(t), p_2(t), \ldots, p_n(t))$ is the local form of a solution of Hamilton's equations then

$$dH(\gamma(t))(\dot{\gamma}(t)) = \sum_{i=1}^{n} \frac{\partial H}{\partial q^i}\dot{q}^i(t) + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\dot{p}_i(t)$$

$$= \sum_{i=1}^{n} \frac{\partial H}{\partial q^i}\,\frac{\partial H}{\partial p_i} + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\left(-\frac{\partial H}{\partial q^i}\right) = 0.$$

The equations of Hamilton have a global formulation on T^*M in the sense that a solution is the integral curve of a smooth vector field defined globally on T^*M.

Recall that the Liouville canonical 1-form θ on T^*M has a local expression

$$\theta|_{locally} = \sum_{i=1}^{n} p_i dq^i$$

Let $\omega = -d\theta$, so that

$$\omega|_{locally} = \sum_{i=1}^{n} dq^i \wedge dp_i$$

Since for every smooth vector field Y on T^*M we have

$$Y = \sum_{i=1}^{n} \omega\left(Y, \frac{\partial}{\partial p_i}\right)\frac{\partial}{\partial q^i} - \sum_{i=1}^{n} \omega\left(Y, \frac{\partial}{\partial q^i}\right)\frac{\partial}{\partial p_i}$$

it follows that ω is a non-degenerate, closed 2-form on T^*M. Thus, given a smooth function $H: T^*M \to \mathbb{R}$, there exists a unique smooth vector field X on T^*M such that $i_X\omega = dH$, called the Hamiltonian vector field. Locally this global equation takes the form

$$\sum_{i=1}^{n} dq^i(X)\, dp_i - \sum_{i=1}^{n} dp_i(X) dq^i = \sum_{i=1}^{n} \frac{\partial H}{\partial q^i}\, \partial q^i + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \partial p_i$$

and therefore

$$dp_i(X) = -\frac{\partial H}{\partial q^i}$$

and

$$dq^i(X) = \frac{\partial H}{\partial p_i}.$$

These are precisely Hamilton's equations. [5]

2.4 Geometrical Relationship

Example 2.4.1 Let us take $M = \mathbb{R}^{2n}$ with canonical coordinates (q,p) equal to ($q_1, \dots, q_n, p_1, \dots, p_n$) and standard (translation invariant) symplectic form $\omega = \sum d\, p_j \wedge d\, q_j$. Hence $i_v\omega = d\, q_j$ if $v = \partial/\partial p_j$ and $i_v\omega = -d\, p_j$ if $v = \partial/\partial q_j$, and so $v_{qj} = -\partial/\partial p_j$ and $v_{pj} = \partial/\partial q_j$. If f is a smooth function on $\mathbb{R}^{2n}$ then

$$d\, f = \sum_j \left(\frac{\partial f}{\partial q_j}\, d\, q_j + \frac{\partial f}{\partial p_j}\, d\, p_j\right),$$

and therefore the Hamiltonian vector field of f becomes

$$v_f = \sum_j \left(\frac{\partial f}{\partial p_j} \frac{\partial}{\partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial}{\partial p_j}\right). \qquad (2.4.1)$$

In other words, the integral curves of the Hamiltonian flow of the function f are solutions of the system of first order differential equations

$$\dot{q}_j = \frac{\partial f}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial f}{\partial q_j}.$$

This is Hamilton's equation for the Hamiltonian flow of the function f on M, which is the reason for the sign convention in the equation

$$i_v\, \omega = -\, d\, f$$

for the relation between the Hamiltonian vector field v and the Hamiltonian function f. In the particular case that Hamiltonian function f = H is of the classical form

$$H(q, p) = K(p) + V(q)$$

With kinetic term $K(p) = \sum p_j^2/2m$ and potential function V(q) Hamilton's equation $m\dot{q} = p$, $\dot{p} = -\text{grad}\, V$ by elimination of p boils down to

$$F(q) = m\ddot{q} \tag{2.4.2}$$

which is the famous Newton equation for the motion of a point particle in R^n with mass m >0 in a conservative force field F(q) = $-\text{grad}\, V(p)$.

The exterior derivative

$$\omega = d\theta$$

Of the tautological 1-form on M is a closed (even exact) 2-form on M. in local coordinates $(x_1, \dots, x_n)$ on N with corresponding dual coordinates $(\xi_1, \dots, \xi_n)$ the projection map $\pi : M \to N$ sends $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ to $(x_1, \dots, x_n)$,and therefore the tautological 1-form θ takes the form

$$\theta = \sum_{j=1}^{n} \xi_j\, d\, x_j$$

In turn we get

$$\omega = \sum_{j=1}^{n} d\, \xi_j \wedge d\, x_j \ , \tag{2.4.4}$$

Which shows that $\omega = \sum d\, p_j \wedge d\, q_j$ is the standard symplectic form of Example 2.2.11 if we substitute $x_j = q_j$ and $\xi_j = p_j$. The conclusion is that $\omega = d\theta$ is a symplectic form on $M = T^* N$.

Definition 2.4.2 The form $\omega = d\theta$ is called the canonical symplectic form on the cotangent bundle $M = T^* N$.

Suppose N_1 and N_2 are smooth manifolds of dimension n with cotangent bundles $M_1 = T^*N_1$ and $M_2 = T^*N_2$ and with tautological 1-forms θ_1 and θ_2. A diffeomorphism $\emptyset : N_1 \rightarrow N_2$ induces a diffeomorphism

$$\phi : M_1 \rightarrow M_2 , \qquad \phi(\xi) = ((T_x \emptyset)^*)^{-1} \xi$$

for all $\xi \epsilon T_x^* N_1$, which is called the lift of $\emptyset$, and in such a way that the diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{\Phi} & M_2 \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ N_1 & \xrightarrow{\phi} & N_2 \end{array}$$

is commutative. The vertical arrows are the two projection maps.

Definition 2.4.3 Let (M, ω) be a symplectic manifold. A closed submanifold L → M is called Lagrangian if for each $x \in L$ the tangent space T_xL is a Lagrangian subspace of T_xM.

In other words, a submanifold ι : L → M is a Lagrangian submanifold if the dimension of L is half the dimension of M and $\iota^* \omega = 0$.

Example 2.4.4 Let M = T^*N with canonical symplectic form $\omega = d\, \theta$. If we consider a smooth 1-form α on N as a smooth section α: N → M of the cotangent bundle π: M → N, then the submanifold α : N → M is Lagrangian if and only if α is closed. [7]

2.5 Hamilton Formalism

Any smooth real-valued function H on a symplectic manifold can be used to define a Hamiltonian system. The function H is known as the Hamiltonian or the energy function. The symplectic manifold is then called the phase space. The Hamiltonian induces a special vector field on the symplectic manifold, known as the Hamiltonian vector field.

The Hamiltonian vector field (a special type of symplectic vector field) induces a Hamiltonian flow on the manifold. This is a one-parameter family of transformations of the manifold (the parameter of the curves is commonly called the time); in other words an isotopy of symplectomorphisms, starting with the identity. In addition to, each symplectomorphism preserves the volume form on the phase space. The collection of symplectomorphisms induced by the Hamiltonian flow is commonly called the Hamiltonian mechanics of the Hamiltonian system.

The symplectic structure induces a Poisson bracket. The Poisson bracket gives the space of functions on the manifold the structure of a Lie algebra.

Poisson Brackets

Suppose (M, ω) is a symplectic manifold. For $f \in \boldsymbol{F}(M)$ a smooth function on M there is a unique vector field $v_f \in$ Ӽ(M)

$$i_{v_f}\,\omega = -d f$$

and v_f is called the Hamiltonian vector field of the function f. The integral curves of v_f are called the solution curves for the Hamiltonian system defined by f, and the corresponding one parameter group $\emptyset_t$: M → M is called the Hamiltonian flow of f.

In local coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$ the integral curves of v_f are given by

$$\dot{q}_j = \frac{\partial f}{\partial p_j}\,, \qquad \dot{p}_j = -\frac{\partial f}{\partial q_j}$$

which is called Hamilton's equation for the Hamiltonian system defined by f in canonical coordinates.

Definition 2.5.1 *For $f, g \in \boldsymbol{F}(M)$ the Poisson bracket {f, g} is the smooth function on M defined by*

$$\{f, g\} = L_{v\,f}(g) = i_{v\,f}(d\,g) = -i_{v\,f}(i_{v\,f}\,\omega) = \omega(v\,f, v_g) \qquad (2.5.1)$$

by using the Cartan formula in the second identity and the antisymmetry of ω in the fourth identity.

The right hand side of the above definition shows that the Poisson bracket is antisymmetric in the sense that

$$\{g, f\} = -\{f, g\}$$

for all $f, g \in \boldsymbol{F}(M)$. In canonical local coordinates the Poisson brackets are given by

$$\{f, g\} = \sum_{j=1}^{n}\left(\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j}\right) \qquad (2.5.2)$$

as discussed previously. The function space $\boldsymbol{F}$ (M) can be thought of as the space of classical observables on the phase space (*M*, *ω*). For an observable $g \in \boldsymbol{F}(M)$ the equation

$$\frac{d\,g}{d\,t} = \{f, g\}$$

is the abstract form of Hamilton's equation for the Hamiltonian f. The infinitesimal change of a given observable g under the Hamiltonian flow of f is equal to $\{f, g\}$. If $\{f, g\} = 0$ on all of M then the observable g is called a constant of motion for the Hamiltonian system defined by f. Since $\{f, g\} = 0$ the observable f is a constant of motion for the Hamiltonian f itself. If f is the total energy of a classical mechanical system, then this is the law of conservation of total energy. [7]