Chapter 1

Introduction

Dynamical systems are the mathematical study of long-term behavior in systems that evolve in time, usually under unchanging rules. The broadness of this description corresponds to the multitude of specialties within dynamics that make up its contemporary spectrum of research.

The purpose of this chapter is to introduce many of basic ideas of mechanics, control, and optimal control, together with a number of illustrative physical examples that will be used throughout this research to illustrate the theory and how to use it. These examples are simple to write down in general and to understand at an elementary level, but they are also useful for the understanding of deeper parts of the theory.

In some mechanical and engineering problems one encounters different kinds of additional conditions, constraining and restricting motions of mechanical systems. Such conditions are called constraints. Constraints in dynamics are restrictions on positions and velocities of the system. Phenomenological constraints are introduced instead of unknown forces to describe observed motions. For example, a rigid body is a system of material points with fixed distances between each pair of points. Another example is the no slip condition in the motion of a rolling body. This constraint requires that the relative velocity of the point of contact of the rolling body vanishes. In the first example, the constraint depends only on the position of the material point. Such constraints are called holonomic. In the second example, the no slip condition is a linear relation on the velocity of the motion. Such conditions are called linear nonholonomic constraints. Examples of holonomic constraints are length constraints for simple pendula and rigidity constraints for rigid body motion. The rolling disk and ball are archetypal nonholonomic systems: systems with *nonintegrable* constraints on their velocities.

In this chapter we discuss both the rolling disk and ball, as well as many other nonholonomic systems such as the roller racer, and the rattleback. We can define a holonomic system as a system between whose possible positions all conceivable continuous motions are also possible motions. The point is that nonholonomic constraints restrict types of motion but not position. The meaning of this statement should become clearer as we continue through the research.

Other examples discussed here include the free rigid body and the somewhat more complex satellite with momentum wheels. These are (holonomic) examples of free and coupled rigid body motion respectively, the motion of bodies with nontrivial spatial extent, as opposed to the motion of point particles.

1.1 Generalized Coordinates

The most basic goal of analytical mechanics is to provide formalism for describing motion. This is often done in terms of a set of ***generalized coordinates***, which may be interpreted as coordinates for the system's ***configuration space***, often denoted by Q. This is a set of variables whose values uniquely specify the location in 3-space of each physical point of the mechanism. A set of generalized coordinates is minimal in the sense that no set of fewer variables suffices to determine the locations of all points on the mechanism. The number of variables in a set of generalized coordinates for a mechanical system are called the number of ***degrees of freedom*** of the system.

1.1.1 Example (A Simple Kinematic Chain)

Kinematic chain refers to an assembly of rigid bodies connected by joints that is the mathematical model for a mechanical system. As in the familiar use of the word chain, the rigid bodies, or links, are constrained by their connections to other links.

Here the simple kinematic chain shown in Figure 1.1.1

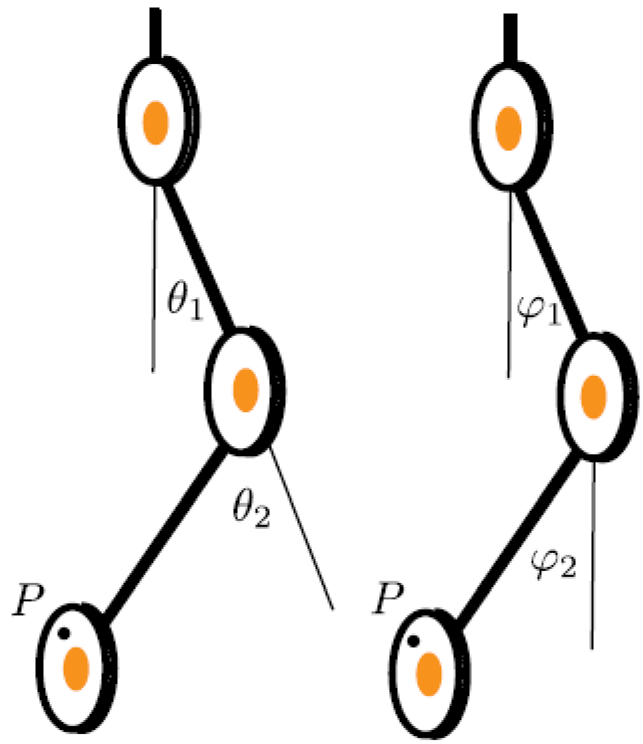

FIGURE 1.1.1. Kinematic chains.

These are two copies of the same mechanism. This mechanism consists of planar rigid bodies connected by massless rods, and the joints are free to rotate in a fixed plane. In the first, the motion of a typical point P is described in terms of coordinate variables (θ_1, θ_2), where θ_2 is the relative angle between the two links in the chain. In Figure 1.1.1 (b), the motion of the typical point P is described in terms of coordinate variables (φ_1, φ_2), which are the (absolute) angles of the links with respect to the vertical direction.

Specifically, in this case, the inertial frame is chosen so that its origin is at the hinge point of the upper link. The y-axis is directed parallel and opposite to the gravitational field, and the x-axis is chosen so as to give the coordinate frame the standard orientation. Suppose the point P is located on the second link, as depicted. If this has coordinates (x_l, y_l) with respect to a local frame fixed in the second link, then the coordinates with respect to the inertial frame are given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r_1 \sin\theta_1 + x_l \sin(\theta_1 + \theta_2) + y_l \cos(\theta_1 + \theta_2) \\ -r_1 \cos\theta_1 - x_l \cos(\theta_1 + \theta_2) + y_l \sin(\theta_1 + \theta_2) \end{pmatrix} \tag{1.1.1}$$

where r_1 is the length of the first link, or equivalently by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r_1 \sin\varphi_1 + x_l \sin\varphi_2 + y_l \cos\varphi_2 \\ -r_1 \cos\varphi_1 - x_l \cos\varphi_2 + y_l \sin\varphi_2 \end{pmatrix} \tag{1.1.2}$$

The mappings $(\theta_1, \theta_2) \to (x, y)$ are examples of functions that associate values of the generalized coordinate variables (θ_1, θ_2) (respectively (φ_1, φ_2)) to inertial coordinates of the point P. In this example, the configuration manifold is given by $Q = S^1 \times S^1$ and is parameterized by the two angles θ_1, θ_2, which serve as generalized coordinates. One can also make the alternative choice of φ_1, φ_2 as generalized coordinates that provide a different set of coordinates on Q.

1.1.2 Example (A Simple Pendulum)

Consider, the problem of a simple pendulum moving in the x-y plane. The pendulum has a length l and moves under the action of gravity, so that its potential energy is mgh. The system is illustrated in the figure on the next page.

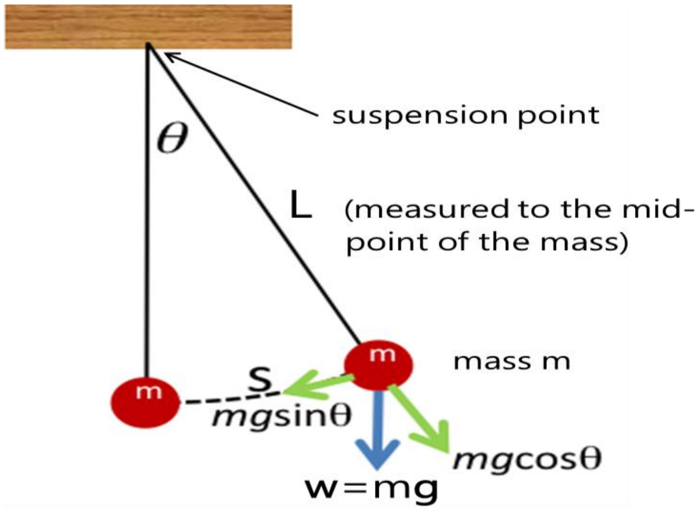

FIGURE 1. 1. 2. Simple pendulum

We could use Cartesian coordinates *x* and *y* to describe the location of the pendulum bob, but *x* and *y* are not independent. In fact, since the length of the pendulum is constant, they are related by

$$x^2 + y^2 = L^2$$

This condition would need to be imposed as a constraint on the system, which can be inconvenient. It is more natural to use the angle θ that the pendulum makes with respect to the vertical to describe the motion. But what would be the equation of motion for θ ? In order to find out what this is, we only need to express the Lagrangian in terms of θ .

Now, the Lagrangian in terms of *x* and *y* is given by

$$L(x, y, \dot{x}, \dot{y}) = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - U(x, y) \qquad (1.1.3)$$

where we have introduced a general potential function, however, for this example, we know that the potential is given by $U(x,y) = -mgy$

The Cartesian coordinates *x* and *y* are related to θ by a set of *transformation equations*:

$$x = l \sin\theta$$

$$y = l \cos\theta$$

In order to transform the kinetic energy, we need the time derivatives of the transformation equations:

$$\dot{x} = l \cos\theta\, \dot{\theta}$$

$$\dot{y} = -l \sin\theta\, \dot{\theta}$$

Substituting the transformations and their derivatives into the Lagrangian gives

$$L(\theta, \dot{\theta}) \dot{=} \frac{1}{2}\, m[(l \cos\theta\, \dot{\theta})^2 + (-\, l \sin\theta\, \dot{\theta})^2\,] + mgl \cos\theta$$

$$= \frac{1}{2}\, m\, l^2 \dot{\theta}^2 [\, cos^2\theta + \; sin^2\theta] + \; mgl \cos\theta$$

$$= \frac{1}{2}\, m\, l^2 \dot{\theta}^2 + \; mgl \cos\theta \qquad (1.1.4)$$

Now, given the Lagrangian, we just turn the crank on the Euler-Lagrange equation and derive the equation of motion for :

$$\frac{\partial L}{\partial \dot{\theta}} = m\, l^2\, \dot{\theta}$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}}) = m\, l^2 \bar{\theta}$$

$$\frac{\partial L}{\partial \theta} = \text{-} m\; g\; l\; sin\,\theta$$

so that the equation of motion is

$$m\, l^2 \bar{\theta} + m\, g\, l \sin\theta = 0$$

$$\bar{\theta} + \frac{g}{l} \sin\theta = 0$$

1.1.3 Example (The Motion of a Rigid Body in the Plane)

In the following part we will derive expressions that describe the general motion of a rigid body in the plane. As rigid bodies are viewed as collections of particles, this may appear an insurmountable task, requiring a description of the motion of each particle. However, the assumption that the body does not deform is a very strong one, requiring that the distance between every pair of particles comprising the body remains unchanged. To satisfy this, the particles that comprise a rigid body must move in concert, making the kinematics almost trivial. So far the particle motion has been described using position vectors that were referred to fixed reference frames. The positions, velocities and accelerations determined in this way are referred to as absolute. Often it isn't possible or convenient to use a fixed set of axes for the observation of motion. Many problems are simplified considerably by the use of a moving reference frame.

In the following we will restrict our attention to moving reference frames that translate but do not rotate.

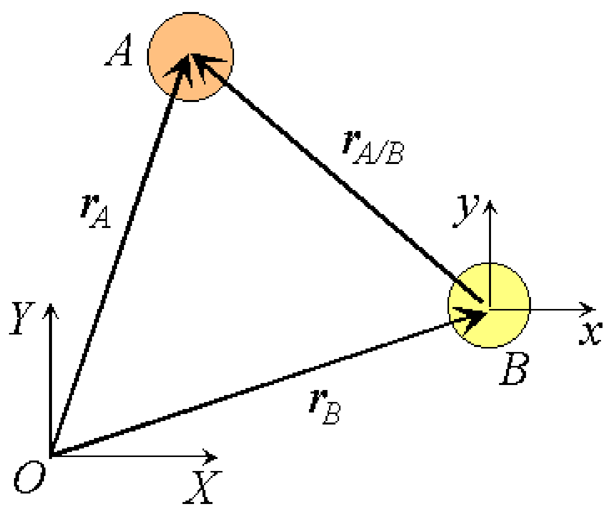

Figure 1.1.3: Observation of particle motion using a translating reference.

Consider two particles A and B moving along independent trajectories in the plane, and a fixed reference O. Let r_A and r_B be the positions of particles A and B in the fixed reference. Instead of observing the motion of particle A relative to the fixed reference, we will attach a non-rotating reference to particle B and observe the motion of A relative to the moving reference at B. Let $\boldsymbol{i}$ and $\boldsymbol{j}$ be basis vectors of the moving reference, then the position vector of A relative to the reference at B, denoted $r_{A/B}$ is,

$$r_{A/B} = x\boldsymbol{i} + y\boldsymbol{j}$$

where the subscript stands for A with respect to B or A relative to B. Observe that, as the moving frame does not rotate, basis vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ do not change in time. Therefore, taking time derivatives, we obtain simply,

$$v_{A/B} = \frac{dx}{dt}\,\boldsymbol{i} + \frac{dy}{dt}\,\boldsymbol{j}$$

which can be interpreted as the velocity of A relative to B. Now we can express the absolute position vector of A as,

$$r_A = r_B + r_{A/B}$$

Differentiating the equation in time to obtain expressions for the absolute velocity and acceleration of particle *A*:

$$v_A = v_B + v_{A/B}$$

$$\alpha_A = \alpha_B + \alpha_{A/B}$$

or the absolute velocity of *A* equals the absolute velocity of *B* plus the velocity of *A* relative to *B*, $v_{A/B}$, and similarly for the acceleration. The relative terms are the velocity or acceleration measured by an observer attached to the moving reference at particle *B*.

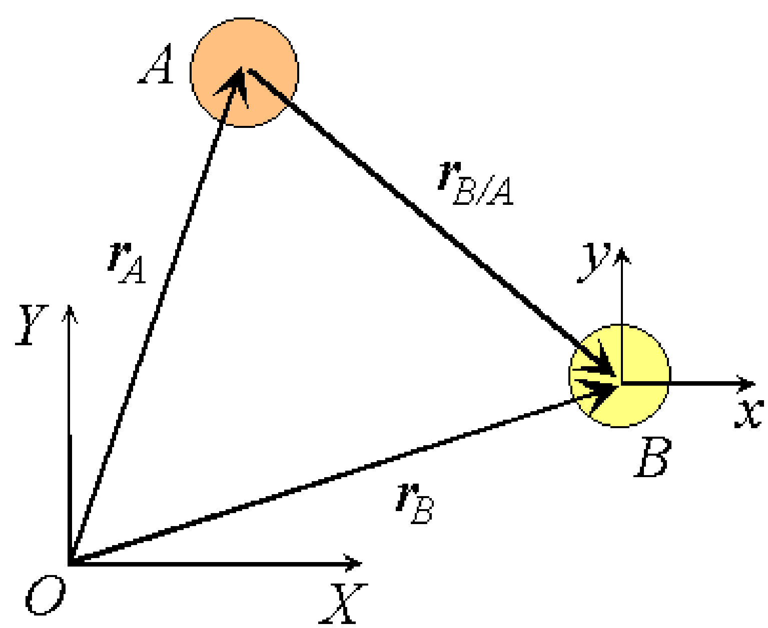

Figure 1.1.4: Relative velocities under change of translating reference.

What would happen if the moving reference were attached to *A* instead?

$$r_B = r_A + r_{B/A}$$

$$v_B = v_A + v_{B/A}$$

$$\alpha_B = \alpha_A + \alpha_{B/A}$$

By comparison with expressions derived previously:

$$r_{B/A} = - r_{A/B}$$

$$v_{B/A} = - v_{A/B}$$

$$\alpha_{B/A} = - \alpha_{A/B}$$

For the motion relative to a translation reference:

$$r_B = r_A + r_{B/A}$$

$$v_B = v_A + v_{B/A}$$

$$\alpha_B = \alpha_A + \alpha_{B/A}$$

which describes the motion of particle A observed relative to a translating reference at *B*.[1]

1.2 The Vertical Rolling Disk

The vertical rolling disk is a basic and simple example of a system subject to nonholonomic constraints: a homogeneous disk rolling without slipping on a horizontal plane. In the first instance we consider the vertical disk, a disk that, unphysically of course, may not tilt away from the vertical; it is not difficult to generalize the situation to the falling disk. It is helpful to think of a coin such as a penny, since we are concerned with orientation and the roll angle of the disk.

Let S^1 denote the circle of radius 1 in the plane. It is parameterized by an angular variable of 2π-periodic. The configuration space for the vertical rolling disk is $Q = R^2 \times S^1 \times S^1$ and is parameterized by the generalized coordinates $q = (x, y, \theta, \varphi)$, denoting the position of the contact point in the xy-plane, the rotation angle of the disk, and the orientation of the disk, respectively, as in Figure 1.2.1.

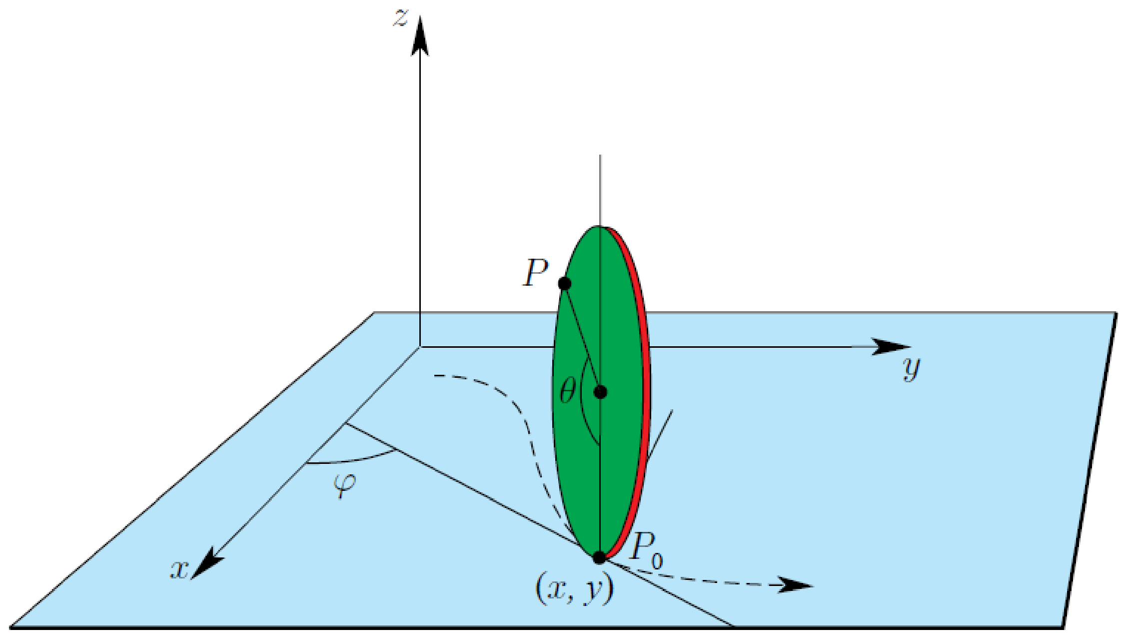

FIGURE 1.2.1. The geometry of the rolling disk.

The variables (x, y, φ) may also be regarded as giving a translational position of the disk together with a rotational position; that is, we may regard (x, y, φ) as an element of the ***Euclidean group*** in the plane. This group, denoted by SE(2), is the group of translations and rotations in the plane, that is, the group of rigid motions in the plane. Thus, $SE(2) = R^2 \times S^1$ (as a set).

In summary, the configuration space of the vertical rolling disk is given by $Q = SE(2) \times S^1$ and this space has coordinates (generalized coordinates) given by $((x, y, \varphi), \theta)$.

The Lagrangian for the vertical rolling disk is taken to be the total kinetic energy of the system, namely

$$L(x, y, \varphi, \theta, \dot{x}, \dot{y}, \dot{\varphi}, \dot{\theta}) = \frac{1}{2}\, m\,(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}\, I\dot{\theta}^2 + \frac{1}{2}\, J\dot{\varphi}^2 , \qquad (1.2.1)$$

where m is the mass of the disk, I is the moment of inertia of the disk about the axis perpendicular to the plane of the disk, and J is the moment of inertia about an axis in the plane of the disk (both axes passing through the disk's center).

If R is the radius of the disk, the nonholonomic constraints of rolling without slipping are

$$\dot{x} = R(\cos\varphi)\dot{\theta} ,$$

$$\dot{y} = R(\sin\varphi)\,\dot{\theta} , \qquad (1.\ 2.\ 2)$$

which state that the point P_0 fixed on the rim of the disk has zero velocity at the point of contact with the horizontal plane. Thus, we can write the constraints as:

$$\dot{x} = R(\cos\varphi)\dot{\theta} = 0 ,$$

$$\dot{y} = R(\sin\varphi)\,\dot{\theta} = 0 .$$

We can write these equations as two constraint equations

$$a^1 \cdot (\dot{x}, \dot{y}, \dot{\varphi}, \dot{\theta})^T = 0 ,$$

$$a^2 \cdot (\dot{x}, \dot{y}, \dot{\varphi}, \dot{\theta})^T = 0 ,$$

where ^{T} denotes the transpose and where

$$a^1 = (1, 0, 0, -R\cos\varphi),\ a^2 = (0, 1, 0, -R\sin\varphi) .$$

we can use the notation ,

$$a_1^1 = 1,\ a_2^1 = 0,\ a_3^1 = 0,\ a_4^1 = -R\cos\varphi ,$$

and similarly for a^2:

$$a_1^2 = 0,\ a_2^2 = 1,\ a_3^2 = 0,\ a_4^2 = -R\sin\varphi.$$

We will compute the dynamical equations for this system with controls later in this research. In particular, when there are no controls, we will get the dynamical equations for the uncontrolled disk. As we shall see, these free equations can be explicitly integrated.

Consider the case where we have two controls, one that can steer the disk and another that determines the roll torque. Now we shall down the equations for the controlled vertical rolling disk. According to these equations, we add the forces to the right-hand side of the Euler–Lagrange equations for the given Lagrangian along with Lagrange multipliers to enforce the constraints and to represent the reaction forces. In our case, L is cyclic in the configuration variables $q = (x, y, \varphi, \theta)$, and so the required dynamical equations become

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = u_\varphi f^\varphi + u_\theta f^\theta + \lambda_1 a^1 + \lambda_2 a^2 \qquad (1.2.3)$$

where, from (1.2.1), we have

$$\frac{\partial L}{\partial \dot{q}} = (m\,\dot{x}, m\dot{y}, J\dot{\varphi}\,, I\dot{\theta}),$$

and where

$$f^\phi = (0, 0, 1, 0), \quad f^\theta = (0, 0, 0, 1),$$

corresponding to assumed controls in the directions of the two angles φ and θ, respectively. Here u_φ and u_θ are control functions, so the external control forces are $F = u_\varphi f^\varphi + u_\theta f^\theta$, and the λ_i are Lagrange multipliers, chosen to ensure satisfaction of the constraints (1.2.2).

We eliminate the multipliers as follows. Consider the first two components of (1.2.3) and substitute the constraints (1.2.2) to eliminate $\dot{x}$ and $\dot{y}$ to give

$$\lambda_1 = m\,\frac{d}{dt}\left(R\cos\varphi\dot{\theta}\right),$$

$$\lambda_2 = m\,\frac{d}{dt}\left(R\sin\varphi\dot{\theta}\right).$$

Substitution of these expressions for λ_1 and λ_2 into the last two components of (1.2.3) and noticing the simple identities

$$\lambda_1 a_3^1 + \lambda_2 a_3^2 = 0,$$

$$\lambda_1 a_4^1 + \lambda_2 a_4^2 = -\,m\,R^2\,\ddot{\theta},$$

gives the dynamic equations

$$J\ddot{\varphi} = u_\varphi$$
$$(I + mR^2)\,\ddot{\varphi} = u_\theta\ , \qquad (1.2.4)$$

which, together with the constraints

$$\dot{x} = R(\cos\varphi)\dot{\theta}\ ,$$
$$\dot{y} = R(\sin\varphi)\,\dot{\theta}\ , \qquad (1.2.5)$$

(and some specification of the control forces), determine the dynamics of the system.

The *free equations*, in which we set $u_\varphi = u_\theta = 0$, are easily integrated. In fact, in this case, the dynamic equations (1.2.4) show that $\dot{\varphi}$ and $\dot{\theta}$ are constants; calling these constants ω and Ω, respectively, we have

$$\varphi = \omega t + \varphi_0\ ,$$
$$\theta = \Omega t + \theta_0\ .$$

Using these expressions in the constraint equations (1.2.5) and integrating again gives

$$x = \frac{\Omega}{\omega} R\sin(\omega t + \varphi_0) + x_0\ ,$$
$$y = -\frac{\Omega}{\omega} R\cos(\omega t + \varphi_0) + y_0\ .$$

Consider next the controlled case, with nonzero controls u_1, u_2. Call the variables θ and φ "base" or "controlled" variables and the variables x and y "fiber" variables. The distinction is that while θ and φ are controlled directly, the variables x and y are controlled indirectly via the constraints. It is clear that the base variables are controllable in any sense we can imagine. Also the full system is controllable, in a precise sense as we shall show later, by virtue of the nonholonomic nature of the constraints.

1.2.1 Example (Two wheels roll without slipping)

Two wheels of radius a are mounted on the ends of an axle of length b such that the wheels rotate independently. The whole combination rolls without slipping on a plane.

Consider first a single wheel (a disk) as in the previous problem. We have seen we can describe the system using four coordinates $x, y, \theta, \emptyset$ constrained by two differential equations (The no-slip condition):

$$\dot{x} = a\,\dot{\emptyset}\,sin\,\theta$$

$$\dot{y} = -a\,\dot{\emptyset}\,cos\,\theta$$

We can also write the constraint using dx, dy:

$$dx - a\,sin\,\theta\,d\emptyset = 0$$

$$dy + a\,cos\,\theta\,d\emptyset = 0$$

In a problem with two wheels, each wheel satisfies the same constraints than the single rolling disk. We use r_1, v_1 for the center of the first wheel and r_2, v_2 for the second wheel. Since the wheels are connected by a common axle, the angles θ_1, $\theta_2 = 0$ that define each wheel's axis are the same: $\theta_1, \theta_2 = 0$, the angle of the common axle. The rotation angles $\emptyset_1, \emptyset_2$ are different, since the wheels can rotate independently.

Thus, we can write the constraints as:

$$dx_1 - a\,sin\,\theta d\emptyset_1 = 0$$

$$dy_1 + a\,cos\,\theta d\emptyset_1 = 0$$

$$dx_2 - a\,sin\,\theta d\emptyset_2 = 0$$

$$dy_2 + a\,cos\,\theta d\emptyset_2 = 0$$

The center of the axle (which is the center of mass) has a position vector $r = (r_2 + r_1)/2$, so $x = (x_1 + x_2)/2$ and $x = (y_1 + y_2)/2$.

Thus, we can write constraints for $, dy$:

$$dx - a\,sin\,\theta(d\emptyset_1 + d\emptyset_2)/2 = 0$$

$$dy + a\,cos\,\theta(d\emptyset_1 + d\emptyset_2)/2 = 0$$

Multiplying each equation by trigonometric factors $sin\,\theta$, $cos\,\theta$ and adding or subtracting them, we can write the equations as

$$cos\,\theta dx + sin\,\theta dy = 0$$

$$sin\,\theta dx - cos\,\theta dy = \frac{a}{2}(\,d\emptyset_1 + d\emptyset_2)$$

So far, these have been the equations for the disks not rolling, but we also have the constraint that the centers of the wheels are at a constant distance b. The constraint can be written as:

$$\boldsymbol{r_2} - \boldsymbol{r_1} = b(cos\,\theta\hat{\imath} + sin\,\theta\hat{\jmath}),$$

or

$$x_2 - x_1 = b\cos\theta\,,\; y_2 - y_1 = b\sin\theta\,.$$

Taking derivatives, the x- constraint is

$$\dot{x}_2 - \dot{x}_1 = -b\dot{\theta}\,sin\,\theta$$

$$a\,sin\,\theta\left(\dot{\emptyset}_2 - \dot{\emptyset}_1\right) = -b\dot{\theta}\sin\theta$$

$$\dot{\theta} = -\frac{a}{b}\left(\dot{\emptyset}_2 - \dot{\emptyset}_1\right)$$

$$\dot{\theta} = C - \frac{a}{b}\,(\emptyset_2 - \emptyset_1)$$

(If we follow on with the y-constraint, we get the same equation: the constraint is only on the magnitude of the distance, not on direction; the constraint on direction was used to define the angle θ for the axle’s direction, perpendicular to the wheels). Notice that the constraint was holonomic to begin with ($|\boldsymbol{r_2} - \boldsymbol{r_1}| = b$), we transformed into one on velocities in order to involve θ, $\emptyset$, but we then were able to integrate those equations to get a holonomic constraint on $\theta, \emptyset$. This is to say, we can have constraints involving velocities that are holonomic, if they are integrable. [6]

1.3 The Falling Rolling Disk

A more realistic disk is of course one that is allowed to fall over (i.e., it is permitted to deviate from the vertical). This turns out to be a very instructive example to analyze. See Figure 1.3.1. As the figure indicates, we denote the coordinates of contact of the disk in the *xy*-plane by (x, y) and let θ, φ, and ψ denote the angle between the plane of the disk and the vertical axis, the "heading angle" of the disk, and "self-rotation" angle of the disk, respectively. Note that the notation ψ for the falling rolling disk corresponds to the notation θ in the special case of the vertical rolling disk.

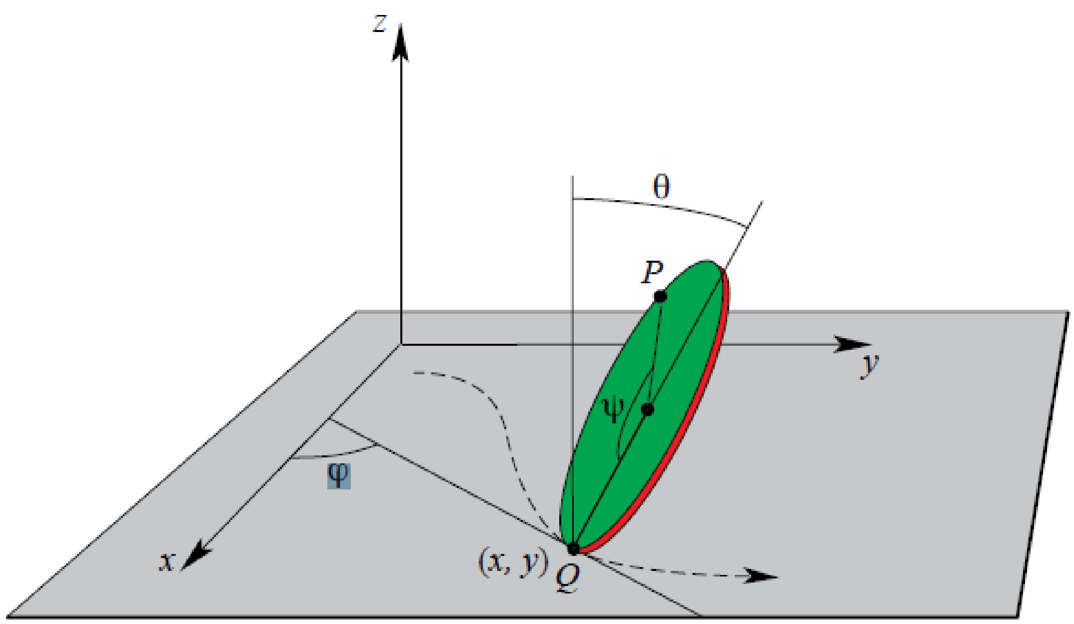

FIGURE 1.3.1. The geometry for the rolling disk.

Denote the mass and radius of the disk by m and R, respectively; let I be, as in the case of the vertical rolling disk, the moment of inertia about the axis through the disk's "axle" and J the moment of inertia about any diameter. The Lagrangian is given by the kinetic minus potential energies:

$$L = \frac{m}{2}\,[(\xi - R\,(\dot{\varphi}\sin\theta + \dot{\psi}))^2 + \eta^2\sin^2\theta + (\eta\cos\theta + R\dot{\theta})^2]$$
$$+\ \frac{1}{2}\,[J\,(\dot{\theta}^2 + \dot{\varphi}^2\cos^2\theta) + I\,(\dot{\varphi}\sin\theta + \dot{\psi})^2] - mg\,R\cos\theta,$$

where $\xi = \dot{x}\cos\varphi + \dot{y}\sin\varphi + R\,\dot{\psi}$ and $\eta = -\dot{x}\sin\varphi + \dot{y}\cos\varphi$, while the constraints are given by

$$\dot{x} = -\dot{\psi}\,R\cos\varphi,$$
$$\dot{y} = -\dot{\psi}\,R\sin\varphi.$$

Note that the constraints may also be written as $\xi = 0$, $\eta = 0$.

1.3.1 Example (Unicycle with Rotor)

An interesting generalization of the falling disk is the "unicycle with rotor," (see Figure 1.3.2).

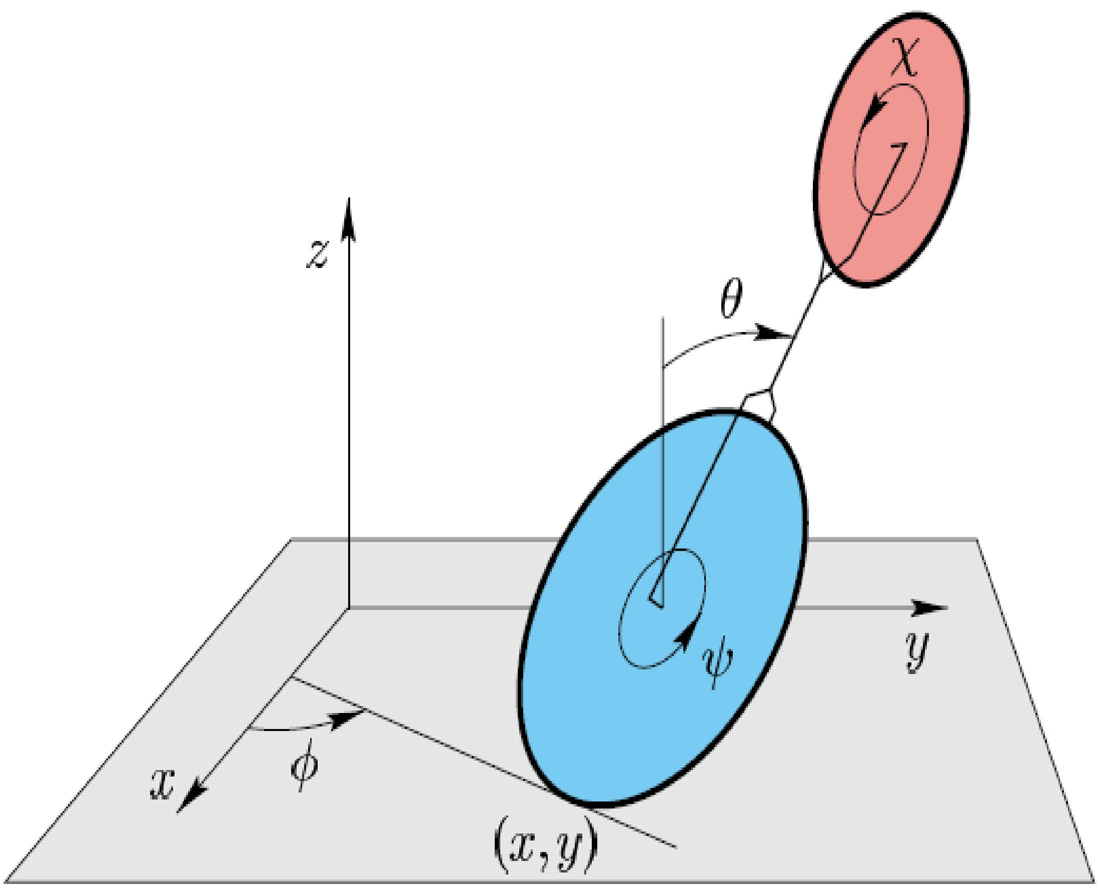

FIGURE 1.3.2. The configuration variables for the unicycle with rotor.

This is a homogeneous disk on a horizontal plane with a rotor. The rotor is free to rotate in the plane orthogonal to the disk. The rod connecting the centers of the disk and rotor keeps the direction of the radius of the disk through the contact point with the plane. We may view this system as a simple model of unicycle with rider whose arms are represented by the rotor. The configuration space for this system is $Q = S^1 \times S^1 \times S^1 \times \mathrm{SE}(2)$, which we parameterize with coordinates $(\theta, \chi, \psi, \varphi, x, y)$. As in Figure 1.3.2, θ is the tilt of the unicycle itself, and ψ and χ are the angular positions of the wheel of the unicycle and the rotor, respectively. The variables (φ, x, y), regarded as a point in SE(2), represent the angular orientation of the overall system and position of the point of contact of the wheel with the ground. [2]

1.4 The Roller Racer

We now consider a tricycle-like mechanical system called the ***roller racer***, or the ***Tennessee racer***, that is capable of locomotion by oscillating the front handlebars. Analysis of this system may be a useful guide for modeling and studying the stability of other systems, such as aircraft landing gears and train wheels.

The roller racer is modeled as a system of two planar-coupled rigid bodies (the main body and the second body) with a pair of wheels attached on each of the bodies at their centers of mass: a nonholonomic generalization of the coupled planar bodies discussed earlier. We assume that the mass and the linear momentum of the second body are negligible, but that the moment of inertia about the vertical axis is not. See Figure 1.4.1. on the next page.

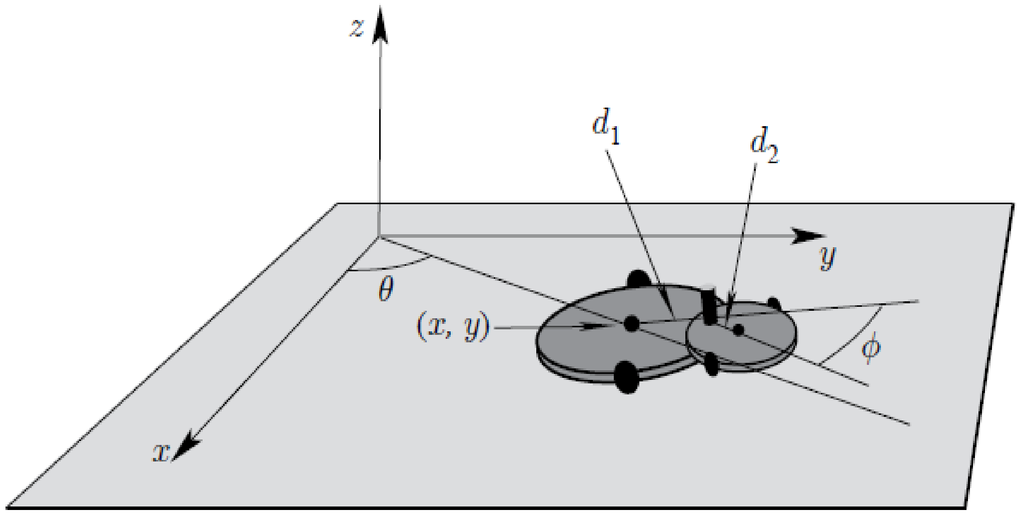

FIGURE 1. 4 .1. The geometry for the roller racer.

Let (x, y) be the location of the center of mass of the first body and denote the angle between the inertial reference frame and the line passing through the center of mass of the first body by θ, the angle between the bodies by $\emptyset$, and the distances from the centers of mass to the joint by d_1 and d_2. The mass of body 1 is denoted by m, and the inertias of the two bodies are written as I_1 and I_2.

The Lagrangian and the constraints are

$$L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} I_2(\dot{\theta} + \dot{\emptyset})^2$$

and

$$\dot{x} = \cos\theta \left(\frac{d_1 \cos\emptyset + d_2}{\sin\emptyset} \dot{\theta} + \frac{d_2}{\sin\emptyset} \dot{\emptyset} \right)$$

$$\dot{y} = \sin\theta \left(\frac{d_1 \cos\emptyset + d_2}{\sin\emptyset} \dot{\theta} + \frac{d_2}{\sin\emptyset} \dot{\emptyset} \right)$$

The configuration space is SE(2) × SO(2). The Lagrangian and the constraints are invariant under the left action of SE(2) on the first factor of the configuration space. [36]

1.5 The Rattleback

We end with a brief discussion of one of the most fascinating nonholonomic system sthat is the rattleback top or Celtic stone. A rattleback is a convex asymmetric rigid body rolling without sliding on a horizontal plane. It is known for its ability to spin in one direction and to resist spinning in the opposite direction for some parameter values, and for other values to exhibit multiple reversals in clear violation of conservation of angular momentum or of damped angular momentum. In fact, this phenomenon may be viewed as a remarkable demonstration of the nontriviality of the momentum equation. Moreover, the stable spin direction is in fact asymptotically stable. See Figure 1.5.1.

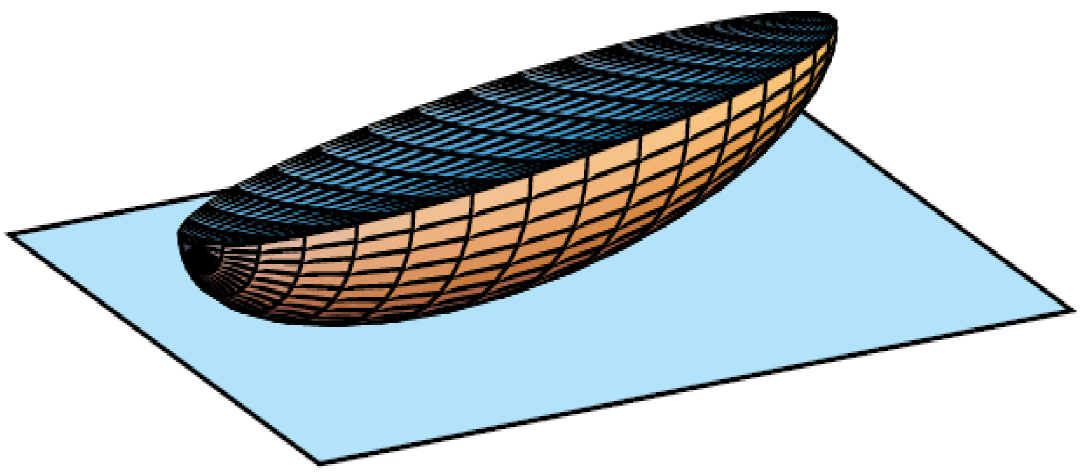

FIGURE 1.5.1. The rattleback.

We adopt the ideal model (with no energy dissipation and no sliding), and within that context no approximations are made. In particular, the shape need not be ellipsoidal.

The Lagrangian of the rattleback is computed to be

$$L = \frac{1}{2}\,[A\cos^2\psi + B\sin^2\psi + m(\gamma_1\cos\theta - \zeta\sin\theta)^2]\,\dot{\theta}^2$$

$$+\,\frac{1}{2}\,[(A\sin^2\psi + B\cos^2\psi)\sin^2\theta + C\cos^2\theta]\,\dot{\phi}^2$$

$+ \frac{1}{2}$ (C + $m\gamma_2^2 2\sin^2\theta$) $\dot{\psi}^2 + \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$

$+ m(\gamma_1 \cos\theta - \zeta \sin\theta)\gamma_2 \sin\theta\dot{\theta}\dot{\psi} + (A - B)\sin\theta \sin\psi \cos\psi^{\cdot}\dot{\theta}\dot{\emptyset}$

$+ C\cos\theta\,\dot{\emptyset}\dot{\psi} + m\,g\,(\gamma_1 \sin\theta + \zeta\cos\theta)$,

where

A, B, C = the principal moments of inertia of the body,

m = the total mass of the body,

(ξ, η, ζ) = coordinates of the point of contact relative to the body frame,

$$\gamma_1 = \xi \sin\psi + \eta\cos\psi,$$

$$\gamma_2 = \xi\cos\psi - \eta\sin\psi.$$

The shape of the body is encoded by the functions ξ, η, and ζ. The constraints are

$$\dot{x} = \alpha_1\dot{\theta} + \alpha_2\dot{\psi} + \alpha_3\dot{\emptyset},\ \dot{y} = \beta_1\dot{\theta} + \beta_2\dot{\psi} + \beta_3\dot{\emptyset},$$

Where

$$\alpha_1 = -(\gamma_1\sin\theta + \zeta\cos\theta)\sin\emptyset,$$

$$\alpha_2 = \gamma_2\cos\theta\sin\emptyset + \gamma_1\cos\emptyset,$$

$$\alpha_3 = \gamma_2\sin\emptyset + (\gamma_1\cos\theta - \zeta\sin\theta)\cos\emptyset,$$

$$\beta k = -\frac{\partial\alpha_k}{\partial\emptyset},\ k = 1, 2, 3.$$

The Lagrangian and the constraints are SE (2)-invariant, where the action of an element $(a, b, \alpha) \in$ SE (2) is given by

$$(x, y, \emptyset) \longrightarrow (x\cos\alpha - y\sin\alpha + a,\ x\sin\alpha + y\cos\alpha + b,\ \emptyset + \alpha).$$

Corresponding to this invariance, ξ, η, and ζ are functions of the variables θ and ψ only. [1]