Chapter 1

Introduction to the Integral Equation(IE) and Construction of the IE.

1.1. Introduction:

Integral Equation began to appear since the mid-seventeenth century , when some scientists were not able to solve the differential equation .

The integral equation developed with appear Abel kernel after that Volterra integral equation lastly Fredholm integral equation.

In this time we find numerical method played a great role to solve integral equation.

Therefore of great progressing in basic science whether physical or engineering has played essential role.

Topics on integral equations has grown and evolved to its direct association lists the large branches of mathematics, such as account differential and integrative and questions of boundary conditions.

During the twenty-five last year, there is a marked increase in the use of integral equations and formulations for finding scientific solutions to engineering problems and solving differential equations that are difficult to solve by normal methods. In the recent period found that the integral equations give a better solution than give differential equations.

The explosive growth in industry and technology requires constructive adjustments in mathematics text researches.

The integral equation it equation that appear in the unknown function under signal or more, from the signals of integrity.

There more types of integral equation from it linear and nonlinear integral equation.

The general formula linear integral equation it is:

$$y(x) = f(x) + \lambda \int_a^b k(x,t)y(t)\,dt \quad (1-1)$$

Where $y(x)$ unknown function, $f(x)$ known function and $k(x,t)$known function are called kernel integral equation.

We say that integral equation it is linear if that which operations on unknown function in equation it linear operations.

And the general formula nonlinear integral equation it is :

$$y(x) = f(x) + \lambda \int_a^b k(x,t)(y(t))^2\,dt \qquad (1-2)$$

Where unknown function it is nonlinear .

Can be solve of integral equatin by differential , Laplace transformation , progression the converging, and the eigenvalue.

The most standared type of integral equation in $u(x)$ is the form :

$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} k(x,t)u(t)\,dt \qquad (1-3)$$

Where $g(x)$ and $h(x)$ are the limits of the integration, λ is a constant parameter, and $k(x,t)$ is a known function, of two variables x and t , called the kernel or the nucleus of the integral equation. The unknown function $u(x)$ that will be determined appears inside the integral sign. In many other cases , the unknown function $u(x)$ appears inside and outside the integral sign. The function $f(x)$ and $K(x,t)$ are given in advance. It is to be noted that the limits of integration $g(x)$ and $h(x)$ may be both variables, constants or mixed.

Integral equation appear in many forms . Two distinct ways that depend on the limits of integration are used to characterize integral equations , namely :

1. If the limits of the integration are fixed , the integral equation is called a Fredholm integral equation given in the form :

$$u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)\,dt \qquad (1-4)$$

Where a and b are constants .

2. If at least one limit is a variable , the equation is called a Volterra integral equation given the form :

$$u(x) = f(x) + \lambda \int_a^x k(x,t)u(t)\,dt \qquad (1-5)$$

Moreover, two other distinct kinds, that depend on the appearance of the unknown function $u(x)$, are defiend as follows:

1. If the unknown function $u(x)$ appears only under the integral sing of Fredholm or Volterra equation , the integral equation is called a first kind Fredholm or Volterra integral equation respectively .
2. If the unknown function $u(x)$ appears both inside and outside the integral sing of Fredholm or Volterra equation, the integral equation is called a second kind Fredholm or Volterra integral equation respectively.

In the all Fredholm or Volterra integral equations presented above , if $f(x)$ is identically zero the resulting equation:

$$u(x) = \lambda \int_a^b k(x,t)u(t)\,dt \qquad (1-6)$$

$$u(x) = \lambda \int_a^x k(x,t)u(t)\,dt \qquad (1-7)$$

Is called homogenous Fredholm or homogenous Volterra integral equation respectively :

It is interesting to point out that any equation that includes both integrals and derivatives of the unknown function $u(x)$ is called integro−differential *equation.* The Fredholm integro−differential equation is of the form:

$$u^k(x) = f(x) + \lambda \int_a^b k(x,t)u(t)dt\,,\ u^k = \frac{d^k u}{dx^k} \qquad (1-8))$$

However , the Volterra integro−differential *equation* is of the form :

$$u^k(x) = f(x) + \lambda \int_a^x k(x,t)u(t)dt \text{ , } u^k = \frac{d^k u}{dx^k} \qquad (1-9)$$

1.2. Classification of Integral Equation :

Integral equations appear in many types. The types depend mainaly on the limits of integration and the kernel of the equation. We will be concerned on the following types of integral equations .

1.2.1. Fredholm Integral Equations :

For Fredholm integral equations, the limits of the integration are fixed . moreover , the unknown function $u(x)$ may appear only inside integral equation in the form :

$$f(x) = \int_a^b k(x,t)u(t)\, dt \qquad (1-10)$$

This is called Fredholm integral equations , of the first kind .however , for the Fredholm integral equations of the second kind , the unknown function $\mathrm{u(x)}$appear inside and outside the integral sign. The second kind is represented by the form:

$$\mathrm{u(x)} = \mathrm{f(x)} + \lambda \int_a^b k(x,t)u(t)\, dt \qquad (1-11)$$

Example *of the two kinds are given by*

$$\frac{sinx - x\ cosx}{x^2} = \int_0^1 \sin(xt)\, u(t)\, dt,$$

And

$$u(x) = x + \frac{1}{2}\int_{-1}^1 (x-t)u(t)dt \text{ ,}$$

Respectively .

1.2.2 Volterra Integral Equations :

In Volterra integral equations, at the least one of the limits of integration is a variable. For the first kind Volterra integral equations, the unknown function $u(x)$ appears only inside integral sign in the form:

$$f(x) = \int_0^x k(x,t)u(t)\,dt \qquad (1-12)$$

However, Volterra integral equations of the second kind, the unknown function $u(x)$ appears inside and outside the integral sign in the form :

$$u(x) = f(x) + \lambda \int_0^x k(x,t)u(t)\,dt \qquad (1-13)$$

Examples of the Volterra integral equation of the first kind are

$$xe^{-x} = \int_0^x e^{t-x}\,u(t)dt \qquad (1-14)$$

$$and$$

$$5x^2 + x^3 = \int_0^x (5+3x-3t)\,u(t)dt \qquad (1-15)$$

However, examples of the Volterra integral equations of the second kind are

$$u(x) = 1 - \int_0^x u(t)\,dt\,,$$

and

$$u(x) = x + \int_0^x (x-t)\,dt\,.$$

1.2.3 Volterra – Fredhplm Integral Equations :

The Volterra – Fredhplm integral equations arise from parabolic boundary value problems, from the mathematical modeling of the spatio- temporal development of an epidemic, and from various physical and biological models. The Volterra – Fredhplm integral equations appear in the literature in two forms, namely

$$u(x) = f(x) + \lambda_1 \int_a^x K_1(x,t)\,u(t)dt + \lambda_2 \int_a^b K_2(x,t)\,u(t)dt \qquad (1-16)$$

And

$$u(x,t) = f(x,t) + \lambda \int_0^t \int_\Omega F\big(x,t,\xi,\tau,u(\xi,\tau)\big) d\xi d\tau,\ (x,t)\epsilon\Omega \times [0,\mathrm{T}] \quad (1-17)$$

Where $f(x,t)$ and $F\big(x,t,\xi,\tau,u(\xi,\tau)\big)$ are analytic function on $D = \Omega \times [0,\mathrm{T}]$ and Ω is a closed subset of $R^n, n = 1,2,3$. it is interesting to note that (1-17) contains disjoint Volterra and Fredholm integral equations. Moreover, the unknown functions $u(x)\ and\ u(x,t)$ appear inside and outside the integral signs .

Examples of the two types are given by

$$u(x) = 6x + 3x^2 + 2 - \int_0^1 x\, u(t)dt - \int_0^1 t\, u(t)dt$$

And

$$u(x,t) = x + t^3 + \frac{1}{2}t^2 - \frac{1}{2}t - \int_0^t \int_0^1 (\tau - \xi)\, d\xi d\tau$$

1.2.3 Singular Integral Equations:

Volterra integral equation of the first kind

$$f(x) = \lambda \int_{g(x)}^{h(x)} k(x,t)\, u(t)dt$$

Or of the second kind

$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} k(x,t)\, u(t)dt$$

Are called singular if one of the limits of integration $g(x), h(x)$ or both are infinite . Moreover, the previous two equations are called singular if the kernel $k(x,t)$ become unbounded at one or more points in the interval of integration. We will focus our concern on equations of the form:

$$f(x) = \int_0^x \frac{1}{(x-t)^\alpha} u(t)dt\,, \quad 0 < \alpha < 1$$

Or of the second kind:

$$u(x) = f(x) + \int_0^x \frac{1}{(x-t)^\alpha} u(t)dt\,, \quad 0 < \alpha < 1$$

The last two standard forms are called generalized Abel's integral equation and weakly singular integral equation respectively .For $\alpha = \frac{1}{2}$,the equation:

$$f(x) = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt$$

Is called the Abel's singular integral equation. It is to be noted that the kernel in each equation, generalized Abel's integral equation, and the weakly singular integral equation are given by

$$\sqrt{x} = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt$$

$$x^3 = \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}} u(t)dt$$

and

$$u(x) = 1 + \sqrt{x} + \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}} u(t)dt$$

respectively.

1.3 Classification of Integro – Differential Equations :

Integro-differential equations appear in many scientific applications, especially when we convert initial value problems or boundary value problems to integral equations. The integro-differential equations contain both integral and differential operators. The derivatives of the unknown functions may appear to any order . in classifying integro-differential equations , we will follow the same category used before .

1.3.1 Fredholm Integro- Differential Equations :

Fredholm integro- differential equations appear when we convert differential equations to integral equations. The Fredholm integro- differential equations contains the unknown function $u(x)$ and one of its derivatives $u^n(x)$, $n \geq 1$ inside and outside the integral sign respectively. The limits of integration in the case are fixed as in the Fredholm integral equations. The equation is labeled as Integro- differential because it contains differential and integral operators in the same equation.

The Fredholm integro- differential equations appears in the form:

$$u^n(x) = f(x) + \lambda \int_a^b k(x,t)u(t)dt \,, \qquad (1-18)$$

Where u^n indicates the nth derivative of (x). Other derivatives of less order may appear with u^n at the left side. Example of the Fredholm integro-differential equations are given by:

$$u'(x) = 1 - \frac{1}{3}x + \int_0^1 xu(t)dt\,, \quad u(0) = 0$$

And

$$u''(x) + u'(x) = x - sinx - \int_0^{\frac{\pi}{2}} xu(t)dt\,, \quad u(0) = 0\,, u'(0) = 1$$

1.3.2 Volterra Integro- Differential Equations :

Volterra integro- differential equations appear when we convert initial value problems to integral equations. The Volterra integro - differential equation contains the unknown function $u(x)$ and one of its derivatives $u^n(x)\,, n \geq 1$ inside and outside the integral sign. At least one of the limits of integration in this case is a variable as in the Volterea integral equations. The equation is called integro – differential because differential and integral operations are involved in the same equation . The Volterra integro - differential equation appear in the form:

$$u^n(x) = f(x) + \lambda \int_0^x k(x,t)u(t)dt\,, \qquad (1-19)$$

Where u^n indicates the nth derivative of (x). Other derivative of less order may appear with u^n at the left side. Example of the Volterra integro – differential equations are given by

$$u'(x) = -1 + \frac{1}{2}x^2 - xe^x - \int_0^x tu(t)dt\,, \quad u(0) = 0$$

And

$$u''(x) + u'(x) = 1 - x(sinx + cosx) - \int_0^x t\,u(t)dt \quad u(0) = -1\,, u'(0) = 1$$

1.3.3 Volterra – Fredholm Integro – Differential Equations :

The Volterra – Fredholm integro-differential equations arise in the same manner as Volterra–Fredholm integral equations with one or more of ordinary derivatives in addition to the integral operations. The Volterra—Fredholm integro-differential equations appear in the literature in two forms, namely

$$u^n(x) = f(x) + \lambda_1 \int_a^x k_1\,(x,t)u(t)dt + \lambda_2 \int_a^b k_2\,(x,t)u(t)dt\,, \quad (1-20)$$

And

$$u^n(x,t) = f(x,t) + \lambda \int_0^t \int_\Omega F(x,t,\xi,\tau,u(\xi,\tau)d\xi d\tau, \quad (x,t) \in \Omega \times [0,\mathrm{T}] \qquad (1-21)$$

Where $f(x,t)$ and $F(x,t,\xi,\tau,u(\xi,\tau))$ are analytical functions on $\mathrm{D} = \Omega \times [0,\mathrm{T}]$, and Ω is a closed subset of R^n ,$n = 1,2,3$. It is interesting to note that (1-20) contains disjoint Volterra and Fredholm integral equations, where as (1-21) contains mixed integrals. Other derivatives of less order may appear as well.

Moreover, the unknown functions $u(x) and\ u(x,t)$ appear inside and outside the integral sings. This is a characteristic feature of a second kind integral equation. If the unknown functions appear only inside the integral sings , the resulting equation are of the first kind . Initial conditions should be given to determine the particular solution. Examples of the two types are given by

$$u'(x) = 24x + x^4 + 3 - \int_0^x (x-t)\,u(t)dt - \int_0^1 t\,u(t)dt\,, \ \ u(0) = 0$$

And

$$u'(x,t) = 1 + t^3 + \frac{1}{2}t^2 - \frac{1}{2}t - \int_0^t \int_0^1 (\tau - \xi)d\xi d\tau\,,\ u(0,t) = t^3.$$

1.4 Linearity and Homogeneity :

integral equations and integro—differential equations fall into other types of classifications according to linearity and homogeneity concepts. These two concepts play amajor role in the structure of the solutions. In what highlight the definitions of these concepts.

1.4.1 Linearity Concept .

If the exponent of the unknown function $u(x)$ inside the integral sign is one , the integral equations or the integro--differential equation is called linear. If the unknown function $u(x)$ has exponent other than one , or if the equation contains nonlinear functions of $u(x)$, such as $e^u, \sinh u$,$\cos u$,$\ln(1+u)$, the integral equation or the integro—differeential equation is called nonlinear. To explain this concept , we consider the equation :

$$u(x) = 1 - \int_0^x (x - t)\, u(t) dt \, ,$$

$$u(x) = 1 - \int_0^1 (x - t)\, u(t) dt \, ,$$

$$u(x) = 1 - \int_0^x (1 + x - t)\, u^4(t)\, dt \; ,$$

and

$$u'(x) = 1 + \int_0^1 xte^{u(t)}\, dt \, , u(0) = 1 \, .$$

The first two examples are linear Volterra and Fredholm integral equations respectively, whereas the last two are nonlinear Volterra and Fredholm integral equations respectively.
The important to point out that linear equations, except Fredholm integral equation of the first kind, give a unique solution exists. However, solution of nonlinear equation may not be unique. Nonlinear equations usually give more than one solution and it is not usually easy to handle.
Both linear and nonlinear integral equations of any kind will be investigated by using traditional and new methods .

1.4.2 Homogeneity Concept :

Integral equations and integro—differential equations of the second kind are classified as homogeneous or inhomogeneous, if the function $f(x)$ in the second kind of Volterra or Fredholm integral equations or the integro—differential equations is identically zero , the equation is called homogeneous. Otherwise it is called inhomogeneous. Notice that this property holds for equations of the second kind only. To clarify this concept we consider the following equations :

$$u(x) = \sin x + \int_0^x (xt)\, u(t) dt \, ,$$

$$u(x) = x + \int_0^1 (x - t)^2\, u(t) dt \, ,$$

$$u(x)=\int_0^x (1+x-t)\,u^4(t)\,dt\ ,$$

And

$$u''(x)=\int_0^1 xtu(t)\,dt\,,\qquad u(0)=1\quad u'(0)=0\,.$$

The first two equations are inhomogeneous because $f(x)=\sin x\ \ and\ f(x)=x$, where the last two equations are homogenous because $f(x)=0$ for each equation.

1.5 Converting IVP to Volterra Integral Equation :

We will study the technique that will convert an initial value problem(IVP) to an equivalent Volterra integral equation and Volterra integro-differential equation .For the simplicity reasons , we will apply this process to second order initial value problem given by

$$y''(\mathrm{x})+\mathrm{p}(\mathrm{x})\mathrm{y}'(\mathrm{x})+\mathrm{q}(\mathrm{x})\mathrm{y}(\mathrm{x})=\mathrm{g}(\mathrm{x})\qquad(1-22)$$

Subject to the initial conditions:

$$y(0)=\alpha\quad,\quad y'(0)=\beta\qquad\qquad(1-23)$$

Where α and β are constants. The function $\mathrm{p(x)}$ and $\mathrm{q(x)}$ are analytic functions , and $\mathrm{g(x)}$ is continuous through the interval of discussion. The achieve our goal we first set

$$y''(x)=u(x),\qquad(1-24)$$

Where $\mathrm{u(x)}$ is continuous function. Integrating both sides of (2-24) from 0 to x yields

$$y'(x)-y'(0)=\int_0^x u(x)dt\qquad(1-25)$$

Or equivalently

$$y'(x)=\beta+\int_0^x u(t)dt\,.\qquad(1-26)$$

Integrating both sides of (1-26) from 0 to x yields

$$y(x) - y(0) = \beta x + \int_0^x \int_0^x u(t)dt\,dt\,, \qquad (1-27)$$

Or equivalently

$$y(x) = \alpha + \beta x + \int_0^x (x-t)u(t)\,dt \qquad (1-28)$$

Substituting (1-23) ,(1-26) and (1-28) into the initial value problem (1-22) yields the Volterra integral equation:

$$u(x) + p(x)\left[\beta + \int_0^x u(t)dt\right] + q(x)\left[\alpha + \beta x + \int_0^x (x-t)u(t)dt\right] = g(x) \qquad (1-29)$$

The last equation can be written in the standard Volterra integral equation form :

$$u(x) = f(x) - \int_0^x k(x,t)\,u(t)dt\,, \qquad (1-30)$$

where

$$k(x,t) = p(x) + q(x)(x-t)$$

And

$$f(x) = g(x) - [\beta p(x) + \alpha q(x) + \beta x q(x)]$$

It is interesting to point out that by differentiating Volterra equation (1-30) with respect to, using Leibnitz rule , we obtain and equivalent Volterra integro –differential equation in the form:

$$u'(x) + k(x,x)u(x) = f'(x) - \int_0^x \frac{\partial k(x,t)}{\partial x} u(t)dt \;\; u(0) = f(0) \quad 1-31)$$

The technique presented above to convert initial value proplems to equivalent Volterra integral equations can be generalized be considering the general initial value problem:

$$y^n + a_1(x)y^{(n-1)} + \cdots + a_{n-1}y' + a_n(x)y = g(x) \qquad (1-32)$$

Subject to the initial conditions

$$y(0) = c_0\;, y'(0) = c_1\;, y''(0) = c_2\,, \ldots, y^{(n-1)}(0) = c_{n-1}$$

We assume that the functions $a_i(x),\ 1 \leq i \leq n$ are analytic at the origin, and the function $g(x)$ is the continuous through the interval of the discussion. And we then consider the transformation:

$$y^n(x) = u(x) \qquad (1-33)$$

Integrating both sides with respect to x gives

$$y^{(n-1)}(x) = c_{n-1} + \int_0^x u(t)\,dt \qquad (1-34)$$

Integrating again both sides with respect to x yields

$$y^{(n-2)}(x) = c_{n-2} + c_{n-1}x + \int_0^x \int_0^x u(t)\,dtdt$$
$$= c_{n-2} + c_{n-1}x + \int_0^x (x-t)\,u(t)dt \quad (1-35)$$

Obtained by reducing the double integral to a single integral. Proceeding as before we find

$$y^{(n-3)}(x) = c_{n-3} + c_{n-2}x + \frac{1}{2}c_{n-1}x^2 + \int_0^x \int_0^x \int_0^x u(t)\,dtdtdt$$
$$= c_{n-3} + c_{n-2}x + \frac{1}{2}c_{n-1}x^2 + \frac{1}{2}\int_0^x (x-t)^2\,u(t)dt \ (1-36)$$

Continuing the integration process leads to

$$y(x) = \sum_{k=0}^{n-1} \frac{C_k}{k!}x^k + \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1}\,u(t)dt \quad (1-37)$$

Substituting (1-33)—(1-37) into (1-32) gives

$$u(x) = f(x) - \int_0^x K(x,t)\,u(t)dt \qquad (1-38)$$

Where

$$K(x,t) = \sum_{k=1}^{n} \frac{a_n}{(k-1)!}(x-t)^{k-1} \quad (1-39)$$

And

$$f(x) = g(x) - \sum_{j=1}^{n} a_j \left(\sum_{k=1}^{j} \frac{c_{n-k}}{(j-k)!} (x)^{j-k} \right) \quad (1-40)$$

The following examples will highlight the process to convert initial value problem to an equivalent Volterra integral equation.

Example (1.1): Convert the following initial value problem to an equivalent Volterra integral equation:

$$y'(x) - 2xy(x) = e^{x^2} \ , \quad y(0) = 1 \qquad (1-41)$$

We first set

$$y'(x) = u(x) \qquad (1-42)$$

Integrating both side of (1-42) , using the initial condition $y(0) = 1$ gives

$$y(x) - y(0) = \int_0^x u(t)dt \, , \qquad (1-43)$$

or equivalently

$$y(x) = 1 + \int_0^x u(t)\, dt \, , \qquad (1-44)$$

substituting (1-42) and (1-44) into (1-41) gives the equivalent Volterra integral equation:

$$u(x) = 2x + e^{x^2} + 2x \int_0^x u(t)\, dt. \qquad (1-45)$$

Example (1.2): Convert the following initial value problem to an equivalent Volterra integral equation:

$$y''' - y'' - y' + y = 0, \ y(0) = 1 \, , y'(0) = 2, y''(0) = 3 \quad (1-46)$$

We first let

$$y''' = u(x)$$

Integration both side

$$\int_0^x d(y''(x)) = \int_0^x u(t)dt$$

$$y'' = \int_0^x u(t)dt$$

Using the initial condition $y''(0) = 3$ we obtain

$$y'' = 3 + \int_0^x u(t)dt$$

The integration again

$$\int_0^x d\big(y'(x)\big) = \int 3\,dx + \int_0^x \int_0^x u(t)dtdt$$

Using the condition $y'(0) = 2$

$$y'(x) = 2 + 3x + \int_0^x \int_0^x u(t)dt\,dt$$

Integration again

$$\int_0^x d(y(x)) = \int_0^x (2+3x)dx + \int_0^x \int_0^x \int_0^x u(t)dt\,dt\,dt$$

Using initial condition $y(0) = 1$ obtain

$$y(x) = 1 + 2x + \frac{3}{2}x^2 + \int_0^x \int_0^x \int_0^x u(t)dt\,dt\,dt$$

$$= 1 + 2x + \frac{3}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2 u(t)dt$$

1.6 Converting Volterra Integral Equation to IVP

We will present a method that will convert a Volterra integral equation to equivalent IVP. The mothed is achieved simply by differentiating both sides of the Volterra equations with respect to x as many times as we need to get rid of the integral sign and come out a differential equation. The conversing of the Volterra equations requires the ues of Leibnitz rule for differentiating the integral at the right hand side. The initial conditions can be obtained by substituting $x = 0$ into $u(x)$ andits derivatives.

Example (1.3): Find the initial value problem equivalent to the Volterra integral equation :

$$u(x) = x^2 + \int_0^x (x-t)u(t)dt \qquad (1-47)$$

Differentiating both side of (1-47)

$$u'(x) = 2x + \int_0^x u(t)dt \qquad (1-48)$$

The get rid of the integral sing we should differentiate (1-48) and by using Leibnitz rule we obtain the second order ODE :

$$u''(x) = 2 + u(x) \qquad (1-49)$$

To determine the initial conditions , we substitute $x = 0$ into both sides of (1-47) and (1-48) to find $u(0) = 0$ and $u'(0) = 0$ respectively. This is turn gives the initial value problem :

$$u''(x) - u(x) = 2\,, u(0) = 0\,, u'(0) = 0\,.$$

Example (1.4): Find the initial value problem equivalent to the Volterra integral equation:

$$u(x) = \sin x - \frac{1}{2}\int_0^x (x-t)^2 u(t)\,dt \qquad (1-50)$$

Differentiating both sides of the integral equation three times to get rid of the integral sign to find

$$u'(x) = \cos x - \int_0^x (x-t)\,u(t)dt\,,$$

$$u''(x) = -\sin x - \int_0^x u(t)\,dt\,,$$

$$u'''(x) = -\cos x - u(x) \qquad (1-51)$$

Substituting $x = 0$ into (1-50) and into the first two integro—differential equation in (1-51) gives the initial conditions:

$$u(0) = 0 \ , u'(0) = 1 \ , \ \ u''(0) = 0 \quad (1-52)$$

In view of the last results , the initial value problem equivalent to Volterra integral equation (1-50) is the third order inhomogeneous ODE given by

$$u'''(x) + u(x) = -cosx \ ,, u(0) = 0 \ , u'(0) = 1 \ , u''(0) = 0$$

1.7 Converting BVP to Fredholm Integral Equation

We will present a method that will convert a boundary value problem to an equivalent Fredholm integral equation. The method is similar previous converting Volterra equation to IVB, with the exception that boundary conditions will be used instead of ignition values. In this case we will determine another initial condition that is not given in the problem. The technique requires more work if compared with the initial value problems when converted to Volterra integral equations .For this reason , the technique that will be presented is rarely used. We will present two specific distinct boundary value problems (BVPs) to derive two distinct formulas that can be used for converting BVP to an equivalent Fredholm integral equation .

Type 1

We first consider the following boundary value problem:

$$y''(x) + g(x)y(x) = h(x), \quad 0 < x < 1 \ , \quad (1-53)$$

With the boundary conditions :

$$y(0) = \alpha \ , \qquad y(1) = \beta \qquad (1-54)$$

We first set

$$y''(x) = u(x). \qquad (1-55)$$

Integrating both sides of (1-55) from 0 to x we obtain

$$\int_0^x y''(x)dt = \int_0^x u(t)\, dt \qquad (1-56)$$

That gives

$$y'(x) = y'(0) + \int_0^x u(t)\,dt\,, \qquad (1-57)$$

Where the initial condition $y'(0)$ is not given in a boundary value problem. The condition $y'(0)$ will be determined later by using the boundary condition at $x = 1$. Integrating both sides of (1-57) from 0 to x gives

$$y(x) = y(0) + xy'(0) + \int_0^x \int_0^x u(t)\,dt \qquad (1-58)$$

Or equivalently

$$y(x) = \alpha + xy'(0) + \int_0^x (x-t)\,u(t)dt\,, \qquad (1-59)$$

Obtained upon using the condition $y(0) = \alpha$ and by reducing double integral to asingle integral. To determine $y'(1) = \beta$ we find

$$y(1) = \alpha + y'(0) + \int_0^1 (1-t)\,u(t)dt\,, \qquad (1-60)$$

That gives

$$\beta = \alpha y'(0) + \int_0^1 (1-t)u(t)\,dt \qquad (1-61)$$

This in turn gives

$$y'(0) = (\beta - \alpha) - \int_0^1 (1-t)u(t)\,dt \qquad (1-62)$$

Substituting (1-62) into (1-59) gives

$$y(x) = \alpha + (\beta - \alpha)x - \int_0^1 x(1-t)u(t)dt + \int_0^x (x-t)u(t)\,dt\,. \quad (1-63)$$

Substituting (1-55)and (1-63) into (1-53) gives

$$u(x) + \alpha g(x) + (\beta - \alpha)xg(x) - \int_0^1 xg(x)(1-t)u(t)\,dt + \int_0^x g(x)\,(x-t)u(t)dt = h(x) \qquad (1-66)$$

To carry Equation (1-64) to

$$u(x) = h(x) - \alpha g(x) - (\beta - \alpha)xg(x) - g(x)\int_0^x (x-t)u(t)dt + xg(x)[\int_0^x (1-t)u(t)dt + \int_x^1 (1-t)u(t)dt], \quad (1-65)$$

That gives

$$u(x) = f(x) + \int_0^x t(1-x)g(x)u(t)dt + \int_x^1 x(1-t)\, g(x)u(t)dt \quad (1-66)$$

That leads to the Fredholm integral equation :

$$u(x) = f(x) + \int_0^x K(x,t)\, u(t)dt \qquad (1-67)$$

Where

$$f(x) = h(x) - \alpha g(x) - (\beta - \alpha)xg(x), \qquad (1-68)$$

And the kernel $K(x,t)$ is given by

$$K(x,t) = \begin{cases} t(1-x)g(x), & for\ 0 \le t \le x \\ x(1-t)g(x), & for\ \ x \le t \le 1 \end{cases} \qquad (1-69)$$

Example (1.4) Convert the following BVP to an equivalent Fredholm integral equation :

$$y''(x) + 9y(x) = \cos x\ , \quad y(0) = y(1) = 0 \qquad (1-70)$$

We can easily observe that $\alpha = \beta = 0\ , g(x) = 9$ and $h(x) = cosx$. This in turn gives

$$f(x) = \cos x \qquad (1-71)$$

Substituting this into (1-67) gives the Fredholm integral equation

$$u(x) = -2x^2 + \int_0^1 K(x,t)\, u(t)dt\ , \quad (1-72)$$

Where the kernel $k(x,t)$ is given by

$$k(x,t)=\begin{cases} tx(1-x), & for\ 0\le t\le x \\ x^2(1-t), & for\ x\le t\le 1 \end{cases} \qquad (1-73)$$

Type 2

We next consider the following boundary value problem:

$$y''(x)+g(x)y(x)=h(x)\ \ ,\ \ 0<x<1 \qquad (1-74)$$

With the boundary conditions:

$$y(0)=\alpha_1\ , \qquad y'(1)=\beta_1 \qquad (1-75)$$

We again set

$$y''(x)=u(x) \qquad (1-76)$$

Integration both sides of (1-76) from $0\ to\ x$we obtain

$$\int_0^x y''(x)\,dt=\int_0^x u(t)\,dt \qquad (1-77)$$

That gives

$$y'(x)=y'(0)+\int_0^x u(t)\,dt \qquad (1-78)$$

Where the initial condition $y'(0)$ is not given. The condition $y'(0)$ will be derived later by using the boundary condition at $y'(1)=\beta_1$. Integrating both sides of (1-78) from $0\ to\ x$ gives

$$y(x)=y(0)+xy'(0)+\int_0^x\int_0^x u(t)\,dt\,dt \qquad (1-79)$$

Or equivalently

$$y(x)=\alpha_1+xy'(0)+\int_0^x (x-t)\,u(t)dt \qquad (1-80)$$

Obtained upon using the condition $y(0)=\alpha_1$ and by reducing double integral to a single integral. To determine $y'(0)$, we first differentiate (1-80) with respect to x this get

$$y'(x)=y'(0)+\int_0^x u(x)\,dt \qquad (1-81)$$

Where by substituting $x = 1$ into both sides of (1-81) and using the boundary condition at $y'(1) = \beta_1$ we find

$$y'(1) = y'(0) + \int_0^1 u(t)dt \qquad (1-82)$$

That gives

$$y'(0) = \beta_1 - \int_0^1 u(t)\,dt. \qquad (1-83)$$

Using (1-83) into (1-80) gives

$$y(x) = \alpha_1 + x[\beta_1 - \int_0^1 u(t)dt + \int_0^x (x-t)\,u(t)dt \qquad (1-84)$$

Substituting (1-76) and (1-84) into (1-74) yields

$$u(x) + \alpha_1 g(x) + \beta_1 x g(x) - \int_0^1 xg(x)\,u(t)dt + \int_0^x g(x)\,(x-t)u(t)dt$$
$$= h(x). \qquad (1-85)$$

To carry equation (1-85) to

$$u(x) = h(x) - (\alpha_1 + \beta_1 x)g(x) + xg(x)\left[\int_0^x u(t)dt + \int_x^1 u(t)dt\right]$$
$$- g(x)\int_0^x (x-t)\,u(t)dt \qquad (1-86)$$

That last equation can be written as

$$u(x) = f(x) + \int_0^x t\,g(x)u(x)dt + \int_x^1 xg(x)u(t)\,dt \quad (1-87)$$

That leads to the Fredholm integral equation:

$$u(x) = f(x) + \int_0^1 k(x,t)\,u(t)dt \qquad (1-88)$$

where $\qquad f(x) = h(x) - (\alpha_1 + \beta_1 x)g(x), \qquad (1-89)$

and the kernel $K(x,t)$ is given by

$$k(x,t) = \begin{cases} tg(x), & for\ 0 \le t \le x \\ xg(x), & for\ \ x \le t \le 1 \end{cases} \qquad (1-90)$$

Chapter2

Existence of Solution of Linear Integral Equations

2.1 Fredholm Integral Equations

It was stated in chapter 1 that Fredholm Integral Equation arise in many scientific applications. It was also shown that Fredholm integral equation can by derived from boundary value problems.

At stated before, in Fredholm integral equation , the integral containing the unknown function $u(x)$ is characterized by fixed limits of the integration in the form

$$u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)dt \qquad (2-1)$$

Where $a \text{ and } b$ are constants. For the first kind Fredholm integral equation the unknown function $u(x)$ occurs only under the integral sign in the form

$$f(x) = \int_a^b k(x,t)u(t)dt \qquad (2-2)$$

However , Fredholm integral equation of the second kind ,the unknown function $u(x)$ occurs inside and outside the integral sign. The second kind is represented by the form

$$u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)dt$$

The kernel $k(x,t)$ and the function $f(x)$ are given real valued function and λ is a parameter. When $f(x) = 0$, the equation is said to be homogeneous.

In this chapter , we will mostly use degenerate or separable kernels. degenerate or separable kernel is a function that can be expressed as the sum of the product of two functions each depends only on one variable. Such a kernel can be expressed in the form

$$k(x,t) = \sum_{i=1}^{n} f_i(x)g_i(t). \qquad (2-3)$$

examples of separable kernels are $x - t, (x - t)^2, 4xt,\ \ etc.$

Theorem(2-1):(Fredholm Alternative Theorem)

If the homogeneous Fredholm Integral Equation

$$u(x) = \lambda \int_a^b k(x,t)u(t)dt \qquad (2-4)$$

Has only the trivial solution $u(x) = 0$,then the corresponding nonhomogeneous Fredholm equation

$$u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)dt \qquad (2-5)$$

Has always a unique solution. This theorem is known by the Fredholm alternative theorem[1].

Theorem(2-2):(unique solution)

If the kernel $k(x,t)$ in Fredholm integral equation (2-1) is the continuous ,real valued function, then anecessary condition for the existence of a unique solution for Fredholm integral equation (2-1) is given by

$$|\lambda| M(b-a) < 1 \qquad (2-6)$$

Where

$$|k(x,t)| \leq M \in R \qquad (2-7)$$

On the contrary, if the necessary condition (2-6) does not hold, then a continuous solution may exist for FIE. To illustrate this, we consider the FIE

$$u(x) = 2 - 3x + \int_0^1 (3x + t)\, u(t)dt \qquad (2-8)$$

It is clear that $\lambda = 1, |k(x,t)| \leq 4\ and\ (b-a) = 1.$ this gives

$$|\lambda| M(b-a) = 4 > 1 \qquad (2-9)$$

However, the Fredholm equation (2-8) has an exact solution given by

$$u(x) = 6x \qquad (2-10)$$

A variety of analytic and numerical methods have been used to handle FIE. The direct computation method, the successive approximations method, and converting FIE to an equivalent BVP are among many traditional methods that were commonly used. However, we will apply the recently developed methods, namely, the Adomian decomposition method (ADM), the modified Adomian decomposition method (MADM), and the variation iteration method (VIM) to handly the FIE some of the traditional methods.

In what follows we will present the methods , new and traditional,that will be used to handle the FIE (2-1)

2.2 Adomian Decomposition Method:

The Adomian Decomposition Method consist of decomposing the unknown function $u(x)$ of any equation into a sum of an infinite number of components defined by the decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \qquad (2-11)$$

Or equivalently

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots \quad (2-12)$$

Where the components $u_n(x), n \geq 0$ will be determined recurrently. The ADM concerns itself with finding the components $u_0(x), u_1(x), u_2(x), \ldots$ individually. As we have seen before the determination of these components can be achieved in an easy way through a recurrence relation that usually involves simple integrals that can be easily evaluated.

To establish the recurrence relation , we substitute $(2-11)$ into the FIE$(2-1)$ to obtain

$$\sum_{n=1}^{\infty} u_n(x) = f(x) + \lambda \int_a^b k(x,t)(\sum_{n=0}^{\infty} u_n(t))dt \quad (2-13)$$

Or equivalently

$$u_0(x)+u_1(x)+u_2(x)+\cdots$$
$$=\lambda\int_a^b k(x,t)\big(u_0(t)+u_1(t)+u_2(t)\big)dt \quad (2-14)$$

The zeroth component $u_0(x)$ is identified by all terms that are not included under the integral sing.This means that the components $u_j(x)\,, j\geq 0$ of the unknown function $u(x)$ are completely determined by setting the recurrence relation

$$u_0(x)=f(x).u_{n+1}(x)=\lambda\int_a^b k(x,t)\big(u_n(t)\big)dt\ ,n\geq 0 \qquad (2-15)$$

Or equivalently

$$u_0(x)=f(x),$$
$$u_1(x)=\lambda\int_a^b k(x,t)\big(u_0(t)\big)dt,$$
$$u_2(x)=\lambda\int_a^b k(x,t)\big(u_1(t)\big)dt, \qquad (2-16)$$
$$u_3(x)=\lambda\int_a^b k(x,t)\big(u_2(t)\big)dt,$$

And so on other components.

In view of $(2-16)$ the components $u_0(x), u_1(x), u_2(x)$, ... are completely determined .As a result , the solution $u(x)$ of the FIE is readily obtained in series form by using series as sumption in $(2-12)$.

Example (2-1): Solve the following FIE

$$\mathrm{u(x)=e^x-x+x\int_0^1 tu(t)\,dt} \qquad (2-17)$$

The ADM assumes that the solution $u(x)$ has a series form given in $(2-11)$. Substituting the decomposition series $(2-11)$ into both sides of $(2-17)$ gives

$$\sum_{n=0}^{\infty}\mathrm{u_n(x)=e^x-x+x\int_0^1 t\sum_{n=0}^{\infty}u_n(t)\,dt} \qquad (2-18)$$

Or equivalently

$$\mathrm{u_0(x) + u_1(x) + u_2(x) + \cdots}$$
$$\mathrm{= e^x - x}$$
$$\mathrm{+ x \int_a^b t(u_0(t) + u_1(t) + u_2(t) + \cdots)dt} \quad (2-19)$$

We identify the zeroth component by all terms that are not included under the integral sing. Therefore , we obtain the following recurrence relation

$$u_0(x) = e^x - x \,, u_{k+1}(x) = x\int_0^1 t\big(u_k(t)\big)dt \;,\; k \geq 0 \qquad (2-20)$$

Consequently, we obtain

$$u_0(x) = e^x - x$$
$$u_1(x) = x\int_0^1 t\big(u_0(t)\big)dt = x\int_0^1 t(e^t - t)dt = \frac{2}{3}x\,,$$
$$u_2(x) = x\int_0^1 t\big(u_1(t)\big)dt = x\int_0^1 \frac{2}{3}t^2dt = \frac{2}{9}x\,, \qquad (2-21)$$
$$u_3(x) = x\int_0^1 t\big(u_2(t)\big)dt = x\int_0^1 \frac{2}{9}t^2dt = \frac{2}{27}x\,,$$
$$u_4(x) = x\int_0^1 t\big(u_3(t)\big)dt = x\int_0^1 \frac{2}{27}t^2dt = \frac{2}{81}x\,,$$

And so on. Using (2-11) gives the series solution

$$u(x) = e^x - x + \frac{2}{3}x\left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots\right) \qquad (2-22)$$

Notice that the infinite geometric series at the right side has $a_1 = 1$,

and the ratio $r = \frac{1}{3}$. The sum of the infinite series is therefore given by

$$s = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2} \qquad (2-23)$$

The series solution (2--22) converges to the closed form solution

$$u(x) = e^x \qquad (2-24).$$

Example (2-2): Consider the Fredholm integral equation of second kind

$$y(x) = \frac{1}{2}\cos x + \int_0^{\frac{\pi}{2}} \cos x \sin t \; y(t)dt .$$

Applying the ADM we find

$$\sum_{n=0}^{\infty} y_n(x) = \frac{1}{2}\cos x + \int_0^{\frac{\pi}{2}} \cos x \sin t \sum_{n=0}^{\infty} y_n(t)\, dt .$$

To determine the components of $y(x)$, we use the recurrence relation

$$y_0(x) = \frac{1}{2}\cos x \quad , y_{n+1}(x) = \int_0^{\frac{\pi}{2}} \cos x \sin t \; y_n(t)dt . \quad n \geq 0.$$

This in turn gives

$$y_0(x) = \frac{1}{2}\cos x$$

$$y_1(x) = \int_0^{\frac{\pi}{2}} \cos x \sin t \; y_0(t)dt = \frac{1}{4}\cos x .$$

$$y_2(x) = \int_0^{\frac{\pi}{2}} \cos x \sin t \; y_1(t)dt = \frac{1}{8}\cos x .$$

$$y_3(x) = \int_0^{\frac{\pi}{2}} \cos x \sin t \; y_2(t)dt = \frac{1}{16}\cos x .$$

$$y_4(x) = \int_0^{\frac{\pi}{2}} \cos x \sin t \; y_3(t)dt = \frac{1}{32}\cos x .$$

$$y_5(x) = \int_0^{\frac{\pi}{2}} \cos x \sin t \; y_4(t)dt = \frac{1}{64}\cos x .$$

$$y_6(x) = \int_0^{\frac{\pi}{2}} \cos x \sin t \; y_5(t)dt = \frac{1}{128}\cos x .$$

And so on. Using (2-11) gives the series solution

$$y(x) = \frac{1}{4}\cos x + \frac{1}{8}\cos x + \frac{1}{16}\cos x + \frac{1}{32}\cos x + \frac{1}{64}\cos x + \frac{1}{128}\cos x + \cdots$$

Then the gives the exact solution

$$y(x) = \frac{8191}{8192}\cos x$$

2.1.2. Modified Decomposition Method

The Adomain decomposition method (ADM) provides the solutions in an infinite series of components. The components $u_j, j \geq 0$ are essily computed if the inhomogeneous term $f(x)$in the FIE

$$u(x) = f(x) + \lambda \int_a^b k(x,t)\, u(t)dt , \qquad (2-25)$$

Consists of a polynomial of one or two terms. However ,if the function $f(x)$ consists of a combination of two or more of polynomials, trigonometric function, hyperbolic functions, and other, the evaluation of the components $u_j, j \geq 0$ requires more work. A relation modification of the ADMs presented and used in before chapter, and it was shown that this modification facilitates the computational work and accelerates the convergence of the series solution. As presented before, the modified decomposition method(MDM) depends mainly on splitting the function $f(x)$consists of only term. the MDM will be briefly outlined here. The standard ADM employs the recurrence relation

$$u_0(x) = f(x).$$

$$u_{k+1}(x) = \lambda \int_a^b k(x,t)\big(u_k(t)\big)dt \ , k \geq 0 \qquad (2-26)$$

Where the solution $u(x)$ is expressed by an infinite sum of components defined by

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \qquad (2-27)$$

In view of (2-26), the components $u_n(x), n \geq 0$ are readily obtained.

The MDM presents a slight variation to the recurrence relation (2-26) to determine the components of $u(x)$in an easier and faster manner. For many cases, the function $f(x)$ can be set as the sum of two partial functions, namely $f_1(x) and\ f_2(x)$. In other words, we can set

$$f(x) = f_1(x) + f_2(x) \qquad (2-28)$$

In view of (2-28), we introduce a qualitative change in the formation of the recurrence relation (2-26). The MDM identifies the zeroth component u_0 by one part of $f(x)$,namely $f_1(x)\ or\ f_2(x)$. The other part of $f(x)$can be added to the component $u_1(x)$ that exists in the standard recurrence relation. The MDM admits the use of the modified recurrence relation:

$$u_0(x) = f_1(x),$$

$$u_1(x) = f_2(x) + \lambda \int_a^b k(x,t)\, u_0(t)dt,$$

$$u_{k+1}(x) = \lambda \int_a^b k(x,t)\, u_k(t)dt, \qquad k \geq 1 \qquad (2-29)$$

It is obvious that the difference between the standard recurrence relation(2-26) and the modified recurrence relation (2-29)rests only in the formation of the first two components $u_0(x)$ *and* $u_1(x)$only. The other components $u_j, j \geq 2$remain the same in the two recurrence relations. Although this variation in the formation of $u_0(x)$ *and* $u_1(x)$ is slight, however it has been shown that it accelerates the convergence of the solution and minimizes the size of calculations. Moreover , reducing the number of terms in $f_1(x)$ affects not only the component $u_1(x)$, but also the other components as well.

We here emphasize on the two important remarks made in chapter 1. First,by proper selection of the functions$f_1(x)$*and* $f_2(x)$,the exact solution $u(x)$ may be obtained by using very few iterations,and sometimes by evaluating only two components. The success of this can be made through trials only. A rule that may help for proper choice of $f_1(x)$*and* $f_2(x)$ could not be found yet. Second, if $f(x)$consists of one term only, the MDM cannot be used in this case.

Example 2.3: Solve the Fredholm integral equation by using the MDM

$$u(x) = 3x + e^{4x} - \frac{1}{16}(17 + 3e^4) + \int_0^1 t\, u(t)dt. \qquad (2-30)$$

We first decompose $f(x)$ given by

$$f(x) = 3x + e^{4x} - \frac{1}{16}(17 + 3e^4)$$

Into two parts, namely

$$f_1(x) = 3x + e^{4x}\,, \qquad f_2(x) = -\frac{1}{16}(17 + 3e^4)$$

We next use the modified recurrence formula (2-29) to obtain

$$u_0(x) = f_1(x) = 3x + e^{4x},$$

$$u_1(x) = -\frac{1}{16}(17 + 3e^4) + \int_0^1 t\, u_0(x)dt = 0$$

$$u_{k+1}(x) = \int_0^1 k(x,t)\, u_k(x)dt = 0\ , k \geq 1$$

It is obvious that each component of $u_j\ , j \geq 1$ is zero. This in turn gives the exact solution by

$$u(x) = 3x + e^{4x}$$

Example 2.4: Solve the Fredholm integral equation by using the MDM

$$u(x) = \sec^2 x + x^2 + x - \int_0^{\frac{\pi}{4}} \left(\frac{4}{\pi}x^2 + x\, u(t)\right) dt \qquad (2-31)$$

We first decompose $f(x)$ given by

$$f(x) = \sec^2 x + x^2 + x$$

into two parts

$$f_1(x) = \sec^2 x \quad f_2(x) = x^2 + x$$

We next use the modified recurrence formula (2-29) gives

$$u_0(x) = f_1(x) = \sec^2 x,$$

$$u_1(x) = x^2 + x - \int_0^{\frac{\pi}{4}} \left(\frac{4}{\pi}x^2 + x\, u_0(t)\right) dt = 0\ ,$$

$$u_{k+1}(x) = \int_0^{\frac{\pi}{4}} k(x,t)\, u_k(x)dt = 0\ , k \geq 1$$

As result , the exact solution is given by

$$u(x) = \sec^2 x.$$

2.1.3 Noise Terms Phenomenon:

The noice terms are the identical terms with opposite sins that may appear between components. By canceling the noice terms between

$u_0(x)$ *and* $u_1(x)$,even though $u_1(x)$ contains further terms, the remaining non-concled terms of $u_0(x)$ may give the exact solution of the integral equation. The appearance of the noice terms between $u_0(x)$ *and* $u_1(x)$is not always sufficient to obtain the exact solution by canceling these noise terms. Therefore, it necessary to show that the non-canceled terms of $u_0(x)$ satisfy the given integral equation.

The phenomenon of the useful noise terms will be explained by the following illustrative examples.

Example 2.5: Solve the Fredholm integral equation by using the noice terms phenomenon:

$$u(x) = xsin\, x - x + \int_0^{\frac{\pi}{2}} xu(t)dt. \qquad (2-32)$$

Following the standard Adomian method we set the recurrence relation

$$u_0(x) = x\sin x - x\,,$$

$$u_{k+1}(x) = \int_0^{\frac{\pi}{2}} xu_k(t)dt. \qquad k \geq 0 \qquad (2-33)$$

This gives

$$u_0(x) = x\sin x - x$$

$$u_1(x) = \int_0^{\frac{\pi}{2}} xu_0(t)dt = x - \frac{\pi^2}{8}x. \qquad (2-34)$$

The noise terms $\pm x$ appear in $u_0(x)$ *and* $u_1(x)$.canceling this term from the zeroth component $u_0(x)$ gives the exact solution

$$u(x) = x\sin x,$$

That justifies the integral equation. The other terms of $u_1(x)$ vanish in the limit with other terms of the other components.

Example 2.6: Solve the Fredholm integral equation by using the noise terms phenomenon

$$u(x) = x^2 + \frac{\sin x}{1+\cos x} + \frac{\pi^2}{24}x + x\ln 2 - \int_0^{\frac{\pi}{2}} xu(t)dt.$$

The standard Adomian method gives the recurrence relation:

$$u_0(x) = x^2 + \frac{\sin x}{1+\cos x} + \frac{\pi^2}{24}x + x\ln 2$$

$$u_{k+1}(x) = -x\int_0^{\frac{\pi}{2}} u_k(t)dt. \qquad k \geq 0 \qquad (2-35)$$

This give

$$u_0(x) = x^2 + \frac{\sin x}{1+\cos x} + \frac{\pi^2}{24}x + x\ln 2$$

$$u_1(x) = -\int_0^{\frac{\pi}{2}} xu_0(t)dt$$

$$= -\frac{\pi^3}{24}x - x\ln 2 - \frac{\pi^2\ln 2}{8}x - \frac{\pi^5}{192}x. \quad (2-36)$$

The noise terms $\pm\frac{\pi^3}{24}x, \pm x\ln 2$ appear in $u_0(x)$ *and* $u_1(x)$.canceling this term from the zeroth component $u_0(x)$ gives the exact solution

$$u(x) = x^2 + \frac{\sin x}{1+\cos x},$$

That justifies the integral equation. The other terms of $u_1(x)$ vanish in the limit with other terms of the other components.

2.1.4 Fuzzy Integral Equations

The most basic notations used in the fuzzy calculus are introduced.

Definition1 . A fuzzy number is a fuzzy set u:$R' \to [0,1]$which satisfies

i. u is upper semicontinuous.
ii. $u(x) = 0$ outside some interval $[c\,,d]$,and
iii. There are real number a *and* $b\,, c \leq a \leq b \leq d$, for which
   - $u(x)$ is monotonic increasing on $[c\,,d]$,
   - $u(x)$ is monotonic decreasing on $[c\,,d]$,
   - $u(x) = 1\ for\ a \leq xb.$

The set of all fuzzy numbers,as given by Definition 1, is denoted by E'. An alternative definition or parametric form of a fuzzy number which yields the same E' is given kaleva.

Definition 2. A fuzzy number u is a pair $(\underline{u}, \bar{u})$ of functions $\underline{u}(r)$ and $\bar{u}(r), 0 \leq r \leq 1$, satisfying requirements

i. $\underline{u}(r)$ is a bounded monotonic increasing left continuous function,
ii. $\overline{u}(r)$ is a bounded monotonic decreasing left continuous function, and
iii. $\underline{u}(r) \leq \overline{u}(r), 0 \leq r \leq 1$.

For arbitrary $u = (\underline{u}, \bar{u}), v = (\underline{v}, \bar{v}) and\ k > 0$ we define addition $(u + v)$and multiplication by k as

$$\left(\underline{u + v}\right) = \underline{u}(r) + \underline{v}(r)$$

$$\left(\overline{u + v}\right) = \bar{u}(r) + \bar{v}(r)$$

$$(\underline{ku})(r) = k\underline{u}(r)$$

$$\left(\overline{ku}\right)(r) = k\overline{u}(r) \qquad (2 - 37)$$

The collection of all the fuzzy numbers with addition and multiplication as defined by Eqs above is denoted by E' and is a convex cone. It can be shown that Eqs above are equivalent to the addition and multiplication as defined by using the $\alpha - cut$ approach and the extension principles. We will next define the fuzzy function notation and metric D in E'.

Definition3. For arbitrary fuzzy numbers $u = (\underline{u}, \bar{u}) and\ v = (\underline{v}, \bar{v})$the quantity

$$D(u, v) = \max\left\{ \underset{0 \leq r \leq 1}{\text{sub}} |\bar{u}(r) - \bar{v}(r)|\ ,\ \underset{0 \leq r \leq 1}{\text{sub}} \left|\underline{u}(r) - \underline{v}(r)\right| \right\} \quad (2 - 38)$$

Is the distance between $u\ and\ v$.

The metric is equivalent to the one used by puri and ralescu , and kaleva. It is shown that $(E', D)$is a complete metric space. Goteschel and voxman defined the integral of a fuzzy function using the Riemann integral concept. If the fuzzy function $f(t)$ is continuous in the metric D, its definite integral exists. Furthermore,

$$\underline{\left(\int_a^b f(t,r)dt\right)} = \int_a^b \underline{f}(t,r)dt$$

$$\overline{\left(\int_a^b f(t,r)dt\right)} = \int_a^b \overline{f}(t,r)dt \qquad (2-39)$$

The Fredholm integral equation of the second kind is

$$F(t) = f(t) + \lambda \int_a^b k(s,t)\, F(s)ds \qquad (2-40)$$

Wher $\lambda > 0$, $k(s,t)$ is an arbitrary kernel function over the square $a \le s\,, t \le b$ and $f(t)$ is a function of t, $a \le t \le b$. If $f(t)$is a crisp function then the solutios of Eqs (2-40) are crisp as well. However , if $f(t)$ is a fuzzy function these equations may only possess fuzzy solutions. Sufficient conditions for the existence of a unique solution to the fuzzy FIE of the second kind, i.e. to (2-40) where $f(t)$is afuzzy function.

Now we introduce the parametric form of an FFIE-2 withh respect to Definition2. Let $\underline{f}(t,r), \overline{f}(t,r)$ and $\left(\underline{u}(t,r), \overline{u}(t,r)\right)$,$0 \le r \le 1$ and $t \in [a,b]$, be parametric form of $f(t) and\, u(t)$, respectively; then the parametric form of FFIE-2 is as follows

$$\underline{u}(t,r) = \underline{f}(t,r) + \lambda \int_a^b v_1\left(s,t,\underline{u}(s,r),\overline{u}(s,r)\right) ds$$

$$\overline{u}(t,r) = \overline{f}(t,r) + \lambda \int_a^b v_2\left(s,t,\underline{u}(s,r),\overline{u}(s,r)\right) ds\,, \qquad (2-41)$$

Where

$$v_1\left(s,t,\underline{u}(s,r),\overline{u}(s,r)\right) = \begin{cases} k(s,t)\underline{u}(s,r), & k(s,t) \ge 0, \\ k(s,t)\overline{u}(s,r), & k(s,t) < 0, \end{cases}$$

And

$$v_2\left(s,t,\underline{u}(s,r),\overline{u}(s,r)\right)=\begin{cases}k(s,t)\overline{u}(s,r), & k(s,t)\geq 0,\\ k(s,t)\underline{u}(s,r), & k(s,t)<0,\end{cases}$$

For each $0\leq r\leq 1$ and $t\in[a,b]$. We can see that is the system of linear FIE in the crisp case for each $0\leq r\leq 1$ and $t\in[a,b]$. We define the homotopy analysis method(HAM) as an analytical algorithm for approximating the solution of the system linear integral equation in the crisp case. Then we find the approximate solutions for $\underline{u}(t,r) and\ \overline{u}(t,r)$ for each $0\leq r\leq 1$ and $t\in[a,b]$.

The following theorem provides sufficient conditions for the existence of unique solution to Eq(2-40) where $f(x)$ is a fuzzy function , and the rate of convergence of error.

Theorem(2-3). Let $k(s,t)$ be continuous for $a\leq s,t\leq b,\lambda>0$ and $f(t)$ a fuzzy continuous function of $a\leq t\leq b$. If

$$\lambda<\frac{1}{m(b-a)},$$

Where $M=Max_{a\leq s,t\leq b}|K(s,t)|$, then the iterative procedure

$$F_0(t)=f(t)$$

$$F_k(t)=f(t)+\lambda\int_a^b k(s,t)\,F_{k-1}(s)ds\ k\geq 1$$

Converges to the unique solution of (2-40).specifically,

$$\sup_{a\leq t\leq b} D\big(F(t),F_k(t)\big)\leq\frac{L^k}{1-L}\sup_{a\leq t\leq b} D\big(F(t),F_1(t)\big),$$

Where $L=\lambda M(b-a)$. This infers that $F_k(t)$ converges uniformly in $t\ to\ F(t)$,I,e. given arbitrary $\in>0$ we can find N such that

$$D\big(F(t),F_k(t)\big)<\in, a\leq t\leq b, k>N.$$

2.4.1 Homotopy Analysis Method Solution for Linear System of Fredholm Integral Equations:

Consider the system of linear Fredholm integral equations of the form

$$U(t) = F(t) + \int_a^b k(s,t)U(s)ds \quad (2-42)$$

Where

$$U(t) = (u_1(t), \dots, u_n(t))^T,$$

$$F(t) = (f_1(t), \dots, f_n(t))^T$$

$$k(s,t) = [k_{ij}(s,t)], \qquad i = 1,2, \dots, n, j = 1,2, \dots, n.$$

For the purpose, we first give the following definition.

Definition4. Let ϕ be a function of the homotopy –parameter q then

$D_m(\phi) = \frac{1}{m!}\frac{\partial^m \emptyset}{\partial q^m}\Big|_{q=0}$,

Is called the mth-order homotopy-derivative of $\emptyset$, where $m \geq 0$is an integer.

From Eq.(2-42),the nonlinear operator is defined as follows

$$N(t;q) = U(t;q) - F(t) - \int_a^b K(s,t)U(s;q)ds, \quad (2-43)$$

And we choose the auxiliary linear operator as follows

$$L[\emptyset(t;q)] = \emptyset(t;q). \quad (2-44)$$

We consider the so-called zero-order deformation equation

$(1-q)L[\emptyset(t;q) - \psi_0(t)] = qhH(t)N[\emptyset(t;q] \quad (2-45),$

Where $q \in [0,1]$ is the embedding parameter, h is a diagonal matrix of nonzero convergence –parameters, $H(t)$is a diagonal matrix of auxiliary functions, $\psi_0(t)$ is an initial guess of the exact solution $\psi(t)$ and $\phi(t;q)$ is an unknown function which depends also on convergence-parameters and auxiliary functions. Expanding $\phi(t;q)$in Taylor series with respect to q, we have

$$\phi(t;q) = \psi_0(t) + \sum_{m=1}^{+\infty} \psi_m(t)q^m, \quad (2-46)$$

Where

$$\psi_m(t) = [\phi(t;q)].$$

Operating on both sides of Eq.(2-45) with D_m,we have the so-called mth-order deformation equation

$$L[\psi_m(t) - \chi_m\psi_{m-1}(t)] = hH(t)R_m(\vec{\psi}_{m-1}, t), \quad (2-47)$$

Where

$$R_m(\vec{\psi}_{m-1}, t) = D_{m-1}(N[\phi(t;q)]), \quad (2-48)$$

And

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m \geq 2 \end{cases}$$

For Eqs. (2-43) and (2-48), we have

$$R_m(\vec{\psi}_{m-1}, t) = \psi_{m-1}(t) - \int_a^b K(s,t)\,\psi_{m-1}(s)ds - (1-\chi_m)F(t).$$

Test example 2-7:

To show the efficiency of the HAM described in the previous section, we present some examples. For all examples, we choose $H(t) = 1$, $\underline{u}_0(t,r) = \underline{f}(t,r) and\ \overline{u}_0(t,r) = \overline{f}(t,r)$. we use $n+1$ terms in evaluating the approximate solution $u_{approx[n]}(t,r;h) = \sum_{m=0}^{n} u_m(t,r;h)$.

Example1. Consider the Fuzzy Fredholm integral equation

$$\underline{f}(t,r) = \frac{1}{15}(13(r^2+r) + 2(4-r^3-r)\sin\left(\frac{t}{2}\right),$$

$$\overline{f}(t,r) = \frac{1}{15}(2(r^2+r) + 13(4-r^3-r)\sin\left(\frac{t}{2}\right)$$

And

$$k(s,t) = 0.1\sin(s)\sin\left({}^{t}/_{2}\right), \quad 0 \leq s,t \leq 2\pi,$$

And $a = 0, b = 2\pi$. the exact solution in this case is given by

$$\underline{u}(t,r) = (r^2 + r)\sin\left(\frac{t}{2}\right),$$

$$\overline{u}(t,r) = (4 - r^3 - r)\sin\left(\frac{t}{2}\right).$$

Resulte are shown in Figs.1-4

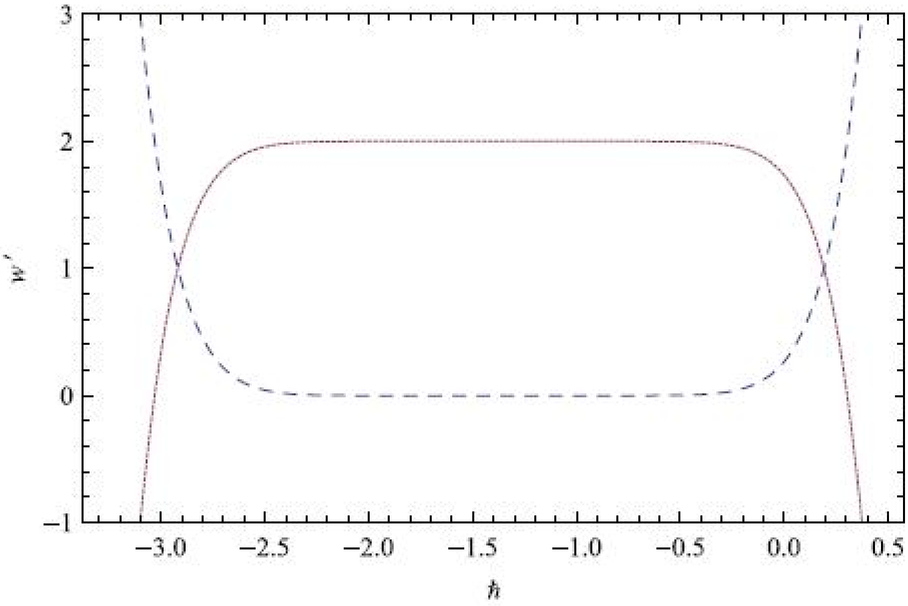

Fig. 1. The $\hbar$-curves of 10th-order of approximation solution given by HAM for the Example 1.

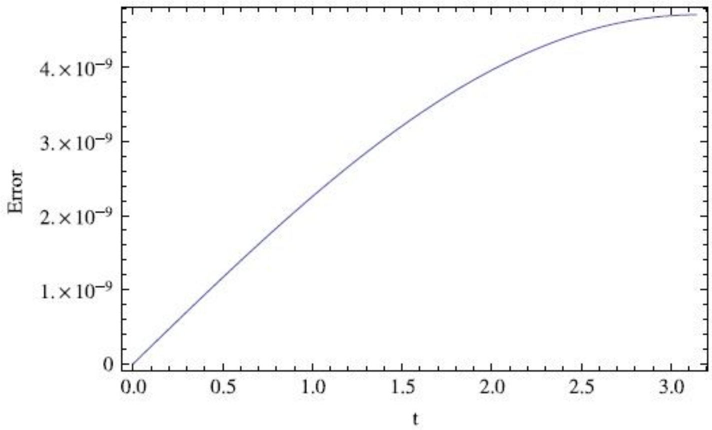

Fig. 2. Comparison between the exact solution and the approximate solution given by HAM ($u_{approx[10]}(t, r; -1.5)$) by metric D for Example 1.

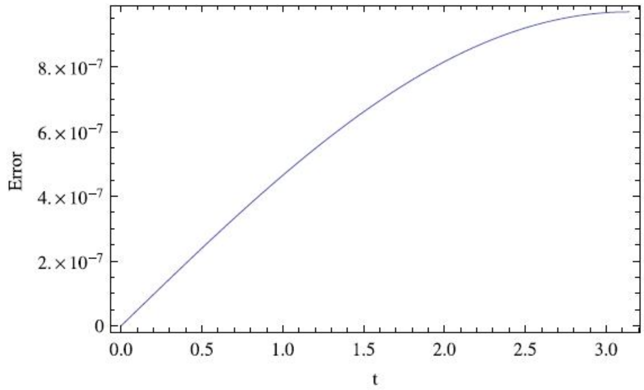

Fig. 3. Comparison between the exact solution and the approximate solution given by HPM and ADM ($u_{approx[10]}(t, r; -1)$) by metric D for Example 1.

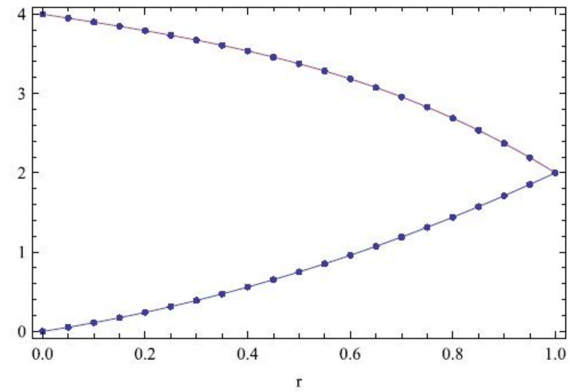

Fig. 4. Comparison between the exact solution and the approximate solution given by HAM ($u_{approx[10]}(t, r; -1.5)$) for Example 1 at $t = \pi$.

2.3 Volterra Integral Equation of the First and the Second Kind:

It was stated in chapter1 that Volterra integral equations arise in many scientific applications such as population dynamics, spread of epidemics, and semi-conductor devices. It was also shown that VIE can be derived from initial value problems.

Volterra integral equations of the first kind or the second kind, are characterized by a variable upper limit of integration. For $u(x)$occurs only under the integral sign in the form:

$$f(x) = \int_a^x k(x,t)u(t)dt \qquad (2-49)$$

However, VIEs of th second kind, the unknown function $u(x)$ occurs inside and outside the integral sign. The second kind is represented in the form:

$$u(x) = f(x) + \lambda \int_a^x k(x,t)u(t)dt \qquad (2-50)$$

The kernel $k(x,t)$ and the function $f(x)$ are given real-valued functions, and λ is a parameter. In what follows we will present the methods,new and traditional, that will be used.

2.3.1 **Adomian Decomposition Method:**

The Adomian Decomposition Method (ADM) consists of decomposing the unknown function $u(x)$ of any equation into a sum of an infinite number of components defined by the decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \qquad (2-51)$$

Or equivalently

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots \quad (2-52)$$

Where the components $u_n(x), n \geq 0$ will be determined recurrently. The ADM concerns itself with finding the components $u_0(x), u_1(x), u_2(x), \ldots$ individually. As will be seen, the determination of these components can be achieved in an easy way through a recurrence relation that usually involves simple integrals that can be easily evaluated.

To establish the recurrence relation , we substitute $(2-51)$ into the VIE$(2-50)$ to obtain

$$\sum_{n=1}^{\infty} u_n(x) = f(x) + \lambda \int_0^x k(x,t)(\sum_{n=0}^{\infty} u_n(t))dt \quad (2-53)$$

Or equivalently

$$u_0(x) + u_1(x) + u_2(x) + \cdots \\ = f(x) \\ + \lambda \int_0^x k(x,t)\big(u_0(t) + u_1(t) + u_2(t)\big)dt \quad (2-54)$$

The zeroth component $u_0(x)$ is identified by all terms that are not included under the integral sing. Consequently, the components $u_j(x)\,, j \geq 0$ of the unknown function $u(x)$ are completely determined by setting the recurrence relation

$$u_0(x) = f(x) \quad ,$$
$$u_{n+1}(x) = \lambda \int_0^x k(x,t)\big(u_n(t)\big)dt \ , n \geq 0 \qquad (2-55)$$

Or equivalently

$$\begin{aligned} u_0(x) &= f(x), \\ u_1(x) &= \lambda \int_0^x k(x,t)\big(u_0(t)\big)dt, \\ u_2(x) &= \lambda \int_0^x k(x,t)\big(u_1(t)\big)dt, \\ u_3(x) &= \lambda \int_0^x k(x,t)\big(u_2(t)\big)dt, \end{aligned} \qquad (2-56)$$

And so on other components.

In view of $(2-56)$ the components $u_0(x), u_1(x), u_2(x)$, ... are completely determined. As a result, the solution $u(x)$ of the VIEs (2-50) is readily obtained in series form by using series assumption in $(2-51)$.

Example(2-8): Solve the Volterra integral equation:

$$u(x) = 1 - \int_0^x u(t)dt. \qquad (2-57)$$

We notice that $f(x) = 1$, $\lambda = -1$ $k(x,t) = 1$. Recall that the solution $u(x)$ is assumed to have a series form given in (2-51).substituting the decomposition series (2-51) into both sides of (2-57) gives

$$\sum_{n=0}^{\infty} u_n(x) = 1 - \int_0^x \sum_{n=0}^{\infty} u_n(x)\, dt, \qquad (2-58)$$

Or equivalently

$$\begin{aligned} &u_0(x) + u_1(x) + u_2(x) + \cdots \\ &\qquad = 1 - \int_0^x [u_0(x) + u_1(x) + u_2(x) + \cdots]dt \ (2-59) \end{aligned}$$

We identify the zeroth component by all terms that are not included under the integral sign. Therefor, we obtain the following recurrence relatin:

$$u_0(x) = 1,$$

$$u_{k+1}(x) = -\int_0^x u_k(t)dt, k \geq 0, \qquad (2-60)$$

So that

$$u_0[\![(x) = 1,$$

$$u_1(x) = -\int_0^x u_0(t)\,dt = -\int_0^x 1\,dt = -x,$$

$$u_2(x) = -\int_0^x u_1(t)\,dt = -\int_0^x -t\,dt = \frac{1}{2!}x^2,$$

$$u_3(x) = -\int_0^x u_2(x)\,dt = -\int_0^x \frac{1}{2!}t^2 dt = -\frac{1}{3!}x^3,$$

$$u_1(x) = -\int_0^x u_0(x)\,dt = -\int_0^x -\frac{1}{3!}x^3\,dt = \frac{1}{4!}x^4,$$

And so on. Using (2-51) gives the series solution:

$$u(x) = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots, \quad (2-61)$$

That converges to the closed form solution:

$$u(x) = e^{-x}$$

Example (2-9): Solve the Volterra integral equation:

$$u(x) = 1 - x - \frac{1}{2}x^2 - \int_0^x (t-x)u(t)dt. \qquad (2-62)$$

Notice that $f(x) = 1 - x - \frac{1}{2}x^2$, $\lambda = -1$, $k(x,t) = t - x$. Substituting the decomposition series (2-51) into both sides of (2-62) gives

$$\sum_{n=0}^{\infty} u_n(x) = 1 - x - \frac{1}{2}x^2 - \int_0^x \sum_{n=0}^{\infty} (t-x)u_n(x)\,dt, \qquad (2-63)$$

Or equivalently

$$\begin{aligned}\boldsymbol{u_0(x) + u_1(x) + u_2(x) + \cdots} \\ \boldsymbol{= 1 - x - \frac{1}{2}x^2} \\ \boldsymbol{- \int_0^x (t-x)[u_0(x) + u_1(x) + u_2(x) + \cdots]dt}\end{aligned}$$

This allows us to set the following recurrence relation:

$$u_0(x) = 1 - \boldsymbol{x} - \boldsymbol{\frac{1}{2}x^2},$$

$$u_{k+1}(x) = -\int_0^x \sum_{n=0}^{\infty} (t-x)u_k(x)\,dt\;, k \geq 0$$

That gives

$$\begin{aligned} u_0(x) &= 1 - \mathbf{x} - \mathbf{\frac{1}{2}x^2}, \\ u_1(x) &= -\int_{\mathbf{0}}^{\mathbf{x}} (\mathbf{t}-\mathbf{x})\mathbf{u_0}(\mathbf{t})\,dt = \frac{1}{2!}x^2 - \frac{1}{3!}x^3 - \frac{1}{4!}x^4 \\ u_2(x) &= -\int_{\mathbf{0}}^{\mathbf{x}} (\mathbf{t}-\mathbf{x})\mathbf{u_1}(\mathbf{t})\,dt = \frac{1}{4!}x^4 - \frac{1}{5!}x^5 - \frac{1}{6!}x^6, \\ u_3(x) &= -\int_{\mathbf{0}}^{\mathbf{x}} (\mathbf{t}-\mathbf{x})\mathbf{u_2}(\mathbf{t})\,dt = \frac{1}{6!}x^6 - \frac{1}{7!}x^7 - \frac{1}{8!}x^8 \end{aligned} \qquad (2-64)$$

And so on. The solution in a series form is given by

$$u(x) = 1 - \left(x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \cdots\right),$$

And in closed form by

$$u(x) = 1 - \sinh x,$$

Obtained upon using the Taylor expansion for $\sinh x$.

Chapter 3

Existence of Solution of Nonlinear Integral Equations

3.1Introduction:

In this chapter to study the nonlinear integral equations. The nonlinear Volterra integral equations are characterized by a least one variable limits of integration. In the nonlinear Volterra integral equation of the second kind, the unknown function $u(x)$ appears inside and outside the integral singn. The nonlinear VIEs of the second kind is represented by the form

$$u(x) = f(x) + \int_0^x k(x,t)\, F\big(u(t)\big) dt, \qquad (3-1)$$

However, the nonlinear VIES of the first kind contains the nonlinear function $F(u(t))$inside the integral sign. The nonlinear VIEs of the first kind is expressed in the form

$$f(x) = \int_0^x K(x,t)\, F\big(u(t)\big) dt. \qquad (3-2)$$

For these two kinds of equations, the kernel $K(x,t)$ and the function $f(x)$ are given real-valued functions, and $F(u(t))$ is a nonlinear function of $u(x)$ such as $u^2(x), \sin\big(u(x)\big), and\ e^{u(x)}$.

3.2 Existence of the Solution for Nonlinear Volterra Integral Equations:

We will present and existence theorem for the solution of nonlinear VIEs. In what follows, we present a brief summary of the conditions under which a solution exists for equation.

We first rewrite the nonlinear VIEs of the second kind by

$$u(x) = f(x) + \int_0^x G\big(x,t,u(t)\big)\, dt. \qquad (3-3)$$

The specific conditions under which a solution exists for the nonlinear VIE are:

i. The function $f(x)$ is integrable and bounded in $a \le x \le b$.

ii. The function $f(x)$ must satisfy the Lipschitz condition in the interval (a,b). This means that

$$|f(x) - f(y)| < k|x - y| \qquad (3-4)$$

iii. The function $G(x,t,u(t))$ is integrable and bounded $|G(x,t,u(t))| < k\ in\ a \le x,t \le b$.

iv. The function $G(x,t,u(t))$ must satisfy the Lipschitz condition

$$|G(x,t,z)-G(x,t,z')| < M|z-z'|. \qquad (3-5)$$

3.3 Nonlinear Volterra Integral Equation of the Second Kind

We being our study on nonlinear VIEs of the second kind given by

$$u(x) = f(x) + \int_0^x k(x,t)\,F(u(t))\mathrm{dt} \qquad (3-6)$$

Where the kernel $k(x,t)$ and the function $f(x)$ are given real-valued function, and $F(u(t))$ is a nonlinear function of $u(x)$ such as $u^3(x), \cos\left(\left(u(x)\right)\right), and\ e^{u(x)}$. The unknown function $f(x)$, that will be determined, occurs inside and outside the integral sign.

The nonlinear VIE $(3-6)$ will be solving by Adomain Decomposition Method,(ADM).

3.3.1 Successive Approximations Method(SAM)

The successive approximation method, or the Picard iteration method was used before in chapter2. This method solves any problem by finding successive approximations to the solution by starting with an initial guess, called the zeroth approximation. As will be used in a recurrence relation to determine the other approximations.

Given the nonlinear VIE of the second kind

$$u(x) = f(x) + \int_0^x k(x,t)\,F(u(t))\mathrm{dt} \qquad (3-7)$$

Where $u(x)$ is unknown function to be determind and $k(x,t)$ is the kernel. The successive approximations methed introduces the recurrence relation

$$u_{n+1}(x) = f(x) + \int_0^x K(x,t)\,F\big(u_n(t)\big)dt, \quad n \geq 0 \qquad (3-8)$$

Where the zeroth approximation $u_0(x)$ can be any selective real valued function. We always start with an initial guess for $u_0(x)$,mostly we select $0,1, or\ x\ for\ u_0(x)$. using the selection of $u_0(x)$ into (3-8), several successive approximations $u_k, k \geq 1$ will be determined as

$$u_1(x) = f(x) + \int_0^x K(x,t)\, F\big(u_0(t)\big)dt,$$

$$u_2(x) = f(x) + \int_0^x K(x,t)\, F\big(u_1(t)\big)dt, \qquad (3-9)$$

$$u_3(x) = f(x) + \int_0^x K(x,t)\, F\big(u_2(t)\big)dt$$

$$\vdots$$

$$u_{n+1}(x) = f(x) + \int_0^x K(x,t)\, F\big(u_n(t)\big)dt,$$

Consequently, the solution $u(x)$is obtained by using

$$u(x) = \lim_{n\to\infty} u_{n+1}(x). \qquad (3-10)$$

Example 3.1: Use the successive approximations method to solve the nonlinear Volterra integral equation

$$u(x) = e^x + \frac{1}{3}x(1-e^{3x}) + \int_0^x x\, u^3(t)dt \qquad (3-11)$$

For the zeroth approximation $u(x)$, we can select

$$u(x) = 1. \qquad (3-12)$$

The method of successive approximations admits the use of the iteration formula

$$u_{n+1}(x) = e^x + \frac{1}{3}x(1-e^{3x}) + \int_0^x x\, u_n{}^3(t)dt \;\; n \geq 0 \qquad (3-13)$$

Substituting (3-12) into (3-13) we obtain the approximations

$$u_0(x) = 1,$$

$$u_1(x) = e^x + \frac{1}{3}x(1 - e^{3x}) + \int_0^x x\, {u_0}^3(t)dt$$

$$= 1 + x + \frac{1}{2!}x^2 - \frac{4}{3}x^3 - \frac{35}{24}x^4 - \frac{67}{60}x^5 + \cdots$$

$$u_2(x) = e^x + \frac{1}{3}x(1 - e^{3x}) + \int_0^x x\, {u_1}^3(t)dt \qquad (3-14)$$

$$= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{67}{60}x^5 + \cdots$$

$$u_1(x) = e^x + \frac{1}{3}x(1 - e^{3x}) + \int_0^x x\, {u_0}^3(t)dt$$

$$= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots$$

And so on. Consequently, the solution $u(x)$ of (3-13) is given by

$$u(x) = \lim_{n\to\infty} u_n(x) = e^x \qquad (3-15$$

Example 3.2: Use the successive approximations method to solve the nonlinear Volterra integral equation

$$\boldsymbol{u(x) = \cos(x) + \frac{1}{8}cos(2x) - \frac{1}{4}x^2 - \frac{1}{8} + \int_0^x (x-t)\, u^2(t)dt \qquad (3-16)}$$

For the zeroth approximation $u_0(x)$, we can select

$$u_0(x) = 1. \qquad (3-17)$$

The method of successive approximations admits the use of the iteration formula

$$u_{n+1}(x) = \cos(x) + \frac{1}{8}cos(2x) - \frac{1}{4}x^2 - \frac{1}{8} + \int_0^x (x-t)\, {u_n}^2(t)dt \qquad n \geq 0 \quad (3-18)$$

Substituting (3-17) into (3-18) we obtain the approximations

$$u_0(x) = 1,$$
$$u_1(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{8}x^4 - \frac{1}{80}x^6 + \cdots,$$
$$u_2(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{240}x^6 + \cdots \qquad (3-19)$$
$$u_3(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6}x^6 + \cdots$$

And so on. Consequently, the solution $u(x)$ of (3-16) is given by

$$u(x) = \lim_{n\to\infty} u_n(x) = cosx \qquad (3-20)$$

3.3.2 Adomian Decomposition Method

The Adomain Decomposition Method has been outlined before in provious chapters and has been applied to a wide class of linear Volterra and Fredholm integral equations. The method usually decomposes the unknown function $u(x)$ into infinite sum of components that will be determined recursively through iterations as discussed before. The ADM will be applied in this chapter and in the coming chapters handle nonlinear integral equations.

Although the linear term $u(x)$ is represented by an infinite sum of components, the nonlinear terms such as $u^2, u^3, u^4, \sin u\, , e^u\, , etc.$ that appear in the equation, should be expressed by a special representation, called the Adomain polynomials $A_n\, , n \geq 0$. Adomain introduced a formal algorithm to establish a reliable representation for all forms of nonlinear terms. Other technique remains the commomonly used one. We will use the Adomain algorithm to evaluate Adomain polynomials. The representation of the nonlinear terms by Adomain polynomials is necessary to handle the nonlinear integral equations in a reliable way.

In the following, the Adomain algorithm for calculating the so-called Adomain polynomials for representing nonlinear terms will be introduced in details. The algorithm will be explaind by illustrative example that will cover a wide variety of nonlinear forms.

Calculation of Adomain polynomials

The Adomain decompostion methed assumes that the unknown linear function $u(x)$ may be represented by the infinite decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \qquad (3-21)$$

Where the components $u_n(x), n \geq 0$ will be computed in a recursive way. However, the nonlinear term $F\big(u(x)\big)$, such as $u^2, u^3, u^4, \sin u\,, e^u\,, etc.$ can be expressed by an infinite series of the so-called Adomain polynomials A_n gven by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{n}\lambda^i u_i\right)\right]_{\lambda=0}\,, n = 0,1,2, \dots\,, \qquad (3-22)$$

Where the so-called Adomain polynomial A_n can be evaluated for all forms of nonlinearity. The general formula (3-22) can be easily used as follows. Assuming that the nonlinear function is $F(u(x))$,therefor by using $(3-22)$, Adomain polynomials are given by

$$A_0 = F(u_0),$$
$$A_1 = u_1 F'(u_0),$$
$$A_2 = u_2 F'^{(u_0)} + \frac{1}{2!}u^2{}_1 F''(u_0) + \frac{1}{3!}$$
$$A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!}u^3{}_1 F'''(u_0),$$
$$A_4 = u_4 F'(u_0) + \left(\frac{1}{2!}u^2{}_2 + u_1 u_3\right) F''(u_0) + \frac{1}{2!}u^2{}_1 F'''(u_0) + \frac{1}{4!}u^4{}_1 F^{iv}(u_0) \qquad (3-.23)$$

Two important observations can be made here. First, A_0 depends only on u_0, A_1depends only on $u_0$and u_1, A_2depends only on u_0, $u_1, and\ u_2$ and so on. Second substituting (3-23) into (3-24) gives

$$F(u) = A_0 + A_1 + A_2 + A_3 + A_4 + \cdots$$

$$= F(u_0) + (u_1 + u_2 + u_3 + u_4 + \cdots)F'(u_0)$$
$$+ \frac{1}{2!}(u^2{}_1 + 2u_1u_2 + 2u_1u_3 + u^2{}_2 + \cdots)F''(u_0) + \cdots$$
$$+ \frac{1}{3!}(u^3{}_1 + 3u^2{}_1u_2 + 3u^2{}_1u_3 + 6u_1u_2u_3$$
$$+ \cdots)F'''(u_0) + \cdots$$
$$= F(u_0) + (u - u_0)F'(u_0) + \frac{1}{2!}(u - u_0)^2F''(u_0)$$
$$+ \cdots \quad (3-24)$$

The last expansion confirms a fact that the series in A_n polynomials is a Taylor series about a function $\boldsymbol{u_0}$ and not about a point as in the standard Taylor series. The few Adomain polynomials given above in (3-23) clearly show that the sum of the subscripts of the components of $u(x)$ of each term of A_n is equal to n.

In the following. We will calculate Adomian polynomials for several nonlinear terms that may arise in nonlinear integral equations.

Case 1.

The first four Adomain polynomials for $F(u) = u^2$ are given by

$$A_0 = F(u_0) = u_0^2$$
$$A_1 = u_1F'(u_0) = 2u_0u_1$$
$$A_2 = u_2F'(u_0) + \frac{1}{2!}u_1^2F''(u_0) = 2u_0u_2 + u_1^2$$
$$A_3 = u_3F'(u_0) + u_1u_2F''(u_0) + \frac{1}{3!}u_1^3F'''(u_0) = 2u_0u_3 + 2u_1u_2,$$

Case 2.

The first four Adomain polynomials for $F(u) = u^3$ are given by

$$A_0 = F(u_0) = u_0^3$$
$$A_1 = u_1F'(u_0) = 3u_0^2u_1$$
$$A_2 = u_2F'(u_0) + \frac{1}{2!}u_1^2F''(u_0) = 3u_0^2u_1 + 3u_0u^2{}_1,$$