Sudan University of Science and Technology

College of Graduate Studies

Twistor Formulation of Some Partial Differential Equations

صيغ الإلتوائيات لبعض المعادلات التفاضلية الجزئية

A Thesis submitted in Fulfillment Requirements for the Degree of Ph.D in Mathematics

By:

Khalid Abdalla Mohammed Zein Babiker

Supervisor:

Prof. Mohamed Ali Bashir

2019

Sudan University of Science & Technology

College of Graduate Studies

بسم الله الرحمن الرحيم

جامعة السودان للعلوم والتكنولوجيا

كلية الدراسات العليا

كلية الدراسات العليا

Approval Page

(To be completed after the college council approval)

Name of Candidate: Khalid Abdalla Mohammedzain Babeker

Thesis title: Twistor Formulation of Some Partial Differential Equations

Degree Examined for: Ph. D.

Approved by:

1. External Examiner

Name: Osman Mohamed EL Mekki

Signature: .... Date: 17/4/2019

2. Internal Examiner

Name: Prof. Shawgy Hussein Abdalla

Signature: .... Date: 17/4/2019

3. Supervisor

Name: Mohammed Ali Bashir

Signature: .... Date: 17/4/2019

cgs @ sustech edu. البريد الالكتروني فاكس/ 83 769363 ص.ب 407

Dedication

To the soul of my mother.

To my father.

To my wife,

To my family.

To my teachers

To my colleagues

To my friends

And

To my students

Acknowledgements

I am so grateful to my supervisor Prof. Dr. Mohammed Ali Bashir for patient supervision and collaboration to achieve this thesis. I highly appreciate Prof. Dr. Bashir's sincerity and generosity and above all his friendly, humanitarian manner.

I would like to express my deep gratitude to my family for the earnest support they avid me throughout my life.

My gratitude extends to all my colleagues and friends for their efforts that have helped my tremendously to accomplish this thesis.

Abstract

Twistor theory has been invented by R. Penrose in order to generalize gravity. He introduced a geometrical model for Minkowski space. This geometrical setup has been generalized in solving particle differential equations. In particular zero-rest-mass field equation have been treated this way as a Contour integral of complex twistor function. In our study we consider geometrical interpretations and solutions of conformal field equations. We also studied Twistors in curved space-time and related this study to the problem of quantum gravity.

الخلاصة

نظرية التويستور اخترعت بواسطة روجربنروز لكي تعمم الجاذبية قام بادخال النموذج الهندسي لفضاء منكوويسكاي 0 هذا النظام الهندسي تم تعميمه في حل معادلات تفاضلية الجسيم 0 بصفة خاصة معادلة حقل كتلة – السكون – الصفر تم علاجها بهذه الطريقة لتكامل كنتور لدالة تويستور المركبة 0في دراستنا اعتبرنا التقسيرات الهندسية وحلول معادلات حقل حافظ الزوايا 0وايضا تمت دراسة التويستورات في منحني المكان – الزمن وصلة هذه الدراسة بمسالة جاذبية الكم.

The Contents

Introduction

In this thesis we introduced geometrical aspect of Minkowiski space which has been generalized to higher dimensions and utilized in solving differential equations.

Chapter 1 An introduction to Twistor theory was proposed by Roger Penrose in 1967 as a possible path to quantum gravity and has evolved into a branch of theoretical and mathematical physics. We proposed that twistor space should be the basic arena for physics from which space-time itself should emerge. It leads to a powerful set of mathematical tools that have applications to Twistor in flat space and Minkowiski geometry.

Chapter 2 We have introduced the the correspondence and Penrose transform which is a complex analogue of the Radon transform that relates massless fields on spacetime twistor space. The twistor space and the twistor transform is also geometrically natural in the sense of integral geometry.

Chapter 3 We have introduced quantum field theory in curved spacetime which is an extension of standard, Minkowski space quantum field theory to curved spacetime. In addition to canonical quantization and conservation low have been introduced.

Chapter 4 Is devoted to twistor in c**urved space** in which we have talked about local twistor, global twistor and quantization.

Chapter 5 Consists of **the physical general relativity** (**GR**, also known as the **general theory of relativity** or **GTR**) which is the geometric theory of gravitation published by Albert Einstein in 1915 and the current description of gravitation in modern physics.

General relativity generalizes special relativity and Newton's law of universal gravitation, providing a unified description of gravity as a geometric property of space and time, or spacetime. In particular, the *curvature of spacetime* is directly related to the energy and momentum of whatever matter and radiation are present. The relation is specified by the Einstein field equations, a system of partial differential equations.

Chapter 1

Introduction to Twistor Theory

1.1. Introduction

It is well known that there are a number of unsatisfactory features of our present ideas about physics. Among these are the infinite divergences of quantum field theory, the lack of a really convincing synthesis of quantum theory and general relativity, and perhaps also our dependence upon the notion of a continuum without any real physical evidence. Twistor theory is an attempt at anew formalism for the description of basic physical processes which has relevance to these problems and it is hoped that when the theory becomes more complete a new outlook on them will be provided. If the attempt is successful, it would of course have very wide implications for all of physics. For everyday purposes our present theories would naturally suffice but our viewpoint would be changed just as the development of relativity modified our view of Newtonian mechanics. Althoughno final assessment of twistor theory's success can yet be made the results have been sufficiently encouraging for us to feel it worthwhile preparing a reasonably up to date and unified account for the use of colleagues in different branches of physics.

The last two of the difficulties mentioned above are clearly related. If space-time is no longer regarded as a continuum, it will no longer be valid to think of either the quantum fields or the gravitational fields in the usual way. One can in fact argue that to accept that there are as many points in 10^{-13} cm or even 10^{-1000} cm as there are in the entire universe is physically unrealistic and that our use of the continuum arises solely from its mathematical utility. We take theview that to encompass quantum theory and general relativity satisfactorily one needs to do more than simply apply some suitable quantization technique to solutions of Einstein's equations. One should rather be thinking of quantising space-time itself. This should not be conceived as simply replacing the continuum by a discrete set of points but rather as seeking a way of treating points as "smeared out" just as quantum theory smears out particles.

It was shown that one could build up the notion of the Euclidean space from the limit of the interaction probabilities of a large network of particles quasi-statically exchanging spin. The Euclidean structure in this development arises from the combinatorial rules satisfied by total angular momentum in non-relativistic quantum mechanics. In the same way that SU(2) spinors provide a basis for the description of non-relativistic angular momentum, twistor theory can be used to describe relativistic angular momentum in a unified way in addition to the concepts of spin and orbital angular momentum uniting together. The hope is that developments of the twistor picture will eventually enable us to construct Lorentzian manifolds to serve as models of space-time. Certainly points of space-time are dependent quantities in the twistor formalism, the twistors themselves playing the basic role. However the complex continuum still plays a large part. Indeed the complex numbers and holomorphic functions which are already basic to modern particle physics now appear mixed up with the structure of space-time itself. Nevertheless, holomorphic functions have a certainrigiditysuggestive of a possible underlying combinatorial structure.

The twistor theory is in fact largely based on ideas of conformal invariance, zero-rest-mass particles and conformally invariant fields being taken as a fundamental aspect of important parts of physics. In this respect twistor theory has a connection with current work by particle physicists, who have been exploring the implications of conformal invariance with considerable vigour.

A twistor of the simplest type can be pictured classically as effectively a zero-rest-mass particle in free motion, where the particle may possess an intrinsic spin, and also a phase which can be realized as a kind of polarization plane. Such twistors form an eight-real-dimensional manifold. Which can be described in a natural way as a vector space of four complex dimensions. This vector space , twistor space, in effect replaces the space-time as the background in terms of which physical phenomena are to be described. Space-time points can then be reconstructed from the twistor space being represented as certain linear subspaces, but they become secondary to the twistors themselves. Furthermore, when general relativity and quantum theory both become involved, it is to be expected that the concept of a space-time point should cease to have precise meaning within the theory. In effect, the space-time points become smeared by the uncertainty principle ratherthan the light cones becoming smeared.

As the theory stands it does not provide a formulation of a quantised general relativity norhas a full treatment of particles with non-vanishing rest-mass emerged. On the other hand, thetheory appears to give correct answers for scattering processes involving massless charged particles and photons, i.e. high energy limit of quantum electrodynamics. The theory so far appears to be successful in avoiding divergences, i.e. the calculations that have been carried out do not lead to infinities in the same way as does the conventional formalism and it seems that such infinities should be absental together. It is hoped that when the theory becomes more complete, this feature will be retained.

The difficulties confronting the theory in respect of gravitational interactions and rest-mass appear to be related to the fact that these are things which *break conformal invariance.* Massless particles and electromagnetic interactions, on the other hand, are conformally invariant concepts. The basic formalism exhibits manifest conformal invariance, so if conformal invariance is to be broken. This must apparently be done explicitly, with the aid of auxiliary elements which do not share in the invariance. One possible method of incorporating such elements is suggested by the work described. The essential act of faith on which the utility of the twistor formalism depends, is that it should be useful to isolate the conformally non-invariant aspects of physics from the conformally invariant ones, and that having done this, a large and important body of physical processes will be seen as possessing conformal invariance.

Twistors are actually the reduced spinors for the proper pseudo-orthogonal group SO(2,4) which is locally isomorphic with, and 2-I homomorphic with, the restricted conformal group of flat space-time. They form a representation space for the pseudo-unitary group SU(2,2), this in turn beinglocally isomorphic and 2-I homomorphic with SO(2,4). Thus, the simplest twistors are four-valued objects with four complex components, which are acted upon by the 15parameter conformal group of flat space-time. The four-valuedness of twistors has not yet played any very importantrole in this theory.

One of the most striking features of complex theory is the way in which complex numbers and holomorphic, i.e. complex analytic, structure emerge as concepts intimately involved in the geometry of space-time. We have become accustomed to the very basic role which complex numbers and holomorphic functions play in quantum theory, particularly that of elementary particles. It seems therefore that complex numbers are very important constituent of the structure of physical laws. The twistor theory carries this further insuggesting that complex numbers may also be very basically involved in defining the nature of space-time itself. In addition, we shall see that the zero rest-mass field equations for each spin all emerge in a very simple way from the complex structure of twistor space, being obtained as contour integrals of holomorphic functions of twistors. The twistor picture geometrises the usual splitting of field amplitudes into positive and negative frequency parts by describing this in terms of the position of singularities of holomorphic functions. Thus the twistor formalism has the effect of uniting various aspects of the role, both quantum and classical, that complex numbers seem to play in physics.

The present introduction should be regarded as to some extent provisional. Many problems remain to be solved in the theory. Even the difficulties involved in merely translating between the twistor formalism and the conventional formalism constitute a serious stumbling block. The twistor scattering diagrams described do not always appear to be directly translatable into conventional terms and this leads to difficulties in interpretation. One must proceed to some extent by guesswork, but here severe problems of actually computing the twistor diagrams constitute another stumbling block. Nevertheless, despite these difficulties, we feel that some new insights into the nature of physical processes may possibly be discernable even in the theory as it stands. For example, if the twistor diagrams are to be taken seriously from the physical point of view and it is tempting to think that they can be then there may be some significance in the twistor lines representing a kind of half particle which can be exchanged in virtual processes. This would appear to be related to the fact that a twistor is really a kind of square root of the structure of a zero rest-mass particle.

1.2 Conformal Transformations

There is a certain confusion in the literature owing to the fact that two quite distinct concepts are both given the name conformal transformation.

The first of these, which we shall refer as a conformal rescaling, consists solely of a replacement

$$g_{ab} \to \hat{g}_{ab} = \Omega^2 \hat{g}_{ab}, \qquad (1.1)$$

of the space-time metric $g_{ab}$by a conformally-related one $\hat{g}_{ab}, \Omega$ being a smooth positive scalarfield on the underlying manifold. Thus the interval ds is transformed to $d\hat{s} = \Omega ds$. If g_{ab} is a flatmetric, then $\hat{g}_{ab}$will in general not be flat, though it will of course be conformally flat. The conformalrescalings of a given space-time form an infinite-parameter Abelian group. The points of thespace-time are unaffected by a conformal rescaling. The null cones, the causal structure of the space-time, are also unchanged.

The other type of conformal transformation is what we shall call a conformal mapping. This is a smooth mapping μ which carries each point of a space-time N to a point of some space-time Ñin such a way that the metric on Ñ induced by μ from that ofN is a conformal rescaling of theoriginal metric on Ñ. In other words the map μ preserves null cone structure Conformal mappings of Minkowski space M' to itself have a particular interest. These include the Poincaré transformations, which are metric-preserving, and the simple overall dilations, whose corresponding rescaling multiplies the metric at each point by a constant factor. The remainder are generated by the in versiotis

$$\tilde{x} = -x^a(x^b x_b)^{-1}, \qquad \tilde{x} = -\hat{x}^a(\hat{x}^b\hat{x}_b)^{-\mathbf{1}} \tag{1.2}$$

Which are a 4-parameter set since the choice of origin is arbitrary. These transformations preserve the time sense but involve spatial reflection. They are conformal mappings since the induced and original metrics are related by

$$d\hat{s}^2 = d\hat{x}^a d\hat{x}_a = dx^a dx_a/(x^b x_b)^2 = \Omega^2 ds^2. \tag{1.3}$$

However, these transformations do not involve only the points of M', because the null cone of theorigin is sent to infinity. We therefore introduce compactified Minkowski space M, which consistsof M’ together with a closed null cone at infinity. We may picture the structure in terms of two cones joined base to base, the interior being M’ and the two bounding cones being identified along opposite generators with future sense preserved (see fig. 1). Thus the “equator” $I_o$must be consideredas a single point. For fuller discussion of the structure of M. Note that one canconsider the equations (1.2) as expressing a coordinate change, rather than a point transformation,on M; and that the null cone at infinity is on the same footing as any other null cone in M as far asconformal mapping symmetry is concerned. In consequence of the latter, the concept ofradiation is not conformally invariant, since it depends on knowing where infinity.

The conformal mapping group of M is of 15 parameters and non-abelian. We shall here concern our selves with the restricted conformal group, i.e. the subgroup of mappings connected with the

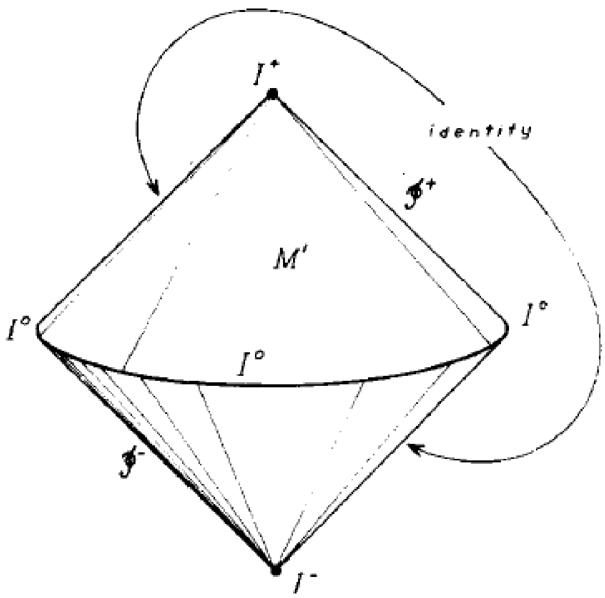


Fig. 1. Identity Map

Compactified Minkowski space M. I^o, I^*, I^- are points at spatial infinity, future time infinity and past time infinity respectively,while $I^+, I^-$are future and past null infinity cones. The compactified space has I^o, I^+, I^-identified and I^-, I^+identified along opposite generators. For typographical reasons, *"I"*replaces the more usual script *I* depicted in fig. 1.identity map. This does not include the actual mappings (1 .2) but does include their products with space reflections. It is 2-1 covered by the six-dimensional pseudo-orthogonal group

SO(2,4) which in turn is 2-1 covered by SU(2,2), a group of unimodularpseudo-unitary matrices. The infinitesimal conformal motions are described by the conformal Killing vectors r and are given, for infinitesimal ϵ, by

$$x^a \rightarrow x^a + \epsilon\xi^a.$$

The vectors ξ^amust satisfy

$$\nabla_{(a\xi_b)} = \frac{1}{4} g_{ab}(\nabla_c \xi^c) \quad (1.4)$$

The general a solution of this is

$$\xi_a = S_{ab}x_b + T_a + Q_a(x^c x_c) - 2x_a(x^c Q_c) + Rx_a (1.5)$$

where $S_{ab} = S_{[ab]}$ generate the Lorentz rotations (6 parameters), $T_a$the translations (4 parameters),$\boldsymbol{R}$ the dilations (1 parameter) and $\boldsymbol{Q_a}$the so-called "uniform acceleration" transformations (4parameters). (This terminology is rather misleading, however, and will be avoided here. A morecorrect use of the terminology uniform acceleration is for a coordinate transformation which makes the Minkowski metric take the form

$$ds^2 = z^2dt^2 - dx^2 - dy^2 - dz^2.$$

Questions of conformal invariance are handled most easily within the framework of conformal rescaling rather than conformal mappings. A physical theory will be said to be conformally invariant if it is possible to attach conformal weights to all the quantities appearing in the theory in such a way that all field equations are preserved under conformal rescalings. A tensor or spinor$A$is said to have conformal weight r if we are to make the replacement $A^{\cdots} \ldots \rightarrow \hat{A}^{\cdots} \ldots = \Omega^r A^{\cdots} \ldots$ under the conformal rescaling$g_{ab} = \hat{g}_{ab}$. A flat space theory which is Poincartinvariantand also conformally invariant in this sense, will then be invariant under the 1 5-parameter conformalgroup. This is because the Poincaré motions of Minkowski space become conformal motionsaccording to any other conformally rescaled flat metric. Conformal motions obtainable in this wayare sufficient to generate the full conformal group. But the type of conformal invariance described above is really more general than this since the conformal rescalings need not apply to flat space-time at all or even to conformally flat space-times. In order to establish conformal invariance of a theory, one needs to know how to transform the (covariant) derivative operator under conformal rescaling. Remarkably enough, this is rather simple rwith in the two-component spinor formalism than within the tensor formalism. Since two-component spinors will also play an essential role in other aspects of twistor theory, we will next briefly summarise the relevant notation and methods.[1]

1.3. Spinors

The essential fact on which the 2-component spinor calculus is based is the local isomorphism between the Lorentz group and the group SL(2, C) of complex unimodular 2×2 matrices whichis the covering group of the identity-connected component of the Lorentz group. It should benoted that this does not mean that spinors can only be used in flat space , since it is possible to usethis isomorphism locally is curved space-time. Representing the Minkowksi components $u^a$of a world vector u^aaccording to the matrix scheme

$$u^a = (u^0, u^1, u^2, u^3) \Leftrightarrow u^{\mathfrak{A}\mathfrak{A}} \begin{pmatrix} u^{00'} & u^{01'} \\ u^{10'} & u^{11'} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} u^0 + u^1 & u^2 + iu^3 \\ u^2 - iu^3 & u^0 - u^1 \end{pmatrix}$$

we find that when the components u^0undergo a restricted Lorentz transformation $L$the $u^{\mathfrak{A}\mathfrak{A}}$undergo

$$u^{\mathfrak{A}\mathfrak{A}} \to \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} u^{00'} & u^{01'} \\ u^{10'} & u^{11'} \end{pmatrix} \begin{pmatrix} \overline{\alpha} & \overline{\gamma} \\ \overline{\beta} & \overline{\delta} \end{pmatrix} \tag{1.7}$$

where $\alpha, \beta, \gamma, \delta$ are complex and their matrix has unit determinant, i.e. $\alpha\delta - \beta\gamma = 1$. The hermiticityof $u^{\mathfrak{U}\mathfrak{U}'}$,i.e. reality of u^a, is preserved and so is

$$\{(u^a)^2 — (u')^2 — (u^2)^2 — (u^3)^2\} = 2\det\left(u^{\mathfrak{U}\mathfrak{U}'}\right). \tag{1.8}$$

We can express (1.7) as

$$u^{\mathfrak{U}\mathfrak{U}'} \to S(L)^{\mathfrak{U}}{}_{\mathfrak{B}} u^{\mathfrak{B}\mathfrak{B}} \overline{S(L)}^{\mathfrak{U}'}{}_{\mathfrak{B}'} (1.9)$$

where $S(L) \in SL(2, \mathbb{C})$. Primed and unprimed indices must be treated as essentially different asregards contractions and permutations, but they are related to each other by complex conjugate on ,which converts a primed index into an unprimed one and vice versa.

The correspondence (1.6) shows how one may relate tensor and spinor components according to a standard scheme, but there is nothing special about this particular correspondence. From the point of view of the abstract index notation', the essential feature is that each abstract tensor index (four-dimensional) is to be equated with a pair of two-dimensional spinor indices, one primed and one unprimed. Thus the abstract tensor indices $a, b, c \ldots, a_0, b_0 \ldots, a_1 \ldots$,may be expressed as$a = AA', b = BB', \ldots, a_0 = A_0A_0' \ldots$ and we can write

$$u^a = u^{AA'}.$$

The reader who prefers to retain a component description such as that of (1.6) can re-express our equations in these terms by use of the in field Waerden symbols $\sigma^a_{\mathfrak{U}\mathfrak{U}'}, \sigma_a^{\mathfrak{U}\mathfrak{U}'}$.)

$$g_{ab} = \epsilon_{AB}\epsilon_{A'B'}, \qquad g^{ab} = \epsilon^{AB}\epsilon^{A'B'} \tag{1.10}$$

where the ϵ's are skew-symmetrical with $\epsilon_{AB'} = \overline{\epsilon_{AB}}\epsilon^{A'B'} = \overline{\epsilon^{AB}}$(i.e. their coordinate representationsunder (1.6) are$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$). We use ϵ's to raise and lower indices, thus

$$\xi_B = \xi^A \epsilon_{AB}; \quad \xi^A = \epsilon^{AB}\xi_B; \quad \eta_{B'} = \eta^{A'}\epsilon_{A'B'}; \quad \eta^{A'} = \epsilon^{A'B'}\eta_{B'}, \qquad (1.11)$$

The tensor and spinor "Kronecker deltas" will be written $g_b^a$and $\epsilon_A^B, \epsilon_{A'}^{B'}$, respectively. Thus

$$g_a^b = \epsilon_A^B \epsilon_{A'}^{B'}$$

and

$$\chi^{\dots a} \therefore g_a^b = \chi^{\dots b} \therefore' \psi_{\dots A \dots}\epsilon_B^A = \psi'_{\dots B \dots}\theta \therefore^{A'} \therefore \epsilon_{A'}^{B'} = \boldsymbol{\theta} \therefore^{B'} \therefore,$$

A complex null vector $u^a(u_b u^a = 0)$ has a spinor form

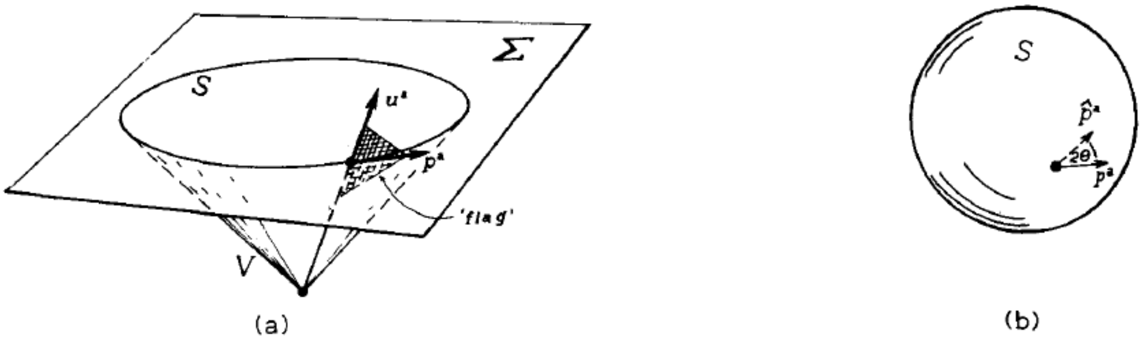
$$u^{AA'} = \xi^A \eta^{A'}$$

Since the matrix of components $u^{\mathfrak{UU}'}$has rank ≤ 1, cf. (1.8)). If u^a is real, then

$$u^{AA'} = \xi^A \eta^{A'}. \qquad (1.12)$$

The plus sign occurs if $u^a$is future-pointing and the minus if $u^a$is past-pointing. Note that $\xi^A \xi_A = 0$(as $\epsilon_{AB}$is skew) so that $u^a u_a = 0$ follows directly from (1.12). Conversely if $\xi^A \xi_A$= 0 then ξ^A is ascalar multiple of ξ^A (or ξ^A = 0).

A spinor ξ^Acontains more information than the corresponding null vector given by (1.12). Anon-zero spinór has a geometrical interpretation, up to an essential sign ambiguity, as a null flag. This consists of the corresponding null vector u^a (flagpole) and a null 2-plane (flag2)



One spatial dimension suppressed. Time dimension suppressed.

Fig. 2.[1]

(a) The spinor ξ_A defines a null flag. This may be pictured as a polarisation vector tangent to the celestial sphere S.

(b) shows how $p^b$is rotated when the phase of is altered.

plane element which contains and is orthogonal to the flagpole. This latter is defined by the bi vector

$$F_{ab} = u_{[\,a}p_{b]} = \epsilon_{AB}\bar{\xi}_{A'}\bar{\xi}_{B'} + \epsilon_{A'B'}\xi_A\xi_B, \qquad (1.13)$$

where $p_a = 2(\xi_A\bar{\lambda}_{A'} + \lambda_A\bar{\xi}_{A'})$for some $\lambda_A$with $\xi_A\lambda^A = 1$. When the phase of $\xi_A$is altered (i.e.$\xi_A \to e^{i\theta}\xi_A$) the vector$u^a$is unaltered, while $p_b$turns through an angle 2θ. We may consider aspacelike hyper-plane $\sum$ intersecting the null cone V, of a point 0, in S a 2-sphere . $\xi_A$then describesa point on S, and a vector tangent to S which defines a polarisation direction. As θ variesthis polarisation vector sweeps out the 2-plane tangent to S, and performs one revolution through

2π as θchanges by π (see fig. 2).

Our covariant derivative operator $\nabla_a = \nabla_{AA'}$satisfies

$$\nabla_a\epsilon_{CB} = 0, \qquad \nabla_a\epsilon_{B'C'} = 0$$

(whence $\nabla_a g_{bc} = 0$), and

$$\nabla_a \nabla_b \phi = \nabla_b \nabla_a \phi.$$

In curved space-time, we have the relation

$$(\nabla_{AA'}\nabla_{BB'} - \nabla_{BB'}\nabla_{\mathrm{AA}'})\xi_C = \Psi_{ABCD}\xi^D \epsilon_{A'B'} - 2\Lambda \xi_{(A\epsilon_B)C}\epsilon_{A'B'} + \Phi_{CDA'B'}\xi^D \epsilon_{AB}$$

where the curvature spinors $\Psi_{ABCD}, \Phi_{ABCD'}$ A have the symmetries

$$\Psi_{ABCD} = \Psi_{(ABCD)}, \qquad \Phi_{ABCD'} = \overline{\Phi}_{ABCD'} = \Phi_{(AB)(C'D')}, \qquad \Lambda = \overline{\Lambda} \quad (1.15)$$

and are related to the curvature tensor R_{abcd}(with sign convention $(\nabla_a \nabla_b — \nabla_b \nabla_{\mathrm{a}})\nabla_d = R_{abcd}V^c$) by

$$R_{abcd} = \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \epsilon_{AB}\epsilon_{CD}\overline{\Psi}_{A'B'C'D'} + 2\Lambda(\epsilon_{AC}\epsilon_{BD}\epsilon_{A'B'}\epsilon_{C'D'} + \epsilon_{AB}\epsilon_{CD}\epsilon_{A'D'}\epsilon_{B'C'}) + \epsilon_{A'B'}\Phi_{ABCD'}\epsilon_{CD} + \epsilon_{AB}\Phi_{CDA'B'}\epsilon_{C'D'}$$

(1.16)

we thus have

$$\Phi_{ABA'B'} = -\frac{1}{2}R_{ab} + \frac{1}{b} Rg_{ab}, \qquad \Lambda = \mathrm{R}/24$$

(where $R_{ab} = R^c_{acb}, R = R^a_a$) and

$$\Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \epsilon_{AB}\epsilon_{CD}\overline{\Psi}_{A'B'C'D'} = C_{abcd}$$

where $C_{abcd}$is Weyl's conformal curvature tensor, whose vanishing is a necessary and sufficient conditionfor the space-time to be conformally flat. We also introduce, for future use, the completely skew tensor

$$\eta^{abcd} = \eta^{[abcd]}$$

defined by $\eta^{123} = \left(\sqrt{-g}\right)^{-1}$,so that $\eta_{0123} = \sqrt{-g}$, in a right-handed coordinate system. Its spinorequivalent is given by

$$\eta_{abcd} = i\,\epsilon_{AD}\epsilon_{BC}\epsilon_{A'C'} - i\epsilon_{AC}\epsilon_{BD}\epsilon_{A'D'}\epsilon_{B'C'}. \tag{1.17}$$

Now we can discuss how conformal rescalings affect spinors. Under the rescaling $g_{ab} \to \hat{g}_{ab} = \Omega^2 g_{ab}$we take

$$\hat{\epsilon}_{AB} = \Omega\epsilon_{AB}, \qquad \hat{\epsilon}_{A'B'} = \Omega\epsilon_{A'B'},$$
$$\hat{\epsilon}^{AB} = \Omega^{-1}\epsilon^{AB}, \hat{\epsilon}^{A'B'}$$
$$= \Omega^{-1}\epsilon^{A'B'}. \tag{1.18}$$

We have

$$\gamma_a = \Omega^{-1}\nabla_{\mathrm{a}}\Omega \tag{1.19}$$

When ∇_aacts on spinors of higher valence we simply treat each index in turn according to the above scheme so there is one term involving $\gamma_{\mathrm{a}}$for each index. For example,

$$\widehat{\nabla}_{CC'}\hat{\epsilon}_{AB} = \widehat{\nabla}_{CC'}(\Omega\epsilon_{AB})$$

$$(\nabla_{CC'}\Omega)\epsilon_{AB} + \Omega(\nabla_{CC'}\epsilon_{AB}) + (\nabla_{AC'}\Omega)\epsilon_{CB} + (\nabla_{BC'}\Omega)\epsilon_{AC}$$

$$= 3\epsilon_{[AB\;\nabla_C]C'}\Omega = 0$$

by the 2-dimensionality of spinor space. The covariant derivative of a vector transforms as

$$\widehat{\nabla}_a V_b = \widehat{\nabla}_{AA'}\widehat{\nabla}_{BB'} = \widehat{\nabla}_{AA'}V_{BB'} - \gamma_{BA'}V_{AB'} - \gamma_{AB'}V_{BA'}$$

$$= \nabla_a V_b - \gamma_a V_b - \gamma_b V_a + (\gamma_{AA'}V_{BB'} + \gamma_{BB'}V_{AA'} - \gamma_{BA'}V_{AB'} - \gamma_{AB'}V_{BA'})$$

$$= \nabla_a V_b - \gamma_a V_b - \gamma_b V_a + \epsilon_{AB'}\epsilon_{A'B'}\gamma_{CC'}V^{CC}$$

$$= \nabla_a V_b - \gamma_a V_b — \gamma_b V_a + g_{ab}(\gamma_c V^c) \quad (1.20)$$

(using $X_{\dots AB\dots} — X_{\dots BA\dots} = \epsilon_{AB}\epsilon^{CD}\chi_{\dots CD\dots}$). Note that this generates the transform under conformal transformations of V applied to tensor indices, as (1.19) does for spinors.

Using this information we find the following transformation laws for the curvature

$$\hat{\psi}_{ABCD} = \Psi_{ABCD}$$

and

$$\hat{P}_{ABCD} — P_{ABC'D'} — \nabla_{AC'}\gamma_{BD'} + \gamma_{AD'}\gamma_{BC'} \quad (1.21)$$

where

$$P_{ABA'B'} = \Phi_{ABA'B'} - \Lambda\epsilon_{AB}\epsilon_{A'B'} = \frac{1}{12}Rg_{ab} - \frac{1}{2}R_{ab}.$$

The Bianchi identities $V_{[aR_{bc}]de} = 0$, which are equivalent to

$$\nabla^a C_{abcd} = 2\,\nabla_{[cP_d]b},$$

become

$$\nabla^D_{A'}\Psi_{ABCD} = -\nabla^{B'}_{(CP_A)BA'B'}$$

which in empty space-time ($P_{ab} = R_{ab} = 0$) simplifies to

$$\nabla^{A'D'}\Psi_{ABCD} = 0. \quad (1.23)$$

Finally let us consider some conformally-invariant theories. For example, Maxwell's equations

$$\nabla_a F^{ba} = 4\pi j^b \text{and} \nabla_{[aF_{bc}]} = 0$$

are conformally invariant ifwe set $\hat{F}_{ab} = F_{ab}$and $\hat{J}_a = \Omega^{-2}$(so $\hat{F}^{ab} = \Omega^2 F^{ab}$; $\hat{J}^a = \Omega^4 J^a$). Thatis to say we get

$$\hat{\nabla}_a \hat{F}^{ba} = 4\pi \hat{j}^b \text{and} \hat{\nabla}_{[a\hat{F}_{bc}]} = 0$$

This may be verified in various ways, e.g. by using the tensor formula for $\hat{\nabla}_a$ above, or by usingthe spinor formulae applied to the spinor version

$$\nabla^A_{B'} \phi_{AB} = -2\pi J_{BB'} \tag{1.24}$$

of Maxwell's equations, where

$$F_{ab} = \phi_{AB} \epsilon_{AB} + \epsilon_{AB} \overline{\phi}_{A'B'}$$

with $\phi_{AB} = \phi_{(AB)}$, $\hat{\phi}_{AB} = \Omega^{-1} \phi_{AB}$ When $J_a = 0$, (1.24) becomes a particular case of the zero-restmassfree-field equations for spin $\frac{1}{2}n$

$$\nabla^{AP'} \phi_{AB\ldots L} = 0 \tag{1.25}$$

where $\phi_{AB\ldots L}$ is symmetric with n indices. If $n = 0$ we adopt the second-order equation

$$\left(\nabla_a \nabla^a + \frac{1}{6} R\right) \phi = 0. \tag{1.26}$$

For each n, these equations are conformally invariant if $\hat{\phi}_{AB\ldots L} = \Omega^{-1} \phi_{AB\ldots L}$ as is readily verifiedusing (1.19). The case $n = 4$ has particular interest since the tensor

$$K_{abcd} = \phi_{ABCD}\epsilon_{C'D'} + \epsilon_{AB}\epsilon_{CD}\overline{\phi}_{A'B'C'D'}$$

defined from a solution of (1.25) in Minkowski space, satisfies

$$K_{abcd} = K_{[ab][cd]} = K_{cdab}, \qquad K_{a[bcd]} = 0,$$

(1.27)

$K_{abcd} = 0,\ \nabla^a K_{abcd} = 0$ (whence$\nabla_{[aK_{bc}]de} = 0$),

and represents a source-free gravitational field in the linearised theory. In this it is assumed that

$$g_{ab} = \eta_{ab} + \epsilon h_{ab}$$

where η_{ab} is the Minkowski metric, ϵ is infinitesimal and h_{ab}, is some symmetric tensor.K_{abcd} is locally the Weyl or in empty space, Riemann tensor for some metric of this form, provided it satisfies (1.27). It should be noted that conformal rescaling of the metric gives $\hat{C}_{abcd} = \Omega^{-2}C_{abcd}$while $\widehat{K}_{abcd} = \Omega^{-1}K_{abcd}$. For further details of spinor calculus. We note that to describe states in quantum mechanical systems, complex vectors and tensors are used. If the operator ih $\partial/\partial t$ has positive eigenvalue, the quantum state has positiveenergy. We will describe these and the corresponding classical states as having positive frequency. It turns out the solutions of (1 .25) with positive energy represent negative helicity particles and the positive energy solutions of the conjugate equation

$$\nabla^{A'P}\theta_{A'B'\ldots L'} = 0, \qquad (1.28)$$

Have the other helicity. This essential difference reappears. Raising and lowering of indices only alters the conformal weight, but complex conjugation of the spinor reverses the helicity.

For example a free photon wave function is described by a complex Fob. When translated into spinor form this gives rise to independent spinors $\phi_{AB}, \theta_{A'B'}$by

$$F_{ab} = \epsilon_{A'B'}\phi_{AB} + \epsilon_{AB}\theta_{A'B'}$$

and $\phi_{AB}, \theta_{A'B'}$ satisfy respectively (1.25) and (1.28). Considering a plane wave, we find

$$\phi_{AB} = \alpha_A\alpha_B \exp\{i(\alpha_P\overline{\alpha}_P, \chi^{PP'})\}$$

$$\theta_{A'B'} = \overline{\alpha}_{A'}\overline{\alpha}_{B'} \exp\{-i(\alpha_P\overline{\alpha}_P, \chi^{PP'})\}.$$

The ϕ_{AB} thus derived corresponds to the ϕ_{AB} for a real circularly-polarised wave, which is in factleft-handed. Thus the spinor representation of complex states splits the states so that the positive energy part of the spinor with unprimed indices has negative helicity, while the positive energy part of the spinor with primed indices has positive helicity.[1]

1.4. Momentum And Angular Momentum

In special relativistic dynamics any finite system possesses a total momentum p^a (a 4-vector) and a total angular momentum M^{ab}skew tensor dependent on the origin O.If $O \to \tilde{O}$, then$p^a = \tilde{p}^a$ and$M^{ab} \to \widetilde{M}^{ab} = M^{ab} - 2\chi^{[aP^b]}$where X^a is the displacement $\tilde{O}$.We may define the spin vector

$$S_a = -\frac{1}{2}\eta_{abcd}P^bM^{cd}. \qquad (1.29)$$

Then

$$\tilde{S}_d = S_d.$$

(i) Assume $P_cP^c > 0$. Then the relativistic centre of mass of the system is defined to move on theworldline which isthe locus of origins $\tilde{O}$ such that

$$\tilde{P}_a \widetilde{M}^{ab} = 0. \tag{1.30}$$

Then, as regards its total momentum and angular momentum the system behaves as a single particle moving along this woridline with momentum $\tilde{P}_a$ and intrinsic spin $\widetilde{M}^{ab}$.Equation (1.30) maybe solved for X^a as

$$X^a = M^a_b P^b / (P_c P^c) + \lambda P^a$$

and this gives a unique time like worldline. The intrinsic spin is

$$\widetilde{M}^{ab} = \frac{\eta_{abcd} S_c P_d}{P^e P_e} \tag{1.31}$$

(ii) However we wish to consider zero rest mass (i.e. $P^c P_c = 0$) to be the more fundamental case.Then (1.30) has no solution unless

$$M_{ab} P^b = p(P^b X_b)$$

Thus there is no solution unless

$$M^{ab} P_b \ \alpha \ P^a \Leftrightarrow P^{[aM^b]c} P_c = 0 \Leftrightarrow P^{[aM^{bc}]} P_c = 0$$

$$\Leftrightarrow P_{[cS_d]} = 0$$

$$\Leftrightarrow S_d = s P_d$$

for some constant s, the helicity whose modulus $[s]$ is the spin . This equation may also be deduced from other points of view about particles. $X^b P_b = k, k$ being a constant, is a null hyperplane$K$, so it appearr that the centre of mass line has become a 3-dimensional region. We can say a little more by considering two cases separately:

(a) Spin $[s] = 0$. Then$M^{ab} = 2A^{[aP^b]}$where$A^a$is some vector, and the centre of mass line can be defined as

$$X^a = A^a + \lambda P^a,$$

the angular momentum about a point on this line being

$$\widetilde{M}^{ab} = 0.$$

Thus we may pick a specific generator of the hypersurface as the centre of mass line, and, as in the case where $P_a P^a > 0$ and $[s] = 0$, the system is completely characterised by this line.

b) Spin $[s] \neq 0$. In this case all points on the null hyperplane $P^a X_a = k$ turn out to be on anequal footing. That is, one can find Poincar6 transformations under which any two given pointsare dynamically equivalent. In this sense the particle is not localised. However, if we take two points $a, b \in K$, the necessary Poincaré transformation demonstrating the equivalence of a and bis not simply a translation, but a translation plus a specific null rotation.[1]

The null vector pa corresponds to a spinor π_A

$$P_a = \overline{\pi}_A \pi_A. \qquad (1.32)$$

uniquely up to phase; $\pi_A \to e^{i\theta}\pi_A$preserves$\left(P_a . M^{ab} = M^{[ab]}\right)$ is represented by a symmetric spinor,$\mu^{AB} = \mu^{(AB)}$

$$M^{ab} = \mu^{AB}\epsilon^{A'B'} + \epsilon^{AB}\overline{\mu}^{A'B'}.$$

The equation $S^a = sP^a$takes the form

$$S_{DD'} = -i\overline{\pi}_D \pi^{A'}\overline{\mu}_{A'D'} + i\overline{\pi}_{AD}\pi_{D'} = s\overline{\pi}_D \pi_{D'}. \qquad (1.33)$$

If we transvect this with $\overline{\pi}_D$ we find $\mu_A \overline{\pi}^A \overline{\pi}^B = 0$ which implies that

$$\mu_{AB} = i\omega_{(A\,\overline{\pi}_B)}$$

for some ω^A. Since any symmetric 2-index spinor is the symmetrised outer product of two spinors, the only new information concerning μ_{AB} is that one of the factors is $\overline{\pi}_A$. Thus

$$M^{AA'BB'} = i\overline{\pi}^{(A\omega^B)}\epsilon^{A'B'} - i\pi^{\left(A'\overline{\omega}^{B'}\right)}\epsilon^{AB}. \tag{1.34}$$

We can now characterise the pair $(P_a\,, M^{ab})$ by the two spinors (ω^A, π_A) (but not uniquely, forthe same pair is represented by $\left(e^{i\theta}\omega^A, e^{i\theta}\pi_{A'}\right)$.This pair is a representation of a twistor Z^a.We do not choose to define the twistor Z^α as the pair of spinors $(\omega^A, \pi_{A'})$ since under change oforigin and under conformal rescaling the ω^A and $\pi_{A'}$ become transformed, whereas the twistor Z^α is supposed to remain unaffected. Thus we must think of$(\omega^A, \pi_{A'})$ simply as a representationof the twistor Z^α. In fact twistors have two stages of representation. The first, in terms of theabove pair of spinors, is specified by a given origin and choice of conformal scale, i.e. of one ofthe conformally related flat metrics. (The spinor indices are here just abstract labels'.) The second is in terms of the coordinates of these spinors with respect to some spinor frame. Such coordinates will be indicated by the presence of Hebrew indices.

If $Z^\alpha$is represented by $(\omega^A, \pi_{A'})$ then we can take twistor components

$$Z^{\mathcal{N}} = (\omega^0, \omega^1, \pi^0, \pi^1,). \tag{1.35}$$

We define a conjugate twistor $\overline{Z}^\alpha$, to have components

$$\overline{Z}_{\mathcal{N}} = \left(\overline{\pi}^0, \overline{\pi}^1\overline{\omega}^{0'}, \overline{\omega}^{1'}\right). \tag{1.36}$$

Dropping the spinor frame, we have the representations $Z^\alpha \Leftrightarrow (\omega^A, \pi_{A'}), \overline{Z}_\alpha \leftrightarrow \left(\overline{\pi}_{A'}, \overline{\omega}^{A'}\right)$and so

$$Z^{\alpha}\overline{Z}^{\alpha} = \omega^{A}\overline{\pi}_{A} + \pi_{A'}\overline{\omega}^{A'} = 2s \quad (1.37)$$

using (1.33). Note that the Hermitian form used on the twistor Z^{α}in (1.37) has signature $(++--)$ and that positive $(s > 0)$ and negative $(s < 0)$ helicities are thus both possible. When $s = 0$ the twistor is said to be null and represents a null worldline. When $s \neq 0$, the twistor represents a particle with intrinsic spin and there is a sense in which this means that the worldline is displaced into the complex. The particle ceases to be localised in M. The remaining will develop the twistor concept.

1.5. Twistors In Flat Space

The concept of a twistor Z^{α} and its complex conjugate $\bar{Z}_{\alpha}$ were introduced, Z^{α} being represented by a pair of spinors $(\omega^{A}, \pi_{A'})$ which define the momentum and angular momentum of a massless particle by (1.32) and (1.34), $\bar{Z}_{\alpha}$being correspondingly represented by $(\overline{\pi}_{A}, \overline{\omega}^{A})$. The helicity of the particle is one-half the Hermitian norm $Z^{\alpha}\bar{Z}_{\alpha}$of the twistor. Twistors also have a linear structure (i.e. $\lambda Z^{\alpha} \leftrightarrow (\lambda\omega^{A}, \lambda\pi_{A}); Z^{\alpha} + Z^{\alpha}(\omega_{1}^{A} + \omega_{2}^{A}, \pi_{A'}^{1} + \pi_{A'}^{2})$) so we may expect the group of transformations which preserves these structures to have some significance. Since the signature of the form $Z^{\alpha}\bar{Z}_{\alpha}$is $(++--)$, this group is U(2,2). But if we wish to retain the geometrical significance of the phase of a twistor in terms of a polarisation plane i.e. the flag plane direction of $\pi_{A'}$ then we are led to consider the group SU(2,2) this being actually 4-1 homomorphic with the restricted conformal group. It turns out that twistors form a 4-1 representation space for the identity-connected component of the conformal group. The algebra of twistors is discussed in detail. Twistor space is 8-real-dimensional 4 complex dimensions . We may regard these dimensions as arising as follows; there is a five-dimensional set of null geodesics in M consider the generators of the light cones with vertices at the points of any fixed space like surface and on each geodesic one may give the momentum scaling one parameter the seventh dimension is the polarisation phase of$\pi_{A'}$ and the eighth the intrinsic spin. The non-vanishing of intrinsic spin ($s \neq 0$) implies we do not have a uniquely-defined null geodesic, nor can we easily extend our interpretation to curved space-times. Twistors are a sort of “square root” of the momentum and angular momentum in the same sense in which spinors are a square root of vectors. We said that a twistor was represented by a pair of spinors in a way dependent on choice of origin and conformal scale. We now need to know how the representation alters on change of origin and or scale. Let us first consider the effect of a change of origin on the spinors which represent the twistor. When $O \rightarrow \bar{O}$, we have $P_\alpha \rightarrow \tilde{P}_\alpha = P_\alpha; M^{ab} - \widetilde{M}^{\alpha b} = M^{ab} - 2X^{[aP^b]}$. If we further insist that the phase of $\pi_{A'}$ be unaltered on translation owing to its interpretation as a polarisation plane (flag plane) we find

$$\tilde{\pi}_{A'} = \pi_{A'}; \widetilde{\omega}^A = \omega^A - iX^{AA'}\pi_{A'}, \tag{1.38}$$

given $g_{ab}, \widetilde{\omega}^A$ is a function of position ($\tilde{O}$) and may be regarded as a spinor field. Actually it corresponds, in the case of null twistors, to a field of null directions of straight lines intercepting the worldline. By (1.38)

$$\nabla^{A'(A}\widetilde{\omega}^{B)} = 0. \tag{1.39}$$

In fact the form (1.38) follows from (1.39). For (1.39) implies $\nabla^A_{A'}\widetilde{\omega}^B$ is skew inAB so also is $\nabla^C_{C'}\nabla^A_{A'}\widetilde{\omega}^B$and hence (since in flat space we may commute the derivative operators) this latter is skew in CB and therefore in CAB. Thus it is zero, so $\nabla^A_{A'}\widetilde{\omega}^B$is constant. If this constant is written $i\epsilon^{AB}\pi_{A'}$. (being skew in AB), the general solution of (1.39) is seen to be (1.38).

Since

$$\pi_{A'} = \frac{1}{2} i \nabla_{AA'} \widetilde{\omega}^B \qquad (1.40)$$

the field $\widetilde{\omega}^B$ completely defines the twistor. Moreover, by (1.19), we see that (1.39) is invariant under a conformal rescaling with $\widehat{\widetilde{\omega}}^B = \widetilde{\omega}^B$ Thus a spinor field $\widetilde{\omega}^B$ satisfying (1.39) can be used as a conformally invariant definition of a twistor, which therefore tells us how a spinor representation of a twistor behaves under change of origin or scale. It should be noted that change of origin preserves $\pi_{A'}$ but alters ω^A,while conformal rescaling preserves ω^A but alters $\pi_{A'}$, the point being that, as viewed from the origin, conformal rescalings make infinity appear to be in a different place; the spinor $\pi_{A'}$ is associated with the vertex I_o of the null cone at infinity in the same way that ω_A is associated with the origin O. If we define dropping the from here onward

$$W^\alpha = \omega^A \overline{\omega}^{A'}$$

Then

$$\nabla^{(a} W^{b)} = \frac{1}{4} i g^{ab} (\nabla_c W^c) \qquad (1.41)$$

so that W" is a conformal Killing vector. However,

$$M^{ab} = \nabla^{[a} W^{b]}$$

is not conformally invariant. Thus the angular momentum is not a conformal invariant, although W^a(orω^A) 15. In fact, from (1.20)

$$\widehat{M}^{ab} = \Omega^{-2} \left(M^{ab} + 2\mathcal{T}^{[a} W^{b]} \right).$$

This also follows from

$$\widehat{\omega}^A = \omega^A; \;\; \pi_{A'} = \pi_{A'} + \mathcal{T}_{AA'} \omega^A \qquad (1.42)$$

and (1 .34) which further implies that

$$s = \frac{1}{2}\left(\omega^A \bar{\pi}_A + \pi_{A'} \bar{\omega}^{A'}\right) \tag{1.43}$$

is conformally invariant. It is the conformal invariance of (1.39) and (1.43), together with linearity, which shows that twistors form a representation space (locally) for the conformal group. Now let us consider the equation

$$\nabla_{p'}^{(M} \alpha^{AB\ldots L)} = 0 \tag{1.44}$$

which may be regarded as the many-index spinor equation generalising $\nabla_{p'}^{(M}\omega^{A)} = 0$. The equation is conformally invariant if its solution obeys

$$\hat{\alpha}^{AB\ldots L} = \alpha^{AB\ldots L},$$

If we now form

$$\Psi_{AB} = \alpha^{E\ldots L}\phi_{AB\ldots DE\ldots L} \tag{1.45}$$

where ϕ is a solution to (1.25) we find that $\Psi_{..}$ satisfies (1.25) for a lower spin. In fact (1.44) has (3^{3+n}) linearly independent solutions if α has n indices, as shown explicitly. Each solution in turn, for a given n, may be substituted in (1.45). For example, in the case of linearised gravitation $\phi_{\ldots}$ we can form

$$\Psi_{AB} = \alpha^{CD}\phi_{ABCD}$$

which is a Maxwell field. We may ask what charge integralsthis gives. There are 10 independent solutions for α^{CD}, so we will obtain 10 conserved complex quantities. These are in fact the energy, momentum and angular momentum. These quantities would be complex for a general solution of (l.25) but we get only real quantities for a ϕ_{ABCD} derivable from a potential. If the integrations are performed at infinity, these quantities give the Bondi-Sachs definition of mass, as applied to a general (shearing) retarded null hypersurface in Minkowski space, for linearised theory, so that it becomes clear that the "correction terms" which distinguish this mass measure from the Newman-Unti mass are really necessary even in linearised theory.

The equation

$$\nabla_{B'}^{(B}\omega^{A)} = 0 \tag{1.46}$$

which defines a twistor has 4 linearly independent solutions in M'. There isa difficulty at infinity because to form M we stick the past and future light cones together and the one-superfix twistors differ at those points by a factor i, essentially because the representation of the conformal group in twistor space is via a four-fold covering. For a many-index twistor, one must allow a factor i for each superfix and one factor$i$ for each suffix. We could remove this difficulty by taking a fourfold covering of M but instead we simply adopt the rule of multiplying by i every time we complete a circuit passing through infinity. The problem is an illustration of the fact that twistors are like spinors in not being local geometric objects (for odd-indexed spinors are multiplied by i when they are rotated through 2π). [1]

1.6. TwistorSpace And Minkowski Geometry

A twistor with $2s = Z^{\alpha}\bar{Z}_{\alpha} = 0$ represents a null real straight line (i.e. the worldline of some particle of zero spin). If $s \neq 0$ there is no such real line, but there is in a certain sense a "complex line". Clearly when $s = 0$, Z^{α} and $\lambda Z^{\alpha}(\lambda \neq 0)$ represent the same line so that the most directly geometrically interpretable twistor space is the space N of equivalence classes $\{\lambda Z^{\alpha}\}$ when $s = 0, Z^{\alpha} \neq 0$, i.e.

$$N = \left\{\{\lambda Z^{\alpha}: \lambda \neq 0, \lambda \in \mathbb{C}\}: Z^{\alpha}\bar{Z}_{\alpha} = 0, Z^{\alpha} \neq 0\right\}, \tag{1.47}$$

which represents the set of null lines in M. We shall therefore consider the space C of equivalence classes of twistors, defined like N but without the requirement $s = 0$ $(fig. 3)$. This is complex projective three-space$\mathbb{C}P(3)$ which has three complex or six real dimensions. It is not just the com

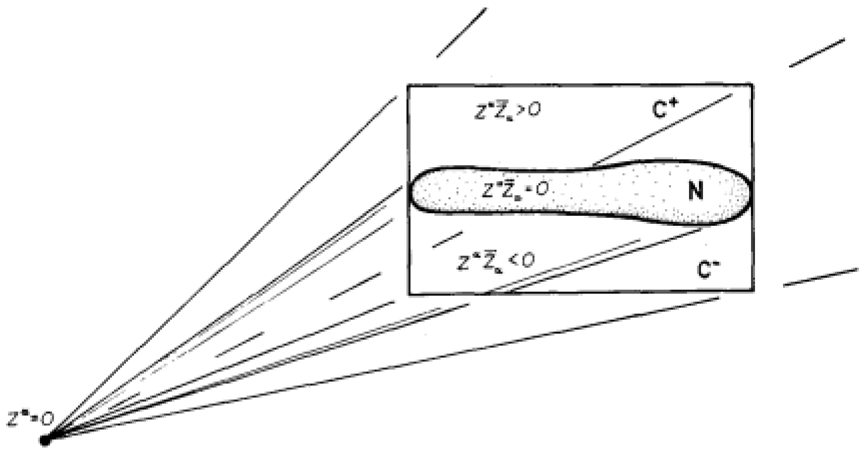


Fig. 3. Projection of twistor space into C.

plexification of N, which would have ten real dimensions. In fact even the complex points of C may be represented as real structures Robinson congruence in M. The conformal transformations of M correspond to projective point transformationsof C preserving N. Let us now consider what a point in M corresponds to in C. We may define a point in M by the intersection of null lines. Suppose two lines are described by twistors $Z^\alpha \leftrightarrow (\omega^A, \pi_{A'}), Y^\alpha \leftrightarrow (\xi^A, \eta_{A'})$.Then the lines meet if there is a common solution to

$$\omega^A = ip^{AA'}\pi_{A'}; \qquad\qquad \xi^A = ip^{AA'}\eta_{A'}.$$

Formally this is

$$P^{AA'} = -\frac{i}{\pi_{B'}\eta^{B'}}\left(\omega^A \eta^{A'} - \xi \pi^{A'}\right) \tag{1.48}$$

but of course the corresponding point need not be real, i.e. $p^{AA'}$ need not be "Hermitian". If the lines do meet (i.e. p^{α} real) then

$$\bar{\eta}_{\alpha}\omega^{A} = i\bar{\eta}_{A}p^{AA'}\pi_{A'} = i\bar{\eta}_{A}\bar{p}^{AA'}\pi_{A'}\pi_{A'} = -\bar{\xi}^{A'}\pi_{A'}$$

$$i.e. Z^{\alpha}\bar{Y}_{\alpha} = 0.$$

Thus the necessary conditions for the two twistors to represent real intersecting lines are

$$Y^{\alpha}\bar{Y}_{\alpha} = 0; \quad Z^{\alpha}\bar{Z}_{\alpha} = 0; \quad Z^{\alpha}\bar{Y}_{\alpha}0. \qquad (1.49)$$

We interpret the condition appropriately when Y and Z are parallel, in which case they meet at infinity, i.e. lie in a null hyperplane. This can be shown, assuming Y and Z to be non-parallel so that π^{A} and η^{A} are not proportional and then taking a limit for the parallel cas, by testing the Hermiticity of (1.48) by taking components with respect to $\pi^{A}$and η^{A}. The three conditions thus derived for Hermiticity are simply (1.49). If (1.49) holds, then it is also satisfied if Y^{α} (or Z^{α}) is replaced by

$$X^{\alpha} = \mu Z^{\alpha} + \lambda Y^{\alpha}$$

for any complex numbers μ, λ . Thus the line X meets each of Y and Z and so belongs to the null cone through the point P with position vectorp^{α}.This null cone can be used to represent P. Thus **P** is represented in **N** by the linear set $\mu X^{\alpha} + \lambda Y$^$\alpha$,i.e. by the complex line **P** joining points **Z** and **Y** which has 2 real dimensions and topology S^{2}.We may therefore represent this point **P** by the 2-index twistor

$$P^{\alpha\beta} = Z^{\alpha}Y^{\beta} \leftrightarrow \begin{pmatrix} \omega^A\xi^B - \xi^A\omega^B & \omega^A\eta_{B\prime} - \xi^A\pi_{B'} \\ \pi_{A'}\xi^B - \eta_{A'}\omega^B & \pi_{A'}\eta_{B'} - \eta_{A'}\pi_{B'} \end{pmatrix}$$

$$= \pi_{D'}\eta^{D'} \begin{pmatrix} -\frac{1}{2}\epsilon^{AB}p_{CC'}p^{CC'} & ip^{A}{}_{B'} \\ -ip_{A'}{}^{B} & \epsilon_{A'B'} \end{pmatrix} \quad (1.50)$$

where $p^a = p^{AA'}$ is the position vector of the point **P**. Thus the points of **M** correspond up to proportionality to simple skew 2-index twistors, i.e. twistors obteying

$$p^{\alpha\beta} = p^{[\alpha\beta]}; \quad P^{[\alpha\beta P\gamma]\delta} = 0 \ \left(i.e. P^{[\alpha\beta P^{\gamma\delta}]} = 0\right).$$

We may define the dual twistor $P_{\alpha\beta}$which gives the geometrically dual description of the same line by

$$P_{\alpha\beta} = \frac{1}{2}P^{\rho\sigma}\epsilon_{\alpha\beta\rho\sigma}. \quad (1.51)$$

One may verify that $p^{\alpha}$is real $\leftrightarrow P_{\alpha\beta} = \bar{P}_{\alpha\beta}$where $\bar{P}_{\alpha\beta}$is the twistor complex conjugate of$P^{\alpha\beta}$.More generally, ifp^a is complex, then its complex conjugate$\bar{p}^{\alpha}$corresponds to $\bar{P}^{\alpha\beta}$ in the same way that p^a corresponds to$P^{\alpha\beta}$.In fact the imaginary part of $p^a$is spacelike, timelike or null respectively according as **P** intersects **N** in a one real-dimensional set (a curve: topology S), in a point, or not at all. If null or time like, the imaginary part of p^a is future pointing or past-pointing according as **p** lies in $C^- \cup \boldsymbol{N}$ or $\boldsymbol{C}^+ \cup \boldsymbol{N}$. Now we recall that we are working in compactified Minkowski space (fig. 1). Suppose **P** is in fact I, the vertex of the null cone at infinity. Null lines at infinity have $\pi_{A'} = 0 = \pi_{\eta'}$ and so the point I corresponds to the twistor

$$I^{\alpha\beta} \leftrightarrow \begin{pmatrix} \epsilon^{AB} & 0 \\ 0 & 0 \end{pmatrix}. \quad (1.52)$$

This we shall call the ‘infinity twistor’. Its dual is

$$I_{\alpha\beta} \leftrightarrow \begin{pmatrix} 0 & 0 \\ 0 & \epsilon^{A'B'} \end{pmatrix}.$$

We can normalise skew twistors by

$$P^{\alpha\beta} I_{\alpha\beta} = 2 \qquad \left(i.e. \pi_{A'} \eta^{A'} = 1\right). \tag{1.53}$$

This fails only if $\pi_{A'}$. And $\eta_{A'}$ are proportional, i.e. only if **P** is at infinity.

Suppose now that $P^{\alpha\beta} I_{\alpha\beta} = Q^{\alpha\beta} I_{\alpha\beta} = 2$. Then by direct calculation we obtain

$$P^{\alpha\beta} Q_{\alpha\beta} = -(p^a - q^a)(p_a - q_a) = (PQ)^2 \tag{1.54}$$

or for non-normalised twistors

$$4P^{\alpha\beta} Q_{\alpha\beta} / \left(I_{\rho\sigma} P^{\rho\sigma}\right)\left(Q^{\delta\gamma} I_{\delta\gamma}\right) = (p^a - q^a)(p_a - q_a).$$

This is clearly a Poincaré invariant quantity. In fact the subgroup of the conformal group which leaves $I_{\alpha\beta}$ invariant is just the Poincaré group. We can form a conformal invariant from the twistors of four points, namely

$$\Phi = \frac{\left(P^{\alpha\beta} Q_{\alpha\beta}\right)\left(R^{\rho\sigma} S_{\rho\sigma}\right)}{\left(P^{\lambda\mu} S_{\lambda\mu}\right)\left(R^{\tau v} Q_{\tau v}\right)} = \frac{(PQ)^2 (RS)^2}{(PS)^2 (RQ)^2}$$

which defines a sort of cross-ratio for any four points in M.

To sum up, a general complex projective line in the projective twistor space C corresponds to a point in $\mathbb{C}\boldsymbol{M}$, the complexification of M; a line in N corresponds to a real point in **M**; a point in **N** corresponds to a null line in **M**. Starting from the space C we can reconstruct $\mathbb{C}\boldsymbol{M}$ as the Klein representation” of lines in the complex three-dimensional projective space C, giving $\mathbb{C}\boldsymbol{M}$ as a quadric fourfold in five-dimensional projective space. In this case all dimensions are complex, so $\mathbb{C}\boldsymbol{M}$ has eight real dimensions.[1]

Chapter 2

Correspondence And Penrose Transforms (Zero-Rest- Mass Field)

2.1. Solutions Of The Zero-Rest-Mass Field Equations

The question we now discuss is how fields in M are represented in twistor space. We shall find that the general zero-rest-mass free fields can be remarkably concisely represented by holomorphic (complex analytic) functions$g(Z^{\alpha})$ and$f(W_{\alpha})$ on the twistor space and its dual, C^{*}. But in order to make the correspondence we must take suitable contour integrals. Thus only the residues at the poles off will be physically meaningful; consequently the subsequent formalism will be based on contour integration in C.

The solutions of the equations (1.25) can be represented by a set of quantities $\phi_r(\boldsymbol{P}, o^A, \iota^B)$ where $r = 0,1\,, \dots n$; $o^A\iota^B$ are a pair of basis spinors at the point **P**, and

$$\phi_r = \phi_{AB\dots L}\, \underbrace{t^A \dots \iota^D}_{r}\, \underbrace{o^E \dots o^L}_{n-r}$$

Now o_A and ι_B define null twistors through $\boldsymbol{P}$, namely U_α, V_βsay, i.e.$U_\alpha \leftrightarrow (o_A, iP^{AA'}o_A), V_\beta \leftrightarrow \left(\iota_B, ip^{BB'}\iota^B\right)$. Thus we have the quantities

$$\Phi_{\mathrm{r}}\left(U_\alpha, V_\beta\right) = \phi_r(P; o^A, \iota^B), \qquad r = 0, \dots n.$$

If U_α and $V_\alpha$are restricted to be null twistors with real intersection, $\Phi_r$rrepresent a zero-rest-mass field in M. Such a field may be regarded as defined on some three-parameter initial set (Cauchy hypersurface) and thence extended over the rest of space by the field equations. In twistor terms it would be economical if we could describe the field on M by some field on the (complex) 3-space **C**, or $\boldsymbol{C}^{*}$. So far it appears that we must define the field on pairs of points U, V in $\boldsymbol{C}^{*}$.

Let us take the point **P** and define a standard tensor and spinor reference frame (cf. (1.6)) such that

$$u = p^{00'} = -\frac{p^0 + P^1}{\sqrt{2}}; \qquad \xi = p^{01'} = \frac{p^2 + ip^3}{\sqrt{2}}$$

$$\tilde{\zeta} = p^{10'} = \frac{p^2 - ip^3}{\sqrt{2}}, \qquad v = p^{11'} = \frac{p^0 - p^1}{\sqrt{2}}.$$

$\bar{\xi} = \tilde{\zeta}, u = \bar{u}, v = \bar{v}$if and only if$p^a$ is real.

$$\frac{\partial \phi_r}{\partial \tilde{\zeta}} = \frac{\partial \phi_{r+1}}{\partial u}; \qquad \frac{\partial \phi_r}{\partial v} = \frac{\partial \phi_{r+1}}{\partial \zeta}; \qquad r = 0, .. n-1. \quad (2.1)$$

These equations are automatically satisfied if

$$\phi_r = \frac{1}{2\pi i} \oint_K \lambda^r F\big(\lambda, u + \lambda\tilde{\zeta}, \zeta + \lambda v\big) d\lambda \qquad (2.2)$$

where F is a holomorphic (i.e. analytic or regular in the complex sense) function ofthree complex variables, the contour K being taken to surround the poles ofF in a suitable way. The resulting fields will always be analytic in the real sense with respect to u, v, ζ, ξ, but we may represent nonanalytic fields as limits of analytic ones.

A real null vector at $p^a = (u, u, \zeta, \xi)$ has direction given by $du{:}\, dv{:}\, d\zeta{:}\, d\xi$ where

$$du + \lambda d\xi = 0 = d\xi + \lambda dv$$

for some complex λ possibly infinite. For the Minkowski metric is $2(dudv - d\zeta d\xi)$so that $dudv = d\zeta d\bar{\zeta}$foranull direction. Thus$du{:}\, dv{:}\, d\zeta{:}\, d\bar{\zeta} = \lambda\bar{\lambda}{:}\, 1{:} -\lambda{:} -\bar{\lambda}$.The corresponding (null) twistor is $U_\alpha + \lambda V_\alpha = W_\alpha \leftrightarrow \left(\bar{\pi}_A, \bar{\varpi}^{A'}\right)$ where

$$\bar{\pi}_{\mathfrak{A}}\pi_{\mathfrak{A}'} \propto \begin{pmatrix} dv & -d\bar{\zeta} \\ -d\zeta & du \end{pmatrix}$$

and $\lambda = \bar{\pi}_1/\bar{\pi}_0 = W_1/W_0$. Thence, as $\bar{\omega}^{A'} = -ip^{AA'}\pi_{A'}$.

$$(W_2, W_3) = \left(\bar{\omega}^{0'}, \bar{\omega}^{1'}\right) = -i(\bar{\pi}_0, \bar{\pi}_1)\begin{pmatrix} u & \zeta \\ \bar{\zeta} & v \end{pmatrix} = -i\, W_0(u + \lambda\bar{\zeta}, \zeta + \lambda v).$$

Thus$(W_0, W_1, W_2, W_3) = W_0\left(1, \lambda, -i(u + \lambda\bar{\zeta}), -i(\zeta + \lambda v)\right)$.If we therefore set

$$f(W_\alpha) = (W_0)^{-n-2} F\left(\frac{W_1}{W_0}, \frac{iW_2}{W_0}, \frac{iW_3}{W_0}\right)$$

Then$f(W_\alpha)$ is homogeneous of degree $-n-2$ in W_α. (We can now check that this has the correct

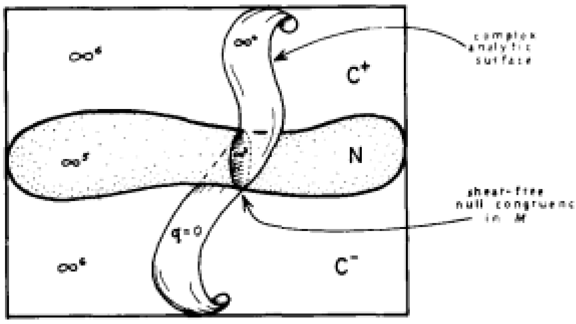


Fig. 1. The Kerr theorem.[2]

transformation properties under rotation for spin $\frac{1}{2}n$.) The final formula is

$$\Phi_r\left(U_\alpha, V_\beta\right) = \frac{1}{2\pi i}\oint_K \lambda^r f(U_\alpha + \lambda V_\alpha)d\lambda. \qquad (2.3)$$

We may now generalize by taking any U_α, V_β no longer necessarily null thus defining complex fields on complex points $U_{1\alpha}V_{\beta 1}$.It seems although there is as yet no completely satisfactory theorem that the set of such fields is extremely general. For a particular field it is clear that f is not unique since all the contour integrals remain the same under$f \rightarrow f + h$ where h is regular inside the contour. We may regard this as a sort of gauge invariance. This non-uniqueness off would clearly lead to difficulties for any proposed explicit formula giving f in terms of $\phi_{A\ldots L}$. It is however easy to construct special types of solution forf. For example $\phi_{A..L}$. is called null if

$$\phi_{AB\ldots L} = \alpha_{(A^{\alpha}B^{\beta}C^{\ldots}\lambda_L)}$$

and such a field arises when the contour surrounds only a single simple pole. Note that a general symmetric spinor may be written as a symmetrised product of one-spinors. More generally, the algebraically special fields

$$\phi_{AB\ldots L} = \alpha_{(A^{\alpha}B^{\beta}C\ldots\lambda_L)}$$

appear as integrals round contours surrounding a pole of order $\leq (n-1)$. E.g. to obtain type {2,2} (i.e. Petrov type D) linearized Weyl tensor fields we may require that has two triple poles and that the contour separates one of them from the other. Since heref has homogeneity degree, it follows that such anf is in fact the inverse cube of a quadratic form. If ϕ is algebraically special (e.g. null) there is associated with it a shearfree null congruence. If

$$f(W_{\alpha}) = p(W_{\alpha})/q(W_{\alpha})$$

then $q(W_{\alpha}) = 0$ is a four (real) dimensional surface in a six dimensional space (C), and intersects the 5-dimensional surface N in a 3-dimensional set of points . This represents a 3-parameter null congruence in M. By a theorem of R.P. Kerr, this congruence must be shearfree. The theorem is that a congruence of null lines is shearfree if and only if it is representable in C as the intersection of N with a complex analytic surface S in C or as a limiting case of such an intersection. It was partly this theorem that motivated the study of holomorphic functions in twistor space.

If we suppose $q = U$ is a plane (i.e. $q(W_\alpha) = A^\alpha W_\alpha$)then we obtain by the above method a “linear” system of null lines in M,which we may consider to be a geometrical picture of the (complex) twistor A^α (which previously had no intuitively obvious picture associated with it). These “Robinson” congruences are largely what led to the name twistor, for they are shearfree, and twist with a handedness dependent on the sign of $A^\alpha \bar{A}_\alpha$.

If we consider a sourcefree spin $\frac{1}{2}n$ massless field in M compactified Minkowski space, which has the correct peeling-off behaviour towardsinfinity, then the field will not match at infinity unless we take a fourfold covering for odd n (twofold for $n \equiv 0$ mod 4). (This is reflected in the behaviour of the integrals introduced above since the homogeneity degree of$f(Z)$ is $-n-2$ and twistors are 4-valued.) Rather than work with awkward covering spaces, however, we shall make the convention that a source-free field with the correct peeling-offproperties is to be regarded as continuous across infinity if it has the right “Grgin discontinuity” at infinity (i.e. a general free wave of spin ~n should jump by a factor of i^{n+2}). Consider then fields with the correct peeling-off and Grgin behaviour which momentum eigenstates, for example, do not have . These may be (uniquely) split into positive and negative energy fields. A process equivalent to Grgin’s harmonic analysis technique applied to the positive energy fields is the following. Instead of $\bar{Z}_0 = \bar{Z}^2$ etc., let us take twistor coordinates so that we get the more natural-looking $\bar{Z}_\aleph = (\bar{Z}^0, \bar{Z}^1, -\bar{Z}^2, -\bar{Z}^3)$, the Hermitian form $Z^\alpha \bar{Z}_\alpha$,of signature $(+ + - -)$, being now diagonalised. The orthonormal basis $\{E_\alpha\}$then has two vectors of positive and two of negative length. These points give us four planes (fig. 2) and the simplest possible function of positive frequency has as its singular region just the planes shaded in fig. 2. A general function for spin $\frac{1}{2}n$ fields of positive frequency is

$$f(\bar{Z}_\alpha) = \sum_{a_0 a_1 a_2 a_3} \frac{(\bar{Z}_0)^{a_0}(\bar{Z}_1)^{a_1}}{(\bar{Z}_2)^{a_2+1}(\bar{Z}_3)^{a_3+1}} f_{a_0 a_1 a_2 a_2 a_3} \qquad (2.4)$$

Where $f_{a_0 a_1 a_2 a_3}$is a constant and $a_0, a_1, a_2, a_3$are nonnegative integers satisfying $a_0 + a_1 + n = a_2 + a_3$. IfS is the set of singularities ofthis runction then assuming suitable convergence $S \cap C^{-*}$ is disconnected in two pieces, and so will yield a positive frequency field. The individual terms in (2.4) will in fact form an orthogonal basis according to the scalar product.[2]

2.2.Quantization

We start out by considering how to connect the spin s of relativistic dynamics, which appeared in the classical twistor picture of angular momentum discussed above with the spin s of the zero-rest-mass fieldsjust considered.

The momentum of a particle with zero-spin was described by $\pi_{A'}(\bar{\pi}_A \pi_{A'} = p_a)$ while the position of the centre of mass is then determined by $\omega^A = iX^{AA'}\pi_{A'}$. As

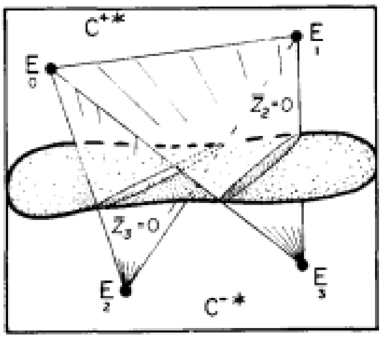


Fig. 2.[2]

$$Z^\alpha \leftrightarrow (\omega^A, \pi_{A'})\bar{Z}_\alpha \leftrightarrow \left(\bar{\pi}_{A'}, \bar{\omega}^{A'}\right)$$

we find that

$$iZ^{\alpha}d\bar{Z}_{\alpha} \leftrightarrow i\omega^{A}d\bar{\pi}_{A} + i\pi_{A'}d\bar{\omega}^{A'} = -X^{AA'}\pi_{A'}.d\bar{\pi}_{A} + \pi_{A'}d\left(X^{AA'}\bar{\pi}_{A}\right)$$
$$= -X^{AA'}\pi_{A'}d\bar{\pi}_{A} + \pi_{A'}(dX^{AA})\bar{\pi}_{A} + \pi_{A'}X^{AA'}d\bar{\pi}_{A} = \pi_{A'}\bar{\pi}_{A}\,dX^{AA'}$$
$$= P_{\alpha}dX^{\alpha} \tag{2.5}$$

If$X^{AA'}$is real. Thus, taking the exterior derivative,

$$idZ^{\alpha d\bar{Z}_0} = dP_{\alpha}^{dX^{\alpha}} \tag{2.6}$$

and the right-hand side isjust the two-form preserved under canonical transformations, i.e. by Hamiltonian equations. For a fuller account of this correspondence. This suggests that we should regard $-iZ^{\alpha}, \bar{Z}_{\alpha}$ as canonically conjugate variables. Thus in the passage to a quantum theory we would expect $-iZ^{\alpha}, \bar{Z}_{\alpha}$ to become canonically conjugate operators with $\bar{Z}_{\alpha} \propto \partial/\partial Z^{\alpha}$.

In the operator form

$$\mathcal{P}_{a} = i\,\partial/\partial x^{a}\left(\text{and} X^{a} = -i\frac{\partial}{\partial P_{a}}\right)$$

$$\mathcal{P}_{a}X^{b} - X^{b}\mathcal{P}_{a} = i\,\delta_{a}^{b}, \tag{2.7}$$

units being chosen so that $\hbar = 1$. Thus we shall want

$$Z^{\alpha} = \frac{\partial}{\partial\bar{Z}_{\alpha}}\left(\bar{Z}_{\alpha} = -\frac{\partial}{\partial Z^{\alpha}}\right)$$

and

$$Z^{\alpha}\bar{Z}_{\beta} - \bar{Z}_{\beta}Z^{\alpha} = \delta_{\beta}^{\alpha}, \tag{2.8}$$

where these operators are taken to act on functions$f(\bar{Z}_{\alpha})$. Now in the method of, ϕ is essentially given by $f(\bar{Z}_{\alpha})$, and it is clear from taking complex conjugates that solutions of(1 .28) are similarly described by a function g(Z^α). Now

$$Z^\alpha f(\bar{Z}) = \frac{\partial}{\partial \bar{Z}_\alpha} f(\bar{Z}); \qquad \bar{Z}_\alpha f(\bar{Z}) = \bar{Z}_\alpha f(\bar{Z})$$

$$Z^\alpha g(Z) = Z^\alpha g(Z); \qquad \bar{Z}_\alpha g(Z) = -\frac{\partial}{\partial Z^\alpha} g(Z). \qquad (2.9)$$

Previously we had $Z^\alpha \bar{Z}_\alpha = 2s$, where $S^a = sP^a, s$ being the spin parallel to the direction of motion. So consider the operator S defined by

$$4S{:} = Z^\alpha \bar{Z}_\alpha + \bar{Z}_\alpha Z^{\alpha'} = 2(\bar{Z}_\alpha Z^\alpha + 2) = 2(Z^\alpha + 2) = 2(Z^\alpha \bar{Z}_\alpha - 2). \ (2.10)$$

then

$$Sg(Z^\alpha) = \frac{1}{2}\big((n+2) - 2\big)g(Z^\alpha) = sg(Z^\alpha)$$

Forg is homogeneous of degree $(-n-2)$ and $2s = n$ whereas $Z^\alpha \partial g(Z)/\partial Z^\alpha$gives $kg(Z^\alpha)$where k is the homogeneity degree. One may, incidentally, say that the fact that $\delta^\alpha_\alpha = 4$ in twistor space, i.e. its 4-dimensionality, is related to the need for the degree $(-n-2)$ in the definition off. We also find $Sf(\bar{Z}_\alpha) = sf(\bar{Z}_\alpha)$ if$n = -2s$, so that the twistor fields corresponding to spinors with primed indices are of opposite helicity, as we expect. The fact that the spin is half-integral is a consequence of the one-valuedness of *f*. We may inquire what is the effect of $Z^\alpha.\bar{Z}_\alpha$ when acting on the fields ϕ ... consider

$$f(W_\alpha) \rightarrow (Q^\alpha W_\alpha) f(W_\alpha), \qquad (2.11)$$

which is the result of $Q^\alpha \bar{Z}_\alpha$. If $Q^\alpha \leftrightarrow (Q^A, Q_{A'})$, eq. (2.11) corresponds to

$$\phi_{AB\ldots L} \rightarrow \tilde{Q}^A \phi_{AB\ldots L} = \psi_{B\ldots L} \qquad (2.12)$$

where $\tilde{Q}^A = Q^A - iX^{AA'}Q_{A'}$,and $\psi_{B\ldots L}$ satisfies the zero-rest-mass field equations for spin $(n-1)$. Similarly, if$R_\alpha \leftrightarrow \left(R_A, R^{A'}\right)$, the operator$R_\alpha Z^\alpha$ acts so that

$$f(W_\alpha) \to R_\beta \frac{\partial}{\partial W_\beta} f(W_\alpha); \quad (2.13)$$

$\phi_{AB\ldots L} \to \frac{1}{2} i(n+1)\phi_{(AB\ldots L\nabla_M)M'}\tilde{R}^{M'} + i\tilde{R}^{M'}\nabla_{M'M}\phi_{AB\ldots L} = X_{AB\ldots M}$, where $X_{AB\ldots M}$ is a solution of the zero rest mass field equations for spin $(n+I)$. Thus $\bar{Z}_\alpha$ raises, and Z^αlowers, the helicity by one half.

2.3.The Linear Penrose Transform

We want to pull back cohomology from P to F and then push it down to M. The composition will then be the Penrose transform mapping cohomology groups on P to solutions of the massless field equations on M. first we shall study the pullback of cohomology from P to F. This is essentially a topological problem, as it turns out. Let W' be an open set in F and let $W = \mu(W')$, which is an open subset of P. We want to determine conditions on W and W' so that data can be transformed from $W$to W'without loss of information. If S is any sheaf on W, then the topological pullback sheaf $\mu^{-1}S$ is a well-defined on W,then the topological Pullback sheaf $\mu^{-1}S$ is a well-defined sheaf on W'. We have an isomorphism $(\mu^{-1}S)_q \cong S_{\mu(q)}$, for all $q \in W'$, and this gives rise to a natural mapping

$$\mu^*: \Gamma(W,S) \to \Gamma(W', \mu^{-1}S)$$

Defined by $\mu^* f \coloneqq f \circ \mu$. we have identified $(\mu^{-1}S)_q$with $S_{\mu(q)}$in this definition. We want to extend this pullback of S to pullback of cohomology with coefficient in S. We can use either Cech cohomology or suitable resolutions of $S$to effect the pullback. For a Cech convering $\mathfrak{U}$ of W, simply pull back the cover $\mathfrak{U}$to a covering $\mathfrak{U}'$of $W'$and pullback representatives of cocyles of a given degree. We want to use specific resolutions, however and we shall develop that point of view.

We are interested in pulling back $H^q\big(W, O, (E)\big)$ for some holomorphic vector bundle E. Dolbeault's Theorem allows us to compute cohomology in terms of the fine resolution

$$0 \to O(E) \to \mathcal{E}^{0,0}(E) \to \mathcal{E}^{0,0}(E) \overset{\bar{\vartheta}}{\to} \mathcal{E}^{0,1}(E) \overset{\bar{\vartheta}}{\to} \ldots .. \tag{2.14}$$

On W, i,e.,

$$H^q\big(W, O(E)\big) \cong H^q\big(W, \mathcal{E}^{0,1}(E)\big),$$

we want to use this resolutio'n to effect the pullback of cohomology. Suppose we look at this same situation a little more abstractly let

$$0 \to S \to \mathcal{R}^0 \to \mathcal{R}^1 \to \cdots$$

Be a resolution of a sheaf S on W. we denote this as

$$0 \to S \to \mathcal{R}^*,$$

we denote the index of the complex of sheaves to avoid overusing the symbol which we shall use for pullback. Now we suppose that we can compute the cohomology of S in terms of the resolution

$$H^P(W, S) \cong H^P\big(\Gamma(W, \mathcal{R}^*)\big) \tag{2.15}$$

As in the example of the Dolpeault representation of $H^{P\{}(W, O(E))$ given above. That is, as in the above example,$\mathcal{R}^*$is a complex of flabby, soft, or fine sheaves or more genrally acyclic sheaves; the Dolbeault resolution is, for instance fine, and therefore acyclic. Note that resolutions of S with the property (2.15) always exist. We can pullback the resolution $\mathcal{R}^*$obtaining on W'

$$0 \to \mu^{-1}S \to \mu^{-1}\mathcal{R}^*, \tag{2.16}$$

The differential sheaf (2.16) will be a resolution of $\mu^{-1}S$, since it is still exact at the stalk level, but it will not necessarily calculate cohomology. for in tstance, if $\mathcal{R}^*$ is fine, then $\mu^{-1}\mathcal{R}^*$ certainly will not be, $\mu^{-1}\mathcal{R}^*$will be constant along the fibers, and this would not be preserved by multiplication by a cutoff function along the fiber direction. Even thouth the resolution (2.16) may not compute the cohomology groups$H^P(W,\mu^{-1}S)$, there is relation given by the spectral sequence of the resolution (2.16) namely, there is a canonical homomorphism

$$H^P\big(\Gamma(W',\mu^{-1}\mathcal{R}^*)\big) \to H^{P\{}(W',\mu^{-1}S), \tag{2.17}$$

This is generalization of the abstract de Rham theorem. Namely, if (2.16) is acyclic, then (2.17) is an isomorphism, and this is simply the abstract, de Rehm. Suppose nowin addition that

$$\mu^*: \mathcal{R}^* \to \mu^{-1}\mathcal{R}^* \text{is a homorphism of complexes,} \tag{2.18}$$

That is to say the diagrams,

$$\begin{array}{ccc} \mu^{-1}\mathcal{R}^p & \longrightarrow & \mu^{-1}\mathcal{R}^{p+1} \\ \uparrow \mu^* & & \uparrow \mu^* \\ \mathcal{R}^p & \longrightarrow & \mathcal{R}^{p+1} \end{array}$$

Are commutative. It follows from (2.18) that there is an homomrphism of complexes

$$\Gamma(W,\mathcal{R}^*) \xrightarrow{\mu^*} \Gamma(W',\mu^{-1}\mathcal{R}^*),$$

And hence a homorphism of the associated cohomolgy

$$H^P\big(\Gamma(W,\mathcal{R}^*)\big) \to H^P(W',\mu^{-1}\mathcal{R}^*), \tag{2.19}$$

Now using (2.15),(2.17), and (2.19) we see that we then have an induced canonical mapping

$$H^P(W,S) \overset{\mu^*}{\to} \Gamma(W',\mu^{-1}S), \qquad (2.20)$$

Now that we have the desired pullback mapping (2.20) of cohomologywe want to investigate its behavior. If $\mu: W' \longrightarrow W$ has connected fibers, then it is clear that

$$H^0(W,S) = H^0(W',\mu^{-1}S),$$

We want to give similar but higher topological conditions on the fibers of μ so that (2.20) is an isomorphism for $p \geq 1$. our major interest for our applications will be the case $p = 1$. let us say that the mapping $\mu: W' \longrightarrow W$ is elementary if the fibers of this mapping $\mu^{-1}(p) = Y_p \subset W'$are all connected, and, moreover have vanishing first Betti number, i.e., $H^0\left(Y_p, C\right) = 0$, for all $p \in W$.

Remark(2.3.1). Suppose that $U \subset M, U' = v^{-1}(U), \widehat{U} = \mu \circ v^{-1}(U)$, as in, then each fiber Y_p of mapping $\mu: U' \longrightarrow \widehat{U}$, for $p \in \widehat{U}$is biholomorphic to $\hat{p} \cap U$. suppose, for instance that U is convex in therefore convex. Hence $\mu: U' \longrightarrow \widehat{U}$ for this U will be an elementary mapping .This shows that the condition of being elementary is not too difficult to check in various cases. We now have the following important lemma. If $V$is a smooth vector bundle over$W \subset P$, then let $\mathcal{E}(V)$ be the sheaf of V.

Lemma (2.3.2). Suppose that $\mu: W' \longrightarrow W$ is elementary, and that V is a smooth vector bundle over W, then

$$H^1\Gamma(W',\mu^{-1}\, \mathcal{E}(V)) = 0,$$

Proof: We define a fine resolution of $\mu^{-1}\mathcal{E}(V)$by the sheaves of smooth relative p −forms $\mathcal{E}_\mu^p(V)$on W'in analogy with (7.1.3):

$$0 \longrightarrow \mu^{-1}\mathcal{E}(V) \longrightarrow \mathcal{E}_\mu^0(V) \xrightarrow{d_\mu} \mathcal{E}_\mu^1(V) \overset{d_\mu}{\to} \mathcal{E}_\mu^2(V) \longrightarrow \cdots$$

This is affine resolution, so by the abstract de Rehm

$$H^P\left(W', \mu^{-1}\mathcal{E}(V)\right) \cong H^P\left(\Gamma(W', \mathcal{E}_\mu^*(V)\right),$$

thus we need to show that

$$\Gamma\left(W', \mathcal{E}_\mu^0(V)\right) \xrightarrow{d_\mu} \Gamma\left(W', \mathcal{E}_\mu^1(V)\right) \xrightarrow{d_\mu} \Gamma\left(W', \mathcal{E}_\mu^2(V)\right), \qquad (2.22)$$

Is exact. Suppose for each $p \in W$there is a neighborhood $N$of $p$such that the sequence

$$\Gamma\left((\mu^{-1}(\mathrm{N}) \cap W', \mathcal{E}_\mu^0(V)\right) \xrightarrow{d_\mu} \Gamma\left((\mu^{-1}(\mathrm{N}) \cap W', \mathcal{E}_\mu^1(V)\right)$$

$$\xrightarrow{d_\mu} \Gamma\left((\mu^{-1}(\mathrm{N}) \cap W', \mathcal{E}_\mu^2(V)\right) \quad (2.23)$$

Is exact. Let $\mathfrak{U} = \{U_\alpha\}$ be an open convering of $W$with the property that (2.23) is exact for each $N \in \mathfrak{U}$. Then let $\omega \in \Gamma\left(W', \mathcal{E}_\mu^1(V)\right)$, such that $d_\mu \omega = 0$, now let $f_\alpha \in \Gamma\left((\mu^{-1}(U_\alpha), \mathcal{E}_\mu^1(V)\right)$ satisfy $d_\mu f_\alpha = \omega$, for all α. Now let $\{\varphi_\alpha\}$be a partition of unity with respect to the convering $\mathfrak{U}$, then $\mu^*\varphi_\alpha$is a partition of unity with respect to the convering $\mathfrak{U}' = \{\mu^{-1}U_\alpha\}$ of W'. Define

$$f = \sum_\alpha (\mu^*\varphi_\alpha) f_\alpha.$$

We see that

$$d_\mu f = \sum_\alpha \left((\mu^*\varphi_\alpha) d_\mu f_\alpha + d_\mu(\mu^*\varphi_\alpha) f_\alpha\right)$$

$$= \sum_\alpha (\mu^*\varphi_\alpha \omega + 0) = \omega,$$

Where $d_\mu(\mu^*\varphi_\alpha) = 0$ since $\mu^*\varphi_\alpha$is constant along the fibers of μ.

Thus (2.22) is exact if (2.23) is exact for neighborhods of arbitrary points of M.

Now we shall check that (2.23) is exact for suitable neighborhoods of a given Point p. Choose such a neighborhood N so that $\mu^{-1}(N) \cong N \times P_2$i.e., the fibration μ is trivial over N. Now let $N' \coloneqq \mu^{-1}(N) \cap W'$, then we see that

$$H^1(N', \mu^{-1}\, \mathcal{E}(V)) \cong H^1(N', \mu^{-1}\, \mathcal{E})^r,$$

Where r is the rank of V. Thus we need to show that (using (2.21) in the case where V is the trivial line bundle)

$$\Gamma(N', \mathcal{E}_\mu^0) \xrightarrow{d_\mu} \Gamma(N', \mathcal{E}_\mu^1) \xrightarrow{d_\mu} \Gamma(N', \mathcal{E}_\mu^2) \tag{2.24}$$

Is exact choose $\omega \in \Gamma(N', \mathcal{E}_\mu^1)$ such that $d_\mu \omega = 0$. then letting $q \in N$, we can consider ω as a smooth family of one-forms on open subsets of p_2 with

Coefficients depending smooth on the parameter $q \in N$. we write $\omega \in N$. we write $\omega(q)$ as the one-form defined on $(\{q\} \times P_2) \cap W'$. now define, for $(q, s) \in N \times P_2$,

$$f(q, s) = \int_{(q, s_0)}^{(q, s)} \omega\,(q),$$

Where the integral is along any path in $(\{q\} \times P_2) \cap W'$, joining $(q, s_0)$to (q, s). By the assumption that μ is elementary, we see that integral is independent of the path, and hence we obtain a solution of the equation

$$d_\mu f = \omega$$

On N', and (2.24) is exact, as desired.

We now have the following fundamental result.

Theorem (2.3.3)

suppose $W' \subset F$ is an open set, $W = \mu(W')$, and E is a holomorphic vector bundle on W. If $\mu: W' \longrightarrow W$is elementary, then

$$\mu^*: H^1(W, O\ (E)) \longrightarrow H^1(W', \mu^{-1}\ O(E))$$

Is a canonical isomorphism.

Proof: The spectral sequence of the resolution

$$0 \longrightarrow \mu^{-1}O(E) \longrightarrow \mu^{-1}\mathcal{E}^{0,*}(\mathrm{E})$$

Has the form

$$E_2^{pq} = H^p\big(H^q(W', \mu^{-1}\ \mathcal{E}^{0,*})\big) \Rightarrow H^r(W', \mu^{-1}\ O(E))$$

Since $H^1(W', \mu^{-1}\ \mathcal{E}^{0,*}) = 0$, by Lemma (2.3.2), the E_2-term of the spectral sequence has the form, letting $\mathcal{A}^* = \mu^{-1}\ \mathcal{E}^{0,*}(E)$,

$$\begin{array}{ccc} * & * & * \\ 0 & 0 & 0 \\ H^0\big(\Gamma(W', \mathcal{A}^*)\big) & H^1\big(\Gamma(W', \mathcal{A}^*)\big) & H^2\big(\Gamma(W', \mathcal{A}^*)\big) \end{array}$$

It follows that $E_2^{10} = H^1\big(\Gamma(W', \mathcal{A}^*)\big), E_\infty^{01} = 0$, and hence that

$$H^1(W', \mu^{-1}\ O(E)) \cong H^1\big(\Gamma(W', \mu^{-1}\ \mathcal{E}^{0,*})\big),$$

i.e. that (2.17) is an isomorphism. Now $\mu: W' \longrightarrow W$has connected fibers implies that

$$\Gamma\big(W', \mu^{-1}\ \mathcal{E}^{0,\mathrm{q}}\ (E)\big) = \Gamma\big(W, \mathcal{E}^{0,\mathrm{q}}\ (E)\big).$$

Hence ,$\mu^*$is the composition of three isomorphisms

$$H^1\big(W, O(E)\big) \xrightarrow{\cong} H^1\big(\Gamma(W, \mathcal{E}^{0,*}(\mathrm{E}))\big)$$

$$\xrightarrow{\cong} H^1\big(\Gamma(W', \mu^{-1}\ \mathcal{E}^{0,*}(\mathrm{E}))\big) \xrightarrow{\cong} H^1(W, \mu^{-1}\mathrm{O}(\mathrm{E})),$$

And the theorem follows.

Remark (2.3.4)

Buchdahl has a generalization of this theorem for higher degree cohomology.

Now let X be any open subset of F, then there is a spectral sequence associated to the resolution of the form

$$E_1^{pq} = H^q\left(X, \Omega_\mu^p(E)\right) \Rightarrow H^r(\mathrm{X}, \mu^{-1}O(E)). \qquad (2.25)$$

This follows and will be a principal tool in the Penrose transform, as we shall see below. We have now Pulled back $H^1(W, O(E))$ to

$H^1(W', \mu^{-1}\, \mathrm{O(E)})$ isomorphically, if $\mu: W' \longrightarrow W$ is elementary. We now want to push down $H^1(W', \mu^{-1}\, \mathrm{O(E)})$ to M. We shall do this for different special choices of $W'$and the vector bundle E. our first relates to right-handed massless fileds on M. let U. be an open subset of M. We define

$$Z_n' \coloneqq ker\nabla_A^{A'}: o_{(A'B'\ldots D')}[1]' \to o_{A(B'\ldots D')}[2]',$$

Where the differential operator $\nabla_A^{A'}$maps conformally weighted spinor fields to conformally weighteted spinor fields. We call $Z_n'$the sheaf of holomorphic right-handed massless free fields of helicity $n/2$. If we consider of Z_n'on an open subset of M'we have the right-handed massless field described, i.e., symmetric spinor fields which are solutions of

$$\nabla^{AA'}\varphi_{(A'B'\ldots D')} = 0$$

Now for any open set $U \subset M$ we let $U' \coloneqq \mu^{-1}(U), \widehat{U} \coloneqq \mu \circ v^{-1}(U)$,

As before. following We then have the basic result.

Theorem (2.3.5). Let Ube open in M, and suppose $n \geq 1$, then there is a canonical linear transformation

$$\mathcal{P}: H^1(\widehat{U}, \mathrm{O}(-\mathrm{n}-2)) \to \Gamma(U, Z_n').$$

If $\mu: U' \longrightarrow \widehat{U}$ has connected fibers, then $p$is injective, and if $M: U' \longrightarrow \widehat{U}$ is elementary, then p is an isomorphism.

Proof: We already know from Theorem (2.3.3) that

$$\mu^*: H^1(\widehat{U}, \mathrm{O}(-\mathrm{n}-2)) \to H^1(\widehat{U}, \mu^{-1}(-\mathrm{n}-2))$$

Exicts, and is injective or an isomorphism to the topological hypotheses of the theorem. We shall show that there is a natural isomorphism

$$H^1(U', \mu^{-1}\mathrm{O}(-\mathrm{n}-2)) \cong \Gamma(U, Z'_n).$$

And the theorem will be proved. We shall use the spectral sequence (2.25) to compute $(U', \mu^{-1}O(-\mathrm{n}-2))$ in terms of the cohomology groups $(U', \Omega^s_\mu(-\mathrm{n}-2))$. Then we shall use the Leray spectral sequence of the fibration $\mu: U' \longrightarrow U$,

$$\tilde{E}_2^{pq} = H^p(U, \mu^q\, \Omega^s_\mu(-\mathrm{n}-2)) \Longrightarrow H^r(U', \Omega^s_\mu(-\mathrm{n}-2)) \qquad (2.26)$$

To relate the cohomology groups $H^r(U', \Omega^s_\mu(-\mathrm{n}-2))$to cohomology groups on U. Let us handle the Leray spectral sequence first, since it is somewhat simpler.

For a fixed s

$$\mathrm{v}^q_*\, \Omega^s_\mu(-\mathrm{n}-2) = 0, \quad q \neq 1,$$

and $\mathrm{v}^q_*\, \Omega^s_\mu(-\mathrm{n}-2)$are particular nontrivial spinor sheaves which are given specifically. Thus the spectral sequence $\tilde{E}_2^{pq}$ is degenerate at the second level, i.e., we have $\tilde{E}_2^{pq} = 0$, for $q \neq 1$. Hence $\tilde{E}_\infty^{pq} = \tilde{E}_2^{pq}$, and it follows that

$$H^r(U', \Omega^s_\mu(-\mathrm{n}-2)) \cong H^{r-1}(U\mathrm{v}^1_*\, \Omega^s_\mu(-\mathrm{n}-2)), \qquad (2.27)$$

For $r \geq 1$, and $H^0(U', \Omega^s_\mu(-\mathrm{n}-2)) = 0$. Thus is a version of integration over the fibers of the mapping v.

Now returning the spectral sequence (2.25) we can write down the E_1-term as follows (letting $m = -n - 2$, for simplicity)

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$H^2\big(U',\Omega^0_\mu(\mathrm{m})\big) \overset{d_\mu}{\to} H^2\big(U',\Omega^1_\mu(\mathrm{m})\big) \overset{d_\mu}{\to} H^2\big(U',\Omega^2_\mu(\mathrm{m})\big) \to$$

$$H^1\big(U',\Omega^0_\mu(\mathrm{m})\big) \overset{d_\mu}{\to} H^1\big(U',\Omega^1_\mu(\mathrm{m})\big) \overset{d_\mu}{\to} H^1\big(U',\Omega^2_\mu(\mathrm{m})\big) \to$$

$$H^0\big(U',\Omega^0_\mu(\mathrm{m})\big) \overset{d_\mu}{\to} H^0\big(U',\Omega^1_\mu(\mathrm{m})\big) \overset{d_\mu}{\to} H^0\big(U',\Omega^2_\mu(\mathrm{m})\big) \to$$

Replacing the terms in this array by the corresponding groups on U under the isomorphism(2.26), the mappings by the induced mappings on the direct image sheaves, and we see that the E_1-term

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$H^1\big(U,v^1_*\Omega^0_\mu(\mathrm{m})\big) \underset{\longrightarrow}{\check{d}_\mu} H^1\big(U,v^1_*\Omega^1_\mu(\mathrm{m})\big) \underset{\longrightarrow}{\check{d}_\mu} H^1\big(U,v^1_*\Omega^2_\mu(\mathrm{m})\big) \to$$

$$H^0\big(U,v^1_*\Omega^0_\mu(\mathrm{m})\big) \underset{\longrightarrow}{\check{d}_\mu} H^0\big(U,v^1_*\Omega^1_\mu(\mathrm{m})\big) \underset{\longrightarrow}{\check{d}_\mu} H^0\big(U,v^1_*\Omega^2_\mu(\mathrm{m})\big) \to$$

$$0 \qquad\qquad 0 \qquad\qquad 0$$

Now we see that

$$E_2^{01} = \ker\left\{H^0\big(U,v^1_*\Omega^0_\mu(\mathrm{m})\big) \underset{\longrightarrow}{\check{d}_\mu} H^0\big(U,v^1_*\Omega^1_\mu(\mathrm{m})\big)\right\},$$

and$E_1^{10} = E_2^{10} = E_\infty^{10} = 0$. moreover, we see from the above diagram that $E_2^{10} = E_\infty^{10}$. We see that

$$H^1(U',\mu^{-1}O(\mathrm{m})) \cong E_2^{01}.$$

Finally we obtain

$$H^1(U',\mu^{-1}O(\mathrm{m})) \cong \ker\Gamma\big(U,o_{(A'B'\ldots D')}[1]'\big) \xrightarrow{\nabla_A^{A'}} \Gamma\big(U,o_{(A'B'\ldots D')}[2]'\big)$$
$$\cong \Gamma(U,Z_n').$$

If we let $U = M^{\pm}$orM^{1}, then we see that topological conditions in Theorem (2.3.5) are satisfied. Thus we have the following corollary.

Corollary (2.3.6). For $n \geq 1$,

$$\text{(a)}\ \ \rho: H^{1}(P^{\pm}, \mathrm{O}(-\mathrm{n}-2)) \to \Gamma(M^{\pm}, Z'_{n}),$$
$$\text{(b)}\ \ \rho: H^{1}(P^{I}, \mathrm{O}(-\mathrm{n}-2)) \to \Gamma(M^{I}, Z'_{n}),$$

Are canonical isomorphisms.

Remark (2.3.7). Theorem (2.3.5) describes every right-handed holomorphic massless field locally as each point of M has convex neighborhoods which will satisfy the topological hypotheses of the theorem.

We now turn our attention to the solution of the wave equation, the helicity zero case. This is less straightforward then the positive helicity case treated above, as it involves second order differential equations. Recall that the wave equation has the form on M^{1}

$$\Box\varphi = 0, \qquad (2.28)$$

Where $\Box \coloneqq \nabla^{a}\nabla_{a} = \nabla^{AA'}\nabla_{AA'}$, and $\varphi$is a scalar function. This is the helicity zero massless field equation. We want to extend this to act on conformally weighted scalar fields or functions on all of M. We note that

$$O \xrightarrow{\nabla_{AA'}} O_{AA'},$$

$$O \xrightarrow{\nabla_{AA'}} O^{AA'}[1][1]',$$

Since $\nabla^{AA'} = \varepsilon^{AB}\varepsilon^{A'B'}\nabla_{BB'}$, and form this we see that $\Box = \nabla^{AA'}\nabla_{AA'}$is a well-defined mapping

$$\Box: O \to O[1][1]'.$$

We can now tensor this with $o[1]'$and obtain the mapping $\Box \otimes id$ which we still denote by $\Box$ and we have

$$\square : O \rightarrow O[1][2]'. \qquad (2.29),$$

We see that (2.29) is a well-defined differential operator of conformally weighted buondlesand the solutions of this equation $\square\varphi = 0$ on open sets of M^1 can be identified with the solutions of (2.28) on that same set. The mapping (2.29) is conformally with invariant. There are no choices of coordinates in its definition. It depends only on exterior differention$(\nabla_{AA'})$ and the global εs, all of which are conformally invariant operators. We now use (2.29) to define an appropriate sheaf of solutions of the global wave equation. Namely, we define

$$Z_0' \coloneqq \ker\{\square : O[1]' \rightarrow O[1][2]'\}$$

And we call $Z_0'$the sheaf of massless fields of helicity zero on $M. Z_0'$ on open subsets U of M^1will then be solutions of the wave equation(2.28) on U. we now have the following result.

Theorem (2.3.8). Let U be open in M, then there is a canonical linear transformation

$$\rho : H^1(\widehat{U}, O(-2)) \rightarrow \Gamma(U, Z_0').$$

If $M: U' \rightarrow \widehat{U}$has connected fibers, then ρ is injective, and if $\mu: U' \rightarrow \widehat{U}$is elementary, then ρ is an isomorphism.

Proof: the proof is identical to the proof of Theorem (2.3.5) except that we have to treat the spectral sequence (2.25) for the case $n = 0$ somewhat differently, as it does not degenerate to first order as it did in that case, the

E_1-term of (2.25) for $m = 0$ has the form

$$H^2(U', \Omega_\mu^0(-2)) \longrightarrow H^2(U', \Omega_\mu^1(-2)) \longrightarrow H^2(U', \Omega_\mu^2(-2)) \longrightarrow$$

$$H^1(U', \Omega_\mu^0(-2)) \longrightarrow H^1(U', \Omega_\mu^1(-2)) \longrightarrow H^1(U', \Omega_\mu^2(-2)) \longrightarrow$$

$$H^0(U', \Omega_\mu^0(-2)) \longrightarrow H^0(U', \Omega_\mu^1(-2)) \longrightarrow H^0(U', \Omega_\mu^2(-2)) \longrightarrow \qquad (2.30)$$

We need to calculate

Each column of (2.30) separately in terms of cohomology on U. For $E_2^{0q} = H^q\big(U', \Omega_\mu^0(-2)\big)$, we have the only nonvanishing direct image of $H^q\big(U', \Omega_\mu^0(-2)\big)$is $v_*^1 \Omega_\mu^0(-2) \cong \mathrm{O}[1]'$, from theorem (2.3.5) therefore we see from the Leray spectral sequence (2.26) that

$$H^q\big(U', \Omega_\mu^0(-2)\big) \cong H^{q-1}(U, O[1]') \quad for\ q \geq 1,$$

$$H^0\big(U', \Omega_\mu^0(-2)\big) = 0.$$

For the second column of (2.30) theorem 7.1.5 implies that

$$v_*^r \Omega_\mu^0(-2) = 0 \quad r \geq 0,$$

And hence from (2.26) we find that

$$H^q\big(U', \Omega_\mu^1(-2)\big) = 0, \qquad q \geq 0.$$

For the third column, we see, again by theorem that the only nonvanishing direct image of $\Omega_\mu^2(-2)$ is given by

$$v_*^0 \Omega_\mu^2(-2) \cong H^q(U, O[2]'[1]), \quad \mathrm{q} \geq 0.$$

Now there are only nonzero columns in (2.30) and they are separated by zero column. It follows that $d_1$vanishes identically on E_1^{0q}, and hence that $E_2^{pq} = E_1^{pq}$. we define the mapping D to be the mapping induced on $E_2^{01}$by the spectral sequence mapping d_2, i.e.,

$$D \coloneqq d_2 : E_2^{01} \to E_2^{20}.$$

Taking into account our calculations above, we see that following array represents the E_2-term of the spectral sequence and the mapping D has been singled out from the family of mapping which constitute

$$H^1\big(U^0, [1]' \qquad 0 \qquad H^2(U, O[1][2]')\big),$$

$$H^0(U, O[1]') \qquad 0 \qquad H^1(U, O[1][2]O),$$

$$0 \qquad 0 \qquad H^0(U, O[1][2]'). \tag{2.31}$$

Thus we see that

$$D: H^0(U, O[1]') \to H^0(U, O[1][2]').$$

One can see that D is the composition of the first order differential operators, and hence is asecond order differential operator. It is also conformally invariant as all of the operators in the spectral sequence are conformally invariant. We shall identify it with the wave operator shortly. First we note that

$$E_3^{01} = \ker\{D: H^0(U, O[1]') \to H^0(U, O[1][2]')\}. \tag{2.32}$$

Moreover, we find that $E_3^{01} = E_\infty^{01}$, and $E_1^{01} = E_\infty^{01} = 0$, so by theorem

$$H^1(U', \mu^{-1}O(-2)) \cong E_3^{01}. \tag{2.33}$$

One we have identified D with a constant multiple of $\square$, we see that (2.32) and (2.33) will prove the theorem.We now turn to the verification of this last fact. Consider the exact sequence of sheaves on F

$$0 \longrightarrow O(-2) \overset{\pi_{A'}}{\longrightarrow} O_{A'}(-1) \overset{\pi^{A'}}{\longrightarrow} O[1]' \longrightarrow 0, \tag{2.34}$$

This is a complex of sheaves, and we can map it linearly to a second complex in the following manner:

$$O(-2) \overset{\pi_{A'}}{\longrightarrow} O_{A'}(-1) \overset{\pi_{A'}}{\longrightarrow} O[1]'$$

$$\downarrow \qquad\qquad \downarrow \nabla_{a'}^{A} \qquad\qquad \downarrow \frac{1}{2}\square$$

$$O(-2) \xrightarrow{\pi_{A'}\nabla_A^{A'}} O_{A'}(-1) \xrightarrow{\pi_{A'}\nabla^{AA'}} O[1][2]'$$

Where the second sequence is the same as the relative de Rham sequence

$$\Omega_\mu^0(-2) \xrightarrow{\mu} \Omega_\mu^1(-2) \overset{d_\mu}{\rightarrow} \Omega_\mu^2(-2).$$

Each of the horizontal sequences is a differential sheaf and as such each has a spectral sequence associated with it.

The spectral sequence of the botoom sequence is the one we have been considering ((2.25) for $n = 0$). The vertical mapping induces mappings on the spectral sequences, and, in particular, at the E_2 level. Let $\tilde{E}_r^{pq}$ be the spectral sequence of the top horizontal complex of sheaves. Noting that $H^0(U', O[1]') \cong H^0(U, O[1]')$, and computing $\tilde{E}_r^{pq}$ in terms of cohomology on U, we can express the mapping between the two spectral sequences at the E_2-level for the relevant terms in the following manner:

$H^0(U, O[1]')$

$0 \searrow \alpha$

$\tilde{E}_2$ $\quad 0 \downarrow \beta$ $\quad H^0(U, O[1]')$

$\downarrow$ $\quad H^0(U, O[1]')$ $\quad \downarrow \frac{1}{2}\square$

$0 \searrow 0D$

E_2 $\quad 0$ $\quad H^0(U, O[1][2]')$

We see that since (2.34) is exact, this particular differential sheaf is a resolution of 0,and hence $\tilde{E}_\infty^{01} = 0$, which implies that$\tilde{E}_3^{01} = 0$, and hence that $\alpha$is an isomorphism. In fact, αmust be the identity mapping, since it is a canonical isomorphism, we see that β is also the identity, and it follows from the commutatvitity of the diagram that $D = \frac{1}{2}\square$.

We have now represented all massless fields of nonnegative helicity in terms of cohomological data on P. We now want to consider the case of negative helicity. One approach to studying negative helicity is to use the same methodology above to represent left-handed fields in terms of cohomology on dual projective twistor space $P^* = P(T^*)$.

However, it is also possible to define Penrose transform directly on P. let, for $n \geq 1$,

$$Z'_{-n} \coloneqq \ker\{\nabla^{DD'}: O_{(AB\dots CD)}[1] \to O_{(AB\dots C)}[2][1]'\}, \tag{2.36}$$

Be the sheaf of left-handed massless fields on $M$of helicity$-n/2$. what we want to describe is a Penrose transform of the form, for $n > 0$, and for U open in M,

$$\rho: H^1\big(\widehat{U}, O(n-2)\big) \to \Gamma(U, Z'_{-n}). \tag{2.37}$$

Let us reconsider the positive helicity case for a moment to illustrate the difficulty in defining (2.37). we shall contrast the cases of helicity 2 and-2 for simplicity self-dual and anti-self-dual Maxwell fields, respectively . If

$$\rho: H^1\left(\widehat{U}, O(-4)\right) \to \Gamma(U, Z'_2)$$

Is given by Theorem (2.3.5), then let $\in H^1\left(\widehat{U}, O(-4)\right)$, and consider the value of $\rho(\omega)(x)$, for a specific point x in U. The fiber of the bundle whose are massless fields is given by which, by the results, is isomorphic to$H^1\big(\hat{x}, O_{\hat{x}}(-4)\big)$ using this isomorphism, we see that is obtained by taking the cohomology class ω and restricting it to the projective line $\hat{x}$ obtaining an element of the vector space. The Serre duality argument then gives the more normal spinor representation of this same vector, which is the value of the field$\rho(\omega)$ at$x$. Thus, in summary, restricting the cohomology class to $\hat{x}$gives the value of the field. If ω is represented in terems of differential forms, then this is the usual restriction of a differential form to a submanifold. Now we consider the same situation for the case of negative helicity. Take an element$\omega \in H^1(\hat{U}, O)$, and consider the restriction $\omega|_{\hat{x}} H^1(\hat{x}, O_{\hat{x}})$, then we see that since $H^1(P_1, O_{P_1}) = 0$, then the restriction is necessarily zero, and so this cannot be the value of the field. So simple restrictions of cohomology groups will not yield anything in this case. The field at $x$defined by ω turns out to depend on the restriction of ω to infinitesimal neighborhoods of the submanifold $\hat{x} \subset P$. the original integral formulae of Penrose for this negative helicity case involve not only integration that is, the Serre duality part, but also differentiation of the integrand, if we think of the cohomology class ω as represented by a differential form say, then we can expand the differential form in aTaylor series with respect to coordinates normal to the submanifold $\hat{x}$(at least locally this makes sense in terms of usual power series). The fact that$\omega|_{\hat{x}}$vanishes says the leading term of this series vanishes, it turns out that the value of the field is given by the first nonvanishes. Coefficient of this expansion. If we have helicity $-n/2$, then the first nonvanishing coefficient turns out to be at the nth order. This has to all be given meaning in turms of the cohomological data. We shall use ideal sheaves to represent the normal bundle and essentially expand cohomology classes in terms of powers of the conormal bundle let us describe this briefly here in a general context, and we shall see a specific paramertrized version of this in the proof of the next theorem.

Suppose that Y is a complex submanifold of a complex manifold X. let $\mathcal{T}$be the ideal sheaf of the submanifold Y, and consider the exact sequence

$$0 \longrightarrow \mathcal{T} \longrightarrow O \longrightarrow O/\mathcal{T} \longrightarrow 0.$$

We shall see below that the quotient sheaf$O/\mathcal{T}$can be identified with the sheafO_Y, the sheaf of holomorphic functions on the submanifold Y. more generally there is an isomorphism which has the form

$$O|_Y \cong O/\mathcal{T} \oplus \mathcal{T}/\mathcal{T}^2 \oplus \mathcal{T}^2/\mathcal{T}^3 \bigoplus \quad \dots, \tag{2.38}$$

Which is given by the Taylor expansion of a germ of f at any point $x \in Y$. Namely, if we choose local coordinates (z, ω) for X at a point $x \in Y$ such that

$$Y = \{(z, \omega); \omega_1 = \cdots = \omega_r = 0\},$$

Then we have for any $f \in O_x$ the expansion

$$f(x) = f(z, \omega) = \sum_{i_1+\cdots+i_r=p} a_{i_1+\cdots+i_r}(z)\, \omega_1^{i_1} \dots \omega_r^{i_r}. \tag{2.39}$$

We see that the monomials $\omega_1^{i_1} \dots \omega_r^{i_r}$for $i_1 + \cdots + i_r = p$, give abasis for the quotient stalk $\mathcal{T}_x^p/\mathcal{T}_x^{p+1}$, which is a finite-dimensional vector space, for each p, the constant term' has the form $a_0(z) \in O/\mathcal{T}$ and this shows that $O/\mathcal{T} \cong O_Y$; these are the intrinsic holomorphic functions on Y. the normal bundle $N = N_Y$of the embedding Y in X is defined by the quotient bundle

$$0 \longrightarrow T(Y) \longrightarrow T(X)|_Y \longrightarrow N_Y \longrightarrow 0.$$

The dual of the normal bundle is N^*, and one can show that

$$O(N^*) \cong \mathcal{T}/\mathcal{T}^2. \tag{2.40}$$

Thus we can express the expansion (2.38) in the form

$$O|_Y \cong \sum_p O_Y(\odot^p N^*). \tag{2.41}$$

Moreover, the symmetric powers have the form

$$O(\odot^p N^*) \cong \mathcal{T}^p/\mathcal{T}^{p+1}.$$

Where$O_Y(\odot^0 N^*) \coloneqq O_Y$, the expansion (2.41) is a formal version of (2.39) in coordinates, where there the homomgeneous powers ofω_i in (2.39) are replaced by the symmetric powers of the vector bundle N^*. The left hand side of (2.41) will correspond to a sequence of intrinsic to Y of the bundles on the right hand side. This is the same as the local correspondence

$$f \longmapsto \left\{a_{i_1}, \dots i_r(z)\right\}$$

In the expansion (2.39).

There are two approaches to understanding (2.37). The first is with potentials. If we apply the spectral sequence machine to the relative de Rham complex for the sheaf $O(n-2)$ we obtain fields on M which do not satisfy the massless field equations. However, if we differentiate these fields appropriately differentiate to n th order then we find that we generate massless fields of negative helicity, and, at least locally, all such fields can be so represented t. So the cohomological machine plus differentiation yields the desired Penrose transform. On the other hand, one can obtain the transformdirectly without potentials in the following manner (Wells). We shall show how to calculate the field at a point $x \in U$ by expanding a cohomology class in a power series about the submanifold $\hat{x} \subset P$. Let $\omega \in H^1\left(\widehat{U}, O(n-2)\right)$, then ωdefines naturally by restriction an element of$H^1(\hat{x}, O(n-2))$, which we still call ω. Using the expansion (2.41) we can write formally

$$\omega = \omega_0 + \omega_1 + \cdots + \omega_p + \cdots,$$

Where

$$\omega_p \in H^1\left(\hat{x}, O_{\hat{x}}\left(H^{n-2} \otimes \odot^p(N_{\hat{x}}^*)\right)\right). \tag{2.42}$$

But one can calculate that $N_{\hat{x}} \cong H \oplus H$ (where here H is the hyperplane bundle of $\hat{x} \cong P_1$). Recalling that $H^* = H^{-1}$,we find that

$$H^{n-2} \otimes \odot^p(H^{-1} \oplus H^{-1}) = H^{n-2} \otimes H^{-p} \otimes \odot^p(C^2) = H^{n-2-p} \otimes \odot^p(C^2).$$

But on P_1 we have$O(-2) \cong \Omega^1$, so we find that, making the substitutions in (2.42),

$$\omega_p \in H^1\big(\hat{x}, \Omega^1_{\hat{x}}(n-p)\big) \otimes \odot^p(C^2).$$

We can evaluate the cohomology groups and we obtain

$$\omega_p \in H^1\big(\hat{x}, \Omega^1_{\hat{x}}(n-p)\big) \otimes \odot^p(C^2).$$

Thus we define

$$H^1\big(\hat{x}, \Omega^1_{\hat{x}}(n-p)\big) \otimes \odot^p(C^2) \cong \begin{cases} 0, & p < n, \\ \odot^n(C^2), & p = n. \end{cases}$$

Then this is a well defined symmetric n-spinor at$x$, and is the value of the field. As we shall see in the proof of the next theorem, in fact we have

$$\rho(\omega)(x) \in [\odot^n S - \otimes det S_+]_x = \big[S_{(AB\ldots D)}[1]'\big]_x.$$

Here we just wanted to illustrate how to get the field at a single point in terms of an expansion. Of course, there's no indication there that this field so defined should satisfy any equations. That's a much more difficult issue which we'll consider later.

To study the negative helicity case we shall need an additional analytical hypothesis which was not necessary in the nonnegative case. We shall summarize some important facts from the theory of function of several complex variables, which is relevant in this regard. A stein manifold X is a complex manifold with the property that there exists a function $\varphi: X \to R$, such that $X_c: \{x \in X: \varphi(x) < c\}$ is relatively compact for all $c > 0$ and such that $\partial\varphi$has a posistive-definite coefficient matrix at each point of X. this is a generalization of the notion of a convex set in R^n, and any convex set inC^n is Stein, for instance, although the defineition is much broader. For instance, for a thorough analysis of these important complex manifolds. They are never compact, and they have the following important property, which we shall state as a theorems.

Theorem (2.3.9). Let X be a Stein manifold, then for any holmlrphic vector bundle $\to X$,

$$H^q(X, O(E)) = 0, \quad q \geq 1.$$

The proof of this can be found in the references above. it is also true for more general sheaves coherent sheaves, which are natural generalizations of the locally free sheaves appearing in the theorm. Using Dolbeault's theorem, the vanishing of cohomology in the in theorem means that we can solve the $\bar{\partial}$-equation

$$\bar{\partial}u = \omega,$$

For$u$, if$\bar{\partial}u = 0$, where u and ω are E-valued $(0, q)$-and $(0, q+1)$- forms on X, respectively. In the book by Hormander we find that these differential equations are solved by means of the L^2-methods of partial differential equations. The more recent works of Krantz and Range show how to solve these same equations using generalizations of the Cauchy integral formula in several variables which have been developed by a number of mathematicans over the past 15years. The simplest example of a Stein manifold is the unit ball in C^ncentered at the origin$X = \{x \in C^n : |x| < 1\}$, where we can take$\varphi(x) := -\log|x|^2$ as is easy to check. However, the difference of two such balls centered at the origin (an annulus in several variables) is not a Stein manifold, provided that $n > 1$. there are also domains which are topologically equivalent to the ball but which are not Stein (take the ball and press a dimple).

After all of this digression on Stein manifolds we can now state the following theorem

Theorem (2.3.10). Let U be an open Stein submanifold of M, then there is a canonical linear transformation, for $n > 0$,

$$\rho: H^1\left(\hat{U}, O(n-2)\right) \to Z'_{-n}(U).$$

Moreover, if $\mu: U' \to \widehat{U}$ has connected fibers, then ρ is injective, and if $\mu: U' \to \widehat{U}$ is elementary, then $\rho$is an isomorpgism. We shall outline the proof in a moment, but first we give a corollary. Since M^1 and $M^{\pm}$are all convex domains in the coordinate chart M^1, we see that they are necessarily Stein, and they also clearly satisfy the topological hypotheses as we have seen before. Thus we have the following immediate consequence of Theorem (2.3.10).

Corollary (2.3.11). For $, > 0$,

(a) $\rho: H^1(P^{\pm}, O(n-2)) \to \Gamma(M^{\pm}, Z'_{-n})$,

(b) $\rho: H^1(P^1, O(n-2)) \to \Gamma(M^1, Z'_{-n})$,

are canonical isomphisms.

Proof: We shall give the essential steps of the proof, Penrose and Wells (1981), Wells (1979b), Penrose (1977b) and Ward (1977b) for some of the details which we leave out. Let us first describe the mapping ρ. As we said before, there are two descriptions. We shall give both, and we shall first give the direct approach with power series expansions, and then we shall turn to potentials later.

We want to expand cohomlogy calasses in $H^1(\widehat{U}, n-2)$ about projective lines of the form $\hat{x}$ for $x \in U$.. But we want to do this in a uniform manner for all points, so we shall work on the product space $P \times M$. The correspondence space F is naturally embedded in$P \times M$ by the mapping$(L_1, L_2) \longmapsto (L_1, L_2)$, and this defines F as a two-codimensional submanifold of $P \times M$. Let be the ideal sheaf of the submanifold Fin $P \times M$ we recall from (2.40) that the conormal sheaf of the embedding is given by

$$O(N^*) \cong \mathcal{T}/\mathcal{T}^2|_F.$$

Let us relate this conformal sheaf to the spinor sheaves on $P \times M$. consider the sheaf $O^A(1)$ on $P \times M$ i.e., the Pullback from P of $O_P(1)$ tensored with the pullback from M of under the natural projections to each of the factors in the Cartesian product. There is a natural on $P \times M$of this sheaf which we shall call, whose vanishing will define F as a submanifold of $P \times M$

$$F = \{(L_1, L_2) \in P \times M : \omega^A = 0\}. \tag{2.43}$$

Let $H \otimes S^A$be the vector bundle associated to the sheaf $O^A(1)$. then at the point $(L_1, L_2) \in P \times M$the fiber of $H \otimes S^A$is given by $L_1^* \otimes (T, L_2)$. Choose $l \in L_1$, and let l^*be the dual element of L_1^*(i.e.,$l^*(l) = 1$),, and let $[l]$be the image of l under the quotient mapping$T \to T/L_2$.Then $l^* \otimes [l] \in L_1^* \otimes (T, L_2)$ choice of l. This defines the value of ω^Aat the point(L_1, L_2). it is easy to verify that ω^Asatisfies (2.43). Thus we have on $P \times M$the mapping

$$\begin{array}{ccc} O_A(-1) & \xrightarrow{\omega^A} & \mathcal{T} \\ \downarrow & & \downarrow \\ fA & \longmapsto & fA\omega^A, \end{array} \tag{2.44}$$

Is a surjective mapping. In terms of components, if $\omega^A = \{\omega^0, \omega^1\}$ thenω^0 and $\omega^1$are generators of the ideal $\mathcal{T}$, and any function $f \in \mathcal{T}$has the form$f = f_1\omega^0 + f_2\omega^1$. the mapping (2.44) expresses this in an abstract form. The surjectivitity is equivalent to saying that the Jacobian matrix of the two components ω^0andω^1 have maximal rank on the submanifold F. we also have the exact sequence on $P \times M$:

$$\begin{array}{ccccccc} 0 \to \mathcal{T}^2 & \to & \mathcal{T} & \to & FO_A(-1) & \to & 0. \\ & & \psi & & \psi & & \\ & & f_A\omega^A & \longmapsto & f_A|_M & & \end{array} \tag{2.45}$$

We are writing X^Oinstead ofO_X to avoid confusion with spinor indices. The sequence (2.45) shows that $M^O A(-1)$can be identified with the conormal sheaf

$\mathcal{O}(N^*) = \mathcal{T}/\mathcal{T}^2$ (this is a parametrized version of the assertion earlier that $N_{\hat{x}} = H \oplus H$) . Now suppose that U is an open subset of M, an d we consider the construction applied to U' as a submanifold of $\widehat{U} \times U$. we claim that

$$H^1\big(U' \times U, \mathcal{T}^n(n-2)\big) \to H^1\left(\widehat{U} \times U, \mathcal{O}(n-2)\right) \qquad (2.46)$$

Is an isomorphism. We shall verify (2.45) in the case $n = 2$ for simplicity. On $\widehat{U} \times U$ we have the short exact sequence

$$0 \to \mathcal{T} \to \mathcal{O} \to \mathcal{O}_{U'} \to 0,$$

Which yields the long exact sequence on cohomology

$$\Gamma\big(\widehat{U} \times U, \mathcal{O}\big) \to \Gamma\big(\widehat{U}, \mathcal{O}\big) \to$$

$$H^1\big(\widehat{U} \times U, \mathcal{O}\big) \to H^1(U', \mathcal{O}), \qquad (2.47)$$

Where we recall that the sheaf$\mathcal{O}_{U'}$is supported on $U' \subset \widehat{U} \times U$, and we have the isomorphism

$$H^q\big(\widehat{U} \times U, \mathcal{O}_{U'}\big) \cong H^q(U', \mathcal{O}_{U'}).$$

The first arrow in (2.46) is surjecctive, namely one can check that

$$\Gamma(U', \mathcal{O}_{U'}) \cong \Gamma(U, \mathcal{O}_U),$$

$$\Gamma\big(\widehat{U} \times U, \mathcal{O}_{\widehat{U} \times U}\big) \cong \Gamma(U, \mathcal{O}).$$

This follows since any holomorphic on $\widehat{U}$ is constant since it is constant on each of the projective lines, for $x \in U$, and for any two points P, Q in$\widehat{U}$, there is an $x \in U$ so that $P, Q \in \hat{x}$. similarly, any holomorphic function on is constant on the fibers $v^{-1}(x)$, for $s \in U$, and hence corresponds to the pullback of a holomorphic function from U. now$H^1(U', \mathcal{O}) \cong H^1(U, \mathcal{O})$ by the Leray spectral sequence (2.26). Since U is Stein we have that $H^1(U, \mathcal{O}) = 0$. Thus we find that

$$H^1\big(\widehat{U} \times U, \mathcal{T}\big) \overset{\cong}{\to} H^1\big(\widehat{U} \times U, \mathcal{O}\big)$$

Now consider the sequence (2.45) and its associated long exact sequence. We find

$$\Gamma\big(U', O_A(-1)\big) \to H^1\big(\widehat{U} \times U, \mathcal{T}^2\big)$$

$$\to H^1\big(\widehat{U} \times U, \mathcal{T}\big) \to H^1(U', O_A(-1)).$$

using the Leray spectral sequence(2.26), we see that the first and fourth terms vanish, and we conclude that

$$H^1\big(\widehat{U} \times U, \mathcal{T}^2\big) \overset{\cong}{\to} H^1\big(\widehat{U} \times U, \mathcal{T}\big). \tag{2.49}$$

From(2.47) and (2.48) we deduce (2.45) In the case $n = 2$.

The general case is a continuation of this argument using successively higher powers of $\mathcal{T}$. intuitively (2.45) being an isomorphism says that the information contained within an element of $H^1\left(\widehat{U} \times U, O(n-2)\right)$depends only on normal derivatives to of order at least n, to 'evaluate the field' in effect, we just take the n th normal derivative by factoring out $\mathcal{T}^{n+1}(n-2)$. that is dividing out by this power leaves all of the lower order information intact. This process also restricts to U' since the sheaf $\mathcal{T}^n(n-2)/\mathcal{T}^{n+2}(n-2)$ is supported on U'. More explicitly we have the exact sequence on $\widehat{U} \times U$:

$$0 \longrightarrow \mathcal{T}^{n+1} \longrightarrow \mathcal{T}^n \longrightarrow U' O_{(AB\ldots D)}(-2) \longrightarrow 0$$

$$\psi \qquad\qquad \psi$$

$$f_{AB\ldots D}\omega^A\omega^B \ldots \omega^D \longmapsto f_{AB\ldots D|U'}$$

Where $f_{AB\ldots D}$ is the Taylor coefficient of the monomials$\omega^A\omega^B \ldots \omega^D$ (which are generators of the nth symmetric power of the conormal bundle). It follows from (2.49) that

$$\mathcal{T}^n(n-2)/\mathcal{T}^{n+1}(n-2) \cong O_{(AB\ldots D)}(-2)$$

On U'. We now have the Penrose transform as originally given in Wells of the form

$$H^1\left(\widehat{U}\times U, O(n-2)\right)\overset{\cong}{\leftarrow} H^1\left(\widehat{U}\times U, O(n-2)\right)$$

$$\downarrow T$$

$$\uparrow \qquad\qquad H^1\left(U', O_{(AB\ldots D)}(-2)\right) \tag{2.50}$$

$$\downarrow L$$

$$H^1\big(\widehat{U}, O(n-2)\big)\overset{p}{\rightarrow} H^0\big(U, O_{(AB\ldots D)}[1]\big)$$

Where T is the Taylor coefficient mapping given by(2.49), L is the Leray spectral sequence mapping which is an isomorphism, the top isomorphism is given by (2.45), the left-hand vertical mapping is the natural pullback mapping, and ρ, the Penrose transform, is defined as the composition. This is the desired mapping from cohomology on $P$to spinor fields on M, but it is not clear from this that the image must satisfy any field equations. One way to see that they do would be lift the spectral sequence of the relative de Rham sequence to this product picture, and deduce the differential equations, as before. We shall not carry this out here. The alternative is to see what one gets from the relative de Rham sequence itself. This leads to potentials.

A potential for a negative heliccity field$\psi_{AB\ldots D}\in\Gamma(U, Z'_{-n})$ is a spinor field

$$\varphi^{B'\ldots D'}\in\Gamma\left(U, O_A^{(B'\ldots D')}[1]'\right)$$

Such that

$$\nabla^{A(A'\ldots.B'\ldots D')}=0$$

$$\psi_{AB\ldots D}=\nabla_{D'(D\ldots\nabla_{B'B})}\varphi^{B'\ldots D'}, \tag{2.52}$$

Where the symmetrization in the definition of ψ refers to the unprimed indices. One can check in a straightforward but tedious manner that if $\psi = \psi_{AB\ldots D}$ is given by (2.51), then ψ satisfies the negative helicity massless field equations (2.36), there is agauge freedom in these potentials, namely, if is a potential for $\psi_{AB\ldots D}$ then so is

$$\varphi_A^{B'\ldots D'} + \nabla_A^{(B'}\gamma^{C'D')}. \qquad (2.53)$$

For any spinor field$\gamma^{C'\ldots D'} \in \Gamma\big(U, O^{(C'\ldots D')}\big)$. it follows from constructions in Penrose and ward , that locally one can always find such a potential, and that (2.53) is the only gauge freedom.

Let us specialize to the case of a left-hand Maxwell field, i.e., a solution of

$$\nabla^{BB'}\psi_{AB\ldots D} = 0. \qquad (2.54)$$

Consider the anti-self-dual two-form

$$F_{ab} = \psi_{AB}\varepsilon_{A'B'},$$

Then (2.53) become simply as we have seen before $df = 0$,, and a potential is any one-formω such that $d\omega = F$. the gauge freedom is simply that $\omega \longmapsto d\omega + d\gamma$, for any holomorphic functionγ. thus the local equivalence of field potentials modulo guage is simply the de Rham sequence. One can rewrite the Rham sequence in terms of spinor in a manner that one can see how to show that any locally massless field has the description of potential gauge. We shall show that if we use the spectral sequence of the relative de Rham complex, then we get the potentials which satisfy (2.51). specifically, there is a Penrose transform of the form

$$\rho\colon H^1\left(\widehat{U}, O(n-2)\right) \to$$

$$\{\text{potential for helicicity } -n/2 \text{ massless fields on } U\}/\{\text{gauge freedom}\} \quad (2.55)$$

The mapping ρ is an injection or isomorphism depending on the topological properties of μ as hypothesized, as before. More precisely, the E_1-term in the spectral sequence (2.25) for The case in question, has the following form after using the Leray spectral sequence. We also use the fact that

$H^q\left(U, O^{(C'\dots D')}\right) = H^q\left(U, O_A^{(B'\dots D')}[1]'\right) = 0$, for $q \geq 1$,since $U$is Stein.

⋮	⋮	⋮
0	0	0
0	0	0

$$\Gamma\left(U, O^{(C'\dots D')}\right) \xrightarrow{\nabla_A^{B'}} \Gamma\left(U, O^{(B'\dots D')}[1]'\right) \xrightarrow{\nabla^{AA'}} \Gamma\left(U, O^{(B'\dots D')}[1]'[2]'\right)$$

where $(C' \dots D'), (B' \dots D')$, and$(A' \dots D')$have$(n-2), (n-1)$ and(n) indices, respectively. Calculating the E_2 term from this E_1-term, we see that (using definition in (2.51))

$$E_2^{10} = \{\text{potentials}\}/\{\text{gauge}\}.$$

Also$E_\infty^{01} = E_2^{01} = 0$, and $E_1^{10} = E_2^{10}$,thus it follows from theorem 2.6.2, as before, that

$$H^1(U', \mu^{-1}O(n-2)) \cong \{\text{potentials}\}/\{\text{gauge}\}.$$

By calculating the field at a point of U one can check that the two Penrose transforms given by (2.50) and (2.54) differ by a factor of (-2). This uses the helicity raising and lowering operations of Penrose and Eastwood . We refer the reader To Eastwood, Penrose and Wells , for this final point, and with this we can conclude that the image of (2.50) does satisfy the field equations (2.36).

2.4 Integral Formulas For Massless Fields

We saw how massless fields correspond to elements of wistor cohomology groups. Historically, this result arose out of something more down-to-earth, namely contour integral expressions for massless fields. In fact, for the scalar case the wave equation, an equivalent integral formula was given by Bateman , who in turn was developing work of Whittaker. We shall work backward, by showing how the Penrose transform. given rise to contour integral formulas. An element of the sheaf cohomlogy group $H^1\left(\widehat{U}, O(-2s-2)\right)$can be represented by the (Cech) cocycle $f_{ij}(Z^\alpha)$ defined on overlap regions$W_i \cap W_j$, where $\{W_i\}$is an open cover of $\widehat{U}$. The corresponding massless field can be expresses as a branched contour integral of f_{ij}. To keep things simple, let us consider one only the special case where$\widehat{U}$ is covered by two coordinate charts W and $\underline{W}$, so that there is only one overlap region $W \cap \underline{W}$, and the representative cocycle consists of a single function $f(Z^\alpha)$ defined on this overlap. Then we shall get an ordinary contour integral, rather than a branched one. $\widehat{U}$ is a region of twistor space Pderived from a region U in M. The two open sets$W$ and $\underline{W}$are chosen so that for any line $\hat{x}$ in$\widehat{U}$, the intersections

$$W_x = W \cap \hat{x},$$

$$\underline{W_x} = \underline{W} \cap \hat{x},$$

Overlap in an annular region of $\hat{x}$ (i.e.,$W_x \cap \underline{W_x}$ is homeomorphic to $S^1 \times R$). The function $f(Z^\alpha)$ is holomorphic on $W \cap \underline{W}$, and homogeneous of degree$-n-2$in Z^α. of course, if h and $\underline{h}$ are functions holomorphic on $W$and$\underline{h}$, respectively, then$\underline{h} - h$ represents a coboundary, and

$$f' = f + \underline{h} - h \tag{2.56}$$

Is regarded as being equivalent to f. We are thinking in terms of the double-fibrstion picture

$$\widehat{U}^{\mu \swarrow^{U'} \searrow \nu} U$$

And we want to use f to construct a massless field on U. The first step is to pull f back to U', and this is easily done: the pullback of $f(\omega^{A}, \pi_{A'})$is

$$g(x^{a}, \pi_{A'}) := f\left(ix^{AA'}\pi_{A'}, \pi_{A'}\right). \tag{2.57}$$

Here$(x^{a}, \pi_{A'})$ are being thought of as coordinates on U'. The function g is holomorphic on the intersection of the two sets $\mu^{-1}(W)$ and $\mu^{-1}(W)$, which cover U'. and g homogeneous of degree $-n-2$ in $\pi_{A'}$. The second, and final, step is to integrate out the π-dependence. For the moment, take $n \geq 0$. then we

$$\varphi_{A'} \ldots C'(x^{d}) = \frac{1}{2\pi i}\oint_{\gamma} \pi_{A'} \ldots \pi_{C'}\, g(x^{d}, \pi_{D'})\Delta\pi. \tag{2.58}$$

The integrand of (2.58) contains n factors of $\pi_{A'}$, and $\Delta\pi$is defined by

$$\Delta\pi = \pi_{E'}d\pi^{E'}. \tag{2.59}$$

The canonical holomorphic one-form $\Delta\pi$ on the fibers of v, is homogeneous of degree two; so the integrand is homogeneous of degree zero in $\pi_{A'}$. This means that, although expressed in terms of the homogeneous coordinates $\pi_{A'}$, the integrand of (2.58) is actually defined on the projective π −space $\hat{x}$. Think of x as being fixed, and consider the integration γ as depicted in by Cauchy's theorem, the value of the integral does not change if we continuously deformγ. In fact, it depends only on the winding number of γ, which we take to be unity. Also by Cauchy's theorem, the value of the integral does not change if we make the co boundary transformation (2.56). So $\varphi_{A'} \ldots C'$satisfies the massless free-field equations

$$\nabla^{AA'}\varphi_{A'} \ldots C' = 0, \quad or \qquad \Box\varphi = 0,$$

Chapter 3

Quantum Fields In Minkowki Space-Time

The theory of quantum fields in curved space-time is a generalization of the well-established theory of quantum fields in Minkowski space-time. To a great extent, the behavior of quantum fields in curved space-time is a direct consequence of the corresponding flat space-time theory. Local entities, such as the field equations and commutation relations, are to a large extent determined by the principle of general covariance and the principle of equivalence.

It is logical, therefore, to review the relevant aspects of flat space-time quantum field theory. This will serve to establish the necessary background, to fix our notation, and to highlight those aspects of the theory which can be carried over to curved space-time, as well as those which lose their meaning in curved space-time.

We discuss the canonical formulation, including the Schwinger action principle and the relation between symmetry transformations and conserved currents (Schwinger). We review the dynamical descriptions known as the Heisenberg picture, Schrodinger picture, and the interaction picture. We introduce the Fock representations, in which states are described in terms of their particle content, and the Schrodinger representation, in which the states are described field configurations, we include discussions of the Maxwell and Yang-Mills gauge fields, as well as the Dirac field, and the definitions of spin and angular momentum.

3.1 Canonical Formulation

Recall that in classical mechanics the equations of a particle or system of particles having independent generalized coordinates $q_i(t)$and velocities $\dot{q}_i(t)$ are given by the principle of stationary action. This principle states that the action

$$S = \int_{t_1}^{t_2} dt\, L(q, \dot{q}) \tag{3.1}$$

Is stationary under arbitrary variations of the q_i which vanish on the boundary of the region of integration. Here L is the Lagrangian of the system. The Hamiltonian is defined by

$$H(q, p) = \sum_i p_i q_i - L,$$

where

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{3.2}$$

Is the momentum conjugate to q_i. The system is quantized by taking the qs and ps to be Hermition operators acting on a Hilbert space, and by imposing the canonical commutation relations

$$[q_i, q_j] = 0, \qquad [p_i, p_j] = 0,$$

$$[q_i, q_j] = i\delta_{ij}. \tag{3.3}$$

Here δ_{ij} is the Kronecker delta. We are using units with $h = c = 1$. form (3.3) it follows that $F(q, p)$ is a function of the coordinate and momentum operators, then assuming F can Taylor expanded in p

$$[q_i, F] = i\frac{\partial F}{\partial p_i}. \tag{3.4}$$

The above commutation relations imply that the q_i are a complete set of commuting observables with continuous spectra consisting ,in the absence of impenetrable (walls) of all real numbers. The same can be said for the p_i an observable is an Hermitian operator with a complete set of eigenstates, the eigenstates of the q_i are the kets $|q' >$, where q' denotes the set of eigenvalues $q_i'$of the operators q_i. Thus,

$$q_i|q' \geq q_i'|q' >,$$

With the normalization

$$< q'|q'' \geq \delta(q' - q''),$$

We use the conventions of Dirac in distinguishing the eigenvalue of an operator by using. Where $\delta(q' - q'')$ is the Dirac δ-function. It also follows that

$$< q'|q'' \geq -i\frac{\partial\delta(q' - q'')}{\partial q_i'},$$

and

$$< q'|q'' \geq (2\pi)^{-n/2}\exp\left(\sum_{i=1}^{n} p_i'q_i'\right),$$

Where $|p' >$is an eigenket of the p_i having delta function normalization, and we have taken the index i to run from 1 to n. If $F(q,p)$ is a function of the $q_i$and $p_i$with any well-defined ordering of factors, then

$$< q'|F(q,p)|q'' \geq F\left(q', -i\frac{\partial}{\partial q'}\right)\delta(q' - q''), \tag{3.5}$$

Where $F\big(q', -i(\partial/\partial\, q')\big)$is the same ordered function with $-i(\partial/\partial\, q_i')$ replacing p_i in each position.

In the Schr□dinger or configuration space representation, the abstract operators are represented by matrix elements based on the $< q'|\psi >$, such as that of p_i above, and the states are represented by functions. For example, a state $|\psi >$is represented by the Schr□dinger wave function $\psi(q') =< q'|\psi >$. an example is the wave function $< q'|p' >$above, representing a particle of definite momentum. Similarly, in the momentum-space representation the operators are represented by matrix elements formed the $|p'>$and the states are represented by functions such as $< p'|\psi >$. Up to now the description has been purely kinematical, with time playing no role. The dynamical evolution of the system is governed by the Hamiltonian $H(q,p,t)$.we have allowed for the possibility that H may have explicit time dependence, as through an interaction with an external field. The time evolution may be described in several physically equivalent ways, known as pictures. In the Spichrödinger picture, the fundamental observable q and p do not change with time, rather, the dynamical evolution of measurable quantities, such as expectation values of observables, is expressed through the time dependence of the ket describing the state of the system at each time. The fundamental dynamical equation is that of Schrödinger,

$$i\frac{d}{dt}|\psi(t) \geq H(q,p,t)|\psi(t) >. \tag{3.6}$$

Because Hmay have explicitly time-dependent terms involving $p$and q, in general $H(t')$and $H(t'')$ May not commute. for brevity, we suppress the dependence of Hon $q$and p. We note in passing that in the Schrodinger representation, the Schrödinger equation (3.6) becomes (using the completeness of the $|q'>$)

$$i < q' \left|\frac{d}{dt}\right|\psi(t) \geq \int < q'|H(q,p,t)|q'' > dq'' < q''|\psi(t) >$$

Or

$$i\frac{\partial}{\partial t}\psi(q',t) = H\left(q', i\frac{\partial}{\partial q'}\right)\psi(q',t), \tag{3.7}$$

Where $\psi(q',t) =< q'|\psi(t) >$.

The solution of (3.6) is

$$|\psi(t) \geq U(t, t_0)\psi(t_0) >,$$

with $U(t, t_0)$satisfiying

$$i\frac{d}{dt}U(t, t_0) = H(t)U(t, t_0), \qquad (3.8)$$

with the boundary condition

$$U(t, t_0) = 1.$$

The evolution operator $U(t, t_0)$preserves the norm of the state vector and is thus unitary, satisfying

$$UU^\dagger = U^\dagger U.$$

From $U(t, t_0)U(t_0, t) = U(t_0, t_0) = 1.$ it then follows that $U(t, t_0)^\dagger = U(t_0, t).$

In the Heisenberg picture, the describing the state of the system does not change with time, while the dynamical evolution of the system is expressed through the time dependence of the fundamental observables $q(t)$And $p(t)$. by applying $U(t, t_0)^\dagger$to the Schrödinger picture ket describing the state of the system, we obtain a time-independent ket, which can be taken as the ket describing the state of the system in the Heisenberg picture. Thus, denoting quantities in the Schrödinger picture by subscript S and those in the Heisenberg picture by subscript H, we have

$$|\psi_H \geq U(t, t_0)^\dagger |\psi_S(t) \geq |\psi_S(t_0) >. \qquad (3.9)$$

In order that measurable expectation values remain the same as in the Schrödinger picture, the Heisenberg operator

$$F_H(t) = U(t, t_0)^\dagger F_S U(t, t_0). \qquad (3.10)$$

Note that the Hamiltonian Hin (3.8) is F_S. When $F_H$has no explicit time dependence, the solution of (3.8) is

$$U(t, t_0) = exp[-i, (t - t_0)H],$$

And it follows that H commutes with U so that $H_H = H_S$. When $H_S$does have explicit time dependence, $H_H \neq H_S$, and we must use (3.10) to define $H_H$in terms of H_S. In general, $F_H$will depend on time through $q_H(t), p_H(t)$, and through further explicit appearance of t. denoting by $\partial F_H(t)/\partial t$ the derivative only with respect to this further explicit appearance of t, it follows from (3.8) and (3.10) that

$$i\frac{d}{dt}F_H = [F_H, H_H]i\frac{d}{dt}F_H. \tag{3.11}$$

This resembles the classical equation of motion with the Poisson bracket replaced by-$i$times the commutator. Of course, we do not need to define the Schr□dinger the ket describing the state of the system is independent of time and that operators F, constructed from the q_i, p_i(dropping subscript H), and $t$obey the Heisenberg stateequation of motion (3.11). When two systems interact through a term in the Hamiltonian, which can be regarded as a perturbative interaction, it is useful to introduce another picture of the dynamical evolution, known as the ‘’interaction picture.

Is in this picture, the ket describing the state of the system evolves as in the Schordinger picture, but only the under the influence of the interaction term in the Hamiltonian, while operators evolve as in the Heisenberg representation, but only under the influence of the unperturbed term in the Hamiltonian.

Let us now turn to the canonical quantization of a system of independent real fields $\phi_a(x)$, where $x$refers to the Minkowski space and time coordinates x^μ, and the index $a$includes tensor or spinor indices and internal quantum numbers of the field multiplet. We will deal here with bosons and discuss later .The modification of canonical quantization required with fermions. For brevity, the index a will often be suppressed; we can usually thing of ϕ as a column or row matrix, depending on where it appears in an expression. Canonical quantization proceeds as in the previously discussed quantization of a particle. One thinks of the classical field $\phi(x)$ as analogous to the classical $q_i(t)$ with the spatial coordinates $\vec{x}$ regarded as labels like i. Because we are now dealing with a continuous label, Dirac δ-functions involving $\vec{x}$ will appear where Kronecker deltas involving the label i previously appeared. As before, we assume that the system is described by an action

$$S = \int_{t_1}^{t_2} dt\, L[\phi, \partial\phi]_t, \tag{3.12}$$

Where the Lagrangian $L$is now a functional of the field ϕ and its first derivatives $\partial\phi/\partial x^\mu \equiv \partial_\mu\phi$, which are denoted collectively by $\partial\phi$. the subscript $t$indicates that L is a function of t. The Lagrangian Lcan be expressed in terms of a Lagrangian density $\mathcal{L}$ as

$$L[\phi, \partial\phi]_t = \int d\,V_x \mathcal{L}\big(\phi(\vec{x}, t), \partial\phi(\vec{x}, t)\big),$$

Where $dV_x$is the spatial volume element, Because L is now a functional, the momentum πconjugate to the field $\phi(\vec{x}, t)$ is defined in analogy with (3.2) through the following functional derivative regarding $\partial_0\phi$ as independent of ϕat time t

$$\pi(\vec{x}, t) = \delta L\,[\phi, \partial\phi]_t/\delta\big(\partial_0\phi(\vec{x}, t)\big)$$

$$= \partial\mathcal{L}\,(\vec{x}, t), \partial\phi(\vec{x}, t)/\partial\big(\partial_0\phi(\vec{x}, t)\big). \tag{3.13}$$

Here, we have used the definition of the functional derivative, which states that if $F[\chi]$is a functional of $\chi(\vec{x})$, then under a variation $\delta\chi(\vec{x})$ of χ, which vanishes sufficiently fast at spatial infinity, we have

$$\delta F[\chi] = \int d\,V_x \frac{\delta}{\delta\chi(\vec{x})} F[\chi]\delta\chi(\vec{x}). \tag{3.14}$$

It follows that if the functional F has form

$$F[\chi] = \int d\,V_x f\left(\chi(\vec{x}), \vec{\partial}\chi(\vec{x})\right),$$

Then

$$\frac{\delta}{\delta\chi(\vec{x})} F[\chi] = \frac{\partial}{\partial\chi} f(\chi, \vec{\partial}\chi) - \partial_i \left(\frac{\partial}{\partial(\partial_i \chi)} f(\chi, \vec{\partial}\chi) \right),$$

where χ is evaluated at $\vec{x}$. The result in (3.13) follows from this with $\chi \to \partial_0 \phi$.

Another consequence is that

$$\frac{\delta\chi(\vec{x}')}{\delta\chi(\vec{x})} = \delta(\vec{x}' - \vec{x}).$$

One can regard the action in (3.12) as a functional depending on the space and time dependence of ϕ. Then, the Euler-Lagrange field equation below will be reconnized as another application of the above result, but for a functional depending on one more dimension. The Hamiltonian defined by

$$H[\phi, \pi]_t = \int d\,V_x \pi_a(\vec{x}, t)\partial_0 \phi_a(\vec{x}, t) - L[\phi, \partial\phi]_t. \tag{3.15}$$

A though we write $H[\phi, \pi]$, dependence on spatial derivatives of ϕ or π is permitted. The principle of stationary action yields upon variation of the fields in (3.12), the Euler-Lagrange field equations

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0, \tag{3.16}$$

where the repeated spacetime coordinate index μ is summed over its full range of values, in acoordance with the Einstein summation convecention.

The field is quantized in analogy with the canonical commutators of (3.3). thus, we postulate that

$$[\phi_a(\vec{x},t),\phi_b(\vec{x}',t)]=0,[\pi_a(\vec{x},t),\pi_b(\vec{x}',t)]=0$$

$$[\phi_a(\vec{x},t),\pi_b(\vec{x}',t)]=i\delta_{a,b}\delta(\vec{x}-\vec{x}'), \tag{3.17}$$

Where, as noted earlier, $\delta(\vec{x}-\vec{x}')$is the Dirac δ-function. in a theory of interacting fields, if we deal directly with the renormalized fields, then (3.17) is somewhat altered by a normalization factor. We are thus dealing here with the bare fields, which by definition satisfy the field equations with unrenormalized

Masses and coupling constants, the t dependence is included in the

commutation relations to emphasize that in a dynamical picture like the Heinberg picture, in which the operators ϕ and π depend on time as do the classical fields, the canonical commutation relations must be imposed on the fields and conjugate momenta evaluated at the same time. For it follows from (3.10) that $q_H(t)$ and $p_H(t)$evaluated at the same time do satisfy (3.3), while that is not true in general if they are evaluated at different times. Of course, in the Schrodinger picture the fields and conjugate have no time dependence, and t would not appear in (3.17). The functional analogue of (3.4) follows from the above commmutators;

$$\left[\phi_a(\vec{x}),F[\phi,\pi]\right]=i\frac{\delta}{\delta\pi_a(\vec{x})}F[\phi,\pi], \tag{3.18}$$

where we are suppressing the time dependence, taking all fields to be a single time t. one can now set up a Schrödinger or field representation using the eigenstates of $\phi(\vec{x})$ defined by

$$\phi(\vec{x})|\phi' \geq \phi'(\vec{x})|\phi' >. \tag{3.19}$$

The ket $|\phi' >$ corresponds to a state of the system in which the field has configuration $\phi'(\vec{x})$, where $\phi'$is an ordinary or-c-number function, unlike the field

Operator ϕ. Thus, we are using the analogue of the Dirac notation in which eigenvalues of the operator $\phi(\vec{x})$ are functions denoted by $\phi'(\vec{x})$. Here the prime does not denote derivative, but instead distinguishes a c-number from an operator, in ordinary quantum mechanics, it follows from (3.5) that if $|\psi>$ is an element of the Hilbert space spanned by the eigenkets $|q'>$, then

$$< q'|F(q,p)|\psi> = F\left(q', -i\frac{\partial}{\partial q'}\right) < q'|\psi>.$$

Similarly, we can show that if $|\Psi>$is a state in the space spanned by the eignstates $|\phi'>$,and $F[\phi,\pi]$is a functional formed from the field operator and conjuguate momentum, then

$$< \phi' F[\phi,\pi]|\Psi> = F\left(\phi', -i\frac{\delta}{\delta\phi'}\right) < \phi'|\Psi>. \qquad (3.20)$$

Here $< \phi'|\Psi> \equiv \Psi[\phi']$is a complex number which is a functional of ϕ'. It is interpreted as the probability amplitude for finding the field observable ϕ to have the configuration or set of values given by $\phi'(\vec{x})$when the system is in the state described by the vector $|\Psi>$. If we work in the Schrodinger picture, then (3.20)can be used to turn the Schr�dinger equation

$$i\frac{d}{dt}|\Psi \geq H[\phi,\pi]|\Psi>$$

Into the functional differential equation

$$i\frac{\partial}{\partial t}\Psi[\phi',t] = H\left[\phi', -i\frac{\delta}{\delta\phi'}\right]\Psi[\phi',t], \qquad (3.21)$$

Where $\Psi[\phi',t] \equiv \phi'|\Psi(\mathrm{t})>$, and ϕ,π depend on $\vec{x}$, but not on t.

On the other hand, in the Heisenberg picture the state describing the evolving system is independent of time, while a general functional F of $\phi(\vec{x},t)$ and $\pi(\vec{x},t)$ will depend on time through its dependence on ϕ and π, as well through a possible further explicit dependence on t. Then Heisenberg equation of motion is

$$i\frac{dy}{dx}F[\phi,\pi;t] = \big[F[\phi,\pi;t],H[\phi,\pi]\big] + i\frac{\partial}{\partial t}F[\phi,\pi;t]. \tag{3.22}$$

The consistency of (3.22) with the Euler-Lagrange equation (3.16) can be proved in Minkowski space-time and in the more general curved space-time context, we expect that when there are on time-or space-dependent external parameters in the Lagrangian, then there should exist a conserved vector observable P^μ corresponding to the total energy and momentum of the system. For such a Lagrangian, by multiplying the Euler-Lagrange equations by $\partial_\mu\phi$ and using

$$\partial_\mu\mathcal{L} = \frac{\partial\mathcal{L}}{\partial(\partial_v\phi)}\partial_\mu\partial_v\phi + \frac{\partial\mathcal{L}}{\partial\phi}\partial_\mu\phi,$$

we find immediately that

$$\partial_\mu T^\mu{}_v = 0, \tag{3.23}$$

where

$$T^\mu{}_v = \frac{\partial\mathcal{L}}{\partial(\partial_v\phi_a)}\partial_v\phi_a - \delta^\mu{}_v\mathcal{L}. \tag{3.24}$$

Here, summation over internal indices$a$ of ϕ and, when the fields are treated as operators, symmetrization over field and their conjugate momwnta are understood. The tensor $T_{\mu v}$is called the canonical energy-momentum or stress tensor, we mention in passing that in order to serve as the source in the Einstein gravitational field equations $T_{\mu v}$should be symmetric under interchange of indices.This assumes that Lagrangian does not have any explicit dependence on the coordinates x^μ .This symmetry is also required if we wish to define a conserved angular momentum in terms of the energy-momentum tensor.

However, expect in the case of particular forms of $\mathcal{L}$, the expression given in (3.24) after lowering the index v with Minkowski metric $\eta_{\mu v}$ is not a symmetric tensor . A general manifestly symmetric expression for $T_{\mu v}$ will be given later, when we deal with curved spacetime. Since both of these expressions for $T^{\mu}{}_{v}$ will satisfy (3.23) for any system with no explicit time or space dependence, we expect on physical grounds that they must each yield the same conserved energy and momentum P^{μ} to within a constant. Furthermore, a modification of the canonical $T_{\mu v}$that makes it symmetric, and yields the same P^{μ} and angular momentum as the original canonical$T_{\mu v}$, has been given by Belinfante.

From(3.23) we have

$$\int dv_x\, \partial_\mu T^{\mu}{}_{v} = 0,$$

Where dv_x denotes the space-time volume element, and the intergration is over a space-time volume bounded by spatial infinity and two constant –time hypersurfaces. Assuming that matrix elements physical interest will be between states in which the physical field configuration is of finite spatial extent, we obtain by the Gauss divergence theorem the conservation law

$$\frac{d}{dt}\int dV_x\, T^{0}{}_{v} = 0,$$

Where dV_x denotes the special volume element and the integration is over any constant-time hypersurface. Hence,

$$P_v = \int dV_x\, T^{0}{}_{v} \tag{3.25}$$

Is the conserved energy-momentum vector. The sign in this definition is chosen so that $P_v = H$, as can be verified by comparing (3.15) with (3.24).

As a special case of the Heisenberg field equation (3.22), suppose that the functional F is an ordinary function $f(\phi(x), \partial_i\phi(x), \pi(x))$with no explicit time dependence. Then, (3.22) can be written as

$$i\partial_0 f = [f, P_0], \tag{3.26}$$

Where the partial derivative symbol is used here in the conventional manner to indicate that the x^i are held fixed. In (3.22), the partial derivative symbol denoted derivation only with respect to explicit t dependence not coming from ϕ and π. The partial derivative in (3.26) includes all $t$dependence. The result in (3.26) is the 0-component of the more general relation

$$i\partial_\mu f = \left[f, P_\mu\right], \tag{3.27}$$

The $\mu = 0$ component, as noted above, follows from (3.22). for $\mu = i$, we easily verify (3.27) for powers of ϕ and π and thus for functions which can be expanded in power series in ϕ and π. For example, suppressing the t-dependence, which is the same in all arguments, we have

$$[\pi(\vec{x})^n, P_i] = \int dV_x' \, \pi\left(\vec{x}'\right)\left[\pi(\vec{x})^n, \partial_i'\phi\left(\vec{x}'\right)\right]$$

$$= -i \int dV_x' \, \pi\left(\vec{x}'\right) n\pi(\vec{x})^{n-1}, \partial_i'\delta\left(\vec{x} - \vec{x}'\right)$$

$$= i\partial_i(\pi(\vec{x})^n).$$

Equation (3.27) also follows from a powerful generalization of action principle and of Noether's theorem, known as the Schwinger operator action principle. The action of (3.12) is integrated over a space-time volume v bounded by two constant-time hypersurfaces at $t_1$and t_2. originally stated, the principle deals with arbitrary spacelike hypersurfaces, but we work with constant-time hypersurfacesfor simplicity at this stage. Consider arbitrary infinitesimal variations, δx^μ and $\delta_0\phi(x)$, of the coordinates field operators,

$$x^{\mu} \to x^{\mu'} = x^{\mu} + \delta x^{\mu}, \tag{3.28}$$

$$\phi(x) \to \phi'(x) = \phi(x)\phi + \delta_0\phi(x), \tag{3.29}$$

where $\delta_0\phi(x)$vanishes on the spatial boundary of integration at each time (i.e., it vanishes everywhere on the boundary of v except on the interior of the constant time hypersurfaces that bound v at $t_1$and t_2). Then, the Schwinger action principle states that the variation of the action S of (3.12) has form

$$\delta S = G(t_1) - G(t_2), \tag{3.30}$$

Where the operator $G(t)$ is the generator of the above variation of the coordinates and fields at time t. To say that Ggenerates the variation means the following. For an operator functional $F[\phi, \pi]$, we have

$$i\delta_0 F = [F, G], \tag{3.31}$$

Where all quantities are evaluated at the same time, and $\delta_0 F$is the infinitesimal variation of Fproduced by (3.28) and (3.29). that is,

$$\delta_0 F = F[\phi + \delta_0\phi, \partial(\phi + \delta_0\phi)] - F[\phi, \partial\phi].$$

One can show form (3.30) that the generator has the form

$$G(t) = \int d\, V_x[\pi_a \delta\phi_a - T^0{}_v x^v], \tag{3.32}$$

Where $T^{\mu}{}_{v}$ is the energy-momentum tensor of (3.24), and

$$\delta\phi(x) \equiv \phi'(x') - \phi(x) \tag{3.33}$$

is the change of the field at a given physical point in space-time (i.e., the local variation).The derivation is as follows. The variation δSabove is defined by

$$\delta S = S' - S,$$

where

$$S' = \int d\, v'_x \mathcal{L}\big(\phi'(x'), \partial' \phi'(x')\big)$$

and

$$S = \int d\, v_x \mathcal{L}\big(\phi(x), \partial \phi(x)\big).$$

Here, both integrals are taken over the same physical volume v, with x and x' denoting the same physical point of the system in space-time, and thus related by (3.28). this way of viewing the transformation is the coordinates undergo a transformation from x and x', but the physical system does not change.

Alternatively, it is possible to adopt the active viewpoint taken by Messiah in which $x$and x' are regarded as describing a change of the physical system relative to a single coordinate system, such that event at x is dragged to x'. In the active viewpoint, the volume $v$is infinitesimally displaced to a new physical volume v', related to v by (3.28). we can write

$$S' = \int_{v'} d\, v_x \mathcal{L}\big(\phi'(x), \partial' \phi'(x)\big)$$

$$= \int_{v'} d\, v_x \mathcal{L}\big(\phi(x) + \delta_0 \phi(x), \partial \phi(x) + \partial \delta_0 \phi(x)\big)$$

$$= \int_{v'} d\, v_x \mathcal{L}(\phi(x), \partial \phi) + \int_{v} d\, v_x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta_0 \phi + \frac{\partial \mathcal{L}}{\partial \big(\partial_\mu \phi\big)} \partial_\mu \delta_0 \phi \right\}.$$

The volume of integration in the second integral has changed to v because the integrand is already of first order. Corresponding events on the boundaries of v and $v'$are related by the infinitesimal displacement δx^μ. Let $d\sigma^\mu$ denote $n^\mu d\sigma$, where $n$is an outward normal to the boundary of v at a point x, and $d\sigma$ is an element of surface area (ie., hypersurface volume) on the boundary of v at x.

Then, the scalar product $d\sigma_\mu \delta x^\mu$ is the volume of the cylindrical four-volume element capped at x by the surface area $d\sigma$ and extending along δx^μ from the boundary of v to that of v'. Then, denoting the boundary of v by ∂v,, the surface integral,

$$\int_{\partial v} d\sigma_\mu \, \delta x^\mu \mathcal{L}(\phi, \partial\phi),$$

Is equal to following difference of volume integrals over v and v':

$$\int_{v'} d\, v_x \mathcal{L}(\phi, \partial\phi) - \int_{v} d\, v_x \mathcal{L}(\phi, \partial\phi)$$

Because the difference is just the integral of $\mathcal{L}$over the infinitesimal volume lying between the boundaries of v and v'.

Hence,

$$\delta S = S' - S$$

$$= \int_{\partial v} d\, v_x \delta x^\mu \mathcal{L}(\phi, \partial\phi) + \int_{v} d\, v_x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta_0 \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu \delta_0 \phi \right\}.$$

Converting the surface integral to a volume integral by the Gauss divergence theorem, and using the identity,

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu \delta_0 \phi = \partial_\mu \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_0 \phi \right\} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta_0 \phi$$

In the second term, we are left with the result

$$\delta S = \int_{v} d\, v_x \partial_\mu \left\{ \delta x^\mu \mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_0 \phi \right\}.$$

Here, the Euler-Lagrange equations (3.16) have beenused . It is convenient to express our result in terms of the local variation $\phi\delta$ defined in (3.33), one has

$$\phi'(x) = \phi'(x' - \delta x) = \phi'(x') - \partial_\mu \phi(x) \delta x^\mu$$

To first order. Consequently, (3.29) gives

$$\delta_0 \phi(x) = \delta\phi - (\partial_\mu \phi) \delta x^\mu.$$

Hence

$$\delta S = \int_v d\, v_x \partial_\mu \left\{ \frac{\partial \mathcal{L}}{\partial \phi \partial(\partial_\mu \phi)} \delta\phi - T^\mu{}_v \delta x^v \right\}, \qquad (3.34)$$

Where $T^\mu{}_v$ is the energy-momentum tensor of (3.24). it follows that $\delta S = G(t_2) - G(t_1)$, with $G(t)$given by (3.32), which completes the derivation.

Under an infinitesimal translation, we have $\phi'(x') = \phi(x)$, so that $\delta\phi$ vanishes. therefore,

$$\delta_0 \phi = -(\partial_\mu \phi) \delta x^\mu.$$

Using the fact that for a translation δx^μ is constant, we now find that the generator is

$$G = -P_\mu \delta x^\mu,$$

Where P_v is the momentum operator of (3.31) for the commutator of G with an operator functional $F[\phi, \pi]$.let us write (3.31) for the case when F is a function f formed from$\phi$and the $\partial_\mu \phi$.then,

$$\delta_0 f = \frac{\partial f}{\partial \phi} \delta_0 \phi \; + \frac{\partial f}{\partial(\partial_\mu \phi)} \partial_\mu \delta_0 \phi.$$

The Euler-Lagrange equations follow, as usual, by demanding that δS is zero for the subset of variations such that $\delta_0 \phi$ vanishes on the boundary ∂_v.

Upon using the form of $\delta_0\phi$ for a translation, we find

$$\delta_0\phi = (\partial_\mu f)\delta x^\mu.$$

Using these results in (3.31), we obtain immediately (3.27).The transformations of (3.28) and (3.29) are said to be a symmetry or invariance of the Lagrangian density $\mathcal{L}$ if

$$\mathcal{L}(\phi'(x'), \partial'\phi'(x')) = \mathcal{L}(\phi(x), \partial\phi(x)). \tag{3.35}$$

Recall that in S and S', the integrals are taken over the same physical volume x and x'refer to the same physical point in space-time. Therefore, the action is invariant when (3.35) holds, and we have $\delta S = 0$. in that case, (3.30) implies that G is a constant of the motion. Furthermore, (3.34) implies that

$$\partial_\mu J^\mu = 0, \tag{3.36}$$

With the conserved current

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a - T^\mu{}_v\delta x^v. \tag{3.37}$$

It follows that when a translation in the x^μ direction is a symmetry of the Lagrangian density, then the operator P_μ is a constant of the motion. In Mikowski space-time, the Lagrangian density of a free field is symmetric under space and time translations, which implies that energy and momentum are conserved. In curved space-time, because of the presence in the Lagrangian density of the metric, we will find that for a free field (i.e., one influenced by gravitation alone), only the component of $P_\mu$along the direction of an isometry of the spacetime is conserved.

3.2 Particle

The Schr�dinger representation based on the eigenvectors of the fields operator ϕ emphasizes the field aspect of the quantized field theory. Let us now trun to a representation which emphasizes the dual particle aspect of the quantum field. one expects that since the particle aspect and wave properties of a system are known to be complementary, rather than simultaneously measurable, the observable corresponding to particle number, for example, will be constructed from both the field ϕ and its non-commuting momentum π, we must first understand how to describe a system of free particles. Then, mutually interacting particles can be described through a perturbative description in which the free particles appear at early times and emerge again at late times. Therefore, we consider for now a single Hermitian free-scalar(or pseudo-scalar) field described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\left(\eta^{\mu v}\partial_\mu\phi\partial_v\phi - m^2\phi^2\right). \tag{3.38}$$

for example, this equation described a free neutral pion (which is a pseudo-scalar, i.e., odd under parity). The charged field is discussed below. The free spin-1massless (Maxwell) and massive (Proca) fields and the free spin-1/2 (Dirac) field will be described. Non-abelian massless gauge fields of spin-1 as the Euler-Lagrange field equation (3.16) gives

$$(\Box + m^2)\phi = 0, \tag{3.39}$$

where $\Box = \eta^{\mu v}\partial_\mu\partial_v$. we shall work in the Heisenberg picture, in which the quantum field satisfies (3.39), which is known as the Klien-Gordon equation. The canonical momentum of (3.13) is

$$\pi = \partial_0\phi = \phi. \tag{3.40}$$

The field and conjugate momentum are assumed to be operators satisfying the canonical commutation relations of (3.17). from (3.24) for the stress tensor and (3.25), we find the energy and momentum observables

$$P_0 = \frac{1}{2}\int d\,V_x\left(\phi^2 + \left(\vec{\partial}\phi\right)^2 + m^2\phi^2\right), \tag{3.41}$$

$$P_i = \frac{1}{2}\int d\,V_x(\phi\partial_i\phi + (\partial_i\phi)\phi), \tag{3.42}$$

recall that symmetrization over non-commuting fields was understood in (3.24), and hence in P_μ because the Lagrangian density $\mathcal{L}$ is a scalar under space and time translations, the energy and momentum of the system are conserved. One can also check, using the commutation relations, that $P_\mu$commutes with P_0 because $\mathcal{L}$ is also a scalar under the group of homogeneous Lorentz transformations, there are six further conserved generators in four-dimensional space-time, corresponding to rotations about axes perpendicular to the six planes determined by pairs taken from the four coordinate axes (i.e., three spatial rotations and three boosts or velocity transformations).

If $f_1(x)$and $f_2(x)$are two solutions of (3.39), then the following scalar product is conserved,

$$(f_1, f_2) = iP_0 = \frac{1}{2}\int d\,V_x\{f_1{}^*(\vec{x},t)\partial_0 f_2(\vec{x},t)\} - \{\partial_0 f_1{}^*(\vec{x},t) f_2(\vec{x},t)\}$$

$$i \equiv \int d\,V_x f_1{}^*\overleftrightarrow{\partial}_0 f_2 \tag{3.43}$$

The scalar product is linear with respect to the second argument and antilinear with respect to the first. Furthermore, we have $(f_1, f_2)^* = (f_2, f_1) = -(f_1{}^*, f_2{}^*)$.

A complete set of positive energy or positive frequency solutions of the Klein-Gordon equation (3.39) in a space-time of dimension $n$is

$$g_{\vec{k}}(x) = (2\pi)^{-(n-1)/2}(2\omega_k)^{-1/2}\exp\big[i\big(\vec{k}.\vec{x} - \omega_k t\big)\big], \tag{3.44}$$

3.3 Basics Of Quantum Field In Curved Space-Times

The successful predictions of general relativity are convincing evidence that gravitational phenomena are most clearly it gravitational influence by curving pace-time, and we study the propagation of particle and wave on this curved background. It I then natural to study the propagation. Of quantum fields in curved pacetime in order to search for new effect of gravitation. At this level, the gravitational field itself Is not quantized, and the methods of Minkwoski pace time quantum field theory are carried over as much as possible. As we shall see, this modest extension of quantum field theory has turned out to be richer in consequences then we could have anticipated. Among other thing, it gives rise to the physically important processes of particle creation in cosmological and black hole spacetimes. The same amplification process that creates particles in an expanding universe is responsible for creating, in the context of an early inflationary expansion, the primordial fluctuation that are now observed with astonishing accuracy in the cosmic microwave background (CMB) radiation. These same primordial fluctuations also appear responsible for the large-scale structure of the universe. The creation of particles by black holes in necessary for maintaining the second law of thermodynamics in their presence. This process of radiation and cvaporation of black holes is an important facet in the fundamental search for a microscopic explanation of the entropy of black holes; a search which appears to be leading to new and exciting physics connecting gravitation and quantum theory. Quantum field theory in curved space-time also provides a new dynamical explanation of the connection between pin and statistics, and brings out some new features of Minkowski pace-time physics, such as the excitation of accelerated detectors. This partial listing is sufficient motivation to turn now to the basis of the theory of quantized fields in curved space-time.

3.4 Canonical Quantization And Conservation Law

Consider a set of fields $\phi_a(x)$ propagating in a curved space-time with invariant line element

$$ds^2 = g_{\mu v}(x)dx^{\mu}dx^{v}. \tag{3.45}$$

The metric $g_{\mu v}(x)$ will be treated as a given unquantized external field. We will assume that the spacetime has a well-defined causal structure and set of Cauchy hypersurfaces. The set of fields $\phi_a(x)$ to be quantized may include linearized gravitational wave perturbations propagating on the background $g_{\mu v}(x)$. Let n denote the dimension of spacetime, with x^0 being the time coordinate and $x^1, \ldots x^{n-1}$ being the spatial coordinates. The action $S$is constructed from the field ϕ_a, so that it is invariant under general coordinate transformations (diffeomorphisms):

$$S\left[\phi'(x'), \nabla'\phi'(x'), g'_{\mu v}(x')\right] = S\left[\phi(x), \nabla\phi(x), g_{\mu v}(x)\right]. \qquad (3.46)$$

The simplest way to construct such an action is to start with the Minkowski space-time action and replace ordinary derivatives ∂_μ by covariant derivatives $\nabla_\mu, \eta^{\mu v}$by $g^{\mu v}$, and $d^n x$ by the invariant volume element $d^n x|g|^{1/2}$, where $g = det\left(g_{\mu v}\right)$. This is called the minimal coupling prescription, and is consistent with the Einstein principle of equivalence, according to which local gravitational effects are not present in a neighborhood of the space-time origin of a locally inertial frame of reference. Occasionally, we can further increase the symmetry by the addition to the Lagragian of a term which does not vanish at the origin of a locally inertial frame, and such a possibility will also be considered. The action already involves $\partial_\lambda g_{\mu v}$ through $\nabla_\mu \phi_a$; when additional terms are included, it may also involve higher derivatives of $g_{\mu v}$.

The requirement that variations of the action

$$S = \int d^n x\, \mathcal{L}\left(\phi, \nabla\phi, g_{\mu v}\right) \qquad (3.47)$$

Vanish with respect to variations of the fields ϕ_a which are zero on the boundary of integration, then yields the Euler-Lagrange equations

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_a\right)}\right)-\frac{\partial\mathcal{L}}{\partial\phi_a}=0. \tag{3.48}$$

The general covariance of (3.48) is insured by the invariance of the action S. Note that $\mathcal{L}$ is a scalar density, transforming like $|g|^{1/2}$. Variation of the action with respect to the external field $g_{\mu v}$does not in general vanish because we have not include an additional term proportional to the scalar curvature R in (3.47). Such a term given rise to the geometric part of the Einstein gravitational field equations. However, because of the invariance of S under general coordinate transformations, δS will be zero under the change in $g_{\mu v}$ induced by an infinitesimal coordinate transformation

$$x^\mu \to x'^\mu = x^\mu - \epsilon^\mu(x), \tag{3.49}$$

Where x and x'refer to the same event in spacetime. Under this transformation we have

$$g_{\mu v}(x) \to g'_{\mu v}(x') = \frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^v}g_{\lambda\sigma}(x),$$

Which yields the variation,

$$\delta_\sigma g_{\mu v}(x) \equiv g'_{\mu v}(x) - g_{\mu v}(x) = \mathcal{L}_\epsilon g_{\mu v}, \tag{3.50}$$

Where $\mathcal{L}_\epsilon$ denotes the Lie derivative,

$$\mathcal{L}_\epsilon g_{\mu v}(x) = \nabla_\mu \epsilon_v + \nabla_v \epsilon_\mu. \tag{3.51}$$

Let us assume that $\epsilon^\mu(x)$ and $\partial_\lambda\epsilon^\mu(x)$ are zero on the boundary of the region of interagtion defining the action S of (3.47). then , under the infinitesimal coordinate transformation, we have

$$\delta S = \int d^n x \frac{\delta S}{\delta g_{\mu v}(x)} \delta_0 g_{\mu v}(x),$$

Because variations in Sproduced by the changes in the dynamical fields $\phi_a$vanish as a consequence of (3.48)and the boundary conditions on ϵ_μ. With $\delta_0 g_{\mu v}$ given by (3.50)and (3.51), the invariance of S under coordinate transformations requires that $\delta S = 0$. Hence, with $dv_x \equiv d^n x|g|^{1/2}$,

$$\delta S = -\int dv_x \, T^{\mu v} \nabla_\mu \epsilon_v = 0, \tag{3.52}$$

Where we have defined the tensor

$$T^{\mu v} \equiv -2|g|^{1/2} \frac{\delta S}{\delta g_{\mu v}(x)}, \tag{3.53}$$

And have used its symmetry under interchange of indices. Then from (3.52) and

$$\nabla_\mu (T^{\mu v} \epsilon_v) = |g|^{1/2} \partial_\mu \left(|g|^{1/2} T^{\mu v} \epsilon_v \right) = \left(\nabla_\mu T^{\mu v} \right) \epsilon_v + T^{\mu v} \nabla_\mu \epsilon_v,$$

It follows that

$$\int dv_x \left(T^{\mu v} \nabla_\mu \right) \epsilon_v = 0,$$

And because ϵ_v is arbitrary, we must have

$$\nabla_\mu T^{\mu v} = 0. \tag{3.54}$$

This is generally covariant generalization of (3.23). further more, $T^{\mu v}$as defined by (3.53)symmetric, which is not true in general for the canonical energy-momentum tensor $\ominus^{\mu v}$ of (3.24), the symbol $T^{\mu v}$ was used for the canonical energy-momentum tensor, $\ominus^{\mu v}$,but from now on $T^{\mu v}$ will refer to the symmetric energy-momentum tensor, from $\delta\left(g^{\mu v} g_{\mu\lambda}\right) = 0$, it follows that

$$g_{\mu\lambda} g_{v\sigma} \frac{\delta}{\delta g_{v\sigma}} = \frac{\delta}{\delta g^{\mu v}},$$

$$T_{\mu v} = \frac{2}{|g|^{1/2}} \frac{\delta S}{\delta g^{\mu v}}. \tag{3.55}$$

The sign convention in our definition of $T^{\mu v}$ in (3.53) is chosen so that T_{00}will be positive for the classical electromagnetic field. If we were to use the opposite metric signature, then the signs on the right-hand sides of (3.53) and (3.55) would also be opposite. We can calculate the symmetric energy-momentum tensor $T^{\mu v}$ in curved space-time and then go to the flat space-time limit, theory obtaining a symmetric energy-momentum tensor satisfying $\partial_\mu T^{\mu v} = 0$in Minkowski spacetime. For any isolated system in Minkowski spacetime, both $\ominus^{\mu v}$ and $T^{\mu v}$ yield a conserved energy-momentum vector p_μ, which, as noted earlier, is unique. In curved space-time it is the symmetric energy-momentum tensor $T^{\mu v}$ defined in (3.53) which describes the matter and radiation and couples to the gravitational field through the Einstein field equations. In a general curved space-time, the tensor density $\ominus^{\mu v}$ of weight $1/2$ defined by (3.24) does not satisfy $\partial_\mu \ominus^{\mu}{}_{v} = 0$as it does in Minkowski spacetime, nor any simple generalization of that equation. The Schwinger operator action principle continues to hold in curved pace-time for an arbitrary infinitesimal transformation of the form given in (3.28)and(3.29), provided that under the transformation $\delta_0 g_{\mu v}(x) = 0$, or

$$g'_{\mu v}(x) = g_{\mu v}(x). \tag{3.56}$$

In that case, the derivation leading to (3.34) goes through as before with the canonical stress tensor now called $\ominus^{\mu}{}_{v}$, because we have $\delta_0 g_{\mu v} = 0$, it is not necessary for the Euler-Lagrange equation to hold for the external field $g_{\mu v}$.

note that the metric $g_{\mu v}$ is not included here among the fields $\phi_a$of (3.34).Thus, under a transformation of the coordinates and fields given in (3.28)and (3.29), and for which (3.56)holds, we have

$$\delta S = G(t_2) - G(t_1), \tag{3.57}$$

With

$$G(t) = \int d^{n-1}\, x\left[\pi_a \delta\phi_a - \ominus^o{}_v\, \delta x^v\right], \tag{3.58}$$

Where, as noted above, $\ominus^o{}_v$is defined by the right-hand side of (3.24). Here the integration is on constant time hypersurface, and

$$\pi_a = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_a)}. \tag{3.59}$$

As before, with suitable operator ordering and regularization, G as the generator of the transformation, satisfying

$$i\delta_0 F = [F, G], \tag{3.60}$$

Where F is a functional of the ϕ_a and π_a.

As in Minkowski space-time, (.16) will hold with

$$[\phi_a(\vec{x}, t), \phi_b(\vec{x}', t)] = 0 \quad [\pi_a(\vec{x}, t), \pi_b(\vec{x}', t)] = 0,$$

$$[\phi_a(\vec{x}, t), \pi_b(\vec{x}', t)] = i\delta_{a,b}\delta(\vec{x} - \vec{x}') \tag{3.61}$$

Which I appropriate for bosons, or

$$\{\phi_a(\vec{x}, t), \phi_b(\vec{x}', t)\} = 0 \quad \{\pi_a(\vec{x}, t), \pi_b(\vec{x}', t)\} = 0,$$

$$\{\phi_a(\vec{x}, t), \pi_b(\vec{x}', t)\} = i\delta_{a,b}\delta(\vec{x} - \vec{x}') \tag{3.62}$$

Which is appropriate for fermions. Here $\delta(\vec{x}-\vec{x}')$is the Dirac δ-function satisfying $\int d^{n-1}\, x\delta(\vec{x}-\vec{x}')f(\vec{x}') = f(\vec{x}')$with the integral being over the spacelike hypersurface t-constant. The above commutation relations are imposed on independent field components. They may be modified when gauge conditions are present. one can show that $\pi(\vec{x}',t)$ and $\delta(\vec{x}-\vec{x}')$each transform as spatial scalar densities under transformations of the spatial coordinates on the constant hypersurface. Hence, (3.61) and (3.62) are covariant under transformations of the spatial coordinates on the hypersurfase, and are therefore the spatially covariant generalization of the corresponding relations thathold in

flat space-time. One can also show that they are consistent with the equations of motion of the fields, in the sense that if they hold on one constant-t spatial hypersurface, then they also will hold on the other constant-t spatial hypersurfaces. Furthermore, for bosons we can define a complete set of commuting Hermitian fields, and define a basis $|\phi'>$for the Hilbert space of state vectors, as in Mickowski spacetime, through (3.19). this gives a field or Schr�dinger representation. For a system of fermions, we can define a set of Hermitian commuting quantities which are bilinear in the fields and build a Hilbert space of state vectors spanned by the simultaneous eigenvectors of these bilinear operators. There are several cases of interest when G is conserved. The simplest is when $\delta x^{\mu} = 0$and $\delta_0\phi_a$ is a symmetry of $\mathcal{L}$. Then (3.56) is trivially satisfied, and the symmetry of $\mathcal{L}$ implies that $\delta S = 0$so that G is independent of time. One also has, in that case, $\partial_\mu J^\mu = 0$,with

$$J^\mu = \frac{\partial y}{\partial(\partial_\mu \phi_a)}\delta\phi_a. \qquad (3.63)$$

note that $J^\mu$is a vector density of weight 1/2, so that we have $\nabla_\mu\left((|g|^{-1/2}J^\mu) = 0.\right)$thus, in curved space-time the electric charge and the generators of internal symmetries continue to be conserved. The generator G is also conserved when $\delta x^{\mu} \neq 0$,but is such that (3.56) holds, and the fields $\phi_a$are components of a spacetime tensor. Then, invariance under coordinate transformations implies that $\delta S = 0$, and (3.57) implies that $G$is constant. For example, if it is possible to choose a coordinate system in which a particular coordinate, say x^{λ}, does not appear in $g_{\mu v}(x)$, then under a translation in the x^{λ} direction (3.56) holds, and furthermore we have $\delta\phi(x) = 0$. It follows that

$$p_{\lambda} = \int d^{n-1}\, x \ominus^{0} \lambda \tag{3.64}$$

Is constant since $g^{\lambda\mu}$ and the other components of p_{μ} may not be constants, it does not follow that $p^{\lambda} = g^{\lambda\mu} p_{\mu}$ is constant. a similar expression involving the symmetric energy-momentum tensor $T_{\mu v}$ also holds. In deriving that result, we will also work in a more covariant language. A coordinate transformation for which (3.56) holds is called an isometry of the spacetime. For an infinitesimal isometry of the form

$$x^{\mu} \to x'^{\mu} = x^{\mu} - \epsilon \xi x^{\mu}(x), \tag{3.65}$$

With $\epsilon \ll 1$. it follows from (3.50) and (3.56) that

$$\mathcal{L}_{\xi} g_{\mu v} \equiv \nabla_{\mu}\xi_{v} + \nabla_{v}\xi_{\mu} = 0. \tag{3.66}$$

A vector field satisfying (3.66) is called a Killing vector of the space time. As a consequence of the symmetry of $T^{\mu v}$ and (3.54) we have

$$\nabla_{\mu}(T^{\mu v}\xi_{v}) = 0. \tag{3.67}$$

But since $T^{\mu v}\xi_{v}$ is a vector, we have$\nabla_{\mu}(T^{\mu v}\xi_{v}) = |g|^{-1/2}\partial_{\mu}\left(|g|^{-1/2} T^{\mu v}\xi_{v}\right)$, and it follows that

$$\partial_\mu\left(|g|^{-1/2}T^{\mu v}\xi_v\right) = 0. \tag{3.68}$$

Hence

$$P_\xi \equiv \int d\,V_x T^0{}_v(x)\xi^v(x) \tag{3.69}$$

Is constant, where $dV_x = d^{n-1}x|g|^{-1/2}$. In the case when the coordinates are such that $g_{\mu v}(x)$is independent of a particular coordinate, say x^λ, then $\xi^v = \delta^v\lambda$ is a Killing vector field, and (3.69) reduces to $P_\lambda = \int d\,V_x T^0{}_\lambda$being constant. In such a case, (3.64) should yield the same p_λ to within possibly an additive constant independent of the configuration. Thus, in a general curved spacetime, we have shown that $\nabla_\mu T^{\mu v} = 0$, and in a space-time having special symmetries or isometries as implied by the existence of one or more Killing vector fields ξ^μ, we have $\partial_\mu\left(|g|^{1/2}T^\mu{}_v\xi^v\right) = 0$ and $P_\xi \equiv \int d\,V_x T^0{}_v\xi^v$is constant. Let us now consider again a general curved space-time without special isomtries. An invariance of the action which is of interest is curved space-time conformal invariance . certain fields, such as the electromagnetic and massless Dirac fields in curved space-time, exhibit an invariance of the action under conformal transformation of the metric and field, as defined below. An action $S[\phi, g_{\mu v}]$(suppressing derivatives for brevity) is conformally invariant if

$$S[\phi, g_{\mu v}] = S[\tilde{\phi}, \tilde{g}_{\mu v}] + \text{surface integral}, \tag{3.70}$$

Where

$$\tilde{g}_{\mu v}(x) = \Omega^2(x)g_{\mu v}(x), \tag{3.71}$$

$$\tilde{\phi}(x) = \Omega^{2p}(x)\phi(x). \tag{3.72}$$

Here $\Omega(x)$ is an arbitrary function and p is a dimensionless constant. Consider an infinitesimal conformal transformation with

$$\Omega^2(x) = 1 + \lambda(x), \qquad |\lambda(x)| \ll 1. \tag{3.73}$$

Then $\delta x^\mu = 0$, and from (3.71) and (3.72),

$$\delta_0 g_{\mu v}(x) = \tilde{g}_{\mu v}(x) - g_{\mu v}(x) = \lambda(x) g_{\mu v}(x), \tag{3.74}$$

$$\delta_0 \phi(x) = \tilde{\phi}(x) - \phi(x) = p\lambda(x)\phi(x). \tag{3.75}$$

From (3.70), if $\lambda(x)$ vanishes sufficiently rapidly on the boundary of the region of integration,

$$0 = \delta S = \int d^n x \left\{ \frac{\delta S}{\delta \phi} \delta_0 \phi + \frac{\delta S}{\delta g_{\mu v}} \delta_0 g_{\mu v} \right\}.$$

If we assume that ϕ satisfies the Euler-Lagrange equation, $\delta S/\delta\phi = 0$,that it follows that

$$0 = \int d^n x \frac{\delta S}{\delta g_{\mu v}(x)} g_{\mu v}(x) \lambda(x),$$

And because $\lambda(x)$ is arbitrary,

$$\frac{\delta S}{\delta g_{\mu v}(x)} g_{\mu v}(x) = 0,$$

From which we have

$$g_{\mu v} T^{\mu v} = 0. \tag{3.76}$$

Thus, conformal invariance of the action implies that trace of the energy-momentum tensor is zero. We will find later that a theory based on a classical or bare action which is conformally invariant will in general lose its conformal invariance in the quantum theory as a result of renormalization. The energy-momentum tensor thus acquires a non-vanishing trace, known as the trace or conformal anomaly. We have discussed the action, field equations, symmetric energy-momentum tensor, generators of field transformations, commutation or anti commutation relations, Hilbert space of state vectors in the field representation, isometries and conservation laws, and conformal invariance. Having built a foundation for our discussion, it us consider next the curved space-time generalization of the free neutral scalar field.

3.5 Scalar Field

Following the minimal coupling prescription based on the principle of equivalence and described. The action of the free neutral scalar field based on the Lagrangian density of (3.38) becomes in curved space-time

$$S = \int d^n x\mathcal{L} \tag{3.77}$$

with

$$\mathcal{L} = \frac{1}{2}|g|^{1/2}\left(g^{\mu v}\partial_\mu\phi\partial_v\phi - m^2\phi^2\right). \tag{3.78}$$

For the scalar field we have $\nabla_\mu\phi = \partial_\mu\phi$,this is a special case of the more general Lagrangian density

$$\mathcal{L} = \frac{1}{2}|g|^{1/2}\left(g^{\mu v}\partial_\mu\phi\partial_v\phi - m^2\phi^2 - \xi R\phi^2\right), \tag{3.79}$$

Where ξ is a dimensionless constant and R is the scalar curvature of the spacetime, $R \equiv g^{\mu v}R_{\mu v}$. We use the following conventions metric signature-2, $R^{\mu}{}_{v\lambda\sigma} = \partial_\sigma\Gamma^{\mu}{}_{v\lambda} - \partial_\lambda\Gamma^{\mu}{}_{v\sigma} + \Gamma^{\mathcal{T}}{}_{v\lambda}\Gamma^{\mu}{}_{\sigma\mathcal{T}} - \Gamma^{\mathcal{T}}{}_{v\sigma}\Gamma^{\mu}{}_{\lambda\mathcal{T}}$, and

$R_{\mu v} = R^{\lambda}{}_{\mu\lambda v}$, in agreement with Birral and Davies or (-,-,-) in the notation of Misner, we will carry out discussion for this more general Lagrangian. One reason is that when we include an interaction term such as $|g|^{-1/2}\lambda\phi^4$, withλa constant (dimensionless in four-dimensional space-time), then a term of the form $|g|^{-1/2}\xi R\phi^2$is needed for renormalization. Another reason is that when $m = 0$and $\xi = 1/6$(in four dimensions), the action is invariant under the curved spacetime conformal transformations. The case $\xi = 0$is referred to as minimal coupling. The magnitude of ξcannot be very large, because if ξ has a non-zero value, then the term in the Lagrangian that is proportional to $R\phi^2$ can cause the effective gravitational constant to very with time and position as a result of such variations in ϕ. Consider an infinitesimal conformal transformation of the form of (3.73) and (3.75) with $p = -\frac{1}{2}$:

$$g_{\mu v}(x) \to \tilde{g}_{\mu v}(x) = \big(1 + \lambda(x)\big) g_{\mu v}(x), \tag{3.80}$$

$$\phi(x) \to \tilde{\phi}(x) = \left(1 - \frac{1}{2}\lambda(x)\right)\phi(x). \tag{3.81}$$

Under this transformation, we have in four dimensions to first order in λ

$$|g|^{1/2} \to |\tilde{g}|^{1/2} = (1 + 2\lambda)|g|^{1/2},$$

$$g^{\mu v} \to \tilde{g}^{\mu v} = (1 - \lambda) g^{\mu v},$$

$$\Gamma^{\mu}{}_{v\mathcal{T}} \to \Gamma^{\mu}{}_{v\mathcal{T}} = \Gamma^{\mu}{}_{v\mathcal{T}} + \frac{1}{2}(\delta^{\mu}{}_{\mathcal{T}}\partial_v\lambda + \delta^{\mu}{}_{v}\partial_{\mathcal{T}}\lambda - g^{\mu\sigma} g_{v\mathcal{T}}\partial_\sigma\lambda),$$

$$R \to R = (1 - \lambda)R + 3\lambda. \tag{3.82}$$

Then, working to first order in λ, the transformed Lagrangian density with $\xi = 1/6$and $m = 0$ becomes

$$\tilde{\mathcal{L}} = \frac{1}{2}|\tilde{g}|^{1/2}\left(\tilde{g}^{\mu v}\partial_\mu\tilde{\phi}\partial_v\phi - \frac{1}{6}\tilde{R}\tilde{\phi}^2\right)$$

$$= \frac{1}{2}(1+2\lambda)|g|^{1/2}\left\{(1-\lambda)g^{\mu v}\partial_\mu\left[\left(1-\frac{1}{2}\lambda\right)\phi\partial_v\right]\left[\left(1-\frac{1}{2}\lambda\right)\phi\right]\right.$$
$$\left. - \frac{1}{6}[(1-\lambda)R + 3\lambda](1-\lambda)\phi^2\right\}$$

$$= \frac{1}{2}|g|^{1/2}\left\{g^{\mu v}\partial_\mu\phi\partial_v\phi - g^{\mu v}\phi\partial_\mu\lambda\partial_v\phi - \frac{1}{2}R\phi^2 - \frac{1}{2}(\Box\lambda)\phi^2\right\}$$

$$\mathcal{L} = \frac{1}{2}|g|^{1/2}g^{\mu v}\phi\partial_\mu\lambda\partial_v\phi - \frac{1}{4}|g|^{1/2}(\lambda)\phi^2$$

$$= \mathcal{L} - \partial_\mu\left(\frac{1}{4}|g|^{1/2}g^{\mu v}\phi^2\partial_v\lambda\right), \tag{3.83}$$

Since

$$|g|^{1/2}\lambda = \partial_\mu\left(|g|^{1/2}g^{\mu v}\partial_v\lambda\right).$$

Hence, the conformally related actions differ only by a surface term and the field equation for ϕ is form invariant under conformal transformations. Since a finite conformal transformation with $\Omega^2(x) = \exp[\lambda(x)]$can be built from an infinite product of infinitesimal transformations (with λ being the sum of the infinitesimal λ), we must have for a finite conformal transformation with

$$g_{\mu v} \to \tilde{g}_{\mu v} = \Omega^2 g_{\mu v}$$

$$\phi \to \tilde{\phi} = \Omega^{-1}\phi$$

that

$$\tilde{\mathcal{L}} = \mathcal{L} - \partial_\mu\left(|g|^{1/2}g^{\mu v}\phi^2\partial_v\ln\Omega\right). \tag{3.84}$$

This has the same form in terms of λas in the infinitesimal case; and again the field equation is form invariant. In n dimensions, the value of ξ in (3.79) which makes the classical theory conformally invariant under the transformation of (3.71) and (2.28(, with $p = (2-n)/2$, is $\xi = [4(n-1)]^{-1}(n-2)$.

Working with ξ and m arbitrary, the Euler-Lagrange equation for the Lagrangian density (3.79) reads

$$(\square + m^2 + \xi R)\phi = 0, \tag{3.85}$$

and (3.53) gives the symmetric energy-momentum tensor,

$$T^{\mu v} = \nabla^{\mu}\phi\nabla^{v}\phi - \frac{1}{2}g^{\mu v}\nabla^{p}\phi\nabla_{p}\phi + \frac{1}{2}g^{\mu v}m^2\phi^2 - \xi\left(R^{\mu v} - \frac{1}{2}g^{\mu v}R\right)\phi^2$$
$$+ \xi[g^{\mu v}(\phi^2) - \nabla^{\mu}\phi\nabla^{v}(\phi^2)]. \tag{3.86}$$

This satisfies, as a consequence of (3.85),

$$\nabla_{\mu}T^{\mu v} = 0, \tag{3.87}$$

and in four dimensions

$$T^{\mu}{}_{\mu} = 0, \text{when } \xi = \frac{1}{6} \text{and} \quad m = 0. \tag{3.88}$$

As we know from the previous, (3.87) is a consequence of consequence of coordinate invariance and (3.88) of conformal invariance for the unquantized field. In the quantized theory, (3.88). The calculation of (3.86) proceeds briefy as follows. Using the identities

$$\delta g^{\mu v} = -g^{\mu p}g^{v\sigma}\delta g_{p\sigma}, \tag{3.89}$$

$$\delta|g|^{1/2} = \frac{1}{2}|g|^{1/2}\delta g^{\mu v}\delta g_{\mu v}, \tag{3.90}$$

$$\delta R = -R^{\mu v}\delta g_{\mu v} + g^{p\sigma}g^{\mu v}\big(\delta g_{p\sigma;\mu v} - \delta g_{p\mu;\sigma v}\big). \tag{3.91}$$

This gives

$$\delta S = \frac{1}{2}\int d^n\, x|g|^{1/2}\Big\{\frac{1}{2}g^{\mu v}\delta g_{\mu v}\big(g^{p\sigma}\partial_p\phi\partial_\sigma\phi - m^2\phi^2 - \xi R\phi^2\big)$$
$$- \delta g_{\mu v}\nabla^\mu\phi\nabla^v\phi$$
$$- \xi\big[-R^{\mu v}\delta g_{\mu v} + g^{p\sigma}g^{\mu v}\big(\delta g_{p\sigma;\mu v} - \delta g_{p\mu;\sigma v}\big)\big]\phi^2\Big\}.$$

Using the identities (taking $\delta g_{\mu v}$and its first derivative to be zero on the boundary of the region of integration)

$$\int d^n\, x|g|^{1/2}g^{p\sigma}g^{\mu v}\delta g_{p\sigma;\mu v}\phi^2 = \int d^n\, x|g|^{1/2}g^{p\sigma}\delta g_{p\sigma}(\phi^2)$$

and

$$\int d^n\, x|g|^{1/2}g^{p\sigma}g^{\mu v}\delta g_{p\mu;\sigma v}\phi^2 = \int d^n\, x|g|^{1/2}g^{\sigma\mu}g^{\lambda v}\delta g_{\mu v}\nabla_\sigma\nabla_\lambda(\phi^2),$$

We then obtain

$$\delta S = -\frac{1}{2}\int d^n\, x|g|^{1/2}T^{\mu v}\delta g_{\mu v},$$

With $T^{\mu v}$given by (3.86), as was to be shown. The scalar field is quantized by imposing the canonical commutation relation of (3.61). the appropriate generalization of the scalar product (3.43) is

$$(f_1, f_2) \equiv i\int d^{n-1}\, x|g|^{1/2}g^{0v}f_1^*(\vec{x},t)\overleftrightarrow{\partial}_v f_2(\vec{x},t) \tag{3.92}$$

Where the integral is taken over a constant $t$hyp ersurface. If $f_1$and $f_2$are solutions of the field equation (3.85) which vanish at spatial infinity, then (f_1, f_2) is conserved, since

$$\frac{d}{dt}(f_1, f_2) = i\int d^{n-1}\, x\partial_0\left(|g|^{1/2} g^{0v} f_1^* \overleftrightarrow{\partial}_v f_2\right)$$

$$= i\int d^{n-1}\, x|g|^{1/2}\nabla_\mu\left(g^{\mu v} f_1^* \overleftrightarrow{\partial}_v f_2\right)$$

$$-i\int d^{n-1}\, x\,\partial_i\left(|g|^{1/2} g^{iv} f_1^* \overleftrightarrow{\partial}_v f_2\right) = 0.$$

The first term of the second equality above is zero by virtue of the field equation, and second term gives a contribution at spatial infinity which vanishes (this also vanishes if we use normalization in a cube with periodic boundary conditions on f_1 and f_2). We have used the basic identity $\nabla_\mu V^\mu = |g|^{-1/2}\partial_\mu\left(|g|^{1/2}V^\mu\right)$,valid for any vector fieldV^μ, in the derivation.

In terms of a general space-like hyper surface σ with future-directed unit normal $n^\mu$and hypersurface element $d\sigma$, the scalar product (3.92) is

$$(f_1, f_2) = i\int_\sigma d\sigma|g|^{1/2}\, n^v f_1^* \overleftrightarrow{\partial}_v f_2. \qquad (3.93)$$

We can show that this scalar product is conserved under deformations of σ. Suppose $\sigma \to \sigma'$ such that $\sigma$and σ' form the spacelike boundaries of a volume v there may also be timelike boundaries of v at spatial infinity. Then by the Gauss divergence theorem

$$(f_1, f_2)_{\sigma'} - (f_1, f_2)_\sigma = i\int_v d^n\, x\partial^\mu\left(|g|^{1/2} f_1^* \overleftrightarrow{\partial}_\mu f_2\right)$$

$$-i\int_{v} d^n x|g|^{1/2}\nabla^\mu\left(f_1^*\overleftrightarrow{\nabla}_\mu f_2\right) = 0,$$

As a consequence of the field equation. We discussion of the scalar field in a general curved space-time, it will be instructive to look at a specific model , the amplitude of the initial perturbations is taken as a parameter set by the observations. Thus, this amplitude is precisely known, but understandably difficult to predict from our current limited knowledge of fundamental physics at the very high energies relevant to cosmological inflation . As in any theory where infinites occur, we could regard the infinities as a manifestation of the incompleteness of the theory. This may be the case here. However, once the amplitude of the initial nearly scale-invariant perturbations can be predicted from fundamental physics, the correctness of including the adiabatic subtractions will become a question that can be tested by observation. As noted earlier, we have only considered here inflationary potentials having non-negative effective mass, $m^2 \geq 0$. If $V''\left(\phi_0(t)\right) < 0$were negative, as occurs for some inflationary potentials, the effective value of m^2evidently would be negative. A separate analysis would be necessary in such a case. We do not discuss that further here. A word on interacting quantized fields and on algebraic quantum field theory in curved space-time. Canonical quantization of quantum field theory in curved space-time has opened our eyes to new phenomena in which general relativity quantum field theory are intertwined in a convincing and sometimes remarkable manner. As we have explained above, particle creation in the early inflationary universe is the most likely source of the perturbations that led to the observed anisotropies of the CMB radiation and of the observed statistical feature of the larg-scale structure of the universe. Accurate measurements of these features continue to support this explanation. A second phenomenon that probes the deepest reaches of current physics is the remarkable temperature of a black hole. This is a consequence of quantum field theory in curved spacetime and is necessary for the second law of thermodynamics to encompass systems in general relativity in which black holes exist, such systems currently serve as attesting ground for developing new fundamental physical theories that combine gravitation and quantum theory. The fact that the particle concept itself becomes ambiguous in a general curved space-time was also discovered in the above cited works of Parker and related of the vacuum and the irreducible uncertainty in the measurement of particle number. These result were obtained for the spacetimes of isotropically expanding universes; and therefore must be a feature of general curved space-times in the absence of special symmetries. The adiabatic condition serves as a requirement on physically acceptable states, and determines the singularity structure of expectation values of products of fields. The Hadamard condition also specifies the singularity structure of the symmetrized two-function of the quantized field. Both conditions give natural generalizations of the singularity structure that is found in Minkowski spacetime are they equivalent? It has been shown, by comparing the expansion of the Hadamard from of such expectation values and the cooresponding expectation values formed states satisfying the adiabatic condition (i.e. adiabatic vacua), in space-times where they are both defined, that the expansions are the same to all orders (Pirk 1993). As we will see in the next chapter, these series expansions are in general asymptotic and not convergent, so an adiabatic vacum or a Hadamard state’’is not unique, but corresponds to alarge class of acceptable states. The Hadamard condition appears to be a natural generalization of the adiabatic to arbitrary cureved space-times. When sufficient symmetries are present, as in the Robertson-Walker space-times, the adiabatic condition gives a relatively simple and direct way to deal with the infinities that appear in the expectation values of products of fields in the limit that the fields are evaluated (with a suitable measure) at the same point.

Luders and Roberts (1990) showed that there is aunique local quasi – equivalence class of physically relevant states, and that this class can be specified in a Robertson-Walker spacetime by using the concept of an adiabatic vacuum state. It was shown in Junker and Schrohe (2002) that the definition of adiabatic vacuum states can be generalized to a general curved spacetime manifold by using the Sobolev wavefront set, and that this definition is also applicable to interacting field theories.Interestingly, Hadamard states form a special subclass of the adiabatic vacuum states defined by this method.

The fact that there is no unique vacuum state in a curved spacetime having insufficient symmetries motivates the search for a way to formulate quantum field theory in curved space-time in a way that does not out any particular state to serve as the basis of a Fock or Hilbert space formulation. In streater and Wightman (1964), it was rigorously shown that a field theory in Minkowski space-time is defined by the vacuum expectation values of products of field operators. The algebraic approach to quantum field theory in Minkowski space-time makes use of such expectation values to define the theory starting from the algebra of products of field operators and from the symmetries of the vacuum state in Minkowski space time . Arthur Wightman, around 1971. suggested that the methods of algebraic quantum field theory that had been developed in flat space-time, when suitably generalized to curved space-time, would give a rigorous way to formulate quantum field theory in curved space-time without reference to a particular vacuum state in the absence of sufficient symmetries of the curved space-time. This suggestion of Wightman was the motivation for the discussion following in Parker and Fulling (1973), where the algebraic approach to quantum field theory in curved space-time was described. Aspects of the algebraic approach to quantum field theory in curved space-time are also discussed by Fulling (1989). A satisfactory understanding of how to formulate the theory of a free quantized field in curved space-time by means of the algebraic approach was obtained by the mid-1980s, as developed in the works of Fulling (1972), Ashteker and others. The algebraic formulation does not supplant the canonical formulation of quantum field theory in curved space-time, but serves to frame the theory in a way that does not single out any particular state vector.

Starting in the early 1970s, the canonical and path integral approaches were used to define quantities, such as expectation values of energy-momentum tensors of free fields, that involve formal products of fields evaluated at the same spacetime point. Analogous formal products of field operators also appear in interacting quantum field theory. They present many more problems than do free fields when renormalization is considered. At about the time as the energy momentum tensor of free fields were being studied, investigations of interacting quantized fields in various curved space-times were undertaken. In order to use the momentum-space methods of quantum field theory that had been developed in flat space-time, Bunch and Parker (1979) introduced a local momentum-space representation using Riemann normal coordinates in a general curved space-time. This local momentum-space representation. Using this method, they carried out the renormalization of a scalar field with a quartic self-interaction to one-loop order in a general curved space-time. This is non-trivial because there are curvature terms that are not present in flat space-time and cannot be canceled by means of counterterms in the Lagrangian. Nevertheless the curvature terms do cancel one another. This raised the conjecture that physically viable interacting theories that are renormalizable in Minkowski spacetime are also renormalizable in curved spacetime. For the quartically self-interacting scalar field, this was proved to all orders in a general curved spacetime by Bunch(1981b). We will discuss the use of the local momentum-space method to analyze $\lambda\phi^4$theory to two-loop order.

3.6 Accelerated Detector In Minkowski Space -Time

A fundamental process related to quantum field theory in curved space-time is the radiation detected by an accelerating observer in Minkowski space-time.

Quantum field theory in the coordinate system appropriate to a set of uniformaly accelerated observers in Minkowski space-time was first studied by Fulling . The Minkowski metric when expressed in Rindler coordinates remains static, permitting the definition the creation and annihilation operator appropriate to the spacetime of the accelerated observers. Fulling defined these operators and found the Bogolubov transformation relating them to the usual Minkowski creation and annihilation operators of a set of interial observers. He discovered that Minkowski space-time vacuum, having no particles with respect to the inertial creation and annihilation operators of the accelerated observers. It remained unclear how to interpet those ‘’particles.’’ In Davied (1975), it was pointed out that the spectrum of the latter particles (obtained from the Bogolubov transformation found by Fulling) was a blackbody specttrum having a temperature given by $a/(2\pi)$, where a is the constant acceleration of the observers. The correct interpretation of these ‘’particles’’ was discovered by Unruh. By considering an accelerated detector coupled to a quantized scalar field Unruh showed that the detector would be excited with the same probability distribution as a similar detector bathed in blackbody radiation. Therefore, we will refer to the thermal response of an accelerated detector as the Fulling-davies-Unrnh effect. There are excellent treatments of these accelerated detectors in the literature, beginning with Unruh’s own exposition (Unruh (1976). We derive the Fulling- Davies-Unrnh effect by applying the page approximation to a zero-temperature field in Minkowski space-time. The page approximation is exact in Minkowski spacetime and shows that a uniformly accelerated observer detects a local temperature given by $a/(2\pi)$. See the derivation leading.

Chapter 4

Twistors In Curved Space-Time

Local twistors, on the other hand, have a well-defined existence (whether null or non-null) and a linear and complex analytic structure. But they are necessarily defined relative to points in the space-time so they cannot in themselves be regarded as a satisfactory generalisation of the flat-space twistors, adequate to form a basis for a formalism in which space-time points are to be regarded as derived objects. Local twistors give rise to a conformally invariant calculus on a space-time manifold. This may have some utility as such; but the main value of the local twistor concept lies in its use in the definition of asymptotic twistors.

The asymptotic twistor concept is one which applies only to a space-time which is asymptotically flat. However this is the situation appropriate to an S-matrix theory of gravitation and consequently has great relevance for the twistor quantisation programme. The space of asymptotic twistors has a complex analytic structure with a non-linear Hermitian scalar product defined giving rise to a (pseudo-) Kählerian (and hence also a symplectic) structure. A brief discription of local twistor theory will be given here and global twistor theory and its relation to gravitational scattering will be discussed.

4.1 Local Twistors

We define a twistor space at each point of space-time. This twistor space may be thought of as the direct sum of a spin-space and a conjugate spin space. However the exact way in which the twistor space splits up as a direct sum depends on the choice of conformal scaling. More explicitly, a local twistor Z^α at a point p can be represented, with respect to the metric $g_{ab'}$ by a pair of spinors $(\omega^A, \pi_{A'})$ at p. Under a conformal rescaling we will have

$$\hat{g}_{ab} = \Omega^2 g_{ab}; \hat{\omega}^A = \omega^A; \ \hat{\pi}_{A'} = \pi_{A'} + \ i\gamma_{AA'}\omega^A. \tag{4.1}$$

This is consistent with the behaviour already encountered in flat space-time since in that case we have

$$\hat{\nabla}_{AA'}\hat{\omega}^B = \hat{\nabla}_{AA'}\omega^B = \nabla_{AA'}\omega^B + \epsilon_A^B\gamma_{CA'}\omega^C \quad (4.2)$$

whence

$$i\hat{\epsilon}_A^B\hat{\pi}_{A'} = i\epsilon_A^B\pi_{A'} - \gamma_{CA'}\omega^C\epsilon_A^B \quad (4.3)$$

follows from

$$\nabla_{AA'}\omega^B = -i\epsilon_A^B\pi_{A'}. \quad (4.4)$$

However this last equation can only be maintained in conformally-flat space-time. Nevertheless it supplies motivation for (4.1) which will be retained for local twistors in curved space-time. The calculation of covariant derivatives of local twistors is most easily accomplished by introducing projection and injection operators from the twistor space to the two spin-spaces which represent it. Thus we define

$$e_\alpha^A, e_{\alpha A'}, e_A^\alpha, e^{\alpha A'}$$

sucn that $Z^\alpha e_\alpha^A = \omega^A, Z^\alpha e_{\backslash\alpha A'} = \pi_{A'}$etc. and

$$e_\alpha^A e_B^\alpha = \epsilon_B^A, \qquad e_\alpha^A e^{\alpha A'} = 0, \qquad e_{\alpha A'}e^{\alpha B'} = \epsilon_{A'}^{B'}, e_{\alpha A'}e_A^\alpha = 0,$$
$$e_\alpha^A e_A^\beta + e_{\alpha A'}e^{\beta A'} = \delta_\alpha^\beta, \qquad \bar{e}_\alpha^A = e^{\alpha A'}, \qquad \bar{e}^\alpha{}_A = e_{\alpha A'}.$$

Under conformal rescaling we will have $Z^\alpha \to \hat{Z}^\alpha = Z^\alpha$while ω_A, $\pi^{A'}$ transform by (4.1). Thus we see that

$$\hat{e}_\alpha^A = e_\alpha^A; \qquad \hat{e}_{\alpha A'} = e_{\alpha A'} + i\gamma_{AA'}e_\alpha^A,$$

$$\hat{e}^{\alpha A'} = e^{\alpha A'}; \qquad \hat{e}_A^\alpha = e_A^\alpha - i\gamma_{AA'}e^{\alpha A'}.$$

We now have to decide how the projection operators vary as we pass from one point to another. We find

$$\nabla_{RS'} e^{\alpha A'} = i\epsilon_{S'}^{A'} e_R^{\alpha}, \qquad \nabla_{RS'} e_{\alpha}^{A} = -i\epsilon_R^A e_{\alpha S'},$$

$$\nabla_{RS'} e_A^{\alpha} = iP_{RAS'A'} e^{\alpha A'}, \nabla_{RS'} = -iP_{RAS'A'} e_{\alpha}^{A}.$$

We have $\omega^A = e_{\alpha}^A Z^{\alpha}, \pi_{A'} = e_{\alpha A'} Z^{\alpha}$ so that

$$\nabla_{\rho}^{\sigma} Z^{\alpha} = e^{\sigma S'} e_{\rho}^{R} \nabla_{RS'} \left(e_A^{\alpha} \omega^A + e^{\alpha A'} \pi_{A'} \right)$$
$$= e_{\rho}^{R} e^{\sigma S'} \{ e_A^{\sigma} (\nabla_{RS'} \omega^A + i\epsilon_R^A \pi_{S'}) + (\nabla_{RS'} \pi_{A'} + iP_{RBS'A'} \omega^B) \}.$$

These forms (4.7), (4.8) are required in order to give a conformally invariant twistor derivative as can be checked using (4.1) and (4.6), and because in flat space, constant local twisters (i.e. those annihilated by $\nabla_{RS'}$) will now correspond to our former global flat-space twistors. When referred to a basis (4.8) has, in all, 64 components, 48 of them being zero. ∇_{ρ}^{σ} satisfies the usual requirements of a derivative (linearity and the Leibniz rule) and it commutes with complex conjugation and contraction. We may now consider

$$\nabla_{\rho}^{\sigma} \nabla_{\tau}^{\mu} - \nabla_{\tau}^{\mu} \nabla_{\rho}^{\sigma} = \left[\nabla_{\rho}^{\sigma}, \nabla_{\tau}^{\mu} \right].$$

Acting on a scalar function ϕ, this gives us

$$\left[\nabla_{\lambda}^{\mu}, \nabla_{\rho}^{\sigma} \right] \phi = i \left(\delta_{\lambda}^{\sigma} \nabla_{\rho}^{\mu} - \delta_{\rho}^{\mu} \nabla_{\lambda}^{\sigma} \right) \phi = T_{\lambda\rho\beta}^{\mu\sigma\alpha} \nabla_{\alpha}^{\beta} \phi \qquad (4.9)$$

Where

$$T_{\lambda\rho\beta}^{\mu\sigma\alpha} = i \left(\delta_{\lambda}^{\sigma} \delta_{\rho}^{\alpha} \delta_{\beta}^{\mu} - \delta_{\rho}^{\mu} \delta_{\beta}^{\sigma} \delta_{\lambda}^{\alpha} \right).$$

Then

$$K_{\lambda\rho\beta}^{\mu\sigma\alpha} Z^{\beta} = i \left\{ \left[\nabla_{\lambda}^{\mu}, \nabla_{\rho}^{\sigma} \right] - T_{\lambda\rho\beta}^{\mu\sigma\alpha} \nabla_{\gamma}^{\beta} \right\} Z^{\alpha} \qquad (4.10)$$

Where

$$K_{\lambda\rho\beta}^{\mu\sigma\alpha} = e_\lambda^L e^{\mu M'} e_\rho^R e^{\sigma S'} \left[e_\beta^B \{ e_A^\alpha i\epsilon_{M'S'} \psi_{BLR}^A \right.$$
$$- e^{\alpha A'} \left(\epsilon_{RL} \nabla_B^{B'} \bar{\psi}_{B'M'S'A'} - \epsilon_{M'S'} \nabla_{A'}^A \psi_{ARLB} \right) \}$$
$$\left. + e_{\beta B'} e^{\alpha A'} i\epsilon_{RL} \bar{\psi}^{B'}{}_{A'M'S'} \right].$$

These define a torsion twistor $T_{\lambda\rho\beta}^{\mu\sigma\alpha}$ and a curvature twistor $K_{\lambda\rho\beta}^{\mu\sigma\alpha}$ The spinor components of $K_{\lambda\rho\beta}^{\mu\sigma\alpha}$ given by (4.11) involve $\psi_{ABCD}$and$\nabla_{P'}^A \psi_{ABCD}$.

Note that, by the conformal transformation rules for twistors, any local twistor$Q_{\alpha\beta}^\gamma$ for example, defines a conformally invariant spinor $Q^{A'B'G}$ If thisspinor vanishes, then each of $Q_A^{B'G}$, $Q^{A'}{}_B^G Q_{G'}^{A'B'}$, is conformally invariant. If$Q_A^{B'G} = 0$ and $Q_B^{A'G} = 0$ then Q_{AB}^G is conformally invariant, etc. etc.. Thus any non-vanishing local twistor defines at least one non-vanishing conformally invariant spinor. In particular we can apply this to $K_{\kappa\mu\upsilon}^{\alpha\beta\gamma}$ (to obtain $\epsilon_{B'G'}$, ψ_{LMN}^A)or to derivatives$\nabla_\rho^\sigma K_{\kappa\mu\upsilon}^{\alpha\beta\gamma}$, $\nabla_\xi^\tau \nabla_\sigma^\rho K_{\kappa\mu\upsilon}^{\alpha\beta\gamma}$etc., or to such derivatives to which symmetry operations have been applied. (K. Dighton has shown how to obtain the Bach tensor and other conformally invariant tensors in this way.)

4.2. Global Twistors

Consider a null geodesic **Z** with a parallelly propagated spinor $\pi_{A'}$ defined along it whose flagpole direction is tangent to **Z**. By analogy with the situation in flat space we may reasonably identify such a structure as a null (global) twistor in curved space. This description is conformally invariant. (We do not give a spinor representation of a twistor relative to each point, nor can we use our former description of a non-null twistor.) Null twistors form a seven-dimensional manifold N (5 dimensions for the set of null geodesics, 2 for the set of spinors $\pi_{A'}$). We shall consider N to be embedded in an abstract 8-dimensional manifold C, the points of C—N representing, formally, the non-null twistors. This is done because the structure of N is most easily described as that induced from the embedding of N in C, the structure of C being describable in simple terms. No precise geometrical definition of the elements of C—N will be given. The space C will have a symplectic structure. Symplectic structures are only possible in even dimensional spaces, and symplectic manifolds of the same dimension are locally congruent. The symplectic structure of C induces on N a structure which has geometric significance in the space-time. This structure on N expresses relations between neighbouring points of N, these relations representing geometrical connections between the null geodesics (and $\pi_{A'}$ spinors) that the points of N represent. Such geometrical connections must refer to properties of null geodesics. For example, the fact that a congruence of null geodesics has vanishing rotation, i.e. is (null) hypersurface forming, is such a property and it turns out that this property is simply describable in terms of the symplectic structure of C. The shear of a null congruence, on the other hand, is something which. in a general curved space-time, can be defined only in relation to points on the null geodesics. A null congruence which is shear-free at one point will, in the presence of conformal curvature, generally be shearing at other points. Recall that the Kerr theorem established a close connection between the shear-free condition for null congruences in flat space-time and the complex analytic structure of the C-picture. The fact that the concept "shear-free" cannot, in general curved space-times, beapplied to null geodesics in their entirety, strongly indicates that C cannot generally be given a geometrically meaningful complex analytic structure. Let us investigate the structure of C for a particular type of curved space-time M, namely one which possesses two regions M_1 and M_2 of flat space-time separated by a curved region of M, through which null geodesics can pass from M_1 to M_2. This will enable us to examine the structure of C in relation to the structure we have previously obtained for flat space-time. By involving two flat metrics we shall be able to isolate the structure of C as that which is common to the structures induced by each of M_1 and M_2. Now in each of M_1 and M_2 we can represent twistors in terms of pairs of spinors and hence in terms of four complex components Z^0, Z^1, Z^2, Z^3 (subject to $Z^{\mathcal{N}}\bar{Z}_{\mathcal{N}} = 0$). The expressions $Z^{\mathcal{N}}d\bar{Z}_{\mathcal{N}}$and $dZ^{\mathcal{N}} \wedge d\bar{Z}_{\mathcal{N}}$define forms on N which as we shall show are the same whether the coordinates $Z^{\mathcal{N}}$are defined in M_1 or in M_2. Each of the forms$\phi = iZ^{\mathcal{N}}d\bar{Z}_{\mathcal{N}}$and $\omega = d\phi = idZ^{\mathcal{N}} \wedge d\bar{Z}_{\mathcal{N}}$ define structure of geometrical significance in M. It turns out, in fact, that 4, measures time separation between neighboring geodesics, while ω measures rotation. Let us consider two examples, both of flat spaces M_1, M_2 joined across a null hypersurface K, the (degenerate) metric of K being the same whether induced by M_1 or M_2. The curvature resides entirely within K, having the form of a 6-function on K. Take two flat spaces

$$M_1: ds^2 = 2\left(du\, dv - d\xi d\bar{\xi}\right), \qquad v \leq 0$$

$$M_2: ds^2 = 2\left(du^* dv^* - d\xi^* d\bar{\xi}^*\right), \qquad v \geq 0 \tag{4.12}$$

joined on the null hyperplane $v = 0 = v^*$ where $\xi^* = \xi, u^* = u — q\left(\xi\bar{\xi}\right)$.This has a δ-function in curvature on the join rather as the surface of a cylinder of finite extent has, at the join of the end and the side - both of which are flat. The Ricci curvature is (essentially) $\delta(v)\partial^2 q/\partial\xi\partial\bar{\xi}$,while the conformal curvature is (essentially) $\delta(v)\partial^2 q/\partial\xi^2, \delta(v)\partial^2 q/\partial\bar{\xi}^2$.Einstein's empty space field equations yield

$$\frac{\partial^2 q}{\partial\xi\partial\bar{\xi}} = 0, \quad \text{whence q} = r(\xi) + \bar{r}\left(\bar{\xi}\right), \tag{4.13}$$

r being a holomorphic (i.e. complex analytic) function. Similarly join flat spaces

$$M_1: ds^2 = du\, dv - u^2 d\xi d\bar{\xi}, \qquad v \leq 0$$

$$M_2: ds^2 = du^* dv^* - u^{*2} d\xi^* d\bar{\xi}^*, \qquad v \geq 0 \tag{4.14}$$

along $v = 0$ (a null cone) with $\xi^* = f(\xi)$(f being a holomorphic function); $u^* = u/|f'(\xi)|$.It turns out that this automatically satisfies Einstein's vacuum field equations. In these examples the passage of a null geodesic through $v = 0$ is determined by the condition that it is orthogonal to the same vectors within $v = 0$ on each side of the join. (The behaviour can also be found by considering an appropriate limit of C^∞spaces.) This tells us how a twistor is affected by an impulsive wave. In both cases the null geodesic is scattered in a way that can be formulated in Hamiltonian terms. Let use this explicitly in terms of example A. A twistor Z^αrepresenting a null line, with coordinates, has

$$Z^0{:}Z^1{:}Z^2{:}Z^3 = -iud\xi + i\xi du : -i\bar{\xi}d\xi + ivdu{:} -d\xi{:}\, du$$

$$= -iudv + i\xi d\bar{\xi} : -i\bar{\xi}dv + ivd\bar{\xi}{:} -dv{:}\, d\bar{\xi}.$$

Thus it satisfies

$$-Z^3 d\xi = Z^2 du\,, \qquad -Z^3 dv = Z^2 d\bar{\xi}, \tag{4.15a}$$

$$Z^1 = i\bar{\xi}Z^2, \qquad Z^0 = i\xi Z^3 + iuZ^2 \tag{4.15b}$$

since we are considering a point on K where $v = 0$. The starred version also holds. Thus

$$Z^{*1} = i\bar{\xi}Z^{*2}, \qquad Z^{*0} = i\xi Z^{*3} + i(u-q)Z^{*2}. \tag{4.16}$$

In order to write the remainder of the starred version of (4.15) in terms of du, dv, $d\xi$we need to use the fact that Z^* and Z are orthogonal to the same vectors in K at the point $Z \cap$ K. Denoting a direction at $Z \cap$ K by $\delta u{:}\,\delta y{:}\,\delta\xi$in the u, v, ξsystem we have $\delta v = 0$ if the direction lies in K. For the direction to be orthogonal to that of Z we require

$$\delta u dv + 0du = \delta\xi d\bar{\xi} + \delta\bar{\xi}d\xi \tag{4.17}$$

whence, from (4.15)

$$\delta u = -\frac{\delta\xi Z^3}{Z^2} - \frac{\delta\bar{\xi}\bar{Z}^3}{Z^3}. \tag{4.18}$$

The starred version of this gives, from (4.12),

$$\delta u - \frac{\partial q}{\partial \xi}\delta\xi - \frac{\partial q}{\partial \bar{\xi}}\delta\bar{\xi} = \frac{\delta\xi Z^{*3}}{Z^{*2}} - \frac{\delta\bar{\xi}\overline{Z^{*3}}}{\overline{Z^{*2}}}. \tag{4.19}$$

Equations (4.18) and (4.19) must represent identical conditions $\delta u : \delta\xi : \delta\bar{\xi}$since they must give the same 2-plane element. Hence

$$Z^3 : Z^2 = Z^{*3} - Z^{*2}\frac{\partial q}{\partial \xi} : Z^{*2}. \tag{4.20}$$

Equations (4.15), (4.16) and (4.20) define the ratios of the $Z^{*\alpha}$ components in terms of the ratios of the Z^α components, by elimination of ξ and u. With the most convenient choice of scale factor we can set

$$Z^{*3} = Z^3 + Z^2\frac{\partial Q}{\partial \xi}; \qquad Z^{*2} = Z^2$$

$$Z^{*1} = Z1\,, \qquad Z^{*0} = Z^0 - iZ^2\left(q - \xi\frac{\partial q}{\partial \xi}\right) \tag{4.21}$$

where $\xi \; = \; i\bar{Z}^1/\bar{Z}^2$. Setting

$$H(Z^\alpha, \bar{Z}_\alpha) \equiv |Z^2|^2 q \tag{4.22}$$

we can write (4.21) comprehensively as

$$Z^{*\alpha} = Z^\alpha - i\,\partial H/\partial Z_\alpha. \tag{4.23}$$

The same formula, with H real and homogeneous of degree one separately in $Z^\alpha$and in $\bar{Z}_\alpha$, is also valid for case B, though H now depends on $(f(\xi)$, rather than q. In the infinitesimal change case we find

$$\delta Z^{\alpha} = Z^{*\alpha} - Z^{\alpha} = -i\frac{\partial H}{\partial \bar{Z}_{\alpha}}; \qquad \delta \bar{Z}_{\alpha} = i\frac{\partial H}{\partial Z^{\alpha}} \tag{4.24}$$

which are equations of the Hamiltonian type and so preserve the symplectic structure.

$Z^{\alpha}\bar{Z}_{\alpha}\phi = iZ^{\alpha}d\bar{Z}_{\alpha}; Z^{\alpha}\frac{\partial}{\partial Z^{\alpha}}; i\left(\frac{\partial}{\partial Z^{\alpha}}\right)\otimes \quad \left(\frac{\partial}{\partial \bar{Z}_{\alpha}}\right) - i\left(\frac{\partial}{\partial \bar{Z}_{\alpha}}\right)\otimes \quad \left(\frac{\partial}{\partial Z^{\alpha}}\right); idZ^{\alpha} \wedge d\bar{Z}_{\alpha} = \omega$are all preserved in the sense that $\delta(Z^{\alpha}Z_{\alpha}) = 0; \delta(Z^{\alpha}dZ_{\alpha}) = 0;$ $\delta \circ Z^{\alpha}\frac{\partial}{\partial Z^{\alpha}} = Z^{\alpha}\frac{\partial}{\partial Z^{\alpha}} \circ \delta$ and so on. If we define

$$[\chi, \psi\,] = -i\frac{\partial \chi}{\partial Z^{\alpha}}\frac{\partial \psi}{\partial \bar{Z}_{\alpha}} - i\frac{\partial \chi}{\partial \bar{Z}_{\alpha}}\frac{\partial \psi}{\partial Z^{\alpha}} \tag{4.25}$$

then

$$\delta\psi = [\psi, H] \tag{4.26}$$

$$-\delta(dZ^{\alpha}) = d\left(i\frac{\partial H}{\partial \bar{Z}_{\alpha}}\right) = i\;0\;dZ^{\beta} + i\frac{\partial^2 H}{\partial \bar{Z}_{\beta}\,\partial \bar{Z}_{\alpha}}D\bar{Z}_{\beta} \tag{4.27}$$

and from these one can check the invariances mentioned above. If we consider any weak gravitational wave of any shape whatever, which separates two regions of flat space-time, then we are led to equations of exactly similar form to the above. This is because weak gravitational waves can be superposed linearly and can be broken down into a superposition of waves of the above types only. (Actually plane waves alone will suffice for this.) The corresponding$H$ functions are likewise linearly composed of those above.

We must define what we mean by ϕ, ω on the N associated with a general curved space.

$$\phi \equiv iZ^{\alpha}d\bar{Z}_{\alpha} = p_a dx^a \tag{4.28a}$$

$$\omega = isZ^{\alpha} \wedge d\bar{Z}_{\alpha} = dP_a \wedge dx^a = \nabla_{[a}P_{b]} \quad dx^a \wedge dx^b \tag{4.28b}$$

where for the right hand expressions of (4.28b) we take P_a to be the tangent vector field of a congruence of geodesics. In curved space we use these as definitions of ϕ, ω. This is possible because $P_a dx^a$ and $dP_a \wedge dx^a$, as forms applied to connecting vectors of neighbouring geodesics, are constant (the constancy in this sense of $dP_a \wedge dx^a$abeing the well-known Lagrange identity and since N (modulo the phase factors) has been identified with the space of null geodesics the forms ϕ, ω will be invariantly defined on N. The expressions (4.28) lead to the interpretations ofϕ , and ω mentioned before as respectively, time-displacement and rotation of neighbouring null geodesics.

In both the examples A and B above we find, provided Einstein's vacuum equations hold, that we may write

$$H(Z^\alpha, \bar{Z}_\alpha) = H^+ + H^-; \qquad H^+ = \overline{H^-}$$

$$H^+ = \bar{Z}_\alpha I^{\alpha\beta} \frac{\partial g}{\partial Z^\beta} \tag{4.29}$$

where $g(Z^\alpha)$ is holomorphic and homogeneous of degree 2 in Z^α.Explicitly, for the case A, we have

$$H^+ = |Z^2|^2 \bar{r}\left(-\frac{iZ^1}{Z^2}\right) = \bar{Z}_\alpha I^{\alpha\beta} \partial\big(g(Z^\alpha)\big)/\partial Z^\beta, \tag{4.30}$$

so we obtain (4.29) if

$$g(Z^\alpha) = i(Z^2)^2 \int_{x_0}^{\frac{iZ^1}{Z^2}} \bar{r}(x)dx.$$

The infinity twistor appears in (4.29) because gravitation is not conformally invariant:$I^{\alpha\beta}$ is the conformal-symmetry breaking term which tells us "where" infinity is.

We can similarly treat electromagnetic scattering, introducing charged zero-rest-mass particles, with momentum P^aThe acceleration of such a particle is

$$P_a^{a\delta} P^b = eF^{ab} P_a$$

where e is the charge and $F^{ab}$the Maxwell field. This gives well-defined equations of motion for the particle even though its rest-mass is zero. We can consider an idealised situation similar to that of the gravitational impulse waves considered above. Here we take two regions of field-free space separated by an electromagnetic plane or spherical wave of 6-function amplitude. A zero-rest-mass particle on either side of the wave may be described by a null twistor. The wave imparts an impulse to the particle and so defines a transformation of the twistor space from one side to the other. The transformation again turns out to be of Hamiltonian form, hut now, in the infinitesimal case H turns out to be homogeneous of degree zero separately in $Z^\alpha$and in $\bar{Z}_\alpha$, where $H = H^+ + \bar{H}^+$with $H^+ = f(Z^\beta)$ holomorphic and of degree zero in Z^α(assumingF^{ab}satisfies the free-space Maxwell equations). The treatment may

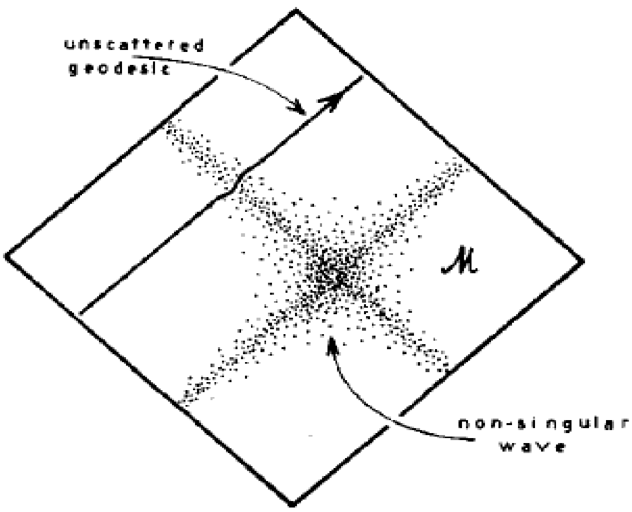


Fig 6.

likewise be extended to any infinitesimal scattering by linear superposition of such waves, and hence of the corresponding H functions. We have encountered holomorphic functions in both the gravitational and electromagnetic cases.

But in general these have poles, and if not, then H is a bilinear function of $Z^{\alpha}, \bar{Z}_{\alpha}$. For the fields of this latter simple type the null geodesics emerge ultimately unscattered although waves can come in and go out (fig. 6). However, in general cases where the wave possesses singularities, singularities in the function can exist and scattering occurs. For instance in example B above one can see from the fact that there is no non-singular non-constant harmonic function on a 2-sphere that the behaviour of $f(\xi)$ on the null cone joining the two flat spaces must be singular on at least one generator of the cone. This singularity could be cancelled by putting together an appropriate set of such cones but we would then be back to the situation of (fig. 6) wherethere is no scattering.

4.3. Quantization

We wish to pass from the scattering of zero-rest-mass particles by a (weak) gravitational or electromagnetic wave to the scattering of zero rest mass fields. In general a zero rest mass field $\phi_{A'B'...L'}$ is defined by a holomorphic function$f(Z^{\alpha})$.We may now ask how to transform$f$in order to represent the scattering of the $\phi_{...}$field. A somewhat formal answer is provided if we use the correspondence $\bar{Z}_{\alpha} \to -\partial/\partial Z^{\alpha}$suggested by the fact that $Z^{\alpha}$and $\bar{Z}_{\alpha}$ are canonically conjugate variables. Thus, we write

$$H\left(Z^{\alpha}, -\frac{\partial}{\partial \bar{Z}^{\alpha}}\right) \text{for } \mathrm{H}(\mathrm{Z}^{\alpha}, \bar{\mathrm{Z}}_{\alpha}),$$

and apply it to f. We are here regarding H as describing the effect of a fixed given gravitational field. Now with the H considered above for the scattering of massless particles by weak gravity we get

$$H^{+} \rightarrow \frac{\partial}{\partial Z^{\alpha}} \circ I^{\alpha\beta} \frac{\partial g}{\partial Z^{\beta}} = -\left(I^{\alpha\beta} \frac{\partial g}{\partial Z^{\beta}}\right) \frac{\partial}{\partial Z^{\alpha}} \tag{4.32}$$

the commutation being possible because \$I^{\alpha\beta}$ is skew. Thus no factor ordering problem arises. Similarly we would have

$$H^{-} \rightarrow Z^{\alpha} I_{\alpha\beta} \left[\frac{\partial \bar{g}}{\partial \bar{Z}_{\beta}}\right]_{\bar{Z}_{\beta} \rightarrow -\frac{\partial}{\partial Z^{\beta}}}$$

which is more awkward! However we aim to consider matrix elements $\langle g|H|f\rangle$ and therefore need not evaluate $H^{-}\,|f\rangle$ as such, for

$$\langle g|H|f\rangle = \langle g|H^{-}|f\rangle + \langle g|H^{+}|f\rangle$$

and we may takeH^{-} to act on $\langle g|$ writing$H^{-} \rightarrow \left(\frac{\partial}{\partial \bar{Z}_{\alpha}}\right) I_{\alpha\beta}\ \partial \bar{g}/\partial \bar{Z}_{\beta}$. So far we have not defined what we mean by $\langle g|\ f\rangle$ and our next task is therefore to set up a Hilbert space of functions$f$. In doing so we can be guided by the need for suitably nice formal formulae and agreement with the scalar product used by Fierz. Form $\phi_{A\ldots L}$we may construct a series of potentials $\phi^{(k)\overbrace{aA'B'\ldots D'}^{k\ indices}}_{EF\ldots L}$ satisfying

$$\nabla_{\mathrm{AA'}} \overset{(I)}{\phi^{A'}_{B\ldots L}} = \phi_{A\ldots L}$$

$$\begin{cases} \nabla_{\mathrm{DD'}} \overset{(K)}{\phi^{A'\ldots D'}_{E\ldots L}} = \overset{(K-1)}{\phi^{A'\ldots C'}_{D\ldots L}} \\ \nabla^{EE'} \overset{(K)}{\phi^{A'B'\ldots D'}_{EF\ldots L}} = 0 \end{cases}$$

$$\nabla^{\mathrm{m}} \nabla_{\mathrm{m}} \overset{(n)}{\phi^{A'\ldots L'}} = 0. \tag{4.33}$$

At each step there is a gauge freedom in choice of $\overset{(K)}{\phi}_{\ldots}$. Following Fierz, we may now define (with suitable numerical constant k)

$$\langle\chi|\phi\rangle : k\int_S \overset{(n-1)}{\phi} L^{A'B'\ldots K'}\chi_{A'B'\ldots L'}dS^{LL'} \tag{4.34}$$

where S is a spacelike surface. One must (and can) check that this is gauge independent, independent of the choice of surface, that one may interchange χ and ϕ yielding a Hermitian symmetry, and that the product is conformally invariant. Our next task is to express the scalar product in terms of $f(Z^\alpha), g(W_\alpha)$. Physically meaningful answers must be contour integrals since if$f$ is replaced by f' where $f - f'$ is nonsingular inside the integration contour of eq. (2.20) the field is not changed. Let us investigate the form that such a contour integral must take. Suppose we have $\beta(Z^\alpha, X^\alpha, \ldots, W_\alpha \ldots)$ which is a function homogeneous of degree (-4) in each variable. To integrate one must define a differential form $\mathcal{D}ZX..W...$, so that the integral depends only on the homology class (relative to the space less regions of singularity) ofthe region of integration (i.e. we require that the resulting object is a genuine contour integral). For this we use

$$\left.\begin{aligned} \mathcal{D}Z &= \frac{1}{6}\epsilon_{\alpha\beta\gamma\delta}Z^\alpha dZ^\beta \wedge fZ^\gamma \wedge dZ^\delta \\ \mathcal{D}W &= \frac{1}{6}\epsilon^{\alpha\beta\gamma\delta}W_\alpha dW_\beta \wedge fW\_\gamma \wedge dW_\delta \end{aligned}\right\}$$

$$\mathcal{D}ZX \ldots W \ldots = \mathcal{D}Z \wedge \mathcal{D}X \wedge .. \mathcal{D}W \wedge \ldots$$

We then find that $d(\beta\mathcal{D}ZX..W..) = 0$ as required for$\oint \beta\mathcal{D}ZX..W..$ to be dependent on the homology class of the contour, and that the integral is a scalar, as we desire. To illustrate the value of such contour integrals let us digress for a moment. If we take $\beta(Z^\alpha)$ to represent an electromagnetic field, then $\oint \beta(Z^\alpha)\mathcal{D}Z$gives the charge integral for a source for the field. In a gravitational field the function$f(Z^\alpha)$introduced before is of degree -6, so we take

$$\oint \quad Z^\alpha Z^\beta f(Z) \mathcal{D}Z$$

and find this is the twistor describing the energy momentum and angular momentum of a source for the field. The same differential forms are now used for the scalar products in terms off and g. We must insert additional factors so that$f(Z^\alpha)g(W_\alpha)$has the correct homogeneity degree i.e. $(-4,-4)$. For $n = 0, 1$ this may be done by

$$\oint \quad \underbrace{f(Z^\alpha)}_{degree-n-2} \underbrace{g(W_\alpha)}_{degree-n-2} \left(W_\beta Z^\beta\right)^{n-2} \mathcal{D}ZW \tag{4.36}$$

since there is then a 6-dimensional contour not homologous to zero in the 16-dimensional subspace of $C \otimes C^*$ where$f, g, (Z^\alpha W_\alpha)^{n-2}$are non-singular (i.e. there is a contour surrounding the singularities). However for $n \geq 2$ this formula is no longer satisfactory since the $(Z^\alpha W_\alpha)^{n-2}$singularity disappears. If we consider defining successive factors (as n increases) by integrating the previous ones, the factor for $n = 0$ will be $\log(W_\alpha Z^\alpha)$,which is not homogeneous. So we take

$$\log \quad \left\{\frac{(W_\alpha Z^\alpha)}{W_{|\alpha} C_\beta D_{\gamma|} Z^\alpha A^\beta B^\gamma}\right\} \tag{4.38}$$

and one does then find an answer which is independent of the auxiliary twistors $A^\gamma \ldots D_\alpha$.) With these definitions it can be checked that the basis functions used in (2.21) are orthogonal. The functions $(W \cdot Z)_k$ do satisfy the formal property

$$\frac{\partial (W \cdot Z)_k}{\partial Z^\alpha} = W_{\alpha (W \cdot Z)_{k+1}} \tag{4.39}$$

which in fact is what is really used in actual calculations.

4.4. Curved Twistor Spaces

The theory of twisters described is relevant only to flat space- time; can it be generalized to a curved space-time background? The theory of curved twistor spaces arises out of this problem. Despite considerable progress, no completely satisfactory generalization exists; we shall look at some of the different approaches to the problem. For a more detailed review, the reader is referred to penrose & ward (1979). We shall also review topic which, although it does not involve curved space-time, is closely related: the use of curved twistor space techniques in self-dual theories.

(a) Twisters in curved space-time

Let me be a curved space-time; we would like to define twistor in M. one possibility is to have a separate twistor space T_X for each pointx of M; in effect, such local twisters are constructed in the tangent space at x, which of course is a copy of Minkowski space-time. If we move a long a curve in the space-time, there a natural way of propagating a local twistor along the curve: this is known as local twistor transport. In general, this transport is not Integrable (i.e. if we propagate a local twistor around a closed loop, then it may not return to its original value).

A more global approach is based on the correspondence between null twisters and null geodesics. Let PN' be the space of null geodesics in M. in general PN' might fail to a have a Hausdorff manifold structure, because of the occurrence of pairs of conjugate points on the on the null geodesics in M.

Let us for simplicity avoid this difficulty by taking M to me the whole space-time, but a suitable subregion of the space-time. The following question now arises:is PN' naturally embedded in a three-complex-dimensional projective twistor space PT'? To throw lifgt on this question, recall the kerr theorem: shear-free congruences of null geodesics in Minkowski space-time correspond to the intersection with PN of a holomorphic surface in PT. However, if a shear-free congruenceenters a region of space-time where is conformal curvature, then it picks up shear (penrose 1968a). in a general curved space-time, therefore, there are no shear-free congruences (although there are congruences which are shear-free 'for an instant' e.g. where they intersect some space like hypersurface).

Thus the kerr theorem suggests that the complex structure of twistor space is destroyed by conformal curvature. An investigation of that happens when null geodesics pass through an impusive plane-fronted gravitational wave reveals this phenomenon explicitly.

Thus it appears that the above question has to be answered in the negative. However, there are ways of avoiding the difficulties. One of these is to fix a point on each null geodesic, so that the shearing effect of gravitation no longer matters; this leads to the theory of hypersurface twistores. Another way is to consider complex space-times in which the right-handed (or left-handed) half of the conformal curvature vanishes, so that there exist ''half-shear-free null congruences. This leads to the on linear graviton construction. These two theories will now be described.

(b) hypersurface tiwstors

Details of the theories of hypersurface twisters and a symptotic twisters may be found in Penrose & ward MacCall (1972); Penrose (1975); ko, Newman & Penrose (1977); Penrose & ward (1977b). the essential idea is that one picks a point on each null geodesic by taking its intersection with a fixed hypersurface S. the theory is simplest if S is taken to be Space like and to intersect each null geodesic exactly once. But one may also consider the case when S is null, even though one then has to think of tow to deal with the null geodesics lying entirely in S. in particular, in an a symptotically flat space-time (hawking &Ellis 1973) one can take $s = g^+$ or $s = g^-$; the hypersurface twisters one then obtains are called asymptotic twistore.

The precise definition of hypersurface twisters will not be given here. Briefly, a hypersurface twistor is a certain type of complex curve in the complexification of the hypersurface S. the space $PT(S)$ of hypersurface twisters with respect to S is a three –dimensional complex manifold containing PN' as a real hypersurface. The complex structure of $PT(S)$depends on S and contains some (but not all) of the information about the space-time metric on S. it is an unsolved problem to determine exactly how much information about the metric is contained in the structure of $PT(S)$.

Remarks (4.4.1). (1) the kerr theorem is replaced by a hypersurface kerr theorem; an analytic congruence of null geodesics which is shear-free at S corresponds in the twistor picture to the intersection with $PN'$of a holmorphic 2-surface in $PT(S)$.

(2) since hypersurface twisters are localized to the hypersurface S, one cannot expect there to be a contour integral formula which solves the massless field equations on the curved space-time background. In any event, the existence in curved space-time of ordinary massless fields of helicity greater than one is severely constrained by the Buchdahl conditions.

(3) If S is null, then there is a naturally defined scalar product $\pi(Z, \bar{Z})$ on $PT(S)$; it is the generalization of the flat-space scalar product $Z^{\alpha}UZ_{\alpha}$. The definition for the case $S = g$ is given in Penrose & MacCallum (I972)and the definition for the case of a general null hypersurface is essential1 the same; see Penrose & Ward (1979) . A twistor Z in $PT(S)$ is null (i .e.lies in PN' and so corresponds to a null geodesic) if and only if $\pi(Z, \bar{Z}) = 0$ the scalar product in turn leads to a KMhler structure on PT(S) : the KMhlerform is

$$\frac{\partial^2 \pi}{\partial Z^2\, \partial \bar{Z}_\beta} dz^\alpha \wedge \partial \bar{Z}_\beta \qquad (4.1)$$

The Kyihler curvature is related to the space-time curvature; see Ko, Newman & Penrose (1977),

(4) Hypersurface twistor spaces have been constructed in some specific space-times, for example in plane-fronted waves $[dS^2 = 2du = dy - dx^2 - -dy^2 + 2h(v\,, x, y\,)dv^2]$and in Schwarzschlld space-time $[2dS^2 = (1 - 2mr^{-1} - dv^2 + 2dvdr - r^2(d\theta^2\ + \sin^2\theta\, d\phi^2)]$,taking S in both cases to be the null hypersurface v = constant (Ward 1977) . In both cases, $PT(S)$ turns out to be flat. i.e . to be an open submanifold of cp^3 . If one takes two hypersurfaces S_1 (given by $v = v_1$) and S_2 (given by $v = v_2$). one can relate the twistors in $PT(S_1)$ to those in $PT(S_2)$ as follows . There is a natural map from the twistors in $PT(S_1)$to those in $PT(S_2)$since null twistors in either) or $PT(S_2)$correspond to null geodesics in the space- time. Now extend map to the whole space $PT(S_1)$; one then obtains a canonical translation, preserving the 2- form (4.1) . The map is non - holomorphic and the way in which the gravitational curvature between the S_1 and S_2 "shifts" the complex structure of twistor space.

(c) The Nonlinear Gravitonthe curvature tensor of a space- time is made up of two parts: the Wey lconformal curvature tensor C_{abcd} and the Ricci tensor R_{ab} . The Weyl tensor can be split up into its self- dual and anti -self- dual parts :

$$C_{abcd} = C^{+}_{abcd} + C^{-}_{abcd},$$

$$C^{+}_{abcd} = \frac{1}{2}\left(C_{abcd} \mp \frac{1}{2} i\varepsilon_{ab}\ efC_{ebcd}\right).$$

C^{+}_{abcd} and $C^{-}_{abcd}$are complex conjugates of each other. However, if we complexify the space-time by allowing the coordinates to become complex andanalytically extending the space-time metric (assuming it to be real analytic to begin with), then R_{ab} and C_{abcd} become complex tensors; C^{+}_{abcd} and $C^{-}_{abcd}$are then no longer complex conjugates. More generally , 1·le can consider complex-Riemannian 4-spaces which are not necessarily the complexifications of real space- times. Such a complex space is said to be right- (or anti -self- dual) if$C^{+}_{abcd} = 0, R_{ab} = 0$. It is clear from the above discussion that a right- flat space cannot be the complexification of a real space - time, unless it is flat. However, it is possible for a complex space , or a real space with signature ++++ or ++-- , to be right- flat without being flat . Such spaces have arisen in two differ areas of genera l relativity :

(1) The theory of H-space (Newman 1976; Hansen, Newman , Penrose & Tod 1978; Ko, Ludvi9sen, Newman & Tod 1978) . H-space is a complex right-flat space which is naturally associated with theg^{+} or g^{-} of an asymptotically flat space- time . The theory of If-space is closely related to asymptotic twistor theory.

(2) Gravitational instantons and the approach to quantum gravity which involves functional integration ;n positive definite 4-space.

H-space and asymptotic twistor theory is closely associated with Penrose's (1976) nonlinear graviton construction . Penrose proved essentially the following result.

Theorem (4.4.2). There is a one- to -one correspondence between

(i) right- flat spaces M; and

(ii) curved twistor spaces PT , t0gether with a certain pairof differential forms on PT .

The construction tell s one how to go back and forth between (i) and (ii).

It is a generalization of the flat -space correspondence$o$- plane in space- time⟷ point in PT. The concept of an a -plane is generalized to that of an α- Surf ace in the curved space- time ~1. The condition for a three - parameter family of a-surfaces to exist is precisely $C^{+}_{abcd} = O$. The lines in PT generalize to compact holomorhic curves (satisfying certain topological conditions) in PT.

Thus given a right- flat space M, we can take the corresponding curved twistor space PT to be the space of a -surfaces in fl. Conversely, given PT , take M to be the space of compact holomorphic curves in PT . Thiscorrespondence as it stands is between

(i) complex 4-man i folds M with conformal metric satisfying $C^{+}_{abcd} = 0$; and

(ii) curved twistor spaces PT .

The extra structure on H represented by having? a right- flat metric (as opposed to just a conformal metric) corresponds in PT to having the two differential forms mentioned in Penrose ' s theorem above .

In principle the construction can be used to find all right - flat metrics. In practice, only certain special cases and classes have so far proved tractable. One of the first examples was the right- flat analogue of the Schwarzschild solution; this space is also known as the antiself-dual Taub -NUT space (Hawking 1977) . For more recent progress on this problem.

(c) Original problem was that of defining twistors in curved space- time . Clearly neither hypersurface twistors nor the nonlinear graviton can be regarded as a completely satisfactory solution to this problem . The former theory suffers as a result of being localized t o a particular hypersurface , the latter applies only to right -flat spaces . What one really needs is something which is analogous to the nonlinear graviton , but which applies to a general space- time . Some preliminary ideas aimed at this have been put forward, but the problem is as yet unsolved.

(e) Self- Dual Gauge Theories

The gravitational field is usually thought of in terms of geometry, and this geometric interpretation is a crucial feature of the nonlinear graviton construction described above . There is a large class of field theories which have a neat geometric interpretation; they are called gauge theories. from the mathematical point of view, gauge theories are described in terms of connections on principal bundles or vector bundles.

Let G be a Lie group (the gauge group) and let B be a principal G-bundle (or its associated vector bundle) over space- time. Let A be a connection on B;

A is called the gauge potential . The corresponding gauge field is the curvature F of the connection. If we choose some of B. to act as a basis . then A and F may be represented (respectively) as a 1-form and a_2- form on M. taking values in the Lie algebra of G. The form F is then given by

$$F = dA + [A, A]. \qquad (4.2)$$

The Bianchi identities $dF + [A, F] = 0$ follow from (4.2). The Yang-Mill/;Equation are

$$d * F + [A, * F] = 0 ,$$

where $* F_{ab} = \frac{1}{2} \varepsilon_{abcd} F^{cd}$ is the dual of F. A Yang-Mills field is a curvature form F which satisfies the Yang-Mills equations . If the gauge group G is U(l) . then the Yang - Hills theory is the same as Maxwell theory . The case $G = SU(2)$ pives the theory originally introduced by Yang and Hills.

We are interested here in special types of Yang - Mills field, namely those which are self- dual or anti - self-dual (i . e . $* F = iF$ or $* F = -i F$) . Clearly it follows from the Bianchi identities that any self- dual or anti - self- dual curvature automatically satisfies the Yang the Yang -Mlills equations . There is ac:orrespondence between anti - self-dual (" left-handed") Yang -Ni11s fields and certain holomorphic vector bundles over projective twistor space: this is described. This is preceded by a discussion of the Max~lel1 case. A study is made of a particular 11axwell field , namely the anti - self-dual part of the Coulomb field . The Maxwell construction can also be looked at from the point of view of asymptotic twistor theory and H-space theory.

More details about the t wistor technique for solving.

The problem of dealing with Yang-Mills fields which are not self- dual or anti - self-dual. using twistor methods . is still unsolved. There have been two approaches to the problem . One of these may be found in Witten (1978) and Isenberg.

Chapter 5

The physical Meaning Of General Relativity

GR is the discovery that spacetime and the gravitational field are the same entity. What we call"spacetime" is itself a physical object, in many respects similar to the electromagnetic field. We cansay that GR is the discovery that there is no spacetime at all. What Newton called "space", andMinkowski called "spacetime", is unmasked: it is nothing but a dynamical object -the gravitationalfield- in a regime in which we neglect its dynamics .

In Newtonian and special relativistic physics, if we take away the dynamical entities -particlesand fields- what remains is space and time. In general relativistic physics, if we take away thedynamical entities, nothing remains. The space and time of Newton and Minkowski are reinterpretedas a configuration of one of the fields, the gravitational field. This implies that physical entities-particles and fields- are not all immersed in space, and moving in time. They do not live on space-time. They live, so to say, on one another .

It is as if we had observed in the ocean many animals living on an island: animals on the island .Then we discover that the island itself is in fact a great whale. Not anymore animals on the island, just animals on animals. Similarly, the universe is not made by fields on space-time; it is made by fields on fields. This book studies the far reaching effect that this conceptual shift has on QFT .

One consequence is that the quanta of the field cannot live in space-time: they must build" space-time" themselves. This is precisely what the quanta of space do in loop quantum gravity .

We may continue to use the expressions "space" and "time" to indicate aspects of the- gravitational field, and I do so in the book. We are used to this in classical GR. But in the quantum theory ‹where the field has quantized "granular" properties and its dynamics is quantized and therefore only probabilistic, most of the "spatial" and "temporal" features of the gravitational field are lost .Therefore for understanding the quantum gravitational field we must abandon some of the emphasis on geometry. Geometry represents well the classical gravitational field, not quantum space-time. This is not a betrayal of Einstein's legacy: to the contrary, it is a step in the direction of "relativity" in the precise sense meant by Einstein. Alain Connes has beautifully described the existence of two points of view on space: the geometrical one, centered on the space points, and the algebraic, or "spectral" one, centered on the algebra of the dual spectral quantities. As emphasized by Alain, quantum theory forces us to a complete shift to this second point of view, because of noncommutativity. In the light of quantum theory, continuous space-time cannot be anything else than an approximation in which we disregard quantum noncommutativity. In loop gravity, the physical features of space appear as spectral properties of quantum operators that describe our interactions with the gravitational field .The key conceptual difficulty of quantum gravity is therefore to accept the idea that we can do physics in the absence of the familiar stage of space and time. We need to free ourselves from the prejudices associated with the habit of thinking of the world as "inhabiting space" and "evolving in time". We describe a general language for describing mechanical systems in this generalized conceptual framework .

5.1 Background Independent Quantum Field Theory

Is quantum mechanics compatible with the general relativistic notions of space and time, It is provided that we choose a sufficiently general formulation. For instance, the Schrödinger picture is only viable for theories where there is a a global observable time variable t; this conflicts with GR, where no such variable exists. Therefore the Schrodinger picture makes little sense in a background independent context. But there are formulations of quantum theory that are more general than the Schrödinger picture. I describe a formulation of

QM sufficiently general to deal with general relativistic systems. (For another relativistic formulation of QM.) Formulations of this kind are sometimes denoted "generalized quantum mechanics". I prefer denoting quantum mechanics" any formulation of quantum theory, irrespectively of its generality, as "classical mechanics" is used to designate formalisms with different degrees of generality, such as Newton's, Lagrange's, Hamilton's or symplectic mechanics. On the other hand, most of the conventional machinery of perturb bative QFT is profoundly incompatible with the general relativistic framework. There are many reasons for this:

• The conventional formalism of QFT relies on Poincare in variance. In particular, it relies on the notion of energy and on the existence of the non vanishing hamiltonian operator that generates unitary time evolution. The vacuum, for instance, is the state that minimizes the energy. Generally, there is no global Poincare invariance, no general notion of energy and no nonvanishing hamiltonian operator in a general relativistic theory.

• At the roots of conventional QFT is the physical notion of particle. The theoretical experience with QFT on curved space-time and on the relation between acceleration and temperature in QFT indicates that in a generic gravitational situation the notion of particle can be quite delicate. (This point is discussed.)

• Consider a conventional renormalized QFT. The physical content of the theory can be ex-expressed in terms of its n-point functions W(xi,... ,xn). The n-point functions reflect the in variances of the classical theory. In a general relativistic theory, invariance under a coordinate transformation $x \rightarrow x' = x'(x)$ implies immediately that the n-point functions must satisfy

$$W(x_1, \ldots, x_n) = W\big(x'(x_1), \ldots, x'(x_n)\big) \qquad (5.1)$$

and therefore (if the points in the argument are distinct) it must be a constant!

$$W(x_1, \dots, x_n) = \text{constant}. \quad (5.2)$$

Clearly we are immediately in a very different framework from conventional QFT.

•Similarly, the behavior for small $|x - y|$ of the two point function of a conventional QFT

$$W(x, y) = \frac{constant}{|x - y|^{\delta}}. \quad (5.3)$$

expresses the short distance structure of the QFT. More generally, the short distance structure of the QFT is reflected in the operator product expansion

$$O(x)O'(y) = \sum_n \frac{O_n(x)}{|x - y|^n}. \quad (5.4)$$

Here $|x — y|$ is the distance measured in the spacetime metric. On flat space for instance $|x - y|^2 = \eta_{\mu^v}(x^\mu - y^\mu)(x^v — y^v)$. In a general relativistic context these expressions make no sense, since there is no background Minkowski (or other) metric η_{μ^v}. In its place, there is the gravitational field, namely the quantum field operator itself. But then, if standard operator product expansion becomes meaningless, the short distance structure of a quantum gravitational theory must be profoundly different from that of conventional QFT. As we shall is precisely the case. There is a tentative escape strategy to circumvent these difficulties: write the gravitational field $e(x)$ as the sum of two terms

$$e(x) = e_{\text{background}}(x) + h(x); \quad (5.5)$$

where $e_{\text{background}}(x)$ is a background field configuration. This may be Minkowski, or any other. Assume that $e_{\text{background}}(x)$ defines spacetime, namely it defines location and causal relations. Then consider $h(x)$ as the gravitational field, governed by a QFT on the space-time background defined by

$e_{\text{background}}$. For instance the field operator $h(x)$ is assumed to commute at spacelike separations, where spacelike is defined in the geometry determined by $e_{\text{background}}(x)$. As a second step onemay then consider conditions on $e_{\text{background}}(x)$ or relations between the formulations of the theory defined by different choices of $e_{\text{background}}(x)$. This escape strategy leads to three orders of difficulties: (i) Conventional perturbative QFT of GR based on A.5) leads to a non renormalizable theory. To get rid of the uncontrollable ultraviolet divergences one has to get to the complications of string theory, (ii) As mentioned, loop quantum gravity shows that the structure of spacetime at the Planck scale is discrete. Therefore physical spacetime doesn't have a short distance structure at all. The unphysical assumption of a smooth background $e_{\text{background}}(x)$ implicit in may be precisely the cause of the ultraviolet divergences, (iii) The separation of the gravitational field from spacetime is in strident contradiction with the very physical lesson of GR. If GR is of any guide in searching for a quantum theory of gravity, the relevant spacetime geometry is the one determined by the full gravitational field $e(x)$, and the separation A.5) is misleading. A formulation of quantum gravity that does not take the escape strategy is a background independent, or general covariant QFT. The main is develop the formalism for background independent QFT.

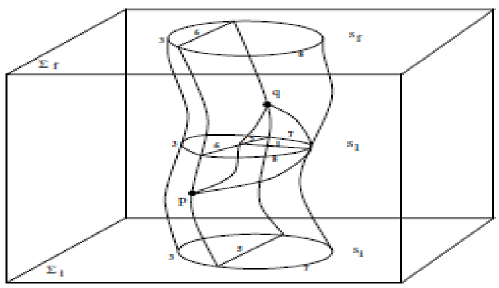


Fig. 1

5.2 Quantum Space-Time: Spinfoam

To be able to compute anything we want from a theory, it is not sufficientto have the general definition of a theory. A road towards the calculation of generic transition amplitudes in quantum gravity is provided by the spinfoam formalism. Following Feynman's ideas, we can give $W(s, s')$ a representation as a sum over paths. This representation can be obtained in various manners. In particular, it can be intuitively derived froma perturbative expansion, summing over different histories of sequences of actions of H that send s' into s.

A path is then the worldhistory of a graph, with interactions happening at the nodes. This worldhistory is a two-complex, as namely a collection of faces (the world-histories of the links); faces join at edges (the world histories of the nodes); in turn, edges join at vertices. A vertex represents an individual action of H. An example of vertex, corresponding to the action of H, is illustrated in Figure 1. Notice that moving from the bottom to the top, the two-complex goes precisely from the graph on the left hand side to the one on the right hand side. Thus, a two-complex is like a Feynman graph, but with one additional structure. A Feynman graph is composed by vertices and edges, a spinfoam by vertices, edges and faces. Faces are labelled by the area quantum numbers ji and edges by the volume quantum numbers i_n. A two-complex with faces and edges labelled in this manner is called a "spinfoam" and denoted σ. Thus, a spinfoam is a Feynman graph of spin networks, or a world history of spin networks. A history going from s' to s is a spinfoam a bounded by s' and s.In the perturbative expansion of $W(s, s')$, there is a term associated to each spinfoam a bounded by s and s'. This term is called the amplitude of a. The amplitude of a spinfoam turns out to be given by (a measure term $\mu(\sigma)$ times) the product over the vertices v of a vertex amplitude $A_v(\sigma)$. The vertex amplitude is determined by the matrix element of H between the incoming and the outgoing spin networks and is a function of the labels of the faces and the edges adjacent to the vertex.

This is analogous to the amplitude of a conventional Feynman vertex, which is determined

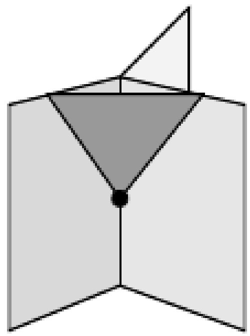


Figure 1.5: The vertex of a spinfoam.

by the matrix element of the Hamiltonian between the incoming and outgoing states. The physical transition amplitudes $W(s,s')$ are then obtained by summing over spinfoams bounded by the spin networks s and s'.

$$W(s,s') \sim \sum_{\substack{\sigma \\ \partial\sigma = s \cup s'}} \mu(\sigma) \prod_{v} A_v(\sigma). \qquad (5.6)$$

More generally for a spin network s representing a closed surface

$$W(s) \sim \sum_{\substack{\sigma \\ \partial\sigma = s}} \mu(\sigma) \prod_{v} A_v(\sigma) \qquad (5.7)$$

In general, the Feynman path integral can be derived from Schrodinger theory by exponentiating the Hamiltonian operator, but it can also be directly interpreted as a sum over classical trajectories of the particle. Similarly, the spinfoam sum (5.6) can be interpreted as a sum over spacetimes. That is, the sum (5.6) can be seen as a concrete and mathematically well defined realization of the (illdefined) Wheeler-Misner-Hawking representation of quantum gravity as a sum over four-geometries

$$W(\,{}^3g,\ {}^3g') \sim \int_{\partial g = {}^3g \,\cup\, {}^3g'} [Dg]\,\mu(g) e^{\frac{1}{\hbar}S[g]}. \qquad (5.8)$$

Because of their foamy structure at the Planck scale, spinfoams can be viewed as a mathemat-mathematically precise realization of Wheeler's intuition of a spacetime "foam". I describe various concrete realizations of equation (5.6), as well as the possibility to directly relate (5.6)with a discretization of (5.8).

5.3. General Relativity

Lev Landau has called GR "the most beautiful" of the scientific theories. The theory is first of all a description of the gravitational force. Nowadays it is very extensively supported by terrestrial and astronomical observations, and so far it has never been questioned by an empirical observation.

But GR is far more than that. It is a complete modification of our understating of the basic grammar of nature. This modification does not regard the sole gravitational interaction: it regards all aspects of physics. In fact, the extent to which Einstein's discovery of this theory has modified our understanding of the physical world and the full reach of its consequences have not been completely unraveled yet.

Nor an exhaustive description of the theory. For this we refer the reader to the classic on the subject. Here, we give a short presentation of the formalism in a compact and modern form, emphasizing the reading of the theory which is most useful for quantum gravity. We discuss in detail the physical and conceptual basis of the theory, and the way it has modified our understanding of the physical world. Let M be the "space-time" four-dimensional manifold. Coordinates on M are written as $x, x', \ldots,$ where $x = (x^\mu) = (x^0, x^1, x^2, x^3)$. Indices $\mu, v, \ldots = 0{,}1{,}2{,}3$ are spacetime tangent indices. The gravitational field e is a one-form

$$e^I(x) = e^I_\mu(x) dx^\mu \qquad (5.9)$$

with values in Minkowski space. Indices $I, J, \ldots = 0,1,2,3$ label the components of a Minkowski vector. They are raised and lowered with the Minkowski metric $\eta_{I,J}$. The reason that led Einsteinto understand that the gravitational field has this form are discussed. We call "gravitational field" the tetrad field rather than Einstein's metric field $g_{\mu^\nu}(x)$. There are three reasons for this: (i) the standard model cannot be written in terms of g because fermions require the tetrad formalism; (ii) the tetrad field e is nowadays more utilized than g in quantum gravity; and (iii) we think that e represents the gravitational field in a more conceptually clean way than g. The relation with the metric formalism is given.

The spin connection ω is a one-form with values in the Lie algebra of the Lorentz group

$$\omega^I_J(x) = \omega^I_{\mu,J}(x)\, dx^\mu, \tag{5.10}$$

where $\omega^{IJ} = —\omega^{JI}$. It defines a covariant partial derivative D_μ on all fields that have Lorentz (I) indices:

$$D_\mu v^I = \partial_\mu v^I + \omega^I_{\mu,J} \tag{5.11}$$

and a gauge covariant exterior derivative D on forms. For instance, for a one-form u^I with a Lorentz index,

$$Du^I = du^I + \omega^I{}_J \wedge u^J . \tag{5.12}$$

The torsion two-form is defined as

$$T^I = De^I = de^I + \omega^I{}_J \wedge e^j = 0. \tag{5.13}$$

A tetrad field e determines uniquely a torsion free spin connection $\omega = \omega[e]$, called compatible with e,by

$$T^I = de^I + \omega[e]^J \wedge e^J = 0. \tag{5.14}$$

The explicit solution of this equation is given below in the curvature R of ω is the Lorentz algebra valued two-form

$$R^I{}_J = R^I_{J\mu^v} dx^\mu \wedge dx^v \quad (5.15)$$

defined by

$$R^I_J = d\omega^I_J + \omega^I_K \wedge \omega^K_J. \quad (5.16)$$

A region where the curvature is zero is called "flat". Equations (5.13) and (5.16) are called the Cartan structure equations. The Einstein equations "in vacuum" are

$$\epsilon_{IJKL}(e^1 \wedge R^{JK} + \lambda\, e^I \wedge e^J \wedge e^K) = 0. \quad (5.17)$$

The equation relating e and to and the Einstein equations (5.17) are the field equations of GR in the absence of other fields. They are the Euler-Lagrange equations of the action

$$S[e,\omega] = \frac{1}{16\pi G}\int \epsilon_{IJKL}(e^I \wedge e^J \wedge R[\omega]^{KL} + \lambda\, e^I \wedge e^J \wedge e^K \wedge e^L). \quad (5.18)$$

G is the Newton constant3; A is the cosmological constant, which I often set to zero below. Inverse tetrad**.** Using the matrix $e^I_\mu(x)$, defined as the inverse of the matrix we define the Ricci tensor and the Ricci scalar

$$R^I_\mu = R^{IJ}_{\mu^v} e^v_J, \quad (5.19)$$

And the Ricci scalar

$$R = R^I_\mu e^\mu_I, \quad (5.20)$$

and write the vacuum Einstein equations (5.17) as

$$R^I_\mu - \frac{1}{2}(R + \lambda) e^I_\mu = 0. \quad (5.21)$$

Second order formalism. Replacing ω with$\omega[e]$ in (5.10) we get the equivalent action

$$S[e] = \frac{1}{16\pi G}\int \epsilon_{IJKL}\left(e^I \wedge e^J \wedge R\big[\omega[e]\big]^{KL} + \lambda\, e^I \wedge e^J \wedge e^K \wedge e^L\right). \qquad (5.22)$$

The formalism in (5.18) where e and ω are independent is called the first order formalism. The two formalism are not equivalent in the presence of fermions; we do not know which one is physically correct, because the effect of gravity on single fermions is hard to measure. Self dual formalism**.** Consider the self dual "projector" $P^i_{I,J}$

$$P^i_{jk} = \frac{1}{2}\epsilon^i_{jk}, \qquad P^i_{j0} = -P^i_{j0} = \frac{i}{2}\delta^i_{j-}. \qquad (5.23)$$

where $i = 1,2,3.$[4] Define the the complex SO(3) connection

$$A^i_\mu = P^i_{I,J}\omega^{I,J}_\mu, \qquad (5.24)$$

Equivalently,

$$A^i = \omega^i + i\omega^{0i}, \qquad (5.25)$$

We can use the complex self dual connection A^i(3 complex one-forms), instead of the real connection $\omega^I{}_J$ (6 real one-forms), as the dynamical variable for GR. (This is equivalent to describing a system with two real degrees of freedom x and y in terms of a single complex variable $z = x + iy$.) In terms of A1, the Einstein equations read

$$P_{i\,IJ}e^I \wedge \left(F^i + \lambda\, P^i_{KL}e^K \wedge e^L\right) = 0. \qquad (5.26)$$

where $F^i = dA^i + \epsilon^i_{jk}A^j A^k$is the curvature of A.[5] These are the Euler-Lagrange equations of the action

$$S[e] = \frac{1}{16\pi G}\int \left(iP_{iIJ}e^I \wedge e^J \wedge F^i + \lambda \epsilon_{IJKL} e^I \wedge e^J \wedge e^K \wedge e^L\right), \qquad (5.27)$$

which can be obtained adding to the action (5.18) an imaginary term that does not change the equations of motion. The self dual formalism is often used in canonical quantization, because it simplifies the form of the hamiltonian theory.

Plebanski formalism**.** The Plebanski self dual two-form is defined as

$$\sum^{i} = P^i_{IJ} e^I \wedge e^J. \qquad (5.28)$$

That is

$$\sum^{1} = e^2 \wedge e^3 + ie^0 \wedge e^1. \qquad (5.29)$$

and so on cyclically. A straightforward calculation shows that $\sum$ satisfies

$$D\sum^{1} = d\sum^{i} + A^i_j \wedge \sum^{j} = 0. \qquad (5.30)$$

The algebraic equations for a triplet a of complex two-forms S*

$$3\sum^{i} \wedge \sum^{j} = \delta^{ij} \sum_{k} \wedge \sum^{k} = \delta^{ij} \overline{\sum_{k}} \wedge \overline{\sum^{k}},$$

$$\sum^{i} \wedge \overline{\sum^{j}} = 0 \qquad (5.31)$$

are solved by equation (5.28), where e1 is an arbitrary real tetrad. The GR action can thus be written as

$$S[\Sigma, A] = \frac{i}{16\pi G}\int \left(\sum_{i} \wedge F^i + \lambda \sum_{k} \wedge \overline{\sum^{k}}\right), \qquad (5.32)$$

where $\sum^i$ satisfies the Plebanski constraints (5.31). The Plebanski formalism is often used as starting point for spinfoam models.

5.4 Field Theory

There are several ways in which a field theory can be cast in hamiltonian form. One possibility is to take the space of the fields at fixed time as the nonrelativistic configuration space Q. This strategy badly breaks special and, in a general covariant theory, general relativistic invariance. Lorentz covariance is broken by the fact that one has to choose a Lorentz frame for the t variable. Far more disturbing is the conflict with general covariance. The very foundation of general covariant physics is the idea that the notion of a simultaneity surface all over the universe is devoid of physical meaning. It is better not to found hamiltonian mechanics on a notion devoid of physical significance.

A second alternative is to formulate mechanics on the space of the solutions of the equations of motion. The idea goes back to Lagrange. In the generally covariant context, a symplectic structure can be defined over this space using a spacelike surface, but one can show that the definition is surface independent and therefore it is well defined. This strategy as been explored. The structure is viable in principle and has the merit of showing that the hamiltonian formalism is intrinsically covariant. In practice, it is difficult to work with the space of the solutions of the field equations, in the case of an interacting theory. Therefore we must either work over a space that we can't even coordinatize, or coordinatize the space with initial data on some instantaneity surface, and therefore, effectively, go back to the conventional fixed time formulation.

The third possibility, which we consider here, is to use a covariant finite dimensional space for formulating hamiltonian mechanics. we noticed above that in the relativistic context the double role of the phase space, as the arena of mechanics and the space of the states, is lost. The space of the states, namely the phase space T is infinite dimensional in field theory, essentially by definition of field theory. But this does not imply that the arena of hamil-hamiltonian mechanics has to be infinite dimensional as well. The natural arena for relativistic mechanics is the extended configuration space C of the partial observables. Is the space of the partial observables of a field theory finite or infinite dimensional Partial observables in field theory consider a field theory for a field $\phi\ (x)$ with N components. The field is defined over spacetime M with coordinates x, and takes value in a N dimensional target space T

$$\phi: M \rightarrow\ T$$

$$x \rightarrow \phi(x). \qquad (5.33)$$

For instance, this could be Maxwell theory for the electric and magnetic fields $0\ =\ (\vec{E},\vec{B})$, where $N\ =\ 6$. In order to make physical measurements on the field described by this theory we need N measuring devices to measure the components of the field ϕ, and 4 devices to determine the spacetime position x.

Field values ϕ and positions x are therefore the partial observables of a field theory. Therefore the operationally motivated relativistic configuration space for a field theory is the finite dimensional space

$$C\ =\ M \times T \qquad (5.34)$$

which has dimension $4 + N$. A correlation is a point (x, ϕ) in C. It represents a certain value (ϕ) of the fields at a certain spacetime point (x). This is the obvious generalization of the (t, α) correlations of the pendulum of the example. A physical motion is a physically realizable ensemble of correlations. A motion is determined by a solution $\phi(x)$ of the field equations. Such a solution determines a 4-dimensional surface in the $D + N$ dimensional) space C: the surface is the graph of the function (5.33). Namely the ensemble of the points $(x, \phi(x))$. The space of the solutions of the field equations, namely the phase space Γ, is therefore an (infinite dimensional) space of 4d surfaces γ in the $(4+N)$)-dimensional configuration space C. Each state in Γ determines a surface γ in C. Hamiltonian formulations of field theory defined directly on $\mathcal{C} = M \times T$ are possible and have been studied. The main reason is that in a local field theory the equations of motion are local, and therefore what happens at a point depends only on the neighborhood of that point. There is no need, therefore, to consider full spacetime to find the hamiltonian structure of the field equations. we refer the reader to the beautiful and the ample references therein, for a discussion of this kind of approach. we give a simple and self-contained illustration of the formalism below, with the emphasis on its general covariance.

5.5 Quantization And Classical Limit

In general, a quantum system $(\mathcal{K}, A_i, H)$ has a classical limit which is a relativistic mechanical system $(\mathcal{C}, H)$ describing the results of observations on the system at scales and with accuracy larger than the Planck constant. In the classical limit, Heisenberg uncertainty can be neglected and a commuting set of partial observables A_i can be taken as coordinates of a commutative relativistic configuration space $\mathcal{C}$.If we are given a classical system defined by a non relativistic configuration space $\mathcal{C}$ with co-coordinates q^a and by a relativistic hamiltonian $H(q^a, p_a)$, a solution of the quantization problem is provided by the multiplicative operators q^a, the derivative operators

$$p_a = -ih\frac{\partial}{\partial q^a} \tag{5.35}$$

and the Hamiltonian operator

$$H + H\left(q^a, -ih\ \frac{\partial}{\partial q^a}\right) \tag{5.36}$$

on the Hilbert space $\mathcal{K} — L^2[\mathcal{C}, dq^a]$, or more precisely, the Gelfand triple determined by $\mathcal{C}$ and the measure dq^a. The physics is entirely contained in the transition amplitudes

$$W(q^a, q^{ta}) = (q^a|P|q^{ta}) \tag{5.37}$$

where the states $|q^a\rangle$ are the eigenstates of the multiplicative operators q^a.

In turn, the space $\mathcal{K}$ has the structure

$$K = L^2[\mathcal{G}]. \tag{5.38}$$

As we shall see, this remains true in field theory and in quantum gravity. The space $\mathcal{G}$ was defined for finite dimensional systems, for field theories and in the case of gravity. In the limit $h \to 0$ the Wheeler-DeWitt equation becomes the relativistic Hamilton-Jacobi equation (3.59) and the propagator has the form (writing $q = (q^a)$)

$$W(q, q') \sim \sum_i \quad A_i(q, q') e^{\frac{1}{h} S_i(q,q')} \tag{5.39}$$

where $S_i(q, q')$ are the different branches of the Hamilton function, as in (3.89). Now, the reverse of each path is still a path. The Hamilton function and the amplitude of a reversed path acquiresa minus, giving

$$W(q, q') \sim \sum_i \quad A_i(q, q') \sin\left(\frac{1}{h} S_i(q, q')\right) \tag{5.40}$$

and W is real. Assuming only one path matters,

$$W(q, q') \sim A(q, q') \sin\left(\frac{1}{h}\, S(q, q')\right) \tag{5.41}$$

and we can write for instance

$$\lim_{h \to 0} \quad \frac{1}{W}\, ih \frac{\partial}{\partial q^a} ih \frac{\partial}{\partial q^b} W(q, q') = \frac{\partial S(q, q')}{\partial q^a} \frac{\partial S(q, q')}{\partial q^b}\,. \tag{5.43}$$

This equation provides a precise relation between a quantum theory (entirely defined by the prop-propagator $W(q,q')$) and a classical theory (entirely defined by the Hamilton function $S(q,q')$). Using this equation can be written in the suggestive form

$$\lim_{h\to 0} \quad \frac{1}{W} p_a p_b W(q,q') \;=\; p^a(q,q')p^b(q,q'). \tag{5.43}$$

Examples (5.5.1): pendulum and timeless double pendulum

Pendulum**.** An example of relativistic formalism is provided by the quantization of the pendulum described in the previous: The kinematical state space is $\mathcal{K} \;=\; L_2[R^2, d\alpha dt]$. The partial observable operators are the multiplicative operators α and t acting on the functions $\psi(\alpha,t)$ in $\mathcal{K}$. Dynamics is defined by the operator H given in (5.18). The Wheeler-DeWitt equation is therefore

$$\left(-ih\frac{\partial}{\partial t} - \frac{h^2}{2m}\frac{\partial^2}{\partial\alpha^2} + \frac{m\omega^2}{2}\alpha^2\right)\Psi(\alpha,t) = 0. \tag{5.44}$$

$\mathcal{H}$ is a space of solutions of this equation. The "projector" operator $P : \mathcal{K} \to \mathcal{H}$ defined by H is given in (5.23), and defines the scalar product in $\mathcal{H}$. Its matrix elements $W(\alpha,t,\alpha',t')$ between the common eigenstates of a and t are given by the propagator (5.11). They express all predictions of the theory. Because of the specific form of H, these define a probability density in a but not in t, as explained. Equivalently, the quantum theory can be defined by the boundary state space $K \;=\; L_2[\mathcal{G}]$, where $\mathcal{G}$ is the boundary space of the classical theory, with coordinates (α,t,α',t'), and the covariant vacuum state $\langle\alpha,t,\alpha',t'|0\rangle \;=\; W(\alpha,t,\alpha',t')$, which determines the amplitude $A \;=\; \langle 0|\psi\rangle$ of any possible outcome $\psi \in$ K of a preparation/measurement experiment. Timeless double pendulum.An example of a relativistic quantum system which cannot be expressed in terms of conventional relativistic quantum mechanics is provided by the quantum theory of the timeless system (5.40). The kinematical Hilbert space $\mathcal{K}$ is $L^2[R^2, dadb]$, and the Wheeler-DeWitt equation is

$$\frac{1}{2}\left(-h^2\frac{\partial^2}{\partial a^2} - h^2\frac{\partial^2}{\partial b^2} + a^2 + b^2 - 2E\right)\Psi(a,b) = 0. \qquad (5.45)$$

5.6. Quantum Field Theory

We assume the reader is familiar with standard quantum field theory (QFT). Here we illustrate the connection between QFT and the relativistic formalism developed above, and I recall a few techniques that will be used in are particular importance are the distinction between Minkowski vacuum and covariant vacuum, the functional representation of a field theory, and the construction of the physical Hilbert space of lattice Yang-Mills theory.

We have seen that a classical field theory can be defined covariantly by the boundary space $\mathcal{G}$ of closed surfaces a in a finite dimensional space $\mathcal{C}$ and a relativistic hamiltonian H on $T*\mathcal{G}$. For instance, in a scalar field theory $\mathcal{C} = M \times R$ has coordinates (x^μ, ϕ), where x^μ is a point in Minkowski space and ϕ a field value. A surface a is determined by the two functions

$$\alpha = [x^\mu(\vec{\tau}), \varphi(\vec{\tau})]. \qquad (5.46)$$

and determines a boundary 3-surface $x^\mu(\vec{\tau})$in Minkowski space M and boundary values $\phi\big(x(\tau)\big) = \varphi(\tau)$ of the field on this surface.

A quantization of the theory can be obtained, precisely as in the finite dimensional case, in terms of a boundary state space $\mathcal{K}$ of functionals $\Psi[\alpha]$ on $\mathcal{G}$. Notice however that the difference between the kinematical state space $\mathcal{K}$ and the boundary state space $\mathcal{K}$ is far less significative in field theory than for finite dimensional systems. In the finite dimensional case, the states $\psi(q^a)$ in $\mathcal{K}$ are functions on the extended configuration space $\mathcal{C}$, while the states $\psi(q^a, q^{a\prime})$ in $\mathcal{K}$ are functions on the boundary space $\mathcal{G} = C \times C$. In the field theoretical case, both states have the form $\Psi[\alpha]$. The difference is that the states in $\mathcal{K}$ are functions of an "initial" surface a, where $x^{\mu}(\vec{\tau})$ can be for instance the spacelike surface $x^0 = 0$; in this case a contains only half of the data needed to determine a solution of the field equations. On the other hand, the states $\psi[\alpha]$ in $\mathcal{K}$ are functions of a closed surface a. In fact, the only difference between $\mathcal{K}$ and $\mathcal{K}$ is in the global topology of α. If we disregard this, and consider local equations, we can confuse $\mathcal{K}$ and K . The relativistic hamiltonian is given . The complete solution of the classical dynamics is known if we know the Hamilton function $S[\alpha]$, which is the value of the action

$$S[\alpha] = S[\mathcal{R}, \phi] = \int_{\mathcal{R}} \mathcal{L}((p(x), \partial_{\mu}\phi(x)) \; d^4x, \qquad (5.47)$$

where $\mathcal{R}$ is the four-dimensional region bounded by $x(\vec{\tau})$ and $\phi(x)$ is the solution of the equations of motion in this region, determined by the boundary data $\phi\big(x(\vec{\tau})\big) = \varphi(\vec{\tau})$. If there is more than one of these solutions, we write them as $\phi_i(x)$ and the Hamilton function is multivalued

$$S_i[\alpha] = S[\mathcal{R}, \phi_i] = \int_{\mathcal{R}} \mathcal{L}\left(\phi_i(x), \partial_{\mu}\phi_i(x)\right) d^4x. \qquad (5.48)$$

The relativistic Hamiltonian gives rise to the Wheeler-DeWitt equation

$$H\left[x^{it}, \phi, -ih\,\frac{\delta}{\delta x^{\mu}} - ih\frac{\partial}{\delta\varphi}\right](\vec{\tau})\psi[\alpha] = 0. \qquad (5.49)$$

precisely as in the finite dimensional case. The Hamilton-Jacobi equation can be interpreted as the ikonal approximation for this wave equation.

The complete solution of the dynamics is known if we know the propagator $W[\alpha]$, which is a solution of this equation. Formally, the field propagator can be written as a functional integral

$$W[\alpha] = \int_{\phi(x(\vec{\tau}))=\varphi(\vec{\tau})} [D\phi] e^{-\frac{i}{h}S[\mathcal{R},\phi]}. \qquad (5.50)$$

Of course one should not confuse the field propagator $W[\alpha]$ with the Feynman propagator. The first propagates field, the second the particles of a QFT. The first is a functional of a surface and the value of the field on this surface, the second is a function of two spacetime points. To the lowest

order in h, the saddle point approximation gives

$$W[\alpha] \sim \sum_i A_i[\alpha] e^{-\frac{i}{h} S_i[\alpha]}. \tag{5.51}$$

There are two characteristic difficulties in the field theoretical context that are absent in finite dimensions: the definition of the scalar product and the need to regularize operator products. In finite dimensions, a measure dq^a on $\mathcal{C}$ is sufficient to define an associated L_2 Hilbert space of wave functions. In the field theoretical case, we have to define the scalar product otherwise. The scalar product must respects the invariances of the theory and must be such that real classical variables be represented by self-adjoint operators. This is because self-adjoint operators have real spectrum, and the spectrum determines the values that a quantity can take in a measurement. Given a set of linear operator on a linear space, the requirement that they are self-adjoint put stringent conditions on the scalar product. As we shall see, in all cases of interest these requirements are sufficient to determine the scalar product.

Second, local operators are in general distributions and their products are ill defined. Operator products arise in physical observable quantities as well as in the dynamical equation, namely in the Wheeler-DeWitt equation. In particular, functional derivatives are distributions. In the classical Hamilton-Jacobi equation we have products of functional derivatives of the Hamilton-Jacobi func- functional, which are well defined products of functions. In the corresponding quantum Wheeler-DeWitt equation, these become products of functional derivatives, which are ill-defined without an appro-appropriate renormalization procedure.

5.7. Functional Representation

Consider a simple free scalar theory, where $V = 0$. we describe this well known QFT in some detail in order to illustrate certain techniques that play a role in quantum gravity. In particular, I illustrate the functional representation of quantum field theory, a simple form of the Wheeler-DeWitt equation, the general form of $W[\alpha]$, and its physical interpretation. The functional representation is the representation in which the field operator is diagonal. The quantum states will be represented as functionals $\Psi[\phi] = \langle\phi|\Psi\rangle$, where $|\phi\rangle$ is the (generalized) eigenstate of the field operator with eigenvalue $\phi(\vec{x})$. The relation between this representation and the conventional one on the Fock basis $|\vec{k}_1, \dots, \vec{k}_1 \quad\rangle$ is precisely the same as the relation between the Schrodinger representation $\psi(x)$ and the one on the energy basis $|n \quad\rangle$, for a simple harmonic oscillator. I also illustrate the way in which the scalar product on the space of the solutions of the Wheeler-DeWitt equation is determined by the reality properties of the field operators.

To start with, and to connect the generally covariant formalism described above with conven-conventional QFT, let's restrict the surface $x(\vec{\tau})$ in a to a spacelike surface $x^\mu(\vec{\tau}) = (t, \vec{\tau})$ in Minkowski space. Then $\alpha = [t, \phi(x)]$ and $\Psi[\alpha] = \Psi[t, \phi(x)]$. The Hamilton-Jacobi equation

$$ih \frac{\partial}{\partial t} \psi = H_0 \psi \qquad (5.52)$$

5.8. Space-Time Relational Versus Quantum Relational

We discussed, the main idea underlying GR is the relational interpretation of localization: objects are not located in space-time. They are located with respect to one another. I have observed that the lesson of QM is that quantum events and states of systems are relational: they make sense only with respect to another system. Thus, both GR and QM are characterized by a form of relationalism. Is there a connection between these two forms of relationalism?

Let us look closer at the two relations. In GR, the localization of an object S in spacetime is relative to another object (or field) O, to which S is contiguous. Contiguity, or, equivalently, Einstein's "spacetime coincidence" is the basic relation that constructs space-time. In QM, there are no absolute properties or facts: properties of a system S are relative to another system O with which S is interacting. Facts are interactions. Thus, interactions form the basic relations between systems.

But there is a strict connection between contiguity and interaction. On the one hand, S and O can interact only if they are contiguous. If they are nearby in spacetime. This is locality. Interaction requires contiguity. On the other hand, what does it mean that S and O are contiguous? What else does it mean besides the fact that they can interact?[6] Therefore contiguity is manifested by interacting. In a sense, the fact that interactions are local means that there is a sort of identity between being contiguous and interacting.

Thus, locality ties together very strictly the spacetime relationalism of GR with the relation-relationalism underlying QM. It is tempting to try to develop a general conceptual scheme based on this observation. This could be a conceptual scheme in which contiguity is nothing else than a man-manifestation, or can be identified with the existence of a quantum interaction. The spatiotemporal structure of the world would then be directly determined by who is interacting with whom. This is of course very vague, and might lead nowhere, but I find the idea intriguing.

5.9 Quantum Space

It is time to begin to put together the tools developed in the first part of the book, and build the quantum theory of spacetime. The strategy is simple. We "quantize" the canonical formulation of GR described at the beginning, according to the relativistic QM formalism detailed. This deals with the kinematical part of the theory: states, partial observables and their eigenvalues. The deals with dynamics, namely with the transition amplitudes.

Structure of quantum gravity, we have seen that GR can be formulated as the dynamical system defined by the Hamilton-Jacobi equation (5.9)

$$F_{ab}^{ij}(\vec{\tau})\frac{\delta S[A]}{\delta A_a^i(\vec{\tau})}\frac{\delta S[A]}{\delta A_b^j(\vec{\tau})} = 0, \tag{5.53}$$

where the functional $S[A]$ is defined on the space $\mathcal{G}$ of the 3d $SU(2)$ connections $A_\alpha^i(\vec{\tau})$, and is invariant under internal gauge transformations and 3d diffeomorphisms, that is

$$\delta_f A_a^i(\vec{\tau})\frac{\delta S[A]}{\delta A_a^i(\vec{\tau})} = 0, \qquad \delta_\lambda A_a^i(\vec{\tau})\frac{\delta S[A]}{\delta A_a^i(\vec{\tau})} = 0, \tag{5.54}$$

where the variations $\delta A_a^i(\vec{\tau})$ and $\delta_\lambda A$ are given in D.12). Equivalently, the theory is defined by the hamiltonian $H[A,E] = F_{ab}^{ij}E_i^a E_j^b$on $T * \mathcal{G}$.

Following the prescription, a quantization of the theory can be obtained in terms of complex valued Schrodinger wave functionals $\Psi[A]$ on $\mathcal{G}$. The quantum dynamics is inferred from the classical dynamics by interpreting $S[A]$ as H times the phase of $\Psi[A]$. Namely interpreting the classical Hamilton-Jacobi theory as the ikonal approximation of a quantum wave equations; semi classical "wave packets" will then behave according to the classical theory. This can be obtained defining the quantum dynamic by replacing derivatives of the Hamilton-Jacobi functional $S[A]$ with derivative operators. The two equations

(5.54) remain unchanged: they simply force $\Psi[A]$ to be invariant under $SU(2)$ gauge transformations and diffeomorphisms. Equation (5.53) gives

$$F_{ab}^{ij}(\vec{\tau})\frac{\delta}{\delta A_a^i(\vec{\tau})}\frac{\delta}{\delta A_b^j(\vec{\tau})}\Psi[A] = 0, \tag{5.55}$$

This is the Wheeler-DeWitt equation, or Einstein-Schrodinger equation. It governs the quantum dynamics of spacetime. In other words, the dynamics is defined by the hamiltonian operator $H = H[A, -ih\partial/\partial A]$.

More in detail, we want a rigged Hilbert space $S \subset K \subset S'$, where S is a suitable space of functionals $\Psi[A]$. Partial observables are represented by self-adjoint operators on $\mathcal{K}$. Theireigenvalues describe the quantization of physical quantities. The operator P, formally given by the Pfield theoretical generalization of (5.58)

$$P \sim \int [DN] e^{-i\int d^3\tau\, N(\vec{\tau})H(\vec{\tau})}, \tag{5.56}$$

sends S to the space of the solutions of (5.55). Its matrix elements between eigenstates of partial observables define the transition amplitudes of quantum gravity. These determine all (probabilistic) dynamical relations between any measurement that we can perform.

A preferred state in $\mathcal{K}$ is $|\emptyset\rangle$, the eigenstate of the geometry with zero volume and zero area. The covariant vacuum is given by $|\emptyset\rangle = P|\emptyset\rangle$. If we assume that the surface $\vec{\tau}$ coordinatized by t is the entire boundary of a finite spacetime region, then we can identify K with the boundary space K. The correlation probability amplitude associated to a measurement of partial observables on the boundary surface is $A = W(s) = \langle 0|s\rangle$, where $|s\rangle$is the eigenstate if the partial observables corresponding to the measured eigenvalues.

References

[1] R. Penrose, Twistor theory:*An Approach To The Quantisation of Fields And Space- Time,* North-Holland Publishing Campany (1972).

[2]Atiyah, M.F., Drinfeld, V., Hitchin, N. and Manin, Y. (1978) Phys. Lett. A65 185

[3]Atiyah, M.F. and Ward, R.S. (1977) Comm. Math. Phys. 55 117

[4]Bailey, T.N. (1985) Proc. Roy. Soc. Lond. A397 143

[5]Bailey, T.N. and Baston, R.J. eds (1990) Twistors in Mathematics and Physics, LMS Lecture Notes 156, CUP, Cambridge

[6]Bailey, T.N., Ehrenpreis, L. and Wells, R.O. (1982) Proc. Roy. Soc. Lond. A384 403

[7]Bartnik, R. (1989) Phys. Rev. Lett. 62 2346

[8]Baston, R.J. (1984) Twistor Newsletter 17

[9]Baston, R.J. and Eastwood, M.G. (1989) The Penrose Transform: Its Interaction With Representation Theory, OUP, Oxford

[10]Baston, R.J. and Mason, L.J. (1987) Class. Quant. Grav. 4 815-826

[11]Besse, A. (1987) Einstein Manifolds, Springer-Verlag, Berlin

[12]Bott, R. and Tu, L. (1982) Differential Forms in Algebraic Topology, Springer-Verlag

[13]Burns, D. (1979) in Complex Manifold Techniques in Theoretical Physics, eds. D. Lerner and P.D. Sommers, Pitman, London

[14]Chern, S.S. (1967) Complex Manifolds Without Potential Theory, van Nostrand Reinhold, New York

[15]Corrigan, E., Fairlie, D., Yates, R. and Goddard, P. (1978) Comm. Math. Phys. 58 223

[16]Curtis, W.D., Lerner, D.E. and Miller, F.R. (1978) J. Math. Phys. 19 2024-2027

[17]Do Carmo, M.P. (1976) Differential Geometry of Curves and Surfaces, Prentice-Hall

[18]Do Carmo, M.P. (1992) Riemannian Geometry, Birkhauser

[19]Donaldson, S.K. and Kronheimer, P.B. (1990) Geometry of Four-Manifolds, Clarendon Press, Oxford

[20]Dougan, A.J. (1991) Definitions of Quasi-Local Momentum, D.Phil, thesis: Oxford University

[21]Dougan, A.J. and Mason, L.J. (1991) Phys. Rev. Lett. 67 2119-2122

[22]Eastwood, M.G. (1981) Proc. Roy. Soc. Lond. A374 431-445

[23]Eastwood, M.G. (1983) Twistor Newsletter 15

[24]Eastwood, M.G. and Ginsberg, M.L. (1981) Duke Math. J. 48 177

[25]Eastwood, M.G., Penrose, R. and Wells, R.O. (1981) Comm. Math. Phys. 78 305

[26]Eastwood, M.G. and Pilato, A. (1991) Pac. J. Math. 151 201-215

[27]Eastwood, M.G. and Tod, K.P. (1982) Math. Proc. Camb. Phil. Soc. 92 317-330

[28]Field, M . (1982) Several Complex Variables and Complex Manifolds, CUP

[29]Geroch, R. (1968) J. Math. Phys. 9 1739

[30]Geroch, R., Held, A. and Penrose, R. (1973) J. Math. Phys. 14 874

[31]Griffiths, P. A. and Harris, J. (1978) Principles of Algebraic Geometry, Wiley, New York

[32]Gross, L. (1964) J. Math. Phys. 5 687-695

[33]Grothendieck, A. (1957) Am. J. Math. 79 121

[34]Gunning, R.C. (1966) Lectures on Riemann Surfaces, Princeton U.P.

[35]Gunning, R.C. and Rossi, H. (1965) Analytic Functions of Several Complex Variables, Prentice-Hall

[36]Hansen, R.O., Newman, E.T., Penrose, R. and Tod, K.P. (1978) Proc. Roy. Soc. Lond. A363 445

[37]Hawking, S.W. and Ellis, G.F.R. (1973) The Large-Scale Structure of Space-Time, CUP

[38]Henkin, G.M. and Manin, Yu. I. (1980) Phys. L«tt. B95 405

[39]Hicks, N.J. (1965) Notes on Differential Geometry, van Nostrand Reinhold

[40]Hirzebruch, F. (1966) Topological Methods in Algebraic Geometry, Springer-Verlag

[41]Hitchin, N.J. (1979) Math. Proc. Camb. Phil. Soc. 85 465

[42]Hitchin, N.J. (1982) Comm. Math. Phys. 83 579

[43]Hitchin, N.J. (1983) Comm. Math. Phys. 89 145

[44]Hodges, A.P. (1982) Physica A114 157

[45]Hodges, A.P. (1983a) Proc. Roy. Soc. Lond. A385 207

[46]Hodges, A.P. (1983b) Proc. Roy. Soc. Lond. A386 185

[47]Hodges, A.P. (1985a) Proc. Roy. Soc. Lond. A397 341

[48]Hodges, A.P. (1985b) Proc. Roy. Soc. Lond. A397 375

[49]Hodges, A.P., Penrose, R. and Singer, M.A. (1989) Phys. Lett. B216 48

[50]Hormander, L. (1966) An Introduction to Complex Analysis in Several Variables, van Nostrand Reinhold

[51]Horowitz, G.T. and Perry, M.J. (1982) Phys. Rev. Lett. 48 371-374

[52]Huggett, S.A. (1990) in Twistors in Mathematics and Physics, eds. T.N. Bailey and R.J.Baston, CUP

[53]Huggett, S.A. (1992) Class. Quant. Grav. 9 127-135

[54]Huggett, S.A. ed (1994) Twistor Theory, Marcel Dekker

[55]Huggett, S.A. and Singer, M.A. (1990) Twistor Newsletter 30

[56]Huggett, S.A. and Singer, M.A. (1991) Trans. Amer. Math. Soc. 324 41

[57]Hughston, L.P. (1979) Twistors and Particles, Springer Lecture Notes in Physics 79

[58] R. Penrose, *Advances In Twistor Theory,* Pitman Advanced Publishing Program(1979).

[59] R. S. WARD, *Twistor Geometry And Field Theory,* Cambridge University Press(1980).

[60] C. Kiefer, *Quantum Gravity* (Carendon Press, Oxford 2004).

[61] S. W. Hawking and R. Penrose, *The nature of the space-time* (Princeton University press, Princeton, 1996).

[62] A. Borde, A. H. Guth, and A. Vilenkin, Phys. Rev. Lett. 90. 1511301 (2003).

[63] C, Rovelli and S. Speziale, Phys. Rev. D 67, 064019 (2003).

[64] C.J. Isham, in: *Canonical Gravity : from Classical to Quantum*, edited by J. Ehlers and H, Friedrich (Springer, Berlin, 1994).

[65] C. Kiefer and C. Weber, Ann. Phys. (Leipzig) 14,253 (2005).

[66] B.J. Carr. in :*Quantum Gravity: From Theory to Experimental Search*, edited by D. Giulini, C.Kiefer and C. Lammerzahl (Springer, Berlin, 2003).

[67] D. Blais, T. Bringmann. C. Kiefer and D. Polarski, Phys. Rev. D 67. 024024 (2003).

[68] L. Rosenfeld, Z. Phys. 65 589 (1930).

[69] M.H. Goroff and A. Sagnotti, Phys. Lett. B 160. 81 (1985).

[70] R.W. Carroll, Fluctuations, Information, Garvity and the Quantum potential, 2006 Springer.

[71] R. S. Ward, Raymond O. Wells, JR., Twistor Geometry and Field Theory, Cambridge University Press, New York, 1989.

[72] Robert J. Baston and Michael G.Eastwood, the Penrose Transform, clarndon press. Oxford, 1989.

[73] M. Amarzguioui. Theory of cosmological perturbations. Cand. Scient. Thesis, University of Oslo, 2003.

[74] C. Armendariz-Picon, V. Mukhanov, and P.J. Steinhardt. Essentials of k-essence. astro-ph/0006373, 2000.

[75] A. Albrecht and P.J. Steinhardt. Cosmology for grand unified theories with radiately induced symmetry breaking. Phys. Rev. Lett., 48:1220, 1982.

[76] U. Alam, V. Sahni, T.D. Saini, and A.A. Starobinsky. Exploring the expanding universe and dark energy using the statefinderdiagnosic. Mon. Not. Roy. Astron. Soc., 344:1057, 2003.

[77] J.D. Barrow and S. Hervik. Magnetic brane-worlds. Class. Quantum Grav. 19:155–172, 2002.

[78] Eastwood, M.G., Penrose, R., and Wells, R.O. 1979. *CohomoZogy*{IfldMasr-;ZessFieZds. To be published.

[79] Hitchin, N. 1979. *Linear Field quations on Self-TJual Spaces*. To bepublished.

[80] Rawnsley, J.H. 1979. *On the Atiyah-Singer Vanishing Theorem for CertainCohomology Groups of InstantDn Bundles.* f\1ath. Ann. 241, 43-5

[81] Wells, R.O. 1979. *Complex Manifolds and Mathematical Physics*. To appearin Bulletin of the American Mathematical Society.

[82]Griffiths, P. &Adams, J .• 1974 *Atgehraican.dAnatytic Geometry.* Princeton University Press.

[83]Gunning, R.C. 1966 *Lectures on Niemann*Sv:t˜faces. Princeton UniversityPress.

[84]Penrose, R. 1965 Proc. Roy. Soc. A284, 159-203.

Spanier) E.H. 1966 *Atgebraic Topology.* McGraw-Hill.

[85]Morrow, J. &Kodaira, K. 1971 *Complex ManifoZds.* Holt˜ Rinehart &Winston.

[86]Penrose, R. 1965 Proc. Roy. Soc. A28˜, 159-203.

[87] C.J. Isham, R. Penrose &D.W. Sciama, pp. 268-407. Clarendon: Oxford.

[88]Morrow, J. and Kodaira, K. (1971) Complex Manifolds, Holt, Rinehart and Winston
[89]Nester, J.M. (1981) Phys. Lett. A83 241-242
[90]Newman, E.T. (1976) Gen. Rel. Grav. 7 107
[91]Palais, R.S. ed. (1965) Seminar on the Atiyah-Singer Index Theorem, Princeton U.P.
[92]Penrose, R. (1955) Math. Proc. Camb. Phil. Soc. 51 406
Penrose, R. (1956) Math. Proc. Camb. Phil. Soc. 52 17
[93]Penrose, R. (1965a) Proc. Roy. Soc. Lond. A284 159
[94]Penrose, R. (1965b) in Relativity, Groups and Topology, eds. B. and C.M. deWitt, Gordon and Breach
[95]Penrose, R. (1966) in Perspectives in Geometry and Relativity, ed. B. Hoffmann, Indiana U.P.
[96]Penrose, R. (1967) J. Math. Phys. 8 345
[97]Penrose, R. (1968a) Int. J. Theor. Phys. 1 61
[98]Penrose, R. (1968b) in Battelle Recontres, eds. C.M. deWitt and J.A. Wheeler, Benjamin, New York
[99]Penrose, R. (1969) J. Math. Phys. 10 38
[100]Penrose, R. (1972) Techniques of Differential Topology in Relativity, S.I.A.M., Philadelphia
[101]Penrose, R. (1974) in Group Theory in Non-Linear Problems, ed. A.O. Barut, Reidel, Dordrecht
[102]Penrose, R. (1975) in Quantum Gravity : An Oxford Symposium, eds. C.J. Isham, R. Penrose and D.W. Sciama, OUP
[103]Penrose, R. (1976) Gen. Rel. Grav. 7 31
[104]Penrose, R. (1982) Proc. Roy. Soc. Lond. A381 53
[105]Penrose, R. (1983) in The Mathematical Heritage of H. Poincare, Proc. Symp. Pure Math. 39, Am. Math. Soc.
[106]Penrose, R. (1984) in Twistor Newsletter 18
[107]Tod, K.P. (1990b) in Geometry of Low-Dimensional Manifolds, vol. 1, eds. S.K. Donaldson and C.B. Thomas, CUP
[108]Tod, K.P. (1992) in Recent Advances in General Relativity, eds. A.I. Janis and J.R. Porter, Birkhauser, Boston
[109]Tod, K.P. and Perjes, A. (1976) Gen. Rel. Grav. 7 903
[110]Tod, K.P. and Ward, R.S. (1979) Proc. Roy. Soc. Lond. A368 411
[111]Ward, R.S. (1977) Phys. Lett. A61 81
[112]Ward, R.S. (1978) Proc. Roy. Soc. Lond. A363 289
[113]Ward, R.S. (1980) Comm. Math. Phys. 78 1
[114]Ward, R.S. (1981) Comm. Math. Phys. 79 317
[115]Ward, R.S. and Wells, R.O. (1990) Twistor Geometry and Field Theory, CUP
[116]Wells, R.O. (1980) Differential Analysis on Complex Manifolds, Springer-Verlag, New York

[117]Wells, R.O. (1981) Comm. Math. Phys. 78 567
[118]Witten, E. (1978) Phys. Lett. B77 394
[119]Witten, E. (1981) Comm. Math. Phys. 80 381-402
[120]Woodhouse, N.M.J. (1983) Phys. Lett. 94A 269
[121]Woodhouse, N.M.J. (1985) Class. Quant. Grav. 2 257
[122]Woodhouse, N.M.J. (1987) Class. Quant. Grav. 4 L121-L123
[123]Woodhouse, N.M.J, and Mason, L.J. (1988) Nonlinearity 1 73-114