Sudan University of Science and Technology
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Boundedness of Multi-Parameter Fourier Multiplier Operators with Characterizations of Logarithmic Besov and Triebel–Lizorkin Spaces

محدودية مؤثرات مضاعف فورير متعدد-الوسيط مع تشخيصات فضاءات بيسوف و تريبل- لزوركن اللوغريثمية

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Dedication

To the soul of my father

To my mother and

Family

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I am so grateful to my supervisor Prof. Dr. Shawgy Hussein AbdAlla, for patient supervision and cooperation to achieve this work. I highly appreciate his sincerity, generosity and above all his humanitarian manner.

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Abstract

Some developments in the theory of function spaces involving differences are shown. Difference and new characterizations of Besov, Sobolev and Triebel-Lizorkin spaces on metric measure spaces, on the Euclidean space and on averages balls are studied. We obtain directly and perfectly the dual multi-parameter, singular integrals and boundedness of the composition operators on Triebel-Lizorkin and Besov spaces associated with singular integrals with different homogeneities and of regular distributions. We introduce the method of Hörmander type theorems for multi-linear and boundedness of multi-parameter Fourier multiplier operators with limited smoothness and on Triebel-Lizorkin and Besov-Lipschitz spaces of logarithmic smoothness, approximation spaces and limiting interpolation. We explain the treatments of the characterizations of generalized and logarithmic Besov spaces in terms of differences, Fourier-analytical decompositions, wavelets bases and semi-groups.

الخــــــلاصـــــة

قمنا بتوضيح بعض التطورات في نظرية فضاءات الدالة المتضمنة الفروقات. تم دراسة الفرق والتشخيصات الجديدة لفضاءات بيسوف و سوبوليف و تريبل- لزوركن على فضاءات القياس المتري وعلى الفضاء الإقليدي وعلى كرات المتوسطات. تم الحصول مباشرة وتماماً على المميز- المتعدد الثنائي وعلى التكاملات الشاذة والمحدودية لمؤثرات التركيب على فضاءات تريبل- لزوركن وبيسوف المشاركة مع التكاملات الشاذة ومع التجانسات المختلفة والتوزيعات المنتظمة. أدخلنا طريقة مبرهنات نوع هورماندر لأجل متعددة- الخطية والحدودية إلى مؤثرات مضاعف فورير متعدد- الوسيط مع الملسان المنتهي وعلى فضاءات تريبل- لزوركن و بيسوف- ليبشيتز إلى الملسان اللوغريثمي وفضاءات التقريب والإستكمال المنتهي. أوضحنا المعالجات إلى تشخيصات فضاءات بيسوف المعممة واللوغريثمية بدلالات الفروقات وتفكيكات فورير- التحليلية وأساس المويجات وشبه الزمر.

Introduction

On a metric measure space satisfying the doubling property, we establish several optimal characterizations of Besov and Triebel-Lizorkin spaces, including a pointwise characterization.

The theory of one-parameter Triebel–Lizorkin and Besov spaces has been very well developed in the past decades, the multi–parameter counterpart of such a theory is still absent. The main purpose is to develop a theory of multi-parameter Triebel–Lizorkin and Besov spaces using the discrete Littlewood–Paley–Stein analysis in the setting of implicit multi–parameter structure. It is motivated by Han and Lu in which they established a satisfactory theory of multi–parameter Littlewood–Paley–Stein analysis and Hardy spaces associated with the flag singular integral operators studied by Muller–Ricci–Stein and Nagel–Ricci–Stein. The theory of Triebel–Lizorkin and Besov spaces in one-parameter has been developed satisfactorily, not so much has been done for the multi-parameter counterpart of such a theory. We introduce the weighted Triebel–Lizorkin and Besov spaces with an arbitrary number of parameters and prove the boundedness of singular integral operators on these spaces using discrete Littlewood–Paley theory and Calderón's identity. This is inspired by the work of discrete Littlewood–Paley analysis with two parameters of implicit dilations associated with the flag singular integrals recently developed by Han and Lu. We introduce new Triebel–Lizorkin and Besov Spaces associated with the different homogeneities of two singular integral operators.

We obtain a wavelet representation in (inhomogeneous) Besov spaces of generalized smoothness via interpolation techniques. We establish conditions on the parameters which are both necessary and sufficient in order that Besov and Triebel–Lizorkin spaces of generalized smoothness contain only regular distributions.

We characterize the Triebel-Lizorkin space $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ via a new square function

$$S_{\alpha,q}(f)(x) = \left\{\sum_{k\in\mathbb{Z}} 2^{k\alpha q}\left|\frac{1}{|B(x,2^{-k})|}\int_{B(x,2^{-k})}[f(x)-f(y)]dy\right|^q\right\}^{1/q},$$

where $\in L^1_{loc}(\mathbb{R}^n)\cap S'(\mathbb{R}^n)$, $x\in\mathbb{R}^n, \alpha\in(0,2)$ and $p,q\in(1,\infty]$. We show several equivalent characterizations of Sobolev spaces of even integer orders on $\mathbb{R}^n$, using the average

$$B_t f(x) := \frac{1}{|B(x,t)|}\int_{B(x,t)} f(y)dy,$$

of a function f over the ball $B(x,t) := \{y\in\mathbb{R}^n\colon |y-x|<t\}$ with $x\in\mathbb{R}^n$ and $t\in(0,\infty)$. Let $\ell\in\mathbb{N}$ and $p\in(1,\infty]$. We show that the sequence $\{f-B_{\ell,2^{-k}}f\}_{k\in\mathbb{Z}}$ consisting of the differences between f and the ball average $B_{\ell,2^{-k}}f$ characterizes the Besov space $\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)$ with $q\in(0,\infty]$ and the Triebel–Lizorkin space $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ with $q\in(1,\infty]$ when the smoothness order $\alpha\in(0,2\ell)$. It is shown that $f-B_{\ell,2^{-k}}f$ plays the same role as the approximation to the identity $\varphi_{2^{-k}}*f$ appearing in the definitions of $\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)$ and $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$.

We are concerned with the limited smoothness conditions in the spirit of Hörmander on the multi-linear and multi-parameter Coifman–Meyer type Fourier multipliers studied by C. Muscalu, J. Pipher, T. Tao, C. Thiele where they established the L^r estimates for the multiplier operators under the assumption that the multiplier has smoothness of sufficiently large order. We study the duality theory of the multi-parameter Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}(\mathbb{R}^m)$ associated with the composition of two singular integral operators on $\mathbb{R}^m$ of different homogeneities. Such composition of two singular operators was considered by Phong and Stein in 1982. For $1 < p < \infty$, we establish the dual spaces of such spaces as $(\dot{F}_p^{\alpha,q}(\mathbb{R}^m))^* = \dot{F}_{p'}^{\alpha,q'}(\mathbb{R}^m)$, and for $0 < p \leq 1$ we prove $(\dot{F}_p^{\alpha,q}(\mathbb{R}^m))^* = CMO_p^{-\alpha,q'}(\mathbb{R}^m)$. We then show the boundedness of the composition of two Calderón-Zygmund singular integral operators with different homogeneities on the spaces $CMO_p^{-\alpha,q'}$. Surprisingly, such dual spaces are substantially different from those for the classical one-parameter Triebel-Lizorkin spaces $\dot{\mathcal{F}}_p^{\alpha,q}(\mathbb{R}^m)$. We show that under the limited smoothness conditions, multi-parameter Fourier multiplier operators are bounded on multi-parameter Triebel–Lizorkin and Besov–Lipschitz spaces by the Littlewood–Paley decomposition and the strong maximal operator.

We compare Besov spaces $B_{p,q}^{0,b}$ with zero classical smoothness and logarithmic smoothness b defined by using the Fourier transform with the corresponding spaces $\mathbf{B}_{p,q}^{0,b}$ defined by means of the modulus of smoothness. With the help of limiting interpolation we determine the spaces obtained by iteration of approximation constructions. We work with Besov spaces $\mathbf{B}_{p,q}^{0,b}$ defined by means of differences, with zero classical smoothness and logarithmic smoothness with exponent b.

The Contents

Chapter 1
Developments in Theory and Characterizations of Besov Spaces

We show some recent developments of distributional Sobolev–Besov spaces and Sobolev–Besov spaces of measurable functions of positive smoothness which can be characterized in terms of differences. We discuss their (non) triviality under a Poincaré inequality.

Section (1.1) Function Spaces Involving Differences

For $1 < p < \infty$. The classical Sobolev spaces $W_p^1(\mathbb{R}^n)$ can be characterized as the collection of all $f \in L_p(\mathbb{R}^n)$ such that there exists a function $0 \le g \in L_p(\mathbb{R}^n)$ with

$$|f(x) - f(y)| \le |x - y|\big(g(x) + g(y)\big), \qquad x, y \in \mathbb{R}^n \qquad a.e. \tag{1}$$

Furthermore,

$$\left\|f|W_p^1(\mathbb{R}^n)\right\| \sim \left\|f|L_p(\mathbb{R}^n)\right\| + \inf \left\|g|L_p(\mathbb{R}^n)\right\|, \tag{2}$$

where the infimum is taken over all g with (1). The idea of dealing with Sobolev spaces of first order in terms of pointwise inequalities goes back to Bojarski and Hajłasz [7, 26]. This approach has been extended afterwards by Bojarski and coauthors to higher-order Sobolev spaces. See [3, 4, 5, 6, 8, 25]. It is quite clear that pointwise estimates of type (1) and even more their counterparts for higher-order Sobolev spaces, usually defined as distributional spaces or spaces of measurable functions, require some care and justification. This new approach attracted a lot of attention. (1), (2) have been extended in [69] to spaces $\mathbf{W}_p^k(\mathbb{R}^n)$ with $k \in N$ and $0 < p \le \infty$. If $1 < p < \infty$, then $\mathbf{W}_p^k(\mathbb{R}^n) = W_p^k(\mathbb{R}^n)$ are the classical Sobolev spaces. If $0 < p < 1$, then $\mathbf{W}_p^k(\mathbb{R}^n)$ are no longer spaces of distributions, but of measurable functions. One may understand $W_p^k(\mathbb{R}^n)$ as a proposal how the classical Sobolev spaces $W_p^k(\mathbb{R}^n)$ can be extended naturally from $1 < p < \infty$ to $0 < p \le \infty$. These pointwise characterizations can be used to have a new look at embedding assertions of Besov–Sobolev spaces in terms of differences, [30], limits of Besov norms and reformulations of some Gagliardo–Nirenberg inequalities, [61, 63]. We concentrate on definitions, explanations and assertions referring for proofs to the above mentioned. However, we wish to make this presentation accessible to a larger audience interested in function spaces and how the outlined specific topic is located within.

We introduce distributional function spaces and smoothness spaces of measurable functions. We recall some known relations. Deals with properties of the spaces $\boldsymbol{L}_p^s(\mathbb{R}^n)^k, 0 < p \le \infty, 0 < s \le k \in N$, covering in particular the above-mentioned spaces $\boldsymbol{W}_p^k(\mathbb{R}^n)$, $0 < p \le \infty, k \in N$. Afterwards we apply these assertions to embeddings and to limits of Besov norms and to Gagliardo–Nirenberg inequalities.

Let $\mathbb{N}$ be the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}^n$ be Euclidean n-space, where $n \in \mathbb{N}$. Put $\mathbb{R} = \mathbb{R}^1$, whereas $\mathbb{C}$ is the complex plane. Let $S(\mathbb{R}^n)$ be the usual Schwartz space and $S'(\mathbb{R}^n)$ the space of all tempered distributions on $\mathbb{R}^n$. Furthermore, $L_p(\mathbb{R}^n)$, with $0 < p \le \infty$, is the standard complex quasi-Banach space with respect to the Lebesgue measure in $\mathbb{R}^n$, quasi-normed by

$$\left\|f|L_p(\mathbb{R}^n)\right\| = \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx\right)^{1/p} \tag{3}$$

with the natural modification if $p = \infty$. As usual $\mathbb{Z}$ is the collection of all integers; and $\mathbb{Z}^n$, $n \in \mathbb{N}$, denotes the lattice of all points $m = (m_1, \dots, m_n) \in \mathbb{R}^n$ with $m_j \in \mathbb{Z}$. Let $\mathbb{N}_0^n$, $n \in \mathbb{N}$, be the set of all multi-indices,

$$\alpha = (\alpha_1, \ldots, \alpha_n) \text{ with } \alpha_j \in \mathbb{N}_0 \text{ and } |\alpha| = \sum_{j=1}^{n} \alpha_j. \tag{4}$$

If $\varphi \in S(\mathbb{R}^n)$, then

$$\hat{\varphi}(\xi) = (F\varphi)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi}\, \varphi(x)dx, \qquad \xi \in \mathbb{R}^n, \tag{5}$$

denotes the Fourier transform of φ. As usual, $F^{-1}\varphi$ and $\varphi^\vee$ stand for the inverse Fourier transform, given by the right-hand side of (5) with i in place of $-i$. Here $x\xi$ stands for the scalar product in $\mathbb{R}^n$. Both F and F^{-1} are extended to $S'(\mathbb{R}^n)$ in the standard way. Let $\varphi_0 \in S(\mathbb{R}^n)$ with

$$\varphi_0(x) = 1 \text{ if } |x| \le 1 \quad \text{and} \quad \varphi_0(y) = 0 \text{ if } |y| \ge 3/2, \tag{6}$$

and let

$$\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x), \qquad x \in \mathbb{R}^n, \qquad k \in \mathbb{N}. \tag{7}$$

Since

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1 \quad \text{for} \quad x \in \mathbb{R}^n, \tag{8}$$

the φ_j's form a dyadic resolution of unity. The entire analytic functions $\left(\varphi_j \hat{f}\right)^\vee(x)$ make sense pointwise in $\mathbb{R}^n$ for any $f \in S'(\mathbb{R}^n)$.

Definition (1.1.1) [74] Let $\varphi = \{\varphi_j\}_{j=0}^{\infty}$ be the above dyadic resolution of unity.

(i) Let

$$0 < p \le \infty, \qquad 0 < q \le \infty, \qquad s \in \mathbb{R}. \tag{9}$$

Then $B^s_{p,q}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\left\| f|B^s_{p,q}(\mathbb{R}^n) \right\|_\varphi = \left(\sum_{j=0}^{\infty} 2^{jsq} \left\| (\varphi_j \hat{f})^\vee | L_p(\mathbb{R}^n) \right\|^q \right)^{1/q} < \infty \tag{10}$$

(with the usual modification if $q = \infty$).

(ii) Let

$$0 < p < \infty, \qquad 0 < q \le \infty, \qquad s \in \mathbb{R}. \tag{11}$$

Then $F^s_{p,q}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\left\| f|F^s_{p,q}(\mathbb{R}^n) \right\|_\varphi = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} \left| (\varphi_j \hat{f})^\vee(\cdot) \right|^q \right)^{1/q} | L_p(\mathbb{R}^n) \right\| < \infty \tag{12}$$

(with the usual modification if $q = \infty$).

Remark (1.1.2) [74] The theory of these spaces may be found in [65, 66, 67]. The above spaces are independent of admitted resolutions of unity according to (5)–(8) (equivalent quasi-norms). This justifies our omission of the subscript φ (in (10), (12)) in what follows. We remind a few special cases and properties referring for details to the above, especially to [67, Section 1.2].

(i) Let $1 < p < \infty$. Then

$$L_p(\mathbb{R}^n) = F^0_{p,2}(\mathbb{R}^n) \tag{13}$$

is a well-known Littlewood–Paley theorem.

(ii) Let $1 < p < \infty$ and $k \in \mathbb{N}_0$.Then

$$W^k_p(\mathbb{R}^n) = F^k_{p,2}(\mathbb{R}^n) \tag{14}$$

are the classical Sobolev spaces usually equivalently normed by

$$\left\|f|W_p^k(\mathbb{R}^n)\right\| = \left(\sum_{|\alpha|\le k} \left\|\mathrm{D}^\alpha f|L_p(\mathbb{R}^n)\right\|^p\right)^{1/p}. \tag{15}$$

(iii) We denote by

$$C^s(\mathbb{R}^n) = B^s_{\infty,\infty}(\mathbb{R}^n), \qquad s \in \mathbb{R}, \tag{16}$$

the Hölder–Zygmund spaces. Let

$$(\Delta_h^1 f)(x) = f(x+h) - f(x), \qquad \left(\Delta_h^{l+1} f\right)(x) = \Delta_h^1\left(\Delta_h^l f\right)(x), \tag{17}$$

where $x \in \mathbb{R}^n, h \in \mathbb{R}^n, l \in \mathbb{N}$, be the iterated differences in $\mathbb{R}^n$. Let $0 < s < m \in \mathbb{N}$. Then

$$\|f|C^s(\mathbb{R}^n)\|_m = \sup_{x\in\mathbb{R}^n} |f(x)| + \sup|h|^{-s}\,|\Delta_h^m f(x)| \tag{18}$$

are equivalent norms in $C^s(\mathbb{R}^n)$ (for the continuous representatives), where the second supremum in (18) is taken over all $x \in \mathbb{R}^n$ and $h \in \mathbb{R}^n$ with $0 < |h| \le 1$.

(iv) This assertion can be generalized to the spaces $B^s_{p,q}(\mathbb{R}^n)$, with

$$0 < p, q \le \infty \ \text{ and } \ s > \sigma_p = n\left(\max\left(\frac{1}{p}, 1\right) - 1\right), \tag{19}$$

as follows. Let $s < k \in \mathbb{N}$. Then

$$\left\|f|B^s_{p,q}(\mathbb{R}^n)\right\|_k = \left\|f|L_p(\mathbb{R}^n)\right\| + \left(\int_0^1 t^{-sq} \sup_{|h|\le t} \left\|\Delta_h^k f|L_p(\mathbb{R}^n)\right\|^q \frac{dt}{t}\right)^{1/q} \tag{20}$$

and

$$\left\|f|B^s_{p,q}(\mathbb{R}^n)\right\|_k^* = \left\|f|L_p(\mathbb{R}^n)\right\| + \left(\int_{|h|\le 1} |h|^{-sq} \left\|\Delta_h^k f|L_p(\mathbb{R}^n)\right\|^q \frac{dh}{|h|^n}\right)^{1/q} \tag{21}$$

(with the natural modifications if $q = \infty$) are equivalent quasi-norms in $B^s_{p,q}(\mathbb{R}^n)$. See [65, Theorem 2.5.12, p. 110]. The spaces $B^s_{p,q}(\mathbb{R}^n)$ with $1 \le p, q \le \infty$ and $s > 0$ are the classical Besov spaces.

We deal with function spaces not only in the framework of distributions but also of Lebesgue-measurable functions in $\mathbb{R}^n$. Let $\mathbf{M}$ $(\mathbb{R}^n)$ be the collection of the equivalence classes of all almost everywhere finite complex-valued functions with respect to the Lebesgue measure in $\mathbb{R}^n$. This linear space, furnished with the convergence in measure, can be converted into a complete metric space. A short description may be found in [60, p. 19] see [43, Section I.5]. One may consider $\mathbf{M}$ $(\mathbb{R}^n)$ as the largest space covering everything that will be treated of measurable functions including the definition of the spaces $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$, $\mathbf{L}^s_p(\mathbb{R}^n)^k$, and the convergence of series. It is the substitute of $S'(\mathbb{R}^n)$ of the distributional spaces according above. But for our purpose it is sufficient to remark that the convergence in the quasi-Banach space $L_p(\mathbb{R}^n)$, $0 < p \le \infty$, is stronger than in $\mathbf{M}$ $(\mathbb{R}^n)$, [60, p. 19]. Let again $\Delta_h^k f(x) = \left(\Delta_h^k f\right)(x)$, with $k \in \mathbb{N}$, $x \in \mathbb{R}^n, h \in \mathbb{R}^n$, and $f \in \mathbf{M}$ $(\mathbb{R}^n)$, be the iterated differences as introduced in (17).

Definition (1.1.3) [74] (i) Let $0 < p, q \le \infty$ and $s > 0$. Let $k \in \mathbb{N}$ with $s < k$.Then $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ is the collection of all $f \in L_p(\mathbb{R}^n)$ (or likewise $f \in \mathbf{M}$ $(\mathbb{R}^n)$, such that

$$\left\|f|\mathbf{B}^s_{p,q}(\mathbb{R}^n)\right\|_k = \left\|f|L_p(\mathbb{R}^n)\right\| + \left(\int_0^1 t^{-sq} \sup_{|h|\le t} \left\|\Delta_h^k f|L_p(\mathbb{R}^n)\right\|^q \frac{dt}{t}\right)^{1/q} \tag{22}$$

is finite (with the usual modification if $q = \infty$).

(ii) Let $0 < p \le \infty$ and $s > 0$. Let $k \in \mathbb{N}$ with $s \le k$. Then $\mathbf{L}^s_p(\mathbb{R}^n)^k$ is the collection of all $f \in L_p(\mathbb{R}^n)$ (or likewise $f \in \mathbf{M}$ $(\mathbb{R}^n)$, for which there exists a function $g \in L_p(\mathbb{R}^n)$ with $g(x) \ge 0$ a. e. such that for all $h \in \mathbb{R}^n$ with $0 < |h| \le 1$,

$$|h|^{-s}\left|\Delta_h^k f(x)\right| \le \sum_{l=0}^{k} g(x+lh) \quad a.e.\ in\ \mathbb{R}^n. \tag{23}$$

Let

$$\left\|f|\mathbf{L}_p^s(\mathbb{R}^n)^k\right\| = \left\|f|L_p(\mathbb{R}^n)\right\| + \inf \left\|g|L_p(\mathbb{R}^n)\right\|, \tag{24}$$

where the infimum is taken over all g with (23).

(iii) Let $0 < p \le \infty$ and $k \in \mathbb{N}$. Then

$$\mathbf{W}_p^k(\mathbb{R}^n) = \mathbf{L}_p^k(\mathbb{R}^n)^k. \tag{25}$$

Remark (1.1.4) [74] The spaces $\mathbf{B}_{p,q}^s(\mathbb{R}^n)$ have some history, when $1 \le p, q \le \infty$, $s > 0$. The study for all admitteds, p and q goes back to [57], relevant comments and references may be found in [67, pp. 387–389]. See [1, Chapter 5, Definition 4.3] and [15, Chapter 2, Section 10]. An approach including atomic and quarkonial characterizations is given in [29, 31, 49], see also [52, 71]. $\mathbf{B}_{p,q}^s(\mathbb{R}^n)$ are quasi-Banach spaces which are independent of $k \in \mathbb{N}$ with $s < k$ (equivalent quasi-norms). The equivalence of (20) and (21) can be extended to the spaces $\mathbf{B}_{p,q}^s(\mathbb{R}^n)$ by the same arguments as in [65, Theorem 2.5.12, p. 110]. In other words, if $0 < p, q \le \infty$ and $0 < s < k \in \mathbb{N}$, then

$$\left\|f|\mathbf{B}_{p,q}^s(\mathbb{R}^n)\right\|_k^* = \left\|f|L_p(\mathbb{R}^n)\right\| + \left(\int_{|h|\le 1} |h|^{-sq} \left\|\Delta_h^k f|L_p(\mathbb{R}^n)\right\|^q \frac{dh}{|h|^n}\right)^{1/q} \tag{26}$$

is an equivalent quasi-norm in $\mathbf{B}_{p,q}^s(\mathbb{R}^n)$. By (19), (20) one has

$$B_{p,q}^s(\mathbb{R}^n) = \mathbf{B}_{p,q}^s(\mathbb{R}^n) \quad \text{if } 0 < p, q \le \infty, \qquad s > \sigma_p \tag{27}$$

(appropriately interpreted). Recall that all spaces in the above definition must be understood of $\mathbf{M}(\mathbb{R}^n)$, hence in terms of equivalence classes, especially

$$\mathbf{L}_p^s(\mathbb{R}^n)^k \hookrightarrow L_{\boldsymbol{p}}(\mathbb{R}^n) \hookrightarrow \mathbf{M}(\mathbb{R}^n). \tag{28}$$

This applies to (23) for any fixed $h \in \mathbb{R}^n$ with $0 < |h| \le 1$. We remark that $\mathbf{L}_p^s(\mathbb{R}^n)^k$ are quasi-Banach spaces. Related arguments may be found in [69, p. 73] which will not be repeated here. It is sufficient to restrict (1) to $x \in \mathbb{R}^n, y \in \mathbb{R}^n$ with $|x-y| \le 1$. Then it follows from the above definition that

$$\mathbf{W}_p^1(\mathbb{R}^n) = W_p^1(\mathbb{R}^n), \qquad 1 < p < \infty. \tag{29}$$

We complement the spaces $\mathbf{B}_{p,q}^s(\mathbb{R}^n)$ by the corresponding spaces $\mathbf{F}_{p,q}^s(\mathbb{R}^n)$.

Let A_0, A_1 be two complex quasi-Banach spaces with $A_1 \hookrightarrow A_0$ (continuous embedding). Let $0 < \theta < 1$ and $0 < q \le \infty$. Then $(A_0, A_1)_{\theta,q}$ are the usual real interpolation spaces, quasi-normed by

$$\left\|a|(A_0, A_1)_{\theta,q}\right\| = \left(\int_0^\infty t^{-\theta q} \inf_{\substack{a=a_0+a_1,\\ a_j \in A_j}} (\|a_0|A_0\| + t\|a_1|A_1\|)^q \frac{dt}{t}\right)^{1/q} \tag{30}$$

(with the usual modification if $q = \infty$). Basic information may be found in [2, 62]. Let $\mathrm{Lip}^k(\mathbb{R}^n)$ with $k \in \mathbb{N}$ be the Lipschitz spaces consisting of all $f \in L_\infty(\mathbb{R}^n)$ such that

$$\left\|f|\mathrm{Lip}^k(\mathbb{R}^n)\right\| = \sup_{x\in\mathbb{R}^n} |f(x)| + \sup |h|^{-k} \left|\Delta_h^k f(x)\right| \tag{31}$$

is finite, where the second supremum is taken over all $x \in \mathbb{R}^n$ and all $h \in \mathbb{R}^n$ with $0 < |h| \le 1$. See [69, p. 73] where one finds some discussion about these spaces and equivalent norms. Recall that the near by Hölder–Zygmund spaces $\mathcal{C}^k(\mathbb{R}^n)$ according to (16) can be normed by (18). As usual, $C^k(\mathbb{R}^n), k \in \mathbb{N}$, collects all functions f having bounded classical derivatives $\mathrm{D}^\alpha f$ with $|\alpha| \le k$, normed by

$$\|f|C^k(\mathbb{R}^n)\| = \sum_{|\alpha|\le k} \sup_{x\in\mathbb{R}^n} |\mathrm{D}^\alpha f(x)|. \tag{32}$$

It is well known that for $k \in \mathbb{N}$,

$$C^k(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}^k(\mathbb{R}^n) \hookrightarrow \mathcal{C}^k(\mathbb{R}^n) \tag{33}$$

and

$$C^k(\mathbb{R}^n) \neq \mathrm{Lip}^k(\mathbb{R}^n), \qquad \mathrm{Lip}^k(\mathbb{R}^n) \neq \mathcal{C}^k(\mathbb{R}^n). \tag{34}$$

One may consult [69, p. 75] and [68, pp. 170–172] as far as (34) with $k = 1$ is concerned. This can be extended to $k \in \mathbb{N}$. As before " $\hookrightarrow$ " indicates continuous embeddings.

Theorem (1.1.5) [74]

(i) Let $0 < s \le k \in \mathbb{N}$ and $0 < p \le \infty$. Then

$$\mathbf{B}^s_{p,\min(p,1)}(\mathbb{R}^n) \hookrightarrow \mathbf{L}^s_p(\mathbb{R}^n)^k \hookrightarrow \mathbf{B}^s_{p,\infty}(\mathbb{R}^n). \tag{35}$$

(ii) Let $0 < s \le k \in \mathbb{N}$, $0 < p \le \infty$, $0 < \theta < 1$ and $0 < q \le \infty$. Then

$$\left(L_p(\mathbb{R}^n), \mathbf{L}^s_p(\mathbb{R}^n)^k\right)_{\theta,q} = \mathbf{B}^{s\theta}_{p,q}(\mathbb{R}^n). \tag{36}$$

(iii) Let $k \in \mathbb{N}$ and $1 < p < \infty$. Then

$$\mathbf{W}^k_p(\mathbb{R}^n) = W^k_p(\mathbb{R}^n). \tag{37}$$

Furthermore,

$$\mathbf{W}^k_\infty(\mathbb{R}^n) = \mathrm{Lip}^k(\mathbb{R}^n) \hookrightarrow \mathcal{C}^k(\mathbb{R}^n), \qquad \mathrm{Lip}^k(\mathbb{R}^n) \neq \mathcal{C}^k(\mathbb{R}^n). \tag{38}$$

Remark (1.1.6) [74] This coincides essentially with the main theorem in [69], where one finds detailed proofs. The most complicated assertion of the above theorem is the left-hand side of (35). Its proof is based on subatomic decompositions of $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$. Afterwards one can reduce (36) to the reiteration theorem of interpolation theory and the remarkable formula

$$\left(L_p(\mathbb{R}^n), \mathbf{B}^s_{p,q_1}(\mathbb{R}^n)\right)_{\theta,q_2} = \mathbf{B}^{s\theta}_{p,q_2}(\mathbb{R}^n), \tag{39}$$

where $p, q_1, q_2 \in (0,\infty], s > 0$ and $0 < \theta < 1$. See [18, Theorem 6.3, p. 859] and the related comments in [68, pp. 373–374] and [69, p. 74].

Remark (1.1.7) [74] The assertion (37) extends (29), and hence (1), (2), from $k = 1$ to $k \in \mathbb{N}$. This will be used later on. One obtains (38) from (23)–(25) and (34). Recently Bojarski proved in [5] that (37) remains valid if one replaces (23) by

$$\left|\Delta^k_h f(x)\right| \le |x-y|^k\big(g(x)+g(y)\big)\ a.e., 0 \le g \in L_p(\mathbb{R}^n), 1 < p < \infty, k \in \mathbb{N}, \tag{40}$$

where $x \in \mathbb{R}^n, h \in \mathbb{R}^n$ and $y = x + kh$. This is the direct generalization of (1), (2).

The spaces $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ according to Definition (1.1.3) (i) are independent of $k \in \mathbb{N}$ with $s < k$. We have no assertion of this type for the spaces $\mathbf{L}^s_p(\mathbb{R}^n)^k$ introduced in Definition (1.1.3) (ii).

Problem (1.1.8) [74] Let $0 < p \le \infty$ and $s > 0$. The question arises whether the spaces $\mathbf{L}^s_p(\mathbb{R}^n)^k$ depend on $k \in \mathbb{N}$ if $s < k$.

Remark (1.1.9) [74] If $p = \infty$, then it follows from Definition (1.1.3) and from (16), (18) that

$$\mathbf{L}^s_\infty(\mathbb{R}^n)^k = \mathbf{B}^s_{\infty,\infty}(\mathbb{R}^n) = B^s_{\infty,\infty}(\mathbb{R}^n) = \mathcal{C}^s(\mathbb{R}^n), \qquad 0 < s < k \in \mathbb{N}. \tag{41}$$

It is not clear whether assertions of this type can be extended to $p < \infty$. Based on (23), (24) there is a temptation to ask whether $\mathbf{L}^s_p(\mathbb{R}^n)^k$ with $0 < p < \infty$ and $0 < s < k \in \mathbb{N}$ coincides with $\mathbf{B}^s_{p,\infty}(\mathbb{R}^n)$. But it will be seen below that in general this is not the case and that there is a more promising candidate.

We complement the spaces $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$. according to Definition (1.1.3) (i) by their F-counterparts. Let $f \in L_p(\mathbb{R}^n)$ with $0 < p < \infty$ and let $\left(\Delta^k_h f\right)(x)$ be the differences as introduced in (17). Let

$$\mathrm{d}^k_{t,p} f(x) = \left(t^{-n} \int_{|h|\le t} \left| (\Delta^k_h f)(x) \right|^p dh \right)^{1/p}, \qquad 0 < t < \infty, \qquad x \in \mathbb{R}^n, \tag{42}$$

be the related local means. Let $0 < p < \infty, 0 < q \le \infty$ and $0 < s < k \in \mathbb{N}$. Then $\mathbf{F}^s_{p,q}(\mathbb{R}^n)$. is the collection of all $f \in L_p(\mathbb{R}^n)$ (or likewise $f \in \mathbf{M}(\mathbb{R}^n)$) such that

$$\left\| f | \mathbf{F}^s_{p,q}(\mathbb{R}^n) \right\|_k = \left\| f | L_p(\mathbb{R}^n) \right\| + \left\| \left(\int_0^1 t^{-sq} \mathrm{d}^k_{t,p} f(\cdot)^q \frac{dt}{t} \right)^{1/q} | L_p(\mathbb{R}^n) \right\| \tag{43}$$

is finite, where

$$\left\| f | \mathbf{F}^s_{p,\infty}(\mathbb{R}^n) \right\|_k = \left\| f | L_p(\mathbb{R}^n) \right\| + \left\| \sup_{0<t<1} t^{-s} \mathrm{d}^k_{t,p} f(\cdot) | L_p(\mathbb{R}^n) \right\|. \tag{44}$$

The theory of these spaces has been developed in [67, Chapter 9], [53, 54], and complemented in [69]. They are independent of $k \in \mathbb{N}$ with $s < k$. Of interest for us are the embeddings

$$\mathbf{B}^s_{p,\min(p,q)}(\mathbb{R}^n) \hookrightarrow \mathbf{F}^s_{p,q}(\mathbb{R}^n) \hookrightarrow \mathbf{B}^s_{p,\max(p,q)}(\mathbb{R}^n). \tag{45}$$

for $s > 0, 0 < p < \infty, 0 < q \le \infty$, see [53] (extending the well-known result [65, Proposition 2.3.2/2] to spaces of type $\mathbf{B}^s_{p,q}$ and $\mathbf{F}^s_{p,q}$), and

$$\mathbf{F}^s_{p,\infty}(\mathbb{R}^n) = F^s_{p,\infty}(\mathbb{R}^n), \qquad 0 < p < \infty, \qquad s > \sigma_p. \tag{46}$$

Here $F^s_{p,\infty}(\mathbb{R}^n)$ are the distributional spaces according to Definition (1.1.1) (ii) and σ_p has the same meaning as in (19). $\mathbf{F}^s_{p,\infty}(\mathbb{R}^n)$ is smaller than $\mathbf{B}^s_{p,\infty}(\mathbb{R}^n)$According to [69, (4.7), p. 80], the right-hand side of (35) can be strengthened by

$$\mathbf{L}^s_p(\mathbb{R}^n)^k \hookrightarrow \mathbf{F}^s_{p,\infty}(\mathbb{R}^n), \qquad 0 < p < \infty, \qquad 0 < s < k \in \mathbb{N}. \tag{47}$$

This suggests complementing Problem (1.1.8) and Remark (1.1.9) as follows.

Problem (1.1.10) [74] Let $0 < p < \infty$ and $s > 0$. The question arises whether

$$\mathbf{L}^s_p(\mathbb{R}^n)^k = \mathbf{F}^s_{p,\infty}(\mathbb{R}^n) \text{ for all } k \in \mathbb{N} \text{ with } s < k. \tag{48}$$

Remark (1.1.11) [74] A first affirmative answer was given by Yang who proved in [72, Corollary 1.3, p. 686] that

$$\mathbf{L}^s_p(\mathbb{R}^n)^1 = \mathbf{F}^s_{p,\infty}(\mathbb{R}^n) = F^s_{p,\infty}(\mathbb{R}^n), \qquad 1 < p < \infty, \qquad 0 < s < 1, \tag{49}$$

where the second equality is covered by (46). In this context we mention also the remarkable observation in [38],

$$\mathbf{W}^1_p(\mathbb{R}^n) = H^1_p(\mathbb{R}^n) = F^1_{p,2}(\mathbb{R}^n), \qquad \frac{n}{n+1} < p < \infty \tag{50}$$

(Hardy–Sobolev spaces). Both (49) and (50) have been extended substantially in [39, 40].

We deal with necessary and sufficient conditions for the Sobolev-type embeddings

$$B^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_r(\mathbb{R}^n) \quad with \ s - \frac{n}{p} = -\frac{n}{r}, \tag{51}$$

where $0 < p, q \le \infty, 1 < r < \infty, \ s > 0$, employing (37), based on (23)–(25). First we fix an easy consequence of (25). Recall that we normed the classical Sobolev spaces $W^k_p(\mathbb{R}^n)$ with $1 < p < \infty, k \in \mathbb{N}$, according to (15).

Proposition (1.1.12) [74] Let $1 < p < \infty$ and $k \in \mathbb{N}$. Then

(i) the norms

$$\left\| f | W^k_p(\mathbb{R}^n) \right\| \sim \left\| f | L_p(\mathbb{R}^n) \right\| + \sup_{0<|h|\le 1} |h|^{-k} \left\| \Delta^k_h f | L_p(\mathbb{R}^n) \right\| \tag{52}$$

are equivalent in $W^k_p(\mathbb{R}^n)$;

(ii) the seminorms

$$\sum_{|\alpha|=k} \left\| \mathrm{D}^\alpha f | L_p(\mathbb{R}^n) \right\| \sim \sup_{h \in \mathbb{R}^n \setminus \{0\}} |h|^{-k} \left\| \Delta^k_h f | L_p(\mathbb{R}^n) \right\| \tag{53}$$

are equivalent in $W_p^k(\mathbb{R}^n)$;

Theorem (1.1.13) [74] Let $0 < p < \infty, 0 < q \le \infty, 1 < r < \infty$ and $s > 0$ with $s - (n/p) = -n/r$. The following assertions are equivalent.

(i) $B_{p,q}^s(\mathbb{R}^n) \hookrightarrow L_r(\mathbb{R}^n)$.

(ii) $0 < q \le r$.

(iii) For all $k \in \mathbb{N}_0$, $K \in \mathbb{N}$ with $s + k < K$ there is a constant $c = c_{k,K} > 0$ such that

$$\|f|L_r(\mathbb{R}^n)\| + \sup_{0<|h|\le 1} |h|^{-k} \left\|\Delta_h^k f|L_r(\mathbb{R}^n)\right\|$$
$$\le c\left\|f|L_p(\mathbb{R}^n)\right\| + c\left(\int_{0<|h|\le 1} |h|^{-(s+k)q} \left\|\Delta_h^K f|L_p(\mathbb{R}^n)\right\|^q \frac{dh}{|h|^n}\right)^{1/q} \tag{54}$$

for all $f \in \mathbf{M}(\mathbb{R}^n)$ (with the usual modification if $q = \infty$).

(iv) For all $k \in \mathbb{N}_0$, $K \in \mathbb{N}$ with $s + k < K$ there is a constant $C = C_{k,K} > 0$ such that

$$\sup_{|h|>0} |h|^{-k} \left\|\Delta_h^k \varphi|L_r(\mathbb{R}^n)\right\| \le C\left(\int_{|h|>0} |h|^{-(s+k)q} \left\|\Delta_h^K \varphi|L_p(\mathbb{R}^n)\right\|^q \frac{dh}{|h|^n}\right)^{1/q} \tag{55}$$

for all $\varphi \in D(\mathbb{R}^n)$ (with the usual modification if $q = \infty$).

Remark (1.1.14) [74] This is the main assertion of [30]. The equivalence of (i) and (ii) is well known and covered, for instance, by [68, Theorem 11.4, p. 170], based on [55]. Corresponding assertion for $\mathbf{B}_{p,q}^s(\mathbb{R}^n)$ can be found in [29] and, in case of $1 \le p, q \le \infty$ and $s > 0$, in [1, Chapter 5, Theorem 4.6, Corollaries 4.20, 4.21], [24] and [36]. Roughly speaking, one lifts (i) to the level of Sobolev spaces, hence

$$B_{p,q}^{s+k}(\mathbb{R}^n) \hookrightarrow W_r^k(\mathbb{R}^n), \qquad k \in \mathbb{N}, \tag{56}$$

and applies Proposition (1.1.12) (i). But the details require some effort. Part (iv) is a homogeneity assertion similarly as (53).

In the homogeneous case (and dealing with generalized moduli of smoothness) there are related results in [59, Theorem 2.4]. Moreover, assertions of type (iii) and (iv) can also be regarded as inequalities of Ul'yanov type referring to the first observation of this kind [70]; for more recent works (in the periodic case) see [19, 58].

Dealing with spaces of generalized smoothness, sharp (limiting) embeddings were studied in some detail in [12, 14] with forerunners in [11, 13, 47]; see also [42, Theorem D.4.1.7]. The most general result may be found in [27], but this only concerns criteria in the sense of (i) and (ii). Though characterizations of such spaces by differences, the full counterpart of Theorem (1.1.13) as presented above has apparently not yet been obtained.

We describe two further applications of the above considerations. we deal with limits of Besov norms and in the following we deal with some related aspects of Gagliardo–Nirenberg inequalities.

Let $1 < p < \infty$. Then it had been observed in [9, 10] that there is some constant $c > 0$ such that

$$\lim_{s\uparrow 1}(1 - s)\int_{\mathbb{R}^{2n}} \frac{|f(x) - f(y)|^p}{|x - y|^{sp+n}} dxdy = c\int_{\mathbb{R}^n} |\nabla f(x)|^p \, dx \tag{57}$$

for all $f \in D(\mathbb{R}^n)$. Limiting assertions of this type attracted afterwards some attention. A few references will be given later on. Otherwise we follow [63] where we applied the above theory to problems of this type. Recall that the spaces $\mathbf{B}_{p,q}^s(\mathbb{R}^n)$ can be quasi-normed by (22) and (26), in analogy to (20), (21). Let $\mathbf{W}_p^k(\mathbb{R}^n)$ be the Sobolev spaces according to (25) with (37) if $1 < p < \infty$. Let

$$\omega_k(f, t)_p = \sup_{|h|\le t}\left\|\Delta_h^k f|L_p(\mathbb{R}^n)\right\|, \qquad 0 < p \le \infty, \qquad t > 0, \qquad k \in \mathbb{N}, \tag{58}$$

be the usual moduli of continuity.

Proposition (1.1.15) [74] (i) Let $0 < p \le \infty$ and $k \in \mathbb{N}$. Then there is a constant $c > 0$ such that for all s with $0 < s < k$ and all q with $0 < q \le \infty$,

$$\left\|f|\mathbf{B}_{p,q}^{s}(\mathbb{R}^n)\right\|_k \le c\big(q(k-s)\big)^{-1/q}\left\|f|\mathbf{W}_p^k(\mathbb{R}^n)\right\| \tag{59}$$

(with 1 in place of $(q(k-s))^{-1/q}$ $if\ q = \infty$).

(ii) Let $1 \le p < \infty$ and $k \in \mathbb{N}$. Let $f \in \mathbf{W}_p^k(\mathbb{R}^n)$. Then $t^{-k}\omega_k(f,t)_p$ is continuous on the interval (0,1] and can be extended continuously to the closed interval [0,1] with

$$\sup_{t>0} t^{-k}\omega_k(f,t)_p = \sup_{0<t\le 1} t^{-k}\omega_k(f,t)_p = \lim_{t\downarrow 0} t^{-k}\omega_k(f,t)_p. \tag{60}$$

Recall again that $\left\|f|\mathbf{B}_{p,q}^{s}(\mathbb{R}^n)\right\|_k$ is given by (22).

Theorem (1.1.16) [74] (i) Let $1 \le p < \infty, 0 < q < \infty$ and $k \in N$. Let $f \in \mathbf{W}_p^k(\mathbb{R}^n)$. Then

$$\lim_{s\uparrow k}(k-s)^{1/q}\left\|f|\mathbf{B}_{p,q}^{s}(\mathbb{R}^n)\right\|_k = q^{-1/q}\lim_{t\downarrow 0} t^{-k}\omega_k(f,t)_p. \tag{61}$$

(ii) Let $1 < p < \infty, 0 < q < \infty$ and $k \in \mathbb{N}$. Then there are positive equivalence constants which are independent of q and $f \in \mathbf{W}_p^k(\mathbb{R}^n)$. (but may depend on p, k) such that

$$\lim_{s\uparrow k}(k-s)^{1/q}\left\|f|\mathbf{B}_{p,q}^{s}(\mathbb{R}^n)\right\|_k \sim q^{-1/q}\sum_{|\alpha|=k}\left\|\mathrm{D}^\alpha f|L_p(\mathbb{R}^n)\right\|. \tag{62}$$

Remark (1.1.17) [74] We refer to [63]. From

$$\lim_{s\uparrow k}(k-s)\left\|f|L_p(\mathbb{R}^n)\right\| + \lim_{s\uparrow k}(k-s)\int_1^\infty t^{-sq}\omega_k^q(f,t)_p\frac{dt}{t} = 0 \tag{63}$$

it follows that one can replace (61) by the more handsome homogeneous version

$$\lim_{s\uparrow k}(k-s)^{1/q}\left(\int_0^\infty t^{-sq}\omega_k^q(f,t)_p\frac{dt}{t}\right)^{1/q} = q^{-1/q}\lim_{t\downarrow 0} t^{-k}\omega_k(f,t)_p. \tag{64}$$

This is also the basis to prove (62) which can be rewritten as

$$\lim_{s\uparrow k}(k-s)^{1/q}\left(\int_0^\infty t^{-sq}\omega_k^q(f,t)_p\frac{dt}{t}\right)^{1/q} \sim q^{-1/q}\sum_{|\alpha|=k}\left\|\mathrm{D}^\alpha f|L_p(\mathbb{R}^n)\right\|. \tag{65}$$

Gagliardo–Nirenberg inequalities go back to [22, 50]. In 1959, Nirenberg proved in [50, Theorem, p. 125] that for $1 \le u, p \le \infty, k \in \mathbb{N}_0$ and $m \in \mathbb{N}$ with $k < m$,

$$\sum_{|\beta|=k}\left\|\mathrm{D}^\beta f|L_v(\mathbb{R}^n)\right\| \le c\|f|L_u(\mathbb{R}^n)\|^{1-\theta}\left(\sum_{|\alpha|=m}\left\|\mathrm{D}^\alpha f|L_p(\mathbb{R}^n)\right\|\right)^\theta \tag{66}$$

for smooth functions f in $\mathbb{R}^n$ with compact support, where

$$k - \frac{n}{v} = -(1-\theta)\frac{n}{u} + \theta\left(m - \frac{n}{p}\right), \qquad \frac{k}{m} \le \theta \le 1 \tag{67}$$

(with some additional conditions in limiting cases). Here $-\infty < 1/v < \infty$ where $L_v(\mathbb{R}^n)$ with $s = -n/v > 0$ refers to the Hölder spaces $C^s(\mathbb{R}^n)$ (an ingenious notation but not in common use nowadays). In the same year, 1959, Gagliardo published in [22] inequalities which are equivalent to Nirenberg's observation, but formulated differently. The assertion (66) with (67) is dimension balanced (differential dimensions on both sides of (67)). This suggests formulating assertions of type (66) preferably in terms of homogeneous (semi-) norms. Gagliardo–Nirenberg inequalities, sometimes also called refined Sobolev embeddings, attracted a lot of attention up to our time. We contributed to this topic in [61] and [64, Chapter 4]. We formulate essentially only one assertion which is directly related to (66), (67) and to the above considerations, especially (53). Recall that the Hölder–Zygmund spaces $C^s(\mathbb{R}^n),\ s > 0$, can be normed according to (18).

Theorem (1.1.18) [74] Let $1 < p < \infty,\ 1 \le u \le \infty,\ m \in \mathbb{N}$ and

$$m - \frac{n}{p} > \sigma = -(1-\theta)\frac{n}{u} + \theta\left(m - \frac{n}{p}\right) > 0. \tag{68}$$

Then there is a constant $c > 0$ such that for all $f \in W_p^m(\mathbb{R}^n) \cap L_u(\mathbb{R}^n)$,

$$\sup_{\substack{h\in\mathbb{R}^n\backslash\{0\},\\ x\in\mathbb{R}^n}} |h|^{-\sigma}|\Delta_h^m f(x)| \le c\|f|L_u(\mathbb{R}^n)\|^{1-\theta}\left(\sum_{|\alpha|=m}\left\|\mathrm{D}^\alpha f|L_p(\mathbb{R}^n)\right\|\right)^\theta$$

$$\sim \|f|L_u(\mathbb{R}^n)\|^{1-\theta}\sup_{h\in\mathbb{R}^n\backslash\{0\}}|h|^{-\theta m}\ \left\|\Delta_h^m f|L_p(\mathbb{R}^n)\right\|^\theta. \tag{69}$$

From

$$0 < \theta = \frac{\sigma + \frac{n}{u}}{m - \frac{n}{p} + \frac{n}{u}} < 1 \tag{70}$$

and

$$\theta m = \sigma + \frac{n}{u} + \theta\left(\frac{n}{p} - \frac{n}{u}\right) = \sigma + \theta\frac{n}{p} + (1-\theta)\frac{n}{u} > \sigma \tag{71}$$

it follows that (69) fits in the usual scheme of Gagliardo–Nirenberg inequalities. The equivalence in (69) comes form (53). Otherwise the above theorem coincides essentially with [61, Remark 3.7] where it is a comment on Gagliardo–Nirenberg inequalities. See [35, 41, 51].

Section (1.2) Triebel-Lizorkin Spaces on Metric Measure Spaces

For (X, d) a metric space and μ be a regular Borel measure on X such that all balls defined by d have finite and positive measures, and assume that μ satisfies a doubling property: there exist constants $C_1 > 1$ and $n > 0$ such that for all $x \in X, r \in (0,\infty)$ and $\lambda \in (1,\infty)$,

$$\mu(B(x,\lambda r)) \le C_1\lambda^n\mu(B(x,r)).$$

The following definition of Besov spaces from [23].

Definition (1.2.1) [95] Let $s \in (0,\infty)$ and $p, q \in (0,\infty]$. The homogeneous Besov space $\dot{B}^s_{p,q}(\mathrm{X})$ is defined to be the collection of all $u \in L^p_{loc}(\mathrm{X})$ such that

$$\|u\|_{\dot{B}^s_{p,q}(\mathrm{X})} \equiv \left(\int_0^\infty\left(\int_X \fint_{B(x,t)}|u(x)-u(y)|^p\,d\mu(y)d\mu(x)\right)^{q/p}\frac{dt}{t^{1+sq}}\right)^{1/q} < \infty$$

with the usual modification made when $p = \infty$ or $q = \infty$.

Above, $u \in L^p_{loc}(X)$ requires that $u \in L^p(B)$ for each ball B.

Observe that functions in $\dot{B}^s_{p,q}(\mathrm{X})$ have the smoothness of order s as measured by

$$t^{-s}\left(\fint_{B(x,t)}|u(x)-u(y)|^p\,d\mu(y)\right)^{1/p}$$

Recall that, there are several ways to measure the smoothness of functions. For example, letting $s \in [0,\infty), \epsilon \in [0,s]$ and $\sigma \in (0,\infty)$, for all measurable functions u, set

$$C_t^{s,\sigma}(u)(x) \equiv t^{-s}\left(\fint_{B(x,t)}|u(x)-u(y)|^\sigma\,d\mu(y)\right)^{\frac{1}{\sigma}},$$

$$A_t^{s,\sigma}(u)(x) \equiv t^{-s}\left(\fint_{B(x,t)} \left|u(x) - u_{B(x,t)}\right|^{\sigma} d\mu(y)\right)^{\frac{1}{\sigma}},$$

$$I_t^{s,\sigma}(u)(x) \equiv t^{-s}\left(\inf_{c\in R} \fint_{B(x,t)} |u(x) - c|^{\sigma} d\mu(y)\right)^{\frac{1}{\sigma}},$$

$$S_t^{s,\epsilon,\sigma}(u)(x) \equiv t^{(\epsilon-s)} \sup_{r\in(0,t)} r^{-\epsilon}\left(\inf_{c\in R} \fint_{B(x,t)} |u(x) - c|^{\sigma} d\mu(y)\right)^{1/\sigma}.$$

for all $x \in X$ and $t \in (0, \infty)$.

We show that the smoothness of functions in Besov, spaces can be measured by the above quantatives with optimal parameters. We introduce the following spaces of Besov type. In what follows, we denote by $\vec{C}^{s,\sigma}$ the operator that maps each $u \in L^{\sigma}_{loc}(x)$ into a measurable function $\vec{C}^{s,\sigma}(u)$ on $X \times (0, \infty)$ defined by $\vec{C}^{s,\sigma}\ (u)(x,t) \equiv C_t^{s,\sigma}\ (u)(x)$ for all $x \in X$ and $t \in (0, \infty)$. We define $\vec{A}^{s,\sigma}$, $\vec{I}^{s,\sigma}$ and $\vec{S}^{s,\epsilon,\sigma}$.

Definition (1.2.2) [95] Let $s, \sigma \in (0, \infty), \epsilon \in [0, s]$ and $p, q \in (0, \infty]$. For $\vec{E} = \vec{C}^{s,\sigma}, \vec{A}^{s,\sigma}, \vec{I}^{s,\sigma}$ or $\vec{S}^{s,\epsilon,\sigma}$ the homogeneous space $\vec{E}\dot{B}_{p,q}(x)$ of Besov type is defined to be the collection of all $u \in L^{\sigma}_{loc}(x)$ such that

$$\|u\|_{\vec{E}\dot{B}_{p,q}(x)} \ \equiv \ \left(\int_0^{\infty} \left\|\vec{E}(u)(\,.\,,t)\right\|^q_{L^p(x)} \frac{dt}{t}\right)^{1/q} < \infty$$

with the usual modification made when $p = \infty$ or $q = \infty$.

For our convenience, for $s \in (0, \infty)$ and $p \in (0, \infty]$, we always set

$$p_*(s) \equiv \begin{cases} np/(n-ps), & if\ \ p < n/s\,; \\ \infty\,, & if\ \ p \geq n/s\,. \end{cases} \tag{72}$$

Definition (1.2.3) [95] Let $s \in (0, \infty)$ and let u be a measurable function on X. A sequence of nonnegative measurable functions, $\vec{g} \equiv \{g_k\}_{k\in\mathbb{Z}}$, is called a fractional s-Hajłasz gradient of u if there exists $E \subset X$ with $\mu(E) = 0$ such that for all $k \in \mathbb{Z}$ and $x, y \in X\backslash E$ satisfying $2^{-k-1} \leq d(x, y) < 2^{-k}$,

$$|u(x) - u(y)| \leq [d(x, y)]^s[g_k(x) + g_k(y)].$$

Denote by $\mathbb{D}^s(u)$ the collection of all fractional s-Hajłasz gradients of u.

In fact, $\vec{g} \equiv \{g_k\}_{k\in\mathbb{Z}}$, above is not really a gradient. One should view it, in the Euclidean setting (at least when $g_k = g_j$ for all k, j), as a maximal function of the usual gradient.

The characterizes the Besov spaces in Definition (1.2.1) via the fractional Hajłasz gradient. In what follows, for $p, q \in (0, \infty]$ and a sequence $\vec{g} \equiv \{g_k\}_{k\in\mathbb{Z}}$ of nonnegative functions, we always write $\left\|\{g_j\}_{j\in\mathbb{Z}}\right\|_{l^q} \equiv \left\{\sum_{j\in z} |g_j|^q\right\}^{1/q}$ when $q < \infty$ and $\left\|\{g_j\}_{j\in\mathbb{Z}}\right\|_{l^\infty} \equiv \sup_{j\in\mathbb{Z}} |g_j|$, $\left\|\{g_j\}_{j\in\mathbb{Z}}\right\|_{l^q(L^p(x))} \equiv \left\|\left\{\|g_j\|_{L^p(x)}\right\}_{j\in\mathbb{Z}}\right\|_{l^q}$.

Definition (1.2.4) [95] Let $s \in (0, \infty)$ and $p, q \in (0, \infty]$. The homogeneous Hajłasz-Besov space $\dot{N}^s_{p,q}(x)$ is the space of all measurable functions u such that

$$\|u\|_{\dot{N}^s_{p,q}(x)} \equiv \inf_{\vec{g}\in\mathbb{D}^s(u)} \|\vec{g}\|_{l^q(L^p(x))} < \infty.$$

Theorem (1.2.6) and (i) through (iv) of Theorem (1.2.5) follow from Theorem (1.2.12) below, whose proof relies on an inequality of Poincaré type established in Lemma (1.2.7) and a pointwise inequality given by Lemma (1.2.10). The proof of (v) through (vii) of Theorem (1.2.5) will be given at the end.

We state the corresponding results for Triebel-Lizorkin spaces (see Theorem (1.2.16)). As a special case, we also establish the equivalence between Hajłasz-Sobolev spaces and the Sobolev type spaces of Calderón and DeVore-Sharpley (see Corollary (1.2.18)).

Applying the above characterizations, we prove the triviality of Besov and Triebel-Lizorkin spaces under a suitable Poincaré inequality (see Theorem (1.2.19) and Theorem (1.2.20)), and also give some examples of nontrivial Besov and Triebel-Lizorkin spaces to show the "necessity" of such a Poincaré inequality (see Theorem (1.2.21)).

The notation $A \lesssim B$ or $B \gtrsim A$ means that $A \le CB$. If $A \gtrsim B$ and $B \lesssim A$, we then write $A \sim B$. For two spaces X and Y endowed with (semi-) norms, the notation $X \subset Y$ means that $u \in X$ implies that $u \in Y$ and $\|u\|_Y \lesssim \|u\|_X$, and the notation $X = Y$ means that $X \subset Y$ and $Y \subset X$. Denote by $\mathbb{Z}$ the set of integers and $\mathbb{N}$ the set of positive integers. For any locally integrable function f, we denote by $\fint_E f\, d\mu$ the average of f on E, namely, $\fint_E f\, d\mu \equiv \frac{1}{\mu(E)} \int_E f\, d\mu$.

Theorem (1.2.5) [95] Let $s \in (0,\infty)$ and $p, q \in (0,\infty]$.

(i) If $\sigma \in (0,p]$, then $\dot{B}^s_{p,q}(x) = \vec{C}^{s,\sigma}\dot{B}^s_{p,q}(x)$.

(ii) If $\sigma \in (0,p_*(s))$, then $\dot{B}^s_{p,q}(x) = \vec{I}^{s,\sigma}\dot{B}^s_{p,q}(x)$.

(iii) If $\epsilon \in [0,s)$ and $\sigma \in (0,p_*(s))$, then $\dot{B}^s_{p,q}(x) = \vec{S}^{s,\epsilon,\sigma}\dot{B}^s_{p,q}(x)$.

(iv) If $p \in (n/(n+s),\infty]$ and $\sigma \in (0,p_*(s))$, then $\dot{B}^s_{p,q}(x) = \vec{A}^{s,\sigma}\dot{B}^s_{p,q}(x)$.

Moreover, the ranges of ϵ and σ above are optimal in the following sense.

(v) Let $s \in (0,1), p \in (0,n/s)$ and $\sigma > p_*(s)$. Then there exists a function u such that for all $q \in (0,\infty], u \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ but $u \notin L^\sigma_{loc}(\mathbb{R}^n)$, and hence, for $\vec{E} = \vec{A}^{s,\sigma}, \vec{I}^{s,\sigma}$ or $\vec{S}^{s,\epsilon,\sigma}, u \notin \vec{E}\dot{B}^s_{p,q}(\mathbb{R}^n)$.

(vi) Let $p \in (0,\infty), \sigma \in (p,\infty)$ and $s \in (0,n/p - n/\sigma) \cap (0,1)$. Then there exists a function u such that for all $q \in (0,\infty], u \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ but $u \notin \vec{C}^{s,\sigma}\dot{B}_{p,q}(\mathbb{R}^n)$.

(vii) Let $s \in (0,1)$ and $p \in (0,\infty)$. Then there exists a function $u \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ with $u \notin \vec{S}^{s,s,p}\dot{B}_{p,p}(\mathbb{R}^n)$.

It is natural and necessary to consider the full range of s due to the nontrivial example of nontrivial Besov spaces $\dot{B}^s_{n/s,\,n/s}(x)$ for all $s \in (0,\infty)$ given by Theorem (1.2.21).

A fractional pointwise gradient was introduced in [40] to measure the smoothness of functions.

Theorem (1.2.6) [95] Let $s \in (0,\infty)$ and $p, q \in (0,\infty]$. Then $\dot{N}^s_{p,q}(x) = \dot{B}^s_{p,q}(x)$.

Under the additional assumptions that μ also satisfies a reverse doubling condition, $0 < s < 1$ and $p > n/(n+1)$, $\dot{B}^s_{p,q}(x)$ also allows for a kernel function characterization [90].

Proofs of Theorem (1.2.5) and Theorem (1.2.6)

We begin with a Poincaré type inequality.

Lemma (1.2.7) [95] Let $s \in (0,\infty)$ and $p \in (0,n/s)$. Then for every pair of $\epsilon, \epsilon' \in (0,s)$ with $\epsilon < \epsilon'$, there exists a positive constant C such that for all $x \in \mathbb{R}^n, k \in \mathbb{Z}$, measurable functions u and $\vec{g} \in \mathbb{D}^s(u)$,

$$\inf_{c\in R}\left(\fint_{B(x,2^{-k})}|u(y)-c|^{p_*(\epsilon)}d\mu(y)\right)^{1/p_*(\epsilon)} \le C2^{-k\epsilon'}\sum_{j\ge k-2}2^{-j(s-\epsilon')}\left\{\fint_{B(x,2^{-k+1})}[g_j(y)]^p d\mu(y)\right\}^{1/p}$$

where $p_*(\epsilon)$ is as in (72).

Recall that when $s \in (0,1]$ and $X = \mathbb{R}^n$, Lemma (1.2.7) was established in [40, Lemma 2.3]. Generally, Lemma (1.2.7) can be proved by an argument similar to that of [40, Lemma 2.3] with the aid of the following variant of [85, Theorem 8.7]. In what follows, for every $s \in (0,\infty)$ and measurable function u on X, a non-negative function g is called an s gradient of u if there exists a set $E \subset X$ with $\mu(E) = 0$ such that for all $x, y \in X \setminus E$,

$$|u(x) - u(y)| \leq [d(x,y)]^s [g(x) + g(y)]. \quad (73)$$

Denote by $\mathcal{D}^s(u)$ the collection of all s-gradients of u.

Lemma (1.2.8) [95] Let $s \in (0,\infty), p \in (0, n/s)$ and let $p_*(s)$ be as in (72). Then there exists a positive constant C such that for all $x \in X, r \in (0,\infty)$, and all measurable functions u and $g \in \mathcal{D}^s(u)$,

$$\inf_{c\in R}\left(\fint_{B(x,r)} |u(y) - c|^{p_*(s)} d\mu(y)\right)^{1/p_*(s)} \leq Cr^s \left\{\fint_{B(x,2r)} [g_j(y)]^p d\mu(y)\right\}^{1/p}.$$

When $s \in (0,1]$, since d^s is also a distance on X, Lemma (1.2.8) follows from [85, Theorem 8.7]. When $s \in (1,\infty)$, with $p < n/s$ in mind, checking the proof of [85, Theorem 8.7] line by line, we still have Lemma (1.2.8).

We still need the following pointwise inequality, which is a variant of the pointwise inequality established in [40, (5.7)].

Lemma (1.2.9) [95] For every real-valued measurable function u, there exists a measurable set $E \subset X$ with $\mu(E) = 0$ such that for all $z \in X \backslash E$,

$$u(z) = \lim_{\mu(B)\to 0, B\ni z} m_u(B).$$

Lemma (1.2.9) was proved in [79, Lemma 2.2] for $X = \mathbb{R}^n$, and the very same argument gives Lemma (1.2.9).

Lemma (1.2.10) [95] Let $\sigma \in (0,\infty)$. Then there exists a positive constant C such that, for each function $u \in L^\sigma_{loc}(X)$, one can find a set E with $\mu(E) = 0$ so that for each pair of points $x, y \in X\backslash E$ with $d(x,y) \in [2^{-k-1}, 2^{-k})$,

$$|u(x) - u(y)| \leq C \sum_{j\geq k-2}\left\{\inf_{c\in\mathbb{R}}\left[\left(\fint_{B(x,2^{-j})} |u(w) - c|^\sigma dw\right)\right]^{1/\sigma} + \inf_{c\in\mathbb{R}}\left[\fint_{B(y,2^{-j})} |u(w) - c|^\sigma dw\right]^{1/\sigma}\right\}. \quad (74)$$

To prove Lemma (1.2.10), we need Lemma (1.2.9) above. In what follows, for a real valued measurable function u and a ball B, define the median value of u on B by

$$m_u(B) \equiv \max\left\{a \in \mathbb{R}, \mu(\{x \in B: u(x) < a\}) \leq \frac{\mu(B)}{2}\right\}. \quad (75)$$

Proof. Let u be a real-valued measurable function and E be the set given by Lemma (1.2.9). Then for all $z \in X\backslash E$, by Lemma (1.2.9), $m_u(B(z, 2^{-j})) \to u(z)\ as\ j \to \infty$, and hence

$$\left|u(z) - m_u(B(z,2^{-k}))\right| \leq \sum_{j\geq k}\left|m_u(B(z,2^{-j})) - m_u(B(z,2^{-j-1}))\right|$$
$$\leq \sum_{j\geq k}\left[\left|m_u(B(z,2^{-j})) - c_{B(z.2^{-j})}\right| + \left|m_u(B(z,2^{-j-1})) - c_{B(z.2^{-j})}\right|\right],$$

where $c_{B(z.2^{-j})}$ is a real number such that

$$\fint_{B(z,2^{-j})} \left|u(w) - c_{B(z.2^{-j})}\right|^\sigma d\mu(w) \leq 2\inf_{c\in\mathbb{R}} \fint_{B(z.2^{-j})} |u(w) - c|^\sigma d\mu(w).$$

We claim that for every ball B and each $c \in \mathbb{R}$

$$|m_u(B)-c| \le \left\{2 \fint_B |u(w)-c|^\sigma d\mu(w)\right\}^{1/\sigma}. \tag{76}$$

Assume that this claim holds for a moment. We have

$$\left|m_u\left(B\left(z,2^{-j}\right)\right)-c_{B\left(z.2^{-j}\right)}\right| \le \left\{2 \fint_{B\left(z.2^{-j}\right)} \left|u(w)-c_{B\left(z.2^{-j}\right)}\right|^\sigma d\mu(w)\right\}^{1/\sigma}$$

$$\lesssim \inf_{c\in\mathbb{R}} \left\{\fint_{B\left(z,2^{-j}\right)} |u(w)-c|^\sigma d\mu(w)\right\}^{1/\sigma} \tag{77}$$

and

$$\left|m_u\left(B\left(z,2^{-j-1}\right)\right)-c_{B\left(z.2^{-j}\right)}\right| \le \left\{2 \fint_{B\left(z.2^{-j-1}\right)} \left|u(w)-c_{B\left(z.2^{-j}\right)}\right|^\sigma dw\right\}^{1/\sigma}$$

$$\lesssim \left\{\fint_{B\left(z,2^{-j}\right)} \left|u(w)-c_{B\left(z,2^{-j}\right)}\right|^\sigma dw\right\}^{1/\sigma} \lesssim \inf_{c\in\mathbb{R}} \left\{\fint_{B\left(z,2^{-j}\right)} |u(w)-c|^\sigma dw\right\}^{1/\sigma} \tag{78}$$

Therefore,

$$\left|u(z)-m_u(B(z,2^{-k}))\right| \le \sum_{j\ge k} \inf_{c\in\mathbb{R}} \left\{\fint_{B\left(z,r^j\right)} |u(w)-c|^\sigma d\mu(w)\right\}^{1/\sigma}. \tag{79}$$

For $x,y\in X\backslash E$ with $2^{-k-1}\le d(x,y)<2^{-k}$, we write

$$|u(x)-u(y)| \le \left|u(x)-m_u\left(B(y,2^{-k})\right)\right| + \left|m_u\left(B(x,2^{-k+1})\right)-c_{B\left(x,2^{-k+1}\right)}\right| + \left|c_{B(x,2^{-k+1})}-m_u(B(y,2^{-k}))\right| + \left|u(y)-m_u(B(y,2^{-k}))\right|.$$

By an argument similar to that of (78), we have

$$\left|c_{B(x,2^{-k+1})}-m_u(B(y,2^{-k}))\right| \lesssim \inf_{c\in\mathbb{R}} \left\{\fint_{B\left(x,2^{-k+1}\right)} |u(w)-c|^\sigma dw\right\}^{1/\sigma},$$

which together with (79) and (77) gives (74).

Now we prove the claim (76). For every ball B and each $c\in\mathbb{R}$, observing that $m_{u-c}(B) = m_u(B)-c$ and recalling that $|m_u(B)|\le m_{|u|}(B)$ as proved in [79, Lemma 2.1], we have $|m_u(B)-c\,| \le m_{|u-c|}(B)$. By this, (76) is reduced to

$$m_{|u-c|}(B) \le \left\{2 \fint_B |u(w)-c|^\sigma d\mu(w)\right\}^{1/\sigma}. \tag{80}$$

To see this, letting $\delta\equiv \fint_B |u(w)-c|^\sigma dw$, by Chebyshev's inequality, for every $a>2$, we have

$$\mu\left(\left\{w\in B: |u(w)-c|\ge (a\delta)^{1/\sigma}\right\}\right) = \mu(\{w\in B: |u(w)-c|^\sigma \ge a\delta\})$$

$$\le (a\delta)^{-1} \int_B |u(w)-c|^\sigma dw < \frac{\mu(B)}{2},$$

which yields that

$$\mu(\{w \in B: |u(w) - c| < (a\delta)^{1/\sigma}\}) > \frac{\mu(B)}{2}$$

and hence by (75), $m_{|u-c|}(B) \leq (a\delta)^{1/\sigma}$. Then letting $a \to 2$, we obtain (80) and hence prove the claim (76). This finishes the proof of Lemma (1.2.10).

We also use the following lemma.

Lemma (1.2.11) [95] Let $s, \sigma \in (0, \infty), \epsilon \in [0, s]$ and $p, q \in (0, \infty]$. Let $\vec{E} = \vec{C}^{s,\sigma}, \vec{A}^{s,\sigma}, \vec{I}^{s,\sigma}$ or $\vec{S}^{s,\epsilon,\sigma}$. Then for each measurable function u,

$$\|u\|_{\vec{E}\dot{B}_{p,q}(x)} \sim \left\| \{\vec{E}(u)(x, 2^{-k})\}_{k\in\mathbb{Z}} \right\|_{l^q(L^p(x))}. \tag{81}$$

Proof. Observe that $\vec{E}(u)(x,t) \lesssim \vec{E}(u)(x, 2^{-k+1})$ for all $t \in (2^{-k}, 2^{-k+1}]$ and $x \in X$, from which (81) follows by a simple computation. This finishes the proof of Lemma (1.2.11).

With the aid of Lemma (1.2.7), Lemma (1.2.10) and Lemma (1.2.11), we obtain the following result, which, together with the fact $\vec{C}^{s,p}\dot{B}_{p,q}(x) = \dot{B}_{p,q}(x)$, implies Theorem (1.2.6) and (i) through (iv) of Theorem (1.2.5).

Theorem (1.2.12) [95] Let $s \in (0, \infty)$ and $p, q \in (0, \infty]$.

(i) If $\sigma \in (0, p]$, then $\dot{N}^s_{p,q}(x) = \vec{C}^{s,\sigma}\dot{B}_{p,q}(x)$.

(ii) If $\sigma \in (0, p_*(s))$, then $\dot{N}^s_{p,q}(x) = \vec{I}^{s,\sigma}\dot{B}_{p,q}(x)$.

(iii) If $\epsilon \in [0, s)$ and $\sigma \in (0, p_*((s))$, then $\dot{N}^s_{p,q}(x) = \vec{S}^{s,\epsilon,\sigma}\dot{B}_{p,q}(x)$.

(iv) If $p \in (n/(n+s), \infty]$ and $\sigma \in (0, p_*((s))$, then $\dot{N}^s_{p,q}(x) = \vec{A}^{s,\sigma}\dot{B}_{p,q}(x)$.

Proof. First, notice that if $\mu(X) < \infty$, then

$$diam\, X \equiv \sup_{x,y\in X} d(x,y) < \infty. \tag{82}$$

Indeed, suppose that diam $X = \infty$. Fix a ball $B(x_0, r_0) \subset X$. By our assumptions on μ, we have $\mu(B(x_0, r_0)) > 0$. Notice that for any $x_1 \in X$ with $d(x_1, x_0) \geq 2r_0$, by the doubling property and $B(x_0, r_0) \subset B(x_1, 2d(x_1, x_0))$, we have

$$\mu\left(B\left(x_1, \frac{1}{2}d(x_1, x_0)\right)\right) \geq (C_1)^{-1}4^{-n}\mu\left(B(x_1, 2d(x_1, x_0))\right) \geq (C_1)^{-1}4^{-n}\mu(B(x_0, r_0)).$$

Let $r_1 = 2d(x_1, x_0)$. Since $B\left(x_1, \frac{1}{2}d(x_1, x_0)\right) \cap B(x_0, r_0) = \emptyset$, we have

$$\mu(x) > \mu(B(x_1, r_1)) \geq [1 + (C_1)^{-1}4^{-n}]\mu(B(x_0, r_0)).$$

Repeating this procedure for N times, we can find $x_N \in X$ and $r_N > 0$ such that

$$\mu(x) > \mu(B(x_N, r_N)) \geq [1 + (C_1)^{-1}4^{-n}]\mu(B(x_{N-1}, r_{N-1}))$$
$$\geq \cdots \geq \mu(x) > [1 + (C_1)^{-1}4^{-n}]\mu(B(x_0, r_0)),$$

which tends to infinity as $N \to \infty$. This is a contradiction. Thus diam $X < \infty$.

Assume that $2^{-k_0-1} \leq$ diam $X < 2^{-k_0}$ for some $k_0 \in \mathbb{Z}$. Observe that

$$\|u\|_{\vec{E}\dot{B}_{p,q}(x)} \sim \left(\sum_{k \geq k_0 - 2} \|\vec{E}(u)(\cdot, 2^{-k})\|^q_{L^p(x)} \right)^{1/q}$$

and that for any $\vec{g} \in \mathbb{D}^s(u)$, we can always take $g_k \equiv 0$ for $k < k_0 - 2$. Because of this, the proof of Theorem (1.2.6) for the case $\mu(X) < \infty$ is a slight modification of that for the case $\mu(X) = \infty$ below. In what follows, we only consider the case $\mu(X) = \infty$.

We first prove (ii) and (iii). Observing that

$$I_t^{s,\sigma}(u)(x) \leq S_t^{s,\epsilon,\sigma}(u)(x) \tag{83}$$

for all $t \in (0, \infty)$ and $x \in X$, we have $\vec{S}^{s,\epsilon,\sigma}\dot{B}_{p,q}(x) \subset \vec{I}^{s,\sigma}\dot{B}_{p,q}(x)$. So it suffices to prove that $\vec{I}^{s,\sigma}\dot{B}_{p,q}(x) \subset \dot{N}^{s}_{p,q}(x) \subset \vec{S}^{s,\epsilon,\sigma}\dot{B}_{p,q}(x)$.

To prove $\vec{I}^{s,\sigma}\dot{B}_{p,q}(x) \subset \dot{N}^{s}_{p,q}(x)$, let $u \in \vec{I}^{s,\sigma}\dot{B}_{p,q}(x)$ and E with $\mu(E) = 0$ be as in Lemma (1.2.10). By Lemma (1.2.10), it is easy to see that for $x, y \in \mathbb{R}^n \setminus E$ and $d(x, y) \in [2^{-k+1}, 2^{-k})$,

$$|u(x) - u(y)| \le C[d(x, y)]^s \sum_{j \ge k-2} 2^{(k-j)s}\left[I^{s,\sigma}_{2^{-j}}(u)(x) + I^{s,\sigma}_{2^{-j}}(u)(y)\right]. \tag{84}$$

For $k \in \mathbb{Z}$, set

$$g_k \equiv \sum_{j \ge k-2} 2^{(k-j)s}\, I^{s,\sigma}_{2^{-j}}(u). \tag{85}$$

Then $\vec{g} \equiv \{g_k\}_{k\in\mathbb{Z}} \in \mathbb{D}^s(u)$ modulo a fixed constant and it is easy to check that

$$\|\vec{g}\|_{\ell^q(L^p(x))} \lesssim \left\|\left\{I^{s,\sigma}_{2^{-k}}(u)\right\}_{k\in\mathbb{Z}}\right\|_{\ell^q(L^p(x))};$$

see the proof of [40, Theorem 2.1] for details. So, by Lemma (1.2.11), $u \in \dot{N}^{s}_{p,q}(x)$ and

$$\|u\|_{\dot{N}^s_{p,q}(x)} \le \|\vec{g}\|_{\ell^q(L^p(x))} \lesssim \left\|\left\{I^{s,\sigma}_{2^{-k}}(u)\right\}_{k\in z}\right\|_{\ell^q(L^p(x))} \sim \|u\|_{\vec{I}^{s,\sigma}\dot{B}_{p,q}(x)}. \tag{86}$$

This leads to $\vec{I}^{s,\sigma}\dot{B}_{p,q}(x) \subset \dot{N}^{s}_{p,q}(x)$.

To prove that $\dot{N}^{s}_{p,q}(x) \subset \vec{S}^{s,\epsilon,\sigma}\dot{B}_{p,q}(x)$, since $\sigma < p_*(s)$, we can choose $\epsilon' \in (0, s)$ and $\delta \in (0, p)$ such that $\sigma \le \delta_*(\epsilon') = n\delta/(n - \epsilon'\delta)$. We also let $\epsilon'' \in (\epsilon', s)$ and $\epsilon''' \in (0, min\{s - \epsilon'', s - \epsilon\})$. For given $u \in \dot{N}^{s}_{p,q}(x)$, take $\vec{g} \in \mathbb{D}^s(u)$ with $\|\vec{g}\|_{\ell^q(L^p(x))} \le 2\|u\|_{\dot{N}^s_{p,q}(x)}$. Set

$$h_k \equiv 2^{k\epsilon'''} \sum_{i \ge k} 2^{-i\epsilon'''}\, g_i$$

for $k \in \mathbb{Z}$. Then $\vec{h} \equiv \{h_k\}_{k\in z} \in \mathbb{D}^s(u), h_i \le 2^{(i-k)\epsilon'''} h_k$ for any $i \ge k$, and moreover, it is easy to check

$$\left\|\vec{h}\right\|_{\ell^q(L^p(x))} \lesssim \|\vec{g}\|_{\ell^q(L^p(x))} \lesssim \|u\|_{\dot{N}^s_{p,q}(x)}; \tag{87}$$

see the proof of [40, Theorem 2.1] for details. Then by Lemma (1.2.7), for all $j \in \mathbb{Z}$ and $j \ge k$,

$$\begin{aligned}
I^{\epsilon,\sigma}_{2^{-j}}(u)(x) &= 2^{j\epsilon}\left(\inf_{c\in\mathbb{R}} \fint_{B(x,2^{-j})} |u(z) - c|^{\sigma}\, d\mu(z)\right)^{1/\sigma} \\
&\le 2^{j\epsilon}\left(\inf_{c\in\mathbb{R}} \fint_{B(x,2^{-j})} |u(z) - c|^{n\delta/(n-\epsilon'\delta)}\, d\mu(z)\right)^{(n-\epsilon'\delta)/n\delta} \\
&\lesssim 2^{j\epsilon} 2^{-j\epsilon''} \sum_{i \ge j-2} 2^{-i(s-\epsilon'')}\left(\fint_{B(x,2^{-j+1})} [h_i(z)]^{\delta}\, d\mu(z)\right)^{1/\delta} \\
&\lesssim 2^{j(\epsilon-s)} \sum_{i \ge j-2} 2^{(j-i)(s-\epsilon'')}\left(\fint_{B(x,2^{-j+1})} [h_i(z)]^{\delta}\, d\mu(z)\right)^{1/\delta} \\
&\lesssim 2^{j(\epsilon-s)} \textstyle\sum_{i \ge j-2} 2^{(j-i)(s-\epsilon'')}\, \mathcal{M}_{\delta}(h_i)(x) \\
&\lesssim 2^{j(\epsilon-s)} \displaystyle\sum_{i \ge j-2} 2^{(j-i)(s-\epsilon'')} 2^{(i-k)\epsilon'''}\, \mathcal{M}_{\delta}(h_k)(x)
\end{aligned}$$

$$\lesssim 2^{j(\epsilon-s)} 2^{(i-k)\epsilon'''} \mathcal{M}_\delta(h_k)(x). \tag{88}$$

Here and in what follows $\mathcal{M}$ denotes the Hardy-Littlewood maximal operator and $\mathcal{M}_\delta(u) \equiv \left[\mathcal{M}\left(|u|^\delta\right)\right]^{1/\delta}$ for all $u \in L^\delta_{loc}(x)$ and $\delta \in (0,\infty)$. Thus for all $k \in \mathbb{Z}$,

$$S^{s,\epsilon,\sigma}_{2^{-k}}(u)(x) \lesssim 2^{k(s-\epsilon)} \sup_{j\geq k} 2^{j(\epsilon-s)} I^{\epsilon,\sigma}_{2^{-j}}(u)(x)$$

$$\lesssim \sup_{j\geq k} 2^{-(j-k)(s-\epsilon)} 2^{(j-k)\epsilon'''} \mathcal{M}_\delta(h_k)(x) \lesssim \mathcal{M}_\delta(h_k)(x). \tag{89}$$

So, by the $L^{p/\delta}(x)$ boundedness of $\mathcal{M}$, Lemma (1.2.11) and (87), we have $u \in \vec{S}^{s,\epsilon,\sigma}\dot{B}_{p,q}(x)$ and

$$\|u\|_{\vec{S}^{s,\epsilon,\sigma}\dot{B}_{p,q}(x),} \lesssim \left\|\left\{S^{s,\epsilon,\sigma}_{2^{-k}}(u)\right\}_{k\in z}\right\|_{\ell^q\left(L^p(x)\right)}$$

$$\lesssim \|\{\mathcal{M}_\delta(h_k)\}_{k\in z}\|_{\ell^q\left(L^p(x)\right)} \lesssim \left\|\vec{h}\right\|_{\ell^q\left(L^p(x)\right)} \lesssim \|u\|_{\dot{N}^s_{p,q}(x)}$$

This yields $\dot{N}^s_{p,q}(x) \subset \vec{S}^{s,\epsilon,\sigma}\dot{B}_{p,q}(x)$ and thus finishes the proofs of (ii) and (iii).

Now we prove (i). Since

$$I^{s,\sigma}_t(u)(x) \leq C^{s,\sigma}_t(u)(x) \tag{90}$$

for all $t \in (0,\infty)$ and $x \in X$, we have $\vec{C}^{s,\sigma}\dot{B}_{p,q}(x) \subset \vec{I}^{s,\sigma}\dot{B}_{p,q}(x)$, and hence by (ii), $\vec{C}^{s,\sigma}\dot{B}_{p,q}(x) \subset \dot{N}^s_{p,q}(x)$. So we only need to show that $\dot{N}^s_{p,q}(x) \subset \vec{C}^{s,\sigma}\dot{B}_{p,q}(x)$. For given $u \in \dot{N}^s_{p,q}(x)$, take $\vec{g} \in \mathbb{D}^s(u)$ with $\|\vec{g}\|_{\ell^q(L^p(x))} \leq 2\|u\|_{\dot{N}^s_{p,q}(x)}$. Then by Lemma (1.2.7), for all $k \in \mathbb{Z}$,

$$C^{s,\sigma}_{2^{-k}}(u)(x) = 2^{ks}\left(\sum_{j\geq k} \frac{1}{\mu B(x,2^{-k})} \int_{B(x,2^{-j})/B(x,2^{-j-1})} |u(z)-u(x)|^\sigma \, d\mu(z)\right)^{1/\sigma}$$

$$\leq 2^{ks}\left(\sum_{j\geq k-2} 2^{-js\sigma} \fint_{B(x,2^{-j})} \left(\left[g_j(z)\right]^\sigma + \left[g_j(x)\right]^\sigma\right) d\mu(z)\right)^{1/\sigma}$$

$$= \left(\sum_{j\geq k-2} 2^{-(j-k)s\delta}\left[g_j(x)\right]^\sigma + \sum_{j\geq k-2} 2^{-(j-k)s\sigma} \fint_{B(x,2^{-j})} \left[g_j(z)\right]^\sigma d\mu(z)\right)^{1/\sigma}.$$

If $p > \sigma$, then when $\sigma \in (0,1)$, applying the Hölder inequality, we have

$$C^{s,\sigma}_{2^{-k}}(u)(x) \lesssim \left(\sum_{j\geq k} 2^{-(j-k)s\sigma}\left[\mathcal{M}_\sigma(g_j)(x)\right]^\sigma\right)^{1/\sigma}$$

$$\lesssim \sum_{j\geq k} 2^{-(j-k)s/2} \mathcal{M}_\sigma(g_j)(x) \left(\sum_{j\geq k} 2^{-(j-k)s/2(1-\sigma)}\right)^{(1-\sigma)/\sigma}$$

$$\lesssim \sum_{j\geq k} 2^{-(j-k)s/2} \mathcal{M}_\sigma(g_j)(x), \tag{91}$$

and when $\sigma \in [1,p)$, by $1/\sigma \leq 1$,

$$C^{s,\sigma}_{2^{-k}}(u)(x) \lesssim \sum_{j\geq k} 2^{-(j-k)s} \mathcal{M}_\sigma(g_j)(x).$$

From this, it is easy to deduce that

$$\left\|\left\{C^{s,\sigma}_{2^{-k}}(u)\right\}_{k\in\mathbb{Z}}\right\|_{\ell^q\left(L^p(x)\right)} \lesssim \|\{\mathcal{M}_\sigma(g_k)\}_{k\in\mathbb{Z}}\|_{\ell^q\left(L^p(x)\right)}.$$

By this, the $L^{p/\sigma}(x)$-boundedness of $\mathcal{M}$ and Lemma (1.2.11), we have $u \in \vec{C}^{s,\sigma}\dot{B}_{p,q}(x)$ and

$$\|u\|_{\vec{C}^{s,\sigma}\dot{B}_{p,q}(x)} \lesssim \left\|\left\{C^{s,\sigma}_{2^{-k}}(u)\right\}_{k\in\mathbb{Z}}\right\|_{\ell^q(L^p(x))} \lesssim \|\vec{g}\|_{\ell^q(L^p(x))} \lesssim \|u\|_{\dot{N}^s_{p,q}(x)}.$$

If $\sigma = p$, then

$$\|u\|_{\vec{C}^{s,\sigma}\dot{B}_{p,q}(x)} \lesssim \left\{\sum_{k\in\mathbb{Z}}\left(\int_x \sum_{j\geq k} 2^{-(j-k)sp}\left[g_j(x)\right]^p d\mu(x)\right)^{q/p}\right\}^{1/q}$$

$$+\left\{\sum_{k\in\mathbb{Z}}\left(\int_x \sum_{j\geq k} 2^{-(j-k)sp} \fint_{B(x,2^{-j})}\left[g_j(z)\right]^p d\mu(z)d\mu(x)\right)^{q/p}\right\}^{1/q}$$

$$\lesssim \left\{\sum_{k\in\mathbb{Z}}\left(\int_x \sum_{j\geq k} 2^{-(j-k)sp}\left[g_j(x)\right]^p d\mu(x)\right)^{q/p}\right\}^{1/q} \lesssim \|\vec{g}\|_{\ell^q(L^p(x))} \lesssim \|u\|_{\dot{N}^s_{p,q}(x)}.$$

This gives $\dot{N}^s_{p,q}(x) \subset \vec{C}^{s,\sigma}\dot{B}_{p,q}(x)$ and thus finishes the proof of (i).

Finally, we prove (iv). Trivially,

$$I^{s,\sigma}_t(u)(x) \leq A^{s,\sigma}_t(u)(x) \tag{92}$$

for all $x \in X$ and $t \in (0,\infty)$, which implies that $\vec{A}^{s,\sigma}\dot{B}_{p,q}(x) \subset \vec{I}^{s,\sigma}\dot{B}_{p,q}(x)$. On the other hand, since $p > n/(n+s)$ and $\sigma < p_*(s)$, we can find $\sigma' \in (max\{\sigma,1\}, p_*(s))$. Notice that, for any $c \in \mathbb{R}$, by the Minkowski inequality and the Hölder inequality,

$$\left(\fint_{B(x,t)}\left|u-u_{B(x,t)}\right|^{\sigma'} d\mu(z)\right)^{1/\sigma'} \leq \left(\fint_{B(x,t)}|u-c|^{\sigma'} d\mu(z)\right)^{1/\sigma'} + \left|c-u_{B(x,t)}\right|$$

$$\leq 2\left(\fint_{B(x,t)}|u-c|^{\sigma'} d\mu(z)\right)^{1/\sigma'},$$

which together with the Hölder inequality again implies that

$$A^{s,\sigma}_t(u)(x) \leq A^{s,\sigma'}_t(u)(x) \leq 2I^{s,\sigma'}_t(u)(x) \tag{93}$$

for all $u \in L^{\sigma}_{loc}(x), x \in X$ and $t \in (0,\infty)$. Then $\vec{I}^{s,\sigma'}\dot{B}_{p,q}(x) \subset \vec{A}^{s,\sigma}\dot{B}_{p,q}(x)$. Recall that we have proved that $\vec{I}^{s,\sigma'}\dot{B}_{p,q}(x) = \dot{N}^s_{p,q}(x) = \dot{B}^s_{p,q}(x) = \vec{I}^{s,\sigma}\dot{B}_{p,q}(x)$. So we obtain (iv). The proof of Theorem (1.2.12) is finished.

One can derive the following inequality from the proof of (88).

Corollary (1.2.13) [95] For $s \in (0,\infty), \delta \in (0,\infty), \sigma \in (0,\delta_*(s))$ and $\sigma' \in (0,\infty)$, there exist $\epsilon > 0$ satisfying $\sigma < \delta_*(s-\epsilon)$ and constant C such that for all $u \in L^{\sigma}_{loc}(x), k \in \mathbb{Z}$ and $x \in X$,

$$I^{s,\sigma}_{2^{-k}}(u)(x) \leq C2^{-k\epsilon}\mathcal{M}_{\delta}\left(\sum_{j\geq k-2} I^{s-\epsilon,\sigma'}_{2^{-j}}(u)\right)(x). \tag{94}$$

Proof. If $\sigma' \geq \sigma$, then (94) is trivial or follows from the Hölder inequality. If $\sigma' < \sigma$, then we employ the argument for (88) with the special choice $g_j \equiv I^{s,\sigma'}_{2^{-j}}(u)$ for all $j \geq k-2$.

We close by proving the optimality of the ranges of ϵ and σ in Theorem (1.2.5).

Proofs of (v) though (vii) of Theorem (72). (v) For $\alpha \in (0,\infty)$, define

$$u_\alpha(x) \equiv |x|^{-\alpha}\chi_{B(0,1)}(x) + \chi_{\mathbb{R}^n\setminus B(0,1)}(x).$$

We first claim that $u_\alpha \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ when $\alpha \in (0, n/p - s)$. To see this, for $j \le 0$, we set

$$g_j(x) \equiv 2^{js}|x|^{-\alpha}\chi_{B(0,1)}(x),$$

and, for $j \ge 1$, we set

$$g_j(x) \equiv 2^{js}|x|^{-\alpha}\chi_{B\left(0,2^{-j-s}\right)}(x) + 2^{-j(1-s)}|x|^{-\alpha-1}\chi_{B(0,1)\setminus B\left(0,2^{-j-s}\right)}(x).$$

Then it is easy to check that $\vec{g} \equiv \{g_j\}_{j\in z} \in \mathbb{D}^s(u_\alpha)$ modulo a fixed constant. Moreover, since $p\alpha < n$, for $j \le 0$, we have that

$$\|g_j\|^p_{L^p(\mathbb{R}^n)} \le \int_{B(0,1)} 2^{jps}|x|^{-p\alpha}dx \le 2^{jps},$$

and for $j \ge 1$,

$$\|g_j\|^p_{L^p(\mathbb{R}^n)} \le \int_{B\left(0,2^{-j-s}\right)} 2^{jps}|x|^{-p\alpha}dx + \int_{B(0,1)\setminus B\left(0,2^{-j-s}\right)} 2^{-jp(1-s)}|x|^{-p(\alpha+1)}dx$$
$$\lesssim 2^{j[p(s+\alpha)-n\}} + 2^{-jp(1-s)}.$$

Observing that $s+\alpha < n/p$ and recalling that $s < 1$, we have $\|\vec{g}\|_{\ell^q(L^p(\mathbb{R}^n))} < \infty$, which is as desired. Now, taking $\alpha = n/\sigma$ and noticing $s + \alpha < s + n/p_*(s) = n/p$, we have $u_\alpha \in \dot{N}^s_{p,q}(\mathbb{R}^n)$ and hence by Theorem (1.2.6), $u_\alpha \in \dot{B}^s_{p,q}(\mathbb{R}^n)$. This yields (v) since $u_\alpha \notin L^\sigma_{loc}(\mathbb{R}^n)$.

(vi) Let $\alpha = n/\sigma$. Since $\alpha + s < n/p$, as shown in (v), $u_\alpha \in \dot{N}^s_{p,q}(\mathbb{R}^n)$. Let us check that $\|u_\alpha\|_{\vec{C}^{s,\sigma}\dot{B}_{p,q}(\mathbb{R}^n)} = \infty$. Indeed, if $1/2 \le |x| \le 3/4$, then $B(0,1/4) \subset B(x,1)$ and for all $z \in B(0,1/4)$,

$$|x|^{-\alpha} \le (1/2)^{-\alpha} = 2^{-\alpha}(1/4)^{-\alpha}2^{-\alpha}|z|^{-\alpha},$$

which implies that $|u(x) - u(z)| \ge (1-2^{-\alpha})|z|^{-\alpha}$, and hence by $n = \alpha\sigma$,

$$\fint_{B(x,1)} |u(x)-u(z)|^{-\alpha}\, dz \gtrsim \fint_{B(x,1/4)} |u(z)|^{-\alpha}\, dz = \infty.$$

Therefore

$$\|u\|_{\vec{C}^{s,\sigma}\dot{B}_{p,q}(\mathbb{R}^n)} \gtrsim \left\{\int_{B(0,3/4)\setminus B(0,1/2)} \left(\fint_{B(x,1)} |u(x)-u(z)|^{-\alpha}\, dz\right)^{p/\sigma} dx\right\}^{1/p} = \infty$$

as desired. This gives (vi).

(vii) Let $\alpha = n/p - s$ and $\beta \in (-2/p, -1/p)$, and define

$$u(x) \equiv |x|^{-\alpha}\left(\log\frac{2}{|x|}\right)^\beta \chi_{B(0,1)}(x) + \chi_{\mathbb{R}^n\setminus B(0,1)}(x).$$

We claim that $u \in \dot{N}^s_{p,q}(\mathbb{R}^n)$. To see this, similarly to (v), for $j \le 0$, we set

$$g_j(x) \equiv 2^{js}|x|^{-\alpha}\left(\log\frac{2}{|x|}\right)^\beta \chi_{B(0,1)}(x),$$

and for $j \ge 1$, we set

$$g_j(x) \equiv 2^{js}|x|^{-\alpha}\left(\log\frac{2}{|x|}\right)^\beta \chi_{B\left(0,2^{-j-s}\right)}(x) + 2^{-j(1-s)}|x|^{-\alpha-1}\left(\log\frac{2}{|x|}\right)^\beta \chi_{B(0,1)\setminus B\left(0,2^{-j-3}\right)}(x).$$

Then $\vec{g} \equiv \{g_j\}_{j\in\mathbb{Z}} \in \mathbb{D}^s(u)$ modulo a fixed constant. Since $\alpha + s = n/p$ and $p\beta < -1$, we have that

$$\sum_{j\ge1} \|g_j\|^p_{L^p(\mathbb{R}^n)} \lesssim \sum_{j\ge1}\int_{B(0,2^{-j-s})} 2^{jps}|x|^{-p\alpha}\left(\log\frac{2}{|x|}\right)^{p\beta}dx$$
$$+\sum_{j\ge1}\int_{B(0,1)\backslash B(0,2^{-j-3})} 2^{-jp(1-s)}|x|^{-p(\alpha+1)}\left(\log\frac{2}{|x|}\right)^{p\beta}dx$$
$$\lesssim \int_{B(0,1)}|x|^{-p(\alpha+s)}\left(\log\frac{2}{|x|}\right)^{p\beta}dx \lesssim \int_0^1\left(\log\frac{2}{t}\right)^{p\beta}\frac{dt}{t}<\infty.$$
and that
$$\sum_{j\le0} \|g_j\|^p_{L^p(\mathbb{R}^n)} \lesssim \int_{B(0,1)}|x|^{-p(\alpha+s)}\left(\log\frac{2}{|x|}\right)^{p\beta}dx<\infty.$$
Thus $u\in \dot N^s_{p,q}(\mathbb{R}^n)$. On the other hand, for any $x\in B(0,1/2)$ and all $t>|x|$,
$$S^{s,s,p}_t(u)(x)\ge |x|^{-s}\left(\inf_{c\in\mathbb{R}} \fint_{B(x,|x|)}|u(z)-c|^p\,dx\right)^{1/p}$$
$$\ge (2|x|)^{-s}\left(\fint_{B(x,|x|)}|u(z)-u(x_0)|^p\,dx\right)^{1/p},$$
where may choose $x_0\in B(x,|x|)$. Moreover, up to a rotation, we can assume that $x_0=x|x_0|/|x|$. Observe that if $|x_0|\ge|x|$, then for $z\in B(x/2,|x|/4)\subset B(0,3|x|/4)$,
$$|u(z)-u(x_0)|\ge|u(3x/4)-u(x)|\gtrsim u(x).$$
Moreover, if $|x_0|\le|x|$, then for $z\in B(3x/2,|x|/4)\subset B(0,1)\backslash B(0,5|x|/4)$,
$$|u(z)-u(x_0)|\ge|u(5x/4)-u(x)|\gtrsim u(x).$$
Hence by $B(x/2,|x|/4)\cup B(3x/2,|x|/4)\subset B(x,|x|)$ and $\alpha+s=n/p$, we have
$$S^{s,s,p}_t(u)(x)\gtrsim |x|^{-\alpha-s}\left(\log\frac{2}{|x|}\right)^{\beta}\sim |x|^{-n/p}\left(\log\frac{2}{|x|}\right)^{\beta},$$
from which together with $p\beta+1>-1$, it follows that
$$\|u\|^p_{\dot S^{s,s,p}\dot B_{p,q}(\mathbb{R}^n)} \gtrsim \sum_{j\ge0}\left\|S^{s,s,p}_{2^{-j}}(u)\right\|^p_{L^p\left(B\left(0,2^{-j-1}\right)\right)} \gtrsim \sum_{j\ge0}\int_{B\left(0,2^{-j-1}\right)}|x|^{-n}\left(\log\frac{2}{|x|}\right)^{p\beta}dx$$
$$\sim \int_{B(0,1)}|x|^{-n}\left(\log\frac{2}{|x|}\right)^{p\beta+1}dx\sim\int_0^{1/2}\left(\log\frac{2}{t}\right)^{p\beta+1}\frac{dt}{t}=\infty.$$
The proof of Theorem (1.2.12) is finished.

In what follows, for $p,q\in(0,\infty]$ and a sequence $\vec g$ of measurable functions, we set $\|\vec g\|_{L^p(x,\ell^q)}\equiv\|\|\vec g\|_{\ell^q}\|_{L^p(X)}$.

Definition (1.2.14) [95] Let $s\in(0,\infty)$ and $p,q\in(0,\infty]$. The homogeneous Hajłasz-Triebel Lizorkin space $\dot M^s_{p,q}(x)$ is the space of all measurable functions u such that $\|u\|_{\dot M^s_{p,q}(x)}<\infty$, where when $p\in(0,\infty)$ or $p,q=\infty$,

$$\|u\|_{\dot M^s_{p,q}(x)}\equiv\inf_{\vec g\in\mathbb{D}^s(u)}\|\vec g\|_{L^p(x,\ell^q)},$$
and when $p=\infty$ and $q\in(0,\infty)$,
$$\|u\|_{\dot M^s_{p,q}(x)}\equiv\inf_{\vec g\in\mathbb{D}^s(u)}\sup_{k\in Z}\sup_{x\in X}\left\{\sum_{j\ge k}\fint_{B(x,2^{-k})}\left[g_j(y)\right]^q d\mu(y)\right\}^{1/q}.$$

Definition (1.2.15) [95] Let $s,\sigma \in (0,\infty), \epsilon \in [0,s]$ and $p,q \in (0,\infty]$. For $\vec{E} = \vec{C}^{s,\sigma}, \vec{A}^{s,\sigma}, \vec{I}^{s,\sigma}$ or $\vec{S}^{s,\epsilon,\sigma}$ the homogeneous space $\vec{E}\dot{F}_{p,q}(x)$ of Triebel-Lizorkin type is defined to be the collection of all $u \in L^{\sigma}_{loc}(x)$ such that $\|u\|_{\vec{E}\dot{F}_{p,q}(x)} < \infty$, where when $p \in (0,\infty)$,

$$\|u\|_{\vec{E}\dot{F}_{p,q}(x)} \equiv \left\| \left(\int_0^\infty [\vec{E}(u)(\cdot,t)]^q \frac{dt}{t} \right)^{1/q} \right\|_{L^p(x)}$$

with the usual modification made when $q = \infty$, and when $p = \infty$ and $q \in (0,\infty)$,

$$\|u\|_{\vec{E}\dot{F}_{p,q}(x)} \equiv \sup_{x\in X} \sup_{r>0} \left(\int_0^r \fint_{B(x,r)} [\vec{E}(u)(y,t)]^q d\mu(y) \frac{dt}{t} \right)^{1/q}$$

and when $p,q = \infty$, $\|u\|_{\vec{E}\dot{F}_{\infty,\infty}(x)} \equiv \|u\|_{\vec{E}\dot{B}_{\infty,\infty}(x)}$.

Theorem (1.2.16) [95] Let $s \in (0,\infty)$ and $p,q \in (0,\infty]$. Let $r \equiv min\{p,q\}$.

(i) If $\sigma \in (0,r)$, then $\dot{M}^s_{p,q}(x) = \vec{C}^{s,\sigma}\dot{F}_{p,q}(x)$.

(ii) If $\sigma \in (0,r_*(s))$, then $\dot{M}^s_{p,q}(x) = \vec{I}^{s,\sigma}\dot{F}_{p,q}(x)$.

(iii) If $\epsilon \in [0,s)$ and $\sigma \in (0,r_*((s))$, then $\dot{M}^s_{p,q}(x) = \vec{S}^{s,\epsilon,\sigma}\dot{F}_{p,q}(x)$.

(iv) If $r \in (n/(n+s),\infty]$ and $\sigma \in (0,r_*((s))$, then $\dot{M}^s_{p,q}(x) = \vec{A}^{s,\sigma}\dot{F}_{p,q}(x)$.

Proof. The proof of Theorem (1.2.16) is similar to that of Theorem (1.2.12). We only sketch it. By (83), we have $\vec{S}^{s,\epsilon,\sigma}\dot{F}_{p,q}(x) \subset \vec{I}^{s,\sigma}\dot{F}_{p,q}(x)$.

For $u \in \vec{I}^{s,\sigma}\dot{F}_{p,q}(x)$, by taking $\vec{g} \equiv \{g_k\}_{k\in Z}$ as in (85), similarly to (86), we can show that $\|u\|_{\dot{M}^s_{p,q}(x)} \lesssim \|u\|_{\vec{I}^{s,\sigma}\dot{F}_{p,q}(x)}$. Hence, $\vec{I}^{s,\sigma}\dot{F}_{p,q}(x) \subset \dot{M}^s_{p,q}(x)$.

The result $\dot{M}^s_{p,q}(x) \subset \vec{S}^{s,\epsilon,\sigma}\dot{F}_{p,q}(x)$ follows from an argument similar to that $\dot{N}^s_{p,q}(x) \subset \vec{S}^{s,\epsilon,\sigma}\dot{B}_{p,q}(x)$ where the inequality (89) plays an important role. The restriction $\sigma \in (0,r_*(s))$ ensures the existence of $\delta \in (0,r)$, and $\epsilon' \in (0,\varrho)$ such that $\sigma \le \delta_*(\epsilon')$. Moreover, by $\delta \in (0,r)$, we can use the Fefferman-Stein maximal inequality (see [81]) to obtain

$$\|\{\mathcal{M}_\delta(h_k)\}_{k\in Z}\|_{L^p(x,\ell^q)} \lesssim \|\vec{h}\|_{L^p(x,\ell^q)}.$$

This gives (ii) and (iii).

For (i), by (90), we have $\vec{C}^{s,\sigma}\dot{F}_{p,q}(x) \subset \vec{I}^{s,\sigma}\dot{F}_{p,q}(x) \subset \dot{M}^s_{p,q}(x)$. The converse result $\dot{M}^s_{p,q}(x) \subset \vec{C}^{s,\sigma}\dot{F}_{p,q}(x)$ follows from (91) and an argument similar to the proof of $\dot{N}^s_{p,q}(x) \subset \vec{C}^{s,\sigma}\dot{B}_{p,q}(x)$ for $\delta \in (0,p)$. Here the restriction $\delta \in (0,r)$ comes from the Fefferman-Stein maximal inequality used to prove

$$\|\{\mathcal{M}_\sigma(g_k)\}_{k\in Z}\|_{L^p(x,\ell^q)} \lesssim \|\vec{g}\|_{L^p(x,\ell^q)}.$$

This gives (i).

For (iv), the equivalence $\vec{A}^{s,\sigma}\dot{F}_{p,q}(x) = \dot{M}^s_{p,q}(x)$. follows from (92), (93) with $\sigma' \in (\max\{\sigma,1\}, r^*(s))$ and (ii). This gives (iv) and hence finishes the proof of Theorem (1.2.16).

In Theorem (1.2.16) (iii), we have the restriction $\epsilon \in [0,s)$. However, when $\epsilon = s$ and $q = \infty$, we have the following result.

Theorem (1.2.17) [95] Let $s \in (0,\infty)$ and $p \in (0,\infty]$. If $\sigma \in (0,p_*(s))$, then $\dot{M}^s_{p,\infty}(x) = \vec{S}^{s,\epsilon,\sigma}\dot{F}_{p,\infty}(x)$.

Proof. To see $\vec{S}^{s,\epsilon,\sigma}\dot{F}_{p,\infty}(x) \subset \dot{M}^s_{p,\infty}(x)$, let $u \in \vec{S}^{s,\epsilon,\sigma}\dot{F}_{p,\infty}(x)$. By (84) and taking $\vec{g} \equiv \{g_k\}_{k\in Z}$ with $g_k = S^{s,s,\sigma}_{2^{-k+1}}(u)$ we have $\vec{g} \in \mathbb{D}^s(u)$ and

$$\|\vec{g}\|_{L^p(x,\ell^\infty)} \le \left\| \{S^{s,s,\sigma}_{2^{-k}}(u)\}_{k\in z} \right\|_{L^p(x,\ell^\infty)} \sim \|u\|_{\vec{S}^{s,s,\sigma}\dot{F}_{p,\infty}(x)},$$

which implies that $u \in \dot{M}^s_{p,\infty}(x)$ and $\|u\|_{\dot{M}^s_{p,\infty}(x)} \lesssim \|u\|_{\vec{S}^{s,s,\sigma}\dot{F}_{p,\infty}(x)}$.

Conversely, let $u \in \dot{M}^s_{p,\infty}(x)$ and $\vec{g} \in \mathbb{D}^s(u)$ with $\|\vec{g}\|_{L^p(x,\ell^\infty)} \le 2\|u\|_{\dot{M}^s_{p,\infty}(x)}$. Taking $g \equiv \sup_{k\in Z} g_k = \|\vec{g}\|_{\ell^\infty}$, we have $g \in \mathcal{D}^s(u)$ and $\|g\|_{L^p(x)} \lesssim \|u\|_{\dot{M}^s_{p,\infty}(x)}$. By $\sigma \in (0, p_*(s))$, let $\delta \in (0,p)$ such that $\sigma = \delta_*(s)$. Then by Lemma (1.2.8), for all $x \in X$ and $k \in \mathbb{Z}$,

$$S^{s,s,\sigma}_{2^{-k+2}}(u)(x) \le \mathcal{M}_\delta(g)(x),$$

which together with the $L^{p/\delta}(X)$-boundedness of $\mathcal{M}$ implies that $\vec{S}^{s,\epsilon,\sigma}\dot{F}_{p,\infty}(X)$ and $\|u\|_{\vec{S}^{s,s,\sigma}\dot{F}_{p,\infty}(X)} \lesssim \|u\|_{\dot{M}^s_{p,\infty}(X)}$ This finishes the proof of Theorem (1.2.17).

Let $s, \sigma \in (0,\infty)$ and recall the classical fractional sharp maximal functions

$$u^{\delta,s}(x) \equiv \sup_{t\in(0,\infty)} t^{-s} \fint_{B(x,t)} \left|u(z) - u_{B(x,t)}\right| d\mu(z)$$

and

$$u^{\delta,s}_\sigma(x) \equiv \sup_{t\in(0,\infty)} t^{-s} \inf_{c\in\mathbb{R}} \left(\fint_{B(x,t)} |u(z)-c|^\sigma d\mu(z) \right)^{1/\sigma}.$$

The Sobolev-type space $\dot{C}^{s,p}(X)$ of Calderón and DeVore-Sharpley is defined as the collection of all locally integrable functions u such that $\|u\|_{\dot{C}^{s,p}(X)} \equiv \left\|u^{\delta,s}\right\|_{L^p(X)} < \infty$; see [78, 93]. Also observe that

$$u^{\delta,s}_\sigma(u)(x) = \sup_{t\in(0,\infty)} S^{s,s,\sigma}_t(u)(x) \sim \left\|\vec{S}^{s,s,\sigma}(u)(x)\right\|_{\ell^\infty},$$

and hence $\|u\|_{\vec{S}^{s,s,\sigma}\dot{F}_{p,\infty}(X)} = \left\|u^{\delta,s}_\sigma\right\|_{L^p(X)}$. On the other hand, recall from [40] that $\dot{M}^s_{p,\infty}(X)$ is simply the Hajłasz-Sobolev space $\dot{M}^{s,p}(X)$. Here $\dot{M}^{s,p}(X)$ is the collection of all functions u such that

$$\|u\|_{\dot{M}^{s,p}(X)} \equiv \inf_{g\in\mathcal{D}^s(u)} \|g\|_{L^p(X)} < \infty,$$

where $\mathcal{D}^s(u)$ is the set of all s-gradients of u as in (73). Then, as a consequence of Theorem (1.2.16) and Theorem (1.2.17), we have the following corollary.

Corollary (1.2.18) [95] Let $s \in (0,\infty)$ and $p \in (0,\infty]$.

(i) If $\sigma \in (0, p^*(s))$, then $u \in \dot{M}^{s,p}(X)$ if and only if $u^{\delta,s}_\sigma \in L^p(X)$, and moreover, for every $u \in \dot{M}^{s,p}(X)$, $\|u\|_{\dot{M}^{s,p}(X)} \sim \left\|u^{\delta,s}_\sigma(u)\right\|_{L^p(X)}$.

(ii) If $p \in (n/(n+s),\infty]$, $\dot{C}^{s,p}(X) = \dot{M}^{s,p}(X)$.

We say that X supports a weak $(1,p)$-Poincaré inequality with $p \in [1,\infty)$ if there exist positive constants C and $\lambda > 1$ such that for all functions u, p-weak upper gradients g of u and balls B with radius $r > 0$,

$$\fint_B |u(x) - u_B| d\mu(x) \le Cr\left\{\fint_{\lambda B} [g(x)]^p \, d\mu(x)\right\}^{1/p}.$$

Recall that a nonnegative Borel function g is called a p-weak upper gradient of u if

$$|u(x) - u(y)| \le \int_\gamma g \, ds \tag{95}$$

for all $\gamma \in \Gamma_{\text{rect}} \backslash \Gamma$, where x and y are the endpoints of γ, Γ_{rect} denotes the collection of non-constant compact rectifiable curves and Γ has p-modulus zero. If X is complete, the above Poincaré inequality holds if and only if it holds for each Lipschitz function with the pointwise Lipschitz constant

$$\mathrm{Lip}(\mathrm{u})(\mathrm{x}) = \limsup_{r\to 0} \sup_{y\in B(x,r)} \frac{|u(x)-u(y)|}{r}$$

on the right-hand side. See [84].

By triviality of $\dot{N}^s_{p,q}(X)$ or $\dot{M}^s_{p,q}(X)$ below we mean that they only contain constant functions. In order to obtain such a conclusion, one needs some connectivity assumption on X; simply consider $B(0,1) \cup B(x_0,1)$ where $x_0 \in \mathbb{R}^n$ and $|x_0| > 3$, equipped with the Euclidean distance and Lebesgue measure. Then $\chi_{B(0,1)} \in \dot{M}^s_{p,q}(X) \cap \dot{N}^s_{p,q}(X)$ for all s, p, q. Notice that X does not support any Poincaré inequality.

Theorem (1.2.19) [95] Suppose that X supports a weak $(1,p)$-Poincaré inequality with $p \in (1,\infty)$. Then for all $q \in (0,\infty)$, $\dot{N}^1_{p,q}(X)$ and $\dot{M}^1_{p,q}(X)$ are trivial.

Proof. Since for $q \in (0,p)$, $\dot{M}^1_{p,q}(X) \subset \dot{M}^1_{p,p}(X)$ and $\dot{N}^1_{p,q}(X) \subset \dot{N}^1_{p,p}(X) = \dot{M}^1_{p,p}(X)$, we only need to prove that for $q \in [p,\infty)$, $\dot{M}^1_{p,q}(X)$ and $\dot{N}^1_{p,q}(X)$ are trivial. Assume that $q \in [p,\infty)$. Notice that $\dot{M}^1_{p,q}(X) \subset \dot{M}^1_{p,\infty}(X) = \dot{M}^{1,p}(X)$, where $\dot{M}^{1,p}(X)$ is the Hajłasz-Sobolev space [26]. Moreover, under the weak $(1,p)$-Poincaré inequality, it is known that $\dot{M}^{1,p}(X) = \dot{N}^{1,p}(X)$ (see [92, Theorem 4.9] and [87]), where $\dot{N}^{1,p}(X)$ is the Newtonian Sobolev space introduced in [92]. So $\dot{M}^s_{p,q}(X) \subset \dot{N}^{1,p}(X)$. Let $u \in \dot{M}^1_{p,q}(X)$. Then. $u \in \dot{N}^{1,p}(X)$ The proof of the trivialilty of $\dot{M}^1_{p,q}(X)$ is reduced to proving $\|u\|_{\dot{N}^{1,p}(X)} = 0$. To this end, it suffices to find a sequence $\{\rho_k\}_{k\in\mathbb{N}}$ of p-weak upper gradients of u such that $\|\rho_k\|_{L^p(X)} \to 0$ as $k \to \infty$.

For $k \in \mathbb{N}$, set

$$\rho_k(x) \equiv \sup_{j\geq k} 2^j \fint_{B(x,2^{-j})} \left|u(z) - u_{B(x,2^{-j})}\right| d\mu(z).$$

Then ρ_k is nonnegative Borel measurable function for all $k \in \mathbb{N}$. Moreover, we have that $\lim_{k\to\infty} \rho_k(x) = 0$ for almost all $x \in X$. Indeed, by a discrete variant of Theorem (1.2.16) (ii),

$$\left\|\left\{I^{1,1}_{2^{-j}}(u)\right\}_{j\in Z}\right\|_{L^p(X,\ell^q)} \sim \|u\|_{\dot{M}^1_{p,q}(X)} < \infty,$$

which implies that $\left\|\left\{I^{1,1}_{2^{-j}}(u)(x)\right\}_{j\in Z}\right\|_{\ell^q} < \infty$ and hence $\rho_k(x) \leq \left\|\left\{I^{1,1}_{2^{-j}}(u)(x)\right\}_{j\geq k}\right\|_{\ell^q} \to 0$ as $k \to \infty$ for almost all $x \in X$. Moreover, applying the Lebesgue dominated convergence theorem, we have $\|\rho_k\|_{L^p(X)} \to 0$ as $\mathrm{k} \to \infty$.

Now it suffices to check that ρ_k is a p-weak upper gradient of u. Observe that if $\rho_k(x) < \infty$, then $\lim_{j\to\infty} u_{B(x,2^{-j})}$ exists. In fact, we have

$$\left|u_{B(x,2^{-j})} - u_{B(x,2^{-i})}\right| \lesssim 2^{-\min(j,\ell)}\rho_k(x) \to 0$$

as $j, \ell \to \infty$. For such an x, we define $\tilde{u}(x) \equiv \lim_{j\to\infty} u_{B(x,2^{-j})}$. Generally, for $x \in X$, if $\lim_{j\to\infty} u_{B(x,2^{-j})}$ exists, then we define $\tilde{u}(x) \equiv \lim_{j\to\infty} u_{B(x,2^{-j})}$; otherwise, $\tilde{u}(x) \equiv 0$. Obviously, $u(x) = \tilde{u}(x)$ for almost all $x \in X$, and hence u and $\tilde{u}$ generate the same element of $\dot{N}^{1,p}(X)$. Therefore we only need to check that ρ_k is a p-weak upper gradient of $\tilde{u}$. To this end, notice that for all $x, y \in X$ with $d(x,y) \leq 2^{-k-2}$, we have

$$|\tilde{u}(x) - \tilde{u}(y)| \leq d(x,y)[\rho_k(x) + \rho_k(y)].$$

Moreover, by [92, Proposition 3.1], u is absolutely continuous on p-almost every curve, namely, $u \circ \gamma$ is absolutely continuous on $[0, \ell(\gamma)]$ for all arc-length parameterized paths $\gamma \in \Gamma_{\mathrm{rect}} \setminus \Gamma$, where Γ has p-modulus zero. For every $\gamma \in \Gamma_{\mathrm{rect}} \setminus \Gamma$, we will show that (95) holds. To see this, by the absolute continuity of u on γ, it suffices to show that for j large enough,

$$2^j\left|\int_0^{2^{-j}} u\circ\gamma(t)\,dt-\int_{\ell(\gamma)-2^{-j}}^{\ell(\gamma)} u\circ\gamma(t)\,dt\right|\lesssim\int_0^{\ell(\gamma)}\rho_k\circ\gamma(t)dz.$$

But, borrowing some ideas from [75], for j large enough, we have

$$\begin{aligned}2^j\left|\int_0^{2^{-j}} u\circ\gamma(t)\,dt-\int_{\ell(\gamma)-2^{-j}}^{\ell(\gamma)} u\circ\gamma(t)\,dt\right|&=2^j\left|\int_0^{\ell(\gamma)-2^{-j}}\left[u\circ\gamma\left(t+2^{-j}\right)-u\circ\gamma(t)\right]dt\right|\\&\le 2^j\int_0^{\ell(\gamma)-2^{-j}}\left|u\circ\gamma\left(t+2^{-j}\right)-u\circ\gamma(t)\right|dt\\&\lesssim\int_0^{\ell(\gamma)-2^{-j}}\left[\rho_k\circ\gamma\left(t+2^{-j}\right)-\rho_k\circ\gamma(t)\right]dt\lesssim\int_0^{\ell(\gamma)}\rho_k\circ\gamma(t)dt.\end{aligned}$$

This means that ρ_k is a p-weak upper gradient of $\tilde{u}$.

To prove the triviality of $\dot{N}^1_{p,q}(X)$ with $q\in(p,\infty)$, for $u\in\dot{N}^1_{p,q}(X)$, applying Theorem (1.2.12), we have

$$\left\|\left\{I^{1,1}_{2^{-k}}(u)\right\}_{k\in Z}\right\|_{\ell^q(L^p(X))}\sim\|u\|_{\dot{N}^1_{p,q}(X)}<\infty,$$

which implies that $\left\|I^{1,1}_{2^{-k}}(u)\right\|_{L^p(X)}\to 0$ as $k\to\infty$. For every $k\in\mathbb{Z}$, let $\{x_k,i\}_i$ be a maximal set of X with $d(x_{k,i},x_{k,j})\ge 2^{-k-2}$ for all $i\ne j$. Then $B_k=\left\{B\left(x_{k,i},2^{-k}\right)\right\}_i$ is a covering of X with bounded overlap. Let $\left\{\varphi_{k,i}\right\}_i$ be a partition of unity with respect to B_k as in [86, Lemma 5.2]. We define a discrete convolution approximation to u by $u_{B_k}=\sum_i u_{B(x_{k,i},2^{-k})}\varphi_{k,i}$. By an argument similar to that of [86, Lemma 5.3], we have that $u_{B_k}\to u$ in $L^p_{loc}(X)$ and hence in $L^1_{loc}(X)$ as $k\to\infty$, and that $\operatorname{Lip} u_{B_k}(x)\le CI^{1,1}_{2^{-k+N}}(u)(x)$ for all $x\in X$, where $C\ge 1$ and $N\in\mathbb{N}$ are constants independent of k,x and u. Now $CI^{1,1}_{2^{-k+N}}(u)$ is an upper gradient of u_{B_k}. So, for every ball $B=B(x_B,r_B)$, by the weak $(1,p)$-Poincaré inequality, we have

$$\begin{aligned}\fint_B|u(z)-u_B|\,d\mu(z)&=\lim_{k\to\infty}\fint_B\left|u_{B_k}(z)-\left(u_{B_k}\right)_B\right|d\mu(z)\\&\le\liminf_{k\to\infty}r_B\left\{\fint_{B(x_B,\lambda r_B)}\left[I^{1,1}_{2^{-k+N}}(u)(z)\right]^p d\mu(z)\right\}^{1/p}=0,\end{aligned}$$

which implies that u is a constant on B and hence is a constant function on X. This finishes the proof of Theorem (1.2.19).

Theorem (1.2.20) [95] Suppose that X supports a weak $(1,p)$-Poincaré inequality with $p\in(1,\infty)$. Let $s\in(1,\infty)$. Then for $q\in(0,\infty]$, $\dot{M}^s_{np/(n+ps-p),q}(X)$ is trivial, and for $q\in(0,np/(n+ps-p)]$, $\dot{N}^s_{np/(n+ps-p),q}(X)$ is trivial. Moreover, if either X is complete or X supports a weak $(1,p-\epsilon)$-Poincaré inequality for some $\epsilon\in(0,p-1)$ then for $q\in(np/(n+ps-p),\infty]$, $\dot{N}^s_{np/(n+ps-p),q}(X)$ is trivial.

Proof. We first prove the triviality of $\dot{M}^s_{np/(n+ps-p),\infty}(X)=\dot{M}^{s,np/(n+ps-p)}(X)$ by considering the following three cases: Case $\mu(X)<\infty$, Case $\mu(X)=\infty$ and X is Ahlfors n-regular, and Case $\mu(X)=\infty$ but X is not Ahlfors n-regular.

Case $\mu(X)<\infty$. Notice that by (81), $2^{-k_0-1}\le\operatorname{diam}X<2^{-k_0}$ for some $k_0\in\mathbb{Z}$. In this case, it suffices to prove that $\dot{M}^{s,np/(n+ps-p)}(X)\subset\dot{M}^1_{p,\sigma}(X)$ for some $\sigma\in(0,p)$; then the triviality of $\dot{M}^{s,np/(n+ps-p)}(X)$ follows from that of $\dot{M}^1_{p,\sigma}(X)$ as proved by Theorem (1.2.19). let $u \in \dot{M}^{s,np/(n+ps-p)}(X)$ and let $g \in \mathcal{D}^s(u)$ with $\|g\|_{L^{np/(n+ps-p)}(X)} \le 2\|u\|_{\dot{M}^{s,np/(n+ps-p)}(X)}$. We claim that there exists $\sigma \in (0,p)$ such that

$$\left\|\left\{I^{1,\sigma}_{2^{-k}}(u)\right\}_{k\ge k_0-2}\right\|_{L^p(X,\ell^q)} \lesssim \|g\|_{L^{np/(n+ps-p)}(X)}. \tag{96}$$

Assume that this claim holds for a moment. By Theorem (1.2.16) (ii) and a variant of Lemma (1.2.11), we have $u \in \dot{M}^1_{p,\sigma}(X)$ and $\|u\|_{\dot{M}^1_{p,\sigma}(X)} \lesssim \|u\|_{\dot{M}^{s,np/(n+ps-p)}(X)}$.

To prove (96), by Lemma (1.2.8),

$$\begin{aligned}\left\|\left\{I^{1,\sigma}_{2^{-k}}(u)(x)\right\}_{k\ge k_0-2}\right\|^{\sigma}_{\ell^\sigma} &= \sum_{k\ge k_0-2} 2^{k\sigma} \inf_{c\in\mathbb{R}} \fint_{B(x,2^{-k})} |u(z)-c|^\sigma \, d\mu(z)\\ &\lesssim \sum_{k\ge k_0-2} 2^{-k(s-1)\sigma} \fint_{B(x,2^{-k})} [g(z)]^\sigma \, d\mu(z)\\ &\lesssim \sum_{k\ge k_0-2} \frac{2^{-k(s-1)\sigma}}{\mu(B(x,2^{-k}))} \sum_{j\ge k} \int_{B(x,2^{-j})\setminus B(x,2^{-j-1})} [g(z)]^\sigma \, d\mu(z).\end{aligned}$$

Notice that there exists $0<\kappa\le n$ such that for $j \ge k$,

$$\mu\left(B(x,2^{-k})\right) \gtrsim \mu\left(B(x,2^{-j})\right) 2^{-(k-j)k};$$

see [94]. Choosing $\sigma \in (0,p)$ such that $\kappa-(s-1)\sigma>0$, we have

$$\begin{aligned}&\left\|\left\{I^{1,\sigma}_{2^{-k}}(u)(x)\right\}_{k\ge k_0-2}\right\|^{\sigma}_{\ell^\sigma}\\ &\lesssim \sum_{k\ge k_0-2} \frac{1}{\mu(B(x,j))} \sum_{k\ge k_0-2}^{j} 2^{-k(s-1)\sigma} 2^{(k-j)k} \int_{B(x,2^{-j})\setminus B(x,2^{-j-1})} [g(z)]^\sigma d\mu(z)\\ &\lesssim \sum_{k\ge k_0-2} \frac{2^{-j(s-1)\sigma}}{\mu(B(x,2^{-j}))} \int_{B(x,2^{-j})\setminus B(x,2^{-j-1})} [g(z)]^\sigma d\mu(z) \lesssim \mathcal{I}_{(s-1)\sigma}(g^\sigma)(x).\end{aligned} \tag{97}$$

where for $\alpha \in (0,n)$, $\mathcal{I}_\alpha$ denotes the fractional integral defined by

$$\mathcal{I}_\alpha(u)(x) \equiv \int_X \frac{[d(x,y)]^\alpha}{\mu\left(B(x,d(x,y))\right)} u(y) d\mu(y).$$

Therefore,

$$\left\|\left\{I^{1,\sigma}_{2^{-k}}(u)\right\}_{k\ge k_0-2}\right\|_{L^p(X,\ell^q)} \lesssim \left\|\left[\mathcal{I}_{(s-1)\sigma}(g^\sigma)\right]^{1/\sigma}\right\|_{L^p(X)} \sim \left\|\mathcal{I}_{(s-1)\sigma}(g^\sigma)\right\|^{1/\sigma}_{L^{p/\sigma}(X)}.$$

Notice that for all $x \in X$ and $r \le \operatorname{diam} X$,

$$\mu(B(x,r)) \ge C\mu(X) \frac{r^n}{(diam\, X)^n} \gtrsim r^n.$$

Recall that $\mathcal{I}_\alpha$ is bounded from $L^p(X)$ to $L^{p_*(\alpha)}(X)$ for all $p \in (1, n/\alpha)$; see, for example, [83, Theorem 3.22]. We have

$$\left\|\left\{I^{1,\sigma}_{2^{-k}}(u)\right\}_{k\ge k_0-2}\right\|_{L^p(X,\ell^q)} \lesssim \|g^\sigma\|^{1/\sigma}_{L^{np/(n+ps-p)\sigma}(X)} \sim \|g\|_{L^{np/(n+ps-p)}(X)},$$

which gives (96).

Case $\mu(X)=\infty$ and X is Ahlfors n-regular. Recall that X is Ahlfors n-regular if for all $x \in X$ and $r>0$,

$$\mu(B(x,r)) \sim r^n.$$

Observe that the fractional integral $\mathcal{J}_\alpha$ is still bounded from $L^p(X)$ to $L^{p_*(\alpha)}(X)$, and hence by an argument similar to above, we have $\dot{M}^{s,np/(n+ps-p)}(X) \subset \dot{M}^1_{p,\sigma}(X)$ for some $\sigma \in (0,p)$, which implies the triviality of $\dot{M}^{s,np/(n+ps-p)}(X)$.

Case $\mu(X) = \infty$ but X is not Ahlfors n-regular. Notice that, in this case, we do not have the boundedness from $L^p(X)$ to $L^{p_*(\alpha)}(X)$ of the fractional integral $\mathcal{J}_\alpha$ and hence we cannot prove $\dot{M}^{s,np/(n+ps-p)}(X) \subset \dot{M}^1_{p,\sigma}(X)$ for some $\sigma \in (0,p)$ as above. But the ideas of an imbedding as above and the proof of Theorem (1.2.19) still work here for a localized version. Indeed, we will show that any function $u \in \dot{M}^{s,np/(n+ps-p)}(X)$ is constant on every ball of X, which implies that u is a constant function on whole X.

To this end, let $x_0 \in X, k_0$ be a negative integer and let η be a cutoff functions such that $\eta(x) = 1 - \operatorname{dist}(x, B(x_0, 2^{-k_0+1}))$ on $B(x_0, 2^{-k_0+1})$ and $\eta(x) = 0$ on $X \setminus B(x_0, 2^{-k_0+1})$. Observe that $\eta(x) = 1$ on $B(x_0, 2^{-k_0})$.

For every $u \in \dot{M}^{s,np/(n+ps-p)}(X)$ with $g \in \mathcal{D}^s(u) \cap L^{np/(n+ps-p)}(X)$, we first claim that $u\eta \in \dot{M}^{1,p}(X)$. Indeed, if $x, y \in B(x_0, 2^{-k_0+1})$,

$$\begin{aligned} |u(x)\eta(x) - u(y)\eta(y)| &= |u(x) - u(y)|\eta(x) + |u(x)||\eta(x) - \eta(y)| \\ &\le |u(x) - u(y)| + d(x,y)[|u(x)| + |u(y)|] \\ &\le d(x,y)[|u(x)| + |u(y)| + h(x) + h(y)] \end{aligned}$$

with $h = \chi_{B(0,2^{-k_0+2})} \sum_{j \ge k_0-4} 2^{(k_0-j)} I^{1,\sigma}_{2^{-j}}(u)$; if $x, y \in X \setminus B(x_0, 2^{-k_0+1}), u(x)\chi(x) = 0 = u(y)\chi(y)$; if $x \in B(x_0, 2^{-k_0+1})$ and $y \in X \setminus B(x_0, 2^{-k_0+2})$ or $y \in B(x_0, 2^{-k_0+1})$ and $x \in X \setminus B(x_0, 2^{-k_0+2})$, then by $d(x,y) \ge 2^{-k_0}$, we have

$$|u(x)\chi(x) - u(y)\chi(y)| = |u(x)| + |u(y)| \le 2^{k_0} d(x,y)[|u(x)| + |u(y)|].$$

This means that $(u\chi_{B(x_0,2^{-k_0+1})} + h) \in \mathcal{D}^1(u)$ modulo a constant depending on k_0. Notice that, by Lemma (1.2.8), $u \in L^p_{loc}(X)$. So to obtain $\left(u\chi_{B(x_0,2^{-k_0+1})} + h\right) \in L^p(X)$, it suffices to prove that $h \in L^p(X)$. For $\alpha \in (0,n)$, define the local fractional integral by

$$\mathcal{J}_\alpha(g)(x) = \int_{d(x,y) \le 2^{-k_0+4}} \frac{[d(x,y)]^\alpha}{\mu\left(B\left(x, d(x,y)\right)\right)} g(y)\,d\mu(y).$$

By an argument similar to that of (97), for $x \in B(x_0, 2^{-k_0+2})$ we still have

$$h(x) \le \left\| \left\{ I^{1,\sigma}_{2^{-k}}(u)(x) \right\}_{k \ge k_0-4} \right\|_{\ell^\sigma} \lesssim \left[\mathcal{J}_{(s-1)\sigma}\left(g^\sigma \chi_{B(x_0,2^{-k_0+4})} \right)(x) \right]^{1/\sigma}.$$

Obviously, $\mathcal{J}_{(s-1)\sigma}\left(g^\sigma \chi_{B(x_0,2^{-k_0+4})}\right)$ is supported in $B(x_0, 2^{-k_0+8})$. Moreover, by an argument similar to that of [83, Theorem 3.22], for $\alpha \in (0,n)$ one can prove that $\mathcal{J}_\alpha$ is bounded from $L^p\left(B(x_0, 2^{-k_0+8})\right)$ to $L^{p_*(\alpha)}(X)$ with its operator norm depending on k_0, x_0, α and X. This together with an argument similar to that for the case $\mu(X) < \infty$ implies that $h \in L^p(X)$ and hence the claim that $u\eta \in \dot{M}^{1,p}(X)$.

For $k \in \mathbb{N}$, set

$$\tilde{\rho}_k(x) \equiv \max_{j \ge k} 2^j \left(\inf_{c \in \mathbb{R}} \fint_{B(x,2^{-j})} |(u\eta)(z) - c|^\sigma d\mu(z) \right)^{1/\sigma}.$$

Since $u\eta \in \dot{M}^{1,p}(X)$, as what we did in the proof of Theorem (1.2.19), we can show that $\tilde{\rho}_k$ is a p-weak upper gradient of $\widetilde{u\eta}$. Notice that $\widetilde{u\eta}(x) = u(x)\eta(x) = u(x)$ for almost all $x \in B(x_0, 2^{-k_0})$, and that for all $x \in B(x_0, 2^{-k_0-1})$ and $j \ge k \ge k_0, (u\eta)_{B(x_0,2^{-j})} = u_{B(x_0,2^{-j})}$, and hence $\tilde{\rho}_k(x) = \sup_{j \ge k} I^{1,\sigma}_k(u)(x)$. Moreover,

$$\left\|\left\{I^{1,\sigma}_{2^{-k}}(u)\right\}_{k\geq k_0}\right\|_{L^p\left(B\left(x_0,2^{-k_0-1}\right),\ell^q\right)} \lesssim \left\|\left[\mathcal{J}_{(s-1)\sigma}\left(g^\sigma\chi_{B\left(x_0,2^{-k_0+4}\right)}\right)\right]^{1/\sigma}\right\|_{L^p(X)}$$
$$\lesssim \|g\|_{L^{np/(n+ps-p)}\left(B\left(x_0,2^{-k_0+4}\right)\right)} < \infty,$$
which implies that $\left\|\left\{I^{1,\sigma}_{2^{-k}}(u)(x)\right\}_{k\geq k_0}\right\|_{\ell^\sigma} < \infty$ and hence $\tilde{\rho}_k(x) \leq \left\|\left\{I^{1,\sigma}_{2^{-j}}(u)(x)\right\}_{j\geq k_0}\right\|_{\ell^\sigma} \to 0$ as $k\to\infty$ for almost all $x\in B(x_0,2^{-k_0-1})$. Then by the Lebesgue dominated convergence theorem, we have $\|\tilde{\rho}_k\|_{L^p\left(B\left(x_0,2^{-k_0-1}\right)\right)} \to 0 \; as \; k\to\infty$. Applying the Poincaré inequality, we obtain

$$\inf_{c\in\mathbb{R}} \fint_{B\left(x_0,2^{-k_0-1}/\lambda\right)} \left|u(z)-u_{B\left(x_0,2^{-k_0-1}/\lambda\right)}\right| d\mu(z)$$
$$= \inf_{c\in\mathbb{R}} \fint_{B\left(x_0,2^{-k_0-1}/\lambda\right)} \left|(u\eta)(z)-(u\eta)_{B\left(x_0,2^{-k_0-1}/\lambda\right)}\right| d\mu(z)$$
$$\lesssim \left(\fint_{B\left(x_0,2^{-k_0-1}\right)} \tilde{\rho}^p_k(z)\,d\mu(z)\right)^{1/p} \to 0.$$

This means that u is a constant on $B(x_0,2^{-k_0-1}/\lambda)$. Since k_0 is arbitrary, we conclude that u is a constant function.

Moreover, for q ∈ (0, ∞], the triviality of $\dot{M}^s_{np/(n+ps-p),q}(X)$ follows from
$$\dot{M}^s_{np/(n+ps-p),q}(X) \subset \dot{M}^{s,np/(n+ps-p)}(X).$$
Meanwhile, for $q\in(0,np/(n+ps-p)]$, the triviality of $\dot{N}^s_{np/(n+ps-p),q}(X)$ follows from $\dot{N}^s_{np/(n+ps-p),q}(X) \subset \dot{M}^s_{np/(n+ps-p),np/(n+ps-p)}(X) \subset \dot{M}^{s,np/(n+ps-p)}(X)$.

Finally, we prove the triviality of $\dot{N}^s_{np/(n+ps-p),q}(X)$ for $q\in(np/(n+ps-p),\infty]$. In fact, it follows from the triviality of $\dot{N}^s_{np/(n+ps-p),\infty}(X)$ since $\dot{N}^s_{np/(n+ps-p),q}(X)\subset \dot{N}^s_{np/(n+ps-p),\infty}(X)$. To see the triviality of $\dot{N}^s_{np/(n+ps-p),\infty}(X)$, we need the additional condition that X supports a weak $(1,p-\epsilon)$-Poincaré inequality for some $\epsilon\in(0,p-1)$. Recall from [87] that if X is complete and supports the weak $(1,p)$-Poincaré inequality, then X supports a weak $(1,p-\epsilon)$-Poincaré inequality for some $\epsilon\in(0,p-1)$. Without loss of generality, we can ask ϵ close to 0 such that
$$t \equiv s+\frac{n}{p}-\frac{n}{p-\epsilon} > 1.$$
Observe that
$$\frac{np}{n+p(s-1)} = \frac{n(p-\epsilon)}{n+(p-\epsilon)(t-1)} \tag{98}$$
Now we will consider the following two cases: $\mu(X)<\infty$ and $\mu(X)=\infty$.

Case $\mu(X) < \infty$. Assume that $2^{-k_0-1}\leq diam\,X < 2^{-k_0}$ for some $k_0\in\mathbb{Z}$. We claim that $\dot{N}^s_{np/(n+ps-p),\infty}(X) \subset \dot{M}^t_{np/(n+ps-p),\infty}(X)$ for any $t\in(1,s)$. Indeed, for every $u\in \dot{N}^s_{np/(n+ps-p),\infty}(X)$,
$$\left\|\sup_{k\geq k_0-2} I^{t,\sigma}_{2^{-k}}(u)\right\|_{L^p(X)} \sim \left\|\sup_{k\geq k_0-2} 2^{-k(s-t)} I^{t,\sigma}_{2^{-k}}(u)\right\|_{L^p(X)} \leq \left\|S^{s,t,\sigma}_{2^{-k_0+2}}(u)\right\|_{L^p(X)}. \tag{99}$$
Since
$$\|u\|_{\dot{N}^s_{np/(n+ps-p),\infty}(X)} \sim \sup_{k\geq k_0-2}\left\|S^{s,t,\sigma}_{2^{-k}}(u)\right\|_{L^p(X)}$$
and

$$\|u\|_{\dot{M}^s_{np/(n+ps-p),\infty}(X)} \sim \left\| \sup_{k\ge k_0-2} I^{t,\sigma}_{2^{-k}}(u) \right\|_{L^p(X)}$$

we conclude that $\|u\|_{\dot{M}^t_{np/(n+ps-p),\infty}(X)} \lesssim \|u\|_{\dot{N}^s_{np/(n+ps-p),\infty}(X)}$ and hence our claim. Then the triviality of $\dot{N}^s_{np/(n+ps-p),\infty}(X)$ follows from that of $\dot{M}^t_{n(p-\epsilon)/[n+(p-\epsilon)(t-1)],\infty}(X)$ and (98).

Case $\mu(X) = \infty$. Since the constant in (99) depends on k_0 and hence the diameter of X, we can not get the imbedding $\dot{N}^s_{np/(n+ps-p),\infty}(X) \subset \dot{M}^t_{np/(n+ps-p),\infty}(X)$ for $t \in (1, s)$. But for any fixed $x_0 \in X$ and $k_0 \in \mathbb{Z}$, we still have

$$\left\| \sup_{k\ge k_0-16} I^{t,\sigma}_{2^{-k}}(u) \right\|_{L^p\left(B\left(x_0,2^{-k_0+8}\right)\right)} \sim \left\| \sup_{k\ge k_0-16} 2^{-k(s-t)} I^{t,\sigma}_{2^{-k}}(u) \right\|_{L^p\left(B\left(x_0,2^{-k_0+8}\right)\right)}$$
$$\lesssim \left\| S^{s,t,\sigma}_{2^{-k_0+16}}(u) \right\|_{L^p\left(B\left(x_0,2^{-k_0+8}\right)\right)} < \infty,$$

which further means that $u \in M^{t,(p-\epsilon)/[n+(p-\epsilon)(t-1)]}\left(B(x_0, 2^{-k_0+8})\right)$. With the weak $(1, p-\epsilon)$-Poincaré inequality in hand, by adapting the arguments in Case *μ(X)* but X is not Ahlfor n-regular as above, we still can prove that *u* is constant on ball $B(x_0, 2^{k_0-1}/\lambda)$. Hence u is a constant function on whole *X*. We omit the details. This finishes the proof of Theorem (1.2.20).

Finally, we give an example to show the "necessity" of the weak $(1, n)$-Poincaré inequality to ensure the triviality of $\dot{B}^s_{n/s,n/s}(X)$ for $s \in [1, \infty)$.

Theorem (1.2.21) [95] For each $p \in (2, \infty)$, there exists an Ahlfors 2-regular space *X* such that *X* supports a weak $(1, p)$-Poincaré inequality but for every $s \in (0, \infty)$, $\dot{B}^s_{2/s,2/s}(X)$ is not trivial.

Proof. Let $\alpha \in (0,1)$ and E_α be the cantor set in [0,1] obtained by first removing an interval of length $1-\alpha$ and leaving two intervals of length $\alpha/2$ and then continuing inductively. The Hausdorff dimension d_α of E_α is $\log 2/\log(2/\alpha)$. The space X_α is obtained by replacing each of the complementary intervals of E_α by a closed square having that interval as one of its diagonals. Then X_α is Ahlfors 2-regular with respect to Euclidean distance and by [88, Theorem 3.1], for any

$$p > \frac{2-d_\alpha}{1-d_\alpha} = 2 + \frac{\log 2}{-\log \alpha},$$

$X_\alpha$supports the $(1, p)$-Poincaré inequality.

So for any $p > 2$, choosing $\alpha \in \left(0, 2^{-1/(p-2)}\right)$, we know that X_α supports the weak $(1, p)$-Poincaré inequality. Moreover, for any $x = (x_1, x_2) \in X_\alpha$, define the Cantor function by $u(x) = \mathcal{H}^{d_\alpha}([0, x_1] \cap E_\alpha)$. Then u is constant on each square generating X_α and moreover, $|u(x) - u(y)| \le |x_1 - y_1|^{d_\alpha} \lesssim [d(x, y)]^{d_\alpha}$ for all $x, y \in X_\alpha$ (see [86]). For $s > d_\alpha$, taking $g(x) = 2[d(x, E_\alpha)]^{d_\alpha - s}$ for all $x \in X_\alpha$, we have $g \in \mathcal{D}^s(u)$. We claim that $g \in L^q(X_\alpha)$ if

$$0 < q < \frac{2-d_\alpha}{s-d_\alpha} = \frac{\log 2 - 2\log\alpha}{(s-1)\log 2 - s\log\alpha}.$$

Indeed, on each square $Q \subset X_\alpha$ with diagonal length $2^{-j}\alpha^j(1-\alpha)$, we have

$$\int_Q [g(x)]^q dx \lesssim \left|2^{-j}\alpha^j\right|^{(d_\alpha - s)q+2}$$

since $(d_\alpha - s)q + 1 > -1$, namely, $q < 2/(s-d_\alpha)$ which is given by $q < (2-d_\alpha)/(s-d_\alpha)$. Observing that there are 2^j such squares, we have

$$\int_{X_\alpha}[g(x)]^q dx \lesssim \sum_{j\ge1} 2^j\left|2^{-j}\alpha^j\right|^{(d_\alpha-s)q+2} \lesssim \sum_{j\ge1} 2^{j-j[(d_\alpha-s)q+2](1-\log\alpha/\log 2)} < \infty,$$

where in the last inequality we use

$$1-[(d_\alpha-s)q+2](1-\log\alpha/\log 2) = 1-[(d_\alpha-s)q+2]/d_\alpha < 0,$$

which is equivalent to $q<(2-d_\alpha)/(s-d_\alpha)$. Thus $u\in \dot{M}^{s,q}(X_\alpha)$. Taking $q=2/s$ for each $s\in(d_\alpha,\infty)$, we know that $\dot{M}^{s,2/s}(X_\alpha)$ are nontrivial. Notice that $\dot{M}^{s,2/s}(X_\alpha)\subset \dot{B}^1_{2,2}(X_\alpha)$ when $s>1$. Similarly, when $0<s<1, \dot{M}^{1,2}(X_\alpha)\subset \dot{B}^s_{2/s,2/s}(X_\alpha)$, and moreover, $\dot{B}^s_{2/s,2/s}(X_\alpha)$ contains the restriction of any function in $\dot{B}^s_{2/s,2/s}(\mathbb{R}^n)$ to X_α.Then $\dot{B}^s_{2/s,2/s}(X_\alpha)$ for all $s\in(0,\infty)$ are nontrivial. This finishes the proof of Theorem (1.2.21).

Corollary (1.2.22) [314] For each $0\le\epsilon<\infty$, there exists an Ahlfors 2-regular space X such that X supports a weak $(1,1+2\epsilon)$-Poincaré inequality but for every $0\le\epsilon<\infty$, $\dot{B}^{1+\epsilon}_{2/1+\epsilon,2/1+\epsilon}(X)$ is not trivial.

Proof. Let $\alpha\in(0,1)$ and E_α be the cantor set in [0,1] obtained by first removing an interval of length $1-\alpha$ and leaving two intervals of length $\alpha/2$ and then continuing inductively. The Hausdorff dimension d_α of E_α is $\log 2/\log(2/\alpha)$. The space X_α is obtained by replacing each of the complementary intervals of E_α by a closed square having that interval as one of its diagonals. Then X_α is Ahlfors 2-regular with respect to Euclidean distance and by [88, Theorem 3.1], for any

$$2+\epsilon > \frac{2-d_\alpha}{1-d_\alpha} = 2+\frac{\log 2}{-\log\alpha},$$

X_α supports the $(1,1+2\epsilon)$-Poincaré inequality.

So for any $\epsilon\ge0$, choosing $\alpha\in\left(0,2^{-1/\epsilon}\right)$, we know that X_α supports the weak $(1,1+2\epsilon)$-Poincaré inequality. Moreover, for any $x^m=(x_1^m,x_2^m)\in X_\alpha$, define the Cantor function by $u_m(x^m)=\mathcal{H}^{d_\alpha}([0,x_1^m]\cap E_\alpha)$. Then u_m is constant on each square generating X_α and moreover, $\sum|u_m(x^m)-u_m(y^m)|\le\sum|x_1^m-y_1^m|^{d_\alpha}\lesssim\sum[d(x^m,y^m)]^{d_\alpha}$ for all $x^m,y^m\in X_\alpha$ (see [86]). For $\epsilon>d_\alpha-1$, taking $g(x^m)=2[d(x^m,E_\alpha)]^{d_\alpha-(1+\epsilon)}$ for all $x^m\in X_\alpha$, we have $g\in\mathcal{D}^s(u_m)$. We claim that $g\in L^q(X_\alpha)$ if

$$0<q<\frac{2-d_\alpha}{(1+\epsilon)-d_\alpha} = \frac{\log 2-2\log\alpha}{(\epsilon)\log 2-(1+\epsilon)\log\alpha}.$$

Indeed, on each square $Q\subset X_\alpha$ with diagonal length $2^{-j}\alpha^j(1-\alpha)$, we have

$$\int_Q\sum[g(x^m)]^q dx^m \lesssim \left|2^{-j}\alpha^j\right|^{(d_\alpha-(1+\epsilon))q+2}$$

since $\left(d_\alpha-(1+\epsilon)\right)q+1>-1$, namely, $q<2/((1+\epsilon)-d_\alpha)$ which is given by $q<(2-d_\alpha)/\left((1+\epsilon)-d_\alpha\right)$. Observing that there are 2^j such squares, we have

$$\int_{X_\alpha}\sum[g(x^m)]^q dx^m \lesssim \sum_{j\ge1} 2^j\left|2^{-j}\alpha^j\right|^{(d_\alpha-(1+\epsilon))q+2} \lesssim \sum_{j\ge1} 2^{j-j\left[\left(d_\alpha-(1+\epsilon)\right)q+2\right](1-\log\alpha/\log 2)} < \infty,$$

where in the last inequality we use

$$1-\left[\left(d_\alpha-(1+\epsilon)\right)q+2\right](1-\log\alpha/\log 2) = 1-\left[\left(d_\alpha-(1+\epsilon)\right)q+2\right]/d_\alpha < 0,$$

which is equivalent to $q<(2-d_\alpha)/((1+\epsilon)-d_\alpha)$. Thus $u_m\in\dot{M}^{1+\epsilon,q}(X_\alpha)$. Taking $q=2/1+\epsilon$ for each $d_\alpha-1\le\epsilon<\infty$, we know that $\dot{M}^{1+\epsilon,2/1+\epsilon}(X_\alpha)$ are nontrivial. Notice that $\dot{M}^{1+\epsilon,2/1+\epsilon}(X_\alpha)\subset\dot{B}^1_{2,2}(X_\alpha)$ when $\epsilon>0$. Similarly, when $-1<\epsilon<0$, $\dot{M}^{1,2}(X_\alpha)\subset\dot{B}^{1+\epsilon}_{2/1+\epsilon,2/1+\epsilon}(X_\alpha)$, and moreover, $\dot{B}^{1+\epsilon}_{2/1+\epsilon,2/1+\epsilon}(X_\alpha)$ contains the restriction of any function in $\dot{B}^{1+\epsilon}_{2/1+\epsilon,2/1+\epsilon}(\mathbb{R}^n)$ to X_α.Then $\dot{B}^{1+\epsilon}_{2/1+\epsilon,2/1+\epsilon}(X_\alpha)$ for all $0\le\epsilon<\infty$ are nontrivial. This finishes the proof of Corollary (1.2.22).

Chapter 2
Multi-Parameter Triebel-Lizorkin and Singular Integrals with Boundedness of Composition Operators

We show the boundedness of flag singular integral operators on Triebel–Lizorkin space and Besov space. The methods here can be applied to develop easily the theory of multi-parameter Triebel–Lizorkin and Besov spaces in the pure product setting. The derivation of the boundedness of singular integrals on the spaces is substantially different from those used where atomic decomposition on the one-parameter Triebel–Lizorkin and Besov spaces. The discrete Littlewood–Paley analysis allows us to avoid using the atomic decomposition or deep Journe's covering lemma in multi-parameter setting. We then establish the boundedness of composition of two Calderón–Zygmund singular integral operators with different homogeneities on these Triebel–Lizorkin and Besov spaces.

Section (2.1) Besov Spaces Associated with Flag Singular Integrals

The multi-parameter pure product theory has been developed. This theory includes the boundedness on $L^p(1 < p < \infty)$ and multi-parameter Hardy spaces $H^p(0 < p \leq 1)$ of singular integral operators of the form $Tf = K * f$, where K is homogeneous, that is, $\delta_1 \cdots \delta_n K(\delta \cdot x) = K(x)$, or, more generally, $K(x)$ satisfies a certain differential inequalities and cancellation conditions such that $\delta_1 \cdots \delta_n K(\delta \cdot x)$ also satisfy the same bounds. This theory also includes the atomic decomposition of Hardy spaces, duality and interpolation theorems on product spaces, maximal function characterization of Hardy spaces, etc. See Gundy and Stein [96], Carleson [97], Fefferman and Stein [98], Fefferman [99], Chang and Fefferman [100–102], Journe [103–104], Pipher [105], etc.

Substantial attention has been paid to the theory of flag singular integral operators in the multi-parameter setting. This is an implicit multi-parameter structure which arises in a number of occasions such as Marcinkiewcz multiplier operators associated with the sub-laplacian L and the centralizer T on the Heisenberg group (see Muller–Ricci–Stein [106]) and flag singular integrals (see Nagel–Ricci–Stein [107]). We focus on the case that the implicit multi-parameter structure is induced by the flag singular integrals on $\mathbb{R}^n \times \mathbb{R}^m$ studied by Nagel–Ricci–Stein [107]. The simplest form of flag singular integral kernel $K(x, y)$ on $\mathbb{R}^n \times \mathbb{R}^m$ is defined through a projection of a product kernel $\widetilde{K}(x, y, z)$ defined on $\mathbb{R}^{n+m} \times \mathbb{R}^m$ given by

$$K(x, y) = \int_{\mathbb{R}^m} \widetilde{K}(x, y - z, z) dz. \tag{1}$$

Discrete Littlewood–Paley–Stein analysis and multi-parameter Hardy space theory has been developed in the framework of flag singular integral operators by Han and Lu [108]. One of the main ideas of the program in [108] is to develop a discrete version of Calderón reproducing formula associated with the given multi-parameter structure, and thus prove a Min-Max comparison inequality in this setting. This discrete scheme of Littlewood–Paley–Stein analysis is particularly useful in dealing with the Hardy spaces H^p for $0 < p \leq 1$. Using this method of discretizing, they are able to show that the flag singular integral operators are bounded on H_F^p for all $0 < p \leq 1$ and then further show that these operators are also bounded from H_F^p to L^p for all $0 < p \leq 1$. This method offers an alternate approach of Fefferman's method of restricting singular integral operator's action on the rectangle atoms. Thus, they bypass the use of Journe's covering lemma in proving the H_F^p to L^p boundedness for all $0 < p \leq 1$. The duality theory of the Hardy space, Calderón–Zygmund decomposition and interpolation theorems have also been established in the setting of multi-parameter flag setting in [108]. Multi-parameter Hardy space theory associated with the Zygmund dilation has also been developed in [109] in which endpoint results of singular integral operators (introduced by Ricci and Stein in [110]) have been established.

We study initiated in [108], and introduce the multi-parameter Triebel–Lizorkin and Besov spaces and prove the boundedness of the flag singular integral operators on such spaces. Though the theory of one-parameter Triebel–Lizorkin and Besov spaces has been very well developed in the past decades, the multi-parameter counterpart of such a theory is still absent. We develop a theory of multi-parameter Triebel–Lizorkin and Besov spaces using the discrete Littlewood–Paley–Stein analysis in the setting of implicit multi-parameter structure.

We first introduce the continuous version of the Littlewood–Paley–Stein square function gF. Inspired by the idea of lifting method of proving the $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ boundedness given in [106], we will use a lifting method to construct a test function defined on $\mathbb{R}^n \times \mathbb{R}^m$, given by the non-standard convolution $*_2$ on the second variable only:

$$\psi(x,y) = \psi^{(1)} *_2 \psi^{(2)}(x,y) = \int_{\mathbb{R}^m} \psi^{(1)}(x, y-z) *_2 \psi^{(2)}(z)dz, \qquad (2)$$

where $\psi^{(1)} \in \mathcal{S}(\mathbb{R}^{n+m}), \psi^{(2)} \in \mathcal{S}(\mathbb{R}^m)$, and satisfy

$$\sum_j \left|\widehat{\psi^{(1)}}\left(2^{-j}\xi_1, 2^{-j}\xi_2\right)\right|^2 = 1$$

for all $(\xi_1, \xi_2) \in \mathbb{R}^n \times \mathbb{R}^m \backslash \{(0,0)\}$, and

$$\sum_k \left|\widehat{\psi^{(1)}}(2^{-k}\eta)\right|^2 = 1$$

for all $\eta \in \mathbb{R}^m \backslash \{0\}$, and the moment conditions

$$\int_{\mathbb{R}^{n+m}} x^\alpha y^\beta \psi^{(1)}(x,y)dxdy = \int_{\mathbb{R}^m} z^\gamma \psi^{(2)}(z)dz = 0$$

for all multi-indices α, β, and γ. We remark here that this idea of considering such a convolution was introduced in [108]. It is this subtle convolution $*_2$ which provides a rich theory for the implicit multi-parameter analysis.

We now recall some definitions given in [107]. Following closely from [107], we begin with the definitions of a class of distributions on an Euclidean space $\mathbb{R}^N$. A k-normalized bump function on a space $\mathbb{R}^N$. is a C^k-function supported on the unit ball with C^k-norm bounded by 1. As pointed out in [107], the definitions given below are independent of the choices of k, and thus we will simply refer to “normalized bump function” without specifying k.

For the sake of simplicity of presentations, we will restrict our considerations to the case $\mathbb{R}^N = \mathbb{R}^{n+m} \times \mathbb{R}^m$. We will rephrase Definition 2.1.1 in [107] of product kernel in this case as follows:

Definition (2.1.1) [120] A product kernel on $\mathbb{R}^{n+m} \times \mathbb{R}^m$ is a distribution K on $\mathbb{R}^{n+m+m}$ which coincides with a C^∞ function away from the coordinate subspaces $(0,0,z)$ and $(x,y,0)$, where $(0,0) \in \mathbb{R}^{n+m}$ and $(x,y) \in \mathbb{R}^{n+m}$, and satisfies

(i) (Differential Inequalities) For any multi-indices $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_m)$ and $\gamma m = (\gamma_1, \dots, \gamma_m)$,

$$|\partial_x^\alpha \partial_y^\beta \partial_z^\gamma K(x,y,z)| \le C_{\alpha,\beta,\gamma}(|x| + |y|)^{-n-m-|\alpha|-|\beta|} \cdot |z|^{-m-|\gamma|}$$

for all $(x,y,z) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$ with $|x| + |y| \ne 0$ and $|z| \ne 0$.

(ii) (Cancellation Condition)

$$\left|\int_{\mathbb{R}^m} \partial_x^\alpha \partial_y^\beta K(x,y,z)\varphi_1(\delta z)dz\right| \le C_{\alpha,\beta}(|x|+|y|)^{-n-m-|\alpha|-|\beta|}$$

for all multi-indices α, β and every normalized bump function φ_1 on $\mathbb{R}^m$ and every $\delta > 0$;

$$\left|\int_{\mathbb{R}^{n+m}} \partial_z^\gamma K(x,y,z)\varphi_2(\delta x, \delta y)dxdy\right| \le C_\gamma |z|^{-m-|\gamma|}$$

for every multi-index γ and every normalized bump function φ_2 on $\mathbb{R}^{n+m}$ and every $\delta > 0$; and

$$\left|\int_{\mathbb{R}^{n+m+m}} K(x,y,z)\varphi_3(\delta_1 x, \delta_1 y, \delta_2 z)dxdydz\right| \le C$$

for every normalized bump function φ_3 on $\mathbb{R}^{n+m+m}$and every $\delta_1 > 0$ and $\delta_2 > 0$.

Definition (2.1.2) [120] A flag kernel on $\mathbb{R}^n \times \mathbb{R}^m$ is a distribution on $\mathbb{R}^{n+m}$ which coincides with a C^∞ function away from the coordinate subspace $\{(0,y)\} \subset \mathbb{R}^{n+m}$ where $0 \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ and satisfies

(i) (Differential Inequalities) For any multi-indices $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_m)$,

$$|\partial_x^\alpha \partial_y^\beta K(x,y)| \le C_{\alpha,\beta}|x|^{-n-|\alpha|} \cdot (|x|+|y|)^{-m-|\beta|}$$

for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$ with $|x| \neq 0$.

(ii) (Cancellation Condition)

$$\left|\int_{\mathbb{R}^m} \partial_x^\alpha K(x,y)\varphi_1(\delta y)dy\right| \le C_\alpha |x|^{-n-|\alpha|}$$

for every multi-index α and every normalized bump function φ_1 on $\mathbb{R}^m$ and every $\delta > 0$;

$$\left|\int_{\mathbb{R}^n} \partial_y^\beta K(x,y)\varphi_2(\delta x)dx\right| \le C_\gamma |y|^{-m-|\beta|}$$

for every multi-index β and every normalized bump function φ_2 on $\mathbb{R}^n$ and every $\delta > 0$; and

$$\left|\int_{\mathbb{R}^{n+m}} K(x,y)\varphi_3(\delta_1 x, \delta_2 y)dxdy\right| \le C$$

for every normalized bump function φ_3 on $\mathbb{R}^{n+m}$ and every $\delta_1 > 0$ and $\delta_2 > 0$.

By a result in [106], we may assume first that a flag kernel K lies in $L^1(\mathbb{R}^{n+m})$. Thus, there exists a product kernel K^* on $\mathbb{R}^{n+m} \times \mathbb{R}^m$ such that

$$K(x,y) = \int_{\mathbb{R}^m} K^*(x, y-z, z)dz\,.$$

Conversely, if a product kernel K^* lies in $L^1(\mathbb{R}^{n+m} \times \mathbb{R}^m)$, then $K(x,y)$ defined as above is a flag kernel on $\mathbb{R}^n \times \mathbb{R}^m$. As pointed out in [106], we may always assume that $K(x,y)$, a flag kernel, is integrable on $\mathbb{R}^n \times \mathbb{R}^m$ by using a smooth truncation argument.

In order to use the Littlewood–Paley–Stein square function g_F to define the Hardy space, one needs to extend the Littlewood–Paley–Stein square function to be defined on a suitable distribution space. We will recall several definitions introduced in [108] concerning the test function space on $\mathbb{R}^n \times \mathbb{R}^m$associated with the flag singular integral operators.

Definition (2.1.3) [120] A Schwartz test function $f(x,y,z)$ defined on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$ is said to be a product test function on $\mathbb{R}^{n+m} \times \mathbb{R}^m$ if

$$\int f(x,y,z)x^\alpha y^\beta dxdy = \int f(x,y,z)z^\gamma dz = 0 \qquad (3)$$

for all multi-indices α, β, γ of nonnegative integers.

If f is a product test function on $\mathbb{R}^{n+m} \times \mathbb{R}^m$we denote $f \in \mathcal{S}_\infty(\mathbb{R}^{n+m} \times \mathbb{R}^m)$ and the norm of f is defined by the norm of Schwartz test function.
Furthermore, the test function space $\mathcal{S}_F$ on $\mathbb{R}^n \times \mathbb{R}^m$associated with the flag structure can be defined as follows:
Definition (2.1.4) [120] A function $f(x,y)$ defined on $\mathbb{R}^n \times \mathbb{R}^m$ is said to be a test function in $\mathcal{S}_F(\mathbb{R}^n \times \mathbb{R}^m)$ if there exists a function $f \in \mathcal{S}_\infty(\mathbb{R}^{n+m} \times \mathbb{R}^m)$ such that

$$f(x,y) = \int_{\mathbb{R}^m} f^*(x, y-z, z)dz. \tag{4}$$

If $f \in \mathcal{S}_F(\mathbb{R}^n \times \mathbb{R}^m)$, then the norm of f is defined by
$\|f\|_{\mathcal{S}_F(\mathbb{R}^n \times \mathbb{R}^m)} = \inf \|f^*\|_{\mathcal{S}_\infty(\mathbb{R}^{n+m} \times \mathbb{R}^m)}$: for all representations of f in (4)}.
We denote by $(\mathcal{S}_F)'$ the dual space of $\mathcal{S}_F$.

For $f \in (\mathcal{S}_F), g_F(f)$, the Littlewood–Paley–Stein square function of f is defined by

$$g_F(f)(x,y) = \left\{ \sum_j \sum_k \left| \psi_{j,k} * f(x,y) \right|^2 \right\}^{\frac{1}{2}},$$

where functions

$$\psi_{j,k}(x,y) = \psi_j^{(1)} *_2 \psi_k^{(2)}(x,y),$$
$$\psi_j^{(1)}(x,y) = 2^{(n+m)j} \psi^{(1)}\left(2^j x, 2^j y\right) \text{ and } \psi_k^{(2)}(z) = 2^{mk} \psi(2)(2^k z). \tag{5}$$

By taking the Fourier transform, it is easy to see the following continuous version of the Calderón reproducing formula holds on $L^2(\mathbb{R}^{n+m})$,

$$f(x,y) = \sum_j \sum_k \psi_{j,k} * \psi_{j,k} * f(x,y). \tag{6}$$

Formally, in [108] the flag Hardy space is defined as follows:
Definition (2.1.5) [120] Let $0 < p \leq 1$. $H_F^p(\mathbb{R}^n \times \mathbb{R}^m) = \{f \in (\mathcal{S}_F)' : g_F(f) \in L^p(\mathbb{R}^n \times \mathbb{R}^m)\}$.

If $f \in H_F^p(\mathbb{R}^n \times \mathbb{R}^m)$the norm of f is defined by

$$\|f\|_{H_F^p} = \|g_F(f)\|_p. \tag{7}$$

It is proved in [108] that this definition is independent of the choice of functions $\psi_{j,k}$ and the following boundedness results of flag singular integral operators are established.
Theorem (2.1.6) [120] Suppose that T is a flag singular integral defined on $\mathbb{R}^n \times \mathbb{R}^m$ with the flag kernel $K(x,y) = \int_{\mathbb{R}^m} K(x, y-z, z)dz$, where the product kernel K^* satisfies the conditions in Definition (2.1.2). Then T is bounded on H_F^p and from H_F^p to L^p for $0 < p \leq 1$. Namely, for all $0 < p \leq 1$ there exists a constant C_p such that

$$\|T(f)\|_{H_F^p} \leq C_p \|f\|_{H_F^p}, \quad \|T(f)\|_{H_F^p} \leq C_p \|f\|_{L^p}.$$

Moreover, T is bounded on BMO_F. Namely, there exists a constant C such that

$$\|T(f)\|_{BMO_F} \leq C \|f\|_{BMO_F}.$$

Having obtained the boundedness of flag singular integral operators on multi-parameter Hardy spaces, a natural question arises: Are these operators bounded on more general function spaces? We will use the approach in [108] to develop a satisfactory theory of the Triebel–Lizorkin space and Besov space associated with the implicit multi-parameter structures induced by the flag singular integrals. Indeed, our ideas and methods apply easily to the pure product theory of Triebel–Lizorkin and Besov spaces. This pure product theory appears to be new and has not been studied.

We now describe our approach and results in more details.

Definition (2.1.7) [120] Let $0 < p, q \leq \infty, s = (s_1, s_2) \in \mathbb{R}^2$. The Triebel–Lizorkin type space $\dot{F}^{s,q}_{p,F}(\mathbb{R}^n \times \mathbb{R}^m)$ associated with the flag singular integrals is defined by

$$\dot{F}^{s,q}_{p,F}(\mathbb{R}^n \times \mathbb{R}^m) = \{f \in (\mathcal{S}_F)' : \|f\|_{\dot{F}^{s,q}_{p,F}(\mathbb{R}^n \times \mathbb{R}^m)} < \infty\},$$

where

$$\|f\|_{\dot{F}^{s,q}_{p,F}} = \left\| \left(\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} 2^{-js_1 q} 2^{-(j+k)s_2 q} \left|\psi_{j,k} * f(\cdot)\right|^q \right)^{\frac{1}{q}} \right\|_p . \tag{8}$$

The Besov space $\dot{B}^{s,q}_{p,F}(\mathbb{R}^n \times \mathbb{R}^m)$ associated with the flag singular integrals is defined as

$$\dot{B}^{s,q}_{p,F}(\mathbb{R}^n \times \mathbb{R}^m) = \{f \in (\mathcal{S}_F) : \|f\|_{\dot{B}^{s,q}_{p,F}} < \infty\},$$

where

$$\|f\|_{\dot{B}^{s,q}_{p,F}} = \left(\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} 2^{-js_1 q} 2^{-(j+k)s_2 q} \left\|\psi_{j,k} * f\right\|_p^q \right)^{\frac{1}{q}} . \tag{9}$$

If $s = (0, 0)$, then $\dot{F}^{0,2}_{p,F} = L^p$ when $1 < p < \infty$, and $\dot{F}^{0,2}_{p,F} = H^p_F$ when $0 < p \leq 1$ defined in [108]. A natural question arises whether this definition is independent of the choice of functions $\psi_{j,k}$. To study the $\dot{F}^{s,q}_{p,F}$-boundedness of flag singular integrals we need to discretize the norm of $\dot{F}^{s,q}_{p,F}$. In order to obtain such a discrete $\dot{F}^{s,q}_{p,F}$ norm we will prove the Min-Max comparison principle. To prove such principle is the Calderón reproducing formula (6). To be more specific, in [108] they have proved that the formula (6) still holds on test function space $\mathcal{S}_F(\mathbb{R}^n \times \mathbb{R}^m)$ and its dual space $(\mathcal{S}_F)'$ (see Theorem 3.6 in [108]). Furthermore, using an approximation procedure and the almost orthogonality argument, the following discrete Calderón reproducing formula is proved in [108].

Theorem (2.1.8) [120] [108, Theorem 1.8] Suppose that $\psi_{j,k}$. are the same as in (5). Then

$$f(x, y) = \sum_j \sum_k \sum_J \sum_I |I||J| \tilde{\psi}_{j,k}(x, y, x_I, y_J) \psi_{j,k} * f(x_I, y_J), \tag{10}$$

where $\tilde{\psi}_{j,k}(x, y, x_I, y_J) \in \mathcal{S}_F(\mathbb{R}^n \times \mathbb{R}^m), I \subset \mathbb{R}^n, J \subset \mathbb{R}^m$, are dyadic cubes with side-length $l(I) = 2^{-j-N}$ and $(J) = 2^{-j-N} + 2^{-j-N} - j - N$ for a fixed large integer $N; x_I, y_J$ are any fixed points in I, J, respectively; and the series in (10) converges in the norm of $\mathcal{S}_F(\mathbb{R}^n \times \mathbb{R}^m)$ and in the dual space $(\mathcal{S}_F)'$.

The discrete Calderón reproducing formula (10) provides the following Min-Max comparison principle in Treibel–Lizorkin spaces. We use the notation $A \approx B$ to denote that two quantities A and B are comparable independent of other substantial quantities involved.

We show the boundedness of the flag singular integral on Triebel–Lizorkin spaces and Besov spaces. We outline the corresponding results for Triebel–Lizorkin spaces and Besov spaces in the pure product setting and boundedness of singular integral operators on these spaces without any proof. Their proofs can be carried out easily following the more complicated case in the implicit flag multi-parameter structure dealt.

We establish the Min-Max comparison principles in Triebel–Lizorkin and Besov spaces. These principles are important in proving that the Triebel–Lizorkin and Besov spaces are well defined.

We first recall some decay estimates proved in [108]. If $\psi^*(x,y,z,u,v,w)$ for $(x,y,z),(u,v,w) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$ is a smooth function and satisfies the differential inequalities

$$\left|\partial_x^{\alpha_1}\partial_y^{\beta_1}\partial_z^{\gamma_1}\partial_u^{\alpha_2}\partial_v^{\beta_2}\partial_w^{\gamma_2}\psi^\#(x,y,z,u,v,w)\right| \\ \leq A_{N,M,\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2}(1+|x-u|+|y-v|)^{-N}(1+|z-w|)^{-M} \quad (11)$$

and the cancellation conditions

$$\int \psi^*(x,y,z,u,v,w)x^{\alpha_1}y^{\beta_1}dxdy = \int \psi^*(x,y,z,u,v,w)z^{\gamma_1}dz \\ = \int \psi^*(x,y,z,u,v,w)u^{\alpha_2}v^{\beta_2}dudv = \int \psi^*(x,y,z,u,v,w)w^{\gamma_2}dw = 0, \quad (12)$$

and for fixed $x_0 \in \mathbb{R}^n, y_0 \in \mathbb{R}^m, \varphi(x,y,z,x_0,y_0) \in S_\infty(\mathbb{R}^{n+m} \times \mathbb{R}^m)$ and satisfies

$$|\partial_x^{\alpha_1}\partial_y^{\beta_1}\partial_z^{\gamma_1}\,\varphi(x,y,z,x_0,y_0)| \\ \leq B_{N,M,\alpha_1,\beta_1,\gamma_1}(1+|x-x_0|+|y-y_0|)^{-N}(1+|z|)^{-M}, \quad (13)$$

for all positive integers N, M and multi-indices $\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2$of nonnegative integers, then we have the following almost orthogonality estimate:

Lemma (2.1.9) [120] [108] For any given positive integers L_1, L_2 and K_1, K_2, there exists a constant $C = C(L_1,L_2,K_1,K_2)$ depending only on L_1,L_2,K_1,K_2 and the constants in (11) and (13) such that for all positive t,s,t',s', we have

$$\left|\iint_{\mathbb{R}^{n+m+m}} \psi^*_{t,s}(x,y,z,u,v,w)\varphi^*_{t',s'}(u,v,w,x_0,y_0)dudvdw\right| \\ \leq C\left(\frac{t}{t'}\wedge\frac{t'}{t}\right)^{L_1}\left(\frac{s}{s'}\wedge\frac{s'}{s}\right)^{L_2}\frac{(t\vee t')^{K_1}}{(t\vee t'+|x-x_0|+|y-y_0|)^{(n+m+K_1)}}\cdot\frac{(s\vee s')^{K_2}}{(s\vee s'+|z|)^{m+K_2}}, \quad (14)$$

where $\psi^*_{t,s}(x,y,z,u,v,w) = t^{-n-m}s^{-m}\psi^*(\frac{x}{t},\frac{y}{t},\frac{z}{s},\frac{u}{t},\frac{v}{t},\frac{w}{s})$ and

$$\varphi^*_{t,s}(x,y,z,x_0,y_0) = t^{-n-m}s^{-m}\varphi^*\left(\frac{x}{t},\frac{y}{t},\frac{z}{s},\frac{x_0}{t},\frac{y_0}{t}\right).$$

Lemma (2.1.10) [120] [108] For any given positive integers L_1, L_2 and K_1, K_2, there exists a constant $C = C(L_1,L_2,K_1,K_2)$ depending only on L_1,L_2,K_1,K_2 such that if $t\vee t' \leq s \vee s'$, then

$$\left|\psi_{t,s} * \varphi_{t',s'}(x,y)\right| \leq C\left(\frac{t}{t'}\wedge\frac{t'}{t}\right)^{L_1}\left(\frac{s}{s'}\wedge\frac{s'}{s}\right)^{L_2}\cdot\frac{(t\vee t')^{K_1}}{(t\vee t'+|x|)^{(n+K_1)}}\frac{(s\vee s')^{K_2}}{(s\vee s'+|y|)^{m+K_2}},$$

and if $t\vee t' \geq s\vee s'$, then

$$\left|\psi_{t,s} * \varphi_{t',s'}(x,y)\right| \leq C\left(\frac{t}{t'}\wedge\frac{t'}{t}\right)^{L_1}\left(\frac{s}{s'}\wedge\frac{s'}{s}\right)^{L_2}\cdot\frac{(t\vee t')^{K_1}}{(t\vee t'+|x|)^{(n+K_1)}}\frac{(t\vee t')^{K_2}}{(t\vee t'+|y|)^{m+K_2}},$$

Lemma (2.1.11) [120] [108] Suppose that $\psi_{\mathrm{j,k}}$ are the same as in (5). Then

$$f(x,y) = \sum_j\sum_k \psi_{j,k} * \psi_{j,k} * f(x,y), \quad (15)$$

where the series converges in the norm of $\mathcal{S}_F$ and in dual space $(\mathcal{S}_F)'$.

Before we prove the Min-Max comparison principles (Theorems (2.1.13) and (2.1.14)), we also give the following lemma.

Lemma (2.1.12) [120] [108] Let I, I', J, J' be dyadic cubes in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively such that $l(I) = 2^{-j-N}, l(J) = 2^{-j-N} + 2^{-k-N}, l(I') = 2^{-j'-N}$ and $l(J') = 2^{-j'-N} + 2^{-k'-N}$. Thus for any $u, u^* \in I$ and $v, v^* \in J$, we have, when $j \wedge j' \geq k \wedge k'$,

$$\sum_{I',J'} \frac{2^{-|j-j'|L_1-|k-k'|L_2} 2^{-(j\wedge j')K_1-(k\wedge k')K_2} |I'||J'|}{\left(2^{-j\wedge j'} + |u - x_{I'}|\right)^{n+K_1} \left(2^{-k\wedge k'} + |v - y_{J'}|\right)^{m+K_2}} |\varphi_{j',k'} * f(x_{I'}, y_{J'})|$$
$$\leq C_1(N, r, j, j', k, k') 2^{-|j-j'|L_1} 2^{-|k-k'|L_2}$$
$$\times \left\{ \left[\left(\sum_{J'} \sum_{I'} M_s |\varphi_{j',k'} * f(x_{I'}, y_{J'})| \chi_{J'} \chi_{I'} \right)^r \right] \right\}^{\frac{1}{r}} (u^*, v^*);$$

and when $j \wedge j' \leq k \wedge k'$,

$$\sum_{I',J'} \frac{2^{-|j-j'|L_1-|k-k'|L_2} 2^{-(j\wedge j')K_1-(j\wedge j')K_2} |I'||J'|}{\left(2^{-j\wedge j'} + |u - x_{I'}|\right)^{n+K_1} \left(2^{-j\wedge j'} + |v - y_{J'}|\right)^{m+K_2}} |\varphi_{j',k'} * f(x_{I'}, y_{J'})|$$
$$\leq C_2(N, r, j, j', k, k') 2^{-|j-j'|L_1} 2^{-|k-k'|L_2}$$
$$\times \left\{ \left[\left(\sum_{J'} \sum_{I'} M |\varphi_{j',k'} * f(x_{I'}, y_{J'})| \chi_{J'} \chi_{I'} \right)^r \right] \right\}^{\frac{1}{r}} (u^*, v^*);$$

where M is the Hardy–Littlewood maximal function on $\mathbb{R}^{n+m}$, M_s is the strong maximal function on $\mathbb{R}^n \times \mathbb{R}^m$ (see [111]), and $\max\{\frac{n}{n+K_1}, \frac{m}{m+K_2}\} < r$ and

$$C_1(N, r, j, j', k, k') = 2^{\left(\frac{1}{r}-1\right)N(n+m)} \cdot 2^{[n(j\wedge j'-j')+m(k\wedge k'-k')]\left(1-\frac{1}{r}\right)},$$
$$C_2(N, r, j, j', k, k') = 2^{(\frac{1}{r}-1)N(n+m)} \cdot 2^{[n(j\wedge j'-j')+m(j\wedge j'-j'\wedge k')](1-\frac{1}{r})}.$$

We now are ready to give the

Theorem (2.1.13) [120] Suppose $\psi^{(1)}, \varphi^{(1)} \in \mathcal{S}(\mathbb{R}^{n+m}), \psi^{(2)}, \varphi^{(2)} \in \mathcal{S}(\mathbb{R}^m)$ and

$$\psi(x, y) = \int_{\mathbb{R}^m} \psi^{(1)}(x, y - z) \psi^{(2)}(z) dz,$$
$$\varphi(x, y) = \int_{\mathbb{R}^m} \varphi^{(1)}(x, y - z) \varphi^{(2)}(z) dz,$$

and ψ_{jk}, φ_{jk} satisfy the conditions in (5). Then for $f \in (\mathcal{S}_F), 0 < p, q \leq \infty$ and $s = (s_1, s_2) \in \mathbb{R}^2$,

$$\left\| \left\{ \sum_j \sum_k 2^{-js_1 q} 2^{-(j+k)s_2 q} \sum_J \sum_I \sup_{u\in I, v\in J} \left|\psi_{j,k} * f(u, v)\right|^q \chi_I(x) \chi_J(y) \right\}^{\frac{1}{q}} \right\|_p$$
$$\approx \left\| \left\{ \sum_j \sum_k 2^{-js_1 q} 2^{-(j+k)s_2 q} \sum_J \sum_I \inf_{u\in I, v\in J} |\varphi_{j,k} * f(u, v)|^q \chi_I(x) \chi_J(y) \right\}^{\frac{1}{q}} \right\|_p,$$

where $I \subset \mathbb{R}^n, J \subset \mathbb{R}^m$ are dyadic cubes with side-length $l(I) = 2^{-j-N}$ and $l(J) = 2^{-k-N} + 2^{-j-N}$ for a fixed large integer N; χ_I and χ_J are indicator functions of I and J, respectively.

Similarly, we have the Min-Max comparison principle in Besov spaces.

Proof. By Theorem (2.1.8), $f \in \mathcal{S}_F$ can be represented by

$$f(x, y) = \sum_{j'} \sum_{k'} \sum_{J'} \sum_{I'} |J'||I'| \tilde{\varphi}_{j',k'}\left(x, y, x_{I'}, y_{J'}\right)\left(\varphi_{j',k'} * f\right)(x_{I'}, y_{J'}).$$

We write

$$\left(\psi_{j,k} * f\right)(u,v)$$
$$= \sum_{j'}\sum_{k'}\sum_{J'}\sum_{I'} |I'||J'| \left(\psi_{j,k} * \tilde{\varphi}_{j',k'}\left(\cdot,\cdot, x_{I'}, y_{J'}\right)\right)(u,v)\left(\varphi_{j',k'} * f\right)\left(x_{I'}, y_{J'}\right).$$

By the almost orthogonality estimates in Lemma (2.1.10), for any given positive integer K, taking $L_1 = L_2 = K_1 = K_2 = K$, we have, if $j' \geq k'$,

$$|\psi_{j,k} * \tilde{\varphi}_{j',k'}(\cdot,\cdot)(u,v)|$$
$$\leq C 2^{-|j-j'|K} \cdot 2^{-|k-k'|K} \cdot \frac{2^{-j'K}}{(2^{-j'} + |u - x_{I'}|)^{n+K}} \cdot \frac{2^{-k'K}}{(2^{-k'} + |u - y_{J'}|)^{m+K}};$$

and if $j' \leq k'$, we have

$$|\psi_{j,k} * \tilde{\varphi}_{j',k'}(\cdot,\cdot)(u,v)|$$
$$\leq C 2^{-|j-j'|K} \cdot 2^{-|k-k'|K} \cdot \frac{2^{-j'K}}{(2^{-j'} + |u - x_{I'}|)^{n+K}} \cdot \frac{2^{-j'K}}{(2^{-j'} + |u - y_{J'}|)^{m+K}}.$$

Using Lemma (2.1.12), for any $u, u^* \in I, v, v^* \in J, x_{I'} \in I', y_{J'} \in J'$,

$$\left|\psi_{j,k} * f(u,v)\right| \leq C \sum_{k' \leq j'}\sum_{I'}\sum_{J'} 2^{-|j-j'|K} 2^{-|k-k'|K} |I'||J'|$$
$$\times \frac{2^{-j'K}}{\left(2^{-j'} + |u - x_{I'}|\right)^{n+K}} \cdot \frac{2^{-k'K}}{\left(2^{-k'} + \left|v - y_{J'}\right|\right)^{m+K}} \left|\varphi_{j',k'} * f\left(x_{I'}, y_{J'}\right)\right|$$
$$+ C \sum_{k' > j'}\sum_{J'}\sum_{I'} 2^{-|j-j'|K} 2^{-|k-k'|K} |I'||J'|$$
$$\times \frac{2^{-j'K}}{\left(2^{-j'} + |u - x_{I'}|\right)^{n+K}} \cdot \frac{2^{-j'K}}{\left(2^{-j'} + \left|v - y_{J'}\right|\right)^{m+K}} \left|\varphi_{j',k'} * f\left(x_{I'}, y_{J'}\right)\right|$$
$$\leq C \sum_{k' \leq j'} 2^{-|j-j'|K} \cdot 2^{-|k-k'|K} \left\{ M_s \left(\sum_{J'}\sum_{I'} \left|\varphi_{j',k'} * f\left(x_{I'}, y_{J'}\right)\right| \chi_{I'} \chi_{J'} \right)^r \right\}^{\frac{1}{r}} (u^*, v^*)$$
$$+ C \sum_{k' \leq j'} 2^{-|j-j'|K} 2^{-|k-k'|K} \left\{ M \left(\sum_{J'}\sum_{I'} \left|\varphi_{j',k'} * f(x_{I'}, y_{J'})\right| \chi_{I'} \chi_{J'} \right)^r \right\}^{\frac{1}{r}} (u^*, v^*),$$

where M is the Hardy–Littlewood maximal function on $\mathbb{R}^{n+m}$, M_s is the strong maximal function on $\mathbb{R}^n \times \mathbb{R}^m$, and $max\{\frac{n}{n+K}, \frac{m}{m+K}\} < r < min\{p, q\}$ by taking K large enough.

Applying the Hölder's inequality and summing over j, k, I, J yields

$$\left\{ \sum_j \sum_k \sum_J \sum_I 2^{-js_1q} 2^{-(j+k)s_2q} \sup_{u \in I, v \in J} |\psi j, k * f(u,v)|^q \chi_I \chi_J \right\}^{\frac{1}{q}}$$
$$\leq C \left\{ \sum_{j'}\sum_{k'} 2^{-j's_1q} 2^{-(j'+k')s_2q} \left\{ M_s \left(\sum_{J'}\sum_{I'} \left|\varphi_{j',k'} * f(x_{I'}, y_{J'})\right| \chi_{I'} \chi_{J'} \right)^r \right\}^{\frac{q}{r}} \right\}^{\frac{1}{q}}.$$

Since $x_{I'}$ and $y_{J'}$ are arbitrary points in I' and J', respectively, we have

$$\left\{\sum_j\sum_k\sum_J\sum_I 2^{-js_1q}2^{-(j+k)s_2q}\sup_{u\in I,v\in J}\left|\psi_{j,k}*f(u,v)\right|^q\chi_I\chi_J\right\}^{\frac{1}{q}}$$

$$\leq C\left\{\sum_{j'}\sum_{k'}2^{-j's_1q}2^{-(j'+k')s_2q}\left\{M_s\left(\sum_{J'}\sum_{I'}\inf_{u\in I',v\in J'}\left|\varphi_{j',k'}*f(u,v)\right|\chi_{I'}\chi_{J'}\right)^r\right\}^{\frac{q}{r}}\right\}^{\frac{1}{q}}.$$

and hence, by the Fefferman–Stein vector-valued maximal function inequality [112] with $r < min\{p,q\}$ we get

$$\left\|\left\{\sum_j\sum_k\sum_J\sum_I 2^{-js_1q}2^{-(j+k)s_2q}\sup_{u\in I,v\in J}\left|\psi_{j,k}*f(u,v)\right|^q\chi_I\chi_J\right\}^{\frac{1}{q}}\right\|_p$$

$$\leq C\left\|\left\{\sum_{j'}\sum_{k'}\sum_{J'}\sum_{I'} 2^{-js_1q}2^{-(j+k)s_2q}\inf_{u\in I',v\in J'}\left|\varphi_{j',k'}*f(u,v)\right|^q\chi_{I'}\chi_{J'}\right\}^{\frac{1}{q}}\right\|_p.$$

This ends the proof of Theorem (2.1.13).

Theorem (2.1.14) [120] For $f\in(\mathcal{S}_F)', 0<p,q\leq\infty$ and $s=(s_1,s_2)\in\mathbb{R}^2$, we have

$$\left(\sum_{j,k}2^{-js_1q}2^{-(j+k)s_2q}\left\|\sum_{I,J}\sup_{u\in I,v\in J}\left|\psi_{j,k}*f(u,v)\right|\chi_I\chi_J\right\|_p^q\right)^{\frac{1}{q}}$$

$$\approx\left(\sum_{j,k}2^{-js_1q}2^{-(j+k)s_2q}\left\|\sum_{I,J}\inf_{u\in I,v\in J}\left|\varphi_{j,k}*f(u,v)\right|\chi_I\chi_J\right\|_p^q\right)^{\frac{1}{q}},$$

where $\psi_{j,k}(x,y)$ and $\varphi_{j,k}(x,y)$ are defined as in (5), $I\subset\mathbb{R}^n, J\subset\mathbb{R}^m$ are dyadic cubes with side-length $l(I)=2^{-j-N}$ and $l(J)=2^{-k-N}+2^{-j-N}$ for a fixed large integer N.

Theorem (2.1.13) implies that the Triebel–Lizorkin space $\dot{F}^{s,q}_{p,F}(\mathbb{R}^n\times\mathbb{R}^m)$ in (8) is well defined and Theorem (2.1.14) implies that the Besov space $\dot{B}^{s,q}_{p,F}(\mathbb{R}^n\times\mathbb{R}^m)$ in (9) is well defined.

By use of the Min-Max comparison principle, we will prove the boundedness of flag singular integrals on $\dot{F}^{s,q}_{p,F}$ and on $\dot{B}^{s,q}_{p,F}$.

Proof. As in the proof of Theorem (2.1.13), $f\in S_F$ can be represented by

$$f(x,y)=\sum_{j'}\sum_{k'}\sum_{J'}\sum_{I'}|J||I|\,\tilde{\varphi}_{j',k'}(x,y,x_I,y_J)\,(\varphi_{j',k'}*f)(x_{I'},y_{J'}).$$

Arguing as in the proof of Theorem (2.1.13), we have

$$\left|\psi_{j,k}*f(u,v)\right|$$

$$\leq C\sum_{k'\leq j'}2^{-|j-j'|K}\cdot 2^{-|k-k'|K}\left\{M_s\left(\sum_{J'}\sum_{I'}\left|\varphi_{j',k'}*f\left(x_{I'},y_{J'}\right)\right|\chi_{J'}\chi_{I'}\right)^r(w,z)\right\}^{\frac{1}{r}}$$

$$+C\sum_{k'>j'}2^{-|j-j'|K}2^{-|k-k'|K}\left\{M\left(\sum_{J'}\sum_{I'}\left|\varphi_{j',k'}*f\left(x_{I'},y_{J'}\right)\right|\chi_{I'}\chi_{J'}\right)^r(w,z)\right\}^{\frac{1}{r}}$$

$$\leq C\sum_{j',k'}2^{-|j-j'|K}\cdot 2^{-|k-k'|K}\left\{M_s\left(\sum_{J'}\sum_{I'}\left|\varphi_{j',k'}*f\left(x_{I'},y_{J'}\right)\right|\chi_{J'}\chi_{I'}\right)^r(w,z)\right\}^{\frac{1}{r}}.$$

Therefore,

$$\sup_{u\in I,v\in J}\left|\psi_{j,k}*f(u,v)\right|\chi_I(w)\chi_J(z)$$

$$\leq C\sum_{j',k'}2^{-|j-j'|K}\cdot 2^{-|k-k'|K}\left\{M_s\left(\sum_{J'}\sum_{I'}\left|\varphi_{j',k'}*f\left(x_{I'},y_{J'}\right)\right|\chi_{J'}\chi_{I'}\right)^r(w,z)\right\}^{\frac{1}{r}}.$$

When $1\leq p<\infty$, by the Fefferman–Stein vector-valued maximal function inequality [112] with $r<p$, we have

$$\left\|\sum_{I,J}\sup_{u\in I,v\in J}\left|\psi_{j,k}*f(u,v)\right|\chi_I\chi_J\right\|_p$$

$$\leq C\left\|\sum_{j',k'}2^{-|j-j'|K}\cdot 2^{-|k-k'|K}\left\{M_s\left(\sum_{J'}\sum_{I'}\left|\varphi_{j',k'}*f\left(x_{I'},y_{J'}\right)\right|\chi_{J'}\chi_{I'}\right)^r\right\}^{\frac{1}{r}}\right\|_p$$

$$\leq C\sum_{j',k'}2^{-|j-j'|K}\cdot 2^{-|k-k'|K}\left\|\left\{M_s\left(\sum_{J'}\sum_{I'}\left|\varphi_{j',k'}*f\left(x_{I'},y_{J'}\right)\right|\chi_{J'}\chi_{I'}\right)^r\right\}^{\frac{1}{r}}\right\|_p$$

$$\leq C\sum_{j',k'}2^{-|j-j'|K}\cdot 2^{-|k-k'|K}\left\|\sum_{J'}\sum_{I'}\left|\varphi_{j',k'}*f\left(x_{I'},y_{J'}\right)\right|\chi_{J'}\chi_{I'}\right\|_p.$$

If $1\leq q$, applying the Hölder's inequality and if $0<q<1$ by using the usual inequality, and summing over j,k yields

$$\left(\sum_{j,k}2^{-js_1q}2^{-(j+k)s_2q}\left\|\sum_{I,J}\sup_{u\in I,v\in J}\left|\psi_{j,k}*f(u,v)\right|\chi_I\chi_J\right\|_p^q\right)^{\frac{1}{q}}$$

$$\leq\left(\sum_{j,k}2^{-js_1q}2^{-(j+k)s_2q}\left(\sum_{j',k'}2^{-|j-j'|K}\cdot 2^{-|k-k'|K}\cdot\left\|\sum_{J'}\sum_{I'}\left|\varphi_{j',k'}*f\left(x_{I'},y_{J'}\right)\right|\chi_{I'}\chi_{J'}\right\|_p\right)^q\right)^{\frac{1}{q}}$$

$$\leq C\left(\sum_{j',k'}2^{-j's_1q}2^{-(j'+k')s_2q}\left\|\sum_{J'}\sum_{I'}\left|\varphi_{j',k'}*f\left(x_{I'},y_{J'}\right)\right|\chi_{I'}\chi_{J'}\right\|_p^q\right)^{\frac{1}{q}}.$$

When $0<p<1$, the Fefferman–Stein vector-valued maximal function inequality [112] with $r<p$ yields

$$\int \left(\sum_{I,J} \sup_{u \in I, v \in J} |\psi_{j,k} * f(u,v)| \chi_I(w) \chi_J(z) \right)^p dwdz$$

$$\leq C \sum_{j',k'} 2^{-|j-j'|Kp} \cdot 2^{-|k-k'|Kp} \int \left(M_s \left(\sum_{J'} \sum_{I'} |\varphi_{j',k'} * f(x_{I'}, y_{J'})| \chi_{J'} \chi_{I'} \right)^r (w,z) \right)^{\frac{p}{r}} dwdz$$

$$\leq C \sum_{j',k'} 2^{-|j-j'|Kp} \cdot 2^{-|k-k'|Kp} \int \left(\sum_{J'} \sum_{I'} |\varphi_{j',k'} * f(x_{I'}, y_{J'})| \chi_{I'}(w) \chi_{J'}(z) \right)^p dwdz.$$

Since $1 < 1/p$, taking $\varepsilon > 0$, by the Hölder's inequality, we have

$$\left(\int \left(\sum_{I,J} \sup_{u \in I, v \in J} |\psi_{j,k} * f(u,v)| \chi_I(w) \chi_J(z) \right)^p dwdz \right)^{\frac{1}{p}}$$

$$\leq C 2^{-|j-j'|K(1-\varepsilon)} \cdot 2^{-|k-k'|K(1-\varepsilon)} \cdot \left(\int \left(\sum_{J'} \sum_{I'} |\varphi_{j',k'} * f(x_{I'}, y_{J'})| \chi_{I'}(w) \chi_{J'}(z) \right)^p dwdz \right)^{\frac{1}{p}}$$

So if $1 \leq q/p$, then applying the Hölder's inequality and if $0 < q/p < 1$ by using the usual inequality, we have

$$\left(\sum_{j,k} 2^{-js_1 q} 2^{-(j+k)s_2 q} \left\| \sum_{I,J} \sup_{u \in I, v \in J} |\psi_{j,k} * f(u,v)| \chi_I \chi_J \right\|_p^q \right)^{\frac{1}{q}}$$

$$\leq \left(\sum_{j,k} 2^{-js_1 q} 2^{-(j+k)s_2 q} \left(\sum_{j',k'} 2^{-|j-j'|K(1-\varepsilon)} \cdot 2^{-|k-k'|K(1-\varepsilon)} \times \left\| \sum_{J'} \sum_{I'} |\varphi_{j',k'} * f(x_{I'}, y_{J'})| \chi_{I'} \chi_{J'} \right\|_p \right)^q \right)^{\frac{1}{q}}$$

$$\leq C \left(\sum_{j',k'} 2^{-j's_1 q} 2^{-(j'+k')s_2 q} \left\| \sum_{J'} \sum_{I'} |\varphi_{j',k'} * f(x_{I'}, y_{J'})| \chi_{I'} \chi_{J'} \right\|_p^q \right)^{\frac{1}{q}}.$$

Since $x_{I'}$ and $y_{J'}$ are arbitrary points in I' and J', respectively, we have the desired inequalities in Theorem (2.1.13).

To establish the boundedness of flag singular integral operators on Triebel–Lizorkin and Besov spaces associated with the flag multi-parameter structure using the results we have proved. As a consequence of Theorems (2.1.13) and (2.1.14), it is easy to see that the Triebel–Lizorkin spaces $\dot{F}_{p,F}^{s,q}$ and Besov spaces are independent of the choice of the functions ψ. Moreover, we have the following characterization of $\dot{F}_{p,F}^{s,q}$ and $\dot{B}_{p,F}^{s,q}$ by using the discrete norm.

Proposition (2.1.15) [120] Let $0 < p, q \leq \infty$ and $s = (s_1, s_2) \in \mathbb{R}^2$. Then we have

$$\|f\|_{\dot{B}_{p,F}^{s,q}} \approx \left\| \left(\sum_j \sum_k \sum_J \sum_I 2^{-js_1 q} 2^{-(j+k)s_2 q} |\psi_{j,k} * f(x_I, y_J)|^q \chi_I(x) \chi_J(y) \right)^{\frac{1}{q}} \right\|_p, (16)$$

where $j, k, \psi, \chi_I, \chi_J, x_I, y_J$ are the same as in Theorem (2.1.13).

Proposition (2.1.16) [120] Let $0 < p, q \leq \infty$ and $s = (s_1, s_2) \in \mathbb{R}^2$. Then we have

$$\|f\|_{\dot{B}^{s,q}_{p,F}} \approx \left(\sum_j \sum_k 2^{-js_1q} 2^{-(j+k)s_2q} \left\| \sum_J \sum_I |\psi_{j,k} * f(x_I, y_J)| \chi_I(x)\chi_J(y) \right\|_p^q \right)^{\frac{1}{q}}, (17)$$

where $j, k, \psi, \chi_I, \chi_J, x_I, y_J$ are the same as in Theorem (2.1.14).

Before we give the proof of the boundedness of flag singular integrals on $\dot{F}^{s,q}_{p,F}$ and $\dot{B}^{s,q}_{p,F}$. we show several properties of them.

Proposition (2.1.17) [120] $\mathcal{S}_F(\mathbb{R}^n \times \mathbb{R}^m)$ is dense in $\dot{F}^{s,q}_{p,F}$ and in $\dot{B}^{s,q}_{p,F}$.

Proof. Suppose $f \in \dot{F}^{s,q}_{p,F}$, and set $W = \{(j,k,I,J): |j| \leq L, |k| \leq M, I \times J \subseteq B(0,r)\}$, where I, J are dyadic cubes in $\mathbb{R}^n, \mathbb{R}^m$ with side length $2^{-j-N}, 2^{-k-N} + 2^{-j-N}$, respectively, and $B(0,r)$ are balls in $\mathbb{R}^{n+m}$ centered at the origin with radius r. It is easy to see that

$$\sum_{(j,k,I,J) \in W} |I||J| \, \tilde{\psi}_{j,k}(x, y, x_I, y_J) \psi_{j,k} * f(x_I, y_J)$$

is a test function in $\mathcal{S}_F(\mathbb{R}^n \times \mathbb{R}^m)$ for any fixed L, M, r. To show the proposition, it suffices to prove

$$\sum_{(j,k,I,J) \in W^c} |I||J| \, \tilde{\psi}_{j,k}(x, y, x_I, y_J) \psi_{j,k} * f(x_I, y_J)$$

tends to zero in the $\dot{F}^{s,q}_{p,F}$ norm as L, M, r tend to infinity. This follows from (16) and a similar proof to that of Theorem (2.1.13). In fact, repeating the same proof as in Theorem (2.1.13) yields

$$\left\| \sum_{(j,k,I,J) \in W^c} |I||J| \tilde{\psi}_{j,k}\big(x, y, x_I, y_J\big) \psi_{j,k} * f\big(x_I, y_J\big) \right\|_{\dot{F}^{s,q}_{p,F}}$$

$$\leq C \left\| \left\{ \sum_{(j,k,I,J) \in W^c} 2^{-js_1q} 2^{-(j+k)s_2q} \left|\psi_{j,k} * f(x_I, y_J)\right|^q \chi_I \chi_J \right\}^{\frac{1}{q}} \right\|_p,$$

where the last term tends to zero as L, M, r tend to infinity whenever $f \in \dot{F}^{s,q}_{p,F}$.

Suppose $f \in \dot{B}^{s,q}_{p,F}$, set $W_1 = \{(j,k): |j| \leq L, |k| \leq M\}$ and $W_2 = \{(I,J): I \times J \subseteq B(0,r)\}$, where I, J are dyadic cubes in $\mathbb{R}^n, \mathbb{R}^m$ with side length $2^{-j-N}, 2^{-k-N} + 2^{-j-N}$, respectively. Then

$$\left\| \sum_{(j,k) \in W_1^c, (I,J) \in W_2^c} |I||J| \, \tilde{\psi}_{j,k}(x, y, x_I, y_J) \psi_{j,k} * f(x_I, y_J) \right\|_{\dot{B}^{s,q}_{p,F}}$$

$$\leq C \left(\sum_{(j,k) \in W_1^c} 2^{-js_1q} 2^{-(j+k)s_2q} \left\| \sum_{(I,J) \in W_2^c} \left|\psi_{j,k} * f(x_I, y_J)\right| \chi_I \chi_J \right\|_p^q \right)^{\frac{1}{q}},$$

where the last term tends to zero as L, M, r tend to infinity whenever $f \in \dot{B}^{s,q}_{p,F}$.

As a consequence of Proposition (2.1.16), $L^2(\mathbb{R}^{n+m})$ is dense in $\dot{F}^{s,q}_{p,F}(\mathbb{R}^n \times \mathbb{R}^m)$ and also in $\dot{B}^{s,q}_{p,F}(\mathbb{R}^n \times \mathbb{R}^m)$.

Theorem (2.1.18) [120] Suppose that T is a flag singular integral defined on $\mathbb{R}^n \times \mathbb{R}^m$ with the flag kernel $K(x,y) = \int_{\mathbb{R}^m} K^*(x, y - z, z) dz$, where the product kernel K satisfies the conditions in Definition (2.1.2). Then T is bounded on $\dot{F}_{p,F}^{s,q}$. Namely, for all $0 < p, q \leq \infty$ and $s = (s_1, s_2) \in \mathbb{R}^2$ there exists a constant C_p such that

$$\|T(f)\|_{\dot{F}_{p,F}^{s,q}} \leq C_p \|f\|_{\dot{F}_{p,F}^{s,q}}.$$

Proof. We assume that K is the kernel of T. Applying the discrete Calderón reproducing formula in Theorem 3.4 in [108] implies that for $f \in L^2(\mathbb{R}^{n+m}) \cap \dot{F}_{p,F}^{s,q}$,

$$\left\| \left\{ \sum_j \sum_k \sum_J \sum_I 2^{-js_1 q} 2^{-(j+k)s_2 q} \left| \varphi_{j,k} * K * f(x,y) \right|^q \chi_I(x) \chi_J(y) \right\}^{\frac{1}{q}} \right\|_p$$

$$= \left\| \left\{ \sum_j \sum_k \sum_J \sum_I 2^{-js_1 q} 2^{-(j+k)s_2 q} \left| \sum_{j'} \sum_{k'} \sum_{J'} \sum_{I'} |J'||I'| \varphi_{j,k} * K * \tilde{\varphi}_{j',k'}(\cdot - x_{I'}, \right. \right. \right.$$

$$\left. \left. \left. \cdot - y_{J'}) \times (x,y) \varphi_{j',k'} * f(x_{I'}, y_{J'}) \right|^q \chi_I(x) \chi_J(y) \right\}^{\frac{1}{q}} \right\|_p ,$$

where the discrete Calderón reproducing formula in $L^2(\mathbb{R}^{n+m})$ is used.

Noting that φ_{jk} are test functions as defined in (5), one can easily check that

$$\left| \varphi_{j,k} * K * \tilde{\varphi}_{j',k'}(\cdot - x_{I'}, \cdot - y_{J'})(x,y) \right| \leq C 2^{-|j-j'|K} \cdot 2^{-|k-k'|K}$$

$$\times \int_{\mathbb{R}^m} \frac{2^{-(j\wedge j')K}}{\left(2^{-(j\wedge j')} + |x - x_I| + |y - z - y_J|\right)^{n+m+K}} \cdot \frac{2^{-(k\wedge k')K}}{\left(2^{-(k\wedge k')} + |z|\right)^{m+K}} dz,$$

where K depends on M_0 given in Theorem 3.4 in [108] and M_0 is chosen to be large enough. Repeating a similar proof to that of Theorem (2.1.8) (see [108]), we obtain

$$\|Tf\|_{\dot{F}_{p,F}^{s,q}} \leq C \left\| \left\{ \sum_{j'} \sum_{k'} 2^{-j's_1 q} 2^{-(j'+k')s_2 q} \left\{ M_s \left(\sum_{J'} \sum_{I'} \left| \varphi_{j',k'} * (x_{I'}, y_{J'}) \right| \chi_{J'} \chi_{I'} \right)^r \right\}^{\frac{q}{r}} (x,y) \right\}^{\frac{1}{q}} \right\|_p$$

$$\leq C \left\| \left\{ \sum_{j'} \sum_{k'} \sum_{J'} \sum_{I'} 2^{-j's_1 q} 2^{-(j'+k')s_2 q} \left| \varphi_{j',k'} * (x_{I'}, y_{J'}) \right|^q \chi_{J'}(y) \chi_{I'}(x) \right\}^{\frac{1}{q}} \right\| \leq C \|f\|_{\dot{F}_{p,F}^{s,q}},$$

where the last inequality follows.

Since $L^2(\mathbb{R}^{n+m})$ is dense in $\dot{F}_{p,F}^{s,q}, T$ can be extended to a bounded operator on $\dot{F}_{p,F}^{s,q}$. This ends the proof of Theorem (2.1.18).

Theorem (2.1.19) [120] Suppose that T is a flag singular integral with the kernel $K(x,y)$ satisfying the same conditions as in Theorem (2.1.18). Then T is bounded on $\dot{B}_{p,F}^{s,q}$. Namely, for all $0 < p, q \leq \infty$ and $s = (s_1, s_2) \in \mathbb{R}^2$ there exists a constant C_p such that

$$\|T(f)\|_{\dot{B}_{p,F}^{s,q}} \leq C_p \|f\|_{\dot{B}_{p,F}^{s,q}}.$$

From the above of the proof of Theorem (2.1.18), it is obvious that Theorem (2.1.19) follows from similar proof of Theorem (2.1.18) immediately.

We some remarks on how our methods can be applied to derive results of Triebel–Lizorkin and Besov spaces in the simplest case of product spaces of two Euclidean spaces.

We shall not provide any proofs since they can be given easily by following those in the flag singular integral case.

To state the realization of our main results on $\mathbb{R}^n \times \mathbb{R}^m$. Let $\mathcal{S}(\mathbb{R}^n)$ denote Schwartz functions in $\mathbb{R}^n$. Then the test function defined on $\mathbb{R}^n \times \mathbb{R}^m$ can be given by

$$\psi(x, y) = \psi^{(1)}(x)\psi^{(2)}(y),$$

where $\psi^{(1)} \in \mathcal{S}(\mathbb{R}^n), \psi^{(2)} \in \mathcal{S}(\mathbb{R}^m)$, and satisfy $\sum_{j\in Z}\left|\hat{\psi}^{(1)}(2^{-j}\xi_1)\right|^2 = 1$ for all $\xi_1 \in \mathbb{R}^n\backslash\{(0)\}$, and $\sum_{k\in Z}\left|\hat{\psi}^{(2)}(2^{-j}\xi_2)\right|^2 = 1$ for all $\xi_2 \in \mathbb{R}^m\backslash\{0\}$, and the moment conditions

$$\int_{\mathbb{R}^n} x^\alpha \psi^{(1)}(x)dx = \int_{\mathbb{R}^m} \psi^{(2)}(y)y^\beta dy = 0$$

for all nonnegative integers α and β.

Let $f \in L^p, 1 < p < \infty$. Thus $g(f)$, the Littlewood–Paley–Stein square function of f, is defined by

$$g(f)(x, y) = \left\{\sum_j \sum_k \left|\psi_{j,k} * f(x, y)\right|^2\right\}^{\frac{1}{2}},$$

where functions

$$\psi_{j,k}(x, y) = 2^{jn+km}\psi^{(1)}(2^j x)\psi^{(2)}(2^k y). \tag{18}$$

By taking the Fourier transform, it is easy to see the following continuous version of Calderón's identity holds on $L^2(\mathbb{R}^n \times \mathbb{R}^m)$,

$$f(x, y) = \sum_j \sum_k \psi_{j,k} * \psi_{j,k} * f(x, y).$$

Using the orthogonal estimates and together with Calderón's identity on L^2, one can easily obtain the L^p estimates of g for $1 < p < \infty$. Namely, there exist constants C_1 and C_2 such that for $1 < p < \infty$,

$$C_1\|f\|_p \le \|g(f)\|_p \le C_2\|f\|_p.$$

In order to use the Littlewood–Paley–Stein square function g to define the Treibel–Lizorkin Besov spaces in pure product setting, one needs to extend the Littlewood–Paley–Stein square function to be defined on a suitable distribution space. We introduce the product test function space on $\mathbb{R}^n \times \mathbb{R}^m$.

Definition (2.1.20) [120] A Schwartz test function $f(x, y)$ defined on $\mathbb{R}^n \times \mathbb{R}^m$ is said to be a product test function on $\mathbb{R}^n \times \mathbb{R}^m$ if $f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$ and

$$\int_{\mathbb{R}^n} f(x, y)x^\alpha dx = \int_{\mathbb{R}^m} f(x, y)y^\beta dy = 0$$

for all indices α, β of nonnegative integers.

If f is a product test function on $\mathbb{R}^n \times \mathbb{R}^m$ we denote $f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$ and the norm of f is defined by the norm of Schwartz test function.

We denote by $(\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m))'$ the dual of $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$.

Since the functions $\psi_{j,k}$ constructed above belong to $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$, so the Littlewood–Paley–Stein square function g can be defined for all distributions in $(\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m))'$. Formally, we can define the multi-parameter Triebel–Lizorkin and Besov spaces in the pure product setting as follows:

Definition (2.1.21) [120] Let $0 < p, q \le \infty, s = (s_1, s_2) \in \mathbb{R}^2$. The Triebel–Lizorkin type space $\dot{F}_p^{s,q}(\mathbb{R}^n \times \mathbb{R}^m)$ in the pure product setting is defined by

$$\dot{F}_p^{s,q}(\mathbb{R}^n \times \mathbb{R}^m) = \{f \in (\mathcal{S}_F)': \|f\|_{\dot{F}_p^{s,q}} < \infty\},$$

where

$$\|f\|_{\dot{F}_p^{s,q}} = \left(\sum_{j\in Z}\sum_{k\in Z} 2^{-js_1q}2^{-ks_2q}\left\|\psi_{j,k} * f(\cdot)\right\|_p^q\right)^{\frac{1}{q}}. \quad (19)$$

The Besov space $\dot{B}_p^{s,q}(\mathbb{R}^n \times \mathbb{R}^m)$ in the pure product setting is defined as

$$\dot{B}_p^{s,q}(\mathbb{R}^n \times \mathbb{R}^m) = \{f \in (\mathcal{S})': \|f\|_{\dot{B}_p^{s,q}} < \infty\},$$

where

$$\|f\|_{\dot{B}_p^{s,q}} = \left(\sum_{j\in Z}\sum_{k\in Z} 2^{-js_1q}2^{-ks_2q}2q\left\|\psi_{j,k} * f\right\|_p^q\right)^{\frac{1}{q}}. \quad (20)$$

If $s = (0,0)$, then $\dot{F}_p^{0,2} = L^p$ when $1 < p < \infty$ and $\dot{F}_p^{0,2} = H^p$ when $0 < p \le 1$, the product Hardy spaces introduced in [100–102]. It can be shown that this definition is independent of the choice of functions $\psi_{j,k}$.

To establish the Triebel–Lizorkin and Besov space theory on $\mathbb{R}^n \times \mathbb{R}^m$ we need the following discrete Calderón's identity.

Theorem (2.1.22) [120] Suppose that $\psi_{j,k}$ are the same as in (18). Then

$$f(x,y) = \sum_{j,k}\sum_{I,J} |I||J|\, \tilde{\psi}_{j,k}(x,y,x_I,y_J)(\psi_{j,k} * f)(x_I,y_J), \quad (21)$$

where $\tilde{\psi}_{j,k}(x,y,x_I,y_J) \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m), I \subset \mathbb{R}^n, J \subset \mathbb{R}^m$ are dyadic intervals with interval length $\ell(I) = 2^{-j-N}, \ell(J) = 2^{-k-N}$ for a fixed large integer N; x_I, y_J are any fixed points in I, J respectively, and the series in (22) converges in the norm of $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$ and in the dual space $(\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m))'$.

By use of the Min-Max comparison principle in the pure product setting similar to Theorems (2.1.13) and (2.1.14), we can prove the boundedness of singular integrals of product kernels on $\mathbb{R}^n \times \mathbb{R}^m$ defined as in Definition (2.1.2) (one notes that it is $\mathbb{R}^n \times \mathbb{R}^m$ instead of $\mathbb{R}^{n+m} \times \mathbb{R}^m$) on $\dot{F}_p^{s,q}$ and on $\dot{B}_p^{s,q}$

Theorem (2.1.23) [120] Suppose that T is a product singular integral defined on $\mathbb{R}^n \times \mathbb{R}^m$ where the product kernel K satisfies the conditions in Definition (2.1.2). Then T is bounded on $\dot{F}_p^{s,q}$ and $\dot{B}_p^{s,q}$. Namely, for all $0 < p,q \le \infty$ and $s = (s_1,s_2) \in \mathbb{R}^2$ there exists a constant C_p such that

$$\|T(f)\|_{\dot{F}_p^{s,q}} \le C_p\|f\|_{\dot{F}_p^{s,q}}$$

and

$$\|T(f)\|_{\dot{B}_{p,F}^{s,q}} \le C_p\|f\|_{\dot{B}_{p,F}^{s,q}}.$$

We also remark that Hardy space theory associated with non-isotropic flag singular integrals induced by the non-isotropic dilation $(\delta x, \delta^2 y)$ on $\mathbb{R}^{n+m}$ has been carried out by Ruan in [117–118]. Therefore, it can be applied to develop the multi-parameter Treibel–Lizorkin and Besov space theory in that setting. See Han and Lu [118] for more comprehensive summaries of multi-parameter Hardy space theory and discrete Littlewood–Paley–Stein analysis.

Section (2.2) Weighted Triebel–Lizorkin and Besov Spaces of Arbitrary Number of Parameters

The multi-parameter pure product theory has been developed over the past decades. This theory includes the boundedness on multi-parameter L^p spaces $(1 < p < \infty)$ and multi-parameter Hardy spaces $H^p(0 < p \leq 1)$ of singular integral operators. This theory also includes the atomic decomposition of multi-parameter Hardy spaces, duality and interpolation theorems on product spaces, and maximal function characterizations, etc. See Gundy and Stein [96], Carleson [97], Fefferman and Stein [98], Fefferman [99, 121], Chang and Fefferman [100–102], Journe [103, 104] and Pipher [105], etc.

[108, 122], developed a theory of discrete Calderón reproducing formula and Littlewood–Paley analysis and then developed the implicit multi-parameter Hardy space theory associated with the flag singular integrals. Adapting ideas from [108, 123] established the boundedness of singular integral operators on weighted multi-parameter Hardy spaces $H^p_w(\mathbb{R}^n \times \mathbb{R}^m)$ and from $H^p_w(\mathbb{R}^n \times \mathbb{R}^m)$ to $L^p_w(\mathbb{R}^n \times \mathbb{R}^m)$ when the weight w is in A_∞. The boundedness of singular integral operators on weighted multi-parameter Hardy spaces of arbitrary number of parameters was derived in [124].

The theory of one-parameter Triebel–Lizorkin and Besov spaces has been very well developed. The multi-parameter counterpart of such theory is still very little explored. [120] studied the nonweighted Triebel–Lizorkin and Besov spaces associated with the flag singular integral operators and proved the boundedness of the flag singular integrals on these spaces.

To use the discrete multi-parameter Littlewood–Paley–Stein analysis to establish the theory of weighted Triebel–Lizorkin and Besov spaces on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_k}$ and the boundedness of singular integral operators on $\dot{F}^{s,q}_p(w; \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_k})$ and $\dot{B}^{s,q}_p(w; \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_k})$ for all $w \in A_\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_k}), 0 < p, q \leq \infty$ and $s = (s_1, s_2, \ldots, s_k) \in \mathbb{R}^k$. For the simplicity of the presentation, we assume $k = 3$ and denote $n = n_1, m = n_2, d = n_3$.

We now recall some definitions of product weights in three parameter setting.

For $1 < p < \infty$, a nonnegative locally integrable function $w \in A_p(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$ if there exists a constant $C > 0$ such that

$$\left(\frac{1}{|R|}\int_R w(x)dx\right)\left(\frac{1}{|R|}\int_R w(x)^{-1/(p-1)}dx\right)^{p-1} \leq C$$

for any dyadic cuboid $R = I \times J \times K$, where I, J, K are cubes in $\mathbb{R}^n, \mathbb{R}^m$ and $\mathbb{R}^d$ respectively. We say $w \in A_1(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$ if there exists a constant $C > 0$ such that

$$M_s w(x) \leq C w(x)$$

for almost every $x \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d$, where M_s is the strong maximal function defined by

$$M_s f(u) = \sup_{u \in R} \frac{1}{|R|}\int_R |f(v)|dv$$

for any dyadic cuboid $R = I \times J \times K$ on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d$. We define $w \in A_\infty(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$ by

$$A_\infty(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d) = \bigcup_{1 \leq p < \infty} A_p(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d).$$

If $w \in A_\infty$, then $q_w = \inf\{q : w \in A_q\}$ is called the critical index of w. Notice that if $w \in A_\infty$, then $q_w < \infty$. The $A_p(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$ has the following restriction property: $w \in A_p(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$ implies $w(\cdot, y, z) \in A_p(\mathbb{R}^n), w(x, \cdot, z) \in A_p(\mathbb{R}^m), w(x, y, \cdot) \in A_p(\mathbb{R}^d)$.

We will use an appropriate Littlewood–Paley square function to characterize the weighted Triebel–Lizorkin space $\dot{F}^{s,q}_p(w; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$, and Besov spaces $\dot{B}^{s,q}_p(w; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d), w \in A_\infty(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$. To approach this, we first introduce the test function space $S_M(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$, where M is a positive integer.

Definition (2.2.1) [128] We call f defined on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d$ a test function of order M if

(i)For $|\alpha|, |\beta|, |\gamma| \le M-1$,

$$|D_x^\alpha D_y^\beta D_z^\gamma f(x,y,z)| \le C \frac{1}{(1+|x|)^{n+M+|\alpha|}} \frac{1}{(1+|y|)^{m+M+|\beta|}} \frac{1}{(1+|z|)^{d+M+|\gamma|}};$$

(ii)For $|x-x'| \le \frac{1}{2}(1+|x|)$ and $|\alpha| = M, |\beta|, |\gamma| \le M-1$,

$$|D_x^\alpha D_y^\beta D_z^\gamma f(x,y,z) - D_x^\alpha D_y^\beta D_z^\gamma f(x',y,z)| \le C \frac{|x-x'|}{(1+|x|)^{n+2M}} \frac{1}{(1+|y|)^{m+M+|\beta|}} \frac{1}{(1+|z|)^{d+M+|\gamma|}};$$

(iii)For $|y-y'| \le \frac{1}{2}(1+|y|)$ and $|\beta| = M, |\alpha|, |\gamma| \le M-1$,

$$|D_x^\alpha D_y^\beta D_z^\gamma f(x,y,z) - D_x^\alpha D_y^\beta D_z^\gamma f(x,y',z)| \le C \frac{|y-y'|}{(1+|y|)^{m+2M}} \frac{1}{(1+|x|)^{n+M+|\alpha|}} \frac{1}{(1+|z|)^{d+M+|\gamma|}};$$

(iv)For $|z-z'| \le \frac{1}{2}(1+|z|)$ and $|\gamma| = M, |\alpha|, |\beta| \le M-1$,

$$|D_x^\alpha D_y^\beta D_z^\gamma f(x,y,z) - D_x^\alpha D_y^\beta D_z^\gamma f(x,y,z')| \le C \frac{|z-z'|}{(1+|z|)^{d+2M}} \frac{1}{(1+|x|)^{n+M+|\alpha|}} \frac{1}{(1+|y|)^{m+M+|\beta|}};$$

(v)For $|x-x'| \le \frac{1}{2}(1+|x|), |y-y'| \le \frac{1}{2}(1+|y|)$ and $|\alpha| = |\beta| = M, |\gamma| \le M-1$,

$$|D_x^\alpha D_y^\beta D_z^\gamma f(x,y,z) - D_x^\alpha D_y^\beta D_z^\gamma f(x',y,z) - D_x^\alpha D_y^\beta D_z^\gamma f(x,y',z) + D_x^\alpha D_y^\beta D_z^\gamma f(x',y',z)| \le C \frac{|x-x'|}{(1+|x|)^{n+2M}} \frac{|y-y'|}{(1+|y|)^{m+2M}} \frac{1}{(1+|z|)^{d+M+|\gamma|}};$$

(vi)For $|x-x'| \le \frac{1}{2}(1+|x|), |z-z'| \le \frac{1}{2}(1+|z|)$ and $|\alpha| = |\gamma| = M, |\beta| \le M-1$,

$$|D_x^\alpha D_y^\beta D_z^\gamma f(x,y,z) - D_x^\alpha D_y^\beta D_z^\gamma f(x',y,z) - D_x^\alpha D_y^\beta D_z^\gamma f(x,y,z') + D_x^\alpha D_y^\beta D_z^\gamma f(x',y,z')| \le C \frac{|x-x'|}{(1+|x|)^{n+2M}} \frac{|z-z'|}{(1+|z|)^{d+2M}} \frac{1}{(1+|y|)^{m+M+|\beta|}};$$

(vii)For $|y-y'| \le \frac{1}{2}(1+|y|), |z-z'| \le \frac{1}{2}(1+|z|)$ and $|\beta| = |\gamma| = M, |\alpha| \le M-1$,

$$|D_x^\alpha D_y^\beta D_z^\gamma f(x,y,z) - D_x^\alpha D_y^\beta D_z^\gamma f(x,y',z) - D_x^\alpha D_y^\beta D_z^\gamma f(x,y,z') + D_x^\alpha D_y^\beta D_z^\gamma f(x,y',z')| \le C \frac{|y-y'|}{(1+|y|)^{m+2M}} \frac{|z-z'|}{(1+|z|)^{d+2M}} \frac{1}{(1+|x|)^{n+M+|\alpha|}};$$

(viii) For $|x-x'| \le \frac{1}{2}(1+|x|), |y-y'| \le \frac{1}{2}(1+|y|), |z-z'| \le \frac{1}{2}(1+|z|)$ and $|\nu| = M$,

$$\Big|\big[D_x^\nu D_y^\nu D_z^\nu f(x,y,z) - D_x^\nu D_y^\nu D_z^\nu f(x',y,z) - D_x^\nu D_y^\nu D_z^\nu f(x,y',z) + D_x^\nu D_y^\nu D_z^\nu f(x',y',z)\big] - \big[D_x^\nu D_y^\nu D_z^\nu f(x,y,z') - D_x^\nu D_y^\nu D_z^\nu f(x',y,z') - D_x^\nu D_y^\nu D_z^\nu f(x,y',z') + D_x^\nu D_y^\nu D_z^\nu f(x',y',z')\big]\Big| \le C \frac{|x-x'|}{(1+|x|)^{n+2M}} \frac{|y-y'|}{(1+|y|)^{m+2M}} \frac{|z-z'|}{(1+|z|)^{d+2M}};$$

(iX) For $|\alpha|, |\beta|, |\gamma| \le M-1$,

$$\int_{\mathbb{R}^n} f(x,y,z) x^\alpha dx = \int_{\mathbb{R}^m} f(x,y,z) y^\beta dy = \int_{\mathbb{R}^d} f(x,y,z) z^\gamma dz = 0.$$

If f is a test function of order M on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d$ we denote $f \in S_M(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$ and the norm of f is defined by

$$\|f\|_{S_M(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^d)} = \inf\{C: \text{(i)– (iX) hold}\}.$$

It is easy to check that $S_M(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$ with this norm is a Banach space. The dual space of $S_M(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$ is denoted by $S_M(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$). Note that a Schwartz function with condition (iX) belongs to S_M.

Let $\psi^{(1)} \in S(\mathbb{R}^n), \psi^{(2)} \in S(\mathbb{R}^m), \psi^{(3)} \in S(\mathbb{R}^d)$ and satisfy

$$\sum_{j\in\mathbb{Z}} \left|\widehat{\psi^{(1)}}(2^{-j}\xi_1)\right| = 1 \text{ for all } \xi_1 \in \mathbb{R}^n \setminus \{0\},$$

$$\sum_{k\in\mathbb{Z}} \left|\widehat{\psi^{(2)}}(2^{-k}\xi_2)\right| = 1 \text{ for all } \xi_2 \in \mathbb{R}^m \setminus \{0\},$$

$$\sum_{\ell\in\mathbb{Z}} \left|\widehat{\psi^{(3)}}(2^{-\ell}\xi_3)\right| = 1 \text{ for all } \xi_3 \in \mathbb{R}^d \setminus \{0\},$$

and the moment conditions

$$\int_{\mathbb{R}^n} \psi^{(1)} x^\alpha dx = 0, \qquad \int_{\mathbb{R}^m} \psi^{(2)} y^\beta dy = 0, \qquad \int_{\mathbb{R}^d} \psi^{(3)} z^\gamma dz = 0$$

for all multi-indices α, β and γ. Denote

$$\psi_{j,k,\ell}(x,y,z) = \left(\psi_j^{(1)} \otimes \psi_k^{(2)} \otimes \psi_\ell^{(3)}\right)(x,y,z) = \psi_j^{(1)}(x)\psi_k^{(2)}(y)\psi_\ell^{(3)}(z), \qquad (22)$$

where

$$\psi_j^{(1)}(x) = 2^{jn}\psi^{(1)}\left(2^j x\right), \qquad \psi_k^{(2)}(y) = 2^{km}\psi^{(2)}(2^k x), \qquad \psi_\ell^{(3)}(z) = 2^{\ell d}\psi^{(3)}\left(2^\ell x\right).$$

By taking the Fourier transform, it is easy to see the following continuous version of Calderón reproducing formula holding on $L^2(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$:

$$f(x,y,z) = \sum_{j,k,l} \left(\psi_{j,k,l} * \psi_{j,k,l} * f\right)(x,y,z). \qquad (23)$$

For $f \in (S_M)'(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$, we define the Littlewood–Paley square function on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d$ by

$$\mathcal{G}(f)(x,y,z) = \left(\sum_{j,k,\ell\in\mathbb{Z}} \left|\psi_{j,k,\ell} * f(x,y,z)\right|^2\right)^{1/2}, \qquad (24)$$

where $\psi_{j,k,\ell}$, satisfies the same conditions as in (22).

Definition (2.2.2) [128] A product kernel K is a distribution on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d$ which coincides with a C^∞ function away from the coordinate subspaces $x = 0, y = 0$ and $z = 0$ and satisfies

(i) (Differential Inequalities) For any multi-indices $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_m)$ and $\gamma = (\gamma_1, \dots, \gamma_m)$,

$$|\partial_x^\alpha \partial_y^\beta \partial_z^\gamma K(x,y,z)| \le C_{\alpha,\beta,\gamma}|x|^{-(n+|\alpha|)}|y|^{-(m+|\beta|)}|z|^{-(d+|\gamma|)};$$

(ii) (Cancellation Condition) For all normalized bump functions $\varphi_1, \varphi_2, \varphi_3$ on $\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^d$ respectively and for any $R_1, R_2, R_3 > 0$,

$$\left|\int_{\mathbb{R}^m\times\mathbb{R}^d} \partial_x^\alpha K(x,y,z)\varphi_2(R_2 y)\varphi_3(R_3 z)dydz\right| \le C|x|^{-n-|\alpha|},$$

$$\left|\int_{\mathbb{R}^n\times\mathbb{R}^d} \partial_y^\beta K(x,y,z)\varphi_1(R_1 x)\varphi_3(R_3 z)dxdz\right| \le C|y|^{-m-|\beta|},$$

$$\left|\int_{\mathbb{R}^n\times\mathbb{R}^m} \partial_z^\gamma K(x,y,z)\varphi_1(R_1 x)\varphi_2(R_2 y)dxdy\right| \le C|z|^{-d-|\gamma|},$$

$$\left|\int_{\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^d} \partial_y^\beta K(x,y,z)\varphi_1(R_1 x)\varphi_2(R_2 y)\varphi_3(R_3 z)dxdydz\right| \le C,$$

We describe the approach and results.

Definition (2.2.3) [128] Let $0 < p, q < \infty, s = (s_1, s_2, s_3) \in \mathbb{R}^3, w \in A_\infty$. The weighted Triebel–Lizorkin type space $\dot{F}_p^{s,q}(w; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$ is defined by

$$\dot{F}_p^{s,q}(w; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d) = \{f \in (S_M)' : \|f\|_{\dot{F}_p^{s,q}(w;\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^d)} < \infty\},$$

where

$$\|f\|_{\dot{F}_p^{s,q}(w;\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^d)} = \left\| \left(\sum_{j,k,l\in\mathbb{Z}} 2^{-(js_1+ks_2+ls_3)q} |\psi_{j,k,l} * f|^q \right)^{\frac{1}{q}} \right\|_{L^p(w)}. \tag{25}$$

The weighted Besov space $\dot{B}_p^{s,q}(w; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$ is defined by

$$\dot{B}_p^{s,q}(w; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d) = \{f \in (S_M)' : \|f\|_{\dot{B}_p^{s,q}(w;\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^d)} < \infty\},$$

where

$$\|f\|_{\dot{B}_p^{s,q}(w;\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^d),} = \left(\sum_{j,k,l\in\mathbb{Z}} 2^{-(js_1+ks_2+ls_3)q} \left\| \psi_{j,k,l} * f \right\|_{L^p(w)}^q \right)^{\frac{1}{q}}. \tag{26}$$

By Littlewood–Paley theory, if $s = (0, 0, 0)$, then $\dot{F}_p^{0,2}(w) = L^p(w)$ when $1 < p < \infty$, and $\dot{F}_p^{0,2}(w) = H^p(w)$ when $0 < p \le 1$ as defined in [124]. To proceed further, a natural question arises whether this definition is independent of the choice of function $\psi_{j,k,l}$. Moreover, to study the $\dot{F}_p^{s,q}(w; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$-boundedness and $\dot{B}_p^{s,q}(w; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$-boundedness of singular integral operators we need to discretize the norm of $\dot{F}_p^{s,q}(w; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$ and $\dot{B}_p^{s,q}(w; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$. To do that we shall prove a Min-Max comparison principle. The main tool to prove such principle is the Calderón reproducing formula (23). Furthermore, using an approximation procedure and the almost orthogonality estimate, the following discrete Calderón reproducing formula is proved in [124].

Theorem (2.2.4) [128] Let $\psi_{j,k,l}$ be the same as in (22). Then

$$f(x,y,z) = \sum_{j,k,l} \sum_{I,J,K} |I||J||K|\, \tilde{\psi}_{j,k,l}(x,y,z,x_I,y_J,z_K)\psi_{j,k,l} * f(x_I,y_J,z_K), \tag{27}$$

where $\tilde{\psi}_{j,k,l} \in S_M(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d), I \subset \mathbb{R}^n, J \subset \mathbb{R}^m$ *and* $K \subset \mathbb{R}^d$ are dyadic cubes with $l(I) = 2^{-j-N}$, where $l(I)$ is the side length of $I, l(J) = 2^{-k-N}$ and $l(K) = 2^{-l-N}$ for a fixed large integer N, x_I, y_J, z_K are any fixed points in I, J, K respectively, and the series in (27) converges in the norm of $S_M(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$ and in the dual space $(S_M)'(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$.

The discrete Calderón reproducing formula (27) provides the following Min-Max comparison principle in weighted Triebel–Lizorkin spaces and Besov spaces.

Lemma (2.2.5) [128] Given positive integers M_1, M_2, M_3, N_1, N_2 and N_3, there exists a constant $C = C(M_1, M_2, M_3, N_1, N_2, N_3)$ such that for all positive numbers t, s, r, t', s', r',

$$|\psi_{t,s,r} * \phi_{t',s',r'} t(x,y,z)|$$
$$\le C\left(\frac{t}{t'} \wedge \frac{t'}{t}\right)^{M_1} \left(\frac{s}{s'} \wedge \frac{s'}{s}\right)^{M_2} \left(\frac{r}{r'} \wedge \frac{r'}{r}\right)^{M_3}$$
$$\cdot \frac{(t \vee t')^{N_1}}{(t \vee t' + |x|)^{n+N_1}} \frac{(s \vee s')^{N_2}}{(s \vee s' + |y|)^{m+N_2}} \frac{(r \vee r')^{N_3}}{(r \vee r' + |z|)^{d+N_3}}, \tag{28}$$

where $\psi_{t,s,r}$ and $\phi_{t',s',r'}$ are the same as in (22) and $t \vee s = \max\{t,s\}, t \wedge s = \min\{t,s\}$.

Lemma (2.2.6) [128] Given any positive integers $M_1, M_2, M_3, N_1, N_2, N_3$, let $I, I'; J, J'$ and K, K' be dyadic cubes in $\mathbb{R}^n, \mathbb{R}^m$ and $\mathbb{R}^d$ respectively such that $l(I) = 2^{-j-N}, l(J) = 2^{-k-N}, l(K) = 2^{-l-N}$ and $l(I') = 2^{-j'-N}, l(J') = 2^{-k'-N}, l(K') = 2^{-l'-N}$. Then for any $u, u^* \in I, v, v^* \in J$ and $w, w^* \in K$, we have

$$\sum_{I',J',K'} \left\{ \frac{2^{-|j-j'|M_1} 2^{-|k-k'|M_2} 2^{-|l-l'|M_3} 2^{-(j\wedge j')N_1} 2^{-(k\wedge k')N_2} 2^{-(l\wedge l')N_3} |I'||J'||K'|}{\left(2^{-j\wedge j'} + |u - x_{I'}|\right)^{n+N_1} \left(2^{-k\wedge k'} + |v - y_{J'}|\right)^{m+N_2} \left(2^{-l\wedge l'} + |w - z_{K'}|\right)^{d+N_3}} \cdot \left|\phi_{j',k',l'} * f(x_{I'}, y_{J'}, z_{K'})\right| \right\}$$
$$\leq 2^{-|j-j'|M_1} \cdot 22^{-|k-k'|M_2} \cdot 2^{-|l-l'|M_3}$$
$$\cdot \left\{ M_s \left(\sum_{I'} \sum_{J'} \sum_{K'} |\phi_{j',k',l'} * f(x_{I'}, y_{J'}, z_{K'})| \chi_{I'} \chi_{J'} \chi_{J'} \right)^r (u^*, v^*, w^*) \right\}^{\frac{1}{r}},$$

where M_s is the strong maximal function on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d, 0 < r \leq min\{1, \frac{p}{q_w}\}, q_w$ is the critical index of w, and

$$C = 2^{N(\frac{1}{r}-1)(n+m+d)} 2^{(1-\frac{1}{r})[n(j\wedge j'-j')+m(k\wedge k'-k')+d(l\wedge l'-l')]}.$$

The proof of Lemmas (2.2.5) and (2.2.6) can be found in [108] and [124].

We need the following weighted Fefferman–Stein vector-valued inequality in the multi-parameter setting:

$$\int_{\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d} |M_s(f)(x,y,z)|_{l^q}^p w(x,y,z) dxdydz$$
$$\leq C \int_{\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d} |x,y,z|_{l^q}^p w(x,y,z) dxdydz, \qquad 1 < p, q < \infty,$$

where $w \in A_p(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$ and $f = (f_1, f_2, f_3, \dots) \in l^q$. The one-parameter version of this inequality corresponding to the Hardy–Littlewood maximal function instead of the strong maximal function, was proved in [126]. From one-parameter version together with the observation

$$M_s(f)(x,y,z) \leq M^{(1)} M^{(2)} M^{(3)}(f)(x,y,z),$$

where $M^{(i)}$ denotes the Hardy–Littlewood maximal operator acting on the i-th variable, the above multi-parameter inequality follows (see, e.g., [127]).

Theorem (2.2.7) [128] Let $\psi^{(1)}, \phi^{(1)} \in S(\mathbb{R}^n), \psi^{(2)}, \phi^{(2)} \in S(\mathbb{R}^m)$ and $\psi^{(3)}, \phi^{(3)} \in S(\mathbb{R}^d)$. Suppose that $\psi_{j,k,l}$ and $\phi_{j,k,l}$ satisfy the same conditions as in (22). Then for $0 < p, q < \infty, s = (s_1, s_2, s_3) \in \mathbb{R}^3$, and $w \in A_\infty(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$ and $f \in (S_M)'(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$,

$$\left\| \left(\sum_{j,k,l} 2^{-(js_1+ks_2+ls_3)q} \sum_{I,J,K} \sup_{u\in I, v\in J, w\in K} \left|\psi_{j,k,l} * f(u,v,w)\right|^q \chi_I(x)\chi_J(y)\chi_K(z) \right)^{1/q} \right\|_{L^p(w)}$$
$$\approx \left\| \left(\sum_{j,k,l} 2^{-(js_1+ks_2+ls_3)q} \sum_{I,J,K} \inf_{u\in I, v\in J, w\in K} \left|\psi_{j,k,l} * f(u,v,w)\right|^q \chi_I(x)\chi_J(y)\chi_K(z) \right)^{1/q} \right\|_{L^p(w)}, \quad (29)$$

$$\left(\sum_{j,k,l} 2^{-(js_1+ks_2+ls_3)q} \left\| \sum_{I,J,K} \sup_{u\in I, v\in J, w\in K} \left|\psi_{j,k,l} * f(u,v,w)\right|^q \chi_I\chi_J\chi_K \right\|_{L^p(w)}^q \right)^{1/q}$$

$$\approx \left(\sum_{j,k,l} 2^{-(js_1+ks_2+ls_3)q} \left\| \sum_{I,J,K} \inf_{u\in I, v\in J, w\in K} \left|\psi_{j,k,l} * f(u,v,w)\right|^q \ \chi_I \chi_J \chi_K \right\|_{L^p(w)}^q \right)^{1/q} . \qquad (30)$$

Theorem (2.2.7) ensures that the Triebel–Lizorkin space $\dot{F}_p^{s,q}(w;\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$ and the Besov space $\dot{B}_p^{s,q}(w;\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$ in Definition (2.2.3) are well defined.

By use of the Min-Max comparison principle, we shall prove the boundedness of singular integrals on both of $\dot{F}_p^{s,q}(w;\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$ and $\dot{F}_p^{s,q}(w;\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$.

Proof. By Theorem (2.2.4), $f \in S_M(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$ can be represented by

$$f(x,y,z) = \sum_{j',k',l'} \sum_{I',J',K'} |I'||J'||K'| \, \tilde{\phi}_{j',k',l'}(x,y,z,x_{I'},y_{J'},z_{K'}) \phi_{j',k',l'} * f(x_{I'},y_{J'},z_{K'}) .$$

We write

$$\left(\psi_{j,k,l} * f\right)(u,v,w) = \sum_{j',k',l'} \sum_{I',J',K'} |I'||J'||K'| \left(\psi_{j,k,l} * \tilde{\phi}_{j',k',l'}\right)(u,v,w,x_{I'},y_{J'},z_{K'}) \phi_{j',k',l'} * f(x_{I'},y_{J'},z_{K'}) .$$

By the almost orthogonality estimate in Lemma (2.2.5) by choosing $t = 2^{-j}, s = 2^{-k}, r = 2^{-l}, t' = 2^{-j'}, s' = 2^{-k'}$ and $r' = 2^{-l'}$, we have from Lemma (2.2.6) that for any given positive integers $M_1, M_2, M_3, N_1, N_2, N_3$, and for any $u, u^* \in I, v, v^* \in J$ and $w, w^* \in K$,

$$\left|\left(\psi_{j,k,l} * f\right)(u,v,w)\right| \le C \sum_{j',k',l'} \sum_{I',J',K'} \left(\frac{2^{-|j-j'|M_1} 2^{-|k-k'|M_2} 2^{-|l-l'|M_3} 2^{-(j\wedge j')N_1} 2^{-(k\wedge k')N_2}}{\left(2^{-j\wedge j'} + |u - x_{I'}|\right)^{n+N_1} \left(2^{-k\wedge k'} + \left|v - y_{J'}\right|\right)^{m+N_2}} \cdot \frac{2^{-(l\wedge l')N_3} |I'||J'||K'|}{\left(2^{-k\wedge k'} + |w - z_{K'}|\right)^{d+N_3}} \left|\phi_{j',k',l'} * f\left(x_I, y_{J'}, z_{K'}\right)\right| \right)$$

$$\le C \sum_{j',k',l'} 2^{-|j-j'|M_1} 2^{-|k-k'|M_2} 2^{-|l-l'|M_3} \cdot \left\{ M_s \left[\left(\sum_{I'} \sum_{J'} \sum_{K'} |\phi_{j',k',l'} * f(x_{I'},y_{J'},z_{K'})| \chi_{I'} \chi_{J'} \chi_{J'} \right)^r (u^*,v^*,w^*) \right] \right\}^{\frac{1}{r}},$$

where M_s is the strong maximal function on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d$ and $\max\{\frac{n}{n+N_1}, \frac{m}{m+N_2}, \frac{d}{d+N_3}\} < r < min\{\frac{p}{q_w}, 1\}$. Applying Hölder's inequality and summing over j, k, l, I, J and K, we have

$$\left\{ \sum_{j,k,l} \sum_{I,J,K} 2^{-(js_1+ks_2+ls_3)q} \sup_{u\in I, v\in J, w\in K} \left|(\psi_{j,k,l} * f)(u,v,w)\right|^q \chi_I \chi_J \chi_K \right\}^{\frac{1}{q}}$$

$$\le C \left\{ \sum_{j',k',l'} 2^{-(j's_1+k's_2+l's_3)q} \left(M_s \left[\sum_{I',J',K'} |\phi_{j',k',l'} * f(x_{I'},y_{J'},z_{K'})| \chi_{I'} \chi_{J'} \chi_{K'} \right]^r \right)^{\frac{q}{r}} \right\}^{\frac{1}{q}} .$$

Since $\chi_{I'}, \chi_{J'}$ and $\chi_{K'}$ are arbitrary points in I', J' and K' respectively, we have by weighted Fefferman–Stein vector-valued inequality that

$$\left\|\left\{\sum_{j,k,l}\sum_{I,J,K}2^{-(js_1+ks_2+ls_3)q}\sup_{u\in I,v\in J,w\in K}\left|(\psi_{j,k,l}*f)(u,v,w)\right|^q\chi_I\chi_J\chi_K\right\}^{\frac{1}{q}}\right\|_{L^p(w)}$$

$$\leq C\left\|\left\{\sum_{j',k',l'}2^{-(j's_1+k's_2+l's_3)q}\left(M_s\left[\sum_{I',J',K'}\inf_{u\in I',v\in J',w\in K'}\left|\varphi_{j',k',l'}*f(u,v,w)\right|\chi_{I'}\chi_{J'}\chi_{K'}\right]^r\right)^{\frac{q}{r}}\right\}^{\frac{1}{q}}\right\|_{L^p(w)}$$

$$\leq C\left\|\left\{\sum_{j',k',l'}\sum_{I',J',K'}2^{-(j's_1+k's_2+l's_3)q}\inf_{u\in I',v\in J',w\in K'}\left|\varphi_{j',k',l'}*f(u,v,w)\right|^q\chi_{I'}\chi_{J'}\chi_{K'}\right\}^{\frac{1}{q}}\right\|_{L^p(w)},$$

the last inequality follows from the fact that $r<\min\{\frac{p}{q_w},1\}$ and $w\in A_{p/r}$. Thus (29) follows and the proof of the first part of Theorem (2.2.7) is completed.

Next, we will show the second part of Theorem (2.2.7), i.e., (30). As in the proof of (29), we have

$$\sup_{u\in I,v\in J,w\in K}\left|(\psi_{j,k,l}*f)(u,v,w)\right|\chi_I\chi_J\chi_K$$

$$\leq C\sum_{j',k',l'}2^{-|j-j'|M_1}2^{-|k-k'|M_2}2^{-|l-l'|M_3}\cdot\left\{M_s\left(\sum_{I',J',K'}\left|\varphi_{j',k',l'}*f\left(x_{I'},y_{J'},z_{K'}\right)\right|\chi_{I'}\chi_{J'}\chi_{K'}\right)^r(u^*,v^*,w^*)\right\}^{\frac{1}{r}}.$$

When $1<p<\infty$, by the weighted boundedness of strong maximal function M_s with $r<min\{\frac{p}{q_w},1\}$ and $w\in A_{p/r}$, we have

$$\left\|\sum_{I,J,K}\sup_{u\in I,v\in J,w\in K}\left|(\psi_{j,k,l}*f)(u,v,w)\right|\chi_I\chi_J\chi_K\right\|_{L^p(w)}$$

$$\leq\left\|\sum_{j',k',l'}2^{-|j-j'|M_1}2^{-|k-k'|M_2}2^{-|l-l'|M_3}\cdot\left\{M_s\left(\sum_{I',J',K'}\left|\phi_{j',k',l'}*f\left(x_{I'},y_{J'},z_{K'}\right)\right|\chi_{I'}\chi_{J'}\chi_{J'}\right)^r\right\}^{\frac{1}{r}}\right\|_{L^p(w)}$$

$$\leq\sum_{j',k',l'}2^{-|j-j'|M_1}2^{-|k-k'|M_2}2^{-|l-l'|M_3}\cdot\left\|\left\{M_s\left(\sum_{I',J',K'}\left|\phi_{j',k',l'}*f\left(x_{I'},y_{J'},z_{K'}\right)\right|\chi_{I'}\chi_{J'}\chi_{J'}\right)^r\right\}^{\frac{1}{r}}\right\|_{L^p(w)}$$

$$\leq\sum_{j',k',l'}2^{-|j-j'|M_1}2^{-|k-k'|M_2}2^{-|l-l'|M_3}\cdot\left\|\sum_{I',J',K'}|\phi_{j',k',l'}*f(x_{I'},y_{J'},z_{K'})|\chi_{I'}\chi_{J'}\chi_{K'}\right\|_{L^p(w)}.$$

If $q>1$, applying Hölder's inequality and if $0<q\leq 1$, using the inequality $(\sum_l a_l)^q\leq\sum_l a_l^q$, and summing over j,k,l yields

$$\left(\sum_{j,k,l}2^{-(js_1+ks_2+ls_3)q}\left\|\sup_{u\in I,v\in J,w\in K}\left|(\psi_{j,k,l}*f)(u,v,w)\right|\chi_I\chi_J\chi_K\right\|_{L^p(w)}^q\right)^{\frac{1}{q}}$$

$$\leq C\left(\sum_{j',k',l'}2^{-(j's_1+k's_2+l's_3)q}\left\|\sum_{I',J',K'}|(\psi_{j,k,l}*f(x_{I'},y_{J'},z_{K'})|\chi_{I'}\chi_{J'}\chi_{J'}\right\|_{L^p(w)}^q\right)^{\frac{1}{q}}.$$

When $0 < p \le 1$, by the fact that $w \in A_{p/r}$, we have

$$\int \left(\sup_{u\in I, v\in J, w\in K} \left|(\psi_{j,k,l} * f)(u,v,w)\right| \chi_I(x)\chi_J(y)\chi_K(z) \right)^p w(x,y,z)dxdydz$$

$$\le C \sum_{j',k',l'} 2^{-|j-j'|M_1 p} 2^{-|k-k'|M_2 p} 2^{-|l-l'|M_3 p}$$

$$\cdot \int \left\{ M_s \left(\sum_{I',J',K'} \left|\phi_{j',k',l'} * f(x_{I'}, y_{J'}, z_{K'})\right| \chi_{I'}\chi_{J'}\chi_{K'} \right)^r (x,y,z) \right\}^{p/r} w(x,y,z)dxdydz$$

$$\le C \sum_{j',k',l'} 2^{-|j-j'|M_1 p} 2^{-|k-k'|M_2 p} 2^{-|l-l'|M_3 p}$$

$$\cdot \int \left(\sum_{I',J',K'} \left|\phi_{j',k',l'} * f(x_{I'}, y_{J'}, z_{K'})\right| \chi_{I'}\chi_{J'}\chi_{K'} \right)^p w(x,y,z)dxdydz.$$

By Hölder's inequality with exponents $\frac{1}{p}$ and $\frac{1}{1-p}$, we have

$$\left\{ \iint \left(\sup_{u\in I, v\in J, w\in K} \left|(\psi_{j,k,l} * f)(u,v,w)\right| \chi_I(x)\chi_J(y)\chi_K(z) \right)^p w(x,y,z)dxdydz \right\}^{\frac{1}{p}}$$

$$\le C \sum_{j',k',l'} 2^{-(|j-j'|M_1 + |k-k'|M_2 + |l-l'|M_3)\varepsilon}$$

$$\cdot \left\{ \iint \left(\sum_{I',J',K'} \left|\phi_{j',k',l'} * f(x_{I'}, y_{J'}, z_{K'})\right| \chi_{I'}\chi_{J'}\chi_{K'} \right)^p w(x,y,z)dxdydz \right\}^{\frac{1}{p}},$$

where $\varepsilon \in (0,1)$. Then, if $q > 1$ by using Hölder's inequality and if $0 < q \le 1$ by using inequality $(\sum_l a_l)^q \le \sum_l a_l^q$ again, we have

$$\left(\sum_{j,k,l} 2^{-(js_1+ks_2+ls_3)q} \left\| \sup_{u\in I, v\in J, w\in K} \left|(\psi_{j,k,l} * f)(u,v,w)\right| \chi_I\chi_J\chi_K \right\|_{L^p(w)}^q \right)^{\frac{1}{q}}$$

$$\le \left[\sum_{j,k,l} 2^{-(js_1+ks_2+ls_3)q} \left(\sum_{j',k',l'} 2^{-(|j-j'|M_1 + |k-k'|M_2 + |l-l'|M_3)\varepsilon} \right.\right.$$

$$\left.\left. \cdot \left\| \sum_{I',J',K'} \left|\phi_{j',k',l'} * f(x_{I'}, y_{J'}, z_{K'})\right| \chi_{I'}\chi_{J'}\chi_{K'} \right\|_{L^p(w)} \right)^q \right]^{\frac{1}{q}}$$

$$\le C \left(\sum_{j',k',l'} 2^{-(j's_1+k's_2+l's_3)q} \left\| \sum_{I',J',K'} |\phi_{j',k',l'} * f(x_{I'}, y_{J'}, z_{K'})| \chi_{I'}\chi_{J'}\chi_{J'} \right\|_{L^p(w)}^q \right)^{\frac{1}{q}}.$$

Since $x_{I'}, y_{J'}$ *and* $z_{K'}$ are arbitrary points in I', J' and K' respectively, we have the desired inequality (30).

As a consequence of Theorem (2.2.7), it is easy to see that the weighted Triebel–Lizorkin and Besov Spaces are independent of the choice of the function ψ. Moreover, we have the following discrete characterization of $\dot{F}_p^{s,q}(w;\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^d)$ and $\dot{B}_p^{s,q}(w;\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^d)$.

Proposition (2.2.8) [128] Let $0<p,q<\infty, s=(s_1,s_2,s_3)\in\mathbb{R}^3$ and $w\in A_\infty$. Then we have

$$\|f\|_{\dot{F}_p^{s,q}(w)} \approx \left\|\left(\sum_{j,k,l}\sum_{I,J,K}2^{-(js_1+ks_2+ls_3)q}\left|(\psi_{j,k,l}*f)(x_I,y_J,z_K)\right|^q\chi_I\chi_J\chi_K\right)^{\frac{1}{q}}\right\|_{L^p(w)}, \tag{31}$$

$$\|f\|_{\dot{B}_p^{s,q}(w)} \approx \left(\sum_{j,k,l}2^{-(js_1+ks_2+ls_3)q}\left\|\sum_{I,J,K}\left|(\psi_{j,k,l}*f)(x_I,y_J,z_K)\right|\chi_I\chi_J\chi_K\right\|_{L^p(w)}^q\right)^{\frac{1}{q}}, \tag{32}$$

where $\psi_{j,k,l}, j,k,l,\chi_I,\chi_J,\chi_K$ are the same as in Theorem (2.2.7).

Before we prove the boundedness of singular integrals on $\dot{F}_p^{s,q}(w)$ and $\dot{B}_p^{s,q}(w)$ we introduce several properties of them.

Proposition (2.2.9) [128] $S_M(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^d)$ is dense in $\dot{F}_p^{s,q}(w;\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^d)$ and in $\dot{B}_p^{s,q}(w;\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^d)$.

Proof. Suppose that $f\in\dot{F}_p^{s,q}(w;\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^d)$. By Theorem (2.2.4),

$$f(x,y,z)=\sum_{j,k,l}\sum_{I,J,K\in W}|I||J||K|\tilde{\psi}_{j,k,l}(x,y,z,x_I,y_J,z_K)(\psi_{j,k,l}*f)(x_I,y_J,z_K).$$

Set

$$W=\{(j,k,l,I,J,K):|j|\le L_1,|k|\le L_2,|l|\le L_3,I\times J\times K\subset B(0,R)\},$$

where I,J,K are dyadic cubes in $\mathbb{R}^n,\mathbb{R}^m,\mathbb{R}^d$ with side-length $2^{-j-N},2^{-k-N},2^{-l-N}$, respectively, and $B(0,R)$ is a ball in $\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^d$ centered at the origin with radius R. Obviously,

$$\sum_{j,k,l,I,J,K\in W}|I||J||K|\tilde{\psi}_{j,k,l}(x,y,z,x_I,y_J,z_K)(\psi_{j,k,l}*f)(x_I,y_J,z_K)$$

is a test function in $S_M(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^d)$ for any fixed L_1,L_2,L_3 and R, where $\tilde{\psi}_{j,k,l}\in S_M(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^d)$. Repeating the same proof as in Theorem (2.2.7), we have

$$\left\|\sum_{(j,k,l,I,J,K)\in W^c}|I||J||K|\tilde{\psi}_{j,k,l}(x,y,z,x_I,y_J,z_K)(\psi_{j,k,l}*f)(x_I,y_J,z_K)\right\|_{\dot{F}_p^{s,q}(w)}$$
$$\le C\left\|\left\{\sum_{(j,k,l,I,J,K)\in W^c}2^{-(js_1+ks_2+ls_3)q}\left|\psi_{j,k,l}*f(x_I,y_J,z_K)\right|^q\chi_I\chi_J\chi_K\right\}^{\frac{1}{q}}\right\|_{L^p(w)},$$

the last term tends to zero as L_1,L_2,L_3 and R tends to infinity whenever $f\in\dot{F}_p^{s,q}(w;\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^d)\subset(S_M(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^d))'$.

Suppose that $f\in\dot{B}_p^{s,q}(w;\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^d)$. Set

$W_1=\{(j,k,l):|j|\le L_1,|k|\le L_2,|l|\le L_3\}$ and $W_2=\{(I,J,K):I\times J\times K\subset B(0,R)\}$, where I,J,K are dyadic cubes in $\mathbb{R}^n,\mathbb{R}^m,\mathbb{R}^d$ with side-length $2^{-j-N},2^{-k-N},2^{-l-N}$, respectively. Then

$$\left\|\sum_{(j,k,l)\in W_1^c}\sum_{(I,J,K)\in W_2^c}|I||J||K|\tilde{\psi}_{j,k,l}(x,y,z,x_I,y_J,z_K)(\psi_{j,k,l}*f)(x_I,y_J,z_K)\right\|_{\dot{B}_p^{s,q}(w)}$$

$$\leq \left(\sum_{(j,k,l)\in W_1^c} 2^{-(js_1+ks_2+ls_3)q} \left\| \sum_{(I,J,K)\in W_2^c} \left|(\psi_{j,k,l} * f)(\,x_I, y_J, z_K)\right| \chi_I \chi_J \chi_K \right\|_{L^p(w)}^q \right)^{\frac{1}{q}},$$

the last term tends to zero as L_1, L_2, L_3 and R tends to infinity whenever $f \in \dot{B}_p^{s,q}(w; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d) \subset (S_M(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d))'$.

As a consequence of Proposition (2.2.9), $L^2(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$ is dense in $\dot{F}_p^{s,q}(w; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$ and in $\dot{B}_p^{s,q}(w; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$.

Theorem (2.2.10) [128] Suppose that T is a singular integral defined on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d$ by $(Tf)(x,y,z) = (K * f)(x,y,z)$, where the kernel satisfies conditions in Definition (2.2.2). Then T is bounded on $\dot{F}_p^{s,q}(w; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$ and $\dot{B}_p^{s,q}(w; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$. Namely, for all $0 < p, q < \infty,\ s = (s_1, s_2, s_3) \in \mathbb{R}^3$ and $w \in A_\infty$, there exists a constant C, such that

$$\|T(f)\|_{\dot{F}_p^{s,q}(w;\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^d)} \leq C\|f\|_{\dot{F}_p^{s,q}(w;\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^d)}, \tag{33}$$

$$\|T(f)\|_{\dot{B}_p^{s,q}(w;\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^d)} \leq C\|f\|_{\dot{B}_p^{s,q}(w;\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^d)}. \tag{34}$$

We end with the following. Rychkov [125] characterized the weighted Besov–Lipschitz and Triebel–Lizorkin spaces on $\mathbb{R}^n$ with weights that are locally in A_p but may grow or decrease exponentially at infinity. A certain local variant of the Calderón reproducing formula is also constructed and used in [125]. We get the boundedness of singular integrals on the weighted Triebel–Lizorkin and Besov spaces with A_∞ weights. This theorem is new even in one parameter case.

We will prove Theorem (2.2.7), namely, the Min-Max comparison principle which implies that the weighted Triebel–Lizorkin and Besov spaces are well defined as given in Definition (2.2.3). Provides the proof of the boundedness of singular integrals on the weighted Triebel–Lizorkin and Besov spaces, namely, Theorem (2.2.10).

We establish the Min-Max comparison principle in weighted Triebel–Lizorkin and Besov spaces. We first recall the almost orthogonality estimates on S_M.

Proof. We assume now that K is the kernel of T. Applying the discrete Calderón reproducing formula, we get, for $f \in L^2 \cap \dot{F}_p^{s,q}(w; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$,

$$\left\| \left\{ \sum_{j,k,l} \sum_{I,J,K} 2^{-(js_1+ks_2+ls_3)q} \left|\psi_{j,k,l} * K * f(x_I, y_J, z_K)\right|^q \chi_I(x)\chi_J(y)\chi_K(z) \right\}^{\frac{1}{q}} \right\|_{L^p(w)}$$

$$= \left\| \left(\sum_{j,k,l} \sum_{I,J,K} 2^{-(js_1+ks_2+ls_3)q} \left| \sum_{j',k',l'} \sum_{I',J',K'} |I'||J'||K'| \psi_{j,k,l} * K * \tilde{\psi}_{j',k',l'}(\cdot - x_I, \cdot - y_{J'}, \cdot - z_{K'})(x,y,z) \psi_{j',k',l'} * f(x_{I'}, y_{J'}, z_{K'}) \right|^q \chi_I(x)\chi_J(y)\chi_K(z) \right)^{\frac{1}{q}} \right\|_{L^p(w)}.$$

Noting that $\psi_{j,k,l}$ are dilation of bump functions, one can easily get from almost orthogonality estimate that

$$|\psi_{j,k,l} * K * \tilde{\psi}_{j',k',l'}(\cdot - x_I, \cdot - y_{J'}, \cdot - z_{K'})(x,y,z)|$$
$$\le C 2^{-|j-j'|L_1} 2^{-|k-k'|L_2} 2^{-|l-l'|L} \frac{2^{-(j\wedge j')L}}{\left(2^{-(j\wedge j')} + |x - x_{I'}|\right)^{n+L}}$$
$$\cdot \frac{2^{-(k\wedge k')L}}{\left(2^{-(k\wedge k')} + |y - y_{J'}|\right)^{m+L}} \cdot \frac{2^{-(l\wedge l')L}}{\left(2^{-(l\wedge l')} + |z - z_{K'}|\right)^{d+L}}, \tag{35}$$

where L only depends on M_0 and M_0 is chosen to be large enough. Then following a similar proof in Theorem (2.2.7), we have

$$\|Tf\|_{\dot{F}_p^{s,q}(w;\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^d)}$$
$$\le C \left\| \left(\sum_{j',k',l'} 2^{-(j's_1+k's_2+l's_3)q} \cdot \left\{ Ms \left(\sum_{I',J',K'} \left|\psi_{j',k',l'} * f\left(x_{I'}, y_{J'}, z_{K'}\right)\right| \chi_{I'}\chi_{J'}\chi_{K'} \right)^r \right\}^{\frac{q}{r}} (x,y,z) \right)^{\frac{1}{q}} \right\|_{L^p(w)}$$
$$\le \left\| \left(\sum_{j',k',l'} \sum_{I',J',K'} 2^{-(j's_1+k's_2+l's_3)q} \left|\psi_{j',k',l'} * f\left(x_{I'}, y_{J'}, z_{K'}\right)\right|^q \chi_{I'}\chi_{J'}\chi_{K'} \right)^{\frac{1}{q}} \right\|_{L^p(w)}$$
$$\le C\|f\|_{\dot{F}_p^{s,q}(w;\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^d)}.$$

Since $L^2(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d) \cap \dot{F}_p^{s,q}(w;\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$ is dense in $\dot{F}_p^{s,q}(w;\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$, we could yield (33) in Theorem (2.2.10) by a limiting argument.

Similarly, for $f \in L^2 \cap \dot{B}_p^{s,q}(w;\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d)$, applying the Calderón reproducing formula,

$$\left\{ \sum_{j,k,l} 2^{-(js_1+ks_2+ls_3)q} \left\| \sum_{I,J,K} \left|\psi_{j,k,l} * K * f(x,y,z)\right| \chi_I(x)\chi_J(y)\chi_K(z) \right\|_{L^p(w)}^q \right\}^{\frac{1}{q}}$$
$$= \left(\sum_{j,k,l} 2^{-(js_1+ks_2+ls_3)q} \left\| \sum_{I,J,K} \left| \sum_{j',k',l'} \sum_{I',J',K'} |I'||J'||K'| \psi_{j,k,l} * K * \tilde{\psi}_{j',k',l'}(\right.\right.\right.$$
$$\left.\left.\left. \cdot - x_I, \cdot - y_{J'}, \cdot - z_{K'}) \psi_{j',k',l'} * f(x_I, y_{J'}, z_{K'}) \right| \chi_I\chi_J\chi_K \right\|_{L^p(w)}^q \right)^{\frac{1}{q}}.$$

By (35), we obtain

$$\|Tf\|_{\dot{B}_p^{s,q}(w;\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^d)}$$
$$\le C \sum_{j',k',l'} 2^{-(j's_1+k's_2+l's_3)q} \cdot \left\| \left(M_s \left(\sum_{I',J',K'} \left|\psi_{j',k',l'} * f\left(x_{I'}, y_{J'}, z_{K'}\right)\right| \chi_{I'}\chi_{J'}\chi_{K'} \right)^r \right)^{\frac{1}{r}} \right\|_{L^p(w)}^q$$
$$\le \left\{ \sum_{j',k',l'} 2^{-(j's_1+k's_2+l's_3)q} \left\| \psi_{j',k',l'} * f\left(x_{I'}, y_{J'}, z_{K'}\right) \chi_{I'}\chi_{J'}\chi_{K'} \right\|_{L^p(w)}^q \right\}^{\frac{1}{q}}$$
$$\le C\|f\|_{\dot{B}_p^{s,q}(w;\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^d)}.$$

Section (2.3) Triebel–Lizorkin and Besov Spaces with Different Homogeneities

The multi-parameter pure product theory has been developed. This theory includes the boundedness of singular integral operators on multi-parameter L^p spaces $(1 < p < \infty)$ and multi-parameter Hardy spaces $H^p (0 < p \le 1)$. This theory also includes the atomic decomposition of multi-parameter Hardy spaces, duality and interpolation theorems on product spaces, and maximal function characterizations, etc. See [96, 97, 99, 100–105, 121, 129].

[108, 119, 122], developed a theory of discrete Calderón reproducing formula and Littlewood–Paley analysis, and then applied it to establish the implicit multi-parameter Hardy space and dual space theory associated with the flag singular integrals both in Euclidean spaces and Heisenberg groups. Their results lead to the endpoint estimates of the Marcinkiewitz multipliers on the Heisenberg group where the L^p estimates were established by Muller–Ricci–Stein [106]. Ideas and methods have inspired much subsequent works using the discrete Littlewood–Paley theory in various multi-parameter settings, see [123, 130, 131, 133, 134, 136], etc. Using the discrete Littlewood–Paley analysis developed in [108] and [119], [128] and [120] introduced the theory of multi-parameter Triebel–Lizorkin and Besov spaces associated with the flag singular integrals [120] and weighted Triebel–Lizorkin and Besov spaces of arbitrary number of parameters [128] and proved the boundedness of singular integral operators on these spaces. These Triebel–Lizorkin spaces include the multi-parameter Hardy spaces when the index p is less than or equal to 1. (See also [138] for Triebel–Lizorkin and Besov spaces associated with the flag singular integrals when the indices p and q are strictly bigger than 1.)

[132] established the boundedness of composition singular integrals on Hardy spaces associated with different homogeneities. To describe their result, we begin with some brief review of composition of two operators with different homogeneities.

The composition of operators was considered by Calderón and Zygmund when introducing the first generation of Calderón–Zygmund convolution operators. Let $e(\xi)$ be a function on $\mathbb{R}^m$ homogeneous of degree 0 in the isotropic sense and smooth away from the origin. Similarly, suppose that $h(\xi)$ is a function on $\mathbb{R}^m$ homogeneous of degree 0 in the nonisotropic sense similar the one in situation of the heat equation, and also smooth away from the origin. Then the classical Calderón–Zygmund theory tells us that the Fourier multipliers T_1 defined by $\widehat{T_1(f)}(\xi) = e(\xi)\hat{f}(\xi)$ and T_2 given by $\widehat{T_2(f)}(\xi) = h(\xi)\hat{f}(\xi)$ are both bounded on L^p for $1 < p < \infty$, and satisfy various other regularity properties such as being of weak-type (1, 1). It was also well known that T_1 and T_2 are bounded on the classical isotropic and non-isotropic Hardy spaces, respectively. The following question raised by Rivieré in [137] is highly nontrivial: Is the composition $T_1 \circ T_2$ still of weak-type (1, 1)? It was Phong and Stein who answered in [135] this question and gave a necessary and sufficient condition such that the composition operator $T_1 \circ T_2$ is of weak-type (1, 1). In fact, the operators Phong and Stein studied are compositions with different homogeneities and such a composition operator arises naturally in the study of $\bar{\partial}$-Neumann problem.

It is well known that any Calderón–Zygmund singular integral operator associated with the isotropic homogeneity is bounded on the classical Hardy space $H^p(\mathbb{R}^m)$ with $0 < p \le 1$. A Calderón–Zygmund singular integral operator associated with the non-isotropic homogeneity is not bounded on the classical Hardy space but bounded on the non-isotropic Hardy space. However, the composition operator is bounded on neither the classical Hardy space nor the non-isotropic Hardy space. This motivates the authors of [132] to introduce a new Hardy space associated with the different homogeneities.

Motivated by [120, 128, 132], we consider new Triebel–Lizorkin and Besov spaces associated with the composition of two operators with different homogeneities. All functions and operators are always defined on $\mathbb{R}^m$. We write $\mathbb{R}^m = \mathbb{R}^{m-1} \times \mathbb{R}$ with $x = (x', x_m)$ where $x' \in \mathbb{R}^{m-1}$ and $x_m \in \mathbb{R}$. We consider two kinds of homogeneities

$$\delta: (x', x_m) \to (\delta x', \delta x_m), \qquad \delta > 0$$

and

$$\delta: (x', x_m) \to (\delta x', \delta^2 x_m), \qquad \delta > 0.$$

The first is the classical isotropic dilations occurring in the classical Calderón–Zygmund singular integrals, while the second is non-isotropic and related to the heat equations (also Heisenberg groups).

For $x = (x', x_m) \in \mathbb{R}^{m-1} \times \mathbb{R}$ we denote $|x|_e = (|x'|^2 + |x_m|^2)^{\frac{1}{2}}$ and $|x|_h = (|x'|^2 + |x_m|)^{\frac{1}{2}}$. We also use notations $j \wedge k = \min\{j, k\}$ and $j \vee k = \max\{j, k\}$. The singular integrals considered are defined in the following.

Definition (2.3.1) [139] A locally integrable function K_1 on $\mathbb{R}^m \backslash \{0\}$ is said to be a Calderón–Zygmund kernel associated with the isotropic homogeneity if

$$\left|\frac{\partial^\alpha}{\partial x^\alpha} K_1(x)\right| \le A|x|_e^{-m-|\alpha|} \quad \text{for all } |\alpha| \ge 0, \tag{36}$$

$$\int_{r_1<|x|_e<r_2} K_2(x)dx = 0 \tag{37}$$

for all $0 < r_1 < r_2 < \infty$.

We say that an operator T_1 is a Calderón–Zygmund singular integral operator associated with the isotropic homogeneity if $T_1(f)(x) = p.v.(K_1 * f)(x)$, where K_1 satisfies conditions in (36) and (37).

Definition (2.3.2) [139] Suppose $K_2 \in L^1_{loc}(\mathbb{R}^m \backslash \{0\}). K_2$ is said to be a Calderón–Zygmund kernel associated with the non-isotropic homogeneity if

$$\left|\frac{\partial^\alpha}{\partial (x')^\alpha} \frac{\partial^\beta}{\partial (x_m)^\beta} K_2(x', x_m)\right| \le B|x|_h^{-m-1-|\alpha|-2\beta}, \qquad \forall |\alpha| \ge 0, \beta \ge 0, \tag{38}$$

$$\int_{r_1<|x|_h<r_2} K_2(x)dx = 0 \tag{39}$$

for all $0 < r_1 < r_2 < \infty$.

We say that an operator T_2 is a Calderón–Zygmund singular integral operator associated with the non-isotropic homogeneity if $T_2(f)(x) = p.v.(K_2 * f)(x)$, where K_2 satisfies the conditions in (38) and (39).

Denote $S_0(\mathbb{R}^m) = \{f \in S(\mathbb{R}^m): \int_{\mathbb{R}^m} f(x)x^\alpha dx = 0, \ \forall |\alpha| \ge 0\}$. Let $\psi^{(1)} \in S(\mathbb{R}^m)$ with

$$\operatorname{supp}\widehat{\psi^{(1)}} \subseteq \left\{(\xi', \xi_m) \in \mathbb{R}^{m-1} \times \mathbb{R}: \frac{1}{2} \le |\xi|_e \le 2\right\}, \tag{40}$$

and

$$\sum_{j\in\mathbb{Z}} \left|\widehat{\psi^{(1)}}(2^{-j}\xi', 2^{-j}\xi_m)\right|^2 = 1 \ \ \text{forall } (\xi', \xi_m) \in \mathbb{R}^m \backslash \{0\}. \tag{41}$$

Let $\psi^{(2)} \in S(\mathbb{R}^m)$ with

$$\operatorname{supp}\widehat{\psi^{(2)}} \subseteq \left\{(\xi', \xi_m) \in \mathbb{R}^{m-1} \times \mathbb{R}: \frac{1}{2} \le |\xi|_h \le 2\right\}, \tag{42}$$

and

$$\sum_{k\in\mathbb{Z}}\left|\widehat{\psi^{(2)}}(2^{-k}\xi', 2^{-2k}\xi_m)\right|^2 = 1 \quad \text{for all } (\xi',\xi_m)\in\mathbb{R}^m\{0\}. \tag{43}$$

Denote $\psi_{j,k}(x)=\psi_j^{(1)}*\psi_k^{(2)}(x)$, where $\psi_j^{(1)}(x',x_m)=2^{jm}\psi^{(1)}(2^jx',2^jx_m), \psi_k^{(2)}(x',x_m)=2^{k(m+1)}\psi^{(2)}(2^kx',2^{2k}x_m)$. Using an approximation procedure and the almost orthogonality estimate, the following discrete Calderón reproducing formula is proved in [132].

Theorem (2.3.3) [139] Suppose that $\psi^{(1)}$ and $\psi^{(2)}$are functions satisfying conditions in (40)–(41) and (42)–(43), respectively. Then

$$f(x',x_m)=\sum_{j,k\in\mathbb{Z}}\sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} 2^{-(m-1)(j\wedge k)}2^{-(j\wedge 2k)}\,(\psi_{j,k}*f)(2^{-(j\wedge k)}\ell',2^{-(j\wedge 2k)}\ell_m)$$
$$\times\,\psi_{j,k}(x'-2^{-(j\wedge k)}\ell',x_m-2^{-(j\wedge 2k)}\ell_m)$$

where the series converges in $L^2(\mathbb{R}^m), S_0(\mathbb{R}^m)$ and $S_0'(\mathbb{R}^m)$.

With the discrete Calderón reproducing formula, we can define Triebel–Lizorkin spaces and Besov spaces with different homogeneities.

Definition (2.3.4) [139] Let $0<p,q<\infty, s=(s_1,s_2)\in\mathbb{R}^2$. The Triebel–Lizorkin type space with different homogeneities $\dot{F}_p^{s,q}(\mathbb{R}^m)$ is defined by

$$\dot{F}_p^{s,q}(\mathbb{R}^m)=\{f\in S_0'(\mathbb{R}^m): \|f\|_{\dot{F}_p^{s,q}(\mathbb{R}^m)}<\infty\},$$

where

$$\|f\|_{\dot{F}_p^{s,q}(\mathbb{R}^m)}=\left\|\left(\sum_{j,k\in\mathbb{Z}}2^{-[(j\wedge k)s_1+(j\wedge 2k)s_2]q}\right.\right.$$
$$\left.\left.\times\sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}}\left|\psi_{j,k}*f(2^{-(j\wedge k)}\ell',2^{-(j\wedge 2k)}\ell_m)\right|^q\chi_I(x')\chi_J(x_m)\right)^{\frac{1}{q}}\right\|_{L^p(\mathbb{R}^m)},$$

where I are dyadic cubes in $\mathbb{R}^{m-1}$ and J are dyadic intervals in $\mathbb{R}$ with the side length $l(I)=2^{-(j\wedge k)}\ell'$ and $l(J)=2^{-(j\wedge 2k)}$, and the left lower corners of I and the left end points of J are $2^{-(j\wedge k)}$ and $2^{-(j\wedge 2k)}\ell_m$, respectively.

The Besov space with different homogeneities $\dot{B}_p^{s,q}(\mathbb{R}^m)$ is defined by

$$\dot{B}_p^{s,q}(\mathbb{R}^m)=\{f\in S_0'(\mathbb{R}^m): \|f\|_{\dot{B}_p^{s,q}(\mathbb{R}^m)}<\infty\},$$

where

$$\|f\|_{\dot{B}_p^{s,q}(\mathbb{R}^m)}=\left(\sum_{j,k\in\mathbb{Z}}2^{-[(j\wedge k)s_1+(j\wedge 2k)s_2]q}\right.$$
$$\left.\times\left\|\sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}}\psi_{j,k}*f(2^{-(j\wedge k)}\ell',2^{-(j\wedge 2k)}\ell_m)\chi_I(x')\chi_J(x_m)\right\|_{L^p(\mathbb{R}^m)}^q\right)^{\frac{1}{q}}$$

where I,J are the same as the above description.

Note that, the multi-parameter structures are involved in the discrete Calderón's identity and also in the new Triebel–Lizorkin spaces and Besov spaces. To see that these spaces are well defined, we need to show that $\dot{F}_p^{s,q}(\mathbb{R}^m)$and $\dot{B}_p^{s,q}(\mathbb{R}^m)$ are independent of the choice of the functions ψ^1 and ψ^2. This will directly follow from the following.

We now state the main results.

Lemma (2.3.5) [139] Suppose that $\psi_{j,k}$ and $\varphi_{j',k'}$ satisfy the same conditions in (40)–(43). Then for any given integers L and M, there exists a constant $C = C(L,M) > 0$ such that

$$\left|\psi_{j,k} * \varphi_{j',k'}(x',x_m)\right| \le C2^{-|j-j'|L}2^{-|k-k'|L} \frac{2^{(j\wedge j'\wedge k\wedge k')(m-1)}}{\left(1+2^{j\wedge j'\wedge k\wedge k'}|x'|\right)^{(M+m-1)}} \times \frac{2^{j\wedge j'\wedge 2(k\wedge k')}}{(1+2^{j\wedge j'\wedge 2(k\wedge k')}|x_m|)^{(M+1)}}.$$

Modifying the proof of Lemma 3.2 in [132] slightly, we have the following discrete version of the strong maximal function.

Lemma (2.3.6) [139] Let I, I' be dyadic cubes in $\mathbb{R}^{m-1}$ and J, J' be dyadic intervals in $\mathbb{R}$ with the side lengths $\ell(I) = 2^{-(j\wedge k)}, \ell(I') = 2^{-(j'\wedge k')-N}$ and $\ell(J) = 2^{-(j\wedge 2k)}, \ell(J') = 2^{-(j'\wedge 2k')-N}$ for an integer $N \ge 0$, and the left lower corners of I, I' and the left end points of J, J' are $2^{-(j\wedge k)}\ell', 2^{-(j'\wedge k')-N}\ell'', 2^{-(j\wedge 2k)}\ell_m$ and $2^{-(j'\wedge 2k')-N}\ell'_m$ for $(\ell',\ell_m),(\ell'',\ell_m) \in \mathbb{Z}^m$, respectively.

Then for any $u', v' \in I, u_m, v_m \in J$, and any $\frac{m-1}{M+m-1} < \delta \le 1$,

$$\sum_{(\ell'',\ell'_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} \frac{2^{(m-1)(j\wedge j'\wedge k\wedge k')}2^{j\wedge j'\wedge 2k\wedge 2k'}2^{-(m-1)[(j'\wedge k')+N]}2^{-(j'\wedge 2k')-N}}{\left(1+2^{j\wedge j'\wedge k\wedge k'}\left|u'-2^{-(j'\wedge k')-N}\ell''\right|\right)^{(M+m-1)}} \times \frac{\left|\left(\phi_{j',k'} * f\right)\left(2^{-(j'\wedge k')-N}\ell'', 2^{-(j'\wedge 2k')-N}\ell'_m\right)\right|}{\left(1+2^{j\wedge j'\wedge 2k\wedge 2k'}\left|u_m-2^{-(j'\wedge 2k')-N}\ell'_m\right|\right)^{(M+1)}}$$

$$\le C_1\left\{M_s\left(\sum_{(\ell'',\ell'_m)} \varphi_{j',k'} * f)(2^{-(j'\wedge k')-N}\ell'', 2^{-(j'\wedge 2k')-N}\ell'_m)|\chi_{I'}\chi_{J'}\right)^{\delta}(v',v_m)\right\}^{1/\delta},$$

Where $C_1 = C2^{-mN(1-\frac{1}{\delta})}2^{(m-1)(\frac{1}{\delta}-1)(j'\wedge k'-j\wedge k)} + 2^{\left(\frac{1}{\delta}-1\right)(j'\wedge 2k'-j\wedge 2k)+}$, here $(a-b)_+ = \max\{a-b,0\}$, and M_s is the strong maximal function.

Theorem (2.3.7) [139] Let $\psi^{(1)}, \phi^{(1)} \in S(\mathbb{R}^m)$ satisfy conditions (40)–(41), $\psi^{(2)}, \phi^{(2)} \in S(\mathbb{R}^m)$ satisfy conditions (42)–(43). Then for $0 < p,q < \infty, s = (s_1,s_2) \in \mathbb{R}^2$ and $f \in S'_0(\mathbb{R}^m)$,

$$\left\|\left(\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)s_1+(j\wedge 2k)s_2]q} \sum_{(\ell',\ell_m)} \left|\psi_{j,k} * f\left(2^{-(j\wedge k)}\ell', 2^{-(j\wedge 2k)}\ell_m\right)\right|^q \chi_I(x')\chi_J(x_m)\right)^{\frac{1}{q}}\right\|_{L^p(\mathbb{R}^m)}$$

$$\approx \left\|\left(\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)s_1+(j\wedge 2k)s_2]q} \cdot \sum_{(\ell',\ell_m)} \left|\varphi_{j,k} * f\left(2^{-(j\wedge k)}\ell', 2^{-(j\wedge 2k)}\ell_m\right)\right|^q \chi_I(x')\chi_J(x_m)\right)^{\frac{1}{q}}\right\|_{L^p(\mathbb{R}^m)}$$

and

$$\left(\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)s_1+(j\wedge 2k)s_2]q} \cdot \left\|\sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} \psi_{j,k} * f\left(2^{-(j\wedge k)}\ell', 2^{-(j\wedge 2k)}\ell_m\right)\chi_I(x')\chi_J(x_m)\right\|_{L^p(\mathbb{R}^m)}^q\right)^{\frac{1}{q}}$$

$$\approx \left(\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)s_1+(j\wedge 2k)s_2]q} \cdot \left\|\sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} \varphi_{j,k} * f(2^{-(j\wedge k)}\ell', 2^{-(j\wedge 2k)}\ell_m)\chi_I(x')\chi_J(x_m)\right\|_{L^p(\mathbb{R}^m)}^q\right)^{\frac{1}{q}},$$

where $\phi_{j,k}$ is constructed as $\psi_{j,k}$.

Proof. Let $f \in S_0(\mathbb{R}^m)$. Denote $x_I = 2^{-(j\wedge k)}\ell', x_J = 2^{-(j\wedge 2k)}\ell_m, x_{I'} = 2^{-(j'\wedge k')}\ell''$ and $x_{J'} = 2^{-(j'\wedge 2k')}\ell'_m$. By Theorem (2.3.3), Lemma (2.3.6) and the almost orthogonality estimates, for $\frac{m-1}{M+m-1} < \delta \le 1$ and any $v' \in I, v_m \in J$, we have

$$\left|\left(\psi_{j,k} * f\right)\left(x_I, x_J\right)\right|$$

$$= \left|\sum_{j',k'}\sum_{(\ell'',\ell'_m)} 2^{(m-1)(j'\wedge k')}2^{-(j'\wedge 2k')} \cdot \left(\psi_{j,k} * \varphi_{j',k'}\right)\left(x_I - x_{I'}, x_J - x_{J'}\right)\left(\varphi_{j',k'} * f\right)\left(x_{I'}, x_{J'}\right)\right|$$

$$\le C\sum_{j',k'} 2^{-|j-j'|L}2^{-|k-k'|L} \times \sum_{(\ell'',\ell'_m)} \frac{2^{(m-1)(j\wedge j'\wedge k\wedge k')}2^{j\wedge j'\wedge 2k\wedge 2k'}2^{-(m-1)(j'\wedge k')}2^{-(j'\wedge 2k')}}{\left(1 + 2^{j\wedge j'\wedge k\wedge k'}|x_I - x_{I'}|\right)^{(M+m-1)}}$$

$$\times \frac{\left|\left(\varphi_{j',k'} * f\right)\left(x_{I'}, x_{J'}\right)\right|}{\left(1 + 2^{j\wedge j'\wedge 2k\wedge 2k'}|x_J - x_{J'}|\right)^{(M+1)}}$$

$$\le C\sum_{j',k'} 2^{-|j-j'|L}2^{-|k-k'|L} \cdot C_1\left\{M_s\left(\sum_{(\ell'',\ell'_m)}\left|\left(\varphi_{j',k'} * f\right)\left(x_{I'}, x_{J'}\right)\right|\chi_{I'}\chi_{J'}\right)^{\delta}(v', v_m)\right\}^{\frac{1}{\delta}}. \quad (44)$$

Summing over j, k and (ℓ', ℓ_m), for any $v' \in I, v_m \in J$, we have

$$\left(\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)s_1 + (j\wedge 2k)s_2]q}\sum_{(\ell',\ell_m)} (\psi_{j,k} * f)(x_I, x_J)\chi_I(x')\chi_J(x_m)\right)^{\frac{1}{q}}$$

$$\le C\left(\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)s_1 + (j\wedge 2k)s_2]q}\left[\sum_{j',k'} 2^{-|j-j'|L}2^{-|k-k'|L}\right.\right.$$

$$\left.\left.\cdot C_1\left\{M_s\left(\sum_{(\ell'',\ell'_m)}\left|(\varphi_{j',k'} * f)(x_{I'}, x_{J'})\right|\chi_{I'}\chi_{J'}\right)^{\delta}(v', v_m)\right\}^{1/\delta}\right]^q\right)^{\frac{1}{q}}.$$

When $0 < q \le 1$, using the inequality $(\sum_l a_l)^q \le \sum_l a_l^q$,

$$\left(\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)s_1 + (j\wedge 2k)s_2]q}\sum_{(\ell',\ell_m)}\left|\psi_{j,k} * f\left(x_I, x_J\right)\right|^q\chi_I(x')\chi_J(x_m)\right)^{\frac{1}{q}}$$

$$\lesssim \left(\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)s_1 + (j\wedge 2k)s_2]q}\sum_{j',k'} 2^{-|j-j'|Lq}2^{-|k-k'|Lq} \cdot C_1^q\left\{M_s\left[\sum_{(\ell'',\ell'_m)}\left|\left(\varphi_{j',k'} * f\right)\left(x_{I'}, x_{J'}\right)\right|\chi_{I'}\chi_{J'}\right]^{\delta}(v', v_m)\right\}^{\frac{q}{\delta}}\right)^{\frac{1}{q}}$$

$$\lesssim \left(\sum_{j',k'\in\mathbb{Z}} 2^{-\left[\left(j'\wedge k'\right)s_1 + \left(j'\wedge 2k'\right)s_2\right]q} \cdot \left\{M_s\left[\sum_{(\ell'',\ell_m)}\left|(\varphi_{j',k'} * f)(x_{I'}, x_{J'})\right|\chi_{I'}\chi_{J'}\right]^{\delta}(v', v_m)\right\}^{q/\delta}\right)^{\frac{1}{q}},$$

where in the last inequality we use the facts that: $(j' \wedge k' - j \wedge k)_+ \le |j - j'| + |k - k'|, (j' \wedge 2k' - j \wedge 2k)_+ \le |j - j'| + 2|k - k'|$ and if choose L big enough such that $L > (m + 1)(\frac{1}{\delta} - 1) + |s_1| + |s_2|$ then

$$\sum_{j,k} 2^{-|j-j'|Lq}\, 2^{-|k-k'|Lq} 2^{-(j\wedge k-j'\wedge k')s_1 q} 2^{-(j\wedge 2k-j'\wedge 2k')s_2 q} C_1^q \le C.$$

When $q > 1$, by Hölder's inequality with exponents $q, q', \frac{1}{q}+\frac{1}{q'} = 1$,

$$\left(\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)s_1+(j\wedge 2k)s_2]q} \sum_{(\ell',\ell_m)} \left|\psi_{j,k} * f\left(x_I, x_J\right)\right|^q \chi_I(x')\chi_J(x_m)\right)^{\frac{1}{q}}$$

$$\lesssim \left(\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)s_1+(j\wedge 2k)s_2]q} \left[\sum_{j',k'} 2^{-(|j-j'|Lq')/2} 2^{-(|k-k'|Lq')/2}\, C_1^{q'}\right]^{q/q'}\right.$$

$$\cdot \sum_{j',k'} 2^{-(|j-j'|Lq')/2} 2^{-(|k-k'|Lq')/2}$$

$$\left.\cdot\left\{M_s\left(\sum_{(\ell'',\ell'_m)} \left|\left(\varphi_{j',k'} * f\right)\left(x_{I'}, x_{J'}\right)\right|\chi_{I'}\chi_{J'}\right)^{\delta} (v', v_m)\right\}^{q/\delta}\right)^{\frac{1}{q}}$$

$$\lesssim \left(\sum_{j',k'\in\mathbb{Z}} 2^{-[(j'\wedge k')s_1+(j'\wedge 2k')s_2]q} \left\{M_s\left[\sum_{(\ell'',\ell'_m)} \left|(\varphi_{j',k'} * f)(x_{I'}, x_{J'})\right|\chi_{I'}\chi_{J'}\right]^{\delta} (v', v_m)\right\}^{q/\delta}\right)^{\frac{1}{q}},$$

where in the last inequality we use similar estimates as in the case of $0 < q \le 1$, since

$$\sum_{j',k'} 2^{-(|j-j'|Lq')/2} 2^{-(|k-k'|Lq')/2}\, C_1^{q'} \le C$$

and

$$\sum_{j,k} 2^{-(|j-j'|Lq)/2}\, 2^{-(|k-k'|Lq)/2} 2^{-(j\wedge k-j'\wedge k')s_1 q} 2^{-(j\wedge 2k-j'\wedge 2k')s_2 q} \le C$$

by choosing L big enough. At last, applying Fefferman–Stein's vector-valued strong maximal inequality on $L^{p/\delta}(\ell^{q/\delta})$ provided $\delta < \min\{p, q, 1\}$ yields the first part of Theorem (2.3.7).

Next, we show the second part of Theorem (2.3.7).

When $1 \le p < \infty$, with (44), we have

$$\left\|\sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} \psi_{j,k} * f\left(2^{-(j\wedge k)} l', 2^{-(j\wedge 2k)} l_m\right)\chi_I(x')\chi_J(x_m)\right\|_{L^p(\mathbb{R}^m)}$$

$$\le C\left\|\sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} \sum_{j',k'} 2^{-|j-j'|L} 2^{-|k-k'|L}\cdot C_1 \left\{M_s\left(\sum_{(\ell'',\ell'_m)} \left|\left(\varphi_{j',k'} * f\right)\left(x_{I'}, x_{J'}\right)\right|\chi_{I'}\chi_{J'}\right)^{\delta} (v', v_m)\right\}^{\frac{1}{\delta}}\right\|_{L^p(\mathbb{R}^m)}$$

$$\le C\sum_{j',k'} 2^{-|j-j'|L} 2^{-|k-k'|L} 2^{(m-1)\left(\frac{1}{\delta}-1\right)(j'\wedge k'-j\wedge k)_+} 2^{\left(\frac{1}{\delta}-1\right)(j'\wedge 2k'-j\wedge 2k)_+}$$

$$\cdot\left\|\left\{M_s\left(\sum_{(\ell'',\ell'_m)} \left|\left(\varphi_{j',k'} * f\right)\left(x_{I'}, x_{J'}\right)\right|\chi_{I'}\chi_{J'}\right)^{\delta} (v', v_m)\right\}^{\frac{1}{\delta}}\right\|_{L^p(\mathbb{R}^m)}$$

$$\leq C\sum_{j',k'} 2^{-|j-j'|L}2^{-|k-k'|L}2^{(m-1)(\frac{1}{\delta}-1)(j'\wedge k'-j\wedge k)_+}2^{(\frac{1}{\delta}-1)(j'\wedge 2k'-j\wedge 2k)_+}$$

$$\cdot\left\|\sum_{(\ell'',\ell'_m)}\left|(\varphi_{j',k'}*f)(x_{I'},x_{J'})\right|\chi_{I'}\chi_{J'}\right\|_{L^p(\mathbb{R}^m)}$$

by the $L^{p/\delta}(\mathbb{R}^m)$ boundedness of strong maximal function M_s for a small δ. If $q>1$, applying Hölder's inequality and if $0<q\leq 1$, using the inequality $(\sum_l a_l)^q\leq\sum_l a_l^q$, and summing over j,k yields

$$\left(\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)s_1+(j\wedge 2k)s_2]q}\cdot\left\|\sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}}\psi_{j,k}*f\left(2^{-(j\wedge k)}\ell',2^{-(j\wedge 2k)}\ell_m\right)\chi_I(x')\chi_J(x_m)\right\|_{L^p(\mathbb{R}^m)}^q\right)^{\frac{1}{q}}$$

$$\leq C\left(\sum_{j',k'\in\mathbb{Z}} 2^{-[(j'\wedge k')s_1+(j'\wedge 2k')s_2]q}\cdot\left\|\sum_{(\ell'',\ell'_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}}\varphi_{j',k'}*f(2^{-(j'\wedge k')}\ell'',2^{-(j'\wedge 2k')}\ell'_m)\chi_I(x')\chi_J(x_m)\right\|_{L^p(\mathbb{R}^m)}^q\right)^{\frac{1}{q}}.$$

When $0<p<1$, using (44) again,

$$\int\left(\sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}}\psi_{j,k}*f(2^{-(j\wedge k)}\ell',2^{-(j\wedge 2k)}\ell_m)\chi_I(x')\chi_J(x_m)\right)^p dx'd\,x_m$$

$$\leq C\sum_{j',k'} 2^{-|j-j'|pL}2^{-|k-k'|pL}2^{p(m-1)(\frac{1}{\delta}-1)(j'\wedge k'-j\wedge k)_+}2^{p(\frac{1}{\delta}-1)(j'\wedge 2k'-j\wedge 2k)_+}$$

$$\cdot\int\left\{M_s\left(\sum_{(\ell'',\ell'_m)}\left|(\varphi_{j',k'}*f)(x_{I'},x_{J'})\right|\chi_{I'}\chi_{J'}\right)^{\delta}(v',v_m)\right\}^{p/\delta}dx'dx_m$$

$$\leq C\sum_{j',k'} 2^{-|j-j'|pL}2^{-|k-k'|pL}2^{p(m-1)(\frac{1}{\delta}-1)(j'\wedge k'-j\wedge k)_+}2^{p(\frac{1}{\delta}-1)(j'\wedge 2k'-j\wedge 2k)_+}$$

$$\cdot\int\left(\sum_{(\ell'',\ell'_m)}\left|(\varphi_{j',k'}*f)(x_{I'},x_{J'})\right|\chi_{I'}\chi_{J'}\right)^p(v',v_m)dx'dx_m.$$

By Hölder's inequality with exponents $\frac{1}{p}$ and $\frac{1}{1-p}$, we have

$$\left\{\int\left(\sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}}\psi_{j,k}*f\left(2^{-(j\wedge k)}\ell',2^{-(j\wedge 2k)}\ell_m\right)\chi_I(x')\chi_J(x_m)\right)^p dx'dx_m\right\}^{1/p}$$

$$\lesssim\left(\sum_{j',k'}\left[2^{-|j-j'|L}2^{-|k-k'|L}2^{(m-1)\left(\frac{1}{\delta}-1\right)(j'\wedge k'-j\wedge k)_+}2^{\left(\frac{1}{\delta}-1\right)(j'\wedge 2k'-j\wedge 2k)_+}\right]^{p/2(1-p)}\right)^{\frac{1-p}{p}}$$

$$\cdot\sum_{j',k'}\left[2^{-|j-j'|L}2^{-|k-k'|L}2^{(m-1)\left(\frac{1}{\delta}-1\right)(j'\wedge k'-j\wedge k)_+}2^{\left(\frac{1}{\delta}-1\right)(j'\wedge 2k'-j\wedge 2k)_+}\right]^{1/2}$$

$$\cdot\left\{\int\left(\sum_{(\ell'',\ell'_m)}\left|\left(\varphi_{j',k'}*f\right)\left(x_{I'},x_{J'}\right)\right|\chi_{I'}\chi_{J'}\right)^p(v',v_m)dx'dx_m\right\}^{1/p}$$

$$\lesssim \sum_{j',k'} \left[2^{-|j-j'|L}2^{-|k-k'|L}2^{(m-1)\left(\frac{1}{\delta}-1\right)(j'\wedge k'-j\wedge k)_+}2^{\left(\frac{1}{\delta}-1\right)(j'\wedge 2k'-j\wedge 2k)_+}\right]^{1/2}$$

$$\cdot\left\{\iint\left(\sum_{(\ell'',\ell'_m)}\left|\left(\varphi_{j',k'}*f\right)\left(x_{I'},x_{J'}\right)\right|\chi_{I'}\chi_{J'}\right)^p(v',v_m)dx'dx_m\right\}^{1/p}.$$

At last, if $q>1$ by using Hölder's inequality and if $0<q\le 1$ by using inequality $(\sum_l a_l)^q \le \sum_l a_l^q$ again, we have

$$\left(\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)s_1+(j\wedge 2k)s_2]q}\cdot\left\|\sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}}\psi_{j,k}*f\left(2^{-(j\wedge k)}\ell',2^{-(j\wedge 2k)}\ell_m\right)\chi_I(x')\chi_J(x_m)\right\|_{L^p(\mathbb{R}^m)}^q\right)^{\frac{1}{q}}$$

$$\lesssim\left(\sum_{j,k\in\mathbb{Z}}2^{-[(j\wedge k)s_1+(j\wedge 2k)s_2]q}\right.$$

$$\cdot\left\{\sum_{j',k'}\left[2^{-|j-j'|L}2^{-|k-k'|L}2^{(m-1)\left(\frac{1}{\delta}-1\right)(j'\wedge k'-j\wedge k)_+}2^{\left(\frac{1}{\delta}-1\right)(j'\wedge 2k'-j\wedge 2k)_+}\right]^{1/2}\right.$$

$$\left.\left.\cdot\left\{\left(\sum_{(\ell'',\ell'_m)}\left|\left(\varphi_{j',k'}*f\right)\left(x_{I'},x_{J'}\right)\right|\chi_{I'}\chi_{J'}\right)^p(v',v_m)dx'dx_m\right\}^{1/p}\right\}^q\right)^{\frac{1}{q}}$$

$$\lesssim\left(\sum_{j',k'\in\mathbb{Z}}2^{-[(j'\wedge k')s_1+(j'\wedge 2k')s_2]q}\right.$$

$$\left.\cdot\left\|\sum_{(\ell'',\ell'_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}}\varphi_{j',k'}*f(2^{-(j'\wedge k')}\ell'',2^{-(j'\wedge 2k')}\ell'_m)\chi_I(x')\chi_J(x_m)\right\|_{L^p(\mathbb{R}^m)}^q\right)^{\frac{1}{q}},$$

which completes the proof.

Before we prove the boundedness of singular integrals on $\dot{F}_p^{s,q}(\mathbb{R}^m)$ and $\dot{B}_p^{s,q}(\mathbb{R}^m)$ we introduce several properties of them.

Proposition (2.3.8) [139] $S_0(\mathbb{R}^m)$ is dense in $\dot{F}_p^{s,q}(\mathbb{R}^m)$ and in $\dot{B}_p^{s,q}(\mathbb{R}^m)$.

Proof. Let $f\in\dot{F}_p^{s,q}(\mathbb{R}^m)$. For any fixed $N>0$, denote

$$E=\{(j,k,\ell',\ell_m):|j|\le N,|k|\le N,|\ell'|\le N,|\ell_m|\le N\}$$

and

$$f_N(x',x_m):=\sum_{(j,k,\ell',\ell_m)\in E}2^{-(m-1)(j\wedge k)}2^{-(j\wedge k)}(\psi_{j,k}*f)(2^{-(j\wedge k)}\ell',2^{-(j\wedge 2k)}\ell_m)$$
$$\times\psi_{j,k}(x'-2^{-(j\wedge k)}\ell',x_m-2^{-(j\wedge 2k)}\ell_m),$$

where $\psi_{j,k}$ is the same as in Theorem (2.3.3).

Since $\psi_{j,k}\in S_0(\mathbb{R}^m)$, we obviously have $f_N\in S_0(\mathbb{R}^m)$. Repeating the same proof as that of Theorem (2.3.7), we have

$$\|f_N\|_{\dot{F}_p^{s,q}(\mathbb{R}^m)}\le C\|f\|_{\dot{F}_p^{s,q}(\mathbb{R}^m)};$$

moreover,

$\|f - f_N\|_{\dot{F}_p^{s,q}(\mathbb{R}^m)}$

$$\lesssim \left\| \left(\sum_{(j,k,\ell',\ell_m)\in E^c} 2^{-[(j\wedge k)s_1+(j\wedge 2k)s_2]q} \cdot |\psi_{j,k} * f(2^{-(j\wedge k)}\ell', 2^{-(j\wedge 2k)}\ell_m)|^q \chi_I(x')\chi_J(x_m) \right)^{\frac{1}{q}} \right\|_{L^p},$$

where the last term tends to 0 as N tends to infinity.

Suppose that $f \in \dot{B}_p^{s,q}(\mathbb{R}^m)$. Set

$$E_1 = \{(j,k): |j| \le N_1, |k| \le N_1\}, \qquad E_2 = \{(\ell', \ell_m): |\ell'| \le N_2, |\ell_m| \le N_2\},$$

and

$$f_N(x', x_m) := \sum_{(j,k)\in E_1} \sum_{(\ell',\ell_m)\in E_2} 2^{-(m-1)(j\wedge k)} 2^{-(j\wedge k)} (\psi_{j,k} * f)(2^{-(j\wedge k)}\ell', 2^{-(j\wedge 2k)}\ell_m)$$
$$\times \psi_{j,k}(x' - 2^{-(j\wedge k)}\ell', x_m - 2^{-(j\wedge 2k)}\ell_m).$$

Then $f_N \in S_0(\mathbb{R}^m)$ too. Proceeding as above, we have

$$\|f_N\|_{\dot{B}_p^{s,q}(\mathbb{R}^m)} \le C\|f\|_{\dot{B}_p^{s,q}(\mathbb{R}^m)}$$

and

$\|f - f_N\|_{\dot{B}_p^{s,q}(\mathbb{R}^m)}$

$$\le C \left(\sum_{(j,k)\in E_1^c} 2^{-[(j\wedge k)s_1+(j\wedge 2k)s_2]q} \cdot \left\| \sum_{(\ell',\ell_m)\in E_2^c} \psi_{j,k} * f(2^{-(j\wedge k)}\ell', 2^{-(j\wedge 2k)}\ell_m)\chi_I(x')\chi_J(x_m) \right\|_{L^p(\mathbb{R}^m)}^q \right)^{\frac{1}{q}},$$

where the last term tends to 0 as N_1, N_2 tends to infinity.

As a consequence of Proposition (2.3.8), $L^2(\mathbb{R}^m) \cap F_p^{s,q}(\mathbb{R}^m)$ and $L^2(\mathbb{R}^m) \cap \dot{B}_p^{s,q}(\mathbb{R}^m)$ are dense in $\dot{F}_p^{s,q}(\mathbb{R}^m)$ and $\dot{B}_p^{s,q}(\mathbb{R}^m)$ respectively.

In order to obtain almost orthogonality estimates with the kernel of T, we need a discrete Calderón-type identity on $L^2(\mathbb{R}^m) \cap \dot{F}_p^{s,q}(\mathbb{R}^m)$ and $L^2(\mathbb{R}^m) \cap \dot{B}_p^{s,q}(\mathbb{R}^m)$. To do this, let $\phi^{(1)} \in S(\mathbb{R}^m)$ with supp $\phi^{(1)} \subseteq B(0,1)$,

$$\sum_{j\in Z} \left|\widehat{\phi^{(1)}}(2^{-j}\xi)\right|^2 = 1 \quad \text{for all} \quad \xi \in \mathbb{R}^m \backslash \{0\}, \tag{45}$$

and

$$\int_{\mathbb{R}^m} \phi^{(1)}(x) x^\alpha dx = 0 \quad \text{for all} \quad |\alpha| \le 10M, \tag{46}$$

where M is a fixed large positive integer depending on p, q, s. We also let $\phi^{(2)} \in S(\mathbb{R}^m)$ with supp $\phi^{(2)} \subseteq B(0,1)$,

$$\sum_{k\in Z} \left|\widehat{\phi^{(2)}}(2^{-k}\xi', 2^{-2k}\xi_m)\right|^2 = 1 \quad \text{for all } (\xi', \xi_m) \in \mathbb{R}^{m-1} \times \mathbb{R}\backslash\{(0,0)\}, \tag{47}$$

and

$$\int_{\mathbb{R}^m} \phi^{(2)}(x) x^\beta dx = 0 \quad \text{for all} \quad |\beta| \le 10M. \tag{48}$$

Set $\phi_{j,k} = \phi_j^{(1)} * \phi_k^{(2)}$, where $\phi_j^{(1)}(x) = 2^{jm}\phi^{(1)}(2^j x)$ and $\phi_k^{(2)}(x', x_m) = 2^{k(m+1)}\phi^{(2)}(2^k x', 2^{2k} x_m)$.

For simplicity, let X be one of $\dot{F}_p^{s,q}(\mathbb{R}^m)$ and $\dot{B}_p^{s,q}(\mathbb{R}^m)$. The discrete Calderón-type identity is then given by the following.

Proposition (2.3.9) [139] Let $\phi^{(1)}$ and $\phi^{(2)}$ satisfy conditions (45)–(48). Then for any $f \in L^2(\mathbb{R}^m) \cap X$, there exists $h \in L^2(\mathbb{R}^m) \cap X$ such that for a sufficiently large $N \in \mathbb{N}$,

$$f(x',x_m)=\sum_{j,k\in\mathbb{Z}}\sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}}|I||J|\phi_{j,k}(x'-2^{-(j\wedge k)-N}\ell',x_m-2^{-(j\wedge 2k)-N}\ell_m)\times(\phi_{j,k}*h)(2^{-(j\wedge k)-N}\ell',2^{-(j\wedge 2k)-N}\ell_m),$$

where the series converges in L^2 and X, I are dyadic cubes in $\mathbb{R}^{m-1}$ and J are dyadic intervals in $\mathbb{R}$ with side-length $\ell(I)=2^{-(j\wedge k)-N}$ and $\ell(J)=2^{-(j\wedge 2k)-N}$, and the left lower corners of I and the left end points of J are $2^{-(j\wedge k)-N}\ell'$ and $2^{-(j\wedge 2k)-N}\ell_m$, respectively. Moreover,

$$\|f\|_{L^2(\mathbb{R}^m)}\approx\|h\|_{L^2(\mathbb{R}^m)}$$

and

$$\|f\|_X\approx\|h\|_X.$$

Proof. The proof is similar to that of Theorem 4.1 in [132], and we only provide a brief outline. For any $f\in L^2(\mathbb{R}^m)$, from the continuous Calderón identity,

$$f(x',x_m)=\sum_{j,k}\phi_{j,k}*\phi_{j,k}*f(x',x_m).$$

Applying Coifman's decomposition of the identity operator, we obtain

$$\begin{aligned}f(x',x_m)&=\sum_{j,k}\sum_{(\ell',\ell_m)}|I||J|\phi_{j,k}\left(x'-2^{-(j\wedge k)-N}\ell',x_m-2^{-(j\wedge 2k)-N}\ell_m\right)\\&\quad\times\left(\phi_{j,k}*f\right)\left(2^{-(j\wedge k)-N}\ell',2^{-(j\wedge 2k)-N}\ell_m\right)+\mathcal{R}_N(f)(x',x_m)\\&:=T_N(f)(x',x_m)+\mathcal{R}_N(f)(x',x_m),\end{aligned}$$

where

$$\begin{aligned}&\mathcal{R}_N(f)(x',x_m)\\&=\sum_{j,k}\sum_{(\ell',\ell_m)}\int_{I\times J}[\phi_{j,k}(x'-y',x_m-y_m)(\phi_{j,k}*f)(y',y_m)\\&\quad-\phi_{j,k}\left(x'-2^{-(j\wedge k)-N}\ell',x_m-2^{-(j\wedge 2k)-N}\ell_m\right)\left(\phi_{j,k}*f\right)\left(x'-2^{-(j\wedge k)-N}\ell',x_m\right.\\&\quad\left.-2^{-(j\wedge 2k)-N}\ell_m\right)\big]\,dy\\&=\sum_{j,k}\sum_{(\ell',\ell_m)}\int_{I\times J}\left[\varphi_{j,k}(x'-y',x_m-y_m)-\varphi_{j,k}\left(x'-2^{-(j\wedge k)-N}\ell',x_m-2^{-(j\wedge 2k)-N}\ell_m\right)\right.\\&\quad\left.\times\left(\varphi_{j,k}*f\right)(y',y_m)\right]dy'dy_m\\&\quad+\sum_{j,k}\sum_{(\ell',\ell_m)}\int_{I\times J}\varphi_{j,k}\left(x'-2^{-(j\wedge k)-N}\ell',x_m-2^{-(j\wedge 2k)-N}\ell_m\right)\\&\quad\times\left[\left(\varphi_{j,k}*f\right)(y',y_m)-\varphi_{j,k}\right.\\&\quad\left.*f\left(x'-2^{-(j\wedge k)-N}\ell',x_m-2^{-(j\wedge 2k)-N}\ell_m\right)\right]dy'dy_m\\&:=\mathcal{R}_N^1(f)(x',x_m)+\mathcal{R}_N^2(x',x_m),\end{aligned}$$

here I are dyadic cubes in $\mathbb{R}^{m-1}$ and J are dyadic intervals in $\mathbb{R}$ with side-length $\ell(I)=2^{-(j\wedge k)-N}$ and $\ell(J)=2^{-(j\wedge 2k)-N}$ and the left lower corners of I and the left end points of J are $2^{-(j\wedge k)-N}\ell'$ and $2^{-(j\wedge 2k)-N}\ell_m$, respectively.

Applying discrete Calderón's identity,

$$\begin{aligned}f(x',x_m)&=\sum_{j'',k''}\sum_{(\ell''',\ell''_m)}2^{-(m-1)(j''\wedge k'')}2^{-(j''\wedge 2k'')}(\psi_{j'',k''}*f)(2^{-(j''\wedge k'')}\ell''',2^{-(j''\wedge 2k'')}\ell''_m)\\&\quad\times\psi_{j'',k''}(x'-2^{-(j''\wedge k'')}\ell''',x_m-2^{-(j''\wedge 2k'')}\ell''_m).\end{aligned}$$

Then

$$\left|\psi_{j',k'} * \mathcal{R}_N^1(f)(x', x_m)\right|$$
$$\lesssim 2^{-N} \sum_{j'',k'',(\ell''',\ell_m'')} 2^{-|j'-j''|3M} 2^{-|k'-k''|3M} \frac{|I''||J''|2^{(j'\wedge j''\wedge k'\wedge k'')(m-1)}}{\left(1+2^{j'\wedge j''\wedge k'\wedge k''}\left|x'-2^{-(j''\wedge k'')}\ell'''\right|\right)^{(M+m-1)}}$$
$$\cdot \frac{2^{(j'\wedge j'')\wedge 2(k\wedge k')}}{\left(1+2^{(j'\wedge j'')\wedge 2(k\wedge k')}\left|x_m-2^{-(j''\wedge 2k'')}\ell_m''\right|\right)^{(M+1)}} \left|(\psi_{j'',k''} * f)(2^{-(j''\wedge k'')}\ell''', 2^{-(j''\wedge 2k'')}\ell_m'')\right|$$
$$\lesssim 2^{-N} \sum_{j'',k''} C_1 \left\{ M_s \left(\sum_{(\ell''',\ell_m'')\in\mathbb{Z}^{m-1}\times\mathbb{Z}} \left|(\varphi_{j'',k''} * f)(2^{-(j''\wedge k'')}\ell''', 2^{-(j''\wedge 2k'')}\ell_m'')\right| \chi_{I''}\chi_{J''} \right)^{\delta} (v', v_m) \right\}^{1/\delta},$$
which follows
$$\|\mathcal{R}_N^1(f)\|_{L^2} \le C\|\mathcal{R}_N^1(f)\|_{\dot{F}_2^{0,2}} \le C2^{-N}\|f\|_{L^2}$$
and
$$\|\mathcal{R}_N^1(f)\|_X \le C2^{-N}\|f\|_X,$$
by repeating the same proof as that of Theorem (2.3.7). With a similar proof, one also has
$$\|\mathcal{R}_N^2(f)\|_{L^2} \le C2^{-N}\|f\|_{L^2}, \qquad \|\mathcal{R}_N^2(f)\|_X \le C2^{-N}\|f\|_X.$$
By choosing sufficiently large $N, T_N^{-1} = \sum_{n=0}^{\infty}(\mathcal{R}_N)^n$ is bounded in both L^2 and X, which implies that
$$\|T_N^{-1}(f)\|_{L^2(\mathbb{R}^m)} \approx \|f\|_{L^2(\mathbb{R}^m)}$$
and
$$\|T_N^{-1}(f)\|_X \approx \|f\|_X.$$
Moreover, for any $f \in L^2(\mathbb{R}^m) \cap X$, set $h = T_N^{-1}(f)$. Then
$$\begin{aligned} f(x', x_m) &= T_N(T_N^{-1}(f)(x', x_m) \\ &= \sum_{j,k\in\mathbb{Z}} \sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} |I||J|\phi_{j,k}(x'-2^{-(j\wedge k)-N}\ell', x_m-2^{-(j\wedge 2k)-N}\ell_m) \\ &\quad \cdot (\varphi_{j,k} * h)(2^{-(j\wedge k)-N}\ell', 2^{-(j\wedge 2k)-N}\ell_m), \end{aligned}$$
where the series converges in L^2 and in X.

Repeating the same proof as that of Theorem (2.3.7), we have

Corollary (2.3.10) [139] Let $0 < p, q < \infty,\ s = (s_1, s_2) \in \mathbb{R}^2$.Suppose $\phi_{j,k}$ satisfies the same conditions as in Proposition (2.3.9) with a large M. For a large N as in Proposition (2.3.9), if $f \in L^2 \cap \dot{F}_p^{s,q}(\mathbb{R}^m)$,
$$\|f\|_{\dot{F}_p^{s,q}(\mathbb{R}^m)} \approx \left\| \left(\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)s_1+(j\wedge 2k)s_2]q} \cdot \sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} \left|\phi_{j,k} * h(2^{-(j\wedge k)-N}\ell', 2^{-(j\wedge 2k)-N}\ell_m)\right|^q \chi_I(x')\chi_J(x_m) \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^m)},$$
and if $f \in L^2 \cap \dot{B}_p^{s,q}(\mathbb{R}^m)$,
$$\|f\|_{\dot{B}_p^{s,q}(\mathbb{R}^m)} \approx \left(\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)s_1+(j\wedge 2k)s_2]q} \cdot \left\| \sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} \phi_{j,k} * h(2^{-(j\wedge k)-N}\ell', 2^{-(j\wedge 2k)-N}\ell_m)\chi_I(x')\chi_J(x_m) \right\|_{L^p(\mathbb{R}^m)}^q \right)^{\frac{1}{q}}.$$

Theorem (2.3.11) [139] Let T_1 and T_2 be Calderón–Zygmund singular integral operators with the isotropic and non-isotropic homogeneity, respectively. Then for all $0 < p, q < \infty, s = (s_1, s_2) \in \mathbb{R}^2$, the composition operator $T = T_1 \circ T_2$ is bounded on $\dot{F}_p^{s,q}(\mathbb{R}^m)$ and $\dot{B}_p^{s,q}(\mathbb{R}^m)$.

We first recall two important lemmas in [132], almost orthogonality estimates and discrete version of the strong maximal function, which play an important role in the theory developed in [108, 122].

Proof. We assume that K_i is the kernel of the convolution operator T_i, $i = 1,2$, and K is the kernel of the composition operator $T = T_1 \circ T_2$. Then $T(f) = K * f$ and $K = K_1 * K_2$. Suppose that $\psi^{(1)}$ and $\psi^{(2)}$ are functions satisfying conditions in (40)–(41) and (42)–(43), respectively. Then by definition

$$\|T(f)\|_{\dot{F}_p^{s,q}(\mathbb{R}^m)} = \left\| \left(\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)s_1+(j\wedge 2k)s_2]q} \sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} \left|\psi_{j,k} * K * f(2^{-(j\wedge k)-N}\ell', 2^{-(j\wedge 2k)-N}\ell_m)\right|^q \chi_I(x')\chi_J(x_m) \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^m)}.$$

Applying the discrete Calderón-type reproducing formula in Proposition (2.3.9), for $f \in L^2 \cap \dot{F}_p^{s,q}(\mathbb{R}^m)$,

$$\|T(f)\|_{\dot{F}_p^{s,q}(\mathbb{R}^m)} = \Bigg\| \Bigg(\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)s_1+(j\wedge 2k)s_2]q} \sum_{(\ell',\ell_m)} \Bigg| \sum_{j',k'} \sum_{(\ell'',\ell'_m)} 2^{-(m-1)[(j'\wedge k')-N]} 2^{-(j'\wedge 2k')-N}$$
$$\cdot (\phi_{j',k'} * h)(2^{-(j'\wedge k')-N}\ell'', 2^{-(j'\wedge 2k')-N}\ell'_m)(K * \psi_{j,k} * \phi_{j',k'})$$
$$\cdot \left(2^{-(j\wedge k)}\ell' - 2^{-(j'\wedge k')-N}\ell'', 2^{-(j\wedge 2k)}\ell_m - 2^{-(j'\wedge 2k')-N}\ell'_m\right) \Bigg|^q \chi_I\chi_J \Bigg)^{\frac{1}{q}} \Bigg\|_{L^p},$$

where $\phi^{(1)}$ and $\phi^{(2)}$ satisfy (45)–(48). It is easy to see that

$$K * \psi_{j,k} * \phi_{j',k'}(x', x_m) = [K_1 * \phi_{j'}^{(1)} * \psi_j^{(1)}] * [K_2 * \phi_{k'}^{(2)} * \psi_k^{(2)}](x', x_m).$$

For $K_1 * \varphi_{j'}^{(1)} * \psi_j^{(1)}$, since $\varphi^{(1)}$ has compact support and satisfies the cancellation condition in (46), one can obtain the following estimates from [132]:

$$|K_1 * \phi_{j'}^{(1)}(x', x_m)| \le C \frac{2^{j'm}}{(1 + 2^{j'}|x'|)^{M+m-1}(1 + 2^{j'}|x_m|)^{M+1}}. \tag{49}$$

Note that, $\forall|\alpha| \ge 0$, $\partial^\alpha\phi^{(1)} \in S(\mathbb{R}^m)$ satisfies the same conditions of $\phi^{(1)}$, that is, compact support and satisfies the cancellation condition in (46) with order not less than $10M$. Then

$$\left|K_1 * \left(\partial^\alpha\phi^{(1)}\right)_{j'}(x', x_m)\right| \le C \frac{2^{j'm}}{\left(1 + 2^{j'}|x'|\right)^{M+m-1}\left(1 + 2^{j'}|x_m|\right)^{M+1}} \tag{50}$$

and $\partial^\alpha(K_1 * \phi_{j'}^{(1)}(x', x_m)) = 2^{j'|\alpha|}K_1 * (\partial^\alpha\phi^{(1)})_{j'}(x', x_m))$. By classical methods of the almost orthogonality estimates, if $j' \le j$, using $\psi^{(1)} \in S_0(\mathbb{R}^m), \partial^\alpha(K_1 * \phi_{j'}^{(1)}(x', x_m)) = 2^{j'|\alpha|}K_1 * (\partial^\alpha\phi^{(1)})_{j'}(x', x_m))$ and (50), and if $j' > j$, using cancellation condition (46) to $K_1 * \phi_{j'}^{(1)}$ and (49), we obtain

$$\left|K_1 * \phi_{j'}^{(1)} * \psi_j^{(1)}(x', x_m)\right| \le C \frac{2^{-|j-j'|L}\, 2^{m(j\wedge j')}}{(1+2^{(j\wedge j')}|x'|)^{(M+m-1)}(1+2^{(j\wedge j')}|x_m|)^{(M+1)}}. \quad (51)$$

Similarly, for $K_2 * \phi_{k'}^{(2)} * \psi_k^{(2)}$,

$$|K_2 * \phi_{k'}^{(2)} * \psi_k^{(2)}(x', x_m)| \le C \frac{2^{-|k-k'|L}\, 2^{(k\wedge k')(M+1)}}{(1+2^{(k\wedge k')}|x'|)^{(M+m-1)}(1+2^{2(k\wedge k')}|x_m|)^{(M+1)}}. \quad (52)$$

Estimates (51) and (52) yield that

$$\begin{aligned}|K * \psi_{j,k} * \phi_{j',k'}(x', x_m)| &= |[K_1 * \phi_{j'}^{(1)} * \psi_j^{(1)}] * [K_2 * \phi_{k'}^{(2)} * \psi_k^{(2)}](x', x_m)](x', x_m)| \\ &\le C \frac{2^{-|j-j'|L} 2^{-|k-k'|L} 2^{(j\wedge j'\wedge k\wedge k')(m-1)} 2^{j\wedge j'\wedge 2k\wedge 2k'}}{(1+2^{(j\wedge j'\wedge k\wedge k')}|x'|)^{(M+m-1)}(1+2^{j\wedge j'\wedge 2k\wedge 2k'}|x_m|)^{(M+1)}}.\end{aligned}$$

Together with Lemma (2.3.6), applying the same proof as that of Theorem (2.3.7) yields that

$$\|T(f)\|_{\dot F_p^{s,q}(\mathbb{R}^m)} \lesssim \left\| \left(\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)s_1+(j\wedge 2k)s_2]q} \left[\sum_{j',k'} 2^{-|j-j'|L} 2^{-|k-k'|L} C_1 \right.\right.\right.$$

$$\left.\left.\left. \cdot \left\{ M_s \left(\sum_{(\ell'',\ell'_m)} \left|(\phi_{j',k'} * h)\left(2^{-(j'\wedge k')-N}\ell'', 2^{-(j'\wedge 2k')-N}\ell'_m\right)\right| \chi_{I'}\chi_{J'} \right)^{\delta} (v', v_m) \right\}^{\frac{1}{\delta}} \right]^q \right)^{\frac{1}{q}} \right\|_{L^p}$$

$$\lesssim \left\| \left(\sum_{j',k'\in\mathbb{Z}} 2^{-[(j'\wedge k')s_1+(j'\wedge 2k')s_2]q} \right.\right.$$

$$\left.\left. \cdot \sum_{(\ell'',\ell'_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} \left|\phi_{j',k'} * h\left(2^{-(j'\wedge k')-N}\ell'', 2^{-(j'\wedge 2k')-N}\ell'_m\right)\right|^q \chi_{I'}(x')\chi_{J'}(x_m) \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^m)} \approx \|f\|_{\dot F_p^{s,q}(\mathbb{R}^m)},$$

where in the last step we use Corollary (2.3.10). Since $L^2 \cap \dot F_p^{s,q}(\mathbb{R}^m)$ is dense in $\dot F_p^{s,q}(\mathbb{R}^m)$, we conclude the boundedness of T on $\dot F_p^{s,q}(\mathbb{R}^m)$.

Similarly, for $f \in L^2 \cap \dot B_p^{s,q}(\mathbb{R}^m)$,

$$\|Tf\|_{\dot B_p^{s,q}(\mathbb{R}^m)}$$

$$= \left(\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)s_1+(j\wedge 2k)s_2]q} \cdot \left\| \sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} \psi_{j,k} * K * f\left(2^{-(j\wedge k)}\ell', 2^{-(j\wedge 2k)}\ell_m\right)\chi_I(x')\chi_J(x_m) \right\|_{L^p(\mathbb{R}^m)}^q \right)^{\frac{1}{q}}$$

$$\le \left(\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)s_1+(j\wedge 2k)s_2]q} \left\| \sum_{(\ell',\ell_m)} \sum_{j',k'} \sum_{(\ell'',\ell'_m)} 2^{-(m-1)[(j'\wedge k')-N]} 2^{-(j'\wedge 2k')-N} \right.\right.$$

$$\cdot \left(\phi_{j',k'} * h\right)\left(2^{-(j'\wedge k')-N}\ell'', 2^{-(j'\wedge 2k')-N}\ell'_m\right)\left(K * \psi_{j,k} * \phi_{j',k'}\right)$$

$$\left.\left. \cdot \left(2^{-(j\wedge k)}\ell' - 2^{-(j'\wedge k')-N}\ell'', 2^{-(j\wedge 2k)}\ell_m - 2^{-(j'\wedge 2k')-N}\ell'_m\right)\chi_I(x')\chi_J(x_m) \right\|_{L^p(\mathbb{R}^m)}^q \right)^{\frac{1}{q}}$$

$$\lesssim \left(\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)s_1+(j\wedge 2k)s_2]q} \left\| \sum_{j',k'} 2^{-|j-j'|L} 2^{-|k-k'|L} C_1 \left\{ M_s \left(\sum_{(\ell'',\ell'_m)} |(\phi_{j',k'} \right.\right.\right.\right.$$

$$\left.\left.\left.\left. * h)(2^{-(j'\wedge k')-N}\ell'', 2^{-(j'\wedge 2k')-N}\ell'_m)|\, \chi_{I'}\, \chi_{J'} \right)^{\delta} (v', v_m) \right\}^{1/\delta} \right\|_{L^p(\mathbb{R}^m)}^{q} \right)^{\frac{1}{q}}$$

$$\lesssim \left(\sum_{j',k'\in\mathbb{Z}} 2^{-[(j'\wedge k')s_1+(j'\wedge 2k')s_2]q} \right.$$

$$\left. \cdot \left\| \sum_{(\ell'',\ell'_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} \phi_{j',k'} * h\left(2^{-(j'\wedge k')-N}\ell'', 2^{-(j'\wedge 2k')-N}\ell'_m\right)\chi_{I'}(x')\chi_{J'}(x_m) \right\|_{L^p(\mathbb{R}^m)}^{q} \right)^{\frac{1}{q}}$$

$$\lesssim \|f\|_{\dot{B}_p^{s,q}(\mathbb{R}^m)},$$

which can be extended to the entire $\dot{B}_p^{s,q}(\mathbb{R}^m)$ spaces.

Chapter 3
Wavelets Bases and Generalized Besov Spaces

We show that compactly supported wavelets of Daubechies type provide an unconditional Schauder basis in these spaces when the integrability parameters are finite. We also connect this with the possibility of embedding such spaces in some particular Lebesgue spaces.

Section(3.1) Generalized Besov Spaces

Wavelets have many applications into mathematics and other areas, such as engineering and physics. For instance, wavelet bases are used in the numerical resolution of some PDE's with the advantage of providing fast and efficient algorithms. Concerning functions spaces, wavelet bases give us the possibility of describing their elements in terms of basic and simple "building blocks." In general, an important point is that we can characterize the original (quasi-)norm by means of certain sums involving the wavelet coefficients. On the other hand, wavelet bases can be quite useful to study some intrinsic questions related to functions spaces. For example, they were successfully used to estimate entropy numbers of compact embeddings between weighted spaces (see [148]).

Motivated by Triebel on wavelet bases in function spaces, we deal with wavelet representations in Besov spaces with generalized smoothness. [151] proved, that compactly supported wavelets of Daubechies type form an unconditional Schauder basis in the "classic" Besov spaces B^s_{pq}. The aim is to extend this result to the "generalized" Besov spaces B^{ϕ}_{pq}, showing that the same wavelet system also provides an unconditional Schauder basis in these spaces. We would like to remark that function spaces of generalized smoothness have applications in other fields such as probability theory and stochastic processes (see [144]).

It is possible to get the result without repeating the approach suggested in [151]. Hence, instead of making use of all that powerful tools (atomic decompositions, local means, maximal functions, duality theory), we try mainly to take advantage of the classic case by means of suitable interpolation techniques. We would like to remark that interpolation tools were recently used by Caetano (see [141]) in order to get subatomic representations of Bessel potential spaces modelled on Lorentz spaces from the corresponding ones for the usual spaces H^s_p.

As long as wavelet bases is concerned, see [143, 151, 150, 155]. And [146], where wavelet decompositions of Besov spaces were studied in a multiresolution analysis framework.

We give the definition of Besov spaces of generalized smoothness and compare them to other well-known function spaces. We also discuss some interpolation properties which will play a key role later on, is devoted to the wavelet representation of Besov spaces. For convenience, we contextualize the problem recalling what is already done in the "classic" case, and then we formulate the main result as well as some of its consequences.

For $\mathbb{R}^n$ be then-dimensional Euclidean space and $\mathbb{Z}^n$ the usual lattice of all points with integers components ($n \in \mathbb{N}$). For $0 < p \le \infty$, $L_p(\mathbb{R}^n)$ denotes the well-known quasi-Banach space with respect to the Lebesgue measure, quasi-normed by

$$\|f\,|L_p(\mathbb{R}^n)\| = \left(\int_{\mathbb{R}^n} |f(x)|^p dx\right)^{1/p},$$

with the usual modification if $p = \infty$. Let $C(\mathbb{R}^n)$ be the space of all complex-valued uniformly continuous bounded functions in $\mathbb{R}^n$ and let, for $r \in \mathbb{N}$,

$$\{C^r(\mathbb{R}^n) = f \in C(\mathbb{R}^n) : D^\alpha f \in C(\mathbb{R}^n), |\alpha| \le r\}, \tag{1}$$

normed by

$$\|f|C^r(\mathbb{R}^n)\| = \sum_{|\alpha|\le r} \|D^\alpha f\, |L_\infty(\mathbb{R}^n)\|.$$

By $S(\mathbb{R}^n)$ we denote the Schwartz space of all rapidly decreasing and infinitely differentiable functions on $\mathbb{R}^n$, and by $S'(\mathbb{R}^n)$ its topological dual, that is, the space of all tempered distributions. If $\varphi \in S(\mathbb{R}^n)$, then $\mathcal{F}\varphi$(or $\hat{\varphi}$) stands for the Fourier transform of φ,

$$(\mathcal{F}\varphi)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi}\varphi(x)dx, \qquad \xi \in \mathbb{R}^n, \tag{2}$$

where as $\mathcal{F}^{-1}\varphi$(or $\varphi^\vee$) denotes its inverse Fourier transform, given by the right-hand side of (2) with i in place of $-i$. Both the Fourier transform and its inverse are extended to $S'(\mathbb{R}^n)$ in the usual way.

Let $\varphi_0 \in S(\mathbb{R}^n)$ be such that

$$\varphi_0(x) = 1 \quad \text{if } |x| \le 1 \text{ and } \quad \operatorname{supp}\varphi_0 \subset \{x \in \mathbb{R}^n: |x| \le 2\}. \tag{3}$$

Putting

$$\varphi_1(x) := \varphi_0(x/2) - \varphi_0(x) \quad \text{and} \quad \varphi_j(x) := \varphi_1(2^{-j+1}x), \qquad x \in \mathbb{R}^n, \qquad j \in \mathbb{N}, \tag{4}$$

then

$$\operatorname{supp}\varphi_j \subset \{x \in \mathbb{R}^n: 2^{j-1} \le |x| \le 2^{j+1}\}, \qquad j \in \mathbb{N},$$

and

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1, \qquad x \in \mathbb{R}^n.$$

Hence $\{\varphi_j\}_{j\in\mathbb{N}_0}$ forms a dyadic smooth resolution of unity. We recall that, for $s \in \mathbb{R}, 0 < p \le \infty, 0 < q \le \infty$, the usual Besov and Triebel–Lizorkin spaces are defined as the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\left\|f\, \middle| B^s_{pq}(\mathbb{R}^n)\right\| := \left(\sum_{j=0}^{\infty} 2^{jsq}\left\|(\varphi_j\hat{f})^\vee\middle|L_p(\mathbb{R}^n)\right\|^q\right)^{1/q} \tag{5}$$

and

$$\left\|f\, \middle| F^s_{pq}(\mathbb{R}^n)\right\| := \left\|\left(\sum_{j=0}^{\infty} 2^{jsq}\left|(\varphi_j\hat{f})^\vee(\cdot)\right|^q\right)^{1/q} \middle| L_p(\mathbb{R}^n)\right\| \tag{6}$$

(with the usual modification if $q = \infty$ and $p < \infty$ in the F-case) are finite, respectively. They are quasi-Banach spaces and are independent of the system $\{\varphi_j\}_{j\in\mathbb{N}_0}$ chosen according to (3) and (4) (with equivalent quasi-norms). We refer to [65] for a systematic theory on these spaces. It is well known that these scales contain some classic spaces as special cases. For instance,

$$F^s_{p,2}(\mathbb{R}^n) = H^s_p(\mathbb{R}^n), \qquad s \in \mathbb{R}, \qquad 1 < p < \infty,$$

are the fractional Sobolev spaces (they are the classic Sobolev spaces when $s \in \mathbb{N}$) and

$$F^0_{p,2}(\mathbb{R}^n) = h_p(\mathbb{R}^n), \qquad 0 < p < \infty, \tag{7}$$

are the local (or inhomogeneous) Hardy spaces introduced by Goldberg (see [147]).

We need to deal with some sequence spaces into a general context as follows. Let E be a quasi-normed space, I a countable set and $0 < q \le \infty$. We denote by $\ell_q(I, E)$ the "sequence" spaces of all E-valued families $a \equiv \{a_i\}_{i\in I}$ such that $\|a|\ell_q(I,E)\|$ is finite, where

$$\|a|\ell_q(I,E)\| := \left(\sum_{i\in I}\|a_i|E\|^q\right)^{1/q}, \qquad 0<q<\infty, \tag{8}$$

and

$$\|a|\ell_\infty(I,E)\| := \sup_{i\in I}\|a_i|E\| \tag{9}$$

define quasi-norms. If the set I is clear from the context, we shall omit it. Besides, we may omit E from the notation if $E = C$.

We obtain Besov spaces of generalized smoothness replacing the usual regularity index s in (5) by a certain function with given properties. We consider a sufficiently wide class of such functions, which allows us to cover many cases.

Definition (3.1.1) [157] We say that a function $\phi: (0,\infty) \to (0,\infty)$ belongs to the class B if it is continuous, $\phi(1) = 1$, and

$$\bar{\phi}(t) := \sup_{s>0}\frac{\phi(ts)}{\phi(s)}, \qquad t\in(0,\infty).$$

See [142, 149] for more details concerning this class. For a function $\phi \in B$, the Boyd upper and lower indices $\alpha_{\bar{\phi}}$ and $\beta_{\bar{\phi}}$ are then well-defined, respectively, by

$$\alpha_{\bar{\phi}} = \lim_{t\to+\infty}\frac{\log\bar{\phi}(t)}{\log t} \text{ and } \beta_{\bar{\phi}} = \lim_{t\to 0}\frac{\log\bar{\phi}(t)}{\log t} \text{ with } -\infty<\beta_{\bar{\phi}}\le\alpha_{\bar{\phi}}<+\infty.$$

If E is a quasi-normed space, $0<q\le\infty$ and $\phi\in B$, one can consider the spaces $\ell_q^\phi(E)$ of all sequences $\{a_j\}_{j\in\mathbb{N}_0}$ such that $\{\phi(2^j)a_j\}_{j\in\mathbb{N}_0}\in\ell_q(E)$, equipped with the quasi-norms$\|\cdot|\ell_q(E)\|$ according to (8) and (9) (with $I=\mathbb{N}_0$). When $\phi(t)=t^s, t\in(0,\infty), s\in\mathbb{R}$, we simply write $\ell_q^s(E)$ instead of $\ell_q^\phi(E)$ for short.

Let $\{\varphi_j\}_{j\in\mathbb{N}_0}\subset S(\mathbb{R}^n)$ be a system with the following properties:

$$\operatorname{supp}\varphi_0\subset\{\xi\in\mathbb{R}^n: |\xi|\le 2\}; \tag{10}$$

$$\operatorname{supp}\varphi_j\subset\left\{\xi\in\mathbb{R}^n: 2^{j-1}\le|\xi|\le 2^{j+1}\right\}, \qquad j\in\mathbb{N}; \tag{11}$$

$$\sup_{\xi\in\mathbb{R}^n}\left|D^\alpha\varphi_j(\xi)\right|\le c_\alpha 2^{-j|\alpha|}, \qquad j\in\mathbb{N}_0, \qquad \alpha\in\mathbb{N}_0^n; \tag{12}$$

$$\sum_{j=0}^{\infty}\varphi_j(\xi)=1, \qquad \xi\in\mathbb{R}^n. \tag{13}$$

Definition (3.1.2) [157] Let $\{\varphi_j\}_{j\in\mathbb{N}_0}$ be a dyadic resolution of unity with the properties (10)–(13) above. For $\phi\in B, 0<p\le\infty$, and $0<q\le\infty$, we define $B_{pq}^\phi(\mathbb{R}^n)$ as being the class of all $f\in S'(\mathbb{R}^n)$ such that $\{(\varphi_j\hat{f})^\vee\}_{j\in\mathbb{N}_0}\in\ell_q^\phi(L_p(\mathbb{R}^n))$ with

$$\|f\,|B_{pq}^\phi(\mathbb{R}^n)\| := \|\{(\varphi_j\hat{f})^\vee\}_{j\in\mathbb{N}_0}|\ell_q^\phi(L_p(\mathbb{R}^n))\|.$$

These spaces were studied by Merucci (see [149]) as a result of real interpolation with function parameter between Sobolev spaces and then by Cobos and Fernandez in [142]. Such as in the classic case according to (5), they are quasi-Banach spaces and are independent of the system $\{\varphi_j\}_{j\in\mathbb{N}_0}$ chosen, up to equivalent quasi-norms. We point out that the spaces $B_{pq}^s(\mathbb{R}^n)$ can be obtained as a particular case of the spaces $B_{pq}^\phi(\mathbb{R}^n)$ by taking $\phi(t)=t^s, t\in(0,\infty),\ s\in\mathbb{R}$.

In general, we are only dealing with functions spaces on $\mathbb{R}^n$. Hence, from now on, we shall omit the $\mathbb{R}^n$ in their notation. For convenience, we will refer to the spaces B_{pq}^s as classic Besov spaces.

Besov spaces with generalized smoothness have been considered and studied. See [144]. In [144] we can also find a general and unified approach for these spaces, as well as the counterpart for the Triebel–Lizorkin scale. As far as Besov spaces are concerned, it is possible to define generalized spaces B^{σ}_{pq} by replacing $\phi(2^j)$ by $\sigma_j, j \in \mathbb{N}_0$, in Definition (3.1.2), where σ is a certain admissible sequence of positive real numbers in the sense of [7]:

$$B^{\sigma}_{pq} = \{f \in S' : \|f \,|B^{\sigma}_{pq}\| := \|\{\sigma_j(\varphi_j \hat{f})^{\vee}\}_{j\in\mathbb{N}_0} |\, \ell_q(L_p)\| < \infty\}, \tag{14}$$

where $\sigma \equiv \{\sigma_j\}_{j\in\mathbb{N}_0}$satisfies the condition

$$d_0\sigma_j \leq \sigma_{j+1} \leq d_1\sigma_j, \qquad \forall j \in \mathbb{N}_0, \tag{15}$$

for some $d_0, d_1 > 0$. The definition given in [144] is even more general: it is introduced a fourth parameter $N \equiv \{N_j\}_{j\in\mathbb{N}_0}$related to generalized resolutions of unity, namely, allowing different sizes for the support of the involved functions. We restrict ourselves here to the standard decomposition, that is, with $N \equiv \{2^j\}_{j\in\mathbb{N}_0}$.

Some "other" generalized spaces of Besov type were introduced by Edmunds and Triebel. They are usually denoted by $B^{(s,\psi)}_{pq}$ and are defined as in (5) with $2^{jsq}\psi(2^{-j})^q$ in place of 2^{jsq}. The parameter ψ here represents a perturbation on the smoothness index s, and, of course, it fulfills certain conditions. See [47] for a systematic study on spaces $B^{(s,\psi)}_{pq}$.

As it was remarked in [144], the spaces $B^{(s,\psi)}_{pq}$ are covered by the general formulation (14), by taking $\sigma_j = 2^{js}\psi(2^{-j}), j \in \mathbb{N}_0$. Since we have $\bar{\phi}(1/2)^{-1}\phi(2^j) \leq \phi(2^{j+1}) \leq \bar{\phi}(2)\phi(2^j), j \in \mathbb{N}_0$, the spaces $B^{\phi}_{pq}, \phi \in B$, are also a particular case of the spaces defined in (14). However, we would like to point out that is enough to consider the spaces B^{ϕ}_{pq}. This fact may be justified by the following result, which was suggested to us by Caetano.

Proposition (3.1.3) [157] Let σ be an admissible sequence in the sense of (15) and $0 < p, q \leq \infty$. Then there exists a function $\varphi_\sigma \in B$ such that

$$B^{\phi\sigma}_{pq} = B^{\sigma}_{pq}.$$

Proof. Let σ be admissible. First, we remark that one can always assume $\sigma_0 = 1$ without loss the generality. In fact, the sequence σ' defined as $\sigma'_0 = 1$ and $\sigma'_j = \sigma_j$, $j \in \mathbb{N}$, is equivalent to σ, so $B^{\sigma}_{pq} = B^{\sigma'}_{pq}$.

We can construct a function $\phi_\sigma \in B$ as follows:

$$\phi_\sigma(t) = \begin{cases} \dfrac{\sigma_{j+1} - \sigma_j}{2^j}(t - 2^j) + \sigma_j, & t \in [2^j, 2^{j+1}), \qquad j \in \mathbb{N}_0 \\ \sigma_0, & t \in (0,1) \end{cases}$$

(cf. [12, Section 2.2]). Hence, $\phi_\sigma(2^j) = \sigma_j$ for all $j \in \mathbb{N}_0$ and we get the result.

Taking into account this proposition, from now on we will only deal with Besov spaces from Definition (3.1.2). Such as in the classic case (cf. [65, pp. 47-48]) one proves the following embeddings related to the spaces B^{ϕ}_{pq}.

Proposition (3.1.4) [157]

(i) Let $\phi \in B, 0 < p \leq \infty, 0 < q \leq \infty$. Then

$$S(\mathbb{R}^n) \hookrightarrow B^{\phi}_{pq}(\mathbb{R}^n) \hookrightarrow S'(\mathbb{R}^n).$$

(ii) Let $\phi \in B, 0 < p \leq \infty, 0 < q_0 \leq q_1 \leq \infty$. Then

$$B^{\phi}_{pq_0}(\mathbb{R}^n) \hookrightarrow B^{\phi}_{pq_1}(\mathbb{R}^n).$$

(iii) Let $\phi, \psi \in B, 0 < p \leq \infty, 0 < q_0, q_1 \leq \infty$. If $\left\{\frac{\phi(2^j)}{\psi(2^j)}\right\}_{j\in\mathbb{N}_0} \in \ell_{min\{q_1,1\}}$, then

$$B^{\psi}_{pq_0}(\mathbb{R}^n). \hookrightarrow B^{\phi}_{pq_1}(\mathbb{R}^n).$$

As usually, the symbol "$\hookrightarrow$" above indicates that the corresponding embedding is continuous. Property (iii) is important, in particular, to derive Lemma (3.1.6) bellow.

As it was referred before, the spaces B^{ϕ}_{pq}, $\phi \in B$, can be obtained from real interpolation between Sobolev spaces with an appropriate function parameter. Interpolation of this kind fits well into these generalized Besov spaces framework if the function parameter belongs to the same class B. See [142,149]. In [142] several interpolation results were obtained for the spaces B^{ϕ}_{pq} in the Banach case ($1 \leq p, q \leq \infty$). The approach followed there was based on interpolation properties of sequence spaces. Those properties were then transferred to the spaces B^{ϕ}_{pq}, by means of the so-called method of retraction and co-retraction(cf. [2, p. 150] and [153, p. 22], for example). Briefly, let E be a quasi-Banach space, $\phi \in B$ and $0 < q_0, q_1, q \leq \infty$. Taking into account [142, Theorem 5.1 and Remark 5.4], one can write

$$\left(\ell^{s_0}_{q_0}(E), \ell^{s_1}_{q_1}(E)\right)_{\gamma,q} = \ell^{\phi}_{q}(E) \tag{16}$$

if $s_0, s_1 \in \mathbb{R}$ with $s_1 < \beta_{\overline{\phi}} \leq \alpha_{\overline{\phi}} < s_0$ and

$$\gamma(t) = \frac{t^{\frac{s_0}{s_0-s_1}}}{\phi\left(t^{\frac{1}{s_0-s_1}}\right)}, \qquad t \in (0,\infty). \tag{17}$$

It is possible to show that B^{ϕ}_{pq} is a retract of $\ell^{\phi}_{q}(L^p)$ if $p \geq 1$ by constructing certain applications (retraction and co-retractions) based on the Fourier transform. But this does not work if $0 < p < 1$. However, as it was remarked in [142, Remark 5.4], some of the interpolation results obtained hold in the quasi-Banach case as well. We do not intend to go into too many details, but we give here a brief description how this question in the general case can be dealt with. Following [152, Theorem 2.2.10], one can prove the result bellow.

Proposition (3.1.5) [157] Let $f \in S', 0 < p < \infty$ and $\{\varphi_j\}_{j\in\mathbb{N}_0}$ satisfying the conditions (10)–(13). Then $\mathcal{F}^{-1}(\varphi_j \mathcal{F}f) \in L_p$ if and only if $\mathcal{F}^{-1}(\varphi_j \mathcal{F}f) \in h_p, j \in \mathbb{N}_0$. Moreover, there are constants $c_1, c_2 > 0$ independent of f and j such that

$$c_1 \left\|\mathcal{F}^{-1}(\varphi_j \mathcal{F}f) \mid L_p\right\| \leq \left\|\mathcal{F}^{-1}(\varphi_j \mathcal{F}f) \mid h_p\right\| \leq c_2 \left\|\mathcal{F}^{-1}(\varphi_j \mathcal{F}f) \mid L_p\right\|.$$

Note that h_p is the local Hardy space from (7). Using these estimates, one can replace L_p by h_p in Definition (3.1.2) when $p < \infty$ (note that $h_p = L_p$ if $1 < p < \infty$). With this change, one avoids the mentioned troubles caused by the Fourier transform. On the other hand, we can prove that B^{ϕ}_{pq} is a retract of $\ell^{\phi}_{q}(h_p)$: if $\{\varphi_j\}_{j\in\mathbb{N}_0} \subset S$ is a system with the properties (10)–(13), then

$$R\{f_j\}_{j\in\mathbb{N}_0} := \sum_{j=0}^{\infty} \mathcal{F}^{-1}(\tilde{\varphi}_j \mathcal{F}f_j), \qquad \text{with } \tilde{\varphi}_j = \sum_{r=-1}^{1} \varphi_{j+r},$$

is a retraction from $\ell^{\phi}_{q}(h_p)$ to B^{ϕ}_{pq} and $Sf := \{\mathcal{F}^{-1}(\varphi_j \mathcal{F}f)\}_{j\in\mathbb{N}_0}$ is the corresponding co-retraction. We remark that R is well-defined with the help of the following lemma, which can be proved following similar techniques as in [156, Theorem 3.6],

Lemma (3.1.6) [157] Let $\phi \in B, 0 < p \le 1, 0 < q \le \infty$. Assume that $\{g_j\}_{j\in\mathbb{N}_0} \subset S'$ fulfills the conditions

$$\operatorname{supp} \mathcal{F} g_0 \subset \{x: |x| \le 2\} \quad \text{and} \quad \operatorname{supp} \mathcal{F} g_j \subset \{x: 2^{j-1} \le |x| \le 2^{j+1}\}, \qquad j \in \mathbb{N}.$$

If $\|\phi(2^j) g_j \mid \ell_q(h_p)\| < \infty$, then $\sum_{j=0}^{\infty} g_j$ converges in S'.

Hence, it is possible to get the result bellow.

Proposition (3.1.7) [157] Let $\phi \in B, 0 < p \le \infty$, and $0 < q_0, q_1, q \le \infty$. Assume $s_0, s_1 \in \mathbb{R}$ satisfy $s_1 < \beta_{\overline{\phi}} \le \alpha_{\overline{\phi}} < s_0$ and γ as in (17). Then

$$\left(B_{pq_0}^{s_0}, B_{pq_1}^{s_1}\right)_{\gamma,q} = B_{pq}^{\phi}.$$

Proposition (3.1.7) shows, that spaces $B_{pq}^{(s,\psi)}$ mentioned can be obtained by interpolation of classic Besov spaces with a suitable function parameter. This fact was already observed in [140].

The aim is to obtain wavelet representations for the generalized Besov spaces under consideration. We will make use of the system considered in [151] and follow the same notation.

Let $L_j = L = 2^n - 1$ if $j \in \mathbb{N}$ and $L_0 = 1$. It is known that, for any $r \in \mathbb{N}$, there are real compactly supported functions

$$\psi_0 \in C^r, \qquad \psi^l \in C^r, \qquad l = 1, \dots, L, \tag{18}$$

with

$$\int_{\mathbb{R}^n} x^\alpha \psi^l(x) dx = 0, \qquad \alpha \in \mathbb{N}_0^n, \qquad |\alpha| \le r, \tag{19}$$

such that

$$\left\{2^{jn/2} \psi_{jm}^l : j \in \mathbb{N}_0, \qquad 1 \le l \le L_j, \qquad m \in \mathbb{Z}^n\right\} \tag{20}$$

with

$$\psi_{jm}^l(\cdot) = \begin{cases} \psi_0(\cdot - m), & j = 0, m \in \mathbb{Z}^n, l = 1, \\ \psi^l\left(2^{j-1} \cdot - m\right), & j \in N, m \in \mathbb{Z}^n, 1 \le l \le L, \end{cases} \tag{21}$$

is an orthonormal basis in L_2. As mentioned in [151], an example of such a system of functions is the (inhomogeneous) Daubechies wavelet basis (see [143, 150, 155]).

Wavelets with the properties above are sufficiently good to provide unconditional bases in many classical spaces. For instance, it was known that the mentioned Daubechies system forms an unconditional Schauder basis in the Sobolev spaces H_p^s if $1 < p < \infty$, $r > |s|$, and in the Besov spaces B_{pq}^s if $1 \le p, q < \infty$, $r > |s|$. These two examples show, that the smoothness required on the wavelets in (18) should be large enough, depending on the regularity of the functions that we pretend to represent. This fact can also be observed.

The main aim in [151] was to extend the results above about Sobolev spaces and some Besov spaces to the entire scales B_{pq}^s and F_{pq}^s. For convenience, we recall here the main result related to Besov spaces. Let $I = \{(l, j, m): j \in \mathbb{N}_0, 1 \le l \le L_j, m \in \mathbb{Z}^n\}$ and $I' = \{(l, j): j \in \mathbb{N}_0, 1 \le l \le L_j\}$.

Theorem (3.1.8) [157] Let $s \in \mathbb{R}, 0 < p \le \infty, 0 < q \le \infty$, and

$$r(s,p) := \max\left(s, \frac{2n}{p} + \frac{n}{2} - s\right). \tag{22}$$

(i) Assume $r \in \mathbb{N}$ with $r > r(s,p)$ and let $f \in S'$. Then $f \in B_{pq}^s$ if and only if it can be represented as

$$f = \sum_{(l,j,m)\in I} \lambda_{jm}^l \psi_{jm}^l \quad \textit{with } \lambda \in b_{pq}^s, \tag{23}$$

unconditional convergence in S' and in any space B^t_{pu} if $t < s$. Moreover, the representation (23) is unique:

$$\lambda = \lambda(f) \quad with \quad \lambda^l_{jm}(f) := 2^{jn}\langle f, \psi^l_{jm}\rangle. \tag{24}$$

Furthermore, $f \longmapsto \{2^{jn}\langle f, \psi^l_{jm}\rangle\}_{(l,j,m)\in I}$ defines an isomorphic map of B^s_{pq} onto b^s_{pq} and

$$\|f \mid B^s_{pq}\| \sim \|\lambda(f) \mid b^s_{pq}\| \tag{25}$$

(equivalent quasi-norms).

(ii) In addition, if $\max(p,q) < \infty$, then (23) with (24) converges unconditionally in B^s_{pq} and $\{\psi^l_{jm}\}_{(l,j,m)\in I}$is an unconditional Schauder basis in B^s_{pq}.

Here, b^s_{pq} is the space of all complex-valued sequences $\lambda \equiv \{\lambda^l_{jm}\}_{(l,j,m)\in I}$ such that

$$\|\lambda \mid b^s_{pq}\| := \left(\sum_{(l,j)\in I'} 2^{j(s-n/p)q}\left(\sum_{m\in\mathbb{Z}^n}\left|\lambda^l_{jm}\right|^p\right)^{q/p}\right)^{1/q} < \infty,$$

with standard modifications if $p = \infty$ and/or $q = \infty$.

The proof of Theorem (3.1.8) was based on atomic decompositions, characterizations by local means, and duality theory (see [68, 154]). An important point there was that the wavelets considered were simultaneously atoms and kernels of those local means. It was also commented in [151] the possibility of getting a similar result of other scales of function spaces. To do that, it would be enough to have the same tools available. However, as we mentioned before, we will not follow this approach. Instead, we will consider a scheme based on interpolation techniques in order to take advantage of the already known wavelet decompositions for the classic case.

Let $\phi \in B, 0 < p \le \infty, 0 < q \le \infty$. For our purposes we need to introduce the sequence spaces b^ϕ_{pq}, consisting of all complex-valued sequences $\lambda \equiv \{\lambda^l_{jm}\}_{(l,j,m)\in I}$ such that the quasi-norm

$$\left\|\lambda \mid b^\phi_{pq}\right\| := \left(\sum_{(l,j)\in I'} \left(\phi(2^j)2^{-jn/p}\right)^q \left(\sum_{m\in\mathbb{Z}^n}\left|\lambda^l_{jm}\right|^p\right)^{q/p}\right)^{1/q} \tag{26}$$

(with the usual modifications if $p = \infty$ and/or $q = \infty$) is finite. When $\phi(t) = t^s, t \in (0,\infty), s \in \mathbb{R}$, then b^ϕ_{pq} coincides with the space b^s_{pq} defined in [151]. We would like to remark that sequence spaces with this structure were introduced by Frazier and Jawerth in [115, 145] in connection with atomic decompositions of (classic) Besov and Triebel–Lizorkin spaces and they have been used afterwards by many.

The interpolation property bellow will be very useful in proving the main result.

Proposition (3.1.9) [157] Let $\phi \in B$ and $0 < p, q, q_0, q_1 \le \infty$. If s_0, s_1 are real numbers fulfilling $s_1 < \beta_{\overline{\phi}} \le \alpha_{\overline{\phi}} < s_0$, then we have

$$\left(b^{s_0}_{pq_0}, b^{s_1}_{pq_1}\right)_{\gamma,q} = b^\phi_{pq},$$

where γ is defined as in (17).

Proof. Firstly, we note that spaces $b^{s_0}_{pq_0}$ and $b^{s_1}_{pq_1}$ form an interpolation couple since they are both continuously embedded in $b^{s_1}_{p\infty}$, for example. We can interpret b^ϕ_{pq} as the sequence space $\ell^{\phi_1}_q(\ell_p(\mathbb{Z}^n))$ where $\phi_1(t) := \phi(t)t^{-n/p}, t \in (0,\infty)$. In fact, the index l does not bring any trouble. It is not hard to see that formula (16) remains valid for these spaces. On the other hand, the Boyd indices of ϕ_1 are given by

$$\beta_{\overline{\phi}_1} = \beta_{\overline{\phi}} - \frac{n}{p} \text{ and } \alpha_{\overline{\phi}_1} = \alpha_{\overline{\phi}} - \frac{n}{p}.$$

Taking $\sigma_i = s_i - n/p \; (i = 0,1)$, then we have

$$\sigma_1 < \beta_{\overline{\phi}_1} \leq \alpha_{\overline{\phi}_1} < \sigma_0 \text{ and } \gamma_1(t) := t^{\frac{\sigma_0}{\sigma_0 - \sigma_1}} \Big/ \phi_1\left(t^{\frac{1}{\sigma_0 - \sigma_1}}\right) = \gamma(t), \qquad t \in (0, \infty).$$

Hence, attending to formula (16), we have

$$\left(\ell_{q_0}^{\sigma_0}\left(\ell_p(\mathbb{Z}^n)\right), \ell_{q_1}^{\sigma_1}\left(\ell_p(\mathbb{Z}^n)\right)\right)_{\gamma,q} = \ell_q^{\phi_1}\left(\ell_p(\mathbb{Z}^n)\right),$$

that is, $\left(b_{pq_0}^{s_0}, b_{pq_1}^{s_1}\right)_{\gamma,q} = b_{pq}^{\phi}$.

Lemma (3.1.10) [157] Let $s \in \mathbb{R}, 0 < p < \infty$, and $0 < q \leq \infty$. If $\{\lambda_{jm}^l\}_{(l,j,m)\in I} \in b_{pq}^s$ and r is a natural number such that $r > \max(s, \sigma_p - s)$, then $\sum_{(l,j,m)\in I} \lambda_{jm}^l \psi_{jm}^l$ converges unconditionally in B_{pq}^s if $q < \infty$ and in any B_{pq}^t with $t < s$, if $q = \infty$.

Proof. First, we assume that $q < \infty$. From properties (18), (19), and (21), we see that, for each l, the functions $2^{-j(s-n/p)}\psi_{jm}^l$ are 1_r-atoms ($j = 0$) or $(s,p)_{r,r}$-atoms ($j \in \mathbb{N}$) according to [154, Definition 13.3], ignoring constants which are independent of, ℓ, j, and m. Using the Atomic Decomposition Theorem (cf. [154, pp. 75–76]), we arrive at the conclusion that there exists $c > 0$ such that the estimate

$$\left\| \sum_{(l,j,m)\in K} \lambda_{jm}^l \psi_{jm}^l \mid B_{pq}^s \right\|^q \leq c \sum_{l,j} 2^{j(s-n/p)q} \left(\sum_m \left|\lambda_{jm}^l\right|^p \right)^{q/p} \tag{27}$$

holds for all finite subsets K of I (the sums on the right-hand side run over all indices (l,j) and m such that $(l,j,m) \in K$). From this estimate and from the summability of the two families of positive real numbers involved in (26), we conclude that the partial sums on the left-hand side of (27) constitute a generalized Cauchy sequence in the complete space B_{pq}^s, thus converge in this space.

Now, let $t < s$ and $q = \infty$. We reduce this case to the previous one by using the atomic decomposition result as before (with t in place of s) and remarking that $b_{p\infty}^s \hookrightarrow b_{pu}^t$ with $0 < u < \infty$.

We formulate our main result related to wavelet representation and some of its consequences.

Theorem (3.1.11) [157] Let $\phi \in B$, $0 < p < \infty$, and $0 < q \leq \infty$. Consider the system $\{\psi_{jm}^l\}_{(l,j,m)\in I}$ as previously. Then there exists $r(\phi, p)$ such that, for any $r \in \mathbb{N}$ with $r > r(\phi, p)$, the following holds:

Given $f \in S'$, then $f \in B_{pq}^{\phi}$ if and only if it can be represented as

$$f = \sum_{(l,j,m)\in K} \lambda_{jm}^l \psi_{jm}^l \quad \text{with} \quad \lambda \in b_{pq}^{\phi} \tag{28}$$

(unconditional convergence in S'). Moreover, the "wavelet coefficients" λ_{jm}^l are uniquely determined by

$$\lambda_{jm}^l = \lambda_{jm}^l(f) := 2^{jn}\langle f, \psi_{jm}^l \rangle, \qquad (l,j,m) \in I. \tag{29}$$

Further,

$$\left\| f \mid B_{pq}^{\phi} \right\| \sim \left\| \lambda(f) \mid b_{pq}^{\phi} \right\| \tag{30}$$

(equivalent quasi-norms), where $\lambda(f) \equiv \{\lambda_{jm}^l(f)\}_{(l,j,m)\in I}$.

Proof.

Step 1. Assume that $f \in S'$ can be represented as

$$f = \sum_{(l,j,m)\in I} \lambda_{jm}^l \psi_{jm}^l \text{ (unconditional convergence in } S')$$

for some $\lambda \in b_{pq}^{\phi}$. Let $s_0, s_1 \in \mathbb{R}$. Attending to Lemma (3.1.10), we conclude that the operator

$$T: b_{p,1}^{s_0} + b_{p,1}^{s_1} \longrightarrow B_{p,1}^{s_0} + B_{p,1}^{s_1},$$

given by

$$T\mu = \sum_{(l,j,m)\in I} \mu_{jm}^l \psi_{jm}^l \quad \text{(unconditional convergence in } S'),$$

is well-defined and linear if $r > \max\left(r(s_0,p), r(s_1,p)\right)$, for example, where $r(s_i,p)(i = 0,1)$ is given by (22). Moreover, by Theorem (3.1.8) one concludes that the restriction of T to each $b_{p,1}^{s_i}$ is a bounded linear operator into $B_{p,1}^{s_i}$. Choosing s_0, s_1 above such that $s_1 < \beta_{\overline{\Phi}} \le \alpha_{\overline{\Phi}} < s_0$ and attending to the interpolation property and to Propositions (3.1.7) and (3.1.9), we arrive at the conclusion that the restriction of T to b_{pq}^{ϕ} is also a bounded linear operator into B_{pq}^{ϕ}. Thus, $f \in B_{pq}^{\phi}$ and

$$\left\| f \mid B_{pq}^{\phi} \right\| = \left\| T\lambda \mid B_{pq}^{\phi} \right\| \le c \left\| \lambda \mid b_{pq}^{\phi} \right\|$$

for some $c > 0$ independent of λ and f.

Step 2. Now, let $f \in B_{pq}^{\phi}$. Assume that s_0, s_1, and r fulfill the same conditions as in Step 1. Consider the operator

$$S: B_{p,1}^{s_0} + B_{p,1}^{s_1} \longrightarrow b_{p,1}^{s_0} + b_{p,1}^{s_1}$$

defined by

$$Sg = \lambda(g) := \left\{2^{jn}\left(\langle g_0, \psi_{jm}^l\rangle + \langle g_1, \psi_{jm}^l\rangle\right)\right\}_{(l,j,m)\in I}, \tag{31}$$

where $g = g_0 + g_1$ with $g_i \in B_{p,1}^{s_i}$, $i = 0,1$. Theorem (3.1.8) shows that S is well-defined, it is linear and its restriction to each $B_{p,1}^{s_i}$ is a bounded linear operator into $b_{p,1}^{s_i}$. Taking into account the interpolation property as previously, one concludes that the restriction of S to B_{pq}^{ϕ} is a bounded linear operator into b_{pq}^{ϕ} as well. Therefore,

$$\left\| Sf \mid b_{pq}^{\phi} \right\| = \left\| \lambda(f) \mid b_{pq}^{\phi} \right\| \le c \left\| f \mid B_{pq}^{\phi} \right\|, \tag{32}$$

where $c > 0$ does not dependent on f. So, $\lambda(f) \in b_{pq}^{\phi}$ and hence

$$g := \sum_{(l,j,m)\in I} \lambda_{jm}^l(f) \psi_{jm}^l \tag{33}$$

(unconditional convergence in S') belongs to the space B_{pq}^{ϕ} by Step 1. But Theorem (3.1.8) once again allows us to conclude that TS is the identity operator, so $g = f$. But we have (by Step 1)

$$\left\| f \mid B_{pq}^{\phi} \right\| \le c \left\| \lambda(f) \mid b_{pq}^{\phi} \right\|, \tag{34}$$

$c > 0$ independent of f. Therefore, equivalence (30) follows from estimates (32) and (34). It remains to show that representation (28) is unique. We do this next. Suppose that $f \in B_{pq}^{\phi}$ admits the representation

$$f = \sum_{(l,j,m)\in I} \lambda_{jm}^l \psi_{jm}^l \quad \text{with } \lambda \in b_{pq}^{\phi} \text{ (unconditional convergence in } S').$$

Since $B^{\phi}_{pq} \hookrightarrow B^{s_0}_{p,1} + B^{s_1}_{p,1} \hookrightarrow B^{s_1}_{p,1}$ and $b^{\phi}_{pq} \hookrightarrow b^{s_0}_{p,1} + b^{s_1}_{p,1} \hookrightarrow b^{s_1}_{p,1}$ (note that $s_0 > s_1$), then $f \in B^{s_1}_{p,1}$ has the representation

$$f = \sum_{(l,j,m)\in I} \lambda^l_{jm}\psi^l_{jm} \quad \text{with } \lambda \in b^{s_1}_{pq} \text{ (unconditional convergence in } S'\text{)},$$

which is unique by Theorem (3.1.8). The proof of the theorem is completed.

Corollary (3.1.12) [157] Let $\phi, p,$ and q be as in Theorem (3.1.11). If $r \in \mathbb{N}$ is large enough and $q < \infty$, then $\{\psi^l_{jm}\}_{(l,j,m)\in I}$ forms an unconditional Schauder basis in B^{ϕ}_{pq}.

Proof. Attending to Theorem (3.1.11), all we need to do is to check that the series in (28) converges unconditionally in B^{ϕ}_{pq}(if $p, q < \infty$). We proceed as in the first part of the proof of Lemma (3.1.10): observe that $\phi(2^j)^{-1}2^{jn/p}\psi^l_{jm}$ are 1_r-N-atoms ($l = 1, j = 0$) or $(\sigma, p)_{r,r}$-N-atoms ($j \in \mathbb{N}$) according to [144, Definition 4.4.1], with $\sigma = \{\phi(2^j)\}_{j\in\mathbb{N}_0}$ and $N = \{2^j\}_{j\in\mathbb{N}_0}$. Hence, it is possible to use the Atomic Decomposition Theorem from [144, Section 4.4.2], in order to get the counterpart of estimate (27), that is,

$$\left\| \sum_{(l,j,m)\in I} \lambda^l_{jm}\psi^l_{jm} \,\middle|\, B^{\phi}_{pq} \right\| \le c \sum_{l,j} \left(\phi(2^j)2^{-jn/p}\right)^q \left(\sum_m \left|\lambda^l_{jm}\right|^p\right)^{q/p}$$

with $c > 0$ independent of K. To do that we have to assume that $r > r(\phi, p)$ satisfies also the conditions mentioned in that theorem restricted to our particular case. We conclude now as in Lemma (3.1.10).

Corollary (3.1.13) [157] Let $\phi, p, q,$ and r as in Theorem (3.1.11). Then

$$\mathcal{J}: f \longmapsto \left\{2^{jn}\langle f, \psi^l_{jm}\rangle\right\}_{(l,j,m)\in I}$$

establishes a topological isomorphism from $B^{\phi}_{p,q}$ onto $b^{\phi}_{p,q}$.

Proof. This result follows at once from the properties of the operators T and S studied in the proof of Theorem (3.1.11).

Corollary (3.1.14) [314] Let $\phi^2 \in B$ and $0 \le \epsilon \le \infty$. If s_{r-1}, s_r are real numbers fulfilling $s_r < \beta_{\overline{\phi^2}} \le \alpha_{\overline{\phi^2}} < s_{r-1}$, then we have

$$\left(b^{s_{r-1}}_{(1+\epsilon)(1+3\epsilon)}, b^{s_r}_{(1+\epsilon)(1+4\epsilon)}\right)_{\gamma,(1+2\epsilon)} = b^{\phi^2}_{(1+\epsilon)(1+2\epsilon)},$$

where γ is defined as in (17).

Proof. Firstly, we note that spaces $b^{s_{r-1}}_{(1+\epsilon)(1+3\epsilon)}$ and $b^{s_r}_{(1+\epsilon)(1+4\epsilon)}$ form an interpolation couple since they are both continuously embedded in $b^{s_r}_{(1+\epsilon)\infty}$, for example. We can interpret $b^{\phi^2}_{(1+\epsilon)(1+2\epsilon)}$ as the sequence space $\ell^{\phi_1^2}_{1+2\epsilon}(\ell_{1+\epsilon}(\mathbb{Z}^n))$ where $\phi_1^2(1+\epsilon) := \phi^2(1+\epsilon)(1+\epsilon)^{-n/1+\epsilon}, 0 \le \epsilon < \infty$. In fact, the index l does not bring any trouble. It is not hard to see that formula (16) remains valid for these spaces. On the other hand, the Boyd indices of ϕ_1^2 are given by

$$\beta_{\overline{\phi_1^2}} = \beta_{\overline{\phi^2}} - \frac{n}{1+\epsilon} \text{ and } \alpha_{\overline{\phi_1^2}} = \alpha_{\overline{\phi^2}} - \frac{n}{1+\epsilon}.$$

Taking $\sigma_i = s_i - n/1 + \epsilon$ $(i = r-1, r)$, then we have

$$\sigma_r < \beta_{\overline{\phi_1^2}} \le \alpha_{\overline{\phi_1^2}} < \sigma_{r-1} \text{ and } \gamma_1(1+\epsilon) := \frac{(1+\epsilon)^{\frac{\sigma_{r-1}}{\sigma_{r-1}-\sigma_r}}}{\phi_1^2\left((1+\epsilon)^{\frac{1}{\sigma_{r-1}-\sigma_r}}\right)} = \gamma(1+\epsilon), \qquad 0 \le \epsilon < \infty.$$

Hence, attending to formula (16), we have

$$\left(\ell^{\sigma_{r-1}}_{1+3\epsilon}(\ell_{1+\epsilon}(\mathbb{Z}^n)), \ell^{\sigma_r}_{1+4\epsilon}(\ell_{1+\epsilon}(\mathbb{Z}^n))\right)_{\gamma,1+2\epsilon} = \ell^{\phi_1^2}_{1+2\epsilon}(\ell_{1+\epsilon}(\mathbb{Z}^n)),$$

that is, $\left(b^{s_{r-1}}_{(1+\epsilon)(1+3\epsilon)}, b^{s_r}_{(1+\epsilon)(1+4\epsilon)}\right)_{\gamma,1+2\epsilon} = b^{\phi^2}_{(1+\epsilon)(1+2\epsilon)}$.

Section (3.2) Triebel–Lizorkin Spaces of Regular Distributions

We describe completely, in terms of their parameters, when the generalized Besov and Triebel–Lizorkin spaces $B^{\sigma,N}_{p,q}(\mathbb{R}^n)$ and $F^{\sigma,N}_{p,q}(\mathbb{R}^n)$ contain only regular distributions. In other terms, we aim to characterize the relations

$$B^{\sigma,N}_{p,q}(\mathbb{R}^n) \subset L^{loc}_1(\mathbb{R}^n)$$

and

$$F^{\sigma,N}_{p,q}(\mathbb{R}^n) \subset L^{loc}_1(\mathbb{R}^n)$$

in terms of the behaviour of σ, N, p and q.

Besides the intrinsic interest of such a question within the theory of those spaces, such a characterization might also be useful when calculating with distributions belonging to them, as the possibility of representing distributions by functions naturally leads to simplifications.

A final answer to such a question of classical spaces $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ was given in [55, Thm. 3.3.2]:

Theorem (3.2.1) [169]

(i) Let $s \in \mathbb{R}, 0 < p < \infty$ and $0 < q \leq \infty$. Then the following two assertions are equivalent:

$$F^s_{p,q} \subset L^{loc}_1,$$

$$\begin{aligned} &either\ 0 < p < 1, && s \geq n\left(\frac{1}{p} - 1\right), && 0 < q \leq \infty, \\ &or\ 1 \leq p < \infty, && s > 0, && 0 < q \leq \infty, \\ &or\ 1 \leq p < \infty, && s = 0, && 0 < q \leq 2. \end{aligned}$$

(ii) Let $s \in \mathbb{R}, 0 < p \leq \infty$ and $0 < q \leq \infty$. Then the following two assertions are equivalent:

$$B^s_{p,q} \subset L^{loc}_1,$$

$$\begin{aligned} &either\ \ 0 < p \leq \infty, && s > n\left(\frac{1}{p} - 1\right)_+, && 0 < q \leq \infty, \\ &or\ \ 0 < p \leq 1, && s = n\left(\frac{1}{p} - 1\right), && 0 < q \leq 1, \\ &or\ \ 1 < p \leq \infty, && s = 0, && 0 < q \leq min\{p, 2\}. \end{aligned}$$

The spaces of generalized smoothness $B^{\sigma,N}_{p,q}(\mathbb{R}^n)$ and $F^{\sigma,N}_{p,q}(\mathbb{R}^n)$ in which we intend to study the same problem are natural generalizations of the classical Besov and Triebel–Lizorkin spaces in the direction of generalizing the smoothness and the partition in frequency. Now, instead of $(2^{sj})_j$, for some $s \in \mathbb{R}$, the smoothness will be controlled by a general so-called admissible sequence $\sigma := (\sigma_j)_j$, whereas the splitting in frequency will also be controlled by an admissible sequence $N := (N_j)_j$ more general than the classical $(2^j)_j$.

Originally they were introduced by Goldman and Kalyabin in the middle of the seventies of the last century on the basis of expansions in series of entire analytic functions and coverings. Another approach used differences and general weight functions. In all these cases the function spaces were subspaces of $L_p(\mathbb{R}^n), 1 < p < \infty$, by definition, therefore the question under which conditions they can contain or not contain singular distributions was pointless. See [32] or [42]. Here we just briefly recall the approaches just mentioned:

Let $(N_j)_j$ be strongly increasing, $(\alpha_j)_j$ be of bounded growth and $(\alpha_j^{-1})_j \in l_{q'}$. Then a space of generalized smoothness $\mathbf{B}^{\alpha,N}_{p,q}(\mathbb{R}^n)$ (resp. $\mathbf{F}^{\alpha,N}_{p,q}(\mathbb{R}^n)$) was defined as the collection of all $f \in L_p(\mathbb{R}^n)$ such that $f = \sum_{j=1}^{\infty} f_j$ in $L_p(\mathbb{R}^n)$ with $\operatorname{supp}(\mathcal{F}f_j) \subset \{\xi \in \mathbb{R}^n : |\xi| \le N_j\}$ and $\left\|(\alpha_j f_j)_j \mid l_q(L_p)\right\| < \infty$ (resp. $(\alpha_j f_j)_j \mid L_p(l_q) < \infty$).

And, respectively, a space of generalized smoothness denoted by $B^{\lambda(\cdot)}_{p,q}(\mathbb{R}^n)$ was defined by

$$B^{\lambda(\cdot)}_{p,q}(\mathbb{R}^n) = \left\{ f \in L_p(\mathbb{R}^n) : \left(\int_0^1 \left(\frac{\omega_p^M(f,t)}{\lambda(t)} \right)^q \frac{d\lambda(t)}{\lambda(t)} \right)^{1/q} < \infty \right\}$$

where $\lambda : (0,1) \to \mathbb{R}^+$ is a non-decreasing, continuous function with $\lim_{t\downarrow 0} \lambda(t) = 0, M \in \mathbb{N}$ and $\omega_p^M(f,t) = \sup_{|h|<t} \left\| \Delta_h^M u(\cdot) \right| L_p \right\|$.

For the connection with the spaces we describe in Definition (3.2.8), cf. [144].

Later on spaces of generalized smoothness appeared naturally by real interpolation with a function parameter (cf. [149] and [142]). Putting it briefly, for example for given $1 < p, q < \infty, k \in \mathbb{N}$ and ρ a suitable function parameter one has the interpolation result

$$\left(L_p(\mathbb{R}^n), W_p^k(\mathbb{R}^n) \right)_{\rho,q} = B^{\sigma}_{p,q}(\mathbb{R}^n), \qquad \text{where } \sigma_j = \left(\rho\left(2^{-jk}\right) \right)^{-1}.$$

Function spaces of generalized smoothness have been used to describe compact and limiting embeddings with the help of the finer tuning given by the smoothness parameter, for example with additional 'logarithmic' smoothness for function spaces either on bounded domains in [165] or in [167] to describe general embeddings of Pohozhaev–Trudinger type.

Moreover they have shown up also in connection with generalized d-sets and h-sets (special fractals) and function spaces defined on them as trace spaces. For example, in the case of the so-called (d,ψ)-sets Γ one has

$$B^{(s,\psi^a)}_{p,q}(\Gamma) := tr_\Gamma B^{\left(s+\frac{n-d}{p}, \psi^{\frac{1}{p}+a}\right)}_{p,q}(\mathbb{R}^n),$$

see [68, Chap. 22], [47] and [158].

In probability theory they have been used as generalized Bessel potential spaces defined by pseudo-differential operators with negative definite functions as symbols [161]. If these negative definite functions are suitably constructed with the help of Bernstein functions, then those generalized Bessel potential spaces belong to the scale of the spaces $F^{\sigma,N}_{p,q}(\mathbb{R}^n)$ we consider.

For a more complete historical survey up to the end of 2000, see [144].

As can be noticed by comparing our main assertions in Theorems (3.2.28) and (3.2.29) below with the classical counterpart recalled in Theorem (3.2.1) above, it is not at all clear why the latter should generalize in that way. As a matter of fact, it was somewhat of a surprise to us that the characterization could be done in such a neat way, specially in the cases where a comparison between the numbers p, q and 2 seemed to be in order. We stress that we get a characterization, and not mere sufficient conditions. The bulk of the work has, indeed, to do with the proof that the guessed conditions are necessary. The tools used there rely heavily on the useful Proposition (3.2.26), which we denote by "a reverse Hölder's inequality result", and on the consideration of suitable sets of extremal functions. These are, for most of the cases, inspired by the possibility of representing the elements of the functions spaces under study by means of infinite linear combinations of atoms. For the tricky cases given by the last lines in Theorems (3.2.28) and (3.2.29) we had to resort to lacunary Fourier series (and standardization) for that effect (by the way, Theorem (3.2.25) might also have independent interest).

Still with respect to techniques used, specially the consideration of extremal functions built as atomic representations in the functions spaces under study, one first difficulty faced is the possibility of having such representations with the required moment conditions. This is solved by Proposition (3.2.27), also used in the somewhat related question of characterizing the growth envelopes of the same spaces. In fact, a construction of this kind appears already in [68, Cor. 13.4] and was used in the study of the growth envelopes of the classical spaces $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ (cf. [68, 163]). For the study of growth envelopes of the generalized spaces considered, see [12, 14]. These contain also sufficient conditions for those spaces to be constituted by regular distributions alone, as this is a requirement for the consideration of the concept of growth envelope itself. So, from this point of view, our results here broaden the class of sufficient conditions to the point that they also become necessary. The fact that in both situations extremal functions are pursued is related to the fact that in both cases one is looking for the validity of embeddings – in $L_1^{loc}(\mathbb{R}^n)$ in our case and in local spaces of integrable functions in the case of growth envelopes.

As a by-product of the main results, we also extend to our framework the classical result [55, Cor. 3.3.1], which states that the Besov and Triebel–Lizorkin spaces of integrability parameter $p \neq \infty$ which are completely formed by regular distributions are exactly those which continuously embed in the Lebesgue spaces of power $\max\{1,p\}$– cf. Corollary (3.2.31).

Since all the Besov and Triebel–Lizorkin spaces under consideration are spaces on $\mathbb{R}^n$, we shall omit the $\mathbb{R}^n$ from the notation.

Given any $r \in (0,\infty]$, we denote by r' the number, possibly ∞, defined through the expression $\frac{1}{r'} := (1-\frac{1}{r})_+$; in case when $1 \le r \le \infty, r'$ is the same as the conjugate exponent usually defined through $\frac{1}{r} + \frac{1}{r'} = 1$.

The symbol $\hookrightarrow$ is used for continuous embedding from one space into other.

Unimportant positive constants might be denoted generically by the same letter, usually c, with additional indices to distinguish them in case they appear in the same or close expression.

Before introducing the spaces we want to consider, we define and make some comments about the type of sequences which will be used as parameters.

Definition (3.2.2) [169] A sequence $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$, with $\sigma_j > 0$, is called an admissible sequence if there are two constants $0 < d_0 = d_0(\sigma) \le d_1 = d_1(\sigma) < \infty$ such that

$$d_0\sigma_j \le \sigma_{j+1} \le d_1\sigma_j \quad \text{for any } j \in \mathbb{N}_0. \tag{35}$$

Definition (3.2.3) [169] Two admissible sequences $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ and $\tau = (\tau_j)_{j\in\mathbb{N}_0}$ are called equivalent if there exist constants C_1 and C_2 such that

$$0 < C_1 \le \frac{\sigma_j}{\tau_j} \le C_2 < \infty \quad \text{for any} \quad j \in \mathbb{N}_0.$$

To illustrate the flexibility of (35) we refer the reader to some examples discussed in [144] or [158, Chap. 1].

The following definition, of Boyd indices of a given admissible sequence, is taken from [159]:

Definition (3.2.4) [169] Let $\overline{\sigma}_j := \sup_{k\geq 0} \frac{\sigma_{j+k}}{\sigma_k}$ and $\underline{\sigma}_j := \inf_{k\geq 0} \frac{\sigma_{j+k}}{\sigma_k}, j \in \mathbb{N}_0$. Then

$$\alpha_\sigma := \inf_{j\in\mathbb{N}} \frac{\log_2 \overline{\sigma}_j}{j} = \lim_{j\to\infty} \frac{\log_2 \overline{\sigma}_j}{j} \quad \text{and} \quad \beta_\sigma := \sup_{j\in\mathbb{N}} \frac{\log_2 \underline{\sigma}_j}{j} = \lim_{j\to\infty} \frac{\log_2 \underline{\sigma}_j}{j}$$

are the (upper and respectively lower) Boyd indices of the sequence σ.

Remark (3.2.5) [169] (i) It is easy to see that the Boyd indices of an admissible sequence σ remain unchanged when replacing σ by an equivalent sequence in the sense of Definition (3.2.2).

(ii) Given an admissible sequence σ with Boyd indices α_σ and β_σ then it is possible to find for any $\varepsilon > 0$ a sequence τ which is equivalent to σ with $d_0(\tau) = 2^{\beta_\sigma - \varepsilon}$ and $d_1(\tau) = 2^{\alpha_\sigma - \varepsilon}$, i.e.

$$2^{\beta_\sigma - \varepsilon}\tau_j \leq \tau_{j+1} \leq 2^{\alpha_\sigma - \varepsilon}\tau_j \qquad \text{for any} \qquad j \in \mathbb{N}_0. \tag{36}$$

Assumption (3.2.6) [169] From now on we will denote $N = (N_j)_{j\in\mathbb{N}_0}$ a sequence of real positive numbers such that there exist two numbers $1 < \lambda_0 \leq \lambda_1$ with

$$\lambda_0 N_j \leq N_{j+1} \leq \lambda_1 N_j \qquad \text{for any} \qquad j \in \mathbb{N}_0. \tag{37}$$

N is a so-called strongly increasing sequence – compare Definition 2.2.1 and Remark 4.1.2 in [144]. We would like to point out that the condition $\lambda_0 > 1$ played a key role in [144, Assump. 4.1.1] in order to get atomic decompositions in function spaces of generalized smoothness.

Moreover we choose a natural number κ_0 in such a way that $2 \leq \lambda_0^{\kappa_0}$ and consequently $2N_j \leq N_k$ for any $j, k \in \mathbb{N}_0$ such that $j + \kappa_0 \leq k$ holds. We will fix such a κ_0 in the following.

Definition (3.2.7) [169] For a fixed sequence $N = (N_j)_{j\in\mathbb{N}_0}$ as in Assumption (3.2.6), let ϕ^N be the collection of all function systems $\varphi^N = (\varphi_j^N)_{j\in\mathbb{N}_0}$ such that:

(i)

$$\varphi_j^N \in C_0^\infty(\mathbb{R}^n) \quad \text{and} \quad \varphi_j^N(\xi) \geq 0 \quad \text{if } \xi \in (\mathbb{R}^n) \quad \text{for any} \quad j \in \mathbb{N}_0;$$

(ii)

$$\operatorname{supp} \varphi_j^N \subset \{\xi \in \mathbb{R}^n : |\xi| \leq N_{j+\kappa_0}\}, \qquad j = 0,1,\ldots,\kappa_0 - 1,$$
$$\operatorname{supp} \varphi_j^N \subset \{\xi \in \mathbb{R}^n : N_{j-\kappa_0} \leq |\xi| \leq N_{j+\kappa_0}\} \qquad \text{if} \quad j \geq \kappa_0;$$

(iii) for any $\gamma \in \mathbb{N}_0^n$ there exists a constant $c_\gamma > 0$ such that for any $j \in \mathbb{N}_0$

$$|D^\gamma \varphi_j^N(\xi)| \leq c_\gamma (1 + |\xi|^2)^{-|\gamma|/2} \qquad \text{for any} \quad \xi \in \mathbb{R}^n;$$

(iv) there exists a constant $c_\varphi > 0$ such that

$$0 < \sum_{j=0}^{\infty} \varphi_j^N(\xi) = c_\varphi < \infty \qquad \text{for any} \qquad \xi \in \mathbb{R}^n.$$

In what follows S stands for the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$ equipped with the usual topology, S' denotes its topological dual, the space of all tempered distributions on $\mathbb{R}^n$, and $\mathcal{F}$ and $\mathcal{F}^{-1}$ stand respectively for the Fourier transformation and its inverse.

Let $(\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence $(N_j)_{j\in\mathbb{N}_0}$ be an admissible sequence satisfying Assumption (3.2.6) and let $\varphi^N \in \phi^N$.

Definition (3.2.8) [169]

(i) Let $0 < p \leq \infty$ and $0 < q \leq \infty$. The Besov space $B_{p,q}^{\sigma,N}$ of generalized smoothness is defined as

$$\left\{f \in S' : \|f \mid B^{\sigma,N}_{p,q}\| := \left(\sum_{j=0}^{\infty} \sigma_j^q \left\|\mathcal{F}^{-1}\left(\varphi_j^N \mathcal{F} f\right) \mid L_p(\mathbb{R}^n)\right\|^q\right)^{1/q} < \infty\right\}.$$

(ii) Let $0 < p < \infty$ and $0 < q \le \infty$. The Triebel–Lizorkin space $F^{\sigma,N}_{p,q}$ of generalized smoothness is defined as

$$\left\{f \in S' : \|f \mid F^{\sigma,N}_{p,q}\| := \left\|\left(\sum_{j=0}^{\infty} \sigma_j^q \left|\mathcal{F}^{-1}\left(\varphi_j^N \mathcal{F} f\right)(\cdot)\right|^q\right)^{1/q} \mid L_p(\mathbb{R}^n)\right\| < \infty\right\}.$$

In both cases one should use the usual modification when $q = \infty$.

Both $B^{\sigma,N}_{p,q}$ and $F^{\sigma,N}_{p,q}$ are Banach spaces which are independent of the choice of the system $(\varphi^N)_{j\in\mathbb{N}_0}$, in the sense of equivalent quasi-norms. As in the classical case, the embeddings $S \hookrightarrow B^{\sigma,N}_{p,q} \hookrightarrow S'$ and $S \hookrightarrow F^{\sigma,N}_{p,q} \hookrightarrow S'$ hold true for all admissible values of the parameters and sequences. If $p, q < \infty$ then S is dense in $B^{\sigma,N}_{p,q}$ and in $F^{\sigma,N}_{p,q}$. Moreover, it is clear that $B^{\sigma,N}_{p,q} = F^{\sigma,N}_{p,q}$.

Note also that if $N_j = 2^j$ and $\sigma = \sigma^s := (2^{js})_{j\in\mathbb{N}_0}$ with s real, then the above spaces coincide with the usual function spaces $B^s_{p,q}$ and $F^s_{p,q}$ on $\mathbb{R}^n$, respectively. We shall use the simpler notation $B^s_{p,q}$ and $F^s_{p,q}$ in the more classical situation just mentioned. Even for general admissible σ, when $N_j = 2^j$ we shall write simply $F^\sigma_{p,q}$ and $B^\sigma_{p,q}$ instead of $F^{\sigma,N}_{p,q}$ and $B^{\sigma,N}_{p,q}$, respectively.

We have the following relation between B and F spaces, the proof of which can be done similarly as in the classical case (cf. [65, Prop. 2.3.2/2. (iii), p. 47]):

Proposition (3.2.9) [169] Let $0 < p < \infty, 0 < q \le \infty$. Let N and σ be admissible sequences with N satisfying also Assumption (3.2.6). Then

$$B^{\sigma,N}_{p,min\{p,q\}} \hookrightarrow F^{\sigma,N}_{p,q} \hookrightarrow B^{\sigma,N}_{p,max\{p,q\}}.$$

Of intrinsic interest are also embedding results involving such spaces. Here we present two which will, moreover, be of great service to us later on. In the case of Besov spaces, this is taken from [12, Thm. 3.7]:

Proposition (3.2.10) [169] Let $N = (N_j)_{j\in\mathbb{N}_0}$ be an admissible sequence as in Assumption (3.2.6) and let $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ and $\tau = (\tau_j)_{j\in\mathbb{N}_0}$ be two further admissible sequences. Let $0 < p_1 \le p_2 \le \infty, 0 < q_1, q_2 \le \infty$ and $\frac{1}{q^*} := (\frac{1}{q_2} - \frac{1}{q_1})_+$. If

$$\left(\sigma_j^{-1} \tau_j N_j^{n(\frac{1}{p_1} - \frac{1}{p_2})}\right)_{j\in\mathbb{N}_0} \in \ell_{q^*} \tag{38}$$

then $B^{\sigma,N}_{p_1,q_1} \hookrightarrow B^{\tau,N}_{p_2,q_2}$.

The following partial counterpart for the F-spaces (which will be enough for our purposes) can be proved similarly (cf. also [166, Prop. 1.1.13. (iv), (vi)]):

Proposition (3.2.11) [169] Let N be an admissible sequence as in Assumption (3.2.6) and let σ and τ be two further admissible sequences. Let $0 < p < \infty, 0 < q_1, q_2 \le \infty$ and $\frac{1}{q^*} := (\frac{1}{q_2} - \frac{1}{q_1})_+$. If

$$\left(\sigma_j^{-1} \tau_j\right)_{j\in\mathbb{N}_0} \in \ell_{q^*} \tag{39}$$

then $F^{\sigma,N}_{p,q_1} \hookrightarrow F^{\tau,N}_{p,q_2}$.

We state now sufficient conditions, already known to us, in order that $B^{\sigma,N}_{p,q}$ and $F^{\sigma,N}_{p,q}$ contain only regular distributions.

Proposition (3.2.12) [169] (See [12, Cor. 3.18].) Let $0 < p \le \infty, 0 < q \le \infty$. Let N and σ be admissible sequences with N satisfying also Assumption (3.2.6). If

$$\left(\sigma_j^{-1} N_j^{n(\frac{1}{p}-1)_+}\right)_{j\in\mathbb{N}_0} \in \ell_{q^*}$$

then $B^{\sigma,N}_{p,q} \hookrightarrow L_{\max\{1,p\}}$.

Proposition (3.2.13) [169] (See [14, Sec. 4, Prop. 3].) Let $0 < p < \infty, 0 < q \le \infty$. Let N and σ be admissible sequences with N satisfying also Assumption (3.2.6). If

$$\begin{cases} \left(\sigma_j^{-1} N_j^{\delta}\right)_{j\in\mathbb{N}_0} \in \ell_{p'}, & \text{for some } \delta > 0, \ \ if \ \ 1 \le p < \infty, \\ (\sigma_j^{-1} N_j^{n(\frac{1}{p}-1)})_{j\in\mathbb{N}_0} \in \ell_\infty, & if \quad 0 < p < 1, \end{cases}$$

then $F^{\sigma,N}_{p,q} \subset L_1^{loc}$.

In order to deal with the main question formulated, we need to introduce some technical tools and derive some results which will be required later on.

In our setting, standardization is the ability to identify our generalized spaces with spaces where N_j has the classical form 2^j.

Let N and σ be admissible sequences, N satisfying also the Assumption (3.2.6) as before, and let κ_0 be the fixed natural number with $\lambda_0^{\kappa_0} \ge 2$. Define

$$\beta_j := \sigma_{k(j)}, \qquad \text{with } k(j) := \min\left\{k \in \mathbb{N}_0 : 2^{j-1} \le N_{k+\kappa_0}\right\}, \qquad j \in \mathbb{N}_0. \tag{40}$$

Then we have that

$$\mu_0\beta_j \le \beta_{j+1} \le \mu_1\beta_j, \qquad j \in \mathbb{N}_0,$$

with $\mu_0 = \min\left\{1, d_0^{\kappa_0}\right\}$, $\mu_1 = \max\left\{1, d_1^{\kappa_0}\right\}$.

Under these conditions we proved in [14, Thm. 1] the following standardization:

Theorem (3.2.14) [169] Let N and σ be admissible sequences, N satisfying also the Assumption (3.2.6). Let, further, $0 < p, q \le \infty$ (with $p \ne \infty$ in the F-case). Then

$$F^{\sigma,N}_{p,q} = F^{\beta}_{p,q} \quad \text{and} \quad B^{\sigma,N}_{p,q} = B^{\beta}_{p,q},$$

where $\beta := (\beta_j)_{j\in\mathbb{N}_0}$ is determined by (40).

As a consequence of this we obtain in case $\sigma_j = \sigma_j^0 = 1$ for all $j \in \mathbb{N}_0$:

Corollary (3.2.15) [169] Let $(\sigma_j)_{j\in\mathbb{N}_0}$ and $(N_j)_{j\in\mathbb{N}_0}$ be as before and $0 < p, q \le \infty$ (with $p \ne \infty$ in the F-case). Then

$$B^{(1),N}_{p,q} = B^{\sigma^0,N}_{p,q} = B^0_{p,q} \tag{41}$$

and

$$F^{(1),N}_{p,q} = F^{\sigma^0,N}_{p,q} = F^0_{p,q}. \tag{42}$$

This extends [144, Thm. 3.1.7] also to the F-spaces and to the case $0 < p \le 1$. The corollary will be useful to prove the sufficiency of the conditions in Theorems (3.2.28) and (3.2.29).

One of the most significant ingredients in the proof of the following theorem, which is Lemma 1 in [14] and will also be useful later on, was again the above standardization theorem.

Theorem (3.2.16) [169] Let $0 < p_1 < p < p_2 \le \infty, 0 < q \le \infty$. Let $N := (N_j)_{j\in\mathbb{N}_0}$ and $\sigma := (\sigma_j)_{j\in\mathbb{N}_0}$ be admissible sequences with N satisfying also Assumption (3.2.6). Let σ' and σ'' be the admissible sequences defined respectively by

$$\sigma_j' = N_j^{n(\frac{1}{p_1}-\frac{1}{p})}\sigma_j, \qquad \sigma_j'' = N_j^{n(\frac{1}{p_2}-\frac{1}{p})}\sigma_j, \qquad j \in \mathbb{N}_0.$$

Then

$$B_{p_1,u}^{\sigma',N} \hookrightarrow F_{p,q}^{\sigma,N} \hookrightarrow B_{p_2,v}^{\sigma'',N}$$

if, and only if, $0 < u \leq p \leq v \leq \infty$.

As we shall see, the main results will be established in terms of the behaviour of the sequences σ and N. Sometimes it is useful to deal with the case of general N after having dealt with the more classical situation when $N = (2^j)_{j\in\mathbb{N}_0}$, through standardization. The problem afterwards then might be that the criteria obtained are expressed in terms of $(\sigma_{k(j)}^{-1})_{j\in\mathbb{N}_0}$, for the $k(j)$ defined in (40), instead of the original sequence $(\sigma_j^{-1})_{j\in\mathbb{N}_0}$. This difficulty can, however, be circumvented by the following observations.

Remark (3.2.17) [169] From the definition of $k(j)$ the following two properties easily follow:

(i) For κ_0 the fixed natural number such that $\lambda_0^{\kappa_0} \leq 2$, it holds

$$k(j+1) \leq k(j) + \kappa_0, \qquad j \in \mathbb{N}_0.$$

(ii) There is $c_0 \in \mathbb{N}$ such that

$$k(j + c_0) > k(j), \qquad j \in \mathbb{N}_0;$$

for example, $c_0 = \kappa_1 + j_0$, where $\kappa_1 \in \mathbb{N}$ satisfies $\lambda_1 \leq 2^{\kappa_1}$ and $j_0 \in \mathbb{N}_0$ is chosen such that $2^{j_0-1} > \lambda_1^{\kappa_0} N_0$.

Proposition (3.2.18) [169] Let σ be an admissible sequence and $0 < r \leq \infty$. Let $k(j)$ be defined as in (40). Then

$$\sigma^{-1} \in r \qquad \text{if, and only if,} \qquad (\sigma_{k(j)}^{-1})_{j\in\mathbb{N}_0} \in \ell_r.$$

Proof. We deal only with the main case when $0 < r < \infty$. The case $r = \infty$ can be dealt with usual modifications.

Consider the numbers κ_0 and c_0 as in Remark (3.2.17).

On one hand,

$$\sum_{j=0}^{\infty} \sigma_{k(j)}^{-r} = \sum_{l=0}^{\infty}\sum_{m=0}^{c_0-1} \sigma_{k(lc_0+m)}^{-r} = \sum_{m=0}^{c_0-1}\sum_{l=0}^{\infty} \sigma_{k(lc_0+m)}^{-r} \leq \sum_{m=0}^{c_0-1}\sum_{j=0}^{\infty} \sigma_j^{-r} = c_0 \sum_{j=0}^{\infty} \sigma_j^{-r}, \tag{43}$$

where the inequality is justified by the fact that, for each fixed $m = 0,\dots,c_0 - 1$, $(\sigma_k(lc_0 + m))_{l\in\mathbb{N}_0}$ is a subsequence of σ, as follows from Remark (3.2.17) (ii).

On the other hand,

$$\kappa_0 \sum_{j=0}^{\infty} \sigma_{k(j)}^{-r} = \sum_{j=0}^{\infty}\sum_{m=0}^{\kappa_0-1} \sigma_{k(j)}^{-r} \geq c \sum_{j=0}^{\infty}\sum_{m=0}^{\kappa_0-1} \sigma_{k(j)+m}^{-r} \geq c \sum_{l=k(0)}^{\infty} \sigma_l^{-r}, \tag{44}$$

where the first inequality is a direct consequence of the admissibility of σ (with the factor c depending on κ_0) and the second inequality comes from the fact that the term following each $\sigma_{k(j)+\kappa_0-1}^{-r}$ in the middle line above, being $\sigma_{k(j+1)}^{-r}$ is, by Remark (3.2.17) (i), either the next term in the sequence σ^{-r} or a term already considered before and that we can discard, turning the total sum smaller, though not smaller than the sum in the last line (because of Remark (3.2.17) (ii)).

Combining (43) and (44), we get the required result.

One of the tools we shall need is the atomic representation of functions in spaces of generalized smoothness. In order to present the atomic decomposition theorem see also [144, Sect. 4.4].

Let $\mathbb{Z}^n$ be the lattice of all points in $\mathbb{R}^n$ with integer-valued components.

If $\nu \in \mathbb{N}_0$ *and* $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$ we denote by $Q_{\nu m}$ the cube in $\mathbb{R}^n$ centred at $N_\nu^{-1}\, m = (N_\nu^{-1}\, m_1, \dots, N_\nu^{-1}\, m_n)$ which has sides parallel to the axes and side length N_ν^{-1}.

If $Q_{\nu m}$ is such a cube in $\mathbb{R}^n$ and $c > 0$ then $cQ_{\nu m}$ denotes the cube in $\mathbb{R}^n$ concentric with $Q_{\nu m}$ and with side length cN_ν^{-1}.

Definition (3.2.19) [169] (i) Let $M \in \mathbb{N}_0, c^* > 1$ and $\kappa > 0$. A function $\rho: \mathbb{R}^n \to \mathbb{C}$ which is M times differentiable (continuous if $M = 0$) is called a 1_M-N-atom if:

$$\operatorname{supp} \rho \subset c^* Q_{0m} \quad \text{for some} \quad m \in \mathbb{Z}^n, \tag{45}$$

$$|D^\alpha \rho(x)| \le \kappa \quad if\ |\alpha| \le M. \tag{46}$$

(ii) Let $\sigma = (\sigma_j)_{j \in \mathbb{N}_0}$ be an admissible sequence, let $0 < p \le \infty$, $M, L + 1 \in \mathbb{N}_0, c^* > 1$ and $\kappa > 0$. A function $\rho: \mathbb{R}^n \to C$ which is M times differentiable (continuous if $M = 0$) is called a $(\sigma, p)_{M,L}$-N-atom if:

$$\operatorname{supp} \rho \subset c^* Q_{0m} \quad \text{for some} \quad \nu \in \mathbb{N}, m \in \mathbb{Z}^n, \tag{47}$$

$$|D^\alpha \rho(x)| \le \kappa \sigma_\nu^{-1} N_\nu^{\frac{n}{p} + |\alpha|} \qquad \text{if } |\alpha| \le M, \tag{48}$$

$$\int_{\mathbb{R}^n} x^\gamma \rho(x) dx = 0 \quad \text{if } |\gamma| \le L. \tag{49}$$

If the atom ρ is located at $Q_{\nu m}$ (that means $\operatorname{supp} \rho \subset c^* Q_{0m}$ with $\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n, c^* > 1$) then we will denote it by $\rho_{\nu m}$.

As in the classical case, the N-atoms (associated to the sequence N) are normalized building blocks satisfying some moment conditions.

The value of the number $c^* > 1$ in (45) and (47) is unimportant. It simply makes clear that at the level ν some controlled overlapping of the supports of $\rho_{\nu m}$ must be allowed.

The moment conditions (49) can be reformulated as $D^\gamma \hat{\rho}(0) = 0 \text{ if } |\gamma| \le L,$ which shows that a sufficiently strong decay of $\hat{\rho}$ at the origin is required. If $L < 0$ then (49) simply means that there are no moment conditions required.

The reason for the normalizing factor in (46) and (48) is that then there exists a constant $c > 0$, depending on κ, such that for all these atoms we have $\| \rho \mid B_{p,q}^{\sigma,N} \| \le c$ and $\| \rho \mid F_{p,q}^{\sigma,N} \| \le c$, provided M and L are large enough – see Theorem (3.2.21) below. In [144] κ was fixed to 1 but we can use any other κ to the effect of normalization.

If $\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n$ and $Q_{\nu m}$ is a cube as above, let $\chi_{\nu m}$ be the characteristic function of $Q_{\nu m}$; if $0 < p \le \infty$ let

$$\chi_{\nu m}^{(p)} = N_\nu^{n/p} \chi_{\nu m}$$

(obvious modification if $p = \infty$) be the L_p-normalized characteristic function of $Q_{\nu m}$.

Definition (3.2.20) [169] Let $0 < p \le \infty, 0 < q \le \infty$. Then:

(i) $b_{p,q}$ is the collection of all sequences $\lambda = \{\lambda_{\nu m} \in \mathbb{C}: \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$ such that

$$\|\lambda \mid b_{p,q}\| = \left(\sum_{\nu=0}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^p \right)^{q/p} \right)^{1/q}$$

(with the usual modification if $p = \infty$ and/or $q = \infty$) is finite;

(ii) $f_{p,q}^N$ is the collection of all sequences $\lambda = \{\lambda_{\nu m} \in \mathbb{C}: \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$ such that

$$\|\lambda \mid f_{p,q}^N\| = \left\| \left(\sum_{v=0}^{\infty} \sum_{m\in\mathbb{Z}^n} |\lambda_{vm}\chi_{vm}^{(p)}(\cdot)|^q \right)^{1/q} \mid L_p \right\|$$

(with the usual modification if $p = \infty$ and/or $q = \infty$) is finite.

One can easily see that $b_{p,q}$ and $f_{p,q}^N$ are quasi-Banach spaces and using $\|\chi_{vm}^{(p)} \mid L_p\| = 1$ it is clear that comparing $\|\lambda \mid b_{p,q}\|$ and $\|\lambda \mid f_{p,q}^N\|$ the roles of the quasi-norms in L_p and l_q are interchanged.

In [144] it was proved the following atomic decomposition theorem.

Theorem (3.2.21) [169] Let $N = (N_j)_{j\in\mathbb{N}_0}$ be an admissible sequence from Assumption (3.2.6) with $\lambda_0 > 1$ and let $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$be an admissible sequence.

Let $0 < p < \infty$, respectively $0 < p \le \infty, 0 < q \le \infty$, and let $M, L + 1 \in \mathbb{N}_0$ be such that

$$M > \frac{\log_2 d_1}{\log_2 \lambda_0} \tag{50}$$

and

$$L > -1 + n\left(\frac{\log_2 \lambda_1}{\log_2 \lambda_0}\frac{1}{\min(1,p,q)} - 1\right) - \frac{\log_2 d_0}{\log_2 \lambda_0}, \tag{51}$$

respectively

$$L > -1 + n\left(\frac{\log_2 \lambda_1}{\log_2 \lambda_0}\frac{1}{\min(1,p)} - 1\right) - \frac{\log_2 d_0}{\log_2 \lambda_0}. \tag{52}$$

Then $g \in S'$ belongs to $F_{p,q}^{\sigma,N}$, respectively to $B_{p,q}^{\sigma,N}$, if and only if it can be represented as

$$g = \sum_{v=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{vm}\rho_{vm}, \tag{53}$$

convergence being in S', where ρ_{vm} are 1_M-N-atoms $(\nu = 0)$ or $(\sigma, p)_{\mathrm{M,L}}$-$N$-atoms $(\nu \in \mathbb{N})$ and $\lambda \in f_{p,q}^N$, respectively $\lambda \in b_{p,q}$, where $\lambda = \{\lambda_{vm}: \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$.

Furthermore, for any fixed $c^* > 1$, any fixed $\kappa > 0$, and any M and L as above, $\inf \|\lambda \mid f_{p,q}^N\|$, respectively in $\|\lambda \mid b_{p,q}\|$, where the infimum is taken over all admissible representations (53), is an equivalent quasi-norm in $F_{p,q}^{\sigma,N}$, respectively $B_{p,q}^{\sigma,N}$.

See [144]. The use of arbitrary $\kappa > 0$ instead of $\kappa = 1$ changes only the equivalence constants for the quasi-norm.

Remark (3.2.22) [169] Let $\tilde{N} = (\tilde{N}_j)_{j\in\mathbb{N}_0}$ be an admissible sequence as in Assumption (3.2.6) which is equivalent to the sequence N.

Let also $\tilde{\sigma} = (\tilde{\sigma}_j)_{j\in\mathbb{N}_0}$be an admissible sequence equivalent to σ.

It follows directly from Definition (3.2.19) that for arbitrary fixed $c^* > 1$ and $\kappa > 0$ there exist $\widetilde{c^*} > 1$ and $\tilde{\kappa} > 0$ such that any 1_M-N-atom is a 1_M-$\tilde{N}$-atom and such that any $(\sigma, p)_{M,L}$-N-atom is a $(\tilde{\sigma}, p)_{M,L}$-$\tilde{N}$-atom with respect to the numbers $\widetilde{c^*}$ and $\tilde{\kappa}$.

Clearly $\widetilde{c^*}$ and $\tilde{\kappa} > 0$ depend on c^*, κ, M, p and on the equivalence constants for the sequences σ and N.

Let us denote by α_σ and β_σ, respectively α_N and β_N, the Boyd indices of σ and N respectively.

According to Remark (3.2.22), Remark (3.2.5) and taking into account the definition of Boyd indices, conditions (50)–(52) can be reformulated and improved as

$$M > \frac{\alpha_\sigma}{\beta_N} \qquad \text{(replacement for (50))}, \tag{54}$$

and

$$L > -1 + n\left(\frac{\alpha_N}{\beta_N}\frac{1}{\min(1,p,q)} - 1\right) - \frac{\beta_\sigma}{\beta_N} \text{ (replacement for (51))}, \tag{55}$$

$$L > -1 + n\left(\frac{\alpha_N}{\beta_N}\frac{1}{\min(1,p)} - 1\right) - \frac{\beta_\sigma}{\beta_N} \text{ (replacement for (52))}. \tag{56}$$

At one point we shall need a specific result about lacunary Fourier series, that is, of Fourier series of the form

$$\sum_{j=1}^{\infty} b_j e^{i\lambda_j t}$$

where $(\lambda_j)_j$ is some given sequence of positive integers for which there exists q such that $\frac{\lambda_{j+1}}{\lambda_j} > q > 1, j \in \mathbb{N}$.

For the following result, see [164, p. 204].

Proposition (3.2.23) [169] If $\sum_{j=1}^{\infty} b_j e^{i\lambda_j t}$, with $(\lambda_j)_j$ as above, is the Fourier series of a function of $L_1([0,2\pi])$, then $(b_j)_j \in \ell_2$.

Related to this, we shall also need the following technical lemma of [160, Lem.5.5.2] and the theorem which we state and prove afterwards, though the proof follows along the same lines of a corresponding result in [160, Thm. 4.2.1; see also Rem. 4.2.2.(c)].

Lemma (3.2.24) [169] Let $N = (2^j)_{j\in\mathbb{N}_0}$ and consider a function system $\varphi = (\varphi_j)_{j\in\mathbb{N}_0} \in \phi^N$ as in Definition (3.2.7) built in the following way: for each $j \in \mathbb{N}\backslash\{1\}, \varphi_j = \varphi_1(2^{-j+1}\,\cdot)$, where, for some suitable $a > 0, \varphi_1 \in S$ is chosen such that

$$\varphi_1(\xi) + \varphi_1(2^{-1}\xi) = 1 \quad \text{if } 2 \le |\xi| \le 4,$$
$$\varphi_1(\xi) = 1 \quad \text{if } 2(1-a) \le |\xi| \le 2(1+a)$$

and

$$\operatorname{supp} \varphi_1 \subset \{\xi \in \mathbb{R}^n : (1+a) \le |\xi| \le 4(1-a)\};$$

$\varphi_0 \in S$ is chosen so that $\varphi_0(\xi) + \varphi_1(\xi) = 1$ if $|\xi| \le 2$ and $\operatorname{supp} \varphi_0 \subset \{\xi \in \mathbb{R}^n : |\xi| \le 2\}$. Consider $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$. Given $\zeta \in S, (b_j)_{j\in\mathbb{N}} \subset \mathbb{C}$ with $|b_j| \le c_r 2^{jr}$ for some $r > 0$ and $k \in \mathbb{N}$, the function

$$V_k := \sum_{j=1}^{\infty} b_j\, \zeta(\cdot - 2^j e_1)\varphi_k \tag{57}$$

is well-defined with convergence in S and, for any given $d > 0$,

$$\lim_{k\to\infty} 2^{kd}\left(V_k - b_k\zeta(\cdot - 2^k e_1)\right) = 0 \quad \text{in } S. \tag{58}$$

Theorem (3.2.25) [169] Let $0 < p \le \infty$ ($0 < p < \infty$ in the case of F-spaces), $0 < q \le \infty$ and σ be admissible. Let $\psi \in S\backslash\{0\}$ and $(b_k)_{k\in\mathbb{N}} \subset \mathbb{C}$ with $|b_k| c_r 2^{kr}$ for some $r > 0$. Then

$$W(x_1, \dots, x_n) := \sum_{j=1}^{\infty} b_j e^{i2^j x_1}$$

converges in S' and

$$\psi W \in B^{\sigma}_{p,q} \Leftrightarrow (\sigma_k b_k)_{k\in\mathbb{N}} \in \ell_q \Leftrightarrow \psi W \in F^{\sigma}_{p,q}. \tag{59}$$

Proof. The hypothesis on the sequence $(b_k)_{k\in\mathbb{N}}$ immediately guarantees that W makes sense in S' and is indeed a periodic distribution on $\mathbb{R}^n$ (cf. [168, Sect. 3.2]). Then it is a straightforward calculation to see that

$$\mathcal{F}(\psi W) = \sum_{j=1}^{\infty} b_j \mathcal{F}\left(\psi e^{i2^j x_1}\right) = \sum_{j=1}^{\infty} b_j (\mathcal{F}\psi) * \delta_{(2^j,0,\dots,0)} = \sum_{j=1}^{\infty} b_j (\mathcal{F}\psi)\left(\cdot - 2^j e_1\right),$$

where e_1 stands for $(1, 0, \dots, 0) \in \mathbb{R}^n$.

Considering a system φ as in Lemma (3.2.24), then

$$\varphi_k \mathcal{F}(\psi W) = \sum_{j=1}^{\infty} b_j (\mathcal{F}\psi)\left(\cdot - 2^j e_1\right)\varphi_k$$

can be taken as the V_k in (57), $k \in \mathbb{N}$, for the choice $\zeta = \mathcal{F}\psi$. Therefore the conclusion (58) reads here as

$$\lim_{k\to\infty} 2^{kd}\left(\varphi_k \mathcal{F}(\psi W) - b_k \mathcal{F}\left(\psi\, e^{i2^k x_1}\right)\right) = 0 \quad \text{in } S,$$

where $d > 0$ is at our disposal. Applying the inverse Fourier transformation we get

$$\lim_{k\to\infty} 2^{kd}\left(\mathcal{F}^{-1}(\varphi_k \mathcal{F}(\psi W)) - b_k \psi\, e^{i2^k x_1}\right) = 0 \qquad in\ S \tag{60}$$

and, using $S \hookrightarrow L_p$, also

$$\lim_{k\to\infty} 2^{kd}\left\|\mathcal{F}^{-1}(\varphi_k \mathcal{F}(\psi W)) - b_k \psi\, e^{i2^k x_1} \mid L_p\right\| = 0. \tag{61}$$

Notice now that (with the usual modification in the case $q = \infty$) we have

$$\left(\sum_{k=1}^{\infty} \sigma_k^q \left\|b_k \psi\, e^{i2^k x_1} \mid L_p\right\|^q\right)^{1/q}$$

$$\leq c\left(\sum_{k=1}^{\infty} \sigma_k^q \left\|\mathcal{F}^{-1}\left(\varphi_k \mathcal{F}(\psi\, W)\right) \mid L_p\right\|^q\right)^{1/q} + c\left(\sum_{k=1}^{\infty} \sigma_k^q \left\|\mathcal{F}^{-1}\left(\varphi_k \mathcal{F}(\psi\, W)\right) - b_k \psi\, e^{i2^k x_1} \mid L_p\right\|^q\right)^{1/q} \tag{62}$$

and a corresponding estimation obtained by interchanging the roles of $b_k \psi\, e^{i2^k x_1}$ and $\mathcal{F}^{-1}(\varphi_k \mathcal{F}(\psi W))$. Since the last term in (62) can be estimated from above by

$$\left(\sum_{k=1}^{\infty} \sigma_k^q 2^{-kdq}\right)^{1/q} \sup_{k\in\mathbb{N}} 2^{kd}\left\|\mathcal{F}^{-1}\left(\varphi_k \mathcal{F}(\psi\, W)\right) - b_k \psi e^{i2^k x_1} \mid L_p\right\|$$

and, from (35), $\sigma_k \leq \sigma_0 2^{k \log_2 d_1}$, by choosing $d > \log_2 d_1$ we get, also with the help of (61), that the above expression is finite and therefore, from (62) and the corresponding estimate referred to above,

$$\sum_{k=1}^{\infty} \sigma_k^q \left\|\mathcal{F}^{-1}\left(\varphi_k \mathcal{F}(\psi\, W)\right) \mid L_p\right\|^q \qquad \text{is finite}$$

if, and only if,

$$\sum_{k=1}^{\infty} \sigma_k^q \left\|b_k \psi e^{i2^k x_1} \mid L_p\right\|^q \qquad \text{is finite.}$$

That is, and after simplifying the last expression (taking also into consideration the hypothesis $\psi \in S \setminus \{0\}$),

$$\psi W \in B_{p,q}^{\alpha} \qquad \text{if, and only if,} \qquad (\sigma_k b_k)_{k\in\mathbb{N}} \in \ell_q.$$

As for $F_{p,q}^{\alpha}$, with $0 < p, q < \infty$, we start by observing that from (60) it follows, in particular, that for any $m \in \mathbb{N}$ and any $d > 0$

$$\lim_{k\to\infty} \sup_{x\in\mathbb{R}^n} \left\{(1 + |x|)^m 2^{kd}\left|\mathcal{F}^{-1}\left(\varphi_k \mathcal{F}(\psi\, W)\right) - b_k \psi e^{i2^k x_1}\right|\right\} = 0.$$

Then we have, pointwisely, with $d' > d$, that

$$(1 + |x|)^{mq} \sum_{k=1}^{\infty} 2^{kdq}\left|\mathcal{F}^{-1}\left(\varphi_k \mathcal{F}(\psi\, W)\right) - b_k \psi\, e^{i2^k x_1}\right|^q$$

$$\leq \left(\sum_{k=1}^{\infty} 2^{k(d-d')q}\right)\left(\sup_{k\in\mathbb{N}} \sup_{x\in\mathbb{R}^n} (1 + |x|)^m 2^{kd'}\left|\mathcal{F}^{-1}\left(\varphi_k \mathcal{F}(\psi\, W)\right) - b_k \psi e^{i2^k x_1}\right|\right)^q$$

is finite and therefore the series of functions above converges pointwisely and, moreover,

$$\sup_{x\in\mathbb{R}^n}\left\{(1+|x|)^{mq}\sum_{k=1}^{\infty}2^{kdq}\left|\mathcal{F}^{-1}\big(\varphi_k\mathcal{F}(\psi W)\big)-b_k\psi\, e^{i2^k x_1}\right|^q\right\}<\infty. \qquad (63)$$

Using now that $\sigma_k \le \sigma_0 2^{k\log_2 d_1}$ – cf. (35) – and choosing $m\in\mathbb{N}$ large enough and $d\ge \log_2 d_1$ in (63), we get that

$$\int_{\mathbb{R}^n}\left(\sum_{k=1}^{\infty}\sigma_k^q\left|\mathcal{F}^{-1}\big(\varphi_k\mathcal{F}(\psi W)\big)-b_k\psi\, e^{i2^k x_1}\right|^q\right)^{p/q} dx$$

$$\le \int_{\mathbb{R}^n}(1+|x|)^{-mp}dx\times\sup_{x\in\mathbb{R}^n}\left((1+|x|)^{mq}\sum_{k=1}^{\infty}2^{k(\log_2 d_1)q}\left|\mathcal{F}^{-1}\big(\varphi_k\mathcal{F}(\psi\, W)\big)-b_k\psi\, e^{i2^k x_1}\right|^q\right)^{p/q}$$

$$<\infty. \qquad (64)$$

The counterpart of (62) is now

$$\left\|\left(\sum_{k=1}^{\infty}\sigma_k^q\left|b_k\psi\, e^{i2^k x_1}\right|^q\right)^{1/q}\ \Big|\ L_p\right\|$$

$$\le c\left\|\left(\sum_{k=1}^{\infty}\sigma_k^q\left|\mathcal{F}^{-1}\big(\varphi_k\mathcal{F}(\psi\, W)\big)\right|^q\right)^{1/q}\ \Big|\ L_p\right\|+c\left\|\left(\sum_{k=1}^{\infty}\sigma_k^q\left|\mathcal{F}^{-1}\big(\varphi_k\mathcal{F}(\psi\, W)\big)-b_k\psi\, e^{i2^k x_1}\right|^q\right)^{1/q}\ \Big|\ L_p\right\|,$$

and, again, a corresponding estimation obtained by interchanging the roles of $b_k\psi\, e^{i2^k x_1}$ and $\mathcal{F}^{-1}\big(\varphi_k\mathcal{F}(\psi\, W)\big)$ also holds. Therefore, taking (64) into account,

$$\left\|\left(\sum_{k=1}^{\infty}\sigma_k^q\left|\mathcal{F}^{-1}\big(\varphi_k\mathcal{F}(\psi\, W)\big)\right|^q\right)^{1/q}\ \Big|\ L_p\right\| \quad \text{is finite}$$

if, and only if,

$$\left\|\left(\sum_{k=1}^{\infty}\sigma_k^q\left|b_k\psi\, e^{i2^k x_1}\right|^q\right)^{1/q}\ \Big|\ L_p\right\| \quad \text{is finite.}$$

That is, and after simplifying the last expression (taking also into consideration the hypothesis $\psi\in S\setminus\{0\}$),

$$\psi\, W\in F_{p,q}^{\sigma} \qquad \text{if, and only if,} \qquad (\sigma_k b_k)_{k\in\mathbb{N}}\in\ell_q.$$

We have been assuming, in this case of F-spaces, that both p and q are finite. However, with the usual modifications the preceding arguments also work out for $q=\infty$.

We start by considering a reverse Hölder's inequality result which will be used as a backbone for the proof of the necessity of most of the conditions in Theorems (3.2.28) and (3.2.29) below.

Proposition (3.2.26) [169] Let $0<r\le\infty$ and $\left(a_j\right)_{j\in\mathbb{N}},\left(b_j\right)_{j\in\mathbb{N}}\subset\mathbb{C}.\,If\left(a_jb_j\right)_{j\in\mathbb{N}}$ belongs to ℓ_1 for all sequences $\left(b_j\right)_{j\in\mathbb{N}}$belonging to ℓ_r, then $\left(a_j\right)_{j\in\mathbb{N}}\in\ell_{r'}$.

The case $1<r<\infty$ is contained in [162, Thm. 161, p. 120]. The case $r=\infty$ is trivial (just take all b_j's equal to 1), though something stronger is true, namely the conclusion still holds merely by drawing $\left(b_j\right)_{j\in\mathbb{N}}$ from c_0, as follows from [162, Thm. 162 (i), pp. 120–121]. Finally, the case $0<r\le 1$ (then $r'=\infty$) can be proved by contradiction. Indeed, assume $\left(b_j\right)_{j\in\mathbb{N}}\notin\ell_\infty$. Then for each natural number l there exists an index $j_l>j_{l-1}$ such that $|a_{j_l}|\ge l^{\frac{1}{r}+1}$, where j_0 can, e.g., be taken equal to 1. Define

$$b_j := \begin{cases} l^{-\frac{1}{r}-1} & \text{if } j = j_l \\ 0 & \text{otherwise.} \end{cases}$$

Then $(b_j)_{j\in\mathbb{N}} \in \ell_r$ but $\sum_{j=1}^{\infty} |a_j| b_j = \infty$.

To prove the necessity of some conditions in the next theorem we will construct so-called extremal functions starting from a smooth basic function ϕ with compact support and vanishing moment conditions, which we describe next:

Proposition (3.2.27) [169] For every $L \in \mathbb{N}$ and $\lambda_0 > 1$ there exist a C^∞-function ϕ on $\mathbb{R}^n$ and suitable positive constants C_1, C_2 and C_3, these constants depending only on λ_0 and n, such that $C_1 < C_3 < \lambda_0 C_1$,

$$\phi(x) \geq C_2 \quad if\ |x|_\infty \leq C_1, \qquad \phi(x) = 0 \quad \text{if } |x|_\infty \geq C_3$$

and

$$\int x^\gamma \phi(x) dx = 0 \quad \text{whenever} \quad \gamma \in \mathbb{N}_0^n \ \text{ and } \ |\gamma|_\infty \leq L.$$

A construction of such functions was described in [12, Lem.4.6].

Theorem (3.2.28) [169] Let $0 < p, q \leq \infty$. Let N and σ be admissible sequences with N satisfying also Assumption (3.2.6). The following are necessary and sufficient conditions for $B_{p,q}^{\sigma,N} \subset L_1^{loc}$, where $\ell_{\frac{p\infty}{\infty-p}}$ should be understood as ℓ_p:

(i) $(\sigma_j^{-1} N_j^{n(\frac{1}{p}-1)})_{j\in\mathbb{N}_0} \in \ell_{q'}$, in case $0 < p \leq 1$ and $0 < q \leq \infty$;

(ii) $(\sigma_j^{-1})_{j\in\mathbb{N}_0} \in \ell_\infty$, in case $1 < p \leq \infty$ and $0 < q \leq \ min\{p, 2\}$;

(iii) $(\sigma_j^{-1})_{j\in\mathbb{N}_0} \in \ell_{\frac{pq}{q-p}}$, in case $1 < p \leq 2$ and $min\{p, 2\} < q \leq \infty$;

(iv) $(\sigma_j^{-1})_{j\in\mathbb{N}_0} \in \ell_{\frac{2q}{q-2}}$, in case $2 < p \leq \infty$ and $min\{p, 2\} < q \leq \infty$.

Proof. (i) First we prove the sufficiency of the given conditions in each case.

In case 1, it follows directly.

For each one of the remaining cases we use, in sequence, Proposition (3.2.10), Corollary (3.2.15) and Theorem (3.2.1). This explains why we can write, assuming the condition in each one of the cases, that in case 2

$$B_{p,q}^{\sigma,N} \hookrightarrow B_{p,q}^{(1),N} = B_{p,q}^0 \subset L_1^{loc},$$

in case 3

$$B_{p,q}^{\sigma,N} \hookrightarrow B_{p,p}^{(1),N} = B_{p,p}^0 \subset L_1^{loc}$$

and in case 4

$$B_{p,q}^{\sigma,N} \hookrightarrow B_{p,2}^{(1),N} = B_{p,2}^0 \subset L_1^{loc}.$$

(ii) Here we prove the necessity of the condition stated in case 1.

Let L be chosen in dependency of $(N_j)_{j\in\mathbb{N}_0}$and $(\sigma_j)_{j\in\mathbb{N}_0}$ by (52) or (56), respectively, and let ϕ be a corresponding basic function depending on L, n and λ_0 from Proposition (3.2.27) Let $(\rho_j)_{j\in\mathbb{N}}$ be a sequence belonging to ℓ_q and

$$f^\rho(x) := \sum_{j=1}^{\infty} |\rho_j| \sigma_j^{-1} N_j^{n/p} \phi(N_j x), \tag{65}$$

convergence in S'. For $x \neq 0$ this is always a finite sum and for each j the functions $\sigma_j^{-1} N_j^{n/p} \phi(N_j x)$ are $(\sigma, p)_{M,L}$-N-atoms located at Q_{j0} in the sense of Definition (3.2.19) and Theorem (3.2.21). Then f^ρ belongs to $B_{p,q}^{\sigma,N}$ and

$$\| f^\rho \mid B_{p,q}^{\sigma,N} \| \leq c \| (\rho_j)_{j\in\mathbb{N}} \mid \ell_q \|.$$

Now we assume $B^{\sigma,N}_{p,q} \subset L^{loc}_1$. Then

$$\int_{|x|_\infty \le C_1 N_1^{-1}} |f^\rho(x)| dx \ < \infty$$

and, actually, f^ρ will also be given by (65) in the pointwise sense a. e. We will split part of the set $\{x : |x|_\infty \le C_1 N_1^{-1}\}$ in a non-overlapping way to obtain simple passages

$$P_m := \{\, x : C_3 \lambda_0^{-1} N_m^{-1} \le |x|_\infty \le C_1 N_m^{-1}\},$$

because on these passages we have

$$\phi(N_j x) \ge C_2 \qquad \text{if } j \le m$$

and

$$\phi(N_j x) = 0 \qquad \text{if } j > m.$$

For each $k \in \mathbb{N}$ we have

$$\infty > \int_{|x|_\infty \le C_1 N_1^{-1}} |f^\rho(x)| dx \ge \int_{C_3 \lambda_0^{-1} N_k^{-1} \le |x|_\infty \le C_1 N_1^{-1}} \left| \sum_{j=1}^{\infty} |\rho_j| \sigma_j^{-1} N_j^{n/p} \phi(N_j x) \right| dx$$

$$\ge \sum_{m=1}^{k} \int_{C_3 \lambda_0^{-1} N_m^{-1} \le |x|_\infty \le C_1 N_m^{-1}} \left| \sum_{j=1}^{\infty} |\rho_j| \sigma_j^{-1} N_j^{n/p} \phi(N_j x) \right| dx$$

$$\ge C_2 \sum_{m=1}^{k} \int_{C_3 \lambda_0^{-1} N_m^{-1} \le |x|_\infty \le C_1 N_m^{-1}} \sum_{j=1}^{m} |\rho_j| \sigma_j^{-1} N_j^{n/p} \, dx$$

$$\ge C_2 \sum_{m=1}^{k} |\rho_m| \sigma_m^{-1} N_m^{n/p} 2^n (C_1^n - C_3^n \lambda_0^{-n}) N_m^{-n} = c \sum_{m=1}^{k} |\rho_m| \sigma_m^{-1} N_m^{n(\frac{1}{p}-1)}.$$

The sum on the right-hand side is monotone increasing and the left-hand side is independent of k. So we have

$$\sum_{j=1}^{\infty} |\rho_j| \sigma_j^{-1} N_j^{n(\frac{1}{p}-1)} < \infty \qquad (66)$$

for any sequence $(\rho_j)_{j \in \mathbb{N}} \in \ell_q$ if $B^{\sigma,N}_{p,q} \subset L^{loc}_1$.

Now by Proposition (3.2.26) it follows

$$(\sigma_j^{-1} N_j^{n(\frac{1}{p}-1)})_{j \in \mathbb{N}_0} \in \ell_{q'}.$$

(iii) Now we prove the necessity of the conditions stated in cases 2 and 3.

Let $(\gamma_j)_{j \in \mathbb{N}_0}$ be an arbitrary sequence belonging to ℓ_1. For technical reasons we consider now the sequence $(\tilde{\gamma}_j)_{j \in \mathbb{N}_0}$with

$$\tilde{\gamma}_j := \max\left(|\gamma j|, 10^3 N_0^{-1} \lambda_0^{-j}\right), \qquad j = 0,1, \ldots. \qquad (67)$$

It is clear that $(\tilde{\gamma}_j)_{j \in \mathbb{N}_0}$also belongs to ℓ_1. Define

$$\kappa_0 := 0 \qquad \text{and} \qquad \kappa_j := \sum_{l=1}^{j} \tilde{\gamma}_l \qquad \text{for } j \in \mathbb{N}.$$

Then $\kappa_j > 0$ if $j \in \mathbb{N}$ and $\lim_{j\to\infty} \kappa_j = \kappa$, where κ is equal to $\left\| (\tilde{\gamma}_j)_{j \in \mathbb{N}_0} | \ell_1 \right\|$.

For all $j = 1,2, \ldots$ put

$$R_j := \{x = (x_1, x_2, \ldots, x_n) : \kappa_{j-1} < x_1 \le \kappa_j, 0 < xi \ < 1, i = 2,3, \ldots, n\}.$$

We obtain rectangles in $\mathbb{R}^n$ which become narrower in the x_1-direction. Inside each R_j we consider cubes $\mathcal{Q}_{jm}$ of the type considered. There exist M_j such cubes inside R_j, centred in $N_j^{-1}m_r, j=1,\ldots,M_j$. Because of

$$10^3 N_j^{-1} \le 10^3 N_0^{-1}\lambda_0^{-j} \le \tilde{\gamma}_j$$

and assuming, without loss of generality, that $N_0 > 2$, we have

$$M_j \sim N_j^{n-1}(\kappa_j - \kappa_{j-1})N_j \; = N_j^n\,\tilde{\gamma}_j.$$

In dependency of $(N_j)_{j\in\mathbb{N}_0}$and $(\sigma_j)_{j\in\mathbb{N}_0}$ choose L which fulfil (52) or (56), respectively. Furthermore let ϕ be a basic function depending on L, n and λ_0 from Proposition (3.2.27) and put $\tilde{\phi}(x) := \phi(2C_3 x)$. Let

$$h^\rho(x) := \sum_{j=1}^{\infty}\sum_{r=1}^{M_j} \rho_j \tilde{\phi}\left(N_j\left(x - N_j^{-1}m_r\right)\right) \tag{68}$$

(pointwise convergence) be a compactly supported function where $(\rho_j)_{j\in\mathbb{N}}$ is an arbitrary sequence of non-negative numbers which will be specified later. Notice that by construction for each $x\in\mathbb{R}^n$ in the double sum appears at most one summand which is not zero and that $\sigma_j^{-1}N_j^{n/p}\tilde{\phi}\left(N_j\left(x-N_j^{-1}m_r\right)\right)$ are $(\sigma,p)_{M,L}$-N-atoms located at $\mathcal{Q}_{jm_r}$ in the sense of Definition (3.2.19) and Theorem (3.2.21).

If $(\rho_j^{(m)}\sigma_j^{-1}N_j^{n/p})_{j\in\mathbb{N}_0, m\in\mathbb{Z}^n} \in b_{p,q}$, where $\rho_j^{(m)} = \rho_j$ if $\mathcal{Q}_{jm}\subset R_j$ and $\rho_j^{(m)} = 0$ otherwise, then the double sum in (68) converges in S' to some g^ρ which, by Theorem (3.2.21), belongs to $B_{p,q}^{\sigma,N}$ and which, moreover, satisfies (assuming further that both p and q are finite)

$$\left\|g^\rho \mid B_{p,q}^{\sigma,N}\right\| \le c\left(\sum_{j=1}^{\infty}\left(\sum_{r=1}^{M_j}\left|\rho_j\sigma_j N_j^{-\frac{n}{p}}\right|^p\right)^{q/p}\right)^{\frac{1}{q}} \sim c\left(\sum_{j=1}^{\infty}\rho_j^q\sigma_j^q N_j^{-\frac{nq}{p}} M_j^{\frac{q}{p}}\right)^{\frac{1}{q}}$$

$$\sim c\left(\sum_{j=1}^{\infty}\rho_j^q\sigma_j^q N_j^{-\frac{nq}{p}} M_j^{\frac{nq}{p}}\tilde{\gamma}_j^{\frac{q}{p}}\right)^{\frac{1}{q}} \sim c\left(\sum_{j=1}^{\infty}\rho_j^q\sigma_j^q\tilde{\gamma}_j^{\frac{q}{p}}\right)^{\frac{1}{q}} < \infty.$$

For each given sequence $(\gamma_j)_{j\in\mathbb{N}_0}\in\ell_1$ we choose

$$\rho_j^q := \sigma_j^{-q}\,\tilde{\gamma}_j^{\frac{q}{p}+1}$$

and obtain

$$\left\|g^\rho \mid B_{p,q}^{\sigma,N}\right\| \le c\left\|(\tilde{\gamma}_j)_{j\in\mathbb{N}_0} \mid \ell_1\right\|^{1/q} < \infty.$$

With this special choice of $(\rho_j)_{j\in\mathbb{N}}$ we also have

$$\int_{[0,\kappa]\times[0,1]^{n-1}} |h^\rho(x)|dx = \sum_{j=1}^{\infty}\int_{R_j}|h^\rho(x)|dx = \sum_{j=1}^{\infty}\sigma_j^{-1}\tilde{\gamma}_j^{-\frac{1}{p}+\frac{1}{q}}\sum_{r=1}^{M_j}\int_{\mathcal{Q}_{jm_r}}\left|\tilde{\phi}\left(N_j\left(x-N_j^{-1}m_r\right)\right)\right|dx$$

$$\sim\sum_{j=1}^{\infty}\sigma_j^{-1}\tilde{\gamma}_j^{-\frac{1}{p}+\frac{1}{q}}N_j^{-n}M_j \sim \sum_{j=1}^{\infty}\sigma_j^{-1}\tilde{\gamma}_j^{-\frac{1}{p}+\frac{1}{q}}N_j^{-n}N_j^n\tilde{\gamma}_j \sim \sum_{j=1}^{\infty}\sigma_j^{-1}\tilde{\gamma}_j^{1-\frac{1}{p}+\frac{1}{q}},$$

where the equivalence constants might depend on ϕ. Now we assume $B_{p,q}^{\sigma,N}\subset L_1^{loc}$. Then h^ρ and g^ρ coincide a. e. and for every sequence $(\gamma_j)_{j\in\mathbb{N}_0}\in\ell_1$

$$\sum_{j=1}^{\infty} \sigma_j^{-1} \tilde{\gamma}_j^{1-\frac{1}{p}+\frac{1}{q}} \sim \int_{[0,\kappa]\times[0,1]^{n-1}} |g^{\rho}(x)| dx < \infty.$$

Moreover by (67)

$$\sum_{j=1}^{\infty} \sigma_j^{-1} |\gamma_j|^{1-\frac{1}{p}+\frac{1}{q}} \leq \sum_{j=1}^{\infty} \sigma_j^{-1} \tilde{\gamma}_j^{1-\frac{1}{p}+\frac{1}{q}}$$

whenever $1 - \frac{1}{p} + \frac{1}{q} > 0$. But this is the case if $1 < p$.

Therefore $B_{p,q}^{\sigma,N} \subset L_1^{loc}$ implies $\sum_{j=1}^{\infty} \sigma_j^{-1} |\gamma_j|^{1-\frac{1}{p}+\frac{1}{q}} < \infty$ for all sequences $(\gamma_j)_{j\in\mathbb{N}_0} \in \ell_1$. But this is equivalent to $\sum_{j=1}^{\infty} \sigma_j^{-1} |\beta_j| < \infty$ for all sequences $(\beta_j)_{j\in\mathbb{N}_0} \in \ell_r$, where $\frac{1}{r} = 1 - \frac{1}{p} + \frac{1}{q} > 0$. Then it follows $(\sigma_j^{-1})_{j\in\mathbb{N}_0} \in \ell_{r'}$ by Proposition (3.2.26). In case $1 < r < \infty$ (this is equivalent to $p < q$) we get $\frac{1}{r'} = \frac{1}{p} + \frac{1}{q}$ and in case $0 < r \leq 1$ (this is equivalent to $q \leq p$) we have $r' = \infty$. Consequently we obtain that $B_{p,q}^{\sigma,N} \subset L_1^{loc}$ implies

$$(\sigma_j^{-1})_{j\in\mathbb{N}_0} \in \ell_{\frac{pq}{q-p}} \text{ if } 1 < p < q < \infty,$$

$$(\sigma_j^{-1})_{j\in\mathbb{N}_0} \in \ell_{\infty} \text{ if } 1 < p < \infty \text{ and } 0 < q \leq p.$$

Adapting the above arguments to the cases where p or q are infinite, we get the same conclusions as long as we interpret $\ell_{\frac{p\infty}{\infty-p}}$ *as* ℓ_p.

(iv) Finally we prove the necessity of the condition stated in case 4.

Let $B_{p,q}^{\sigma,N}$ be given. Then by Theorem (3.2.14) we find a sequence $(\beta_j)_{j\in\mathbb{N}_0} := (\sigma_{k(j)})_{j\in\mathbb{N}_0}$ determined by (40) with $B_{p,q}^{\sigma,N} = B_{p,q}^{\beta}$. Furthermore by Theorem (3.2.25) we can construct for each sequence $(b_j)_{j\in\mathbb{N}} \subset \mathbb{C}$ with $|b_j| \leq c_r 2^{jr}$ (for some $r > 0$) a distribution

$$W(x_1, \dots, x_n) := \sum_{j=1}^{\infty} b_j e^{i2^j x_1}$$

such that

$$\psi\, W \in B_{p,q}^{\beta} \Leftrightarrow (\beta_k b_k)_{k\in\mathbb{N}} \in \ell_q, \qquad \text{for any given } \psi \in S\backslash\{0\}.$$

If we assume $B_{p,q}^{\sigma,N} = B_{p,q}^{\sigma} \subset L_1^{loc}$, then it follows $\psi\, W \in L_1^{loc}(\mathbb{R}^n)$ whenever $(\beta_k b_k)_{k\in\mathbb{N}} \in \ell_q$. With a choice of ψ different from 0 everywhere, then also $W \in L_1^{loc}(\mathbb{R}^n)$ and, consequently, the one variable version w (that is,$w(t) := \sum_{j=1}^{\infty} b_j e^{i2^j t}$) is locally integrable too. $\sum_{j=1}^{\infty} b_j e^{i2^j t}$ is the Fourier series of a function in $L_1([0, 2\pi])$ and by Proposition (3.2.23) it follows $(b_j)_{j\in\mathbb{N}} \in \ell_2$.

Since the assumption $(\beta_k b_k)_{k\in\mathbb{N}} \in \ell_q$ implies that $|b_j| \leq c_r 2^{jr}$ for some $r > 0$, then we have shown that $(b_j)_{j\in\mathbb{N}} \in \ell_2$ for all sequences $(b_k)_{k\in\mathbb{N}} \subset C$ such that $(\beta_k b_k)_{k\in\mathbb{N}} \in \ell_q$. Given any $(\gamma_j)_{j\in\mathbb{N}} \in \ell_{\frac{q}{2}}$ and defining

$$b_j := |\gamma_j|^{\frac{1}{2}} \beta_j^{-1},$$

the assumption $(\beta_k b_k)_{k\in\mathbb{N}} \in \ell_q$ is satisfied and therefore $(\beta_k^{-2} |\gamma_k|)_{k\in\mathbb{N}} \in \ell_1$. If $q > 2$, then again by Proposition (3.2.26) we have $(\beta_k^{-2})_{k\in\mathbb{N}} \in \ell_{\frac{q}{2}'}$, i.e., $(\beta_k^{-1})_{k\in\mathbb{N}} \in \ell_{\frac{2q}{q-2}}$ (with the understanding that $\ell_{\frac{2\infty}{\infty-2}}$ should be read as ℓ_2).

Finally, by Proposition (3.2.18) we can transfer this to the original sequence σ with arbitrary $\sigma_0 > 0$ and obtain $(\sigma_k^{-1})_{k\in\mathbb{N}_0} \in \ell_{\frac{2q}{q-2}}$.

Theorem (3.2.29) [169] Let $0 < p < \infty, 0 < q \le \infty$. Let N and σ be admissible sequences with N satisfying also Assumption (3.2.6). The following are necessary and sufficient conditions for $F_{p,q}^{\sigma,N} \subset L_1^{loc}$, where $\ell_{\frac{2\infty}{\infty-2}}$ should be understood as 2:

(i) $(\sigma_j^{-1} N_j^{n(\frac{1}{p}-1)})_{j\in\mathbb{N}_0} \in \ell_\infty$, in case $0 < p < 1$ and $0 < q \le \infty$;
(ii) $(\sigma_j^{-1})_{j\in\mathbb{N}_0} \in \ell_\infty$, in case $1 \le p < \infty$ and $0 < q \le 2$;
(iii) $(\sigma_j^{-1})_{j\in\mathbb{N}_0} \in \ell_{\frac{2q}{q-2}}$, in case $1 \le p < \infty$ and $2 < q \le \infty$.

Proof. (i) First we prove the sufficiency of the given conditions in each case.

In case 1, it follows directly from Proposition (3.2.13).

For both remaining cases we use, in sequence, Proposition (3.2.11), Corollary (3.2.15) and Theorem (3.2.1). This explains why we can write, assuming the condition in each one of the cases, that in case 2

$$F_{p,q}^{\sigma,N} \hookrightarrow F_{p,q}^{(1),N} = F_{p,q}^0 \subset L_1^{loc}$$

and in case 3

$$F_{p,q}^{\sigma,N} \hookrightarrow F_{p,2}^{(1),N} = F_{p,2}^0 \subset L_1^{loc}.$$

(ii) Now we prove the necessity of the conditions stated in cases 1 and 2.

If we assume $F_{p,q}^{\sigma,N} \subset L_1^{loc}$, by Proposition (3.2.9) it follows

$$B_{p,\min\{p,q\}}^{\sigma,N} \subset L_1^{loc}.$$

In case $0 < p \le 1$ and $0 < q \le \infty$ it holds $0 < \min\{p,q\} \le 1$ and by Theorem (3.2.28), part 1, we have $(\sigma_j^{-1} N_j^{n(\frac{1}{p}-1)})_{j\in\mathbb{N}_0} \in \ell_\infty$.

In case $1 < p < \infty$ and $0 < q \le 2$ it holds $0 < \min\{p,q\} \le \min\{p,2\}$ and by Theorem (3.2.28), part 2, we have $(\sigma_j^{-1})_{j\in\mathbb{N}_0} \in \ell_\infty$.

(iii) Finally, the proof of the necessity of the condition stated in case 3 is the same, mutatis mutandis, as in the last part in Theorem (3.2.28) because, under the conditions of Theorem (3.2.25),

$$\psi W \in B_{p,q}^\beta \Leftrightarrow (\beta_k b_k)_{k\in\mathbb{N}} \in \ell_q \Leftrightarrow \psi W \in F_{p,q}^\beta.$$

Example (3.2.30) [169] Let $\sigma_j := 2^{sj}(1+j)^b$, where $b > 0$ and $N_j = 2^j$. Then

$$B_{p,q}^{s_1} \hookrightarrow B_{p,q}^\sigma \hookrightarrow B_{p,q}^s$$

for any $s_1 > s$, that is, we have a scale in smoothness finer than in the classical case.

Naturally we obtain in some cases also really finer results concerning the embedding of $B_{p,q}^\sigma$ in L_1^{loc}.

(i) Let $0 < p \le 1$ and $1 < q \le \infty$. Then the classical result gives the embedding if and only if $s > n(\frac{1}{p} - 1)$, while in our example the embedding is still true if $s = n(\frac{1}{p} - 1)$ and in addition $b > \frac{q-1}{q}$.

(ii) Let $1 < p \le 2$ and $\min\{p,2\} < q \le \infty$. Then, in contrast to the classical case, $s = 0$ is possible if and only if $b > \frac{q-p}{pq}$ (meaning $b > \frac{1}{p}$ if $q = \infty$).

(iii) Let $2 < p \le \infty$ and $\min\{p,2\} < q \le \infty$. Then again $s = 0$ is possible if and only if $b > \frac{q-2}{2q}$ (meaning $b > \frac{1}{2}$ if $q = \infty$).

(iv) The same is true for the F-spaces in the case $1 \le p < \infty$ and $2 < q \le \infty$, where instead of $s > 0$ now $s = 0$ together with $b > \frac{q-2}{2q}$ (meaning $b > \frac{1}{2}$ if $q = \infty$) is permitted.

The following extends [55, Cor. 3.3.1] to our setting:

Corollary (3.2.31) [169] Let N and σ be admissible sequences with N satisfying also Assumption (3.2.6).

(i) Let $0 < p < \infty, 0 < q \le \infty$. The following two assertions are equivalent:

$$B_{p,q}^{\sigma,N} \subset L_1^{loc}$$

and

$$B_{p,q}^{\sigma,N} \hookrightarrow L_{max\{1,p\}}.$$

(ii) Let $0 < q \le \infty$. The following two assertions are equivalent:

$$B_{\infty,q}^{\sigma,N} \subset L_1^{loc}$$

and

$$B_{\infty,q}^{\sigma,N} \hookrightarrow bmo.$$

(iii) Let $0 < p < \infty, 0 < q \le \infty$. The following two assertions are equivalent:

$$F_{p,q}^{\sigma,N} \subset L_1^{loc}$$

and

$$F_{p,q}^{\sigma,N} \hookrightarrow L_{max\{1,p\}}.$$

Proof. Since the implication in which one concludes that $B_{p,q}^{\sigma,N}$ or $F_{p,q}^{\sigma,N}$ is in L_1^{loc} is obvious, we concentrate on the reverse one. So, let us assume that $B_{p,q}^{\sigma,N} \subset L_1^{loc}$ when proving (i) and (ii) above and that $F_{p,q}^{\sigma,N} \subset L_1^{loc}$ when proving (iii).

In what follows we shall use the following classical facts without further notice:

$$F_{p,2}^0 = L_p, \qquad 1 < p < \infty \quad (\text{by } [65, \text{Thm. } 2.5.6(\text{i})]);$$
$$F_{1,2}^0 = h_1 \qquad (\text{by } [65, \text{Thm. } 2.5.8/1]);$$
$$h_1 \hookrightarrow L_p \qquad (\text{by } [65, \text{Rem. } 2.5.8/4]);$$
$$F_{\infty,2}^0 = bmo \qquad (\text{by } [65, \text{Thm. } 2.5.8/2]);$$
$$B_{\infty,2}^0 \to F_{\infty,2}^0 \qquad (\text{cf. } [65, \text{Prop. } 2.3.2/2(\text{iii}), \text{Thm. } 2.11.2]).$$

(i) The B case when $0 < p < \infty$.

First let $0 < p \le 1$ and $0 < q \le \infty$.

We have, by Theorem (3.2.28), that $(\sigma_j^{-1} N_j^{n(\frac{1}{p}-1)})_{j\in\mathbb{N}_0} \in \ell_{q'}$ and by Proposition (3.2.10) and Corollary (3.2.15) it follows

$$B_{p,q}^{\sigma,N} \hookrightarrow B_{1,1}^{(1),N} = B_{1,1}^0 = F_{1,1}^0 \hookrightarrow F_{1,2}^0 = h_1 \hookrightarrow L_p = L_{max\{1,p\}}.$$

In case $1 < p < \infty$ and $0 < q \le \min\{p, 2\}$ Theorem (3.2.28) implies $(\sigma_j^{-1})_{j\in\mathbb{N}_0} \in \ell_\infty$ and by Propositions (3.2.9), (3.2.11) and Corollary (3.2.15) we have

$$B_{p,q}^{\sigma,N} \hookrightarrow F_{p,2}^{\sigma,N} \hookrightarrow F_{p,2}^{(1),N} = F_{p,2}^0 = L_p = L_{max\{1,p\}}.$$

If $1 < p \le 2$ and $\min\{p, 2\} < q \le \infty$, then Theorem (3.2.28) implies $(\sigma_j^{-1})_{j\in\mathbb{N}_0} \in \ell_{\frac{2q}{q-p}}$ and combining Proposition (3.2.10), Corollary (3.2.15) and Proposition (3.2.11) we get

$$B_{p,q}^{\sigma,N} \hookrightarrow B_{p,p}^{(1),N} = B_{p,p}^0 = F_{p,p}^0 \hookrightarrow F_{p,2}^0 = L_p = L_{max\{1,p\}}.$$

Finally in case $2 < p < \infty$ and $\min\{p, 2\} < q \le \infty$ Theorem (3.2.28) gives $(\sigma_j^{-1})_{j\in\mathbb{N}_0} \in \ell_{\frac{2q}{q-2}}$ and again Proposition (3.2.10), Corollary (3.2.15) and Proposition (3.2.9) lead to

$$B_{p,q}^{\sigma,N} \hookrightarrow B_{p,2}^{(1),N} = B_{p,2}^0 \hookrightarrow F_{p,2}^0 = L_p = L_{max\{1,p\}}.$$

(ii) The B case when $p = \infty$.

Then we have, similarly as above, that, in case $0 < q \le \min\{p, 2\}$, $(\sigma_j^{-1})_{j\in\mathbb{N}_0} \in \ell_\infty$ and

$$B_{\infty,q}^{\sigma,N} \hookrightarrow B_{\infty,2}^{(1),N} = B_{\infty,2}^{0} \hookrightarrow F_{\infty,2}^{0} = bmo;$$

in case $\min\{p, 2\} < q \le \infty$, $(\sigma_j^{-1})_{j\in\mathbb{N}_0} \in \ell_{\frac{2q}{q-2}}$ and

$$B_{\infty,q}^{\sigma,N} \hookrightarrow B_{\infty,2}^{(1),N} = B_{\infty,2}^{0} \hookrightarrow F_{\infty,2}^{0} = bmo.$$

(iii) The F case.

Let first $0 < p < 1$ and $0 < q \le \infty$. Then by Theorem (3.2.29) it holds $\left(\sigma_j^{-1} N_j^{n\left(\frac{1}{p}-1\right)}\right)_{j\in\mathbb{N}_0} \in \ell_\infty$. By Theorem (3.2.16) we obtain

$$F_{p,q}^{\sigma,N} \hookrightarrow B_{1,p}^{\sigma'',N} \quad \text{with} \quad \sigma_j'' = \sigma_j N_j^{n\left(\frac{1}{p}-1\right)}.$$

Moreover by Proposition (3.2.10) and Corollary (3.2.15) we get

$$B_{1,p}^{\sigma'',N} \hookrightarrow B_{1,1}^{(1),N} = B_{1,1}^{0} = F_{1,1}^{0} \hookrightarrow F_{1,2}^{0} = h_1 \hookrightarrow L_1 = L_{\max\{1,p\}}.$$

If $1 \le p < \infty$ and $0 < q \le 2$ we obtain $(\sigma_j^{-1})_{j\in\mathbb{N}_0} \in \ell_\infty$and by Proposition (3.2.11) and Corollary (3.2.15) we have

$$F_{p,q}^{\sigma,N} \hookrightarrow F_{p,2}^{(1),N} = F_{p,2}^{0} = L_p = L_{max\{1,p\}} \quad \text{in case } 1 < p.$$

and

$$F_{1,q}^{\sigma,N} \hookrightarrow F_{1,2}^{(1),N} = F_{1,2}^{0} = h_1 \hookrightarrow L_1 = L_{max\{1,p\}} \quad \text{in case } p = 1.$$

At last, in case $1 \le p < \infty$ and $2 < q \le \infty$ we get $(\sigma_j^{-1})_{j\in\mathbb{N}_0} \in \ell_{\frac{2q}{q-2}}$and in a similar way by Proposition (3.2.11) and Corollary (3.2.15)

$$F_{p,q}^{\sigma,N} \hookrightarrow F_{p,2}^{0} \hookrightarrow L_p = L_{max\{1,p\}}.$$

Chapter 4
New Characterizations of Sobolev and Besov Spaces

We establish similar characterizations are also established for Triebel-Lizorkin spaces $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ with $\alpha \in (0,\infty) \setminus 2\mathbb{N}$ and $p,q \in (1,\infty]$, and for Besov spaces $\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)$ with $\alpha \in (0,\infty) \setminus 2\mathbb{N}, p \in (1,\infty]$ and $q \in (0,\infty]$. The characterizations rely only on the metric and the Lebesgue measure on $\mathbb{R}^n$ and are simpler than those obtained recently by Alabern et al. These results may shed new light on the theory of high order Sobolev spaces on spaces of homogeneous type. The corresponding results for inhomogeneous Besov and Triebel–Lizorkin spaces are also obtained. These results, for the first time, give a way to introduce Besov and Triebel–Lizorkin spaces with any smoothness order in $(0,2\ell)$ on spaces of homogeneous type, where $\ell \in \mathbb{N}$.

Section (4.1) Triebel–Lizorkin Spaces on $\mathbb{R}^n$

The fractional Sobolev space $\dot{W}^{\alpha,p}(\mathbb{R}^n)$ with $\alpha \in (0,1)$ and $p \in (0,\infty)$ can be characterized by the square function s_α, defined by setting, for all $x \in (\mathbb{R}^n)$ and $f \in L^1_{loc}(\mathbb{R}^n) \cap S'(\mathbb{R}^n)$,

$$s_\alpha(f)(x) := \left\{ \int_0^\infty \left[\fint_{B(x,t)} [f(x) - f(y)] dy \right]^2 \frac{dt}{t^{1+2\alpha}} \right\}^{1/2},$$

where above and in what follows, for any $g \in L^1_{loc}(\mathbb{R}^n)$ and ball $B \subset \mathbb{R}^n$,

$$\fint_B g(y)dy := \frac{1}{|B|} \int_B g(y)dy$$

and $B(x,t)$ denotes the ball of $\mathbb{R}^n$ with the center $x \in \mathbb{R}^n$ and $t \in (0,\infty)$; see, for example, [56], [65], [72], [174]. However, when $\alpha \geq 1$ and $p \in (0,\infty)$, the above square function fails to characterize $\dot{W}^{\alpha,p}(\mathbb{R}^n)$; indeed, if $f \in L^1_{loc}(\mathbb{R}^n)$ and $\|s_\alpha(f)\|_{L^p(\mathbb{R}^n)} < \infty$, then f must be a constant function (see [6, Section 4]).

Alabern, Mateu and Verdera [170] characterized the fractional Sobolev space $\dot{W}^{\alpha,p}(\mathbb{R}^n)$ for $\alpha \in (0,2)$ and $p \in (1,\infty)$ via a new square function defined by setting, for all $f \in L^1_{loc}(\mathbb{R}^n) \cap S'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$s_\alpha(f)(x) := \left\{ \int_0^\infty \left| \fint_{B(x,t)} [f(x) - f(y)] dy \right|^2 \frac{dt}{t^{1+2\alpha}} \right\}^{1/2}.$$

S_1-function characterizes the Sobolev space $\dot{W}^{1,p}(\mathbb{R}^n)$. Comparing S_α with s_α, we see that the only difference is that $|f(x) - f(y)|$ appearing in the definition of $s_\alpha(f)$ is replaced by $f(x) - f(y)$ in that of $S_\alpha(f)$. Such a slight difference leads to a quite different conclusion in the characterization of (fractional) Sobolev spaces. The main point, as first observed by Wheeden in [173] (see [174]), when studying the Lipschitz-type (Besov) spaces, and later independently by Alabern, Mateu and Verdera in [170], is that S_α-function provides smoothness up to order 2 in the following sense: for all $f \in C^2(\mathbb{R}^n)$ and $t \in (0,1)$,

$$\fint_{B(x,t)} [f(x) - f(y)] dy = O(t^2), \qquad x \in \mathbb{R}^n.$$

which follows from the Taylor expansion of order 2

$$f(y) = f(x) + \nabla f(x) \cdot (x - y) + O(|x - y|^2), \qquad x, y \in \mathbb{R}^n.$$

The purpose is to show that the above observation further leads to a new characterization of Triebel-Lizorkin spaces with reasonable parameters. We denote by $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ the classical homogeneous Triebel-Lizorkin space while $F^{\alpha}_{p,q}(\mathbb{R}^n)$ the inhomogeneous Triebel-

Lizorkin space for all reasonable parameters; see for their definitions. Moreover, we introduce the following function spaces of Triebel-Lizorkin type via a variant of the above square function S_α.

Definition (4.1.1) [177] Let $\alpha \in (0,2)$ and $q \in (0,\infty]$.

(i) If $p \in (0,\infty)$, the space $S\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ is defined as the collection of all functions $f \in L^1_{loc}(\mathbb{R}^n) \cap S'(\mathbb{R}^n)$ such that $\|f\|_{S\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)} := \|S_{\alpha,q}(f)\|_{L^p(\mathbb{R}^n)} < \infty$, where, for all $x \in \mathbb{R}^n$,

$$S_{\alpha,q}(f)(x) := \left\{\sum_{k\in\mathbb{Z}} 2^{k\alpha q} \left| \fint_{B(x,2^{-k})} [f(x)-f(y)]dy \right|^q \right\}^{1/q}$$

with the usual modification made when $q = \infty$.

(ii) The space $S\dot{F}^{\alpha}_{\infty,q}(\mathbb{R}^n)$ is defined as the collection of all functions $f \in L^1_{loc}(\mathbb{R}^n) \cap S'(\mathbb{R}^n)$ such that

$$\|f\|_{S\dot{F}^{\alpha}_{\infty,q}(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n} \sup_{\ell\in\mathbb{Z}} \left\{ \fint_{B(x,2^{-k})} \sum_{k\geq\ell} 2^{k\alpha q} \left| \fint_{B(x,2^{-k})} [f(y)-f(z)]dz \right|^q dy \right\}^{1/q} < \infty$$

with the usual modification made when $q = \infty$.

(iii) If $p \in (1,\infty]$, the inhomogeneous space $SF^{\alpha}_{p,q}(\mathbb{R}^n)$ is defined by

$$SF^{\alpha}_{p,q}(\mathbb{R}^n) := L^p(\mathbb{R}^n) \cap S\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$$

with its norm $\|f\|_{SF^{\alpha}_{p,q}(\mathbb{R}^n)} := \|f\|_{L^p(\mathbb{R}^n)} + \|f\|_{S\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)}$ for all $f \in SF^{\alpha}_{p,q}(\mathbb{R}^n)$.

In the above definition, $S(\mathbb{R}^n)$ denotes the space of all Schwartz functions and $S'(\mathbb{R}^n)$ its topological dual, namely, the space of all Schwartz distributions. Recall that $f \in L^1_{loc}(\mathbb{R}^n) \cap S'(\mathbb{R}^n)$ means that $L^1_{loc}(\mathbb{R}^n)$ and the natural pair $\langle f,\varphi\rangle$ given by the integral $\int_{\mathbb{R}^n} f(x)\varphi(x)dx$ exists for all $\varphi \in S(\mathbb{R}^n)$ and induces an element of $S'(\mathbb{R}^n)$.

Then the first result reads as follows.

Theorem (4.1.2) [177] Let $\alpha \in (0,2)$ and $p \in (1,\infty]$. Then $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n) = S\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$, with equivalent norms, and also $F^{\alpha}_{p,q}(\mathbb{R}^n) = SF^{\alpha}_{p,q}(\mathbb{R}^n)$, with equivalent norms.

Remark (4.1.3) [177] Notice that to obtain Theorem (4.1.2), it is necessary to make the a priori assumption $f \in L^1_{loc}(\mathbb{R}^n) \cap S'(\mathbb{R}^n)$ in Definition (4.1.1). Indeed, let $f(x_1,x_2) := e^{x_1}\sin x_2$ for $(x_1,x_2) \in \mathbb{R}^2$. Then f is a harmonic function in the plane and hence by the mean value property,

$$\fint_{B(x,2^{-k})} [f(x)-f(y)]dy = f(x) - \fint_{B(x,2^{-k})} f(y)dy = 0$$

for all $x \in \mathbb{R}^2$ and $k \in \mathbb{Z}$. So $f \in L^1_{loc}(\mathbb{R}^2)$ and $S_{\alpha,q}(f) = 0 \in L^p(\mathbb{R}^n)$ for all, α, p, q as in Theorem (4.1.2). However, let $\varphi(x_1,x_2) := e^{-x_1/2}e^{-x_2/2}\sin x_2$. Then $\varphi \in S(\mathbb{R}^2)$ and $\int_{\mathbb{R}^2} f(x)\varphi(x)dx = \infty$, which implies that $f \notin S'(\mathbb{R}^2)$. Since $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^2)$ is a subspace of $S'(\mathbb{R}^2)$ (or $S'(\mathbb{R}^2)$ modulo polynomials), we then conclude that $f \notin \dot{F}^{\alpha}_{p,q}(\mathbb{R}^2)$. In this sense, the assumption $f \in L^1_{loc}(\mathbb{R}^n) \cap S'(\mathbb{R}^n)$ in Definition (4.1.1) is necessary.

In what follows, the space $\dot{W}^{2N,1}_{loc}(\mathbb{R}^n)$ denotes the set of all functions that are locally in the homogeneous Sobolev space $\dot{W}^{2N,1}(\mathbb{R}^n)$. When $\alpha \in (2N, 2N+2)$ with $N \in \mathbb{N} := \{1,2,\dots\}$ and $q \in (0,\infty]$, as motivated by higher order Taylor expansions, for all $f \in \dot{W}^{2N,1}_{loc}(\mathbb{R}^n) \cap S'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we set

$$S_{\alpha,q}(f)(x) := \left\{\sum_{k\in\mathbb{Z}} 2^{k\alpha q} \left| \fint_{B(x,2^{-k})} R_N(y;x,2^{-k})dy \right|^q \right\}^{1/q}, \tag{1}$$

where, for all $x, y \in \mathbb{R}^n$,

$$R_N(y; x, 2^{-k}) := f(y) - f(x) - \sum_{j=1}^{N} \frac{1}{L_j} \Delta^j f(x)|y-x|^{2j} \quad (2)$$

with $L_j := \Delta^j |x|^{2j}$ for $j \in f\{1, \dots, N\}$; see [170] (also [173]) for more details. Similar to Definition (4.1.1), we introduce its following higher-order variant.

Definition (4.1.4) [177] Let $\alpha \in (2N, 2N+2)$ with $N \in \mathbb{N}, q \in (0, \infty]$ and $S_{\alpha,q}(f)$ be as in (1).

(i) If $p \in (0, \infty)$, the space $S\dot{F}^{\alpha}_{\infty,q}(\mathbb{R}^n)$, is defined as the collection of all functions $f \in \dot{W}^{2N,1}_{loc}(\mathbb{R}^n) \cap S'(\mathbb{R}^n)$ such that $\|f\|_{S\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)} := \|S_{\alpha,q}(f)\|_{L^p(\mathbb{R}^n)} < \infty$ with the usual modification made when $q = \infty$.

(ii) The space $S\dot{F}^{\alpha}_{\infty,q}(\mathbb{R}^n)$ is defined as the collection of all functions $f \in \dot{W}^{2N,1}_{loc}(\mathbb{R}^n) \cap S'(\mathbb{R}^n)$ such that

$$\|f\|_{S\dot{F}^{\alpha}_{\infty,q}(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n} \sup_{\ell\in\mathbb{Z}} \left\{ \fint_{B(x,2^{-k})} \sum_{k\geq\ell} 2^{k\alpha q} \left| \fint_{B(x,2^{-k})} R_N(y; x, 2^{-k}) dz \right|^q dy \right\}^{1/q} < \infty$$

with the usual modification made when $q = \infty$.

(iii) If $p \in (1, \infty]$, the inhomogeneous space $SF^{\alpha}_{p,q}(\mathbb{R}^n)$ is defined by

$$SF^{\alpha}_{p,q}(\mathbb{R}^n) := L^p(\mathbb{R}^n) \cap S\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$$

with its norm $\|f\|_{SF^{\alpha}_{p,q}(\mathbb{R}^n)} := \|f\|_{L^p(\mathbb{R}^n)} + \|f\|_{S\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)}$ for all $f \in SF^{\alpha}_{p,q}(\mathbb{R}^n)$.

Recall that, via the square function $S_{\alpha,2}$, Alabern, Mateu and Verdera [170] also characterized the higher order Sobolev space $\dot{W}^{\alpha,p}$ for all $\alpha \in (2N, 2N+2)$ with $N \in \mathbb{N}$ and $p \in (1, \infty)$. We extend this as follows.

Theorem (4.1.5) [177] Let $N \in \mathbb{N}, \alpha \in (2N, 2N+2)$ and $p, q \in (1, \infty]$. Then $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n) = S\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$, with equivalent norms, and also $F^{\alpha}_{p,q}(\mathbb{R}^n) = SF^{\alpha}_{p,q}(\mathbb{R}^n)$, with equivalent norms.

We prove Theorems (4.1.2) and (4.1.5). We extend the above results to Besov spaces and also give some further remarks on the case $\alpha \in 2\mathbb{N}$ and on the higher order Triebel-Lizorkin spaces on metric measure spaces.

Finally, we point out that the proofs of Theorems (4.1.2) and (4.1.5) below are totally different from the method used in [170]. The method in [170] strongly depends on the theory of Fourier transforms and vector-valued singular integrals, while our approach heavily depends on some Calderon reproducing formulae, one of which is from Peetre [91] (see also Frazier and Jawerth [145] and Frazier, Jawerth and Weiss [171], or Lemma (4.1.7) below) and some others are constructed.

Let $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$. Denote by $S(\mathbb{R}^n)$ the space of all Schwartz functions, whose topology is determined by a family of seminorms, $\{\|\cdot\|_{S_{k,m}(\mathbb{R}^n)}\}_{k,m\in\mathbb{Z}_+}$, where, for all $k \in \mathbb{Z}_+, m \in (0, \infty)$ and $\varphi \in S(\mathbb{R}^n)$,

$$\|\varphi\|_{S_{k,m}(\mathbb{R}^n)} := \sup_{\alpha\in\mathbb{Z}^n_+, |\alpha|\leq k} \sup_{x\in\mathbb{R}^n} (1+|x|)^m |\partial^\alpha \varphi(x)|.$$

Here, for any $\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_+, |\alpha| := \alpha_1 + \dots + \alpha_n$ and $\partial^\alpha := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$. It is known that $S(\mathbb{R}^n)$ forms a locally convex topological vector space. Denote by $S'(\mathbb{R}^n)$ the topological dual space of $S(\mathbb{R}^n)$ endowed with the weak topology. In what follows, for every $\varphi \in S(\mathbb{R}^n), t > 0$ and $x \in \mathbb{R}^n$, set $\varphi_t(x) := t^{-n}\varphi(t^{-1}x)$.

For $p \in (0, \infty]$, denote by $L^p(\mathbb{R}^n)$ the Lebesgue space of order p. For $N \in \mathbb{N}$ and $p \in (0, \infty)$, denote by $\dot{W}^{N,p}(\mathbb{R}^n)$ the homogeneous Sobolev space of order N, namely, the collection of all measurable functions f with their distributional derivatives $\partial^\alpha f \in L^p(\mathbb{R}^n)$, where $\alpha \in \mathbb{Z}_+^n$ and $|\alpha| = N$. Moreover, let

$$\|f\|_{\dot{W}^{N,p}(\mathbb{R}^n)} := \sum_{|\alpha|=N} \|\partial^\alpha f\|_{L^p(\mathbb{R}^n)}.$$

Set $\dot{W}^{N,p}(\mathbb{R}^n) := L^p(\mathbb{R}^n) \cap \dot{W}^{N,p}(\mathbb{R}^n)$ as the inhomogeneous Sobolev space with norm

$$\|f\|_{W^{N,p}(\mathbb{R}^n)} := \|f\|_{L^p(\mathbb{R}^n)} + \|f\|_{\dot{W}^{N,p}(\mathbb{R}^n)}$$

for all $f \in W^{N,p}(\mathbb{R}^n)$. Denote by the space $L^1_{loc}(\mathbb{R}^n)$ the locally integrable function and similarly the space $\dot{W}^{N,1}_{loc}(\mathbb{R}^n)$.

Now we recall the notions of Triebel-Lizorkin and Besov spaces; see [65], [66]. In what follows, for any $\varphi \in L^1(\mathbb{R}^n), \hat{\varphi}$ denotes the Fourier transform of φ, namely, for all $\xi \in (\mathbb{R}^n)$,

$$\hat{\varphi}(\xi) := \int_{\mathbb{R}^n} e^{-i\xi\cdot x}\varphi(x)dx.$$

Definition (4.1.6) [177] Let $\alpha \in (0,\infty)$, $p, q \in (0,\infty]$ and $\varphi \in S(\mathbb{R}^n)$ satisfy that

$$\operatorname{supp}\hat{\varphi} \subset \{\xi \in (\mathbb{R}^n) : 1/2 \le |\xi| \le 2\} \text{ and} |\hat{\varphi}(\xi)| \ge \text{constant} > 0 \text{ if } 3/5 \le |\xi| \le 5/3. \quad (3)$$

(i) The homogeneous Triebel-Lizorkin space $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$ is defined as the collection of all $f \in S'(\mathbb{R}^n)$ such that $\|f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)} < \infty$, where, when $p \in (0,\infty)$,

$$\|f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)} := \left\| \left(\sum_{k\in\mathbb{Z}} 2^{k\alpha q} \left|\varphi_{2^{-k}} * f\right|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)},$$

with the usual modification made when $q = \infty$, and

$$\|f\|_{\dot{F}^\alpha_{\infty,q}(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n} \sup_{\ell\in\mathbb{Z}} \left\{ \fint_{B(x,2^{-\ell})} \sum_{k\ge\ell} 2^{k\alpha q} \left|\varphi_{2^{-k}} * (f)\right|^q dy \right\}^{1/q},$$

with the usual modification made when $q = \infty$.

When $p \in (1,\infty]$, the inhomogeneous Triebel-Lizorkin space $F^\alpha_{p,q}(\mathbb{R}^n)$ is defined by

$$F^\alpha_{p,q}(\mathbb{R}^n) := L^p(\mathbb{R}^n) \cap \dot{F}^\alpha_{p,q}(\mathbb{R}^n)$$

with the norm $\|f\|_{F^\alpha_{p,q}(\mathbb{R}^n)} := \|f\|_{L^p(\mathbb{R}^n)} + \|f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)}$ for all $f \in F^\alpha_{p,q}(\mathbb{R}^n)$.

(ii) The homogeneous Besov space $\dot{B}^\alpha_{p,q}(\mathbb{R}^n)$ is defined as the collection of all $f \in S'(\mathbb{R}^n)$ such that $\|f\|_{\dot{B}^\alpha_{p,q}(\mathbb{R}^n)} < \infty$, where

$$\|f\|_{\dot{B}^\alpha_{p,q}(\mathbb{R}^n)} := \left(\sum_{k\in\mathbb{Z}} 2^{k\alpha q} \left\|\varphi_{2^{-k}} * f\right\|^q_{L^p(\mathbb{R}^n)} \right)^{1/q}$$

with the usual modifications made when $p = \infty$ or $q = \infty$.

When $p \in (1,\infty]$, the inhomogeneous Besov space $B^\alpha_{p,q}(\mathbb{R}^n)$ is defined by

$$B^\alpha_{p,q}(\mathbb{R}^n) := L^p(\mathbb{R}^n) \cap \dot{B}^\alpha_{p,q}(\mathbb{R}^n)$$

with the norm $\|f\|_{B^\alpha_{p,q}(\mathbb{R}^n)} := \|f\|_{L^p(\mathbb{R}^n)} + \|f\|_{\dot{B}^\alpha_{p,q}(\mathbb{R}^n)}$ for all $f \in B^\alpha_{p,q}(\mathbb{R}^n)$.

we need the following Calderón reproducing formula established in [91, pp. 52-54] (see also [145, Remark 2.2]).

Lemma (4.1.7) [177] For any $\varphi \in S(\mathbb{R}^n)$ satisfying (3), there exists $\psi \in S(\mathbb{R}^n)$ satisfying (3) such that, for a $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$\sum_{j\in\mathbb{Z}} \hat{\varphi}(2^j\xi)\hat{\psi}(2^j\xi) = 1.$$

Moreover, for ever $f \in S'(\mathbb{R}^n)$, there exist polynomials $\{P_j\}_{j\in\mathbb{Z}}$ and P_f such that

$$f+P_f=\lim_{i\to-\infty}\left\{\sum_{j=i}^{\infty}\varphi_{2^{-j}}*\psi_{2^{-j}}*f+P_i\right\} \tag{4}$$

in $S'(\mathbb{R}^n)$.

Theorem (4.1.8) [177] Let $\alpha\in(0,\infty)/2\mathbb{N}$ and $p,q\in(0,\infty]$. If $f\in\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ then there exists a polynomial P_f such that $f+P_f\in S\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$; moreover, $\|f+P_f\|_{S\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)}\le C\|f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)}$ where C is a positive constant independent of f.

Proof. We first assume that $\alpha\in(0,2)$ and $p,q\in(0,\infty]$. Notice that $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)\subset L^1_{loc}(\mathbb{R}^n)$; see, for example, [90, Proposition 4.2] or [176, Proposition 5.1] for a proof. Let $f\in\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$. Then $f\in S'(\mathbb{R}^n)\cap L^1_{loc}(\mathbb{R}^n)$. Let φ and ψ be as in Lemma (4.1.7). Then (4) holds for f. Observe that $f\in\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ further implies that the degrees of the polynomials $\{P_i\}_{i\in\mathbb{Z}}$ in (3) do not exceed $\lfloor\alpha-n/p\rfloor\le1$; see [115, pp. 153-155] and [145]. Moreover, since P_i has at most degree 1 for each i, we have

$$P_i(x)-⨍_{B(x,2^{-k})}P_i(z)dz=0$$

for all $x\in\mathbb{R}^n$ and $k\in\mathbb{Z}$. Moreover, as shown in [115, pp. 153-155], $f+P_f$ is the canonical representative of f in the sense that if for $i\in\{1,2\}$, $\varphi^{(i)}$ and $\psi^{(i)}$ satisfy (3) and

$$\sum_{k\in\mathbb{Z}}\widehat{\varphi^{(i)}}(2^{-k}\xi)\widehat{\psi^{(i)}}(2^{-k}\xi)=1$$

for all $\xi\in\mathbb{R}^n\setminus\{0\}$, then $P_f^{(1)}-P_f^{(2)}$ is a polynomial of degree not more than $\lfloor\alpha-n/p\rfloor\le1$, where $P_f^{(i)}$ is as in (4) corresponding to $\{\varphi^{(i)},\psi^{(i)}\}$ for $i\in\{1,2\}$. Also notice that for all $x\in\mathbb{R}^n$ and $k\in\mathbb{Z}$,

$$P_f^{(1)}(x)-P_f^{(2)}(x)-⨍_{B(x,2^{-k})}\left[P_f^{(1)}(z)-P_f^{(2)}(z)\right]dz=0.$$

Let $\tilde{f}:=f+P_f$. Then by (4), we have

$$\tilde{f}-\tilde{f}_{B(\cdot,2^{-k})}=\sum_{j\in\mathbb{Z}}\left(\varphi_{2^{-j}}-\chi_k*\varphi_{2^{-j}}\right)*\psi_{2^{-j}}*f \tag{5}$$

in $S'(\mathbb{R}^n)$. Here $\chi:=\frac{\chi_{B(0,1)}}{|B(0,1)|}$ and $\chi_k:=2^{kn}\chi(2^k\,\cdot)$. From the above discussion, it follows that $\tilde{f}-\tilde{f}_{B(\cdot,2^{-k})}$ is independent of the choices of φ and ψ satisfying (3). Then it suffices to prove that, when $p\in(1,\infty)$ and $q\in(1,\infty]$,

$$\left\{\int_{\mathbb{R}^n}\left[\sum_{k\in\mathbb{Z}}2^{k\alpha q}\left(\sum_{j\in\mathbb{Z}}\left|\left(\varphi_{2^{-j}}-\chi_k*\varphi_{2^{-j}}\right)*\psi_{2^{-j}}*f(x)\right|\right)^q\right]^{\frac{p}{q}}dx\right\}^{\frac{1}{p}}\lesssim\|f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)} \tag{6}$$

and that, when $p=\infty$ and $q\in(1,\infty]$, for all $x\in\mathbb{R}^n$ and $\ell\in\mathbb{Z}$,

$$\left\{⨍_{B(x,2^{-\ell})}\sum_{k\ge\ell}2^{k\alpha q}\left(\sum_{j\in\mathbb{Z}}\left|\left(\varphi_{2^{-j}}-\chi_k*\varphi_{2^{-j}}\right)*\psi_{2^{-j}}*f(y)\right|\right)^q dy\right\}^{\frac{1}{q}}\lesssim\|f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)} \tag{7}$$

Indeed, if (6) holds, then for each $k\in\mathbb{Z}$, we have

$$\int_{\mathbb{R}^n}\left[\sum_{j\in\mathbb{Z}}\left|\left(\varphi_{2^{-j}}-\chi_k*\varphi_{2^{-j}}\right)*\psi_{2^{-j}}*f(x)\right|\right]^p dx<\infty,$$

which further implies that (5) holds in $L^p(\mathbb{R}^n)$ and hence almost everywhere. Therefore, for every $k\in\mathbb{Z}$,

$$\left|\tilde{f}-\tilde{f}_{B(\cdot,2^{-k})}\right|\le\sum_{j\in\mathbb{Z}}\left|\left(\varphi_{2^{-j}}-\chi_k*\varphi_{2^{-j}}\right)*\psi_{2^{-j}}*f\right|$$

almost everywhere, and hence $\|\tilde{f}\|_{S\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)}$ is less than the left hand side of (6), which further implies that kef $\|\tilde{f}\|_{S\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)}\lesssim\|f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)}$ Similarly, if (7) holds, then (5) holds in $L^1_{loc}(\mathbb{R}^n)$ and hence almost everywhere and, moreover, an argument similar to above leads to $\|\tilde{f}\|_{S\dot{F}^{\alpha}_{\infty,q}(\mathbb{R}^n)}\lesssim\|f\|_{\dot{F}^{\alpha}_{\infty,q}(\mathbb{R}^n)}$.

To prove (6), we consider $\sum_{j\le k}$ and $\sum_{j>k}$ separately. Notice that for any smooth function ϕ on $\mathbb{R}$,

$$\phi(1)=\phi(0)+\int_0^1\phi'(s)ds=\phi(0)+\phi'(0)+\int_0^1(1-s)\phi''(s)ds. \tag{8}$$

Let $\phi(s):=\varphi(2^jx+sz)$ for $s\in[0,1]$ and $x,z\in\mathbb{R}^n$. Then

$$\varphi\left(2^jx+z\right)=\varphi\left(2^jx\right)+(\nabla\varphi)\left(2^jx\right)z^t+\int_0^1(1-s)z(\nabla^2\varphi)\varphi\left(2^jx+sz\right)z^t ds,$$

where z^t denotes the transpose of z. Therefore, when $j\le k$, for all $x\in\mathbb{R}^n$,

$$\left|\chi_k*\varphi_{2^{-j}}(x)-\varphi_{2^{-j}}(x)\right|=\left|\fint_{B(0,1)}2^{jn}\left[\varphi\left(2^jx+2^{j-k}z\right)-\varphi\left(2^jx\right)\right]dz\right|$$

$$=\left|\fint_{B(0,2^{j-k})}2^{jn}\left[\varphi\left(2^jx+z\right)-\varphi\left(2^jx\right)\right]dz\right|$$

$$=\left|\fint_{B(0,2^{j-k})}2^{jn}\int_0^1(1-s)z(\nabla^2\varphi)\varphi\left(2^jx+sz\right)z^t ds\,dz\right|\lesssim 2^{2(j-k)}\frac{2^{jn}}{(1+|2^jx|)^L}, \tag{9}$$

where $L>n$. Hence

$$\left|\left(\chi_k*\varphi_{2^{-j}}-\varphi_{2^{-j}}\right)*\psi_{2^{-j}}*f(x)\right|\lesssim 2^{2(j-k)}\int_{\mathbb{R}^n}\frac{2^{jn}}{(1+|2^jy|)^L}\left|\psi_{2^{-j}}*f(x-y)\right|dy$$
$$\lesssim 2^{2(j-k)}M\left(\left|\psi_{2^{-j}}*f\right|\right)(x),$$

where M denotes the Hardy-Littlewood maximal function. Then, choosing $\delta\in(0,2-\alpha)$, by Hölder's inequality and $\alpha\in(0,2)$, we see that

$$\mathrm{I}_1:=\left\{\int_{\mathbb{R}^n}\left[\sum_{k\in\mathbb{Z}}2^{k\alpha q}\left(\sum_{j\le k}\left|\left(\varphi_{2^{-j}}-\chi_k*\varphi_{2^{-j}}\right)*\psi_{2^{-j}}*f(x)\right|\right)^q\right]^{p/q}dx\right\}^{1/p}$$

$$\lesssim\left\{\int_{\mathbb{R}^n}\left[\sum_{k\in\mathbb{Z}}2^{k\alpha q}\sum_{j\le k}2^{(2-\delta)(j-k)}\left[M\left(\left|\psi_{2^{-j}}*f\right|\right)(x)\right]^q\right]^{p/q}dx\right\}^{1/p}$$

$$\lesssim\left\{\int_{\mathbb{R}^n}\left[\sum_{j\in\mathbb{Z}}2^{j\alpha q}\left[M\left(\left|\psi_{2^{-j}}*f\right|\right)(x)\right]^q\right]^{p/q}dx\right\}^{1/p},$$

which, together with the Fefferman-Stein vector-valued maximal inequality (see [112]), further implies that

$$\mathrm{I}_1 \lesssim \left\{\int_{\mathbb{R}^n}\left[\sum_{j\in\mathbb{Z}} 2^{j\alpha q}\left|\psi_{2^{-j}} * f(x)\right|^q\right]^{p/q} dx\right\}^{1/p} \lesssim \|f\|_{\dot F^{\alpha}_{p,q}(\mathbb{R}^n)}.$$

Notice that when $j > k$, for all $x \in \mathbb{R}^n$, we always have

$$\begin{aligned}\left|\left(\chi_k * \varphi_{2^{-j}} - \varphi_{2^{-j}}\right) * \psi_{2^{-j}} * f(x)\right| &\le \left|\chi_k * \varphi_{2^{-j}} - \varphi_{2^{-j}} * f(x)\right| + \left|\varphi_{2^{-j}} * \psi_{2^{-j}} * f(x)\right| \\ &\le \chi_k * \left[M\left(\left|\psi_{2^{-j}} * f\right|\right)\right](x) + M\left(\left|\psi_{2^{-j}} * f\right|\right)(x)| \\ &\lesssim M \circ M\left(\left|\psi_{2^{-j}} * f\right|\right)(x)\end{aligned}$$

where $M \circ M$ denotes the composition of M and M. Then by $\alpha > 0$, taking $\delta \in (0, \alpha\ )$ and applying Hölder's inequality, we obtain

$$\begin{aligned}\mathrm{I}_2 &:= \left\{\int_{\mathbb{R}^n}\left[\sum_{k\in\mathbb{Z}} 2^{k\alpha q}\left(\sum_{j>k}\left|\left(\varphi_{2^{-j}} - \chi_k * \varphi_{2^{-j}}\right) * \psi_{2^{-j}} * f(x)\right|\right)^q\right]^{p/q} dx\right\}^{1/p} \\ &\lesssim \left\{\int_{\mathbb{R}^n}\left[\sum_{k\in\mathbb{Z}} 2^{k(\alpha-\delta)q}\sum_{j>k} 2^{j\delta q}\left[M \circ M\left(\left|\psi_{2^{-j}} * f\right|\right)(x)\right]^q\right]^{p/q} dx\right\}^{1/p} \\ &\lesssim \left\{\int_{\mathbb{R}^n}\left[\sum_{j\in\mathbb{Z}} 2^{j\alpha q}\left[M \circ M\left(\left|\psi_{2^{-j}} * f\right|\right)(x)\right]^q\right]^{p/q} dx\right\}^{1/p},\end{aligned}$$

which, together with the Fefferman-Stein vector-valued maximal inequality, further implies that $\mathrm{I}_2 \lesssim \ \|f\|_{\dot F^{\alpha}_{p,q}(\mathbb{R}^n)}$. This proves (6).

To prove (7), we consider $\sum_{j<\ell\le k}$,$\sum_{\ell\le j\le k}$ and $\sum_{j>k\ge\ell}$ separately. If $j \le \ell \le k$, from (9) and Hölder's inequality, we deduce that for all $y \in \mathbb{R}^n$,

$$\begin{aligned}\left|\left(\chi_k * \varphi_{2^{-j}} - \varphi_{2^{-j}}\right) * \psi_{2^{-j}} * f(y)\right| &\lesssim 2^{2(j-k)}\int_{\mathbb{R}^n}\frac{2^{jn}}{(1+|2^j z|)^L}\left|\psi_{2^{-j}} * f(y-z)\right| dz \\ &\lesssim 2^{2(j-k)}\sum_{i=0}^{\infty} 2^{i(n-L)}\fint_{B\left(y,2^{i-j}\right)}\left|\psi_{2^{-j}} - f(z)\right|^q dz \\ &\lesssim 2^{2(j-k)}\sum_{i=0}^{\infty} 2^{i(n-L)}\left\{\fint_{B\left(y,2^{i-j}\right)}\left|\psi_{2^{-j}} - f(z)\right|^q dz\right\}^{1/q} \\ &\lesssim 2^{2(j-k)}\sum_{i=0}^{\infty} 2^{i(n-L)}2^{-j\alpha}\|f\|_{\dot F^{\alpha}_{\infty,q}(\mathbb{R}^n)} \lesssim 2^{2(j-k)}2^{-j\alpha}\|f\|_{\dot F^{\alpha}_{\infty,q}(\mathbb{R}^n)},\end{aligned}$$

where we used the following trivial estimate that

$$\left\{\fint_{B\left(y,2^{i-j}\right)}\left|\psi_{2^{-j}} - f(z)\right|^q dz\right\}^{1/q} \lesssim 2^{-j\alpha}\|f\|_{\dot F^{\alpha}_{\infty,q}(\mathbb{R}^n)}.$$

Hence

$$\left\{\fint_{B\left(x,2^{-\ell}\right)}\sum_{k\ge\ell} 2^{k\alpha q}\left(\sum_{j\le\ell}\left|\left(\varphi_{2^{-j}} - \chi_k * \varphi_{2^{-j}}\right) * \psi_{2^{-j}} * f(y)\right|\right)^q dy\right\}^{1/q}$$

$$\lesssim \left\{ \fint_{B(x,2^{-\ell})} \sum_{k\geq\ell} 2^{k\alpha q} \left(\sum_{j\leq\ell} 2^{2(j-k)} 2^{-j\alpha} \right)^q dy \right\}^{1/q} \|f\|_{\dot{F}^{\alpha}_{\infty,q}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{F}^{\alpha}_{\infty,q}(\mathbb{R}^n)}.$$

If $\ell \leq j \leq k$, then, for all $y \in \mathbb{R}^n$,

$$\begin{aligned}
&\left|\left(\chi_k * \varphi_{2^{-j}} - \varphi_{2^{-j}}\right) * \psi_{2^{-j}} * f(y)\right| \\
&\quad \lesssim 2^{2(j-k)} M\left(\left|\psi_{2^{-j}} * f\right| \chi_{B(x,2^{-\ell})}\right)(y) + \sum_{i\geq j-\ell} \fint_{B(y,2^{i-j})} \left|\psi_{2^{-j}} * f(z)\right| dz \\
&\quad \lesssim 2^{2(j-k)} M\left(\left|\psi_{2^{-j}} * f\right| \chi_{B(x,2^{-\ell})}\right)(y) + 2^{2(j-k)} 2^{-j\alpha} 2^{2(j-k)} 2^{(j-\ell)(n-L)} \|f\|_{\dot{F}^{\alpha}_{\infty,q}(\mathbb{R}^n)}
\end{aligned}$$

and hence, by Hölder's inequality,

$$\begin{aligned}
&\left\{ \fint_{B(x,2^{-\ell})} \sum_{k\geq\ell} 2^{k\alpha q} \left[\sum_{\ell<j\leq k} \left|\left(\varphi_{2^{-j}} - \chi_k * \varphi_{2^{-j}}\right) * \psi_{2^{-j}} * f(y)\right| \right]^q dy \right\}^{1/q} \\
&\quad \lesssim \left\{ \fint_{B(x,2^{-\ell})} \sum_{k\geq\ell} 2^{k\alpha q} \left[\sum_{\ell<j\leq k} 2^{2(j-k)} 2^{-j\alpha} M\left(\left|\psi_{2^{-j}} * f\right| \chi_{B(x,2^{-\ell})}\right)(y) \right]^q dy \right\}^{1/q} \\
&\qquad + \left\{ \fint_{B(x,2^{-\ell})} \sum_{k\geq\ell} 2^{k\alpha q} \left[\sum_{\ell<j\leq k} 2^{2(j-k)} 2^{(n-L)(j-\ell)} 2^{-j\alpha} \right]^q dy \right\}^{1/q} \|f\|_{\dot{F}^{\alpha}_{\infty,q}(\mathbb{R}^n)} \\
&\quad \lesssim \left\{ \fint_{B(x,2^{-\ell})} \sum_{k\geq\ell} 2^{k\alpha q} \left[\sum_{\ell<j\leq k} 2^{2(j-k)} \left|\psi_{2^{-j}} * f(y)\right| \right]^q dy \right\} + \|f\|_{\dot{F}^{\alpha}_{\infty,q}(\mathbb{R}^n)} \\
&\quad \lesssim \|f\|_{\dot{F}^{\alpha}_{\infty,q}(\mathbb{R}^n)}.
\end{aligned}$$

Similarly, if $j > k \geq \ell$, then we have

$$\begin{aligned}
&\left|\left(\chi_k * \varphi_{2^{-j}} - \varphi_{2^{-j}}\right) * \psi_{2^{-j}} * f(y)\right| \\
&\quad \lesssim M\left(\chi_{B(x,2^{-\ell})} M\left(\left|\psi_{2^{-j}} * f\right| \chi_{B(x,2^{-\ell})}\right)\right)(y) + 2^{-j\alpha} 2^{(j-\ell)(n-L)} \|f\|_{\dot{F}^{\alpha}_{\infty,q}(\mathbb{R}^n)},
\end{aligned}$$

which further implies that

$$\left\{ \fint_{B(x,2^{-\ell})} \sum_{k\geq\ell} 2^{k\alpha q} \left(\sum_{j>k} \left|\left(\varphi_{2^{-j}} - \chi_k * \varphi_{2^{-j}}\right) * \psi_{2^{-j}} * f(y)\right| \right)^q dy \right\}^{1/q} \lesssim \|f\|_{\dot{F}^{\alpha}_{\infty,q}(\mathbb{R}^n)}.$$

This proves (7).

Now we consider the case $\alpha \in (2N, 2N+2)$ with $N \in \mathbb{N}$. Since the idea of the proof is similar to the case $\alpha \in (0,2)$, we only sketch the main steps. First we observe that $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n) \subset \dot{W}^{2N,1}_{loc}(\mathbb{R}^n)$, which follows from the lifting properties of Triebel-Lizorkin spaces (see [13]) and the fact that $\dot{F}^{\alpha-2N}_{\infty,q}(\mathbb{R}^n) \subset L^1_{loc}(\mathbb{R}^n)$ mentioned above. Moreover, similar to the above, $f \in \dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ implies that the degrees of the polynomials $\{P_i\}_{i\in\mathbb{Z}}$ in (3) do not exceed $\lfloor \alpha - n/p \rfloor \leq 2N+1$, and also that the polynomial P_f is unique modulo a polynomial with degree no more than $\lfloor \alpha - n/p \rfloor \leq 2N+1$; see [115, pp. 153-155] and [145]. In what follows, we set $\tilde{f} := f + P_f$ and let $\tilde{R}_N(y;\, x, 2^{-k})$ be defined as in (3) with f replaced by $\tilde{f}$.

Notice that for $i \in N, \fint_{B(0,1)} |y|^{2i} dy = \frac{n}{n+2i}$. Then from (4), it follows that for all $x \in \mathbb{R}^n$,

$$\fint_{B(x,2^{-k})} \tilde{R}_N(y;\ x, 2^{-k}) dy = \chi_k * \tilde{f}(x) - \sum_{i=0}^{N} 2^{-2ik} \frac{1}{L_i} \frac{n}{n+2i} \Delta^i \tilde{f}(x)$$
$$= \sum_{j \in \mathbb{Z}} \left[\chi_k * \varphi_{2^{-j}} - \sum_{i=0}^{N} 2^{-2ik} \frac{1}{L_i} \frac{n}{n+2i} \Delta^i \varphi_{2^{-j}} \right] * \psi_{2^{-j}} * f(x).$$

We now consider $\sum_{j \le k}$ and $\sum_{j>k}$ separately.

By an argument similar to (8), we see that, for any smooth function ϕ on $\mathbb{R}$ and $N \in \mathbb{N}$,

$$\phi(1) = \phi(0) + \sum_{i=1}^{2N+1} \frac{\phi^{(i)}(0)}{i!} + \frac{1}{(2N+1)!} \int_0^1 (1-s)^{2N+1} \phi^{(2N+2)}(s) ds. \qquad (10)$$

As above, choosing $\phi(s) := \varphi(2^j x + sz)$ for $s \in [0,1]$ and $x, z \in \mathbb{R}^n$, we have

$$\varphi\left(2^j x + z\right) = \varphi\left(2^j x\right) + (\nabla\varphi)\left(2^j x\right) z^t + \frac{1}{2!} z (\nabla^2 \varphi)\left(2^j x\right) z^t + \cdots$$
$$+ \frac{1}{(2N+1)!} \int_0^1 (1-s)^{2N+1} \mathrm{z}^{N+1} (\nabla^{2N+2} \varphi)\left(2^j x + sz\right) (z^t)^{N+1} ds.$$

Notice that in this expansion, except the terms $\frac{1}{L_i} \Delta^i \varphi\left(2^j x\right) \left|2^{j-k} z\right|^{2i}$ for $0 \le i \le N$ and the last term, the other terms are harmonic and then have average 0 on any ball centered at 0. When $j \le k$, applying these facts, we conclude that, for all $x \in \mathbb{R}^n$,

$$\left| \chi_k * \varphi_{2^{-j}}(x) - \sum_{i=0}^{N} 2^{-2ik} \frac{1}{L_i} \frac{n}{n+2i} \Delta^i \varphi_{2^{-j}}(x) \right|$$
$$= \left| \fint_{B(0,1)} 2^{jn} \left[\varphi\left(2^j x + 2^{j-k} z\right) - \sum_{i=0}^{N} \frac{1}{L_i} \Delta^i \varphi\left(2^j x\right) \left|2^{j-k} z\right|^{2i} \right] dz \right|$$
$$= \left| \fint_{B\left(0, 2^{j-k}\right)} 2^{jn} \int_0^1 (1-s)^{2N+1} \mathrm{z}^{N+1} (\nabla^{2N+2} \varphi)\left(2^j x + sz\right) (z^t)^{N+1} ds dz \right|$$
$$\lesssim 2^{2(N+1)(j-k)} \frac{2^{jn}}{(1 + |2^j x|)^L},$$

where $L \in \mathbb{N}$ is larger than n. Here the decay factor $2^{2(N+1)(j-k)}$ is crucial. Indeed, when $j \le k$, we see that

$$\left| \left(\chi_k * \varphi_{2^{-j}}(x) - \sum_{i=0}^{N} 2^{-2ik} \frac{1}{L_i} \frac{n}{n+2i} \Delta^i \varphi_{2^{-j}}(x) \right) * \psi_{2^{-j}} * f(x) \right|$$
$$\lesssim 2^{2(N+1)(j-k)} M\left(\left| \psi_{2^{-j}} * f \right| \right)(x),$$

while, when $j > k$, we also see that

$$\left| \left(\chi_k * \varphi_{2^{-j}}(x) - \sum_{i=0}^{N} 2^{-2ik} \frac{1}{L_i} \frac{n}{n+2i} \Delta^i \varphi_{2^{-j}}(x) \right) * \psi_{2^{-j}} * f(x) \right|$$
$$\lesssim 2^{2N(j-k)} M \circ M\left(\left| \psi_{2^{-j}} * f \right| \right)(x).$$

Since $2N < \alpha < 2(N+1)$, for all $p \in (1, \infty)$ and $q \in (1, \infty]$, by exactly the same procedure as above, we conclude that

$$\|S_{\alpha,q}(\tilde{f})\|_{L^p(\mathbb{R}^n)} \lesssim \left\{\int_{\mathbb{R}^n}\left(\sum_{k\in\mathbb{Z}} 2^{k\alpha q}\left[\sum_{j\le k} 2^{2(N+1)(j-k)} M(|\psi_{2^{-j}} * f|)(x)\right]^q\right)^{p/q} dx\right\}^{1/p}$$

$$+\left\{\int_{\mathbb{R}^n}\left(\sum_{k\in\mathbb{Z}} 2^{k\alpha q}\left[\sum_{j>k} 2^{2(N+1)(j-k)} M(|\psi_{2^{-j}} * f|)(x)\right]^q\right)^{p/q} dx\right\}^{1/p} \lesssim \|f\|_{\dot{F}^{\alpha}_{\infty,q}(\mathbb{R}^n)}.$$

When $2N < \alpha < 2(N+1)$, for all $p = \infty$ and $q \in (1,\infty]$, from a much more complicated argument, similar to the case $0 < \alpha < 2, p = \infty$ and $q \in (1,\infty]$, we also deduce that $\|\tilde{f}\|_{S\dot{F}^{\alpha}_{\infty,q}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{F}^{\alpha}_{\infty,q}(\mathbb{R}^n)}$. We omit the details. This finishes the proof of Theorem (4.1.8).

Lemma (4.1.9) [177] Let $\chi := \frac{\chi_{B(0,1)}}{|B(0,1)|}$, $L \in \mathbb{Z}_+ \cup \{-1\}$ and $N \in \mathbb{N}$.

(i) There exist $\phi, \psi \in S(\mathbb{R}^n)$ satisfying that $\operatorname{supp}\phi \subset B(0,1)$, $\int_{\mathbb{R}^n}\phi(x)x^\gamma dx = 0$ for all $|\gamma| < L$ and $\operatorname{supp}\hat{\psi} \subset \{\xi \in (\mathbb{R}^n): 1/64 \le |\xi| \le 1/16\}$ such that, for all $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$\sum_{j\in\mathbb{Z}} \hat{\psi}_{2^{-j}}(\xi)\hat{\phi}_{2^{-j}}(\xi)\left[\hat{\chi}_{2^{-j}}(\xi) - \hat{\chi}_{2^{1-j}}(\xi)\right] = 1. \quad (11)$$

Moreover, for every $f \in L^1_{loc}(\mathbb{R}^n) \cap S'(\mathbb{R}^n)$. there exist polynomials $\{P_j\}_{j\in\mathbb{Z}}$ and P_f such that

$$f + P_f = \lim_{i\to-\infty}\left\{\sum_{j=i}^{\infty} \phi_{2^{-j}} * \psi_{2^{-j}} * \left(f_{B(\cdot,2^{-j})} - f_{B(\cdot,2^{1-j})}\right) + P_i\right\} \quad (12)$$

in $S'(\mathbb{R}^n)$.

(ii) There exist $\phi, \psi \in S(\mathbb{R}^n)$ satisfying the same conditions as in (i) such that, for all $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$\sum_{j\in\mathbb{Z}} \hat{\psi}_{2^{-j}}(\xi)\hat{\phi}_{2^{-j}}(\xi)\left\{\left[\hat{\chi}_{2^{-j}}(\xi) - \sum_{i=1}^{N} 2^{-2ij}\frac{1}{L_i}\frac{n}{n+2i}|\xi|^{2i}\right]\right.$$
$$\left. - \left[\hat{\chi}_{2^{1-j}}(\xi) - \sum_{i=1}^{N} 2^{-2i(j-1)}\frac{1}{L_i}\frac{n}{n+2i}|\xi|^{2i}\right]\right\} = 1. \quad (13)$$

Moreover, for every $f \in \dot{W}^{2N,1}_{loc}(\mathbb{R}^n) \cap S'(\mathbb{R}^n)$, there exist polynomials $\{P_j\}_{j\in\mathbb{Z}}$ and P_f such that

$$f + P_f = \lim_{m\to-\infty}\left\{\sum_{j=m}^{\infty} \phi_{2^{-j}} * \psi_{2^{-j}}\right.$$
$$\left. * \left[\left(f_{B(\cdot,2^{-j})} - \sum_{i=1}^{N} 2^{-2ij}\frac{1}{L_i}\frac{n}{n+2i}\Delta^i f\right) - \left(f_{B(\cdot,2^{1-j})} - \sum_{i=1}^{N} 2^{-2i(j-1)}\frac{1}{L_i}\frac{n}{n+2i}\Delta^i f\right)\right] + P_m\right\} \quad (14)$$

in $S'(\mathbb{R}^n)$.

Proof: (i) It suffices to show (11). The proof of (11) follows from (12) and an argument similar to the arguments in [91, pp. 52-54].

First we show that there exists a positive constant C_0 such that for all $1/64 \le |\xi| \le 1/16$,

$$|\hat{\chi}(\xi) - \hat{\chi}_2(\xi)| \ge C_0 > 0. \quad (15)$$

By [172, p. 429], we know that $\hat{\chi}_{B(0,1)}(\xi) = J_{n/2}(2\pi\xi)/|\xi|^{n/2}$, where $J_{n/2}$ is the Bessel function of order $n/2$. Thus,

$$\hat{\chi}(\xi) = \frac{1}{|B(0,1)|}\frac{J_{n/2}(2\pi\xi)}{|\xi|^{n/2}} \text{ and } \hat{\chi}_2(\xi) = \frac{1}{|B(0,1)|}\frac{J_{n/2}(4\pi\xi)}{|2\xi|^{n/2}}$$

Therefore,

$$\hat{\chi}(\xi) - \hat{\chi}_2(\xi) = \frac{\pi^{n/2}}{|B(0,1)|\Gamma(n/2+1/2)\Gamma(1/2)} \times \left\{\int_{-1}^{1}\left[e^{2\pi i|\xi|s} - e^{4\pi i|\xi|s}\right](1-s^2)^{n/2-1/2}\,ds\right\}.$$

Notice that if $1/64 \le |\xi| \le 1/16$ and $s \in [-1,1]$, then $4\pi|\xi|s \in [-\pi/4, \pi/4]$ and hence $\cos(2\pi|\xi|s) \ge \cos(4\pi|\xi|s)$. Then we conclude that

$$|\hat{\chi}(\xi) - \hat{\chi}_2(\xi)| \ge \frac{\pi^{n/2}}{|B(0,1)|\Gamma(n/2+1/2)\Gamma(1/2)} \times \left\{\int_{-1}^{1}[\cos(2\pi|\xi|s) - \cos(4\pi|\xi|s)](1-s^2)^{n/2-1/2}\,ds\right\}.$$

By the fact that $1/64 \le |\xi| \le 1/16$ and $s \in [-1,1]$ again, we see that $\pi^2|\xi|^2 s^2 \ge 10\pi^4|\xi|^4 s^4$. Thus, by the Taylor expansion of the cosine function, we know that

$$\cos(2\pi|\xi|s) - \cos(4\pi|\xi|s) \ge 5\pi^2|\xi|^2 s^2$$

and hence

$$|\hat{\chi}(\xi) - \hat{\chi}_2(\xi)| \ge \frac{5\pi^2|\xi|^2 s^2}{|B(0,1)|\Gamma(n/2+1/2)\Gamma(1/2)}\left\{\int_{-1}^{1} s^2(1-s^2)^{n/2-1/2}\,ds\right\}.$$

From the properties of Gamma functions (see [172, Appendix A]), it follows that

$$\begin{aligned}
&\frac{\pi^{n/2}}{|B(0,1)|\Gamma(n/2+1/2)\Gamma(1/2)}\left\{\int_{-1}^{1}(1-s^2)^{n/2-1/2}\,ds\right\} \\
&\quad = \frac{n\Gamma(n/2)}{2\Gamma(n/2+1/2)\Gamma(1/2)}\left\{\int_{0}^{1}(1-t)^{n/2-1/2}t^{-1/2}\,dt\right\} \\
&\quad = \frac{n\Gamma(n/2)}{2\Gamma(n/2+1/2)\Gamma(1/2)}\frac{\Gamma(n/2+1/2)\Gamma(1/2)}{\Gamma(n/2+1)} = 1.
\end{aligned}$$

Thus,

$$\begin{aligned}
|\hat{\chi}(\xi) - \hat{\chi}_2(\xi)| &\ge 5\pi^2|\xi|^2\frac{\int_{-1}^{1} s^2(1-s^2)^{n/2-1/2}\,ds}{\int_{-1}^{1}(1-s^2)^{n/2-1/2}\,ds} = 5\pi^2|\xi|^2\frac{\int_{0}^{1} t^{1/2}(1-t)^{n/2-1/2}\,dt}{\int_{0}^{1} t^{-1/2}(1-t)^{n/2-1/2}\,dt} \\
&= 5\pi^2|\xi|^2\frac{\Gamma(3/2)\Gamma(n/2+1/2)\Gamma(n/2+1)}{\Gamma(1/2)\Gamma(n/2+1/2)\Gamma(n/2+2)} = \frac{5\pi^2|\xi|^2}{n+2}. \qquad (16)
\end{aligned}$$

Therefore, for all $1/64 \le |\xi| \le 1/16$, we have

$$|\hat{\chi}(\xi) - \hat{\chi}_2(\xi)| \ge 2^{-12}\frac{5\pi^2}{n+2} > 0,$$

namely, (15) holds.

For any fixed $L \in \mathbb{Z}_+ \cup \{-1\}$, select a smooth function ϕ on $\mathbb{R}^n$ such that $\operatorname{supp}\phi \subset B(0,1)$, $\int_{\mathbb{R}^n}\phi(x)x^\gamma dx = 0$ for all $|\gamma| < L$, and $|\phi(\xi)| \ge C > 0$ for all $1/64 \le |\xi| \le 1/16$, where C is a positive constant. Then $C_c^\infty(\mathbb{R}^n)$, has vanishing moments till order L and satisfies that

$$\left|\hat{\phi}(\xi)[\hat{\chi}(\xi) - \hat{\chi}_2(\xi)]\right| \ge C > 0 \qquad (17)$$

for all $1/64 \le |\xi| \le 1/16$.

Let $g \in S(\mathbb{R}^n)$ such that $\hat{g}$ is nonnegative, $\operatorname{supp}\hat{g} \subset \{\xi \in \mathbb{R}^n : 1/64 \le |\xi| \le 1/16\}$ and $\hat{g}(\xi) \ge C > 0$ if $3/128 \le |\xi| \le 7/128$, where C is a positive constant. Let

$$F := \sum_{j\in\mathbb{Z}} \hat{g}(2^{-j}\cdot).$$

Then F is a bounded smooth function satisfying that $F(\xi) \geq C > 0$ for all $\xi \neq 0$ and $F(2^{-j}\cdot) \equiv F$.

Now define $h := \hat{g}/F$. Then $h \in S(\mathbb{R}^n)$, $\operatorname{supp} h \subset \{\xi \in \mathbb{R}^n : 1/64 \leq |\xi| \leq 1/16\}$, $h(\xi) \geq C > 0$ for all $3/128 \leq |\xi| \leq 7/128$, and $\sum_{j\in\mathbb{Z}} h(2^{-j}\xi)$ for all $\xi \neq 0$. By (17), we define a Schwartz function by setting $\hat{\psi} := h\{\hat{\phi}[\hat{\chi} - \hat{\chi}_2]\}^{-1}$. Then

$$\operatorname{supp} \hat{\psi} \subset \{\xi \in \mathbb{R}^n : 1/64 \leq |\xi| \leq 1/16\}$$

and, for all $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$\sum_{j\in\mathbb{Z}} \hat{\psi}_{2^{-j}}(\xi)\hat{\phi}_{2^{-j}}(\xi)\left[\hat{\chi}_{2^{-j}}(\xi) - \hat{\chi}_{2^{1-j}}(\xi)\right] = \sum_{j\in\mathbb{Z}} \hat{h}(2^{-j}\xi) = 1,$$

which completes the proof of (i).

(ii) Similar to the argument in (i), it suffices to show that there exists a positive constant C_0 such that, for all $1/64 \leq |\xi| \leq 1/16$,

$$\left|\left[\hat{\chi}(\xi) - \sum_{i=1}^{N} \frac{1}{L_i}\frac{n}{n+2i}|\xi|^{2i}\right] - \left[\hat{\chi}_2(\xi) - \sum_{i=1}^{N} 2^{2i}\frac{1}{L_i}\frac{n}{n+2i}|\xi|^{2i}\right]\right| \geq C_0 > 0. \quad (18)$$

From (16), we deduce that, for all $1/64 \leq |\xi| \leq 1/16$,

$$|\hat{\chi}(\xi) - \hat{\chi}_2(\xi)| \geq \frac{5\pi^2|\xi|^2}{n+2},$$

while

$$\left|\sum_{i=1}^{N} 2^{2i}\frac{1}{L_i}\frac{n}{n+2i}|\xi|^{2i} - \sum_{i=1}^{N} \frac{1}{L_i}\frac{n}{n+2i}|\xi|^{2i}\right|$$

$$\leq |\xi|^2 \sum_{i=1}^{N} \frac{(2^{2i}-1)n}{n+2i} \frac{2^{-4(2i-2)}}{\sum_{m_1+\cdots+m_n=i}(2m_1)! + \cdots + (2m_n)!} \leq 4|\xi|^2/(n+2).$$

Thus, (18) holds in this case, which completes the proof of (ii) and hence Lemma (4.1.9).

Theorem (4.1.10) [177] Let $\alpha \in (0,\infty) \setminus 2\mathbb{N}$ and $p, q \in (0,\infty]$. If $f \in S\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$, then $f \in \dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ and there exists a positive constant C, independent of f, such that

$$\|f\|_{\dot{F}^{\alpha}_{\infty,q}(\mathbb{R}^n)} \leq C\|f\|_{S\dot{F}^{\alpha}_{\infty,q}(\mathbb{R}^n)}.$$

Proof. We first consider the case $\alpha \in (0,2)$. Let $f \in S\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$. By Lemma (4.1.9) (i) and $f \in L^1_{loc}(\mathbb{R}^n) \cap S'(\mathbb{R}^n)$, we conclude that

$$f = \sum_{k\in\mathbb{Z}} \phi_{2^{-k}} * \psi_{2^{-k}} * (\chi_{2^{-k}} - \chi_{2^{-k+1}}) * f = \sum_{k\in\mathbb{Z}} \phi_{2^{-k}} * \psi_{2^{-k}} * \left(f_{B(\cdot,2^{-k})} - f_{B(\cdot,2^{1-k})}\right),$$

which, modulo polynomials, holds in $S'(\mathbb{R}^n)$. Here ϕ and ψ are as in Lemma (4.1.9) (i). Let φ be as in (3). For $k \in \mathbb{Z}$, we have

$$\varphi_{2^{-k}} * f = \sum_{j\in\mathbb{Z}} \varphi_{2^{-k}} * \phi_{2^{-j}} * \psi_{2^{-j}} * \left(f_{B(\cdot,2^{-j})} - f_{B(\cdot,2^{1-j})}\right).$$

Notice that for all $k, j \in \mathbb{Z}$, and $x \in \mathbb{R}^n$,

$$\left|\varphi_{2^{-k}} * \phi_{2^{-j}} * \psi_{2^{-j}}(x)\right| = \left|\varphi_{2^{-k}} * (\phi * \psi)_{2^{-j}}(x)\right| \lesssim 2^{-s|j-k|} \frac{2^{n(min(j,k))}}{\left(1 + \left|2^{n(min(j,k))}x\right|\right)^L}, \quad (19)$$

where s, L can be chosen large enough as we need; see, for example, [175, Lemma 2.2]. Thus,

$$\left|\varphi_{2^{-k}} * \phi_{2^{-j}} * \psi_{2^{-j}} * g\right| = \left|\varphi_{2^{-k}} * (\phi * \psi)_{2^{-j}} * g\right| \lesssim 2^{-2|j-k|} M(g).$$

Therefore, when $p \in (1, \infty)$, from Definition (4.1.6), Hölder's inequality and the Fefferman-Stein vector-valued maximal inequality, we infer that

$$\begin{aligned}
\|f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)} &\lesssim \left\| \left\{ \sum_{k\in\mathbb{Z}} 2^{k\alpha q} \left| \sum_{j\le k} 2^{2(k-j)} M\left(f_{B(\cdot,2^{-j})} - f_{B(\cdot,2^{1-j})}\right) \right|^q \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\
&\quad + \left\| \left\{ \sum_{k\in\mathbb{Z}} 2^{k\alpha q} \left| \sum_{j> k} 2^{2(j-k)} M\left(f_{B(\cdot,2^{-j})} - f_{B(\cdot,2^{1-j})}\right) \right|^q \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\
&\lesssim \left\| \left\{ \sum_{j\in\mathbb{Z}} 2^{j\alpha q} \left[M\left(f_{B(\cdot,2^{-j})} - f_{B(\cdot,2^{1-j})}\right) \right]^q \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\
&\lesssim \left\| \left\{ \sum_{j\in\mathbb{Z}} 2^{j\alpha q} \left| f_{B(\cdot,2^{-j})} - f_{B(\cdot,2^{1-j})} \right|^q \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\
&\lesssim \left\| \left\{ \sum_{j\in\mathbb{Z}} 2^{j\alpha q} \left| f_{B(\cdot,2^{-j})} - f \right|^q \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} \lesssim \left\| S_{\alpha,q}(f) \right\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

When $p = \infty$, we need to show that

$$\left\{ \fint_{B(x,2^{-\ell})} \sum_{k\ge\ell} 2^{k\alpha q} \left(\sum_{j\in\mathbb{Z}} \left| \varphi_{2^{-k}} * \phi_{2^{-j}} * \psi_{2^{-j}} * \left[f_{B(\cdot,2^{-j})} - f_{B(\cdot,2^{1-j})} \right](y) \right| \right)^q dy \right\}^{1/q}$$

is controlled by $\|f\|_{S\dot{F}^{\alpha}_{\infty,q}(\mathbb{R}^n)}$ uniformly in $x \in \mathbb{R}^n$ and $\ell \in \mathbb{Z}$. The proof of this is quite similar to that of (7). Indeed, we consider $\sum_{j<\ell\le k}$,$\sum_{\ell\le j\le k}$ and $\sum_{j>k\ge\ell}$ separately. With the help of (19) and some necessary calculus, we arrive at $\|f\|_{\dot{F}^{\alpha}_{\infty,q}(\mathbb{R}^n)} \lesssim \|f\|_{S\dot{F}^{\alpha}_{\infty,q}(\mathbb{R}^n)}$. We omit the details.

When $\alpha \in (2N, 2(N+1))$, by Lemma (4.1.6) (ii), we see that

$$f = \sum_{k\in\mathbb{Z}} \phi_{2^{-k}} * \psi_{2^{-k}} * \left[f_{B(\cdot,2^{-k})} - \sum_{i=1}^{N} 2^{-2ij} \frac{1}{L_i} \frac{n}{n+2i} \Delta^i f - f_{B(\cdot,2^{1-k})} - \sum_{i=1}^{N} 2^{-2i(j-1)} \frac{1}{L_i} \frac{n}{n+2i} \Delta^i f \right].$$

By (19) with $s = 2(N+1)$ and $L > n$, repeating the above argument for the case $\alpha \in (0,2)$, we then conclude that $\|f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{S\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)}$. This finishes the proof of Theorem (4.1.10).

We first establish a similar characterization for Besov spaces and then make some remarks for the case $\alpha \in 2\mathbb{N}$.

Let $N \in \mathbb{N} \cup \{0\}, \alpha \in (2N, 2N+2)$ and $p, q \in (0, \infty]$. The space $S\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)$ of Besov type is defined as the collection of functions $f \in \dot{W}^{2N,1}_{loc}(\mathbb{R}^n) \cap S'(\mathbb{R}^n)$, such that

$$\|f\|_{S\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)} := \left\{ \sum_{k\in\mathbb{Z}} 2^{k\alpha q} \left\| \fint_{B(\cdot,2^{-k})} R_N(y; \cdot, 2^{-k})\, dy \right\|^q_{L^p(\mathbb{R}^n)} \right\}^{1/q} < \infty.$$

Here $\dot{W}^{0,1}_{loc}(\mathbb{R}^n) = L^1_{loc}(\mathbb{R}^n), R_N$ with $N \ge 1$ is as in (2) and, for all $x, y \in \mathbb{R}^n$,

$$R_0(y;\ x, 2^{-k}) := f(y) - f(x).$$

Also the space $S\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)$ is similarly defined as above.

Then Theorems (4.1.8) and (4.1.10) admit Besov space versions; indeed, by similar arguments, Theorems (4.1.8) and (4.1.10) still hold with spaces $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ and $S\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ replaced by Besov spaces $\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)$ and $S\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)$ and, moreover, with the indices α, p and q replaced, respectively, by $\alpha \in (0,\infty) \setminus 2\mathbb{N}, p \in (1,\infty]$ and $q \in (0,\infty]$. Namely, we have the following characterization on Besov spaces.

Theorem (4.1.11) [177] Let $\alpha \in (0,\infty) \setminus 2\mathbb{N}, p \in (1,\infty]$ and $q \in (0,\infty]$. Then $\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n) = S\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)$, with equivalent norms, and $B^{\alpha}_{p,q}(\mathbb{R}^n) = SB^{\alpha}_{p,q}(\mathbb{R}^n)$, with equivalent norms.

It should be pointed out that $B^{\alpha}_{p,q}(\mathbb{R}^n) \subset SB^{\alpha}_{p,q}(\mathbb{R}^n)$ when $\alpha \in (0,\infty)$ and $p, q \in [1,\infty]$ was obtained by Wheeden [173, Theorem 5] via a totally different approach.

Finally, we make some remarks. The first remark is on the missing indexes $\alpha \in 2\mathbb{N}$ in Theorems (4.1.2) and (4.1.5) while the second one is on the higher order Besov and Triebel-Lizorkin spaces on metric measure spaces.

Remark (4.1.12) [177] We point out that when $\alpha \in 2\mathbb{N}$, it was proved in [170] that a variant of Theorems (4.1.2) and (4.1.5) when $p \in (1,\infty)$ and $q = 2$ still holds. However, it is not clear that when $\alpha \in 2\mathbb{N}$, whether there exists a similar variant of Theorems (4.1.2) and (4.1.5) when $q \neq 2$ and Theorem (4.1.11) for all $q \in (0,\infty]$. Indeed, as pointed out in [170], $S_{2,2}$-function as in (1) fails to characterize $\dot{F}^{2}_{p,2}(\mathbb{R}^n) = \dot{W}^{2,p}(\mathbb{R}^n)$. To overcome this drawback, Alabern, Mateu and Verdera [170] then introduced a variant of (1) to characterize $\dot{F}^{2}_{p,2}(\mathbb{R}^n)$. Precisely, for $N \in \mathbb{Z}_+, k \in \mathbb{Z}$, and $x, y \in \mathbb{R}^n$, let

$$\overline{R}_N(y;\ x, 2^{-k}) := f(y) - f(x) - \sum_{j=1}^{N-1} \frac{1}{L_j} \Delta^j f(x)|y-x|^{2j} - \frac{1}{L_N}\left(\fint_{B(x,2^{-k})} \Delta^j f(z)\, dz\right)|y-x|^{2N}, \quad (20)$$

where L_N is as in (2). Let $\overline{S}_{\alpha,q}(f)$ be as in (1) with R_N replaced by $\overline{R}_N$ and, similarly, the spaces $\overline{S}\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ for $\alpha \in (2N, 2N+2)$ and $p, q \in (1,\infty]$ are similarly defined to the spaces $S\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$. Then it was proved in [170] that $\overline{S}\dot{F}^{2N}_{p,2}(\mathbb{R}^n) = \dot{F}^{\alpha}_{p,2}(\mathbb{R}^n) = \dot{W}^{2N,p}(\mathbb{R}^n)$ for $N \in \mathbb{N}$ and $p \in (1,\infty)$.

When $\alpha \in (2N, 2N+2)$ and $p, q \in (1,\infty]$, by modifying the proofs, we can also show that $\overline{S}\dot{F}^{2N}_{p,q}(\mathbb{R}^n) = \dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ with equivalent norms. But our above proof can only show $\overline{S}\dot{F}^{2N}_{p,q}(\mathbb{R}^n) \subset \dot{F}^{2N}_{p,q}(\mathbb{R}^n)$ for $N \in \mathbb{Z}_+$ and $p, q \in (1,\infty]$. It is still unknown whether the relation $\dot{F}^{2N}_{p,q}(\mathbb{R}^n) \subset \overline{S}\dot{F}^{2N}_{p,q}(\mathbb{R}^n)$ is still true for $N \in \mathbb{Z}_+$ and $p, q \in (1,\infty]$ but $q \neq 2$ or not.

Section (4.2) Sobolev Spaces via Averages on Balls

The problem of introducing Sobolev spaces on metric measure spaces where differential structures are not available is one of the central topics in analysis. A very important progress on this problem was achieved by Hajłasz [26], who successfully introduced a concept of gradients (now widely known as the Hajłasz gradients) and used it to introduce the first order Sobolev spaces on metric measure spaces. The Hajłasz gradients have become a powerful tool in the study of the first order Sobolev spaces on metric measure spaces; see [26, 72, 108, 184, 186]. After the pioneering work of Hajłasz [26], several different approaches were proposed to introduce and study first-order Sobolev spaces on metric measure spaces (see [72, 108, 118, 183, 184, 185]). Indeed, great success has been achieved on the theory of the first order Sobolev spaces on metric measure spaces. On the other hand, however, the problem of developing a successful theory of higher order Sobolev spaces on metric measure spaces remains open.

Alabern, Mateu and Verdera [170] obtained an interesting new characterization of Sobolev spaces on $\mathbb{R}^n$, which relies only on the metric and the Lebesgue measure on $\mathbb{R}^n$ and hence provides a possible way to introduce Sobolev spaces of arbitrary order of smoothness on any metric measure space. To describe this new characterization, we first recall that the (inhomogeneous) Sobolev spaces $W^{\alpha,p}(\mathbb{R}^n)$ on $\mathbb{R}^n$ consist of all functions f on $\mathbb{R}^n$ such that $\| f \|_{W^{\alpha,p}(\mathbb{R}^n)} := \| f \|_{L^p(\mathbb{R}^n)} + \| (-\Delta)^{\alpha/2} f \|_{L^p(\mathbb{R}^n)} < \infty$. Here, the smoothness index α is any positive real number, $p \in (1,\infty), \Delta := \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i}\right)^2$ is the Laplacian, and $(-\Delta)^{\alpha/2}$ is the fractional Laplacian defined in terms of the distributional Fourier transform via $((-\Delta)^{\alpha/2} f) \wedge (\xi) := |\xi|^{\alpha} \hat{f}(\xi)$ for any tempered distribution f. Next, we recall a well-known classical characterization of $W^{\alpha,p}(\mathbb{R}^n)$ via square functions (see [174, 56, 65, 72]), which asserts that, for $\alpha \in (0,1)$ and $p \in (1,\infty), f \in W^{\alpha,p}(\mathbb{R}^n)$ if and only if $f \in L^p(\mathbb{R}^n)$ and $s_\alpha(f) \in L^p(\mathbb{R}^n)$, where $s_\alpha(f)$ is the square function given by

$$s_\alpha(f)(\cdot) := \left\{ \int_0^\infty \left[\fint_{B(\cdot,t)} |f(\cdot) - f(y)| dy \right]^2 \frac{dt}{t^{1+2\alpha}} \right\}^{1/2}. \tag{21}$$

For $g \in L^1_{loc}(\mathbb{R}^n), x \in \mathbb{R}^n$ and $t \in (0,\infty)$,

$$B(x,t) := \{ y \in \mathbb{R}^n : |y - x| < t \},$$

$$\fint_{B(x,t)} g(y)\, dy := \frac{1}{|B(x,t)|} \int_{B(x,t)} g(y)\, dy =: B_t g(x). \tag{22}$$

Such a characterization, however, fails for $\alpha \geq 1$. Indeed, it is known that, if $\alpha \geq 1$, then $\| f \|_{L^p(\mathbb{R}^n)} + \| s_\alpha(f) \|_{L^p(\mathbb{R}^n)} < \infty$ implies $f \equiv 0$ on $\mathbb{R}^n$ (see [95, Section 4]).

In order to have a similar characterization for $W^{\alpha,p}(\mathbb{R}^n)$ with $\alpha \geq 1$, Alabern, Mateu and Verdera [170] introduced a new square function S_α, with a slight modification of the definition of $s_\alpha(f)$ via dropping the absolute value in $|f(\cdot) - f(y)|$ of (21), given by

$$S_\alpha(f)(\cdot) := \left\{ \int_0^\infty \left| \fint_{B(\cdot,t)} |f(\cdot) - f(y)| dy \right|^2 \frac{dt}{t^{1+2\alpha}} \right\}^{1/2}, f \in L^1_{loc}(\mathbb{R}^n) \tag{23}$$

It turns out that such a modification is significant enough for [170] to establish a characterization for all Sobolev spaces of smoothness orders $\alpha \in (0,2)$: for $\alpha \in (0,2)$ and $p \in (1,\infty), f \in W^{\alpha,p}(\mathbb{R}^n)$ if and only if $f \in L^p(\mathbb{R}^n)$ and $S_\alpha(f) \in L^p(\mathbb{R}^n)$. The key point here is that, unlike the classical square function s_α in (21), this new function S_α in (23) provides smoothness up to order 2, namely, for $f \in C^2(\mathbb{R}^n)$ and $t \in (0,1)$,

$$\fint_{B(x,t)} [f(x) - f(y)]\, dy \;=\; O(t^2), \qquad x \in \mathbb{R}^n. \tag{24}$$

This phenomenon, followed directly from the Taylor expansion, was first observed by Wheeden in [173] (see [174]) and later independently by Alabern, Mateu and Verdera in [170].

A more complicated characterization of $W^{\alpha,p}(\mathbb{R}^n)$ for higher orders of smoothness (i.e., $\alpha \geq 2$) was also established in [170, Theorems 2 and 3]. Assume that $\alpha \in [2N, 2N+2)$ with $N \in \mathbb{N} := \{1, 2, \ldots\}$. For $f, g_1, \ldots, g_N \in L^1_{loc}(\mathbb{R}^n)$, define

$$S_\alpha(f, g_1, \ldots, g_N)(x) := \left\{ \int_0^\infty \left| \fint_{B(x,t)} t^{-\alpha} R_N(y,x) dy \right|^2 \frac{dt}{t} \right\}^{1/2}, \qquad x \in \mathbb{R}^n, \tag{25}$$

where, for all $x, y \in \mathbb{R}^n$,

$$R_N(y,x) := R_N(y,x,t) = f(y) - f(x) - \sum_{j=1}^{N-1} g_j(x)|y-x|^{2j}$$
$$-\begin{cases} g_N(x)|y-x|^{2N}, & if\ \alpha \in (2N, 2N+2), \\ B_t g_N(x)|y-x|^{2N}, & if\ \alpha = 2N. \end{cases} \tag{26}$$

Here we recall that B_t is the average operator given in (22). With the above notation, it is shown in [170] that $f \in W^{\alpha,p}(\mathbb{R}^n)$, with $\alpha \in [2N, 2N+2), N \in \mathbb{N}$ and $p \in (1,\infty)$, if and only if $f \in L^p(\mathbb{R}^n)$ and

$$S_\alpha(f, g_1, \ldots, g_N) \in L^p(\mathbb{R}^n)$$

for some functions $g_1, \ldots, g_N \in L^p(\mathbb{R}^n)$. Indeed, according to [170, Theorem 3], the functions g_j can be taken as $\frac{1}{L_j}\Delta^j f$ almost everywhere with $L_j := \Delta^j |x|^{2j}$, where $j \in \{1, \ldots, N\}$. Of particular interest is the case when $\alpha = 2$ and $N = 1$, where the characterization can be formulated more explicitly as follows (see [170, Theorem 2]).

Theorem (4.2.1) [188] Let $p \in (1,\infty)$. Then $f \in W^{2,p}(\mathbb{R}^n)$ if and only if $f \in L^p(\mathbb{R}^n)$ and there exists $g \in L^p(\mathbb{R}^n)$ such that

$$S_2(f,g)(\cdot) := \left\{ \int_0^\infty \left| \frac{B_t f(\cdot) - f(\cdot)}{t^2} - B_t g(\cdot) \right|^2 \frac{dt}{t} \right\}^{1/2} \in L^p(\mathbb{R}^n), \tag{27}$$

where B_t is as in (22). Moreover, the function g can be chosen such that $\| f \|_{L^p(\mathbb{R}^n)} + \| S_2(f,g) \|_{L^p(\mathbb{R}^n)}$ is equivalent to $\| f \|_{W^{2,p}(\mathbb{R}^n)}$ with equivalent positive constants independent of f.

See [170], and the pointwise characterizations of the first-order Sobolev spaces via the Hajłasz gradients established in [26] (see also [186, 72]). Recall that a non-negative measurable function g on $\mathbb{R}^n$ is called a Hajłasz gradient of a measurable function f on $\mathbb{R}^n$ if the inequality

$$|f(x) - f(y)| \le |x-y|[g(x) + g(y)] \tag{28}$$

holds true for almost every $x, y \in \mathbb{R}^n$. Hajłasz [26] proved that a function $f \in L^p(\mathbb{R}^n)$ belongs to the first order Sobolev space $W^{1,p}(\mathbb{R}^n)$, with $p \in (1,\infty)$, if and only if it has a Hajłasz gradient in $L^p(\mathbb{R}^n)$. Also recall that $W^{2,p}(\mathbb{R}^n)$ (or even higher order Sobolev spaces) can also be characterized via the second order difference (or higher order differences) (see Haroske and Triebel [74, Proposition 4.1] and also [30, 63, 69]); but it is still unclear how to introduce higher order differences on spaces of homogeneous type in the sense of Coifman and Weiss [179, 180].

The aim is to use the average operator B_t in (22) to establish pointwise characterizations of the higher-order Sobolev spaces $W^{\alpha,p}(\mathbb{R}^n)$ that are analogous to (28). A novel aspect of our characterizations of $W^{\alpha,p}(\mathbb{R}^n)$ for $\alpha \ge 2$ lies in that they look much simpler than those in [170]. We will state the main result for the second order Sobolev spaces $W^{2,p}(\mathbb{R}^n)$ only. Pointwise characterizations of the higher-order Sobolev spaces will be given. The main result for the second order Sobolev spaces can be stated as follows.

Lemma (4.2.2) [188] Let $\varphi \in S(\mathbb{R}^n)$ and $\tilde{C}$ be a given positive constant. Then

$$\lim_{t\to 0^+} \frac{\varphi - B_t\varphi}{t^2} = -\frac{1}{2(n+2)}\Delta\varphi \quad \text{in } S(\mathbb{R}^n) \tag{29}$$

and

$$\lim_{t\to 0^+} \fint_{B(\cdot,t)} \frac{\varphi(y) - B_{\tilde{C}t}\varphi(y)}{t^2} dy = -\frac{\tilde{C}^2}{2(n+2)}\Delta\varphi \quad \text{in } S(\mathbb{R}^n), \tag{30}$$

where the convergences are with respect to the topology of $S(\mathbb{R}^n)$.

Proof. By the Taylor expansion of $\varphi \in S(\mathbb{R}^n)$, for any given $x, y \in \mathbb{R}^n$, there exists a point $\xi_{x,y}$ on the line segment connecting x and y such that

$$\varphi(y) = \varphi(x) + \sum_{0<|\alpha|\le 2} \frac{1}{\alpha!}\partial^\alpha \varphi(x)(y-x)^\alpha + \sum_{|\alpha|=3} \frac{1}{\alpha!}\partial^\alpha \varphi(\xi_{x,y})(y-x)^\alpha,$$

here and hereafter, for any $\alpha := (\alpha_1, \dots, \alpha_n) \in (\mathbb{Z}_+)^n, |\alpha| := |\alpha_1| + \cdots + |\alpha_n|, \alpha! := \alpha_1! \cdots \alpha_n!$ and $\partial^\alpha := (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n}$. Fixing $x \in \mathbb{R}^n$ and $t \in (0,\infty)$, and taking average over $y \in B(x,t)$ on both sides of the last equation, we see that

$$\varphi(x) - B_t\varphi(x) = -\frac{1}{2(n+2)}\Delta\varphi(x)t^2 - \sum_{|\alpha|=3}\frac{1}{\alpha!}\fint_{B(x,t)} \partial^\alpha \varphi(\xi_{x,y})(y-x)^\alpha dy. \quad (31)$$

Hence, for any $\beta \in (\mathbb{Z}_+)^n, m \in \mathbb{Z}_+$, and all $x \in \mathbb{R}^n$ and $t \in (0, \infty)$, we have

$$\left|\partial^\beta\left(\frac{\varphi - B_t\varphi}{t^2} - \left[-\frac{1}{2(n+2)}\Delta\varphi\right]\right)(x)\right|(1+|x|)^m$$
$$= \left|\left(\frac{\partial^\beta\varphi - B_t(\partial^\beta\varphi)}{t^2} + \frac{1}{2(n+2)}\Delta(\partial^\beta\varphi)\right)(x)\right|(1+|x|)^m$$
$$\lesssim \|\varphi\|_{3+|\beta|,m} \fint_{B(x,t)} \frac{|y-x|^3}{(1+|\xi_{x,y}|)^m}\frac{(1+|x|)^m}{t^2}dy \lesssim t\|\varphi\|_{3+|\beta|,m}(1+t)^m,$$

which converges to 0 as $t \to 0^+$. This proves (29).

The proof of (30) is similar to that of (29), the details being omitted. This finishes the proof of Lemma (4.2.2).

Recall that, for any $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the Hardy–Littlewood maximal function Mf is defined by

$$Mf(x) := \sup_{B\ni x}\fint_B |f(y)|\,dy,$$

where the supremum is taken over all balls B containing x.

Theorem (4.2.3) [188] Let $p \in (1,\infty)$. The following statements are equivalent:

(i) $f \in W^{2,p}(\mathbb{R}^n)$;

(ii) $f \in L^p(\mathbb{R}^n)$ and there exists a function $g_1 \in (\mathbb{R}^n)$ such that

$$\lim_{t\to 0^+}\frac{f - B_t f}{t^2} = g_1 \text{ in } S'(\mathbb{R}^n); \quad (32)$$

(iii) $f \in L^p(\mathbb{R}^n)$ and there exists a non-negative $g_2 \in (\mathbb{R}^n)$ such that, for all $t \in (0,\infty)$ and almost every $x \in \mathbb{R}^n$,

$$|f(x) - B_t f(x)| \le t^2 g_2(x). \quad (33)$$

(iv) $f \in L^p(\mathbb{R}^n)$ and there exist a non-negative $g_3 \in L^p(\mathbb{R}^n)$ and positive constants C_1 and C_2 (depending only on n) such that, for all $t \in (0,\infty)$ and almost every $x \in \mathbb{R}^n$,

$$\fint_{B(x,t)}|f(y) - B_{C_1 t}f(y)|\,dy \le t^2\fint_{B(x,C_2 t)} g_3(y)dy; \quad (34)$$

(v) $f \in L^p(\mathbb{R}^n)$ and there exist a non-negative $g_4 \in L^p(\mathbb{R}^n)$ and a positive constant C_3 (depending only on n) such that, for all $t \in (0,\infty)$ and almost every $x \in \mathbb{R}^n$,

$$\left|\fint_{B(x,t)}[f(y) - B_{C_3 t}f(y)]dy\right| \le t^2 g_4(x). \quad (35)$$

Furthermore, if $f \in W^{2,p}(\mathbb{R}^n)$, then the functions $g_i, i \in \{1,2,3,4\}$ in the above statements, can be chosen so that $\| g_i \|_{L^p(\mathbb{R}^n)}$ are equivalent to $\| \Delta f \|_{L^p(\mathbb{R}^n)}$ with the implicit equivalent positive constants depending only on n and p.

Clearly, the pointwise inequality (33) can be considered as a second-order analogue of the pointwise characterization (28).

Proof. “(i) $\Longrightarrow$ (ii)”: This implication is a direct consequence of (29) and the facts that the operators $\frac{f-B_t f}{t^2}$ and Δ are both self-adjoint with respect to the inner product of $L^2(\mathbb{R}^n)$. Indeed, if $f \in W^{2,p}(\mathbb{R}^n)$, we choose the function g_1 as $g_1 := -\frac{1}{2(n+2)}\Delta f$.

“ (ii) $\Longrightarrow$ (iii) ”: We first observe that, if $f \in C^2(\mathbb{R}^n) \cap W^{2,p}(\mathbb{R}^n)$, then

$$|f(x) - B_t f(x)| \lesssim t^2 M\left(\sum_{|\alpha|=2} |\partial^\alpha f|\right)(x), \qquad x \in \mathbb{R}^n, \qquad t \in (0,\infty). \tag{36}$$

Indeed, by the Taylor expansion of f, we see that

$$\begin{aligned} |f(x) - B_t f(x)| &= \left| \fint_{B(x,t)} [f(x) - f(y)]\, dy \right| \\ &\lesssim \sum_{|\alpha|=2} \fint_{B(x,t)} \left| \int_0^1 (1 - s)\partial^\alpha f\big(x + s(y - x)\big)\, ds \right| |y - x|^2\, dy \\ &\lesssim \sum_{|\alpha|=2} t^2 \int_0^1 \fint_{(0,st)} |\partial^\alpha f(x+z)|\, dz\, ds, \end{aligned}$$

from which (36) follows.

Now assume that (ii) is satisfied. Then Lemma (4.2.2) implies that, for all $\varphi \in S(\mathbb{R}^n)$,

$$\langle f, \Delta\phi\rangle = \lim_{t\to 0} \langle f, -\frac{1}{2(n+2)}\frac{\varphi - B_t\varphi}{t^2}\rangle = \lim_{t\to 0} \langle -\frac{1}{2(n+2)}\frac{f - B_t f}{t^2}, \varphi\rangle = \langle -\frac{1}{2(n+2)} g_1, \varphi\rangle.$$

This means that $\Delta f = -\frac{1}{2(n+2)} g_1 \in L^p(\mathbb{R}^n)$ in $S'(\mathbb{R}^n)$. Since $f \in L^p(\mathbb{R}^n)$, this further implies that $f \in W^{2,p}(\mathbb{R}^n)$.

Now (iii) follows from (36) and a standard limiting argument. To see this, let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be such that $\int_{\mathbb{R}^n} \phi(x)dx = 1$. Let $f_k = f * \phi_{2^{-k}}$ for all $k \in \mathbb{N}$. Since $\{\phi_{2^{-k}}\}_{k\in\mathbb{N}}$ is an approximation to the identity, it follows from (36) that, for almost every $x \in \mathbb{R}^n$,

$$\begin{aligned} \left|\frac{f(x) - B_t f(x)}{t^2}\right| &= \lim_{k\to\infty} \left|\frac{\phi_{2^{-k}} * f(x) - B_t\big(\phi_{2^{-k}} * f\big)(x)}{t^2}\right| \\ &\lesssim M\left(\sum_{|\alpha|=2} \sup_{k\in\mathbb{N}} \left|\phi_{2^{-k}} * \partial^\alpha f\right|\right)(x) \lesssim M\left(\sum_{|\alpha|=2} M(\partial^\alpha f)\right)(x), \end{aligned}$$

where M denotes the Hardy–Littlewood maximal operator. Now letting

$$g_2 := \tilde{C} M\left(\sum_{|\alpha|=2} M(\partial^\alpha f)\right),$$

where $\tilde{C}$ is the implicit positive constant in the above inequality, we deduce from the boundedness of M on $L^p(\mathbb{R}^n)$, with $p \in (1,\infty)$, that

$$\| g_2 \|_{L^p(\mathbb{R}^n)} \lesssim \sum_{|\alpha|=2} \|\partial^\alpha f\|_{L^p(\mathbb{R}^n)} \lesssim \| f \|_{W^{2,p}(\mathbb{R}^n)}. \tag{37}$$

This shows (iii).

"(iii) $\Longrightarrow$ (iv)": The implication (iii) $\Longrightarrow$ (iv) is straightforward. Indeed, letting $C_1 = C_2 = 1$ and $g_3 := g_2$, we obtain by (33) that

$$\text{LHS of (34)} \le t^2 \fint_{B(x,t)} g_2(y)\,dy = t^2 \fint_{B(x,t)} g_3(y)\,dy.$$

"(iv) $\Longrightarrow$ (v)": (v) follows directly from (iv) with $C_3 = C_1$ and $g_4 := Mg_3$.

"(v) $\Longrightarrow$ (i)": Notice that LHS of (35) can be written as $|B_t(f - B_{C_3 t} f)(x)|$. Thus, (v) implies that

$$\sup_{t>0} \frac{\| B_t(f - B_{C_3 t} f) \|_{L^p(\mathbb{R}^n)}}{t^2} \le \| g_4 \|_{L^p(\mathbb{R}^n)} < \infty.$$

By the Banach–Alaoglu theorem (see [187, p. 70, Theorem 3.17]), there exist a subsequence $\{k_j\}_j^\infty = 1$ of positive integers and a function $h \in L^p(\mathbb{R}^n)$ such that $\| h \|_{L^p(\mathbb{R}^n)} \lesssim \| g_4 \|_{L^p(\mathbb{R}^n)}$ and, for all $\varphi \in S(\mathbb{R}^n)$,

$$\lim_{j\to\infty} \langle 2^{2k_j} B_{2^{-k_j}} \left(f - B_{C_3 2^{-k_j}} f\right), \varphi\rangle = \langle h, \phi\rangle.$$

Since the operators $\{B_t\}_{t\in(0,\infty)}$ are self-adjoint with respect to the inner product of $L^2(\mathbb{R}^n)$, it follows that, for all $\varphi \in S(\mathbb{R}^n)$,

$$2^{2k_j} \langle B_{2^{-k_j}} \left(f - B_{C_3 2^{-k_j}} f\right), \varphi\rangle = 2^{2k_j} \langle B_{2^{-k_j}} \left(f, \varphi - B_{C_3 2^{-k_j}} \varphi\right)\rangle.$$

However, by (30), we find that

$$\lim_{j\to\infty} 2^{2k_j} B_{2^{-k_j}} \left(\varphi - B_{C_3 2^{-k_j}} \varphi\right) = -\frac{C_3^2}{2(n+2)} \Delta\varphi \ \text{ in } S(\mathbb{R}^n).$$

Thus, for all $\varphi \in S(\mathbb{R}^n)$, we have

$$\langle f, \Delta\varphi\rangle = -\frac{2(n+2)}{C_3^2} \langle h, \varphi\rangle.$$

This implies that $\Delta f = -\frac{2(n+2)}{C_3^2} h \in L^p(\mathbb{R}^n)$ and hence $f \in W^{2,p}(\mathbb{R}^n)$.

Finally, it is easily seen from the above proof that all the functions $g_i, i \in \{1, 2, 3, 4\}$, can be chosen so that each norm $\| g_i \|_{L^p(\mathbb{R}^n)}$ is equivalent to $\| \Delta f \|_{L^p(\mathbb{R}^n)}$. This finishes the proof of Theorem (4.2.3).

From the above proof of Theorem (4.2.3), we further deduce the following equivalent descriptions of $W^{2,p}(\mathbb{R}^n)$.

Corollary (4.2.4) [188] Let $p \in (1, \infty)$. The following statements are equivalent:

(i) $f \in W^{2,p}(\mathbb{R}^n)$;

(ii) $f \in L^p(\mathbb{R}^n)$ and there exists $g \in L^p(\mathbb{R}^n)$ such that

$$\liminf_{t\to 0^+} \frac{f - B_t f}{t^2} = g \quad in \ S'(\mathbb{R}^n);$$

(iii) $f \in L^p(\mathbb{R}^n)$ and there exist $g \in L^p(\mathbb{R}^n)$ and a sequence $\{t_k\}_{k\in\mathbb{N}}$ of positive numbers such that $\lim_{k\to\infty} t_k = 0$ and

$$\lim_{k\to\infty} \frac{f - B_{t_k} f}{t_k^2} = g \quad in \ S'(\mathbb{R}^n);$$

(iv) $f \in L^p(\mathbb{R}^n)$ and

$$\sup_{t\in(0,\infty)} \frac{\left\|f - B_{t_k} f\right\|_{L^p(\mathbb{R}^n)}}{t_k^2} =: C_4 < \infty.$$

In (ii) and (iii), the function g can be chosen such that $\| g \|_{L^p(\mathbb{R}^n)}$ is equivalent to $\| \Delta f \|_{L^p(\mathbb{R}^n)}$ with the equivalent positive constants independent of f, which also holds true for C_4 in (iv).

Corollary (4.2.5) [188] Let $p \in (1, \infty), q \in [1, p), c \in (0, \infty)$ and $K \in (0, \infty]$. Then $f \in W^{2,p}(\mathbb{R}^n)$, if and only if $f \in L^p(\mathbb{R}^n)$ and $f_{c,q}^{\delta,K} \in L^p(\mathbb{R}^n)$. Moreover, $\| \Delta f \|_{L^p(\mathbb{R}^n)} \sim \| f_{c,q}^{\delta,K} \|_{L^p(\mathbb{R}^n)}$ with the implicit equivalent positive constants depending only on c, p, q, K and n.

One novel idea used in these proofs is the convergence of $\lim_{t \to 0^+} \frac{\varphi - B_t \varphi}{t^2}$ to $-\frac{1}{2(n+2)} \Delta \varphi$ in $S(\mathbb{R}^n)$ (see Lemma (4.2.2)). After that, we show how to establish similar characterizations for the higher order Sobolev spaces $W^{\alpha,p}(\mathbb{R}^n)$. In our discussion the concept of higher-order average operators $B_{\ell,t}$. Ideas behind the introduction of the operators $B_{\ell,t}$ are also explained in detail. Finally, several remarks on how to introduce higher order Sobolev spaces on spaces of homogeneous type are given and an advantage of these definitions is that they are simper than those of [170] and, in the case of Euclidean spaces, all these definitions are equivalent. Also, several related open problems in this direction are raised.

Let $S(\mathbb{R}^n)$ denote the collection of all Schwartz functions on $\mathbb{R}^n$, endowed with the usual topology, and $S'(\mathbb{R}^n)$ its topological dual, namely, the collection of all bounded linear functionals on $S(\mathbb{R}^n)$ endowed with the weak $*$-topology. For all $m \in \mathbb{Z}_+ := \mathrm{N} \cup \{0\}, \alpha \in \mathbb{Z}_+^n$ and $\varphi \in S(\mathbb{R}^n)$, let

$$\| \varphi \|_{\alpha,m} := \sup_{|\beta| \le |\alpha|, x \in \mathbb{R}^n} (1 + |x|)^m |\partial^\beta \varphi(x)|.$$

For any $\varphi \in S(\mathbb{R}^n)$ and $t \in (0, \infty)$, we let $\varphi_t(\cdot) := t^{-n} \varphi(\cdot/t)$.

The symbol C denotes a positive constant which depends only on the fixed parameters n, ℓ, p and probably on auxiliary functions, unless otherwise stated; its value may vary from line to line. We use the symbol $f \lesssim g$ to denote that there exists a positive constant C such that $f \le Cg$. The symbol $f \sim g$ is used as an abbreviation of $f \lesssim g \lesssim f$.

The following simple lemma plays a key role in our proofs. In what follows, $t \to 0^+$ means $t > 0$ and $t \to 0$.

Now we are ready to prove Corollary (4.2.5).

Proof. If $f \in W^{2,p}(\mathbb{R}^n)$, then, by Theorem (4.2.3) (ii), we know that, for all $x \in \mathbb{R}^n$,

$$f_{c,q}^{\delta,K}(x) \lesssim \sup_{t \in (0,\infty)} \left[\fint_{B(x,t)} |g_1(y)|^q \right]^{\frac{1}{q}} \lesssim [M(|g_1|^q)(x)]^{1/q},$$

where M denotes the Hardy–Littlewood maximal operator. Since $q < p$, by the boundedness of M on $L^{p/q}(\mathbb{R}^n)$ with $q \in [1, p)$, it follows that

$$\left\| f_{c,q}^{\delta,K} \right\|_{L^p(\mathbb{R}^n)} \lesssim \left\| [M(|g_1|^q)]^{1/q} \right\|_{L^p(\mathbb{R}^n)} = \| M(|g_1|^q) \|_{L^p(\mathbb{R}^n)}^{1/q} \lesssim \|g_1\|_{L^p(\mathbb{R}^n)} \lesssim \|\Delta f\|_{L^p(\mathbb{R}^n)},$$

which is the desired estimate.

Conversely, assume that $f_{c,q}^{\delta,K} \in L^p(\mathbb{R}^n)$. By the definition of $f_{c,q}^{\delta,K}$ and the Hölder inequality, we see that, for all $t \in (0, K)$ and $x \in \mathbb{R}^n$,

$$\fint_{B(x,t)} |f(y) - B_{ct} f(y)| \, dy \lesssim t^2 f_{c,q}^{\delta,K}(x).$$

From the proof of the implication "(v) $\Longrightarrow$ (i)" in the proof of Theorem (4.2.3), it follows that this implies that $f \in W^{2,p}(\mathbb{R}^n)$ and $\|\Delta f\|_{L^p(\mathbb{R}^n)} \lesssim \left\| f_{c,q}^{\delta,K} \right\|_{L^p(\mathbb{R}^n)} < \infty$. This finishes the proof of Corollary (4.2.5).

We discuss how to establish similar characterizations of the higher order Sobolev spaces $W^{2\ell,p}(\mathbb{R}^n)$ with $\ell \in \mathbb{N}$ and $p \in (1, \infty)$. The crucial idea is to replace the average operator B_t with its higher order invariant $B_{\ell,t}$ defined via higher order symmetric differences.

To illustrate the idea behind the definition of $B_{\ell,t}$, we first recall that the r-th order directional symmetric difference $\Delta_h^r f$ of a function $f:\mathbb{R}^n\to\mathbb{R}$ along a vector $h\in\mathbb{R}^n$ is defined by

$$\Delta_h^1 f(x):=f(x+h/2)-f(x-h/2),\qquad \Delta_h^r:=\Delta_h^1\Delta_h^{r-1},\qquad r\in\{2,3,\dots\}.$$

Letting $T_h f(\cdot):=f(\cdot+h)$ for all $h\in\mathbb{R}^n$, we can write $\Delta_h^r f$ more explicitly as

$$\Delta_h^r f(x)=T_{\frac{-hr}{2}}(T_h-I)^r f(x)=\sum_{k=0}^{r}(-1)^k\binom{r}{k}f\left(x+\left(\frac{r}{2}-k\right)h\right),\qquad x\in\mathbb{R}^n,\qquad (38)$$

where I denotes the identity operator.

Next, we observe that, for all $t\in(0,\infty)$ and $x\in\mathbb{R}^n$,

$$\begin{aligned} f(x)-B_t f(x)&:=f(x)-\frac{1}{2}\fint_{B(0,1)}[f(x+ty)+f(x-ty)]\,dy\\ &=-\frac{1}{2}\fint_{B(0,1)}[f(x+ty)+f(x-ty)-2f(x)]\,dy=-\frac{1}{2}\fint_{B(0,1)}\Delta_{ty}^2 f(x)\,dy.\qquad (39)\end{aligned}$$

This means that $f(x)-B_t f(x)$ can be considered as a constant multiple of the integral average of the second order symmetric difference $\Delta_{ty}^2 f(x)$ of f at x with respect to y over the unit ball $B(0,1)$! In view of the characterizations of Sobolev spaces via differences (see [74, 69, 63, 30]), this, in some sense, explains the reason why $f-B_t f$ can be used to characterize the second order Sobolev spaces on $\mathbb{R}^n$.

Given $\ell\in\mathbb{N}$, according to (39), it is very natural to introduce a higher order average operator $B_{t,\ell}$ via the identity

$$f(x)-B_{\ell,t}f(x)=\frac{1}{C_\ell}\fint_{B(0,1)}\Delta_{ty}^{2\ell}f(x)\,dy,\qquad x\in\mathbb{R}^n,\qquad (40)$$

Where C_ℓ is a normalization constant to be specified later. To obtain an explicit formulation of $B_{\ell,t}$, we deduce from (38) that

$$\begin{aligned}\frac{1}{C_\ell}\fint_{B(0,1)}\Delta_{ty}^{2\ell}f(x)\,dy&=\frac{1}{C_\ell}\sum_{k=0}^{2\ell}(-1)^k\binom{2\ell}{k}\fint_{B(0,1)}f(x+(\ell-k)ty)\,dy\\ &=\frac{(-1)^\ell\binom{2\ell}{\ell}}{C_\ell}f(x)+\frac{2(-1)^\ell}{C_\ell}\sum_{j=1}^{\ell}(-1)^j\binom{2\ell}{\ell-j}B_{jty}f(x).\qquad (41)\end{aligned}$$

Comparing (40) with (41), we let $C_\ell:=(-1)^{\ell+1}\binom{2\ell}{\ell}$ and then obtain

$$B_{\ell,t}f(x):=\frac{-2}{\binom{2\ell}{\ell}}\sum_{j=1}^{\ell}(-1)^j\binom{2\ell}{\ell-j}B_{jt}f(x),\quad t\in(0,\infty),\qquad x\in\mathbb{R}^n.\qquad (42)$$

Notice that (42) relies only on the metric and the Lebesgue measure of $\mathbb{R}^n$. We point out that the higher order average operator $B_{\ell,t}$ was previously used and studied in approximation theory (see [178, 181]).

Another way to look at (42) is to consider $B_{\ell,t}f$ as a 2ℓ-th order symmetric difference of $B_t f$ with respect to t. To be precise, for any fixed $f\in L^1_{loc}(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$, let

$$g_{x,f}(t):=\begin{cases}B_t f(x), & t\in(0,\infty),\\ f(x), & t=0,\\ B_{-t}f(x), & t\in(-\infty,0).\end{cases}\qquad (43)$$

Then a straightforward calculation shows that, for all $\ell\in\mathbb{N}$ and $t\in(0,\infty)$,

$$f(x)-B_{\ell,t}f(x)=\frac{(-1)^\ell}{\binom{2\ell}{\ell}}\sum_{j=0}^{2\ell}(-1)^j\binom{2\ell}{j}g_{x,f}\big((\ell-j)t\big)=\frac{(-1)^\ell}{\binom{2\ell}{\ell}}\Delta_t^{2\ell}g_{x,f}(0),x\in\mathbb{R}^n.\quad (44)$$

The results of Theorem (4.2.3) can be easily extended to the case of higher order Sobolev spaces, with $\frac{f-B_t f}{t^2}$ replaced by $\frac{f-B_{\ell,t} f}{t^{2\ell}}$. We have the following conclusion.

Theorem (4.2.6) [188] Let $\ell \in \mathbb{N}$ and $p \in (1, \infty)$. Then the conclusion of Theorem (4.2.3) remains valid when $W^{2,p}(\mathbb{R}^n), B_t f$ and t^2 therein are replaced by $W^{2\ell,p}(\mathbb{R}^n), B_{\ell,t} f$ and $t^{2\ell}$, respectively.

The proof of Theorem (4.2.6) is very close to that of Theorem (4.2.3) given. The crucial step is to show the following analogue of Lemma (4.2.2): for each $\ell \in \mathbb{N}, t \in (0, \infty)$, $\tilde{C} \in (0, \infty)$ and $\varphi \in S(\mathbb{R}^n)$,

$$\lim_{t\to 0^+} \frac{\varphi - B_{\ell,t}\varphi}{t^{2\ell}} = a_\ell (-\Delta)^\ell \varphi$$

and

$$\lim_{t\to 0^+} \fint_{B(\cdot,t)} \frac{\varphi(y) - B_{\ell,\tilde{C}t}\varphi(y)}{t^{2\ell}} dy = b_\ell (-\Delta)^\ell \varphi$$

with convergences in $S(\mathbb{R}^n)$, where

$$a_\ell := \frac{1}{\binom{2\ell}{\ell}} \frac{1 \times 3 \times \cdots \times (2\ell - 3) \times (2\ell - 1)}{(n+2)(n+4)\cdots(n+2\ell-2)(n+2\ell)},$$

with $\binom{2\ell}{\ell}$ being the binomial coefficient, and $b_\ell := \tilde{C}^{2\ell} a_\ell$.

Remark (4.2.7) [188] Recall that, for $\alpha \in (0,2)$ and $p \in (1, \infty)$, the Sobolev spaces $W^{\alpha,p}(\mathbb{R}^n)$ can be characterized via the square function $S_\alpha f$ given in (23) (see [170]). Motivated by this characterization, very naturally, one can introduce a higher-order analogue of the square function $S_\alpha f$, using $f - B_{\ell,t} f$ to replace $f - B_t f$. More precisely, given $\ell \in \mathbb{N}$ and $\alpha \in (0, 2\ell)$, we define, for all $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$S_{\alpha,\ell}(f)(x) := \left\{ \int_0^\infty \left| f(x) - B_{\ell,t} f(x) \right|^2 \frac{dt}{t^{1+2\alpha}} \right\}^{1/2}. \tag{45}$$

It turns out that such a square function can be used to characterize higher-order Sobolev spaces. It is possible to show that, for all $\ell \in \mathbb{N}, \alpha \in (0, 2\ell)$ and $p \in (1, \infty), f \in W^{\alpha,p}(\mathbb{R}^n)$ if and only if $f \in L^p(\mathbb{R}^n)$ and $S_{\alpha,\ell} f \in L^p(\mathbb{R}^n)$. Indeed, a discrete version of this assertion was proved (see [196]).

Is devoted to some related questions on spaces of homogeneous type in the sense of Coifman and Weiss [179, 180]. Recall that a triple $(\mathcal{X}, d, \mu)$ is called a space of homogeneous type in the sense of Coifman and Weiss [179, 180] if d is a quasi-metric on $\mathcal{X}$, namely, d satisfies

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{X}$;

(iii) there exists a constant $A \in [1, \infty)$ such that, for all $x, y, z \in \mathcal{X}$,

$$d(x, y) \le A[d(x, z) + d(z, y)], \tag{46}$$

and μ is a non-trivial regular Borel measure on $\mathcal{X}$ satisfying the following doubling condition: there exists a constant $C_0 \in [1, \infty)$ such that, for all $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$\mu(B(x, 2r)) \le C_0 \mu(B(x, r)) \text{ (doubling property)}. \tag{47}$$

Every quasi-metric d on $\mathcal{X}$ determines a topology on $\mathcal{X}$, for which the class of all balls,

$$B(x, r) := \{y \in \mathcal{X}: d(x, y) < r\}, \qquad (x \in \mathcal{X}, r > 0),$$

forms a basis on $\mathcal{X}$. A space $(\mathcal{X}, d, \mu)$ of homogeneous type is called a metric measure space of homogeneous type if $A = 1$ in (46); namely, if $(\mathcal{X}, d)$ is a metric space.

As in the Euclidean case, the average operator is defined by

$$B_t f(x) := \fint_{B(x,t)} f(y)\, d\mu(y) := \frac{1}{\mu(B(x,t))}\int_{B(x,t)} f(y)\, d\mu(y),$$

where $f \in L^1_{loc}(\mathcal{X}), x \in \mathbb{R}^n$ and $t \in (0,\infty)$. Similarly, given $\ell \in \mathbb{N}$, we define the 2ℓ-th order average operator as in (42).

In what follows, $(\mathcal{X}, d, \mu)$ always denotes a space of homogeneous type. We have the following analogue of Theorem (4.2.6).

Theorem (4.2.8) [188] Let $\ell \in \mathbb{N}$ and $p \in (1,\infty)$. Then the following two statements are equivalent:

(i) $f \in L^p(\mathcal{X})$ and there exist a non-negative $g_1 \in L^p(\mathcal{X})$ and a positive constant C_1 such that, for all $t \in (0,\infty)$ and almost every $x \in \mathcal{X}$,

$$\fint_{B(x,t)} \left|f(y) - B_{\ell,C_1 t} f(y)\right| d\mu(y) \le t^{2\ell} g_1(x); \tag{48}$$

(ii) $f \in L^p(\mathcal{X})$ and there exist a non-negative $g_2 \in L^p(\mathcal{X})$ and positive constants C_2, C_3 such that, for all $t \in (0,\infty)$ and almost every $x \in \mathcal{X}$,

$$\fint_{B(x,t)} \left|f(y) - B_{\ell,C_2 t} f(y)\right| d\mu(y) \le t^{2\ell} \fint_{B(x,C_3 t)} g_2(y)\, d\mu(y). \tag{49}$$

Proof. The implication (ii) $\Longrightarrow$ (i) is obvious as we may choose $g_1 := M g_2$ and $C_1 = C_2$, where M denotes the Hardy–Littlewood maximal operator on $\mathcal{X}$.

To show the inverse implication (i) $\Longrightarrow$ (ii), we first notice that, for any $y \in B(x,t)$,

$$B(x,t) \subset B(y,(1+A)t) \subset B(x,(1+2A)t).$$

Thus, by the doubling condition of the measure μ, it follows that, for all $y \in B(x,t)$,

$$\fint_{B(x,t)} \left|f(y) - B_{\ell,C_1 t} f(y)\right| d\mu(y) \le C \fint_{B(y,(1+A)t)} \left|f(z) - B_{\ell,C_1 t} f(z)\right| d\mu(z),$$

which, by (i), is controlled by $t^2 g_1(y)$ modulus a positive constant C. Thus,

$$\fint_{B(x,t)} \left|f(y) - B_{\ell,C_1 t} f(y)\right| d\mu(y) \le C t^2 \inf_{y \in B(x,t)} g_1(y) \le C t^2 \fint_{B(x,t)} g_1(y)\, d\mu(y).$$

This yields (ii) with $g_2 := C g_1, C_3 = 1$ and $C_2 = C_1$. Thus, the proof of Theorem (4.2.8) is complete.

According to Theorem (4.2.8) and the characterizations in [170], it is very natural to introduce the following notion of Sobolev spaces on $(\mathcal{X}, d, \mu)$.

Definition (4.2.9) [188] Let $\ell \in \mathbb{N}$ and $p \in (1,\infty)$.

(i) The Sobolev space $W^{2\ell,p}(\mathcal{X})$ is defined to be the set of all functions $f \in L^p(\mathcal{X})$ for which either of the condition (i) or (ii) in Theorem (4.2.8) is satisfied. For any $f \in W^{2\ell,p}(\mathcal{X})$, write

$$\|f\|_{W^{2\ell,p}(\mathcal{X})} := \|f\|_{L^p(\mathcal{X})} + \inf_g \left\{\|g\|_{L^p(\mathcal{X})}\right\},$$

where g in the infimum is taken over either all functions g_1 satisfying (48) or functions g_2 satisfying (49).

(ii) For $\alpha \in (0, 2\ell)$, the Sobolev space $\mathcal{W}^{\alpha,p}(\mathcal{X})$ is defined to be the set of all functions $f \in L^p(\mathcal{X})$ for which

$$S_{\alpha,\ell}(f) := \left\{\int_0^\infty \left|f - B_{\ell,t} f\right|^2 \frac{dt}{t^{1+2\alpha}}\right\}^{1/2} \in L^p(\mathcal{X}).$$

For any $f \in \mathcal{W}^{\alpha,p}(\mathcal{X})$, define

$$\|f\|_{\mathcal{W}^{\alpha,p}(\mathcal{X})} := \|f\|_{L^p(\mathcal{X})} + \left\|S_{\alpha,\ell}(f)\right\|_{L^p(\mathcal{X})}.$$

Several remarks are in order.

Lemma (4.2.10) [188] For $\varphi \in S(\mathbb{R}^n)$, it holds true that

$$\lim_{t\to 0^+} \frac{\varphi - B_{t,K}\varphi}{t^2} = P(\partial)\varphi$$

and

$$\lim_{t\to 0^+} \fint_{\cdot + K} \frac{\varphi(y) - B_{t,K}\varphi(y)}{t^2} dy = P(\partial)\varphi$$

in the sense of $S(\mathbb{R}^n)$, where

$$P(\partial) := \fint_K (u\cdot \nabla)^2\, du := \sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}\frac{\partial^2}{\partial x_i \partial x_j}, a_{i,j} := \fint_K x_i x_j dx, i,j \in \{1,\ldots,n\}.$$

Since $P(x) := \sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}x_i x_j$ is an elliptical homogeneous polynomial, it follows that

$$\|P(\partial)f\|_{L^p(\mathbb{R}^n)} \sim \|\Delta f\|_{L^p(\mathbb{R}^n)}, \qquad p \in (1,\infty).$$

Similar to the proof of Theorem (4.2.3), we can show the following conclusion.

Theorem (4.2.11) [188] The conclusions of Theorem (4.2.3) remain true with $B_{t,K}f$ in place of $B_t f$.

Next, for any $\ell \in \mathbb{N}, f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we define

$$B_{\ell,t,K}f(x) := \frac{-2}{(\ell^{2\ell})}\sum_{j=1}^{\ell}(-1)^j \binom{2\ell}{\ell - j} B_{jt,K}f(x).$$

As the higher order variant of Theorem (4.2.11), we have the following result.

Theorem (4.2.12) [188] The conclusion of Theorem (4.2.6) remains valid with $B_{\ell,t}$ therein replaced by $B_{\ell,t,K}$.

Section (4.3) Triebel–Lizorkin Spaces via Averages on Balls

The theory of function spaces with smoothness is a central topic of the analysis on spaces of homogeneous type in the sense of Coifman and Weiss [179, 180]. The first order Sobolev space on spaces of homogeneous type was originally introduced by Hajłasz in [85] and later Shanmugalingam [118] introduced another kind of a first order Sobolev space which has strong locality and hence is more suitable for problems related to partial differential equations on spaces of homogeneous type. Alabern et al. [170] gave a way to introduce Sobolev spaces of any order bigger than 1on spaces of homogeneous type in spirit closer to the square function and Dai et al. [190] gave several other ways, different from [170], to introduce Sobolev spaces of order 2on spaces of homogeneous type in spirit closer to the pointwise characterization as in [85], where $\ell \in \mathbb{N} := \{1, 2, \ldots\}$. Later, motivated by [170], Yang et al. [178] gave a way to introduce Besov and Triebel–Lizorkin spaces with smoothness order in $(0, 2)$ on spaces of homogeneous type. It is still an open question how to introduce Besov and Triebel–Lizorkin spaces with smoothness order not less than 2 on spaces of homogeneous type.

We establish a characterization of Besov and Triebel–Lizorkin spaces which can have any positive smoothness order on $\mathbb{R}^n$ via the difference between functions themselves and their ball averages. Since the average operator used is also well defined on spaces of homogeneous type, this characterization can be used to introduce Besov and Triebel–Lizorkin spaces with any positive smoothness order on any space of homogeneous type and hence our results give an answer to the above open question.

It is well known that a locally integrable function f belongs to the Sobolev space $W^{\alpha,p}(\mathbb{R}^n)$, with $\alpha \in (0,1)$ and $p \in (1,\infty)$, if and only if $f \in L^p(\mathbb{R}^n)$ and

$$s_\alpha(f) := \left\{ \int_0^\infty \left[\fint_{B(\cdot,t)} |f(\cdot) - f(y)|\, dy \right]^2 \frac{dt}{t^{1+2\alpha}} \right\}^{1/2} \in L^p(\mathbb{R}^n)$$

(see, for example, [174, 56, 65, 72]). Here, $B(x,t)$ denotes an open ball with center at $x \in \mathbb{R}^n$ and radius $t \in (0,\infty)$, and $\fint_{B(x,t)} f(y)\, dy$ denotes the integral average of $f \in L^1_{loc}(\mathbb{R}^n)$ on the ball $B(x,t) \subset \mathbb{R}^n$, namely,

$$\fint_{B(x,t)} f(y)\, dy := \frac{1}{|B(x,t)|} \int_{B(x,t)} f(y)\, dy =: B_t f(x). \tag{50}$$

However, when $\alpha \in [1,\infty), s_\alpha(f)$ is not able to characterize $W^{\alpha,p}(\mathbb{R}^n)$, since, in this case, $f \in L^1_{loc}(\mathbb{R}^n)$ and $\|s_\alpha(f)\|_{L^p(\mathbb{R}^n)} < \infty$ imply that f must be a constant function (see [95, Section 4]).

Alabern et al. [170] established a remarkable characterization of Sobolev spaces of smooth order bigger than 1 and they proved that a function $W^{\alpha,p}(\mathbb{R}^n)$, with $\alpha \in (0,2)$ and $p \in (1,\infty)$, if and only if $f \in L^p(\mathbb{R}^n)$ and the square function $S_\alpha f \in L^p(\mathbb{R}^n)$ where

$$S_\alpha(f)(\cdot) := \left\{ \int_0^\infty \left| \fint_{B(\cdot,t)} |f(\cdot) - f(y)|\, dy \right|^2 \frac{dt}{t^{1+2\alpha}} \right\}^{1/2}, \qquad f \in L^1_{loc}(\mathbb{R}^n)$$

(see [170, Theorem 1 and p. 591]). Comparing S_α and s_α, we see that the only difference exists in that the absolute value $|f(\cdot) - f(y)|$ in $s_\alpha(f)$ is replaced by $f(\cdot) - f(y)$ in $S_\alpha(f)$. However, this slight change induces a quite different behaviour between $s_\alpha(f)$ and S_α when characterizing Sobolev spaces. The former characterizes Sobolev spaces only with smoothness order less than 1, while the later characterizes Sobolev spaces with smoothness order less than 2. Such a difference follows from the following observation: for all $f \in C^2(\mathbb{R}^n)$ and $t \in (0,1)$,

$$\fint_{B(x,t)} [f(x) - f(y)] dy = O(t^2), \qquad x \in \mathbb{R}^n, \tag{51}$$

which follows from the Taylor expansion of f up to order 2:

$$f(y) = f(x) + \nabla f(x) \cdot (y - x) + O(|y - x|^2), \qquad x, y \in \mathbb{R}^n;$$

in other words, the S_α-function provides smoothness up to order 2. We point out that this phenomenon was first observed by Wheeden in [173] (see also [174]), and later independently by Alabern, Mateu and Verdera [170].

By means of the fact (51), Alabern et al. [170, Theorems 2 and 3] also characterized Sobolev spaces of higher smoothness order and showed that $f \in W^{\alpha,p}(\mathbb{R}^n)$, with $\alpha \in [2N, 2N+2), N \in \mathbb{N}$ and $p \in (1,\infty)$, if and only if $f \in L^p(\mathbb{R}^n)$ and there exist functions $g_1, \ldots, g_N \in L^p(\mathbb{R}^n)$ such that $S_\alpha(f, g_1, \ldots, g_N) \in L^p(\mathbb{R}^n)$, where

$$S_\alpha(f, g_1, \ldots, g_N)(\cdot) := \left\{ \int_0^\infty \left| \fint_{B(\cdot,t)} t^{-\alpha} R_N(y,\cdot)\, dy \right|^2 \frac{dt}{t} \right\}^{1/2}$$

with

$$R_N(y;\cdot) := f(y) - f(\cdot) - \sum_{j=1}^{N} g_j(\cdot) |y - \cdot|^{2j} \tag{52}$$

when $\alpha \in (2N, 2N+2)$, and

$$R_N(y;\cdot) := f(y) - f(\cdot) - \sum_{j=1}^{N-1} g_j(\cdot)|y-\cdot|^{2j} - B_t g_N(\cdot)|y-\cdot|^{2N} \tag{53}$$

when $\alpha = 2N$. Indeed, the function g_j was proved in [170, Theorems 2 and 3] to equal to $\frac{1}{L_j}\Delta^j f$ almost everywhere, where $L_j := \Delta^j |x|^{2j}$ for $j \in \{1, \ldots, N\}$. As the corresponding results for Triebel–Lizorkin spaces, Yang et al. [170, Theorems 1.1, 1.3 and 4.1] further proved that, for all $\alpha \in (2N, 2N+2), N \in \mathbb{N}$ and $p \in (1, \infty]$, the Besov space $\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)$ with $q \in (0, \infty]$ and the Triebel–Lizorkin space $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ with $q \in (1, \infty]$ can be characterized via the function

$$S_{\alpha,q}(f)(x) := \left\{ \sum_{k\in\mathbb{Z}} \left| \fint_{B(x,2^{-k})} \tilde{R}_N(y;\, x)dy \right|^q \right\}^{1/q}, \qquad x \in \mathbb{R}^n, \tag{54}$$

where, for all $x, y \in \mathbb{R}^n$ and $k \in \mathbb{Z}$,

$$\tilde{R}_N(y;\, x) := f(y) - f(x) - \sum_{j=1}^{N} \frac{1}{L_j} \Delta^j f(x) |y-x|^{2j}. \tag{55}$$

It is an open question, posed in [178, Remark 4.1], whether there exists a corresponding characterization for $\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)$ and $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ when $\alpha = 2N$ with $N \in \mathbb{N}$. Moreover, only when $\alpha \in (0, 2)$, [178, Theorems 1.1 and 4.1] provide a way to introduce Besov and Triebel–Lizorkin spaces with smoothness order α on spaces of homogeneous type.

Via higher order differences, Triebel [69, 63] and Haroske and Triebel [30, 74] obtained another characterization of Sobolev spaces with order bigger than 1 on $\mathbb{R}^n$ without involving derivatives. Recall that, for $\ell \in \mathbb{N}$, the ℓ-th order (forward) difference operator $\widetilde{\Delta}^{\ell}_h$ with $h \in \mathbb{R}^n$ is defined by setting, for all functions f and $x \in \mathbb{R}^n$,

$$\widetilde{\Delta}^1_h f(x) := f(x+h) - f(x), \qquad \widetilde{\Delta}^{\ell}_h := \widetilde{\Delta}^1_h \widetilde{\Delta}^{\ell-1}_h, \qquad \ell \geq 2.$$

By means of $\widetilde{\Delta}^{\ell}_h f$, Triebel [69, 63] and Haroske and Triebel [30, 74] proved that the Sobolev space $W^{\ell,p}(\mathbb{R}^n)$ with $\ell \in \mathbb{N}$ and $p \in (1, \infty)$ can be characterized by a pointwise inequality in the spirit of Hajłasz [26] (see also Hu [85] and Yang [72]). Recall that the difference $\widetilde{\Delta}^{\ell}_h f$ can also be used to characterize Besov spaces and Triebel–Lizorkin spaces with smoothness order no more than ℓ. See Triebel's monograph [66, Section 3.4] for these difference characterizations of Besov and Triebel–Lizorkin spaces; see also [193, Section 3.1]. However, it is still unclear how to define higher than 1 order differences on spaces of homogeneous type.

On the other hand, recall that the averages of a function f can be used to approximate f itself in some function spaces; see [178, 182]. Motivated by (51) and the pointwise characterization of Sobolev spaces with smoothness order no more than 1 (see Hajłasz [26], Hu [85] and Yang [72]), it established in [190] some pointwise characterizations of Sobolev spaces with smoothness order 2ℓ on $\mathbb{R}^n$ via ball averages of f, where $\ell \in \mathbb{N}$. To be precise, as the higher order variants of B_t in (50), for all $\ell \in \mathbb{N}$, $t \in (0, \infty)$ and $x \in \mathbb{R}^n$, we define the 2ℓ-th order average operator $B_{\ell,t}$ by setting, for all $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$B_{\ell,t}f(x) := -\frac{2}{\binom{2\ell}{\ell}}\sum_{j=1}^{\ell}(-1)^j\binom{2\ell}{\ell-j}B_{jt}f(x), \tag{56}$$

here and hereafter, $\binom{2\ell}{\ell-j}$ denotes the binomial coefficients. Obviously, $B_{\ell,t}f = B_t f$. Moreover, it was observed in [190] that $f - B_{\ell,t}f$ is a 2ℓ-th order central difference of the function $t \longmapsto B_t f(x)$ with step t at the origin, namely, for all $\ell \in \mathbb{N}, t \in (0,\infty), f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$f(x) - B_{\ell,t}f(x) = \frac{(-1)^\ell}{\binom{2\ell}{\ell}}\Delta_t^{2\ell}g(0) \tag{57}$$

with

$$g(t) := \begin{cases} B_t f(x), & t \in (0,\infty), \\ f(x), & t = 0, \\ B_{-t}f(x), & t \in (-\infty, 0). \end{cases} \tag{58}$$

Here and hereafter, for all functions h on $\mathbb{R}$ and $\theta, t \in \mathbb{R}$, let $T_\theta h(t) := h(t+\theta)$, and the central difference operators Δ_t^r are defined by setting

$$\Delta_\theta^1 h(t) := \Delta_\theta h(t) := h\left(t+\frac{\theta}{2}\right) - h\left(t-\frac{\theta}{2}\right) = \left(T_{\theta/2} - T_{-\theta/2}\right)h(t),$$

$$\Delta_\theta^r h(t) := \Delta_\theta\left(\Delta_\theta^{r-1}h\right)(t) = \sum_{j=0}^{r}\binom{r}{j}(-1)^j\left(t+\frac{r\theta}{2}-j\theta\right), \qquad r \in \{2,3,\ldots\}.$$

It is proved in [190] that $f \in W^{2\ell,p}(\mathbb{R}^n)$, with $\ell \in \mathbb{N}$ and $p \in (1,\infty)$, if and only if $f \in L^p(\mathbb{R}^n)$ and there exist a non-negative $g \in L^p(\mathbb{R}^n)$ and a positive constant C such that $\left|f(x) - B_{\ell,t}f(x)\right| \leq Ct^{2\ell}g(x)$ for all $t \in (0,\infty)$ and almost every $x \in \mathbb{R}^n$. Various variants of this pointwise characterization were also presented in [190]. Recall that centered averages or their combinations were used to measure the smoothness and to characterize the K-functionals in [189, 191, 192].

Comparing the difference $f - B_{\ell,t}f$ with the usual difference $\tilde{\Delta}_h^{2\ell}f$, we find that the former has an advantage that it involves only averages of f over balls, and hence can be easily generalized to any space of homogeneous type, whereas the difference operator $\tilde{\Delta}_h^{2\ell}f$ cannot. We can also see their difference via (57). Indeed, it follows from (57) that $f - B_{\ell,t}f$ is a 2ℓ-th order central difference of a function g and the parameter related to such a difference is the radius $t \in (0,\infty)$ of the ball $B(x,t)$ with $x \in \mathbb{R}^n$, while the parameter related to $\tilde{\Delta}_h^{2\ell}f$ is $h \in \mathbb{R}^n$, which also curbs the extension of $\tilde{\Delta}_h^{2\ell}f$ to spaces of homogeneous type.

Although there exist differences between $f - B_{\ell,t}f$ and the usual difference $\tilde{\Delta}_h^{2\ell}f$, the characterizations of $W^{2\ell,p}(\mathbb{R}^n)$ via $f - B_{\ell,t}f$ obtained in [190] imply that, in some sense, $f - B_{\ell,t}f$ also plays the role of 2ℓ-order derivatives. Therefore, it is natural to ask whether we can use $f - B_{\ell,t}f$ to characterize Besov and Triebel–Lizorkin spaces with smoothness order less than 2ℓ or not.

Let $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ and $S(\mathbb{R}^n)$ denote the collection of all Schwartz functions on $\mathbb{R}^n$, endowed with the usual topology, and $S'(\mathbb{R}^n)$ its topological dual, namely, the collection of all bounded linear functionals on $S(\mathbb{R}^n)$ endowed with the weak $*$-topology. Let $S_\infty(\mathbb{R}^n)$ be the set of all Schwartz functions φ such that $\int_{\mathbb{R}^n} x^\gamma\varphi(x)dx = 0$ for all $\gamma \in \mathbb{Z}_+^n$, and $S'_\infty(\mathbb{R}^n)$ its topological dual. For all $\alpha \in \mathbb{Z}_+^n$, $m \in \mathbb{Z}_+$ and $\varphi \in S(\mathbb{R}^n)$, let

$$\|\varphi\|_{\alpha,m} := \sup_{x\in\mathbb{R}^n,|\beta|\le|\alpha|} (1 + |x|)^m \left|\partial^\beta \varphi(x)\right|.$$

For all $\varphi \in S'_\infty(\mathbb{R}^n)$, we use $\hat{\varphi}$ to denote its Fourier transform. For any $\varphi \in S(\mathbb{R}^n)$ and $t \in (0,\infty)$, we let $\varphi_t(\cdot) := t^{-n}\varphi(\cdot/t)$.

For all $a \in \mathbb{R}$, $\lfloor a \rfloor$ denotes the maximal integer no more than a. For any $E \subset \mathbb{R}^n$, let χ_E be its characteristic function.

We now recall the notions of Besov and Triebel–Lizorkin spaces; see [65, 66, 115, 195].

Definition (4.3.1) [196] Let $\alpha \in (0,\infty), p, q \in (0,\infty]$ and $\varphi \in S(\mathbb{R}^n)$ satisfy that

$$\operatorname{supp} \hat{\varphi} \subset \{\xi \in \mathbb{R}^n : 1/2 \le |\xi| \le 2\} \text{ and } |\hat{\varphi}(\xi)| \ge \text{constant} > 0 \text{ if } 3/5 \le |\xi| \le 5/3. \quad (59)$$

(i) The homogeneous Besov space $\dot{B}^\alpha_{p,q}(\mathbb{R}^n)$ is defined as the collection of all $f \in S'_\infty(\mathbb{R}^n)$ such that $\|f\|_{\dot{B}^\alpha_{p,q}(\mathbb{R}^n)} < \infty$, where

$$\|f\|_{\dot{B}^\alpha_{p,q}(\mathbb{R}^n)} := \left[\sum_{k\in\mathbb{Z}} 2^{k\alpha q} \left\|\varphi_{2^{-k}} * f\right\|^q_{L^p(\mathbb{R}^n)}\right]^{1/q}$$

with the usual modifications made when $p = \infty$ or $q = \infty$.

(ii) The homogeneous Triebel–Lizorkin space $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$ is defined as the collection of all $f \in S'_\infty(\mathbb{R}^n)$ such that $\|f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)} < \infty$, where, when $p \in (0,\infty)$,

$$\|f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)} := \left\|\left[\sum_{k\in\mathbb{Z}} 2^{k\alpha q} \left|\varphi_{2^{-k}} * f(y)\right|^q\right]^{1/q}\right\|_{L^p(\mathbb{R}^n)}$$

with the usual modification made when $q = \infty$, and

$$\|f\|_{\dot{F}^\alpha_{\infty,q}(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n} \sup_{m\in\mathbb{Z}} \left\{ \fint_{B(x,2^{-m})} \sum_{k=m}^{\infty} 2^{k\alpha q} \left|\varphi_{2^{-k}} * f(y)\right|^q dy \right\}^{1/q}$$

with the usual modification made when $q = \infty$.

It is well known that the spaces $\dot{B}^\alpha_{p,q}(\mathbb{R}^n)$ and $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$ are independent of the choice of functions φ satisfying (59); see [171].

We also recall the corresponding inhomogeneous spaces.

Definition (4.3.2) [196] Let $\alpha \in (0,\infty), p, q \in (0,\infty], \varphi \in S(\mathbb{R}^n)$ satisfy (59) and $\phi \in S(\mathbb{R}^n)$ satisfy that

$$\operatorname{supp} \hat{\phi} \subset \{\xi \in \mathbb{R}^n : |\xi| \le 2\} \text{ and } \left|\hat{\phi}(\xi)\right| \ge \text{constant} > 0 \text{ if } |\xi| \le 5/3. \quad (60)$$

(i) The inhomogeneous Besov space $B^\alpha_{p,q}(\mathbb{R}^n)$ is defined as the collection of all $f \in S'(\mathbb{R}^n)$ such that $\|f\|_{B^\alpha_{p,q}(\mathbb{R}^n)} < \infty$, where

$$\|f\|_{B^\alpha_{p,q}(\mathbb{R}^n)} := \left[\sum_{k\in\mathbb{Z}_+} 2^{k\alpha q} \left\|\varphi_{2^{-k}} * f\right\|^q_{L^p(\mathbb{R}^n)}\right]^{1/q}$$

with the usual modifications made when $p = \infty$ or $q = \infty$, where, when $k = 0, \varphi_{2^{-k}}$ is replaced by ϕ.

(ii) The inhomogeneous Triebel–Lizorkin space $F^\alpha_{p,q}(\mathbb{R}^n)$ is defined as the collection of all $f \in S'(\mathbb{R}^n)$ such that $\|f\|_{F^\alpha_{p,q}(\mathbb{R}^n)} < \infty$, where, when $p \in (0,\infty)$,

$$\|f\|_{F^\alpha_{p,q}(\mathbb{R}^n)} := \left\|\left[\sum_{k\in\mathbb{Z}_+} 2^{k\alpha q} \left|\varphi_{2^{-k}} * f(y)\right|^q\right]^{1/q}\right\|_{L^p(\mathbb{R}^n)}$$

with the usual modification made when $q=\infty$, and

$$\|f\|_{F^{\alpha}_{\infty,q}(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n}\sup_{m\in\mathbb{Z}_+}\left\{\fint_{B(x,2^{-m})}\sum_{k=m}^{\infty}2^{k\alpha q}\left|\varphi_{2^{-k}}*f\right|^q dy\right\}^{1/q}$$

with the usual modification made when $q=\infty$, where, when $k=0, \varphi_{2^{-k}}$ is replaced by ϕ.

It is also well known that the spaces $B^{\alpha}_{p,q}(\mathbb{R}^n)$and $F^{\alpha}_{p,q}(\mathbb{R}^n)$ are independent of the choice of functions φ and ϕ satisfying (59) and (60), respectively; see, for example, [65].

We prove that the difference $f-B_{\ell,2^{-k}}f$ with $k\in\mathbb{Z}$ plays the same role of the approximation to the identity $\varphi_{2^{-k}}*f$ in the definitions of Besov and Triebel–Lizorkin spaces in the following sense.

We need some technical lemmas. Let, for all $t\in(0,\infty)$ and $x\in\mathbb{R}^n$, $I(x) := \frac{1}{|B(0,1)|}\chi_{B(0,1)}(x)$ and $I_t(x) := t^{-n}I(x/t)$. Then

$$\left(B_{\ell,t}f\right)(x) = \frac{-2}{\binom{2\ell}{\ell}}\sum_{j=1}^{\ell}(-1)^j\binom{2\ell}{\ell-j}\left(f*I_{jt}\right)(x), \qquad x\in\mathbb{R}^n, \qquad t\in(0,\infty),$$

and hence

$$\left(B_{\ell,t}f\right)^{\wedge}(\xi) = m_\ell(t\xi)\hat{f}(\xi), \qquad \xi\in\mathbb{R}^n, \tag{61}$$

where

$$m_\ell(x) := \frac{-2}{\binom{2\ell}{\ell}}\sum_{j=1}^{\ell}(-1)^j\binom{2\ell}{\ell-j}\hat{I}(jx), \qquad x\in\mathbb{R}^n. \tag{62}$$

A straightforward calculation shows that

$$\hat{I}(x) = \gamma_n\int_0^1\cos(u|x|)\,(1-u^2)^{\frac{n-1}{2}}du, \qquad x\in\mathbb{R}^n, \tag{63}$$

with $\gamma_n := [\int_0^1(1-u^2)^{\frac{n-1}{2}}du]^{-1}$ (see also Stein's book [194, p. 430, Section 6.19]).

Lemma (4.3.3) [196] For all $\ell\in\mathbb{N}$ and $x\in\mathbb{R}^n$,

$$m_\ell(x) = 1-A_\ell(|x|), \tag{64}$$

where

$$A_\ell(s) := \gamma_n\frac{4^\ell}{\binom{2\ell}{\ell}}\int_0^1(1-u^2)^{\frac{n-1}{2}}\left(\sin\frac{us}{2}\right)^{2\ell}du, \qquad s\in\mathbb{R}. \tag{65}$$

Furthermore, $s^{-2\ell}A_\ell(s)$ is a smooth function on $\mathbb{R}$ satisfying that there exist positive constants c_1 and c_2 such that

$$0<c_1\le\frac{A_\ell(s)}{s^{2\ell}}\le c_2, \qquad s\in(0,4] \tag{66}$$

and

$$\sup_{s\in\mathbb{R}}\left|\left(\frac{d}{ds}\right)^i\left(\frac{A_\ell(s)}{s^{2\ell}}\right)\right|<\infty, \qquad i\in\mathbb{N}.$$

Proof. Combining (62) with (63), we obtain

$$m_\ell(x) = \frac{-2\gamma_n}{\binom{2\ell}{\ell}}\int_0^1\left[\sum_{j=1}^{\ell}(-1)^j\binom{2\ell}{\ell-j}\cos(ju|x|)\right](1-u^2)^{\frac{n-1}{2}}du, \qquad x\in\mathbb{R}^n. \tag{67}$$

However, a straightforward calculation shows that, for all $s\in\mathbb{R}$,

$$4^\ell\left(\sin\frac{s}{2}\right)^{2\ell}=\binom{2\ell}{\ell}+\sum_{j=1}^{\ell}(-1)^j\binom{2\ell}{\ell-j}\cos(js).$$

This, together with (67), implies (64).

Next we show (66). By the mean value theorem, we know that, for all $u\in(0,1)$ and $s\in\mathbb{R}$, there exists $\theta\in(0,1)$ such that

$$\left(\sin\frac{us}{2}\right)^{2\ell}=\left(\frac{1}{2}us\right)^{2\ell}\left(\cos\frac{us\theta}{2}\right)^{2\ell}.$$

From this and (65), we deduce that, for all $s\in(0,4]$,

$$\frac{A_\ell(s)}{s^{2\ell}}\le\gamma_n\frac{4^\ell}{2\binom{2\ell}{\ell}}\int_0^1(1-u^2)^{\frac{n-1}{2}}u^{2\ell}du=:c_2<\infty$$

and

$$\frac{A_\ell(s)}{s^{2\ell}}\ge\gamma_n\frac{4^\ell}{2\binom{2\ell}{\ell}}\int_0^{\min\left\{1,\frac{2\pi}{3s}\right\}}(1-u^2)^{\frac{n-1}{2}}u^{2\ell}\left(\cos\frac{us\theta}{2}\right)^{2\ell}du$$

$$\ge\gamma_n\frac{1}{2\binom{2\ell}{\ell}}\int_0^{\min\left\{1,\frac{2\pi}{3s}\right\}}(1-u^2)^{\frac{n-1}{2}}u^{2\ell}\,du\ge\gamma_n\frac{1}{2\binom{2\ell}{\ell}}\int_0^{\frac{\pi}{6}}(1-u^2)^{\frac{n-1}{2}}u^{2\ell}\,du=:c_1>0.$$

These prove (66).

Finally, by the mean value theorem again, an argument similar to the above also implies that

$$\sup_{s\in\mathbb{R}}\left|\left(\frac{d}{ds}\right)^i\left(\frac{A_\ell(s)}{s^{2\ell}}\right)\right|<\infty$$

for all $i\in\mathbb{N}$. This finishes the proof of Lemma (4.3.3).

Recall that the Hardy–Littlewood maximal operator M is defined by setting, for all $f\in L^1_{loc}(\mathbb{R}^n)$,

$$Mf(x):=\sup_{B\subset\mathbb{R}^n}\fint_B|f(y)|\,dy,\qquad x\in\mathbb{R}^n,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$ containing x. The following two lemmas can be verified straightforwardly.

Lemma (4.3.4) [196] Let $\{T_t\}_{t\in(0,\infty)}$ be a family of multiplier operators given by setting, for all $f\in L^2(\mathbb{R}^n)$,

$$(T_tf)^\wedge(\xi):=m(t\xi)\hat{f}(\xi),\qquad \xi\in\mathbb{R}^n,\qquad t\in(0,\infty)$$

for some $m\in L^\infty(\mathbb{R}^n)$. If

$$\|\nabla^{n+1}m\|_{L^1(\mathbb{R}^n)}+\|m\|_{L^1(\mathbb{R}^n)}\le C_1<\infty,$$

then there exists a positive constant C such that, for all $f\in L^2(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$,

$$\sup_{t\in(0,\infty)}|T_tf(x)|\le CC_1Mf(x).$$

Proof. For all $t\in(0,\infty)$, $f\in L^2(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$, by the Fubini theorem, we see that

$$|T_tf(x)|=\left|\int_{\mathbb{R}^n}m(t\xi)\hat{f}(\xi)e^{ix\cdot\xi}d\xi\right|=\left|\int_{\mathbb{R}^n}f(y)\int_{\mathbb{R}^n}m(t\xi)\hat{f}(\xi)e^{i(x-y)\cdot\xi}d\xi\,dy\right|$$

$$\leq \left| \int_{|x-y|<t} f(y) \int_{\mathbb{R}^n} m(t\xi)\hat{f}(\xi)e^{i(x-y)\cdot\xi} d\xi\, dy \right| + \left| \int_{|x-y|\geq t} \cdots \right| =: \mathrm{I} + \mathrm{II}.$$

It is easy to see that $\mathrm{I} \lesssim \|m\|_{L^1(\mathbb{R}^n)} Mf(x)$.

For II, via the Fubini theorem and the integration by parts, we also have

$$\mathrm{II} \lesssim \int_{|x-y|\geq t} \frac{|f(y)|}{|x-y|^{n+1}} \int_{\mathbb{R}^n} t^{n+1} |\nabla^{n+1} m(t\xi)| d\xi\, dy$$

$$\lesssim \|\nabla^{n+1} m\|_{L^1(\mathbb{R}^n)} \sum_{j=1}^{\infty} t \int_{2^j t \leq |x-y| < 2^{j+1} t} \frac{|f(y)|}{|x-y|^{n+1}} dy$$

$$\lesssim \|\nabla^{n+1} m\|_{L^1(\mathbb{R}^n)} \sum_{j=1}^{\infty} 2^{-j} Mf(x) \lesssim \|\nabla^{n+1} m\|_{L^1(\mathbb{R}^n)} Mf(x),$$

which completes the proof of Lemma (4.3.4).

Lemma (4.3.5) [196] Let $\{a_j\}_{j\in\mathbb{Z}} \subset \mathbb{C}, q \in (0,\infty]$ and $\beta \in (0,\infty)$. Then there exists a positive constant C, independent of $\{a_j\}_{j\in\mathbb{Z}}$, such that

$$\left[\sum_{k\in\mathbb{Z}} 2^{k\beta q} \left(\sum_{j=k}^{\infty} |a_j|\right)^q\right]^{1/q} \leq C \left(\sum_{k\in\mathbb{Z}} 2^{k\beta q} |a_k|^q\right)^{1/q}$$

and

$$\left[\sum_{k\in\mathbb{Z}} 2^{-k\beta q} \left(\sum_{j=-\infty}^{k} |a_j|\right)^q\right]^{1/q} \leq C \left(\sum_{k\in\mathbb{Z}} 2^{-k\beta q} |a_k|^q\right)^{1/q}$$

Theorem (4.3.6) [196] Let $\ell \in \mathbb{N}$ and $\alpha \in (0, 2\ell)$.

(i) Let $p \in (1,\infty]$ and $q \in (0,\infty]$. If $f \in \dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)$, then there exists $g \in L^1_{loc}(\mathbb{R}^n) \cap S'_\infty(\mathbb{R}^n)$ such that $g = f$ in $S'_\infty(\mathbb{R}^n)$ and $|||g|||_{\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)} \leq C\|f\|_{\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)}$ for some positive constant C independent of f, where

$$|||g|||_{\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)} := \left\{\sum_{k\in\mathbb{Z}} 2^{k\alpha q} \left\|g - B_{\ell,2^{-k}} g\right\|^q_{L^p(\mathbb{R}^n)}\right\}^{1/q}.$$

Conversely, if $f \in L^1_{loc}(\mathbb{R}^n) \cap S'_\infty(\mathbb{R}^n)$ and $|||f|||_{\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)} < \infty$, then $f \in \dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)$ and $\|f\|_{\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)} \leq C|||f|||_{\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)}$ for some positive constant C independent of f.

(ii) Let $p \in (1,\infty]$ and $q \in (1,\infty]$. If $f \in \dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$, then there exists $g \in L^1_{loc}(\mathbb{R}^n) \cap S'_\infty(\mathbb{R}^n)$ such that $g = f$ in $S'_\infty(\mathbb{R}^n)$ and $|||g|||_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)} \leq C\|f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)}$ for some positive constant C independent of f, where, when $p \in (1,\infty)$,

$$|||g|||_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)} := \left\|\left\{\sum_{k\in\mathbb{Z}} 2^{k\alpha q} \left|g - B_{\ell,2^{-k}} g\right|^q\right\}^{1/q}\right\|_{L^p(\mathbb{R}^n)}$$

and, when $p = \infty$,

$$\|g\|_{\dot{F}^{\alpha}_{\infty,q}(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n} \sup_{m\in\mathbb{Z}} \left\{ \fint_{B(x,2^{-m})} \sum_{k=m}^{\infty} 2^{k\alpha q} \left|g(y) - B_{\ell,2^{-k}} g(y)\right|^q dy \right\}^{1/q}.$$

Conversely, if $f \in L^1_{loc}(\mathbb{R}^n) \cap S'_\infty(\mathbb{R}^n)$ and $|||f|||_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)} < \infty$, then $f \in \dot{F}^\alpha_{p,q}(\mathbb{R}^n)$ and $\|f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)} \le C\|f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)}$ for some positive constants C independent of f.

Proof. We only prove (ii), the proof of (i) being similar and easier.

To show (ii), let $\varphi \in S(\mathbb{R}^n)$ satisfy (59) and $\sum_{j\in\mathbb{Z}} \hat{\varphi}_j \equiv 1$ on $\mathbb{R}^n\backslash\{0\}$. Assume first that $\alpha \in (0, 2\ell), p \in (1,\infty)$ and $q \in (1,\infty]$. Let $f \in \dot{F}^\alpha_{p,q}(\mathbb{R}^n)$. We know that $\dot{F}^\alpha_{p,q}(\mathbb{R}^n) \hookrightarrow L^1_{loc}(\mathbb{R}^n)$ in the sense of distributions; see, for example, [90, Proposition 4.2], [176, Proposition 5.1] or [195, Proposition 8.2] for a proof. Indeed, it was proved therein that there exists a sequence $\{P_j\}_{j\in\mathbb{Z}}$ of polynomials of degree not more than $\alpha - n/p$ such that the summation $\sum_{j\in\mathbb{Z}}(\varphi_j * f + P_j)$ converges in $L^1_{loc}(\mathbb{R}^n)$ and $S'_\infty(\mathbb{R}^n)$ to a function $g \in L^1_{loc}(\mathbb{R}^n)$, which is known to be the Calderón reproducing formula (see, for example, [115, 171]). The function g serves as a representative of f. Thus, in the below proof, we identify f with g. Then $g \in L^1_{loc}(\mathbb{R}^n) \cap S'_\infty(\mathbb{R}^n)$. Now we show $|||g|||_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)}$, namely,

$$\left\|\left\{\sum_{k\in\mathbb{Z}} 2^{k\alpha q}\left|g - B_{\ell,2^{-k}}g\right|^q\right\}^{1/q}\right\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)}. \tag{68}$$

To this end, for all $k, j \in \mathbb{Z}$ and $\xi \in \mathbb{R}^n\backslash\{0\}$, define $T_{k,j}$ as

$$\left(T_{k,j}f\right)^\wedge(\xi) := \hat{\varphi}\left(2^{-j}\xi\right)A_\ell(2^{-k}|\xi|)\hat{f}(\xi), \qquad \xi \in \mathbb{R}^n. \tag{69}$$

Noticing that the degree of each P_j is not more than $\lfloor \alpha - n/p \rfloor < 2\ell$ and $P - B_{\ell,2^{-k}}P = 0$ for all polynomials P of degree less than 2ℓ, we then find that

$$g - B_{\ell,2^{-k}}g = \sum_{j\in\mathbb{Z}} T_{k,j}f. \tag{70}$$

We split the sum $\sum_{j\in\mathbb{Z}}$ in this last equation into two parts $\sum_{j\ge k}$ and $\sum_{j<k}$. The first part is relatively easy to deal with. Indeed, for $j \ge k$, by (69), we see that, for all $x \in \mathbb{R}^n$,

$$\begin{aligned}\left|T_{k,j}f(x)\right| &= \left|(I - B_{\ell,2^{-k}})(f * \varphi_{2^{-j}})(x)\right| \le \left|f * \varphi_{2^{-j}}(x)\right| + C_\ell \sum_{i=1}^{\ell}\left|B_{i2^{-k}}(f * \varphi_{2^{-j}})(x)\right| \\ &\lesssim M(f * \varphi_{2^{-j}})(x). \end{aligned} \tag{71}$$

From this and Lemma (4.3.5), it follows that

$$\sum_{k\in\mathbb{Z}} 2^{k\alpha q}\left|\sum_{j\ge k} T_{k,j}f\right|^q \lesssim \sum_{k\in\mathbb{Z}} 2^{k\alpha q}\left[\sum_{j\ge k} M\left(f * \varphi_{2^{-j}}\right)\right]^q \lesssim \sum_{j\ge k} 2^{jq\alpha}\left[M\left(f * \varphi_{2^{-j}}\right)\right]^q \tag{72}$$

Now we handle the sum $\sum_{j<k}$. Since φ satisfies (59), by [171, Lemma (6.9)], there exists $\psi \in S(\mathbb{R}^n)$ satisfying (59) such that

$$\sum_{j\in\mathbb{Z}} \hat{\varphi}\left(2^{-j}\xi\right)\hat{\psi}\left(2^{-j}\xi\right) = 1, \qquad \xi \in \mathbb{R}^n\backslash\{0\}.$$

Thus, for all $\xi \in \mathbb{R}^n\backslash\{0\}$,

$$\left(T_{k,j}f\right)^\wedge(\xi) = \hat{\varphi}\left(2^{-j}\xi\right)A_\ell(2^{-k}|\xi|)\hat{f}_j(\xi) = m_{k,j}(\xi)\hat{f}_j(\xi),$$

where $f_j := \sum_{i=-1}^{1} f * \psi_{2^{i-j}}$ and

$$m_{k,j}(\xi) := \hat{\varphi}\left(2^{-j}\xi\right)\frac{A_\ell(2^{-k}|\xi|)}{(2^{-k}|\xi|)^{2\ell}}(2^{-k}|\xi|)^{2\ell}, \qquad \xi \in \mathbb{R}^n\backslash\{0\}.$$

Write $\tilde{m}_{k,j}(\xi) := m_{k,j}\left(2^j\xi\right)$. From Lemma (4.3.3), it follows that, for all $j < k$ and $\xi \in \mathbb{R}^n\backslash\{0\}$,

$$\left|\partial^\beta \widetilde{m}_{k,j}(\xi)\right| \lesssim 2^{2\ell(j-k)}\chi_{\overline{B(0,2)}\backslash B(0,1/2)}(\xi), \qquad \beta \in \mathbb{Z}_+^d, \tag{73}$$

and hence $\left\|\widetilde{m}_{k,j}\right\|_{L^1(\mathbb{R}^n)} + \left\|\nabla^{n+1}\widetilde{m}_{k,j}\right\|_{L^1(\mathbb{R}^n)} \lesssim 2^{2\ell(j-k)}$, which, together with Lemma (4.3.4), implies that

$$\left|T_{k,j}f(x)\right| \lesssim 2^{2\ell(j-k)}Mf_j(x), \quad x \in \mathbb{R}^n.$$

Thus, by Lemma (4.3.5), for $\alpha \in (0, 2\ell)$, we have

$$\sum_{k\in\mathbb{Z}} 2^{k\alpha q}\left|\sum_{j=-\infty}^{k} T_{k,j}f\right|^q \lesssim \sum_{k\in\mathbb{Z}} 2^{k(\alpha-2\ell)q}\left(\sum_{j=-\infty}^{k} 2^{2\ell j}Mf_j\right)^q \lesssim \sum_{j\in\mathbb{Z}} 2^{jq\alpha}\left[Mf_j\right]^q. \tag{74}$$

Combining (72) and (74) with (70), and using the Fefferman–Stein vector-valued maximal inequality (see [112] or [194]), we see that

$$\left\|\left\{\sum_{k\in\mathbb{Z}} 2^{k\alpha q}\left|g - B_{\ell,2^{-k}}g\right|^q\right\}^{1/q}\right\|_{L^p(\mathbb{R}^n)} \lesssim \left\|\left\{\sum_{k\in\mathbb{Z}} 2^{k\alpha q}\left[M\left(f * \varphi_{2^{-k}}\right)\right]^q\right\}^{1/q}\right\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)}.$$

This proves (68) and hence finishes the proof of the first part of Theorem (4.3.6) (ii).

To see the inverse conclusion, we only need to prove

$$\|f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)} \lesssim \left\|\left\{\sum_{k\in\mathbb{Z}} 2^{k\alpha q}\left|f - B_{\ell,2^{-k}}f\right|^q\right\}^{1/q}\right\|_{L^p(\mathbb{R}^n)} \tag{75}$$

whenever $f \in L^1_{loc}(\mathbb{R}^n) \cap S'_\infty(\mathbb{R}^n)$ and the right-hand side of (75) is finite. To this end, we first claim that

$$\left|f * \varphi_{2^{-j}}(x)\right| \lesssim M\left(f - B_{\ell,2^{-k}}f\right)(x), \qquad j \in \mathbb{Z}, \qquad x \in \mathbb{R}^n. \tag{76}$$

Indeed, we see that, for all $j \in \mathbb{Z}$ and $\xi \in \mathbb{R}^n\backslash\{0\}$,

$$(f * \varphi_{2^{-j}})^\wedge(\xi) = \frac{\hat{\varphi}\left(2^{-j}\xi\right)}{A_\ell(2^{-j}|\xi|)}(f - B_{\ell,2^{-j}}f)^\wedge(\xi) =: \eta(2^{-j}\xi)(f - B_{\ell,2^{-j}}f)^\wedge(\xi),$$

where $\eta(\xi) := \frac{\hat{\varphi}(\xi)}{A_\ell(|\xi|)}$ for all $\xi \in \mathbb{R}^n\backslash\{0\}$, which is well defined due to (66). By Lemma (4.3.3), we know that $\eta \in C_c^\infty(\mathbb{R}^n)$ and $\operatorname{supp}\eta \subset \left\{\xi \in \mathbb{R}^n : \frac{1}{2} \le |\xi| \le 2\right\}$. The claim (76) then follows from Lemma (4.3.4).

Now, using the claim (76) and the Fefferman–Stein vector-valued maximal inequality (see [112] or [194]), we find that

$$\|f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)} \lesssim \left\|\left\{\sum_{j\in\mathbb{Z}} 2^{j\alpha q}\left[M\left(f - B_{\ell,2^{-j}}f\right)\right]^q\right\}^{1/q}\right\|_{L^p(\mathbb{R}^n)} \lesssim \left\|\left\{\sum_{j\in\mathbb{Z}} 2^{j\alpha q}\left|f - B_{\ell,2^{-j}}f\right|^q\right\}^{\frac{1}{q}}\right\|_{L^p(\mathbb{R}^n)}$$
$$\sim \; |||f|||_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)}.$$

This proves the desired conclusion when $\alpha \in (0, 2\ell), p \in (1, \infty)$ and $q \in (1, \infty]$.

It remains to consider the case that $\alpha \in (0, 2\ell), p = \infty$ and $q \in (1, \infty]$. The proof is similar to that of the case $p \in (1, \infty)$ but more subtle. Assume first that $f \in \dot{F}^\alpha_{\infty,q}(\mathbb{R}^n)$. By an argument similar to the above, in this case, we need to show

$$\sup_{x\in\mathbb{R}^n}\sup_{m\in\mathbb{Z}}\left\{\fint_{B(x,2^{-m})}\sum_{k=m}^{\infty} 2^{k\alpha q}\left|g(y) - B_{\ell,2^{-k}}g(y)\right|^q dy\right\}^{1/q} \lesssim \|f\|_{\dot{F}^\alpha_{\infty,q}(\mathbb{R}^n)}. \tag{77}$$

Notice that, if $y \in B(x, 2^{-m})$ and $z \in B(y, i2^{-k})$ with $k \geq m$ and $i \in \{1, \ldots, \ell\}$, then $z \in B(x, (\ell+1)2^{-m})$. Then, similar to (71), we know that, for $j \geq k$ and $y \in B(x, 2^{-m})$,

$$\left|T_{k,j}f(y)\right| = \left|\left(I - B_{\ell,2^{-k}}\right)\left(f * \varphi_{2^{-j}}\right)(y)\right| \leq \left|f * \varphi_{2^{-j}}(y)\right| + C_\ell \sum_{i=1}^{\ell} \left|B_{i2^{-k}}\left(f * \varphi_{2^{-j}}\right)(y)\right|$$
$$\lesssim M\left(\left|f * \varphi_{2^{-j}}\right| \chi_{B(x,(\ell+1)2^{-m})}\right)(y),$$

which, together with (70) and Lemma (4.3.5), implies that

$$\sum_{k \geq m} 2^{k\alpha q} \left|\sum_{j \geq k} T_{k,j}f(y)\right|^q \lesssim \sum_{k \geq m} 2^{k\alpha q} \left[\sum_{j \geq k} M\left(\left|f * \varphi_{2^{-j}}\right| \chi_{B(x,(\ell+1)2^{-m})}\right)(y)\right]^q$$
$$\lesssim \sum_{j \geq m} 2^{j\alpha q} \left[M\left(\left|f * \varphi_{2^{-j}}\right| \chi_{B(x,(\ell+1)2^{-m})}\right)(y)\right]^q. \tag{78}$$

When $m \leq j < k$, by (73), instead of Lemma (4.3.4), we find that, for all $k \geq m$, integer $l \geq n+1$ and $y \in B(x, 2^{-m})$,

$$\left|T_{k,j}f(y)\right| \lesssim 2^{2\ell(j-k)} \sum_{i=0}^{\infty} 2^{-i(l-n)} M\left(f_j \chi_{B(x,2^{i-j}+2^{-m})}\right)(y)$$

and hence, by Lemma (4.3.5) and the Minkowski inequality, we see that

$$\left\{\sum_{k \geq m} 2^{k\alpha q} \left|\sum_{j=m}^{k} T_{k,j}f(y)\right|^q\right\}^{1/q}$$
$$\lesssim \sum_{i=0}^{\infty} 2^{-i(l-n)} \left\{\sum_{k \geq m} 2^{k(\alpha-2\ell)q} \left[\sum_{j=m}^{k} M\left(f_j \chi_{B\left(x,\left(2^i+1\right)2^{-m}\right)}\right)(y)\right]^q\right\}^{1/q}$$
$$\lesssim \sum_{i=0}^{\infty} 2^{-i(l-n)} \left\{\sum_{j=m}^{k} 2^{j\alpha q} \left[M\left(f_j \chi_{B(x,(2^i+1)2^{-m})}\right)(y)\right]^q\right\}^{1/q}. \tag{79}$$

When $j < m \leq k$, we invoke (73) to find that, for all $y \in \mathbb{R}^n$,

$$\left|T_{k,j}f(y)\right| \leq \left|\int_{|z-y|<2^{-j}} f_j(z) \int_{\mathbb{R}^n} \widetilde{m}_{k,j}\left(2^{-j}\xi\right) e^{i(x-y)\cdot\xi} d\xi dz\right| + \left|\int_{|z-y|\geq 2^{-j}} \cdots\right|$$
$$\lesssim 2^{2\ell(j-k)} \fint_{|z-y|<2^{-j}} \left|f_j(z)\right| dz + \int_{|z-y|\geq 2^{-j}} \frac{\left|f_j(z)\right|}{|z-y|^l} \int_{\mathbb{R}^n} \left|\nabla^l \widetilde{m}_{k,j}\left(2^{-j}\xi\right)\right| d\xi \, dz$$
$$\lesssim 2^{2\ell(j-k)} \sum_{i=0}^{\infty} 2^{-i(l-n)} \fint_{|z-y|\sim 2^{i-j}} \left|f_j(z)\right| dz$$
$$\lesssim 2^{2\ell(j-k)} \sum_{i=0}^{\infty} 2^{-i(l-n)} 2^{-j\alpha} \|f\|_{\dot{F}^\alpha_{\infty,q}(\mathbb{R}^n)} \lesssim 2^{2\ell(j-k)} 2^{-j\alpha} \|f\|_{\dot{F}^\alpha_{\infty,q}(\mathbb{R}^n)}, \tag{80}$$

where $|z-y| \sim 2^{i-j}$ means that $2^{i-j-1} \leq |z-y| < 2^{i-j}$ and we chose $l > n$.

Combining (70), (78), (79) and (80), and applying the Minkowski inequality and the boundedness of M on $L^q(\mathbb{R}^n)$ with $q \in (1, \infty]$, we know that, for all $m \in \mathbb{Z}$ and $x \in \mathbb{R}^n$,

$$\left\{ \fint_{B(x,2^{-m})} \sum_{k=m}^{\infty} 2^{k\alpha q} \left| g(y) - B_{\ell,2^{-k}} g(y) \right|^q dy \right\}^{1/q}$$

$$\lesssim \left\{ \fint_{B(x,2^{-m})} \sum_{j \geq m} 2^{j\alpha q} \left[M\left(\left| f * \varphi_{2^{-j}} \right| \chi_{B(x,(\ell+1)2^{-m})} \right)(y) \right]^q dy \right\}^{1/q}$$

$$+ \sum_{i=0}^{\infty} 2^{-i(l-n)} \left\{ \fint_{B(x,2^{-m})} \sum_{j \geq m} 2^{j\alpha q} \left[M\left(f_j \chi_{B(x,(2^i+1)2^{-m})} \right)(y) \right]^q dy \right\}^{1/q}$$

$$+ \left\{ \fint_{B(x,2^{-m})} \sum_{j \geq m} 2^{k\alpha q} \left[\sum_{j \leq m-1} 2^{2\ell(j-k)} 2^{-j\alpha} \right]^q dy \right\}^{1/q} \|f\|_{\dot{F}^{\alpha}_{\infty,q}(\mathbb{R}^n)}$$

$$\lesssim \left\{ \fint_{B(x,(\ell+1)2^{-m})} \sum_{j \geq m} 2^{j\alpha q} \left| f * \varphi_{2^{-j}}(y) \right| dy \right\}^{1/q}$$

$$+ \sum_{i=0}^{\infty} 2^{-i(l-n)} 2^{in/q} \left\{ \fint_{B\left(x,(2^i+1)2^{-m}\right)} \sum_{j \geq m} 2^{j\alpha q} \left[f_j(y) \right]^q dy \right\}^{1/q} + \|f\|_{\dot{F}^{\alpha}_{\infty,q}(\mathbb{R}^n)}$$

$$\lesssim \|f\|_{\dot{F}^{\alpha}_{\infty,q}(\mathbb{R}^n)},$$

where we took $l > n(1 + 1/q)$. This proves (77).

Finally, the inverse estimate of (77) is deduced from an argument similar to that used in the above proof for (77), with $\widetilde{m}_{k,j}$ and f_j therein replaced by $\eta := \frac{\widehat{\varphi}}{A_\ell(|\cdot|)}$ and $f - B_{\ell,2^{-j}} f$, respectively. This finishes the proof for the case $\alpha \in (0,2), p = \infty$ and $q \in (1,\infty]$, and hence Theorem (4.3.6).

We first present the inhomogeneous version of Theorem (4.3.6). As a further generalization, we show that the conclusions of Theorems (4.3.6) and (4.3.7) remain valid on Euclidean spaces with non-Euclidean metrics.

It is known that, when $p \in (1,\infty)$ and $\alpha \in (0,\infty)$, then $B^{\alpha}_{p,q}(\mathbb{R}^n) \cup F^{\alpha}_{p,q}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$, while when $p = \infty$ and $\alpha \in (0,\infty)$, then $B^{\alpha}_{\infty,q}(\mathbb{R}^n) \cup F^{\alpha}_{\infty,q}(\mathbb{R}^n) \subset C(\mathbb{R}^n)$, where $C(\mathbb{R}^n)$ denotes the set of all complex-valued uniformly continuous functions on $\mathbb{R}^n$ equipped with the sup-norm; see, for example, [55, Theorem 3.3.1] and [193, Chapter 2.4, Corollary 2].

Theorem (4.3.7) [196] Let $\ell \in \mathbb{N}$ and $\alpha \in (0,2)$.

(i) Let $q \in (0,\infty]$. Then $f \in B^{\alpha}_{p,q}(\mathbb{R}^n)$ if and only if $f \in L^p(\mathbb{R}^n)$ when $p \in (1,\infty)$ or $f \in C(\mathbb{R}^n)$ when $p = \infty$, and

$$|||f|||_{B^{\alpha}_{p,q}(\mathbb{R}^n)} := \|f\|_{L^p(\mathbb{R}^n)} + \left\{ \sum_{k=1}^{\infty} 2^{k\alpha q} \left\| f - B_{\ell,2^{-k}} f \right\|^q_{L^p(\mathbb{R}^n)} \right\}^{1/q} < \infty.$$

Moreover, $|||\cdot|||_{B^{\alpha}_{p,q}(\mathbb{R}^n)}$ is equivalent to $\|f\|_{B^{\alpha}_{p,q}(\mathbb{R}^n)}$.

(ii) Let $p \in (1,\infty]$ and $q \in (1,\infty]$. Then $f \in F^{\alpha}_{p,q}(\mathbb{R}^n)$ if and only if $f \in L^p(\mathbb{R}^n)$ when $p \in (1,\infty)$ or $f \in C(\mathbb{R}^n)$ when $p = \infty$, and $|||f|||_{F^{\alpha}_{p,q}(\mathbb{R}^n)} < \infty$, where, when $p \in (1,\infty)$,

$$|||f|||_{F^{\alpha}_{p,q}(\mathbb{R}^n)} := \|f\|_{L^p(\mathbb{R}^n)} + \left\| \left\{ \sum_{k=1}^{\infty} 2^{k\alpha q} \left| f - B_{\ell,2^{-k}} f \right|^q \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)}$$

and, when $p = \infty$,

$$|||f|||_{F^{\alpha}_{\infty,q}(\mathbb{R}^n)} := \|f\|_{L^{\infty}(\mathbb{R}^n)} + \sup_{x\in\mathbb{R}^n} \sup_{m\geq 1} \left\{ \fint_{B(x,2^{-m})} \sum_{k=m}^{\infty} 2^{k\alpha q} \left| f(y) - B_{\ell,2^{-k}} f(y) \right|^q dy \right\}^{1/q}.$$

Moreover, $|||\cdot|||_{F^{\alpha}_{p,q}(\mathbb{R}^n)}$ is equivalent to $\|\cdot\|_{F^{\alpha}_{p,q}(\mathbb{R}^n)}$.

Proof. By similarity, we only consider (ii). The proof is similar to that of Theorem (4.3.6), and we mainly describe the difference. We need to use the following well-known result: when $\alpha \in (0,\infty)$ and $p, q \in (1,\infty]$, then, for all $f \in F^{\alpha}_{p,q}(\mathbb{R}^n)$,

$$\|f\|_{F^{\alpha}_{p,q}(\mathbb{R}^n)} \sim \|f\|_{L^p(\mathbb{R}^n)} + \|\widetilde{f}\|_{F^{\alpha}_{p,q}(\mathbb{R}^n)}, \tag{81}$$

where $\|\widetilde{f}\|_{F^{\alpha}_{p,q}(\mathbb{R}^n)}$ is defined as $\|f\|_{F^{\alpha}_{p,q}(\mathbb{R}^n)}$ in Definition (4.3.2) with $k \in \mathbb{Z}_+$ and $m \in \mathbb{Z}_+$ therein replaced, respectively, by $k \in \mathbb{N}$ and $m \in \mathbb{N}$ (which can be easily seen from [55, Theorem 3.3.1]and [193, Chapter 2.4, Corollary 2]).

Assume first that $f \in F^{\alpha}_{p,q}(\mathbb{R}^n)$. By [55, Theorem 3.3.1] and [193, Chapter 2.4, Corollary 2], we know that $f \in L^p(\mathbb{R}^n)$ when $p \in (1,\infty)$ or $f \in C(\mathbb{R}^n)$ when $p = \infty$. On the other hand, repeating the proof of Theorem (4.3.6), we see that, when $p \in (1,\infty)$,

$$\left\| \left\{ \sum_{k=1}^{\infty} 2^{k\alpha q} \left| f - B_{\ell,2^{-k}} f \right|^q \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{F^{\alpha}_{p,q}(\mathbb{R}^n)}$$

and, when $p = \infty$,

$$\sup_{x\in\mathbb{R}^n} \sup_{m\geq 1} \left\{ \fint_{B(x,2^{-m})} \sum_{k=m}^{\infty} 2^{k\alpha q} \left| f(y) - B_{\ell,2^{-k}} f(y) \right|^q dy \right\}^{1/q} \lesssim \|f\|_{F^{\alpha}_{\infty,q}(\mathbb{R}^n)}$$

which show $|||f|||_{F^{\alpha}_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{F^{\alpha}_{p,q}(\mathbb{R}^n)}$.

Conversely, assume that $f \in L^p(\mathbb{R}^n)$ when $p \in (1,\infty)$ or $f \in C(\mathbb{R}^n)$ when $p = \infty$, and $|||f|||_{F^{\alpha}_{p,q}(\mathbb{R}^n)} < \infty$. Again the proof of Theorem (4.3.6) shows that, when $p \in (1,\infty)$,

$$\|\widetilde{f}\|_{F^{\alpha}_{p,q}(\mathbb{R}^n)} \lesssim \left\| \left\{ \sum_{k=1}^{\infty} 2^{k\alpha q} \left| f - B_{\ell,2^{-k}} f \right|^q \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)}$$

and, when $p = \infty$,

$$\|\widetilde{f}\|_{F^{\alpha}_{\infty,q}(\mathbb{R}^n)} \lesssim \sup_{x\in\mathbb{R}^n} \sup_{m\geq 1} \left\{ \fint_{B(x,2^{-m})} \sum_{k=m}^{\infty} 2^{k\alpha q} \left| f(y) - B_{\ell,2^{-k}} f(y) \right|^q dy \right\}^{1/q} \lesssim \|f\|_{F^{\alpha}_{\infty,q}(\mathbb{R}^n)}$$

This, together with (81), further implies that $\|f\|_{F^{\alpha}_{\infty,q}(\mathbb{R}^n)} \lesssim |||f|||_{F^{\alpha}_{\infty,q}(\mathbb{R}^n)}$, and hence finishes the proof of Theorem (4.3.7).

Finally, we point out that the conclusions of Theorems (4.3.6) and (4.3.7) are independent of the choice of the metric in $\mathbb{R}^n$. To be precise, let $\|\cdot\|$ be a norm in $\mathbb{R}^n$, which is not necessarily the usual Euclidean norm. Then $(\mathbb{R}^n, \|\cdot\|)$ is a finite dimensional normed vector space with the unit ball

$$K := \{x \in \mathbb{R}^n : \|x\| \leq 1\}.$$

Clearly, K is a compact and symmetric convex set in $\mathbb{R}^n$ satisfying that $-K = K$ and $B(0,\delta_1) \subset K \subset B(0,\delta_2)$ for some $\delta_1, \delta_2 \in (0,\infty)$.

For all $\ell \in \mathbb{N}$, $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, define

$$B_{\ell,t,K}f(x) := \frac{-2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell - j} B_{jt,K}f(x) .$$

Then we have the following conclusion.

Theorem (4.3.8) [196] The conclusions of Theorems (4.3.6) and (4.3.7) remain valid with $B_{\ell,t}$ therein replaced by $B_{\ell,t,K}$.

Since the proof of Theorem (4.3.8) is essentially similar to the proofs of Theorems (4.3.6) and (4.3.7), we only describe the main differences, the other details being omitted.

We first observe that

$$\left(B_{\ell,t,K}f\right)^{\wedge}(\xi) := m_{\ell,K}(t\xi)\hat{f}(\xi), \qquad \xi \in \mathbb{R}^n,$$

where

$$m_{\ell,K}(x) := \frac{-2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell - j} \widehat{I_K}(jx) , \qquad x \in \mathbb{R}^n.$$

Similar to the proof of Lemma (4.3.3), by means of the symmetry property of K, a straightforward calculation shows that, for all $x \in \mathbb{R}^n$,

$$m_{\ell,K}(x) = \fint_K \frac{-2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell - j} \cos(jx \cdot y)\, dy = : 1 - A_{\ell,K}(x),$$

where

$$A_{\ell,K}(x) := \frac{4^\ell}{\binom{2\ell}{\ell}} \fint_K \left(\sin \frac{x \cdot u}{2}\right)^{2\ell} du.$$

Furthermore, we have the following estimates: for all $x \in \mathbb{R}^n$ with $|x| \le 4$,

$$0 < C_1 \le \frac{A_{\ell,K}(x)}{|x|^{2\ell}} \le C_2 \tag{82}$$

and

$$\left|\nabla^i A_{\ell,K}(x)\right| \le C \min\left\{|x|^{2\ell - i}, 1\right\}, \qquad i \in \mathbb{N}, \tag{83}$$

where C_1, C_2 and C are positive constants independent of x. Similar to the proof of Lemma (4.3.4), by (82) and (83), we observe that

$$\sup_{t \in (0,\infty)} \fint_K |f(x + ty)|\, dy \lesssim Mf(x)$$

for all $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$.

Finally, notice that, by the equivalence of norms on finite-dimensional vector spaces, the spaces $\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n), \dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ and their inhomogeneous counterparts are essentially independent of the choice of the norm of the underlying space $\mathbb{R}^n$. By means of this observation and using (82), (83) in place of Lemma (4.3.3), we obtain Theorem (4.3.8) via some arguments similar to those used in the proofs of Theorems (4.3.6) and (4.3.7), the details being omitted.

Corollary (4.3.9) [314] Let $\{T_{1+\epsilon}\}_{0 \le \epsilon < \infty}$ be a family of multiplier operators given by setting, for all $f_j \in L^2(\mathbb{R}^n)$,

$$\sum \left(T_{1+\epsilon} f_j\right)^{\wedge}(\xi) := m\big((1+\epsilon)\xi\big) \sum \hat{f}_j(\xi) , \qquad \xi \in \mathbb{R}^n, \qquad 0 \le \epsilon < \infty$$

for some $m \in L^{\infty}(\mathbb{R}^n)$. If

$$\|\nabla^{n+1}m\|_{L^1(\mathbb{R}^n)} + \|m\|_{L^1(\mathbb{R}^n)} \le C_1 < \infty,$$

then there exists a positive constant C such that, for all $f_j \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\sup_{0\le\epsilon<\infty} \sum \left|T_{(1+\epsilon)} f_j(x)\right| \le CC_1 \sum M f_j(x).$$

Proof. For all $0 \le \epsilon < \infty, f_j \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, by the Fubini theorem, we see that

$$\begin{aligned}\sum \left|T_{1+\epsilon} f_j(x)\right| &= \sum \left| \int_{\mathbb{R}^n} m((1+\epsilon)\xi) \hat{f}_j(\xi) e^{ix\cdot\xi} d\xi \right| \\ &= \sum \left| \int_{\mathbb{R}^n} f_j(y) \int_{\mathbb{R}^n} m((1+\epsilon)\xi) \hat{f}_j(\xi) e^{i(x-y)\cdot\xi} d\xi\, dy \right| \\ &\le \sum \left| \int_{|x-y|-1<\epsilon} f_j(y) \int_{\mathbb{R}^n} m((1+\epsilon)\xi) \hat{f}_j(\xi) e^{i(x-y)\cdot\xi} d\xi\, dy \right| + \left| \int_{|x-y|-1\ge\epsilon} \cdots \right| \\ &=: \mathrm{I} + \mathrm{II}.\end{aligned}$$

It is easy to see that $\mathrm{I} \lesssim \|m\|_{L^1(\mathbb{R}^n)} \sum M f_j(x)$.

For II, via the Fubini theorem and the integration by parts, we also have

$$\begin{aligned}\mathrm{II} &\lesssim \sum \int_{|x-y|-1\ge\epsilon} \frac{\left|f_j(y)\right|}{|x-y|^{n+1}} \int_{\mathbb{R}^n} (1+\epsilon)^{n+1} \left|\nabla^{n+1} m\big((1+\epsilon)\xi\big)\right| d\xi\, dy \\ &\lesssim \|\nabla^{n+1}m\|_{L^1(\mathbb{R}^n)} \sum_{j=1}^{\infty} (1+\epsilon) \int_{2^{j(1+\epsilon)}\le|x-y|<2^{j+1}(1+\epsilon)} \frac{\left|f_j(y)\right|}{|x-y|^{n+1}} dy \\ &\lesssim \|\nabla^{n+1}m\|_{L^1(\mathbb{R}^n)} \sum_{j=1}^{\infty} 2^{-j} M f_j(x) \lesssim \|\nabla^{n+1}m\|_{L^1(\mathbb{R}^n)} \sum M f_j(x),\end{aligned}$$

which completes the proof of Corollary (4.3.9).

Corollary (4.3.10) [314] Let $\ell \in \mathbb{N}$ and $0 \le \epsilon < 1$.

(i) Let $\epsilon \ge 0$. Then $f_j \in B^{1+\epsilon}_{1+\epsilon,1+\epsilon}(\mathbb{R}^n)$ if and only if $f_j \in L^{1+\epsilon}(\mathbb{R}^n)$ when $0 < \epsilon < \infty$ or $f_j \in C(\mathbb{R}^n)$ when $\epsilon = \infty$, and

$$\sum |||f_j|||_{B^{1+\epsilon}_{1+\epsilon,1+\epsilon}(\mathbb{R}^n)} := \sum \left\|f_j\right\|_{L^{1+\epsilon}(\mathbb{R}^n)} + \sum \left\{ \sum_{k=1}^{\infty} 2^{k(1+\epsilon)^2} \left\|f_j - B_{\ell,2^{-k}} f_j\right\|^{1+\epsilon}_{L^{1+\epsilon}(\mathbb{R}^n)} \right\}^{1/1+\epsilon} < \infty.$$

Moreover, $|||\cdot|||_{B^{1+\epsilon}_{1+\epsilon,1+\epsilon}(\mathbb{R}^n)}$ is equivalent to $\left\|f_j\right\|_{B^{1+\epsilon}_{1+\epsilon,1+\epsilon}(\mathbb{R}^n)}$.

(ii) Let $0 < \epsilon \le \infty$. Then $f_j \in F^{1+\epsilon}_{1+\epsilon,1+\epsilon}(\mathbb{R}^n)$ if and only if $f_j \in L^{1+\epsilon}(\mathbb{R}^n)$ when $0 < \epsilon < \infty$ or $f_j \in C(\mathbb{R}^n)$ when $\epsilon = \infty$, and $\sum |||f_j|||_{F^{1+\epsilon}_{1+\epsilon,1+\epsilon}(\mathbb{R}^n)} < \infty$, where, when $0 < \epsilon < \infty$,

$$\sum |||f_j|||_{F^{1+\epsilon}_{1+\epsilon,1+\epsilon}(\mathbb{R}^n)} := \sum \left\|f_j\right\|_{L^{1+\epsilon}(\mathbb{R}^n)} + \sum \left\| \left\{ \sum_{k=1}^{\infty} 2^{k(1+\epsilon)^2} \left|f_j - B_{\ell,2^{-k}} f_j\right|^{1+\epsilon} \right\}^{1/1+\epsilon} \right\|_{L^{1+\epsilon}(\mathbb{R}^n)}$$

and, when $\epsilon = \infty$,

$$\sum |||f_j|||_{F^{1+\epsilon}_{\infty,1+\epsilon}(\mathbb{R}^n)} := \sum \left\|f_j\right\|_{L^{\infty}(\mathbb{R}^n)} + \sup_{x\in\mathbb{R}^n} \sup_{m\ge1} \sum \left\{ \fint_{B(x,2^{-m})} \sum_{k=m}^{\infty} 2^{k(1+\epsilon)^2} \left|f_j(y) - B_{\ell,2^{-k}} f_j(y)\right|^{1+\epsilon} dy \right\}^{1/1+\epsilon}.$$

Moreover, $|||\cdot|||_{F^{1+\epsilon}_{1+\epsilon,1+\epsilon}(\mathbb{R}^n)}$ is equivalent to $\|\cdot\|_{F^{1+\epsilon}_{1+\epsilon,1+\epsilon}(\mathbb{R}^n)}$.

Proof. By similarity, we only consider (ii). The proof is similar to that of Theorem (4.3.6), and we mainly describe the difference. We need to use the following well-known result: when $0 < \epsilon \le \infty$, then, for all $f_j \in F^{1+\epsilon}_{1+\epsilon,1+\epsilon}(\mathbb{R}^n)$,

$$\sum \|f_j\|_{F^{1+\epsilon}_{1+\epsilon,1+\epsilon}(\mathbb{R}^n)} \sim \sum \|f_j\|_{L^{1+\epsilon}(\mathbb{R}^n)} + \sum \|\tilde{f}_j\|_{F^{1+\epsilon}_{1+\epsilon,1+\epsilon}(\mathbb{R}^n)}, \tag{84}$$

where $\sum \|\tilde{f}_j\|_{F^{1+\epsilon}_{1+\epsilon,1+\epsilon}(\mathbb{R}^n)}$ is defined as $\sum \|f_j\|_{F^{1+\epsilon}_{1+\epsilon,1+\epsilon}(\mathbb{R}^n)}$ in Definition (4.3.2) with $k \in \mathbb{Z}_+$ and $m \in \mathbb{Z}_+$ therein replaced, respectively, by $k \in \mathbb{N}$ and $m \in \mathbb{N}$ (which can be easily seen from [55, Theorem 3.3.1]and [193, Chapter 2.4, Corollary 2]).

Assume first that $f_j \in F^{1+\epsilon}_{1+\epsilon,1+\epsilon}(\mathbb{R}^n)$. By [55, Theorem 3.3.1] and [193, Chapter 2.4, Corollary 2], we know that $f_j \in L^{1+\epsilon}(\mathbb{R}^n)$ when $0 < \epsilon < \infty$ or $f_j \in C(\mathbb{R}^n)$ when $\epsilon = \infty$. On the other hand, repeating the proof of Theorem (4.3.6), we see that, when $0 < \epsilon < \infty$,

$$\left\| \left\{ \sum_{k=1}^{\infty} 2^{k(1+\epsilon)^2} \sum \left| f_j - B_{\ell,2^{-k}} f_j \right|^{1+\epsilon} \right\}^{1/1+\epsilon} \right\|_{L^{1+\epsilon}(\mathbb{R}^n)} \lesssim \sum \|f_j\|_{F^{1+\epsilon}_{1+\epsilon,1+\epsilon}(\mathbb{R}^n)}$$

and, when $\epsilon = \infty$,

$$\sup_{x\in\mathbb{R}^n} \sup_{m\ge 1} \left\{ \fint_{B(x,2^{-m})} \sum_{k=m}^{\infty} 2^{k(1+\epsilon)^2} \sum \left| f_j(y) - B_{\ell,2^{-k}} f_j(y) \right|^{1+\epsilon} dy \right\}^{1/1+\epsilon} \lesssim \sum \|f_j\|_{F^{1+\epsilon}_{\infty,1+\epsilon}(\mathbb{R}^n)}$$

which show $\sum |||f_j|||_{F^{1+\epsilon}_{1+\epsilon,1+\epsilon}(\mathbb{R}^n)} \lesssim \sum \|f_j\|_{F^{1+\epsilon}_{1+\epsilon,1+\epsilon}(\mathbb{R}^n)}$.

Conversely, assume that $f_j \in L^{1+\epsilon}(\mathbb{R}^n)$ when $0 < \epsilon < \infty$ or $f_j \in C(\mathbb{R}^n)$ when $\epsilon = \infty$, and $\sum |||f_j|||_{F^{1+\epsilon}_{1+\epsilon,1+\epsilon}(\mathbb{R}^n)} < \infty$. Again the proof of Theorem (4.3.6) shows that, when $0 < \epsilon < \infty$,

$$\sum \|\tilde{f}_j\|_{F^{1+\epsilon}_{1+\epsilon,1+\epsilon}(\mathbb{R}^n)} \lesssim \left\| \left\{ \sum_{k=1}^{\infty} 2^{k(1+\epsilon)^2} \sum \left| f_j - B_{\ell,2^{-k}} f_j \right|^{1+\epsilon} \right\}^{1/1+\epsilon} \right\|_{L^{1+\epsilon}(\mathbb{R}^n)}$$

and, when $\epsilon = \infty$,

$$\begin{aligned} \sum \|\tilde{f}_j\|_{F^{1+\epsilon}_{\infty,1+\epsilon}(\mathbb{R}^n)} &\lesssim \sup_{x\in\mathbb{R}^n} \sup_{m\ge 1} \left\{ \fint_{B(x,2^{-m})} \sum_{k=m}^{\infty} 2^{k(1+\epsilon)^2} \sum \left| f_j(y) - B_{\ell,2^{-k}} f_j(y) \right|^{1+\epsilon} dy \right\}^{1/1+\epsilon} \\ &\lesssim \sum \|f_j\|_{F^{1+\epsilon}_{\infty,1+\epsilon}(\mathbb{R}^n)} \end{aligned}$$

This, together with (84), further implies that $\sum \|f_j\|_{F^{1+\epsilon}_{\infty,1+\epsilon}(\mathbb{R}^n)} \lesssim \sum |||f_j|||_{F^{1+\epsilon}_{\infty,1+\epsilon}(\mathbb{R}^n)}$, and hence finishes the proof of Corollary (4.3.10).

Chapter 5
Hörmander Type Theorems and Duality and Boundedness of Multi-Parameter Triebel-Lizorkin Spaces and Fourier Multiplier Operators

We show that $L^{p_1} \times \cdots \times L^{p_n} \to L^r$ boundedness with $\frac{1}{p_1} + \cdots + \frac{1}{p_n} = \frac{1}{r}$ for $1 < p_1, \ldots, p_n < \infty$ and $0 < r < \infty$. The proof of L^r estimates also offers a different and more direct approach than the one given in Muscalu et al. where they use the deep analysis of multi-linear and multi-parameter paraproducts. We also show a Hörmander type multiplier theorem in the weighted Lebesgue spaces for such operators when the Fourier multiplier is only assumed with limited smoothness. The work requires more complicated analysis associated with the underlying geometry generated by the multi-parameter structures of the composition of two singular integral operators with different homogeneities. Therefore, it is more difficult to deal with than the duality result of the Triebel-Lizorkin spaces in the one-parameter settings. We note that for $0 < p \le 1, q = 2$ and $\alpha = 0, \dot{F}_p^{\alpha,q}(\mathbb{R}^m)$ is the Hardy space associated with the composition of two singular operators considered. The work appears to be the first effort on duality for Triebel-Lizorkin spaces in the multi-parameter setting. We offer a different and more direct method to deal with the boundedness instead of transforming Fourier multiplier operators into multi-parameter Calderón–Zygmund operators. We also show the boundedness of multi-parameter Fourier multiplier operators on weighted multi-parameter Triebel–Lizorkin and Besov–Lipschitz spaces when the Fourier multiplier is only assumed with limited smoothness.

Section (5.1) Multi-Linear and Multi-Parameter Fourier Multiplier Operators with Limited Smoothness

We consider the limited smoothness condition on the Fourier multipliers in the multi-parameter and multi-linear setting. This is an analogue of the well-known Hörmander–Mihlin type theorem in the linear and multi-linear cases.

Let $\mathcal{S}(\mathbb{R}^d)$ denote the space of Schwartz functions, and $\mathcal{S}'(\mathbb{R}^d)$ denote tempered distributions. The Fourier transform $\hat{f}$ and the inverse Fourier transform $\check{f}$ of $f \in \mathcal{S}(\mathbb{R}^d)$ are defined by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\,dx \text{ and } \mathcal{F}^{-1}f(\xi) = \check{f}(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(\xi)\,d\xi. \quad (1)$$

In the linear case, we first recall the following Mihlin theorem (see, e.g., [197, Corollary 8.11]):

Theorem (5.1.1) [224] If a multiplier $m \in C^{\left[\frac{n}{2}\right]+1}(\mathbb{R}^n \backslash \{0\})$ satisfies the following condition

$$|\partial^\alpha m(\xi)| \le C_\alpha |\xi|^{-|\alpha|} \quad \text{for all} \quad |\alpha| \le \left[\frac{n}{2}\right] + 1 \quad (2)$$

then the Fourier multiplier operator $m(D)f = \mathcal{F}^{-1}[m\hat{f}]$ defined with the symbol $m(\xi)$ is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$.

Hörmander reformulated and improved Mihlin's theorem using the Sobolev regularity of the multiplier [198]. To describe Hörmander's theorem, we let $\psi \in \mathcal{S}(\mathbb{R}^d)$ be a Schwartz function satisfying

$$\operatorname{supp}\psi \subset \left\{\xi \in \mathbb{R}^d : \frac{1}{2} \le |\xi| \le 2\right\}, \qquad \sum_{j\in\mathbb{Z}} \psi\left(\frac{\xi}{2^j}\right) = 1, \text{ for all } \xi \in \mathbb{R}^d \backslash \{0\}. \quad (3)$$

For $s \in \mathbb{R}$, the Sobolev space $\mathrm{H}^s(\mathbb{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{\mathrm{H}^s} \triangleq \left\|(I - \Delta)^{s/2} f\right\|_{L^2} < \infty, \quad (4)$$

where $(I - \Delta)^{s/2} f = \mathcal{F}^{-1}\left[(1 + |\xi|^2)^{s/2}\hat{f}(\xi)\right]$. Then the Hörmander multiplier theorem says.

Theorem (5.1.2) [224] If $m \in L^{\infty}(\mathbb{R}^n)$ satisfies

$$\sup_{j\in\mathbb{Z}} \left\| m(2^j \cdot)\psi \right\|_{\mathrm{H}^s(\mathbb{R}^n)} < \infty, \qquad \text{for all } s > \frac{n}{2},$$

where ψ is the same as in (3) when $d = n$ and $\mathrm{H}^s(\mathbb{R}^n)$ is the Sobolev space, then the Fourier multiplier operator $m(D)$ defined with the symbol m is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$.

Clearly, Hörmander's theorem is stronger than Mihlin's and the number $\frac{n}{2}$ cannot be improved in Hörmander's theorem.

The weighted estimates for Fourier multipliers. We first introduce the notion of Muckenhoupt's A_p weights [199]. Let $1 < p < \infty$ and denote $p' = \frac{p}{p-1}$. We say that a weight $w \geq 0$ belongs to the Muckenhoupt class $A_p(\mathbb{R}^n)$, if

$$\sup_R \left(\frac{1}{|R|}\int_R w(x)dx\right)\left(\frac{1}{|R|}\int_R w(x)^{1-p'}dx\right)^{p-1} < \infty \tag{5}$$

where the supremum is taken over all cubes R in $\mathbb{R}^n$. We also use

$$\|f\|_{L^p_w(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)dx\right)^{\frac{1}{p}}.$$

Then, Kurtz and Wheeden [200] extended Hörmander's theorem to weighted Lebesgue spaces and proved the following:

Theorem (5.1.3) [224] Let $\frac{n}{2} < s \leq n$ and $1 < p < \infty$. Assume $\frac{n}{s} < p < \infty$ and $w \in A_{\frac{ps}{n}}$. If $m \in L^{\infty}(\mathbb{R}^n)$ satisfies

$$\sup_{j\in\mathbb{Z}} \left\| m(2^j \cdot)\psi \right\|_{\mathrm{H}^s(\mathbb{R}^n)} < \infty,$$

then the Fourier multiplier operator $m(D)$ defined with the symbol m is bounded from $L^p_w(\mathbb{R}^n)$ to $L^p_w(\mathbb{R}^n)$ for all $1 < p < \infty$.

We now turn to the discussion of multi-linear Coifman–Meyer Fourier multiplier operators. We only state the bilinear case as an example for simplicity of its presentation. For $m \in L^{\infty}(\mathbb{R}^{2n})$, the bilinear Coifman–Meyer Fourier multiplier operator T_m is defined by

$$T_m(f,g)(x) = \frac{1}{(2\pi)^{(2n)}}\int_{\mathbb{R}^{2n}} m(\xi\,,\eta)e^{ix(\xi+\eta)}\hat{f}(\xi)\hat{g}(\eta)d\xi\, d\eta \tag{6}$$

for all $f, g \in \mathcal{S}(\mathbb{R}^n)$.

Coifman and Meyer [5–7] first proved that if $m \in C^L(\mathbb{R}^{2n}\backslash\{0\})$ satisfies

$$\left|\partial_\xi^\alpha \partial_\eta^\beta m(\xi,\eta)\right| \leq C_{\alpha\beta}(|\xi| + |\eta|)^{-(|\alpha|+|\beta|)} \tag{7}$$

for all $|\alpha| + |\beta| \leq L$, where L is a sufficiently large natural number, then T_m is bounded from $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$ for all $1 < p,q,r < \infty$ satisfying $1/p + 1/q = 1/r$. Results in [201–203] have been extended to multi-linear Calderón–Zygmund operators by Kenig and Stein [204], Grafakos and Kalton [205], Grafakos and Torres [206], [207] to include $0 < r \leq 1$ (see also work of generalizations to bilinear square functions and vector-valued Calderón–Zygmund operators of Hart [208]). However, in many cases where m has only limited smoothness, we cannot use this result since L is not an explicit number. Finding the best possible number of L thus becomes an interesting problem. By reducing the bilinear Fourier multiplier operators to linear Calderón–Zygmund operators, Coifman–Meyer obtained the L^r estimates under the assumption $L = 2n + 1$. [206] also proved the condition (7) with $L=2n+1$ assures the boundedness of T_m by using the bilinear $T1$ theorem. However this number seems to be too large in view of the linear case.

Tomita [209] improved this theorem for multipliers with limited smoothness in terms of the Sobolev regularity. To state the result in [209], for $m \in L^\infty(\mathbb{R}^{2n})$, we set $m_k(\xi,\eta) = m(2^k\xi, 2^k\eta)\psi(\xi_1,\eta_1)$, where ψ is the same as the (3) with $d = 2n$.

Theorem (5.1.4) [224] Let $s > n, 1 < p, q, r < \infty$ and $1/p + 1/q = 1/r$. If $m \in L^\infty(\mathbb{R}^{2n})$ satisfies

$$\sup_{k\in\mathbb{Z}} \|m_k\|_{H^s(\mathbb{R}^{2n})} < \infty$$

then T_m is bounded from $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$.

For further improvement in this direction in the case $0 < r \le 1$ or the case where p or q can be smaller than or equal to 1, see the works in Grafakos, Miyachi and Tomita [210], Miyachi and Tomita [211] and Grafakos and Si [212].

Fujita and Tomita [213] considered the weighted norm inequalities for multilinear Fourier multiplier operators, for simplicity we only state their result in the bilinear case.

Theorem (5.1.5) [224] Let $1 < p, q < \infty, 1/p + 1/q = 1/r$ and $n < s \le 2n$. Assume
(i) $\min\{p,q\} > 2n/s$ and $w \in A_{\min\{ps/2n, qs/2n\}}$ or
(ii) $\min\{p,q\} < (2n/s)'$, $1 < r < \infty$ and $w^{1-r'} \in A_{r's/2n}$.
If $m \in L^\infty(\mathbb{R}^{2n})$ satisfies

$$\sup_{k\in\mathbb{Z}} \|m_k\|_{H^s(\mathbb{R}^{2n})} < \infty.$$

Then T_m is bounded from $L^p(w) \times L^q(w)$ to $L^r(w)$.

This theorem can be understood as bilinear version of the results by Kurtz and Wheeden [200].

We discuss the L^r estimates for the multi-linear and multi-parameter Fourier multiplier operators. In the bilinear and bi-parameter case, Muscalu, Pipher, Tao, and Thiele [214] proved the following

Theorem (5.1.6) [224] Let $1 < p, q < \infty, 1/r = 1/p + 1/q, 0 < r < \infty$ and $m \in L^\infty(\mathbb{R}^{4n})$ satisfy

$$\left|\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\eta_1}^{\beta_1}\partial_{\eta_2}^{\beta_2} m(\xi_1,\xi_2,\eta_1,\eta_2)\right| \le C_{\alpha_1\alpha_2\beta_1\beta_2}(|\xi_1| + |\eta_1|)^{-(|\alpha_1|+|\beta_1|)}(|\xi_2| + |\eta_2|)^{-(|\alpha_2|+|\beta_2|)} \quad (8)$$

for $|\alpha_1| + |\beta_1| \le M$, and $|\alpha_2| + |\beta_2| \le N$, where M, N are sufficiently large natural numbers.

Then T_m is bounded from $L^p(\mathbb{R}^{2n}) \times L^q(\mathbb{R}^{2n}) \mapsto L^r(\mathbb{R}^{2n})$, where T_m is defined by

$$T_{m^1}(f,g)(x_1,x_2) = \frac{1}{(2\pi)^{(4n)}}\int_{\mathbb{R}^{4n}} m(\xi_1,\xi_2,\eta_1,\eta_2)e^{ix_1(\xi_1+\eta_1)+ix_2(\xi_2+\eta_2)}\hat{f}(\xi_1,\xi_2)\hat{g}(\eta_1,\eta_2)d\xi_1 d\xi_2 d\eta_1 d\eta_2. \quad (9)$$

This theorem was extended to the case of multi-linear and multi-parameter setting in [215]. The method of proof of the above theorem in [214, 215] is to decompose the multi-linear and multi-parameter Fourier multiplier operator into discretized multi-linear and multi-parameter paraproducts. By proving the L^r estimates for the discretized paraproducts, they establish the L^r estimates for the Fourier multipliers. The difficult part of their proof is in the quasi-Banach case when $0 < r \le 1$ where the standard duality argument for the paraproducts does not work (see [216]). Therefore, [214, 215] establish the desired result by using a new duality lemma of $L^{r,\infty}$ for $(0 < r \le 1)$, the stopping-time decompositions arguments and multi-linear interpolation. We mention in passing that the endpoint estimates of results in [214, 215] were obtained by Lacey and Metcalfe [217] and

L^r estimates in the above Theorem (5.1.6) have also been established recently in the case of multi-linear and multi-parameter pseudo-differential operators by W. Dai [218]. Furthermore, symbolic calculus has been carried out and boundedness of multi-parameter and multi-linear pseudo-differential operators in the Hörmander classes have been established by Q. Hong [219]. L^p estimates for modified bilinear and multi-parameter Hilbert transforms have also been established by W. Dai [220], where we address the open question raised in [214].

It is worth noting that the smoothness condition for the Fourier multiplier $m(\xi_1, \xi_2, \eta_1, \eta_2)$ in [214, 215] requires M and N to be sufficiently large. Thus, it is interesting to know what the limited smoothness assumption is on m to assure the L^r estimates. This is one of the main purposes.

To establish the L^r estimates of the multi-linear and multi-parameter Fourier multipliers with limited smoothness, we need to introduce the two-parameter Sobolev spaces. For $s_1, s_2 \in \mathbb{R}$, the two-parameter Sobolev space $\mathrm{H}^{s_1,s_2}(\mathbb{R}^{4n})$ consists of all $f \in \mathcal{S}'(\mathbb{R}^{4n})$ such that

$$\|f\|_{\mathrm{H}^{s_1,s_2}} = \left\|(I - \Delta)^{s_1/2, s_2/2} f\right\|_{L^2} < \infty, \tag{10}$$

where

$$(I - \Delta)^{s_1/2, s_2/2} f = \mathcal{F}^{-1}\left[(1 + |\xi_1|^2 + |\eta_1|^2)^{s_1/2}(1 + |\xi_2|^2 + |\eta_2|^2)^{s_2/2} \hat{f}(\xi_1, \xi_2, \eta_1, \eta_2)\right]$$

where $\xi_1, \xi_2, \eta_1, \eta_2 \in \mathbb{R}^n$.

We first establish a Hörmander's type theorem in the bilinear and bi-parameter setting. One of the main theorems states that:

From the theorem above, we have

Theorem (5.1.7) [224] Let $1 < p, q < \infty$ and $1/p + 1/q = 1/r$. If $m \in C^{2n+1}(\mathbb{R}^{2n}\backslash\{0\} \times \mathbb{R}^{2n}\backslash\{0\})$ satisfies

$$\left|\partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} \partial_{\eta_1}^{\beta_1} \partial_{\eta_2}^{\beta_2} m(\xi_1, \xi_2, \eta_1, \eta_2)\right| \le C_{\alpha_1\alpha_2\beta_1\beta_2}(|\xi_1| + |\eta_1|)^{(-|\alpha_1|+|\beta_1|)}(|\xi_2| + |\eta_2|)^{(-|\alpha_2|+|\beta_2|)} \tag{11}$$

for all $|\alpha_1| + |\beta_1| \le n + 1, |\alpha_2| + |\beta_2| \le n + 1$ and $(\xi_1, \xi_2, \eta_1, \eta_2) \in \mathbb{R}^{2n}\backslash\{0\} \times \mathbb{R}^{2n}\backslash\{0\}$, then T_m is bounded from $L^p(\mathbb{R}^{2n}) \times L^q(\mathbb{R}^{2n})$ to $L^r(\mathbb{R}^{2n})$.

Finally, we consider the weighted norm inequalities for the bilinear and bi-parameter Fourier multipliers. To this end, we first introduce the notion of product A_p weights (see [221]).

Let $1 < p < \infty$. We say that a weight $w \ge 0$ belongs to the product Muckenhoupt class $A_p(\mathbb{R}^n \times \mathbb{R}^n)$, if

$$\sup_R \left(\frac{1}{|R|}\int_R w(x, y)dxdy\right)\left(\frac{1}{|R|}\int_R w(x, y)^{1-p'}dxdy\right)^{p-1} < \infty \tag{12}$$

where the supremum is taken over all rectangles $R = I \times J, I$ and J are both cubes in $\mathbb{R}^n$.

We define $A_\infty(\mathbb{R}^n \times \mathbb{R}^n) = \bigcup_{p>1} A_p(\mathbb{R}^n \times \mathbb{R}^n)$ as usual.

Then we can establish the following

Theorem (5.1.8) [224] Let $m \in L^\infty\left(\mathbb{R}^{tn\ell}\right)$. Set

$$m_{j_1,\dots,j_t}(\overline{\xi}_1, \dots, \overline{\xi}_t) = m(2^{j_1}\overline{\xi}_1, \dots, 2^{j_t}\overline{\xi}_t)\psi(\overline{\xi}_1) \cdots \psi(\overline{\xi}_t),$$

where $\psi_1, \dots, \psi_t$ are the same as in (3) with $d = n\ell$ there. For any $n \ge 1, t \ge 2$, the n-linear, t-parameter multiplier operator $T_m^{(t)}$ maps $L^{p_1}\left(\mathbb{R}^{t\ell}\right) \times \cdots \times L^{p_n}\left(\mathbb{R}^{t\ell}\right)$ to $L^r\left(\mathbb{R}^{t\ell}\right)$, provided that $1 < p_1, \dots, p_n < \infty, p_1 > \frac{t\ell}{s}, \dots, p_n > \frac{t\ell}{s}$, where $s_1 > \frac{t\ell}{2}, \dots, s_t > \frac{t\ell}{2}$ and $s = \min(s_1, \dots, s_t)$ and $\frac{1}{r} = \frac{1}{p_1} + \cdots + \frac{1}{p_n} > 0$ and the multiplier m satisfies

$$\sup_{j_1,\dots,j_t\in\mathbb{Z}}\|m_{j_1,\dots,j_t}\|_{H^{s_1,\dots,s_t}(\mathbb{R}^{tn\ell})}<\infty.$$

We can also establish the following weighted estimates.

Theorem (5.1.9) [224] Let $1<p_1,\dots,p_n<\infty, \frac{1}{p_1}+\cdots+\frac{1}{p_n}=\frac{1}{r}$ and $\frac{t\ell}{2}<s_1,\dots,s_t\le t\ell$, $s=\min\{s_1,\dots,s_t\}$. Assume one of the following two conditions (i) and (ii) holds, namely,

(i) $p_j>\frac{t\ell}{s},\ w_j\in A_{\frac{p_js}{t\ell}},\ j=1,\dots,n,$ or (13)

(ii) $\min\{p_1,\dots,p_n\}<\left(\frac{t\ell}{s}\right)', 1<r<\infty, w_j^{1-r'}\in A_{\frac{r's}{t\ell}}$. (14)

If $m\in L^\infty(\mathbb{R}^{tn\ell})$ satisfies

$$\sup_{j_1,\dots,j_t\in\mathbb{Z}}\|m_{j_1,\dots,j_t}\|_{H^{s_1,\dots,s_t}(\mathbb{R}^{tn\ell})}<\infty. \tag{15}$$

Then T_m is bounded from $L^{p_1}(w_1)\times\cdots\times L^{p_n}(w_n)$ to $L^r(w)$ where $w=w_1^{\frac{r}{p_1}}\cdots w_n^{\frac{r}{p_n}}$.

We prove Theorem (5.1.17), namely, the L^r estimates for the multi-linear and multi-parameter Coifman–Meyer multiplier operators with limited smoothness. We give the proof of Theorem (5.1.18), i.e., the weighted version of Theorem (5.1.17).

The strong maximal operator M_s for a function f on $\mathbb{R}^{2n}$ is defined by

$$M_sf(x,y)=\sup_{r_1,r_2>0}\frac{1}{r_1^n}\frac{1}{r_2^n}\int_R|f(u,v)|dudv, \tag{16}$$

where $R=\{(u,v)\in\mathbb{R}^{2n}|\ |u-x|<r_1,|v-y|<r_2\}$ and f is a locally integrable function on $\mathbb{R}^{2n}$. It is well known that M_s is bounded on $L^p(\mathbb{R}^{2n})$ for all $1<p<\infty$.

Lemma (5.1.10) [224] Let $\epsilon_1,\epsilon_2>0$. Then there exists a constant $C>0$ such that

$$\sup_{r_1,r_2>0}\left(r_1^nr_2^n\int_{\mathbb{R}^{2n}}\frac{|f(u,v)|}{(1+r_1|x-u|)^{n+\epsilon_1}(1+r_2|y-v|)^{n+\epsilon_2}}dudv\right)\le CM_sf(x,y) \tag{17}$$

for all locally integrable functions f on $\mathbb{R}^{2n}$.

Proof. Note that

$$r_1^nr_2^n\int_{(u,v):|u-x|<r_1^{-1},|v-y|<r_2^{-1}}\frac{|f(u,v)|}{(1+r_1|x-u|)^{n+\epsilon_1}(1+r_2|y-v|)^{n+\epsilon_2}}dudv\le CM_sf(x,y)$$

and

$$\int_{(u,v):|u-x|\ge r_1^{-1},|v-y|\ge r_2^{-1}}\frac{|f(u,v)|}{(1+r_1|x-u|)^{n+\epsilon_1}(1+r_2|y-v|)^{n+\epsilon_2}}dudv$$

$$\le\sum_{k=0}^{\infty}\int_{(u,v):2^kr_1^{-1}\le|u-x|<2^{k+1}r_1^{-1},2^kr_2^{-1}\le|v-y|<2^{k+1}r_2^{-1}}\frac{|f(u,v)|}{(1+r_1|x-u|)^{n+\epsilon_1}(1+r_2|y-v|)^{n+\epsilon_2}}dudv$$

$$\le\sum_{k=0}^{\infty}\frac{1}{(1+2^k)^{n+\epsilon}(1+2^k)^{n+\epsilon}}\int_{(u,v):|u-x|<2^{k+1}r_1^{-1},|v-y|<2^{k+1}r_2^{-1}}|f(u,v)|dudv.$$

Then it follows immediately that

$$\sup_{r_1,r_2>0}\left(r_1^nr_2^n\int_{\mathbb{R}^{2n}}\frac{|f(u,v)|}{(1+r_1|x-u|)^{n+\epsilon_1}(1+r_2|y-v|)^{n+\epsilon_2}}dudv\right)\le CM_sf(x,y).$$

Using the inequality for vector-valued Hardy–Littlewood maximal functions of C. Fefferman and Stein [112], and the fact that $M_sf(x,y)\le M_1M_2f(x,y)$, where M_1 and M_2 are the Hardy–Littlewood maximal functions with respect to the x and y variables respectively, we have the following inequality for the vector-valued strong maximal functions:

Lemma (5.1.11) [224] Let $1 < p, q < \infty$. Then there exists a constant $C > 0$ such that

$$\left\| \left\{ \sum_{k \in \mathbb{Z}} (M_s f_k)^q \right\}^{1/q} \right\|_{L^p} \leq C \left\| \left\{ \sum_{k \in \mathbb{Z}} |f_k|^q \right\}^{1/q} \right\|_{L^p} \tag{18}$$

for all sequences $\{f_k\}_{k \in \mathbb{Z}}$ of locally integrable functions on $\mathbb{R}^{2n}$.

Using the Littlewood–Paley inequality of L^p estimates in the product space of R. Fefferman and Stein [98], we can deduce immediately the following

Lemma (5.1.12) [224] Let $1 < p < \infty$, and let $\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}^n)$ be such that $\operatorname{supp} \psi_1 \subset \{\xi \in \mathbb{R}^n : 1/a \leq |\xi| \leq a\}$ for some $a > 1$, $\operatorname{supp} \psi_2 \subset \{\eta \in \mathbb{R}^n : 1/b \leq |\eta| \leq b\}$ for some $b > 1$. Then there exists a constant $C > 0$ such that

$$\left\| \left\{ \sum_{j,k \in \mathbb{Z}} \left| \psi_1(D/2^j) \psi_2(D/2^k) f \right|^2 \right\}^{1/2} \right\|_{L^p} \leq C \|f\|_{L^p} \text{ for all } f \in L^p(\mathbb{R}^{2n}), \tag{19}$$

where $[\psi_1(D/2^j)\, \psi_2(D/2^k) f](\xi_1, \xi_2) = \mathcal{F}^{-1}[\hat{\psi}_1(\,\cdot\,/2^j)\, \hat{\psi}_2(\,\cdot\,/2^k)\, \hat{f}(\,\cdot\,,\,\cdot\,)]\,(\xi_1, \xi_2)$. Moreover, if $\sum_{j \in \mathbb{Z}} \psi_i(\xi_i/2^j) = 1$ for all $\xi_i \neq 0$, for $i = 1, 2$, then

$$\left\| \left\{ \sum_{j,k \in \mathbb{Z}} \left| \psi_1(D/2^j) \psi_2(D/2^k) f \right|^2 \right\}^{1/2} \right\|_{L^p} \approx \|f\|_{L^p} \text{ for all } f \in L^p(\mathbb{R}^{2n}). \tag{20}$$

Let ϕ_0 be a C^∞-function on $[0, \infty)$ satisfying

$$\phi_0(t) = 1 \text{ on } [0, 1/8], \operatorname{supp} \phi_0 \subset [0, 1/4] \tag{21}$$

we set $\phi_1(t) = 1 - \phi_0(t)$, and set for $\xi, \eta \in \mathbb{R}^n$ the following notations:

$$\Phi_{(1)}(\xi, \eta) = \phi_0(|\xi|/|\eta|) \qquad \Phi_{(2)}(\xi, \eta) = \phi_1(|\eta|/|\xi|) \tag{22}$$

$$\Phi_{(3)}(\xi, \eta) = (1 - \phi_0(|\xi|/|\eta|))(1 - \phi_1(|\eta|/|\xi|)). \tag{23}$$

Lemma (5.1.13) [224] ([213]).

(i) For $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(0, 0)\}$,

$$\Phi_{(1)}(\xi, \eta) + \Phi_{(2)}(\xi, \eta) + \Phi_{(3)}(\xi, \eta) = 1. \tag{24}$$

(ii) Each $\Phi_{(i)}$ satisfies

$$\left| \partial_\xi^{\alpha_1} \partial_\eta^{\alpha_2} \Phi_{(i)}(\xi, \eta) \right| \leq C_{\alpha_1, \alpha_2} (|\xi| + |\eta|)^{-(|\alpha_1| + |\alpha_2|)} \tag{25}$$

for all multi-indices α_1, α_2.

(iii) $\operatorname{supp} \Phi_{(3)} \subset \{|\xi|/8 \leq |\eta| \leq 8|\xi|\}$, $\operatorname{supp} \Phi_{(1)} \subset \{|\xi| \leq |\eta|/2\}$ and $\operatorname{supp} \Phi_{(2)} \subset \{|\eta| \leq |\xi|/2\}$.

With a similar proof to that of Lemma 3.2 in [209] with a little modification, we can obtain the following:

Lemma (5.1.14) [224] Assume that $m \in C^{N+M}(\mathbb{R}^{2n}\setminus\{0\} \times \mathbb{R}^{2n}\setminus\{0\})$ satisfies

$$\left| \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} \partial_{\eta_1}^{\beta_1} \partial_{\eta_2}^{\beta_2} m(\xi_1, \xi_2, \eta_1, \eta_2) \right| \leq C_{\alpha_1 \alpha_2 \beta_1 \beta_2} (|\xi_1| + |\eta_1|)^{(-|\alpha_1| + |\beta_1|)} (|\xi_2| + |\eta_2|)^{(-|\alpha_2| + |\beta_2|)} \tag{26}$$

for all $|\alpha_1| + |\beta_1| \leq N$, $|\alpha_2| + |\beta_2| \leq M$ and $(\xi_1, \xi_2, \eta_1, \eta_2) \in \mathbb{R}^{2n}\setminus\{0\} \times \mathbb{R}^{2n}\setminus\{0\}$, where N, M are non-negative integers. Let Φ_1 and $\Phi_2 \in \mathcal{S}(\mathbb{R}^{2n})$ be such that none of $\operatorname{supp} \Phi_1, \operatorname{supp} \Phi_2$ contains the origin, and set

$$\widetilde{m}_{s,t}(\xi_1, \xi_2, \eta_1, \eta_2) = m(s\xi_1, t\xi_2, s\eta_1, t\eta_2) \Phi_1(\xi_1, \eta_1) \Phi_2(\xi_2, \eta_2). \tag{27}$$

Then $\sup_{s,t>0} \|\widetilde{m}_{s,t}\|_{\mathrm{H}^{N,M}(\mathbb{R}^{4n})} < \infty$.

Lemma (5.1.15) [224] ([210]). Let $2 \le q < \infty, r > 0$ and $s \ge 0$. Then there exists a constant $C > 0$ such that

$$\left\|\hat{f}\right\|_{L^q(w_{s,q})} \triangleq \left(\int_{\mathbb{R}^{4n}} |f(x,y)|^q (1+x^2)^s (1+y^2)^s dxdy\right)^{1/q} \le C\|f\|_{\mathrm{H}^{s,s}(\mathbb{R}^{2n}\times\mathbb{R}^{2n})}. \quad (28)$$

We need to establish the following

Lemma (5.1.16) [224] Let $s_1, s_2 \in \mathbb{R}$, and let $\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}^n)$ be such that supp ψ_1, supp ψ_2 are compact and none of them contains the origin. Assume that $\Phi \in C^\infty(\mathbb{R}^{2n}\backslash\{0\} \times \mathbb{R}^{2n}\backslash\{0\})$ satisfies

$$\left|\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\eta_1}^{\beta_1}\partial_{\eta_2}^{\beta_2}\Phi(\xi_1,\xi_2,\eta_1,\eta_2)\right| \le C_{\alpha_1\alpha_2\beta_1\beta_2}(|\xi_1|+|\eta_1|)^{-(|\alpha_1|+|\beta_1|)}(|\xi_2|+|\eta_2|)^{-(|\alpha_2|+|\beta_2|)}$$

for all $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{N}_0^n$. Then there exists a constant $C > 0$ such that

$$\sup_{t,s>0}\|m(t\xi_1, s\xi_2, t\eta_1, s\eta_2)\Phi(t\xi_1, s\xi_2, t\eta_1, s\eta_2)\psi_1(\xi_1,\eta_1)\psi_2(\xi_2,\eta_2)\|_{\mathrm{H}^{s_1,s_2}} \le C \sup_{j,k\in\mathbb{Z}}\left\|m_{j,k}\right\|_{\mathrm{H}^{s_1,s_2}}$$

for all $m \in L^\infty(\mathbb{R}^{4n})$ satisfies $\sup_{j,k\in\mathbb{Z}}\left\|m_{j,k}\right\|_{\mathrm{H}^{s_1,s_2}(\mathbb{R}^{4n})} < \infty$, where $m_{j,k}$ is defined by (29).

Proof. We mimic the proof of Lemma (3.4) in [210]. First, we assume that supp $\psi_1 \subset \{1/2^{j_0} \le |(\xi_1,\eta_1)| \le 2^{j_0}\}$ and supp $\psi_2 \subset \{1/2^{k_0} \le |(\xi_2,\eta_2)| \le 2^{k_0}\}$ for some $j_0, k_0 \in \mathbb{N}$. Given $t, s > 0$, take $j, k \in \mathbb{Z}$ satisfying $2^{j-1} \le t \le 2^j, 2^{k-1} \le s \le 2^k$. Then, since $1 < 2^j/t \le 2, 1 < 2^k/t \le 2$, by change of variables,

$$\|m(t\,\cdot, s\,\cdot)\Phi(t\,\cdot, s\,\cdot)\psi_1(\cdot)\psi_2(\cdot)\|_{\mathrm{H}^{s_1,s_2}} \le C\left\|m(2^j\cdot, 2^k\cdot)\Phi(2^j\cdot, 2^k\cdot)\psi_1(2^j t^{-1}\cdot)\psi_2(2^k s^{-1}\cdot)\right\|_{\mathrm{H}^{s_1,s_2}}.$$

Let $\psi(\xi_1,\eta_1), \psi(\xi_2,\eta_2)$ be as in (3) with $d = 2n$. Using supp $\psi_1(2^j t^{-1}\cdot) \subset \{1/2^{j_0+1} \le |(\xi_1,\eta_1)| \le 2^{j_0}\}$ and supp $\psi_2(2^k s^{-1}\cdot) \subset \{1/2^{k_0+1} \le |(\xi_2,\eta_2)| \le 2^{k_0}\}$, we have

$$\begin{aligned}
&\left\|m(2^j\cdot, 2^k\cdot)\Phi(2^j\cdot, 2^k\cdot)\psi_1(2^j t^{-1}\cdot)\psi_2(2^k s^{-1}\cdot)\right\|_{\mathrm{H}^{s_1,s_2}}\\
&\le C\sum_{j_1=-(j_0+1)}^{j_0}\sum_{k_1=-(k_0+1)}^{k_0}\left\|m(2^j\cdot, 2^k\cdot)\Phi(2^j\cdot, 2^k\cdot)\psi_1(2^j t^{-1}\cdot)\psi_2(2^k s^{-1}\cdot)\psi(\cdot/2^{j_1})\psi(\cdot/2^{k_1})\right\|_{\mathrm{H}^{s_1,s_2}}\\
&\le C\sum_{j_1=-(j_0+1)}^{j_0}\sum_{k_1=-(k_0+1)}^{k_0}\left\|m(2^j\cdot, 2^k\cdot)\psi(\cdot/2^{j_1})\psi(\cdot/2^{k_1})\right\|_{\mathrm{H}^{s_1,s_2}}\left\|\Phi(2^j\cdot, 2^k\cdot)\psi_1(2^j t^{-1}\cdot)\psi_2(2^k s^{-1}\cdot)\right\|_{\mathrm{H}^{s_1,s_2}}\\
&\le C\sum_{j_1=-(j_0+1)}^{j_0}\sum_{k_1=-(k_0+1)}^{k_0}\left\|m(2^{j+j_1}\cdot, 2^{k+k_1}\cdot)\psi(\cdot)\psi(\cdot)\right\|_{\mathrm{H}^{s_1,s_2}}\|\Phi(t\,\cdot, s\,\cdot)\psi_1\psi_2\|_{\mathrm{H}^{s_1,s_2}}\\
&\le C\left(\sup_{j,k\in\mathbb{Z}}\left\|m(2^{j+j_1}\cdot, 2^{k+k_1}\cdot)\psi\psi\right\|_{\mathrm{H}^{s_1,s_2}}\right)\left(\sup_{j,s>0}\|\Phi(t\,\cdot, s\,\cdot)\psi_1\psi_2\|_{\mathrm{H}^{s_1,s_2}}\right).
\end{aligned}$$

By Lemma (5.1.14), $\sup_{j,s>0}\|\Phi(t\,\cdot, s\,\cdot)\psi_1\psi_2\|_{\mathrm{H}^{s_1,s_2}} < \infty$,

The proof is then complete.

The main effort is to establish the first main theorem on L^r estimates for the multi-linear and multi-parameter Fourier multipliers with limited smoothness, namely, Theorem (5.1.17). The proof is quite complicated and involved due to the multi-parameter structure of the Fourier multiplier m. Therefore, we will divide the proof into several steps. The main idea is to decompose the multiplier into different pieces and handle them separately in each piece.

Theorem (5.1.17) [224] Let $m \in L^\infty(\mathbb{R}^{4n})$. Set

$$m_{j,k}(\xi_1,\xi_2,\eta_1,\eta_2) = m(2^j\xi_1, 2^k\xi_2, 2^j\eta_1, 2^k\eta_2)\psi_1(\xi_1,\eta_1)\psi_2(\xi_2,\eta_2), \quad (29)$$

where ψ_1, ψ_2 are the same as (3) with $d = 2n$. Let $s_1, s_2 > n$, $s = \min(s_1, s_2)$, $1 < p, q < \infty, p > \frac{2n}{s}, q > \frac{2n}{s}$ and $1/p + 1/q = 1/r$ with $0 < r < \infty$. If $m \in L^\infty(\mathbb{R}^{4n})$ satisfies

$$\sup_{j,k\in\mathbb{Z}} \|m_{j,k}\|_{\mathrm{H}^{s_1,s_2}(\mathbb{R}^{4n})} < \infty \tag{30}$$

then T_m is bounded from $L^p(\mathbb{R}^{2n}) \times L^q(\mathbb{R}^{2n})$ to $L^r(\mathbb{R}^{2n})$.

Proof. Let $s_1, s_2 > n$ and $m \in L^\infty(\mathbb{R}^{4n})$ satisfy $\sup_{j,k\in\mathbb{Z}} \|m_{j,k}\|_{\mathrm{H}^{s_1,s_2}} < \infty$, where $m_{j,k}$ is defined by (29). Since $\mathrm{H}^{s_1,s_2}(\mathbb{R}^{4n}) \hookrightarrow \mathrm{H}^{min\{s_1,s_2\},min\{s_1,s_2\}}(\mathbb{R}^{4n})$, it is sufficient to consider $\mathrm{H}^{s,s}(\mathbb{R}^{4n})$, where $s = min\{s_1, s_2\} > n$. We rewrite m as follows:

$$m(\xi_1,\xi_2,\eta_1,\eta_2) = m(\xi_1,\xi_2,\eta_1,\eta_2)\left(\sum_{i=1}^{3} \Phi_{(i)}(\xi_1,\eta_1)\right)\left(\sum_{j=1}^{3} \Phi_{(j)}(\xi_2,\eta_2)\right)$$

$$= \sum_{i,j=1}^{3} m(\xi_1,\xi_2,\eta_1,\eta_2)\Phi_{(i)}(\xi_1,\eta_1)\Phi_{(j)}(\xi_2,\eta_2) = \sum_{i,j=1}^{3} m_{i,j}(\xi_1,\xi_2,\eta_1,\eta_2) \tag{31}$$

where Φ_i, Φ_j $(1 \le i,j \le 3)$ are defined by (22) and (23).

By Lemma (5.1.13), we divide these $m_{j,k}$ into four groups and estimate the bilinear and bi-parameter Fourier multiplier operator defined by each symbol $m_{j,k}$. Since the Fourier multiplier operator corresponding to every symbol $m_{j,k}$ in the same group can be estimated in the similar way, we just choose one to handle in each group.

• Group 1:
– $m_{1,1}$, where supp $m_{1,1} \in \{|\xi_1| \le |\eta_1|/2, |\xi_2| \le |\eta_2|/2\}$
– $m_{2,2}$, *where supp* $m_{1,1} \in \{|\eta_1| \le |\xi_1|/2, |\eta_2| \le |\xi_2|/2\}$.
• Group 2:
– $m_{1,3}$, where supp $m_{1,3} \in \{|\xi_1| \le |\eta_1|/2, |\eta_2|/8 \le |\xi_2| \le 8|\eta_2|\}$
– $m_{2,3}$, where supp $m_{1,3} \in \{|\eta_1| \le |\xi_1|/2, |\eta_2|/8 \le |\xi_2| \le 8|\eta_2|\}$
– $m_{3,1}$, where supp $m_{1,3} \in \{|\eta_1|/8 \le |\xi_1| \le 8|\eta_1|, |\xi_2| \le |\eta_2|/2\}$
– $m_{3,2}$, where supp $m_{1,3} \in \{|\eta_1|/8 \le |\xi_1| \le 8|\eta_1|, |\eta_2| \le |\xi_2|/2\}$.
• Group 3:
– $m_{1,2}$, where supp $m_{1,2} \in \{|\xi_1| \le |\eta_1|/2, |\eta_2| \le |\xi_2|/2\}$
– $m_{2,1}$, where supp $m_{2,1} \in \{|\eta_1| \le |\xi_1|/2, |\xi_2| \le |\eta_2|/2\}$.
• Group 4:
– $m_{3,3}$, where supp $m_{3,3} \in \{|\eta_1|/8 \le |\xi_1| \le 8|\eta_1|, |\eta_2|/8 \le |\xi_2| \le 8|\eta_2|\}$.

In the following proof, we assume $2n/s < p, q$.

Estimates for Fourier multiplier corresponding to a symbol $m_{j,k}$ in Group 1.

First, we consider $m_{2,2}$, for simplicity we denote it as m^1 instead of $m_{2,2}$. Using the fact that L^p norm is bounded by the H^p norm in the multi-parameter setting established, e.g., in [108, 119, 222], and the equivalence of the definition of the multi-parameter Hardy space, we have for all $0 < r < \infty$

$$\|T_m(f,g)\|_{L^p} \le \left\|\sup_{s,t>0}\left|\Phi_{s,t}T_m(f,g)\right|\right\|_{L^r} \approx \left\|\left\{\sum_{j,k\in\mathbb{Z}}\left|\psi_1(D/2^j)\psi_2(D/2^k)T_m(f,g)\right|^2\right\}^{1/2}\right\|_{L^r} \tag{32}$$

for $0 < p < \infty$, where $\Phi_{s,t}(x,y) = 2^{sn}\phi(2^{sn}x)2^{tn}\phi(2^{tn}y), \phi \in \mathcal{S}(\mathbb{R}^n)$ and $\hat{\phi}$ does not contain the origin, ψ is the same as (3) with $d = n$.

Let $f, g \in \mathcal{S}(\mathbb{R}^{2n})$ since $\sum_{j\in\mathbb{Z}}\psi_j(\xi) = 1$, for all $\xi \in \mathbb{R}^n\backslash\{0\}$, we have

$$A_{j,k} \triangleq \psi(D/2^j)\psi(D/2^k)T_{m^1}(f,g)(x_1,x_2)$$
$$= \frac{1}{(2\pi)^{(4n)}}\int_{\mathbb{R}^{4n}} m^1(\xi_1,\xi_2,\eta_1,\eta_2)e^{ix_1(\xi_1+\eta_1)+ix_2(\xi_2+\eta_2)}\psi_j(\xi_1 + \eta_1)\hat{f}(\xi_1,\xi_2)\psi_k(\xi_2+\eta_2)\hat{g}(\eta_1,\eta_2)\,d\xi_1 d\xi_2 d\eta_1 d\eta_2$$
$$= \frac{1}{(2\pi)^{(4n)}}\int_{\mathbb{R}^{4n}} m^1(\xi_1,\xi_2,\eta_1,\eta_2)e^{ix_1(\xi_1+\eta_1)+ix_2(\xi_2+\eta_2)} \times \psi_j(\xi_1+\eta_1)\tilde{\psi}_j(\xi_1)\hat{f}(\xi_1,\xi_2)\psi_k(\xi_2+\eta_2)\tilde{\psi}_k(\xi_2)\hat{g}(\eta_1,\eta_2)d\xi_1 d\xi_2 d\eta_1 d\eta_2$$
$$= \frac{1}{(2\pi)^{(4n)}}\int_{\mathbb{R}^{4n}} m^1(\xi_1,\xi_2,\eta_1,\eta_2)e^{ix_1(\xi_1+\eta_1)+ix_2(\xi_2+\eta_2)} \times \psi_j(\xi_1+\eta_1)\tilde{\psi}_j(\xi_1)\hat{f}(\xi_1,\xi_2)\psi_k(\xi_2+\eta_2)\tilde{\psi}_k(\xi_2)\hat{g}(\eta_1,\eta_2)d\xi_1 d\xi_2 d\eta_1 d\eta_2$$
$$= \int_{\mathbb{R}^{4n}} 2^{(2jn+2kn)}(\mathcal{F}^{-1}m^1_{j,k})(2^j(x_1-y_1),2^k(x_2-y_2),2^j(x_1-z_1),2^k(x_2 - z_2)) \times (\tilde{\psi}_j(D)\tilde{\psi}_k(D)f)(y_1,y_2)g(z_1,z_2)dy_1dy_2dz_1dz_2 \quad (33)$$
where $\psi_k(\xi)=\psi(\xi/2^k)$ and $\tilde{\psi}(\xi_1)\in \mathcal{S}(\mathbb{R}^n)$ such that $\tilde{\psi}(\xi_1)\psi(\xi_1+\eta_1)=\psi(\xi_1+\eta_1)$, on the supp m^1, since $|\xi_1+\eta_1|\approx|\xi_1|$. The same is true for $\tilde{\psi}(\xi_2)$, i.e.,$\tilde{\psi}(\xi_2)\psi(\xi_2+\eta_2)=\psi(\xi_2+\eta_2)$, on the supp m^1, since $|\xi_2+\eta_2|\approx|\xi_2|$.
$$m^1_{j,k} = m^1(2^j\xi_1,2^k\xi_2,2^j\eta_1,2^k\eta_2)\psi(\xi_1+\eta_1)\psi(\xi_2+\eta_2). \quad (34)$$
Take $1<t<2$ satisfying $2n/s<t<\min\{2,p,q\}$.
$$|A|_{j,k} \le 2^{2jn+2kn}\int_{\mathbb{R}^{4n}}\left(1+2^j|x_1-y_1|+2^j|x_1-z_1|\right)^s(1+2^k|x_2-y_2|+2^k|x_2-z_2|)^s \times \left(\mathcal{F}^{-1}m^1_{j,k}\right)\left(2^j(x_1-y_1),2^k(x_2-y_2),2^j(x_1-z_1),2^k(x_2-z_2)\right) \times \left(1+2^j|x_1-y_1|+2^j|x_1-z_1|\right)^{-s}(1+2^k|x_2-y_2|+2^k|x_2-z_2|)^{-s} \times \left(\tilde{\psi}_j(D)\tilde{\psi}_k(D)f\right)(y_1,y_2)g(z_1,z_2)dy_1dy_2dz_1dz_2$$
$$\le \left(\int_{\mathbb{R}^{4n}}(1+|y_1|+|z_1|)^{t's}(1+|y_2| + |z_2|)^{t's}\left|\left(\mathcal{F}^{-1}m^1_{j,k}\right)(y_1,y_2,z_1,z_1)\right|^{t'}\right)^{1/t'} dy_1dy_2dz_1dz_2 \times\left(\int_{\mathbb{R}^{4n}}2^{2jn+2kn}\left(1+2^j|x_1-y_1|+2^j|x_1-z_1|\right)^{-ts}(1+2^k|x_2-y_2| + 2^k|x_2-z_2|)^{-ts}\times\left|\left(\tilde{\psi}_j(D)\tilde{\psi}_k(D)f\right)(y_1,y_2)g(z_1,z_2)\right|^t dy_1dy_2dz_1dz_2\right)^{1/t}$$
$$\lesssim \left\|m^1_{j,k}\right\|_{L^{t'}\left(w_{s,t'}\right)}\left(\int_{\mathbb{R}^{4n}}2^{jn+kn}|g(z_1,z_2)|^t(1+2^k|x_2-z_2|)^{-st/2}\left(1 + 2^j|x_1-z_1|\right)^{-st/2}dz_1dz_2\right)^{1/t} \times\left(\int_{\mathbb{R}^{4n}}2^{jn+kn}\left|\left(\tilde{\psi}_j(D)\tilde{\psi}_k(D)f\right)(y_1,y_2)\right|^t\left(1+2^j|x_1-y_1|\right)^{-st/2}(1 + 2^k|x_2-y_2|)^{-st/2}dy_1dy_2\right)^{1/t}$$
$$\lesssim \left\|m^1_{j,k}\right\|_{\mathrm{H}^{s,s}}\left(M_s\left(\left|(\tilde{\psi}_j(D)\tilde{\psi}_k(D)f)\right|^t\right)(x_1,x_2)\right)^{1/t}(M_s(|g|^t)(x_1,x_2))^{1/t}. \quad (35)$$
The last inequality is from Lemmas (5.1.10) and (5.1.11) since $st/2>n$.
Then by Hölder's inequality, (32) and (35), we have

$\|T_m^1(f,g)(x_1,x_2)\|_{L^r}$

$$\lesssim \sup_{j,k\in\mathbb{Z}}\|m_{j,k}^1\|_{\mathrm{H}^{s,s}}\left\|\left\{\sum_{j,k}\left(M_s\left(\left|(\tilde{\psi}_j(D)\tilde{\psi}_k(D)f)\right|^t\right)\right)^{2/t}\right\}^{1/2}\right\|_{L^p}\left\|\left\{\left(M_s(|g|^t)\right)^{2/t}\right\}^{1/2}\right\|_{L^q}$$

$$\lesssim \sup_{j,k\in\mathbb{Z}}\|m_{j,k}^1\|_{\mathrm{H}^{s,s}}\left\|\left\{\sum_{j,k}\left(M_s\left(\left|(\tilde{\psi}_j(D)\tilde{\psi}_k(D)f)\right|^t\right)\right)^{2/t}\right\}^{t/2}\right\|_{L^{p/t}}^{1/t}\left\|\left\{\left(M_s(|g|^t)\right)^{2/t}\right\}^{t/2}\right\|_{L^{q/t}}^{1/t}$$

$$\lesssim \sup_{j,k\in\mathbb{Z}}\|m_{j,k}^1\|_{\mathrm{H}^{s,s}}\|f\|_{L^p}\|g\|_{L^q}. \tag{36}$$

Using $\operatorname{supp} m^1 \in \{1/a \le \sqrt{|\xi_1|^2+|\eta_1|^2} \le a, 1/b \le \sqrt{|\xi_2|^2+|\eta_2|^2} \le b\}$ for some $a,b>1$, by Lemma (5.1.11) we have

$$\sup_{j,k\in\mathbb{Z}}\|m_{j,k}^1\|_{\mathrm{H}^{s,s}} \lesssim \sup_{j,k\in\mathbb{Z}}\|m_{j,k}\|_{\mathrm{H}^{s,s}}. \tag{37}$$

Consequently

$$\|T_{m^1}\|_{L^p\times L^q\to L^r} \lesssim \sup_{j,k\in\mathbb{Z}}\|m_{j,k}\|_{s_1,s_2}. \tag{38}$$

Changing the roles ξ_1,η_1 and ξ_2,η_2, we can prove

$$\|T_{m^1}\|_{L^p\times L^q\to L^r} \lesssim \sup_{j,k\in\mathbb{Z}}\|m_{j,k}\|_{\mathrm{H}^{s_1,s_2}} \tag{39}$$

where $m^1 = m_{1,1}$ this time.

Estimates for the Fourier multiplier operators with a symbol in Group 2:

We write m^2 instead of $m_{1,3}$ for simplicity. Since $\operatorname{supp} m_{1,3} \in \{|\xi_1| \le |\eta_1|/2, |\eta_2|/8 \le |\xi_2| \le 8|\eta_2|\}$, then there exists $\psi^1 \in \mathcal{S}(\mathbb{R}^n)$, such that $\psi(\xi_2)\psi^1(\eta_2) = \psi(\xi_2)$ on $\{|\eta_2|/8 \le |\xi_2| \le 8|\eta_2|\}$, where ψ is the function which is the same as case 1. Hence,

$$\begin{aligned}
&\psi(D/2^j)T_{m^2}(f,g)(x_1,x_2)\\
&= \frac{1}{(2\pi)^{(4n)}}\int_{\mathbb{R}^{4n}} m^2(\xi_1,\xi_2,\eta_1,\eta_2)e^{ix_1(\xi_1+\eta_1)+ix_2(\xi_2+\eta_2)}\psi_j(\xi_1\\
&\quad +\eta_1)\hat{f}(\xi_1,\xi_2)\hat{g}(\eta_1,\eta_2)\,d\xi_1 d\xi_2 d\eta_1 d\eta_2\\
&= \frac{1}{(2\pi)^{(4n)}}\sum_k\int_{\mathbb{R}^{4n}} m^2(\xi_1,\xi_2,\eta_1,\eta_2)e^{ix_1(\xi_1+\eta_1)+ix_2(\xi_2+\eta_2)}\\
&\quad \times\psi_j(\xi_1+\eta_1)\tilde{\psi}_j(\eta_1)\psi_k(\xi_2)\hat{f}(\xi_1,\xi_2)\psi_k^1(\eta_2)\hat{g}(\eta_1,\eta_2)d\xi_1 d\xi_2 d\eta_1 d\eta_2\\
&= \frac{1}{(2\pi)^{(4n)}}\sum_k\int_{\mathbb{R}^{4n}} m^2(\xi_1,\xi_2,\eta_1,\eta_2)e^{ix_1(\xi_1+\eta_1)+ix_2(\xi_2+\eta_2)}\psi_j(\xi_1+\eta_1)\\
&\quad \times\tilde{\psi}_j(\xi_1)\psi_k(\xi_2)\psi_k^2(\xi_2)\hat{f}(\xi_1,\xi_2)\psi_k^1(\eta_2)\hat{g}(\eta_1,\eta_2)d\xi_1 d\xi_2 d\eta_1 d\eta_2\\
&= \sum_k\int_{\mathbb{R}^{4n}} 2^{(2jn+2kn)}(\mathcal{F}^{-1}m_{j,k}^2)(2^j(x_1-y_1),2^k(x_2-y_2),2^j(x_1\\
&\quad -z_1),2^k(x_2-z_2))\\
&\quad \times(\tilde{\psi}_j(D)\psi_k^2(D)f)(y_1,y_2)(\psi_k^1(D)g)(z_1,z_2)dy_1dy_2dz_1dz_2 \triangleq \sum_k A_{j,k}
\end{aligned} \tag{40}$$

where $\tilde{\psi}$ is the same as we used in Estimates for symbols in Group 1 and $\psi(\xi_2)\,\psi^2(\xi_2) = \psi(\xi_2)$.

$$m_{j,k}^2 = m^2(2^j\xi_1,2^k\eta_1,2^j\xi_2,2^k\eta_2)\psi(\xi_1+\eta_1)\psi(\xi_2). \tag{41}$$

Take $1 < t < 2$ satisfying $2n/s < t < \min\{2, p, q\}$. Arguing in the same way as deriving (35), we can prove

$$|A_{j,k}| \lesssim \|m_{j,k}^2\|_{\mathrm{H}^{s,s}} \left(M_s\left(\left|(\tilde{\psi}_j(D)\psi_k^2(D)f)\right|^t\right)(x_1, x_2)\right)^{1/t} (M_s(\psi_k^1(D)|g|^t)(x_1, x_2))^{1/t}. \quad (42)$$

Moreover we can assume $f(\xi_1, \xi_2) = f_1(\xi_1)f_2(\xi_2)$, where $f_1, f_2 \in \mathcal{S}(\mathbb{R}^{2n})$, since $f_1 \otimes f_2$ is dense in $L^p(\mathbb{R}^{2n}), 1 \leq p < \infty$. Then we have

$$|A_{j,k}| \lesssim \|m_{j,k}^2\|_{\mathrm{H}^{s,s}} \left(M(|g_1|^t)(x_1) M\left(\left|(\tilde{\psi}_j(D)f_1)\right|^t\right)(x_1)\right)^{1/t}$$
$$\times \left(M\left(\left|\psi_k^1(D)g_2\right|^t\right)(x_2) M\left(\left|\psi_k^2(D)f_2\right|^t\right)(x_2)\right)^{1/t}. \quad (43)$$

Then from (40) and (43), we have

$$\left|\psi(D/2^j)T_{m^2}(f,g)(x_1,x_2)\right|$$
$$\lesssim \sum_k \|m_{j,k}^2\|_{\mathrm{H}^{s,s}} \left(M(|g_1|^t)(x_1) M\left(\left|(\tilde{\psi}_j(D)f_1)\right|^t\right)(x_1)\right)^{1/t}$$
$$\times \left(M(|\psi_k^1(D)g_2|^t)(x_2) M(|\psi_k^2(D)f_2|^t)(x_2)\right)^{1/t}$$
$$\lesssim \sup_{j,k\in\mathbb{Z}} \|m_{j,k}^2\|_{\mathrm{H}^{s,s}} \left(M(|g_1|^t)(x_1) M\left(\left|(\tilde{\psi}_j(D)f_1)\right|^t\right)(x_1)\right)^{1/t}$$
$$\times \left\{\sum_k [M(|\psi_k^1(D)g_2|^t)(x_2) M(|\psi_k^2(D)f_2|^t)(x_2)]^{1/t}\right\}. \quad (44)$$

Then

$$\left(\sum_j \left|\psi(D/2^j)T_m^2(f,g)(x_1,x_2)\right|^2\right)^{1/2}$$
$$\lesssim \sup_{j,k\in\mathbb{Z}} \|m_{j,k}^2\|_{\mathrm{H}^{s,s}} \left\{\sum_j \left[M(|g_1|^t)(x_1) M\left(\left|(\tilde{\psi}_j(D)f_1)\right|^t\right)(x_1)\right]^{2/t}\right.$$
$$\left.\times \left[\sum_k \left(M(|\psi_k^1(D)g_2|^t)(x_2) M(|\psi_k^2(D)f_2|^t)(x_2)\right)^{1/t}\right]^2\right\}^{1/2}$$
$$= \sup_{j,k\in\mathbb{Z}} \|m_{j,k}^2\|_{\mathrm{H}^{s,s}} \left\{\sum_j \left[M(|g_1|^t)(x_1) M\left(\left|(\tilde{\psi}_j(D)f_1)\right|^t\right)(x_1)\right]^{2/t}\right\}^{1/2}$$
$$\times \left\{\sum_k [M(|\psi_k^1(D)g_2|^t)(x_2) M(|\psi_k^2(D)f_2|^t)(x_2)]^{1/t}\right\}. \quad (45)$$

Since $p/t, q/t, 2/t > 1$, by Hölder's inequality, Lemmas (5.1.11), (5.1.12) and (45)

$$\|T_m^2(f,g)(x_1,x_2)\|_{L^r} \lesssim \left\|\left(\sum_j \left|\psi(D/2^j)T_m^2(f,g)(x_1,x_2)\right|^2\right)^{1/2}\right\|_{L^r}$$
$$\lesssim \sup_{j,k\in\mathbb{Z}} \|m_{j,k}^2\|_{\mathrm{H}^{s,s}} \left\|\left\{\sum_j \left[M(|g_1|^t)(x_1) M\left(\left|(\tilde{\psi}_j(D)f_1)\right|^t\right)(x_1)\right]^{2/t}\right\}^{1/2}\right\|_{L^r(\mathbb{R}^n)}$$
$$\times \left\|\sum_k [M(|\psi_k^1(D)g_2|^t)(x_2) M(|\psi_k^2(D)f_2|^t)(x_2)]^{1/t}\right\|_{L^r(\mathbb{R}^n)}$$

$$\lesssim \sup_{j,k\in\mathbb{Z}} \|m_{j,k}^2\|_{\mathrm{H}^{s,s}} \left\| \left\{ \sum_j \left[M\left(\left|(\tilde{\psi}_j(D) f_1)\right|^t \right)(x_1) \right]^{2/t} \right\}^{1/2} \right\|_{L^p} \left\| \left(M(|g_1|^t) \right)^{1/t} \right\|_{L^q}$$

$$\times \left\| \left(\sum_k \left(M(|\psi_k^1(D) g_2|^t)(x_2) \right)^{2/t} \right)^{1/2} \left(\sum_k \left(M(|\psi_k(D) f_2|^t)(x_2) \right)^{2/t} \right)^{1/2} \right\|_{L^r(\mathbb{R}^n)}$$

$$\lesssim \sup_{j,k\in\mathbb{Z}} \|m_{j,k}^2\|_{\mathrm{H}^{s,s}} \|f_1\|_{L^p} \|g_1\|_{L^q}$$

$$\times \left\| \left(\sum_k \left(M(|\psi_k^1(D) g_2|^t)(x_2) \right)^{2/t} \right)^{1/2} \right\|_{L^q} \left\| \left(\sum_k \left(M(|\psi_k^1(D) f_2|^t)(x_2) \right)^{2/t} \right)^{1/2} \right\|_{L^p}$$

$$\lesssim \sup_{j,k\in\mathbb{Z}} \|m_{j,k}^2\|_{\mathrm{H}^{s,s}} \|f_1\|_{L^p} \|f_2\|_{L^p} \|g_1\|_{L^q} \|g_2\|_{L^q}. \tag{46}$$

Using $\operatorname{supp} m_{j,k}^2 \in \{1/a \le \sqrt{|\xi_1|^2 + |\eta_1|^2} \le a, 1/b \le \sqrt{|\xi_2|^2 + |\eta_2|^2} \le b\}$ for some $a, b > 1$, by Lemma (5.1.16) we have

$$\sup_{j,k\in\mathbb{Z}} \|m_{j,k}^2\|_{\mathrm{H}^{s,s}} \lesssim \sup_{j,k\in\mathbb{Z}} \|m_{j,k}\|_{\mathrm{H}^{s,s}}. \tag{47}$$

Consequently

$$\|T_{m^2}\|_{L^p \times L^q \to L^r} \lesssim \sup_{j,k\in\mathbb{Z}} \|m_{j,k}\|_{\mathrm{H}^{s,s}}. \tag{48}$$

By changing the roles of ξ_1 and η_1 or (ξ_1, η_1) and (ξ_2, η_2), we can prove other situations in Group 2.

Estimates for Fourier multiplier with symbols in Group 3:

We write m^3 instead of $m_{1,2}$, the proof is similar to case 1 with necessary modification. Since $|\xi_1 + \eta_1| \approx |\eta_1|$ and $|\xi_2 + \eta_2| \approx |\xi_2|$we have

$$\psi(D/2^j) T_{m^3}(f, g)(x_1, x_2)$$

$$= \frac{1}{(2\pi)^{(4n)}} \int_{\mathbb{R}^{4n}} m^3(\xi_1, \xi_2, \eta_1, \eta_2) e^{ix_1(\xi_1+\eta_1)+ix_2(\xi_2+\eta_2)}$$

$$\times \psi_j(\xi_1 + \eta_1) \hat{f}(\xi_1, \xi_2) \psi_k(\xi_2 + \eta_2) \hat{g}(\eta_1, \eta_2)\, d\xi_1 d\xi_2 d\eta_1 d\eta_2$$

$$= \frac{1}{(2\pi)^{(4n)}} \int_{\mathbb{R}^{4n}} m^3(\xi_1, \xi_2, \eta_1, \eta_2) e^{ix_1(\xi_1+\eta_1)+ix_2(\xi_2+\eta_2)}$$

$$\times \psi_j(\xi_1 + \eta_1) \tilde{\psi}_j(\eta_1) \hat{f}(\xi_1, \xi_2) \psi_k(\xi_2 + \eta_2) \tilde{\psi}_k(\xi_2) \hat{g}(\eta_1, \eta_2) d\xi_1 d\xi_2 d\eta_1 d\eta_2$$

$$= \frac{1}{(2\pi)^{(4n)}} \int_{\mathbb{R}^{4n}} m^3(\xi_1, \xi_2, \eta_1, \eta_2) e^{ix_1(\xi_1+\eta_1)+ix_2(\xi_2+\eta_2)}$$

$$\times \psi_k(\xi_2 + \eta_2) \tilde{\psi}_k(\xi_2) \hat{f}(\xi_1, \xi_2) \psi_j(\xi_1 + \eta_1) \tilde{\psi}_j(\eta_1) \hat{g}(\eta_1, \eta_2) d\xi_1 d\xi_2 d\eta_1 d\eta_2$$

$$= \int_{\mathbb{R}^{4n}} 2^{(2jn+2kn)} (\mathcal{F}^{-1} m_{j,k}^3)(2^j(x_1 - y_1), 2^k(x_2 - y_2), 2^j(x_1 - z_1), 2^k(x_2 - z_2))$$

$$\times (\tilde{\psi}_k(D) f)(y_1, y_2) \tilde{\psi}_j(D) g(z_1, z_2) dy_1 dy_2 dz_1 dz_2 \triangleq A_{j,k} \tag{49}$$

where $\psi, \tilde{\psi}$ are defined the same way as we deal with symbols in Group 1 and

$$m_{j,k}^3 = m^3(2^j \xi_1, 2^k \eta_1, 2^j \xi_2, 2^k \eta_2) \psi(\xi_1 + \eta_1) \psi(\xi_2 + \eta_2). \tag{50}$$

As we did in dealing with symbols in Group 1, we can easily prove

$$|A_{j,k}| \lesssim \|m_{j,k}^3\|_{\mathrm{H}^{s,s}} \left(M_s\left(\left|(\tilde{\psi}_j(D) f)\right|^t \right)(x_1, x_2) \right)^{1/t} \left(M_s\left(\left|(\tilde{\psi}_k(D) f)\right|^t \right)(x_1, x_2) \right)^{1/t}. \tag{51}$$

where $\max\{1, 2n/s\} < t < 2$.

Since the rest of the proof is similar to that of case 1, we omit the details. Thus we obtain

$$\|T_{m^3}\|_{L^p \times L^q \to L^r} \lesssim \sup_{j,k\in\mathbb{Z}} \left\| m_{m_{j,k}}^3 \right\|_{\mathrm{H}^{s,s}} \lesssim \sup_{j,k\in\mathbb{Z}} \|m_{j,k}\|_{\mathrm{H}^{s,s}}. \tag{52}$$

By changing the roles of $(\xi_1 + \eta_1)$ and $(\xi_2 + \eta_2)$, we can get the same conclusion for $m_{2,1}$.

Estimates for Fourier multipliers with symbols in Group 4:

We write m^4 instead of $m_{3,3}$. Since the proof is similar to the case dealing with symbols in Group 2, we will outline the main estimates and omit the details here.

First, we can easily prove

$$|T_{m^4}(f,g)(x_1,x_2)|$$
$$\lesssim \sup_{j,k\in\mathbb{Z}} \|m_{j,k}^4\|_{\mathrm{H}^{s,s}} \left\{\sum_{j,k}\left(M_s\left(\left|(\tilde{\tilde{\psi}}_j(D)\tilde{\tilde{\psi}}_k(D)f)\right|^t\right)(x_1,x_2)\right)^{2/t}\right\}^{1/2}$$
$$\times\left\{\sum_{j,k}\left(M_s\left(\left|(\tilde{\tilde{\psi}}_j(D)\tilde{\tilde{\psi}}_k(D)f)\right|^t\right)(x_1,x_2)\right)^{t/2}\right\}^{1/2} \quad (53)$$

where $\max\{1,2n/s\}<t<2$.

$$m_{j,k}^4 = m^4(2^j\xi_1,2^k\eta_1,2^j\xi_2,2^k\eta_2)\psi(\xi_1+\eta_1)\tilde{\psi}(\xi_1)\psi(\xi_2+\eta_2)\tilde{\psi}(\xi_2). \quad (54)$$

Since $p/t, q/t, 2/t > 1$, by Hölder's inequality, Lemmas (5.1.11) and (5.1.12), we have

$$\|T_m^4(f,g)(x_1,x_2)\|_{L^r}$$
$$\lesssim \sup_{j,k\in\mathbb{Z}} \|m_{j,k}^4\|_{\mathrm{H}^{s,s}} \left\|\left(\sum_{j,k}\left(M_s\left(\left|(\tilde{\tilde{\psi}}_j(D)\tilde{\tilde{\psi}}_k(D)f)\right|^t\right)\right)^{2/t}\right)^{1/2}\right\|_{L^p}$$
$$\times\left\|\left(\sum_{j,k}\left(M_s\left(\left|(\tilde{\tilde{\psi}}_j(D)\tilde{\tilde{\psi}}_k(D)g)\right|^t\right)\right)^{2/t}\right)^{1/2}\right\|_{L^q}$$
$$\lesssim \sup_{j,k\in\mathbb{Z}} \|m_{j,k}^4\|_{\mathrm{H}^{s,s}} \left\|\left\{\sum_{j,k}\left(M_s\left(\left|(\tilde{\tilde{\psi}}_j(D)\tilde{\tilde{\psi}}_k(D)f)\right|^t\right)\right)^{2/t}\right\}^{t/2}\right\|_{L^{p/t}}^{1/t}$$
$$\times\left\|\left\{\sum_{j,k}\left(M_s\left(\left|(\tilde{\tilde{\psi}}_j(D)\tilde{\tilde{\psi}}_k(D)g)\right|^t\right)\right)^{2/t}\right\}^{t/2}\right\|_{L^{p/t}}^{1/t}$$
$$\lesssim \sup_{j,k\in\mathbb{Z}} \|m_{j,k}^4\|_{\mathrm{H}^{s,s}} \left\|\left\{\sum_{j,k}\left|(\tilde{\tilde{\psi}}_j(D)\tilde{\tilde{\psi}}_k(D)f)\right|^2\right\}^{1/2}\right\|_{L^p}\left\|\left\{\sum_{j,k}\left|(\tilde{\tilde{\psi}}_j(D)\tilde{\tilde{\psi}}_k(D)g)\right|^2\right\}^{1/2}\right\|_{L^q}$$
$$\lesssim \sup_{j,k\in\mathbb{Z}} \|m_{j,k}^4\|_{\mathrm{H}^{s,s}} \|f\|_{L^p}\|g\|_{L^q}. \quad (55)$$

Since $\operatorname{supp} m^4 \in \{1/a \le \sqrt{|\xi_1|^2+|\eta_1|^2} \le a, 1/b \le \sqrt{|\xi_2|^2+|\eta_2|^2} \le b\}$ for some $a,b>1$, by Lemma (5.1.16) we have

$$\|T_{m^4}\|_{L^p\times L^q\to L^r} \lesssim \sup_{j,k\in\mathbb{Z}} \left\|m_{m_{j,k}}^3\right\|_{\mathrm{H}^{s,s}} \lesssim \sup_{j,k\in\mathbb{Z}} \|m_{j,k}\|_{\mathrm{H}^{s,s}}. \quad (56)$$

Next, we consider $T_{m^{*1}}, T_{m^{*2}}$, the dual operator of T_m, which are defined by

$$\int_{\mathbb{R}^{2n}} T_m(f,g)h\,dx = \int_{\mathbb{R}^{2n}} T_{m^{*1}}(h,g)f\,dx = \int_{\mathbb{R}^{2n}} T_{m^{*2}}(f,h)g\,dx \quad (57)$$

for all $f,g,h\in\mathcal{S}(\mathbb{R}^{2n})$.

If we have proved the same conclusion for $T_{m^{*1}}, T_{m^{*2}}$, as T_m, then using the same proof as in the bilinear case in [209], we complete the proof of Theorem (5.1.17) by multi-linear and multi-parameter duality and interpolation.

To finish the proof of Theorem (5.1.17), we only need to show

$$\sup_{j,k\in\mathbb{Z}} \left\| m^{*1}_{m_{j,k}} \right\|_{\mathrm{H}^{s_1,s_2}(\mathbb{R}^{4n})} \lesssim \sup_{j,k\in\mathbb{Z}} \left\| m_{j,k} \right\|_{\mathrm{H}^{s_1,s_2}(\mathbb{R}^{4n})}$$
$$\sup_{j,k\in\mathbb{Z}} \left\| m^{*2}_{m_{j,k}} \right\|_{\mathrm{H}^{s_1,s_2}(\mathbb{R}^{4n})} \lesssim \sup_{j,k\in\mathbb{Z}} \left\| m_{j,k} \right\|_{\mathrm{H}^{s_1,s_2}(\mathbb{R}^{4n})} \quad (58)$$

where $m^{*1}(\xi_1,\eta_1,\xi_2,\eta_2) = m(-(\xi_1+\eta_1),\eta_1,-(\xi_2+\eta_2),\eta_2)$ and $m(\xi_1,\eta_1,\xi_2,\eta_2) = m^{*1}(\xi_1,-(\xi_1+\eta_1),\xi_2,-(\xi_2+\eta_2))$.

We only choose one case to prove, the remaining cases are the same.

By a change of variables,

$$\begin{aligned}\left\| m^{*1}_{m_{j,k}} \right\|_{\mathrm{H}^{s_1,s_2}} &= \left\| m\left(-2^j(\xi_1+\eta_1),-2^k(\xi_2+\eta_2),2^j\eta_1,2^k\eta_2\right)\psi_1(\xi_1+\eta_1)\psi_2(\xi_2+\eta_2)\right\|_{\mathrm{H}^{s_1,s_2}} \\ &\approx \left\| m\left(2^j\xi_1,2^k\xi_2,2^j\eta_1,2^k\eta_2\right)\psi_1(-(\xi_1+\eta_1),\eta_1)\psi_2(-(\xi_2+\eta_2),\eta_2)\right\|_{\mathrm{H}^{s_1,s_2}}. \end{aligned} \quad (59)$$

Since $\sqrt{|\xi+\eta|^2+|\eta|^2} \approx \sqrt{|\xi|^2+|\eta|^2}$, then we can obtain

$$\sup_{j,k\in\mathbb{Z}} \left\| m^{*1}_{m_{j,k}} \right\|_{\mathrm{H}^{s_1,s_2}} \lesssim \sup_{j,k\in\mathbb{Z}} \left\| m^{*1}_{m_{j,k}} \right\|_{\mathrm{H}^{s_1,s_2}}. \quad (60)$$

Therefore, we have finished the proof of Theorem (5.1.17).

Theorem (5.1.18) [224] Let $1 < p, q < \infty, 1/p + 1/q = 1/r$ and $n < s_1, s_2 \le 2n, s = \min\{s_1, s_2\}$. Assume

(i) $p > 2n/s_1 \ w_1 \in A_{ps_1/2n}$ (61)

$q > 2n/s_1 \ w_2 \in A_{ps_2/2n}$ *or* (62)

(ii) $\min\{p,q\} < (2n/s)', \quad 1 < r < \infty$ (63)

$w_1^{1-r'} \in A_{r's/2n}, \qquad w_2^{1-r'} \in A_{r's/2n}$. (64)

If $m \in L^\infty(\mathbb{R}^{4n})$ satisfies

$$\sup_{j,k\in\mathbb{Z}} \left\| m_{j,k} \right\|_{\mathrm{H}^{s_1,s_2}(\mathbb{R}^{4n})} < \infty, \quad (65)$$

then T_m is bounded from $L^p(w_1) \times L^q(w_2)$ to $L^r(w)$, where w = $w_1^{r/p} w_2^{r/q}$.

The statements and their proofs of Theorems (5.1.17) and (5.1.18) can be easily generalized to multi-linear and multi-parameter settings. We also remark that the proofs of our main theorems can be viewed as alternative ones different from those given in [214, 215]. Moreover, we provide weighted estimates for the multi-linear and multi-parameter Coifman–Meyer multiplier operators considered in [214, 215]. We only state these results here.

In general, any collection of n generic vectors $\xi_1 = (\xi_1^i)_{i=1}^t, \dots, \xi_n = (\xi_n^i)_{i=1}^t$ in $\mathbb{R}^{t\ell}$ generates naturally the following collection of t vectors in $\mathbb{R}^{n\ell}$:

$$\overline{\xi}_1 = (\xi_j^1)_{j=1}^n, \overline{\xi}_2 = (\xi_j^2)_{j=1}^n, \dots, \overline{\xi}_t = (\xi_j^t)_{j=1}^n. \quad (66)$$

Let $m = m(\xi) = m(\overline{\xi})$ be a bounded symbol in $L^\infty(\mathbb{R}^{tn\ell})$ that is smooth away from the subspaces $\{\overline{\xi}_1 = 0\} \cup \cdots \cup \{\overline{\xi}_t = 0\}$ and satisfying

$$\left| \partial_{\overline{\xi}_1}^{\alpha_1} \cdots \partial_{\overline{\xi}_t}^{\alpha_t} m(\overline{\xi}) \right| \le C_{\alpha_1,\dots,\alpha_t} \prod_{i=1}^t \left|\overline{\xi}_i\right|^{-|\alpha_i|} \quad (67)$$

for sufficiently many multi-indices $\alpha_1, \dots, \alpha_t$. We will naturally want to investigate the L^r estimates of the n-linear multiplier operator $T_m^{(t)}$ defined by

$$T_m^{(t)}(f_1,\dots,f_n)(x) := \int_{\mathbb{R}^{tn}} m(\xi)\hat{f}_1(\xi_1)\cdots\hat{f}_n(\xi_n)e^{2\pi i x\cdot(\xi_1+\cdots+\xi_n)d\xi}. \quad (68)$$

Thus, we can prove the following L^r estimates for general n-linear, t-parameter multiplier operator $T_m^{(t)}$ with limited smoothness.

Proof. Is devoted to establishing the second main theorem on weighted estimates for the multi-linear and multi-parameter Fourier multipliers with limited smoothness, namely, Theorem (5.1.18). Before we prove Theorem (5.1.18), we recall some useful facts about product $A_p(\mathbb{R}^n \times \mathbb{R}^n)$ weights.

Lemma (5.1.19) [224] ([223]). Let $1 < p < \infty$ and $w \in A_p(\mathbb{R}^n \times \mathbb{R}^n)$. Then

(i) $w^{1-p'} \in A_{p'}(\mathbb{R}^n \times \mathbb{R}^n)$

(ii) there exists $1 < q < p$ such that $w \in A_q(\mathbb{R}^n \times \mathbb{R}^n)$.

Lemma (5.1.20) [224] Suppose that $w_j \in A_{p_j}(\mathbb{R}^n \times \mathbb{R}^n)$ with $1 \le j \le m$ for some $1 \le p_1, \dots, p_m \le \infty$ and let $0 < \theta_1, \dots, \theta_m < 1$ be such that $\theta_1 + \cdots + \theta_m = 1$. Then

$$w_1^{\theta_1} \cdots w_m^{\theta_m} \in A_{\max\{p_1,\dots,p_m\}}. \tag{69}$$

Proof. First note that $w_j \in A_{\max\{p_1,\dots,p_m\}}$, for $j = 1, \dots, m$, then apply Hölder's inequality, we can obtain the conclusion.

Lemma (5.1.21) [224] ([112]). Let $1 < p, q < \infty$ and $w \in A_p(\mathbb{R}^n \times \mathbb{R}^n)$. Then there exists a constant $C > 0$ such that

$$\left\| \left\{ \sum_{k\in\mathbb{Z}} (M_s f_k)^q \right\}^{1/q} \right\|_{L^p(w)} \le C \left\| \left\{ \sum_{k\in\mathbb{Z}} |f_k|^q \right\}^{1/q} \right\|_{L^p(w)} \tag{70}$$

for all sequences $\{f_k\}_{k\in\mathbb{Z}}$ of locally integrable functions on $\mathbb{R}^{2n}$.

Lemma (5.1.22) [224] ([98]). Let $1 < p < \infty, w \in A_p(\mathbb{R}^n \times \mathbb{R}^n)$, and let $\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}^n)$ be such that $\operatorname{supp} \psi_1 \subset \{\xi \in \mathbb{R}^n : 1/a \le |\xi| \le a\}$ for some $a > 1, \operatorname{supp} \psi_2 \subset \{\xi \in \mathbb{R}^n : 1/b \le |\xi| \le b\}$ for some $b > 1$. Then there exists a constant $C > 0$ such that

$$\left\| \left\{ \sum_{j,k\in\mathbb{Z}} \left| \psi_1(D/2^j) \psi_2(D/2^k) f \right|^2 \right\}^{1/2} \right\|_{L^p(w)} \le C \|f\|_{L^p(w)} \ for\ all\ f \in L_w^p(\mathbb{R}^n). \tag{71}$$

Moreover, $if \sum_{j\in\mathbb{Z}} \psi_i(\xi/2^j) = 1$ for all $\xi \ne 0$, for $i = 1, 2$, then

$$\left\| \left\{ \sum_{j,k\in\mathbb{Z}} \left| \psi_1(D/2^j) \psi_2(D/2^k) f \right|^2 \right\}^{1/2} \right\|_{L^p(w)} \approx \|f\|_{L^p(w)} \ for\ all\ f \in L^p(w). \tag{72}$$

Lemma (5.1.23) [224] ([124]). If $0 < p < \infty, w \in A_\infty(\mathbb{R}^n \times \mathbb{R}^n), f$ is a local integrable function in $H_w^p(\mathbb{R}^n \times \mathbb{R}^n)$. Then

$$\|f\|_{L^p(w)} \le \left\| \left\{ \sum_{j,k\in\mathbb{Z}} \left| \psi_1(D/2^j) \psi_2(D/2^k) f \right|^2 \right\}^{1/2} \right\|_{L^p(w)}. \tag{73}$$

We first prove Theorem (5.1.18) under assumption (i) in Theorem (5.1.18). Since $2n/s_1 < \min\{2, p\}$ and $w_1 \in A_{ps_1/2n}$, by Lemma (5.1.19), we can take $2n/s_1 < p_1 < \min\{2, p\}$ satisfying $w_1 \in A_{p/p_1}$, the same is for w_2. Then

$$\|T_{m^1}(f, g)\|_{L^p(w)} \le \left\| \left\{ \sum_{j,k\in\mathbb{Z}} \left| \psi_1(D/2^j) \psi_2(D/2^k) T_m(f, g) \right|^2 \right\}^{1/2} \right\|_{L^p(w)}$$

$$\lesssim \sup_{j,k\in\mathbb{Z}}\|m^1_{j,k}\|_{\mathrm{H}^{s_1,s_2}}\left\|\left\{\sum_{j,k}\left(M_s\left(\left|(\tilde{\psi}_j(D)\tilde{\psi}_k(D)f)\right|^t\right)\right)^{2/t}\right\}^{1/2}\right\|_{L^p(w_1)}\left\|\left\{\left(M_s(|g|^t)\right)^{2/t}\right\}^{1/2}\right\|_{L^q(w_2)}$$

$$\lesssim \sup_{j,k\in\mathbb{Z}}\|m^1_{j,k}\|_{\mathrm{H}^{s_1,s_2}}\left\|\left\{\sum_{j,k}\left(M_s\left(\left|(\tilde{\psi}_j(D)\tilde{\psi}_k(D)f)\right|^t\right)\right)^{2/t}\right\}^{t/2}\right\|^{1/t}_{L^{p/t}\left(w_1^{p/t}\right)}\left\|\left\{\left(M_s(|g|^t)\right)^{2/t}\right\}^{t/2}\right\|^{1/t}_{L^{q/t}\left(w_2^{q/t}\right)}$$

$$\lesssim \sup_{j,k\in\mathbb{Z}}\|m^1_{j,k}\|_{\mathrm{H}^{s,s}}\|f\|_{L^p(w_1)}\|g\|_{L^q(w_2)} \tag{74}$$

where we take $t=\max\{p_1,q_1\}$, then $w_1\in A_{p/t}$, and $w_2\in A_{q/t}$.

To conclude the weighted estimates for the Fourier multipliers m, we need to do estimates corresponding to other symbols. Since the estimates for the remaining symbols in other groups are similar to that of m^1.

We give the proof of Theorem (5.1.18) under condition (ii) we consider case $p=\min\{p,q\}$. Since $p'<(2n/s)'$, then $\max\{1/r',1/q\}<1/r'+1/q=1/p<s/2n$, that is, $r',q>2n/s$. Hence $2n/s<\min\{2,r',q\}$.

Since $1/2<s/2n\le 1$ and $w_1^{1-r'}\in A_{r's/(2n)}, w_2^{1-r'}\in A_{r's/(2n)}$, by Lemma (5.1.19) we have

$$w_1^{1-r'}\in A_{r's/(2n)}\subset A_{r'},\qquad \text{then}\ \ w_1\in A_r \tag{75}$$

$$w_2^{1-r'}\in A_{r's/(2n)}\subset A_{r'},\qquad \text{then}\ \ w_2\in A_r \tag{76}$$

$$w^{1-r'}=w_1^{(1-r')r/p}w_2^{(1-r')r/q}\in A_{r's/(2n)} \tag{77}$$

where (77) is from Lemma (5.1.20).

It is from the assumption that $p\le q$, we also have $r\le q/2$, then $w_2\in A_r\subset A_{q/2}\subset A_{qs/2n}$. Since $w^{1-r'}\in A_{r's/(2n)}, w_2\in A_r\subset A_{qs/2n}$, by Lemma (5.1.20) we can take $2n/s<t<\min\{2,r',q\}$ such that

$$w^{1-r'}\in A_{r'/t},\qquad w_2\in A_{q/t}. \tag{78}$$

By duality and (59), it is enough to prove

$$\|T_{m^{*1}}\|_{L^{r'}(w^{1-r'})\times L^q(w_2)\to L^{p'}(w_1^{1-p'})}\le C\sup_{j,k\in\mathbb{Z}}\|m^1_{j,k}\|_{\mathrm{H}^{s_1,s_2}}. \tag{79}$$

From the proof of Theorem (5.1.17), we have

$$\|T_{m^{*1}}(f,g)\|_{L^{p'}\left(w_1^{1-p'}\right)}\le\left\|\left\{\sum_{j,k\in\mathbb{Z}}\left|\psi_1(D/2^j)\psi_2(D/2^k)T_m(f,g)\right|^2\right\}^{1/2}\right\|_{L^p(w)}$$

$$\lesssim \sup_{j,k\in\mathbb{Z}}\|m^1_{j,k}\|_{\mathrm{H}^{s_1,s_2}}\left\|\left\{\sum_{j,k}\left(M_s\left(\left|(\tilde{\psi}_j(D)\tilde{\psi}_k(D)f)\right|^t\right)(x_1,x_2)\right)^{2/t}\right\}^{1/2}w^{-1/r}\times\left(M_s(|g|^t)(x_1,x_2)\right)^{1/t}w_2^{1/q}\right\|_{L^{r'}}$$

$$\lesssim \sup_{j,k\in\mathbb{Z}}\|m^1_{j,k}\|_{\mathrm{H}^{s_1,s_2}}\left\|\left\{\sum_{j,k}\left(M_s\left(\left|(\tilde{\tilde{\psi}}_j(D)\tilde{\tilde{\psi}}_k(D)f)\right|^t\right)(x_1,x_2)\right)^{2/t}\right\}^{1/2}\right\|_{L^{r'}\left(w^{1-r'}\right)}\times\left\|\left\{\left(M_s(|g|^t)(x_1,x_2)\right)^{2/t}\right\}^{1/2}\right\|_{L^q(w_2)}$$

$$\lesssim \sup_{j,k\in\mathbb{Z}}\|m^1_{j,k}\|_{\mathrm{H}^{s_1,s_2}}\left\|\left\{\sum_{j,k}\left(M_s\left(\left|(\tilde{\psi}_j(D)\tilde{\psi}_k(D)f)\right|^t\right)(x_1,x_2)\right)^{2/t}\right\}^{t/2}\right\|^{1/t}_{L^{r'/t}\left(w^{1-r'}\right)}$$

$$\times\left\|\left\{\left(M_s(|g|^t)(x_1,x_2)\right)^{2/t}\right\}^{t/2}\right\|^{1/t}_{L^{q/t}(w_2)}\lesssim \sup_{j,k\in\mathbb{Z}}\|m^1_{j,k}\|_{\mathrm{H}^{s_1,s_2}}\|f\|_{L^p(w_1)}\|g\|_{L^q(w_2)}. \tag{80}$$

The weighted estimates for the Fourier multiplier operators corresponding to the remaining symbols are the same as with T_{m^1}, thus we finish the proof of Theorem (5.1.18).

Section (5.2) Composition of Two Singular Integral Operators

The classical theory of one-parameter harmonic analysis may be considered as centering around the Hardy-Littlewood maximal operator and its relationship with certain singular integral operators which commute with the usual one-parameter dilations on $\mathbb{R}^m$, given by $\delta(x) = (\delta x_1, \cdots, \delta x_m), \delta > 0$. If this isotropic dilation is replaced by more general non-isotropic groups of dilations, then many nonisotropic variants of the classical theories can be produced, such as the strong maximal functions, multi-parameter singular integral operators, corresponding to the multi-parameter dilations $\delta : x \to (\delta_1 x_1, \delta_2 x_2), x = (x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^m, \delta = (\delta_1, \delta_2), \delta_1, \delta_1 > 0$. Such a multi-parameter theory has been developed extensively over the past decades. See [96–105, 110, 116, 121, 123, , 128, 129, 136, 233, 234, 237]. Multi-parameter flag singular integrals and their boundedness on L^p and H^p spaces have been studied in [106–108, 118, 120, 235, 239, 240, 242], multi-parameter and multi-linear Coifman-Meyer Fourier multipliers have been investigated in [214, 215, 219, 220, 224], and a theory of multi-parameter singular Radon transforms have been developed in [244–246].

[132] developed a theory of new multi-parameter Hardy space associated with the composition of two singular integral operators with different homogeneities and established the boundedness of the composition of such singular integrals on this space. For $\mathbb{R}^m = \mathbb{R}^{m-1} \times \mathbb{R}$ with $x = (x', x_m)$ where $x \in \mathbb{R}^{m-1}$ and $x_m \in \mathbb{R}$, they consider two kinds of homogeneities:

$$\delta: (x', x_m) \to (\delta x', \delta x_m), \delta > 0,$$

and

$$\delta: (x', x_m) \to (\delta x', \delta^2 x_m), \delta > 0.$$

The first is the classical isotropic dilations occurring in the classical Calderón-Zygmund singular integrals, while the second is non-isotropic and related to the heat equations (also Heisenberg groups). For $x = (x', x_m) \in \mathbb{R}^{m-1} \times \mathbb{R}$, denote $|x|_e = (|x'|^2 + |x_m|^2)^{\frac{1}{2}}$ and $|x|_h = (|x'|^2 + |x_m|^2)^{\frac{1}{2}}$. The singular integrals considered in [132] are defined in the following.

Definition (5.2.1) [250] A locally integrable function K_1 on $\mathbb{R}^m \backslash \{0\}$ is said to be a Calderón-Zygmund kernel associated with the isotropic homogeneity if

$$\left|\frac{\partial^\alpha}{\partial x^\alpha}\mathrm{K}_1(x)\right| \le A|x|_e^{-m-|\alpha|} \quad \text{for all } \ |\alpha| \ge 0, \tag{81}$$

$$\int_{r_1<|x|_e<r_2} \mathrm{K}_1(x)dx = 0 \tag{82}$$

for all $0 < r_1 < r_2 < \infty$.

An operator T_1 is said to be a Calderón-Zygmund singular integral operator associated with the isotropic homogeneity if $T_1(f)(x) = p.v.(\mathrm{K}_1 * f)(x)$, where K_1 satisfies conditions in (81) and (82).

Definition (5.2.2) [250] Suppose $\mathrm{K}_2 \in L^1_{loc}(\mathbb{R}^m \backslash \{0\})$. K_2 is said to be a Calderón-Zygmund kernel associated with the non-isotropic homogeneity if

$$\left|\frac{\partial^\alpha}{\partial (x')^\alpha}\frac{\partial^\beta}{\partial (x_m)^\beta}\mathrm{K}_2(x', x_m)\right| \le B|x|_h^{-m-1-|\alpha|-2\beta} \quad \forall \ |\alpha| \ge 0, \beta \ge 0, \tag{83}$$

$$\int_{r_1<|x|_h<r_2} \mathrm{K}_2(x)dx = 0 \tag{84}$$

for all $0 < r_1 < r_2 < \infty$.

An operator T_2 is said to be a Calderón-Zygmund singular integral operator associated with the non-isotropic homogeneity if $T_2(f)(x) = p.v.(\mathrm{K}_2 * f)(x)$, where K_2 satisfies the conditions in (83) and (84).

Both the classical Calderón-Zygmund theory and theory of singular integral operators associated with the non-isotropic dilations indicate that both the operators T_1 and T_2 are bounded on L^p for $1 < p < \infty$ and of weak-type (1,1). It is shown by Phong and Stein in [135] that in general the composition operator $T_1 \circ T_2$ is not of weak-type (1,1). [135] gave a necessary and sufficient condition such that the composition operator $T_1 \circ T_2$ is of weak-type (1,1). This answers the question raised by Rivieré in [249]. In fact, the operators studied in [135] are compositions with different homogeneities, and such a composition operator arises naturally in the study of the $\bar{\partial}$-Neumann problem.

It is also well-known that any Calderón-Zygmund singular integral operator associated with the isotropic homogeneity is bounded on the classical Hardy space $H^p(\mathbb{R}^m)$ with $0 < p \leq 1$. A Calderón-Zygmund singular integral operator associated with the non-isotropic homogeneity is not bounded on the classical Hardy space but bounded on the non-isotropic Hardy space (see [230]). The composition operator $T_1 \circ T_2$ is bounded on neither the classical Hardy space nor the non-isotropic Hardy space. Thus, the natural question is to ask on what Hardy space can the composition operator $T_1 \circ T_2$ be bounded? [132] introduced a new Hardy space $H^p_{hom}(\mathbb{R}^n)$ associated with the composition of these two different homogeneities and proved that $T_1 \circ T_2$ is indeed bounded on such spaces. Developed in [139] the theory of the Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}(\mathbb{R}^m)$ associated with the composition of these different homogeneities. Such Triebel-Lizorkin spaces for $0 < p \leq 1, \alpha_1 = \alpha_2 = 0$ and $q = 2$ are the Hardy spaces $H^p_{hom}(\mathbb{R}^n)$ considered in [132]. Triebel-Lizorkin spaces form a unifying class of function spaces encompassing many well studied classical function spaces such as Lebesgue spaces, Hardy spaces, the Lipschitz spaces, and the space BMO [65, 115]. Boundedness of singular integrals and pseudo-differential operators on the Triebel-Lizorkin spaces have also been extensively studied; see, Frazier and Jawerth [115] and Torres [247].

The main goals are to identify the dual spaces $CMO_p^{-\alpha,q'}$ of the new Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}(\mathbb{R}^m)$.

We now introduce the new Triebel-Lizorkin spaces associated with different homogeneities. Denote $S_0(\mathbb{R}^m) = \{f \in S(\mathbb{R}^m): \int_{\mathbb{R}^m} f(x)x^\alpha dx = 0, \forall\, |\alpha| \geq 0\}$. Let $\psi^{(1)} \in S(\mathbb{R}^m)$ with

$$\operatorname{supp} \widehat{\psi^{(1)}} \subseteq \left\{(\xi', \xi_m) \in \mathbb{R}^{m-1} \times \mathbb{R} : \frac{1}{2} \leq |\xi|_e \leq 2\right\} \tag{85}$$

and

$$\sum_{j\in\mathbb{Z}} \left|\widehat{\psi^{(1)}}(2^{-j}\xi', 2^{-j}\xi_m)\right|^2 = 1 \text{ for all } (\xi', \xi_m) \in \mathbb{R}^m \backslash \{0\}. \tag{86}$$

Let $\psi^{(2)} \in S(\mathbb{R}^m)$ with

$$\operatorname{supp} \widehat{\psi^{(2)}} \subseteq \left\{(\xi', \xi_m) \in \mathbb{R}^{m-1} \times \mathbb{R} : \frac{1}{2} \leq |\xi|_h \leq 2^{1/2}\right\} \tag{87}$$

and

$$\sum_{j\in\mathbb{Z}} \left|\widehat{\psi^{(2)}}(2^{-k}\xi', 2^{-2k}\xi_m)\right|^2 = 1 \text{ for all } (\xi', \xi_m) \in \mathbb{R}^m \backslash \{0\}. \quad (88)$$

Denote $\psi_{j,k}(x) = \psi_j^{(1)} * \psi_k^{(2)}(x)$, where $\psi_j^{(1)}(x', x_m) = 2^{jm}\psi^{(1)}(2^j x', 2^j x_m)$, $\psi_k^{(2)}(x', x_m) = 2^{k(m+1)}\psi^{(2)}(2^k x', 2^{2k} x_m)$, and $j \wedge k = \min\{j, k\}, j \vee k = \max\{j, k\}$. The following discrete Calderón reproducing formula is from [132].

Theorem (5.2.3) [250] Suppose that $\psi^{(1)}$ and $\psi^{(2)}$ are functions satisfying conditions in (85)-(86) and (87)-(88), respectively. Then

$$f(x', x_m) = \sum_{j,k\in\mathbb{Z}} \sum_{(\ell', \ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} 2^{-(m-1)(j\wedge k)} 2^{-(j\wedge 2k)} (\psi_{j,k} * f)(2^{-(j\wedge k)}\ell', 2^{-(j\wedge 2k)}\ell_m)$$
$$\times \psi_{j,k}(x' - 2^{-(j\wedge k)}\ell', x_m - 2^{-(j\wedge 2k)}\ell_m), \quad (89)$$

where the series converges in $L^2(\mathbb{R}^m), S_0(\mathbb{R}^m)$ and $S_0'(\mathbb{R}^m)$.

Definition (5.2.4) [250] Let $0 < p, q < \infty, \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$. The multi-parameter Triebel-Lizorkin type space with different homogeneities $\dot{F}_p^{\alpha,q}(\mathbb{R}^m)$ is defined by

$$\dot{F}_p^{\alpha,q}(\mathbb{R}^m) = \{f \in S_0'(\mathbb{R}^m) : \|f\|_{\dot{F}_p^{\alpha,q}(\mathbb{R}^m)} < \infty\},$$

where

$$\|f\|_{\dot{F}_p^{\alpha,q}(\mathbb{R}^m)} = \left\|\left(\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)\alpha_1 + (j\wedge 2k)\alpha_2]q} \times \sum_{(\ell', \ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} \left|\psi_{j,k} * f(2^{-(j\wedge k)}\ell', 2^{-(j\wedge 2k)}\ell_m)\right|^q \chi_I(x')\chi_J(x_m)\right)^{\frac{1}{q}}\right\|_{L^p(\mathbb{R}^m)},$$

where I are dyadic cubes in $\mathbb{R}^{m-1}$ and J are dyadic intervals in R with the side lengths $l(I) = 2^{-(j\wedge k)}$ and $l(J) = 2^{-(j\wedge 2k)}$, and the left lower corners of I and the left end points of J are $2^{-(j\wedge k)}\ell'$ and $2^{-(j\wedge 2k)}\ell_m$, respectively.

This multi-parameter Triebel-Lizorkin space is well defined, since it has been proved in [139] that $\dot{F}_p^{\alpha,q}(\mathbb{R}^m)$ is independent of the choice of the functions ψ^1 and ψ^2. This space can also be characterized by its continuous form, that is,

$$\left\|\left(\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)\alpha_1 + (j\wedge 2k)\alpha_2]q} \times \sum_{(\ell', \ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} \left|\psi_{j,k} * f(2^{-(j\wedge k)}\ell', 2^{-(j\wedge 2k)}\ell_m)\right|^q \chi_I(x')\chi_J(x_m)\right)^{\frac{1}{q}}\right\|_{L^p(\mathbb{R}^m)}$$
$$\approx \left\|\left(\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)\alpha_1 + (j\wedge 2k)\alpha_2]q} \left|\psi_{j,k} * f\right|^q\right)^{\frac{1}{q}}\right\|_{L^p(\mathbb{R}^m)}; \quad (90)$$

for a rigorous proof, see [229].

In Definition (5.2.4), setting $\alpha_1 = \alpha_2 = 0, q = 2, 0 < p \leq 1$, one obtains Hardy spaces associated with different homogeneities $H_{com}^p(\mathbb{R}^m)$, which was introduced in [132] to study the boundedness of composition operators with different homogeneities.

Note that the multi-parameter structure with different homogeneities is involved in (90). If $\psi_{j,k}(x,y)$ in (90) is the form $\psi_j^1(x)\cdot\ \psi_k^2(y)$, then we obtain the Triebel-Lizorkin space of multi-parameter pure product $\dot{F}_p^{\alpha,q}(\mathbb{R}^n\times\mathbb{R}^m)$with the norm

$$\left\|\left(\sum_{j,k\in\mathbb{Z}}2^{-[(j\wedge k)\alpha_1+(j\wedge 2k)\alpha_2]q}\left|\psi_{j,k}*f\right|^q\right)^{\frac{1}{q}}\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}$$

for $f\in\dot{F}_p^{\alpha,q}(\mathbb{R}^n\times\mathbb{R}^m), 0<p,q<\infty, \alpha=(\alpha_1,\alpha_2)\in\mathbb{R}^2$. It has been introduced in [128].

Let q' denote the conjugate of q, so that $1/q+1/q'=1$ when $1\le q\le\infty$. If $0<q<1$, it is also convenient to let $q=\infty$. The first main theorem concerns the duality of the spaces $\dot{F}_p^{\alpha,q}$ when $p>1$.

Definition (5.2.5) [250] For $0<p\le 1, 1\le q\le\infty, \alpha=(\alpha_1,\alpha_2)\in\mathbb{R}^2$ and with I,J and x_I,x_J being the same as before, the space $CMO_p^{\alpha,q}(\mathbb{R}^m)$ is defined by

$$CMO_p^{\alpha,q}(\mathbb{R}^m)=\{f\in S_0'(\mathbb{R}^m):\|f\|_{CMO_p^{\alpha,q}(\mathbb{R}^m)}<\infty\},$$

where

$$\|f\|_{CMO_p^{\alpha,q}(\mathbb{R}^m)}$$
$$=\sup_{\Omega}\left(\frac{1}{|\Omega|^{\frac{q}{p}-\frac{q}{q'}}}\int_\Omega\sum_{j,k\in\mathbb{Z}}2^{-[(j\wedge k)\alpha_1+(j\wedge 2k)\alpha_2]q}\times\sum_{R\in\Pi_{j,k},R\subseteq\Omega}\left(\left|\psi_{j,k}*f(x_I,x_J)\right|^q\chi_R(x)\right)dx\right)^{1/2}. \quad (91)$$

In order to prove the duality theorems, following Frazier and Jawerth in the one-parameter case [115] (see also [247]), we should first do these in the corresponding discrete multi-parameter Triebel-Lizorkin sequence spaces. For any $R\in\Pi_{j,k}$, setting $\psi_R(x)=|R|^{1/2}\psi_{j,k}(x'-x_I,x_m-x_J)$, then by (89), it's easy to have

$$f(x)=\sum_{R\in\mathcal{D}}\langle f,\psi_R\rangle\psi_R(x). \quad (92)$$

Definition (5.2.6) [250] Suppose that $\psi^{(1)}$ and $\psi^{(2)}$ are functions satisfying conditions in (85)-(86) and (87)-(88), respectively. Define the multi-parameter ψ-transform S_ψ as the map taking $f\in S_0'(\mathbb{R}^m)$ to the sequence $S_\psi f=\{(S_\psi f)_R\}_R$, where $(S_\psi f)_R=\langle f,\psi_R\rangle$. Define the inverse multi-parameter ψ-transform T_ψ as the map taking a sequence $s=\{s_R\}_R$ to $T_\psi s=\sum_R s_R\psi_R(x)$.

By (92), for $f\in S_0, g\in S_0'$ one has

$$\langle f,g\rangle=\langle\sum_{R\in\mathcal{D}}(S_\psi f)_R\psi_R(x),g\rangle=\langle S_\psi f,S_\psi g\rangle. \quad (93)$$

For a sequence $s=s_R$, one also has the following identity:

$$\langle S_\psi f,s\rangle=\sum_{R\in\mathcal{D}}\langle f,\psi_R\rangle s_R=\langle f,\sum_{R\in\mathcal{D}}s_R\psi_R\rangle=\langle f,T_\psi s\rangle. \quad (94)$$

The discrete Triebel-Lizorkin sequence space $\dot{f}_p^{\alpha,q}$ is defined as follows.

Definition (5.2.7) [250] For $0<p<\infty, 0<q\le\infty, \alpha=(\alpha_1,\alpha_2)\in\mathbb{R}^2$, define $\dot{f}_p^{\alpha,q}$ to be the collection of all complex-valued sequences $s=\{s_R\}_R$ such that

$$\|s\|_{\dot{f}_p^{\alpha,q}}=\left\|\left(\sum_{R\in\mathcal{D}}\left(|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|s_R|\tilde{\chi}_R(x)\right)^q\right)^{1/q}\right\|_{L^p}<\infty \quad (95)$$

where $\tilde{\chi}_R(x)=|R|^{-1/2}\chi_R(x)$.

Definition (5.2.8) [250] For $0 < p \leq 1, 1 \leq q \leq \infty, \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, define $C_p^{\alpha,q}$ to be the collection of all complex-valued sequences $t = \{t_R\}_R$ such that

$$\|t\|_{C_p^{\alpha,q}} = \sup_{\Omega} \left(\frac{1}{|\Omega|^{\frac{q}{p} - \frac{q}{q'}}} \int_{\Omega} \sum_{R \subseteq \Omega, R \in \mathcal{D}} \left(|I|^{\alpha_1/(m-1)} |J|^{\alpha_2} |t_R| \tilde{\chi}_R(x) \right)^q dx \right)^{1/q} \tag{96}$$

where $\tilde{\chi}_R(x)$ is the same as the form defined in Definition (5.2.7).

Theorem (5.2.9) [250] Suppose $0 < p < \infty, 0 < q \leq \infty, \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, and $\psi^{(1)}$ and $\psi^{(2)}$ are functions satisfying conditions in (85)-(86) and (87)-(88), respectively. The operators $S_\psi : \dot{F}_p^{\alpha,q} \to \dot{f}_p^{\alpha,q}$ and $T_\psi : \dot{f}_p^{\alpha,q} \to \dot{F}_p^{\alpha,q}$ are bounded, and $T_\psi \circ S_\psi$ is the identity on $\dot{F}_p^{\alpha,q}$.

Proof. The boundedness of S_ψ is immediate since

$$\left\| S_\psi(f) \right\|_{\dot{f}_p^{\alpha,q}} = \|f\|_{\dot{F}_p^{\alpha,q}}$$

from the definition.

We now outline the proof of T_ψ's boundedness. For a sequence $s = \{s_R\}_{R \in \mathcal{D}}$, let $f(x) = T_\psi s = \sum_R s_R \psi_R(x)$. Then by almost orthogonality estimates (e.g. see Lemma 3.1 in [132]), one has

$$\left| \psi_{j',k'} * \psi_{j,k}(x_{I'} - x_I, x_{J'} - x_J) \right|$$
$$\lesssim 2^{-|j-j'|L} 2^{-|k-k'|L} \frac{2^{(j \wedge j' \wedge k \wedge k')(m-1)}}{(1 + 2^{j \wedge j' \wedge k \wedge k'} |x_{I'} - x_I|)^{(M+m-1)}} \times \frac{2^{j \wedge j' \wedge 2(k \wedge k')}}{(1 + 2^{j \wedge j' \wedge 2(k \wedge k')} |x_{J'} - x_J|)^{(M+1)}}.$$

Hence for any $v'' \in x_I, v'_m \in x_J$,

$$\left| f * \psi_{j',k'}(x_{I'}, x_{J'}) \right| \lesssim \sum_{j,k} 2^{-|j-j'|L} 2^{-|k-k'|L} C_1 \left\{ \mathcal{M}_s \left(\sum_{R \in \Pi_{j,k}} |R|^{-1/2} |s_R| \chi_I \chi_J \right)^{\delta} (v'', v'_m) \right\}^{1/\delta}$$

for a $\delta > 0$ which can be sufficiently small if one chooses M big enough by Lemma 3.2 in [132]. Summing over j', k' and (ℓ'', ℓ'_m), one has

$$\left(\sum_{j',k' \in \mathbb{Z}} 2^{-[(j' \wedge k')\alpha_1 + (j' \wedge 2k')\alpha_2]q} \sum_{(\ell'', \ell'_m)} \left| \psi_{j',k'} * f(x_{I'}, x_{J'}) \right|^q \chi_{I'}(x'') \chi_{J'}(x'_m) \right)^{\frac{1}{q}}$$
$$\leq C \left(\sum_{j',k' \in \mathbb{Z}} 2^{-[(j' \wedge k')\alpha_1 + (j' \wedge 2k')\alpha_2]q} \left[\sum_{j,k} 2^{-|j-j'|L} 2^{-|k-k'|L} \right.\right.$$
$$\left.\left. \times C_1 \left\{ \mathcal{M}_s \left(\sum_{R \in \Pi_{j,k}} |R|^{-1/2} |s_R| \chi_I \chi_J \right)^{\delta} (v'', v'_m) \right\}^{1/\delta} \right]^q \right)^{\frac{1}{q}}.$$

Then by the inequality $(\sum_l a_l)^q \leq \sum_l a_l^q$, if $0 < q \leq 1$, or Cauchy's inequality with exponents $q, q', \frac{1}{q} + \frac{1}{q'} = 1$, if $q > 1$, we obtain

$$\left(\sum_{j',k' \in \mathbb{Z}} 2^{-[(j' \wedge k')\alpha_1 + (j' \wedge 2k')\alpha_2]q} \sum_{(\ell'', \ell'_m)} \left| \psi_{j',k'} * f(x_{I'}, x_{J'}) \right|^q \chi_{I'}(x'') \chi_{J'}(x'_m) \right)^{\frac{1}{q}}$$

$$\leq C\left(\sum_{j,k\in\mathbb{Z}} 2^{-[(j'\wedge k')\alpha_1+(j'\wedge 2k')\alpha_2]q}\left\{\mathcal{M}_s\left(\sum_{R\in\Pi_{j,k}}|R|^{-1/2}|s_R|\chi_I\chi_J\right)^{\delta}(v'',v'_m)\right\}^{q/\delta}\right)^{\frac{1}{q}}.$$

Applying Fefferman-Stein's vector-valued strong maximal inequality on $L^{p/\delta}(\ell^{q/\delta})$ provided $\delta<\min\{p,q,1\}$, we complete the proof.

We will obtain a similar correspondence between $CMO_p^{\alpha,q}$ and $C_p^{\alpha,q}$. Following the proof of Lemma 3.1 in [132], one can obtain the following almost orthogonality estimates.

Lemma (5.2.10) [250] Suppose that $\psi_{j,k}$ and $\varphi_{j',k'}$ satisfy the same conditions in (85)-(88). Then for any given integers L_1, L_2 and M, there exists a constant $C=C(L,M)>0$ such that

$$\left|\psi_{j,k}*\varphi_{j',k'}(x',x_m)\right|\leq C2^{-|j-j'|L_1}2^{-|k-k'|L_2}\frac{2^{(j\wedge k)(m-1)}2^{j\wedge 2k}}{(1+2^{j\wedge k}|x'|)^{(M+m-1)}(1+2^{j\wedge 2k}|x_m|)^{(M+1)}}.$$

Proof. One can write

$$(\psi_{j,k}*\varphi_{j',k'})(x',x_m)=\int_{\mathbb{R}^{m-1}\times\mathbb{R}}(\psi_j^{(1)}*\varphi_{j'}^{(1)})(x'-y',x_m-y_m)(\psi_k^{(2)}*\varphi_{k'}^{(2)})(y',y_m)dy'dy_m\,.$$

Then by classical almost orthogonality estimates, one has

$$\left|\psi_j^{(1)}*\varphi_{j'}^{(1)}(u',u_m)\right|\leq C\frac{2^{(j\wedge j')m}2^{-|j-j'|L_1}}{\left(1+2^{(j\wedge j')}|u'|\right)^{(M+m-1)}\left(1+2^{(j\wedge j')}|u_m|\right)^{(M+1)}}\qquad(97)$$

and

$$\left|\psi_k^{(2)}*\varphi_{k'}^{(2)}(y',y_m)\right|\leq C\frac{2^{(k\wedge k')(m+1)}2^{-|k-k'|L_2}}{\left(1+2^{(k\wedge k')}|y'|\right)^{(M+m-1)}\left(1+2^{2(k\wedge k')}|y_m|\right)^{(M+1)}}\qquad(98)$$

for any positive integer L_1, L_2 and M. With the same process as in the proof of Lemma 3.1 in [132], we have

$$\left|\psi_{j,k}*\varphi_{j',k'}(x',x_m)\right|$$
$$\leq C2^{-|j-j'|L_1}2^{-|k-k'|L_2}\frac{2^{(j\wedge j'\wedge k\wedge k')(m-1)}}{(1+2^{j\wedge j'\wedge k\wedge k'}|x'|)^{(M+m-1)}}\times\frac{2^{j\wedge j'\wedge 2(k\wedge k')}}{(1+2^{j\wedge j'\wedge 2(k\wedge k')}|x_m|)^{(M+1)}},$$

which gives

$$\left|\psi_{j,k}*\varphi_{j',k'}(x',x_m)\right|$$
$$\lesssim 2^{-|j-j'|L_1}2^{-|k-k'|L_2}\frac{2^{(j\wedge k)M-(j\wedge j'\wedge k\wedge k')M}2^{(j\wedge k)(m-1)}}{(1+2^{j\wedge k}|x'|)^{(M+m-1)}}\times\frac{2^{(j\wedge 2k)M-(j\wedge j'\wedge 2(k\wedge k'))M}2^{j\wedge j'\wedge 2(k\wedge k')}}{(1+2^{j\wedge 2k}|x_m|)^{(M+1)}}.$$

After observing that

$$j\wedge k-j\wedge j'\wedge k\wedge k'\leq|j-j'|+|k-k'|$$

and

$$j\wedge 2k-j\wedge j'\wedge 2(k\wedge k')\leq|j-j'|+2|k-k'|,$$

we obtain the desired result.

The next theorem concerns the actions of the multi-parameter ψ-transform S_ψ and its inverse ψ-transform T_ψ on the space $CMO_p^{\alpha,q}$ and its discrete sequence form $C_p^{\alpha,q}$. We prove that operators $S_\psi: CMO_p^{\alpha,q}\to C_p^{\alpha,q}$ and $T_\psi: C_p^{\alpha,q}\to CMO_p^{\alpha,q}$ are bounded, and $T_\psi\circ S_\psi$ is the identity on $CMO_p^{\alpha,q}$. The proof of this theorem is rather involved, and the underlying geometry of the multi-parameter structures of the dyadic rectangles associated with the composition of two operators with different homogeneities plays an important role. These sorts of ideas have been initially used in [108] and then [132] for duality of flag Hardy spaces, and similar ideas have been used subsequently for Hardy spaces in different multi-parameter settings (see [130], [131], [133], etc.). It is more difficult and complicated to carry out our multi-parameter Triebel-Lizorkin spaces.

Theorem (5.2.11) [250] Suppose $0 < p \le 1 \le q \le \infty, \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, and $\psi^{(1)}$ and $\psi^{(2)}$ are functions satisfying conditions in (85)-(86) and (87)-(88), respectively. Then the operators $S_\psi : CMO_p^{\alpha,q} \to C_p^{\alpha,q}$ and $T_\psi : C_p^{\alpha,q} \to CMO_p^{\alpha,q}$ are bounded, and $T_\psi \circ S_\psi$ is the identity on $CMO_p^{\alpha,q}$.

Proof. We only prove T_ψ is bounded since the rest is obvious. Let $t = \{t_{R'}\}_{R'} \in C_p^{\alpha,q}$ and $f = \sum t_{R'}\psi_{R'}$. When $1 \le q < \infty$, we are going to prove

$$\sup_\Omega \frac{1}{|\Omega|^{\frac{1}{p}-\frac{1}{q'}}}\left(\sum_{R=I\times J\subseteq\Omega, R\in\mathcal{D}} \left(|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}\left|\varphi_{j,k} * f(x_I,x_J)\right|\right)^q |R|\right)^{1/q}$$

$$\lesssim \sup_\Omega \frac{1}{|\Omega|^{\frac{1}{p}-\frac{1}{q'}}}\left(\sum_{R'=I'\times J'\subseteq\Omega, R'\in\mathcal{D}} \left(|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2}\right)^q |R'|\right)^{1/q}. \quad (99)$$

For any $R' \in \Pi_{j',k'}$, by Lemma (5.2.10), one has

$$\left|\varphi_{j,k} * \psi_{R'}(x_I,x_J)\right| \le C|R'|^{1/2}2^{-|j-j'|(L_1+L_2)}2^{-|k-k'|(L_1+2L_2)} \times \frac{2^{(j'\wedge k')(m-1)}}{\left(1+2^{j'\wedge k'}|x_I - x_{I'}|\right)^{(M+m-1)}} \times \frac{2^{j'\wedge 2k'}}{\left(1+2^{j'\wedge 2k'}\left|x_J - x_{J'}\right|\right)^{(M+1)}}.$$

Since $|j'\wedge k' - j\wedge k| \le |j-j'| + |k-k'|$, $|j'\wedge 2k' - j\wedge 2k| \le |j-j'| + |2k-k'|$, one has

$$\left|\varphi_{j,k} * \psi_{R'}(x_I,x_J)\right| \le C|R'|^{-1/2}\frac{2^{-L_1|j'\wedge k' - j\wedge k|}}{(1+2^{j'\wedge k'}|x_I - x_{I'}|)^{(M+m-1)}} \times \frac{2^{-L_2|j'\wedge 2k' - j\wedge 2k|}}{(1+2^{j'\wedge 2k'}\left|x_J - x_{J'}\right|)^{(M+1)}}$$

for any sufficiently larger L_1, L_1. Using conditions (85), (87), it is easy to see that

$$\left(\varphi_{j,k} * \psi_{j',k'}\left(\cdot - 2^{-(j'\wedge k')}\ell', \cdot - 2^{-(j'\wedge 2k')}\ell_m\right)\right)^\wedge(\xi',\xi_m)$$
$$= \widehat{\varphi_{j,k}}(\xi',\xi_m)\widehat{\psi_{j',k'}}(\xi',\xi_m)\exp\left(-2\pi i\left[2^{-(j'\wedge k')}\ell'\xi' + 2^{-(j'\wedge 2k')}\ell_m\xi_m\right]\right) = 0 \text{ if } |j'-j| > 1 \text{ or } |k'-k| > 1,$$

from which follows

$$\left|\varphi_{j,k} * f(x_I,x_J)\right|^q \lesssim \sum_{R'}|t_{R'}|^q\left|\varphi_{j,k} * \psi_{R'}(x_I,x_J)\right|^q.$$

Hence

$$\sum_{R=I\times J\subseteq\Omega, R\in\mathcal{D}} \left(|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}\left|\varphi_{j,k} * f\left(x_I,x_J\right)\right|\right)^q |R|$$
$$\lesssim \sum_{R=I\times J\subseteq\Omega, R\in\mathcal{D}}\ \sum_{R'=I'\times J'\in\mathcal{D}} |I'|^{q\alpha_1/(m-1)}|J'|^{q\alpha_2}|t_{R'}|^q|R'|^{-q/2}|R|$$
$$\times\left(\frac{|I|}{|I'|}\wedge\frac{|I'|}{|I|}\right)^{qL_1/(m-1)}\frac{1}{\left(1+\frac{\operatorname{dist}(I,I')}{\ell(I')}\right)^M}\left(\frac{|J|}{|J'|}\wedge\frac{|J'|}{|J|}\right)^{qL_2}\frac{1}{\left(1+\frac{\operatorname{dist}(J,J')}{\ell(J')}\right)^M}$$
$$= \sum_{R=I\times J\subseteq\Omega, R\in\mathcal{D}}\ \sum_{R'=I'\times J'\in\mathcal{D}} \left(\frac{|I|}{|I'|}\right)^{q\alpha_1/(m-1)}\left(\frac{|J|}{|J'|}\right)^{qL_2}\frac{|R|}{|R'|}$$
$$\times\left(\frac{|I|}{|I'|}\wedge\frac{|I'|}{|I|}\right)^{qL_1/(m-1)}\left(\frac{|J|}{|J'|}\wedge\frac{|J'|}{|J|}\right)^{qL_2}\frac{1}{\left(1+\frac{\operatorname{dist}(I,I')}{\ell(I')}\right)^M}\frac{1}{\left(1+\frac{\operatorname{dist}(J,J')}{\ell(J')}\right)^M}$$
$$\left(|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2}\right)^q|R'|$$

$$\lesssim \sum_{R=I\times J\subseteq\Omega, R\in\mathcal{D}} \sum_{R'=I'\times J'\in\mathcal{D}} \left(\frac{|I|}{|I'|}\vee\frac{|I'|}{|I|}\right)^{q|\alpha_1|/(m-1)+1} \left(\frac{|J|}{|J'|}\vee\frac{|J'|}{|J|}\right)^{q|\alpha_2|+1}$$
$$\times \left(\frac{|I|}{|I'|}\wedge\frac{|I'|}{|I|}\right)^{qL_1/(m-1)} \left(\frac{|J|}{|J'|}\wedge\frac{|J'|}{|J|}\right)^{qL_2} \frac{1}{\left(1+\frac{\operatorname{dist}(I,I')}{\ell(I')}\right)^M} \frac{1}{\left(1+\frac{\operatorname{dist}(J,J')}{\ell(J')}\right)^M}$$
$$\times \left(|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2}\right)^q|R'|$$
$$= \sum_{R=I\times J\subseteq\Omega, R\in\mathcal{D}} \sum_{R'=I'\times J'\in\mathcal{D}} \left(|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2}\right)^q|R'|$$
$$\times \left(\frac{|I|}{|I'|}\wedge\frac{|I'|}{|I|}\right)^{L} \left(\frac{|J|}{|J'|}\wedge\frac{|J'|}{|J|}\right)^{L} \frac{1}{\left(1+\frac{\operatorname{dist}(I,I')}{\ell(I')}\right)^M} \frac{1}{\left(1+\frac{\operatorname{dist}(J,J')}{\ell(J')}\right)^M}$$

by setting $q\,L_1/(m-1) - q\,|\alpha_1|/(m-1) - 1 = L = qL_2 - q|\alpha_2| - 1$. Thus

$$\sup_\Omega \frac{1}{|\Omega|^{\frac{1}{p}-\frac{1}{q'}}} \left(\sum_{R=I\times J\subseteq\Omega, R\in\mathcal{D}} \left(|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}\left|\psi_{j,k}*f(x_I,x_J)\right|\right)^q|R|\right)^{1/q}$$
$$\lesssim \sup_\Omega \frac{1}{|\Omega|^{\frac{1}{p}-\frac{1}{q'}}} \left(\sum_{R=I\times J\subseteq\Omega, R\in\mathcal{D}} \sum_{R'=I'\times J'\in\mathcal{D}} r(R,R')p(R,R')\times\left(|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2}\right)^q|R'|\right)^{1/q}, \quad (100)$$

where

$$r(R,R') = \left(\frac{|I|}{|I'|}\wedge\frac{|I'|}{|I|}\right)^{L} \left(\frac{|J|}{|J'|}\wedge\frac{|J'|}{|J|}\right)^{L}$$

and

$$p(R,R') = \frac{1}{\left(1+\frac{\operatorname{dist}(I,I')}{\ell(I')}\right)^M} \frac{1}{\left(1+\frac{\operatorname{dist}(J,J')}{\ell(J')}\right)^M}.$$

In order to prove inequality (99), using (100), we only need to prove

$$\sup_\Omega \frac{1}{|\Omega|^{\frac{1}{p}-\frac{1}{q'}}} \left(\sum_{R=I\times J\subseteq\Omega, R\in\mathcal{D}} \sum_{R'=I'\times J'\in\mathcal{D}} r(R,R')p(R,R')\times\left(|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2}\right)^q|R'|\right)^{1/q}$$
$$\lesssim \sup_\Omega \frac{1}{|\Omega|^{\frac{1}{p}-\frac{1}{q'}}} \left(\sum_{R'=I'\times J'\subseteq\Omega, R'\in\mathcal{D}} \left(|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2}\right)^q|R'|\right)^{1/q}. \quad (101)$$

To do this, define

$$\Omega^{0,0} = \bigcup_{R=I\times J\subseteq\Omega} 3(I\times J).$$

For any $R\subseteq\Omega$, let $A_{i,l}(R)$ be the collection of dyadic rectangles R' so that

$$A_{0,0}(R) = \{R'=I'\times J'\subseteq\Omega : \operatorname{dist}(I,I')\le\ell(I)\vee\ell(I'), \operatorname{dist}(J,J')\le\ell(J)\vee\ell(J')\},$$

and for $i\ge 1$,

$$A_{i,0}(R) = \{R'=I'\times J'\subseteq\Omega : (2^{i-1}\ell(I))\vee\ell(I') < \operatorname{dist}(I,I') \le (2^i\ell(I))\vee\ell(I'), \operatorname{dist}(J,J') \le \ell(J)\vee\ell(J')\},$$

and for $l\ge 1$,

$$A_{0,l}(R) = \{R'=I'\times J'\subseteq\Omega : \operatorname{dist}(I,I')\le\ell(I)\vee\ell(I'), (2^{l-1}\ell(J'))\vee\ell(J) < \operatorname{dist}(J,J') \le (2^l\ell(J'))\vee\ell(J)\},$$

and for $i,l\ge 1$,

$$A_{i,l}(R) = \{R'=I'\times J'\subseteq\Omega : (2^{i-1}\ell(I'))\vee\ell(I) < \operatorname{dist}(I,I') \le (2^i\ell(I'))\vee\ell(I), (2^{l-1}\ell(J'))\vee\ell(J) < \operatorname{dist}(J,J') \le (2^l\ell(J'))\vee\ell(J)\},$$

and $i,l\geq 0$,

$$A_{i,j}=\{R'=I'\times J'\in\mathcal{D}:3(2^iI'\times 2^lJ')\cap\Omega^{0,0}\neq\emptyset\}.$$

It is easy to see that for any $R\subseteq\Omega,\bigcup_{i,l\geq 0}A_{i,l}(R)=\mathcal{D},A_{i,l}(R)\cap A_{i',l'}(R)=\emptyset$ if $(i,l)\neq(i',l')$ and $A_{i,l}(R)\subseteq A_{i,l}$. Note that for $R'\in A_{i,j}(R),\ i,l\geq 0$,

$$1+\frac{\operatorname{dist}(I,I')}{\ell(I')}\gtrsim 2^i,\qquad 1+\frac{\operatorname{dist}(J,J')}{\ell(J')}\gtrsim 2^l,$$

from which follows

$$p(R,R)\gtrsim 2^{-(i+l)M}.$$

Hence

$$\frac{1}{|\Omega|^{\frac{1}{p}-\frac{1}{q'}}}\left(\sum_{R=I\times J\subseteq\Omega,R\in\mathcal{D}}\sum_{R'=I'\times J'\in\mathcal{D}}r(R,R')p(R,R')\times\left(|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2}\right)^q|R'|\right)^{1/q}$$

$$=\frac{1}{|\Omega|^{\frac{1}{p}-\frac{1}{q'}}}\left(\sum_{R=I\times J\subseteq\Omega,R\in\mathcal{D}}\sum_{i,l\geq 0}\sum_{R'\in A_{i,l}(R)}r(R,R')p(R,R')\times\left(|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2}\right)^q|R'|\right)^{1/q}$$

$$=\frac{1}{|\Omega|^{\frac{1}{p}-\frac{1}{q'}}}\left(\sum_{R=I\times J\subseteq\Omega,R\in\mathcal{D}}\sum_{i,l\geq 0}\sum_{R'\in A_{i,l}}\chi_{R'\in A_{i,l}(R)}r(R,R')p(R,R')\times\left(|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2}\right)^q|R'|\right)^{1/q}$$

$$\lesssim\frac{1}{|\Omega|^{\frac{1}{p}-\frac{1}{q'}}}\Bigg(\left(\sum_{R'\in A_{0,0}}+\sum_{i\geq 1}\sum_{R'\in A_{i,0}}+\sum_{l\geq 1}\sum_{R'\in A_{0,l}}+\sum_{i,l\geq 1}\sum_{R'\in A_{i,l}}\right)2^{-(i+l)M}$$

$$\times\sum_{R=\subseteq\Omega,R\in\mathcal{D}}\chi_{R'\in A_{i,l}(R)}r(R,R')\left(|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2}\right)^q|R'|\Bigg)^{1/q}$$

$$=(I_1+I_2+I_3+I_4)^{1/q}.$$

We only estimate I_4 since estimates of estimate I_1,I_2 and I_3 can be concluded by applying the same techniques.

For each integer $h\geq 1$, let $\mathcal{F}_h^{i,l}=\{R'=I'\times J'\in A_{i,l}:\left|3(2^iI'\times 2^lJ')\cap\Omega^{0,0}\right|\geq\frac{1}{2^h}\left|2^iI'\times 2^lJ'\right|\}$. Let

$$\mathcal{D}_h^{i,l}=\mathcal{F}_h^{i,l}\backslash\mathcal{F}_{h-1}^{i,l}\qquad\text{and}\qquad\Omega_h^{i,l}=\bigcup_{R'\in\mathcal{D}_h^{i,l}}R'.$$

Then

$$I_4=\frac{1}{|\Omega|^{\frac{q}{p}-\frac{q}{q'}}}\sum_{i,l\geq 1}\sum_{h\geq 1}\sum_{R'\in\mathcal{D}_h^{i,l}}2^{-(i+l)M}\sum_{R\subseteq\Omega,R\in\mathcal{D}}\chi_{R'\in A_{i,l}(R)}r(R,R')$$
$$\times\left(|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2}\right)^q|R'|.$$

To estimate the right-hand side of the above inequality, we only need to estimate

$$\sum_{R\subseteq\Omega,R\in\mathcal{D}}\chi_{R'\in A_{i,l}(R)}r(R,R').$$

Firstly, because $R'\in A_{i,l}(R)$, one has $3R\cap 3(2^iI'\times 2^jJ')\neq\emptyset$. For $R\subseteq\Omega$, there are four cases:

Case 1: $\left|2^iI'\right|\geq|I|,\left|2^lJ'\right|\geq|J|$; Case 2: $\left|2^iI'\right|\geq|I|,\left|2^lJ'\right|\leq|J|$;

Case 3: $\left|2^iI'\right|\leq|I|,\left|2^lJ'\right|\geq|J|$; Case 4: $\left|2^iI'\right|\leq|I|,\left|2^lJ'\right|\leq|J|$.

From the definition of $A_{i,l}(R)$, one can see that if $R'\in$ Case 2, then

$$\ell(J)=(2^{l-1}\ell(J'))\vee\ell(J)<\mathrm{dist}(J,J')\le(2^{l}\ell(J'))\vee\ell(J)=\ell(J),$$

which implies Case 2 is an empty set. For the same reason, Case 3 is also an empty set. We split I_4 into two terms:

$$I_4=\frac{1}{|\Omega|^{\frac{q}{p}-\frac{q}{q'}}}\sum_{i,l\ge1}\sum_{h\ge1}\sum_{R'\in\mathcal{D}_h^{i,l}}2^{-(i+l)M}\Big(\sum_{R\in\mathrm{Case1}}+\sum_{R\in\mathrm{Case4}}\Big)\chi_{R'\in A_{i,l}(R)}r(R,R')$$
$$\times\left(|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2}\right)^q|R'|=I_4^1+I_4^4.$$

In Case 1, since $R'\in A_{i,l}(R)$ and $R'\in\mathcal{D}_h^{i,l}$, one has

$$|R|\le\left|3R\cap3\left(2^iI'\times2^lJ'\right)\right|\le\left|3\left(2^iI'\times2^lJ'\right)\cap\Omega^{0,0}\right|\le\frac{1}{2^{h-1}}\left|3\left(2^iI'\times2^lJ'\right)\right|.\quad(102)$$

In $r(R,R')$, we should compare the side-length of R with the side-length of R'. We divide $R\subseteq\Omega$ into four categories:

Category 1.1 $|I|\le|I'|,|J|\le|J'|$; Category 1.2 $|I|\le|I'|,|J|>|J'|$;

Category 1.3 $|I|>|I'|,|J|\le|J'|$; Category 1.4 $|I|>|I'|,|J|>|J'|$.

For Category 1.1, (102) gives $2^{i(m-1)+l}|R'|=2^{h-1-2m+\eta}|R|$ for some integer $\eta>0$ since I,I' are all dyadic, where 2^{-2m} is used to offset 3^m. For each fixed $\eta>0$, the number of such R's must be less than $7^m2^{h-1-2m+\eta}$ since $R\subseteq7(2^iI'\times2^lJ')$. Therefore

$$\sum_{R\in\mathrm{Case1},|I|\le|I'|,|J|\le|J'|}\chi_{R'\in A_{i,l}(R)}r(R,R')\le\ 7^m\sum_{\eta>0}\frac{\left(2^{i(m-1)+l}\right)^L}{\left(2^{h-1-2m+\eta}\right)^{L-1}}\lesssim2^{iLm-h(L-1)+lL}.$$

For Category 1.2, $|I|\le|I'|,|J|>|J'|$. From (102), one has

$$|I||J'|\le|R|\le\left|3R\cap3(2^iI'\times2^lJ')\right|\le\left|3(2^iI'\times2^lJ')\cap\Omega^{0,0}\right|\le\frac{1}{2^{h-1}}\left|3(2^iI'\times2^lJ')\right|.$$

It follows that

$$|I|\le\frac{3^m2^{i(m-1)+l}}{2^{h-1}}|I'|;$$

hence $2^{i(m-1)+l}|I'|=2^{h-1-2m+\theta}|I|$ for some integer $\theta>0$. For each fixed $\theta>0$, the number of such I's must be less than $7^{m-1}2^{h-1-2m-l+\theta}$ since $I\subseteq7(2^iI')$. Moreover from $|J'|<|J|\le\left|2^lJ'\right|$ we have $\left|2^\beta J'\right|=|J|$ for some positive integer β with $1\le\beta\le l$. For each fixed $\beta>0$, the number of such J's must be less than $72^{(l-\beta)}$ since $J\subseteq7(2^lJ')$. Hence

$$\sum_{R\in\mathrm{Case1},|I|\le|I'|,|J|>|J'|}\chi_{R'\in A_{i,l}(R)}r(R,R')\le\ 7^m\sum_{\theta>0}\sum_{\beta=1}^{l}\frac{\left(2^{i(m-1)}\right)^L}{\left(2^{h-1-2m-l+\theta}\right)^{L-1}}\frac{2^{l-\beta}}{2^{L\beta}}$$
$$\lesssim l2^{iLm-h(L-1)+3lL}.$$

With a similar argument to Category 1.2, for Category 1.3 one can obtain the following estimates:

$$\sum_{R\in\mathrm{Case1},|I|>|I'|,|J|\le|J'|}\chi_{R'\in A_{i,l}(R)}r(R,R')\lesssim i2^{2iLm-h(L-1)+2lL}.$$

For Category 1.4, from (102) one has

$$|R'|\le\frac{1}{2^{h-1}}\left|3(2^iI'\times2^lJ')\right|,$$

from which follows that $2^{h-1}\le3^m2^{i(m-1)+l}$. On the other hand, with $|R'|\le|R|\le\left|2^iI'\times2^lJ'\right|$, one has $2^{i(m-1)+l}|R'|=2^\lambda|R|$ for some integer $0\le\lambda\le i(m-1)+l$. For each fixed $\lambda\ge0$, the number of such R's must be less than 7^m2^λ since $R\subseteq7(2^iI'\times2^lJ')$. So

$$\sum_{R\in \text{Case1},|I|>|I'|,|J|>|J'|} \chi_{R'\in A_{i,l}(R)} r(R,R') = \sum_{\lambda=0}^{i(m-1)+l} 7^m 2^\lambda \left(\frac{2^\lambda}{2^{i(m-1)+l}}\right)^L \lesssim 2^{im+l}.$$

Therefore

$$\begin{aligned} I_4^1 &= \frac{1}{|\Omega|^{\frac{q}{p}-\frac{q}{q'}}} \sum_{i,l\geq 1}\sum_{h\geq 1}\sum_{R'\in \mathcal{D}_h^{i,l}} 2^{-(i+l)M} \sum_{R\in \text{Case4}} \chi_{R'\in A_{i,l}(R)} r(R,R') \times \left(|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2}\right)^q |R'| \\ &\lesssim \frac{1}{|\Omega|^{\frac{q}{p}-\frac{q}{q'}}} \sum_{i,l\geq 1}\left(\sum_{h\geq 1}\left(2^{iLm-h(L-1)+lL} + l2^{iLm-h(L-1)+3lL} + i2^{2iLm-h(L-1)+2lL}\right)\right. \\ &\qquad \left. + \sum_{h:2^{h-1}\leq 3^m 2^{i(m-1)+l}} 2^{iM+l}\right) 2^{-(i+l)M} |\Omega_h^{i,l}|^{\frac{q}{p}-\frac{q}{q'}} \\ &\qquad \times \frac{1}{|\Omega_h^{i,l}|^{\frac{q}{p}-\frac{q}{q'}}} \sum_{R'\subseteq \Omega_h^{i,l}} \left(|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2}\right)^q |R'| \\ &\lesssim \sum_{i,l\geq 1}\left(\sum_{h\geq 1} 2^{4iLm-h(L-1)+4lL} + \sum_{h:2^{h-1}\leq 3^m 2^{i(m-1)+l}} 2^{iM+l}\right) \\ &\qquad \times 2^{-(i+l)M}(2^{2h})^{\frac{q}{p}-\frac{q}{q'}} \sup_{\bar{\Omega}} \frac{1}{|\bar{\Omega}|^{\frac{q}{p}-\frac{q}{q'}}} \sum_{R'\subseteq\bar{\Omega}} \left(|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2}\right)^q |R'| \\ &\lesssim \sup_{\bar{\Omega}} \frac{1}{|\bar{\Omega}|^{\frac{q}{p}-\frac{q}{q'}}} \sum_{R'\subseteq\bar{\Omega}} \left(|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2}\right)^q |R'| \end{aligned}$$

since $\left|\Omega_h^{i,l}\right| \lesssim 2^{2h}|\Omega^{0,0}| \leq 2^{2h}|\Omega|, 0<p\leq 1\leq q\leq\infty$ and choosing $M>4mL$ with L large enough.

In Case 4, firstly, since $R'\in A_{i,j}(R)$ and $R'\in \mathcal{D}_h^{i,l}$, one has

$$\left|2^i I' \times 2^l J'\right| \leq \left|3R\cap 3(2^iI'\times 2^lJ')\right| \leq \left|3(2^iI'\times 2^lJ')\cap\Omega^{0,0}\right| \leq \frac{1}{2^{h-1}}\left|3(2^iI'\times 2^lJ')\right|,$$

which follows $h\leq 2^{2m+1}$. Moreover, from $\left|2^iI'\right|\leq|I|, \left|2^lJ'\right|\leq|J|$, one has $2^{i(m-1)+l+\sigma}|R'|=|R|$ for some integer $\sigma\geq 0$. For each fixed $\sigma\geq 0$ and any R', the number of such R's must be less than 7^m. In this situation, we have the following estimates:

$$\sum_{R\in\text{Case4}} \chi_{R'\in A_{i,l}(R)} r(R,R') = \sum_{\sigma\geq 0} 7^m \left(\frac{1}{2^{i(m-1)+l+\sigma}}\right)^L \lesssim 2^{im+l}.$$

Then with the same process, one has

$$I_4^4 \lesssim \sup_{\bar{\Omega}} \frac{1}{|\bar{\Omega}|^{\frac{q}{p}-\frac{q}{q'}}} \sum_{R'\subseteq\bar{\Omega}} \left(|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2}\right)^q |R'|.$$

We then complete the proof of (100).

When $q=\infty$, for $t=\{t_{R'}\}_{R'}\in C_p^{\alpha,q}$ and $f=\sum t_{R'}\psi_{R'}$, we are going to prove

$$\begin{aligned} &\sup_{\Omega} \frac{1}{|\Omega|^{\frac{1}{p}-1}} \sup_{R=I\times J\subseteq\Omega, R\in\mathcal{D}} |I|^{\alpha_1/(m-1)}|J|^{\alpha_2}\left|\varphi_{j,k}*f(x_I,x_J)\right| \\ &\qquad \lesssim \sup_{\Omega} \frac{1}{|\Omega|^{\frac{1}{p}-1}} \sup_{R'=I'\times J'\subseteq\Omega, R'\in\mathcal{D}} |I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2} \end{aligned}$$

Its proof is similar to the case of $0<q<\infty$; hence we only give an outline.

We use the same symbols as above. With the same process, one has

$$\frac{1}{|\Omega|^{\frac{1}{p}-1}}\sup_{R=I\times J\subseteq\Omega,R\in\mathcal{D}}|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}\left|\varphi_{j,k}*f(x_I,x_J)\right|$$

$$\lesssim\frac{1}{|\Omega|^{\frac{1}{p}-1}}\sup_{R=I\times J\subseteq\Omega,R\in\mathcal{D}}\sum_{R'=I'\times J'\in\mathcal{D}}|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2}$$

$$\times\left(\frac{|I|}{|I'|}\wedge\frac{|I'|}{|I|}\right)^{L_1/(m-1)}\frac{1}{\left(1+\frac{\operatorname{dist}(I,I')}{\ell(I')}\right)^M}\left(\frac{|J|}{|J'|}\wedge\frac{|J'|}{|J|}\right)^{L_2}\frac{1}{\left(1+\frac{\operatorname{dist}(J,J')}{\ell(J')}\right)^M}$$

$$=\frac{1}{|\Omega|^{\frac{1}{p}-1}}\sup_{R=I\times J\subseteq\Omega,R\in\mathcal{D}}\sum_{R'=I'\times J'\in\mathcal{D}}\left(\frac{|I|}{|I'|}\right)^{\alpha_1/(m-1)}\left(\frac{|J|}{|J'|}\right)^{\alpha_2}$$

$$\times\left(\frac{|I|}{|I'|}\wedge\frac{|I'|}{|I|}\right)^{L_1/(m-1)}\frac{1}{\left(1+\frac{\operatorname{dist}(I,I')}{\ell(I')}\right)^M}\left(\frac{|J|}{|J'|}\wedge\frac{|J'|}{|J|}\right)^{L_2}\frac{1}{\left(1+\frac{\operatorname{dist}(J,J')}{\ell(J')}\right)^M}$$

$$|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2}$$

$$\lesssim\frac{1}{|\Omega|^{\frac{1}{p}-1}}\sup_{R=I\times J\subseteq\Omega,R\in\mathcal{D}}\sum_{R'=I'\times J'\in\mathcal{D}}|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2}$$

$$\times\left(\frac{|I|}{|I'|}\wedge\frac{|I'|}{|I|}\right)^{L}\left(\frac{|J|}{|J'|}\wedge\frac{|J'|}{|J|}\right)^{L}\frac{1}{\left(1+\frac{\operatorname{dist}(I,I')}{\ell(I')}\right)^M}\frac{1}{\left(1+\frac{\operatorname{dist}(J,J')}{\ell(J')}\right)^M}$$

$$=\frac{1}{|\Omega|^{\frac{1}{p}-1}}\sup_{R=I\times J\subseteq\Omega,R\in\mathcal{D}}\sum_{R'=I'\times J'\in\mathcal{D}}r(R,R')p(R,R')|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2}$$

$$=\frac{1}{|\Omega|^{\frac{1}{p}-1}}\sup_{R=I\times J\subseteq\Omega,R\in\mathcal{D}}\left(\sum_{R'\in A_{0,0}(R)}+\sum_{i\geq1}\sum_{R'\in A_{i,0}(R)}+\sum_{l\geq1}\sum_{R'\in A_{0,l}(R)}+\sum_{i,l\geq1}\sum_{R'\in A_{i,l}(R)}\right)$$

$$\cdot\,2^{-(i+l)M}r(R,R')|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2}=B_1+B_2+B_3+B_4.$$

We only estimate B_4 since estimates of B_1, B_2 and B_3 can be concluded by applying the same techniques.

For each integer $h\geq 1$, let $\mathcal{F}_h^{i,l}=\{R'=I'\times J'\in A_{i,l}(R):\ \left|3(2^iI'\times 2^lJ')\cap R\right|\geq\frac{1}{2^h}\left|2^iI'\times 2^lJ'\right|\}$. Let

$$\mathcal{D}_h^{i,l}=\mathcal{F}_h^{i,l}\backslash\mathcal{F}_{h-1}^{i,l}\qquad\text{and}\qquad\Omega_h^{i,l}=\bigcup_{R'\in\mathcal{D}_h^{i,l}}R'.$$

Then

$$B_4=\frac{1}{|\Omega|^{\frac{1}{p}-1}}\sup_{R=I\times J\subseteq\Omega,R\in\mathcal{D}}\sum_{i,l\geq1}\sum_{h\geq1}\sum_{R'\in\mathcal{D}_h^{i,l}}2^{-(i+l)M}r(R,R')\times|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2}.$$

To estimate the right-hand side of the above inequality, we only need to estimate

$$\sum_{R'\in\mathcal{D}_h^{i,l}}r(R,R').$$

Firstly, because $R'\in A_{i,l}(R)$, one has $3R\cap 3(2^iI'\times 2^lJ')\neq\emptyset$. For $R'\in\mathcal{D}_h^{i,l}$, there are also four cases:

Case 1: $\left|2^iI'\right|\geq|I|, \left|2^lJ'\right|\geq|J|$; Case2: $\left|2^iI'\right|\geq|I|, \left|2^lJ'\right|\leq|J|$;

Case 3: $|2^i I'| \le |I|, |2^l J'| \ge |J|$; Case4: $|2^i I'| \le |I|, |2^l J'| \le |J|$;
It is easy to see that Case 2, Case 3 are none. Then $B_4 = B_4^1 + B_4^4$.

For $R' \in$ Case 1, one has

$$|R| \le |3R \cap 3(2^i I' \times 2^l J')| \le |3(2^i I' \times 2^l J') \cap \Omega^{0,0}| \ge \frac{1}{2^{h-1}} |3(2^i I' \times 2^l J')|.$$

We divide $R' \in$ Case 1 into four categories:

Category 1.1 $|I| \le |I'|, |J| \le |J'|$; Category 1.2 $|I| \le |I'|, |J| > |J'|$;

Category 1.3 $|I| > |I'|, |J| \le |J'|$; Category 1.4 $|I| > |I'|, |J| > |J'|$;

In Category 1.1, we have $2^{i(m-1)+l}|R'| = 2^{h-1-2m+\eta}|R|$ for some integer $\eta > 0$. Moreover, for any fixed $\eta > 0$ and R, the number of such R''s is less than $7^m 2^{i(m-1)+l}$ since $3R \cap 3(2^i I' \times 2^l J') \neq \emptyset$ and $|2^i I'| \ge |I|, |2^l J'| \ge |J|$. Hence

$$\sum_{R' \in \text{Category 1.1}} r(R,R') = \sum_{\eta \ge 0} 7^m 2^{i(m-1)+l} \left(\frac{2^{i(m-1)+l}}{2^{h-1-2m+\eta}} \right)^L \lesssim 2^{2imL+2lL-hL}.$$

In Category 1.2, one has $2^{i(m-1)+l}|I'| = 2^{h-1-2m+\theta}|I|$ for some integer $\theta > 0$. For each fixed $\theta > 0$ and R, the number of such I''s must be less than $7^{m-1}2^{i(m-1)}$. Moreover, from $|J'| < |J| \le |2^l J'|$, we have $|2^\beta J'| = |J|$ for some positive integer β with $1 \le \beta \le l$. For each fixed $\beta > 0$ and J, the number of such J''s must be less than $72^{(l-\beta)}$. Hence

$$\sum_{R \in \text{Category 1.2}} r(R,R') \le 7^m \sum_{\eta > 0} \sum_{\beta=1}^{l} \frac{2^{i(m-1)} \left(2^{i(m-1)+l}\right)^L}{(2^{h-1-2m+\theta})^L} \frac{2^{l-\beta}}{2^{L\beta}} \lesssim l 2^{2iLm-hL+2lL}.$$

With a similar argument, one has

$$\sum_{R \in \text{Category 1.3}} r(R,R') \lesssim i 2^{2iLm-hL+2lL}.$$

In Category 1.4, one has $2^{h-1} \le 3^m 2^{i(m-1)+l}$, and with $|R'| \le |R| \le |2^i I' \times 2^l J'|$, one has $2^{i(m-1)+l}|R'| = 2^\lambda |R|$ for some integer $0 \le \lambda \le i(m-1)+l$. For each fixed $\lambda \ge 0$, the number of such R''s must be less than $7^m 2^{i(m-1)+l}$. So

$$\sum_{R \in \text{Category 1.4}} r(R,R') = \sum_{\lambda=0}^{i(m-1)+l} 7^m 2^{i(m-1)+l} \left(\frac{2^\lambda}{2^{i(m-1)+l}} \right)^L \lesssim 2^{im+l}.$$

Therefore

$$\begin{aligned}
B_4^1 &= \frac{1}{|\Omega|^{\frac{1}{p}-1}} \sup_{R=I\times J \subseteq \Omega, R \in \mathcal{D}} \sum_{i,l \ge 1} \sum_{h \ge 1} \sum_{R' \in \text{Case1}} 2^{-(i+l)M} r(R,R') \, |I'|^{\alpha_1/(m-1)} \times |J'|^{\alpha_2} |t_{R'}| |R'|^{-1/2} \\
&\le \frac{1}{|\Omega|^{\frac{1}{p}-1}} \sup_{R=I\times J \subseteq \Omega, R \in \mathcal{D}} \sum_{i,l \ge 1} \sum_{h \ge 1} \sum_{R' \in \text{Case1}} 2^{-(i+l)M} r(R,R') |\Omega_h^{i,l}| \\
&\qquad \times \frac{1}{|\Omega_h^{i,l}|^{\frac{1}{p}-1}} \sup_{R \in \Omega_h^{i,l}} |I'|^{\alpha_1/(m-1)} |J'|^{\alpha_2} |t_{R'}| |R'|^{-1/2} \\
&\lesssim \sup_{\Omega} \frac{1}{|\Omega|^{\frac{1}{p}-1}} \sup_{R'=I'\times J' \subseteq \Omega, R' \in \mathcal{D}} |I'|^{\alpha_1/(m-1)} |J'|^{\alpha_2} |t_{R'}| |R'|^{-1/2}.
\end{aligned}$$

With a similar argument for the rest, we can obtain the desired result. We then have completed the proof.

Theorem (5.2.12) [250] Suppose that $\psi_{j,k}$ and $\varphi_{j',k'}$ satisfy the same conditions in (85)-(88). Then if $0 < p \leq 1, 1 \leq q \leq \infty, \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, one has

$$\sup_{\Omega}\left(\frac{1}{|\Omega|^{\frac{q}{p}-\frac{q}{q'}}}\int_{\Omega}\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)\alpha_1+(j\wedge 2k)\alpha_2]q} \times \sum_{R\in\Pi_{j,k},R\subseteq\Omega}\left(\left|\psi_{j,k} * f\left(x_I, x_J\right)\right|^q \chi_R(x)\right) dx\right)^{1/q}$$

$$\approx \sup_{\Omega}\left(\frac{1}{|\Omega|^{\frac{q}{p}-\frac{q}{q'}}}\int_{\Omega}\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)\alpha_1+(j\wedge 2k)\alpha_2]q} \times \sum_{R\in\Pi_{j,k},R\subseteq\Omega}\left(\left|\varphi_{j,k} * f(x_I, x_J)\right|^q \chi_R(x)\right) dx\right)^{1/2}$$

for $f \in S_0'$.

The proof of this theorem can follow from the ψ-transforms that correspond between the multi-parameter Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}$ and the discrete multi-parameter Triebel-Lizorkin sequence spaces $\dot{f}_p^{\alpha,q}$ indexed by the multi-parameter dyadic rectangles in $\mathbb{R}^m$ associated with the underlying structures of the composition of two singular integrals. Since the definition of $\dot{F}_p^{\alpha,q}(\mathbb{R}^m)$ is independent of the choice of the functions ψ^1 and ψ^2, this theorem is immediate once we prove the following duality theorem (Theorem (5.2.20)). Nevertheless, we offer another proof following the proof of Theorem (5.2.11). The dual spaces for $\dot{F}_p^{\alpha,q}$ when $0 < p \leq 1$ are considerably different from those for $1 < p < \infty$ and more difficult to get, in the multi-parameter settings. Therefore, the following duality result is the third main theorem.

Proof. Suppose that $\varphi^{(1)}$ and $\varphi^{(2)}$ are functions satisfying conditions in (85)-(86) and (87)-(88), respectively. For $f \in CMO_p^{\alpha,q}$, setting $\varphi_R(x) = |R|^{1/2}\varphi_{j,k}(x' - x_I, x_m - x_J)$ with $R \in \Pi_{j,k}$, and $t_R = \langle f, \varphi_R\rangle = \varphi_{j,k} * f(x_I, x_J)$, by Theorem (5.2.11), we have $f = \sum_R t_R\varphi_R$ and $t = \{t_R\}_R \in C_p^{\alpha,q}$. Then (100) gives

$$\sup_{\Omega}\frac{1}{|\Omega|^{\frac{1}{p}-\frac{1}{q'}}}\left(\sum_{R=I\times J\subseteq\Omega,R\in\mathcal{D}}\left(|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}\left|\psi_{j,k} * f\left(x_I, x_J\right)\right|\right)^q|R|\right)^{1/q}$$

$$\lesssim \sup_{\Omega}\frac{1}{|\Omega|^{\frac{1}{p}-\frac{1}{q'}}}\left(\sum_{R=I\times J\subseteq\Omega,R\in\mathcal{D}}\left(|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2}\right)^q|R|\right)^{1/q}$$

The conclusion of Theorem (5.2.12) follows immediately.

We give a characterization of imbedding of ℓ^r spaces into $\dot{f}_p^{\alpha,q}$ and imbedding of $\dot{f}_p^{\alpha,q}$ into ℓ^r spaces. This result was first established by Verbitsky [248] in the dyadic cubes with respect to an arbitrary positive locally finite measure on the Euclidean space and was generalized by Bownik [225] to discrete anisotropic Triebel-Lizorkin sequence spaces.

Lemma (5.2.13) [250] (Theorem 1 (i)(ii) of [248]). Let $0 < p < r \leq q \leq \infty$. Then

$$\left\|\left(\sum_{i\in I}|s_i|^q\varphi_i^q\right)^{1/q}\right\|_{L^p} \leq C\|s\|_{\ell^r}$$

holds if

$$\int \sup_i \left[\left(\phi_i^{r-p}(x)\|\phi_i\|_{L^p}^p\right)\right]^{p/(r-p)} dx < \infty.$$

Suppose $0 < q \le r < p < \infty$. Then

$$\left\| \left(\sum_i |s_i|^q \varphi_i^q \right)^{1/q} \right\|_{L^p} \ge C \|s\|_{\ell^r}$$

holds if

$$\int \sup_i \left[\left(\phi_i^{p-r}(x) / \|\phi_i\|_{L^p}^p \right) \right]^{p/(p-r)} dx < \infty.$$

Lemma (5.2.14) [250] (Theorem 1.1 of [243]). Let $0 < p < r < \infty$, I be any index set, and $\{\varphi_i\}_{i \in I}$ be a family in L^p. Then, the inequality

$$\left\| \sup_{i \in I} (|s_i| \varphi_i) \right\|_{L^p} \le C \|s\|_{\ell^r}$$

holds for all scalar sequences $s = \{s_i\}_{i \in I} \in \ell^r$ if and only if there exists a nonnegative measurable function $F \ge 0$ with $\int F(x)dx \le 1$, such that

$$\sup_{i \in I} \left\| F^{-1/p} \varphi_i \right\|_{L^{r,\infty}(\mu)} < \infty,$$

where $L^{r,\infty}(\mu)$ is a weak-L^r with respect to the measure $du(x) = F(x)dx$ defined by

$$\|f\|_{L^{r,\infty}(\mu)} = \left(\sup_{t > 0} t^r \mu(\{x \in R^m : |f(x)| > t\}) \right)^{1/r} < \infty$$

for $f \in L^{r,\infty}(\mu)$.

Lemma (5.2.15) [250] (Remark 3 of [248]). If $0 < q = r < p < \infty$, then

$$\left\| \left(\sum_{i \in I} |s_i|^q \phi_i^q \right)^{1/q} \right\|_{L^p} \ge C \|s\|_{\ell^r}$$

holds if and only if there exists $F \ge 0$ such that

$$\int F(x)dx \le 1 \text{ and } \inf_i \left\| F^{-1/p} \phi_i \right\|_{L^r(\mu)} > 0,$$

where $d\mu(x) = F(x)dx$.

Theorem (5.2.16) [250] Assume that Π is any subfamily $\mathcal{D}$ and $\{c_R\}_{R \in \Pi}$ is any positive sequence.

(i) Suppose $0 < p < r \le q \le \infty$. Then the inequality

$$\left\| \left(\sum_{R \in \Pi} |s_R|^q (c_R)^q \chi_R \right)^{1/q} \right\|_{L^p} \le C \|s\|_{\ell^r} \tag{103}$$

holds for all scalar sequences $s = \{s_R\}_{R \in \Pi}$ if and only if

$$\int \sup_{R \in \Pi} \left((c_R)^r |R| \right)^{p/(r-p)} \chi_R(x) dx < \infty. \tag{104}$$

(ii) Suppose $0 < q \le r < p < \infty$. Then the inequality

$$\left\| \left(\sum_{R \in \Pi} |s_R|^q (c_R)^q \chi_R \right)^{1/q} \right\|_{L^p} \ge C \|s\|_{\ell^r} \tag{105}$$

holds for all scalar sequences $s = \{s_R\}_{R \in \Pi}$ if and only if (105) holds.

To establish Theorem (5.2.16) we will follow the original approach of Verbitsky [248].

Proof. We begin with the proof of part (i). Firstly (104) ⇒ (103) is a direct consequence of Lemma (5.2.13) since $\int (c_R \chi_R(x))^p dx = (c_R)^p |R|$.

Now suppose that (103) holds for $p < r$. By imbedding $\ell^q \hookrightarrow \ell^\infty$ and Lemma (5.2.14), there exists a non-negative measurable function $F \ge 0$ with $\int F(x)dx \le 1$, such that

$$\sup_{R\in\Pi}\left\|F^{-1/p}c_R\chi_R\right\|_{L^{r,\infty}(\mu)} = \sup_{R\in\Pi} c_R\left\|F^{-1/p}\chi_R\right\|_{L^{r,\infty}(\mu)} < \infty, \tag{106}$$

where $d\mu = Fdx$. Let $f = F^{-1/p}\chi_R$; then $\|f\|_{L^p(\mu)} = |R|^{1/p}$. Suppose $p < s < r$ and $1/s = t/p + (1-t)/r$ with $0 < t < 1$. Applying the well-known interpolation inequality (e.g. Proposition 1.1.14 of [231])

$$\|f\|_{L^s(\mu)} \le C\|f\|^t_{L^{p,\infty}(\mu)}\|f\|^{1-t}_{L^{r,\infty}(\mu)},$$

one has for any $R \in \Pi$,

$$\left(\int_R F^{-s/p+1} - s/p + 1dx\right)^{1/s} \le C|R|^{t/p}\left\|F^{-1/p}\chi_R\right\|^{1-t}_{L^{r,\infty}(\mu)}.$$

Letting $\delta = s/p - 1$ and combining the above inequality with (106), we obtain

$$\left(c_R|R|^{1/r}\right)^{pr/(r-p)}\left(\frac{1}{|R|}\int_R F^{-\delta}dx\right)^{1/\delta} \le C < \infty.$$

On the other hand, by Hölder's inequality with exponents $\frac{\delta+\varepsilon}{\varepsilon}, \frac{\delta+\varepsilon}{\delta}$ one has

$$\left(\frac{1}{|R|}\int_R F^{-\delta}dx\right)^{1/\delta}\left(\frac{1}{|R|}\int_R F^{\varepsilon}dx\right)^{1/\varepsilon} \ge 1,$$

for all $\delta, \varepsilon > 0$. Hence

$$\left(c_R|R|^{1/r}\right)^{pr/(r-p)} \le C\left(\frac{1}{|R|}\int_R F^{\varepsilon}dx\right)^{1/\varepsilon} \le C(M_s(F^\varepsilon)(x))^{1/\varepsilon}$$

for $x \in R$, where M_s denotes the strong maximal operator. Since M_s is bounded on $L^{1/\varepsilon}$ for $0 < \varepsilon < 1$, we have

$$\int \sup_{R\in\Pi}\ ((c_R)^r|R|)^{p/(r-p)}\,dx \lesssim \int (M_s(F^\varepsilon)(x))^{1/\varepsilon}dx \lesssim \int F(x)dx < \infty.$$

We then have completed the proof of part (i) of Theorem (5.2.16).

We now give the proof of part (ii). The second part of Lemma (5.2.13) gives the proof of (104) ⇒ (105). Now suppose that (105) holds. We first prove (104) for $q = r$ following the original argument of Verbitsky [248]. By Lemma (5.2.15), there exists $F \in L^1, F \ge 0$, such that

$$\inf_{R\in\Pi}\int F^{1-r/p}(c_R\chi_R)^r dx = \inf_{R\in\Pi}(c_R)^r\int F^{1-r/p}dx > 0.$$

It follows from the above inequality that

$$\int \sup_{R\in\Pi}\ ((c_R)^r|R|)^{p/(r-p)}\chi_R(x)\,dx \le \int \sup_{R\in\Pi}\left(\frac{1}{|R|}\int_R F^{1-r/p}dy\right)^{p/(p-r)}\chi_R(x)\,dx$$
$$\le \int (M_s(F^{1-r/p})(x))^{p/(p-r)}dx \le\ C\int F(x)dx < \infty.$$

When $q < r$, we use the argument of Bownik [225] by taking advantage of the already established duality of $\dot{f}^{\alpha,1}_p, p > 1$. Note that by duality

$$\|s\|_{\ell^r} = \sup_{t=\{t_R\}}\frac{\left(\sum|s_R|^q|t_R|^q\right)^{1/q}}{\|t\|_{\ell^{rq/(r-q)}}}.$$

Hence (105) is equivalent to the inequality

$$\left(\sum_{R\in\Pi}|s_R|^q|t_R|^q\right)^{1/q} \le C\left\|\left(\sum_{R\in\Pi}|s_R|^q(c_R)^q\chi_R\right)^{1/q}\right\|_{L^p}\|t\|_{\ell^{rq/(r-q)}}. \tag{107}$$

On the other hand, since $1 < p/q < \infty$, by the already established duality $(\dot{f}_{p/q}^{\alpha,1})^* = \dot{f}_{p/(p-q)}^{-\alpha,\infty}$, one has for $\alpha = (\frac{1}{2}(m-1), \frac{1}{2})$,

$$\sup_{u=\{u_R\}} \frac{|\sum u_R \bar{v}_R|}{\|\sum u_R \chi_R\|_{L^{p/q}}} = \sup_{u=\{u_R\}} \frac{|\langle u, v\rangle|}{\|u\|_{\dot{f}_{p/q}^{\alpha,1}}} = \|v\|_{\dot{f}_{p/(p-q)}^{-\alpha,\infty}} = \left\|\sup_{R\in\mathcal{D}} |v_R||R|^{-1}\chi_R\right\|_{L^{p/(p-q)}}. \quad (108)$$

Let

$$u_R = \begin{cases} |t_R|^q (c_R)^{-q}, & R \in \Pi; \\ 0, & R \in \mathcal{D}/\Pi, \end{cases}$$

and

$$u_R = \begin{cases} |s_R|^q (c_R)^{q}, & R \in \Pi; \\ 0, & R \in \mathcal{D}/\Pi. \end{cases}$$

Then (108) may be rewritten in the following form by taking the qth roots:

$$\sup_{u=\{u_R\}} \frac{|\sum_{R\in\Pi} |s_R|^q |t_R|^q|^{1/q}}{\|(\sum_{R\in\Pi} |s_R|^q (c_R)^q \chi_R)^{1/q}\|_{L^p}} = \left\|\sup_{R\in\Pi} |t_R|(c_R)^{-1}|R|^{-1/q}\chi_R\right\|_{L^{pq/(p-q)}}. \quad (109)$$

Let $p_1 = pq/(p-q)$, $r_1 = rq/(r-q)$ and $\tilde{c}_R = (c_R)^{-1}|R|^{-1/q}$. Combining (107) with (109) yields

$$\left\|\sup_{R\in\Pi} |t_R|(\tilde{c}_R)^{-1}\chi_R\right\|_{L^{p_1}} \le C\|t\|_{\ell^{r_1}}$$

for all $t = \{t_R\}_R$. Using the facts that $p_1 r_1/(r_1 - p_1) = pr/(r-p)$, $p_1 < r_1$, and applying (i) of Theorem (5.2.16), we get from the preceding inequality

$$\int \sup_{R\in\Pi} ((\tilde{c}_R)^{r_1}|R|)^{p_1/(r_1-p_1)}\chi_R(x)\,dx = \int \sup_{R\in\Pi} ((c_R)^{r}|R|)^{p/(r-p)}\chi_R(x)\,dx < \infty.$$

Hence (104) holds for $q < r$. We thus have completed the proof.

Theorem (5.2.17) [250] Suppose $1 < p < \infty, 0 < q < \infty, \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$. Then

$$(\dot{f}_p^{\alpha,q})^* = \dot{f}_{p'}^{-\alpha,q'}.$$

Proof. For any $s \in \dot{f}_p^{\alpha,q}, t \in \dot{f}_{p'}^{\alpha,q'}$ we have

$$\left|\sum_{R\in\mathcal{D}} s_R \bar{t}_R\right| \le \int \sum_{R\in\mathcal{D}} |I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|s_R|\tilde{\chi}_R(x)|I|^{-\alpha_1/(m-1)}|J|^{-\alpha_2}|t_R|\tilde{\chi}_R(x)dx$$

$$\le \int \left(\sum_{R\in\mathcal{D}} \left(|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|s_R|\tilde{\chi}_R(x)\right)^q\right)^{1/q} \times \left(\sum_{R\in\mathcal{D}} \left(|I|^{-\alpha_1/(m-1)}|J|^{-\alpha_2}|t_R|\tilde{\chi}_R(x)\right)^{q'}\right)^{1/q'} dx$$

$$\le \|s\|_{\dot{f}_p^{\alpha,q}} \|t\|_{\dot{f}_{p'}^{-\alpha,q'}}$$

by duality if $1 \le q < \infty$ or by imbedding $\ell^q \hookrightarrow \ell^1$ if $0 < q < 1$. This yields that t is a continuous linear functional on $\dot{f}_p^{\alpha,q}$ and

$$\|t\|_{(\dot{f}_p^{\alpha,q})^*} \le \|t\|_{\dot{f}_{p'}^{-\alpha,q'}}.$$

For the converse direction, we split its proof into 2 cases: $(p,q) \in (1,\infty) \times [1,\infty)$ and $(1,\infty) \times (0,1)$.

Case1: $(p,q) \in (1,\infty) \times [1,\infty)$. This case is elementary. Take any $l \in (\dot{f}_p^{\alpha,q})^*$. Then there exists some sequence $t = t_R$ such that $l(s) = \sum_R s_R \bar{t}_R$ for any $s = \{s_R\}_R \in \dot{f}_p^{\alpha,q}$. Now we need a well-known result that

$$(L^p(l^q))^* = L^{p'}(l^{q'}) \quad (110)$$

if $1 < p < \infty, 1 < q < \infty$, where

$$L^p(l^q)=\left\{f=\{f_v\}:\|f\|_{L^p(l^q)}=\left\|\left(\sum_v|f_v|^q\right)^{1/q}\right\|_{L^p}<\infty\right\},$$

with the pairing $\langle f,g\rangle=\int\sum_v f_v\bar{g}_v$ for $f\in L^p(l^q), g\in L^{p'}(l^{q'})$ (see e.g. [65]). Let $I:\dot{f}_p^{\alpha,q}\to L^p(l^q)$ be defined by

$$I(s)=\{f_{j,k}\}_{j,k\in\mathbb{Z}},\qquad \text{where } f_{j,k}=\sum_{R\in\Pi_{j,k}}|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}s_R\tilde{\chi}_R(x).$$

Clearly, the map I is a linear isometry onto a subspace of $L^p(l^q)$. By the Hahn-Banach Theorem, there exists $\tilde{l}\in(L^p(l^q))^*$ such that $\tilde{l}\circ I=l$ and $\|\tilde{l}\|=\|l\|$. By (110), $\tilde{l}(f)=\langle f,g\rangle$ for some $g\in L^{p'}(l^{q'})$ with $\|g\|_{L^{p'}(l^{q'})}\le l$. Hence

$$l(s)=\tilde{l}(I(s))=\int\sum_{j,k}f_{j,k}\bar{g}_{j,k}=\int\sum_{j,k}\left(\sum_{R\in\Pi_{j,k}}|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}s_R\tilde{\chi}_R(x)\right)\bar{g}_{j,k}(x)\,dx$$

$$=\sum_{j,k}\sum_{R\in\Pi_{j,k}}s_R\left(|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|R|^{-1/2}\int_R\bar{g}_{j,k}(x)dx\right)=\sum_{j,k}\sum_{R\in\Pi_{j,k}}s_Rt_R=\langle t,s\rangle,$$

for all $s\in\dot{f}_p^{\alpha,q}$, where $t=\{t_R\}_R$ with $t_R=|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|R|^{-1/2}\int_R\bar{g}_{j,k}(x)dx$ for $R\in\Pi_{j,k}$. Then

$$\|t\|_{\dot{f}_{p'}^{-\alpha,q'}}=\left\|\sum_{j,k}\sum_{R\in\Pi_{j,k}}\left(\frac{1}{|R|}\int_R g_{j,k}\right)\chi_R(x)\right\|_{L^{p'}}\le\left\|\{M_s(g_{j,k})\}\right\|_{L^{p'}(l^{q'})}\lesssim\|g\|_{L^{p'}(l^{q'})}\le\|l\|.$$

This completes the proof of Case 1.

Case 2: $(p,q)\in(1,\infty)\times(0,1)$. In this case, $L^p(l^q)$ is not a normed space; hence we can't use the Hahn-Banach theorem.

Take $l\in(\dot{f}_p^{\alpha,q})^*$. Then there exists some sequence $t=t_R$ such that for any $s=\{s_R\}_R\in\dot{f}_p^{\alpha,q}$,

$$|l(s)|=\left|\sum_R s_R\bar{t}_R\right|\le C\|s\|_{\dot{f}_p^{\alpha,q}}=C\left\|\left(\sum_{R\in\mathcal{D}}\left(|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|s_R|\tilde{\chi}_R(x)\right)^q\right)^{1/q}\right\|_{L^p}.\quad(111)$$

If we prove the estimates

$$\|t\|_{\dot{f}_{p'}^{-\alpha,\infty}}=\left\|\sup_{R\in\mathcal{D}}\left(|I|^{-\alpha_1/(m-1)}|J|^{-\alpha_2}|t_R|\tilde{\chi}_R(x)\right)\right\|_{L^{p'}}<\infty,$$

we then complete the proof.

Define $\Pi=\{R\in\mathcal{D},t_R\neq0\}$, and let $u_R=s_R\bar{t}_R, c_R=\frac{|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}}{|R|^{1/2}|t_R|}$ for $R\in\Pi$. We may assume that $s_R\bar{t}_R\ge0$ for all $R\in\mathcal{D}$ by choosing proper s_R. Moreover we can assume $s_R=0$ if $R\notin\Pi$. Then (111) can be rewritten as

$$\|u\|_{\ell^1}\le c\left\|\left(\sum_{R\in\Pi}|u_R|^q(c_R)^q\chi_R\right)^{1/q}\right\|_{L^p}$$

for all $u=\{u_R\}_R\in\Pi$. Then (ii) of Theorem (5.2.16) with $0<q<r=1<p<\infty$ yields

$$\int\sup_{R\in\Pi}\left((c_R\chi_R)|R|\right)^{p/(1-p)}dx<\infty,$$

that is,

$$\int \sup_{R\in\Pi} \left(\left(|I|^{-\alpha_1/(m-1)}|J|^{-\alpha_2}|t_R||R|^{-1/2}\chi_R\right)\right)^{p/(p-1)} dx < \infty.$$

We thus have completed the proof.

Theorem (5.2.18) [250] Suppose $1 < p \leq \infty, 0 < q < \infty, \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$. Then

$$(\dot{f}_p^{\alpha,q})^* = C_p^{-\alpha,q'}.$$

Proof. We first assume $1 \leq q < \infty$. Suppose $t \in C_p^{-\alpha,q'}$. For any $s \in \dot{f}_p^{\alpha,q}$, set

$$h(x) = \left(\sum_{R\in\mathcal{D}} \left(|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|s_R|\tilde{\chi}_R(x)\right)^q\right)^{1/q},$$

and for $k \in \mathbb{Z}$,

$$\Omega_k = \{x \in \mathbb{R}^m : h(x) > 2^k\},$$

$$B_k = \{R \in \mathcal{D} : |R \cap \Omega_k| > \frac{1}{2}|R|, |R \cap \Omega_{k+1}| \leq \frac{1}{2}|R|\}.$$

One can obtain

$$\left|\sum_{R\in\mathcal{D}} s_R t_R\right| = \sum_k \sum_{R\in B_k} \left(|I|^{-\alpha_1/(m-1)}|J|^{-\alpha_2}|t_R||R|^{-\frac{1}{2}}|R|^{\frac{1}{q'}}\right) \times \left(|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|s_R||R|^{\frac{1}{2}}|R|^{-\frac{1}{q'}}\right)$$

$$\leq \left\{\sum_k \left[\sum_{R\in B_k} \left(|I|^{-\alpha_1/(m-1)}|J|^{-\alpha_2}|t_R||R|^{-\frac{1}{2}}|R|^{\frac{1}{q'}}\right)^{q'}\right]^{p/q'}\right.$$

$$\left.\times \left[\sum_{R\in B_k} \left(|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|s_R||R|^{\frac{1}{2}}|R|^{-\frac{1}{q'}}\right)^q\right]^{p/q}\right\}^{1/p}. \quad (112)$$

Let $\widetilde{\Omega}_k = \{x \in \mathbb{R}^m, \mathcal{M}_s(\chi_{\Omega_k})(x) > \frac{1}{2}\}$; then $|\widetilde{\Omega}_k| \lesssim |\Omega_k|$. One sees that if $R \in B_k$, then one has $R \subseteq \widetilde{\Omega}_k$. So

$$\left[\sum_{R\in B_k} \left(|I|^{-\alpha_1/(m-1)}|J|^{-\alpha_2}|t_R||R|^{-\frac{1}{2}}|R|^{\frac{1}{q'}}\right)^{q'}\right]^{1/q'}$$

$$= \frac{1}{|\widetilde{\Omega}|^{\frac{1}{p}-\frac{1}{q}}}\left[\sum_{R\in B_k} \left(|I|^{-\alpha_1/(m-1)}|J|^{-\alpha_2}|t_R||R|^{-\frac{1}{2}}|R|^{\frac{1}{q'}}\right)^{q'}\right]^{p/q'} |\widetilde{\Omega}|^{\frac{1}{p}-\frac{1}{q}} \lesssim \|t\|_{C_p^{-\alpha,q'}} |\Omega_k|^{\frac{1}{p}-\frac{1}{q}}.$$

On the other hand, using the fact that if $R \in B_k, R \subseteq \widetilde{\Omega}_k$, one also obtains

$$\frac{1}{2}|R| < |R\backslash\Omega_{k+1}| = |R \cap \widetilde{\Omega}_k\backslash\Omega_{k+1}|.$$

Hence

$$\left[\sum_{R\in B_k} \left(|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|s_R||R|^{\frac{1}{2}}|R|^{-\frac{1}{q'}}\right)^q\right]^{1/q} = \left[\sum_{R\in B_k} \left(|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|s_R||R|^{\frac{1}{2}-\frac{1}{q'}-\frac{1}{q}}\right)^q |R|\right]^{1/q}$$

$$\lesssim \left[\sum_{R\in B_k} \left(|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|s_R||R|^{1/2}\right)^q |R \cap \widetilde{\Omega}_k\backslash\Omega_{k+1}|\right]^{1/q}$$

$$= \left(\int_{\widetilde{\Omega}_k\backslash\Omega_{k+1}} \sum_{R\in B_k} \left(|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|s_R|\tilde{\chi}_R(x)\right)^q dx\right)^{1/q} \leq \left(\int_{\widetilde{\Omega}_k\backslash\Omega_{k+1}} h^q(x)dx\right)^{1/q} \lesssim 2^k|\Omega_k|^{1/q}.$$

Combining (112) with the above inequality, one obtains

$$\left|\sum_{R\in\mathcal{D}} s_R t_R\right| \lesssim \|t\|_{C_p^{-\alpha,q'}} \left(\sum_k 2^{kp}|\Omega_k|\right)^{1/p} \lesssim \|t\|_{C_p^{-\alpha,q'}} \|s\|_{\dot f_p^{\alpha,q}}.$$

Next, we will prove $C_p^{-\alpha,q'} \supseteq \left(\dot f_p^{\alpha,q}\right)^*$. Let $\ell \in (\dot f_p^{\alpha,q})^*$. Then there exists some $t = \{t_R\}_R$ such that for every $s = \{s_R\}_R \in \dot f_p^{\alpha,q}$, $\ell(s) = \sum_R s_R t_R$ and

$$\left|\sum_R s_R t_R\right| \leq \|t\|_{(\dot f_p^{\alpha,q})^*} \|s\|_{\dot f_p^{\alpha,q}}.$$

Once having shown $t \in C_p^{-\alpha,q'}$, we will then complete the proof. For any open set $\Omega \subseteq \mathbb{R}^m$ with finite measure, let $X = \{R \in \mathcal{D} : R \subseteq \Omega\}$, and let μ be a measure on X such that the μ-measure of R is $|R|$ if $1 \leq q < \infty$ or $\mu(R) = 1$ if $q = 1$. Then by the above inequality, one has

$$\begin{aligned}
\left(\sum_{R\subseteq\Omega, R\in\mathcal{D}} \left(|I|^{-\alpha_1/(m-1)}|J|^{-\alpha_2}|t_R||R|^{-1/2}\right)^{q'}|R|\right)^{1/q'} &= \left\||I|^{-\alpha_1/(m-1)}|J|^{-\alpha_2}|t_R||R|^{-1/2}\right\|_{l^{q'}(X,d\mu)} \\
&= \sup_{\|s\|_{l^q(X,d\mu)}\leq 1} \left|\sum_{R\subseteq\Omega,R\in\mathcal{D}} |I|^{-\alpha_1/(m-1)}|J|^{-\alpha_2}|t_R||R|^{-1/2}s_R|R|\right| \\
&\leq \|\ell\|_{(\dot f_p^{\alpha,q})^*} \sup_{\|s\|_{l^q(X,d\mu)}\leq 1} \left\||I|^{-\alpha_1/(m-1)}|J|^{-\alpha_2}|t_R||R|^{-1/2}s_R|R|\right\|_{\dot f_p^{\alpha,q}}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\left\||I|^{-\alpha_1/(m-1)}|J|^{-\alpha_2}|t_R||R|^{-1/2}s_R|R|\right\|_{\dot f_p^{\alpha,q}} \\
&= \left\|\left(\sum_{R\subseteq\Omega,R\in\mathcal{D}} \left(|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|I|^{-\alpha_1/(m-1)}|J|^{-\alpha_2}|R|^{-1/2}|s_R||R|\tilde\chi_R(x)\right)^q\right)^{1/q}\right\|_{L^p} \\
&= \left\|\left(\sum_{R\subseteq\Omega,R\in\mathcal{D}} (|s_R|\chi_R(x))^q\right)^{1/q}\right\|_{L^p} \leq \left\{\int_\Omega \sum_{R\subseteq\Omega,R\in\mathcal{D}} (|s_R|\chi_R(x))^q\,dx\right\}^{1/q} |\Omega|^{\frac1p-\frac1q}
\end{aligned}$$

by Hölder’s inequality since $0 < p \leq 1 \leq q < \infty$. So

$$\begin{aligned}
&\left(\sum_{R\subseteq\Omega,R\in\mathcal{D}} \left(|I|^{-\alpha_1/(m-1)}|J|^{-\alpha_2}|t_R||R|^{-1/2}\right)^{q'}|R|\right)^{1/q'} \\
&\leq \|\ell\|_{(\dot f_p^{\alpha,q})^*} \sup_{\|s\|_{l^q(X,d\mu)}\leq 1} \left\{\int_\Omega \sum_{R\subseteq\Omega,R\in\mathcal{D}} (|s_R|\chi_R(x))^q\,dx\right\}^{1/q} |\Omega|^{\frac1p-\frac1q} \leq \|\ell\|_{(\dot f_p^{\alpha,q})^*}|\Omega|^{\frac1p-\frac1q},
\end{aligned}$$

that is, $t \in C_p^{-\alpha,q'}$.

When $0 < q < 1$, by the trivial imbedding $\dot f_p^{\alpha,q} \to \dot f_p^{\alpha,1}$, one has

$$(\dot f_p^{\alpha,q})^* \supseteq (\dot f_p^{\alpha,1})^* = C_p^{-\alpha,\infty}.$$

To show the other direction, as above, let $\ell \in (\dot f_p^{\alpha,q})^*$. Then there exists some $t = \{t_R\}_R$ such that for every $s = \{s_R\}_R \in \dot f_p^{\alpha,q}$, $\ell(s) = \sum_R s_R t_R$ and

$$\left|\sum_R s_R t_R\right| \leq \|t\|_{(\dot f_p^{\alpha,q})^*} \|s\|_{\dot f_p^{\alpha,q}}.$$

We now prove $\|t\|_{C_p^{-\alpha,\infty}} = \sup_\Omega \frac{1}{|\Omega|^{\frac{1}{p}-1}} \sup_{R\subseteq\Omega, R\in\mathcal{D}} |I|^{-\alpha_1/(m-1)} |J|^{-\alpha_2} |t_R| |R|^{-1/2} < \infty$. For any fixed $R = I \times J \in \mathcal{D}$, let $\delta_{Q,R} = 1$ if $Q = R$; otherwise $\delta_{Q,R} = 0$. So

$$\sup_{R\subseteq\Omega, R\in\mathcal{D}} |I|^{-\alpha_1/(m-1)} |J|^{-\alpha_2} |t_R| |R|^{-1/2} = \sup_{R\subseteq\Omega, R\in\mathcal{D}} \sum_{Q=\tilde{I}\times\tilde{J}\subseteq\Omega} \delta_{Q,R} \left|\tilde{I}\right|^{-\alpha_1/(m-1)} \left|\tilde{J}\right|^{-\alpha_2} |Q|^{-1/2} \left|t_Q\right|$$
$$\leq \sup_{R\subseteq\Omega, R\in\mathcal{D}} \|\ell\|_{\left(\dot{f}_p^{\alpha,q}\right)^*} \left\| \delta_{Q,R} \left|\tilde{I}\right|^{-\alpha_1/(m-1)} \left|\tilde{J}\right|^{-\alpha_2} |Q|^{-1/2} \right\|_{\dot{f}_p^{\alpha,q}} = \sup_{R\subseteq\Omega, R\in\mathcal{D}} \|\ell\|_{\left(\dot{f}_p^{\alpha,q}\right)^*} \| |R|^{-1} \chi_R \|_{L^p}$$
$$= \sup_{R\subseteq\Omega, R\in\mathcal{D}} \|\ell\|_{(\dot{f}_p^{\alpha,q})^*} |R|^{\frac{1}{p}-1} \leq \sup_{R\subseteq\Omega, R\in\mathcal{D}} \|\ell\|_{(\dot{f}_p^{\alpha,q})^*} |\Omega|^{\frac{1}{p}-1}$$

since $0 < p \leq 1$, which implies our desired results, and we thus have completed the proof of Theorem (5.2.18).

We derive the duality of Theorem (5.2.19) and Theorem (5.2.20) from Theorem (5.2.17) and Theorem (5.2.18), respectively, in the sequence space cases. It is known from Proposition 3.1 in [139] that $S_0(\mathbb{R}^m)$ is dense in $\dot{F}_p^{\alpha,q}(\mathbb{R}^m)$ for $0 < p, q < \infty$.

Theorem (5.2.19) [250] Suppose $1 < p < \infty, 0 < q < \infty, \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$; then

$$(\dot{F}_p^{\alpha,q})^* = \dot{F}_{p'}^{-\alpha,q'}.$$

Namely, if $g \in \dot{F}_{p'}^{-\alpha,q'}$, then the map l_g, given by $l_g(f) = \langle f, g\rangle$, defined initially for $f \in S_0$, extends to a continuous linear functional on $\dot{F}_p^{\alpha,q}$ with $\|l_g\| \lesssim \|g\|_{\dot{F}_{p'}^{-\alpha,q'}}$.

Conversely, every $l \in (\dot{F}_p^{\alpha,q})^*$ satisfies $l = l_g$ for some $g \in \dot{F}_{p'}^{-\alpha,q'}$ with $\|l_g\| \approx \|g\|_{\dot{F}_{p'}^{-\alpha,q'}}$.

On dual spaces of multi-parameter Hardy spaces (see [102], [108], [129], [130], [131], [132], etc.), the duality of Triebel-Lizorkin spaces has only been studied in the one-parameter settings started in [65, 115]; see also [225] for anisotropic Triebel-Lizorkin spaces, [236] for weighted anisotropic Triebel-Lizorkin spaces. For $0 < p, q < \infty, \alpha \in \mathbb{R}$, the Triebel-Lizorkin space of one-parameter $\dot{\mathcal{F}}_p^{\alpha,q}(\mathbb{R}^m)$ with the norm

$$\left\| \left(\sum_{j\in\mathbb{Z}} 2^{-jq} \left|\psi_j * f\right|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^m)}$$

was investigated in [115, 65]. There it was shown that the dual space of $\dot{\mathcal{F}}_p^{\alpha,q}(\mathbb{R}^m)$ is

$$(\dot{\mathcal{F}}_p^{\alpha,q}(\mathbb{R}^m))^* = \begin{cases} \dot{\mathcal{F}}_{p'}^{\alpha,q'}(\mathbb{R}^m), & 1 \leq p < \infty; \\ \dot{\mathcal{F}}_\infty^{-\alpha+m(1/p-1),\infty}(\mathbb{R}^m), & 0 < p < 1, \end{cases} \tag{113}$$

where $\dot{\mathcal{F}}_\infty^{\alpha,q}(\mathbb{R}^m)$ is defined to be the set of all $f \in S_0'(\mathbb{R}^m)$ such that

$$\|f\|_{\dot{\mathcal{F}}_\infty^{\alpha,q}(\mathbb{R}^m)} = \sup_{\text{Q dyadic cubes}} \left(\frac{1}{|\mathrm{Q}|} \int_{\mathrm{Q}} \sum_{j=-\log_2 l(\mathrm{Q})}^{\infty} 2^{-j\alpha q} \left|\psi_j * f\right|^q dx \right)^{1/q} < \infty.$$

It is well known that $\dot{\mathcal{F}}_p^{0,2}(\mathbb{R}^m)$ is the classical Hardy space $\mathcal{H}_p, 0 < p \leq 1$. From (113), one has

$$(\mathcal{H}_p)^* = \dot{\mathcal{F}}_\infty^{m(1/p-1),\infty}(\mathbb{R}^m).$$

The method to obtain (113) no longer works in multi-parameter cases when $0 < p \leq 1$. By using techniques of discrete Littlewood-Paley theory developed in [108, 235] for flag Hardy spaces, established the dual spaces for flag Hardy spaces. Using similar ideas of discrete Littlewood-Paley theory, the dual spaces for Hardy spaces on product spaces of homogeneous type and on weighted multi-parameter Hardy spaces have been obtained in [130, 131] and [133]. To give an idea of such dual spaces in the simplest form, we state the dual space of multi-parameter pure product Hardy space $\mathrm{H}_p = \dot{F}_p^{0,2}(\mathbb{R}^n \times \mathbb{R}^m)$ by another form, for $0 < p \le 1$,

$$(\mathrm{H}_p)^* = CMO_p,$$

where $f \in CMO_p$ is defined by

$$\|f\|_{CMO_p} = \sup_{\Omega} \left(\frac{1}{|\Omega|^{\frac{2}{p}-1}} \int_{\Omega} \sum_{j,k\in\mathbb{Z}} \sum_{R\in\Pi_{j,k}, R\subseteq\Omega} \left(|\psi_j * f(x_I, x_J)|^q \chi_R(x) \right) dx \right)^{1/2}$$

for all open sets $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}^m$ with finite measure (see [133]). Combining the techniques developed in [65, 115] for one-parameter Triebel-Lizorkin spaces and [235] for multi-parameter Hardy spaces, we investigate the dual spaces of the multi-parameter Triebel-Lizorkin spaces associated with different homogeneities when $0 < p \le 1$. Before we state the duality result, we first give the definition of $CMO_p^{\alpha,q}(\mathbb{R}^m)$.

Proof. Let $g \in \dot{F}_{p'}^{-\alpha,q'}(\mathbb{R}^m), f \in S_0(\mathbb{R}^m)$ and $1 < p < \infty, 0 < q < \infty$. Then by the identity (93) one has $\langle f, g\rangle = \langle S_\psi f, S_\psi g\rangle$. Hence

$$|\langle f, g\rangle| \le \|S_\psi f\|_{\dot{f}_p^{\alpha,q}(\mathbb{R}^m)} \|S_\psi g\|_{\dot{f}_{p'}^{-\alpha,q'}(\mathbb{R}^m)} \lesssim \|f\|_{\dot{F}_p^{\alpha,q}(\mathbb{R}^m)} \|g\|_{\dot{F}_{p'}^{-\alpha,q'}(\mathbb{R}^m)}$$

by Theorem (5.2.17) and Theorem (5.2.9). This proves that $\|l_g\| \lesssim \|g\|_{\dot{F}_{p'}^{-\alpha,q'}(\mathbb{R}^m)}$.

Conversely, suppose $l \in \left(\dot{F}_p^{\alpha,q}(\mathbb{R}^m)\right)^*$. Then $l_1 \equiv l \circ T_\psi \in (\dot{f}_p^{\alpha,q})^*$, so by Theorem (5.2.17), there exists $t = \{t_R\}_R \in \dot{f}_{p'}^{-\alpha,q'}$ such that

$$l_1(s) = \langle s, t\rangle = \sum_R s_R \bar{t}_R$$

for all $s = \{s_R\}_R \in \dot{f}_p^{\alpha,q}(\mathbb{R}^m)$. Moreover $\|t\|_{\dot{f}_{p'}^{-\alpha,q'}} \approx \|l_1\| \lesssim \|l\|$ for the boundedness of T_ψ. Note that $l_1 \circ S_\psi = l \circ T_\psi \circ S_\psi = l$ since $T_\psi \circ S_\psi$ is an identity by Theorem (5.2.9). Then letting $g = T_\psi(t)$ and $f \in S_0(\mathbb{R}^m)$, one has

$$l(f) = l_1(S_\psi(f)) = \langle S_\psi(f), t\rangle = \langle f, T_\psi(t)\rangle = \langle f, g\rangle$$

by (94), which implies that $l = l_g$ and by Theorem (5.2.9) again, one has

$$\|g\|_{\dot{F}_{p'}^{-\alpha,q'}} = \|T_\psi(t)\|_{\dot{F}_{p'}^{-\alpha,q'}} \lesssim \|t\|_{\dot{f}_{p'}^{-\alpha,q'}} \lesssim \|l\|.$$

We have then completed the proof of Theorem (5.2.19).

Theorem (5.2.20) [250] Suppose $0 < p \le 1, 0 < q < \infty, \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$. Then

$$(\dot{F}_p^{\alpha,q})^* = CMO_p^{-\alpha,q'}$$

where q' is defined to be ∞ when $0 < q \le 1$.

By duality, one can obtain the boundedness of $T_1 \circ T_2$ on $CMO_p^{-\alpha,q'}$.

Proof. One can go through the same process as above to finish the proof of Theorem (5.2.20).

Theorem (5.2.21) [250] Suppose $0 < p \le 1$, $1 < q \le \infty$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$. Then the composition operator $T = T_1 \circ T_2$ is bounded on $CMO_p^{\alpha,q}$.

Proof. We assume that $\mathcal{K}_i$ is the kernel of the convolution operator $T_i, i = 1,2$, and T^* is the conjugate operator of T with the kernel $\mathcal{K}^*$. One may check that

$$\mathcal{K}^* * f(x) = \widetilde{\mathcal{K}}_2 * \widetilde{\mathcal{K}}_1 * f(x) = \widetilde{\mathcal{K}}_1 * \widetilde{\mathcal{K}}_2 * f(x)$$

for $f \in C_c^\infty$, where $\widetilde{\mathcal{K}}_i(x) = \mathcal{K}_i(-x), i = 1,2$. Hence T^* is bounded on $\dot{F}_p^{\alpha,q}$ for all $0 < p, q < \infty, \alpha \in \mathbb{R}^2$ by Theorem 1.5 in [139] since $\widetilde{\mathcal{K}}_i$ satisfy Definition (5.2.1), Definition (5.2.2), respectively.

For $\forall\ 1 \le q < \infty$, there exists a $0 < \tilde{q} < \infty$ such that $\tilde{q}' = q$. Then by Theorem (5.2.20),

$$\begin{aligned}\|T(f)\|_{CMO_p^{\alpha,q}} &= \sup_{\|g\|_{\dot{F}_p^{-\alpha,\tilde{q}'}} \le 1} |\langle T(f), g\rangle| = \sup_{\|g\|_{\dot{F}_p^{-\alpha,\tilde{q}'}} \le 1} |\langle f, T^*(g)\rangle| \\ &\le \sup_{\|g\|_{\dot{F}_p^{-\alpha,\tilde{q}'}} \le 1} \|f\|_{CMO_p^{\alpha,q}} \|T^*(g)\|_{\dot{F}_p^{-\alpha,\tilde{q}'}} \lesssim \sup_{\|g\|_{\dot{F}_p^{-\alpha,\tilde{q}'}} \le 1} \|f\|_{CMO_p^{\alpha,q}} \|g\|_{\dot{F}_p^{-\alpha,\tilde{q}'}} \\ &\le \|f\|_{CMO_p^{\alpha,q'}}.\end{aligned}$$

We thus have completed the proof of Theorem (5.2.21).

The multi-parameter Triebel-Lizorkin spaces we study here are associated with the composition of two singular integral operators with the specific dilations

$$\delta : (x', x_m) \to (\delta x', \delta x_m), \delta > 0$$

and

$$\delta : (x', x_m) \to (\delta x', \delta^2 x_m), \delta > 0.$$

The first is the classical isotropic dilations occurring in the classical Calderón-Zygmund singular integrals, while the second is non-isotropic and related to the heat equations (also Heisenberg groups).

These two dilations are motivated by the study of weak-(1, 1) boundedness of the composition of two singular integrals by Phong and Stein [135]. This composition of such two singular integral operators is particularly interesting because they essentially arise naturally in the study of the $\bar{\partial}$-Neumann problem (see [135], [232], [241], [242]). This motivates us to study the function spaces associated with the composition of two such dilations and then the boundedness of relevant operators. It is to note that the underlying multi-parameter structure we study is intrinsic to the composition of these two dilations. Nevertheless, the multi-parameter structures we consider are still in the framework of the translation-invariant environment. The more general case of translation non-invariant dilations will be studied in a forthcoming project.

Though we restrict our attention to the above two very specific dilations, all results can be carried out to the composition with more singular integral operators associated with more general non-isotropic homogeneities. To see this, let

$$\delta_i : (x_1, x_2, \cdots, x_m) \to (\delta_i^{\lambda_{i,1}} x_1, \delta_i^{\lambda_{i,2}} x_2, \cdots, \delta_i^{\lambda_{i,m}} x_m)$$

for $\delta_i > 0, \lambda_{i,t} > 0, 1 \le i \le n$ and $1 \le t \le m$.

For $x \in \mathbb{R}^m$ we denote $|x|_i = \sqrt{|x_1|^{\frac{2}{\lambda_{i,1}}} + |x_2|^{\frac{2}{\lambda_{i,2}}} + \cdots + |x_m|^{\frac{2}{\lambda_{i,m}}}}$. Let $\psi^{(i)} \in S(\mathbb{R}^m)$ with

$$\operatorname{supp} \widehat{\psi^{(i)}} \subseteq \{(\xi_1, \xi_2, \cdots, \xi_m) \in \mathbb{R}^m : \frac{1}{2} \le |\xi|_i \le 2\}$$

and

$$\sum_{j_i \in \mathbb{Z}} \left|\widehat{\psi^{(i)}}(2^{-j_i\lambda_{i,1}}\xi_1, 2^{-j_i\lambda_{i,2}}\xi_2, \cdots, 2^{-j_i\lambda_{i,m}}\xi_m)\right|^2 = 1 \quad \forall\ (\xi_1, \xi_2, \cdots, \xi_m) \in \mathbb{R}^m/\{0\}.$$

Set $\psi_{j_1,j_2,\cdots,j_n}(x) = \psi_{j_1}^{(1)} * \psi_{j_2}^{(2)} * \cdots * \psi_{j_n}^{(n)}(x)$, where

$$\psi_{j_i}^{(i)}(x) = 2^{j_i(\lambda_{i,1}+\lambda_{i,2}+\cdots+\lambda_{i,m})} \psi^{(i)}(2^{j_i\lambda_{i,1}}x_1, 2^{j_i\lambda_{i,2}}x_2, \cdots, 2^{j_i\lambda_{i,m}}x_m).$$

Then we can obtain the following general discrete Calderón reproducing formula:

Theorem (5.2.22) [250] Suppose that $\psi^{(i)}, i = 1, \cdots, n$, are functions satisfying the above conditions, respectively. Then

$$f(x_1,x_2,\cdots,x_m) = \sum_{j_1,\cdots,j_n\in\mathbb{Z}} \sum_{(\ell_1,\cdots,\ell_m)\in\mathbb{Z}^m} \prod_{t=1}^{m} 2^{-(j_1\lambda_{1,t}\wedge j_2\lambda_{2,t}\wedge\cdots\wedge j_n\lambda_{n,t})} \times (\psi_{j_1,j_2,\cdots,j_n} * f)(2^{-(j_1\lambda_{1,1}\wedge j_2\lambda_{2,1}\wedge\cdots\wedge j_n\lambda_{n,1})}\ell_1,\cdots,2^{-(j_1\lambda_{1,m}\wedge j_2\lambda_{2,m}\wedge\cdots\wedge j_n\lambda_{n,m})}\ell_m) \times \psi_{j_1,j_2,\cdots,j_n}(x_1 - 2^{-(j_1\lambda_{1,1}\wedge j_2\lambda_{2,1}\wedge\cdots\wedge j_n\lambda_{n,1})}\ell_1,\cdots,x_m - 2^{-(j_1\lambda_{1,m}\wedge j_2\lambda_{2,m}\wedge\cdots\wedge j_n\lambda_{n,m})}\ell_m),$$

where the series converges in $L^2(\mathbb{R}^m), S_0(\mathbb{R}^m)$ and $S_0'(\mathbb{R}^m)$.

With the above discrete Calderón reproducing formula, the multi-parameter Triebel-Lizorkin spaces with different homogeneities can be introduced as follows:

Definition (5.2.23) [250] Let $0 < p,q < \infty, \alpha = (\alpha_1,\alpha_2,\cdots,\alpha_m) \in \mathbb{R}^m$. The multi-parameter Triebel-Lizorkin type spaces with different homogeneities $\dot{F}_p^{\alpha,q}(\mathbb{R}^m)$ are defined by

$$\dot{F}_p^{\alpha,q}(\mathbb{R}^m) = \{f \in S_0'(\mathbb{R}^m) : \|f\|_{\dot{F}_p^{\alpha,q}(\mathbb{R}^m)} < \infty\},$$

where

$$\|f\|_{\dot{F}_p^{\alpha,q}(\mathbb{R}^m)} = \left\| \left(\sum_{j_1,\cdots,j_n\in\mathbb{Z}} \prod_{t=1}^{m} 2^{-(j_1\lambda_{1,t}\wedge j_2\lambda_{2,t}\wedge\cdots\wedge j_n\lambda_{n,t})\alpha_t p} \times \sum_{(\ell_1,\cdots,\ell_m)\in\mathbb{Z}^m} \left|(\psi_{j_1,j_2,\cdots,j_n} * f)(2^{-(j_1\lambda_{1,1}\wedge j_2\lambda_{2,1}\wedge\cdots\wedge j_n\lambda_{n,1})}\ell_1,\cdots,2^{-(j_1\lambda_{1,m}\wedge j_2\lambda_{2,m}\wedge\cdots\wedge j_n\lambda_{n,m})}\ell_m)\right|^q \chi_{I_1}(x_1)\chi_{I_2}(x_2)\cdots\chi_{I_m}(x_m) \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^m)},$$

where $\{I_t\}_{t=1,\cdots,m}$ are dyadic intervals in $\mathbb{R}$ with the side length $l(I_t) = 2^{-(j_1\lambda_{1,t}\wedge j_2\lambda_{2,t}\wedge\cdots\wedge j_n\lambda_{n,t})}$, and the left end points of I_t are $2^{-(j_1\lambda_{1,t}\wedge j_2\lambda_{2,t}\wedge\cdots\wedge j_n\lambda_{n,t})}\ell_1$, respectively.

Applying the same techniques, one can establish the duality theory (Theorem (5.2.19)) of the multi-parameter Triebel-Lizorkin spaces associated with these more general non-isotropic dilations. The details of the proofs appear to be very lengthy and complicated to present in the more general situation. Therefore, we shall not discuss these in more detail.

Corollary (5.2.24) [314] Suppose $0 < \epsilon < 1$, $\alpha_{r-1} = (\alpha_r,\alpha_{r+1}) \in \mathbb{R}^2$. Then the composition operator $T_{r-1} = T_r \circ T_{r+1}$ is bounded on $CMO_{1-\epsilon}^{\alpha_{r-1},1+\epsilon}$.

Proof. We assume that $\mathcal{K}_i$ is the kernel of the convolution operator $T_i, i = r, r+1$, and T_{r-1}^* is the conjugate operator of T_{r-1} with the kernel $\mathcal{K}^*$. One may check that

$$\mathcal{K}_{r-1}^* * \sum f_j(x) = \widetilde{\mathcal{K}}_{r+1} * \widetilde{\mathcal{K}}_r * \sum f_j(x) = \widetilde{\mathcal{K}}_r * \widetilde{\mathcal{K}}_{r+1} * \sum f_j(x)$$

for $f_j \in C_c^\infty$, where $\widetilde{\mathcal{K}}_i(x) = \mathcal{K}_i(-x), i = r, r+1$. Hence T_{r-1}^* is bounded on $\dot{F}_{1+\epsilon}^{\alpha_{r-1},1+2\epsilon}$ for all $0 \le \epsilon < \infty, \alpha_{r-1} \in \mathbb{R}^2$ by Theorem 1.5 in [139] since $\widetilde{\mathcal{K}}_i$ satisfy Definition (5.2.1), Definition (5.2.2), respectively.

For $\forall\ 0 \le \epsilon < \infty$. Then by Theorem (5.2.20),

$$\sum_j \left\|T_{r-1}(f_j)\right\|_{CMO_{1+\epsilon}^{\alpha_{r-1},1+\epsilon}} = \sup_{\|g_j\|_{\dot{F}_{1+\epsilon}^{-\alpha_{r-1},1+\epsilon}}\le 1} \sum \left|\langle T_{r-1}(f_j), g_j\rangle\right|$$

$$= \sup_{\|g_j\|_{\dot{F}_{1+\epsilon}^{-\alpha_{r-1},1+\epsilon}}\le 1} \sum \left|\langle f_j, T_{r-1}^*(g_j)\rangle\right| \le \sup_{\|g_j\|_{\dot{F}_{1+\epsilon}^{-\alpha_{r-1},1+\epsilon}}\le 1} \sum \left\|f_j\right\|_{CMO_{1+\epsilon}^{\alpha_{r-1},1+\epsilon}} \left\|T_{r-1}^*(g_j)\right\|_{\dot{F}_{1+\epsilon}^{-\alpha_{r-1},1+\epsilon}}$$

$$\lesssim \sup_{\|g_j\|_{\dot{F}_{1+\epsilon}^{-\alpha_{r-1},1+\epsilon}}\le 1} \sum \|f_j\|_{CMO_{1+\epsilon}^{\alpha_{r-1},1+\epsilon}} \|g_j\|_{\dot{F}_{1+\epsilon}^{-\alpha_{r-1},1+\epsilon}} \le \sum \|f_j\|_{CMO_{1+\epsilon}^{\alpha_{r-1},1+\epsilon}}.$$

We thus have completed the proof of Corollary (5.2.24).

Section (5.3) Triebel–Lizorkin and Besov–Lipschitz Spaces

Many have worked on proving boundedness of the one-parameter Fourier multipliers. Mihlin [252] obtained the L^p-boundedness of the one-parameter Fourier multiplier operators with minimal smooth condition. Hörmander [198] reformulated and improved Mihlin's theorem using the Sobolev regularity of the multiplier. Peetre [253] considered the boundedness of the one-parameter Fourier multiplier operators on Triebel–Lizorkin spaces.

The multi-parameter singular integral and Hardy space theory play an important role in many aspects of harmonic analysis. Many have studied the pure product theory. This theory includes the boundedness on multi-parameter L^p spaces $(1 < p < \infty)$ and multi-parameter Hardy spaces $H^p(0 < p \le 1)$, etc. For more results related to multi-parameter theory, see [96–105, 107, 108, 121, 129, 131, 133, 233, 235, 240, 242, 244, 245, 246, 254]. [120, 128, 139] established the boundedness of singular integral operators on multi-parameter Triebel–Lizorkin and Besov–Lipschitz spaces. The atomic decomposition and dual spaces for multi-parameter Triebel–Lizorkin spaces associated with the composition of two singular operators studied by Phong and Stein [135] were given in [229, 250].

We establish the sufficient conditions for boundedness of multi-parameter Fourier multiplier operators on multi-parameter Triebel–Lizorkin and Besov–Lipschitz spaces (Theorem (5.3.11)). Furthermore, we also consider the weighted cases (Theorem (5.3.13)). For the sake of simplicity of presentations, we will restrict our consideration to the bi-parameter case of $\mathbb{R}^n \times \mathbb{R}^n$.

We denote by $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ the n-dimension multi-index, by $S(\mathbb{R}^n \times \mathbb{R}^n)$ the spaces of Schwartz functions on $\mathbb{R}^n \times \mathbb{R}^n$ and by $S^*(\mathbb{R}^n \times \mathbb{R}^n)$ the spaces of all tempered distributions on $\mathbb{R}^n \times \mathbb{R}^n$. The bi-parameter Fourier transform and the Fourier inverse transform of f are defined respectively by

$$\hat{f}(\xi_1, \xi_2) = \int_{\mathbb{R}^{2n}} e^{-2\pi i(x_1\cdot\xi_1 + x_2\cdot\xi_2)} f(x_1, x_2) dx_1 dx_2$$

and

$$f^{\vee}(\xi_1, \xi_2) = \int_{\mathbb{R}^{2n}} e^{-2\pi i(x_1\cdot\xi_1 + x_2\cdot\xi_2)} f(x_1, x_2) dx_1 dx_2$$

for $f \in S(\mathbb{R}^n \times \mathbb{R}^n)$.

Definition (5.3.1) [255] A bi-parameter Fourier multiplier operator is defined as follows

$$T(f)(x_1, x_2) = \int_{\mathbb{R}^{2n}} m(\xi_1, \xi_2) e^{-2\pi i(x_1\cdot\xi_1 + x_2\cdot\xi_2)} \hat{f}(\xi_1, \xi_2) d\xi_1 d\xi_2,$$

where $m(\xi_1, \xi_2) \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$.

We denote by $\psi^{(1)}(x)$ and $\psi^{(2)}(y)$ Schwartz functions whose Fourier transforms are supported in $\{1/2 \le |\xi| \le 2\}$. Moreover, we require that they satisfy the vanishing moment condition

$$\int_{\mathbb{R}^n} x^\alpha \psi^{(1)}(x) dx = \int_{\mathbb{R}^n} y^\alpha \psi^{(2)}(y) dy = 0$$

for all multi-indices α and β. The test function defined on $\mathbb{R}^n \times \mathbb{R}^n$ can be given by

$$\psi(x, y) = \psi^{(1)}(x)\psi^{(2)}(y),$$

where $\psi^{(1)}(x)$ satisfies $\sum_{j\in\mathbb{Z}}\left|\widehat{\psi^{(1)}}(2^{-j}x)\right| = 1$ for all $x \in \{\mathbb{R}^n \backslash (0)\}$ and $\psi^{(2)}(y)$ satisfies $\sum_{k\in\mathbb{Z}}\left|\widehat{\psi^{(2)}}(2^{-k}x)\right| = 1$ for all $y \in \{\mathbb{R}^n \backslash (0)\}$.

Let $S^{\infty}(\mathbb{R}^n \times \mathbb{R}^n)$ denote the space of Schwartz functions whose Fourier transform are supported away from origin and $(S^{\infty}(\mathbb{R}^n \times \mathbb{R}^n))^*$ be its dual. For $f \in (S^{\infty}(\mathbb{R}^n \times \mathbb{R}^n))^*$, the Littlewood–Paley–Stein square function of f is defined by

$$g(f)(x,y) = \left(\sum_{j,k}\left|\psi_{j,k} * f(x,y)\right|^2\right)^{\frac{1}{2}},$$

where the function

$$\psi_{j,k}(x,y) = 2^{(j+k)n}\psi^{(1)}(2^j x)\psi^{(2)}(2^k y).$$

Definition (5.3.2) [255] Let $0 < p, q < \infty, \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$. The bi-parameter Triebel–Lizorkin space $\dot{F}_p^{\alpha,q}(\mathbb{R}^n \times \mathbb{R}^n)$ is defined by

$$\dot{F}_p^{\alpha,q}(\mathbb{R}^n \times \mathbb{R}^n) = \{f \in (S^{\infty}(\mathbb{R}^n \times \mathbb{R}^n))^* : \|f\|_{\dot{F}_p^{\alpha,q}} < \infty\},$$

where

$$\|f\|_{\dot{F}_p^{\alpha,q}} = \left\|\left(\sum_{j,k} 2^{j\alpha_1 q + k\alpha_2 q}\left|\psi_{j,k} * f(\cdot)\right|^q\right)^{\frac{1}{q}}\right\|_{L^p}.$$

Definition (5.3.3) [255] Let $0 < p, q < \infty, \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$. The bi-parameter Besov–Lipschitz space $\dot{B}_p^{\alpha,q}(\mathbb{R}^n \times \mathbb{R}^n)$ is defined by

$$\dot{B}_p^{\alpha,q}(\mathbb{R}^n \times \mathbb{R}^n) = \{f \in (S^{\infty}(\mathbb{R}^n \times \mathbb{R}^n))^* : \|f\|_{\dot{B}_p^{\alpha,q}} < \infty\},$$

where

$$\|f\|_{\dot{B}_p^{\alpha,q}} = \left(\sum_{j,k} 2^{j\alpha_1 q + k\alpha_2 q}\left\|\psi_{j,k} * f(\cdot)\right\|_{L^p}^q\right)^{\frac{1}{q}}.$$

Here we need to emphasize that the definitions of bi-parameter Besov–Lipschitz and Triebel–Lizorkin spaces are independent of $\psi^{(1)}$ and $\psi^{(2)}$. See [128, 250] for the detailed proof. Therefore, we can choose $\psi^{(1)} = \psi^{(2)} = \psi$ in our proofs.

Definition (5.3.4) [255] The strong maximal operator is defined as follows

$$M_s f(x,y) = \sup_{r_1,r_2>0}\frac{1}{(r_1)^n}\frac{1}{(r_2)^n}\int_R |f(u,v)|dudv,$$

where $R = \{(u,v) \in \mathbb{R}^n \times \mathbb{R}^n : |u-x| < r_1, |v-y| < r_2\}$, and f is a locally integrable function on $\mathbb{R}^n \times \mathbb{R}^n$.

Definition (5.3.5) [255] A nonnegative locally integrable function w is said to be in the class $A_p(\mathbb{R}^n \times \mathbb{R}^n), 1 < p < \infty$ if there exists a constant $C > 0$ such that

$$\left(\frac{1}{|R|}\int_R w(x_1,x_2)dx_1dx_2\right)\left(\frac{1}{|R|}\int_R w(x_1,x_2)^{-\frac{1}{p-1}}dx_1dx_2\right)^{p-1} \le C$$

for any rectangle $R = I \times J$.

A nonnegative locally integrable function w is said to be in $A_1(\mathbb{R}^n \times \mathbb{R}^n)$ if there exists a constant $C > 0$ such that

$$M_s w(x_1,x_2) \le C w(x_1,x_2)$$

for almost every $(x_1,x_2) \in \mathbb{R}^n \times \mathbb{R}^n$, where M_s denotes the strong maximal operator.

The class $A_\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is defined by

$$A_\infty(\mathbb{R}^n \times \mathbb{R}^n) = \bigcup_{1\le p<\infty} A_p(\mathbb{R}^n \times \mathbb{R}^n).$$

Definition (5.3.6) [255] Let $0 < p, q < \infty, \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$. The weighted bi-parameter Triebel–Lizorkin space $\dot{F}_p^{\alpha,q}(w, \mathbb{R}^n \times \mathbb{R}^n)$ is defined by

$$\dot{F}_p^{\alpha,q}(w, \mathbb{R}^n \times \mathbb{R}^n) = \{f \in (S^\infty(\mathbb{R}^n \times \mathbb{R}^n))^*: \|f\|_{\dot{F}_p^{\alpha,q}(w)} < \infty\},$$

where

$$\|f\|_{\dot{F}_p^{\alpha,q}(w)} = \left\| \left(\sum_{j,k} 2^{j\alpha_1 q + k\alpha_2 q} \left|\psi_{j,k} * f(\cdot)\right|^q \right)^{\frac{1}{q}} \right\|_{L^p(w)}.$$

Definition (5.3.7) [255] Let $0 < p, q < \infty, \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$. The weighted bi-parameter Besov–Lipschitz space $\dot{B}_p^{\alpha,q}(w, \mathbb{R}^n \times \mathbb{R}^n)$ is defined by

$$\dot{B}_p^{\alpha,q}(w, \mathbb{R}^n \times \mathbb{R}^n) = \{f \in (S^\infty(\mathbb{R}^n \times \mathbb{R}^n))^*: \|f\|_{\dot{B}_p^{\alpha,q}(w)} < \infty\},$$

where

$$\|f\|_{\dot{B}_p^{\alpha,q}(w)} = \left(\sum_{j,k} 2^{j\alpha_1 q + k\alpha_2 q} \left\|\psi_{j,k} * f(\cdot)\right\|_{L^p(w)}^q \right)^{\frac{1}{q}}.$$

Corollary (5.3.8) [255] Let $0 < p < \infty, N = \left[\frac{n}{2} + \frac{n}{min(p,2)}\right] + 1$. Assume that $m(\xi, \eta)$ is a $L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ function that satisfies

$$\left|\partial_\xi^\alpha \partial_\eta^\beta m(\xi, \eta)\right| \le A \frac{1}{|\xi|^{|\alpha|}|\eta|^{|\beta|}}$$

for all $|\alpha| \le N, |\beta| \le N$ and $(\xi, \eta) \in (\mathbb{R}^n \times \mathbb{R}^n)$ with $|\xi||\eta| \ne 0$. Then T is bounded on L^p for $1 < p < \infty$ and on H^p for $0 < p \le 1$.

We need to point out that we offer a different way to deal with the boundedness of Fourier multiplier operators on bi-parameter Triebel–Lizorkin and Besov–Lipschitz spaces (mainly with index $0 < p \le 1$) instead of transforming Fourier multiplier operators into bi-parameter Calderón–Zygmund operators. This way also allows us to avoid using atomic decomposition and Journé's covering lemma. We can obtain the boundedness of the multi-parameter Fourier multiplier operators with limited smoothness assumption on the multiplier $m(\xi, \eta)$. We also remark here that using the multi-parameter Littlewood–Paley theory to establish the $L^p(p > 1)$ boundedness of the multi-parameter and multilinear Fourier multipliers with limited smoothness has been done in [224].

We will establish the boundedness of Fourier multiplier operators on weighted bi-parameter Triebel–Lizorkin and Besov–Lipschitz spaces when the multiplier has only limited smoothness. We only require the bi-parameter weight w to be in $A_\infty(\mathbb{R}^n \times \mathbb{R}^n)$ (see [123] for weighted boundedness for Calderón–Zygmund operators and [234] for Journé type operators and [128] on weighted Triebel–Lizorkin and Besov spaces in product spaces).

Lemma (5.3.9) [255] ([198]). Let $1 < p, q < \infty$. Then there exists a constant $C > 0$ such that

$$\left\| \left\{ \sum_{k\in\mathbb{Z}} \left(M_s(f_k)\right)^q \right\}^{\frac{1}{q}} \right\|_{L^p} \le C \left\| \left\{ \sum_{k\in\mathbb{Z}} |f_k|^q \right\}^{\frac{1}{q}} \right\|_{L^p}$$

for all sequences $\{f_k\}$ of locally integrable functions on $\mathbb{R}^n \times \mathbb{R}^n$.

The following lemma is an extension of one-parameter version due to Peetre [253] (see also [251]).

Lemma (5.3.10) [255] Let $0 < c < \infty$ and $0 < r < \infty$. For any $C^1(\mathbb{R}^n \times \mathbb{R}^n)$ function u whose Fourier transform is supported in $\{|\xi| \le ct_1, |\eta| \le ct_2\}$, assume that $|u(x,y)| \le B(1+|x|)^{\frac{n}{r}}(1+|y|)^{\frac{n}{r}}$ for some $B > 0$. Then there exist two constants C_1 and C_2 such that the following inequality is valid.

$$\sup_{z_1\in\mathbb{R}^n, z_2\in\mathbb{R}^n}\left\{\frac{1}{t_1}\frac{|\nabla_x u(x-z_1,y-z_2)|}{(1+t_1|z_1|)^{\frac{n}{r}}(1+t_2|z_2|)^{\frac{n}{r}}}+\frac{1}{t_2}\frac{|\nabla_y u(x-z_1,y-z_2)|}{(1+t_1|z_1|)^{\frac{n}{r}}(1+t_2|z_2|)^{\frac{n}{r}}}\right\}$$
$$\le C_1 \sup_{z_1\in\mathbb{R}^n, z_2\in\mathbb{R}^n}\frac{|u(x-z_1,y-z_2)|}{(1+t_1|z_1|)^{\frac{n}{r}}(1+t_2|z_2|)^{\frac{n}{r}}} \le C_2 M_s(|u|^r)(x,y)^{\frac{1}{r}},$$

where M_s denotes the strong maximal operator and the constants C_1 and C_2 depend only on n, c and r.

Proof. This proof is divided into two parts.

Part 1. We first show that

$$\sup_{z_1\in\mathbb{R}^n, z_2\in\mathbb{R}^n}\left\{\frac{1}{t_1}\frac{|\nabla_x u(x-z_1,y-z_2)|}{(1+t_1|z_1|)^{\frac{n}{r}}(1+t_2|z_2|)^{\frac{n}{r}}}+\frac{1}{t_2}\frac{|\nabla_y u(x-z_1,y-z_2)|}{(1+t_1|z_1|)^{\frac{n}{r}}(1+t_2|z_2|)^{\frac{n}{r}}}\right\}$$
$$\le C_1 \sup_{z_1\in\mathbb{R}^n, z_2\in\mathbb{R}^n}\frac{|u(x-z_1,y-z_2)|}{(1+t_1|z_1|)^{\frac{n}{r}}(1+t_2|z_2|)^{\frac{n}{r}}}. \tag{114}$$

Select a Schwartz function ψ on $\mathbb{R}^n$ whose Fourier transform is supported in $\{\xi : |\xi| \le 2c\}$ and is equal to 1 on $\{\xi : |\xi| \le c\}$. Then $\hat{\psi}\left(\frac{\xi}{t_1}\right)\hat{\psi}\left(\frac{\eta}{t_2}\right)$ is equal to 1 on the support of the Fourier transform of $u(\xi,\eta)$ and we can write

$$u(x-z_1,y-z_2) = \left(\hat{u}(\cdot,\cdot)\hat{\psi}\left(\frac{\cdot}{t_1}\right)\hat{\psi}\left(\frac{\cdot}{t_2}\right)\right)^{\vee}(x-z_1,y-z_2)$$
$$= \int_{\mathbb{R}^{2n}} t_1^n\psi(t_1(x-z_1-\xi))t_2^n\psi(t_2(y-z_2-\eta))u(\xi,\eta)d\xi d\eta\,.$$

We use the partial derivative to obtain

$$|\nabla_x u(x-z_1,y-z_2)|$$
$$\le \int_{\mathbb{R}^{2n}} t_1^{n+1}\left|(\nabla_x\psi)\big(t_1(x-z_1-\xi)\big)\right|\left|t_2^n\psi\big(t_2(y-z_2-\eta)\big)u(\xi,\eta)\right|d\xi d\eta$$
$$\le C_N\int_{\mathbb{R}^{2n}} t_1^{n+1}(1+t_1|x-z_1-\xi|)^{-N}t_2^n(1+t_2|y-z_2-\eta|)^{-N}|u(\xi,\eta)|d\xi d\eta\,.$$

Therefore

$$\frac{1}{t_1}\frac{|\nabla_x u(x-z_1,y-z_2)|}{(1+t_1|z_1|)^{\frac{n}{r}}(1+t_2|z_2|)^{\frac{n}{r}}}$$
$$\lesssim \int_{\mathbb{R}^{2n}} t_1^n t_2^n(1+t_1|x-z_1-\xi|)^{-N}(1+t_2|y-z_2-\eta|)^{-N}\times\frac{|u(\xi,\eta)|}{(1+t_1|z_1|)^{\frac{n}{r}}(1+t_2|z_2|)^{\frac{n}{r}}}d\xi d\eta\,,$$

where N is an arbitrarily large positive integer and C_N is a constant which depends only on N.

Since

$$1 \le (1+t_1|x-z_1-\xi|)^{\frac{n}{r}}\frac{(1+t_1|z_1|)^{\frac{n}{r}}}{(1+t_1|x-\xi|)^{\frac{n}{r}}}$$

and

$$1 \le (1+t_2|y-z_2-\eta|)^{\frac{n}{r}} \frac{(1+t_2|z_2|)^{\frac{n}{r}}}{(1+t_2|y-\eta|)^{\frac{n}{r}}},$$

we obtain that

$$\begin{aligned}
&\frac{1}{t_1} \frac{|\nabla_x u(x-z_1, y-z_2)|}{(1+t_1|z_1|)^{\frac{n}{r}}(1+t_2|z_2|)^{\frac{n}{r}}} \\
&\qquad \le C_N \int_{\mathbb{R}^{2n}} \frac{|u(\xi,\eta)|}{(1+t_1|x-\xi|)^{\frac{n}{r}}(1+t_2|y-\eta|)^{\frac{n}{r}}} \\
&\qquad \times \frac{t_1^n}{(1+t_1|x-z_1-\xi|)^{-N-\frac{n}{r}}} \frac{t_2^n}{(1+t_2|y-z_2-\eta|)^{-N-\frac{n}{r}}} d\xi d\eta \\
&\qquad \le C_1 \sup_{\xi\in\mathbb{R}^n, \eta\in\mathbb{R}^n} \frac{|u(x-\xi, y-\eta)|}{(1+t_1|\xi|)^{\frac{n}{r}}(1+t_2|\eta|)^{\frac{n}{r}}}. \qquad (115)
\end{aligned}$$

Similarly, we also obtain that

$$\frac{1}{t_2} \frac{|\nabla_y u(x-z_1, y-z_2)|}{(1+t_1|z_1|)^{\frac{n}{r}}(1+t_2|z_2|)^{\frac{n}{r}}} \le C_1 \sup_{\xi\in\mathbb{R}^n, \eta\in\mathbb{R}^n} \frac{|u(x-\xi, y-\eta)|}{(1+t_1|\xi|)^{\frac{n}{r}}(1+t_2|\eta|)^{\frac{n}{r}}}. \qquad (116)$$

Combining (115) and (116), we conclude the proof of Part 1.

Part 2. We show that

$$\sup_{z_1\in\mathbb{R}^n, z_2\in\mathbb{R}^n} \frac{|u(x-z_1, y-z_2)|}{(1+t_1|z_1|)^{\frac{n}{r}}(1+t_2|z_2|)^{\frac{n}{r}}} \le C M_s(|u|^r)(x,y)^{\frac{1}{r}}. \qquad (117)$$

Let $|\xi| \le \delta_1$ and $|\eta| \le \delta_2$ for some $\delta_1, \delta_2 > 0$ to be chosen later. We use the mean value theorem to obtain the estimate

$$\begin{aligned}
u(x-z_1, y-z_2) - u(x-z_1-\xi, y-z_2-\eta) &= (\nabla_x u)(x-z_1-\theta\xi, y-z_2-\theta\eta)\cdot\xi \\
&+ (\nabla_y u)(x-z_1-\theta\xi, y-z_2-\theta\eta)\cdot\xi \qquad (0<\theta<1)
\end{aligned}$$

for all $z_1, z_2 \in \mathbb{R}^n$. Therefore

$$\begin{aligned}
&|u(x-z_1, y-z_2)| \\
&\qquad \le \sup_{\substack{|w_1|\le|z_1|+\delta_1 \\ |w_2|\le|z_2|+\delta_2}} \left(|(\nabla_x u)(x-w_1, y-w_2)|\delta_1 + \left|(\nabla_y u)(x-w_1, y-w_2)\right|\delta_2\right) \\
&\qquad + |u(x-z_1-\xi, y-z_2-\eta)|,
\end{aligned}$$

where $w_1 = z_1 + \theta\xi$ and $w_2 = z_2 + \theta\eta$. Let Q_1 be a ball with radius equal to δ_1 and z_1 being the center, and Q_2 be a ball with radius equal to δ_2 and z_2 being the center. By raising to the power of r, averaging over $Q_1 \times Q_2$ and raising to the power $\frac{1}{r}$, we derive

$$\begin{aligned}
&|u(x-z_1, y-z_2)| \\
&\qquad \le C_r^2 \sup_{\substack{|w_1|\le|z_1|+\delta_1 \\ |w_2|\le|z_2|+\delta_2}} |(\nabla_x u)(x-w_1, y-w_2)|\delta_1 + \left|(\nabla_y u)(x-w_1, y-w_2)\right|\delta_2 \\
&\qquad + \left(\frac{1}{v_n^2 \delta_1^n \delta_2^n} \int_{Q_1}\int_{Q_2} |u(x-z_1-\xi, y-z_2-\eta)|^r \, d\xi d\eta\right)^{\frac{1}{r}},
\end{aligned}$$

where $C_r = \max\{2^r, 2^{\frac{1}{r}}\}$ and v_n is the volume of the unit ball in $\mathbb{R}^n$.

Set $\delta_1 = \frac{\varepsilon}{t_1}$ and $\delta_2 = \frac{\varepsilon}{t_2}$ for some $\varepsilon \le 1$. Then we have $|w_1| \le |z_1| + \frac{\varepsilon}{t_1}, |w_2| \le |z_2| + \frac{\varepsilon}{t_2}, \frac{1}{1+t_1|z_1|} \le \frac{2}{1+t_1|w_1|}$ and $\frac{1}{1+t_2|z_2|} \le \frac{2}{1+t_2|w_2|}$. Therefore we can write

$$\frac{|u(x-z_1,y-z_2)|}{(1+t_1|z_1|)^{\frac{n}{r}}(1+t_2|z_2|)^{\frac{n}{r}}}$$
$$\leq C_{r,n}\left(\sup_{|w_i|\leq|z_i|+\frac{\varepsilon}{t_i}}\frac{|\nabla_x u(x-w_1,y-w_2)|\varepsilon}{t_1(1+t_1|w_1|)^{\frac{n}{r}}(1+t_2|w_2|)^{\frac{n}{r}}}+\frac{|\nabla_y u(x-w_1,y-w_2)|\varepsilon}{t_2(1+t_1|w_1|)^{\frac{n}{r}}(1+t_2|w_2|)^{\frac{n}{r}}}\right.$$
$$\left.+\frac{\left(\frac{t_1^n t_2^n}{v_n^2\varepsilon^{2n}}\int_{|\xi|\leq\frac{1}{t_1}+|z_1|}\int_{|\eta|\leq\frac{1}{t_2}+|z_2|}|u(x-z_1-\xi,y-z_2-\eta)|^r\,d\xi d\eta\right)^{\frac{1}{r}}}{(1+t_1|z_1|)^{\frac{n}{r}}(1+t_2|z_2|)^{\frac{n}{r}}}\right)$$
$$\leq C_{r,n}\left(\sup_{w_i\in\mathbb{R}^n}\frac{|\nabla_x u(x-w_1,y-w_2)|\varepsilon}{t_1(1+t_1|w_1|)^{\frac{n}{r}}(1+t_2|w_2|)^{\frac{n}{r}}}+\frac{|\nabla_y u(x-w_1,y-w_2)|\varepsilon}{t_2(1+t_1|w_1|)^{\frac{n}{r}}(1+t_2|w_2|)^{\frac{n}{r}}}+\varepsilon^{-\frac{2n}{r}}M_s(|u|^r)(x,y)^{\frac{1}{r}}\right).$$

Using (114) and setting $\varepsilon=\frac{1}{2C_{r,n}C_1}\leq 1$ with $C_1=\ 2\varepsilon^{-\frac{n}{r}}$, we can achieve (117), where we used the hypothesis

$$\sup_{\substack{|z_1|\in\mathbb{R}^n\\|z_2|\in\mathbb{R}^n}}\frac{|u(x-z_1,y-z_2)|}{(1+t_1|z_1|)^{\frac{n}{r}}(1+t_2|z_2|)^{\frac{n}{r}}}\leq\sup_{\substack{|z_1|\in\mathbb{R}^n\\|z_2|\in\mathbb{R}^n}}\frac{B(1+|x|+|z_1|)^{\frac{n}{r}}(1+|y|+|z_2|)^{\frac{n}{r}}}{(1+t_1|z_1|)^{\frac{n}{r}}(1+t_2|z_2|)^{\frac{n}{r}}}<\ \infty.$$

Combining (114) and (117), we conclude the proof of Lemma (5.3.10).

We are now ready to state our first main theorem.

Theorem (5.3.11) [255] Suppose that T is a Fourier multiplier operator defined on $\mathbb{R}^n\times\mathbb{R}^n$. Let $\rho=(\rho_1,\rho_2)\in\mathbb{R}^2, 0<p,q<\infty, N=\left[\frac{n}{2}+\frac{n}{min(p,q)}\right]+1$. Assume that $m(\xi,\eta)$ is a $L^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ function that satisfies

$$\left|\partial_\xi^\alpha\partial_\eta^\beta m(\xi,\eta)\right|\leq\ A\frac{1}{|\xi|^{|\alpha|}|\eta|^{|\beta|}}$$

for all $|\alpha|\leq N, |\beta|\leq N$ and $(\xi,\eta)\in(\mathbb{R}^n\times\mathbb{R}^n)$ with $|\xi||\eta|\neq 0$. Then there exists a constant C such that

$$\|T(f)\|_{\dot{B}_p^{\rho,q}}\leq C\|f\|_{\dot{B}_p^{\rho,q}}$$

and

$$\|T(f)\|_{\dot{F}_p^{\rho,q}}\leq C\|f\|_{\dot{F}_p^{\rho,q}},$$

where the constant C is independent of f.

If $\rho=(0,0)$ and $q=2$, then $\dot{F}_p^{0,2}(\mathbb{R}^n\times\mathbb{R}^n)=L^p(\mathbb{R}^n\times\mathbb{R}^n)$ when $1<p<\infty$ and $\dot{F}_p^{0,2}(\mathbb{R}^n\times\mathbb{R}^n)=H^p(\mathbb{R}^n\times\mathbb{R}^n)$ when $0<p<1$. Then we can easily obtain the boundedness of Fourier multiplier operators on $L^p(\mathbb{R}^n\times\mathbb{R}^n)$ and $H^p(\mathbb{R}^n\times\mathbb{R}^n)$ from Theorem (5.3.11).

Proof. The proof is divided into two parts.

Part 1. We prove that

$$\|T(f)\|_{\dot{B}_p^{\rho,q}}\leq C\|f\|_{\dot{B}_p^{\rho,q}}. \tag{118}$$

Let $\psi(\xi)$ be a Schwartz function whose Fourier transform is supported in $\left\{\frac{1}{2}\leq|\xi|\leq 2\right\}$ and satisfies $\sum_{k\in\mathbb{Z}}\left|\hat{\psi}(2^{-k}\xi)\right|=1$ for all $\xi\in\{\mathbb{R}^n(0)\}$. Denote $\psi_{j,k}(\xi,\eta)=\psi_{2^{-j}}\psi_{2^{-k}}(\eta)$, $\Delta_{j,k}(f)=\psi_{j,k}*f$ and $m_{j,k}(\xi,\eta)=\hat{\psi}(2^{-j}\xi)\hat{\psi}(2^{-k}\eta)m(\xi,\eta)$. It is easy to see that $m=\sum_{j,k}m_{j,k}$. By the orthogonality estimate, we have

$$\Delta_{j,k}(T(f))(x,y)=\psi_{j,k}*T(f)(x,y)=\psi_{j,k}*(m\hat{f})^{\vee}(x,y)$$
$$=\left(\hat{\psi}(2^{-j}\xi)\hat{\psi}(2^{-k}\eta)\sum_{j',k'}\hat{\psi}\big(2^{-j'}\xi\big)\hat{\psi}\big(2^{-k'}\eta\big)m(\xi,\eta)\hat{f}(\xi,\eta)\right)^{\vee}(x,y)$$
$$=\left(\hat{\psi}(2^{-j}\xi)\hat{\psi}(2^{-k}\eta)\sum_{i_1,i_2}\hat{\psi}\big(2^{-(j-i_1)}\xi\big)\hat{\psi}\big(2^{-(k-i_2)}\eta\big)m(\xi,\eta)\hat{f}(\xi,\eta)\right)^{\vee}(x,y)$$
$$=\left(\hat{\psi}(2^{-j}\xi)\hat{\psi}(2^{-k}\eta)\sum_{i_1,i_2}m_{j-i_1,k-i_2}(\xi,\eta)\hat{f}(\xi,\eta)\right)^{\vee}(x,y),$$
where ($i_1,i_2\in\{-1,0,1\}$). We only estimate that
$$\big(\hat{\psi}(2^{-j}\xi)\hat{\psi}(2^{-k}\xi)m_{j,k}(\xi,\eta)\hat{f}(\xi,\eta)\big)^{\vee}(x,y)\lesssim\big(M_s(|\Delta_{j,k}(f)|)^r(x,y)\big)^{\frac{1}{r}}.$$
Pick $r<\min(p\cdot q)$ such that $N>\frac{n}{2}+\frac{n}{r}$. By Lemma (5.3.10), we have
$$\left|\Big(\hat{\psi}(2^{-j}\xi)\hat{\psi}(2^{-k}\xi)m_{j,k}(\xi,\eta)\hat{f}(\xi,\eta)\Big)^{\vee}(x,y)\right|=\left|\int_{\mathbb{R}^{2n}}m_{j,k}^{\vee}(\xi,\eta)\Delta_{j,k}(f)(x-\xi,y-\eta)d\xi d\eta\right|$$
$$\le\sup_{z_1,z_2\in\mathbb{R}^n}\frac{|\Delta_{j,k}(f)(x-z_1,y-z_2)|}{(1+2^j|z_1|)^{\frac{n}{r}}(1+2^k|z_2|)^{\frac{n}{r}}}\int_{\mathbb{R}^{2n}}|m_{j,k}^{\vee}(\xi,\eta)|(1+2^j|\xi|)^{\frac{n}{r}}(1+2^k|\eta|)^{\frac{n}{r}}d\xi d\eta$$
$$\le C_2M_s\big(|\Delta_{j,k}(f)|^r(x,y)\big)^{\frac{1}{r}}\int_{\mathbb{R}^{2n}}|m_{j,k}^{\vee}(\xi,\eta)|(1+2^j|\xi|)^{\frac{n}{r}}(1+2^k|\eta|)^{\frac{n}{r}}d\xi d\eta\,.$$
Next it suffices to estimate
$$\int_{\mathbb{R}^{2n}}|m_{j,k}^{\vee}(\xi,\eta)|\big(1+2^j|\xi|\big)^{\frac{n}{r}}(1+2^k|\eta|)^{\frac{n}{r}}d\xi d\eta.$$
We can use the Hölder's inequality to write
$$\int_{\mathbb{R}^{2n}}|m_{j,k}^{\vee}(\xi,\eta)|(1+2^j|\xi|)^N(1+2^k|\eta|)^N\frac{1}{(1+2^j|\xi|)^{N-\frac{n}{r}}}\frac{1}{(1+2^k|\eta|)^{N-\frac{n}{r}}}d\xi d\eta$$
$$\le\left\|m_{j,k}^{\vee}(\xi,\eta)(1+2^j|\xi|)^N(1+2^k|\eta|)^N\right\|_{L^2}\left\|\frac{1}{(1+2^j|\xi|)^{N-\frac{n}{r}}}\frac{1}{(1+2^k|\eta|)^{N-\frac{n}{r}}}\right\|_{L^2}.$$
It is easy to see that
$$\left\|\frac{1}{(1+2^j|\xi|)^{N-\frac{n}{r}}}\frac{1}{(1+2^k|\eta|)^{N-\frac{n}{r}}}\right\|_{L^2}=C2^{-\frac{jn}{2}}2^{-\frac{kn}{2}}.$$
We only estimate $\left\|m_{j,k}^{\vee}(\xi,\eta)\big(1+2^j|\xi|\big)^N(1+2^k|\eta|)^N\right\|_{L^2}$.

We use the obvious fact that
$$\big(1+2^j|\xi|\big)^N\le C(n)\sum_{|\alpha|\le N}\left|\big(2^j\xi\big)^{\alpha}\right|\ \text{ and }\ (1+2^k|\eta|)^N\le C(n)\sum_{|\beta|\le N}|(2^k\eta)^{\beta}|$$
to obtain
$$\left\|m_{j,k}^{\vee}(\xi,\eta)\big(1+2^j|\xi|\big)^N(1+2^k|\eta|)^N\right\|_{L^2}=\left(\int_{\mathbb{R}^{2n}}|m_{j,k}^{\vee}(\xi,\eta)|^2\big(1+2^j|\xi|\big)^{2N}(1+2^k|\eta|)^{2N}d\xi d\eta\right)^{\frac{1}{2}}$$
$$\le C\sum_{|\alpha|\le N,|\beta|\le N}\left(\int_{\mathbb{R}^{2n}}|m_{j,k}^{\vee}(\xi,\eta)|^2|\xi^{\alpha}|^2|\eta^{\beta}|^2 2^{2j|\alpha|}2^{2k|\beta|}d\xi d\eta\right)^{\frac{1}{2}}$$

$$= C \sum_{|\alpha|\le N, |\beta|\le N} \left(\int_{\mathbb{R}^{2n}} \left| \left(\partial_\xi^\alpha \partial_\eta^\beta m_{j,k} \right)^\vee (\xi,\eta) \right|^2 2^{2j|\alpha|} 2^{2k|\beta|} d\xi d\eta \right)^{\frac{1}{2}}$$

$$= C \sum_{|\alpha|\le N, |\beta|\le N} 2^{j|\alpha|} 2^{k|\beta|} \left(\int_{\mathbb{R}^{2n}} \left| \left(\partial_\xi^\alpha \partial_\eta^\beta m_{j,k} \right)^\vee (\xi,\eta) \right|^2 d\xi d\eta \right)^{\frac{1}{2}}$$

$$= C \sum_{|\alpha|\le N, |\beta|\le N} 2^{j|\alpha|} 2^{k|\beta|} \left(\int_{\mathbb{R}^{2n}} \left| \partial_\xi^\alpha \partial_\eta^\beta (m_{j,k})(\xi,\eta) \right|^2 d\xi d\eta \right)^{\frac{1}{2}}.$$

For multi-indices $\delta = (\delta_1, \delta_2, \ldots, \delta_n)$ and $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$, we denote $\delta \le \alpha$ to mean $\delta_i \le \alpha_i$ and $\gamma \le \beta$ to mean $\gamma_i \le \beta_i$ for all $i = 1, 2, \ldots, n$. Considering that $\hat{\psi}(\xi)$ is supported on $\{\xi |\ \frac{1}{2} \le |\xi| \le 2\}$, we can use Leibniz's rule to obtain

$$\int_{\mathbb{R}^{2n}} \left| \partial_\xi^\alpha \partial_\eta^\beta (m_{j,k})(\xi,\eta) \right|^2 d\xi d\eta$$

$$\le \sum_{\substack{\delta\le\alpha \\ \gamma\le\beta}} C_{\delta,\gamma} \int_{\mathbb{R}^{2n}} \left| 2^{-j|\alpha-\delta|-k|\beta-\gamma|} \left(\partial_\xi^{\alpha-\delta} (\hat{\psi}) \right) (2^{-j}\xi) \left(\partial_\eta^{\beta-\gamma} (\hat{\psi}) \right) (2^{-k}\eta) \partial_\xi^\delta \partial_\beta^\gamma m(\xi,\eta) \right|^2 d\xi d\eta$$

$$\le \sum_{\substack{\delta\le\alpha \\ \gamma\le\beta}} C_{\delta,\gamma} 2^{-2j|\alpha|} 2^{2j|\delta|} 2^{-2k|\beta|} 2^{2k|\gamma|} \int_{\substack{2^{j-1}\le|\xi|\le 2^{j+1} \\ 2^{k-1}\le|\eta|\le 2^{k+1}}} \left| (\partial_\xi^\delta \partial_\beta^\gamma m(\xi,\eta) \right|^2 d\xi d\eta$$

$$\le \sum_{\substack{\delta\le\alpha \\ \gamma\le\beta}} C_{\delta,\gamma} 2^{-2j|\alpha|} 2^{2j|\delta|} 2^{-2k|\beta|} 2^{2k|\gamma|} 2^{jn} 2^{kn} 2^{-2j|\delta|} 2^{-2k|\gamma|}.$$

Then

$$\left\| m_{j,k}^\vee (\xi,\eta)(1+2^j|\xi|)^N (1+2^k|\eta|)^N \right\|_{L^2} \left\| \frac{1}{(1+2^j|\xi|)^{N-\frac{n}{r}}} \frac{1}{(1+2^k|\eta|)^{N-\frac{n}{r}}} \right\|_{L^2}$$

$$= C 2^{-\frac{jn}{2}} 2^{-\frac{kn}{2}} \sum_{|\alpha|\le N, |\beta|\le N} C_{\alpha,\beta} 2^{j|\alpha|} 2^{2k|\beta|} \left(\int_{\mathbb{R}^{2n}} \left| (\partial_\xi^\delta \partial_\beta^\gamma (m_{j,k})(\xi,\eta) \right|^2 d\xi d\eta \right)^{\frac{1}{2}}$$

$$\le C 2^{-\frac{jn}{2}} 2^{-\frac{kn}{2}} \sum_{|\alpha|\le N, |\beta|\le N} C_{\alpha,\beta} 2^{j|\alpha|} 2^{2k|\beta|} \left(\sum_{\delta\le\alpha, \gamma\le\beta} C_{\delta,\gamma} 2^{jn} 2^{kn} 2^{-2j|\alpha|} 2^{-2k|\beta|} \right)^{\frac{1}{2}} < \infty,$$

where C depends only on n, N, A, r. Therefore

$$\psi_{j,k} * (m_{j,k}\hat{f})^\vee \lesssim \left(M_s \left(|\Delta_{j,k}(f)| \right)^r (x,y) \right)^{\frac{1}{r}}.$$

Similarly we obtain

$$\psi_{j,k} * \left(m_{j-i_1,k-i_2} \hat{f} \right)^\vee \lesssim \left(M_s \left(|\Delta_{j,k}(f)| \right)^r (x,y) \right)^{\frac{1}{r}} \ (i \in \{1, 0, -1\}).$$

Then

$$\psi_{j,k} * (m\hat{f})^\vee \lesssim \left(M_s \left(|\Delta_{j,k}(f)| \right)^r (x,y) \right)^{\frac{1}{r}}.$$

Therefore we conclude that

$$\|T(f)\|_{\dot{B}_p^{\rho,q}} = C \left\| (m\hat{f})^\vee \right\|_{\dot{B}_p^{\rho,q}} \le C \left(\sum_{j,k} \left(2^{j\rho_1 + k\rho_2} \left\| \left(M_s \left(|\Delta_{j,k}(f)|^r \right) \right)^{\frac{1}{r}} \right\|_{L^p} \right)^q \right)^{\frac{1}{q}}$$

$$\leq C\left(\sum_{j,k}\left(2^{j\rho_1+k\rho_2}\left\|\left(\left|\Delta_{j,k}(f)\right|^r\right)\right\|_{L^{\frac{p}{r}}}^{\frac{1}{r}}\right)^q\right)^{\frac{1}{q}} = \left(\sum_{j,k}\left(2^{j\rho_1+k\rho_2}\left\|\Delta_{j,k}(f)\right\|_{L^p}\right)^q\right)^{\frac{1}{q}} = C\|f\|_{\dot{B}_p^{\rho,q}}$$

for all $f \in S^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Since $S^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ are dense in $\dot{B}_p^{\rho,q}$, the operator T can be extended to a bounded operator on $\dot{B}_p^{\rho,q}$. Then the proof of Part 1 is accomplished.

Part 2. We show that

$$\|T(f)\|_{\dot{F}_p^{\rho,q}} \leq C\|f\|_{\dot{F}_p^{\rho,q}}. \tag{119}$$

In Part 1, we have obtained

$$\psi_{j,k} * (m\hat{f})^\vee \lesssim \left(M_s\left(\left|\Delta_{j,k}(f)\right|\right)^r\right)^{\frac{1}{r}}$$

for all $f \in S^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Pick $r < \min\{p, q\}$ such that $\frac{p}{r}, \frac{q}{r} > 1$. According to Fefferman–Stein vector-valued inequality in the bi-parameter setting, we obtain

$$\|T(f)\|_{\dot{F}_p^{\rho,q}} = C\left\|\left(m\hat{f}\right)^\vee\right\|_{\dot{F}_p^{\rho,q}} \leq C\left\|\left(\sum_{j,k}\left(2^{j\rho_1+k\rho_2}\left(M_s\left(\left|\Delta_{j,k}(f)\right|^r\right)\right)^{\frac{1}{r}}\right)^q\right)^{\frac{1}{q}}\right\|_{L^p}$$

$$\leq C\left\|\left(\sum_{j,k}\left(2^{j\rho_1+k\rho_2}\left|\Delta_{j,k}(f)\right|^r\right)^{\frac{q}{r}}\right)^{\frac{r}{q}}\right\|_{L^{\frac{p}{r}}}^{\frac{1}{r}} = C\left\|\left(\sum_{j,k}\left(2^{j\rho_1+k\rho_2}\left|\Delta_{j,k}(f)\right|\right)^q\right)^{\frac{1}{q}}\right\|_{L^p} = C\|f\|_{\dot{F}_p^{\rho,q}},$$

which concludes the proof of Theorem (5.3.11).

We prove the bi-parameter Fourier multiplier operators are bounded on weighted bi-parameter Besov–Lipschitz and Triebel–Lizorkin spaces. First we state a well-known vector-valued weighted maximal inequality.

Lemma (5.3.12) [255] Assume $1 < p, q < \infty$ and $w \in A_p(\mathbb{R}^n \times \mathbb{R}^n)$. Then there exists a constant $C > 0$ such that

$$\left\|\left\{\sum_{k\in\mathbb{Z}}\left(M_s(f_k)\right)^q\right\}^{\frac{1}{q}}\right\|_{L^p(w)} \leq C\left\|\left\{\sum_{k\in\mathbb{Z}}|f_k|^q\right\}^{\frac{1}{q}}\right\|_{L^p(w)}$$

for all sequences $\{f_k\}$ of locally integrable functions on $\mathbb{R}^n \times \mathbb{R}^n$.

Theorem (5.3.13) [255] Suppose that T is a bi-parameter Fourier multiplier operator defined on $(\mathbb{R}^n \times \mathbb{R}^n)$. Let $\rho = (\rho_1, \rho_2) \in \mathbb{R}^2, 0 < p, q < \infty, N = \left[\frac{n}{2} + \frac{n}{min(p,q)}\right] + 1$. Assume that $m(\xi, \eta)$ is a $L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ function that satisfies

$$\left|\partial_\xi^\alpha \partial_\eta^\beta m(\xi, \eta)\right| \leq A\frac{1}{|\xi|^{|\alpha|}|\eta|^{|\beta|}}$$

for all $|\alpha| \leq N, |\beta| \leq N$ and $(\xi, \eta) \in (\mathbb{R}^n \times \mathbb{R}^n)$ with $|\xi||\eta| \neq 0$. If $w \in A_\infty(\mathbb{R}^n \times \mathbb{R}^n)$, then the following inequalities are valid with the constant C independent of f :

(i) $\|T(f)\|_{\dot{B}_p^{\rho,q}(w)} \leq C\|f\|_{\dot{B}_p^{\rho,q}(w)}$,

(ii) $\|T(f)\|_{\dot{F}_p^{\rho,q}(w)} \leq C\|f\|_{\dot{F}_p^{\rho,q}(w)}$.

We extend the above theorems to the cases where the multipliers $m(\xi, \eta)$ have weaker decay condition and worse singularity at the origin.

Proof. In the proof of Theorem (5.3.11), we have obtained the pointwise estimate

$$\left|\Delta_{j,k}(T(f))\right| \lesssim \left(M_s\left(\left|\Delta_{j,k}(f)\right|^r\right)\right)^{\frac{1}{r}}.$$

Pick r such that $1 < \frac{p}{r}, \frac{q}{r} < \infty$ and $w \in A_{\frac{p}{r}}$, we have

$$\left\|\Delta_{j,k}\left((m\hat{f})^{\vee}\right)\right\|_{L^p(w)} \le C \left\|\left(M_s\left(\left|\Delta_{j,k}(f)\right|^r\right)\right)^{\frac{1}{r}}\right\|_{L^p(w)} \le C\left\|\Delta_{j,k}(f)\right\|_{L^p(w)}. \qquad (120)$$

By Lemma (5.3.12) and (120), we can derive that

$$\|T(f)\|_{\dot{B}_p^{\rho,q}(w)} = C\left\|\left(m\hat{f}\right)^{\vee}\right\|_{\dot{B}_p^{\rho,q}(w)} = \left(\sum_{j,k}\left(2^{j\alpha_1+k\alpha_2}\left\|\Delta_{j,k}\left(\left(m\hat{f}\right)^{\vee}\right)\right\|_{L^p(w)}\right)^q\right)^{\frac{1}{q}}$$

$$\le C\left(\sum_{j,k}\left(2^{j\alpha_1+k\alpha_2}\left\|\left(M_s\left(\left|\Delta_{j,k}(f)\right|^r\right)\right)^{\frac{1}{r}}\right\|_{L^p(w)}\right)^q\right)^{\frac{1}{q}}$$

$$\le \left(\sum_{j,k}\left(2^{j\alpha_1+k\alpha_2}\left\|\Delta_{j,k}(f)^r\right\|_{L^{\frac{p}{r}}(w)}^{\frac{1}{r}}\right)^q\right)^{\frac{1}{q}} = C\|f\|_{\dot{B}_p^{\rho,q}(w)}$$

and

$$\|T(f)\|_{\dot{F}_p^{\rho,q}(w)} = C\left\|\left(m\hat{f}\right)^{\vee}\right\|_{\dot{F}_p^{\rho,q}(w)} = \left\|\left(\sum_{j,k}\left(2^{j\alpha_1+k\alpha_2}\left|\Delta_{j,k}\left(\left(m\hat{f}\right)^{\vee}\right)\right|\right)^q\right)^{\frac{1}{q}}\right\|_{L^p(w)}$$

$$\le C\left\|\left(\sum_{j,k}\left(2^{j\alpha_1+k\alpha_2}\left(M_s\left(\left|\Delta_{j,k}(f)\right|^r\right)\right)^{\frac{1}{r}}\right)^q\right)^{\frac{1}{q}}\right\|_{L^p(w)}$$

$$\le C\left\|\left(\sum_{j,k}\left(2^{(j\alpha_1+k\alpha_2)r}\left|\Delta_{j,k}(f)\right|^r\right)^{\frac{q}{r}}\right)^{\frac{r}{q}}\right\|_{L^{\frac{p}{r}}(w)}^{\frac{1}{r}} = C\|f\|_{\dot{F}_p^{\rho,q}(w)},$$

which concludes the proof of Theorem (5.3.13).

We have proved Theorem (5.3.11) and Theorem (5.3.13). Now we generalize the results of Theorems (5.3.11) and (5.3.13) to more general cases where the multipliers have the worse singularity at the origin.

Theorem (5.3.14) [255] Let $\rho = (\rho_1, \rho_2) \in \mathbb{R}^2, m = (m_1, m_2) \in \mathbb{R}^+ \times \mathbb{R}^+, 0 < p, q < \infty$ and $N = \left[\frac{n}{2} + \frac{n}{min(p,q)}\right] + 1$. Assume that $m(\xi, \eta)$ satisfies

$$\left|\partial_\xi^\alpha \partial_\eta^\beta m(\xi,\eta)\right| \le A\frac{1}{|\xi|^{|\alpha|+m_1}|\eta|^{|\beta|+m_2}}$$

for all $|\alpha| \le N, |\beta| \le N$ and $(\xi, \eta) \in (\mathbb{R}^n \times \mathbb{R}^n)$ with $|\xi||\eta| \ne 0$. If $w \in A_\infty(\mathbb{R}^n \times \mathbb{R}^n)$, Then there exists a constant C such that

$$\|T(f)\|_{\dot{B}_p^{\rho+m,q}} \le C\|f\|_{\dot{B}_p^{\rho+m,q}}$$

and

$$\|T(f)\|_{\dot{F}_p^{\rho+m,q}} \le C\|f\|_{\dot{F}_p^{\rho+m,q}},$$

where the constant C is independent of f.

Proof. By the Littlewood–Paley decomposition, we can derive that

$$\Delta_{j,k}(T(f)) = \psi_{j,k} * T(f) = \left(\hat{\psi}(2^{-j}\xi)\hat{\psi}(2^{-k}\eta)\sum_{i_1,i_2} m_{j-i_1,k-i_2}(\xi,\eta)\hat{f}(\xi,\eta)\right)^{\vee}(x,y),$$

where $(i_1, i_2 \in \{-1, 0, 1\})$. We only verify that

$$\left(\hat{\psi}(2^{-j}\xi)\hat{\psi}(2^{-k}\xi)m_{j,k}(\xi,\eta)\hat{f}(\xi,\eta)\right)^{\vee}(x,y) \lesssim 2^{-jm_1-km_2}\left(M_s(|\Delta_{j,k}(f)|)^r(x,y)\right)^{\frac{1}{r}}.$$

Pick $r < \min\{p.q\}$ such that $N > \frac{n}{2} + \frac{n}{r}$. We write

$$\left|\left(\hat{\psi}(2^{-j}\xi)\hat{\psi}(2^{-k}\xi)m_{j,k}(\xi,\eta)\hat{f}(\xi,\eta)\right)^{\vee}(x,y)\right| = \left|\int_{\mathbb{R}^{2n}} m_{j,k}^{\vee}(\xi,\eta)\Delta_{j,k}(f)(x-\xi,y-\eta)d\xi d\eta\right|$$

$$\leq C_2 M_s\left(\left|\Delta_{j,k}(f)\right|^r(x,y)\right)^{\frac{1}{r}}\int_{\mathbb{R}^{2n}}\left|m_{j,k}^{\vee}(\xi,\eta)\right|\left(1+2^j|\xi|\right)^{\frac{n}{r}}(1+2^k|\eta|)^{\frac{n}{r}}d\xi d\eta. \quad (121)$$

Next we estimate

$$\int_{\mathbb{R}^{2n}}\left|m_{j,k}^{\vee}(\xi,\eta)\right|\left(1+2^j|\xi|\right)^{\frac{n}{r}}(1+2^k|\eta|)^{\frac{n}{r}}d\xi d\eta.$$

We use the Hölder's inequality to obtain

$$\int_{\mathbb{R}^{2n}}\left|m_{j,k}^{\vee}(\xi,\eta)\right|\left(1+2^j|\xi|\right)^{N}(1+2^k|\eta|)^{N}\frac{1}{(1+2^j|\xi|)^{N-\frac{n}{r}}}\frac{1}{(1+2^k|\eta|)^{N-\frac{n}{r}}}d\xi d\eta$$

$$\leq C2^{-\frac{jn}{2}}2^{-\frac{kn}{2}}\left\|m_{j,k}^{\vee}(\xi,\eta)\left(1+2^j|\xi|\right)^{N}(1+2^k|\eta|)^{N}\right\|_{L^2}.$$

It suffices to estimate $\left\|m_{j,k}^{\vee}(\xi,\eta)\left(1+2^j|\xi|\right)^{N}(1+2^k|\eta|)^{N}\right\|_{L^2}$.

$$\left\|m_{j,k}^{\vee}(\xi,\eta)\left(1+2^j|\xi|\right)^{N}(1+2^k|\eta|)^{N}\right\|_{L^2}$$

$$\leq C\sum_{|\alpha|\leq N,|\beta|\leq N}\left(\int_{\mathbb{R}^{2n}}\left|m_{j,k}^{\vee}(\xi,\eta)\right|^2|\xi^{\alpha}|^2\left|\eta^{\beta}\right|^2 2^{2j|\alpha|}2^{2k|\beta|}d\xi d\eta\right)^{\frac{1}{2}}$$

$$= C\sum_{|\alpha|\leq N,|\beta|\leq N}C_{\alpha,\beta}2^{j|\alpha|}2^{k|\beta|}\left(\int_{\mathbb{R}^{2n}}\left|(\partial_{\xi}^{\alpha}\partial_{\eta}^{\beta}(m_{j,k})(\xi,\eta)\right|^2 d\xi d\eta\right)^{\frac{1}{2}}. \quad (122)$$

We use Leibniz's rule to obtain

$$\int_{\mathbb{R}^{2n}}\left|\partial_{\xi}^{\alpha}\partial_{\eta}^{\beta}(m_{j,k})(\xi,\eta)\right|^2 d\xi d\eta$$

$$\leq \sum_{\substack{\delta\leq\alpha\\ \gamma\leq\beta}}C_{\delta,\gamma}\int_{\mathbb{R}^{2n}}\left|2^{-j|\alpha-\delta|-k|\beta-\gamma|}\left(\partial_{\xi}^{\alpha-\delta}(\hat{\psi})\right)(2^{-j}\xi)\left(\partial_{\eta}^{\beta-\gamma}(\hat{\psi})\right)(2^{-k}\eta)\partial_{\xi}^{\delta}\partial_{\beta}^{\gamma}m(\xi,\eta)\right|^2 d\xi d\eta$$

$$\leq \sum_{\substack{\delta\leq\alpha\\ \gamma\leq\beta}}C_{\delta,\gamma}2^{-2j|\alpha|}2^{2j|\delta|}2^{-2k|\beta|}2^{2k|\gamma|}2^{jn}2^{kn}2^{-2j(|\delta|+m_1)}2^{-2k(|\gamma|+m_2)}. \quad (123)$$

According to (122) and (123), we derive

$$\left\|m_{j,k}^{\vee}(\xi,\eta)\left(1+2^j|\xi|\right)^{N}(1+2^k|\eta|)^{N}\right\|_{L^2}$$

$$\leq C\sum_{|\alpha|\leq N,|\beta|\leq N}C_{\alpha,\beta}2^{j|\alpha|}2^{k|\beta|}\left(\sum_{\substack{\delta\leq\alpha\\ \gamma\leq\beta}}C_{\delta,\gamma}2^{jn}2^{kn}2^{-2j(|\delta|+m_1)}2^{-2k(|\gamma|+m_2)}\right)^{\frac{1}{2}}$$

$$= 2^{-jm_1-km_2}C < \infty. \quad (124)$$

Combining (121) and (124), we have

$$\psi_{j,k} * (m_{j,k}\hat{f})^{\vee} \lesssim 2^{-jm_1-km_2}\left(M_s\left(\left|\Delta_{j,k}(f)\right|\right)^r(x,y)\right)^{\frac{1}{r}}.$$

Similarly we also have

$$\psi_{j,k} * \left(m_{j-i_1,k-i_2}\hat{f}\right)^{\vee} \lesssim 2^{-jm_1-km_2}\left(M_s\left(\left|\Delta_{j,k}(f)\right|\right)^r(x,y)\right)^{\frac{1}{r}} \quad (i \in \{1,0,-1\}).$$

Therefore

$$\|T(f)\|_{\dot{B}_p^{\rho+m,q}} = C\left\|\left(m\hat{f}\right)^{\vee}\right\|_{\dot{B}_p^{\rho+m,q}} \le C\left(\sum_{j,k}\left(2^{j\rho_1+k\rho_2}\left\|\left(M_s\left(\left|\Delta_{j,k}(f)\right|^r\right)\right)^{\frac{1}{r}}\right\|_{L^p}\right)^q\right)^{\frac{1}{q}}$$

$$\le C\left(\sum_{j,k}\left(2^{j\rho_1+k\rho_2}\left\|\left(\left|\Delta_{j,k}(f)\right|^r\right)\right\|_{L^{\frac{p}{r}}}^{\frac{1}{r}}\right)^q\right)^{\frac{1}{q}} \le C\|f\|_{\dot{B}_p^{\rho,q}}$$

and

$$\|T(f)\|_{\dot{F}_p^{\rho+m,q}} = C\left\|\left(m\hat{f}\right)^{\vee}\right\|_{\dot{F}_p^{\rho+m,q}} \le C\left\|\left(\sum_{j,k}\left(2^{j\rho_1+k\rho_2}\left(M_s\left(\left|\Delta_{j,k}(f)\right|^r\right)\right)^{\frac{1}{r}}\right)^q\right)^{\frac{1}{q}}\right\|_{L^p}$$

$$\le C\left\|\left(\sum_{j,k}\left(2^{j\rho_1+k\rho_2}\left|\Delta_{j,k}(f)\right|^r\right)^{\frac{q}{r}}\right)^{\frac{r}{q}}\right\|_{L^{\frac{p}{r}}}^{\frac{1}{r}} \le C\|f\|_{\dot{F}_p^{\rho,q}},$$

which conclude the proof of Theorem (5.3.14).

Theorem (5.3.15) [255] Let $\rho = (\rho_1, \rho_2) \in \mathbb{R}^2, m = (m_1, m_2) \in \mathbb{R}^+ \times \mathbb{R}^+, 0 < p, q < \infty$ and $N = \left[\frac{n}{2} + \frac{n}{min(p,q)}\right] + 1$. Assume that $m(\xi,\eta)$ satisfies

$$\left|\partial_\xi^\alpha \partial_\eta^\beta m(\xi,\eta)\right| \le A\frac{1}{|\xi|^{|\alpha|+m_1}|\eta|^{|\beta|+m_2}}$$

for all $|\alpha| \le N, |\beta| \le N$ and $(\xi,\eta) \in (\mathbb{R}^n \times \mathbb{R}^n)$ with $|\xi||\eta| \neq 0$. If $w \in A_\infty(\mathbb{R}^n \times \mathbb{R}^n)$, Then there exists a constant C such that

$$\|T(f)\|_{\dot{B}_p^{\rho+m,q}(w)} \le C\|f\|_{\dot{B}_p^{\rho+m,q}(w)}$$

and

$$\|T(f)\|_{\dot{F}_p^{\rho+m,q}(w)} \le C\|f\|_{\dot{F}_p^{\rho+m,q}(w)},$$

where the constant C is independent of f.

We prove that the bi-parameter Fourier multiplier operators are bounded on bi-parameter Besov–Lipschitz and Triebel–Lizorkin spaces (Theorem (5.3.11)). We are concerned with the weighted cases (Theorem (5.3.13)). We extend the results to more general cases where the multipliers have the worse singularity at the origin (see Theorems (5.3.14) and (5.3.15)).

Proof. Since the proof is similar to that of Theorem (5.3.13).

Corollary (5.3.16) [314] Let $0 \le \epsilon < \infty$. Then there exists a constant $C_{r-1} > 0$ such that

$$\sum_d \left\|\left\{\sum_{k\in\mathbb{Z}}\left(M_{s_d}\left(f_k^d\right)\right)^{1+2\epsilon}\right\}^{\frac{1}{1+2\epsilon}}\right\|_{L^{1+\epsilon}} \le C_{r-1}\sum_d\left\|\left\{\sum_{k\in\mathbb{Z}}\left|f_k^d\right|^{1+2\epsilon}\right\}^{\frac{1}{1+2\epsilon}}\right\|_{L^{1+\epsilon}}$$

for all double sequences $\{f_k^d\}$ of locally integrable functions on $\mathbb{R}^n \times \mathbb{R}^n$.

Corollary (5.3.17) [314] Let $0 \le \epsilon < \infty$. For any $C_{r-1}^1(\mathbb{R}^n \times \mathbb{R}^n)$ function u_{r-1} whose Fourier transform is supported in $\{|\xi_{r-1}| \le (1+\epsilon)t_r, |\eta_{r-1}| \le (1+\epsilon)t_{r+1}\}$, assume that $|u_{r-1}(x_{r-1}, y_{r-1})| \le (1+\epsilon)(1+|x_{r-1}|)^{\frac{n}{1+\epsilon}}(1+|y_{r-1}|)^{\frac{n}{1+\epsilon}}$ for some $\epsilon \ge 0$. Then there exist two constants C_r and C_{r+1} such that the following inequality is valid.

$$\sup_{z_r\in\mathbb{R}^n, z_{r+1}\in\mathbb{R}^n}\left\{\frac{1}{t_r}\frac{\left|\nabla_{x_{r-1}}u_{r-1}(x_{r-1}-z_r,y_{r-1}-z_{r+1})\right|}{(1+t_r|z_r|)^{\frac{n}{1+\epsilon}}(1+t_{r+1}|z_{r+1}|)^{\frac{n}{1+\epsilon}}}+\frac{1}{t_{r+1}}\frac{\left|\nabla_{y_{r-1}}u_{r-1}(x_{r-1}-z_r,y_{r-1}-z_{r+1})\right|}{(1+t_r|z_r|)^{\frac{n}{1+\epsilon}}(1+t_{r+1}|z_{r+1}|)^{\frac{n}{1+\epsilon}}}\right\}$$
$$\leq C_r\sup_{z_r\in\mathbb{R}^n, z_{r+1}\in\mathbb{R}^n}\frac{|u_{r-1}(x_{r-1}-z_r,y_{r-1}-z_{r+1})|}{(1+t_r|z_r|)^{\frac{n}{1+\epsilon}}(1+t_{r+1}|z_{r+1}|)^{\frac{n}{1+\epsilon}}}$$
$$\leq C_{r+1}M_{s_d}(|u_{r-1}|^{1+\epsilon})(x_{r-1},y_{r-1})^{\frac{1}{1+\epsilon}},$$

where M_{s_d} denotes the strong maximal operators and the constants C_r and C_{r+1} depend only on n and $1+\epsilon$.

Proof. This proof is divided into two parts.

Part 1. We first show that

$$\sup_{z_r\in\mathbb{R}^n, z_{r+1}\in\mathbb{R}^n}\left\{\frac{1}{t_r}\frac{\left|\nabla_{x_{r-1}}u_{r-1}(x_{r-1}-z_r,y_{r-1}-z_{r+1})\right|}{(1+t_r|z_r|)^{\frac{n}{1+\epsilon}}(1+t_{r+1}|z_{r+1}|)^{\frac{n}{1+\epsilon}}}+\frac{1}{t_{r+1}}\frac{\left|\nabla_{y_{r-1}}u_{r-1}(x_{r-1}-z_r,y_{r-1}-z_{r+1})\right|}{(1+t_r|z_r|)^{\frac{n}{1+\epsilon}}(1+t_{r+1}|z_{r+1}|)^{\frac{n}{1+\epsilon}}}\right\}$$
$$\leq C_r\sup_{z_r\in\mathbb{R}^n, z_{r+1}\in\mathbb{R}^n}\frac{|u_{r-1}(x_{r-1}-z_r,y_{r-1}-z_{r+1})|}{(1+t_r|z_r|)^{\frac{n}{1+\epsilon}}(1+t_{r+1}|z_{r+1}|)^{\frac{n}{1+\epsilon}}}. \tag{125}$$

Select a Schwartz function ψ on $\mathbb{R}^n$ whose Fourier transform is supported in $\{\xi_{r-1} : |\xi_{r-1}| \leq 2(1+\epsilon)\}$ and is equal to 1 on $\{\xi_{r-1} : |\xi_{r-1}| \leq 1+\epsilon\}$. Then $\hat{\psi}\left(\frac{\xi_{r-1}}{t_r}\right)\hat{\psi}\left(\frac{\eta_{r-1}}{t_{r+1}}\right)$ is equal to 1 on the support of the Fourier transform of $u_{r-1}(\xi_{r-1},\eta_{r-1})$ and we can write

$$u_{r-1}(x_{r-1}-z_r,y_{r-1}-z_{r+1})=\left(\hat{u}_{r-1}(\cdot,\cdot)\hat{\psi}\left(\frac{\cdot}{t_r}\right)\hat{\psi}\left(\frac{\cdot}{t_{r+1}}\right)\right)^{\vee}(x_{r-1}-z_r,y_{r-1}-z_{r+1})$$
$$=\int_{\mathbb{R}^{2n}}t_r^n\psi(t_r(x_{r-1}-z_r-\xi_{r-1}))t_{r+1}^n\psi(t_{r+1}(y_{r-1}-z_{r+1}-\eta_{r-1}))u_{r-1}(\xi_{r-1},\eta_{r-1})d\xi_{r-1}d\eta_{r-1}.$$

We use the partial derivative to obtain

$$\left|\nabla_{x_{r-1}}u_{r-1}(x_{r-1}-z_r,y_{r-1}-z_{r+1})\right|$$
$$\leq\int_{\mathbb{R}^{2n}}t_r^{n+1}\left|(\nabla_{x_{r-1}}\psi)\big(t_r(x_{r-1}-z_r-\xi_{r-1})\big)\right|\left|t_{r+1}^n\psi\big(t_{r+1}(y_{r-1}-z_{r+1}-\eta_{r-1})\big)u_{r-1}(\xi_{r-1},\eta_{r-1})\right|d\xi_{r-1}d\eta_{r-1}$$
$$\leq (C_{r-1})_N\int_{\mathbb{R}^{2n}}t_r^{n+1}(1+t_r|x_{r-1}-z_r-\xi_{r-1}|)^{-N}t_{r+1}^n(1+t_{r+1}|y_{r-1}-z_{r+1}-\eta_{r-1}|)^{-N}|u_{r-1}(\xi_{r-1},\eta_{r-1})|d\xi_{r-1}d\eta_{r-1}.$$

Therefore

$$\frac{1}{t_r}\frac{\left|\nabla_{x_{r-1}}u_{r-1}(x_{r-1}-z_r,y_{r-1}-z_{r+1})\right|}{(1+t_r|z_r|)^{\frac{n}{1+\epsilon}}(1+t_{r+1}|z_{r+1}|)^{\frac{n}{1+\epsilon}}}$$
$$\lesssim\int_{\mathbb{R}^{2n}}t_r^nt_{r+1}^n(1+t_r|x_{r-1}-z_r-\xi_{r-1}|)^{-N}(1+t_{r+1}|y_{r-1}-z_{r+1}-\eta_{r-1}|)^{-N}$$
$$\times\frac{|u_{r-1}(\xi_{r-1},\eta_{r-1})|}{(1+t_r|z_r|)^{\frac{n}{1+\epsilon}}(1+t_{r+1}|z_{r+1}|)^{\frac{n}{1+\epsilon}}}d\xi_{r-1}d\eta_{r-1},$$

where N is an arbitrarily large positive integer and $(C_{r-1})_N$ is a constant which depends only on N.

Since

$$1\leq(1+t_r|x_{r-1}-z_r-\xi_{r-1}|)^{\frac{n}{1+\epsilon}}\frac{(1+t_r|z_r|)^{\frac{n}{1+\epsilon}}}{(1+t_r|x_{r-1}-\xi_{r-1}|)^{\frac{n}{1+\epsilon}}}$$

and

$$1\leq(1+t_{r+1}|y_{r-1}-z_{r+1}-\eta_{r-1}|)^{\frac{n}{1+\epsilon}}\frac{(1+t_{r+1}|z_{r+1}|)^{\frac{n}{1+\epsilon}}}{(1+t_{r+1}|y_{r-1}-\eta_{r-1}|)^{\frac{n}{1+\epsilon}}},$$

we obtain that

$$\frac{1}{t_r}\frac{\left|\nabla_{x_{r-1}}u_{r-1}(x_{r-1}-z_r,y_{r-1}-z_{r+1})\right|}{(1+t_r|z_r|)^{\frac{n}{1+\epsilon}}(1+t_{r+1}|z_{r+1}|)^{\frac{n}{1+\epsilon}}}$$

$$\le (C_{r-1})_N \int_{\mathbb{R}^{2n}} \frac{|u_{r-1}(\xi_{r-1},\eta_{r-1})|}{(1+t_r|x_{r-1}-\xi_{r-1}|)^{\frac{n}{1+\epsilon}}(1+t_{r+1}|y_{r-1}-\eta_{r-1}|)^{\frac{n}{1+\epsilon}}}$$
$$\times \frac{t_r^n}{(1+t_r|x_{r-1}-z_r-\xi_{r-1}|)^{-N-\frac{n}{1+\epsilon}}} \frac{t_{r+1}^n}{(1+t_{r+1}|y_{r-1}-z_{r+1}-\eta_{r-1}|)^{-N-\frac{n}{1+\epsilon}}} d\xi_{r-1}d\eta_{r-1}$$
$$\le C_r \sup_{\xi_{r-1}\in\mathbb{R}^n,\eta_{r-1}\in\mathbb{R}^n} \frac{|u_{r-1}(x_{r-1}-\xi_{r-1},y_{r-1}-\eta_{r-1})|}{(1+t_r|\xi_{r-1}|)^{\frac{n}{1+\epsilon}}(1+t_{r+1}|\eta_{r-1}|)^{\frac{n}{1+\epsilon}}}. \quad (126)$$

Similarly, we also obtain that

$$\frac{1}{t_{r+1}} \frac{|\nabla_{y_{r-1}}u_{r-1}(x_{r-1}-z_r,y_{r-1}-z_{r+1})|}{(1+t_r|z_r|)^{\frac{n}{1+\epsilon}}(1+t_{r+1}|z_{r+1}|)^{\frac{n}{1+\epsilon}}} \le C_r \sup_{\xi_{r-1}\in\mathbb{R}^n,\eta_{r-1}\in\mathbb{R}^n} \frac{|u_{r-1}(x_{r-1}-\xi_{r-1},y_{r-1}-\eta_{r-1})|}{(1+t_r|\xi_{r-1}|)^{\frac{n}{1+\epsilon}}(1+t_{r+1}|\eta_{r-1}|)^{\frac{n}{1+\epsilon}}}. \quad (127)$$

Combining (126) and (127), we conclude the proof of Part 1.

Part 2. We show that

$$\sup_{z_r\in\mathbb{R}^n,z_{r+1}\in\mathbb{R}^n} \frac{|u_{r-1}(x_{r-1}-z_r,y_{r-1}-z_{r+1})|}{(1+t_r|z_r|)^{\frac{n}{1+\epsilon}}(1+t_{r+1}|z_{r+1}|)^{\frac{n}{1+\epsilon}}} \le C_{r-1} M_{s_d}(|u_{r-1}|^{1+\epsilon})(x_{r-1},y_{r-1})^{\frac{1}{1+\epsilon}}. \quad (128)$$

Let $|\xi_{r-1}| \le \delta_r$ and $|\eta_{r-1}| \le \delta_{r+1}$ for some $\delta_r,\delta_{r+1} > 0$ to be chosen later. We use the mean value theorem to obtain the estimate

$$\begin{aligned} &u_{r-1}(x_{r-1}-z_r,y_{r-1}-z_{r+1}) - u_{r-1}(x_{r-1}-z_r-\xi_{r-1},y_{r-1}-z_{r+1}-\eta_{r-1}) \\ &\quad = (\nabla_{x_{r-1}}u_{r-1})(x_{r-1}-z_r-(1-\epsilon)\xi_{r-1},y_{r-1}-z_{r+1}-(1-\epsilon)\eta_{r-1})\cdot\xi_{r-1} \\ &\quad + (\nabla_{y_{r-1}}u_{r-1})(x_{r-1}-z_r-(1-\epsilon)\xi_{r-1},y_{r-1}-z_{r+1}-(1-\epsilon)\eta_{r-1})\cdot\xi_{r-1} \; (0<\epsilon<1) \end{aligned}$$

for all $z_r, z_{r+1} \in \mathbb{R}^n$. Therefore

$$\begin{aligned} &|u_{r-1}(x_{r-1}-z_r,y_{r-1}-z_{r+1})| \\ &\le \sup_{\substack{|w_r|\le|z_r|+\delta_r \\ |w_{r+1}|\le|z_{r+1}|+\delta_{r+1}}} \left(|(\nabla_{x_{r-1}}u_{r-1})(x_{r-1}-w_r,y_{r-1}-w_{r+1})|\delta_r + |(\nabla_{y_{r-1}}u_{r-1})(x_{r-1}-w_r,y_{r-1}-w_{r+1})|\delta_{r+1}\right) \\ &\quad + |u_{r-1}(x_{r-1}-z_r-\xi_{r-1},y_{r-1}-z_{r+1}-\eta_{r-1})|, \end{aligned}$$

where $w_r = z_r + (1-\epsilon)\xi_{r-1}$ and $w_{r+1} = z_{r+1} + (1-\epsilon)\eta_{r-1}$. Let Q_r be a ball with radius equal to δ_r and z_r being the center, and Q_{r+1} be a ball with radius equal to δ_{r+1} and z_{r+1} being the center. By raising to the power of $(1+\epsilon)$, averaging over $Q_r \times Q_{r+1}$ and raising to the power $\left(\frac{1}{1+\epsilon}\right)$, we derive

$$\begin{aligned} &|u_{r-1}(x_{r-1}-z_r,y_{r-1}-z_{r+1})| \\ &\le (C_{r-1})^2_{1+\epsilon} \sup_{\substack{|w_r|\le|z_r|+\delta_r \\ |w_{r+1}|\le|z_{r+1}|+\delta_{r+1}}} |(\nabla_{x_{r-1}}u_{r-1})(x_{r-1}-w_r,y_{r-1}-w_{r+1})|\delta_r + |(\nabla_{y_{r-1}}u_{r-1})(x_{r-1}-w_r,y_{r-1}-w_{r+1})|\delta_{r+1} \\ &\quad + \left(\frac{1}{v_{n+r-1}^2\delta_r^n\delta_{r+1}^n}\int_{Q_r}\int_{Q_{r+1}} |u_{r-1}(x_{r-1}-z_r-\xi_{r-1},y_{r-1}-z_{r+1}-\eta_{r-1})|^{1+\epsilon}\, d\xi_{r-1}d\eta_{r-1}\right)^{\frac{1}{1+\epsilon}}, \end{aligned}$$

where $(C_{r-1})_{1+\epsilon} = \max\{2^{1+\epsilon}, 2^{\frac{1}{1+\epsilon}}\}$ and v_{n+r-1} is the volume of the unit ball in $\mathbb{R}^n$.

Set $\delta_r = \frac{\epsilon}{t_r}$ and $\delta_{r+1} = \frac{\epsilon}{t_{r+1}}$ for some $\epsilon \le 1$. Then we have $|w_r| \le |z_r| + \frac{\epsilon}{t_r}, |w_{r+1}| \le |z_{r+1}| + \frac{\epsilon}{t_{r+1}}, \frac{1}{1+t_r|z_r|} \le \frac{2}{1+t_r|w_r|}$ and $\frac{1}{1+t_{r+1}|z_{r+1}|} \le \frac{2}{1+t_{r+1}|w_{r+1}|}$. Therefore we can write

$$\begin{aligned} &\frac{|u_{r-1}(x_{r-1}-z_r,y_{r-1}-z_{r+1})|}{(1+t_r|z_r|)^{\frac{n}{1+\epsilon}}(1+t_{r+1}|z_{r+1}|)^{\frac{n}{1+\epsilon}}} \\ &\le (C_{r-1})_{1+\epsilon,n}\left(\sup_{|w_{i+r-1}|\le|z_{i+r-1}|+\frac{\epsilon}{t_{i+r-1}}} \frac{|\nabla_{x_{r-1}}u_{r-1}(x_{r-1}-w_r,y_{r-1}-w_{r+1})|\epsilon}{t_r(1+t_r|w_r|)^{\frac{n}{1+\epsilon}}(1+t_{r+1}|w_{r+1}|)^{\frac{n}{1+\epsilon}}} + \frac{|\nabla_{y_{r-1}}u_{r-1}(x_{r-1}-w_r,y_{r-1}-w_{r+1})|\epsilon}{t_{r+1}(1+t_r|w_r|)^{\frac{n}{1+\epsilon}}(1+t_{r+1}|w_{r+1}|)^{\frac{n}{1+\epsilon}}} \right. \\ &\left. + \frac{\left(\frac{t_r^n t_{r+1}^n}{v_{n+r-1}^2\epsilon^{2n}}\int_{|\xi_{r-1}|\le\frac{1}{t_r}+|z_r|}\int_{|\eta_{r-1}|\le\frac{1}{t_{r+1}}+|z_{r+1}|} |u_{r-1}(x_{r-1}-z_r-\xi_{r-1},y_{r-1}-z_{r+1}-\eta_{r-1})|^{1+\epsilon}\, d\xi_{r-1}d\eta_{r-1}\right)^{\frac{1}{1+\epsilon}}}{(1+t_r|z_r|)^{\frac{n}{1+\epsilon}}(1+t_{r+1}|z_{r+1}|)^{\frac{n}{1+\epsilon}}}\right) \end{aligned}$$

$$\leq (C_{r-1})_{1+\epsilon,n}\left(\sup_{w_{i+r-1}\in\mathbb{R}^n}\frac{|\nabla_{x_{r-1}}u_{r-1}(x_{r-1}-w_r,y_{r-1}-w_{r+1})|\epsilon}{t_r(1+t_r|w_r|)^{\frac{n}{1+\epsilon}}(1+t_{r+1}|w_{r+1}|)^{\frac{n}{1+\epsilon}}}+\frac{|\nabla_{y_{r-1}}u_{r-1}(x_{r-1}-w_r,y_{r-1}-w_{r+1})|\epsilon}{t_{r+1}(1+t_r|w_r|)^{\frac{n}{1+\epsilon}}(1+t_{r+1}|w_{r+1}|)^{\frac{n}{1+\epsilon}}}\right.$$
$$\left.+\epsilon^{-\frac{2n}{1+\epsilon}}M_{s_d}(|u_{r-1}|^{1+\epsilon})(x_{r-1},y_{r-1})^{\frac{1}{1+\epsilon}}\right).$$

Using (125) and setting $\epsilon=\frac{1}{2(C_{r-1})_{1+\epsilon,n}C_r}\leq 1$ with $C_r = 2\epsilon^{-\frac{n}{1+\epsilon}}$, we can achieve (128), where we used the hypothesis

$$\sup_{\substack{|z_r|\in\mathbb{R}^n\\|z_{r+1}|\in\mathbb{R}^n}}\frac{|u_{r-1}(x_{r-1}-z_r,y_{r-1}-z_{r+1})|}{(1+t_r|z_r|)^{\frac{n}{1+\epsilon}}(1+t_{r+1}|z_{r+1}|)^{\frac{n}{1+\epsilon}}}\leq\sup_{\substack{|z_r|\in\mathbb{R}^n\\|z_{r+1}|\in\mathbb{R}^n}}\frac{(1+\epsilon)(1+|x_{r-1}|+|z_r|)^{\frac{n}{1+\epsilon}}(1+|y_{r-1}|+|z_{r+1}|)^{\frac{n}{1+\epsilon}}}{(1+t_r|z_r|)^{\frac{n}{1+\epsilon}}(1+t_{r+1}|z_{r+1}|)^{\frac{n}{1+\epsilon}}}<\infty.$$

Combining (125) and (128), we conclude the proof of Corollary (5.3.17).

Corollary (5.3.18) [314] Suppose that T_d are a Fourier multiplier operators defined on $\mathbb{R}^n\times\mathbb{R}^n$. Let $\rho_{r-1}=(\rho_r,\rho_{r+1})\in\mathbb{R}^2, 0\leq\epsilon<\infty, N=\left[\frac{n}{2}+\frac{n}{min(1+\epsilon,1+2\epsilon)}\right]+1$. Assume that $m_{r-1}(\xi_{r-1},\eta_{r-1})$ is a $L^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ function that satisfies

$$\left|\partial_{\xi_{r-1}}^{\alpha_{r-1}}\partial_{\eta_{r-1}}^{\beta_{r-1}}m_{r-1}(\xi_{r-1},\eta_{r-1})\right|\leq A_{r-1}\frac{1}{|\xi_{r-1}|^{|\alpha_{r-1}|}|\eta_{r-1}|^{|\beta_{r-1}|}}$$

for all $|\alpha_{r-1}|\leq N, |\beta_{r-1}|\leq N$ and $(\xi_{r-1},\eta_{r-1})\in(\mathbb{R}^n\times\mathbb{R}^n)$ with $|\xi_{r-1}||\eta_{r-1}|\neq 0$. Then there exists a constant C_{r-1} such that

$$\sum_d\|T_d(f^d)\|_{\dot{B}_{1+\epsilon}^{\rho_{r-1},1+2\epsilon}}\leq C_{r-1}\sum_d\|f^d\|_{\dot{B}_{1+\epsilon}^{\rho_{r-1},1+2\epsilon}}$$

and

$$\sum_d\|T_d(f^d)\|_{\dot{F}_{1+\epsilon}^{\rho_{r-1},1+2\epsilon}}\leq C_{r-1}\sum_d\|f^d\|_{\dot{F}_{1+\epsilon}^{\rho_{r-1},1+2\epsilon}},$$

where the constant C_{r-1} is independent of the sequence f^d.

If $\rho_{r-1}=(0,0)$ and $\epsilon=1/2$, then $\dot{F}_{1+\epsilon}^{0,2}(\mathbb{R}^n\times\mathbb{R}^n)=L^{1+\epsilon}(\mathbb{R}^n\times\mathbb{R}^n)$ when $0<\epsilon<\infty$ and $\dot{F}_{1+\epsilon}^{0,2}(\mathbb{R}^n\times\mathbb{R}^n)=H^{1+\epsilon}(\mathbb{R}^n\times\mathbb{R}^n)$ when $-1<\epsilon<0$. Then we can easily obtain the boundedness of Fourier multiplier operators on $L^{1+\epsilon}(\mathbb{R}^n\times\mathbb{R}^n)$ and $H^{1+\epsilon}(\mathbb{R}^n\times\mathbb{R}^n)$ from Corollary (5.3.18).

Proof. The proof is divided into two parts.

Part 1. We prove that

$$\sum_d\|T_d(f^d)\|_{\dot{B}_{1+\epsilon}^{\rho_{r-1},1+2\epsilon}}\leq C_{r-1}\sum_d\|f^d\|_{\dot{B}_{1+\epsilon}^{\rho_{r-1},1+2\epsilon}}. \tag{129}$$

Let $\psi(\xi_{r-1})$ be a Schwartz function whose Fourier transform is supported in $\left\{\frac{1}{2}\leq|\xi_{r-1}|\leq 2\right\}$ and satisfies $\sum_{k\in\mathbb{Z}}|\hat{\psi}(2^{-k}\xi_{r-1})|=1$ for all $\xi_{r-1}\in\{\mathbb{R}^n(0)\}$. Denote $\psi_{j,k}(\xi_{r-1},\eta_{r-1})=\psi_{2^{-j}}(\xi_{r-1})\psi_{2^{-k}}(\eta_{r-1}),\sum_d\Delta_{j,k}(f^d)=\sum_d(\psi_{j,k}*f^d)$ and $m_{j+r-1,k+r-1}(\xi_{r-1},\eta_{r-1})=\hat{\psi}(2^{-j}\xi_{r-1})\hat{\psi}(2^{-k}\eta_{r-1})m_{r-1}(\xi_{r-1},\eta_{r-1})$. It is easy to see that $m_{r-1}=\sum_{j+r-1,k+r-1}m_{j+r-1,k+r-1}$. By the orthogonality estimate, we have

$$\sum_d\Delta_{j,k}(T_d(f^d))(x_{r-1},y_{r-1})=\sum_d\psi_{j,k}*T_d(f^d)(x_{r-1},y_{r-1})=\sum_d\psi_{j,k}*(m_{r-1}\widehat{f^d})^\vee(x_{r-1},y_{r-1})$$
$$=\sum_d\left(\hat{\psi}(2^{-j}\xi_{r-1})\hat{\psi}(2^{-k}\eta_{r-1})\sum_{j',k'}\hat{\psi}(2^{-j'}\xi_{r-1})\hat{\psi}(2^{-k'}\eta_{r-1})m_{r-1}(\xi_{r-1},\eta_{r-1})\widehat{f^d}(\xi_{r-1},\eta_{r-1})\right)^\vee(x_{r-1},y_{r-1})$$
$$=\sum_d\left(\hat{\psi}(2^{-j}\xi_{r-1})\hat{\psi}(2^{-k}\eta_{r-1})\sum_{i_1,i_2}\hat{\psi}(2^{-(j-i_1)}\xi_{r-1})\hat{\psi}(2^{-(k-i_2)}\eta_{r-1})m_{r-1}(\xi_{r-1},\eta_{r-1})\widehat{f^d}(\xi_{r-1},\eta_{r-1})\right)^\vee(x_{r-1},y_{r-1})$$
$$=\sum_d\left(\hat{\psi}(2^{-j}\xi_{r-1})\hat{\psi}(2^{-k}\eta_{r-1})\sum_{i_1,i_2}m_{(j-i_1)+r-1,(k-i_2)+r-1}(\xi_{r-1},\eta_{r-1})\widehat{f^d}(\xi_{r-1},\eta_{r-1})\right)^\vee(x_{r-1},y_{r-1}),$$

where $(i_1,i_2\in\{-1,0,1\})$. We only estimate that

$$\sum_d\left(\hat{\psi}(2^{-j}\xi_{r-1})\hat{\psi}(2^{-k}\xi_{r-1})m_{j+r-1,k+r-1}(\xi_{r-1},\eta_{r-1})\widehat{f^d}(\xi_{r-1},\eta_{r-1})\right)^\vee(x_{r-1},y_{r-1})$$

$$\lesssim \sum_d \left(M_{s_d}(|\Delta_{j,k}(f^d)|)^{1+\epsilon}(x_{r-1},y_{r-1})\right)^{\frac{1}{1+\epsilon}}.$$

Pick $(1+\epsilon) < \min((1+\epsilon)\cdot(1+2\epsilon))$ such that $N > \frac{n}{2} + \frac{n}{1+\epsilon}$. By Corollary (5.3.17), we have

$$\sum_d \left|\left(\hat{\psi}(2^{-j}\xi_{r-1})\hat{\psi}(2^{-k}\xi_{r-1})m_{j+r-1,k+r-1}(\xi_{r-1},\eta_{r-1})\widehat{f^d}(\xi_{r-1},\eta_{r-1})\right)^{\vee}(x_{r-1},y_{r-1})\right|$$

$$= \sum_d \left|\iint_{\mathbb{R}^{2n}} m^{\vee}_{j+r-1,k+r-1}(\xi_{r-1},\eta_{r-1})\Delta_{j,k}(f^d)(x_{r-1}-\xi_{r-1},y_{r-1}-\eta_{r-1})d\xi_{r-1}d\eta_{r-1}\right|$$

$$\leq \sup_{z_r,z_{r+1}\in\mathbb{R}^n}\sum_d \frac{|\Delta_{j,k}(f^d)(x_{r-1}-z_r,y_{r-1}-z_{r+1})|}{(1+2^j|z_r|)^{\frac{n}{1+\epsilon}}(1+2^k|z_{r+1}|)^{\frac{n}{1+\epsilon}}}\int_{\mathbb{R}^{2n}}|m^{\vee}_{j+r-1,k+r-1}(\xi_{r-1},\eta_{r-1})|(1+2^j|\xi_{r-1}|)^{\frac{n}{1+\epsilon}}(1+2^k|\eta_{r-1}|)^{\frac{n}{1+\epsilon}}d\xi_{r-1}d\eta_{r-1}$$

$$\leq C_{r+1}\sum_d M_{s_d}\left(|\Delta_{j,k}(f^d)|^{1+\epsilon}(x_{r-1},y_{r-1})\right)^{\frac{1}{1+\epsilon}}\int_{\mathbb{R}^{2n}}|m^{\vee}_{j+r-1,k+r-1}(\xi_{r-1},\eta_{r-1})|(1+2^j|\xi_{r-1}|)^{\frac{n}{1+\epsilon}}(1+2^k|\eta_{r-1}|)^{\frac{n}{1+\epsilon}}d\xi_{r-1}d\eta_{r-1}.$$

Next it suffices to estimate

$$\int_{\mathbb{R}^{2n}}|m^{\vee}_{j+r-1,k+r-1}(\xi_{r-1},\eta_{r-1})|\left(1+2^j|\xi_{r-1}|\right)^{\frac{n}{1+\epsilon}}(1+2^k|\eta_{r-1}|)^{\frac{n}{1+\epsilon}}\,d\xi_{r-1}d\eta_{r-1}.$$

We can use the Hölder's inequality to write

$$\int_{\mathbb{R}^{2n}}|m^{\vee}_{j+r-1,k+r-1}(\xi_{r-1},\eta_{r-1})|(1+2^j|\xi_{r-1}|)^N(1+2^k|\eta_{r-1}|)^N\frac{1}{(1+2^j|\xi_{r-1}|)^{N-\frac{n}{1+\epsilon}}}\frac{1}{(1+2^k|\eta_{r-1}|)^{N-\frac{n}{1+\epsilon}}}d\xi_{r-1}d\eta_{r-1}$$

$$\leq \left\|m^{\vee}_{j+r-1,k+r-1}(\xi_{r-1},\eta_{r-1})\left(1+2^j|\xi_{r-1}|\right)^N(1+2^k|\eta_{r-1}|)^N\right\|_{L^2}\left\|\frac{1}{(1+2^j|\xi_{r-1}|)^{N-\frac{n}{1+\epsilon}}}\frac{1}{(1+2^k|\eta_{r-1}|)^{N-\frac{n}{1+\epsilon}}}\right\|_{L^2}.$$

It is easy to see that

$$\left\|\frac{1}{(1+2^j|\xi_{r-1}|)^{N-\frac{n}{1+\epsilon}}}\frac{1}{(1+2^k|\eta_{r-1}|)^{N-\frac{n}{1+\epsilon}}}\right\|_{L^2} = C_{r-1}2^{-\frac{jn}{2}}2^{-\frac{kn}{2}}.$$

We only estimate $\left\|m^{\vee}_{j+r-1,k+r-1}(\xi_{r-1},\eta_{r-1})\left(1+2^j|\xi_{r-1}|\right)^N(1+2^k|\eta_{r-1}|)^N\right\|_{L^2}$.

We use the obvious fact that

$$(1+2^j|\xi_{r-1}|)^N \leq C_{r-1}(n)\sum_{|\alpha_{r-1}|\leq N}\left|(2^j\xi_{r-1})^{\alpha_{r-1}}\right| \text{ and } (1+2^k|\eta_{r-1}|)^N \leq C_{r-1}(n)\sum_{|\beta_{r-1}|\leq N}\left|(2^k\eta_{r-1})^{\beta_{r-1}}\right|$$

to obtain

$$\left\|m^{\vee}_{j+r-1,k+r-1}(\xi_{r-1},\eta_{r-1})\left(1+2^j|\xi_{r-1}|\right)^N(1+2^k|\eta_{r-1}|)^N\right\|_{L^2}$$

$$= \left(\int_{\mathbb{R}^{2n}}|m^{\vee}_{j+r-1,k+r-1}(\xi_{r-1},\eta_{r-1})|^2\left(1+2^j|\xi_{r-1}|\right)^{2N}(1+2^k|\eta_{r-1}|)^{2N}d\xi_{r-1}d\eta_{r-1}\right)^{\frac{1}{2}}$$

$$\leq C_{r-1}\sum_{|\alpha_{r-1}|\leq N,|\beta_{r-1}|\leq N}\left(\int_{\mathbb{R}^{2n}}|m^{\vee}_{j+r-1,k+r-1}(\xi_{r-1},\eta_{r-1})|^2\left|\xi_{r-1}^{\alpha_{r-1}}\right|^2\left|\eta_{r-1}^{\beta_{r-1}}\right|^2 2^{2j|\alpha_{r-1}|}2^{2k|\beta_{r-1}|}d\xi_{r-1}d\eta_{r-1}\right)^{\frac{1}{2}}$$

$$= C_{r-1}\sum_{|\alpha_{r-1}|\leq N,|\beta_{r-1}|\leq N}\left(\int_{\mathbb{R}^{2n}}\left|\left(\partial^{\alpha_{r-1}}_{\xi_{r-1}}\partial^{\beta_{r-1}}_{\eta_{r-1}}m_{j+r-1,k+r-1}\right)^{\vee}(\xi_{r-1},\eta_{r-1})\right|^2 2^{2j|\alpha_{r-1}|}2^{2k|\beta_{r-1}|}d\xi_{r-1}d\eta_{r-1}\right)^{\frac{1}{2}}$$

$$= C_{r-1}\sum_{|\alpha_{r-1}|\leq N,|\beta_{r-1}|\leq N}2^{j|\alpha_{r-1}|}2^{k|\beta_{r-1}|}\left(\int_{\mathbb{R}^{2n}}\left|\left(\partial^{\alpha_{r-1}}_{\xi_{r-1}}\partial^{\beta_{r-1}}_{\eta_{r-1}}m_{j+r-1,k+r-1}\right)^{\vee}(\xi_{r-1},\eta_{r-1})\right|^2 d\xi_{r-1}d\eta_{r-1}\right)^{\frac{1}{2}}$$

$$= C_{r-1}\sum_{|\alpha_{r-1}|\leq N,|\beta_{r-1}|\leq N}2^{j|\alpha_{r-1}|}2^{k|\beta_{r-1}|}\left(\int_{\mathbb{R}^{2n}}\left|\partial^{\alpha_{r-1}}_{\xi_{r-1}}\partial^{\beta_{r-1}}_{\eta_{r-1}}\left(m_{j+r-1,k+r-1}\right)(\xi_{r-1},\eta_{r-1})\right|^2 d\xi_{r-1}d\eta_{r-1}\right)^{\frac{1}{2}}.$$

For multi-indices $\delta_{r-1} = (\delta_r,\delta_{r+1},\ldots,\delta_{n+r-1})$ and $\gamma_{r-1} = (\gamma_r,\gamma_{r+1},\ldots,\gamma_{n+r-1})$, we denote $\delta_{r-1} \leq \alpha_{r-1}$ to mean $\delta_{i+r-1} \leq \alpha_{i+r-1}$ and $\gamma_{r-1} \leq \beta_{r-1}$ to mean $\gamma_{i+r-1} \leq \beta_{i+r-1}$ for all $i = 1,2,\ldots,n$. Considering that $\hat{\psi}(\xi_{r-1})$ is supported on $\{\xi_{r-1}|\ \frac{1}{2} \leq |\xi_{r-1}| \leq 2\}$, we can use Leibniz's rule to obtain

$$\int_{\mathbb{R}^{2n}}\left|\partial^{\alpha_{r-1}}_{\xi_{r-1}}\partial^{\beta_{r-1}}_{\eta_{r-1}}\left(m_{j+r-1,k+r-1}\right)(\xi_{r-1},\eta_{r-1})\right|^2 d\xi_{r-1}d\eta_{r-1}$$

$$\leq \sum_{\substack{\delta_{r-1}\leq\alpha_{r-1}\\ \gamma_{r-1}\leq\beta_{r-1}}} (C_{r-1})_{\delta_{r-1},\gamma_{r-1}} \int_{\mathbb{R}^{2n}} \left| 2^{-j|\alpha_{r-1}-\delta_{r-1}|-k|\beta_{r-1}-\gamma_{r-1}|} \left(\partial_{\xi_{r-1}}^{\alpha_{r-1}-\delta_{r-1}}(\hat{\psi})\right)(2^{-j}\xi_{r-1}) \left(\partial_{\eta_{r-1}}^{\beta_{r-1}-\gamma_{r-1}}(\hat{\psi})\right) \right.$$

$$\left. \times (2^{-k}\eta_{r-1}) \partial_{\xi_{r-1}}^{\delta_{r-1}} \partial_{\beta_{r-1}}^{\gamma_{r-1}} m_{r-1}(\xi_{r-1},\eta_{r-1}) \right|^2 d\xi_{r-1} d\eta_{r-1}$$

$$\leq \sum_{\substack{\delta_{r-1}\leq\alpha_{r-1}\\ \gamma_{r-1}\leq\beta_{r-1}}} (C_{r-1})_{\delta_{r-1},\gamma_{r-1}} 2^{-2j|\alpha_{r-1}|} 2^{2j|\delta_{r-1}|} 2^{-2k|\beta_{r-1}|} 2^{2k|\gamma_{r-1}|}$$

$$\times \int_{\substack{2^{j-1}\leq|\xi_{r-1}|\leq 2^{j+1}\\ 2^{k-1}\leq|\eta_{r-1}|\leq 2^{k+1}}} \left| (\partial_{\xi_{r-1}}^{\delta_{r-1}} \partial_{\beta_{r-1}}^{\gamma_{r-1}} m_{r-1}(\xi_{r-1},\eta_{r-1}) \right|^2 d\xi_{r-1} d\eta_{r-1}$$

$$\leq \sum_{\substack{\delta_{r-1}\leq\alpha_{r-1}\\ \gamma_{r-1}\leq\beta_{r-1}}} (C_{r-1})_{\delta_{r-1},\gamma_{r-1}} 2^{-2j|\alpha_{r-1}|} 2^{2j|\delta_{r-1}|} 2^{-2k|\beta_{r-1}|} 2^{2k|\gamma_{r-1}|} 2^{jn} 2^{kn} 2^{-2j|\delta_{r-1}|} 2^{-2k|\gamma_{r-1}|}.$$

Then

$$\left\| m_{j+r-1,k+r-1}^{\vee}(\xi_{r-1},\eta_{r-1})(1+2^j|\xi_{r-1}|)^N (1+2^k|\eta_{r-1}|)^N \right\|_{L^2} \left\| \frac{1}{(1+2^j|\xi_{r-1}|)^{N-\frac{n}{1+\epsilon}}} \frac{1}{(1+2^k|\eta_{r-1}|)^{N-\frac{n}{1+\epsilon}}} \right\|_{L^2}$$

$$= C_{r-1} 2^{-\frac{jn}{2}} 2^{-\frac{kn}{2}} \sum_{|\alpha_{r-1}|\leq N, |\beta_{r-1}|\leq N} (C_{r-1})_{\alpha_{r-1},\beta_{r-1}} 2^{j|\alpha_{r-1}|} 2^{2k|\beta_{r-1}|} \left(\int_{\mathbb{R}^{2n}} \left| (\partial_{\xi_{r-1}}^{\delta_{r-1}} \partial_{\beta_{r-1}}^{\gamma_{r-1}} (m_{j+r-1,k+r-1})(\xi_{r-1},\eta_{r-1}) \right|^2 d\xi_{r-1} d\eta_{r-1} \right)^{\frac{1}{2}}$$

$$\leq C_{r-1} 2^{-\frac{jn}{2}} 2^{-\frac{kn}{2}} \sum_{|\alpha_{r-1}|\leq N, |\beta_{r-1}|\leq N} (C_{r-1})_{\alpha_{r-1},\beta_{r-1}} 2^{j|\alpha_{r-1}|} 2^{2k|\beta_{r-1}|} \left(\sum_{\delta_{r-1}\leq\alpha_{r-1}, \gamma_{r-1}\leq\beta_{r-1}} (C_{r-1})_{\delta_{r-1},\gamma_{r-1}} 2^{jn} 2^{kn} 2^{-2j|\alpha_{r-1}|} 2^{-2k|\beta_{r-1}|} \right)^{\frac{1}{2}} < \infty,$$

where C_{r-1} depends only on $n, N, A_{r-1}, (1+\epsilon)$. Therefore

$$\sum_d \psi_{j,k} * (m_{j+r-1,k+r-1}\widehat{f^d})^{\vee} \lesssim \sum_d \left(M_{s_d}\left(|\Delta_{j,k}(f^d)|\right)^{1+\epsilon}(x_{r-1},y_{r-1}) \right)^{\frac{1}{1+\epsilon}}.$$

Similarly we obtain

$$\sum_d \psi_{j,k} * \left(m_{(j-i_1)+r-1,(k-i_2)+r-1}\widehat{f^d}\right)^{\vee} \lesssim \sum_d \left(M_{s_d}\left(|\Delta_{j,k}(f^d)|\right)^{1+\epsilon}(x_{r-1},y_{r-1}) \right)^{\frac{1}{1+\epsilon}} \ (i \in \{1,0,-1\}).$$

Then

$$\sum_d \psi_{j,k} * (m_{r-1}\widehat{f^d})^{\vee} \lesssim \sum_d \left(M_{s_d}\left(|\Delta_{j,k}(f^d)|\right)^{1+\epsilon}(x_{r-1},y_{r-1}) \right)^{\frac{1}{1+\epsilon}}.$$

Therefore we conclude that

$$\sum_d \|T_d(f^d)\|_{\dot{B}_{1+\epsilon}^{\rho_{r-1},1+2\epsilon}} = C_{r-1} \sum_d \left\| \left(m_{r-1}\widehat{f^d}\right)^{\vee} \right\|_{\dot{B}_{1+\epsilon}^{\rho_{r-1},1+2\epsilon}}$$

$$\leq C_{r-1} \sum_d \left(\sum_{j,k} \left(2^{j\rho_r+k\rho_{r+1}} \left\| \left(M_{s_d}\left(|\Delta_{j,k}(f^d)|^{1+\epsilon} \right) \right)^{\frac{1}{1+\epsilon}} \right\|_{L^{1+\epsilon}} \right)^{1+2\epsilon} \right)^{\frac{1}{1+2\epsilon}}$$

$$\leq C_{r-1} \sum_d \left(\sum_{j,k} \left(2^{j\rho_r+k\rho_{r+1}} \left\| \left(|\Delta_{j,k}(f^d)|^{1+\epsilon} \right) \right\|_{L}^{\frac{1}{1+\epsilon}} \right)^{1+2\epsilon} \right)^{\frac{1}{1+2\epsilon}}$$

$$= \sum_d \left(\sum_{j,k} \left(2^{j\rho_r+k\rho_{r+1}} \|\Delta_{j,k}(f^d)\|_{L^{1+\epsilon}} \right)^{1+2\epsilon} \right)^{\frac{1}{1+2\epsilon}} = C_{r-1} \sum_d \|f^d\|_{\dot{B}_{1+\epsilon}^{\rho_{r-1},1+2\epsilon}}$$

for the sequence $f^d \in S^{\infty}(\mathbb{R}^n \times \mathbb{R}^n)$. Since $S^{\infty}(\mathbb{R}^n \times \mathbb{R}^n)$ are dense in $\dot{B}_{1+\epsilon}^{\rho_{r-1},1+2\epsilon}$, the operators T_d can be extended to a bounded operators on $\dot{B}_{1+\epsilon}^{\rho_{r-1},1+2\epsilon}$. Then the proof of Part 1 is accomplished.

Part 2. We show that

$$\sum_d \|T_d(f^d)\|_{\dot{F}_{1+\epsilon}^{\rho_{r-1},1+2\epsilon}} \leq C_{r-1} \sum_d \|f^d\|_{\dot{F}_{1+\epsilon}^{\rho_{r-1},1+2\epsilon}}. \tag{130}$$

In Part 1, we have obtained

$$\sum_d \psi_{j,k} * (m_{r-1}\widehat{f^d})^\vee \lesssim \sum_d \left(M_{s_d}\left(\left|\Delta_{j,k}(f^d)\right|\right)^{1+\epsilon}\right)^{\frac{1}{1+\epsilon}}$$

for the sequence $f^d \in S^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Pick $(1+\epsilon) < \min\{1+2\epsilon, 1+3\epsilon\}$ such that $\left(\frac{1+2\epsilon}{1+\epsilon}, \frac{1+3\epsilon}{1+\epsilon}\right) > 1$. According to Fefferman–Stein vector-valued inequality in the bi-parameter setting, we obtain

$$\sum_d \|T_d(f^d)\|_{\dot{F}_{1+\epsilon}^{\rho_{r-1},1+2\epsilon}} = C_{r-1}\sum_d \left\|\left(m_{r-1}\widehat{f^d}\right)^\vee\right\|_{\dot{F}_{1+\epsilon}^{\rho_{r-1},1+2\epsilon}}$$

$$\le C_{r-1}\sum_d \left\|\left(\sum_{j,k}\left(2^{j\rho_r+k\rho_{r+1}}\left(M_{s_d}\left(\left|\Delta_{j,k}(f^d)\right|^{1+\epsilon}\right)\right)^{\frac{1}{1+\epsilon}}\right)^{1+2\epsilon}\right)^{\frac{1}{1+2\epsilon}}\right\|_{L^{1+\epsilon}}$$

$$\le C_{r-1}\sum_d \left\|\left(\sum_{j,k}\left(2^{j\rho_r+k\rho_{r+1}}\left|\Delta_{j,k}(f^d)\right|^{1+\epsilon}\right)^{\frac{1+2\epsilon}{1+\epsilon}}\right)^{\frac{1+\epsilon}{1+2\epsilon}}\right\|_{L}^{\frac{1}{1+\epsilon}}$$

$$= C_{r-1}\sum_d \left\|\left(\sum_{j,k}\left(2^{j\rho_r+k\rho_{r+1}}\left|\Delta_{j,k}(f^d)\right|\right)^{1+2\epsilon}\right)^{\frac{1}{1+2\epsilon}}\right\|_{L^{1+\epsilon}} = C_{r-1}\sum_d \|f^d\|_{\dot{F}_{1+\epsilon}^{\rho_{r-1},1+2\epsilon}},$$

which concludes the proof of Corollary (5.3.18).

Corollary (5.3.19) [314] Let $-1 < \epsilon < \infty, N = \left[\frac{n}{2} + \frac{n}{min(1+\epsilon,2)}\right] + 1$. Assume that $m_{r-1}(\xi_{r-1}, \eta_{r-1})$ is a $L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ function that satisfies

$$\left|\partial_{\xi_{r-1}}^{\alpha_{r-1}}\partial_{\eta_{r-1}}^{\beta_{r-1}} m_{r-1}(\xi_{r-1}, \eta_{r-1})\right| \le A_{r-1}\frac{1}{|\xi_{r-1}|^{|\alpha_{r-1}|}|\eta_{r-1}|^{|\beta_{r-1}|}}$$

for all $|\alpha_{r-1}| \le N, |\beta_{r-1}| \le N$ and $(\xi_{r-1}, \eta_{r-1}) \in (\mathbb{R}^n \times \mathbb{R}^n)$ with $|\xi_{r-1}||\eta_{r-1}| \ne 0$. Then T_d is bounded on $L^{1+\epsilon}$ for $0 < \epsilon < \infty$ and on $H^{1+\epsilon}$ for $-1 < \epsilon \le 0$.

We show the bi-parameter Fourier multiplier operators are bounded on weighted bi-parameter Besov–Lipschitz and Triebel–Lizorkin spaces. First we state a well-known vector-valued weighted maximal inequality.

Corollary (5.3.20) [314] Assume $0 \le \epsilon < \infty$ and $w_{r-1} \in A_{1+\epsilon}(\mathbb{R}^n \times \mathbb{R}^n)$. Then there exists a constant $C_{r-1} > 0$ such that

$$\sum_d \left\|\left\{\sum_{k\in\mathbb{Z}}\left(M_{s_d}(f_k^d)\right)^{1+2\epsilon}\right\}^{\frac{1}{1+2\epsilon}}\right\|_{L^{1+2\epsilon}(w_{r-1})} \le C_{r-1}\sum_d \left\|\left\{\sum_{k\in\mathbb{Z}}\left|f_k^d\right|^{1+2\epsilon}\right\}^{\frac{1}{1+2\epsilon}}\right\|_{L^{1+\epsilon}(w_{r-1})}$$

for all double sequences $\{f_k^d\}$ of locally integrable functions on $\mathbb{R}^n \times \mathbb{R}^n$.

Corollary (5.3.21) [314] Suppose that T_d are a bi-parameter Fourier multiplier operators defined on $(\mathbb{R}^n \times \mathbb{R}^n)$. Let $\rho_{r-1} = (\rho_r, \rho_{r+1}) \in \mathbb{R}^2, 0 \le \epsilon < \infty, N = \left[\frac{n}{2} + \frac{n}{min(1+\epsilon,1+2\epsilon)}\right] + 1$. Assume that $m_{r-1}(\xi_{r-1}, \eta_{r-1})$ is a $L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ function that satisfies

$$\left|\partial_{\xi_{r-1}}^{\alpha_{r-1}}\partial_{\eta_{r-1}}^{\beta_{r-1}} m_{r-1}(\xi_{r-1}, \eta_{r-1})\right| \le A_{r-1}\frac{1}{|\xi_{r-1}|^{|\alpha_{r-1}|}|\eta_{r-1}|^{|\beta_{r-1}|}}$$

for all $|\alpha_{r-1}| \le N, |\beta_{r-1}| \le N$ and $(\xi_{r-1}, \eta_{r-1}) \in (\mathbb{R}^n \times \mathbb{R}^n)$ with $|\xi_{r-1}||\eta_{r-1}| \ne 0$. If $w_{r-1} \in A_\infty(\mathbb{R}^n \times \mathbb{R}^n)$, then the following inequalities are valid with the constant C_{r-1} independent of the sequence f^d :

(i) $\sum_d \|T_d(f^d)\|_{\dot{B}_{1+\epsilon}^{\rho_{r-1},1+2\epsilon}(w_{r-1})} \le C_{r-1}\sum_d \|f^d\|_{\dot{B}_{1+\epsilon}^{\rho_{r-1},1+2\epsilon}(w_{r-1})}$,

(ii) $\sum_d \|T_d(f^d)\|_{\dot{F}_{1+\epsilon}^{\rho_{r-1},1+2\epsilon}(w_{r-1})} \le C_{r-1}\sum_d \|f^d\|_{\dot{F}_{1+\epsilon}^{\rho_{r-1},1+2\epsilon}(w_{r-1})}$.

Finally, we extend the above theorems to the cases where the multipliers $m_{r-1}(\xi_{r-1}, \eta_{r-1})$ have weaker decay condition and worse singularity at the origin.

Proof. In the proof of Corollary (5.3.18), we have obtained the pointwise estimate

$$\sum_d \left|\Delta_{j,k}(T_d(f^d))\right| \lesssim \sum_d \left(M_{s_d}\left(\left|\Delta_{j,k}(f^d)\right|^{1+\epsilon}\right)\right)^{\frac{1}{1+\epsilon}}.$$

Pick $(1+\epsilon)$ such that $1 < \frac{1+2\epsilon}{1+\epsilon}, \frac{1+3\epsilon}{1+\epsilon} < \infty$ and $w_{r-1} \in A_{\frac{1+2\epsilon}{1+\epsilon}}$, we have

$$\sum_d \|\Delta_{j,k}((m_{r-1}\widehat{f^d})^\vee)\|_{L^{1+\epsilon}(w_{r-1})} \leq C_{r-1} \sum_d \left\| \left(M_{s_d}\left(|\Delta_{j,k}(f^d)|^{1+\epsilon} \right) \right)^{\frac{1}{1+\epsilon}} \right\|_{L^{1+\epsilon}(w_{r-1})} \leq C_{r-1} \sum_d \|\Delta_{j,k}(f^d)\|_{L^{1+\epsilon}(w_{r-1})}. \quad (131)$$

By Corollary (5.3.20) and (131), we can derive that

$$\sum_d \|T_d(f^d)\|_{\dot{B}_{1+\epsilon}^{\rho_{r-1},1+2\epsilon}(w_{r-1})} = C_{r-1} \sum_d \left\| \left(m_{r-1}\widehat{f^d} \right)^\vee \right\|_{\dot{B}_{1+\epsilon}^{\rho_{r-1},1+2\epsilon}(w_{r-1})}$$

$$= \sum_d \left(\sum_{j,k} \left(2^{j\alpha_r + k\alpha_{r+1}} \left\| \Delta_{j,k}\left(\left(m_{r-1}\widehat{f^d} \right)^\vee \right) \right\|_{L^{1+\epsilon}(w_{r-1})} \right)^{1+2\epsilon} \right)^{\frac{1}{1+2\epsilon}}$$

$$\leq C_{r-1} \sum_d \left(\sum_{j,k} \left(2^{j\alpha_r + k\alpha_{r+1}} \left\| \left(M_{s_d}\left(|\Delta_{j,k}(f^d)|^{1+\epsilon} \right) \right)^{\frac{1}{1+\epsilon}} \right\|_{L^{1+\epsilon}(w_{r-1})} \right)^{1+2\epsilon} \right)^{\frac{1}{1+2\epsilon}}$$

$$\leq \sum_d \left(\sum_{j,k} \left(2^{j\alpha_r + k\alpha_{r+1}} \|\Delta_{j,k}(f^d)^{1+\epsilon}\|_{L(w_{r-1})}^{\frac{1}{1+\epsilon}} \right)^{1+2\epsilon} \right)^{\frac{1}{1+2\epsilon}} = C_{r-1} \sum_d \|f^d\|_{\dot{B}_{1+\epsilon}^{\rho_{r-1},1+2\epsilon}(w_{r-1})}$$

and

$$\sum_d \|T_d(f^d)\|_{\dot{F}_{1+\epsilon}^{\rho_{r-1},1+2\epsilon}(w_{r-1})} = C_{r-1} \sum_d \left\| \left(m_{r-1}\widehat{f^d} \right)^\vee \right\|_{\dot{F}_{1+\epsilon}^{\rho_{r-1},1+2\epsilon}(w_{r-1})}$$

$$= \sum_d \left\| \left(\sum_{j,k} \left(2^{j\alpha_r + k\alpha_{r+1}} \left| \Delta_{j,k}\left(\left(m_{r-1}\widehat{f^d} \right)^\vee \right) \right| \right)^{1+2\epsilon} \right)^{\frac{1}{1+2\epsilon}} \right\|_{L^{1+\epsilon}(w_{r-1})}$$

$$\leq C_{r-1} \sum_d \left\| \left(\sum_{j,k} \left(2^{j\alpha_r + k\alpha_{r+1}} \left(M_{s_d}\left(|\Delta_{j,k}(f^d)|^{1+\epsilon} \right) \right)^{\frac{1}{1+\epsilon}} \right)^{1+2\epsilon} \right)^{\frac{1}{1+2\epsilon}} \right\|_{L^{1+\epsilon}(w_{r-1})}$$

$$\leq C_{r-1} \sum_d \left\| \left(\sum_{j,k} \left(2^{(j\alpha_r + k\alpha_{r+1})(1+\epsilon)} |\Delta_{j,k}(f^d)|^{1+\epsilon} \right)^{\frac{1+2\epsilon}{1+\epsilon}} \right)^{\frac{1+\epsilon}{1+2\epsilon}} \right\|_{L(w_{r-1})}^{\frac{1}{1+\epsilon}} = C_{r-1} \sum_d \|f^d\|_{\dot{F}_{1+\epsilon}^{\rho_{r-1},1+2\epsilon}(w_{r-1})},$$

which concludes the proof of Corollary (5.3.21).

Corollary (5.3.22) [314] Let $\rho_{r-1} = (\rho_r, \rho_{r+1}) \in \mathbb{R}^2, m_{r-1} = (m_r, m_{r+1}) \in \mathbb{R}^+ \times \mathbb{R}^+, 0 \leq \epsilon < \infty$ and $N = \left[\frac{n}{2} + \frac{n}{min(1+\epsilon, 1+2\epsilon)}\right] + 1$. Assume that $m_{r-1}(\xi_{r-1}, \eta_{r-1})$ satisfies

$$\left| \partial_{\xi_{r-1}}^{\alpha_{r-1}} \partial_{\eta_{r-1}}^{\beta_{r-1}} m_{r-1}(\xi_{r-1}, \eta_{r-1}) \right| \leq A_{r-1} \frac{1}{|\xi_{r-1}|^{|\alpha_{r-1}|+m_r} |\eta_{r-1}|^{|\beta_{r-1}|+m_{r+1}}}$$

for all $|\alpha_{r-1}| \leq N, |\beta_{r-1}| \leq N$ and $(\xi_{r-1}, \eta_{r-1}) \in (\mathbb{R}^n \times \mathbb{R}^n)$ with $|\xi_{r-1}||\eta_{r-1}| \neq 0$. If $w_{r-1} \in A_\infty(\mathbb{R}^n \times \mathbb{R}^n)$, Then there exists a constant C_{r-1} such that

$$\sum_d \|T_d(f^d)\|_{\dot{B}_{1+\epsilon}^{\rho_{r-1}+m_{r-1},1+2\epsilon}} \leq C_{r-1} \sum_d \|f^d\|_{\dot{B}_{1+\epsilon}^{\rho_{r-1}+m_{r-1},1+2\epsilon}}$$

and

$$\sum_d \|T_d(f^d)\|_{\dot{F}_{1+\epsilon}^{\rho_{r-1}+m_{r-1},1+2\epsilon}} \leq C_{r-1} \sum_d \|f^d\|_{\dot{F}_{1+\epsilon}^{\rho_{r-1}+m_{r-1},1+2\epsilon}},$$

where the constant C_{r-1} is independent of the sequence f^d.

Proof. By the Littlewood–Paley decomposition, we can derive that

$$\sum_d \Delta_{j,k}\left(T_d(f^d) \right) = \sum_d \psi_{j,k} * T_d(f^d)$$

$$= \sum_d \left(\hat{\psi}(2^{-j}\xi_{r-1})\hat{\psi}(2^{-k}\eta_{r-1}) \sum_{i_1,i_2} m_{(j-i_1)+r-1,(k-i_2)+r-1}(\xi_{r-1},\eta_{r-1})\widehat{f^d}(\xi_{r-1},\eta_{r-1}) \right)^{\vee} (x_{r-1},y_{r-1}),$$

where $(i_1, i_2 \in \{-1, 0, 1\})$. We only verify that

$$\sum_d \left(\hat{\psi}(2^{-j}\xi_{r-1})\hat{\psi}(2^{-k}\xi_{r-1}) m_{j+r-1,k+r-1}(\xi_{r-1},\eta_{r-1})\widehat{f^d}(\xi_{r-1},\eta_{r-1}) \right)^{\vee} (x_{r-1},y_{r-1})$$
$$\lesssim 2^{-jm_r - km_{r+1}} \sum_d \left(M_{s_d}(|\Delta_{j,k}(f^d)|)^{1+\epsilon}(x_{r-1},y_{r-1}) \right)^{\frac{1}{1+\epsilon}}.$$

Pick $(1+\epsilon) < \min\{(1+\epsilon),(1+2\epsilon)\}$ such that $N > \frac{n}{2} + \frac{n}{1+\epsilon}$. We write

$$\sum_d \left| \left(\hat{\psi}(2^{-j}\xi_{r-1})\hat{\psi}(2^{-k}\xi_{r-1}) m_{j+r-1,k+r-1}(\xi_{r-1},\eta_{r-1})\widehat{f^d}(\xi_{r-1},\eta_{r-1}) \right)^{\vee} (x_{r-1},y_{r-1}) \right|$$
$$= \sum_d \left| \int_{\mathbb{R}^{2n}} m^{\vee}_{j+r-1,k+r-1}(\xi_{r-1},\eta_{r-1}) \Delta_{j,k}(f^d)(x_{r-1}-\xi_{r-1}, y_{r-1}-\eta_{r-1}) d\xi_{r-1} d\eta_{r-1} \right|$$
$$\le C_{r+1} \sum_d M_{s_d}\left(|\Delta_{j,k}(f^d)|^{1+\epsilon}(x_{r-1},y_{r-1}) \right)^{\frac{1}{1+\epsilon}} \int_{\mathbb{R}^{2n}} |m^{\vee}_{j+r-1,k+r-1}(\xi_{r-1},\eta_{r-1})| (1+2^j|\xi_{r-1}|)^{\frac{n}{1+\epsilon}} (1$$
$$+ 2^k|\eta_{r-1}|)^{\frac{n}{1+\epsilon}} d\xi_{r-1} d\eta_{r-1}. \tag{132}$$

Next we estimate

$$\int_{\mathbb{R}^{2n}} |m^{\vee}_{j+r-1,k+r-1}(\xi_{r-1},\eta_{r-1})| (1+2^j|\xi_{r-1}|)^{\frac{n}{1+\epsilon}} (1+2^k|\eta_{r-1}|)^{\frac{n}{1+\epsilon}} d\xi_{r-1} d\eta_{r-1}.$$

We use the Hölder's inequality to obtain

$$\int_{\mathbb{R}^{2n}} |m^{\vee}_{j+r-1,k+r-1}(\xi_{r-1},\eta_{r-1})| (1+2^j|\xi_{r-1}|)^N (1+2^k|\eta_{r-1}|)^N \frac{1}{(1+2^j|\xi_{r-1}|)^{N-\frac{n}{1+\epsilon}}} \frac{1}{(1+2^k|\eta_{r-1}|)^{N-\frac{n}{1+\epsilon}}} d\xi_{r-1} d\eta_{r-1}$$
$$\le C_{r-1} 2^{-\frac{jn}{2}} 2^{-\frac{kn}{2}} \left\| m^{\vee}_{j+r-1,k+r-1}(\xi_{r-1},\eta_{r-1})(1+2^j|\xi_{r-1}|)^N (1+2^k|\eta_{r-1}|)^N \right\|_{L^2}.$$

It suffices to estimate $\left\| m^{\vee}_{j+r-1,k+r-1}(\xi_{r-1},\eta_{r-1})(1+2^j|\xi_{r-1}|)^N (1+2^k|\eta_{r-1}|)^N \right\|_{L^2}$.

$$\left\| m^{\vee}_{j+r-1,k+r-1}(\xi_{r-1},\eta_{r-1})(1+2^j|\xi_{r-1}|)^N (1+2^k|\eta_{r-1}|)^N \right\|_{L^2}$$
$$\le C_{r-1} \sum_{|\alpha_{r-1}|\le N, |\beta_{r-1}|\le N} \left(\int_{\mathbb{R}^{2n}} |m^{\vee}_{j+r-1,k+r-1}(\xi_{r-1},\eta_{r-1})|^2 |\xi_{r-1}^{\alpha_{r-1}}|^2 |\eta_{r-1}^{\beta_{r-1}}|^2 2^{2j|\alpha_{r-1}|} 2^{2k|\beta_{r-1}|} d\xi_{r-1} d\eta_{r-1} \right)^{\frac{1}{2}}$$
$$= C_{r-1} \sum_{|\alpha_{r-1}|\le N, |\beta_{r-1}|\le N} (C_{r-1})_{\alpha_{r-1},\beta_{r-1}} 2^{j|\alpha_{r-1}|} 2^{k|\beta_{r-1}|} \left(\int_{\mathbb{R}^{2n}} \left| (\partial^{\alpha_{r-1}}_{\xi_{r-1}} \partial^{\beta_{r-1}}_{\eta_{r-1}}(m_{j+r-1,k+r-1})(\xi_{r-1},\eta_{r-1}) \right|^2 d\xi_{r-1} d\eta_{r-1} \right)^{\frac{1}{2}}. \tag{133}$$

We use Leibniz's rule to obtain

$$\int_{\mathbb{R}^{2n}} \left| \partial^{\alpha_{r-1}}_{\xi_{r-1}} \partial^{\beta_{r-1}}_{\eta_{r-1}}(m_{j+r-1,k+r-1})(\xi_{r-1},\eta_{r-1}) \right|^2 d\xi_{r-1} d\eta_{r-1}$$
$$\le \sum_{\substack{\delta_{r-1}\le\alpha_{r-1} \\ \gamma_{r-1}\le\beta_{r-1}}} (C_{r-1})_{\delta_{r-1},\gamma_{r-1}} \int_{\mathbb{R}^{2n}} \left| 2^{-j|\alpha_{r-1}-\delta_{r-1}|-k|\beta_{r-1}-\gamma_{r-1}|} \left(\partial^{\alpha_{r-1}-\delta_{r-1}}_{\xi_{r-1}}(\hat{\psi}) \right) (2^{-j}\xi_{r-1}) \right.$$
$$\left. \times \left(\partial^{\beta_{r-1}-\gamma_{r-1}}_{\eta_{r-1}}(\hat{\psi}) \right) (2^{-k}\eta_{r-1}) \partial^{\delta_{r-1}}_{\xi_{r-1}} \partial^{\gamma_{r-1}}_{\beta_{r-1}} m_{r-1}(\xi_{r-1},\eta_{r-1}) \right|^2 d\xi_{r-1} d\eta_{r-1}$$
$$\le \sum_{\substack{\delta_{r-1}\le\alpha_{r-1} \\ \gamma_{r-1}\le\beta_{r-1}}} (C_{r-1})_{\delta_{r-1},\gamma_{r-1}} 2^{-2j|\alpha_{r-1}|} 2^{2j|\delta_{r-1}|} 2^{-2k|\beta_{r-1}|} 2^{2k|\gamma_{r-1}|} 2^{jn} 2^{kn} 2^{-2j(|\delta_{r-1}|+m_r)} 2^{-2k(|\gamma_{r-1}|+m_{r+1})}. \tag{134}$$

According to (133) and (134), we derive

$$\left\| m^{\vee}_{j+r-1,k+r-1}(\xi_{r-1},\eta_{r-1})(1+2^j|\xi_{r-1}|)^N (1+2^k|\eta_{r-1}|)^N \right\|_{L^2}$$

$$\leq C_{r-1}\sum_{|\alpha_{r-1}|\leq N,|\beta_{r-1}|\leq N}(C_{r-1})_{\alpha_{r-1},\beta_{r-1}}2^{j|\alpha_{r-1}|}2^{k|\beta_{r-1}|}\left(\sum_{\substack{\delta_{r-1}\leq\alpha_{r-1}\\ \gamma_{r-1}\leq\beta_{r-1}}}(C_{r-1})_{\delta_{r-1},\gamma_{r-1}}2^{jn}2^{kn}2^{-2j(|\delta_{r-1}|+m_r)}2^{-2k(|\gamma_{r-1}|+m_{r+1})}\right)^{\frac{1}{2}}$$

$$= 2^{-jm_r-km_{r+1}}C_{r-1}<\infty. \tag{135}$$

Combining (132) and (135), we have

$$\sum_d \psi_{j,k}*(m_{j+r-1,k+r-1}\widehat{f^d})^\vee \lesssim 2^{-jm_r-km_{r+1}}\sum_d\left(M_{s_d}\left(\left|\Delta_{j,k}(f^d)\right|\right)^{1+\epsilon}(x_{r-1},y_{r-1})\right)^{\frac{1}{1+\epsilon}}.$$

Similarly we also have

$$\sum_d \psi_{j,k}*\left(m_{(j-i_1)+r-1,(k-i_2)+r-1}\widehat{f^d}\right)^\vee \lesssim 2^{-jm_r-km_{r+1}}\sum_d\left(M_{s_d}\left(\left|\Delta_{j,k}(f^d)\right|\right)^{1+\epsilon}(x_{r-1},y_{r-1})\right)^{\frac{1}{1+\epsilon}}\ (i\in\{1,0,-1\}).$$

Therefore

$$\sum_d\|T_d(f^d)\|_{\dot B_{1+\epsilon}^{\rho_{r-1}+m_{r-1},1+2\epsilon}} = C_{r-1}\sum_d\left\|(m_{r-1}\widehat{f^d})^\vee\right\|_{\dot B_{1+\epsilon}^{\rho_{r-1}+m_{r-1},1+2\epsilon}}$$

$$\leq C_{r-1}\sum_d\left(\sum_{j,k}\left(2^{j\rho_r+k\rho_{r+1}}\left\|\left(M_{s_d}\left(\left|\Delta_{j,k}(f^d)\right|^{1+\epsilon}\right)\right)^{\frac{1}{1+\epsilon}}\right\|_{L^{1+\epsilon}}\right)^{1+2\epsilon}\right)^{\frac{1}{1+2\epsilon}}$$

$$\leq C_{r-1}\sum_d\left(\sum_{j,k}\left(2^{j\rho_r+k\rho_{r+1}}\left\|\left(\left|\Delta_{j,k}(f^d)\right|^{1+\epsilon}\right)\right\|_L^{\frac{1}{1+\epsilon}}\right)^{1+2\epsilon}\right)^{\frac{1}{1+2\epsilon}} \leq C_{r-1}\sum_d\|f^d\|_{\dot B_{1+\epsilon}^{\rho_{r-1},1+2\epsilon}}$$

and

$$\sum_d\|T_d(f^d)\|_{\dot F_{1+\epsilon}^{\rho_{r-1}+m_{r-1},1+2\epsilon}} = C_{r-1}\sum_d\left\|\left(m_{r-1}\widehat{f^d}\right)^\vee\right\|_{\dot F_{1+\epsilon}^{\rho_{r-1}+m_{r-1},1+2\epsilon}}$$

$$\leq C_{r-1}\sum_d\left\|\left(\sum_{j,k}\left(2^{j\rho_r+k\rho_{r+1}}\left(M_{s_d}\left(\left|\Delta_{j,k}(f^d)\right|^{1+\epsilon}\right)\right)^{\frac{1}{1+\epsilon}}\right)^{1+2\epsilon}\right)^{\frac{1}{1+2\epsilon}}\right\|_{L^{1+\epsilon}}$$

$$\leq C_{r-1}\sum_d\left\|\left(\sum_{j,k}\left(2^{j\rho_r+k\rho_{r+1}}\left|\Delta_{j,k}(f^d)\right|^{1+\epsilon}\right)^{\frac{1+2\epsilon}{1+\epsilon}}\right)^{\frac{1+\epsilon}{1+2\epsilon}}\right\|_L^{\frac{1}{1+\epsilon}} \leq C_{r-1}\sum_d\left\|f^d\right\|_{\dot F_{1+\epsilon}^{\rho_{r-1},1+2\epsilon}},$$

which conclude the proof of Corollary (5.3.22).

Corollary (5.3.23) [314] Let $\rho_{r-1}=(\rho_r,\rho_{r+1})\in\mathbb{R}^2, m_{r-1}=(m_r,m_{r+1})\in\mathbb{R}^+\times\mathbb{R}^+, 0\leq\epsilon<\infty$ and $N=\left[\frac{n}{2}+\frac{n}{min(1+\epsilon,1+2\epsilon)}\right]+1$. Assume that $m_{r-1}(\xi_{r-1},\eta_{r-1})$ satisfies

$$\left|\partial_{\xi_{r-1}}^{\alpha_{r-1}}\partial_{\eta_{r-1}}^{\beta_{r-1}}m_{r-1}(\xi_{r-1},\eta_{r-1})\right|\leq A_{r-1}\frac{1}{|\xi_{r-1}|^{|\alpha_{r-1}|+m_r}|\eta_{r-1}|^{|\beta_{r-1}|+m_{r+1}}}$$

for all $|\alpha_{r-1}|\leq N, |\beta_{r-1}|\leq N$ and $(\xi_{r-1},\eta_{r-1})\in(\mathbb{R}^n\times\mathbb{R}^n)$ with $|\xi_{r-1}||\eta_{r-1}|\neq 0$. If $w_{r-1}\in A_\infty(\mathbb{R}^n\times\mathbb{R}^n)$, Then there exists a constant C_{r-1} such that

$$\sum_d\|T_d(f^d)\|_{\dot B_{1+\epsilon}^{\rho_{r-1}+m_{r-1},1+2\epsilon}(w_{r-1})}\leq C_{r-1}\sum_d\|f^d\|_{\dot B_{1+\epsilon}^{\rho_{r-1}+m_{r-1},1+2\epsilon}(w_{r-1})}$$

and

$$\sum_d\|T_d(f^d)\|_{\dot F_{1+\epsilon}^{\rho_{r-1}+m_{r-1},1+2\epsilon}(w_{r-1})}\leq C_{r-1}\sum_d\|f^d\|_{\dot F_{1+\epsilon}^{\rho_{r-1}+m_{r-1},1+2\epsilon}(w_{r-1})},$$

where the constant C_{r-1} is independent of the sequence f^d.

Since the proof of Corollary (5.3.23) is similar to that of Corollary (5.3.21), we omit the details.

Chapter 6
Characterizations of Logarithmic Besov Spaces in Terms of Differences

We show that $B_{2,2}^{0,b+1/2} = \mathbf{B}_{2,2}^{0,b}$ for $b > -1/2$. We also determine the dual of $\mathbf{B}_{p,q}^{0,b}$ with the help of logarithmic Lipschitz spaces $\mathrm{Lip}_{p,q}^{(1,-\alpha)}$. We show embeddings between spaces $\mathrm{Lip}_{p,q}^{(1,-\alpha)}$ and $B_{p,q}^{1,b}$ which complement and improve embeddings established. We apply the reiteration formula and limiting interpolation to investigate several problems on Besov spaces, including embeddings in Lorentz–Zygmund spaces and distribution of Fourier coefficients. We characterize $\mathbf{B}_{p,q}^{0,b}$ by means of Fourier-analytical decompositions, wavelets and semi-groups. We also compare those results with the well-known characterizations for classical Besov spaces $B_{p,q}^{s}$.

Section (6.1) Logarithmic Smoothness and Lipschitz Spaces

Besov spaces $B_{p,q}^{s}$ play a central role in the theory of function spaces as can be seen by Triebel [65–67]. For the complete solution of some natural questions as compactness in limiting embeddings [165, 261] or spaces on fractals [270, 271], more general spaces have been introduced where smoothness of functions is considered in a more delicate manner than in $B_{p,q}^{s}$. These spaces of generalized smoothness have been studied for long and from different points of view. See, DeVore, Riemenschneider and Sharpley [226], Brézis and Wainger [258], Gol'dman [275], Merucci [149], Kalyabin and Lizorkin [32], Cobos and Fernandez [142], Edmunds and Haroske [268], Haroske and Moura [28], Farkas and Leopold [144], Triebel [67, pp. 52–55].

As in the case of $B_{p,q}^{s}$, spaces of generalized smoothness on $\mathbb{R}^n$ can be introduced by following the Fourier analytic approach or by means of the modulus of smoothness. If we take classical smoothness sand additional logarithmic smoothness with exponent b, the first way leads to spaces $B_{p,q}^{s,b}$ and the second to spaces $\mathbf{B}_{p,q}^{s,b}$. If $1 \le p \le \infty$ and $s > 0$, it turns out that $B_{p,q}^{s,b} = \mathbf{B}_{p,q}^{s,b}$ with equivalence of norms (see [28, Theorem 2.5] and [65, 2.5.12]; but if $0 < p < 1$ and $0 < q \le 1$ then $B_{p,q}^{n(1/p-1)} \neq \mathbf{B}_{p,q}^{n(1/p-1)}$ as it is shown in [52, Corollary 3.10]). However, the relation between these two kinds of spaces when $s = 0$ has not been described yet. This problem is stated in the report of Triebel [282, p. 6], where first results on this question have been shown: Working with spaces on the unit cube $\mathbb{Q}^n$ in $\mathbb{R}^n$ and using the Haar basis, Triebel established in [282, Proposition 9] that $\mathbf{B}_{p,q}^{0,b}(\mathbb{Q}^n) \hookrightarrow B_{p,q}^{0,b}(\mathbb{Q}^n)$ provided that $1 < p \le 2$ and $b \ge 0$ or $2 < p < \infty$ and $b > 1/2 - 1/p$. See also Besov [257], where spaces $B_{p,q}^{0}, 1 \le p, q \le \infty$, are compared with certain spaces defined by first differences.

We compare spaces $B_{p,q}^{s,b}$ and $\mathbf{B}_{p,q}^{s,b}$ with the help of the limiting real method

$$(A_0, A_1)_{(\theta,\eta),q} = \left\{ a \in A_0 : \|a\|_{\overline{A}_{(\theta,\eta),q}} = \left(\int_0^1 \left(\frac{K(t,a)}{t^{\theta}(1-\log t)^{\eta}} \right)^q \frac{dt}{t} \right)^{1/q} < \infty \right\}.$$

Here $\theta = 0$ or $1, A_0 \hookrightarrow A_1$ and $K(t,a)$ is the K-functional of Peetre. Among other things, for $b > -1/p$ we show that

$$B_{p,p}^{0,b+1/p} \hookrightarrow \mathbf{B}_{p,p}^{0,b} \hookrightarrow B_{p,p}^{0,b+1/2} \qquad \text{if } 1 < p \le 2,$$
$$B_{p,p}^{0,b+1/2} \hookrightarrow \mathbf{B}_{p,p}^{0,b} \hookrightarrow B_{p,p}^{0,b+1/p} \qquad \text{if } 2 \le p < \infty.$$

Therefore, $B_{2,2}^{0,b+1/2} = \mathbf{B}_{2,2}^{0,b}$. This implies that the classical space $\mathbf{B}_{2,2}^{0}$, defined by the modulus of continuity, coincides with the space $B_{2,2}^{0,1/2}$, defined by the Fourier transform, with zero classical smoothness and logarithmic smoothness with exponent $1/2$.

We also consider embeddings between spaces $\mathbf{B}_{p,q}^{s,b}$. According to [62, Theorem 2.8.1] or [1, Corollary 5.4.21], if $1 \le p \le r < \infty, 1 \le q \le \infty$ and $s > 0$, then $\mathbf{B}_{p,q}^{n(1/p-1/r)+s} \hookrightarrow \mathbf{B}_{r,q}^{s}$. Note that in the embedding the two spaces have the same differential dimension. The limit case where $s = 0$ has been studied by DeVore, Riemenschneider and Sharpley [266, Corollary 5.3 (ii)], where they showed that the embedding holds with a loss of a unit in the exponent of the logarithmic smoothness. To be more precise, if $1 \le p \le r \le \infty, 1 \le q \le \infty$ and $b > -1/q$. then $\mathbf{B}_{p,q}^{n(1/p-1/r),b+1} \hookrightarrow \mathbf{B}_{r,q}^{0,b}$. This result has been improved recently by Gogatishvili, Opic, Tikhonov and Trebels [274, Corollary 2.8] by showing that the embedding holds with the loss of only $1/\min\{q,r\}$ in the exponent of the logarithmic smoothness. We use limiting interpolation to derive the embedding $\mathbf{B}_{p,q}^{n(1/p-1/r),b+1/\min\{q,r\}} \hookrightarrow \mathbf{B}_{r,q}^{0,b}$ following a more simple approach than in [274].

In addition we determine the dual of $\mathbf{B}_{p,q}^{0,b}$ for $1 < p < \infty, 1 \le q < \infty$ and $b > -1/q$. This is done with the help of logarithmic Lipschitz spaces $\mathrm{Lip}_{p,q}^{(1,-\alpha)}$ introduced by Haroske in [277] (see also [279, Definition 2.16], [267, p. 149]).

Finally we study embeddings between Lipschitz spaces $\mathrm{Lip}_{p,q}^{(1,-\alpha)}$ and Besov spaces $B_{p,q}^{1,b}$. This problem was considered by Haroske [277, 279] and Neves [280] among other authors. Our approach allows us to cover some critical cases which come up for the techniques used in [277]. As a consequence, we complement and improve several results of Haroske [277].

Subsequently, given two quasi-Banach spaces X, Y, we put $X \hookrightarrow Y$ to mean that X is continuously embedded in Y.

If U, V are non-negative quantities depending on certain parameters, we write $U \lesssim V$ if there is a constant $c > 0$ independent of the parameters in U and V such that $U \le cV$. We put $U \sim V$ if $U \lesssim V$ and $V \lesssim U$.

Let $\bar{A} = (A_0, A_1)$ be a quasi-Banach couple, that is to say, two quasi-Banach spaces A_0, A_1 which are continuously embedded in some Hausdorff topological vector space. The Peetre's K-functional is given by

$$K(t,a) = K(t,a;A_0,A_1) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1}\}, \qquad t > 0\,, a \in A_0 + A_1,$$

where the infimum is taken over all representations $a = a_0 + a_1$ with $a_0 \in A_0$ and $a_1 \in A_1$.

For $0 < \theta < 1$ and $0 < q \le \infty$, the real interpolation space $\bar{A}_{\theta,q} = (A_0, A_1)_{\theta,q}$ is formed by all $a \in A_0 + A_1$ having a finite quasi-norm

$$\|a\|_{\bar{A}_{\theta,q}} = \left(\int_0^\infty \left(t^{-\theta} K(t,a)\right)^q \frac{dt}{t} \right)^{1/q}$$

(as usual, when $q = \infty$ the integral should be replaced by the supremum). See [2, 259] or [62].

For $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$, let $\ell(t) = 1 + |\log t|$ and

$$\ell^{\mathbb{A}}(t) = \begin{cases} \ell^{\alpha_0}(t) & \text{for } t \in (0,1], \\ \ell^{\alpha_\infty}(t) & \text{for } t \in (1,\infty). \end{cases}$$

Replacing t^θ by $t^\theta/\ell^{\mathbb{A}}(t)$ we obtain the spaces

$$\bar{A}_{\theta,q,\mathbb{A}} = (A_0, A_1)_{\theta,q,\mathbb{A}} = \left\{ a \in A_0 + A_1 : \|a\|_{\bar{A}_{\theta,q,\mathbb{A}}} = \left(\int_0^\infty \left(t^{-\theta} \ell^{\mathbb{A}}(t) K(t,a) \right)^q \frac{dt}{t} \right)^{1/q} < \infty \right\}$$

(see [272, 273]). Under suitable assumptions on $\mathbb{A}$ and q, spaces $(A_0, A_1)_{\theta,q,\mathbb{A}}$ are well-defined even if $\theta = 0$ or $\theta = 1$. In the special case $\alpha_0 = \alpha_\infty = \alpha$, we simply write $(A_0, A_1)_{\theta,q,\alpha}$ instead of $(A_0, A_1)_{\theta,q,(\alpha,\alpha)}$.

We shall also need the following limiting real spaces. Let $A_0 \hookrightarrow A_1, 0 < q \leq \infty$ and $-\infty < \eta < \infty$. For $\theta = 1$ or $\theta = 0$, the space $\bar{A}_{(\theta,\eta),q} = (A_0, A_1)_{(\theta,\eta),q}$ consists of all those $a \in A_0$ with

$$\|a\|_{\bar{A}_{(\theta,\eta),q}} = \left(\int_0^1 \left(\frac{K(t,a)}{t^\theta (1 - \log t)^\eta} \right)^q \frac{dt}{t} \right)^{1/q} < \infty$$

(see [276, 260, 262]). To avoid that $\overline{A}_{(1,\eta),q} = \{0\}$, if $\theta = 1$, we assume that $\eta > 1/q$ if $q < \infty$, and $\eta \geq 0$ if $q = \infty$.

The following result is established in [305, Lemma 2.5] by using the connection between limiting real spaces $\overline{A}_{(\theta,\eta),q}$ and logarithmic spaces $\bar{A}_{\theta,q,\mathbb{A}}$ [269, Proposition 1], and reiteration results for logarithmic spaces [272, Theorems 5.9*, 4.7*, 5.7 and 4.7]. It will be important in our later considerations (we follows, [272] and so it is slightly different from [305]).

Lemma (6.1.1) [283] Let A_0, A_1 be quasi-Banach spaces with $A_0 \hookrightarrow A_1$. Assume that $0 < \theta < 1, 0 < p, q \leq \infty$ and $\gamma < -1/q < \eta$. The following continuous embeddings hold:

(a) $(A_0, A_1)_{\theta,q,\gamma+1/\min\{p,q\}} \hookrightarrow \left(A_0, (A_0, A_1)_{\theta,p}\right)_{(1,-\gamma),q} \hookrightarrow (A_0, A_1)_{\theta,q,\gamma+1/\max\{p,q\}}$,

(b) $(A_0, A_1)_{\theta,q,\eta+1/\min\{p,q\}} \hookrightarrow \left((A_0, A_1)_{\theta,p}, A_1\right)_{(0,-\eta),q} \hookrightarrow (A_0, A_1)_{\theta,q,\eta+1/\max\{p,q\}}$.

Remark (6.1.2) [283] Note that if $\gamma = -1/q$ then $\left(A_0, (A_0, A_1)_{\theta,p}\right)_{(1,1/q),q} = \{0\}$, so none of the embeddings in statement (a) of Lemma (6.1.1) hold in this case. As for statement (b) when $\eta = -1/q$, if $p = q$ we can determine explicitly $\left((A_0, A_1)_{\theta,p}, A_1\right)_{(0,1/p),p}$. Indeed, by Holmstedt's formula [279, Remark 2.1]

$$K\left(t, a; (A_0, A_1)_{\theta,p}, A_1\right) \sim \left(\int_0^{t^{1/(1-\theta)}} \left(\frac{K(t, a; A_0, A_1)}{s^\theta} \right)^p \frac{ds}{s} \right)^{1/p}.$$

Hence, we obtain

$$\|a\|_{((A_0,A_1)_{\theta,p},A_1)_{(0,1/p),p}} \sim \left(\int_0^1 \frac{1}{1 - \log t} \int_0^{t^{1/(1-\theta)}} \left(\frac{K(t, a; A_0, A_1)}{s^\theta} \right)^p \frac{ds}{s} \frac{dt}{t} \right)^{1/p}$$

$$= \left(\int_0^1 \left(\frac{K(t, a; A_0, A_1)}{s^\theta} \right)^p \int_{s^{1-\theta}}^1 \frac{1}{1 - \log t} \frac{dt}{t} \frac{ds}{s} \right)^{1/p} \sim \left(\int_0^1 \left(\frac{K(t, a; A_0, A_1)}{s^\theta} \left(\log(1 - \log s)\right)^{1/p} \right)^p \frac{ds}{s} \right)^{1/p}.$$

Therefore, if $\eta = -1/q$ and $p = q$ we still have the embedding of the right-hand side in statement (b) of Lemma (6.1.1) because $\eta + 1/\max\{p, q\} = 0$ and so $(A_0, A_1)_{\theta,q,\eta+1/\max\{p,q\}} = (A_0, A_1)_{\theta,q}$. But the embedding of the left-hand side in (b) fails.

Other kind of limiting reiteration formulae can be seen in [263].

Let S and S' be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$, and the space of tempered distributions on $\mathbb{R}^n$, respectively. By F we denote the Fourier transform on S' and by $\mathcal{F}^{-1}$ the inverse Fourier transform.

Take $\varphi_0 \in S$ such that

$$\operatorname{supp} \varphi_0 \subset \{x \in \mathbb{R}^n : |x| \le 2\} \text{ and } \varphi_0(x) = 1 \text{ if } |x| \le 1.$$

For $j \in \mathbb{N}$ and $x \in \mathbb{R}^n$ let $\varphi_j(x) = \varphi_0(2^{-j}x) - \varphi_0(2^{-j+1}x)$. Then the sequence $(\phi_j)_{j=0}^{\infty}$ forms a dyadic resolution of unity, $\sum_{j=0}^{\infty} \varphi_j(x) = 1$ for all $x \in \mathbb{R}^n$.

For $1 \le p \le \infty, 0 < q \le \infty$ and $s, b \in R$, the space $B_{p,q}^{s,b}$ consists of all $f \in S'$ having a finite quasi-norm

$$\|f\|_{B_{p,q}^{s,b}} = \left(\sum_{j=0}^{\infty} \left(2^{js}(1+j)^b \left\| \mathcal{F}^{-1}(\varphi_j \mathcal{F} f) \right\|_{L^p} \right)^q \right)^{1/q}$$

(with the usual modification if $q = \infty$). See [149, 142, 165, 28]. Note that if $b = 0$ then $B_{p,q}^{s,0}$ coincides with the usual Besov space $B_{p,q}^{s}$.

Besov spaces of generalized smoothness can be also introduced by using the modulus of smoothness as we recall next. Let f be a function on $\mathbb{R}^n$, let $h \in \mathbb{R}^n$ and $k \in \mathbb{N}$. We put

$$(\Delta_h^1 f)(x) = f(x+h) - f(x) \text{ and } \left(\Delta_h^{k+1} f\right)(x) = \Delta_h^1\left(\Delta_h^k f\right)(x).$$

The k-th order modulus of smoothness of a function $f \in L^p$ is defined by

$$\omega_k(f,t)_p = \sup_{|h| \le t} \left\| \Delta_h^k f \right\|_{L^p}, \qquad t > 0.$$

If k=1we simply write $\omega(f,t)_p$ instead of $\omega_1(f,t)_p$.

For $1 \le p \le \infty$, the following connection holds between the K-functional for the couple (L_p, W_p^k) and the k-th order modulus of smoothness: There are positive constants c_1 and c_2 such that

$$c_1 K\left(t^k, f; L_p, W_p^k\right) \le \min(1, t^k)\|f\|_{L_p} + \omega_k(f,t)_p \le c_2 K\left(t^k, f; L_p, W_p^k\right) \tag{1}$$

for all $f \in L^p$ and $t > 0$ (see [256, Theorem 5.4.12]).

For $1 \le p \le \infty, 0 < q \le \infty, -\infty < b < \infty, s \ge 0$ and $k \in \mathbb{N}$ with $k > s$, the space $\mathbf{B}_{p,q}^{s,b}$ consists of all $f \in L^p$ such that

$$\|f\|_{\mathbf{B}_{p,q}^{s,b}} = \|f\|_{L_p} + \left(\int_0^1 \left[t^{-s}(1-\log t)^b \omega_k(f,t)_p\right]^q \frac{dt}{t} \right)^{1/q} < \infty.$$

See [266, 28]. Note that if $s = 0$ and $b < -1/q$, then $\mathbf{B}_{p,q}^{s,b} = L_p$.

If $s > 0$ it is well-known that the definition of $\mathbf{B}_{p,q}^{s,b}$ does not depend on the choice of $k > s$ (see [28, Theorem 2.5]). Next we show that the same property holds for $\mathbf{B}_{p,q}^{0,b}$. We also characterize spaces $\mathbf{B}_{p,q}^{0,b}$ by interpolation.

Theorem (6.1.3) [283] Let $1 \le p \le \infty, 0 < q \le \infty, -\infty < b < \infty$ and $k \in \mathbb{N}$.

(a) The space $\mathbf{B}_{p,q}^{0,b}$ does not depend on the choice of $k \in \mathbb{N}$.

(b) We have $\mathbf{B}_{p,q}^{0,b} = (L_p, W_p^k)_{(0,-b),q}$ with equivalence of quasi-norms.

Proof. Let $k \in \mathbb{N}, k > 1$. Put

$$\|f\|_{\mathbf{B}_{p,q}^{0,b}} = \|f\|_{L_p} + \left(\int_0^1 \left((1-\log t)^b \omega(f,t)_p\right)^q \frac{dt}{t} \right)^{1/q} \tag{2}$$

and

$$\|f\|^{(k)}_{\mathbf{B}^{0,b}_{p,q}} = \|f\|_{L_p} + \left(\int_0^1 \left((1-\log t)^b \omega_k(f,t)_p \right)^q \frac{dt}{t} \right)^{1/q}.$$

Our aim is to show the equivalence between the quasi-norms $\|\cdot\|_{\mathbf{B}^{0,b}_{p,q}}$ and $\|\cdot\|^{(k)}_{\mathbf{B}^{0,b}_{p,q}}$. Since $\omega_k(f,t)_p \le 2^{k-1}\omega(f,t)_p$, it is clear that $\|f\|^{(k)}_{\mathbf{B}^{0,b}_{p,q}} \lesssim \|f\|_{\mathbf{B}^{0,b}_{p,q}}$. Let us check the converse inequality. Using Marchaud's inequality [1, Theorem 5.4.4], for $0 < t \le 1$ we obtain

$$\frac{\omega(f,t)_p}{t} \lesssim \int_t^\infty \frac{\omega_k(f,s)_p}{s}\frac{ds}{s} \lesssim \int_t^1 \frac{\omega_k(f,s)_p}{s}\frac{ds}{s} + \|f\|_{L_p}\int_1^\infty s^{-1}\frac{ds}{s} \sim \int_t^1 \frac{\omega_k(f,s)_p}{s}\frac{ds}{s} + \|f\|_{L_p}.$$

Therefore, since $\omega_k(f,s)_p/s^k$ is equivalent to a decreasing function, applying Hardy's inequality [256, Theorem 6.4], we get

$$\left(\int_0^1 \left((1-\log t)^b \omega(f,t)_p \right)^q \frac{dt}{t} \right)^{1/q}$$

$$\lesssim \left(\int_0^1 \left(t(1-\log t)^b \right)^q \frac{dt}{t} \right)^{1/q} \|f\|_{L_p} + \left(\int_0^1 \left[t(1-\log t)^b \int_t^1 \frac{\omega_k(f,s)_p}{s^k} s^{(k-1)-1} ds \right]^q \frac{dt}{t} \right)^{1/q}$$

$$\lesssim \|f\|_{L_p} + \left(\int_0^1 \left[t^2(1-\log t)^b \frac{\omega_k(f,t)_p}{t^2} \right]^q \frac{dt}{t} \right)^{1/q} = \|f\|^{(k)}_{\mathbf{B}^{0,b}_{p,q}}.$$

This proves statement (a).

As for (b), using (1), we obtain

$$\|f\|_{(L_p,W^k_p)_{(0,-b),q}} = \left(\int_0^1 \left[(1-\log t)^b K(t,f;L_p,W^k_p) \right]^q \frac{dt}{t} \right)^{1/q}$$

$$\sim \left(\int_0^1 \left[(1-\log t)^b K(t^k,f;L_p,W^k_p) \right]^q \frac{dt}{t} \right)^{1/q} \sim \left(\int_0^1 \left[(1-\log t)^b \left(t^k \|f\|_{L_p} + \omega_k(f,t)_p \right) \right]^q \frac{dt}{t} \right)^{1/q}$$

$$\sim \|f\|_{L_p} + \left(\int_0^1 \left[(1-\log t)^b \omega_k(f,t)_p \right]^q \frac{dt}{t} \right)^{1/q} \sim \|f\|_{\mathbf{B}^{0,b}_{p,q}}$$

where we have used (a) in the last equivalence. This completes the proof.

In what follows we assume that $\mathbf{B}^{0,b}_{p,q}$ is quasi-normed by (2). Next we compare $B^{0,b}_{p,q}$ and $\mathbf{B}^{0,b}_{p,q}$.

Theorem (6.1.4) [283] Let $1 < p < \infty, 0 < q \le \infty$ and $b > -1/q$. Then

$$B^{0,b+1/\min\{2,p,q\}}_{p,q} \hookrightarrow \mathbf{B}^{0,b}_{p,q} \hookrightarrow B^{0,b+1/\max\{2,p,q\}}_{p,q}.$$

Proof. Recall that

$$B^s_{p,\min\{p,q\}} \hookrightarrow F^s_{p,q} \hookrightarrow B^s_{p,\max\{p,q\}}$$

where $F^s_{p,q}$ stands for the Triebel–Lizorkin space (see [65, Proposition 2.3.2/2 (iii)]). Moreover, $F^s_{p,2} = H^s_p$ [65, Theorem 2.5.6 (i)] and so $F^0_{p,2} = L_p$. According to Theorem (6.1.3) (b), Lemma (6.1.1) and [142, Theorem 5.3 and Remark 5.4], we derive

$$\mathbf{B}^{0,b}_{p,q} = (L_p, W^1_p)_{(0,-b),q} \hookrightarrow \left(B^0_{p,\max\{2,P\}}, H^1_P\right)_{(0,-b),q} = \left((H^{-1}_P, H^1_P)_{1/2,\max\{2,P\}}, H^1_P\right)_{(0,-b),q}$$
$$\hookrightarrow (H^{-1}_P, H^1_P)_{1/2,q,b+1/\max\{2,P,q\}} = B^{0,b+1/\max\{2,p,q\}}_{p,q}.$$
Similarly, we have
$$B^{0,b+1/\min\{2,p,q\}}_{p,q} = (H^{-1}_P, H^1_P)_{1/2,q,b+1/\min\{2,P,q\}} \hookrightarrow \left((H^{-1}_P, H^1_P)_{1/2,\min\{2,P\}}, H^1_P\right)_{(0,-b),q}$$
$$= \left(B^0_{p,\min\{2,P\}}, H^1_P\right)_{(0,-b),q} \hookrightarrow (L_p, W^1_p)_{(0,-b),q} = \mathbf{B}^{0,b}_{p,q}.$$
Corollary (6.1.5) [283] Let $1 < p < \infty$ and $b > -1/p$.

(a) If $1 < p \le 2\ then\ B^{0,b+1/p}_{p,p} \hookrightarrow \mathbf{B}^{0,b}_{p,p} \hookrightarrow B^{0,b+1/2}_{p,p}$.

(b) If $2 \le p < \infty\ then\ B^{0,b+1/2}_{p,p} \hookrightarrow \mathbf{B}^{0,b}_{p,p} \hookrightarrow B^{0,b+1/p}_{p,p}$.

In particular, for $b > -1/2$ we obtain with equivalence of norms
$$B^{0,b+1/2}_{2,2} = \mathbf{B}^{0,b}_{2,2}.$$
Theorem (6.1.6) [283] Let $1 \le p < r < \infty, 0 < q \le \infty, b > -1/q$ and $\alpha = n(1/p - 1/r)$. Then
$$\mathbf{B}^{\alpha,b+1/\min\{q,r\}}_{p,q} \hookrightarrow \mathbf{B}^{0,b}_{r,q}.$$
Proof. According to [1, Corollary 5.4.20], we have
$$\mathbf{B}^{\alpha}_{p,r} \hookrightarrow L_r. \tag{3}$$
On the other hand, let $k \in \mathbb{N}$ such that $k > \alpha$ and $0 < \theta < 1$ such that $\theta k > \alpha$. By [1, Corollaries 5.4.13 and 5.4.21], we derive
$$W^k_p \hookrightarrow (L_p, W^k_p)_{\theta,p} = \mathbf{B}^{\theta k}_{p,p} \hookrightarrow \mathbf{B}^{\theta k-\alpha}_{r,p}. \tag{4}$$
Interpolating embeddings (3) and (4) by the limiting real method we get
$$(\mathbf{B}^{\alpha}_{p,r}, W^k_p)_{(0,-b),q} \hookrightarrow (L_r, \mathbf{B}^{\theta k-\alpha}_{r,p})_{(0,-b),q}.$$
The target space in this embedding can be determined by using [305, Lemma 2.2(b)] and Theorem (6.1.3) (b). Indeed,
$$(L_r, \mathbf{B}^{\theta k-\alpha}_{r,p})_{(0,-b),q} = (L_r, W^k_r)_{(0,-b),q} = (L_r, (L_r, W^k_r)_{\frac{\theta k-\alpha}{k},p})_{(0,-b),q} = \mathbf{B}^{0,b}_{r,q}.$$
As for the domain space, Lemma (3.1.1) (b) yields
$$\mathbf{B}^{\alpha,b+1/\min\{q,r\}}_{p,q} = (L_p, W^k_p)_{\alpha/k,q,b+1/\min\{q,r\}} \hookrightarrow \left((L_p, W^k_p)_{\alpha/k,r}, W^k_p\right)_{(0,-b),q} = (\mathbf{B}^{\alpha}_{p,r}, W^k_p)_{(0,-b),q}.$$
This completes the proof.

Let $1 < p < \infty, 1 \le q < \infty$ and $-\infty < b < \infty$. Since $B^{0,b}_{p,q} = (H^{-1}_P, H^1_P)_{1/2,q,b}$, using the duality formula for spaces $(A_0, A_1)_{\theta,q,\mathbb{A}}$ (see [265, Theorem 3.1] or [281, Theorem 2.4]) and that $(H^s_p)' = H^{-s}_{p'}$ [62, Theorem 2.6.1], it follows that
$$\left(B^{0,b}_{p,q}\right)' = B^{0,-b}_{p',q'} \quad \text{where} \quad 1/p + 1/p' = 1 = 1/q + 1/q'$$
(see also [144, Theorem 3.1.10]).

In order to determine the dual space of $\mathbf{B}^{0,b}_{p,q}$, we first establish an auxiliary result and recall the definition of logarithmic Lipschitz spaces (see [277] and [279]).

Lemma (6.1.7) [283] Let A_0, A_1 be Banach spaces with A_1 continuously and densely embedded in A_0. Assume that $1 \le q < \infty, 1/q + 1/q' = 1$, and $\eta > -1/q$. Then we have with equivalence of norms
$$(A_0, A_1)'_{(0,-\eta),q} = (A'_1, A'_0)_{(1,\eta+1),q'}.$$
Proof. Since $A_1 \hookrightarrow A_0$, we have that $K(t, a;\ A_0, A_1) \sim \|a\|_{A_0}$ for $t \ge 1$. Take any $\tau < -1/q$. It follows that
$$\left(\int_1^\infty [(1-\log t)^\tau K(t, f; A_0, A_1)]^q \frac{dt}{t}\right)^{1/q} \sim \left(\int_1^\infty (1-\log t)^{\tau q} \frac{dt}{t}\right)^{1/q} \|a\|_{A_0} \sim \|a\|_{A_0} \lesssim \|a\|_{(A_0,A_1)_{(0,-\eta),q}}.$$

This yields that

$$(A_0, A_1)_{(0,-\eta),q} = (A_0, A_1)_{0,q,(\eta,r)} = (A_1, A_0)_{1,q,(r,\eta)}.$$

Since $\tau + 1/q < 0 < \eta + 1/q$, we can apply the duality formula established in [264, Theorem 5.6] to derive

$$(A_0, A_1)'_{(0,-\eta),q} = (A_1, A_0)'_{1,q,(r,\eta)} = (A'_1, A'_0)_{1,q',(-\eta-1,-r-1)}.$$

Density of the embedding $A_1 \hookrightarrow A_0$ implies that $A'_0 \hookrightarrow A'_1$. So $K(t, g; A'_1, A'_0) \sim \|g\|_{A'_1}$ for $t \geq 1$. Now, using that $K(t, g)/t$ is a decreasing function we get

$$\left(\int_1^\infty [t^{-1}(1-\log t)^{-\tau-1} K(t, g; A'_1, A'_0)]^{q'} \frac{dt}{t}\right)^{1/q'} \sim \|g\|_{A'_1}$$

$$\sim K(1, g; A'_1, A'_0) \left(\int_0^1 (1-\log t)^{(-\eta-1)q'} \frac{dt}{t}\right)^{1/q'} \leq \|g\|_{(A'_1, A'_0)_{(1,\eta+1),q'}}.$$

Consequently, $(A_0, A_1)'_{(0,-\eta),q} = (A'_1, A'_0)_{(1,\eta+1),q'}$.

Definition (6.1.8) [283] Let $1 \leq p \leq \infty, 0 < q \leq \infty$ and $\alpha > 1/q$ ($\alpha \geq 0$ if $q = \infty$). The space $\mathrm{Lip}_{p,q}^{(1,-\alpha)}$ is formed by all functions $f \in L_p$ having a finite quasi-norm

$$\|f\|_{\mathrm{Lip}_{p,q}^{(1,-\alpha)}} = \|f\|_{L_p} + \left(\int_0^1 \left[\frac{\omega(f, t)_p}{t(1-\log t)^\alpha}\right]^q \frac{dt}{t}\right)^{1/q}.$$

Now we are ready to describe the dual space of $\mathbf{B}_{p,q}^{0,b}$. Recall that the usual lift operator I_s is defined by

$$I_s f = \mathcal{F}^{-1}(1+|x|^2)^{s/2}\mathcal{F}f, \qquad -\infty < s < \infty.$$

Theorem (6.1.9) [283] Let $1 < p < \infty, 1 \leq q < \infty$ and $b > -1/q$. The space $\left(\mathbf{B}_{p,q}^{0,b}\right)'$ consists of all $f \in H_{p'}^{-1}$ such that $I_{-1}f \in \mathrm{Lip}_{p',q'}^{(1,-b-1)}$ with $1/p + 1/p' = 1 = 1/q + 1/q'$. Moreover,

$$\|f\|_{\left(\mathbf{B}_{p,q}^{0,b}\right)'} \sim \|I_{-1}f\|_{\mathrm{Lip}_{p',q'}^{(1,-b-1)}}.$$

Proof. By Theorem (6.1.3) (b) and Lemma (6.1.7), we derive

$$\left(\mathbf{B}_{p,q}^{0,b}\right)' = \left(\left(L_p, W_p^1\right)_{(0,-b),q}\right)' = \left(H_{P'}^{-1}, L_{p'}\right)_{(1,b+1),q'}.$$

On the other hand, lift operators

$$I_{-1}: H_{P'}^{-1} \longrightarrow L_{p'}, \qquad I_{-1}: L_{p'} \longrightarrow W_{P'}^{-1}$$

are bijective and bounded. Hence

$$K\left(t, f; H_{P'}^{-1}, L_{p'}\right) \sim K\left(t, I_{-1}f; L_{p'}, W_{P'}^{-1}\right) \sim \min(1,t)\|I_{-1}f\|_{L_{p'}} + \omega(I_{-1}f, t)_{p'}$$

where we have used (1)for the last equivalence. Consequently

$$\|f\|_{\left(\mathbf{B}_{p,q}^{0,b}\right)'} \sim \left(\int_0^1 (1-\log t)^{(-b-1)q'} \frac{dt}{t}\right)^{1/q'} \|I_{-1}f\|_{L_{p'}} + \left(\int_0^1 \left[\frac{\omega(I_{-1}f, t)_{p'}}{t(1-\log t)^{b+1}}\right]^{q'} \frac{dt}{t}\right)^{1/q'}$$

$$\sim \|I_{-1}f\|_{\mathrm{Lip}_{p',q'}^{(1,-b-1)}}.$$

We start by showing that Lipschitz spaces can be generated by interpolation from the couple (L_p, W_p^{-1}).

Lemma (6.1.10) [283] Let $1 \leq p \leq \infty, 0 < q \leq \infty$ and $\alpha > 1/q$ ($\alpha \geq 0$ if $q = \infty$). Then

$$(L_p, W_p^1)_{(1,\alpha),q} = \mathrm{Lip}_{p,q}^{(1,-\alpha)}$$

with equivalent quasi-norms.

Proof. Using (1) we derive

$$\|f\|_{(L_p,W_p^{-1})_{(1,\alpha),q}} = \left(\int_0^1 \left[\frac{K(t,I-1f;L_p,W_p^1)}{t(1-\log t)^\alpha}\right]^q \frac{dt}{t}\right)^{1/q}$$

$$\sim \left(\int_0^1 (1-\log t)^{-\alpha q}\frac{dt}{t}\right)^{1/q} \|f\|_{L_p} + \left(\int_0^1 \left[\frac{\omega(f,t)_p}{t(1-\log t)^\alpha}\right]^q \frac{dt}{t}\right)^{1/q} \sim \|f\|_{\mathrm{Lip}_{p,q}^{(1,-\alpha)}}.$$

The next result describes the position of Lipschitz spaces between Besov spaces with classical smoothness 1 and additional logarithmic smoothness.

Theorem (6.1.11) [283] Let $1 < p < \infty, 0 < q \le \infty$ and $\alpha > 1/q$. Then

$$B_{p,q}^{1,-\alpha+1/\min\{2,p,q\}} \hookrightarrow \mathrm{Lip}_{p,q}^{(1,-\alpha)} \hookrightarrow B_{p,q}^{1,-\alpha+1/\max\{2,p,q\}}.$$

Proof. By Lemmata (6.1.10), (6.1.1) (a) and [142, Theorem 5.3 and Remark 5.4], we obtain

$$\mathrm{Lip}_{p,q}^{(1,-\alpha)} = (L_p,W_p^1)_{(1,\alpha),q} \hookrightarrow (L_p,B_{p,\max\{2,p\}}^1)_{(1,\alpha),q} = (L_p,(L_p,W_p^2)_{1/2,\max\{2,p\}})_{(1,\alpha),q}$$

$$\hookrightarrow (L_p,W_p^2)_{1/2,q,-\alpha+1/\max\{2,p,q\}} = B_{p,q}^{1,-\alpha+1/\max\{2,p,q\}}.$$

Similarly, we derive

$$B_{p,q}^{1,-\alpha+1/\min\{2,p,q\}} = (L_p,W_p^2)_{1/2,q,-\alpha+1/\min\{2,p,q\}} \hookrightarrow (L_p,(L_p,W_p^2)_{1/2,\min\{2,p\}})_{(1,\alpha),q}$$

$$= (L_p,B_{p,\min\{2,p\}}^1)_{(1,\alpha),q} \hookrightarrow (L_p,W_p^1)_{(1,\alpha),q} = \mathrm{Lip}_{p,q}^{(1,-\alpha)}.$$

Corollary (6.1.12) [283] Let $1 < p < \infty$ and $\alpha > 1/p$.

(a) If $1 < p \le 2$ *then* $B_{p,p}^{1,-\alpha+1/\mathrm{p}} \hookrightarrow \mathrm{Lip}_{p,p}^{(1,-\alpha)} \hookrightarrow B_{p,p}^{1,-\alpha+1/2}$.

(b) If $2 \le p < \infty$ *then* $B_{p,p}^{1,-\alpha+1/2} \hookrightarrow \mathrm{Lip}_{p,p}^{(1,-\alpha)} \hookrightarrow B_{p,p}^{1,-\alpha+1/\mathrm{p}}$.

If $\alpha > 1/2$ we have

$$B_{2,2}^{1,-\alpha+1/2} = \mathrm{Lip}_{2,2}^{(1,-\alpha)}.$$

Next we recall a result of Haroske [267, Proposition 16].

Proposition (6.1.13) [283] Let $1 \le p \le \infty, 0 < q, v \le \infty, \alpha > 1/q$ and $\beta > 1/v$. Then

$$\mathrm{Lip}_{p,q}^{(1,-\alpha)} \hookrightarrow \mathrm{Lip}_{p,v}^{(1,-\beta)} \quad if, and\ only\ if, \qquad f(x) = \begin{cases} \beta - \frac{1}{v} \ge \alpha - \frac{1}{q} & and \quad v \ge q, \\ \beta - \frac{1}{v} > \alpha - \frac{1}{q} & and \quad v < q. \end{cases}$$

We show that combining Proposition (6.1.13) with the previous results we can derive some complements and improvements of the results of [267].

Theorem (6.1.14) [283] Let $1 < p < \infty, 0 < q, v \le \infty$ and $\alpha > 1/v$. Then

$$B_{p,q}^1 \hookrightarrow \mathrm{Lip}_{p,v}^{(1,-\alpha)}\ if \begin{cases} 0 < q \le \min\{2,p\}, \\ \min\{2,p\} < q, v < q \quad and \quad \alpha > \frac{1}{v} + \frac{1}{\min\{2,p\}} - \frac{1}{q}, \\ min\{2,p\} < q \le v \qquad and \quad \alpha \ge \frac{1}{v} + \frac{1}{\min\{2,p\}} - \frac{1}{q}. \end{cases}$$

Proof. If $0 < q \le \min\{2,p\}$, we obtain

$$B_{p,q}^1 \hookrightarrow B_{p,\min\{2,p\}}^1 \hookrightarrow (L_p,B_{p,\min\{2,p\}}^1)_{(1,\alpha),v} \hookrightarrow (L_p,W_p^1)_{(1,\alpha),v} = \mathrm{Lip}_{p,v}^{(1,-\alpha)}.$$

If $\min\{2,p\} < q$, let $\beta = 1/\min\{2,p\}$. By Theorem (6.1.11) and Proposition (6.1.13), we derive

$$B^1_{p,q} = B^{1,-\beta+\frac{1}{\min\{2,p,q\}}}_{p,q} \hookrightarrow \mathrm{Lip}^{(1,-\beta)}_{p,v} \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}_{p,v}.$$

Corollary (6.1.15) [283] Let $1 < p < \infty, 1 \le q \le \infty$ and $0 < \beta = \alpha - 1/q$. Then

$$B^{1,-\beta}_{p,1} \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}_{p,q}.$$

Proof. Using again Theorem (6.1.11) and Proposition (6.1.13) we obtain

$$B^{1,-\beta}_{p,1} \hookrightarrow \mathrm{Lip}^{(1,-\beta-1)}_{p,1} \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}_{p,q}.$$

Proceeding as in Corollary (6.1.15), we can also derive the embedding

$$B^{1,-\alpha+1/q}_{p,\min\{2,p,q\}} \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}_{p,q}$$

provided that $1 < p < \infty, 0 < q \le \infty$ and $\alpha > 1/q$. This improves [267, embedding (29), p. 793] because

$$B^{1,-\alpha+1/q}_{p,\min\{1,q\}} \hookrightarrow B^{1,-\alpha+1/q}_{p,\min\{2,p,q\}}.$$

Note also that from Theorem (6.1.11) we can recover [267, embeddings (41), p. 796] for $1 < p < \infty$. Besides, Theorem (6.1.11) also yields that if $\alpha > 1/q$ then

$$\mathrm{Lip}^{(1,-\alpha)}_{p,q} \hookrightarrow B^{1,-\alpha+1/q}_{p,q} \quad \text{if} \quad \max\{2,p\} \le q,$$

and

$$B^{1,-\alpha+1/q}_{p,\mathrm{q}} \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}_{p,q} \quad \text{if} \quad q \le \min\{2,p\}.$$

Corollary (6.1.16) [314] Let $\epsilon > 0$. The space $\left(\mathbf{B}^{0,\frac{\epsilon^2+\epsilon-1}{1+\epsilon}}_{1+\epsilon,1+\epsilon}\right)'$ consists of all $f_r \in H^{-1}_{\frac{1+\epsilon}{\epsilon}}$ such that $I_{-1}f_r \in \mathrm{Lip}^{(1,\frac{-\epsilon(2+\epsilon)}{1+\epsilon})}_{(\frac{1+\epsilon}{\epsilon},\frac{1+\epsilon}{\epsilon})}$ with $\epsilon = -1/2$. Moreover,

$$\sum \|f_r\|_{\left(\mathbf{B}^{\left(0,\frac{\epsilon^2+\epsilon-1}{1+\epsilon}\right)}_{(1+\epsilon,1+\epsilon)}\right)'} \sim \sum \|I_{-1}f_r\|_{\mathrm{Lip}^{(1,\frac{-\epsilon(2+\epsilon)}{1+\epsilon})}_{(\frac{1+\epsilon}{\epsilon},\frac{1+\epsilon}{\epsilon})}}.$$

Proof. By Theorem (6.1.3) (b) and Lemma (6.1.7), we derive

$$\left(\mathbf{B}^{0,\frac{\epsilon^2+\epsilon-1}{1+\epsilon}}_{(1+\epsilon,1+\epsilon)}\right)' = \left((L_{1+\epsilon}, W^1_{1+\epsilon})_{(0,-\frac{\epsilon^2+\epsilon-1}{1+\epsilon}),1+\epsilon}\right)' = \left(H^{-1}_{\frac{1+\epsilon}{\epsilon}}, L_{\frac{1+\epsilon}{\epsilon}}\right)_{\left(1,\frac{\epsilon(2+\epsilon)}{1+\epsilon}\right),\frac{1+\epsilon}{\epsilon}}.$$

On the other hand, lift operators

$$I_{-1}: H^{-1}_{\frac{1+\epsilon}{\epsilon}} \longrightarrow L_{\frac{1+\epsilon}{\epsilon}}, \qquad I_{-1}: L_{\frac{1+\epsilon}{\epsilon}} \longrightarrow W^{-1}_{\frac{1+\epsilon}{\epsilon}}$$

are bijective and bounded. Hence

$$\sum K\left(t, f_r; H^{-1}_{\frac{1+\epsilon}{\epsilon}}, L_{\frac{1+\epsilon}{\epsilon}}\right) \sim \sum K\left(t, I_{-1}f_r; L_{\frac{1+\epsilon}{\epsilon}}, W^{-1}_{\frac{1+\epsilon}{\epsilon}}\right)$$
$$\sim \sum \min(1,t)\|I_{-1}f_r\|_{L_{\frac{1+\epsilon}{\epsilon}}} + \sum \omega(I_{-1}f_r, t)_{\frac{1+\epsilon}{\epsilon}}$$

where we have used (1) for the last equivalence. Consequently

$$\sum \|f_r\|_{\left(\mathbf{B}^{0,\frac{\epsilon^2+\epsilon-1}{1+\epsilon}}_{1+\epsilon,1+\epsilon}\right)'} \sim \left(\int_0^1 (1-\log t)^{-2+\epsilon} \frac{dt}{t}\right)^{\frac{\epsilon}{1+\epsilon}} \sum \|I_{-1}f_r\|_{L_{\frac{1+\epsilon}{\epsilon}}} + \sum \left(\int_0^1 \left[\frac{\omega(I_{-1}f_r, t)_{\frac{1+\epsilon}{\epsilon}}}{t(1-\log t)^{\frac{\epsilon(2+\epsilon)}{1+\epsilon}}}\right]^{\frac{1+\epsilon}{\epsilon}} \frac{dt}{t}\right)^{\frac{\epsilon}{1+\epsilon}}$$
$$\sim \sum \|I_{-1}f_r\|_{\mathrm{Lip}^{(1,\frac{-\epsilon(2+\epsilon)}{1+\epsilon})}_{\frac{1+\epsilon}{\epsilon},\frac{1+\epsilon}{\epsilon}}}.$$

We start by showing that Lipschitz spaces can be generated by interpolation from the couple $(L_{1+\epsilon}, W^{-1}_{1+\epsilon})$.

Section (6.2) Approximation Spaces and Limiting Interpolation

There is a symbiotic relationship between approximation theory and interpolation theory as can be seen by Bergh and Löfstörm [2], Triebel [62] and Petrushev and Popov [299]. The real interpolation method $(A_0, A_1)_{\theta,q}$ plays an important role in this matter. Usually $0 < \theta < 1$ but to cover some extreme cases, limiting versions have been also used, where $\theta = 0, 1$ and logarithmic weights may be included. See Gomez and Milman [276], Evans, Opic and Pick [273], Cobos, Fernández-Cabrera, Kühn and Ullrich [260], Cobos and Kühn [262] and Edmunds and Opic [269].

Given a quasi-Banach space X and an approximation family $(G_n)_{n\in\mathbb{N}_0}$ of subsets of X, approximation spaces X_p^α are defined by selecting those elements of X such that $(n^{\alpha-1/p}E_n(f))$ belongs to ℓ_p. Here $0 < \alpha < \infty, 0 < p \le \infty$ and $E_n(f)$ is the error of best approximation to f by the elements of G_{n-1}. These spaces have been studied by Butzer and Scherer [287], Pietsch [300, 301], Petrushev and Popov [299], and DeVore and Lorentz [15]. Limiting approximation spaces $X_q^{(0,\gamma)}$ are defined by doing $\alpha = 0$ and inserting the weight $(1 + \log n)^\gamma$. They have been investigated by Cobos and Resina [292], Cobos and Milman [291], Cobos and Kühn [290], Fehér and Grässler [295]. As it is shown in [292], even when $\gamma = 0$, the theory of limiting approximation spaces does not follow from the theory of spaces X_p^α by taking $\alpha = 0$. Spaces X_p^α and $X_q^{(0,\gamma)}$ allow to establish in an elegant and clear way a number of important results on function spaces, sequence spaces and spaces of operators.

We continue the investigation on limiting interpolation and approximation spaces, applying the results to problems on Besov spaces.

After reviewing basic concepts from approximation spaces and interpolation theory, we establish some connections between limiting methods and the real method with a function parameter. Is devoted to reiteration of approximation constructions. The construction $(\cdot)_p^\alpha$ is stable by iteration [300, Theorem 3.2] and a similar property holds for $(\cdot)_q^{(0,\gamma)}$ [295, Theorem 2]. We study the stability properties when we apply first the construction $(\cdot)_p^\alpha$ and then $(\cdot)_q^{(0,\gamma)}$ or vice versa. As we will show, outside the case where $p = q$, the constructions do not commute. The space $(X_q^{(0,\gamma)})_p^\alpha$ consists of those $f \in X$ such that $(n^{\alpha-1/p}(1 + \log n)^{\gamma+1/q}E_n(f))$ belongs to ℓ_p while the space $(X_p^\alpha)_q^{(0,\gamma)}$ has a different shape that we determine with the help of limiting interpolation.

We apply the previous results to investigate several problems on function spaces. We first consider embeddings of Besov spaces $B_{p,q}^{\alpha,b}$ with $\alpha > 0$ into Lorentz–Zygmund spaces. In the Banach case where parameters p, q are greater than or equal to 1, this question was studied by DeVore, Riemenschneider and Sharpley [266] by means of weak type interpolation. Our approach is different and results cover the whole range of parameters. Then we consider Besov spaces $B_{p,q}^{0,\gamma}$ with zero classical smoothness and logarithmic smoothness with exponent γ. We show embeddings of $B_{p,q}^{0,\gamma}$ into spaces of the kind of the small Lebesgue space $L^{(p}$ which include the embeddings into Lorentz–Zygmund spaces established in [289]. We also study the relationship between smoothness of derivatives of f and smoothness of f, and the behaviour of the conjugate-function operator on $B_{1,q}^{0,\gamma}$.

We investigate the distribution of Fourier coefficients of functions of $B^{0,\gamma}_{p,q}$. Our results complement and improve those of [266] for the Banach case. We also show estimates for Fourier coefficients of functions in spaces close to L_1 and L_2, which extend the estimates of Hardy and Littlewood and of Bennett for functions in $L(\log L)_\gamma$ [286, Theorem 1.6/(a)] and the estimate of Cobos and Segurado for functions in $L_2(\log L)_{-1/2}$ [294, Theorem 8.5].

Let $(X, \|\cdot\|_X)$ be a quasi-Banach space. We say that a sequence $(G_n)_{n\in\mathbb{N}_0}$ of subsets of X is an approximation family in X if the following conditions hold

$$G_0 = \{0\} \quad \text{and} \quad \lambda G_n \subseteq G_n \quad \text{for any scalar } \lambda \text{ and } n \in \mathbb{N}, \tag{5}$$

$$G_n \subseteq G_{n+1} \qquad \text{for any} \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \tag{6}$$

$$G_n + G_m \subseteq G_{n+m} \text{ for any } n, m \in \mathbb{N}. \tag{7}$$

Given any $f \in X$ and $n \in \mathbb{N}$, we put

$$E_n(f) = E_n(f;\ X) = \inf\{\|f - g\|_X : g \in G_{n-1}\}.$$

Let $\alpha \geq 0, 0 < p \leq \infty$ and $-\infty < \gamma < \infty$. The approximation space $X_p^{(\alpha,\gamma)} = (X, G_n)_p^{(\alpha,\gamma)}$ is formed by all those $f \in X$ which have a finite quasi-norm

$$\|f\|_{X_p^{(\alpha,\gamma)}} = \left(\sum_{n=1}^{\infty} (n^\alpha (1 + \log n)^\gamma E_n(f))^p n^{-1}\right)^{1/p} \quad \text{if } 0 < p < \infty$$

$$\|f\|_{X_p^{(\alpha,\gamma)}} = \sup_{n\geq 1}\{n^\alpha (1 + \log n)^\gamma E_n(f)\} \text{ if } \ p = \infty.$$

The space $X_p^{(\alpha,\gamma)}$ is a quasi-Banach space with $X_p^{(\alpha,\gamma)} \hookrightarrow X$, where the symbol $\hookrightarrow$ means continuous embedding. The case $\alpha > 0$ and $\gamma = 0$ corresponds to the classical approximation spaces, which have been studied in [292, 299, 300]. We write simply X_p^α and $\|\cdot\|_{X_p^\alpha}$ if $\gamma = 0$. If $\alpha = 0$ we obtain the limiting approximation spaces considered in [14, 23, 292]. Note that $X_q^{(0,\gamma)}$ coincides with X if $\gamma < -1/q$. Moreover, $X_p^\alpha \hookrightarrow X_q^{(0,\gamma)}$ for any choice of parameters. See [302] for properties of spaces $X_p^{(\alpha,\gamma)}$ if $\alpha > 0$.

Let us give a concrete example. Let $X = \ell_\infty$, the space of bounded sequences and let $G_n = F_n$, the subset of sequences having at most n coordinates different from 0. Then, for any $\xi \in \ell_\infty$, the sequence $(E_n(\xi;\ \ell_\infty))$ is the non-increasing rearrangement (ξ_n^*) of the sequence ξ. The space X_p^α coincides with the Lorentz sequence space $\ell_{1/\alpha,p}$, the space $X_q^{(0,\gamma)}$ is the Lorentz–Zygmund space $\ell_{\infty,q}(\log \ell)_\gamma$ and $X_p^{(\alpha,\gamma)}$ is $\ell_{1/\alpha,p}(\log \ell)_\gamma$. Recall that for $0 < r, q \leq \infty$ and $-\infty < \gamma < \infty$,

$$\ell_{r,q}(\log \ell)_\gamma = \left\{\xi \in \ell_\infty : \|\xi\|_{\ell_{r,q}(\log \ell)_\gamma} = \left(\sum_{n=1}^{\infty} (n^{1/r}(1 + \log n)^q \xi_n^*) n^{-1}\right)^{1/q} < \infty\right\}$$

(the sum should be replaced by the supremum if $q = \infty$) and $\ell_{r,q} = \ell_{r,q}(\log \ell)_0$ (see [256, 300, 266]).

Let $\mu_n = 2^{2^n}, n \in \mathbb{N}_0$. It is shown in [292, 295] that $f \in X_q^{(0,\gamma)}$ if and only if there is a representation $f = \sum_{n=0}^{\infty} g_n$ with $g_n \in G_{\mu_n}$ and

$$\left(\sum_{n=0}^{\infty} (2^{n(\gamma+1/q)} \|g_n\|_X)^q\right)^{1/q} < \infty. \tag{8}$$

Besides, taking the infimum of the values (8) we obtain an equivalent quasi-norm to $\|\cdot\|_{X_q^{(0,\gamma)}}$. This property is important for the proof of the following embedding result.

In what follows, if U, V are non-negative quantities depending on certain parameters, we write $U \lesssim V$ if there is a constant $c > 0$ independent of the parameters in U and V such that $U \leq cV$. If $U \lesssim V$ and $V \lesssim U$, we put $U \sim V$.

Lemma (6.2.1) [305] Let X, Y be quasi-Banach spaces which are continuously embedded in a Hausdorff topological vector space. Let $(G_n)_{n\in\mathbb{N}_0}$ be an approximation family such that $G_n \subseteq X \cap Y$ for any $n \in \mathbb{N}_0$. Assume that there are constants $c, \beta > 0$ such that

$$\|g\|_Y \leq c(\log(1+n))^\beta \|g\|_X, \qquad g \in G_n, n \in \mathbb{N}.$$

Then for $0 < q \leq \infty$ and $\gamma < -1/q$ we have that

$$X_q^{(0,\beta+\gamma)} \hookrightarrow Y_q^{(0,\gamma)}.$$

Proof. By (8), there is a constant $c_1 > 0$ such that for any $f \in X_q^{(0,\beta+\gamma)}$, we can find a representation $f = \sum_{n=0}^{\infty} g_n$ (in X) with $g_n \in G_{\mu_n}$ such that

$$\left(\sum_{n=0}^{\infty}(2^{n(\beta+\gamma+1/q)}\|g_n\|_X)^q\right)^{1/q} \leq c_1\|f\|_{X_q^{(0,\beta+\gamma)}}.$$

Since

$$\|g_n\|_Y \lesssim (\log\mu_n)^\beta \|g_n\|_X \sim 2^{n\beta}\|g_n\|_X,$$

we obtain that

$$\left(\sum_{n=0}^{\infty}(2^{n(\gamma+1/q)}\|g_n\|_Y)^q\right)^{1/q} \leq c_2\left(\sum_{n=0}^{\infty}(2^{n(\beta+\gamma+1/q)}\|g_n\|_X)^q\right)^{1/q} < \infty.$$

Now it is not hard to show that $\sum_{n=0}^{\infty} g_n$ is convergent in Y. By compatibility, $f = \sum_{n=0}^{\infty} g_n$ also in Y. Therefore, taking the infimum over all possible representations and using again (8) we conclude that

$$\|f\|_{Y_q^{(0,\gamma)}} \leq c_3\|f\|_{X_q^{(0,\beta+\gamma)}}.$$

We now review some notions from interpolation theory. By a quasi-Banach couple $\bar{A} = (A_0, A_1)$ we mean two quasi-Banach spaces A_0, A_1 which are continuously embedded in some Hausdorff topological vector space. Given $t > 0$, Peetre's K-functional is defined by

$$\begin{aligned} K(t,a) &= K(t,a;\ \bar{A}) = K(t,a;\ A_0, A_1) \\ &= \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} :\ a = a_0 + a_1, a_j \in A_j\}, \qquad a \in A_0 + A_1. \end{aligned}$$

Let $0 < \theta < 1$ and $0 < q \leq \infty$. The real interpolation space $\bar{A}_{\theta,q} = (A_0, A_1)_{\theta,q}$ consists of all $a \in A_0 + A_1$ having a finite quasi-norm

$$\|a\|_{\bar{A}_{\theta,q}} = \left(\int_0^\infty \left(t^{-\theta}K(t,a)\right)^q\right)^{1/q}$$

(when $q = \infty$ the integral should be replaced by the supremum). See [2, 62] or [259].

It is shown in [298] that we have

$$(X_{p_0}^{\alpha_0}, X_{p_1}^{\alpha_1})_{\theta,q} = X_p^\alpha$$

with equivalence of quasi-norms. Here $0 < \alpha_0 \neq \alpha_1 < \infty, 0 < p_0, p_1, q \leq \infty, 0 < \theta < 1$ and $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$. The choice $(X, G_n) = (\ell_\infty, F_n)$ and $r_j = p_j = 1/\alpha_j$ yields that

$$(\ell_{r_0}, \ell_{r_1})_{\theta,q} = \ell_{r,q} \tag{9}$$

provided that $0 < r_0 \neq r_1 \leq \infty$ and $1/r = (1-\theta)/r_0 + \theta/r_1$ (see [2] or [62] for another proof). Moreover,

$$(X, X_{p_1}^{\alpha_1})_{\theta,q} = X_q^\beta \tag{10}$$

with $\beta = \theta\alpha_1$ (a more general formula is established in Proposition (6.2.7)).

Replacing t^θ by a more general function $\rho(t)$, we obtain the space $\bar{A}_{\rho,q} = (A_0, A_1)_{\rho,q}$, Its quasi-norm is

$$\|a\|_{\bar{A}_{\rho,q}} = \left(\int_0^\infty \left(\frac{K(t,a)}{\rho(t)}\right)^q \frac{dt}{t}\right)^{1/q}$$

(see [281]). Let $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$, put $\ell(t) = 1 + |\log t|$ and write

$$\ell^{\mathbb{A}}(t) = \begin{cases} \ell^{\alpha_0}(t) & \text{for } t \in (0,1], \\ \ell^{\alpha_\infty}(t) & \text{for } t \in (1,\infty). \end{cases}$$

If $\rho(t) = t^\theta \ell^{\mathbb{A}}(t)$, we put

$$(A_0, A_1)_{\rho,q} = (A_0, A_1)_{\theta,q,\mathbb{A}}.$$

See [272, 273] for details on these spaces which, under suitable conditions on $\mathbb{A}$, are well defined even if $\theta = 0$ or $\theta = 1$. Note that our notation is slightly different from [272]. In the special case $\alpha_0 = \alpha_\infty = \alpha$, we simply put $(A_0, A_1)_{\rho,q} = (A_0, A_1)_{\theta,q,\alpha}$.

The following limiting real methods will be also very useful in our considerations. Let $A_1 \hookrightarrow A_0, 0 < q \le \infty$ and $-\infty < \eta < \infty$. For $\theta = 1$ or $\theta = 0$, the space $\bar{A}_{(\theta,\eta),q} = (A_0, A_1)_{(\theta,\eta),q}$ is formed by all those $a \in A_0$ having a finite quasi-norm

$$\|a\|_{\bar{A}_{(\theta,\eta),q}} = \left(\int_0^1 \left(\frac{K(t,a)}{t^\theta (1 - \log t)^\eta}\right)^q \frac{dt}{t}\right)^{1/q}$$

(see [276, 260, 262]). To avoid that $\bar{A}_{(1,\eta),q} = \{0\}$, when $\theta = 1$ we suppose that $\eta > 1/q$ if $q < \infty$, and $\eta \ge 0$ if $q = \infty$.

It is clear that the (ρ, q)-method and the $((\theta,\eta), q)$-method have the interpolation property for bounded linear operators.

We establish now some connections among all these interpolation methods and we also determine some concrete interpolation spaces.

We put $K(t,a)$ for the K-functional of (A_0, A_1). If we work with a different couple, then we write it explicitly in the notation of the K-functional.

Lemma (6.2.2) [305] Let A_0, A_1 be quasi-Banach spaces with $A_1 \hookrightarrow A_0$. Suppose that $0 < \theta < 1, 0 < q, r \le \infty, \gamma < -1/q$ and $-\infty < \eta < \infty$. Then we have with equivalence of quasi-norms

(a) $\left((A_0, A_1)_{\theta,r}, A_1\right)_{(1,-\gamma),q} = (A_0, A_1)_{(1,-\gamma),q}$.

(b) $\left(A_0, (A_0, A_1)_{\theta,r}\right)_{(1,-\eta),q} = (A_0, A_1)_{(1,-\eta),q}$.

Proof. Since $(A_0, A_1)_{\theta,r} \hookrightarrow A_0$, we have that

$$\left((A_0, A_1)_{\theta,r}, A_1\right)_{(1,-\gamma),q} \hookrightarrow (A_0, A_1)_{(1,-\gamma),q}.$$

To check the converse embedding, assume first that $0 < r < q$. By Holmstedt's formula [279, Remark 2.1],

$$K(t, a; (A_0, A_1)_{\theta,r}, A_1) \sim \left(\int_0^{t^{1/(1-\theta)}} (s^{-\theta} K(s,a))^r \frac{ds}{s}\right)^{1/r}.$$

Hence, we obtain

$$\|a\|_{\left((A_0,A_1)_{\theta,r},A_1\right)_{(1,-\gamma),q}} \sim \left(\int_0^1 \left[\frac{(1-\log t)^\gamma}{t}\left(\int_0^{t^{1/(1-\theta)}} \left(s^{-\theta} K(s,a)\right)^r \frac{ds}{s}\right)^{1/r}\right]^q \frac{dt}{t}\right)^{1/q}$$

$$\sim \left(\int_0^1 \left[\frac{(1-\log t)^{\gamma r}}{t^{(1-\theta)r}} \int_0^t \left(s^{-\theta} K(s,a)\right)^r \frac{ds}{s}\right]^{q/r} \frac{dt}{t}\right)^{1/q}$$

$$\lesssim \left(\int_0^1 \left[\frac{t(1-\log t)^{\gamma r}}{t^{(1-\theta)r}}(t^{-\theta}K(s,a))^r t^{-1}\right]^{q/r} \frac{dt}{t}\right)^{1/q}$$

where we have used Hardy's inequality [256, Theorem 6.4] in the last inequality. Therefore $\|a\|_{((A_0,A_1)_{\theta,r},A_1)_{(1,-\gamma),q}} \lesssim \|a\|_{(A_0,A_1)_{(1,-\gamma),q}}$ and so

$$(A_0,A_1)_{(1,-\gamma),q} \hookrightarrow \left((A_0,A_1)_{\theta,r},A_1\right)_{(1,-\gamma),q}.$$

If $q \le r$, take $0 < r_0 < q$. Using that$(A_0,A_1)_{\theta,r_0} \hookrightarrow (A_0,A_1)_{\theta,r}$ and the previous case, we derive

$$(A_0,A_1)_{(1,-\gamma),q} \hookrightarrow \left((A_0,A_1)_{\theta,r_0},A_1\right)_{(1,-\gamma),q} \hookrightarrow \left((A_0,A_1)_{\theta,r},A_1\right)_{(1,-\gamma),q}.$$

This completes the proof of equality (a). The statement (b) follows from similar arguments.

In order to give some examples, recall that for $0 < p < \infty, 0 < q \le \infty$ and $-\infty < \gamma < \infty$, the Lorentz–Zygmund space $L_{p,q}(\log L)_\gamma$ on the unit circle $\mathbb{T}$ is formed by all (classes of) measurable functions f on $\mathbb{T}$ having a finite quasi-norm

$$\|f\|_{L_{p,q}(\log L)_\gamma} = \left(\int_0^{2\pi} \left[t^{1/p}(1+|\log t|)^\gamma f^*(t)\right]^q \frac{dt}{t}\right)^{1/q}$$

where f^* is the non-increasing rearrangement of f. See [1, 256]. If $p = q$, then $L_{p,q}(\log L)_\gamma$ coincides with the Zygmund space $L_p(\log L)_\gamma$, and if in addition $\gamma = 0$, then the space becomes the Lebesgue space L_p If $\gamma = 0$ but $p \ne q$, we obtain the Lorentz space $L_{p,q}$.

Lemma (6.2.3) [305] Let $0 < p < \infty, 0 < q \le \infty$ and $\gamma < -1/q$. Then we have with equivalence of quasi-norms

$$(L_p,L_\infty)_{(1,-\gamma),q} = L_{\infty,q}(\log L)_\gamma.$$

Proof. The K-functional of (L_p,L_∞) is given by

$$K(t,f) \sim \left(\int_0^{t^p} f^*(s)^p ds\right)^{1/p} \tag{11}$$

(see [2, Theorem 5.2.1]). Assume that $p \le q$. Using Hardy's inequality [256, Theorem 6.4], we obtain

$$\|f\|_{(L_p,L_\infty)_{(1,-\gamma),q}} \sim \left(\int_0^1 \left[\frac{(1-\log t)^\gamma}{t}\left(\int_0^{t^p} f^*(s)^p ds\right)^{1/p}\right]^q \frac{dt}{t}\right)^{1/q}$$

$$\sim \left(\int_0^1 \left[\frac{(1-\log t)^{\gamma p}}{t}\int_0^t f^*(s)^p ds\right]^{q/p} \frac{dt}{t}\right)^{1/q}$$

$$\sim \left(\int_0^1 [(1-\log t)^\gamma f^*(t)]^q \frac{dt}{t}\right)^{1/q} \sim \|f\|_{L_{\infty,q}(\log L)_\gamma}.$$

Suppose now $q < p$. Take $0 < r < q$. Since $L_p = (L_r,L_\infty)_{1-r/p,p}$ (see [2, Theorem 5.2.1]), by Lemma (6.2.2) (a) and the previous case, we derive

$$\left(L_p,L_\infty\right)_{(1,-\gamma),q} = \left((L_r,L_\infty)_{1-r/p,p},L_\infty\right)_{(1,-\gamma),q} = (L_r,L_\infty)_{(1,-\gamma),q} = L_{\infty,q}(\log L)_\gamma.$$

For $0 < p < \infty, 0 < r \le \infty$ and $\gamma > -1/r$, we designate by $Y_{p,r,\gamma}$ the collection of all (classes of) measurable functions f on $\mathbb{T}$ such that

$$\|f\|_{Y_{p,r,\gamma}} = \left(\int_0^{2\pi} \left[(1-\log t)^\gamma \left(\int_0^t f^*(s)^p ds\right)^{1/p}\right]^r \frac{dt}{t}\right)^{1/r} < \infty.$$

Note that if $1 < p < \infty, r = 1$ and $\gamma = -1/p$, the space $Y_{p,1,-1/p}$ coincides with the small Lebesgue space $L^{(p}$ (see [296]).

Lemma (6.2.4) [305] Let $0 < p < \infty, 0 < r \le \infty$ and $\gamma > -1/r$. Then we have with equivalence of quasi-norms

$$(L_p, L_\infty)_{(0,-\gamma),r} = Y_{p,r,\gamma}.$$

Proof. Inserting (11) in the quasi-norm of $(L_p, L_\infty)_{(0,-\gamma),r}$ and making a change of variables we obtain

$$\|f\|_{(L_p,L_\infty)_{(0,-\gamma),r}} \sim \left(\int_0^1 \left[(1-\log t)^\gamma \left(\int_0^{t^{1/p}} f^*(s)^p ds\right)^{1/p}\right]^r \frac{dt}{t}\right)$$

$$\sim \left(\int_0^1 \left[(1-\log t)^\gamma \left(\int_0^t f^*(s)^p ds\right)^{1/p}\right]^r \frac{dt}{t}\right)^{1/r} \sim \|f\|_{Y_{p,r,\gamma}}.$$

Lemma (6.2.5) [305] Let A_0, A_1 be quasi-Banach spaces with $A_1 \hookrightarrow A_0$. Assume that $0 < \theta < 1, 0 < p, q \le \infty$ and $\gamma < -1/q < \eta$. The following continuous embeddings hold

(a) $(A_0, A_1)_{\theta,q,-\gamma-1/\min\{p,q\}} \hookrightarrow \left(A_0, (A_0, A_1)_{\theta,p}\right)_{(1,-\gamma),q} \hookrightarrow (A_0, A_1)_{\theta,q,-\gamma-1/\max\{p,q\}}$,

(b) $(A_0, A_1)_{\theta,q,-\eta-1/\min\{p,q\}} \hookrightarrow \left((A_0, A_1)_{\theta,p}, A_1\right)_{(1,-\eta),q} \hookrightarrow (A_0, A_1)_{\theta,q,-\eta-1/\max\{p,q\}}$.

Proof. Take $\alpha > -1/q$. According to [269, Proposition 1] (which also works in the quasi-normed case), we have that

$$\left(A_0, (A_0, A_1)_{\theta,p}\right)_{(1,-\gamma),q} = \left(A_0, (A_0, A_1)_{\theta,p}\right)_{1,q,(-\gamma,-\alpha)}.$$

By [272, Theorems 5.9* and 4.7*], we derive that

$$(A_0, A_1)_{\theta,q,\left(-\gamma-\frac{1}{\min\{p,q\}},-\alpha-\frac{1}{\min\{p,q\}}\right)} \hookrightarrow \left(A_0, (A_0, A_1)_{\theta,p}\right)_{(1,-\gamma),q} \hookrightarrow (A_0, A_1)_{\theta,q,\left(-\gamma-\frac{1}{\max\{p,q\}},-\alpha-\frac{1}{\max\{p,q\}}\right)}.$$

Now applying again [269, Proposition 1], we see that the space to the left is $(A_0, A_1)_{\theta,q,-\gamma-1/\min\{p,q\}}$ and the space to the right is $(A_0, A_1)_{\theta,q,-\gamma-1/\max\{p,q\}}$. This completes the proof of the embeddings (a). The proof of (b) can be carried out in the same way but using [272, Theorems 5.7 and 4.7].

Next we return to interpolation of approximation spaces. First we establish an auxiliary result.

Lemma (6.2.6) [305] Let X be a quasi-Banach space, let $(G_n)_{n\in\mathbb{N}_0}$ be an approximation family and $0 < p < \infty$. Then we have

$$K(n^{1/p}, f;\, X_p^{1/p}, X) \sim \left(\sum_{k=1}^n E_k(f)^p\right)^{1/p}, \qquad n \in \mathbb{N}, f \in X.$$

Proof. Take any $f \in X$ and let $g \in X$ such that $f - g \in X_p^{1/p}$. Then

$$\left(\sum_{k=1}^n E_k(f)^p\right)^{1/p} \lesssim \left(\sum_{k=1}^n (E_k(f-g) + \|g\|_X)^p\right)^{1/p} \lesssim \left(\sum_{k=1}^n E_k(f-g)^p\right)^{1/p} + n^{1/p}\|g\|_X$$

$$\le \|f-g\|_{X_p^{1/p}} + n^{1/p}\|g\|_X.$$

Taking the infimum over all $g \in X$ with $f - g \in X_p^{1/p}$, we obtain that

$$\left(\sum_{k=1}^{n} E_k(f)^p\right)^{1/p} \lesssim K(n^{1/p}, f;\ X_p^{1/p}, X).$$

Conversely, choose $g \in G_{n-1}$ satisfying that $\|f-g\|_X \le 2E_n(f)$. It follows that

$$K\left(n^{1/p}, f;\ X_p^{1/p}, X\right) \le \|g\|_{X_p^{1/p}} + n^{1/p}\|f-g\|_X \lesssim \left(\sum_{k=1}^{n} E_k(g)^p\right)^{1/p} + n^{1/p}E_n(f)$$

$$\le \left(\sum_{k=1}^{n} E_k(g)^p\right)^{1/p} + \left(\sum_{k=1}^{n} E_k(f)^p\right)^{1/p}$$

because $(E_n(f))$ is decreasing. Besides, for $1 \le k \le n$, we get

$$E_k(g) \lesssim E_k(f) + \|f-g\|_X \le 3E_k(f).$$

Consequently,

$$K(n^{1/p}, f;\ X_p^{1/p}, X) \lesssim \left(\sum_{k=1}^{n} E_k(f)^p\right)^{1/p}.$$

Now we can prove the following interpolation formula.

Proposition (6.2.7) [305] Let X be a quasi-Banach space and let $(G_n)_{n\in\mathbb{N}_0}$ be an approximation family. Suppose that $\alpha > 0, 0 < r, q \le \infty, 0 < \theta < 1, -\infty < \gamma < \infty$ and put $\rho(t) = t^\theta(1+|\log t|)^{-\gamma}$. Then we have with equivalence of quasi-norms

$$(X_r^\alpha, X)_{\rho,q} = X_q^{((1-\theta)\alpha,\gamma)}.$$

Proof. Let $p > 0$. We claim that

$$(X_p^{1/p}, X)_{\rho,q} = X_q^{((1-\theta)\alpha,\gamma)}. \tag{12}$$

Indeed, since $X_p^{1/p} \hookrightarrow X$, we have that $K(t,f;\ X_p^{1/p},X) \sim t\|f\|_X$ for $0 < t < 1$. This yields that

$$\|f\|_{(X_p^{1/p},X)_{\rho,q}} \sim \left(\int_1^\infty \left[\frac{K(t,f;\ X_p^{1/p},X)}{t^\theta(1+\log t)^{-\gamma}}\right]^q \frac{dt}{t}\right)^{1/q}.$$

Whence, using Lemma (6.2.6) and Hardy's inequality [293, Theorem 1.2], we get

$$\|f\|_{\left(X_p^{1/p},X\right)_{\rho,q}} \sim \left(\sum_{n=1}^{\infty}\left[\frac{K\left(n^{1/p},f;\ X_p^{1/p},X\right)}{n^{\theta/p}(1+\log n)^{-\gamma}}\right]^q \frac{1}{n}\right)^{1/q}$$

$$\sim \left(\sum_{n=1}^{\infty}\left[n^{(1-\theta)/p}(1+\log n)^{\gamma}\left(\sum_{k=1}^{n}\frac{E_k(f)^p}{n}\right)^{1/p}\right]^q \frac{1}{n}\right)^{1/q} \sim \left(\sum_{n=1}^{\infty}\left[n^{(1-\theta)/p}(1+\log n)^{\gamma}E_n(f)\right]^q \frac{1}{n}\right)^{1/q}.$$

Take $0 < p < 1/\alpha$. Let $(1-\lambda)/p = \alpha$ and $\mu(t) = t^{(1-\lambda)\theta+\lambda}(1+|\log t|)^{-\gamma}$. By (12) and [281, Corollary 4.4], we get

$$(X_r^\alpha, X)_{\rho,q} = ((X_p^{1/p}, X)_{\lambda,r}, X)_{\rho,q} = (X_p^{1/p}, X)_{\mu,q} = X_q^{((1-\theta)\alpha,\gamma)}$$

where we have used again (12) in the last equality.

We close with a Hardy-type inequality which can be proved as [1, Lemma 3.3.9].

Lemma (6.2.8) [305] Let ψ be a non-negative measurable function on $(0,\infty)$, let $-\infty < \lambda < 1$, $-\infty < \gamma < \infty$ and $1 \le q \le \infty$. Then

$$\left(\int_0^\infty \left[t^{\lambda-1}(1+|\log t|)^\gamma \int_0^t \psi(s)ds\right]^q \frac{dt}{t}\right)^{1/q} \lesssim \left(\int_0^\infty \left[t^\lambda(1+|\log t|)^\gamma \psi(t)\right]^q \frac{dt}{t}\right)^{1/q}.$$

Let $(G_n)_{n\in\mathbb{N}_0}$ be an approximation family in the quasi-Banach space X. Since $G_n \subseteq X_p^\alpha$ and $G_n \subseteq X_q^{(0,\gamma)}$ for any $n \in \mathbb{N}_0$, the sequence $(G_n)_{n\in\mathbb{N}_0}$ is also an approximation family in X_p^α and $X_q^{(0,\gamma)}$. Hence, we can apply again these constructions. Suppose $0 < \alpha, \beta < \infty$ and $0 < p, r \le \infty$. It was shown by Pietsch [300, Theorem 3.2] that

$$(X_p^\alpha)_r^\beta = X_r^{\alpha+\beta}. \tag{13}$$

On the other hand, Fehér and Grässler [295, Theorem 2] proved that

$$(X_q^{(0,\gamma)})_r^{(0,\delta)} = X_r^{(0,\gamma+1/q+\delta)} \tag{14}$$

provided that $0 < q, r \le \infty, \gamma > -1/q$ and $\delta > -1/r$.

We determine the space that arises applying the construction $(\cdot)_p^\alpha$ to $X_q^{(0,\gamma)}$. For this aim, we first establish an inequality of Jackson's type.

Lemma (6.2.9) [305] Let X be a quasi-Banach space and let $(G_n)_{n\in\mathbb{N}_0}$ be an approximation family in X. Suppose that $0 < q \le \infty$ and $\gamma > -1/q$. Then there is a constant $c > 0$ such that

$$E_{2n-1}(f;\, X) \le c(1+\log n)^{-(\gamma+1/q)} E_n(f; X_q^{(0,\gamma)})$$

for all $f \in X_q^{(0,\gamma)}$ and $n \in \mathbb{N}$.

Proof. We can find $g_1, g_2 \in G_{n-1}$ such that

$$\|f - g_1\|_{X_q^{(0,\gamma)}} \le 2E_n(f; X_q^{(0,\gamma)})$$

and

$$\|f - g_1 - g_2\|_X \le 2E_n(f - g_1; X).$$

So, $g_1 + g_2 \in G_{2n-2}$ and

$$E_{2n-1}(f;\, X) \le \|f - g_1 - g_2\|_X \le 2E_n(f - g_1; X). \tag{15}$$

Since the sequence $(E_n(f - g_1))$ is monotone, we obtain

$$\|f - g_1\|_{X_q^{(0,\gamma)}} \ge \left(\sum_{k=1}^{n} [(1+\log k)^\gamma E_k(f - g_1; X)]^q k^{-1}\right)^{1/q}$$

$$\ge E_n(f - g_1; X)\left(\sum_{k=1}^{n} (1+\log k)^{\gamma q} k^{-1}\right)^{1/q} \ge c_1(1+\log n)^{\gamma+1/q} E_n(f - g_1; X).$$

Consequently, by (15) and the choice of g_1, we conclude that

$$E_{2n-1}(f;\, X) \le 2E_n(f - g_1; X) \le c_2(1+\log n)^{-(\gamma+1/q)} \|f - g_1\|_{X_q^{(0,\gamma)}}$$

$$\le c(1+\log n)^{-(\gamma+1/q)} E_n\left(f; X_q^{(0,\gamma)}\right).$$

Theorem (6.2.10) [305] Let X be a quasi-Banach space and let $(G_n)_{n\in\mathbb{N}_0}$ be an approximation family. Suppose that $\alpha > 0, 0 < p, q \le \infty$ and $\gamma > -1/q$. Then

$$(X_q^{(0,\gamma)})_p^\alpha = X_p^{(\alpha,\gamma+1/q)}$$

with equivalence of quasi-norms.

Proof. Take any $f \in (X_q^{(0,\gamma)})_p^\alpha$. Using Lemma (6.2.9) we obtain

$$\|f\|_{X_p^{(\alpha,\gamma+1/q)}} = \left(\sum_{n=1}^{\infty} \left[n^\alpha (1+\log n)^{\gamma+1/q} E_n(f; X)\right]^p n^{-1}\right)^{1/p}$$

$$\lesssim \left(\sum_{n=1}^{\infty} \left[n^\alpha (1+\log n)^{\gamma+1/q} E_{2n-1}(f; X)\right]^p n^{-1}\right)^{1/p}$$

$$\leq c\left(\sum_{n=1}^{\infty}\left[n^{\alpha}E_n(f;X_q^{(0,\gamma)})\right]^p n^{-1}\right)^{1/p} = c\|f\|_{(X_q^{(0,\gamma)})_p^{\alpha}}.$$

Conversely, according to [302, Theorem 3.3], there is a constant $c > 0$ such that any $f \in X_p^{(\alpha,\gamma+1/q)}$ can be represented as $f = \sum_{n=0}^{\infty} g_n$ with $g_n \in G_{2^n}$ and

$$\left(\sum_{n=0}^{\infty}\left[2^{n\alpha}(1+n)^{\gamma+1/q}\|g_n\|_X\right]^p\right)^{1/p} \leq c\|f\|_{X_p^{(\alpha,\gamma+1/q)}}.$$

Since

$$\|g_n\|_{X_q^{(0,\gamma)}} = \left(\sum_{k=1}^{2^n}[(1+\log k)^{\gamma}E_k(g_n;X)]^q k^{-1}\right)^{1/q}$$

$$\leq \left(\sum_{k=1}^{2^n}[(1+\log k)^{\gamma}]^q k^{-1}\right)^{1/q} \|g_n\|_X \leq c_1(1+n)^{\gamma+1/q}\|g_n\|_X,$$

we derive that

$$\left(\sum_{n=0}^{\infty}\left[2^{n\alpha}\|g_n\|_{X_q^{(0,\gamma)}}\right]^p\right)^{1/p} \leq c_1\left(\sum_{n=0}^{\infty}\left[2^{n\alpha}(1+n)^{\gamma+1/q}\|g_n\|_X\right]^p\right)^{1/p} \leq c_2\|f\|_{X_p^{(\alpha,\gamma+1/q)}}.$$

This yields that the series $\sum_{n=0}^{\infty} g_n$ converges to f in $X_q^{(0,\gamma)}$. Finally, by [300, Theorem 3.1], we get

$$\|f\|_{(X_q^{(0,\gamma)})_p^{\alpha}} \leq c_3\left(\sum_{n=0}^{\infty}\left[2^{n\alpha}\|g_n\|_{X_q^{(0,\gamma)}}\right]^p\right)^{1/p} \leq c_4\|f\|_{X_p^{(\alpha,\gamma+1/q)}}.$$

Writing down Theorem (6.2.10) in the special case $1 \leq p = q \leq \infty$, with X being a Banach space and $(G_n)_{n\in\mathbb{N}_0}$ being a sequence of subspaces of X, we recover a result of Almira and Luther [284, Theorem 6.1].

Example (6.2.11) [305] If $X = \ell_{\infty}$ and $G_n = F_n$, the subset of sequences having at most n coordinates different from 0, then for $\alpha > 0, 0 < p, q \leq \infty$ and $\gamma > -1/q$, we obtain

$$(\ell_{\infty,q}(\log \ell)_{\gamma})_p^{\alpha} = ((\ell_{\infty})_q^{(0,\gamma)})_p^{\alpha} = (\ell_{\infty})_p^{(\alpha,\gamma+1/q)} = \ell_{1/\alpha,p}(\log \ell)_{\gamma+1/q}.$$

It is more difficult to determine the resulting space when we apply the approximation constructions in reverse order. We shall do it with the help of interpolation techniques.

Theorem (6.2.12) [305] Let X be a quasi-Banach space and let $(G_n)_{n\in\mathbb{N}_0}$ be an approximation family. Suppose $0 < p < \infty, 0 < q \leq \infty$ and $\gamma \geq -1/q$. Then we have with equivalence of quasi-norms

$$(X, X_p^{1/p})_{(0,-\gamma),q} = X_q^{(0,\gamma)}.$$

Proof. Doing a change of variable, using Lemma (6.2.6) and Hardy's inequality (see [293, Theorem 1.2]), we derive

$$\|f\|_{\left(X,X_p^{1/p}\right)_{(0,-\gamma),q}} = \left(\int_0^1\left[(1-\log t)^{\gamma}tK\left(t^{-1},f;\ X_p^{1/p},X\right)\right]^q\frac{dt}{t}\right)^{1/q}$$

$$\sim \left(\int_1^{\infty}\left[(1+\log s)^{\gamma}s^{-1/p}K\left(s^{1/p},f;\ X_p^{1/p},X\right)\right]^q\frac{ds}{s}\right)^{1/q}$$

$$\sim \left(\sum_{n=1}^{\infty}\left[(1+\log n)^{\gamma} n^{-1/p} K\left(n^{1/p}, f;\ X_p^{1/p}, X\right)\right]^q n^{-1}\right)^{1/q}$$

$$\sim \left(\sum_{n=1}^{\infty}\left[(1+\log n)^{\gamma}\left(\frac{1}{n}\sum_{k=1}^{n} E_k(f)^p\right)^{1/p}\right]^q n^{-1}\right)^{1/q}$$

$$\sim \left(\sum_{n=1}^{\infty}[(1+\log n)^{\gamma} E_n(f)]^q n^{-1}\right)^{1/q} = \|f\|_{X_q^{(0,\gamma)}}.$$

Corollary (6.2.13) [305] Let X be a quasi-Banach space and let $(G_n)_{n\in\mathbb{N}_0}$ be an approximation family. Suppose $\alpha > 0, 0 < p, q \le \infty$ and $\gamma \ge -1/q$. Then

$$(X, X_p^{\alpha})_{(0,-\gamma),q} = X_q^{(0,\gamma)}$$

with equivalence of quasi-norms.

Proof. Take $0 < \rho < 1/\alpha$ and put $\theta = \alpha\rho$. By (10), we have that $X_p^{\alpha} = (X_{\rho}^{1/\rho}, X)_{\theta,p}$. Whence, according to Lemma (6.2.2) (b) and Theorem (6.2.12), we derive

$$(X, X_p^{\alpha})_{(0,-\gamma),q} = (X, (\,X, (X_{\rho}^{1/\rho}, X)_{\theta,p})_{(0,-\gamma),q} = (X, X_{\rho}^{1/\rho})_{(0,-\gamma),q} = X_q^{(0,\gamma)}.$$

Now we can determine $(X_p^{\alpha})_q^{(0,\gamma)}$ by means of an auxiliary sequence space.

Theorem (6.2.14) [305] Let X be a quasi-Banach space and let $(G_n)_{n\in\mathbb{N}_0}$ be an approximation family. Suppose $\alpha > 0, 0 < p, q \le \infty$ and $\gamma \ge -1/q$. Take any $0 < r < 1/r$. Then we have

$$f \in \left(X_p^{\alpha}\right)_q^{(0,\gamma)} \quad \text{if and only if} \quad (E_n(f)) \in (\ell_{1/\alpha,p}, \ell_r)_{(0,-\gamma),q}.$$

Proof. Using Corollary (6.2.13) and (13), we get

$$\left(X_p^{\alpha}\right)_q^{(0,\gamma)} = (X_p^{\alpha}, \left(X_p^{\alpha}\right)_r^{1/r-\alpha})_{(0,-\gamma),q} = (X_p^{\alpha}, X_r^{1/r})_{(0,-\gamma),q}.$$

In order to estimate the K-functional for the couple $(X_p^{\alpha}, X_r^{1/r})$, take $0 < \theta < 1$ such that $\alpha = (1-\theta)/r$. By (10) we have $X_p^{\alpha} = (X_r^{1/r}, X)_{\theta,p}$. Whence, according to Holmstedt's formula [279, Remark 2.1] and Lemma (6.2.6), we derive

$$K\left(2^{n\theta/r}, f;\ X_r^{1/r}, X_p^{\alpha}\right) = K\left(2^{n\theta/r}, f;\ X_r^{1/r}, \left(X_r^{1/r}, X\right)_{\theta,p}\right)$$

$$\sim 2^{n\theta/r}\left(\int_{2^{n/r}}^{\infty}\left[u^{-\theta} K\left(u, f;\ X_r^{1/r}, X\right)\right]^p \frac{du}{u}\right)^{1/p}$$

$$\sim 2^{n\theta/r}\left(\sum_{k=n}^{\infty}\left[2^{-\theta k/r} K\left(2^{k/r}, f;\ X_r^{1/r}, X\right)\right]^p\right)^{1/p}$$

$$\sim 2^{n\theta/r}\left(\sum_{k=n}^{\infty}\left[2^{-\theta k/r}\left(\sum_{j=1}^{2^k} E_j(f)^r\right)^{1/r}\right]^p\right)^{1/p}$$

$$\sim 2^{n\theta/r}\left(\sum_{k=n}^{\infty}[2^{-\theta k/r} K(2^{k/r}, (E_j(f));\ \ell_r, \ell_\infty)]^p\right)^{1/p}$$

where we have used again Lemma (6.2.6) in the last equivalence but now with the couple (ℓ_r, ℓ_∞), viewing ℓ_r as $(\ell_\infty)_r^{1/r}$. Hence, reversing the steps and using (9), we conclude that

$$K(2^{n\theta/r}, f;\ X_r^{1/r}, X_p^{\alpha}) \sim 2^{n\theta/r}\left(\int_{2^{n/r}}^{\infty}\left[u^{-\theta}K(u,(E_j(f));\ \ell_r,\ell_\infty)\right]^p \frac{du}{u}\right)^{1/p}$$
$$\sim K(2^{n\theta/r},(E_j(f));\ \ell_r,(\ell_r,\ell_\infty)_{\theta,p}) \sim K(2^{n\theta/r},(E_j(f));\ \ell_r,\ell_{1/\alpha,p}).$$
This yields that
$$K(t,f;\ X_r^{1/r},X_p^{\alpha}) \sim K(t,(E_j(f));\ \ell_r,\ell_{1/\alpha,p}), \qquad 1 \le t < \infty.$$
Reversing the couple, we obtain that
$$K(t,f;\ X_p^{\alpha},X_r^{1/r}) \sim K(t,(E_j(f));\ \ell_{1/\alpha,p},\ell_r), \qquad 0 < t \le 1.$$
Therefore
$$\|f\|_{(X_p^{\alpha})_q^{(0,\gamma)}} \sim \|f\|_{(X_p^{\alpha},X_r^{1/r})_{(0,-\gamma),q}} \sim \left\|(E_j(f))\right\|_{(\ell_{1/\alpha,p},\ell_r)_{(0,-\gamma),q}}$$
which completes the proof.

We proceed now to study the sequence space that arose in Theorem (6.2.14).

Definition (6.2.15) [305] Let $\alpha > 0, 0 < p,q \le \infty$ and $-\infty < \gamma < \infty$. We put
$$Z = Z_{\alpha,p,\gamma,q} = \left\{\xi \in \ell_\infty : \|\xi\|_Z = \left(\sum_{n=1}^{\infty}\left[(1+\log n)^{\gamma}\left(\sum_{j=n}^{\infty}(j^{\alpha}\xi_j^*)^p j^{-1}\right)^{1/p}\right]^q n^{-1}\right)^{1/q} < \infty\right\}.$$
Note that when $q = 1, Z_{\alpha,p,\gamma,1}$ is a small Lorentz sequence space in the terminology of Fiorenza and Karadzhov [296].

Lemma (6.2.16) [305] Let $\alpha > 0, 0 < p,q \le \infty, \gamma \ge -1/q$ and $0 < r < \min\{1/\alpha, q\}$. Then we have with equivalent quasi-norms
$$\left(\ell_{1/\alpha,p},\ell_r\right)_{(0,-\gamma),q} = Z \cap \ell_{1/\alpha,q}(\log \ell)_\gamma$$
where $\|\xi\|_{Z\cap \ell_{1/\alpha,q}(\log \ell)_\gamma} = \|\xi\|_Z + \|\xi\|_{\ell_{1/\alpha,q}(\log \ell)_\gamma}$.

Proof. Let $1/\delta = 1/r - \alpha$. According to [279, Theorem 4.2], we have that
$$K(t,\xi;\ \ell_r,\ell_{1/\alpha,p}) \sim \left(\int_0^{t^\delta}\xi^*(v)^r dv\right)^{1/r} + t\left(\int_{t^\delta}^{\infty}\left(v^{\alpha}\xi^*(v)\right)^p \frac{dv}{v}\right)^{1/p}, 0 < t < \infty.$$
Here
$$\xi^*(t) = \begin{cases}\xi_1^* & \text{for } t \in (0,1),\\ \xi_n^* & \text{for } t \in [n-1,n), n = 2,3,\ldots.\end{cases}$$
Therefore,
$$\|\xi\|_{(\ell_{1/\alpha,p},\ell_r)_{(0,-\gamma),q}} = \left(\int_0^1\left[(1-\log t)^{\gamma} tK(t^{-1},\xi;\ \ell_r,\ell_{1/\alpha,p})\right]^q \frac{dt}{t}\right)^{1/q}$$
$$= \left(\int_1^{\infty}\left[(1-\log t)^{\gamma} t^{-1}K(t,\xi;\ \ell_r,\ell_{1/\alpha,p})\right]^q \frac{dt}{t}\right)^{1/q}$$
$$\sim \left(\int_1^{\infty}\left[(1-\log t)^{\gamma} t^{-1}\left(\int_0^{t^\delta}\xi^*(v)^r dv\right)^{1/r}\right]^q \frac{dt}{t}\right)^{1/q}$$
$$+ \left(\int_1^{\infty}\left[(1-\log t)^{\gamma}\left(\int_{t^\delta}^{\infty}\left(v^{\alpha}\xi^*(v)\right)^p \frac{dv}{v}\right)^{1/p}\right]^q \frac{dt}{t}\right)^{1/q} = I_1 + I_2.$$
For the term I_1, a change of variables yields
$$I_1 \sim \left(\int_1^{\infty}\left[t^{-1/\delta}(1-\log t)^{\gamma}\left(\int_0^{t}\xi^*(v)^r dv\right)^{1/r}\right]^q \frac{dt}{t}\right)^{1/q}.$$

Since

$$\left(\int_0^1\left[t^{-1/\delta}(1-\log t)^{\gamma}\left(\int_0^t \xi^*(v)^r dv\right)^{1/r}\right]^q\frac{dt}{t}\right)^{1/q} \leq \xi_1^*\left(\int_0^1\left[t^{1/r-1/\delta}(1-\log t)^{\gamma}\right]^q\frac{dt}{t}\right)^{1/q} \lesssim \xi_1^*$$

$$\lesssim \left(\int_0^1 \xi^*(v)^r dv\right)^{1/r}\left(\int_1^\infty\left[t^{-1/\delta}(1+\log t)^{\gamma}\right]^q\frac{dt}{t}\right)^{1/q} \leq \left(\int_1^\infty\left[t^{-1/\delta}(1+\log t)^{\gamma}\left(\int_0^t \xi^*(v)^r dv\right)^{1/r}\right]^q\frac{dt}{t}\right)^{1/q},$$

we obtain that

$$I_1 \sim \left(\int_0^\infty\left[t^{-1/\delta}(1+|\log t|)^{\gamma}\left(\int_0^t \xi^*(v)^r dv\right)^{1/r}\right]^q\frac{dt}{t}\right)^{1/q}$$

$$= \left(\int_0^\infty\left[t^{-r/\delta}(1+|\log t|)^{\gamma r}\left(\int_0^t \xi^*(v)^r dv\right)^{q/r}\right]^q\frac{dt}{t}\right)^{1/q}.$$

Now using the Hardy-type inequality given in Lemma (6.2.8), we derive

$$I_1 \sim \left(\int_0^\infty\left[t^{1-r/\delta}(1+|\log t|)^{\gamma r}\xi^*(t)^r\right]^{q/r}\frac{dt}{t}\right)^{1/q} = \left(\int_0^\infty\left[t^{\alpha}(1+|\log t|)^{\gamma}\xi^*(t)\right]^q\frac{dt}{t}\right)^{1/q}$$

$$\sim \left(\sum_{n=1}^\infty\left[n^{\alpha}(1+\log n)^{\gamma}\xi_n^*\right]^q\frac{1}{n}\right)^{1/q}.$$

As for I_2, we get

$$I_2 \sim \left(\int_1^\infty\left[(1+\log t)^{\gamma}\left(\int_t^\infty\left(v^{\alpha}\xi^*(v)\right)^p\frac{dv}{v}\right)^{1/p}\right]^q\frac{dt}{t}\right)^{1/q}$$

$$= \left(\sum_{n=1}^\infty\int_n^{n+1}\left[(1+\log t)^{\gamma}\left(\int_t^\infty\left(v^{\alpha}\xi^*(v)\right)^p\frac{dv}{v}\right)^{1/p}\right]^q\frac{dt}{t}\right)^{1/q}.$$

Hence

$$I_2 \lesssim \left(\sum_{n=1}^\infty\left[(1+\log n)^{\gamma}\left(\sum_{j=n}^\infty\int_j^{j+1}\left(v^{\alpha}\xi^*(v)\right)^p\frac{dv}{v}\right)^{1/p}\right]^q\frac{1}{n}\right)^{1/q}$$

$$\sim \left(\sum_{n=1}^\infty\left[(1+\log n)^{\gamma}\left(\sum_{j=n}^\infty\left((j+1)^{\alpha}\xi_{j+1}^*\right)^p\frac{1}{j+1}\right)^{1/p}\right]^q\frac{1}{n}\right)^{1/q}.$$

Similarly,

$$I_2 \gtrsim \left(\sum_{n=1}^\infty\left[(1+\log n)^{\gamma}\left(\sum_{j=n+1}^\infty\left((j+1)^{\alpha}\xi_{j+1}^*\right)^p\frac{1}{j+1}\right)^{1/p}\right]^q\frac{1}{n}\right)^{1/q}.$$

Consequently,

$$\|\xi\|_{(\ell_{1/\alpha,p},\ell_r)_{(0,-\gamma),q}} \sim I_1 + I_2$$

$$\sim \left(\sum_{n=1}^\infty\left[n^{\alpha}(1+\log n)^{\gamma}\xi_n^*\right]^q\frac{1}{n}\right)^{1/q} + \left(\sum_{n=1}^\infty\left[(1+\log n)^{\gamma}\left(\sum_{j=n+1}^\infty\left((j+1)^{\alpha}\xi_{j+1}^*\right)^p\frac{1}{j+1}\right)^{1/p}\right]^q\frac{1}{n}\right)^{1/q}$$

$$\sim \|\xi\|_{\ell_{1/\alpha,q}(\log \ell)_{\gamma}} + \|\xi\|_Z.$$

As a direct consequence of Theorem (6.2.14) and Lemma (6.2.16), we can now show the following explicit description of $\left(X_p^\alpha\right)_q^{(0,\gamma)}$.

Theorem (6.2.17) [305] Let X be a quasi-Banach space and let $(G_n)_{n\in\mathbb{N}_0}$ be an approximation family. Suppose $\alpha > 0, 0 < p, q \leq \infty$ and $\gamma \geq -1/q$. Put

$$W = W_{\alpha,p,\gamma,q} = \left\{f \in X : (E_n(f)) \in Z_{\alpha,p,\gamma,q}\right\}$$

with $\|f\|_W = \|(E_n(f))\|_Z$. Then we have with equivalence of quasi-norms

$$\left(X_p^\alpha\right)_q^{(0,\gamma)} = X_q^{(\alpha,\gamma)} \cap W.$$

Corollary (6.2.18) [305] Let X be a quasi-Banach space and let $(G_n)_{n\in\mathbb{N}_0}$ be an approximation family. Suppose $\alpha > 0, 0 < q \leq \infty$ and $\gamma > -1/q$. Then we have with equivalence of quasi-norms

$$\left(X_p^\alpha\right)_q^{(0,\gamma)} = X_q^{(\alpha,\gamma+1/q)}$$

Proof. Using equality of the lower parameters, we obtain

$$\|f\|_W = \left(\sum_{j=1}^{\infty}\left(j^\alpha E_j(f)\right)^q j^{-1}\sum_{n=1}^{j}(1+\log n)^{\gamma q} n^{-1}\right)^{1/q}$$

$$\sim \left(\sum_{j=1}^{\infty}\left(j^\alpha (1+\log n)^{\gamma+1/q} E_j(f)\right)^q j^{-1}\right)^{1/q} = \|f\|_{X_q^{(\alpha,\gamma+1/q)}}.$$

Therefore, applying Theorem (6.2.17), we derive

$$\left(X_p^\alpha\right)_q^{(0,\gamma)} = X_q^{(0,\gamma)} \cap X_q^{(\alpha,\gamma+1/q)} = X_q^{(\alpha,\gamma+1/q)}.$$

Corollary (6.2.18) and Theorem (6.2.10) show that in the “diagonal case” where $p = q$ the order of application of the approximation constructions is not important.

As we have seen in Theorem (6.2.17), in general $\left(X_p^\alpha\right)_q^{(0,\gamma)}$ cannot be realized as a space $X_p^{(\alpha,\omega)}$. In applications it is important to know the biggest (respectively, smallest) space $X_p^{(\alpha,\omega)}$ which is contained in (respectively, which contains to) $\left(X_p^\alpha\right)_q^{(0,\gamma)}$. Next we determine those spaces.

Theorem (6.2.19) [305] Let $\alpha > 0, 0 < p, q \leq \infty$ and $\gamma > -1/q$. Then

$$X_q^{\left(\alpha,\gamma+\frac{1}{\min\{p,q\}}\right)} \hookrightarrow \left(X_p^\alpha\right)_q^{(0,\gamma)} \hookrightarrow X_q^{\left(\alpha,\gamma+\frac{1}{\max\{p,q\}}\right)}.$$

Proof. Let $\beta > \alpha$. By Corollary (6.2.13) and (13), we obtain that

$$\left(X_p^\alpha\right)_q^{(0,\gamma)} = (X_p^\alpha, X_p^\beta)_{(0,-\gamma),q}.$$

Moreover, according to (10), $X_p^\alpha = (X, X_p^\beta)_{\theta,p}$ for $\alpha = \theta\beta$. Therefore, Lemma (6.2.5) (b) and Proposition (6.2.7) yield the wanted embeddings.

Remark (6.2.20) [305] Embeddings in Theorem (6.2.19) are the best possible in the sense that in general for any $\tau > 0$ embeddings

$$X_q^{\left(\alpha,\gamma+\frac{1}{\min\{p,q\}}-\tau\right)} \hookrightarrow \left(X_p^\alpha\right)_q^{(0,\gamma)}, \tag{16}$$

$$\left(X_p^\alpha\right)_q^{(0,\gamma)} \hookrightarrow X_q^{\left(\alpha,\gamma+\frac{1}{\max\{p,q\}}+\tau\right)} \tag{17}$$

do not hold. We show it now by means of counterexamples.

Take $X=\ell_\infty$ and $G_n=F_n$, so $E_n(\xi;\ \ell_\infty)=\xi_n^*$. As for (16), suppose that $\min\{p,q\}=p$. Given any $\tau>0$, choose $\epsilon>0$ such that $\tau-\epsilon>0$. The sequence

$$\xi=\left(n^{-\alpha}(1+\log n)^{-(\gamma+1/p-\tau+1/q+\epsilon)}\right)$$

belongs to $\ell_{1/\alpha,p}(\log \ell)_{\gamma+1/p-\tau}=X_q^{(\alpha,\gamma+1/p-\tau)}$. However, since

$$\left(\sum_{j=n}^{\infty}(1+\log j)^{-(\gamma+1/p-\tau+1/q+\epsilon)p}j^{-1}\right)^{1/p}\sim(1+\log n)^{-(\gamma-\tau+1/q+\epsilon)}$$

if $\tau<\gamma+1/q+\epsilon$ and the series diverges otherwise, it follows that

$$\|\xi\|_{(X_p^\alpha)_q^{(0,\gamma)}}\geq\|\xi\|_W\sim\left(\sum_{n=1}^{\infty}\left[(1+\log n)^{\gamma}(1+\log n)^{-(\gamma-\tau+1/q+\epsilon)}\right]^q n^{-1}\right)^{1/q}$$
$$=\left(\sum_{n=1}^{\infty}(1+\log n)^{q(\tau-\epsilon)-1}n^{-1}\right)^{1/q}=\infty.$$

Hence, (16) does not hold.

As for (17), assume that $\max\{p,q\}=p$. Given any $\tau>0$, let $0<\epsilon<\tau$ and put

$$\xi=\left(n^{-\alpha}(1+\log n)^{-(\gamma+1/p+1/q+\epsilon)}\right).$$

We claim that $\xi\in\left(X_p^\alpha\right)_q^{(0,\gamma)}$. Indeed,

$$\|\xi\|_{X_q^{(0,\gamma)}}=\|\xi\|_{\ell_{1/\alpha,q}(\log\ell)_\gamma}=\left(\sum_{n=1}^{\infty}(1+\log n)^{-q/p-\epsilon q-1}n^{-1}\right)^{1/q}<\infty,$$

and

$$\|\xi\|_W=\left(\sum_{n=1}^{\infty}\left[(1+\log n)^{\gamma}\left(\sum_{j=n}^{\infty}(1+\log j)^{-(\gamma+1/q+\epsilon)p-1}j^{-1}\right)^{1/p}\right]^q n^{-1}\right)^{1/q}$$
$$\sim\left(\sum_{n=1}^{\infty}(1+\log n)^{-\epsilon q-1}n^{-1}\right)^{1/q}<\infty.$$

Hence, according to Theorem (6.2.17), $\xi\in\left(X_p^\alpha\right)_q^{(0,\gamma)}$. However, $\xi\notin\ell_{1/\alpha,q}(\log\ell)_{\gamma+1/q+\tau}+X_q^{(\alpha,\gamma+\frac{1}{\max\{p,q\}}+\tau)}$.

We apply the previous results to study several problems on Besov spaces.

In what follows we take $X=L_p(\mathbb{T})$, the Lebesgue space of periodic measurable functions defined on the unit circle $\mathbb{T}$, and we choose G_n as the set O_n of all trigonometric polynomials of degree less than or equal to n. Then X_r^α is the (classical) Besov space $B_{p,r}^\alpha$, $X_q^{(0,\gamma)}$ coincides with the Besov space of logarithmic smoothness $B_{p,q}^{0,\gamma}$ and $X_r^{(\alpha,\eta)}$ with $B_{p,r}^{\alpha,\eta}$ (see [34,19,10]).

The following interpolation formulae follow from Proposition (6.2.7) and Corollary (6.2.13):

$$\left(L_p,B_{p,r}^\alpha\right)_{\rho,q}=B_{p,q}^{\theta\alpha,b}\quad\text{where }\rho(t)=t^\theta(1+|log\ t|)^{-b},\tag{18}$$
$$\left(L_p,B_{p,r}^\alpha\right)_{(0,-\gamma),q}=B_{p,q}^{0,\gamma}\quad\text{if }\gamma\geq-1/q.\tag{19}$$

DeVore, Riemenschneider and Sharpley have established in [266, Corollary 5.5] embeddings of spaces $B_{p,r}^{\alpha,b}$ with $\alpha > 0$ into Lorentz–Zygmund spaces. They dealt with the Banach case where the parameters are greater than or equal to 1. Next we extend those results to the whole range of parameters. We start with known embeddings into L_∞ (see [168, Theorem 3.5.5]).

Theorem (6.2.21) [305] Let $0 < p < \infty$. Then

$$B_{p,1}^{1/p} \hookrightarrow L_\infty.$$

Proof. According to [300, Theorem 3.1], there is a constant $c > 0$ such that for any $f \in B_{p,1}^{1/p}$ there is a representation $f = \sum_{n=0}^{\infty} g_n$ with $g_n \in O_{2^n}$ and

$$\sum_{n=0}^{\infty} 2^{n/p}\|g_n\|_{L_p} \le c\|f\|_{B_{p,1}^{1/p}}.$$

Hence, using the Nikolskiĭ inequality for trigonometric polynomials (see [297, 3.4.3] and [284]), we derive that

$$\|f\|_{L_\infty} \le \sum_{n=0}^{\infty}\|g_n\|_{L_\infty} \lesssim \sum_{n=0}^{\infty} 2^{n/p}\|g_n\|_{L_p} \le c\|f\|_{B_{p,1}^{1/p}}.$$

Theorem (6.2.22) [305] Let $0 < p < q < \infty, s = 1/p - 1/q\,, 0 < r \le \infty$ and $-\infty < b < \infty$. Then

$$B_{p,r}^{s,b} \hookrightarrow L_{q,r}(\log L)_b.$$

Proof. Combining Theorem (6.2.21) with the natural embedding $L_p \hookrightarrow L_p$ and interpolating with the function parameter $\rho(t) = t^\theta(1 + |log\, t|)^{-b}$ where $\theta = 1 - p/q$, we derive the continuous embedding

$$\left(L_p, B_{p,1}^{1/p}\right)_{\rho,r} \hookrightarrow \left(L_p, L_\infty\right)_{\rho,r}.$$

By (18), the space to the left is $B_{p,r}^{s,b}$ and, according to [281, Lemma 6.1], the space to the right is $L_{q,r}(\log L)_b$. This completes the proof.

Theorem (6.2.23) [305] Let $0 < p < \infty, 0 < r \le \infty$ and $b + 1/r < 0$. Then

$$B_{p,r}^{1/p,b+1/\min\{1,r\}} \hookrightarrow L_{\infty,r}(\log L)_b.$$

Proof. We interpolate by the limiting method with $\theta = 1$ the embeddings

$$B_{p,1}^{1/p} \hookrightarrow L_\infty, \qquad L_p \hookrightarrow L_p$$

to obtain that

$$\left(L_p, B_{p,1}^{1/p}\right)_{(1,-b),r} \hookrightarrow \left(L_p, L_\infty\right)_{(1,-b),r}.$$

Lemma (6.2.24) [305] Yields that $\left(L_p, L_\infty\right)_{(1,-b),r} = L_{\infty,r}(\log L)_b$. On the other hand, take $0 < \alpha < p$ and put $\rho(t) = t^{\alpha/p}(1 + |\log t|)^{-b-1/\min\{1,r\}}$. Using (18) and Lemma (6.2.5) (a), we get

$$B_{p,r}^{1/p,b+1/\min\{1,r\}} = \left(L_p, B_{p,\alpha}^{1/\alpha}\right)_{\rho,r} \hookrightarrow \left(L_p, \left(L_p, B_{p,\alpha}^{1/\alpha}\right)_{\alpha/p,1}\right)_{(1,-b),r} = \left(L_p, B_{p,1}^{1/p}\right)_{(1,-b),r}.$$

Consequently,

$$B_{p,r}^{1/p,b+1/\min\{1,r\}} \hookrightarrow L_{\infty,r}(\log L)_b.$$

We focus the attention on Besov spaces $B_{p,r}^{0,b}$. Embeddings of $B_{p,r}^{0,b}$ into Lorentz–Zygmund spaces have been established by [289] (see also Caetano, Gogatishvili and Opic [288] for the case of Besov spaces on $\mathbb{R}^n$ and by Triebel [282]). Now we show embeddings into spaces $Y_{p,r,\gamma}$ introduced.

Theorem (6.2.25) [305] Let $0<p<\infty, 0<r\leq\infty, b>-1/r, r\leq q$ and $\gamma=b+1/r-1/q$. Then $B_{p,r}^{0,b}\hookrightarrow Y_{p,q,\gamma}$.

Proof. This time we interpolate the embeddings

$$B_{p,1}^{1/p}\hookrightarrow L_\infty,\qquad L_p\hookrightarrow L_p$$

by the limiting method with $\theta=0$. Since $\gamma>-1/q$, using (19) and Lemma (6.2.4), it turns out that

$$B_{p,q}^{0,\gamma}=\left(L_p,B_{p,1}^{1/p}\right)_{(0,-\gamma),q}\hookrightarrow\left(L_p,L_\infty\right)_{(0,-\gamma),q}=Y_{p,q,\gamma}.$$

Besides, by [295, Lemma 2], we have $B_{p,r}^{0,b}\hookrightarrow B_{p,q}^{0,\gamma}$. Therefore $B_{p,r}^{0,b}\hookrightarrow Y_{p,q,\gamma}$.

Corollary (6.2.26) [305] Let $1<p<\infty$. Then $B_{p,1}^{0,-1/p}$ is continuously embedded in the small Lebesgue space $L^{(p)}$.

Now we study the relationship between smoothness of derivatives of f and the smoothness of f. Let $k\in\mathbb{N}$. According to [266, page 70], if $f\in B_{p,q}^{k,\gamma+1}$ then $D^kf\in B_{p,q}^{0,\gamma}$. The following result shows that sometimes the loss of smoothness is less than a logarithm.

Theorem (6.2.27) [305] Let $k\in\mathbb{N}, 1<p<\infty, 0<q\leq\infty$ and $\gamma>-1/q$. If $f\in B_{p,q}^{k,\gamma+\frac{1}{\min\{2,p,q\}}}$ then $D^kf\in B_{p,q}^{0,\gamma}$.

Proof. Clearly $D^k: W_p^k\longrightarrow L_p$ is bounded and $D^k(O_n)\subseteq O_n$. Then $E_m(D^kf;L_p)\leq E_m(f;W_p^k)$, and so

$$D^k:\ (W_p^k)_q^{(0,\gamma)}\longrightarrow(L_p)_q^{(0,\gamma)}=B_{p,q}^{0,\gamma}$$

is bounded. Moreover, it follows from the embedding $B_{p,\min\{p,2\}}^k\hookrightarrow W_p^k$ (see [168, Remark 4, page 164 and Theorem 3.5.4, page 169]) that $(B_{p,\min\{p,2\}}^k)_q^{(0,\gamma)}\hookrightarrow(W_p^k)_q^{(0,\gamma)}$. Finally, by Theorem (6.2.19),

$$B_{p,q}^{k,\gamma+\frac{1}{\min\{2,p,q\}}}=(L_p)_q^{(k,\gamma+\frac{1}{\min\{q,\min\{p,2\}\}})}\hookrightarrow((L_p)_{\min\{p,2\}}^k)_q^{(0,\gamma)}=(B_{p,\min\{p,2\}}^k)_q^{(0,\gamma)}.$$

Therefore, $D^k: B_{p,q}^{k,\gamma+\frac{1}{\min\{2,p,q\}}}\longrightarrow B_{p,q}^{0,\gamma}$ is bounded.

We close with a result on the conjugate-function operator H, which is defined on $L_1(\mathbb{T})$ by the principal-value integral

$$Hf(e^{ix})=\lim_{\epsilon\to0+}\int_\epsilon^{2\pi-\epsilon}f(e^{i(x-y)})\cot(y/2)\,dy$$

(see [38, Chapter IV]).

Theorem (6.2.28) [305] Let $0<q\leq\infty$ and $\gamma>-1/q$. If $f\in B_{1,q}^{0,\gamma+1}$ then $Hf\in B_{1,q}^{0,\gamma}$

Proof. According to [304, Theorem IV.3.16], we have that $H: L_1\longrightarrow L_{1,\infty}$ is bounded. Besides, by [1, Lemma 3.6.9], H maps any trigonometric polynomial in another trigonometric polynomial with the same degree. Hence,

$$H: B_{1,q}^{0,\gamma+1}=(L_1)_q^{(0,\gamma+1)}\longrightarrow(L_{1,\infty})_q^{(0,\gamma+1)}$$

is bounded.

Using the Nikolskiĭ inequality in Lorentz spaces [303, Theorem 3], we obtain that

$$\|g\|_{L_1}\lesssim\log(1+n)\|g\|_{L_{1,\infty}},\qquad g\in O_n.$$

Therefore, Lemma (6.2.1) yields that

$$(L_{1,\infty})_q^{(0,\gamma+1)}\hookrightarrow(L_1)_q^{(0,\gamma)}=B_{1,q}^{0,\gamma}.$$

Consequently,

$$H : B_{1,q}^{0,\gamma+1} \longrightarrow B_{1,q}^{0,\gamma}$$

is bounded.

The result above in the case $1 \le q \le \infty$ can be found in [266, Corollary 6.3 and Remark 8.4]. Other estimates for H can be seen in [256, Section IV.16] and [1, Chapter 3].

Given any integrable function f on $\mathbb{T}$, its Fourier coefficients are defined by

$$\hat{f}(m) = c_m = \frac{1}{2\pi}\int_0^{2\pi} f(e^{ix})e^{-imx}dx\,, \qquad m \in \mathbb{Z}.$$

We write $\mathfrak{F}$ for the operator assigning to any function f the sequence $\mathfrak{F}(f) = (\hat{f}(m))$ of its Fourier coefficients.

We use the reiteration results to study Fourier coefficients of functions in Besov spaces with logarithmic smoothness.

Theorem (6.2.29) [305] Let $1 \le p \le 2, 1/p' = 1 - 1/p\,, 0 < q \le \infty$ and $\gamma > -1/q$. If $f \in B_{p,q}^{0,\gamma}$ then $(\hat{f}(m))$ belongs to $\ell_{p',q}(\log \ell)_{\gamma+\frac{1}{\max\{p',q\}}}$.

Proof. By the Hausdorff–Young inequality, $\mathfrak{F} : L_p \longrightarrow \ell_{p'}$ is bounded. Besides $\mathfrak{F}(O_n) \subseteq F_{2n+1}$, the subset of sequences having at most $2n+1$ coordinates different from 0. Therefore $E_{2(n+1)}(\mathfrak{F}(f);\, \ell_{p'}) \lesssim E_{n+1}(f;\, L_p)$ for $n \in \mathbb{N}$. It follows that

$$\mathfrak{F} : B_{p,q}^{0,\gamma} = (L_p, O_n)_q^{(0,\gamma)} \longrightarrow (\ell_{p'}, F_n)_q^{(0,\gamma)}$$

is bounded. Now, according to Theorem (6.2.19), we have that

$$(\ell_{p'}, F_n)_q^{(0,\gamma)} = ((\ell_\infty)_{p'}^{1/p'})_q^{(0,\gamma)} \hookrightarrow (\ell_\infty)_q^{(1/p',\gamma+\frac{1}{\max\{p',q\}})} = \ell_{p',q}(\log \ell)_{\gamma+\frac{1}{\max\{p',q\}}}.$$

This completes the proof.

The distribution of the Fourier coefficients of functions of $B_{p,q}^{0,\gamma}$ was considered by DeVore, Riemenschneider and Sharpley in [266, Corollary 7.3/(i)]. They proved that if $1 \le q \le \infty$ and $f \in B_{p,q}^{0,\gamma}$ then $(\hat{f}(m)) \in \ell_{p',q}(\log \ell)_\gamma$. Theorem (6.2.29) improves this result in two ways: On the one hand, if $p \ne 1$ and $q \ne \infty$,

$$\ell_{p',q}(\log \ell)_{\gamma+\frac{1}{\max\{p',q\}}} \subsetneq \ell_{p',q}(\log \ell)_\gamma,$$

on the other hand q can now take values less than 1.

We study Fourier coefficients of functions in spaces close to L_1 and to L_2. For this end, we do not need the approximation constructions but limiting interpolation results established.

It was shown by Hardy and Littlewood (case $\gamma = 1$) and Bennett (case $\gamma > 0$) that if $f \in L(\log L)_\gamma$ then $\sum_{n=1}^{\infty}(1 + \log n)^{\gamma-1}c_n^* n^{-1} < \infty$ (see [286, Theorem 1.6/(a)]). Next we extend this result to functions in $L_{1,q}(\log L)_\gamma$.

Theorem (6.2.30) [305] Let $0 < q \le \infty$ and $\gamma > -1/q$. If $f \in L_{1,q}(\log L)_{\gamma+1/\min\{1,q\}}$ then

$$\sum_{n=1}^{\infty}((1 + \log n)^\gamma c_n^*)^q n^{-1} < \infty.$$

Proof. Since $\mathfrak{F} : L_1 \longrightarrow \ell_\infty$ and $\mathfrak{F} : L_2 \longrightarrow \ell_2$ are bounded operators, we obtain that

$$\mathfrak{F} : (L_1, L_2)_{(0,-\gamma),q} \longrightarrow (\ell_1, \ell_2)_{(0,-\gamma),q} \text{ boundedly.}$$

Take $0 < p < 1$. By Lemma (6.2.2) (b) we have

$$(L_1, L_2)_{(0,-\gamma),q} = (L_1, (L_1, L_\infty)_{1/2,2})_{(0,-\gamma),q} = (L_1, L_\infty)_{(0,-\gamma),q} = ((L_p, L_\infty)_{1-p,1}, L_\infty)_{(0,-\gamma),q}.$$

On the other hand, [281, Lemma 6.1] and Lemma (6.2.5) (b) yield that

$$L_{1,q}(\log L)_{\gamma+1/\min\{1,q\}} = (L_p, L_\infty)_{1-p,q,-\gamma-1/\min\{1,q\}} \hookrightarrow ((L_p, L_\infty)_{1-p,1}, L_\infty)_{(0,-\gamma),q}.$$

Hence

$$\mathfrak{F}: L_{1,q}(\log L)_{\gamma+1/\min\{1,q\}} \longrightarrow (\ell_\infty, \ell_2)_{(0,-\gamma),q}$$

is bounded. Now we work with the space at the right side. We have

$$(\ell_\infty, \ell_2)_{(0,-\gamma),q} = (\ell_\infty, (\ell_\infty, \ell_1)_{1/2,2})_{(0,-\gamma),q} = (\ell_\infty, \ell_1)_{(0,-\gamma),q} = \ell_{\infty,q}(\log \ell)_\gamma$$

where the last equality follows by using that $K(n, \xi; \ell_1, \ell_\infty) = \sum_{k=1}^n \xi_k^*$ and Hardy's inequality. Consequently,

$$\mathfrak{F}: L_{1,q}(\log L)_{\gamma+1/\min\{1,q\}} \longrightarrow \ell_{\infty,q}(\log \ell)_\gamma$$

is bounded, which completes the proof.

Note that when $q = 1$ and $\gamma > 0$, Theorem (6.2.30) recovers [286, Theorem 1.6/(a)].

Theorem (6.2.31) [305] Let $0 < q \leq \infty$ and $\gamma < -1/q$. If $f \in L_{2,q}(\log L)_{\gamma+1/\min\{2,q\}}$ then

$$\sum_{n=1}^{\infty} \left(n^{1/2}(1+\log n)^{\gamma+1/\max\{2,q\}} c_n^*\right)^q n^{-1} < \infty.$$

Proof. This time we interpolate by the other limiting method to obtain that

$$\mathfrak{F}: (L_1, L_2)_{(1,-\gamma),q} \longrightarrow (\ell_\infty, \ell_2)_{(1,-\gamma),q}$$

is bounded. By [281, Lemma 6.1] and Lemma (6.2.5) (a), we get that

$$L_{2,q}(\log L)_{\gamma+1/\min\{2,q\}} = (L_1, L_\infty)_{1/2,q,-\gamma-1/\min\{2,q\}} \hookrightarrow (L_1, (L_1, L_\infty)_{1/2,2})_{(1,-\gamma),q} = (L_1, L_2)_{(1,-\gamma),q}.$$

Besides, Lemma (6.2.5) (a) and Proposition (6.2.7) yield that

$$(\ell_\infty, \ell_2)_{(1,-\gamma),q} = (\ell_\infty, (\ell_\infty, \ell_1)_{1/2,2})_{(1,-\gamma),q} \hookrightarrow (\ell_\infty, \ell_1)_{1/2,q,-\gamma-1/\max\{2,q\}} = \ell_{2,q}(\log \ell)_{\gamma+1/\max\{2,q\}}.$$

Therefore

$$\mathfrak{F}: L_{2,q}(\log L)_{\gamma+1/\min\{2,q\}} \longrightarrow \ell_{2,q}(\log \ell)_{\gamma+1/\max\{2,q\}}$$

boundedly.

Writing down Theorem (6.2.31) in the special case $q = 2$ and $\gamma = -1$, we recover a result of Cobos and Segurado [294, Theorem 8.5].

Corollary (6.2.32) [314] Let X be a quasi-Banach space and let $(G_n)_{n\in\mathbb{N}_0}$ be an approximation family. Suppose $\epsilon \geq 0$. Then we have with equivalence of quasi-norms

$$(X, X_{1+\epsilon}^{1/1+\epsilon})_{(0,-\frac{\epsilon^2+\epsilon-1}{1+\epsilon}),1+\epsilon} = X_{1+\epsilon}^{(0,\frac{\epsilon^2+\epsilon-1}{1+\epsilon})}.$$

Proof. Doing a change of variable, using Lemma (6.2.6) and Hardy's inequality (see [293, Theorem 1.2]), we derive

$$\begin{aligned}
\sum \|f_r\|_{(X,X_{1+\epsilon}^{1/1+\epsilon})_{(0,-\frac{\epsilon^2+\epsilon-1}{1+\epsilon}),1+\epsilon}} &= \sum \left(\int_0^1 \left[(1-\log t)^{\frac{\epsilon^2+\epsilon-1}{1+\epsilon}} tK(t^{-1}, f_r;\, X_{1+\epsilon}^{1/1+\epsilon}, X)\right]^{1+\epsilon} \frac{dt}{t}\right)^{1/1+\epsilon} \\
&\sim \sum \left(\int_1^\infty \left[(1+\log s)^{\frac{\epsilon^2+\epsilon-1}{1+\epsilon}} s^{-1/1+\epsilon} K(s^{1/1+\epsilon}, f_r;\, X_{1+\epsilon}^{1/1+\epsilon}, X)\right]^{1+\epsilon} \frac{ds}{s}\right)^{1/1+\epsilon} \\
&\sim \left(\sum_{n=1}^{\infty} \sum \left[(1+\log n)^{\frac{\epsilon^2+\epsilon-1}{1+\epsilon}} n^{-1/1+\epsilon} K(n^{1/1+\epsilon}, f_r;\, X_{1+\epsilon}^{1/1+\epsilon}, X)\right]^{1+\epsilon} n^{-1}\right)^{1/1+\epsilon} \\
&\sim \left(\sum_{n=1}^{\infty} \sum \left[(1+\log n)^{\frac{\epsilon^2+\epsilon-1}{1+\epsilon}} \left(\frac{1}{n}\sum_{k=1}^{n} E_k(f_r)^{1+\epsilon}\right)^{1/1+\epsilon}\right]^{1+\epsilon} n^{-1}\right)^{1/1+\epsilon} \\
&\sim \left(\sum_{n=1}^{\infty} \sum \left[(1+\log n)^{\frac{\epsilon^2+\epsilon-1}{1+\epsilon}} E_n(f_r)\right]^{1+\epsilon} n^{-1}\right)^{1/1+\epsilon} = \sum \|f_r\|_{X_{1+\epsilon}^{(0,\frac{\epsilon^2+\epsilon-1}{1+\epsilon})}}.
\end{aligned}$$

Corollary (6.2.33) [314] Let X be a quasi-Banach space and let $(G_n)_{n\in\mathbb{N}_0}$ be an approximation family. Suppose $\epsilon \geq 0$. Then we have with equivalence of quasi-norms

$$\left(X_p^{1+\epsilon}\right)_{1+\epsilon}^{\left(0,\frac{\epsilon^2+\epsilon-1}{1+\epsilon}\right)} = X_{1+\epsilon}^{(1+\epsilon,\epsilon)}$$

Proof. Using equality of the lower parameters, we obtain

$$\sum_r \|f_r\|_W = \left(\sum_{j=1}^{\infty}\sum_r \left(j^{1+\epsilon}E_j(f_r)\right)^{1+\epsilon} j^{-1}\sum_{n=1}^{j}(1+\log n)^{\frac{\epsilon^2-1}{1+\epsilon}}n^{-1}\right)^{1/1+\epsilon}$$

$$\sim \left(\sum_{j=1}^{\infty}\sum_r \left(j^{1+\epsilon}(1+\log n)^{\epsilon}E_j(f_r)\right)^{1+\epsilon} j^{-1}\right)^{1/1+\epsilon} = \sum_r \|f_r\|_{X_{1+\epsilon}^{(1+\epsilon,\epsilon)}}.$$

Therefore, applying Theorem (6.2.17), we derive

$$(X_{1+\epsilon}^{1+\epsilon})_{1+\epsilon}^{\left(0,\frac{\epsilon^2+\epsilon-1}{1+\epsilon}\right)} = X_{1+\epsilon}^{\left(0,\frac{\epsilon^2+\epsilon-1}{1+\epsilon}\right)} \cap X_{1+\epsilon}^{(1+\epsilon,\epsilon)} = X_{1+\epsilon}^{(1+\epsilon,\epsilon)}.$$

Corollary (6.2.34) [314] Let $0 \leq \epsilon < \infty$. Then

$$B_{1+\epsilon,1}^{1/1+\epsilon} \hookrightarrow L_\infty.$$

Proof. According to [300, Theorem 3.1], there is a constant $c > 0$ such that for any $f_j \in B_{1+\epsilon,1}^{1/1+\epsilon}$ there is a representation $\sum_j f_j = \sum_{n=0}^{\infty}\sum_j g_n^j$ with $g_n^j \in O_{2^n}$ and

$$\sum_{n=0}^{\infty}\sum_j 2^{n/1+\epsilon}\left\|g_n^j\right\|_{L_{1+\epsilon}} \leq c\sum_j \left\|f_j\right\|_{B_{1+\epsilon,1}^{1/1+\epsilon}}.$$

Hence, using the Nikolskiĭ inequality for trigonometric polynomials (see [297, 3.4.3] and [284]), we derive that

$$\sum_j \left\|f_j\right\|_{L_\infty} \leq \sum_{n=0}^{\infty}\sum_j \left\|g_n^j\right\|_{L_\infty} \lesssim \sum_{n=0}^{\infty}\sum_j 2^{n/1+\epsilon}\left\|g_n^j\right\|_{L_{1+\epsilon}} \leq c\sum_j \left\|f_j\right\|_{B_{1+\epsilon,1}^{1/1+\epsilon}}.$$

Corollary (6.2.35) [314] Let $0 \leq \epsilon \leq \infty$ If $f \in B_{1+\epsilon,1+\epsilon}^{0,\frac{\epsilon^2+\epsilon-1}{1+\epsilon}}$ then $(\hat{f}(m))$ belongs to $\ell_{\frac{1+\epsilon}{\epsilon},1+\epsilon}(\log \ell)_{\frac{\epsilon^2+\epsilon-1}{1+\epsilon}+\frac{1}{\max\{\frac{1+\epsilon}{\epsilon},1+\epsilon\}}}$.

Proof. By the Hausdorff–Young inequality, $\mathfrak{F} : L_{1+\epsilon} \longrightarrow \ell_{\frac{1+\epsilon}{\epsilon}}$ is bounded. Besides $\mathfrak{F}(O_n) \subseteq F_{2n+1}$, the subset of sequences having at most $2n+1$ coordinates different from 0. Therefore $E_{2(n+1)}(\mathfrak{F}(f);\ \ell_{\frac{1+\epsilon}{\epsilon}}) \lesssim E_{n+1}(f;\ L_{1+\epsilon})$ for $n \in \mathbb{N}$. It follows that

$$\mathfrak{F} : B_{1+\epsilon,1+\epsilon}^{0,\frac{\epsilon^2+\epsilon-1}{1+\epsilon}} = (L_{1+\epsilon}, O_n)_{1+\epsilon}^{(0,\frac{\epsilon^2+\epsilon-1}{1+\epsilon})} \longrightarrow (\ell_{\frac{1+\epsilon}{\epsilon}}, F_n)_{1+\epsilon}^{(0,\frac{\epsilon^2+\epsilon-1}{1+\epsilon})}$$

is bounded. Now, according to Theorem (6.2.19), we have that

$$(\ell_{\frac{1+\epsilon}{\epsilon}}, F_n)_{1+\epsilon}^{(0,\frac{\epsilon^2+\epsilon-1}{1+\epsilon})} = ((\ell_\infty)_{\frac{1+\epsilon}{\epsilon}}^{\frac{\epsilon}{1+\epsilon}})_{1+\epsilon}^{(0,\frac{\epsilon^2+\epsilon-1}{1+\epsilon})} \hookrightarrow (\ell_\infty)_{1+\epsilon}^{(\frac{\epsilon}{1+\epsilon},\frac{\epsilon^2+\epsilon-1}{1+\epsilon}+\frac{1}{\max\{\frac{1+\epsilon}{\epsilon},1+\epsilon\}})}$$

$$= \ell_{\frac{1+\epsilon}{\epsilon},1+\epsilon}(\log \ell)_{\frac{\epsilon^2+\epsilon-1}{1+\epsilon}+\frac{1}{\max\{\frac{1+\epsilon}{\epsilon},1+\epsilon\}}}.$$

This completes the proof.

Section (6.3) Fourier-Analytical Decompositions with Wavelets and Semi-Groups

For f be a (complex-valued) Lebesgue measurable function in $\mathbb{R}^n, n \in \mathbb{N}, h \in \mathbb{R}^n$ and $k \in \mathbb{N}$. We put

$$(\Delta_h^1 f)(x) = f(x+h) - f(x) \quad \text{and} \quad (\Delta_h^{k+1} f)(x) = \Delta_h^1(\Delta_h^k f)(x),$$

$x \in \mathbb{R}^n$. The k-th order modulus of smoothness of a function $f \in L_p, 1 \leq p \leq \infty$, is defined by

$$\omega_k(f,t)_p = \sup_{|h| \leq t} \left\| \Delta_h^k f \middle| L_p \right\|, \qquad t > 0. \tag{20}$$

Let

$$1 < p < \infty, 0 < q \leq \infty \quad \text{and} \quad s > 0. \tag{21}$$

Let $s < k \in \mathbb{N}$. The classical Besov space $B_{p,q}^s$ on $\mathbb{R}^n$ consists of all $f \in L_p$ such that

$$\left\| f \middle| B_{p,q}^s \right\|_{k_+} = \left\| f \middle| L_p \right\| + \left(\int_0^1 \left[t^{-s} \omega_k(f,t)_p \right]^q \frac{dt}{t} \right)^{1/q} \tag{22}$$

is finite (as usual, if $q = \infty$, the integral should be replaced by the supremum). See [65, p. 110] or [1, p. 332]. Besov spaces $B_{p,q}^s$ have a central role in many aspects of the theory of function spaces as can be seen, for example, by Triebel [62, 65, 66, 67, 311]. There are numerous characterizations of these spaces in terms of other means. Let

$$\varphi = \{\varphi_j\}_{j=0}^\infty, \qquad \sum_{j=0}^\infty \varphi_j(\xi) = 1, \qquad \xi \in \mathbb{R}^n, \tag{23}$$

be the usual dyadic resolution of unity in $\mathbb{R}^n$. Then $B_{p,q}^s$ consists of all $f \in L_p$ such that

$$\left\| f \middle| B_{p,q}^s \right\|_\varphi = \left(\sum_{j=0}^\infty \left\| \left(\varphi_j \hat{f} \right)^\vee \middle| L_p \right\|^q \right)^{1/q} \tag{24}$$

is finite (the sum should be replaced by the supremum if $q = \infty$). Furthermore $\left\| \cdot \middle| B_{p,q}^s \right\|_{k_+}$ and $\left\| \cdot \middle| B_{p,q}^s \right\|_\varphi$ are equivalent quasi-norms (see [62, 66]). Let

$$W(t)f(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} f(y)dy = \left(e^{-t|\xi|^2} \hat{f}(\xi) \right)^\vee (x) \tag{25}$$

$t > 0, x \in \mathbb{R}^n$, be the Gauss–Weierstrass semi-group, $W(0) = \text{id}$ (identity). Let p, q, s be as in (21) and $s/2 < m \in \mathbb{N}$. Then $B_{p,q}^s$ consists of all $f \in L_p$ such that

$$\left\| f \middle| B_{p,q}^s \right\|_{(m)} = \left\| f \middle| L_p \right\| + \left(\int_0^1 t^{-\frac{s}{2}q} \left\| t^m \frac{\partial^m W(t) f}{\partial t^m} \middle| L_p \right\|^q \frac{dt}{t} \right)^{1/q} \tag{26}$$

is finite. Furthermore, $\left\| \cdot \middle| B_{p,q}^s \right\|_{(m)}$ is an equivalent quasi-norm in the space $B_{p,q}^s$. See [62, Theorem 2.5.2, p. 191], [66, Theorem 2.6.4, p. 152]. Another equivalent quasi-norm is

$$\left\| f \middle| B_{p,q}^s \right\|_{(m)}^* = \left\| f \middle| L_p \right\| + \left(\int_0^1 t^{-\frac{s}{2}q} \left\| [W(t) - \text{id}]^m \middle| L_p \right\|^q \frac{dt}{t} \right)^{1/q} \tag{27}$$

(see [306, Theorem 3.4.6 and Section 4.3.2] and [62, Section 1.13.2]). Finally we recall the characterizations of the spaces $B_{p,q}^s$ in $\mathbb{R}^n$ in terms of wavelets. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and

$$\Psi = \left\{ \Psi_{G,m}^j : j \in \mathbb{N}_0, G \in G^j, m \in \mathbb{Z}^n \right\} \tag{28}$$

be the same real orthonormal wavelet basis in L_2 as [311, Section 1.2.1, pp. 13–14]. Let $b_{p,q}^s$ be the collection of all sequences

$$\lambda = \{\lambda_m^{j,G} \in \mathbb{C} : j \in \mathbb{N}_0, G \in G^j, m \in \mathbb{Z}^n\} \tag{29}$$

such that

$$\|\lambda | b_{p,q}^s\| = \left(\sum_{j=0}^{\infty} 2^{j\left(s-\frac{n}{p}\right)q} \sum_{G \in G^j} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_m^{j,G}|^p \right)^{q/p} \right)^{1/q} \tag{30}$$

is finite. Let again p, q, s be as in (21). Then $B_{p,q}^s$ consists of all $f \in L_p$ which can be represented as

$$f = \sum_{j,G,m} \lambda_m^{j,G} 2^{-jn/2} \Psi_{G,m}^j, \qquad \lambda \in b_{p,q}^s, \tag{31}$$

unconditional convergence being in L_p. This representation is unique,

$$\lambda_m^{j,G} = \lambda_m^{j,G}(f) = 2^{jn/2}\left(f, \Psi_{G,m}^j\right) = 2^{jn/2} \int_{\mathbb{R}^n} f(x) \Psi_{G,m}^j(x) dx, \tag{32}$$

and

$$I : f \mapsto \{2^{jn/2}\left(f, \Psi_{G,m}^j\right)\} \tag{33}$$

is an isomorphic map of $B_{p,q}^s$ onto $b_{p,q}^s$. See [311, Theorem 1.20, pp. 15–16].

The complete solution of some natural questions has motivated the introduction of Besov spaces where smoothness of functions is considered in a more delicate manner than in $B_{p,q}^s$ (see, DeVore, Riemenschneider and Sharpley [266], Kalyabin and Lizorkin [32], Farkas and Leopold [144]). Among them, logarithmically perturbed Besov spaces are receiving a growing interest in recent times as can be seen Caetano, Gogatishvili and Opic [288, 307], Besov [257] or Cobos and Domínguez [283, 289, 305]. These spaces have classical smoothness zero and logarithmic smoothness with exponent b. They are near L_p but they have additional properties than L_p due to their structure of Besov spaces and their logarithmic smoothness (see, [305, Theorem 5.1]). The most popular version imitates (22). Let

$$1 < p < \infty, 0 < q \le \infty \quad \text{and} \quad b > -1/q. \tag{34}$$

Let $\omega(f,t)_p = \omega_1(f,t)_p$ according to (20) (first differences). Then $\mathbf{B}_{p,q}^{0,b}$ consists of all $f \in L_p$ such that

$$\|f|\mathbf{B}_{p,q}^{0,b}\| = \|f|L_p\| + \left(\int_0^1 \left[(1-\log t)^b \omega(f,t)_p\right]^q \frac{dt}{t} \right)^{1/q} \tag{35}$$

is finite. According to [283, Theorem 3.1] one can replace $\omega(f,t)_p$ in (35) by $\omega_k(f,t)_p, k \in \mathbb{N}$, which means that $\mathbf{B}_{p,q}^{0,b}$ consists of all $f \in L_p$ such that

$$\|f|\mathbf{B}_{p,q}^{0,b}\|_{k_+} = \|f|L_p\| + \left(\int_0^1 \left[(1-\log t)^b \omega_k(f,t)_p\right]^q \frac{dt}{t} \right)^{1/q}$$

is finite. This property justifies to deal with this type of logarithmic Besov spaces of perturbed main smoothness.

It is natural to investigate if $\mathbf{B}_{p,q}^{0,b}$ admits characterizations in terms of Fourier-analytical decompositions, wavelets and heat kernels. Accordingly, we establish those characterizations. They are new even for spaces $\mathbf{B}_{p,q}^{0,b}$ where $b = 0$.

Let φ be as in (23) and p, q, b be as in (34). The natural candidate of quasi-norm

$$\left(\sum_{j=0}^{\infty}(1+j)^{bq}\left\|\left(\varphi_j\hat{f}\right)^{\vee}\middle|L_p\right\|^q\right)^{1/q}$$

does not characterize $\mathbf{B}^{0,b}_{p,q}$ but the Fourier-analytically defined space $\mathbf{B}^{0,b}_{p,q}$ also with logarithmic smoothness of exponent b but which is different from $\mathbf{B}^{0,b}_{p,q}$ (see [283, Theorem 3.3]). We show that $\mathbf{B}^{0,b}_{p,q}$ consists of all $f \in L_p$ such that

$$\left\|f\middle|\mathbf{B}^{0,b}_{p,q}\right\|_{\varphi_+} = \left(\sum_{j=0}^{\infty}(1+j)^{bq}\left\|\left(\sum_{\nu=j}^{\infty}\left|\left(\varphi_\nu\hat{f}\right)^{\vee}(\cdot)\right|^2\right)^{1/2}\middle|L_p\right\|^q\right)^{1/q}$$

is finite and that $\left\|\cdot\middle|\mathbf{B}^{0,b}_{p,q}\right\|_{k_+}$ is an equivalent quasi-norm to $\left\|\cdot\middle|\mathbf{B}^{0,b}_{p,q}\right\|_{\varphi_+}$. Therefore, in contrast to (24), the Fourier-analytical characterization of $\mathbf{B}^{0,b}_{p,q}$ requires now an additional truncated Littlewood–Paley construction.

Such new ingredient also appears in the characterization of $\mathbf{B}^{0,b}_{p,q}$ by means of wavelets. Let $\chi_{j,m}$ be the characteristic function of the cube $Q_{j,m} = 2^{-j}m + 2^{-j}(0,1)^n$ in $\mathbb{R}^n$ where $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$. Let $\mathbf{b}^{0,b}_{p,q}$ be the collection of all sequences λ according to (29) such that

$$\left\|\lambda\middle|\mathbf{b}^{0,b}_{p,q}\right\| = \left(\sum_{j=0}^{\infty}(1+j)^{bq}\left\|\left(\sum_{\nu=j}^{\infty}\sum_{\substack{G\in G^\nu \\ m\in\mathbb{Z}^{n}}}\left|\lambda_m^{\nu,G}\right|^2\chi_{\nu,m}(\cdot)\right)^{1/2}\middle|L_p\right\|^q\right)^{1/q} \tag{36}$$

is finite. Let p,q,b be as in (34). Then $\mathbf{B}^{0,b}_{p,q}$ consists of all $f \in L_p$ which can be represented as

$$f = \sum_{j,G,m}\lambda_m^{j,G}2^{-jn/2}\Psi^j_{G,m}, \qquad \lambda \in \mathbf{b}^{0,b}_{p,q},$$

unconditional convergence being in L_p. This representation is unique with $\lambda_m^{j,G}$ as in (32). Furthermore, I in (33) is an isomorphic map of $\mathbf{B}^{0,b}_{p,q}$ onto $\mathbf{b}^{0,b}_{p,q}$.

However the truncated Littlewood–Paley construction does not appear in the following characterization of $\mathbf{B}^{0,b}_{p,q}$ by means of heat kernels. Let $W(t)$ be as in (25) and let p,q,b be as in (34) and $m \in \mathbb{N}$. Then $\mathbf{B}^{0,b}_{p,q}$ consists of all $f \in L_p$ such that

$$\left\|f\middle|\mathbf{B}^{0,b}_{p,q}\right\|^*_{(m)} = \left\|f\middle|L_p\right\| + \left(\int_0^1(1-\log t)^b\left\|[W(t)-\mathrm{id}]^m\middle|L_p\right\|^q\frac{dt}{t}\right)^{1/q}$$

is finite (equivalent quasi-norms).

The arguments rely decisively on real interpolation in limiting situations. We introduce the spaces $\mathbf{B}^{0,b}_{p,q}$ and show another characterization by differences, as well as some results on their structure. We deal with the indicated characterizations of $\mathbf{B}^{0,b}_{p,q}$ in terms of Fourier-analytical decompositions, wavelets and heat kernels. Results on heat kernels are derived from abstract results on semi-groups of operators which are of independent interest and apply to the Cauchy–Poisson semi-group as well. We also ask for embeddings between

$\mathbf{B}^{0,b}_{p,q}$ and their Fourier-analytically defined counterparts $B^{0,b}_{p,q}$. In addition we discuss the structural differences of diverse quasi-norms of $\mathbf{B}^{0,b}_{p,q}$ and their counterparts in $B^{s}_{p,q}$ and $B^{0,b}_{p,q}$.

Given two non-negative quantities U, V depending on certain parameters, we put $U \lesssim V$ if there is a constant $c > 0$ independent of all parameters such that $U \le cV$. If $U \lesssim V$ and $V \lesssim U$, we write $U \sim V$.

Let A_0, A_1 be Banach spaces with $A_1 \hookrightarrow A_0$, where $\hookrightarrow$ means continuous embedding. The Peetre K-functional is defined by

$$K(t,a) = K(t,a;A_0,A_1) = \inf\{\|a - a_1|A_0\| + t\|a_1|A_1\| : a_1 \in A_1\}, t > 0, a \in A_0.$$

As in [283], given $0 < q \le \infty$ and $\eta \in \mathbb{R}$, we define the limiting real method with $\theta = 0$ by

$$(A_0,A_1)_{(0,\eta),q} = \left\{ a \in A_0 : \left\|a\middle|(A_0,A_1)_{(0,\eta),q}\right\| = \left(\int_0^1 \left(\frac{K(t,a)}{(1-\log t)^{\eta}}\right)^q \frac{dt}{t}\right)^{1/q} < \infty \right\}.$$

It is easy to check that the interpolation theorem for bounded linear operators holds for the construction $(\cdot,\cdot)_{(0,\eta),q}$.

The space $(A_0,A_1)_{(0,\eta),q}$ is quasi-normed and complete. It is a Banach space if $1 \le q \le \infty$.

Proceeding as in [283, p. 79], if $1 \le q < \infty$ and $\eta \le 1/q$, we can compare $(A_0,A_1)_{(0,\eta),q}$ with logarithmic interpolation spaces $(A_0,A_1)_{\theta,q,(\eta,\tau)}$ studied in [272, 264]. Indeed, take $\tau < -1/q$. It turns out that

$$(A_0,A_1)_{(0,\eta),q} = (A_0,A_1)_{0,q,(-\eta,\tau)} = (A_1,A_0)_{1,q,(\tau,-\eta)}. \tag{37}$$

Consequently, since $\tau + 1/q < 0 \le -\eta + 1/q$, it follows from [264, Corollary 3.7] and (37) that

$$A_1 \text{ is dense in } (A_0,A_1)_{(0,\eta),q} \text{ if } 1 \le q < \infty \text{ and } \eta \le 1/q. \tag{38}$$

We establish now some auxiliary results. The first one refers to interpolation of couples of vector-valued L_p-spaces. Our measure space is $\mathbb{R}^n$ with the Lebesgue measure. So, $L_p(A) = L_p(\mathbb{R}^n, A)$. The proof is similar to [62, Theorem 1.18.4]. Recall that

$$K(t,a;A_0,A_1) \sim K_p(t,a;A_0,A_1) = \inf_{\substack{a = a_0 + a_1 \\ a_j \in A_j}} \{\|a_0|A_0\|^p + t^p\|a_1|A_1\|^p\}^{1/p}.$$

Lemma (6.3.1) [313] Let A_0, A_1 be Banach spaces with $A_1 \hookrightarrow A_0$. Let $1 \le p < \infty$ and $b \in \mathbb{R}$ with $b \ge -1/p$. Then we have with equivalence of norms

$$\big(L_p(A_0), L_p(A_1)\big)_{(0,-b),p} = L_p\big((A_0,A_1)_{(0,-b),p}\big).$$

Proof. Consider the collection S of all functions $v(x) = \sum_{j=1}^{N} a_j \chi_{\Omega_j}(x)$, where $N \in \mathbb{N}$, $a_j \in A_1$, the measure of $\Omega_j \subset \mathbb{R}^n$ is finite and $\Omega_j \cap \Omega_k = \emptyset$ if $j \ne k$. By (38), the set S is dense in $\big(L_p(A_0), L_p(A_1)\big)_{(0,-b),p}$ and in $L_p\big((A_0,A_1)_{(0,-b),p}\big)$. For $v \in S$ we have

$$\left\|v\middle|\big(L_p(A_0),L_p(A_1)\big)_{(0,-b),p}\right\|^p$$

$$\sim \int_0^1 \left((1-\log t)^b \inf_{\substack{v = v_0 + v_1 \\ v_j \in L_p(A_j)}} \left\{\left\|v_0\middle|L_p(A_0)\right\|^p + t^p\left\|v_1\middle|L_p(A_1)\right\|^p\right\}^{1/p}\right)^p \frac{dt}{t}$$

$$= \int_0^1 (1-\log t)^{bp} \int_{\mathbb{R}^n} \inf_{\substack{v(x) = v_0(x) + v_1(x) \\ v_j(x) \in (A_j)}} (\|v_0(x)|A_0\|^p + t^p\|v_1(x)|A_1\|^p)\, dx \frac{dt}{t}$$

$$= \int_{\mathbb{R}^n} \int_0^1 (1-\log t)^{bp} K_p(t, v(x); A_0, A_1)^p \frac{dt}{t} dx \sim \int_{\mathbb{R}^n} \left\| v(x) \middle| (A_0, A_1)_{(0,-b),p} \right\|^p dx$$

$$= \left\| v \middle| L_p\big((A_0, A_1)_{(0,-b),p}\big) \right\|^p.$$

This completes the proof.

Next consider the sequence space ℓ_2 on $\mathbb{N}_0$ and for $\lambda > 0$ write

$$\ell_2^\lambda = \left\{ \xi = (\xi_j)_{j\in\mathbb{N}_0} : \left\|\xi\middle|\ell_2^\lambda\right\| = \left(\sum_{j=0}^{\infty} (2^{\lambda j} |\xi_j|)^2 \right)^{1/2} < \infty \right\}.$$

If A is a Banach space, the vector-valued sequence space $\ell_2^\lambda(A)$ is defined by

$$\ell_2^\lambda(A) = \left\{ x = (x_j)_{j\in\mathbb{N}_0} : \left\|x\middle|\ell_2^\lambda(A)\right\| = \left(\sum_{j=0}^{\infty} (2^{\lambda j} \|x_j|A\|)^2 \right)^{1/2} < \infty \right\}$$

Lemma (6.3.2) [313] Let $0 < q < \infty$ and $b \in \mathbb{R}$. Then we have with equivalence of quasi-norms

$$(\ell_2, \ell_2^\lambda)_{(0,-b),q} = \left\{ \xi = (\xi_j) : \|\xi\| = \left(\sum_{j=0}^{\infty} \left[(1+j)^b \Big(\sum_{\nu=j}^{\infty} |\xi_\nu|^2\Big)^{1/2} \right]^q \right)^{1/q} < \infty \right\}$$

and $\|\cdot\|$ is an equivalent quasi-norm on $(\ell_2, \ell_2^\lambda)_{(0,-b),q}$.

Proof. Consider first the case $\lambda = 1$. Since

$$K_2(t, \xi; \ell_2, \ell_2^1) \sim \left(\sum_{\nu=0}^{\infty} (\min(1, t2^\nu) |\xi_\nu|)^2 \right)^{1/2},$$

we have for $j \geq 0$ that

$$K_2(2^{-j}, \xi; \ell_2, \ell_2^1) \sim \left(\sum_{\nu=0}^{\infty} (2^{\nu-j} |\xi_\nu|)^2 \right)^{1/2} + \left(\sum_{\nu=j+1}^{\infty} |\xi_\nu|^2 \right)^{1/2}.$$

Hence

$$\left\|\xi \left| (\ell_2, \ell_2^\lambda)_{(0,-b),q} \right.\right\| \sim \left(\sum_{j=0}^{\infty} \left[(1+j)^b K_2(2^{-j}, \xi) \right]^q \right)$$

$$\sim \left(\sum_{j=0}^{\infty} \left[(1+j)^b \Big(\sum_{\nu=0}^{j} (2^{\nu-j}|\xi_\nu|)^2\Big)^{1/2} \right]^q \right)^{1/q} + \left(\sum_{j=0}^{\infty} \left[(1+j)^b \Big(\sum_{\nu=j}^{\infty} |\xi_\nu|^2\Big)^{1/2} \right]^q \right)^{1/q}.$$

In the last expression, the first term is dominated by the second term. Indeed, if $q/2 \leq 1$, we obtain

$$\left(\sum_{j=0}^{\infty} \left[(1+j)^{2b} \sum_{\nu=0}^{j} \big(2^{\nu-j}|\xi_\nu|\big)^2 \right]^{q/2} \right)^{1/q} \leq \left(\sum_{j=0}^{\infty} (1+j)^{qb} 2^{-qj} \sum_{\nu=0}^{j} 2^{q\nu} |\xi_\nu|^q \right)^{1/q}$$

$$= \left(\sum_{\nu=0}^{\infty} 2^{q\nu} |\xi_\nu|^q \sum_{j=\nu}^{\infty} (1+j)^{qb} 2^{-qj} \right)^{1/q} \lesssim \left(\sum_{\nu=0}^{\infty} (1+\nu)^{qb} |\xi_\nu|^q \right)^{1/q}$$

$$\leq \left(\sum_{j=0}^{\infty} \left[(1+j)^b (\sum_{\nu=j}^{\infty} |\xi_\nu|^2)^{1/2} \right]^q \right)^{1/q}.$$

If $1 < q/2$ we use the variant of Hardy's inequality given in [309, Lemma 3.1] (see also [299, Lemma 3.10]). We get

$$\left(\sum_{j=0}^{\infty} \left[(1+j)^b 2^{-j} \left(\sum_{\nu=0}^{j} (2^\nu |\xi_\nu|)^2 \right)^{1/2} \right]^q \right)^{1/q} = \left(\sum_{j=0}^{\infty} \left[(1+j)^{2b} 2^{-2j} \sum_{\nu=0}^{j} (2^\nu |\xi_\nu|)^2 \right]^{q/2} \right)^{1/q}$$

$$\lesssim \left(\sum_{j=0}^{\infty} \left[(1+j)^b 2^{-j} 2^j |\xi_j| \right]^q \right)^{1/q} \leq \left(\sum_{j=0}^{\infty} \left[(1+j)^b \left(\sum_{\nu=j}^{\infty} |\xi_\nu|^2 \right)^{1/2} \right]^q \right)^{1/q}.$$

Consequently,

$$\| \xi | (\ell_2, \ell_2^1)_{(0,-b),q} \| \sim \left(\sum_{j=0}^{\infty} \left[(1+j)^b \left(\sum_{\nu=j}^{\infty} |\xi_\nu|^2 \right)^{1/2} \right]^q \right)^{1/q}.$$

The general case $0 < \lambda$ follows by appropriate modifications.

Let $\mathbf{B}_{p,q}^{0,b}$ be the Besov spaces on $\mathbb{R}^n$ with logarithmic smoothness defined in (35) by means of differences. Note that definition makes sense in a wider range of parameters than (34). Namely

$$1 \leq p \leq \infty, \qquad 0 < q \leq \infty, \qquad \text{and} \quad b \geq -1/q$$

but the extreme value $b = -1/q$ may give rise to jumps in the scale (see [308, Theorem 3.2 and Corollary 3.3] and Remark (6.3.8) below), and the cases $p = 1, \infty$ sometimes require different type of arguments (see [308, Theorem 3.8]).

As we have pointed out, the quasi-norms

$$\| f | \mathbf{B}_{p,q}^{0,b} \|_{k_+} = \| f | L_p \| + \left(\int_0^1 \left[(1 - \log t)^b \omega_k(f,t)_p \right]^q \frac{dt}{t} \right)^{1/q}, \qquad k \in \mathbb{N},$$

are equivalent on $\mathbf{B}_{p,q}^{0,b}$. This is the log-version of the quasi-norms (22) on the classical Besov spaces $B_{p,q}^s$ with $s < k$. According to [65, Theorem 2.5.12, p. 110], the quasi-norm (22) is equivalent to

$$\| f | B_{p,q}^s \|_k = \| f | L_p \| + \left(\int_{|h| \leq 1} |h|^{-sq} \| \Delta_h^k f | L_p \|^q \frac{dh}{|h|^n} \right)^{1/q}. \tag{39}$$

The next result shows the characterization by differences in $\mathbf{B}_{p,q}^{0,b}$ corresponding to (39).

Theorem (6.3.3) [313] Let $1 \leq p \leq \infty, 0 < q \leq \infty, b \geq -1/q$ and $k \in \mathbb{N}$. Then

$$\| f | \mathbf{B}_{p,q}^{0,b} \|_k = \| f | L_p \| + \left(\int_{|h| \leq 1} (1 - \log|h|)^{bq} \| \Delta_h^k f | L_p \|^q \frac{dh}{|h|^n} \right)^{1/q}$$

is an equivalent quasi-norm on $\mathbf{B}_{p,q}^{0,b}$.

Proof. We shall use that

$$\omega_k(f,t)_p \sim \left(t^{-n} \int_{|h| \leq t} \| \Delta_h^k f | L_p \|^q dh \right)^{1/q}$$

(see [33, (2.4) and Appendix A]). We have

$$\left\|f\middle|\mathbf{B}^{0,b}_{p,q}\right\|_{k_+} = \left\|f\middle|L_p\right\| + \left(\int_0^1 \left[(1-\log t)^b \omega_k(f,t)_p\right]^q \frac{dt}{t}\right)^{1/q}$$

$$\sim \left\|f\middle|L_p\right\| + \left(\int_0^1 (1-\log t)^{bq} t^{-n} \int_{|h|\le t} \left\|\Delta_h^k f\middle|L_p\right\|^q dh \frac{dt}{t}\right)^{1/q}$$

$$= \left\|f\middle|L_p\right\| + \left(\int_{|h|\le 1} \left\|\Delta_h^k f\middle|L_p\right\|^q \int_{|h|}^1 t^{-n}(1-\log t)^{bq} \frac{dt}{t} dh\right)^{1/q}$$

$$\lesssim \left\|f\middle|L_p\right\| + \left(\int_{|h|\le 1} \left\|\Delta_h^k f\middle|L_p\right\|^q |h|^{-n}(1-\log|h|)^{bq} dh\right)^{1/q} = \left\|f\middle|\mathbf{B}^{0,b}_{p,q}\right\|_k.$$

Conversely,

$$\left\|f\middle|\mathbf{B}^{0,b}_{p,q}\right\|_k \le \left\|f\middle|L_p\right\| + \left(\int_{|h|\le 1} (1-\log|h|)^{bq} \omega_k(f,|h|)_p^q \frac{dh}{|h|^n}\right)^{1/q}$$

$$= \left\|f\middle|L_p\right\| + \left(\sum_{j=0}^{\infty} \int_{2^{-j-1}<|h|\le 2^{-j}} (1-\log|h|)^{bq} \omega_k(f,|h|)_p^q \frac{dh}{|h|^n}\right)^{1/q}$$

$$\lesssim \left\|f\middle|L_p\right\| + \left(\sum_{j=0}^{\infty} (1+j)^{bq} \omega_k(f,2^{-j})_p^q\right)^{1/q}$$

$$\lesssim \left\|f\middle|L_p\right\| + \left(\int_0^1 (1-\log t)^{bq} \omega_k(f,t)_p^q \frac{dt}{t}\right)^{1/q} = \left\|f\middle|\mathbf{B}^{0,b}_{p,q}\right\|_{k_+}.$$

Remark (6.3.4) [313] The special case of the semi-quasi-norm

$$\left(\int_{|h|\le 1} (1-\log|h|)^{bq} \left\|\Delta_h^k f\middle|L_p\right\|^q \frac{dh}{|h|^n}\right)^{1/q}$$

when $b = 0$ and $k = 1$ has been used by Besov [257].

In our later characterizations the structure of $\mathbf{B}^{0,b}_{p,q}$ as approximation space will be useful. We describe it now.

Let $G_0 = \{0\}$ and for $j \in \mathbb{N}$ put

$$G_j = \left\{g \in L_p : \operatorname{supp} \hat{g} \subseteq \{x : |x| \le j\}\right\}. \tag{40}$$

So if $g \in G_j$ then g is an entire analytic function of exponential type j. Given $f \in L_p$ and $j \in \mathbb{N}$, let

$$E_j(f)_p = \inf\left\{\left\|f-g\middle|L_p\right\| : g \in G_{j-1}\right\}.$$

For $1 \le p \le \infty, \alpha > 0, 0 < q \le \infty$ and $-\infty < b < \infty$ put

$$(L_p)_q^\alpha = (L_p,\{G_j\})_q^\alpha = \left\{f \in L_p : \left\|f\middle|(L_p)_q^\alpha\right\| = \left(\sum_{j=1}^{\infty} (j^\alpha E_j(f)_p)^q j^{-1}\right)^{1/q} < \infty\right\}$$

and

$$\left(L_p\right)_q^{(0,b)} = \left(L_p, \{G_j\}\right)_q^{(0,b)}$$
$$= \left\{ f \in L_p : \left\| f \left| (L_p)_q^{(0,b)} \right\| = \left(\sum_{j=1}^{\infty} ((1+\log j)^b E_j(f)_p)^q j^{-1} \right)^{1/q} < \infty \right\} \quad (41)$$
(see [62, 300, 295, 305]).

The following result can be proved using ideas of [297, Section 5.6] and [62, Section 2.5.4] but also by means of interpolation as we do.

Lemma (6.3.5) [313] Let $1 \le p \le \infty, 0 < q \le \infty$ and $b \ge -1/q$. Then we have
$$\left(L_p\right)_q^{(0,b)} = \mathbf{B}_{p,q}^{0,b}.$$
Proof. Take any $0 < \alpha < 1$. By [65, Theorem 2.5.3] (see also [62, Theorem 2.5.4]), we know that $(L_p)_q^{\alpha} = B_{p,p}^{\alpha}$. Whence, using [305, Corollary 3.5 and Lemma 2.2/(b)], we derive
$$\left(L_p\right)_q^{(0,b)} = \left(L_p, \left(L_p\right)_q^{\alpha}\right)_{(0,-b),q} = \left(L_p, B_{p,p}^{\alpha}\right)_{(0,-b),q} = \left(L_p, \left(L_p, W_p^1\right)_{\alpha,p}\right)_{(0,-b),q}$$
$$= \left(L_p, W_p^1\right)_{(0,-b),q} = \mathbf{B}_{p,q}^{0,b}$$
where the last equality follows from [283, Theorem 3.1].

In the proof of the previous lemma we have used that $\mathbf{B}_{p,q}^{0,b} = \left(L_p, W_p^k\right)_{(0,-b),q}$ [283, Theorem 3.1]. We can show now that in this interpolation formula the Sobolev space W_p^k can be replaced by any fractional Sobolev space $H_p^s = F_{p,2}^s$, any Triebel–Lizorkin space $F_{p,u}^s$, or any Besov space $B_{p,u}^s$.

Theorem (6.3.6) [313] Let $s > 0, 1 \le p < \infty, 0 < u \le \infty, 0 < q \le \infty$ and $b \ge -1/q$. For $A_{p,u}^s = F_{p,u}^s$ or $B_{p,u}^s$ we have with equivalence of quasi-norms $\mathbf{B}_{p,q}^{0,b} = \left(L_p, A_{p,u}^s\right)_{(0,-b),q}$.

Proof. By [65, Proposition 2, p. 47], we have
$$B_{p,\min(p,u)}^s \hookrightarrow F_{p,u}^s \hookrightarrow B_{p,\max(p,u)}^s.$$
Hence, using [65, Theorem 2.5.3], [305, Corollary 3.5] and Lemma (6.3.5), we derive
$$\left(L_p, A_{p,u}^s\right)_{(0,-b),q} \hookrightarrow \left(L_p, B_{p,\max(p,u)}^s\right)_{(0,-b),q} = \left(L_p, (L_p)_{\max(p,u)}^s\right)_{(0,-b),q} = \left(L_p\right)_q^{(0,b)} = \mathbf{B}_{p,q}^{0,b}.$$
For the converse embedding, we obtain
$$\mathbf{B}_{p,q}^{0,b} = \left(L_p\right)_q^{(0,b)} = \left(L_p, (L_p)_{\min(p,u)}^s\right)_{(0,-b),q} = \left(L_p, B_{p,\min(p,u)}^s\right)_{(0,-b),q} \hookrightarrow \left(L_p, A_{p,u}^s\right)_{(0,-b),q}.$$
With the help of the limiting real method for $\theta = 0$ we are going to characterize $\mathbf{B}_{p,q}^{0,b}$ by using smooth dyadic resolutions of unity and the Fourier transform.

Subsequently S and S' stand for the Schwartz space of all (complex-valued) rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$, and the space of tempered distributions on $\mathbb{R}^n$, respectively. If $f \in S'$, we write $\hat{f}$ for its Fourier transform and $f^{\vee}$ for its inverse Fourier transform. Take $\varphi_0 \in S$ such that
$$\operatorname{supp} \varphi_0 \subset \{x \in \mathbb{R}^n : |x| \le 2\} \text{ and } \varphi_0(x) = 1 \quad \text{if} \quad |x| \le 1.$$
For $j \in \mathbb{N}$ and $x \in \mathbb{R}^n$ let $\varphi_j(x) = \varphi_0(2^{-j}x) - \varphi_0(2^{-j+1}x)$. The sequence $(\varphi_j)_{j\in\mathbb{N}_0}$ is a smooth dyadic resolution of unity, $\sum_{j=0}^{\infty} \varphi_j(x) = 1$ for all $x \in \mathbb{R}^n$.

Imitating the quasi-norm (24), for $1 \le p \le \infty, 0 < q \le \infty, s \ge 0$ and $b \in \mathbb{R}$, the Besov spaces $B_{p,q}^{s,b}$ are defined by

$$B_{p,q}^{s,b} = \left\{ f \in S' : \|f|B_{p,q}^{s,b}\| = \left(\sum_{j=0}^{\infty} \left(2^{js}(1 + j)^b \|(\varphi_j \hat{f})^\vee|L_p\| \right)^q \right)^{1/q} < \infty \right\} \quad (42)$$

(see [142, 47, 28, 144, 169]). We are mainly interested in the case $s = 0$. Spaces $B_{p,q}^{0,b}$ also have logarithmic smoothness but they are different from $\mathbf{B}_{p,q}^{0,b}$ although they are closely related (see [283, Theorem 3.3] and [308, Theorem 3.2]). The characterization of $\mathbf{B}_{p,q}^{0,b}$ in terms of Fourier-analytical decompositions has not been studied yet, even for the classical space $\mathbf{B}_{p,q}^{0}$. Next we establish it. We start with the diagonal case where $p = q$.

Theorem (6.3.7) [313] Let $1 < p < \infty$ and $b \geq -1/p$. Then $f \in L_p$ belongs to $\mathbf{B}_{p,q}^{0,b}$ if and only if

$$\| \cdot |\mathbf{B}_{p,p}^{0,b}\|_{\varphi_+} = \left(\sum_{j=0}^{\infty} (1+j)^{bp} \left\| \left(\sum_{\nu=j}^{\infty} \left| (\varphi_\nu \hat{f})^\vee(\cdot) \right|^2 \right)^{1/2} |L_p \right\|^p \right)^{1/p} < \infty.$$

Moreover, $\| \cdot |\mathbf{B}_{p,p}^{0,b}\|_{\varphi_+}$ is an equivalent norm on $\mathbf{B}_{p,p}^{0,b}$. If $b = 0$ we obtain

$$\mathbf{B}_{p,p}^{0} = \left\{ f \in L_p : \|f|\mathbf{B}_{p,p}^{0}\|_{\varphi_+} = \left(\sum_{j=0}^{\infty} \left\| \left(\sum_{\nu=j}^{\infty} \left| (\varphi_\nu \hat{f})^\vee(\cdot) \right|^2 \right)^{1/2} |L_p \right\|^p \right)^{1/p} < \infty \right\}.$$

Proof. We know that $L_p = F_{p,2}^0$ is a retract of $L_p(\ell_2)$ and that $W_p^1 = F_{p,2}^1$ is a retract of $L_p(\ell_2^1)$, the corresponding coretraction operator being $\mathfrak{J}f = \left((\varphi_\nu \hat{f})^\vee \right)_{\nu \in \mathbb{N}_0}$ (see [62, p. 185]). For the vector-valued L_p-spaces, using Lemmata (6.3.1) and (6.3.2), we obtain

$$\left(L_p(\ell_2), L_p(\ell_2^1) \right)_{(0,-b),p} = L_p\left((\ell_2, \ell_2^1)_{(0,-b),p} \right)$$

$$= \left\{ f = (f_j) : \|f\| = \left\| \left(\sum_{j=0}^{\infty} (1+j)^{bp} \left(\sum_{\nu=j}^{\infty} |f_\nu(x)|^2 \right)^{p/2} \right)^{1/p} |L_p \right\| < \infty \right\}.$$

Since $\mathbf{B}_{p,p}^{0,b} = \left(L_p, W_p^1 \right)_{(0,-b),q}$ (see [283, Theorem 3.1/(b)]or Theorem 3.4), and

$$\|\mathfrak{J}f\|_{L_p\left((\ell_2,\ell_2^1)_{(0,-b),p} \right)} = \left(\sum_{j=0}^{\infty} (1+j)^{bp} \int_{\mathbb{R}^n} \left(\sum_{\nu=j}^{\infty} \left| (\varphi_\nu \hat{f})^\vee(x) \right|^2 \right)^{p/2} dx \right)^{1/p}$$

$$= \left(\sum_{j=0}^{\infty} (1+j)^{bp} \left\| \left(\sum_{\nu=j}^{\infty} \left| (\varphi_\nu \hat{f})^\vee(\cdot) \right|^2 \right)^{1/2} |L_p \right\|^p \right)^{1/p},$$

the wanted characterization follows from [62, Theorem 1.2.4].

Remark (6.3.8) [313] It is shown in [283, Corollary 3.5] that

$$\mathbf{B}_{2,2}^{0,b} = B_{2,2}^{0,b+1/2} \text{ if } b > -1/2.$$

In the more recent paper [308, Corollary 3.3], it has been proved that in the limit case $b = -1/2$ we have

$$\mathbf{B}_{2,2}^{0,-1/2} = B_{2,2}^{0,0,1/2} = \left\{ f \in S' : \left(\sum_{j=0}^{\infty} \left((1 + \log(1 \ + j))^{1/2} \left\| \left(\varphi_j \hat{f}\right)^{\vee} | L_2 \right\| \right)^2 \right)^{1/2} < \infty \right\}.$$

Using Theorem (6.3.7) we can see clearly the reason for this jump in the scale: It is owing to the asymptotic behaviour of $d_\nu = \sum_{j=0}^{\nu} (1+j)^{2b}$ which behaves as $(1+\nu)^{2b+1}$ if $b > -1/2$ and as $\log(1+\nu)$ if $b \ = -1/2$. Indeed

$$\left\| f \middle| \mathbf{B}_{2,2}^{0,b} \right\|^2 \sim \sum_{j=0}^{\infty} (1+j)^{2b} \left\| \left(\sum_{\nu=j}^{\infty} \left| \left(\varphi_\nu \hat{f}\right)^{\vee}(\cdot) \right|^2 \right)^{1/2} | L_2 \right\|$$

$$= \sum_{j=0}^{\infty} (1+j)^{2b} \int_{\mathbb{R}^n} \left(\sum_{\nu=j}^{\infty} \left| \left(\varphi_\nu \hat{f}\right)^{\vee}(x) \right|^2 \right) dx = \int_{\mathbb{R}^n} \left(\sum_{\nu=0}^{\infty} \left| \left(\varphi_\nu \hat{f}\right)^{\vee}(x) \right|^2 \sum_{j=0}^{\nu} (1+j)^{2b} \right) dx$$

$$= \sum_{\nu=0}^{\infty} \left(\mathrm{d}_\nu^{1/2} \left(\int_{\mathbb{R}^n} \left| \left(\varphi_\nu \hat{f}\right)^{\vee}(x) \right|^2 dx \right)^{1/2} \right)^2 \sim \begin{cases} \left\| f \middle| B_{2,2}^{0,b+1/2} \right\|^2 & \text{if } b > -1/2\,, \\ \left\| f \middle| B_{2,2}^{0,0,1/2} \right\|^2 & \text{if } b = -1/2\,. \end{cases}$$

Next we study the non-diagonal case $p = q$. We work with the vector-valued sequence spaces

$$\ell_p^s(A_k) = \left\{ (a_k)_{k \in \mathbb{N}_0} : \left\| (a_k) \middle| \ell_p^s(A_k) \right\| = \left(\sum_{k=0}^{\infty} (2^{ks} \| a_k | A_k \|)^p \right)^{1/p} < \infty \right\}.$$

Here $s \in \mathbb{R}$ and $(A_k)_{k \in \mathbb{N}_0}$ is a sequence of Banach spaces. It is well-known that for $0 < \theta < 1, -\infty < s_0 \neq s_1 < \infty$ and $0 < q \leq \infty$, we have with equivalence of quasi-norms

$$\left(\ell_p^{s_0}(A_k), \ell_p^{s_1}(A_k) \right)_{\theta,q} = \ell_q^s(A_k), \qquad s = (1-\theta)s_0 + \theta s_1 \tag{43}$$

(see [62, Theorem 1.18.2] where (43) is proved for $A_k = A, k \in \mathbb{N}_0$; arguments work also in the general case).

Theorem (6.3.9) [313] Let $1 < p < \infty, 0 < q \leq \infty$ and $b > -1/q$. Then $f \in L_p$ belongs to $\mathbf{B}_{p,q}^{0,b}$ if and only if

$$\left\| f \middle| \mathbf{B}_{p,q}^{0,b} \right\|_{\varphi_+} = \left(\sum_{j=0}^{\infty} \left[(1+j)^b \left\| \left(\sum_{\nu=j}^{\infty} \left| \left(\varphi_\nu \hat{f}\right)^{\vee}(\cdot) \right|^2 \right)^{1/2} \middle| L_p \right\| \right]^q \right)^{1/q} < \infty \tag{44}$$

(usual modification if $q = \infty$). Furthermore, $\| \cdot | \mathbf{B}_{p,q}^{0,b} \|_{\varphi_+}$ is an equivalent quasi-norm on $\mathbf{B}_{p,q}^{0,b}$.

Proof. Take b_0, b_1 such that $-1/p < b_0 < b + 1/q - 1/p < b_1$. We can find $0 < \theta < 1$ such that

$$b + 1/q = (1-\theta)(b_0 + 1/p) + \theta(b_1 + 1/p). \tag{45}$$

With this choice of parameters, Lemma (6.3.5) and [295, Theorem 5] yield that

$$\mathbf{B}_{p,q}^{0,b} = (\mathbf{B}_{p,p}^{0,b_0}, \mathbf{B}_{p,p}^{0,b_1})_{\theta,q}. \tag{46}$$

Moreover, it follows from Theorem (6.3.7) and Lemma (6.3.2) that $\mathbf{B}_{p,p}^{0,b_i}$ is a retract of $L_p\left((\ell_2, \ell_2^1)_{(0,-b_i),p}\right), i = 0, 1$. Note also that

$$\left\| \left(\left(\varphi_j \hat{f}\right)^{\vee} \right) \middle| L_p\left((\ell_2, \ell_2^1)_{(0,-b_i),p}\right) \right\| \sim \left(\sum_{j=0}^{\infty} (1+j)^{b_i p} \left\| \left(\sum_{\nu=j}^{\infty} \left| \left(\varphi_\nu \hat{f}\right)^{\vee}(\cdot) \right|^2 \right)^{1/2} \middle| L_p \right\|^p \right)^{1/p}$$

$$= \left(\sum_{k=0}^{\infty} \sum_{j=2^k}^{2^{k+1}-1} j^{b_i p} \left\| \left(\sum_{\nu=j-1}^{\infty} \left| (\varphi_\nu \hat{f})^\vee (\cdot) \right|^2 \right)^{1/2} \left| L_p \right\| \right\|^p \right)^{1/p}.$$

If $2^k \leq j \leq 2^{k+1} - 1$ then $j \sim 2^k$ and so $\sum_{j=2^k}^{2^{k+1}-1} j^{b_i p} \sim 2^{k b_i p} 2^k = 2^{k(b_i + 1/p)p}$. Whence,

$$\left\| \left((\varphi_j \hat{f})^\vee \right) \left| L_p \left((\ell_2, \ell_2^1)_{(0,-b_i),p} \right) \right\| \sim \left(\sum_{k=0}^{\infty} 2^{k(b_i+1/p)p} \left\| \left(\sum_{\nu=2^k-1}^{\infty} \left| (\varphi_\nu \hat{f})^\vee (\cdot) \right|^2 \right)^{1/2} \left| L_p \right\| \right\|^p \right)^{1/p}.$$

Write $\Re f = (((\varphi_\nu \hat{f})^\vee)_{\nu \geq 2^k - 1})_{k \in \mathbb{N}_0}$ and $A_k = L_p(\ell_2)$ for $k \in \mathbb{N}_0$. The previous considerations and Theorem (6.3.7) show that

$$\Re : \mathbf{B}_{p,p}^{0,b_i} \longrightarrow \boldsymbol{\ell}_p^{b_i + 1/p}(A_k)$$

is bounded for $i = 0, 1$. Interpolating this operator and using (43), (45) and (46), we derive that

$$\Re : \mathbf{B}_{p,q}^{0,b} \longrightarrow \boldsymbol{\ell}_q^{b+1/q}(A_k)$$

is also bounded. Therefore one has by (44)

$$\|f | \mathbf{B}_{p,q}^{0,b}\|_{\varphi_+} \sim \left(\sum_{j=0}^{\infty} \left[2^{k(b+1/q)} \left\| \left(\sum_{\nu=2^k-1}^{\infty} \left| (\varphi_\nu \hat{f})^\vee (\cdot) \right|^2 \right)^{1/2} \left| L_p \right\| \right] ^q \right)^{1/q} = \|\Re f | \boldsymbol{\ell}_q^{b+1/q}(A_k)\| \lesssim \|f | \mathbf{B}_{p,q}^{0,b}\|.$$

To check the converse inequality, note that using $L_p(\ell_2)$-Fourier multipliers and Littlewood–Paley theorem based on (φ_τ) where $\varphi_\tau \varphi_\nu = 0$ if $|\tau - \nu| > 1$ one has

$$\left\| \sum_{\nu=j+1}^{\infty} (\varphi_\nu \hat{f})^\vee \left| L_p \right\| \lesssim \left\| \left(\sum_{\nu=j}^{\infty} \left| (\varphi_\tau \hat{f})^\vee (\cdot) \right|^2 \right)^{1/2} \left| L_p \right\|.$$

Hence

$$\|f | \mathbf{B}_{p,q}^{0,b}\|_{\varphi_+}^q = \sum_{j=0}^{\infty} \left[(1+j)^b \left\| \left(\sum_{\nu=j}^{\infty} \left| (\varphi_\nu \hat{f})^\vee (\cdot) \right|^2 \right)^{1/2} \left| L_p \right\| \right]^q$$

$$\sim \|f | L_p\|^q + \sum_{j=0}^{\infty} 2^{j(b+1/q)q} \left\| \left(\sum_{\nu=2^j}^{\infty} \left| (\varphi_\nu \hat{f})^\vee (\cdot) \right|^2 \right)^{1/2} \left| L_p \right\|^q$$

$$\gtrsim \|f | L_p\|^q + \sum_{j=0}^{\infty} 2^{j(b+1/q)q} \left\| \sum_{\nu=2^j+1}^{\infty} (\varphi_\nu \hat{f})^\vee \left| L_p \right\|^q$$

$$= \|f | L_p\|^q + \sum_{j=0}^{\infty} 2^{j(b+1/q)q} \left\| f - \sum_{\nu=0}^{2^j} (\varphi_\nu \hat{f})^\vee \left| L_p \right\|^q.$$

Let $\mu_j = 2^{2^j}$ and consider the sets G_j defined in (40). Since

$$\operatorname{supp} \left(\sum_{\nu=0}^{2^j} (\varphi_\nu \hat{f})^\vee \right)^\wedge = \operatorname{supp} \sum_{\nu=0}^{2^j} \varphi_\nu \hat{f} \subseteq \{x : |x| \leq \mu_{j+1}\},$$

we have $\sum_{\nu=0}^{2^j} (\varphi_\nu \hat{f})^\vee \in G_{\mu_{j+1}}$ and

$$\left\| f - \sum_{\nu=0}^{2^j} (\varphi_\nu \hat{f})^\vee \left| L_p \right\| \geq \inf_{g \in G_{\mu_{j+2}-1}} \|f - g | L_p\| = E_{\mu_{j+2}}(f)_p.$$

Consequently, using [295, Lemma 1] and Lemma (6.3.5), we derive that

$$\left\|f\middle|\mathbf{B}^{0,b}_{p,q}\right\|_{\varphi_+} \gtrsim \left(\left\|f\middle|L_p\right\|^q + \sum_{j=0}^{\infty} 2^{j(b+1/q)q} E_{\mu_j}(f)_p^q\right)^{1/q} \sim \left\|f\left|\left(L_p\right)_q^{(0,b)}\right.\right\| \sim \left\|f\middle|\mathbf{B}^{0,b}_{p,q}\right\|.$$

This completes the proof.

We collect basic notation of wavelets as needed below following closely [311, Section 1.2.1, pp. 13–14]. As usual, $C^u(\mathbb{R})$ with $u \in \mathbb{N}$ collects all (complex-valued) continuous functions on $\mathbb{R}$ having continuous bounded derivatives up to order u (inclusively). Let

$$\psi_F \in C^u(\mathbb{R}), \qquad \psi_M \in C^u(\mathbb{R}), \qquad u \in \mathbb{N}, \tag{47}$$

be real compactly supported Daubechies wavelets with

$$\int_{\mathbb{R}} \psi_M(x) x^v dx = 0 \quad \text{for all} \quad v \in \mathbb{N}_0 \quad with \quad v < u.$$

Recall that ψ_F is called the scaling function (father wavelet) and ψ_M the associated wavelet (mother wavelet). The extension of these wavelets from $\mathbb{R}$ to $\mathbb{R}^n, 2 \leq n \in \mathbb{N}$, is based on the usual tensor procedure. Let

$$G = (G_1, \ldots, G_n) \in G_0 = \{F, M\}^n,$$

which means that G_r is either For M. Let

$$G = (G_1, \ldots, G_n) \in G_j = \{F, M\}^{n*}, \qquad j \in \mathbb{N},$$

which means that G_r is either For M where * indicates that at least one of the components of G must be an M. Hence G^0 has 2^n elements, whereas G^j with $j \in \mathbb{N}$ has 2^{n-1} elements. Let

$$\Psi^j_{G,m}(x) = 2^{jn/2} \prod_{r=1}^{n} \psi G_r(2^j x_r - m_r), \qquad G \in G^j, \qquad m \in \mathbb{Z}^n, \tag{48}$$

where (now) $j \in \mathbb{N}_0$. We always assume the ψ_F and ψ_M in (47) have L_2-norm 1. Then

$$\Psi = \left\{\Psi^j_{G,m} : j \in \mathbb{N}_0, G \in G^j, m \in \mathbb{Z}^n\right\}$$

is an orthonormal basis in $L_2(\mathbb{R}^n)$ and

$$f = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \lambda^{j,G}_m 2^{-jn/2} \Psi^j_{G,m}$$

with

$$\lambda^{j,G}_m = \lambda^{j,G}_m(f) = 2^{jn/2} \int_{\mathbb{R}^n} f(x) \Psi^j_{G,m}(x) dx = 2^{jn/2}\left(f, \Psi^j_{G,m}\right)$$

in the corresponding expansion, adapted to our needs, where $2^{-jn/2}\Psi^j_{G,m}$, are uniformly bounded functions (with respect to j and m). The spaces $B^s_{p,q}$ with p, q, s according to (21) can be expanded in terms of Ψ as described above in (31)–(33) where $s < u \in \mathbb{N}$. This is a special case of [311, Theorem 1.20, pp. 15–16].

Let $\chi_{j,m}$ be the characteristic function of the dyadic cube $Q_{j,m} = 2^{-j}m + 2^{-j}(0,1)^n$ in $\mathbb{R}^n$ with sides of length 2^{-j} parallel to the axes of coordinates and $2^{-j}m$ as the lower left corner. For $s = 0, 1$ and $1 < p < \infty$, we write $f^s_{p,2}$ for the space of all sequences $\lambda = (\lambda^{j,G}_m)$ with $j \in \mathbb{N}_0, G \in G^j$and $m \in \mathbb{Z}^n$ such that

$$\left\|\lambda | f^s_{p,2}\right\| = \left\|\left(\sum_{j,G,m} 2^{js2} \left|\lambda^{j,G}_m \chi_{j,m}(\cdot)\right|^2\right)^{1/2} \left|L_p\right.\right\| < \infty.$$

We put

$$\ell_2^s(\ell_2)=\left\{((\mu_m^{j,G})_{\substack{G\in G^j\\ m\in\mathbb{Z}^n}})_{j\in\mathbb{N}_0}:\left\|((u_m^{j,G})_{\substack{G\in G^j\\ m\in\mathbb{Z}^n}})|\ell_2^s(\ell_2)\right\|=\left(\sum_j 2^{js2}\sum_{G,m}\left|\mu_m^{j,G}\right|^2\right)^{1/2}<\infty\right\}.$$

Lemma (6.3.10) [313] For $s=0,1$ and $1<p<\infty$, the space $f_{p,2}^s$ can be identified with a complemented subspace $\Delta_{p,2}^s$ of $L_p(\ell_2^s(\ell_2))$. The projection onto $\Delta_{p,2}^s$ associates to each $h(\cdot)=(h_j(\cdot))_{j\in\mathbb{N}_0}=((h_m^{j,G}(\cdot))_{\substack{G\in G^j\\ m\in\mathbb{Z}^n}})_{j\in\mathbb{N}_0}\in L_p(\ell_2^s(\ell_2))$ the function Ph defined by

$$Ph(x)=((2^{jn}\int_{Q_{j,m}}h_m^{j,G}(y)dy\chi_{j,m}(x))_{\substack{G\in G^j\\ m\in\mathbb{Z}^n}})_{j\in\mathbb{N}_0}.$$

Proof. Given any $\lambda\in f_{p,2}^s$, let $R(\lambda)$ be the function defined by $R(\lambda)(x)=(g_j(x))$ where $g_j(x)=(\lambda_m^{j,G}\chi_{j,m}(x))_{\substack{G\in G^j\\ m\in\mathbb{Z}^n}}$. Since

$$\|\lambda|f_{p,2}^s\|=\left\|\left(\sum_{j\in\mathbb{N}_0}2^{js2}\sum_{\substack{G\in G^j\\ m\in\mathbb{Z}^n}}\left|\lambda_m^{j,G}\right|^2\chi_{j,m}(\cdot)\right)^{1/2}\Big|L_p\right\|=\left\|\left(\sum_{j\in\mathbb{N}_0}2^{js2}\left\|g_j(\cdot)|L_p\right\|^2\right)^{1/2}\Big|L_p\right\|$$
$$=\left\|R(\lambda)|L_p(\ell_2^s(\ell_2))\right\|$$

we have that $f_{p,2}^s$ is isometric to the subspace $\Delta_{p,2}^s=\{R(\lambda):\lambda\in f_{p,2}^s\}$ of $L_p(\ell_2^s(\ell_2))$. It is easy to check that $Ph=h$ for any $h\in\Delta_{p,2}^s$. Let us show that P is bounded in $L_p(\ell_2^s(\ell_2))$. We have that

$$2^{jn}\int_{Q_{j,m}}\left|h_m^{j,G}(y)\right|dy\chi_{j,m}(x)\lesssim\left(\mathcal{M}h_m^{j,G}\right)(x),\qquad x\in\mathbb{R}^n,$$

where $\mathcal{M}$ is the Hardy–Littlewood maximal operator. Using the vector-valued estimate for $\mathcal{M}$ (see [194, Theorem 1.1.1, p. 51]), we obtain

$$\left\|Ph|L_p\left(\ell_2^s(\ell_2)\right)\right\|\le\left\|\left(\sum_{j\in\mathbb{N}_0}\sum_{\substack{G\in G^j\\ m\in\mathbb{Z}^n}}\left(2^{jn}\int_{Q_{j,m}}\left|2^{js}h_m^{j,G}(y)\right|dy\right)^2\chi_{j,m}(\cdot)\right)^{1/2}\Big|L_p\right\|$$
$$\lesssim\left\|\left(\sum_{j\in\mathbb{N}_0}\sum_{\substack{G\in G^j\\ m\in\mathbb{Z}^n}}\left(\mathcal{M}(2^{js}\left|h_m^{j,G}\right|)(\cdot)\right)^2\right)^{1/2}\Big|L_p\right\|\lesssim\left\|\left(\sum_{j\in\mathbb{N}_0}\sum_{\substack{G\in G^j\\ m\in\mathbb{Z}^n}}2^{2js}\left|h_m^{j,G}(\cdot)\right|^2\right)^{1/2}\Big|L_p\right\|$$
$$=\left\|h|L_p(\ell_2^s(\ell_2))\right\|.$$

In addition, this also shows that if $h\in L_p(\ell_2^s(\ell_2))$ then $Ph\in\Delta_{p,2}^s$. The proof is completed.

Lemma (6.3.11) [313] Let $1<p<\infty$ and $b\ge -1/p$. Then $\lambda=(\lambda_m^{j,G})$ belongs to $(f_{p,2}^0,f_{p,2}^1)_{(0,-b),p}$ if and only if

$$\|\lambda\|=\left(\int_{\mathbb{R}^n}\sum_{j=0}^{\infty}(1+j)^{bp}\left(\sum_{\nu=j}^{\infty}\sum_{\substack{G\in G^j\\ m\in\mathbb{Z}^n}}\left|\lambda_m^{\nu,G}\right|^2\chi_{\nu,m}(x)\right)^{p/2}dx\right)^{1/p}$$

is finite. Moreover, $\|\lambda\|$ defines an equivalent norm in $(f^0_{p,2}, f^1_{p,2})_{(0,-b),p}$.

Proof. By Lemmata (6.3.1), (6.3.2), we have that

$$(L_p(\ell_2(\ell_2)), L_p(\ell^s_2(\ell_2)))_{(0,-b),p} = L_p((\ell_2, \ell^1_2)_{(0,-b),p}(\ell_2)).$$

Hence, according to Lemma (6.3.10) and the theorem on interpolation of complemented subspaces [62, Theorem 1.17.1], we derive that

$$\left\|\lambda|(f^0_{p,2}, f^1_{p,2})_{(0,-b),p}\right\| \sim \left(\int_{\mathbb{R}^n}\sum_{j=0}^{\infty}(1+j)^{bp}\left(\sum_{\nu=j}^{\infty}\sum_{\substack{G\in G^j\\ m\in\mathbb{Z}^n}}\left|\lambda^{\nu,G}_m\right|^2\chi_{\nu,m}(x)\right)^{p/2} dx\right)^{1/p}.$$

In what follows we work with the sequence of wavelets $(\Psi^j_{G,m})$ defined in (48) with $u > 1$ and the sequence space $\mathbf{b}^{0,b}_{p,q}$ defined in (36) but allowing now $b \geq -1/q$.

Theorem (6.3.12) [313] Let $1 < p < \infty$ and $b \geq -1/p$. Then f belongs to $\mathbf{B}^{0,b}_{p,p}$ if, and only if, it can be represented as $f = \sum_{j,G,m}\lambda^{j,G}_m 2^{-jn/2}\Psi^j_{G,m}$ (unconditional convergence being in L_p) with $\lambda^{j,G}_m = \lambda^{j,G}_m(f) = 2^{jn/2}(f, \Psi^j_{G,m})$ and

$$\left\|f\left|\mathbf{B}^{0,b}_{p,p}\right\|_{\Psi_+} = \left\|(\lambda^{j,G}_m)\right|\mathbf{b}^{0,b}_{p,p}\right\| = \left(\sum_{j=0}^{\infty}(1+j)^{bp}\left\|\left(\sum_{\nu=j}^{\infty}\sum_{\substack{G\in G^\nu\\ m\in\mathbb{Z}^n}}\left|\lambda^{\nu,G}_m\right|^2\chi_{\nu,m}(\cdot)\right)^{1/2}\middle|L_p\right\|^p\right)^{1/p}$$

is finite. Moreover, $\left\|\cdot\left|\mathbf{B}^{0,b}_{p,p}\right\|_{\Psi_+}$ defines an equivalent norm in $\mathbf{B}^{0,b}_{p,p}$.

Proof. The unconditional convergence in L_p for any sequence $(\lambda^{j,G}_m) \in \mathbf{b}^{0,b}_{p,p}$ follows from a corresponding assertion for L_p based on $f^0_{p,2}$ according to [311, Theorem 1.20] and $\mathbf{b}^{0,b}_{p,p} \hookrightarrow f^0_{p,2}$ as a consequence of Lemma (6.3.11).

Let D be the operator defined by

$$D((\lambda^{j,G}_m)) = \sum_{j,G,m}\lambda^{j,G}_m 2^{-jn/2}\Psi^j_{G,m}.$$

According to [311, Theorem 1.20], the restrictions

$$D : f^0_{p,2} \longrightarrow L_p \text{ and } D : f^0_{p,2} \longrightarrow W^1_p$$

are isomorphisms. Interpolating and using Theorem (6.3.6) or [283, Theorem 3.1], we obtain that

$$D : (f^0_{p,2}, f^1_{p,2})_{(0,-b),p} \longrightarrow (L_p, W^1_p)_{(0,-b),p} = \mathbf{B}^{0,b}_{p,p}$$

is also an isomorphism. As for the source space of this operator, by Lemma (6.3.11), we know that

$$\left\|(\lambda^{j,G}_m)\middle|(f^0_{p,2}, f^1_{p,2})_{(0,-b),p}\right\| \sim \left(\int_{\mathbb{R}^n}\sum_{j=0}^{\infty}(1+j)^{bp}\left(\sum_{\nu=j}^{\infty}\sum_{\substack{G\in G^\nu\\ m\in\mathbb{Z}^n}}\left|\lambda^{\nu,G}_m\right|^2\chi_{\nu,m}(x)\right)^{p/2} dx\right)^{1/p}$$

$$= \left(\sum_{j=0}^{\infty}(1+j)^{bp}\left\|\left(\sum_{\nu=j}^{\infty}\sum_{\substack{G\in G^\nu\\ m\in\mathbb{Z}^n}}\left|\lambda^{\nu,G}_m\right|^2\chi_{\nu,m}(\cdot)\right)^{1/2}\middle|L_p\right\|^p\right)^{1/p}.$$

Furthermore, $\lambda^{j,G}_m = \lambda^{j,G}_m(f)$ is again covered by [311, Theorem 1.20]. This completes the proof.

In order to study the case $p \neq q$, we first introduce some notation and we establish an auxiliary result.

For $1 < p < \infty$ and $j \in \mathbb{N}$, let $P_{2^j} : L_p \longrightarrow L_p$ be the operator defined by

$$P_{2^j} f = \sum_{\nu=0}^{j-1} \sum_{\substack{G \in G^\nu \\ m \in \mathbb{Z}^n}} \lambda_m^{\nu,G}(f) 2^{-\nu n/2} \Psi_{G,m}^\nu .$$

As $\{\Psi_{G,m}^\nu\}$ is an unconditional Schauder basis in L_p, we have that

$$\sup\{\|P_{2^j}|L_p \longrightarrow L_p\| : j \in \mathbb{N}\} < \infty.$$

Let $V_0 = \{0\}$ and for $j = 2, 3, \ldots$ with $2^m \leq j < 2^{m+1}, m \in \mathbb{N}$, put

$$V_{j-1} = P_{2^m}(L_p) = \left\{ g \in L_p : g = \sum_{\nu=0}^{m-1} \sum_{\substack{G \in G^\nu \\ m \in \mathbb{Z}^n}} c_m^{\nu,G} 2^{-\nu n/2} \Psi_{G,m}^\nu \text{ with } c_m^{\nu,G} \in \mathbb{C} \right\}.$$

Then $(L_p, \{V_j\})$ is an approximation scheme in the sense of Pietsch [300]. Put

$$E_j^\Psi(f)_p = \inf\{\|f - g|L_p\| : g \in V_{j-1}\}, \qquad j \in \mathbb{N},$$

and define the space $(L_p, \{V_j\})_q^{(0,b)}$ as (41) but replacing the sequence $(E_j(f)_p)$ by $(E_j^\Psi(f)_p)$.

Note that

$$E_{2^j}^\Psi(f)_p \sim \|f - P_{2^j} f|L_p\|, \qquad j \in \mathbb{N}.$$

Indeed, given any $g \in V_{2^j-1}$, we have

$$\|f - P_{2^j} f|L_p\| \leq \|f - g|L_p\| + \|g - P_{2^j} f|L_p\| = \|f - g|L_p\| + \|P_{2^j}(g - f)|L_p\|$$
$$\lesssim \|f - g|L_p\|.$$

Lemma (6.3.13) [313] Let $1 < p < \infty, 0 < q \leq \infty$ and $b > -1/q$. Then we have with equivalence of norms

$$(L_p, \{V_j\})_q^{(0,b)} = \mathbf{B}_{p,q}^{0,b}.$$

Proof. We start with the case $1 < p = q < \infty$. Put $\mu_j = 2^{2^j}, j \in \mathbb{N}_0$. We have

$$\begin{aligned}
\left\|f \left|(L_p, \{V_j\})_q^{(0,b)}\right.\right\|^p &\sim \|f|L_p\|^p + \sum_{j=0}^\infty \left(2^{j(b+1/p)} E_{\mu_j}^\Psi(f)_p\right)^p \\
&\sim \|f|L_p\|^p + \sum_{j=0}^\infty \left(2^{j(b+1/p)} \|f - P_{\mu_j} f|L_p\|\right)^p \\
&= \|f|L_p\|^p + \sum_{j=0}^\infty 2^{j(b+1/p)p} \left\| \sum_{\nu=2^j}^\infty \sum_{\substack{G \in G^\nu \\ m \in \mathbb{Z}^n}} \lambda_m^{\nu,G}(f) 2^{-\nu n/2} \Psi_{G,m}^\nu \left| L_p \right.\right\|^p \\
&\sim \|f|L_p\|^p + \sum_{j=0}^\infty 2^{j(b+1/p)p} \left\| \left(\sum_{\nu=2^j}^\infty \sum_{\substack{G \in G^\nu \\ m \in \mathbb{Z}^n}} |\lambda_m^{\nu,G}(f)|^2 \chi_{\nu,m}(\cdot) \right)^{1/2} \left| L_p \right.\right\|^p
\end{aligned}$$

where we have used [67, Theorem 1.64](or [311, Theorem 1.20]) in the last equivalence. Now the result follows from Theorem (6.3.12). Note that the above argument works even if $b = -1/p$.

To establish the remaining case $p \neq q$, choose b_0, b_1 such that $-1/p < b_0 < b + 1/q - 1/p < b_1$ and take $0 < \theta < 1$ with

$$b + 1/q = (1-\theta)(b_0 + 1/p) + \theta(b_1 + 1/p).$$

According to (46), [295, Theorem 5] and the result just proved for the diagonal case, we obtain that

$$\mathbf{B}^{0,b}_{p,q} = (\mathbf{B}^{0,b_0}_{p,p}, \mathbf{B}^{0,b_1}_{p,p})_{\theta,q} = ((L_p,\{V_j\})^{(0,b_0)}_q, (L_p,\{V_j\})^{(0,b_1)}_q)_{\theta,q} = (L_p,\{V_j\})^{(0,b)}_q.$$

Theorem (6.3.14) [313] Let $1 < p < \infty, 0 < q \le \infty$ and $b > -1/q$. Then f belongs to $\mathbf{B}^{0,b}_{p,q}$ if, and only if, it can be represented as $f = \sum_{j,G,m} \lambda^{j,G}_m 2^{-jn/2}\Psi^j_{G,m}$ (unconditional convergence in L_p) with $\lambda^{j,G}_m = \lambda^{j,G}_m(f) = 2^{jn/2}(f, \Psi^j_{G,m})$ and

$$\|f|\mathbf{B}^{0,b}_{p,q}\|_{\Psi_+} = \|(\lambda^{j,G}_m)|\mathbf{b}^{0,b}_{p,q}\| = \left(\sum_{j=0}^{\infty}(1+j)^{bp}\left\|\left(\sum_{\nu=j}^{\infty}\sum_{\substack{G\in G^\nu \\ m\in\mathbb{Z}^n}}|\lambda^{\nu,G}_m|^2\chi_{\nu,m}(\cdot)\right)^{1/2}|L_p\right\|^p\right)^{1/p} < \infty.$$

Moreover, $\|\cdot|\mathbf{B}^{0,b}_{p,q}\|_{\Psi_+}$ defines an equivalent quasi-norm in $\mathbf{B}^{0,b}_{p,q}$.

Proof. The unconditional convergence in L_p is covered by the related argument at the beginning of the proof of Theorem (6.3.12) and the above interpolation (46).

Using Lemma (6.3.13), we obtain

$$\|f|\mathbf{B}^{0,b}_{p,q}\| \sim \left\|f\left|(L_p,\{V_j\})^{(0,b)}_q\right.\right\| \sim \left(\|f|L_p\|^p + \sum_{j=0}^{\infty}2^{j(b+1/q)q}E^{\Psi}_{\mu_j}(f)^q_p\right)^{1/q}$$

$$\sim \left(\|f|L_p\|^p + \sum_{j=0}^{\infty}2^{j(b+1/q)q}\left\|f - P_{\mu_j}f|L_p\right\|^q\right)^{1/q}$$

$$\sim \left(\|f|L_p\|^p + \sum_{j=0}^{\infty}2^{j(b+1/q)q}\left\|\left(\sum_{\nu=2^j}^{\infty}\sum_{\substack{G\in G^\nu \\ m\in\mathbb{Z}^n}}|\lambda^{\nu,G}_m|^2\chi_{\nu,m}(\cdot)\right)^{1/2}|L_p\right\|^q\right)^{1/q}$$

$$\sim \left(\sum_{j=0}^{\infty}(1+j)^{bp}\left\|\left(\sum_{\nu=j}^{\infty}\sum_{\substack{G\in G^\nu \\ m\in\mathbb{Z}^n}}|\lambda^{\nu,G}_m|^2\chi_{\nu,m}(\cdot)\right)^{1/2}|L_p\right\|^q\right)^{1/q}.$$

Remark (6.3.15) [313] Comparing Theorem (6.3.14) with the corresponding result for classical Besov spaces given in (30)–(33), we observe again an additional truncated Littlewood–Paley-type construction. The corresponding sequence space being $\mathbf{b}^{0,b}_{p,q}$ quasi-normed by (36) in contrast to $b^s_{p,q}$ in (30).

In order to take a closer look into these sequence spaces, consider the Banach case $1 \le q \le \infty$ and notice that the norm (30) of $b^s_{p,q}$ can be rewritten as

$$\|\lambda|b^s_{p,q}\| = \left(\sum_{j=0}^{\infty}2^{jsq}\sum_{G\in G^j}\left\|\sum_{m\in\mathbb{Z}^n}|\lambda^{j,G}_m|\chi_{j,m}(\cdot)\,|L_p\right\|^q\right)^{1/q}. \qquad (49)$$

Then it follows for some $0 < c_1 < c_2 < \infty$,

$$c_1\|\lambda|b^s_{p,q}\| \le \left(\sum_{j=0}^{\infty}2^{jsq}\left\|\left(\sum_{\nu=j}^{\infty}\sum_{\substack{G\in G^\nu \\ m\in\mathbb{Z}^n}}|\lambda^{\nu,G}_m|^2\chi_{\nu,m}(\cdot)\right)^{1/2}|L_p\right\|^q\right)^{1/q}$$

$$\leq c_2 \big\| \lambda | b^s_{p,q} \big\|, \qquad \lambda \in b^s_{p,q}, \tag{50}$$

where we used again $s > 0$. Hence one can replace $b^s_{p,q}$ in (30) by the middle term in (50). Afterwards one can compare $b^s_{p,q}$ with $\mathbf{b}^{0,b}_{p,q}$ according to (36).

To continue with the description of relationships between classical and logarithmic spaces, let

$$(\lambda^j f)(x) = \sum_{\substack{G \in G^j \\ m \in \mathbb{Z}^n}} \big| \lambda^{j,G}_m(f) \big| \chi_{j,m}(x)$$

and

$$(\lambda^j f)_+(x) = \left(\sum_{\nu=j}^{\infty} \sum_{\substack{G \in G^\nu \\ m \in \mathbb{Z}^n}} \big| \lambda^{\nu,G}_m(f) \big|^2 \chi_{\nu,m}(x) \right)^{1/2}.$$

Then

$$\big\| f | B^s_{p,q} \big\|_\Psi = \left(\sum_{j=0}^{\infty} 2^{jsq} \big\| \lambda^j f | L_p \big\|^q \right)^{1/q} \tag{51}$$

and

$$\big\| f | B^s_{p,q} \big\|_{\Psi+} = \left(\sum_{j=0}^{\infty} 2^{jsq} \, \big\| \big(\lambda^j f \big)_+ | L_p \big\|^q \right)^{1/q} \tag{52}$$

are equivalent norms in $B^s_{p,q}$. This is covered by (30), (33) combined with (49), (50). The norm-generating basic ingredient in the refined norm (52) is monotonically decreasing in j, in contrast to their original counterpart in (51).

If one switches from $B^s_{p,q}$ to their logarithmic counterpart $\mathbf{B}^{0,b}_{p,q}$, then as it is shown in Theorem (6.3.14), the corresponding norm to (52) is

$$\big\| f | \mathbf{B}^{0,b}_{p,q} \big\|_{\Psi+} = \left(\sum_{j=0}^{\infty} (1+j)^{bq} \, \big\| \big(\lambda^j f \big)_+ | L_p \big\|^q \right)^{1/q}.$$

On the other hand, according to [157, Theorem 13] and the comments in [60, Section 1.3.3, pp. 54–60], the counterparts of (51), hence

$$\big\| f | B^{0,b}_{p,q} \big\|_\Psi = \left(\sum_{j=0}^{\infty} (1+j)^{bq} \big\| \lambda^j f | L_p \big\|^q \right)^{1/q}$$

is an equivalent norm in the space $B^{0,b}_{p,q}$ which does not coincide with $\mathbf{B}^{0,b}_{p,q}$.

To finish we consider the embeddings

$$B^{0,b+1/\min\{2,p,q\}}_{p,q} \hookrightarrow \mathbf{B}^{0,b}_{p,q} \hookrightarrow B^{0,b+1/\max\{2,p,q\}}_{p,q},$$

established in [283, Theorem 3.3] for $1 < p < \infty, 0 < q \leq \infty$ and $b > -1/q$. As an application of the characterizations by means of wavelets, we show next two results on the optimality of the embeddings above.

Remark (6.3.16) [313] Let $1 < p < \infty, 0 < q \leq \infty$ and $b > -1/q$. Suppose that $p = \max\{2, p, q\}$. We are going to show that for any $\varepsilon > 0$ we have $\mathbf{B}^{0,b}_{p,q} \not\hookrightarrow B^{0,b+1/p+\varepsilon}_{p,q}$.

Given ε choose βsuch that $b + 1/q + 1/p < \beta \leq b + 1/q + 1/p + \varepsilon$ and put

$$\lambda_m^{j,G}=\begin{cases}2^{jn/p}(1+j)^{-\beta} & \text{if } j\in\mathbb{N}_0, G=(M,\dots,M) \\ & \quad \text{and } m=(0,\dots,0), \\ 0 & \text{otherwise.}\end{cases}$$

Then $f=\sum_{j,G,m}\lambda_m^{j,G}2^{-jn/2}\Psi_{G,m}^j$ belongs to $\mathbf{B}_{p,q}^{0,b}$. Indeed, according to Theorem (6.3.14), it is enough to show that

$$\left\|f\middle|\mathbf{B}_{p,q}^{0,b}\right\|_{\Psi_+}=\left(\sum_{j=0}^{\infty}(1+j)^{bp}\left\|\left(\sum_{\nu=j}^{\infty}\left(2^{\nu n/p}(1+\nu)^{-\beta}\right)^2\chi_{\nu,(0,\dots,0)}(\cdot)\right)^{1/2}\middle|L_p\right\|^q\right)^{1/q}<\infty.$$

We claim that if $x\neq 0$ and $j\in\mathbb{N}_0$, then

$$\left(\sum_{\nu=j}^{\infty}\left(2^{\nu n/p}(1+\nu)^{-\beta}\right)^2\chi_{\nu,(0,\dots,0)}(x)\right)^{1/2}\sim\left(\sum_{\nu=j}^{\infty}\left(2^{\nu n/p}(1+\nu)^{-\beta}\right)^p\chi_{\nu,(0,\dots,0)}(x)\right)^{1/p}\qquad(53)$$

with constants in the equivalence which are independent of x and j. Indeed, assume $2\leq p$. The case $p<2$ can be carried out similarly. Since $\ell_2\to\ell_p$, it is clear that

$$\left(\sum_{\nu=j}^{\infty}\left(2^{\nu n/p}(1+\nu)^{-\beta}\right)^p\chi_{\nu,(0,\dots,0)}(x)\right)^{1/p}\leq\left(\sum_{\nu=j}^{\infty}\left(2^{\nu n/p}(1+\nu)^{-\beta}\right)^2\chi_{\nu,(0,\dots,0)}(x)\right)^{1/2}.$$

To check the converse inequality, we distinguish two cases. If $x\notin 2^{-j}(0,1)^n=Q_{j,(0,\dots,0)}$, then $\chi_{\nu,(0,\dots,0)}(x)=0$ for $\nu\geq j$. So

$$\left(\sum_{\nu=j}^{\infty}\left(2^{\nu n/p}(1+\nu)^{-\beta}\right)^2\chi_{\nu,(0,\dots,0)}(x)\right)^{1/2}=0=\left(\sum_{\nu=j}^{\infty}\left(2^{\nu n/p}(1+\nu)^{-\beta}\right)^p\chi_{\nu,(0,\dots,0)}(x)\right)^{1/p}.$$

Suppose now that $x\in 2^{-j}(0,1)^n$ and let $\nu_x\in\mathbb{N}_0$ the bigger value such that $x\in Q_{\nu_x,(0,\dots,0)}$. We have

$$\left(\sum_{\nu=j}^{\infty}\left(2^{\nu n/p}(1+\nu)^{-\beta}\right)^2\chi_{\nu,(0,\dots,0)}(x)\right)^{1/2}=\left(\sum_{\nu=j}^{\nu_x}\left(2^{\nu n/p}(1+\nu)^{-\beta}\right)^2\right)^{1/2}$$
$$\leq\left(\sum_{\nu=0}^{\nu_x}\left(2^{\nu n/p}(1+\nu)^{-\beta}\right)^2\right)^{1/2}\lesssim 2^{\nu_x n/p}(1+\nu_x)^{-\beta}\leq\left(\sum_{\nu=j}^{\nu_x}\left(2^{\nu n/p}(1+\nu)^{-\beta}\right)^p\right)^{1/p}$$
$$=\left(\sum_{\nu=j}^{\infty}\left(2^{\nu n/p}(1+\nu)^{-\beta}\right)^p\chi_{\nu,(0,\dots,0)}(x)\right)^{1/p}$$

which establishes (53).

Consequently, since $-\beta p+1<0$ and $(b-\beta+1/p)q<-1$, we get

$$\left\|f\middle|\mathbf{B}_{p,q}^{0,b}\right\|_{\Psi_+}\sim\left(\sum_{j=0}^{\infty}(1+j)^{bq}\left\|\left(\sum_{\nu=j}^{\infty}\left(2^{\nu n/p}(1+\nu)^{-\beta}\right)^p\chi_{\nu,(0,\dots,0)}(\cdot)\right)^{1/p}\middle|L_p\right\|^q\right)^{1/q}$$
$$=\left(\sum_{j=0}^{\infty}(1+j)^{bq}\left(\int_{\mathbb{R}^n}\sum_{\nu=j}^{\infty}\left(2^{\nu n/p}(1+\nu)^{-\beta}\right)^p\chi_{\nu,(0,\dots,0)}(x)\,dx\right)^{q/p}\right)^{1/q}$$

$$= \left(\sum_{j=0}^{\infty} (1+j)^{bq} \left(\sum_{\nu=j}^{\infty} \left(2^{\nu n/p}(1+\nu)^{-\beta}\right)^p 2^{-\nu n} \right)^{q/p} \right)^{1/q}$$

$$= \left(\sum_{j=0}^{\infty} (1+j)^{bq} \left(\sum_{\nu=j}^{\infty} (1+\nu)^{-\beta p} \right)^{q/p} \right)^{1/q} \sim \left(\sum_{j=0}^{\infty} (1+j)^{bq} (1+j)^{(-\beta p+1)q/p} \right)^{1/q}$$

$$= \left(\sum_{j=0}^{\infty} (1+j)^{(b-\beta+1/p)q} \right)^{1/q} < \infty.$$

Therefore, $f \in \mathbf{B}_{p,q}^{0,b}$. But, by our choice of β, we have

$$\left(\sum_{j=0}^{\infty} 2^{jnq/p} (1+j)^{(b+1/p+\varepsilon)q} \left(2^{jn/p}(1+j)^{-\beta}\right)^q \right)^{1/q} = \left(\sum_{j=0}^{\infty} (1+j)^{(b+1/p+\varepsilon-\beta)q} \right)^{1/q} = \infty.$$

Hence, according to [157, Theorem 13], we derive that $f \notin B_{p,q}^{0,b+1/\mathrm{p}+\varepsilon}$.

Assume this time that $p = \min\{2, p, q\}$. Let us show that for any $\varepsilon > 0$ we have $B_{p,q}^{0,b+1/\mathrm{p}-\varepsilon} \nrightarrow \mathbf{B}_{p,q}^{0,b}$. Take β such that $b + 1/p + 1/q - \varepsilon < \beta \leq b + 1/p + 1/q$ and put as before,

$$\lambda_m^{\nu,G} = \begin{cases} 2^{jn/p}(1+j)^{-\beta} & \text{if } \; j \in \mathbb{N}_0, G = (M, \dots, M) \; and \; m = (0, \dots, 0), \\ 0 & \text{otherwise,} \end{cases}$$

and $f = \sum_{j,G,m} \lambda_m^{j,G} 2^{-jn/2} \Psi_{G,m}^j$. Using (53) we derive

$$\left\| f \middle| \mathbf{B}_{p,q}^{0,b} \right\|_{\Psi_+} = \left(\sum_{j=0}^{\infty} (1+j)^{bq} \left\| \left(\sum_{\nu=j}^{\infty} \left(2^{\nu n/p}(1+\nu)^{-\beta}\right)^2 \chi_{\nu,(0,\dots,0)}(\cdot) \right)^{1/2} \middle| L_p \right\|^q \right)^{1/q}$$

$$\sim \left(\sum_{j=0}^{\infty} (1+j)^{bq} \left\| \left(\sum_{\nu=j}^{\infty} \left(2^{\nu n/p}(1+\nu)^{-\beta}\right)^p \chi_{\nu,(0,\dots,0)}(\cdot) \right)^{1/p} \middle| L_p \right\|^q \right)^{1/q}$$

$$= \left(\sum_{j=0}^{\infty} (1+j)^{bq} \left(\sum_{\nu=j}^{\infty} \left(2^{\nu n/p}(1+\nu)^{-\beta}\right)^p 2^{-\nu n} \right)^{q/p} \right)^{1/q} = \left(\sum_{j=0}^{\infty} (1+j)^{bq} \left(\sum_{\nu=j}^{\infty} (1+\nu)^{-\beta p} \right)^{q/p} \right)^{1/q}.$$

This sum is ∞ if $-\beta p + 1 \geq 0$. If $-\beta p + 1 < 0$, we have

$$\left\| f \middle| \mathbf{B}_{p,q}^{0,b} \right\|_{\Psi_+} \sim \left(\sum_{j=0}^{\infty} (1+j)^{bq} \left(\sum_{\nu=j}^{\infty} (1+\nu)^{-\beta p} \right)^{q/p} \right)^{1/q} \sim \left(\sum_{j=0}^{\infty} (1+j)^{(b-\beta+1/p)q} \right)^{1/q} = \infty.$$

So, Theorem (6.3.14) yields that $f \notin \mathbf{B}_{p,q}^{0,b}$. However

$$\left(\sum_{j=0}^{\infty} 2^{-jnq/p} (1+j)^{(b+1/p-\varepsilon)q} \left(2^{jn/p}(1+j)^{-\beta}\right)^q \right)^{1/q} = \left(\sum_{j=0}^{\infty} (1+j)^{(b+1/p-\varepsilon-\beta)q} \right)^{1/q} < \infty$$

which means, according to [157, Theorem 13], that $f \in B_{p,q}^{0,b+1/\mathrm{p}-\varepsilon}$.

First we show an abstract result on semi-groups of operators and limiting real interpolation, and then we apply it to heat kernels and spaces $\mathbf{B}_{p,q}^{0,b}$.

Let A be a (complex) Banach space and let $\{T(t) : 0 \le t < \infty\}$ be a family of linear and bounded operators from A into itself. The family $\{T(t)\}_{t\ge 0}$ is called to be a strongly continuous equi-bounded semi-group of operators in A if

(i) $T(t+s) = T(t)T(s), t, s \ge 0$.
(ii) $T(0) = \mathrm{id}$(identity in A).
(iii) $\|T(t)a|A\| \le M\|a\,|A\|, t \ge 0, a \in A$.
(iv) $\lim_{t\to 0+} T(t)a = a, a \in A$.

The infinitesimal generator Λ of the semi-group $\{T(t)\}_{t\ge 0}$ is defined by

$$\Lambda a = \lim_{t\downarrow 0}(t^{-1}T(t)a - a)$$

whenever that the limit exists. The domain $D(\Lambda)$ of Λ consists of all those $a \in A$ for which the limit exists. The domain $D(\Lambda^m)$ of the m-th power of Λ is a Banach space endowed with the norm

$$\|a|D(\Lambda^m)\| = \|a|A\| + \|\Lambda^m a|A\|, \qquad m = 1,2,\ldots.$$

The semi-group $\{T(t)\}_{t\ge 0}$ is said to be analytic (or holomorphic) if in addition to (i)–(iv), it satisfies

(v) $T(t)a \in D(\Lambda)$, for all $a \in A$ and $t > 0$.
(vi) $t\|\Lambda T(t)a|A\| \le N\|a\,|A\|, 0 < t < \infty, a \in A$.

See [5,31,33] for more details on semi-groups of operators.

Consider the following modulus of continuity of order $m \in \mathbb{N}$

$$\bar{\omega}_m(t^m, a) = \sup_{0\le s\le t} \|[T(s) - \mathrm{id}]^m a|A\|.$$

This modulus is related to the K-functional of the couple $A, D(\Lambda^m)$. Indeed, it is shown in [306, Proposition 3.4.1] that

$$K\big(t^m, a; A, D(\Lambda^m)\big) \lesssim \bar{\omega}_m(t^m, a) + \min(1, t^m)\|a\,|A\| \tag{54}$$

and

$$\bar{\omega}_m(t^m, a) \lesssim K(t^m, a; A, D(\Lambda^m)). \tag{55}$$

On the other hand, if the semi-group is analytic and we consider the modified K-functional given by

$$\widetilde{K}(t, a) = \widetilde{K}(t, a; A, D(\Lambda^m)) = \inf_{a_1\in D(\Lambda^m)}\{\|a - a_1|A\| + t\|\Lambda^m a_1|A\|\},$$

then it follows from [191, Theorem 5.1] that

$$\widetilde{K}(t^m, a; A, D(\Lambda^m)) \sim \|[T(s) - \mathrm{id}]^m a|A\|. \tag{56}$$

Theorem (6.3.17) [313] Let A be a Banach space, let $\{T(t)\}_{t\ge 0}$ be a strongly continuous equi-bounded semi-group of operators in A, let $m \in \mathbb{N}, 0 < q \le \infty$ and $b \ge -1/q$. The quasi-norm

$$\|a\|_1 = \|a|A\| + \left(\int_0^1 \big((1 - \log t)^b \bar{\omega}_m(t^m, a)\big)^q \frac{dt}{t}\right)^{1/q}$$

is equivalent to the interpolation quasi-norm $\big\|\cdot|(A, D(\Lambda^m))_{(0,-b),q}\big\|$ on

$$(A, D(\Lambda^m))_{(0,-b),q}.$$

In addition, if the semi-group $\{T(t)\}_{t\ge 0}$ is analytic then

$$\|a\|_2 = \|a|A\| + \left(\int_0^1 ((1 - \log t)^b \|[T(t) - \mathrm{id}]^m a|A\|)^q \frac{dt}{t}\right)^{1/q}$$

is also an equivalent quasi-norm on $(A, D(\Lambda^m))_{(0,-b),q}$.

Proof. Making a change of variable and using (54), we obtain

$$\left\|a|(A,D(\Lambda^m))_{(0,-b),q}\right\| = \left(\int_0^1 \left((1-\log t)^b K(t,a)\right)^q \frac{dt}{t}\right)^{1/q} \sim \left(\int_0^1 \left((1-\log t)^b K(t^m,a)\right)^q \frac{dt}{t}\right)^{1/q}$$
$$\lesssim \left(\int_0^1 \left((1-\log t)^b \overline{\omega}_m(t^m,a)\right)^q \frac{dt}{t}\right)^{1/q} + \left(\int_0^1 ((1-\log t)^b t^m)^q \frac{dt}{t}\right)^{1/q} \|a|A\| \sim \|a\|_1.$$

To check the converse inequality, note that $(A,D(\Lambda^m))_{(0,-b),q} \hookrightarrow A$. Moreover, by (55), we have

$$\left(\int_0^1 \left((1-\log t)^b \overline{\omega}_m(t^m,a)\right)^q \frac{dt}{t}\right)^{1/q} \lesssim \left(\int_0^1 \left((1-\log t)^b K(t^m,a)\right)^q \frac{dt}{t}\right)^{1/q} \sim \left\|a|(A,D(\Lambda^m))_{(0,-b),q}\right\|.$$

Consequently, $\left\|a|(A,D(\Lambda^m))_{(0,-b),q}\right\| \sim \|a\|_1$.

Assume now that the semi-group $\{T(t)\}_{t\geq 0}$ is analytic. To complete the proof we first show that

$$K(t,a) \sim t\|a|A\| + \widetilde{K}(t,a), \qquad 0 < t \leq 1, \qquad a \in A. \tag{57}$$

Indeed, take any $a_1 \in D(\Lambda^m)$. Using the triangle inequality in A and that $t \leq 1$, we obtain

$$\begin{aligned} K(t,a) &\leq \|a-a_1|A\| + t\|a_1|A\| + t\|\Lambda^m a_1|A\| \\ &\leq 2\|a-a_1|A\| + t\|a|A\| + t\|\Lambda^m a_1|A\|. \end{aligned}$$

Taking the infimum over all $a_1 \in D(\Lambda^m)$ it follows that $K(t,a) \lesssim t\|a|A\| + \widetilde{K}(t,a)$. Conversely,

$$\begin{aligned} t\|a|A\| + \widetilde{K}(t,a) &\leq t\|a|A\| + \|a-a_1|A\| + t\|\Lambda^m a_1|A\| \\ &\leq t\|a-a_1|A\| + t\|a_1|A\| + \|a-a_1|A\| + t\|\Lambda^m a_1|A\| \\ &\leq 2\|a-a_1|A\| + t\|a_1|D(\Lambda^m)\|. \end{aligned}$$

Therefore, we derive (57).

Now (57) and (56) yield that $\|\cdot\|_2 \sim \left\|\cdot|(A,D(\Lambda^m))_{(0,-b),q}\right\|$. This completes the proof.

Next we specify Theorem (6.3.17) for the case of the Gauss–Weierstrass semi-group

$$W(t)f(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} f(y)dy = \left(e^{-t|\xi|^2}\hat{f}(\xi)\right)^{\vee}(x), t > 0, x \in \mathbb{R}^n,$$
$$W(0) = \mathrm{id}.$$

Basic information about the use of $\{W(t)\}_{t\geq 0}$ in connection with function spaces may be found in [62, Section 2.5.2, pp. 190–192]. See also [306, Section 4.3.2], [66, Section 2.6.4] and [312, Section 3.6.6]. The semi-group $\{W(t)\}_{t\geq 0}$ is analytic in L_p for $1 < p < \infty$ and its infinitesimal generator is the Laplacian operator $\Delta = \sum_{j=1}^n \partial^2/\partial x_j^2$. Hence, $D(\Lambda^m) = W_p^{2m}$ for $m = 1,2,\ldots$.

Since $\mathbf{B}_{p,q}^{0,b} = (L_p, W_p^{2m})_{(0,-b),q}$ (see [283, Theorem 3.1] or Theorem 3.4) as a direct consequence of Theorem (6.3.17) we obtain the following characterization of $\mathbf{B}_{p,q}^{0,b}$ by means of heat kernels.

Theorem (6.3.18) [313] Let $1 < p < \infty, 0 < q \leq \infty, b \geq -1/q$ and $m \in \mathbb{N}$. Then $f \in L_p$ belongs to $\mathbf{B}_{p,q}^{0,b}$ if, and only if,

$$\|f|\mathbf{B}_{p,q}^{0,b}\|_{(m)}^* = \|f|L_p\| + \left(\int_0^1 (1-\log t)^b \left\|[W(t)-\mathrm{id}]^m f|L_p\right\|^q \frac{dt}{t}\right)^{1/q}$$

is finite. Furthermore, $\left\|\cdot\left|\mathbf{B}_{p,q}^{0,b}\right\|_{(m)}^*$ is an equivalent quasi-norm in $\mathbf{B}_{p,q}^{0,b}$.

Comparing Theorem (6.3.18) with the corresponding result for classical Besov spaces $B^s_{p,q}$ given in (27) we observe that the truncated Littlewood–Paley construction does not appear this time.

We consider harmonic extensions, that is, the case of the Cauchy–Poisson semi-group

$$P(t)f(x) = c_n \int_{\mathbb{R}^n} \frac{t}{(|x-y|^2+t^2)^{\frac{n+1}{2}}} f(y)dy = \left(e^{-t|\xi|}\hat{f}(\xi)\right)^{\vee}(x), t > 0, x \in \mathbb{R}^n,$$

where $c_n \left\|(1+|x|^2)^{-(n+1)/2}\middle|L_1\right\| = 1$, with $P(0) = \mathrm{id}$. This is also an analytic semi-group in L_p for $1 < p < \infty$ (see [62, Section 2.5.3] and [66, Section 2.6.4]). The corresponding characterization reads as follows.

Theorem (6.3.19) [313] Let $1 < p < \infty, 0 < q \le \infty, b \ge -1/q$ and $m \in \mathbb{N}$. Then $f \in L_p$ belongs to $\mathbf{B}^{0,b}_{p,q}$ if, and only if,

$$\left\|f\middle|\mathbf{B}^{0,b}_{p,q}\right\|^{\diamondsuit}_{(m)} = \left\|f\middle|L_p\right\| + \left(\int_0^1 (1-\log t)^b \left\|[P(t)-\mathrm{id}]^m f\middle|L_p\right\|^q \frac{dt}{t}\right)^{1/q}$$

is finite. Furthermore, $\left\|\cdot\middle|\mathbf{B}^{0,b}_{p,q}\right\|^{\diamondsuit}_{(m)}$ is an equivalent quasi-norm in $\mathbf{B}^{0,b}_{p,q}$.

Proof. For the semi-group $\{P(t)\}_{t\ge 0}$ we have

$$\Lambda^{2m} f = (-1)^m \Delta^m f \quad \text{and} \quad D(\Lambda^{2m}) = W^{2m}_p.$$

Using again that $\mathbf{B}^{0,b}_{p,q} = (L_p, W^{2m}_p)_{(0,-b),q}$ and Theorem (6.3.17), we obtain that the wanted result holds for any even natural number m. To complete the proof, write

$$\|f\|_m = \left\|f\middle|L_p\right\| + \left(\int_0^1 (1-\log t)^b \left\|[P(t)-\mathrm{id}]^m f\middle|L_p\right\|^q \frac{dt}{t}\right)^{1/q}.$$

It suffices to show that for any $m \in \mathbb{N}$ the quasi-norms $\|\cdot\|_m$ and $\|\cdot\|_{m+1}$ are equivalent on $\mathbf{B}^{0,b}_{p,q}$.

Take any $f \in \mathbf{B}^{0,b}_{p,q}$. Using (56), we obtain

$$\left\|[P(t)-\mathrm{id}]^m f\middle|L_p\right\| \sim K(t^m, f; L_p, D(\Lambda^m)) \sim \sup_{0\le s\le t} \left\|[P(s)-\mathrm{id}]^m f\middle|L_p\right\| = \bar{\omega}_m(t^m, f).$$

By [266, (4.10)], we have that $\bar{\omega}_{m+1}(t^{m+1}, f) \lesssim \bar{\omega}_m(t^m, f)$ Hence

$$\|f\|_{m+1} \sim \left\|f\middle|L_p\right\| + \left(\int_0^1 \left((1-\log t)^b \bar{\omega}_{m+1}(t^{m+1}, f)\right)^q \frac{dt}{t}\right)^{1/q}$$

$$\lesssim \left\|f\middle|L_p\right\| + \left(\int_0^1 \left((1-\log t)^b \bar{\omega}_m(t^m, f)\right)^q \frac{dt}{t}\right)^{1/q} \sim \|f\|_m.$$

In order to establish the converse inequality, note that

$$\bar{\omega}_m(t^m, f) \lesssim t^m \int_t^\infty s^{-m-1} \bar{\omega}_{m+1}(s^{m+1}, f) ds$$

(see [310, Theorem 1.4, (1.7)]). Therefore

$$\|f\|_m \sim \left\|f\middle|L_p\right\| + \left(\int_0^1 \left((1-\log t)^b \bar{\omega}_m(t^m, f)\right)^q \frac{dt}{t}\right)^{1/q}$$

$$\lesssim \|f|L_p\| + \left(\int_0^1 \left((1-\log t)^b t^m \int_t^\infty \frac{\overline{\omega}_{m+1}(s^{m+1},f)}{s^{m+1}}ds\right)^q \frac{dt}{t}\right)^{1/q}$$

$$\sim \|f|L_p\| + \left(\int_0^1 \left((1-\log t)^b t^m \int_t^1 \frac{\overline{\omega}_{m+1}(s^{m+1},f)}{s^{m+1}}ds\right)^q \frac{dt}{t}\right)^{1/q}$$

$$+ \left(\int_0^1 \left((1-\log t)^b t^m \int_1^\infty \frac{\overline{\omega}_{m+1}(s^{m+1},f)}{s^{m+1}}ds\right)^q \frac{dt}{t}\right)^{1/q} = \|f|L_p\| + J_1 + J_2.$$

Since $\overline{\omega}_{m+1}(s^{m+1},f)/s^{m+1}$ is equivalent to the decreasing function $\widetilde{K}(s^{m+1},f)/s^{m+1}$, we can still apply the extension of the Hardy inequality established in [256, Theorem 6.4] to derive that

$$J_1 \lesssim \left(\int_0^1 \left(t^{m+1}(1-\log t)^b \frac{\overline{\omega}_{m+1}(t^{m+1},f)}{t^{m+1}}\right)^q \frac{dt}{t}\right)^{1/q}$$

$$\sim \left(\int_0^1 (1-\log t)^b \left\|[P(t)-\mathrm{id}]^{m+1} f|L_p\right\|^q \frac{dt}{t}\right)^{1/q} \leq \|f\|_{m+1}.$$

As for J_2, using that

$$\overline{\omega}_{m+1}(s^{m+1},f) \sim \widetilde{K}(s^{m+1},f;L_p,D(\Lambda^{m+1})) \leq \|f|L_p\|$$

we get

$$J_2 \lesssim \left(\int_0^1 ((1-\log t)^b t^m)^q \frac{dt}{t}\right)^{1/q} \|f|L_p\| \lesssim \|f\|_{m+1}.$$

This yields that $\|f\|_m \lesssim \|f\|_{m+1}$ and completes the proof.

Comparing (27) with Theorem (6.3.18), one might think that the counterpart of the quasi-norm (26) for logarithmically perturbed Besov spaces is given by replacing in (26) the term $t^{-\frac{s}{2}q}$ by $(1-\log t)^{bq}$. However, the involved spaces are not $\mathbf{B}^{0,b}_{p,q}$ but $B^{0,b}_{p,q}$. We shall need the spaces $B^{s,b}_{p,q}, s>0$, introduced in (42) and the logarithmic interpolation spaces $(A_0,A_1)_{\theta,q,b}$ formed by all those $a \in A_0 + A_1$ such that the quasi-norm

$$\|a|(A_0,A_1)_{\theta,q,b}\| = \left(\int_0^\infty \left(t^{-\theta}(1+|\log t|)^b K(t,a)\right)^q \frac{dt}{t}\right)^{1/q}$$

is finite. Here $0<\theta<1, 0<q\leq\infty$ and $b \in \mathbb{R}$. See [272, 264].

The following result refers to abstract semi-groups.

Theorem (6.3.20) [313] Let A be a Banach space, let $\{T(t)\}_{t\geq 0}$ be an analytic semi-group of operators in A, let $0<s/2<m \in \mathbb{N}, 0<q\leq\infty$ and $b \in \mathbb{R}$. The quasi-norm

$$\|a\|_3 = \|a|A\| + \left(\int_0^1 \left(t^{m-\frac{s}{2}}(1-\log t)^b \|\Lambda^m T(t)a|A\|\right)^q \frac{dt}{t}\right)^{1/q}$$

is equivalent to the interpolation quasi-norm on $(A,D(\Lambda^m))_{s/2m,q,b}$.

Proof. Since $\|a|A\| \leq \|a|D(\Lambda^m)\|$ for any $a \in D(\Lambda^m)$, we have that $K(t,a) = \|a|A\|$ for any $a \in A$ and $t \geq 1$. This yields that

$$\left\|a|\left(A,D(\Lambda^m)\right)_{s/2m,q,b}\right\| \sim \|a|A\| + \left(\int_0^1 \left(t^{-s/2m}(1-\log t)^b K(t,a)\right)^q \frac{dt}{t}\right)^{1/q}$$
$$\sim \|a|A\| + \left(\int_0^1 \left(t^{-s/2}(1-\log t)^b K(t^m,a)\right)^q \frac{dt}{t}\right)^{1/q}.$$

Using that

$$t^m\|\Lambda^m T(t)a|A\| \lesssim K(t^m,a) \tag{58}$$

(see [306, Lemma 3.5.4]), we get that $\|a\|_3 \lesssim \left\|a|(A,D(\Lambda^m))_{s/2m,q,b}\right\|$. To check the converse inequality, note that

$$\|[T(t)-\mathrm{id}]^m a|A\| \lesssim \int_0^t \tau^{m-1}\|\Lambda^m T(\tau)a|A\| d\tau \tag{59}$$

(see [306, Lemma 3.5.5]). Hence, using (54), we obtain for $0<t<1$

$$K(t^m,a) \lesssim \bar{\omega}_m(t^m,a) + t^m\|a|A\| \lesssim \int_0^t \tau^{m-1}\|\Lambda^m T(\tau)a|A\| d\tau + t^m\|a|A\|.$$

This implies that

$$\left\|a|(A,D(\Lambda^m))_{s/2m,q,b}\right\| \lesssim \|a|A\| + \left(\int_0^1 \left(t^{-s/2}(1-\log t)^b \int_0^t \tau^{m-1}\|\Lambda^m T(\tau)a|A\| d\tau\right)^q \frac{dt}{t}\right)^{1/q}.$$

By (iii), we have for $0<\tau<\mu$ that

$$\|\Lambda^m T(\mu)a|A\| = \|T(\mu-\tau)\Lambda^m T(\tau)a|A\| \le M\|\Lambda^m T(\tau)a|A\|.$$

Hence, using the extension of the Hardy inequality established in [256, Theorem 6.4], we derive that

$$\left\|a|(A,D(\Lambda^m))_{s/2m,q,b}\right\| \lesssim \|a|A\| + \left(\int_0^1 \left(t^{1-\frac{s}{2}}(1-\log t)^b t^{m-1}\|\Lambda^m T(t)a|A\|\right)^q \frac{dt}{t}\right)^{1/q} = \|a\|_3.$$

Next we apply Theorem (6.3.20) to the Gauss–Weierstrass semi-group $\{W(t)\}_{t\ge 0}$. Subsequently I_s stands for the usual lift operator defined by

$$I_s f = \left((1+|x|^2)^{s/2}\hat{f}\right)^{\vee}, s \in \mathbb{R}.$$

Theorem (6.3.21) [313] Let $1<p<\infty, 0<q\le\infty, b\in\mathbb{R}$ and $m\in\mathbb{N}$. Then

$$\|f|B^{0,b}_{p,q}\|_{(m)} = \|I_{-2}f|L_p\| + \left(\int_0^1 \left(t^m(1-\log t)^b \left\|\frac{\partial^m W(t)f}{\partial t^m}|L_p\right\|\right)^q \frac{dt}{t}\right)^{1/q}$$

is an equivalent quasi-norm on $B^{0,b}_{p,q}$.

Proof. According to [47, Proposition 1.8], the operator I_{-2} is an isomorphism from $B^{0,b}_{p,q}$ onto $B^{2,b}_{p,q}$. The classical smoothness of $B^{2,b}_{p,q}$ is $2>0$ so, by [28, Theorem 2.5], we know that

$$\|f|B^{2,b}_{p,q}\| \sim \|f|L_p\| + \left(\int_0^1 \left(t^{-2}(1-\log t)^b \omega_{2(m+1)}(f,t)_p\right)^q \frac{dt}{t}\right)^{1/q}. \tag{60}$$

Moreover, for $f\in L_p$ and $t>0$, we have

$$K(t^{2(m+1)}, f; L_p, W_p^{2(m+1)}) \sim \min(1, t^{2(m+1)})\|f|L_p\| + \omega_{2(m+1)}(f,t)_p \tag{61}$$
(see [1, Theorem 5.4.12]). It follows from (60) and (61) that
$$B_{p,q}^{2,b} = (L_p, W_p^{2(m+1)})_{1/(m+1),q,b}, \quad \text{so} \quad B_{p,q}^{2,b} = (L_p, D(\Lambda^{m+1}))_{1/(m+1),q,b} \tag{62}$$
where $\Lambda = \Delta$ is the infinitesimal generator of the semi-group $\{W(t)\}_{t\geq 0}$. Applying Theorem (6.3.20) we get
$$\|f|B_{p,q}^{0,b}\| \sim \|I_{-2}f|B_{p,q}^{2,b}\| \sim \|I_{-2}f|L_p\| + \left(\int_0^1 \left(t^m(1-\log t)^b\|\Delta^{m+1}W(t)I_{-2}f|L_p\|\right)^q \frac{dt}{t}\right)^{1/q}.$$
Next we use that
$$\Delta^{m+1}W(t)I_{-2}f = \Delta^{m+1}I_{-2}W(t)f = \Delta I_{-2}\Delta^m W(t)f$$
and that, according to [56, p. 133],
$$\|I_{-2}g|L_p\| = \|\Delta(\mathrm{id}-\Delta)^{-1}g|L_p\| \sim \|g|L_p\|.$$
This yields that
$$\|f|B_{p,q}^{0,b}\| \sim \|I_{-2}f|L_p\| + \left(\int_0^1 \left(t^m(1-\log t)^b \left\|\frac{\partial^m W(t)f}{\partial t^m}\Big|L_p\right\|\right)^q \frac{dt}{t}\right)^{1/q}.$$
Note that the operator I_{-2} is necessary because in general $B_{p,q}^{0,b}$ may not contain only regular distributions (see [169, Theorem 4.3]).

Finally we consider the case of the Cauchy–Poisson semi-group $\{P(t)\}_{t\geq 0}$.

Theorem (6.3.22) [313] Let $1 < p < \infty, 0 < q \leq \infty, b \in \mathbb{R}$ and $m \in \mathbb{N}$. Then
$$\|f|B_{p,q}^{0,b}\|_{(m)}^{\diamondsuit} = \|I_{-2}f|L_p\| + \left(\int_0^1 \left(t^m(1-\log t)^b \left\|\frac{\partial^m P(t)f}{\partial t^m}\Big|L_p\right\|\right)^q \frac{dt}{t}\right)^{1/q}$$
is an equivalent quasi-norm on $B_{p,q}^{0,b}$.

Proof. This time $\Lambda^{2(m+1)}f = (-1)^{m+1}\Delta^{m+1}f$ and $D(\Lambda^{2(m+1)}) = W_p^{2(m+1)}$. By (62), we get
$$B_{p,q}^{2,b} = (L_p, W_p^{2(m+1)})_{1/(m+1),q,b} = (L_p, D(\Lambda^{m+1}))_{1/(m+1),q,b}.$$
Therefore, applying Theorem (6.3.20) with $s = 4$, we obtain
$$\|f|B_{p,q}^{2,b}\| \sim \|f|L_p\| + \left(\int_0^1 \left(t^{2(m+1)-2}(1-\log t)^b\|\Lambda^{2(m+1)}P(t)f|L_p\|\right)^q \frac{dt}{t}\right)^{1/q}.$$
This means that for any even natural number m with $m \geq 4$ we have
$$\|f|B_{p,q}^{2,b}\| \sim \|f|L_p\| + \left(\int_0^1 \left(t^{m-2}(1-\log t)^b\|\Lambda^m P(t)f|L_p\|\right)^q \frac{dt}{t}\right)^{1/q}. \tag{63}$$
Write $\|f\|_m^{\diamondsuit}$ for the quasi-norm on the right-hand side of (63). We claim that for any $m \in \mathbb{N}$ with $m > 2$ we have
$$\|\cdot\|_m^{\diamondsuit} \sim \|\cdot\|_{m+1}^{\diamondsuit} \quad \text{on} \quad B_{p,q}^{2,b}. \tag{64}$$
Indeed, by (vi), given any $f \in L_p$ we have
$$\|\Lambda^{m+1}P(t)f|L_p\| = \|\Lambda P(t/2)\Lambda^m P(t/2)f|L_p\| \lesssim t^{-1}\|\Lambda^m P(t/2)f|L_p\|.$$
Whence
$$\|f\|_{m+1} \lesssim \|f|L_p\| + \left(\int_0^1 \left(t^{m-2}(1-\log t)^b\|\Lambda^m P(t/2)f|L_p\|\right)^q \frac{dt}{t}\right)^{1/q} \sim \|f\|_{m+1}$$

$$\lesssim \|f|L_p\| + \left(\int_0^{1/2} \left(t^{m-2}(1-\log t)^b \|\Lambda^m P(t) f|L_p\|\right)^q \frac{dt}{t}\right)^{1/q} \le \|f\|_m^\diamond.$$

Conversely, by (58) and (57)

$$\|f\|_m^\diamond \lesssim \|f|L_p\| + \left(\int_0^1 \left(t^{-2}(1-\log t)^b K\left(t^m, f; L_p, D(\Lambda^m)\right)\right)^q \frac{dt}{t}\right)^{1/q}$$

$$\sim \|f|L_p\| + \left(\int_0^1 \left(t^{-2}(1-\log t)^b \widetilde{K}\left(t^m, f; L_p, D(\Lambda^m)\right)\right)^q \frac{dt}{t}\right)^{1/q}.$$

As we have seen in the proof of Theorem (6.3.19),

$$\widetilde{K}\left(t^m, f; L_p, D(\Lambda^m)\right) \sim \overline{\omega}_m(t^m, f).$$

Moreover, by [310, Theorem 1.4, (1.7)]

$$\overline{\omega}_m(t^m, f) \lesssim t^m \int_t^\infty \frac{\overline{\omega}_{m+1}(s^{m+1}, f)}{s^{m+1}} ds\,.$$

Now proceeding as in the proof of Theorem (6.3.19), using that $m > 2$ and the extension of the Hardy inequality [256, Theorem 6.4], we obtain

$$\|f\|_m \lesssim \|f|L_p\| + \left(\int_0^1 \left(t^{m-2}(1-\log t)^b \int_t^\infty \frac{\overline{\omega}_{m+1}(s^{m+1}, f)}{s^{m+1}} ds\right)^q \frac{dt}{t}\right)^{1/q}$$

$$\lesssim \|f|L_p\| + \left(\int_0^1 \left(t^{m-2}(1-\log t)^b \int_t^1 \frac{\overline{\omega}_{m+1}(s^{m+1}, f)}{s^{m+1}} ds\right)^q \frac{dt}{t}\right)^{1/q}$$

$$+ \left(\int_0^1 (t^{m-2}(1-\log t)^b)^q \frac{dt}{t}\right)^{1/q} \|f|L_p\|$$

$$\lesssim \|f|L_p\| + \left(\int_0^1 \left(t^{m-1}(1-\log t)^b \frac{\overline{\omega}_{m+1}(t^{m+1}, f)}{t^{m+1}}\right)^q \frac{dt}{t}\right)^{1/q}$$

$$\sim \|f|L_p\| + \left(\int_0^1 \left(t^{-2}(1-\log t)^b \|[P(t) - \mathrm{id}]^{m+1} f|L_p\|\right)^q \frac{dt}{t}\right)^{1/q}$$

$$\lesssim \|f|L_p\| + \left(\int_0^1 \left(t^{-2}(1-\log t)^b \int_0^t s^m \|\Lambda^{m+1} P(s) f|L_p\|\right)^q \frac{dt}{t}\right)^{1/q}$$

where we have used (59) in the last inequality. The extension of the Hardy inequality implies now that

$$\|f\|_m^\diamond \lesssim \|f|L_p\| + \left(\int_0^1 \left(t^{-1}(1-\log t)^b t^m \|\Lambda^{m+1} P(t) f|L_p\|\right)^q \frac{dt}{t}\right)^{1/q} = \|f\|_{m+1}^\diamond.$$

This proves (64).

Now to complete the proof of the theorem we can proceed as in Theorem (6.3.21) with the help of the lift operator I_{-2}. Indeed, given any natural number $m > 2$, since

$$\Lambda^m P(t) I_{-2} f = \Lambda^2 \Lambda^{m-2} I_{-2} P(t) f = -\Delta I_{-2} \Lambda^{m-2} P(t) f,$$

by (63) and (64) we obtain

$$\left\| f \middle| B_{p,q}^{0,b} \right\| \sim \left\| I_{-2} f \middle| B_{p,q}^{2,b} \right\| \sim \| I_{-2} f \|_m$$

$$= \left\| I_{-2} f \middle| L_p \right\| + \left(\int_0^1 \left(t^{m-2} (1 - \log t)^b \left\| \Delta I_{-2} \Lambda^{m-2} P(t) f \middle| L_p \right\| \right)^q \frac{dt}{t} \right)^{1/q}$$

$$\sim \left\| I_{-2} f \middle| L_p \right\| + \left(\int_0^1 \left(t^{m-2} (1 - \log t)^b \left\| \Lambda^{m-2} P(t) f \middle| L_p \right\| \right)^q \frac{dt}{t} \right)^{1/q}$$

$$= \left\| I_{-2} f \middle| L_p \right\| + \left(\int_0^1 \left(t^{m-2} (1 - \log t)^b \left\| \frac{\partial^{m-2} P(t) f}{\partial t^{m-2}} \middle| L_p \right\| \right)^q \frac{dt}{t} \right)^{1/q}.$$

This finishes the proof.

Corollary (6.3.23) [314] Let A_{r-1}, A_r be Banach spaces with $A_r \hookrightarrow A_{r-1}$. Let $0 \le \epsilon < \infty$ and $(\epsilon^2 + \epsilon - 1)/1 + \epsilon \in \mathbb{R}$ with $\epsilon \ge 0$. Then we have with equivalence of norms

$$\left(L_{1+\epsilon}(A_{r-1}), L_{1+\epsilon}(A_r) \right)_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon} = L_{1+\epsilon}\left((A_{r-1}, A_r)_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon} \right).$$

Proof. Consider the collection S of all functions $v_{r-2}(x_n) = \sum_{j=1}^N a_{j+r-1} \chi_{\Omega_{j+r-1}}(x_n)$, where $N \in \mathbb{N}$, $a_{j+r-1} \in A_r$, the measure of $\Omega_{j+r-1} \subset \mathbb{R}^n$ is finite and $\Omega_{j+r-1} \cap \Omega_{1+2\epsilon} = \emptyset$ if $j \ne 1 + 2\epsilon$. By (38), the set S is dense in $\left(L_{1+\epsilon}(A_{r-1}), L_{1+\epsilon}(A_r) \right)_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon}$ and in $L_{1+\epsilon}\left((A_{r-1}, A_r)_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon} \right)$. For $v_{r-2} \in S$ we have

$$\left\| v_{r-2} \middle| \left(L_{1+\epsilon}(A_{r-1}), L_{1+\epsilon}(A_r) \right)_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon} \right\|^{1+\epsilon}$$

$$\sim \int_0^1 \left((1 - \log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon} \inf_{\substack{v_{r-2} = v_{r-1} + v_r \\ v_{j+r-1} \in L_{1+\epsilon}(A_{j+r-1})}} \{ \| v_{r-1} | L_{1+\epsilon}(A_{r-1}) \|^{1+\epsilon} + (1+\epsilon)^{1+\epsilon} \| v_r | L_{1+\epsilon}(A_r) \|^{1+\epsilon} \}^{1/1+\epsilon} \right)^{1+\epsilon} \frac{d(1+\epsilon)}{(1+\epsilon)}$$

$$= \int_0^1 (1 - \log(1+\epsilon))^{\epsilon^2+\epsilon-1} \int_{\mathbb{R}^n} \inf_{\substack{v_{r-2}(x_n) = v_{r-1}(x_n) + v_r(x_n) \\ v_{j+r-1}(x_n) \in A_{j+r-1}}} \left(\| v_{r-1}(x_n) | A_{r-1} \|^{1+\epsilon} + (1+\epsilon)^{1+\epsilon} \| v_r(x_n) | A_r \|^{1+\epsilon} \right) dx_n \frac{d(1+\epsilon)}{(1+\epsilon)}$$

$$= \int_{\mathbb{R}^n} \int_0^1 (1 - \log(1+\epsilon))^{\epsilon^2+\epsilon-1} K_{1+\epsilon}(1+\epsilon, v_{r-2}(x_n); A_{r-1}, A_r)^{1+\epsilon} \frac{d(1+\epsilon)}{(1+\epsilon)} dx_n$$

$$\sim \int_{\mathbb{R}^n} \left\| v_{r-2}(x_n) \middle| (A_{r-1}, A_r)_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon} \right\|^{1+\epsilon} dx_n = \left\| v_{r-2} \middle| L_{1+\epsilon}\left((A_{r-1}, A_r)_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon} \right) \right\|^{1+\epsilon}.$$

This completes the proof.

Next consider the sequence space ℓ_2 on $\mathbb{N}_0$ and for $\lambda > 0$.

$$\ell_2^\lambda = \left\{ \xi = (\xi_j)_{j \in \mathbb{N}_0} : \left\| \xi \middle| \ell_2^\lambda \right\| = \left(\sum_{j=0}^\infty (2^{\lambda j} |\xi_j|)^2 \right)^{1/2} < \infty \right\}.$$

If A_{r-2} is a Banach space, the vector-valued sequence space $\ell_2^\lambda(A_{r-2})$ is defined by

$$\ell_2^\lambda(A_{r-2}) = \left\{ x_n = ((x_n)_j)_{j \in \mathbb{N}_0} \subset A_{r-2} : \left\| x_n \middle| \ell_2^\lambda(A_{r-2}) \right\| = \left(\sum_{j=0}^\infty (2^{\lambda j} \| (x_n)_j | A_{r-2} \|)^2 \right)^{1/2} < \infty \right\}$$

Corollary (6.3.24) [314] Let $-1 < \epsilon < \infty$ and $(\epsilon^2 + \epsilon - 1)/1 + \epsilon \in \mathbb{R}$. Then we have with equivalence of quasi-norms

$$(\ell_2, \ell_2^\lambda)_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon} = \left\{ \xi = (\xi_j) : \| \xi \| = \left(\sum_{j=0}^\infty \left[(1+j)^{(\epsilon^2+\epsilon-1)/1+\epsilon} (\sum_{\nu=j}^\infty |\xi_\nu|^2)^{1/2} \right]^{1+\epsilon} \right)^{1/1+\epsilon} < \infty \right\}$$

and $\|\cdot\|$ is an equivalent quasi-norm on $(\ell_2,\ell_2^\lambda)_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon}$.

Proof. Consider first the case $\lambda=1$. Since

$$K_2(1+\epsilon,\xi;\ell_2,\ell_2^1)\sim\left(\sum_{\nu=0}^{\infty}(\min(1,(1+\epsilon)2^\nu)|\xi_\nu|)^2\right)^{1/2},$$

we have for $j\geq 0$ that

$$K_2(2^{-j},\xi;\ell_2,\ell_2^1)\sim\left(\sum_{\nu=0}^{j}(2^{\nu-j}|\xi_\nu|)^2\right)^{1/2}+\left(\sum_{\nu=j+1}^{\infty}|\xi_\nu|^2\right)^{1/2}.$$

Hence

$$\left\|\xi\left|(\ell_2,\ell_2^\lambda)_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon}\right.\right\|\sim\left(\sum_{j=0}^{\infty}\left[(1+j)^{(\epsilon^2+\epsilon-1)/1+\epsilon}K_2(2^{-j},\xi)\right]^{1+\epsilon}\right)^{1/1+\epsilon}$$

$$\sim\left(\sum_{j=0}^{\infty}\left[(1+j)^{(\epsilon^2+\epsilon-1)/1+\epsilon}(\sum_{\nu=0}^{j}(2^{\nu-j}|\xi_\nu|)^2)^{1/2}\right]^{1+\epsilon}\right)^{1/1+\epsilon}+\left(\sum_{j=0}^{\infty}\left[(1+j)^{(\epsilon^2+\epsilon-1)/1+\epsilon}(\sum_{\nu=j}^{\infty}|\xi_\nu|^2)^{1/2}\right]^{1+\epsilon}\right)^{1/1+\epsilon}.$$

In the last expression, the first term is dominated by the second term. Indeed, if $(1+\epsilon)/2\leq 1$, we obtain

$$\left(\sum_{j=0}^{\infty}\left[(1+j)^{2(\epsilon^2+\epsilon-1)/1+\epsilon}\sum_{\nu=0}^{j}\left(2^{\nu-j}|\xi_\nu|\right)^2\right]^{(1+\epsilon)/2}\right)^{1/1+\epsilon}$$

$$\leq\left(\sum_{j=0}^{\infty}(1+j)^{\epsilon^2+\epsilon-1}2^{-(1+\epsilon)j}\sum_{\nu=0}^{j}2^{(1+\epsilon)\nu}|\xi_\nu|^{1+\epsilon}\right)^{1/1+\epsilon}$$

$$=\left(\sum_{\nu=0}^{\infty}2^{(1+\epsilon)\nu}|\xi_\nu|^{1+\epsilon}\sum_{j=\nu}^{\infty}(1+j)^{\epsilon^2+\epsilon-1}2^{-(1+\epsilon)j}\right)^{1/1+\epsilon}$$

$$\lesssim\left(\sum_{\nu=0}^{\infty}(1+\nu)^{\epsilon^2+\epsilon-1}|\xi_\nu|^{1+\epsilon}\right)^{1/1+\epsilon}\leq\left(\sum_{j=0}^{\infty}\left[(1+j)^{(\epsilon^2+\epsilon-1)/1+\epsilon}(\sum_{\nu=j}^{\infty}|\xi_\nu|^2)^{1/2}\right]^{1+\epsilon}\right)^{1/1+\epsilon}.$$

If $1<(1+\epsilon)/2$ we use the variant of Hardy's inequality given in [309, Lemma 3.1] (see also [299, Lemma 3.10]). We get

$$\left(\sum_{j=0}^{\infty}\left[(1+j)^{(\epsilon^2+\epsilon-1)/1+\epsilon}2^{-j}\left(\sum_{\nu=0}^{j}(2^\nu|\xi_\nu|)^2\right)^{1/2}\right]^{1+\epsilon}\right)^{1/1+\epsilon}=\left(\sum_{j=0}^{\infty}\left[(1+j)^{2(\epsilon^2+\epsilon-1)/1+\epsilon}2^{-2j}\sum_{\nu=0}^{j}(2^\nu|\xi_\nu|)^2\right]^{(1+\epsilon)/2}\right)^{1/1+\epsilon}$$

$$\lesssim\left(\sum_{j=0}^{\infty}\left[(1+j)^{(\epsilon^2+\epsilon-1)/1+\epsilon}2^{-j}2^j|\xi_j|\right]^{1+\epsilon}\right)^{1/1+\epsilon}\leq\left(\sum_{j=0}^{\infty}\left[(1+j)^{(\epsilon^2+\epsilon-1)/1+\epsilon}\left(\sum_{\nu=j}^{\infty}|\xi_\nu|^2\right)^{1/2}\right]^{1+\epsilon}\right)^{1/1+\epsilon}.$$

Consequently,

$$\left\|\xi\left|(\ell_2,\ell_2^1)_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon}\right.\right\|\sim\left(\sum_{j=0}^{\infty}\left[(1+j)^{(\epsilon^2+\epsilon-1)/1+\epsilon}\left(\sum_{\nu=j}^{\infty}|\xi_\nu|^2\right)^{1/2}\right]^{1+\epsilon}\right)^{1/1+\epsilon}.$$

Corollary (6.3.25) [314] Let $0\leq\epsilon\leq\infty$ and $(1+2\epsilon)\in\mathbb{N}$. Then

$$\left\|f^2\left|\mathbf{B}_{1+\epsilon,1+\epsilon}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}\right.\right\|_{1+2\epsilon}=\|f^2|L_{1+\epsilon}\|+\left(\int_{|h^2|\leq 1}(1-\log|h^2|)^{\epsilon^2+\epsilon-1}\left\|\Delta_{h^2}^{1+2\epsilon}f^2|L_{1+\epsilon}\right\|^{1+\epsilon}\frac{dh^2}{|h^2|^n}\right)^{1/1+\epsilon}$$

is an equivalent quasi-norm on $\mathbf{B}_{1+\epsilon,1+\epsilon}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}$.

Proof. We shall use that

$$\omega_{1+2\epsilon}(f^2,1+\epsilon)_{1+\epsilon}\sim\left((1+\epsilon)^{-n}\int_{|h^2|\le 1+\epsilon}\left\|\Delta_{h^2}^{1+2\epsilon}f^2|L_{1+\epsilon}\right\|^{1+\epsilon}dh^2\right)^{1/1+\epsilon}$$

(see [33, (2.4) and Appendix A]). We have

$$\left\|f^2\left|\mathbf{B}_{1+\epsilon,1+\epsilon}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}\right\|_{(1+2\epsilon)+}=\|f^2|L_{1+\epsilon}\|+\left(\int_0^1\left[(1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}\omega_{1+2\epsilon}(f^2,1+\epsilon)_{1+\epsilon}\right]^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}$$

$$\sim\|f^2|L_{1+\epsilon}\|+\left(\int_0^1(1-\log(1+\epsilon))^{\epsilon^2+\epsilon-1}(1+\epsilon)^{-n}\int_{|h^2|\le 1+\epsilon}\left\|\Delta_{h^2}^{1+2\epsilon}f^2|L_{1+\epsilon}\right\|^{1+\epsilon}dh^2\,\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}$$

$$=\|f^2|L_{1+\epsilon}\|+\left(\int_{|h^2|\le 1}\left\|\Delta_{h^2}^{1+2\epsilon}f^2|L_{1+\epsilon}\right\|^{1+\epsilon}\int_{|h^2|}^1(1+\epsilon)^{-n}(1-\log(1+\epsilon))^{\epsilon^2+\epsilon-1}\frac{d(1+\epsilon)}{(1+\epsilon)}dh^2\right)^{1/1+\epsilon}$$

$$\lesssim\|f^2|L_{1+\epsilon}\|+\left(\int_{|h^2|\le 1}\left\|\Delta_{h^2}^{1+2\epsilon}f^2|L_{1+\epsilon}\right\|^{1+\epsilon}|h^2|^{-n}(1-\log|h^2|)^{\epsilon^2+\epsilon-1}dh^2\right)^{1/1+\epsilon}=\left\|f^2\left|\mathbf{B}_{1+\epsilon,1+\epsilon}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}\right\|_{1+2\epsilon}.$$

Conversely,

$$\left\|f^2\left|\mathbf{B}_{1+\epsilon,1+\epsilon}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}\right\|_{1+2\epsilon}\le\|f^2|L_{1+\epsilon}\|+\left(\int_{|h^2|\le 1}(1-\log|h^2|)^{\epsilon^2+\epsilon-1}\omega_{1+2\epsilon}(f^2,|h^2|)_{1+\epsilon}^{1+\epsilon}\frac{dh^2}{|h^2|^n}\right)^{1/1+\epsilon}$$

$$=\|f^2|L_{1+\epsilon}\|+\left(\sum_{j=0}^{\infty}\int_{2^{-j-1}<|h^2|\le 2^{-j}}(1-\log|h^2|)^{\epsilon^2+\epsilon-1}\omega_{1+2\epsilon}(f^2,|h^2|)_{1+\epsilon}^{1+\epsilon}\frac{dh^2}{|h^2|^n}\right)^{1/1+\epsilon}$$

$$\lesssim\|f^2|L_{1+\epsilon}\|+\left(\sum_{j=0}^{\infty}(1+j)^{\epsilon^2+\epsilon-1}\omega_{1+2\epsilon}\left(f^2,2^{-j}\right)_{1+\epsilon}^{1+\epsilon}\right)^{1/1+\epsilon}$$

$$\lesssim\|f^2|L_{1+\epsilon}\|+\left(\int_0^1(1-\log(1+\epsilon))^{\epsilon^2+\epsilon-1}\omega_{1+2\epsilon}(f^2,1+\epsilon)_{1+\epsilon}^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}$$

$$=\left\|f^2\left|\mathbf{B}_{1+\epsilon,1+\epsilon}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}\right\|_{(1+2\epsilon)+}.$$

Corollary (6.3.26) [314] Let $0\le\epsilon\le\infty$. Then we have

$$(L_{1+\epsilon})_{1+\epsilon}^{(0,(\epsilon^2+\epsilon-1)/1+\epsilon)}=\mathbf{B}_{1+\epsilon,1+\epsilon}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}.$$

Proof. Take any $\epsilon<1$. By [65, Theorem 2.5.3] (see also [62, Theorem 2.5.4]), we know that $(L_{1+\epsilon})_{1+\epsilon}^{1-\epsilon}=B_{1+\epsilon,1+\epsilon}^{1-\epsilon}$. Whence, using [305, Corollary 3.5 and Lemma 2.2/(b)], we derive

$$(L_{1+\epsilon})_{1+\epsilon}^{(0,(\epsilon^2+\epsilon-1)/1+\epsilon)}=(L_{1+\epsilon},(L_{1+\epsilon})_{1+\epsilon}^{1-\epsilon})_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon}=\left(L_{1+\epsilon},B_{1+\epsilon,1+\epsilon}^{1-\epsilon}\right)_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon}$$

$$=\left(L_{1+\epsilon},(L_{1+\epsilon},W_{1+\epsilon}^1)_{1-\epsilon,1+\epsilon}\right)_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon}=(L_{1+\epsilon},W_{1+\epsilon}^1)_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon}=\mathbf{B}_{1+\epsilon,1+\epsilon}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}$$

where the last equality follows from [283, Theorem 3.1].

We have used that $\mathbf{B}_{1+\epsilon,1+\epsilon}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}=(L_{1+\epsilon},W_{1+\epsilon}^{1+2\epsilon})_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon}$ [283, Theorem 3.1]. We can show now that in this interpolation formula the Sobolev space $W_{1+\epsilon}^{1+2\epsilon}$ can be replaced by any fractional Sobolev space $H_{1+\epsilon}^{1+\epsilon}=F_{1+\epsilon,2}^{1+\epsilon}$, any Triebel–Lizorkin space $F_{1+\epsilon,v_{r-2}+\epsilon}^{1+\epsilon}$, or any Besov space $B_{1+\epsilon,v_{r-2}+\epsilon}^{1+\epsilon}$.

Corollary (6.3.27) [314] Let $0\le\epsilon\le\infty$ and $-(v_{r-2})<\epsilon\le\infty$. For $A_{1+\epsilon,v_{r-2}+\epsilon}^{1+\epsilon}=F_{1+\epsilon,v_{r-2}+\epsilon}^{1+\epsilon}$ or $B_{1+\epsilon,v_{r-2}+\epsilon}^{1+\epsilon}$ we have with equivalence of quasi-norms $\mathbf{B}_{1+\epsilon,1+\epsilon}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}=\left(L_{1+\epsilon},A_{1+\epsilon,v_{r-2}+\epsilon}^{1+\epsilon}\right)_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon}$.

Proof. By [65, Proposition 2, p. 47], we have

$$B_{1+\epsilon,\min(1+\epsilon,v_{r-2}+\epsilon)}^{1+\epsilon}\hookrightarrow F_{1+\epsilon,v_{r-2}+\epsilon}^{1+\epsilon}\hookrightarrow B_{1+\epsilon,\max(1+\epsilon,v_{r-2}+\epsilon)}^{1+\epsilon}.$$

Hence, using [65, Theorem 2.5.3], [305, Corollary 3.5] and Lemma 3.3, we derive

$$\left(L_{1+\epsilon}, A^{1+\epsilon}_{1+\epsilon,v_{r-2}+\epsilon}\right)_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon} \hookrightarrow \left(L_{1+\epsilon}, B^{1+\epsilon}_{1+\epsilon,\max(1+\epsilon,v_{r-2}+\epsilon)}\right)_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon}$$
$$= \left(L_{1+\epsilon}, (L_{1+\epsilon})^{1+\epsilon}_{\max(1+\epsilon,v_{r-2}+\epsilon)}\right)_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon} = (L_{1+\epsilon})^{(0,(\epsilon^2+\epsilon-1)/1+\epsilon)}_{1+\epsilon} = \mathbf{B}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon}.$$

For the converse embedding, we obtain

$$\mathbf{B}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon} = (L_{1+\epsilon})^{(0,(\epsilon^2+\epsilon-1)/1+\epsilon)}_{1+\epsilon} = \left(L_{1+\epsilon}, (L_{1+\epsilon})^{1+\epsilon}_{\min(1+\epsilon,v_{r-2}+\epsilon)}\right)_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon}$$
$$= \left(L_{1+\epsilon}, B^{1+\epsilon}_{1+\epsilon,\min(1+\epsilon,v_{r-2}+\epsilon)}\right)_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon} \hookrightarrow \left(L_{1+\epsilon}, A^{1+\epsilon}_{1+\epsilon,v_{r-2}+\epsilon}\right)_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon}.$$

Corollary (6.3.28) [314] Let $0 < \epsilon < \infty$. Then $f^2 \in L_{1+\epsilon}$ belongs to $\mathbf{B}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon}$ if and only if

$$\left\| \cdot \left| \mathbf{B}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon} \right\|_{\varphi+} = \left(\sum_{j=0}^{\infty} (1+j)^{\epsilon^2+\epsilon-1} \left\| \left(\sum_{\nu=j}^{\infty} \left| \left(\varphi_\nu^2 \hat{f}^2\right)^\vee(\cdot) \right|^2 \right)^{1/2} \middle| L_{1+\epsilon} \right\|^{1+\epsilon} \right)^{1/1+\epsilon} < \infty.$$

Moreover, $\left\| \cdot \left| \mathbf{B}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon} \right\|_{\varphi+}$ is an equivalent norm on $\mathbf{B}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon}$.

If $\epsilon = \left(-1 \pm \sqrt{5}\right)/2$ we obtain

$$\mathbf{B}^{0}_{1+\epsilon,1+\epsilon} = \left\{ f^2 \in L_{1+\epsilon} : \left\| f^2 \middle| \mathbf{B}^0_{1+\epsilon,1+\epsilon} \right\|_{\varphi+} = \left(\sum_{j=0}^{\infty} \left\| \left(\sum_{\nu=j}^{\infty} \left| \left(\varphi_\nu^2 \hat{f}^2\right)^\vee(\cdot) \right|^2 \right)^{1/2} \middle| L_{1+\epsilon} \right\|^{1+\epsilon} \right)^{1/1+\epsilon} < \infty \right\}.$$

Proof. We know that $L_{1+\epsilon} = F^0_{1+\epsilon,2}$ is a retract of $L_{1+\epsilon}(\ell_2)$ and that $W^1_{1+\epsilon} = F^1_{1+\epsilon,2}$ is a retract of $L_{1+\epsilon}(\ell_2^1)$, the corresponding coretraction operator being $\mathfrak{J} f^2 = \left(\left(\varphi_\nu^2 \hat{f}^2\right)^\vee\right)_{\nu \in \mathbb{N}_0}$ (see [62, p. 185]). For the vector-valued $L_{1+\epsilon}$-spaces, using Corollary (6.3.23) and (6.3.24), we obtain

$$\left(L_{1+\epsilon}(\ell_2), L_{1+\epsilon}(\ell_2^1)\right)_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon} = L_{1+\epsilon}\left((\ell_2, \ell_2^1)_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon}\right)$$
$$= \left\{ f^2 = \left(f_j^2\right) : \|f^2\| = \left\| \left(\sum_{j=0}^{\infty} (1+j)^{\epsilon^2+\epsilon-1} \left(\sum_{\nu=j}^{\infty} \left| f_\nu{}^2(x_n) \right|^2 \right)^{(1+\epsilon)/2} \right)^{1/1+\epsilon} \middle| L_{1+\epsilon} \right\| < \infty \right\}.$$

Since $\mathbf{B}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon} = (L_{1+\epsilon}, W^1_{1+\epsilon})_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon}$ (see [283, Theorem 3.1/(b)] or Theorem 3.4), and

$$\|\mathfrak{J} f^2\|_{L_{1+\epsilon}\left((\ell_2,\ell_2^1)_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon}\right)} = \left(\sum_{j=0}^{\infty} (1+j)^{\epsilon^2+\epsilon-1} \int_{\mathbb{R}^n} \left(\sum_{\nu=j}^{\infty} \left| \left(\varphi_\nu^2 \hat{f}^2\right)^\vee(x_n) \right|^2 \right)^{(1+\epsilon)/2} dx_n \right)^{1/1+\epsilon}$$
$$= \left(\sum_{j=0}^{\infty} (1+j)^{\epsilon^2+\epsilon-1} \left\| \left(\sum_{\nu=j}^{\infty} \left| \left(\varphi_\nu^2 \hat{f}^2\right)^\vee(\cdot) \right|^2 \right)^{1/2} \middle| L_{1+\epsilon} \right\|^{1+\epsilon} \right)^{1/1+\epsilon},$$

Corollary (6.3.29) [314] Let $0 < \epsilon < \infty$. Then $f^2 \in L_{1+\epsilon}$ belongs to $\mathbf{B}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon}$ if and only if

$$\left\| f^2 \left| \mathbf{B}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon} \right\|_{\varphi+} = \left(\sum_{j=0}^{\infty} \left[(1+j)^{(\epsilon^2+\epsilon-1)/1+\epsilon} \left\| \left(\sum_{\nu=j}^{\infty} \left| \left(\varphi_\nu^2 \hat{f}^2\right)^\vee(\cdot) \right|^2 \right)^{1/2} \middle| L_{1+\epsilon} \right\| \right]^{1+\epsilon} \right)^{1/1+\epsilon} < \infty \quad (65)$$

(usual modification if $\epsilon = \infty$). Furthermore, $\left\| \cdot \left| \mathbf{B}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon} \right\|_{\varphi+}$ is an equivalent quasi-norm on $\mathbf{B}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon}$.

Proof. Take b_{r-1}, b_r such that $-1/1+\epsilon < b_{r-1} < (\epsilon^2+\epsilon-1)/1+\epsilon < b_r$. We can find $0 < \epsilon < 1$ such that

$$\epsilon = (\epsilon)(b_{r-1} + (1/1+\epsilon)) + (1-\epsilon)(b_r + (1/1+\epsilon)). \quad (66)$$

With this choice of parameters, Corollary (6.3.26) and [295, Theorem 5] yield that

$$\mathbf{B}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon} = (\mathbf{B}^{0,b_{r-1}}_{1+\epsilon,1+\epsilon}, \mathbf{B}^{0,b_r}_{1+\epsilon,1+\epsilon})_{1-\epsilon,1+\epsilon}. \quad (67)$$

Moreover, it follows from Corollary (6.3.28) and Corollary (6.3.24) that $\mathbf{B}^{0,b_{i+r-1}}_{1+\epsilon,1+\epsilon}$ is a retract of $L_{1+\epsilon}\big((\ell_2,\ell_2^1)_{(0,-b_{i+r-1}),1+\epsilon}\big), i=0,1$. Note also that

$$\left\|\left(\left(\varphi_j^2\hat{f}^2\right)^{\vee}\right)|L_{1+\epsilon}\big((\ell_2,\ell_2^1)_{(0,-b_{i+r-1}),1+\epsilon}\big)\right\|$$

$$\sim\left(\sum_{j=0}^{\infty}(1+j)^{b_{i+r-1}(1+\epsilon)}\left\|\left(\sum_{\nu=j}^{\infty}\left|\left(\varphi_\nu^2\hat{f}^2\right)^{\vee}(\cdot)\right|^2\right)^{1/2}|L_{1+\epsilon}\right\|^{1+\epsilon}\right)^{1/1+\epsilon}$$

$$=\left(\sum_{\epsilon=-1/2}^{\infty}\sum_{j=2^{1+2\epsilon}}^{2^{2+2\epsilon}-1}j^{b_{i+r-1}(1+\epsilon)}\left\|\left(\sum_{\nu=j-1}^{\infty}\left|\left(\varphi_\nu^2\hat{f}^2\right)^{\vee}(\cdot)\right|^2\right)^{1/2}|L_{1+\epsilon}\right\|^{1+\epsilon}\right)^{1/1+\epsilon}.$$

If $2^{1+2\epsilon}\le j\le 2^{2+2\epsilon}-1$ then $j\sim 2^{1+2\epsilon}$ and so $\sum_{j=2^{1+2\epsilon}}^{2^{2+2\epsilon}-1}j^{b_{i+r-1}(1+\epsilon)}\sim 2^{b_{i+r-1}(1+2\epsilon)(1+\epsilon)}2^{1+2\epsilon}=2^{(1+2\epsilon)(b_{i+r-1}+(1/1+\epsilon))(1+\epsilon)}$. Whence,

$$\left\|\left(\left(\varphi_j^2\hat{f}^2\right)^{\vee}\right)|L_{1+\epsilon}\big((\ell_2,\ell_2^1)_{(0,-b_{i+r-1}),1+\epsilon}\big)\right\|$$

$$\sim\left(\sum_{k=0}^{\infty}2^{(1+2\epsilon)(b_{i+r-1}+(1/1+\epsilon))(1+\epsilon)}\left\|\left(\sum_{\nu=2^{1+2\epsilon}-1}^{\infty}\left|\left(\varphi_\nu^2\hat{f}^2\right)^{\vee}(\cdot)\right|^2\right)^{1/2}|L_{1+\epsilon}\right\|^{1+\epsilon}\right)^{1/1+\epsilon}.$$

Write $\Re f^2=\big(((\varphi_\nu^2\hat{f}^2)^{\vee})_{\nu\ge 2^{1+2\epsilon}-1}\big)_{(1+2\epsilon)\in\mathbb{N}_0}$ and $A_{1+2\epsilon}=L_{1+\epsilon}(\ell_2)$ for $(1+2\epsilon)\in\mathbb{N}_0$. The previous considerations and Corollary (6.3.28) show that

$$\Re:\mathbf{B}^{0,b_{i+r-1}}_{1+\epsilon,1+\epsilon}\longrightarrow\boldsymbol{\ell}^{b_{i+r-1}+(1/1+\epsilon)}_{1+\epsilon}(A_{1+2\epsilon})$$

is bounded for $i=0,1$. Interpolating this operator and using (43), (66) and (67), we derive that

$$\Re:\mathbf{B}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon}\longrightarrow\boldsymbol{\ell}^{\epsilon}_{1+\epsilon}(A_{1+2\epsilon})$$

is also bounded. Therefore one has by (65)

$$\left\|f^2\left|\mathbf{B}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon}\right.\right\|_{\varphi+}\sim\left(\sum_{\epsilon=-1/2}^{\infty}\left[2^{(1+2\epsilon)\epsilon}\left\|\left(\sum_{\nu=2^{1+2\epsilon}-1}^{\infty}\left|\left(\varphi_\nu^2\hat{f}^2\right)^{\vee}(\cdot)\right|^2\right)^{1/2}|L_{1+\epsilon}\right\|\right]^{1+\epsilon}\right)^{1/1+\epsilon}$$

$$=\|\Re f^2|\ell^{\epsilon}_{1+\epsilon}(A_{1+2\epsilon})\|\lesssim\left\|f^2\left|\mathbf{B}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon}\right.\right\|.$$

To check the converse inequality, note that using $L_{1+\epsilon}(\ell_2)$-Fourier multipliers and Littlewood–Paley theorem based on $(\varphi^2_{-(\epsilon^2+\epsilon+1)/1+\epsilon})$ where $\varphi^2_{-(\epsilon^2+\epsilon+1)/1+\epsilon}\varphi_\nu^2=0$ if $|(-(\epsilon^2+\epsilon+1)/1+\epsilon)-\nu|>1$ one has

$$\left\|\sum_{\nu=j+1}^{\infty}\left(\varphi_\nu^2\hat{f}^2\right)^{\vee}|L_{1+\epsilon}\right\|\lesssim\left\|\left(\sum_{-(\epsilon^2+\epsilon+1)/1+\epsilon=j}^{\infty}\left|\left(\varphi^2_{-(\epsilon^2+\epsilon+1)/1+\epsilon}\hat{f}^2\right)^{\vee}(\cdot)\right|^2\right)^{1/2}|L_{1+\epsilon}\right\|.$$

Hence

$$\left\|f^2\left|\mathbf{B}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon}\right.\right\|^{1+\epsilon}_{\varphi+}=\sum_{j=0}^{\infty}\left[(1+j)^{(\epsilon^2+\epsilon-1)/1+\epsilon}\left\|\left(\sum_{\nu=j}^{\infty}\left|\left(\varphi_\nu^2\hat{f}^2\right)^{\vee}(\cdot)\right|^2\right)^{1/2}|L_{1+\epsilon}\right\|\right]^{1+\epsilon}$$

$$\sim\|f^2|L_{1+\epsilon}\|^{1+\epsilon}+\sum_{j=0}^{\infty}2^{j\epsilon(1+\epsilon)}\left\|\left(\sum_{\nu=2^j}^{\infty}\left|\left(\varphi_\nu^2\hat{f}^2\right)^{\vee}(\cdot)\right|^2\right)^{1/2}|L_{1+\epsilon}\right\|^{1+\epsilon}$$

$$\gtrsim\|f^2|L_{1+\epsilon}\|^{1+\epsilon}+\sum_{j=0}^{\infty}2^{j\epsilon(1+\epsilon)}\left\|\sum_{\nu=2^j+1}^{\infty}\left(\varphi_\nu^2\hat{f}^2\right)^{\vee}|L_{1+\epsilon}\right\|^{1+\epsilon}$$

$$= \|f^2|L_{1+\epsilon}\|^{1+\epsilon} + \sum_{j=0}^{\infty} 2^{j\epsilon(1+\epsilon)} \left\| f^2 - \sum_{\nu=0}^{2^j} \left(\varphi_\nu^2 \hat{f}^2\right)^\vee |L_{1+\epsilon} \right\|^{1+\epsilon}.$$

Let $\mu_j = 2^{2^j}$ and consider the sets G_j defined in (40). Since

$$\operatorname{supp}\left(\sum_{\nu=0}^{2^j} \left(\varphi_\nu^2 \hat{f}^2\right)^\vee\right)^\wedge = \operatorname{supp}\sum_{\nu=0}^{2^j} \varphi_\nu^2 \hat{f}^2 \subseteq \{x_n : |x_n| \le \mu_{j+1}\},$$

we have $\sum_{\nu=0}^{2^j}\left(\varphi_\nu^2 \hat{f}^2\right)^\vee \in G_{\mu_{j+1}}$ and

$$\left\| f^2 - \sum_{\nu=0}^{2^j} \left(\varphi_\nu^2 \hat{f}^2\right)^\vee |L_{1+\epsilon} \right\| \ge \inf_{g^2 \in G_{\mu_{j+2}-1}} \|f^2 - g^2|L_{1+\epsilon}\| = E_{\mu_{j+2}}(f^2)_{1+\epsilon}.$$

Consequently, using [295, Lemma 1] and Corollary (6.3.26), we derive that

$$\left\|f^2 \left|\mathbf{B}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon}\right.\right\|_{\varphi+} \gtrsim \left(\|f^2|L_{1+\epsilon}\|^{1+\epsilon} + \sum_{j=0}^{\infty} 2^{j\epsilon(1+\epsilon)} E_{\mu_j}(f^2)^{1+\epsilon}_{1+\epsilon}\right)^{1/1+\epsilon}$$

$$\sim \left\|f^2 \left|(L_{1+\epsilon})^{(0,(\epsilon^2+\epsilon-1)/1+\epsilon)}_{1+\epsilon}\right.\right\| \sim \left\|f^2 \left|\mathbf{B}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon}\right.\right\|.$$

This completes the proof.

Corollary (6.3.30) [314] For $\epsilon = -1, 0$ and $0 < \epsilon < \infty$, the space $f^{2(1+\epsilon)}_{1+\epsilon,2}$ can be identified with a complemented subspace $\Delta^{1+\epsilon}_{1+\epsilon,2}$ of $L_{1+\epsilon}(\ell_2^{1+\epsilon}(\ell_2))$. The projection onto $\Delta^{1+\epsilon}_{1+\epsilon,2}$ associates to each $h^2(\cdot) = (h_j^2(\cdot))_{j\in\mathbb{N}_0} = (((h^2_{m^2})^{j,G}(\cdot))_{\substack{G\in G^j \\ m^2\in\mathbb{Z}^n}})_{j\in\mathbb{N}_0} \in L_{1+\epsilon}(\ell_2^{1+\epsilon}(\ell_2))$ the function Ph^2 defined by

$$Ph^2(x_n) = ((2^{jn} \int_{Q_{j,m^2}} \left(h^2_{m^2}\right)^{j,G}(y_n) dy_n \chi_{j,m^2}(x_n))_{\substack{G\in G^j \\ m^2\in\mathbb{Z}^n}})_{j\in\mathbb{N}_0}.$$

Proof. Given any $\lambda \in f^{2(1+\epsilon)}_{1+\epsilon,2}$, let $R(\lambda)$ be the function defined by $R(\lambda)(x_n) = (g_j^2(x_n))$ where $g_j^2(x_n) = (\lambda^{j,G}_{m^2}\chi_{j,m^2}(x_n))_{\substack{G\in G^j \\ m^2\in\mathbb{Z}^n}}$. Since

$$\left\|\lambda|f^{2(1+\epsilon)}_{1+\epsilon,2}\right\| = \left\|\left(\sum_{j\in\mathbb{N}_0} 2^{j(1+\epsilon)2} \sum_{\substack{G\in G^j \\ m^2\in\mathbb{Z}^n}} \left|\lambda^{j,G}_{m^2}\right|^2 \chi_{j,m^2}(\cdot)\right)^{1/2} |L_{1+\epsilon}\right\|$$

$$= \left\|\left(\sum_{j\in\mathbb{N}_0} 2^{j(1+\epsilon)2} \left\|g_j^2(\cdot)|\ell_2\right\|^2\right)^{1/2} |L_{1+\epsilon}\right\| = \|R(\lambda)|L_{1+\epsilon}(\ell_2^{1+\epsilon}(\ell_2))\|$$

we have that $f^{2(1+\epsilon)}_{1+\epsilon,2}$ is isometric to the subspace $\Delta^{1+\epsilon}_{1+\epsilon,2} = \left\{R(\lambda) : \lambda \in f^{2(1+\epsilon)}_{1+\epsilon,2}\right\}$ of $L_{1+\epsilon}(\ell_2^{1+\epsilon}(\ell_2))$. It is easy to check that $Ph^2 = h^2$ for any $h^2 \in \Delta^{1+\epsilon}_{1+\epsilon,2}$. Let us show that P is bounded in $L_{1+\epsilon}(\ell_2^{1+\epsilon}(\ell_2))$. We have that

$$2^{jn} \int_{Q_{j,m^2}} \left|\left(h^2_{m^2}\right)^{j,G}(y_n)\right| dy_n \chi_{j,m^2}(x_n) \lesssim \left(\mathcal{M}\left(h^2_{m^2}\right)^{j,G}\right)(x_n), \qquad x_n \in \mathbb{R}^n,$$

where $\mathcal{M}$ is the Hardy–Littlewood maximal operator. Using the vector-valued estimate for $\mathcal{M}$ (see [194, Theorem 1.1.1, p. 51]), we obtain

$$\left\|Ph^2|L_{1+\epsilon}\left(\ell_2^{1+\epsilon}(\ell_2)\right)\right\| \le \left\|\left(\sum_{j\in\mathbb{N}_0} \sum_{\substack{G\in G^j \\ m^2\in\mathbb{Z}^n}} \left(2^{jn} \int_{Q_{j,m^2}} \left|2^{j(1+\epsilon)}\left(h^2_{m^2}\right)^{j,G}(y_n)\right| dy_n\right)^2 \chi_{j,m^2}(\cdot)\right)^{1/2} |L_{1+\epsilon}\right\|$$

$$\lesssim \left\|\left(\sum_{j\in\mathbb{N}_0} \sum_{\substack{G\in G^j \\ m^2\in\mathbb{Z}^n}} \left(\mathcal{M}\left(2^{j(1+\epsilon)}\left|\left(h^2_{m^2}\right)^{j,G}\right|\right)(\cdot)\right)^2\right)^{1/2} |L_{1+\epsilon}\right\| \lesssim \left\|\left(\sum_{j\in\mathbb{N}_0} \sum_{\substack{G\in G^j \\ m^2\in\mathbb{Z}^n}} 2^{2j(1+\epsilon)} \left|\left(h^2_{m^2}\right)^{j,G}(\cdot)\right|^2\right)^{1/2} |L_{1+\epsilon}\right\|$$

$$= \|h^2|L_{1+\epsilon}(\ell_2^{1+\epsilon}(\ell_2))\|.$$

In addition, this also shows that if $h^2 \in L_{1+\epsilon}(\ell_2^{1+\epsilon}(\ell_2))$ then $Ph^2 \in \Delta_{1+\epsilon,2}^{1+\epsilon}$. The proof is completed.

Corollary (6.3.31) [314] Let $0 < \epsilon < \infty$. Then $\lambda = (\lambda_{m^2}^{j,G})$ belongs to $((f^2)_{1+\epsilon,2}^0, (f^2)_{1+\epsilon,2}^1)_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon}$ if and only if

$$\|\lambda\| = \left(\int_{\mathbb{R}^n} \sum_{j=0}^{\infty} (1+j)^{\epsilon^2+\epsilon-1} \left(\sum_{\nu=j}^{\infty} \sum_{\substack{G\in G^\nu \\ m^2\in\mathbb{Z}^n}} \left|\lambda_{m^2}^{\nu,G}\right|^2 \chi_{\nu,m^2}(x_n)\right)^{(1+\epsilon)/2} dx_n\right)^{1/1+\epsilon}$$

is finite. Moreover, $\|\lambda\|$ defines an equivalent norm in $((f^2)_{1+\epsilon,2}^0, (f^2)_{1+\epsilon,2}^1)_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon}$.

Proof. By Corollary (6.3.23), Corollary (6.3.24), we have that

$$(L_{1+\epsilon}(\ell_2(\ell_2)), L_{1+\epsilon}(\ell_2^1(\ell_2)))_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon} = L_{1+\epsilon}((\ell_2, \ell_2^1)_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon}(\ell_2)).$$

Hence, according to Corollary (6.3.30) and the theorem on interpolation of complemented subspaces [62, Theorem 1.17.1], we derive that

$$\left\|\lambda|((f^2)_{1+\epsilon,2}^0, (f^2)_{1+\epsilon,2}^1)_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon}\right\| \sim \left(\int_{\mathbb{R}^n} \sum_{j=0}^{\infty} (1+j)^{\epsilon^2+\epsilon-1} \left(\sum_{\nu=j}^{\infty} \sum_{\substack{G\in G^\nu \\ m^2\in\mathbb{Z}^n}} \left|\lambda_{m^2}^{\nu,G}\right|^2 \chi_{\nu,m^2}(x_n)\right)^{(1+\epsilon)/2} dx_n\right)^{1/1+\epsilon}.$$

Corollary (6.3.32) [314] Let $0 < \epsilon < \infty$. Then f^2 belongs to $\mathbf{B}_{1+\epsilon,1+\epsilon}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}$ if, and only if, it can be represented as $f^2 = \sum_{j,G,m^2} \lambda_{m^2}^{j,G} 2^{-jn/2} \Psi_{G,m^2}^j$ (unconditional convergence being in $L_{1+\epsilon}$) with $\lambda_{m^2}^{j,G} = \lambda_{m^2}^{j,G}(f^2) = 2^{jn/2}(f^2, \Psi_{G,m^2}^j)$ and

$$\left\|f^2\left|\mathbf{B}_{1+\epsilon,1+\epsilon}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}\right\|_{\Psi+} = \left\|(\lambda_{m^2}^{j,G})\left|\mathbf{b}_{1+\epsilon,1+\epsilon}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}\right\|\right.$$

$$= \left(\sum_{j=0}^{\infty} (1+j)^{\epsilon^2+\epsilon-1} \left\|\left(\sum_{\nu=j}^{\infty} \sum_{\substack{G\in G^\nu \\ m^2\in\mathbb{Z}^n}} \left|\lambda_{m^2}^{\nu,G}\right|^2 \chi_{\nu,m^2}(\cdot)\right)^{1/2} |L_{1+\epsilon}\right\|^{1+\epsilon}\right)^{1/1+\epsilon}$$

is finite. Moreover, $\left\|\cdot\left|\mathbf{B}_{1+\epsilon,1+\epsilon}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}\right\|_{\Psi+}$ defines an equivalent norm in $\mathbf{B}_{1+\epsilon,1+\epsilon}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}$.

Proof. The unconditional convergence in $L_{1+\epsilon}$ for any sequence $(\lambda_{m^2}^{j,G}) \in \mathbf{b}_{1+\epsilon,1+\epsilon}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}$ follows from a corresponding assertion for $L_{1+\epsilon}$ based on $(f^2)_{1+\epsilon,2}^0$ according to [311, Theorem 1.20] and $\mathbf{b}_{1+\epsilon,1+\epsilon}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon} \hookrightarrow (f^2)_{1+\epsilon,2}^0$ as a consequence of Corollary (6.3.31).

Let D be the operator defined by

$$D((\lambda_{m^2}^{j,G})) = \sum_{j,G,m^2} \lambda_{m^2}^{j,G} 2^{-jn/2} \Psi_{G,m^2}^j.$$

According to [311, Theorem 1.20], the restrictions

$$D : (f^2)_{1+\epsilon,2}^0 \longrightarrow L_{1+\epsilon} \text{ and } D : (f^2)_{1+\epsilon,2}^1 \longrightarrow W_{1+\epsilon}^1$$

are isomorphisms. Interpolating and using Corollary (6.3.27) or [183, Theorem 3.1], we obtain that

$$D : ((f^2)_{1+\epsilon,2}^0, (f^2)_{1+\epsilon,2}^1)_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon} \longrightarrow (L_{1+\epsilon}, W_{1+\epsilon}^1)_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon} = \mathbf{B}_{1+\epsilon,1+\epsilon}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}$$

is also an isomorphism. As for the source space of this operator, by Corollary (6.3.31), we know that

$$\left\|(\lambda_{m^2}^{j,G})|((f^2)_{1+\epsilon,2}^0, (f^2)_{1+\epsilon,2}^1)_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon}\right\|$$

$$\sim \left(\int_{\mathbb{R}^n} \sum_{j=0}^{\infty} (1+j)^{\epsilon^2+\epsilon-1} \left(\sum_{\nu=j}^{\infty} \sum_{\substack{G\in G^\nu \\ m^2\in\mathbb{Z}^n}} \left|\lambda_{m^2}^{\nu,G}\right|^2 \chi_{\nu,m^2}(x_n)\right)^{(1+\epsilon)/2} dx_n\right)^{1/1+\epsilon}$$

$$= \left(\sum_{j=0}^{\infty} (1+j)^{\epsilon^2+\epsilon-1} \left\|\left(\sum_{\nu=j}^{\infty} \sum_{\substack{G\in G^\nu \\ m^2\in\mathbb{Z}^n}} \left|\lambda_{m^2}^{\nu,G}\right|^2 \chi_{\nu,m^2}(\cdot)\right)^{1/2} |L_{1+\epsilon}\right\|^{1+\epsilon}\right)^{1/1+\epsilon}.$$

Furthermore, $\lambda^{j,G}_{m^2} = \lambda^{j,G}_{m^2}(f^2)$ is again covered by [311, Theorem 1.20]. This completes the proof.

Corollary (6.3.33) [314] Let $0 < \epsilon < \infty$. Then we have with equivalence of norms

$$(L_{1+\epsilon}, \{V_j\})^{(0,(\epsilon^2+\epsilon-1)/1+\epsilon)}_{1+\epsilon} = \mathbf{B}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon}.$$

Proof. We start with the case $0 < \epsilon < \infty$. Put $\mu_j = 2^{2^j}, j \in \mathbb{N}_0$. We have

$$\left\| f^2 \left| (L_{1+\epsilon}, \{V_j\})^{(0,(\epsilon^2+\epsilon-1)/1+\epsilon)}_{1+\epsilon} \right\| \right.^{1+\epsilon} \sim \|f^2|L_{1+\epsilon}\|^{1+\epsilon} + \sum_{j=0}^{\infty} \left(2^{j\epsilon} E^{\Psi}_{\mu_j}(f^2)_{1+\epsilon}\right)^{1+\epsilon}$$

$$\sim \|f^2|L_{1+\epsilon}\|^{1+\epsilon} + \sum_{j=0}^{\infty} \left(2^{j\epsilon} \left\| f^2 - P_{\mu_j} f^2 |L_{1+\epsilon} \right\|\right)^{1+\epsilon}$$

$$= \|f^2|L_{1+\epsilon}\|^{1+\epsilon} + \sum_{j=0}^{\infty} 2^{j\epsilon(1+\epsilon)} \left\| \sum_{\nu=2^j}^{\infty} \sum_{\substack{G\in G^\nu \\ m^2 \in \mathbb{Z}^n}} \lambda^{\nu,G}_{m^2}(f^2) 2^{-\nu n/2} \Psi^{\nu}_{G,m^2} \,|L_{1+\epsilon} \right\|^{1+\epsilon}$$

$$\sim \|f^2|L_{1+\epsilon}\|^{1+\epsilon} + \sum_{j=0}^{\infty} 2^{j\epsilon(1+\epsilon)} \left\| \left(\sum_{\nu=2^j}^{\infty} \sum_{\substack{G\in G^\nu \\ m^2 \in \mathbb{Z}^n}} \left|\lambda^{\nu,G}_{m^2}(f^2)\right|^2 \chi_{\nu,m^2}(\cdot) \right)^{1/2} |L_{1+\epsilon} \right\|^{1+\epsilon}$$

where we have used [67, Theorem 1.64](or [311, Theorem 1.20]) in the last equivalence. Now the result follows from Corollary (6.3.32). Note that the above argument works even if $\epsilon = 0$.

To establish the remaining case $\epsilon \neq 0$, choose b_{r-1}, b_r such that $-1/1+\epsilon < b_{r-1} < (\epsilon^2+\epsilon-1)/1+\epsilon < b_r$ and take $0 < \epsilon < 1$ with

$$\epsilon = (\epsilon)(b_{r-1} + 1/1+\epsilon) + (1-\epsilon)(b_r + 1/1+\epsilon).$$

According to (67), [295, Theorem 5] and the result just proved for the diagonal case, we obtain that

$$\mathbf{B}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon} = (\mathbf{B}^{0,b_{r-1}}_{1+\epsilon,1+\epsilon}, \mathbf{B}^{0,b_r}_{1+\epsilon,1+\epsilon})_{1-\epsilon,1+\epsilon} = ((L_{1+\epsilon},\{V_j\})^{(0,b_{r-1})}_{1+\epsilon}, (L_{1+\epsilon},\{V_j\})^{(0,b_r)}_{1+\epsilon})_{1-\epsilon,1+\epsilon}$$
$$= (L_{1+\epsilon},\{V_j\})^{(0,(\epsilon^2+\epsilon-1)/1+\epsilon)}_{1+\epsilon}.$$

Corollary (6.3.34) [314] Let $0 < \epsilon < \infty$. Then f^2 belongs to $\mathbf{B}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon}$ if, and only if, it can be represented as $f^2 = \sum_{j,G,m^2} \lambda^{j,G}_{m^2} 2^{-jn/2} \Psi^j_{G,m^2}$ (unconditional convergence in $L_{1+\epsilon}$) with $\lambda^{j,G}_{m^2} = \lambda^{j,G}_{m^2}(f^2) = 2^{jn/2}(f^2, \Psi^j_{G,m^2})$ and

$$\left\| f^2 \left| \mathbf{B}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon} \right\|_{\Psi+} \right. = \left\| (\lambda^{j,G}_{m^2}) \left| \mathbf{b}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon} \right\| \right.$$

$$= \left(\sum_{j=0}^{\infty} (1+j)^{\epsilon^2+\epsilon-1} \left\| \left(\sum_{\nu=j}^{\infty} \sum_{\substack{G\in G^\nu \\ m^2\in\mathbb{Z}^n}} \left|\lambda^{\nu,G}_{m^2}\right|^2 \chi_{\nu,m^2}(\cdot) \right)^{1/2} |L_{1+\epsilon} \right\|^{1+\epsilon} \right)^{1/1+\epsilon} < \infty.$$

Moreover, $\left\| \cdot \left| \mathbf{B}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon} \right\|_{\Psi+}\right.$ defines an equivalent quasi-norm in $\mathbf{B}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon}$.

Proof. The unconditional convergence in $L_{1+\epsilon}$ is covered by the related argument at the beginning of the proof of Corollary (6.3.32) and the above interpolation (67).

Using Lemma (6.3.13), we obtain

$$\left\| f^2 \left| \mathbf{B}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon} \right\| \right. \sim \left\| f^2 \left| (L_{1+\epsilon},\{V_j\})^{(0,(\epsilon^2+\epsilon-1)/1+\epsilon)}_{1+\epsilon} \right\| \right.$$

$$\sim \left(\|f^2|L_{1+\epsilon}\|^{1+\epsilon} + \sum_{j=0}^{\infty} 2^{j\epsilon(1+\epsilon)} E^{\Psi}_{\mu_j}(f^2)^{1+\epsilon}_{1+\epsilon} \right)^{1/1+\epsilon}$$

$$\sim \left(\|f^2|L_{1+\epsilon}\|^{1+\epsilon} + \sum_{j=0}^{\infty} 2^{j\epsilon(1+\epsilon)} \left\| f^2 - P_{\mu_j} f^2 |L_{1+\epsilon} \right\|^{1+\epsilon} \right)^{1/1+\epsilon}$$

$$\sim\left(\|f^2|L_{1+\epsilon}\|^{1+\epsilon}+\sum_{j=0}^{\infty}2^{j\epsilon(1+\epsilon)}\left\|\left(\sum_{\nu=2^j}^{\infty}\sum_{\substack{G\in G^\nu\\ m^2\in\mathbb{Z}^n}}\left|\lambda_{m^2}^{\nu,G}\right|^2\chi_{\nu,m^2}(\cdot)\right)^{1/2}|L_{1+\epsilon}\right\|^{1+\epsilon}\right)^{1/1+\epsilon}$$

$$\sim\left(\sum_{j=0}^{\infty}(1+j)^{\epsilon^2+\epsilon-1}\left\|\left(\sum_{\nu=j}^{\infty}\sum_{\substack{G\in G^\nu\\ m^2\in\mathbb{Z}^n}}\left|\lambda_{m^2}^{\nu,G}\right|^2\chi_{\nu,m^2}(\cdot)\right)^{1/2}|L_{1+\epsilon}\right\|^{1+\epsilon}\right)^{1/1+\epsilon}.$$

Corollary (6.3.35) [314] Let A_{r-2} be a Banach space, let $\{T(1+\epsilon)\}_{\epsilon\geq-1}$ be a strongly continuous equi-bounded semi-group of operators in A_{r-2}, let $m^2\in\mathbb{N}, -1<\epsilon\leq\infty$. The quasi-norm

$$\|a_{r-2}\|_1=\|a_{r-2}|A_{r-2}\|+\left(\int_0^1\left((1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}\overline{\omega}_{m^2}\left((1+\epsilon)^{m^2},a_{r-2}\right)\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}$$

is equivalent to the interpolation quasi-norm $\left\|\cdot|(A_{r-2},D(\Lambda^{m^2}))_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon}\right\|$ on

$$(A_{r-2},D(\Lambda^{m^2}))_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon}.$$

In addition, if the semi-group $\{T(1+\epsilon)\}_{\epsilon\geq-1}$ is analytic then

$$\|a_{r-2}\|_2=\|a_{r-2}|A_{r-2}\|+\left(\int_0^1\left((1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}\left\|[T(1+\epsilon)-\mathrm{id}]^{m^2}a_{r-2}|A_{r-2}\right\|\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}$$

is also an equivalent quasi-norm on $(A_{r-2},D(\Lambda^{m^2}))_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon}$.

Proof. Making a change of variable and using (54), we obtain

$$\left\|a_{r-2}|(A_{r-2},D(\Lambda^{m^2}))_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon}\right\|$$

$$=\left(\int_0^1\left((1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}K(1+\epsilon,a_{r-2})\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}$$

$$\sim\left(\int_0^1\left((1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}K\left((1+\epsilon)^{m^2},a_{r-2}\right)\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}$$

$$\lesssim\left(\int_0^1\left((1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}\overline{\omega}_{m^2}\left((1+\epsilon)^{m^2},a_{r-2}\right)\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}$$

$$+\left(\int_0^1\left((1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}(1+\epsilon)^{m^2}\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}\|a_{r-2}|A_{r-2}\|$$

$$\sim\|a_{r-2}\|_1.$$

To check the converse inequality, note that $(A_{r-2},D(\Lambda^{m^2}))_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon}\hookrightarrow A_{r-2}$. Moreover, by (55), we have

$$\left(\int_0^1\left((1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}\overline{\omega}_{m^2}\left((1+\epsilon)^{m^2},a_{r-2}\right)\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}$$

$$\lesssim\left(\int_0^1\left((1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}K\left((1+\epsilon)^{m^2},a_{r-2}\right)\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}$$

$$\sim\left\|a_{r-2}|(A_{r-2},D(\Lambda^{m^2}))_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon}\right\|.$$

Consequently, $\left\|a_{r-2}|(A_{r-2},D(\Lambda^{m^2}))_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon}\right\|\sim\|a_{r-2}\|_1$.

Assume now that the semi-group $\{T(1+\epsilon)\}_{\epsilon\geq-1}$ is analytic. To complete the proof we first show that

$$K(1-\epsilon, a_{r-2}) \sim (1-\epsilon)\|a_{r-2}|A_{r-2}\| + \widetilde{K}(1-\epsilon, a_{r-2}), 0 < \epsilon \le 1, \qquad a_{r-2} \in A_{r-2}. \tag{68}$$

Indeed, take any $a_r \in D(\Lambda^{m^2})$. Using the triangle inequality in A_{r-2} and that $\epsilon \ge 0$, we obtain

$$K(1-\epsilon, a_{r-2}) \le \|a_{r-2} - a_r|A_{r-2}\| + (1-\epsilon)\|a_r|A_{r-2}\| + (1-\epsilon)\left\|\Lambda^{m^2} a_r|A_{r-2}\right\|$$
$$\le 2\|a_{r-2} - a_r|A_{r-2}\| + (1-\epsilon)\|a_{r-2}|A_{r-2}\| + (1-\epsilon)\left\|\Lambda^{m^2} a_r|A_{r-2}\right\|.$$

Taking the infimum over all $a_r \in D(\Lambda^{m^2})$ it follows that $K(1-\epsilon, a_{r-2}) \lesssim (1-\epsilon)\|a_{r-2}|A_{r-2}\| + \widetilde{K}(1-\epsilon, a_{r-2})$. Conversely,

$$(1-\epsilon)\|a_{r-2}|A_{r-2}\| + \widetilde{K}(1-\epsilon, a_{r-2})$$
$$\le (1-\epsilon)\|a_{r-2}|A_{r-2}\| + \|a_{r-2} - a_r|A_{r-2}\| + (1-\epsilon)\left\|\Lambda^{m^2} a_r|A_{r-2}\right\|$$
$$\le (1-\epsilon)\|a_{r-2} - a_r|A_{r-2}\| + (1-\epsilon)\|a_r|A_{r-2}\| + \|a_{r-2} - a_r|A_{r-2}\|$$
$$+ (1-\epsilon)\left\|\Lambda^{m^2} a_r|A_{r-2}\right\| \le 2\|a_{r-2} - a_r|A_{r-2}\| + (1-\epsilon)\left\|a_r\middle|D(\Lambda^{m^2})\right\|.$$

Therefore, we derive (68).

Now (68) and (56) yield that $\|\cdot\|_2 \sim \left\|\cdot|(A_{r-2}, D(\Lambda^{m^2}))_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon}\right\|$. This completes the proof.

Corollary (6.3.36) [314] Let $0 < \epsilon < \infty$ and $m^2 \in \mathbb{N}$. Then $f^2 \in L_{1+\epsilon}$ belongs to $\mathbf{B}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon}$ if, and only if,

$$\left\|f^2\middle|\mathbf{B}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon}\right\|_{(m^2)} = \|f^2|L_{1+\epsilon}\| + \left(\int_0^1 (1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}\left\|[P(1+\epsilon)-\mathrm{id}]^{m^2} f^2|L_{1+\epsilon}\right\|^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}$$

is finite. Furthermore, $\left\|\cdot\middle|\mathbf{B}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon}\right\|^{\diamondsuit}_{(m^2)}$ is an equivalent quasi-norm in $\mathbf{B}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon}$.

Proof. For the semi-group $\{P(1+\epsilon)\}_{\epsilon\ge -1}$ we have

$$\Lambda^{2m^2} f^2 = (-1)^{m^2}\Delta^{m^2} f^2 \quad \text{and} \quad D(\Lambda^{2m^2}) = W^{2m^2}_{1+\epsilon}.$$

Using again that $\mathbf{B}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon} = (L_{1+\epsilon}, W^{2m^2}_{1+\epsilon})_{(0,-(\epsilon^2+\epsilon-1)/1+\epsilon),1+\epsilon}$ and Corollary (6.3.35), we obtain that the wanted result holds for any even natural number m^2. To complete the proof, write

$$\|f^2\|_{m^2} = \|f^2|L_{1+\epsilon}\| + \left(\int_0^1 (1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}\left\|[P(1+\epsilon)-\mathrm{id}]^{m^2} f^2|L_{1+\epsilon}\right\|^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}.$$

It suffices to show that for any $m^2 \in \mathbb{N}$ the quasi-norms $\|\cdot\|_{m^2}$ and $\|\cdot\|_{m^2+1}$ are equivalent on $\mathbf{B}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon}$.

Take any $f^2 \in \mathbf{B}^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon}$. Using (56), we obtain

$$\left\|[P(1+\epsilon)-\mathrm{id}]^{m^2} f^2|L_{1+\epsilon}\right\| \sim K\left((1+\epsilon)^{m^2}, f^2; L_{1+\epsilon}, D\left(\Lambda^{m^2}\right)\right)$$
$$\sim \sup_{\epsilon\ge -1}\left\|[P(1+\epsilon)-\mathrm{id}]^{m^2} f^2|L_{1+\epsilon}\right\| = \overline{\omega}_{m^2}\left((1+\epsilon)^{m^2}, f^2\right).$$

By [266, (4.10)], we have that $\overline{\omega}_{m^2+1}\left((1+\epsilon)^{m^2+1}, f^2\right) \lesssim \overline{\omega}_{m^2}\left((1+\epsilon)^{m^2}, f^2\right)$ Hence

$$\|f^2\|_{m^2+1} \sim \|f^2|L_{1+\epsilon}\| + \left(\int_0^1 \left((1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}\overline{\omega}_{m^2+1}\left((1+\epsilon)^{m^2+1}, f^2\right)\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}$$

$$\lesssim \|f^2|L_{1+\epsilon}\| + \left(\int_0^1 \left((1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}\overline{\omega}_{m^2}\left((1+\epsilon)^{m^2}, f^2\right)\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon} \sim \|f^2\|_{m^2}.$$

In order to establish the converse inequality, note that

$$\overline{\omega}_{m^2}\left((1+\epsilon)^{m^2}, f^2\right) \lesssim (1+\epsilon)^{m^2}\int_{1+\epsilon}^{\infty}(1+\epsilon)^{-(m^2)-1}\overline{\omega}_{m^2+1}\left((1+\epsilon)^{m^2+1}, f^2\right)d(1+\epsilon)$$

(see [310, Theorem 1.4, (1.7)]). Therefore

$$\|f^2\|_{m^2} \sim \|f^2|L_{1+\epsilon}\| + \left(\int_0^1 \left((1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}\overline{\omega}_{m^2}\left((1+\epsilon)^{m^2}, f^2\right)\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}$$

$$\lesssim \|f^2|L_{1+\epsilon}\| + \left(\int_0^1 \left((1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}(1+\epsilon)^{m^2}\int_{1+\epsilon}^{\infty}\frac{\overline{\omega}_{m^2+1}\left((1+\epsilon)^{m^2+1},f^2\right)}{(1+\epsilon)^{m^2+1}}d(1+\epsilon)\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}$$

$$\sim \|f^2|L_{1+\epsilon}\| + \left(\int_0^1 \left((1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}(1+\epsilon)^{m^2}\int_{1+\epsilon}^{1}\frac{\overline{\omega}_{m^2+1}\left((1+\epsilon)^{m^2+1},f^2\right)}{(1+\epsilon)^{m^2+1}}d(1+\epsilon)\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}$$

$$+ \left(\int_0^1 \left((1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}(1+\epsilon)^{m^2}\int_{1}^{\infty}\frac{\overline{\omega}_{m^2+1}\left((1+\epsilon)^{m^2+1},f^2\right)}{(1+\epsilon)^{m^2+1}}d(1+\epsilon)\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}$$

$$= \|f^2|L_{1+\epsilon}\| + J_1 + J_2.$$

Since $\overline{\omega}_{m^2+1}\left((1+\epsilon)^{m^2+1},f^2\right)/(1+\epsilon)^{m^2+1}$ is equivalent to the decreasing function $\widetilde{K}\left((1+\epsilon)^{m^2+1},f^2\right)/(1+\epsilon)^{m^2+1}$, we can still apply the extension of the Hardy inequality established in [256, Theorem 6.4] to derive that

$$J_1 \lesssim \left(\int_0^1 \left((1+\epsilon)^{m^2+1}(1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}\frac{\overline{\omega}_{m^2+1}\left((1+\epsilon)^{m^2+1},f^2\right)}{(1+\epsilon)^{m^2+1}}\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}$$

$$\sim \left(\int_0^1 (1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}\left\|[P(1+\epsilon)-\mathrm{id}]^{m^2+1}f^2|L_{1+\epsilon}\right\|^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon} \leq \|f^2\|_{m^2+1}.$$

As for J_2, using that

$$\overline{\omega}_{m^2+1}\left((1+\epsilon)^{m^2+1},f^2\right) \sim \widetilde{K}((1+\epsilon)^{m^2+1},f^2;L_{1+\epsilon},D(\Lambda^{m^2+1})) \leq \|f^2|L_{1+\epsilon}\|$$

we get

$$J_2 \lesssim \left(\int_0^1 \left((1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}(1+\epsilon)^{m^2}\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}\|f^2|L_{1+\epsilon}\| \lesssim \|f^2\|_{m^2+1}.$$

This yields that $\|f^2\|_{m^2} \lesssim \|f^2\|_{m^2+1}$ and completes the proof.

Corollary (6.3.37) [314] Let A_{r-2} be a Banach space, let $\{T(1+\epsilon)\}_{\epsilon\geq -1}$ be an analytic semi-group of operators in A_{r-2}, let $0 < 1+\epsilon/2 < m^2 \in \mathbb{N}, -1 < \epsilon \leq \infty$ and $(\epsilon^2+\epsilon-1)/1+\epsilon \in \mathbb{R}$. The quasi-norm

$$\|a_{r-2}\|_3 = \|a_{r-2}|A_{r-2}\| + \left(\int_0^1 \left((1+\epsilon)^{m^2-\frac{1+\epsilon}{2}}(1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}\left\|\Lambda^{m^2}T(1+\epsilon)a_{r-2}|A_{r-2}\right\|\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}$$

is equivalent to the interpolation quasi-norm on $(A_{r-2},D(\Lambda^{m^2}))_{1+\epsilon/2m^2,1+\epsilon,(\epsilon^2+\epsilon-1)/1+\epsilon}$.

Proof. Since $\|a_{r-2}|A_{r-2}\| \leq \left\|a_{r-2}|D(\Lambda^{m^2})\right\|$ for any $a_{r-2} \in D(\Lambda^{m^2})$, we have that $K(1+\epsilon,a_{r-2}) = \|a_{r-2}|A_{r-2}\|$ for any $a_{r-2} \in A_{r-2}$ and $\epsilon \geq 0$. This yields that

$$\left\|a_{r-2}|\left(A_{r-2},D\left(\Lambda^{m^2}\right)\right)_{(1+\epsilon)/2m^2,1+\epsilon,(\epsilon^2+\epsilon-1)/1+\epsilon}\right\|$$

$$\sim \|a_{r-2}|A_{r-2}\| + \left(\int_0^1 \left((1+\epsilon)^{-(1+\epsilon)/2m^2}(1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}K(1+\epsilon,a_{r-2})\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}$$

$$\sim \|a_{r-2}|A_{r-2}\| + \left(\int_0^1 \left((1+\epsilon)^{-(1+\epsilon)/2}(1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}K\left((1+\epsilon)^{m^2},a_{r-2}\right)\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}.$$

Using that

$$(1+\epsilon)^{m^2}\left\|\Lambda^{m^2}T(1+\epsilon)a_{r-2}|A_{r-2}\right\| \lesssim K\left((1+\epsilon)^{m^2},a_{r-2}\right) \tag{69}$$

(see [306, Lemma 3.5.4]), we get that $\|a_{r-2}\|_3 \lesssim \left\|a_{r-2}|(A_{r-2},D(\Lambda^{m^2}))_{1+\epsilon/2m^2,1+\epsilon,(\epsilon^2+\epsilon-1)/1+\epsilon}\right\|$. To check the converse inequality, note that

$$\left\|[T(1+\epsilon)-\mathrm{id}]^{m^2}a_{r-2}|A_{r-2}\right\|$$

$$\lesssim \int_0^{1+\epsilon} -((\epsilon^2+\epsilon+1)/1+\epsilon)^{m^2-1} \left\|\Lambda^{m^2} T(-(\epsilon^2+\epsilon+1)/1+\epsilon) a_{r-2} | A_{r-2}\right\| d(-(\epsilon^2+\epsilon+1)/1+\epsilon) \quad (70)$$

(see [306, Lemma 3.5.5]). Hence, using (54), we obtain for $\epsilon < 1$

$$K((1+\epsilon)^{m^2}, a_{r-2}) \lesssim \bar{\omega}_{m^2}((1+\epsilon)^{m^2}, a_{r-2}) + (1+\epsilon)^{m^2} \|a_{r-2}|A_{r-2}\|$$

$$\lesssim \int_0^{1+\epsilon} -((\epsilon^2+\epsilon+1)/1+\epsilon)^{m^2-1} \left\|\Lambda^{m^2} T(-(\epsilon^2+\epsilon+1)/1+\epsilon) a_{r-2}|A_{r-2}\right\| d(-(\epsilon^2+\epsilon+1)/1+\epsilon) + (1+\epsilon)^{m^2} \|a_{r-2}|A_{r-2}\|.$$

This implies that

$$\left\| a_{r-2} | \left(A, D(\Lambda^{m^2})\right)_{(1+\epsilon)/2m^2, 1+\epsilon, (\epsilon^2+\epsilon-1)/1+\epsilon} \right\|$$

$$\lesssim \|a_{r-2}|A_{r-2}\| + \left(\int_0^1 \left((1+\epsilon)^{-(1+\epsilon)/2} (1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon} \right.\right.$$

$$\left.\left. \times \int_0^{1+\epsilon} (-(\epsilon^2+\epsilon+1)/1+\epsilon)^{m^2-1} \left\|\Lambda^{m^2} T(-(\epsilon^2+\epsilon+1)/1+\epsilon) a_{r-2}|A_{r-2}\right\| d(-(\epsilon^2+\epsilon+1)/1+\epsilon) \right)^{1+\epsilon} \frac{d(1+\epsilon)}{(1+\epsilon)} \right)^{1/1+\epsilon}.$$

By (iii), we have for $0 < -(\epsilon^2+\epsilon+1)/1+\epsilon < \mu$ that

$$\left\|\Lambda^{m^2} T(\mu) a_{r-2}|A_{r-2}\right\| = \left\|T(\mu - (-(\epsilon^2+\epsilon+1)/1+\epsilon)) \Lambda^{m^2} T(-(\epsilon^2+\epsilon+1)/1+\epsilon) a_{r-2}|A_{r-2}\right\|$$
$$\leq M \left\|\Lambda^{m^2} T(-(\epsilon^2+\epsilon+1)/1+\epsilon) a_{r-2}|A_{r-2}\right\|.$$

Hence, using the extension of the Hardy inequality established in [256, Theorem 6.4], we derive that

$$\left\| a_{r-2} | \left(A_{r-2}, D(\Lambda^{m^2})\right)_{(1+\epsilon)/2m^2, 1+\epsilon, (\epsilon^2+\epsilon-1)/1+\epsilon} \right\|$$

$$\lesssim \|a_{r-2}|A_{r-2}\| + \left(\int_0^1 \left((1+\epsilon)^{\frac{1-\epsilon}{2}} (1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon} (1+\epsilon)^{m^2-1} \left\|\Lambda^{m^2} T(1+\epsilon) a_{r-2}|A_{r-2}\right\| \right)^{1+\epsilon} \frac{d(1+\epsilon)}{(1+\epsilon)} \right)^{1/1+\epsilon}$$

$$= \|a_{r-2}\|_3.$$

Corollary (6.3.38) [314] Let $0 < \epsilon < \infty$, $(\epsilon^2+\epsilon-1)/1+\epsilon \in \mathbb{R}$ and $m^2 \in \mathbb{N}$. Then

$$\left\| f^2 \left| B^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon} \right\|_{(m^2)}\right.$$

$$= \|I_{-2} f^2 | L_{1+\epsilon}\| + \left(\int_0^1 \left((1+\epsilon)^{m^2} (1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon} \left\| \frac{\partial^{m^2} W(1+\epsilon) f^2}{\partial (1+\epsilon)^{m^2}} | L_{1+\epsilon} \right\| \right)^{1+\epsilon} \frac{d(1+\epsilon)}{(1+\epsilon)} \right)^{1/1+\epsilon}$$

is an equivalent quasi-norm on $B^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon}$.

Proof. According to [47, Proposition 1.8], the operator I_{-2} is an isomorphism from $B^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon}$ onto $B^{2,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon}$. The classical smoothness of $B^{2,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon}$ is $2 > 0$ so, by [28, Theorem 2.5], we know that

$$\left\| f^2 \left| B^{2,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon} \right\|\right.$$

$$\sim \|f^2|L_{1+\epsilon}\| + \left(\int_0^1 \left((1+\epsilon)^{-2} (1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon} \omega_{2(m^2+1)}(f^2, 1+\epsilon)_{1+\epsilon} \right)^{1+\epsilon} \frac{d(1+\epsilon)}{(1+\epsilon)} \right)^{1/1+\epsilon}. \quad (71)$$

Moreover, for $f^2 \in L_{1+\epsilon}$ and $\epsilon > -1$, we have

$$K\left((1+\epsilon)^{2(m^2+1)}, f^2; L_{1+\epsilon}, W^{2(m^2+1)}_{1+\epsilon}\right) \sim \min\left(1, (1+\epsilon)^{2(m^2+1)}\right) \|f^2|L_{1+\epsilon}\| + \omega_{2(m^2+1)}(f^2, 1+\epsilon)_{1+\epsilon} \quad (72)$$

(see [1, Theorem 5.4.12]). It follows from (71) and (72) that

$$B^{2,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon} = \left(L_{1+\epsilon}, W^{2(m^2+1)}_{1+\epsilon}\right)_{1/(m^2+1), 1+\epsilon, (\epsilon^2+\epsilon-1)/1+\epsilon},$$

$$\text{so} \quad B^{2,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon} = \left(L_{1+\epsilon}, D(\Lambda^{m^2+1})\right)_{1/(m^2+1), 1+\epsilon, (\epsilon^2+\epsilon-1)/1+\epsilon} \quad (73)$$

where $\Lambda = \Delta$ is the infinitesimal generator of the semi-group $\{W(1+\epsilon)\}_{\epsilon \geq -1}$. Applying Corollary (6.3.36) we get

$$\left\| f^2 \left| B^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon} \right\| \right. \sim \left\| I_{-2} f^2 \left| B^{2,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon} \right\| \right.$$

$$\sim \|I_{-2}f^2|L_{1+\epsilon}\| + \left(\int_0^1 \left((1+\epsilon)^{m^2}(1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}\|\Delta^{m^2+1}W(1+\epsilon)I_{-2}f^2|L_{1+\epsilon}\|\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}.$$

Next we use that

$$\Delta^{m^2+1}W(1+\epsilon)I_{-2}f^2 = \Delta^{m^2+1}I_{-2}W(1+\epsilon)f^2 = \Delta I_{-2}\Delta^{m^2}W(1+\epsilon)f^2$$

and that, according to [56, p. 133],

$$\|I_{-2}g^2|L_{1+\epsilon}\| = \|\Delta(\mathrm{id}-\Delta)^{-1}g^2|L_{1+\epsilon}\| \sim \|g^2|L_{1+\epsilon}\|.$$

This yields that

$$\left\|f^2\left|B^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon}\right\|\right.$$

$$\sim \|I_{-2}f^2|L_{1+\epsilon}\| + \left(\int_0^1 \left((1+\epsilon)^{m^2}(1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}\left\|\frac{\partial^{m^2}W(1+\epsilon)f^2}{\partial(1+\epsilon)^{m^2}}|L_{1+\epsilon}\right\|\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}.$$

Note that the operator I_{-2} is necessary because in general $B^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon}$ may not contain only regular distributions (see [169, Theorem 4.3]).

Corollary (6.3.39) [314] Let $0<\epsilon<\infty$, $(\epsilon^2+\epsilon-1)/1+\epsilon\in\mathbb{R}$ and $m^2\in\mathbb{N}$. Then

$$\left\|f^2\left|B^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon}\right\|^{\diamond}_{(m^2)}\right.$$

$$= \|I_{-2}f^2|L_{1+\epsilon}\| + \left(\int_0^1 \left((1+\epsilon)^{m^2}(1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}\left\|\frac{\partial^{m^2}P(1+\epsilon)f^2}{\partial(1+\epsilon)^{m^2}}|L_{1+\epsilon}\right\|\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}$$

is an equivalent quasi-norm on $B^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon}$.

Proof. This time $\Lambda^{2(m^2+1)}f^2 = (-1)^{m^2+1}\Delta^{m^2+1}f^2$ and $D(\Lambda^{2(m^2+1)}) = W^{2(m^2+1)}_{1+\epsilon}$. By (73), we get

$$B^{2,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon} = (L_{1+\epsilon}, W^{2(m^2+1)}_{1+\epsilon})_{1/(m^2+1),1+\epsilon,(\epsilon^2+\epsilon-1)/1+\epsilon} = (L_{1+\epsilon}, D(\Lambda^{m^2+1}))_{1/(m^2+1),1+\epsilon,(\epsilon^2+\epsilon-1)/1+\epsilon}.$$

Therefore, applying Corollary (6.3.37) with $\epsilon=3$, we obtain

$$\left\|f^2\left|B^{2,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon}\right\|\right.$$

$$\sim \|f^2|L_{1+\epsilon}\| + \left(\int_0^1 \left((1+\epsilon)^{2(m^2+1)-2}(1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}\|\Lambda^{2(m^2+1)}P(1+\epsilon)f^2|L_{1+\epsilon}\|\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}.$$

This means that for any even natural number m^2 with $m^2\geq 4$ we have

$$\left\|f^2\left|B^{2,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon}\right\|\right.$$

$$\sim \|f^2|L_{1+\epsilon}\| + \left(\int_0^1 \left((1+\epsilon)^{m^2-2}(1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}\|\Lambda^{m^2}P(1+\epsilon)f^2|L_{1+\epsilon}\|\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}. \quad (74)$$

Write $\|f^2\|^{\diamond}_{m^2}$ for the quasi-norm on the right-hand side of (74). We claim that for any $m^2\in\mathbb{N}$ with $m^2>2$ we have

$$\|\cdot\|^{\diamond}_{m^2} \sim \|\cdot\|^{\diamond}_{m^2+1} \quad \text{on} \quad B^{2,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon}. \quad (75)$$

Indeed, by (vi), given any $f^2\in L_{1+\epsilon}$ we have

$$\|\Lambda^{m^2+1}P(1+\epsilon)f^2|L_{1+\epsilon}\| = \|\Lambda P((1+\epsilon)/2)\Lambda^{m^2}P((1+\epsilon)/2)f^2|L_{1+\epsilon}\| \lesssim (1+\epsilon)^{-1}\|\Lambda^{m^2}P((1+\epsilon)/2)f^2|L_{1+\epsilon}\|.$$

Whence

$$\|f^2\|^{\diamond}_{m^2+1} \lesssim \|f^2|L_{1+\epsilon}\| + \left(\int_0^1 \left((1+\epsilon)^{m^2-2}(1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}\|\Lambda^{m^2}P((1+\epsilon)/2)f^2|L_{1+\epsilon}\|\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}$$

$$\sim \|f^2|L_{1+\epsilon}\| + \left(\int_0^{1/2} \left((1+\epsilon)^{m^2-2}(1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}\|\Lambda^{m^2}P(1+\epsilon)f^2|L_{1+\epsilon}\|\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon} \leq \|f^2\|^{\diamond}_{m^2}.$$

Conversely, by (69) and (68)

$$\|f^2\|^{\diamond}_{m^2} \lesssim \|f^2|L_{1+\epsilon}\| + \left(\int_0^1 \left((1+\epsilon)^{-2}(1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}K\left((1+\epsilon)^{m^2}, f^2; L_{1+\epsilon}, D(\Lambda^{m^2})\right)\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}$$

$$\sim \|f^2|L_{1+\epsilon}\| + \left(\int_0^1 \left((1+\epsilon)^{-2}(1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}\widetilde{K}\left((1+\epsilon)^{m^2}, f^2; L_{1+\epsilon}, D\left(\Lambda^{m^2}\right)\right)\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}.$$

As we have seen in the proof of Corollary (6.3.36),

$$\widetilde{K}\left((1+\epsilon)^{m^2}, f^2; L_{1+\epsilon}, D\left(\Lambda^{m^2}\right)\right) \sim \overline{\omega}_{m^2}((1+\epsilon)^{m^2}, f^2).$$

Moreover, by [310, Theorem 1.4, (1.7)]

$$\overline{\omega}_{m^2}((1+\epsilon)^{m^2}, f^2) \lesssim (1+\epsilon)^{m^2}\int_{1+\epsilon}^{\infty}\frac{\overline{\omega}_{m^2+1}((1+\epsilon)^{m^2+1}, f^2)}{(1+\epsilon)^{m^2+1}}d(1+\epsilon).$$

Now proceeding as in the proof of Corollary (6.3.36), using that $m^2 > 2$ and the extension of the Hardy inequality [256, Theorem 6.4], we obtain

$$\|f^2\|^{\diamond}_{m^2}$$

$$\lesssim \|f^2|L_{1+\epsilon}\| + \left(\int_0^1\left((1+\epsilon)^{m^2-2}(1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}\int_{1+\epsilon}^{\infty}\frac{\overline{\omega}_{m^2+1}\left((1+\epsilon)^{m^2+1}, f^2\right)}{(1+\epsilon)^{m^2+1}}d(1+\epsilon)\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}$$

$$\lesssim \|f^2|L_{1+\epsilon}\| + \left(\int_0^1\left((1+\epsilon)^{m^2-2}(1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}\int_{1+\epsilon}^{1}\frac{\overline{\omega}_{m^2+1}\left((1+\epsilon)^{m^2+1}, f^2\right)}{(1+\epsilon)^{m^2+1}}d(1+\epsilon)\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}$$

$$+\left(\int_0^1\left((1+\epsilon)^{m^2-2}(1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}\|f^2|L_{1+\epsilon}\|$$

$$\lesssim \|f^2|L_{1+\epsilon}\| + \left(\int_0^1\left((1+\epsilon)^{m^2-1}(1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}\frac{\overline{\omega}_{m^2+1}\left((1+\epsilon)^{m^2+1}, f^2\right)}{(1+\epsilon)^{m^2+1}}\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}$$

$$\sim \|f^2|L_{1+\epsilon}\| + \left(\int_0^1\left((1+\epsilon)^{-2}(1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}\left\|[P(1+\epsilon)-\mathrm{id}]^{m^2+1}f^2|L_{1+\epsilon}\right\|\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}$$

$$\lesssim \|f^2|L_{1+\epsilon}\| + \left(\int_0^1\left((1+\epsilon)^{-2}(1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}\int_0^{1+\epsilon}(1+\epsilon)^{m^2}\left\|\Lambda^{m^2+1}P(1+\epsilon)f^2|L_{1+\epsilon}\right\|\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}$$

where we have used (70) in the last inequality. The extension of the Hardy inequality implies now that

$$\|f^2\|^{\diamond}_{m^2}$$

$$\lesssim \|f^2|L_{1+\epsilon}\| + \left(\int_0^1\left((1+\epsilon)^{-1}(1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}(1+\epsilon)^{m^2}\left\|\Lambda^{m^2+1}P(1+\epsilon)f^2|L_{1+\epsilon}\right\|\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}$$

$$= \|f^2\|^{\diamond}_{m^2+1}.$$

This proves (75).

Now to complete the proof of the theorem we can proceed as in Corollary (6.3.38) with the help of the lift operator I_{-2}. Indeed, given any natural number $m^2 > 2$, since

$$\Lambda^{m^2}P(1+\epsilon)I_{-2}f^2 = \Lambda^2\Lambda^{m^2-2}I_{-2}P(1+\epsilon)f^2 = -\Delta I_{-2}\Lambda^{m^2-2}P(1+\epsilon)f^2,$$

by (74) and (75) we obtain

$$\left\|f^2\left|B^{0,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon}\right.\right\| \sim \left\|I_{-2}f^2\ \left|B^{2,(\epsilon^2+\epsilon-1)/1+\epsilon}_{1+\epsilon,1+\epsilon}\right.\right\| \sim \|I_{-2}f^2\|^{\diamond}_{m^2}$$

$$= \|I_{-2}f^2\ |L_{1+\epsilon}\| + \left(\int_0^1\left((1+\epsilon)^{m^2-2}(1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}\left\|\Delta I_{-2}\Lambda^{m^2-2}P(1+\epsilon)f^2|L_{1+\epsilon}\right\|\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}$$

$$\sim \|I_{-2}f^2\ |L_{1+\epsilon}\| + \left(\int_0^1\left((1+\epsilon)^{m^2-2}(1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}\left\|\Lambda^{m^2-2}P(1+\epsilon)f^2|L_{1+\epsilon}\right\|\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}$$

$$= \|I_{-2}f^2\ |L_{1+\epsilon}\| + \left(\int_0^1\left((1+\epsilon)^{m^2-2}(1-\log(1+\epsilon))^{(\epsilon^2+\epsilon-1)/1+\epsilon}\left\|\frac{\partial^{m^2-2}P(1+\epsilon)f^2}{\partial(1+\epsilon)^{m^2-2}}|L_{1+\epsilon}\right\|\right)^{1+\epsilon}\frac{d(1+\epsilon)}{(1+\epsilon)}\right)^{1/1+\epsilon}.$$

This finishes the proof.

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