Sudan University of Science and Technology

College of Graduate Studies

A Solution of Nonlinear Models for Concrete Beam

حل النماذج غير خطية للعارضة الخرسانية

A Thesis submitted in fulfillment for Ph.D.

Degree in Mathematics

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Dedication

To …

My lovely beloved father

To …

My sweetest beloved mother

I dedicate this research.

‘‘‘‘

Acknowledgements

All thanks to Allah the Almighty who gave me the strength, determination, health and helped me to successfully complete this research.

I cannot express enough thanks to my supervisors prof. Osman Mohammed El Mekki and Dr. Emadeldeen Abdallah Abdalrahim, for their continued guidance, support, unlimited help and encouragement, I offer my sincere appreciation for the learning opportunities provided to me by them.

My completing of this research could not have been possible without the support of my parents, family and friends. My deepest gratitude, and special appreciation and thanks to mathematical department staff for their encouragement when the times got very rough.

ABSTRACT

This thesis provides solutions for nonlinear partial differential equations, obtained from Concrete Beams Design (CBD), by extracting three partial differential equations of the beam model.

Including beam definition, types of beams, calculation of the equation of the elastic curve, and extract the equations from structure model by using Euler-Lagrange equation. The equations are solved by using Adomian's decomposition method, homotopy perturbation method, variational iteration method and finally it solved numerically by using finite differences method and spectral method, with design algorithms using MATLAB program.

The Adomian's decomposition method, homotopy perturbation method and variational iteration method give the solution in power series form, but the finite differences method and spectral method give approximation solution, it gave satisfactory results in agreement with the analytical solutions.

الخلاصة

هذا البحث عبارة عن حلول لمعادلات تفاضلية جزئية غير خطية تم الحصول عليها من تصميم العارضات الخرسانية، حيث تحصلنا على ثلاثة معادلات تفاضلية جزئية من نموذج للعارضة الخرسانية.

تم تعريف العارضة الخرسانية وانواعها ومعامل المرونة لها وحساب معادلة الانحاء لها، والحصول على المعادلات التفاضلية الجزئية من النموذج الانشائي للعارضة الخرسانية ذلك باستخدام معادلة اويلر لاجرانج، ومن ثم حلها عن طريقة ادوميان التفكيكية، طريقة هوموتوبي الاضطرابية وطريقة التكرار التغيرية. اخيرا تم الحل عدديا باستخدام الفروقات المحددة وطريق الطيف مع تصميم خوارزميات باستخدام برنامج الماتلاب، ذلك لان طريقة ادوميان التفكيكية، طريقة هوموتوبي الاضطرابية وطريقة التكرار التغيرية عبارة عن طرق تحليلية تعطي الحل في شكل متسلسلة لكن طريقة الفروقات المحددة وطريقة الطيف تعطيان حلول تقريبية وقد وجدنا توافقا بين الحلول العددية والحلول التحليلية.

INTRODUCTION

A beam is a bar-like structural member whose primary function is to support transverse loading and carry it to the supports and are the most common type of structural component, particularly in Civil Engineering.

The objective of a beam link the columns concrete, distribution the loads from the slab to columns concrete and dividing the slabs to parts section, so we obtain the best distribution moments.

One-dimensional mathematical models of structural beams are constructed on the basis of beam theories. Because beams are actually three-dimensional bodies, all models necessarily involve some form of approximation to the underlying physics.

The previous study calculation the deflection curve by approximate the bending to be small, so that the first derivative of deflection curve is equal to zero, but in this thesis we need to give the new solutions of bending of elastic beam without approximate the first derivative of deflection curve is equal to zero.

This study has been applied in three partial differential equations (PDEs) obtained from Concrete Beams Design (CBD), the equations solving by analytical methods and numerical methods. Comparison the analytical methods with exact solution when we choose the second term from series, which gave satisfied error.

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List of Abbreviations and Notation

Notation	Meaning
CBD	Concrete Beam Design
PDEs	Partial Differential Equations
BVP	Boundary Value Problem
Ω	The Problem's Domain
ADM	Adomian Decomposition Method
ADTM	Adomian Decomposition Transform Method
HPM	Homotopy Perturbation Method
HPTM	Homotopy Perturbation Transform Method
VPM	Variation of Parameter Method
FDM	Finite Differences Method
EEM	Euler's Explicit Method
EIM	Euler's Implicit Method
SM	Spectral Method
DFT	Discrete Fourier Transform

1 Introduction to Bending of Elastic Beams

1.1 Beams

A beam is a bar-like structural member whose primary function is to support transverse loading and carry it to the supports and are the most common type of structural component, particularly in Civil Engineering and Mechanical Engineering [1].

The 1700's and early 1800's were a productive period in which the mechanics of simple elastic structural elements were developed well before the beginnings in the 1820's of the general three-dimensional theory. The development of beam theory by Euler, who generally modeled beams as elastic lines which resist bending, and by several members of the Bernoulli family and by Coulomb, remains among the most immediately useful aspects of solid mechanics, in part for its simplicity and in part because of the pervasiveness of beams and columns in structural technology. James Bernoulli proposed in 1705 that the curvature of a beam was proportional to bending moment. Euler in 1744 and John's son, Daniel Bernoulli (1700-1782) in 1751 used the theory to address the transverse vibrations of beams, and Euler gave in 1757 his famous analysis of the buckling of an initially straight beam subjected to a compressive loading; the beam is then commonly called a column. Following a suggestion of Daniel Bernoulli in 1742, Euler in 1744 introduced the strain energy per unit length for a beam, proportional to the square of its curvature, and regarded the total strain energy as the quantity analogous to the potential energy of a discrete mechanical system [2].

By adopting procedures that were becoming familiar in analytical mechanics, and following from the principle of virtual work as introduced by John Bernoulli for discrete systems such as pin-connected rigid bodies in 1717, Euler rendered the energy stationary and in this way developed the calculus of variations as an approach to the equations of equilibrium and motion of elastic structures. That same variational approach played a major role in the development by French mathematicians in the early 1800's of a theory of small transverse displacements and vibrations of elastic plates. This theory was developed in preliminary form by Sophie Germain and partly improved upon by Simeon Denis Poisson in the early 1810's; they considered a flat plate as an elastic plane which resists curvature. Navier gave a definitive development of the correct energy expression and governing differential equation a few years later. An uncertainty of some duration in the theory arose from the fact that the final partial differential equation for the transverse displacement is such that it is impossible to prescribe, simultaneously, along an unsupported edge of the plate, both the twisting moment per unit length of middle surface and the transverse shear force per unit length. This was finally resolved in 1850 by German physicist Gustav Robert Kirchhoff in an application of virtual work and variational calculus procedures, in the framework of simplifying kinematic assumptions that fibers initially perpendicular to the plate middle surface remain so after deformation of that surface. As first steps in the theory of thin shells, in the 1770's Euler addressed the deformation of an initially curved beam, as an elastic line, and provided a simplified analysis of vibration of an elastic bell as an array of annular beams. John's grandson, through a son and mathematician also named John, James Bernoulli "the younger" (1759-1789) further developed this model in the last year of his life as a two dimensional network of elastic lines, but could not develop an acceptable treatment, Shell theory was not to attract attention for a century after Euler's work, as the outcome of many researches following the first consideration of shells from a three dimensional elastic viewpoint by H. Aron in 1873. Acceptable thin-shell theories for general situations, appropriate for cases of small deformation, were developed by English mathematician, mechanician and geophysicist A. E. H. Love in 1888 and mathematician and physicist Horace Lamb in 1890 (there is no uniquely correct theory as the Dutch applied mechanician and engineer W.T.Koiter and Russian mechanician V.V.Novozhilov were to clarify in the 1950's; the difference between predictions of acceptable theories is small when the ratio of shell thickness to a typical length scale is small). Shell theory remained of immense interest well beyond the mid-1900's in part because so many problems lay beyond the linear theory (rather small transverse displacements often dramatically alter the way that a shell supports load by a combination of bending and membrane action), and in part because of the interest in such light-weight structural forms for aeronautical technology [3].

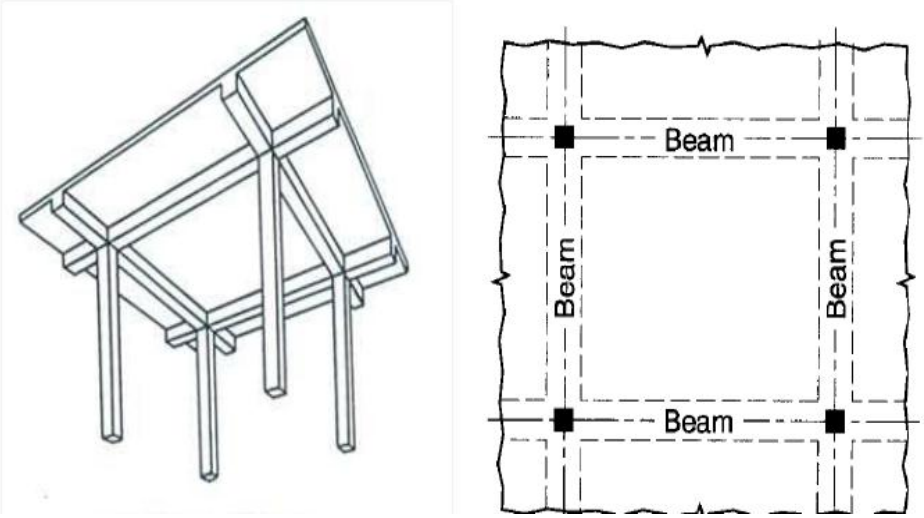

Figure 1.1: Beam with slab

1.1.1 Types of Beams

i. T Beam

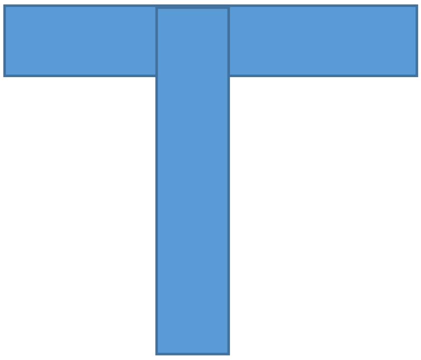

Figure 1.2: T beam

ii. L Beam

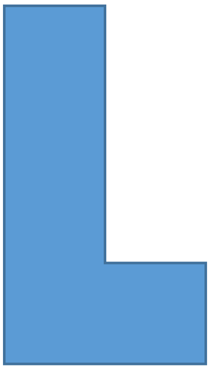

Figure 1.3: L beam

iii. I Beam

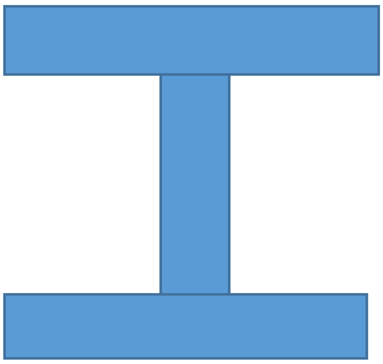

Figure 1.4: I beam

iv. H Beam

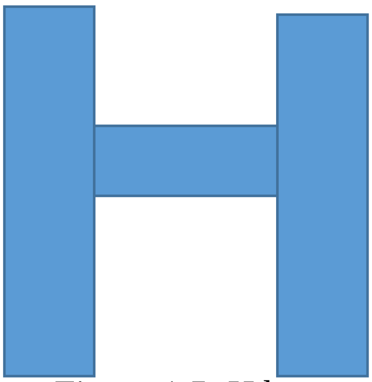

Figure 1.5: H beam

v. Box Beam

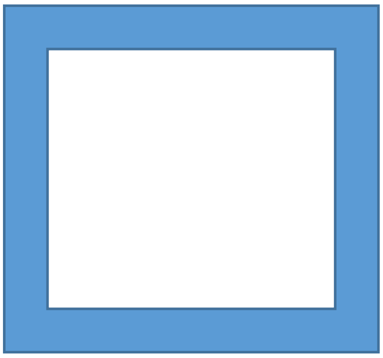

Figure 1.6: Box beam

1.1.2 The Uses of Beams

i. Link the concrete columns
ii. Distribution the loads from the slab to concrete columns
iii. Dividing the slabs to parts section, so we obtain the beast distribution moments.

1.2 Strength of Materials (Elasticity)

Linear elasticity as a general three-dimensional theory was in hand in the early 1820's based on Cauchy's work. Simultaneously, Navier had developed an elasticity theory based on a simple corpuscular, or particle, model of matter in which particles interacted with their neighbors by a central-force attraction between particle pairs. As was gradually realized following works by Navier, Cauchy and Poisson in the 1820's and 1830's the particle model is too simple and makes predictions concerning relations among elastic moduli which are not met by experiment. In the isotropic case it predicts that there is only one elastic constant and that the Poisson ratio has the universal value of 1/4.Most subsequent development of the subject was in terms of the continuum theory. Controversies concerning the maximum possible number of independent elastic moduli in the most general anisotropic solid were settled by English mathematician George Green in 1837, through pointing out that the existence of an elastic strain energy required that of the 36 elastic constants, relating the six stress components to the six strains, at most 21 could be independent. Scottish physicist Lord Kelvin (William Thomson) put this consideration on sounder ground in 1855 as part of his development of macroscopic thermodynamics, in much the form as it is known today, showing that a strain energy function must exist for reversible isothermal or adiabatic (isentropic) response, and working out such results as the (very modest) temperature changes associated with isentropic elastic deformation. The middle and late 1800's were a period in which many basic elastic solutions were derived and applied to technology and to the explanation of natural phenomena. French mathematician Barre de Saint-Venant derived in the 1850's solutions for the torsion of non-circular cylinders, which explained the necessity of warping displacement of the cross section in the direction parallel to the axis of twisting, and for flexure of beams due to transverse loadings. The German physicist Heinrich Rudolph Hertz developed solutions for the deformation of elastic solids as they are brought into contact, and applied these to model details of impact collisions. Solutions for stress and displacement due to concentrated forces acting at an interior point of a full space were derived by Kelvin, and on the surface of a half space by mathematicians J. V. Bousinesq (French) and V. Cerruti(Italian). The Prussian mathematician L. Pochhammer analyzed the vibrations of an elastic cylinder and Lamb and the Prussian physicist P. Jaerisch derived the equations of general vibration of an elastic sphere in the 1880, an effort that was continued by many seismologists in the 1900's to describe the vibrations of the Earth. Kelvin derived in 1863 the basic form of the solution of the static elasticity equations for a spherical solid, and these were applied in following years to such problems as deformation of the Earth due to rotation and to tidal forcing, and to effects of elastic deformability on the motions of the Earth's rotation axis.

The classical development of elasticity never fully confronted the problem of finite elastic straining, in which material fibers change their lengths by other than very small amounts. Possibly this was because the common materials of construction would remain elastic only for very small strains before exhibiting either plastic straining or brittle failure. However, natural polymeric materials show elasticity over a far wider range (usually also with enough time or rate effects that they would more accurately be characterized as viscoelastic), and the widespread use of natural rubber and like materials motivated the development of finite elasticity. While many roots of the subject were laid in the classical theory, especially in the work of Green, G. Piola and Kirchhoff in the mid-1800's, the development of a viable theory with forms of stress-strain relations for specific rubbery elastic materials, and an understanding of the physical effects of the nonlinearity in simple problems like torsion and bending, is mainly the achievement of British American engineer and applied mathematician Ronald S. Rivlin in the 1940’s and 1950’s [3, 4, 5].

1.2.1 Moduli of Elasticity

Material properties that indicates stiffness and rigidity, Values of E for many materials are readily available in table some common values are.

Table 1.1: Elasticity modulus for same materials

Material	Modulus of Elasticity (E)
Steel	30×10^6
Aluminum	10×10^6
Wood	2×10^6

1.3 Simple Bending Theory

The simple theory of elastic bending states that

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

where M is the applied bending moment (B.M.) at a transverse section, I is the second moment of area of the beam cross-section about the neutral axis (N.A.), σ is the stress at distance y from the N.A. of the beam cross-section, E is the Young’s modulus of elasticity for the beam material, and R is the radius of curvature of the N.A. at the section. In order for this to be achieved it is necessary to make certain simplifying assumptions, and for this reason the theory introduced below is often termed the simple theory of bending. The assumptions are as follows:

1. The beam is initially straight and unstressed.
2. The material of the beam is perfectly homogeneous and isotropic, i.e. of the same density and elastic properties throughout.
3. The elastic limit is nowhere exceeded
4. Young's modulus for the material is the same in tension and compression.
5. Plane cross-sections remain plane before and after bending.

6. Every cross-section of the beam is symmetrical about the plane of bending, i.e. about an axis perpendicular to the N.A.
7. There is no resultant force perpendicular to any cross-section.

If we now consider a beam initially unstressed and subjected to a constant B.M. along its length, i.e. pure bending, as would be obtained by applying equal couples at each end, it will bend to a radius R as shown in (Figure 1.7b**).** As a result of this bending the top fibres of the beam will be subjected to tension and the bottom to compression. It is reasonable to suppose, therefore, that somewhere between the two there are points at which the stress is zero. The locus of all such points is termed the neutral axis. The radius of curvature R is then measured to this axis. For symmetrical sections the N.A. is the axis of symmetry, but whatever the section the N.A. will always pass through the centre of area or centroid [6].

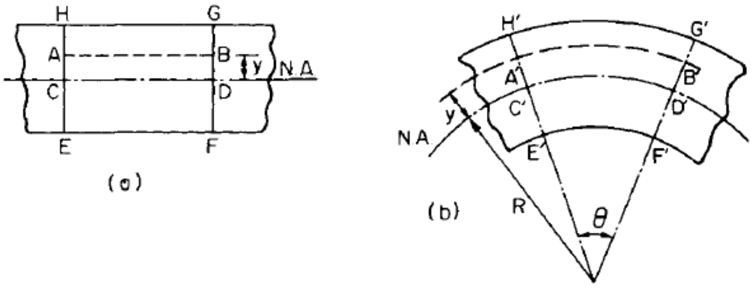

Fig (1.7): Beam subjected to pure bending (a) before, and (b) after, the moment *M* has been applied.

Consider now two cross-sections of a beam, HE and GF, originally parallel (Fig. 1.7a**).** When the beam is bent (Fig. 1.7b) it is assumed that these sections remain plane; i.e. H E and GF', the final positions of the sections, are still straight lines. They will then subtend some angle θ.

Consider now some fibre *AB* in the material, distance ***y*** from the N.A. When the beam is bent this will stretch to $A'B'$**.**

$$strain\ fibre\ AB = \frac{extension}{original\ length} = \frac{A'B' - AB}{AB}$$

But $A'B' = CD$ and since the N.A is unstressed $CD = C'D'$

$$strain = \frac{A'B' - C'D'}{C'D'} = \frac{(R+y)\theta - y\theta}{R\theta} = \frac{y}{R}$$

But $$\frac{stress}{strain} = young'smoduls\ (E)$$

Than $$strain = \frac{\sigma}{E}$$

Equating the two equations for strain

$$\frac{y}{R} = \frac{\sigma}{E} \tag{1.1}$$

Consider now cross section of the beam (Figure 1.7) for equation (1.1) the stress is $\sigma = \frac{E}{R} y$

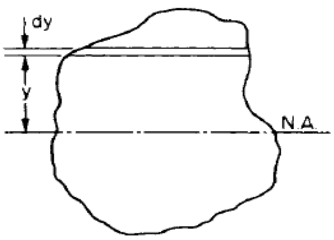

Figure 1.8: Beam cross section

If the strip is of area, δA the force on the strip is

$$F = \sigma \delta A = \frac{E}{R} y \delta A$$

This has a moment about the N.A. of

$$Fy = \sigma y \delta A = \frac{E}{R} y^2 \delta A$$

The total moment for the whole cross-section is therefore

$$M = \sum \frac{E}{R} y^2 \delta A = \frac{E}{R} \sum y^2 \delta A$$

Since E and R constants.

The term $\sum \ y^2 \delta A$ is called the second moment of area of the cross-section and given the symbol I

$$M = \frac{E}{R} I \quad or \quad \frac{M}{I} = \frac{E}{R} \tag{1.2}$$

Combining equations (1.1) and (1.2) we have the bending theory equations

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R} \tag{1.3}$$

From Eq. (1.2) it will be seen that if the beam is of uniform section, the material of the beam is homogeneous and the applied moment is constant, the values of I, E and M remain constant and hence the radius of curvature of the bent beam will also be constant. Thus for pure bending of uniform sections, beams will deflect into circular arcs and for this reason the term circular bending is often used. From Eq. (1.2) the radius of curvature to which any beam is bent by an applied moment M is given by:

$$R = \frac{EI}{M}$$

Thus directly related to the value of the quantity ***EI*.** Since the radius of curvature is a direct indication of the degree of flexibility of the beam (the larger the value of R, the smaller the deflection and the greater the rigidity) the quantity *EI* is often termed the jexural rigidity or flexural stiflness of the beam. The relative stiffnesses of beam sections can then easily be compared by their *El* values. It should be observed here that the above proof has involved the assumption of pure bending without any shear being present. [6]

1.4 The Euler–Bernoulli Law of Linear and Nonlinear Deformations for Structural Members

The first public work regarding the large deformation of flexible members was given by L. Euler (1707–1783) in 1744, according to Euler, when a member is subjected to bending, we cannot neglect**,** the slope of the deflection curve in the expression of the curvature unless the deflections are small. Euler has published about 75 substantial volumes, he was a dominant figure during the 18th century, and his contributions to both pure and applied mathematics made him worthy of inclusion in the short list of giants of mathematics Archimedes, I. Newton (1642–1727), and C. Gauss (1777–1855).

We should point out, however, that the development of this theory took place in the 18th century, and the credits for this work should be given to Jacob Bernoulli (1654–1705), his younger brother Johann Bernoulli (1667–1748), and Leonhard Euler (1707–1783). Both Bernoulli brothers have contributed heavily in the mathematical sciences and related areas. They also worked on the mathematical treatment of the Greek problems of isochrone, brahistochrone, isoperimetric figures, and geodesies, which led to the development of the new calculus known as the calculus of variations**.** Jacob also introduced the word integral in suggesting the name calculus integrals. G.W. Leibniz (1646–1716) used the name calculus summatorius for the inverse of the calculus differential [5].

The Euler–Bernoulli law states that the bending moment M is proportional to the change in the curvature produced by the action of the load. This law may be written mathematically as follows:

$$\frac{1}{\rho} = \frac{d\theta}{dx_0} = \frac{M}{EI} \tag{1.4}$$

where ρ is radius of curvature, θ is the slope at any point x_0, x_0 is measured along the arc length of the member as shown in Fig: 1.1a, M the bending moment, E the modulus of elasticity, and I the moment of inertia of cross section about its neutral axis.

When a beam is subjected to a transverse loading, Eq. (1.4) remains valid for any given transverse section, provided that saint-venant's principle applies. However, both the bending moment and the curvature of the neutral of the section from the left end of the beam, we write

$$\frac{1}{\rho} = \frac{M(x)}{EI} \tag{1.5}$$

1.4.1 The Classical Model of Elastic Beam-Theory

$$EI\frac{d^4y}{dx^4} + p\frac{d^2y}{dx^2} = q \quad , \; 0 < x < L \tag{1.6}$$

where EI, y, P and q represent the flexural rigidity, the lateral deflection, the axial compressive force, and the intensity of lateral load, respectively. Equation (1.6) is usually studied with one of the following pairs of boundary conditions, which depend on how the ends of the beam are supported.

- Embedded at both ends

$$y(0) = y'(0) = y(L) = y'(L) = 0 \tag{1.7a}$$

- Embedded at one end, free at the other (cantilever)

$$y(0) = y'(0) = y''(L) = y'''(L) = 0 \tag{1.7b}$$

- Simply supported at both ends

$$y(0) = y''(0) = y(L) = y''(L) = 0 \tag{1.7c}$$

- Periodic boundary conditions

$$y^{(i)}(0) = y^{(i)}(L), \; i = 0,1,2,3 \tag{1.7d}$$

If $P = 0$, Eq (1.6) is called the Euler-Bernoulli equation which usually desc--ribes the relationship between the deflection of the beam and applied load.

1.5 Equation of the Elastic Curve

We first recall from elementary calculus that the curvature of a plane curve at a point $Q(x, y)$ of the curve expressed as:

$$\frac{1}{\rho} = \frac{y''}{(1 + (y')^2)^{\frac{3}{2}}} \tag{1.8}$$

where y' and y'' are the first and second derivatives of the function $y(x)$ represented by that curve, but in the cases of the elastic curve of a beam the slope y' is very small, and its square is negligible compared to unity. We may write, therefore,

$$\frac{1}{\rho} = \frac{d^2y}{dx^2} \tag{1.9}$$

Substituting for $\frac{I}{\rho}$ from (1.8) into (1.9), we have

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI} \tag{1.9}$$

The equation obtained is a second order linear differential equation, it is the governing differential equation for the elastic curve.

The product EI is known as the flexural rigidity and, if it varies along the beam, as in the case of a beam of varying depth, we must express it as a function of x before proceeding to integration Eq. (1.9). However, in the case of a prismatic beam, which is the case considered here, the flexural rigidity is constant. We may thus multiply both members of Eq. (1.9). by EI and integration in x. We write

$$EI\frac{dy}{dx} = \int_0^x M(x)dx + c_1 \tag{1.10}$$

where c_1 is constant of integration. Denoting by $\theta(x)$ the angle, measured in radians, that the tangent at Q to the elastic curve forms with the horizontal Fig 1.13, and recalling that this angle is very small, we have

$$\frac{dy}{dx} = \tan(\theta) \cong \theta(x) \tag{1.11}$$

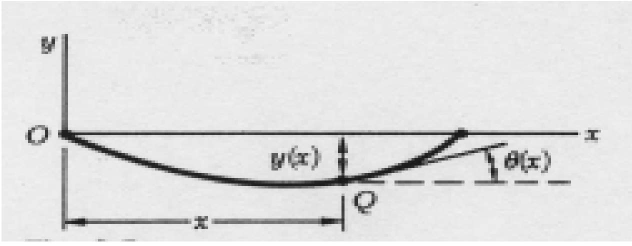

Figure 1.9: Deflection beam

Thus, we may write Eq. (1.10) in the alternate form

$$EI\theta(x) = \int_0^x M(x)dx + c_1 \tag{1.12}$$

Integration both members of Eq. (1.10) in x, we have

$$EIy(x) = \int_0^x \left(\int_0^x M(x)dx + c_1 \right) + c_2 \tag{1.13}$$

where c_2 is second constant of integration, and where the first term in the right hand side represents the function of x obtained by integration twice in x the bending moment $M(x)$. If it were not for the fact the constants $c_1$and c_2 are as yet undetermined, Eq. (1.13) would define the deflection of the beam at any given point Q, and Eq. (1.2) would similarly define the slope of the beam at Q.

The constants $c_1$and c_2 are determined from boundary conditions or more precisely, from the conditions imposed on the beam by its supports. Limiting our analysis in this section to statically determinate beams, i.e. to beams supported in such a way that the reactions at the supports may be obtained by the methods of statics, we note that only three types of beams need to be considered here (Figure 1.10):

i. The simply supported beam
ii. The overhanging beam
iii. The cantilever beam

In the first two cases, the supports consist of a pin and bracket at A and of a roller at B, and require that the deflection be zero at each of these points. Letting first $x = x_A$, $y = y_A = 0$ in Eq. (1.13), and then $x = x_B$,

$y = y_B = 0$ in the same equation, we obtain two equations which may be solved for $c_1$and $c_2$in the case of the cantilever beam (Figure 1.10c), we note that both the deflection and the slope at A must be zero. Letting $x = x_A$, $y = y_A = 0$ in Eq. (1.13), and $x = x_A$, $\theta = \theta_A = 0$ in Eq. (1.11), we obtain again two equations which may be solved for $c_1$and c_2 [5].

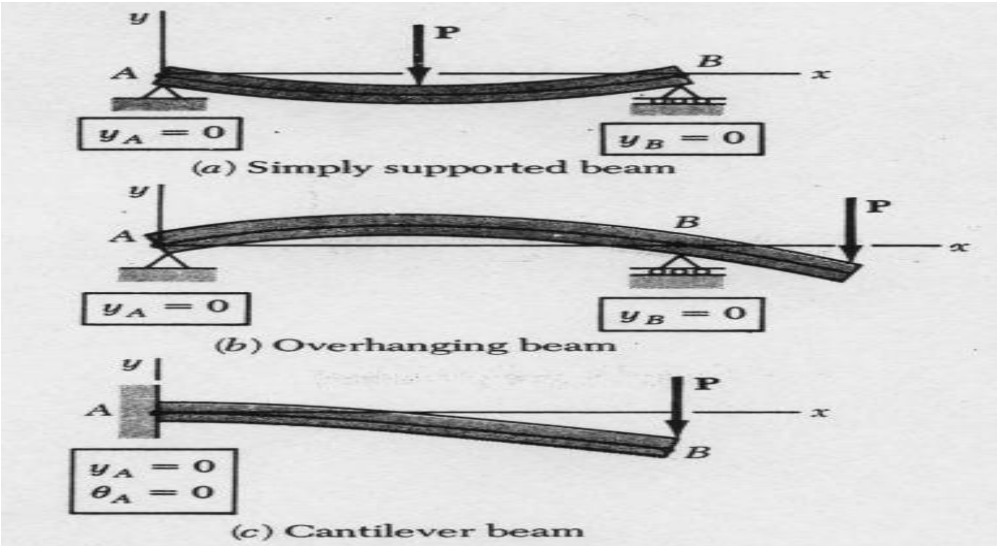

Figure 1.10 Boundary conditions for statically determinate beams.

1.6 Equation of Woinwsky-Krieger

Its nonlinear beam equation denoded by

$$\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial^4 u}{\partial x^4} - \left(\beta + k \int_0^L \left(u_\xi(\xi, t) \right)^2 d\xi \right) \frac{\partial^2 u}{\partial x^2} = 0 \qquad (1.14)$$

where the constants α , β and k are positive. Equation (1.14) was proposed by Woinwsky-Krieger as a model for the transverse deflection $u(x, t)$ of an extensible beam of natural length L whose ends are held a fixed distance apart. The nonlinear term represents the change in the tension of the beam due to its extensibility. The model has also beam discussed by Eisley. Dickey recently considered the initial-boundary value problem for (1.14) in the case when the ends of beam are hinged, so that

$$u(0, t) = u(L, t) = u_{xx}(0, t) = u_{xx}(L, t) = 0$$

Example 1.1: the cantilever beam AB is of uniform cross section and carries a load P at its free end A (Figure 1.11). Determine the equation of the elastic curve, deflection and slope at A.

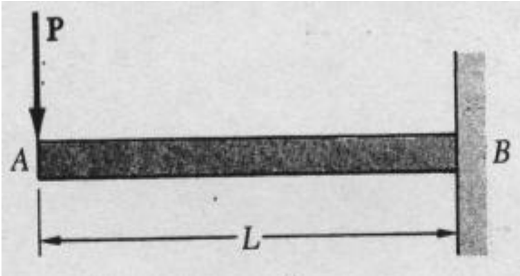

Figure 1.11

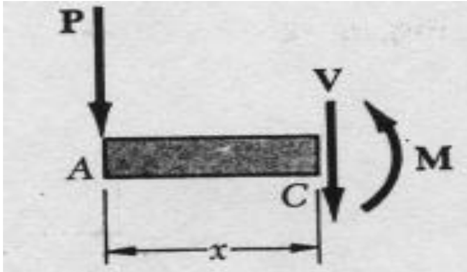

Figure 1.12

Using the free body diagram of the portion AC of the beam (Figure 1.12), where C is located at a distance x from end A, we find

$$M = -Px$$

Substituting for M into Eq. (1.4) and multiply both members by the EI, we write

$$EI\frac{d^2y}{dx^2} = -Px$$

Integration in x, we obtain

$$EI\frac{dy}{dx} = -0.5Px^2 + c_1 \qquad (1.15)$$

We now observe that at the fixed end B we have $x = L$ and $\theta = \frac{dy}{dx} = 0$

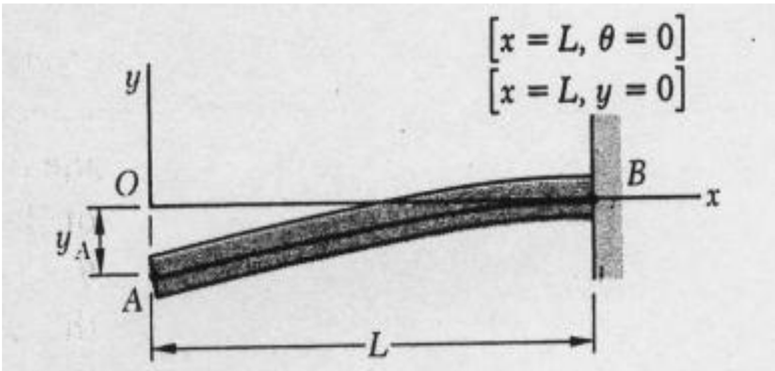

Figure 1.13

Substituting theses values into Eq. (1.15), and solving for c_1, we have

$$c_1 = 0.5PL^2$$

Then

$$EI\frac{dy}{dx} = -0.5 * Px^2 + 0.5PL^2 \qquad (1.16)$$

Integration both members, we write

$$EIy(x) = -\frac{1}{6}Px^3 + \frac{1}{2}PL^2x + c_2 \qquad (1.17)$$

But, at B we have $x = L$, $y = 0$. Substituting into Eq. (1.17) we have

$$c_2 = \frac{1}{3}PL^3$$

Substituting the value of c_2 in Eq. (1.17), we obtain the equation of elastic curve

$$y(x) = \frac{1}{EI}(-\frac{1}{6}Px^3 + \frac{1}{2}PL^2x + \frac{1}{3}PL^3)$$

$$y(x) = \frac{P}{6EI}(-x^3 + 3L^2x - 2L^3) \qquad (1.18)$$

The deflection and slope at A are obtained by letting $x = 0$ in Eq. (1.18), and (1.16). We find

$$y_A = -\frac{PL^3}{3EI} \quad and \quad \theta_A = \frac{PL^2}{2EI}$$

Example 1.2: the simply supported prismatic beam AB carries a uniformly distributed load w per unit length (Fig: 1.14). Determine the equation of elastic curve.

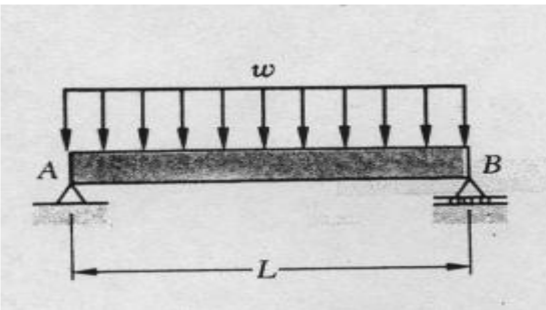

Figure 1.14

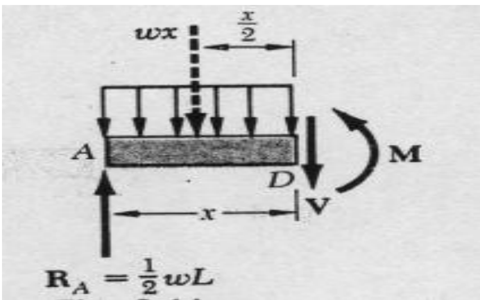

Figure 1.15

Drawing the free body diagram of the portion AD of the beam (Figure 1.15), and taking moments about D, we find that

$$M = \frac{1}{2}wLx - \frac{1}{2}Pwx^2$$

Substituting for M into Eq. (1.4) and multiply by EI, we get

$$EI\frac{d^2y}{dx^2} = \frac{1}{2}wLx - \frac{1}{2}Pwx^2$$

Integration twice in x, we have

$$EI\frac{dy}{dx} = \frac{1}{4}wLx^2 - \frac{1}{6}Pwx^3 + c_1 \qquad (1.19)$$

We now observe that at the fixed end B we have $x = L$ and $\theta = \frac{dy}{dx} = 0$

Substituting theses values into Eq. (1.19), and solving for c_1, we have

$$c_1 = -\frac{1}{12}PwL^3$$

Then

$$EI\frac{dy}{dx} = \frac{1}{4}wLx^2 - \frac{1}{6}Pwx^3 - \frac{1}{12}PwL^3 \qquad (1.20)$$

Integration both members, we write

$$EIy(x) = \frac{1}{12}wLx^3 - \frac{1}{24}Pwx^4 - \frac{1}{12}PwL^3x + c_2 \qquad (1.21)$$

But, at B we have $x = L$, $y = 0$. Substituting into Eq. (1.21) we have

$$c_2 = \frac{1}{8}PwL^4 - \frac{1}{12}wL^4$$

Substituting the value of c_2 in Eq. (1.17), we obtain the equation of elastic curve

$$y(x) = \frac{1}{EI}\left(\frac{1}{12}wLx^3 - \frac{1}{24}Pwx^4 - \frac{1}{12}PwL^3x + \frac{1}{8}PwL^4 - \frac{1}{12}wL^4\right)$$

2 Model of Bending of Elastic Beams

Consider a straight beam on an elastic foundation with length L, a cross-section A, a mass per unit length μ, moment of inertia I, ρ is radius of curvature and modulus of elasticity E that subjected to an axial force of magnitude F as shown in Fig. 2.1. We first recall from elementary calculus that the curvature of a plane curve at a point $Q(x, u)$ as shown in Fig. 2.1, of the curve expressed as

$$\frac{1}{\rho} = \frac{u_{xx}}{(1 + (u_x)^2)^{\frac{3}{2}}}$$

where u_x *and* u_{xx} are the first and second derivatives of function u represented by that curve but, in the case of the elastic curve of a beam the slope u_x is very small.

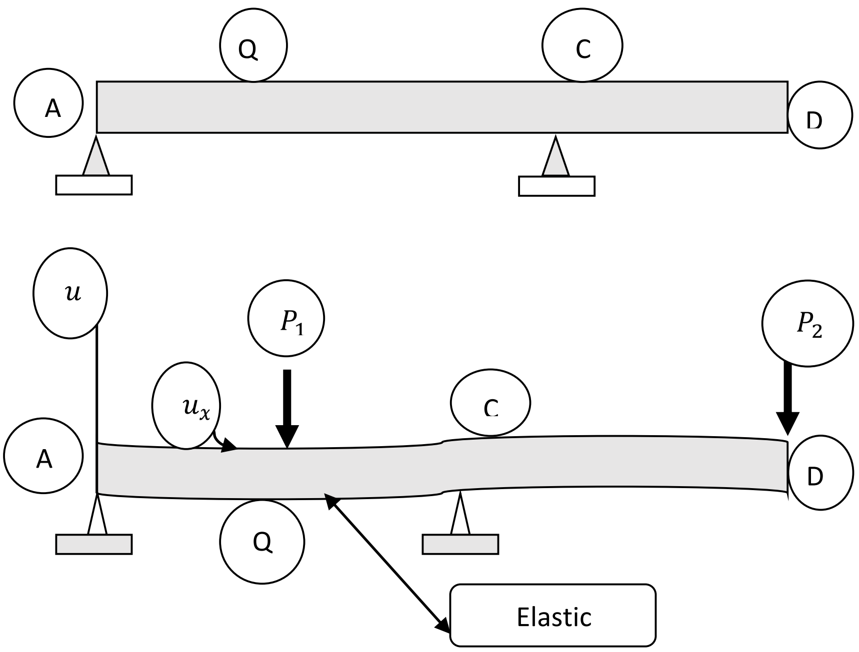

Figure 2.1: the elastic curve of a beam

2.1 Hamilton's Principle

$$H = \int L dt$$

where L is called Lagrange's function defined by

$$L = T - V$$

T is Kinetic Energy defined by

$$T = \frac{1}{2}\mu \int (u_t)^2 dx$$

V is Potential Energy defined by

$$V = \frac{1}{2}k \int \frac{(u_{xx})^2}{(1 + (u_x)^2)^{\frac{3}{2}}} dx$$

Substituting the kinetic energy and potential energy in Hamilton principle, we obtain.

$$J[u] = \frac{1}{2}\iint_S \left\{\mu(u_t)^2 - \frac{k(u_{xx})^2}{(1 + (u_x)^2)^{\frac{3}{2}}}\right\} dxdt$$

2.1.1 First Case: Obtain the Linear Equation of Transvers Vibration of Beam.

In this case, we assume the bending to be small so that the deflection $u(x,t)$ and its derivative are Small, i.e $(u_x)^2 = 0$

Using the Euler - Lagrange equation. Suppose $f = \mu(u_t)^2 - k(u_{xx})^2$

Therefor the Euler – Lagrange equation

$$\frac{\partial f}{\partial u} - \frac{d}{dx}\left(\frac{\partial f}{\partial u_x}\right) - \frac{d}{dt}\left(\frac{\partial f}{\partial u_t}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial u_{xx}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial f}{\partial u_{tt}}\right) = 0$$

$$-\frac{d}{dt}(2\,\mu(u_t) - \frac{d^2}{dx^2}(2ku_{xx}) = 0$$

$$-2\,\mu u_{tt} - 2ku_{xxxx} = 0$$

$$\frac{\partial^2 u}{\partial t^2} + \frac{k}{\mu}\frac{\partial^4 u}{\partial x^4} = 0$$

u is arbitrary variable the transverse vibration of beam, and $k = \frac{EI}{A}$

$$\frac{\partial^2 u}{\partial t^2} + \frac{EI}{A\mu}\frac{\partial^4 u}{\partial x^4} = 0$$

$Let\, us \quad \alpha^2 = \frac{EI}{A\mu}$

$$\frac{\partial^2 u}{\partial t^2} + \alpha^2 \frac{\partial^4 u}{\partial x^4} = 0 \qquad (2.1)$$

2.1.2 Second Case: Obtain the Nonlinear Equation of Transvers Vibration of Beam.

In this case we assumption ${u'}^2 \ll 1$. Suppose $f = \mu(u_t)^2 - \frac{k(u_{xx})^2}{(1+(u_x)^2)^{\frac{3}{2}}}$.

Therefor the Euler – Lagrange equation

$$\frac{\partial f}{\partial u} - \frac{d}{dx}\left(\frac{\partial f}{\partial u_x}\right) - \frac{d}{dt}\left(\frac{\partial f}{\partial u_t}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial u_{xx}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial f}{\partial u_{tt}}\right) = 0$$

$$-\frac{d}{dt}(2\,\mu(u_t) - \frac{d}{dx}\left(k(u_{xx})^2(u_x)\left(-3(1+(u_x)^2)^{\frac{-5}{2}}\right)\right)$$
$$-\frac{d^2}{dx^2}\left(\frac{2ku_{xx}}{(1+(u_x)^2)^{\frac{3}{2}}}\right) = 0$$

$$-\frac{d}{dt}(2\,\mu(u_t) + 3k\frac{d}{dx}\left((u_{xx})^2 u_x(1+(u_x)^2)^{\frac{-5}{2}}\right)$$
$$+ 2k\frac{d}{dx}\left(\frac{(1+(u_x)^2)^{\frac{3}{2}}(u_{xxx}) - 3u_x(u_{xx})^2(1+(u_x)^2)^{\frac{1}{2}}}{(1+(u_x)^2)^3}\right)$$

$$-2\,\mu u_{tt} + 3k\Big(-5(u_x)^2(u_{xx})^3(1+(u_x)^2)^{\frac{-7}{2}} + (u_{xx})^3(1+(u_x)^2)^{\frac{-5}{2}}$$
$$+ 2u_x u_{xx} u_{xxx}(1+(u_x)^2)^{\frac{-5}{2}}\Big)$$
$$+ 2k\frac{d}{dx}\Big((u_{xxx})(1+(u_x)^2)^{\frac{-3}{2}}$$
$$- 3u_x(u_{xx})^2(1+(u_x)^2)^{\frac{-5}{2}}\Big) = 0$$

$$-2\,\mu u_{tt} + 3k\left(-5(u_x)^2(u_{xx})^3(1+(u_x)^2)^{\frac{-7}{2}} + (u_{xx})^3(1+(u_x)^2)^{\frac{-5}{2}} + 2u_x u_{xx} u_{xxx}(1+(u_x)^2)^{\frac{-5}{2}}\right) + 2k\left(-3u_x u_{xx} u_{xxx}(1+(u_x)^2)^{\frac{-5}{2}} + u_{xxxx}(1+(u_x)^2)^{\frac{-3}{2}}\right) + 15(u_x)^2(u_{xx})^3 - 3(u_{xx})^3(1+(u_x)^2)^{\frac{-5}{2}} - 6u_x u_{xx} u_{xxx}(1+(u_x)^2)^{\frac{-5}{2}} = 0$$

$$u_{tt} = \frac{k}{\mu}\left[\frac{u_{xxxx}}{(1+(u_x)^2)^{\frac{3}{2}}} - \frac{3(u_{xx})^3}{2(1+(u_x)^2)^{\frac{5}{2}}} - \frac{6u_x u_{xx} u_{xxx}}{(1+(u_x)^2)^{\frac{5}{2}}} - \frac{45(u_x)^2(u_{xx})^3}{2(1+(u_x)^2)^{\frac{7}{2}}}\right]$$

When ${u'}^2 \ll 1$ all derivatives we obtain the Nonlinear Transverse Vibrations of a beam equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{k}{\mu}\left[\frac{\partial^4 u}{\partial x^4} - \frac{3}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)^3 - 6\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 u}{\partial x^3} - \frac{45}{2}\left(\frac{\partial u}{\partial x}\right)^2\left(\frac{\partial^2 u}{\partial x^2}\right)^3\right]$$

Let us $a = \frac{k}{\mu}$, $b = -\frac{3k}{2\mu}$, $c = -\frac{6k}{\mu}$, $d = -\frac{45k}{2\mu}$

$$\frac{\partial^2 u}{\partial t^2} = a\frac{\partial^4 u}{\partial x^4} + b\left(\frac{\partial^2 u}{\partial x^2}\right)^3 + c\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 u}{\partial x^3} + d\left(\frac{\partial u}{\partial x}\right)^2\left(\frac{\partial^2 u}{\partial x^2}\right)^3 \qquad (2.2)$$

2.1.3 Third Case: Obtain the Nonlinear Equation of Transvers Vibration of a Beam.

In this case we assumption without approximation ${u'}^2 \ll 1$.

Suppose $f = \mu(u_t)^2 - \frac{k(u_{xx})^2}{(1+(u_x)^2)^{\frac{3}{2}}}$. The Euler – Lagrange equation

$$\frac{\partial f}{\partial u} - \frac{d}{dx}\left(\frac{\partial f}{\partial u_x}\right) - \frac{d}{dt}\left(\frac{\partial f}{\partial u_t}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial u_{xx}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial f}{\partial u_{tt}}\right) = 0$$

$$-\frac{d}{dt}(2\,\rho\mu) - \frac{d}{dx}\left(k(u_{xx})^2(u_x)\left(-3(1+(u_x)^2)^{\frac{-5}{2}}\right)\right) - \frac{d^2}{dx^2}\left(\frac{2ku_{xx}}{(1+(u_x)^2)^{\frac{3}{2}}}\right) = 0$$

$$-\frac{d}{dt}(2\,\mu(u_t)+3k\frac{d}{dx}\left((u_{xx})^2u_x(1+(u_x)^2)^{\frac{-5}{2}}\right)$$
$$-2k\frac{d}{dx}\left(\frac{(1+(u_x)^2)^{\frac{3}{2}}(u_{xxx})-3u_x(u_{xx})^2(1+(u_x)^2)^{\frac{1}{2}}}{(1+(u_x)^2)^3}\right)$$

$$-2\,\mu u_{tt}+3k\Big(-5(u_x)^2(u_{xx})^3(1+(u_x)^2)^{\frac{-7}{2}}+(u_{xx})^3(1+(u_x)^2)^{\frac{-5}{2}}$$
$$+2u_xu_{xx}u_{xxx}(1+(u_x)^2)^{\frac{-5}{2}}\Big)$$
$$+2k\frac{d}{dx}\Big((u_{xxx})(1+(u_x)^2)^{\frac{-3}{2}}$$
$$-3u_x(u_{xx})^2(1+(u_x)^2)^{\frac{-5}{2}}\Big)=0$$

$$-2\,\mu u_{tt}+3k\Big(-5(u_x)^2(u_{xx})^3(1+(u_x)^2)^{\frac{-7}{2}}+(u_{xx})^3(1+(u_x)^2)^{\frac{-5}{2}}$$
$$+2u_xu_{xx}u_{xxx}(1+(u_x)^2)^{\frac{-5}{2}}\Big)$$
$$+2k\Big(-3u_xu_{xx}u_{xxx}(1+(u_x)^2)^{\frac{-5}{2}}+u_{xxxx}(1+(u_x)^2)^{\frac{-3}{2}}\Big)$$
$$+15(u_x)^2(u_{xx})^3-3(u_{xx})^3(1+(u_x)^2)^{\frac{-5}{2}}$$
$$-6u_xu_{xx}u_{xxx}(1+(u_x)^2)^{\frac{-5}{2}}=0$$

$$u_{tt}=\frac{k}{\mu}\left[\frac{u_{xxxx}}{(1+(u_x)^2)^{\frac{3}{2}}}-\frac{3(u_{xx})^3}{2(1+(u_x)^2)^{\frac{5}{2}}}-\frac{6u_xu_{xx}u_{xxx}}{(1+(u_x)^2)^{\frac{5}{2}}}-\frac{45(u_x)^2(u_{xx})^3}{2(1+(u_x)^2)^{\frac{7}{2}}}\right]$$

Let us $a=\frac{k}{\mu}$, $b=-\frac{3k}{2\mu}$, $c=-\frac{6k}{\mu}$, $d=-\frac{45k}{2\mu}$

$$u_{tt}=a\frac{u_{xxxx}}{(1+(u_x)^2)^{\frac{3}{2}}}+b\frac{(u_{xx})^3}{(1+(u_x)^2)^{\frac{5}{2}}}+c\frac{u_xu_{xx}u_{xxx}}{(1+(u_x)^2)^{\frac{5}{2}}}+d\frac{(u_x)^2(u_{xx})^3}{(1+(u_x)^2)^{\frac{7}{2}}}\quad(2.3)$$

2.2 Conclusions of this Chapter

1. Equation (1.1) is Called Linear Equation of Transverse Vibration of a Beam, but Equations (1.2) and (1.3) are Called Nonlinear Equations of Transverse Vibration of a Beam.

2. E is the modulus of elasticity, I is the moment of inertia of any cross section about the x-axis, A is the area of cross section and μ mass per unit length.
3. If $a = -\alpha^2$ and others constants are equal zeros in equation (1.2) we obtain the equation (1.1)
4. The bending to be small, so that the deflection $u(x, t)$ and its derivative are small i.e $(u_x)^2 \ll 1$, so the equation (1.3) is became equation (1.2)

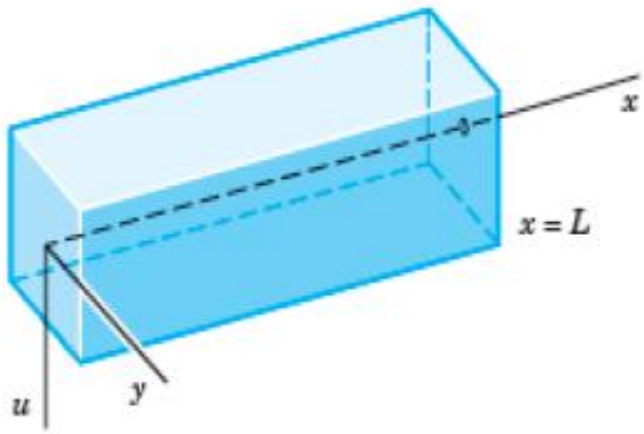

Figure 2.2: Elastic beam

3 Adomain's Decomposition Method

We discuss the Adomain's decomposition method for solving models of bending of elastic beams.

3.1 Adomian's Decomposition Method (ADM)

George Adomian established the Adomian decomposition method (ADM) in 1980s, the ADM has been paid much attention in the recent years in applied mathematics, and in the field of series solution particularly. Moreover, it is a fact that this method is powerful, effective, as well easily solves many types of linear or nonlinear ordinary or partial differential equations, and integral equations. The ADM solves the problems in direct way and in an uncomplicated fashion without using linearization, perturbation or any other assumptions that may change the physical behavior of the model. The technique is based on a decomposition of a solution of a nonlinear operator equation in series of function each term of the series is obtained from a polynomial generated from an expansion of an analytic function into a power series. The Adomian technique is very simple in an abstract formulation but difficulty a rises in calculating the polynomials and in proving the convergence of the series of function. Convergence of the Adomian method when applied to some classes of ordinary and partial differential equation is discussed by many authors for example, K. Abbauoi and Y.Cherruault proved the convergence of the Adomian method for differential and operator equations.

Lesnic investigated convergence of the ADM when applied to time dependent heat wave and beam equations for both forward and backward time evolution.

He should that the convergence was faster for forward problem than backward problems. Al-khaled and Allan implemented the Adomian method for variable –depth shallow water equations with source term and illustrated the convergence numerically.

Adomian decomposition method introduces the solution of any equation in a series form, where the components of the solution are elegantly computed by a recursive manner. Further, the resulting series may converge to a closed form solution if exact solution exists.

In the case where a closed form solution is not obtainable, a truncated n-term approximation is usually used for approximations and numerical purposes. It was formally proved by many researchers that the method provides the solution in a rapidly convergent power series. An important point can be made here in that the method attacks the problem, homogeneous or inhomogeneous, in a straightforward manner without any need for transformation formulas [7, 8].

$$Lu + Nu = g(x,t) \tag{3.1}$$

where L and N are linear and nonlinear operators respectively, and $g(x,t)$ is the source inhomogeneous term.

$$Lu = g(x,t) - Nu$$

$$L^{-1}Lu = L^{-1}(g(x,t) - Nu)$$

$$u(x,t) = \varphi_0 - L^{-1}(Nu)$$

where

$$\varphi_0 = \begin{cases} u(x,0) & for\ L = \dfrac{\partial}{\partial t} \\ u(x,0) + tu_t(x,0) & for\ L = \dfrac{\partial^2}{\partial t^2} \\ u(x,0) + tu_t(x,0) + \dfrac{t^2}{2}u_{tt}(x,0) & for\ L = \dfrac{\partial^3}{\partial t^3} \end{cases}$$

The solution is given by:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$

If the equation is having nonlinear term $f(u)$, the solution is given by:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \varphi_0 + L^{-1}(\sum_{n=0}^{\infty} u_n(x,t) + \sum_{n=0}^{\infty} A_n)$$

where

$$F(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, \dots)$$

A_n is called Adomian's polynomials, can be evaluated for all forms of nonlinearity. Several schemes have been introduced in the literature by researchers to calculate Adomian polynomials. Adomian introduced a scheme for the calculation of Adomian polynomials that was formally justified. A_n alternative reliable method that is based on algebraic and trigonometric identities and on Taylor series has been developed and will be examined later. The alternative method employs only elementary operations and does not require specific formulas. The Adomian polynomials A_n for the nonlinear term *F*(*u*) can be evaluated by using the following expression

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{j=0}^{n}\lambda^j u_j\right)\right]_{\lambda=0}, n = 0,1,2,\dots$$

where

$$A_0 = F(u_0)$$

$$A_1 = u_1 F'(u_0)$$

$$A_2 = u_2 F'(u_0) + \frac{1}{2}{u_1}^2 F''(u_0)$$

$$A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{6}{u_1}^3 F'''(u_0)$$

$$A_4 = u_4 F'(u_0) + \left(\frac{1}{2}{u_2}^2 + u_1 u_3\right) F''(u_0) + \frac{1}{2}{u_1}^2 u_2 F'''(u_0) + \frac{1}{24}{u_4}^3 F''''(u_0)$$

Other polynomials can be generated in a similar manner.
Two important observations can be made here. First, A_0 depends only on u_0, A_1 depends only on u_0 and u_1, A_2 depends only on u_0, u_1 and u_2, and so on. Second, substituting [7].

$$F(u) = A_0 + A_1 + A_2 + \cdots$$

$$= F(u_0) + (u_1 + u_2 + u_3 + \cdots)F'^{(u_0)}$$
$$+ \frac{1}{2!}({u_1}^2 + 2u_1u_2 + {u_2}^2 + \cdots)F''^{(u_0)}$$
$$+ \frac{1}{3!}({u_1}^3 + 3{u_1}^2u_2 + 3{u_1}^2u_3 + 6u_1u_2u_3 \ldots)F'''^{(u_0)} + \cdots$$

3.2 Adomian's Decomposition Transform Method (ADTM)

In this method, we using Laplace's transform of equation (3.1), we obtain

$$\mathcal{L}(L(u)) + \mathcal{L}(Nu) = \mathcal{L}(\mathrm{g})$$
$$u(x,s) = \frac{1}{s^n}\sum_{k=0}^{n-1} s^{(k+n-1)}\frac{\partial^k u(x,0)}{\partial t^k} - \frac{1}{s^n}\mathcal{L}((Nu - \mathrm{g}))$$

Finally, we use Laplace's inverse

$$u(x,t) = \mathcal{L}^{-1}\left(\frac{1}{s^n}\sum_{k=0}^{n-1} s^{(k+n-1)}\frac{\partial^k u(x,0)}{\partial t^k} - \frac{1}{s^n}\mathcal{L}\big((Nu - \mathrm{g})\big)\right)$$

Put $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$

$$\sum_{n=0}^{\infty} u_n = \mathcal{L}^{-1}\left(\frac{1}{s^n}\sum_{k=0}^{n-1} s^{(k+n-1)}\frac{\partial^k u(x,0)}{\partial t^k} - \frac{1}{s^n}\mathcal{L}\left(\left(N\sum_{n=0}^{\infty} u_n - \mathrm{g}\right)\right)\right)$$

To illustrate this method we give some examples.

Example 3.1: Solving the Linear Equation of Transverse Vibration of Beam.

$$\frac{\partial^2 u}{\partial t^2} + \alpha^2\frac{\partial^4 u}{\partial x^4} = 0$$

where $\alpha^2 = \frac{EI}{A\mu}$

with initial conditions

$$u(x,0) = \sin(x) \quad \& \quad u_t(x,0) = x^2$$

Using the Adomian's Decomposition Method, we have

$$L_t u(x,t) + \alpha^2 L_x u(x,t) = 0$$

$$L_t^{-1}L_t u(x,t) = -\alpha^2 L_t^{-1}L_x u(x,t)$$

where

$$L_t = \frac{\partial^2}{\partial t^2} \quad , \quad L_x = \frac{\partial^4}{\partial x^4} \quad \& \quad L_t^{-1}(.) = \int_0^t \int_0^t (.)dtdt$$

$$u(x,t) = u(x,0) + tu_t(x,0) - \alpha^2 L_t^{-1} L_x u(x,t)$$

$$u(x,t) = \sin(x) + x^2 t - \alpha^2 L_t^{-1} L_x u(x,t)$$

We next define the unknown function $u(x,t)$ by a sum of components defined by the series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$

$$\sum_{n=0}^{\infty} u_n(x,t) = \sin(x) + x^2 t - \alpha^2 L_t^{-1} L_x (\sum_{n=0}^{\infty} u_n(x,t))$$

or equivalently

$$u_0 + u_1 + u_2 + \cdots = \sin(x) + x^2 t - \alpha^2 L_t^{-1} L_x (u_0 + u_1 + u_2 + \cdots)$$

Let us $\qquad u_0 = \sin(x) + x^2 t$

$$u_1 = L_t^{-1} L_x(u_0) = -\alpha^2 \int_0^t \int_0^t \frac{\partial^4}{\partial x^4}(u_0)dtdt = -\alpha^2 \sin(x)\frac{t^2}{2!}$$

$$u_2 = -\alpha^2 L_t^{-1} L_x(u_1) = \alpha^4 \int_0^t \int_0^t \frac{\partial^4}{\partial x^4}(\sin(x)\frac{t^2}{2!})dtdt = \sin(x)\frac{(\alpha t)^4}{4!}$$

$$u_3 = L_t^{-1} L_x(u_2) = -\alpha^6 \int_0^t \int_0^t \frac{\partial^4}{\partial x^4}\left(\sin(x)\frac{t^4}{4!}\right)dtdt = -\sin(x)\frac{(\alpha t)^6}{6!}$$

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$$u_k = -\alpha^2 {L_t}^{-1} L_x(u_{k-1}) = -\alpha^2 \int_0^t \int_0^t \frac{\partial^4}{\partial x^4}(u_{k-1}) dt dt$$

$$= (-1)^k \sin(x) \frac{(\alpha t)^{2k}}{(2k)!}$$

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = u_0 + u_1 + u_2 + \cdots$$

$$u(x,t) = x^2 t + \sum_{n=0}^{\infty} (-1)^n \sin(x) \frac{(\alpha t)^{2n}}{(2n)!} = x^2 t + \sin(x)\cos(\alpha t)$$

Using Adomian's Decomposition Transform Method (ADTM). By using Laplace's transform of both sides

$$s^2 u(x,s) - s u(x,0) - u_t(x,0) = -\alpha^2 \mathcal{L}(\frac{\partial^4 u}{\partial x^4})$$

$$u(x,s) = \frac{u(x,0)}{s} + \frac{u_t(x,0)}{s^2} - \alpha^2 \left(\frac{1}{s^2} \mathcal{L} \left(\frac{\partial^4 u}{\partial x^4} \right) \right)$$

Given $u(x,0) = \sin(x)$ & $u_t(x,0) = 0$, so

$$u(x,t) = \sin(x) + x^2 t - \alpha^2 \mathcal{L}^{-1} \left(\frac{1}{s^2} \mathcal{L} \left(\frac{\partial^4 u}{\partial x^4} \right) \right)$$

Put $u = \sum_{n=0}^{\infty} u_n$ and integration of both sides of t variable

$$\sum_{n=0}^{\infty} u_n = \sin(x) + x^2 t - \alpha^2 \mathcal{L}^{-1} \left(\frac{1}{s^2} \mathcal{L} \left(\frac{\partial^4 (\sum_{n=0}^{\infty} u_n(x,t))}{\partial x^4} \right) \right)$$

Compare we obtain

$$u_0 = \sin(x) + x^2 t$$

$$u_1 = -\alpha^2 \mathcal{L}^{-1} \left(\frac{1}{s^2} \mathcal{L} \left(\frac{\partial^4}{\partial x^4}(u_0) \right) \right)$$

$$u_1 = -\alpha^2 \mathcal{L}^{-1}\left(\frac{\sin(x)}{s^3}\right) = -\alpha^2 \sin(x)\frac{t^2}{2!}$$

$$u_2 = -\alpha^2 \mathcal{L}^{-1}\left(\frac{1}{s^2}\mathcal{L}\left(\frac{\partial^4}{\partial x^4}(u_1)\right)\right)$$

$$u_2 = \alpha^4 \mathcal{L}^{-1}\left(\frac{\sin(x)}{s^5}\right) = \sin(x)\frac{(\alpha t)^4}{4!}$$

$$u_3 = -\alpha^2 \mathcal{L}^{-1}\left(\frac{1}{s^2}\mathcal{L}\left(\frac{\partial^4}{\partial x^4}(u_2)\right)\right)$$

$$u_3 = -\alpha^6 \mathcal{L}^{-1}\left(\frac{\sin(x)}{s^7}\right) = -\sin(x)\frac{(\alpha t)^6}{6!}$$

$$u_4 = -\alpha^2 \mathcal{L}^{-1}\left(\frac{1}{s^2}\mathcal{L}\left(\frac{\partial^4}{\partial x^4}(u_3)\right)\right)$$

$$u_4 = -\alpha^6 \mathcal{L}^{-1}\left(\frac{\sin(x)}{s^9}\right) = \sin(x)\frac{(\alpha t)^8}{8!}$$

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$$u_k = -\alpha^2 \mathcal{L}^{-1}\left(\frac{1}{s^2}\mathcal{L}\left(\frac{\partial^4}{\partial x^4}(u_{k-1})\right)\right)$$

$$u_k = (-1)^k \sin(x)\frac{(\alpha t)^{2k}}{(2n)!}$$

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = x^2 t + \sum_{n=0}^{\infty} (-1)^n \sin(x)\frac{(\alpha t)^{2n}}{(2n)!}$$

$$= x^2 \mathrm{t} + \sin(x)\sum_{n=0}^{\infty} (-1)^n \frac{(\alpha t)^{2n}}{(2n)!}$$

$$u(x,t) = x^2\text{t} + \sin(x)\cos(\alpha t)$$

Example 3.2: Solving the Linear Equation of Transverse Vibration of Beam.

$$\frac{\partial^2 u}{\partial t^2} - (\frac{1}{x} + \frac{x^4}{120})\frac{\partial^4 u}{\partial x^4} = 0$$

With initial conditions

$$u(x,0) = 0 \quad ,u_t(x,0) = \left(1 + \frac{x^5}{120}\right)$$

Using the Adomian's Decomposition Method, we have

$$L_t u(x,t) - (\frac{1}{x} + \frac{x^4}{120})L_x u(x,t) = 0$$

$$L_t{}^{-1} L_t u(x,t) = (\frac{1}{x} + \frac{x^4}{120})L_t{}^{-1} L_x u(x,t)$$

where

$$L_t = \frac{\partial^2}{\partial t^2} \quad , \quad L_x = \frac{\partial^4}{\partial x^4} \quad \& \quad L_t{}^{-1}(.) = \int_0^t \int_0^t (.)dtdt$$

$$u(x,t) = u(x,0) + tu_t(x,0) + L_t{}^{-1}\{\left(\frac{1}{x} + \frac{x^4}{120}\right)L_x u(x,t)\}$$

$$u(x,t) = \left(1 + \frac{x^5}{120}\right)t + L_t{}^{-1}\{\left(\frac{1}{x} + \frac{x^4}{120}\right)L_x u(x,t)\}$$

We next define the unknown function $u(x,t)$ by a sum of components defined by the series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$

$$\sum_{n=0}^{\infty} u_n(x,t) = \left(1 + \frac{x^5}{120}\right)t + L_t{}^{-1}\{\left(\frac{1}{x} + \frac{x^4}{120}\right)L_x(\sum_{n=0}^{\infty} u_n(x,t))\}$$

$$u_0 + u_1 + u_2 + \cdots$$
$$= \left(1 + \frac{x^5}{120}\right)t + {L_t}^{-1}\{\left(\frac{1}{x} + \frac{x^4}{120}\right)L_x(u_0 + u_1 + u_2 + \cdots)\}$$

Let us $$u_0 = \left(1 + \frac{x^5}{120}\right)t$$

$$u_1 = \int_0^t \int_0^t (\frac{1}{x} + \frac{x^4}{120}) \frac{\partial^4 u_0(x,t)}{\partial x^4} dtdt$$

$$u_1 = \int_0^t \int_0^t (\frac{1}{x} + \frac{x^4}{120}) xtdtdt$$

$$u_1 = \left(1 + \frac{x^5}{120}\right)\int_0^t tdtdt = \left(1 + \frac{x^5}{120}\right)\frac{t^3}{3!}$$

$$u_2 = \int_0^t \int_0^t (\frac{1}{x} + \frac{x^4}{120}) \frac{\partial^4 u_1(x,t)}{\partial x^4} dtdt$$

$$u_2 = \int_0^t \int_0^t \left(\frac{1}{x} + \frac{x^4}{120}\right)\left(-x\frac{t^3}{3!}\right)dtdt = \left(1 + \frac{x^5}{120}\right)\frac{t^5}{5!}$$

$$u_3 = \int_0^t \int_0^t (\frac{1}{x} + \frac{x^4}{120}) \frac{\partial^4 u_2(x,t)}{\partial x^4} dtdt$$

$$u_3 = \int_0^t \int_0^t \left(\frac{1}{x} + \frac{x^4}{120}\right)\left(x\frac{t^5}{5!}\right)dtdt = \left(1 + \frac{x^5}{120}\right)\frac{t^7}{7!}$$

$$u_4 = \int_0^t \int_0^t \left(\frac{1}{x} + \frac{x^4}{120}\right)\left(x\frac{t^7}{7!}\right)dtdt = \left(1 + \frac{x^5}{120}\right)\frac{t^9}{9!}$$

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$$u_n = \int_0^t \int_0^t \left(\frac{1}{x} + \frac{x^4}{120}\right) \frac{\partial^4 u_{n-1}(x,t)}{\partial x^4} dt dt$$

$$u_n = \left(1 + \frac{x^5}{120}\right) \frac{t^{2n+1}}{(2n+1)!}$$

We know that

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$

$$u(x,t) = \sum_{n=0}^{\infty} \left(1 + \frac{x^5}{120}\right) \frac{t^{2n+1}}{(2n+1)!}$$

$$u(x,t) = \left(1 + \frac{x^5}{120}\right) \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!}$$

$$u(x,t) = \left(1 + \frac{x^5}{120}\right) \sinh(t)$$

Example 3.3: Solving the Nonlinear Equation of Transverse Vibration of Beam.

$$\frac{\partial^2 u}{\partial t^2} = a \frac{\partial^4 u}{\partial x^4} + b \left(\frac{\partial^2 u}{\partial x^2}\right)^3 + c \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} \frac{\partial^3 u}{\partial x^3} + d \left(\frac{\partial u}{\partial x}\right)^2 \left(\frac{\partial^2 u}{\partial x^2}\right)^3$$

with initial conditions

$$u(x,0) = -0.5x^2 \quad \& \quad u_t(x,0) = 0$$

$$a = 1\,, b = -1.5\,, c = -6 \;\; \& \; d = -22.5$$

Using the Adomian's Decomposition Method, we have

$$L_t u(x,t) = L_x u(x,t)$$

$$L_t^{-1} L_t u(x,t) = L_t^{-1} L_x u(x,t)$$

where

$$L_t = \frac{\partial^2}{\partial t^2} \quad , \quad L_x = \frac{\partial^4}{\partial x^4} \quad \& \quad {L_t}^{-1}(.) = \int_0^t \int_0^t (.)dtdt$$

$$u(x,t) = u(x,0) + tu_t(x,0) + {L_t}^{-1} L_x u(x,t)$$

$$u(x,t) = -0.5x^2 + {L_t}^{-1} L_x u(x,t)$$

We next define the unknown function $u(x,t)$ by a sum of components defined by the series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$

$$\sum_{n=0}^{\infty} u_n(x,t) = -0.5x^2 + {L_t}^{-1} L_x (\sum_{n=0}^{\infty} u_n(x,t)) - {L_t}^{-1} (\sum_{n=0}^{\infty} A_n)$$

$$\sum_{n=0}^{\infty} u_n(x,t) = -0.5x^2 + {L_t}^{-1} (\sum_{n=0}^{\infty} L_x(u_n) - A_n)$$

or equivalently

$$\begin{aligned} u_0 + u_1 + u_2 + \cdots \\ &= -0.5x^2 + {L_t}^{-1}((L_x u_0 - A_0) + (L_x u_1 - A_1) \\ &+ (L_x u_2 - A_2) + \cdots) \end{aligned}$$

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[F\left(\sum_{j=0}^{n} \lambda^j \, u_j \right) \right]_{\lambda=0} \quad , n = 0,1,2, \ldots$$

where

$$F(u) = \frac{3}{2} \left(\frac{\partial^2 u}{\partial x^2} \right)^3 + 6 \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} \frac{\partial^3 u}{\partial x^3} + \frac{45}{2} \left(\frac{\partial u}{\partial x} \right)^2 \left(\frac{\partial^2 u}{\partial x^2} \right)^3$$

So

$$F\left(\sum_{j=0}^{n}\lambda^j u_j\right)$$

$$= \frac{3}{2}\left(\sum_{j=0}^{n}\lambda^j u_{j_{xx}}\right)^3$$

$$+ 6\left(\sum_{j=0}^{n}\lambda^j u_{j_x}\right)\left(\sum_{j=0}^{n}\lambda^j u_{j_{xx}}\right)\left(\sum_{j=0}^{n}\lambda^j u_{j_{xxx}}\right)$$

$$+ \frac{45}{2}\left(\sum_{j=0}^{n}\lambda^j u_{j_x}\right)^2\left(\sum_{j=0}^{n}\lambda^j u_{j_{xx}}\right)^3$$

when $n = 0$

$$u_0 = -0.5x^2$$

$$A_0 = F(u_0) = \frac{3}{2}\left(\frac{\partial^2 u_0}{\partial x^2}\right)^3 + 6\frac{\partial u_0}{\partial x}\frac{\partial^2 u_0}{\partial x^2}\frac{\partial^3 u_0}{\partial x^3} + \frac{45}{2}\left(\frac{\partial u_0}{\partial x}\right)^2\left(\frac{\partial^2 u_0}{\partial x^2}\right)^3$$

$$A_0 = -\frac{45x^2 + 3}{2}$$

when $n = 1$

$$u_1 = {L_t}^{-1}(L_x u_0 - A_0)$$

$$u_1 = {L_t}^{-1}(L_x u_0 - A_0) = \int_0^t\int_0^t\left(\frac{\partial^4}{\partial x^4}(u_0) - \left(\frac{45x^2 + 3}{2}\right)\right)dtdt$$

$$= \left(\frac{45x^2 + 3}{2}\right)\frac{t^2}{2!}$$

$$A_1 = \frac{d}{d\lambda}\left[\frac{3}{2}\left(\sum_{j=0}^{1}\lambda^j u_{j_{xx}}\right)^3 + 6\left(\sum_{j=0}^{1}\lambda^j u_{j_x}\right)\left(\sum_{j=0}^{1}\lambda^j u_{j_{xx}}\right)\left(\sum_{j=0}^{1}\lambda^j u_{j_{xxx}}\right) + \frac{45}{2}\left(\sum_{j=0}^{1}\lambda^j u_{j_x}\right)^2\left(\sum_{j=0}^{1}\lambda^j u_{j_{xx}}\right)^3\right]_{\lambda=0}$$

$$A_1 = \frac{9}{2}\left(u_{0_{xx}}\right)^2\left(u_{1_{xx}}\right) + 6\left(u_{0_x}\right)\left(u_{0_{xx}}\right)\left(u_{1_{xxx}}\right) + 6\left(u_{0_x}\right)\left(u_{1_{xx}}\right)\left(u_{0_{xxx}}\right) + 6\left(u_{1_x}\right)\left(u_{0_{xx}}\right)\left(u_{0_{xxx}}\right) + \frac{45}{2}\left(u_{0_x}\right)^2\left(3u_{1_{xx}}\right)\left(u_{0_{xx}}\right)^2 + \frac{45}{2}\left(u_{0_{xx}}\right)^3\left(2u_{1_x}u_{0_x}\right)$$

$$A_1 = -405\left(\frac{25x^2+1}{2}\right)\frac{t^2}{2!}$$

when $n = 2$

$$u_2 = L_t^{-1}((L_x u_1 - A_1))$$

$$u_2 = L_t^{-1}(L_x u_1 - A_1) = \int_0^t \int_0^t \left(\frac{\partial^4}{\partial x^4}(u_1) - A_1\right) dtdt$$

$$u_2 = L_t^{-1}(L_x u_1 - A_1) = \int_0^t \int_0^t \left(\frac{\partial^4}{\partial x^4}(u_1) + 405\left(\frac{25x^2+1}{2}\right)\frac{t^2}{2!}\right) dtdt$$

$$u_2 = 405\left(\frac{25x^2+1}{2}\right)\frac{t^4}{4!}$$

$$A_2 = \frac{1}{2!}\frac{d^2}{d\lambda^2}\left[\frac{3}{2}\left(\sum_{j=0}^{2}\lambda^j u_{j_{xx}}\right)^3 + 6\left(\sum_{j=0}^{2}\lambda^j u_{j_x}\right)\left(\sum_{j=0}^{2}\lambda^j u_{j_{xx}}\right)\left(\sum_{j=0}^{2}\lambda^j u_{j_{xxx}}\right) + \frac{45}{2}\left(\sum_{j=0}^{2}\lambda^j u_{j_x}\right)^2\left(\sum_{j=0}^{2}\lambda^j u_{j_{xx}}\right)^3\right]_{\lambda=0}$$

$$A_2 = \frac{1}{2}\frac{d}{d\lambda}\left[\frac{9}{2}\left(\sum_{j=0}^{2}\lambda^j u_{j_{xx}}\right)^2 (u_{1_{xx}} + 2\lambda u_{2_{xx}}) + 6\left(\sum_{j=0}^{2}\lambda^j u_{j_x}\right)\left(\sum_{j=0}^{2}\lambda^j u_{j_{xx}}\right)(u_{1_{xxx}} + 2\lambda u_{2_{xxx}}) + 6\left(\sum_{j=0}^{2}\lambda^j u_{j_x}\right)\left(\sum_{j=0}^{2}\lambda^j u_{j_{xxx}}\right)(u_{1_{xx}} + 2\lambda u_{2_{xx}}) + 6\left(\sum_{j=0}^{2}\lambda^j u_{j_{xx}}\right)\left(\sum_{j=0}^{2}\lambda^j u_{j_{xxx}}\right)(u_{1_x} + 2\lambda u_{2_x}) + \frac{135}{2}\left(\sum_{j=0}^{2}\lambda^j u_{j_x}\right)^2\left(\sum_{j=0}^{2}\lambda^j u_{j_{xx}}\right)^2 (u_{1_{xx}} + 2\lambda u_{2_{xx}}) + \frac{90}{2}\left(\sum_{j=0}^{2}\lambda^j u_{j_{xx}}\right)^3 (\sum_{j=0}^{2}\lambda^j u_{j_x})(u_{1_x} + 2\lambda u_{2_x})\right]_{\lambda=0}$$

$$A_2 = -\left(\frac{4829625x^2 - 88695}{2}\right)\frac{t^4}{4!}$$

when $n = 3$

$$u_3 = {L_t}^{-1}(L_x u_2 - A_2)$$

$$u_3 = L_t^{-1}(L_x u_2 - A_2) = \int_0^t \int_0^t (L_x u_2 - A_2) dtdt$$

$$u_3 = L_t^{-1}(L_x u_2 - A_2) = \int_0^t \int_0^t \left(0 + \left(\frac{4829625x^2 - 88695}{2}\right)\frac{t^4}{4!}\right) dtdt$$

$$u_3 = \left(\frac{4829625x^2 - 88695}{2}\right)\frac{t^6}{6!}$$

$$A_3 = \frac{1}{3!}\frac{d^3}{d\lambda^3}\left[\frac{3}{2}\left(\sum_{j=0}^{2}\lambda^j u_{j_{xx}}\right)^3 + 6\left(\sum_{j=0}^{2}\lambda^j u_{j_x}\right)\left(\sum_{j=0}^{2}\lambda^j u_{j_{xx}}\right)\left(\sum_{j=0}^{2}\lambda^j u_{j_{xxx}}\right) + \frac{45}{2}\left(\sum_{j=0}^{2}\lambda^j u_{j_x}\right)^2\left(\sum_{j=0}^{2}\lambda^j u_{j_{xx}}\right)^3\right]_{\lambda=0}$$

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The solution is given by

$$u = \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + u_3 + \cdots$$

The approximation solution is

$$u(x,t) = -\frac{x^2}{2} + \left(\frac{45x^2 + 3}{2}\right)\frac{t^2}{2!} + 405\left(\frac{25x^2 + 1}{2}\right)\frac{t^4}{4!} + \left(\frac{4829625x^2 - 88695}{2}\right)\frac{t^6}{6!}$$

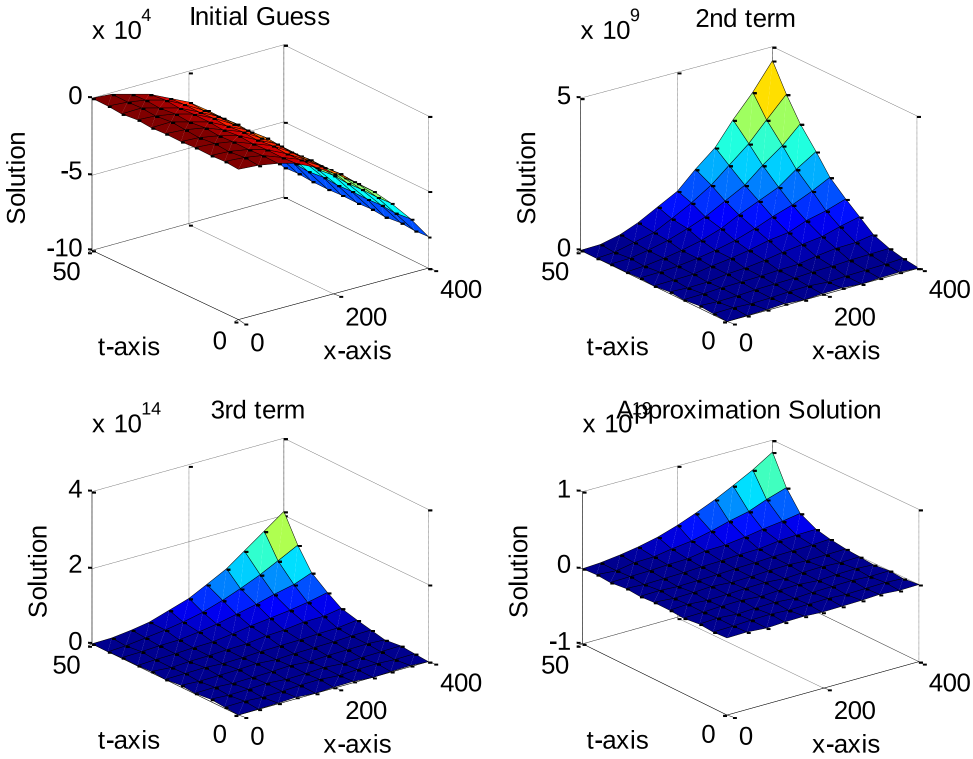

Figure 3.1 Graphical Representation of Example 3.3

4 Homotopy Perturbation Method

We discuss the homotopy perturbation method for solving models of bending of elastic beams.

4.1 Homotopy Perturbation Method (HPM)

The homotopy perturbation method (HPM) was first proposed by He J. Huan in 1999, where the solution of this method is considered as the sum of an infinite series which is very rapidly converge to the accurate solution. The homotopy perturbation method, presents some advantages: obtaining exact solutions with higher accuracy, minimal calculations without loss of physical verification. This method has found application in different fields of nonlinear equations. Many authors and researchers studied the homotopy perturbation method, say. He J. In 1999, used the homotopy perturbation method for solving nonlinear ordinary differential equations of the first and second orders. He J. in 2003, solved the Nonlinear ordinary differential equations with nth order, He J. in 2004, solved the oscillators equation with discontinuities via the homotopy perturbation method, He J. in 2005, studied the homotopy perturbation method for solving one dimensional nonlinear wave equation. Li-Na Z. and He J. in 2006, solved the electrostatic potential differential equation. The homotopy perturbation method gives the solution by using initial conditions only. The fact that the proposed homotopy perturbation method solves nonlinear problems without using Adomian's polynomials can be considered as a clear advantage of this technique over the decomposition method. To explain the homotopy perturbation method. We consider a general equation of type.

$$Lu + Nu = \mathrm{g}(\mathrm{x},\mathrm{t}) \quad ,x \in \Omega \tag{4.1}$$

where L is linear operator, N is nonlinear operator and g is known analytical function. We define a convex homotopy $H(u,p)$

$$H(u,p): \Omega \times [0,1] \to R$$

by

$$H(u,p) = (1-p)F(u) + p(Lu + Nu - \mathrm{g}) = 0 \tag{4.2}$$

where $F(u)$ is a functional operator. We have

$$H(u,0) = F(u) \qquad \& \qquad H(u,1) = Lu + Nu - \mathrm{g} \tag{4.3}$$

The embedding parameter monotonically increases from zero to unity as the trivial problem, $F(u) = 0$, continuously deforms the original problem

$Lu = 0$, the embedding parameter $p \in [0,1]$, can be considered as expanding parameter. The homotopy perturbation method use homotopy parameter p as an expanding parameter to obtain [9, 10].

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \tag{4.4}$$

If $p \to 1$, then equation (4.2) corresponds to equation (4.1) and became the approximate solution of the form

$$\mathrm{u} = \lim_{p\to 1} \sum_{n=0}^{\infty} p^n u_n(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = u_0 + u_1 + u_2 + \cdots \tag{4.5}$$

The coupling of the perturbation method and the homotopy method is called the homotopy perturbation method (HPM), which has eliminated limitations of the traditional perturbation methods. In the other hand, the proposed technique can take full advantage of the traditional perturbation techniques [9].

The series (4.5) is convergent for most cases; however, the convergent rate depends upon the nonlinear operator.

i. The second derivative of $N(\xi)$ with respect to ξ must be small, because the parameter p may be relatively large, i.e. $p \to 1$.
ii. The norm of $L^{-1}(\frac{\partial N}{\partial \xi})$ must be smaller than one, in order that the series converges.

4.2 Homotopy Perturbation Transform Method (HPTM)

In this method, we using Laplace's transform of equation (4.2) and we suppose $F(u)$ is linear operator subset form original equation (4.1), we obtain

$$\mathcal{L}(L(u)) + p\mathcal{L}((Nu - \mathrm{g})) = 0$$

$$u(x,s) = \frac{1}{s^n} \sum_{k=0}^{n-1} s^{(k+n-1)} \frac{\partial^k u(x,0)}{\partial t^k} - p\frac{1}{s^n}\mathcal{L}((Nu - \mathrm{g}))$$

Finally, we use Laplace's inverse

$$u(x,t) = \mathcal{L}^{-1}\left(\frac{1}{s^n} \sum_{k=0}^{n-1} s^{(k+n-1)} \frac{\partial^k u(x,0)}{\partial t^k} - p\frac{1}{s^n}\mathcal{L}\big((Nu - \mathrm{g})\big)\right)$$

Put $u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t)$

$$\sum_{n=0}^{\infty} p^n u_n = \mathcal{L}^{-1}\left(\frac{1}{s^n}\sum_{k=0}^{n-1} s^{(k+n-1)}\frac{\partial^k u(x,0)}{\partial t^k} - p\frac{1}{s^n}\mathcal{L}\left(\left(N\sum_{n=0}^{\infty} p^n u_n - \mathrm{g}\right)\right)\right)$$

To illustrate this method we provide some examples.

Example 4.1: Solving the Linear Equation of Transverse Vibration of Beam.

$$\frac{\partial^2 u}{\partial t^2} + \alpha^2\frac{\partial^4 u}{\partial x^4} = 0$$

where $\alpha^2 = \frac{EI}{A\mu}$

with initial conditions: $u(x,0) = \sin(x)$ & $u_t(x,0) = x^2$

Using the Homotopy Perturbation Method (HPM), we have

$$(1-p)\frac{\partial^2 u}{\partial t^2} + p\left(\frac{\partial^2 u}{\partial t^2} + \alpha^2\frac{\partial^4 u}{\partial x^4}\right) = 0$$

$$\frac{\partial^2 u}{\partial t^2} = -\alpha^2 p\frac{\partial^4 u}{\partial x^4}$$

Put $u = \sum_{n=0}^{\infty} p^n u_n$ and integration of both sides of t variable

$$\frac{\partial^2\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)}{\partial t^2} = -\alpha^2 p\frac{\partial^4\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)}{\partial x^4}$$

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = u(x,0) + u_t(x,0)t - \alpha^2 p\int_0^t\int_0^t \frac{\partial^4\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)}{\partial x^4}dtdt$$

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = \sin(x) + x^2 t - \alpha^2 p\int_0^t\int_0^t \frac{\partial^4\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)}{\partial x^4}dtdt$$

Compare the coefficients of power of p

$$p^0: \qquad u_0 = \sin(x) + x^2 t$$

$$p^1: \qquad u_1 = -\alpha^2 \int_0^t \int_0^t \frac{\partial^4}{\partial x^4}(u_0)\,dtdt$$

$$u_1 = -\alpha^2 \int_0^t \int_0^t \frac{\partial^4}{\partial x^4}(\sin(x))\,dtdt = -\alpha^2 \sin(x)\frac{t^2}{2!}$$

$$p^2: \qquad u_2 = \alpha^4 \int_0^t \int_0^t \frac{\partial^4}{\partial x^4}(\sin(x)\frac{t^2}{2!})\,dtdt = \sin(x)\frac{(\alpha t)^4}{4!}$$

$$p^3: \qquad u_3 = -\alpha^6 \int_0^t \int_0^t \frac{\partial^4}{\partial x^4}\left(\sin(x)\frac{t^4}{4!}\right)dtdt = -\sin(x)\frac{(\alpha t)^6}{6!}$$

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$$p^n: \qquad u_n = -\alpha^2 \int_0^t \int_0^t \frac{\partial^4}{\partial x^4}(u_{n-1})\,dtdt = (-1)^n \sin(x)\frac{(\alpha t)^{2n}}{(2n)!}$$

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t) = x^2 t + \sum_{k=0}^{\infty} (-1)^k \sin(x)\frac{(\alpha t)^{2k}}{(2k)!}$$

$$u(x,t) = x^2 t + \sin(x)\cos(\alpha t)$$

Using the Homotopy Perturbation Transform Method (HPTM), we have

$$(1-p)\frac{\partial^2 u}{\partial t^2} + p\left(\frac{\partial^2 u}{\partial t^2} + \alpha^2 \frac{\partial^4 u}{\partial x^4}\right) = 0$$

$$\frac{\partial^2 u}{\partial t^2} = -\alpha^2 p \frac{\partial^4 u}{\partial x^4}$$

Using Laplace's Transform of both sides

$$s^2u(x,s) - su(x,0) - u_t(x,0) = -\alpha^2 p\mathcal{L}(\frac{\partial^4 u}{\partial x^4})$$

$$u(x,s) = \frac{u(x,0)}{s} + \frac{u_t(x,0)}{s^2} - \alpha^2 p\left(\frac{1}{s^2}\mathcal{L}\left(\frac{\partial^4 u}{\partial x^4}\right)\right)$$

Given $u(x,0) = \sin(x)$ & $u_t(x,0) = x^2$, so

$$u(x,t) = \sin(x) + x^2 t - \alpha^2 p\mathcal{L}^{-1}\left(\frac{1}{s^2}\mathcal{L}\left(\frac{\partial^4 u}{\partial x^4}\right)\right)$$

Put $u = \sum_{n=0}^{\infty} p^n u_n$ and integration of both sides of t variable

$$\sum_{n=0}^{\infty} p^n u_n = \sin(x) + x^2 t - \alpha^2 p\mathcal{L}^{-1}\left(\frac{1}{s^2}\mathcal{L}\left(\frac{\partial^4(\sum_{n=0}^{\infty} p^n u_n(x,t))}{\partial x^4}\right)\right)$$

Compare the coefficients of power of p

p^0: $$u_0 = \sin(x) + x^2 t$$

p^1: $$u_1 = -\alpha^2\mathcal{L}^{-1}\left(\frac{1}{s^2}\mathcal{L}\left(\frac{\partial^4}{\partial x^4}(u_0)\right)\right)$$

$$u_1 = -\alpha^2\mathcal{L}^{-1}\left(\frac{\sin(x)}{s^3}\right) = -\alpha^2\sin(x)\frac{t^2}{2!}$$

p^2: $$u_2 = -\alpha^2\mathcal{L}^{-1}\left(\frac{1}{s^2}\mathcal{L}\left(\frac{\partial^4}{\partial x^4}(u_1)\right)\right)$$

$$u_2 = \alpha^4\mathcal{L}^{-1}\left(\frac{\sin(x)}{s^5}\right) = \sin(x)\frac{(\alpha t)^4}{4!}$$

p^3: $$u_3 = -\alpha^2\mathcal{L}^{-1}\left(\frac{1}{s^2}\mathcal{L}\left(\frac{\partial^4}{\partial x^4}(u_1)\right)\right)$$

$$u_3 = -\alpha^6\mathcal{L}^{-1}\left(\frac{\sin(x)}{s^7}\right) = -\sin(x)\frac{(\alpha t)^6}{6!}$$

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$$p^n: \qquad u_n = -\alpha^2 \mathcal{L}^{-1}\left(\frac{1}{s^2}\mathcal{L}\left(\frac{\partial^4}{\partial x^4}(u_{n-1})\right)\right)$$

$$u_n = (-1)^n \sin(x)\frac{(\alpha t)^{2n}}{(2n)!}$$

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t) = x^2 t + \sum_{k=0}^{\infty}(-1)^k \sin(x)\frac{(\alpha t)^{2k}}{(2k)!}$$

$$u(x,t) = x^2\mathrm{t} + \sin(x)\cos(\alpha t)$$

Example 4.2: Solving the Linear Equation of Transverse Vibration of Beam.

$$\frac{\partial^2 u}{\partial t^2} - (\frac{2}{x^2} + \frac{x^4}{360})\frac{\partial^4 u}{\partial x^4} = 0$$

With initial conditions

$$u(x,0) = 0 \quad ,u_t(x,0) = \left(2 + \frac{x^6}{720}\right)$$

Using the Homotopy Perturbation Method (HPM), we have

$$(1-p)\frac{\partial^2 u}{\partial t^2} + p\left(\frac{\partial^2 u}{\partial t^2} - \left(\frac{2}{x^2} + \frac{x^4}{360}\right)\frac{\partial^4 u}{\partial x^4}\right) = 0$$

Put $u = \sum_{n=0}^{\infty} p^n u_n$ and integration of both sides of t variable

$$\frac{\partial^2 u}{\partial t^2} = p(\frac{2}{x^2} + \frac{x^4}{360})\frac{\partial^4 u}{\partial x^4}$$

$$\frac{\partial^2(\sum_{n=0}^{\infty} p^n u_n(x,t))}{\partial t^2} = p\left(\frac{2}{x^2} + \frac{x^4}{360}\right)\frac{\partial^4(\sum_{n=0}^{\infty} p^n u_n(x,t))}{\partial x^4}$$

$$\sum_{n=0}^{\infty} p^n u_n(x,t)$$
$$= u(x,0) + u_t(x,0)t + 2p\int_0^t\int_0^t \left\{\left(\frac{1}{x^2}+\frac{x^4}{720}\right)\frac{\partial^4\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)}{\partial x^4}\right\} dtdt$$

$$\sum_{n=0}^{\infty} p^n u_n(x,t)$$
$$= \left(2+\frac{x^6}{720}\right)t + 2p\int_0^t\int_0^t \left\{\left(\frac{1}{x^2}+\frac{x^4}{720}\right)\frac{\partial^4\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)}{\partial x^4}\right\} dtdt$$

Compare the coefficients of power of p

p^0: $$u_0 = \left(2+\frac{x^6}{720}\right)t$$

p^1: $$u_1 = 2\int_0^t\int_0^t \left(\frac{1}{x^2}+\frac{x^4}{720}\right)\frac{\partial^4 u_0(x,t)}{\partial x^4} dtdt$$

$$u_1 = 2\int_0^t\int_0^t \left(\frac{1}{x^2}+\frac{x^4}{720}\right)\left(\frac{x^2 t}{2}\right) dtdt = (1+\frac{x^4}{720})\frac{t^3}{6}$$

p^2: $$u_1 = 2\int_0^t\int_0^t \left(\frac{1}{x^2}+\frac{x^4}{720}\right)\frac{\partial^4 u_1(x,t)}{\partial x^4} dtdt$$

$$u_2 = 2\int_0^t\int_0^t \left(\frac{1}{x^2}+\frac{x^4}{720}\right)\left(-\frac{x^2 t^3}{12}\right) dtdt = (1+\frac{x^4}{720})\frac{t^5}{5!}$$

p^3: $$u_3 = 2\int_0^t\int_0^t \left(\frac{1}{x^2}+\frac{x^4}{720}\right)\frac{\partial^4 u_2(x,t)}{\partial x^4} dtdt$$

$$u_3 = 2\int_0^t\int_0^t \left(\frac{1}{x^2}+\frac{x^4}{720}\right)\left(\frac{x^2t^5}{2*5!}\right)dtdt = (1+\frac{x^4}{720})\frac{t^7}{7!}$$

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p^n: $$u_n = 2\int_0^t\int_0^t \left(\frac{1}{x^2}+\frac{x^4}{720}\right)\frac{\partial^4 u_{n-1}(x,t)}{\partial x^4}dtdt$$

$$u_n = (1+\frac{x^4}{720})\frac{t^{2n+1}}{(2n+1)!}$$

$$u = \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + \cdots$$

$$u(x,t) = \left(1+\frac{x^6}{6!}\right)\sum_{n=0}^{\infty}\frac{t^{2n+1}}{(2n+1)!}$$

$$u(x,t) = \left(1+\frac{x^6}{6!}\right)\sinh(t)$$

Example 4.3: Solving the Nonlinear Equation of Transverse Vibration of beam.

$$\frac{\partial^2 u}{\partial t^2} = a\frac{\partial^4 u}{\partial x^4} + b\left(\frac{\partial^2 u}{\partial x^2}\right)^3 + c\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 u}{\partial x^3} + d\left(\frac{\partial u}{\partial x}\right)^2\left(\frac{\partial^2 u}{\partial x^2}\right)^3$$

with initial conditions

$$u(x,0) = \frac{x^3}{3!} \qquad \& \qquad u_t(x,0) = 0$$

Suppose $a = 1, b = 1, c = 1 \ \& \ d = 0$

Using the Homotopy Perturbation Method (HPM), we have

$$(1-p)\frac{\partial^2 u}{\partial t^2} + p\left(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^4 u}{\partial x^4} - \left(\frac{\partial^2 u}{\partial x^2}\right)^3 - \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 u}{\partial x^3}\right) = 0$$

$$\frac{\partial^2 u}{\partial t^2} = p\left(\frac{\partial^4 u}{\partial x^4} + \left(\frac{\partial^2 u}{\partial x^2}\right)^3 + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 u}{\partial x^3}\right)$$

Put $u = \sum_{n=0}^{\infty} p^n u_n$ and integration of both sides of t variable

$$\sum_{n=0}^{\infty} p^n u_n = u(x,0) + u_t(x,0)t + p\int_0^t\int_0^t (\frac{\partial^4(\sum_{n=0}^{\infty} p^n u_n)}{\partial x^4} + \left(\frac{\partial^2(\sum_{n=0}^{\infty} p^n u_n)}{\partial x^2}\right)^3 + \frac{\partial^3(\sum_{n=0}^{\infty} p^n u_n)}{\partial x^3}\frac{\partial^2(\sum_{n=0}^{\infty} p^n u_n)}{\partial x^2}\frac{\partial(\sum_{n=0}^{\infty} p^n u_n)}{\partial x})dtdt$$

$$\sum_{n=0}^{\infty} p^n u_n = \left(\frac{x^3}{3!}\right) + p\int_0^t\int_0^t (\frac{\partial^4(\sum_{n=0}^{\infty} p^n u_n)}{\partial x^4} + \left(\frac{\partial^2(\sum_{n=0}^{\infty} p^n u_n)}{\partial x^2}\right)^3 + \frac{\partial^3(\sum_{n=0}^{\infty} p^n u_n)}{\partial x^3}\frac{\partial^2(\sum_{n=0}^{\infty} p^n u_n)}{\partial x^2}\frac{\partial(\sum_{n=0}^{\infty} p^n u_n)}{\partial x})dtdt$$

Compare the coefficients of power of p

p^0: $$u_0 = \left(\frac{x^3}{3!}\right)$$

p^1: $$u_1 = \int_0^t\int_0^t (\frac{\partial^4 u_0}{\partial x^4} + \left(\frac{\partial^2 u_0}{\partial x^2}\right)^3 + \frac{\partial u_0}{\partial x}\frac{\partial^2 u_0}{\partial x^2}\frac{\partial^3 u_0}{\partial x^3})dtdt$$

$$u_1 = \int_0^t\int_0^t \left(0 + (x)^3 + \left(\frac{x^2}{2!}\right)(x)(1)\right)dtdt = \left(\frac{3x^3}{2!}\right)\frac{t^2}{2!}$$

p^2: $$u_2 = \int_0^t\int_0^t ((u_1)_{xxxx} + (u_0)_{xx}(u_1)_{xx} + ((u_0)_x(u_1)_{xx}(u_0)_{xxx} + (u_1)_x(u_0)_{xx}(u_0)_{xxx} + (u_1)_x(u_0)_{xxx}))dtdt$$

$$u_2 = 9\left(\frac{3x^3 + 2x^2}{2}\right)\frac{t^4}{4!}$$

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$$u = \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + \cdots$$

The approximation solution is

$$u(x,t) = \left(\frac{x^3}{3!}\right) + \left(\frac{3x^3}{2!}\right)\frac{t^2}{2!} + 9\left(\frac{3x^3 + 2x^2}{2!}\right)\frac{t^4}{4!}$$

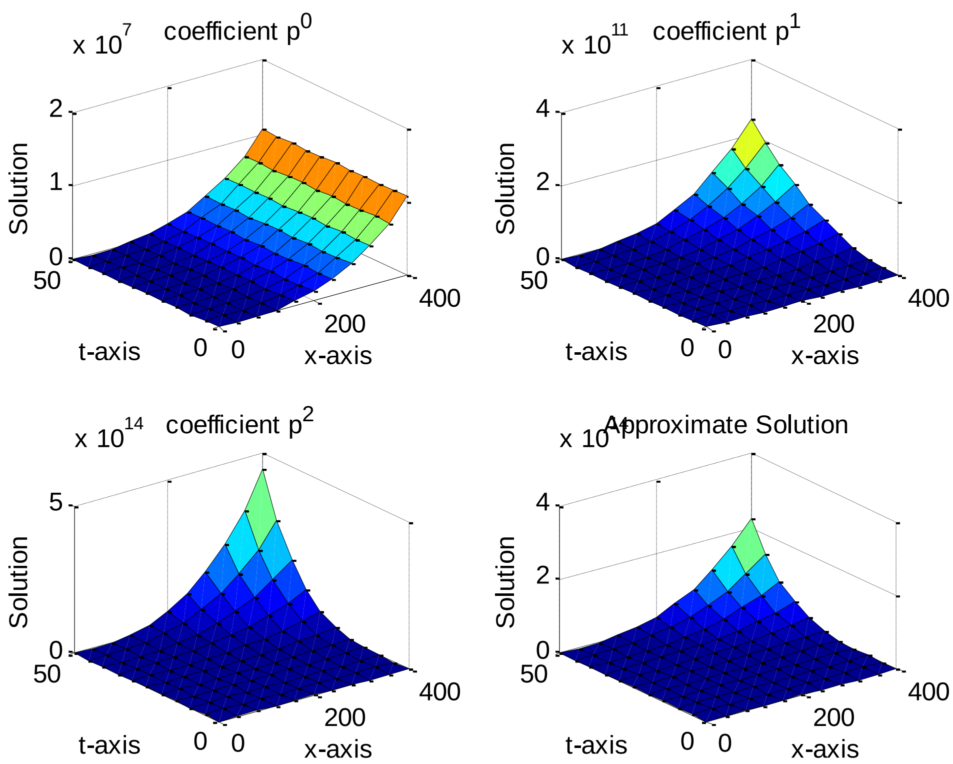

Figure 4.1: Graphical Representation of Example 4.3

Example 4.4: Solving the Nonlinear Equation of Transverse Vibration of Beam.

$$\frac{\partial^2 u}{\partial t^2} = a\frac{\partial^4 u}{\partial x^4} + b\left(\frac{\partial^2 u}{\partial x^2}\right)^3 + c\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 u}{\partial x^3} + d\left(\frac{\partial u}{\partial x}\right)^2\left(\frac{\partial^2 u}{\partial x^2}\right)^3$$

with initial conditions

$$u(x,0) = -0.5x^2 - 1.5 \quad \& \quad u_t(x,0) = 0$$

$$a = 1\,, b = -1.5\,, c = -6 \;\; \& \; d = -22.5$$

Using the Homotopy Perturbation Method (HPM), we have

$$(1-p)\frac{\partial^2 u}{\partial t^2} + p\left(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^4 u}{\partial x^4} - \frac{3}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)^3 - 6\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 u}{\partial x^3} - \frac{45}{2}\left(\frac{\partial u}{\partial x}\right)^2\left(\frac{\partial^2 u}{\partial x^2}\right)^3\right) = 0$$

$$\frac{\partial^2 u}{\partial t^2} = p\left(\frac{\partial^4 u}{\partial x^4} - \frac{3}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)^3 - 6\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 u}{\partial x^3} - \frac{45}{2}\left(\frac{\partial u}{\partial x}\right)^2\left(\frac{\partial^2 u}{\partial x^2}\right)^3\right)$$

Put $u = \sum_{n=0}^{\infty} p^n u_n$ and integration of both sides of t variable

$$\sum_{n=0}^{\infty} p^n u_n = u(x,0) + u_t(x,0)t + p\int_0^t\int_0^t \Bigg(\frac{\partial^4(\sum_{n=0}^{\infty} p^n u_n)}{\partial x^4} - \frac{3}{2}\left(\frac{\partial^2(\sum_{n=0}^{\infty} p^n u_n)}{\partial x^2}\right)^3 - 6\frac{\partial^3(\sum_{n=0}^{\infty} p^n u_n)}{\partial x^3}\frac{\partial^2(\sum_{n=0}^{\infty} p^n u_n)}{\partial x^2}\frac{\partial(\sum_{n=0}^{\infty} p^n u_n)}{\partial x} - \frac{45}{2}\left(\frac{\partial(\sum_{n=0}^{\infty} p^n u_n)}{\partial x}\right)^2\left(\frac{\partial^2(\sum_{n=0}^{\infty} p^n u_n)}{\partial x^2}\right)^3\Bigg)dtdt$$

$$\sum_{n=0}^{\infty} p^n u_n = -\left(\frac{x^2+3}{2}\right) + p\int_0^t\int_0^t \left(\frac{\partial^4(\sum_{n=0}^{\infty} p^n u_n)}{\partial x^4} - \frac{3}{2}\left(\frac{\partial^2(\sum_{n=0}^{\infty} p^n u_n)}{\partial x^2}\right)^3 - 6\frac{\partial^3(\sum_{n=0}^{\infty} p^n u_n)}{\partial x^3}\frac{\partial^2(\sum_{n=0}^{\infty} p^n u_n)}{\partial x^2}\frac{\partial(\sum_{n=0}^{\infty} p^n u_n)}{\partial x} - \frac{45}{2}\left(\frac{\partial(\sum_{n=0}^{\infty} p^n u_n)}{\partial x}\right)^2\left(\frac{\partial^2(\sum_{n=0}^{\infty} p^n u_n)}{\partial x^2}\right)^3\right)dtdt$$

Compare the coefficients of power of p

p^0: $$u_0 = -\left(\frac{x^2+3}{2}\right)$$

p^1: $$u_1 = \int_0^t\int_0^t \left(\frac{\partial^4 u_0}{\partial x^4} - \frac{3}{2}\left(\frac{\partial^2 u_0}{\partial x^2}\right)^3 - 6\frac{\partial u_0}{\partial x}\frac{\partial^2 u_0}{\partial x^2}\frac{\partial^3 u_0}{\partial x^3} - \frac{45}{2}\left(\frac{\partial u_0}{\partial x}\right)^2\left(\frac{\partial^2 u_0}{\partial x^2}\right)^3\right)dtdt$$

$$u_1 = \int_0^t\int_0^t \left(\frac{3+45x^2}{2}\right)dtdt = \left(\frac{45x^2+3}{2}\right)\frac{t^2}{2!}$$

p^2: $$u_2 = \int_0^t\int_0^t \Big((u_1)_{xxxx} - 4.5(u_0)_{xx}(u_1)_{xx} - 6\big((u_0)_x(u_1)_{xx}(u_0)_{xxx} + (u_1)_x(u_0)_{xx}(u_0)_{xxx} + (u_1)_x(u_0)_{xxx}\big) - 22.5\big(2(u_0)_x(u_1)_x(u_0)_{xx} + 3{(u_0)_x}^2(u_0)_{xx}(u_1)_{xx}\big)\Big)dtdt$$

$$u_2 = 45\left(\frac{45x^2+3}{2}\right)\frac{t^4}{4!}$$

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$$u = \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + \cdots$$

$$u(x,t) = -\left(\frac{x^2+3}{2}\right) + \left(\frac{45x^2+3}{2}\right)\frac{t^2}{2!} + 45\left(\frac{45x^2+3}{2}\right)\frac{t^4}{4!}$$

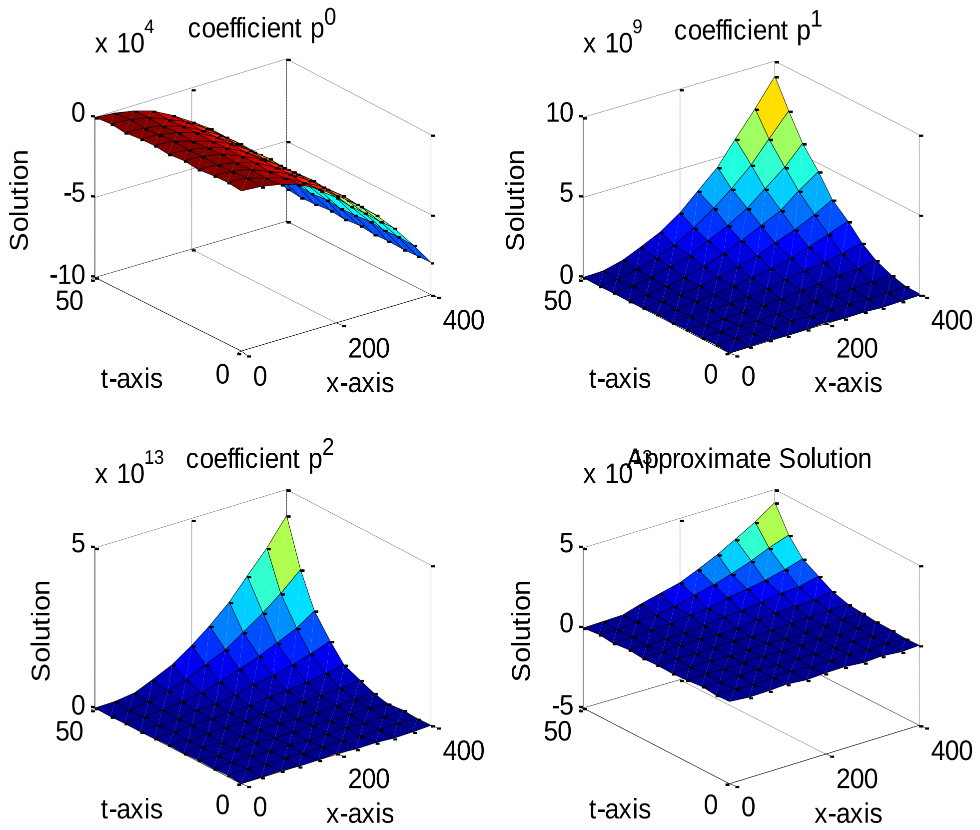

Figure 4.2: Graphical Representation of Example 4.4

5 Variational Iteration Method

We investigate the variational iteration method for solving modeling of bending of elastic beams.

5.1 Variational Iteration Method (VPM)

It was stated before that Adomian decomposition method, with is modified form and the noise terms phenomenon. The variational iteration method (VIM), established by He J. Huan, it is thoroughly used by many researchers to handle linear and nonlinear models. The method gives rapidly convergent successive approximations of the exact solution if such a solution exists. In what follows, we summarize the main steps of this method. for the differential equation

$$Lu + Nu = f(x,t)$$

where L and N are linear and nonlinear operators respectively, and $f(x,t)$ is the source inhomogeneous term, variational iteration method admits the use of the correction functional for equation which can be written as

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)(L\,u_n(\xi) + Nu_n(\xi) - f(x,\xi))d\xi$$

It is obvious that the successive approximations $u_n, n \geq 0$ can be established by determining $\lambda(\xi)$, a general Lagrange multiplier, which can be identified optimally via the variational theory. The function u_n is a restricted variation which means $\delta u_n = 0$. Using the obtained $\lambda(\xi)$, and selecting $u_0(x,t)$, the successive approximations $u_{n+1}(x,t), n \geq 0$, of the solution $u(x,t)$ will follow immediately.

In order to illustrate this method we give some examples.

Example 5.1: Solving the Linear Equation of Transverse Vibration of Beam.

$$\frac{\partial^2 u}{\partial t^2} + \alpha^2 \frac{\partial^4 u}{\partial x^4} = 0$$

with initial conditions

$$u(x,0) = \sin(x) \quad , \quad u_t(x,0) = 0 \quad \& \quad \alpha^2 = -1$$

Using the variational iteration method, we have

$$u_{n+1} = u_n + \int_0^t \lambda(\xi)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^4 u_n(x,\xi)}{\partial x^4}\right) d\xi$$

The stationary conditions give $\lambda(\xi) = \xi - t$, substituting this value of the Lagrange multiplier. We can select

$$u_0 = \sin(x)$$

when $n = 0$

$$u_1(x,t) = u_0(x,t) + \int_0^t (\xi - t)\left(\frac{\partial^2 u_0(x,\xi)}{\partial \xi^2} - \frac{\partial^4 u_0(x,\xi)}{\partial x^4}\right) d\xi$$

$$u_1(x,t) = \sin(x) + \int_0^t (\xi - t)\left(\frac{\partial^2(\sin(x))}{\partial \xi^2} - \frac{\partial^4(\sin(x))}{\partial x^4}\right) d\xi$$

$$u_1 = sin(x) + \sin(x)\frac{t^2}{2!}$$

when $n = 1$

$$u_2(x,t) = u_1(x,t) + \int_0^t (\xi - t)\left(\frac{\partial^2 u_1(x,\xi)}{\partial \xi^2} - \frac{\partial^4 u_1(x,\xi)}{\partial x^4}\right) d\xi$$

$$u_2(x,t) = sin(x) + \sin(x)\frac{t^2}{2!} + \int_0^t (\xi - t)\left(\frac{\partial^2(sin(x)\left(1 + \frac{\xi^2}{2!}\right))}{\partial \xi^2} - \frac{\partial^4(sin(x)\left(1 + \frac{\xi^2}{2!}\right))}{\partial x^4}\right) d\xi$$

$$u_2(x,t) = sin(x) + \sin(x)\frac{t^2}{2!} + \sin(x)\frac{t^4}{4!}$$

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$$u_n = sin(x) + \sin(x)\frac{t^2}{2!} + \sin(x)\frac{t^4}{4!} + \cdots + \sin(x)\frac{t^{2n}}{(2n)!}$$

$$u_n = sin(x)\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots + \frac{t^{2n}}{(2n)!}\right) = \sin(x)\sum_{k=0}^{n}\frac{t^{2k}}{(2k)!}$$

when $n \to \infty$

$$u(x,t) = \sin(x)\sum_{k=0}^{\infty}\frac{t^{2k}}{(2k)!}$$

$$u(x,t) = \sin(x)\cosh(t)$$

Example 5.2: Solving the Linear Equation of Transverse Vibration of Beam.

$$\frac{\partial^2 u}{\partial t^2} - (\frac{1}{x} + \frac{x^4}{120})\frac{\partial^4 u}{\partial x^4} = 0$$

with initial conditions

$$u(x,0) = 0 \quad ,u_t(x,0) = \left(1 + \frac{x^5}{120}\right)$$

Using the variational iteration method, we have

$$u_{n+1} = u_n + \int_0^t \lambda(\xi)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - (\frac{1}{x} + \frac{x^4}{120})\frac{\partial^4 u_n(x,\xi)}{\partial x^4}\right)d\xi$$

The stationary conditions give $\lambda(\xi) = \xi - t$, substituting this value of the Lagrange multiplier. We can select

$$u_0 = \left(1 + \frac{x^5}{120}\right)t$$

when $n = 0$

$$u_1 = u_0 + \int_0^t (\xi - t)\left(\frac{\partial^2 u_0(x,\xi)}{\partial \xi^2} - \left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{\partial^4 u_0(x,\xi)}{\partial x^4}\right)d\xi$$

$$u_1 = \left(1+\frac{x^5}{120}\right)t + \int_0^t (\xi - t)\left(0 - (\frac{1}{x}+\frac{x^4}{120})x\xi\right)d\xi$$

$$u_1 = \left(1+\frac{x^5}{120}\right)t - \int_0^t (\xi - t)\left(1+\frac{x^5}{120}\right)\xi d\xi$$

$$u_1 = \left(1+\frac{x^5}{120}\right)t + \left(1+\frac{x^5}{120}\right)\frac{t^3}{3!}$$

$$u_1 = \left(1+\frac{x^5}{120}\right)\left(t+\frac{t^3}{3!}\right)$$

when $n = 1$

$$u_2 = u_1 + \int_0^t (\xi - t)\left(\frac{\partial^2 u_1(x,\xi)}{\partial \xi^2} - (\frac{1}{x}+\frac{x^4}{120})\frac{\partial^4 u_1(x,\xi)}{\partial x^4}\right)d\xi$$

$$u_2 = \left(1+\frac{x^5}{120}\right)\left(t+\frac{t^3}{3!}\right) + \int_0^t (\xi - t)\left(\left(1+\frac{x^5}{120}\right)(\xi) - (\frac{1}{x}+\frac{x^4}{120})x(\xi+\frac{\xi^3}{3!})\right)d\xi$$

$$u_2 = \left(1+\frac{x^5}{120}\right)\left(t+\frac{t^3}{3!}\right) + \int_0^t (\xi - t)\left(\left(1+\frac{x^5}{120}\right)(\xi) - \left(1+\frac{x^5}{120}\right)(\xi+\frac{\xi^3}{3!})\right)d\xi$$

$$u_2 = \left(1+\frac{x^5}{120}\right)\left(t+\frac{t^3}{3!}\right) + \int_0^t (\xi - t)\left(\left(1+\frac{x^5}{120}\right)(-\frac{\xi^3}{3!})\right)d\xi$$

$$u_2 = \left(1+\frac{x^5}{120}\right)\left(t+\frac{t^3}{3!}\right) + \left(1+\frac{x^5}{120}\right)\frac{t^5}{5!}$$

$$u_2 = \left(1+\frac{x^5}{120}\right)\left(t+\frac{t^3}{3!}+\frac{t^5}{5!}\right)$$

.

.

.

when $n = k - 1$

$$u_k = u_{k-1} + \int_0^t (\xi - t)\left(\frac{\partial^2 u_{k-1}(x,\xi)}{\partial \xi^2} - (\frac{1}{x} + \frac{x^4}{120})\frac{\partial^4 u_{k-1}(x,\xi)}{\partial x^4}\right) d\xi$$

$$u_k = \left(1 + \frac{x^5}{120}\right)\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots \frac{t^{2k+1}}{(2k+1)!}\right)$$

$$u_k = \left(1 + \frac{x^5}{120}\right)\sum_{n=0}^{k} \frac{t^{2n+1}}{(2n+1)!}$$

Taking limit at $k \to \infty$

$$u(x,t) = \left(1 + \frac{x^5}{120}\right)\sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!}$$

$$u(x,t) = \left(1 + \frac{x^5}{120}\right)\sinh(t)$$

Example 5.3: Solving the Nonlinear Equation of Transverse Vibration of Beam.

$$\frac{\partial^2 u}{\partial t^2} = a\frac{\partial^4 u}{\partial x^4} + b\left(\frac{\partial^2 u}{\partial x^2}\right)^3 + c\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 u}{\partial x^3} + d\left(\frac{\partial u}{\partial x}\right)^2\left(\frac{\partial^2 u}{\partial x^2}\right)^3$$

with initial conditions

$$u(x,0) = \sin(x) \qquad \& \qquad u_t(x,0) = 0$$

$$a = 1, \quad b = -1.5, \qquad c = -6 \quad \& \; d = -22.5$$

Using the variational iteration method, we have

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^4 u_n(x,\xi)}{\partial x^4} + \frac{3}{2}\left(\frac{\partial^2 u_n(x,\xi)}{\partial x^2}\right)^3 + 6\frac{\partial u_n(x,\xi)}{\partial x}\frac{\partial^2 u_n(x,\xi)}{\partial x^2}\frac{\partial^3 u_n(x,\xi)}{\partial x^3} + \frac{45}{2}\left(\frac{\partial u_n(x,\xi)}{\partial x}\right)^2\left(\frac{\partial^2 u_n(x,\xi)}{\partial x^2}\right)^3\right)d\xi$$

The stationary conditions give $\lambda(\xi) = \xi - t$, substituting this value of the Lagrange multiplier. We can select

$$u_0(x,t) = \sin(x)$$

when $n = 0$

$$u_1(x,t) = u_0(x,t) + \int_0^t (\xi - t)\left(\frac{\partial^2 u_0(x,\xi)}{\partial \xi^2} - \frac{\partial^4 u_0(x,\xi)}{\partial x^4} + \frac{3}{2}\left(\frac{\partial^2 u_0(x,\xi)}{\partial x^2}\right)^3 + 6\frac{\partial u_0(x,\xi)}{\partial x}\frac{\partial^2 u_0(x,\xi)}{\partial x^2}\frac{\partial^3 u_0(x,\xi)}{\partial x^3} + \frac{45}{2}\left(\frac{\partial u_n(x,\xi)}{\partial x}\right)^2\left(\frac{\partial^2 u_n(x,\xi)}{\partial x^2}\right)^3\right)d\xi$$

$$u_1(x,t) = sin(x) + \int_0^t (\xi - t)\left(\frac{\partial^2 u_0(x,\xi)}{\partial \xi^2} - \frac{\partial^4 u_0(x,\xi)}{\partial x^4} + \frac{3}{2}\left(\frac{\partial^2 u_0(x,\xi)}{\partial x^2}\right)^3 + 6\frac{\partial u_0(x,\xi)}{\partial x}\frac{\partial^2 u_0(x,\xi)}{\partial x^2}\frac{\partial^3 u_0(x,\xi)}{\partial x^3} + \frac{45}{2}\left(\frac{\partial u_0(x,\xi)}{\partial x}\right)^2\left(\frac{\partial^2 u_0(x,\xi)}{\partial x^2}\right)^3\right)d\xi$$

$$u_1(x,t) = sin(x) + \int_0^t (\xi - t)\left(\frac{\partial^2 (\sin(x))}{\partial \xi^2} - \frac{\partial^4 (\sin(x))}{\partial x^4} + \frac{3}{2}\left(\frac{\partial^2 (\sin(x))}{\partial x^2}\right)^3 + 6\frac{\partial (\sin(x))}{\partial x}\frac{\partial^2 (\sin(x))}{\partial x^2}\frac{\partial^3 (\sin(x))}{\partial x^3} + \frac{45}{2}\left(\frac{\partial (\sin(x))}{\partial x}\right)^2\left(\frac{\partial^2 (\sin(x))}{\partial x^2}\right)^3\right)d\xi$$

$$u_1(x,t) = sin(x) + \int_0^t (\xi - t)\left(0 - \sin(x) - \frac{3}{2}sin^3(x) + 6cos^2(x)\sin(x) - \frac{45}{2}cos^2(x)sin^3(x)\right)d\xi$$

$$u_1(x,t) = \sin(x) + \left(-5\sin(x) + 30sin^3(x) - \frac{45}{2}sin^5(x)\right)\frac{t^2}{2}$$

when $n = 1$

$$u_2(x,t) = \sin(x) + \left(-5\sin(x) + 30sin^3(x) - \frac{45}{2}sin^5(x)\right)\frac{t^2}{2}$$

$$+ \int_0^t (\xi$$

$$- t)\left(\frac{\partial^2 u_1(x,\xi)}{\partial \xi^2} - \frac{\partial^4 u_1(x,\xi)}{\partial x^4} + \frac{3}{2}\left(\frac{\partial^2 u_1(x,\xi)}{\partial x^2}\right)^3\right.$$

$$+ 6\frac{\partial u_1(x,\xi)}{\partial x}\frac{\partial^2 u_1(x,\xi)}{\partial x^2}\frac{\partial^3 u_1(x,\xi)}{\partial x^3}$$

$$\left. + \frac{45}{2}\left(\frac{\partial u_1(x,\xi)}{\partial x}\right)^2\left(\frac{\partial^2 u_1(x,\xi)}{\partial x^2}\right)^3\right)d\xi$$

$$u_2(x,t) = \sin(x) + \left(-5\sin(x) + 30sin^3(x) - \frac{45}{2}sin^5(x)\right)\frac{t^2}{2}$$

$$+ \int_0^t (\xi - t)\{(\sin(x)$$

$$+ \left(-1805\sin(x) + 8280sin^3(x) - \frac{14625}{2}sin^5(x)\right)\frac{\xi^2}{2}$$

$$- \frac{3}{2}\left(-\sin(x)\right.$$

$$+ \left(5si\,n(x) - 90sin^3(x) + 180\sin(x)\,cos^2(x)\right.$$

$$\left.\left. + \frac{225}{2}sin^5(x) - 450sin^3(x)cos^2(x)\right)\frac{\xi^2}{2}\right)^3$$

$$- 6\left(\cos(x) + \left(-5\cos(x) + 90\cos(x)\,sin^2(x) - \right.\right.$$

$$\left.\left.\frac{225}{2}sin^4(x)\cos(x)\right)\frac{\xi^2}{2}\right)\left(-\sin(x) + \left(5si\,n(x) - 90sin^3(x) + \right.\right.$$

$$180\sin(x)\,cos^2(x) + \frac{225}{2}sin^5(x) -$$

$$\left.\left.450sin^3(x)cos^2(x)\right)\frac{\xi^2}{2}\right)\left(-\cos(x) + \left(5\cos(x) - \right.\right.$$

$$660sin^2(x)\cos(x) + 180cos^3(x) + \frac{2925}{2}sin^4(x)\cos(x) -$$

$$1350 sin^2(x) cos^3(x)\Big)\frac{\xi^2}{2}\Big) - -\frac{45}{2}\Big(\cos(x) + \Big(-5\cos(x) + 90\cos(x)\, sin^2(x) - \frac{225}{2} sin^4(x)\cos(x)\Big)\frac{\xi^2}{2}\Big)^2 \Big(-\sin(x) + \Big(5 si\, n(x) - 90 sin^3(x) + 180\sin(x)\, cos^2(x) + \frac{225}{2} sin^5(x) - 450 sin^3(x) cos^2(x)\Big)\frac{\xi^2}{2}\Big)^3 \}d\xi$$

Algorithm 5.1

To obtain u_2 , u_3 , ...

Input: u_1

Output: u

Step 1: for i=1,2,…..

$$u(i+1) = u(i) + int((s-t) * diff(u(i), 4, x) - 1.5 * \mathrm{diff}((\mathrm{u}(i))^\wedge 3{,}2, \mathrm{x})) - 6 * \mathrm{diff}(\mathrm{u}(i), 1, \mathrm{x}) * \mathrm{diff}(\mathrm{u}(i), 2, \mathrm{x}) * \mathrm{diff}(\mathrm{u}(i), 3, \mathrm{x}) - 22.5 * \mathrm{diff}((\mathrm{u}(i))^\wedge 2{,}1, \mathrm{x}) * \mathrm{diff}((\mathrm{u}(i))^\wedge 3{,}2, \mathrm{x})), '\mathrm{s}', 0, \mathrm{t})$$

Step 2: output
Stop

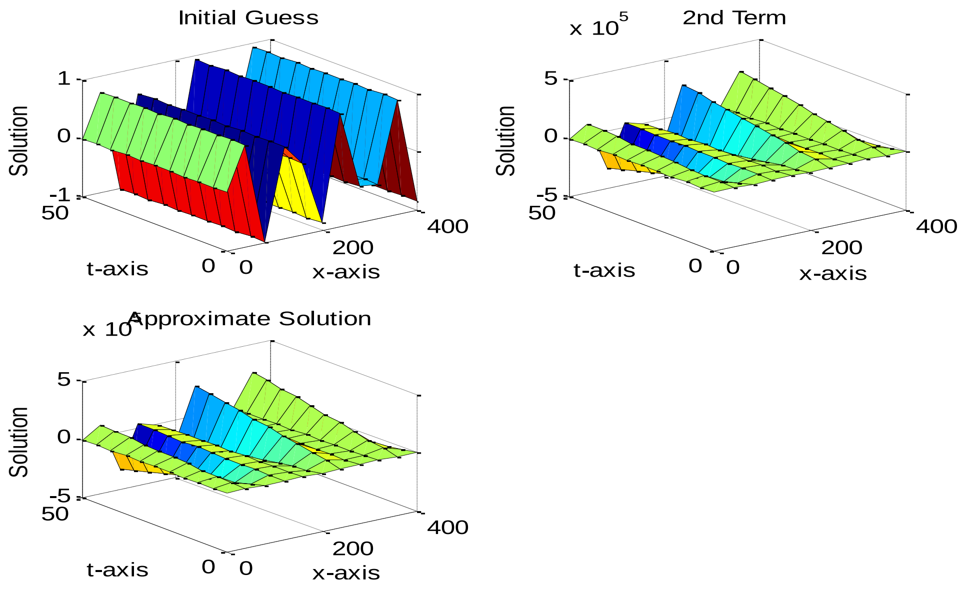


Figure 5.1: Graphical Representation of Example 5.3

Table 5.1: Comparison between two methods, of the transverse of vibration beam equation when the time equal one, of Example 5.1

	Approximation Solution				Error		
X	EXACT	ADM	HPM	VIM	ADM	HPM	VIM
0.0000	0.0000	0.0000	0.0000	0.0000	0.000	0.000	0.000
0.1000	0.1541	0.1541	0.1541	0.1541	0.000	0.000	0.000
0.2000	0.3066	0.3066	0.3066	0.3066	0.000	0.000	0.000
0.3000	0.4560	0.4560	0.4560	0.4560	0.000	0.000	0.000
0.4000	0.6009	0.6009	0.6009	0.6009	0.000	0.000	0.000
0.5000	0.7398	0.7398	0.7398	0.7398	0.000	0.000	0.000
0.6000	0.8713	0.8713	0.8713	0.8713	0.000	0.000	0.000
0.7000	0.9941	0.9941	0.9941	0.9941	0.000	0.000	0.000
0.8000	1.1069	1.1069	1.1069	1.1069	0.000	0.000	0.000
0.9000	1.2087	1.2087	1.2087	1.2087	0.000	0.000	0.000
1.0000	1.2985	1.2985	1.2985	1.2985	0.000	0.000	0.000

*Error=Exact Solution – Approximation Solution.

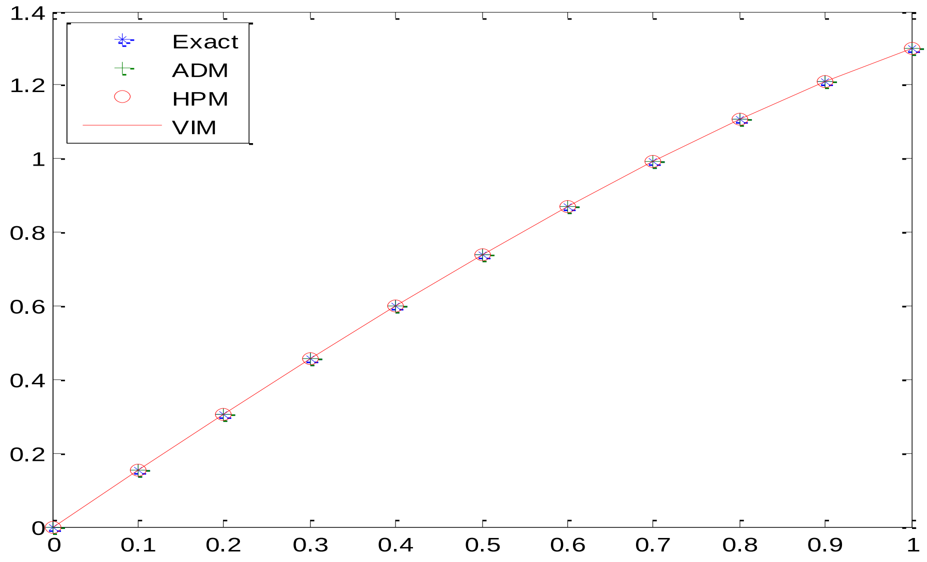


Figure 5.2: Comparison between Exact Solution, ADM, HPM and VIM of Example 5.1

Table 5.2: Comparison between two methods, of the transverse of vibration beam equation when the time equal one, of Example 5.2

	Approximation Solution				Error		
X	EXACT	ADM	HPM	VIM	ADM	HPM	VIM
0.0000	1.1752	1.1752	1.1752	1.1752	0.000	0.000	0.000
0.1000	1.1752	1.1752	1.1752	1.1752	0.000	0.000	0.000
0.2000	1.1752	1.1752	1.1752	1.1752	0.000	0.000	0.000
0.3000	1.1752	1.1752	1.1752	1.1752	0.000	0.000	0.000
0.4000	1.1753	1.1753	1.1753	1.1753	0.000	0.000	0.000
0.5000	1.1755	1.1755	1.1755	1.1755	0.000	0.000	0.000
0.6000	1.1760	1.1760	1.1760	1.1760	0.000	0.000	0.000
0.7000	1.1768	1.1768	1.1768	1.1768	0.000	0.000	0.000
0.8000	1.1784	1.1784	1.1784	1.1784	0.000	0.000	0.000
0.9000	1.1810	1.1810	1.1810	1.1810	0.000	0.000	0.000
1.0000	1.1850	1.1850	1.1850	1.1850	0.000	0.000	0.000

*Error=Exact Solution – Approximation Solution.

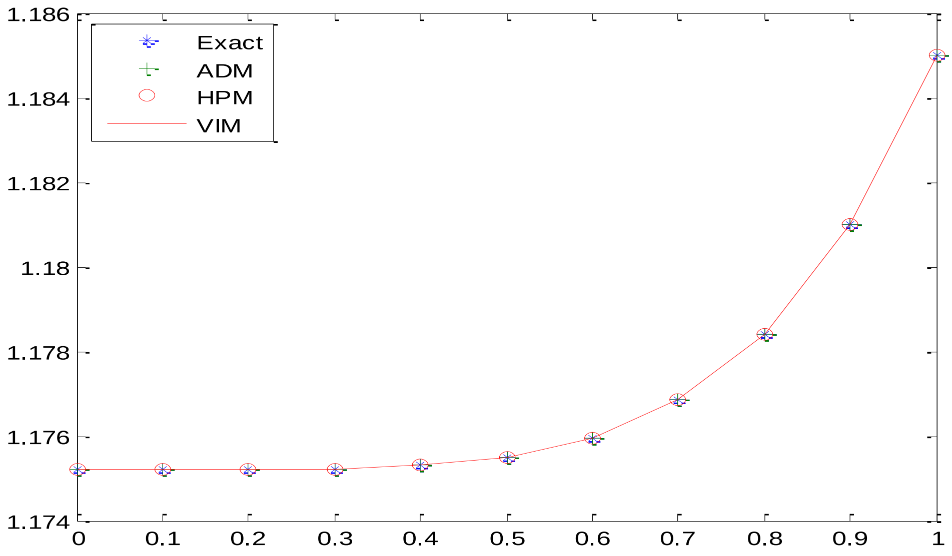


Figure 5.3: Comparison between Exact Solution, ADM, HPM and VIM of Example 5.2

Table 5.3: Comparison between two methods, of the nonlinear transverse of vibration beam equation when the time equal one and the initial conditions are: $u(x,0) = -0.5x^2$ & $u_t(x,0) = 0$, of Example 5.3

		Approximation Solution			Error		
X	EXACT	ADM	HPM	VIM	ADM	HPM	VIM
0.0000	0.0000	9.1875	3.5625	8.6875	9.1875	3.5625	8.6875000
0.1000	-0.0077	11.4044	4.0919	10.9044	11.4121	4.0996	10.9121000
0.2000	-0.0309	18.0550	5.6800	17.5550	18.0859	5.7109	17.5859000
0.3000	-0.0694	29.1394	8.3269	28.6394	29.2088	8.3963	28.708800
0.4000	-0.1234	44.6575	12.0325	44.1575	44.7809	12.1559	44.280900
0.5000	-0.1929	64.6094	16.7969	64.1094	64.8023	16.9898	64.302300
0.6000	-0.2778	88.9950	22.6200	88.4950	89.2728	22.8978	88.772800
0.7000	-0.3781	117.8144	29.5019	117.314	118.1924	29.8799	117.69240
0.8000	-0.4938	151.0675	37.4425	150.5675	151.5613	37.9363	151.0613
0.9000	-0.6249	188.7544	46.4419	188.2544	189.3793	47.0668	188.8793
1.0000	-0.7715	230.8750	56.5000	230.3750	231.6465	57.2715	231.1465

*Error=Exact Solution – Approximation Solution

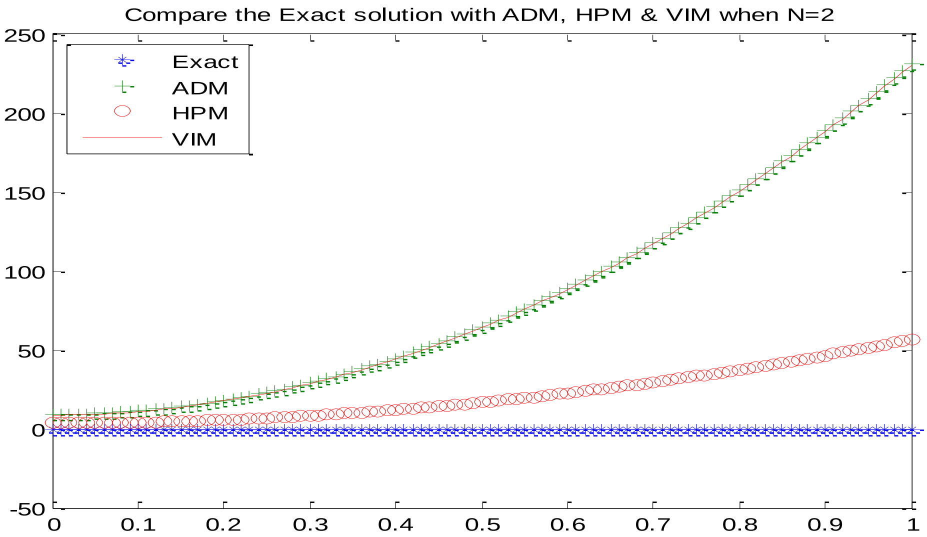


Figure 5.4: Comparison between Exact Solution, ADM, HPM and VIM of Example 5.3

Conclusions of this Chapter

In this chapter we Comparison the three analytical methods, Adomian's Decomposition Method (ADM), Homotopy Perturbation Method (HPM) and Variational Iteration Method (VIM). We get the Homotopy Perturbation Method is best method and fast to convergent for exact solution.

6 Numerical Solutions

There are a great many methods used for finding the solutions of differential equations numerically. These methods usually depend on the sort of discretization and transformation of the continuous equations into a matrix problem. Then we can solve it by using the existing matrix solver. The idea of the discretization method is to transform a differential equation problem of infinite dimension to finite dimension; this is possible to solve approximate solutions. The common known discretization methods for obtaining the approximate solutions to differential equations are finite difference method, finite element method, finite volumes method, and spectral methods. These methods lead to linear or nonlinear systems of polynomial equations according to the proposed problem. In general, numerical discretization techniques use a method, which can discretize the continuity of a proposed system into finite points. we discuss the finite difference method (FDM) for solving modeling of bending of elastic beams and design algorithms by using MATLAB Program [8].

6.1 The Finite Differences Approximations

The finite differences approximations for derivatives are one of the simplest and of the oldest methods to solve differential equations. It was already known by L. Euler 1768, in one dimension of space and was probably extended to dimension two by C. Runge 1908. The advent of finite difference techniques in numerical applications began in the early 1950s and their development was stimulated by the emergence of computers that offered a convenient framework for dealing with complex problems of science and technology. Theoretical results have been obtained during the last five decades regarding the accuracy, stability and convergence of the finite difference method for partial differential equations. The principle of finite difference methods is close to the numerical schemes used to solve ordinary differential equations. It consists in approximating the differential operator by replacing the derivatives in the equation using differential quotients. The domain is partitioned in space and in time and approximations of the solution are computed at the space or time points.

The error between the numerical solution and the exact solution is determined by the error that is committed by going from a differential operator to a difference operator. This error is called the discretization error or truncation error. The term truncation error reflects the fact that a finite part of a Taylor series is used in the approximation [8, 13, 14].

6.1.1 Taylor's Series

Suppose the function u is continuous in the neighborhood of x, for any $h > 0$ we have

$$u(x+h) = u(x) + hu'(x) + \frac{h^2}{2}u''(x+\xi)$$

where ξ is number between 0 and h. For the treatment of problems, it is convenient to retain only the first two terms of the previous expression:

$$u(x+h) = u(x) + hu'(x) + O(h^2)$$

$$u'(x) = \frac{u(x+h) - u(x)}{h} - O(h)$$

where the term $O(h)$ indicates that the error of the approximation is proportional to h.

6.1.2 Euler's Explicit Method (EEM)

The explicit method by using central difference operator to approximate the derivatives

$$\frac{\partial u(x_i,t_j)}{\partial x} \cong \frac{u(x_{i+1},t_j) - u(x_{i-1},t_j)}{2(\Delta x)} - \frac{(\Delta x)^2}{6}\frac{\partial^3 u(\xi_i,t_j)}{\partial x^3}$$

$$\frac{\partial u(x_i,t_j)}{\partial t} \cong \frac{u(x_i,t_{j+1}) - u(x_i,t_{j-1})}{2(\Delta t)} - \frac{(\Delta t)^2}{6}\frac{\partial^3 u(x_i,\eta_j)}{\partial t^3}$$

$$\frac{\partial^2 u(x_i,t_j)}{\partial x^2} = \frac{u(x_{i+1},t_j) - 2u(x_i,t_j) + u(x_{i-1},t_j)}{(\Delta x)^2} - \frac{(\Delta x)^2}{12}\frac{\partial^4 u(\xi_i,t_j)}{\partial x^4}$$

$$\frac{\partial^2 u(x_i,t_j)}{\partial t^2} = \frac{u(x_i,t_{j+1}) - 2u(x_i,t_j) + u(x_i,t_{j-1})}{(\Delta t)^2} - \frac{(\Delta t)^2}{12}\frac{\partial^4 u(x_i,\eta_j)}{\partial t^4}$$

$$\frac{\partial^3 u(x_i,t_j)}{\partial x^3} = \frac{u(x_{i+2},t_j) - 2u(x_{i+1},t_j) + 2u(x_{i-1},t_j) - u(x_{i-2},t_j)}{2(\Delta x)^3} - \frac{2(\Delta x)^2}{6!}\frac{\partial^6 u(\xi_i,t_j)}{\partial x^6}$$

$$\frac{\partial^4 u(x_i,t_j)}{\partial x^4} = \frac{u(x_{i+2},t_j) - 4u(x_{i+1},t_j) + 6u(x_i,t_j) - 4u(x_{i-1},t_j) + u(x_{i-2},t_j)}{(\Delta x)^4} - \frac{2(\Delta x)^2}{8!}\frac{\partial^8 u(\xi_i,t_j)}{\partial x^8}$$

6.1.3 Euler's Implicit Method (EIM)

The implicit method by using central difference operator to approximate the derivatives

$$\frac{\partial u(x_i,t_{j+1})}{\partial x} \cong \frac{u(x_{i+1},t_{j+1}) - u(x_{i-1},t_{j+1})}{2(\Delta x)} - \frac{(\Delta x)^2}{6}\frac{\partial^3 u(\xi_i,t_{j+1})}{\partial x^3}$$

$$\frac{\partial u(x_i,t_j)}{\partial t} \cong \frac{u(x_i,t_{j+1}) - u(x_i,t_{j-1})}{2(\Delta t)} - \frac{(\Delta t)}{6}\frac{\partial^3 u(x_i,\eta_j)}{\partial t^3}$$

$$\frac{\partial^2 u(x_i,t_{j+1})}{\partial x^2} = \frac{u(x_{i+1},t_{j+1}) - 2u(x_i,t_{j+1}) + u(x_{i-1},t_{j+1})}{(\Delta x)^2} - \frac{(\Delta x)^2}{12}\frac{\partial^4 u(\xi_i,t_{j+1})}{\partial x^4}$$

$$\frac{\partial^2 u(x_i,t_j)}{\partial t^2} = \frac{u(x_i,t_{j+2}) - 2u(x_i,t_j) + u(x_i,t_{j-1})}{(\Delta t)^2} - \frac{(\Delta t)^2}{12}\frac{\partial^4 u(x_i,\eta_j)}{\partial t^4}$$

$$\frac{\partial^3 u(x_i,t_{j+1})}{\partial x^3} = \frac{u(x_{i+2},t_{j+1}) - 2u(x_{i+1},t_{j+1}) + 2u(x_{i-1},t_{j+1}) - u(x_{i-2},t_{j+1})}{2(\Delta x)^3} - \frac{2(\Delta x)^2}{6!}\frac{\partial^6 u(\xi_i,t_{j+1})}{\partial x^6}$$

$$\frac{\partial^4 u(x_i, t_{j+1})}{\partial x^4} = \frac{u(x_{i+2}, t_{j+1}) - 4u(x_{i+1}, t_{j+1}) + 6u(x_i, t_{j+1}) - 4u(x_{i-1}, t_{j+1}) + u(x_{i-2}, t_{j+1})}{(\Delta x)^4} - \frac{2(\Delta x)^2}{8!} \frac{\partial^8 u(\xi_i, t_{j+1})}{\partial x^8}$$

To design function by using MATLAB program, we consider

$$u(x_{i+2}, t_j) = u_{i+2,j} \ , \ u(x_{i+1}, t_j) = u_{i+1,j} \ , u(x_i, t_j) = u_{i,j} \ ,$$
$$u(x_{i-1}, t_j) = u_{i-1,j} \ , \ u(x_{i-2}, t_j) = u_{i-2,j}$$

$$u(x_i, t_{j+1}) = u_{i,j+1} \ , \ u(x_i, t_{j-1}) = u_{i,j-1}, \qquad \Delta x = h \ \ \& \ \Delta t = k$$

Table 6.1: The derivatives by using Euler explicit method for central differences operator

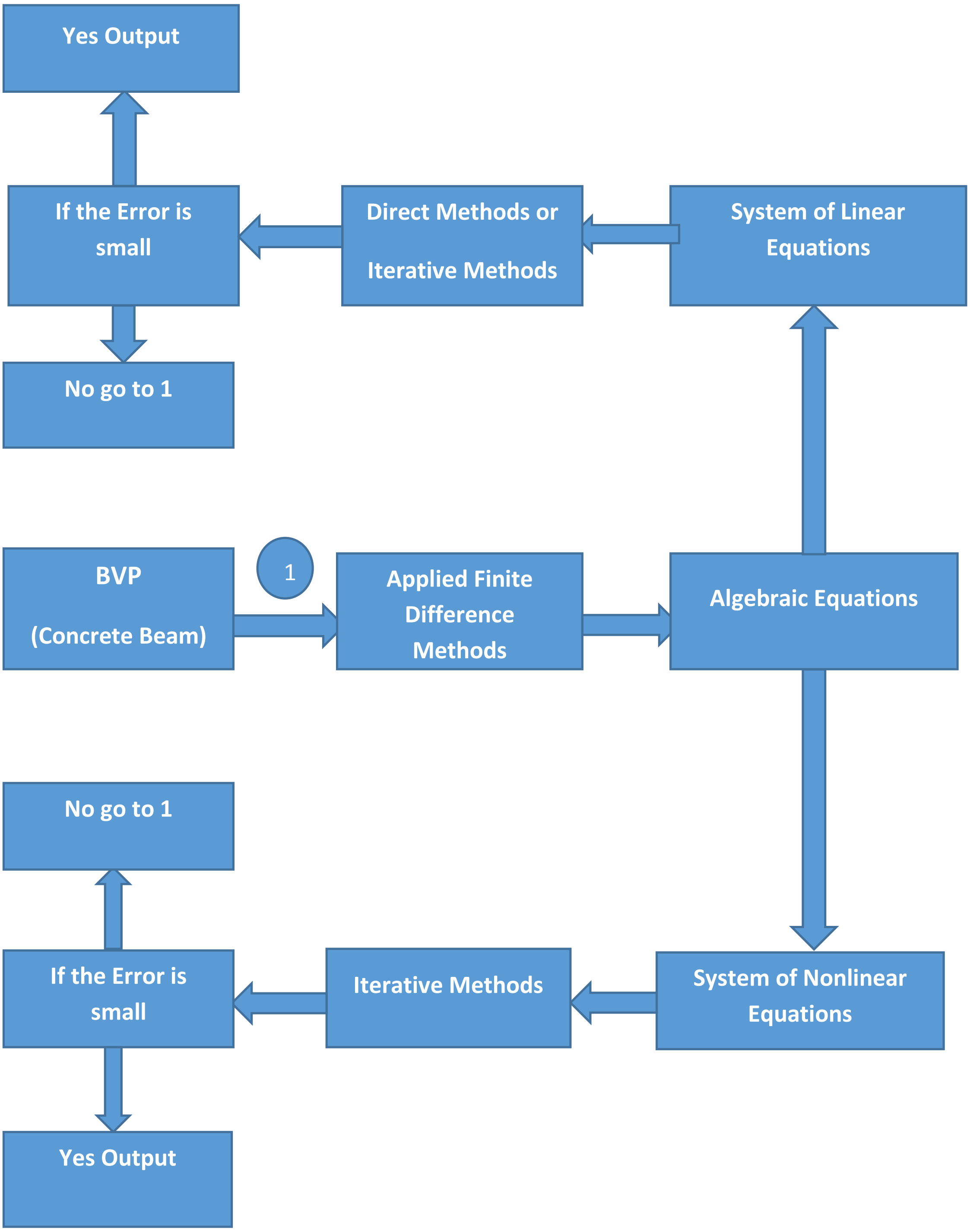
Method	Formula	Truncation Error
Two Point Central Difference	$\frac{\partial u(x_i, t_j)}{\partial x} = \frac{u_{i+1,j} - u_{i-1,j}}{2h}$ $\frac{\partial u(x_i, t_j)}{\partial t} = \frac{u_{i,j+1} - u_{i,j-1}}{2k}$	$O(h^2)$ $O(k^2)$
Three Point Central Difference	$\frac{\partial^2 u(x_i, t_j)}{\partial x^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$ $\frac{\partial^2 u(x_i, t_j)}{\partial t^2} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}$	$O(h^2)$ $O(k^2)$
Four Point Central Difference	$\frac{\partial^3 u(x_i, t_j)}{\partial x^3} = \frac{u_{i+2,j} - 2u_{i+1,j} + 2u_{i-1,j} - u_{i-2,j}}{2h^3}$ $\frac{\partial^3 u(x_i, t_j)}{\partial t^3} = \frac{u_{i,j+2} - 2u_{i,j+1} + 2u_{i,j-1} - u_{i,j-2}}{2k^3}$	$O(h^2)$ $O(k^2)$
Five Point Central Difference	$\frac{\partial^4 u(x_i, t_j)}{\partial x^4} = \frac{u_{i+2,j} - 4u_{i+1,j} + 6u_{i,j} - 4u_{i-1,j} - u_{i-2,j}}{h^4}$	$O(h^2)$

	$$\frac{\partial^4 u(x_i,t_j)}{\partial t^4} = \frac{u_{i,j+2} - 4u_{i,j+1} + 6u_{i,j} - 4u_{i,j-1} - u_{i,j-2}}{k^4}$$	$O(k^2)$

Table 6.2: The derivatives by using Euler implicit method for central difference operator

Method	Formula	Truncation Error
Two Point Central Difference	$$\frac{\partial u(x_i,t_j)}{\partial x} = \frac{u_{i+1,j+1} - u_{i-1,j+1}}{2h}$$ $$\frac{\partial u(x_i,t_j)}{\partial t} = \frac{u_{i,j+1} - u_{i,j-1}}{2k}$$	$O(h^2)$ $O(k^2)$
Three Point Central Difference	$$\frac{\partial^2 u(x_i,t_j)}{\partial x^2} = \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2}$$ $$\frac{\partial^2 u(x_i,t_j)}{\partial t^2} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}$$	$O(h^2)$ $O(k^2)$
Four Point Central Difference	$$\frac{\partial^3 u(x_i,t_j)}{\partial x^3} = \frac{u_{i+2,j+1} - 2u_{i+1,j+1} + 2u_{i-1,j+1} - u_{i-2,j+1}}{2h^3}$$ $$\frac{\partial^3 u(x_i,t_j)}{\partial t^3} = \frac{u_{i,j+2} - 2u_{i,j+1} + 2u_{i,j-1} - u_{i,j-2}}{2k^3}$$	$O(h^2)$ $O(k^2)$
Five Point Central Difference	$$\frac{\partial^4 u(x_i,t_j)}{\partial x^4} = \frac{u_{i+2,j+1} - 4u_{i+1,j+1} + 6u_{i,j+1} - 4u_{i-1,j+1} - u_{i-2,j+1}}{h^4}$$ $$\frac{\partial^4 u(x_i,t_j)}{\partial t^4} = \frac{u_{i,j+2} - 4u_{i,j+1} + 6u_{i,j} - 4u_{i,j-1} - u_{i,j-2}}{k^4}$$	$O(h^2)$ $O(k^2)$

Diagram of Steps for Solving BVPs by Using Finite Difference Methods



6.2 Stability Analysis

Consider the transition equation is

$$\underline{u}_i^{j+1} = A\underline{u}_i^j$$

where

$$\underline{u}_i^j = \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ u_{3,j} \\ \cdot \\ \cdot \\ \cdot \\ u_{n-1,j} \end{bmatrix} , \quad \underline{u}_i^{j+1} = \begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \\ \cdot \\ \cdot \\ \cdot \\ u_{n-1,j+1} \end{bmatrix}$$

$A \in IR^{n-1 \times n-1}$ is a matrix.

when $j = 0$

$$\underline{u}_i^1 = A\underline{u}_i^0$$

when $j = 1$

$$\underline{u}_i^2 = A\underline{u}_i^1 = AA\underline{u}_i^0 = A^2\underline{u}_i^0$$

when $j = 1$

$$\underline{u}_i^3 = A\underline{u}_i^2 = AA^2\underline{u}_i^0 = A^3\underline{u}_i^0$$

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when $j = k$

$$\underline{u}_i^k = A\underline{u}_i^2 = AA^{k-1}\underline{u}_i^0 = A^k\underline{u}_i^0$$

$$\left\|\underline{u}_i^k\right\| \leq \|A\|^k \left\|\underline{u}_i^0\right\|$$

i. for the influence of the initial conditions and rounding errors in the initial condition to decay with time, it must be the case that $\|A\| < 1$
ii. if $\|A\| > 1$, some eigenvalues of the matrix A can grow without bound generating ridiculous results. In such cases the method is said to be unstable.
iii. Taking $\lambda = \|A\| = \|A\|_2$ equal maximum eigenvalues of A for symmetric A (the spectral norm), the maximum eigenvalues describes the stability of the method.

6.3 Spectral Method (SM)

Spectral methods are approximation techniques for the computation of the solutions to ordinary and partial differential equations. They are based on a polynomial expansion of the solution; the precision of these methods is limited only by the finite difference method and the finite element method. In this method, we use discrete Fourier transform and difference operator to gather to solve partial differential equations. Spectral method approximate the solution, as a linear combination of continuous functions that are generally nonzero throughout the domain (Chebychev polynomials) is global approach.

6.3.1 Discrete Fourier Transform (DFT)

The discrete Fourier transform of $x[n]$ denoted by

$$x[k] = \sum_{k=0}^{2n-1} x[n]e^{-jx_i k}$$

However, inverse of discrete Fourier transform denoted by

$$x[n] = \frac{1}{2n} \sum_{k=-n+1}^{n} x[k]e^{jx_i k}, \quad i = 0,1,\dots,22-1$$

With this definition the spatial derivative are

$$\frac{\partial u}{\partial x} = \frac{1}{2n} \sum_{k=-n+1}^{n} (jk)u[k]e^{jx_i k}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2n} \sum_{k=-n+1}^{n} (jk)^2 u[k]e^{jx_i k}$$

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$$\frac{\partial^n u}{\partial x^n} = \frac{1}{2n} \sum_{k=-n+1}^{n} (jk)^n u[k]e^{jx_i k}$$

To clarify this procedure we provide some examples.

Example 6.1: Solving the Linear Equation of Transverse Vibration of Beam.

$$\frac{\partial^2 u}{\partial t^2} + c^2 \frac{\partial^4 u}{\partial x^4} = 0 \quad , \quad c^2 = \frac{EI}{A\mu}$$

with conditions

$$u(0,t) = \alpha \quad , \quad u(L,t) = \beta \quad , u_x(0,t) = 0 \, , \qquad u_x(L,t) = 0$$

$$u(x,0) = f(x) \quad \& \quad u_t(x,0) = \text{g}(x)$$

Firstly: by using Euler explicit method

$$\frac{\partial^2 u}{\partial t^2} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} - \frac{k^2}{12} \frac{\partial^4 u(x_i, \eta_j)}{\partial t^4}$$

$$\frac{\partial^4 u}{\partial x^4} = \frac{u_{i+2,j} - 4u_{i+1,j} + 6u_{i,j} - 4u_{i-1,j} + u_{i-2,j}}{h^4} - \frac{2h^2}{8!} \frac{\partial^8 u(\xi_i, t_j)}{\partial x^8}$$

Substituting in transverse vibration beam equation we get

$$\left(\frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2}\right) + c^2 \left(\frac{u_{i+2,j} - 4u_{i+1,j} + 6u_{i,j} - 4u_{i-1,j} + u_{i-2,j}}{h^4}\right)$$
$$= O(k^2) + O(h^2)$$

The local truncation error for this differential equation is

$$t_{ij} = \frac{k^2}{12} \frac{\partial^4 u(x_i, \eta_j)}{\partial t^4} + c^2 \frac{2h^2}{8!} \frac{\partial^8 u(\xi_i, t_j)}{\partial x^8} = O(k^2) + O(h^2)$$

6.2.1 The Stability of Euler Explicit Method

$$u_{i,j+1} = -ru_{i+2,j} + 4ru_{i+1,j} + 2(1-3r)u_{i,j} + 4ru_{i-1,j} - ru_{i-2,j} - u_{i,j-1} \tag{6.1}$$

where $r = \frac{c^2 k^2}{h^4}$

Let us: $\quad u_{i,j} = (-1)^i \lambda^j$ or $\quad u_{i,j} = \lambda^j e^{in\Delta x\theta}$

Applied in equation (6.1)

$$(-1)^i \lambda^{j+1} = -r(-1)^{i+1}\lambda^j + 4r(-1)^{i+1}\lambda^j + 2(1-3r)(-1)^i\lambda^j + 4r(-1)^{i-1}\lambda^j - r(-1)^{i-2}\lambda^j - (-1)^i\lambda^{j-1}$$

Multiply both sides by, $(-1)^{-i}\lambda^{-j}$, we obtain

$$\lambda = -r - 4r + 2(1-3r) - 4r - r - \lambda^{-1}$$

$$\lambda + \lambda^{-1} = 2 - 16r \quad \Rightarrow \quad \frac{\lambda^2 + 1}{\lambda} = 2 - 16r$$

Suppose $w = 1 - 8r \;\; \Rightarrow \quad \lambda^2 - 2w\lambda + 1 = 0$

$$\Rightarrow \quad \lambda_{1,2} = w \pm \sqrt{w^2 - 1}$$

Now since λ_1 and λ_2 are roots of this quadratic equation, we may conclude that $\lambda_1 \lambda_2 = 1$. However, for stability of solutions we require

$|\lambda_1| \leq 1$ and $|\lambda_2| \leq 1$. Given the constraint $\lambda_1 \lambda_2 = 1$, the only possibility if the solution to be stable is $|\lambda_1| = |\lambda_2| = 1$, thus λ must fall on the unit disk, which implies

$$|w| = |1 - 8r| < 1 \quad \Rightarrow |8r - 1| < 1 \quad \Rightarrow \quad r < \frac{1}{4}$$

$r = \frac{c^2 k^2}{h^4} < 0.25 \quad \Rightarrow k^2 < \frac{h^4}{4c^2}$ Then

$$k < \frac{h^2}{2c}$$

The necessary condition implies k is small than h. the stability is conditionally.

Now substituting $i = 1,2, \ldots, n - 1$ in below equation.

$$u_{i,j+1} = -ru_{i+2,j} + 4ru_{i+1,j} + 2(1 - 3r)u_{i,j} + 4ru_{i-1,j} - ru_{i-2,j} - u_{i,j-1}$$

when $i = 1$

$$u_{1,j+1} = -ru_{3,j} + 4ru_{2,j} + 2(1 - 3r)u_{1,j} + 4ru_{0,j} - ru_{-1,j} - u_{1,j-1} \tag{6.2}$$

But $\frac{\partial u(0,t)}{\partial x} = \frac{u_{0,j} - u_{-1,j}}{h}$ $\quad then \;\; u_{-1,j} = u_{0,j} - h * \frac{\partial u(0,t_j)}{\partial x} = \alpha$

Given $u_{0,j} = \alpha$ then

$$u_{1,j+1} = -ru_{3,j} + 4ru_{2,j} + 2(1 - 3r)u_{1,j} + 3r\alpha - u_{1,j-1} \tag{6.3}$$

when $i = 2$

$$u_{2,j+1} = -ru_{4,j} + 4ru_{3,j} + 2(1-3r)u_{2,j} + 4ru_{1,j} - ru_{0,j} - u_{2,j-1} \quad (6.4)$$

$$u_{2,j+1} = -ru_{4,j} + 4ru_{3,j} + 2(1-3r)u_{2,j} + 4ru_{1,j} - r\alpha - u_{2,j-1} \quad (6.5)$$

when $i = 3$

$$u_{3,j+1} = -ru_{5,j} + 4ru_{4,j} + 2(1-3r)u_{3,j} + 4ru_{2,j} - ru_{1,j} - u_{3,j-1} \quad (6.6)$$

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when $i = n-1$

$$u_{n-1,j+1} = -ru_{n+1,j} + 4ru_{n,j} + 2(1-3r)u_{n-1,j} + 4ru_{n-2,j} - ru_{n-3,j} - u_{n-1,j-1} \quad (6.7)$$

Given $u_{n,j} = \beta$

But $\frac{\partial u(x_n,t)}{\partial x} = \frac{u_{n+1,j} - u_{n,j}}{h}$ $\quad then \;\; u_{n+1,j} = h * \frac{\partial u(x_n,t_j)}{\partial x} + u_{n,j} = \beta$

$$u_{n-1,j+1} = 3r\beta + 2(1-3r)u_{n-1,j} + 4ru_{n-2,j} - ru_{n-3,j} - u_{n-1,j-1} \quad (6.8)$$

Now put the equations (6.3), (6.4), (6.5), (6.8) & (6.7) in matrix notation

$$\underline{u}_{j+1} = A\underline{u}_j - \underline{u}_{j-1} + \underline{b}$$

Where

$$A = \begin{bmatrix} 2(1-3r) & 4r & -r & 0 & 0 & 0 \\ 4r & 2(1-3r) & 4r & r & 0 & 0 \\ -r & 4r & 2(1-3r) & 4r & -r & 0 & 0 \\ & & & . & & \\ & & & . & & \\ & & & . & & \\ 0 & 0 & 0 & 0 \ldots & -r & 4r & 2(1-3r) \end{bmatrix}$$

$$\underline{u}_j = \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ u_{3,j} \\ . \\ . \\ . \\ u_{n-1,j} \end{bmatrix}, \underline{u}_{j+1} = \begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \\ . \\ . \\ . \\ u_{n-1,j+1} \end{bmatrix}, \underline{u}_{j-1} = \begin{bmatrix} u_{1,j-1} \\ u_{2,j-1} \\ u_{3,j-1} \\ . \\ . \\ . \\ u_{n-1,j-1} \end{bmatrix}, \underline{b} = \begin{bmatrix} 3r\alpha \\ -r\alpha \\ 0 \\ . \\ 0 \\ -r\beta \\ 3r\beta \end{bmatrix}$$

Algorithm 6.1

To obtain the numerical solution of Example 6.1.

Input: endpoint L; maximum time T; constants α, β, c; integers n and m

Output: approximations $u(x_i, t_j)$, for each i=0,1,….,n and j=0,1,….,m

Step 1: h=L/n

k=Tmax/m

r= (c*k)^2/h^4

Step 2: for i=0,1,…..,n

for j=0,1,……,m

Do step 3 and step 4

Step 3: $u(x_0, t_j) = \alpha$

$u(x_n, t_j) = \beta$

Step 4: $u(x_i, t_0) = f(x_i)$

Step 5: for i=1,…..,n-1

for j=1,……,m-1

$u(x_i, t_{j+1}) = -ru(x_{i+2}, t_j) + 4ru(x_{i+1}, t_j) + 2(1-3r)u(x_i, t_j) + 4ru(x_{i-1}, t_j) - ru(x_{i-2}, t_j) - u(x_i, t_{j-1})$

Step 6: output $u_{00}, u_{01}, u_{02}, \ldots\ldots u_{nm}$

Step 7: Stop (the producer is complete)

Secondly: by using Euler implicit method

$$\frac{\partial^2 u}{\partial t^2} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} - \frac{k^2}{12}\frac{\partial^4 u(x_i,\eta_j)}{\partial t^4}$$

$$\frac{\partial^4 u}{\partial x^4} = \frac{u_{i+2,j+1} - 4u_{i+1,j} + 6u_{i,j+1} - 4u_{i-1,j+1} + u_{i-2,j+1}}{h^4} - \frac{2h^2}{8!}\frac{\partial^8 u(\xi_i,t_{j+1})}{\partial x^8}$$

Substituting in transverse vibration beam equation we get

$$\left(\frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2}\right) + c^2\left(\frac{u_{i+2,j+1} - 4u_{i+1,j} + 6u_{i,j+1} - 4u_{i-1,j+1} + u_{i-2,j+1}}{h^4}\right) = O(k^2) + O(h^2)$$

The local truncation error for this differential equation is

$$t_{ij} = \frac{k^2}{12}\frac{\partial^4 u(x_i,\eta_j)}{\partial t^4} + c^2\frac{2h^2}{8!}\frac{\partial^8 u(\xi_i,t_{j+1})}{\partial x^8} = O(k^2) + O(h^2)$$

6.2.2 The Stability of Euler Implicit Method

$$ru_{i+2,j+1} - 4ru_{i+1,j+1} + (1+6r)u_{i,j+1} - 4ru_{i-1,j+1} + ru_{i-2,j+1} = 2u_{i,j} - u_{i,j-1} \qquad (6.9)$$

where $r = \frac{c^2k^2}{h^4}$

Let us: $u_{i,j} = (-1)^i\lambda^j$ or $u_{i,j} = \lambda^j e^{in\Delta x\theta}$

Applied in equation (6.9)

$$r(-1)^{i+2}\lambda^{j+1} - 4r(-1)^{i+1}\lambda^{j+1} + (1+6r)(-1)^i\lambda^{j+1} - 4r(-1)^{i-1}\lambda^{j+1} + r(-1)^{i-2}\lambda^{j+1} = 2(-1)^i\lambda^j - (-1)^i\lambda^{j-1}$$

Multiply both sides by, $(-1)^{-i}\lambda^{-j}$, we obtain

$$r\lambda + 4r\lambda + (1+6r)\lambda + 4r\lambda + r\lambda + \lambda^{-1} = 2$$

$$2 = (1+16r)\lambda + \lambda^{-1} \quad \Longrightarrow \quad \lambda^2 - 2w\lambda + w = 0$$

Since $w = \frac{1}{1+16r}$

$$\Rightarrow \quad \lambda_{1,2} = w \pm \sqrt{w^2 - w}$$

Now since λ_1 and λ_2 are roots of this quadratic equation. However, for stability of solutions we require $|\lambda_1| \leq 1$ and $|\lambda_2| \leq 1$. The only possibility, if the solution to be stable is $|\lambda_1| = |\lambda_2| = 1$, thus λ must fall on the unit disk, which implies

$$|w| = \left|\frac{1}{1+16r}\right| < 1 \quad \Rightarrow \left|\frac{1}{1+16r}\right| < 1 \quad \Rightarrow \; 1 + 16\,r > 1$$

$$\Rightarrow \; r > 0 \;\; \Rightarrow \quad r = \frac{c^2k^2}{h^4} > 0 \quad \Rightarrow \; k^2 > \frac{h^4}{c^2}$$

Then, the stability unconditionally.

Now substituting $i = 1,2, \ldots, n-1$ in below equation.

$$2u_{i,j} = ru_{i+2,j+1} - 4ru_{i+1,j+1} + (1+6r)u_{i,j+1} - 4ru_{i-1,j+1} + ru_{i-2,j+1} + u_{i,j-1}$$

when $i = 1$

$$2u_{1,j} = ru_{3,j+1} - 4ru_{2,j+1} + (1+6r)u_{1,j+1} - 4ru_{0,j+1} + ru_{-1,j+1} + u_{1,j-1} \tag{6.10}$$

But $\frac{\partial u(0,t)}{\partial x} = \frac{u_{0,j+1} - u_{-1,j+1}}{h}$ $\quad then \;\; u_{-1,j+1} = \; u_{0,j+1} - h * \frac{\partial u(0,t_{j+1})}{\partial x} = \alpha$

Given $u_{0,j} = \alpha$ then

$$2u_{1,j} = ru_{3,j+1} - 4ru_{2,j+1} + (1+6r)u_{1,j+1} - 3r\alpha + u_{1,j-1} \tag{6.11}$$

when $i = 2$

$$2u_{2,j} = ru_{4,j+1} - 4ru_{3,j+1} + (1+6r)u_{2,j+1} - 4ru_{1,j+1} + ru_{0,j+1} + u_{2,j-1} \tag{6.12}$$

$$2u_{2,j} = ru_{4,j+1} - 4ru_{3,j+1} + (1+6r)u_{2,j+1} - 4ru_{1,j+1} + r\alpha + u_{2,j-1} \tag{6.13}$$

when $i = 3$

$$2u_{3,j} = ru_{5,j+1} - 4ru_{4,j+1} + (1+6r)u_{3,j+1} - 4ru_{2,j+1} + ru_{1,j+1} + u_{3,j-1} \quad (6.14)$$

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when $i = n-1$

$$2u_{n-1,j} = ru_{n+1,j+1} - 4ru_{n,j+1} + (1+6r)u_{n-1,j+1} - 4ru_{n-2,j+1} + ru_{n-3,j+1} + +u_{n-1,j-1} \quad (6.15)$$

Given $u_{n,j} = \beta$

But $\frac{\partial u(x_n,t)}{\partial x} = \frac{u_{n+1,j+1} - u_{n,j+1}}{h}$

So $u_{n+1,j+1} = h * \frac{\partial u(x_n,t_{j+1})}{\partial x} + u_{n,j+1} = \beta$

$$2u_{n-1,j} = -3r\beta + (1+6r)u_{n-1,j+1} - 4ru_{n-2,j+1} + ru_{n-3,j+1} + u_{n-1,j-1} \quad (6.16)$$

Now put the equations (6.11), (6.12), (6.13), (6.14) and (6.16) in matrix notation

$$A\underline{u}_{j+1} = 2\underline{u}_j - \underline{u}_{j-1} + \underline{b}$$

where

$$A = \begin{bmatrix} (1+6r) & -4r & r & 0 & 0 & 0 \\ -4r & (1+6r) & -4r & r & 0 & 0 \\ r & -4r & (1+6r) & -4r & r & 0 & 0 \\ & & & . & & & \\ & & & . & & & \\ & & & . & & & \\ 0 & 0 & 0 & 0 \ldots & r & -4r & (1+6r) \end{bmatrix}$$

$$\underline{u}_{j+1} = \begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \\ \cdot \\ \cdot \\ u_{n-2,j+1} \\ u_{n-1,j+1} \end{bmatrix}, \underline{u}_j = \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ u_{3,j} \\ \cdot \\ \cdot \\ u_{n-2,j} \\ u_{n-1,j} \end{bmatrix}, \underline{u}_{j-1} = \begin{bmatrix} u_{1,j-1} \\ u_{2,j-1} \\ u_{3,j-1} \\ \cdot \\ \cdot \\ u_{n-2,j-1} \\ u_{n-1,j-1} \end{bmatrix}, \underline{b} = \begin{bmatrix} 3r\alpha \\ -r\alpha \\ 0 \\ \cdot \\ \cdot \\ -r\beta \\ 3r\beta \end{bmatrix}$$

Algorithm 6.2

To obtain the numerical solution of Example 6.1

Input: endpoint L; maximum time T; constants α, β, c; integers n and m

Output: approximations $u(x_i, t_j)$, for each i=0,1,….,n and j=0,1,….,m

Step 1: h=L/n

k=Tmax/m

r= (c*k)^2/h^4

Step 2: for i=0,1,…..,n

for j=0,1,……,m

Do step 3 and step 4

Step 3: $u(x_0, t_j) = \alpha$

$u(x_n, t_j) = \beta$

Step 4: $u(x_i, t_0) = f(x_i)$

Step 5: for i=1,…..,n-1

for j=1,……,m-1

$2u(x_i, t_j) = ru(x_{i+2}, t_j) - 4ru(x_{i+1}, t_j) + (1 + 6r)u(x_i, t_j) - 4ru(x_{i-1}, t_j) + ru(x_{i-2}, t_j) + u(x_i, t_{j-1})$

Step 6: output $u_{00}, u_{01}, u_{02}, \ldots\ldots u_{nm}$

Step 7: Stop (the producer is complete)

Thirdly: by using spectral method

We use finite difference of time and discrete Fourier transform of space x, we obtain

$$\frac{u_{ij+1} - 2u_{ij} + u_{ij-1}}{(\Delta t)^2} + \frac{4}{2n}\sum_{k=-n+1}^{n}(ik)^4 u[k]e^{jx_ik} = 0$$

$$u_{ij+1} = 2u_{ij} - u_{ij-1} - \frac{4(\Delta t)^2}{2n}\sum_{k=-n+1}^{n}(ik)^4 u[k]e^{jx_ik}$$

Algorithm 6.3

To obtain the numerical solution of Example 6.1.

Input: endpoint L; maximum time T; constants α, β, c; integers n and m

Output: approximations $u(x_i, t_r)$, for each i=0,1,….,2n-1 and r=0,1,….,m

Step 1: Δt =Tmax/m

Step 2: for r=0,1,…..,m

Do step 3.

Step 3: $u(x_0, t_r) = \alpha$

$u(x_n, t_r) = \beta$

Step 4: for i=0:2n-1

$x_i = \cos(\frac{i\pi}{n})$

Step 5: for i=1:2n-2

$u(x_i, t_0) = f(x_i)$

Step 6: for r=1,…..,m-1

for i=1,……,2n-1

$$u(x_i, t_{r+1}) = 2u(x_i, t_r) - u(x_i, t_{r-1}) - \frac{4(\Delta t)^2}{2N}\sum_{k=-n+1}^{n}(ik)^4 u[k]e^{jx_ik}$$

Step 7: output $u_{00}, u_{01}, u_{02}, \dots\dots u_{nm}$

Step 8: Stop (the producer is complete)

Example 6.2: Solving the Linear Equation of Transverse Vibration of Beam.

$$\frac{\partial^2 u}{\partial t^2} + (\frac{1}{x} + \frac{x^4}{120})\frac{\partial^4 u}{\partial x^4} = 0$$

with conditions

$$u(0,t) = \alpha \quad , \quad u(L,t) = \beta \quad , u_x(0,t) = 0 \,, \qquad u_x(L,t) = 0$$

$$u(x,0) = f(x) \quad \& \quad u_t(x,0) = \mathrm{g}(x)$$

Firstly: by using Euler explicit method

$$\frac{\partial^2 u}{\partial t^2} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} - \frac{k^2}{12}\frac{\partial^4 u(x_i, \eta_j)}{\partial t^4}$$

$$\frac{\partial^4 u}{\partial x^4} = \frac{u_{i+2,j} - 4u_{i+1,j} + 6u_{i,j} - 4u_{i-1,j} + u_{i-2,j}}{h^4} - \frac{2h^2}{8!}\frac{\partial^8 u(\xi_i, t_j)}{\partial x^8}$$

Substituting in transverse vibration beam equation we get

$$\frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} + \left(\frac{1}{ih} + \frac{(ih)^4}{120}\right)\left(\frac{u_{i+2,j} - 4u_{i+1,j} + 6u_{i,j} - 4u_{i-1,j} + u_{i-2,j}}{h^4}\right) = O(k^2) + O(h^2)$$

The local truncation error for this differential equation is

$$t_{ij} = \frac{k^2}{12}\frac{\partial^4 u(x_i, \eta_j)}{\partial t^4} + \left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{2h^2}{8!}\frac{\partial^8 u\left(\xi_i, t_j\right)}{\partial x^8} = O(k^2) + O(h^2)$$

$$u_{i,j+1} = \left(\frac{1}{ih} + \frac{(ih)^4}{120}\right)\left(-ru_{i+2,j} + 4ru_{i+1,j} - 6ru_{i,j} + 4ru_{i-1,j} - ru_{i-2,j}\right) + 2u_{i,j} - u_{i,j-1} \qquad (6.17)$$

where $r = \frac{k^2}{h^4}$

when $i = 1$

$$u_{1,j+1} = \left(\frac{1}{h} + \frac{(h)^4}{120}\right)\left(-ru_{3,j} + 4ru_{2,j} - 6ru_{1,j} + 4ru_{0,j} - ru_{-1,j}\right) + 2u_{1,j} - u_{1,j-1} \qquad (6.18)$$

But $\frac{\partial u(0,t)}{\partial x} = \frac{u_{0,j} - u_{-1,j}}{h} \quad then\ u_{-1,j} = u_{0,j} - h * \frac{\partial u(0,t_j)}{\partial x} = \alpha$

Because $u_{0,j} = \alpha$ then

$$u_{1,j+1} = \left(\frac{1}{h} + \frac{(h)^4}{120}\right)\left(-ru_{3,j} + 4ru_{2,j} - 6ru_{1,j} + 3r\alpha\right) + 2u_{1,j} - u_{1,j-1} \qquad (6.19)$$

$$u_{1,j+1} = \left(\frac{1}{h} + \frac{(h)^4}{120}\right)\left(-ru_{3,j} + 4ru_{2,j} + 3r\alpha\right) + 2(1 - 3r\left(\frac{1}{h} + \frac{(h)^4}{120}\right))u_{1,j} - u_{1,j-1} \qquad (6.20)$$

when $i = 2$

$$u_{2,j+1} = \left(\frac{1}{2h} + \frac{(2h)^4}{120}\right)\left(-ru_{4,j} + 4ru_{3,j} + 4ru_{1,j} - r\alpha\right) + 2(1 - 3r\left(\frac{1}{2h} + \frac{(2h)^4}{120}\right))u_{2,j} - u_{2,j-1} \qquad (6.21)$$

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when $i = n - 1$

$$u_{n-1,j+1} = \left(\frac{1}{(n-1)h} + \frac{((n-1)h)^4}{120}\right)\left(-ru_{n+1,j} + 4ru_{n,j} + 4ru_{n-2,j} - ru_{n-3,j}\right) + 2(1 - 3r\left(\frac{1}{(n-1)h} + \frac{((n-1)h)^4}{120}\right))u_{n-1,j} - u_{n-1,j-1} \qquad (6.22)$$

Given $u_{n,j} = \beta$

But $\frac{\partial u(x_n,t)}{\partial x} = \frac{u_{n+1,j} - u_{n,j}}{h} \quad then\ u_{n+1,j} = h * \frac{\partial u(x_n,t_j)}{\partial x} + u_{n,j} = \beta$ then

$$u_{n-1,j+1} = \left(\frac{1}{(n-1)h} + \frac{((n-1)h)^4}{120}\right)\left(3r\beta + 4ru_{n-2,j} - ru_{n-3,j}\right) + 2(1 - 3r\left(\frac{1}{(n-1)h} + \frac{((n-1)h)^4}{120}\right))u_{n-1,j} - u_{n-1,j-1} \quad (6.23)$$

Now put the equations (6.20), (6.21) and (6.23) in matrix notation

$$\underline{u}_{j+1} = A\underline{u}_j - \underline{u}_{j-1} + \underline{b}$$

where

$$A = \begin{bmatrix} 2\left(1-3rw(1)\right) & 4r & -r & 0 & 0 & 0 \\ 4r & 2\left(1-3rw(2)\right) & 4r & -r & 0 & 0 & 0 \\ & & & \cdot & & & \\ & & & \cdot & & & \\ & & & \cdot & & & \\ & & & \cdot & & & \\ 0 & 0 & 0 & 0 & -r & 4r & 2\left(1-3rw(n-1)\right) \end{bmatrix}$$

$$\underline{u}_j = \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ u_{3,j} \\ \cdot \\ \cdot \\ \cdot \\ u_{n-1,j} \end{bmatrix}, \underline{u}_{j+1} = \begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \\ \cdot \\ \cdot \\ \cdot \\ u_{n-1,j+1} \end{bmatrix}, \underline{u}_{j-1} = \begin{bmatrix} u_{1,j-1} \\ u_{2,j-1} \\ u_{3,j-1} \\ \cdot \\ \cdot \\ \cdot \\ u_{n-1,j-1} \end{bmatrix},$$

$$\underline{b} = \begin{bmatrix} 3r\alpha w(1) \\ r\alpha w(2) \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ -r\beta w(n-2) \\ 3r\beta w(n-1) \end{bmatrix} \quad and \quad \underline{w} = \begin{bmatrix} \frac{1}{h} + \frac{h^4}{120} \\ \frac{1}{2h} + \frac{(2h)^4}{120} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \frac{1}{(n-1)h} + \frac{((n-1)h)^4}{120} \end{bmatrix}$$

Algorithm 6.4

To obtain the numerical solution of Example 6.2

Input: endpoint L; maximum time T; constants α, ; integers n and m

Output: approximations $u(x_i, t_j)$, for each i=0,1,….,n and j=0,1,….,m

Step 1: h=L/n

k=Tmax/m

r=k^2/h^4

Step 2: for i=0,1,…..,n

for j=0,1,……,m

Do step 3 and step 4

Step 3: $u(x_0, t_j) = \alpha$

$u(x_n, t_j) = \beta$

Step 4: $u(x_i, t_0) = f(x_i)$

Step 5: for i=1,…..,n-1

for j=1,……,m-1

$u(x_i, t_{j+1}) = \left(\frac{1}{ih} + \frac{(ih)^4}{120}\right)\left(ru(x_{i+2}, t_j) - 4ru(x_{i+1}, t_j) + 6u(x_i, t_j) - 4ru(x_{i-1}, t_j) + ru(x_{i-2}, t_j)\right) + 2u(x_i, t_j) - u(x_i, t_{j-1})$

Step 6: output $u_{00}, u_{01}, u_{02}, \ldots\ldots u_{nm}$

Step 7: Stop (the producer is complete)

Secondly: by using Euler implicit method

$$\frac{\partial^2 u}{\partial t^2} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} - \frac{k^2}{12}\frac{\partial^4 u(x_i, \eta_j)}{\partial t^4}$$

$$\frac{\partial^4 u}{\partial x^4} = \frac{u_{i+2,j+1} - 4u_{i+1,j+1} + 6u_{i,j+1} - 4u_{i-1,j+1} + u_{i-2,j+1}}{h^4} - \frac{2h^2}{8!}\frac{\partial^8 u(\xi_i, t_{j+1})}{\partial x^8}$$

Substituting in transverse vibration beam equation we get

$$\frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} + \left(\frac{1}{ih} + \frac{(ih)^4}{120}\right)\left(\frac{u_{i+2,j+1} - 4u_{i+1,j+1} + 6u_{i,j+1} - 4u_{i-1,j+1} + u_{i-2,j+1}}{h^4}\right) = O(k^2) + O(h^2)$$

The local truncation error for this differential equation is

$$t_{ij} = \frac{k^2}{12}\frac{\partial^4 u(x_i, \eta_j)}{\partial t^4} + \left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{2h^2}{8!}\frac{\partial^8 u\left(\xi_i, t_j\right)}{\partial x^8} = O(k^2) + O(h^2)$$

$$u_{i,j+1} + \left(\frac{1}{ih} + \frac{(ih)^4}{120}\right)\left(ru_{i+2,j+1} - 4ru_{i+1,j+1} + 6ru_{i,j+1} - 4ru_{i-1,j+1} + ru_{i-2,j}\right) = 2u_{i,j} - u_{i,j-1} \qquad (6.24)$$

where $r = \frac{k^2}{h^4}$

when $i = 1$

$$u_{1,j+1} + \left(\frac{1}{h} + \frac{(h)^4}{120}\right)\left(ru_{3,j+1} - 4ru_{2,j+1} + 6ru_{1,j+1} - 4ru_{0,j+1} + ru_{-1,j+1}\right) = 2u_{1,j} - u_{1,j-1} \qquad (6.25)$$

But $\frac{\partial u(0,t)}{\partial x} = \frac{u_{0,j} - u_{-1,j}}{h}$ $\quad then \ u_{-1,j} = u_{0,j} - h * \frac{\partial u(0,t_j)}{\partial x} = \alpha$

Because $u_{0,j} = \alpha$ then

$$u_{1,j+1} + \left(\frac{1}{h} + \frac{(h)^4}{120}\right)\left(ru_{3,j+1} - 4ru_{2,j+1} + 6ru_{1,j+1} - 3r\alpha\right) = 2u_{1,j} - u_{1,j-1} \qquad (6.26)$$

when $i = 2$

$$u_{2,j+1} + \left(\frac{1}{2h} + \frac{(2h)^4}{120}\right)\left(ru_{4,j+1} - 4ru_{3,j+1} + 6ru_{2,j+1} - 4ru_{1,j+1} + r\alpha\right) = 2u_{2,j} - u_{2,j-1} \qquad (6.27)$$

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when $i = n - 1$

$$u_{n-1,j+1} + \left(\frac{1}{(n-1)h} + \frac{((n-1)h)^4}{120}\right)\left(ru_{n+1,j+1} - 4ru_{n,j+1} + 6ru_{n-1,j+1} - 4ru_{n-2,j+1} + ru_{n-3,j+1}\right) = 2u_{n-1,j} - u_{n-1,j-1} \qquad (6.28)$$

Given $u_{n,j} = \beta$

But $\frac{\partial u(x_n,t)}{\partial x} = \frac{u_{n+1,j} - u_{n,j}}{h}$ $\quad then\ u_{n+1,j} = h * \frac{\partial u(x_n,t_j)}{\partial x} + u_{n,j} = \beta$ then

$$u_{n-1,j+1} + \left(\frac{1}{(n-1)h} + \frac{((n-1)h)^4}{120}\right)\left(-3r\beta - 4ru_{n-2,j+1} + 6ru_{n-1,j+1} + ru_{n-3,j+1}\right) = 2u_{n-1,j} - u_{n-1,j-1} \qquad (6.29)$$

Now put the equations (6.25), (6.27) and (6.29) in matrix notation

$$\underline{Au}_{j+1} = 2\underline{u}_j - \underline{u}_{j-1} + \underline{b}$$

where

$$A = \begin{bmatrix} (1+6rw(1)) & -4r & r & 0 & 0 & 0 \\ -4r & (1+6rw(2)) & -4r & r & 0 & 0 & 0 \\ & & & \cdot & & & \\ & & & \cdot & & & \\ & & & \cdot & & & \\ & & & \cdot & & & \\ 0 & 0 & 0 & 0 & r & -4r & (1+6rw(n-1)) \end{bmatrix}$$

$$\underline{u}_j = \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ u_{3,j} \\ \cdot \\ \cdot \\ \cdot \\ u_{n-1,j} \end{bmatrix}, \underline{u}_{j+1} = \begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \\ \cdot \\ \cdot \\ \cdot \\ u_{n-1,j+1} \end{bmatrix}, \underline{u}_{j-1} = \begin{bmatrix} u_{1,j-1} \\ u_{2,j-1} \\ u_{3,j-1} \\ \cdot \\ \cdot \\ \cdot \\ u_{n-1,j-1} \end{bmatrix},$$

$$\underline{b} = \begin{bmatrix} 3r\alpha w(1) \\ r\alpha w(2) \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ -r\beta w(n-2) \\ 3r\beta w(n-1) \end{bmatrix} \quad and \quad \underline{w} = \begin{bmatrix} \frac{1}{h} + \frac{h^4}{120} \\ \frac{1}{2h} + \frac{(2h)^4}{120} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \frac{1}{(n-1)h} + \frac{((n-1)h)^4}{120} \end{bmatrix}$$

Algorithm 6.5

To obtain the numerical solution of Example 6.2

Input: endpoint L; maximum time T; constants α, ; integers n and m

Output: approximations $u(x_i, t_j)$, for each i=0,1,….,n and j=0,1,….,m

Step 1: h=L/n

k=Tmax/m

r=k^2/h^4

Step 2: for i=0,1,…..,n

for j=0,1,……,m

Do step 3 and step 4

Step 3: $u(x_0, t_j) = \alpha$

$u(x_n, t_j) = \beta$

Step 4: $u(x_i, t_0) = f(x_i)$

Step 5: for i=1,…..,n-1

for j=1,……,m-1

$u(x_i, t_{j+1}) + \left(\frac{1}{ih} + \frac{(ih)^4}{120}\right)\left(ru(x_{i+2}, t_j) - 4ru(x_{i+1}, t_j) + 6ru(x_i, t_j) - 4ru(x_{i-1}, t_j) + ru(x_{i-2}, t_j)\right) = 2u(x_i, t_j) - u(x_i, t_{j-1})$

Step 6: output $u_{00}, u_{01}, u_{02}, \dots\dots u_{nm}$

Step 7: Stop (the producer is complete)

Example 6.3: Solving the Nonlinear Equation of Transverse Vibration of Beam**.**

$$\frac{\partial^2 u}{\partial t^2} = a\frac{\partial^4 u}{\partial x^4} + b\left(\frac{\partial^2 u}{\partial x^2}\right)^3 + c\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 u}{\partial x^3} + d\left(\frac{\partial u}{\partial x}\right)^2\left(\frac{\partial^2 u}{\partial x^2}\right)^3$$

with conditions

$$u(0,t) = \alpha \quad , \quad u(L,t) = \beta \quad , u_x(0,t) = 0 \, , \qquad u_x(L,t) = 0$$

$$u(x,0) = f(x) \quad \& \quad u_t(x,0) = \mathrm{g}(x)$$

Using EEM; let us a=1, b= -1.5, c= 6 and d= -22.5.

$$\frac{\partial u}{\partial x} = \frac{u_{i+1,j} - u_{i-1,j}}{2h} - \frac{h^2}{6}\frac{\partial^3 u(\xi_i, t_j)}{\partial x^3}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} - \frac{k^2}{12}\frac{\partial^4 u(x_i, \eta_j)}{\partial t^4}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} - \frac{h^2}{12}\frac{\partial^4 u(\xi_i, t_j)}{\partial x^4}$$

$$\frac{\partial^3 u}{\partial x^3} = \frac{u_{i+1,j} - 2u_{i+1,j} + 2u_{i-1,j} - u_{i-1,j}}{2h^3} - \frac{2h^2}{6!}\frac{\partial^6 u(\xi_i, t_j)}{\partial x^6}$$

$$\frac{\partial^4 u}{\partial x^4} = \frac{u_{i+2,j} - 4u_{i+1,j} + 6u_{i,j} - 4u_{i-1,j} + u_{i-2,j}}{h^4} - \frac{2h^2}{8!}\frac{\partial^8 u(\xi_i, t_j)}{\partial x^8}$$

Substituting in nonlinear transverse vibration beam equation we get

$$\left(\frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2}\right)$$
$$= \frac{u_{i+2,j} - 4u_{i+1,j} + 6u_{i,j} - 4u_{i-1,j} + u_{i-2,j}}{h^4}$$
$$- \frac{3}{2}\frac{\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right)^3}{h^6}$$
$$- \frac{3\left(u_{i+1,j} - u_{i-1,j}\right)\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right)\left(u_{i+2,j} - 2u_{i+1,j} + 2u_{i-1,j} - u_{i-2,j}\right)}{h^6}$$
$$- \frac{45}{8}\frac{\left(u_{i+1,j} - u_{i-1,j}\right)^2\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right)^3}{h^8} + O(k^2) + O(h^2)$$

The local truncation error for this differential equation in order

$$t_{ij} = O(k^2) + O(h^2)$$

Multiply by k^2 and transform the terms $u_{i,j-1}$, $u_{i,j}$ to right side we get

$$u_{i,j+1}$$
$$= k^2 \left[\frac{u_{i+2,j} - 4u_{i+1,j} + 6u_{i,j} - 4u_{i-1,j} + u_{i-2,j}}{h^4}\right.$$
$$- \frac{3}{2}\frac{\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right)^3}{h^6}$$
$$- \frac{3}{2}\frac{\left(u_{i+1,j} - u_{i-1,j}\right)\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right)\left(u_{i+2,j} - 2u_{i+1,j} + 2u_{i-1,j} - u_{i-2,j}\right)}{h^6}$$
$$\left.- \frac{45}{8}\frac{\left(u_{i+1,j} - u_{i-1,j}\right)^2\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right)^3}{h^8}\right] + 2u_{i,j} - u_{i,j-1}$$

Put the above equation in matrix notation

$$\underline{u}_{j+1} = \emptyset + 2\underline{u}_j - \underline{u}_{j-1}$$

where $\emptyset$ is nonlinear terms

$$\underline{u}_j = \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ u_{3,j} \\ \cdot \\ \cdot \\ \cdot \\ u_{n-1,j} \end{bmatrix}, \underline{u}_{j+1} = \begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \\ \cdot \\ \cdot \\ \cdot \\ u_{n-1,j+1} \end{bmatrix}, \underline{u}_{j-1} = \begin{bmatrix} u_{1,j-1} \\ u_{2,j-1} \\ u_{3,j-1} \\ \cdot \\ \cdot \\ \cdot \\ u_{n-1,j-1} \end{bmatrix}$$

Algorithm 6.6

To obtain the numerical solution of Example 6.3

Input: endpoint L; maximum time T; constants α, ; integers n and m

Output: approximations $u(x_i, t_j)$, for each i=0,1,….,n and j=0,1,….,m

Step 1: h=L/n

k=T/m

r=k^2/h^4

p=-3*k^2/2*h^6

q=-45*k^2/8*h^8

Step 2: for i=0,1,…..,n

for j=0,1,……,m

Do step 3 and step 4

Step 3: $u(x_0, t_j) = \alpha$

$u(x_n, t_j) = \beta$

Step 4: $u(x_i, t_0) = f(x_i)$

Step 5: for i=1,…..,n-1

for j=1,……,m-1

$u(x_i, t_{j+1}) = ru(x_{i+2}, t_j) - 4ru(x_{i+1}, t_j) + 6ru(x_i, t_j) - 4ru(x_{i-1}, t_j) - 4ru(x_{i-2}, t_j) + p\left(u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)\right)^3 - p\left(u(x_{i+1}, t_j) - u(x_{i-1}, t_j)\right)\left(u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)\right)\left(u(x_{i+2}, t_j) - 2u(x_{i+1}, t_j) +\right.$

$2u(x_{i-1},t_j) - u(x_{i-2},t_j)\Big) - q(u(x_{i+1},t_j) - u(x_{i-1},t_j))^2 \Big(u(x_{i+1},t_j) -$
$2u(x_i,t_j) + u(x_{i-1},t_j)\Big)^3 + 2u(x_i,t_j) - u(x_i,t_{j-1})$

Step 6: output $u_{00}, u_{01}, u_{02}, \ldots \ldots u_{nm}$

Step 7: Stop (the producer is complete)

Example 6.4: Solving the Nonlinear Equation of Transverse Vibration of Beam.

$$u_{tt} = a\frac{u_{xxxx}}{(1+(u_x)^2)^{\frac{3}{2}}} + b\frac{(u_{xx})^3}{(1+(u_x)^2)^{\frac{5}{2}}} + c\frac{u_x u_{xx} u_{xxx}}{(1+(u_x)^2)^{\frac{5}{2}}} + d\frac{(u_x)^2(u_{xx})^3}{(1+(u_x)^2)^{\frac{7}{2}}}$$

with conditions

$$u(0,t) = \alpha \quad , \quad u(L,t) = \beta \quad , \quad u_x(0,t) = 0 \,, \qquad u_x(L,t) = 0$$

$$u(x,0) = f(x) \quad \& \quad u_t(x,0) = \mathrm{g}(x)$$

Using EEM; let us a=1, b= -1.5, c= 6 and d= -22.5.

$$\frac{\partial u}{\partial x} = \frac{u_{i+1,j} - u_{i-1,j}}{2h} - \frac{h^2}{6}\frac{\partial^3 u(\xi_i,t_j)}{\partial x^3}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} - \frac{k^2}{12}\frac{\partial^4 u(x_i,\eta_j)}{\partial t^4}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} - \frac{h^2}{12}\frac{\partial^4 u(\xi_i,t_j)}{\partial x^4}$$

$$\frac{\partial^3 u}{\partial x^3} = \frac{u_{i+1,j} - 2u_{i+1,j} + 2u_{i-1,j} - u_{i-1,j}}{2h^3} - \frac{2h^2}{6!}\frac{\partial^6 u(\xi_i,t_j)}{\partial x^6}$$

$$\frac{\partial^4 u}{\partial x^4} = \frac{u_{i+2,j} - 4u_{i+1,j} + 6u_{i,j} - 4u_{i-1,j} + u_{i-2,j}}{h^4} - \frac{2h^2}{8!}\frac{\partial^8 u(\xi_i,t_j)}{\partial x^8}$$

Substituting in nonlinear transverse vibration beam equation we get

$$\left(\frac{u_{i,j-1}-2u_{i,j}+u_{i,j+1}}{k^2}\right)$$
$$=\frac{u_{i+2,j}-4u_{i+1,j}+6u_{i,j}-4u_{i-1,j}+u_{i-2,j}}{h^4\left(1+\left(\frac{u_{i+1,j}-u_{i-1,j}}{2h}\right)^2\right)^{\frac{3}{2}}}$$
$$-\frac{3}{2}\frac{\left(u_{i+1,j}-2u_{i,j}+u_{i-1,j}\right)^3}{h^6\left(1+\left(\frac{u_{i+1,j}-u_{i-1,j}}{2h}\right)^2\right)^{\frac{5}{2}}}$$
$$-\frac{3\left(u_{i+1,j}-u_{i-1,j}\right)\left(u_{i+1,j}-2u_{i,j}+u_{i-1,j}\right)\left(u_{i+2,j}-2u_{i+1,j}+2u_{i-1,j}-u_{i-2,j}\right)}{h^6\left(1+\left(\frac{u_{i+1,j}-u_{i-1,j}}{2h}\right)^2\right)^{\frac{5}{2}}}$$
$$-\frac{45}{8}\frac{\left(u_{i+1,j}-u_{i-1,j}\right)^2\left(u_{i+1,j}-2u_{i,j}+u_{i-1,j}\right)^3}{h^8\left(1+\left(\frac{u_{i+1,j}-u_{i-1,j}}{2h}\right)^2\right)^{\frac{7}{2}}}+O(k^2)+O(h^2)$$

The local truncation error for this differential equation in order

$$t_{ij}=O(k^2)+O(h^2)$$

$$u_{i,j+1}$$
$$=k^2\left[\frac{u_{i+2,j}-4u_{i+1,j}+6u_{i,j}-4u_{i-1,j}+u_{i-2,j}}{h^4\left(1+\left(\frac{u_{i+1,j}-u_{i-1,j}}{2h}\right)^2\right)^{\frac{3}{2}}}\right.$$
$$-\frac{3}{2}\frac{\left(u_{i+1,j}-2u_{i,j}+u_{i-1,j}\right)^3}{h^6\left(1+\left(\frac{u_{i+1,j}-u_{i-1,j}}{2h}\right)^2\right)^{\frac{5}{2}}}$$
$$-\frac{3\left(u_{i+1,j}-u_{i-1,j}\right)\left(u_{i+1,j}-2u_{i,j}+u_{i-1,j}\right)\left(u_{i+2,j}-2u_{i+1,j}+2u_{i-1,j}-u_{i-2,j}\right)}{h^6\left(1+\left(\frac{u_{i+1,j}-u_{i-1,j}}{2h}\right)^2\right)^{\frac{5}{2}}}$$
$$\left.-\frac{45}{8}\frac{\left(u_{i+1,j}-u_{i-1,j}\right)^2\left(u_{i+1,j}-2u_{i,j}+u_{i-1,j}\right)^3}{h^8\left(1+\left(\frac{u_{i+1,j}-u_{i-1,j}}{2h}\right)^2\right)^{\frac{7}{2}}}\right]+2u_{i,j}-u_{i,j-1}$$

Put the above equation in matrix notation

$$\underline{u}_{j+1} = \emptyset + 2\underline{u}_j - \underline{u}_{j-1}$$

where $\emptyset$ is nonlinear terms

$$\underline{u}_j = \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ u_{3,j} \\ \cdot \\ \cdot \\ \cdot \\ u_{n-1,j} \end{bmatrix}, \underline{u}_{j+1} = \begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \\ \cdot \\ \cdot \\ \cdot \\ u_{n-1,j+1} \end{bmatrix}, \underline{u}_{j-1} = \begin{bmatrix} u_{1,j-1} \\ u_{2,j-1} \\ u_{3,j-1} \\ \cdot \\ \cdot \\ \cdot \\ u_{n-1,j-1} \end{bmatrix}$$

Algorithm 6.7

To obtain the numerical solution of Example 6.4

Input: endpoint L; maximum time T; constants α, ; integers n and m.

Output: approximations $u(x_i, t_j)$, for each i=0,1,….,n and j=0,1,….,m

Step 1: h=L/n

k=T/m

r=k^2/h^4

p=-3*k^2/2*h^6

q=-45*k^2/8*h^8

Step 2: for i=0,1,…..,n

for j=0,1,……,m

Do step 3 and step 4

Step 3: $u(x_0, t_j) = \alpha$

$u(x_n, t_j) = \beta$

Step 4: $u(x_i, t_0) = f(x_i)$

Step 5: for i=1,…..,n-1

for j=1,……,m-1

$$u(x_i,t_{j+1}) = r(u(x_{i+2},t_j) - 4ru(x_{i+1},t_j) + 6ru(x_i,t_j) - 4ru(x_{i-1},t_j) - 4ru(x_{i-2},t_j))/(1+(\frac{u(x_{i+1},t_j)-u(x_{i-1},t_j)}{2h})^2)^\wedge\frac{3}{2} + p\left(u(x_{i+1},t_j) - 2u(x_i,t_j) + u(x_{i-1},t_j)\right)^3)/(1+(\frac{u(x_{i+1},t_j)-u(x_{i-1},t_j)}{2h})^2)^\wedge\frac{5}{2} - 2p\left(u(x_{i+1},t_j) - u(x_i,t_j)\right)\left(u(x_{i+1},t_j) - 2u(x_i,t_j) + u(x_{i-1},t_j)\right)\left(u(x_{i+2},t_j) - 2u(x_{i+1},t_j) + 2u(x_{i-1},t_j) - u(x_{i-2},t_j)\right)/(1+(\frac{u(x_{i+1},t_j)-u(x_{i-1},t_j)}{2h})^2)^\wedge\frac{5}{2} - q(\left(u(x_{i+1},t_j) - u(x_{i-1},t_j)\right)^2)\left(u(x_{i+1},t_j) - 2u(x_i,t_j) + u(x_{i-1},t_j)\right)^3)/(1+(\frac{u(x_{i+1},t_j)-u(x_{i-1},t_j)}{2h})^2)^\wedge\frac{7}{2} + 2u(x_i,t_j) - u(x_i,t_{j-1})$$

Step 6: output $u_{00}, u_{01}, u_{02}, \ldots\ldots u_{nm}$

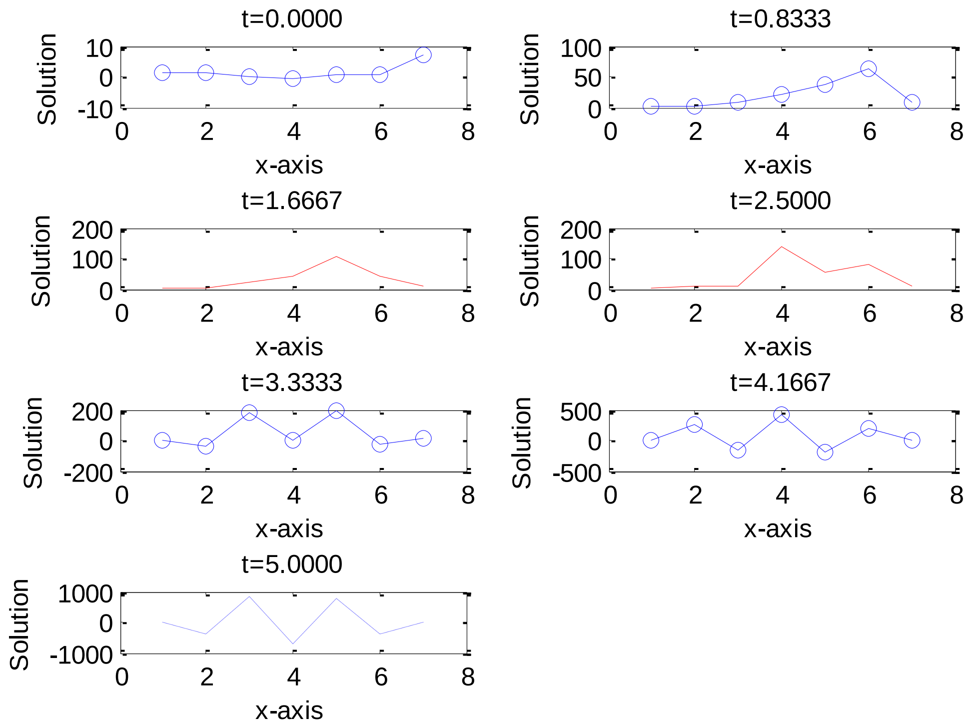
Step 7: Stop (the producer is complete)

6.4 Numerical Results

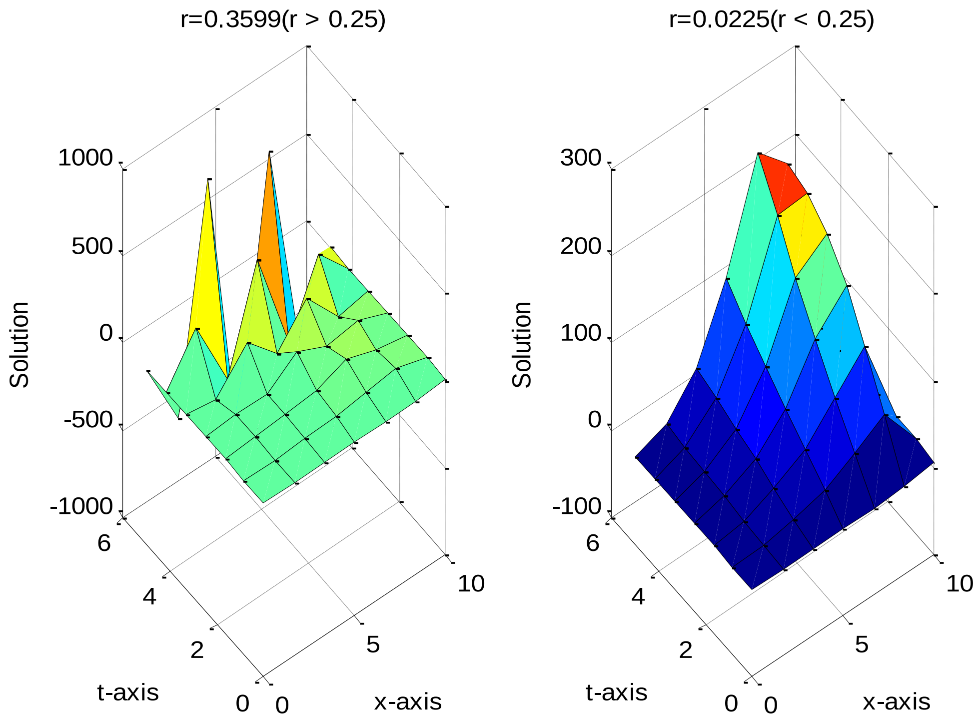
We input the initial conditions and boundary conditions in above algorithms and using MATLAB program.

Table 6.3: approximation solution of example 6.1, by using Euler explicit method, if the length of x-axis equal 10 and width equal 5. Let c=2, n=6, m=6, $\alpha = 1, \beta = 7$, $f(x) = \sin(x)$ and $g(x) = x^2$

Approximation Solution of Example 6.1							
x	t=0.00	t=0.83	t=1.66	t=2.50	t=3.33	t=4.16	t=5.00
0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2.00	0.9954	2.8106	5.2186	11.3745	-38.3144	249.5011	-388.6383
4.00	-0.1906	9.0534	19.3793	12.2539	175.9102	-166.4545	868.1506
6.00	-0.9589	20.7033	38.7948	138.8930	-0.9854	412.2724	-728.7675
8.00	0.3742	35.7298	107.2534	52.1533	191.9536	-180.0952	787.6867
10.0	0.8873	62.0214	40.7470	79.4979	-20.8075	208.1592	-412.6931
12.0	7.0000	7.0000	7.0000	7.0000	7.0000	7.0000	7.0000



(a)



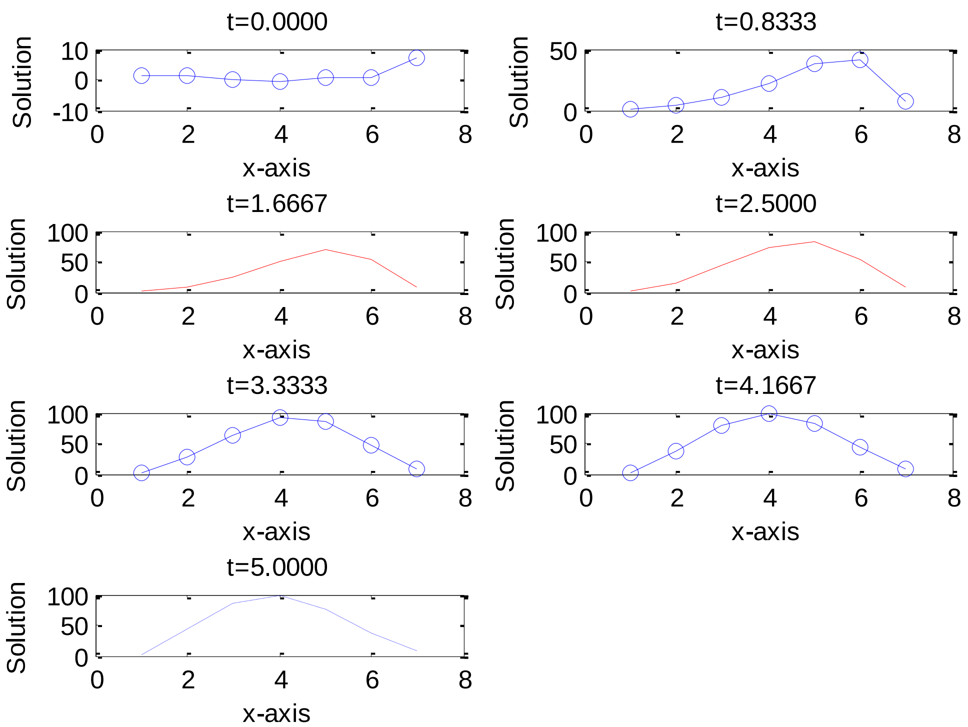
(b)

Figure 6.1: Graphical Representation is Unstable when r > 0.25, but Stable r < 0.25

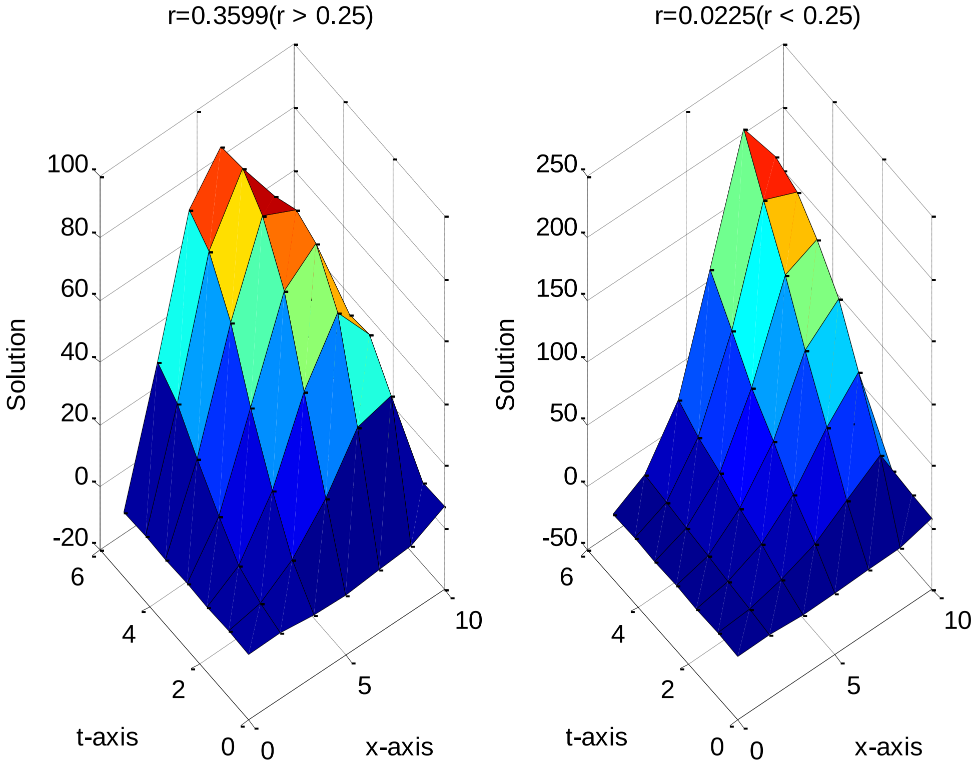
Table 6.4: approximation solution of example 6.1, by using Euler implicit method, if the length of x-axis equal 10 and width equal 5. Let c=2, n=6, m=6, $\alpha = 1, \beta = 7$, $f(x) = \sin(x)$ and $g(x) = x^2$

Approximation Solution of Example **6.1**

x	t=0.00	t=0.83	t=1.66	t=2.50	t=3.33	t=4.16	t=5.00
0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2.00	0.9954	3.3173	7.6696	15.6856	26.4596	36.6238	42.5445
4.00	-0.1906	9.8545	24.5782	43.8667	63.5681	78.3332	84.5051
6.00	-0.9589	22.4748	49.6547	74.1768	90.8535	98.2836	97.5407
8.00	0.3742	38.6758	67.8322	82.5691	85.6969	82.3423	76.1807
10.0	0.8873	41.5090	53.6372	52.3801	47.2719	42.0616	37.6572
12.0	7.0000	7.0000	7.0000	7.0000	7.0000	7.0000	7.0000



(a)

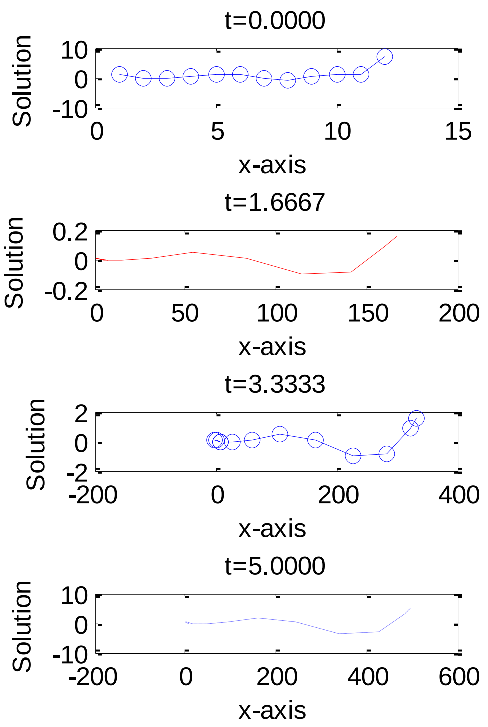


(b)

Figure 6.2: Graphical Representation is Stable for all values r.

Table 6.5: approximation solution of example 6.1, by using spectral method, if the length of x-axis equal 10 and width equal 5. Let us c=0.005, n=6, m=6, $\alpha = 1, \beta = 7$, $f(x) = \sin(x)$ and $g(x) = x^2$

Approximation Solution of Example 6.1						
1.0e+002 *						
0.0100	0.8383 + 0.0000i	1.6662 + 0.0015i	2.4932 + 0.0060i	3.3189 + 0.0150i	4.1427 + 0.0300i	4.9643 + 0.0524i
-0.0054	0.7972 - 0.0000i	1.6011 + 0.0008i	2.4075 + 0.0033i	3.2177 + 0.0082i	4.0329 + 0.0164i	4.8545 + 0.0287i
-0.0036	0.7045 + 0.0000i	1.4136 - 0.0009i	2.1247 - 0.0034i	2.8389 - 0.0086i	3.5571 - 0.0173i	4.2803 - 0.0303i
0.0022	0.5716 - 0.0000i	1.1406 - 0.0010i	1.7088 - 0.0039i	2.2756 - 0.0098i	2.8407 - 0.0195i	3.4035 - 0.0341i
0.0091	0.4219 + 0.0000i	0.8339 + 0.0001i	1.2444 + 0.0004i	1.6526 + 0.0010i	2.0577 + 0.0021i	2.4589 + 0.0036i
0.0071	0.2754 + 0.0000i	0.5436 + 0.0005i	0.8117 + 0.0020i	1.0795 + 0.0051i	1.3470 + 0.0101i	1.6140 + 0.0176i
-0.0054	0.1505 - 0.0000i	0.3068 + 0.0001i	0.4635 + 0.0005i	0.6210 + 0.0012i	0.7796 + 0.0025i	0.9395 + 0.0043i
-0.0091	0.0666 + 0.0000i	0.1425 - 0.0001i	0.2185 - 0.0004i	0.2948 - 0.0010i	0.3714 - 0.0020i	0.4485 - 0.0035i
0.0022	0.0259 - 0.0000i	0.0496 - 0.0000i	0.0733 - 0.0002i	0.0968 - 0.0004i	0.1203 - 0.0008i	0.1437 - 0.0014i
0.0099	0.0102 + 0.0000i	0.0105 + 0.0000i	0.0107 + 0.0000i	0.0110 + 0.0000i	0.0112 + 0.0001i	0.0114 + 0.0001i
0.0071	0.0039 + 0.0000i	0.0007 + 0.0000i	-0.0025 + 0.0000i	-0.0058 + 0.0000i	-0.0090 + 0.0000i	-0.0122 + 0.0000i
0.0700	0.0700	0.0700	0.0700	0.0700	0.0700	0.0700



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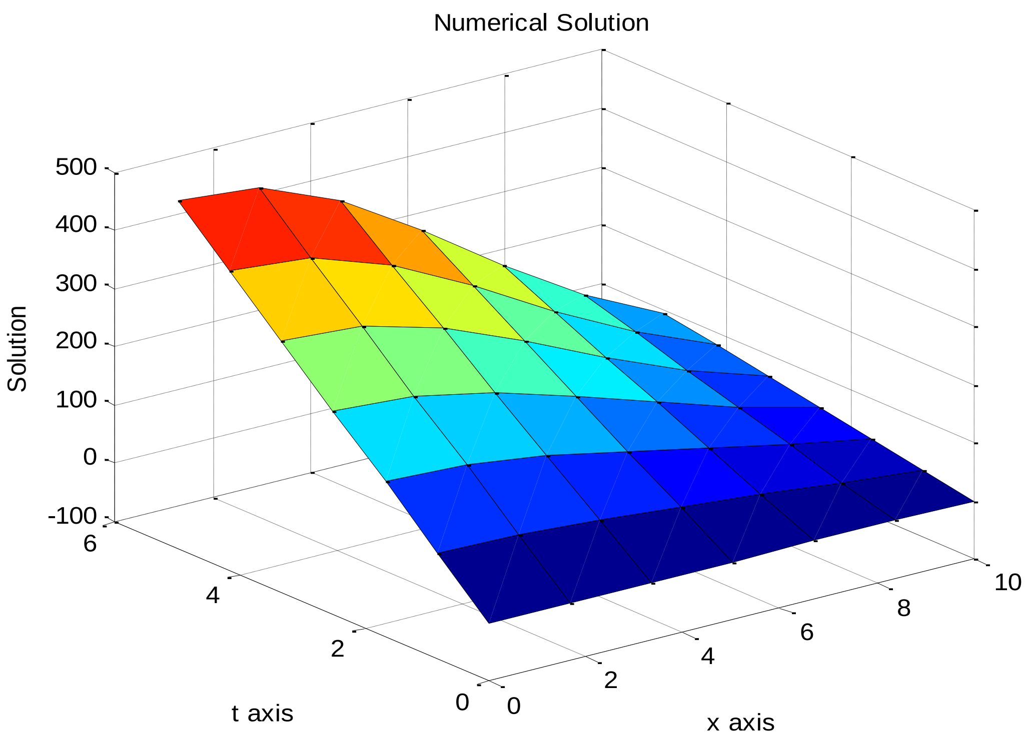


(b)

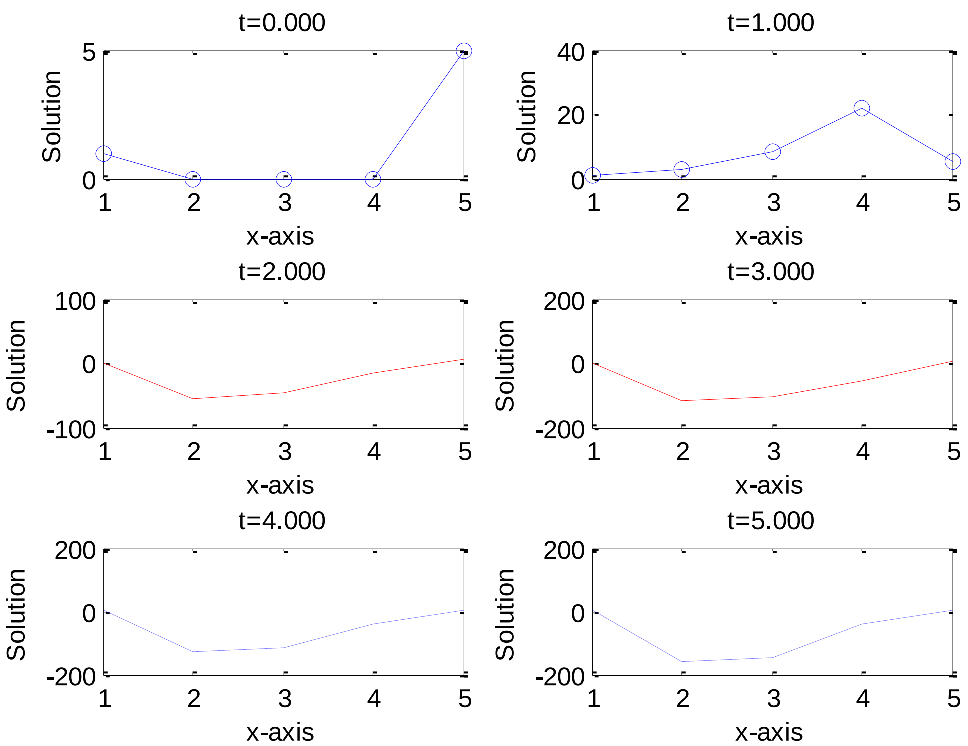
Figure 6.3: Graphical Representation of Example 6.1

Table 6.6: approximation solution of Example 6.2 by using Euler explicit method, if the length of x-axis equal 6 and width equal 5. n=4 , m=5,

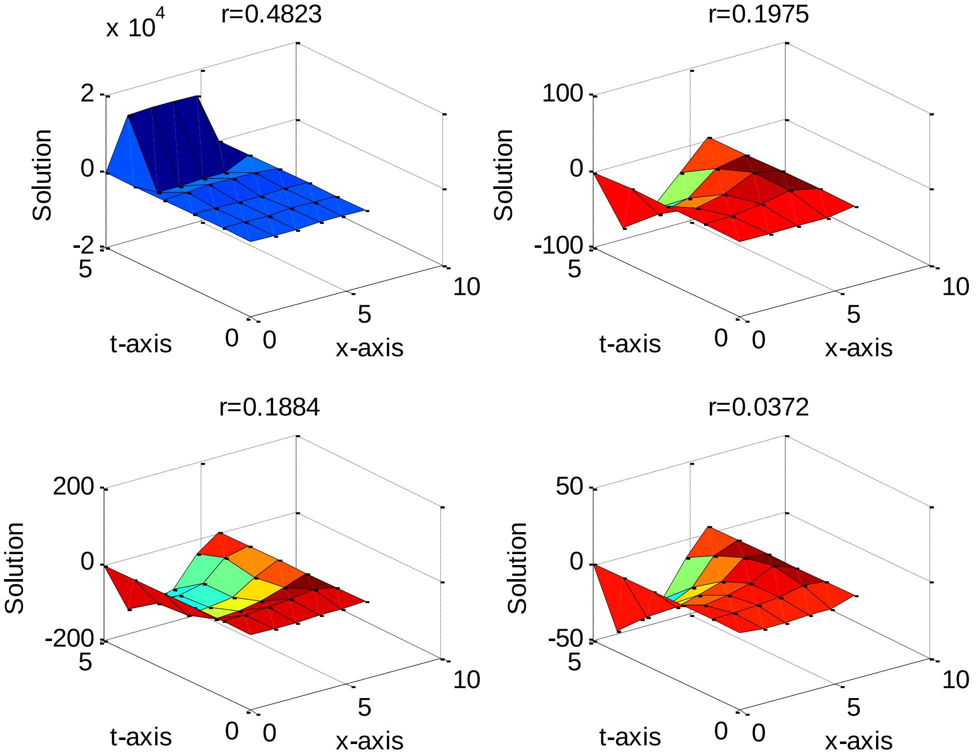
$$\alpha = 1 , \beta = 5 , f(x) = 0 \text{ and } g(x) = \left(1 + \frac{x^5}{120}\right)$$

Approximation Solution of Example 6.2

x	t=0.00	t=1.00	t=2.00	t=3.00	t=4.00	t=5.00
0.000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.500	0.0000	2.5463	-56.6622	-117.6549	-128.7829	-159.2308
3.000	0.0000	8.5062	-46.3226	-104.5160	-114.4249	-145.2339
4.500	0.0000	21.7315	-15.9214	-52.9882	-37.8200	-39.6011
5.000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000



(a)



(b)

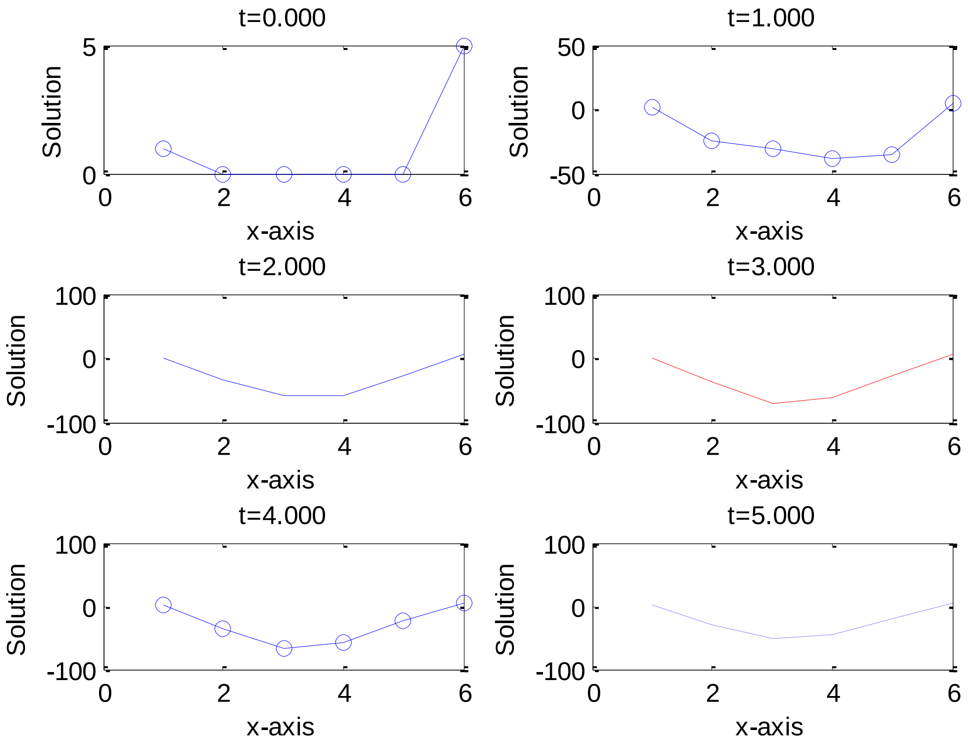
Figure 6.4: Graphical Representation of Example 6.2

Table 6.7: approximation solution of Example 6.2 by using Euler implicit method, if the length of x-axis equal 6 and width equal 5. n=5 , m=5,

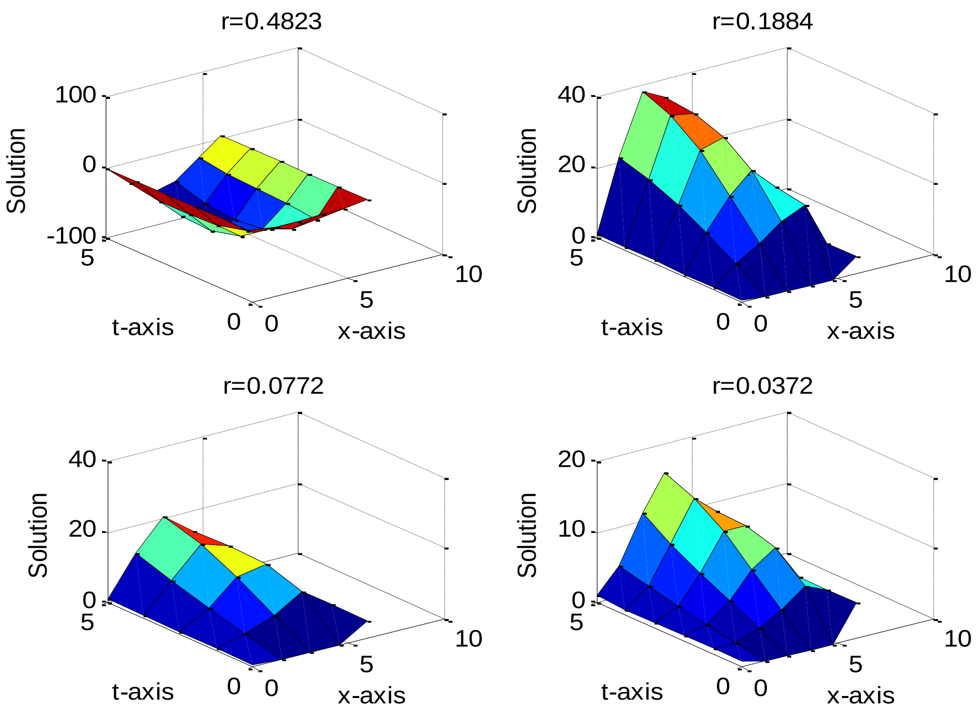
$$\alpha = 1 \, , \beta = 5 \, , f(x) = 0 \text{ and } g(x) = \left(1 + \frac{x^5}{120}\right)$$

Approximation Solution of Example 6.2

x	t=0.00	t=1.00	t=2.00	t=3.00	t=4.00	t=5.00
0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.00	0.0000	-24.4444	-34.2530	-38.1550	-36.4719	-28.9064
2.00	0.0000	-30.0031	-57.4790	-69.7748	-66.3489	-51.7295
3.00	0.0000	-38.9636	-57.0771	-61.5682	-57.1792	-44.8195
4.00	0.0000	-34.8407	-29.0064	-26.6564	-24.5436	-19.0131
5.00	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000



(a)



(b)

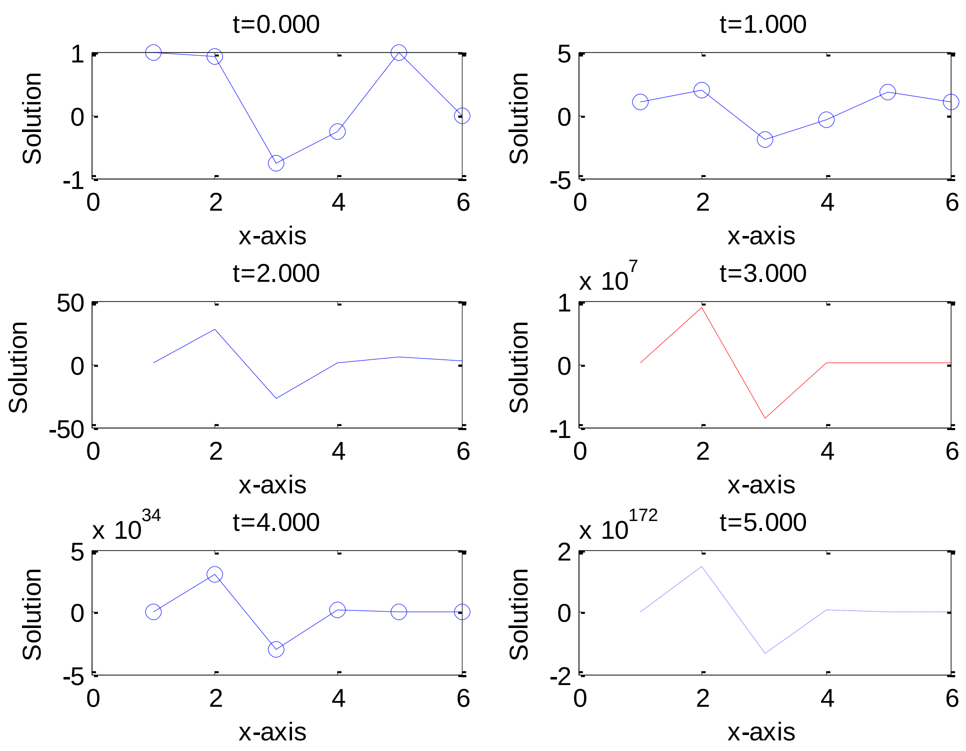
Figure 6.5: Graphical Representation of Example 6.2

Table 6.8: approximation solution of Example 6.3 by using Euler explicit method if the length of x-axis equal 10 and width equal 5. n=5, m=5, $\alpha = 1 , \beta = t ,$

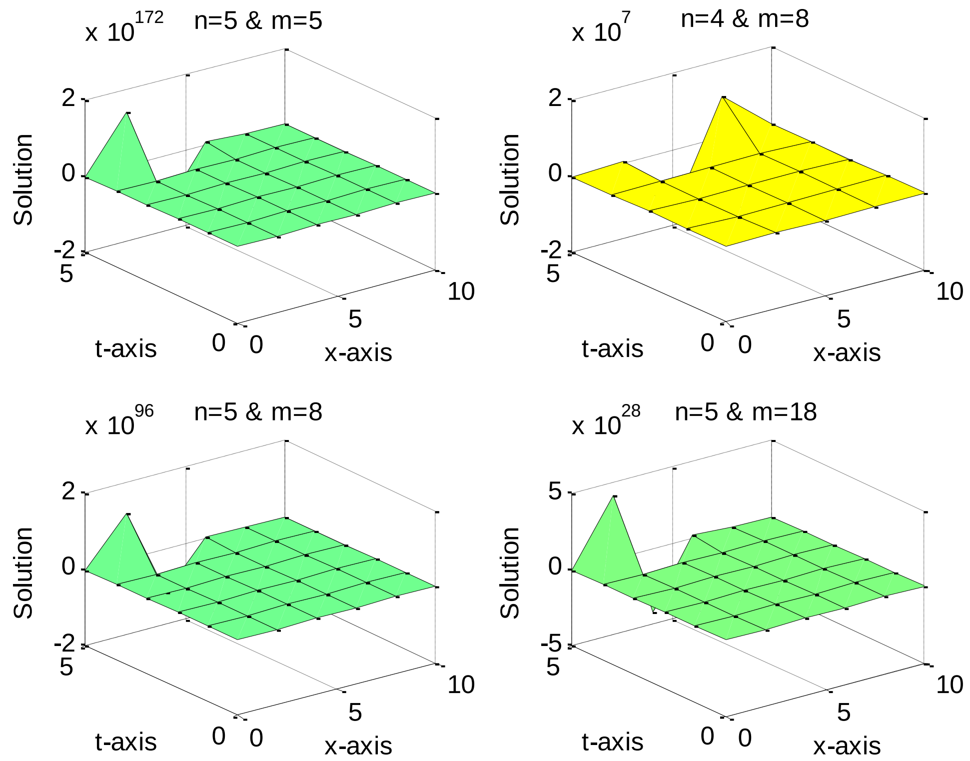
$$f(x) = \sin(x) \text{ and } g(x) = 0$$

Approximation Solution of Example 6.3

x	t=0.00	t=1.00	t=2.00	t=3.00	t=4.00	t=5.00
	1.0e+172 *					
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1.0	0.0000	0.0000	0.0000	0.0000	0.0000	1.4490
2.0	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-1.3557
3.0	-0.0000	-0.0000	0.0000	0.0000	0.0000	0.0705
4.0	0.0000	0.0000	0.0000	0.0000	-0.0000	-0.0000
5.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000



(a)

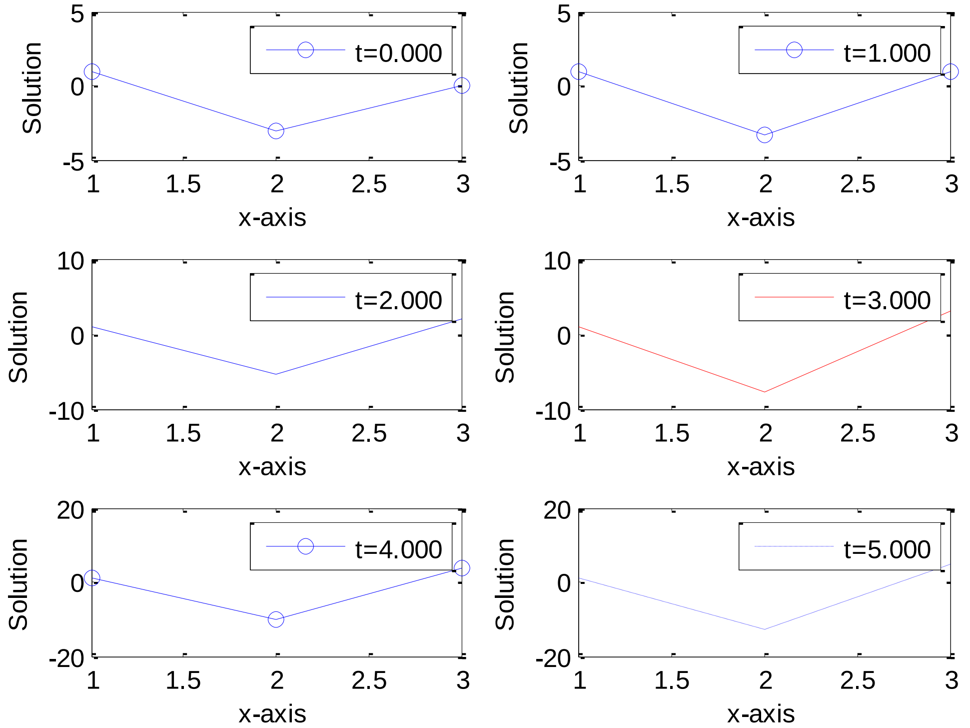


(b)

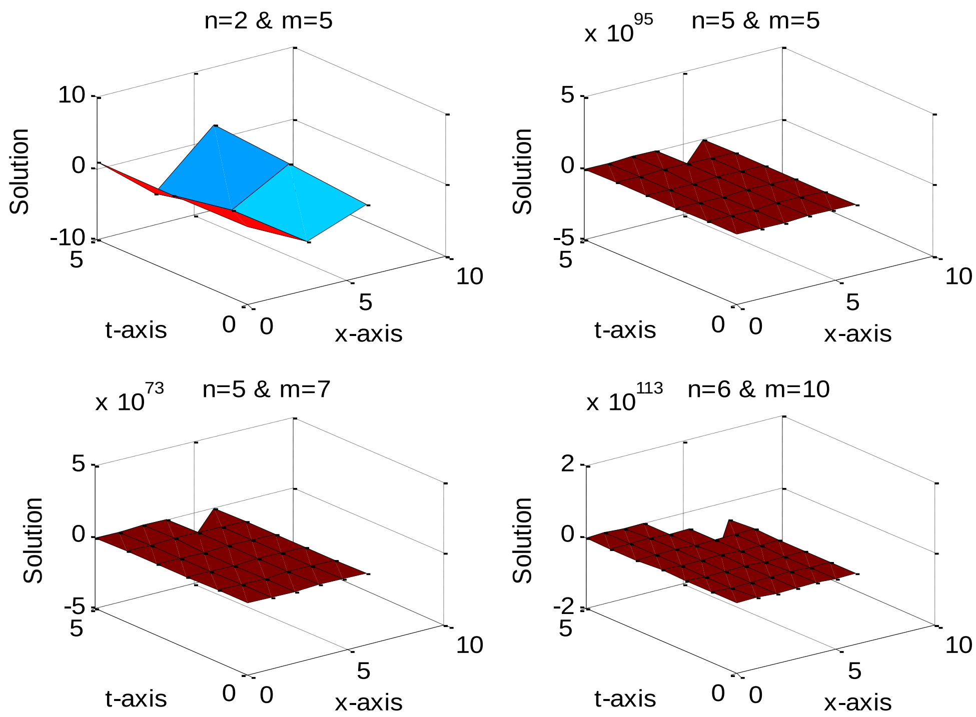
Figure 6.6: Graphical Representation of Example 6.3

Table 6.9: approximation solution of Example 6.4 by using Euler explicit method if the length of x-axis equal 6 and width equal 5. n=2, m=5, $\alpha = 1$, $\beta = t$, $g(x) = 0$ and $f(x) = \sin(x)$.

Approximation Solution of Example 6.4						
x	t=0.00	t=1.00	t=2.00	t=3.00	t=4.00	t=5.00
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
3.0	-3.0000	-3.3750	-5.4247	-7.8537	-10.3521	-12.8742
6.0	0.0000	1.0000	2.0000	3.0000	4.0000	5.0000



(a)



(b)

Figure 6.7: Graphical Representation of Example 6.4

6.5 Conclusions of this Chapter

Comparing graphical results of the numerical solutions and analytical solutions, we find that the agreement is almost complete.

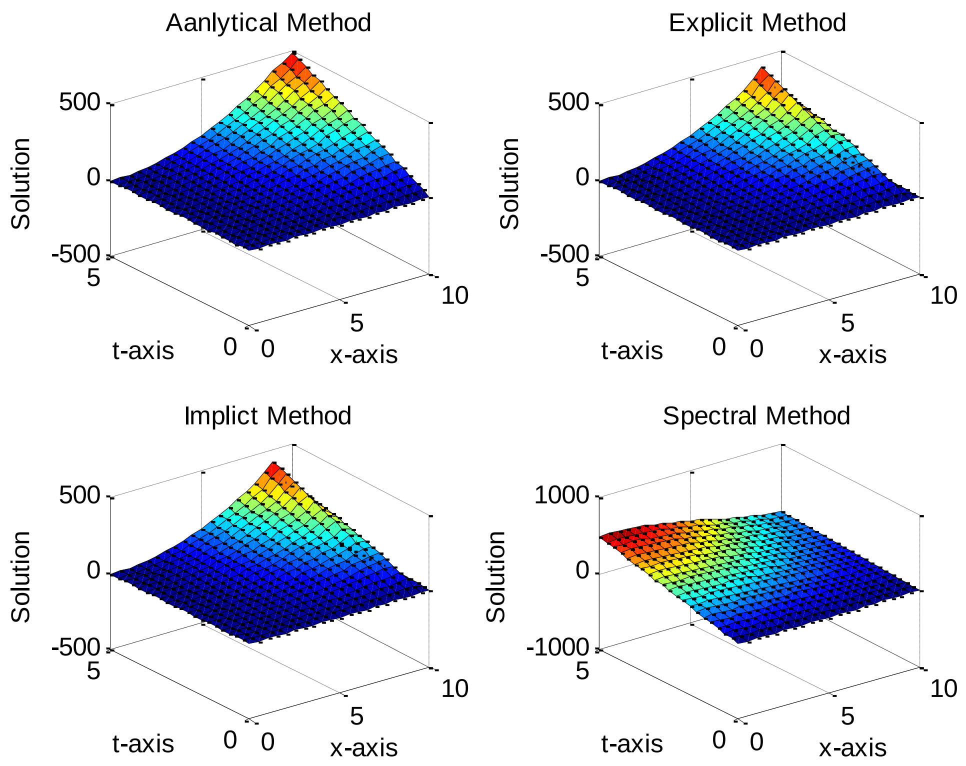


Figure 6.8: the Analytical Solution and Numerical Solution of Linear Equation of Transvers Vibration of Beam. If Initial Conditions $u(x,0) = \sin(x)$ & $u_t(x,0) = x^2$

7 Conclusions and Recommendations

7.1 The Conclusions

To design any building, Civil engineer need firstly calculate the moment, reaction, loads, and elastic curve equation of building, then implement the construction.

This study has been applied in three partial differential equations (PDEs) obtained from Concrete Beams Design (CBD), first and second partial differential equations solving by analytical methods and numerical method and then compare the iteration methods with exact solution, which gave satisfied error, and then plot the solutions by using MATLAB program.

The third equation solving numerically by using finite difference method and design algorithm using MATLAB program, because the third equation required calculation in several steps, and difficult to apply analytical solving method.

Future outlook engineers used to the transvers of vibration beams became of its simplify. We tried to go about further by considering a non-unity first derivative, which appears in the curvature equation. We hope that future workers would follow the live.

7.2 The Recommendations

The recommendations: when applying the finite differences methods one should use the implicit method because the stability of this method is unconditional but the explicit method is conditional. If the BVP is nonlinear, one should use the iterative method. For example, Homotopy and Continuation methods give higher accuracy.

References

[1] Lars Anderson and R. K. Nielson, Lecture 10: Beams Deflections Second order Methods.

[2] A.Barari, H. D. kaliji, M. Ghodimi and G. Domairrt, Nonlinear Vibration of Euler-Bernoulli Beams, Latin American Journal of Solids and Structures, 2011, 8: 139-148.

[3] S. Tisnoshenko of Theory of Elasticity, the maple bars company, New York, 1951.

[4] Lars Anderson and R. K. Nielson, Elastic Beams in three dimensions, DCE lecture notes no.23, Alaborg university, Department Civil Engineering Structural Mechanics.

[5] A. Barari, B. Ganjavi, M. Ghanbari Jeloudar, and G. Domairry. Assessment of two analytical methods in solving the linear and nonlinear elastic beam deformation problems. Journal of Engineering, Design and Technology, 2010, 8(2):127–145.

[6] Fertis, D. G, Nonlinear Structural Engineering with unique theories and methods to solve effectively complex nonlinear problems, 2006.

[7] Abdul Majid Wazwaz, Partial Differential Equations and Solitary Waves Theory, 2009.

[8] Abdrhaman Mahmoud, Numerical Solutions for Fourth Order-Ordinary Differential Equations with Polynomial Nonlinearity, 2018.

[9] Dennis G. Zill, Differential Equations with Boundary-Value Problems, Seventh Edition, 2009.

[10] He J. H., "Homotopy Perturbation Technique", Comput. Math. Appl. Mech. Engrg, 1999, Vol.178, pp.257-262.

[11] Allahviranloo T., Khezerloo M., Ghanbari M. and Khezerloo S, The Homotopy Perturbation Method for Solving Fuzzy Volttera Integral Equations", Int. J. of Comp. Co, 2010, Vol.8, No.2, pp.31-37.

[12] Syed Tauseef Mohyud-Din,Muhammad Aslam Noor and Asif Waheed, Variation of parameters for Initial and Boundary Vaule problem, World Applied Sciences Journal, 2010, 11 (5): 622-639.

[13] Abdou, M. A. and A. A. Soliman, Variational iteration method for solving Burger’s and coupled Burger’s equations. J.Comput. Appl. Math, 2005,181: 245-251.

[14] Randall J. Leveque, Finite Difference Methods for Ordinary Differential Equations and Partial Differential Equations, 2007.

[15] J. W. Thomas, Numerical Partial Differential Equations, Conservation Laws and Elliptic Equations, New York, 1999.

[16] John C. Strikwerda, Finite Difference Schemes and Partial Differential Equations, Second Edition, 2004.

[17] Abdou, M. A. and A. A. Soliman, New applications of variational iteration method. Phys, 2005, 211(1-2): 1-8.
[18] Ghorbani, A. and J. S. Nadjfi, He's homotopyperturbation method for calculating Adomian's polynomials. Int. J. Nonlin. Sci. Num. Simul, 2007, 8(2): 229-332.
[19] He, J. H, Variational iteration method-some recent results and new interpretations. J. Comput.Appl. Math, 2007, 207: 3-17.
[20] Noor, M. A. and S. T. Mohyud-Din, Variational iteration method for solving fifth-order boundary value problems using He's polynomials. Math. Prob. Engg, 2008.
[21] Abdul-majid Wazwaz, partial Differential Equations and solitary wave theory.
[22] Tyn Myint-U Lokenath Debnath, Linear Partial Differential Equations for Scientists and Engineers, four edition.
[23] Travis J. Moore, Application of Variation of Parameters to Solve Nonlinear Multimode Heat Transfer Problems, 2014.
[24] M. J. Ablowitz and P. A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, Cambridge, 1991.
[25] G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer, Boston, 1994.
[26] D. J. Evans and B. B. Sanugi, Numerical solution of the Goursat problem by a nonlinear trapezoidal formula, Appl. Math. Lett, 1988, 1(3): 221 – 223.
[27] R. Hirota, the Direct Method in Soliton Theory, Cambridge University Press, Cambridge, 2004.
[28] R. Hirota, Exact solutions of the Sine-Gordon equation for multiple collisions of solitons, J. Phys. Soc. Japan, 1972, 33(5): 1459–1463.
[29] A. M. Wazwaz, The decomposition method for approximate solution of the Goursat problem, Appl. Math. Comput, 1995, 69: 299–311.
[30] J. H. He, a variational iteration method-a kind of nonlinear analytical technique: Some examples, Int. J. Nonlinear Mech, 1999, 34: 699–708.
[31] G. Adomian, A review of the decomposition method and some recent results for nonlinear equation, Math. Comput. Modelling, 1992, 13(7): 17–43.
[32] S. J. Farlow, Partial Differential Equations for Scientists and Engineers, Dover, New York, 1993.
[33] J. H. He, Variational iteration method: some recent results and new interpretations, J. Comput. and Appl. Math., 2007, 207(1): 3–17.
[34] A. M. Wazwaz, Partial Differential Equations: Methods and Applications, Balkema Publishers, The Netherlands, 2002.

[35] A. M. Wazwaz, A reliable modification of Adomian's decomposition method, Appl. Math. And Comput., 92(1), 1–7, 1998.
[36] J. M. Cooper, Introduction to Partial Differential Equations with MATLAB, Birkhauser, Boston, 1998.
[37] Love, A. E. H., A Treatise on the Mathematical Theory of Elasticity, 4th ed.,1927, reprinted by Dover Publications, New York, 1947.
[38] Timoshenko, S. P., History of Strength of Materials: With a Brief Account of the Theory of Elasticity and Theory of Structures, 1953, reprinted by Dover Publications, New York, 1983.
[39] T. Stratford and C. Burgoyne, "Shear Analysis of Concrete with Brittle Reinforcement," Journal of Composite Construction, 2003, vol. 7, no. 4.
[40] P. Balagury, A. Nanni, and J. Giancaspro, FRP Composites for Reinforced And Prestressed Concrete Structures, 1st ed. Taylor & Francis, 2009.
[41] J. F. Chen and J. G. Teng, "Shear capacity of FRP-strengthened RC beams, FRP debonding," Construction and Building Materials, 2003, vol. 17, no. 1, pp. 27–41.
[42] A. Shah, "Evaluation of Shear Strength of High Strength Concrete Beams," University of Engineering & Technology Taxila Pakistan, Pakistan, 2009.
[43] M. Y. Mansour, M. Dicleli, J. Y. Lee, and J. Zhang, "Predicting the shear strength of reinforced concrete beams using artificial neural networks," Engineering Structures, May 2004, vol. 26, no. 6, pp. 781–799.
[44] S. J. Farlow, Partial Differential Equations for Scientists and Engineers, Dover, New York, 1993.
[45] D. Bleecker and G. Csordas, Basic Partial Differential Equations, Chapman and Hall, New York, 1995.
[46] J. D. Logan, An Introduction to Nonlinear Partial Differential Equations, John Wiley, New York, 1994.
[47] A. M. Wazwaz, A reliable modification of Adomian's decomposition method, Appl. Math. Comput, 1998, 92(1): 1–7.
[48] A. M. Wazwaz, A new algorithm for solving differential equations of the Lane-Emden type, Appl. Math. Comput, 2001, 118(2): 287–310.
[49] A. M. Wazwaz, The modified decomposition method for analytic treatment of differential equations, Appl. Math. Comput, 2006, 173(1): 165–176.
[50] J. H. He, A variational iteration method-a kind of nonlinear analytical technique: Some examples, Int. J. Nonlinear Mech, 1999, 34(4): 699–70

APPENDIX

APPENDIX A

A: Derivation of Euler-Lagrange Equation

We derive Euler-Lagrange Equation for the simplest case, that in which the functional $I[u]$ has the form

$$I[u] = \iint\limits_{s} f(t, x, u, u_t, u_x, u_{tt}, u_{xx}) dxdt$$

And in which u must satisfy the conditions $\eta(s) = \eta_t(s) = \eta_x(s) = 0$, we assume that f has continuous first partial derivatives with respect to each of its variables.

Suppose that a minimizing (or maximizing) function does indeed exist, and denote it by $\bar{u}$ (so that $\bar{u}$ are analogous to a critical point in calculus).

The functions η and $\bar{u}$ remain fixed, and ε is allowed to very small. The new function $\bar{u} + \delta\bar{u}$, or $\bar{u} + \varepsilon\eta(x)$ and $\bar{u} + \varepsilon\eta(t)$, also yields a value of I, which is formed by substituting $\bar{u} + \delta\bar{u} = \bar{u} + \varepsilon\eta$ for $\bar{u}$, and derivatives $(\bar{u} + \delta\bar{u})' = (\bar{u} + \varepsilon\eta)'$. This procedure determines a real function of ε.

$$\frac{\partial f}{\partial \varepsilon} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial \varepsilon} + \frac{\partial f}{\partial u_t}\frac{\partial u_t}{\partial \varepsilon} + \frac{\partial f}{\partial u_x}\frac{\partial u_x}{\partial \varepsilon} + \frac{\partial f}{\partial u_{tt}}\frac{\partial u_{tt}}{\partial \varepsilon} + \frac{\partial f}{\partial u_{xx}}\frac{\partial u_{xx}}{\partial \varepsilon}$$

$$\frac{\partial f}{\partial \varepsilon} = \eta\frac{\partial f}{\partial u} + \eta_t\frac{\partial f}{\partial u_t} + \eta_x\frac{\partial f}{\partial u_x} + \eta_{tt}\frac{\partial f}{\partial u_{tt}} + \eta_{xx}\frac{\partial f}{\partial u_{xx}}$$

$$\frac{\partial}{\partial \varepsilon} I[u] = \frac{\partial}{\partial \varepsilon}\iint\limits_{s} f(t, x, u, u_t, u_x, u_{tt}, u_{xx}) dxdt$$

$$\frac{\partial}{\partial \varepsilon} I[u] = \iint\limits_{s} (\eta\frac{\partial f}{\partial u} + \eta_t\frac{\partial f}{\partial u_t} + \eta_x\frac{\partial f}{\partial u_x} + \eta_{tt}\frac{\partial f}{\partial u_{tt}} + \eta_{xx}\frac{\partial f}{\partial u_{xx}}) dxdt$$

Now integral by parts of second, third, fourth and fifth terms.

$$\int \eta_t \frac{\partial f}{\partial u_t} dt = \eta \frac{d}{dt}\frac{\partial f}{\partial u_t} - \int \eta \frac{d}{dt}\frac{\partial f}{\partial u_t} dt = -\int \eta \frac{d}{dt}\frac{\partial f}{\partial u_t} dt$$

$$\int \eta_x \frac{\partial f}{\partial u_x} dx = \eta \frac{d}{dx}\frac{\partial f}{\partial u_x} - \int \eta \frac{d}{dx}\frac{\partial f}{\partial u_x} dx = -\int \eta \frac{d}{dx}\frac{\partial f}{\partial u_x} dx$$

$$\int \eta_{tt} \frac{\partial f}{\partial u_{tt}} dt = \eta_t \frac{d}{dt}\frac{\partial f}{\partial u_{tt}} - \eta \frac{d}{dt}\frac{\partial f}{\partial u_{tt}} + \int \eta \frac{d^2}{dt^2}\frac{\partial f}{\partial u_{tt}} dt$$

$$\int \eta_{tt} \frac{\partial f}{\partial u_{tt}} dt = \int \eta \frac{d^2}{dt^2}\frac{\partial f}{\partial u_{tt}} dt$$

$$\int \eta_{xx} \frac{\partial f}{\partial u_{xx}} dx = \eta_x \frac{d}{dx}\frac{\partial f}{\partial u_{xx}} - \eta \frac{d}{dx}\frac{\partial f}{\partial u_{xx}} + \int \eta \frac{d^2}{dx^2}\frac{\partial f}{\partial u_{xx}} dx$$

$$\int \eta_{xx} \frac{\partial f}{\partial u_{xx}} dx = \int \eta \frac{d^2}{dx^2}\frac{\partial f}{\partial u_{xx}} dx$$

Substitute we get

$$\frac{\partial}{\partial \varepsilon} I[u] = \iint_S \left(\eta \frac{\partial f}{\partial u} - \eta \frac{d}{dt}\frac{\partial f}{\partial u_t} - \eta \frac{d}{dx}\frac{\partial f}{\partial u_x} + \eta \frac{d^2}{dt^2}\frac{\partial f}{\partial u_{tt}} + \eta \frac{d^2}{dx^2}\frac{\partial f}{\partial u_{xx}}\right) dxdt$$

$$\frac{\partial}{\partial \varepsilon} I[u] = \iint_S \eta \left(\frac{\partial f}{\partial u} - \frac{d}{dt}\frac{\partial f}{\partial u_t} - \frac{d}{dx}\frac{\partial f}{\partial u_x} + \frac{d^2}{dt^2}\frac{\partial f}{\partial u_{tt}} + \frac{d^2}{dx^2}\frac{\partial f}{\partial u_{xx}}\right) dxdt$$

From necessary condition $\frac{\partial}{\partial \varepsilon} I[u] = 0$

$$\iint_S \eta \left(\frac{\partial f}{\partial u} - \frac{d}{dt}\frac{\partial f}{\partial u_t} - \frac{d}{dx}\frac{\partial f}{\partial u_x} + \frac{d^2}{dt^2}\frac{\partial f}{\partial u_{tt}} + \frac{d^2}{dx^2}\frac{\partial f}{\partial u_{xx}}\right) dxdt = 0$$

This condition must hold for all functions η that are continuously differentiable and satisfy $\eta(s) = 0$ hence the factor inside the parentheses in the integrand must equal zero. Thus, any minimizing (or maximizing) function u must be a solution of the differential equation.

$$\frac{\partial f}{\partial u} - \frac{d}{dt}\frac{\partial f}{\partial u_t} - \frac{d}{dx}\frac{\partial f}{\partial u_x} + \frac{d^2}{dt^2}\frac{\partial f}{\partial u_{tt}} + \frac{d^2}{dx^2}\frac{\partial f}{\partial u_{xx}} = 0$$

This is Euler – Lagrange Equation for the functional $I[u]$.

APPENDIX B

B: MATLAB Programs

1. MATLAB program of figure 3.1

```
Clc;  clear
[x,t]=meshgrid(0:40:400,0:5:50);
subplot(2,2,1)
u=-0.5*x.^2;
surf(x,t,u);
xlabel('x-axis')
ylabel('t-axis')
zlabel('Solution')
title('Initial Guess')
subplot(2,2,2)
u=0.5*(45*x.^2+3).*(t.^2)/2;
surf(x,t,u);
xlabel('x-axis')
ylabel('t-axis')
zlabel('Solution')
title('2nd term')
subplot(2,2,3)
u=202.5*(25*x.^2+1).*(t.^4)/24;
surf(x,t,u);
xlabel('x-axis')
ylabel('t-axis')
zlabel('Solution')
title('3rd term')
subplot(2,2,4)
u=-
0.5*x.^2+0.5*(45*x.^2+3).*(t.^2)/2+202.5*(25*x.^2+1)
.*(t.^4)/24+0.5*(4829625*x.^2-88695).*(t.^6)/720;
surf(x,t,u);
xlabel('x-axis')
```

```
ylabel('t-axis')
zlabel('Solution')
title('Approximation Solution')
```

2. MATLAB program of figure 4.1

```
clc; clear
[x,t]=meshgrid(0:40:400,0:5:50);
subplot(2,2,1)
u=(1/6)*x.^3;
surf(x,t,u);
xlabel('x-axis')
ylabel('t-axis')
zlabel('Solution')
title('coefficient p^0')
subplot(2,2,2)
u=0.5*(3*x.^3).*(t.^2)/2;
surf(x,t,u);
xlabel('x-axis')
ylabel('t-axis')
zlabel('Solution')
title('coefficient p^1')
subplot(2,2,3)
u=0.5*(3*x.^3+2*x.^2).*(t.^4)/24;
surf(x,t,u);
xlabel('x-axis')
ylabel('t-axis')
zlabel('Solution')
title('coefficient p^2')
subplot(2,2,4)
u=(1/6)*x.^3+0.5*(3*x.^3).*(t.^2)/2+0.5*(3*x.^3+2*x.
^2).*(t.^4)/24;
surf(x,t,u);
xlabel('x-axis')
ylabel('t-axis')
zlabel('Solution')
title('Approximate Solution')
```

3. MATLAB program of figure 4.2

```
Clc; clear
[x,t]=meshgrid(0:40:400,0:5:50);
subplot(2,2,1)
u=-0.5*x.^2-1.5;
surf(x,t,u);
xlabel('x-axis')
ylabel('t-axis')
zlabel('Solution')
title('coefficient p^0')
subplot(2,2,2)
```

```
u=0.5*(45*x.^2+3).*(t.^2)/2;
surf(x,t,u);
xlabel('x-axis')
ylabel('t-axis')
zlabel('Solution')
title('coefficient p^1')
subplot(2,2,3)
u=22.5*(45*x.^2+3).*(t.^4)/24;
surf(x,t,u);
xlabel('x-axis')
ylabel('t-axis')
zlabel('Solution')
title('coefficient p^2')
subplot(2,2,4)
     u=-0.5*x.^2-
1.5+0.5*(45*x.^2+3).*(t.^2)/2+22.5*(45*x.^2+3).*(t.^
4)/24;
surf(x,t,u);
xlabel('x-axis')
ylabel('t-axis')
zlabel('Solution')
title('Approximate Solution')
```

4. MATLAB program of figure 5.1

```
clc; clear
[x,t]=meshgrid(0:40:400,0:5:50);
subplot(2,2,1)
u=sin(x);
surf(x,t,u);
xlabel('x-axis')
ylabel('t-axis')
zlabel('Solution')
title('Initial Guess')
subplot(2,2,2)
     u=sin(x)+(-
5*sin(x)+30*(sin(x)).^3+22.5*(sin(x)).^5).*t^2/2;
surf(x,t,u);
xlabel('x-axis')
ylabel('t-axis')
zlabel('Solution')
title('2nd Term')
subplot(2,2,3)
u=sin(x)+sin(x)+(-
5*sin(x)+30*(sin(x)).^3+22.5*(sin(x)).^5).*t^2/2;
surf(x,t,u);
xlabel('x-axis')
ylabel('t-axis')
zlabel('Solution')
title('Approximate Solution')
```

5. MATLAB program of figure 5.2

```
clc
clear
x=0:0.1:1; t=1;
Exact=sin(x)*cosh(t);
HPM=sin(x)*cosh(t);
VIM=sin(x)*cosh(t);
ADM=sin(x)*cosh(t);
u=[x',Exact',ADM',HPM',VPM',abs(Exact'-
ADM'),abs(Exact'-HPM'),abs(Exact'-VPIM')]
plot(x,Exact,'*',x,ADM,'+',x,HPM,'o',x,VIM,'r')
legend('Exact','ADM','HPM','VIM',2)
```

6. MATLAB program of figure 5.3

```
Clc; clear
x=0:0.1:1; t=1;
Exact=(1+x.^5/120)*sinh(t);
HPM=(1+x.^5/120)*sinh(t);
VIM=(1+x.^5/120)*sinh(t);
ADM=(1+x.^5/120)*sinh(t);
u=[x',Exact',ADM',HPM',VIM',abs(Exact'-
ADM'),abs(Exact'-HPM'),abs(Exact'-VIM')]
plot(x,Exact,'*',x,ADM,'+',x,HPM,'o',x,VIM,'r')
legend('Exact','ADM','HPM','VIM',2)
```

7. MATLAB program of figure 5.4

```
Clc; clear
x=0:0.01:1; t=1;
Exact=(-0.5*x.^2)*(cosh(t));
HPM=-0.5*x.^2+0.5*(45*x.^2+3)*
t.^2/2+22.5*(45*x.^2+3)* t.^4/24;
VIM=-0.5*x.^2+0.5*(45*x.^2+1)*
t.^2/2+202.5*(25*x.^2+1)*(t.^4/24);
ADM=-0.5*x.^2+0.5*(45*x.^2+3)*
t.^2/2+202.5*(25*x.^2+1)*(t.^4/24);
u=[x',Exact',ADM',HPM',VPM',abs(Exact'-
ADM'),abs(Exact'-HPM'),abs(Exact'-VIM')]
plot(x,Exact,'*',x,ADM,'+',x,HPM,'o',x,VIM,'r')
legend('Exact','ADM','HPM','VIM',2)
title('Compare the Exact solution with ADM, HPM &
VIM when N=2')
```

8. MATLAB Program by Using Explicit Method of Example 6.1

```
function u=Exprom61(c,L,Tmax,m,n)
% L & Tmax are length of x and t axes
% m is number of divided of x axis & n is number of
divided of t axis
% alpha & beta are boundary conditions
% f(x) & g(x) are initial conditions
h=L/m;
k=Tmax/n;
r=c^2*k^2/h^2;
u=zeros(m+1,n+1);
for j=1:n+1
    u(1,j)=alpha;
    u(m+1,j)=beta;
end
for i=2:m
    u(i,1)=f((i-1)*h); % u(x,0)=f(x)
end
A=zeros(m-1,m-1);
b=zeros(m-1,1); b(1)=3*r*alpha; b(2)=-r*alpha; b(m-
2)=-r*beta; b(m-1)=3*r*beta;
for j=1:m-1
    for i=1:m-1
       if i==j
           A(i,j)=2-6*r;
       elseif abs(i-j)==1
           A(i,j)=4*r;
       elseif abs(i-j)==2
           A(i,j)=-r;
       end
    end
end
d=zeros(m-1,1);
for i=1:m-1
    d(i)=g(i*h); % u'(x,0)=g(x)
end
u(2:m,2)=0.5*(A*u(2:m,1)+2*k*d+b);
for j=2:n
u(2:m,j+1)=A*u(2:m,j)-u(2:m,j-1)+b;
end
```

9. MATLAB Program by Using Implicit Method of Example 6.1

```
function u=Improm61(c,L,Tmax,m,n)
% L & Tmax are length of x and t axes
% m is number of divided of x axis & n is number of
divided of t axis
% alpha & beta are boundary conditions
% f(x) & g(x) are initial conditions
h=L/m;
k=Tmax/n;
r=c^2*k^2/h^2;
```

```
u=zeros(m+1,n+1);
for j=1:n+1
    u(1,j)=alpha;
    u(m+1,j)=beta;
end
for i=2:m
    u(i,1)=f((i-1)*h); % u(x,0)=f(x)
end
A=zeros(m-1,m-1); B=zeros(m-1,m-1);
b=zeros(m-1,1); b(1)=3*r*alpha; b(m-1)=3*r*beta;
for j=1:m-1
    for i=1:m-1
       if i==j
           A(i,j)=1+6*r;
           B(i,j)=2+6*r;
       elseif abs(i-j)==1
           A(i,j)=-3*r;
           B(i,j)=-3*r;
       end
    end
end
    d=zeros(m-1,1);
for i=1:m-1
    d(i)=g(i*h); % u'(x,0)=g(x)
end
u(2:m,2)=inv(B)*(2*u(2:m,1)+2*k*d+b);
for j=2:n
u(2:m,j+1)=inv(A)*(2*u(2:m,j)-u(2:m,j-1)+b);
end
```

10. MATLAB Program by Using Spectral Method of Example 6.1

```
function u=Spectral_Beam(c,L,Tmax,n,m)
k=Tmax/m;
u=zeros(1:2*n,1:m+1);
[D,xx] = cheb(2*n-1);
x = L*(xx+1)/2;
for j=1:m+1
   u(1,j)=1; % u(0,t)=alpha
   u(2*n,j)=7; % u(L,t)=beta
end
for i=2:2*n-1
    u(i,1)=sin(x(i-1)); % u(x,0)=f(x)
end
for i=1:2*n-1
    for r=-n+1:n
   u(i,2)=0.5*(u(i,1)+2*k*(x(i))^2-
((c*k)^2/(2*n))*sum((sqrt(-1)*r)^4*u(i,1)*exp(sqrt(-
1)*L*i*r/(2*n))));
    end
end
```

```
    for j=2:m
        for i=1:2*n-1
            for r=-n+1:n
            u(i,j+1)=2*u(i,j)-u(i,j-1)-
((c*k)^2/(2*n))*sum((sqrt(-1)*r)^4*u(i,j)*exp(sqrt(-
1)*L*i*r/(2*n)));
            end
        end
    end

    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    function [D,x] = cheb(N)
    if N==0, D=0; x=1;
    return
    end
    x = cos(pi*(0:N)/N)';
    c = [2; ones(N-1,1); 2].*(-1).^(0:N)';
    X = repmat(x,1,N+1);
    dX = X-X';
    D  = (c*(1./c)')./(dX+(eye(N+1)));
    D  = D - diag(sum(D')); % diagonal entries
```

11. MATLAB Program by Using Explicit Method of Example 6.2

```
function u=Exprom62(L,Tmax,m,n)
% L & Tmax are length of x and t axes
% m is number of divided of x axis & n is number of
divided of  t axis
% alpha & beta are boundary conditions
% f(x) & g(x) are initial conditions
h=L/m;
k=Tmax/n;
r=k^2/h^4;
u=zeros(m+1,n+1);
for j=1:n+1
    u(1,j)=alpha;
    u(m+1,j)=beta;
end
for i=2:m
    u(i,1)=f((i-1)*h); % u(x,0)=f(x)
end
A=zeros(m-1,m-1);
     b=zeros(m-1,1); b(1)=3*r*alpha; b(2)=-r*alpha;
b(m-2)=-r*beta; b(m-1)=3*r*beta;
for j=1:m-1
    for i=1:m-1
       if i==j
           A(i,j)=-6*r;
       elseif abs(i-j)==1
           A(i,j)=4*r;
       elseif abs(i-j)==2
```

```
            A(i,j)=-r;
        end
    end
end
d=zeros(m-1,1); w=zeros(1,m-1);
for i=1:m-1
    d(i)=g(i*h); % u'(x,0)=g(x)
    w(i)=(1/(i*h)+(i*h)^4/120);
end
u(2:m,2)=0.5*(w*A*u(2:m,1)+2*u(2:m,1)+2*k*d+b);
for j=2:n
u(2:m,j+1)=w*A*u(2:m,j)+2*u(2:m,j)-u(2:m,j-1)+b;
end
```

12. MATLAB Program by Using Implicit Method of Example 6.2

```
function u=Improm62(L,Tmax,m,n)
% L & Tmax are length of x and t axes
% m is number of divided of x axis & n is number of
divided of t axis
% alpha & beta are boundary conditions
% f(x) & g(x) are initial conditions
h=L/m;
k=Tmax/n;
r=k^2/h^4;
u=zeros(m+1,n+1);
alpha=1; beta=5;
for j=1:n+1
    u(1,j)=alpha;
    u(m+1,j)=beta;
end
for i=2:m
    u(i,1)=f((i-1)*h); % u(x,0)=f(x)
end
d=zeros(m-1,1); w=zeros(1,m-1);
for i=1:m-1
    d(i)=g(i*h); % u'(x,0)=g(x)
    w(i)=(1/(i*h)+(i*h)^4/120);
end
      b=zeros(m-1,1); b(1)=3*r*alpha*w(1); b(2)=-
r*alpha*w(2); b(m-2)=-r*beta*w(m-2); b(m-
1)=3*r*beta*w(m-1);
A=zeros(m-1,m-1); B=zeros(m-1,m-1);
for j=1:m-1
for i=1:m-1
       if i==j
           A(i,j)=1+6*w(i)*r;
           B(i,j)=2++6*w(i)*r;
       elseif abs(i-j)==1
           A(i,j)=-4*r*w(i);
           B(i,j)=-4*r*w(i);
```

```
            elseif abs(i-j)==2
                A(i,j)=r*w(i);
                B(i,j)=-4*r*w(i);
            end
    end
    end
    u(2:m,2)=inv(B)*(2*u(2:m,1)+2*k*d+b);
    for j=2:n
    u(2:m,j+1)=inv(A)*(2*u(2:m,j)-u(2:m,j-1)+b);
    end
```

13. MATLAB Program by Using Explicit Method of Example 6.3

```
function u=Exprom63(L,Tmax,n,m)
% L & Tmax are length of x and t axes
% m is number of divided of x axis & n is number of
divided of t axis
 % alpha & beta are boundary conditions
% f(x) & g(x) are initial conditions
h=L/n;
k=Tmax/m;
u=zeros(n+1,m+1);
% f and g are initial conditions when t is equal to
zero
%  alpha and beta are boundary conditions
for i=2:n
    u(i,1)=f((i-1)*h);
end
for j=1:m+1
    u(1,j)=alpha;
    u(n+1,j)=beta;
end
for i=2:n
    u(i,2)=0.5*((k^2/h^4)*(-3*u(i+1,j)+6*u(i,1)-
3*u(i-1,1))-(3/2)*(k^2/h^6)*((u(i+1,1)-2*u(i,1)+u(i-
1,1))^3))-(6*k^2/(4*h^5))*((u(i+1,1)-u(i-
1,1))*(u(i+1,1)-2*u(i,1)+u(i-1,1))*(-u(i+1,1)+u(i-
1,1)))-(22.5*k^2/(4*h^8))*((u(i+1,1)-u(i-
1,1))^2*(u(i+1,1)-2*u(i,1)+u(i-
1,1))^3)+2*u(i,1)+g((i-1)*h);
end
for j=2:m
for i=2:n
        u(i,j+1)=(k^2/h^4)*(-3*u(i+1,j)+6*u(i,1)-
3*u(i-1,j))-(3/2)*(k^2/h^6)*((u(i+1,j)-2*u(i,j)+u(i-
1,j))^3)-(6*k^2/(2*h^5))*((u(i+1,j)-u(i-
1,j))*(u(i+1,j)-2*u(i,j)+u(i-1,j))*(-u(i+1,j)+u(i-
1,j)))-(22.5*k^2/(4*h^8))*((u(i+1,j)-u(i-
1,j))^2*(u(i+1,j)-2*u(i,j)+u(i-1,j))^3)+2*u(i,j)-
u(i,j-1);
end
```

```
end
```

14. MATLAB Program by Using Explicit Method of Example 6.4

```
function u=Exprom64(L,Tmax,n,m)
% L & Tmax are length of x and t axes
% m is number of divided of x axis & n is number of
divided of t axis
% alpha & beta are boundary conditions
% f(x) & g(x) are initial conditions
h=L/n;
k=Tmax/m;
u=zeros(n+1,m+1);
% f and g are initial conditions when t is equal to
zero
%  alpha and beta are boundary conditions
for i=2:n
    u(i,1)=f((i-1)*h);
end
for j=1:m+1
    u(1,j)=alpha;
    u(n+1,j)=beta;
end
for i=2:n
   u(i,2)=0.5*((k^2/h^4)*(-3*u(i+1,j)+6*u(i,1)-
3*u(i-1,1))/(1+((1/2*h)*(u(i+1,1)-u(i-
1,1)))^2)^(3/2)-(3/2)*(k^2/h^6)*((u(i+1,1)-
2*u(i,1)+u(i-1,1))^3)/(1+((1/2*h)*(u(i+1,1)-u(i-
1,1)))^2)^(5/2)-(3*k^2/(4*h^5))*((u(i+1,1)-u(i-
1,1))*(u(i+1,1)-2*u(i,1)+u(i-1,1))*(-u(i+1,1)+u(i-
1,1)))/(1+((1/2*h)*(u(i+1,1)-u(i-1,1)))^2)^(5/2)-
(22.5*k^2/h^8)*((u(i+1,1)-u(i,1))^2*(u(i+1,1)-
2*u(i,1)+u(i-1,1))^3)/(1+(1/h)*(u(i+1,1)-
u(i,1))^(7/2))+2*u(i,1))+g((i-1)*h);
end
for j=2:m
    for i=2:n
        u(i,j+1)=(k^2/h^4)*(-3*u(i+1,j)+6*u(i,1)-
3*u(i-1,j))/(1+((1/2*h)*(u(i+1,j)-u(i-
1,j)))^2)^(3/2)-(3/2)*(k^2/h^6)*((u(i+1,j)-
2*u(i,j)+u(i-1,j))^3)/(1+((1/2*h)*(u(i+1,j)-u(i-
1,j)))^2)^(5/2)-(3*k^2/(4*h^5))*((u(i+1,j)-u(i-
1,j))*(u(i+1,j)-2*u(i,1)+u(i-1,j))*(-u(i+1,j)+u(i-
1,j)))/(1+((1/2*h)*(u(i+1,j)-u(i-1,j)))^2)^(5/2)-
(22.5*k^2/(4*h^8))*((u(i+1,j)-u(i-1,j))^2*(u(i+1,j)-
2*u(i,j)+u(i-1,j))^3)/(1+((1/2*h)*(u(i+1,j)-u(i-
1,j)))^2)^(7/2)+2*u(i,j)-u(i,j-1);
    end
end
```

15. MATLAB Program of Figures 6.1

```
Clc;  clear
x=linspace(0,12,7);
t=linspace(0,4,7);
u=Exprom61(10,5,6,6)
subplot(4,2,1)
plot(u(:,1),'o-')
xlabel('x-axis')
ylabel('Solution')
title('t=0.0000')
subplot(4,2,2)
plot(u(:,2),'o-')
xlabel('x-axis')
ylabel('Solution')
title('t=0.8333')
subplot(4,2,3)
plot(u(:,3),'b')
xlabel('x-axis')
ylabel('Solution')
title('t=1.6667')
subplot(4,2,4)
plot(u(:,4),'r')
xlabel('x-axis')
ylabel('Solution')
title('t=2.5000')
subplot(4,2,5)
plot(u(:,5),'o-')
xlabel('x-axis')
ylabel('Solution')
title('t=3.3333')
subplot(4,2,6)
plot(u(:,6),':')
xlabel('x-axis')
ylabel('Solution')
title('t=4.1667')
subplot(4,2,7)
plot(u(:,7),':')
xlabel('x-axis')
ylabel('Solution')
title('t=5.0000')
figure
subplot(1,2,1)
surf(x,t,u');
xlabel('x-axis')
ylabel('t-axis')
zlabel('Solution')
title('c=2 or r=0.3599')
subplot(1,2,2)
u=Exprom61(0.5,10,5,6,6);
```

```
surf(x,t,u');
xlabel('x-axis')
ylabel('t-axis')
zlabel('Solution')
title('c=0.5 or r=0.0225')
```

16. MATLAB Program of Figures 6.2

```
Clc;  clear
x=linspace(0,10,7);
t=linspace(0,5,7);
u=Improm61(2,10,5,6,6)
subplot(4,2,1)
plot(u(:,1),'o-'
xlabel('x-axis')
ylabel('Solution')
title('t=0.0000')
subplot(4,2,2)
plot(u(:,2),'o-')
xlabel('x-axis')
ylabel('Solution')
title('t=0.8333')
subplot(4,2,3)
plot(u(:,3),'r')
xlabel('x-axis')
ylabel('Solution')
title('t=1.6667')
subplot(4,2,4)
plot(u(:,4),'r')
xlabel('x-axis')
ylabel('Solution')
title('t=2.5000')
subplot(4,2,5)
plot(u(:,5),'o-')
xlabel('x-axis')
ylabel('Solution')
title('t=3.3333')
subplot(4,2,6)
plot(u(:,6),'o-')
xlabel('x-axis')
ylabel('Solution')
title('t=4.1667')
subplot(4,2,7)
plot(u(:,7),':')
xlabel('x-axis')
ylabel('Solution')
title('t=5.0000')
figure
subplot(1,2,1)
surf(x,t,u');
```

```
xlabel('x-axis')
ylabel('t-axis')
zlabel('Solution')
title('r=0.3599')
subplot(1,2,2)
u=Improm61(0.5,10,5,6,6);
surf(x,t,u');
xlabel('x-axis')
ylabel('t-axis')
zlabel('Solution')
title('r=0.0225')
```

17. MATLAB Program of Figures 6.3

```
Clc;  clear
x=linspace(0,10,7);
t=linspace(0,5,7);
u=Spectral_Beam(0.005,10,5,6,6)
subplot(4,2,1)
plot(u(:,1),'o-')
xlabel('x-axis')
ylabel('Solution')
title('t=0.0000')
subplot(4,2,2)
plot(u(:,2),'o-')
xlabel('x-axis')
ylabel('Solution')
title('t=0.8333')
subplot(4,2,3)
plot(u(:,3),'r')
xlabel('x-axis')
ylabel('Solution')
title('t=1.6667')
subplot(4,2,4)
plot(u(:,4),'r')
xlabel('x-axis')
ylabel('Solution')
title('t=2.5000')
subplot(4,2,5)
plot(u(:,5),'o-')
xlabel('x-axis')
ylabel('Solution')
title('t=3.3333')
subplot(4,2,6)
plot(u(:,6),'o-')
xlabel('x-axis')
ylabel('Solution')
title('t=4.1667')
subplot(4,2,7)
plot(u(:,7),':')
```

```
xlabel('x-axis')
ylabel('Solution')
title('t=5.0000')
figure
surf(x(1:7),t,(real(u(1:7,:)))')
xlabel('x axis'); ylabel('t axis');
zlabel('Solution')
title('Numerical Solution')
```

18. MATLAB Program of Figures 6.4

```
Clc;  clear
u=Exprom62(6,5,5,5)
subplot(3,2,1)
plot(u(:,1),'o-')
title('t=0.000')
xlabel('x-axis')
ylabel('Solution')
subplot(3,2,2)
plot(u(:,2),'o-')
title('t=1.000')
xlabel('x-axis')
ylabel('Solution')
subplot(3,2,3)
plot(u(:,3),'r')
title('t=2.000')
xlabel('x-axis')
ylabel('Solution')
subplot(3,2,4)
plot(u(:,4),'r')
title('t=3.000')
xlabel('x-axis')
ylabel('Solution')
subplot(3,2,5)
plot(u(:,5),':')
title('t=4.000')
xlabel('x-axis')
ylabel('Solution')
subplot(3,2,6)
plot(u(:,6),':')
title('t=5.000')
xlabel('x-axis')
ylabel('Solution')
figure
subplot(2,2,1)
x=linspace(0,6,6);
t=linspace(0,5,6);
u=ExProm62(6,5,5,5)
surf(x,t,u');
xlabel('x-axis')
```

```
ylabel('t-axis')
zlabel('Solution')
title('r=0.4823')
subplot(2,2,2)
x=linspace(0,6,5);
t=linspace(0,5,5);
u=ExProm62(6,5,4,8)
surf(x,t,u(:,1:5)');
xlabel('x-axis')
ylabel('t-axis')
zlabel('Solution')
title('r=0.1975')
subplot(2,2,3)
x=linspace(0,6,6);
t=linspace(0,5,6);
u=ExProm62(6,5,5,8)
surf(x,t,u(:,1:6)');
xlabel('x-axis')
ylabel('t-axis')
zlabel('Solution')
title('r=0.1884')
subplot(2,2,4)
x=linspace(0,6,6);
t=linspace(0,5,6);
u=ExProm62(6,5,5,18)
surf(x,t,u(:,1:6)');
xlabel('x-axis')
ylabel('t-axis')
zlabel('Solution')
title('r=0.0372')
```

19. MATLAB Program of Figures 6.5

```
Clc; clear
u=Improm62(6,5,5,5)
subplot(3,2,1)
plot(u(:,1),'o-')
title('t=0.000')
xlabel('x-axis')
ylabel('Solution')
subplot(3,2,2)
plot(u(:,2),'o-')
title('t=1.000')
xlabel('x-axis')
ylabel('Solution')
subplot(3,2,3)
plot(u(:,3),'b')
title('t=2.000')
xlabel('x-axis')
ylabel('Solution')
```

```
subplot(3,2,4)
plot(u(:,4),'r')
title('t=3.000')
xlabel('x-axis')
ylabel('Solution')
subplot(3,2,5)
plot(u(:,5),'o-')
title('t=4.000')
xlabel('x-axis')
ylabel('Solution')
subplot(3,2,6)
plot(u(:,6),':')
title('t=5.000')
xlabel('x-axis')
ylabel('Solution')
figure
x=linspace(0,6,6);
t=linspace(0,5,6);
subplot(2,2,1)
u=ImProm62(6,5,5,5)
surf(x,t,u');
xlabel('x-axis')
ylabel('t-axis')
zlabel('Solution')
title('r=0.4823')
subplot(2,2,2)
x=linspace(0,6,6);
t=linspace(0,5,6);
u=ImProm62(6,5,5,8)
surf(x,t,u(:,1:6)')
xlabel('x-axis')
ylabel('t-axis')
zlabel('Solution')
title('r=0.1884')
subplot(2,2,3)
x=linspace(0,6,5);
t=linspace(0,5,5);
u=ImProm62(6,5,4,8)
surf(x,t,u(:,1:5)');
xlabel('x-axis')
ylabel('t-axis')
zlabel('Solution')
title('r=0.0772')
subplot(2,2,4)
x=linspace(0,6,6);
t=linspace(0,5,6);
u=ImProm62(6,5,5,18)
surf(x,t,u(:,1:6)');
xlabel('x-axis')
ylabel('t-axis')
zlabel('Solution')
```

```
title('r=0.0372')
```

20. MATLAB Program of Figures 6.6

```
Clc; clear
u=Exprom63(10,5,5,5)
subplot(3,2,1)
plot(u(:,1),'o-')
legend('t=0.000',3)
xlabel('x-axis')
ylabel('Solution')
subplot(3,2,2)
plot(u(:,2),'o-')
legend('t=1.000',3)
xlabel('x-axis')
ylabel('Solution')
subplot(3,2,3)
plot(u(:,3),'b')
legend('t=2.000',1)
xlabel('x-axis')
ylabel('Solution')
subplot(3,2,4)
plot(u(:,4),'r')
legend('t=3.000',1)
xlabel('x-axis')
ylabel('Solution')
subplot(3,2,5)
plot(u(:,5),'o-')
legend('t=4.000',1)
xlabel('x-axis')
ylabel('Solution')
subplot(3,2,6)
plot(u(:,6),':')
legend('t=5.000',1)
xlabel('x-axis')
ylabel('Solution')
figure
subplot(2,2,1)
x=linspace(0,10,6);
t=linspace(0,5,6);
u=ExProm63(10,5,5,5)
surf(x,t,u');
xlabel('x-axis')
ylabel('t-axis')
zlabel('Solution')
title('n=5 & m=5')
subplot(2,2,2)
x=linspace(0,10,5);
t=linspace(0,5,5);
u=ExProm63(10,5,4,8)
```

```
surf(x,t,u(:,1:5)');
xlabel('x-axis')
ylabel('t-axis')
zlabel('Solution')
title('n=4 & m=8')
subplot(2,2,3)
x=linspace(0,10,6);
t=linspace(0,5,6);
u=ExProm63(10,5,5,8)
surf(x,t,u(:,1:6)');
xlabel('x-axis')
ylabel('t-axis')
zlabel('Solution')
title('n=5 & m=8')
subplot(2,2,4)
x=linspace(0,10,6);
t=linspace(0,5,6);
u=ExProm63(10,5,5,18)
surf(x,t,u(:,1:6)');
xlabel('x-axis')
ylabel('t-axis')
zlabel('Solution')
title('n=5 & m=18')
```

21. MATLAB Program of Figures 6.7

```
Clc; clear
u=Exprom64(6,5,2,5)
subplot(3,2,1)
plot(u(:,1),'o-')
legend('t=0.000',1)
xlabel('x-axis')
ylabel('Solution')
subplot(3,2,2)
plot(u(:,2),'o-')
legend('t=1.000',2)
xlabel('x-axis')
ylabel('Solution')
subplot(3,2,3)
plot(u(:,3),'b')
legend('t=2.000',2)
xlabel('x-axis')
ylabel('Solution')
subplot(3,2,4)
plot(u(:,4),'r')
legend('t=3.000',1)
xlabel('x-axis')
ylabel('Solution')
subplot(3,2,5)
plot(u(:,5),'o-')
```

```
legend('t=4.000',1)
xlabel('x-axis')
ylabel('Solution')
subplot(3,2,6)
plot(u(:,6),':')
legend('t=5.000',1)
xlabel('x-axis')
ylabel('Solution')
figure
subplot(2,2,1)
x=linspace(0,6,3);
t=linspace(0,5,3);
u=ExProm64(6,5,2,5)
surf(x,t,u(:,1:3)');
xlabel('x-axis')
ylabel('t-axis')
zlabel('Solution')
title('n=2 & m=5')
subplot(2,2,2)
x=linspace(0,6,3);
t=linspace(0,5,3);
u=ExProm64(6,5,2,7)
surf(x,t,u(:,1:3)');
xlabel('x-axis')
ylabel('t-axis')
zlabel('Solution')
title('n=2 & m=7')
subplot(2,2,3)
x=linspace(0,6,3);
t=linspace(0,5,3);
u=ExProm64(6,5,2,10)
surf(x,t,u(:,1:3)');
xlabel('x-axis')
ylabel('t-axis')
zlabel('Solution')
title('n=2 & m=10')
subplot(2,2,4)
x=linspace(0,6,3);
t=linspace(0,5,3);
u=ExProm64(6,5,2,15)
surf(x,t,u(:,1:3)');
xlabel('x-axis')
ylabel('t-axis')
zlabel('Solution')
title('n=5 & m=15')
```

22. MATLAB Program of Figure 6.7

```
Clc; clear
subplot(2,2,1)
```

```
[x,t]=meshgrid(0:0.5:10,0:0.25:5);
u=sin(x).*cos(0.05*t)+x.^2.*t;
surf(x,t,u)
xlabel('x-axis')
ylabel('t-axis')
zlabel('Solution')
title('Aanlytical Method')
subplot(2,2,2)
x=linspace(0,10,21);
t=linspace(0,5,21);
u=Exprom61(0.05,10,5,20,20);
surf(x,t,u');
xlabel('x-axis')
ylabel('t-axis')
zlabel('Solution')
title('Explicit Method')
subplot(2,2,3)
x=linspace(0,10,21);
t=linspace(0,5,21);
u=Improm61(0.05,10,5,20,20);
surf(x,t,u');
xlabel('x-axis')
ylabel('t-axis')
zlabel('Solution')
title('Implict Method')
subplot(2,2,4)
x=linspace(0,10,21);
t=linspace(0,5,21);
u=Spectral_Beam(0.0005,10,5,20,20);
surf(x,t,(real(u(1:21,:)))')
xlabel('x-axis')
ylabel('t-axis')
zlabel('Solution')
title('Spectral Method')
```

APPENDIX C

C: Some Images from Concrete Lab.

Figure: Casting of beams specimens

Figure: Crack pattern for de-bonding failure

Figure: Beam with Bridge

Published Academic Articles during PhD Period

1. Khaled A. Ishag, O. M. El Mekki and Emadeldeen A. Abdalrahim, Adomian's Decomposition Method of Concrete Beams Equations, Sudan Journal for Basic Sciences (SJBS),M 2018, Vol. 14.

2. Khaled A. Ishag, O. M. El Mekki and Emadeldeen A. Abdalrahim, Homotopy Perturbation Method for Solving Transverse Vibration of a Beam, Sudan Journal for Basic Sciences (SJBS),M 2019, Vol. 15.

3. Khaled A. Ishag, O. M. El Mekki and Emadeldeen A. Abdalrahim, Numerical Solution for Concrete Beams Equations, Sudan Journal for Basic Sciences (SJBS),M 2019, Vol. 15.

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