Sudan University of Science & Technology

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Conformal Mapping and its Applications

التخريط الإمتثالي وتطبيقاته

A thesis Subrited in fulillements for The Degree of PH.D in Mathematics

By

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إستهلال

قَالَ تَعَالَىٰ:

﴿ يَرۡفَعِ ٱللَّهُ ٱلَّذِينَ ءَامَنُواْ مِنكُمۡ وَٱلَّذِينَ أُوتُواْ ٱلۡعِلۡمَ دَرَجَٰتٖۚ وَٱللَّهُ بِمَا تَعۡمَلُونَ خَبِيرٞ ١١ ﴾

صدق الله العظيم

(سورة المجادلة الآية - 11)

Dedication

To soul of my Mother,

To my Father,

To my lovely wife,

To my sons Mohammed, Abdelbari , Tamim, and to my daughter hidaya,

To all who helped me ...

ACKNOWLEDGEMENTS

First and foremost I would like to thank my God how gave me love and grace and gave me the resolve to finish this thesis.

Beyond that, I would like to thank my supervisor Prof. Mohamed Ali Bashir for all his guidance and support throughout this work and for doing so much to mature my approach to mathematics and physics.

Last, but by no means least I owe a great debt to my father, my wife, my sons and my daughter.

Abstract

Complex analysis and conformal mapping play a central role in mathematical sciences and theoretical physics. The traditional applications include differential equations, harmonic analysis, potential theory and fluid mechanics. Of particular interest to this study is the complexfied Minkowski space and its corresponding spin space model which is appropriate for the description of quantum field theory. Moreover, for an ambitious scheme to incorporate gravitational field in a quantized form, we introduced the three-dimensional complex projective space as an advanced model whereby points of the complexified Minkowski space are not prime but secondary. In the light of Penrose correspondence these points are complex lines in Twistor space. It has been shown that the conformally invariant zero-rest mass fields, such as weak gravitational field, are represented by contour integrals of holomorphic functions on twistor space.

المستخلـص

لقد لعب كل من التحليل المركب والتحويل الإمتثالي دوراً مركزياً في العلوم الرياضية والفيزياء النظرية. من التطبيقات التقليدية لذلك المعادلات التفاضلية ، التحليل التوافقي ، نظرية الطاقة الكامنة وميكانيكا الموانع. على وجه الخصوص فإن فضاء منكاوسكي المركب وفضاء الغزل المقابل ، والملائم لوصف الظواهر الكمية ، يُشكلان إهتماماً لهذه الدراسة. بالإضافة إلى ذلك ، ومن أجل خطة طموحة تستوعب المجال التثاقلي في صياغة مكممة فقد تمّ تقديم فضاء إسقاطي مركب ذي ثلاثة أبعاد ، حيث يمثل هذا الفضاء نموذجاً متقدماً على فضاء منكاوسكي المركب. في هذا الفضاء تكون نقاط فضاء منكاوسكي المركب ثانوية لا أولية. وعلى ضوء تقابل بنروس فإن هذه النقاط تقابل خطوط مركبة في فضاء توستر. لقد بينا أن المجالات ذات الكتل الصفرية عند السكون والتي تكون ثابتة تحت التحويلات الإمتثالية كالمجال التثاقلي الضعيف ، يمكن تمثيلها بتكاملات كنتورية لدوال مركبة على فضاء توستر.

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Introduction

The role of complex analysis and complex algebra in mathematical sciences can hardly be exaggerated. It was already used to facilitate solutions of algebraic and differential equations. For instance complex integration handles some integrals which are hard to solve on real spaces. The concept of the analytic function has also been used in describing smooth variations in a general way and also integrate many problems.

Complex analysis has been used in fluid dynamics and harmonic analysis. In particular conformal mapping was utilized to transform problems, usually differential equations, to conformal domains that facilitate the solution of many problems. Following this strategy, we generalized the utilization of conformal mapping to physical fields. In our thesis we study the extension of the applications of conformal mapping to particular physical fields, the so called zero-rest-mass fields. The conformality is apparently manifested via using the complex analysis and complex spaces, such as complex projective space, whose geometrical structure naturally reflects the concept of conformality.

A conformal rescaling, consists solely of a replacement

$$g_{ab} \rightarrow \hat{g}_{ab} = \Omega^2 \hat{g}_{ab}$$

Of the space time metric g_{ab} by a conformally-related one $\hat{g}_{ab}$, Ω being a smooth positive scalar filed on the underlying manifold. Thus the interval ds are transformed to$d\hat{s} = \Omega ds$. If g_{ab} is a flat metric, then $\hat{g}_{ab}$ will in general not be flat, though it will of course be conformally flat. The conformal rescaling of a given space form infinite-parameter a belian group. The points of the space time are unaffected by a conformal rescaling. The null cones, and, therefore, the causal structure of the space time, are also unchanged.

Conformal mappings of Minkowski space M to itself have a particular interest. These include the Poincare transformations, which are metric preserving, and the simple overall dilations, whose corresponding rescaling multiplies the metric at each point by a constant factor. The remainders are generated by the (involuntary) inversiotis

$$\tilde{x} = -x^a (x^a x_b)^{-1} \quad , \quad \tilde{x} = -\tilde{x}^a (\tilde{x}^a \tilde{x}_b)^{-1}$$

Which are a 4- parameter set since the choice of origin is arbitrary. These transformations preserve the time sense but involve spatial reflection. They are conformal mapping since the induced and original metrics are related by:

$$d\hat{s}^2 = d\hat{x}^a d\hat{x}_a = \frac{dx^a dx_a}{(x^b x_b)^2} = \Omega^2 ds^2$$

However, these transformations do not involve only the points of M', because the null cone of the origin is sent to infinity. We therefore introduce compactified Minkowski space M, which consist of M together with closed null cone infinity. We may picture the structure in term of two cone joined base to base, the interior being M and the two bounding cones being identified along opposite generators with future sense preserved. (Thus the "equator" I_0 must be considered as a single point). For fuller discussion of the structure of M. note that one can consider the equations as expressing a coordinate change, rather than a point transformation, on M; and that the null cone at infinity is the same footing as any other null cone in M as far as conformal mapping symmetry is concerned. In consequence of the latter, the concept of radiation is not conformal invariant, since it depends on knowing where infinity.

The conformal mapping group of M is of 15 parameter (and non-abelian). We shall here concern ourselves with the restricted conformal group, i.e. the subgroup of mappings connected with the Compactified Minkowski space M, $I^0, I^*, I^-$are point at spatial infinity, future time infinity and past time infinity respectively, while I^+, I^- are future and past null infinity cones. The compactified space has I^0, I^+, I^-identified and I^+, I^-identified along opposite generator for typographical reasons," I" replaces the more usual script depicted.

These do not include the actual mappings but does include their products with space reflections. It is 2-1 covered by (and so locally isomorphic with) the six-dimensional pseudo-orthogonal group SO (2,4) which in turn is 2-1 covered by SU (2.2) group of unimodular pseudo-unitary matrices.

The infinitesimal conformal motion are described by the conformal killing vector r and are given, for infinitesimal ε by

$$x^a \to x^a + \epsilon \xi^a$$

The vector ξ^amust satisfy

$$\nabla_{(a\xi_b)} = \frac{1}{4} g_{ab}(\nabla_c \xi^c)$$

The general solution of this

$$\xi_a = S_{ab}x_b + T_a + Q_a(x^c x_c) - 2x_a(x^c Q_c) + Rx_a$$

Where $S_{ab} = S_{[ab]}$ generate the Lorentz rotations (6 parameters), T_a the translation (4 parameters), R the dilation (1 parameter) and$Q_a$the so-called "uniform acceleration" transformation (4 parameters). (This terminology is rather misleading, however, and will be avoided here Amore correct use of the terminology "uniform acceleration is for a coordinate transformation which makes the Minkowski metric take the form.

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

Questions of conformal invariance are handled most easily within framework of conformal rescaling rather than conformal mappings .Physical theory will be said to be conformal invariant if it is possible to attach conformal weights to all the quantities appearing in the theory in such a way that all field equations are preserved under conformal rescaling. (A tensor or spinor A is said to have conformal weight rescaling γ under the 15-parameter conformal group. this is because the Poincare motions of Minkowski space become conformal motions according to any other conformal rescaled flat metric. Conformal motions obtainable in this way are sufficient to generate the full conformal group. But the type of conformal invariance described above is really more general than this since the conformal rescaling need not apply to flat space-time at all or even to conformal flat space times.

In order to establish conformal invariance of a theory, one needs to know how to transform the (covariant) derivative operator under conformal rescaling. Remarkably enough, this is rather simpler within the two-component spinor formalism than within the tensor formalism.since two-component spinsor will also play an essential role in other aspects of twistor theory, briefly summaries' the relevant notation and methods.

In this thesis the solution of zero-rest-mass field appeared as a complex function in four-dimensional complexified Minkowski space or in complex three-dimensional projective space.We have achieved the goal of this theirs via five chapters.

In chapter one, we have reviewed complex analysis, starting from algebra and differential analysis of complex functions and then we proceeded to complex integration of these functions, where we have compared some analytic properties to those of the real case.

Chapter two gave a short study of conformal mapping, providing some theorems and examples. Also we refered to the significance of the conformal properties in solving differential equations. Thus we have introduced in some sense harmonic analysis.

In chapter three we aimed at extending the utility of the conformal mapping to physical fields. So we first studied the structure of physical fields in space-time, the variational principle and the resulting Euler-Lagrange equations. One then considered the effect of conformal transformations on these fields. This effect is physically accepted.

We then thought of studying Zero-rest mass fields as conformal fields. We found out that the most appropriate space for such a study is the so called Twistor space. For this reason we have introduced Radan transform and John transform in chapter four, as an initiation trans that chapter justified our later consideration of Penrose transform.

Lastly in chapter five we introduced Twistor theory. We have shown that the complexified Minkowski space can be modeled by the complex projective three-dimensional space. We have arrived at Penrose correspondence whereby a point in Minkowski space is not prime, it is secondary and this infact appropriate to quantum effects. The Penrose correspondence is used to construct Penrose transform, where the later has been utilized to transform Zero-rest mass field to algebraic data in complex three-dimensional space which is the first cohomology class.

Chapter One

Analytic Functions on the Complex Plane

1. Complex Differentiation

1.1. Functions of a Complex Variable

Let S be a set of complex numbers. A *function f* defined on a sets S of complex numbers is a rule that assigns to each z in S a complex number w. the number w is called the *value* of *f* at *z* and is denoted by *f(z);* that is, $\omega = f(z)$. The set S is called the *domain of definition* of *f* (z) .Z can be expressed interms of a pair of real-valued function of the real variables *x* and *y* as

$$z = x + iy \tag{1.1}$$

And $f(z)$ is can be expressed as:

$$f(z) = u(x, y) + iv(x, y) \tag{1.2}$$

Where u and v are functions of x and y.

If polar coordinates *r* and θ, instead of *x* and *y*, are used, then:

$w = f\left(re^{i\theta}\right) = u + iv.$

Where $z = re^{i\theta}$. In that case, we may write

$$f(z) = u(r, \theta) + iv(r, \theta) \tag{1.3}$$

Example 1.1.1: If $f(z) = z^2$, then

$$f(x + iy) = (x + iy)^2 = x^2 - y^2 + i2xy.$$

Hence

$$u(x, y) = x^2 - y^2 \text{ and } v(x, y) = 2xy$$

When polar coordinates are used,

$$f\left(re^{i\theta}\right) = \left(re^{i\theta}\right)^2 = r^2 cos2\,\theta + ir^2 sin2\,\theta.$$

Consequently,

$$u(r,\theta) = r^2 cos2\theta \text{ and } v(r,\theta) = r^2 sin2\theta.$$

Ifthe function *v*always has value zero, then*f* is a *real-valued function* of a complex variable.

Example 1.1.2: Areal-valued function that is used to illustrate some important concepts later in this chapter is:

$$f(z) = |z|^2 = x^2 + y^2 + i0.$$

If *n* is zero or a positive integer, and if $a_0, a_1, a_2, \ldots, a_n$ are complex constants, where $a_n \neq 0$, the function

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \tag{1.4}$$

Is a *polynomial* of degree *n*. note that the sum here has a finite number of terms and that the domain of definition is the entire z plane. Quotients $P(z)/Q(z)$ of polynomials are called *rational functions* and are defined at each point z where $Q(z) \neq 0$. Polynomials and rational functions constitute elementary, but importnt, classes of functions of a complex variable.

A generalization of the concept of function is the rule that assigns more than one value to a point z in the domain of definition. The submultiple-*valued functions* occur in the theory of functions of a complex variable, just as they do in the case of real variables. When multiple-valued functions are studied, usually just one of the possible values assigned to each point is taken, in a systematic manner, and a (single-valued) function is constructed from the multiple-valued function.

Example 1.1.3: Let z denote any nonzero complex number. Then $z^{1/2}$ has the two values

$$z^{1/2} = \pm\sqrt{r} exp\left(i\frac{\Theta}{2}\right) \tag{1.5}$$

Where $r = |z|$ and $\Theta: (-\pi < \Theta \leq \pi)$ is the *principal value* of arg z. but, if we choose only the positive value of $\pm\sqrt{r}$ and write

$$f(z) = \sqrt{r}\exp\left(i\frac{\Theta}{2}\right)(r > 0, -\pi < \Theta \leq \pi).$$

The (single-valued) function is well defined on the set of nonzero numbers in the z plane. Since zero is the only square root of zero, we also write$f(0) = 0$. The function f is then well defined on the entire plane.

1.2 Mappings

The properties of a real-valued function of a real variable are often exhibited by the graph of the function. But when w = *f(z)*, no such convenient graphical representation of the function *f* is available, because each of the numbers *z* and *w* is located in a plane rather than on a line. One can, however, display some information about the function by indicating pairs of corresponding points z = (*x, y*) and *w* = (*u, v*). To do this, it is generally simpler to draw the *z* and *w* planes separately.

When a function *f* is thought of in this way, it is often referred to as a *mapping*, or transformation. The *image* of a point *z* in the domain of definition *S* is the point *w* = *f (z)* and the set of images of all points in a set *T* that is contained in *S and* is called the *range* of the image to *T*. The image of the entire domain of definition *S* is called the *range* of *f*. The *inverse image* of a point *w* is the set of all points *z* in the domain of definition of *f* that have *w*as their image.
The inverse image of a point may contain just one point, many points, or none at all. The last case occurs, of course, when *w* is not in the range of *f*.

Terms such as *translation*, *rotation*, and *reflection* are used to convey dominant geometric characteristics of certain mappings. In such cases, it is sometimes convenient to consider *z* and *w* to be the same. for example, the mapping

$$w = f(z) = z + 1 = (x + 1) + iy,$$

Where $z = x + iy$,can be thought of as a translation of each point z one unit to the right .Since $i = e^{i\pi/2}$, the mapping

$$w = f(z) = iz = rexp\left[i\left(\theta + \frac{\pi}{2}\right)\right],$$

Where $z = re^{i\theta}$, rotates the radius vector for each nonzero point z through

a right angle about the origin in the counterclockwise direction; and the mapping

$$w = \bar{z} = x - iy \qquad (1.6)$$

Transforms each point $z = x + iy$ into its reflection in the real axis.

The complex exponential function is given by:

$$e^x = e^x \cos y + ie^x \sin y \tag{1.7}$$

The complex trigonometric functions are then given by:

$$\cos z = \frac{e^{iz}+e^{-iz}}{2} \tag{1.8}$$

$$\sin z = \frac{e^{iz}-e^{-iz}}{2} \tag{1.9}$$

also the logarithmic complex function has the form:

$$\log z = \log re^{i\theta} = \log r + i\theta \tag{1.10}$$

More information is usually exhibited by sketching images of curves and regions than by simply indicating images of individual points. In the following examples, we illustrate this with the transformation $w = z^2$.

We begin by finding the images of some curves in the z plane.

Example 1.2.1:

The mapping $w = z^2$ can be thought of as the transformation

$$u = x^2 - y^2, \qquad v = 2xy \tag{1.11}$$

From the *x*- *y* plane to the *u v* plane. This form of mapping is especially useful in finding the images of certain hyperbolas.

It is easy to show, for instance, that each branch of a hyperbola

$$x^2 - y^2 = c_1 (c_1 > 0) \tag{1.12}$$

is mapped in a one manner onto the vertical line $u = c_1$. We start by noting from the first of equations (1.11) that u = *c* when (*x*, *y*) is a point lying on either branch. When, in particular, it lies on the right-hand branch, the second of equation (1.11) tells us that $v = 2y\sqrt{y^2 + c_1}$. Thus, the image of the right-hand branch can be expressed parametrically as.

$$u = c_1, \qquad v = 2y\sqrt{y^2 + c_1}\,(-\infty < y < \infty);$$

And it is evident that the image of a point (*x*, *y*) on that branch moves upward along the entire line as (*x*, *y*) traces out the branch in the upward direction, since the above pair of equations furnishes a parametric representation for the image of the left-hand branch of the hyperbola, the image of a point going *downward* along the entire left-hand branch is seen to move up the entire line $u = c_1$.

On the other hand, each branch of a hyperbola

$$2xy = c_2 \qquad (c_2 > 0)$$

Transformed into the line $v = c_2$ as indicated in Fig. (1). To verify this, we note from the second of equations (1.11) that $v = c_2$ when (*x*, *y*) is a point on either branch. Suppose

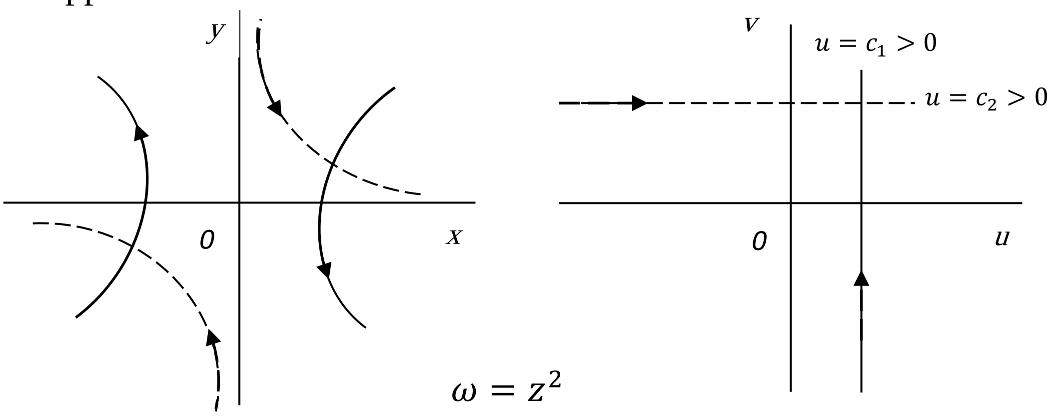

Figure (1)

That it lies on the branch lying in the first quadrant. Then, since

$y = c_2/(2x)$, the first of equations (1.11) reveals that the branch's image has parametric representation

$$u = x^2 - \frac{c_2^2}{4x^2}\ , \qquad v = c_2 (0 < x < \infty) \tag{1.13}$$

Observe that

$$\lim_{\substack{x \to 0 \\ x > 0}} u = -\infty \ and \ \lim_{x \to \infty} u = \infty$$

Since *u* depends continuously on *x*, then, it is clear that as (*x*, *y*) travels down the entire upper branch of hyperbola (1.13), its image moves to the right along the entire horizontal line $v = c_2$. In as much as the image of the lower branch has parametric representation

$$u = \frac{c_2^2}{4y^2} - y^2, \quad v = c_2 \qquad (-\infty < y < 0)$$

And since

$$\lim_{\substack{x\to 0\\ x>0}} u = -\infty \; and \; \lim_{\substack{x\to 0\\ y<0}} u = \infty.$$

1.3. Mappings by the exponential function

We shall start with the exponential function

$$e^z = e^x e^{iy} \; (z = x + iy), \tag{1.14}$$

It is suggested by the familiar property

$$e^{x_1 + x_2} = e^{x_1} e^{x_2}$$

Of the exponential function in calculus.

The object of this section is to use the function e^z to provide additional examples of simple mappings. We begin by examining the images of vertical and horizontal lines.

Example1.3.1: The transformation

$$w = e^z \tag{1.15}$$

Can be written as $w = e^x e^{iy}$, or equivalently $w = \rho e^{i\theta}$. Thus,

$\rho = e^x$ and $\theta = y + 2n\pi$, where *n* is some integer and (1.15) can be expressed in the form

$$\rho = e^x, \qquad \theta = y \tag{1.16}$$

The image of a typical point $z = (c_1, y)$ on a vertical line $x = c_1$ has polar coordinates $\rho = \exp c_1$ and $\theta = y$ in the new plane. That image moves counterclockwise around the circle shown in Fig. (2) as *z* moves up the line. The image of the line is evidently the entire circle; and each point on the circle is the image of an infinite number of points, spaced 2π units apart, along the line.

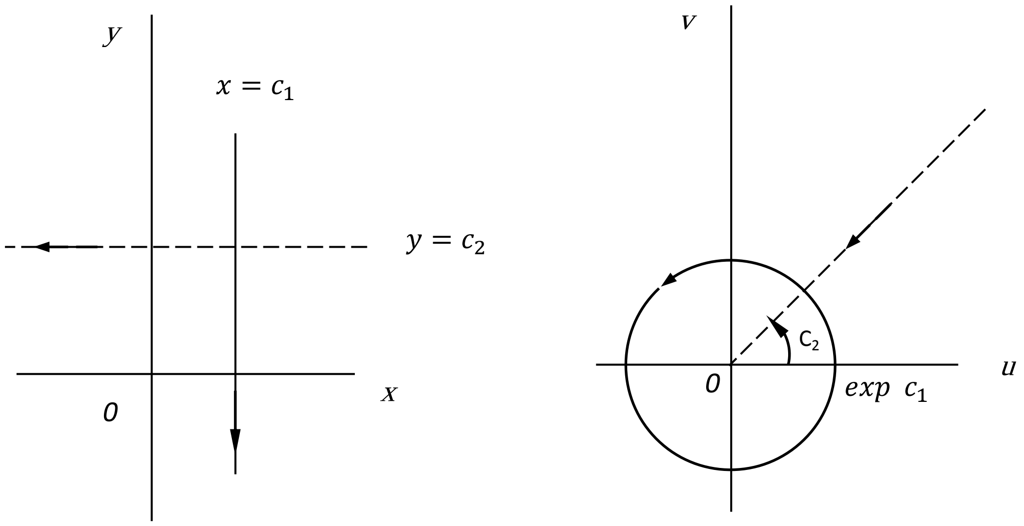

Figure (2)

A horizontal line $y = c_2$is mapped in a one to one manner onto the ray

$\emptyset = c_2$. To see that this so, we note that the image of a point $z = (x, c_2)$ has polar coordinates $\rho = e^x$and $\emptyset = c_2$. Evidently, then, as that point z moves along the entire line from left to right, its image moves outward along the entire ray $\emptyset = c_2$, as indicated in Fig (2).

Vertical and horizontal line *segments* are mapped onto portions of circles and rays, respectively, and images of various regions are readily obtained from observations made in Example (1.2.1). This is illustrated in the following example.

Example 1.3.2: Let us show that the transformation $w = e^z$ maps the rectangular region: $a \leq x \leq b \quad , \quad c \leq y \leq d$ onto the region

$e^a \leq \rho \leq e^b, c \leq \theta \leq d$. The vertical line segment *AD* is mapped onto the arc $\rho = e^a, c \leq \theta \leq d$, which is labeled *A'D'*. The image of vertical line segments to the right of *AD* and joining the horizontal parts of the boundary are larger arcs; eventually, the image of the line segment *BC* is the arc $\rho = e^b$,

$c \leq \emptyset \leq d$, labeled $B'C'$. The mapping is one to one if $d - c < 2\pi$. In particular, if $c = 0$ and $d = \pi$, then $0 \leq \theta \leq \pi$, and the rectangular region is mapped onto half of a circular ring.

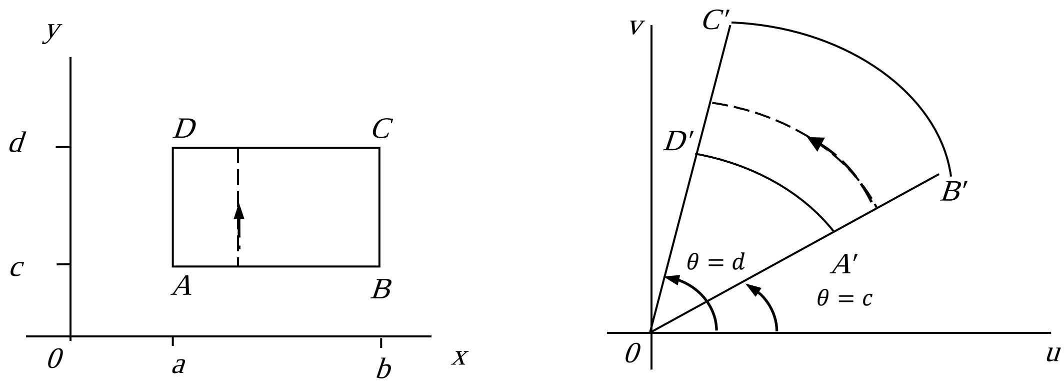

Figure (3) w = *exp z.*

Our final example here uses the images of *horizontal* lines to find the image of a horizontal strip.

Example 1.3.3: When $w = e^z$, the image of the infinite strip $0 \leq y \leq \pi$ is the upper half $v \geq 0$ of the plane (Fig. 2). This is seen by recalling from example 1.3.2 how a horizontal line $y = c$ is transformed into a ray $\theta = c$ from the origin.

As the real number c increases from $c = 0$ to $c = \pi$, the y intercepts of the lines increase from 0 to π and the angles of inclination of the rays increase from $\theta = 0$ to $\theta = \pi$.

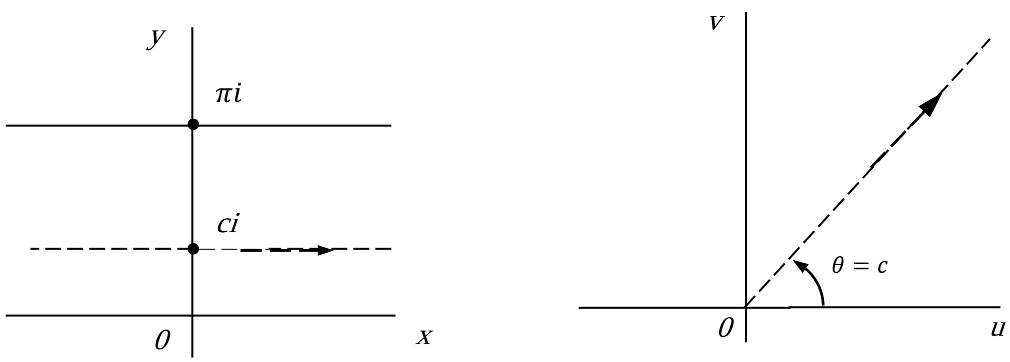

Figure (4)w = *exp z*

1.4 Differentiability of a complex function

Definition 1.4.1: Acomplex functions $f(z)$ is differentiable at a point $z \in$C if and only if the following limiting difference quotient exists:

$$f'(z) = \lim_{w \to z} \frac{f(w) - f(z)}{w - z} \tag{1.17}$$

The key feature of this definition is that the limiting value $f^{'}(z)$ of the difference quotient must be independent of how ω converges to z. On the real line, there are only two directions to approach a limiting point-either from the left or from the right. These lead to the concepts of left-and right-handed derivatives and their equality is required for the existence of the usual derivative of a real function. In the complex plane, there are an infinite variety of directions approaching the point$z$, and the definition requires that all.

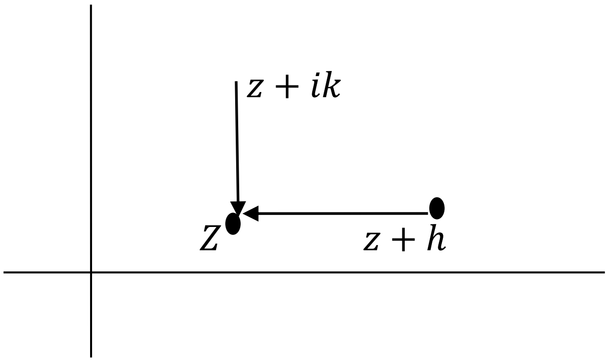

Figure (1.5) Complex Derivative Directions.

Of these ''directional derivatives'' must agree. This requirement imposes severe restrictions on complex derivatives, and is the source of their remarkable properties. To understand the consequences of this definition, let us first see what happens when we approach z along the two simplest directions_ vertical and horizontal. If we set

$$w = z + h = (x + h) + iy, \text{ where h is real}$$

Then w→ z along a horizontal line as h→ $o,$ as sketched in figure 1.5.

If we write out

$$f(z) = u(x, y) + iv(x, y)$$

In terms of its real and imaginary parts, then we must have

$$f'(z) = \lim_{h\to o} \frac{f(z+h)-f(z)}{h} = \lim_{h\to o} \frac{f(x+h+iy)-f(x+iy)}{h} = \lim_{h\to o} \left[\frac{u(x+h,y)-u(x,y)}{h} + i\,\frac{v(x+h,y)-v(x,y)}{h}\right] = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = \frac{\partial f}{\partial x},$$

Which follow from the usual definition of the (real) partial derivative.

On the other hand, if we set

$$w = z + ik = x + i(y + k), \text{ Where k is real}$$

Then w→ z along a vertical line as k→ o.Therefore, we must also have

$$f'(z) = \lim_{k\to o} \frac{f(z+ik)-f(z)}{ik} = \lim_{k\to o} \left[-i\frac{f\big(x+i(y+k)\big)-f(x+iy)}{k}\right] =$$
$$\lim_{k\to o} \left[\frac{v(x,y+k)-v(x,y)}{k} - i\frac{u(x,y+k)-u(x,y)}{k}\right] = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = -i\frac{\partial f}{\partial y}.$$

When we equate the real and imaginary parts of these two distinct formulae for the complex derivative $f'(z)$, we discover that the real and imaginary components of $f(z)$ must satisfy a certain homogeneous linear system of partial differential equations, named after Augustine-Louis Cauchy and Bernhard Riemann, two of the founders of modern complex analysis.

Theorem 1.4.2:A complex function $f(z) = u(x,y) + iv(x,y)$ depending on $z = x + iy$ has a complex derivative $f'(z)$ only if its real and imaginary parts are continuously differentiable and satisfy the Cauchy-Riemann equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{1.18}$$

In this case, the complex derivative of $f(z)$is equal to any of the following expressions:

$$f'(z) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial f}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \tag{1.19}$$

Satisfaction of the Cauchy-Riemann equations at a point $z_0 = (x_0, y_0)$ is not sufficient to ensure the existence of the derivative of a function $f(z)$ at that point. But, with certain continuity conditions, we have the following useful theorem.

Theorem.1.4.3: Let the function

$$f(z) = u(x,y) + iv(x,y)$$

be defined throughout some ε neiborhood of a point $z_0 = x_0 + iy_0$, and suppose that the first-order partial derivatives of the functions u and v with respect to x and y exist everywhere in that neighborhood. If those partial derivatives are continuous at (x_0, y_0) and satisfy the Cauchy-Riemann equations

$$u_x = v_y, \qquad u_y = -v_x$$

At (x_0, y_0), then $f'(z_0)$ exists.

Proof: we write $\Delta z = \Delta x + i\Delta y,\ where\ 0 < |\Delta z| < \varepsilon$, and

$$\Delta w = f(z_0 + \Delta z) - f(z_0).$$

Thus:

(1)$\Delta w = \Delta u + i\Delta v,$

Where:

$$\Delta u = u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)$$

And

$$\Delta v = v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0).$$

The assumption that the first-order partial derivatives of u and v are continuous at point(x_0, y_0)enables us to write

(2)$\Delta u = u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \varepsilon_1\sqrt{(\Delta x)^2 + (\Delta y)^2},$

And

(3)$\Delta v = v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \varepsilon_2\sqrt{(\Delta x)^2 + (\Delta y)^2},$

Where:

ε_1 andε_2 tend to 0 as $(\Delta x, \Delta y)$ approaches (0, 0) in the Δz plane.

Substitution of expressions (2) and (3) into equation (1) now tells us that

(4) $\Delta w = u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \varepsilon_1\sqrt{(\Delta x)^2 + (\Delta y)^2}$

$$+i\left[v_x(v_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \varepsilon_2\sqrt{(\Delta x)^2 + (\Delta y)^2}\right].$$

Assuming that the Cauchy-Riemann equations are satisfied at (x_0, y_0), we can replace u_y (x_0, y_0) by $-v_x(x_0, y_0)$ and $v_y(x_0, y_0)$ by $u_x(x_0, y_0)$ in equation (4) and then divide through by Δz to get

(5) $\frac{\Delta w}{\Delta z} = u_x(x_0, y_0) + iv_x(x_0, y_0) + (\varepsilon_1 + i\varepsilon_2)\frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\Delta z}$

But $\sqrt{(\Delta x)^2 + (\Delta y)^2} = |\Delta z|$, and so

$$\left|\frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\Delta z}\right| = 1.$$

Also, $\varepsilon_1 + i\varepsilon_2$ tends to 0 as $(\Delta x, \Delta y)$ approaches (0, 0). So, the last term on the right in equation (5) tends to 0 as the variable $\Delta z = \Delta x + i\Delta y$ tends to 0.

This means that the limit of the left-hand side of equation (5) exists and that

(6)$f'(z_0) = u_x + iv_x$,

Where the right hand side is to be evaluated at (x_0, y_0).

Example 1.4.4: Consider the exponential function

$$f(z) = e^x = e^x e^{iy} \quad (z = x + iy),$$

In view of Euler's formula, this function can, of course, be written

$$f(z) = e^x cosy + ie^x siny,$$

Where y is to be taken in radians when cos y and sin y are evaluated. Then

$u(x, y) = e^x cosy, \quad v(x, y) = e^x siny.$

Since $u_x = v_x$ everywhere and since these derivatives are everywhere continuous, the conditions in the theorem are satisfied at all points in the complex plane. Thus $f'(z)$ exists everywhere, and

$$f(z) = u_x + iv_x = e^x cosy + ie^x siny,$$

Note that$f'(z) = f(z)$.

Example 1.4.5:

It also follows from the theorem in this section that the function$f(z) = |z|^2$, whose components are

$$u(x, y) = x^2 + y^2 \; and \; v(x, y) = 0,$$

Has a derivative at z = 0. In fact, $f'(0) = 0 + i0 = 0$, However that this function *cannot* have a derivative at any nonzero point since the Cauchy-Riemann Equations are not satisfied at such points.

Remark: It is worth pointing out that Cauchy Riemann equations (1.18) imply that f satisfies $\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$, which, reassuringly, agree with the first equation in (1.19).

Example 1.4.6: Consider the elementary function

$$z^3 = (x^3 - 3xy^2) + i(3x^2y - y^3).$$

Its real part $u = x^3 - 3xy^2$ and imaginary part $v = 3x^2y - y^3$ satisfy the Cauchy -Riemann equations (1.18), sine

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -6xy = -\frac{\partial v}{\partial x}.$$

Theorem (1.4. 3) implies that $f(z) = z^3$ is complex differentiable.

Not surprisingly, its derivative turns out to be

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = (3x^2 - 3y^2) + i(6xy) = 3z^2.$$

Fortunately, the complex derivative obeys all of the usual rules in real-variable calculus. For example,

$$\frac{d}{dz}z^n = nz^{n-1}, \qquad \frac{d}{dz}e^{cz} = ce^{cz}, \qquad \frac{d}{dz}\log z = \frac{1}{z}, \tag{1.20}$$

And so on. Here, power n can be non-integral-or oven, in view of the identity $z^n = e^{n \log z}$.

The exponential formulae (1.14) for the complex trigonometric functions implies that they also satisfy the standard rules

$$\frac{d}{dz}\cos z = -\sin z, \qquad \frac{d}{dz}\sin z = \cos z \tag{1.21}$$

The formulae for differentiating sums, products, ratios, inverses, and compositions of complex functions are all identical to their real counterparts, with similar proofs.

Remark: There are many examples of seemingly reasonable functions which do not have a complex derivative. The simplest is the complex conjugate function

$$f(z) = \bar{z} = x - iy.$$

Its real and imaginary parts do not satisfy the Cauchy-Riemann equation, and hence $\bar{z}$ does not have a complex derivative. More generally, any function

$f(z, \bar{z})$ that explicitly depends on the complex conjugate variable $\bar{z}$ is not complex- differentiable.

1.5 Power Series and Analyticity

A remarkable feature of complex differentiation is that the existence of one complex derivative automatically implies the existence of infinity many! All complex function $f(z)$are infinitely differentiable and, in fact, analytic where defined. The reason for this surprising and profound fact will, however, not become evident until we learn the basics of complex integration in Section 1.9 In this section, we shall take analyticity as a given, and investigate some of its principal consequences.

Definition 1.5.1: A complex function$f(z)$ is called analytic at a point $z_0 \in \mathbb{C}$ if it has a power series expansion

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \cdots$$
$$= \textstyle\sum_{n=0}^{\infty} a_n(z - z_0)^n, \qquad (1.22)$$

That converges for all z sufficiently close to z_0.

In practice, the standard ratio or root tests for convergence of (real) series can be applied to determine where a given (complex) power series converges. We note that if $f\ (z)$ and $g\ (z)$ are analytic at a point $z_{0,}$so is their sum $f(z) + g(z)$, product $f(z)g(z)$and, provided $g(z_0) \neq 0$, ratio $f(z)\ /\ g(z)$.

Example 1.5.2: All of the real power series found in elementary calculus carry over to the complex versions of the functions. For complex,

$$e^z = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad (1.23)$$

Is the power series for the exponential function based at $z_0 = 0$.

A straightforward application of the ratio test proves that the series converges for all z. On the other hand, the power series

$$\frac{1}{z^2 + 1} = 1 - z^2 + z^4 - z^6 + \cdots = \sum_{k=o}^{\infty} (-1)^k\, z^{2k}, \qquad (1.24)$$

Converges inside the unit disk, where |z|< 1, and diverges outside, Where |z|> 1. A gain convergence is established by the ratio test.

The ratio test is inconclusive when |z|=1,and we shall leave the more delicate question of precisely where on the unit disk this complex series converges.

In general there are three possible options for the domain of convergence of a complex power series (1.22):

(a) The series converges for all z.

(b) The series converges inside a disk $|z - z_0| < p$ of radius $p > 0$ centered at z_0 and diverges for all $|z - z_0| > \rho$ outside the disk. The series may converge at some (but not all) of the points on the boundary of the disk where $|z - z_0| = \rho$.

(c) The series only converges, trivially, at $z = z_0$.

The number ρ is known as the *radius of convergence* of the series.

In case (a), we say $\rho = \infty$, while in case (c), ρ= 0 and the series does not represent an analytic function. An example that has $\rho = 0$ is the power series $\sum n!\, z^n$

Remarkably, the radius of convergence for the power series of known analytic function $f(z)$ can be determined by inspection, without recourse to any fancy convergence tests! Namely, ρ is equal to the distance from z_0 to the nearest singularity of $f(z)$, meaning apoint where the function fails to be analytic. In particular, the radius of convergence$\rho = \infty$ if and only if $f(z)$is an *entire function*, meaning that it is analytic for all $z \in \mathbb{C}$ and has no singularities; examples include polynomials, e^z, $\cos z$, and $\sin z$.

On the other hand, the rational function

$$f(z) = \frac{1}{z^2 + 1} = \frac{1}{(z + i)(z - i)}$$

Has singularities at $z = \pm i$, and so its power series (1.24) has radius of convergence $\rho = 1$, which is the distance from $z_0 = 0$ to the singularities. Thus, the extension of the theory of power series to the complex plane serves to explain the apparent mystery of why, as a real function, $(1 + x^2)^{-1}$ is well-defined and analytic for all real x, but its power series only converges on the interval$(-1,1)$. It is the complex singularities that prevent its convergence when $|x| > 1$. If we expand $(z^2 + 1)^{-1}$ in a power series at some other point, say

$z_0 = 1 + 2i$, then we need to determine which singularity is closest. We compute $|i - z_0| = |-1 - i| = \sqrt{2}$, while$|-i - z_0| = |-1 - 3i| = \sqrt{10}$, and so the radius of convergence ρ=$\sqrt{2}$ is the smaller. This allows us to determine the radius of convergence in the absence of any explicit formula for its (rather complicated) Taylor expansion at $z_0 = 1 + 2i$.

There are, in fact, only three possible types of singularities of a complex function$f(z)$:

(1) **Pole**: Asingular point $z = z_0$ is called a pole of order $0 < n \in \mathbb{z}$ if and only if

$$f(z) = \frac{h(z)}{(z - z_0)^n} \tag{1.25}$$

Where $h(z)$is analytic at $z = z_0$ and $h\ (z_0) \neq 0$.the simplest example of such a function is $f(z) = a(z - z_0)^{-n}$for a $\neq 0$ a complex constant.

(2)**Branch point**: We have already encountered the two basic types: algebraic branch points, such as the function$\sqrt[n]{z}$ at $z_0 = 0$, and logarithmic branch points such as log z at $z_0 = 0$.the degree of the branch point is n in the first case and ∞ in the second. In general, the power function $z^a = e^{a \log z}$is analytic at $z_0 = 0$ if a $\in \mathbb{z}$ is an integer; has a algebraic branch point of degree q at the origin if $a = p/q \in \mathbb{Q}\backslash\mathbb{Z}$ is rational, non-integral with $0 \neq p \in \mathbb{Z}$ and $2 \leq q \in \mathbb{Z}$ haning no common factors, and a logarithmic branch point of infinite degree at$z = 0$ when $a \in \mathbb{C}\backslash\mathbb{Q}$ is not rational.

Essential singularity: By definition. A singularity is essential if it is not a pole or a branch point. The quintessential example is the essential singularity of the function $e^{1/z}$ at $z_0 = 0$.The behaviour of a complex function near an essential singularity is quite complicated.

Example 1.5.3: The complex function

$$f(z) = \frac{e^z}{z^3 - z^2 - 5z - 3} = \frac{e^z}{(z - 3)(z + 1)^2}$$

is analytic everywhere except for singularities at the points

$z = 3$ and $z = -1$, wheren its denominator vanishes. Since

$$f(z) = \frac{h_1(z)}{z-3}, \quad \text{where} \quad h_1(z) = \frac{e^z}{(z+1)^2}$$

Is analytic at $z = 3$and $h_1(3) = \frac{1}{16}e^3 \neq 0$, we conclude that $z = 3$ is a simple (order 1) pole. Similarly,

$$f(z) = \frac{h_2(z)}{(z+1)^2}, \qquad wher\ eh_2(z) = \frac{e^z}{z-3}$$

Is analytic at $z = -1$ with $h_2(-1) = -\frac{1}{4}e^{-1} \neq 0$, we see that the point $z = -1$ is a double (order 2) pole.

A complicated complex function can have a variety of singularities.

For example, the function

$$f(z) = \frac{e^{-1(z-1)^2}}{(z^2+1)(z+2)^{2/3}} \tag{1.26}$$

Has a simple pole at $z = \pm i$, a branch point of degree 3 at $z = -2$, and an essential singularity at $z = 1$.

As in the real case, and unlike Fourier, convergent power series can always be repeatedly term-wise differentiated. Therefore, given the convergent series (1.22), we have the corresponding series

$$f'(z) = a_1 + 2a_2(z-z_0) + 3a_3(z-z_0)^2 + 4a_4(z-z_0)^3 + \cdots$$

$$= \sum_{n=0}^{\infty}(n+1)a_{n+1}(z-z_0)^n,$$

$$f''(z) = 2a_2 + 6a_3(z-z_0) + 12a_4(z-z_0)^2 + 20a_5(z-z_0)^3 + \cdots$$

$$= \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}\,(z-z_0)^n, \tag{1.27}$$

And so on, for its derivatives. The differentiated series all have the same radius of convergence as the original. As a consequence, we deduce the following important result

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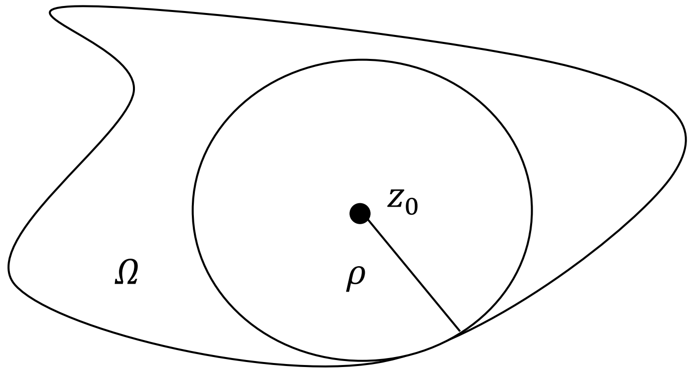


Figure (6) Radius of Convergence.

Theorem1.5.4: Any analytic function is infinitely differentiable.

In particular, when we substitute $z = z_0$ into the successively differentiated series, we get $a_0 = f(z_0), a_1 = f'(z_0), a_2 = \frac{1}{2}f''(z_0),$

and in general

$$a_n = \frac{f^{(n)}(z)}{n!} \tag{1.28}$$

Therefore, a convergent power series (1.22) is, inevitably, the usual Taylor series

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \tag{1.29}$$

For the function $f(z)$at the point , z_0.

Let us conclude this section by summarizing the fundamental theorem that characterizes complex functions. A complete, rigorous proof relies on complex integration theory.

Theorem 1.5.5: Let $\Omega \subset \mathbb{C}$ be an open set. The following properties are equivalent:

(a) The function $f(z)$ has a continuous complex derivative $f'(z)$ for all $z \in \Omega$.

The real and imaginary parts of $f(z)$ have continuous partial derivatives and satisfy the Cauchy-Riemann equations (1.18) in Ω.

(b) The function $f(z)$ is analytic for all $z \in \Omega$, and so is infinitely differentiable and has a convergent power series expansion at each point $z_0 \in \Omega$.the radius of convergence ρ is at least as large as the distance from z_o to the boundary $\partial\Omega$, as in Figure (6).

From now on, we reserve the term complex function to signify one that satisfies the conditions of Theorem (1.4. 5) Sometimes one of the equivalent adjectives "analytic" or "holomorphic" is added for emphasis. From now on, all complex functions are assumed to be analytic everywhere on their domain of definition, except, possibly, at certain singularities.

1.6 The Open Mapping Theorem

Definition 1.6.1: A map from an open set $\Omega \subset \mathbb{C}\ to\ \mathbb{C}$ is an open mapping when the image by f of any open subset of Ω is open.

Proposition 1.6.2: A map is open if and only if for each $z \in \Omega$

The image of any open set containing z contains a neighbourhood of $f(z)$.

Examples 1.6.3:

1. The map $f(z) = az + b$ with $a, b \in \mathbb{C}$ is open when $a \neq 0$ and not open when $a = 0$.

2. The map $x + iy \to \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} (x, y)$ is open when the matrix M is invertible and not open otherwise. The same is true of $a\bar{z} + b$.

3. If f is analytic with $f^{'}(z) \neq 0$ at all z then f is locally invertible by the inverse function theorem so satisfies the criterion of proposition (1.6.2.)

4. The mapping $f(z) = z^k$ with $k \geq 1$ integer is open. To prove this it suffices to show that the image by f of a small open set about $z = 0$ contains a neighbourhood of 0. For that it suffices to observe that the image by f of

$\{|z| < r\}$is exactly the disk of radius $r^{1/k}$. Each point in the latter disk has k preimages in the former disk located at the vertices of regular $k - g\ on$.

5. The map $x + iy \mapsto x^2 + iy$ is not open. The image of the open set $\mathbb{C}$ is not open. The restriction of this map to any set Ω that does not meet the imaginary axis is an open mapping. The mapping is not analytic.

Proposition 1.6.4: The composition of open maps is open. Precisely, if g is open on $\Omega_{1,} f$ is open on Ω_2 and $g(\Omega_1) \subset \Omega_2$ then $f(g(z))$ is open on $\Omega_{1.}$

Theorem 1.6.5: If $f(z)$ is a nonconstant analytic function on an open connected set Ω, then f is an open mapping.

Proof of the Open Mapping Theorem:

Verify the condition of proposition 1.6.2 at $\underline{z} \in \Omega$ where $f'(\underline{z}) \neq 0$ the map is locally invertible and the condition is automatic.

Suppose on the other hand that $\underline{z} \in \Omega$ and $f'(\underline{z}) = 0.$ since f is not constant there is a small lest $k \geq 2$ so that $f^{(k)}(\underline{z}) \neq 0$. Taylor's theorem yields for z near $\underline{z}$

$$f(z) - f(\underline{z}) = c_k(z - \underline{z})^k + c_{k+1}(z - \underline{z})^{k+1} + \cdots c_k \neq 0.$$

Factor to find

$$\begin{aligned} f(z) &= c_k(z - \underline{z})^k \left(1 + a_1(z - \underline{z}) + a_2(z - \underline{z})^2 + \cdots\right) \\ &= c_k(z - \underline{z})^k h(z), \end{aligned}$$

With

$$h(z) = 1 + a_1(z - \underline{z}) + a_2(z - \underline{z})^2 + \cdots$$

Then $h(z)$ is analytic as the sum of a convergent power series.

Define $g(w)$ on a neighbourhood of $w = 1$ to be a local inverse of the map $w = z^k$ on a neighbourhood of $z = 1$ where the derivative $w'(1) = k \neq 0$.

Then $g'(1) = 1/k$, $g(h(z))^k = h(z)$ for z near 1, and

$$f(z) - f(\underline{z}) = c_k \left((z - \underline{z}) g(h(z))\right)^k \tag{1.30}$$

Since

$$\frac{d}{dz}\left((z-\underline{z})g(h(z))\right)\Big|_{z=\underline{z}} = g\left(h(\underline{z})\right) = 1 \neq 0,$$

Is follows that the map $z \mapsto (z-\underline{z})g(h(z))$ is open on a neighbourhood of $\underline{z}$.

Therefore equation (1.30) expresses$f(z)$ as the composition of the mappings

$(z-\underline{z})g(h(z))z^kandz \mapsto c_k z + f(\underline{z})$. Thus the image by f of an open set containing $\underline{z}$ contains an open neighbourhood of $f(\underline{z})$ verifying the criterion of proposition 1.6.4 at points wheref' vanishes.

Remark: Examining (1.30) one sees that the preimage of a point $w \approx f(\underline{z})$ consists of k points near $\underline{z}$ nearly positioned at the vertices of a regular k −gon centered at $\underline{z}$. In this sense the behaviour of $f(z) - f(\underline{z})$ near $\underline{z}$ is well modelled by the k to one open mapping $c_k(z-\underline{z})^k$.

1.7 The Maximum Modulus Principle

If f is a non-constant, analytic function in a domain D,

Then $|f|$ can have no local maximum in D.

PROOF:

Suppose there exists a $z_0 \in D$, local maximum of $|f|$,i.e.

$$(\exists)r > 0: D_r(z_0) \subset D \quad \text{with} \quad |f(z)| \leq |f(z_o)| (\forall) z \in D_r(z_0).$$

By the Open Mapping Theorem $w_0 = f(z_0)$ is an inner point of $f(D_r(z_0))$,i.e.

$$(\exists)\rho > 0: D_\rho(w_0) \subset f(D_r(z_0)).$$

In the disk$D_\rho(w_0)$there are point w, such that $|w| > |w_0|$, e.g.

For $w = w_0 + \frac{\rho}{2}e^{iArgw_0}$ we get

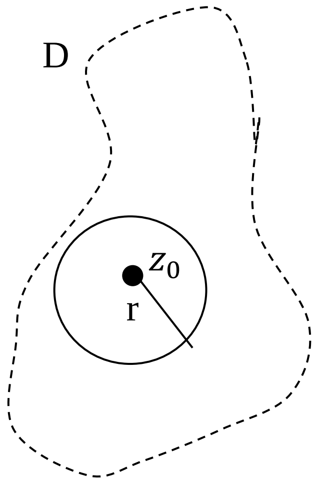
$$|w| = \left|w_0 + \frac{\rho}{2}e^{iArgw_0}\right| = |w_0| + \frac{\rho}{2} > |w_0|.$$

But

$$w \in D_\rho(w_0) \subset f(D_r(z_0)) \Rightarrow (\exists) z \in D_r(z_0): |f(z)| > |f(z_0)|$$

in contradiction with (1.6.5).

Thus our supposition was false and consequently $|f|$ cannot have a local maximum inD.



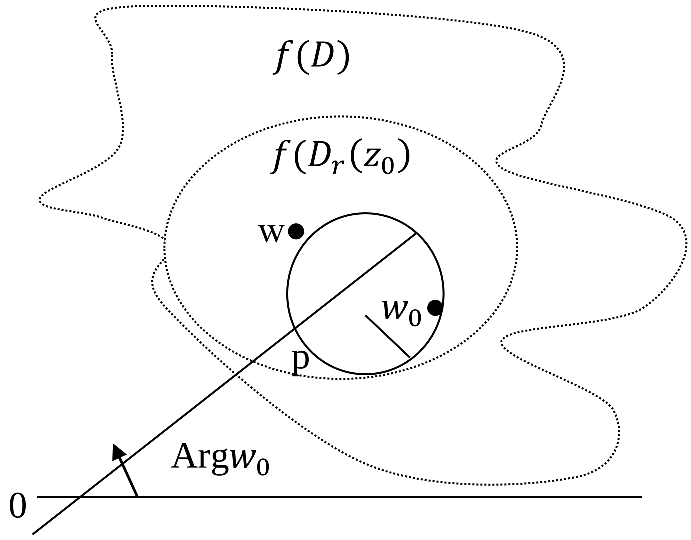


Figure (1.7a) A supposed local max

Of$|f|$.i.e. there exists a $r > 0$

such that $\left| f(z) \leq |f(z_0|\right|$ for

every z in$D_r(z_0)$.

Figure (1.7b) The image under f of

D and $D_r(z_0)$.

$w_0 = f(z_0) and\ w = f(z) \in f\left(D_r(z_0)\right)$.

$|w| > |w_0| \Leftrightarrow |\ f(z)| > |\ f\ (z_0)|$

1.7.1 Corollary: The Minimum Modulus Principle

If f is a nowhere zero, non-constant, analytic function in a domain D, then $|f|$ can have no local minimum in D.

PROOF:

Suppose there exists a $z_0 \in D,$ local minimum of $|f|$, then:

$(\exists) r > o: D_r(z_0) \subset D$ with $|f(z)| \geq |f(z_0)| (\forall) z \in D_r(z_0)$.

Define the function g: D→ $\mathbb{C}, g(Z) = \frac{1}{f(Z)}, g$ is analytic since $f(z) \neq 0$ in D.

A local minimum of $|f|$corresponds to a local maximum of $|g|$,since

$|f(z)| \geq |f(z_0)| \Leftrightarrow |\ g(z)| \leq |g(z_0)| (\forall) z \in D_r(z_0)$.

g is a non-constant, analytic function in the domain D and by the Maximum Modulus Principle$|g|$ can have no local maximum in consequently $|f|$can have no local minimum in D.

1.7.2 Corollary:

If f is a non-constant, analytic function in a domain D, then Re (f) have no local maxima and no local minima in D.

PROOF:

Define the functions: $g: D \to c, g(z) = e^{f(z)}$

$$u: D \to R, u(z) = Re\big(f(z)\big)$$

Because f is non-constant and analytic in D, so is g and also $\frac{1}{g}: D \to c,$ since $g \neq 0$.

By the Maximum Modulus Principle, neither $|g|$ nor $\left|\frac{1}{g}\right|$ can have local maxima in D.

$|g| = \left|e^{f(z)}\right| = e^{Re(f(z))=}e^{u}$ and $\left|\frac{1}{g}\right| = e^{-u}$, and it follows that $u = Re(f)$ can have no local maximum and no local minimum in D.

1.7.3 The Maximum Modulus Theorem:

Let f be non- constant function defined and continuous on a bounded, closed region K and analytic in the interior of K. Then the maximum value of $|f(z)|$in K must occur on the boundary of K.

PROOF:

By a theorem of Weierstrass, since K is compact,

$$(\exists) z_0 \in k: |f(z)| \leq |f(z_0)| (\forall) z \in K.$$

Suppose that this maximum is attained in an interior point, i.e. $z_0 \in k^0$.

It would then be the global maximum of the restriction of f to k^0. In contradiction with the Maximum Modulus Principle.

Thus we have $z_0 \in k \backslash k^0 = \partial k,$ the boundary of k.

Remark: A maximum at an interior point is possible if we drop the condition of f being non-constant.

Then f (z) =f (z_0) for each$z \in k^0$.

By continuity of, f (z) =f(z_0) also on ∂k,and thus f is constant for every $z \in k$.

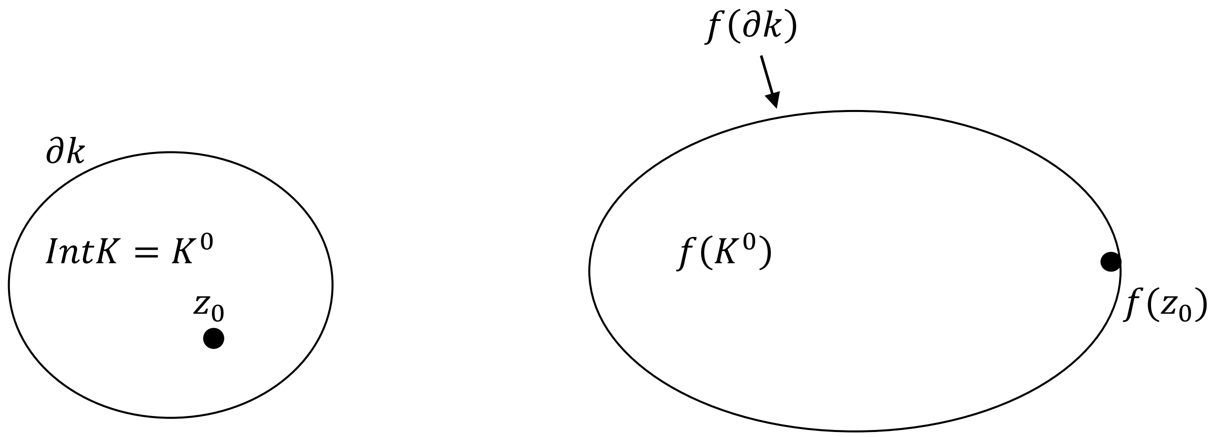


Fig (1.8). Example of a bounded closed region K, and its image under a non-constant function f, analytic on the interior of K and continuous on K. The maximum of $|f|$ is attained on ∂K.

1.7.4 Schwarz's Lemma

If f is analytic in the disk $D_1 = \{z \in c: |z| < 1\}$and satisfies the conditions $|f(z)| \leq 1$ and $f(0) = 0$, then$|f(z)| \leq |z|$ and $|f'(0)| \leq 1$.

If $|f(z)| = |z|$ for some $z \neq 0$,or if $|f'(0)| = 1$, then there exists a constant $c \in C, |c| = 1$ such that $f(z) = cz$ for every $z \in D_1$.

PROOF:

Define the function$g: D_1 \to$ C$, by\ g\ (z) = \frac{f(z)}{z}$.

The singularity at $z = 0$ is removable and

$$g(0) = \lim_{z \to 0} \frac{f(z)}{z} = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = f'(0).$$

Thus the function $g: D_1 \to$ C$, g(z) = \begin{Bmatrix} \frac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0 \end{Bmatrix}$ is analytic.

On the circle $|z| = r, 0 < r < 1$, we have

$$|g(z)| = \frac{|f(z)|}{|z|} = \frac{|f(z)|}{r} \leq \frac{1}{r}.$$

In the closed disk $|z| \leq r$ the maximum of $|g(z)|$ is attained on the boundary$|z| = r$, By the Maximum Modulus Theorem, i.e. $|z| \le r \Rightarrow |g(z)| \le \frac{1}{r}$.

Let $r_n = 1 - \frac{1}{2n}(n \ge 1)$.We have $r_n \to 1$ as $n \to \infty$.

$(\forall) z \in D_1, (\exists) m \ge 1: |z| \le r_m \le r_n < 1 \ (\forall) n \ge m \Rightarrow |g(z)| \le \frac{1}{r_n} (\forall) n \ge m.$

$n \to \infty \Rightarrow \frac{1}{r_n} \to 1$And we have $|g(z)| \le \lim_{n\to\infty} \frac{1}{r_n} = 1$ for every $z \in D_1$.

$|g(z)| = \frac{|f(z)|}{|z|} \le 1$ and $f(0) = 0$, so $|f(z)| \le |z| (\forall) z \in D_1$ and

$$|f'(0)| = |g(0)| \le 1.$$

If $|f(z)| = |z|$ for some $z \ne 0$ or if $|f'(0)| = 1$, then $|g(z)| = 1$ inside the disk $|z| < 1$, i.e. the maximum is attained at an interior point of D_1.

Then, by the Maximum Modulus Principle, g is a constant function:

$$g(z) = c, c \in C, |c| = 1.$$

In D_1 we get $f(z) = cz = e^{i\theta} z, \theta \in R$.

1.8 The Group of Möbius Transformations of $\hat{\mathrm{C}}$

A Möbius transformation is a function, $\hat{\mathrm{C}} \to \hat{\mathrm{C}} = C \cup \{\infty\}$,

$$\mathrm{M: z} \to m(z) = \frac{az+b}{cz+d};\ a, b, d \in C$$

And $\quad ad - bc \ne 0, M_m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \det M_m \ne 0.$

$c = 0 \Rightarrow m(z) = az + \beta; a, b \in C$ and $m(\infty) = \infty$

$c \ne 0 \Rightarrow m\left(-\frac{d}{c}\right) = \infty$ and $m(\infty) = \lim_{z\to\infty} \frac{az+b}{cz+d} = \frac{a}{c}$.

The set of all Mobius transformations is denoted Möb$^+$.

We will show that Möb$^+$is a group under composition of functions:

First we check that $Möb^+$ is stable under multiplication i.e.

$$m_1, m_2 \in Möb \Rightarrow m_1 \circ m_2 \in Möb.$$

Let $m_1(z) = \frac{az+b}{cz+d}; a, b, c, d \in C$ and

$$ad - bc \neq 0, M_{m1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ \det M_{m1} \neq 0.$$

And $m_2(z) = \frac{ez+f}{gz+h}; e, f, g, h \in C$and $eh - fg \neq 0$ $M_{m2} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$,

$\det M_{m2} \neq 0$.

We get$(m_1 \circ m_2)(z) = m_1(m_2(z)) = \frac{(ae+g)z+(af+h)}{(ce+dg)z+(cf+dh)}$,

$M_{m_1 \circ m_2} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix} = M_{m_1} \times M_{m_2}$and

$$det M_{m_1 \circ m_2} = det(M_{m_1} \times M_{m_2}) = det\ M_{m_1} . det\ M_{m_2} \neq 0.$$

We see that $m_1 \circ m_2 \in$ Möb and since m_1 and m_2 was arbitrary elements of Möb, the set of Möbius transformations is stable under composition of functions.

The neutral element of $M\ddot{o}b^+$ is $n(z) = z, M_n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

It remains to show that each element of $M\ddot{o}b^+$ is invertible.

Let $w = m(z) = \frac{az+b}{cz+d}; a, b, c, d \in$ C $and\ ad - bc \neq 0$.

Then:

$$z = m^{-1}(w) = \frac{dw - b}{-cw + a}; da - bc \neq 0, M_{m^{-1}} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, det\ M_{m^{-1}} \neq 0.$$

We have $m^{-1}(w) = \frac{dw-b}{-cw+a} \in M\ddot{o}b^+$ for every $m \in M\ddot{o}b^+$.

Thus $(M\ddot{o}b^+,\circ)$ is a group.

1.9 Complex Integration.

The magic and power of calculus ultimately rests on the amazing fact that differentiation and integration are mutually inverse operations. And, just as complex functions enjoy remarkable differentiability properties not shared by their real counterparts, so the sublime beauty of complex integration goes far beyond its more mundane real progenitor.

Let's begin by motivating the definition of the complex integral. As you know, the (definite) integral of a real function, $\int_a^b f(t)dt,$ is evaluated on an interval

$[a, b] \subset R$. In complex function theory, integrals are taken along curves in the complex plane, and are akin to the line integrals appearing in real vector calculus. Indeed, the identification of a complex number r$z = x + iy$with a planar vector $X = (x, y)^T$ will serve to connect the two theories.

Consider a curve C in the complex plane, parameterized by $z(t) = x(t) + iy(t)$ for $a \leq t \leq b$. we define the integral of the complex function $f(z)$ along C to be the complex number

$$\int_c f(z)dz = \int_a^b f\big(z(t)\big)\frac{dz}{dt}dt, \tag{1.30}$$

The right hand side being an ordinary real integral of a complex-valued function. We shall always assume that the integrand $f(z)$ is a well-defined complex function at each point on the curve, and hence the integral is well-defined. Let us write out the integrand

$$f(z) = u(x, y) + iv(x, y)$$

In terms of its real and imaginary parts, as well as the differential

$$dz = \frac{dz}{dt}dt = \left(\frac{dx}{dt} + i\frac{dy}{dt}\right)dt = dx + idy.$$

As a result, the complex integral (1.30) splits up into a pair of real line integrals:

$$\int_c f(z)dz = \int_c (u + iv)(dx + i\,dy)$$

$$= \int_c (u\,dx - v\,dy) + i\int_c (v\,dx + u\,dy). \tag{1.31}$$

Example 1.9.1: Suppose n is an integer. Let us compute the complex integrals

$$\int_c z^n\,dz \tag{1.32}$$

Of the monomial function $f(z) = z^n$along several different curves.

We begin with a straight line segment I along the real axis connecting the point -1 and 1, which we parameterize by $z(t) = t$ for $-1 \le t \le 1$.

The defining formula (1.30) implies that the complex integral (1.32) reduces to an elementary real integral:

$$\int_I z^n \, dz = \int_{-1}^{1} t^n \, dt = \begin{cases} 0, & 0 < n = 2k+1 \quad odd, \\ \dfrac{2}{n+1}, & 0 \le n = 2k \quad even. \end{cases}$$

If $n \le -1$ is negative, then the singularity of the integrand at the origin implies that the integral diverges, and so the complex integral is not defined.

Let us evaluate the same complex integral, but now along a parabolic arc p parametrized by

$$z(t) = t + i(t^2 - 1), \qquad -1 \le t \le 1.$$

We again refer back to the basic equation (1.31) to evaluate the integral, so

$$\int_P z^n \, dz = \int_{-1}^{1} [t + i(t^2-1)]^n \, (1 + 2it) dt.$$

We could, at this point, expand the resulting complex polynomial integrand, and then integrate term by term. A more elegant approach is to recognize that it is an exact derivative:

$$\frac{d}{dt} \frac{[t + i(t^2-1)]^{n+1}}{n+1} = [t + i(t^2-1)]^n (1 + 2it),$$

As long as $n \ne -1$. Therefore, we can use the Fundamental Theorem of Calculus (which works equally well for real integrals of complex-valued functions), to evaluate

$$\int_P z^n \, dz = \frac{[t + i(t^2-1)]^{n+1}}{n+1} \Big|_{t=-1}^{1} = \begin{cases} 0, & -1 \ne n = 2k+1 \quad odd, \\ \dfrac{2}{n+1}, & n = 2k \quad even. \end{cases}$$

Thus, when $n \ge 0$ is a positive integer, we obtain the same result as before. Interestingly, in this case the complex integral is well-defined even when n is a negative integer because, unlike the real line segment, the parabolic path does not go through the singularity of z^n at $z = 0$. The case$n = -1$needs to be done slightly differently, and integration of $1/z$ along the parabolic path requires some care of basic complex integration techniques.

Finally, let us try integrating around a semi-circular arc, again with the same endpoints -1 and 1. If we parameterize the semi-circle S^+ by $z(t) = e^{it}, 0 \leq t \leq \pi$,we find:

$$\int_{S^+} z^n \, dz = \int_0^{\pi} z^n \frac{dz}{dt} dt = \int_0^{\pi} e^{int} \, ie^{it} \, dt$$

$$= \int_0^{\pi} ie^{i(n+1)t} dt = \frac{e^{i(n+1)t}}{n+1} |_{t=0}^{\pi} = \frac{1 - e^{i(n+1)t}}{n+1}$$

$$= \left\{ \begin{matrix} 0, & -1 \neq n = 2k+1 & odd, \\ -\frac{2}{n+1}, & n = 2k & even. \end{matrix} \right\}$$

This value is the negative of the previous cases – but this can be explained by the fact that the circular are is oriented to go from 1 to -1 whereas the line segment and parabola both go from -1 to 1. Just as with line integrals, the direction of the curve determines the sign of the complex integral; if we reverse direction, replacing t by $-t$, we end up with the same value as the preceding two complex integrals. Moreover – again provided $n \neq -1$ - it does not matter whether we use the upper semicircle or lower semicircle to go from -1 to 1 – the result is exactly the same. However, the case $n = -1$ is an exception to this "rule". Integrating along the upper semicircle S^+ from 1 to -1 yields

$$\int_{S^+} \frac{dz}{z} = \int_0^{\pi} i \, dt = \pi i, \tag{1.33}$$

Whereas integrating along the lower semicircle S^- from 1 to -1 yields the negative

$$\int_{S^-} \frac{dz}{z} = \int_0^{-\pi} i \, dt = -\pi i. \tag{1.34}$$

Hence, when integrating the function$1/z$, it makes a difference which direction we go around the origin.

Integrating z^n for any integer $n \neq -1$ around an entire circlegiveszero – irrespective of the radius. This can be seen as follows.

We parameterize a circle of radius$r$ by $z(t) = re^{it}$ for 0$\leq t \leq 2\pi$. then, by the same computation,

$$\oint_C z^n\,dz = \int_0^{2\pi} \left(r^n e^{i\,nt}\right)\left(rie^{it}\right)dt = \int_0^{2\pi} ir^{n+1}e^{i(n+1)t}\,dt$$

$$= \frac{r^{n+1}}{n+1} e^{i(n+1)t} \,\Big|\,_{t=0}^{2\pi} = 0 \qquad (1.35)$$

Provided $n \neq -1$. The circle on the integral sign serves to remind us that we are integrating around a closed curve. The case $n = -1$ remains special. Integrating once around the circle in the counter-clockwise direction yields a nonzero result

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} i\,dt = 2\pi i. \qquad (1.36)$$

Let us note that a complex integral does not depend on the particular parameterization of the curve C. It does, however, depend upon its orientation: if we traverse the curve in the reverse direction, then the complex integral changes its sign:

$$\int_{-C} f(z)dz = -\int_C f(z)dz. \qquad (1.37)$$

Moreover, if we chop up the curve into two non-overlapping pieces, $C = C_1 \cup C_{2,}$ with a common orientation, then the complex integral can be decomposed into a sum over the pieces:

$$\int_{C_1 \cup C_2} f(z) = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz \qquad (\,1.38)$$

For instance, the integral (1.36) of $1/z$ around the circle is the difference of the individual semi-circular integrals (1.33, 34); the lower semi-circular integral acquires a negative sign to flip its orientation so as to agree with that of the entire circle. All these facts are immediate consequences of the well-known properties of line integrals, or can be proved directly from the defining formula (1.30).

Note: In complex integration theory, a simple closed curve is often referred to as a contour, and so complex integration is sometimes referred to as contour integration.

Unless explicitly started otherwise, we always go around contours in the counter-clockwise direction.

Further experiments lead us to suspect that complex integrals are usually path-independent, and hence evaluate to zero around closed curves. One must be careful, though, as the integral (1.36) makes clear. Path independence, in fact, follows from the complex version of the Fundamental Theorem of Calculus.

Theorem 1.9.2: Let $f(z) = F'(z)$ be the derivative of a single-valued complex function $F(z)$ difined on a domain $\Omega \subset C$. let $C \subset \Omega$ be any curve with initial point α and final point β. Then

$$\int_C f(z)dz = \int_C F'(z)dz = F(\beta) - f(\alpha). \tag{1.39}$$

Proof: This follows immediately from the definition and the chain rule:

$$\int_C F'(z)dz = \int_a^b F'\big(z(t)\big)\frac{dz}{dt}dt = \int_a^b \frac{d}{dt}F\big(z(t)\big)dt = F\big(z(b)\big) - F\big(z(a)\big)$$
$$= F(\beta) - F(\alpha),$$

Where $\alpha = z(a)$ and $\beta = z(b)$ are the endpoints of the curve. Q. E.D. for example, when $n \neq -1$, the function$f(z) = z^n$is the derivative of the single-valued function $F(z) = \frac{1}{n+1}z^{n+1}$. *Hence*

$$\int_C z^n dz = \frac{\beta^{n+1}}{n+1} - \frac{\alpha^{n+1}}{n+1}$$

Whenever C is (almost) any curve connecting $\alpha\ to\ \beta$. Then only restriction is that, when $n < 0$, the curve is not allowed to pass through the singularity at the origin $z = 0$. In contrast, the function$f(z) = 1/z$ is the derivative of the complex logarithm

$$\log z = \log|z| + i\ ph\ z,$$

Which is not single-valued on all of $C\{0\}$.If our curve is contained within a simply connected sub domain that does not include the origin, $0 \notin \Omega \subset C$, then we can use any single-valued branch of the complex logarithm to evaluate the integral

$$\int_C \frac{dz}{z} = \log\beta - \log\alpha,$$

Where α, β are the endpoints of the curve. Since the common multiples of $2\pi\, i$ cancel, the answer does not depend upon which particular branch of the complex logarithm is selected as long as we are consistent in our choice.

For example, on the upper semicircle S^+ of radius 1 going from 1 to -1,

$$\int_{S^+} \frac{dz}{z} = \log(-1) - \log 1 = \pi\, i,$$

Where we take the branch of $\log z = \log|z| + i$ ph z with $0 \le ph\, z \le \pi$. On the other hand, if we integrate on the lower semi-circle S^- going from 1 to -1, we need to adopt a different branch, say that with $-\pi \le ph\, z \le 0$.

With this choice, the integral becomes

$$\int_{S^-} \frac{dz}{z} = \log(-1) - \log 1 = -\pi\, i$$

Thus reproducing (1.33, 34). Pay particular attention to the different values of $\log(-1)$ used in the two cases!

Cauchy's Theorem 1.9.3:

The preceding considerations suggest the following fundamental theorem, due in its general form to Cauchy. Before stating it, we introduce the convention that a complex function $f(z)$ is to be deemed analytic on a domain $\Omega \subset C$ provided it is analytic at everyPoint inside Ω and, in addition, remains (at least) continuous on the boundary$\partial\Omega$. When Ω is bounded, its boundary$\partial\Omega$consists of one or more simple closed curves. In general, as in Green's Theorem, we orient $\partial\Omega$so that domain is always on our left hand side. This means that the outermost boundary curve is traversed in the counter-clockwise direction, but those around interior holes take on a clockwise orientation.

Theorem 1.9.4: If $f(z)$ is analytic on a bounded domain $\Omega \subset C,$ then

$$\oint_{\partial\Omega} f(z)dz = 0. \qquad (1.40)$$

Proof: Application of Green's Theorem to the two real line integrals in (1.40) yields

$$\oint_{\partial\Omega} u\, dx - v\, dy = \iint_{\Omega} \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\, dy = 0,$$

$$\oint_{\partial\Omega} v\,dx + u\,dy = \iint_{\Omega} (\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}) = 0,$$

Both of which vanish by virtue of the Cauchy-Riemann equations (1.18) Q.E.D.

If the domain of definition of our complex function$f(z)$is simply connected, then by definition, the interior of any closed curve $C \subset \Omega$is contained in Ω, and hence Cauchy's Theorem (1.40) implies path independence of the complex integral within Ω.

Corollary 1.9.5: If $f(z)$is analytic on a simply connected domain $\Omega \subset C$, then its complex integral $\int_C f(z)dz$ for $C \subset \Omega$ is independent of path. In particular,

$$\oint_C f(z)dz = 0 \tag{1.41}$$

For any closed curve $S \subset \Omega$.

$$\oint_C f(z)dz = \oint_S f(z)dz. \tag{1.42}$$

Proof: If C and S do not cross each other, we let Ω denote the domain contained between them, so that $\partial\Omega = C \cup S$; According to Cauchy's Theorem 1.9.4, $\oint_{\partial\Omega} f(z) = 0$. Now, our orientation convention for $\partial\Omega$means that the outer curve, say C, is traversed in the counter-clockwise direction, while the inner curve S assumes the opposite, clockwise orientation. Therefore, if we assign both curves the same counter-clockwise orientation,

$$0 = \oint_{\partial\Omega} f(z) = \oint_C f(z)dz - \oint_S f(z)dz,$$

proving (1.42).

If the two curves cross, we can construct a near by curve $K \subset \Omega$ that neither crosses. By the preceding paragraph, each integral is equal to that over the third curve,

$$\oint_C f(z)dz = \oint_K f(z)dz = \oint_S f(z)dz,$$

And formula (1.40) remains valid. Q. E. D.

Example 1.9.6: Consider the function $f(z) = z^n$ where n is an integer.

In (1.35), we already computed

$$\oint_C z^n dz = \begin{Bmatrix} 0, & n \neq -1, \\ 2\pi\, i, & n = -1, \end{Bmatrix} \tag{1.43}$$

When C is a circle centered at $z = 0$. When $n \geq 0$, Corollary (1.9) immediately implies that the integral of z^n is 0 over any closed curve in the plane. The same applies in the cases $n \leq -2$ provided the curve does not pass through the singular point $z = 0$. In particular, the integral is zero around closed curve encircling the origin, even though for $n \leq -2$ has a singularity inside the curve and so Cauchy's Theorem does not apply as stated.

The case $n = -1$ has particular significance. Here, proposition 1.31 implies that the integral is the same as the integral around a circle-provided the curve C also goes once around the origin in a counter-clockwise direction.

Thus (1.36) holds for any closed curve that goes counter-clockwise once around the origin. More generally, if the curve goes several times around the origin, then

$$\oint_C \frac{dz}{z} = 2k\pi\, i \tag{1.44}$$

Is an integer multiple of $2\pi\, i$. the integer k is called the winding number of the curve C, and measures the total number of time C goes around the origin.

For instance, if Cwinds three times around 0 in a counter-clockwise fashion, then $k = 3$, while $k = -5$ indicates that the curve winds 5 times around 0 in a clockwise direction.

In particular, a winding number $k = 0$ indicates that C is not wrapped around the origin. If C represents a loop of string wrapped around a pole (the pole of $1/z$ at 0) then a winding number $k = 0$ would indicate that the string can be disentangled from the pole without cutting; nonzero winding numbers would indicate that the string is truly entangled.

Lemma 1.9.7: If C is a simple closed curve, and a is any point not lying on C, then.

$$\oint_C \frac{dz}{z-a} = \begin{cases} 2\pi\, i, & a \ \ inside\ \ C \\ 0, & a\ \ outside\ \ C. \end{cases} \tag{1.45}$$

I f$a \in C,$ then the integral does not converge.

Proof: Note that the integrand $f(z) = 1/(z-a)$ is analytic everywhere except at $z = a,$ where it has a simple pole. If a is outside C, then Cauchy's Theorem (1.9.3) applies, and the integral is zero. On the other hand, if a is inside C, then corollary (1.9.5) implies that the integral is equal to the integral around a circle centered at $z = a.$ the latter integral can be computed directly by using the parameterization $z(t) = a + re^{i\,t}$ for $0 \le t \le 2\pi,$ as in (1.36) Q. E.D.

Example 1.9.8: Let $D \subset C$ be a closed and connected domain. Let $a, b \in D$ be two points in D. Then

$$\oint_C \left(\frac{1}{z-a} - \frac{1}{z-b}\right) dz = \oint_C \frac{dz}{z-a} - \oint_C \frac{dz}{z-b} = 0$$

For any closed curve $C \subset \Omega = \mathbb{C}\backslash D$ lying outside the domain D. This is because, by connectivity of D, either C contains both points in its interior, in which case both integrals equal $2\pi\, i,$ or C contains neither point, in which case both integrals are 0. The conclusion is that, while the individual logarithms are multiply-valued, their difference

$$F(z) = \log(z-a) - \log(z-b) = \log\frac{z-a}{z-b} \tag{1.46}$$

Is a consistent, single-valued complex function on all of $\Omega = \mathbb{C}\backslash D.$ The function (1.46) has, in fact, an infinite number of possible values, differing by integer multiples of $2\pi\, i$;the ambiguity can be resolved by choosing one of its values at a single point in Ω. These conclusions rest on the fact that D is connected, and are not valid, say, for the twice-punctured plane $\mathbb{C}\ /\ \{a, b\}$.

Circulation and lift 1.9.9:

In fluid mechanical applications, the complex integral can be assigned an important physical interpretation. We consider the steady state flow of an incompressible, irrotational fluid.

Let$f(z) = u(x,y) - i\, v(x,y)$ denote the complex velocity corresponding to the real velocity vector $V = (u(x,y), v(x,y))^T$ at the point $(x,y)^T$.

The integral of the complex velocity $f(z)$ along a curve C can be written as a pair of real line integrals:

$$\int_C f(z)dz = \int_C (u - i\, v)(dx + i\, dy)$$

$$= \int_C (u\, dx + v\, dy) - i \int_C (v\, dx - u\, dy). \qquad (1.47)$$

$$\left(-\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\, dy$$

The real part is the circulation integral

$$\int_C V.dx = \int_C u\, dx + v\, dy, \qquad (1.48)$$

While the imaginary part is minus the flux integral

$$\int_C V.n\, ds = \int_C V \times dx = \int_C v\, dx - u\, dy. \qquad (1.49)$$

If the complex velocity admits a single-valued complex potential

$$X(z) = \varphi(z) - i\, \psi(z), \qquad where \qquad X'(z) = f(z),$$

Which is always the case if its domain of definition is simply connected, then the complex integral is independent of path, and one can use the Fundamental Theorem (1.9.2) to evaluate it

$$\int_C f(z)dz = X(\beta) - X(\alpha) \qquad (1.50)$$

For any curve C connecting $\alpha\ to\ \beta$.Path independence of the complex integral reconfirms the path independence of the circulation and flux integrals for ideal fluid flow. The real part of formula (1.50) evaluates the circulation integral

$$\int_C V.dx = \int_C \nabla\varphi.dx = \varphi(\beta) - \varphi(\alpha), \qquad (1.51)$$

As the difference in the values of the (real) potential at the endpoints α, β of the curve C. On the other hand, the imaginary part of formula (1.50) computes the flux integral

$$\int_C V.n\, ds = \int_C \nabla\psi . dx = \psi(\beta) - \psi(\alpha), \qquad (1.52)$$

As the difference in the values of the stream function at the endpoints of the curve. The stream function acts as a "flux potential" for the flow.

Thus, for ideal flows, the fluid flux through a curve depends only upon its endpoints. In particular, if C is a closed contour, and $X(z)$ is analytic on its interior, then

$$\oint_C V.\, dx = 0 = \oint_C V.n\, dx, \qquad (1.53)$$

And so there is no net circulation or flux along any closed curve in this scenario.

Typically, lift on a body requires a nonzero circulation around it.

Let $D \subset \mathbb{C}$ a bounded, simply connected domain representing the cross-section of a cylindrical body, e.g., an airplane wing. The velocity vector field V of a steady state flow around the exterior of the body is defined on the domain $\Omega = \mathbb{C}\backslash\overline{D}$. the no flux boundary conditions $V.N = 0 \quad on \quad \partial\Omega = \partial D$ indicate that there is no fluid flowing across the boundary of the soild body.

The resulting circulation of the fluid around the body is given by the integral $\oint_C V.dx$, Where $C \subset \Omega$ is any simple closed contour encircling the body. (Cauchy's theorem, tells us that the value does not depend upon the choice of contour.) However, if the corresponding complex velocity $f(z)$admits a single-valued complex potential in Ω, then (1.53) tells us that the circulation integrals is automatically zero, and so the body will not experience any lift!

The only way to introduce lift into the picture is through a (single-valued) complex velocity with a non-zero circulation integral, and this requires that its complex potential be multiply-valued. The one function that we know that has such a property is the complex logarithm

$$\lambda(z) = \log(a\,z + b), \quad \text{Whose derivative} \quad \lambda'(z) = \frac{a}{a\,z+b}$$

Is single-valued away from the singularity at $z = -b/a$. Thus, we are naurally led to introduce the family of complex potentials

$$X_{\gamma}(z) = z + \frac{1}{z} + i\,y \log z. \tag{1.54}$$

The coefficientγmust be real in orders to maintain the no flux boundary condition on the unit circle. By (1.47), the circulation is equal to the real part of the integral of the complex velocity

$$f_{\gamma}(z) = \frac{dx_{\gamma}}{dz} 1 - \frac{1}{z^2} + \frac{iy}{z}, \tag{1.55}$$

Which remains asymptotically 1 at large distance. By Cauchy's Theorem 1.40 coupled with formula (1.45), if C is a curve going once around the disk in a counter-clockwise direction, then

$$\oint_C f_{\gamma}(z) dz = \oint_C \left(1 - \frac{1}{z^2} + \frac{i\,\gamma}{z}\right) dz = -2\pi\gamma$$

In particular, assuming $|\gamma| \leq 2$, the stagnation points have moved from

$$\pm 1 \text{ to } \pm\sqrt{1 - \tfrac{1}{4}\gamma^2} - \tfrac{1}{2}\, i\,\gamma.$$

When we compose the modified potentials (1.54) with the Joukowski transformation we obtain a complex potential for flow around the corresponding airfoil-the image of the unit disk. The conformal mapping does not affect the value of the complex integrals, and hence, for any$\gamma \neq 0$, there is a nonzero circulation around the airfoil under the modified fluid flow, and at last our airplane will fly!

Cauchy's Integral Formula 1.9.10:

Cauchy's Theorem (1.9.4) forms the cornerstone of almost all applications of complex integration. The fact that we can move the contours of complex integrals around freely-as long as we do not cross over singularities of the integrand grants us great flexibility in their evaluation. An important consequence of Cauchy's Theorem is the justly famous Cauchy integral formula, which enables us to compute the value of an analytic function at a point by evaluating a contour integral around a closed curve encircling the point.

Theorem 1.9.11: Let$\Omega \subset \mathbb{C}$ be a bounded domain with boundary $\partial\Omega$, and let $a \in \Omega$. If $f(z)$ is analytic on Ω, then

$$f(a) = \frac{1}{2\pi\, i} \oint_{\partial\Omega} \frac{f(z)}{z-a} dz. \tag{1.56}$$

Remark: As always, we traverse the boundary curve $\partial\Omega$ so that domain Ω lies on our left. In most applications, $\Omega$is simply connected, and so $\partial\Omega$ is a simple closed curve oriented in the counter-clockwise direction.

It is worth emphasizing that Cauchy's formula (1.56) is not a form of the Fundamental Theorem of Calculus, since we are reconstructing the function by integration- not its anti-derivative! The closed real counterpart is the Poisson Integral Formula expressing the value of a harmonic function in a disk in terms of its values on the boundary of a circle. Indeed, there is a direct and harmonic function.

Proof: We first prove that the difference quotient

$$g(z) = \frac{f(z) - f(a)}{z-a}$$

Is an analytic function on all of Ω. The only problematic point is at $z = a$ where the denominator vanishes. First, by the definition of complex derivative,

$$g(a) = \lim_{z\to a} \frac{f(z) - f(a)}{z-a} = f'(a)$$

Exists and therefore $g(z)$ is well-defined and, in fact, continuous at

$z = a$ directly from the definition:

$$g'(a) = \lim_{z\to a} \frac{g(z) - g(a)}{z-a} = \lim_{z\to a} \frac{f(z) - f(a) - f'(a)(z-a)}{(z-a)^2} = \frac{1}{2} f''(a),$$

Which follows from Taylor's Theorem (or I' Hôpital's rule).

Knowing that g is differentiable at $z = a$ suffices to establish that it is analytic on all of Ω. Thus, we may appeal to Cauchy's Theorem (1.40) and conclude that

$$0 = \oint_{\partial\Omega} g(z)dz = \oint_{\partial\Omega} \frac{f(z) - f(a)}{z-a} dz = \oint_{\partial\Omega} \frac{f(z)}{z-a} dz - f(a) \oint_{\partial\Omega} \frac{dz}{z-a}$$
$$= \oint_{\partial\Omega} \frac{f(z)}{z-a} dz - 2\pi\, i\, f(a),$$

Where the second integral was evaluated using (1.45).

Rearranging terms completes the proof of the Cauchy formula.

Remark: The proof shows that if, in contrast, $a \notin \bar{\Omega}$, then the Cauchy integral vanishes:

$$\frac{1}{2\pi i}\oint_{\partial\Omega} \frac{f(z)}{z-a} dz = 0.$$

If $a \in \partial\Omega$, then the integral does not converge.

Let us see how we can apply this result to evaluate seemingly intractable complex integrals.

Example1.9.12: Suppose that you are asked to compute the contour integral

$$\oint_C \frac{e^z dz}{z^2 - 2z - 3}$$

Where C is a circle of radius 2 centred at the origin. A direct evaluation is not easy, since the integrand does not have an elementary anti-derivative.

However, we note that

$$\frac{e^z}{z^2 - 2z - 3} = \frac{e^z}{(z+1)(z-3)} = \frac{f(z)}{z+1} \qquad where \quad f(z) = \frac{e^z}{z-3}$$

Is analytic in the disk $|z| \le 2$ since its only singularity, at$z = 3$, lies outside the contour C. Therefore, by Cauchy's formula (1.56), we immediately obtain the integral

$$\oint_C \frac{e^z dz}{z^2-2z-3} = \oint_C \frac{f(z)}{z+1} dz = 2\pi\, i\, f(-1) = -\frac{\pi\, i}{2e}.$$

Note: Path independence implies that the integral has the same value on any other simple closed contour, provided it is oriented in the usual counter-clockwise direction and encircles the point $z = 1$ but not the point $z = 3$.

1.10 Derivatives by Integration

The fact that we can recover values of complex functions by integration is noteworthy.

Even more amazing is the fact that we can compute derivatives of complex functions by integration-turning the Fundamental Theorem on its head! Let us differentiate both sides of Cauchy's formula (1.56) with respect to a.

The integrand in the Cauchy formula is sufficiently nice so as to allow us to bring the derivative inside the integral sign. Moreover, the derivative of the Cauchy integrand with respect to a is easily found:

$$\frac{\partial}{\partial a}\left(\frac{f(z)}{z-a}\right)=\frac{f(z)}{(z-a)^2}$$

In this manner, we deduce integral formulae for the derivative of an analytic function:

$$f'(a)=\frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2}dz, \tag{1.57}$$

Where, as before, C is any simple closed curve that goes once around the point $z=a$ in a counter-clockwise direction. Further differentiation yields the general integral formulae

$$f^{(n)}(a)=\frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}dz \tag{1.58}$$

That expresses the n^{th} order derivative of a complex function in terms of a contour integral.

These remarkable formulae can be used to prove our earlier claim that an analytic function is infinitely differentiable, and thereby complete the proof of Theorem (1.9.11)

Example 1.10.1: Let us compute the integral

$$\oint_C \frac{e^z dz}{z^3-z^2-5z-3}=\oint_C \frac{e^z dz}{(z+1)^2(z-3)},$$

Around the circle of radius 2 centred at the origin. We use (1.57) with

$$f(z)=\frac{e^z}{z-3}, \quad whereby \quad f'(z)=\frac{(z-4)e^z}{(z-3)^2}.$$

And therefore we get

$$\oint_C \frac{e^z\, dz}{z^3 - z^2 - 5z - 3} = \oint_C \frac{f(z)}{(z+1)^2} dz = 2\pi\, i\, f'(-1) = -\frac{5\pi\, i}{8\, e}.$$

1.11 Harmonic Functions.

We began this chapter by motivating the analysis of complex functions through applications to the solution of the two –dimensional Laplace equation.

Let us now formalize the precise relationship between the two subjects.

Theorem 1.11.1: If $f(z) = u(x,y) + iv(x,y)$is any complex analytic function, then its real and imaginary parts, $u(x,y), v(x,y)$, are both harmonic functions.

Proof: Differentiating the Cauchy-Riemann equations (1.18), and invoking the equality of mixed partial derivatives, we find that

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = \frac{a^2 u}{\partial y^2}.$$

Therefore, u is a solution to the Laplace equation$u_{xx} + u_{yy} = 0$.

The proof for v is similar.

Thus, every complex function give srise to two harmonic functions.

It is of course, of interest to know whether we can invert this procedure.

Given a harmonic function $u(x,y)$, dose there exists a harmonic function $v(x,y)$

Such that$f = u + iv$ is a complex analytic function? If so, the harmonic function $v(x,y)$ is known as a *harmonic conjugate*to u. The harmonic conjugate is found by solving the Cauchy-Riemann equations

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \qquad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}, \tag{1.59}$$

Which, for a prescribed function $u(u,y)$, constitutes an inhomogeneous linear system of partial differential equations for $v(x,y)$. As such, it is usually not hard to solve, as the following example illustrates.

Example 1.11.2: As the reader can verify, the harmonic polynomial

$$u(x, y) = x^3 - 3x^2y - 3xy^2 + y^3$$

Satisfies the Laplace equation everywhere. To find a harmonic conjugate, we solve the Cauchy-Riemann equations (1.59). First of all,

$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 3x^2 + 6xy - 3y^2,$

And hence, by direct integration with respect to x,

$$v(x, y) = x^3 + 3x^2y - 3xy^2 + f(y),$$

Where$h(y)$-the “constant of integration” - is a function of y alone.

To determine h we substitute our formula into the second Cauchy-Riemann equation:

$$3x^2 - 6xy + h'(y) = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3x^2 - 6xy - 3y^2.$$

Theorem (1.9) allows us to differentiate u and v as often as desired Therefore $h'(y) = -3y^2$, and so $h(y) = -y^3 + c$, where c is a real constant. We conclude that every harmonic conjugate to $u(x, y)$ has the from

$$v(x, y) = x^3 + 3x^2y - 3xy^2 - y^3 + c.$$

Note that the corresponding complex function

$$\begin{aligned} &u(x, y) + iv(x, y) \\ &= (x^3 - 3x^2y - 3xy^2 + y^3) + i(x^3 + 3x^2y - 3xy^2 - y^3 + c) \\ &= (1 - i)z^3 + c \end{aligned}$$

Turns out to be a complex cubic polynomial.

Remark**:** On a connected domain $\Omega \subset R^2$,the harmonic conjugates (if any) to a given function $u(x, y)$ differ from each other by a constant:

$$\tilde{v}(x, y) = v(x, y) + c$$

Although most harmonic functions have harmonic conjugates, unfortunately this is not always the case. Interestingly, the existence or non-existence of a harmonic conjugate can depend on the underlying topology of its domain of definition. If the domain is simply connected, and so contains no holes, then one can *always* find a harmonic conjugate. On non-simply connected domains, there may not exist a single- valued harmonic conjugate to serve as the imaginary part of a complex function $f(z)$.

Example 1.11.3: The simplest example where the latter possibility occurs is the logarithmic potential

$$u(x, y) = \log r = \frac{1}{2}\log(x^2 + y^2).$$

This function is harmonic on the non-simply connected domain

$\Omega = \mathbb{C}\backslash\{0\}$,but is not the real part of any single-valued complex function. Indeed, according to (1.15), the logarithmic potential is the real part of the multiply- valued complex logarithm $\log z$, and so its harmonic conjugate is $ph\, z = \theta$, which cannot be consistently and continuously defined on all of Ω.On the other hand, on any simply connected sub domain$\tilde{\Omega} \subset \Omega$, one can select a continuous, single-valued branch of the angle $\theta = ph\, z$,which is then a bona fide harmonic conjugate to $\log r$ restrict to this subdomain.

The harmonic function

$$u(x, y) = \frac{x}{x^2 + y^2}$$

Is also defined on the same non-simply connected domain$\Omega = c\backslash\{0\}$ with a singularity at$x = y = 0$. In this case, there is a single-valued harmonic conjugate, namely

$$v(x, y) = -\frac{y}{x^2+y^2},$$

Which is defined on all of Ω.? Indeed, according to (1.59), these functions define the real and imaginary parts of the complex function $u + iv = 1/z$. Alternatively, one can directly check that they satisfy the Cauchy-Riemann equations (1.18).

We can, by the preceding remark, add in any constant to the harmonic conjugate, but this does not affect the subsequent argument.

Theorem 1.11.4: Every harmonic function $u(x, y)$ defined on a simply connected domain Ω is the real part of a complex valued function

$f(z) = u(x, y) + iv(x, y)$ which is defined for all $z = x + iy \in \Omega$.

Proof: We first rewrite the Cauchy-Riemann equations (1.18) in vectorial from as an equation for the gradient of v:

$$\nabla_v = \nabla^{\perp} \quad u, \quad where \quad \nabla^{\perp} u = \begin{pmatrix} -u_y \\ u_x \end{pmatrix} \tag{1.60}$$

Is known as the *skew gradient* of u. it is everywhere orthogonal to the gradient of u. And of the same length:

$$\nabla_u = \nabla^{\perp} u = 0, \qquad \|\nabla_{\mathrm{u}}\| = \|\nabla^{\perp} u\|.$$

Thus, we have established the important observation that the gradient of a harmonic function and that of its harmonic conjugate are mutually orthogonal vector fields having the same Euclidean lengths:

$$\nabla u . \nabla v \equiv 0, \qquad \| \nabla u \| \equiv \| \nabla v \| . \tag{1.61}$$

Now, given the harmonic function u, our goal is to construct a solution v to the gradient equation (1.61). A well-known result from vector calculus states the vector field defined by $\nabla^{\perp} u$ has a potential function v if and only if the corresponding line integral is independent of path, which means that

$$0 = \oint_c \nabla v . dx = \oint_c \nabla^{\perp} u . dx = \oint_c \nabla u . n \, ds, \tag{1.62}$$

For every closed curve $C \subset \Omega$. Indeed, if this holds, then a potential function can be devised† by integrating the vector field:

$$v(x, y) = \int_a^x \nabla v . dx = \int_a^x \nabla u . n \, ds. \tag{1.63}$$

Here a$\in \Omega$ is any fixed point, and, in view of path independence, the line integral can be taken over any curve that connects a to x =$(x, y)^T$.

If the domain Ω is simply connected then every simple closed curve $C \subset \Omega$bounds a sub domain$D \subset \Omega$with $C = \partial D$. Applying the divergence from of Green's Theorem (6.80), we find

$$\oint_c \nabla u . n \, ds = \iint_D \nabla . \nabla u \, dx \, dy = \iint_D \nabla u \, dx \, dy = 0,$$

Because u is harmonic. Thus, in this situation, we have proved the existence of a harmonic conjugate function. Q.E.D.

This assumes that the domain Ω is connected; if not, we apply our reasoning to each connected component.

Technically, we have only verified path-independence (1.33)when C is a simple closed curve, but this suffices to establish it for arbitrary closed curves; see the proof of proposition (1.51) for details.

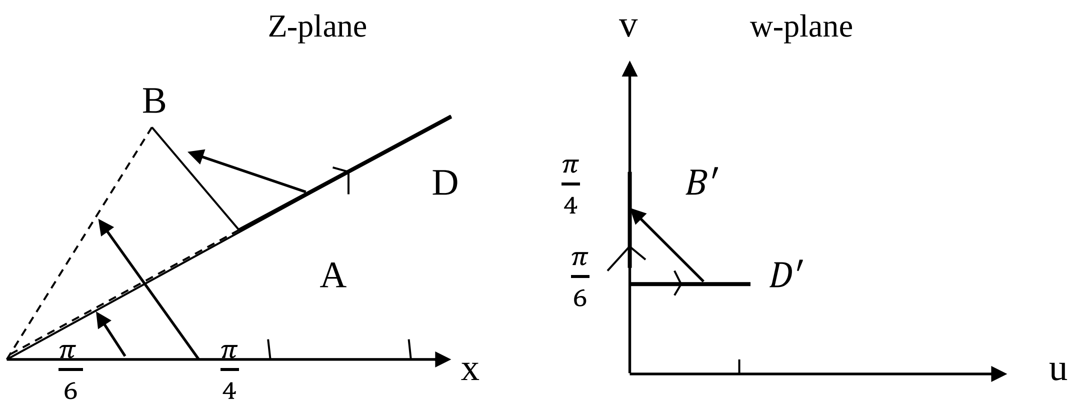
Chapter Two

Conformal Mapping

Conformal mapping can be utilized in different situation to facilitate the solution of partial differential equations. For instance it has been used in harmonic analysis and related topics such as fluid mechanics. In this chapter we shall introduce the concept of conformal mapping. Also bilinear transformations has been studied.

2.1 DEFINITION:

A mapping $w = f(z)$ that preserves the size and sense of the angle of intersection between any two curves intersecting at z_0 is said to be conformal atz_0. A mapping that is conformal at every point in a domain D is called conformal in D.



(a) (b) log 2

Figure 2.1

In a moment we will be able to show why$w = \log z$is conformal at the point A and also decide when functions $f(z)$ are conformal in general. The following theorem will be proved and used.

THEOREM 2.1.1: (Condition for Conformal Mapping)

Let $f(z)$be analytic in a domain D. Then $f(z)$ is conformal at every point in D where $f'(z) \neq 0$.

The proof requires our considering a curve C that is a smooth arc in the z-plane.

The curve is generated by a parameter t, which we might think of as time. Thus

$$z(t) = x(t) + iy(t)$$

Traces out the curve C as t increases (see Fig.2.1 (a)). We assume $x(t)$ and $y(t)$to be differentiable functions of t. The curve C can be transformed into an image curve C' (see Fig.2.1(b)) by means of the analytic function

$$w = f(z) = u(x, y) + iv(x, y).$$

$\lim_{\Delta t \to 0} \Delta z/\Delta t = dz/dtand\lim_{\Delta t \to 0} \Delta w/\Delta t = dw/dt$ are tangent to C and C' at $z_0$and w_0, respectively. Note that

$$\left.\frac{dz}{dt}\right|_{z_0} = \left.\frac{dx}{dt}\right|_{z_0} + i\left.\frac{dy}{dt}\right|_{z_0}$$

And that the slope of this vector is$(dy/dx)|_{z_0}$, the slope of the curve C at z_0.Similarly, dv/du_{w_0} is the slope of the curve C'at w_0.

From the chain rule for differentiation,

$$\frac{dw}{dt} = \frac{dw}{dz}\frac{dz}{dt} = f'(z)\frac{dz}{dt}.$$

Setting $t = t_0$ so that $z = z_0$ and $w = w_0$ in the preceding, we have

$$\left.\frac{dw}{dt}\right|_{w_0} = f'(z_0)\left.\frac{dz}{dt}\right|_{z_0}.$$

Equating the arguments of each side of the above, we obtain

$$arg\left.\frac{dw}{dt}\right|_{w_0} = \arg f'(z_0) + \arg\left.\frac{dz}{dt}\right|_{z_0}. \tag{2.1}$$

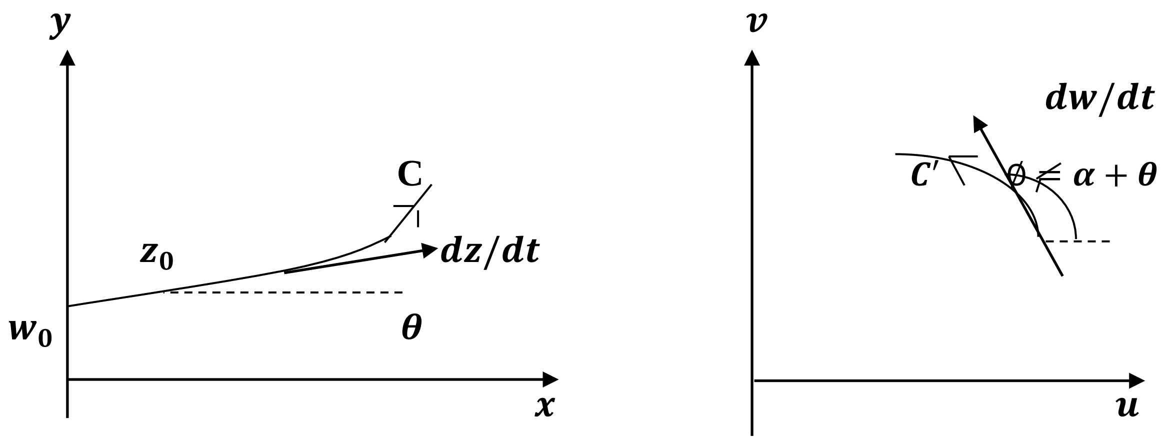
Let

$$\phi = \arg\left.\frac{dw}{dt}\right|_{w_0}, \quad \alpha = \arg f'(z_0), \qquad \theta = arg\frac{dz}{dt_{z_0}}.$$

Thus Eq.(2.1) becomes

$$\phi = \alpha + \theta. \tag{2.2}$$

We should recall that θ and ϕ specify the directions of the tangents to the curves C and C' at $z_0$and w_0, respectively (see Fig.2.1). Using Eq.(2.2), we realize that under the mapping $w = f(z)$ the directed tangent to the curve C, at z_0, is rotated through an angle $\alpha = \arg f'(z_0)$. The rotation of the tangent is shown in Fig.2.2(b)



(a) **(b)**

Figure 2.2

Another smooth arc, say C_1, intersecting C at the point z_0 with angle ψ (the angle between the tangents to the curves) can be mapped by $w = f(z)$ into the image curve C_1':the tangent to C_1 at z_0 is also rotated through the angle $f'(z_0) = \alpha$ by the mapping

The mapping $w = f(z)$ rotates the tangents to C and C_1 by identical amounts in the same direction. Thus the image curves C' and C_1' have the same angle of intersection ψ as do C and C_1. The sense (direction) of the intersection is also preserved, as shown in Fig.2.2.

If $f'(z_0) = 0$, the preceding discussion will break down since the angle

$\alpha = \arg\big(f'(z_o)\big)$,Through which tangents are rotated, is undefined.

There is no guarantee of a conformal mapping where $f'(z_0) = 0$.

one can show that if $f'(z_0) = 0$ the mapping cannot be conformal at z_0.

A value of z for which $f'(z) = 0$ is known as a critical point of the transformation.

EXAMPLE 2.1.2: Consider the contour C defined by $x = y, x > 0$ and the contour C_1 defined by $x = 1, y \geq 1$.We shall map these two curves using $w = 1/z$and verify that their angle of intersection is preserved in size and direction.

Our transformation is

$$w = \frac{1}{z} = \frac{1}{u+iv} = \frac{1}{x+iy} = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2},$$

So that

$$u = \frac{x}{x^2 + y^2}, \tag{2.3}$$

$$v = \frac{-y}{x^2 + y^2}. \tag{2.4}$$

On $C, y = x,$ which, when substituted in Eqs. (2.3) and (2.4), yields

$$u = \frac{1}{2x} = -v. \tag{2.5}$$

Since $x > 0$, we have $u \geq 0$and $v \leq 0$.the line defined by Eq. (2.5) is shown as C'in Fig.2.2(b). As we move outward from the origin along C, the corresponding image point moves toward the origin on C'since, according to Eq.(2.5), both u *and* v tend to zero with increasing x.

O n$C_1, x = 1$which, when used in Eqs. (2.5) and (2.6), yields

$$u = \frac{1}{1 + y^2}, \tag{2.6}$$

$$v = -\frac{y}{1 + y^2}. \tag{2.7}$$

This implies that

$$v = -uy \tag{2.8}$$

From Eq.(2.7) we easily obtain $y = \sqrt{1/u - 1}$ which, combined Eq.(2.8), yields$v = -\sqrt{u - u^2}$. We can square both sides of this equation and make some algebraic rearrangements to show that $(u - 1/2)^2 + v^2 = (1/2)^{2.}$

Thus points on C_1have their images on a circle of radius $1/2$, centered at $(1/2,0)$ in the $w-$plane. As y increases from 1 to ∞ along C_1, then, according to Eq.(2.8), the $u-$coordinate of the image point varies from $1/2$ to 0 along the circle. Since v remains negative (see Eq.(2.8), the image of $C_1$is the arc C_1' shown in Fig (2.3).

From plane geometry we recall that the angle between a tangent and a chord of a circle is $1/2$ the angle of the intercepted arc. Thus the angle of intersection between $C_1'$and C' in Fig.2.2(b) is 45̊, the same angle existing between C_1 and C. Observe in Figs.2.2(a,b) that sense of the angular displacement between the tangents to C and C_1 is the same as for C' and C_1'.

Suppose a small line segment, not necessarily straight, connecting the points z_0 and $z_0 + \Delta z$is mapped by means of the analytic transformation $w = f(z)$(see Fig (2.4). The image line segment connects the point $w_0 = f(z_0)$ with the point $w = f(z_0 + \Delta z)$.

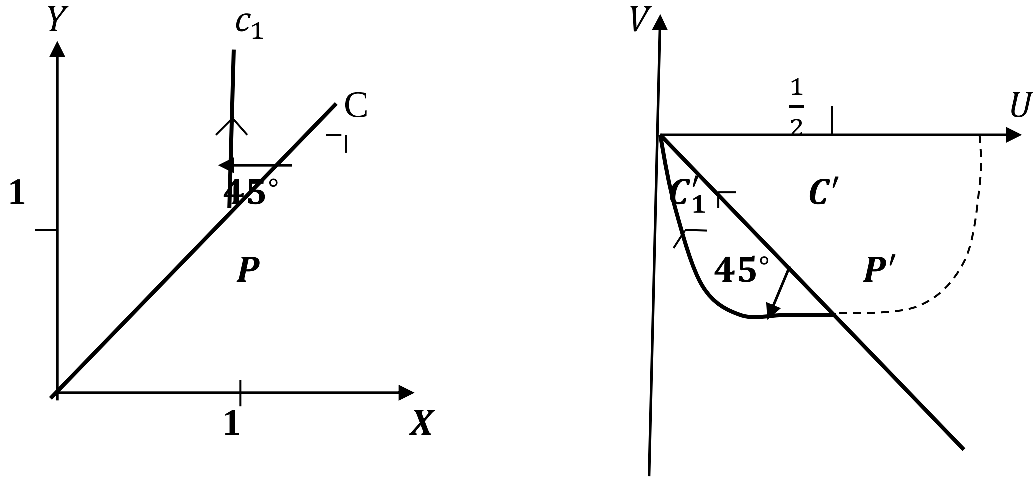


Figure 2.3

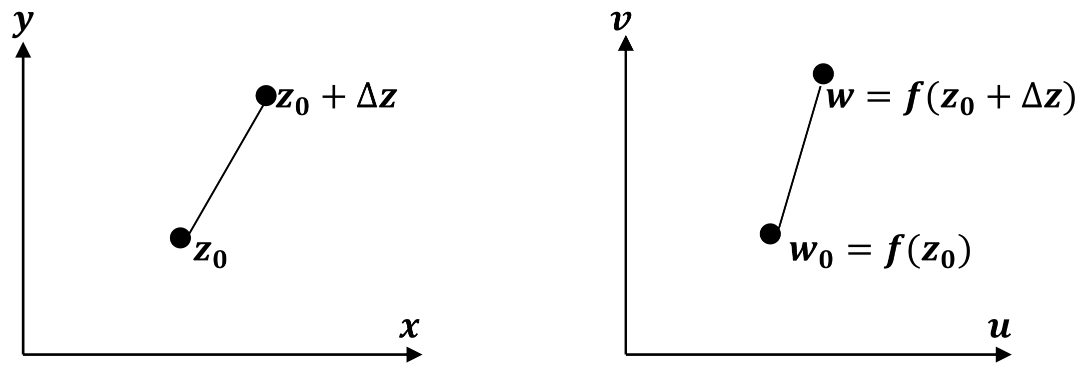


Figure 2.4

Now consider

$$|f'(z_0)| = \lim_{\Delta z \to 0} \left| \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \right| \tag{2.9}$$

Equation (2.9) follows from the definition of the derivative, Eq.(2.9), and this easily proved fact: $\left|\lim_{z\to z_0} g(z)\right| = \lim_{z\to z_0}|g(z)|$ when $\lim_{z\to z_0} g(z)$ exists.

The expression $\left|f\big((z_0+\Delta z) - f(z_0)\big)/\Delta z\right|$ is the approximate ratio of the lengths of the line segments in Figs. 2.3(b, a). Thus a small line segment straight at z_0 is magnified in length by approximately $|f'(z_0)|$ under the transformation $w = f(z)$.

As the length of this segment approaches zero, the amount of magnification tends to the limit $|f'(z_0)|$.

We see that if $f'(z_0) \neq 0$, all small line segments passing through z_0 are approximately magnified under the mapping by the same nonzero factor

$R = |f'(z_0)|$. A "small" figure composed of line segments and constructed near z_0 will, when mapped into the w-plane, have each of its sides approximately magnified by the same factor $|f'(z_0)|$. The shape of the new figure will conform to the shape of the old one although its size and orientation will typically have been altered. Because of the magnification in lengths, the image figure in the w-plane will have an area approximately $|f'(z_0)|^2$ times as large as that of the original figure. The conformal mapping of a small figure is shown in Fig. 2.5 The similarity in shapes and the magnification of areas need not hold if we map a "large figure since $f'(z)$ may deviate signification from $f'(z_0)$over the figure.

EXAMPLE 2.1.3: We discuss the way in which $w = z^2$ maps the grid

$x = x_1, x = x_{2,\ldots,}$

$y = y_1, y = y_{2,\ldots}$ (See Fig. 2.6(a) into the w-plane. We also verify that the angles of intersection are preserved and that a small rectangle is approximately preserved in shape under the transformation.

With $w = u + iv, z = x + iy,$ the transformation is

$u + iv = (x+iy)^2 = x^2 - y^2 + i2xy$ So that

$$u = x^2 - y^2, \tag{2.10}$$

$$v = 2xy. \tag{2.11}$$

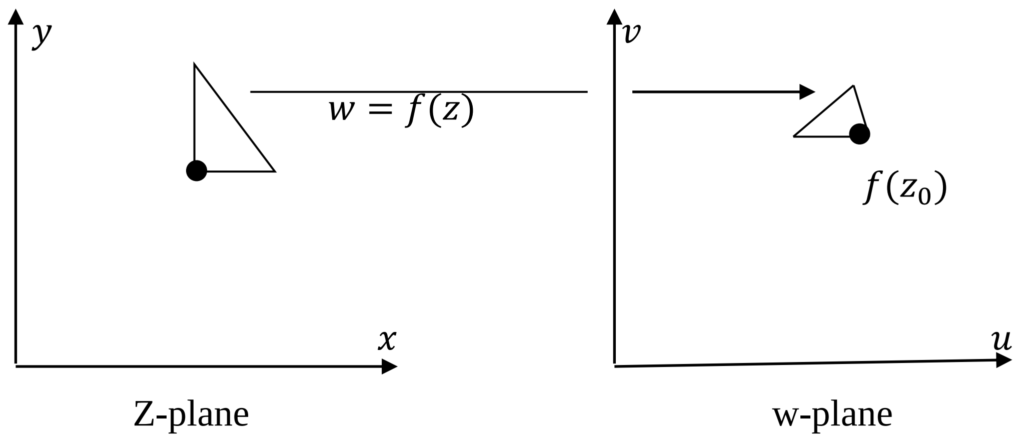


Figure 2.5

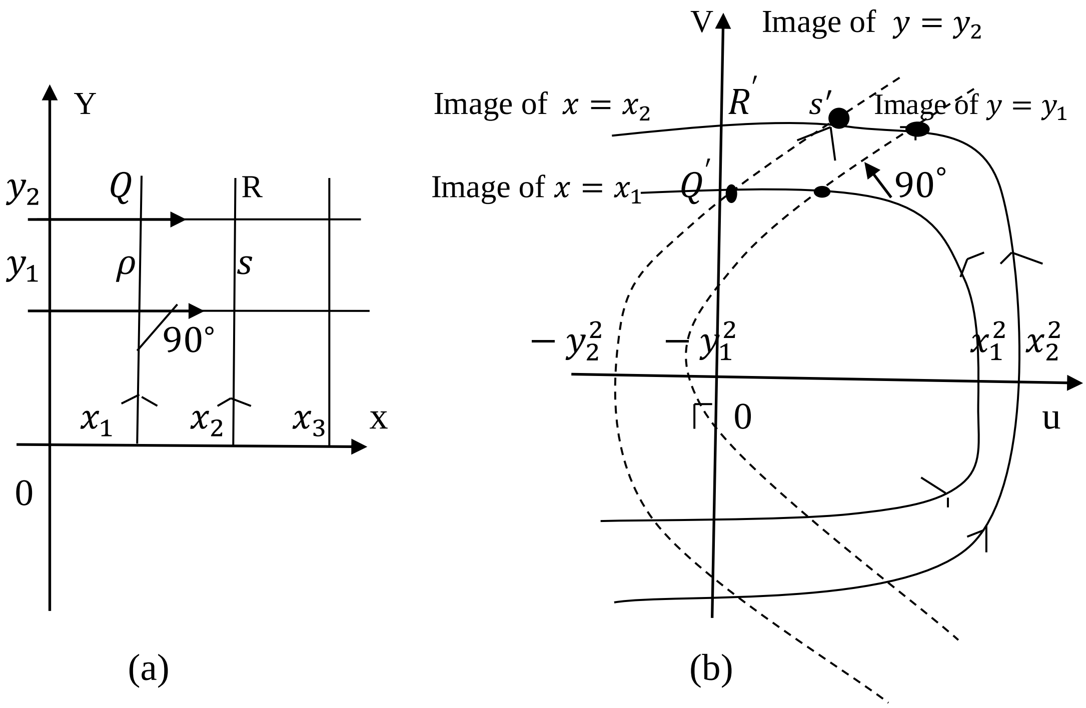


Figure 2.6

The line $x = x_1, -\infty \leq \boldsymbol{y} \leq \infty$,we have

$$u = x_1^2 - y^2, \tag{2.12}$$

$$v = 2x_1 y. \tag{2.13}$$

Can use Eq.(2.12) to eliminate y from Eq.(2.13) with the result that

$$u = x_1^2 - \frac{v^2}{4x_1^2} \tag{2.14}$$

Y-coordinate of a point on $x = x_1$ increases from $-\infty$ to ∞,Eq.(2.14) that v progresses from $-\infty$ to ∞ (if $x_1 > 0$).A parabola described by Eq. (2.14) is generated. This curve, which passes through $u = x_1^2, v = 0$, is shown by the solid line in Fig. 2.6. This parabola is the image of $x = x_1$. Also illustrated is the image of $x = x_2$, where $x_2 > x_1$.

Mapping a horizontal line $y = y_1, -\infty \leq x \leq \infty$, we have from Eqs. (2.15) and (2.16) that

$$u = x^2 - y_1^2, \tag{2.15}$$

$$v = 2xy_1. \tag{2.16}$$

Using Eq.(2.15) to eliminate x from Eq. (2.16), we have

$$u = \frac{v^2}{4y_1^2} - y_1^2. \tag{2.17}$$

This is also the equation of a parabola-one opening to the right. One can easily show that, as the x-coordinate of a point moving along $y = y_1$ increases from $-\infty\ to\ \infty$, its image traces out a parabola shown by the broken line in Fig. (2.6(b)). The direction of progress is indicated by the arrow. Also shown in Fig (2.6(b)) is the image of the line $y = y_2$. The point ρ at (x_1, y_1) is mapped by $w = z^2$into the image $u_1 = x_1^2 - y_1^2, v_1 = 2x_1, y_1$ shown as ρ' in Fig. 2.6(b). ρ'lies at the intersection of the images 0f $x = x_1$ and $y = y_1$. Although these curves have two intersections, only the upper one corresponds to p' since Eq.(2.17) indicates that $v > 0$ when $x > 0$ and $y > 0$.

The slope of the image of $x = x_1$is found from Eq. (2.17). Differentiating implicitly, we have

$$du = -\frac{2v\, dv}{4x_1^2},$$

Or:

$$\frac{du}{dv} = \frac{-v}{2x_1^2}. \tag{2.18}$$

Similarly, from Eq. (2.18), the slop of the image of $y = y_1$ is

$$\frac{du}{dv} = \frac{v}{2y_1^2}. \tag{2.19}$$

Substituting $v_1 = 2x_1y_1$, which is valid at the point of intersection, into Eqs. (2.18) and (2.19), we find that the respective slopes are $-y_1/x_1$ and x_1/y_1.

As these values are negative reciprocals of each other, we have established the orthogonality of the intersection of the two parabolas atp'. Since $x = x_1$ intersection. Notice that the rectangular region with corners at $p, Q, R,$ and S shown shaded in Fig.2.6(a) is mapped onto the rectangular region having corner at nearly p', Q', R', and S' shown shaded in Fig.2.6(b).

With $f(z) = z^2$, we have $f'(z) = 0$ at $z = 0$.our theorem on conformal mapping no longer guarantees a conformal transformation at $z = 0$. Lines intersecting here require special attention. The vertical line $x = 0, -\infty < y < \infty$is transformed (see Eqs. (2.18 and 2.19)) into $u = -y^2, v = 0$, the negative real axis.

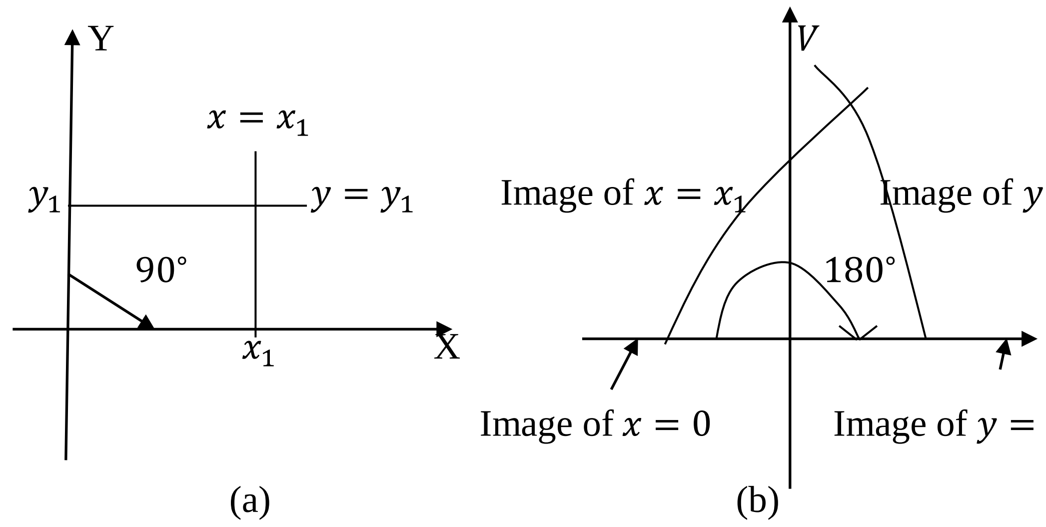


Figure 2.7

The horizontal line $y = 0, -\infty < x < \infty$ is, by the same equations, mapped into $u = x^2, v = 0$,the positive u-axis.

The lines $x = 0$ and $y = 0$, which intersect at the origin at 90, have images in the $u\ v$-plane intersecting at 180 (see Fig. 2.7(b)).

Notice that the small rectangle $0 \leq x \leq x_1, 0 \leq y \leq y_1$ in Fig. (2.8 (a)) is mapped onto the nonrectangular shape in Fig. 2.7(b). The breakdown of the conformal property is again evident.

EXAMPLE 2.1.4: We shall discuss the way in which the infinite strip $0 \leq Im\ z \leq a$, is mapped by the transformation

$$w = u + iv = e^z = e^{x+iy}. \qquad (2.20)$$

Take $0 \leq a < 2\pi$.

Into we first note the desirability of taking $0 \le a < 2\pi$.It arises from the periodic property $e^z = e^{z+2\pi i}$. By making the width of the strip (see Fig. 2.7(a)) less than 2π, we avoid having two points inside with identical real parts and imaginary parts that differ by 2π. A pair of such points are mapped identical locations in the w-plane and a one-to-one mapping of the strip becomes.

The bottom boundary of the strip,$y = 0. \ -\infty < x < \infty$, is mapped by our setting $y = 0$ in Eq. (2.20) to yield $e^x = u + iv$. As x ranges from $-\infty$ to ∞, the entire line $v = 0 \ 0 \le u \le \infty$ is generated. This line is shown in Fig. (2.7(b)). The points A', b', and c' are the images of A, B and C in Fig.(2.7(a).)

The upper boundary of the strip is mapped by our putting $y = a$ in Eq. (2.20) so that

$$u = e^x \cos a, \tag{2.21}$$

$$v = e^x \sin a. \tag{2.22}$$

Dividing the second equation by the first, we have $v/u = \tan a$ or

$$v = u \tan a, \tag{2.23}$$

Which is the equation of a straight line through the origin in the uv-plane. If $\sin a$ and $\cos a$ are both positive (0< $a < \pi/2$), we see from Eqs.(2.21) and (2.22) that, as x ranges from $-\infty \ to \ \infty$, only that portion of the line lying in the first quadrant of the w-plane is generated. Such a line is shown in Fig. (2.7(b)).

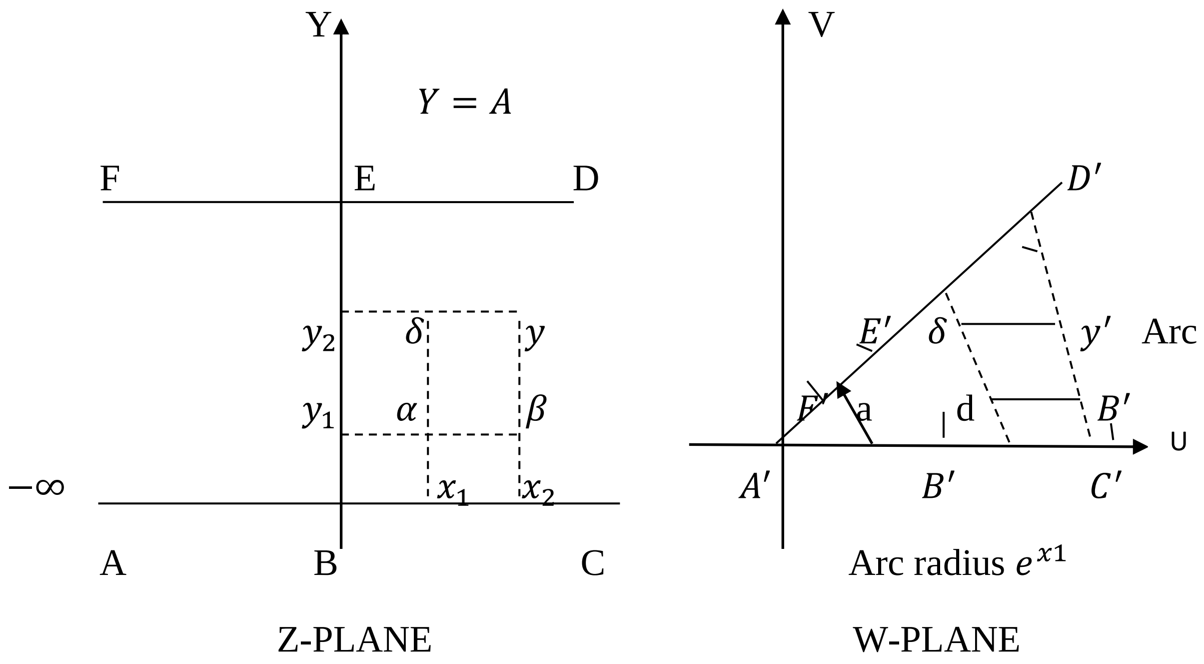
It is labeled with the points $D', E', and\ F'$, which are the images of D,E, and f in Fig.(2.7(a)). The slope of the lines is $\tan a$, and it makes an angle a with the real axis.

If a satisfied the condition

$$\pi/2 < a < \pi \ or\ \pi < a < 3\pi/2 \ or\ \ 3\pi/2 < a < 2\pi,$$

Lines lying in, respectively, the second or third or fourth quadrant would have been obtained.

The cases $\pi/2 = a, 3\pi/2 = a, and\ \pi = a$ yield lines along the coordinate axes.



(a) (b)

Figure2.8

The strip in Fig.(2.8(a)) is mapped onto the wedge –Shaped region shown in Fig.(2.8(b)). An important mapping occurs if the strip is chosen to have width $a = \pi$. The upper boundary passing through F,E,D in Fig.(2.8(a)) is transformed into the negative real axis in the w-plane. The wedge shown in Fig.(2.8 (b)) evolves into the half plane $v \geq 0$, which is now the image of the strip.

The inverse transformation of Eq. (2.23), that is,

$$z = \log w \tag{2.24}$$

Can be used to obtain the image in the z-plane of any point in the wedge of Fig. (2.8 (b)). Of course, $\log w$ is multivalued, but there is only one value of $\log w$ that lies in the strip of Fig. (2.8(a)). The shaded rectangular area boundary by the lines $x = x_1, x = x_2, y = y_1, y = y_2$ shown in Fig. 2.8(a) is readily mapped on to a region in the w-plane. With $x = x_1$ we have Eq.(2.23) that

$$u = e^{x_1} \cos y,$$

$$v = e^{x_1} \cos y,$$

So that:

$$u^2 + v^2 = e^{2x_1},$$

Which is the equation of a circle of radius e^{x_1}.

The line segment $x = x_1, 0 \leq y \leq a$ is transformed into an arc lying on this circle and illustrated in fig. 2.8(b).

The line segment $x = x_2, 0 \leq y \leq a(x_2 > x_1)$ is transformed into an arc of larger radius, which is also shown.

The images of the lines $y = y_1$ and $y = y_2$ are readily found from

Eqs.(2.23) and (2.24) if we replace a by $y_1$or y_2. Rays are obtained with slopes $\tan y_1$and $\tan y_2$, respectively. These rays (see Fig. 2.8(b) together with the arcs of radius e^{x_1} and e^{x_2} from the boundary of a nonrectangular shape (shaded in Fig. 2.8(b) that is the image of the rectangle shown in Fig 2.8(a). Notice that the corners of the nonrectangular shape have right angles as in the original rectangle.

EXAMPLE 2.1.5: We investigate the way in which $w = \sin z$ maps the strip $y \geq 0, -\pi/2 \leq x \leq 2$.Because $\sin z$ is periodic, that is, $\sin z = \sin(z + 2\pi)$, any two points the z-plane having identical imaginary parts and real parts differing by 2π (or as multiple) will be mapped into identical locations in the w-plane. This situation cannot occur for points in the given strip (see Fig. 8.3-5(a)) because its width isπ.

Rewriting the given transformation using Eq. (2.24), we have

$$w = (u + iv) = \sin(x + iy) \sin x \, cosh\, y + i \cos x \ \sin h\, y,$$

Means

$$u = \sin x \cosh y \tag{2.25}$$

$$v = \cos x \ \sin h\, y \tag{2.26}$$

Is bottom boundary of the strip is $y = 0, -\pi/2 \leq x \leq \pi/2$.Hereu= $\sin x$. As we move from $x = -\pi/2$ to $x = \pi/2$along this bottom boundary, the

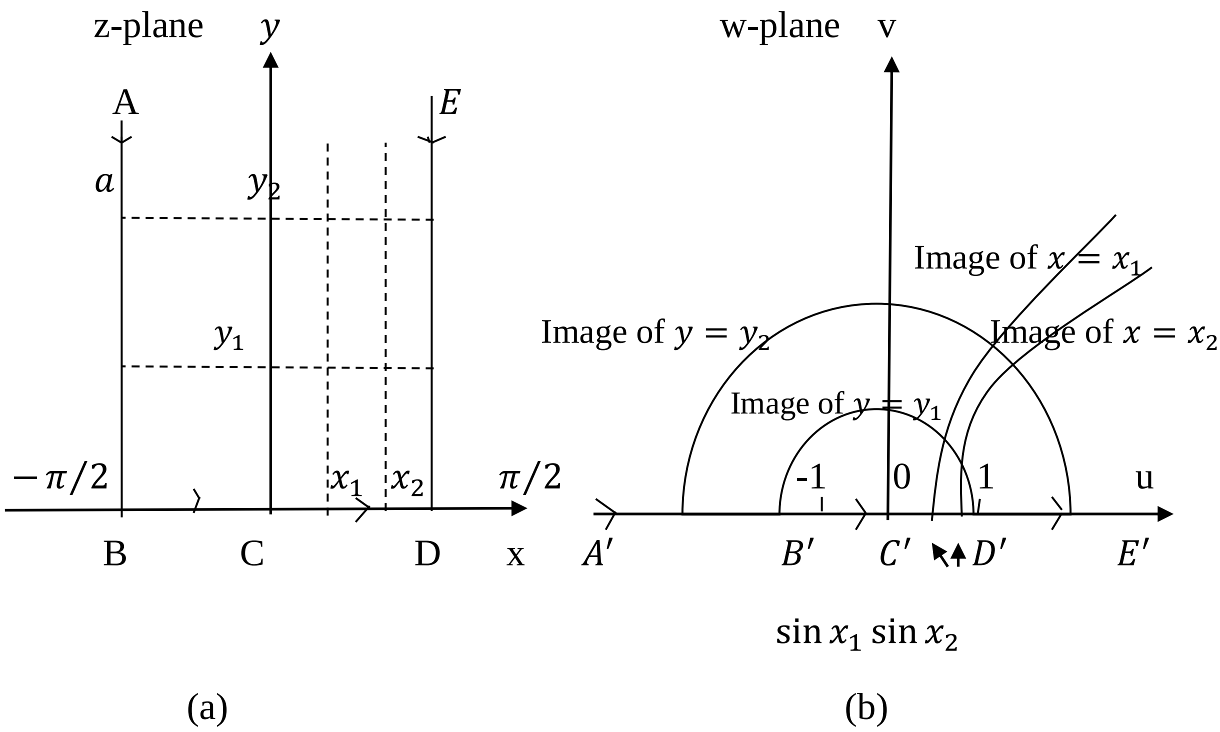


Figure 2.9

Image point in the w-plane advances from -1 to +1 along the line $v = 0$.

The image of the line segment B, C, D of Fig. (2.9(a)) is the line B', C', D' in Fig. (2.9(b)).

Along the left boundary of the given strip, $x = -\pi/2, y \geq 0$. From Eqs. (2.25) and (2.26), we have:

$$u = \sin x_1 \cosh y, \tag{2.27}$$

$$v = \cos x_1 \sin hy. \tag{2.28}$$

Recalling that $cosh^2 y - \sin h^2\, y = 1$, we find that

$$\frac{u^2}{sin^2 x_1} - \frac{v^2}{cos^2 x_1} = 1,$$

Which is the equation of a hyperbola. We will assume that

$0 < x_1 < \pi/2$. Because $y \geq 0$, Eqs. (2.27) and (2.28) reveal that only that portion of the hyperbola lying in the first quadrant of the w-plane is obtained by this mapping. This curve is shown in Fig. (2.10(b)); also indicated is the image of $x = x_2, y \geq 0$,where$x_2 > x_1$. If or x_2 had been negative, the portions of the hyperbolas obtained would be in the second quadrant of the w-plane.

The horizontal line segment $y = y_1 (y_1 > 0), -\pi/2 \leq x \leq \pi/2$ in Fig. (2.10(a)) can be mapped into the w-plane with the aid of Eqs. (2.28), which yield

$$u = \sin x \cosh y_1, \tag{2.29}$$

$$v = \cos x \sin h\, y_{1.} \tag{2.30}$$

Since $sin^2 x + cos^2 x = 1$, we have

$$\frac{u^2}{cosh^2 y_1} + \frac{v^2}{sinh^2 y_1} = 1,$$

Which describes an ellipse. Because $y_1 > 0$ and $-\pi/2 \leq x \leq \pi/2$, Eq. (2.29) indicates that $v \geq 0$, that is, only the upper half of the ellipse is the image of the given segment. In Fig.(2.10(b)) we have shown elliptic arcs that are the images of the two horizontal line segments inside the strip in Fig.(2.10(a)).

The rectangular area $x_1 \leq x \leq x_2, y_1 \leq y \leq y_2$ in the z-plane is mapped onto the four-sided figure bounded by two ellipse and two hyperbolas, which we see shaded in Fig. (2.10(b)). The four corner of this figure have right angles.

It should be evident that interior of our semi infinite strip, in the z-plane, is mapped by $w = \sin z$ on to the upper half of the w-plane.

The transformation of other strip is considered in this section.

The transformation in $w = \sin z$ fails to be conformal where $d \sin z/dz = \cos z = 0$.This occurs at $z = \pm\pi/2$. The line segments AB and BC in Fig. (2.10(a)) intersect at $z = -\pi/2$ at right angles. However, their images intersect in the w-plane at 180 angle. The same phenomenon occurs for segments CD and DE.

Suppose we needed to map a large number of vertical line$x = x_1, x = x_2$, etc., and horizontal lines $y = y_1, y = y_2, etc.$, using $w = \sin z$ Using this transformation, we could find and laboriously plot in the w-plane the image of each line, as was just done in a few cases. However, there is some useful computer software available that can save us much work. Using the complex variables program entitled $f(z)$available lying in the space $-\pi/2 \leq x \leq \pi/2, 0 \leq y \leq \pi/2$ into the u, v-plane. The result as illustrated in Fig. (2.8.) The large horizontal and vertical "tic" marks are at$x = \pm 1, y = \pm 1, u = \pm 1, v = \pm 1$ and serve to establish the scale.

Other functions are available with the software to perform different mappings.

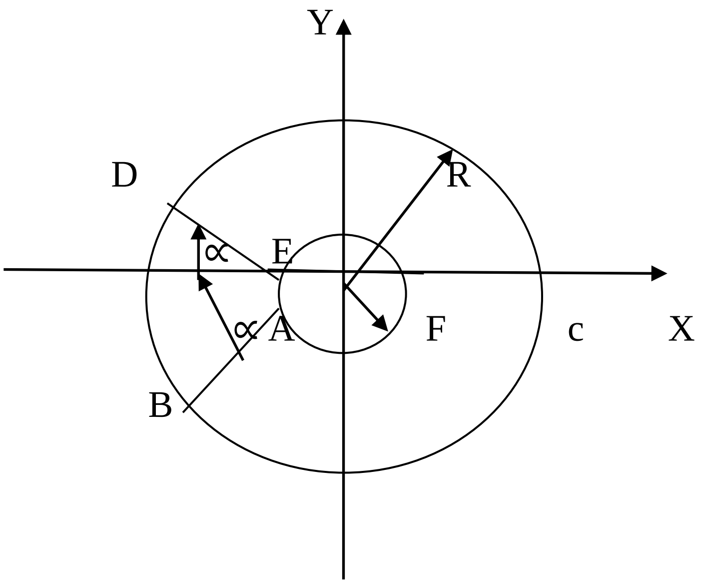


Figure 2.10

(a)Consider the half-disc-shaped domain $|z| < 1, im\ z > 0$. Find the image of this domain under the transformation

$$w = \left(\frac{z-1}{z+1}\right)^2.$$

Hint: Map the semicircular arc bounding the top of the disc by putting $z = e^{i\theta}$ in the above formula. The resulting expression reduces to a simple trigonometric function.

(b) What inverses transformation $z = g(w)$ will map the domain found in part (a) back onto the half-disc? State the appropriate branches of any square roots.

In the Theorem 2.1.1 there is a remark asserting that if $f'(z) = 0$ at any point in a domain, then $w = f(z)$ cannot map that domain one to one. However, in Example 2.1.2 we found that a wedge containing $z = 0$ can be mapped one to one by $f(z) = z^0$ even though $f'(0) = 0$.

2.2 The Bilinear Transformation

The bilinear transformation defined by

$$w = \frac{az+b}{cz+d}, \quad where\ a, b, d\ are\ complex\ constants, \tag{2.31}$$

Which is also known as the linear fractional transformation or the Möbius transformation is especially useful in the solution of a number of physical problems, some of which are discussed in this chapter. The utility of this transformation arises from way in which it maps straight lines and circles.

Equation (2.31) defines a finite value of w for all $z \neq -d/c$. One generally mimes that

$$ad \neq bc. \tag{2.32}$$

If we take $ad = bc,$ we can readily show that Eq. (2.32) reduces to a constant value of w, that is, $dw/dz = 0$ for all $z,$ and the mapping is not conformal nor especially interesting since all points in the z-plane are mapped into one point in the w-plane.

In general, from Eq. (2.32) we have

$$\frac{dw}{dz} = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{ad - bc}{(cz+d)^2}, \tag{2.33}$$

Which is nonzero if Eq. (2.33) is satisfied.

The inverse transformation of Eq. (2.33) is obtained by our solving this equation for z. We have

$$z = \frac{-dw + b}{cw - a}, \tag{2.34}$$

Which is also a bilinear transformation and defines a finite value of z for all $w \neq a/c$.

For reasons that will soon be evident, we now employ the extended w-plane and the extended z-plane (see section 1.5), that is, planes that include the "points" $z = \infty$ and $w = \infty$.

Consider Eq. (2.34) for the case $c = 0$.we have

$$z = \frac{d}{a}w - \frac{b}{a}, \tag{2.35}$$

Which also indicates that $z = \infty$ and $w = \infty$ are images (for $c = 0$). If $c \neq 0$, Eq. (2.35) shows that $w = \infty$ has image $z = -d/c$, where as $w = a/c$ has image $z = \infty$. In summary, Eq. (2.35) provides a one-to-one mapping of the extended z-plane onto the extended w-plane.

Suppose now we regard infinitely long straight lines in the complex plane as being circles of infinite radius (see Fig. 2.11). Thus we use the word "circle" to mean not only circles in the conventional sense but in finite straight lines as well.

Circle (without the quotation marks) will mean a circle in the conventional sense.

We will now prove the following theorem.

THEOREM2.2.1: The bilinear transformation always transformscirclesinto circles.

Our proof of Theorem 2.2.1 begins with a restatement of Eq. (2.35):

$$w = \frac{a}{c} + \frac{bc - cd}{c}\frac{1}{cz + d}, \qquad (2.36)$$

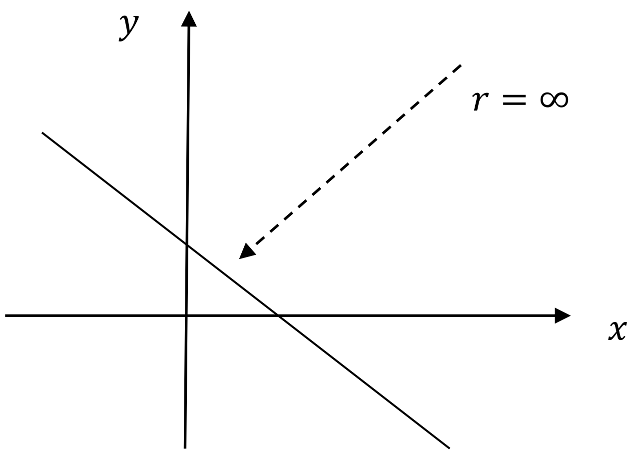


Figure 2.11

Where we assume $c \neq 0$.If we put Eq. (2.36) over a common denominator, its equivalence to Eq. (2.36) becomes apparent.

The transformation described by Eq. (2.36) can be treated as a sequence of mappings. Consider a transformation involving a mapping from the z-plane into the w_1-plane, from the w_1-plane into the w_2-plane, and so on, according to the following scheme:

$$w_1 = cz, \qquad (2.37a)$$

$$w_2 = w_1 + d = cz + d, \qquad (2.37b)$$

$$w_3 = \frac{1}{w2} = \frac{1}{cz + d}, \tag{2.37c}$$

$$w_4 = \frac{bc - ad}{c} e = w_3 = \frac{bc - ad}{c(cz + d)}, \tag{2.37d}$$

$$w = \frac{a}{c} + w_4 = \frac{a}{c} + \frac{bc - ad}{c(cz + d)}. \tag{2.37e}$$

Equation (2.37e) confirms that these five mappings are together equivalent to Eq. (2.37).

There are three distinctly different kinds of operations contained in Eqs. (2.37a). Let k be a complex constant. There are translations of the form

$$w = z + k, \tag{2.38}$$

In Eqs. (2.37b). There are rotation-magnifications of the form

$$w = kz, \tag{2.39}$$

In Eqs. (2.37a,d). And there are inversions of the form

$$w = \frac{1}{z}, \tag{2.40}$$

this manner the entire left side of Eq. (2.40) can be written in terms of $z_1, z_2, z_3$and z_4. After some simple algebra, we obtain the right side of Eq. (2.40). The reader should supply the details.

If one of the points $z_1, z_2, ...$ is at infinite, the invariance of the cross-ratio must be proved differently. If, say, $z_1 = \infty$, its image is

$w_1 = a/c$ (See, for example, Eq. 2.40 as $z_i = z_1 \to \infty$). Thus the left side of Eq. (2.40) becomes

$$\frac{\left(\frac{a}{c} - w_2\right)(w_3 - w_4)}{\left(\frac{a}{c} - w_4\right)(w_3 - w_2)}.$$

If Eqs. (2.39) and (2.40) are used in this expression, the values w_2, w_3, and w_4 can be rewritten in terms of $z_2, z_3, . z_4$. After some manipulation, the expression $(z_3 - z_4)/(z_3 - z_2)$ is obtained. This is the cross-ratio(z_1, z_2, z_3, z_4) when $z_1 = \infty$.

The invariance of the cross-ratio is useful when we seek the bilinear transformation capable of mapping three specific points z_1, z_2, z_3 into three specific images w_1, w_2, w_3. The point z_4 Eq. (2.40) is taken as a general point z whose image is w (instead ofw_4). Thus our working formula becomes

$$\frac{(w_1 - w_2)(w_3 - w)}{(w_1 - w)(w_3 - w_2)} = \frac{(z_1 - z_2)(z_3 - z)}{(z_1 - z)(z_3 - z_2)}. \tag{2.41}$$

Which must be suitably modified if any point is at ∞. The solution of w in terms of z yields the required transformation.

EXAMPLE 2.2.2: We Find the bilinear transformation that maps
$z_1 = 1, z_2 = i, z_3 = 0$ into $w_1 = 0, w_2 = -1, w_3 = -i$.
First we substitute these six complex numbers into the appropriate location in Eq. (2.41) and obtain

$$\frac{\big(0 - (-1)\big)(-i - w)}{(0 - w)(-i + 1)} = \frac{(1 - i)(0 - z)}{(1 - z)(0 - i)}$$

With some minor algebra, we get

$$w = \frac{i(z - 1)}{z + 1} \tag{2.42}$$

This result can be checked by our letting z assume the three given values 1, i, and 0.

The desired values of w are obtained.

EXAMPLE 2.2.3: For the transformation found in Example 2.2.2, what is the image of the circle passing through $z_1 = 1, z_2 = i, z_3 = 0$, and what is the image of the interior of this circle? To answer these questions, be given circle is shown in Fig. 2.12(a). From elementary geometry, is center is found to beat$(1 + i)/2$, and its radius is $1/\sqrt{2}$. The circle is described by

$$\left| z - \frac{(1 + i)}{2} \right| = \frac{1}{\sqrt{2}}$$

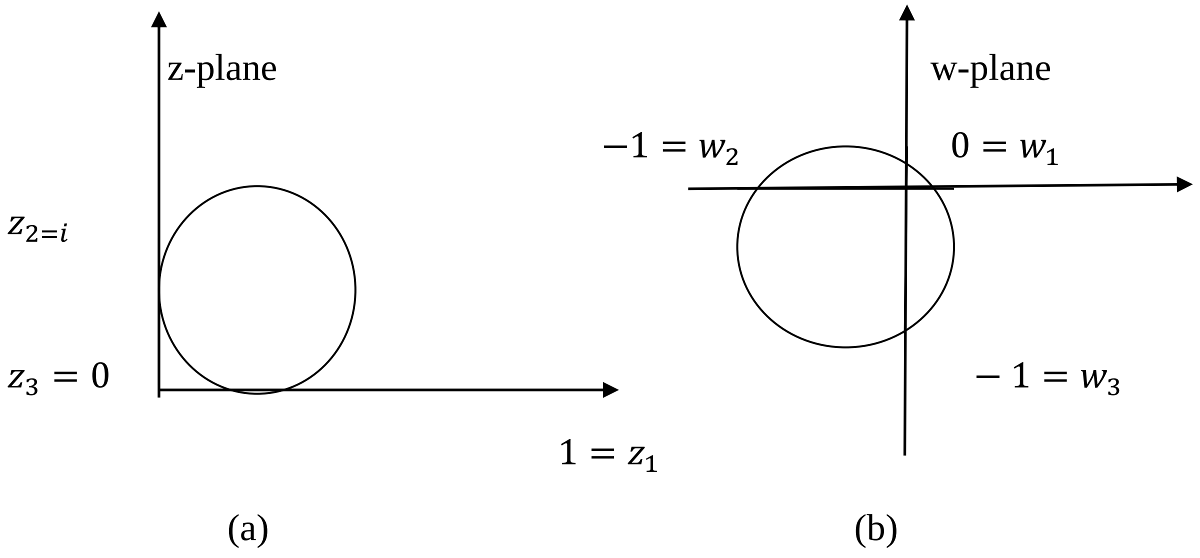


Figure2.12

The image of the circle under Eq. (2.42) must be a straight line or circle in the w-plane. The image is known to pass through

$w_1 = 0, w_2 = -1, w_3 = -i$. The circle determined by these three points is shown in Fig. 2.42(b) (no straight line can connect $w_1, w_2$and w_3) and is described by

$$\left|w + \frac{(1+i)}{2}\right| = \frac{1}{\sqrt{2}} \qquad (2.43)$$

This disc-shaped domain

$$\left|z - \frac{(1+i)}{2}\right| < \frac{1}{\sqrt{2}},$$

Is mapped onto the domain

$$\left|w + \frac{(1+i)}{2}\right| < \frac{1}{\sqrt{2}}.$$

EXAMPLE 2.2.4: Let us Find the bilinear transformation that maps

$z_1 = 1, z_2 = i, z_3 = 0$ in to $w_1 = 0, w_2 = \infty, w_3 = -i.$
Note that $z_1, z_2,$ and z_3 are same as in Example 3. We again employ Eq. (2.43). However, since $w_2 = \infty$, the ratio $(w_1 - w_2)/(w_3 - w_2)$ on the left must be replaced by1. Thus

$$\frac{-\boldsymbol{i}-\boldsymbol{w}}{-\boldsymbol{w}} = \frac{(\mathbf{1}-\boldsymbol{i})(-\boldsymbol{z})}{(\mathbf{1}-\boldsymbol{z})(-\boldsymbol{i})},$$

Whose solution is

$$w = \frac{1-z}{i-z}. \qquad (2.44)$$

Note that the circle in Fig. (2.12(a)) passing through 1 and I and 0 is transformed into a "circle" passing through 0 and ∞ and $-i,$ that is, an infinite straight line lying along the imaginary axis in the w-plane. The half of this line is the image of the interior of the circle in Fig.(2.12(a)), as the reader can readily verify.

EXAMPLE 2.2.5: We find the transformation that will map the domain $0 < \arg z < \frac{\pi}{2}$from the z-plane on to $|w| < 1$ in the w-plane (see Fig. 2.13).

The boundary of the given domain in the z-plane, that is, the positive x- and y-axes, must be transformed into the unit circle $|w| = 1$ by the required formula. A bilinear transmutation will map an infinite straight line into a circle but cannot transform a line with a bend into a circle. (why?) Hence our answer cannot be a bilinear transformation.

Notice, however, that the transformation

$$s = z^2 \qquad (2.45)$$

(See Example 2.2.2, section 2.2) will map our sector onto the upper half of the s-plane. If we can find a second transformation that will map the upper half of the s-plane onto the interior of the unit circle in the w-plane, we can combine the two mappings into the required transformation.

We observe that

$$w = e^{iy}\frac{(s-p)}{(s)-\bar{p}}, \quad where\ y\ is\ a\ real\ number\ \ and\ \ im\, p \qquad (2.46)$$

Will transform the real axis from the s-plane into the circle $|w| = 1$ and map the domain im$s > 0$ onto the interior of this circle.

Combining Eqs. (2.45) and (2.46), we have as our result

$$w = e^{iy}\frac{(z^2-p)}{(z^2-p)}, \ where\ y\ is\ a\ real\ number\ \ and\ \ Im\, p \quad > 0 \qquad (2.47)$$

$$w = \frac{z^2-i}{z^2+i}.$$

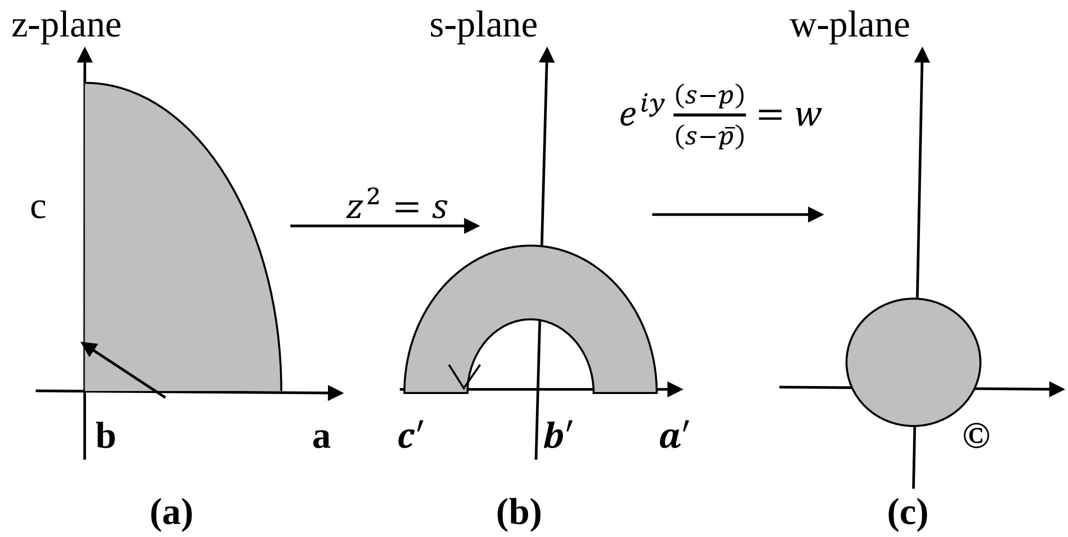
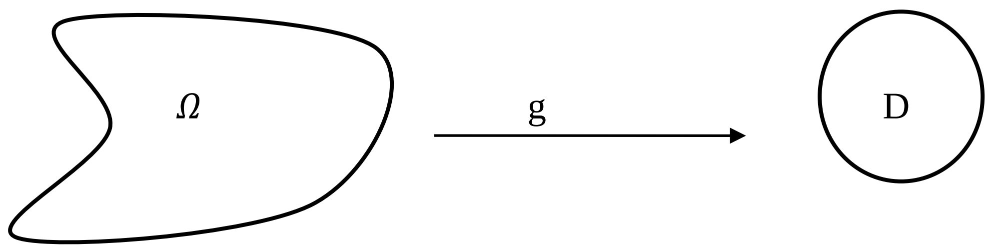


Figure 2.13

As we now know, complex functions provide an almost inexhaustible supply of harmonic functions, that, is, solutions of the two-dimensional Laplace equation. Thus, to solve an associated boundary value problem, we "merely" find the complex function whose real part matches the prescribed boundary conditions. Unfortunately, even for relatively simple domains, this remains a daunting task.

The one case where we do have an explicit solution is that of a circular disk, where the Poisson integral formula provides a complete solution to the Dirichlet boundary value problem. Thus, an evident solution strategy for the corresponding boundary value problem on a more complicated domain would be to transform it into a solved case by an inspired change of variables.

The intimate connection between complex analysis and solutions to the Laplace equation inspires us to look at changes of variables defined by complex functions. To this end



Mapping to the Unit Disk.

We will re-interpret complex analytic function

$$\varsigma = g(z) \quad or \quad \xi + in = p(x,y) + iq(x,y) \tag{2.48}$$

As a mapping that takes a point$z = x + iy$ belonging to a prescribed domain $\Omega \subset \mathbb{C}$ to a point $\varsigma = \xi + in$ belong to the image domain $D = g(\Omega) \subset \mathbb{C}$.in many cases, the image domain D is the unit disk, as in Mapping to the Unit Disk, but the method can also be applied to more general domains. In order to unambiguously relate functions on Ω to functions on D, we require that the analytic mapping to the Unit Disk be one-to-one, so that each point ς ∈ D comes from a unique point $z \in \Omega$. As a result, the inverse function $z = g^{-1}(\boldsymbol{\varsigma})$is a well-defined map from D back to Ω, which we assume is also analytic on D. The calculus formula for the derivative of the inverse function

$$\frac{d}{d\varsigma} g^{-1}(\boldsymbol{\varsigma}) = \frac{1}{g'(\boldsymbol{z})} at \qquad \varsigma = g(z), \tag{2.49}$$

Remains valid for complex functions. It implies that derivative of $g(z)$ must be nonzero everywhere in order that $g^{-1}(\boldsymbol{\varsigma})$be differentiable. This condition,

$$g'(z) \neq 0 \qquad at\ every\ point \qquad z \in \Omega, \tag{2.50}$$

will play a crucial role in the development of the method. Finally, in order to match the boundary conditions, we will assume that mapping extends continuously to the boundary $\partial\Omega$ and maps it, one-to-one, to the boundary ∂D of the image domain.

Before trying to apply this idea to solve boundary value problems for the Laplace equation, let us look at some of the most basic examples of analytic mapping.

Example 2.2.6: The simplest nontrivial analytic maps are the translations

$$\varsigma = z + \beta = (x + a) + i(y + b), \tag{2.51}$$

Where $\beta = a + ib$ is a fixed complex number the effect of (2.38) is to translate the entire complex plane in the direction and distance prescribed by the vector $(a, b)^T$.In particular, (2.37) maps the disk $\Omega = \{|z + \beta| < 1\}$ of radius 1 and centre at the point-β to the unit disk $D = \{|\boldsymbol{\varsigma}| < 1\}$**.**

Example 2.2.7: There are two types of linear analytic maps. First is the scaling

$$\varsigma = pz = px + ipy, \tag{2.52}$$

Figure 2.14 The mapping $\varsigma = e^z$.

Where$p \neq 0$ is a fixed nonzero real number. This maps the disk $|z| < 1 / |p|$ to the unit disk $|\boldsymbol{\varsigma}| < 1$.second are the rotations

$$\varsigma = e^{i\phi}z = e^{i\emptyset}(x, y) \qquad (2.53)$$

This rotates the complex plane around the origin by a fixed (real) angleϕ.

These all map the unit disk to itself.

Example2.2.8: Any non-constant *affine transformation*

$$\varsigma = \alpha z + \beta, \alpha \neq 0, \qquad (2.54)$$

Defines an invertible analytic map on all of $\mathbb{C}$, whose inverse

$z = \alpha^{-1}(\varsigma - \beta)$is also affine.

Writing $\alpha = \rho e^{i\Phi}$ in polar coordinates, we see that the affine map (2.54) can be viewed as the composition of a rotation (2.53), followed by a scaling (2.52), followed by a translation (2.51). As such, it takes the disk $|\alpha z + \beta| < 1$ of radius1\$|\alpha| = \frac{1}{|\rho|}$ and center $-\frac{\beta}{\alpha}$ to the unit disk$|\varsigma| < 1$.

Example 2.2.9: A more interesting example is the complex function

$$\boldsymbol{\varsigma} = g(z) = \frac{1}{z}, \qquad or \qquad \xi = \frac{x}{x^2 + y^2},$$

$$\eta = -\frac{y}{x^2 + y^2}, \qquad (2.55)$$

Which defines an *inversion* of the complex plane. The inversion is a one-to-one analytic map everywhere except at the origin$z = 0$; indeed $g(z)$ is its own inverse:$g^{-1}(\boldsymbol{\varsigma}) = \mathbf{1} / \boldsymbol{\varsigma}$**.**

Since $g'(z) = -1/z^2$ is never zero, the derivative condition (2.50) is satisfied everywhere.

Note that $|\boldsymbol{\varsigma}|=1/|z|$,while $\text{ph}\,\varsigma = -ph\,z$. thus, if= $\{|z| > p\}$ denotes the exterior of the circle of radius p, then the image points$\varsigma = 1/z$Satisfy$|\boldsymbol{\varsigma}| = 1/|\boldsymbol{z}|$,and hence the image domain is the punctured disk$D = \{0 < |\boldsymbol{\varsigma}| < 1/P\}$.In particular, the inversion maps outside of the unit disk to its inside, but with the origin removed, and vice versa. We may enjoy seeing what the inversion dose to other domains, e.g., the unit square$S = \{z = x + iy|0 < x, y < 1\}$.

Example 2.2.10: The complex exponential

$$\varsigma = g(z) = e^z, \quad or \quad \xi = e^x \cos y, \quad \eta = e^x \sin y, \tag{2.56}$$

Satisfies the condition $g'(z) = e^z \neq 0$ everywhere. Nevertheless, it is not one-to-one because$e^{z+2\pi i} = e^z$, and so points that differ by an integer multiple of $2\pi i$ are all mapped to the same point. We deduce that condition (2.50) is necessary, but not sufficient for invertibility.

Under the exponential map, the horizontal line $\text{Im}\, z = b$ is mapped to the curve $\varsigma = e^{x+ib} = e^x(\cos b + i \sin b)$, which, as x varies from$-\infty\ to\ \infty$, traces out the ray emanating from the origin that makes an angle ph$\varsigma = b$ with the real axis. Therefore, the exponential map will map a horizontal strip

$$S_{a,b} = \{a < Im\, z < b\} \quad to\ a\ wedge - shaped\ domain$$

$$\Omega_{a,b} = \{a < ph\boldsymbol{\varsigma} < b\},$$

And is one-to-one provided$|b - a| < 2\pi$. In particular, the horizontal strip

$$S_{-\pi/2,\pi/2} = \{-\frac{1}{2}\pi < Im\, z < \frac{1}{2}\pi\}$$

Of width π centered around the real axis is mapped, in a one-to-one manner, to the right half plane

$$R = \Omega_{-\pi/2,\pi/2} = \left\{-\frac{1}{2}\pi < ph\,\varsigma < \frac{1}{2}\pi\right\} = \{Im\,\varsigma > 0\},$$

While the horizontal strip $S_{-\pi,\pi} = \{-\pi < Im\, z < \pi\}$width 2π is mapped onto the domain

$$\Omega_* = \Omega_{-\pi,\pi} = \{-\pi < ph\,\boldsymbol{\varsigma} < \pi\} = \mathbb{C}\backslash\{Im\, z = 0, Re\, z \leq 0\}$$

Obtained by slitting the complex plane along negative real axis.

On the other hand, vertical lines $Re\ z = a$ are mapping to circles$|\boldsymbol{\varsigma}| = \boldsymbol{e^a}$**.** Thus, a vertical strip $a < Re\ z < b$ is mapped to an annulus

$e^a < |\boldsymbol{\varsigma}| < e^b$, albeit many-to-one, since the strip is effectively wrapped around the annulus. The rectangle$R = \{a < x < b, -\pi < y < \pi\}$Of height 2π is mapped in a one-to-one fashion on an annulus that has been cut along the negative real axis, as illustrated in Figure 1.15. Finally, we not that no domain is mapped to the unit disk $D = \{|\boldsymbol{\varsigma}| < 1\}$ (or, indeed, any other domain that contains 0) because the exponential function is never zero:$\varsigma = e^z \neq 0$.

Example 2.2.11: The squaring map

$$\varsigma = g(z) = z^2, \quad or \quad \xi = x^2 - y^2, \quad \eta = 2xy, \tag{2.57}$$

Is analytic on all of $\mathbb{C}$, but is not one-to-one. Its inverse is the square root function $z = \sqrt{\varsigma}$, which, as we noted in section 2.2, is doubly-valued, except at the origin $z = 0$. Furthermore, its derivative

$g'(z) = 2z$ Vanashes at $z = o,$ violating the inevitability condition (2.50). However, once we restrict $g(z)$ to a simply connected subdomain Ω that does not contain o, the function $g(z) = z^2$ does define a one-to-one mapping, whose inverse $z = g^{-1}(\boldsymbol{\varsigma}) = \sqrt{\boldsymbol{\varsigma}}$ is a well-defined, analytic and single-valued branch of the square root function.

The effect of the squaring map on a point z is to square its modulus, $|\boldsymbol{\varsigma}| = |\boldsymbol{z}|^2$**,** while doubling its phase, $ph\ \boldsymbol{\varsigma} = phz^2 = 2ph\ z$. Thus, for example, the upper right quadrant

$$Q = \{x > 0, y > 0\} = \{0 < ph\ z < \frac{1}{2}\pi\}$$

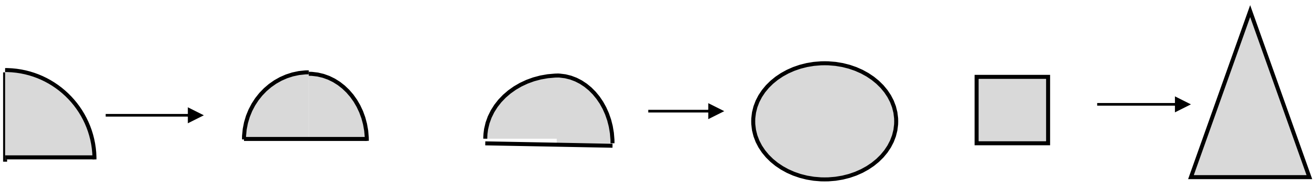


Figure 2.15 The Effect of $\varsigma = z^2$ on Various Domains.

Is mapped onto the upper half plane

$$U = g(Q) = \{\eta = Im\boldsymbol{\varsigma} > 0\} = \{0 < ph\boldsymbol{\varsigma} < \pi\}.$$

The inverse function maps a point $\varsigma \in U$ back to its unique root $z = \sqrt{\varsigma}$ that lies in the quadrant Q. Similarly, a quarter disk

$$Q_P = \{0 < |Z| < P, 0 < ph\, z < \frac{1}{2}\pi\}$$

Of radius p is mapped to a half disk

$$U_{p^2} = g(\Omega) = \{0 < |\boldsymbol{\varsigma}| < p^{\mathbf{2}}, Im\varsigma > 0\}$$

of radius p^2. On the other hand, the unit square

$S = \{0 < x < 1, 0 < y < 1\}$is mapped to a curvilinear triangular domain, as indicated in Figure 2.15. The Effect of $\varsigma = z^2$ on Various Domains; the edges of the square on the real and imaginary axes map to the two halves of the straight base of the triangle, while the other two edges become its curved sides.

Example 2.2.12: A particularly important example is the analytic map

$$\boldsymbol{\varsigma} = \frac{z-1}{\boldsymbol{z}+1} = \frac{x^{\mathbf{2}}+y^{\mathbf{2}}-1}{(x+1)^{\mathbf{2}}+y^{\mathbf{2}}} + \boldsymbol{i}\frac{2y}{(x+1)^{\mathbf{2}}+y^{\mathbf{2}}}, \quad (2.58)$$

The map is one-to-one with analytic inverse

$$z = \frac{1+\varsigma}{1-\varsigma} = \frac{1-\xi^2+\eta^2}{(1-\xi)^2+\eta^2} + i\frac{2\eta}{(1-\xi)^2+\eta^2}, \quad (2.59)$$

provided $z \neq -1$ and $\varsigma \neq 1$. This particular analytic map has the important property of mapping the right half plane $R = \{x = Re\, z > 0\}$ to the unit disk $D = \{|\varsigma|^2 < 1\}$. Indeed, by Eqs.(2.59)

$$|\varsigma|^2 = \xi^2 + \eta^2 < 1 \qquad \text{If and only if} \quad x = \frac{1-\xi^2-\eta^2}{(1-\xi)^2+\eta^2} > 0.$$

Note that the denominator does not vanish on the interior of the disk D.

The complex functions (2.54,2.55, and 2.58) are all particular examples of linear fractional transformations

$$\varsigma = \frac{az+\beta}{yz+\delta} \quad (2.60)$$

Which form one of the most important classes of analytic maps. Here α, β, y, δ are complex constants, subject only to the restriction

$$\alpha\delta - \beta y \neq 0,$$

Since otherwise (2.60) reduces to a trivial constant (and non-invertible) map. The map is well defined except when $y \neq 0$and $z = -\frac{\delta}{y}$, which, by convention, is said to be mapped to the point $\varsigma = \infty$. On the other hand, the linear fractional transformation maps $z = \infty \; to \;\; \varsigma = \alpha/\boldsymbol{y}(or \;\; \infty \, when \, y = 0)$, the value following from an evident limiting process. Thus, every linear fractional transformation defined a one-to-one, analytic map from the Riemann sphere $S \equiv \mathbb{C} \cup \{\infty\}$Obtained by adjoining the point at infinity to the complex plane. The resulting space is identified with a two-dimensional sphere via stereographic projection $\pi: S \to \mathbb{C}$, which is one-to-one (and conformal) except at the North Pole, where it is not defined and which is thus identified with the point ∞. In complex analysis, one treats the point at infinity on an equal footing with all other complex points, using the map $\varsigma = 1/z$, say, to analyze the behaviour of analytic functions there.

Example 2.2.13: The linear fractional transformation

$$\varsigma = \frac{z - \alpha}{\bar{\alpha}\, z - 1}, \quad with \quad |\alpha| < 1, \tag{2.61}$$

Maps the unit disk to itself, moving the origin $z = 0$ to the point $\varsigma = \alpha$.to prove this, we note that

$$|z - \alpha|^2 = (z - \alpha)(\bar{z} - \bar{\alpha}) \qquad = |z|^2 - \alpha\bar{z} - \bar{\alpha}z + |\alpha|^2,$$

$$|\bar{\alpha}z - 1|^2 = (\bar{\alpha}z - 1)(\alpha\bar{z} - 1) = |\alpha|^2|z|^2 - \alpha\bar{z} - \bar{\alpha}z + 1.$$

Subtracting these two formulae,

$$|z - \alpha|^2 - |\bar{\alpha}z - 1|^2 = (1 - |\alpha|^2)(|z|^2 - 1) < 0, \text{Whenever} |z| < 1, |\alpha| < 1$$

Remark: Simple connectivity of the domain is an essential hypothesis of our evaluation (1.53) of the integral of $1/z$ around the unit circle provides a simple counterexample to (1.58) for the non-simply connected domain $\Omega = C\{0\}$.interestingly, this result also admits a converse: a continuous complex-valued function that satisfies (1.58) for all closed curves is necessarily analytic; see [4] for a proof.

We will also require a slight generalization of this result.

Proposition 2.2.14: If $f(z)$ is analytic in a domain that contains two simple closed curves $S$and$C$, and the entire region lying between them, then, assuming they are oriented in the same direction.

2.3 More on Harmonic Functions

The real and imaginary parts of an analytic function are conjugate harmonic functions. Therefore, all theorems on analytic functions are also theorems on pairs of conjugate harmonic functions. However, harmonic functions are important in their own right, and their treatment is not always simplified by the use of complex methods. This is particularly true when the conjugate harmonic function is not single-valued.

We assemble in this section some facts about harmonic functions that are intimately connected with Cauchy's theorem.

2.3.1 Definitions and basic properties

A real-valued function $u(z)$ or $u(x, y)$, defined and single-valued in a region Ω, is said to be *harmonic* in Ω, or a *potential function*, if it continuous together with its partial derivatives of the first two orders and satisfies *Laplace's equation.*

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{2.62}$$

We shall see later that the regularity conditions can be weakened, but this is a point of relatively minor importance.

The sum of two harmonic functions and a constant multiple of a harmonic function are again harmonic; this is due to the linear character of Laplace's equation. The simplest harmonic functions are the linear functions $ax + by$. In polar coordinates (r, θ) equation (2.62) takes the form

$$r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial \theta^2} = 0 \tag{2.63}$$

This shows that $\log r$ is a harmonic function and that any harmonic function which depends only on r must be of the form $a \log r + b$. The argumentθ is harmonic whenever it can be uniquely defined.

If u is harmonic in Ω, then

$$f(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \tag{2.64}$$

is analytic, for writing $U = \frac{\partial u}{\partial x}, V = \frac{\partial u}{\partial y}$ we have

$$\frac{\partial U}{\partial x} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} = \frac{\partial V}{\partial y}$$

$$\frac{\partial U}{\partial y} = \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial V}{\partial x}$$

This, it should be remembered, is the most natural way of passing from harmonic to analytic functions.

From (2.64) we pass to the differential

$$f dz = \left(\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy\right) + i\left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy\right) \tag{2.65}$$

In this expression, the real part is the differential of *u*,

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy.$$

If *u* has a conjugate harmonic function *v*, then the imaginary part can be written as

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy.$$

In general, however, there is no single-valued conjugate function, and in these circumstances, it is better not to use the notation *dv*. Instead, we write:

$$^*du = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

And call $^*du$ the *conjugate differential of du*. We have by (2.65))

$$f\, dz = du + i\ ^*du. \tag{2.66}$$

By Cauchy's theorem, the integral of *fdz* vanishes along any cycle, which is homologous to zero in **Ω**. On the other hand, the integral of the exact differential *du* vanishes along all cycles. It follows by (2.66) that

$$\int_r\ {*du} \int_r\ = \int_r\ -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy = 0 \tag{2.67}$$

For all cycles r which are homologous to zero in Ω.

The integral in (2.67) has an important interpretation, which cannot be left unmentioned. If r is a regular curve with the equation $z = z(t)$, the direction of the tangent is determined by the angle $\propto = \arg z'(t)$, and we can write

$dx = |dz| \cos \propto$. The normal which points to the right of the tangent has the direction $\beta = \propto - \pi/2$, and thus

$$\cos \propto = -\sin \beta, \sin \propto = \cos \beta$$

The expression

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x} \cos \beta + \frac{\partial u}{\partial y} \sin \beta$$

Isa directional derivative of u, the right-hand *normal derivative* with respect to the curve r. We obtain $^*du = (\partial u/\partial n)\,|dz|$, and (2.67)) can be written in the form

$$\int_r \frac{\partial u}{\partial n} |dz| = 0 \tag{2.68}$$

This is the classical notation. Its main advantage is that $\partial u/\partial n$ actually represents a rate of change in the direction prependicular to r. For instance, if r is the circle$|z| = r$, described in the positive sense, $\partial u/\partial n$ can be replaced by the partial derivative $\partial u/\partial r$. It has no disadvantage that (2.68)) is not expressed as an ordinary line integral, but as an integral with respect to arc length. For this reason, the classical notation is less natural in connection with homology theory, and we prefer to use the notation $^*du$.

In simply connected region, the integral of $^*du$ vanishes over all cycles, and u has a single-valued conjugate function v which is determined up to an additive constant. In the multiply connected case the conjugate function has *periods*

$$\int_r {}^*du = \int_r \frac{\partial u}{\partial} |dz|$$

Corresponding to the cycles in a homology basis.

There is an important generalization of (2.67) which deals with a pair of harmonic functions. If u_1 and u_2 are harmonic in Ω, we claim that

$$\int_r u_1 \; {}^*du_2 - u_2 \; {}^*du_1 = 0 \qquad (2.69)$$

for every cycle r which is homologous to zero in Ω. It is sufficient to prove (2.68) for $r = \partial R$, where R is a rectangle contained in Ω. In R, $u_1$and $u_2$have single-valued conjugate functions v_1, v_2 and we can write

$$u_1 \; {}^*du_1 - u_2 \; {}^*du_2 = u_1 du_2 - u_2 du_2 = u_1 dv_2 + v_1 dv_2 - d(u_2 v_1).$$

Here $d(u_2 v_1)$ is an exact differential, and $u_1 dv_2 + v_1 du_2$is the imaginary part of

$$(u_1 + iv_1)(du_2 + idv_2)$$

Can be written in the form $F_1 F_2 dz$ where $F_1(z)$ and $F_2(z)$ are analytic on R. The integral of $F_1 F_2 dz$ vanishes by Cauchy's theorem, and so does therefore the integral of its imaginary part. We conclude that (2.68) holds for $r = \partial R$, and we have proved:

Theorem 2.3.2: *If u_1 and u_2 are harmonic in a region* Ω, *then*

$$\int_r u_1 \; {}^*du_2 - u_2 \; {}^*du_1 = 0 \qquad (2.70)$$

For every cycle r which is homologous to zero in Ω.

For $u_1 = 1,\ u_2 = u$ the formula reduces to (2.66). In the classical notation (2.68) would be written as

$$\int_r \left(u_1 \frac{\partial u_2}{\partial n} - u_2 \frac{\partial u_1}{\partial n}\right) |dz| = 0$$

.

The Mean-Value Property 2.3.3:

Let us apply Theorem 19 with u_1 = log r and $u_2$equal toa function u, harmonic in $|z| < \rho$.For Ω we choose the punctured disk $0 < |z| < \rho$, and for r we take the cycle C_1-C_2 where C_i is a circle $|z| = r_i < \rho$ described in the possitive sense. On a circle $|z| = r$ we have ${}^*du = r(\partial u/\partial r)\, d\theta$ and hence (2.68) yields

$$\log r_1 \int_{c_1} r_1 \frac{du}{\partial n} d\theta - \int_{c_2} u \, d\theta = \log r_2 \int_{c_2} r_2 \frac{\partial u}{\partial r} d\theta - \int_{c_2} u d\theta.$$

In other words, the expression

$$\int_{|z|=r} u\, d\theta - \log r \int_{|z|=r} r \frac{\partial u}{\partial r} d\theta$$

is constant, and this is true even if *u* is only known to be harmonic in an annulus. By (2.66) we find the same way that

$$\int_{|z|=r} r \frac{\partial u}{\partial r}\, d\theta$$

is constant in the case of an annulus and zero if *u* is harmonic in the whole disk. Combining these results we obtain:

Theorem 2.3.4:*The arithmetic mean of a harmonic function over concentric circles* $|z| = r$is a linear function of$\log r$,

$$\frac{1}{2\pi} \int_{|z|=r} u\ d\theta = \alpha \log r + \beta, \tag{2.71}$$

It is clear that (2.71) could also have been derived from the corresponding formula for analytic functions. It leads directly to the *maximum principle* for harmonic functions:

Theorem 2.3.5: *A non-constant harmonic function has neither a maximum nor a minimum in its region of definition. Consequently, the maximum and the minimum on a closed bounded set E are taken on the boundary of F.*

The proof of the same as for the maximum principle of analytic functions and will not be repeated. It applies also to the minimum for the reason that –*u* is harmonic together with *u*. In the case of analytic functions the corresponding procedure would have been to apply the maximum principle to 1/*f*(*z*), which is illegitimate unless *f*(z) ≠ 0.

Observe that the maximum principle for analytic functions follows from the maximum principle for harmonic functions by applying the latter to log $|f(z)|$ which is harmonic when *f*(*z*) ≠ 0.

Poisson's Formula2.3.6:

The maximum principle has the following important consequence: If $u(z)$ is continuous on a closed bounded set E and harmonic on the interior of E. Indeed, in u_1 and u_2 are two such functions with the same boundary values, then u_1-u_2 is harmonic with the boundary values 0.

By the maximum and minimum principle the differences u_1-u_2 must thenbe identically zero on E.

There arises the problem of finding u when its boundary values are given.

At this point, we shall solve the problem only in the simplest case, namely for a closed disk.

Formula (2.71) determines the value of u at the center of the disk. But this is all we need, for there exists a linear transformation, which carries any point to the center. To be explicit, suppose that $u(z)$ is harmonic in the closed disk $|z| \leq R$.The linear transformation

$$z = s(\zeta) = \frac{R(R\zeta + a)}{R + \bar{a}\zeta}$$

Maps $|\zeta| \leq 1$ onto $|z| \leq R$ with $\zeta = 0$ corresponding to $z = a$.The function $u(S(\zeta))$is harmonic in$|\zeta| \leq 1$ and by (10) we obtain

$$u(a) = \frac{1}{2\pi}\int_{|\zeta|=1} u\big(S(\zeta)\big) d \arg \zeta.$$

$$\zeta = \frac{R(z-a)}{R^2 - \bar{a}z}$$

We compute

$$d \arg \zeta = -i\left(\frac{1}{z-a} + \frac{\bar{a}}{R^2 - \bar{a}z}\right)dz = \left(\frac{z}{z-a} + \frac{\bar{a}z}{R^2 - \bar{a}z}\right)d\theta.$$

On substituting$R^2 = z\bar{z}$ the coefficient of $d\theta$ in the last expression can be rewritten as

$$\frac{z}{z-a} + \frac{\bar{a}}{\bar{z} - \bar{a}} = \frac{R^2 - |a|^z}{|z-a|^z}$$

Or equivalently, as

$$\frac{1}{2}\left(\frac{z+a}{z-a} + \frac{\bar{z} + \bar{a}}{\bar{z} - \bar{a}}\right) = Re\ \frac{z+a}{z-a}$$

We obtain two forms

$$u(a) = \frac{1}{2\pi}\int_{|z=R|} \frac{R^2 - |a|^z}{|z-a|^z}\ u(z) d\theta = \frac{1}{2\pi}\int_{|z=R|} Re \frac{z+a}{z-a}\ u(z) d\theta \qquad (2.72)$$

Of *Poisson's formula*. I n polar coordinates,

$$u(re^{ie}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2rR\ \cos(\theta - \varphi) + r^2} u(Re^{i\theta})d\theta$$

In the derivation we have assumed that $u(z)$ is harmonic in the closed disk. However, the result remains true under the weaker condition that $u(z)$ is harmonic in the open disk and continuous in the closed disk.

Indeed, if $0 < r < 1$,then $u(rz)$ is harmonic in the closed disk, and we obtain

$$u(ra) = \frac{1}{2\pi}\int_{|z=R|} \frac{R^2-|a|^z}{|z-a|^z} u(rz)d\theta.$$

Now all we need to do is to let r tend to 1. Because $u(z)$ is uniformly continuous on $|z| \le R$ it is true that $u(rz) \to u(z)$ uniformly for $|z| \le R$, and we conclude that (2.72) remains valid.

We shall formulate the result as a theorem:

Theorem V: *Suppose that*$u(z)$*is harmonic* $for\ |z| < R,$ continuous for $|z| \le R$. *Then*

$$u(a) = \frac{1}{2\pi}\int_{|z|=R} \frac{R^2-|a|^z}{|z-a|^z} u(z)d\theta \qquad (2.73)$$

For all $|a| < R$.

The theorem leads at once to an explicit expression for the conjugate function of u.

Indeed formula (2.72) gives

$$u(z) = Re\left[\frac{1}{2\pi}\int_{|\zeta|=R} \frac{\zeta+z}{\zeta-z}\ u(\zeta)\frac{d\zeta}{\zeta}\right]. \qquad (2.74)$$

The bracketed expression is an analytic function of z for $|z| < R$.

Tt follows that $u(z)$ is real part of

$$f(z) = \frac{1}{2\pi}\int_{|\zeta|=R} \frac{\zeta + z}{\zeta - z}\ u(\zeta)\frac{d\zeta}{\zeta} + iC$$

Where C is an arbitrary real constant. This formula is known as Schwarz's formula.

As a special case of (2.73), note that $u = 1$ yields

$$\int_{|z|=R} \frac{R^2 - |a|^2}{|z - a|^2} d\theta = 2\pi$$

For all $|a| < R$.

Chapter Three

Conformal Field Theory

Classical field theory is a very important theory in theoretical physics. Most of these theories are conformal field. We started first with classical field and then proceed to quantum field theory to consider some equation aspects.

3.1 Euclidean Space Formulation:

We consider first Euclidean world sheets. Though we could also formulate everything in Minkowski space.

Let $(\sigma^1, \sigma^2) = (\sigma^1, i\sigma^0)$be Euclidean world sheets.Using the complex coordinates,

$$z = \sigma^1 + i\sigma^2 \qquad and \qquad \bar{z} = \sigma^1 - i\sigma^2$$

As Euclidean analogues of the light cone coordinates and motivated by this analogy, it is common to refer to holomorphic functions as “left-moving” and

Anti-holomorphic functions as “right-moving”.

The Holomorphy derivatives are

$$\partial_z = \frac{1}{2}(\partial_1 - i\partial_2) \qquad and \qquad \partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2)$$

These obey $\partial_z = \bar{\partial}_{\bar{z}} = 1$ and $\partial_{\bar{z}} = \bar{\partial}_z = 0$ .we shall work in flat Euclidean space, with metric

$$ds^2 = (d\sigma^1)^2 + (d\sigma^2)^2 = dz\, d\overline{z} \tag{3.1}$$

In components, this flat metric reads

$$g_{zz} = g_{\bar{z}\bar{z}} = 0 \qquad and \qquad g_{z\bar{z}} = \frac{1}{2}$$

With this convention, the measure factor is $dzd\bar{z} = 2d\sigma^1 d\sigma^2$. We define the delta-function such that $\int d^2z\, \delta(z, \bar{z}) = 1$. Notice that because we also have $\int d^2\, \sigma\, \delta(\sigma) = 1,$ this means that there is a factor of 2 differences between the two delta functions.We also have the vectors

$$v_z = \tfrac{1}{2}(v^1 - iv^2) \text{ and } v_{\bar{z}} = \tfrac{1}{2}(v^1 + iv^2)$$

3.2 Conformal Transformations:

The conformal transformations in the complex Euclidean space Z and E are defined by any holomorphic change of coordinates,

$$z \to z' = f(z) \qquad and \qquad \bar{z} \to \bar{z}' = \bar{f}(\bar{z})$$

Under this transformation, $\bar{z} = \left|\frac{df}{dz}\right|^2 ds$ .Notethat we have an infinite number of conformal transformations.This is special to conformal field theories in two dimensions. In higher dimensions, the space of conformal transformations is a finite dimensional group. For theories defined on $R^{p,q}$, the conformal group is

$SO(p+1, q+1)$when$p+q > 2$.

A couple of particularly simple and important examples of 2d conformal transformations are

- $z \to z + a$: This is a translation.
- z $\to \varsigma z$:This is a rotation for $|\varsigma| = 1$ and a scale transformation (dilatation) for real $\varsigma \neq 1$.

Treating z and$\bar{z}$as independent variables. We can extend the world sheet from R^2 toC^2.this will allow us to make use of various theorems from complex methods. We can then consider $R^2 \subset C^2$ defined by $\bar{z} = z^\star$.

3.3 Classical Aspects:

We start by deriving some properties of classical theories which are invariant under conformal transformations.

3.3.1 The Stress-Energy Tensor:

One of the most important objects in any field theory is the stress- energy tensor

(energy-momentum tensor). This is defined in the usual way as the matrix of conserved currents which arise from translational invariance,

$$\delta\sigma^\alpha = \epsilon^\alpha.$$

In flat spacetime, a translation is a special case of a conformal transformation.

There's a cute way to derive the stress-energy tensor in any theory. Suppose for the moment that we are in flat space $g_{\alpha B} = \eta_{\alpha B}$.

Recall that we can usually derive conserved currents by promoting the constant parameter ϵ that appears in the symmetry to a function of the space time coordinates. The change in the action must then be of the form,

$$\delta S = \int d^2 \sigma\, J^\alpha \partial_\alpha \epsilon \tag{3.2}$$

For some function of the fields, J^α.This ensures that the variation of the action vanishes when ϵ is constant, which is of course the definition of symmetry. But when the equations of motion are satisfied, we must have $\delta S = 0$ for all variations $\epsilon(\sigma)$, not just constant ϵ. This means that when the equations of motion are obeyed, J^α must satisfy

$$\partial_\alpha J^\alpha = 0$$

The function J^α is our conserved current.

Let's see how this works for translational invariance. If we promote ϵ to a function of the world sheet variables, the change of the action must be of the form (3.2).but what is J^α?at this point we do the cute thing.

Consider the same theory, but now coupled to a dynamical background metric $g_{\alpha B}(\sigma)$. In other words, coupled to gravity.

Then we could view the transformation

$$\delta \sigma^\alpha = \epsilon^\alpha(\sigma)$$

As a diffeomorphism and we know that the theory is invariant as long as we make the corresponding change to the metric.

$$\delta g_{\alpha\beta} = \partial_\alpha \epsilon_\beta + \partial_\beta \epsilon_\alpha.$$

This means that if we just make the transformation of the coordinates in our original theory, then the change in the action must be the opposite of what we get if we justtransform the metric. (Because doing both together leaves the action invariant). So we have

$$\delta S = -\int d^2 \sigma \frac{\partial S}{\partial g_{\alpha\beta}} \delta g_{\alpha\beta} = -2 \int d^2 \sigma \frac{\partial S}{\partial g_{\alpha\beta}} \partial_\alpha \epsilon_\beta$$

Note that $\partial S / \partial g_{\alpha\beta}$ in this expression is really functional derivatives but we now have the conserved current arising from translational invariance.

We will add a normalization constant which is standard in string theory (although not necessarily in other areas) and define the stress-energy tensor to be

$$T_{\alpha\beta} = -\frac{4\pi}{\sqrt{g}}\frac{\partial S}{\partial g^{\alpha\beta}} \tag{3.3}$$

If we have a flat world sheet, we evaluate $T_{\alpha\beta}$ on $g_{\alpha\beta} = \delta_{\alpha\beta}$ and the resulting expression obeys $\partial^{\alpha}T_{\alpha\beta} = 0$.if we're working on a curved world sheet, then the energy-momentum tensor is covariantly conserved, $\nabla^{\alpha}T_{\alpha\beta} = 0$.

(i)The Stress-Energy Tensor is Traceless:

In conformal theories, $T_{\alpha\beta}$ has a very important property: its trace vanishes.

To see this, let's vary the action with respect to a scale transformation which is a special case of a conformal transformation,

$$\delta g_{\alpha\beta} = \epsilon g_{\alpha\beta} \tag{3.4}$$

Then we have

$$\delta S = \int d^2\sigma \frac{\partial S}{\partial g_{\alpha\beta}}\delta g_{\alpha\beta} = -\frac{1}{4\pi}\int d^2\,\sigma\sqrt{g}\,\epsilon T^{\alpha}_{\alpha}$$

But this must vanish in a conformal theory because of scaling transformations symmetry. So

$$T^{\alpha}_{\alpha} = 0$$

This is the key feature of a conformal field theory in any dimension.

Many theories have this feature at the classical level, including Maxwell theory and Yang-Mills theory in four-dimensions. However, it is much harder to preserve at the quantum level. (The weight of the world rests on the fact that Yang-Mills theory fails to be conformal at the quantum level). Technically the difficulty arises due to the need to introduce a scale when regulating the theories. Here we will be interested in two-dimensional theories which succeed in preserving the conformal symmetry at the quantum level.

Even when the conformal invariance survives in a 2d quantum theory, the vanishing trace $T^{\alpha}_{\alpha} = 0$ will only turn out to hold in flat space.

(ii)The Stress-Tensor in Complex Coordinates:

In complex coordinates, $z = \sigma^1 + i\sigma^2$, the vanishing of the trace $T^\alpha_\alpha = 0$ becomes

$$T_{z\bar{z}} = 0$$

Meanwhile, the conservation equation $\partial_\alpha T^{\alpha\beta} = 0$ becomes

$\partial T^{zz} = \bar{\partial} T^{\bar{z}\bar{z}} = 0.$ or, lowering the indices on T,

$$\bar{\partial} T_{zz} = 0 \qquad and \qquad \partial T_{\bar{z}\bar{z}} = 0$$

In other words, $T_{zz} = T_{zz}(z)$ is a Holomorphy function while

$T_{\bar{z}\bar{z}} = T_{\bar{z}\bar{z}}(\bar{z})$ is an anti-Homomorphy function. We will often use the simplified notation

$$T_{zz}(z) \equiv T(z) \qquad and \qquad T_{\bar{z}\bar{z}}(\bar{z}) \equiv \bar{T}(\bar{z})$$

3.3.2 Noether Currents:

The stress-energy tensor $T_{\alpha\beta}$ provides the No ether currents for translations. What are the currents associated to the other conformal transformations? Consider the infinites- imal change,

$$z' = z + \epsilon(z), \quad \bar{z}' = \bar{z} + \bar{\epsilon}(\bar{z})$$

Where, making contact with the two examples above, constant ϵ corresponds to a translation while $\epsilon(z) \sim z$ corresponds to a rotation and dilation.

To compute the current, we'll use the same trick that we saw before:

We promote the parameter ϵ to depend on the world sheet coordinates.

But it's already a function of half of the world sheet coordinates, so this now means $\epsilon(z) \to \epsilon(z, \bar{z})$. Then we can compute the change in the action, again using the fact that we can make a compensating change in the metric,

$$\delta S = -\int d^2\sigma \frac{\partial S}{\partial g^{\alpha\beta}} \delta g^{\alpha\beta}$$

$$= \frac{1}{2\pi} \int d^2\, \sigma\, T_{\alpha\beta} \left(\partial^\alpha \delta\sigma^\beta\right)$$
$$= \frac{1}{2\pi} \int d^2\, z\, \frac{1}{2}\, [T_{zz}(\partial^z \delta z) + T_{\bar{z}\bar{z}}(\partial^{\bar{z}} \delta \bar{z})]$$
$$= \frac{1}{2\pi} \int d^2 z\, [T_{zz} \partial_{\bar{z}} \epsilon + T_{\bar{z}\bar{z}} \partial_z \bar{\epsilon}] \tag{3.5}$$

Firstly note that ifϵ is Holomorphy and$\bar{\epsilon}$is anti-Homomorphy, then we immediately have $\delta S = 0$.this, of course, is the statement that we have symmetry on our hands. (You may wonder where in the above derivation we used the fact that the theory was conformal. It lies in the transition to the third line where we needed $T_{z\bar{z}} = 0$).

At this stage, let's use the trick of treating z and $\bar{z}$ as independent variables.

We look at separate currents that come from shifts in z and shifts$\bar{z}$.

Let's first look at the symmetry

$$\delta z = \epsilon(z) \quad , \quad \delta \bar{z} = 0$$

We can read off the conserved current from by using the standard trick of letting the small parameter depend on position. Since $\epsilon(z)$ already depends on position, this means promoting $\epsilon \to \boldsymbol{\epsilon}(z) f(\bar{z})$ for some function$f$ and thenlooking at the$\bar{\partial}$f terms in (3.6).this gives us the current

$$J^z = 0 \quad and \quad J^{\bar{z}} = T_{zz}(z)\epsilon(z) \equiv T(z)\epsilon(z) \tag{3.6}$$

Importantly, we find that the current itself is also Homomorphy. We can cheek that this is indeed a conserved current: it should satisfy

$$\partial_\alpha J^\alpha = \partial_z J^z + \partial_{\bar{z}} J^{\bar{z}} = 0.$$

But in fact it does so with room to spare: it satisfies the much stronger condition $\partial_{\bar{z}} J^{\bar{z}} = 0$.

Similarly, we can look at transformations $\delta \bar{z} = \bar{\epsilon}(\bar{z})$ with $\delta z = 0$.

We get the anti-Homomorphy current $\bar{J}$,

$$\bar{J}^z = \bar{T}(\bar{z})\bar{\epsilon}(\bar{z}) \quad and \quad \bar{J}^{\bar{z}} = 0 \tag{3.7}$$

3.3.3The Free Scalar Field:

Let's illustrate some of these ideas about classical conformal theories with the free scalar field,

$$S = \frac{1}{4\pi\alpha'}\int d^2\,\sigma\partial_\alpha X\partial^\alpha\,X$$

Notice that there's no overall minus sign, in contrast to our earlier action (3.30).That's because we're now working with a Euclidean world sheet metric. The theory of a free scalar field is, of course, dead easy.

We can compute anything we like in this theory.

Nonetheless, it will still exhibit enough structure to provide an example of all the abstract concepts that we will come across in CFT (conformal field Theory). Firstly, let's just check that this free scalar field is actually conformal.

In particular, we can look at rescaling $\sigma^\alpha \to \lambda\sigma^\alpha$. If we view this in the sense of an active transformation, the coordinates remain fixed but the value of the field at point σ gets moved to point $\lambda\sigma$. This means,

$$X(\sigma) \to X(\lambda^{-1}\sigma) \qquad and \qquad \frac{\partial X(\sigma)}{\partial\sigma^\alpha} \to \frac{\partial X(\lambda^{-1}\sigma)}{\partial\sigma^\alpha} = \frac{1}{\lambda}\frac{\partial X(\tilde{\sigma})}{\partial\tilde{\sigma}}$$

Where we've defined $\tilde{\sigma} = \lambda^{-1}\sigma$.the factor of λ^{-2}coming from the two derivatives in the Lagrangian then cancels the Jacobin factor from the measure $d^2\sigma = \lambda^2 d^2\tilde{\sigma}$, leaving the action invariant. Note that any polynomial interaction term for X would break conformal invariance.

The stress-energy tensor for this theory is defined using (3.4),

$$T_{\alpha\beta} = -\frac{1}{\alpha'}\left(\partial_\alpha\,X\partial_\beta X - \frac{1}{2}\delta_{\alpha\beta}(\partial X)^2\right), \tag{3.8}$$

Which indeed satisfies $T_\alpha^\alpha = 0$ as it should. The stress-energy tensor looks much simpler in complex coordinates. It is simple to check that $T_{z\bar{z}} = 0$ while

$$T = -\frac{1}{\alpha'}\partial X\partial X \qquad and \qquad \bar{T} = -\frac{1}{\alpha'}\bar{\partial}X\bar{\partial}X$$

The equation of motion for X is $\partial\bar{\partial}X = 0$.The general classical solution decomposes as,

$$X(z,\bar{z}) = X(z) + \bar{X}(\bar{z})$$

When evaluated on this solution, $T$and $\bar{T}$ become holomorphic and anti-Homomorphy functions respectively.

3.4 Quantum Aspects:

So far our discussion has been entirely classical. We now turn to the quantum theory. The first concept that we want to discuss is actually a feature of any quantum field theory. But it really comes into its own in the context of CFT: It is the operator product expansion.

3.4.1 Operator Product Expansion:

Let's first describe what we mean by a local operator in a CFT. We will also refer to these objects as fields. There is a slight difference in terminology between CFTs and more general quantum field theories. Usually in quantum field theory, one reserves the term "field" for the objects Φ which sit in the action and are integrated over in the path integral. In contrast, in CFT the term "field" refer to any local expression that we can write down. This includes Φ, but also includes derivatives $\partial^n\Phi$ or composite operators such as $e^{i\Phi}$.

All of these are thought of as different fields in a CFT. It should be clear from this that the set of all "field" in a CFT is always infinite even though, if you were used to working with quantum field theory, you would talk about only a finite number of fundamental objects Φ. Obviously, this is nothing to be scared about.

It's just a change of language: it doesn't mean that our theory got harder.

We now define the operator product expansion (OPE). It is a statement about what happens as local operators approach each other. The idea is that two local operators inserted at nearby points can be closely approximated by a string of operators at one of these points. Let's denote all the local operators of the CFT by$\mathcal{O}_i$, where i runs over the set of all operators. Then OPE is

$$\mathcal{O}_i(z,\bar{z})\mathcal{O}_j(\omega,\bar{\omega}) = \sum_k C_{ij}^k\,(z-w,\bar{z}-\bar{w})\mathcal{O}_k(\omega,\bar{\omega}) \qquad (3.9)$$

Here$C_{ij}^{k}(z-w,\bar{z}-\bar{w})$are a set of functions which,

On grounds of translational invariance, depend only on the Separation between the two operators. We will write a lot of operator equations of the form (4.10) and it's important to clarify exactly what they mean: they are always to be Understood as statements hold as operator insertions inside Time-ordered correlation functions,

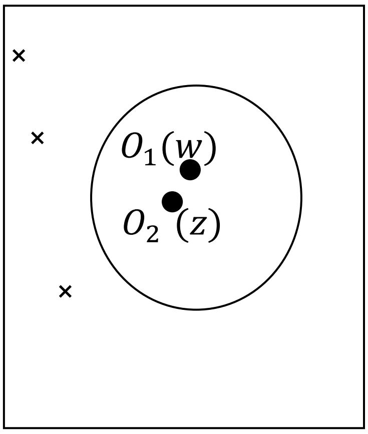


Figure 19:

$$\langle \mathcal{O}_i(z,\bar{z})\mathcal{O}_j(w,\bar{w}) \ldots \rangle = \sum_k C_{ij}^{k}\,(z-w,\bar{z}-\bar{w})\langle \mathcal{O}_k(w,\bar{w}) \ldots \rangle$$

Where the . . . can be any other operator insertions that we choose. Obviously it would be tedious to continually write $\langle \ldots \rangle$.so we don't. But it's always implicitly there. There are further caveats about the OPE that are worth stressing

●The correlation functions are always assumed to be time-ordered.

(Or something similar that we will discuss in Section 3.5.1). This means that as far as the OPE is concerned, everything commutes since the ordering of operators is determined inside the correlation function anyway. So we must have $\mathcal{O}_i(z,\bar{z})\mathcal{O}_j(w,\bar{w}) = \mathcal{O}_j(w,\bar{w})\mathcal{O}_i(z,\bar{z})$. (There is a caveat here: if the operators are Grassmann objects, then they pick up an extra minus sign when commuted, even inside time-ordered products).

●The other operator insertions in the correlation function (denoted . . . above) are arbitrary. Except they should be at a distance large compared to$|z-w|$.

It turns out- rather remarkably- that in a CFT the OPEs are exact statements and have a radius of convergence equal to the distance to the nearest other insertion. We will return to this in Section 3.6. The radius of convergence is denoted in the figure by the dotted line.

●The OPEs have singular behaviour as $z \to w$. In fact, this singular behaviour will really be the only thing we care about! It will turn out to contain the same information as commutation relations, as well as telling us how operators transform under symmetries. Indeed, in many equations we will simply write the singular terms in the OPE and denote the non-singular terms as $+ \ldots$.

3.4.2 Ward Identities:

The spirit of Noether's theorem in quantum field theories is captured by operator equations known as Ward Identities. Here we derive the Ward identities associated to conformal invariance. We start by considering a general theory with symmetry. Later we will restrict to conformal symmetries.

3.4.3 Games with Path Integrals:

We'll take this opportunity to get comfortable with some basic techniques using path integrals. Schematically, the integral takes the form

$$Z = \int \mathcal{D}\phi e^{-s[\phi]}$$

Where ϕ collectively denote all the fields (in the path integral sense . . . not the CFT sense!). Asymmetry of the quantum theory is such that an infinitesimal transformation

$$\phi' = \phi + \epsilon\delta\boldsymbol{\phi}$$

Leaves both the action and the measure invariant,

$$S[\phi'] = S[\phi] \qquad and \qquad \mathcal{D}\phi' = \mathcal{D}\phi$$

(In fact, we only really need the combination $\mathcal{D}\phi e^{-s[\phi]}$ to be invariant but this subtlety won't matter in this course). We use the same trick that we employed earlier in the classical theory and promote $\epsilon \rightarrow \epsilon(\sigma)$. Then, typically, neither the action nor the measure are invariant but, to leading order in ϵ, the change has to be proportional to $\partial\epsilon$. We have

$$Z \rightarrow \int \mathcal{D}\phi' \ exp(-S[\phi'])$$

$$= \int \mathcal{D}\phi \quad \exp\left(-S[\phi] - \frac{1}{2\pi}\int J^\alpha \partial_\alpha \epsilon\right)$$

$$= \int \mathcal{D}\phi e^{-S[\phi]}\left(1 - \frac{1}{2\pi}\int J^\alpha \partial_\alpha \epsilon\right)$$

Where the factor of $1/2\pi$ is merely a convention and f is shorthand for $f\, d^2\sigma\sqrt{g}$**.** Noticethat the current J^α may now also have contributions from the measure transformation as well as the action.

Now comes the clever step. Although the integrand has changed, the actual value of the partition function can't have changed at all. After all, we just redefined a dummy integration variable ϕ. So the expression above must be equal to the original Z. Or, in other words,

$$\int \mathcal{D}\phi e^{-S[\phi]} \left(\int J^\alpha\, \partial_\alpha \epsilon \right) = 0$$

Moreover, this must hold for all ϵ. This gives us the quantum version of Noether's **theorem:** the vacuum expectation value of the divergence of the current vanishes:

$$\langle \partial_\alpha J^\alpha \rangle = 0.$$

We can repeat these tricks of this sort to derive some stronger statements.

Let's see what happens when we have other insertions in the path integral.

The time-ordered correlation function is given by

$$\langle \mathcal{O}_1(\sigma_1) \ldots\, \mathcal{O}_n(\sigma_n) \rangle = \frac{1}{Z} \int \mathcal{D}\phi e^{-S[\phi]}\, \mathcal{O}_1(\sigma_1)\, \ldots\, \mathcal{O}_n(\sigma_n)$$

We can think of these as operators inserted at particular points on the plane as shown in the figure. As we described above, the operators $\mathcal{O}_i$ are any general expressions that we can form the ϕ fields. Under the symmetry of interest, the operator will change in some way, say

$$\mathcal{O}_i \rightarrow \mathcal{O}_i + \epsilon \delta \mathcal{O}_i$$

We once has again promote $\epsilon \rightarrow \epsilon(\sigma)$.as our first pass, let's pick a choice of $\epsilon(\sigma)$ which only has support away from the operator insertionsThen,

$$\delta \mathcal{O}_i(\sigma_i) = 0$$

And the above derivation goes through in exactly theSame way to give $\epsilon \neq 0$

$\langle \partial_\alpha J^\alpha(\sigma) \mathcal{O}_1(\sigma_1) \; \ldots \; \mathcal{O}_n(\sigma_n) \rangle = 0 \qquad for \quad \sigma \neq \sigma_i$

Because this holds for any operator insertion away$\epsilon \neq 0$ from σ,from the discussion in Section 3.2.1 we are Entitled to write the operator equation

$\partial_\alpha J^\alpha = 0$

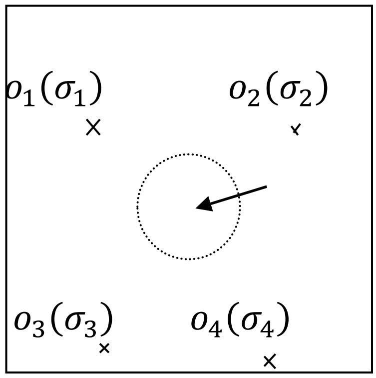


Figure 20

Butwhat if there are operator insertions that lie at Thesame point as J^α? In other words, what happens as σApproaches one of the insertion points? The resulting ϵ≠ 0 Formulae are called Ward identities. To derive$\epsilon \neq 0$ these,

Let's take $\epsilon(\sigma)$ to have support in some region that includes the point $\sigma_{1,}$ but not the other points as shown in Figure 22. The simplest choice is just to take $\epsilon(\sigma)$ to be constant inside the shaded region and zero outside.

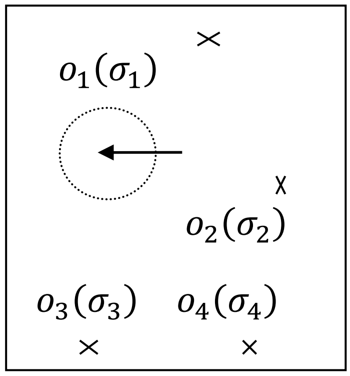


Figure 21

Now using the same procedure as before, we find that the original correlation function is equal to,

$$\frac{1}{Z} \int \mathcal{D}\phi e^{-S[\phi]} \left(1 - \frac{1}{2\pi} \int J^\alpha \, \partial_\alpha \epsilon \right) (\mathcal{O}_1 + \epsilon \delta \mathcal{O}_1) \mathcal{O}_2 \; \ldots \; \mathcal{O}_n$$

Working to leading order in ϵ, this gives

$$-\frac{1}{2\pi} \int_\epsilon \; \partial_\alpha \langle \, J^\alpha \, (\sigma) \mathcal{O}_1(\sigma_1) \; \ldots \; \rangle = \langle \delta \mathcal{O}_1(\sigma_1) \; \ldots \rangle \qquad (3.10)$$

Where the integral on the left-hand-side is only over the region of non-zero ϵ. This is the Ward Identity.

3.4.4 Ward Identities for Conformal Transformations:

Ward identities (3.11) hold for any symmetries. Let's now see what they give when applied to conformal transformations. There are two further steps needed in the derivation. The first simply comes from the fact that we're working in two dimensions and we can use Stokes' theorem to convert the integral on the left-hand-side of (3.11) to a line integral around the boundary. Let $\hat{n}^{\alpha}$ be the unit vector normal to the boundary. For any vector J^{α}, we have

$$\int_{\epsilon} \partial_{\alpha} J^{\alpha} = \oint_{\partial\epsilon} J_{\alpha}\, \hat{n}^{\alpha} = \oint_{\partial\epsilon} (J_1 d\sigma^2 - J_2 d\sigma^1) = -i \oint_{\partial\epsilon} (J_z dz - J_{\bar{z}} d\bar{z})$$

Where we have written the expression both in Cartesian coordinates σ^{α} and complex coordinates on the plane. As described in Section 3.0.1, the complex components of the vector with indices down are defined as

$J_z = \frac{1}{2}(J_1 - i\,J_2)\ and\ J_{\bar{z}} = \frac{1}{2}(J_1 + i\,J_2)$. So, applying this to the Ward identity (3.11), we find for two dimensional theories

$$\frac{i}{2\pi}\oint_{\partial\epsilon} dz\, \langle J_z(z,\bar{z})\mathcal{O}_1(\sigma_1)\ \dots \rangle - \frac{i}{2\pi}\oint_{\partial\epsilon} d\bar{z}\, \langle J_{\bar{z}}(z,\bar{z})\mathcal{O}_1(\sigma_1)\ \dots\rangle = \langle \delta\mathcal{O}_1(\sigma_1)\ \dots\rangle$$

So far our derivation holds for any conserved current J in two dimensions.

At this stage we specialize to the currents that arise from conformal transformations (3.7) and (3.8).

Here something nice happens J_z is Holomorphy while $J_{\bar{z}}$ is anti-Holomorphy.

This means that the contour integral simply pick up the residue,

$$\frac{i}{2\pi}\oint_{\partial\epsilon} dz\, J_z\,(z)\mathcal{O}_1(\sigma_1) = -Res[\,J_z\,\mathcal{O}_1]$$

Where this means the residue in the OPE between the two operators,

$$J_z(z)\mathcal{O}_1(w,\bar{w}) = \ \dots + \frac{Res[\,J_z\mathcal{O}_1(w,\bar{w})]}{z-w} + \ \dots$$

So we find a rather nice way of writing the ward identities for conformal transformations. If we again view z and $\bar{z}$ as independent variables, the Ward identities split into two pieces. From the change $\delta z = \epsilon(z)$, we get

$$\delta \mathcal{O}_1(\sigma_1) = -Res[J_z(z)\mathcal{O}_1(\sigma_1)] = -Res[\epsilon(z)T(z)\mathcal{O}_1(\sigma_1)] \tag{3.11}$$

Where, in the second equality, we have used the expression for the conformal current (3.7). Meanwhile, from the change $\delta \bar{z} = \bar{\epsilon}(\bar{z})$, we have

$$\delta \mathcal{O}_1(\sigma_1) = -Res[\bar{J}_{\bar{z}}(\bar{z})\mathcal{O}_1(\sigma_1)] = -Res[\bar{\epsilon}(\bar{z})\bar{T}(\bar{z})\mathcal{O}_1(\sigma_1)]$$

Where the minus sign comes from the fact that $\oint d\bar{z}$ boundary integral is taken in the opposite direction.

This result means that if we know the OPE between an operator and the stress-tensors $T(z)$and$\bar{T}(\bar{z})$, then we immediately know the operator transform under conformal symmetry. Or, standing this on its head, if we know how an operator transform then we know at least some part of it's OPE with T and$\bar{T}$.

3.4.5 Primary Operators:

The Ward identity allows us to start piecing together some OPEs by looking at how operators transform under conformal symmetries. Although we don't yet know the action of general conformal symmetries, we can start to make progress by looking at the two simplest examples.

(i) Translations: If $\delta z = \epsilon, a$ constant, then all operators transform as

$$\mathcal{O}(z-\epsilon) = \mathcal{O}(z) - \epsilon\partial\mathcal{O}(z) + \cdots$$

The Noether current for translations is the stress-energy tensor T.

The Ward identity in the form (3.12) tells us that the OPE of T with any operator $\mathcal{O}$ must be the form,

$$T(z)\mathcal{O}(w,\bar{w}) = \cdots + \frac{\partial\mathcal{O}(w,\bar{w})}{z-w} + \cdots \tag{3.12}$$

Similarly, the OPE with $\bar{T}$ is

$$\bar{T}(\bar{z})\mathcal{O}(w,\bar{w}) = \cdots + \frac{\bar{\partial}\mathcal{O}(w,\bar{w})}{\bar{z}-\bar{w}} + \cdots \tag{3.13}$$

(ii)Rotations and Scaling: The transformation

$$z \to z + \epsilon z \quad and \quad \bar{z} \to \bar{z} + \bar{\epsilon}\bar{z} \tag{3.14}$$

Describes rotation for ϵ purely imaginary and scaling (dilatation) for ϵ real.

Not all operators have good transformation properties under these actions.

This is entirely analogous to the statement in quantum mechanics that not all states transform nicely under the Hamiltonian H and angular momentum operator L. However, in quantum mechanics we know that the eigenstates of $H$and L can be chosen as a basis of the Hilbert space provided, of course, that $[H, L] = 0$.

The same statement holds for operators in a CFT: we can choose a basis of local operators that have good transformation properties under rotations and dilatations. In fact, we will see in Section 3.6 that the statement about local operators actually follows from the statement about states.

Definition (**3.4.6**): An operator $\mathcal{O}$is said to have weight $(h, \tilde{h})$if, under $\delta z = \epsilon z$ and $\delta\bar{z} = \bar{\epsilon}\bar{z}, \mathcal{O}$transforms as

$$\delta\mathcal{O} = -\epsilon(h\mathcal{O} + z\partial\mathcal{O}) - \bar{\epsilon}\left(\tilde{h}\mathcal{O} + \bar{z}\bar{\partial}\mathcal{O}\right) \tag{3.15}$$

The terms $\partial\mathcal{O}$ in this expression would be there for any operator. They simply come from expanding $\mathcal{O}(z - \epsilon z, \bar{z} - \bar{\epsilon}\bar{z})$. The terms $h\mathcal{O}$ and $\tilde{h}\mathcal{O}$ are special to operators which are eigenstates of dilatations and rotations.

Some comment:

1. Both h and$\tilde{h}$ are real numbers. In a unitary CFT, all operators have $h, \tilde{h} \geq 0$.wewill prove this is Section 3.4.5.
2. The weights are not as unfamiliar as they appear. They simply tell us how operators transform under rotations and scaling. But we already have names for these concepts from undergraduate days. The eigenvalue under rotation is usually called the spin, s, and is given in terms of the weights as

$$s = h - \tilde{h}$$

Meanwhile, the scaling dimension Δ of an operator i

$$\Delta = h + \tilde{h}$$

3. To motivate these definitions, it's worth recalling how rotations and scale transformations act on the underlying coordinates. Rotations are implemented by the operator

$$L = -i(\sigma^1\partial_2 - \sigma^2\partial_1) = z\partial - \bar{z}\bar{\partial}$$

While the dilation operator D which gives rise to scaling is

$$D = \sigma^\alpha\partial_\alpha = z\partial + \bar{z}\bar{\partial}$$

4. The scaling dimension is nothing more that the familiar "dimension" that we usually associate to fields and operators by dimensional analysis.

For example, world sheet derivatives always increase the dimension of an operator by one**:**

$\Delta[\partial] = +1$. The tricky part is that the naive dimension that fields have in the classical theory is not necessarily the same as the dimension in the quantum theory.

Let's compare the transformation law (3.16) with the Ward identity (3.12).

The Nether current arising from rotations and scaling $\delta z = \epsilon z$ was given in (3.7): it is$J(z) = zT(z)$.this means that the residue of the $J\mathcal{O}$ OPE will determine the $1/z^2$ term in the $T\mathcal{O}$ OPE. Similar arguments hold, of course, for $\delta\bar{z} = \bar{\epsilon}\bar{z}$ and$\bar{T}$.so, the upshot of this is that, for an operator $\mathcal{O}$ with weight $(h, \tilde{h})$, the OPE with T and $\bar{T}$ takes the form

$$T(z)\mathcal{O}(w,\bar{w}) = \cdots + h\frac{\mathcal{O}(w,\bar{w})}{(z-w)^2} + \frac{\partial\mathcal{O}(w,\bar{w})}{z-w} + \cdots$$

$$\bar{T}(\bar{z})\mathcal{O}(w,\bar{w}) = \cdots\ \tilde{h}\frac{\mathcal{O}(w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\mathcal{O}(w,\bar{w})}{\bar{z}-\bar{w}} + \cdots$$

3.4.6 Definition of Primary Operators:

A primary operator is one whose OPE with T and $\bar{T}$ truncates at order $(z-w)^{-2}$ or order $(\bar{z}-\bar{w})^{-2}$ respectively. There are no higher singularities:

$$T(z)\mathcal{O}(w,\bar{w}) = h\frac{\mathcal{O}(w,\bar{w})}{(z-w)^2} + \frac{\partial\mathcal{O}(w,\bar{w})}{z-w} + non-singular$$

$$\bar{T}(\bar{z})\mathcal{O}(w,\bar{w}) = \tilde{h}\frac{\mathcal{O}(w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\mathcal{O}(w,\bar{w})}{\bar{z}-\bar{w}} + non-singular$$

Since we now know all singularities in the $T\mathcal{O}$ OPE, we can reconstruct the transformation under all conformal transformations. The importance of primary operator is that they have particularly simple transformation properties. Focussing on $\delta = \epsilon(z)$, we have

$$\delta\mathcal{O}(w,\bar{w}) = -Res[\epsilon(z)T(z)\mathcal{O}(w,\bar{w}]$$

$$= -Res\left[\epsilon(z)\left(h\frac{\mathcal{O}(w,\bar{w})}{(z-w)^2} + \frac{\partial\mathcal{O}(w,\bar{w})}{z-w} + \cdots\right)\right]$$

We want to look at smooth conformal transformations and so require that $\epsilon(z)$ it self has no singularities at $z = w$.we can then Taylor expand

$$\epsilon(z) = \epsilon(w) + \epsilon'(w)(z-w) + \cdots$$

We learn that the infinitesimal change of a primary operator under a general conformal transformation $\delta z = \epsilon(z)$ is

$$\delta\mathcal{O}(w,\bar{w}) = -h\,\epsilon'(w)\mathcal{O}(w,\bar{w}) - \epsilon(w)\,\partial\mathcal{O}(w,\bar{w}) \qquad (3.16)$$

There is a similar expression for the anti-holomorphic transformations $\delta\bar{z}$=$\bar{\epsilon}(\bar{z})$.

Equation (3.16) holds for infinitesimal conformal transformations.

It is a simple matter to integrate up to find how primary operators change under a finite conformal transformation,

$$z \to \tilde{z}(z) \quad and \quad \bar{z} \to \tilde{\bar{z}}(\bar{z})$$

The general transformation of a primary operator is given by

$$\mathcal{O}(z,\bar{z}) \to \tilde{\mathcal{O}}(\tilde{z},\tilde{\bar{z}}) = \left(\frac{\partial\tilde{z}}{\partial z}\right)^{-h}\left(\frac{\partial\tilde{\bar{z}}}{\partial\bar{z}}\right)^{-\tilde{h}}\mathcal{O}(z,\bar{z}) \qquad (3.17)$$

It will turn out that one of the main objects of interest in a CFT is the spectrum of weights $(h,\tilde{h})$ of primary fields. This will be equivalent to computing the particle mass spectrum in a quantum field theory. In the context of statistical mechanics, the weights of primary operators are the critical exponents.

3.5 An Example: The Free Scalar Field:

Let's look at how all of this works for the free scalar field. We'll start by familiarizing ourselves with some techniques using the path integral.

The action is,

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\, \partial_\alpha X\, \partial^\alpha X \tag{3.18}$$

The classical equation of motion is $\partial^2 X = 0$.let'sstarting by seeing how to derive the analogous statement in the quantum theory using the path integral.

The key fact that we'll need is that the integral of a total derivative vanishes in the path integral just as it does in an ordinary integral. From this we have,

$$0 = \int \mathcal{D}X \frac{\delta}{\delta X(\sigma)} e^{-S} = \int \mathcal{D}X e^{-S}\left[\frac{1}{2\pi\alpha'}\partial^2 X(\sigma)\right]$$

But this is nothing more than the Ehrenfest theorem which stares that expectation values of operators obey the classical equations of motion,

$$\langle \partial^2 X(\sigma)\rangle = 0$$

3.5.1 The Propagator:

The next thing that we want to do is compute the propagator for X.

We could do this using canonical quantization, but it will be useful to again see how it works using the path integral. This time we look at,

$$0 = \int \mathcal{D}X \frac{\delta}{\delta X(\sigma)}[e^{-S}X(\sigma')] = \int \mathcal{D}X e^{-S}\left[\frac{1}{2\pi\alpha'}\partial^2 X(\sigma)X(\sigma') + \delta(\sigma-\sigma')\right]$$

So this time we learn that

$$\langle \partial^2 X(\sigma)X(\sigma')\rangle = -2\pi\alpha'\delta(\sigma-\sigma') \tag{3.19}$$

Note that if we'd computed this in the canonical approach, we would have found the same answer**:** the $\delta -$ function arises in this calculation because all correlation functions are time-ordered.

We can now treat (3.20) as a differential equation for the propagator $\langle X(\sigma)X(\sigma')\rangle$.

To solve this equation, we need the following standard result

$$\partial^2 In(\sigma - \sigma')^2 = 4\pi\delta(\sigma - \sigma') \tag{3.20}$$

Since this is important, let's just quickly cheek that it's true. It's a simple application of Stokes' theorem. Set $\sigma' = 0$ and integrate over $\int d^2\,\sigma$.

We obviously get 4π from the right-hand-side. The left-hand-side gives

$$\int d^2\sigma\,\partial^2 In(\sigma_1^2 + \sigma_2^2) = \int d^2\,\sigma\partial^\alpha\left(\frac{2\sigma_\alpha}{\sigma_1^2 + \sigma_2^2}\right) = 2\oint \frac{(\sigma_1 d\sigma^2 - \sigma_2 d\sigma^1)}{\sigma_1^2 + \sigma_2^2}$$

Switching to polar coordinates $\sigma_1 + i\sigma_2 = re^{i\theta,}$ we can rewrite this expression as

$$2\int \frac{r^2 d\theta}{r^2} = 4\pi$$

Confirming (3.20). Applying this result to our equation (3.19), we get the propagator of a free scalar in two-dimensions,

$$\langle X(\sigma)X(\sigma')\rangle = -\frac{\alpha'}{2} In(\sigma - \sigma')^2$$

The propagator has a singularity as$\sigma \to \sigma'$.this is an ultra-violet divergence and is common to all field theories. It also has a singularity as $|\sigma - \sigma'| \to \infty$.this is telling us something important that we will mention in Section 3.3.2.

Finally, we could repeat our trick of looking at total derivatives in the path integral, now with other operator insertions $\mathcal{O}_1(\sigma_1), \dots \mathcal{O}_n(\sigma_n)$ in the path integral. As long as$\sigma, \sigma' \neq \sigma_i$, then the whole analysis goes through as before. But this is exactly our criterion to write the operator product equation,

$$X(\sigma)X(\sigma') = -\frac{\alpha'}{2} In(\sigma - \sigma')^2 + \cdots \tag{3.21}$$

We can also write this in complex coordinates. The classical equation of motion $\partial\bar{\partial}X = 0$ allows us to split the operator X into left-moving and right-moving pieces,

$$X(z, \bar{z}) = X(z) + \bar{X}(\bar{z})$$

We'll focus just on the left-moving piece. This has the operator product expansion,

$$X(z)X(w) = -\frac{\alpha'}{2} In(z - w) + \cdots$$

The logarithm means that $X(z)$ doesn't have any nice properties under the conformal transformations. For this reason, the "fundamental field" X is not really the object of interest in this theory! However, we can look at the derivative of X. This has a rather nice looking OPE,

$$\partial X(z)\partial X(w) = -\frac{\alpha'}{2}\frac{1}{(z-w)^2} + non - singular \qquad (3.22)$$

3.5.2 An Aside: No Goldstone Bosons in Two Dimensions:

The infra-red divergence in the propagator has an important physical implication. Let's start by pointing out one of the big difference between quantum mechanics and quantum field theory in $d = 3 + 1$ dimensions. Since the language used to describe these two theories is rather different, you may not even be aware that this difference exists.

Consider the quantum mechanics of a particle on a line. This is a $d = 0 + 1$ dimensional theory of a free scalar field X. Let's prepare the particle in some localized state-say a Gaussian wave function$\Psi(X) \sim exp(-X^2/L^2)$.

What then happens?

The wave function starts to spread out. And the spreading doesn't stop. In fact, the would-be ground state of the system is a uniform wave function of infinite width, which isn't a state in the Hilbert space because it is non-normalizable.

Let's now compare this to the situation of a free scalar without potential. The physics is very different: The theory has an infinite number of ground state, determined by the expectation value$\langle X \rangle$. Small fluctuations around this vacuum are mass less: they are Goldstone bosons for broken translational invariance $X \to X + c$.

We see that the physics is very different in field theories in$d = 0 + 1$ and $d = 3 + 1$ dimensions. The wave function spreads along flat directions in quantum mechanics, but not in higher dimensional field theories. But what happens in $d = 1 + 1$ and $d = 2 + 1$ dimensions? It turns out that field theories in $d = 1 + 1$ dimensions are more like quantum mechanics: the wave function spreads. Theories in $d = 2 + 1$ dimensions and higher exhibit the opposite behaviour: they have Goldstone bosons. The place to see this is the propagator. In d spacetime dimensions, it takes the form

$$\langle X(r)X(0)\rangle \sim \begin{Bmatrix} 1/r^{d-2} & d \neq 2 \\ Inr & d = 2 \end{Bmatrix}$$

Which diverges at large r only for $d = 1$ and $d = 2$.If we perturb the vacuum slightly by inserting the operator $X(0)$, this correlation function tells us how this perturbation falls off with distance. The infra-red divergence in low dimensions is telling us that the wave function wants to spread.

The spreading of the wave function in low dimensions means that there is no spontaneous symmetry breaking and no Goldstone bosons. It is usually referred to as the Coleman-Mermin-Wagner theorem. Note, however, that it certainly doesn't prohibit massless excitations in two dimensions: it only prohibits Goldstone-like massless excitations.

3.5.3 The Stress-Energy Tensor and Primary Operators:

We want to compute the OPE of T with other operators. Firstly, what is T?

We computed it in the classical theory in (3.9). It is,

$$T = -\frac{1}{\alpha'}\partial X \partial X \tag{3.23}$$

But we need to be careful about what this means in the quantum theory.

It involves the product of two operators defined at the same point and this is bound to mean divergences if we just treat it natively.

In canonical quantization, we would be tempted to normal order by putting all annihilation operators to the right. This guarantees that the vacuum has zero energy.

Here we do something that is basically equivalent, but without reference to creation and annihilation operators.

We write

$$T = -\frac{1}{\alpha'}:\partial X \partial X:\equiv -\frac{1}{\alpha'}\lim_{z\to w}(\partial X(z)\partial X(w) - \langle \partial X(z)\partial X(w)\rangle) \tag{3.24}$$

Which, by construction, has$\langle T\rangle = 0$.

With this definition of T, let's start to compute the OPEs to determine the primary fields in the theory.

Claim 1: ∂X is a primary field with weight $h = 1$ and $\tilde{h}$=0.

Proof: We need to figure out how to take products of normal ordered operators

$$T(z)\,\partial\,X(w) = -\frac{1}{\alpha'}:\partial\,X\,(z)\,\partial\,X\,(z):\partial\,X\,(w)$$

The operators on the left-hand side are time-ordered (because all operator expressions of this type are taken to live inside time-ordered correlation functions). In contrast, the right-hand side is a product of normal-ordered operators: But we know how to change normal ordered products into time ordered products: this is the content of Wick's theorem. Although we have defined normal ordering in (3.25) without reference to creation and annihilation operators, Wick's theorem still holds. We must sum over all possible contractions of pairs of operators, where the term "contraction" means that we replace the pair by the propagator,

$$\overbrace{\partial X(z)\partial X(w)} = \frac{\alpha'}{2}\frac{1}{(z-w)^2}$$

Using this, we have

$$T(z)\partial X(w) = -\frac{2}{\alpha'}\partial X(z)\left(-\frac{\alpha'}{2}\frac{1}{(z-w)^2} + non - singular\right)$$

Here the "non-singular" piece includes the totally normal ordered term: $T(z)\partial X(w)$ ∴ It is only the singular part that interests us. Continuing, we have

$$T(z)\partial X(w) = \frac{\partial X(z)}{(z-w)^2} + \cdots = \frac{\partial X(w)}{(z-w)^2} + \frac{\partial^2 X(w)}{z-w} + \cdots$$

This is indeed the OPE for a primary operator of weight $h = 1$.

Note that higher derivatives $\partial^n X$ are not primary for $n > 1$. For example, $\partial^2 X$ has weight$(h, \tilde{h}) = (2{,}0)$, but is not a primary operator, as we see from the OPE,

$$T(z)\partial^2 X(w) = \partial_w\left[\frac{\partial X(w)}{(z-w)^2} + \cdots\right] = \frac{2\partial X(w)}{(z-w)^3} + \frac{2\partial^2 X(w)}{(z-w)^2} + \cdots$$

The fact that the field $\partial^n X$ has weight $(h, \tilde{h}) = (n, 0)$ fits our natural intuition: each derivative provides spin $s = 1$ and dimension $\Delta = 1$, while the field X does not appear to be contributing, presumably reflecting the fact that it has native, classical dimension zero. However, in the quantum theory, it is not correct to say that X has vanishing dimension: it has an ill-defined dimension due to the logarithmic behaviour of its OPE (3.22). This is responsible for the following, more surprising, and result

Claim 2: The field:e^{ikX}: is primary with weight $h = \tilde{h} = \alpha' k^2/4$.

This result is not what we would guess from the classical theory.

Indeed, it's obvious that it has a quantum origin because the weight is proportional to α', which sits outside the action in the same place that h would (if we hadn't set it to one). Note also that this means that the spectrum of the free scalar field is continuous. This is related to the fact that the range of X is non-compact. Generally, CFTs will have a discrete spectrum.

Proof: Let's first compute the OPE with ∂X. We have

$$\partial X(z) \colon e^{ikX(w)} \colon \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \partial X(z) \colon X(w)^n \colon$$

$$= \sum_{n=1}^{\infty} \frac{(ik)^n}{(n-1)!} \colon X(w)^{n-1} \colon \left(-\frac{\alpha'}{2} \frac{1}{z-w} \right) + \cdots$$

$$= -\frac{i\alpha' k}{2} \frac{\colon e^{ikX(w)} \colon}{z-w} + \cdots \tag{3.25}$$

From this, we can compute the OPE with T.

$$T(z) \colon e^{ikX(w)} \colon = -\frac{1}{\alpha'} \colon \partial X(z) \partial X(z) \colon\colon e^{ik(w)} \colon$$

$$= \frac{\alpha' k^2}{4} \frac{\colon e^{ikX(w)} \colon}{(z-w)^2} + ik \frac{\colon \partial X(z) e^{ikX(w)} \colon}{z-w} + \cdots$$

Where the first term comes from two contractions, while the second term comes from a single contraction. Replacing ∂_z by ∂_w in the final term we get

$$T(z) \colon e^{ikX(w)} \colon = \frac{\alpha' k^2}{4} \frac{\colon e^{ikX(w)} \colon}{(z-w)^2} + \frac{\partial_w \colon e^{ikX(w)}}{z-w} + \cdots \tag{3.26}$$

Showing that:$e^{ikX(w)}$:is indeed primary. We will encounter this operator frequently later, but will choose to simplify notation and drop the normal ordering colons. Normal ordering will just be assumed from now on.

Finally, let's check to see the OPE of T with itself. This is again just an exercise in Wick contractions.

$$T(z)T(w)\frac{1}{\alpha'^2}:\partial X(z)\partial X(z)::\partial X(w)\partial X(w):$$

$$=\frac{2}{\alpha'^2}\left(-\frac{\alpha'}{2}\frac{1}{(z-w)^2}\right)^2-\frac{4}{\alpha'^2}\frac{\alpha'}{2}\frac{:\partial X(z)\partial X(w):}{(z-w)^2}+\cdots$$

The factor of 2 in front of the first term comes from the two ways of performing two contractions; the factor of 4 in the second term comes from the number of ways of performing a single contraction. Continuing,

$$T(z)T(w)=\frac{1/2}{(z-w)^4}+\frac{2T(w)}{(z-w)^2}-\frac{2}{\alpha'}\frac{\partial^2X(w)\partial X(w)}{z-w}+\cdots$$

$$=\frac{1/2}{(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{\partial T(w)}{z-w}+\cdots \qquad (3.27)$$

We learn that T is not a primary operator in the theory of a single free scalar field. It is an operator of weight$(h,\tilde{h})=(2,0)$, but it fails the primary test on account of the $(z-w)^{-4}$ term. In fact, this property of the stress energy tensor a general feature of all CFTs which we now explore in more detail.

3.6 The Central Charge:

In any CFT, The most prominent example of an operator which is not primary is the stress-energy tensor itself.

For the free scalar field, we have already seen that T is an operator of weight$(h,\tilde{h})=(2,0)$. This remains true in any CFT. The reason for this is simple: $T_{\alpha\beta}$ has dimension$\Delta=2$ because we obtain the energy by integrating over space. It has spun$s=2$ because it is a symmetric 2-tensor. But these two pieces of information are equivalent

To the statement that T is has weight (2, 0). Similarly, $\bar{T}$ has weight (0, 2).

This means that the TT OPE takes the form,

$$T(z)T(w) = \cdots + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots$$

And similar for $\bar{T}\bar{T}$, what other terms could we have in this expansion? Since each term has dimension $\Delta = 4$, any operators that appear on the right-hand-side must be of the form

$$\frac{\mathcal{O}_n}{(z-w)^n} \tag{3.28}$$

Where $\Delta[\mathcal{O}_n] = 4 - n$. But, in a unitary CFT there are no operators with h,

$\tilde{h} < 0$. (We will prove this shortly). So the most singular term that we can have is of order$(z-w)^{-4}$. Such a term must be multiplied by a constant. We write,

$$T\,(z)\,T\,(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots$$

And, similarly,

$$\bar{T}(\bar{z})\bar{T}(\bar{w}) = \frac{\tilde{c}/2}{(\bar{z}-\bar{w})^4} + \frac{2\bar{T}(\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\bar{T}(w)}{\bar{z}-\bar{w}} + \cdots$$

The constants c and $\tilde{c}$ are called the central charges. (Sometimes they are referred to as left-moving and right-moving central charges). They are perhaps the most important numbers characterizing the CFT. We can already get some intuition for the information contained in these two numbers. Looking back at the free scalar field (3.28) we see that it has $c = \tilde{c} = 1$. If we instead considered D non-interacting free scalar fields, we would get $c = \tilde{c} = D$. This gives us a hint: c and $\tilde{c}$ are somehow measuring the number of degrees of freedom in the CFT. This is true in a deep sense! However, be warned: c is not necessarily an integer.

Before moving on, it's worth pausing to explain why we didn't include a $(z-w)^{-3}$ term in the TT OPE. The reason is that OPE must obey

$T(z)T(w) = T(w)T(z)$ because, as explained previously, these operator equations are all taken to hold inside time-ordered correlation functions.

So the quick answer is that a $(z-w)^{-3}$ term would not be invariant under $z \leftrightarrow w$. However, you may wonder how the $(z-w)^{-1}$ term manages to satisfy this property. Let's see how this works:

$$T(w)T(z) = \frac{c/2}{(z-w)^4} + \frac{2T(z)}{(z-w)^2} + \frac{\partial T(z)}{w-z} + \cdots$$

Now we can Taylor expand $T(z) = T(w) + (z-w)\partial T(w) + \cdots$

and$\partial T(z) = \partial T(w) + \cdots$using this in the above expression, we find

$$T(w)T(z) = \frac{c/2}{(z-w)^4} + \frac{2T(w) + 2(z-w)\partial T(w)}{(z-w)^2} - \frac{\partial T(w)}{z-w} + \cdots$$
$$= T(z)T(w)$$

This trick of Taylor expanding saves the $(z-w)^{-1}$ term. It wouldn't work for the $(z-w)^{-3}$ term.

3.6.1 The Transformation of Energy:

So T is not primary unless $c = 0$. And we will see shortly that all theories have $c > 0$.what does this mean for the transformation of T?

$$\delta T\ (w) = -Res[\epsilon(z)T(z)T(w)]$$

$$= -Res\left[\epsilon(z)\left(\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots\right)\right]$$

If $\epsilon(z)$ contains no singular terms, we can expand

$$\epsilon(z) = \epsilon(w) + \epsilon'(w)(z-w) + \frac{1}{2}\epsilon''(z-w)^2 + \frac{1}{6}\epsilon'''(w)(z-w)^3 + \ldots$$

From which we find

$$\delta T(w) = -\epsilon(w)\partial T(w) - 2\epsilon'(w)T(w) - \frac{c}{12}\epsilon'''(w) \tag{3.29}$$

This is the infinitesimal version. We would like to know what becomes of T under the finite conformal transformation $z \to \tilde{z}(z)$. The answer turns out to be

$$\tilde{T}(\tilde{z}) = \left(\frac{\partial \tilde{z}}{\partial z}\right)^{-2}\left[T(z) - \frac{c}{12}S(\tilde{z}, z)\right] \tag{3.30}$$

Where $S(\tilde{z}, z)$ is known as the Schwaezian and is defined by

$$S(\tilde{z}, z) = \left(\frac{\partial^3 \tilde{z}}{\partial z^3}\right)\left(\frac{\partial \tilde{z}}{\partial z}\right)^{-1} - \frac{3}{2}\left(\frac{\partial^2 \tilde{z}}{\partial z^2}\right)^2\left(\frac{\partial \tilde{z}}{\partial z}\right)^{-2} \tag{3.31}$$

It is simple to check that the Schwarzian has the right infinitesimal form to give (3.30). Its key property is that it preserves the group structure of successive conformal transformations.

3.6.2C is for Casmir

Note that the extra term in the transformation (3.31) of T does not depend on T itself. In particular, it will be the same evaluated on all states. It only affects the constant term-or zero mode-in the energy. In other words, it is the Casimir energy of the system.

Let's look at an example that will prove to be useful later for the string. Consider the Euclidean cylinder, parameterizes by

$$w = \sigma + iT \quad , \quad \sigma \in [0,2\pi)$$

We can make a conformal Transformation from the Cylinder to the complex Plane by

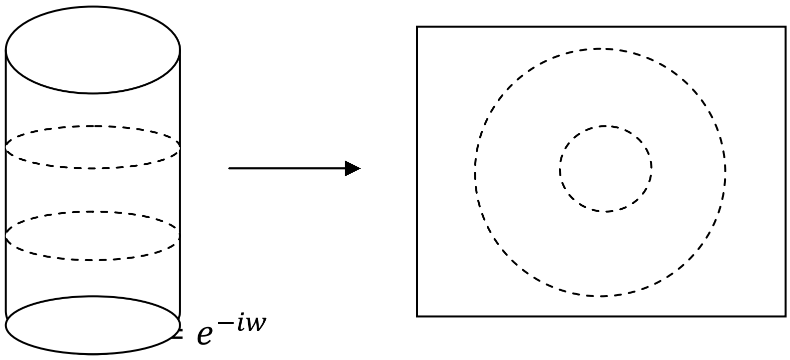


Figure 22:

The fact that the cylinder and the planeare related by a conformal map means that if we understand a given CFT on The cylinder, then we immediately understand it on the plane. And vice- versa. Notice that constant time slices on the cylinder are mapped to circles of constant radius. The origin, $z = 0$, is the distant past, $T \to -\infty$.

What becomes of T under this transformation? The Schwarzian can be easily calculated to be $S(z, w) = 1/2$.so we find,

$$T_{cylinder}(w) = -z^2 T_{pane}(z) + \frac{c}{24} \tag{3.32}$$

Suppose that the ground state energy vanishes when the theory is defined on the plane: $\langle T_{plane} \rangle = 0$. What happens on the cylinder? We want to look at the Hamiltonian, which is defined by

$$H \equiv \int d\sigma\, T_{TT} = -\int d\sigma\, (T_{ww} + \bar{T}_{\bar{w}\bar{w}})$$

The conformal transformation then tells us that the ground state energy on the cylinder is

$$E = -\frac{2\pi(c + \tilde{c})}{24}$$

This is indeed the (negative) Casimir energy on a cylinder. For a free scalar field, we have $c = \tilde{c} = 1$ and the energy density $E/2\pi - 1/12$.this is the same result that we got in Section 2.2.2, but this time with no funny business where we throw out infinities.

3.7An Application: the Luscher Term

If we're looking at a physical system, the cylinder will have a radiusL.

In thiscase,the Casimir energy is given by$E = -2\pi(c, \tilde{c})24L$. There is an application of this to QCD −like theories. Consider two quarks in a confining theory, separated by a distance L. If the tension of the confining flux tube is T, then the string will be stable as long as $TL \lesssim m$, the mass of the lightest quark. The energy of the stretched string as a function of L is given by

$$E(L) = TL + a - \frac{\pi c}{24L} + \cdots$$

Here $a$is an undetermined constant, while c counts the number of degrees of freedom of the QCD flux tube. (There is no analog of $\tilde{c}$ here because of the reflecting boundary conditions at the end of the string). If the string has no internal degrees of freedom, then $c = 2$ for the two transverse fluctuations. This contribution to the string energy is known as the luscher term.

3.7.1 The Weyl Anomaly:

There is another way in which the central charge affects the stress-energy tensor. Recall that in the classical theory, one of the defining features of a CFT was the vanishing of the trace of the stress tensor,

$$T_{\alpha}^{\alpha} = 0$$

However, things are more subtle in the quantum theory. While $\langle T_{\alpha}^{\alpha} \rangle$ indeed vanishes in flat space, it will not longer be true if we place the theory on a curved background. The purpose of this section is to show that

$$\langle T_{\alpha}^{\alpha} \rangle = -\frac{c}{12} R \qquad (3.33)$$

Where R is the Ricci scalar of the 2d world sheet? Before we derive this formula, some quick comments:

1. Equation (3.33) holds for any state in the theory- not just the vacuum.

This reflects the fact that it comes from regulating short distant divergences in the theory. But, at short distances all finite energy states look basically the same.

2. Because$\langle T_\alpha^\alpha \rangle$ is the same for any state it must be equal to something that depends only on the background metric. This something should be local and must be dimension 2. The only candidate is the Ricci scalar R.for this reason; the formula $\langle T_\alpha^\alpha \rangle \sim R$ is the most general possibility. The only question is: what is the coefficient. And, in particular, is it non-zero?

3. By a suitable choice of coordinates, we can always put any 2d metric in the form$g_{\alpha\beta} = e^{2w}\delta_{\alpha\beta}$. In these coordinates, the Ricci scalar is given by

$$R = -2e^{-2w}\partial^2 w \tag{3.34}$$

Which depends explicitly on the function w. Equation (3.33) is then telling us that any conformal theory with $c \neq 0$ has at least one physical observable, $\langle T_\alpha^\alpha \rangle$, which takes different values on backgrounds related by a Weyl transformation w. This result is referred to as the Weyl anomaly, or sometimes as the trace anomaly.

4. There is also a Weyl anomaly for conformal field theories in higher dimensions. For example, 4d CFTs are characterizes by two numbers, a and c, which appear as coefficients in the Weyl anomaly,

$$\langle T_\mu^\mu \rangle_{4d} = \frac{c}{16\pi^2} C_{\rho\sigma k\lambda} C^{\rho\sigma k\lambda} - \frac{a}{16\pi^2} \tilde{R}_{\rho\sigma k\lambda} \tilde{R}^{\rho\sigma k\lambda}$$

Where C is the Weyl tensor and $\tilde{R}$ is the dual of the Riemann tensor.

5. Equation (3.33) involves only the left-moving central charge c.

You might wonder what's special about the left-moving sector. The answer, of course, is nothing. We also have

$$\langle T_\alpha^\alpha \rangle = -\frac{\tilde{c}}{12} R$$

In flat space, conformal field theories with different c and $\tilde{c}$ are perfectly acceptable. However, if we wish these theories to be consistent in fixed, curved backgrounds, then we require $c = \tilde{c}$. This is an example of a gravitational anomaly.

6. The fact that Weyl invariance requires $c = 0$ will prove crucial in string theory. We shall return to this in Chapter5.

We will now prove the Weyl anomaly formula (3.33). Firstly, we need to derive an intermediate formula: the $T_{z\bar{z}}T_{w\bar{w}}$ OPE. Of course, in the classical theory we found that conformal invariance requires $T_{z\bar{z}} = 0$.we will now show that it's a little more subtle in the quantum theory.

Our starting point is the equation for energy conservation,

$$\partial T_{z\bar{z}} = -\bar{\partial} T_{zz}$$

Using this, we can express our desired OPE in terms of the familiar TT OPE,

$$\begin{aligned} \partial_z T_{z\bar{z}}(z,\bar{z})\partial_w T_{w\bar{w}}(w,\bar{w}) &= \bar{\partial}_{\bar{z}} T_{zz}(z,\bar{z})\bar{\partial}_{\bar{w}} T_{ww}(w,\bar{w}) \\ &= \bar{\partial}_{\bar{z}}\bar{\partial}_{\bar{w}}\left[\frac{c/2}{(z-w)^4} + \cdots\right] \end{aligned} \tag{3.35}$$

Now you might think that the right-hand-side just vanishes: after all, it is an anti-Holomorphy derivative $\bar{\partial}$ of a Holomorphy quantity. But we shouldn't be so cavalier because there is a singularity at $z = w$. For example, consider the follow equation,

$$\bar{\partial}_{\bar{z}}\partial_z \, In \, |z-w|^2 = \bar{\partial}_{\bar{z}} \frac{1}{z-w} = 2\pi\delta(z-w, \bar{z}-\bar{w}) \tag{3.36}$$

We proved this statement after equation (3.20). (The factor of 2 differences from (3.20) can be traced to the conventions we defined for complex coordinates in Section 3.0.1). Looking at the intermediate step in (3.36), we again have an anti-Holomorphy derivative of a Holomorphy function and you might be tempted to say that this also vanishes. But you'd be wrong: subtle things happen because of the singularity and equation (3.37) tells us that the function $1/z$ secretly depends on $\bar{z}$. (This should really be understood as a statement about distributions, with the delta function integrated against arbitrary test functions). Using this result, we can write

$$\bar{\partial}_{\bar{z}}\bar{\partial}_{\bar{w}}\frac{1}{(z-w)^4}=\frac{1}{6}\bar{\partial}_{\bar{z}}\bar{\partial}_{\bar{w}}\left(\partial_z^2\partial_w\frac{1}{z-w}\right)=\frac{\pi}{3}\partial_z^2\partial_w\bar{\partial}_{\bar{w}}\delta(z-w,\bar{z}-\bar{w})$$

Inserting this into the correlation function (3.36) and stripping off the $\partial_z\partial_w$ derivatives on both sides, we end up with what we want,

$$T_{z\bar{z}}(z,\bar{z})T_{w\bar{w}}(w,\bar{w})=\frac{c\pi}{6}\partial_z\bar{\partial}_{\bar{w}}\delta(z-w,\bar{z}-\bar{w}) \tag{3.37}$$

So the OPE of $T_{z\bar{z}}$ and $T_{w\bar{w}}$ almost vanishes, but there's some strange singular behaviour going on as $z\to w$. This is usually referred to as a contact term between operators and, as we have shown, it is needed to ensure the conservation of energy-momentum. We will now see that this contact term is responsible for the Weyl anomaly.

We assume that$\langle T_\alpha^\alpha\rangle=0$ in flat space. Our goal is to derive an expression for $\langle T_\alpha^\alpha\rangle$ close to flat space. Firstly, consider the change of $\langle T_\alpha^\alpha\rangle$ under a general shift of the metric$\delta g_{\alpha\beta}$. Using the definition of the energy-momentum tensor (3.4), we have

$$\delta\langle T_\alpha^\alpha(\sigma)\rangle=\delta\int\mathcal{D}\phi e^{-S}T_\alpha^\alpha\,(\sigma)$$

$$=\frac{1}{4\pi}\int\mathcal{D}\phi e^{-S}\left(T_\alpha^\alpha(\sigma)\int d^2\,\sigma'\sqrt{g}\,\delta g^{\beta\gamma}T_{\beta\gamma}(\sigma')\right)$$

If we now restrict to a Weyl transformation, the change to a flat metric is

$\delta g_{\alpha\beta}=2\omega\delta_{\alpha\beta}$, so the change in the inverse metric is$\delta g^{\alpha\beta}-2\omega\delta^{\alpha\beta}$.

This gives

$$\delta\langle T_\alpha^\alpha(\sigma)\rangle=-\frac{1}{2\pi}\int\mathcal{D}\phi e^{-S}\left(T_\alpha^\alpha(\sigma)\int d^2\sigma'\,\omega(\sigma')T_\beta^\beta(\sigma')\right) \tag{3.38}$$

Now we see why the OPE (3.38) determines the Weyl anomaly. We need to change between complex coordinates and Cartesian coordinates, keeping track of factors of 2. We have

$$T_\alpha^\alpha(\sigma)T_\beta^\beta(\sigma')=16T_{z\bar{z}}(z,\bar{z})T_{w\bar{w}}(w,\bar{w})$$

Meanwhile, using the conventions laid down in 2.0.1, we have

$8\partial_z \bar{\partial}_{\bar{w}} \delta(z-w, \bar{z}-\bar{w}) = -\partial^2 \delta(\sigma - \sigma')$.This gives us the OPE in Cartesian coordinates

$$T^{\alpha}_{\alpha}(\sigma) T^{\beta}_{\beta}(\sigma') = -\frac{c\pi}{3} \partial^2 \delta(\sigma - \sigma')$$

We now plug this into (3.38) and integrate by parts to move the two derivatives onto the conformal factor **ω**. We're left with,

$$\delta\langle T^{\alpha}_{\alpha} \rangle = \frac{c}{6} \partial^2 \omega \quad \Rightarrow \quad \langle T^{\alpha}_{\alpha} \rangle = -\frac{c}{12} R$$

Where, to get to the final step, we've used (3.35) and, since we're working infinitesimally, we can replace $e^{-2w} \approx 1$.thiscomplete the proof of the Weyl anomaly, at least for spaces infinitesimally close to flat space. The fact that R remains on the right-hand-side for general 2d surfaces follows simply from the comments after equation (3.33), most pertinently the need for the expression to be reparameterization invariant.

3.7.2C is for Cardy:

The Casmir effect and the Weyl anomaly have a similar smell. In both, the central change provides an extra contribution to the energy.

We now demonstrate a different avatar of the central change: it tells us the density of high energy states.

We will study conformal field theory on a Euclidean torus. We'll keep our normalization $\sigma \in [0, 2\pi)$, but now we also take T to be periodic, lying in the range

$$T \in [0, \beta)$$

The partition function of a theory with periodic Euclidean time gas a very natural interpretation: it is related to the free energy of the theory at temperature $T = \frac{1}{\beta}$.

$$Z[\beta] = Tre^{-\beta H} = e^{-\beta F} \tag{3.39}$$

At very low temperatures, $\beta \to \infty$, the free energy is dominated by the lowest energy state. All other states are exponentially suppressed. But we saw in 3.4.1 that the vacuum state on the cylinder has Casimir energy$H = -c/12$. In the limit of low temperature, the partition function is therefore approximated by

$$Z \to e^{c\beta/12} \qquad as \qquad \beta \to \infty \tag{3.40}$$

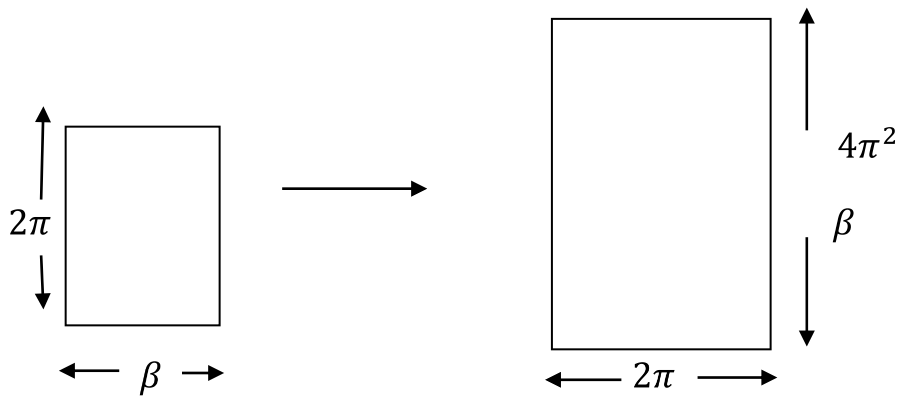


Figure2.3:

Now comes the trick. In Euclidean space, both directions of the torus are on equal footing. We're perfectly at liberty to decide that $\sigma$is "time" and τ is "space". This can't change the value of the partition function. So let's make the swap. To compare to our original partition function, wewant the spatial direction to have range [0,2π). Happily, due to the conformal nature of our theory, we arrange this through the scaling

$$\tau \to \frac{2\pi}{\beta}\tau \qquad , \qquad \sigma \to \frac{2\pi}{\beta}\sigma$$

Now we're back where we started, but with the temporal direction taking values in$\sigma \in [0,4\pi^2/\beta)$. This tells us that the high-temperature and low-temperature partition functions are related,

$$Z[4\pi^2/\beta] = Z[\beta]$$

This is called modular invariance. We'll come across it again in Section 3.4. Writing $\beta' = 4\pi^2/\beta,$ this tells us the very high temperature behaviour of the partition function

$$Z[\beta'] \to e^{c\pi 2/3\beta'} \qquad as \qquad \beta' \to 0$$

But the very high temperature limit of the partition function is sampling all states in the theory. On entropic grounds, this sampling is dominated by the high energy states. So this computation is telling us how many high energy states there are.

To see this more explicitly, let's do some elementary manipulations in statistical mechanics. Any system has a density of states

$\rho(E) = e^{S(E)}$,where $S(E)$ is the entropy. The free energy is given by

$$e^{-\beta F} = \int dE\, \rho(E)\, e^{-\beta E} = \int dE\, e^{S(E)-\beta E}$$

In two dimensions, all systems have an entropy which scales at large energy as

$$S(E) \to N\sqrt{E} \tag{3.41}$$

The coefficient N counts the number of degrees of freedom. The fact that $S \sim \sqrt{E}$ is equivalent to the fact that $F{\sim}T^2$, as befits an energy density in a theory with one spatial dimension. To see this, we need only approximate the integral by the saddle point $S'(E_\star) = \beta$. From (3.42), this gives us the free energy

$$F{\sim}N^2T^2$$

We can now make the statement about the central charge more explicit.

In a conformal field theory, the entropy of high energy states is given by

$$S(E){\sim}\sqrt{cE}$$

This is Cardy's formula.

Chapter Four

Radon Transform and John Transform

4.1. Radon Transform:

4.1.1. Introduction:

Integral geometry goes back to Radon who considered the following problem**:**

Let f: $R^2 \rightarrow R$ is a smooth function with suitable decay condition at∞, and let L C R be an oriented line. Define a function on the space of oriented lines in R^2 by

$$\emptyset(L) = \int_l F \qquad (4.1)$$

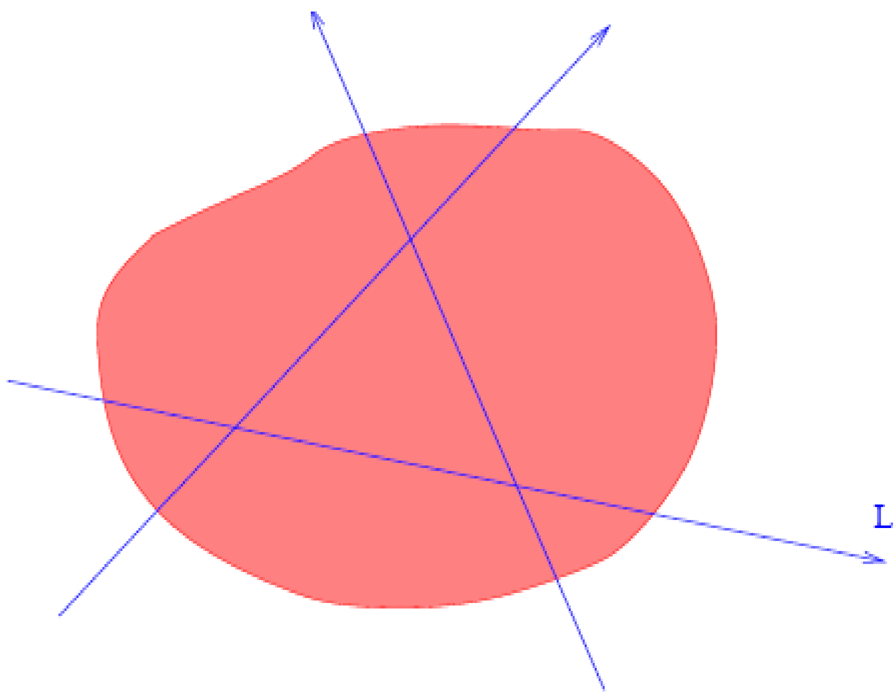


Figure (4.1)

Radon has demonstrated that there exists an inversion formula $\emptyset \rightarrow F$.

Radon's contraction can be generalized in many ways and it will become clear that Penrose's twistor theory is its far reaching generalization.

Be for moving on, it is however worth remarking that an extension of Radon's work has lead to Nobel Prize a warded (in medicine) for pour mathematical research.

It was given in 1979 to Cormack, who unaware of Radon's results had rediscovered the inversion formula (4.1), and had explored the setup allowing the function f to be defined on anon-simply connected region in R^2 with a convex boundary. If one only allows the lines which do not pass through the black region.

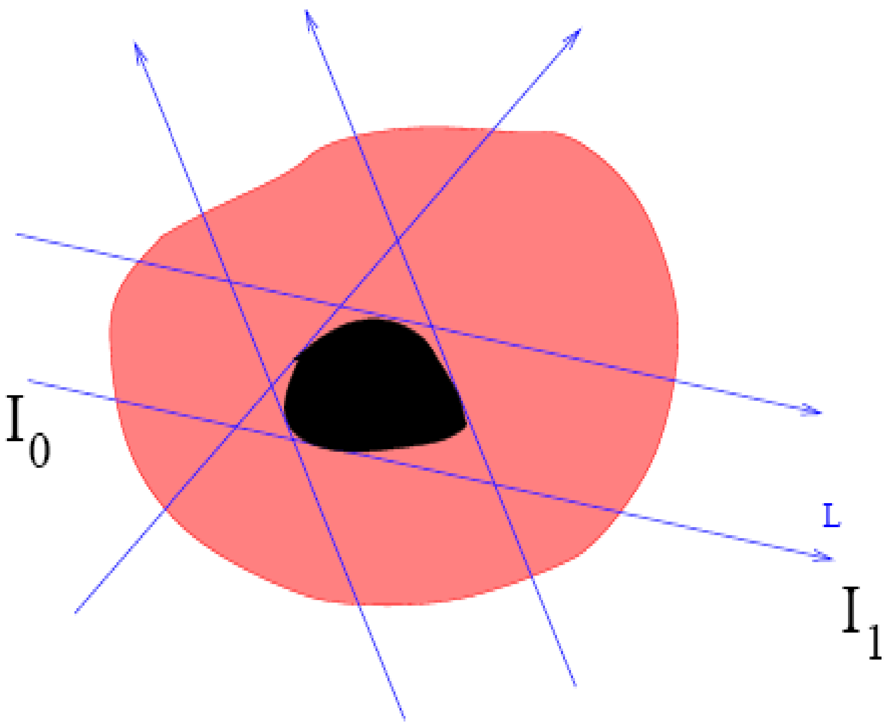


Figure (4.2)

Or are tangent to the boundary of this region, the original function f may still be reconstruction from its integrals along such lines (this is called the support theorem). In the application to computer tomography on takes a number of 2D (Un known) density of these objects. The input data given to a radiologist consist of the intensity of the incoming and outgoing X- rays passing through the object with intensities I_0 and I_1 respectively

$$\emptyset(L) = \int_L \frac{dI}{I} = \log I_1 S - \log I_0 = -\int_L f \qquad (4.2)$$

Where $\frac{dI}{I}$ = - f(s) d s is the relative infinitesimal intensity lass inside the body on interval of length d s.

The Radon transform then allows to recover f from this data , and the generalization provided by the support theorem becomes important if not all regions in the object can be X-raged .

4.1.2. Definition:

Let f(x) = f(x ,y) be a compactly supported continuous function on R^2 . The Radon transform R f is a function defined on the space of straight lines L in R^2 by the line integral along each such line.

$$R f (L) = \int_l f(x) \,|d x| . \tag{4.3}$$

Concretely, the parametrization of any straight line L with respect to arc length Z can always be written

$$(x(z), y(z)) =((z \sin \alpha + s \cos \alpha), (-z \cos \alpha + s \sin \alpha) \tag{4.4}$$

Where s is the distance of L from the origin and α is the angle the normal vector to L makes with the x axis.

It follows that the quantities (a ,s) can be considered as coordinates on the space of all lines in R^2 , and the Radon transform can be expressed in these coordinates by

$$Rf(\alpha, s) = \int_{-\infty}^{\infty} f\big(x(z), y(z)\big)dz \tag{4.5}$$

$$Rf(\alpha, s) = \int_{-\infty}^{\infty} ((z \sin\alpha + s \cos\alpha), (-z \cos\alpha + s \sin\alpha))dz$$

More generally, in then – dimensional Euclidean space R^n the Radon transform of a compactly supported continuous function f is a function R f on the space $\sum n$ of all hyper planes in R^n . It is defined by

$$R f (\xi) = \int_{\xi} f(x)d\sigma(x) \tag{4.6}$$

For $\xi \in \sum n$ where the integral is taken with respect to the natural hyper surface measure, d σ (generalizing the $|d x|$ term from the 2- dimensional case) .Observe that any element of $\sum n$ is characterized as the solution locus of an equation

$$X . \alpha = S \tag{4.7}$$

Where $a \in S^{n-1}$ is a unit vector and $S \in R$. Thus the n- dimensional Radon transform may be rewritten as a function on $S^{n-1} \times R$ via.

$$R\,f(\alpha, s) = \int_{x.\alpha=s} f(x)d\sigma(x) \tag{4.8}$$

It is also possible to generalize the Radon transform still further by integrating instead over k-dimensional affine subspace of R^n the x-ray transform is the most widely used special case of this construction, and is obtained by integrating over starting lines.

4.1.3. Generalized Radon Transform:

Let $s \in R$, $\varphi \in [0, 2\pi]$. We let $\mu(\varphi, x)$ be a smooth function, 2π periodic in φ defined on $R \times R^2$.

Then the generalized radon transform is defined as follows.

$$R\mu f(\emptyset,s) = \int_{x\in L(\emptyset,s)} f(x)\ \mu(\emptyset, x)dx \tag{4.9}$$

Where d x is the arc length measure on the line $L(\varphi,s)$.

Note, that the transform integrates the weighted function f(x) along the lines $L(\varphi,s)$.

4.1.4. Relationship with the Fourier Transform:

The Radon transform is closely related to the Fourier transform.

For a function of one variable the Fourier transform is defined by

$$F(w) = \int_{-\infty}^{\infty} f(x)e^{-2\pi\, i\, x\, w}\ dx \tag{4.10}$$

And for a function of a 2-vector x = (x,y) .

$$F(w) = \iint_{-\infty}^{\infty} f(x)\ e^{-2\pi\, i\, x\, w}dxdy \tag{4.11}$$

For convenience define $R\alpha[\,f\,](s) = R\,[\,f\,](\alpha, s)$ as it is only meaningful to take the Fourier Transform in the s variable . the Fourier slice theorem then states

$$R\alpha[\,f\,](\sigma) = f(\sigma n(\alpha)) \tag{4.12}$$

Where $n(\alpha) = (\cos\alpha, \sin\alpha)$.

Thus the tow-dimensional Fourier transform of the initial function is the one variable Fourier transform of the Radon transform of that function.

More generally, one has the result valid in n-dimensions.

$$F(r\alpha) = \int_{-\infty}^{\infty} Rf(\alpha, s)\, e^{-2\pi i\, x\, t} ds \qquad (4.13)$$

Indeed, the result follows at once by computing the two variable Fourier integral along appropriate sliees:

$$F(r\alpha) = \int_{-\infty}^{\infty} ds \int_{x.\alpha=s} e^{-2\pi i\ t(x.\alpha)} dm(x) \qquad (4.14)$$

An application of the Fourier inversion formula also gives an explicit inversion formula for the Radon transform, and thus shows that it is invertible on suitably chosen spaces of function. However this form is not particularly useful for numerical inversion, and faster discrete inversion methods exist.

4.1.5 Dual Transform:

The dual Radon transform is a kind of adjoint to the Radon transform Beginning with a function g on the space $\sum n$ the dual Radon transform is the function R^*g on R^n defined by

$$R^*g(x) = \int_{x\in\xi} g(\xi) d\mu(\xi). \qquad (4.15)$$

The integral here is taken over the set of all lines incident with the point $x \in R^n$, and the measure $d\mu$ is the unique probability measure on the set $\{\xi/x \in \xi\}$ invariant under rotations about the point x.

Concretely, for the two-dimensional Radon transform the dual transform is given by

$$R^*g(x) = \frac{1}{2\pi}\int_{\alpha=0}^{2\pi} g(\alpha, n(\alpha).x) dx \qquad (4.16)$$

In the context of image processing, the dual transform is commonly called back projection as it takes a function defined on each line in the plane and smears or projects it back over the line to produce an image.

Computationally efficient inversion formulas reconstruct the image from the points where the back-projection lines meet.

4.1.6. Intertwining property

Let Δ denote the Laplacian on R^n:

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2} \quad (4.17)$$

This is a natural rotationally invariant second – order differential operator.

On $\sum n$ the radial second derivative

$$Lf(\alpha, s) = \frac{\partial^2}{\partial s^2} f(\alpha, s) \quad (4.18)$$

Is also rotationally invariant. The Radon transform and its dual are intertwining operators for these two differential operators in the sense that.

$$R(\Delta f) = L(R f) \quad , \quad R^*(L g) = \Delta (R^* g) \quad (4.19)$$

4.1.7. Inversion Formula:

Explicit and computationally efficient inversion formula for the Radon transform and its dual are available. The Radon transform in n-dimensions can be inverted by the formula

$$C^{n f} = (-\Delta)^{(n-1)2} R^* R f \quad (4.20)$$

Where $C_n = (4\pi)^{(n-1)2} \frac{\Gamma(n/2)}{\Gamma(1/2)}$.

And the power of the Laplacian $(-\Delta)^{(n-1)2}$ is defined as a pseudo differential operator if necessary by the Fourier transform. Radon transform by

4.1.8 Theorem: The function f can be recovered from themeans of the following inversion formula

$$C f = (-L)\, (^{n-1)/2} ((\hat{f})) \quad f \in \varepsilon(R^n), \quad (4.21)$$

Provided $f(x) = o(|x|^{-N})$ for some N>n. Here C is the constant

$$C = (4\pi)^{(n-1)2} \frac{\Gamma(n/2)}{\Gamma(1/2)} \quad .$$

Proof: We use the connection between the powers of L and the Riesz potential. We in fact have

$$(\hat{f}) = 2^{n-1} \quad (4.22)$$

We thus obtain the desired formula (4.21). For n odd one can give a more geometric proof of (4.21). We start with some general useful facts about the mean value operator M^r. It is a familiar fact that iff$\in c^2(R^n)$ is αradial function,i.e.

$F(x) = F(r)$, $r = |x|$, then

$$(Lf)(x) = \frac{d2F}{dr2} + \frac{n-1}{r}\frac{dF}{dr}. \qquad (4.23)$$

This is immediate form the relations

$$\frac{\partial^2 f}{\partial x_i^2} = \frac{\partial^2 f}{\partial r^2}\left(\frac{\partial r}{\partial x_i}\right)^2 + \frac{\partial f}{\partial r}\frac{\partial^2 r}{\partial x_i^2}.$$

4.1.9Lemma: (i) $LM^r = M^r L$ for each $r > 0$.

(ii) For $f \in C^2(R^n)$ the mean value $(M^r f)(x)$ satisfies the "Drobox equation"

$$L_x((M^r f)(x)) = \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right)(M^r f(x)),$$

That is, the function $F(x, y) = (M^{|y|} f)(x)$ satisfies

Proof: We prove this group theoretically using expression

$$(M^r f)(x) = \int_k f(x + k.y)\, dk$$

For the mean value. For $z \in R^n$, $k \in k$ let T_z denote the translation $x \to x+z$ and R_k the rotation $x \to k.x$. Since L is invariant these transformation, we have if

$r = |y|$,

$$(L M^r f)(x) = \int_k L_x f(x + k.y)\, dk = \int_k (Lf)(x + k.y) dk = (M^r L f)(x)$$

$$= \int_k [(Lf) \circ T_x \circ R_k](y) = \int_k [L(f \circ T_x \circ R_k)](y) dk$$

$$= L_y \left(\int_k f(x + k.y) dk\right.$$

Which proves the lemma.

Now suppose $f \in s(R^n)$. Fix a hyper plan ξ through o , and n isometry $g \in M(n)$ As k runs through O(n) , g k . $\xi_0$runs through the set of hyper plan through go and we have $\check{\varphi}(g.os) = \int_k \varphi(gk.\xi_0) dk$

And there for

$$(\hat{f})^{\vee}(g.o) \int_k \left(\int_{\xi_0} f(gk.g)dm(y)\right)dk = \int_{\xi_0} dm(y) \int_k f(gk.y)dk$$

$$= \int_{\xi_0} \left(M^{|y|} f\right) g.o)dm(y)$$

$$\text{Hence } ((\hat{f}))^{\vee}(x) = \Omega_{n-1} \int_0^{\infty} ((M^r f)(x) r^{n-2} dr \qquad (4.24)$$

Where Ω_{n-1} is the area of the unit sphere in R^{n-1}.

Applying L to (4.24) using (4.23) and lemma (4.1.8) we obtain

$$L((\hat{f}))^{\vee} = \Omega_{n-1} \int_0^{\infty} \left(\frac{d^2F}{dr^2} + \frac{n-1}{r}\frac{dF}{dr}\right) r^{n-2} dr \qquad (4.25)$$

Where $F(r) = (M^r f)(x)$ Integrating by parts and using

$$F(0) = F(x) \ , \ \lim_{r\to\infty} r^k F(r) = 0 \ ,$$

We get

$$L((\hat{f}))^{\vee} = \begin{cases} -\Omega\, n-1\, f(x) & n = 3 \\ -\Omega\, n-1\,(n-3) \int_0^{\infty} F(r) r^{n-4}\, dr & (k < 1) \end{cases} .$$

More generally

$$L_x\left(\int_0^{\infty} (M\, r\, f)(x) r^k\right) = \begin{cases} -(n-2)f(x) \text{ if } k = 1 \\ -(n-1-k)(k-1) \int_0^{\infty} F(r) r^{k-2}\, dr, (k > 1) \end{cases}$$

If n is odd the formula in theorem (4.1.8) follows by iteration. Although we assumed $f \in s(R^n)$ the proof is valid under much weaker assumptions .

4.1.10Remark: the condition $f(x) = O(|x|^{-N})$ for some $N > n$ cannot in general be dropped . In [1982] zaleman has give an example of a smooth function f on R^n satisfying $f(x) = O(|x|^{-2})$ on all lines with $\hat{f}(\xi) = 0$ for all lines ξ and yet $f \neq 0$. The function is even $f(x) = O(|x|^{-3})$ on each line which is not the x-axis.

4.1.11 Remark: It is interesting to observe that while the inversion formula requires $f(x) = O(|x|^{-N})$ for one $N > n$ the support theorem requires

$F(x) = O(|x|^{-N})$ for all N as mentioned .

We shall now prove a similar inversion formula for the dual transform $\varphi \to \check{\varphi}$ on the subspace $S^*(p^n)$.

4.1.12 Application of the Radon Transform In partial Differential Equation:

The inversion formula is very well suited for applications to partial differential equation. To explain the underlying principle we write the inversion formula in the form.

$$F(x) = \gamma L_x^{\frac{n-1}{2}} \left(\int_{S^{n-1}} \hat{f}(w, \langle x, w\rangle) dw\right). \qquad (4.26)$$

Where the constant $\gamma = \frac{1}{2}(2\pi i)^{1-n}$. Note that the function$f_w(x) = \hat{f}(w, \langle x, w\rangle)$ is a plane wave with normal w, that is, it is constant on each hyper plane perpendicular to w.

Consider now a differential operator

$$D = \sum_{(k)} ak_1 \ldots k_n \, \partial_1^{k_1} \ldots \partial_n^{k_n}$$

With constant coefficients $ak_1 \ldots k_n$ and suppose we want to solve the differential equation

$$D_u = f \qquad (4.27)$$

Where f is a given function in $s(R^n)$.

To simplify the use of (4.26) we assume n to be odd. We begin by considering the differential equation

$$D_v = f_w \qquad (4.28)$$

Where f_w is the plane wave defined above and we look for a solution v which is also a plane wave with normal w.

But a plane wave with normal w is just a function of one variable, also if v is a plane wave with normal w so is the functionD_v. The differential equation (4.28) (with v a plane wave) is there for an ordinary differential equation with constant coefficients suppose v = u_w is a solution and assume that this choice can be made smoothly in w. Then the function

$$U = \gamma L^{\frac{n-1}{2}} \int_{S^{n-1}} u_w \, dw \qquad (4.29)$$

Is a solution to the differential equation (4.27) In fact, since D and $L^{\frac{n-1}{2}}$ commute we have

$$D_u = \gamma L^{\frac{n-1}{2}} \int_{s^{n-1}} Du_w \, dw = \gamma L^{\frac{n-1}{2}} \int_{s^{n-1}} f_w \, dw = f.$$

This method only assumes that the plane wave solution u_w to the ordinary differential equation $D_v = f_w$ exist and can be chosen so as to depend smoothly on w . This cannot always be done because D might annihilate all plane waves with normal w. (For example, take D = $\partial^2 / \partial x_1 \, \partial x_2$ and w = (1, 0)).

However , if his restriction to plane waves is never 0 it follows from a theorem of Treves [1963] that the solution u_w can be chosen depending smoothly on w. Thus we can state.

4.1.13Theorem: Assuming the restriction D_w of D to the space of plane waves with normal w is $\neq 0$ for each w formula (4.29) gives a solution to the differential equation $D_u = f$ ($f \in s(R^n)$) .

The method of plane waves can also be used to solve the Cauchy problem for hyperbolic differential equations with constant coefficients.

We illustrations this method by means of the wave equation R^n.

$$L_u = \frac{\partial^2 u}{\partial t^2} \quad , \quad u(x,0) = f_0(x) \quad , \quad u_t(x,0) = f_1(x) \qquad (4.30)$$

f_0 , f_1 being given function in $D(R^n)$.

4.1.14 Lemma: Let $h \in c^2(R)$ and $w \in s^{n-1}$, then the function $V(x, t) = h(\langle x, w \rangle + t)$ satisfies

$$L_{sv} = \left(\frac{\partial^2}{\partial t^2}\right) v.$$

4.1.15Theorem: The solution to (4.30) is given by

$$u(x,t) = \int_{s^{n-1}} (sf)(w, \langle x, w \rangle + t) \, dw \qquad (4.31)$$

$$\text{Where } s f \begin{cases} c\left(\partial^{n-1}\hat{f}_0 + \partial^{n-2}\hat{f}_1\right), & n \text{ odd} \\ cH\left(\partial^{n-1}\hat{f}_0 + \partial^{n-2}\hat{f}_1\right), & n \text{ even} \end{cases}$$

Here $\partial = \partial / \partial p$ and the constant c equals

$$c = \frac{1}{2} (2\pi i)^{1-n} \quad .$$

4.1.16: Lemma: shows that (4.31) is annihilated by the operator L - $\frac{\partial^2}{\partial t^2}$ so we just have to check the initial conditions in (4.30).

(a)if n < 1 is odd then w → $(\partial^{n-1}\hat{f}_0)$(w, $\langle x, w \rangle$) is an even function on s^{n-1} but the other term in Sf , that is the function w → $(\partial^{n-2}\hat{f}_1)$ (w , $\langle x, w \rangle$) , is odd. Thus u(x,0) = $f_0(x)$. Applying $\partial/_{\partial t}$ to (4.31) and putting t = 0 gives u_t(x,0) = f_1(x) this time because the function W→ $(\partial^{n}\hat{f}_0)$ (w,$\langle x, w \rangle$) is odd and the function W→ $(\partial^{n-1}\hat{f}_1)$(w, $\langle x, w \rangle$) is even.

(b) If n is even the same proof works if we take into account the fact that Ƕ interchanges odd and even function on R .

4.1.17 Definition: For the pair f = { f_0, f_1 } we refer to the function Sf in (4.31) as the source .

In the terminology of Lax – Philips [1967] the wave u(x,t) is said to be

(a) Outgoing if u(x ,t) = 0 in the forward cone $|x| < t$,

(b) Incoming if u(x ,t) = in the backward cone $|x| < -t$.

The notation is suggestive because "outgoing" means that the function x → $u(x, t)$ vanishes in larger balls around the origin as t increases.

4.1.18Corollary: The solution u(x,t) to (4.30) is:

(i) Outgoing if and only if (S f)(w ,s) = 0 for s > 0 , all w .

(ii) Incoming if and only if (S f)(w ,s) = 0 for s < o all w .

Theorem 1.1.16 Given f_0 , $f_1 \in \varepsilon(R^n)$ the function

$$U(x, t) = (f_0 * \grave{T}_t)(x) + (f_1 * T_t)(x) \qquad (4.32)$$

Satisfies (4.30). Here $\grave{T}_t$stands for $\partial_t(T_t)$.

Not that (4.30) implies (4.32) if $f_0$and $f_1$have compact support. The converse holds without this support condition.

4.1.19 Corollary: If $f_0$and $f_1$have support in B_R (0) then u has support in the region $|x| \leq |t| + R$.

In fact by (3.32) and support property of convolution, the function x $\rightarrow$u(x, t) has support in $B_{R+|t|}(0)$. While corollary (4.1.15) implies that forf_0, $f_1 \in$ D(R^n)u has support in a suitable solid cone we shall how see that theorem (4.1.14) implies that if n is odd u has support in a conical shall.

4.2. John Transform:

The inversion formula for Radon transform (4.2) can exist because both $R^2$and the space of oriented lines in $R^2$are two dimensional. Thus, at least naively, one function of two variables can be constructed from anther such function (albeit defined on a different space). Thus symmetry does not hold in higher important result of John let $f:R^3 \rightarrow R$ be a function (again, subject to some decay conditions which makes the integrals well defined) and let $L \subset R^3$ be an oriented line.

Define $\emptyset(L)= \int_L f$, or

$$\emptyset(\alpha_1 , \alpha_2 , \beta_1 , \beta_2) = \int_{-\infty}^{\infty} f(\alpha_1 s + \beta_1, \alpha_2 s + \beta_2 , s) ds$$

Where (α , β) parametrise the four-dimensional space T of oriented lines inR^3(Not that this parameterization misses out the lines paralled to the plane x_3 = const. The whole construction can be done invariantly without choosing any parameterization, but here we choose the explicit approach for clarity).

The space of oriented lines is four dimensional, and 4 > 3 so expect one condition on $\emptyset$.

Differentiating under the integral sing yields the ultra-hyperbolic wave equation

$$\frac{\partial^2 \emptyset}{\partial \alpha_1 \partial \beta_2} - \frac{\partial^2 \emptyset}{\partial \alpha_2 \partial \beta_1} = 0,$$

And John has shown that all smooth solutions to this equation arise from some function onR^3. This is a feature of twistor space (which in this case is identified withR^3) yields a solution to a differential equation an space-time (in this case locally R^4 with a metric of signature).

After the change of coordinates

$$\alpha_1 = x + y \quad , \qquad \alpha_2 = t + z$$

$$\beta_1 = t - z \quad , \qquad \beta_2 = x - y$$

The equation becomes

$$\frac{\partial^2 \emptyset}{\partial x^2} + \frac{\partial^2 \emptyset}{\partial z^2} - \frac{\partial^2 \emptyset}{\partial y^2} - \frac{\partial^2 \emptyset}{\partial t^2} = 0$$

Which may be relevant to physics with two times, the integral formula given in the next section corrects the wrong signature to that of the Minkowski space and is a starting point of Twistor theory.

4.3 Introduction to Twistor Space

Had established a correspondence twice complexfied Minkowski space.

Let as first start with Minkowski space time. This is the space of all events given by {x, y, z, t}. This space is endowed with a metric stmcruve

$$dS^2 = dx^2 + dy^2 + dz^2 - c^2dt^2$$

In fact this wither medical model descried the physical phenomenon of special theory of relativity with its two concepts:

1. The speed of light is constant C.
2. The laws of physics are covariant relative to the inertial frames of reference.

Now Penrose had developed the model of special theory of relativity to encode quantum mechanics. The new resulting space is called Twistor space.

Penrose twistor space. This correspondence is demonstrated in the following section.

4.4 Twistor Programme:

Penrose gave formula for solution to wave equation in Minkowski space

$$\emptyset(x, y, z, t) = \int_{\Gamma \subset cp^1} f\big((z+t) + (x+iy)\lambda, (x-iy) - (z-t)\lambda, \lambda\big) d\lambda \quad (4.33)$$

Here $\Gamma \subset \mathbb{C}\,\mathbb{p}^1$ is aclosed contour and the function f is holomorphic on $\mathbb{C}\,\mathbb{p}^1$ except some number of poles.

According to the Twistor philosophy this appearance of complex number should be understood at a fundamental, rather than technical, level.

In quantum physics the complex numbers are regarded as fundamental: The complex wave function is an element of a complex Hilbert space.

In twistor theory Penrose aimed to bring the classical physics at the equal footing, where the complex number play a role from the start. This already takes place in special relativity, where the complex numbers appear on the celestial sphere visible to an observer on a night sky.

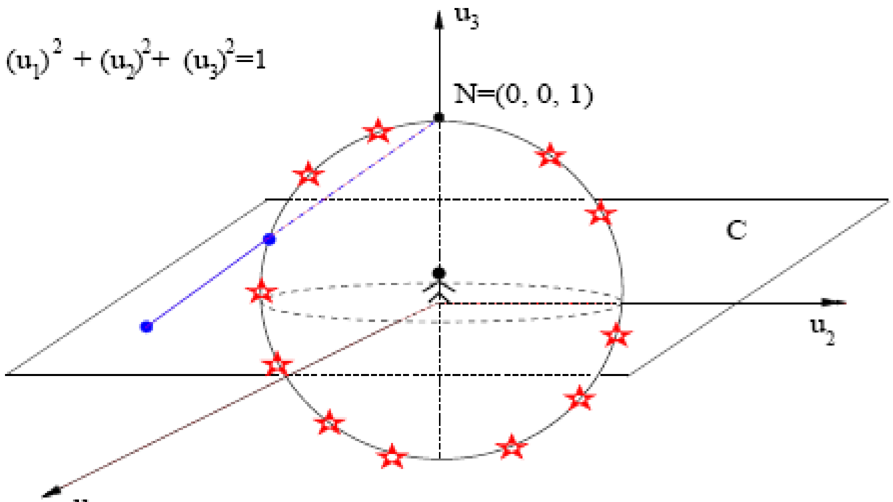


Figure (4.3)

The two-dimensional sphere is the simplest example of a non-trivial complex manifold. Stereographic projection from the North Pole (0, 0, 1) gives a complex coordinate:

$$\lambda = \frac{u_1 + iu_2}{1 - u_3} \ .$$

Projecting form the South Pole (0, 0, -1) gives another coordinate

$$\tilde{\lambda} = \frac{u_1 - iu_2}{1 + u_3} \, .$$

On the overlap $\tilde{\lambda} = {}^1/_{\lambda}$. Thus the transition function is holomorphic and this makes s^2 into a complex manifold CP^1. The double covering $SL(2, C) \rightarrow SO(3,1)$ can be understood in this context . If worldliness of two observers travelling with relative constant velocity intersects at a point in space-time, the celestial spheres these observers see are related by a Möbius transformation

$$\lambda = \frac{\alpha\lambda + \beta}{\gamma\lambda + \delta} \, .$$

where the unit-determinant matrix $\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \in SL(2,C)$. Corresponds to the Lorentz transformation relating the two observers.

The celestial sphere is a past light cone of an observer **O** which consist of light rays through an event **O** at a given moment. In the twistor approach the light rays are regarded as more fundamental than events in space-time. The five dimensional space of light rays $\mathcal{PN}$ in the Minkowski space is a hyper-surface in a three dimensional complex manifold PT = $CP^3 - CP^1$ called the projective twistor space.

$$\text{Let } (Z^0, Z^1, Z^2, Z^3) \sim (cZ^0, cZ^1, cZ^2, cZ^3)\ c \in C^*$$

With($Z^2.Z^3$) $\neq$(0 ,0) be homogeneous coordinates of a twistor (a point in P T). The twistor space and the Minkowski space are linked by the incidence relation:

$$\begin{pmatrix} Z^0 \\ Z^1 \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix} \begin{pmatrix} Z^2 \\ Z^3 \end{pmatrix} \tag{4.34}$$

Where $x^\mu = (t,x,y,z)$ are coordinates of a point in Minkowski space.

4.4.1 Definition: The Hermitian inner product

$$\sum(Z, \bar{Z}) = Z^0\overline{Z^2} + Z^1\overline{Z^3} + Z^2\overline{Z^0} + Z^3\overline{Z^1}$$

On the non-projective twistor space $\mathcal{T} = \mathbb{C}^4 - \mathbb{C}^2$. The signature of $\sum$ is (+ + - -) so that the orientation-preserving endomorphisms of $\mathcal{T}$ preserving $\sum$ form a group SU(2, 2). This group has fifteen parameters and is locally isomorphic to the conformal group SO(4, 2) of the Minkowski space.

We divide the twistor space into three parts depending on whether $\sum$ is positive,negative or zero. This partition descends to the projective twistor space. In particular the hyper surface

$$\mathcal{PN} = \{[Z] \in \mathcal{PT}, \sum(Z, \bar{Z}) = 0\} \subset \mathcal{PT}$$

Is preserved by the conformal transformation of the Minkowski space which can be verified directly using (2.2) fixing the coordinates x^μ of a space-time point in (2.2) given a plane in the non-projective twistor space $\mathbb{C}^4 - \mathbb{C}^2$ or a projective line $\mathbb{C}\,\mathbb{P}^1$ in. If the coordinates x^μ are real this lies in the hyper surface. Conversely, fixing a twistor in $\mathcal{PN}$ gives a light-ray in the Minkowski space.

So far only the null twistor (point in) has been relevant in this discussion. General points in $\mathcal{PT}$ can be interpreted in terms of the complexfied Minkowski space$\mathbb{C}^4$where they correspond to null two-dimensional planes with self-dual tangent bi-vector. This again is a direct consequence of (2.2) where now the coordinates$x^\mu$are complex. There is also an interpretation of non-null twistor in the real Minkowski space, but this is far less obvious. The Hermitian inner product $\sum$ defines a vector space $\mathcal{T}^*$dual to the non-projective twistor space. The element of the corresponding projective space $\mathcal{P}\,\mathcal{T}^*$are called dual twistor. Now take a non-null twistor $Z \in \mathcal{PT}$. Its dual $\bar{Z} \in \mathcal{P}\,\mathcal{T}^*$ corresponds to a projective two plane $\mathbb{C}\,\mathbb{P}^2$in $\mathcal{PT}$. A holomorphic two-plane intersects the hyper-surface $\mathcal{PN}$ in a real three-dimensional locus. This locus corresponds to a three-parameter family of light-rays in the real Minkowski space. This family representing a single twister is called the Robinson congruence. A picture of this configuration which appears on the front cover of shows a system of twisted oriented circles in the Euclidean space$\mathbb{R}^3$, the point being that any light-ray is represented by a point in $\mathbb{R}^3$together with an arrow indicating the direction of the ray's motion. This configuration originally gave rise to a name twistor.

Finally we can give a twistor interpretation of the contour integral formula (2.1) Consider a function f = f (Z^0/Z^2, Z^1/Z^2, Z^3/Z^2) which is holomorphic on intersection of two open sets covering $\mathcal{PT}$ (one of this sets is defined by $Z^2 \neq 0$ and the other by $Z^3 \neq 0$) and restrict this function to a rational curve (2.2) in $\mathcal{PN}$. Now integrate f along a contour in this curve. This gives (2.1) with $\lambda = Z^3/Z^2$.

4.5. Non-Abelian Monopoles and Euclidean Mini-Twisters

It is well known that the problem of finding harmonic function in $\mathbb{R}^2$can be solved in one line by introducing complex numbers: Any solution of a two-dimensional Laplace equation $\emptyset_{xx} + \emptyset_{yy} = 0$is a real part of a function holomorphic in x + iy. This technique fails when applied to the Laplace equation in three dimensions as $\mathbb{R}^3$can not be identified with $\mathbb{C}^n$ for any n.

Following Hitchin we shall associate a two-dimensional complex manifold to the three-dimensional Euclidean space.

Definition 2.3.1: The Twistor space $\mathbb{T}$ to be space of oriented lines in $\mathbb{R}^3$.

Any oriented line is of the form **v** +s**u**.s∈ℝ where **u** is a unit vector giving the direction of the line and **v** is orthogonal to **u** and joints the line with some chosen point (say the origin) in $\mathbb{R}^3$.

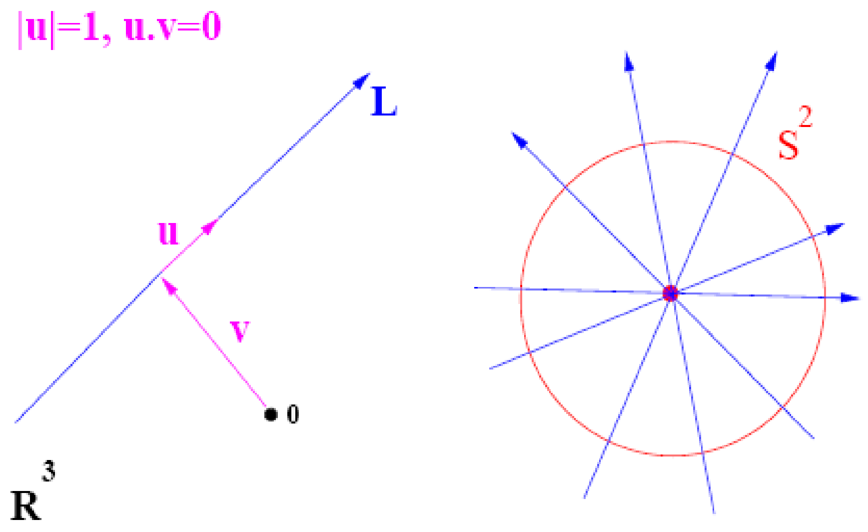


Figure (4.4)

Thus$\mathbb{T}$ = {(**u**,**v**) ∈$S^2 \times \mathbb{R}^3$, **u. v = 0**} and the dimension of $\mathbb{T}$ is four. For each fixed **u** ∈S^2this space restricts to a tangent plane to S^2. The twistor space is the union of all tangent planes – the tangent bundle T S^2.

This is a topologically nontrivial manifold: Locally it is diffeomorphic to $S^2 \times \mathbb{R}^3$but globally it is twisted in a way analogous to Möbius strip.

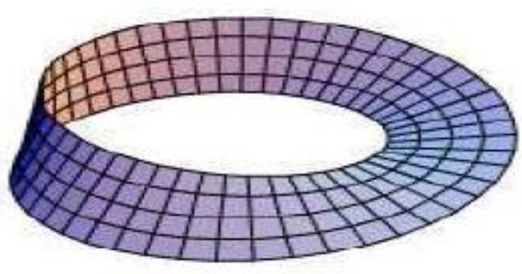


Figure (4.5)

Reversing the orientation of lines induces a map: $\mathbb{T}\to\mathbb{T}$ given by

$$\tau\,(u, v) = (-u, v).$$

The point **p** = (x, y, z) in $\mathbb{R}^3$correspond to – spheres in $\mathbb{T}$. Given by τ – invariant maps

$$u \to (u, v(u) = p - (p\,.\,u) \in \mathbb{T} \qquad (4.\,35)$$

which are section of the projection $\mathbb{T} \to S^2$

4.5.1 Twistor Space as a Complex Manifold:

Introduce the local holomorphic coordinates on an open set U C $\mathbb{T}$where**u**≠(0, 0, 1) by

$$\lambda = \frac{u_1 + iu_2}{1 - u_3} \in \mathbb{CP}^1 = S^2,\ \eta = \frac{v_1 + iv_2}{1 - u_3} + \frac{u_1 + iu_2}{(1 - u_3)^2} v_3,$$

and analogous complex coordinates ($\tilde{\lambda}$, $\tilde{\eta}$) in an open set $\tilde{U}$ containing u= (0, 0, 1). On the overlap

$$\tilde{\lambda} = 1/\lambda, \qquad \tilde{\eta} = -\eta/\lambda^2.$$

This endows $\mathbb{T}$ with a structure of complex manifold $T\mathbb{CP}^1$. It is a holomorphic tangent bundle to the Riemann sphere.

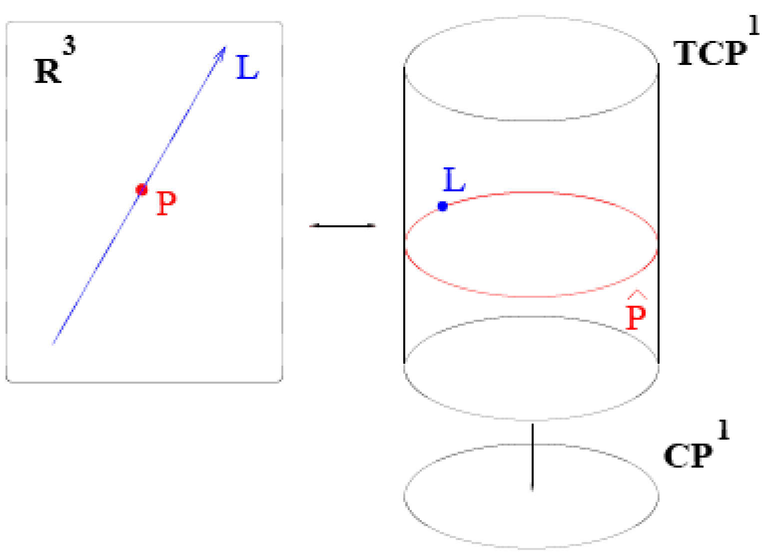


Figure (4.6)

In the holomorphic coordinates the line orientation reversing involution $\mathcal{T}$ is given by

$$(\lambda, \eta) = \left(-\frac{1}{\lambda}, -\frac{\bar{\eta}}{\bar{\lambda}^2}\right). \tag{4.36}$$

This is an antipodal map lifted from a two-sphere to the total space of the tangent bundle the formula (2. 3) implies that the point in$\mathbb{R}^3$are τ –invariant holomorphic maps $\mathbb{CP}^1 \to T\mathbb{CP}^1$ given by

$$\lambda \to (\lambda, \eta = (x + iy) + 2\lambda z - \lambda^2 (x - iy) \tag{4.37}$$

4.5.2 Harmonic Function and Abelian Monopoles:

Finally we can return to our original problem. To find a harmonic function at

p = (x, y, z),

Write: .1

$$\emptyset(x, y, z) = \oint f\big(\lambda, (x + iy) + 2\lambda z - \lambda^2 (x - iy)\big) d\lambda, \tag{4.38}$$

3. Differentiate under the integral to verify

$$\frac{\partial^2 \emptyset}{\partial x^2} + \frac{\partial^2 \emptyset}{\partial y^2} + \frac{\partial^2 \emptyset}{\partial z^2} = 0.$$

This formula was already known to Whittaker in 1903, albeit Whittaker's formulation does not make any use complex numbers and his formula is given in terms of a real integral.

Small modification of this formula can be used to solve a 1st order linear equation for a function $\emptyset$and a magnetic potential **A** = (A_1, A_2, A_3) of the form $\nabla\emptyset = \nabla \wedge \mathbf{A}$.

This is the abelian monopole equation. Geometrically, the one – form $A = A_j dx^j$ is a connection on a U(1) principal bundle over $\mathbb{R}^3$, and $\emptyset$is a section of the adjoint bundle. Taking the curl of both sides of this equation implies that $\emptyset$is harmonic, and conversely given a harmonic function $\emptyset$locally one can always find a one-form A (defined up to addition of a gradient of some function) such that the abelian monopole equation holds.

4.5.3 Non – Abelian Monopoles and Hitchin Correspondence:

Replacing U(1) by a non -abelian Lie group generalizes this picture to some equation on $\mathbb{R}^3$in the following way: Let $(A_i, \emptyset)$ be anti – hermitian traceless n by n matrices on $\mathbb{R}^3$. Define the non – abelian magnetic field

$$F_{jk} = \frac{\partial A_k}{\partial x^j} + \frac{\partial A_j}{\partial x^k} + [A_j, A_k], \quad j, k = 1, 2, 3.$$

The non – abelian monopole equation is a system of non – linear PDEs

$$\frac{\partial \emptyset}{\partial x^j} + [A_j, \emptyset] = \frac{1}{2} \varepsilon_{jkl} F_{kl}. \tag{4.39}$$

These are three equation for three unknowns as $(A, \emptyset)$ are defined up to gauge transformations

$$A \to gAg^{-1} - dg\, g^{-1}, \emptyset \to g\emptyset g^{-1}, \; g = g(x, y, t) \in SU(n) \tag{4.44}$$

And one component of A (say A_1)can always be set to zero. The twistor solution to the monopole equation consists of the following steps:

- Given (A_j (x), $\emptyset(x)$) solve a matrix ODE along each oriented linex(s) = **v** + s **u**

$$\frac{dV}{ds} + (u^j A_j + i\emptyset)V = 0.$$

Space of solution at $p \in \mathbb{R}^3$is a complex vector space $\mathbb{C}^n$.

- This assigns a complex vector space $\mathbb{C}^n$ to each point of $\mathbb{T}$, thus giving rise to a complex vector bundle over $\mathbb{T}$ with patching matrix

$$F(\lambda, \bar{\lambda}, \eta, \bar{\eta}) \in GL(n, \mathbb{C}).$$

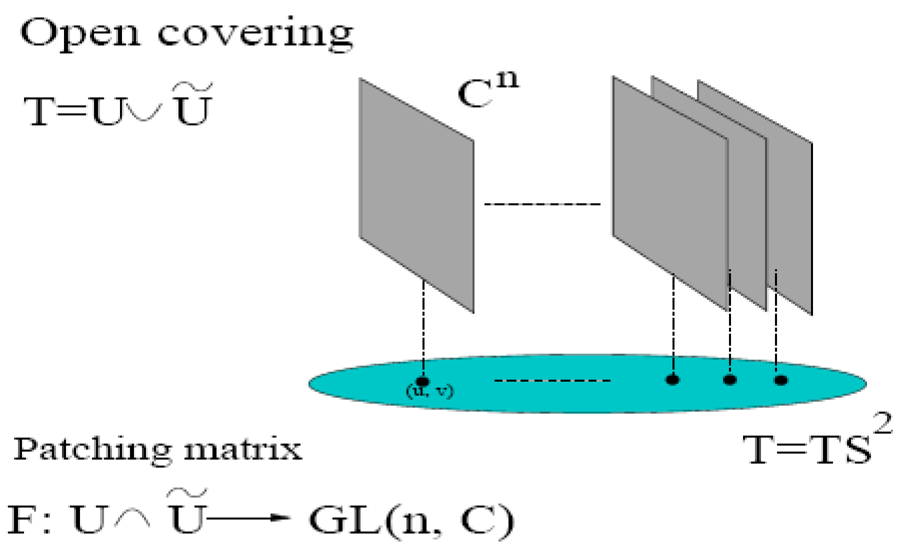


Figure (4.7)

- The monopole equation (4. 7) on $\mathbb{R}^3$holds if and only if this vector bundles is holomorphic, i.e. the Cauchy – Riemann equation

$$\frac{\partial F}{\partial \lambda} = 0, \quad \frac{\partial F}{\partial \eta} = 0 \qquad \text{holds.}$$

- Holomorphic vector bundles over $T\mathbb{CP}^1$ are well understood. Take one and work back – wards to construct a monopole. We shall work through the details of this reconstruction (albeit in complexfied settings) in the proof of Theorem 4. 3. 1.

4.6. The Ward model and Lorenztian Mini – Twistor

In this Section we shall demonstrate how mini – twistor theory can be used to solve non – linear equations in 2 + 1 dimensions. Let $A = A_\mu dx^\mu$and $\emptyset$ be a one – form and a function respectively on the Minkowski space $\mathbb{R}^{2,1}$ with values in a Lie algebra of the general linear group. They are defined up to gauge transformation (4. 8) where g takes values in GL(n, $\mathbb{R}$).

Let $D_\mu = \partial_\mu + A_\mu$ be a covariant derivative, and define $D\emptyset = d\emptyset + [A, \emptyset]$. The Ward model is a system of PDEs (4.7) where now the indices are raised using the metric on $\mathbb{R}^{2,1}$. If the metric and the volume form are chosen to be

$$h = dx^2 - 4dudv, \quad vol = du \wedge dx \wedge dv$$

where the coordinates (x, u, v) are real the equation become

$$D_x \emptyset = \frac{1}{2} F_{uv}, \; D_u \emptyset = F_{ux}, \; D_v \emptyset = F_{xv} \qquad (4.40)$$

Where $F_{\mu v} = [D_\mu, D_v]$. These equation arise as the integrability conditions foran over determined system of linear Lax equations $L_0 \Psi = 0, \; L_1 \Psi = 0$,

Where

$$L_0 = D_u - \lambda(D_x + \emptyset), \; L_1 = D_x - \emptyset - \lambda D_v, \qquad (4.41)$$

And$\Psi = \Psi(x, u, v, \lambda)$ takes values in GL(n,$\mathbb{C}$).We shall follow and solve the system by establishing one-to-one correspondence between its solution and certain holomorphic vector bundles over the twistor space $\mathbb{T}$.

This construction is of interest in solution theory as many known integrable models arise as symmetry reduction and/or choosing a gauge in (4.9). To this end we note a few example of such reduction.

For a much more complex list:

– Choose the unitary gauge group G = U(n). The integrability conditions for (4.10) imply the existence of a gauge $A_v = 0$, and $A_x = -\emptyset$, and a matrixj:$\mathbb{R}^{2,1} \rightarrow U(n)$ such that

$$A_u = j^{-1}\partial_u j. \qquad A_x = -\emptyset = \frac{1}{2} j^{-1}\partial_x j.$$

With this gauge choice the equations (4.9) become the integrable chiral model

$$\partial_v(j^{-1}\partial_u j) - \partial_x(j^{-1}\partial_x j) = 0.$$

This formulation breaks the Lorenz invariance of (4.9) but it allows an introduction of a positive definite energy functional. Where more details can be found.

– Solution to equation (4.9) with the gauge group SL(2,ℝ) which are invariant under a null translation given by a Killing vector K ⌟ A is nilpotent are characterized by the KdV equation. The direct calculation shows that the Ward equations with the gauge group SL(3,ℝ) are solve by the ansatz

$$\emptyset = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -e^{\Psi} & 0 & 0 \end{pmatrix} \qquad (4.42)$$

$$A = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ e^{\Psi} & 0 & 0 \end{pmatrix} dx + \begin{pmatrix} \Psi_u & 0 & 0 \\ 1 & -\Psi_u & 0 \\ 0 & 1 & 0 \end{pmatrix} du$$

$$+ \begin{pmatrix} 0 & e^{-2\Psi} & 0 \\ 0 & 0 & e^{\Psi} \\ 0 & 0 & 0 \end{pmatrix} dv$$

if $\Psi(u, v)$ satisfies the Tzitzeica equation

$$\frac{\partial^2\Psi}{\partial u\,\partial v} = e^{\Psi} - e^{-2\Psi}. \qquad (4.43)$$

This reduction can also be characterized in a gauge invariant manner using the Jordan normal forms for the Higgs fields.

4.6.1 Null Planes and Ward Correspondence:

The geometric interpretation of the lax representation (4.10) is the following. For any fixed pair of real number (η, λ) the plane

$$\eta = v + x\lambda + u\lambda^2 \qquad (4.44)$$

is null with respect to the Minkowski metric on $\mathbb{R}^{2,1}$,

$$\delta_0 = \partial_u - \lambda\,\partial_x, \qquad \delta_1 = \partial_x - \lambda\,\partial_v \qquad (4.45)$$

span this null plane. Thus the Lax equation (4.10) implies that the generalized connection $(A,\emptyset)$ is flat on null planes. This underlies the twistor approach, where one works in a complexfied Minkowski space $M = \mathbb{C}^3$, and interprets (η, λ) as coordinates in a patch of the twistor space $\mathbb{T} = T\mathbb{CP}^1$, with $\eta \in \mathbb{C}$ being a coordinate on the fibers and $\lambda \in \mathbb{CP}^1$ being an affine coordinate on the base. We shall adopt this complexfied point of view from now on. It is convenient to make use of the spinor formalism based on the isomorphism TM = S ʘ S where S is rank two complex vector bundle (spin bundle) over M and ʘ is the symmetries tensor product. The fiber coordinates of this bundle are denoted by

(π^0,π^1) and the sections M →S are called spinors. We shall regard S as a symplectic bundle with anti – symmetric product

$$\kappa \,.\, \rho = {}^0\rho^1 - \kappa^1\rho^0 = \varepsilon(\kappa, \rho)$$

On its sections. The constant symplectic form ε is represented by a matrix

$$\varepsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

This gives an isomorphism between S and its dual bundle, and thus can be used to rise and lower the indices according to $\kappa_A = \kappa^B\varepsilon_{AB}$, $\kappa_A = \varepsilon_{AB}\kappa_B$, where $\varepsilon AB\varepsilon^{CB}$ is an identity endomorphism.

Rearrange the space time coordinates (u, x, v) of a displacement vector as a symmetric two-spinor

$$X_{AB} = \begin{pmatrix} u & x/2 \\ x/2 & v \end{pmatrix},$$

such that the space – time metric is

$$h = -2dx_{AB}dx^{AB}.$$

The twistor space of M is the two-dimensional complex manifold $\mathbb{T} = T\mathbb{CP}^1$. Point of $\mathbb{T}$ correspond to null 2-planes in M via the incidence relation

$$X_{AB}\pi_A\pi_B = \omega \,. \quad (4.46)$$

Here (ω, π_0, π_1) are homogeneous coordinates on $\mathbb{T}$ as $(\omega, \pi_A) \sim (c^2\omega, c\pi_A)$, where $c \in \mathbb{C}^*$. In the affine coordinates $\lambda := \pi_0/\pi_1, \eta := \omega/(\pi_1)^2$ equation (4.16) gives (4.14).

The projective spin space p(S)is the complex projective line $\mathbb{CP}^1$.The homogeneous coordinates are denoted by$\pi_A = (\pi_0, \pi_1)$, and the two set covering of $\mathbb{CP}^1$ lifts to a covering of the twistor space $\mathbb{T}$

$$U = \{(\omega, \pi_A), \pi_1 \neq 0\}. \quad \tilde{U} = \{(\omega, \pi_A), \pi_0 \neq 0\}. \quad (4.47)$$

The function $\lambda = \pi_0/\pi_1$, $\tilde{\lambda} = 1/\lambda$ are the inhomogeneous coordinates in U and $\tilde{U}$ respectively. It then follows that $\lambda = -\pi^1/\pi^0$.

Fixing (ω, π_A) gives a null plane in M. An alternative interpretation of (4.16) is to fix x^{AB}. This determines ω as a function of π_A i.e. a section of $\mathbb{T} \to \mathbb{CP}^1$ when factored out by the relation $(\omega, \pi_A) \sim (c^2\omega, c\pi_A)$.

These are embedded rational curves with self-intersection number 2.2; as infinitesimally perturbed curve $\eta + \delta$ with $\delta\eta = \delta v + \lambda\delta x + \lambda^2\delta u$ generically intersects (4.14) at two point. Two curve intersect at one point if the corresponding points in M are null separated. This defines a conformal structure on M.

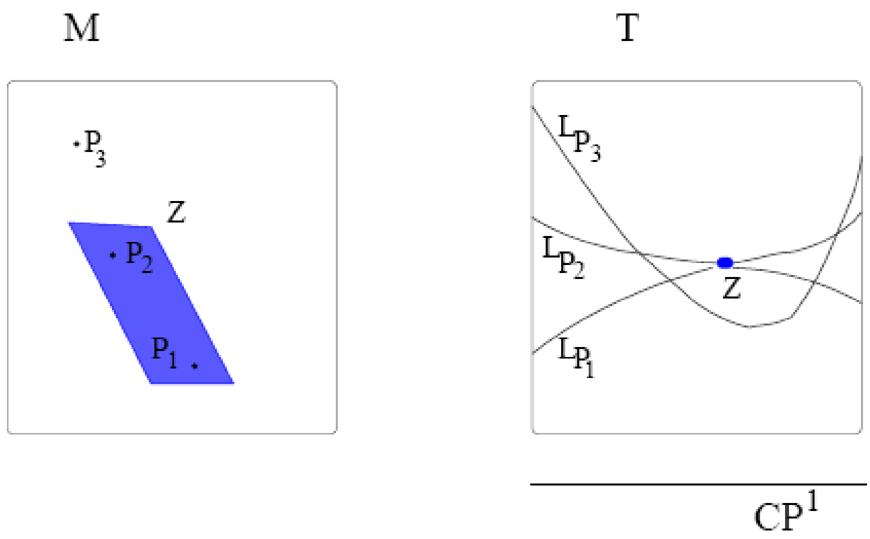


Figure (4.8)

The space of holomorphic sections of $\mathbb{T}\to\ \mathbb{CP}^1$is $M = \mathbb{C}^3$. The real space-time $\mathbb{R}^{2+1}$ arises as the module space of those sections that same invariant under the conjugation

$$(\omega\ ,\pi_A) = (\ \tilde{\omega},\tilde{\pi}_A), \qquad (4.48)$$

Which corresponds to real x^{AB}. The points in $\mathbb{T}$ fixed by τ correspond to real null planes in $\mathbb{R}^{2+1}$.

The following result makes the mini-twistors worthwhile:

4.6.2Theorem:

There is a one-to-one correspondence between:

1.The gauge equivalence classes of complex solutions to (4.9) in complexfied Minkowski space M with the gauge $GL(n, \mathbb{C})$.

2.Holomorphic rank n vector bundles E over the twistor space $\mathbb{T}$ which are trivial on the holomorphic sections of $T\mathbb{CP}^1\to \mathbb{CP}1$**.**

Proof:

Let $(A,\emptyset)$ be a solution to (4.9). Therefore we can integrate a pair of linear PDEs $L_0V = L_1V = 0$, where L_0, L_1 are given by (4.10). This assigns an n-dimensional vector space to each null plane Z in complexfied Minkowski space, and so to each point $Z \in\mathbb{T}$. It is a fiber of a holomorphic vector bundle $\mu : E\to\mathbb{T}$. The bundle E is trivial on each section, since we can identify fibers of $E|_{L_p}$ atZ_1, Z_2 because covariantly constant vector fields at null planes Z_1, Z_2 coincide at a common point$p\in$ M.

Conversely, assume that we are given a holomorphic vector bundle E over $\mathbb{T}$ which is trivial on each section. Since $E|_{L_p}$is trivial, and $L_p \cong \mathbb{CP}1$.The Birkhoff – Grothendieck theorem (Appendix) gives

$$E|_{L_p} = O + O + \ldots + O$$

And the space of sections of E restricted to L_p is $\mathbb{C}^n$. This gives us a holomorphic rank n vector bundle $\hat{E}$over the complexified three-dimensional Minkowski space. We shall give a concrete method of constructing a pair (A,∅) on this bundle which satisfies (4.9).

Let us cover the twistor space with two open sets U and $\tilde{U}$ as in (4.17). Let

$$x : \mu^{-1}(U) \to U \times \mathbb{C}^n, \ \tilde{x} : \mu^{-1}(\tilde{U}) \to \tilde{U} \times \mathbb{C}^n$$

be local trivializations of E, and let $F = \tilde{x} \circ x^{-1} : \mathbb{C}^n \to \mathbb{C}^n$ be a holomorphic patching matrix for a vector bundle E over $T\mathbb{CP}^1$ defined on $U \cap \tilde{U}$. Restrict F to a section (4.16) where the bundle is trivial, and therefore F can be split.

$$F = \tilde{H}H^{-1}, \tag{4.49}$$

where the matrices H and $\tilde{H}$are defined on $M \times \mathbb{CP}^1$and holomorphic in π^A around

$\pi^A = o^A = (1, 0)$ and $\pi^A = l^A = (0, 1)$ respectively. As a consequence of $\delta_A F = 0$ the splitting matrices satisfy

$$H^{-1}\delta_A H = \tilde{H}^{-1}\delta_A \tilde{H} = \pi^B \Phi_{AB}, \tag{4.50}$$

For some $\Phi_{AB}(x^\mu)$ which does not depend on λ. This is because the RHS and LHS are homogeneous of degree one in $\pi^A$and holomorphic around λ = 0 andλ =∞ respectively. Decomposing

$$\Phi_{AB} = \Phi_{(AB)} + \varepsilon_{AB}\emptyset$$

Given a one-form $A = \Phi_{AB} dx^{AB}$and a scalar field $\emptyset = (1/2)\varepsilon^{AB}\Phi_{AB}$ on the complexified Minkowski space, i. e.

$$\Phi_{AB} = \begin{pmatrix} A_u & A_x + \emptyset \\ A_x - \emptyset & A_v \end{pmatrix}.$$

The Lax pair (2.10) becomes

$$L_A = \delta_A + H^{-1}\delta_A H$$

Where$\delta_A = \pi^B \partial_{AB}$, so that

$L_A(H^{-1}) = -H^{-1}(\delta_A H)H^{-1} + H^{-1}(\delta_A H)H^{-1} = 0$ and $\Psi = H^{-1}$

is a solution to the Lax equation regular around λ = 0. Let us show explicitly that (2.9) holds. Differentiating both sides to (2.20) yields

$$\delta^A(H^{-1}\delta_A H) = -(H^{-1}\delta^A H)(H^{-1}\delta_A H)$$

which holds for all π^Aif

$$D_{A(c}\Phi^A{}_{B)} = 0 \tag{4.51}$$

Where$D_{Ac} = \partial_{Ac} + \Phi_{Ac}$. This is the spinor form of the Yang – Mills – Higgs system (4.9).

Chapter Five

Penrose Transform

5.1. Twistor Geometry:

5.1.1 Introduction:

Twistors were introduced by R. 0. Wells in 1966 to describe the geometry of Minkowski space where the ordinary space-time concepts can be translated into Twistor terms. The primary geometrical object is not a point in Minkowski space but a null straight line (a Twistor) or, more generally, a twisting congruence of null lines. It turns out that Twistor algebra has the same type of universality in relation to the Lorentz group. Thus, Twistor theory is applicable to quantum field theory and free fields of zero-rest-mass. It is also used to formulate other fields such as Yang Mill's fields.

In section (5.2) of this chapter we describe a Twistor geometrically as a null line in Minkowski space. In section (5.5) we give a necessary and sufficient condition for two null lines to intersect in Twistor terms. This incidence will be very useful in algebraic manipulation of Twistor. Section (5.6) is devoted two alternative pictures of visualizing the Twistors, namely null lines in Minkowski space or points of complex projective three space, we also get other geometrical correspondences between the two pictures (Penrose correspondence).

5.2. Spinors Review

Since the discussion here depends essentially on the use of spinors, a very brief review of the ideas required and already introduced will be given:

(a) The translation from world tensors to spinors is achieved using a quantity:

$$\sigma_j^{jj^1} \text{(ahermitian (2×2) matrix for each j)}$$

and its inverse $\sigma_{jj^1}^{j}$ subject to

$$\sigma_j^{jj^1}\sigma_k^{kk^1} = \varepsilon_{jk}\varepsilon_{j^1k^1} = g_{jk}$$

$$\sigma_j^{jj^1}\sigma_{jj^1}^{j} = \delta_j^i$$

The ε_{jk} are skew-symmetric Levi-Civita symbols and are used for raising and lowering spinor indices:

$$\text{e.g.,}\ \xi^A\varepsilon_{AB} = \xi_B\ ,\ \ \varepsilon^{AB}\xi_B = \xi^A$$

(b) Any tensor (e.g., X_k^{ij}) has a spinor translation which is written using the same base symbol, but with each tensor index replaced by the corresponding pair of spinor indices, e.g.,

$$X_k^{ij} \leftrightarrow X_{kk^1}^{II^1jj^1} = X_k^{ij}\sigma_i^{II^1}\sigma_j^{jj^1}\sigma_{kk^1}^{k}$$

(c) Under complex conjugation, the roles of primed and unprimed indices are interchangeable, so that reality of tensors is expressed as Hermiticity of Spinors.

(d) Since our Minkowski space M will be flat, we can chose $\sigma_j^{jj^1}$ constant and = $2^{-1/2}$ times the unit matrix and Pauli matrices.

Then:

$$(x^1, x^2, x^3, x^4) \longleftrightarrow \begin{bmatrix} x^{11^1} & x^{12^1} \\ x^{21^1} & x^{22^1} \end{bmatrix}$$

$$= 2^{-1/2}\begin{bmatrix} x^1 + x^2 & x^3 + ix^4 \\ x^3 - ix^4 & x^1 - x^2 \end{bmatrix}$$

(e) The spinor translation of a complex null vector of α^j(i-c $\alpha^j\alpha_j = 0$) has the form $\alpha^j \leftrightarrow \alpha^{jj^1} = \beta^j\gamma^{j^1}$. If α^j is real and future pointing, then we can take γ^{j^1} to be the complex conjugate of β^j i-e $\alpha^j \leftrightarrow \beta^j\beta^{-j^1}$

5.3. Geometric Description of a Twistor:

Consider a null straight line L in Minkowski space M. choose a set (x^i) of Minkowski coordinates with origin 0 and let ℓ^i be the position vector of some point p on L (see figure next page). To assign a set of coordinates for L we may select a vector n^i along L and the moment

$$m^{ij} = \ell^i n^j - \ell^j n^i \qquad (5.1)$$

of the vector n^i (acting at p) about 0.

Then the ratios of the 10 quantities (n^i, m^{ij}) will uniquely define L. in addition to the requirement here that n^i be null

$$n^i n_i = 0. \qquad (5.2)$$

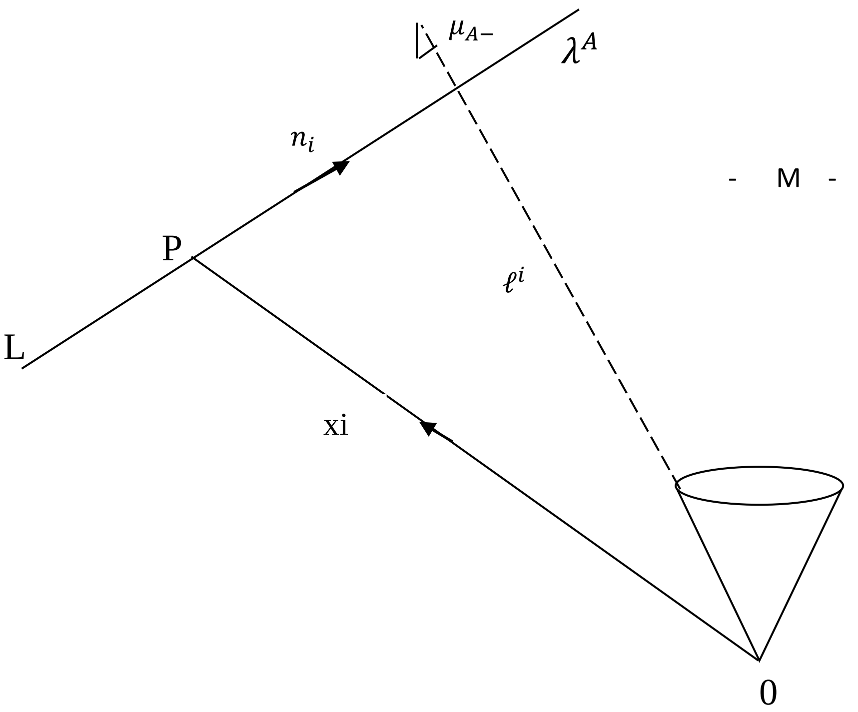


Figure (5.1)

There are the consistency relations for (1)

$$\varepsilon_{ijk1} m^{ij} n^k = 0. \qquad (5.3)$$

Equation (5.3) represents just three independent conditions which together with (5.2) reduce the set of nine ratios in (n^i, m^{ij}) to just five independent real numbers. (This is consistent with the fact that null lines in M form an∞^5 system, for choosing P as the intersection of L with a fixed space like hyperplane, we have ∞^3choices for p and ∞^2 for the null direction at p).

Let us now represent in spinor terms the quantities n^i, m^{ij}which define the null line L. we have

$$n^j \leftrightarrow n^{jj} = \lambda^j \lambda^{-j^1}$$

From (1)$m^{jk} \leftrightarrow \ell^{jj^1} \lambda^k \lambda^{-k^1} - \lambda^j \lambda^{-j^1} \ell^{kk^1}$

$= i\, \varepsilon^{jk} \mu^{j^1} \lambda^{-k^1} - i\, \mu^{(j\lambda k)} \varepsilon^{j^1 k^1}$

Where $\mu_{A^1} = -i\, \lambda^A \ell_{AA^1}$ (5.4)

Thus λ^A and μ_{A^1} together determine ℓ^jandm^{jk}. We may think of λ^Aas defining the direction of L and μ_{A^1} as effectively giving us the moment of λ^A (acting at p) abut.

5.4. Some Remarks:

(i) From (4) if λ^A is multiplied by any complex factor, then L is unchanged if μ_{A^1} is multiplied by the same factor.

(ii) (4) implies also that $\mu_{A^1}$is independent of the choice of p on L since if $\ell_{AA^1} \leftrightarrow \ell_{AA^1} + a\lambda_A\bar{\lambda}_{A^1}$ then μ_{A^1} is unchanged since $\lambda^A\lambda_A = 0$.

(iii) A particular choice of p which is of interest the intersection of L with the null cone of 0. In fact that $\ell^j$is real null it follows that $\ell_{AA^1} = i\,(\lambda^B\bar{\mu}_B)^{-1}\bar{\mu}_A\ \mu_{A^1}$.

Thus the null direction defined by μ_{A^1} is that of the null line through 0 which meets L (see the figure).

(iv) The exceptional case $\lambda^B\bar{\mu}_B = 0$ corresponds to L lying in a null hyperplane thrugh 0. This follows from (4) since n^jwould be necessarily orthogonal to any choice of ℓ_j. In this case λ^A and μ^{-A} are proportional, so that null direction of $\bar{\mu}_A$ is that of L.

The null line L can now be assigned for coordinates, the three complex ratios of four complex quantities

$L^0 = \lambda^0, L^1 = \lambda^1, L^2 = \mu_{0-}$, $L^3 = \mu_{1|}$

$$\text{which we write as} L^\alpha = (\lambda^A, \mu_{A^1}) \quad \alpha = 0, 1, 2, 3 \qquad (5.5)$$

There are six real parameters, so we expect to find one real relation connecting $\lambda^A$and μ_{A^1}. This is obtained from (4) since the reality of ℓ^jimplies ℓ_{AA^1} is hermition, whence

$$(\lambda^A \neq 0) \qquad \mathrm{Re}\,(\lambda^A\bar{\mu}_A) = 0. \qquad (5.6)$$

Condition (6) is also sufficient to ensure the existence of null line L associated with λ^A and μ_{A^1} . For if $\lambda^A\bar{\mu}$ is purely imaginary then $\ell_{AA^1} = i\,(\lambda^B\bar{\mu}_B)^{-1}\bar{\mu}_A\ \mu_{A^1}$, gives a point through which we choose L with direction λ^A.

5.4.1. Definition:

A null Twistor of valence $\begin{bmatrix}1\\0\end{bmatrix}$ is a quantity with components (L^α) given as in (5) with condition (6).

If $Re(\lambda^A\bar{\mu}_A) > 0$ then $L^\alpha$is called a right-handed twistor (valence $\begin{bmatrix}1\\0\end{bmatrix}$) and if $Re(\lambda^A\bar{\mu}_A) < 0$ it is called a left-handed Twistor.

5.5. Incidence of Null Lines in Twistor Terms:

Algebraic rules for manipulation of twisters will have as their basis the idea of incidence between null lines.

Start with two null lines

$$L^{\alpha} = (\lambda^{A}, \bar{\mu}_{A}), X^{\alpha} = (\xi^{A}\eta_{A^1}) \quad \alpha = 0,1,2,3$$

we have by (4)

$$\mu_{A^1} = -i\,\lambda^{A}\ell_{AA^1}, \eta_{A^1} = -i\,\xi^{A}x_{AA^1} \qquad (5.7)$$

Where ℓ_i and x_i are the position vectors of the lines L and x respectively. Suppose now that x and L do intersect, then $\ell^i = x^i$ for the coordinate vector of the intersection point.

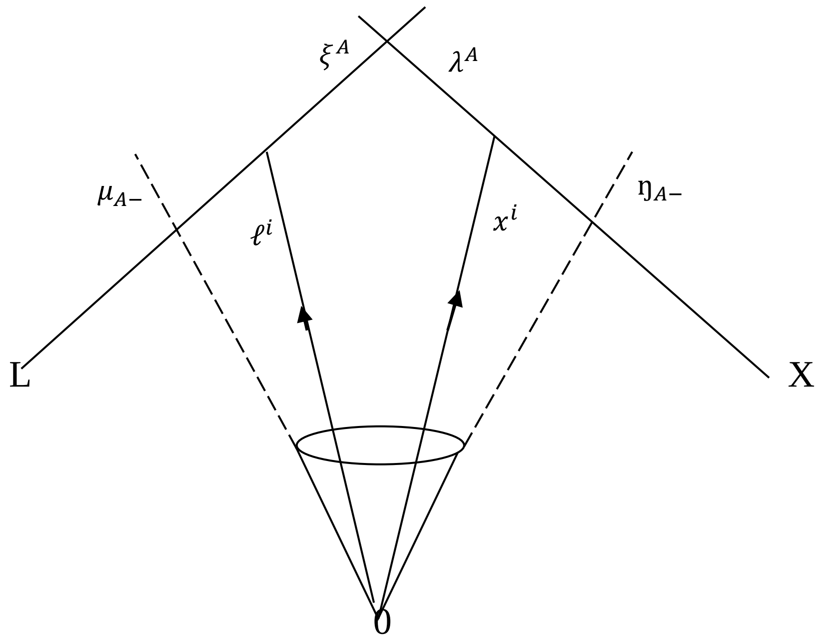


Figure (5.2)

Incidence of two null lines in Minkowski space. Then by Equation (5.7):

$$\xi^{A}\bar{\mu}_{A} = i\,\xi^{A}\ell_{AA}\lambda^{-A^1} = i\,\xi^{A}x_{AA^1}\lambda^{-A^1} = -\eta_{A^1}\lambda^{-A^1}$$

Define the complex conjugate of a twistor L^{α} to be $\bar{L}^{\alpha}$ (valence $\begin{bmatrix}1\\0\end{bmatrix}$) where

$$(\bar{L}^{\alpha}) = (\bar{\mu}_{A}, \bar{\lambda}^{A^1}) \qquad (5.8)$$

or in component form:

$$\bar{L}_0 = \bar{L}^2, \quad \bar{L}_1 = \bar{L}^3, \quad \bar{L}_2 = \bar{L}^0 \bar{L}_3 = \bar{L}^1$$

Then Equation (5.8) tells us that a necessary condition for L and X to meet is

$$X^{\alpha}\bar{L}_{\alpha} = 0 \qquad (5.9)$$

since

$$X^{\alpha}\bar{L}_{\alpha} = \xi^{A}\bar{\mu}_{A} + \eta_{A^1}\lambda^{-A^1} \qquad (5.10)$$

Note that condition (5.6) for $L^{\alpha}$to represent a real null line is

$$L^{\alpha}\bar{L}_{\alpha} = 0 \qquad (5.11)$$

So L intersects itself.

Condition (4.10) is also sufficient for L and X to intersect. Suppose X and L are not parallel so λ^{A} and ξ^{A} are not proportional then $\xi^{A}\lambda_{A} \neq 0$. Construct the complex vector:

$$p_{j} \leftrightarrow p_{jj^1} = \left({}^{i}/_{\xi^{A}\lambda_{A^1}}\right)(\lambda_{j}\eta_{j^1} - \xi_{j}\mu_{j^1})$$

We observe that:$\mu_{A^1} = i\,\lambda^{A}p_{AA^1}$, $= -i\,\xi^{A}p_{AA^1}$. Thus, when p_j is real, we can satisfy (3.7) by putting $\ell_{j} = p_{j} = x_{j}$ whence X and L must intersect.

Now p_j is real $\leftrightarrow$ p_{jj^1} is Hermition. Since λ^{A} and ξ^{A} are not proportional, we can test Hermiticity of p_{jj^1} by taking components with respect toλ^{A}, ξ^{A} :

$\lambda^{A}\lambda^{-A^1}(p_{AA^1} - \bar{p}_{A^1A}) = i\,\mu_{A^1}\lambda^{-A^1} + i\,\lambda^{A}\bar{\mu}_{A} = i\,L^{\alpha}\bar{L}_{\alpha}\xi^{A}\xi^{A^1}(p_{AA^1} - \bar{p}_{A^1A})$
$= i\,\eta_{A^1}\xi^{A^1} + i\,\xi^{A}\bar{\eta}_{A} = i\,X^{\alpha}X_{\alpha}\xi^{A}\lambda^{-A^1}(p_{AA^1} - \bar{p}_{A^1A}) = i\,\eta_{A^1}\lambda^{-A^1} +$
$i\,\xi^{A}\bar{\mu}_{A} = i\,X^{\alpha}\bar{L}_{\alpha}$

Therefore if$X^{\alpha}\bar{L}_{\alpha} = 0$ then p_j is real.

5.6. Two Alternative Pictures of Visualizing Twistors:

The members of the ∞^{6}system of twisters introduced in the last section can be given complex projective coordinates $L^{\alpha} = (L^{0}, L^{1}, L^{2}, L^{3})$. That is, it is only the three complex ration$L^{0}\!:L^{1}\!:L^{2}\!:L^{3}$which are significant. This ∞^{6}system we may think of as constituting a three-dimensional complex projective space C. the points of C are just the "complexified" null lines (and null lines) of M. In fact we have two alternative pictures of any given situation .For example, we may think of an object L with projective coordinates (L) either as, say, a "complexfied" null line of M (M-picture) or simply as a point in a certain projected three-space C (C-picture). In order that the two pictures be completely equivalent, we need to be able to interpret in C, the condition of reality of a null line in M, of incidence between null lines in M and, finally, of points in M. in effect this requires that regard to the conjugation relation $L^{\alpha} \leftrightarrow \bar{L}_{\alpha}$should have a meaning with regard to the C picture. We have seen that a twistor L^{α} (valence$\begin{bmatrix}1\\0\end{bmatrix}$) refers to a point in C, a twistor R_{α} (valence$\begin{bmatrix}0\\1\end{bmatrix}$) therefore refers to the dual concept of a plane R in C, namely the plane of all points X for which $X^{\alpha}R_{\alpha} = 0$. The conjugation relation $L^{\alpha} \leftrightarrow \bar{L}_{\alpha}$ therefore describes a point

↔plane correspondence in C; which we may refer to as a Hermitian Correlation of signature (++--). The signature here refers to the Hermition form

$$X^{\alpha}\overline{X}^{\alpha} = X^0\overline{X}^2 + X^1\overline{X}^2 + X^2\overline{X}^0 + X^3\overline{X}^1.$$

This Hermitian Correlation is an intrinsic part of the geometric structure of C.

The real null lines in M are the points of the five-real –dimensional subset N (top- $S^3 \times S^2$) of C defined by $X^{\alpha}\overline{X}$= 0. Thus N is a hypyersurface if C is regarded as real 6-dimensional manifold (but not in the sense of the complex structure of C).

We refer to the subset of C for which $X^{\alpha}\overline{X} > 0$ holds as C^+ , we may regard the plane $\overline{L}$ as the polar plane of L , with respect to N , since polarizing $X^{\alpha}\overline{X}$ with L^{α} yields $X^{\alpha}\overline{L}_{\alpha}$ = 0 , which is the equation of the plane $\overline{L}$.

The Robinson Congruence associated with L is the intersection (top S^3) of the plane $\overline{L}$ with N . When L lies on N, the plane $\overline{L}$ can be thought of as a complex tangent plane to N at L. this is just the case when L lies on its polar plane.

To represent a point of M in term of the C-picture, we do these using incidence properties of null lines in M. Any point in M can be uniquely represented by an ∞^2 system of null lines in M, namely the generators of the null cone of p. Let k and L be two null lines in M through p. the generators of the null cone of p are then the null lines common to both $\overline{k}$ and $\overline{L}$ (i.e., the generators must meet both $\overline{K}$ and $\overline{L}$). In the C-picture this is an ∞^2system of lines on N which lie on the intersection of$\overline{k}$and $\overline{L}$. This intersection is a complex projective line in C.

Conversely, any line p in M. To see this consider the C-picture and let the line p lie entirely on N. let k and L be two points on p. Then we have $k^{\alpha}\overline{k}_{\alpha}$= 0, $L^{\alpha}\overline{L}_{\alpha}$= 0 and, more generally, $(k^{\alpha} + \beta L^{\alpha})\left(\overline{k} + \overline{\beta}\overline{L}_{\alpha}\right) = 0$

Hence $L^{\alpha}\overline{k}_{\alpha}$= 0. So the intersection of L^{α} and k_{α} will be the point p. Note that, in the C-picture, that a point L lies on a line p , both L and p lying on N is interpreted in the M-picture as the condition that the null line L passes through the point p.

Thus we get the following correspond:

(i) There is a one-to-one correspondence between the null lines in M and the points in N.

(ii) There is a one-to-one relation between the point in M and the complex lines in N.

(iii) The condition for a point to lie on a null line in M is interpreted in N, as the condition for the corresponding line to pass through the corresponding point.

5.7. Penrose Transform

In chapter three we introduced a Twistor as a null line or a congruence of null lines in Minkowski space M. We also gave a one-to-one correspondence between lines in Minkowski space and points of the projective complex three spaces. In this chapter we will consider the basic geometric properties of the above correspondence—Penrose correspondence--using complex manifolds techniques. The geometry of Penrose correspondence will then be used to describe the Penrose transform which transfers Cohomology on subsets of p_3 to spinor fields on subsets ofM_c.

5.8. Penrose Correspondence:

Let M be compactified Minkowski space. M = M_o U (light cone at ∞) where M_o is the flat Minkowiski space. Let M_c be the complexification of M which is of the complex dimension. We had, the Penrose correspondence between M and N p(T) ≡projective twistor space; namely

{Complex lines in N} $|-|$ ⟷ {points in M}

{points in N} $|-|$ ⟷ {Null lines in M} The above correspondence can be extended toM_c:

{Complex lines in P (T)} ⟷ M_c

{Complex lines in N} ↑ M ⟷ ↑

This correspondence is used to transfer problems in mathematical physics in Minkowiski space into problems of several complex variables on subsets of $p_3(\ell)$. Now $M_c$turns out to be the Grassmanian manifold of two-dimensional complex planes in ℓ^4, G_{24} which is clearly equivalent to the set of all complex lines in $p_3(\ell)$.

Accordingto R 0. Wells the above remarkable correspondence of Penrose between $M_c$and subsets of $p_3(\ell)$ can be described in terms of a basic double fibration. Briefly, the idea is this: Considerℓ^4. Define the flag manifold $F_{d_1 \dots d_r}$ where o$< d_1 < \dots < d_r < 4$ as follows:

$$F_{d_1 \dots d_r} = [(L_1, L_2, \dots, L_r): L_1 \subset L_2 \dots \subset L_r \subset \dots \ell^4$$

is a nested sequence of subspaces of ℓ^4 with dimension

$$c^{L}j = d_j \qquad j = 1, \dots \dots, r\,]$$

If r = 1, d_1 = 1 then $F_1 = p_3$, and if r = 1,d_1 = 2, then $F_2 = G_{2,4}(\ell)$. We then consider with respect to ℓ^4 the three flag manifolds F_{12}, F_1 and F_2 and get the following diagram:

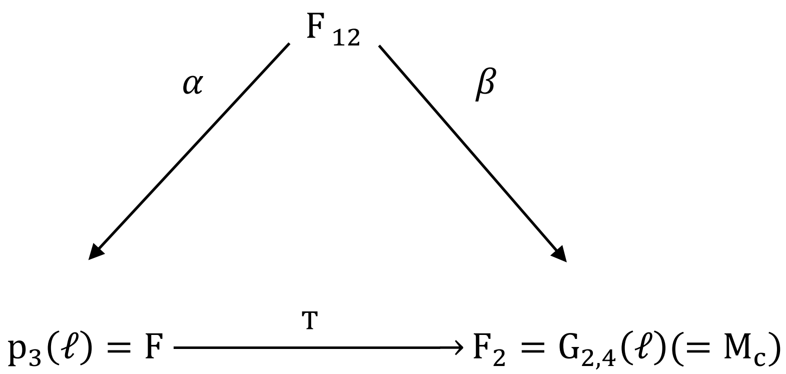


Figure (5.3)

We want this diagram to describe Penrose correspondence. So define the analogue of Penrose correspondence T (which is again called Penrose correspondence) between F_1 and F_2 by (p) = $\beta\alpha^{-1}$(p) andT^{-1}(p) = $\alpha\,\beta^{-1}$(p). One can then prove the following:

5.8.1 Proposition:

(1) (p) is two-complex-dimensional projective plane ≅ p_2(c) embedded in F_2.

(2) τ^{-1}(p)is a one-complex dimensional projective line embedded in F_1.

Proof: (1) By definition:

α^{-1}(p) = {flags (L_1^0, L_2) : $L_1^0 \subset L_2$, L_1^0 fixed, L_2 variable} therefor $\beta\,\alpha^{-1}$(p) = { $L_2 \subset \ell^4$: $L_2 \supset L_1^0$ – fixed } i.e., $\beta\alpha^{-1}$(p) is the set of all two-dimensional subspaces of ℓ^4 which contain a fixed one-dimensional subspace L_1^0**.** This is simply an embedding of p_2(c)in F_2. Since, if we fix one vector e_1, and let e_2 vary in a three-dimensional subspace $\frac{1}{e_1}$ perpendicular to $e_1$with respect to some metric on ℓ^4, then the span of $\{e_1, e_2\}$ will span all subspace $L_2 \supset L_1^0$. But the set of all such e_2 span the set of all complex lines in $\frac{1}{e_1}$, and hence is isomorphic to $p_2(\ell)$.

(3) By the same reasoning as (1):

$$\alpha\,\beta^{-1}(p) = \{L_1 \subset \ell^4 \ : \ L_1 \subset L_2^0, L_2^0 \ -\text{fixed}\}$$

But L_2^0 is two-complex dimensional, and hence $\alpha\beta^{-1}$(p) ≅ $p_1(\ell)$.

5.9. Twistor Structure:

Let Φ be a non-degenerate Hermitian bilinear form on ℓ^4 of signature 0, i.e., {+, +, -, -, }.

In an appropriate coordinate system the matrix for Φ can

$$\Phi = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}.$$

As a quadratic form we can write

$$\Phi(z) = z^0 z^{\overline{2}} + z^1 z^{\overline{3}} + z^2 z^{\overline{0}} + z^3 z^{\overline{1}}.$$

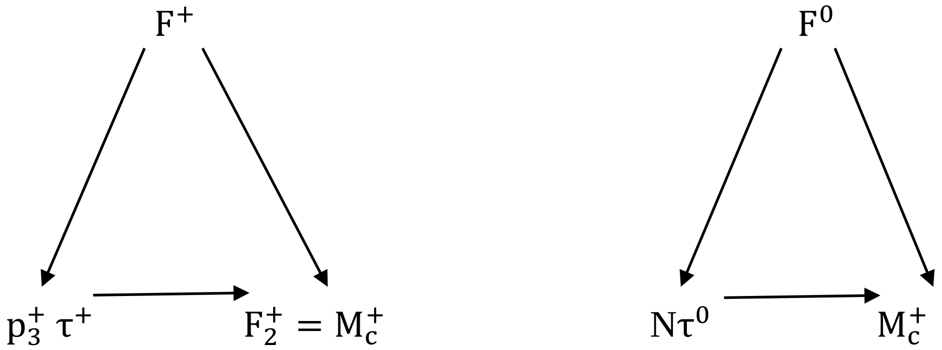
We can define the space of Twistor s T to be T $= (\ell^4$ with the Hermition form Φ). This space is a representation space for SU(2,2) which is a 4-1 covering of conformal group acting on M_c.

Now let: $T^+ = \{ Z_\varepsilon T : \Phi(Z) > 0 \}$positive twisters

$T^o = \{ Z_\varepsilon T : \Phi(Z) = 0 \}$Null twisters

$T^- = \{ Z_\varepsilon T : \Phi(Z) < 0 \}$Negative twisters

In the projectivizedTwistor space p(T) = $p_3(\ell)$ we have the corresponding portions p_3^+, p_3^o and p_3^- . Let N = p_3^0 . So as we have seen in chapter 3 N is a real five-dimensional hyper surface in p(T) which divides p(T) into two complex-analytically equivalent parts p_3^+ and p_3^-. We then got the induced correspondence



(compactified Minkowski space)

Figure (5.4)

The basic geometric properties of the above spaces are given by the following proposition.

Proposition: 5.9.1

(1) p_3^+ contains a four-complex dimensional family of projective complex lines parameterized by M_c^+.

(2) M_c^+ is biholomorphically equivalent to the domain of a 2×2 complex matrices whose Hermitian imaginary part is positive definite.

(3) Let M_0 be the Hermition2×2 matrices, then M_0 is a boundary component of M_c^+.

(4) M is a compact four-dimensional real-analytic submanifold of M_c which is diffeomorphic to $S^1 \times S^3$.

(5) N is a compact five-dimensional real-analytic hyper surface in p_3 diffeomorphic to $s^2 \times S^3$.

(6) F^+ is biholomorphic to $p_1 \times M_c^+$. Now considering the correspondenceτ^+ and τ^0, one can prove the following

Proposition 5.9.2:

(1)$\tau^+(p)$ is the intersection of an affine complex two-plane with M_c^+.

(2) $\tau^0(p)$ is a circle S' embedded in M.

(3) $(\tau^+)^{-1}(p)$ and $(\tau^0)^{-1}(p)$ are complex projective lines embedded in p_3^+ and N respectively.

Suppose we have Twistor homogeneous coordinates$Z^\alpha = (Z^0, Z^1, Z^2, Z^3)$. In these coordinates $\Phi(z) = z^0\overline{z^2} + z^1\overline{z^3} + z^3\overline{z^1}$. We define the dual variables with respect to the Hermitian from Φ by

$$\bar{Z}_0 = \bar{Z}^2, \bar{Z}_1 = \bar{Z}^3, \bar{Z}_2 = \bar{Z}^0, \bar{Z}_3 = \bar{Z}^1$$

and thus $\Phi(Z^\alpha) = Z^\alpha \bar{Z}_\alpha$.

Again, as in caper 3, with $\omega^A = (\omega^0, \omega^1) = (Z^0, Z^1)\Pi_{A'} = (\Pi_{0'}, \Pi_{1'}) = (Z^2, Z^3)$

$Z^\alpha = (Z^0, Z^1, Z^2, Z^3)$ becomes the pair of spinors $(\omega^A, \Pi_{A'})$ which corresponds to $\ell^4 = \ell^2 \oplus \ell^2$. The dual coordinates become:

$\bar{Z}_\alpha = (\bar{\Pi}_A, \omega^{-A'})and\Phi(Z^\alpha) = \omega^A \bar{\Pi}_A + \Pi_{A'}\omega^{-A'}$

If a particle of zero – test – mass moves along a light ray then $\Pi^A \Pi^{-A'}$corresponds to momentum and $\omega^A \omega^{-A'}$ corresponds to angular momentum. The form $\Phi(Z^\alpha)$ corresponds to the spin (twist) of the particle. The path of motion of the particle is givenby $\omega([\tau^A, \Pi_{A'}])$. The detail of this interpretation can be found in penrose-MacCallum.

5.10. Penrose Transform:

We will see in this section how holomorphic massless fields of positive helicity on M^+ could be described in terms of cohomology classes on $p_3^+(\ell)$. This result is due to R. Penrose, and is extended to negative helicity and to weak (hyper function) solutions on the real compactified Minkowski space M by R. 0. Wells.

We start with Penrose correspondence given by the diagram:

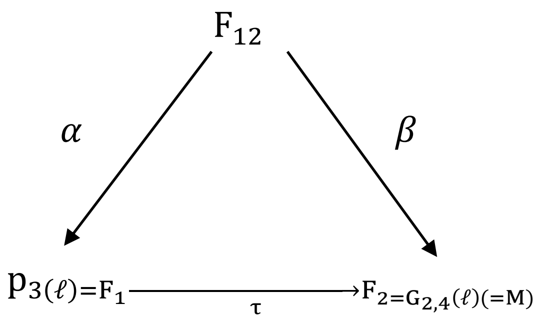


Figure (5.5)

In fact the geometry of the above diagram is used extensively to construct certain fundamental sheaves on p_3, F and M_c .

In general, let X be a complex manifold and let 0_X be the sheaf of local holomorphic function on X. suppose V ⟶ X is a holomorphic vector bundle. Let $0_X(V)$ be the sheaf of local holomorphic section of V. Consider a holomorphic mapping $X \xrightarrow{f} \gamma$ of two complex manifolds. If $G = 0_\gamma(V)$ is a locally free sheaf on γ, then we define the pullback sheaf $F * G$to be $0_X(f*(V))$, the sheaf of holomorphic section of the pullback bundle.

If K is any sheaf of abelian groups on X, then we define a sequence of sheaves $\{f^n * K\}$ on γ called the direct image sheaves underf. The nth direct image sheaf of K under f, denoted by$f_*^n K$ is the sheaf generated by the presheaf.

$$U \to H^n(f^{-1}(U), K) \text{for } U \text{ open in } \gamma$$

The stalk of $f^n * K$ at pεγis the direct limit

$$(f^n * K)_p = \lim_{p\varepsilon\gamma} H^n(f^{-1}(U), K)$$

This is essentially the Cohomology along the fiber $f^{-1}(p)$.

With this preparation Penrose transform can be briefly described:

For nεZ , let $H^{-n-2} \to p_3(\ell)$ be the hyperplane section bundle of p_3 raised to the power -n-2, i.e., the local sections of H^{-n-2} are the homogeneous functions in the homogeneous coordinates of p_3 of degree -n-2. For nεZ define:

$S_{p_3}(n) = 0_{p_3}(H^{-n-2})$The sheaf of holomorphic sections of H^{-n-2}

$$S_F(n) = \alpha^* 0_{p_3}(H^{-n-2})$$

$$S_M(n) = \beta_* \alpha^* 0_{p_3}(H^{-n-2})$$

These are the basic fundamental sheaves on p_3, F and M_cSo the sheaf $S_F(n)$ and $S_M(n)$ are the natural pullback and push-forward of the sheaf $S_{p_3}(n)$. It can be shown that if $n \geq 0$

$$S_M(n) = 0_M(V_n) \text{where rank } \ell\, V_n = \text{rank} \odot^n(\ell^2)$$

$\odot^n$denotes the nth symmetric tensor product.

Moreover, sections of $S_M(n)$ can be identified with holomorhic spinors of primed type on open subsets of M, i.e., of the form A' B' . . . D'. To transform Cohomology on subsets of p_3 to spinor fields on subsets of M_c we need differential operators defined by the geometry of the Penrose correspondence. Let T(F) be the tangent bundle to F and let $T_\alpha(F) \subset T(F)$ be the subbundle of vectors tangent to the fibers of the vibrationα . This induces a canonical surjection of the dual bundles

$$F^*(F) \longrightarrow F_\alpha^*(F).$$

and a differential form $d_\alpha = \Pi_\alpha od$. This acts on a differential form of any degree r on F, i.e.

$$E^r(F)E^r \longrightarrow (F, T^*(F))$$

Exterior differentiation of scalar r-forms to$T^*(F)$-valued r forms. Thus we obtain by composition with Π_α a differential operator

$$E^r(F) \overset{d_\alpha}{\rightarrow} E^r(F, T^*_\alpha(F))$$

$d_\alpha$corresponds to "Differentiation along the fibers of α".

Now use $T^*_\alpha = T^*_\alpha(F)$to introduce two new sheaves on F and M, namely

$$S^\alpha_F(n) = 0_F(\alpha^* H^{-n-2} \times T^*_\alpha)\text{Differential form valued section}$$

$$S^\alpha_M(n) = \beta^1_* 0_F(\alpha^* H^{-n-2} \times T^*_\alpha)$$

Which is the extension of the basic sheaves on F and M by the cotangent bundle along the fibers ofα. Let us represent the sheaf Cohomology in terms of differential via the Dolbealt isomorphic (Cohomology classes being represented by$\bar{\partial}$-closed (0,1) forms as usual).

Then the differential operator d_α extends to differential forms with coefficients in $\alpha^* H^{-n-2}$ can be taken to be constant along the fibers of $\alpha$and thus would be annihilated byd_α. Therefore we get a mapping

$$H^1(F_+, S_F(n) \overset{d_\alpha}{\rightarrow} H^1(F_+, S^\alpha_F(n)$$

We can define a vector bundle $V_n \rightarrow M_{+c}$ by defining the fibers of V_n to be

$$V_{n,X} = H^1(\beta^{-1}(X), S_F(n).$$

One can then use the theory of direct image sheaves to show that V_n defined in this manner is indeed a holomorphic vector bundle.

There is a natural mapping:

$$I : H^1(F_+, S_F(n) \rightarrow H^0(M_+, S_M(n)$$

Obtained by restricting a $\bar{\partial}$-closed differential form representing a cohomology class in $H^1(F_+, S_F(n)$ to a differential form on the submanifold $\beta^{-1}(X)(\cong p_1(\ell)$ which defines a Cohomology class in$H^1(\beta^{-1}(X), S_F(n))$, i.e., a point in the vector space V_n. Similarly there is a mapping

$$I_\alpha : H^1(F_+, S^\alpha_F(n)) \rightarrow H^0(M_+, S^\alpha_M(n));$$

Using all the above data we get the following diagram:

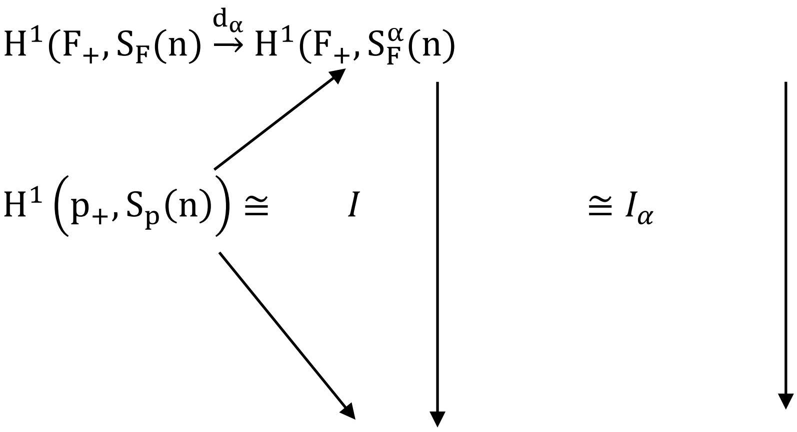


Figure (5.6)

$H^0(M_+, S_M(n) H^0(M_+, S_M^\alpha(n) \quad \xrightarrow{\nabla_\alpha}$

Theorem 5.10.1: Suppose $n > 0$, then

(1) I and I_α are isomorphic and ∇_α is well defined.

(2) $V_{n,X} \cong \otimes^n$ sym C^2.

(3) The induced differential operator ∇_α is the zero-set-mass operator of spin s = $n/2$.

(4) Ker $d_\alpha = \mathrm{Im}\alpha^*$

As a corollary of this theorem we get:

Corollary 5.10.2: (Penrose transform)

If $S > 0$, then

$$H^1\left(p_+, S_p(n)\right) \underset{\cong}{\longrightarrow} \{\text{Ker } \nabla_\alpha: H^0(M_+, S_M(n) \to H^0\left(M, S_M^\alpha(n)\right)\}$$

$\cong$ {Self dual holomorphic solutions of the zero-rest-mass equation of spin s onM_{C+}}.

All the detail of the proof of the above theorem is found in the elegant paper by Eastwood, Penrose and Wells. These people study the pullback of the local data in p_3 involving inverse image sheaves and the relative deRham sequence. Then they solve the problem of integration over the fibers of the mappingβ. This involves direct image sheaves and a fundamental spectral.

Part (1) of the theorem follows from the Leray spectral sequence for direct image sheaves and appropriate standard Cohomology vanishing theorem in several complex variables along either the fibers of β or on M_c. Part (2) is a computation using the theory of compact complex manifolds. Part (3) follows from an appropriate choice of basis for the vector spaces involved. The last part is much deeper and involves solving $\bar{\partial}\mu = f$ problems locally along the fibers ofα, i.e., the inhomogeneous Cauchy-Riemann equation for differential form.

References:

[1] L. Hughston. The wave equation in even Dimensions, in Further Advances in Twister Theory, Research Notes in Mathematics 1 (1990), no. 231 26.
[2] B. Carneiro da Cunba and A. de Queiroz, Killing Horizons and spinors, arxiv; 1312. 4911.
[3] S. A. Huggett and K. P. Tod, An Introduction to Twister Theory. CambrigeUniversty Press, 1994.
[4] E. Cartan, The Theory of Spinors. Dover, 1981.
[5] S. Weinberg, Six-dimensional Methods for Four-dimensional Conformal Field Theories,II: Irreducible Fields, Phys. Rev. D86 (2012) 085013, [arxiv: 1209. 4659].
[6] F. R. Harvey, Spinor and Calibrations, vol. 9 of Perspectives in Mathematics. Academic Press, 1990.
[7] N. Berkovits, Super Poincare covariant quantization of the superstring, JHEP 0004 (2000) 018, [hep-th/0001035].
[8] Alekseevsky, D. V., Marchiafava, S. and Pontecorvo, M.: Compatible almost complex structures on quaternion-kahler manifolds, Preprint ESI 419, 1997.
[9] Salamon, S.: Orthogonal complex siructures, in Janiska, J., Kolar, I. and Slovak, J. (eds), Proceedings of the 6th International Conference on Differential Geometry, Brno, August 28-September 1, 1995, pp. 103-117 (1996).
[10] R. S. Ward and R.O. Wells, Twistor Geometry and Field Theory. Cambridge University Press, 1990.
[11] Abbena, E., Garbiero, S. and Salamon, S., Hermition geometry on the Iwasawa manifold, Boll. Un. Mat. Ital., VII, Ser. B 11(2), (1997), Suppl. 231-249.
[12] Burstall, F., Grantcharov, G., Muskarov, O. and Rawnsley, J.: Hermitian structures on Hermitian symmetric spaces, J. Geom. Phys. 10 (1993), 245-249.
[13] Gauduchon, P.: Complex structures on compact conformal manifolds of negative type, in Ancona, V., Ballico, E. and Silva, S. (eds), Complex analysis and Geometry, Proceedings of the Conference at Trento, Marcel Dekker, New York/Basel/Hong Kong, 1996, pp. 201-212.
[14] Nurowski, P.: Einstein equations and Cauchy-Remann geometry, Ph.D. thesis, SISSA/ISAS. Trieste, 1993.
[15] R. Penrose, Twistor algebra, J. Math. Phys.8(1967) 345.
[16] R. Penrose,Twistor quantization and curved space-time,Int. J. Theor. Phys.1(1968) 61.
[17] R. Penrose,Solutions of the zero-rest-mass equations, J. Math. Phys.10(1969) 38.
[18] R. Penrose and M. A. H. MacCallum, Twistor theory: An approach to the quantiza-tion of fields and space-time,Phys. Rept.6(1972) 241.
[19] R. S. Ward,On self-dual gauge fields,Phys. Lett. A61(1977)81.

[20] R. Penrose,Non-linear gravitons and curved twistor theory,Gen. Rel. Grav.7(1976) 31.M. Atiyah, N. J. Hitchin, and I. Singer, Self-duality in four-dimensional Riemannian geometry,Proc. Roy. Soc. Lond. A362(1978) 425.R. S. Ward, Self-dual space-times with cosmological constant,Commun. Math.Phys.78(1980) 1.
[21] M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld, and Y. I. Manin,Construction of in-stantons,Phys. Lett. A 65 (1978) 185.[8] E. Witten, An interpretation of classical Yang–Mills theory, Phys. Lett. B 77 (1978)394.
[22] J. Isenberg, P. B. Yasskin, and P. S. Green, Non-self-dual gauge fields, Phys. Lett.B78 (1978) 462.[10] J. Isenberg and P. B. Yasskin, Twistor description of non-self-dual Yang–Mills fields,in: 'Complex Manifold Techniques In Theoretical Physics,' 180, Lawrence, 1978.
[23] M. G. Eastwood and C. R. LeBrun, Thickening and supersymmetric extensions of complex manifolds, Amer. J. Math. 108 (1986) 1177. C. R. LeBrun, Thickenings and gauge fields, Class. Quant. Grav. 3 (1986) 1039.
[24] Y. I. Manin, Gauge field theory and complex geometry, Springer Verlag, Berlin, 1988.
[25] C. R. LeBrun, The first formal neighbourhood of ambitwistor space for curved space-time, Lett. Math.Phys. 6 (1982) 345. C. R. LeBrun, Thickenings and conformal gravity, Commun.Math.Phys. 139 (1991) 1. C. R. LeBrun, Ambitwistors and Ein-stein's equations, Class. Quant. Grav. 2 (1985) 555. S. A. Merkulov, Simple super-gravity, supersymmetric non-linear gravitons and supertwistor theory, Class. Quant.Grav. 9 (1992) 2369.
[26] N. J. Hitchin, Monopoles and geodesics, Commun.Math.Phys. 83 (1982) 579.
[27] R. Penrose and W. Rindler, Spinors and space-time. Vol. 1: Two-spinor calculus and relativistic fields, Cambridge University Press, Cambridge, 1984.
[28] R. Penrose and W. Rindler, Spinors and space-time. Vol. 2: Spinor and twistor methods in space-time geometry, Cambridge University Press, Cambridge, 1986.
[29] R. S. Ward and R. O. Wells, Twistor geometry and field theory, Cambridge UniversityPress, Cambridge, 1990.
[30] M. Wolf, A first course on twistors, integrability and gluon scattering amplitudes, J.Phys.A 43 (2010) 393001 [1001.3871 [hep-th].
[31] T. Adamo, M. Bullimore, L. J. Mason, and D. Skinner, Scattering amplitudes and Wilson loops in twistor space, J. Phys. A 44 (2011) 454008 [1104.2890 [hep-th]].
[32] R. S. Ward, Completely solvable gauge field equations in dimension greater than four,Nucl. Phys. B 236 (1984) 381. E. Witten, Twistor-like transform in ten dimen-sions, Nucl. Phys. B 266 (1986) 245. M. G.

Eastwood, The generalized Penrose– Ward transform, Math. Proc. Camb. Phil. Soc. 97 (1985) 165. T. N. Bailey andM. G. Eastwood, Complex paraconformal manifolds - Their differential geometry,Forum Math. 3 (1991) 61.
[33] M. K. Murray, A Penrose transform for the twistor space of an even dimensional conformally flat Riemannian manifold,Ann. Global Anal.Geom. 4 (1986) 71.61.

[34] M. Wolf, A connection between twistors and superstring sigma models on cosetsu-perspaces, JHEP 0909 (2009) 071 [0907.3862 [hep-th]].

[35] J. Bagger and N. D. Lambert, Gauge symmetry and supersymmetry of multiple M2-branes, Phys. Rev. D 77 (2008) 065008 [0711.0955 [hep-th]]. A. Gustavsson,Algebraic structures on parallel M2-branes, Nucl. Phys. B 811 (2009) 66 [0709.1260[hep-th]]. O. Aharony, O. Bergman, D. L. Jafferis, and J. M. Maldacena, N = 6superconformalChern–Simons-matter theories, M2-branes and their gravity duals,JHEP 10 (2008) 091 [0806.1218 [hep-th]].

[36] P. S. Howe, N. D. Lambert, and P. C. West, The self-dual string soliton, Nucl.Phys.B 515 (1998) 203 [hep-th/9709014].

[37] A. Gustavsson, Self-dual strings and loop space Nahm equations, JHEP 04 (2008)083 [0802.3456 [hep-th]]. C. Saemann, Constructing self-dual strings, Com-mun. Math.Phys. 305 (2011) 513 [1007.3301 [hep-th]]. S. Palmer and C. Sae-mann, Constructing generalized self-dual strings, JHEP 10 (2011) 008 [1105.3904[hep-th]].

[38] L. P. Hughston and W. T. Shaw, Minimal curves in six dimensions, Class. Quant.Grav. 4 (1987) 869.

[39] L. P. Hughston, The wave equation in even dimensions, Twistor Newsletter 9(1979) 60.L. P. Hughston, A remarkable connection between the wave equation and spinors in higher dimensions, Twistor Newsletter 14 (1982) 46. L. P. Hughston,Applications of the geometry of SO(8) spinors to Laplace’s equation in six dimen-sions, Twistor Newsletter 17 (1984) 18. L. P. Hughston, Applications of SO(8)spinors, in: ‘Gravitation and Geometry: a volume in honour of Ivor Robinson,’ eds.W. Rindler and A. Trautman, 253, Bibliopolis, Naples, 1987.

[40] L. P. Hughston, Applications of Cartanspinors to differential geometry in higher dimensions, in: 'Spinors in Physics and Geometry,' eds. G. Furlan and A. Trautman,World Scientific Press, 1987.L. P. Hughston and L. J. Mason, A generalized Kerr– Robinson theorem, Class. Quant. Grav. 5 (1988) 275. Y. Inoue, Twistor spaces of even-dimensional Riemannian manifolds, J. Math. Kyoto Univ. 32 (1992) 101.

[41] N. Berkovits and S. A. Cherkis, Higher-dimensional twistor transforms using pure spinors, JHEP 0412 (2004) 049 [hep-th/0409243 [hep-th]].

[42] T. Chern, Superconformal field theory in six dimensions and supertwistor, 0906.0657[hep-th].

[43] N. Lambert and C. Papageorgakis, Non-Abelian (2,0) tensor multiplets and 3-algebras, JHEP 08 (2010) 083 [1007.2982 [hep-th]]. C. PapageorgakisandC. Saemann, The 3-Lie algebra (2,0) tensor multiplet and equations of motion on loop space, JHEP 05 (2011) 099 [1103.6192 [hep-th]]. H. Samtleben, E. Sez-gin, and R. Wimmer, (1,0) superconformal models in six dimensions, 1108.4060[hep-th]. C.-S. Chu, A theory of non-Abelian tensor gauge field with non-Abelian gauge symmetry G×G, 1108.5131 [hep-th].

[44] M. G. Eastwood, R. Penrose, and R. O. Wells, Cohomology and massless fields,Commun.Math.Phys. 78 (1981) 305.

[45] P. Pasti, D. P. Sorokin, and M. Tonin, Note on manifest Lorentz and general coordinate invariance in duality symmetric models, Phys. Lett. B 352 (1995) 59[hep-th/9503182 [hep-th]]. P. Pasti, D. P. Sorokin, and M. Tonin, Duality sym-metric actions with manifest space-time symmetries, Phys. Rev. D 52 (1995) 4277[hep-th/9506109 [hep-th]]. P. Pasti, D. P. Sorokin, and M. Tonin, On Lorentz invariant actions for chiral p-forms, Phys. Rev. D 55 (1997) 6292 [hep-th/9611100[hep-th]]. P. Pasti, D. P. Sorokin, and M. Tonin, Covariant action for a D = 11 five-brane with the chiral field, Phys. Lett.B 398 (1997) 41 [hep-th/9701037].

[46] L. J. Mason, R. A. Reid-Edwards, and A. Taghavi-Chabert, Conformal field theories in six-dimensional twistor space, 1111.2585 [hep-th].

[47] C. Cheung and D. O'Connell, Amplitudes and spinor-helicity in six dimensions,JHEP 0907 (2009) 075 [0902.0981 [hep-th]]. T. Dennen, Y.-t. Huang, andW. Siegel, Supertwistor space for 6d maximal super Yang–Mills, JHEP 04 (2010)127 [0910.2688 [hep-th]]. Y.-t. Huang and A. Lipstein, Amplitudes of 3d and 6d maximal

superconformaltheories in supertwistor space, JHEP 1010 (2010) 007[1004.4735 [hep-th]]. B. Czech, Y.-t. Huang, and M. Rozali, Amplitudes for multiple M5-branes, 1110.2791 [hep-th].
[48] P. Claus, R. Kallosh, and A. Van Proeyen,M five-brane and superconformal (0,2)tensor multiplet in six-dimensions, Nucl. Phys. B 518 (1998) 117 [hep-th/9711161[hep-th]].
[49] E. Witten, Perturbative gauge theory as a string theory in twistor space, Commun.Math.Phys. 252 (2004) 189 [hep-th/0312171 [hep-th]].
[50] R. Bott, Homogeneous vector bundles, Ann. Math. 2 (1957) 66.[39] D. Burns, Some background and examples in deformation theory, in: ‘Complex Mani-fold Techniques in Theoretical Physics,’ eds. D. Lerner and P. Sommers, Pitman, SanFransisco–London–Melbourne, 1979. K. Kodaira, Complex manifolds and deform-ation of complex structures,SpringerVerlag, New York, 1986.
[51] P. Griffiths and J. Harris, Principles of algebraic geometry, John Wiley & Sons, NewYork, 1978.[41] N. P. Buchdahl, Analysis on analytic spaces and non-self-dual Yang–Mills fields,Trans. Amer. Math. Soc. 288 (1985) 431.
[52] R. Pool, Some applications of complex geometry to field theory, PhD Thesis, RiceUniversity Texas, 1981.[43] R. Godement, Topologiealg´ebrique et th´eorie des faisceaux, Hermann, Paris, 1964.63.

[53] M. Eastwood, Supersymmetry, twistors, and the Yang–Mills equations, Trans. Amer.Math.Soc. 301 (1987) 615.
[54] R. O. Wells, Cohomology and the Penrose transform, in: ‘Complex Manifold Tech-niques in Theoretical Physics,’ eds. D. Lerner and P. Sommers, Pitman, San Fransisco–London–Melbourne, 1979.
[55] R. Penrose, Massless fields and sheaf cohomology, Twistor Newsletter 2 (1977) 9.R. S. Ward, Sheaf cohomology and an inverse twistor function, Twistor Newsletter 6 (1977) 13.
[56] A. D. Helfer, A formula for the local solution of the self-dual Yang–Mills equations,
Proc. Roy. Soc. Lond. A 414 (1987) 135.
[57] L. J. Mason and D. Skinner, Gravity, twistors and the MHV formalism, Commun.Math.Phys. 294 (2010) 827 [0808.3907 [hep-th]]. L. J. Mason and D. Skin-ner, Dual superconformal invariance, momentum twistors and Grassmannians, JHEP0911 (2009) 045 [0909.0250 [hep-th]]. T. Adamo and L. J. Mason, MHV dia-grams in twistor space and the twistor action, 1103.1352 [hep-th]. R. Boels,L. J. Mason, and D. Skinner,Supersymmetric gauge theories in twistor space, JHEP02 (2007) 014 [hep-th/0604040].

[58] F. Cachazo, P. Svrˇcek, and E. Witten, MHV vertices and tree amplitudes in gauge theory, JHEP09 (2004) 006 [hep-th/0403047].
[59] L. J. Mason and D. Skinner, An ambitwistor Yang–Mills Lagrangian, Phys. Lett. B636 (2006) 60 [hep-th/0510262].
[60] A. D. Popov and C. Saemann, On supertwistors, the Penrose–Ward transform and N = 4 super Yang–Mills theory, Adv. Theor. Math.Phys. 9 (2005) 931[hep-th/0405123].
[61] M. G. Eastwood, On the twistor description of massive fields, Proc. Roy. Soc. Lond.374 (1981) A431.
[62] C. Saemann, On the mini-superambitwistor space and N = 8 super Yang–Mills the-ory, Adv. Math. Phys. 2009 (2009) 784215 [hep-th/0508137].
[63] R. S. Ward, Twistors in 2+1 dimensions, J. Math. Phys. 30 (1989) 2246.
[64] M. F. Atiyah and N. J. Hitchin, The geometry and dynamics of magnetic monopoles,M. B. Porter lectures, Princeton Univ. Pr., Princeton, 1988. A. D. Popov, On ex-plicit point multi-monopoles in SU (2) gauge theory, J. Math. Phys. 46 (2005) 073506[hep-th/0412042].
[65] A. D. Popov, C. Saemann, and M. Wolf, The topological B model on a mini-supertwistor space and supersymmetricBogomolny monopole equations, JHEP 10(2005) 058 [hep-th/0505161].
[66] C. Saemann and M. Wolf, Work in progress.64.

[67] J. P. Harnad, J. Hurtubise, M. Legare, and S. Shnider, Constraint equations and field equations in supersymmetric N = 3 Yang–Mills theory, Nucl. Phys. B 256 (1985)609.
[68] J. P. Harnad and S. Shnider, Constraints and field equations for ten-dimensional super Yang–Mills theory, Commun. Math.Phys. 106 (1986) 183.
[69] H. Samtleben and R. Wimmer, N = 8 superspace constraints for three-dimensional gauge theories, JHEP 1002 (2010) 070 [0912.1358 [hep-th]].
[70] H. Samtleben and R. Wimmer, N = 6 superspace constraints, SUSY enhancement and monopole operators, JHEP 1010 (2010) 080 [1008.2739 [hep-th]].
[71] L. Breen and W. Messing, Differential geometry of gerbes, Adv. Math. 198 (2005)732 [math.AG/0106083].
[72] P. Aschieri, L. Cantini, and B. Jurco, Non-Abelian bundle gerbes, their dif- ferential geometry and gauge theory, Commun. Math.Phys. 254 (2005) 367[hep-th/0312154]. B. Jurco, Non-Abelian bundle 2-gerbes, 0911.1552 [math.DG].
[73] K. Pohlmeyer, On the Lagrangian theory of anti(-self-)dual fields in four-dimensional Euclidean space, Commun. Math.Phys. 72 (1980) 37. L. Dolan, Kac–Moody algeb-ras and exact solvability in hadronic physics, Phys. Rept. 109 (1984) 1. A. D. Popovand C. R. Preitschopf, Extended

conformal symmetries of the self-dual Yang–Mills equations, Phys. Lett. B 374 (1996) 71 [hep-th/9512130]. T. A. Ivanova, On infinite-dimensional algebras of symmetries of the self-dual Yang–Mills equations,J. Math. Phys. 39 (1998) 79 [hep-th/9702144 [hep-th]].A. D. Popov, Self-dual Yang–Mills: Symmetries and moduli space, Rev. Math.Phys. 11 (1999) 1091[hep-th/9803183]. A. D. Popov, Holomorphic Chern–Simons–Witten theory: From 2D to 4D conformal field theories, Nucl. Phys. B 550 (1999) 585 [hep-th/9806239]. M. Wolf, On hidden symmetries of a super gauge theory and twistor string theory,JHEP 02 (2005) 018 [hep-th/0412163]. M. Wolf, Twistors and aspects of integrabil-ity of self-dual SYM theory, Proc. of the Intern. Workshop on SupersymmetriesandQuantum Symmetries 1 (2005) 448 [hep-th/0511230]. A. D. Popov and M. Wolf,Hidden symmetries and integrable hierarchy of the N = 4 supersymmetric Yang–Mills equations, Commun. Math.Phys. 275 (2007) 685 [hep-th/0608225].

[74] C. Devchand, Integrability on light-like lines in six-dimensional superspace, Z. Phys.C 32 (1986) 233.[66] C. Saemann, R. Wimmer, and M. Wolf, Work in progress.

[75] J. L. Brylinski, Loop spaces, characteristic classes and geometric quantization,Birkh¨auser, Boston, 2007.

[76] N. J. Hitchin, Lectures on special Lagrangian submanifolds, math/9907034 [math].

[77] D.S. Chatterjee, Ongerbs, PhD thesis, Trinity College Cambridge, 1998.M. K. Murray, Bundle gerbes, J. Lond. Math.Soc. 54 (1996) 403 [dg-ga/9407015[dg-ga]].

[78] D. Stevenson, The geometry of bundle gerbes, PhD thesis, University of Adelaide, 2000.