Sudan University of Science and Technology

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Solution of Some Integral and Differential Equations Using Adomian Decomposition Method

حل بعض المعادلات التكاملية والتفاضلية باستخدام طريقة أدوميان التفكيكية

Athesis submitted for fullments of the requirement of the degree of Ph.D in Mathmatics

Submitted by:

Yasir Abeid Hussin Mohammed

Supervisor:

Prof.Mohammed Ali Bashir

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Dedication

I dedicate this research to my parents who have given me love and support .

Acknowledgments

First I thank my teacher

Prof mohammed Ali Bashir

Who helped me to complete this research .I also thank my wife who supported me.

Abstract

Adomian analytical decomposition method is one of the important method in solving integral and differential equations.

In this study we handled the analysis of the method and then the solution of Volterra integral equation .Also we treated the solution of nonlinear integral equations.

The integro–differential equations has been studied with applications to the wave and heat equations in three dimensions, using Adomian method.

الخلاصة

تعتبر طريقة تفكيك ادوميان أحد الطرق التحليلية الهامة في حلول المعادلات التكاملية والتفاضلية لذلك تناول هذا البحث طريقة ادوميان. تمت دراسة تحليل هذه الطريقة و حل معادلة فولتيرا التكاملية بطريقة ادوميان وكذلك علاج حل المعادلات التكاملية غير الخطية. أيضاً درست المعادلات التفاضلية التكاملية مع التطبيق على حلول معادلة الموجة والحرارة في ثلاثة ابعاد بطريقة ادوميان.

Introduction

The Adomian decomposition method (ADM) is a semi analytical method for solving ordinary and partial non linear differential equation the method was developed from 1970 to the 1990 by George Adomian the chair of the center for applied mathematics at the university of Georgia in (USA) [21] it's further extensible to (stochastic systems) by using the integral [21] The aim of the method is to wards a unified theory for the solutions of partial differential equations (PDE) and aim which has been superseded by the more general theory of the homology analysis method[21] the crucial aspect of the method employment of the Adomian polynomial which allow for solution convergence the nonlinear partial of equation. Without simply linearaizing the system. These polynomials mathematically generalize to a MacLoren series about on arbitrary external parameter, which gives the solution method more flexibility than direct Taylor series expansion[21].

The Adomian Decomposition Method has been receiving much attention in recent years in applied mathematical general, and in the area of series solutions in particular. The method proved to be powerfull. Effective and can easily handle a wide class of linear or nonlinear ordinary or partial differential equation, and linear or non linear integral equations the decomposition method demonstrates fast convergence of the solution and therefore provides several significant advantages.

the show that the method will successfully used to handle most types of partial differential equation that appear in several physical models and scientific applications the method attacks the problem in a direct way and straight forword fashion without using the linearization, perturbation or any other restrictive assumption that may change the physical behavior of the model under discussion[21].

Adomian decomposition method was introduced and developed in USA by George Adomian and is well addressed in the literature. Considerable amount of research work has been invested recently in applying this method to ordinary partial differential equations, and partial differential equation and integral equation as well and the non linear partial differential equation can be found in wide variety scientific and engineering application. Many important mathematical models can be expressed in terms of the non linear partial differential equations, the most general form of non linear partial equation is given by:

$$F\left(u, u_t, u_x, u_y, x, y, t\right) = 0$$

With initial and boundary conditions

$$u(x, y, 0) = \emptyset(x, y), \qquad \forall x, y \in \partial\Omega, \qquad \Omega \in \mathbb{R}^2$$

$$u(x, y, t) = f(x, y, t), \ \forall x, y \in \partial\Omega, \ \text{(IC)}$$

where (Ω) the solution region and $\partial\Omega$ is the boundary of Ω.

In the recent years, much research has been focused on the numerical solution of non linear partial equation by using numerical method and developing these methods some persons some of them (Alsaif , 2007; levegue 2006; Rosser & Husner 1997, Wescot & Rizwan- Uddin, 2001)[21].

In the numerical methods, which are commonly used for solving these kinds of equations large size or difficult of computations and appeared and usually the round of error causes the loss of accuracy. The Adomain Decomposition Method which needs less computation was employed to solve in any problems therefore, we applied the Adomian decomposition method to solve some models of non linear partial equation, this study and search reveals that the steps of the Adomian Decomposition for solving and how this method of very efficient for the non linear equation. And most of result gives and shows using given evidence that high accuracy and flexibility for the non linear can be achieved[21].

In this research our scheme is as follows:

In chapter one Adomian technique method , and chapter two solving Volterra integral equation .

Also we treated the solution of nonlinear integral equations in chapter three , also we studied the integro – differential equations with applications to the wave and heat equations in three dimensional , using Adomian method in chapter four and five .

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CHAPTER ONE

ANALYSIS OF ADMOIAN DECOMPOSITION METHOD

we study the analysis of the Adomian decomposition method .

1.1 Adomian Decomposition Formula

The Adomian decomposition method consists of the decomposing the unknown function $u(x,y)$ of an equation it sum of an intuitive number of component defined by the decomposition series.

$$u(x,y) = \sum_{n=0}^{\infty} u_n\,(x,y), \qquad (1-1)$$

where the component $u_n(x,y), n \geq 0$ are to be determined a recursive in manner the decomposition method concerns itself with find the components $u_0, u_1, u_2, \ldots$ individually. As will be seen thorough recursive relation the usually involve simple integrals.

To give a clear an overview of Adomian Decomposition Method, we first consider the linear differential equation write in an operator form by

$$L(u) + R(u) = g \qquad (1-2)$$

Where L is an operator, mostly, the lower order derivative which is assumed to be invertible, R is other linear differential equations will represented in last chapter. We next apply the inverse operator L^{-1} to both side of equation (1-2) we obtain

$$L^{-1}(L(u) + R(u)) = L^{-1}(g)$$

$$(L^{-1}L(u) + L^{-1}R(u) = L^{-1}(g) \quad (1\text{-}3)$$

$$\Rightarrow u + L^{-1}R(u) = f \quad where\, f = L^{-1}(g)$$

Then we get

$$u = f - L^{-1}R(u) \quad (1\text{-}4)$$

Where function f represents the terms arising from integrating the source term g and from using the given condition that are assumed to be prescribed. As indicated before, Adomian method defines the solution by and infinite series of components given by:

$$u = \sum_{n=0}^{\infty} u_n, \quad (1\text{-}5)$$

where the components u_0, u_1, u_2 are usually recurrently determined substituting it to both side of (1-4) and (1-5) lied to

$$\sum_{n=0}^{\infty} u_n = f - L^{-1}\left(R\left(\sum_{n=0}^{\infty} u_n\right)\right) \quad (1-6)$$

For simplicity equation (1-6) can be write as

$$u_0 + u_1 + u_2 \dots = f - L^{-1}(R(u_0 + u_1 + u_2 \dots)\ (1-7)$$

1-2 Formulas of Adomian Decomposition Method:

The normal in differential equations that we seek a closed form solution or a series solution with a proper number of terms.

For the first problem we consider the equation.

$$u^{'}(x) = u(x)\ , u(0) = A \quad (1-8)$$

In an operator form . Equation (1-8) becomes

$$Lu = u\,, \quad (1-9)$$

Where the differential L is given by

$$L = \frac{d}{dx} \quad , \qquad (1-10)$$

and therefore the inverse operator L^{-1} is defined by

$$L^{-1}\,(.) = \int_0^x (.)dx \qquad (1-11)$$

Applying L^{-1} to both sides of (1-9) and using the initial condition we obtain

$$L^{-1}(Lu) = L^{-1}(u) \qquad (1-12)$$

So that

$$u(x) - u(0) = L^{-1}(u) \qquad (1-13)$$

Or equivalently

$$u(x) = A + L^{-1}(u) \qquad (1-14)$$

Substituting the series assumption (1-6) into both sides of (1-14) gives

$$\sum_{n=0}^{\infty} u_n(x) = A + L^{-1}\left(\sum_{n=0}^{\infty} u_n(x)\right) \qquad (1-15)$$

In view of (1-15) , the following recursive relation

$$\left.\begin{aligned} &u_0(x) = A \\ &u_{k+1}(x) = L^{-1}\big(u_k(x)\big) \ \ , k \ \geq 0 \end{aligned}\right\} \qquad (1-16)$$

Follows immediately consequently , we obtain

$$\left.\begin{aligned} u_0(x) &= \; A \\ u_1(x) = L^{-1}\big(u_0(x)\big) &= Ax\,, \\ u_2(x) = L^{-1}(u_1(x)) &= A\,\frac{x^2}{2!} \;\;, \\ u_3(x) = L^{-1}(u_2(x)) &= A\,\frac{x^3}{3!} \;\;, \end{aligned}\right\} \qquad (1-17)$$

Substituting (1-7) into (1-6) gives the solution in a series form by

$$u(x) = A(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots) \qquad (1-18)$$

and in a closed form by

$$u(x) = \; Ae^x \qquad (1-19)$$

We next consider the well-known Air's equation

$$u''(x) = \; x\,u(x), u(0) = A, u'(0) = \; B \qquad (1-20)$$

In an operator form, Equation (1-20) becames

$$Lu = xu \qquad (1-21)$$

where the differential operator L is given by

$$L = \frac{d^2}{dx^2} \qquad (1-22)$$

and therefore the inverse operator $L^{-1}$is defined by

$$L^{-1}(.) = \int_0^x \int_0^x (.)dx\,dx \qquad (1-23)$$

Operating with L^{-1}on both sides of (1-20) and using the initial conditions we obtain.

$$L^{-1}(Lu) = \; L^{-1}(xu) \qquad (1-24)$$

So that

$$u(x) - xu'(0) - u(0) = L^{-1}(xu) \qquad (1-25)$$

Or equivalently

$$u(x) = A + Bx + L^{-1}(xu) \qquad (1-26)$$

Substituting the series assumption (1-6) into both sides of (1-26) yields

$$\sum_{n=0}^{\infty} u_n(x) = A + Bx + L^{-1}\left(x \sum_{n=0}^{\infty} \left(u_n(x)\right)\right) \qquad (1-27)$$

Following the decomposition method we obtain the following recursive relation .

$$\left.\begin{array}{l} u_0(x) = A + Bx \\ u_{k+1}(x) = L^{-1}(xu_k(x)\,, k \geq 0 \end{array}\right\} \qquad (1-28)$$

Consequently , we obtain

$$\left.\begin{array}{l} u_0(x) = A + Bx \\ u_1(x) = L^{-1}(xu_0(x) = A\,\frac{x^3}{6} + B\,\frac{x^4}{12} \\ u_2(x) = L^{-1}(xu_1(x)) = A\,\frac{x^6}{180} + B\,\frac{x^7}{504} \\ \quad\vdots \end{array}\right\} \qquad (1-29)$$

Substituting (1-29) into (1-6) gives the solution in a series form by

$$u(x) = A\left(1 + \frac{x^3}{6} + \frac{x^6}{180} + \cdots\right) + B\left(x + \frac{x^4}{12} + \frac{x^7}{504} + \cdots\right) (1-30)$$

Other components can be easily computed to enhance the accuracy of the approximation .

It seems now reasonable to apply Adomian decomposition method to first –order partial differential equations we consider the inhomogeneous partial differential equation:

$$u_x + u_y = f(x,y), u(0,y) = g(y), u(x,0) = h(x) \qquad (1-31)$$

In an operator form Equation (1-31) can be written as

$$L_x u + L_y u = f(x,y) \qquad (1-32)$$

Where

$$L_x = \frac{\partial}{\partial x} \quad , L_y = \frac{\partial}{\partial y} \qquad (1-33)$$

Where each operator is assumed easily invertible , and thus the inverse operators L_x^{-1} and L_y^{-1} exist and given by

$$\left.\begin{aligned} L_x^{-1}(.) &= \int_0^x (.)dx \\ L_y^{-1}(.) &= \int_0^y (.)dy \end{aligned}\right\} \qquad (1-34)$$

This means that

$$L_x^{-1} L_x u(x,y) = u(x,y) - u(0,y) \qquad (1-35)$$

Applying L_x^{-1} to both sides of (1-32) gives

$$L_x^{-1} L_x u = L_x^{-1}\left(F(x,y)\right) - L_x^{-1}\left(L_y u\right) \qquad (1-36)$$

Or equivalently

$$u(x,y) = g(y) + L_x^{-1}\left(F(x,y)\right) - L_x^{-1}\left(L_y u\right) \qquad (1-37)$$

Obtained by using (1-35)and by using the condition $u(0,y) = g(y)$

As stated above , the decomposition method sets

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y) \qquad (1-38)$$

Substituting (1-38) into both sides of (1-37) we fined

$$\sum_{n=0}^{\infty} u_n(x,y) = g(y) + L_x^{-1}\left(F(x,y)\right) - L_x^{-1}(L_y\left(\sum_{n=0}^{\infty} u_n(x,y)\right) \qquad (1-39)$$

This can be rewritten as

$$u_0 + u_1 + u_2 + \dots = g(y) + L_x^{-1}\left(F(x,y)\right) - L_x^{-1}L_x(u_0 + u_1 + u_2 + \cdots) \qquad (1-40)$$

The zeroth component u_0 as suggested by Adomian method is always identified by the given initial condition and the terms arising from

$L_x^{-1}\left(F(x,y)\right)$, both of which are assumed to be known .Accordingly , we set.

$$u_0(x,y) = g(y) + L_x^{-1}\left(F(x,y)\right) \qquad (1-41)$$

Consequently , the other components $u_{k+1}, k \geq 0$ are defined by using the relation

$$u_{k+1}(x,y) = -L_x^{-1}L_y(u_k), k \geq 0 \qquad (1-42)$$

Combining Equations (1-41) and (1-42) , we obtain the recursive scheme

$$\left.\begin{aligned} u_0(x,y) &= g(y) + L_x^{-1}(F(x,y)) \\ u_{k+1}(x,y) &= -L_x^{-1}L_y(u_k), k \geq 0 \end{aligned}\right\} \qquad (1-43)$$

That forms the basis for complete determination of the components $u_0, u_1, u_2, \ldots$.,the components can be easily obtained by.

$$\left.\begin{aligned} u_0(x,y) &= g(y) + L_x^{-1}(F(x,y) \\ u_1(x,y) &= -L_x^{-1}\left(L_y u_0(x,y)\right) \\ u_2(x,y) &= -L_x^{-1}(L_y u_1(x,y)) \\ u_3(x,y) &= -L_x^{-1}(L_y u_2(x,y)) \end{aligned}\right\} \qquad (1-44)$$

and so on. Thus the components u_n can be determined recursively as far we like .

CHAPTER TOW

VOLTERRA INTEGRAL EQUATIONS

2.1 Volterra Integral Equations

It was stated in this Chapter, that Volterra integral equations arise in many scientific applications such as the population dynamics, spread of epidemics, and semi-conductor devices. It was also shown that Volterra integral equations can be derived from initial value problems. Volterra started working on integral equations in 1884, but his serious study began in 1896. The name integral equation was given by du Bois-Reymond in 1888. However, the name Volterra integral equation was first coined by Lalesco in 1908.

Abel considered the problem of determining the equation of a curve in a vertical plane. In this problem, the time taken by a mass point to slide under the influence of gravity along this curve, from a given positive height, to the horizontal axis is equal to a prescribed function of the height. Abel derived the singular Abel's integral equation. a specific kind of Volterra integral equation. That will be studied in a forthcoming chapter.

Volterra integral equations, of the first kind or the second kind, are characterized by a variable upper limit of integration [1]. For the first kind Volterra integral equations, the unknown function $u(x)$ occurs only under the integral sign in the form:

$$f(x) = \int_{0}^{x} K(x.t)u(t)dt. \qquad (2-1)$$

However, Volterra integral equations of the second kind, the unknown function $u(x)$occurs inside and outside the integral sign. The second kind is represented in the form:

$$u(x) = f(x) + \lambda \int_0^x K(x.t)u(t)dt. \qquad (2-2)$$

The kernel $K(x,t)$ and the function$f(x)$ are given real-valued functions, and λ is a parameter.

A variety of analytic and numerical methods, such as successive approximations method, Laplace transform method, spline collocation method, Runge-Kutta method, and others have been used to handle Volterra integral equations. we apply the recently developed methods. namely, the Adomian decomposition method (ADM), the modified decomposition method (mADM), and the variational iteration method (VIM) to handle Volterra integral equations. Some of the traditional methods, namely, successive approximations method, series solution method, and the Laplace transform method will be employed as well. The emphasis will be on the use of these methods and approaches rather than proving theoretical concepts of convergence and existence. The theorems of uniqueness, existence, and convergence are important and can be found in the literature. The concern will be on the determination of the solution $u(x)$ of the Volterra integral equation of first and second kind.

2.2 Volterra Integral Equations of the Second Kind:

We will first study Volterra integral equations of the second kind given by

$$u(x) = f(x) + \lambda \int_{0}^{x} K(x.t)u(t)dt. \qquad (2-3)$$

The unknown function $u(x)$ that will be determined, occurs inside and outside the integral sign. The kernel $K(x,t)$ and the function $f(x)$ are given real-valued functions, and λ is a parameter. In what follows we will present the methods. new and traditional. that will be used.

2.3 Adomian Decomposition Method

The Adomian decomposition method (ADM) was introduced and developed by George Adomian in [34] and is well addressed in many references. A considerable amount of research work has been invested recently in applying tins method to awide class of linear and nonlinear ordinary differential equations, partial differential equations and integral equations as well.

The Adomian decomposition method consists of decomposing the unknown function $u(x)$ of any equations into a sum of an infinite number of components defined by the decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \qquad (2-4)$$

or equivalently

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots \qquad (2-5)$$

where the components $u_n(x)$, $n \geq 0$ are to be determined in a recursive manner. The decomposition method concerns itself with finding the components.

If the unknown function $u(x)$ appears both inside and outside the integral sign of Fredholm or Volterra equation, the integral equation is called a second kind Fredholm or Volterra equation integral equation respectively.

In all Fredholm or Volterra integral equations presented above, if $f(x)$ is identically zero, the resulting equation:

$$u(x) = \lambda \int_{0}^{b} K(x.t)u(t)dt \qquad (2-6)$$

or

$$u(x) = \lambda \int_{0}^{x} K(x.t)u(t)dt \qquad (2-7)$$

is called homogeneous Fredholm or homogeneous Volterra integral equation respectively.

It is interesting to point out that any equation that includes both integrals and derivatives of the unknown function $u(x)$ is called integro-differential equation. The Fredholrn *integro-differential* equation is of the form:

$$u^{(k)}(x) = f(x) + \lambda \int_{0}^{b} K(x,t)u(t)dt, \qquad u^{(k)} = \frac{d^k u}{dx^k}. \qquad (2-8)$$

However, the Volterra integro-differential equation is of the form:

$$u^{(k)}(x) = f(x) + \lambda \int_{0}^{x} K(x,t)u(t)dt, \qquad u^{(k)} = \frac{d^k u}{dx^k}. \qquad (2-9)$$

The integro-differential equations [6] will be defined and classified.

2.4 Classification of Integral Equations:

Integral equations appear in many types. The types depend mainly on the limits of integration and the kernel of the equation. we will be concerned on the following types of integral equations.

2.4.1 Fredholm Integral Equations

For Fredholm integral equations, the limits of integration are fixed. Moreover, the unknown function $u(x)$ may appear only inside integral equation in the form:

$$f(x) = \int_a^b K(x,t)u(t)dt. \qquad (2-10)$$

This is called Fredholm integral equation of the first kind. However, for Fredholm integral equations of the second kind, the unknown function $u(x)$ appears inside and outside the integral sign. The second kind is represented by the form:

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt. \qquad (2-11)$$

Examples of the two kinds are given by:

$$\frac{\sin x - x\cos x}{x^2} = \int_0^1 \sin(xt)\, u(t)dt, \qquad (2-12)$$

and

$$u(x) = x + \frac{1}{2}\int_{-1}^{1} (x-t)u(t)dt, \qquad (2-13)$$

Respectively.

2.4.2 Volterra Integral Equations

In Volterra integral equations, at least one of the limits of integration is a variable. For the first kind Volterra integral equations, the unknown function $u(x)$ appears only inside integral sign in the form:

$$f(x) = \int_0^x k(x,t)u(t)dt, \qquad (2-14)$$

However, Volterra integral equations of the second kind, the unknown function $u(x)$ appears inside and outside the integral sign. The second kind is represented by the form:

$$u(x) = f(x) + \lambda \int_0^x K(x,t)u(t)dt, \qquad (2-15)$$

Examples of the Volterra integral equations of the first kind are

$$xe^{-x} = \int_0^x e^{t-x}\ u(t)dt, \qquad (2-16)$$

and

$$5x^2 + x^3 = \int_0^x (5 + 3x - 3t)\, u(t)dt. \qquad (2-17)$$

However, examples of the Volterra integral equations of the second kind are:

$$u(x) = 1 - \int_0^x u(t)dt, \qquad (2-18)$$

and

$$u(x) = x + \int_0^x (x - t)\, u(t)dt. \qquad (2-19)$$

2.4.3 Volterra-Fredholm Integral Equations

The Volterra-Fredholm integral equations [34] arise from parabolic boundary value problems, from the mathematical modelling of the development of an epidemic, and from various physical and biological models. The Volterra-Fredholm integral equations appear in the literature in two forms, namely

$$u(x) = f(x) + \lambda_1 \int_0^x K_1(x,t)u(t)dt + \lambda_2 \int_0^b K_2(x,t)u(t)dt, \quad (2-20)$$

and

$$u(x,t) = f(x,t) + \lambda \int_0^t \int_\Omega F\big(x,t,\xi,\mathcal{T},u(\xi,\mathcal{T})\big)d\xi d\mathcal{T}, (x,t) \in \Omega \times (0,T), \quad (2-21)$$

where $f(x,t)$ and $F\big(x,t,\xi,\mathcal{T},u(\xi,\mathcal{T})\big)$ are analytic functions on $\mathrm{D} = \Omega \times (0,\mathrm{T})$and Ω is a closed subset of $\mathbb{R}^{\mathrm{n}}, \mathrm{n} = 1,2.3$. It is interesting to note that (2-20) contains disjoint Volterra and Fredholm integral equations, whereas (2-21) contains mixed Volterra and Fredholm integral equations. Moreover, the unknown functions $u(x)$and $u(x,t)$ appear inside and outside the integral signs. This is a characteristic feature of a second kind integral equation. If the unknown functions appear only inside the integral signs, The resulting equations are of first kind, but will not be examined. Examples of the two types are given by

$$u(x) = 6x + 3x^2 + 2 - \int_0^x xu(t)dt - \int_0^1 tu(t)dt, \quad (2-22)$$

and

$$u(x,t) = x + t^3 + \frac{1}{2}t^2 - \frac{1}{2}t - \int_0^t \int_{\Omega} (\mathcal{T} - \xi) d\xi d\mathcal{T}. \qquad (2-23)$$

2.4.4 Singular Integral Equations

Volterra integral equations of the first kind [4.7]:

$$f(x) = \lambda \int_{g(x)}^{h(x)} k(x,t)u(t)dt \qquad (2-24)$$

or of the second kind:

$$u(x) = f(x) + \int_{g(x)}^{h(x)} k(x,t)u(t)dt \qquad (2-25)$$

are called singular if one of the limits of integration $g(x), h(x)$ or both are infinite. Moreover, the previous two equations are called singular if the kernel $K(x,t)$ becomes unbounded at one or more points in the interval of integration. we will focus our concern on equations of the form:

$$f(x) = \int_0^x \frac{1}{(x-t)^{\alpha}} u(t)dt, \quad 0 < \alpha < 1, \qquad (2-26)$$

or of the second kind:

$$u(x) = f(x) + \int_0^x \frac{1}{(x-t)^{\alpha}}\, u(t)dt, \quad 0 < \alpha < 1, \qquad (2-27)$$

The last two standard forms are called *generalizedAbel'sintegral equation* and weakly singular integral equations respectively. For $\alpha = \frac{1}{2}$, the equation:

$$f(x) = \int_0^x \frac{1}{\sqrt{(x-t)}}\, u(t)dt \qquad (2-28)$$

is called the Abel's singular integral equation. It is to be noted that the kernel in each equation becomes infinity at the upper limit $t = x$. Examples of Abel's integral equation. generalized Abel's integral equation, and the weakly singular integral equation are given by:

$$\sqrt{x} = \int_0^x \frac{1}{\sqrt{(x-t)}}\, u(t)dt, \qquad (2-29)$$

$$x^3 = \int_0^x \frac{1}{\sqrt{(x-t)^{\frac{1}{3}}}}\, u(t)dt, \qquad (2-30)$$

and

$$u(x) = 1 + \sqrt{x} + \int_0^x \frac{1}{\sqrt{(x-t)^{1/3}}}\, u(t)dt, \quad (2-31)$$

respectively.

2.5 Classification of Integro-Differential Equations:

Integro-differential equations appear in many scientific applications, especially when we convert initial value problems or boundary value problems to integral equations, the integro- differential equations contain both integral

$u_0, u_1, u_2, u_3, \ldots$individually. As will be seen, the determination of these components can be achieved in an easy way through a recurrence relation that usually involves simple integrals that can be easily evaluated.

To establish the recurrence relation, we substitute (2-4) into the Volterra integral equation (2-3) to obtain

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_0^x K(x,t)\left(\sum_{n=0}^{\infty} u_n(t)\right)dt, \qquad (2-32)$$

or equivalently

$$u_0(x) + u_1(x) + u_2(x) + \cdots$$
$$= f(x) + \lambda \int_0^x K(x,t)[u_0(t) + u_1(t) + \cdots]dt, \qquad (2-33)$$

The zeroth component $u_0(x)$ is identified by all terms that are not included under the integral sign. Consequently, the components $u_j(x), j \geq 1$ of the unknown function $u(x)$are completely determined by setting the recurrence relation:

$$u_0(x) = f(x),$$

$$u_{n+1}(x) = \lambda \int_0^x K(x,t)u_n(t)dt, \qquad n \geq 0, \qquad (2-34)$$

that is equivalent to

$$u_0(x) = f(x), \quad u_1(x) = \lambda \int_0^x K(x,t)u_0(t)dt,$$

$$u_2(x) = \lambda \int_0^x K(x,t)u_1(t)dt, u_3(x) = \lambda \int_0^x K(x,t)u_2(t)dt \quad (2-35)$$

and so on for other components.

In view of (2-35), the components $u_0(x), u_1(x), u_2(x), u_3(x), \ldots$ are completely determined. As a result, the solution $u(x)$ of the Volterra integral equation (2-7) in a series form is readily obtained by using the series assumption in (2-1).

It is clearly seen that the decomposition method converted the integral equation into an elegant determination of computable components. It was formally shown by many researchers that if an exact solution exists for the problem, then the obtained series converges very rapidly to that solution. The convergence concept of the decomposition series was thoroughly investigated by many researchers to confirm the rapid convergence of the resulting series. However, for concrete problems. Where a closed form solution is not obtainable, a truncated number of terms is usually used for numerical purposes. The more components we use tile higher accuracy we obtain.

2.6 Examples:

Example 2.6.1:

Solve the following Volterra integral equation:

$$u(x) = 1 - \int_0^x u(t)\,dt \qquad (2-36)$$

We notice that $f(x) = 1, \lambda = -1, K(x,t) = 1$. Recall that the solution $u(x)$ is assumed to have a series form given in (2-4) Substituting the decomposition series (2-4) into both sides of (2-36) gives.

$$\sum_{n=0}^{\infty} u_n(x) = 1 - \int_0^x \sum_{n=0}^{\infty} u_n(t)\,dt, \qquad (2-37)$$

or equivalently

$$u_0(x) + u_1(x) + u_2(x) + \cdots$$
$$= 1 - \int_0^x [u_0(t) + u_1(t) + u_2(t) + \cdots]\,dt \qquad (2-38)$$

We identify the zeroth component by all terms that are not included under the integral sign. Therefore, we obtain the following recurrence relation:

$$u_0(x) = 1,$$

$$u_{k+1}(x) = -\int_0^x u_k(t)\,dt, \quad k \geq 0, \qquad (2-39)$$

so that

$$u_0(x) = 1,$$

$$u_1(x) = -\int_0^x u_0(t)dt = -\int_0^x 1dt = -x,$$

$$u_2(x) = -\int_0^x u_1(t)dt = -\int_0^x (-t)dt = \frac{1}{2!}x^2,$$

$$u_3(x) = -\int_0^x u_2(t)dt = -\int_0^x \frac{1}{2!}t^2\,dt = -\frac{1}{3!}x^3,$$

$$u_4(x) = -\int_0^x u_3(t)dt = -\int_0^x \frac{1}{3!}t^3\,dt = -\frac{1}{4!}x^4. \quad (2-40)$$

and so on. Using (2-4) gives the series solution:

$$u(x) = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots, \qquad (2-41)$$

that converges to the closed form solution:

$$u(x) = e^{-x} \qquad (2-42)$$

Example 2.6.2

Solve the following Volterra integral equation:

$$u(x) = 1 + \int_0^x (t-x)u(t)dt. \qquad (2-43)$$

We notice that $f(x) = 1,\ \lambda = 1, K(x,t) = t - x$. Substituting the deconiposition series (2-4) into both sides of (2-34) gives:

$$\sum_{n=0}^{\infty} u_n(x) = 1 + \int_0^x \sum_{n=0}^{\infty} (t-x)\,u_n(t)dt, \qquad (2-44)$$

or equivalently.

$$u_0(x) + u_1(x) + u_2(x) + \ldots$$
$$= 1 + \int_0^x (t - x)[u_0(t) + u_1(t) + u_2(t) + \cdots]dt. \quad (2-45)$$

Proceeding as before we set the following recurrence relation:

$$u_0(x) = 1,$$

$$u_{k+1}(x) = \int_0^x (t - x)\, u_k(t)dt, \quad k \geq 0, \qquad (2-46)$$

that gives

$$u_0(x) = 1,$$

$$u_1(x) = \int_0^x (t - x)\, u_0(t)dt = \int_0^x (t - x)dt = -\frac{1}{2!}x^2,$$

$$u_2(x) = \int_0^x (t - x)\, u_1(t)dt = \frac{1}{2!}\int_0^x (t - x)t^2 dt = \frac{1}{4!}x^4,$$

$$u_3(x) = \int_0^x (t - x)\, u_2(t)dt = \frac{1}{4!}\int_0^x (t - x)t^4 dt = -\frac{1}{6!}x^6,$$

$$u_4(x) = \int_0^x (t - x)\, u_3(t)dt = -\frac{1}{6!}\int_0^x (t - x)t^6 dt = \frac{1}{8!}x^8, \quad (2-47)$$

and so on. The solution in a series form is given by

$$u(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \cdots, \qquad (2-48)$$

and in aclosed form by

$$u(x) = cosx, \qquad (2-49)$$

obtained upon using the Taylor expansion for $cos\ x$.

Example 2.6.3

Solve the following Volterra integral equation:

$$u(x) = 1 - x - \frac{1}{2}x^2 - \int_0^x (t-x)\, u(t)dt. \qquad (2-50)$$

Notice that $f(x) = 1 - x - \frac{1}{2}x^2, \lambda = -1, K\ (x,t) = t - x$. Substituting the decomposition series (2-4) into both sides of (2-50) gives

$$\sum_{n=0}^{\infty} u_n(x) = 1 - x - \frac{1}{2}x^2 - \int_0^x \sum_{n=0}^{\infty} (t-x)u_n(t)dt, \qquad (2-51)$$

or equivalently

$$u_0(x) + u_1(x) + u_2(x) + \cdots$$
$$= 1 - x - \frac{1}{2}x^2 - \int_0^x (t-x)[u_0(t) + u_1(t) + \cdots]dt\,. (2-52)$$

This allows us to set the following recurrence relation:

$$u_0(x) = 1 - x - \frac{1}{2}x^2,$$

$$u_{k+1}(x) = -\int_0^x (t-x)\, u_k(t)dt, \qquad k \geq 0, \qquad (2-53)$$

that gives

$$u_0(x) = 1 - x - \frac{1}{2}x^2,$$

$$u_1(x) = -\int_0^x (t-x)\,u_0(t)dt = \frac{1}{2!}x^2 - \frac{1}{3!}x^3 - \frac{1}{4!}x^4,$$

$$u_2(x) = -\int_0^x (t-x)\,u_1(t)dt = \frac{1}{4!}x^4 - \frac{1}{5!}x^5 - \frac{1}{6!}x^6,$$

$$u_3(x) = -\int_0^x (t-x)\,u_2(t)dt = \frac{1}{6!}x^6 - \frac{1}{7!}x^7 - \frac{1}{8!}x^8,$$

and so on. The solution in a series form is given by

$$u(x) = 1 - \left(x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \cdots\right), \qquad (2-54)$$

and in a closed form by

$$u(x) = 1 - sinhx, \qquad (2-55)$$

obtained upon using the Taylor expansion for $\sinh x$.

Example 2.6.4

We consider here the Volterra integral equation:

$$u(x) = 5x^3 - x^3 + \int_0^x tu(t)dt. \qquad (2-56)$$

Identifying the zeroth component $u_0(x)$ by the first two terms that are not included under the integral sign, and using the ADM we set the recurrence relation as

$$u_0(x) = 5x^3 - x^5,$$

$$u_{k+1}(x) = \int_0^x t u_k(t)dt, k \geq 0. \qquad (2-57)$$

This in turns gives:

$$u_0(x) = 5x^3 - x^5,$$

$$u_1(x) = \int_0^x t u_0(t)dt, x^5 - \frac{1}{7}x^7,$$

$$u_2(x) = \int_0^x t u_1(t)dt, = \frac{1}{7}x^7 - \frac{1}{63}x^9,$$

$$u_3(x) = \int_0^x t u_1(t)dt, = \frac{1}{63}x^9 - \frac{1}{693}x^{11}, \qquad (2-58)$$

The solution in a series form is given by

$$u(x) = (5x^3 - x^5) + \left(x^5 - \frac{1}{7}x^7\right) + \left(\frac{1}{7}x^7 - \frac{1}{63}x^9\right) + \left(\frac{1}{63}x^9 - \frac{1}{693}x^{11}\right) + \cdots \quad (2-59)$$

We can easily notice the appearance of identical terms with opposite signs. Such terms are called noise terms that will be discussed later. Canceling the identical terms with opposite signs gives the exact solution

$$u(x) = 5x^3, \qquad (2-60)$$

that satisfies the Volterra integral equation (2-56).

<u>Example 2.6.5</u>

We now consider the Volterra integral equation:

$$u(x) = x + x^4 + \frac{1}{2}x^2 + \frac{1}{5}x^5 - \int_0^x u(t)dt, \qquad (2-61)$$

identifying the zeroth component $u_0(x)$ by the first four terms that are not included under the integral sign, and using the ADM we set the recurrence relation as

$$u_0(x) = x + x^4 + \tfrac{1}{2}x^2 + \tfrac{1}{5}x^5,$$

$$u_{k+1}(x) = -\int_0^x u_k(t)dt,\ k \geq 0. \qquad (2-62)$$

This in turn gives

$$u_0(x) = x + x^4 + \frac{1}{2}x^2 + \frac{1}{5}x^5,$$

$$u_1(x) = -\int_0^x u_0(t)dt = -\frac{1}{2}x^2 - \frac{1}{5}x^5 - \frac{1}{30}x^6,$$

$$u_2(x) = -\int_0^x u_1(t)dt = \frac{1}{6}x^3 + \frac{1}{30}x^6 + \frac{1}{24}x^4 + \frac{1}{210}x^7,$$

$$u_3(x) = -\int_0^x u_2(t)dt = -\frac{1}{24}x^4 - \frac{1}{210}x^7 + \cdots \qquad (2-63)$$

The solution in a series form is given by

$$u(x) = \left(x + x^4 + \frac{1}{2}x^2 + \frac{1}{5}x^5\right) - \left(\frac{1}{2}x^2 + \frac{1}{5}x^5 + \frac{1}{6}x^3 + \frac{1}{30}x^6\right)$$

$$+ \left(\tfrac{1}{6}x^3 + \tfrac{1}{30}x^6 + \tfrac{1}{24}x^4 + \tfrac{1}{210}x^7\right) - \left(\tfrac{1}{24}x^4 + \tfrac{1}{210}x^7 + \cdots\right) + \cdots \quad (2-64).$$

We can easily notice the appearance of identical terms with opposite signs. This phenomenon of such terms is called noise terms phenomenon that will be presented later. Canceling the identical terms with opposite terms gives the exact solution

$$u(x) = x + x^4. \qquad (2-65)$$

<u>Example 2.6.6</u>

We finally solve the Volterra integral equation:

$$u(x) = 2 + \frac{1}{3}\int_0^x xt^3 u(t)dt. \qquad (2-66)$$

Proceeding as before we set the recurrence relation

$$u_0(x) = 2, u_{k+1}(x) = \frac{1}{3}\int_0^x xt^3 u_k(t)dt, \quad k \geq 0. \qquad (2-67)$$

This in turn gives

$$u_0(x) = 2,$$

$$u_1(x) = \frac{1}{3}\int_0^x xt^3 u_0(t)dt = \frac{1}{6}x^5,$$

$$u_2(x) = \frac{1}{3}\int_0^x xt^3 u_1(t)dt = \frac{1}{162}x^{10}, \qquad (2-68)$$

$$u_3(x) = \frac{1}{3}\int_0^x xt^3 u_2(t)dt = \frac{1}{6804}x^{15},$$

$$u_4(x) = \frac{1}{3}\int_0^x xt^3 u_3(t)dt = \frac{1}{387828}x^{20},$$

and so on. The solution in a series form is given by

$$u_2(x) = 2 + \frac{1}{6}x^5 + \frac{1}{6.3^3}x^{10} + \frac{1}{6.3^4.14}x^{15} + \frac{1}{6.3^5.14.19}x^{20} + \cdots \quad (2-69)$$

It seems that an exact solution is not obtainable. The obtained series solution can be used for numerical purposes. The more components that we determine the higher accuracy level that we can achieve.

2.7 Systems of Volterra Integral Equations

2.7.1 Introduction:

Systems of integral equations, linear or nonlinear, appear in scientific applications in engineering, physics, chemistry and populations growth models [34]. Studies of systems of integral equations have attracted much concern in applied sciences. The general ideas and the essential features of these systems are of wide applicability.

The systems of Volterra integral equations appear in two kinds. For systems of Volterra integral equations of the first kind, the unknown functions appear only under integral sign in the form:

$$f_1(x) = \int_0^x \left(K_1(x,t)u(t) + \tilde{K}_1(x,t)v(t) + \cdots\right)dt,$$

$$f_2(x) = \int_0^x \left(K_2(x,t)u(t) + \tilde{K}_2(x,t)v(t) + \cdots\right)dt, \qquad (2-70)$$

$$\vdots$$

However, systems of Volterra integral equations of the second kind, the unknown functions appear inside and outside the integral sign of the form:

$$u(x) = f_1(x) + \int_0^x \left(K_1(x,t)u(t) + \widetilde{K}_1(x,t)v(t) + \cdots\right)dt,$$

$$v(x) = f_2(x) + \int_0^x \left(K_2(x,t)u(t) + \widetilde{K}_2(x,t)v(t) + \cdots\right)dt, \quad (2-71)$$

$\vdots$

The kernels $K_i(x,t)$ and $\widetilde{K}_i(x,t)$, and the functions $f_i(x), i = 1,2,\dots,n$ are given real-valued functions.

A variety of analytical and numerical methods are used to handle systems of Volterra integral equations. The existing techniques encountered some difficulties in terms of the size of computational work, especially when the system involves several integral equations. To avoid the difficulties that usually arise from the traditional methods, we will use some of the methods presented. The Adomian decomposition method. the Variational iteration method, and the Laplace transform method will form a reasonable basis for studying systems of integral equations. The emphasis will be on the use of these methods rather than proving theoretical concepts of convergence and existence that can be found.

2.7.2 Systems of Volterra Integral Equations of the Second Kind:

We will first study systems of Volterra integral equations of the second kind given:

$$u(x) = f_1(x) + \int_0^x \left(K_1(x,t)u(t) + \widetilde{K}_1(x,t)v(t) + \cdots\right)dt,$$

$$v(x) = f_2(x) + \int_0^x \left(K_2(x,t)u(t) + \widetilde{K}_2(x,t)v(t) + \cdots\right)dt, \quad (2-72)$$

The unknown functions $u(x), v(x), \ldots$, that will be determined, appear inside and outside the integral sign. The kernels $K_i(x,t)$ and $\widetilde{K}_i(x,t)$, and the function $f_i(x)$ are given real-valued functions. In what follows we will present the methods, new and traditional, that will be used to handle these systems.

2.7.3 Adomian Decomposition Method:

The Adomian decomposition method [34] was presented before. The method decomposes each solution as an infinite sum of components, where these components are determined recurrently. This method can be used in its standard form, or combined with the noise terms phenomenon. Moreover, the modified decomposition method will be used wherever it is appropriate. It is interesting to point out that the VIM method can also be used, but we need to transform the system of integral equations to a system of integro-differential equations that will be presented later.

2.7.4 Examples

Example 1

Use the Adomian decomposition method to solve the following system of Volterra integral equations:

$$u(x) = x - \frac{1}{6}x^4 + \int_0^x \left((x-t)^2 u(t) + (x-t)v(t)\right)dt,$$

$$v(x) = x^2 - \frac{1}{12}x^5 + \int_0^x \left((x-t)^3 u(t) + (x-t)^2 v(t)\right)dt, \quad (2-73)$$

The Adomian decomposition method suggests that the linear terms $u(x)$ and $v(x)$ be decomposed by an infinite series of components

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \quad v(x) = \sum_{n=0}^{\infty} u_n(x) \qquad (2-74)$$

where $u_n(x)$ and $v_n(x), n \geq 0$ are the components of $u(x)$ and $v(x)$ that will be elegantly determined in a recursive manner.

Substituting (2-74) into (2-73) gives

$$\sum_{n=0}^{\infty} u_n(x) = x - \frac{1}{6}x^4 + \int_0^x \left((x-t)^2 \sum_{n=0}^{\infty} u_n(t) + (x-t) \sum_{n=0}^{\infty} v_n(t) \right) dt,$$

$$\sum_{n=0}^{\infty} v_n(x) = x^2 - \frac{1}{12}x^5 + \int_0^x \left((x-t)^3 \sum_{n=0}^{\infty} u_n(t) + (x-t)^2 \sum_{n=0}^{\infty} v_n(t) \right) dt \quad ,(2-75)$$

The zeroth components $u_0(x)$ and $v_0(x)$ are defined by all terms that are not included under the integral sign. Following Adomian analysis. The system (2-75) is transformed into a set of recursive relations given by

$$u_0(x) = x - \frac{1}{6}x^4,$$

$$u_{k+1}(x) = \int_0^x \left((x-t)^2 u_k(t) + (x-t) v_k(t) \right) dt, \qquad k \geq 0 \quad (2-76)$$

and

$$v_0(x) = x^2 - \frac{1}{12}x^5,$$

$$v_{k+1}(x) = \int_0^x \left((x-t)^3 u_k(t) + (x-t)^2 v_k(t) \right) dt, \qquad k \geq 0 \quad (2-77)$$

This in turn gives

$$u_0(x) = x - \frac{1}{6}x^4,\ u_1(x) = \frac{1}{6}x^4 - \frac{1}{280}x^7, \qquad (2-78)$$

and

$$v_0(x) = x^2 - \frac{1}{12}x^5,\ v_1(x) = \frac{1}{12}x^5 - \frac{11}{10080}x^8, \quad (2-79)$$

It is obvious that the noise terms $\mp\frac{1}{6}x^4$ appear between $u_0(x)$ and $u_1(x)$. Moreover, the noise terms $\mp\frac{1}{12}x^5$ appear between $v_0(x)$ and $v_1(x)$. By canceling these noise terms from $u_0(x)$ and $v_0(x)$. The non-canceled terms of $u_0(x)$ and $v_0(x)$give the exact solutions

$$\big(u(x), v(x)\big) = (x, x^2) \qquad (2-80)$$

Example 2

We use the Adomian decomposition method to solve the following system of Volterra integral equations

$$u(x) = \cos x - x\sin x + \int_0^x (\sin(x-t)\,u(t) + \cos(x-t)v(t))\,dt,$$

$$v(x) = \sin x - x\cos x + \int_0^x (\cos(x-t)\,u(t) - \sin(x-t)v(t))\,dt, \quad (2-81)$$

We first decompose the linear terms $u(x)$ and $v(x)$ by an infinite series of components

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \quad v(x) = \sum_{n=0}^{\infty} v_n(x) \qquad (2-82)$$

where $u_n(x)$ and $v_n(x), n \geq 0$are the components of $u(x)$ and $v(x)$ that will be elegantly determined in a recursive manner.

Substituting (2-82) into (2-81) gives

$$\sum_{n=0}^{\infty} u_n(x) = \cos x - x \sin x + \int_0^x \left(\sin(x-t) \sum_{n=0}^{\infty} u_n(t) + \cos(x-t) \sum_{n=0}^{\infty} v_n(t) \right) dt,$$

$$\sum_{n=0}^{\infty} v_n(x) = \sin x - x \cos x + \int_0^x \left(\cos(x-t) \sum_{n=0}^{\infty} u_n(t) + \sin(x-t) \sum_{n=0}^{\infty} v_n(t) \right) dt, \quad (2-83)$$

The zeroth components $u_0(x)$ and $v_0(x)$ are defined by all terms that are not included under the integral sign. For this example, we will use the modified decomposition method, therefore we set the recursive relation

$$u_0(x) = cosx,$$

$$u_{k+1}(x) = -x \sin x + \int_0^x (\sin(x-t) u_k(t) + \cos(x-t) v_k(t))\, dt, k \geq 0 \qquad (2-84)$$

and

$$v_0(x) = \sin x\,,$$

$$v_{k+1}(x) = xcos + \int_0^x (\cos(x-t)u_k(t) + \sin(x-t)v_k(t))\, dt, k \geq 0 \quad (2-85)$$

This in turn gives

$$u_0(x) = \cos x, u_1(x) = 0, u_{x+1}(x) = 0, k \geq 1, \qquad (2-86)$$

and

$$v_0(x) = \sin x, v_1(x) = 0, v_{k+1}(x) = 0, k \geq 1, \qquad (2-87)$$

This gives the exact solutions

$$\big(u(x), v(x)\big) = (cosx\,, sinx), \qquad (2-88)$$

that satisfy the system (2-81).

Example 3

We use the Adomian decomposition method to solve the following system of Volterra integral equations

$$u(x) = 1 + x^2 - \frac{1}{3}x^3 - \frac{1}{3}x^4 + \int_0^x \big((x-t)^3 u(t) + (x-t)^2 v(t)\big)dt,$$

$$v(x) = 1 + x - x^3 - \frac{1}{4}x^4 - \frac{1}{4}x^5 + \int_0^x \big((x-t)^4 u(t) + (x-t)^3 v(t)\big)dt\,, \quad (2\text{-}89)$$

Proceeding as before we find

$$\sum_{n=0}^{\infty} u_n(x) = 1 + x^2 - \frac{1}{3}x^3 - \frac{1}{3}x^4 + \int_0^x \left((x-t)^3 \sum_{n=0}^{\infty} u_n(t) + (x-t)^2 \sum_{n=0}^{\infty} v_n(t) \right) dt,$$

$$\sum_{n=0}^{\infty} v_n(x) = 1 + x - x^3 - \frac{1}{4}x^4 - \frac{1}{4}x^5 + \int_0^x \left((x-t)^4 \sum_{n=0}^{\infty} u_n\,(t) + (x-t)^3 \sum_{n=0}^{\infty} v_n\,(t) \right) dt, \quad (2-90)$$

We next set the recursive relations

$$u_0(x) = 1 + x^2 - \frac{1}{3}x^3 - \frac{1}{3}x^4,$$

$$u_{k+1}(x) = \int_0^x \big((x-t)^3 u_k(t) + (x-t)^2 v_k(t)\big)dt,\ k \geq 0, \quad (2-91)$$

and

$$v_0(x) = 1 + x - x^3 - \tfrac{1}{4}x^4 - \tfrac{1}{4}x^5,$$

$$v_{k+1}(x) = \int_0^x \big((x-t)^4 u_k(t) + (x-t)^3 v_k(t)\big)dt, \quad k \geq 0, \quad (2-92)$$

This in turn gives

$$u_0(x) = 1 + x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{3}x^4,$$

$$u_1(x) = \frac{1}{3}x^3 + \frac{1}{3}x^4 + \cdots, \qquad (2-93)$$

and

$$v_0(x) = 1 + x - x^3 - \frac{1}{4}x^4 - \frac{1}{4}x^5,$$

$$v_1(x) = \frac{1}{4}x^4 + \frac{1}{4}x^5 + \cdots, \qquad (2-94)$$

It is obvious that the noise terms $\mp\frac{1}{3}x^3$and $\mp\frac{1}{3}x^4$ appear between $u_0(x)$ and $u_1(x)$. Moreover, the noise terms$\mp\frac{1}{4}x^4$ and $\mp\frac{1}{4}x^5$appear between $v_0(x)$ and $v_1(x)$. By canceling these noise terms from $u_0(x)$ and$v_0(x)$, the non-canceled terms of $u_0(x)$ and $v_0(x)$ give the exact solutions

$$\left(u(x), v(x)\right) = (1 + x^2, 1 + x - x^3). \qquad (2-95)$$

that satisfy the given system (2-89).

Example 4

We use the Adomian decomposition method to solve the following system of Volterra integral equations

$$u(x) = e^x - 2x + \int_0^x \left(e^{-t}u(t) + e^t v(t)\right)dt,$$

$$v(x) = e^{-x} + \sin h\, 2x + \int_0^x \left(e^t u(t) + e^{-t} v(t)\right)dt, \qquad (2-96)$$

Proceeding as before we find

$$\sum_{n=0}^{\infty} u_n(x) = e^x - 2x + \int_0^x \left(e^{-t}\sum_{n=0}^{\infty} u_n(t) + e^t \sum_{n=0}^{\infty} v_n(t)\right)dt,$$

$$\sum_{n=0}^{\infty} v_n(x) = e^{-x} + \sin h\, 2x$$

$$+ \int_0^x \left(e^t \sum_{n=0}^{\infty} u_n(t) + e^{-t} \sum_{n=0}^{\infty} v_n(t)\right)dt, \qquad (2-97)$$

Proceeding as before we set

$$u_0(x) = e^x - 2x, u_1(x) = 2x + \cdots, \qquad (2-98)$$

and

$$v_0(x) = e^{-x} + \sin h\, 2x, v_1(x) = -\sin h\, 2x + \cdots, \quad (2-99)$$

By canceling the noise terms from $u_0(x)$ and $v_0(x)$, the non-canceled terms of $u_0(x)$ and $v_0(x)$ give the exact solutions

$$\big(u(x), v(x)\big) = (e^{x}, e^{-x}), \qquad (2-100)$$

that satisfy the given system (2-96).

CHAPTER THREE

NON LINEAR INTEGRAL EQUATION

3.1 Introduction

We first define a nonlinear integral equation in general , and then cite some particular types of nonlinear integral equations .In general , a nonlinear integral equation is defined as given in the following equation:

$$u(x) = F(x) + \lambda \int_{0}^{x} K(x,t)F(u(t))dt \qquad (3.1)$$

and

$$u(x) = F(x) + \lambda \int_{a}^{b} K(x,t)F(u(t))dt \qquad (3.2)$$

Equations (3.1) and (3.2) are called nonlinear Volterra integral equation and nonlinear Fredholm integral equations , respectively , The function $F(u,(x))$ is nonlinear except $F = a$ const or $F(u(x)) = u(x)$ in which case F is linear.

If $F\big(u(x)\big) = u^n(x),\ for\ n \geq 2$ then the F function in nonlinear.

To clarify this point we cite a few examples below :

$$u(x) = x + \lambda \int_{0}^{x} (x-t)u^2(t)dt \qquad (3.3)$$

$$u(x) = x + \lambda \int_{0}^{1} \cos x\, u^3(t)dt \qquad (3.4)$$

Equation (3.3) and (3.4) are nonlinear Volterra and Fredholm integral equations respectively.

(3.2) The Method of Successive Approximations:

We shall consider three approximate method.

The first one is the Picard's method to obtain successive algebraic approximations . By putting numbers in these, we generally get excellent numerical results . Unfortunately, the method can only be applied to a timited class of equations. In which the successive integration be easily performed.

The second method is the Adomion decomposition method .The method was applied to ordinary and partial differential equations, and was rarely used for integral equations.

The decomposition method can be successfully applied linear and nonlinear integral equations.

The third method , which is extremlly numerical and of much more general applications, is due to Runge with proper precautions it gives good arithmetical calculation.

3.3 Picard's Method of Successive Approximations:

Consider the initial value problem given by the first order nonlinear differential equation $\frac{du}{dx} = F\big(x, u(x)\big)$ with the initial condition $u(a) = b \ at \ x = a$. This initial value problem can be transformed to the nonlinear integral equation and is written as.

$$u(x) = b + \int_a^x F\big(x, u(x)\big)\, dx \qquad (3.5)$$

For a first approximation , we replace the $u(x)$ in $F(x), u(x)$ by b , for a second approximation , we replace it by the first approximation , for the third the second , and soon.

We demonstrate this method by examples .

Example 3.1:

Consider the first order nonlinear differential equation $\frac{du}{dx} = x + u^2$, where $u(0) = 0$ when $x = 0$ determine the approximate analytical solution by Picard's method.

The given differential equation can be written in integral equation form as

$$u(x) = \int_0^x (x + u^2(x)) \ \ dx \qquad (3.6)$$

Zeroth approximation is $u(x) = 0$

First approximation : put $u(x) = 0 \ in \ x + \ u^2$, yielding.

$$u(x) = \int_0^x x \ \ dx = \frac{1}{2}\, x^2 \qquad (3.7)$$

Second approximation : put $u(x) = \frac{x^2}{2}\ in\ x + u^2$, yielding

$$u\ (x) = \int_0^x \left(x + \frac{x^4}{4}\right) dx = \frac{x^2}{2} + \frac{x^5}{20} \qquad (3.8)$$

Third approximation : put $u = \frac{x^2}{2} + \frac{x^5}{20}$ in $x + u^2$, giving

$$u(x) = \int_0^x \left\{ x + \left(\frac{x^2}{2} + \frac{x^5}{20}\right)^2 \right\} dx =$$

Third approximation : put $u = \frac{x^2}{2} + \frac{x^5}{20}$ in $x + u^2$, giving

$$u(x) = \int_0^x \left\{ x + \left(\frac{x^2}{2} + \frac{x^5}{20}\right)^2 \right\} dx =$$

$$\int_0^x \left\{ x + \left(\frac{x^2}{4} + \frac{x^5}{400}\right)^2 \right\} dx =$$

$$\frac{x^2}{2} + \frac{x^5}{20} + \frac{x^8}{160} + \frac{x^{11}}{4400} \qquad (3.9)$$

Fourth approximation :

$$u(x) = \frac{x^2}{2} + \frac{x^5}{20} + \frac{x^8}{160} + \frac{x^{11}}{4400} \qquad (3.10)$$

And so on .This is the solution of the problem in series form ,and it seems from its appearance the series is convergent.

Example 3.2:

Find the solution of the coupled first order nonlinear differential equations by converting them to nonlinear integral equations

$$\frac{du}{dx} = v \quad , \qquad \frac{du}{dx} = x^3 \,(u + v)$$

Subject to the initial condition $u(0) = 1$ and $v(0) = \frac{1}{2}$ when $x = 0$.

The coupled differential equations can be written in the form of coupled integral equations as follows:

$$u(x) = 1 + \int_0^x dx \qquad (3.11)$$

$$v(x) = \frac{1}{2} + \int_0^x x^3 \,(u + v)dx \qquad (3.12)$$

First approximation :

$$u(x) = 1 + \frac{x}{2} \qquad (3.13)$$

$$u(x) = \frac{1}{2} + \frac{3x^4}{8} \qquad (3.14)$$

Second approximation :

$$u(x) = 1 + \frac{x}{2} + \frac{3x^4}{8} \qquad (3.15)$$

$$v\,(x) = \frac{1}{2} + \frac{3x^4}{8} + \frac{3x^5}{10} + \frac{3x^8}{64} \qquad (3.16)$$

Third approximation :

$$u(x) = 1 + \frac{x}{2} + \frac{3x^5}{40} + \frac{x^6}{60} + \frac{x^9}{192} \qquad (3.17)$$

$$v(x) = \frac{1}{2} + \frac{3x^4}{8} + \frac{x^5}{10} + \frac{3x^8}{64} + \frac{7x^4}{360} + \frac{x^{12}}{256} \qquad (3.18)$$

and so on .Thus , the solution is given by the above expressions up to third order.

Example (3.3) :

Find the solution of the nonlinear second order ordinary differential equation

$$u(x)\grave{\grave{u}}\,(x) = (\grave{u}(x))^2$$

With the initial conditions $u(0) = 1$ and $\grave{u}\,(0) = 1$ at $x = 0$ by converting it to the integral equation.

The given equation can be transformed into a couple of first order differential equations

$$u(x) = 1 + \int_0^x v(\,x)dx \qquad (3.19)$$

$$v(x) = 1 + \int_0^x \frac{v^2}{u} dx \qquad (3.20)$$

By Picard's successive approximation method we obtain.

First approximation:

$$u(x) = 1 + \int_0^x dx = 1 + x \qquad (3.21)$$

$$v(x) = 1 + \int_0^x dx = 1 + x \qquad (3.22)$$

Second approximation :

$$u(x) = 1 + \int_0^x (1 + x)\, dx = 1 + x + \frac{x^2}{2!} \qquad (3.23)$$

$$v(x) = 1 + \int_0^x (1 + x)\, dx = 1 + x + \frac{x^2}{2!} \qquad (3.24)$$

Third approximation

$$u(x) = 1 + \int_0^x \left(1 + x + \frac{x^2}{2!}\right) dx = 1 + x + \frac{x^2}{2!} \qquad (3.25)$$

$$v(x) = 1 + \int_0^x \left(1 + x + \frac{x^2}{2!}\right) dx = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \qquad (3.26)$$

So continuing in this way indefinitely we see that.

$$u(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = e^x \qquad (3.27)$$

which is the desired solution of the given nonlinear initial value problem.

Remark:

If $\frac{du}{dx} = f(x,u)$ and $u = b$ and $x = a$, the successive approximations from the values of u as a functions of x are

$$u_1 = b + \int_a^x F(x,b)\, dx$$

$$u_2 = b + \int_a^x F(x, u_1)\, dx$$

$$u_3 = b + \int_a^x F(x, u_2)\, dx$$

$$\vdots \quad = \quad \vdots$$

$$u_{n+1} = b + \int_a^x F(x, u_n)\, dx \qquad (3.28)$$

and so on

3.4 Adomian Decomposition Method:

The decomposition method is similar to the Picard's successive approximation method .

In the decomposition method , we usually express the solution $u(x)$ of the integral equation.

$$u(x) = b + \int_0^x F(x, u) dx \qquad (3.29)$$

in a series from defined by

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \qquad (3.30)$$

Substituting the decomposition equation (3.30) into both sides of equation (3.29) yulds

$$\sum_{n=0}^{\infty} u_n(x) = b + \int_a^x F\left(x, \sum_{n=0}^{\infty} u_n(x)\right) dx \qquad (3.31)$$

The components $u_0(x), u_1(x), u_2(x), \ldots$ of the unknown function $u(x)$are completely determined in a recurrence manner if we set

$$u_0(x) = b$$

$$u_1(x) = \int_a^x F(x, u_0)\, dx$$

$$u_2(x) = \int_a^x F(x, u_1)\, dx \qquad (3.32)$$

$$u_3(x) = \int_a^x F(x, u_2)\, dx$$

and so on .The above decomposition scheme for determination of the components

$u_0(x), u_1(x), u_2(x), \ldots\ldots$ of the solution $u(x)$ of equation (3.29) can be written in a recurrence form by

$$u_0\,(x) = b$$

$$u_{n+1}(x) = \int_a^x F(x, u_n)\, dx \qquad (3.33)$$

In this new decomposition process we expand the solution function in a straight forward infinite series.

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \ldots. + u_n\,(x) + \cdots =$$

$$\sum_{n=n}^{\infty} u_n\,(x) \qquad (3.34)$$

assuming that the series converges to a finite limit as $n \rightarrow \infty$

next we expand the function $F(x,u)$ which contains the solution function $u(x)$ by Taylor's expansion about $u_0(x)$ keeping x as it is such that.

$$F(x,u) = F(x,u_0) + (u-u_0)F\,(x,u_0) + \frac{(u-u_0)^2}{2!}\,F_{uu}(x,u_0) +$$

$$\frac{(u-u_0)^3}{3!}\,F_{uu}(x,u_0) + \frac{(u-u_0)^4}{4!}\,F_{uu}(x,u_0) + \cdots \qquad (3.35)$$

We know that Taylor's expansion is absolutely and uniformly convergent in a given domain now using equation (3.34) into equation (3.35) yields.

$$F(x,u) = F(x,y) + \sum_{n=1}^{\infty} u_n(x) F_u(x,u_0) +$$

$$\frac{1}{2!}\left[\sum_{n=1}^{\infty} u_n(x)\right]^2 F_{uu}(x,u_0) + \frac{1}{3!}\left[\sum_{n=1}^{\infty} u_n(x)\right]^3 F_{uuu}(x,u_0) +$$

$$\frac{1}{4!}\left[\sum_{n=1}^{\infty} u_n(x)\right]^4 F_{uuuu}(x,u_0) + \,.... \qquad (3.36)$$

Which can subsequently be written as

$$F(x,u) = A_0(x) + A_1(x) + A_2(x) + A_3(x) + \cdots + A_n(x) + \cdots$$

$$= \sum_{n=0}^{\infty} A_n(x) \qquad (3.37)$$

We define the different term in $A_n(x\,,u)$ as follows :

$$A_0 = F(x,u_0)$$

$$A_1 = u_1 F_u(x,u_0)$$

$$A_2 = u_2 F_u(x,u_0) + \frac{1}{2}\,u_1^2\,F_{uu}(x,u_0)$$

$$A_3 = u_3 F_u(x,u_0) + \frac{1}{2}\,(2\,u_1 u_2) F_{uu}(x,u_0) + \frac{1}{6}\,u_1^3\,F_{uuu}(x,u_0)$$

$$A_4 = u_4 F_u(x,u_0) + \frac{1}{2}\,(2\,u_1 u_3) F_{uu}(x,u_0) + \frac{1}{6}\,(3\,u_1^2 u_2)\,F_{uuu}(x,u_0)$$

$$+ \frac{1}{24} u_1^4\,F_{uuu}(x,u_0) \qquad (3.38)$$

Substituting equation (3.37) and equation (3.30) into the integral equation.

$$u(x) = b + \int_a^x F(x,u)dx \qquad (3.39)$$

We obtain

$$\sum_{n=0}^{\infty} u_n(x) = b + \int_a^x \sum_{n=0}^{\infty} A_n\,(x)dx \qquad (3.40)$$

or simply

$$u_0(x) + u_1(x) + u_2(x) + \cdots$$
$$= b + \int_a^x [\,A_0(x) + A_1(x) + A_2(x) + .. +]\;dx \quad (3.41)$$

The components $u_0(x) + u_1(x) + u_2(x), ...$

are completely determined by using the recurrence scheme

$u_0(x) = b$

$$\left.\begin{aligned} u_1(x) &= \int_a^x A_0(x)dx = \int_0^x A_0(t)dx \\ u_2(x) &= \int_a^x A_1(x)dx = \int_a^x A_1(t)dt \end{aligned}\right\} \qquad (3.42)$$

$$u_3(x) = \int_a^x A_2(x)dx = \int_a^x A_2(t)dt$$

$$u_4(x) = \int_a^x A_3(x)dx = \int_a^x A_3(t)dt$$

$$\vdots \quad = \quad \vdots$$

$$u_{n+1}(x) = \int_a^x A_n(x)dx = \int_a^x A_n(t)dt \ , \ \ n \geq 1$$

In the following example , we will illustrate the decomposition method as established earlier by Picard's successive approximation method.

Example (3.4):

We solve the integral equation

$$u(x) = \int_0^x (x + u^2)dx = \frac{x^2}{2} + \int_0^x u^2(t)dt \qquad (3.43)$$

by the decomposition method.

Solution:

To do that , we can write the equation in series form.

$$u(x) = \sum_{n=0}^{\infty} u_n(x) = \frac{x^2}{2} + \int_0^x \sum_{n=0}^{\infty} A_n(t)dt \qquad (3.44)$$

In which we can decompose our solution set as

$$u_0\,(x) = \frac{x^2}{2}$$

$$u_1\,(x) = \int_0^x A_0(t)dt$$

$$u_2\,(x) = \int_0^x A_1(t)dt$$

$$\vdots \;=\; \vdots \qquad (3.45)$$

$$u_n\,(x) = \int_0^x A_{n-1}(t)dt$$

We know $F(u) = u^2$, and so $\grave{F}(u) = 2u$ and

$\grave{\grave{F}}(u) = 2$, $\grave{\grave{\grave{F}}}(u) = 0$

thus , we obtain

$$\left.\begin{aligned} F(u_0) &= u_0^2 = \frac{x^4}{4} \\ \acute{F}(u_0) &= 2u_0 = x^2 \\ \acute{\acute{F}}(u_0) &= 2 \\ \acute{\acute{\acute{F}}}(u_0) &= 0 \end{aligned}\right\} \qquad (3.46)$$

Thus , with these information , we obtain

$$\left.\begin{aligned} A_0(x) &= \frac{x^4}{4} \\ A_1(x) &= u_1x^2 \\ A_2(x) &= u_2x^2 + u_1^2 \\ A_3(x) &= u_3x^2 + 2u_1u_2 \end{aligned}\right\} \qquad (3.47)$$

Hence the different components of the series can be obtained as

$$u_0(x) = \frac{x^2}{2}$$

$$\left.\begin{aligned}
u_1(x) &= \int_0^x A_0(t)dt \quad = \int_0^x \frac{t^4}{4}\,dt \;=\; \frac{x^5}{20} \\
u_2(x) &= \int_0^x A_1(t)dt \quad = \int_0^x u_1(t^2)dt \quad = \int_0^x u_1(t)t^2 dt \\
&\int_0^x \left(\frac{t^4}{4}\right) t^2\,dt \;=\; \frac{x^8}{160} \qquad\qquad = \\
u_3(x) &= \int_0^x A_2(t)dt \quad = \int_0^x [u_2(t)t^2 + u_1^2(t)]\,dt \quad = \\
&\int_0^x \left[\left(\frac{t^8}{160}\right) t^2 + \frac{t^{10}}{400}\right] dt \;=\; 7\,\frac{x^{11}}{8800}
\end{aligned}\right\} \qquad (3.48)$$

Example (3.5) :

Find the solution of the initial value problem by

$$\frac{du}{dx} = \; xu \;\; , \;\; u(0) = 1$$

(a) Picard’s method

(b) the decomposition method.

Solution :

To do that

(a) Picard’s method

The initial value problem can be put in the integral equation from

$$u(x) = 1 + \int_0^x t\,u(t)dt \qquad (3.49)$$

The first approximation is

$$u_1(x) = 1 + \int_0^x t\,(1)dt = 1 + \frac{x^2}{2}$$

The second approximation is

$$u_2(x) = 1 + \int_0^x t\left(1 + \frac{x^2}{2}\right)dt = 1 + \frac{1}{2!}\left(\frac{x^2}{2}\right)^2$$

Proceeding in this manner, we can derive the n the approximation as the n the approximation is

$$u_n(x) = 1 + \frac{x^2}{2} + \frac{1}{2!}\left(\frac{x^2}{2}\right)^2 + \cdots + \frac{1}{n!}\left(\frac{x^2}{2}\right)^n. \qquad (3.50)$$

This , when $n \to \infty$

$$u(x) = \lim_{n\to\infty} u_2(x) = e^{\frac{x^2}{2}} \qquad (3.51)$$

(b) The decomposition method

We decompose the solution in a series

$$as\,u(x) = \sum_{n=0}^{\infty} u_n(x) \qquad (3.52)$$

Substituting this series into the integral equation , we obtain

$$\sum_{n=0}^{\infty} u_n(x) = 1 + \int_0^x \left[t \sum_{n=0}^{\infty} u_n(t)\right]dt \qquad (3.53)$$

Equating the different order of terms of this series , we construct a set integral equations.

$$u_0(x) = 1$$

$$u_1(x) = \int_0^x t\, u_0(t)dt = \int_0^x t(1)dt = \frac{x^2}{2}$$

$$u_2(x) = \int_0^x tu_1(t)dt = \int_0^x t\left(\frac{t^2}{2}\right) dt = \frac{x^4}{8} = \frac{1}{2!}\left(\frac{t^2}{2}\right)^2$$

$$u_n(x) = \frac{1}{n!}\left(\frac{x^2}{2}\right)^2 \qquad (3.54)$$

Thus , the solution is

$$u(x) = \sum_{n=0}^{\infty} u_n(x) = 1 + \frac{x^2}{2} + \frac{1}{2!}\left(\frac{t^2}{2}\right)^2 + \cdots = e^{\frac{x^2}{2}} \qquad (3.55)$$

Solution . i,e equation (3.51) and (3.55) are identical and agree with analytical solution $u(x) = e^{\frac{x^2}{2}}$.

Remark :

It worth noting that the decomposition method encounters computation difficulties if the non homogeneous term $f(x)$ is not a polynomial in term of x. When the $f(x)$ is not polynomial we use the modified decomposition method to minimize the calculation .In the following example, we shall illustrate the modified decomposition method for handling the Volterra type integral equation.

Example (3.6):

We use the decomposition method or the modified decomposition method to solve the following non linear Volterra integral equation by finding the exact solution or after terms of the series solution.

$$u(x) = \sin x + \frac{1}{8}\sin(2x) - \frac{x}{4} + \frac{1}{2}\int_0^x u^2(t)dt \qquad (3.56)$$

By inspection it can be easily seen that $u(x) = \sin x$ is an exact solution.

Let us confirm the result by modified decomposition method .for this reason we split $f(x) = \sin x + \frac{1}{8}\sin(2x) - \frac{x}{4}$ between the two components u_0(x) and $u_1(x)$ and here we set $u_0(x) = \sin x$ consequently , the first decomposition component is defined by

$$u_1(x) = \frac{1}{8}\sin(2x) - \frac{x}{4} + \frac{1}{2}\int_0^x A_0(t)dt \qquad (3.57)$$

Here , $A_0(x) = u_0^2 = sin^2x$. Thus , we obtain

$$u_1(x) = \frac{1}{8}\sin(2x) - \frac{x}{4} + \frac{1}{2}\int_0^x sin^2t\, dt \quad =$$

$$\frac{1}{8}\sin(2x) - \frac{x}{4} + \left(\frac{x}{4} - \frac{1}{8}\sin(2x)\right) = 0 \qquad (3.58)$$

This defines the there components by $u_k(x) = 0\, , for\ k \geq 1$

the exact solution $u(x) = \sin x$ following .

Example (3.7) :

We use the decomposition method or the modified decomposition method to solve the following nonlinear Volterra integral equation by finding the exact solution or a few terms of the series solution.

$$u(x) = \tan x - \frac{1}{4}\sin 2x - \frac{x}{2} + \int_0^x \frac{dt}{1 + u^2(t)} \qquad (3.59)$$

By inspection it can be easily seen that $u(x) = \tan x$ is an exact solution. Let us show this result by the modified decomposition method. To accomplish this we split $f(x) \tan x - \frac{1}{4}\sin 2x - \frac{x}{2}$ between the two components $u_0(x)$ and $u_1(x)$, and we set $u_0(x) = \tan x$. Consequently, the first component is defined by.

$$u_1(x) = -\frac{1}{4}\, sin\ 2x - \frac{x}{2} + \int_0^x A_0(t)dt =$$

$$-\frac{1}{4}\sin 2x - \frac{x}{2} + \int_0^x \frac{1}{1 + tan^2}\ dt = -\frac{1}{4}\sin 2x - \frac{x}{2} + \int_0^x cos^2 t\ dt$$

$$= -\frac{1}{4}\, sin\ 2x - \frac{x}{2} + \frac{1}{4}\sin 2x - \frac{x}{2} = 0 \qquad (3.60)$$

This defines that the other components by $u_k(x) = 0, for\ k \geq 1$

Hence the exact solution $u(x) = \tan x$ follows immediately.

In the following example we shall deal with Fredholm type nonlinear integral equation by using the direct computational method.

Example (3.8) :

We use the direct computational method to show the given nonlinear Fredholm integral equation and verify with the method of decomposition.

$$u(x) = \frac{7}{8}x + \frac{1}{2}\int_0^1 x\,t\,u^2(t)dt \qquad (3.61)$$

To do that :

(a) Direct Computational Method :

Setting

$$\ltimes = \int_0^1 t\,u^2(t)\,dt \qquad (3.62)$$

Where ∝ is a constant , the given integral equation can be written as .

$$u(x) = \frac{7}{8}x + \frac{1}{2}x \ltimes \qquad (3.63)$$

But $\ltimes = \int_0^1 t\left(\frac{7}{8}+\frac{\ltimes}{2}\right)^2 t^2\,dt =$

$$\int_0^1 \left(\frac{7}{8}+\frac{\ltimes}{2}\right)^2 t^3\,dt = \frac{1}{4}\left(\frac{7}{8}+\frac{\ltimes}{2}\right)^2 . \qquad (3.64)$$

Now solution this quadrates equation in ⋉ by (3.64) .

$$\ltimes = \frac{1}{4}\left(\frac{7}{8}+\frac{\ltimes}{2}\right)^2 = \left(\frac{1}{2}\left(\frac{7}{8}+\frac{\ltimes}{2}\right)\right)^2 = \left(\frac{7}{16}+\frac{\ltimes}{4}\right)^2,$$

$$16\ltimes^2 - 200\ltimes + 49 = 0$$

$$(4\ltimes - 49)(4\ltimes - 1) = 0$$

either $\ltimes = \frac{49}{4}$, or $\ltimes = \frac{1}{4}$

substituting $\ltimes = \frac{1}{4}$,$\ltimes = \frac{49}{4}$

in equation (3.63)

$$u(x) = \frac{7x}{8} + \frac{x}{8} = \frac{8x}{8} = x \qquad (3.65)$$

$$and \quad u(x) = \frac{7x}{8} + \frac{49\,x}{8} = \frac{56\,x}{8} = 7\,x \qquad (3.66)$$

(b) The Decomposition Method :

In this method the Adomian polynomials for the non linear term $f(u) = u^2$ are expressed as.

$$\left.\begin{aligned} A_0(x) &= u_0^2 \\ A_1(x) &= 2u_0u_1 \\ A_2(x) &= 2u_0u_2 + u_1^2 \\ A_3(x) &= 2u_0u_3 + 2u_1u_2 \\ \vdots &= \vdots \end{aligned}\right\} \qquad (3.67)$$

Where the different components are calculated from

$$f(u) = u^2\,, \acute{f}(u) = 2u, \acute{\acute{f}}(u) = 2, \acute{\acute{\acute{f}}}(u) = 0.$$

under the recurcive algorithm , we have

$$\left.\begin{aligned} u_0(x) &= \frac{7}{8}x \\ u_1(x) &= \frac{x}{2}\int_0^1 t\,A_0(t)\,dt = \frac{49}{512}\,x \\ u_{21}(x) &= \frac{x}{2}\int_0^1 t\,A_1(t)\,dt = \frac{343}{16384}\,x \end{aligned}\right\} \qquad (3.68)$$

And so on .The solution in the series from is given by

$$u(x) = \frac{7}{8}x + \frac{49}{512}\,x + \frac{343}{16384}\,x + \cdots \approx x \qquad (3.69)$$

This is an example where the exact solution is not obtainable ; hence we use a few terms of the series to approximate the solution. We remark that the two solutions $u(x) = x$ and $u(x) = 7x$ were obtained in (a) by the direct computational method.

Thus the given integral equation does not have a unique solution.

CHAPTER FOUR
INTEGRO -DIFFERENTIAL EQUATIONS
4.1 Introduction

We devote this chapter to linear integro-differential equation and we will be concerned with the different solution techniques .To obtain a solution of the integro-differential equation , we need to specify the initial conditions to determine the unknown constants.

The unknown function $u(x)$ and one or more of its derivatives such as $u`(x), u``(x), ...$ appear out and under the integral sign as well. One quick source of intrgro- differential equations can be clearly seen when we convert the differential equation to an integral equation by using Leibnitz rule.

The integro – differential equation can be viewed in this case as an intermediate stage when finding an equivalent Volterra integral equation to the given differential equation.

The following are the examples of linear integro-differential equations:

$$u`(x) = f(x) - \int_{0}^{x} (x-t)u(t)dt \quad , u(0) = 0 \quad (4-1)$$

$$u``(x) = g(x) - \int_{0}^{x} (x-t)u(t)dt \; , u(0) = 0 \, , u`(0) = -1 \quad (4-2)$$

$$u`(x) = e^{x} - x + \int_{0}^{1} xt\, u(t) \quad , u(0) = 0 \qquad (4-3)$$

$$u``(x) = h(x) + \int_{0}^{1} t\, u`(t)dt \quad , u(0) = 0, u`(0) = 1 \qquad (4-4)$$

It is clear from the above examples that the unknoun function $u(x)$or one of its derivatives appear under the integral singe, and the other derivatives appear out the integral sign as well, these examples can be classified as the Volterra and Fredholm integro- differential equation .Equation (4-1) and (4-2) are the Volterra type whereas equations (4-3) and (4-4) are of Fredholm type integro-differentail equation.

It is to be noted that these equations are linear integro-differential equations.

4.2 Volterra Integro-Differential Equations:

we shall present some sophisticated mathematical methods to obtain the solution of the Volterra integro – differential equations .We shall focus our attention to study the integral equation that involves separable kernel of the form

$$k(x,t) = \sum_{k=1}^{n} g_k(x)\, h_k(t) \qquad (4.5)$$

We shall first study the case when $k(x,t)$consists of one product of the functions $g(x)$and $h(t)$such that $k(x,t) = g(x)h(t)$ only where other cases can be generalized in the same manner .

The non separable kernel can be reduced to the separable kernel by using the Taylor's expansion for the kernel involved .We will illustrate the method first and then use the technique to some examples.

4.2.1 The Series Solution Method:

Let us consider a standard form of Volterra ontegro –differential equation of nth order as given below:

$$u^{(n)}(x) = F(x) + g(x) \int_0^x h(t)u(t)dt \, , \; u^{(n)} = b_k \, ,$$

$$0 \leq k \leq (n-1) \qquad (4.6)$$

We shall follow the Frobenius method of series solution used to solve ordinary differential equations a round an ordinary point. To achieve this goal, we first assume that the solution $u(x)$ of equation (4.6) is an analytic function and hence can be represented by a series expansion about the ordinary point $x = 0$ given by

$$u(x) = \sum_{k=0}^{\infty} a_k \, x^k \qquad (4.7)$$

Where the coefficients a_k are the unknown constants and must be determined. It is to be noted that first few coefficients a_k can be determined by using the initial conditions so that

$a_0 = u(0)$, $a_1 = u`(0)$, $a_2 = \frac{1}{2!}u``(0)$, and soon depending on the number of the initial conditions, where as the remaining coefficients a_k will be determined from applying the technique as will be discussed later. Substituting equation (4.7) into both sides of equation (4.6) yields

$$\left(\sum_{k=0}^{\infty} a_k \, x^k\right)^{(n)} = f(x) + g(x) \int_0^x \left(\sum_{k=0}^{\infty} a_k \, t^k\right) dt \qquad (4.8)$$

In view of equation (4.8), equation (4.6) will be reduced to calculable integrals in the right- hand side of equation (4.8) that can be easily evaluated where we have to integrate terms of the form t^n, $n \geq 0$ only. The next step is to write the Taylor's expansion for f(x) evaluate the resulting traditional integrals, i.e. equation.(4.8) and then equating the coefficients of like powers of x in both sides of the equation .This will lead to a complete determination of the coefficients $a_0, a_1, a_2, \dots \dots ....$ of the series in equation (4.7). Consequently, substituting the obtained coefficients a_k , $k \geq 0$ in equation (4.7) produces the solution in the series form. This may give a solution in closed form, the expansion obtained is a Taylor's expansion to a well known elementary function, or we may use the series from solution if a closed from is not attainable.

To give a clear over view of the method just described and how it should be implemented for Volterra integro-differential equations , the series solution method will be illustrated by considering the following example.

Example (4.1)

Solve the following Volterra integro-differential equation.

$$u``(x) = c\cosh x - \int_0^x tu(t)dt \, , u(0) = 0, \qquad u`(0) = 1 \quad (4.9)$$

by using the series solution method

solution:

substitute $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \qquad (4.10)$$

Into both sides of the equation (4.9) and using the Taylor's expansion of cosh x , we obtain.

$$\sum_{n=2}^{\infty} n\,(n-1) a_n x^{n-2} = x\left(\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k!)}\right) - \int_0^x t\left(\sum_{k=0}^{\infty} a_n t^n\right) \; (4.11)$$

Using the initial condition ,we have $a_0 = 0$, and $a_1 = 1$ Evaluating the integrals that involve terms of the form $t^n, n \geq 0$, and using few terms from both sides yield.

$$2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \cdots + =$$

$$x\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} +..\right) - \left(\frac{x^3}{3} + \frac{1}{4}a_2x^4 + \cdots\right) \qquad (4.12)$$

Equating the coefficients of like Powers of x in both sides we find $a_2 = 0$, $a_3 = \frac{1}{3!}$, $a_4 = 0$, and in general $a_{2n} = 0$, for $n \geq 0$ and $a_{2n+1} = \frac{1}{(2n+1)!}$, for $n \geq 0$.

Thus , using the values of these coefficients the solution for $u(x)$ from equation (4.10) can be written in series form as

$$u(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots, \qquad (4.13)$$

And in a closed – form $\quad u(x) = \sinh x \quad$ (4.14)

is the exact solution v of equation (4.9).

Example (4.2):

Solve the following Volterra integro – differential equation

$$u``(x) = \cosh x + \frac{1}{4} - \frac{1}{4}\cosh ex + \int_0^x \sin t\, u(t)dt \qquad (4.15)$$

$$u(0) = 1, u`(0) = 0$$

by using the series solution method

Solution:

Using the same procedure , we obtain the first few terms of the expression $u(x)$ as

$$u(x) = 1 + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots \qquad (4.16)$$

Substituting equation (4.16) into both sides of equation (4.15) yields.

$$2a_2 + 6\,a_3 + 12a_4x^2 + 20\,a_5x^3 + \;.... = \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) +$$

$$\frac{1}{4} - \frac{1}{4}\left(1 + \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + \cdots\right) +$$

$$\int_0^x \left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right)(1 + a_2t^2 + a_3t \propto^3 + \cdots)\,dt \qquad (4.17)$$

Integrating the right – hand side and equating the coefficients of like powers of x we find

$$a_0 = 1\,, a_1 = 0, a_2 = \frac{1}{2!}\,, a_3 = 0\,, a_4 = \frac{1}{4!}\,, a_5 = 0,$$

and so on , where the constants a_0 and $a_1$are defined by the initial condiction consequently , the solution in the series method is given by

$$u(x) = \;1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \;..., \qquad (4.18)$$

Which give $u(x) = \cosh x$, as the exact solution in a closed – form.

4.2.2 The Decomposition Method:

we shall introduce the decomposition method and the modified decomposition method to solve the Volterra integro-differential equations .This method appears to be reliable and effective.

With out loss of generality , we may assume a standard form to Volterra integro-differential equation defined by the standard form.

$$u^{(n)} = f(x) + \int_0^x k(x,t)u(t)dt \quad , u^k(0) = \, b_k \, , 0 \le k \le (n-1) \quad (4.19)$$

Where $u^{(n)}$ is the nth order derivative of $u(x)$ with respect to x and b_k are constants that defines the initial conditions . It is natural to seek an expression for $u(x)$ that will be derived from equation (4.19) .This can be done by integrating both sides of equation (4.19) from o to x as many times as the order of the derivative involved.

Consequently , we obtain

$$u(x) = \sum_{k=0}^{n-1} \frac{1}{k!} \, b_k x^k + L^{-1}\big(f(x)\big) + L^{-1}\left(\int_0^x k\,(x,t)u(t)dt \right) \quad (4.20)$$

Where $\sum_{k=0}^{n-1} \frac{1}{k!} \, b_k x^k$ is obtained by using the initial conditions , and L^{-1} is an n- fold integration operator now we are in a position to apply the decomposition method by defining the solution $u(x)$ of equation (4.20) on a decomposed series.

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \qquad (4.21)$$

Substitution of equation (4.21) into both sides of equation (4.20) we get.

$$\sum_{n=0}^{\infty} u_n(x) = \sum_{k=0}^{n-1} \frac{1}{k!} \, b_k x^k + \, L^{-1} \left(f(x)\right) + L^{-1}$$

$$\left(\int_0^x k(x,t)\left(\sum_{n=0}^{\infty} u_n(t)\right) dt \qquad (4.22)$$

This equation can be explicitly written as

$$u_0(x) + u_1(x) + u_2(x) + \cdots = \sum_{k=0}^{n-1} \frac{1}{k!}\, b_k x^k + L^{-1}(F(x)) +$$

$$L^{-1}\left(\int_0^x k(x,t)u_0(t)dt\right) + L^{-1}\left(\int_0^x k(x,t)u_1(t)dt\right) +$$

$$L^{-1}\left(\int_0^x k(x,t)u_2(t)dt\right) + L^{-1}\left(\int_0^x k(x,t)u_3(t)dt\right) + \cdots \quad (4.23)$$

The components $u_0(x), u_1(x), u_2(x), u_3(x), \ldots$ of the unknown function $u(x)$ are determined in a recursive manner, if we set

$$u_0\,(x) = \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}\big(f(x)\big),$$

$$u_1\,(x) = L^{-1}\left(\int_0^x k(x,t)\, u_0(t)dt\right),$$

$$u_2\,(x) = L^{-1}\left(\int_0^x k(x,t)\, u_1(t)dt\right),$$

$$u_3\,(x) = L^{-1}\left(\int_0^x k(x,t)\, u_2(t)dt\right),$$

$$u_4\,(x) = L^{-1}\left(\int_0^{-1} k(x,t)\, u_3(t)dt\right),$$

And so on .The above equations can be written in a recursive manner as

$$u_0\,(x) = \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}\big(f(x)\big), \qquad (4.24)$$

$$u_{n+1}\,(x) = L^{-1}\left(\int_0^x k(x,t)\,u_n(t)dt\right), n \geq 0 \qquad (4.25)$$

In view of equations (4.24) and (4.25) , the components $u_0(x), u_1(x), u_2(x), \ldots$ are immediately determined. Once these components are determined , the solution $u(x)$ of equation (4.19) is then obtained as a series.

From using equation (4.21) .The series solution may be put into an exact closed –form solution with can be clarified by some illustration as follows .It is to be noted here that the phenomena of self-cancelling noise terms that was introduced before may be applied here if the noise terms appear in $u_0(x)$ and $u_1(x)$ .The following example will explain how we can use the decomposition method .

Example (4.3):

Solve the following Volterra integro-differential equation.

$$u``(x) = x + \int_0^x (x-t)\,u(t)dt \quad , \quad u(0) = 0\; u`(0) = 1 \; (4.26)$$

by suing the decomposition method .Verify the result by the Laplace transform method

Solution:

applying the two – fold integration operator L^{-1}

$$\mathcal{L}^{-1}(.) = \int_0^x \int_0^x (.)dx\, dx \qquad (4.27)$$

To both sides of equation (4.26) . i,e integrating both sides of equation (4.26) twice from 0 to x , and using the give linear conditions yield,

$$u(x) = x + \frac{x^3}{3!} + L^{-1}\left(\int_0^x (x-t)u(t)dt\right) \qquad (4.28)$$

Following the decomposition scheme , i,e equation (4.24) and (4.15) , we find (4.25) , we find.

$$u_0(x) = x + \frac{x^3}{3!}$$

$$u_1(x) = L^{-1}\int_0^x (x-t)u_0(t)dt = \frac{x^5}{5!} + \frac{x^7}{7!}$$

$$u_2(x) = L^{-1}\int_0^x (x-t)u_1(t)dt = \frac{x^9}{9!} + \frac{x^{11}}{11!}$$

With this information the final solution can be written as

$$u(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \frac{x^{11}}{11!} + \cdots \qquad (4.29)$$

and this leads to $u(x) = \sin h\, x$, the exact solution in closed – form.

By using the Laplace transform method with the concept of convolution and using the initial conditions the given equation can be very easily simplified to

$$\mathcal{L}\,\{u(x)\} = \frac{1}{s^2-1}$$

and taking the inverse transform , we obtain $u(x) = \sinh x$ which identical to the previous result.

Example (4-4):

Solve the following Volterra integro differential equation.

$$u``(x) = 1 + \int_0^x (x-t)u(t)dt\,, \qquad u(0) = 1\,, u`(0) = 0 \quad (4.30)$$

by using the decomposition method , then verify it by the Laplace transform method

Solution:

Integrating both sides of equation (4.30) from 0 to x and suing the given initial conditions yield.

$$u(x) = 1 + \frac{x^2}{2!} + L^{-1}\left(\int_0^x (x-t)u(t)dt\right), \qquad (4.31)$$

Where L^{-1} is a two – fold integration operator following the decomposition method , we obtain

$$u_0(x) = 1 + \frac{x^2}{2!}$$

$$x_1(x) = L^{-1}\left(\int_0^x (x-t)u_0(t)dt\right),\ \frac{x^4}{4!} + \frac{x^6}{6!}$$

$$x_2(x) = L^{-1}\left(\int_0^x (x-t)u_1(t)dt\right),\ \frac{x^8}{8!} + \frac{x^{10}}{10!}$$

Using this information the solution $u(x)$ can be written as

$$u(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \frac{x^{10}}{10!} + \cdots \qquad (4.32)$$

and this gives $u(x) \cosh x$ the exact solution by using the Laplace transform method with the concept of convolution and using the initial conditions , we obtain.

$$\mathcal{L}\{u(x)\} = \frac{s}{s^2 - 1}$$

and its inversion is simply $u(x) = \cosh x$ .These two results are identical.

4.2.3 Converting to Volterra Integral Equation :

Concerned with converting to Volterra integral equations , we can easily convert the Volterra integro-differential to equivalent Volterra integral equation, provided the kernel is a difference kernel defined by

$$k(x,t) = k(x - t).$$

This can be easily done by integrating both sides of the equation and using the initial conditions.

We illustrate for the benefit three specific formulas:

$$\int_0^x \int_0^x u(t)dt = \int_0^x (x-t)u(t)dt\,,$$

$$\int_0^x \int_0^x \int_0^x u(t)dt = \frac{1}{2!}\int_0^x (x-t)^2\ u(t)dt\,,$$

$$\int_0^x \int_0^x \ldots . \int_0^x u(t)dt = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1}\, u(t)dt$$

n- fold integration.

To give a clear overview of this method we illustrate the following example.

Example 4.5:

Solve the following Volterra integro-diffential equation

$$u`(x) = 2 - \frac{x^2}{4} + \frac{1}{4}\int_0^x u(t)dt\,,\ \ u(0) = 0 \qquad (4.33)$$

by converting to a standard Volterra integral equation

Solution:

Integrating both sides from 0 to x and using the initial condition and also converting the double integral to the single integral, we obtain.

$$u(x) = 2x - \frac{x^3}{12} + \frac{1}{4}\int_0^x \int_0^x u(t)dt\, dt =$$

$$2x - \frac{x^3}{12} + \frac{1}{4}\int_0^x (x-t)u(t)dt$$

it is clearly seen that the above equation is a standard Volterra integral equation .

It will be solved by the decomposition method following that technique we set.

$$u_0(x) = 2x - \frac{x^3}{12} \quad , \qquad (4.34)$$

Which gives

$$u_1(x) = \frac{1}{4}\int_0^x (x-t)\left(2t - \frac{t^3}{12}\right)dt = \frac{x^3}{12} - \frac{x^5}{240} \qquad (4.35)$$

We can easily observed that $\frac{x^3}{12}$ appears with opposite sign in the components $u_0(x)$ and $u_1(x)$,and by cancelling this noise term from $u_0(x)$ and justifying that $u(x) = 2x$, is the exact solution of equation (4.33).

This result can be easily verified by taking the Laplace transform of equation (4.33) and using the initial condition which simply reduces to $\mathcal{L}\{u(x)\} = \frac{2}{s^2}$ and its inversion is $u(x) = 2x$.

4.2.4 Converting to Initial Value Problems :

We shall investigate how to reduce the Volterra integro-differential equation to an equivalent initial value problem , In this study, we shall mainly focus our attention to the case where the kernel is a difference kernel of the form $k(x,t) = k(x-t)$.

In differentiating the integral involved we shall use the Leibnitz rule to achieve our goal.

The Leibnitz rule is

$$let \quad y(x) = \int_{t=a(x)}^{t=b(x)} F(x,t)dt$$

Then

$$\frac{dy}{dx} = \int_{t=a(x)}^{t=b(x)} \frac{\partial}{\partial x} F(x,t)dt + \frac{db(x)}{dx} F(b(x),x)) - \frac{da(x)}{dx} F(a(x),x)).$$

Having converted the Volterra integro-differential equation to an initial value problem , the various methods that are used in any ordinary differential equation can be used to determine the solution .The concept is easy to implement but requires more calculations in comparison to the integral equation technique .To give a clear over view of this method we illustrate the following example.

Example (4.6)

Solve the following Volterra integro-differential equation

$$u`(x) = 1 + \int_{0}^{x} u(t)dt \quad u(0) = 0\,, \qquad (4.36)$$

by converting it to initial value problem

Solution:

Differentiating both sides of equation (4.36) with respect to x and using the Leibnitz rule to differentiate the integral at the right hand side we obtain.

$u``(x) = \ u(x)$, with the initial conditions

$$u(0) = 0 \quad ,u`(0) = 1 \qquad , \qquad (4.37)$$

where the derivative condition is obtained by substituting $x = 0$ in both sides of the equation (4.36) .the solution of equation (4.37) is simply.

$u(x) = A\cosh x + B\sin h\, x$,

Where A and B are arbitrary constants and using the initial conditions we have A=0 and B=1 and thus the solution becomes.

$$u(x) = \sin h\, x$$

This solution can be verified by the Laplace transform method. By taking the Laplace to equation (4.36) and using the initial condition we have after reduction.

$$\mathcal{L}\,\{u(x)\} = \frac{1}{s^2 - 1}$$

And its inversion gives us $u(x) = \sin hx$ which is identical to the above result .

Example (4.7):

Solve the following Voltarra integro –differential equation.

$$u`(x) = 2 - \frac{x^2}{4} + \frac{1}{4}\int_0^x u(t)dt\,, \qquad u(0) = 0 \qquad (4.38)$$

by reducing the equation to an initial value problem

Solution:

By differentiating the above equation with respect to x it can be reduced to the following initial value problem.

$$u``(x) = \frac{1}{4}u(x) = \frac{-x}{2}\,,\;\; u(0) = 0\,, u`(0) = 2 \qquad (4.39)$$

The general solution is obvious and can be written at once.

$$u(x) = A\cosh\left(\frac{x}{2}\right) + B\sin h\ \left(\frac{x}{2}\right) + \ 2x.$$

Using the initial conditions yields A = B= 0 and the solution reduces to $u(x) = 2x$.

This result can also be obtained by using the Laplace transform method.

4.3 Fredholm Integro-Differential Equations:

We will discuss the reliable methods used to solve Fredholm integro-diferential equations. We remark here that we will focus our attention on the equations that involve separable kernels where the kernel $k(x,t)$ can be expressed as the finite sum of the form.

$$k(x,t) = \sum_{k=1}^{n} g_k(x)h_k(t). \qquad (4.40)$$

Without loss of generality , we will make our analysis on a one– term kernel $k(x,t)$of the form $k(x,t) = g(x)h(t)$, and this can be generalized for other cases .The non separable kernel can be reduced to separable kernel by using the Taylor expansion of the kernel involved we point out that the method to be discussed are introduced before , but we shall focus on how these methods can be implemented in this type of equations ,we shall start with the most practical method.

4.3.1 The Direct Computation Method:

We assume a standard form to the Fredholm integro-differential equation given by.

$$u^{(n)}(x) = f(x) + \int_0^1 (x,t)u(t)dt \, , u^{(k)} = b_k(0) \, , 0 \leq k \leq (n-1), \quad (4.41)$$

Where $u^{(n)}(x)$ is the nth derivative of $u(x)$ with respect to x and b_k are constants that define the initial conditions .substituting $k(x,t) = g(x)h(t)$ into equation (4.41) yields.

$$u^{(n)}(x) = f(x) + g(x)\int_0^1 h(t)u(t)dt \, , u^{(k)} = b_k, 0 \leq k \leq (n-1) \quad (4.42)$$

We can easily see from equation (4.42) that the definite integral on the right - hand side is a constant ∝ , i.e we set.

$$\propto = \int_0^1 h(t)u(t)dt, \qquad (4.43)$$

and so equation (4.42) can be written as

$$u^{(n)} = F(x) + \propto g(x) \qquad (4.44)$$

It remains to determine the constant ∝ to evaluate the exact solution $u(x)$ .To find ∝ we should derive a form for $u(x)$ by using equation (4.44) following by substituting the form in equation (4.43) .To achieve this we integrate both sides of equation (4.44) n times from o to , and by using the given initial conditions

$u^{(k)} = b_k$, $0 \leq k \leq (n-1)$ and we obtain an expression for $u(x)$ in the following form

$$u(x) = p(x\,;\,\propto) \quad , \qquad (4.45)$$

where $p(x;\ \propto)$ is the result derived from integrating equation (4.44) and also by using the given initial conditions, substituting equation (4.45) into the right –hand side of equation (4.43) , integrating and solving the resulting algebraic equation to determine $\propto$ .The exact solution of equation (4.42) follows immediately upon substituting the value of $\propto$ into equation (4.45) we consider here to demonstrate the technique with an example.

Example (4.8)

Solve the following Fredholm integro-differential equation.

$$u```(x) = \ sin\, x - x - \int_0^{\frac{\pi}{2}} x\, t\, u`(t) dt \qquad (4.46)$$

Subject to the initial conditions $u(0) = 1,\ \ u`(0) = 0\,, u``(0) = \ -1$

Solution:

This equation can be written in the form

$$u```\ (x) = \sin x - (1+ \propto)x \ \ ,u(0) = 1\ \ ,u`\ (0) = 0\,, u``(0) = -1\,, \quad (4.47)$$

where

$$\propto = \int_0^{\frac{\pi}{2}} t\, u`(t) dt \qquad (4.48)$$

to determine $\propto$, we should find an expression for $u`(x)$ in terms of x and $\propto$ to be used in equation (4.47). This can be done by integrating equation (4.47) three times from 0 to x and using the initial condition ,hence , we find.

$$u``(x) = -\cos x - \frac{1+\propto}{2!}\, x^2$$

$$u`(x) = -\sin x - \frac{1+\propto}{3!}\, x^3$$

$$u(x) = \cos x - \frac{1+\propto}{4!}\, x^4$$

Substituting the expression for $u`(x)$ into equation (4.48) we obtain.

$$\propto = \int_0^{\frac{\pi}{2}} \left[-t\sin t - \frac{1+\propto}{3!}\, t^4\right] dt = -1 \qquad (4.49)$$

Substituting $\propto = -1$ into $u(x) = \cos x - \frac{1+\propto}{4!}\, x^4$ simply yields $u(x) = \cos x$ which is the required solution of the problem.

Example (4.9)

Solve the following Fredholm integro- differential equation by using the direct computation method:

$$u`(x) = \frac{x}{2} - \int_0^1 x\, t\, u\,(t)\, dt \quad , \qquad u(0) = \frac{1}{6}. \qquad (4.50)$$

Solution:

This equation can be written in the form

$$u`(x) = \frac{x}{2} - x \propto , \qquad u(0) = \frac{1}{6} \qquad (4.51)$$

Where

$$\propto = \int_0^1 t\, u\,(t) dt \quad , \qquad (4.52)$$

Now to integrating (4.51) one time from 0 to x and using the initial conditions.

Hence

$$u(x)\Big|_0^x = \frac{x^2}{4}\Big|_0^x - \frac{x^2 \propto}{2}\Big|_0^x$$

$$u(x) - u(0) = \frac{x^2}{4} - \frac{x^2}{2}\propto$$

$$u(x) - \frac{1}{6} = \frac{x^2}{4} - \frac{x^2}{2}\propto$$

$$u(x) = \frac{1}{6} + \frac{x^2}{4}(1 - 2\propto), \qquad (4.53)$$

Substituting (4.53)into equation (4.52)

$$\propto = \int_0^1 t\left[\frac{1}{6} + \frac{t^2}{4}(1 - 2\propto)\right] dt =$$

$$\int_0^1 \left[\frac{t}{6} + \frac{t^3}{4}(1 - 2\propto)\right] dt = \frac{t^2}{12} + \frac{t^4}{16}(1 - 2\propto)\Big|_0^1$$

$$\propto = \frac{1}{12} + \frac{1}{16}(1 - 2\propto)$$

$$\propto = \frac{1}{12} + \frac{1}{16} - \frac{\propto}{8}$$

$$\propto + \frac{\propto}{8} = \frac{1}{12} + \frac{1}{16}$$

$$\propto\left(\frac{9}{8}\right) = \frac{7}{48}$$

$$\propto = \frac{7}{54} \quad , \qquad (4.54)$$

Substituting (4.54) into equation (4.53)

$$u(x) = \frac{1}{6} + \frac{x^2}{4}\left(1 - 2\left(\frac{7}{54}\right)\right)$$

$$u(x) = \frac{1}{6} + \frac{x^2}{4}\left(1 - \left(\frac{7}{27}\right)\right)$$

$$u(x) = \frac{5x^2}{27} + \frac{1}{6}$$

Example (4.10):

Solve the following Fredholm integro- differential equation by using the direct computation method:

$$u`(x) = 2\sec^2 x \tan x - x + \int_0^{\frac{\pi}{4}} x\, t u(t) dt \quad , \quad u(0) = 1 \quad (4.55)$$

Solution:

This equation can be written in the form

$$u`(x) = 2\sec^2 x \tan x - x + x \propto \qquad (4.56)$$

Were

$$\propto = \int_0^{\frac{\pi}{4}} t\, u(t) dt \qquad (4.57)$$

This can be done by intargeting (4.56) one time from 0 $to\ x$ and using the initial conditions.

Hence

$$u(x)-u(0)=\tan^2 x-\frac{1}{2}x^2+\frac{x^2}{2}\propto$$

$$u(x)=1+\tan^2 x-\frac{1}{2}x^2+\frac{1}{2}x^2\propto \qquad (4-58)$$

Substituting (4.58) into equation (4.57)

$$\propto=\int_0^{\frac{\pi}{4}}\left[t+t\tan^2 t-\frac{t^3}{2}+\frac{t^3}{2}\propto\right]dt$$

$$\propto=\int_0^{\frac{\pi}{4}}\left[t\,[1+\tan^2 t]-\frac{t^3}{2}+\frac{t^3}{2}\propto\right]dt$$

$$\propto=\int_0^{\frac{\pi}{4}}\left[t\sec^2 \mathrm{t}-\frac{t^3}{2}+\frac{t^3}{2}\propto\right]dt$$

$$\propto=t\tan t\,\Big|_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}tant\;-\frac{t^4}{8}+\frac{t^4}{8}\propto$$

$$\propto=\frac{\pi}{4}-\ln\cos\frac{\pi}{4}-\frac{\pi^4}{2048}+\frac{\pi^4\propto}{2048}$$

$$\propto\left(1-\frac{\pi^4}{2048}\right)=1-\frac{\pi^4}{2048}$$

$$\propto\left(1-\frac{2048-x^4}{2048}\right)=\frac{2048-\pi^4}{2048}$$

$$\propto=1 \qquad (4.59)$$

Substituting (4.59) into equation (4.58)

$$u(x) = 1 + \tan^2 x - \frac{1}{2}x^2 + \frac{1}{2}x^2$$

$$u(x) = 1 + tan^2 x$$

$$u(x) = sec^2 x$$

4.3.2 The Decomposition Method:

We shall assume standard form to the Fredholm integro-differential equation as given below .

$$u^{(n)}(x) = F(x) + \int_0^1 k(x,t)u(t)dt \quad , \qquad u^{(k)} = b_k(0),$$

$$0 \le k \le (n-1), \qquad (4.59)$$

Substituting $k(x,t) = g(x)h(t)$ into equation (4.59) yields

$$u^{(n)}(x) = F(x) + g(x) \int_0^1 h(t)u(t)dt \qquad (4.60)$$

Equation (4.60) can be written in the operator form as

$$L\,u(x) = fx) + g(x) \int_0^1 h(t)u\,(t)dt\,, \qquad (4.61)$$

where the differential operator is given by

$L = \frac{d^n}{d_x^n}$, It is clear that L is an invertible operator ; therefore , the integral operator $L^{-1}$is an n-fold integration operator and may be considered as definite integrals from 0 to x for each integral . Applying L^{-1} to both sides of equation (4.61) yields .

$$u(x) = b_0 + b_1 x + \frac{1}{2!} b_2 x^2 + \cdots + \frac{1}{(n-1)} \frac{1}{(b-1)} x^{n-1} +$$

$$L^{-1}(f(x) + \left(\int_0^1 h(t)u(t)dt \right) L^{-1}\big(g(x)\big) \qquad (4.62)$$

In other words , we integrate equation (4.60) n times from 0 to x and we use the initial conditions at every step of integration .It is important to note that the equation obtained in equation (4.62) is a standard Fredholm integral equation , This information will be used later.

In the decomposition method, we usually define the solution $u(x)$ of equation (4.59).

in a series form given by

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \qquad (4.63)$$

Substituting equation (4.63) into both sides of equation (4.62) we get

$$\sum_{n=0}^{\infty} u_n(x) = \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}\big(f(x)\big)$$
$$+ \left(\int_0^1 h(t)u(t)dt \right) \; L^{-1}\big(g(x)\big) \qquad (4.64)$$

This can be written explicitly as follows:

$$u_0(x) + u_{1(x)} + u_2(x) + \cdots$$

$$= \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}\big(f(x)\big)$$

$$+ \left(\int_0^1 h(t)u_0\,(t)dt \right) L^{-1}\big(g(x)\big)$$

$$+ \left(\int_0^1 h(t)u_2(t)dt \right) L^{-1}\big(g(x)\big) + \cdots \qquad (4.65)$$

The components $u_0(x)$,$u_1(x)$, $u_2(x)$, ... of the unknown function $u(x)$are determined in are current weren't manner, in a similar fashion as discussed before, if we set

$$u_0(x) = \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}\big(f(x)\big)$$

$$u_1(x) = \left(\int_0^1 h(t)u_0(t)dt \right) L^{-1}\big(g(x)\big)$$

$$u_2(x) = \left(\int_0^1 h(t)u_1(t)dt \right) L^{-1}\big(g(x)\big)$$

$$u_3(x) = \left(\int_0^1 h(t)u_2(t)dt \right) L^{-1}\big(g(x)\big)$$

……………=………….. (4.66)

The above scheme can be written incompact form as follows:

$$u_0(x) = \sum_{k=0}^{n-1} \frac{1}{k!} b_r x^k + L^{-1}\big(f(x)\big)$$

$$u_{n+1}(x) = \left(\int_0^1 h(t) u_n(t) dt \right) L^{-1}(g(x)) \qquad (4.67)$$

In view of equation (4.67), the components of $u(x)$ are immediately determined and consequently the solution $u(x)$ is determined. We shall now consider to demonstrate this method by an example.

Example (4.11)

Solve the following Fredholm integro –differential equation

$$u```(x) = \sin x - x - \int_0^{\frac{\pi}{2}} x\, t\, u`\,(t) dt$$

$$u(0) = 1\,, u`(0) = 0 \quad , u``\,(0) = -1\ , \qquad (4.68)$$

by using the decomposition method

Solution:

Integrating both sides of equation (4.68) from 0 to x three times and using the initial conditions we obtain.

$$u``(x)\Big|_0^x = -\cos x\Big|_0^x - \frac{x^2}{2}\Big|_0^x - \frac{x^2}{2}\Big|_0^x \int_0^{\frac{\pi}{2}} t\, u`\,(t) dt$$

$$u``(x) - u``(0) = -\cos x + \cos(0) - \frac{x^2}{2} - \frac{x^2}{2} \int_0^{\frac{\pi}{2}} t\, u`\,(t) dt$$

$$u``(x) + 1 = -\cos x + 1 - \frac{x^2}{2} - \frac{x^2}{2} \int_0^{\frac{\pi}{2}} t\, u`\,(t) dt$$

$$u``(x) = -\cos x - \frac{x^2}{2} - \frac{x^2}{2}\int_0^{\frac{\pi}{2}} t\, u`\,(t)dt$$

$$u`(x) = -\sin x \Big|_0^x - \frac{x^3}{3!}\Big|_0^x - \frac{x^3}{3!}\Big|_0^x \int_0^{\frac{\pi}{2}} t\, u`\,(t)dt$$

$$u`(x) - u`(0) = -\sin x + \sin(0) - \frac{x^3}{3!} - \frac{x^3}{3!}\int_0^{\frac{\pi}{2}} t\, u`\,(t)dt$$

$$u`(x) = -\sin x - \frac{x^3}{3!} - \frac{x^3}{3!}\int_0^{\frac{\pi}{2}} t\, u`(t)dt$$

$$u(x) = \cos x \Big|_0^x - \frac{x^4}{4!}\Big|_0^x - \frac{x^4}{4!}\Big|_0^x \int_0^{\frac{\pi}{2}} t\, u`\,(t)dt$$

$$u(x) - 1 = cosx - 1 - \frac{x^4}{4!} - \frac{x^4}{4!}\int_0^{\frac{\pi}{2}} t\, u`\,(t)dt$$

$$u(x) - u(0) = \cos x - \cos(0) - \frac{x^4}{4!} - \frac{x^4}{4!}\int_0^{\frac{\pi}{2}} t\, u`\,(t)dt$$

$$u(x) = \cos x - \frac{x^4}{4!} - \frac{x^4}{4!}\int_0^{\frac{\pi}{2}} t\, u`\,(t)dt$$

$$u(x) = cos\ x - \frac{x^4}{4!} - \frac{x^4}{4!}\int_0^{\frac{\pi}{2}} t\, u`\,(t)dt$$

We use the series solution given by

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \tag{4.69}$$

Substituting equation (4.69) into both sides of equation (4.68) yields

$$\sum_{n=0}^{\infty} u_n(x) = \cos x - \frac{x^4}{4!} - \frac{x^4}{4!}\int_0^{\frac{\pi}{2}} t \left(\sum_{n-0}^{\infty} u_n\ (t)\right) dt. \tag{4.70}$$

This can be explicitly written as

$$u_0(x) + u_1(x) + u_2(x) + \cdots = \cos x - \frac{x^4}{4!} - \frac{x^4}{4!}\left(\int_0^{\frac{\pi}{2}} t\, u_n^{`}\ (t)dt\right) -$$

$$\frac{x^4}{4!}\left(\int_0^{\frac{\pi}{2}} t\, u_1^{`}\ (t)dt\right) - \frac{x^4}{4!}\left(\int_0^{\frac{\pi}{2}} t\, u_2^{`}\ (t)dt\right) + \ \ldots. \tag{4.71}$$

Let us set

$$u_0(x) = \cos x - \frac{x^4}{4!} \tag{4.72}$$

$$u_1(x) = \frac{-x^4}{4!}\int_0^{\frac{\pi}{2}} t\left(-\operatorname{sint} - \frac{t^3}{3!}\right) dt =$$

$$\frac{-x^4}{4!}\left[-\left(\int_0^{\frac{\pi}{2}} t \sin t\, dt + \int_0^{\frac{\pi}{2}} \frac{t^4}{3!}\, dt\right)\right] =$$

$$\frac{x^4}{4!} + \frac{\pi^5}{(5)(3!)(32}\ x^4 \qquad , \tag{4.73}$$

Considering the first two components $u_0(x)$ and $u_1(x)$ in equations (4.72) and (4.73) we observe that the term $\frac{x^4}{4!}$ appears in both components with opposite sigen .

So

$$u(x) = \cos x$$

Example (4.12):

Solve the following Fredholm integro-differential equations by using the decomposition method:

$$u`(x) = xe^x + e^x - x + \int_0^1 x\, u\,(t)dt \quad , \quad u(0) = 0\,, \qquad (4.74)$$

Solution:

Integrating both sides of equation (4.74) from 0 to x one times and using initial condition we obtain.

$$u(x)\Big|_0^x = x\,e^x\Big|_0^x - e^x\Big|_0^x + e^x\Big|_0^x - \frac{x^2}{2}\Big|_0^x + \frac{x^2}{2}\int_0^1 u(t)dt$$

$$u(x) - u(0) = x\,e^x - e^x + 1 + e^x - 1 - \frac{x^2}{2} + \frac{x^2}{2}\int_0^1 u(t)dt$$

$$u(x) = xe^x - \frac{x^2}{2} + \frac{x^2}{2}\int_0^1 u(t)dt \qquad , \qquad (4.75)$$

We use the series solution given by

$$u(x) = \sum_{n=0}^{\infty} u_n(x)\,. \qquad (4.76)$$

Substituting (4.76) into both sides of equation (4.75) yields.

$$\sum_{n=0}^{\infty} u_n(x) = x\,e^x - \frac{x^2}{2} + \frac{x^2}{2}\int_0^1 \left(\sum_{n-0}^{\infty} u_n\,(t)\right) dt. \qquad (4.77)$$

This can be explicitly written as

$$u_0(x) + u_1(x) + u_2(x) + \cdots = xe^x - \frac{x^2}{2} + \frac{x^2}{2}\left(\int_0^1 u_0(t)dt\right) +$$

$$\frac{x^2}{2}\left(\int_0^1 u_1(t)dt\right) + \frac{x^2}{2}\left(\int_0^1 u_2(t)dt\right) + \ \ldots \qquad (4.78)$$

Let us set

$$u_0(x) = \ xe^x \ - \frac{x^2}{2} \qquad (4.79)$$

$$u_1(x) = \frac{x^2}{2}\left(\int_0^1 u_2(t)dt\right)$$

$$u_1(x) = \frac{x^2}{2}\int_0^1 \left(te^t - \frac{t^2}{2}\right) dt$$

$$u_1(x) = \frac{x^2}{2}\left[\int_0^1 te^t\,dt - \int_0^1 \frac{t^2}{2}\,dt\right]$$

$$u_1(x) = \left[te^t \Big|_0^1 - \ e^t\Big|_0^1 - \frac{t^3}{6}\Big|_0^1\right]\frac{x^2}{2}$$

$$u_1(x) = \left(e - e + 1 - \frac{1}{6}\right)\frac{x^2}{2}$$

$$u_1(x) = \frac{5x^2}{12} \quad , \quad (4.80)$$

$$u_2(x) = \frac{x^2}{2}\int_0^1 u_1(t)dt = \frac{x^2}{2}\int_0^1 \left(\frac{5t^2}{12}\right) dt$$

$$= \frac{x^2}{2}\left[\frac{5t^3}{36}\right]_0^1 = \frac{x^2}{2}\left[\frac{5}{36}\right] = \frac{5x^3}{72}$$

$$u_2(x) = \frac{5x^3}{72} \quad , \quad (4.81)$$

So the exact solution is

$$u(x) = xe^x$$

4.3.3 Converting to Fredholm Integral Equations:

This section is concerned about a technique that will reduce Fredholm integro differential equations to an equivalent Fredholm integral equation.

This can be easily done by integrating both sides of the integro-differential equation as many times as the order of the derivative involved in the equation from 0 to x for every time we integrate , and using the given initial conditions.

We illustrate an example below.

Example (4.13):

Solve the following Fredholm integro-differential equation.

$$u``(x) = e^x - x + x\int_0^1 t\,u(t)dt\,, u(0) = 1, u`(0) = 1 \ (4.82)$$

by reducing it to aFredholm integral equation

Solution:

Integrating both sides of equation (4.82) twice from 0 to x and using the initial conditions we obtain.

$$u(x)`\Big|_0^x = e^x\Big|_0^x - \frac{x^2}{2}\Big|_0^x + \frac{x^2}{2}\Big|_0^x \int_0^1 t\,u(t)dt$$

$$u`(x) - u`(0) = e^x - 1 - \frac{x^2}{2} + \frac{x^2}{2}\int_0^1 t\,u(t)dt$$

$$u`(x) - 1 = e^x - 1 - \frac{x^2}{2} + \frac{x^2}{2}\int_0^1 t\,u(t)dt$$

$$u`(x) = e^x - \frac{x^2}{2} + \frac{x^2}{2}\int_0^1 t\,u(t)dt$$

$$u(x)\Big|_0^x = e^x\Big|_0^x - \frac{x^3}{3!}\Big|_0^x + \frac{x^3}{3!}\Big|_0^x \int_0^1 t\,u(t)dt$$

$$u(x) - u(0) = e^x - 1 - \frac{x^3}{3!} + \frac{x^3}{3!}\int_0^1 t\,u\,(t)\;\; dt$$

$$u(x) - 1 = e^x - 1 - \frac{x^3}{3!} + \frac{x^3}{3!} \int_0^1 t\, u\,(t)\ \ dt$$

$$u(x) = e^x - \frac{x^3}{3!} + \frac{x^3}{3!} \int_0^1 t\, u\,(t)\ \ dt$$

atypical Fredholm integral equation . By the direct computational method ,this equation can be written as

$$u(x) = e^x - \frac{x^3}{3!} + \frac{x^3}{3!} \propto\ , \qquad (4.83)$$

where the constant $\propto$ is determined by

$$\propto = \int_0^1 t\ \ u(t)\, dt\,, \qquad (4.84)$$

Substituting equation (4.83) into equation (4.84) we obtain

$$\propto = \int_0^1 t \left[e^t - \frac{t^3}{3!} + \frac{t^3}{3!} \propto \right]\ dt\,,$$

$$\propto = \int_0^1 t\, e^t dt - \int_0^1 \frac{t^4}{3!}\, dt + \int_0^1 \frac{t^4}{3!} \propto\ \ dt$$

$$\propto = e - e + 1\ - \frac{1}{30} + \frac{1}{30}\ \propto$$

$$\propto \left(1 - \frac{1}{30}\right) = 1 - \frac{1}{30}$$

$$\propto \left(\frac{29}{30}\right) = \frac{29}{30}$$

$$\propto\ = 1 \qquad (4.85)$$

Substituting (4.85) into equation (4.83)

$$u(x) = e^x - \frac{x^3}{3!} + \frac{x^3}{3!}\ \ (1)$$

Thus , the solution can be written as $u(x) = e^x$.

CHAPTER FIVE

THREE DIMENSIONAL WAVE EQUATION AND HEAT FLOW

5-1 Introduction:

We discuss the initial - boundary value problems that control the wave equation and heat flow in three dimensioned spaces. the decomposition method decomposes

The solution u of any equation into an infinite series of components $u_0, u_1, u_2, \dots$ where theses components are elegantly computed.

5-2 Homogeneous and inhomogeneous PDES:

Partial differential equations are also classified as homogeneous or in homogeneous .Apartial differential of any order is called homogeneous if every term of the PDE contains the dependent variable u or one of its derivatives, other wise, it is called an in homogeneous PDE .This can be illustrated by the following example.

Example (5-2-1) classify the following partial differential equation as homogeneous or in homogeneous:

$$\left.\begin{array}{l} (a)\ u_t = 2u_{xx} \\ \text{(b) } u_t = u_{xx} + x \\ \text{(c)} u_{xx} + u_{yy} = 0 \\ \text{(d) } u_x + u_y = u + 3 \end{array}\right\} \qquad (5-1)$$

Solution :

(a) The terms of the equation contain partial derivatives of u only, therefore it is a homogeneous PDE.
(b) The equation is an in homogeneous PDE, because one term contains the independent variable x.
(c) The equation is a homogeneous PDE.

(d) The equation is an in homogeneous PDE.

5-3 Adomian Decomposition Method:

The decomposition method consists of decomposing the unknown function (u) into an infinite sum of components u_0, u_1, u_2 ,..., and concerns itself with determining these components recurrently .The zeroth component $u_0$is usually identified by the terms arising from integrating inhomogeneous terms and from initial boundary conditions .The successive components $u_1, u_2, ...$ are determined in a recursive manner .

The decomposition method will be applied to three dimensional wave equation .

5-3-1 Three Dimensional Wave Equation:

The propagation of waves in a three dimensional volume of length a, with b , and height d is governed by the following initial boundary value problem

$$\left.\begin{array}{ll} PDE & u_{tt} = c^2\left(u_{xx} + u_{yy} + u_{zz}\right), t > 0 \\ BC & u\,(0, y, z, t) = u(a, y, z, t) = 0 \\ & u\,(x, 0, z, t) = u(x\,, b, z, t) = 0 \\ & u\,(x, y, 0, t) = u(x, y, d, t) = 0 \\ IC & u(x, y, z, 0) = F(x, y, z) = u_t(x, y, z, 0) = g(x, y, z) \end{array}\right\} \quad (5.2)$$

Where $o < x < a\,, o < y < b\,, o < z < d$, and

$u = u\ (x, y, z, t)$ is the displacement of any point located at the position (x, y, z) of a rectangular volume at any time t and c is the velocity of a propagation wave.

As discussed before , the solution in the t space minimizes the volume of calculations . Accordingly , the operator L_t^{-1} will be applied here , we first rewrite (5-2) in an operator form by

$$L_t u = c^2\left(L_x u + L_y u + L_z u\right) \qquad (5-3)$$

Where the differential operators L_x, L_y and L_z are defined by

$$L_t = \frac{\partial^2}{\partial t^2}\ , L_x = \frac{\partial^2}{\partial x^2}\ , L_y = \frac{\partial^2}{\partial y^2}\ , L_z = \frac{\partial^2}{\partial z^2} \quad (5-4)$$

So that the integral operator L_t^{-1} represents a two – fold integration from o to t given by.

$$L_t^{-1}(.) = \int_0^t \int_0^t (.)dt \qquad (5-5)$$

This means that

$$L_t^{-1}\ L_t u\ (x\ , y, z, t) = u(x, y, z, t) - u(x, y, z, 0) - t u_t(x, y, z, 0) \ (5-6)$$

Applying L_t^{-1} to both sides of (5-2) , noting (5-6) and using the initial conditions we find .

$$u(x, y, z, t) = f(x, y, z) + \ t g(x, y, z) + \ c^2 L_t^{-1}\left(L_x u + L_y u + L_z u\right) \ (5-7)$$

The decomposition method defines the solution $u(x, y, z, t)$ as a series given by.

$$u(x,y,z,t) = \sum_{n=0}^{\infty} u_n(x,y,z,t) \qquad (5-8)$$

Substituting (5-8) into both sides of (5-7) yields

$$\sum_{n=0}^{\infty} u_n = F(x,y,z) + tg(x,y,z) + c^2 L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right) + \left(\sum_{n=0}^{\infty} u_n\right) L_y + L_z\left(\sum_{n=0}^{\infty} u_n\right)\right) \quad (5-9)$$

The components $u_n(x,y,z,t)$, $n \geq o$ can be completely determined by using the recursive relation .

$$\left.\begin{aligned} u_0(x,y,z,t) &= F(x,y,z) + tg\,(x,y,z) \\ u_{k+1}(x,y,z,t) &= c^2 L_t^{-1}\left(L_x u_k + L_y u_k + L_z u_k\right), k \geq 0 \end{aligned}\right\} (5-10)$$

Having determined the components $u_n, n \geq 0$

by applying the scheme (5-10) , the solution in a series form follows immediately .

(5-3-1-1) Homogeneous Wave Equations:

The decomposition method will be used to discuss the following homogeneous wave equations in three dimensional space with homogeneous or in homogeneous boundary conditions.

Example (1) :

We shall use he Adomian decomposition method to solve the initial boundary value problem.

$$\left.\begin{array}{ll} PDE & u_{tt} = 3\left(u_{xx} + u_{yy} + u_{zz}\right), 0 < x, y, z < \pi, t > 0 \\ BC & u(o, y, z, t) = u(\pi, y, z, t) = 0 \\ & u(x, 0, z, t) = u(x, \pi, z, t) = 0 \\ & u(x, y, 0, t) = u(x, y, \pi, t) = 0 \\ IC & u(x, y, z, 0) = 0, u_t(x, y, z, 0) = 3\sin x \sin y \sin z \end{array}\right\} \quad (5-11)$$

To do that

The PDE of (5-11) can be rewritten by

$$L_t u = 3\left(L_x u + L_y u + L_z u\right) \qquad (5-12)$$

Applying the inverse operator L_t^{-1} to (5-12) , using (5-6) and substituting the initial conditions we obtain

$$u(x, y, z, t) = 3t \sin x \sin y \sin z + 3\, L_t^{-1}\left(L_x u + L_y u + L_z u\right) \quad (5-13)$$

Using the decomposition series

$$u(x, y, z, t) = \sum_{n=0}^{\infty} u_n(x, y, z, t) \qquad (5-14)$$

Into both sides of (5-13)

$$\sum_{n=0}^{\infty} u_n = 3t \sin x \sin y \sin z + 3\, L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right) + L_y\left(\sum_{n=0}^{\infty} u_n\right) + L_z\left(\sum_{n=0}^{\infty} u_n\right)\right) (5-15)$$

Identifying the zeroth component as discussed before we then set the relation.

$$\left.\begin{aligned} u_0(x,y,z,t) &= 3t \sin x \sin y \sin z \\ u_{K+1}(x,y,z,t) &= 3\, L_t^{-1}\left(L_x u_k + L_y u_k + L_z u_k\right) \;, k \geq 0 \end{aligned}\right\} (5-16)$$

The first few components of the decomposition of u are given by

$$u_0(x,y,z,t) = 3\, t \sin x \sin y \sin z$$

$$u_1(x,y,z,t) = 3L_t^{-1}\left(L_x u_0 + L_y u_0 + L_z u_0\right) =$$

$$3\, L_t^{-1}\, (\, t \cos x \sin y \sin z + t \sin x \cos y \sin z + t \sin x \sin y \cos z) =$$

$$3\, L_t^{-1}(\, -t \sin x \sin y \sin z - t \sin x \sin y \sin z - t \sin x \sin y =$$

$$3\, L_t^{-1}(\, -3 \sin x \sin y \sin z\,) = -\, 9\, L_t^{-1}(\, t \sin x \sin y \sin z) =$$

$$-9\left(\frac{t^2}{2} \sin x sin\, y \sin z\right) = -9\left(\frac{t^3}{6} \sin x \sin y\, sinz\right)$$

$$= -\frac{(3t)^3}{3!} \sin x \sin y \sin z$$

$$u_2(x,y,z,t) = 3L_z^{-1}\left(L_x u_1 + L_y u_1 + L_z u_1\right) = -\frac{(3t)^5}{5!} \sin x\, sin\, y\, sinz$$

$$u_3(x,y,z,t) = 3L_z^{-1}\left(L_x u_2 + L_y u_2 + L_z u_2\right)$$

$$= -\frac{(3t)^7}{7!} \sin x\, sin\, y\, sinz \quad (5-17)$$

So that

The solution in a series form is given by

$$u(x,y,z,t) = \sin x \sin y \sin z \left(3\, t - \frac{(3t)^3}{3!} + \frac{(3t)^5}{5!} - \frac{(3t)^7}{7!} + \cdots\right) (5-18)$$

And in a closed form by

$$u(x,y,z,t) = \sin x \sin y \sin z \sin(3t) \qquad (5-19)$$

Example (2) :

Now we apply the Adomian decomposition method to solve the initial boundary value problem.

$$\left.\begin{array}{ll} PDE & u_{tt} = u_{xx} + u_{yy} + u_{zz} - u \ , 0 < x\ , y\ , z\ < \pi\ , t > 0 \\ BC & u(0,y,z,t) = -u(\pi\ , y,z,t) = \sin y \sin(z+2t) \\ & u\ (x,0,z,t) = -u(x\ ,\pi,z,t) = \sin x \sin(z+2t) \\ & u(\ x,y,0,t) = -u(x,y,\pi,t) = \sin(x+y)\sin(2t) \\ IC & u(\ x,y,z,0) = \sin(x+y)\sin z\ , u_t(x,y,z,0) = 2\sin(x+y)\cos z \end{array}\right\} (5-20)$$

Applying L_t^{-1} to (5-20) , using (5-6) gives

$$u(x,y,z,t) = \sin(x+y) sinz + 2\,t\sin(x+y)\cos z +$$

$$L_t^{-1}\big(L_x u + L_y u + L_z u - u\big) \quad (5-21)$$

Therefore we find

$$\sum_{n=0}^{\infty} u_n = \sin(x+y)\sin z + 2\,t\sin(x+y)\cos z + L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right)\right.$$

$$\left. + L_y\left(\sum_{n=0}^{\infty} u_n\right) - L_z\left(\sum_{n=0}^{\infty} u_n\right)\right)(5-22)$$

This means that

$$\left.\begin{array}{r} u_0(x,y,z,t) = \sin(x+y)\sin z + 2\,t\sin(x+y)\cos z \\ u_{k+1}(x,y,z,t) = L_t^{-1}\big(L_x u_k + L_y u_k + L_z u_k - u_k\big), k \geq 0 \end{array}\right\} (5-23)$$

and therefore we obtain

$$\left.\begin{array}{l} u_0(x,y,z,t) = \sin(x+y) sinz + 2t\sin(x+y)\cos z \\ u_1(x,y,z,t) = L_t^{-1}\left(L_x u_0 + L_y u_0 + L_z u_0 - u_0\right) = \\ -\frac{(2t)^2}{2!}\sin(x+y)\sin z - \frac{(2t)^3}{3!}\sin(x+y)\cos z \\ u_2(x,y,z,t) = L_t^{-1}\left(L_x u_1 + L_y u_1 + L_z u_1 - u_1\right) = \\ \frac{(2t)^4}{4!}\sin(x+y) sinz + \frac{(2t)^5}{5!}\sin(x+y)\cos z \end{array}\right\} \quad (5-24)$$

The solution in a series form is given by

$$u(x,y,z,t) = \sin(x+y)\sin z\left(1 - \frac{(2t)^2}{2!} + \frac{(2t)^4}{4!} - \dots\right) +$$

$$\sin(x+y)\cos z\ \left(2t - \frac{(2t)^3}{3!} + \frac{(2t)^5}{5!} - \cdots\right) \qquad (5-25)$$

and consequently the exact solution is

$$u(x,y,z) = \sin(x+y)(\sin z\cos(2t) + \cos z\sin(2t)) \quad (5-26)$$

Example (3)

In this example we apply Adomian decomposition method to solve the homogeneous problems.

$$\left.\begin{array}{ll} PDE & u_{tt} = \frac{1}{3}\left(u_{xx} + u_{yy} + u_{zz}\right),\ 0 < x, y, z < \pi, t > 0 \\ BC & u(0,y,z,t) = u(\pi,y,z,t) = 1 \\ & u(x,0,z,t) = u(x,\pi,z,t) = 1 \\ & u(x,y,0,t) = u(x,y,\pi,t) = 1 \\ IC & u(x,y,z,0) = 1,\ u_t(x,y,z,0) = sinx\sin y\sin z \end{array}\right\} (5-27)$$

The PDE of (5-27) can be rewritten by

$$L_t u = \frac{1}{3}\left(L_x u + L_y u + L_z u\right) \qquad (5-28)$$

Applying L_t^{-1} to (5-28) using (5-6) and substituting the initial condition we obtain

$$u(x,y,z,t) = 1 + t\sin x sin\ y sinz + \frac{1}{3}L_t^{-1}\left(L_x u + L_y u + L_z u\right)(5-29)$$

Using the decomposition series

$$u(x,y,z,t) = \sum_{n=0}^{\infty} u_n(x,y,z,t) \qquad (5-30)$$

$$\sum_{n=0}^{\infty} u_n = 1 + t\sin x\ siny\sin z + \frac{1}{3}\,L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right) + L_y\left(\sum_{n=0}^{\infty} u_n\right) + L_z\left(\sum_{n=0}^{\infty} u_n\right)\right)(5-31)$$

This means that

$$u_0(x,y,z,t) = 1 + t\sin x\sin y\sin z \qquad (5-32)$$

$$u_{k+1}(x,y,z,t) = \frac{1}{3}\,L_z^{-1}\left(L_x u_k + L_y u_k + L_z u_k\right) \quad (5-33)$$

The first few components of the decomposition of u are given by

$$\left.\begin{aligned} u_0(x,y,z,t) &= 1 + t\sin x\sin y\sin z \\ u_1(x,y,z,t) &= \frac{1}{3}\,L_t^{-1}\left(L_x u_0 + L_y u_0 + L_z u_0\right) = -\frac{t^3}{3!}\sin x\sin y\ sinz \\ u_2(x,y,z,t) &= \frac{1}{3}\,L_t^{-1}\left(L_x u_1 + L_y u_1 + L_z u_1\right) = \frac{t^5}{5!}\sin x\sin y\sin z \\ u_3(x,y,z,t) &= \frac{1}{3}\,L_t^{-1}\left(L_x u_2 + L_y u_2 + L_z u_2\right) = -\frac{t^7}{7!}\sin x\sin y\sin z \end{aligned}\right\}(5-34)$$

The series solution and the exact solution are given by

$$u(x,y,z,t) = 1 + t\sin x sin\, y\sin z - \frac{t^3}{3!}\sin x siny\; sinz +$$

$$\frac{t^5}{5!}\sin x\sin y\sin z - \frac{t^7}{7!}\sin x\sin y\sin z =$$

$$1 + \sin x\sin y\sin z\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\right) \quad (5-35)$$

(5-3-1-2) Inhomogeneous Wave Equations:

We now consider the inhomogeneous wave equation in a three dimensional space of the form.

$$u_{tt} = c^2\left(u_{xx} + u_{yy} + u_{zz}\right) + h\,(x,y,z) \qquad (5-36)$$

Where $h\,(x,y,z)$is an inhomogeneous term .

The decomposition method can be applied without any need to transform this equation to a homogenous equation.

The following examples will be used to explain the implementation of the method.

Example (1)

In this example we apply Adomian decomposition method to solve the initial boundary value problem.

$$\left.\begin{array}{ll} PDE & u_{tt} = u_{xx} + u_{yy} + u_{zz} + \sin x + \sin y\,, 0 < x,y,z < \pi \\ BC & u(0,y,z,t) = u(\pi,y,z,t) = \sin y + \sin z\sin t, \\ & u(x,0,z,t) = u(x,\pi,z,t) = \sin x + \sin z\sin t \\ & u(x,y,0,t) = u(x,y,\pi,t) = \sin x + \sin y\,, \\ IC & u(x,y,z,0) = \sin x + \sin y\,, u_t(x,y,z,0) = \sin z \end{array}\right\}(5-$$

37)

Operating with L_t^{-1} on (5-37) gives

$$u = \sin x + siny + t\sin z + \frac{t^2}{2!}\sin x + \frac{t^2}{2!}\sin y$$
$$+ L_t^{-1}\left(L_x u + L_y u + L_z u\right)(5-38)$$

We substitute the decomposition series

$$u(x,y,z,t) = \sum_{n=0}^{\infty} u_n(x,y,z,t) \qquad (5-39)$$

into both sides of (5-37) to obtain

$$\sum_{n=0}^{\infty} u_n = \sin x + \sin y + t\sin z + \frac{t^2}{2!}\sin x + \frac{t^2}{2!}\sin y$$
$$+ \ L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right) + L_y\left(\sum_{n=0}^{\infty} u_n\right) + L_z\left(\sum_{n=0}^{\infty} u_n\right)\right) \ (5-40)$$

This leads to

$$\left.\begin{aligned}
&u_0(x,y,z,t) = \sin x + \sin y + t\sin z + \frac{t^2}{2!}\ sin\,x + \frac{t^2}{2!}\sin y \\
&u_1(x,y,z,t) = \ L_t^{-1}\left(L_x u_0 + L_y u_0 + L_z u_0\right) = -\frac{t^2}{2!}\sin x - \frac{t^2}{2!}\sin y - \frac{t^3}{3!}sinz - \frac{t^4}{4!}sinx - \frac{t^4}{4!}siny \\
&u_2(x,y,z,t) = \ L_t^{-1}\left(L_x u_1 + L_y u_1 + L_z u_1\right) = \\
&\frac{t^4}{4!}\sin x + \frac{t^4}{4!}\sin y + \frac{t^5}{5!}\sin z + \frac{t^6}{6!}\sin x + \frac{t^6}{6!}\sin y \\
&u_3(x,y,z,t) = L_t^{-1}\left(L_x u_2 + L_y u_2 + L_z u_2\right) = \\
&-\frac{t^6}{6!}\sin x - \frac{t^6}{6!}\sin y - \frac{t^7}{7!}\sin z - \frac{t^8}{8!}\sin x - \frac{t^8}{8!}\sin y
\end{aligned}\right\}(5-41)$$

The series solution and the exact solution are given by

$$u(x,y,z,t) = \sin x + \sin y + \sin z\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\right) \ (5-42)$$

$$u(x,y,z,t) = \sin x + \sin y + \sin z \sin t \qquad (5-43)$$

Example (2)

In this example we apply Adomian decomposition method to solve the inhomogeneous problem

$$\left.\begin{array}{lll} PDE & u_{tt} = \frac{1}{2}\left(u_{xx} + u_{yy} + u_{zz}\right) - 1\,, 0 < x, y, z < \pi, t\ > 0 \\ BC & u_x(0, y, z, t) = 0\ \ , u_x(\pi, y, z, t) = 2\pi \\ & u_y(x, 0, z, t) = -u_y(x, \pi, z, t) = \sin z \cos t \\ & u_z(x, y, 0, t) = \ -u_z(x, y, \pi, t) = \sin y \cos t \\ IC & u(x, y, z, 0) = x^2 + \sin x \sin z\,, u_t(x, y, z, 0) = 0 \end{array}\right\} (5-44)$$

Operating with L_z^{-1} on (5-44)

$$u = x^2 + \sin x \sin z - \frac{t^2}{2!} + \frac{1}{2} L_t^{-1}\left(L_x u + L_y u + L_z u\right) \quad (5-46)$$

$$u(x, y, z, t) = \sum_{n=0}^{\infty} u_n(x, y, z, t) \qquad (5-47)$$

$$\sum_{n=0}^{\infty} u_n = x^2 + \sin x \sin z - \frac{t^2}{2!} + \frac{1}{2} L_z^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right) + L_y\left(\sum_{n=0}^{\infty} u_n\right) + \ L_z\left(\sum_{n=0}^{\infty} u_n\right)\right) (5-47)$$

$$\left.\begin{array}{r} u_0(x, y, z, t) = x^2 + \sin x \sin z - \frac{t^2}{2!} \\ u_1(x, y, z, t) = \frac{1}{2} L_t^{-1}\left(L_x u_0 + L_y u_0 + L_z u_0\right) = \ -\frac{t^2}{2!} \sin x \sin z \\ u_2(x, y, z, t) = \frac{1}{2} L_t^{-1}\left(L_x u_1 + L_y u_1 + L_z u_1\right) = \ \frac{t^4}{4!} \sin x \sin z \end{array}\right\} (5-48)$$

The series solution and the exact solution are given by

$$u(x,y,z,t) = x^2 + \sin x \sin z - \frac{t^2}{2!}\sin x\, sinz + \frac{t^4}{4!}\sin x \sin z$$

$$= x^2 + \sin x\, sinz\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \ldots\right) \quad (5-49)$$

and

$$u(x,y,z,t) = x^2 + \sin x \sin z \cos t \qquad (5-50)$$

(5-3-2) Three Dimensional Heat Flow :

The distribution of heat flow in a three dimensional space is governed by the following initial boundary value problem.

$$\left.\begin{array}{ll} PDE & u_t = \bar{k}\left(u_{xx} + u_{yy} + u_{zz}\right), t > 0, \\ & 0 < x < a, 0 < y < b, 0 < z < c \\ BC & u(0,y,z,t) = u(a,y,z,t = 0 \\ & u(x,0,z,t) = u(x,b,z,t) = 0 \\ & u(x,y,0,t) = u(x,y,c,t) = 0 \\ IC & u(x,y,z,0) = F(x,y,z) \end{array}\right\} (5-51)$$

Where $u = u(x,y,z,t)$ is the temperature of any point located at the position (x,y,z) of a retangular volume at any time t , and $\bar{k}$ is the thermal diffusivity we first rewrite (5-51) in an operator form by

$$L_t u = \bar{k}\left(L_x u + L_y u + L_z u\right) \qquad (5-52)$$

Where the differential operators L_x, L_y and L_z are defined by

$$L_t = \frac{\partial}{\partial t}, L_x = \frac{\partial^2}{\partial x^2}, L_y = \frac{\partial^2}{\partial y^2}, L_z = \frac{\partial^2}{\partial z^2} \quad (5-53)$$

So that the integral operator L_t^{-1} exists and given by

$$L_t^{-1}(.) = \int_0^t (.)dt \qquad (5-54)$$

Applying L_t^{-1} to both sides of (5-52) and using the initial condition leads to

$$u(x,y,z,t) = F(x,y,z) + \bar{k}\, L_t^{-1}\left(L_x u + L_y u + L_z u\right) \quad (5-55)$$

The decomposition method defines the solution $u(x,y,z,t)$ as a series given by

$$u(x,y,z,t) = \sum_{n=0}^{\infty} u_n(x,y,z,t) \qquad (5-56)$$

Substituting (5-56) into both sides of (5-55)

$$\sum_{n=0}^{\infty} u_n = F(x,y,z) + \bar{k}\, L_t^{-1}\left(L_x\left(\sum_{n=1}^{\infty} u_n\right) + L_y\left(\sum_{n=1}^{\infty} u_n\right) + L_z\left(\sum_{n=1}^{\infty} u_n\right)\right) \quad (5-56$$

The components $u_n(x,y,z,t)$, $n \geq 0$ can be completely determined by using the recursive relationship.

$$\left.\begin{array}{l} u_0(x,y,z,t) = F(x,y,z) \\ u_{k+1}(x,y,z,t) = \bar{k}L_t^{-1}\left(L_x u_k + L_y u_k + L_z u_k\right), k \geq 0 \end{array}\right\} \quad (5-57)$$

The components can be determind recursively as do as we like. consequently , the components u_n $n \geq 0$, are completely determined and the solution in a series form follows immediately.

(5-3-2-1) Homogeneous Heat Equations:

The decomposition method will be used to discuss the following homogeneous heat equations .

Example (1) :

We shall use the Adomian decomposition method to solve the following initial – boundary value problem.

$$\left.\begin{array}{ll} PDE & u_t = u_{xx} + u_{yy} + u_{zz}, 0 < x, y\, , z\; < \pi, t > 0 \\ BC & u(0, y, z, t) = u(\pi, y, z, t) = 0 \\ & u(x, 0, z, t) = u(x, \pi, z, t) = 0 \\ & u(x, y, 0, t) = u(x, y, \pi, t) = 0 \\ IC & u(x, y, z, 0) = 2\sin x\; siny\; sinz \end{array}\right\} \quad (5-58)$$

Applying the inverse operator L_t^{-1} to the operator form of (5-58) gives .

$$u(x, y, z, t) = \; 2\sin x \sin y \sin z + \; L_t^{-1}\big(\, L_x u + L_y u + L_z u\big) \; (5-59)$$

Using the decomposition series

$$u(x, y, z, t) = \sum_{n=0}^{\infty} u_n(x, y, z, t) \qquad\qquad (5-60)$$

Into (5-59)

$$\sum_{n=0}^{\infty} u_n = 2\; sinx \sin y \sin z$$

$$+\; L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right) + L_y\left(\sum_{n=0}^{\infty} u_n\right) + L_z\left(\sum_{n=0}^{\infty} u_n\right)\right) \; (5-61)$$

The components $u_n(x, y, z, t), n\; \geq 0$ can be determined by using the recurrence relation

$u_0(x, y, z, t) = 2\sin x \sin y \sin z$

$$u_{k+1}(x,y,z,t) = L_t^{-1}\left(L_x u_k + L_y u_k + L_z u_k\right), k \geq 0\} \quad (5-62)$$

It follows that the first few terms of the decomposition series of $u(x,y,z,t)$ are given by.

$$u_0(x,y,z,t) = 2\sin x\, siny\, sinz$$

$$\left.\begin{aligned} u_1(x,y,z,t) &= L_t^{-1}\left(L_x u_0 + L_y u_0 + L_z u_0\right) = -2(3t) sinx\, siny\, sinz \\ u_2(x,y,z,t) &= L_t^{-1}\left(L_x u_1 + L_y u_1 + L_z u_1\right) = 2\frac{(3t)^2}{2!} sinx\, siny\, sinz \\ u_3(x,y,z,t) &= L_t^{-1}\left(L_x u_2 + L_y u_2 + L_z u_2\right) = -2\frac{(3t)^3}{3!} sinx\, siny\, sinz \end{aligned}\right\} (5-63)$$

and so on.

The solution in a series form is given by

$$u(x,y,z,t) = 2\sin x siny\, sinz \left(1 - 3t + \frac{(3t)^2}{2!} - \frac{(3t)^2}{3!} + \ldots\right) \quad (5-64)$$

and in a closed form by

$$u(x,y,z,t) = 2e^{-3t}\sin x \sin y \sin z \qquad (5-65)$$

Example (2)

Now we apply the Admion decomposition method to solve the homogeneous initial boundary value problems:

$$\left.\begin{aligned} &PDE \quad u_t = 2\left(u_{xx} + u_{yy} + u_{zz}\right), 0 < x,y,z < \pi, t > 0 \\ &BC \quad u(0,y,z,t) = u(\pi,y,z,t) = 0 \\ &\qquad u\,(x,0,z,t) = u(x,\pi,z,t) = 0 \\ &\qquad u(x,y,0,t) = u(x,y,\pi,t) = 0 \\ &IC \quad u(x,y,z,0) = \sin x \sin y \sin z \end{aligned}\right\} (5-66)$$

Appling the inverse operator L_t^{-1} to (5-66) give

$$u(x,y,z,t) = \sin x \sin y \sin z + 2L_t^{-1}\left(L_x u + L_y u + L_z u\right) \quad (5-67)$$

and hence we find

$$\sum_{n=0}^{\infty} u_n = \sin x \sin y \sin z + 2L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right) + L_y\left(\sum_{n=0}^{\infty} u_n\right) + L_z\left(\sum_{n=0}^{\infty} u_n\right)\right) \quad (5-68)$$

$$u_0(x,y,z,t) = \sin x \sin y \sin x$$

$$u_{k+1}(x,y,z,t) = 2L_t^{-1}\left(L_x u_k + L_y u_k + L_z u_k\right), k \geq 0\} \quad (5-69)$$

Consequently , the first few components

$$u_0(x,y,z,t) = \sin x \sin y \sin z$$

$$\left.\begin{aligned} u_1(x,y,z,t) &= 2L_t^{-1}\left(L_x u_0 + L_y u_0 + L_z u_0\right) = -6t \sin x \sin y \sin z \\ u_2(x,y,z,t) &= 2L_t^{-1}\left(L_x u_1 + L_y u_1 + L_z u_1\right) = \frac{(6t)^2}{2!} \sin x sin\, y \sin z \\ u_3(x,y,z,t) &= 2L_t^{-1}\left(L_x u_2 + L_y u_2 + L_z u_2\right) = -\frac{(6t)^3}{3!} \sin x sin\, y \sin z \end{aligned}\right\} \quad (5-70)$$

The solution in a series form is given by

$$u(x,y,z,t) = u_0(x,y,z,t) + u_1(x,y,z,t) + u_2(x,y,z,t) + u_3(x,y,z,t) + \cdots \quad (5-71)$$

$$u(x,y,z,t) = \sin x sin\, y \sin z \left(1 - 6t + \frac{(6t)^2}{2!} - \frac{(6t)^3}{3!} + \cdots\right) \quad (5-72)$$

and in closed form by

$$u(x,y,z,t) = e^{-6t} \sin x \sin y \sin z \qquad (5-73)$$

Example (3) :

We shall use the Adomian decomposition method to solve the homogeneous initial – boundary value problems.

$$\left.\begin{array}{lll} PDE & u_t = u_{xx} + u_{yy} + u_{zz} & 0 < x,y,z < \pi, t > 0 \\ BC & u(0,y,z,t) = -u(\pi,y,z,t) = e^{-3t}\sin(x+z) & \\ & u(x,0,z,t) = -u\,(x,\pi,z,t) = e^{-3t}\sin(x+z) & \\ & u(x,y,0,t) = -u\,(x,y,\pi,t) = e^{-3t}\sin(x+y) & \\ IC & u(x,y,z,0) = \sin(x+y+z) & \end{array}\right\} (5-74)$$

Applying the inverse operator L_t^{-1} to the operator form (5-74) gives

$$u(x,y,z,t) = \sin(x+y+z) + L_t^{-1}\left(L_x u + L_y u + L_z u\right)(5-75)$$

using the decomposition series

$$u(x,y,z,t) = \sum_{n=0}^{\infty} u_n(x,y,z,t) \qquad (5-76)$$

$$\sum_{n=0}^{\infty} u_n = \sin\,(\mathrm{x+y+z}) + L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right) + L_y\left(\sum_{n=0}^{\infty} u_n\right) + L_z\left(\sum_{n=0}^{\infty} u_n\right)\right) (5-77)$$

$$\left.\begin{array}{l} u_0(x,y,z,t) = \sin(x+y+z) \\ u_{k+1}(x,y,z,t) = L_t^{-1}\left(L_x u_k + L_y u_k + L_z u_k\right), k \geq 0 \end{array}\right\} (5-78)$$

Consequently , we obtain.

$$\left.\begin{aligned} u_0(x,y,z,t) &= \sin(x+y+z) \\ u_1(x,y,z,t) &= L_t^{-1}\big(L_x u_0 + L_y u_0 + L_z u_0\big) = -3\,t\sin(x+y+z) \\ u_2(x,y,z,t) &= L_t^{-1}\big(L_x u_1 + L_y u_1 + L_z u_1\big) = \frac{(3t)^2}{2!}\sin(x+y+z) \\ u_3(x,y,z,t) &= L_t^{-1}\big(L_x u_2 + L_y u_2 + L_z u_2\big) = \frac{(3t)^3}{3!}\sin(x+y+z) \end{aligned}\right\} (5-79)$$

The solution in a series form is given by

$$u(x,y,z,t) = \sin(x+y+z)\left(-3t + \frac{(3t)^2}{2!} - \frac{(3t)^3}{3!} + \cdots\right) \quad (5-80)$$

and in a closed form by

$$u(x,y,z,t) = e^{-3t}\sin(x+y+z) \qquad (5-81)$$

Example (4) :

We shall use the Adomian decomposition method to solve the following initial – boundary value problem.

$$\left.\begin{array}{ll} PDE & u_t = u_{xx} + u_{yy} + u_{zz} - u\,,\, 0 < x,y,z < \pi\,, t > 0 \\ BC & u(0,y,z,t) = u(\pi,y,z,t) = 0 \\ & u\,(x,0,z,t) = u(x,\pi,z,t) = 0 \\ & u(x,y,0,t) = u(x,y,\pi,t) = 0 \\ IC & u(x,y,z,0) = \sin x \sin y \sin z \end{array}\right\} (5-82)$$

Operating with L_t^{-1} on (5-82) we obtain.

$$u(x,y,z,t) = \sin x \sin y \sin z + L_t^{-1}\big(L_x u + L_y u + L_z u - u\big) \quad (5-83)$$

Proceeding as before we find

$$\sum_{n=0}^{\infty} u_n = \sin x \, sin\, y\, sin\, z + L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right) + L_y\left(\sum_{n=0}^{\infty} u_n\right) + L_z\left(\sum_{n=0}^{\infty} u_n - \sum_{n=0}^{\infty} u_n\right)\right) (5-84)$$

Using the assumptions of the decomposition method yields

$$u_0(x, y, z, t) = \sin x \sin y \sin z$$

$$u_{k+1}(x, y, z, t) = L_t^{-1}\left(L_x u_k + L_y u_k + L_z u_k - u_k\right), k \geq 0\} \quad (5-85)$$

Consequently , we obtain

$$\left.\begin{aligned} u_0(x, yz, t) &= \sin x \sin y \sin z \\ u_1(x, y, z, t) &= L_t^{-1}\left(L_x u_0 + L_y u_0 + L_z u_0 - u_0\right) = -4t \sin x \, siny \sin z \\ u_2(x, y, z, t) &= L_t^{-1}\left(L_x u_1 + L_y u_1 + L_z u_1 - u_1\right) = \frac{(4t)^2}{2!} sinx \sin y \sin z \\ u_3(x, y, z, t) &= L_t^{-1}\left(L_x u_2 + L_y u_2 + L_z u_2 - u_2\right) = -\frac{(4t)^3}{3!} sinx \sin y \sin z \end{aligned}\right\} (5-86)$$

The solution in a series form is given by

$$u(x, y, z, t) = \sin x \sin \sin z \left(1 - 4t + \frac{(4t)^2}{2!} - \frac{(4t)^3}{3!} + \cdots\right) \quad (5-87)$$

and in a closed form by

$$u(x, y, z, t) = e^{-4t} \sin x \sin y \sin z \qquad (5-88)$$

(5-3-2-2) Inhomogeneous Heat Equation :

In the following , the Adomian decomposition method will be applied to inhomogeneous heat equations.

The method will be implemented in a like manner to that used inhomogeneous cases .

Example (1):

Now we apply the Adomion decomposition method to solve the following initial – boundary value problem .

$$\left.\begin{array}{lll} PDE & u_t = u_{xx} + u_{yy} + u_{zz} - 4 & ,0 < x,y,z < \pi ,t > 0 \\ BC & u(0,y,z,t) = 0 ,u(\pi,y,z,t) = 2\pi^2 & \\ & u(x,0,z,t) = u\ (x,\pi,z,t) = 2x^2 & \\ & u(x,y,0,t) = u\ (x,y,\pi,t) = 2x^2 & \\ IC & u(x,y,z,0) = 2x^2 + \sin x \sin y \sin z & \end{array}\right\} (5-89)$$

Operating with L_t^{-1} on (5-89) we obtain

$$u(x,y,z,t) = -4t + 2x^2 + sinx\ siny\ sinz + L_t^{-1}\left(L_x u + L_y u + L_z u\right)(5-90)$$

and proceeding as before we fined

$$\sum_{n=0}^{\infty} u_n = -4t + 2x^2 + sinx\ siny\ sinz + L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right) + L_y\left(\sum_{n=0}^{\infty} u_n\right) + L_z\left(\sum_{n=0}^{\infty} u_n\right)\right) \quad (5-91)$$

We next set the recurrence relation.

$$\left.\begin{array}{r} u_0(x,y,z,t) = -4t + 2x^2 + sinx\ siny\ sinz \\ u_{k+1}(x,y,z,t) = L_t^{-1}\left(L_x u_k + L_y u_k + L_z u_k\right), k \geq 0 \end{array}\right\} \quad (5-92)$$

The first few terms of the decomposition series are

$$\left.\begin{array}{l} u_0(x,y,z,t) = -4t + 3x^2 + sinx\ siny\ sinz \\ u_1(x,y,z,t) = L_t^{-1}\left(L_x u_0 + L_y u_0 + L_t u_0\right) = -3t\ sinx\ siny\ sinz + 4t \\ u_2(x,y,z,t) = L_t^{-1}\left(L_x u_1 + L_y u_1 + L_z u_1\right) = \dfrac{(3t)^2}{2!} sinx\ siny\ sinz \\ u_3(x,y,z,t) = L_t^{-1}\left(L_x u_2 + L_y u_2 + L_z u_2\right) = \dfrac{(3t)^3}{3!} sinx\ siny\ sinz \end{array}\right\} (5-93)$$

The solution in a series form is given by

$$u(x,y,z,t) = 2x^2 + sinx\ siny\ sinz\ (1 - 3t + \frac{(3t)^2}{2!} - \frac{(3t)^3}{3!} + \cdots)(5-94)$$

and in a closed form by

$$u(x,y,z,t) = 2x^2 + \ e^{-3t} sinx\ siny\ sinz \qquad (5-95)$$

Example (2) :

In this example we apply the Adomian decomposition method to solve the in homogeneous initial – boundary value problems.

$$\left.\begin{array}{l} PDE \quad u_t = u_{xx} + u_{yy} + u_{zz} + sinx\ , 0\ < x,y,z < \pi, t > 0 \\ BC \quad u(0,y,z,t) = u(\pi,y,z,t) = e^{-2}\sin(y+z) \\ u(x,0,z,t) = sin + e^{-2t} sinz\ , u(x,\pi,z,t) = sinx - e^{-2t} sinz \\ u(x,y,0,t) = sinx + e^{-2t} siny\ , u(x,y,\pi,t) = sinx - e^{-2t} siny \\ IC \quad u\ (x,y,z,0) = sinx + \sin(y+z) \end{array}\right\} (5-96)$$

Applying the inverse operator L_t^{-1} to (5-95) gives

$$u(x,y,z,t) = sinx + \sin(y+z) + t\ sinx + L_t^{-1}(L_x u + L_y u + L_z u)(5-97)$$

and this turn gives

$$\sum_{n=0}^{\infty} u_n = sinx + \sin(y+z) + t\ sinx$$
$$+ L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right) + L_y\left(\sum_{n=0}^{\infty} u_n\right) + L_z\left(\sum_{n=0}^{\infty} u_n\right)\right) (5-98)$$

Accordingly ,we set the recursive relationship.

$$\left.\begin{array}{l} u_0(x,y,z,t) = \ sinx + \sin(y+z) + t\ sinx \\ u_{k+1}(x,y,z,t) = \ L_t^{-1}(L_x u_k + L_y u_k + L_z u_k)\ , k \geq 0 \end{array}\right\} (5-99)$$

Consequently , the first few components

$$\left.\begin{aligned}
u_0(x,y,z,t) &= sinx + \sin(y+z) + tsinx\\
u_1(x,y,z,t) &= L_t^{-1}\left(L_x u_0 + L_y u_0 + L_z u_0\right) = -tsinx - 2tsin(y+z) - \frac{t^2}{3!}sinx\\
u_2(x,y,z,t) &= L_t^{-1}\left(L_x u_1 + L_y u_1 + L_z u_1\right) = \frac{t^2}{2!}sinx + \frac{t^2}{2!}\sin(y+z) + \frac{t^3}{3!}sinx\\
u_3(x,y,z,t) &= L_t^{-1}\left(L_x u_2 + L_y u_2 + L_z u_2\right) = -\frac{t^3}{3!}sinx - \frac{(2t)^3}{3!}\sin(y+z) - \frac{t^4}{4!}sinx
\end{aligned}\right\} (5-100)$$

The solution in a series form is given by

$$u(x,y,z,t) = sinx + \sin(y+z)\left(1 - 2t + \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!} + \cdots\right)(5-101)$$

$$u(x,y,z,t) = sinx + e^{-2t}\sin(y+z) \qquad (5-102)$$

Example (3)

We shall use the Adomian decomposition method to solve inhomogeneous initial – boundary value problems.

$$\left.\begin{aligned}
&PDE \quad u_t = u_{xx} + u_{yy} + u_{zz} - 2 \ ,0 < x,y,z < \pi \ ,t > 0\\
&BC \quad u(0,y,z,t) = e^{-3t}(siny + sinz)\\
&\qquad u(\pi,y,z,t) = \pi^2 + e^{-3t}(siny + sinz)\\
&\quad u(x,0,z,t) = u(x,\pi,z,t) = x^2 + e^{-3t}(sinx + sinz)\\
&\quad u(x,y,0,t) = u(x,y,\pi,t) = x^2 + e^{-3t}(sinx + sinz)\\
&IC \quad u(x,y,z,0) = x^2 + (sinx + siny + sinz)
\end{aligned}\right\} (5-103)$$

Applying the inverse operator L_t^{-1} to (5-103) and using the initial condition we obtain.

$$u(x,y,z,t) = -2t + x^2 + (sinx + siny + sinz) + L_t^{-1}\left(L_x u + L_y u + L_z u\right)(5-104)$$

and proceeding as before we find

$$\sum_{n=0}^{\infty} u_n = -2t + x^2 + \ (sinx + siny + sinz) + \ L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right)\right.$$

$$\left. + L_y\left(\sum_{n=0}^{\infty} u_n\right) + L_z\left(\sum_{n=0}^{\infty} u_n\right)\right) \quad (5-105)$$

We next set the recurrence relation

$$\left.\begin{aligned} u_0(x,y,z,t) &= -2t + x^2 + (sinx + siny + sinz) \\ u_{k+1}(x,y,z,t) &= \ L_t^{-1}\big(L_x u_k + L_y u_k + L_z u_k\big), k \geq 0 \end{aligned}\right\} \quad (5-106)$$

The first few terms of the decomposition series are.

$$\left.\begin{aligned} u_0(x,y,z,t) &= -2t + x^2(sinx + siny + sinz) \\ u_1(x,y,z,t) &= L_t^{-1}\big(L_x u_0 + L_y u_0 + L_z u_0\big) = -3t(sinx + siny + sinz) + 2t \\ u_2(x,y,z,t) &= L_t^{-1}\big(L_x u_1 + L_y u_1 + L_z u_1\big) = \frac{(3t)^2}{2!}(sinx + siny + sinz) \\ u_3(x,y,z,t) &= L_t^{-1}\big(L_x u_2 + L_y u_2 + L_z u_2\big) = \frac{(3t)^3}{3!}(sinx + siny + sinz) \end{aligned}\right\} \quad (5-107)$$

The solution in a series form is given by

$$u(x,y,z,t) = x^2(sinx + siny + sinz)\left(1 - 3t + \frac{(3t)^2}{2!} - \frac{(3t)^3}{3!} + ..\right) \quad (5-108)$$

and in a closed form by

$$u(x,y,z,t) = x^2 + e^{-3t}(sinx + siny + sinz) \qquad (5-109)$$

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