Sudan University of Science and Technology

College of Graduate Studies

Logarithmic Bump with Bilinear T (B) Theorem and Maximal Singular Integral Operators

ثنائية الخطية ومؤثرات T (B)النتوء اللوغريثمي مع مبرهنة التكامل الشاذة الأعظمية

A Thesis Submitted in Partial Fulfillment Requirements for the Degree of M.Sc in Mathematics

By

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Dedication

I would like to dedicate this simple work to my

mother and father who touch me the meaning of life

husband

daughter

brother

sisters

aunt she is my second mother

friends who share me in our roads

and to allwhom help in preparing this simple work

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First I would like to thank Allah for giving me Knowledge .Words cannot express the special appreciation deepest gratitude I feel to words my supervisor Prof. Shawgy Hussein AbdAlla gave me all the help I need. Thank to my friends in mathematics department in college of science in Sudan University and to all my teachers.

Abstract

We show that if a pair of weights (u,v) satisfies a sharp A_p - bump condition in the scale of all log bumps certain loglog bumps , then Haar shifts map L^p (v) into L^p (u) with a constant quadratic in the complexity of the shift . This in turn implies the two weight boundedness for all Calderón – Zygmund operators. We obtain a generalized version of the former theorem valid for a larger family of Calderón – Zygmund operators in any ambient space . We present a bilinear Tb theorem for singular operators Calderón – Zygmund type. Extending the end point results obtained to maximal singular. Another consequence is a quantitative two weight bump estimate.

الخلاصة

أوضحنا أنه إذا كان الزوج المرجح (u, v) يحقق شرط النتوء A_p - القاطع في كل النتوءات اللوغريثمية أو لنتوء لوغريثم اللوغريثم المعين إذاً راسم ازاحات هار $L^p(v)$ إلي $L^p(u)$ مع ثوابت الدرجة الثانية في تعقدية الازاحة . هذا بالمقابل يدل على المحدودية المرجحة الثانية لكل مؤثرات كالدرون – زيجموند . تحصلنا اصداء معممة لمبرهنة سابقة صحيحة لأجل العائلة الكبيرة لمؤثرات كالدرون – زيجموند في أي فضاء محيط . أحضرنا مبرهنة Tb ثنائية الخطية لأجل مؤثرات التكامل الشاذة لنوع كالدرون – زيجموند . تمديد نتائج النقطة النهائية المتحصل عليها الى مؤثرات التكاملات الشاذة الأعظمية . نتيجة أخرى هي تقدير نتوء مرجح اثنين كمي .

The Contents

Chapter 1

Two-Weight Boundedness of Calderón-Zygmund Operators

We give a partial answer to a long-standing conjecture. We also give a partial result for a related conjecture for weak-type inequalities. We combine several different approaches to these problems; we use many of the ideas developed to prove the A_2 conjecture. As a by product of the work we also disprove a conjecture by Muckenhoupt and Wheeden on weak-type inequalities for the Hilbert transform: (The Hilbert transform is a linear operator that takes a function u(t) and produces a function H(u)(t) with the same domain. The Hilbert transform is important in signal processing, where it derives the analytic representation of a signal u(t). This means that the real signal u(t) is extended into the complex plane such that it satisfies the Cauchy–Riemann equations. For example, the Hilbert transform leads to the harmonic conjugate of a given function in Fourier analysis, and harmonic analysis. Equivalently, it is an example of a singular integral operator and of a Fourier multiplier)[5].

We show several partial results related to a pair of long-standing conjectures in the theory of two-weight norm inequalities. To state the conjectures and the results we recall a few facts about Orlicz spaces. Given a Young function A, the complementary function $\bar{A}$ is theYoung function that satisfies

$$t \leq A^{-1}(t)\bar{A}^{-1}(t) \ \leq \ 2t, \qquad t > 0.$$

We will say that a Young function $\bar{A}$ satisfies the $B_{p'}$ condition, 1 < p <∞, if for some c >0,

$$\int_c^\infty \frac{\bar{A}(t)}{t^{p'}}\frac{dt}{t} < \infty.$$

If A and $\bar{A}$ are doubling (i.e., if A(2t) ≤ CA(t), and similarly for $\bar{A}$), then $\bar{A} \in B_p$ if and only if

$$\int_c^\infty \left(\frac{t^p}{A(t)}\right)^{p'-1}\frac{dt}{t} < \infty.$$

Given $p, 1 < p < \infty$, let A and B be Young functions such that $\bar{A} \in B_{p'}$ and $\overline{B} \in B_p$. We say that the pair of weights (u, v) satisfies an A_p bump condition with respect to A and B if

$$\sup_Q \|u^{1/p}\|_{A,Q} \|v^{-1/p}\|_{B,Q} < \infty, \quad \textbf{(1)}$$

where the supremum is taken over all cubes Q in R^d, and the Luxemburg norm is defined by

$$\|f\|_{A,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q A(|f(x)|/\lambda) dx \leq 1 \right\}$$

If (1) holds, then it is conjectured that

$$T: L^p(v) \to L^p(u). \quad \textbf{(2)}$$

Similarly, if the pair *(u, v)* satisfies the weaker condition

$$\sup_Q \|u^{1/p}\|_{A,Q} \|v^{-1/p}\|_{p',Q} < \infty, \quad \textbf{(3)}$$

then the conjecture is that

$$T: L^p(v) \to L^{p,\infty}(u). \quad \textbf{(4)}$$

The conditions (1) and (3) are referred to as A_p bump conditions because they may be thought of as the classical two-weight A_p condition with the localized L^p and $L^{p'}$ norms "bumped up" in the scale of Orlicz spaces. They first appeared in connection with estimates for integral operator related to the spectral theory of Schrödinger operators. The bump condition considered was the Fefferman–Phong condition that used "power" bumps: i.e., Young functions of the form $A(t) = t^{rp}, r > 1$. Power bumps were independently introduced by Neugebauer. Bump conditions in full generality were introduced by Pérez .

It was proved for *p = 2* in any dimension and for any Calderón–Zygmund operator using Bellman function techniques.

Theorem (1.1)[1]: Given *p=2*, suppose the pair of weights *(u,v)* satisfies (1), where $\bar{A} \in B_2$ and $\bar{B} \in B_2$. Then every Calderón-Zygmund singular integral operator T satisfies $\|Tf\|_{L^2(u)} \leq C\|f\|_{L^2(v)}$, where C depends only on T, the dimension d, and the suprema in (1).

Certain additional results are known in the special case that A and B are “logbumps”: that is, of the form

$$A(t) = t^p log(e + t)^{p-1+\delta},\ \bar{A}(t) \approx \frac{t^{p'}}{log(e+t)^{1+\delta'}}, \qquad \textbf{(5)}$$

$$B(t) = t^{p'} log(e + t)^{p'-1+\delta},\ \bar{B}(t) \approx \frac{t^{p}}{log(e+t)^{1+\delta''}}, \qquad \textbf{(6)}$$

where δ > 0, δ′ = δ/(p − 1), δ″ = δ/(p' − 1). But even in this case the result for all Calderón –Zygmund operators was unknown. The weak-type conjecture is only known for log bumps.

One can motivate the conjectures (1) ⇒ (2) and (3) ⇒ (4) (and the related conjectures we consider below) by considering a pair of conjectures due to Muckenhoupt and Wheeden. First, they conjectured that a singular integral operator (in particular, the Hilbert transform) satisfies (2) provided that the Hardy-Littlewood maximal operator: (Maximal functions appear in many forms in harmonic analysis (an area of mathematics). One of the most important of these is the Hardy–Littlewood maximal function. They play an important role in understanding, for example, the differentiability properties of functions, singular integrals and partial differential equations. They often provide a deeper and more simplified approach to understanding problems in these areas than other methods.)[6] satisfies

$$M: L^p(v) \to L^p(u),$$

$$M: L^{p'}(u^{1-p'}) \to L^{p'}(v^{1-p'}).$$

They also conjectured that (4) holds if the maximal operator satisfies the second, $L^{p'}$ inequality. Proved that a sufficient condition for each of these estimates to hold for M is that the pair (*u*, *v*) satisfies

$$\sup_Q \|u^{1/p}\|_{p,Q} \|v^{-1/p}\|_{B,Q} < \infty, \qquad \textbf{(7)}$$

$$sup_Q \|u^{1/p}\|_{A,Q} \|v^{-1/p}\|_{p',Q} < \infty, \qquad \textbf{(8)}$$

in particular, both these conditions hold if (1) holds.

Given the falsity of the Muckenhoupt-Wheeden conjectures (even for p =*2*), the A_p bump conjectures become even more interesting. Moreover, Theorem (1.1) and the other results listed above strongly suggest that it should hold in the full range of p, dimensions, and Calderón–Zygmund operators. Here we consider two even stronger conjectures, motivated by the fact that the "separated" bump conditions (7) and (8) are sufficient for the maximal operator inequalities in the original conjecture.

Conjecture (1.2)[1]: Given p, $1 < p < \infty$, suppose the pair of weights (u,v) satisfies (7) and (8), where $\bar{A} \in B_{p'}$ and $\bar{B} \in B_p$. Then every Calderón-Zygmund singular integral operator T satisfies $\|Tf\|_{L^p(u)} \leq C\|f\|_{L^p(v)}$, where C depends only on T, the dimension d, and the suprema in (7) and (8).

Conjecture (1.3)[1]: Given p, $1 < p < \infty$, suppose the pair of weights (u,v) satisfies (8) where $\bar{A} \in B_{p'}$. Then every Calderón-Zygmund singular integral operator T satisfies $\|Tf\|_{L^{p,\infty}(u)} \leq C\|f\|_{L^p(v)}$, where C depends only on T, the dimension d, and the supremum in (8).

We can prove Conjecture (1.2) in the special case when A, B are log bumps.

Theorem (1.4)[1]: Given p,$1 < p < \infty$, suppose the pair of weights (u,v) satisfies (7) and (8), where A and B are log bumps of the form (5) and (6). Then every Calderón-Zygmund singular integral operator T satisfies $\|Tf\|_{L^p(u)} \leq C\|f\|_{L^p(v)}$, where C depends only on T, the dimension d, and the suprema in (7) and (8).

The techniques also immediately yield Conjecture (1.3) for log bumps. This gives anew proof of the result originally. For completeness we include it here.

Theorem (1.5)[1]: Given p,$1 < p < \infty$, suppose the pair of weights (u,v) satisfies (8) where A is a log bump of the form (5). Then every Calderón-Zygmund singular integral operator T satisfies $\|Tf\|_{L^{p,\infty}(u)} \leq C\|f\|_{L^p(v)}$, where C depends only on T, the dimension d, and the supremum in (8).

The method to prove Conjectures (1.2) and (1.3) for a subset of the class of bump functions referred to as loglog-bumps:

$$A(t) = t^p log(e+t)^{p-1} loglog(e^e+t)^{p-1+\delta} \quad \bar{A}(t) \approx \frac{t^{p'}}{log(e+t)\, loglog(e^e+t)^{1+\delta'}}, \qquad \textbf{(9)}$$

$$\mathrm{B}(t) = t^{\mathrm{p}'} log(e+t)^{\mathrm{p}'-1} loglog(e^e+t)^{\mathrm{p}'-1+\delta}, \qquad \overline{\mathrm{B}}(t) \approx \frac{t^p}{log(e+t)\, loglog(e^e+t)^{1+\delta''}}, \qquad \textbf{(10)}$$

Where δ >0. These bump conditions are well known, but have been difficult to work with. No results were known for loglog-bumps that were not proved for bump conditions in general. However, we can show the following results, both of which are new.

Theorem (1.6)[1]: Given p, $1 < p < \infty$, suppose the pair of weights (u,v) satisfy (7) and (8), where A and B are loglog-bumps of the form (9) and (10) with δ sufficiently large. Then every Calderón-Zygmund singular integral operator T satisfies $\|Tf\|_{L^{\mathrm{p}}(u)} \le C\|f\|_{L^{p}(v)}$, where C depends only on T, the dimension d and the suprema in (7) and (8).

Theorem (1.7)[1]: Given p, $1 < p < \infty$, suppose the pair of weights (u,v) satisfies (8) where A is a loglog-bump of the form (9) with δ sufficiently large. Then every Calderón-Zygmund singular integral operator T satisfies $\|Tf\|_{L^{p,\infty}(u)} \le C\|f\|_{L^{p}(v)}$, where C depends only on T, the dimension d and the supremum in (8).

Reformulate the results and reduce the problem to proving the corresponding results for a general class of dyadic shift operators. It is important to note that in most of the proof we only need to assume that $\bar{A} \in B_{p'}$, $\bar{B} \in B_p$; only at one step are we forced to assume that A, B are log bumps. We will use the notation

$$\langle f \rangle_Q = \frac{1}{|Q|}\int_Q f(x)dx.$$

We also restate the weighted norm inequalities in an equivalent form. Let $\sigma = v^{1-p'}$; then we can rewrite (7) and (8) as

$$\sup_Q \langle u \rangle_Q^{1/p} \left\|\sigma^{1/p'}\right\|_{\mathrm{B,Q}} < \infty, \qquad \textbf{(11)}$$

$$sup_Q \left\|\mathrm{u}^{1/\mathrm{p}}\right\|_{\mathrm{A,Q}} \langle \sigma \rangle_{\mathrm{Q}}^{1/p'} < \infty, \qquad \textbf{(12)}$$

By the properties of the Luxemburg norm we have that either condition implies the two-weight A_p condition:

$$\sup_{Q}\langle u\rangle_Q^{1/p}\langle\sigma\rangle_Q^{1/p'} < \infty, \qquad \textbf{(13)}$$

Similarly, we can restate the conclusions of Theorems (1.1) and (1.4) as

$$\|T(f\sigma)\|_{L^p(u)} \le C\|f\|_{L^p(\sigma)}, \qquad \|T(f\sigma)\|_{L^{p,\infty}(u)} \le C\|f\|_{L^p(\sigma)}$$

The B_p condition is closely connected to a generalization of the maximal operator.

Recall that the Hardy-Littlewood maximal operator is defined to be

$$Mf(x) = \sup_{Q\ni x}\langle|f|\rangle_Q = \sup_{Q\ni x}\|f\|_{1,Q}\,.$$

Given a Young function A, we define the Orlicz maximal operator M_A by

$$M_A f(x) := \sup_{Q\ni x}\|f\|_{A,Q}\,.$$

Theorem (1.8)[1]: Fix p, $1 < p < \infty$, and let A be a Young function such that $A \in B_p$. Then $M_A\colon L^p \to L^p$.

The B_p condition is also sufficient for a two-weight norm inequality for the Hardy-Littlewood maximal operator.

Theorem (1.9)[1]: Fix p, $1 < p < \infty$, and let B be a Young function such that $\bar{B} \in B_p$. If the pair of weights (u,σ) satisfies

$$sup_Q\langle u\rangle_Q^{1/p}\left\|\sigma^{1/p'}\right\|_{B,Q<\infty}, \qquad \textbf{(14)}$$

then

$$\|M(f\sigma)\|_{L^p(u)} \le C\|f\|_{L^p(\sigma)}. \qquad \textbf{(15)}$$

We now turn to the definition of the dyadic Haar shift operators that will replace an arbitrary Calderón-Zygmund operator.

Definition (1.10)[1]: Given a dyadic cube Q, h_Q is a (generalized) Haar function associated to a cube Q if

$$h_Q(x) = \sum_{Q'\in ch(Q)} c_{Q'}\chi_{Q'}(x)\,,$$

Where *ch(Q)* is the set of dyadic children of Q and $|c_{Q'}| \le 1$.

Definition (1.11)[1]: We say that an operator S has a Haar shift kernel of complexity (m,n) if

$$Sf(x) = \sum_{Q} S_Q(f),$$

where

$$S_Q(f) = \frac{1}{|Q|} \sum_{\substack{Q',Q''\subset Q \\ \ell(Q')=2^{-n}\ell(Q) \\ \ell(Q'')=2^{-m}\ell(Q)}} (f, h_{Q'})h_{Q''}$$

and $h_{Q'}$ and $h_{Q''}$ are generalized Haar functions associated to the cubes Q' and Q'' respectively. We say that *S* is a Haar shift of complexity (*m*,*n*) if it has a Haar shift kernel of complexity (*m*, *n*), and it is bounded on $L^2(dx)$.

To prove Theorems (1.4) and (1.5) it will suffice to prove that they hold for Haar shift operators of complexity (*m*,*n*) with a constant that grows polynomially in $\tau = max(m,n) + 1$. We will prove the following.

Theorem (1.12)[1]: Given p, $1 < p < \infty$, suppose A and B are log-bumps of the form (5), (6), and the pair of weights (u, σ) satisfies (11) and (12). Given any dyadic shift S of complexity (*m*,*n*), $\tau = max(m,n) + 1$, $\|S(f\sigma)\|_{L^p(u)} \le C\tau^3\|f\|_{L^p(\sigma)}$, where C depends only on the dimension d and the suprema in (11) and (12).

Theorem (1.13)[1]: Given p, $1 < p < \infty$, suppose A is a log-bump of the form (5), and the pair of weights (u,σ) satisfies (12). Given any dyadic shift S of complexity (*m*,*n*), $\|S(f\sigma)\|_{L^{p,\infty}(u)} \le C\tau^3\|f\|_{L^p(\sigma)}$, where C depends only on the dimension d and the supremum in (12).

To prove the strong-type inequality we follow the argument used by Hytönen in the one-weight case. Fix a function f that is bounded and has compact support. For each $N>0$, let $Q_N = [-2^N, 2^N]^d$. By Fatou's lemma,

$$\|S(f\sigma)\|_{L^p(u)} \leq \liminf_{N\to\infty} \left(\int_{Q_N} |S(f\sigma)(x) - m_{S(f\sigma)}|^p u(x)\, dx\right)^{1/p},$$

where $m_{S(f\sigma)}$ is the median value of $S(f\sigma)$ on Q_N. Fix N. Using the remarkable decomposition theorem of Lerner, they show that there exists a family of dyadic cubes $\mathcal{L} = \{Q_j^k\}$ and pairwise disjoint sets $\{E_j^k\}$ such that $E_j^k \subset Q_j^k, |E_j^k| \geq \frac{1}{2} |Q_j^k|$, and

$$\left(\int_{Q_N} \left|S(f\sigma)(x) - m_{S(f\sigma)}\right|^p u(x) dx\right)^{\frac{1}{p}}$$
$$\leq C\tau\|M(f\sigma)\|_{L^p(u)} + C\tau \sum_{i=1}^{\tau} \left\| \sum_{j,k} \langle |f|\sigma\rangle_{\left(Q_j^k\right)^i} \chi_{Q_j^k} \right\|_{L^p(u)} \qquad \textbf{(16)}$$

(Here, given a dyadic cube Q, Q^i denotes the i-th parent of Q.) Alternatively, we combine it with the unweighted weak-type estimate with the right dependence on τ we precede the weak-type estimate of $1_Q S(1_{Q^\tau} f)$ by a careful pointwise estimate of this function on Q. This lets the reduce the weak-type estimate of this expression to the weak-type estimate of $1_Q S(1_Q f)$ (with an error term that can be controlled).

By Theorem (1.9), $\|M(f\sigma)\|_{L^p(u)} \leq C\|f\|_{L^p(\sigma)}$. Therefore, it remains to estimate the second term in (16). We will show that each term in the sum is bounded by $C\tau\|f\|_{L^p(\sigma)}$. Note that this gives the estimate which is cubic in τ.

Again, we show that this reduces to a two-weight estimate for apositive Haar shift operator. We reorder the sum as follows: fix an integer $i \in [1, \tau]$ and sum over every cube $Q = (Q_j^k)^i$ and then over all cubes $Q_r^s \in \mathcal{L}$ such that $(Q_r^s)^i = Q$. Then we have that

$$\sum_{j,k} \langle |f|\sigma\rangle_{(Q_j^k)^i} \chi_{Q_j^k} = \sum_{Q} \langle |f|\sigma\rangle_Q \sum_{\substack{R\in\mathcal{L} \\ R^i = Q}} \chi_R = \sum_{Q} \chi_Q^i \langle |f|\sigma\rangle_Q = S_{\mathcal{L}}^i(|f|\sigma),$$

where the last sum is taken over all dyadic cubes Q and

$$\chi_Q^i = \sum_{\substack{R\in\mathcal{L} \\ R^i = Q}} \chi_R \,.$$

Clearly, $S_{\mathcal{L}}$(hereafter we omit the superscript *i*) is a positive operator. We claim that it is in fact a positive Haar shift of complexity at most (0,τ− 1). From the definition we have that

$$S_{\mathcal{L}}f = \sum_{Q\in\mathcal{L}} \frac{1}{|Q|}\chi_Q^i \int_Q f\,,$$

and so in the notation used above we have that

$$S_Q = \frac{1}{|Q|}\chi_Q^i \int_Q f\,,$$

$Q' = Q, h_{Q'} = \chi_Q$, the Q″ are all the (*i*−*1*)-children of *Q*, and $h_{Q''} = \sum_{R\in ch(Q'')} c_R\chi_R$, where $c_R = 1\ if\ R \in \mathcal{L}$ and $c_R = 0$ otherwise. Thus $S_{\mathcal{L}}$ has a Haar shift kernel of complexity $(0, i-1), i \leq \tau$. To see that it is bounded on L^2, we use the properties of the cubes Q_j^k. By duality, there exists $g \in L^2, \|g\|_2 = 1$, such that

$$\|S_{\mathcal{L}}f\|_2^2 = \int_{R^d} \sum_{j,k} \langle f\rangle_{(Q_j^k)^i}\chi_{Q_j^k}(x)g(x)\,dx$$

$$\leq 2\sum_{j,k}\langle f\rangle(Q_j^k)^i\langle g\rangle_{Q_j^k}|E_j^k| \leq 2\int_{R^d} Mf(x)Mg(x)\,dx\,.$$

The last integral is bounded by $\|f\|_2\|g\|_2$ by Hölder's inequality and the unweighted L^2 inequality for the maximal operator.

Definition (1.14)[1]: Given a positive Haar shift operator *S*, define the associated maximal singular integral operator by

$$S_\#(x) := \sup_{0<\epsilon\leq v<\infty} S_{\epsilon,v}(x) = \sup_{0<\epsilon\leq v<\infty} \sum_{Q\in D,\epsilon\leq\ell(Q)\leq v} S_Q f(x)\,.$$

To prove that $\|S_{\mathcal{L}}(f\sigma)\|_{L^p(u)} \leq C\tau\|f\|_{L^p(\sigma)}$, we use the following result that is essentially due to Sawyer .

Theorem (1.15)[1]: Let S be a positive Haar shift of complexity (*m*,*n*). Then the associated maximal singular integral $S_\#$ satisfies

$$\|S_{\#}(\cdot\,\sigma)\|_{L^p(\sigma)\to L^p(u)}$$

$$\leq \tau\|M(\cdot\,\sigma)\|_{L^p(\sigma)\to L^p(u)} + \sup_Q \frac{\left\|\chi_Q S(\chi_Q\sigma)\right\|_{L^p(u)}}{\sigma(Q)^{\frac{1}{p}}}$$

$$+ \sup_Q \frac{\left\|\chi_Q S^*(\chi_Q u)\right\|_{L^p(\sigma)}}{u(Q)^{\frac{1}{p}}} \quad \textbf{(17)}$$

Fix a cube Q_0; using the notation from the definition of a Haar shift, we have that

$$\chi Q_0 S(\chi_{Q_0}\sigma) = \sum_{R\subset Q_0} S_R(\sigma) + X_{Q_0}\sum_{R,Q_0\subset R} S_R(\chi Q_0\sigma) \leq \sum_{R\subset Q_0} S_R(\sigma) + \chi Q_0\langle\sigma\rangle_{Q_0} \quad \textbf{(18)}$$

The second inequality is straight forward. As we noted above, the pair (u,σ) satisfies the two-weight $A_p$condition (13). Therefore, the $L^p(u)$ norm of the second term is bounded by

$$\left\|\chi_{Q_0}\right\|_{L^p(u)}\langle\sigma\rangle_{Q_0} = \langle u\rangle_{Q_0}^{1/p}\langle\sigma\rangle_{Q_0}^{1/p'}\sigma(Q_0)^{1/p} \leq C\sigma(Q_0)^{1/p}.$$

To estimate the $L^p(u)$ norm of the first term, we form the following decomposition:

$$K = K_i = \{Q \subset Q_0 : \ell(I) = 2^{i+\tau n}\}, n \in \mathbb{Z}_+;$$

$$K_a = \{Q \in K: 2^a \leq \langle u\rangle_Q^{1/P}\langle\sigma\rangle_Q^{1/p'} < 2^{a+1}\};$$

$$P_0^a = \textit{all maximal cubes in } K_a;$$

$$P_n^a = \left\{\textit{maximal cubes } p' \subset P \in P_{n-1}^a, \textit{such that } \langle\sigma\rangle_{p'} > 2\langle\sigma\rangle_P\right\};$$

$$P^a = \bigcup_{n\geq 0} P_n^a\,.$$

Hereafter we suppress the index *i*; this will give a sum with τ + 1 terms. Given $Q \in K_a$, let $\Pi(Q)$ denote the minimal principal cube that contains it, and define

$$K_a(P) = \{Q \in K_a: \Pi(Q) = P\}.$$

We will estimate the $L^p(u)$ norm of the first sum on the right-hand side of (18) using the exponential decay distributional inequality originated. Below, S is any positive generalized Haar shift that is bounded on unweighted L^2. In particular, we will take S to be one of the positive Haar shifts $S_{\mathcal{L}}$ from above.

Theorem **(1.16)[1]:** There exists a constant c, depending only on the dimension and the unweighted L^2 norm of the shift, such that for any $P \in P^a$,

$$u\left(x \in P: |S_{K_{a(P)}}(\sigma)| > t\frac{\sigma(P)}{|P|}\right) \lesssim e^{-ct}u(P)$$

It follows from Theorem (1.16) that for some positive constant c,

$$\left\|\sum_{R \subsetneq Q} S_R(\sigma)\right\|_{L^p(u)} \leq C\tau \sum_a \left(\sum_{P \in P^a} u(P)\left(\frac{\sigma(P)}{|P|}\right)^p\right)^{1/p}. \qquad \textbf{(19)}$$

We sketch the proof of (19) following the beautiful calculations .

$$\sum_{R \subsetneq Q} S_R(\sigma) = \sum_{\tau=0}^{i} \sum_a \sum_{P \in P^a} S_{K^a(P)}(\sigma),$$

and so

$$\left\|\sum_{R \subsetneq Q} S_R(\sigma)\right\|_{L^p(u)} \leq (\tau + 1)\sum_a \left\|\sum_{P \in P^a} S_{K^a(P)}(\sigma)\right\|_{L^p(u)}.$$

Fix a. Using Fubini's theorem we write

$$\left\|\sum_{P \in P^a} S_{K^a(P)}(\sigma)\right\|_{L^p(u)}$$

$$= \left(\int\left(\sum_j \sum_{P \in P^a} \chi_{\left\{S_{K^a(P)}(\sigma) \in (j,j+1)\frac{v(P)}{|P|}\right\}} S_{K^a(P)}(\sigma)(x)\right)^p u(x)\,dx\right)^{1/p}$$

$$\leq \sum_j (j+1)\left(\int\left[\sum_{P \in P^a} \chi_{\left\{S_{K^a(P)}(\sigma) \in (j,j+1)\frac{\sigma(P)}{|P|}\right\}} \frac{\sigma(P)}{|P|}\right]^p u(x)\,dx\right)^{1/p}$$

By the choice of the stopping cubes $P \in P^a$ we have that

$$\left[\sum_{P\in P^a}\chi_{\left\{S_{K^a(P)}(\sigma)\in(j,j+1)\frac{\sigma(P)}{|P|}\right\}}\frac{\sigma(P)}{|P|}\right]^p \lesssim \sum_{P\in P^a}\chi_{\left\{S_{K^a(P)}(\sigma)\in(j,j+1)\frac{\sigma(P)}{|P|}\right\}}\left(\frac{\sigma(P)}{|P|}\right)^P.$$

This follows because the ratios $\frac{\sigma(P)}{|P|}$ in the sum on the left are super-exponential. This beautiful observation lets a write

$$\left\|\sum_{P\in P^a}S_{K^a(P)}(\sigma)\right\|_{L^p(u)}$$

$$\lesssim \sum_j (j+1)\left(\sum_{P\in P^a}\left(\frac{\sigma(P)}{|P|}\right)^p u\left(S_{K^a(P)}(\sigma)\in (j,j+1)\frac{\sigma(P)}{|P|}\right)\right)^{1/p}.$$

Then by the distributional inequality from Theorem (1.16):

$$\left\|\sum_{P\in P^a}S_{K^a(P)}(\sigma)\right\|_{L^p(u)} \lesssim \sum_j (j+1)e^{-cj/p}\left(\sum_{P\in P^a}\left(\frac{\sigma(P)}{|P|}\right)^p u(P)\right)^{1/p}.$$

This gives (19).

We can no longer assume that a pair of weights (u,σ) satisfies the general A_p bump condition and we must instead make the more restrictive assumption that we have log bumps. Before doing so, however, we want to show how the proof goes and where the problem arises for general bumps.We will then give the modification necessary to make this argument work for log bumps.

Define the sequence

$$\mu_Q = \begin{cases} |P|, & Q = P\ , for some\ cube\ P\in P^a \\ 0, & otherwise; \end{cases}$$

then the inner sum in (19) becomes

$$\sum_{Q\subset Q_0}\frac{u(Q)}{|Q|}\left(\frac{\sigma(Q)}{|Q|}\right)^p \mu_Q\,.$$

But by Hölder's inequality in the scale of Orlicz spaces,

$$\frac{\sigma(Q)}{|Q|} = \langle \sigma^{\frac{1}{p}} \sigma^{\frac{1}{p'}} \rangle_Q \leq C \left\| \sigma^{\frac{1}{p'}} \right\|_{Q,B} \left\| \sigma^{\frac{1}{p}} \right\|_{Q,\bar{B}} \leq \| \sigma^{1/p'} \|_{Q,B} \inf_{x \in Q} M_{\bar{B}}(\sigma^{1/p} \chi_Q) \,. \qquad \textbf{(20)}$$

Therefore, by (11),

$$\sum_{Q \subset Q_0} \frac{u(Q)}{|Q|} \left(\frac{\sigma(Q)}{|Q|} \right)^p \mu_Q \leq K^p \sum_{Q \subset Q_0} \mu_Q \inf_{x \in Q} M_{\bar{B}}(\sigma^{1/p} \chi_Q)^p. \qquad \textbf{(21)}$$

To complete the proof we need two lemmas.

Lemma (1.17)[1]: $\{\mu_Q\}$ is a Carleson sequence.

The second is a folk theorem.

Lemma (1.18)[1]: If $\{\mu_Q\}$ is a Carleson sequence, then

$$\sum_{Q \subset Q_0} \mu_Q \inf_{Q} \chi_{Q_0} F(x) \lesssim \int_{Q_0} F(x) dx \,.$$

Combining these two lemmas with Theorem (1.8) (since $\bar{B} \in B_p$) we see that

$$\sum_{Q} \frac{u(Q)}{|Q|} \left(\frac{\sigma(Q)}{|Q|} \right)^p \mu Q \leq K^p \sum_{Q, Q \subset Q_0} \mu_Q \inf_{x \in Q} M_{\bar{B}}(\sigma^{1/p} \chi_{Q_0})^p \lesssim K^p \left\| M_{\bar{B}}(\sigma^{1/p} \chi_{Q_0}) \right\|_{L^p(dx)}^p$$
$$\lesssim K^p \left\| \sigma^{1/p} \chi_{Q_0} \right\|_{L^p(dx)}^p = K^p \sigma(Q_0).$$

This would complete the proof except that we must now sum over a, and in (19) this sum goes from $-\infty$ to the logarithm of the two-weight A_p constant of the pair (u,σ). We cannot evaluate this sum unless we can modify the above argument to yield a decay constant in a. We use the fact that the parameter a run from 0 to the logarithm of A_p constant: this follows since by Hölder's inequality the A_p constant of any weight is at least 1. In the two-weight case the A_p constant can be arbitrarily small, and therefore we must sum over infinitely many values of a. We are able to get the desired decay constant only by assuming that we are working with log bumps.

We modify the above argument as follows. Essentially, we will use the properties of log bumps to replace $\bar{B}$ with a slightly larger Young function. Define $B_0(t) =$

$t^{p'} log(e + t)^{p'-1+\frac{\delta}{2}}$; then we again have that $\bar{B}_0 \in B_p$. Instead of (21) we will prove that there exists γ, 0 < γ <1, such that

$$\sum_{Q\subset Q_0} \frac{u(Q)}{|Q|}\left(\frac{\sigma(Q)}{|Q|}\right)^p \mu_Q \leq K^{(1-\gamma)p} 2^{a\gamma p} \sum_{Q\subset Q_0} \mu_Q \inf_{x\in Q} M_{\bar{\boldsymbol{B}}_0}(\sigma^{1/p}\chi_{Q_0})^p \quad \textbf{(22)}$$

Given inequality (22), we can repeat the argument above, but we now have the decay term $2^{a\gamma p}$ which allows to sum in *a* and get the desired estimate.

To prove (22) suppose for the moment that there exists γ such that

$$\|\sigma^{1/p'}\|_{Q,B_0} \leq C_1 \|\sigma^{1/p'}\|_{Q,B}^{1-\gamma} \|\sigma^{1/p'}\|_{L^{p'}(Q,dx/|Q|)}^{\gamma}. \quad \textbf{(23)}$$

Given this, fix a cube $Q \in P^a$—we can do this since otherwise $\mu_Q = 0$. Then

$$\frac{u(Q)}{|Q|}\left(\frac{\sigma(Q)}{|Q|}\right)^p \leq \langle u\rangle_Q \left\|\sigma^{\frac{1}{p'}}\right\|_{B_0,Q}^p \left\|\sigma^{\frac{1}{p}}\right\|_{\bar{B}_0,Q}^p \leq \langle u\rangle_Q \left\|\sigma^{\frac{1}{p'}}\right\|_{B,Q}^{(1-\gamma)p} \left\|\sigma^{\frac{1}{p'}}\right\|_{L^{p'}\left(Q,\frac{dx}{|Q|}\right)}^{\gamma p} \left\|\sigma^{\frac{1}{p}}\right\|_{\bar{B}_0,Q}^p$$

$$= \left(\langle u\rangle_Q^{\frac{1}{p}} \left\|\sigma^{\frac{1}{p'}}\right\|_{B,Q}\right)^{(1-\gamma)p} \cdot \left(\langle u\rangle_Q^{\frac{1}{p}} \left\|\sigma^{\frac{1}{p'}}\right\|_{L^{p'}\left(Q,\frac{dx}{|Q|}\right)}\right)^{\gamma p} \cdot \left\|\sigma^{\frac{1}{p}}\right\|_{\bar{B}_0,Q}^p$$

$$\leq K^{(1-\gamma)p} \cdot (\langle u\rangle_Q^{1/p} \langle\sigma\rangle_Q^{1/p'})^{\gamma p} \cdot \|\sigma^{1/p}\|_{\bar{B}_0,Q}^p \leq K^{(1-\gamma)p} \cdot 2^{a\gamma p} \cdot \|\sigma^{1/p}\|_{\bar{B}_0,Q}^p$$

$$\leq K^{(1-\gamma)p} \cdot 2^{a\gamma p} \cdot \inf_{x\in Q} M_{\bar{B}_0}(\sigma^{1/p}\chi_{Q_0})^p.$$

Inequality (22) now follows immediately.

Therefore, to complete the proof we must establish (23). By the rescaling properties of the Luxemburg norm, the right-hand side of this inequality is equal to

$$\left\|\sigma^{\frac{1-\gamma}{p'}}\right\|_{C,Q} \left\|\sigma^{\frac{\gamma}{p'}}\right\|_{\frac{p'}{\gamma},Q},$$

Where $C(t) = B(t^{\frac{1}{1-\gamma}})$. Therefore, by the generalized Hölder's inequality in Orlicz spaces, inequality (23) holds if for all t >1,

$$C^{-1}(t)t^{\frac{\gamma}{p'}} \lesssim B_0^{-1}(t). \quad \textbf{(24)}$$

A straightforward calculation shows that

$$C^{-1}(t) = B^{-1}(t)^{1-\gamma} \approx \frac{t^{\frac{1-\gamma}{p'}}}{log(e+t)^{\frac{1-\gamma}{p}+\frac{\delta(1-\gamma)}{p'}}}, B_0^{-1}(t) \approx \frac{t^{\frac{1}{p'}}}{log(e+t)^{\frac{1}{p}+\frac{\delta}{2p'}}}.$$

By equating the exponents on the logarithm terms, we see that (24) holds if we take

$$\gamma = \frac{\delta}{2(p'-1+\delta)}.$$

Therefore, with this value of γ inequality (23) holds, and this completes the proof.

We give a direct proof of (23). The desire inequality obviously follows from the following lemma.

Lemma (1.19)[1]: Given a probability measure μ, let f be a non-negative measurable function. Let B, B_0 be logarithmic bumps as in (6) with $\delta = \tau$ and $\delta = \frac{\tau}{2}$ respectively. Then there exists an absolute constant C and $\gamma = \gamma(p', \tau) > 0$ such that

$$\|f\|_{B_0,\mu} \leq C\|f\|_{B,\mu}^{1-\gamma}\|f\|_{L^{p'}(\mu)}^{\gamma}. \quad \textbf{(25)}$$

Proof. We will actually show that $\gamma = \frac{1}{2+(p'-1)\frac{2}{\tau}}$. Define $\Delta := \int |f|^{p'} d\mu$. Since inequality (25) is homogeneous, we may assume without loss of generality that

$$\|f\|_{B,\mu} = 1. \quad \textbf{(26)}$$

Moreover, we may assume that $\Delta \leq 1$: otherwise (25) can be achieved by choosing C sufficiently large. Let ϵ <1 and K be constants; we will determine their precise value (in this order) below. Then we have that

$$\int \frac{f^{p'}}{\epsilon^{p'}} log\left(e+\frac{f}{\epsilon}\right)^{p'-1+\frac{\tau}{2}} d\mu \leq \int_{\{f\leq K\epsilon\}} \cdots + \int_{\{f\geq K\epsilon\}} \cdots$$

$$\leq \frac{\Delta}{\epsilon^{p'}}[log(e+ K)]^{p'-1+\frac{\tau}{2}} + \int_{\{f\geq K\epsilon\}} \frac{f^{p'}}{\epsilon^{p'}} \frac{log(e+ \frac{f}{\epsilon})^{p'-1+\tau}}{[log(e+ K)]^{\frac{\tau}{2}}} d\mu$$

$$\leq \frac{\Delta}{\epsilon^{p'}}[log(e+K)]^{p'-1+\frac{\tau}{2}} + \int \frac{f^{p'}}{\epsilon^{p'}} \frac{log(\frac{e}{\epsilon}+ \frac{f}{\epsilon})^{p'-1+\tau}}{[log(e+ K)]^{\frac{\tau}{2}}} d\mu$$

$$\leq \frac{\Delta}{\epsilon^{p'}}[log(e+K)]^{p'-1+\frac{\tau}{2}}$$

$$+ \frac{1}{\epsilon^{p'}[log(e+ K)]^{\tau/2}}\left[\int f^{p'} log^{p'-1+\tau}(e+ f)\, d\mu\right.$$

$$\left. + \int f^{p'} log(\epsilon^{-1})^{p'-1+\tau} d\mu\right]$$

$$\leq \frac{\Delta}{\epsilon^{p'}}[log(e+ K)]^{p'-1+\frac{\tau}{2}} + \frac{1}{\epsilon^{p'}[log(e+ K)]^{\tau/2}}\left[1 + \Delta log(\epsilon^{-1})^{p'-1+\tau}\right].$$

In the last line we used (26). Fix ϵ so that

$$\Delta = (\epsilon^{p'})^{1+c},$$

where $c = 1 + (p' - 1)\, \frac{2}{\tau}$. In other words,

$$\epsilon = (\Delta^{1/p'})^{\gamma} = \|f\|^{\gamma}_{L^{p'}(\mu)}, \qquad \gamma = \frac{1}{1 + c}.$$

Now choose K so that

$$[log(e+ K)]^{\tau/2} \approx \epsilon^{-p'};$$

then

$$[log(e+ K)]^{p'-1+\tau/2} \approx (\epsilon^{-p'})^{1+(p'-1)\frac{2}{\tau}} =: (\epsilon^{-p'})^{c}.$$

If we substitute these values into the above calculation, we see that the right hand side is dominated by a constant. Hence, by the definition of the Luxemburg norm,

$$\|f\|_{B_0,\mu} \leq C\epsilon = C\|f\|^{\gamma}_{L^{p'}(\mu)}.$$

This completes the proof.

The proof of the weak-type inequality uses essentially the same argument as above; here we sketch the changes required. We repeat the argument that yields in equality (16), replacing the $L^p(u)$ norm with the $L^{p,\infty}(u)$ norm. Since the pair (u,σ) satisfies the two-weight A_p condition we have the well known inequality that

$$\|M(f\sigma)\|_{L^{p,\infty}(u)} \leq C\|f\|_{L^p(\sigma)},$$

where the constant C depends only on the A_p constant and the dimension. Therefore it remains to estimate the $L^{p,\infty}(u)$ norm of $S_{\mathcal{L}}(|f|\sigma)$. We have the following analog of Theorem (1.15).

Theorem (1.20)[1]: Let S be a positive Haar shift of complexity (m, n). Then

$$\|S(\cdot\,\sigma)\|_{L^p(\sigma)\to L^{p,\infty}(u)} \leq \tau\|M(\cdot\,\sigma)\|_{L^p(\sigma)\to L^{p,\infty}(u)} + \sup_Q \frac{\left\|\chi_Q S^*(\chi_Q u)\right\|_{L^p(\sigma)}}{u(Q)^{1/p}}.$$

Given Theorem (1.20) the argument now proceeds exactly as before, using the bump condition (12) to bound the testing condition. This completes the proof.

We will describe the principle changes. Following the argument, it will suffice to prove the corresponding results for dyadic shifts.

Theorem (1.21)[1]: Given p, $1 < p < \infty$, suppose A and B are loglog-bumps of the form (9), (10) with $\delta > 0$ sufficiently large, and the pair of weights (u,σ) satisfies (7) and (8). Given any dyadic shift S of complexity (m,n), $\tau = \max(m,n)+1$, $\|S(f\sigma)\|_{L^p(u)} \leq C\tau^3\|f\|_{L^p(\sigma)}$, where C depends only on the dimension d and the suprema in (7) and (8).

Theorem (1.22)[1]: Given $p, 1 < p < \infty$, suppose A is a loglog-bump of the form (9) with $\delta > 0$ sufficiently large, and the pair of weights (u,σ) satisfies (8). Given any dyadic shift S of complexity (m,n), $\|S(f\sigma)\|_{L^{p,\infty}(u)} \leq C\tau^2\|f\|_{L^p(\sigma)}$, where C depends only on the dimension d and the supremum in (8).

We will prove Theorem (1.21) by modifying the proof of Theorem (1.12) above; Theorem (1.22) is proved similarly. The main step is to adapt Lemma (1.19) to work with loglog-bumps. Let B be as in (10), and define B_0 similarly but with δ replaced by *δ/2*. Then arguing almost exactly as we did in the proof of Lemma (1.19), we have that

$$\|f\|_{B_0,\mu} \leq C\|f\|_{B,\mu}\varepsilon\frac{\|f\|_{L^{p'}(\mu)}}{\|f\|_{B,\mu}}, \qquad \textbf{(27)}$$

where $\varepsilon(t) = (log\frac{C}{t})^{-\kappa}, C = C(p,\delta)$, and $\kappa = \kappa(p,\delta)\, with\, p\kappa > 1$ if δ is large enough.

Given (27) we have that

$$\begin{aligned}\frac{u(Q)}{|Q|}\left(\frac{\sigma(Q)}{|Q|}\right)^p &\leq C\langle u\rangle_Q\left\|\sigma^{1/p'}\right\|_{B_0,Q}^p\left\|\sigma^{1/p}\right\|_{\bar{B}_0,Q}^p \\ &\leq C\langle u\rangle_Q\left\|\sigma^{1/p'}\right\|_{B,Q}^p\varepsilon\left(\frac{\langle\sigma\rangle_Q^{1/p'}}{\left\|\sigma^{1/p'}\right\|_{B,Q}}\right)^p\left\|\sigma^{1/p}\right\|_{\bar{B}_0,Q}^p \\ &\leq C\left(\langle u\rangle_Q^{1/p}\left\|\sigma^{1/p'}\right\|_{B,Q}\right)^p\varepsilon\left(\frac{\langle u\rangle_Q^{1/p}\langle u\rangle_Q^{1/p'}}{\langle u\rangle_Q^{1/p}\left\|\sigma^{1/p'}\right\|_{B,Q}}\right)^p\left\|\sigma^{1/p}\right\|_{\bar{B}_0,Q}^p.\end{aligned}$$

To complete the proof, we need a bound in a for the first two terms. Since $Q \in P^a$, we have that $\langle u\rangle_Q^{1/p}\langle u\rangle_Q^{1/p'} \asymp 2^a$. Let $b_0 - 1$ be the logarithm of the supremum in (7). Then there exists $b, a \leq b \leq b_0$, such that $\langle u\rangle_Q^{1/p}\left\|\sigma^{1/p'}\right\|_{B,Q} \asymp 2^b$. The sum in a will go from −∞ to b_0, so it will suffice to consider those terms where a <0.

If b is negative and |b| ≥ |a|/2, then the first term is bounded by $2^{-\frac{p|a|}{2}}$, and the second is bounded by some constant. If *b >0* or if it is negative but $|b| \leq \frac{|a|}{2}$, then we estimate the first term by 2^{b_0}, and the argument in the second term is at most $2^{-\frac{|a|}{2}}$. Hence the second term is bounded by $\frac{C}{|a|^{pk}}$. By the assumption, $p\kappa > 1$ and so the series $\sum_{a<0}[2^{\frac{pa}{2}} + \frac{C}{|a|^{p^k}}]$ converges.

We prove that the weak-type conjecture of Muckenhoupt and Wheeden discussed is false for the Hilbert transform when $p = 2$. We in fact prove a stronger result.

For brevity, we introduce some additional notation. Let $\sigma = v^{-1}$ and let $M_\sigma f = M(f\sigma)$. Define M_u, H_σ and H_u similarly, where H is the Hilbert transform. Then we can reformulate the conjecture as follows: if

$$M_u: L^2(u) \to L^2(\sigma). \quad \textbf{(28)}$$

then

$$H_\sigma: L^2(\sigma) \to L^{2,\infty}(u). \quad \textbf{(29)}$$

We will show by contradiction that this is not true in general. Suppose to the contrary that if the pair (*u*,*v*) satisfies (28), then (29) holds. Then for any $f \in L^2(u)$ and any cube Q,

$$\int_Q H_\sigma f u dx \le \|H_\sigma f\|_{L^{2,\infty}(u)} \|\chi_Q\|_{L^{2,1}(u)} \le \|H_\sigma\|_{L^2(\sigma)\to L^{2,\infty}(u)} \|f\|_{L^2(\sigma)} u(Q)^{1/2}.$$

Let $f = H_u(\chi_Q)$. Then by duality (since H_σ is the adjoint of H_u) we have that the pair (*u*,*v*) satisfies the testing condition

$$\int_Q |H_u(\chi_Q)|^2 \sigma dx \le Cu(Q). \quad \textbf{(30)}$$

The same argument shows that if the pair (*u, σ*) satisfies

$$M_\sigma: L^2(\sigma) \to L^2(u), \quad \textbf{(31)}$$

then this pair also satisfies the testing condition

$$\int_Q |H_\sigma(\chi_Q)|^2 u dx \le C\sigma(Q). \quad \textbf{(32)}$$

However, we have the following testing condition result for the Hilbert transform when $p = 2$.

Theorem (1.23)[1]: Let H be the Hilbert transform. Then

$$\|H(\cdot\,\sigma)\|_{L^2(\sigma)\to L^2(u)}$$
$$\le \|M(\cdot\,\sigma)\|_{L^2(\sigma)\to L^2(u)} + \|M(\cdot\,u)\|_{L^2(u)\to L^2(\sigma)} + \sup_Q \frac{\|H(\chi_Q\sigma)\|_{L^2(u)}}{\sigma(Q)^{1/2}}$$
$$+ \sup_Q \frac{\|H(\chi_Q u)\|_{L^2(\sigma)}}{u(Q)^{1/2}}.$$

Therefore, by the assumption and Theorem (1.23), we have that if a pair of weights (u,σ) satisfies (28) and (31), then $H: L^2(\sigma) \to L^2(u)$. Therefore, the weak-type conjecture of Muckenhoupt and Wheeden cannot hold.

We have proved a stronger result.

Theorem (1.24)[1]: There exists a pair of weights (u,σ) such that (28) and (31) hold, but the Hilbert transform does not satisfy the weak-type inequality $H_\sigma: L^2(\sigma) \to L^{2,\infty}(u)$.

For the convenience we want to explain why these two results are in fact equivalent.

If $M_u, M_\sigma, H_u, H_\sigma$ all satisfy the testing conditions (30) and (32) (replacing H with M when dealing with the maximal operator), then all four of them are bounded in corresponding pairs of weighted spaces. This gives Theorem (1.23). Conversely, suppose that the right-hand side of the inequality in Theorem (1.23) is finite. Then (30) and (32) are both satisfied. Moreover, M_u and M_σ are both bounded on the corresponding pairs of weighted spaces. Therefore, trivially, M_u, M_σ also satisfy the corresponding testing conditions. Thus, all four operators satisfy the testing conditions, and then $H_u : L^2(u) \to L^2(\sigma)$ and $H_\sigma : L^2(\sigma) \to L^2(u)$ as well as the corresponding norm inequalities for the maximal operators.

We noted that the weak-type conjecture we just disproved followed from another conjecture of Muckenhoupt and Wheeden: that

$$u(\{x: |Hf(x)| > t\}) \le \frac{C}{t}\int |f|Mu\,dx \qquad \textbf{(33)}$$

(This implication is a straightforward duality argument: the result above gives another (indirect) proof of this fact.

We note that there is a weaker conjecture than (33) which has also been shown to be false. In the one-weight case it was conjectured that for all $w \in A_1$,

$$w(\{x: Hf(x) > t\}) \leq \frac{C}{t}[w]A_1 \int |f|wdx \ , \qquad \textbf{(34)}$$

where $[w]A_1 = \left\| \frac{Mw}{w} \right\|_{L^\infty}$. The counter example to (33) is not in A_1. Essentially, disproving (34) amounts to finding a "smooth" bad weight, which is even more difficult to build than the weight of Reguera–Thiele. While no explicit example has been constructed, the existence of such a weight has been proved using Bellman function techniques: it was shown that there exist weights in A_1 such that

$$\|H\|_{L^1(w)\to L^{1,\infty}(w)} \geq c[w]_{A_1} log^{1/5}[w]_{A_1} \qquad \textbf{(35)}$$

Chapter 2

Sobolev Spaces on Domains

We need to check the boundedness of not only the characteristic function, but a finite collection of polynomials restricted to the domain. we find a sufficient condition in terms of Carleson measures for $p \leq d$, and, in the particular case $s = 1$.

Section(2.1): Oriented Whitney Convering and Approximating Polynomials with Derivatives of Tf

Given an open set $U \subset R^d$, we say that a function is in the Sobolev space $W^{n,p}(U)$if it has derivatives up to order *n* in the weak sense in *U* and all of them are integrable in the L^p sense.We say that $f \in W^{n,p}_{loc}(U)$ if those derivatives are in the space $L^p_{loc}\,(U)$ instead.

Definition (2.1.1)[2]: We say that a measurable function $K \in W^{n,1}_{loc}\,(R^d \backslash \{0\})$ is a smooth convolution Calderón-Zygmund kernel of order *n* if

$$\left|\nabla^j K(x)\right| \leq \frac{C}{|x|^{d+j}} \quad for\ 0\ \leq j \leq n.$$

and that kernel can be extended to a tempered distribution *W* in R^d in the sense that for any Schwartz function $\phi \in S$ with $0 \notin sup(\phi)$,

$$\langle W, \phi \rangle = (K * \phi)(0).$$

Abusing notation, we will write *K* instead of *W*.

We will use the classical notation $\hat{f}$ for the Fourier transform of a given Schwartz function,

$$\hat{f}(\xi) = \int_{R^d} e^{-2\pi i x.\xi} f(x) dx$$

And $\check{f}$ will denote its inverse. It is well known that the Fourier transform can be extended to the whole space of tempered distributions by duality and it induces an isometry in L^2.

Definition (2.1.2)[2]: We say that an operator $T: S \to \acute{S}$ is a smooth convolution Calderón-Zygmund operator of order n with kernel K if K is a smooth convolution Calderón-Zygmund kernel of order n such that $\widehat{K} \in L^1_{loc}, T$ is defined as

$$T\phi = K * \phi := (\widehat{K} \,.\, \hat{\phi}\,)^{\vee}$$

for any $\phi \in S$ and T extends to an operator bounded in L^p for any $1 < p < \infty$.

For instance, that this boundedness property is equivalent to having $\widehat{K} \in L^{\infty}$.

It is a well-known fact that the Schwartz class is dense in L^p for $p < \infty$. Bearing this in mind, we get that given any $f \in L^p$ and $x \notin supp(f)$,

$$Tf(X) = \int K(x-y)f(y)dy\,.$$

Example (2.1.3) [2]: In the complex plane, the Beurling transform is defined as the principal value

$$Bf(z) := -\frac{1}{\pi}\lim_{\epsilon\to 0}\int_{|w-z|>\epsilon} \frac{f(w)}{(z-w)^2}\,dm(w)$$

It is a smooth convolution Calderón-Zygmund operator of any order associated to the kernel

$$K(z) = \frac{1}{z^2}$$

and its multiplier is

$$\widehat{K}(\xi) = \frac{\bar{\xi}}{\xi}$$

Thus, the Beurling transform is an isometry in L^2.

Definition (2.1.4)[2]: Let $\Omega \subset R^d$ be a domain (open and connected). We say that a cube Q with side-length $R > 0$ and center $x \in \partial\Omega$ is an R-window of the domain if it induces a local parameterization of the boundary, i.e. there exists a continuous function

$A_Q : R^{d-1} \to R$ such that, after a suitable rotation that brings all the faces of Q parallel to the coordinate axes,

$$\Omega \cap Q = \{(\acute{y}, y_d) \in (R^{d-1} \times R) \cap Q : y_d > A_Q(\acute{y})\}$$

We say that a bounded domain Ω it is a (δ, R)- Lipschitz domain: (A Lipschitz domain (or domain with Lipschitz boundary) is a domain in Euclidean space whose boundary is "sufficiently regular" in the sense that it can be thought of as locally being the graph of a Lipschitz continuous function. The term is named after the German mathematician Rudolf Lipschitz.)[7] if for each $x \in \partial\Omega$ there exist an R-window centered in *x* with A_x Lipschitz with a uniform bound $\|\nabla A\|_\infty < \delta$.

We say that an unbounded domain Ω is a special δ-Lipschitz domain if there exists a Lipschitz function *A* such that $\|\nabla A\|_\infty < \delta$ and

$$\Omega = \{(\acute{y}, y_d) \in R^{d-1} \times R: y_d > A(\acute{y})\}$$

With no risk of confusion, we will forget often about the parameters δ and R and we will talk in general of Lipschitz domains and windows without more explanations.

Theorem (2.1.5)[2]: Let Ω be a bounded $C^{1+\varepsilon}$ domain (i.e. a Lipschitz domain with parameterizations in $C^{1+\varepsilon}$) for a given $\varepsilon > 0$, and let $1 < p < \infty$ and $0 < s \leq 1$ such that $sp > 2$. Then the Beurling transform is bounded in $W^{s,p}(\Omega)$ if and only if $B(X_\Omega) \in W^{s,p}(\Omega)$.

This was proved in fact for a wider class of even Calderón-Zygmund operators in the plane. We considered the extension of the Theorem (2.1.5) to higher orders of smoothness *s* and other ambient spaces R^d. We have restricted ourselves to the study of the classical Sobolev spaces, where the smoothness is a natural number.

Theorem (2.1.6)[2]: Let Ω be a Lipschitz domain, *T* a smooth convolution Calderón-Zygmund operator, $n \epsilon N$ and $p > d$. Then the following statements are equivalent.

a) The operator *T* is bounded in $W^{n,p}(\Omega)$.
b) For any polynomial restricted to the domain, $P \in p^{n-1}(\Omega)$, we have that $T(P) \in W^{n,p}(\Omega)$.

This result remind the results by Rodolfo H. Where the characterization of some generalized Calderón-Zygmund operators which are bounded in the homogeneous Triebel-Lizorkin spaces in the whole ambient space is given in terms of its behavior on polynomials. Vähäkangas obtained some T1 theorem for weakly singular integral operators on domains, but in that case, roughly speaking, the image of the characteristic function being in a certain BMO-type space was shown to be equivalent to the boundedness of $T : L^p(\Omega) \to W^{m,p}(\Omega)$ where m is the degree of the singularity of T's kernel.

One can see that, if $\varepsilon > s$ and Ω is a $C^{1+\varepsilon}$ domain then $BX_{\Omega} \in W^{s,p}(\Omega)$, so we have that, assuming the conditions in the Theorem (2.1.5) for Ω, s and p, one always has the Beurling transform bounded in $W^{s,p}(\Omega)$. With this result, they could deduce the next remarkable theorem that we state here as a corollary.

Corollary (2.1.7)[2]: Assuming Ω, s and p to be as in the previous theorem with the restriction $\varepsilon > s$,if we have a function μ such that $supp\ (\mu) \subset \overline{\Omega}$ and $\|\mu\|_{\infty} < 1$, we can define the principal solution of the Beltrami equation

$$\bar{\partial}\phi(z) = \mu(z)\partial\phi(z),$$

as $\phi(z) = z + C(h)(z)$ where C stands for the Cauchy transform. Then

$$\mu \in W^{s,p}(\Omega) \ \to h \in W^{s,p}(\Omega)$$

Víctor Cruz and Xavier Tolsa worked to find a sufficient condition weaker than $\varepsilon > s$, that if $\Omega \subset C$ is a Lipschitz domain and its unitary outward normalvector N is in the Besov space $B_{p,p}^{s-1/P}$, then one has $B(X_{\Omega}) \in W^{s,p}(\Omega)$. Taking into account that for anyn $\epsilon > 0$, $B_{p,p}^{s-1/P} \subset W^{s-1/p-\epsilon,p}, if\ sp > 2$ we can use the Sobolev Embedding Theorem to deduce that the parameterizations are indeed in $C^{1+\varepsilon}$ for some $\varepsilon > s$, leading to the boundedness of the Beurling transform. That this geometric condition is necessary when the Lipschitz constants are small. The result can be formulated similarly for $= n \geq 2$. We are trying to see which conditions can be weakened.

We work with Carleson measures to find a sufficient condition for p ≤ d. This condition is in fact necessary for s = 1.

Theorem (2.1.8)[2]: Given a Calderón-Zygmund smooth operator of order 1, a Lipschitz domain Ω and $1 < p < \infty$, the following statements are equivalent.

i. T is a bounded operator on $W^{1,p}(\Omega)$.

ii. The measure $|\nabla T X_{\Omega}(x)|^p dx$ is a Carleson measure in the sense of Definition (2.2.9)

We write p^n for the vector space of polynomials of degree smaller or equal than n (in R^d). Givena set $U \subset R^d$, we write $p^n(U)$ for the family of functions $p.XU$ with $p \in P^n$.

The polynomials and derivatives that we need to use will be written with the multiindex notation. For any multiindex $\alpha \in N^d$ (where we assume the natural numbers to include the 0), $\alpha = (\alpha_1, \dots, \alpha_d)$, we define its modulus as $|\alpha| = \sum_{j=1}^{d} \alpha_i$ and its factorial $\alpha! := \prod_{j=1}^{d} \alpha_i!$, leading to the usual definitions of combinatorial numbers. For $x \in R^d$ we write $x^\alpha := \prod_{j=1}^{d} x_j^{\alpha_j}$ and for $\phi \in C_c^\infty$(infinitely many times differentiable with compact support), $D^\alpha \phi := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha 1} \dots \partial x_d^{\alpha d}} \phi$.

In general, for any open set Ω, and any distribution $f \in \acute{D}(\Omega)$, we define the α derivative in the sense of distributions, i.e.

$$\langle D^\alpha f, \phi \rangle := (-1)^{|\alpha|} \langle f, D^\alpha \phi \rangle \quad for\ every\ \phi \in C_c^\infty(\Omega)$$

If the distribution is regular, i.e. $D^\alpha f \in L^1_{loc}$, we say it is a weak derivative.

We say that $f \in L^p(\Omega)$ is in the Sobolev space $W^{n,p}(\Omega)$ if it has weak derivatives up to order *n* and $D^\alpha f \in L^p(\Omega)$ for $|\alpha| \le n$. We will use the norm

$$\|f\|_{W^{n,p}(\Omega)} = \sum_{\alpha \le n} \|D^\alpha f\|_{L^p(\Omega)}$$

For Lipschitz domains, it is enough to consider the higher order derivatives,

$$\|f\|_{W^{n,p}(\Omega)} \approx \|f\|_{L^p} + \|\nabla^n f\|_{L^p(\Omega)}$$

Where $|\nabla^n f| = \sum_{|\alpha|=n} |D^\alpha f|$.

We will solve a Neumann problem by means of the Newton potential: given an integrable function with compact support $g \in L^1_0(R^d)$, its Newton potential is

$$Ng(x) = \int \frac{|x-y|^{2-d}}{(2-d)w_d} g(y)dy \quad if\ d > 2\ , \tag{1}$$

$$Ng(x) = \int \frac{\log|x-y|}{2\pi} g(y)dy \quad if\ d = 2,$$

where w_d stands for the surface measure of the unit sphere in R^d. Recall that the gradient of N_g is the (d – 1)-dimensional Riesz transform of g,

$$\nabla Ng(x) = R^{(d-1)}g(x) = \int \frac{x-z}{w_d|x-z|^d} g(z)dz:$$

It is well known that $\Delta Ng(x) = g(x)$ for $x \in R^d$.

We recall now two results that we will use every now and then. The first is the Leibnitz' Formula, which states that for $f \in W^{n,p}(\Omega)$ and $|\alpha| \leq n$, if $\phi \in C_c^\infty(\Omega)$, then $f.\phi \in W^{n,p}(\Omega)$ and

$$D^\alpha(f.\phi) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta \phi D^{\alpha-\beta} f \tag{2}$$

The second is the Sobolev Embedding Theorem for Lipschitz domains, which says that for any Lipschitz domain Ω, we have the continuous embedding

$$W^{1,p}(\Omega) \subset C^{0,1-\frac{d}{p}}(\overline{\Omega}) \tag{3}$$

of the Sobolev space $W^{1,p}(\Omega)$ into the Hölder space $C^{0,1-\frac{d}{p}}(\overline{\Omega})$. Recall that

$$\|f\|_{C^{0,s}(\overline{\Omega})} = \|f\|_{L^\infty} + \sup_{\substack{x,y\in\overline{\Omega} \\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^s}$$

Consider a given dyadic grid of semi-open cubes in R^d.

Definition (2.1.9)[2]: We say that a collection of cubes *W* is a Whitney covering of a Lipschitz domain Ω if

W1. The cubes in *W* are dyadic.

W2. The cubes have pairwise disjoint interiors.

W3. The union of the cubes in W is Ω .

W4. There exists a constant C_W such that

$$C_W\ell(Q) \leq dist(Q, \partial\Omega) \leq 4C_W\ell(Q)$$

W5. Two neighbor cubes Q and R $(i.e. \bar{Q} \cap \bar{R} \neq \phi, Q \neq R)$ satisfy $\ell(Q) \leq 2\ell(R)$.

W6. The family $\{10Q\}_{Q\in W}$ has finite superposition, i.e. $\sum_{Q\in W} X_{10Q} \leq C$.

We do not prove here the existence of such a covering because this kind of coverings are well known.

We consider the R-window Q to be a cube centered in $x \in \partial\Omega$, with side-length R inducing a Lipschitz parameterization of the boundary. Given $a(\delta, R)$-Lipschitz domain, we can choose a number $N \approx H^{d-1}(\partial\Omega)/R^{d-1}$ of windows $\{Q_k\}_{k-1}^{N}$ such that

$$\partial\Omega \subset \bigcup_{k=1}^{N} \frac{\delta_0}{c_0} Q_k, \quad \textbf{(4)}$$

where $\delta_0 < \frac{1}{2}$ and $c_0 > 2$ are values to fix later. Notice that

$\{x \in \Omega: dis(x, \partial\Omega) > \varepsilon\}$ is a connected set for any " small enough. **(5)**

Each window Q_k is associated to a parameterization A_k in the sense that, after a rotation,

$$\Omega \cap Q_k = \{(\acute{y}, y_d) \in (R^{d-1} \times R) \cap Q : y_d > A_k(\acute{y})\}$$

Thus, each Q_k induces a "vertical" direction, given by the eventually rotated y_d axis. The following is an easy consequence of the previous statements and the fact that the domain is Lipschitz.

W7. The number of Whitney cubes in $\frac{1}{2}Q_k$ with the same side-length intersecting a given vertical line is uniformly bounded where the "vertical" direction is the one induced by the window.

This is the last property of the Whitney cubes we want to point out. We give some structure to construct paths connecting Whitney cubes. First, we use that the vertical direction allows us to say that one cube is above another one:

Definition (2.1.10)[2]: We say that a cube S is above Q with respect to Q_k if $Q, S \subset \frac{1}{2} Q_k$, there is aline parallel to the vertical direction induced by Q_k intersecting the interior of both cubes and there exists a point $x \in S$ such that for any $y \in Q, x_d > y_d$ in local coordinates.

In order to give a structure to the covering, we distinguish the cubes in the central region from those which are close to the boundary of the domain.

Definition (2.1.11)[2]: We say that Q is central if $sup_{x\in Q}\, dist(x, \partial\Omega) > \frac{\delta_0}{c_0} R$, We call W_0 to this subcollection of cubes.

We say that Q is peripheral if it is not central.

Taking c_0, c_1 and the Whitney constants big enough, if Q is peripheral, then $Q \subset \delta_0 Q_k$ for some $1 \le k \le N$. We call W_k to each subcollection of peripheral cubes. Those subcollections are not disjoint.

On the other hand we call windowpane to $\delta_0 Q_k \cap \Omega$. We will choose δ_0 in such a way that the cubes contained in a windowpane will have "enough room over them" inside Q_k. Namely, taking δ_0 small enough we can grant that for every peripheral cube $Q \in W_k$ there exists a cube S above Q with respect to Q_k such that $sup_{x\in Q}\, dist(x, \partial\Omega) > \frac{R}{8\delta}$ due to the Lipschitz character of Ω. Choosing δ_0 even smaller, if necessary, $\frac{R}{8\delta} > \frac{\delta_0}{c_1} R$, so we can say that for any peripheral cube $Q \in W_k$ there is another cube S which is at the same time central and above Q with respect to Q_k.

There is a minimal length ℓ_0 such that any central cube $Q \in W_0$ has $\ell(Q) > \ell_0$. There is amaximal side-length ℓ_1 such that any cube $Q \in \cup_k W_k$ has $\ell(Q) \le \ell_1$.We have $\ell_1 \approx \ell_0 \approx R$ with constants depending on the Lipschitz character.

We provide a tree-like structure to the family of cubes so that we can refer to the neighbor cubes easier.

Definition (2.1.12)[2]: We say that $C = (Q_1, Q_2 \dots, Q_M)$ is a chain connecting Q_1 and Q_M if Q_i and Q_{i+1} are neighbors for any $i < M$. We will call the "next" cube to $N_C(Q_i) = Q_{i+1}$.

We want to have a somewhat rigid structure to gain some control on the chains we use, so we need to introduce the "chain function" $[.,.] : W \times W \rightarrow \bigcup_M W^M$. We state three rules. The first one is on the Definition of chain function.

First rule:

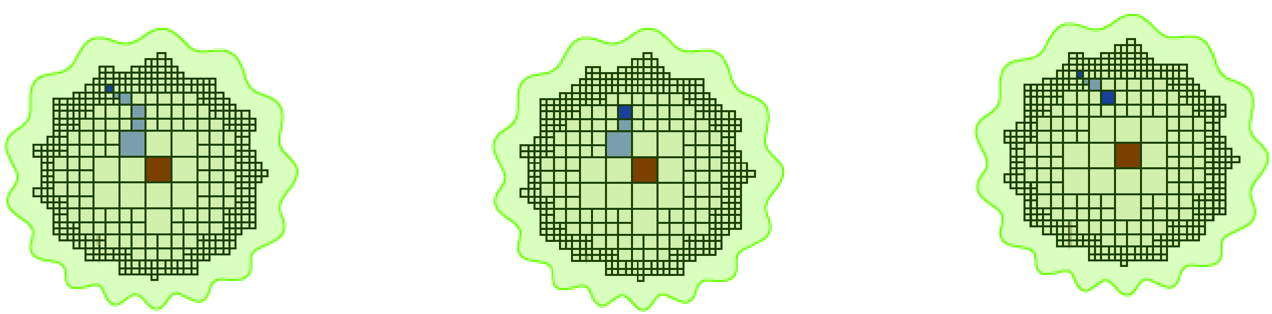
- 1.1: For any cubes $Q, S \in W$, $[Q, S]$ is a chain connecting Q and S.
- 1.2: Given two cubes $Q, S \in W$, if $[Q, S] = (Q_1, Q_2 \dots, Q_M)$ then $[Q, S] = (Q_M, \dots, Q_1)$.

We will also write $[Q, S]$ for the non-ordered collection $\{Q_i\}_{i=1}^{M}$ so that we can say that $Q_i \in [Q, S]$.

Given two cubes Q, S, we will use the open-close interval notation: $(Q, S) := [Q, S] \backslash \{Q, S\}, [Q, S) := [Q, S] \backslash \{S\}, (Q, S] := [Q, S] \backslash \{Q\}$.

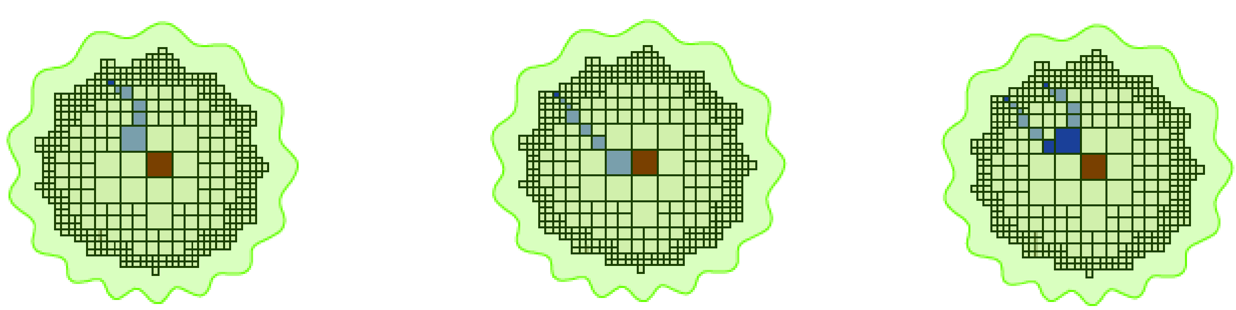
Now we can state the second rule, concerning the central cubes. For that purpose, assume that we have fixed a central cube Q_0.

Figure (2.1): Second rule, 2.2.



(a) $[Q, Q_0]$. (b) $[S, Q_0] \subset [Q, Q_0]$. (c) $[Q, S] \subset [Q, Q_0]$.

Figure (2.2): Second rule, 2.3:



(a) $[Q, Q_0]$ (b)

$[S, Q_0]$ (c)$[Q, S]$

Second rule:

2.1 For any central cube $Q \in W_0$, $[Q, Q_0]$ is a chain of central cubes connecting these two cubes with minimal number of steps.

2.2 For any central cubes $Q, S \in W_0$ with $S \in [Q, Q_0]$, then $[S, Q_0] \subset [Q, Q_0]$. Thus, we can define $[Q, S] = [Q, Q_0] \backslash [S, Q_0]$ (see Figure 2.1).

2.3 Given two different central cubes Q and S, let $Q_S \in [Q, Q_0]$ and $S_Q \in [S, Q_0]$ be the first pair of cubes which are neighbors. Then, $[Q, S] = [Q, Q_S] \cup Y[S_Q, S]$ (see Figure 2.2).

This completes the central structure. For any cube $Q \subset \delta_0 Q_k$, we define $[Q, Q_0]_k$ as a chain connecting Q and Q_0 and such that for any cube $S \in [Q, Q_0]_k$, S is either central or above Q with respect to Q_k, and in case S is central, then $[Q, Q_0]_k = [Q, S]_k \cup [S, Q_0]$, where $[Q, S]_k$ is the subchain of $[Q, Q_0]_k$ limited by Q and S (see Figure 2.3)

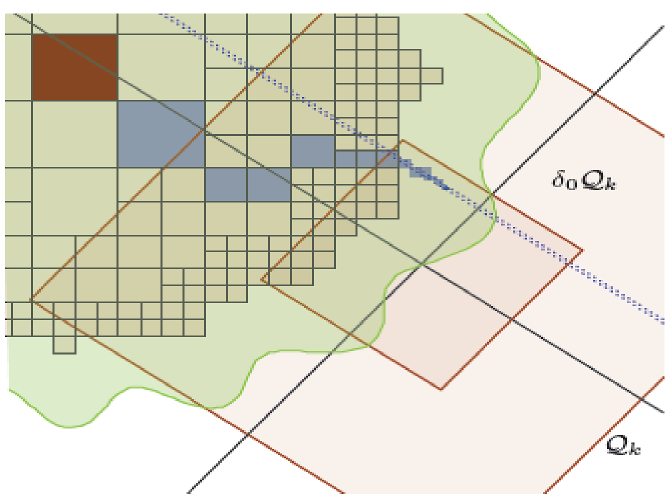


Figure (2.3): $[Q, Q_0]_k$ for $Q \subset \delta_0 Q_k$.

Now we can add the rule for peripheral cubes.

Third rule:

3.1: Given two different peripheral cubes which are both contained in, at least, one common windowpane $Q, S \in W_k, fix\ k$ and use $[\,,]_k$: Call $Q_S \in [Q, Q_0]_k$ and $S_Q \in [S, Q_0]_k$ to the first pair of cubes which are neighbors. Then, $[Q, S] = [Q, Q_S]_k \cup [S_Q, S]_k$ where $[Q, Q_S]_k \subset [Q, Q_0]_k$ and $[S,\, S_Q]_k \subset [S, Q_0]_k$.

3.2: For any peripheral cube *S*, fix any *k* such that $S \in W_k$ and define $[S, Q_0] := [S, Q_0]_k$.

3.3: Given two diferent cubes *Q* and *S* in any situation different from (2.1), use rule 2.3

Definition (2.1.13)[2]: Given a Lipschitz domain Ω, we say that $\{W, \{Q_k\}_{k=1}^N, Q_0, [.,.]\}$is an oriented Whitney covering of Ω if *W* is a Whitney covering of Ω, Q_k are windows as in (4), the cube $Q_0 \in W$ is a central cube of Ω with respect to those windows and $[.,.]$ is a chain function satisfying the three rules explained before.

We say that the covering is properly oriented with respect to a window Q_k if the cubes in the Whitney covering have sides parallel to the faces of Q_k.

Definition (2.1.14)[2]: If $Q, S \in [P, Q_0]$ for some P and $N^j_{[P,Q_0]}(Q) = S$ for some $j \geq 0$, then we say that Q ≤ S. We will say that $Q < S$ if $Q \leq S$ and $Q \neq S$.

Definition (2.1.15)[2]: Given two cubes *Q* and *S* of an oriented Whitney covering, we define the long distance

$$D(Q, S) = \ell(Q) + \ell(S) + dist(Q, S)$$

Remark (2.1.16)[2]: One can see using the Lipschitz condition that, if two Whitney cubes $Q, S \subset \frac{1}{2} Q_k$, then

$$D(Q, S) \approx \ell(Q) + \ell(S) + dist_h(Q, S)$$

where $dist_h$ is the usual horizontal distance between the vertical projections of *Q* and *S* in the window Q_k.

Using that, the properties of the Whitney covering and the chain function rules 2.3, 3.1 and 3.3, one can also prove that, for $P \in [Q, Q_S]$,

$$D(P,S) \approx D(Q,S)$$

and

$$D(P,Q) \approx \ell(P)$$

Now we consider some properties of sums across regions and we relate them to the Hardy-Littlewood maximal operator,

$$Mg(x) = \sup_{Q \ni x} ⨍_Q g(y)dy$$

It is a well known fact that this operator is bounded in L^p for $1 < p \leq \infty$.

Lemma (2.1.17)[2]: Assume that $g \in L^1_{loc}$ and $r > 0$. Then

$i. If\ \eta > 0$,

$$\sum_{D(Q,S)>r} \frac{\int_S g(x)dx}{D(Q,S)^{d+\eta}} \lesssim \frac{inf_Q Mg}{r^\eta}$$

$ii. If\ \eta > 0$,

$$\sum_{D(Q,S)<r} \frac{\int_S g(x)dx}{D(Q,S)^{d-\eta}} \lesssim \inf_Q Mgr^\eta$$

iii. In particular,

$$\sum_{S<Q} \int_S g(x)dx \lesssim \inf_Q Mg\, \ell(Q)^d$$

and, thus, ¸

$$\sum_{S<Q} \ell(S)^d \approx \ell(Q)^d$$

We used the Lipschitz character only to prove the last two inequalities.

Lemma (2.1.18)[2]: Let $a > d - 1$. Then

$$\sum_{S\leq Q} \ell(S)^a \approx \ell(Q)^a$$

with constants depending on a.

Proof. First assume that Q is not central. Selecting the cubes by their side-length, we can write

$$\sum_{S< Q} \ell(S)^a = \sum_{j=1}^{\infty} \sum_{\substack{S< Q \\ \ell(S)=2^{-j}\ell(Q)}} \left(2^{-j}\ell(Q)\right)^a = \ell(Q)^a \sum_{j-1}^{\infty} 2^{-ja} \#\{S < Q : \ell(S) = 2^{-j}\ell(Q)\}$$

Using W7 we get that

$$\#\{S < Q : \ell(S) = 2^{-j}\ell(Q)\} \leq C2^{(d-1)^j}$$

and thus

$$\sum_{S< Q} \ell(S)^a \lesssim \ell(Q)^a \sum_{j=1}^{\infty} 2^{-j(a-(d-1))}$$

This is bounded if $a > d - 1$. By the same token, given an R-window Q_k,

$$\sum_{S\subset\frac{1}{2}Q_k} \ell(S)^a \lesssim R^a \tag{6}$$

If Q is central, use (6) in any region .

Lemma (2.1.19)[2]: Let $b > a > d - 1$. Then

$$\sum_{S\in W} \frac{\ell(S)^a}{D(Q,S)^b} \leq C_{a,b}\ell(Q)^{a-b}$$

Proof. Let us assume that $Q \in W_k$. First of all we consider the cubes contained in $\frac{1}{2}Q_k$ and we classify those cubes by their side-length and their distance to Q:

$$\sum_{S\subset\frac{1}{2}Q_k}\frac{\ell(S)^a}{D(Q,S)^b}\leq\sum_{k=-\infty}^{\infty}\sum_{j=0}^{\infty}\sum_{\substack{S:\ell(S)=2^k\ell(Q)\\ 2^j\ell(Q)\leq D(S,Q)<2^{j+1}\ell(Q)}}\frac{\left(2^k\ell(Q)\right)^a}{\left(2^j\ell(Q)\right)^b}$$

$$\leq\ell(Q)^{a-b}\sum_{k,j}\frac{2^{a^k}}{2^{j^b}}\,\#\{S:\ \ell(S)=\ 2^k\ell(Q),D(S,Q)<\ 2^{j+1}\ell(Q)\}$$

Notice that the value of j in the sum must be greater or equal than k because, otherwise, the last cardinal would be zero.

Using again W7, we only have to bother about how many cubes of side-length $2^k\ell(Q)$ can be fit in the section where one can find cubes at a horizontal distance smaller than $2^{j+1}\ell(Q)$

$$\#\{S:\ \ell(S)=\ 2^k\ell(Q)\ and\ D(S,Q)<\ 2^{j+1}\ell(Q)\}\leq C\left(\frac{\left(2\,.\,2^{j+1}\ +\ 1\right)\ell(Q)}{2^k\ell(Q)}\right)^{d-1}$$

$$\leq C2^{(j+3-k)(d-1)}$$

Thus,

$$\sum_{S\subset\frac{1}{2}Q_k}\frac{\ell(S)^a}{D(Q,S)^b}\lesssim\ell(Q)^{a-b}\sum_{j=0}^{\infty}\sum_{k=-\infty}^{j}2^{k(a+1-d)-j(b+1-d)}\leq C_{a,b}\ell(Q)^{a-b}$$

as soon as $b>a>d-1$.

On the other hand, when $S\not\subset\frac{1}{2}Q_k$ the long distance $D(Q,S\,)$ is always bounded from below by a constant times R (because $Q\subset\delta_0Q_k$), so separating in windows and using Lemma (2.1.19)

$$\sum_{S\not\subset Q_k}\frac{\ell(S)^a}{D(Q,S\,)^b}\lesssim\sum_{S\in W_0}\frac{(diam\Omega)^a}{R^b}+\sum_{j\neq k}\sum_{S\in W_j}\frac{\ell(S)^a}{R^b}\lesssim\ R^{a-b}\lesssim\ \ell(Q)^{a-b}\qquad\textbf{(7)}$$

When it comes to a central cube Q $\in W_0$, just apply an argument analogous to (7).

We will fix a Whitney cube and approximate the function by some mean. Recall that the Poincaré inequality says that, given a function $f \in W^{1,p}(Q)$, with 0 mean in the cube,

$$\|f\|_{L^p(Q)} \lesssim \ell(Q)\|\nabla f\|_{L^p(Q)}$$

with universal constants once we fix *d* and *p*.

If we want to iterate that inequality, we need also the gradient of *f* to have 0 mean on Q. That leads us to define the next approximating polynomials.

Definition (2.1.20)[2]: Let Ω be a domain. Let Q be a cube with $3Q \subset \Omega$. Given $f \in W^{n,p}(3Q)$, we define $\boldsymbol{P}_Q^n(f) \in p^n(\Omega)$ as the unique polynomial (restricted to Ω) of degree smaller or equal than n such that

$$⨍_Q D^\beta \boldsymbol{P}_Q^n f\, dm = ⨍_Q D^\beta f\, dm \tag{8}$$

for any multiindex $\beta \in N^d$ with $|\beta| \leq n$.

The existence of those polynomials is granted in the next lemma.

Lemma (2.1.12)[2]: The polynomial $\boldsymbol{P}_{3Q}^{n-1} f \in p^{n-1}(\Omega)$ exists and is unique as long as we fix Q and $f \in W^{n-1,p}(3Q)$.

Furthermore those polynomials have the next properties.

P1. Let Q be a cube with center x_Q. If we consider the Taylor expansion of $\boldsymbol{P}_{3Q}^{n-1} f$ $at\ x_Q$,

$$\boldsymbol{P}_{3Q}^{n-1} f\ (y) = X\Omega(y) \sum_{\substack{\gamma\in N^d \\ |\gamma|< n}} m_{Q,\gamma}(y - x_Q)^\gamma, \tag{9}$$

then the coefficients $m_{Q,\gamma}$ are bounded by

$$|m_{Q,\gamma}| \leq c_n \sum_{j=|\gamma|}^{n-1} \|\nabla^j f\|_{L^\infty(3Q)} \ell(Q)^{j-|\gamma|}.$$

P2. Given any $0 \leq j < n$, any cube Q and any function $f \in W^{n-1,p}(3Q)$,

$$\int_{3Q} \nabla^j(\boldsymbol{P}_{3Q}^{n-1} f - f) dm = 0.$$

P3. Furthermore, if $f \in W^{n,p}(3Q)$, for $1 \le p \le \infty$ we have

$$\left\|f - \boldsymbol{P}_{3Q}^{n-1}\right\|_{L^{p}(3Q)} \le C\ell(Q)^n \|\nabla^n f\|_{L^p(3Q)}$$

P4. Given a square $Q \subset R^d$, if $p \in P^{n-1}$,

$$|p(y)| \le C\frac{D(y,Q)^{n-1}}{\ell(Q)^{n-1}}\|p\|_{L^\infty(Q)}$$

where $y \in R^d$, and $D(y,Q) = dist(y,Q) + \ell(Q)$.

P5. Given an oriented Whitney covering W with chain function $[.,.]$associated to Ω, and given two Whitney cubes $Q, S \in W$ and $f \in W^{n,p}(\Omega)$,

$$\left\|f - \boldsymbol{P}_{3Q}^{n-1} f\right\|_{L^1(S)} \le \sum_{P\in[S,Q]} \frac{\ell(S)^d D(P,S)^{n-1}}{\ell(P)^{d-1}}\|\nabla^n f\|_{L^1(3P)}$$

$P6.\ If\ |\alpha| < n,$

$$D^\alpha \boldsymbol{P}_{3Q}^{n-1} f(y) = \boldsymbol{P}_{3Q}^{n-1-|\alpha|}(D^\alpha f)(y)$$

Proof. Notice that (8) is a triangular system of equations on the coefficients of the polynomial.

Indeed, if the polynomial exists and has Taylor expansion (9), then

$$D^\gamma \boldsymbol{P}_{3Q}^{n-1} f(y) = \sum_{\beta\ge\gamma} m_{Q,\beta}\frac{\beta!}{(\beta-\gamma)!}\left(y - x_Q\right)^{\beta-\gamma}$$

Fix γ. When we integrate on the cube 3Q,

$$⨍_{3Q} D^\gamma f\, dm = ⨍_{3Q} D^\gamma \boldsymbol{P}_{3Q}^{n-1} f\, dm = \sum_{\beta\ge\gamma} m_{Q,\beta}\frac{\beta!}{(\beta-\gamma)!}\left(\frac{3}{2}\ell(Q)\right)^{|\beta-\gamma|} ⨍_{Q(0,1)} y^{\beta-\gamma} dy$$

$$= \sum_{\beta\ge\gamma} C_{\beta,\gamma} m_{Q,\beta}\ell(Q)^{|\beta-\gamma|}$$

which is a triangular system of equations on the coefficients $m_{Q,\beta}$.

Solving for $m_{Q,\gamma}$, we obtain the explicit expression

$$m_{Q,\gamma} = \frac{1}{C_\gamma} \fint_{3Q} D^\gamma f \, dm \; - \sum_{\beta > \gamma} C_{\beta,\gamma} m_{Q,\beta} \ell(Q)^{|\beta - \gamma|} \qquad \textbf{(10)}$$

For $|\gamma| = n - 1$ this gives the value of $m_{Q,\gamma}$ in terms of $D^\gamma f$,

$$m_{Q,\gamma} = \frac{1}{C^\gamma} \fint_{3Q} D^\gamma f \, dm.$$

Using induction on $n - |\gamma|$ we get the existence and uniqueness of $\boldsymbol{P}_{3Q}^{n-1}$ f. Taking absolute values we obtain P1.

In P2 we write the Definition of the polynomial in a new fashion. This allows us to iterate the Poincaré inequality

$$\left\| f \; - \; \boldsymbol{P}_{3Q}^{n-1} \, f \right\|_{L^p(3Q)} \leq \; C \left\| \nabla f \; - \; P_{3Q}^{n-1} \, f \right\|_{L^p(3Q)} \leq \cdots \leq \; C^n \ell(Q)^n \| \nabla^n f \|_{L^p(3Q)}$$

that is P3.

Property P4 is left.

To prove P5, we consider the chain function in Definition (2.1.13) to write

$$\left\| f - \boldsymbol{P}_{3Q}^{n-1} \, f \right\|_{L^1(S)} \leq \| f - \boldsymbol{P}_{3S}^{n-1} \, f \|_{L^1(S)} \; + \sum_{P \in [S,Q)} \left\| \boldsymbol{P}_{3P}^{n-1} \, f - \; \boldsymbol{P}_{3N(P)}^{n-1} f \right\|_{L^1(S)}$$

where we write $N(P)$ instead of $N_{[S,Q]}(P)$ from Definition (2.1.12). Using the equivalence of norms of polynomials of bounded degree and the property P4,

$$\begin{aligned} \left\| \boldsymbol{P}_{3P}^{n-1} \, f - \; \boldsymbol{P}_{3N(P)}^{n-1} f \right\|_{L^1(S)} &\approx \left\| \boldsymbol{P}_{3P}^{n-1} \, f - \; \boldsymbol{P}_{3N(P)}^{n-1} f \right\|_{L^\infty(S)} \ell(S)^d \\ &\lesssim \left\| \boldsymbol{P}_{3P}^{n-1} \, f - \; \boldsymbol{P}_{3N(P)}^{n-1} f \right\|_{L^\infty(3P \cap 3N(P))} \frac{\ell(S)^d D(P,S)^{n-1}}{\ell(P)^{n-1}} \\ &\approx \left\| \boldsymbol{P}_{3P}^{n-1} \, f - \; \boldsymbol{P}_{3N(P)}^{n-1} f \right\|_{L^1(3P \cap 3N(P))} \frac{\ell(S)^d D(P,S)^{n-1}}{\ell(P)^{n-1} \ell(P)^d} \end{aligned}$$

Taking into account P3 we get

$$\left\|f - \boldsymbol{P}_{3Q}^{n-1} f\right\|_{L^1(S)} \le \sum_{P\in[S,Q]} \left\|f - \boldsymbol{P}_{3Q}^{n-1} f\right\|_{L^1(3P)} \frac{\ell(S)^d D(P,S)^{n-1}}{\ell(P)^{d+n-1}}$$

$$\le \sum_{P\in[S,Q]} \|\nabla^n f\|_{L^1(3P)} \frac{\ell(S)^d D(P,S)^{n-1}}{\ell(P)^{d-1}}$$

Finally, to prove P6, notice that for $|\beta| \le n - |\alpha| - 1$, we have

$$⨍_{3Q} D^\beta\left(D^\alpha \boldsymbol{P}_{3Q}^{n-1} f\right) dm = ⨍_{3Q} D^{\beta+\alpha} \boldsymbol{P}_{3Q}^{n-1} f dm = ⨍_{3Q} D^{\beta+\alpha} f dm = ⨍_{3Q} D^\beta (D^\alpha f)\, dm$$

From now on, we assume *T* to be a smooth convolution Calderón-Zygmund operator of order *n*. Recall that for $f \in L^p$ and $x \notin supp(f)$,

$$Tf(x) = \int K(x - y) f(y)\, dy$$

where the kernel K has derivatives bounded by

$$\left|\nabla^j K(x)\right| \le \frac{C}{|x|^{d+j}} \quad for\ 0 \le j \le n \qquad \textbf{(11)}$$

Given a function $f \in W^{n,p}(\Omega)$, we want to see that its transform $T f$ is in some Sobolev space and, thus, we need to check that its weak derivatives exist up to order *n*. Indeed that is the case.

Remark(2.1.22)[2]: Using that *S* is dense in any Triebel-Lizorkin space $F_{p,q}^s$ with finite exponents *p* and *q* and that $W^{n,p} = F_{p,2}^n$, we conclude that for any $f \in W^{n,p}(R^d)$

$$D^\alpha T(f) = TD^\alpha(f) \qquad \textbf{(12)}$$

and, thus, the operator *T* is bounded in $W^{n,p}(R^d)$

Definition (2.1.23): Let $K \in W_{loc}^{n,1}(R^d \backslash \{0\})$ be a smooth convolution Calderón-Zygmund kernel of order n, $f \in L^p, \alpha \in N^d$ a multiindex with $|\alpha| \le n$ and $x \notin supp(f)$. We define

$$T^{(\alpha)} f(x) = \int D^\alpha K(x - y) f(y)\, dy \qquad \textbf{(13)}$$

Lemma (2.1.24)[2]: Let T be a smooth convolution Calderón-Zygmund kernel of degree n and $f \in L^p$. Then Tf has weak derivatives up to order n in $R^d \backslash suppf$. Moreover, for any multiindex $\alpha \in N^d$ with $|\alpha| \leq n$, and $x \notin suppf$,

$$D^\alpha Tf(x) = T^{(\alpha)} f(x)$$

Proof. Take a compactly supported smooth function $\phi \in C_0^\infty(R^d \backslash suppf)$. We can use Tonelli's Theorem and get

$$\langle T^{(\alpha)} f, \phi \rangle = \int_{supp\phi} \int_{suppf} D^\alpha K(x-y) f(y)\, dy\, \phi(x)\, dx$$

$$= \int_{suppf} \int_{supp\phi} D^\alpha K(x-y) \phi(x)\, dx\, f(y)\, dy.$$

Using the definition of distributional derivative and Tonelli's Theorem again,

$$\langle T^{(\alpha)} f, \phi \rangle = (-1)^\alpha \int_{suppf} \int_{supp\phi} K(x-y) D^\alpha \phi(x)\, dx\ f(y)\, dy$$

$$= (-1)^\alpha \int_{supp\phi} \int_{suppf} K(x-y) f(y)\, dy D^\alpha \phi(x)\, dx = (-1)^\alpha \langle T f, D^\alpha \phi \rangle$$

Lemma (2.1.25)[2]: Given a function $f \in W^{n,p}(\Omega)$, the weak derivatives of Tf in Ω exist up to order *n*.

Before proving this, we consider the functions defined in all R^d.

Proof . Take a classical Whitney covering of Ω, *W*, and for any $Q \in W$, define a bump function $\varphi_Q \in C_0^\infty$ such that $\chi_{2Q} \leq \varphi_Q \leq \chi_{3Q}$. On the other hand, let $\{\psi_Q\}_{Q \in W}$ be a partition of the unity associated to $\left\{\frac{3}{2} Q : Q \in W\right\}$ Consider a multiindex $\alpha\ with\ |\alpha| = n$. Then take $f_1^Q = \varphi_Q \cdot f$, and $f_2^Q = f - f_1^Q$. One can define

$$g(y) = \sum_{Q \in W} \psi_Q(y) \left(T D^\alpha f_1^Q(y) + T^{(\alpha)} f_2^Q(y) \right)$$

This function is defined almost everywhere and is the weak derivative $D^\alpha Tf$.

Indeed, given a test function $\phi \in C_0^\infty(\Omega)$, then, since ϕ is compactly supported in Ω, its suppor tintersects a finite number of Whitney double cubes and, thus, the following additions are finite:

$$\langle g, \phi \rangle = \langle \sum_{Q \in W} \psi_Q \cdot TD^\alpha f_1^Q + \psi_Q \cdot T^{(\alpha)} f_2^Q, \phi \rangle$$

$$= \sum_{Q \in W} \langle TD^\alpha f_1^Q, \phi_Q \rangle + \sum_{Q \in W} \langle T^{(\alpha)} f_2^Q, \phi_Q \rangle, \quad \textbf{(14)}$$

where $\phi_Q = \psi_Q \cdot \phi$

In the local part we can use (12), so

$$\langle TD^\alpha f_1^Q, \phi_Q \rangle = (-1)^{|\alpha|} \langle T f_1^Q, D^\alpha(\phi_Q) \rangle$$

When it comes to the non-local part, bearing in mind that f_2^Q has support away form 2Q and $\phi_Q \in C_0^\infty(2Q)$, we can use the Lemma (2.1.24) and we get

$$\langle T^{(\alpha)} f_2^Q, \phi_Q \rangle = (-1)^{|\alpha|} \langle T f_2^Q, D^\alpha \phi_Q \rangle$$

Back in (14) we have

$$\langle g, \phi \rangle = \sum_{Q \in W} (-1)^{|\alpha|} \langle T f_1^Q, D^\alpha \phi_Q \rangle + \sum_{Q \in W} (-1)^{|\alpha|} \langle T f_2^Q, D^\alpha \phi_Q \rangle$$

$$= \sum_{Q \in W} (-1)^{|\alpha|} \langle T f, D^\alpha \phi_Q \rangle = (-1)^{|\alpha|} \langle T f, D^\alpha \phi \rangle$$

that is $g = D^\alpha T f$ in the weak sense.

Key Lemma (2.1.26)[2]: Let Ω be a Lipschitz domain, W an oriented Whitney covering associated to it, T a smooth convolution Calderón-Zygmund operator of order $n \in N$. Then the following statements are equivalent:

i. For every $f \in W^{n,p}(\Omega)$,

$$\|Tf\|_{W^{n,p}(\Omega)} \leq C \|f\|_{W^{n,p}(\Omega)}$$

ii. For every $f \in W^{n,p}(\Omega)$¸

$$\sum_{Q\in W} \left\|\nabla^n T\left(\boldsymbol{P}_{3Q}^{n-1} f\right)\right\|_{L^p(Q)}^p \leq C\|f\|_{W^{n,p}(\Omega)}^p$$

Proof. Given a multiindex α with $|\alpha| = n$, we will bound the difference¸

$$\sum_{Q\in W} \left\|D^\alpha T\left(f - \boldsymbol{P}_{3Q}^{n-1} f\right)\right\|_{L^p(Q)}^p \lesssim \|\nabla^n f\|_{L^p(\Omega)}^p \quad \textbf{(15)}$$

Given a cube $Q \in W$ we define a bump function $\varphi_Q \in C_0^\infty$ such that $\chi_{\frac{3}{2}Q} \leq \varphi_Q \leq \chi_{2Q}$ and $\left\|\nabla^j \varphi_Q\right\|_\infty \approx \ell(Q)^{-j}$ for any $j \in N$. Then we can break (15) into the local and the non-local parts as follows:

$$\begin{aligned} &\sum_{Q\in W} \left\|D^\alpha T\left(f - \boldsymbol{P}_{3Q}^{n-1} f\right)\right\|_{L^p(Q)}^p \\ &\qquad \lesssim \sum_{Q\in W} \left\|D^\alpha T\left(\varphi_Q\left(f - \boldsymbol{P}_{3Q}^{n-1} f\right)\right)\right\|_{L^p(Q)}^p \\ &\qquad + \sum_{Q\in W} D^\alpha T\left\|\left(\chi_\Omega - \varphi_Q\right)\left(f - \boldsymbol{P}_{3Q}^{n-1} f\right)\right\|_{L^p(Q)}^p = ① + ② \end{aligned} \quad \textbf{(16)}$$

First of all we bound the local term in (16),

$$① = \sum_{Q\in W} \left\|D^\alpha T\left(\varphi_Q\left(f - \boldsymbol{P}_{3Q}^{n-1} f\right)\right)\right\|_{L^p(Q)}^p \lesssim \|\nabla^n f\|_{L^p(\Omega)}^p \quad \textbf{(17)}$$

To do so, notice that $\varphi_Q\left(f - \boldsymbol{P}_{3Q}^{n-1} f\right) \in W^{n,p}(R^d)$ and, by (12) and the boundedness of T in L^p,

$$\begin{aligned} \left\|D^\alpha T\left(\varphi_Q\left(f - \boldsymbol{P}_{3Q}^{n-1} f\right)\right)\right\|_{L^p(Q)}^p &\lesssim \|T\|_{(p,p)}^p \left\|D^\alpha\left(\varphi_Q\left(f - \boldsymbol{P}_{3Q}^{n-1} f\right)\right)\right\|_{L^p(R^d)}^p \\ &= C\left\|D^\alpha\left(\varphi_Q\left(f - \boldsymbol{P}_{3Q}^{n-1} f\right)\right)\right\|_{L^p(2Q)}^p \end{aligned}$$

where $\|.\|_{(p,p)}$ stands for the operator norm in $L^p(R^d)$.

Using first the Leibnitz formula (2), and then using j times the Poincaré inequality (which can be used by the property P2 in Lemma (2.1.21), we get

$$\left\| D^\alpha T\left(\varphi_Q\left(f - \boldsymbol{P}_{3Q}^{n-1}f\right)\right)\right\|_{L^p(Q)}^p \lesssim \sum_{j=1}^{n} \|\nabla^j \varphi_Q\|_{L^\infty(2Q)}^p \|\nabla^{n-j}\left(f - \boldsymbol{P}_{3Q}^{n-1}f\right)\|_{L^p(2Q)}^p$$

$$\lesssim \sum_{j=1}^{n} \frac{1}{\ell(Q)^{jp}}\ \ell(Q)^{jp} \|\nabla^n\left(f - \boldsymbol{P}_{3Q}^{n-1}f\right)\|_{L^p(3Q)}^p = n\|\nabla^n f\|_{L^p(3Q)}^p$$

Summing over all Q we get (17).

For the non-local part in (16),

$$② = \sum_{Q\in W} \|D^\alpha T(\chi_\Omega - \varphi_Q)(f - \boldsymbol{P}_{3Q}^{n-1}f)\|_{L^p(Q)}^p$$

we will argue by duality. We can write

$$②^{\frac{1}{p}} = \sup_{\|g\|_{L^{p'}}\leq 1} \sum_{Q\in W} \int_Q \left|D^\alpha T\left[(\chi_\Omega - \varphi_Q)(f - \boldsymbol{P}_{3Q}^{n-1}f)\right](x)\right| g(x)dx \qquad \textbf{(18)}$$

Notice that given $x \in Q$, by Lemma (2.1.24) one has

$$D^\alpha T\left[(\chi_\Omega - \varphi_Q)(f - \boldsymbol{P}_{3Q}^{n-1}f)\right](x)$$

$$= \int_\Omega D^\alpha K(x - w)\left(1 - \varphi_Q(w)\right)\left(f(w) - \boldsymbol{P}_{3Q}^{n-1}f(w)\right)dw$$

Taking absolute values and using Definition (2.1.1), we can bound

$$\left|D^\alpha T\left[(\chi_\Omega - \varphi_Q)(f - \boldsymbol{P}_{3Q}^{n-1}f)\right](x)\right| \leq \int_{\Omega\setminus\frac{3}{2}Q} \frac{\left|f(w) - \boldsymbol{P}_{3Q}^{n-1}f(w)\right|}{|x - w|^{n+d}}\, dw$$

$$\leq \sum_{S\in W} \frac{\left\|f - \boldsymbol{P}_{3Q}^{n-1}f\right\|_{L^1(S)}}{D(Q,S)^{n+d}} \qquad \textbf{(19)}$$

By property P5 in Lemma (2.1.21) we have that

$$\left\|f - \boldsymbol{P}_{3Q}^{n-1}f\right\|_{L^1(S)} \leq \sum_{P\in[S,Q]} \frac{\ell(S)^d D(P,S)^{n-1}}{\ell(P)^{d-1}} \|\nabla^n f\|_{L^1(3P)}$$

so plugging this expression and (19) into (18), we get

$$②^{\frac{1}{p}} \lesssim \sup_{\|g\|_{\dot{p}} \leq 1} \sum_{Q \in W} \int_Q g(x)dx \sum_{S \in W} \sum_{P \in [S,Q]} \frac{\ell(S)^d D(P,S)^{n-1} \|\nabla^n f\|_{L^1(3P)}}{\ell(P)^{d-1} D(Q,S)^{n+d}}$$

Finally, we use that $P \in [S,Q]$ implies $D(P,S) \lesssim D(Q,S)$ (see Remark (2.1.16),

$$②^{\frac{1}{p}} \lesssim \sup_{\|g\|_{\dot{p}} \leq 1} \sum_{Q,S \in W} \sum_{P \in [S,S_Q]} \int_Q g(x)dx \frac{\ell(S)^d \|\nabla^n f\|_{L^1(3P)}}{\ell(P)^{d-1} D(Q,S)^{n+d}}$$

$$+ \sup_{\|g\|_{\dot{p}} \leq 1} \sum_{Q,S \in W} \sum_{P \in [Q_S,Q,]} \int_Q g(x)dx \frac{\ell(S)^d \|\nabla^n f\|_{L^1(3P)}}{\ell(P)^{d-1} D(Q,S)^{n+d}} = 2.1 + 2.2$$

We consider first the term 2.1 , where $P \in [S, S_Q]$ and, thus, by Remark (2.1.16), $D(Q,S) \approx D(P,Q)$. Rearranging the sum,

$$2.1 \lesssim \sup_{\|g\|_{\dot{p}} \leq 1} \sum_P \frac{\|\nabla^n f\|_{L^1(3P)}}{\ell(P)^{d-1}} \sum_Q \frac{\int_Q g(x)dx}{D(Q,P)^{d+1}} \sum_{S \leq P} \ell(S)^d$$

By Lemma (2.1.17),

$$\sum_{S \leq P} \ell(S)^d \approx \ell(P)^d$$

and

$$\sum_Q \frac{\int_Q g(x)dx}{D(Q,P)^{d+1}} \lesssim \frac{inf_{x \in 3P} Mg(x)}{\ell(P)}$$

We perform a similar argument with 2.2 . Notice that when $P \in [Q, Q_S]$, we have $D(Q,S) \approx D(P,S)$, leading to

$$2.2 \lesssim \sup_{\|g\|_{\dot{p}} \leq 1} \sum_P \frac{\|\nabla^n f\|_{L^1(3P)}}{\ell(P)^{d-1}} \sum_{Q \leq P} \int_Q g(x)dx \sum_S \frac{\ell(S)^d}{D(P,S)^{d+1}}$$

By Lemma (2.1.17),

$$\sum_{Q\le P}\int_Q g(x)dx \lesssim \inf_{x\in 3P} Mg(x)\ell(P)^d$$

and

$$\sum_S \frac{\ell(S)^d}{D(P,S)^{d+1}} \approx \frac{1}{\ell(P)}$$

Thus,

$$2.1 + 2.2 \lesssim \sup_{\|g\|_{\acute{p}}\le 1}\sum_P \frac{\|\nabla^n f\|_{L^1(3P)}}{\ell(P)^{d-1}}\frac{inf_{3P}\, Mg}{\ell(P)}\ell(P)^d \lesssim \sup_{\|g\|_{\acute{p}}\le 1}\sum_p \|\nabla^n f \cdot Mg\|_{L^1(3P)}$$

and, by Hölder inequality and the boundedness of the Hardy-Littlewood maximal operator in $L^{\acute{p}}$

$$②^{\frac{1}{p}} \lesssim \left(\sum_P \|\nabla^n f\|^p_{L^p(P)}\right)^{1\backslash p}\left(\sup_{\|g\|_{L^{\acute{p}}}\le 1}\sum_P \|Mg\|^{\acute{p}}_{L^{\acute{p}}(p)}\right)^{1\backslash \acute{p}}$$

Section (2.2): Carleson Measures and a Necessary Condition

Theorem (2.2.1)[2]: Let Ω be a Lipschitz domain, T a smooth convolution Calderón-Zygmund operator of order $n \in N$ and $p > d$. Then the following statements are equivalent:

a) The operator T is bounded in $W^{n,p}(\Omega)$.
b) For any polynomial restricted to the domain, $P \in p^{n-1}(\Omega)$, we have that $T(P) \in W^{n,p}(\Omega)$

Proof. The implication a)→ b) is trivial.

To see the converse, fix a point $x_0 \in \Omega$. We have a finite number of monomials $P_\lambda(x) = (x - x_0)^\lambda \chi_\Omega(x)$ for multiindices $\lambda \in N^d$ and $|\lambda| < n$, so the hypothesis can be written as

$$\|T(P_\lambda)\|_{W^{n,p}(\Omega)} \le C \qquad \textbf{(20)}$$

Assume $f \in W^{n,p}(\Omega)$. By the Key Lemma, we have to prove that

$$\sum_{Q\in W} \left\|\nabla^n T\left(\boldsymbol{P}_{3Q}^{n-1} f\right)\right\|_{L^p(Q)}^p \lesssim \|f\|_{W^{n,p}(\Omega).}^p$$

We can write the polynomials

$$\boldsymbol{P}_{3Q}^{n-1} f(x) = \chi_\Omega(x) \sum_{|\gamma|< n} m_{Q,\gamma}\left(x - x_Q\right)^\gamma,$$

where x_Q stands for the center of each square Q. Taking the Taylor expansion in $x_0$for each monomial one has

$$\boldsymbol{P}_{3Q}^{n-1} f(x) = \chi_\Omega(x) \sum_{|\gamma|< n} m_{Q,\gamma} \sum_{0\le\lambda\le\gamma} \binom{\gamma}{\lambda} (x - x_0)^\lambda \left(x_0 - x_Q\right)^{\gamma-\lambda}.$$

Thus,

$$\nabla^n T\left(\boldsymbol{P}_{3Q}^{n-1} f\right)(y) = \sum_{|\gamma|< n} m_{Q,\gamma} \sum_{0\le\lambda\le\gamma} \binom{\gamma}{\lambda} \left(x_0 - x_Q\right)^{\gamma-\lambda} \nabla^n (TP_\lambda)(y). \qquad \textbf{(21)}$$

Recall the property P1 in Lemma (2.1.21), which states that

$$|m_{Q,\gamma}| \le C \sum_{j=|\gamma|}^{n-1} \left\|\nabla^j f\right\|_{L^\infty(3Q)} \ell(Q)^{j-|\gamma|} \lesssim \sum_{j=|\gamma|}^{n-1} \left\|\nabla^j f\right\|_{L^\infty(\Omega)} diam\Omega^{j-|\gamma|} \qquad \textbf{(22)}$$

Raising (21) to the power p, integrating in Q and using (22) we get

$$\begin{aligned}\left\|\nabla^n T\left(\boldsymbol{P}_{3Q}^{n-1} f\right)\right\|_{L^p(Q)}^p &\lesssim \sum_{j< n} \left\|\nabla^j f\right\|_{L^\infty(\Omega)}^p \sum_{0\le\lambda\le\gamma} diam\Omega^{(j-|\lambda|)p} \|\nabla^n (TP_\lambda)\|_{L^p(Q)}^p \\ &\lesssim \sum_{j< n} \left\|\nabla^j f\right\|_{L^\infty(\Omega)}^p \sum_{0\le\lambda\le\gamma} \|\nabla^n (TP_\lambda)\|_{L^p(Q)}^p,\end{aligned}$$

with constants depending on the diameter of Ω, p, d and n. By the Sobolev Embedding Theorem, we know that $\left\|\nabla^j f\right\|_{L^\infty(\Omega)} \le C\left\|\nabla^j f\right\|W^{1,p}(\Omega)$ as long as $p > d$. If we add with respect to $Q \in W$ and we use (20) we get

$$\sum_{Q\in W}\left\|\nabla^n T\left(\boldsymbol{P}_{3Q}^{n-1} f\right)\right\|_{L^p(Q)}^p \lesssim \sum_{j<n}\left\|\nabla^j f\right\|_{W^{1,p}(\Omega)}^p \sum_{0\le\lambda\le\gamma}\|\nabla^n(TP_\lambda)\|_{L^p(Q)}^p \lesssim \|f\|_{W^{n,p}(\Omega)}^p$$

Lemma (2.2.2)[2]: Let Ω be a Lipschitz domain, T a smooth convolution Calderón-Zygmund operator of order $n \in N$. Then the following statements are equivalent:

i. For every $f \in W^{n,p}(\Omega)$,

$$\|Tf\|_{W^{n,p}(\Omega)} \le C\|f\|_{W^{n,p}(\Omega)} \tag{23}$$

ii. For every window Q and every $f \in W^{n,p}(\Omega)$ with $f|_{(\delta_0 Q)^c} \equiv 0$,

$$\sum_{Q\in W_Q}\left\|\nabla^n T\left(\boldsymbol{P}_{3Q}^{n-1} f\right)\right\|_{L^p(Q)}^p \le C\|f\|_{W^{n,p}(\Omega)}^p \tag{24}$$

where the whitney covering W_Q is properly oriented with respect to Q, i.e., with the dyadic grid parallel to the local coordinates (see Definition (2.1.13).

Sketch of the proof. To see that i) implies ii) just use the Key Lemma with an appropriate dyadic grid.

To see the converse, one can choose a finite a collection of windows $\{Q_k\}_{k=1}^N\ with\ N \approx H^{d-1}(\partial\Omega)/R^{d-1}$ such that $\frac{\delta_0}{c_0}Q_k$ is a covering of the boundary of Ω, call $Q_0$to the inner region $\Omega\backslash\cup\frac{\delta_0}{2}Q_k$, and let $\{\psi_k\}$ be a partition of the unity related to the covering $\{Q_0\}\cup\{\delta_0 Q_k\}_{k=1}^N$. Consider a function $f \in W^{n,p}(\Omega)$. Since $\psi_0 f \in W^{n,p}(R^d)$, one can see that

$$\|T\,(\psi_0 f)\|_{W^{n,p}(\Omega)} \le C\,\|\psi_0 f\|_{W^{n,p}(\Omega)}$$

Now, following the proof for the Key Lemma but replacing f by $\psi_k f$ and using an appropriate Whitney covering for every single window, one can get

$$\|Tf\|_{W^{n,p}(\Omega)} \le C\sum_k\|\psi_k f\|_{W^{n,p}(\Omega)}$$

Choosing ψ_k as bump functions with the usual estimates on the derivatives $\|\nabla^j \psi_k\|_{L^\infty} \lesssim R^{-j}$, one can get (23) using the Leibnitz formula.

We stated in terms of trees, and relating measures in a Whitney covering with measures in trees.

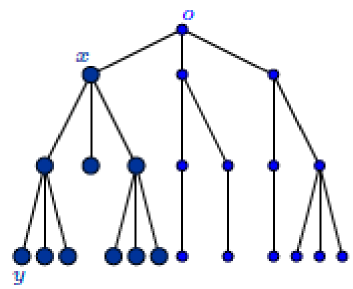


Figure (2.4): $y \in \boldsymbol{Sh}_{\mathcal{T}}(x)$.

Definition (2.2.3)[2]: We say that a connected, loopless graph $\mathcal{T}$ is a tree, and we will fix a vertex $o \in \mathcal{T}$ and call it its root. This choice induces a partial order in $\mathcal{T}$, given by $x \geq y \; if \; x \in [o,y]$ where $[o,y]$ stands for the geodesic path uniting those two vertices of the graph (see Figure (2.4)). We call shadow of x in $\mathcal{T}$ to the collection

$$\boldsymbol{Sh}_{\mathcal{T}}(x) = \{y \in \mathcal{T} : y \leq x\}$$

We say that a function $\rho : \mathcal{T} \to R$ is a weight if it takes positive values (by a function we mean a function defined in the vertices of the tree).

Definition (2.2.4)[2]: Given $h : \mathcal{T} \to R$, we call the primitive Ih the function

$$Ih(y) = \sum_{x \in [o,y]} h(x)$$

Theorem (2.2.5)[2]: Let $1 < p < \infty$ and let ρ be a weight on $\mathcal{T}$. For a nonnegative function μ, the following statements are equivalent:

i. There exists a constant $C = C(\mu)$ such that

$$\|Ih\|_{L^p(\mu)} \leq C\|h\|_{L^p(\rho)}$$

ii. There exists a constant $C = C(\mu)$ such that for any $r \in T$

$$\sum_{x \in \boldsymbol{Sh}_T(r)} \left(\sum_{y \in \boldsymbol{Sh}_T(x)} \mu(y) \right)^{\acute{p}} \rho(x)^{1-\acute{p}} \leq C \sum_{x \in \boldsymbol{Sh}_T(r)} \mu(x)$$

For any $1 \leq p \leq \infty$, we say that a non-negative function μ is a Carleson measure for (I, ρ, p) if there exists a constant $C = C(\mu)$ such that the condition *i)* is satisfied.

Given an R-window Q of a Lipschitz domain Ω with a properly oriented Whitney covering *W*, for any $x \in \frac{1}{2}Q$, we write $x = (\acute{x}, x_d) \in R^{d-1} \times R$. Given a cube $Q \subset \frac{1}{2}Q$, we define the shadow of any point $x \in Q$ as

$$\boldsymbol{Sh}(x) = \left\{ y \in Q \cap \Omega \ : y_d < x_d \ and \ \|\acute{x} - \acute{y}\|_\infty \leq \frac{1}{2}\ell(Q) \right\}.$$

Notice that if x is the center of the upper (n - 1)-dimensional face of *Q*, the vertical projection of $\boldsymbol{Sh}(x)$ (which is a (n - 1)-dimensional square) coincides with the vertical projection of *Q* (see Figure (2.5)). Finally, we define the vertical extension of $\boldsymbol{Sh}(x)$,

$$\widetilde{\boldsymbol{Sh}}(x) = \left\{ y \in Q \cap \Omega \ : \ y_d < x_d + 2\ell(Q) \ and \ \|\acute{x} - \acute{y}\|_\infty \leq \frac{1}{2}\ell(Q) \right\}.$$

More generally, given a set $U \subset \frac{1}{2}Q$ we call its shadow

$$\boldsymbol{Sh}(U) = \{ y \in Q \cap \Omega : there \ exists \ x \in U \ such \ that \ y_d < x_d \ and \ \acute{x} = \acute{y} \}.$$

Notice that we have a proper orientation in the Whitney covering. Thus, given a Whitney cube *Q*, we call the father of $Q, F(Q)$ to the neighbor Whitney cube which is immediately on top of *Q* with respect to the vertical direction. This parental relation induces an order relation ($P \leq Q$ if P is a descendant of *Q*). This would provide a tree structure to the Whitney covering *W* if there was a common ancestor Q_0 for all the cubes. This does not happen, but we can add a “formal” cube Q_0 (root of the tree) and writing $Q \leq Q_0$ for any Q $\subset \frac{1}{2}Q$, since we will only consider functions and measures supported in the windowpane $\delta_0 Q \cap \Omega$. If we call T to the tree with the Whitney cubes as vertices complemented with Q_0 and the structure given by the order relation $\leq$, then for any Whitney cube Q $\subset \frac{1}{2}Q$,

$$\boldsymbol{Sh}(Q) = \bigcup_{P \leq Q} P = \bigcup_{P \in \boldsymbol{Sh}_T(Q)} P$$

(see Figure (2.5)).

Proposition (2.2.6)[2]: Given $1 < p < \infty$ and an R-window Q of a Lipschitz domain Ω with a properly oriented Whitney covering W, consider the weights $\rho(x) = dist(x, \partial\Omega)^{d-p}$, $\rho_W(Q) = \ell(Q)^{d-p}$. For a positive Borel measure μ supported on $\delta_0 Q \cap \Omega$, the following are equivalent:

i. For any $a \in \delta_0 Q \cap \Omega$

$$\int_{\widetilde{\boldsymbol{Sh}}(a)} \rho(x)^{1-\acute{p}} \left(\mu\big(\boldsymbol{Sh}(x) \cap \boldsymbol{Sh}(a)\big)\right)^{\acute{p}} \frac{dx}{dist(x, \partial\Omega)^d} \leq C\mu\big(\boldsymbol{Sh}(a)\big)$$

ii. For any $P \in W$,

$$\sum_{Q \leq P} \left(\sum_{P \leq Q} \mu(P) \right)^{\acute{p}} \rho W(Q)^{1-\acute{p}} \leq C \sum_{Q \leq P} \mu(Q). \quad \textbf{(25)}$$

When $d = 2$ and the domain is a disk, the first condition is equivalent to μ being a Carleson measure for the analytic Besov space $B_p(\rho)$, i.e., for any analytic function defined on the unit disc D,

$$\|f\|_{L^p(\mu)}^p \lesssim \|f\|_{B_p(\rho)}^p = |f(0)|^p + \int_D (1 - |z|^2)^p |\acute{f}(z)|^p \rho(z) \frac{dm(z)}{(1 - |z|^2)^2}.$$

Definition (2.2.7)[2]: We say that a measure satisfying the hypothesis of the previous theorem is a (ρ, p)-Carleson measure for Q (or just a Carleson measure for Q when there is no risk of confusion).

We say that a positive Borel measure μ is a Carleson measure for a Lipschitz domain Ω if it is a Carleson measure for any R-window of the domain.

Now consider a given point $x_0 \in \Omega$. We have a finite number of monomials $P_\lambda(x) = (x - x_0)^\lambda \chi_\Omega(x)$ for multiindices $\lambda \in N^d$ and $|\lambda| < n$. Then,

Theorem (2.2.8)[2]: If for every multiindex $|\lambda| < n$

$$d\mu_\lambda(x) = |\nabla^n T P_\lambda(x)|^p dx$$

defines a Carleson measure, then T is a bounded operator on $W^{n,p}(\Omega)$.

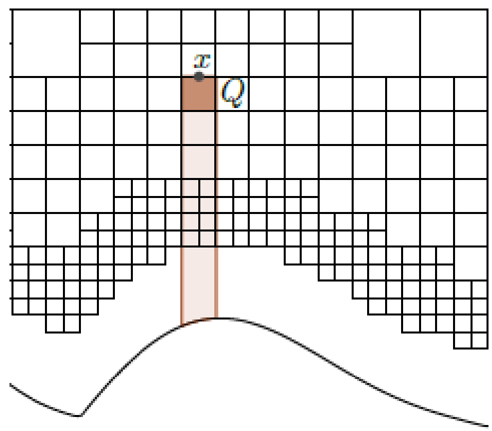


Figure (2.5): The shadows $\boldsymbol{Sh}(x)$ and $\boldsymbol{Sh}$(Q) coincide when x is the center of the upper face of the cube. Furthermore, $P \subset \boldsymbol{Sh}$(Q) if and only if $P \in \boldsymbol{Sh}_{\mathcal{T}}$ (Q).

Proof. Consider a fixed R-window Q and a properly oriented Whitney covering W, i.e. with dyadic grid parallel to the window faces. Making use of Lemma (2.2.2), we only need to bound

$$\sum_{Q\in W} \left\|\nabla^n T\left(p\boldsymbol{P}_{3Q}^{n-1} f\right)\right\|_{L^p(Q)}^p \leq C\|f\|_{W^{n,p}(\Omega)}^p$$

for every $f \in W^{n,p}(\Omega)$ with $f||_{(\delta_0 Q)^c} \equiv 0$.

Consider a function $f \in W^{n,p}(\Omega)$ with $f||_{(\delta_0 Q)^c} \equiv 0$. Using the expression (9) and expanding it as in (21) at a fixed point $x_0 \in \Omega$, we have

$$\sum_{Q\in W} \left\|\nabla^n T\left(\boldsymbol{P}_{3Q}^{n-1} f\right)\right\|_{L^p(Q)}^p \lesssim \sum_{|\gamma|< n} \sum_{\vec{0}\leq\lambda\leq\gamma} C_{\gamma,\lambda,\Omega} \sum_{Q\in W} |m_{Q,\gamma}|^p \|\nabla^n T P_\lambda\|_{L^p(Q)}^p$$

Moreover, taking induction on (10), the coefficients are bounded by

$$m_{Q,\gamma} \lesssim \sum_{|\beta|< n:\, \beta\geq\gamma} \ell(Q)^{|\beta-\gamma|} C_{\beta,\gamma} \left| ⨍_{3Q} D^{\beta} f\, dm \right| \lesssim \sum_{|\beta|< n:\, \beta\geq\gamma} C_{\beta,\gamma,R} \left| ⨍_{3Q} D^{\beta} f\, dm \right|$$

so

$$\sum_{Q\in W} \left\| \nabla^{n} T\left(\boldsymbol{P}_{3Q}^{n-1} f\right) \right\|_{L^{p}(Q)}^{p} \lesssim \sum_{\substack{|\beta|< n \\ \vec{0}\leq\lambda\leq\beta}} \sum_{Q\in W} \left| ⨍_{3Q} D^{\beta} f\, dm \right|^{p} \mu_{\lambda}(Q).$$

Taking into account that $f|_{(\delta_0 Q)^c} \equiv 0$, we have $⨍_{3P} D^{\beta} f\, dm = 0$ for P close enough to the root Q_0. Thus,

$$⨍_{3Q} D^{\beta} f\, dm = \sum_{P\in[Q,Q_0)} \left(⨍_{3P} D^{\beta} f\, dm - ⨍_{3N(P)} D^{\beta} f\, dm \right)$$

and we can use the Poincaré inequality to find that

$$\sum_{Q\in W} \left\| \nabla^{n} T\left(\boldsymbol{P}_{3Q}^{n-1} f\right) \right\|_{L^{p}(Q)}^{p} \lesssim \sum_{\substack{|\beta|< n \\ \vec{0}\leq\lambda\leq\beta}} \sum_{Q\in W} \left(\sum_{P\geq Q} \ell(P) ⨍_{5P} \left| \nabla D^{\beta} f \right| dm \right)^{p} \mu_{\lambda}(Q). \tag{26}$$

By assumption, μ_λ is a Carleson measure for every $|\lambda| < n$, i.e. it satisifies both conditions of Proposition (2.2.6). By Theorem (2.2.5), we have that, for any $h \in\ l^p(\rho w)$,

$$\sum_{Q\in W} \left(\sum_{P\geq Q} h(P) \right)^{p} \mu_{\lambda}(Q) \leq\ C \sum_{Q\in W} h(Q)^{p} \ell(Q)^{d-p,} \tag{27}$$

where $\rho_W(Q) = \ell(Q)^{d-p}$

Consider multiindices β and λ with $|\beta|, |\lambda| < n$ and take $h(P) = \ell(P) ⨍_{3P} |\nabla D^{\beta} f| dm$ in (27).

Using Hölder inequality and the finite overlapping of the triple cubes, we have

(28)

$$\sum_{Q\in W}\left(\sum_{P\geq Q}\ell(P)⨍_{3P}|\nabla D^{\beta}f|dm\right)^{p}\mu_{\lambda}(Q)\leq C\sum_{Q\in W}\left(⨍_{3Q}|\nabla D^{\beta}f|dm\right)^{p}\ell(Q)^{d}$$

$$\lesssim\sum_{Q\in W}\int_{3Q}|\nabla D^{\beta}f|^{p}\,dm\ell(Q)^{\frac{d_p}{p}+d-d_p}\lesssim\int_{\Omega}|\nabla D^{\beta}f|^{p}\,dm.$$

Plugging (28) into (26) for each β and λ, we get

$$\sum_{Q\in W}\left\|\nabla^{n}T\left(\boldsymbol{P}_{3Q}^{n-1}f\right)\right\|_{L^{p}(Q)}^{p}\leq C\|f\|_{W^{n,p}(\Omega)}^{p}.$$

Remark (2.2.9)[2]: Given $g\in L_0^1\left(\overline{R_+^d}\right)$, consider the function

$$F(x):=N\left[\left(R_d^{(d-1)}g\right)d\sigma\right](x)=\int_{\partial R_+^d}\frac{\left(R_d^{(d-1)}g\right)(y)}{(2-d)w_d|x-y|^{d-2}}\,d\sigma(y)\qquad for\ x\in R_+^d \tag{29}$$

For $d>2$, where N denotes the Newton potential (1), $R_d^{(d-1)}$ stands for the vertical component of the vectorial $(d-1)$-dimensional Riesz transform $R^{(d-1)}$ and $d\sigma$ is the hypersurface measure in ∂R_+^d. This function is well defined since

$$\left\|R_d^{(d-1)}g\right\|_{L^1(\sigma)}\leq\int_{\partial R_+^d}\int_{R_+^d}\frac{z_d}{|y-z|^d}\,|g(z)|\,dz\,d\sigma(y)$$

$$=\int_{R_+^d}\left(\int_{\partial R_+^d}\frac{z_d}{|y-z|^d}d\sigma(y)\right)|g(z)|\,dz\lesssim\|g\|_1$$

and, thus, the right-hand side of (29) is an absolutely convergent integral for all $x\in R_+^d$ with $F(x)\leq\frac{\|g\|_1}{|x_d|^{d-2}}$. By the same token, all the derivatives of F are well defined, F is $C^{\infty}(R_+^d)$, harmonic and $\nabla F(x)=R^{(d-1)}\left[\left(R_d^{(d-1)}g\right)d\sigma\right](x)$. When $d=2$ we have to make the usual modifications.

Lemma (2.2.10)[2]: Consider a ball $B_1\subset R^d$ centered at the origin with radius r_1. Let $g\in L^1\left(R_+^d\cap\frac{1}{4}B_1\right)$and

$$h(x):=N\left[\left(R_d^{(d-1)}g\right)d\sigma\right](x)-Ng(x)$$

Then h has weak derivatives in R_+^d *and for any* $\phi\in C_c^{\infty}\left(\overline{R_+^d}\right)$,

$$\int_{R_+^d} \nabla\phi \cdot \nabla h \, dm \; = \int_{R_+^d} \phi g \, dm. \qquad \textbf{(30)}$$

Furthermore, for any $x \notin B_1$, we have

$$|h(x)| \lesssim \begin{cases} \frac{1}{|x|^{d-2}} \|g\|_1 & if\ d > 2, \\ \left(|\log|x|| + 1 + r_1 \frac{x_2|\log x_2|}{|x|^2}\right) \|g\|_1 & if\ d = 2, \end{cases} \qquad \textbf{(31)}$$

and

$$|\nabla h(x)| \lesssim \frac{1}{|x|^{d-1}} \left(1 + \left|log \frac{x_d}{|x|}\right|\right) \|g\|_1 \qquad \textbf{(32)}$$

Remark (2.2.11)[2]: Notice that h can be understood as a weak solution to the Neumann problem

$$\begin{cases} -\nabla h(x) = g(x) & if\ x \in R_+^d, \\ \partial_d h(y) = 0 & if\ y \in \partial R_+^d. \end{cases}$$

Sketch of the proof of Lemma (2.2.10). Let us define F as in (29). Then,

$$\nabla F = R^{(d-1)} \left[\left(R_d^{(d-1)} g \right) d\sigma \right] \qquad \textbf{(33)}$$

and $h = F - Ng$. Using Green's identities we would finish, but we have to check the behavior of the integrands when approaching the boundary.

Given $\varepsilon > 0$, consider the truncated versions $g_\varepsilon(x) = g(x)\chi_{\{x_d > \varepsilon\}}(x)$. Then, writing $F_\varepsilon = N\left[\left(R_d^{(d-1)} g_\varepsilon\right) d\sigma\right]$, it is an exercise to check that F_ε is C^1 up to the boundary, with $\partial_d F_\varepsilon(y) = R_d^{(d-1)} g_\varepsilon(y)$ for all $y \in \partial R_+^d$. Consider $\phi \in C_c^\infty\left(\overline{R^d}\right)$. Using the Green identities, since F_ε is harmonic in R_+^d, we have

$$\int_{R_+^d} \nabla\phi \cdot \nabla F_\varepsilon \, dm - \int_{R_+^d} \nabla\phi \cdot \nabla N g_\varepsilon \, dm$$

$$= \int_{\partial R_+^d} \phi \partial_d F_\varepsilon d\sigma - \int_{\partial R_+^d} \phi \, R_d^{(d-1)} g_\varepsilon d\sigma + \int_{R_+^d} \phi g_\varepsilon \, dm = \int_{R_+^d} \phi g_\varepsilon$$

Taking limits in the previous identity one gets (30).

To prove the pointwise bounds for ∇h, recall that

$$\nabla h(x) = R^{(d-1)}\left[\left(R_d^{(d-1)}g\right)d\sigma\right](x) \; - \; R^{(d-1)}g(x).$$

Given $x \in R_+^d \backslash B_1$, since $supp(g) \subset \frac{1}{4}B_1$,

$$\left|R^{(d-1)}g(x)\right| = c\int_{B_1}\left|\frac{g(z)(x-z)}{|x-\; z|^d}\; dz\right| \lesssim \frac{\|g\|_1}{|x|^{d-1}}. \tag{34}$$

On the other hand, consider $z \in supp(g)$ and $x \notin B_1$. Then, for $y \in \partial R_+^d \cap B(0,|x|/2)$ one has $|x-\; y| \approx |x|$, for $y \in \partial R_+^d \cap B(0,2|x|)\backslash B(0,|x|/2)$ one has $|y-\; z| \approx |x|$ and otherwise $|y\; -\; x| \approx |y-\; z| \approx |y|$. Thus,

$$\begin{aligned}\left|R^{(d-1)}\left[\left(R_d^{(d-1)}\, g\right)d\sigma\right](x)\right| &= c\left|\int_{\partial R_+^d}\left(\int_{B_1}\frac{g(z)z_d\, dz}{|y\; -\; z|^d}\frac{(x\; - y)d\sigma(y)}{|x\; - y|^d}\right)\right| \\ &\lesssim \int_{\partial R_+^d \cap B(0,|x|/2)}\left(\int_{B_1}\frac{|g(z)|z_d\, dz}{|y\; -\; z|^d}\right)\frac{d\sigma(y)}{|x|^{d-1}} \\ &+ \int_{\partial R_+^d \cap B(\acute{x},|x|/2)\backslash B(0,|x|/2)}\left(\int_{B_1}\frac{|g(z)|z_d\, dz}{|x|^d}\right)\frac{d\sigma(y)}{|x-y|^{d-1}} \\ &+ \int_{\partial R_+^d \backslash (B(\acute{x},|x|/2)\cup B(0,|x|/2))}\left(\int_{B_1}|g(z)|z_d\, dz\right)\frac{d\sigma(y)}{|y|^{2d-1}}.\end{aligned} \tag{35}$$

The first term can be bounded by $C\,\frac{\|g\|_1}{|x|^{d-1}}$ because $\int_{\partial R_+^d}\frac{d\sigma(y)}{|y-z|^d} = C\,\frac{1}{z_d}$. The second can be bounded by $C\,\frac{r_1\|g\|_1}{|x|^d}\left|log\,\frac{|x_d|}{|x|}\right|$ using polar coordinates and the last one can be bounded by $C\,\frac{r_1\|g\|_1}{|x|^d}$ trivially. Thus,

$$\left|R^{(d-1)}\left[\left(R_d^{(d-1)}\, g\right)d\sigma\right](x)\right| \lesssim \frac{\|g\|_1}{|x|^{d-1}} + \frac{r_1\|g\|_1}{|x|^d}\left|log\,\frac{x_d}{|x|}\right| + \frac{r_1\|g\|_1}{|x|^d}$$

proving (32) since $r_1 \le |x|$.

To prove the pointwise bounds for *h*, recall that

$$h(x) = N\left[\left(R_d^{(d-1)}\, g\right) d\sigma\right](x) - Ng(x)$$

When $d > 2$ we use the same method as in (34) and (35) using Newton's potential instead of the vectorial $(d-1)$-dimensional Riesz transform to get

$$|h^s(x)| \lesssim \frac{\|g\|_1}{|x|^{d-2}} + \frac{r_1\|g\|_1}{|x|^{d-1}}$$

When $d = 2$ the Newton potential is logarithmic, but the spirit is the same. In this case, arguing as before,

$$|h^s(x)| \lesssim \log|x|\|g\|_1 + r_1\|g\|_1 \frac{|x| + |x| \log|x| + x_2 \log x_2}{|x|^2}$$

Proposition (2.2.12)[2]: Given a window Q of a special Lipschitz domain Ω with a Whitney covering W, a finite positive Borel measure μ supported on $\delta_0 Q$ with

$$\mu(\boldsymbol{Sh}(Q)) \leq C\ell(Q)^{d-p} \quad for\ every\ Q \in W, \tag{36}$$

and given $f \in W^{1,p}(\Omega)$ define the Whitney averaging function

$$Af(x) = \sum_{Q \in W} \chi_Q(x) ⨍_{3Q} f(y)dy \tag{37}$$

If $A : W^{1,p}(\Omega) \to L^p(\mu)$ is bounded, then μ is a (ρ, p)-Carleson measure for $\rho(x) = dist(x, \partial\Omega)^{d-p}$.

Proof. We will argue by duality. Let us assume that the window $Q = Q\left(0, \frac{R}{2}\right)$ is of side-length R and centered at the origin, which belongs to $\partial\Omega$. Notice that the boundedness of A is equivalent to the boundedness of its dual operator

$$A^* : L^{p'}(\mu) \to \left(W^{1,p}(\Omega)\right)^*$$

We also assume that $\mu \equiv 0$ in a neighborhood of $\partial\Omega$. One can prove the general case by means of truncation and taking limits since the constants of the Carleson condition (25) and the norm of the averaging operator will not get worse by this procedure.

Fix a cube P. We apply the boundedness of A^* to the test function $g = \chi_{\boldsymbol{Sh}(P)}$ to get

$$\|A^*g\|^{\acute{p}}_{\left(W^{1,p}(\Omega)\right)^*} \lesssim \|g\|^{\acute{p}}_{L^{\acute{p}}(\mu)} \; = \mu\big(\boldsymbol{Sh}(P)\big).$$

Thus, it is enough to prove that¸

$$\sum_{Q\leq P} \mu\big(\boldsymbol{Sh}(Q)\big)^{\acute{p}} \ell(Q)^{\frac{p-d}{p-1}} \lesssim \|A^*g\|^{\acute{p}}_{\left(W^{1,p}(\Omega)\right)^*} \; + \; \mu\big(\boldsymbol{Sh}(P)\big) \qquad \textbf{(38)}$$

$Given\ any\ f\ \in W^{1,p}(\Omega), using\ (37)$

$$\langle A^*g, f\rangle \; = \int g\ Af\ d\mu = \int_\Omega \ f\ \left(\sum_{Q\in W} \frac{\chi_{3Q}}{3^d} ⨍_Q g\ d\mu\right)\ dm,$$

where we wrote $\langle .,.\rangle$ for the duality pairing. Consider

$$\tilde{g}(x) := \sum_{Q\in W} \frac{\chi_{3Q}(x)}{3^d} ⨍_Q g\ d\mu = \sum_{Q\leq P} \chi_{3Q}(x)\frac{\mu(Q)}{m3Q}. \qquad \textbf{(39)}$$

Then,

$$\langle A^*g, f\rangle = \int_\Omega \tilde{g}f\ \ dm$$

Notice that $\tilde{g}$ is in L^∞ with norm depending on the distance from the support of μ to $\partial\Omega$ by (36), but the norm of $\tilde{g}$ in L^1 is well known:

$$\|\tilde{g}\|_{L^1} \; = \mu\big(\boldsymbol{Sh}(P)\big).$$

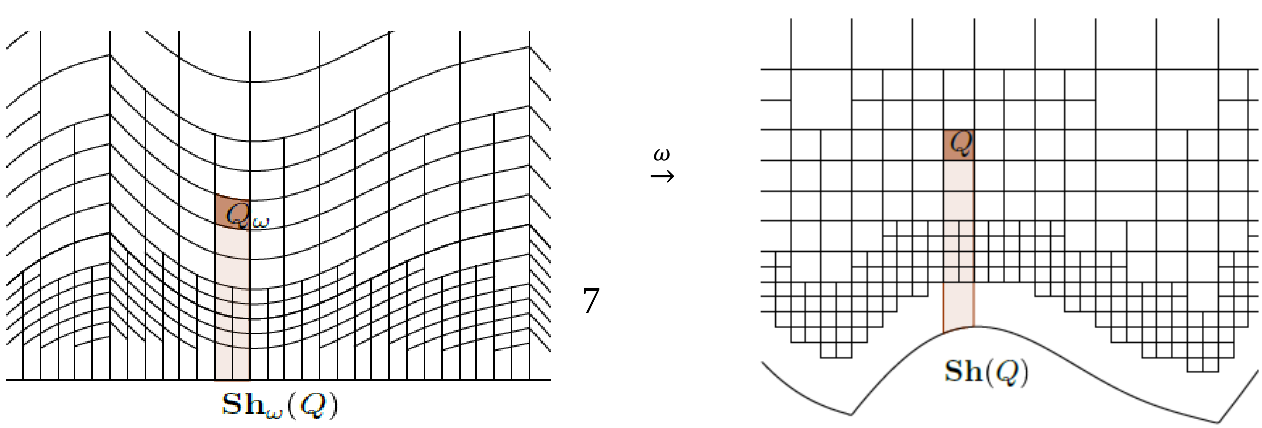


Figure (2.6): We divide R_+^d in pre-images of Whitney cubes.

Consider also the change of variables $\omega : R^d \to R^d, \omega(\acute{x}, x_d) = (\acute{x}, x_d + A(\acute{x}))$ where A is the Lipschitz function whose graph coincides with $\partial\Omega$, and to any Whitney cube Q assign the set $Q_\omega = \omega^{-1}(Q)$ and its shadow $\boldsymbol{Sh}_\omega(Q) = \omega^{-1}(\boldsymbol{Sh}(Q))$ (see Figure (2.6)). Then, for any $x \in R^d$ we define

$$g_0(x) := \tilde{g}(\omega(x))|det(Dw(x))| \quad \textbf{(40)}$$

where $det(Dw(.))$ stands for the determinant of the jacobian matrix. Notice that still $\|g_0\|_{L_1} = \|\tilde{g}\|_{L_1} = \mu(\boldsymbol{Sh}(P))$, and

$$\langle A^* g, f \rangle = \int_\Omega f\, \tilde{g}\, dm = \int_{R_+^d} f \circ \omega \cdot g_0 dm. \quad \textbf{(41)}$$

The key of the proof is using

$$h(x) := N\left[\left(R_d^{(d-1)} g_0\right) d\sigma\right](x) - N g_0(x), \quad \textbf{(42)}$$

which is the solution $h \in L_{loc}^1(R_+^d)$ of the Neumann problem

$$\int_{R_+^d} \nabla\phi \,.\, \nabla h\, dm = \int_{R_+^d} \phi g_0\, dm \quad for\ any\ \phi \in C_c^\infty\left(\overline{R_+^d}\right) \quad \textbf{(43)}$$

provided by Lemma (2.2.10).

We divide the proof in four claims:

Claim (2.2.13)[2]:

$$\langle A^* g, \phi \circ \omega^{-1} \rangle = \int_{R_+^d} \nabla h \,.\, \nabla \phi \qquad for\ any\ \phi \in C_c^\infty\left(\overline{R_+^d}\right)$$

Proof. Since ω is bilipschitz, the Sobolev $W^{1,p}$ norms before and after the change of variables ω are equivalent. In particular, for $\phi \in C_c^\infty\left(\overline{R_+^d}\right)$, $\phi \circ \omega^{-1} \in W^{1,p(\Omega)}$ and we can use (41) and (43).

Now we look for bounds for $\|\partial_d h\|_{L^{\acute{p}}(\boldsymbol{Sh}_\omega(P))}$. The Hölder inequality together with a density argument would give us the bound

$$\|A^* g\|_{\left(W^{1,p}(\Omega)\right)^*} \lesssim \|\nabla h\|_{L^{\acute{p}}} + \mu\left(\boldsymbol{Sh}(P)\right),$$

with constants depending on the window size *R*, but we shall need a kind of converse.

Claim (2.2.14)[2]:

$$\|\partial_d h\|_{L^{\acute{p}}} \lesssim \|A^* g\|_{\left(W^{1,p}(\Omega)\right)^*} + \mu\left(\boldsymbol{Sh}(P)\right)$$

Proof. Take a ball B_1 containing $\omega^{-1}(2Q)$. The duality between L^p and $L^{\acute{p}}$ gives us the bound

$$\|\partial_d h\|_{L^{\acute{p}}(\boldsymbol{Sh}_\omega(P))} \lesssim \sup_{\substack{\phi \in C_c^\infty(B_1 \cap R_+^d) \\ \|\phi\|_p \le 1}} \left| \iint \phi \partial_d \, h \, dm \right|.$$

To avoid problems in the boundary, we will consider h^s to be the symmetric extension of h with respect to the hyperplane $x_d = 0$, $h^s(\acute{x}, x_d) = h(\acute{x}, |x_d|)$. One can see that h^s has global weak derivatives $\partial_j h^s = \left(\partial_j h\right)^s$ for $1 \le j \le d-1$ and $\partial_d h^s(\acute{x}, x_d) = -\partial_d h(\acute{x}, -x_d)$ for any $x_d < 0$. Thus,

$$\|\partial_d h\|_{L^{\acute{p}}(\boldsymbol{Sh}_\omega(P))} \lesssim \sup_{\substack{\phi \in C_c^\infty(B_1) \\ \|\phi\|_p \le 1}} \left| \iint \phi \partial_d \, h^s \, dm \right|. \qquad \textbf{(44)}$$

Given $\phi \in C_c^\infty(B_1)$, one can consider the function $\tilde{\phi}(x) = \phi(x) - \phi(x - r_1 e_d)$, where e_d denotes the unit vector in the *d-th* direction and $r_1 = \frac{1}{2} diam(B_1)$, and take

(45)

$$I_\phi(x) = \int_{-\infty}^{x_d} \tilde{\phi}(\acute{x}, t)\, dt$$

Then, we have $I_\phi \in C_c^\infty(3B_1)$ with $\partial_d I_\phi \equiv \phi$ in the support of ϕ and $\left\|\partial_d I_\phi\right\|_p^p = 2\|\phi\|_p^p$. Thus,

$$\int \phi \partial_d h^s dm = \langle \partial_d I_\phi, \partial_d h^s \rangle - \int_{3B_1 \setminus B_1} \partial_d I_\phi \partial_d h^s\, dm \qquad \textbf{(46)}$$

where we use the brackets for the dual pairing of test functions and distributions. Using Hölder's inequality and the estimate (32) one can see that the error term in (46) is bounded by

$$\int_{3B_1 \setminus B_1} \left|\partial_d I_\phi \partial_d h^s\right| dm \leq \left\|\partial_d I_\phi\right\|_p \|\partial_d h^s\|_{L^{\acute{p}}(3B_1 \setminus B_1)} \leq C\|\phi\|_{L^p} \mu\big(\boldsymbol{Sh}(P)\big) \qquad \textbf{(47)}$$

Notice that the constant *C* only depends on the Lipschitz constant δ_0 and the window side-length *R*.

It is well known that the vectorial d-dimensional Riesz transform,

$$R^{(d)} f(x) = \frac{1}{2w_{d+1}} p.v \int_{R^d} \frac{x - y}{|x - y|^{d+1}} f(y) d(y) \quad for\ any\ f \in S$$

is, in fact, a Calderón-Zygmund operator and, thus, it can be extended to a bounded operator in L^p. Writing $R_i^{(d)}$ for the i-th component of the transform and $R_{ij}^{(d)} := R_i^{(d)} \circ R_j^{(d)}$ for the double Riesz transform in the *i* and *j* direction, one has $\partial_{ii} I_\phi = R_{ii}^{(d)} \Delta I_\phi = \Delta R_{ii}^{(d)} I_\phi$ by a simple Fourier argument. Thus, writing $f_\phi = R_{dd}^{(d)} I_\phi$, we have $\Delta f_\phi = \partial_{dd} I_\phi$, so

$$\langle \partial_d I_\phi, \partial_d h^s \rangle = -\langle \partial_{dd} I_\phi, h^s \rangle = -\langle \Delta f_\phi, h^s \rangle. \qquad \textbf{(48)}$$

We claim that

$$-\langle \Delta f_\phi, h^s \rangle = -\lim_{r\to\infty} \langle \Delta f_r, h^s \rangle = \lim_{r\to\infty} \langle \nabla f_r, \nabla h^s \rangle \qquad \textbf{(49)}$$

where ϕ is a given function in $C_c^\infty(B_1)$, $f_\phi = R_{dd}^{(d)} I_\phi$ with I_ϕ defined as in (45) and $f_r = \varphi_r f_\phi$ with φ_r a bump function in $C_c^\infty(B_{2r}(0))$ such that $\chi_{B_r(0)} \leq \varphi_r \leq \chi_{B_{2r}(0)}$, $|\nabla\varphi_r| \lesssim 1/r$ and $|\Delta\varphi_r| \lesssim 1/r^2$. The advantage of f_r is that it is compactly supported, while only the laplacian of f_ϕ is compactly supported.

Recall that $\Delta f_\phi = \partial_{dd} I_\phi \in C_c^\infty(R^d)$ so, by the hypoellipticity of the Laplacian operator, $f \in C^\infty(R^d)$ itself. Thus, the second equality in (49) is trivial since f_r are C_c^∞ functions. It remains to proof

$$\langle \Delta f_r - \Delta f_\phi, h^s\rangle \xrightarrow{r\to\infty} 0 \qquad \textbf{(50)}$$

To prove (49) recall that Δf_ϕ is compactly supported, so taking *r* big enough we can assume that

$$\Delta\left[(\varphi_r - 1) f_\phi\right] = (\Delta\varphi_r) f_\phi + 2\nabla\varphi_r \cdot \nabla f_{\phi,}$$

so

$$\left|\langle \Delta f_r - \Delta f_\phi, h^s\rangle\right| \lesssim \int_{B_{2r(0)}\setminus B_r(0)} \frac{|f_\phi||h^s|}{r^2} + \frac{|\nabla f_\phi||h^s|}{r}\, dm.$$

It is left for the reader to prove (50) plugging (31) in this expression. One only needs to use that f_ϕ *and* ∇f_ϕ are in any L^q space for $1 < q < \infty$.

We can use $f_r^s(\acute{x}, x_d) = f_r(\acute{x}, -x_d)$ by a change of variables and, by Claim (2.2.13), we obtain

$$\int \nabla f_r \cdot \nabla h^s\, dm = \int_{R_+^d} \nabla f_r \cdot \nabla h\, dm + \int_{R_+^d} \nabla f_r^s \cdot \nabla h\, dm = \langle A^* g, (f_r + f_r^s) \circ \omega^{-1}\rangle. \qquad \textbf{(51)}$$

Summing up, by (46), (47), (48), (49) and (51) and letting r tend to infinity, we get

$$\left|\int \phi\, \partial_d h^s\, dm\right| \lesssim \langle A^* g, \left(f_\phi + f_\phi^s\right) \circ \omega^{-1}\rangle + \|\phi\|_{L^p} \mu\left(\boldsymbol{Sh}(P)\right). \qquad \textbf{(52)}$$

Using Hölder inequality in (45) we have that $\|I_\phi\| \le C\|\phi\|_p$. Now, $\partial_j f_\phi = \partial_j R_{dd}^{(d)} I_\phi = R_{dj}^{(d)} \partial_d I_\phi$, sousing the boundedness of thed-dimensional Riesz transform in L^p we get

$$\|f_\phi\|_{W^{1,p}} = \|f_\phi\|_{L^p} + \|\nabla f_\phi\|_{L^p} \le C\left(\|I_\phi\|_p + \|\partial_d I_\phi\|_p\right) \le \|\phi\|_p. \qquad \textbf{(53)}$$

Summing up, by (44), (52) and (53) we have got that

$$\|\partial_d h\|_{L^{\acute{p}}} \lesssim \sup_{\|f\|_{W^{1,p}(R^d)} \le 1} |\langle A^* g, f \circ \omega^{-1}\rangle| + \mu(\boldsymbol{Sh}(P)).$$

On the other hand, $\|f \circ \omega^{-1}\|_{W^{1,p(\Omega)}} \approx \|f\|_{W^{1,p}(R^d_+)}$ for any f, so we have

$$\|\partial_d h\|_{L^{\acute{p}}} \lesssim \sup_{\|f\|_{W^{1,p}(R^d)} \le 1} |\langle A^* g, f \circ \omega^{-1}\rangle| + \mu(\boldsymbol{Sh}(P)) = \|A^* g\|_{\left(W^{1,p}(\Omega)\right)^*} + \mu(\boldsymbol{Sh}(P))$$

that is Claim (2.2.14).

We can make the big step towards the proof of (38).

Claim (2.2.15)[2]:

$$\sum_{Q \le P} \mu(Sh(Q))^{\acute{p}} \ell(Q)^{\frac{p-d}{p-1}} \lesssim \|\partial_d h\|_{L\acute{p}}^{\acute{p}} + \sum_{Q \le P} \int_{Q_\omega} \left(\int_{\{z: z_d > x_d\}} \frac{z_d - x_d}{|x - z|^d} \tilde{g}(\omega(z)) dz \right)^{\acute{p}} dx$$

$$= ① + ② \qquad \textbf{(54)}$$

Proof. Notice that in (42) we have defined h in such a way that

$$\partial_d h(x) = R_d^{(d-1)}\left[\left(R_d^{(d-1)} g_0\right) d\sigma\right](x) - R_d^{(d-1)} g_0(x)$$

$$= \frac{-1}{w_d} \int_{R^d_+} \left(\frac{2x_d z_d}{w_d} \int_{\partial R^d_+} \frac{d\sigma(y)}{|y - z|^d |x - y|^d} + \frac{x_d - z_d}{|x - z|^d} \right) g_0(z) dz$$

Given $x, z \in R^d_+$, consider the kernel of $R_d^{(d-1)}\left[\left(R_d^{(d-1)}(\cdot)\right) d\sigma\right] - R_d^{(d-1)}(\cdot)$,

$$G(x, z) = \frac{2x_d z_d}{w_d} \int_{\partial R^d_+} \frac{d\sigma(y)}{|y - z|^d |x - y|^d} + \frac{x_d - z_d}{|x - z|^d}, \qquad \textbf{(55)}$$

so that

$$\partial_d h(x) = \frac{-1}{w_d}\int_{R_+^d} G(x,z)g_0(z)dz \tag{56}$$

We have the trivial bound

$$G(x,z) + \frac{z_d - x_d}{|x - y|^d}\chi_{\{z_d>x_d\}}(z) \geq 0, \tag{57}$$

but given any Whitney cube $Q \leq P,\ if\ z \in \boldsymbol{Sh}_\omega(Q)$ we can improve the estimate. In this case,

$$\int_{\partial R_+^d \cap \overline{\boldsymbol{Sh}_\omega(Q)}} \frac{d\sigma(y)}{|y - z|^d} \gtrsim \int_{\partial R_+^d \cap \overline{\omega^{-1}(\boldsymbol{Sh}(z))}} \frac{d\sigma(y)}{|y - z|^d} \approx \frac{1}{z_d},$$

and, thus, when we consider $x \in Q_\omega\ and\ z \in \boldsymbol{Sh}_\omega(Q)$ we have

$$\begin{aligned} G(x,z) + \frac{z_d - x_d}{|x - z|^d}\chi_{\{z_d>x_d\}}(z) &\geq \frac{2x_d z_d}{w_d}\int_{\partial R_+^d} \frac{d\sigma(y)}{|y - z|^d |x - y|^d} \\ &\gtrsim \frac{\ell(Q) z_d}{\ell(Q)^d}\int_{\partial R_+^d \cap \overline{\boldsymbol{Sh}_\omega(Q)}} \frac{d\partial(y)}{|y - z|^d} \gtrsim \frac{\ell(Q)}{\ell(Q)^d}. \end{aligned} \tag{58}$$

By the Lipschitz character of Ω we know that $|\,detD\omega(z)| \approx 1$ for any $z \in R_+^d$. Thus, by (39) and (40), given $Q \leq P$ we have

$$\mu(\boldsymbol{Sh}(Q)) = \sum_{S\leq Q}\mu(S) \lesssim \int_{\boldsymbol{Sh}(Q)} \tilde{g}(w)dw \approx \int_{\boldsymbol{Sh}_\omega(Q)} g_0(z)dz$$

For any $x \in Q_\omega$ using (57) and (58) together with (56) we get

$$\begin{aligned} \mu(\boldsymbol{Sh}(Q)) &\lesssim \int_{R_+^d} G(x,z)g_0(z)\,dz\ell(Q)^{d-1} + \int_{\{z:z_d>x_d\}} \frac{z_d - x_d}{|x - z|^d}\tilde{g}(\omega(z))dz\,\ell(Q)^{d-1} \\ &\lesssim |\partial_d h(x)|\ell(Q)^{d-1} + \int_{\{z:z_d>x_d\}} \frac{z_d - x_d}{|x - z|^d}\tilde{g}(\omega(z))dz\,\ell(Q)^{d-1}. \end{aligned}$$

Then, raising to the power $\acute{p}$, averaging with respect to $x \in Q_\omega$ and summing with respect to $Q \leq P$ with weight $\rho_W(Q) = \ell(Q)^{\frac{p-d}{p-1}}$, since $(d-1)\,\acute{p} + \frac{p-d}{p-1} = 0$, we get Claim (2.2.15).

We bound the negative contribution of the $(d-1)$-dimensional Riesz transform in Claim(2.2.15), i.e. we bound ② .

Claim (2.2.16)[2]:

$$② = \sum_{Q \leq P} \int_{Q_\omega} \left(\int_{\{z: z_d > x_d\}} \frac{z_d - x_d}{|x - z|^d} \tilde{g}\big(\omega(z)\big) dz \right)^{\acute{p}} dx \lesssim \mu\big(\boldsymbol{Sh}(P)\big).$$

Proof. Consider $x, z \in R^d_+ \; with \, x_d < z_d$ and two Whitney cubes Q and S such that $x \in Q_\omega$ and $z \in \omega^{-1}(3S) \backslash \omega^{-1}(3Q)$, then

$$\frac{z_d - x_d}{|x - z|^d} \lesssim \frac{dist(\omega(z), \partial\Omega)}{D(S,Q)^d} \lesssim \frac{\ell(S)}{D(S,Q)^d}.$$

On the other hand, when $3S \cap 3Q \neq \emptyset$,

$$\int_{\omega^{-1}(3Q)} \frac{|z_d - x_d|}{|x - z|^d} dz \lesssim \ell(Q) \approx \ell(S).$$

Bearing this in mind and the fact that, by (39) $\tilde{g}\big(\omega(z)\big) \lesssim \sum_{L \in W} \chi_{3L} \omega(z) \frac{\mu(L)}{m(3L)}$, one gets

$$② \lesssim \sum_{Q \leq P} \ell(Q)^d \left(\sum_{S \leq P} \frac{\mu(S)\ell(S)}{D(S,Q)^d} \right)^{\acute{P}}.$$

Let us consider a fixed $\epsilon > 0$ and apply first the Hölder inequality and then (36). We get

$$② \lesssim \sum_{Q \leq P} \ell(Q)^d \left(\sum_{S \leq P} \frac{\mu(S)\ell(S)^{1-\epsilon\acute{p}}}{D(S,Q)^d} \right) \left(\sum_{S \leq P} \frac{\mu(S)\ell(S)^{1+\epsilon p}}{D(S,Q)^d} \right)^{\frac{\acute{p}}{p}}$$

$$\lesssim \sum_{Q \leq P} \ell(Q)^d \left(\sum_{S \leq P} \frac{\mu(S)\ell(S)^{1-\epsilon\acute{p}}}{D(S,Q)^d} \right) \left(\sum_{S \leq P} \frac{\ell(S)^{d-p+1+\epsilon p}}{D(S,Q)^d} \right)^{\frac{\acute{p}}{p}}.$$

By Lemma (2.1.19), the last sum is bounded by $C\ell(Q)^{-p+1+\epsilon p}$ with C depending on ϵ as long as $d > d-p+1+\epsilon p > d-1$, that is, when $\frac{p-2}{p} < \epsilon < \frac{p-1}{p}$. Thus,

$$②\lesssim \sum_{Q\leq P}\sum_{S\leq P}\frac{\mu(S)\ell(S)^{1-\epsilon\acute{p}}\ell(Q)^{d+(\epsilon-1)\acute{p}+\acute{p}/p}}{D(S,Q)^d} = \sum_{S\leq P}\mu(S)\ell(S)^{1-\epsilon\acute{p}}\sum_{Q\leq P}\frac{\ell(Q)^{d-1+\epsilon\acute{p}}}{D(S,Q)^d}.$$

Again by Lemma (2.1.19), the last sum does not exceed $C\ell(S)^{-1+\epsilon\acute{p}}$ with C depending on ϵ as long as $d > d-1+\epsilon\acute{p} > d-1$, *i.e.*, when $0 < \epsilon < \frac{1}{\acute{p}} = \frac{p-1}{p}$. Summing up, we need

$$max\left\{\frac{p-2}{p},0\right\} < \epsilon < \frac{p-1}{p}.$$

Such a choice of ϵ is possible for any $p > 1$. Thus,

$$②\lesssim \sum_{S\leq P}\mu(S) = \mu(\boldsymbol{Sh}(P)).$$

Being μ a finite measure, $\mu(\boldsymbol{Sh}(P))^{\acute{p}} \leq \mu(\boldsymbol{Sh}(P))\mu(\delta_0 Q)^{\grave{p}-1}$. Thus, the last term in (54) is also bounded due to Claim (2.2.14):

$$①\lesssim \|A^*g\|^{\acute{p}}_{\left(W^{1,p}(R^d_+)\right)^*} + \mu(\boldsymbol{Sh}(P))^{\acute{p}} \lesssim \|A^*\|^{\acute{p}}\|g\|^{\acute{p}}_{L^{\acute{p}}(\mu)} + \mu(\boldsymbol{Sh}(P)) \lesssim \mu(\boldsymbol{Sh}(P)) \qquad \textbf{(59)}$$

Using Claim (2.2.15) together with Claim (2.2.16) and (59), we get that

$$\sum_{Q\leq P}\mu(\boldsymbol{Sh}(P))^{\acute{p}}\ell(Q)^{\frac{p-d}{p-1}} \lesssim \mu(\boldsymbol{Sh}(P)).$$

Theorem (2.2.17)[2]: Given a Calderón-Zygmund smooth operator of order *1* and *a* Lipschitz domain Ω, the following statements are equivalent:

i. Given any window Q with a properly oriented Whitney covering, and given any Whitney cube $P \subset \delta_0 Q$, one has¸

$$\sum_{Q\leq P}\left(\int_{\boldsymbol{Sh}(Q)}|\nabla T(\chi_\Omega)|^p\right)^{\acute{p}}\ell(Q)^{\frac{p-d}{p-1}} \leq C\int_{\boldsymbol{Sh}(P)}|\nabla T(\chi_\Omega)|^p$$

ii. T is a bounded operator on $W^{1,p}(\Omega)$.

Proof. The implication $i \Rightarrow ii$ is Theorem (2.2.8).

To prove that $ii \Rightarrow i$ we will use the previous proposition. Let us assume that we have a properly oriented Whitney covering W associated to an R-window Q of a Lipschitz domain Ω, where we assume that the window $Q = Q\left(0, \frac{R}{2}\right)$ is of side-length R and centered at the origin, and define

$$d\mu(x) := |\nabla T(\chi_\Omega)(x)|^p \chi_{\delta_0 Q}(x)\, dx$$

Notice that if T is bounded in $W^{1,p}(\Omega)$ then, by Lemma (2.2.2),

$$\sum_{Q \in W} \left|⨍_{3Q} f\right|^p \mu(Q) \lesssim \|f\|_{W^{1,p}(\Omega)}$$

so the discrete averaging operator $A_\Omega : W^{1,p}(\Omega) \to L^p(\mu)$ defined as

$$A_\Omega f(x) = \sum_{Q \in W} \chi_Q(x) ⨍_{3Q} f(y)\, dy \qquad \textbf{(60)}$$

is bounded.

Consider the Lipschitz function A whose graph coincides with the boundary of Ω in Q. We say that $\widetilde{\Omega}$ is the special Lipschitz domain defined by the graph of A that coincides with Ω in the window Q. One can consider a Whitney covering $\widetilde{\mathrm{W}}$ associated to $\widetilde{\Omega}$ such that it coincides with W in $\delta_0 Q$. Consider the $\widetilde{\Omega}$ version of the averaging operator,

$$Af(x) := A_{\widetilde{\Omega}} f(x) = \sum_{Q \in W} \chi_Q(x)\, ⨍_{3Q} f(y) dy \quad for\ f \in W^{1,p}(\widetilde{\Omega}).$$

It is easy to see that the boundedness of A_Ω implies the boundedness of

$$A : W^{1,p}(\widetilde{\Omega}) \to L^p(\mu)$$

(consider an appropriate bump function and use the Leibnitz formula).

In order to apply Proposition (2.2.12), we only need to show that $\mu(\boldsymbol{Sh}(Q)) \leq C\ell(Q)^{d-p}$, which in particular implies that μ is finite. Consider a bump function φ_Q such that $\chi_{\boldsymbol{Sh}}(2Q) \leq \varphi_Q \leq \chi_{\boldsymbol{Sh}}(4Q)\ with\ |\nabla \varphi_Q| \lesssim \frac{1}{\ell(Q)}$.

Then,

$$\mu(\boldsymbol{Sh}(P)) = \int_{\boldsymbol{Sh}(Q)} |\nabla T\chi_\Omega(x)|^p dx \leq \int_{\boldsymbol{Sh}(Q)} |\nabla T(\chi_\Omega - \varphi_Q)(x)|^p dx + \int_\Omega |\nabla T\varphi_Q(x)|^p dx$$

With respect to the first term, notice that given $x \in \boldsymbol{Sh}(Q)$, $dist\left(x, sup(\chi_\Omega - \varphi_Q)\right) > \frac{1}{2}\ell(Q)$ so Lemma (2.1.24) together with (11) allows us to write

$$\left|\nabla T(\chi_\Omega - \varphi_Q)(x)\right| \leq \int_{R^d \setminus \boldsymbol{Sh}(2Q)} \frac{1}{|y - x|^{d+1}} dy \lesssim \frac{1}{\ell(Q)}.$$

Being a Lipschitz domain, $m(\boldsymbol{Sh}(Q)) \approx \ell(Q)^d$, so

$$\int_{\boldsymbol{Sh}(Q)} |\nabla T(\chi_\Omega - \varphi_Q)(x)|^p dx \lesssim \ell(Q)^{d-p}$$

The second term is bounded by hypothesis by a constant times $\|\varphi_Q\|^p_{W^{1,p}(\Omega)}$, and

$$\|\varphi_Q\|^p_{W^{1,p}(\Omega)} \approx \|\varphi_Q\|^p_{L^p(\Omega)} + \|\nabla\varphi_Q\|^p_{L^p(\Omega)} \lesssim \ell(Q)^d + \ell(Q)^{d-p} \lesssim (R^p + 1)\ell(Q)^{d-p},$$

Where R is the side-length of the R-window Q, proving that μ satisfies (36).

In the case of the unit disk, we found in the Key Lemma that if T is a smooth convolution Calderón-Zygmund operator of order 1 bounded in $W^{1,p}(D)$, then

$$\sum_{Q\in W} \left|⨍_{3Q} f\right|^p \int_Q |\nabla T\chi_D(z)|^p \, dm(z) \lesssim \|f\|^p_{W^{1,p}(D)} \quad \textbf{(61)}$$

for all $f \in W^{1,p}(D)$. If one considers $d\mu(z) = |\nabla T\chi_D(z)|^p dm(z)$ and $\rho(z) = (1 - |z|^2)^{2-p}$, then, when f is in the Besov space of analytic functions on the unit disk $B_p(\rho)$,

$$\|f\|^p_{B_p(\rho)} := |f(0)|^p + \int_D |f'(z)|^p (1 - |z|^2)^p \rho(z) \frac{dm(z)}{(1 - |z|^2)^2} \approx \|f\|_{W^{1,p}(D)}$$

Using the mean value property (and (36) for the error term), one can see that if T is bounded then for every holomorphic function f, the bound in (61) is equivalent to

$$\int_D |f(z)|^p |\nabla T\chi_D(z)|^p dm(z) \lesssim \|f\|^p_{B_p(\rho)}, \quad \textbf{(62)}$$

i.e., $\|f\|_{L^p(\mu)} \lesssim \|f\|_{B_p(\rho)}$. The measure μ is a Carleson measure for $\left(B_p(\rho), P\right)$, stablishing Theorem (2.2.17) for the unit disk by means in that article.

For $\Omega \subset C$ Lipschitz, we also have

$$\int_{\Omega} |f(z)|^p |\nabla T\chi_D(z)|^p dm(z) \lesssim \|f\|_{W^{1,p}(D)}^p$$

for any analytic function f. If Ω is simply connected, considering a Riemann mapping $F: D \to \Omega$, and using it as a change of variables, one can rewrite the previous inequality as

$$\int_{D} |f \circ F|^p \mu\big(F(\omega)\big)\left|\acute{F}(\omega)\right|^2 dm(\omega)$$

$$\lesssim \left|f\big(F(0)\big)\right|^p + \int_{D} |(f \circ F)'(\omega)|^p \left|\acute{F}(\omega)\right|^{2-p} dm(\omega)$$

for every f analytic in Ω. Writing $d\tilde{\mu}(\omega) = \mu\big(F(\omega)\big)|\acute{F}(\omega)|^2 dm(\omega),\ and\ \rho(\omega) = |\acute{F}(\omega)(1-|\omega|^2|)^{2-p}$, one has that given any g analytic on D,

$$\|g\|_{L^p(\tilde{\mu})} \lesssim \|g\|_{B_p(\rho)}$$

So far so good, we have seen that $\tilde{\mu}$ is a Carleson measure for $\big(B_p(\rho), p\big)$, if two conditions on ρ are satisfied. The first condition is that the weight ρ is “almost constant" in Whitney squares, i.e.,

$$for\ x_1, x_2 \in Q \in W \Rightarrow \rho(x_1) \approx \rho(x_2),$$

and this is a consequence of Koebe distortion theorem, which asserts that

$$|\acute{F}(\omega)|(1 - |\omega|^2) \approx dist(F(\omega), \partial\Omega)$$

The second condition is the Bekollé-Bonami condition, which is

$$\int_{Q} (1 - |z|^2)^{p-2} \rho(z) dm(z) \left(\int_{Q} ((1 - |z|^2)^{p-2} \rho(z))^{1-\acute{p}} dm(z)\right)^{p-1} \lesssim m(Q)^p.$$

If the domain Ω is Lipschitz with small constant depending on p (in particular if it is C^1), then this condition is satisfied.

Chapter 3

Singular Integral Operators

We prove some new accretive type Littlewood-Paley results and construct a bilinear paraproduct for a para-accretive function setting. As an application of a bilinear Tb theorem, we prove product Lebesgue space bounds for bilinear Riesz transforms defined on Lipschitz curves.

Section (3.1) : Almost Orthogonality Estimates and Convergence Results

Tb Theorem: *Let* $b_0, b_1,$ *be para-accretive functions. Assume that T is a singular integral operator of Calderón-Zygmund type associated to* b_0, b_1*. Then T can be extended to a bounded operator on* L^2 *if and only if* $M_{b_0}TM_{b_1}$ *satisfies the weak boundedness property and* $M_{b_0}T(b_1)$, $M_{b_1}T^*(b_0) \in BMO$.

Theorem (3.1.1)[3]: *Let* b_0, b_1, b_2*be para-accretive functions. Assume that T is a bilinear singular integral operator of Calderón-Zygmund type associated to* b_0, b_1, b_2*. Then T can be extended to a bounded operator from* $L^{p_1} \times L^{p_2}$ *into* L^p *for all* $1 < p_1, p_2 < \infty$ *satisfying* $\frac{1}{P1} + \frac{1}{P2} = \frac{1}{P1}$ *if and only if* $M_{b_0}T(M_{b_1}\cdot, M_{b_2}\cdot)$ *satisfies the weak boundedness property and* $M_{b_0}T(b_1, b_2)$, $M_{b_1}T^{*1}(b_0, b_2)$, $M_{b_2}T^{*2}(b_1, b_0) \in BMO$.

Some convergence results for a reproducing formula of the form

$$\int_0^\infty \phi_t * \phi_t * f \frac{dt}{t} = f ,$$

For appropriate functions ϕ_t, which came to be known as Calderón's reproducing formula. The convergence of Calderón's reproducing formula holds in many function space topologies. This formula has since been generalized and reformulated in many ways. For some general formulations of this Calderón reproducing formula. We consider discrete versions of Calderón's formula where we replace convolution with ϕ_t with certain non-convolution integral operators indexed by a discrete parameter $k \in Z$ instead of the continuous parameter $t > 0$. We prove a criterion for extending the convergence of perturbed discrete Calderón reproducing formulas from L^p spaces to the Hardy space H^1. We will prove:

Theorem (3.1.2)[3]: *Let* $b \in L^\infty$ *be para-accretive functions and* θ_k *be a collection of Littlewood-Paley square function kernels such that* $\Theta_k b = \Theta_k^* b = 0$ *for all* $k \in$ Z. *Also assume that*

$$\sum_{k \in Z} M_b \Theta_k M_b f = bf$$

for any $f \in C_0^\delta$ *such that* b f *has mean zero, where the convergence holds in* L^p *for some* $1 < p < \infty$. *If* $\phi \in C_0^\delta$ *for some* $0 < \delta \leq 1$ *such that* $b\phi$ *has mean zero, then* $b\phi \in H^1$ *and*

$$\sum_{k \in Z} M_b \Theta_k M_b \phi = b\phi,$$

where the convergence holds in H^1.

Here $C_0^\delta = C_0^\delta(R^n)$ denotes the collection of compactly supported, δ-Hölder continuous functions from R^n into C. Also we take the typical definition of the Hardy space H^1 with norm $||f||_{H^1} = ||f||_{L^1} + \sum_{\ell=1}^n ||R_\ell f||_{L^1}$, where R_ℓ is the ℓ^{th} Reisz transform in $R^n$for $\ell = 1, \ldots, n, R_\ell f = c_n\, p.v. \frac{y_\ell}{|y|^{n+1}} * f$ and c_n is a dimensional constant. Theorem (3.1.2) tells ust hat anytime we have convergence of Calderón's reproducing formula in L^p for some p, then it also converges in H^1, for appropriate operators and functions.

Proof. Define for $k \in Z$, $f_k(x) = M_b \Theta_k M_b \phi$. It easily follows that

$$\int_{R^n} f_k(x)dx = \int_{R^n} M_b \phi(x) \Theta_k^* b(x) dx = 0.$$

Let R be large enough so that $supp(\phi) \subset B(0, R)$. We estimate

$$| f_k(x)| \leq ||b||_{L^\infty} \left| \int_{R^n} (\theta_k(x,y) - \theta_k(x,0)) b(y) \phi(y) dy \right|$$

$$\lesssim \int_{R^n} (2^k|y|)^\gamma (\Phi_k^N(x-y) + \Phi_k^N(x)) |\phi(y)| dy$$

$$\lesssim 2^{\gamma k} R^\gamma (\Phi_k^N * \Phi_0^N(x) + \Phi_k^N(x)) \lesssim 2^{\gamma k} (\Phi_0^N(x) + \Phi_k^N(x)).$$

We also estimate

$$
\begin{aligned}
|\, f_k(x)| \leq ||b||_{L^\infty} &\left| \int_{R^n} \theta_k(x,y) b(y)(\phi(y) - \phi(x)) dy \right| \\
&\lesssim \int_{R^n} \Phi_k^{N+\gamma}\,(x-y)|x-y|^\gamma (\Phi_0^N(y) + \Phi_0^N(x)) dy \\
&\lesssim\ 2^{-\gamma k} \int_{R^n} \Phi_k^N(x-y)(\Phi_0^N(y) + \Phi_0^N(x)) dy \lesssim 2^{-\gamma k}(\Phi_0^N(x) + \Phi_k^N(x)).
\end{aligned}
$$

So we have proved that $|f(x)| \lesssim\ 2^{-\gamma|k|}(\Phi_0^N(x) + \Phi_k^N(x))$. It follows from Lemma (3.1.18) applied with $j = 0$ that

$$||f_k||_{H^1}\ \lesssim (1 + |k|) 2^{-|k|\gamma}.$$

Therefore

$$\left\| \sum_{|k|<M} f_k \right\|_{H^1} \leq \sum_{|k|<M} ||f_k||_{H^1} \lesssim \sum_{k \in Z} (1 + |k|) 2^{-|k|\gamma} < \infty\,.$$

Hence $\sum_{|k|<M} f_k$ is a Cauchy sequence in H^1, and there exists $\tilde{\phi}\ \in\ H^1$ such that

$$\tilde{\phi} = \sum_{k \in Z} f_k = \sum_{k \in Z} M_b \Theta_k M_b \phi$$

But since the reproducing formula holds for $b\phi$ in L^p for some $1 < p < \infty$, it follows that $\tilde{\phi} =\ b\phi$ and the reproducing formula holds for $b\phi$ in H^1, which completes the proof.

The need of Theorem (3.1.2) to prove Theorem (3.1.1) comes about in a product construction used to decompose a bilinear singular integral operator T (as in Theorem (3.1.1)). To prove Theorem (3.1.1), we follow the ideas to write $T = S + L_0 + L_1 + L_2$,where $M_{b_0} S(b_1, b_2) = M_{b_1} S^{*1}(b_0, b_2) =\ M_{b_2} S^{*2}(b_1, b_0) = 0$ and L_0 , L_1 , L_2 are bilinear paraproducts.We construct these paraproducts so that they satisfy $M_{b_0} L_0(b_1, b_2)\ = M_{b_0} T(b_1, b_2)$ and $M_{b_1} L_0^{*1}\,(b_0, b_2)\ =\ M_{b_2} L_0^{*2}\,(b_1, b_0)\ = 0$ in *BMO*; likewise for L_1 and L_2.The paraproduct L_0 is defined in terms of a generalized Calderón type reproducing formula, like the ones described in Theorem (3.1.2). The H^1 convergence given by Theorem (3.1.2) implies *BMO* convergence of the formula by duality when paired with appropriate elements ofH^1, and eventually this convergence yields $M_{b_0}L_0(b_1,b_2) = M_{b_0}T(b_1,b_2)$ for this paraproduct construction. For more details on the construction of these paraproducts and the decomposition $T = S + L_0 + L_1 + L_2$.

Definition (3.1.3)[3]: A function $b \in L^\infty$ is para-accretive if $b - 1 \in L^\infty$ and there is a $c_0 > 0$ such that for every cube Q, there exists a sub-cube $R \subset Q$ such that

$$\frac{1}{|Q|}\left|\int_R b(x)dx\right| \geq c_0.$$

We introduce the Hölder continuous spaces and para-accretive perturbed Hölder spaces. These are the functions spaces that we use to form an initial weak continuity assumption for T in Theorem (3.1.1), similar to the linear Tb theorem.

Definition (3.1.4)[3]: Define for $0 < \delta \leq 1$ and $f : R^n \to C$

$$||f||_\delta = \sup_{x\neq y}\frac{|f(x) - f(y)|}{|x-y|^\delta},$$

and the space $C^\delta = C^\delta(R^n)$ to be the collection of all functions $f : R^n \to C$ such that $||f||_\delta < \infty$. Also define $C_0^\delta = C_0^\delta(R^n)$ to be the subspace of all compactly supported functions in C^δ.It follows that $||\cdot||_\delta$ is a norm on C_0^δ. Despite conventional notation, we will take C^1 and C_0^1 to be the spaces of Lipschitz continuous functions to keep a notation consistent. Let b be a para-accretive function and define bC_0^δ to be the collection of functions $b\,f$ suchthat $f \in C_0^\delta$ with norm $||bf||_{b,\delta} = ||f||_\delta$. Also let $\left(bC_0^\delta\right)'$ be the collection of all sequentially continuous linear functionals on bC_0^δ, i.e. a linear functional $W : bC_0^\delta \to C$ is in $\left(bC_0^\delta\right)'$ if and only if

$$\lim_{k\to\infty}||f_k - f||_\delta = 0 \ where\ f_k, f \in C_0^\delta \Rightarrow \lim_{k\to\infty}\langle W, bf_k\rangle = \langle W, bf\rangle,$$

where these are both limits of complex numbers. Given a topological space X, we say that an operator $T: X \to \left(bC_0^\delta\right)'$ is continuous if

$$\lim_{k\to\infty} x_k = x\ in\ X \Rightarrow \lim_{k\to\infty}\langle T(x_k), bf\rangle = \langle T(x), bf\rangle\ forall\ f \in C_0^\delta.$$

Given a bilinear operator $T : b_1C_0^\delta \times b_2C_0^\delta \to (b_0C_0^\delta)'$ for some d > 0, define the transposes of T for $f_1, f_{2,3} \in C_0^\delta$

$$\langle T^{1*}(b_0f_0, b_2f_2), b_1f_1\rangle = \langle T^{*2}(b_1f_1, b_0f_0), b_1f_1\rangle = \langle T(b_1f_1, b_2f_2), b_0\, f_0\rangle.$$

Then the transposes of T are bilinear operators acting on the following spaces: $T^{1*}: b_0C_0^\delta \times b_2C_0^\delta \to (b_1C_0^\delta)'$ and $T^{2*}: b_1C_0^\delta \times b_0C_0^\delta \to (b_2C_0^\delta)'$. One could more generally define the transpose $T^{1*} on\ (b_1C_0^\delta)'' \times b_1C_0^\infty$, but this is not necessary for this work. So we restrict the first spot of T^{1*} to $b_1C_0^\delta$ instead of $(b_1C_0^\delta\,)''$. Likewise for T^{2*}.

Definition (3.1.5)[3]: A function $K : R^{3n} \backslash \{(x,x,x) :\ x \in R^n\} \to C$ is a standard bilinear Calderón-Zygmund kernel if

$$|K(x,y_1,y_2)| \lesssim \frac{1}{(|x-y_1|+|x-y_2|)^{2n}}\ when\ |x-y_1|+|x-y_2|\ \neq\ 0$$

$$|K(x,y_1,y_2)-K(x',y_1,y_2)|\ \lesssim \frac{|x-x'|}{(|x-y_1|+|x-y_2|)^{2n+\gamma}},$$
$$when |x-x'|\ < max(|x-y_1|,|x-y_2|)/2$$

$$|K(x,y_1,y_2)-K(x,y'_1,y_2) \lesssim \frac{|y_1-y'_1|}{(|x-y_1|+|x-y_2|)^{2n+\gamma}},$$
$$when |y_1-y'_1|\ < max(|x-y_1|,|x-y_2|)/2$$

$$|K(x,y_1,y_2)-K(x,y_1,y'2)|\ \lesssim \frac{|y_2-y'_2|}{(|x-y_1|+|x-y_2|)^{2n+\gamma}},$$
$$when |y_2-y'_2|\ < max(|x-y_1|,|x-y_2|)/2.$$

Let $b_0, b_1, b_2 \in L^\infty\,(R^n)$ be para-accretive functions. We say a bilinear operator $T :$ $b_1C_0^\delta \times b_2C_0^\delta \to \left(b_0C_0^\delta\right)'$ is a bilinear singular integral operator of Calderón-Zygmund type associated to b_0, b_1, b_2, or for short a bilinear C-Z operator associated to b_0, b_1, b_2, if T is continuous from $b_1C_0^\delta \times b_2C_0^\delta$ into $(b_2C_0^\delta)'$ for some d > 0 and there exists a standard Calderón-Zygmund kernel K such that for all $f_1, f_2, f_3 \in C_0^\delta$ with disjoint support

$$\langle T(M_{b_1}f_1, M_{b_2}f_2), M_{b_0}f_0\rangle = \int_{R^{3n}} K(x,y_1,y_2) \prod_{i=0}^{2} f_i(y_i)b_i(y_i)dy_i\,.$$

Note that this continuity assumption for T from $b_1C_0^\delta \times b_2C_0^\delta$into $(b_0C_0^\delta)'$ is equivalent to the following: For any $f_0, f_1, f_2, g, g_k \in C_0^\delta$ such that $g_k \to g$ in C_0^δ, we have

$$\lim_{k\to\infty}\langle T\big(M_{b_1}g_k, M_{b_2}f_2\big), M_{b_0}f_0\rangle = \langle T\big(M_{b_1}g, M_{b_2}f_2\big), M_{b_0}f_0\rangle,$$

$$\lim_{k\to\infty}\langle T\big(M_{b_1}f_1, M_{b_2}g_k\big), M_{b_0}f_0\rangle = \langle T\big(M_{b_1}f_1, M_{b_2}g\big), M_{b_0}f_0\rangle,$$

$$\lim_{k\to\infty}\langle T(M_{b_1}f_1, M_{b_2}f_2), M_{b_0}g_k\rangle = \langle T(M_{b_1}f_1, M_{b_2}f_2), M_{b_0}g\rangle.$$

It follows that the continuity assumptions for a bilinear singular integral operator T associated to para-accretive functions b_0, b_1, b_2 is symmetric under transposes. That is, T is a bilinear C-Z operator associated to b_0, b_1, b_2 if and only if T^{1*} is a bilinear C-Z operator associated to b_1, b_0, b_2 if and only if T^{2*} is a bilinear C-Z operator associated to b_2, b_1, b_0.

Definition (3.1.6)[3]: A function $\phi \in C_0^\infty$ is a normalized bump of order $m \in$ N if $supp(\phi) \subset B(0,1)$ and

$$\sup_{|\alpha|\le m} ||\partial^\alpha \phi||_{L^\infty} \le 1.$$

Let $b_0, b_1, b_2 \in$ L$^\infty$ be para-accretive functions, and T be an bilinear C-Z operator associated to b_0, b_1, b_2. We say that $M_{b_0}T(M_{b_1}\cdot, M_{b_2}\cdot)$ satisfies the weak boundedness property (written $M_{b_0}T(M_{b_1}\cdot, M_{b_2}\cdot) \in WBP$) if there exists an $m \in N$ such that for all normalize dbumps $\phi_0, \phi_1, \phi_2 \in C_0^\infty$ of order $m, x \in R^n$, and $R > 0$

$$\left|\langle T(M_{b_1}\phi_1^{x,R}, M_{b_2}\phi_2^{x,R}), M_{b_0}\phi_0^{x,R}\rangle\right| \lesssim R^n$$

Where $\phi^{x,R}(u) = \phi(\frac{u-x}{R})$.

It follows by the symmetry of this definition that $M_{b_0}T\big(M_{b_1}\cdot, M_{b_2}\cdot\big) \in WBP$ if and only if $M_{b_1}T^{1*}(M_{b_0}\cdot, M_{b_2}\cdot) \in WBP$ if and only if $M_{b_2}T^{2*}(M_{b_1}\cdot, M_{b_0}\cdot) \in WBP$. We define T on $(b_1C^\delta \cap L^\infty) \times (b_2C^\delta \cap L^\infty)$, so that we can make sense of the testing condition $M_{b_0}T(b_1, b_2) \in BMO$ as well as the transpose conditions. The definition we give is essentially the same as the one given by Torres in the linear setting and by Grafakos-Torres in the multilinear setting. Here we use the definition with the necessary modifications for the accretive functions b_0, b_1, b_2. A benefit of this definition versus the ones is that we define $T(b_1, b_2)$ paired with any element of $b_0C_0^\delta$, not just the ones with mean zero. Although one must still take care to note that the definition of T agrees with the given definition of T when paired with elements of $b_0 C_0^\delta$ with mean zero.

Definition (3.1.7)[3]: Let b_0, b_1, b_2 be para-accretive function, T be a bilinear singular integral operator associated to b_0, b_1, b_2, and $f_1, f_2 \in C^\delta \cap L^\infty$. Also fix functions $\eta_R^i \in C_0^\infty$ for $R > 0, i = 1,2$ such that $\eta_R^i \equiv 1 \text{ on } B(0,R)$ and $supp(\eta_R^i) \subset B(0,2R)$. Then we define

$$\begin{aligned}\langle T(b_1 f_1, b_2 f_2), b_0 f_0\rangle &= \lim_{R\to\infty} \langle T(\eta_R^1 b_1 f_1, \eta_R^2 b_2 f_2), b_0 f_0\rangle \\ &- \int_{R^3} K(0, y_1, y_2) b_0(x)\, f_0(x) \prod_{i=1}^{2} f_i(y_i)(\eta_R^i(y_i) \\ &- \eta_{R_0}^i(y_i)) b_i(y_i) dx dy_1 dy_2,\end{aligned} \quad \textbf{(1)}$$

where $f_0 \in C_0^\delta$ *and* $R_0 > 0$ is minimal such that $supp(f_0) \subset B(0, R_0/2)$. When $R > 2R_0$, we have

$$\begin{aligned}\langle T(\eta_R^1 b_1 f_1, \eta_R^2 b_2 f_2), b_0 f_0\rangle &= \langle T\left(\eta_{R_0}^1 b_1 f_1, \eta_{R_0}^2 b_2 f_2\right), b_0 f_0\rangle \\ &+ \langle T(\eta_{R_0}^1 b_1 f_1, (\eta_R^2 - \eta_{R_0}^2) b_2 f_2), b_0 f_0\rangle \\ &+ \langle T\left(\left(\eta_R^1 - \eta_{R_0}^1\right) b_1 f_1, \eta_{R_0}^2 b_2 f_2\right), b_0 f_0\rangle \\ &+ \langle T((\eta_R^1 - \eta_{R_0}^1) b_1 f_1, (\eta_R^2 - \eta_{R_0}^2) b_2 f_2), b_0 f_0\rangle\end{aligned}$$

$$= \langle T\left(\eta_{R_0}^1 b_1 f_1, \eta_{R_0}^2 b_2 f_2\right), b_0 f_0\rangle$$

$$+ \int_{R^{3n}} K(y_0, y_1, y_2) \eta_{R_0}^1(y_1) \left(\eta_R^2(y_2) - \eta_{R_0}^2(y_2)\right) \prod_{i=0}^{2} b_i(y_i) f_i(y_i) dy_0\, dy_1\, dy_2$$

$$+ \int_{R^{3n}} K(y_0, y_1, y_2) \left(\eta_R^1(y_1) - \eta_{R_0}^1(y_1)\right) \eta_{R_0}^2(y_2) \prod_{i=0}^{2} b_i(y_i) f_i(y_i) dy_0\, dy_1\, dy_2$$

$$+\langle T\left(\left(\eta_R^1 - \eta_{R_0}^1\right)b_1 f_1, \left(\eta_R^2 - \eta_{R_0}^2\right)b_2 f_2\right), b_0 f_0\rangle = I + II + III + IV.$$

The first term *I* is well defined since $\eta_{R_0}^i b_i f_i \in b_i C_0^\delta$ for a fixed R_0(depending on f_0). We check that the first integral term *II* is absolutely convergent: The integrand of *II* is bounded by $||b_0||_{L^\infty} \prod_{i=1}^2 ||b_i||_{L^\infty}|| f_i||_{L^\infty}$ times

$$|K(y_0, y_1, y_2)\eta_{R_0}^1(y_1)(\eta_R^2(y_2) - \eta_{R_0}^2(y_2))\, f_0(y_0)|$$

$$\lesssim \frac{|\eta_{R_0}^1(y_1)(\eta_R^2(y_2) - \eta_{R_0}^2(y_2))\, f_0(y_0)|}{(|y_0 - y_1| + |y_0 - y_2|)^{2n}}$$

$$\leq \frac{|\eta_{R_0}^1(y_1)(\eta_R^2(y_2) - \eta_{R_0}^2(y_2))\, f_0(y_0)|}{(|y_0 - y_1| + |y_0 - y_2|/2 + (R_0 - R_0/2)/2)^{2n}} \lesssim \frac{|\eta_{R_0}^1(y_1)\, f_0(y_0)|}{(R_0 + |y_0 - y_2|)^{2n}}.$$

This is an $L^1(R^{3n})$ function that is independent of *R* (as long as $R > 4R_0$),

$$\int_{R^{3n}} \frac{|\eta_{R_0}^1(y_1)\, f_0(y_0)|}{(R_0 + |y_0 - y_2|)^{2n}}\, dy_0\, dy_1\, dy_2 \lesssim \int_{R^{2n}} \frac{|\eta_{R_0}^1(y1)\, f_0(y_0)|}{R_0^n} dy_0\, dy_1 \lesssim || f_0||_{L^\infty} R_0^n.$$

Since $\eta_R \to 1$ pointwise, by dominated convergence the following limit exists:

$$\lim_{R\to\infty} \int_{R^{3n}} K(y_0, y_1, y_2)\eta_{R_0}^1(y_1)(\eta_R^2(y_2) - \eta_{R_0}^2(y_2)) \prod_{i=0}^{2} b_i(y_i)\, f_i(y_i) dy_0\, dy_1\, dy_2$$

$$= \int_{R^{3n}} K(y_0, y_1, y_2)\eta_{R_0}^1(y_1)(1 - \eta_{R_0}^2(y_2)) \prod_{i=0}^{2} b_i(y_i)\, f_i(y_i) dy_0 dy_1\, dy_2.$$

So $\lim_{R\to\infty} II$ exists. A symmetric argument holds for $\lim_{R\to\infty} III$.We consider *IV* minus the integral term from (1)

$$IV - \int_{R^3} K(0, y_1, y_2) b_0(y_0)\, f_0(y_0) \prod_{i=1}^{2} f_i(y_i)\eta_R^i(y_i) b_i(y_i) dy_0\, dy_1\, dy_2$$

$$= \int_{R^{3n}} (K(y_0, y_1, y_2) - K(0, y_1, y_2)) b_0(y_0)\, f_0(y_0)$$

$$\times \prod_{i=1}^{2} (\eta_R^i(y_i) - \eta_{R_0}^i(y_i))\, f_i(y_i) b_i(y_i) dy_0\, dy_1\, dy_2\,.$$

Again we bound the integrand by $||b_0||_{L^\infty} \prod_{i=1}^2 ||b_i||_{L^\infty}|| fi||_{L^\infty}$ times

$$
\begin{aligned}
|K(y_0, y_1, y_2) - K(0, y_1, y_2)|\,|\,f_0(y_0)|(\eta_R^1(y_1) - \eta_{R_0}^1(y1))& \\
\lesssim \frac{|y_0|^\gamma |\eta_R^1(y_1) - \eta_{R_0}^1(y_1)|}{(|y_0 - y_1| + |y_0 - y_2|)^{2n+\gamma}} |\,f_0(y_0)|& \\
\lesssim \frac{|y_0|^\gamma |\eta_R^1(y_1) - \eta_{R_0}^1(y_1)|}{(|y_0 - y_1|/2 + R_0/4 + |y_0 - y_2|)^{2n+\gamma}} |\,f_0(y_0)|& \\
\lesssim \frac{R_0^\gamma |\,f_0(y_0)|}{(R_0 + |y_0 - y_1| + |y_0 - y_2|)^{2n+\gamma}},&
\end{aligned}
$$

which is an $L^1(R^{3n})$ function:

$$
\int_{R^{3n}} \frac{R_0^\gamma |\,f_0(y_0)|}{(R_0 \ + |y_0 - y_1| + |y_0 - y_2|)^{2n+\gamma}} \, dy_0 \, dy_1 \, dy_2
$$

$$
\lesssim \int_{R^{2n}} \frac{R_0^\gamma |\,f_0(y_0)|}{(R_0 + |y_0 - y_1|)^{n+\gamma}} \, dy_0 \, dy_1 \lesssim \int_{R^n} |\,f_0(y_0)| dy_0 \ \lesssim \ ||\,f_0||_{L^\infty} R_0^n
$$

Then it follows again by dominated convergence that

$$
\lim_{R\to\infty} \langle T\left(\left(\eta_R^1 - \eta_{R_0}^1\right) b_1 f_1, \left(\eta_R^2 - \eta_{R_0}^2\right) b_2 f_2 \right), b_0 f_0 \rangle
$$

$$
- \int_{R^3} K(0, y_1, y_2) b_0(y_0) f_0(y_0) \prod_{i=1}^{2} f_i(y_i)(\eta_R^i(y_i) - \eta_{R_0}^i(y_i)) b_i(y_i) dy_0 \, dy_1 \, dy_2
$$

$$
= \int_{R^{3n}} \left(K(x, y_1, y_2) - K(0, y_1, y_2)\right) b_0(x) f_0(x) \prod_{i=1}^{2} (1 - \eta_{R_0}^i(y_i)) \, f_i(y_i) b_i(y_i) dy_1 \, dy_2 \, dx,
$$

which is an absolutely convergent integral. Therefore $T(b_1 f_1, b_2 f_2)$ is well defined as an element of $\left(b_0 C_0^\delta\right)'$ for $f_1, f_2 \in C^\delta \cap L^\infty$. Furthermore if $f_0, f_1, f_2 \in C_0^\delta$ and $b_0 f_0$ has mean zero, then this definition of T is consistent with the a priori definition of T since

$$
\lim_{R\to\infty} \int_{R^3} K(0, y_1, y_2) b_0(y_0) f_0(y_0) \prod_{i=1}^{2} f_i(y_i) \left(\eta_R^i(y_i) - \eta_{R_0}^i(y_i)\right) b_i(y_i) dy_0 \, dy_1 \, dy_2
$$

$$
= \left(\int_{R^3} K(0, y_1, y_2) \prod_{i=1}^{2} b_i(y_i) f_i(y_i) \left(1 - \eta_{R_0}^i(y_i)\right) dy_1 \, dy_2 \right) \left(\int_{R^n} b_0(y_0) f_0(y_0) dy_0 \right)
$$

$$
= 0,
$$

since both of these integrals are absolutely convergent. Also, when $b_0 f_0$ has mean zero in this way, the definition of $\langle T(b_1,b_2), b_0 f_0\rangle$ is independent of the choice of η_R^1 and η_R^2. We will also use the notation $M_{b_0}T(b_1,b_2) \in BMO$ or $M_{b_0}T(b_1,b_2) = \beta$ for $\beta \in BMO$ to mean that for all $f_0 \in C_0^\delta$ such that $b_0 f_0$ has mean zero, the following holds:

$$\langle T(b_1,b_2), b_0 f_0\rangle = \langle \beta, b_0 f_0\rangle$$

Here the left hand side makes sense since $T(b_1,b_2)$ is defined in $(b_0 C_0^\delta)'$. The right hand side also makes sense since $b_0 f_0 \in H^1$ for $f_0 \in C_0^\delta$ where $b_0 f_0$ has mean zero. The condition $M_{b_0}T(b_1,b_2) \in BMO$ defined here is weaker than (possibly equivalent to) $T(b_1,b_2) \in BMO$ when we can make sense of $T(b_1,b_2)$ as a locally integrable function. This is because a definition of $M_{b_0}T(b_1,b_2) \in BMO$ only requires this equality to hold when paired with a subset of the predual space of *BMO*, namely we require this to hold for $\{b_0\, f : f \in C_0^\delta$ and $b_0\ f$ has mean zero$\} \subsetneq H^1$. It is possible that this is equivalent through some sort of density argument, but that is not of consequence here. So we do not pursue it any further, and use the definition of $M_{b_0}T(b_1,b_2) \in BMO$ that we have provided. Furthermore, if T is bounded from $L^{p_1} \times L^{p_2}$ into L^p for some $1 \leq p_1, p_2, p < \infty$, then T can be defined on $L^\infty \times L^\infty$ and is bounded from $L^\infty \times L^\infty$ into *BMO*. Hence, if T is bounded, then $M_{b_0}T(b_1,b_2), M_{b_1}T^{*1}(b_0,b_2), M_{b_2}T^{*2}(b_1,b_0) \in BMO$.

We say (by definition) that $M_{b_0}Tb_1 \in BMO$ if there is a $\beta \in BMO$ so that $\langle Tb_1, b_0\, f\rangle = \langle \beta, b_0\, f\rangle$ for all $f \in C_0^\delta$ where $b_0\, f$ has integral zero. This isequivalent to the notion of $Tb_1 \in BMO$. We also abuse notation here in the sense that if $M_{b_0}Tb_1 \in BMO$ as we defined it, then the appropriate identification of an element in *BMO* would be $Tb_1 = \beta$, not $M_{b_0}Tb_1 = \beta$ as the notation suggests. We have two reasons for using this notation.

The first is that we felt it necessary to mention the function b_0 in the requirement $Tb_1 \in BMO$ since, as a matter of definition, it does depend on b_0.

The second reason is a bit more involved. Note that we do not assert the following: If $M_{b_0}Tb_1 \in BMO$, then $Tb_1 \in BMO$. We don't make this conclusion because 1) we only deal with pairings of the form $\langle Tb_1, b_0\, f\rangle$ for $f \in C_0^\delta$ where $b_0\, f$ has mean zero, and 2) we have not shown that the collection $\{b_0\, f : f \in C_0^{\delta}$and $b_0\, f$has mean zero$\}$ is dense in H^1. If this collection is in fact dense in H^1, then we conclude that $M_{b_0}Tb_1 \in BMO$(as we've defined it) implies $Tb_1 \in BMO$. It may be the case that this collection is dense in H^1, but it is not of consequence to us in this work. This discussion applies to the bilinear conditions $M_{b_0}T(b_1, b_2) \in BMO$ as well with the appropriate modifications.

Define for $N > 0,\ k \in Z, and\ x \in R^n$

$$\Phi_k^N(x) = \frac{2^{kn}}{(1+2^k|x|)^N}.$$

For $f : R^n \to C$, we use the notation $f_k(x) = 2^{kn} f\,(2^k x)$. We will say indices $0 < p, p_1, p_2 < \infty$ satisfy a Hölder relationship if

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}. \qquad \textbf{(2)}$$

Definition (3.1.8)[3]: Let θ_k be a function from R^{2n} into C for each $k \in$ Z. We call $\{\theta_k\}_{k\in Z}$ a collection of Littlewood-Paley square function kernels of type $LPK(A, N, \gamma)$ for $A > 0$, $N > n$, and $0 < \gamma \leq 1$ if for all $x, y, y' \in$ R^n and $k \in$ Z

$$|\theta_k(x,y)| \leq A\Phi_k^{N+\gamma}(x-y) \qquad \textbf{(3)}$$

$$|\theta_k(x,y) - \theta_k(x,y')| \leq A(2^k|y-y'|)^{\gamma}(\Phi_k^{N+\gamma}(x-y) + \Phi_k^{N+\gamma}(x-y')). \qquad \textbf{(4)}$$

We say that $\{\theta_k\}_{k\in Z}$ is a collection of smooth Littlewood-Paley square function kernels of type $SLPK(A, N, \gamma)$ for $A > 0$, $N > n$, and $0 < \gamma \leq 1$ if it satisfies (3), (4), and for all $x, x', y \in R^n$and $k \in$ Z

$$|\theta_k(x,y) - \theta_k(x',y)| \leq A(2^k|x-x'|)^{\gamma} \prod_{i=1}^{2} \Phi_k^{N+\gamma}(x'-y) - \Phi_k^{N+\gamma}(x'-y), \qquad \textbf{(5)}$$

If $\{\theta_k\}$ is a collection of Littlewood-Paley square function kernels of type $LPK(A, N, \gamma)$ (respectively $SLPK(A, N, \gamma)$) for some $A > 0, N > n$, and $0 < \gamma \leq 1$, then write $\{\theta_k\} \in LPK$ (respectively $\{\theta_k\} \in SLPK$). We also define for $k \in Z$, $x \in R^n$, and $f \in L^1 + L^{\infty}$

$$\Theta_k\, f(x) = \int_{R^n} \theta_k(x,y) f\,(y)dy.$$

Definition (3.1.9)[3]: Let $\theta_k$be a functions from R^{3n} into C for each $k \in$ Z. We call $\{\theta_k\}_{k\in Z}$ a collection of bilinear Littlewood-Paley square function kernels of type $BLPK(A,N,\gamma)$ for $A > 0, N > n$, and $0 < \gamma \leq 1$ if for all $x, y_1, y_2, y'_1, y'_2 \in R^n$ and k $\in$Z

$$|\theta_k(x,y_1,y_m)| \leq A\Phi_k^{N+\gamma}(x-y_1)\Phi_k^{N+\gamma}(x-y_2) \tag{6}$$

$$\left|\theta_k(x,y_1,y_2) - \theta_k\left(x,y'_1,y_2\right)\right| \leq A\left(2^k\left|y_1 - y'_1\right|\right)^\gamma \Phi_k^{N+\gamma}(x-y_2)$$
$$\times\left(\Phi_k^{N+\gamma}(x-y_1) + \Phi_k^{N+\gamma}(x-y'_1)\right) \tag{7}$$

$$\left|\theta_k(x,y_1,y_2) - \theta_k\left(x,y_1,y'_2\right)\right| \leq A\left(2^k\left|y_2 - y'_2\right|\right)^\gamma \Phi_k^{N+\gamma}(x-y_1)$$
$$\times\left(\Phi_k^{N+\gamma}(x-y_2) + \Phi_k^{N+\gamma}(x-y'_2)\right). \tag{8}$$

We say that $\{\theta_k\}_{k\in Z}$ is a collection of smooth Littlewood-Paley square function kernels of type $SBLPK(A,N,\gamma)$ for $A > 0$, $N > n$, and $0 < \gamma \leq 1$ if it satisfies (3)-(5) and for all $x, x', y_1, y_2 \in R^n$ and $k \in$ Z

$$|\theta_k(x,y_1,y_2) - \theta_k(x',y_1,y_2)| \leq A(2^k|x-x'|)^\gamma \prod_{i=1}^2 (\Phi_k^{N+\gamma}(x-y_i) - \Phi_k^{N+\gamma}(x'-y_i)). \tag{9}$$

If $\{\theta_k\}$ is a collection of bilinear Littlewood-Paley square function kernels of type $BLPK(A,N,\gamma)$ (respectively of type $SBLPK(A,N,\gamma)$) for some $A > 0$, $N > n$,and $0 < \gamma \leq 1$, then we write $\{\theta_k\} \in BLPK$(respectively $\{\theta_k\} \in SBLPK$). We also define for $k \in Z$, $x \in R^n$, and $f_1, f_2 \in L^1 + L^\infty$

$$\Theta_k(f_1,f_2)(x) = \int_{R^{2n}} \theta_k(x,y_1,y_2)\, f_1(y_1)\, f_2(y_2) dy_1\, dy_2.$$

Remark (3.1.10)[3]: Let θ_k be a function from R^{3n} to C for each $k \in$ Z. There exists $A_1 > 0$, $N_1 > n$, and $0 < \gamma \leq 1$ such that $\{\theta_k\}$ is a collection of Littlewood-Paley square function kernels of type $SBLPK(A_1,N_1,\gamma)$ if and only if there exist $A_2 > 0, N_2 > n$, and $0 < \gamma \leq 1$ such that for all $x, y_1, y_2, y'_1, y'_2 \in R^n$ *and* $k \in Z$

$$|\theta_k(x,y_1,y_2)| \leq A_2\Phi_k^{N_2}(x-y_1)\Phi_k^{N_2}(x-y_2) \tag{10}$$

$$|\theta_k(x, y_1, y_2) - \theta_k(x, y'_1, y_2)| \le A_2 2^{2nk}(2^k|y_1 - y'_1|)^\gamma \tag{11}$$

$$|\theta_k(x, y_1, y_2) - \theta_k(x, y_1, y'_2)| \le A_2 2^{2nk}(2^k|y_2 - y'_2|)^\gamma \tag{12}$$

$$|\theta_k(x, y_1, y_2) - \theta_k(x', y_1, y_2)| \le A_2 2^{2nk}(2^k|x - x'|)^\gamma. \tag{13}$$

A similar equivalence holds for smooth square function kernels of type $BLPK(A, N, \gamma), LPK(A, N, \gamma)$, and $SLPK(A, N, \gamma)$ with the appropriate modifications.

Proof. Assume that $\{\theta_k\} \in SBLPK(A1, N1, \gamma)$, and define $A_2 = 2A_1$, $N_2 = N_1 + \gamma$, $and\ \gamma = \gamma$ It follows easily that (10) holds. Also

$$\left|\theta_k(x, y_1, y_2) - \theta_k\left(x, y'_1, y_2\right)\right| \le A_1(2^k|y_1 - y'_1|))^\gamma \Phi_k^{N_1+\gamma}(x - y_2)$$

$$\times \left(\Phi_k^{N_1+\gamma}(x - y_1) + \Phi_k^{N_1+\gamma}\left(x - y'_1\right)\right) \le 2A_1 2^{2nk}(2^k|y_1 - y'_1|)^\gamma.$$

A similar argument holds for regularity in the y_2 and *x* spots. Then $\theta_k$satisfies (10)-(13).

Conversely we assume that (10)-(13) hold. Define $\eta = \frac{N_{2-n}}{2(N_2+\gamma)}$, $A_1 = A_2$, $N_1 = N_2(1 - \eta) - \eta\gamma$, and $\gamma = \eta\gamma$. Estimate (6) easily follows since $N_1 + \gamma < N_2$. Estimate (7) also follows since

$$|\theta_k(x, y_1, y_2) - \theta_k(x, y'_1, y_2) \le A_2\left(2^k\left|y_1 - y'_1\right|\right)^{\eta\gamma} \Phi_k^{N_2(1-\eta)}(x - y_2)$$

$$\times \left(\Phi_k^{N_2(1-\eta)}(x - y_1) + \Phi_k^{N_2(1-\eta)}\left(x - y'_1\right)\right)$$

$$\le A_1\left(2^k\left|y_1 - y'_1\right|\right)^\gamma \Phi_k^{N_1+\gamma}(x - y_2)$$

$$\times \left(\Phi_k^{N_1+\gamma}(x - y_1) + \Phi_k^{N_1+\gamma}(x - y'_1)\right).$$

Note that this selection satisfies

$$N_1 = N_2 - \eta(N_2 + \gamma) = \frac{N_2 + n}{2} > n.$$

Then (7) holds for this choice of A_1, N_1, and γ as well. Estimates (8) and (9) follow with a similar argument, and hence $\{\theta_k\}$ is a collection of Littlewood-Paley square function kernel of type $BLPK(A_1, N_1, \gamma)$. The proofs of the other equivalences are contained in the proof of this one.

We first mention a well known almost orthogonalitye stimate for non-negative functions: If $M, N > n$, then for all $j, k \in Z$

$$\int_{R^n} \Phi_j^M(x-u)\Phi_k^N(u-y)du \lesssim \Phi_j^M(x-y) + \Phi_k^N(x-y).$$

Then next result is also a result for integrals with non-negative integrands, but this one involves regularity estimates on the functions.

Proposition (3.1.11)[3]: *If* $\{\theta_k\}_{k\in Z} \in BLPK$, *then for all* $j, k \in Z, x, y_1, y_2 \in R^n$

$$\int_{R^n} \left|\theta_j(x, y_1, y_2) - \theta_j(x, u, y_2)\right| \Phi_k^{N+\gamma}(u - y_1)du$$
$$\lesssim 2^{\gamma(j-k)}\left(\Phi_j^N(x-y_1) + \Phi_k^N(x-y_1)\right)\Phi_j^N(x-y_2),$$

$$\int_{R^n} |\theta_j(x, y_1, y_2) - \theta_j(x, y_1, u)|\Phi_k^{N+\gamma}(u-y_2)du$$
$$\lesssim 2^{\gamma(j-k)}\Phi_j^N(x-y_1)\left(\Phi_j^N(x-y_2) + \Phi_k^N(x-y2)\right),$$

and

$$\int_{R^{2n}} |\theta_j(x, y_1, y_2) - \theta_j(x, u_1, u_2)|\Phi_k^{N+\gamma}(u_1-y_1)\Phi_k^{N+\gamma}(u_2-y_2)du_1\, du_2$$
$$\lesssim 2^{\gamma(j-k)}\prod_{i=1}^{2}\Phi_j^N(x-y_i) + \Phi_k^N(x-y_i).$$

Proof. Since $\{\theta_k\}k \in Z$ is of type $BLPK(A, N, \gamma)$, it follows that

$$\int_{R^n} \left|\theta_j(x, y_1, y_2) - \theta_j(x, u, y_2)\right|\Phi_k^{N+\gamma}(u-y_1)du$$

$$\lesssim \Phi_j^N(x-y_2)\int_{R^n}\left(2^j|u-y_1|\right)^\gamma\left(\Phi_j^{N+\gamma}(x-y_1) + \Phi_j^{N+\gamma}(x-u)\right)\Phi_k^{N+\gamma}(u-y_1)du$$

$$\leq 2^{\gamma(j-k)}\Phi_j^N(x-y_2)\int_{R^n}\Phi_j^{N+\gamma}(x-y_1) + \left(\Phi_j^{N+\gamma}(x-u)\right)\Phi_k^N(u-y_1)du$$

$$\lesssim 2^{\gamma(j-k)}\left(\Phi_j^N(x-y_1) + \Phi_k^N(x-y_1)\right)\Phi_j^N(x-y_2).$$

By symmetry the second estimate holds as well. For the third estimate, we make a similar argument,

$$\int_{R^{2n}} \left|\theta_j(x,y_1,y_2)-\theta_j(x,u_1,u_2)\right|\Phi_k^{N+\gamma}(u_1-y_1)\Phi_k^{N+\gamma}(u_2-y_2)du_1\,du_2$$

$$\leq \int_{R^{2n}} |\theta_j(x,y_1,y_2)-\theta_j(x,y_1,u_2)|\Phi_k^{N+\gamma}(u_1-y_1)\Phi_k^{N+\gamma}(u_2-y_2)du_1\,du_2$$

$$+\int_{R^{2n}} |\theta_j(x,y_1,u_2)-\theta_j(x,u_1,u_2)|\Phi_k^{N+\gamma}(u_1-y_1)\Phi_k^{N+\gamma}(u_2-y_2)du_1\,du_2$$

$$\lesssim 2^{\gamma(j-k)}\int_{R^{2n}} \Phi_j^N(x-y_1)\big(\Phi_j^N(x-y_2)+\Phi_j^N(x-u_2)\big)\Phi_k^{N+\gamma}(u_1-y_1)\Phi_k^N(u_2 - y_2)du_1\,du_2$$

$$+2^{\gamma(j-k)}\int_{R^{2n}} \Phi_j^N(x-y_1)+\big(\Phi_j^N(x-u_1)\big)\Phi_j^N(x-u_2)\Phi_k^N(u_1-y_1)\Phi_k^{N+\gamma}(u_2 - y_2)du_1\,du_2$$

$$\lesssim\ 2^{\gamma(j-k)}\big(\Phi_j^N(x-y_1)+\Phi_k^N(x-y_1)\big)\big(\Phi_j^N(x-y_2)+\Phi_k^N(x-y_2)\big).$$

This completes the proof of the proposition.

It is well-known that if $N > n$ and $f \in L^1 + L^\infty$, then $\Phi_k * |f|(x) \lesssim Mf(x)$ for all k $\in$ Z, where M is the Hardy-Littlewood maximal function

$$Mf(x) = \sup_{x\ni B}\frac{1}{|B|}\int_B |f(y)|dy,$$

and here the supremum is taken over all balls B containing x. We use the kernel function almost orthogonality estimates to prove pointwise estimates for some operators.

Proposition(3.1.12)[3]: *If* $\{\lambda_k\},\{\theta_k\} \in LPK$ *and there exists a para-accretive function b such that* $\Lambda_k(b) = \theta_k(b) = 0$ *for all k* $\in$ Z*, then for all* $f \in L^1 + L^\infty$ *and* $j,k \in Z$

$$|\Theta_j M_b \Lambda_k^* f(x)| \lesssim 2^{-\gamma|j-k|}Mf(x). \tag{14}$$

If $\{\lambda_k\} \in LPK, \{\theta_k\} \in SBLPK$ *and there exists a para-accretive functions* b *such that* $\Lambda_k(b) = 0$ *and*

$$\int_{R^n} \theta_k(x, y_1, y_2) b(x) dx = 0$$

for all $k \in \mathrm{Z}$ *and* $y_1, y_2 \in R^n$*, then for all* $f_1, f_2 \in L^1 + L^\infty$ *and* $j, k \in Z$

$$\left|\Lambda_k M_b \Theta_j(f_1, f_2)(x)\right| \lesssim 2^{-\gamma|j-k|} M(M f_1 \cdot M f_2)(x) \quad \textbf{(15)}$$

If $\{\lambda_k^1\}, \{\lambda_k^2\} \in LPK, \{\theta_k\} \in BLPK$ *and there exist para-accretive functions* b_1, b_2 *and* $i \in \{1,2\}$ *such that* $\Lambda_k^1(b_1) \cdot \Lambda_k^2(b_2) = \Theta_k(b_1, b_2) = 0$ *for all* $k \in \mathrm{Z}$*, then for all* $f_1, f_2 \in L^1 + L^\infty$ *and* $j, k \in Z$

$$|\Theta_j(M_{b_1} \Lambda_k^{1*} f_1, M_{b_2} \Lambda_k^{2*} f_2)(x)| \lesssim 2^{-\gamma|j-k|} M f_1(x) M f_2(x). \quad \textbf{(16)}$$

Here we use capital Λ_k to be the operator defined by integration against the kernel lower case λ_k, just like Θ_k and θ_k.

Proof. We first prove (14). Using that $\Lambda_k^* (b) = 0$ and Proposition (3.1.11).

$$|\Theta_j M_b \Lambda_k^* f(x)| \leq \int_{R^n} \left| \int_{R^n} (\Theta_j(x, u) - \Theta_j(x, y)) b(u) \lambda_k(y, u) du \right| f(y)| dy$$

$$\lesssim \int_{R^{2n}} \left|\Theta_j(x, u) - \Theta_j(x, y)\right| \Phi_k^{N+\gamma}(y - u) |f(y)| du\, dy$$

$$\lesssim 2^{\gamma(j-k)} \left(\Phi_j^N * |f|(x) + \Phi_k^N * |f|(x)\right) \lesssim 2^{\gamma(j-k)} M f(x).$$

With a symmetric argument, the same estimate holds replacing $2^{\gamma(j-k)}$ with $2^{\gamma(k-j)}$. Therefore (14) holds. Now we prove (15). We first use that $\Lambda_k(b) = 0$ to estimate

$$|\Lambda_k M_b \Theta_j(f_1, f_2)(x)|$$

$$\leq \int_{R^{2n}} \left| \int_{R^n} \lambda_k(x, u) b(u) (\theta_j(u, y_1, y_2) - \Theta_j(x, y_1, y_2)) du \right| |f_1(y_1) f_2(y_2)| dy_1\, dy_2$$

$$\lesssim \int_{R^{3n}} \Phi_k^{N+\gamma}(x - u) \left(2^j |x - u|\right)^\gamma (\Phi_j^{N+\gamma}(u - y_1) \Phi_j^{N+\gamma}(u - y_2) + \Phi_j^{N+\gamma}(x - y_1) \Phi_j^{N+\gamma}(x - y_2)) \times |f_1(y_1) f_2(y_2)| du\, dy_1\, dy_2$$

$$\lesssim 2^{\gamma(j-k)} \int_{R^{3n}} \Phi_k^N(x-u)(\Phi_j^{N+\gamma}(u-y_1)\Phi_j^{N+\gamma}(u-y_2) + \Phi_j^{N+\gamma}(x-y_1)\Phi_j^{N+\gamma}(x - y_2)) \times | f_1(y_1) f_2(y_2)| du\, dy_1\, dy_2$$

$$= 2^{\gamma(j-k)} \int_{R^n} \Phi_k^N(x-u) \prod_{i=1}^{2} \left(\Phi_j^N * |f_i|(u) + \Phi_j^N * |f_i|(x)\right) du$$

$$\lesssim 2^{\gamma(j-k)} M(M f_1 \cdot M f_2)(x).$$

We also have

$$\left|\Lambda_k M_b \Theta_j(f_1, f_2)(x)\right|$$

$$\leq \int_{R^{2n}} \left| \int_{R^n} (\lambda_k(x,u) - \lambda_k(x,y_1)) b(u) \theta_j(u, y_1, y_2) du \right| | f_1(y_1) f_2(y_2)| dy_1\, dy_2$$

$$\lesssim 2^{(k-j)\gamma} \int_{R^{3n}} (\Phi_k^N(x-u) + \Phi_k^N(x-y_1)) \prod_{i=1}^{2} \Phi_j^N(u-y_i)| f_i(y_i)| dy_i\, du$$

$$\leq 2^{(k-j)\gamma} \int_{R^n} \Phi_k^N(x-u) \prod_{i=1}^{2} \Phi_j^N * | f_i|(u) du$$

$$+2^{(k-j)\gamma} \int_{|x-y_1| \geq |x-u|/2} \Phi_k^N(x-y_1) \prod_{i=1}^{2} \Phi_j^N(u-y_i)| f_i(y_i)| dy_i\, du$$

$$+2^{(k-j)\gamma} \int_{|x-y_1| < \frac{|x-u|}{2}} \Phi_k^N(x-y_1) \prod_{i=1}^{2} \Phi_j^N(u-y_i)|f_i(y_i)| dy_i\, du$$

$$= 2^{(k-j)\gamma}(I + II + III).$$

Note that $2^{(k-j)\gamma} I \lesssim M(M f_1 \cdot M f_2)(x)$, which is on the right hand side of (15). In II, replace $\Phi_k^N(x-y_1)$ with $\Phi_k^N((x-u)/2)$ and it follows that $II \lesssim I$. So II is bounded appropriately as well. The final term, III is bounded by

$$\int_{|x-y_1| < \frac{|x-u|}{2}} \Phi_k^N(x-y_1) \frac{2^{jn} | f_1(y_1)}{|(1 + 2^j(|x-u| - |x-y_1|))^N} \Phi_j^N * |f_2|(u) dy_1\, du$$

$$\lesssim \int_{|x-y_1|<|x-u|/2} \Phi_k^N(x-y_1)\Phi_j^N(x-u)|\, f_1(y_1)|\Phi_j^N * |\, f_2|(u)dy_1\, du$$

$$\lesssim \left(\int_{R^n} \Phi_k^N(x-y_1)|f_1(y_1)|dy_1\right)\left(\int_{R^n} \Phi_j^N(x-u)\Phi_j^N * |f_2|(u)du\right)$$

$$\lesssim \Phi_k^N * |f_1|(x)\Phi_j^N * |f_2|(x)$$

$$\leq M(M\, f_1 \cdot M\, f_2)(x).$$

This verifies that (15) holds. For estimate (16) when $j \leq k$, we use that $\Lambda_k^1(b_1) \cdot \Lambda_k^2(b_2) = 0$ and Proposition (3.1.11)

$$|\Theta_j(M_{b_1}\Lambda_k^1 * f_1, M_{b_2}\Lambda_k^2 * f_2)(x)|$$

$$\leq \int_{R^{4n}} |\theta_j(x,u_1,u_2) - \theta_j(x,y_1,y_2)| \prod_{i=1}^{2} |b_i(u)\lambda_k^i(y_i,u_i)\, f_i(y_i)|dy_i\, du_i$$

$$\lesssim 2^{\gamma(j-k)} \int_{R^{4n}} \prod_{i=1}^{2} \left(\Phi_j^N(x-u_i) + \Phi_j^N(x-y_i)\right) \Phi_k^N(u_i-y_i)|\, f_i(y_i)|du_i\, dy_i$$

$$\lesssim 2^{\gamma(j-k)} \prod_{i=1}^{2} \int_{R^n} \left(\Phi_j^N(x-y_i) + \Phi_k^N(x-y_i)\right)|\, f_i(y_i)|dy_i$$

$$\lesssim 2^{\gamma(j-k)} M\, f_1(x) M\, f_2(x).$$

Using that $\Theta_j(b_1,b_2) = 0$, it follows that

$$\left|\Theta_j\left(M_{b_1}\Lambda_k^1 * f_1, M_{b_2}\Lambda_k^2 * f_2\right)(x)\right|$$

$$\leq \int_{R^{4n}} |\theta_j(x,u_1,u_2)| \left|\prod_{i=1}^{2} \lambda_k^i(y_i,u_i) - \prod_{i=1}^{m} \lambda_k^i(y_i,x)\right| \prod_{i=1}^{2} |b_i(u)\, f_i(y_i)|dy_i\, du_i$$

$$\lesssim \int_{R^{2n}} \left(\int_{R^{2n}} \prod_{i=1}^{2} \lambda_k^i(y_i,u_i) - \prod_{i=1}^{2} \lambda_k^i(y_i,x) \prod_{i=1}^{2} \Phi_j^{N+\gamma}(u_i-y_i)du_i\right) \prod_{i=1}^{m} |\, f_i(y_i)|dy_i$$

$$\lesssim 2^{\gamma(k-j)}\left(\Phi_j^N * |\, f_1|(x) + \Phi_k^N * |\, f_1|(x)\right)\left(\Phi_j^N * |\, f_2|(x) + \Phi_k^N * |\, f_2|(x)\right)$$

$$\lesssim 2^{\gamma(k-j)} M\, f_1(x) M\, f_2(x).$$

We see that $\lambda_k^1(x,y_1)\lambda_k^2(x,y_2)$ form a collection of kernels of type *BLPK*. Then (16) holds as well.

Proposition (3.1.13)[3]: *Suppose* $p_k : R^{2n} \to C$ *for* $k \in \mathrm{Z}$ *satisfy* $|p_k(x,y)| \lesssim \Phi_k^N(x-y)$*and* $N > n$, *and define* $\mathrm{P_k}$

$$\mathrm{P_k}\ f(x) = \int_{\mathrm{R^n}} \mathrm{P_k}(x,y)\, f(y) dy$$

For $f \in L^1 + L^\infty$. *If*

$$\int_{\mathrm{R^n}} \mathrm{P_k}(x,y) dy = 1$$

for all $k \in \mathrm{Z}$ *and* $x \in R^n$, *then* $\mathrm{P_k} f \to f$ *in* L^p *as* $k \to \infty$ *for all* $f \in L^p$ *when* $1 \le p < \infty$ *and* $P_k f \to 0$ *in* L^p *as* $k \to -\infty$ *for all* $f \in L^p \cap L^q$ *for* $1 \le q < p < \infty$.

Proof. For $f \in L^p$ with $1 \le p < \infty$

$$||P_k f - f||_{L^p} = \left(\int_{R^n} \left|\int_{R^n} P_k(x,y)\, f(y) dy - \int_{R^n} P_k(x,y)\, f(x) dy\right|^p dx\right)^{\frac{1}{p}}$$

$$= \left(\int_{R^n} \left|\int_{R^n} P_k(x, x-2^{-k}y)(f(x-2^{-k}y) - f(x)) 2^{-kn} dy\right|^p dx\right)^{\frac{1}{p}}$$

$$\lesssim \int_{R^n} \left(\int_{R^n} \Phi_0^N(y) |f(x-2^{-k}y) - f(x)|^p dx\right)^{\frac{1}{p}} dy$$

$$\lesssim \int_{R^n} \Phi_0^N(y) ||f(\cdot - 2^{-k}y) - f||_{L^p} dy.$$

Note that $\Phi_0^N(y)||f(\cdot - 2^{-k}y) - f||_{L^p} \le 2||f||_{Lp}\, \Phi_0^N(y)$ which is an$L^1(R^n)$ function independent of *k*. So by Lebesgue dominated convergence and the continuity of translation in $||\,.\,||_{L^p}$

$$\lim_{k\to\infty} ||P_k f - f||_{Lp} \le \int_{R^n} \Phi_0^N(y) \lim_{k\to\infty} ||f(\cdot - 2^{-k}y) - f||_{L^p} dy = 0.$$

We compute

$$|P_k\, f\,(x)| \lesssim ||\Phi_k^N||_{L^{q'}}||f||_{L^q} \;=\; 2^{kn/q}||\Phi_0^N||_{L^{q'}}||f||_{L^q}.$$

So $P_k\, f \to 0$ almost everywhere as $k \to -\infty$. We also have

$$|P_k\, f\,(x)| \;\lesssim\; \Phi_k^N * \,|\, f\,|(x) \;\lesssim M\, f\,(x),$$

and since $f \;\in\; L^p(R^n)$, it follows that $M\, f \;\in\; L^p(R^n)$ as well when $1 < p < \infty$. So by dominated convergence

$$\lim_{k\to-\infty} ||P_k\, f\,||_{L^p}^p \;=\; \int_{R^n} \lim_{k\to-\infty} |P_k\, f\,(x)|^p dx \;=\; 0.$$

This proves the proposition.

Corollary (3.1.14)[3]: *Let b be a para-accretive function. Suppose* $s_k\colon R^{2n} \to C$ *for* $k \in$ Z *satisfy* $|s_k(x,y)| \lesssim \Phi_k^N(x-y)$ *for some N > n, and define* s_k

$$s_k\, f\,(x) \;=\; \int_{R^n} s_k(x,y)\, f\,(y)dy$$

for $f \in L^1 + L^\infty$*. If*

$$\int_{R^n} s_k(x,y)b\,(y)dy \;=\; 1$$

for all $k \in$ Z *and* $x \in R^n$*, then* $S_kM_b\, f \;\to\; f$ *and* $M_bS_k\, f \;\to\; f$ *in* L^p *as* $k\to\infty$ *for all* $f \in L^p(R^n)$*when* $1 \le p < \infty$ *. Also* $S_kM_b\, f \;\to\; 0$ *and* $M_bS_k\, f \;\to 0$ *in* L^p*as* $k \;\to -\infty$ *for all* $f \;\in\; L^p \cap L^q$ *for* $1 \le q < p < \infty$ *.*

Proof. Define $P_k\, f \;=\; S_kM_b\, f$ with kernel p_k. It is obvious that $|P_k(x,y)| \;\lesssim\; \Phi_k^N(x - y)$,and $P_k(1) \;=\; S_k(b) \;=\; 1$. So by Proposition (3.1.13), since $f \;\in\; L^p$ it follows that $S_kM_b\, f \;=\; P_kf \;\to\; f$ in L^p when $f \;\in\; L^p$ and $1 \;\le\; p \;<\; \infty$. Also when $f \;\in\; L^p \cap L^q$ for $1 \;\le\; q \;< p \;< \infty$, it follows that $S_kM_b\, f \;=\; P_k\, f \;\to 0$ as $k \to -\infty$. Also $M_bS_k\, f \;= M_b\, P_k\,(b^{-1}\, f\,)$, so the same convergence properties hold for M_bS_k

These approximation to identities perturbed by para-accretive functions are important to this work.

Definition (3.1.15)[3]: Let $b \;\in\; L^\infty$ be a para-accretive function. A collection of operators $\{S_k\}_{k\in z}$ defined by

$$S_k f(x) = \int_{R^n} s_k(x,y) f(y)dy$$

for kernel functions $s_k : R^{2n} \to C$ is an approximation to identity with respect to b if $\{s_k\} \in SLPK$, and

$$|s_k(x,y) - s_k(x',y) - s_k(x,y') + s_k(x',y')| \le A2^{kn}(2^k|x-x'|)^\gamma(2^k|y-y'|)^\gamma$$
$$\times \left(\Phi_k^{N+\gamma}(x-y) + \Phi_k^{N+\gamma}(x'-y) + \Phi_k^{N+\gamma}(x-y') + \Phi_k^{N+\gamma}(x'-y')\right)$$

$$\int_{R^n} s_k(x,y)b(y)dy = \int_{R^n} s_k(x,y)b(x)dx = 1.$$

We say that an approximation to identity with respect to b has compactly supported kernel if $s_k(x,y) = 0$ whenever $|x-y| > 2^{-k}$.

Proposition (3.1.16)[3]: *Let b be a para-accretive function, $\{S_k\}$ be the approximation to identity with respect to b that has compactly supported kernel, and $\delta_0 > 0$. Then $M_b S_N M_b f \to bf$ and $M_b S_{-N} M_b f \to 0$ in bC_0^δ as $N \to \infty$ for all $f \in C_0^{\delta_0}$ and $0 < \delta < min(\delta_0, \gamma)$, where γ is the smoothness parameter for $\{s_k\} \in SLPK$.*

Proof. Let $f \in C_0^{\delta_0}$ and $0 < \delta < \delta_0$. Without loss of generality assume that $\gamma = \delta$, where γ is the smoothness parameter of s_k. We must check that $||S_N M_b f - f||_\delta \to 0$ as $N \to \infty$. So we start by estimating

$$|(S_N M_b f(x) - f(x)) - (S_N M_b f(y) - f(y))|$$
$$= \left|\int_{R^n} (s_N(x,u)(f(u) - f(x))b(u)du - \int_{R^n} (s_N(y,u)(f(u) - f(y))b(u)du\right|$$
$$\le ||b||_{L^\infty} \int_{R^n} |F_N^x(u) - F_N^y(u)|du$$

Where $F_N^x(u) = s_N(x,u)(f(u) - f(x))$. Consider $u \in B(y, 2^{-N})$, and it follows that

$$|F_N^x(u) - F_N^y(u)| = |s_N(x,u)(f(u) - f(x)) - s_N(y,u)(f(u) - f(y))|$$
$$\le |s_N(x,u)|\,|f(y) - f(x)| + |s_N(x,u) - s_N(y,u)|\,|(f(u) - f(y))| \quad \textbf{(17)}$$

$\lesssim \; || f ||_{\delta_0} 2^{nN} |x-y|^{\delta_0} + || f ||_{\delta_0} 2^{nN} (2^N |x-y|)^{\delta_0} |y-u|^{\delta_0}$

$\lesssim || f ||_{\delta_0} 2^{nN} |x-y|^{\delta_0}$

With a similar argument, it follows that for $u \in B(x, 2^{-N}), |F_N^x(u) - F_N^y(u)| \lesssim ||f||_{\delta_0} 2^{nN} |x-y|_{\delta_0}$. Now we may also estimate $|F_N^x(u)|$ in the following way for $u \in B(x, 2^{-N})$,

$$|F_N^x(u)| \lesssim 2^{nN} | f(u) - f(x)| \leq || f ||_{\delta_0} 2^{nN} |u-x|^{\delta_0} \leq || f ||_{\delta_0} 2^{nN} 2^{-\delta_0 N} \qquad \textbf{(18)}$$

Using the support properties of s_k, we have that $supp(F_N^x - F_N^y) \subset B(x, 2^{-N}) \cup B(y, 2^{-N})$.Then it follows from (17), (18), and $\frac{\delta}{\delta_0} \in (0,1)$ that

$$\begin{aligned} |F_N^x(u) - F_N^y(u)| &\lesssim \left(|| f ||_{\delta_0} 2^{nN} |x-y|^{\delta_0}\right)^{\frac{\delta}{\delta_0}} \left(|| f ||_{\delta_0} 2^{nN} 2^{-\delta_0 N}\right)^{1-\frac{\delta}{\delta_0}} \\ &\lesssim || f ||_{\delta_0} 2^{nN} |x-y|^{\delta} 2^{-(\delta_0 - \delta)N}. \end{aligned}$$

Therefore $S_N M_b f \to f \; in \; || \cdot ||_{\delta}$ since

$$\begin{aligned} \frac{|(S_N M_b f(x) - f(x)) - (S_N M_b f(y) - f(y))|}{|x-y|^{\delta}} &\leq \frac{1}{|x-y|^{\delta}} \int_{R^n} |F_N^x(u) - F_N^y(u)| du \\ &\lesssim || f ||_{\delta_0} 2^{-(\delta_0-\delta)N} \int_{B(x,2^{-N}) \cup B(y,2^{-N})} 2^{nN} du \lesssim || f ||_{\delta_0} 2^{-(\delta_0-\delta)N}. \end{aligned}$$

This proves that $S_N M_b f \to f$ in C_0^{δ} as $N \to \infty$. Now we conider $S_{-N} M_b f$ as $\to \infty$. We also have

$$\begin{aligned} \frac{|S_{-N} M_b f(x) - S_{-N} M_b f(y)|}{|x-y|^{\delta}} &\leq \frac{1}{|x-y|^{\delta}} \int_{R^n} |S_{-N}(x,u) - S_{-N}(y,u)| \, |b(u) f(u)| du \\ &\lesssim \frac{|| f ||_{L^\infty}}{|x-y|^{\delta}} \left(\int_{|x-u|<2^N} + \int_{|y-u|<2^N} \right) 2^{-nN} (2^{-N} |x-y|)^{\delta} du \\ &\lesssim || f ||_{L^\infty} 2^{-\delta N}. \end{aligned}$$

Note that $||f||_{L^\infty} < \infty$ since f is continuous and compactly supported. Therefore $S_N M_b f \to f$ and $S_{-N} M_b f \to 0$ as $N \to \infty$ in the topology of C_0^{δ}.

We state a Calderón type reproducing formula for the para-acretive setting.

Theorem (3.1.17)[3]: *Let $b \in L^{\infty}$ be a para-accretive function and S_k^b for $k \in$ Z be approximation to the identity operators with respect to b. Define $D_k^b = S_{k+1}^b - S_k^b$. There exist operators $\widetilde{D}_k^b$ such that*

$$\sum_{k \in Z} \widetilde{D}_k^b M_b D_k^b M_b f = b f \tag{19}$$

in L^p for all $1 < p < \infty$ and $f \in C_0^{\delta}$ such that b f has mean zero. Furthermore, $\widetilde{D}_k^b(b) = \widetilde{D}_k^{b*}(b) = 0$ and $\widetilde{D}_k^b$ is defined by

$$\widetilde{D}_k^b f(x) = \int_{R^n} \tilde{d}_k^b(x,y) f(y) dy$$

Where $\{\tilde{d}_k^{b*}\} \in LPK$, where $\tilde{d}_k^{b*}(x,y) = \check{d}_k^b(y,x)$ are the kernels associated with $\widetilde{D}_k^{b*}$

We will use this formula extensively, and in fact, we need this formula in H^1 as well to construct the accretive type para-product. We will prove that this reproducing formula holds in H^1 in Theorem (3.1.2) and its Corollary (3.1.19). First we prove a lemma.

Lemma (3.1.18)[3]: *If $f : R^n \to C$ has mean zero and*

$$|f(x)| \lesssim \Phi_j^N(x) + \Phi_k^N(x)$$

for some $N > n$ and $j,k \in$ Z, then $f \in H^1$ and $||f||_{H^1} \lesssim 1 + |j - k|$, where the suppressed constant is independent of j and k.

This is an extension of a result, which is Lemma (3.1.18) when $j = k$. We obtained a quadratic bound, $|j - k|^2$, for Lemma (3.1.18) using an argument involving atomic decompositions in H^1. Such a result suffices for a purposes, but thanks to suggestions from Atanas Stefanov we are able to obtain the linear bound stated here. We present Stefanov's proof, which appears more natural.

Proof. The conclusion of Lemma (3.1.18) is well known for $j = k$. So without loss of generality we take $j \neq k$, and furthermore we suppose that $j < k$. It is easy to see that

$$||f||_{L^1} \lesssim ||\Phi_j^N||_{L^1} + ||\Phi_k^N||_{L^1} \lesssim 1,$$

so we may reduce the problem to proving that $||R_\ell f||_{L^1} \lesssim k - j$ for $\ell = 1, \ldots, n$. The strategy here is to split the norm $||R_\ell f||_{L^1}$ into two sets, where $|x| \leq 2^{-j}$ and where $|x| > 2^{-j}$. We will control the first by $k - j$and the second by 1. Define$p = 1 + \frac{1}{k-j} > 1$, and use that $||R_\ell||_{L^p \to L^p} \lesssim p'$ to estimate

$$||\chi_{|x|\le 2^{-j}} R_\ell f\,||_{L^1} \le ||\chi_{|x|\le 2^{-j}}||_{L^{p'}} ||R_\ell f\,||_{L^p} \lesssim 2^{-n\,j/p'} p' ||\,f\,||_{L^{p_1}}$$
$$\lesssim (k - j) 2^{-nj\,/p'} \left(2^{n\,j/p'} + 2^{nk/p'}\right) \lesssim k - j. \qquad \textbf{(20)}$$

Note that here we use that $p' = k - j + 1$ and hence $2^{n(k-j)/p'} \le 2^n$. Now it remains to control

$$||\chi_{|x|>2^{-j}} R_\ell f\,||_{L^1} \le \sum_{m=-j}^{\infty} ||\chi_{2^m<|x|\le 2^{m+1}} R_\ell\, f\,||_{L^1}$$
$$\le \sum_{m=-j}^{\infty} ||\chi_{2^m<|x|\le 2^{m+1}} R_\ell(\,f\,\chi_{|y|\le 2^{m-1}}\,)||_{L^1}$$
$$+ \sum_{m=-j}^{\infty} ||\chi_{2^m<|x|\le 2^{m+1}} R_\ell(\,f\,\chi_{|y|>2^{m-1}}\,)||_{L^1} \quad = I + II. \qquad \textbf{(21)}$$

In order to estimate I from (21), we bound the terms of the sum by first breaking them into two pieces using the mean zero hypothesis on f:

$$||\chi_{2^m<|x|\le 2^{m+1}} R_\ell(\,f\,\chi_{|y|\le 2^{m-1}})||_{L^1}$$
$$= \int_{2^m<|x|\le 2^{m+1}} \left| R_\ell(\,f\,\chi_{|y|\le 2^{m-1}})(x) - \int_R \frac{x_\ell}{|x|^{n+1}}\, f(y)dy \right| dx$$
$$\le \int_{|x|>2^m} \int_{|y|\le 2^{m-1}} \left| \frac{x_\ell - y_\ell}{|x-y|^{n+1}} - \frac{x_\ell}{|x|^{n+1}} \right| |\,f\,(y)| dydx$$
$$+ \int_{2^m<|x|\le 2^{m+1}} \int_{|y|>2^{m-1}} \frac{|\,f\,(y)|}{|x|^n} dydx \quad = I_a + I_b. \qquad \textbf{(22)}$$

Let $\delta = min(1, (N - n)/2)$ and $N' = N - \delta > n$. Then the first term of (22) is bounded by

$$I_a \leq \int_{|x|>2^m} \int_{|y|\leq 2^{m-1}} \frac{|y|}{|x|^{n+1}} |f(y)| dy dx \leq \int_{|x|>2^m} \int_{|y|\leq 2^{m-1}} \frac{|y|^\delta}{|x|^{n+\delta}} |f(y)| dy dx$$

$$\lesssim 2^{-m\delta} \int_R |y|^\delta \left(\Phi_j^N(y) + \Phi_k^N(y)\right) dy$$

$$\leq 2^{-m\delta} \int_R \left(2^{-j\delta} \Phi_j^{N'}(y) + 2^{-k\delta} \Phi_k^{N'}(y)\right) dy \lesssim 2^{-(j+m)\delta}.$$

Note that we absorb the $2^{-k\delta}$ term into the $2^{-j\delta}$ term since $k > j$. The second term of (22) is bounded by

$$I_b \leq \int_{2^m<|x|\leq 2^{m+1}} \int_{|y|>2^{m-1}} \frac{1}{|x|^n} |f(y)| dy dx$$

$$\leq 2^{-mn} \int_{2^m<|x|\leq 2^{m+1}} \int_{|y|>2^{m-1}} |f(y)| dy dx$$

$$\leq \int_{|y|>2^{m-1}} \left(\frac{2^{-j(N-n)}}{|y|^N} + \frac{2^{-k(N-n)}}{|y|^N} \right) dy \lesssim 2^{-(j+m)(N-n)} + 2^{-(k+m)(N-n)}$$

$$\lesssim 2^{-(j+m)(N-n)}.$$

Again we use that $2^{-k(N-n)} \leq 2^{-j(N-n)}$ since $k > j$ and $N > n$. Now in order to estimate II from (21), we bound the terms of the sum using an L^2 bound for R^ℓ

$$||\chi_{2^m<|x|\leq 2^{m+1}} R_\ell(f \chi_{|y|>2^{m-1}})||_{L^1} \leq ||\chi_{2^m<|x|\leq 2^{m+1}}||_{L^2} ||R_\ell(f \chi_{|y|>2^{m-1}})||_{L^2}$$

$$\lesssim 2^{mn/2} \left(\int_{|y|>2^{m-1}} \left(\Phi_j^N(y) + \Phi_k^N(y)\right)^2 dy \right)^{\frac{1}{2}}$$

$$\leq 2^{mn/2} \left(\int_{|y|>2^{m-1}} \left[\frac{2^{2j(n-N)}}{|y|^{2N}} + \frac{2^{2k(n-N)}}{|y|^{2N}} \right] dy \right)^{\frac{1}{2}}$$

$$\lesssim 2^{mn/2} \left(2^{-j(N-n)} + 2^{-k(N-n)}\right) \left(\int_{|y|>2^{m-1}} \frac{1}{|y|^{2N}} dy \right)^{\frac{1}{2}} \lesssim 2^{-(j+m)(N-n)}.$$

Using these estimates, it follows that (21) is bounded in the following way:

$$I + II \lesssim \sum_{m=-j}^{\infty} 2^{-(j+m)\delta} + \sum_{m=-j}^{\infty} 2^{-(j+m)(N-n)} \lesssim 1.$$

Therefore using (20) and (21), it follows that $||R_\ell f||_{L^1} \lesssim k - j$ for $\ell = 1, \dots, n$ and hence $||\, f\, ||_{H^1} \lesssim k - j$.

Corollary (3.1.19)[3]: *Le $b \in L^\infty$ be a para-accretive function, S_k^b, D_k^b, and $\widetilde{D}_k^b$ be approximation to identity and reproducing formula operator with respect to b as in Theorem (3.1.17). Then for all $\delta > 0$ and $\phi \in C_0^\delta$ such that $b\phi$ has mean zero,*

$$\sum_{k \in Z} M_b \widetilde{D}_k M_b D_k M_b \phi = \sum_{k \in Z} M_b D_k M_b \phi = b\phi$$

in H^1.

Proof. By Theorem(3.1.17), it follows that the kernels of $\widetilde{D}_k M_b D_k$ and D_k are Littlewood-Paley square function kernels of type LPK, that

$$\widetilde{D}_k M_b D_k(b) = (\widetilde{D}_k M_b D_k)^*(b) = D_k(b) = D_k^*(b) = 0,$$

and finally that

$$\sum_{k \in Z} M_b \widetilde{D}_k M_b D_k M_b \phi = \sum_{k \in Z} M_b D_k M_b f = b\phi$$

in L^p for all $1 < p < \infty$ when $\phi \in C_0^\delta$ when $b\phi$ has mean zero. Therefore it follows fromTheorem (3.1.2) that the formula holds in H^1 as well.

Section (3.2): A Square Function-Like Estimate and Singular Integral Operators with Application to Bilinear Reisz Transforms

Theorem (3.2.1)[3]: *If $\{\theta_k\} \in SLPK$ and there exist para-accretive functions b_0, b_1, b_2 such that*

$$\int_{R^n} \theta_k(x, y_1, y_2) b_0(x) dx = \int_{R^{2n}} \theta_k(x, y_1, y_2) b_1(y_1) b_2(y_2) dy_1\, dy_2 = 0$$

for all $x, y_1, y_2 \in R^n$ and $k \in$ Z, then for all $1 < p$, p_1, $p_2 < \infty$ satisfying (2), *$f_i \in L^{p_i}$ for $i = 0,1,2$ where $p_0 = p'$*

$$\sum_{k \in Z} \left| \int_{R^n} \Theta_k(f_1, f_2)(x)\, f_0(x) dx \right| \lesssim \prod_{i=0}^{2} ||\, f_i||_{L^{p_i}}$$

Proof. Since $b_i, b_i^{-1} \in L\infty$, it is sufficient to prove this estimate for $b_i f_i$ in place of $f_i \, for \, i = 0,1,2$. Fix $1 < p,\ p_1,\ p_2 < \infty$ satisfying (2), $f_i \in C_0^\delta$ for $i = 0,1,2$ and some δ where $b_i f_i$ has mean zero for $i = 0,1,2$. Define.

$$\prod_j^1 (f_1, f_2)(x) = M_{b_1} D_k^{b_1} M_{b_1} f_1(x) M_{b_2} S_{k+1}^{b_2} M_{b_2} f_2(x)$$

$$\prod_j^2 (f_1, f_2)(x) = M_{b_1} D_k^{b_1} M_{b_1} f_1(x) M_{b_2} S_k^{b_2} M_{b_2} f_2(x),$$

where $S_k^{b_i}$ and $D_k^{b_i}$ defined as in Theorem (3.1.17). Then it follows that

$$\Theta_k(b_1 f_1, b_2 f_2)$$

$$= \lim_{N\to\infty} \Theta_k(M_{b_1} S_N^{b_1} M_{b_1} f_1, M_{b_2} S_N^{b_2} M_{b_2} f_2) - \Theta_k(M_{b_1} S_{-N}^{b_1} M_{b_1} f_1, M_{b_2} S_{-N}^{b_2} M_{b_2} f_2)$$

$$= \lim_{N\to\infty} \sum_{j=-N}^{N-1} \Theta_k\left(M_{b_1} S_{k+1}^{b_1} M_{b_1} f_1, M_{b_2} S_{k+1}^{b_2} M_{b_2} f_2\right) - \Theta_k\left(M_{b_1} S_k^{b_1} M_{b_1} f_1, M_{b_2} S_k^{b_2} M_{b_2} f_2\right)$$

$$= \lim_{N\to\infty} \sum_{j=-N}^{N-1} \Theta_k \prod_j^1 (f_1, f_2) + \Theta_k \prod_j^2 (f_1, f_2)$$

where the convergence holds in L^p. Then we approximate the above dual pairing in the following way

$$\left|\sum_{k\in Z} \int_{R^n} \Theta_k(b_1 f_1, b_2 f_2)(x) b_0(x) f_0(x) dx\right| \le \sum_{j,k\in Z} \left|\int_{R^n} \Theta_k \prod_j^1 (f_1, f_2)(x) b_0(x) f_0(x) dx\right|$$

$$+ \sum_{j,k\in Z} \left|\int_{R^n} \Theta_k \prod_j^2 (f_1, f_2)(x) b_0(x)\, f_0(x) dx\right|.$$

These two terms are symmetric, so we only bound the first one. The bound for the other term follows with a similar argument. By the convergence in Theorem (3.1.17), we have that

$$\sum_{j,k\in Z}\left|\int_{R^n}\Theta_k\prod_j^1(f_1,f_2)(x)b_0(x)\,f_0(x)dx\right|$$

$$\leq \sum_{j,k,\ell\in Z}\left|\int_{R^n}\Theta_k\prod_j^1(\widetilde{D}_\ell^{b_1}Mb_1D_\ell^{b_1}\,M_{b_1}f_1,f_2)(x)b_0(x)\,f_0(x)dx\right|$$

$$\leq \sum_{j,k,\ell,m\in Z}\left|\int_{R^n}\widetilde{D}_m^{b_0}M_{b_0}D_m^{b0}M_{b_0}\Theta_k\prod_j^1(\widetilde{D}_\ell^{b_1}M_{b_1}D_\ell^{b_1}M_{b_1}f_1,f_2)(x)b_0(x)\,f_0(x)dx\right|$$

$$\leq \sum_{j,k,\ell,m\in Z}\int_{R^n}|D_m^{b_0}M_{b_0}\Theta_k\prod_j^1(\widetilde{D}_\ell^{b_1}M_{b_1}D_\ell^{b_1}M_{b_1}f_1,f_2)(x)M_{b_0}\widetilde{D}_m^{b_0*}\,M_{b_0}f_0(x)|dx.$$

By Proposition (3.1.12) we also have the following three estimates

$$\left|D_m^{b_0}M_{b_0}\Theta_k\prod_j^i\big(\widetilde{D}_\ell^{b_1}M_{b_1}D_\ell^{b_1}M_{b_1}f_1,f_2\big)(x)\right|$$

$$\lesssim\ 2^{-\gamma|m-k|}M\left(\prod_j^1\big(\widetilde{D}_\ell^{b_1}M_{b_1}D_\ell^{b_1}M_{b_1}f_1,f_2\big)\right)(x)$$

$$\lesssim\ 2^{-\gamma|m-k|}M^2\big(M(D_\ell^{b_1}M_{b_1}f_1)\ \cdot M\ f_2\big)(x).$$

$$|D_m^{b_0}M_{b_0}\Theta_k\prod_j^1\big(\widetilde{D}_\ell^{b_1}M_{b_1}D_\ell^{b_1}M_{b_1}f_1,f_2\big)(x)|\ \lesssim M(\Theta_k\prod_j^1(\widetilde{D}_\ell^{b_1}M_{b_1}D_\ell^{b_1}M_{b_1}f_1,f_2))(x)$$

$$\lesssim\ 2^{-\gamma|k-j|}M^2(M(D_\ell^{b_1}M_{b_1}f_1)\ \cdot M\ f_2))(x)$$

$$|D_m^{b_0}M_{b_0}\Theta_k\prod_j^i(\widetilde{D}_\ell^{b_1}M_{b_1}D_\ell^{b_1}M_{b_1}f_1,f_2)(x)|\ \lesssim M^2(\prod_j^1(\widetilde{D}_\ell^{b_1}M_{b_1}D_\ell^{b_1}M_{b_1}f_1,f_2))(x)$$

$$\lesssim 2^{-\gamma|\,j-\ell|}M^2(M(D_\ell^{b_1}M_{b_1}f_1)\ \cdot Mf_2)(x)$$

Taking the geometric mean of these three estimates, we have the following pointwise bound

$$|D_m^{b_0}M_{b_0}\Theta_k\prod_j^i(M_{b_1}\widetilde{D}_\ell^{b_1}M_{b_1}D_\ell^{b_1}M_{b_1}f_1,f_2)(x)\,|$$

$$\lesssim 2^{-\gamma\left(\frac{|m-k|}{3}+\frac{|k-j|}{3}+\frac{|j-\ell|}{3}\right)}M^2\big(M(D_\ell^{b_1}M_{b_1}f_1)\ \cdot M\ f_2\big)(x).$$

Therefore

$$\sum_{j,k,\ell,m\in Z}\int_{R^n}|D_m^{b_0}M_{b_0}\Theta_k\prod_j^1(\widetilde{D}_\ell^{b_1}M_{b_1}D_\ell^{b_1}M_{b_1}f_1,f_2)(x)\,\widetilde{D}_m^{b_0*}M_{b_0}f_0(x)|dx$$

$$\lesssim\int_{R^n}\sum_{j,k,\ell,m\in Z}2^{-\gamma\left(\frac{|m-k|}{3}+\frac{|k-j|}{3}+\frac{|j-\ell|}{3}\right)}M^2\left(M(D_\ell^{b_1}M_{b_1}f_1)\cdot M\,f_2\right)(x)|\widetilde{D}_m^{b_0*}M_{b_0}f_0(x)|dx$$

$$\le\left\|\left(\sum_{j,k,\ell,m\in Z}2^{-\gamma\left(\frac{|m-k|}{3}+\frac{|k-j|}{3}+\frac{|j-\ell|}{3}\right)}M^2\left(M(D_\ell^{b_1}M_{b_1}f_1)\cdot M\,f_2\right)^2\right)^{\frac{1}{2}}\right\|_{L^p}$$

$$\times\left\|\left(\sum_{j,k,\ell,m\in Z}2^{-\gamma\left(\frac{|m-k|}{3}+\frac{|k-j|}{3}+\frac{|j-\ell|}{3}\right)}|\widetilde{D}_m^{b_0*}M_{b_0}f_0|^2\right)^{\frac{1}{2}}\right\|_{L^{p'}}$$

$$\lesssim\left\|\left(\sum_{\ell\in Z}M^2\left(M(D_\ell^{b_1}M_{b_1}f_1)\cdot M\,f_2\right)^2\right)^{\frac{1}{2}}\right\|_{L^p}\left\|\left(\sum_{m\in Z}|\widetilde{D}_m^{b_0*}M_{b_0}f_0|^2\right)^{\frac{1}{2}}\right\|_{L^{p'}}$$

$$\lesssim\left\|\left(\sum_{\ell\in Z}(M(D_\ell^{b_1}M_{b_1}f_1)M\,f_2)^2\right)^{\frac{1}{2}}\right\|_{L^p}||\,f_0||_{L^{p'}}$$

$$\le\left\|\left(\sum_{\ell\in Z}\left(M(D_\ell^{b_1}M_{b_1}f_1)\right)^2\right)^{\frac{1}{2}}\right\|_{L^{p_1}}||M\,f_2||_{L^{p_2}}||\,f_0||_{L^{p'}}\lesssim\prod_{i=0}^{2}||\,f_i||_{L^{p_i}}.$$

In the last three lines, we apply the Fefferman-Stein vector valued maximal inequality, Hölder's inequality, and the square function bounds for $D_\ell^{b_1}$ and $\widetilde{D}_m^{b_0*}$ proved by David-Journéd-Semmes. By symmetry and density, this completes the proof.

Lemma (3.2.2)[3]: *Let $b_0,b_1,b_2\in L^\infty$ be para-accretive functions, and assume that T is a bilinear C-Z operator associated to b_0,b_1,b_2 such that $M_{b_0}T(M_{b_1}\cdot,M_{b_2}\cdot)\in WBP$ for normalized bumps of order m. Then for all normalized bumps $\phi_0,\phi_1,\phi_2,R>0$ of order m, and $y_0,y_1,y_2\in R^n$ such that $|y_0-y_i|\le tR$*

$$\left|\langle T(M_{b_1}\phi_1^{y_1,R},M_{b_2}\phi_2^{y_2,R}),M_{b_0}\phi_0^{y_0,R}\rangle\right|\lesssim(1+t)^{n+3m}R^n.$$

Proof. Let $y_0,y_1,y_2\in R^n$, $R>0$, and define $D=1+2t$. Then it follows that

$$\left|\langle T(M_{b_1}\phi_1^{y_1,R}, M_{b_2}\phi_2^{y_2,R}), M_{b_0}\phi_0^{y_0,R}\rangle\right| = \left|\langle T(M_{b_1}\tilde{\phi}_1^{y_0,DR} M_{b_2}\tilde{\phi}_2^{y_0,DR}), M_{b_0}\tilde{\phi}_0^{y_0,DR}\rangle\right|$$

Where $\tilde{\phi}_0(u) = \phi_0(Du)$ and $\tilde{\phi}_i(u) = \phi_i\left(Du + \frac{|y_0-y_1|}{R}\right)$for $i = 1,2$. If $|u| > 1$, then clearly $D|u| > 1$, and

$$\left|Du + \frac{y_0 - y_1}{R}\right| \geq D|u| - \frac{|y_0 - y_1|}{R} \geq (1 + 2t)|u| - t \geq 1.$$

So we have that $supp(\tilde{\phi}_i) \subset B(0,1)$. It follows that $D^{-m}\tilde{\phi}_i \in C_0^\infty$ are normalized bumps of order *m*, and it follows that

$$\left|\langle T(M_{b_1}\tilde{\phi}_1^{y_0,DR} M_{b_2}\tilde{\phi}_2^{y_0,DR}), M_{b_0}\tilde{\phi}_0^{y_0,DR}\rangle\right| \lesssim D^{3m}(DR)^n \lesssim (1+t)^{n+3m}R^n.$$

This completes the proof.

Lemma (3.2.3)[3]: *Let $b_0, b_1, b_2 \in L^\infty$ be para-accretive functions. Suppose T is an bilinear C-Z operator associated to b_0, b_1, b_2 with standard kernel K, and that $M_{b_0}T(M_{b_1}\cdot, M_{b_2}\cdot) \in WBP$. Also let $S_k^{b_i}$ be approximations to the identity with respect to b_i and $D_k^{b_0} = S_{k+1}^{b_0} - S_k^{b_0}$ with compactly supported kernels $S_k^{b_i}$ and $d_k^{b_i}$ for $k \in$ Z. Then*

$$\theta_k(x, y_1, y_2) = \langle T\left(b_1 S_k^{b_1}(\cdot, y_1), b_2 S_k^{b_2}(\cdot, y_2)\right), b_0 d_k^{b_0}(x,\cdot)\rangle$$

is a collection of Littlewood-Paley square function kernels of type SBLPK. Furthermore θ_k satisfies

$$\int_{R^n} \theta_k(x, y_1, y_2) b_0(x) dx = 0$$

For all $y_1, y_2 \in R^n$.

Proof. Fix $x, y_1, y_2 \in R^n$ and $k \in$ Z. We split estimate (6) into two cases: $|x - y_1| + |x - y_2| \leq 2^{3-k}$ and $|x - y_1| + |x - y_2| > 2^{3-k}$. Note that

$$\phi_1(u) = s_k^{b_1}(u + 2^k y_1, 2^k y_1)$$

is a normalized bump up to a constant multiple and $s_k^{b_1}(u, y_1) = 2^{-kn}\phi_1^{y_1,2^{-k}}(u)$. Likewise $s_k^{b_2}(u, y_2) = 2^{-kn}\phi_2^{y_2,2^{-k}}(u)$ and $d_k^{b_0}(x, u) = 2^{-kn}\phi_0^{x,2^{-k}}(u)$ where ϕ_0 and ϕ_2 are normalize bumps up to a constant multiple. Then

$$|\theta_k(x, y_1, y_2)| = \left|\langle T\left(b_1 s_k^{b_1}(\cdot, y_1), b_2 s_k^{b_2}(\cdot, y_2)\right), b_0 d_k^{b_0}(x,\cdot)\rangle\right|$$
$$= 2^{3kn}\left|T\left(b_1\phi_1^{y_1,2^{-k}}, b_2\phi_2^{y_2,2^{-k}}\right), b_0\phi_0^{x,2^{-k}}\right| \lesssim 2^{2kn}$$

Now if we assume that $|x - y_1| + |x - y_2| > 2^{3-k}$, then it follows that $\left|x - y_{i_0}\right| > 2^{2-k}$for at least one $i_0 \in \{1,2\}$ and hence

$$supp(d_k^{b_0}(x,\cdot)) \cap supp(s_k^{b_i}(\cdot, y_1)) \cap supp(s_k^{b_i}(\cdot, y_2)) \subset B(x, 2^{-k}) \cap B(y_{i_0}, 2^{-k}) = 0.$$

Therefore, we can estimate $\theta_k$the kernel representation of T in the following way

$$|\theta_k(x, y_1, y_2)| =$$
$$| \int_{R^{3n}} (K(u_0, u_1, u_2)$$
$$- K(x, u_1, u_2))b(u_0)d_k^{b_0}(x, u_0)\prod_{i=1}^{2} b_i(u_i)s_k^{b_i}(u_i, y_i)du_0\, du_1\, du_2 \,|$$

$$\lesssim \int_{|x-u_0|<2^{-k}}\int_{|y_1-u_1|<2^{-k}}\int_{|y_2-u_2|<2^{-k}} \frac{|u_0 - x|^{\gamma}2^{3nk}du_0\, du_1\, du_2}{(|x - u_1| + |x - u_2|)2^{n+\gamma}} \, .$$

$$\lesssim \int_{|x-u_0|<2^{-k}}\int_{|y_1-u_1|<2^{-k}}\int_{|y_2-u_2|<2^{-k}} \frac{2^{-\gamma k}2^{3nk}du_0\, du_1\, du_2}{(2^{-k} + |x - y_1| + |x - y_2|)^{2n+\gamma}} \, .$$

$$\lesssim \frac{2^{-\gamma k}}{(2^{-k} + |x - y_1| + |x - y_2|)^{2n+\gamma}}.$$

$$\lesssim \Phi_k^{n+\gamma/2}(x - y_1)\Phi_k^{n+\gamma/2}(x - y_2).$$

For (7), note that by the continuity from $b_1C_0^{\delta} \times b_2C_0^{\delta}$ into $(b_0C_0^{\delta})'$ and that $S_k^b = P_kM_{(P_kb)^{-1}}P_k$ has a C_0^{∞} kernel, we have for $\alpha \in N_0^n$ with $|\alpha| = 1$

$$|\partial_x^{\alpha}\theta_k(x, y, z)| = \left|\langle T\left(b_1 s_k^{b_1}(\cdot, y_1), b_2 s_k^{b_2}(\cdot, y_2)\right), b_0\partial_x^{\alpha}(d_k(x,\cdot))\rangle\right| \lesssim 2^k 2^{2kn}.$$

By symmetry, it follows that $\{\theta_k\}$ is a collection of smooth bilinear Littlewood-Paley square function kernels. Now we verify that θ_k has integral 0 in the x spot: By the continuity of T from $b_1 C_0^\delta \times b_2 C_0^\delta$ into $(b_0 C_0^\delta)'$

$$\int_{R^n} \theta_k(x, y_1, y_2) b_0(x) dx$$

$$= \lim_{R\to\infty} \langle T\left(b_1 s_k^{b_1}(\cdot, y_1), b_2 s_k^{b_2}(\cdot, y_2)\right), b_0 \int_{|x|<R} d_k^{b_0}(x,\cdot) b_0(x) dx\rangle$$

$$= \lim_{R\to\infty} \langle T(b_1 s_k^{b_1}(\cdot, y_1), b_2 s_k^{b_2}(\cdot, y_2)), \lambda_R\rangle$$

where we take this to be the definition of λ_R. Now if we take $R > 2 \cdot 2^{-k}$, then for $|u| < R - 2^{-k}$ it follows that

$$\text{supp}(d_k^{b_0}(\cdot, u)) \subset B(u, 2^{-k}) \subset B(0, |u| + 2^{-k}) \subset B(0, R),$$

and hence for $|u| < R - 2^{-k}$ we have that

$$\lambda_R(u) = b_0(u) \int_{|x|<R} d_k^{b_0}(x, u) b_0(x) dx = b_0(u) D_k^{b_0 *} b_0(u) = 0.$$

Also when $|u| > R + 2^{-k}$, it follows that $supp(d_k^{b_0}(\cdot, u)) \cap B(0, R) =/0$, and hence that $\lambda_R(u) = 0$. So we have $\lambda_R(x) = 0$ for $|x| < R - 2^{-k}$ and for $|x| > R + 2^{-k}$. Finally $||\lambda_R||_{L^\infty} \leq sup_u ||d_k^{b_0}(\cdot, u)||_{L^1} \lesssim 1$. Since $supp(d_k^{b_0}(x,\cdot)) \subset B(0, R + 2^{-k}) \backslash B(0, R - 2^{-k})$, it follows that for $R > 4(2^{-k} + |y_1|)$, we may use the integral representation

$$\left|\langle T(b_1 s_k^{b_1}(\cdot, y_1), b_2 s_k^{b_2}(\cdot, y_2)), \lambda_R\rangle\right|$$

$$\leq \int_{R^{3n}} |K(u, v_1, v_2) b_1(v_1) s_k^{b_1}(v_1, y_1) b_2(v_2) s_k^{b_2}(v_2, y_2) \lambda_R(u)| du dv_1 dv_2$$

$$\lesssim \int_{|v_2 - y_2| < 2^{-k}} \int_{|v_1 - y_1| < 2^{-k}} \int_{supp(\lambda_R)} \frac{2^{2kn}}{(|u - v_1| + |u - v_2|)^{2n}} du dv_1 \, dv_2$$

$$\leq \int_{|v_2-y_2|<2^{-k}} \int_{|v_1-y_1|<2^{-k}} \int_{supp(\lambda_R)} \frac{2^{2kn}}{(|u|-|v_1-y_1|-|y_1|)^{2n}}\, dudv_1\, dv_2$$

$$\leq \int_{|v_2-y_2|<2^{-k}} \int_{|v_1-y_1|<2^{-k}} \int_{supp(\lambda_R)} \frac{2^{2kn}}{(R-2^{-k}-|v_1-y_1|-|y_1|)^{2n}}\, dudv_1\, dv_2$$

$$\leq \int_{|v_2-y_2|<2^{-k}} \int_{|v_1-y_1|<2^{-k}} \int_{supp(\lambda_R)} \frac{2^{2kn}}{R^{2n}}\, dudv_1\, dv_2$$

$$\lesssim |supp(\lambda_R)| R^{-2n}$$

$$\lesssim 2^{-k} R^{-(n+1)}.$$

This tends to zero as $R\to\infty$. Hence $\theta_k(x, y_1, y_2)$ has integral zero in the *x* variable.

Theorem (3.2.4)[3]: *Let T be an bilinear C-Z operator associated to para-accretive functions* b_0, b_1, b_2. *If* $M_{b_0}T(M_{b_1}\cdot, M_{b_2}\cdot) \in WBP$ *and*

$$M_{b_0}T(b_1, b_2) = M_{b_1}T^{*1}(b_0, b_2) = M_{b_2}T^{*2}(b_1, b_0) = 0,$$

then T can be extended to a bounded linear operator from $L^{p_1} \times L^{p_2}$ *into* L^p *for all* $1 < p_1, p_2 < \infty$ *satisfying* (2).

Note that in the hypothesis of Theorem (3.2.4), we take $M_{b_0}T(b_1, b_2) = 0$ in the sense of Definition (3.1.7). For appropriate η_R^1, η_R^2 and all $\phi \in C_0^\delta$ such that $b_0\phi$ has mean zero

$$\lim_{R\to\infty} \langle T(\eta_R^1 b_1, \eta_R^2 b_2), b_0\phi\rangle = 0.$$

The meaning of $M_{b_1}T^{*1}(b_0, b_2) = M_{b_2}T^{*2}(b_1, b_0) = 0$ are expressed in a similar way.

Proof. Let T be as in the hypothesis, $1 < p, p_1, p_2 < \infty$ satisfy (2), and $f_0, f_1, f_2 \in C_0^1$ such that $b_i f_i$ have mean zero. Then by Proposition (3.1.16) and the continuity of T from $b_1 C_0^\delta \times b_2 C_0^\delta$ into $(b_0 C_0^\delta)'$, it follows that

$$|\langle T(b_1 f_1, b_2 f_2), b_0 f_0\rangle|$$

$$= \lim_{N\to\infty} \Big| \langle T(M_{b_1} S_N^{b_1} M_{b_1} f_1, M_{b_2} S_N^{b_2} M_{b_2} f_2), M_{b_0} S_N^{b_0} M_{b_0} f_0\rangle$$

$$- \langle T(S_{-N}^{b_1} M_{b_1} f_1, M_{b_2} S_{-N}^{b_2} M_{b_2} f_2), M_{b_0} S_{-N}^{b_0} M_{b_0} f_0\rangle \Big|$$

$$= \lim_{N\to\infty} \left| \sum_{k=-N}^{N-1} \langle T(M_{b_1} S_{k+1}^{b_1} M_{b_1} f_1, M_{b_2} S_{k+1}^{b_2} M_{b_2} f_2), M_{b_0} S_{k+1}^{b_0} M_{b_0} f_0 \rangle \right.$$

$$\left. - \langle T(M_{b_1} S_k^{b_1} M_{b_1} f_1, M_{b_2} S_k^{b_2} M_{b_2} f_2), M_{b_0} S_k^{b_0} M_{b_0} f_0 \rangle \right|$$

$$\leq \sum_{k\in Z} \left| \int_{R^n} \Theta_k^0(b_1 f_1, b_2 f_2) b_0(x)\, f_0(x) dx \right| + \left| \int_{R^n} \Theta_k^1(b_0 f_0, b_2 f_2) b_1(x)\, f_1(x) dx \right|$$

$$+ \left| \int_{R^n} \Theta_k^2(b_1 f_1, b_0 f_0) b_2(x)\, f_2(x) dx \right|$$

where

$$\Theta_k^0(f_1, f_2) = D_k^{b_0} M_{b_0} T(M_{b_1} S_{k+1}^{b_1} f_1, M_{b_2} S_{k+1}^{b_2} f_2),$$

$$\Theta_k^1(f_1, f_2) = D_k^{b_1} M_{b_1} T^{*1}(M_{b_0} S_k^{b_0} f_1, M_{b_2} S_k^{b_2} f_2),$$

$$\Theta_k^2(f_1, f_2) = D_k^{b_2} M_{b_2} T^{*2}(M_{b_1} S_{k+1}^{b_1} f_1, M_{b_0} S_k^{b_0} f_2).$$

We focus on $\Theta_k^0 = \Theta_k$to simplify notation; the other terms are handled in the same way. Since $M_{b_0} T(M_{b_1} \cdot, M_{b_2} \cdot) \in WBP$and T has a standard kernel, it follows from Lemma (3.2.3) that $\{\theta_k\} \in SBLPK$ and $\theta_k(x, y_1, y_2) b_0(x)$ has mean zero in the *x* variable for all $y_1, y_2 \in R^n$.

Now we show that $\theta_k(b_1, b_2) = 0$, which follows from the assumption that $M_{b_0} T(b_1, b_2) = 0$:

$$\Theta_k(b_1, b_2)(x) = \int_{R^{2n}} \langle M_{b_0} T\left(M_{b_1} s_k^{b_1}(\cdot, y_1) b_1(y_1), M_{b_2} s_k^{b_2}(\cdot, y_2) b_2(y_2)\right), d_k^{b_0}(x, \cdot) \rangle\, dy$$

$$= \lim_{R\to\infty} \langle T(b_1 \eta_R^1, b_2 \eta_R^2), b_0 d_k^{b_0}(x, \cdot) \rangle = 0,$$

where

$$\eta_R^i(u) = \int_{|y|<R} s_k^{b_i}(u, y) b_i(y) dy.$$

We've used that $M_{b_0} T(b_1, b_2) = 0$, and that $\eta_R^i \in C^\infty, \eta_R^i \equiv 1$ on $B(0,R)$, and $supp(\eta_R^i) \subset B(0,2R)$ for R sufficiently large. Then by Theorem (3.2.1), it follows that

$$\sum_{k\in Z}\left|\langle \Theta_k^0(M_{b_1}f_1, M_{b_2}f_2), M_{b_0}f_0\rangle\right| \lesssim ||\, f_0||_{L^{p'}}\, ||\, f_1||_{L^{p_1}}||\, f_2||_{L^{p_2}}.$$

A similar argument holds for Θ_k^i with $i = 1{,}2$ again taking advantage of the facts $\frac{1}{p'} + \frac{1}{p^2} = \frac{1}{p_1'}$ and $\frac{1}{p_1} + \frac{1}{p'} = \frac{1}{p_2'}$. Therefore T can be extended to a bounded operator from $L^{p_1} \times L^{p_2}$ into L^p for all $1 < p, p_1, p_2 < \infty$ satisfying (2).

Lemma (3.2.5)[3]: *Suppose* $\{\theta_k\} \in SBLPK$ *with decay parameter* $N > 2n$, *and define* $K : R^{3n}\backslash\{(x,x,x) : x \in R^n\} \to C$

$$K(x,y_1,y_2) = \sum_{k\in Z}\theta_k(x,y_1,y_2)\,.$$

Then K is a bilinear standard Calderón-Zygmund kernel.

Proof. To prove the size estimate, we take $d = |x-y_1| + |x-y_2| \neq 0$ and compute

$$|K(x,y_1,y_2)| \lesssim \sum_{k\in Z}\frac{2^{2kn}}{(1+2^k|x-y_1|)^{N+\gamma}(1+2^k|x-y_2|)^{N+\gamma}}$$
$$\lesssim \sum_{2^k\le d^{-1}} 2^{2kn} + \sum_{2^k> d^{-1}}\frac{2^{2kn}}{(2^k d)^{N+\gamma}} \lesssim d^{-2n}.$$

For the regularity in x, we take $x, x', y_1, y_2 \in R^n$ with $|x-x'| < max(|x-y_1|, |x-y_2|)/2$ and define $d' = |x'-y_1| + |x'-y_2|$. Then

$$|K(x,y_1,y_2) - K(x',y_1,y_2)|$$
$$\lesssim \sum_{k\in Z}\frac{(2^k|x-x'|)^\gamma 2^{2kn}}{(1+2^k|x-y_1|)^{N+\gamma}(1+2^k|x-y_2|)^{N+\gamma}}$$
$$+ \sum_{k\in Z}\frac{(2^k|x-x'|)^\gamma 2^{2kn}}{(1+2^k|x'-y_1|)^{N+\gamma}(1+2k|x'-y_2|)^{N+\gamma}} = I + II.$$

We first bound I by $|x-x'|^\gamma$ times

$$\sum_{2^k\le d^{-1}} 2^{k(2n+\gamma)} + \sum_{2^k> d^{-1}}\frac{2^{k(2n+\gamma)}}{(2^k d)^{N+\gamma}} \lesssim d^{-(2n+\gamma)} + d^{-(N+\gamma)}\sum_{2^k> d^{-1}} 2^{k(2n-N)}$$
$$\lesssim d^{-(2n+\gamma)}.$$

By symmetry, it follows that $II \lesssim |x-x'|^{\gamma} d'^{-(2n+\gamma)}$, but since $|x-x'| < max(|x-y_1|, |x-y_2|)/2$, without loss of generality say $|x-y_1| \geq |x-y_2|$ it follows that

$$II \lesssim \frac{|x-x'|^{\gamma}}{(|x'-y_1|+|x'-y_2|)^{2n+\gamma}} \leq \frac{|x-x'|^{\gamma}}{(|x-y_1|-|x-x'|)^{2n+\gamma}} \lesssim \frac{|x-x'|^{\gamma}}{|x-y_1|^{2n+\gamma}} \lesssim \frac{|x-x'|^{\gamma}}{d^{2n+\gamma}}$$

With a similar computation for y_1, y_2, it follows that K is a standard kernel.

Theorem (3.2.6)[3]: *Given para-accretive functions* $b_0, b_1, b_2 \in L^{\infty}$ *and* $b \in BMO$, *there exists a bilinear Calderón-Zygmund operator L that is bounded from* $L^{p_1} \times L^{p_2}$ *into* L^p *for all* $1 < p, p_1, p_2 < \infty$ *satisfying* (2) *with* $p = 2$ *such that* $M_{b_0} L(b_1, b_2) = \beta$, $M_{b_1} L^{*1}(b_0, b_2) = M_{b_2} L^{*2}(b_1, b_0) = 0$.

Proof. Let b_0, b_1, b_2 be para-accretive functions, and $S_k^{b_i}, D_k^{b_i}, and\ \widetilde{D}_k^{b_i}$ be the approximation to identity and reproducing formula operators with respect to b_i for $i = 0,1,2$ that have compactly supported kernels as defined in Theorem (3.1.17). Define

$$L(f_1, f_2) = \sum_{k \in Z} L_k(f_1, f_2) = \sum_{k \in Z} D_k^{b_0} M_{b_0} (\widetilde{D}_k^{b_0 *} M_{b_0} \beta)(S_k^{b_1} f_1)(S_k^{b_2} f_2)$$

$$\ell(x, y) = \sum_{k \in Z} \ell_k(x, y) = \sum_{k \in Z} \int_{R^n} d_k^{b_0}(x, u) b_0(u) \widetilde{D}_k^{b_0 *} M_{b_0} \beta(u) s_k^{b_1}(u, y_1) s_k^{b_2}(u, y_2) du.$$

It follows that L is bounded from $L^{p_1} \times L^{p_2}$ into L^2 for all $1 < p_1, p_2 < \infty$ satisfying $\frac{1}{2} = \frac{1}{p_1} + \frac{1}{p_2}$:

$$\left| \int_{R^n} L(f_1, f_2)(x)\, f_0(x) dx \right| \leq \sum_{k \in Z} \int_{R^n} \widetilde{D}_k^{b_0 *} M_{b_0}\, \beta(x) S_k^{b_1} f_1(x) S_k^{b_2} f_2(x) D_k^{b_0} f_0(x) b_0(x) dx$$

$$\lesssim \left\| \left(\sum_{k \in Z} \left| M_{\widetilde{D}_k^{b_0 *} M_{b_0} \beta} S_k^{b_1} f_1 S_k^{b_2} f_2 \right|^2 \right)^{\frac{1}{2}} \right\|_{L^2} \left\| \sum_{k \in Z} \left(\left| D_k^{b_0} f_0 \right|^2 \right)^{\frac{1}{2}} \right\|_{L^2}$$

$$\lesssim \left(\int_{R^n} \sum_{k \in Z} [\Phi_k^N * |f_1|(x) \Phi_k^N * |f_2|(x)]^2 |\widetilde{D}_k^{b_0 *} M_{b_0} \beta(x)|^2 dx \right)^{\frac{1}{2}} ||f_0||_{L^2}.$$

$$\lesssim \left(\int_{R^n}\sum_{k\in Z}[\Phi_k^N * |f_1|(x)]^{p_1}|\widetilde{D}_k^{b_0*}M_{b_0}\beta(x)|^2dx\right)^{\frac{1}{p_1}}$$

$$\times\left(\int_{R^n}\sum_{k\in Z}[\Phi_k^N * |f_2|(x)]^{p_2}|\widetilde{D}_k^{b_0*}M_{b_0}\beta(x)|^2dx\right)^{\frac{1}{p_2}}||f_0||_{L^2}$$

$$\lesssim ||f_0||_{L^2}||f_1||_{L^{p_1}}||f_2||_{L^{p_2}}.$$

Note that in the last line we apply the discrete version of a well-known result relating Carleson measure and square functions due to Carleson and Jones for the Carleson measure

$$d\mu(x,t) = \sum_{k\in Z}|\widetilde{D}_k^{b_0*}b_0\beta(x)|^2\delta_{t=2^{-k}}.$$

For details of the discrete version of this result. This proves that L is bounded from $L^{p_1}\times L^{p_2}$ into L^2 for all $1 < p_1, p_2 < \infty$ satisfying (2) with $p = 2$. It is easy to check that $\{\ell_k\} \in SBLPK$ with size index $N > 2n$: since $d_k^{b_0}$ and $s_k^{b_i}$ are compactly supported kernels, for $i = 1,2$ it follows that

$$|\ell_k(x,y_1,y_2)| \le \left|\left|b_0\widetilde{D}_k^{b_0*}M_{b_0}\beta\right|\right|_{L^\infty}\int_{R^n}\left|d_k^{b_0}(x-u)s_k^{b_1}(u-y_1)s_k^{b_2}(u-y_2)\right|du$$

$$\lesssim 2^{kn}\int_{R^n}\Phi_k^{4(n+1)}(x-u)\Phi_k^{4(n+1)}(u-y_i)du \lesssim 2^{kn}\Phi_k^{4(n+1)}(x-y_i).$$

Hence the size condition (6) with size index $N = 2n+1$ and $\gamma = 1$ follows

$$|\ell_k(x,y_1,y_2)| \lesssim \prod_{i=1}^{2}\left(2^{kn}\Phi_k^{4(n+1)}(x-y_i)\right)^{\frac{1}{2}} = \Phi_k^{2n+2}(x-y_1)\Phi_k^{2n+2}(x-y_2).$$

The regularity estimates (7)-(9) follow easily from the regularity of $d_k^{b_0}, s_k^{b_1}$, and $s_k^{b_2}$. Then by Lemma (3.2.5), L has a standard Calderón-Zygmund kernel ℓ. It follows from a result of Grafakos-Torres that L is bounded from $L^{p_1}\times L^{p_2}$ into L^p where $1 < p_1, p_2 < \infty$ satisfy (2). We compute $M_{b_0}L(b_1,b_2)$: Let $\delta > 0, \phi \in C_0^\delta$ such that $supp(\phi) \subset B(0,N)$ and $b_0\phi$ has mean zero. Let $\eta_R(x) = \eta(x/R)$ where $\eta \in C_0^\infty$ satisfies $\eta \equiv 1$ on $B(0,1)$, and $supp(\eta) \subset B(0,2)$. Then

$\langle L(b_1, b_2), b_0\phi\rangle$

$$= \lim_{R\to\infty} \sum_{2^{-k}<R/4} \int_{R^n} \widetilde{D}_k^{b_0 *} M_{b_0}\beta(x) S_k^{b_1} M_{b_1}\eta_R(x) S_k^{b_2} M_{b_2}\eta_R(x) M_{b_0} D_k^{b_0}(b_0\phi)(x)dx$$

$$+ \lim_{R\to\infty} \sum_{2^{-k}\geq R/4} \int_{R^n} \widetilde{D}_k^{b_0 *} M_{b_0}\beta(x) S_k^{b_1} M_{b_1}\eta_R(x) S_k^{b_2} M_{b_2}\eta_R(x) M_{b_0} D_k^{b_0}\,(b_0\phi)(x)dx,$$

where we may write this only if the two limits on the right hand side of the equation exist. As we are taking $R\to\infty$ and N is a fixed quantity determined by ϕ, without loss of generality assume that $R > 2N$. Note that for $2^{-k} \leq R/4$ and $|x| < N + 2^{-k}$,

$$supp(s_k^{b_i}\,(x,\cdot)) \subset B(x, 2^{-k}) \subset B(0, N + 2^{1-k}) \subset B(0, R).$$

Since $\eta_R \equiv 1$ on $B(0, R)$, it follows that $s_k^{b_i} M_{b_i}\eta_R(x) = 1$ for all $|x| < N + 2^{-k}$ when $2^{-k} \leq R/4$. Therefore

$$\lim_{R\to\infty} \sum_{2^{-k}<R/4} \int_{R^n} \widetilde{D}_k^{b_0 *} M_{b_0}\beta(x) M_{b_0} D_k^{b_0}\,(b_0\phi)(x)dx$$

$$= \int_{R^n} \sum_{k\in Z} M_{b_0}\widetilde{D}_k^{b_0} M_{b_0} D_k M_{b_0}\phi(x)\beta(x)dx = \langle\beta, b_0\phi\rangle$$

Here we use the convergence of the accretive type reproducing formula in H^1 from Corollary(3.1.19). For any $k \in$ Z, we have the estimates

$$||S_k^{b_i} M_{b_i}\eta_R||_{L^1} \lesssim ||\eta_R||_{L^1} \lesssim R^n, \qquad \textbf{(23)}$$

$$||S_k^{b_i} M_{b_i}\eta_R||_{L^\infty} \lesssim ||\eta_R||_{L^\infty} = 1, \qquad \textbf{(24)}$$

and for any $x \in R^n$

$$|D_k^{b_0} M_{b_0}\phi(x)| \leq \int_{R^n} |d_k^{b_0}\,(x,y) - d_k^{b_0}(x,0)||b_0(y)\phi(y)|dy \lesssim \int_{R^n} (2^k|y|)^\gamma |f(y)|dy$$

$$\lesssim N^\gamma ||\phi||_{L^1} 2^{k(n+\gamma)}.$$

Here we know that $\{d_k^{b_0}\} \in LPK$, so without loss of generality we take the corresponding smoothness parameter $\gamma \leq \delta$. Later we will use that $\gamma \leq \delta$ implies $\phi \in C_0^\delta \subset C_0^\gamma$, so we have that $|\phi(x) - \phi(y)| \lesssim |x - y|^\gamma$. Therefore

$$\sum_{2^{-k}>R/4} \int_{R^n} |\widetilde{D}_k^{b_0 *} M_{b_0}\beta(x) S_k^{b_1} M_{b_1}\eta_R(x) S_k^{b_2} M_{b_2}\eta_R(x) M_{b_0} D_k^{b_0}\,(b_0\phi)(x)|dx \leq$$
$$\sum_{2^{-k}>R/4} ||\widetilde{D}_k^{b_0 *} M_{b_0}\beta||_{L^\infty} ||S_k^{b_1} M_{b_1}\eta_R||_{L^1} ||S_k^{b_2} M_{b_2}\eta_R||_{L^\infty} ||M_{b_0} D_k^{b_0}\,(b_0\phi)||_{L^\infty} \lesssim$$
$$R^n \sum_{2^{-k}>R/4} 2^{k(n+\gamma)} \quad \lesssim R^{-\gamma}. \tag{25}$$

Hence the second limit above exists and tends to 0 as $R \to \infty$. Then $\langle L(b_1, b_2), b_0\phi\rangle = \langle \beta, b_0\phi\rangle$ for all $\phi \in C_0^\delta$ such that $b_0\phi$ has mean zero and hence $M_{b_0} L(b_1, b_0) = \beta$. Again for any $\phi \in C_0^\delta$ such that $b_1\phi$ has mean zero and $supp(\phi) \subset B(0, N)$, we have

$$|\langle L^{1*}(b_0, b_2), b_1\phi\rangle|$$
$$= \lim_{R\to\infty} \left| \sum_{k\in Z} \int_{R^n} \widetilde{D}_k^{b_0 *} M_{b_0}\beta(x) S_k^{b_1} M_{b_1}\phi(x) S_k^{b_2} M_{b_2}\eta_R(x) D_k^{b_0} M_{b_0}\eta_R(x) b_0(x) dx \right|$$
$$\lesssim \lim_{R\to\infty} \sum_{k\in Z} ||\widetilde{D}_k^{b_0 *} M_{b_0}\beta||_{L^\infty} ||S_k^{b_1} M_{b_1}\phi||_{L^1} ||S_k^{b_2} M_{b_2}\eta_R||_{L^\infty} ||D_k^{b_0} M_{b_0}\eta_R||_{L^\infty}$$
$$\lesssim \lim_{R\to\infty} \sum_{k\in Z} ||S_k^{b_1} M_{b_1}\phi||_{L^1} ||D_k^{b_0} M_{b_0}\eta_R||_{L^\infty}.$$

We will now show that $||S_k^{b_1} M_{b_1}\phi||_{L^1} ||D_k^{b_0} M_{b_0}\eta_R||_{L^\infty}$ bounded by a in integrable function in k (i.e. summable) independent of R, so that we can bring the limit in R inside the sum. To do this we start by estimating

$$|S_k^{b_1} M_{b_1}\phi(x)| \leq \int_{R^n} |S_k^{b_1}(x,y) - S_k^{b_1}(x,0)||\phi(y) b_1(y)| dy$$
$$\leq N^\gamma ||\phi||_{L^1} ||b_1||_{L^\infty} 2^{\gamma k} (\Phi_0^N(x) + \Phi_k^N(x))$$

and so $\left\|S_k^{b_1} M_{b_1}\phi\right\|_{L^1} \lesssim 2^{\gamma k}$. We also have that $\left\|S_k^{b_1} M_{b_1}\phi\right\|_{L^1} \lesssim ||\phi||_{L^1} \lesssim 1$, so $\left\|S_k^{b_1} M_{b_1}\phi\right\|_{L^1} \lesssim min(1, 2^{\gamma k})$. Also

$$|D_k^{b_0} M_{b_0}\eta_R(x)| \leq \int_{R^n} |d_k^{b_0}\,(x,y)||\eta_R(y) - \eta_R(x)||b_0(y)| dy$$
$$\lesssim 2^{-\gamma k} R^{-\gamma} \int_{R^n} \Phi_k^{N+\gamma}\,(x-y)(2^k|x-y|)^\gamma dy \lesssim 2^{-\gamma k} R^{-\gamma}.$$

It follows that $||D_k^{b_0} M_{b_0}\eta_R||_{L^\infty} \lesssim ||\eta_R||_{L^\infty} \lesssim 1$, and hence $||D_k^{b_0} M_{b_0}\eta_R||_{L^\infty} \lesssim min(1, 2^{-\gamma k})$. So when R >1, we have

$$||D_k^{b_0} M_{b_0}\eta_R||_{L^\infty} ||S_k^{b_1} M_{b_1}\phi||_{L^1} \lesssim min(2^{-\gamma k} R^{-\gamma}, 2^{\gamma k}) \leq 2^{-\gamma|k|},$$

and hence by dominated convergence

$$|\langle L^{1*}(b_0,b_2),b_1\phi\rangle| \lesssim \sum_{k\in Z} \lim_{R\to\infty} ||S_k^{b_1} M_{b_1}\phi||_{L^1} ||D_k^{b_0} M_{b_0}\eta_R||_{L^\infty} \lesssim \sum_{k\in Z} \lim_{R\to\infty} 2^{-k\gamma} R^{-\gamma} \ = \ 0$$

Then $M_{b_1}L^{*1}(b_0,b_2) = 0$ and a similar argument shows that $M_{b_2}L^{*2}(b_1,b_0) = 0$, which concludes the proof.

Now to complete the proof of Theorem (3.1.1) is a standard argument using the reduced Tb Theorem (3.2.4) and paraproducts constructioned in Theorem (3.2.6).

***Proof* .** Assume that $M_{b_0}T(M_{b_1}\cdot, M_{b_2}\cdot)$ satisfies the weak boundedness property and

$$M_{b_0}T(b_1,b_2), M_{b_1}T^{*1}(b_0,b_2), M_{b_2}T^{*2}(b_1,b_0) \in BMO$$

By Theorem (3.2.6), there exist bounded bilinear Calderón-Zygmund operators L_i such that

$$M_{b_0}L_0(b_1,b_2) = M_{b_0}T(b_1,b_2), \qquad M_{b_1}L_0^{*1}(b_0,b_2) = M_{b_2}L_0^{*2}(b_1,b_0) = 0,$$
$$M_{b_1}L_1^{*1}(b_0,b_2) = M_{b_1}T^{*1}(b_0,b_2), \qquad M_{b_0}L_1(b_1,b_2) = M_{b_2}L_1^{*2}(b_1,b_0) = 0,$$
$$M_{b_2}L_2^{*2}(b_1,b_0) = M_{b_2}T^{*2}(b_1,b_0), \qquad M_{b_1}L_2^{*1}(b_0,b_2) = M_{b_0}L_2(b_1,b_2) = 0$$

Now define the operator

$$S = T - \sum_{i=0}^{2} L_i ,$$

which is continuous from $b_1C_0^\delta \times b_2C_0^\delta$ into $(b_0C_0^\delta)'$. Also $M_{b_0}S(M_{b_1}\cdot, M_{b_2}\cdot)$ satisfies the weak boundedness property since $M_{b_0}T(M_{b_1}\cdot, M_{b_2}\cdot)$ and $M_{b_0}L_i(M_{b_1}\cdot, M_{b_2}\cdot)$ for i= 0,1,2 do. We have

$$M_{b_0}S(b_1,b_2) = M_{b_0}T(b_1,b_2) - \sum_{i=0}^{2} M_{b_0}L_i(b_1,b_2) = 0$$
$$M_{b_1}S^{*1}(b_0,b_2) = M_{b_1}T^{*1}(b_0,b_2) - \sum_{i=0}^{2} M_{b_1}L_i^{*1}(b_0,b_2) = 0$$
$$M_{b_2}S^{*2}(b_1,b_0) = M_{b_2}T^{*2}(b_1,b_0) - \sum_{i=0}^{2} M_{b_2}L_i^{*2}(b_1,b_0) = 0$$

Then by Theorem (3.2.4), S can be extended to a bounded linear operator from $L^{p_1} \times L^{p_2}$ into L^p for all $1 < p, p_1, p_2 < \infty$ satisfying (2). Therefore T is bounded on the same indices. T is also bounded without restriction on p.

We prove bounds of the form

$$||T(f_1,f_2)||_{L^p(\Gamma_0)} \lesssim ||f_1||_{L^{p_1}(\Gamma)} ||f_2||_{L^{p_2}(\Gamma)}$$

for parameterized Lipschitz curves Γ and p, p_1, p_2 satisfying Hölder.

Let L be a Lipschitz function with small Lipschitz constant $\lambda < 1$, and define the parameterization $\gamma(x) = x + iL(x)$ of the curve $\Gamma = \{\gamma(x) : x \in R\}$. Define $L^p(\Gamma)$ to be the collection of all measurable functions $f : \Gamma \to C$ such that

$$|| f ||_{L^p(\Gamma)} = \left(\int_\Gamma | f (z)|^p |dz|\right)^{\frac{1}{p}} = \int_R (| f (\gamma(x))|^\gamma |\gamma'(x)| dx)^{\frac{1}{p}} < \infty .$$

The applications are in part motivated by the proof of L^p bounds for the Cauchy integral using the Tb theorem. We define the Cauchy integral operator for appropriate $g : \Gamma \to C \; and \; z \in G$

$$C_\Gamma g(z) = \lim_{\varepsilon \to 0+} \int_\Gamma \frac{g(\xi) d\xi}{(\xi + i\varepsilon) - z},$$

and parameterized Cauchy integral operator for $f : R \to C$ and $x \in$ R

$$\tilde{C}_\Gamma f (z) = \lim_{\varepsilon \to 0+} \int_R \frac{f(y) dy}{(\gamma(y) + i\varepsilon) - \gamma(x)}.$$

The bounds of C_Γ on $L^p(\Gamma)$ can be reduced to the bounds of $\tilde{C}_\Gamma$ on $L^p(R)$. We formally check the Tb conditions for $\tilde{C}_\Gamma$ with $b_0 = b_1 = \gamma'$ needed to apply the Tb theorem of David-Journé-Semmes: We check (1) $\tilde{C}_\Gamma(\gamma') \in BMO$

$$\tilde{C}_\Gamma \gamma'(x) = \lim_{\varepsilon \to 0+} \int_R \frac{\gamma'(y) dy}{(\gamma(y) + i\varepsilon) - \gamma(x)} = \lim_{\varepsilon \to 0+} \int_\Gamma \frac{d\xi}{(\xi + i\varepsilon) - \gamma(x)} = 2\pi i$$

and (2) $\tilde{C}^*_\Gamma(\gamma') \in BMO$, for appropriate $\phi \in C_0^\infty$

$$\tilde{C}^*_\Gamma \gamma'(x) = \lim_{\varepsilon \to 0+} \int_R \frac{\gamma'(y) dy}{(\gamma(x) + i\varepsilon) - \gamma(y)}) = \lim_{\varepsilon \to 0+} - \int_\Gamma \frac{d\xi}{(\xi + i\varepsilon) - \gamma(x)} = 0.$$

The crucial role that Cauchy's formula plays in this argument is to be able to evaluate the limit from the definition of $\tilde{C}_\Gamma$ for nice enough input functions $\gamma'(x) f (x)$. In the application,we use a similar argument except the role of Cauchy's integral formula is replaced with an integration by parts identity to verify the WBP and Tb conditions.

We look at the "flat" bilinear Riesz tranforms, which we generate from a potential function perspective.Consider the potential function

$$F(x, y_1, y_2) = \frac{1}{((x - y_1)2 + (x - y_2)^2)^{1/2}},$$

and the kernels that it generates:

$$K_0(x, y_1, y_2) = \partial_x F(x, y_1, y_2) = \frac{2x - y_1 - y_2}{((x - y_1)^2 + (x - y_2)^2)^{\frac{3}{2}}},$$
$$K_1(x, y_1, y_2) = \partial_{y_2} F(x, y_1, y_2) = \frac{x - y_1}{((x - y_1)^2 + (x - y_2)^2)^{\frac{3}{2}}},$$
$$K_2(x, y_1, y_2) = \partial_{y_2} F(x, y_1, y_2) = \frac{x - y_2}{((x - y_1)^2 + (x - y_2)^2)^{\frac{3}{2}}}.$$

We define the bilinear Reisz transforms as principle value integrals for $f_1, f_2 \in C_0^\infty$

$$R_j(f_1, f_2)(x) = p.v. \int_{R^2} K_j(x, y_1, y_2)\, f_1(y_1)\, f_2(y_2) dy_1\, dy_2 .$$

Here is it only interesting to study two of these three operators since $R_0 = R_1 + R_2$. We can be applied to the bilinear Riesz transforms R_1: We formally check (1) $R_1(1,1) \in BMO$

$$R_1(1,1)(x) = -\int_{R^2} F(x, y_1, y_2) \partial\, y_1(1) dy_1\, dy_2 = 0,$$

(2) R_1^{*1} (1,1) $\in BMO$

$$R_1^{*1}(1,1)(y_1) = \int_{R^2} \partial_{y_1} F(x, y_1, y_2) dx dy_2$$

$$= \int_{R^2} (\partial_x F(x, y_1, y_2) - \partial_{y_2} F(x, y_1, y_2)) dx dy_2$$

$$= -\int_{R^2} F(x, y_1, y_2)(\partial_x(1) - \partial_{y_2}(1)) dx dy_2 = 0,$$

and (3) $R_1^{*2}1$ (1,1) $\in BMO$

$$R_1^{*2}(1,1)(y_2) = -\int_{R^2} F(x, y_1, y_2) \partial y_1\, (1) dx dy_1 = 0.$$

Here we observe that the conditions $R_1(1,1), R_1^{*2}(1,1) = 0$ are identical arguments and rely on the cancellation of the kernel K_1. The R_1^{*1} (1,1) condition relies on more than just the cancellation K_1; it also exploits the symmetry of the kernel via the identity $\partial_{y_1} F(x, y_1, y_2) = \partial_x F(x, y_1, y_2) - \partial_{y_2} F(x, y_1, y_2)$. This is the general argument that we will use to prove L^p bounds for bilinear Riesz transforms defined along Lipschitz curves, which we define now.

For $z, \xi_1, \xi_2 \in \Gamma$, define the potential function

$$F_{\Gamma}(z,\xi_1,\xi_2) = \frac{1}{((z-\xi_1)^2 + (z-\xi_2)^2)^{1/2}},$$

and the Riesz kernels generated by *F*:

$$K_{\Gamma,0}(z,\xi_1,\xi_2) = \partial_z F_{\Gamma}(\xi,\xi_1,\xi_2) = \frac{2z-\xi_1-\xi_2}{((z-\xi_1)^2 + (z-\xi_2)^2)^{3/2}},$$

$$K_{\Gamma,1}(z,\xi_1,\xi_2) = \partial_{\xi_1} F_{\Gamma}(z,\xi_1,\xi_2) = \frac{z-\xi_1}{((z-\xi_1)^2 + (z-\xi_2)^2)^{3/2}},$$

$$K_{\Gamma,2}(z,\xi_1,\xi_2) = \partial_{\xi_2} F_{\Gamma}(z,\xi_1,\xi_2) = \frac{z-\xi_2}{((z-\xi_1)^2 + (z-\xi_2)^2)^{3/2}}.$$

We will keep the notation $z=\gamma(x)$, $\xi_1 = \gamma(y_1)$, $\xi_2=\gamma(y_2)$, $y_0 = x$, and $\xi_0 = z$. Here we define $\sqrt{.}$ on C with the negative real axis for a branch cut, i.e. for $\omega = r\,e^{i\theta} \in C$ with *r* > 0 and $\theta \in (-\pi,\pi]$, we define $\sqrt{\omega} = \sqrt{r}e^{i\theta/2}$. We make this definition to be precise, but it will not cause any issues with computations since we will only evaluate $\sqrt{\omega}$ for $\omega \in C$ with positive real part.

For appropriate $g_1, g_2 : \Gamma \to C$ and $z \in \Gamma$, we define

$$C_{\Gamma,j}(g_1,g_2)(z) = p.v.\int_{\Gamma^2} K_{\Gamma,j}(z,\xi_1,\xi_2) g_1(\xi_1) g_2(\xi_2) d\xi_1\, d\xi_2$$

$$= \lim_{\varepsilon\to 0+} \int_{|Re(z-\xi_1)|,|Re(z-\xi_2)|>\varepsilon} K_{\Gamma,j}(z,\xi_1,\xi_2) g_1(\xi_1) g_2(\xi_2) d\xi_1 d\xi_2. \qquad \textbf{(26)}$$

Initially we take this definition for $g_j = f_j \circ \gamma^{-1}$ for $f_j \in C_0^\infty(R), j = 1,2$, but even for such g_j it is not yet apparent that this limit exists. We will establish that this limit exists, and furthermore that $C_{\Gamma,j}$ can be continuously extended to a bilinear operator from $L^{p_1}(\Gamma) \times L^{p_2}(\Gamma)$ *into* $L^p(\Gamma)$. To prove these things, we will pass through "parameterized" versions of F , K_j, and $C_{\Gamma,j}$ for *j* = 0,1,2 in the same way that David-Journé-Semmes did to apply their Tb theorem to the Cauchy integral operator . For $x, y_1, y_2 \in R$, define

$$\tilde{F}_{\Gamma}(x,y_1,y_2) = F_{\Gamma}\big(\gamma(x),\gamma(y_1),\gamma(y_2)\big), \qquad \widetilde{K}_{\Gamma,j}(x,y_1,y_2) = K_{\Gamma}(\gamma(x),\gamma(y_1),\gamma(y_2)),$$

and for $f_1, f_2 \in C_0^\infty$(R), define for *x* ∈R

(27)

$$M_{\gamma'}\tilde{C}_{\Gamma,j}(\gamma' f_1,\gamma' f_2)(x) = p.v.\int_{R^2} \widetilde{K}_j(x,y_1,y_2)\, f_1(y_1)f_2(y_2)\gamma'(x)\gamma'(y_1)\gamma'(y_2)dy_1\, dy_2$$

$$= \lim_{\varepsilon\to 0+}\int_{|x-y_1|,|x-y_2|>\varepsilon} \widetilde{K}_j(x,y_1,y_2)\, f_1(y_1)\, f_2(y_2)\gamma'(x)\,\gamma'(y_1)\gamma'(y_2)dy_1\, dy_2.$$

for $j = 0,1,2$. We begin by proving that $\tilde{C}_{\Gamma,j}$ for $j = 0,1,2$ is well defined, and find an absolutely convergent integral representation for it that depends on derivatives of the input functions $f_1, f_2 \in C_0^\infty(R)$.

Proposition (3.2.7)[3]: *Let L be a Lipschitz function with Lipschitz constant $\lambda < 1$ such that for almost every $x \in$ R the limits*

$$\lim_{\varepsilon\to 0+}\gamma'(x+\varepsilon) = \gamma'(x+) \quad and \quad \lim_{\varepsilon\to 0+}\gamma'(x-\varepsilon) = \gamma'(x-)$$

exist. Then $M_{\gamma'}\tilde{C}_{\Gamma,j}(\gamma' f_1,\gamma' f_2)$ is an almost everywhere well defined function for $f_1, f_2 \in C_0^\infty(R)$ and $j = 0,1,2$. More precisely, for $f_1, f_2 \in C_0^\infty(R)$ the limit in (27) *converges for almost every $x \in$ R and*

$$M_{\gamma'}\tilde{C}_{\Gamma,1}(\gamma' f_1,\gamma' f_2)(x) = -\int_{R^2}\tilde{F}_\Gamma(x,y_1,y_2)\, f_1'(y_1)\, f_2(y_2)\gamma'(x)\gamma'(y_2)dy_1\, dy_2,$$

$$M_{\gamma'}\tilde{C}_{\Gamma,2}(\gamma' f_1,\gamma' f_2)(x) = -\int_{R^2}\tilde{F}_\Gamma(x,y_1,y_2) f_1(y_1) f_2'(y_2)\gamma'(x)\gamma'(y_1)dy_1\, dy_2.$$

Furthermore $\tilde{C}_{\Gamma,j}$ is continuous from $\gamma' C_0^1(R)\times\gamma' C_0^1(R)$ into $(\gamma' C_0^1(R))'$, and for $f_0, f_1, f_2 \in C_0^\infty(R)$,

$$\langle\tilde{C}_{\Gamma,0}(\gamma' f_1,\gamma' f_2),\gamma' f_0\rangle$$

$$= -\int_{R^3}\tilde{F}_\Gamma(x,y_1,y_2)\, f_0'(x)\, f_1(y_1)f_2(y_2)\gamma'(y_1)\gamma'(y_2)dx dy_1\, dy_2$$

$$\langle\tilde{C}_{\Gamma,1}(\gamma' f_1,\gamma' f_2),\gamma' f_0\rangle = -\int_{R^3}\tilde{F}_\Gamma(x,y_1,y_2)f_0(x)\, f_1'(y_1)\, f_2(y_2)\gamma'(x)\gamma'(y_2)dx dy_1\, dy_2$$

$$\langle\tilde{C}_{\Gamma,2}(\gamma' f_1,\gamma' f_2),\gamma' f_0\rangle = -\int_{R^3}\tilde{F}_\Gamma(x,y_1,y_2)f_0(x)\, f_1(y_1)\, f_2'(y_2)\gamma'(x)\gamma'(y_1)dx dy_1\, dy_2.$$

Proof. Fix $f_1, f_2 \in C_0^\infty$, and we start by showing that $M_{b_0}\tilde{C}_{\Gamma,1}(\gamma' f_1,\gamma' f_2)$ can be realized as a bounded function. Define for $\varepsilon > 0$ and $x \in$ R

$$C_\varepsilon(x) = \int_{|x-y_1|,|x-y_2|>\varepsilon} \widetilde{K}_{\Gamma,1}(x,y_1,y_2)\, f_1(y_1)\, f_2(y_2)\gamma'(x)\gamma'(y_1)\gamma'(y_2)dy_1dy_2$$

Note that $\partial_{y_1}\tilde{F}_\Gamma(x,y_1,y_2) = \widetilde{K}_{\Gamma,1}(x,y_1,y_2)\gamma'(y_1)$, and we integrate by parts to rewrite C_ε

$$C_\varepsilon(x) = \int_{|x-y_1|,|x-y_2|>\varepsilon} \partial_{y_1}\tilde{F}_\Gamma(x,y_1,y_2)\, f_1(y_1)\, f_2(y_2)\gamma'(x)\gamma'(y_2)dy_1\, dy_2$$

$$= -\int_{|x-y_1|,|x-y_2|>\varepsilon} \tilde{F}_\Gamma(x,y_1,y_2)\, f_1'\,(y_1)\, f_2(y_2)\gamma'(x)\gamma'(y_2)dy_1\, dy_2$$

$$+\int_{|x-y_2|>\varepsilon} \tilde{F}_\Gamma(x,x-\varepsilon,y_2)\, f_1(x-\varepsilon)\, f_2(y_2)\gamma'(x)\gamma'(y_2)\, dy_2$$

$$-\int_{|x-y_2|>\varepsilon} \tilde{F}_\Gamma(x,x+\varepsilon,y_2)\, f_1(x+\varepsilon)\, f_2(y_2)\gamma'(x)\gamma'(y_2)\, dy_2$$

$$= -\int_{|x-y_1|,|x-y_2|>\varepsilon} \tilde{F}_\Gamma(x,y_1,y_2)\, f_1'\,(y_1)\, f_2(y_2)\gamma'(x)\gamma'(y_2)dy_1\, dy_2$$

$$+\int_{|x-y_2|>\varepsilon} \left(\tilde{F}_\Gamma(x,x-\varepsilon,y_2) - \tilde{F}_\Gamma(x,x+\varepsilon,y_2)\right) f_1(x-\varepsilon)\, f_2(y_2)\gamma'(x)\gamma'(y_2)dy_2$$

$$+\int_{|x-y_2|>\varepsilon} \tilde{F}_\Gamma(x,x+\varepsilon,y_2)(\, f_1(x-\varepsilon) - \, f_1(x+\varepsilon))\, f_2(y_2)\gamma'(x)\gamma'(y_2)dy_2$$

$$= I_\varepsilon(x) + II_\varepsilon(x) + III_\varepsilon(x).$$

We use that $f_1 \in C_0^\infty\,(R)$ to conclude that the when integrating by parts, the boundary terms at $y_1 = \pm\infty$ vanish, leaving the $y_1 = x \pm \varepsilon$ above. We now verify that the limits of $I_\varepsilon(x),\ II_\varepsilon(x),\ and\ III_\varepsilon(x)$ each exist as $\varepsilon \to 0$.

$\underline{I_\varepsilon converges}$: To compute this limit, we verify that the integrand of I_ε is an integrable function. Note that $||L'||_{L^\infty} = \lambda < 1$ implies

$$
\begin{aligned}
&|\,\tilde{F}_\Gamma(x,y_1,y_2)| \\
&\le \frac{1}{|Re((x-y_1)^2-(L(x)-L(y_1))^2\ +(x-y_2)^2-(L(x)-L(y_2))^2)\,|^{1/2}} \\
&\lesssim \frac{1}{(1-\lambda)^{1/2}}\frac{1}{|x-y_1|+|x-y_2|}.
\end{aligned}
$$

Now let $R_0 > 0$ be large enough so that $supp(\, f_j\,) \subset B(0,R_0)\ for\ j=1,2$, and it follows that

$$
\begin{aligned}
&\int_{R^2} |\,\tilde{F}_\Gamma(x,y_1,y_2)\, f_1'\,(y_1)\, f_2(y_2)\gamma'(x)\gamma'(y_2)|dy_1\, dy_2 \\
&\qquad \le \frac{||\gamma'||_{L^\infty}^2}{(1-\lambda)^{1/2}}\int_{|y_1|,|y_2|\le R_0}\frac{||\, f_1'||_{L^\infty}||\, f_2||_{L^\infty}}{|x-y_1|+|x-y_2|}\ dy_1\, dy_2 \\
&\qquad \lesssim \frac{R_0}{(1-\lambda)^{1/2}}||\, f_1'||_{L^\infty}||\, f_2||_{L^\infty}.
\end{aligned}
$$

Therefore I_ε converges to an absolutely convergent integral as $\varepsilon \to 0$.

$\underline{II_\varepsilon \to 0}$: First we make a change of variables in II_ε to rewrite

$$
II_\varepsilon\ = \int_{|y_2|>1} h_\varepsilon(x,y_2)\, f_1(x-\varepsilon)\, f_2(\,x-\varepsilon y_2)\gamma'(x-\varepsilon y_2)dy_2\,, \qquad \textbf{(28)}
$$

where $h_\varepsilon(x,y_2)\ =\ \varepsilon\gamma'(x)\big(\tilde{F}_\Gamma(x,x-\varepsilon,x-\varepsilon y_2) -\ \tilde{F}_\Gamma(x,x+\varepsilon,x-\varepsilon y_2)\big)$.

We wish to apply dominated convergence to II_ε as it is written in (28). First we show that the integrand converges to zero almost everywhere (in particular for every $y_2 \neq\ 0$ and x such that $\gamma'(x)\ exists$): $For\ y_2 > 0$ it follows that $f_2(x-\varepsilon y_2)\gamma'(x-\varepsilon y_2) \to f_2(x)\gamma'(x-)\ as\ \varepsilon \to 0^+$. When $y_2 < 0$, it follows that $f_2(x-\varepsilon y_2)\gamma'(x-\varepsilon y_2) \to f_2(x)\gamma'(x+)\ as\ \varepsilon \to 0^+$. So either way, the limit exists for $y_2 \neq 0$ and almost every x. Now we show that $h_\varepsilon(x,y_2) \to 0$ for almost every $x, y_2 \in\ R$. For any $x \in$ R such that $\gamma'(x)$ exists and $y_2 \neq 0$, we compute

$$\lim_{\varepsilon\to 0} \varepsilon\gamma'(x)\, \tilde{F}_\Gamma(x, x-\varepsilon, x-\varepsilon y_2) \;=\; \lim_{\varepsilon\to 0} \frac{\gamma'(x)}{\left(\frac{(\gamma(x)-\gamma(x-\varepsilon))^2}{\varepsilon^2} + \frac{y_2^2(\gamma(x)-\gamma(x-\varepsilon y_2))^2}{(\varepsilon y_2)^2}\right)^{1/2}}$$

$$= \frac{\gamma'(x)}{(\gamma'(x)^2 + y_2^2\gamma'(x)^2)^{1/2}} = (1+y_2^2)^{-1/2}.$$

It follows that $\varepsilon\gamma'(x)\, \tilde{F}_\Gamma(x, x-\varepsilon, x-\varepsilon y_2) \to (1+y_2^2)^{-1/2}$ as well. Therefore $h_\varepsilon(x, y_2) \to 0$ as $\varepsilon \to 0$ for all $x, y_2 \in R$ such that $\gamma'(x)$ exists and $y_2 \neq 0$. Now in order to apply dominated convergence to (28), we need only to show that $h_\varepsilon(x, y_2)$ is integrable in y_2 independent of $\varepsilon. Define\ g_t = \gamma(x) - \gamma(x-t)$, which satisfies for all $s, t \in$ R

$$|g_t| = |\gamma(x) - \gamma(x-t)| \le \||\gamma'|\|_{L^\infty}|t| \le 2|t|$$

$$Re(g_s^2 + g_t^2) = Re[(\gamma(x)-\gamma(x-s))^2] + Re[(\gamma(x)-\gamma(x-t))^2]$$
$$\ge (1-\lambda)(s^2+t^2).$$

Also it is easy to verify that if $\omega = re^{i\theta}, \zeta = \rho e^{i\phi} \in C$ both have positive real part, i.e. $\theta, \phi \in (-\pi/2, \pi/2)$, then

$$|\sqrt{\omega} + \sqrt{\zeta}| \ge \sqrt{r}\, cos(\theta/2) + \sqrt{\rho} cos(\phi/2) \ge \sqrt{r\, cos(\theta)} + \sqrt{\rho\, cos(\phi)} = \sqrt{Re(\omega)} + \sqrt{Re(\zeta)}.$$

Here we use that $\sqrt{cos(\theta)} \le cos(\theta/2)\ for\ \theta \in (-\pi/2, \pi/2)$. Using these properties, we bound h_ε

$$|h_\varepsilon(x, y_2)| = \varepsilon \left| \frac{1}{\left(g_\varepsilon^2 + g_{\varepsilon y_2}^2\right)^{1/2}} - \frac{1}{\left(g_{-\varepsilon}^2 + g_{\varepsilon y_2}^2\right)^{1/2}} \right|$$

$$= \varepsilon \frac{|g_{-\varepsilon}^2 - g_\varepsilon^2|}{|g_{-\varepsilon}^2 + g_{\varepsilon y_2}^2|^{1/2} |g_\varepsilon^2 + g_{\varepsilon y_2}^2|^{1/2}} \frac{1}{\left|\left(g_\varepsilon^2 + g_{\varepsilon y_2}^2\right)^{1/2} + \left(g_{-\varepsilon}^2 + g_{\varepsilon y_2}^2\right)^{1/2}\right|}$$

$$\le \varepsilon \frac{|g_{-\varepsilon}|^2 + |g_\varepsilon|^2}{Re\left(g_{-\varepsilon}^2 + g_{\varepsilon y_2}^2\right)^{1/2} Re\left(g_\varepsilon^2 + g_{\varepsilon y_2}^2\right)^{1/2}} \frac{1}{[Re\left(g_\varepsilon^2 + g_{\varepsilon y_2}^2\right)]^{\frac{1}{2}} + Re[\left(g_{-\varepsilon}^2 + g_{\varepsilon y_2}^2\right)]^{1/2}}$$

$$\le \frac{\varepsilon}{(1-\lambda)} \frac{2(2\varepsilon)^2}{\varepsilon^2 + (\varepsilon y_2)^2} \frac{1}{2(1-\lambda)^{1/2}\,(\varepsilon^2 + (\varepsilon y_2)2)^{1/2}} \lesssim \frac{1}{(1-\lambda)^{3/2}} \frac{1}{(1+|y_2|)^3}$$

Therefore $|h_\varepsilon(x, y_2)| \lesssim (1+|y_2|)^{-3}$, and we can apply dominated convergence to (28). Hence $II_\varepsilon \to 0\ as\ \varepsilon \to 0$.

$\underline{III_\varepsilon \to 0}$: For this term, we use the regularity and compact support of f_1 to directly bound

$$|III_\varepsilon| \lesssim \int_{\varepsilon<|x-y_2|<|x|+R_0+\varepsilon} |\widetilde{F}_\Gamma(x, x+\varepsilon, y_2)|\, |f_1(x-\varepsilon) - f_1(x+\varepsilon)|f_2(y_2)|dy_2\,|$$

$$\leq \frac{1}{(1-\lambda)^{1/2}} \int_{\varepsilon<|x-y_2|<|x|+R_0+\varepsilon} \frac{1}{|x-y_2|} (2\varepsilon||\,f_1'||_{L^\infty})||\,f_2||_{L^\infty}\,dy_2$$

$$\lesssim \frac{||\,f_1'||_{L^\infty}\,||\,f_2||_{L^\infty}}{(1-\lambda)^{1/2}}\,\varepsilon|log(|x|+R_0+\varepsilon) - log(\varepsilon)|.$$

Recall we chose $R_0 > 0$ such that $supp(\,f_j\,) \subset B(0, R_0)\, for\, j = 1{,}2$. Hence $III_\varepsilon \to 0\; as\; \varepsilon \to 0$, and so

$$\lim_{\varepsilon\to 0} C_\varepsilon(x) = C(x) = -\int_{R^2} \tilde{F}_\Gamma(x, y_1, y_2)\, f_1'\,(y_1)\, f_2(y_2)\gamma'(x)\gamma'(y_2)dy_1\, dy_2\,,$$

which is an absolutely convergent integral. This verifies the absolutely convergent integral representation for $\tilde{C}_{\Gamma,1}$ in Proposition (3.2.7). It also follows from our estimate of I_ε that $C_\varepsilon(x)$ is bounded uniformly in *x*; hence for $f_0 \in C_0^\infty\,(R)$ and $\varepsilon > 0$

$$|C_\varepsilon(x)\, f_0(x)\gamma'(x)| \lesssim (1-\lambda)^{-1/2}||\,f_1'||_{L^\infty}||\,f_2||_{L^\infty}\, R_0|\,f_0(x)|,$$

and by dominated convergence

$$-\int_{R^3} \tilde{F}_\Gamma(x, y_1, y_2) f_0(x) f_1'(y_1) f_2(y_2)\gamma'(x)\gamma'(y_2)dxdy_1\, dy_2$$

$$= \lim_{\varepsilon\to 0}\int_R C_\varepsilon(x)\gamma'(x) f_0(x)dx = \int_R C(x)\, f_0(x)\gamma'(x)dx$$

$$= \langle \tilde{C}_{\Gamma,1}(\gamma' f_1, \gamma' f_2), \gamma' f_0\rangle$$

Furthermore since the bounds of $I_\varepsilon,\; II_\varepsilon,\; and\; III_\varepsilon$ are in terms of $||f_j||_{L^\infty}, ||f_j'||_{L^\infty}$, and R_0 for $j = 0{,}1{,}2$ it follows that $\tilde{C}_{\Gamma,1}$ is continuous from $\gamma' C_0^1(R) \times \gamma' C_0^1(R)$ into $(\gamma' C_0^1(R))'$. By symmetry, the properties of $\tilde{C}_{\Gamma,2}$ follow as well. Also

$\langle \tilde{C}_{\Gamma,0}(\gamma' f_1, \gamma' f_2), \gamma' f_0\rangle$

$$= \int_R \lim_{\varepsilon\to 0}\int_{|x-y_1|,|x-y_2|>\varepsilon} \widetilde{K}_{\Gamma,0}(x, y_1, y_2)\, f_0(x)\, f_1(y_1)\, f_2(y_2)\gamma'(x)\gamma'(y_1)\gamma'(y_2)dy_1\, dy_2\, dx$$

$$= \int_R \lim_{\varepsilon\to 0}\int_{|x-y_1|,|x-y_2|>\varepsilon} \partial_x \tilde{F}_\Gamma(x, y_1, y_2)\, f_0(x) f_1(y_1) f_2(y_2)\gamma'(y_1)\gamma'(y_2)dy_1\, dy_2\, dx$$

$$= \int_R \lim_{\varepsilon\to 0} \int_{|x-y_1|,|x-y_2|>\varepsilon} \partial y_1 \tilde{F}_\Gamma(x,y_1,y_2)\, f_0(x)\, f_1(y_1)\, f_2(y_2)\gamma'(y_1)\gamma'(y_2) dy_1\, dy_2\, dx$$

$$+ \int_R \lim_{\varepsilon\to 0} \int_{|x-y_1|,|x-y_2|>\varepsilon} \partial y_2 \tilde{F}_\Gamma(x,y_1,y_2)\, f_0(x)\, f_1(y_1)\, f_2(y_2)\gamma'(y_1)\gamma'(y_2) dy_1\, dy_2\, dx$$

$$= \langle \tilde{C}_{\Gamma,1}(\gamma' f_1, \gamma' f_2), \gamma' f_0\rangle + \langle \tilde{C}_{\Gamma,2}(\gamma' f_1, \gamma' f_2), \gamma' f_0\rangle$$

By the absolutely integrable representations of $\tilde{C}_{\Gamma,1}$ and $\tilde{C}_{\Gamma,2}$, it follows that

$$\langle \tilde{C}_{\Gamma,0}(\gamma' f_1, \gamma' f_2), \gamma' f_0\rangle = \langle \tilde{C}_{\Gamma,1}(\gamma' f_1, \gamma' f_2), \gamma' f_0\rangle + \langle \tilde{C}_{\Gamma,2}(\gamma' f_1, \gamma' f_2), \gamma' f_0\rangle$$

$$= -\int_{R^3} \tilde{F}_\Gamma(x,y_1,y_2)\, f_0(x)\, f_1'(y_1)\, f_2(y_2)\gamma'(x)\gamma'(y_2) dx dy_1\, dy_2$$

$$- \int_{R^3} \tilde{F}_\Gamma(x,y_1,y_2)\, f_0(x)\, f_1(y_1)\, f_2'(y_2)\gamma'(x)\gamma'(y_1) dx dy_1\, dy_2$$

$$= \lim_{\varepsilon\to 0} \int_{|x-y_1|,|x-y_2|>\varepsilon} \partial y_1 \tilde{F}_\Gamma(x,y_1,y_2)\, f_0(x)\, f_1(y_1)\, f_2(y_2)\gamma'(x)\gamma'(y_2) dx dy_1\, dy_2$$

$$+ \int_{|x-y_1|,|x-y_2|>\varepsilon} \partial_{y_2} \tilde{F}_\Gamma(x,y_1,y_2) f_0(x) f_1(y_1) f_2(y_2)\gamma'(x)\gamma'(y_1) dx dy_1\, dy_2$$

$$+ \int_{|x-y_2|>\varepsilon} \tilde{F}_\Gamma(x,x+\varepsilon,y_2) f_0(x) f_1(x+\varepsilon) f_2(y_2)\gamma'(x)\gamma'(y_2) dx\, dy_2$$

$$- \int_{|x-y_2|>\varepsilon} \tilde{F}_\Gamma(x,x-\varepsilon,y_2) f_0(x) f_1(x-\varepsilon) f_2(y_2)\gamma'(x)\gamma'(y_2) dx\, dy_2$$

$$+ \int_{|x-y_1|>\varepsilon} \tilde{F}_\Gamma(x,y_1,x+\varepsilon) f_0(x) f_1(y_1) f_2(x+\varepsilon)\gamma'(x)\gamma'(y_1) dx\, dy_1$$

$$- \int_{|x-y_1|>\varepsilon} \tilde{F}_\Gamma(x,y_1,x-\varepsilon) f_0(x) f_1(y_1) f_2(x-\varepsilon)\gamma'(x)\gamma'(y_1) dx\, dy_1$$

$$= \lim_{\varepsilon\to 0} \int_{|x-y_1|,|x-y_2|>\varepsilon} \partial_x \tilde{F}_\Gamma(x,y_1,y_2)\, f_0(x)\, f_1(y_1)\, f_2(y_2)\gamma'(y_1)\gamma'(y_2) dx dy_1\, dy_2 .$$

The boundary terms of the integration by parts here (the last 4 terms) tend to zero as $\varepsilon \to 0$ in the same way as they did for II_ε and III_ε above. Now we will integrate by parts one more time in *x* here to obtain an integral representation for $C_{\Gamma,0}$:

$$\lim_{\varepsilon\to 0}\int_{|x-y_1|,|x-y_2|>\varepsilon} \partial_x \tilde{F}_\Gamma(x,y_1,y_2)\, f_0(x)\, f_1(y_1)\, f_2(y_2)\gamma'(y_1)\gamma'(y_2)dxdy_1\, dy_2$$

$$= \lim_{\varepsilon\to 0} -\int_{|x-y_1|,|x-y_2|>\varepsilon} \tilde{F}_\Gamma(x,y_1,y_2) f_0'(x)\, f_1(y_1)\, f_2(y_2)\gamma'(y_1)\gamma'(y_2)dxdy_1\, dy_2$$

$$+\int_{|y_1-y_2-\varepsilon|>\varepsilon} \tilde{F}_\Gamma(y_1-\varepsilon,y_1,y_2)\;\; f_0(y_1-\varepsilon)\, f_1(y_1)\, f_2(y_2)\gamma'(y_1)\gamma'(y_2)dy_1\, dy_2$$

$$-\int_{|y_1-y_2+\varepsilon|>\varepsilon} \tilde{F}_\Gamma(y_1\;+\varepsilon,y_1,y_2)\, f_0(y_1+\varepsilon)\, f_1(y_1)\, f_2(y_2)\gamma'(y_1)\gamma'(y_2)dy_1\, dy_2$$

$$+\int_{|y_1-y_2+\varepsilon|>\varepsilon} \tilde{F}_\Gamma(y_2-\varepsilon,y_1,y_2)\, f_0(y_2-\varepsilon)\, f_1(y_1)\, f_2(y_2)\gamma'(y_1)\gamma'(y_2)dy_1\, dy_2$$

$$-\int_{|y_1-y_2-\varepsilon|>\varepsilon} \tilde{F}_\Gamma(y_2\;+\varepsilon,y_1,y_2)\, f_0(y_2+\varepsilon)\, f_1(y_1)\, f_2(y_2)\gamma'(y_1)\gamma'(y_2)dy_1\, dy_2$$

$$= \lim_{\varepsilon\to 0} -\int_{|x-y_1|,|x-y_2|>\varepsilon} |\tilde{F}_\Gamma(x,y_1,y_2)\, f_0'\,(x)\, f_1(y_1)f_2(y_2)\gamma'(y_1)\gamma'(y_2)dxdy_1\, dy_2$$

$$+\int_{|y_1-y_2|>\varepsilon} \tilde{F}_\Gamma(y_1,y_1+\varepsilon,y_2)\, f_0(y_1)\, f_1(y_1+\varepsilon)\, f_2(y_2)\gamma'(y_1)\gamma'(y_2)dy_1\, dy_2$$

$$-\int_{|y_1-y_2|>\varepsilon} \tilde{F}_\Gamma(y_1,y_1-\varepsilon,y_2)\, f_0(y_1)\, f_1(y_1-\varepsilon)\, f_2(y_2)\gamma'(y_1)\gamma'(y_2)dy_1\, dy_2$$

$$+\int_{|y_1-y_2|>\varepsilon} \tilde{F}_\Gamma(y_2,y_1,y_2\;+\varepsilon)\, f_0(y_2)\, f_1(y_1)\, f_2(y_2\;+\varepsilon)\gamma'(y_1)\gamma'(y_2)dy_1\, dy_2$$

$$-\int_{|y_1-y_2|>\varepsilon} \tilde{F}_\Gamma(y_2,y_1,y_2-\varepsilon)\, f_0(y_2)\, f_1(y_1)\, f_2(y_2-\varepsilon)\gamma'(y_1)\gamma'(y_2)dy_1\, dy_2$$

$$= -\int_{R^3} \tilde{F}_\Gamma(x,y_1,y_2)\, f_0'(x)\, f_1(y_1)\, f_2(y_2)\gamma'(y_1)\gamma'(y_2)dxdy_1\, dy_2\,.$$

Once again we use the same argument for the II_ε and III_ε terms to verify that these boundary terms (the last 4 terms) tend to zero as $\varepsilon\to 0$. Then the pairing identity for $\tilde{C}_{\Gamma,0}$ holds as well. This completes the proof of Proposition (3.2.7).

In the next proposition we extend $\tilde{C}_\Gamma$ and C_Γ to product Lebesgue spaces.

Proposition (3.2.8)[3]: *Let L be a Lipschitz function with Lipschitz constant* $\lambda < 1$ *such that for almost every* $x \in$ R *the limits*

$$\lim_{\varepsilon\to 0+} \gamma'(x+\varepsilon) = \gamma'(x+) \quad \text{and} \quad \lim_{\varepsilon\to 0+} \gamma'(x-\varepsilon) = \gamma'(x-)$$

exist. If L id differentiable off of some compact set and there exists $c_0 \in R$ *such that*

$$\lim_{|x|\to\infty} L'(x) = c_0 ,$$

then $\tilde{C}_{\Gamma,j}$ *is bounded* $L^{p_1}(R) \times L^{p_2}(R)$ *into* $L^p(R)$ *for all* $1 < p_1, p_2 < \infty$ *satisfying* (2) *for each* $j = 0,1,2$. *Furthermore,* $C_{\Gamma,j}$ *is bounded* $L^{p_1}(\Gamma) \times L^{p_2}(\Gamma)$ *into* $L^p(\Gamma)$ *for all*$1 < p_1, p_2 < \infty$ *satisfying* (2) *for each* $j = 0,1,2$.

Proof. We will apply Theorem (3.1.1) to $\tilde{C}_{\gamma,1}$ with $b_0 = b_1 = b_2 = \gamma'$. Note that $\gamma'$is para-accretive since $Re(\gamma') = 1$ and $\gamma' \in L^\infty$. It is not hard to see that $\widetilde{K}_{\Gamma,1}$ is the kernel function associated to $\tilde{C}_{\Gamma,1}$. It also follows from $||L'||_{L^\infty} = \lambda < 1$ that $\widetilde{K}_{\Gamma,1}$ is a standard bilinear kernel:

$$|\widetilde{K}_{\Gamma,1}(x,y_1,y_2)| \le \frac{1}{(1-\lambda)^{3/2}} \frac{|\gamma(x)-\gamma(y_1)|}{|(x-y_1)^2+(x-y_2)^2|^{3/2}}$$
$$\le \frac{1}{(1-\lambda)^{3/2}} \frac{||\gamma'||_{L^\infty}}{(x-y_1)^2+(x-y_2)^2},$$

and

$$\partial_{y_2}\widetilde{K}_{\Gamma,1}(x,y_1,y_2) \le \frac{3}{(1-\lambda)^{5/2}} \frac{|\gamma(x)-\gamma(y_1)||\gamma(x)-\gamma(y_2)||\gamma'(y_2)|}{((x-y_1)^2+(x-y_2)^2)^{5/2}}$$
$$\le \frac{3}{(1-\lambda)^{5/2}} \frac{||\gamma'||_{L^\infty}^3}{((x-y_1)^2+(x-y_2)^2)^{3/2}}.$$

A simliar estimate holds for $\partial_x\widetilde{K}_{\Gamma,1}(x,y_1,y_2)$ and $\partial_{y_1}\widetilde{K}_{\Gamma,1}(x,y_1,y_2)$, which implies that $\widetilde{K}_{\Gamma,1}(x,y_1,y_2)$ is a standard bilinear kernel. Now it remains to verify that $\tilde{C}_{\Gamma,1}$ satisfies the WBP and the BMO testing conditions for $b_0 = b_1 = b_2 = \gamma'$. Let ϕ_0, ϕ_1, ϕ_2 be normalized bumps of order 1, $u \in$ R, and $R > 0$. By Proposition (3.2.7), we have

$$\left|\langle \tilde{C}_{\Gamma,1}\left(\gamma'\phi_1^{u,R}, \gamma'\phi_2^{u,R}\right), \gamma'\phi_0^{u,R}\rangle\right|$$
$$\le \int_{R^3} \left|\tilde{F}_\Gamma(x,y_1,y_2)\phi_0^{u,R}(x)\left(\phi_1^{u,R}\right)'(y_1)\phi_2^{u,R}(y_2)\gamma'(x)\gamma'(y_2)\right| dx dy_1\, dy_2$$
$$\lesssim \frac{1}{R(1-\lambda)^{1/2}} \int_{[R,R]^3} \frac{dx dy_1\, dy_2}{|x-y_1|+|x-y_2|} \lesssim \frac{R}{(1-\lambda)^{1/2}}$$

So $\tilde{C}_{\Gamma,1}$ satisfies the WBP. Now we check the three *BMO* conditions of Theorem (3.1.1):

$\underline{M_{\gamma'}\tilde{C}_{\Gamma,1}(\gamma',\gamma') = 0 \in BMO}$: Let $\phi \in C_0^\infty(R)$ such that $\gamma'\phi$ has mean zero and $\eta \in C_0^\infty(R)$ such that $0 \le \eta \le 1$, $\eta = 1$ on $[-1,1]$, $supp(\eta) \subset [-2,2]$, and $\eta_R(x) = \eta(x/R)$. Again we use Proposition (3.2.7) and make a change of variables,

$$\begin{aligned}
&\langle \tilde{C}_{\Gamma,1}(\gamma'\eta_R, \gamma'\eta_R), \gamma'\phi\rangle \\
&\qquad = -\int_{R^3} \tilde{F}_\Gamma(x, y_1, y_2)(\eta_R)'(y_1)\gamma'(y_2)\eta_R(y_2)\gamma'(x)\phi(x)dy_1\, dy_2\, dx \\
&\qquad = -\int_{R^3} R\tilde{F}_\Gamma(x, Ry_1, Ry_2)\phi(x)\eta'(y_1)\eta(y_2)\gamma'(x)\gamma'(Ry_2)dxdy_1dy_2.
\end{aligned} \tag{29}$$

Then for $y_1, y_2 \neq 0$, we can compute the pointwise limit

$$\begin{aligned}
&\lim_{R\to\infty} R\tilde{F}_\Gamma(x, Ry_1, Ry_2)\gamma'(Ry_2) \\
&\qquad = \lim_{R\to\infty} \frac{R\gamma'(Ry_2)}{((\gamma(x) - \gamma(Ry_1))^2 + (\gamma(x) - \gamma(Ry_2))^2)^{1/2}} \\
&\qquad = \lim_{R\to\infty} \frac{\gamma'(Ry_2)}{\left(y_1^2 \frac{(\gamma(x)-\gamma(Ry_1))^2}{(Ry_1)^2} + y_2^2 \frac{(\gamma(x)-\gamma(Ry_2))^2}{(Ry_2)^2}\right)^{1/2}} \\
&\qquad = \frac{1 + ic_0}{(y_1^2(1 + ic_0)^2 + y_2^2(1 + ic_0)^2)^{1/2}} = \frac{1}{(y_1^2 + y_2^2)^{1/2}}.
\end{aligned}$$

Here we use that L is differentiable off of a compact set, that $L'(x) \to c_0$ as $|x| \to \infty$. And L'Hospital's rule to conclude that $L(x)/x \to c_0$ as $|x| \to \infty$. Now let $R > 0$ be large enough so that $supp(\phi) \subset B(0, R/4)$, and using that $supp(\eta') \subset [-2,2]\backslash[-1,1]$, we have the estimate

$$\begin{aligned}
|R\tilde{F}_\Gamma(x, Ry_1, Ry_2)\gamma'(x)\gamma'(Ry_2)\phi(x)\eta'(y_1)\eta(y_2)| &\lesssim \frac{1}{(1-\lambda)^{1/2}} \frac{R|\phi(x)\eta'(y_1)|}{|x - Ry_1| + |x - Ry_2|} \\
&\le \frac{1}{(1-\lambda)^{1/2}} \frac{R|\phi(x)\eta'(y_1)|}{R|y_1|/2 + R/2 - 2|x| + R|y_2|} \lesssim \frac{1}{(1-\lambda)^{1/2}} \frac{1}{|y_1| + |y_2|}.
\end{aligned}$$

Then by dominated convergence

$$\lim_{R\to\infty} \langle \tilde{C}_{\Gamma,1}(\gamma'\eta_R, \gamma'\eta_R), \gamma'\phi\rangle = \int_R \left(-\int_{R^2} \frac{\eta'(y_1)\eta(y_2)}{(y_1^2 + y_2^2)^{1/2}} dy_1 dy_2\right) \phi(x)\gamma'(x)dx = 0.$$

Here we also use that $\gamma'\phi$ has mean zero. Therefore $M_{\gamma'}\tilde{C}_{\Gamma,1}(\gamma',\gamma') = 0 \in BMO$ in the sense of Definition (3.1.7).

$\underline{M_{\gamma'}\tilde{C}_{\Gamma}^{*1}(\gamma',\gamma') = 0 \in BMO}$: Note that for every $x, y_1, y_2 \in R$ such that $|x-y_1|+|x-y_2| \neq 0$ we can write

$$\widetilde{K}_{\Gamma,1}(x,y_1,y_2)\gamma'(x)\gamma'(y_1)\gamma'(y_2) = \partial_{y_1}\tilde{F}_{\Gamma,1}(x,y_1,y_2)\gamma'(x)\gamma'(y_2)$$
$$= \partial_x\tilde{F}_{\Gamma,1}(x,y_1,y_2)\gamma'(y_1)\gamma'(y_2) - \partial_{y_2}\tilde{F}_{\Gamma,1}(x,y_1,y_2)\gamma'(x)\gamma'(y_1)$$
$$= (\widetilde{K}_{\Gamma,0}(x,y_1,y_2) - \widetilde{K}_{\Gamma,2}(x,y_1,y_2))\gamma'(x)\gamma'(y_1)\gamma'(y_2).$$

Then it follows that $M_{\gamma'}\tilde{C}_{\Gamma,1}(M_{\gamma'}\cdot, M_{\gamma'}\cdot) = M_{\gamma'}\tilde{C}_{\Gamma,0}(M_{\gamma'}\cdot, M_{\gamma'}\cdot) - M_{\gamma'}\tilde{C}_{\Gamma,2}(M_{\gamma'}\cdot, M_{\gamma'}\cdot)$, and so by Proposition (3.2.7)

$$\langle\tilde{C}_{\Gamma,1}^{*1}(\gamma'\eta_R,\gamma'\eta_R),\gamma'\phi\rangle = \langle\tilde{C}_{\Gamma,1}(\gamma'\phi,\gamma'\eta_R),\gamma'\eta_R\rangle$$
$$= \langle\tilde{C}_{\Gamma,0}(\gamma'\phi,\gamma'\eta_R),\gamma'\eta_R\rangle - \langle\tilde{C}_{\Gamma,2}(\gamma'\phi,\gamma'\eta_R),\gamma'\eta_R\rangle$$
$$= -\int_{R^3}\tilde{F}_{\Gamma}(x,y_1,y_2)\gamma'(y_1)\phi(y_1)\gamma'(y_2)\eta_R(y_2)(\eta_R)'(x)dy_1\,dy_2\,dx$$
$$+\int_{R^3}\tilde{F}_{\Gamma}(x,y_1,y_2)\gamma'(y_1)\phi(y_1)(\eta_R)'(y_2)\gamma'(x)\eta_R(x)dy_1\,dy_2\,dx\,.$$

These two expressions tend to zero by the same argument that (29) tends to zero as $R\to\infty$ in the proof of the $M_{\gamma'}\tilde{C}_{\Gamma,1}(\gamma',\gamma') = 0$ condition. Therefore $M_{\gamma'}\tilde{C}_{\Gamma}^{*1}(\gamma',\gamma') = 0 \in BMO$ as well.

$\underline{M_{\gamma'}\tilde{C}_{\Gamma}^{*2}(\gamma',\gamma') = 0 \in BMO}$: By Proposition (3.2.7), we can compute

$$\langle\tilde{C}_{\Gamma,1}^{*2}(\gamma'\eta_R,\gamma'\eta_R),\gamma'\phi\rangle$$
$$= -\int_{R^3}\tilde{F}_{\Gamma}(x,y_1,y_2)(\eta_R)'(y_1)\gamma'(x)\eta_R(x)\gamma'(y_2)\phi(y_2)dy_1dy_2\,dx.$$

Again, this expression is essentially the same as the one in (29), and hence tends to zero as $R\to\infty$ by the argument. Therefore $M_{\gamma'}\tilde{C}_{\Gamma}^{*2}(\gamma',\gamma') = 0 \in BMO$.

Then by Theorem (3.1.1), $\tilde{C}_{\Gamma,1}$ can be extended to a bounded operator from $L^{p_1}\times L^{p_2}$ into L^p for appropriate p, p_1, p_2. Now it is easy to prove that $C_{\Gamma,1}$ can also be extended to a bounded operator: Let $1<p_1,p_2<\infty$ and $1/2<p<\infty$ satisfy (2). For $g_1 \in L^{p_1}(\Gamma)$ and

$g_2 \in L^{p_2}(\Gamma)$, and it follows that

$$||C_{\Gamma,1}(g_1, g_2)||_{L^p(\Gamma)} = \left(\int_R |C_{\Gamma,1}(g_1, g_2)(\gamma(x))|^p |\gamma'(x)| dx\right)^{1/p}$$

$$\leq ||\gamma'||_{L^\infty}^{1/p} || \tilde{C}_{\Gamma,1}||_{p,p_1,p_2} ||g_1 \circ \gamma^{-1}||_{L^{p_1}(R)} ||g_2 \circ \gamma^{-1}||_{L^{p_2}(R)}$$

$$= ||\gamma'||_{L^\infty}^{1/p} ||\gamma'^{-1}||_{L^\infty}^{1/p} || \tilde{C}_{\Gamma,1}||_{p,p_1,p_2} ||g_1||_{L^{p_1}(\Gamma)} ||g_2||_{L^{p_2}(\Gamma)}.$$

The bounds for $\tilde{C}_{\Gamma,0}, \tilde{C}_{\Gamma,2}, C_{\Gamma,0}$, and $C_{\Gamma,2}$ follow in the same way.

Chapter 4

The L(logL)$^{\epsilon}$ Endpoint Estimate

We show the following estimate for the maximal operator T^* associated to the singular integral operator T:

$$\|T^* f\|_{L^{1,\infty}(w)} \lesssim \frac{1}{\epsilon}\int_{R^n} |f(x)| M_{L(\log L)^{\epsilon}}(w)(x)\, dx\,, \quad w \geq 0, \qquad 0 < \epsilon \leq 1.$$

This follows from the sharp L^P estimate

$$\|T^* f\|_{L^p(w)} \lesssim p'(\frac{1}{\delta})^{1/p'} \|f\|_{L^p\left(M_{L(\log L)^{p-1+\delta}}(w)\right)}, \qquad 1 < p < \infty, w \geq 0, 0 < \delta \leq 1.$$

As a consequence we deduce that

$$\|T^* f\|_{L^{1,\infty}(w)} \lesssim [w]_{A_1} \log\left(e + [w]_{A_\infty}\right) \int_{R^n} |f| w dx,$$

Section (4.1): Basic Definition and Dyadic Theory

The Muckenhoupt–Wheeden conjecture has been disproved by Reguera and Thiele. This conjecture claimed that there exists a constant c such that for any function f and any weight w (i.e. a nonnegative locally integrable function), there holds

$$\|Hf\|_{L^{1,\infty}(w)} \leq c \int_{R} |f| Mw dx \quad , \qquad \textbf{(1)}$$

where H is the Hilbert transform. The failure of the conjecture was previously obtained by M.C. Reguera for a special model operator T instead of H. This conjecture was motivated by a similar inequality by C. Fefferman and E. Stein for the Hardy–Littlewood maximal function:

$$\|Mf\|_{L^{1,\infty}(w)} \leq c \int_{R^n} |f| Mw\, dx. \qquad \textbf{(2)}$$

The importance of this result stems from the fact that it was a central piece in the approach by Fefferman–Stein to derive the following vector-valued extension of the classical L^p Hardy–Littlewood maximal theorem: for every $1 < p, q < \infty$, there is a finite constant $c = c_{p,q}$ such that

$$\left\|\left(\sum_j (Mf_j)^q\right)^{\frac{1}{q}}\right\|_{L^p(R^n)} \le c\left\|\left(\sum_j |f_j|^q\right)^{\frac{1}{q}}\right\|_{L^p(R^n)}. \quad \textbf{(3)}$$

This is a very deep theorem and has been used a lot in modern harmonic analysis explaining the central role of inequality (2).

Inequality (1) was conjectured by B. Muckenhoupt and R. Wheeden during the 70's. That this conjecture was believed to be false was already mentioned where the best positive result in this direction so far can be found, and where M is replaced by $M_{L(logL)^{\epsilon}}$, i.e., a maximal type operator that is "ϵ-logarithmically" bigger than M:

$$\|Tf\|_{L^{1,\infty}(w)} \le c_\varepsilon \int_{R^n} |f|_{ML(log\,L)^{\varepsilon}}(w)dx \qquad w \ge 0,$$

Theorem (4.1.1)[4]: *Let* T *be a Calderón–Zygmund operator with maximal singular integral operator* T^*. *Then for any* $0 < \epsilon \le 1$,

$$\|T^* f\|_{L^{1,\infty}(w)} \lesssim \frac{c_T}{\epsilon}\int_{R^n} |f(x)| M_{L(log\,L)^{\epsilon}}(w)(x)\,dx \qquad w \ge 0 \quad \textbf{(4)}$$

If we formally optimize this inequality in ϵ we derive the following conjecture:

$$\|T^* f\|_{L^{1,\infty}(w)} \le c_T \int_{R^n} |f(x)| M_{L\,log\,log\,L}(w)(x)\,dx \qquad w \ge 0, f \in L_c^{\infty}(R^n). \quad \textbf{(5)}$$

To prove Theorem (4.1.1) we need first an L^p version of this result, which is fully sharp, at least in the logarithmic case. The result will hold for all $p \in (1,\infty)$but for proving Theorem (4.1.1) we only need it when p is close to one.

There are two relevant properties that will be used (see Lemma (4.2.2). The first one establishes that for appropriate A and all $\gamma \in (0,1)$, we have $(M_A f)^{\gamma} \in A_1$with constant $[(M_A f)^{\gamma}]_{A_1}$independent of A and *f*. The second property is that $M_{\bar{A}}$ is a bounded operator on $L^{p'}(R^n)$ where $\bar{A}$ is the complementary Young function of A. The main example is $A(t) = t^p(1 + log^+ t)^{p-1+\delta},\ p \in (1,\infty),\ \delta \in (0,\infty)$ since

$$\|M_{\bar{A}}\|_{B(L^{p'}(R^n))} \lesssim p^2(\frac{1}{\delta})^{1/p'}$$

by (25).

Theorem (4.1.2)[4]: *Let 1 < p < ∞ and let* A *be a Young function, then*

$$\|T^* f\|_{L^p(w)} \leq c_T p' \|M_{\bar{A}}\|_{B(Lp'(R^n))} \|f\|_{L^p(MA(w^{1/p})^p)} \quad w \geq 0. \tag{6}$$

In the particular case $A(t) = t^p(1 + log^+ t)^{p-1+\delta}$ *we have*

$$\|T^* f\|_{L^p(w)} \leq c_T p' p^2 (\frac{1}{\delta})^{1/p'} \|f\|_{L^p\left(M_{L(log\,L)^{p-1+\delta}}(w)\right)} \quad w \geq 0, \quad 0 < \delta \leq 1.$$

Another worthwhile example is given by $M_{L(log\,L)^{p-1}(loglogL)^{p-1+\delta}}$ instead of $M_{L(logL)^{p-1+\delta}}$ for which:

$$\|T^* f\|_{L^p(w)} \leq c_T p' p^2 (\frac{1}{\delta})^{1/p'} \|f\|_{L^p\left(M_{L(log\,L)^{p-1}(log\,log\,L)^{p-1+\delta}}(w)\right)} \quad w \geq 0, \quad 0 < \delta \leq 1.$$

There are some interesting consequences from Theorem (4.1.1), the first one is related to the one weight theory.

Definition (4.1.3)[4]:

$$[w]_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q)\, dx\,.$$

Observe that $[w]_{A_\infty} \geq 1$ by the Lebesgue differentiation theorem.

When specialized to weights $w \in A_\infty$ or $w \in A_1$, Theorem (4.1.1) yields the following corollary. It was formerly known for the linear singular integral T, and this was used in the proof, which proceeded via the adjoint of T; the novelty in the corollary below consists in dealing with the maximal singular integral T^* .

Corollary (4.1.4)[4]:

$$\|T^* f\|_{L^{1,\infty}(w)} \lesssim log(e + [w]_{A_\infty}) \int_{R^n} |f| Mw dx\ , \tag{7}$$

and hence

$$\|T^* f\|_{L^{1,\infty}(w)} \lesssim [w]_{A_1} \log(e + [w]_{A_\infty}) \int_{R^n} |f| w dx. \qquad \textbf{(8)}$$

Proof. To apply (4), we use $\log t \leq \frac{t^\alpha}{\alpha} for\ t > 1$ and $\alpha > 0$ to deduce that

$$M_{L(\log L)^\epsilon}(w) \lesssim \frac{1}{\alpha^\epsilon} M_{L^{1+\epsilon\alpha}}(w)$$

Hence, if $w \in A_\infty$ we can choose αsuch that $\alpha\epsilon = \frac{1}{\tau_n [w]_{A_\infty}}$. Then, applying Theorem (4.1.5)

$$\frac{1}{\epsilon} M_{L(\log L)^\epsilon}(w) \lesssim \frac{1}{\epsilon} (\epsilon\tau\, [w]_{A_\infty})^\epsilon M_{L^{r_w}}(w) \lesssim \frac{1}{\epsilon} [w]_{A_\infty}^\epsilon M(w)$$

and optimizing with $\epsilon \approx 1/\log(e + [w]_{A_\infty})$ we obtain (7).

Theorem (4.1.5)[4]: *Let $w \in A_\infty$, then there exists a dimensional constant τ_n such that*

$$\left(⨍_Q w^{r_w}\right)^{1/r_w} \leq 2⨍_Q w$$

where

$$r_w = 1 + \frac{1}{\tau_n [w]_{A_\infty}}$$

Corollary (4.1.6)[4]: *Let* u, σ *be a pair of weights and let $p \in (1, \infty)$. We also let $\delta, \delta_1, \delta_2 \in (0, 1]$. Then:*

(a) *If*

$$K = \sup_Q \|u^{1/p}\|_{L^p(\log L)^{p-1+\delta}, Q} \left(\frac{1}{|Q|} \int_Q \sigma\, dx\right)^{1/p'} < \infty, \qquad \textbf{(9)}$$

then

$$\|T^* (f\sigma)\|_{L^{p,\infty}(u)} \lesssim \frac{1}{\delta} K \left(\frac{1}{\delta}\right)^{1/p'} \|f\|_{L^p(\sigma)} \qquad \textbf{(10)}$$

(The boundedness in the case $\delta = 0$ is false)

(b) *As consequence*, *if*

$$K = \sup_Q \left\| u^{\frac{1}{p}} \right\|_{L^p(\log L)^{p-1+\delta_1},Q} \left(\frac{1}{|Q|} \int_Q \sigma \, dx \right)^{1/p'}$$

$$+ \sup_Q \left(\frac{1}{|Q|} \int_Q u \, dx \right)^{\frac{1}{p}} \left\| \sigma^{1/p'} \right\|_{L^{p'}(\log L)^{p'-1+\delta_2},Q} < \infty, \qquad \textbf{(11)}$$

then

$$\|T^*(f\sigma)\|_{L^p(u)} \lesssim K \left(\frac{1}{\delta_1} \left(\frac{1}{\delta_1} \right)^{\frac{1}{p'}} + \frac{1}{\delta_2} \left(\frac{1}{\delta_2} \right)^{\frac{1}{p}} \right) \|f\|_{L^p(\sigma)}. \qquad \textbf{(12)}$$

We don't know whether the factors $\frac{1}{\delta^i}, i = 1, 2$ can be removed or improved from the estimate (12). Perhaps the method is not so precise to prove the conjecture formulated. However, it is clear from the arguments that these factors are due to the appearance of the factor $\frac{1}{\epsilon}$ in (4).

By a Calderón–Zygmund operator we mean a continuous linear operator $T: C_0^\infty(R^n) \to \acute{D}(R^n)$ that extends to a bounded operator on $L^2(R^n)$, and whose distributional kernel K coincides away from the diagonal $x = y \; in \; R^n \times R^n$ with a function *K* satisfying the size estimate

$$|K(x,y)| \leq \frac{c}{|x - y|^n}$$

and the regularity condition: for some $\varepsilon > 0$,

$$|K(x,y) - K(z,y)| + |K(y,x) - K(y,z)| \leq c \frac{|x - z|^\varepsilon}{|x - y|^{n+\varepsilon}},$$

Whenever $2|x - z| < |x - y|$, and so that

$$Tf(x) = \int_{R^n} K(x,y) f(y) dy,$$

Whenever $f \in C_0^\infty(R^n) and \; x \notin supp(f)$.

Also we will denote by T^* the associated maximal singular integral:

$$T^*f(x) = \sup_{\varepsilon>0}\left|\int_{|y-x|>\varepsilon} K(x,y)f(y)dy\right| \qquad f \in C_0^\infty(R^n)$$

A Young function is a convex, increasing function $A : [0,\infty) \to [0,\infty) with\ A(0) = 0$, such that A(t) → ∞ as t → ∞. Such a function is automatically continuous. From these properties it follows that $A : [t_0,\infty) \to [0,\infty)$ is a strictly increasing bijection, where $t_0 = sup\{t \in [0,\infty) : A(t) = 0\}$. Thus $A^{-1}(t)$ is well-defined (single-valued) for t > 0, but in general it may happen that $A^{-1}(0) = [0, t_0]$ is an interval.

The properties of A easily imply that for $0 < \varepsilon < 1\ and\ t \geq 0$

$$A(\varepsilon\, t) \leq \varepsilon A(t)\,. \tag{13}$$

The A-norm of a function f over a set E with finite measure is defined by

$$\|f\|_{A,E} = \|f\|_{A(L),E} = inf\{\lambda > 0 : ⨍_E A\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1\}$$

where as usual we define the average of f over a set E, $⨍_E f = \frac{1}{|E|}\int_E f\, dx$.

In many situations the convexity does not play any role and basically the monotonicity is the fundamental property. The convexity is used for proving that $\| \quad \|_{A,E}$ is a norm which is often not required.

We will use the fact that

$$\|f\|_{A,E} \leq 1 \ \ if\ and\ only\ if\ ⨍_E A(|f(x)|)\, dx \leq 1. \tag{14}$$

Associated with each Young function A, one can define a complementary function

$$\bar{A}(s) = \sup_{t>0}\{st - A(t)\} \qquad s \geq 0. \tag{15}$$

Then $\bar{A}$ is finite-valued if and only if $lim_{t\to\infty}A(t)/t = sup_{t>0}A(t)/t = \infty$, which we henceforth assume; otherwise, $\bar{A}(s) = \infty$ for all $s > sup_{t>0}A(t)/t$. Also, $\bar{A}$ is strictly increasing on [0, ∞) if and only if $lim_{t\to 0}A(t)/t = inf_{t>0}A(t)/t = 0$; otherwise $\bar{A}(s) = 0$ for all $s \leq inf_{t>0}A(t)/t$.

Such $\bar{A}$ is also a Young function and has the property that

$$st \leq A(t) + \bar{A}(s), \qquad t, s \geq 0, \tag{16}$$

and also

$$t \leq A^{-1}(t)\bar{A}^{-1}(t) \leq 2\,t, \qquad t > 0. \tag{17}$$

The main property is the following generalized Hölder's inequality

$$\frac{1}{|E|}\int_E |fg|dx \leq 2\|f\|_{A,E}\|g\|_{\bar{A},E}. \tag{18}$$

As we already mentioned, the following Young functions play a main role in the theory:

$$A(t) = t^p(1 + log^+ t)^{p-1+\delta} \qquad t, \delta > 0, p > 1.$$

Given a Young function A or more generally any positive function *A(t)* we define the following maximal operator

$$M_{A(L)}f(x) = M_A f(x) = \sup_{Q \ni x} \|f\|_{A,Q}.$$

This operator satisfies the following distributional type estimate: there are finite dimensional constants c_n, d_n such that

$$|\{x \in R^n : M_A f(x) > t\}| \leq c_n \int_{R^n} A\left(d_n \frac{f}{t}\right) dx \quad f \geq 0, t > 0 \tag{19}$$

A first consequence of this estimate is the following L^p estimate of the operator. A second application will be used in the proof of Lemma (4.2.2).

Lemma (4.1.7)[4]: *Let* A *be a Young function, then*

$$\|M_A\|_{B(Lp(R^n))} \leq c_n \alpha_p(A) \tag{20}$$

where $\alpha_p(A)$ *is the following tail condition that plays a central role in the sequel*

$$\alpha_p(A) = \left(\int_1^{\infty} \frac{A(t)}{t^p}\frac{dt}{t}\right)^{1/p} < \infty. \qquad \textbf{(21)}$$

Examples of functions satisfying condition (21) are $A(t) = t^q, 1 \leq q < p$. More interesting examples are given by

$$A(t) = \frac{t^p}{(1 + log^+ t)^{1+\delta}} \;\; A(t) \approx t^p log(t)^{-1} log\, log(t)^{-(1+\delta)}, \qquad p > 1, \delta > 0.$$

Often we need to consider instead of the function *A* in (21) the complementary $\bar{A}$.

We also record a basic estimate between a Young function and its derivative:

$$A(t) \leq t\acute{A}(t) \qquad \textbf{(22)}$$

which holds for any $t \in (0, \infty)$ such that A' (t) does exist.

There is the following useful alternative estimate of (20) that will be used in the sequel. We would like to stress the fact that we avoid the doubling condition on the Young functions B and $\bar{B}$, which is important in view of the quantitative applications to follow: even if our typical Young functions are actually doubling, we want to avoid the appearance of their (large) doubling constants in the estimates.

Lemma (4.1.8)[4]: *Let* B *be a Young function. Then*

$$\|M_B\|_{B(L^p(R^n))} \leq c_n \beta_p(B) \qquad \textbf{(23)}$$

where

$$\beta_p(B) = \left(\int_{B(1)}^{\infty} \left(\frac{t}{\bar{B}(t)}\right)^p d\bar{B}(t)\right)^{1/p}$$

Proof. We first prove that for $a > 0$

$$\int_{B^{-1}(a)}^{\infty} \frac{dB(t)}{t^p} \leq \int_{\bar{B}^{-1}(a)}^{\infty} \left(\frac{t}{\bar{B}(t)}\right)^p d\bar{B}(t). \qquad \textbf{(24)}$$

We discretize the integrals with a sequence $a_k := \eta^k a$, where $\eta > 1$ and eventually we pass to the limit $\eta \to 1$. Then

$$\int_{B^{-1}(a)}^{\infty} \frac{dB(t)}{t^p} = \sum_{k=1}^{\infty} \int_{B^{-1}(a_k)}^{B^{-1}(a_{k+1})} \frac{dB(t)}{t^p} \leq \sum_{k=1}^{\infty} \frac{1}{B^{-1}(a_k)^p} \int_{B^{-1}(a_k)}^{B^{-1}(a_{k+1})} dB(t)$$

$$= \sum_{k=1}^{\infty} \frac{1}{B^{-1}(a_k)^p} (a_{k+1} - a_k).$$

Similarly,

$$\int_{\bar{B}^{-1}(a)}^{\infty} \left(\frac{t}{\bar{B}(t)}\right)^p d\bar{B}(t) = \sum_{k=0}^{\infty} \int_{\bar{B}^{-1}(a_k)}^{\bar{B}^{-1}(a_{k+1})} \left(\frac{t}{\bar{B}(t)}\right)^p d\bar{B}(t)$$

$$\geq \sum_{k=0}^{\infty} \left(\frac{\bar{B}^{-1}(a_{k+1})}{\bar{B}(\bar{B}^{-1}(a_{k+1}))}\right)^p \int_{\bar{B}^{-1}(a_k)}^{\bar{B}^{-1}(a_{k+1})} d\bar{B}(t)$$

$$= \sum_{k=0}^{\infty} \left(\frac{\bar{B}^{-1}(a_{k+1})}{a_{k+1}}\right)^p (a_{k+1} - a_k),$$

where we used the fact that $t \mapsto \bar{B}(t)/t$ is increasing, so its reciprocal is decreasing. Moreover,

$$\frac{\bar{B}^{-1}(a_{k+1})}{a_{k+1}} \geq \frac{\bar{B}^{-1}(a_k)}{a_{k+1}} \frac{B^{-1}(a_k)}{B^{-1}(a_k)} \overset{(17)}{\geq} \frac{a_k}{a_{k+1}} \frac{1}{B^{-1}(a_k)} = \frac{1}{\eta B^{-1}(a_k)}$$

and hence

$$\int_{B^{-1}(a)}^{\infty} \frac{dB(t)}{t^p} \leq \eta^p \int_{\bar{B}^{-1}(a)}^{\infty} \left(\frac{t}{\bar{B}(t)}\right)^p d\bar{B}(t).$$

Since this is valid for any $\eta > 1$, we obtain (24).

Now, let $t_1 = max(1, t_0)$, where $t_0 = max\{t : B(t) = 0\}$. Using $B(t)dt/t \leq dB(t)$ and applying (24) with $a = B(t_1 + \epsilon) > 0$

$$\alpha_p(B) = \lim_{\epsilon\to 0}\left(\int_{t_1+\epsilon}^{\infty} \frac{B(t)}{t^p}\frac{dt}{t}\right)^{\frac{1}{p}}$$

$$\leq \lim_{\epsilon\to 0}\left(\int_{B^{-1}(B(t_1+\epsilon))}^{\infty} \frac{dB(t)}{t^p}\right)^{\frac{1}{p}} \overset{(24)}{\leq} \lim_{\epsilon\to 0}\left(\int_{\bar{B}^{-1}(B(t_1+\epsilon))}^{\infty} \left(\frac{(t)}{\bar{B}(t)}\right)^p d\bar{B}(t)\right)^{\frac{1}{p}}$$

$$\leq \left(\int_{B(1)}^{\infty} \left(\frac{t}{\bar{B}(t)}\right)^p d\bar{B}(t)\right)^{\frac{1}{p}},$$

where in the last step we used (17) with $t = B(t_1 + \epsilon)$ to conclude that

$$\bar{B}^{-1}(B(t_1 + \epsilon)) \geq \frac{B(t_1+\epsilon)}{t_1+\epsilon} \geq \frac{B(t_1)}{t_1} \geq B(1),$$

since $B(t)/t$ is increasing and $t_1 \geq 1$.

We will consider B so that $\bar{B}(t) = A(t) = t^p(1 + log^+ t)^{p-1+\delta},\ \delta > 0$. Then, for $0 < \delta \leq 1$

$$A'(t) \leq 2p\frac{A(t)}{t} \qquad t > 1$$

and

$$\bar{A}(1) = \sup_{t\in(0,1)}(t - t^p) = (t - t^p)\Big|_{t=p^{-1/(p-1)}} = (p-1)p^{-\acute{p}}.$$

Thus, by the lemma

$$\|M_{\bar{A}}\|_{B(L^{\acute{p}}(R^n))} \leq c_n\left(\int_{(p-1)p^{-\acute{p}}}^{\infty} \left(\frac{t}{A(t)}\right)^{\acute{p}} \acute{A}(t)\, dt\right)^{1/p} \leq c_n p^2 \left(\frac{1}{\delta}\right)^{1/\acute{p}} \qquad \textbf{(25)}$$

Similarly for the smaller functional:

$$\bar{B}(t) = A(t) = t^p(1 + log^+ t)^{p-1}\left(1 + log^+(1 + log^+ t)\right)^{p-1+\delta} \qquad \delta > 0.$$

Then, using that $A'(t) \leq 3\, p\, \frac{A(t)}{t}$, $t > 1$, when $0 < \delta \leq 1$ and hence by the lemma

$$\|M_{\bar{A}}\|_{B(L^{\acute{p}}(R^n))} \leq c_n p^2 \left(\frac{1}{\delta}\right)^{1/\acute{p}}$$

We will need the following variation of the Rubio de Francia algorithm.

Lemma (4.1.9)[4]: *Let $1 < s < \infty$ and let v be a weight. Then there exists a nonnegative sublinear operator R satisfying the following properties:*

a) h ≤ R(h),
b) $\|R(h)\|_{L^s(v)} \leq 2\|h\|_{L^s(v)}$,
c) $R(h)v^{1/s} \in A_1$ *with*

$$[R(h)v^{1/s}]_{A_1} \leq c\acute{s}$$

Proof. We consider the operator

$$S(f) = \frac{M(f\, v^{1/s})}{v^{1/s}}$$

Since $\|M\|_{L^s} \sim \acute{s}$, we have

$$\|S(f)\|_{L^s(v)} \leq c\acute{s}\|f\|_{L^s(v)}.$$

Now, define the Rubio de Francia operator R by

$$R(h) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{S^k(h)}{(\|S\|_{L^s(v)})^k}.$$

It is very simple to check that R satisfies the required properties.

The new result is intimately related to a sharp two weight estimate for M.

Theorem (4.1.10)[4]: *Given a pair of weights u, σ and $p, 1 < p < \infty$, suppose that*

$$K = \sup_Q \left(\frac{1}{|Q|}\int_Q u(y)\, dy\right)^{1/p} \left\|\sigma^{1/\acute{p}}\right\|_{X,Q} < \infty, \quad \textbf{(26)}$$

where X *is a Banach function space such that its corresponding associate space* $\acute{X}$ *satisfies* $M_{\acute{X}}: L^p(R^n) \to L^p(R^n)$. *Then*

$$\|M(f\sigma)\|_{L^p(u)} \lesssim K \|M_{\acute{X}}\|_{B(L^p(R^n))} \|f\|_{L^p(\sigma)} \qquad \textbf{(27)}$$

In particular if $X = L_B$ *with* $B(t) = t^{\acute{p}}(1 + log^+ t)^{\acute{p}-1+\delta}$, $\delta > 0$, *then by* (25)

$$\|M_{\acute{X}}\|_{B(L^p(R^n))} = \|M_{\bar{B}}\|_{B(L^p(R^n))} \approx (\acute{p})^2 \left(\frac{1}{\delta}\right)^{1/p},$$

where the last ≈ *is valid for* δ ≤ 1.

We say that a dyadic grid, denoted by D, is a collection of cubes in R^n with the following properties:

i. each $Q \in D$ $satisfies$ $|Q| = 2^{nk}$ for some $k \in Z$;
ii. if $Q, P \in D$ then $Q \cap P = \emptyset, P,$ or Q;
iii. for each $k \in Z$, the family $D_k = \{Q \in D: |Q| = 2^{nk}\}$ forms a partition of R^n.

We say that a family of dyadic cubes $S \subset D$ is *sparse* if for each $Q \in S$,

$$\left|\bigcup_{\substack{\acute{Q}\in S\\ \acute{Q}\subsetneq Q}} \acute{Q}\right| \le \frac{1}{2}|Q|.$$

Given a sparse family, S, if we define

$$E(Q) := Q \setminus \bigcup_{\substack{\acute{Q}\in S\\ \acute{Q}\subsetneq Q}} \acute{Q},$$

then

i. the family $\{E(Q)\}_{Q\in S}$ is pairwise disjoint,
ii. $E(Q) \subset Q$, and
iii. $|Q| \le 2|E(Q)|$.

If $S \subset D$ is a sparse family we define the sparse Calderón–Zygmund operator associated to S as

$$T^S f := \sum_{Q\in S} ⨍_Q \; f\,dx \cdot \chi_Q \,.$$

As already mentioned the key idea is to "transplant" the continuous case to the discrete version by means of the following theorem.

Theorem (4.1.11)[4]: *Suppose that* X *is a quasi-Banach function space on* R^n *and* T *is a Calderón–Zygmund operator. Then there exists a constant* c_T

$$\|T^*\|_{B(X)} \le c_T \sup_{S\subset D} \|T^S\|_{B(X)} \,.$$

We will not prove this theorem, we will simply mention that a key tool is the decomposition formula for functions found previously using the median. The main idea of this decomposition goes back to the work where the standard average is used instead.

Section (4.2): Proof of Theorems

Lemma (4.2.1)[4]: *Let* $w \in A_\infty$. *Then for any sparse family* $S \subset D$

$$\|T^S f\|_{L^1(w)} \le 8[w]_{A_\infty} \|Mf\|_{L^1(w)} \qquad \textbf{(28)}$$

Proof. The left hand side equals for $f \ge 0$

$$\sum_{Q\in S} ⨍_Q f\,dx w(Q) \le \sum_{Q\in S} \inf_{z\in Q} Mf(z) w(Q) \le \sum_{Q\in S} \left(⨍_Q (Mf)^{1/2} dw\right)^2 w(Q)\,.$$

By the Carleson embedding theorem, applied to $g = (Mf)^{1/2}$, we have

$$\sum_{Q\in S} \left(⨍_Q g\,dw\right)^2 w(Q) \le 4K\|g\|_{L^2(w)}^2 = 4K\|Mf\|_{L^1(w)}$$

provided that the Carleson condition

$$\sum_{\substack{Q\in S\\ Q\subseteq R}} w(Q) \le Kw(R) \qquad \textbf{(29)}$$

is satisfied. To prove (29), we observe that

$$\sum_{\substack{Q\in S\\ Q\subseteq R}} w(Q) = \sum_{\substack{Q\in S\\ Q\subseteq R}} \frac{w(Q)}{|Q|}\,|Q| \le \sum_{\substack{Q\in S\\ Q\subseteq R}} \inf_{z\in Q} M(1_R w)(z)\cdot 2|E(Q)| \le 2\int_R M(1_R w)(z)dz$$

$$\le 2[w]_{A_\infty} w(R).$$

This proves (29) with $K = 2[w]_{A_\infty}$, and the lemma follows.

Actually, in the applications we have in mind we just need this for $w \in A_q \subset A_\infty$ for some fixed finite q.

The second lemma is an extension of the well known Coifman–Rochberg Lemma:

$$If\ \gamma \in (0,1\ \ then\quad M(\mu)^\gamma \in A_1 \quad with\ \ [M(\mu)^\gamma]_{A_1} \le \frac{c_n}{1-\gamma}$$

Lemma (4.2.2)[4]: *Let A be a Young function and* u *be a nonnegative function such that* $M_A u(x) < \infty$ *a.e. For* $\gamma \in (0,1)$*, there is a dimensional constant* c_n *such that*

$$[(M_A u)^\gamma]_{A_1} \le c_n c_\gamma. \qquad \textbf{(30)}$$

Proof. We claim now that for each cube Q and each u

$$⨍_Q M_A(u\chi_Q)(x)^\gamma dx \le c_{n,\gamma}\|u\|_{A,Q}^\gamma. \qquad \textbf{(31)}$$

By homogeneity we may assume $\|u\|_{A,Q} = 1$, and so, in particular, that $⨍_Q A(u(x))\,dx \le 1$.

Now, the proof of (31) is based on the distributional estimate (19). We split the integral at a level $\lambda \ge b_n$, yet to be chosen:

$$⨍_Q M_A(u\chi_Q)(x)^\gamma\,dx = \frac{1}{|Q|}\int_0^\infty \gamma\, t^\gamma\,\left|\{x \in Q : M_A(u\chi_Q)(x) > t\}\right|\frac{dt}{t}$$

$$\le \frac{1}{|Q|}\int_0^\lambda \gamma t^\gamma |Q| \frac{dt}{t} + \frac{1}{|Q|}\int_\lambda^\infty \gamma t^\gamma a_n \int_Q A\left(b_n \frac{|u(x)|}{t}\right) dx \frac{dt}{t}$$

$$\leq \lambda^{\gamma} + \frac{1}{|Q|}\int_{\lambda}^{\infty} \gamma t^{\gamma} a_n \int_{Q} \frac{b_n}{t} A(|u(x)|)\, dx \frac{dt}{t} \leq \lambda^{\gamma} + a_n b_n \gamma \int_{\lambda}^{\infty} t^{\gamma-2} dt$$

$$= \lambda^{\gamma} + a_n b_n \frac{\gamma}{1-\gamma} \lambda^{\gamma-1}.$$

With $\lambda = a_n b_n$, we arrive at

$$⨍_Q M_A(u\chi_Q)(x)^{\gamma} dx \leq \frac{(a_n b_n)^{\gamma}}{1-\gamma},$$

which is (31), in view of a normalization that $\|u\|_{A,Q} = 1$.

$$M_A\left(u\chi_{R^n\backslash 3\,Q}\right)(x) \approx \sup_{P\supset Q}\left\|u\chi_{R^n\backslash 3\,Q}\right\|_{A,P} \qquad x \in Q \tag{32}$$

where the constant in the direction $\leq$ is dimensional (actually 3^n). (32) shows that $M_A\left(f\chi_{R^n\backslash 3\,Q}\right)$ is essentially constant on Q.

Since A is a Young, the triangle inequality combined with (31) and (32) gives for every $y \in Q$,

$$⨍_Q M_A u(x)^{\gamma} dx \leq 3^n ⨍_{3Q} M_A(u\chi_{3Q})(x)^{\gamma} dx + ⨍_Q M_A(u\chi_{R^n\backslash 3\,Q})(x)^{\gamma} dx$$

$$\leq c_{n,\gamma}\|u\|_{A,3Q}^{\gamma} + 3^n \sup_{P\supset Q}\left(\left\|u\chi_{R^n\backslash 3\,Q}\right\|_{A,P}\right)^{\gamma} \leq c_{n,\gamma} M_A u(y)^{\gamma}.$$

This completes the proof of the lemma.

We have to prove

$$\|T^* f\|_{L^p(w)} \leq c_T \acute{p} \|M_{\bar{A}}\|_{B(L^{\acute{p}}(R^n))} \|f\|_{L^p(M_A(w^{1/p})^p)} \qquad w \geq 0,$$

and if we use the notation $A_p(t) = A\left(t^{\frac{1}{p}}\right)$ this becomes

$$\|T^* f\|_{L^p(w)} \leq c_T \acute{p} \|M_{\bar{A}}\|_{B(L^{\acute{p}}(R^n))} \|f\|_{L^p(M_{A_p}(w))}.$$

By Theorem (4.1.11) everything is reduced to proving that

$$\|T^S f\|_{L^p(w)} \lesssim \acute{p} \|M_{\bar{A}}\|_{B(L^{\acute{p}}(R^n))} \|f\|_{L^p(M_{A_p}(w))} \qquad S \subset D. \tag{33}$$

Now, by duality we will prove the equivalent estimate

$$\|T^S(fw)\| L^{\acute{p}}(M_{A_p}(w)^{1-\acute{p}}) \lesssim \acute{p} \|M_{\bar{A}}\|_{B(L^{\acute{p}}(R^n))} \|f\|_{L^{\acute{p}}(w)},$$

because the adjoint of T^S (with respect to the Lebesgue measure) is itself.

The main claim is the following:

Lemma (4.2.3)[4]:

$$\|T^S(g)\|_{L^{\acute{p}}(M_{A_p}(w)^{1-\acute{p}})} \lesssim \acute{p}\|M(g)\|_{L^{\acute{p}}(M_{A_p}(w)^{1-\acute{p}})} \qquad S \subset D \quad g \geq 0. \tag{34}$$

Proof. Now

$$\|T^S(g)\|_{L^{\acute{p}}(M_{A_p}(w)^{1-\acute{p}})} = \left\|\frac{T^S(g)}{M_{A_p}w}\right\|_{L^{\acute{p}}(M_{A_p}w)}$$

and by duality we have that for some nonnegative h with $\|h\|_{L^p(M_{A_p}w)} = 1$

$$\left\|\frac{T^S(g)}{M_{A_p}w}\right\|_{L^{\acute{p}}(M_{A_p}w)} = \int_{R^n} T^S(g)hdx$$

Now, by Lemma (4.1.9) with $s = p$ and $v = M_{A_p}w$ there exists an operator R such that

a) $h \leq R(h)$,

b) $\|R(h)\|_{L^p(M_{A_p}w)} \leq 2\|h\|_{L^p(M_{A_p}w)}$,

c) $[R(h)(M_{A_p}w)^{1/p}]_{A_1} \leq c\acute{p}$.

Hence,

$$\|T^S(g)\|_{L^{\acute{p}}\left(M_{A_p}(w)^{1-\acute{p}}\right)} \leq \int_{R^n} T^S(g)Rh\,dx.$$

We plan to replace T^S by M by using Lemma (4.2.1). To do this we estimate the A_q constant of *Rh*, for a fixed $q > 1$ (in fact, $q = 3$) using property (C) combining the following two facts. The first one is well known, it is the easy part of the factorization theorem: if $w_1, w_2 \in A_1$, then $w = w_1 w_2^{1-p} \in A_p$, and

$$[w]_{A_p} \leq [w_1]_{A_1}[w_2]_{A_1}^{p-1}$$

The second fact is Lemma (4.2.2).

Now if we choose $\gamma = \frac{1}{2}$ in Lemma (4.2.2),

$$[R(h)]_{A_\infty} \lesssim [R(h)]_{A_3} = [R(h)(M_{A_p}w)^{\frac{1}{p}}\left((M_{A_p}w)^{\frac{1}{2p}}\right)^{1-3}]_{A_3}$$

$$\leq [R(h)(M_{A_p}w)^{\frac{1}{p}}]_{A_1}[(M_{A_p}w)^{\frac{1}{2p}}]_{A_1}^{3-1} \leq c_n\acute{p}[M_A(w^{\frac{1}{p}})^{\frac{1}{2}}]_{A_1}^{3-1} \leq c_n\acute{p}$$

by the lemma and since $A_p(t) = A(t^{1/p})$.

Therefore, by Lemma (4.2.1) and by properties (A) and (B) together with Hölder,

$$\int_{R^n} T^S(g)h dx \leq \int_{R^n} T^S(g)R(h)\, dx \lesssim [R(h)]_{A_\infty} \int_{R^n} M(g)R(h)\, dx$$

$$\lesssim \acute{p} \left\|\frac{M(g)}{M_{A_p}w}\right\|_{L^{\acute{p}}(M_{A_p}w)} \|Rh\|_{L^p(M_{A_p}w)} = c_N\acute{p} \left\|\frac{M(g)}{M_{A_p}w}\right\|_{L^{\acute{p}}(M_{A_p}w)}.$$

This proves claim (34).

With (34), the proof of Theorem (4.1.2) is reduced to showing that

$$\|M(fw)\|_{L^{\acute{p}}(M_{A_p}(w)^{1-\acute{p}})} \leq c\|M_{\bar{A}}\|_{B(L^{\acute{p}}(R^n))}\|f\|_{L^{\acute{p}}(w)}$$

for which we can apply the two weight theorem for the maximal function (Theorem (4.1.10) to the couple of weights $(M_{A_p}(w)^{1-\acute{p}}, w)$ with exponent $\acute{p}$. We need then to compute (26).

$$\left(\frac{1}{|Q|}\int_Q M_{A_p}(w)^{1-\acute{p}}dy\right)^{1/\acute{p}} \left\|w^{1/p}\right\|_{A,Q} \leq \|w\|_{A_p,Q}^{-1/p}\left\|w^{1/p}\right\|_{A,Q} = \left\|w^{1/p}\right\|_{A,Q}^{-1}\left\|w^{1/p}\right\|_{A,Q}$$

$$= 1,$$

Since $A_p(t) = A(t^{1/p})$. Hence

$$\|M(fw)\|_{L^{\acute{p}}(M_{A_p}(w)^{1-\acute{p}})} \leq c\|M_{\bar{A}}\|_{B(L^{\acute{p}}(R^n))}\|f\|_{L^{\acute{p}}(w)}$$

concluding the proof of the theorem.

Proposition (4.2.4)[4]: *Let* D *be a dyadic grid and let $S \subset D$ be a sparse family. Then, there is a universal constant* c *independent of* D *and* S *such that for any* $0 < \epsilon \leq 1$

$$\|T^S f\|_{L^{1,\infty}(w)} \leq \frac{c}{\epsilon} \int_{R^n} |f(x)| M_{L(\log L)^{\epsilon}}(w)(x) dx \qquad w \geq 0 \qquad \textbf{(35)}$$

Note that in order to deduce Theorem (4.1.1) from the proposition above, we need the full strength of Theorem (4.1.11) with quasi-Banach function space, because the space $L^{1,\infty}$ is not normable. It is also possible to prove Theorem (4.1.1) directly (without going through the dyadic model); this was an original approach, since the quasi-Banach version of Theorem (4.1.11) was not yet available at that point. However, we now present a proof via the dyadic model, which simplifies the argument.

Recall that the sparse Calderón–Zygmund operator T^S is defined by,

$$T^S f = \sum_{Q \in S} ⨍_Q f \, dx \cdot \chi_Q .$$

By homogeneity on fit would be enough to prove

$$w\{x \in R^n : T^S f(x) > 2\} \leq \frac{c}{\epsilon} \int_{R^n} |f(x)| M_{L(\log L)^{\epsilon}}(w)(x) dx.$$

We consider the CZ decomposition of f with respect to the grid D at level $\lambda = 1$. There is family of pairwise disjoint cubes $\{Q_j\}$ from D such that

$$1 < \frac{1}{|Q_j|} \int_{Qj} |f| \leq 2^n$$

Let $\Omega = \bigcup_j Q_j$ and $\tilde{\Omega} = \bigcup_j 3Q_j$. The "good part" is defined by

$$g = \sum_j f_{Q_j} \chi_{Q_j}(x) + f(x) \chi \Omega^c (x),$$

and it satisfies $\|g\|_{L^\infty} \leq 2^n$ by construction. The "bad part" b is $b = \sum_j b_j \, where \, b_j(x) = (f(x) - f_{Q_j})\chi_{Q_j}(x)$. Then, $f = g + b$ and we split the level set as

$$w\{x \in R^d : T^S f(x) > 2\} \leq w(\tilde{\Omega}) + w\{x \in (\tilde{\Omega})^c : T^S b(x) > 1\} + w\{x \in (\tilde{\Omega})^c : T^S g(x) > 1\} = I + II + III .$$

The most singular term is *III*. We first deal with the easier terms I and *II*, which actually satisfy the better bound

$$I + II \leq c_T \|f\|_{L^1(Mw)}.$$

The first is simply the classical Fefferman–Stein inequality (2).

To estimate $II = w\{x \in (\tilde{\Omega})^c : |T^S g(x)| > 1\}$ we argue as follows:

$$w\{x \in (\tilde{\Omega})^c : |T^S b(x)| > 1\} \leq \int_{R^n \setminus \tilde{\Omega}} |T^S b(x)| w(x) dx \lesssim \sum_j \int_{R^n \setminus \tilde{\Omega}} |T^S (b_j)(x)| w(x) dx$$

$$\lesssim \sum_j \int_{R^n \setminus 3Q_j} |T^S (b_j)(x)| w(x) dx$$

We fix one of these j and estimate now $T^S\big(b_j\big)(x) for \; x \notin 3Q_j$:

$$T^S(b_j)(x) = \sum_{Q \in S} ⨍_Q b_j \, dy \cdot \chi_Q(x) = \sum_{Q \in S, Q \subset Q_j} + \sum_{Q \in S, Q \supset Q_j} = \sum_{Q \in S, Q \supset Q_j}$$

Since $x \notin Q_j$. Now, this expression is equal to

$$\sum_{\mathrm{Q \in S, Q \supset Q_j}} \frac{1}{|\mathrm{Q}|} \int_{\mathrm{Q_j}} (f(y) - f_{Q_j}) \, dy \cdot \chi Q(x)$$

and this expressionis zero by the key cancellation: $\int_{\mathrm{Q_j}} (f(y) - f_{Q_j}) \, dy = 0$. Hence $II = 0$, and we are only left with the singular term *III*.

We now consider the last term *III*, the singular part. We apply Chebyschev's inequality and then (33) with exponent p and functional A, that will be chosen soon:

$$III = w\{x \in (\tilde{\Omega})^c : T^S g(x) > 1\} \leq \|T^S g\|^p_{L^p(w\chi_{(\tilde{\Omega})^c})}$$

$$\lesssim (\acute{p})^p \|M_{\bar{A}}\|^p_{B(L^{\acute{p}}(R^n))} \int_{R^n} |g|^p M_{A_p}(w\chi_{(\tilde{\Omega})^c})dx$$

$$\lesssim (\acute{p})^p \|M_{\bar{A}}\|^p_{B(L^{\acute{p}}(R^n))} \int_{R^n} |g| M_{A_p}(w\chi_{(\tilde{\Omega})^c})dx \ ,$$

using the boundedness of g by $2^n \lesssim 1$, and denoting $A_p(t) = A(t^{1/p})$.

Now, we will make use of (32) again: for an arbitrary Young function B, a nonnegative function w with $M_B w(x) < \infty \ a.e.$, and a cube Q, we have

$$M_B(\chi_{R^n \backslash 3Q} w)(y) \approx M_B(\chi_{R^n \backslash 3Q} w)(z) \qquad \textbf{(36)}$$

for each $y, z \in Q$ with dimensional constants. Hence, combining (36) with the definition of g we have

$$\int_{\Omega} |g| M_{A_p}(w\chi_{(\tilde{\Omega})^c})dx \lesssim \sum_j \int_{Q_j} |f(x)| \, dx \inf_{Q_j} M_{A_p}(w\chi_{(\tilde{\Omega})^c}) \lesssim \int_{\Omega} |f(x)| M_{A_p} w(x) \, dx,$$

and of course

$$\int_{\Omega^c} |g| \, M_{A_p}(w\chi_{(\tilde{\Omega})^c})dx \lesssim .$$

Combining these, we have

$$III \lesssim (\acute{p})^p \|M_{\bar{A}}\|^p_{B(L^{\acute{p}}(R^n))} \int_{R^d} |f| M_{A_p}(w)dx \, .$$

We optimize this estimate by choosing an appropriate A. To do this we apply now Lemma (4.1.8) and more particularly to the example considered in (25), namely B is so that $\bar{B}(t) = A(t) = t^p(1 + log^+ t)^{p-1+\delta}, \ \delta > 0$. Then

$$\|M_{\bar{A}}\|_{B(L^{\acute{p}}(R^n))} \leq c_n \int_1^{\infty} \left(\left(\frac{t}{A(t)}\right)^{\acute{p}} \acute{A}(t) \, dt\right)^{1/\acute{p}} \lesssim p \left(\frac{1}{\delta}\right)^{1/\acute{p}} \qquad 0 < \delta \leq 1$$

Then $A_p(t) = A(t^{1/p}) \leq t(1 + log^+ t)^{p-1+\delta}$ and we have

$$III \lesssim (\acute{p})^p \left(\frac{1}{\delta}\right)^{p-1} \int_{R^d} |f| M_{L(log\ L)^{p-1+\delta}}(w)(x)\, dx\,.$$

Now if we choose p such that

$$p - 1 = \frac{\epsilon}{2} = \delta < 1$$

Then $(\acute{p})^p \left(\frac{1}{\delta}\right)^{p-1} \lesssim \frac{1}{\epsilon}$ if $\epsilon < 1$.

This concludes the proof of (35), and hence of Theorem (4.1.1.)

The essential difference is that we compute in a more precise way the constants involved. We consider the set

$$\Omega = \{x \in R^n : T^*(f\sigma)(x) > 1\}$$

Then by homogeneity it is enough to prove

$$u(\Omega)^{1/p} \lesssim \frac{1}{\delta} K \left(\frac{1}{\delta}\right)^{1/\acute{p}} \|f\|_{L^p(\sigma)} \quad \textbf{(37)}$$

where we recall that

$$K = \sup_Q \|u^{1/p}\|_{L^p(log\ L)^{p-1+\delta},Q} \left(\frac{1}{|Q|}\int_Q \sigma\, dx\right)^{1/\acute{p}} < \infty \quad \textbf{(38)}$$

Now, by duality, there exists a non-negative function $h \in L^{\acute{p}}(R^n)$, $\|h\|_{L^{\acute{p}}(R^n)} = 1$, such that

$$u(\Omega)^{\frac{1}{p}} = \left\|u^{\frac{1}{p}}\chi_\Omega\right\|_{L^p(R^n)} = \int_\Omega u^{\frac{1}{p}} h dx = u^{\frac{1}{p}} h(\Omega) \lesssim \frac{1}{\varepsilon}\int_{R^n} |f| M_{L(\log L)^\varepsilon}\left(u^{\frac{1}{p}} h\right)\sigma dx$$

$$\leq \frac{1}{\varepsilon}\left(\int_{R^n} |f|^p \sigma dx\right)^{1/p} \left(\int_{R^n} M_{L(log\ L)^\varepsilon}(u^{1/p}h)^{\acute{p}}\, \sigma dx\right)^{1/\acute{p}},$$

where we have used inequality (4) from Theorem (4.1.1) and then Hölder's inequality. Therefore everything is reduced to understanding a two weight estimate for $M_{L(\log L)^{\varepsilon}}$.

Lemma (4.2.5)[4]: *Given a Young function A, suppose f is a non-negative function such that* $\|f\|_{A,Q}$ *tends to zero as* $l(Q)$ *tends to infinity. Given* $a > 2^{n+1}$, *for each* $k \in Z$ *there exists a disjoint collection of maximal dyadic cubes* $\{Q_j^k\}$ *such that for each* j,

$$a^k < \|f\|_{A,Q_j^k} \leq 2^n a^k, \qquad \textbf{(39)}$$

and

$$\{x \in R^n : M_A f(x) > 4^n a^k\} \subset \bigcup_j 3Q_j^k$$

Further, let $D_k = \bigcup_j Q_j^k$ *and* $E_j^k = Q_j^k \backslash (Q_j^k \cap D_{k+1})$. *Then the* E_j^k*'s are pairwise disjoint for all j and k and there exists a constant* $\alpha > 1$, *depending only on* a, *such that* $|Q_j^k| \leq \alpha |E_j^k|$.

Fix a function *h* bounded with compact support. Fix $a > 2^{n+1}$; for $k \in Z$ let

$$\Omega_k = \{x \in R^n : 4^n a^k < M_A f(x) \leq 4^n a^{k+1}\}.$$

Then by Lemma (4.2.5),

$$\Omega_k \subset \bigcup_j 3Q_j^k, \quad \text{where} \quad \|f\|_{A,Q_j^k} > a^k.$$

Lemma (4.2.6)[4]: *Let* A, B *and* C *be Young functions such that*

$$B^{-1}(t) C^{-1}(t) \leq \kappa A^{-1}(t), \quad t > 0. \qquad \textbf{(40)}$$

Then for all functions f and g and all cubes Q,

$$\|fg\|_{A,Q} \leq 2\kappa \|f\|_{B,Q} \|g\|_{C,Q}. \qquad \textbf{(41)}$$

Proof. The assumption (40) says that if $A(x) = B(y) = C(z)$, then $yz \leq \kappa x$. Let us derive a more applicable consequence:

Let $y, z \in [0, \infty)$, and assume without loss of generality (by symmetry) that *B(y)* ≤ *C(z)*. Since Young functions are onto, we can find a $\acute{y} \geq y$ *and* $x \in [0, \infty)$ such that $B(\acute{y}) = C(z) = A(x)$. Then (40) tells us that $yz \leq \acute{y}z \leq \kappa x$. Since A is increasing, it follows that.

$$A\frac{yz}{\kappa} \leq A(x) = C(z) = max(B(y), C(z)) \leq B(y) + C(z). \quad \textbf{(42)}$$

Let then $s > \|f\|_B$ and $t > \|g\|_C$. Then, using (42),

$$⨍_Q A\left(\frac{|fg|}{\kappa st}\right) \leq ⨍_Q B\left(\frac{|f|}{s}\right) + ⨍_Q C\left(\frac{|g|}{t}\right) \leq 1 + 1,$$

and hence

$$⨍_Q A\left(\frac{|fg|}{2\kappa st}\right) \leq \frac{1}{2}⨍_Q A\left(\frac{|fg|}{2\kappa st}\right) \leq 1.$$

This proves that $\|fg\|_A \leq 2\kappa st$, and taking the infimum over admissible s and t proves the claim.

If $A(t) = t(1 + log^+ t)^\varepsilon$, the goal is to "break" M_A in an optimal way, with functions B and C so that one of them, for instance B, has to be $B(t) = t^p(1 + log^+ t)^{p-1+\delta}$ coming from (38).

We can therefore estimate M_A using Lemma (4.2.5) as follows:

$$\int_{R^n} \left(M_A\left(u^{\frac{1}{p}}h\right)\right)^{\acute{p}} \sigma\, dx = \sum_k \int_{\Omega k} \left(M_A\left(u^{\frac{1}{p}}h\right)\right)^{\acute{p}} \sigma\, dx$$

$$\leq c\sum_k a^{k\acute{p}}\sigma(\Omega_k)$$

$$\leq c\sum_k a^{k\acute{p}}\sigma(3Q_j^k)$$

$$\leq c\sum_{j,k} \sigma(3Q_j^k)\left\|u^{\frac{1}{p}}h\right\|_{A,Q_j^k}^{\acute{p}}$$

$$\leq c\sum_{j,k} \sigma(3Q_j^k)\|u^{1/p}\|_{B,Q_j^k}^{\acute{p}}\,\|h\|_{C,Q_j^k}^{\acute{p}},$$

by (41). Now since $\|u^{1/p}\|_{B,Q_j^k} \leq 3^n \|u^{1/p}\|_{B,3Q_j^k}$, we can apply condition (38), and since the E_j^k's are disjoint,

$$\leq c \sum_{j,k} \left(\frac{1}{|3Q_j^k|} \int_{3Q_j^k} \sigma \, dx \right) \left\| u^{\frac{1}{p}} \right\|_{B,3Q_j^k}^{\acute{p}} \|h\|_{C,Q_j^k}^{\acute{p}} |E_j^k|$$

$$\leq K^{\acute{p}} \sum_{j,k} \int_{E_j^k} M_C(h)^{\acute{p}} dx$$

$$\leq K^{\acute{p}} \int_{R^n} M_C(h)^{\acute{p}} dx$$

$$\leq K^{\acute{p}} \|M_C\|_{B(L^{\acute{p}}(R^n))}^{\acute{p}} \int_{R^n} h^{\acute{p}} dx \, ,$$

if we choose C such that M_C is bounded on $L^{\acute{p}}(R^n)$, namely it must satisfy the tail condition (21). We are left with choosing the appropriate C. Now, $1 < p < \infty$ and $\delta > 0$ are fixed from condition (38) but $\varepsilon > 0$ is free and will be chosen appropriately close to 0. To be more precise we need to choose $0 < \varepsilon < \frac{\delta}{\mathrm{p}}$ and let $\eta = \delta - p\varepsilon$. Then

$$\begin{aligned} A^{-1}(t) &\approx \frac{t}{(1 + log^+ t)^{\varepsilon}} \\ &= \frac{t^{1/p}}{(1 + log^+ t)^{\varepsilon} + (p - 1 + \eta)/p} \times t^{1/\acute{p}} (1 + log^+ t)^{(p-1+\eta)/p} \\ &= B^{-1}(t) C^{-1}(t), \end{aligned}$$

where

$$B(t) \approx t^p (1 + log^+ t)^{(1+\varepsilon)p - 1 + \eta} = t^p (1 + log^+ t)^{p-1+\delta}$$

and

$$C(t) \approx t^{\acute{p}} (1 + log^+ t)^{-1-(\acute{p}-1)\eta}.$$

It follows at once from Lemma (4.1.7)that

$$\|M_C\|_{B(L^{\acute{p}}(R^n))} \lesssim \left(\frac{1}{\eta}\right)^{1/\acute{p}} = \left(\frac{1}{\delta - p\varepsilon}\right)^{1/\acute{p}},$$

where we suppress the multiplicative dependence on p. If we choose $\varepsilon = \frac{\delta}{2p}$ we get the desired result:

$$u(\Omega)^{1/p} \lesssim \frac{1}{\delta} K \left(\frac{1}{\delta}\right)^{1/\acute{p}} \|f\|_{L^p(\sigma)} \tag{43}$$

This completes the proof of part (a) of Corollary (4.1.6).

To prove part (b) we combine Lerner's Theorem (4.1.11),

$$\|T^* f\|_{L^p(u)} \leq c_T \sup_{S\subset D} \|T^S f\|_{L^p(u)},$$

with the characterization of the two-weight inequalities for T^S by testing conditions: a combination of their characterizations for weak and strong norm inequalities shows in particular that

$$\|T^S(.\sigma)\|_{L^p(\sigma)\to L^p(u)} \approx \|T^S(.\sigma)\|_{L^p(\sigma)\to L^{p,\infty}(u)} + \|T^S(.u)\|_{L^{\acute{p}}(u)\to L^{\acute{p},\infty}(\sigma)}$$

Now, as it is mentioned after the statement of Corollary (4.1.6), since T^Ssatisfies estimate (4) we can apply the same argument as the just given to both summands and since that estimate has to be independent of the grid and we must take the two weight constant K over all cubes, not just for those from the specific grid. This concludes the proof of the corollary.

A conjecture related to Corollary (4.1.6) is as follows:

Conjecture (4.2.7)[4]: *Let* T^*, p, u, σ *be as above. Let* X *be a Banach function space so that its corresponding associate space* $\acute{X}$ *satisfies* $M_{\acute{X}}: L^{\acute{p}}(R^n) \to L^{\acute{p}}(R^n)$. *If*

$$K = \sup_Q \left\|u^{1/p}\right\|_{X,Q} \left(\frac{1}{|Q|}\int_Q \sigma\, dx\right)^{1/\acute{p}} < \infty, \tag{44}$$

then

$$\|T^*(f\sigma)\|_{L^{p,\infty}(u)} \lesssim K\, \|M_{\acute{X}}\|_{B(L^{\acute{p}}(R^n))} \|f\|_{L^p(\sigma)}. \qquad \textbf{(45)}$$

As a consequence, if Y is another Banach function space with $M_{\acute{Y}}: L^p(R^n) \to L^p(R^n)$ *and if*

$$K = \sup_Q \|u^{1/p}\|_{X,Q} \left(\frac{1}{|Q|}\int_Q \sigma\, dx\right)^{1/\acute{p}} + \left(\frac{1}{|Q|}\int_Q u\, dx\right)^{1/p} \|\sigma^{1/\acute{p}}\|_{Y,Q} < \infty, \qquad \textbf{(46)}$$

then

$$\|T^*(f\sigma)\|_{L^p(u)} \lesssim K \left(\|M_{\acute{X}}\|_{B(L^{\acute{p}}(R^n))} + \|M_{\acute{Y}}\|_{B(L^p(R^n))}\right) \|f\|_{L^p(\sigma)} \qquad \textbf{(47)}$$

If we could prove this, we would get as corollary:

Corollary (4.2.8)[4]:

$$\|T^*\|_{B(L^p(w))} \leq c[w]_{A_p}^{1/p}[w]_{A_\infty}^{1/\acute{p}} + [\sigma]_{A_\infty}^{1/p} \qquad \textbf{(48)}$$

This last result itself is known, but not as a corollary of a general two-weight norm inequality.

List of Symbols

min : Minimum	89
Re : Real	113
a.e : Almost Everywhere	141

References:

[1]: David Cruz – Uribe , Two – Weight Boundedness of Calderón – Zygmund Operators, vol 255 ,page 706-729 , 1 April 2014.

[2]: Marti Parts, Sobolev Spaces on Domains, vol 268,page 2946-2989, 15 May 2015.

[3]: Jarod Hart, Singular Integral Operators, vol 268 ,page 3680-3733 ,15 June 2015.

[4]: Tuomas Hytönen, Carlos Pérez,The $L(logL)^{\epsilon}$ End Point Estimate , vol 428, page 605-626,1 August 2015 .

[5]: Kak, Subhash, "Number theoretic Hilbert transform", page 2539–2548 , 2014.

[6]: Grafakos .L ,Educaction Pearson , Analysis Fourier , 2004.

[7]: Dacorogna, Introduction to the Calculus of Variations, 2004.