Decompositions and Characterizations of Besov-Hausdorff and Triebel-Lizorkin Spaces with Variable Smoothness and Integerability

التفكيكات والتشخيصات لفضاءات بيسوف ـ هاوسدورف وتريبل ـ لزوركن مع الملسان المتغير والتكاملية

A Thesis submitted in Fulfillment Requirements for the Degree of Ph.D in Mathematics

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Dedication

To my...

Family,

Teachers,

Friends

&

Students.

Acknowledgments

First and foremost, I must acknowledge my limitless thanks to Allah, for his help and bless.

I am grateful to some people, who worked hard with me from the beginning till the completion of the present research particularly my supervisor Prof. Dr. Shawgy Hussein AbdAlla of Sudan University of Science & Technology, who has been always generous during all phases of the research.

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Abstract

We show the decompositions of Besov-Hausdorff and Triebel-Lizorkin-Hausdorff spaces. We also show the characterizations of Besov-Lipschitz, Besov-Morrey and Triebel-Lizorkin-Morrey spaces by maximal functions, local mean and including $\mathcal{Q}$ spaces. The function spaces, Besov-type spaces and atomic decomposition with variable smoothness and integrability are studied.

الخلاصة

أوضحنا التفكيكات لفضاءات بيسوف ـ هاوسدورف و تريبل ـ لزوركن ـ
هاوسدورف. أيضاً أوضحنا التشخيصات لفضاءات بيسوف ـ ليبشيتز و بيسوف- مورى و تريبل
ـ لزوركن- مورى بواسطة الدوال الأعظمية والأوساط الموضعية والمتضمنة لفضاءات Q. تمت
دراسة دوال الدالة والفضاءات نوع بيسوف والتفكيك الذرى مع الملسان المتغير وقابلية
التكاملية.

Introduction

We consider the Besov-Hausdorff spaces and the Triebel-Lizorkin spaces. We complete earlier results. We find the dual spaces. Many function or distribution spaces have been found to admit decomposition.

We study certain spaces of distribution. They are intimately related to certain spaces studied by Triebel and Lizorkin. We discuss some basic properties of Morrey type Besov-Triebel spaces.

We introduce Triebel-Lizorkin spaces with variable smoothness and integrability. Our new scale covers spaces with variable exponent as well as spaces of variable smoothness that have been studied in recent years. Vector-valued maximal inequalities do not work in the generality which we pursue, and an alternate approach is thus developed. We introduce Besov spaces with variable smoothness and integrability indices. We show independence of the choice of basic functions, as well as several other basic properties.

Also we introduce Besov-type spaces $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ for $p \in (0, \infty]$ and Triebel-Lizorkin-type spaces $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ for $p \in (0, \infty]$ where $s, \tau \in \mathbb{R}$, which unify and generalize the Besov spaces, Triebel-Lizorkin spaces and Q spaces. We then establish the φ-transform characterization of these new spaces in the sense of Frazier and Jawerth. Using the φ-transform characterization of $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$, we obtain their embedding and lifting properties; moreover, for appropriate τ, we also establish the smooth atomic and molecular decomposition characterizations of $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$. For $p \in (1, \infty), q \in [1, \infty), s \in \mathbb{R}$ and $\tau \in [0, 1 - \frac{1}{\max\{p,q\}}]$. We establish the φ-transform characterizations of Besov-Hausdorff spaces $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, and Triebel-Lizorkin-Hausdorff spaces $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ $(q > 1)$; as applications, then we establish their embedding properties (which on $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is also sharp), smooth atomic and molecular decomposition characterizations for suitable τ.

We obtain three independent results on the Besov-Morrey spaces and the Triebel-Lizorkin-Morrey spaces. We establish the maximal function characterizations of the Besov-type space $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ with $p, q \in (0, \infty]$ and $\tau \in [0, \infty)$, the Triebel-Lizorkin-type space $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ with $p \in (0, \infty), q \in (0, \infty]$ and $\tau \in [0, \infty)$, the Besov-Hausdorff space $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ with $p \in (1, \infty), q \in [1, \infty)$ and $\tau \in \left[0, \frac{1}{(\max\{p,q\})'}\right]$ and the Triebel-

Lizorkin-Hausdorff space $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ with $p, q \in (1, \infty)$ and $\tau \in \left[0, \frac{1}{(\max\{p,q\})'}\right]$, where t' denotes the conjugate index of $t \in [1, \infty]$.

Finally we introduce Besov-type spaces with variable smoothness and integrability. We establish their characterizations, respectively, in terms of φ-transforms in the sense of Frazier and Jawerth, smooth atoms or Peetre maximal functions, as well as a Sobolev-type embedding.

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Chapter 1

Some Observations and Decompositions

We determine the trace of Lizorkin-Triebel spaces. We give an extension of Hardy's inequality. We also obtain two types of decompositions for distributions in the homogeneous Besov spaces.

Section (1.1): Besov and Lizorkin-Triebel Spaces

We consider the Besov spaces $\dot{B}_p^{sq}$ and the Lizorkin-Triebel spaces $\dot{F}_p^{sq}$. We complete earlier results. We find the dual spaces when $0 < p < 1$. For $\dot{B}_p^{sq}$ the dual was previously only known when $0 < q \leq 1$ (see [6]). We also determine the trace of $\dot{F}_p^{sq}$, obtaining in this way a result analogous to the ne in [7] for $\dot{B}_p^{sq}$. We give an extension of Hardy's inequality to $\dot{F}_p^{0q}$. In oure treatment, based on Szasz' theorem and an imbedding theorem, this becomes almost a triviality even in special case $q = 2$ corresponding to the Hardy space H_p.

Definition (1.1.1) [17] (Definition of the spaces): To define the spaces to be studied we choose a sequence $\{\varphi_v\}_{v \in \mathbb{Z}}$ of test functions such that

$$\begin{cases} \varphi_v \in \mathcal{S}_0 \\ \text{supp } \hat{\varphi}_v = \{1.5^{-1} \cdot 2^v \leq |\xi| \leq 1.5 \cdot 2^v\} \\ |\hat{\varphi}_v(\xi)| \geq C_\varepsilon > 0 \quad \text{if } 2^v(1.5 - \varepsilon)^{-1} \leq |\xi| \leq 2^v(1.5 - \varepsilon) \\ |D_\alpha \hat{\varphi}_v(\xi)| \leq C_\alpha |\xi|^{-|\alpha|} \quad \text{for every multiindex } \alpha. \end{cases} \tag{1}$$

Here and in what follows $\mathcal{S}_0$ is space of rapidly decreasing whose Fourier transforms vanish together with all their derivatives at the origin. $\mathcal{S}_0'$ is its dual space. It is easy to see that $\mathcal{S}_0'$ in fact can be identified to the space of tempered distributions $\mathcal{S}'$ modulo polynomials.

Definition (1.1.2) [17]: Let s be real, $0 < p, q \leq \infty$. The Besov space $\dot{B}_p^{sq}$ is the space of all $f \in \mathcal{S}_0'$ such that

$$\|f\|_{\dot{B}_p^{sq}} \equiv \left(\sum_v \left(2^{vs} \|\varphi_v * f\|_{L_p} \right)^q \right)^{1/q} < \infty.$$

Definition (1.1.3) [17]: Let s be real, $0 < p \leq \infty$, $0 < q \leq \infty$. The Lizorkin-Triebel space $\dot{F}_p^{sq}$ is the space of all $f \in \mathcal{S}_0'$ such that

$$\|f\|_{\dot{F}_p^{sq}} \equiv \left\| \left(\sum_v 2^{vs} |\varphi_v * f|^q \right)^{1/q} \right\|_{L_p} < \infty.$$

From the definitions we at once get the embeddings

$$\begin{array}{ll} \dot{B}_p^{sq} \to \dot{F}_p^{sq} \to \dot{B}_p^{sp} & \text{if } q \leq p \\ \dot{B}_p^{sp} \to \dot{F}_p^{sq} \to \dot{B}_p^{sq} & \text{if } q \geq p \end{array} \tag{2}$$

and especially

$$\dot{B}_p^{sp} = \dot{F}_p^{sp} \text{ if } 0 < p \le \infty. \tag{3}$$

Furthermore, using Littlewood-Paley theory it is possible to prove (see [8])

$$\dot{F}_p^{02} \approx H_p \tag{4}$$

where H_p is the hardy space on $\mathbb{R}^n$.

Let us list some other known properties of the space (cf. [9], [15]): the embeddings from $\mathcal{S}_0$ and in to $\mathcal{S}_0'$ are continuous:

$$\mathcal{S}_0 \to \dot{B}_p^{sq}, \qquad \dot{F}_p^{sq} \to \mathcal{S}_0'. \tag{5}$$

$$\text{if } 0 < p, q < \infty \text{ then } \mathcal{S}_0 \text{ is dense in } \dot{B}_p^{sq} \text{ and } \dot{F}_p^{sq}. \tag{6}$$

$$\dot{B}_p^{sq} \text{ and } \dot{F}_p^{sq} \text{ are complete.} \tag{7}$$

The Riesz potential $I^s = (-\Delta)^{s/2}$ is an isomorphism from $\dot{B}_p^{s_0 q}$ onto $\dot{B}_p^{s_0-s,q}$ and from $\dot{F}_p^{s_0 q}$ onto $\dot{F}_p^{s_0-s,q}$. $\tag{8}$

Theorem (1.1.4) [17]: Let $s_0 > s_1, 0 < p_0 < p_1 < \infty, 0 < q, r \le \infty$. If

$$s_0 - n/p_0 = s_1 - n/p_1$$

Then

(i) $\dot{B}_{p_0}^{s_0 q} \to \dot{B}_{p_1}^{s_1 p}$

(ii) $\dot{F}_{p_0}^{s_0 q} \to \dot{F}_{p_1}^{s_1 r}$

(iii) $\dot{F}_{p_0}^{s_0 q} \to \dot{F}_{p_1}^{s_1 p_0}$

Proof: We may assume $s_0 = 0$ (cf. (8)).

(i) From the easily verified inequality

$$\|\varphi_v * f\|_{L_\infty} \le C 2^{vn/p} \|\varphi_v * f\|_{L_p} \tag{9}$$

We get by Hölder's inequality

$$\|\varphi_v * f\|_{L_{p_1}} \le C 2^{vn(1/p_0 - 1/p_1)} \|\varphi_v * f\|_{L_{p_0}}.$$

This readily gives (i).

(ii) It suffices to take $q = \infty$ and $\|f\|_{\dot{F}_{p_0}^{s_0 \infty}} = 1$. By (9)

$$\|\varphi_v * f\|_{L_\infty} \le C 2^{vn/p_0} \|f\|_{\dot{F}_{p_0}^{s_0 \infty}} = C 2^{vn/p_0}.$$

Therefore for any fixed integer N

$$\left(\sum_{-\infty}^{N} |2^{vs_1} \varphi_v * f|^r \right)^{1/r} \le C 2^{nN/p_1} \le t \tag{10}$$

If $t \approx C 2^{nN/p_1}$ On the other hand, since $s_1 < 0$

$$\left(\sum_{N}^{\infty} |2^{vs_1} \varphi_v * f|^r \right)^{1/r} \le C 2^{s_1 N} \sup_v |\varphi_v * f| \le C t^{1-p_1/p_0} \sup_v |\varphi_v * f|. \tag{11}$$

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Combining (10) and (11) we get

$$\|f\|_{\dot{F}_{p_1}^{s_1 r}}^{p_1} = p_1 \int_0^\infty t^{p_1-1} \left| \left\{ \left(\sum_{-\infty}^\infty |2^{vs_1}\varphi_v * f|^r \right)^{1/r} > t \right\} \right| dt$$

$$\leq p_1 \int_0^\infty t^{p_1-1} \left| \left\{ \sup_v |\varphi_v * f| > C t^{p_1/p_0} \right\} \right| dt \leq C \int_0^\infty t^{p_1-1} \left| \left\{ \sup_v |\varphi_v * f| > t \right\} \right| dt$$

$$= \|f\|_{\dot{F}_{p_0}^{0\infty}}^{p_1}.$$

This proves (ii).

(iii) By (9) it follows that

$$\dot{F}_{p'}^{s_0 \infty} \to \dot{B}_{p_1}^{s',\infty}$$

if $s' - n/p_1 = s_0 - n/p'$. Hence by interpolation

$$\left(\dot{F}_{p'}^{s_0 \infty}, \dot{F}_{p''}^{s_0 \infty} \right)_{\theta p_0} \to \left(\dot{B}_{p_1}^{s',\infty}, \dot{B}_{p_1}^{s'',\infty} \right)_{\theta p_0}$$

or by the lemma below

$$\dot{F}_{p_0}^{s_0 \infty} \to \dot{B}_{p_1}^{s_1 p_0}$$

if $1/p_0 = (1-\theta)/p' + \theta/p''$, $s_1 = (1-\theta)s' + \theta s''(0 < \theta < 1)$.

Lemma (1.1.5) [17]: Let $0 < p, q \leq \infty$. Concerning real interpolation we have

(i) $\left(\dot{B}_p^{s',q'}, \dot{B}_p^{s'',q''} \right)_{\theta q} = \dot{B}_p^{sq}$, if $s = (1-\theta)s' + \theta s'(0 < \theta < 1; s' \neq s'')$

(ii) $\left(\dot{F}_{p'}^{s\infty}, \dot{F}_{p''}^{s\infty} \right)_{\theta p} = \dot{F}_p^{s\infty}$, if $1/p = (1-\theta)/p' + \theta/p'' (0 < \theta < 1)$.

Proof: (i) is well-known; see [9]. We do not detail the proof of (iii). Roughly speaking, one first shows that $\dot{F}_p^{s\infty}$ is a retract of $H_p(l_\infty)$. Then it is just to invoke a vector valued version of the Fefferman-Riviere-Sagher theorem [4] on interpolation of Hardy spaces.

Now we consider an application of Theorem (1.1.4). Recall the n-dimensional version of Szasz' theorem (cf. [9]):

Lemma (1.1.6) [17]: Let $\hat{f}$ denote the Fourier transform of f. Then

$$\left\| \hat{f} \right\|_{L_p} \leq C\|f\|_{\dot{B}_2^{n(1/p-1/2),p}} \text{ if } 0 < p \leq 2.$$

Theorem (1.1.7) [17]: Let $0 < p < 2$ and $0 < q \leq \infty$. Then

$$\left(\int |\hat{f}(\xi)|^p / |\xi|^{n(2-p)} d\xi \right)^{1/p} \leq C\|f\|_{\dot{F}_p^{0q}}. \tag{12}$$

Proof: Let $g = I^{-n(2-p)/p} f$. By Theorem (1.1.4) :(iii) and (8) we have

$$\|g\|_{\dot{B}_2^{n(1/p-1/2),p}} \leq C\|f\|_{\dot{F}_p^{0q}}.$$

Hence, using the Lemma, we get

$$\|\hat{g}\|_{L_p} \leq C\|f\|_{\dot{F}_p^{0q}}.$$

which is the desired inequality.

The duals of $\dot{B}_p^{sq}$ and $\dot{F}_p^{sq}$ may be considered as sudspaces of S_0' because of (5) and (6). It turns out that to characterize them exactly if $0 < p < 1$ is to a large extent just another application of Theorem (1.1.4).

Theorem (1.1.8) [17]: Let s be real and $0 < p < 1$. Then

(i) $\left(\dot{B}_p^{sq}\right)' \approx \dot{B}_\infty^{-s+n(1/p-1),\infty}$ if $0 < q \le 1$

(ii) $\left(\dot{B}_p^{sq}\right)' \approx \dot{B}_\infty^{-s+n(1/p-1),q'}$ if $1 < q < \infty$
     where $1/q + 1/q' = 1$.

Proof: For simplicity we take $s = 0$. By Theorem (1.1.4): (i) we have

$$\dot{B}_p^{0q} \to \dot{B}_1^{-n(1/p-1),1} \quad \text{if } 0 < q \le 1$$

$$\dot{B}_p^{0q} \to \dot{B}_1^{-n(1/p-1),q} \quad \text{if } 1 < q < \infty$$

If we use the well-known fact (cf. [9])

$$\left(\dot{B}_1^{sq}\right)' \approx \dot{B}_\infty^{-s,q} \quad \text{if } 1 \le q < \infty$$

we therefore find

(i') $\dot{B}_\infty^{n(1/p-1),\infty} \to \left(\dot{B}_p^{0q}\right)'$ if $0 < q \le 1$

(ii') $\dot{B}_\infty^{n(1/p-1),q} \to \left(\dot{B}_p^{0q}\right)'$ if $0 < q < \infty$.

In order to prove the converse inclusions we fix a $f \in \left(\dot{B}_p^{0q}\right)'$ with $\|f\|_{\left(\dot{B}_p^{0q}\right)'} = 1$. (i) Is a consequence of (i') and

$$\|f\|_{\dot{B}_\infty^{n(1/p-1),\infty}} = \sup_v \sup_h \left| \langle f, 2^{vn(1/p-1)} \varphi_v(\cdot - h) \rangle \right| \le C \cdot 1.$$

Thus there only remains to verify the second half of (ii). Let $\{\varphi_v\}$ be a sequence of testfunctions satisfying in addition to (1) also $\sum_v \varphi_v = \delta$ (which is on restriction). Then we obviously have

$$f = \sum_v \varphi_v * f \equiv \sum_v a_v.$$

Assume with no loss of generality that $\text{supp}\, \hat{a}_v$ and $\text{supp}\, \varphi_\mu$ disjoint if $v \ne \mu$. (Just multiply $\hat{f}$ by a suitable function.) For a fixed integer $N > 0$ we define the sequence $\{b_v\}_{v=-N}^N$ by

$$b_v(x) = \varepsilon_v 2^{vn(1/p-1)} \|a_v\|_{\dot{B}_\infty^{n(1/p-1),\infty}}^{q'-1} \varphi_v(x - h_v), \qquad |\varepsilon_v| = 1$$

where $\{h_v\}$ and the argument of $\{\varepsilon_v\}$ are at our disposal. (That indeed $\|a_v\|_{\dot{B}_\infty^{n(1/p-1),\infty}} < \infty$ can be seen in the same way as for (i)). Clearly, in view of our assumption on the supports,

$$f\left(\sum_{-N}^{N}b_v\right) = \sum_{-N}^{N}f(b_v) \ge \sum_{-N}^{N}\|a_v\|_{\dot{B}_\infty^{n(1/p-1),\infty}}^{q'-1}\langle a_v, \varepsilon_v 2^{vn(1/p-1)}\varphi_v(\cdot - h_v)\rangle \ge C\sum_{-N}^{N}\|a_v\|_{\dot{B}_\infty^{n(1/p-1),\infty}}^{q'}$$

$$= \sum_{-N}^{N}\left(2^{vn(1/p-1)}\|a_v\|_{L_\infty}\right)^{q'}$$

If $\{h_v\}$ and the arguments of $\{\varepsilon_v\}$ are chosen properly. But on other hand

$$f\left(\sum_{-N}^{N}b_v\right) \le 1 \cdot \left\|\sum_{-N}^{N}b_v\right\|_{\dot{B}_p^{0q}} \le \left(\sum_{-N}^{N}\left(2^{vn(1/p-1)}\|a_v\|_{L_\infty}\right)^{q'}\right)^{1/q}.$$

Hence,

$$\left(\sum_{-N}^{N}\left(2^{vn(1/p-1)}\|a_v\|_{L_\infty}\right)^{q'}\right)^{1/q'} \le C$$

and the proof is complete if we let $N \to \infty$.

We turn our attention to $\dot{F}_p^{sq}$. To determine its dual almost becomes a triviality when knowing both Theorem (1.1.4) and Theorem (1.1.8):

Theorem (1.1.9) [17]: Let s be real and $0 < p < 1$. Then

$$\left(\dot{F}_p^{sq}\right)' \approx \dot{B}_\infty^{-s+n(1/p-1),\infty} \qquad \text{if } 0 < q < \infty.$$

Proof: From (2) we deduce that

$$\left(\dot{F}_p^{sq}\right)' \to \left(\dot{B}_p^{sq}\right)' \quad \text{if } q \le p$$

$$\left(\dot{F}_p^{sq}\right)' \to \left(\dot{B}_p^{sp}\right)' \quad \text{if } q \ge p.$$

Conversely, Theorem (1.1.4) (ii) or (iii) yields

$$\left(\dot{B}_1^{s-n(1/p-1),1}\right)' \to \left(\dot{F}_p^{sq}\right)'$$

Invoking Theorem (1.1.8) we see that the Besov spaces have the same dual and thus also $\dot{F}_p^{sq}$:

$$\left(\dot{F}_p^{sq}\right)' \approx \dot{B}_\infty^{-s+n(1/p-1),\infty}.$$

Let us denote a point $x \in \mathbb{R}^n$ by $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}^1$. Identify $\mathbb{R}^{n-1}$ with the hyperplance $x_n = 0$ in $\mathbb{R}^n$ and consider the trace operator

$$\text{Tr} : \mathcal{S}_0(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^{n-1})$$

Defined by

$$\text{Tr } f(x') = f(x', 0).$$

Theorem (1.1.10) [17]: Let $0 < p < \infty$, $0 < q \leq \infty$ and $s > 1/p + \max\big(0, (n-1)(1/p - 1)\big)$. Then the trace operator can be extended so that

$$\mathrm{Tr} : \dot{F}_p^{sq}(\mathbb{R}^n) \to \dot{B}_p^{s-1/p,p}(\mathbb{R}^{n-1}). \tag{13}$$

Conversely, there is an operator Sr

$$\mathrm{Sr} : \dot{B}_p^{s-1/p,p}(\mathbb{R}^{n-1}) \to \dot{F}_p^{sq}(\mathbb{R}^n) \tag{14}$$

so that $\mathrm{Tr} \circ \mathrm{Sr} = \mathrm{Id}$.

Proof: In proving (13) we shall for convenience assume $0 < p \leq 1$ (with minor modifications the same proof also works for $p > 1$). It also suffices to take $q = \infty$.

Let $f \in \dot{F}_p^{sq}(\mathbb{R}^n)$. If $q < \infty$ we can extend Tr by continuity, since $\mathcal{S}_0$ is then dense in $\dot{F}_p^{sq}(\mathbb{R}^n)$. For $q = \infty$ this is no longer applicable. However, for all q we can define Tr by

$$\mathrm{Tr}\, f(x') = \sum_{v \in \mathbb{Z}} \varphi_v * f(x', 0) \equiv \sum_{v \in \mathbb{Z}} a_v$$

where $\{\varphi_v\}_{v \in \mathbb{Z}}$ is a sequence of testfunctions on $\mathbb{R}^n$ satisfying (1) and $\sum_v \varphi_v = \delta$. Obviously, it is an extension of our original Tr. That the sum has a limit and thus that Tr is well-defined.

Lemma (1.1.11) [17]: If $f, g \in \mathcal{S}_0'$ and $\mathrm{supp}\,\hat{f}, \mathrm{supp}\,\hat{g} \subset \{|\xi| \leq r\}$ then

$$\|f * g\|_{L_p} \leq C r^{n(1/p-1)} \|f\|_{L_p} \|g\|_{L_p} \qquad \text{if } 0 < p \leq 1$$

$$\|f * g\|_{L_p} \leq \|f\|_{L_1} \|g\|_{L_p} \qquad \text{if } 0 < p \leq \infty.$$

For a proof we refer the reader to [9]. The second lemma is also a result by Peetre [10] (we shall only need it for $q = \infty$):

Set

$$\varphi_v^a f(x) = \sup_{|x-y| \leq 2^{-v} a} |\varphi_v * f(y)|$$

For a fixed $a \geq 0$.

Lemma (1.1.12) [17]: Let s be real and $0 < p < \infty$, $0 < q \leq \infty$. Then

$$\|f\|_{\dot{F}_p^{sq}} \approx \left\| \left(\sum_v |2^{vs} \varphi_v^a f|^q \right)^{1/q} \right\|_{L_p}.$$

Now the proof of (13) is easily accomplished. If $\{\varphi_v'\}_{v \in \mathbb{Z}}$ is sequence of testfunctions on $\mathbb{R}^{n-1}$ satisfying (1), then

$$\mathrm{Tr}\, f * \varphi_v' = \sum_{v \in \mathbb{Z}} a_\mu * \varphi_v'.$$

Consequently,

$$\|\mathrm{Tr}\, f * \varphi_v'\|_{L_p(\mathbb{R}^{n-1})}^p \leq \sum_{\mu \geq v-1} \|a_\mu * \varphi_v'\|_{L_p(\mathbb{R}^{n-1})}^p$$

Since

$$(x + y)^p \leq x^p + y^p$$

when $x, y \geq 0$ and $0 < p \leq 1$. By Lemma (1.1.11) we have

$$\|a_\mu * \varphi_v'\|_{L_p(\mathbb{R}^{n-1})} \leq C2^{\mu(t-1/p)}\|a_\mu\|_{L_p(\mathbb{R}^{n-1})}\|\varphi_v'\|_{L_p(\mathbb{R}^{n-1})}$$

with $t = 1/p + (n-1)(1/p - 1)$. But

$$\|\varphi_v'\|_{L_p(\mathbb{R}^{n-1})} \approx 2^{-v(t-1/p)}.$$

Hence,

$$2^{v(s-1/p)p}\|\varphi_v' * \mathrm{Tr}\, f\|_{L_p(\mathbb{R}^{n-1})}^p \leq C \sum_{\mu \geq v-1} 2^{(v-\mu)(s-t)p}2^{-\mu}\|2^{\mu s}a_\mu\|_{L_p(\mathbb{R}^{n-1})}^p$$

Inserting this in to the definition of Besov spaces (Definition (1.1.1)) and using Minkowski's inequality for sums we find

$$\|f\|_{\dot{B}_p^{s-1/p,p}(\mathbb{R}^{n-1})}^p \leq C \sum_{v \leq 1} 2^{v(s-t)p} \sum_\mu 2^{-\mu}\|2^{\mu s}a_\mu\|_{L_p(\mathbb{R}^{n-1})}^p \leq C \sum_\mu 2^{-\mu}\|2^{\mu s}a_\mu\|_{L_p(\mathbb{R}^{n-1})}^p$$

Since $s > t$. However,

$$\sum_\mu 2^{-\mu}\|2^{\mu s}a_\mu\|_{L_p(\mathbb{R}^{n-1})}^p \leq \sum_\mu \int_{2^{-\mu}}^{2^{-\mu+1}} \|2^{\mu s}\varphi_\mu^2 f(\cdot, x_n)\|_{L_p(\mathbb{R}^{n-1})}^p dx_n \leq C \left\|\sup_\mu |2^{\mu s}\varphi_\mu^2 f|\right\|_{L_p(\mathbb{R}^n)}^p.$$

Thus by Lemma (1.1.12)

$$\|f\|_{\dot{B}_p^{s-1/p,p}(\mathbb{R}^{n-1})} \leq C\|f\|_{\dot{F}_p^{s\infty}(\mathbb{R}^n)}. \tag{15}$$

This concludes the proof of (13).

We turn to (14). Now we can take q very small, at least $q \leq p$. Let $\{\varphi_v'\}_{v \in \mathbb{Z}}$ and $\{\psi_v\}_{v \in \mathbb{Z}}$ be testfunctions on $\mathbb{R}^{n-1}$ and respectively, satisfying (1) as well as

$$\sum_v \varphi_v' = \delta, \qquad \psi_v(x_n) = 2^v \psi_0(2^v x_n), \qquad \psi_0(0) = 1.$$

Again in order to avoid some trivial technical nuisances we assume that

$$\begin{aligned} \varphi_v' * \varphi_\mu' * f &= 0 && \text{if } v \neq \mu \\ \psi_v * \psi_\mu &= 0 && \text{if } v \neq \mu \end{aligned} \tag{16}$$

However, without any loss of generality we may assume that the testfunctions $\{\varphi_v\}_{v \in \mathbb{Z}}$ on $\mathbb{R}^n$ are of the form

$$\varphi_v = \varphi'_v \otimes \sum_{\mu \leq v} \psi_\mu + \sum_{\mu \leq v} \varphi'_\mu \otimes \psi_v. \tag{17}$$

Put

$$\mathrm{Sr}\, f(x', x_n) = \sum_\mu 2^{-\mu} \varphi'_\mu * f(x') \otimes \psi_\mu(x_n).$$

Clearly,

$$\mathrm{Sr}\, f(x', 0) = f(x').$$

That is, $\mathrm{Tr} \circ \mathrm{Sr} = \mathrm{Id}$. Because of (16) and (17) we see that

$$\mathrm{Sr}\, f * \varphi_v = 2^{-v+1} \varphi'_v * \varphi'_v * f \otimes \psi_v * \psi_v.$$

Hence,

$$\|\mathrm{Sr}\, f\|^p_{\dot{F}^{sq}_p(\mathbb{R}^n)} \leq C \left\| \left(\sum_v |2^{v(s-1)} \varphi'_v * \varphi'_v * f \otimes \psi_v * \psi_v|^q \right)^{1/q} \right\|^p_{L_p(\mathbb{R}^n)}$$

$$\leq C \int_{-\infty}^{\infty} \left| \sum_v \left(2^{v(1/p-1)} a_v \psi_v * \psi_v(x_n) \right)^q \right|^{p/q} dx_n$$

with

$$a_v = 2^{v(s-1/p)} \|\varphi'_v * \varphi'_v * f\|_{L_p(\mathbb{R}^{n-1})}.$$

Inserting the trivial estimate

$$|\psi_v * \psi_v(x_n)| \leq C 2^v \min(1, (2^v |x_n|)^{-j})$$

for an arbitrarily large j, gives with $r = p/q$

$$\|\mathrm{Sr}\, f\|^p_{\dot{F}^{sq}_p(\mathbb{R}^n)} \leq C \int_{-\infty}^{\infty} \left| \sum_v (2^{v/p} a_v \min(1, (2^v |x_n|)^{-j}))^q \right|^{p/q} dx_n$$

$$\leq C \sum_\mu \left(\sum_v \left(2^{v-\mu} a_v^p \min(1, (2^{v-\mu})^{-jp}) \right)^{1/r} \right)^r$$

If we use Minkowski again, we see that

$$\|\mathrm{Sr}\, f\|^p_{\dot{F}^{sq}_p(\mathbb{R}^n)} \leq C \sum_v a_v^p \leq C \|f\|^p_{\dot{B}^{s-1/p,p}_p(\mathbb{R}^{n-1})}.$$

This is the desired inequality and thus the proof of the theorem is complete.

Section (1.2): Besov Spaces and Decompositions

Many functions or distribution spaces have been found to admit decomposition, in the sense that every member of the space is a linear combination of basic functions of a particularly elementary form. Such decompositions simplify the analysis of the spaces and

the operators acting on mogeneous Besov spaces $\dot{B}_p^{\alpha q}$, $-\infty < \alpha < +\infty$, $0 < p, q \leq +\infty$, and present some applications of these results.

Defining the Fourier transform by $\hat{f}(\xi) = \int f(x)e^{-ix\cdot\xi}dx$, let $\{\varphi_v\}_{v\in\mathbb{Z}}$ be a family of functions on $\mathbb{R}^n$ satisfying

$$\varphi_v \in \mathcal{S} \tag{18}$$

$$\text{supp } \hat{\varphi}_v \subseteq \left\{ \xi \in \mathbb{R}^n : \frac{1}{2} \leq 2^{-v}|\xi| \leq 2 \right\}, \tag{19}$$

$$|\hat{\varphi}_v(\xi)| \geq c > 0 \text{ if } \frac{3}{5} \leq 2^{-v}|\xi| \leq \frac{5}{3}, \tag{20}$$

and

$$|\partial^\gamma \hat{\varphi}_v(\xi)| \geq c_\gamma 2^{-v|\gamma|}, \qquad \text{for every multi- index } \gamma. \tag{21}$$

The Besov space $\dot{B}_p^{\alpha q}$, $-\infty < \alpha < +\infty$, $0 < p, q \leq +\infty$, is the collection of all $f \in \mathcal{S}'/\mathcal{S}$ (tempered distributions modulo polynomials) such that

$$\|f\|_{\dot{B}_p^{\alpha q}} = \left(\sum_{v\in\mathbb{Z}} (2^{v\alpha}\|\varphi_v * f\|_{L^p})^q \right)^{1/q} < +\infty,$$

with the usual interpretation if $q = +\infty$. This definition is independent of the family $\{\varphi_v\}$ satisfying (18, 19, 20 and 21); see [26].

We show that each $f \in \dot{B}_p^{\alpha q}$ can be decomposed into a sum of simple building blocks. The building blocks in our first decomposition are similar to the atoms in the atomic decomposition of Hardy spaces $H^p(\mathbb{R}^n)$, $0 < p \leq 1$ ([27, 33, 23, 43]). We define an (α, p)-atom $a(x)$ $(-\infty < \alpha < +\infty, 0 < p \leq +\infty)$ to be a function satisfying, for some cube $Q \subseteq \mathbb{R}^n$,

$$\text{supp } a \subseteq 3Q, \tag{22}$$

$$|\partial^\gamma a(x)| \leq |Q|^{-\alpha/n-1/p-|\gamma|/n} \quad \text{if } |\gamma| \leq K, \tag{23}$$

and

$$\int x^\gamma a(x)dx = 0 \quad \text{if } |\gamma| \leq N, \tag{24}$$

where $K \geq ([\alpha] + 1)_+$ and $N \geq \max([n(1/p - 1)_+ - \alpha], -1)$ are fixed integers. Here $x_+ = \max(x, 0)$, $[x]$ is the greatest integer in x, and $3Q$ is the cube in $\mathbb{R}^n$ concentric with Q but with side length three times the side length $\ell(Q)$ of Q. In (24), $N = -1$ means that $a(x)$ is not required to have any vanishing moments.

We write a_Q for an atom satisfying (22, 23 and 24) for a given cube Q, and adopt the convention hereafter that whenever Q appears as a summation index, the sum runs only over dyadic cubes. Our result is the quasi-norm equivalence

$$\|f\|_{\dot{B}_p^{\alpha q}} \approx \inf \left\{ \left(\sum_{v \in \mathbb{Z}} \left(\sum_{\ell(Q)=2^{-v}} |s_Q|^p \right)^{q/p} \right)^{1/q} : f = \lim_{j \to \infty} \sum_{v=-j}^{j} \sum_{\ell(Q)=2^{-v}} s_Q a_Q, \right.$$

$$\left. \text{(in } \mathcal{S}'/\mathcal{S}) \text{ and each } a_Q \text{ is an } (\alpha, p)\text{- atom} \right\}. \tag{25}$$

In our other decomposition of $\dot{B}_p^{\alpha q}$, the building blocks, although not of compact support, are taken from a fixed, explicitly given family of functions which have simple properties. Fixed $\psi \in \mathcal{S}$ satisfying $\operatorname{supp} \hat{\psi}(\xi) \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq \pi\}$, $\int x^\gamma a(x) dx = 0$ if $|\gamma| \leq N$, and $\hat{\psi}(\xi) \geq c > 0$ if $1/2 \leq |\xi| \leq 2$ (N is the fixed integer above). For each $v \in \mathbb{Z}$ and $k = (k_1, \cdots, k_n) \in \mathbb{Z}^n$, set

$$Q_{vk} = \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n : k_i 2^{-v} \leq x_i < (k_i + 1)2^{-v}, \quad i = 1, \cdots, n\}, \tag{26}$$

and define

$$\psi_Q(x) = |Q|^{\alpha/n - 1/p} \psi(2^v x - k) \quad \text{if } Q = Q_{vk} \tag{27}$$

We will show that

$$\|f\|_{\dot{B}_p^{\alpha q}} \approx \inf \left\{ \left(\sum_{v \in \mathbb{Z}} \left(\sum_{\ell(Q)=2^{-v}} |s_Q|^p \right)^{q/p} \right)^{1/q} : f = \lim_{j \to \infty} \sum_{v=-j}^{j} \sum_{\ell(Q)=2^{-v}} s_Q \psi_Q \text{ (in } \mathcal{S}'/\mathcal{S}) \right\} \tag{28}$$

In fact, in our representation $f = \sum_Q s_Q \psi_Q$, each s_Q for $Q = Q_{vk}$, is a multiple of the "sample value" $\varphi_Q * f(x_Q)$ for appropriate $\{\varphi_Q\}$ satisfying (18, 19, 20 and 21), where $x_Q = 2^{-v}k$.

After completing the proofs of (25) and (28), we consider the space of functions of bounded mean oscillation (BMO). Our decomposition (25) of $\dot{B}_p^{\alpha q}$ corresponds to the decomposition of BMO given by Uchiyama [42]. We show the analogue of (28) for BMO.

Both of our decomposition methods utilize a discrete version of Calderón's reproducing formula ([24, 26]), and a classical result of Plancherel-Pólya [36]. The primary difference between these methods is the manner in which a certain convolution is written as a discrete sum. In (28) this is done on the Fourier transform, or frequency, side using a Fourier series, while in (25) it is done directly on the time side.

Each decomposition has advantages. For example, we see that the fact that the (α, p)-atoms in (25) have compact support is convenient for the consideration of "trace" problems. The well-known result that

$$\operatorname{Tr} \dot{B}_p^{\alpha q}(\mathbb{R}^n) = \dot{B}_p^{\alpha - 1/p, q}(\mathbb{R}^{n-1}), \quad \text{if } \alpha - 1/p > (n-1)(1/p - 1)_+$$

is immediate, and the result that

$$\operatorname{Tr} \dot{B}_p^{1/p, 1}(\mathbb{R}^n) = L^p(\mathbb{R}^{n-1}), \quad \text{if } 1 \leq p < +\infty$$

([22, 31, 35]) is extended to show that $\text{Tr}\dot{B}_p^{1/p,1}(\mathbb{R}^n) = L^p(\mathbb{R}^{n-1})$ whenever $0 < p < +\infty$, $q \leq \max(1, p)$.

One advantage of (28) is the simplicity of the building blocks ψ_Q on the frequency side. This is exploited to give a very simple proof that $\dot{B}_p^{\alpha q}$ has the lower majorant property if $0 < p \leq 1$.

Calderón's formula plays a key role in many decomposition results; it is crucial to the simplest known proofs of the atomic decomposition of $H^p(\mathbb{R}^n)$, $0 < p \leq 1$ ([23, 43]). Uchiyama [42], following Chang-Fefferman [25], used a similar formula to show the BMO decomposition mentioned above. Our application of Calderón's formula to $\dot{B}_p^{\alpha q}$ in general was prompted by recent work of Wilson [44], who used it to obtain (25) for the case of the "special atoms" space $\dot{B}_1^{01}(\mathbb{R}^1)$. Another application was made by Cohen [26], who decomposed the spaces $H^{(p,q)}$, which are similar to the Besov spaces. Methods not relying on Calderón's formula have been used by de Souza et al. [29] to obtain (25) for $\dot{B}_1^{\alpha 1}$, $0 < \alpha < 1$, on the circle.

A motivation for our second decomposition is the work of Coifman-Rochberg [28], where Bergman spaces are decomposed into building blocks obtained from the Bergman kernel. Rochberg-Semmes [39] obtain similar results for BMO. Ricci-Taibleson [37] employ related ideas to show (25) in $\mathbb{R}^1$ for $\alpha < 0$; their work is extended to $\mathbb{R}^n$ by Bui [21].

Lemma (1.2.1) [45]: Suppose $f \in \mathcal{S}'/\mathcal{S}$ and that φ and ψ are functions satisfying

$$\varphi, \psi \in \mathcal{S} \tag{29}$$

$$\text{supp}\,\widehat{\varphi} \subseteq \{\xi : |\xi| < \pi\}, \qquad \text{supp}\,\widehat{\psi} \subseteq \{\xi : |\xi| \leq \pi\}, \tag{30}$$

and

$$\sum_{v \in \mathbb{Z}} \widehat{\varphi}(2^v \xi)\widehat{\psi}(2^v \xi) = 1 \text{ if } \xi \in \mathbb{R}^n \backslash \{0\}. \tag{31}$$

If $\varphi_v(x) = 2^{vn}\varphi(2^v x)$ and $\psi_v(x) = 2^{vn}\psi(2^v x)$, then

$$f(\cdot) = \sum_{v \in \mathbb{Z}} 2^{-vn} \sum_{k \in \mathbb{Z}^n} \varphi_v * f(2^{-v}k)\psi_v(\cdot - 2^{-v}k) \tag{32}$$

In (32), the convergence of the right-hand side, as well as the equality, is in $\mathcal{S}'/\mathcal{S}$.

Proof: By (31), we have

$$f = \sum_{v \in \mathbb{Z}} \psi_v * \varphi_v * f \tag{33}$$

Hence, (32) will follow from

$$\psi_v * \varphi_v * f(x) = 2^{-vn} \sum_{k \in \mathbb{Z}^n} \varphi_v * f(2^{-v}k)\psi_v(x - 2^{-v}k) \tag{34}$$

To prove (34), we note that $\varphi_v * f$ is slowly increasing, and hence,

$$f_{v,\delta}(x) = \varphi_v * f(x) \prod_{i=1}^{n} \left(\frac{\sin(\delta x_i)}{\delta x_i} \right)^j \in L^2$$

If j is large enough. Also, by (30) and the fact that

$$\text{supp}[(\sin \delta x)/\delta x]^\wedge \subseteq [-\delta, \delta], \text{supp}\, \hat{f}_{v,\delta}(\xi) \subseteq \{\xi : |\xi| \leq 2^v \pi\}$$

If δ is sufficiently small. We have

$$f_{v,\delta}(x) * \psi_v = (2\pi)^{-n} \int \hat{f}_{v,\delta}(\xi) \widehat{\psi}_v(\xi) e^{ix\cdot\xi} d\xi. \tag{35}$$

Extending $\widehat{\psi}_v(\xi)e^{ix\cdot\xi}$ periodically with period $2^{v+1}\pi$ in each variable and representing $\widehat{\psi}_v(\xi)e^{ix\cdot\xi}$ by its Fourier series, we obtain, by Fourier inversion,

$$\widehat{\psi}_v(\xi)e^{ix\cdot\xi} = 2^{-vn} \sum_{k \in \mathbb{Z}^n} \psi_v(x - 2^{-v}k)e^{i2^{-v}k\cdot\xi} \tag{36}$$

If $|\xi| \leq 2^v\pi$. Inserting (36) in (35) and using Fourier inversion again yields

$$f_{v,\delta} * \psi_v(x) = 2^{-vn} \sum_{k \in \mathbb{Z}^n} \psi_v(x - 2^{-v}k)f_{v,\delta}(2^{-v}k). \tag{37}$$

Letting $\delta \to 0$ in (37) and applying the dominated convergence theorem, (34) follows.

Lemma (1.2.2) [45]: Let $0 < p \leq +\infty$ and $v \in \mathbb{Z}$. Suppose $g \in \mathcal{S}'$ and $\text{supp}\, \hat{g} \subseteq \{\xi : |\xi| \leq 2^{v+1}\}$. If $\mathcal{Q}_{vk}$ is defined by (26), then

$$\left(\sum_{k \in \mathbb{Z}^n} \sup_{z \in \mathcal{Q}_{vk}} |g(z)|^p \right)^{1/p} \leq c_{n,p} 2^{v(n/p)} \|g\|_{L^p}. \tag{38}$$

Proof: By the Paley-Wiener theorem, g is a function of exponential type 2^{v+1}. Also, g is slowly increasing. Let $\psi \in \mathcal{S}$ satisfy $\text{supp}\, \widehat{\psi} \subseteq \{\xi : |\xi| \leq \pi\}$ and $\widehat{\psi}(\xi) = 1$ if $|\xi| \leq 2$. If $\psi_v(x) = 2^{vn}\psi(2^v x)$, and $g(x + y) = g^y(x)$, then, exactly as in the proof of (34),

$$g(x + y) = \psi_v * g^y(x) = 2^{-vn} \sum_{\ell \in \mathbb{Z}^n} g(2^{-v}\ell + y)\psi_v(x - 2^{-v}\ell).$$

Therefore, for any $y \in \mathcal{Q}_{vk}$,

$$\sup_{z \in \mathcal{Q}_{vk}} |g(z)| \leq \sup_{|x| \leq 2^{-v}\sqrt{n}} |g(x + y)| \leq 2^{-vn} \sum_{\ell \in \mathbb{Z}^n} g(2^{-v}\ell + y) \sup_{|x| \leq 2^{-v}\sqrt{n}} |\psi_v(x - 2^{-v}\ell)|.$$

Since $\psi \in \mathcal{S}$,

$$\sup_{|x| \leq 2^{-v}\sqrt{n}} |\psi_v(x - 2^{-v}\ell)| \leq c_M 2^{vn}(1 + |\ell|)^{-M},$$

for any M. Taking M sufficiently large and applying the p-triangle inequality $|a + b|^p \leq |a|^p + |b|^p$ if $0 < p \leq 1$ or Hölder's inequality if $p > 1$, we obtain

$$\sup_{z \in Q_{vk}} |g(z)|^p \leq c_p \sum_{\ell \in \mathbb{Z}^n} g(2^{-v}\ell + y)(1 + |\ell|)^{-n-1}$$

for any $y \in Q_{vk}$. Integrating with respect to y over Q_{vk} yields

$$2^{-vn} \sup_{z \in Q_{vk}} |g(z)|^p \leq c_p \sum_{\ell \in \mathbb{Z}^n} (1 + |\ell|)^{-n-1} \int_{Q_{v,k+\ell}} |g(x)|^p dx. \tag{39}$$

Summing over $k \in \mathbb{Z}^n$, we obtain

$$2^{-vn} \sum_{k \in \mathbb{Z}^n} \sup_{z \in Q_{vk}} |g(z)|^p \leq c_p \|g\|_{L^p}^p \sum_{\ell \in \mathbb{Z}^n} (1 + |\ell|)^{-n-1} \leq c_{p,n} \|g\|_{L^p}^p,$$

which proves (38).

Remark (1.2.3) [45]: An alternate way of proving Lemma (1.2.2), which has the advantage of making the connection with Hardy spaces more obvious, is to observe that

$$\sup_{y \in Q_{vk}} |g(y)| \leq \inf_{z \in Q_{vk}} N_v(g)(z),$$

where $N_v(g)(z) = \sup_{|z-y| < 2^{-v}\sqrt{n}} |g(y)|$. Hence

$$\left(\sum_{k \in \mathbb{Z}^n} \sup_{y \in Q_{vk}} |g(y)|^p \right)^{1/p} \leq 2^{v(n/p)} \|N_v(g)\|_{L^p}.$$

Since $\operatorname{supp} \hat{g} \subseteq \{\xi : |\xi| \leq 2^{v+1}\}$, (37) follows from the well-known inequality $\|N_v(g)\|_{L^p} \leq c\|g\|_{L^p}$ ([5, 27, 32, 41]); if in addition $\hat{g}(\xi) = 0$ for $|\xi| < 2^{v-1}$, this is essentially a restatement of the fact that $\|g\|_{L^p} \approx \|g\|_{H^p}$.

In addition to (38), Plancherel-Pólya [36] that the inequality

$$\|g\|_{L^p} \leq c_{p,n} 2^{-vn/p} \left(\sum_{k \in \mathbb{Z}^n} \inf_{z \in Q_{vk}} |g(z)|^p \right)^{1/p} \tag{40}$$

holds as well if $\operatorname{supp} \hat{g} \subseteq \{\xi : |\xi| \leq \varepsilon 2^{v+1}\}$ for ε sufficiently small. A proof of this can be given, using Lemma (1.2.1) in a manner similar to our proof of Lemma (1.2.2). A more precise statement can be obtained from an interpolation formula in Boas, [20].

Our main decomposition theorem for $\dot{B}_p^{\alpha q}$ now follows readily from Lemmas (1.2.1) and (1.2.2).

Theorem (1.2.4) [45]: Let $-\infty < \alpha < +\infty$, $0 < p, q \leq +\infty$. Then each $f \in \dot{B}_p^{\alpha q}$ can be decomposed as follows:

(i) $f = \sum_{v \in \mathbb{Z}} \sum_{\ell(Q) = 2^{-v}} s_Q \psi_Q$,
   where the ψ_Q's are defined by (37), and
(ii) $f = \sum_{v \in \mathbb{Z}} \sum_{\ell(Q) = 2^{-v}} s_Q a_Q$
   where the a_Q's are (α, p)-atoms.

In both cases the convergence is in $\mathcal{S}'/\mathcal{S}$ and the numbers s_Q satisfy

$$\left(\sum_{v\in\mathbb{Z}}\left(\sum_{\ell(Q)=2^{-v}}|s_Q|^p\right)^{q/p}\right)^{1/p} \leq c\|f\|_{\dot{B}_p^{\alpha q}} \tag{41}$$

for some constant c independent of f.

Proof:

(i) Our assumption on ψ imply that there is a function φ satisfying (18, 19, 20, 21) for $v = 0$, and such that (31) holds. By Lemma (1.2.1),

$$f(\cdot) = \sum_{v\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n}\varphi_v * f(2^{-v}k)2^{vn(\alpha/n-1/p)}\psi_{Q_{vk}}(\cdot)$$

For $Q = Q_{vk}$, define

$$s_Q = 2^{vn(\alpha/n-1/p)}\varphi_v * f(2^{-v}k). \tag{42}$$

Clearly $f = \sum_Q s_Q\psi_Q$, and (40) follows easily by applying (37) to $\varphi_v * f$.

(ii) Select a function $\theta \in \mathcal{S}$ satisfying $\operatorname{supp}\theta \subseteq \{x : |x| \leq 1\}$, $\int x^\gamma\theta(x)dx = 0$ if $|\gamma| \leq N$, and $\hat{\theta}(\xi) \geq c > 0$ if $1/2 \leq |\xi| \leq 2$. (Such a θ is easy to construct: let $\Theta \in \mathcal{S}$ be a real-valued radial function satisfying $\operatorname{supp}\Theta \subseteq \{x : |x| \leq 1\}$ and $\hat{\Theta}(0) = 1$. Then for some $\varepsilon > 0$, $\hat{\Theta}(\xi) = 1/2$ for all ξ satisfying $|\xi| < 2\varepsilon < 1$. Then $\theta(x) = (-\Delta)^N\left(\varepsilon^{-n}\Theta(x/\varepsilon)\right)$ satisfies all of the requirements.) Our conditions on θ guarantee that a function φ exists which satisfies (18, 19, 20, 21) for $v = 0$ and so that (31) holds. Therefore, if $\theta_v(x) = 2^{vn}\theta(2^v x)$,

$$f = \sum_{v\in\mathbb{Z}}\theta_v * \varphi_v * f = \sum_{v\in\mathbb{Z}}\sum_{\ell(Q)=2^{-v}}\int_Q\theta_v(x-y)\,\varphi_v * f(y)dy.$$

Define, for $Q = Q_{vk}$,

$$s_Q = C2^{vn(\alpha/n-1/p)}\sup_{y\in Q}|\varphi_v * f(y)| \tag{43}$$

and

$$a_Q(x) = \frac{1}{s_Q}\int_Q\theta_v(x-y)\varphi_v * f(y)dy$$

Where C is a constant, picked large enough so that every a_Q satisfy (23). The a_Q's are (α, p)-atoms, since (22) and (24) are consequences of our requirements on θ. Finally, (41) follows from Lemma (1.2.2) exactly as in (i).

Remark (1.2.5): There is a simpler proof for $p \geq 1$ of Theorem (1.2.4) (ii), which does not depend on Lemma (1.2.2). Replace (43) by

$$s_Q = C|Q|^{-\alpha/n}\left(\int_Q|\varphi_v * f(y)|^p dy\right)^{1/p} \tag{44}$$

(cf. [42]), and continue with $a_Q(x) = (1/s_Q) \int_Q \theta_\nu(x-y)\varphi_\nu * f(y)dy$ as above. Then (23) follows by Hölder's inequality if C is a large enough constant, and (22) and (24) follow as above. With this definition of s_Q, (41) is trivial.

In each of our decompositions, s_Q, for $Q = Q_{\nu k}$, is determined by the values of $\varphi_\nu * f$ on $Q_{\nu k}$. Up to multiple factors, s_Q in (43) is the sup of $|\varphi_\nu * f|$ on $Q_{\nu k}$, s_Q in (44) is the L^p-average of $\varphi_\nu * f$ on $Q_{\nu k}$. And in (42) s_Q is the sample value $\varphi_\nu * f(2^{-\nu}k)$. By Plancherel-Pólya, (38) and (40), these values are roughly interchangeable.

We call a function m an (α, p)-molecule if there exist $\mu \in \mathbb{Z}$ and a point $x_0 \in \mathbb{R}^n$ such that

$$|\partial^\gamma m(x)| \le 2^{\mu(n/p-\alpha+|\gamma|)}(1 + 2^\mu|x - x_0|)^{-M-|\gamma|} \qquad \text{if } |\gamma| \le K \tag{45}$$

and

$$\int x^\gamma m(x)dx = 0 \qquad \text{if } |\gamma| \le N, \tag{46}$$

Where M is a large, fixed number; $M \ge N + 10n \max(1/p, 1)$ is certainly enough. We recall that $K \ge ([\alpha] + 1)_+$ and $N \ge \max(n[(1/p - 1)_+ - \alpha], -1)$ are fixed integers.

For each $\mu \in \mathbb{Z}$, let $\{x_{\mu,j}\}_j$ be an arbitrary sequence of point in $\mathbb{R}^n$. We will write $m = m_{\mu,j}$ if m satisfies (45, 46) for $x_0 = x_{\mu,j}$.

We will use the notation m_Q for an (α, p)-molecule which is in fact concentrated on the dyadic cube $Q = Q_{\mu,\ell}$ (defined by (26)); i.e. m_Q satisfies (45, 46) with $\ell(Q) = 2^{-\mu}$ and $x_0 = x_Q = 2^{-\mu}\ell$.

This distinction in notation is adopted to emphasize the fact that our estimates require the (α, p)-molecules to be in correspondence with the dyadic cubes only if $1 < p \le +\infty$.

Theorem (1.2.8) [45]: Let $-\infty < \alpha < +\infty$, $0 < q \le +\infty$.

(a) Let $0 < p \le 1$. Suppose $f = \sum_{\mu \in \mathbb{Z}} \sum_j s_{\mu,j} m_{\mu,j}$, where the $m_{\mu,j}$'s are (α, p)-molecules, indexed as above. Then

$$\|f\|_{\dot{B}_p^{\alpha q}} \le c \left(\sum_{\mu \in \mathbb{Z}} \left(\sum_j |s_{\mu,j}|^p \right)^{q/p} \right)^{1/q}.$$

(b) Let $1 < p \le +\infty$. Suppose $f = \sum_{\mu \in \mathbb{Z}} \sum_{\ell(Q)=2^{-\mu}} s_Q m_Q$, where each m_Q is an (α, p)-molecule concentrated on Q. Then

$$\|f\|_{\dot{B}_p^{\alpha q}} \le c \left(\sum_{\mu \in \mathbb{Z}} \left(\sum_{\ell(Q)=2^{-\mu}} |s_Q|^p \right)^{q/p} \right)^{1/q}.$$

In both cases the constant c is independent of f.

Proof: To prove (i), we write

$$\varphi_v * \sum_{\mu \in \mathbb{Z}} \sum_j s_{\mu,j} m_{\mu,j} = \varphi_v * \left(\sum_{\mu=-\infty}^{v} + \sum_{\mu=v+1}^{+\infty} \right) \sum_j s_{\mu,j} m_{\mu,j}$$

and use Lemma (1.2.7) below and the p-triangle inequality $|a + b|^p \le |a|^p + |b|^p$ to see that

$$\|f\|_{\dot{B}_p^{\alpha q}}^q \le c \sum_{v \in \mathbb{Z}} \left(\sum_{\mu=-\infty}^{v} 2^{-(v-\mu)(K-\alpha)p} \sum_j |s_{\mu,j}|^p \right)^{q/p}$$

$$+ c \sum_{v \in \mathbb{Z}} \left(\sum_{\mu=v+1}^{+\infty} 2^{-(\mu-v)(N+1+n-n/p+\alpha)p} \sum_j |s_{\mu,j}|^p \right)^{q/p}.$$

The proof of (ii) is similar. If $\mu > v$ we apply (47) and Lemma (1.2.8) with $\mu = v$, and if $\mu \le v$ we apply (48) and Lemma (1.2.8) with $\eta = \mu$, to obtain

$$\|f\|_{\dot{B}_p^{\alpha q}}^q \le c \sum_{v \in \mathbb{Z}} \left(\sum_{\mu=-\infty}^{v} 2^{-(v-\mu)(K-\alpha)} \left(\sum_{\ell(Q)=2^{-\mu}} |s_Q|^p \right)^{1/p} \right)^q$$

$$+ c \sum_{v \in \mathbb{Z}} \left(\sum_{\mu=v+1}^{\infty} 2^{-(\mu-v)(N+1+\alpha)} \left(\sum_{\ell(Q)=2^{-\mu}} |s_Q|^p \right)^{1/p} \right)^q.$$

Now $K - \alpha > 0$ and $N + 1 + \alpha > 0$ by definition, so (ii) follows by considering $q \ge 1$ and $q < 1$ separately, similarly to the proof of (i).

Lemma (1.2.7) [45]: Let $m_{\mu,j}$ be an (α, p)-molecule. Then

$$\left| \varphi_v * m_{\mu,j}(x) \right| \le 2^{\mu(n/p-\alpha)} 2^{-(\mu-v)(N+1+n)} \left(1 + 2^v |x - x_{\mu,j}| \right)^{N+1+n-M} \tag{47}$$

if $v \le \mu$, and

$$\left| \varphi_v * m_{\mu,j}(x) \right| \le 2^{\mu(n/p-\alpha)} 2^{-(v-\mu)K} \left(1 + 2^\mu |x - x_{\mu,j}| \right)^{N+1+n-M} \tag{48}$$

if $\mu \le v$.

Proof: Consider (47) first. By translation and dilation invariance, we may assume $v = 0$ and $x_{\mu,j} = 0$. Put $m = m_{\mu,j}$ and $\varphi = \varphi_0$. By (46),

$$\varphi * m(x) = \int m(x - y) \left(\varphi(y) - \sum_{|\beta| \le N} \partial^\beta \varphi(x) (y - x)^\beta / \beta! \right) dy.$$

Hence,

$$|\varphi * m(x)| \le \left(\int_{|x-y| \le |x|/2} + \int_{|x-y| > |x|/2} \right) |m(x - y)| |x - y|^{N+1} \Phi(x, y) dy = I + II$$

where $\Phi(x,y) = \sup\limits_{|\beta|=N+1} \sup\limits_{0<\varepsilon<1} |\partial^\beta \varphi(x + \varepsilon(y-x))|/\beta!$. Since $\varphi \in \mathcal{S}$, $\Phi(x,y) = c(1+|x|)^{N+1+n-M}$ if $|x-y| \le |x|/2$. Using this and (44),

$$I \le c \int |m(x-y)||x-y|^{N+1} dy \, (1+|x|)^{N+1+n-M}$$

$$\le c2^{\mu(n/p-\alpha)} \int (1 + 2^\mu|y|)^{-M}|y|^{N+1} dy \, (1+|x|)^{N+1+n-M}$$

$$\le c2^{\mu(n/p-\alpha)} 2^{-\mu(N+1+n)}(1+|x|)^{N+1+n-M}$$

Since Φ is bounded,

$$II \le \int\limits_{|x-y|>|x|/2} |m(x-y)||x-y|^{N+1} dy \le c2^{\mu(n/p-\alpha)} \int\limits_{|x-y|>|x|/2} (1+2^\mu|x-y|)^{-M}|x-y|^{N+1}dy$$

$$\le c2^{\mu(n/p-\alpha)}2^{-\mu(N+1+n)}(1+|x|)^{N+1+n-M}$$

by (45). This proves (47).

Now (48) follows similarly by reversing the role of φ and m in the above proof. By (19), φ has moments of arbitrary order; we subtract a Taylor polynomial of degree $K-1$ from m in the convolution $\varphi * m(x)$ and use (45) with $|\gamma| = K$.

Lemma (1.2.8) [45]: Let $1 \le p \le +\infty$ and $\mu, \eta \in \mathbb{Z}$, $\eta \le \mu$. Suppose $F(x) = \sum_{\ell(Q)=2^{-\mu}} s_Q f_Q(x)$, where

$$\left|f_Q(x)\right| \le 2^{\mu(n/p-\alpha)}\left(1 + 2^\eta|x - x_Q|\right)^{-n-1}$$

Then

$$\|F\|_{L^p} \le c2^{-\mu\alpha}\left(\sum_{\ell(Q)=2^{-\mu}} |s_Q|^p\right)^{1/p} 2^{(\mu-\eta)n}$$

Proof: By our assumption on the f_Q's,

$$\|F\|_{L^p}^p \le c \sum_{\ell(S)=2^{-\mu}} 2^{-\mu n}\left(2^{\mu(n/p-\alpha)} \sum_{\ell(Q)=2^{-\mu}} |s_Q|(1 + 2^\eta|x_S - x_Q|)^{-n-1}\right)^p$$

where $\{S\}$ are the dyadic cubes of side length $2^{-\mu}$. Determine $k, \ell \in \mathbb{Z}^n$ such that $x_Q = 2^{-\mu}k$ and $x_S = 2^{-\mu}\ell$, and $s_Q = s_k$ in this case. Then by Young's inequality,

$$\|F\|_{L^p}^p \le c2^{-\mu\alpha p} \sum_{\ell \in \mathbb{Z}^n}\left(\sum_{k \in \mathbb{Z}^n} |s_k|(1 + 2^{\eta-\mu}|k - \ell|)^{-n-1}\right)^p$$

$$\le c2^{-\mu\alpha p}\left(\sum_{k \in \mathbb{Z}^n} |s_k|^p\right)\left(\sum_{\ell \in \mathbb{Z}^n} (1 + 2^{\eta-\mu}|\ell|)^{-n-1}\right)^p$$

$$\le c2^{-\mu\alpha p}\left(\sum_{\ell(S)=2^{-\mu}} |s_k|^p\right)2^{-(\eta-\mu)np},$$

which prove the lemma.

We should perhaps note that an immediate consequence of Theorem (1.2.6) is that if $q < \infty$, our representations of $f \in \dot{B}_p^{\alpha q}$ in Theorem (1.2.4) converge to f in $\dot{B}_p^{\alpha q}$ (quasi-norm). This may also be seen directly from the proof of the decomposition.

The functions ψ_Q and a_Q in Theorem (1.2.4) may be taken to have any fixed number of vanishing moments (or infinitely many, in the case of ψ_Q). Conversely, though, in Theorem (1.2.6) the assumption that $m_{\mu,j}$, or m_Q, has at least $N = \max([n(1/p - 1)_+ - \alpha], -1)$ vanishing moments cannot be improved. For $0 < p \le 1$, this follows from the general fact that any $f \in \dot{B}_p^{\alpha q}$ satisfies $|\hat{f}(\xi)| \le c|\xi|^{n(1/p-1)\alpha}$; the same holds with $\alpha = 0$ for $f \in H^p$.

A consideration of (45) leads to a remark about the atomic decomposition of $H^p(\mathbb{R}^n)$, $0 < p \le 1$. A distribution $f \in H^p(\mathbb{R}^n)$ satisfies $f = \sum_i \lambda_i a_i$ where $\sum_i |\lambda_i|^p \le c\|f\|_{H^p}^p$ and each "p-atom" a_i satisfies $\int x^\gamma a_i(x) dx = 0$ if $|\gamma| \le [n(1/p - 1)]$, $\operatorname{supp} a_i \subseteq Q_i$ and $\|a_i\|_L \le |Q_i|^{-1/p}$, for some associated cube $Q_i \subseteq \mathbb{R}^n$. Clearly, if a_i also satisfies

$$|x^\gamma a_i(x)| \le |Q_i|^{-1/p-|\gamma|/n} \tag{49}$$

for $|\gamma| = 1$, then a_i is a $(0, p)$-atom. The condition $\sum_i |\lambda_i|^p < \infty$ is precisely the summation condition on the coefficients $\{s_{\mu,j}\}$ in our decomposition of $\dot{B}_p^{0p}$. Since $\dot{B}_p^{0q} \subsetneq H^p$ for $0 < p \le 1$, we see that it is not possible in general to obtain the smoothness estimate (49) for $|\gamma| = 1$ in the atomic decomposition of $H^p(\mathbb{R}^n)$. In our decompositions of $\dot{B}_p^{\alpha q}$. however, our building blocks are C^∞ and may be taken to satisfy (45) for an arbitrarily large K. In particular, then, for $0 < p \le 1$, the space

$$\left\{\sum_i \lambda_i a_i : \sum_i |\lambda_i|^p < \infty, \qquad \text{each } a_i \text{ is a p- atom satisfying (49)for } |\gamma| \le J\right\}$$

is either H^p, if $J = 0$, or $\dot{B}_p^{0p}$, if $J \ge 1$.

The space of functions of bounded mean oscillation is defined by

$$\text{BMO} = \left\{f \in L_{\text{loc}}^1/\mathbb{C} : \|f\|_{\text{BMO}} = \sup_I \frac{1}{|I|} \int_I |f - f_I| < +\infty\right\}$$

where the sup is taken over all cubes $I \subseteq \mathbb{R}^n$ (not necessarily dyadic), and $f_I = (1/|I|) \int_I f$.

For a sequence $\{s_Q\}_Q$, indexed by the dyadic cubes, define

$$\|\{s_Q\}\|_{\wp} = \sup_{J \text{ dyadic}} \left(\frac{1}{|I|} \sum_{Q \subseteq J} |s_Q|^2 |Q|\right)^{1/2}$$

$\|\{s_Q\}\|_{\wp}^2$ is equivalent to the Carleson norm of the measure $\sum_Q |s_Q|^2 \delta_{(x_Q, \ell(Q))}$, where $\delta_{(x,t)}$ is the point mass at $(x, t) \in \mathbb{R}_+^{n+1}$.

Uchiyama's decomposition of BMO functions into $(0,\infty)$-atoms [42] shows the close relation between the $\wp$- and BMO-norms. Recall that a $(0,\infty)$-atoms a_Q, for some cube Q, is a function satisfying $\operatorname{supp} a_Q \subseteq 3Q$, $\left|\partial^\gamma a_Q(x)\right| \le \ell(Q)^{-|\gamma|}$ for $|\gamma| \le K$, and $\int x^\gamma a_Q(x)dx = 0$ for $|\gamma| \le N$, where $K \ge 1$ and $N \ge 0$ are fixed integers. Uchiyama's result that

$$\|f\|_{\mathrm{BMO}} = \inf\left\{\left\|\{s_Q\}\right\|_\wp : f = \sum_Q s_Q a_Q \text{ (in } \mathcal{S}'/\mathbb{C}), \quad \text{where each } a_Q \text{ is } (0,\infty)\text{- atoms}\right\} \quad (50)$$

(In [42], it is in fact the L^2-, rather than the $(\mathcal{S}'/\mathbb{C})$-, version of (50) that is stated.) Clearly, (50) is the BMO-analogue of Theorem (1.2.4) ii).

The following theorem gives a decomposition of BMO corresponding to Theorem (1.2.4) (i).

Theorem (1.2.9) [45]:

a) If $f \in \mathrm{BMO}(\mathbb{R}^n)$, then there exists a sequence $\{s_Q\}_Q$ satisfying $\left\|\{s_Q\}\right\|_\wp \le c\|f\|_{\mathrm{BMO}}$ such that $f = \sum_Q s_Q \psi_Q$ (in $\mathcal{S}'/\mathbb{C}$), where ψ_Q is defined by (27) with $\alpha = 0$ and $p = \infty$.

b) Conversely, suppose m_Q satisfies $\int m_Q(x)dx = 0$ and
$$\left|\partial^\gamma m_Q(x)\right| \le \ell(Q)^{-|\gamma|}\left(1 + \ell(Q)^{-1}|x - x_Q|\right)^{-n-1-|\gamma|}$$
if $|\gamma| \le 1$, for each cube $Q \subseteq \mathbb{R}^n$. If $\left\|\{s_Q\}\right\|_\wp < \infty$, then $\sum_Q s_Q \psi_Q$ converges in $\mathcal{S}'/\mathbb{C}$ and weak-* in BMO (regarded as $(H^1)^*$), with $\left\|\sum_Q s_Q \psi_Q\right\|_{\mathrm{BMO}} \le c\left\|\{s_Q\}\right\|_\wp$.

Proof: The proof of a) is a direct application of the methods above. By Lemma (1.2.1), $f = \sum_{v\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n} \varphi_v * f(2^{-v}k)\psi_{Q_{vk}}(x)$ (in $\mathcal{S}'/\mathbb{C}$). Set $s_Q = \varphi_v * f(2^{-v}k)$ if $Q = Q_{vk}$. It is enough to prove that

$$\sum_{v,k:Q_{vk}\subseteq J} 2^{-vn}|\varphi_v * f(2^{-v}k)|^2 \le c|J|\|f\|_{\mathrm{BMO}}^2 \quad (51)$$

for every dyadic cube J.

In (51), we may clearly assume $J = [0,1]^n$. We apply the estimate (39) of Lemma (1.2.2) to $\varphi_v * f$. (This is appropriate here because (39) is a nearly localized version of Lemma (1.2.2).) This gives

$$\sum_{v=0}^\infty 2^{-vn} \sum_{k\in[0,2^v)^n} |\varphi_v * f(2^{-v}k)|^2 \le c\sum_{v=0}^\infty \sum_{k\in[0,2^v)^n} \sum_{\ell\in\mathbb{Z}^n} (1 + |\ell|)^{-n-1} \int_{Q_{v,k+\ell}} |\varphi_v * f|^2$$

$$= c\sum_{v=0}^\infty \sum_{k\in[0,2^v)^n} \sum_{r\in\mathbb{Z}^n} (1 + |k - r|)^{-n-1} \int_{Q_{vr}} |\varphi_v * f|^2$$

$$= c\sum_{v=0}^\infty \sum_{k\in[0,2^v)^n} \sum_{m\in\mathbb{Z}^n} \sum_{r:Q_{vr}\subseteq Q_{0m}} (1 + |k - r|)^{-n-1} \int_{Q_{vr}} |\varphi_v * f|^2.$$

We claim it if $Q_{vr} \subseteq Q_{0m}$, then $\sum_{k\in[0,2^v)^n}(1 + |k - r|)^{-n-1} \le c(1 + |m|)^{-n-1}$. For $|m|$ small, say $|m| \le 10\sqrt{n}$, this trivial; if $|m| > 10\sqrt{n}$, then

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$$|r| > 2^{v-1}|m| \geq 5 \cdot 2^v \sqrt{n} > 5|k|,$$

so that

$$\sum_{k \in [0,2^v)^n} (1 + |k - r|)^{-n-1} \leq c 2^{vn} (2^v|m|)^{-n-1} \leq c(1 + |m|)^{-n-1}.$$

Hence, using this in the inequality above,

$$\sum_{v=0}^{\infty} 2^{-vn} \sum_{k \in [0,2^v)^n} |\varphi_v * f(2^{-v}k)|^2 \leq c \sum_{v=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \sum_{r: Q_{vr} \subseteq Q_{0m}} (1 + |m|)^{-n-1} \int_{Q_{vr}} |\varphi_v * f|^2$$

$$\leq c \sum_{v=0}^{\infty} \sum_{m \in \mathbb{Z}^n} (1 + |m|)^{-n-1} \int_{Q_{0m}} |\varphi_v * f|^2.$$

Therefore (51) is reduced to

$$\sum_{v=0}^{\infty} \int_Q |\varphi_v * f|^2 \leq c \|f\|_{\mathrm{BMO}}^2, \qquad \text{for } Q = Q_{0m}, m \in \mathbb{Z}^n \tag{52}$$

The proof of (52) is standard (see [5]). Writing

$$f = f_{3Q} + (f - f_{3Q})\chi_{3Q} + (f - f_{3Q})\chi_{\mathbb{R}^n \setminus 3Q} = f_1 + f_2 + f_3,$$

f_1 contributes nothing to (52), while

$$\sum_{v=0}^{\infty} \int_Q |\varphi_v * f_2|^2 \leq \int_{\mathbb{R}^n} \sum_{v=0}^{\infty} |\widehat{\varphi}_v|^2 |\widehat{f}_2|^2 \leq c \|f_2\|_{\mathrm{L}^2}^2 \leq c \|f\|_{\mathrm{BMO}}^2.$$

For $x \in Q$, we have the pointwise estimate

$$|\varphi_v * f_3(x)| \leq \int_{\mathbb{R}^n \setminus 3Q} 2^{vn} |f - f_{3Q}| (1 + 2^v|x - y|)^{-n-1} dy \leq c 2^{-v} \|f\|_{\mathrm{BMO}}$$

[5]. Altogether, this implies (52) and completes the proof of (i).

In (ii), the convergence of $f_m = \sum_{\ell(Q) \leq 2^m} s_Q m_Q$ in $\mathcal{S}'/\mathbb{C}$, to some f (see [45]). If $\sup_m \|f_m\|_{\mathrm{BMO}} = A < +\infty$, then $|\int f_m h| \leq cA \|h\|_{\mathrm{H}^1}$ for any $h \in \mathrm{H}^1 \cap \mathcal{S}$ such that $\widehat{h} = 0$ in neighborhood of the origin. It follows that $|\int fh| \leq c\|h\|_{\mathrm{H}^1}$ for these h, so that $\|f\|_{\mathrm{BMO}} \leq cA$ and f_m converges to f weak-* in BMO, by the H^1- BMO duality theorem [5]. Therefore we need only to prove $\|f_m\|_{\mathrm{BMO}} \leq c\|\{s_Q\}\|_{\wp}$.

The proof of this is contained in Uchiyama's work in [42]. By Lemma 3.5 of [42], each m_Q may be written $m_Q = \sum_{j=0}^{\infty} 2^{-j(n+1)} m_{Q,j}$, where

$$\mathrm{supp}\, m_{Q,j} \subseteq 2^j Q, \qquad \|m_{Q,j}\|_{\mathrm{Lip}\,1} \leq c 2^{-j} \ell(Q)^{-1}, \qquad \text{and } \int m_{Q,j}(x) dx = 0.$$

Then Lemma 3.4 of [42] implies that $\|\sum_{\ell(Q) \leq 2^m} s_Q m_{Q,j}\|_{\mathrm{BMO}} \leq c 2^{jn} \|\{s_Q\}\|_{\wp}$. Therefore

$$\left\|\sum_{\ell(Q)\le 2^m} s_Q m_Q\right\|_{\text{BMO}} \le \sum_{j=0}^{\infty} 2^{-j(n+1)}\left\|\sum_{Q} s_Q m_{Q,j}\right\|_{\text{BMO}} \le c\sum_{j=0}^{\infty} 2^{-j}\|\{s_Q\}\|_{\wp} = c\|\{s_Q\}\|_{\wp}.$$

We may note that the proof of Theorem (1.2.9) shows that

$$\|f\|_{\text{BMO}} \approx \sup_{S\text{ dyadic}}\left(\frac{1}{|S|}\sum_{Q_{vk}\subseteq S} 2^{-vn}\sup_{Q_{vk}}|\varphi_v * f|^2\right)^{\frac{1}{2}} \approx \sup_{S\text{ dyadic}}\left(\frac{1}{|S|}\sum_{v=-\log_2 \ell(S)}\int_S |\varphi_v * f|^2\right)^{\frac{1}{2}} \quad (53)$$

This should be compared with (38), with Theorem 3 (iii) of [5], and with the definition of the $\dot{B}_p^{\alpha q}$ norm.

Now let $f \in \dot{B}_p^{\alpha q}$. In Theorem (1.2.4) ii) we obtained s_Q and a_Q such that $f = \sum_Q s_Q a_Q$ where $\operatorname{supp} a_Q \subseteq 3Q$, $|\partial^\gamma a_Q(x)| \le |Q|^{\alpha/n-1/p-|\gamma|/n}$ if $|\gamma| \le K$, $\int x^\gamma a_Q(x)dx = 0$ if $|\gamma| \le N$, and

$$\left(\sum_{v\in\mathbb{Z}}\left(\sum_{\ell(Q)=2^{-v}} |s_Q|^p\right)^{q/p}\right)^{1/p} < c\|f\|_{\dot{B}_p^{\alpha q}}.$$

Write $x \in \mathbb{R}^n$ as $x = (x', x_n)$, $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$, and let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the natural projection $\pi(x) = x'$. For each dyadic cube j in $\mathbb{R}^{n-1}$ we set

$$t_J = \sum_{\substack{Q:\pi(Q)=J \\ 3Q\cap J\ne\emptyset}} |s_Q| \quad \text{and} \quad h_J(x') = \sum_{\substack{Q:\pi(Q)=J \\ 3Q\cap J\ne\emptyset}} s_Q a_Q(x',0)/t_J.$$

The restriction, or trace, of f to $\mathbb{R}^{n-1}$ is now

$$\operatorname{Tr} f(x') = \sum_{\mu\in\mathbb{Z}}\sum_{\ell(J)=2^{-\mu}} t_J h_J(x') \tag{54}$$

Whenever the sum converges in $\mathcal{S}'/\mathcal{P}$. Clearly,

$$\left(\sum_{\mu\in\mathbb{Z}}\left(\sum_{\ell(J)=2^{-\mu}} |t_J|^p\right)^{q/p}\right)^{1/q} \le c\|f\|_{\dot{B}_p^{\alpha q}} \tag{55}$$

$$\operatorname{supp} h_J \subseteq 3J \tag{56}$$

and

$$|\partial^\gamma h_J(x')| \le |J|^{(\alpha-1/p/(n-1))-1/p-|\gamma|/(n-1)} \quad \text{if } |\gamma| \le K \tag{57}$$

since $|Q| = |J|^{n/(n-1)}$ if $\pi(Q) = J$. In other words, each h_J satisfies all the requirements for an $(\alpha - 1/p, p)$-atom except possibly the moment condition (24). However, for $\alpha - 1/p > (n-1)(1/p - 1)_+$, an $(\alpha - 1/p, p)$-atoms in $\mathbb{R}^{n-1}$ is not required to have any vanishing moments. In these cases, then, by Theorem (1.2.6) and (55),

$$\|\operatorname{Tr} f\|_{\dot{B}_p^{\alpha-1/p,q}(\mathbb{R}^{n-1})} \le c\|f\|_{\dot{B}_p^{\alpha q}(\mathbb{R}^n)} \tag{58}$$

Although $\operatorname{Tr} f$ in (54) is expressed in terms of the decomposition of $f \in \dot{B}_p^{\alpha q}(\mathbb{R}^n)$ given in Theorem (1.2.4) ii), it is clear that $\operatorname{Tr} f(x') = f(x',0)$ if that decomposition converges absolutely and uniformly. This is the case, for instance, if $f \in \dot{B}_\infty^{01}$. In particular, if f is any function of the form $f = \sum_{v=-M}^{N} \sum_{\ell(Q)=2^{-v}} \lambda_Q b_Q$, where $\sup_Q |\lambda_Q| < +\infty$ and b_Q's are (α, p)-atoms satisfying $\int b_Q(x)dx = 0$ and $|\partial^\gamma b(x)| \le \ell(Q)^{(\alpha-n/p-|\gamma|)}$, $|\gamma| \le 1$, for some α, p, M, N then $f \in \dot{B}_\infty^{01}$ and hence $\operatorname{Tr} f(x') = f(x',0)$. Since such sums are dense in $\dot{B}_p^{\alpha q}$ for $q < +\infty$, by (58) the map $\operatorname{Tr} : \dot{B}_p^{\alpha q}(\mathbb{R}^n) \to \dot{B}_p^{\alpha-1/p,q}(\mathbb{R}^{n-1})$ is the unique continuous linear extension to $\dot{B}_p^{\alpha q}$ of pointwise restriction operator if $\alpha - 1/p > (n-1)(1/p-1)_+$. Moreover, Tr extends the restriction operator for $\dot{B}_p^{\alpha\infty}$ as well, since the inclusion $\dot{B}_p^{\alpha\infty} \subseteq \dot{B}_p^{\alpha_0 1} + \dot{B}_p^{\alpha_1 1}$ holds whenever $\alpha_0 < \alpha < \alpha_1$.

It is easy to see that the trace map Tr above is onto $\dot{B}_p^{\alpha-1/p,q}(\mathbb{R}^{n-1})$, since any $h_J(x')$ satisfying (56, 57) can be obtained as the restriction of an (α, p)-atom $a_\alpha(x)$. Hence, we obtain the known fact (which is classical if $p \ge 1$) that $\operatorname{Tr} \dot{B}_p^{\alpha q}(\mathbb{R}^n) = \dot{B}_p^{\alpha-1/p,q}(\mathbb{R}^{n-1})$ when $\alpha - 1/p > (n-1)(1/p-1)_+$ (cf. [17, 41]). The failure of this result for $\alpha - 1/p \le (n-1)(1/p-1)_+$ is clearly due to the failure of the h_J's to have vanishing moments, which is first necessary at this critical index.

The existence, or non-existence, of the trace of $\dot{B}_p^{\alpha q}$ is equivalent to the question whether we can make sense of the sums in (54) whenever (55, 56, 57) holds, since any such expression can arise from a suitable $f \in \dot{B}_p^{\alpha q}(\mathbb{R}^n)$. It is not difficult to see that the sums in (54) always converge, and thus the trace exists, in $\mathcal{S}'/\mathcal{S}$, if and only if $\alpha - 1/p > (n-1)(1/p-1)_+$ or $\alpha - 1/p = (n-1)(1/p-1)_+$ and $0 < q \le 1$. This is also previously known ([35], [41]).

Suppose now $0 < p < 1$. When $0 < \alpha - 1/p < (n-1)(1/p-1)$, $\alpha - 1/p = (n-1)(1/p-1)$ and $q > 1$, or $\alpha = 1/p, q \le p$, (54) does not necessarily converge in $\mathcal{S}'/\mathcal{S}$, but does not converge in $L^p + L^\infty(\mathbb{R}^{n-1})$. This was observed in [17], and me be seen readily from (54, 55, 56, 57). This is best possible in the sense that the sums (54) do not necessarily converge in $L^p + L^\infty$ when $\alpha - 1/p < 0$ or $\alpha = 1/p, q > p$. Let us show this, for example, in the case $\alpha = 1/p, q > p$.

Pick a sequence $\{t_\mu\}_{\mu=2}^{\infty} \in \ell^q \backslash \ell^p$ (since this is not our usual convention, in $c_0 \backslash \ell^p$ if $q = +\infty$) and a collection $\{J_\mu\}_{\mu=2}^{\infty}$ of dyadic cubes satisfying $J_\mu \subseteq [0,1]^{n-1}$, $\ell(J_\mu) = 2^{-\mu}$, and $3J_\mu \cap 3J_v = \emptyset$ if $\mu \ne v$. Set $t_J = t_\mu$ if $J = J_\mu$ and $t_J = 0$ for any $J \notin \{J_\mu\}_{\mu=2}^{\infty}$. Let $\{h_J\}_J$ be functions satisfying (56, 57) and, in addition, $h_J(x') \ge c|J|^{-1/p}$ if $x' \in J$, for some small constant c. Then

$$\left(\sum_{\mu \in \mathbb{Z}} \left(\sum_{\ell(J)=2^{-\mu}}^{\infty} |t_J|^p \right)^{q/p} \right)^{1/q} = \left(\sum_{\mu=2}^{\infty} |t_\mu|^q \right)^{1/q} < +\infty,$$

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and it is clear that $\sum_{\mu \in \mathbb{Z}} \sum_{\ell(J)=2^{-\mu}} t_J h_J$ would arise as the trace of a suitable $f \in \dot{B}_p^{(1/p)q}(\mathbb{R}^n)$ if the trace operator were continuous. But if we let $h_N = \sum_{\mu=2}^{N} \sum_{\ell(J)=2^{-\mu}} t_J h_J$ for large N, then $\mathrm{supp}\, h_N \subset [0,1]^{n-1}$. Hence, $\|h_n\|_{L^p + L^\infty} \geq c\|h_N\|_{L^p} \geq c\left(\sum_{\mu=2}^{N} |t_\mu|^p\right)^{1/p}$, which can be made arbitrarily large. Therefore the sum $\sum_J t_J h_J$ cannot converge in $L^p + L^\infty$.

We know from above that the trace of $\dot{B}_p^{\alpha q}(\mathbb{R}^n)$ in the limiting case $\alpha = 1/p$ exists in $L^p + L^\infty(\mathbb{R}^{n-1})$ only if $0 < q \leq \min(1, p)$. In fact, we then have the following result.

Theorem (1.2.10) [45]: Let $0 < p < +\infty$, $0 < q \leq \min(1, p)$. Then

$$\mathrm{Tr}\, \dot{B}_p^{(1/p)q}(\mathbb{R}^n) = L^p(\mathbb{R}^{n-1}).$$

Proof: It is easy to verify that the sums in (54) converge in $L^p(\mathbb{R}^{n-1})$ if $f \in \dot{B}_p^{(1/p)q}(\mathbb{R}^n)$, $0 < q \leq \min(1, p)$, and that Tr is bounded from $\dot{B}_p^{(1/p)q}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^{n-1})$.

To show that Tr is onto L^p, it is sufficient to show that each $h \in L^p(\mathbb{R}^{n-1})$ has a decomposition

$$h(x') = \sum_{\mu \in \mathbb{Z}} \sum_{\ell(J)=2^{-\mu}} t_J h_J(x')$$

where the h_J's satisfy (56, 57) with $\alpha = 1/p$, and

$$\left(\sum_{\mu \in \mathbb{Z}} \left(\sum_{\ell(J)=2^{-\mu}} |t_J|^p \right)^{q/p} \right)^{1/q} \leq c\|h\|_{L^p(\mathbb{R}^{n-1})}$$

To prove such a decomposition, start by picking a $\Phi \in C_0^\infty$ satisfying $\mathrm{supp}\, \Phi \subseteq [0,1]^{n-1}$, $0 \leq \Phi \leq 1$, and $\|1 - \Phi\|_{L^p([0,1]^{n-1})} \leq \min(1/5, (1/5)^p)$. If

$$J = \{x : k_i 2^{-\mu} \leq x_i < (k_i + 1)2^{-\mu}, \qquad i = 1, \cdots, n-1\}$$

put

$$h_J(x') = C\Phi(2^\mu x' - k)2^{\mu(n-1)/p} \tag{59}$$

where $k = (k_1, \cdots, k_{n-1})$ and C is chosen small enough for h_J to satisfy (57).

Fix a non-negative $h \in L^p(\mathbb{R}^{n-1})$; it is enough to prove the decomposition for such functions.

By choosing the side length $2^{-\mu_1}$ small enough, it is possible to find that a simple function

$$e_1(x') = \sum_{\ell(J)=2^{-\mu_1}} r_J \chi_J(x')$$

such that $e_1 \geq 0$ and $\|h - e_1\|_{L^p} \leq \min(1/4, (1/4)^{1/p})\|h\|_{L^p}$. we define the smooth version

$$\tilde{e}_1(x') = \sum_{\ell(J)=2^{-\mu_1}} t_J h_J(x'),$$

where the h_J's are given by (59) and $t_J = r_J 2^{-\mu_1(n-1)/p}/C$ with the same constant C. If we set $D = \max\left(5/4, (5/4)^{1/p}\right)$, then

$$\left(\sum_{\ell(J)=2^{-\mu_J}} |t_J|^p\right)^{1/p} = \|e_1\|_{L^p}/C \le \|h\|_{L^p}/D \tag{60}$$

and we picked Φ so that $\|e_1 - \hat{e}_1\|_{L^p} \le \min\left(1/5, (1/5)^{1/p}\right)\|e_1\|_{L^p}$. Hence,

$$\|h - \hat{e}_1\|_{L^p} \le \|h\|_{L^p}/2 \tag{61}$$

If this process is repeated with h replaced by $h - \hat{e}_1$, we obtain $\tilde{e}_2 = \sum_{\ell(J)=2^{-\mu_2}} t_J h_J$ such that

$$\left(\sum_{\ell(J)=2^{-\mu_2}} |t_J|^p\right)^{1/p} \le \|h - \hat{e}_1\|_{L^p}/D \le \|h\|_{L^p}/2D$$

and

$$\|h - \hat{e}_1 - \hat{e}_2\|_{L^p} \le \|h - \hat{e}_1\|_{L^p}/2 \le \|h\|_{L^p}/4$$

by (60). We can also arrange so that $\mu_2 > \mu_1$. Continuing this process inductively, we obtain the functions $\tilde{e}_i = \sum_{\ell(J)=2^{-\mu_i}} t_J h_J$, $i = 1, 2, \cdots$, satisfying

$$\left(\sum_{\ell(J)=2^{-\mu_i}} |t_J|^p\right)^{1/p} \le \|h\|_{L^p}/2^{i-1}D \tag{62}$$

$$\left\|h - \sum_{i=1}^m \tilde{e}_i\right\|_{L^p} \le 2^{-m}\|h\|_{L^p}, \qquad m = 1, 2, \cdots \tag{63}$$

and $\mu_{i+1} > \mu_i$ for every i. The required decomposition of h is $h(x') = \sum_{i=1}^\infty \tilde{e}_i(x')$. By (63) this sum converges in L^p and by (62),

$$\left(\sum_{i=1}^\infty \left(\sum_{\ell(J)=2^{-\mu_i}} |t_J|^p\right)^{q/p}\right)^{1/q} \le c\|h\|_{L^p}.$$

Theorem (1.2.10) was previously known when $1 \le p < +\infty$ and $q = 1$; see [33, 35, 31, 22].

A space X of tempered distributions on $\mathbb{R}^n$ is said to have the lower majorant property if, for each $f \in X$, there is a $g \in X$ such that

$$|\hat{f}(\xi)| \le \hat{g}(\xi), \qquad \text{if } \xi \in \mathbb{R}^n \tag{64}$$

and

$$\|g\|_X \leq c\|f\|_X$$

with c independent of f. For instance, the Hardy spaces $H^p(\mathbb{R}^n)$, $0 < p \leq 1$ are known to have this property; $L^p(\mathbb{R}^n)(p \geq 1)$ has the lower majorant property if and only if $p = 1$ or $p = 2k/(2k-1)$, $k = 1, 2, 3, \cdots$ References can be found in [34] and [19].

For a space $X \subseteq \mathcal{S}'/\mathcal{S}$, the definition of the lower majorant property is as above, except that the origin is excluded in (64). For $\dot{B}_p^{\alpha q}(\mathbb{R}^n)$, the following theorem is an immediate consequence of Theorem (1.2.4) i) and Theorem (1.2.6) i).

Theorem (1.2.11) [45]: Let $-\infty < \alpha < +\infty$, $0 < q \leq +\infty$. If $0 < p \leq 1$, then $\dot{B}_p^{\alpha q}$ has the lower majorant property.

Proof: By Theorem (1.2.6) i), each $f \in \dot{B}_p^{\alpha q}$ has a representation $f = \sum_Q s_Q \psi_Q$ where

$$\left(\sum_{v \in \mathbb{Z}} \left(\sum_{\ell(Q)=2^{-v}} |s_Q|^p \right)^{q/p} \right)^{1/q} \leq c\|f\|_{\dot{B}_p^{\alpha q}}$$

The ψ_Q's are defined by (27); up to a multiple, the are obtained by translating the dilates of a fixed ψ by $x_Q = 2^{-v}k$. We choose, as we may, ψ so that $0 \leq \widehat{\psi} \leq 1$. If we put $g(x) = \sum_{v \in \mathbb{Z}} \sum_{\ell(Q)=2^{-v}} |s_Q| \psi_Q(x + x_Q)$ we then have $|\widehat{f}(\xi)| \leq \widehat{g}(\xi)$.

Now since $0 < p \leq 1$, it does not matter that the building blocks $\psi_Q(x + x_Q)$ are not evenly scattered. Indeed, by Theorem (1.2.4) i), we obtain

$$\|g\|_{\dot{B}_p^{\alpha q}} \leq c \left(\sum_{v \in \mathbb{Z}} \left(\sum_{\ell(Q)=2^{-v}} |s_Q|^p \right)^{q/p} \right)^{1/q} \leq c\|f\|_{\dot{B}_p^{\alpha q}}$$

It is not difficult to prove analogues of Theorems (1.2.4) and (1.2.6) for the inhomogeneous Besov spaces $B_p^{\alpha q}(\mathbb{R}^n)$, $-\infty < \alpha < +\infty$, $0 < p, q \leq +\infty$. To define these spaces, let $\{\varphi_v\}_{v=0}^{\infty}$ satisfy (18, 19, 20, 21) and let $\Phi \in \mathcal{S}$ satisfy

$$\text{supp } \widehat{\Phi} \subseteq \{\xi : |\xi| \leq 1\} \quad \text{and} \quad \widehat{\Phi}(\xi) \geq c \text{ if } |\xi| \leq 5/6 \tag{65}$$

Then $B_p^{\alpha q}$ is the set of $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B_p^{\alpha q}} = \|\Phi * f\|_{L^p} + \left(\sum_{v=0}^{\infty} (2^{v\alpha} \|\varphi_v * f\|_{L^p})^q \right)^{1/q} < +\infty$$

This definition is independent of the choice if Φ and $\{\varphi_v\}_{v=0}^{\infty}$ (see [26]).

Suppose $\Psi \in \mathcal{S}$ satisfies supp $\widehat{\Psi} \subseteq \{\xi : |\xi| \leq \pi\}$ and $\widehat{\Psi}(\xi) \geq c > 0$ if $|\xi| \leq 1$. We have then the following decomposition results.

Theorem (1.2.12) [45]: Let $-\infty < \alpha < +\infty$ and $0 < p, q \leq +\infty$.

a) Each $f \in B_p^{\alpha q}$ can be decomposed as follows;

i) $f(\cdot) = \sum_{k \in \mathbb{Z}^n} s_k \Psi(\cdot - k) + \sum_{v=0}^{\infty} \sum_{\ell(Q)=2^{-v}} s_Q \psi_Q(\cdot)$,

Where the ψ_Q's are defined by (26), or

ii) $f = \sum_{k \in \mathbb{Z}^n} s_k b_k + \sum_{v=0}^{\infty} \sum_{\ell(Q)=2^{-v}} s_Q a_Q$,

Where the a_Q's are (α, p)-atoms, and the b_k's satisfy $\operatorname{supp} b_k \subseteq 3Q_{0k}$ and $|\partial^\gamma b_k(x)| \leq 1$ if $|\gamma| \leq K$. In both cases the convergence is in $\mathcal{S}'$, and

$$\left(\sum_{k \in \mathbb{Z}^n} |s_k|^p \right)^{1/p} + \left(\sum_{v=0}^{\infty} \left(\sum_{\ell(Q)=2^{-v}} |s_Q|^p \right)^{q/p} \right)^{1/q} \leq c\|f\|_{B_p^{\alpha q}},$$

with c independent of f.

b) Conversely, suppose $f = \sum_{k \in \mathbb{Z}^n} s_k m_k + \sum_{v=0}^{\infty} \sum_{\ell(Q)=2^{-v}} s_Q m_Q$, where each m_Q is an (α, p)-molecule concentrated on Q, and each m_k satisfies
$$|\partial^\gamma b_k(x)| \leq (1 + |x - k|)^{-M-|\gamma|} \qquad \text{if } |\gamma| \leq K,$$
for some sufficiently large M. Then

$$\|f\|_{B_p^{\alpha q}} \leq c \left(\sum_{k \in \mathbb{Z}^n} |s_k|^p \right)^{1/p} + c \left(\sum_{v=0}^{\infty} \left(\sum_{\ell(Q)=2^{-v}} |s_Q|^p \right)^{q/p} \right)^{1/q}$$

For $0 < p \leq 1$, the conclusion of b) holds even if the m_k's and m_Q's are not centered near k and Q, respectively.

To prove (i), one obtains $\Phi \in \mathcal{S}$ satisfying (65) and φ satisfying (18, 19, 20, 21) for $v = 0$ such that

$$\hat{\Phi}(\xi)\hat{\Psi}(\xi) + \sum_{v=0}^{\infty} \hat{\Phi}(2^v\xi)\hat{\Psi}(2^v\xi) = 1 \qquad \text{for all } \xi \in \mathbb{R}^n$$

and proceeds as in Lemma (1.2.1) and Theorem (1.2.4). The proof of (ii) uses the inequalities above and similar estimates, not requiring the assumption of vanishing moments, however, for Φ and m_k.

The results above also have analogues for $B_p^{\alpha q}$. The standard result that $\operatorname{Tr} B_p^{\alpha q}(\mathbb{R}^n) = B_p^{\alpha-1/p,q}(\mathbb{R}^{n-1})$ if $\alpha - 1/p > (n-1)(1/p-1)_+$, and the analogue of Theorem (1.2.10) that $\operatorname{Tr} B_p^{(1/p)q}(\mathbb{R}^n) = L^p(\mathbb{R}^{n-1})$ if $0 < p < +\infty$ and $0 < q \leq \min(1, p)$, follow from Theorem (1.2.12). Also, remarks about the non-existence of the trace in $\mathcal{S}'$ or in L^p are analogous. Further, $B_p^{\alpha q}$ is proved to have the lower majorant property if $0 < p \leq 1$.

The decompositions in Theorem (1.2.4) provide a natural approach to many of the well-known properties of the Besov spaces, including the standard embedding and interpolation results. Also, they yield a way of comparing the Besov spaces to other spaces known to have a decomposition, such as H^p, $0 < p \leq 1$, or L^p. On the other hand, the main distinction between the building blocks obtained in the decomposition of L^p

(Theorem (1.2.10)) and the p-atoms for H^p, or the $(0, p)$-atoms for $\dot{B}_p^{0p}$, is that no vanishing moments are assumed in the case of L^p. It would be interesting to clarify the relation between $\dot{B}_p^{0p}$ and L^p, H^p and L^p, and $\dot{B}_p^{0p}$ and H^p, by determining the interpolation spaces between each of these couples.

There are a number of directions could possibly be extended. It is straightforward to obtain decompositions similar to (25) and (28) for Besov spaces defined with respect to a measure satisfying the doubling property. In the case of the polydisk, as well, the results generalize in an obvious way. Since the machinery necessary of Calderón's representation formula has been developed by Folland-Stein in [30] for appropriate homogenous groups, it should be possible to extend our approach to this setting. In the case of more general domains in $\mathbb{R}^n$, it may be natural to define Besov spaces for $\alpha > 0$ via the atomic decomposition. This point of view might be useful in the study of differential equations on these domains (cf. [41]), especially since trace Theorems are easy in the atomic context. In the proof of Theorem (1.2.4) (i), it would be interesting to replace the Fourier series expansion with a representation in terms of other bases in L^2, for example certain sets $\left\{e^{i\lambda_k x}\right\}_k$, or the eigenfunctions of some differential operator other than the Laplacian. Similarly, it may be possible to replace Fourier series by appropriate group representations in more abstract settings.

Now, After a normalization and reindexing, we obtain an expansion of the form $f = \sum_i \langle f, \psi_i \rangle \psi_i$, with $\|\psi_i\|_{L^2} \leq c$. The key aspect of this decomposition in our treatment of Besov spaces is that the norm of f is equivalent to the appropriate sequence space norm of the coefficients $\langle f, \psi_i \rangle$. Although the expansion is not orthonormal, it has many of the advantages of an orthonormal expansion. It follows directly from the identity $f = \sum_i \langle f, \psi_i \rangle \psi_i$ that $\|f\|_{L^2} = \left(\sum_i |\langle f, \psi_i \rangle|^2\right)^{1/2}$. Applying this to ψ_j gives $\sup_j \sum_i \left|\langle \psi_j, \psi_i \rangle\right|^2 \leq c$, which is an almost orthogonality property. Writing an operator T in the form

$$\mathrm{T}f = \sum_i \langle \mathrm{T}f, \psi_i \rangle \psi_i = \sum_{i,j} \langle f, \psi_j \rangle \langle \mathrm{T}\psi_j, \psi_i \rangle \psi_i$$

effectively reduces the study of T to the study of the matrix $\left\{\langle \mathrm{T}\psi_j, \psi_i \rangle\right\}$.

Suppose $\{\psi_i\}_i$ is a quasi-orthogonal family, or that the matrix $\left\{\langle \psi_i, \psi_j \rangle\right\}$ is bounded on ℓ^2, and in addition that $\psi_j = \sum_{i,j} \langle \psi_j, \psi_i \rangle \psi_i$, for each j. Then clearly the operator P defined by $\mathrm{P}f = \sum_i \langle f, \psi_i \rangle \psi_i$ is a bounded projection onto $\mathcal{H}$, the closure of the span of $\{\psi_i\}_i$. Also we can write

$$\mathrm{T}f(x) = \sum_i \int f(y) \overline{\psi_i(y)} dy \, \psi_i(x) = \int \mathrm{K}(x, y) f(y) dy$$

for $\mathrm{K}(x, y) = \sum_i \overline{\psi_i(y)} \, \psi_i(x)$. This is reminiscent of the Bergman and Szegö kernels except that ψ_i's are not necessarily orthonormal. If the ψ_i's are sufficiently localized, as in Lemma (1.2.2), then for $f \in \mathcal{H}$ the identity $f = \sum_i \langle f, \psi_i \rangle \psi_i$ can be used to prove an analogue of Plancherel-Pólya.

Chapter 2
Characterization of the Besov-Lipschitz with Some Properties

We show a multiplier theorem of the Mikhlin type, extending the one by Triebel and Lizorkin. We complete the characterization of the weighted Besov-Lipschitz and Triebel-Lizorkin spaces. We give the boundedness of some operators which including pseudo-differential operators of the Hörmander class.

Section (2.1): Spaces of Triebel-Lizorkin Type

We study certain spaces of distributions $F_p^{sq} = F_p^{sq}(\mathbb{R}^n)$ where s real, $0 < p, q \leq \infty$. They are intimately related to certain spaces studied by Triebel [15] and Lizorkin [63] which says that the spaces do not depend on the special sequence of test functions $\{\varphi_v\}_{v \in \mathbb{Z}}$. This extends Triebel's corresponding result. But we have to give an entirely new proof, relying on two deep results by Fefferman & Stein: 1° their real variable characterization of the Hardy classes H_p[5], 2° their sequence valued version of the Hardy & Littewood maximal theorem [60]. (Incidentally it follows from [5] that $F_p^{02} = H_p$ if $0 < p < \infty$ while as $F_\infty^{02} = B.M.O.$!) As an application we prove a multiplier theorem of the Milkhlin type, extending the one by Triebel and Lizorkin. We also give an application to approximation theory related to a theorem of Freud's [61]. Finally we briefly indicate how the result might be extended to the case or a Riemannian manifold.

By L_p where $0 < p \leq \infty$ we denote the space of measurable functions $f = f(x)$ $(x \in \mathbb{R}^n)$ such that

$$\|f\|_{L_p} = \left(\int |f(x)|^p dx \right)^{1/p} < \infty,$$

By ℓ^q where $0 < q \leq \infty$ we denote the space of sequences $t = \{t_v\}_{v \in \mathbb{Z}}$ such that

$$\|t\|_{\ell^q} = \left(\sum_{v \in \mathbb{Z}} |t_v|^q \right)^{1/q} < \infty.$$

We consider also spaces of sequence valued measurable functions $L_p(\ell^q)$ and $\ell^q(L_p)$, defined in the obvious way. If $1 \leq p, q \leq \infty$ these are all Banach spaces, in the general case only quasi-Banach space.

By $\mathcal{S}$ we denote the space of rabidly decreasing functions in $\mathbb{R}^n$ and by $\mathcal{S}'$ the dual space of tempered distributions.

We denote a sequence of test functions $\{\varphi_v\}_{v \in \mathbb{Z}}$, with $\varphi_v(x) = 2^{vn}\varphi(2^v x)$, where $\varphi \in \mathcal{S}$ with $\operatorname{supp} \hat{\varphi} = \{2^{-1} \leq |\xi| \leq 2\}$. For convenience let us also assume that $\{\varphi_v\}_{v \in \mathbb{Z}}$ is normalized in the sense that

$$\sum_{v \in \mathbb{Z}} \left(\hat{\varphi}_v(\xi) \right)^2 = 1 \qquad \left(\text{or} \sum_{v \in \mathbb{Z}} \varphi_v * \varphi_v = \delta \right)$$

We can now define the principal spaces.

Definition (2.1.1) [66]: Let s real, $1 < p, q \leq \infty$. Then we set (poised spaces of Besov type)

$$B_p^{sq}(a) = \{f | f \in \mathcal{S}' \& \{2^{vs}(1 + 2^v|x|)^a \varphi_v * f\}_{v \in \mathbb{Z}} \in \ell^q(L_p)\}.$$

We quip $B_p^{sq}(a)$ with the quasi-norm

$$\|f\|_{B_p^{sq}(a)} = \|\{2^{vs}(1 + 2^v|x|)^a \varphi_v * f\}_{v \in \mathbb{Z}}\|_{\ell^q(L_p)}.$$

If $a = 0$ we simply write $B_p^{sq}(0) = B_p^{sq}$ (Besov space).

Let us now rapidly state some properties of these spaces which can be proven in a more or less standard way (cf. [15]).

(i) The space F_p^{sq} and $B_p^{sq}(a)$ are complete. The embedding from $\mathcal{S}$ and into $\mathcal{S}'$ are continuous. They are thus quasi-Banach (Banach if $1 \le p, q \le \infty$) spaces of tempered distributions.

(ii) $\mathcal{S}$ is a dense subspace of F_p^{sq} and $B_p^{sq}(a)$ if $0 < p, q < \infty$.

(iii) We have embedding theorem, e.g. the embedding $B_p^{sq}(a) \to B_{p_1}^{s_1 q}(a)$ if $s - n/p = s_1 - n/p_1$, $s \ge s_1, q \le q_1$.

(iv) We have duality theorems, e.g. the duality $\left(F_p^{sq}\right)' \approx F_{p'}^{-sq'}$ if $1 \le p, q \le \infty$.

We have the following elementary result.

Lemma (2.1.3) [66]: Let u be any C^1 function on $\mathbb{R}^n$ and let $0 < r \le \infty$. Then we have the inequality

$$u^{**} \le C\{\delta^{-n/r}(Mu^r)^{1/r} + \delta(\nabla u)^{**}\}, \qquad \delta \le 1$$

Where M denotes the Hardy & Littlewood maximal operator and where we have defied u^{**} by

$$u^{**}(x) = \sup_{y \in \mathbb{R}^n} |u(x-y)|/(1+|y|)^{n/r}$$

and $(\nabla u)^{**}$ in a similar fashion.

Proof: By the mean value theorem we have for any $x, z \in \mathbb{R}^n$

$$|u(x-z)| \le C\left\{\delta^{-n/r}\left(\int_{|x-y-z|<\delta} |u(y)|^r dy\right)^{1/r} + \delta \sup_{|x-y-z|<\delta} |\nabla u(y)|\right\}.$$

By definition of M and $(\nabla u)^{**}$ follows

$$|u(x-y)| \le C\left\{\delta^{-n/r}\left(Mu^r(x)\right)^{1/r} + \delta(\nabla u)^{**}(x)\right\}(1 + \delta + |z|)^{n/r}.$$

If $\delta \le 1$ we clearly get the desired inequality.

Lemma (2.1.4) [66]: Let f be any measurable function in $\mathbb{R}^n$ and let $b > n$. Then holds

$$\int |f(y)|/(1+|x-y|)^b \, dy \le CMf(x).$$

Lemma (2.1.5) [66]: Let $f = \{f_v\}_{v \in \mathbb{Z}}$ be a sequence of measurable functions in $\mathbb{R}^n$ and let $1 < p, q \le \infty$. Then holds

$$\|Mf\|_{L_p(\ell^q)} \le C\|f\|_{L_p(\ell^q)}$$

where $Mf = \{Mf_v\}_{v \in \mathbb{Z}}$.

If $f \in F_p^{sq}$ and if $\{\varphi_v\}_{v \in \mathbb{Z}}$ is the sequence of test functions we set

$$\varphi^{**}f(x) = \|\{\varphi_v^{**}f(x)\}_{v \in \mathbb{Z}}\|_{\ell^q},$$
$$\varphi_v^{**}f(x) = \sup_{y \in \mathbb{R}^n} 2^{vs} |\varphi_v * f(x-y)|/(1+2^v|y|)^a.$$

We also set

$$\varphi^+ f(x) = \|\{\varphi_v^+ f(x)\}_{v \in \mathbb{Z}}\|_{\ell^q}$$
$$\varphi_v^+ f(x) = 2^{vs}\varphi_v * f(x).$$

Clearly $\varphi^+ f \in L_P$. Below we show that also $\varphi^{**}f \in L_P$, at least if a is sufficiently large. More generally, let $\{\sigma_v\}_{v \in \mathbb{Z}}$ be a general sequence of test functions, with $\sigma_v(x) = 2^{vn}\sigma(2^v x)$ (but with no restriction on $\operatorname{supp}\widehat{\sigma}$) and define $\sigma^{**}f, \sigma_v^{**}f, \sigma_v^+ f$ as above. Then we have the following

Theorem (2.1.6) [66]: Assume that $\sigma \in B_1^{-sq_1}(a) \cap B_1^{-s+a,q_1}(a)$ with $a > n/\min(p, q)$, $q_1 = \min(1, q)$. Then holds:

$$f \in F_p^{sq} \Rightarrow \sigma^{**}f \in L_P. \tag{1}$$

In particular (1) holds with $\sigma = \varphi$.

Proof: (Cf. Fefferman & Stein [5]) Let us start with the identity

$$\sigma_\mu * f = \sum_{v \in \mathbb{Z}} (\sigma_\mu * \varphi_v) * (\varphi_v * f).$$

We then get

$$2^{\mu s}|\sigma_\mu * f(x-z)| \le \sum 2^{\mu s} \int |(\sigma_\mu * \varphi_v)(y)||\varphi_v * f(x-y-z)|dy$$

$$\le \sum 2^{\mu n} \int |(\sigma * \varphi_{v-\mu})(2^\mu y)|(1+2^v|y|)^a dy\, \varphi_v^{**} f(x)(1+2^v|z|)^a$$

$$\le \sum 2^{(\mu-v)s} \int |(\sigma * \varphi_{v-\mu})(y)|(1+2^{v-\mu}|y|)^a dy\, \varphi_v^{**} f(x)(1+2^{v-\mu})^a(1+2^\mu|z|)^a$$

where we have used the elementary inequality:

$$\max(1+u+v, 1+uv) \le (1+u)(1+v), \qquad u \ge 0, \qquad v \ge 0$$

In other words we have

$$\sigma_\mu^{**} f(x) \le \sum t_{v-\mu} \varphi_v^{**} f(x) \tag{2}$$

with $t_v = \sum 2^{-vs}(1+2^v)^a \int(1+2^v|y|)^a|\sigma * \varphi_v(y)|dy$. Here by hypothesis

$$\left(\sum |t_v|^{q_1}\right)^{1/q_1} \le C$$

Therefore follows

$$\sigma^{**} f \le C\varphi^{**} f \tag{3}$$

Thus we have reduced ourselves to proving (1) with $\sigma = \varphi$. To this end we first note that (3) in particular entails

$$(\nabla\varphi)^{**} f \le C\varphi^{**} f.$$

On the other hand Lemma (2.1.3) implies (with $r = n/a$)

$$\varphi_v^{**} f \le C\{\delta^{-n/r}(M(\varphi_v^+ f)^r)^{1/r} + \delta(\nabla\varphi)_v^{**} f\}, \qquad \delta \le 1$$

Thus we get

$$\|\varphi^{**} f\|_{L_p} \le C\left\{\delta^{-n/r}\left\|(M(\varphi_v^+ f)^r)^{1/r}\right\|_{L_p(\ell^q)} + \delta\|\varphi^{**} f\|_{L_p}\right\}$$

By Lemma (2.1.5) we have (since $r < \min(p,q)$)

$$\left\|(M(\varphi_v^+ f)^r)^{1/r}\right\|_{L_p(\ell^q)} = \|M(\varphi_v^+ f)^r\|_{L_{p/r}(\ell^{q/r})}^{1/r} \le C\|(\varphi_v^+ f)^r\|_{L_{p/r}(\ell^{q/r})}^{1/r} = C\|\varphi_v^+ f\|_{L_p(\ell^q)}$$

$$= C\|f\|_{F_p^{sq}}$$

Thus we have

$$\|\varphi^{**} f\|_{L_p} \le C\left\{\delta^{-n/r}\|f\|_{F_p^{sq}} + \delta\|\varphi^{**} f\|_{L_p}\right\}, \quad \delta \le 1.$$

If we knew already that $\|\varphi^{**} f\|_{L_p} < \infty$ we could, taking δ sufficiently small, conclude that

$$\|\varphi^{**} f\|_{L_p} \le C\|f\|_{F_p^{sq}} \tag{4}$$

But if $\|\varphi^{**} f\|_{L_p} = \infty$ this argument does not apply. To circumvent this difficulty we use an approximation argument. The above proof at least shows that (4) is valid if $f \in \mathcal{S}$. For a general $f \in F_p^{sq}$ we find a sequence $\{f_i\}_{i=1}^\infty$ in $\mathcal{S}$ such that $f_i \to f$ in $\mathcal{S}'$ as $i \to \infty$, with $\sup_i \|f_i\|_{F_p^{sq}} < \infty$. It is easily seen that

$$\|\varphi^{**} f\|_{L_p} \le \lim_{i \to \infty} \|\varphi^{**} f_i\|_{L_p}$$

So an application of (3) to f_i effectively yields $\|\varphi^{**} f\|_{L_p} < \infty$. The proof is complete.

Corollary (2.1.7) [66]: The space F_p^{sq} is independent of the particular sequence of test functions $\{\varphi_v\}_{v \in \mathbb{Z}}$ chosen.

Theorem (2.1.8) [66]: Assume that $\sigma \in B_p^{-sq_1}(a)$ with $a > n/\min(p,q)$, $q_1 = \min(1,q)$. Then holds:

$$f \in F_p^{sq} \Rightarrow \sigma^+ f \in L_p \tag{5}$$

Proof: The proof of Theorem (2.1.6) clearly also gives in place of (2)

$$\sigma_\mu^+ f(x) \le \sum t_{v-\mu}' \varphi_v^{**} f(x)$$

with $t_v' = 2^{-vs} \int (1 = 2^v|y|)^a |\sigma * \varphi_v(y)| dy$. This gives in place of (3):

$$\sigma^+ f \le C\varphi^{**} f.$$

Since we know already that $\varphi^{**} f \in L_p$ it follows that $\sigma^+ f \in L_p$.

Theorem (2.1.9) [66]: Assume that $\sigma \in B_\infty^{-s-n,q_1}(a)$ where $a > n/\min(1,p,q)$, $q_1 = \min(1,q)$. Then holds again (5).

Proof: From Lemma (2.1.4) and Lemma (2.1.5) follows readily that

$$f \in F_p^{sq} \Rightarrow \left\{ 2^{vs} \left(2^{vn} \int |\varphi_v * f(x-y)|^r /(1 + 2^v|y|)^b \, dy \right)^{1/r} \right\} \in L_p(\ell^q)$$

where $r < \min(p,q)$, $b > n$. From this follows again readily

$$f \in F_p^{sq} \Rightarrow \left\{ 2^{v(s+n)} \int |\varphi_v * f(x-y)|/(1 + 2^v|y|)^a \, dy \right\} \in L_p(\ell^q)$$

with a as in the hypothesis of the theorem. The proof of Theorem (2.1.9) now yields

$$\sigma_\mu^+ f(x) \le \sum t_{v-\mu}'' 2^{v(s+n)} \int |\varphi_v * f(x-y)|/(1 + 2^v|y|)^a \, dv$$

with $t_v'' = 2^{-v(s+n)} \int (1 + 2^v|y|)^a |\sigma * \varphi_v(y)| dy$. The rest of the proof is the same.

Theorem (2.1.10) [66]: Assume that $m \in B_1^{0\infty}(a)$ where $a > n/\min(p,q)$. Then $f \in F_p^{sq} \Rightarrow m * f \in F_p^{sq}$.

Proof: Let us set $g = m * f$. We want to estimate $\varphi^+ g$. Choose σ in such a way that Theorem (2.1.6) is applicable and that in addition $\hat{\sigma}_v(\xi) = 1$ in $\text{supp}\,\hat{\sigma}_v$. Then we have

$$\varphi_v * g = (\varphi_v * m) * (\sigma_v * f)$$

and we get

$$2^{vs}|\varphi_v * g(x)| \le \int |\varphi_v * m(y)|(1 + 2^v|y|)^a dy \sigma_v^{**} f(x) \le C\sigma_v^{**} f(x)$$

or

$$\varphi^+ g \le C\sigma^{**} f.$$

Since $\sigma^{**} f \in L_p$ we get $\varphi^+ g \in L_p$ and $g \in F_p^{sq}$.

In order to get a true multiplier theorem we have to express the condition on m in terms of m.

Corollary (2.1.11) [66]: The conclusion of Theorem (2.1.10) is valid in particular if $|D^\alpha \hat{m}(\xi)| \le C|\xi|^{-|\alpha|}$ for all multi-indices α with $|\alpha| \le T$ where T is an integer $> n/2 + a$.

We start by recalling the following known result (in the periodic case with $n = 1$)

Theorem (2.1.12) [61]: Let f belong to the closure of S in $B_\infty^{1\infty}(T^1)$. Then $f'(x)$ exists at a point $x \in T$ iff $\Phi_n f'(x)$ tend to a limit as $n \to \infty$. Here $\Phi_n f$ denote the Fejer sums of f.

We can now show the following analogue of Theorem (2.1.12).

Theorem (2.1.13) [66]: Let f be in the closure of S in $F_p^{0\infty} = F_p^{0\infty}(\mathbb{R}^n)$ where $1 < p \le \infty$. Assume that, for some σ, $\sigma_v * f(x)$ converges as $v \to \infty$ a.e. for x in set of positive measure. Then the same is true for any other kernel such that the difference with the first one belongs to $B_\infty^{-n1}(a)$ where $a > n/\min(1,p)$.

Proof: It suffices of course to prove that $\sigma_v * f$ tend to 0 a.e. throughout $\mathbb{R}^n$, for every $\sigma \in B_\infty^{-n1}(a)$. Since $\hat{\sigma}(0) = 0$ this certainly is true if $f \in S$. On the other hand by Theorem (2.1.9) $\sup|\sigma_v * f(x)| < \infty$ a.e. for a general f. Thus it suffices to apply the usual density argument.

In retrospect we notice that in the preceding treatment only very little of the structure of underlying space $\mathbb{R}^n$ has been utilized. This indicates that there exist

generalizations. In the place of $\mathbb{R}^n$ we may indeed consider any (complete) Riemannian manifold Ω. The spaces $F_p^{sq} = F_p^{sq}(\Omega)$ are then defined by a condition of the type

$$\{2^{vs}\varphi(\sqrt{-\Delta}/2^v)f\}_{v\in\mathbb{Z}} \in L_p(\ell^q)$$

where Δ is the Laplace-Beltrami operator on Ω. (In particular we can thus define Hardy-classes $H_p = H_p(\Omega)$). We plan to return to this topic in a forthcoming publication.

Section (2.2): Triebel-Lizorkin Spaces the Case q < 1

We complete the characterization of the weighted Besov-Lipschitz and Triebel-Lizorkin spaces, which was started in [68]. Let us recall the following theorem:

Theorem (2.2.1) [68]: Let $0 < p, q \leq \infty, p < \infty$. Suppose ω is an A_∞ weight, $\mu \in \mathcal{S}$ satisfies a moment condition of appropriately high order, and $v \in \mathcal{S}$ satisfies the Tauberian condition. Then, there exist a constant C such that

$$\left\|\left(\int_0^\infty (t^{-\alpha}\mu_t^* f)^q \frac{dt}{t}\right)^{1/q}\right\|_{p,\omega} \leq C\|f\|_{\dot{F}_{p,q}^{\alpha,\omega}}$$

and

$$\|f\|_{\dot{F}_{p,q}^{\alpha,\omega}} \leq C\left\|\left(\int_0^\infty (t^{-\alpha}v_t^* f)^q \frac{dt}{t}\right)^{1/q}\right\|_{p,\omega}$$

for all $f \in \mathcal{S}'/\mathcal{P}$, tempered distributions modulo polynomials.

Let us recall briefly that

$$\mu_t^* f(x) = \mu_{t,\lambda}^* f(x) = \sup_{y\in\mathbb{R}^n} \frac{|\mu_t * f(x-y)|}{(1+|y|/t)^\lambda}, \qquad \mu_t(x) = t^{-n}\mu(x/t)$$

is the Peetre maximal function, with λ, in the case of the above theorem, large and dependent on p, q, ω, and the dimension n. The norm $\|\cdot\|_{p,\omega}$ is the weighted Lebesgue L^p norm with the weighted ω. A similar theorem holds for the Besov-Lipschitz spaces $\dot{B}_{p,q}^{\alpha,\omega}$. It has been shown in [68] that under additional assumptions ($q \geq 1$ for $\dot{F}_{p,q}^{\alpha,\omega}$ and $p, q \geq 1$ for $\dot{B}_{p,q}^{\alpha,\omega}$, for example) the second inequality can be the improved:

$$\|f\|_{\dot{F}_{p,q}^{\alpha,\omega}} \leq C\left\|\left(\int_0^\infty |t^{-\alpha}v_t * f|^q \frac{dt}{t}\right)^{1/q}\right\|_{p,\omega}$$

We will show that the improved estimate holds without any addition hypotheses i.e., for $0 < p, q \leq \infty, p < \infty$. In effect, if v satisfies the conditions of the theorem, then the right-hand side in the above inequality begin finite is necessary and sufficient condition for f to be in $\dot{F}_{p,q}^{\alpha,\omega}$, with equivalent norms.

The problem of characterizing these function spaces in terms of the Littlewood-Paley g-functions, when $\hat{v}$ is not compact support, has been an open problem since the appearance of [66] by Peetre in 1975. The interest in this problem partially stems from the historical fact that, classically, the study of function spaces was usually done via the Poisson kernel or the Gaussian kernel, both of which do not have compactly supported Fourier transforms. For the Hardy spaces H^p of Fefferman and Stien, $0 < p < \infty$, Uchiyama [72] proved a general theorem which implies that if $v \in \mathcal{S}$ satisfies the standard Tauberian condition, then there are positive constants c and C such that

$$c\left\|\left(\int_0^\infty |v_t * f|^2 \frac{dt}{t}\right)^{1/2}\right\|_p \leq \|f\|_{H^p} \leq C\left\|\left(\int_0^\infty |v_t * f|^2 \frac{dt}{t}\right)^{1/2}\right\|_p \tag{6}$$

for all $f \in H^p$. The method used by Uchiyama involves the full machinery of the H^p-theory, including the atomic decomposition and duality. Moreover, the prove of the right-hand side inequality in (6) was done under the assumption that $f \in H^p$, and hence the result does not give a complete characterization of H^p. One would get a complete characterization of H^p from (6) by showing that the space of all $f \in \mathcal{S}'$ for which the norm

$$\left\| \left(\int_0^\infty |v_t * f|^2 \frac{dt}{t} \right)^{1/2} \right\|_p$$

is finite, complete and contains a nice dense subspace (which is also dense in H^p). However, the proof of such a density result would use the independence of the function space on the defining function v, which is more or less the characterization one would want to establish in the first place; we note that the proof of the density result for H^p is rather non-trivial (see, e.g., [67] or [71]). Since $H^p = \dot{F}^0_{p,2}$, our result in [68] seems new even for the unweighted Hardy spaces, see [68].

Let us recall some of the notation from [68]. Choose $\psi \in \mathcal{S}$ such that

$$\operatorname{supp} \widehat{\psi} \subset \{1/2 \le |\xi| \le 2\} \quad \text{and} \quad \sum_{j \in \mathbb{Z}} \widehat{\psi}(2^j \xi) = 1 \quad \text{for } \xi \ne 0.$$

For each integer j we let $\psi_j(x) = \psi_{2^{-j}}(x) = 2^{jn}\psi(2^j x)$. Let $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, and in the case of $\dot{F}^{\alpha,\omega}_{p,q}$ suppose additionally that $p < \infty$. For $f \in \mathcal{S}'$ we let

$$\|f\|_{\dot{B}^{\alpha,\omega}_{p,q}} = \left(\sum_{j=-\infty}^{\infty} \left(2^{j\alpha} \|\psi_j * f\|_{p,\omega} \right)^q \right)^{1/q},$$

$$\|f\|_{\dot{F}^{\alpha,\omega}_{p,q}} = \left\| \left(\sum_{j=-\infty}^{\infty} \left(2^{j\alpha} |\psi_j * f| \right)^q \right)^{1/q} \right\|_{p,\omega}.$$

Let $\psi \in \mathcal{S}$ be defined by $\widehat{\psi}(\xi) + \sum_{j \ge 1} \widehat{\psi}_j(\xi) = 1$. The following are the inhomogeneous versions of the above norms:

$$\|f\|_{B^{\alpha,\omega}_{p,q}} = \|\psi * f\|_{p,\omega} + \left(\sum_{j=1}^{\infty} \left(2^{j\alpha} \|\psi_j * f\|_{p,\omega} \right)^q \right)^{1/q},$$

$$\|f\|_{F^{\alpha,\omega}_{p,q}} = \|\psi * f\|_{p,\omega} + \left\| \left(\sum_{j=1}^{\infty} \left(2^{j\alpha} |\psi_j * f| \right)^q \right)^{1/q} \right\|_{p,\omega}.$$

Suppose that $v \in \mathcal{S}$ satisfies the Tauberian condition, that if for each $\xi \ne 0$ there exista a $t > 0$ such that $\widehat{v}(t\xi) \ne 0$. Suppose $-\infty < \alpha < \infty$, $0 < p, q \le \infty$. and additionally, in the case of $\dot{F}^{\alpha,\omega}_{p,q}$, $p < \infty$. Let ω be an A_∞ weight. We then have the following:

Theorem (2.2.2) [73]: Under the above assumptions there exists a constant C independent of $f \in \mathcal{S}'$ such that

$$\|f\|_{\dot{F}^{\alpha,\omega}_{p,q}} \le C \left\| \left(\int_0^\infty |t^{-\alpha}(v_t * f)|^q \frac{dt}{t} \right)^{1/q} \right\|_{p,\omega},$$

$$\|f\|_{\dot{B}^{\alpha,\omega}_{p,q}} \le \left(\int\limits_0^\infty \left(t^{-\alpha} \|v_t * f\|_{p,\omega} \right)^q \frac{dt}{t} \right)^{1/q}.$$

Proof: We will follow the argument from [68]. We use a version of the Calderón's reproducing formula, which is due to Janson and Taibleson [69]. There exists an $\eta \in \mathcal{S}$ with $\hat{\eta}$ supported on an annulus centered at the origin, such that for $f \in \mathcal{S}'$

$$f = \int\limits_0^\infty f * v_t * \eta_t \frac{dt}{t}$$

with the integral converging in $\mathcal{S}'/\mathcal{P}_m$, the tempered distributions modulo polynomials of degree up to m, depending on f. In our setting this implies the pointwise representation

$$(\psi_j * f)(x) = \int\limits_{I_j} v_t * \eta_t * \psi_j * f(x) \frac{dt}{t} \tag{7}$$

where I_j is such that $\eta_t * \psi_j \equiv 0$ unless $t \in I_j$. If $\operatorname{supp} \hat{\eta} \subset \{2^{-A+1} \le |\xi| \le 2^{A-1}\}$ for some $A > 1$, then we can take $I_j = \{2^{-j-A}, 2^{-j+A}\}$. Pick $\lambda, r > 0$ and let $t \in I_j$. By a version of the result of Strömberg and Torchinsky [68], we obtain

$$\left| v_t * \eta_t * \psi_j * f(x) \right|^r$$
$$\le C \int\limits_{I_j} \int\limits_{\mathbb{R}^n} \left| v_s * \eta_s * \psi_j * f(y) \right|^r \left(1 + \frac{|x-y|}{s} \right)^{-\lambda r} \left(1 + \frac{|x-y|}{s} \right)^{-n-1} dy\, s^{-n} \frac{ds}{s}$$

Moreover, since $s, t \in I_j$, then by [68] the inner integral can be estimated by $v_s^* f(x)^r$, and thus

$$\left| (\psi_j * f)(x) \right|^r \le C \int\limits_{I_j} v_s^* f(x)^r \frac{ds}{s} \tag{8}$$

for every j and x. The following lemma is special case of the result by Strömberg and Torchinsky [71].

Lemma (2.2.3) [73]: Let $v \in \mathcal{S}$ satisfy Tauberian condition, $r, \lambda > 0$. Then there exists a constant C such that

$$|v_t * f(x)|^r \le C \int\limits_0^t \int\limits_{\mathbb{R}^n} |(v_s * f)(z)|^r \left(1 + \frac{|x-z|}{s} \right)^{-\lambda r} \left(\frac{s}{t} \right)^{\lambda r} dz\, s^{-n} \frac{ds}{s}$$
$$+ C \int\limits_{\mathbb{R}^n} |(v_t * f)(z)|^r \left(1 + \frac{|x-z|}{t} \right)^{-\lambda r} t^{-n} dz$$

for all $f \in \mathcal{S}'$, $x \in \mathbb{R}^n$ and $t > 0$.

Let us observe the following inequality, which holds for $s \le t$

$$\left(1 + \frac{|x-y-z|}{s} \right)^{-\lambda r} \left(\frac{s}{t} \right)^{\lambda r} \le \left(1 + \frac{|x-y-z|}{t} \right)^{-\lambda r} \left(\frac{s}{t} \right)^{\lambda r} \le \left(1 + \frac{|y|}{t} \right)^{\lambda r} \left(1 + \frac{|x-z|}{t} \right)^{-\lambda r} \left(\frac{s}{t} \right)^{\lambda r}$$
$$\le \left(1 + \frac{|y|}{t} \right)^{\lambda r} \left(1 + \frac{|x-z|}{s} \right)^{-\lambda r/2} \left(\frac{s}{t} \right)^{\lambda r/2}.$$

It follows that

$$|v_t * f(x-y)|^r$$

$$\leq C\left(1 + \frac{|y|}{t}\right)^{\lambda r}\left(\int_0^t \int_{\mathbb{R}^n} |(v_s * f)(z)|^r \left(1 + \frac{|x-z|}{s}\right)^{-\lambda r/2}\left(\frac{s}{t}\right)^{\lambda r/2} dz \, s^{-n}\frac{ds}{s}\right.$$

$$\left. + \int_{\mathbb{R}^n} |(v_t * f)(z)|^r \left(1 + \frac{|x-z|}{t}\right)^{-\lambda r} t^{-n}dz\right)$$

$$\leq C\left(1 + \frac{|y|}{t}\right)^{\lambda r}\left(\int_0^t M(|v_s * f|^r)(x)\left(\frac{s}{t}\right)^{\lambda r/2}\frac{ds}{s} + M(|v_t * f|^r)(x)\right)$$

where M denotes the Hardy-Littlewood maximal operator. The last inequality follows, provided $\lambda r > 2n$, by a standard argument involving decomposing $\mathbb{R}^n$ into a sum of concentric annuli:

$$\mathbb{R}^n = \{z : |x-z| \leq s\} \cup \bigcup_{k=1}^{\infty}\{z : 2^{k-1}s < |x-z| \leq 2^k s\},$$

with s replaced by t for the second integral. We have thus proved the following lemma, which serves as replacement for the well-known pointwise estimate, due to Peetre [66], which holds for functions of exponential type.

Lemma (2.2.4) [73]: For $v \in \mathcal{S}$ satisfying the Tauberian condition and λ, r such that $\lambda r > 2n$ we have

$$\left(v_t^* f(x)\right)^r \leq C\left(\int_0^t M(|v_s * f|^r)(x)\left(\frac{s}{t}\right)^{\lambda r/2}\frac{ds}{s} + M(|v_t * f|^r)(x)\right)$$

Thus, combining the above with (8), we obtain

$$|\psi_j * f(x)| \leq C\left(\int_{I_j}\int_0^t M(|v_s * f|^r)(x)\left(\frac{s}{t}\right)^{\lambda r/2}\frac{ds}{s}\frac{dt}{t}\right)^{1/r} + C\left(\int_{I_j} M(|v_t * f|^r)(x)\frac{dt}{t}\right)^{1/r}$$

$$= J_j^1(x) + J_j^2(x). \tag{9}$$

Choose $0 < r < \min(p/r_0, q)$, where $r_0 = \inf\{s : \omega \in A_s\}$, so that, in particular, $\omega \in A_{p/r}$. Choose λ such that $\lambda + 2\alpha > 0$. Using Hölder's inequality, and then Hardy's inequality, we obtain

$$\left\|\left(\sum_{j=-\infty}^{\infty}(2^{j\alpha}J_j^1)^q\right)^{1/q}\right\|_{p,\omega} \leq C\left\|\left(\sum_{j=-\infty}^{\infty}\int_{I_j}\left(\int_0^t M(|v_s * f|^r)\left(\frac{s}{t}\right)^{\lambda r/2}\frac{ds}{s}\right)^{q/r}2^{-\alpha q}\frac{dt}{t}\right)^{1/q}\right\|_{p,\omega}$$

$$\leq C\left\|\left(\int_0^{\infty}\left(\int_0^t M(|v_s * f|^r)s^{\lambda r/2}\frac{ds}{s}\right)^{q/r}t^{-(\alpha+\lambda/2)q}\frac{dt}{t}\right)^{1/q}\right\|_{p,\omega}$$

$$\leq C\left\|\left(\int_0^{\infty}(t^{-\alpha r}M(|v_t * f|^r))^{q/r}\frac{dt}{t}\right)^{1/q}\right\|_{p,\omega}.$$

We can now use the vector valued maximal inequality [70] to conclude

$$\left\|\left(\sum_{j=-\infty}^{\infty}\left(2^{j\alpha}J_j^1(x)\right)^q\right)^{1/q}\right\|_{p,\omega} \le C\left\|\left(\int_0^\infty (t^{-\alpha}|v_t * f|)^q \frac{dt}{t}\right)^{1/q}\right\|_{p,\omega}$$

A similar, but simpler argument gives us the second inequality

$$\left\|\left(\sum_{j=-\infty}^{\infty}\left(2^{j\alpha}J_j^2(x)\right)^q\right)^{1/q}\right\|_{p,\omega} \le C\left\|\left(\int_0^\infty (t^{-\alpha}|v_t * f|)^q \frac{dt}{t}\right)^{1/q}\right\|_{p,\omega}$$

The second inequality of Theorem (2.2.2), concerning the space $\dot{B}_{p,q}^{\alpha,\omega}$, can be obtained similarly from (9). Applying weighted L^p norm to both side of (9) we obtain

$$\|\psi_j * f\|_{p,\omega} \le C\left(\|J_j^1\|_{p,\omega} + \|J_j^2\|_{p,\omega}\right)$$

Consider the first component. Applying Minkowski's inequality twice for the weighted $L^{p/r}$ norm, and the inequality for the Hardy-Littlewood maximal operator, we obtain

$$\|J_j^1\|_{p,\omega} = \left\|\left(\int_{I_j}\int_0^t M(|v_s * f|^r)(\cdot)\left(\frac{s}{t}\right)^{\lambda r/2}\frac{ds\,dt}{s\,t}\right)^{1/r}\right\|_{p,\omega}$$

$$\le \left(\int_{I_j}\int_0^t \|M(|v_s * f|^r)\|_{p/r,\omega}\left(\frac{s}{t}\right)^{\lambda r/2}\frac{ds\,dt}{s\,t}\right)^{1/r}$$

$$\le C\left(\int_{I_j}\int_0^t \|v_s * f\|_{p,\omega}^r\left(\frac{s}{t}\right)^{\lambda r/2}\frac{ds\,dt}{s\,t}\right)^{1/r}$$

We can now, as in the Triebel-Lizorkin case, multiply both sides by $2^{j\alpha}$, apply discrete ℓ^q norm, and use Hardy's inequality. The same argument, without the Hardy's inequality, is used to handle J_j^2. Observe that in this case we only need to use the scalar valued inequality for the Hardy-Littlewood maximal operator rather than the vector valued one, so we can allow the case $p = \infty$. The proof of Theorem (2.2.2) is complete.

Theorem (2.2.5) [73]: Under the assumptions of Theorem (2.2.2) there exists a $\gamma > 1$ and a constant C, independent of $f \in \mathcal{S}'$ such that

$$\|f\|_{\dot{F}_{p,q}^{\alpha,\omega}} \le C\left\|\left(\sum_{j=-\infty}^{\infty}\left(\gamma^{j\alpha}|v_j * f|\right)^q\right)^{1/q}\right\|_{p,\omega}$$

$$\|f\|_{\dot{B}_{p,q}^{\alpha,\omega}} \le C\left(\sum_{j=-\infty}^{\infty}\left(\gamma^{j\alpha}\|v_j * f\|_{p,\omega}\right)^q\right)^{1/q}.$$

Proof: The starting point for this case is the discrete version of the Calderón's reproducing formula, which can be obtained in the same way as the continuous from [69]. For v as in the theorem, there exist $\gamma > 1$, and $\eta \in \mathcal{S}$, with $\hat{\eta}$ supported in annulus such that

$$f = \sum_{k\in\mathbb{Z}} f * v_k * \eta_k \quad \text{in } \mathcal{S}'/\mathcal{P}$$

where $v_k = v_{\gamma^{-k}}$, and $\eta_k = \eta_{\gamma^{-k}}$. Then

$$(f * \psi_j)(x) = \sum_{\gamma^k \in I_{-j}} v_k * \eta_k * f * \psi_j(x)$$

Point wise ($\psi_j = \psi_{2^{-j}}$). The summation is over a finite set of ks, since both $\widehat{\psi}$ and $\widehat{\eta}$ are supported in annuli centered at the origin, and I_{-j} is the interval, defined previously, depending on the support of $\widehat{\eta}$. We will use the discrete versions of the Strömberg-Torchinsky type estimates that were used in the proof of Theorem (2.2.2):

$$\begin{aligned}
&\left| v_k * \eta_k * \psi_j * f(x) \right|^r \\
&\leq C \sum_{\gamma^l \in I_{-j}} \int_{\mathbb{R}^n} \left| v_l * \eta_l * \psi_j * f(y) \right|^r (1 + \gamma^l |x - y|)^{-\lambda r} (1 + \gamma^l |x - y|)^{-n-1} dy \, \gamma^{ln}
\end{aligned}$$

and

$$|(v_k * f)(x)|^r \leq C \sum_{l \geq k} \int_{\mathbb{R}^n} |(v_l * f)(z)|^r (1 + \gamma^l |x - z|)^{-\lambda r} \gamma^{-(k-l)\lambda r} dz \, \gamma^{ln}.$$

Both of these estimates hold in the setting they were used in the continuous case, and their proof is the same. From the reproducing formula and the first estimate we obtain

$$|(\psi_j * f)(x)|^r \leq C \sum_{\gamma^l \in I_{-j}} \left(v^*_{\gamma^{-l}} f(x) \right)^r$$

and, subsequently, from the second estimate

$$\left(v^*_{\gamma^{-l}} f(x) \right)^r \leq C \sum_{k \geq l} M(v_k * f)(x) \gamma^{(l-k)\lambda r / 2}$$

We, combine these two inequalities, and then use Hölder's inequality, the discrete version of the Hardy's inequality, and the discrete vector valued maximal inequality to obtain

$$\|f\|_{\dot{F}^{\alpha,\omega}_{p,q}} \leq C \left\| \left(\sum_{j=-\infty}^{\infty} (2^{j\alpha} |nu_j * f|)^q \right)^{1/q} \right\|_{p,\omega}$$

The corresponding result for the $\dot{B}^{\alpha,\omega}_{p,q}$ space follows similarly.

The following theorem is the inhomogeneous counterpart of Theorem (2.2.2). Together with Theorem 5.1 from [68] it provides the characterization of the inhomogeneous spaces $\dot{F}^{\alpha,\omega}_{p,q}$ and $\dot{B}^{\alpha,\omega}_{p,q}$ in terms of convolutions with general defining functions.

Theorem (2.2.6) [73]: Let $-\infty < \alpha < \infty$, $0 < p, q \leq \infty$, with $p < \infty$ in the case of $\dot{F}^{\alpha,\omega}_{p,q}$. Suppose $\omega \in A_\infty$. Let $v \in \mathcal{S}$ satisfy the Tauberian condition and let $\Phi \in \mathcal{S}$ satisfy the strong Tauberian condition $\widehat{\Phi}(0) \neq 0$. Then, for b sufficiently large (see the proof) there exists a positive constant C such that

$$\|f\|_{F^{\alpha,\omega}_{p,q}} \leq C \left(\|\Phi * f\|_{p,\omega} + \left\| \left(\int_0^b |t^{-\alpha}(v_t * f)|^q \frac{dt}{t} \right)^{1/q} \right\|_{p,\omega} \right),$$

$$\|f\|_{B^{\alpha,\omega}_{p,q}} \leq C \left(\|\Phi * f\|_{p,\omega} + \left(\int_0^b (t^{-\alpha} \|v_t * f\|_{p,\omega})^q \frac{dt}{t} \right)^{1/q} \right).$$

for every $f \in \mathcal{S}'$.

Proof: As before, we will follow the line of argument from [68]. Let us recall [69] that there exist η and γ in $\mathcal{S}$, with $\hat{\eta}$ supported on an annulus, and $\hat{\gamma}$ compactly supported such that

$$f = \gamma * f + \int_0^b \eta_t * v_t * f \frac{dt}{t}$$

for arbitrary $b > 0$, with γ depending on b. Moreover, taking b large we can make the support of $\hat{\gamma}$ small. The integral in the above formula converge in $\mathcal{S}'$. Pick $j \geq 1$ and we obtain

$$\psi_j * f = \int_{I_j} \psi_j * \eta_t * v_t * f \frac{dt}{t}$$

if b is sufficiently large, so that the supports of $\widehat{\psi}_j$ and $\hat{\gamma}$ are disjoint, and that $2^{A-1} \leq b$. The intervals I_j are the same as those in the previous. This representation is the same as (7), so we can apply the same argument as we did in the homogeneous case. Doing so, we arrive at the estimate

$$\left\| \left(\sum_{j=1}^{\infty} (2^{j\alpha} |\psi_j * f|)^q \right)^{1/q} \right\|_{p,\omega} \leq C \left\| \left(\int_0^b (t^{-\alpha} |v_t * f|)^q \frac{dt}{t} \right)^{1/q} \right\|_{p,\omega} \tag{10}$$

Again, analogous argument yields the corresponding estimate for the Besov-Lipschitz spaces, without the $p < \infty$ restriction:

$$\left(\sum_{j=1}^{\infty} \left(2^{j\alpha} \|\psi_j * f\|_{p,\omega} \right)^q \right)^{1/q} \leq C \left(\int_0^b \left(t^{-\alpha} \|v_t * f\|_{p,\omega} \right)^q \frac{dt}{t} \right)^{1/q}$$

We now consider $\psi * f$

$$\Psi * f = \Psi * \gamma * f + \int_a^b \Psi * \eta_t * v_t * f \frac{dt}{t}$$

for some $0 < a < b$. Recall that $\operatorname{supp} \widehat{\Psi} \subset \{2^{-A+1} \leq |\xi| \leq 2^{A-1}\}$, and $\operatorname{supp} \widehat{\Psi} \subset \{|\xi| \leq 2\}$, so can take $a = 2^{-A}$. The second part is handled exactly as previously. Indeed, even though $\widehat{\Psi}$ does not vanish around 0, $\hat{\eta}$ does, so for the purpose of this integral we may multiply $\widehat{\Psi}$ by a suitable cutoff function. Observe, however, that the b will have to be increased 2^{2A-2} times. For the first part, observe that if we choose b large enough, so that $\operatorname{supp} \hat{\gamma} \subset \{\xi : \widehat{\Phi}(\xi) \neq 0\}$, then we can write
$$\Psi * \gamma * f = u * \Phi * f$$
with $u \in \mathcal{S}$. It follows that for each $\lambda > 0$ there is a constant C, independent of f, such that
$$(\Psi * \gamma)_1^* f(x) \leq C \Phi_1^* f(x)$$
Let f_1 be f multiplied, on the Fourier transform side, by a smooth cutoff function, equal to around the origin, and 0 outside some ball. Then
$$\Phi_1^* f(x) \leq \Phi_1^* f_1(x) + \Phi_1^* f_2(x)$$
Since $\Phi * f_1$ is of exponential type, so we may apply Peetre's estimate to obtain
$$\|\Phi_1^* f_1\|_{p,\omega} \leq C \|\Phi * f_1\|_{p,\omega}$$
provided λ in the definition of $\Phi_1^* f_1$ is large enough (see, [67]). To estimate $\Phi_1^* f_2(x)$ we use the left-hand inequality from [68] (actually only a part of it)
$$\|\Phi_1^* f_2\|_{p,\omega} \leq C \|f_2\|_{F_{p,q}^{\alpha,\omega}}$$

Choose the cutoff function, which defines f_1 and f_2 such that the supports of $\hat{f}_2$ and $\widehat{\psi}$ are disjoint. Then

$$\|f_2\|_{F_{p,q}^{\alpha,\omega}} \le C \left\| \left(\sum_{j=1}^{\infty} (2^{j\alpha} |\psi_j * f|)^q \right)^{1/q} \right\|_{p,\omega}$$

and this has already been estimated in (10). Finally,

$$\|\Phi_1^* f_1\|_{p,\omega} \le C \|\Phi * f_1\|_{p,\omega} \le C \|\Phi * f\|_{p,\omega} + \|\Phi * f_2\|_{p,\omega}$$

Since we already have the estimate for $\|\Phi * f_2\|_{p,\omega} \le \|\Phi_1^* f_2\|_{p,\omega}$, we obtain

$$\|f\|_{F_{p,q}^{\alpha,\omega}} = \|\Psi * f\|_{p,\omega} + \left\| \left(\sum_{j=1}^{\infty} (2^{j\alpha} |\psi_j * f|)^q \right)^{1/q} \right\|_{p,\omega}$$

$$\le C \left(\|\Phi * f\|_{p,\omega} + \left\| \left(\int_0^b (t^{-\alpha} |v_t * f|)^q \frac{dt}{t} \right)^{1/q} \right\|_{p,\omega} \right).$$

The proof of the Besov-Lipschitz case follows similarly.

Theorem (2.2.7) [73]: Let α, p, q, ω, v, and Φ be as in Theorem (2.2.5). Then, for any $a, b, r, N > 0$, there is a positive constant C such that

$$\|f\|_{F_{p,q}^{\alpha,\omega}} \le C \left(\left(\int_0^a (t^N \|\Phi_t * f\|_{p,\omega})^r \frac{dt}{t} \right)^{1/r} + \left\| \left(\int_0^b (t^{-\alpha} |v_t * f|)^q \frac{dt}{t} \right)^{1/q} \right\|_{p,\omega} \right)$$

$$\|f\|_{B_{p,q}^{\alpha,\omega}} \le C \left(\left(\int_0^a (t^N \|\Phi_t * f\|_{p,\omega})^r \frac{dt}{t} \right)^{1/r} + \left(\int_0^b (t^{-\alpha} \|v_t * f\|_{p,\omega})^q \frac{dt}{t} \right)^{1/q} \right)$$

for every $f \in \mathcal{S}'$.

Proof: We first show that for any $\varphi \in \mathcal{S}$,

$$\|\Phi * f\|_{p,\omega} \le C \left(\int_0^a (t^N \|\Phi_t * f\|_{p,\omega})^r \frac{dt}{t} \right)^{1/r} \tag{11}$$

for all $f \in \mathcal{S}'$. Choose $0 < s \le r$ such that $\omega \in A_{p/s}$. Let N' be such that $sN' > n$. By using [6] and decomposing $\mathbb{R}^n$ into concentric annuli, we obtain

$$|\varphi * f(x)|^s \le C \int_0^a \int_{\mathbb{R}^n} |\Phi_t * f(y)|^s \left(1 + \frac{|x-y|}{t} \right)^{sN'} t^{sN'} t^{-n} dy \frac{dt}{t} \le C \int_0^a M(|\Phi_t * f|^s)(x) t^{sN'} \frac{dt}{t}$$

for every $x \in \mathbb{R}^n$. Hence, it follows from Minkowski's inequality and the weighted estimate for the Hardy-Littlewood maximal function that

$$\|\varphi * f\|_{p,\omega} \le C \left(\int_0^a (t^{N'} \|\Phi_t * f\|_{p,\omega})^s \frac{dt}{t} \right)^{1/s}$$

The above and Hölder's inequality imply (11).

We shall show the theorem only for the Triebel-Lizorkin spaces since the proof for the Besov-Lipschitz spaces is similar. As in the proof of Theorem (2.2.6), we start with the representation

$$f = \gamma * f + \int_0^b \eta_t * \nu_t * f \frac{dt}{t} \tag{12}$$

where $\hat{\gamma} \in \mathcal{S}$ has compact support. There is a finite set $\mathcal{S}_1$ such that $\gamma * \psi_j = 0$ for all $j \notin \mathcal{S}_1$, and there is a finite set $\mathcal{S}_2$ such that

$$I_j = \{2^{-j-A}, 2^{-j+A}\} \subseteq [0, b]$$

for all $j \notin \mathcal{S}_2$. Let $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$. Then by (11),

$$\left\| \left(\sum_{j \in \mathcal{S}} (2^{j\alpha} |\psi_j * f|)^q \right)^{1/q} \right\|_{p,\omega} \leq C \sum_{j \in \mathcal{S}} \|\psi_j * f\|_{p,\omega} \leq C|\mathcal{S}| \left(\int_0^a (t^N \|\Phi_t * f\|_{p,\omega})^r \frac{dt}{t} \right)^{1/r}$$

and

$$\|\Psi * f\|_{p,\omega} \leq C \left(\int_0^{ka} (t^N \|\Phi_t * f\|_{p,\omega})^r \frac{dt}{t} \right)^{1/r}$$

where $|\mathcal{S}|$ denotes the number of elements in $\mathcal{S}$. For $j \notin \mathcal{S}$, the representation (12) gives

$$\psi_j * f = \int_0^b \eta_t * \nu_t * \psi_j * f \frac{dt}{t} = \int_{I_j} \eta_t * \nu_t * \psi_j * f \frac{dt}{t}$$

As observed in the proof of Theorem (2.2.6), the above and the method in the proof of Theorem (2.2.2) imply that

$$\left\| \left(\sum_{j \notin \mathcal{S}} (2^{j\alpha} |\psi_j * f|)^q \right)^{1/q} \right\|_{p,\omega} \leq C \left\| \left(\int_0^b (t^{-\alpha} |\nu_t * f|)^q \frac{dt}{t} \right)^q \right\|_{p,\omega}$$

Combining all the above estimates, we obtain the desired inequality for the Triebel-Lizorkin spaces. The proof of Theorem (2.2.7) is hence complete.

Section (2.3): Morrey Type Besov-Triebel Spaces

Many people have been considered problems of partial differential equation based on Morrey space and Morrey type Besov space, (see [78, 79, 81, 82, 83, 84]). It is well-known that Besov spaces $B_{p,q}^s(\mathbb{R}^n)$ and Triebel-Lizokin spaces $F_{p,q}^s(\mathbb{R}^n)$ contain as special cases many classical spaces, for example, the Hölder spaces, the Sobolev spaces, the Bessel-potential spaces, the Zygmund spaces, the local Hardy spaces and the space $BMO(\mathbb{R}^n)$. All the above-mentioned classical spaces have been proved to be useful tools in the study of ordinary and partial differential equations. For detail one can see Triebel's books [41, 56, 57, 58].

We study some properties, such as lifting properties, Fourier multiplier theorem, and discrete characterization of Morrey type Besov-Triebel spaces. We consider the boundedness of a class pseudo-differential operators on these spaces.

As usual, the n-dimensional real Euclidean space and its points are dented by $\mathbb{R}^n$ and $x = (x_1, \cdots, x_n)$ respectively. We purpose that $n > 1$ holds. $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of all rapidly decreasing infinitely differential complex-valued functions on $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ is the space of all complex-valued tempered distributions on $\mathbb{R}^n$. Let

$$(\mathcal{F}\phi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \phi(x) e^{-ix\cdot\xi} dx$$

and let $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse on $\mathcal{S}'(\mathbb{R}^n)$, respectively.

Definition (2.3.1) [55]: If $0 < q \le p < \infty$ and $f \in L^q_{loc}(\mathbb{R}^n)$, we say $f \in M^p_q(\mathbb{R}^n)$ provided that, for any ball $B_R(x)$ of center at x and radius R,

$$\|f\|_{M^p_q} := \sup_{x\in\mathbb{R}^n, R>0} R^{n(1/p-1/q)} \left(\int_{B_R(x)} |f(y)|^q dy \right)^{1/q} < \infty$$

Definition (2.3.2) [55]: Let $\phi(\mathbb{R}^n)$ be the collection of all systems $\phi = \{\phi_j\}_{j=0}^{\infty} \subset \mathcal{S}(\mathbb{R}^n)$ of real-valued even function with respect to the origin, such that $\phi_j(x) = \phi_j(-x)$ if $x \in$ supp ϕ_j where

$$\text{supp } \phi_0 \subset \{x, |x| \le 2\}$$

and

$$\text{supp } \phi_j \subset \{x, 2^{j-1} \le |x| \le 2^{j+1}\} \text{ for } j = 0,1,\cdots,$$

for every multi-index α there exists a positive number C_α such that

$$2^{j|\alpha|}|D^\alpha\phi_j(x)| \le C_\alpha \text{ for all } j = 0,1,\cdots, \quad \text{ and all } x \in \mathbb{R}^n,$$

and

$$\sum_{j=0}^{\infty} \phi_j(x) = 1 \text{ for every } x \in \mathbb{R}^n$$

We introduce Morrey type Besov-Triebel spaces.

Definition (2.3.3) [55]: Let $-\infty < s < \infty, 0 < q \le p < \infty, 0 < \beta \le \infty$, and $\phi = \{\phi_j\}_{j=0}^{\infty} \in \Phi(\mathbb{R}^n)$, then we define:

(i) $MB^{s,\beta}_{p,q} = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|^{\phi}_{MB^{s,\beta}_{p,q}} = \left\{\sum_{k=0}^{\infty} 2^{ks\beta} \|\mathcal{F}^{-1}\phi_j\mathcal{F}f\|^{\beta}_{M^p_q}\right\}^{1/\beta} < \infty \right\}$

(ii) $MF^{s,\beta}_{p,q} = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|^{\phi}_{MF^{s,\beta}_{p,q}} = \left\{\sum_{k=0}^{\infty} 2^{ks\beta} \|\mathcal{F}^{-1}\phi_j\mathcal{F}f\|^{\beta}_{M^p_q}\right\}^{1/\beta} < \infty \right\}$

Obviously, for $s \in \mathbb{R}$, $0 < p = q < \infty$, and $0 < \beta \le \infty$, then $MB^{s,\beta}_{p,q} = B^s_{p,\beta}$ and $MF^{s,\beta}_{p,q} = F^s_{p,\beta}$, standard Besov and Triebel-Lizorkin spaces respectively; see [41].

If $s \in \mathbb{R}$, we write

$$H^s_2(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{H^s_2} = \left\|(1+|x|^2)^{s/2}(\mathcal{F})(x)\right\|_{L^2} < \infty \right\}$$

If Ω is a compact set of $\mathbb{R}^n$ we write

$$L^{q,\Omega} = \{f \in \mathcal{S}'(\mathbb{R}^n) : \text{supp } \mathcal{F}f \subset \Omega, \quad \|f\|_{L^q} < \infty\}$$

Let $0 < \beta \leq \infty$. If $\{f_j\}_{j=0}^{\infty}$ is a sequence of complex-valued Lebesgue measurable function on $\mathbb{R}^n$, then we write

$$\|f_j\|_{M_{p,q}^{\beta}} = \left\|\left(\sum_{j=0}^{\infty}|f_j(\cdot)|^{\beta}\right)^{1/\beta}\right\|_{M_q^p}$$

If $\|f_j\|_{M_{p,q}^{\beta}} < \infty$, we call the sequence $\{f_j\}_{j=0}^{\infty} \in M_{p,q}^{\beta}$. Furthermore, assume $0 < q \leq p < \infty$, $0 < \beta \leq \infty$, and that $\Omega = \{\Omega_j\}_{j=0}^{\infty}$ is a sequence of compact sets on $\mathbb{R}^n$. If $f_j \in L^{q,\Omega_j}$ for $j \in \mathbb{N} \cup \{0\}$, and $\{f_j\}_{j=0}^{\infty} \in M_{p,q}^{\beta}$, then we call the sequence $\{f_j\}_{j=0}^{\infty} \in M_{p,q}^{\beta,\Omega}$. From then, we let $\mathbb{N}_0$ denote $\mathbb{N} \cup \{0\}$.

Lemma (2.3.4) [55]: Let $0 < \beta < \infty$, $1 < q \leq p < \infty$. If $\{f_j\}_{j=0}^{\infty}$ is a sequence of local integral function on $\mathbb{R}^n$, then

$$\left\|\left(\sum_{j=0}^{\infty}|\mathcal{M}f_j|^{\beta}\right)^{1/\beta}\right\|_{\mathcal{M}_q^p} \leq C \left\|\left(\sum_{j=0}^{\infty}|f_j|^{\beta}\right)^{1/\beta}\right\|_{\mathcal{M}_q^p}$$

where the constant C is independent of $\{f_j\}_{j=0}^{\infty}$ and $\mathcal{M}$ denotes Standard Hardy-Littlewood maximal function.

Proof: In fact, the conclusion can be deduced from the weighted version of the Fefferman-Stein vector maximal inequalities [74] and Theorem 3.1 in [75].

It is also interesting to give a direct proof. Let $\left(\sum_{j=0}^{\infty}|f_j|^{\beta}\right)^{1/\beta} \in M_q^p$. Pick any $x_0 \in \mathbb{R}^n$, and write

$$f_j(x) = f_j^0(x) + \sum_{i=0}^{\infty} f_j^i(x)$$

where $f_j^0 = \chi_{B_{2r}(x_0)}f_j$, $f_j^i = \chi_{B_{2^{i+1}r}(x_0)\backslash B_{2^i r}(x_0)}f_j$ for $i \geq 1$. We want to estimate $\left(\sum_{j=0}^{\infty}|\mathcal{M}f_j^0(x)|^{\beta}\right)^{1/\beta}$ on $B_r(x_0)$. By Fefferman-Stein maximal inequality [80] we have

$$\left\|\left(\sum_{j=0}^{\infty}|\mathcal{M}f_j^0|^{\beta}\right)^{1/\beta}\right\|_{L^q} \leq C \left\|\left(\sum_{j=0}^{\infty}|f_j^0|^{\beta}\right)^{1/\beta}\right\|_{L^q} \leq Cr^{n\left(\frac{1}{q}-\frac{1}{p}\right)}\left\|\left(\sum_{j=0}^{\infty}|f_j|^{\beta}\right)^{1/\beta}\right\|_{M_q^p}$$

Thus,

$$\left[\int_{B_r(x_0)} \left(\sum_{j=0}^{\infty} |\mathcal{M}^\beta(f_j^0)(x)|^\beta \right)^{q/\beta} dx \right]^{1/q} \leq \left\| \left(\sum_{j=0}^{\infty} |\mathcal{M}f_j^0|^\beta \right)^{1/\beta} \right\|_{L^q}$$

$$\leq Cr^{n\left(\frac{1}{q}-\frac{1}{p}\right)} \left\| \left(\sum_{j=0}^{\infty} |f_j|^\beta \right)^{1/\beta} \right\|_{M_q^p}$$

Hence, we obtain

$$\left\| \left(\sum_{j=0}^{\infty} |\mathcal{M}f_j^0|^\beta \right)^{1/\beta} \right\|_{M_q^p} \leq C \left\| \left(\sum_{j=0}^{\infty} |f_j|^\beta \right)^{1/\beta} \right\|_{M_q^p}$$

It remains to estimate $\left(\sum_{j=0}^{\infty} |\mathcal{M}f_j^i(x)|^\beta \right)^{1/\beta}$ on $B_r(x_0)$. For $i \geq 1$ and $x \in B_r(x_0)$, by the generalized Minkowski inequality, we have

$$\left(\sum_{j=0}^{\infty} |\mathcal{M}f_j^i(x)|^\beta \right)^{1/\beta} \leq C \left[\sum_{j=0}^{\infty} \left((2^i r)^{-n} \int_{\mathbb{R}^n} |f_j^i(y)| dy \right)^\beta \right]^{1/\beta} \leq C(2^i r)^{-n} \int_{\mathbb{R}^n} \left(\sum_{j=0}^{\infty} |f_j^i(y)|^\beta \right)^{1/\beta} dy$$

Then,

$$\left[\int_{B_r(x_0)} \left(\sum_{j=0}^{\infty} \mathcal{M}^\beta \left(\sum_{i=1}^{\infty} f_j^i(x) \right) \right)^{q/\beta} dx \right]^{1/q} \leq \left[\int_{B_r(x_0)} \left(\sum_{j=0}^{\infty} \left(\sum_{i=1}^{\infty} \mathcal{M}f_j^i(x) \right)^\beta \right)^{q/\beta} dx \right]^{1/q}$$

$$\leq \left[\int_{B_r(x_0)} \left(\sum_{i=1}^{\infty} \left(\sum_{j=0}^{\infty} \mathcal{M}^\beta f_j^i(x) \right)^{1/\beta} \right)^q dx \right]^{1/q}$$

$$\leq \sum_{i=1}^{\infty} \left[\int_{B_r(x_0)} \left(\sum_{j=0}^{\infty} \mathcal{M}^\beta f_j^i(x) \right)^{q/\beta} dx \right]^{1/q}$$

$$\leq \sum_{i=1}^{\infty} (2^i)^{-n/q} \left[\int_{B_{2^{i+1}r}(x_0)} \left(\sum_{j=0}^{\infty} |f_j(x)|^\beta \right)^{q/\beta} dx \right]^{1/q}$$

$$\leq C \sum_{i=1}^{\infty} (2^i)^{-\frac{n}{p}} r^{\left(\frac{1}{q}-\frac{1}{p}\right)n} \left\| \left(\sum_{j=0}^{\infty} |f_j|^\beta \right)^{1/\beta} \right\|_{M_q^p}$$

Thus, we obtain

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$$\left\|\sum_{j=0}^{\infty}\left(\mathcal{M}^{\beta}\left(\sum_{i=1}^{\infty}f_j^i\right)\right)^{1/\beta}\right\|_{M_q^p} \leq C\left\|\left(\sum_{j=0}^{\infty}|f_j|^{\beta}\right)^{1/\beta}\right\|_{M_q^p}.$$

Lemma (2.3.4) is proved.

Theorem (2.3.5) [55]: Let $0 < q \leq p < \infty$, $0 < \beta \leq \infty$, and $\Omega = \{\Omega_j\}_{j=0}^{\infty}$ be a sequence of compact sets $f_j \in L^{q,\Omega_j}$, $j \in \mathbb{N}_0$. Let $d_j > 0$ be the radius of Ω_j. If $0 < r < \min\{q, \beta\}$, then exists a constant C such that

(i) $\left\|\sup_{x \in \mathbb{R}^n} \dfrac{|f_j(\cdot - z)|}{1 + |d_j z|^{\frac{n}{r}}}\right\|_{M_{p,q}^{\beta}} \leq C\|f_j\|_{M_{p,q}^{\beta}}$

(ii) $\left\|\left(\sum_{j=0}^{\infty}\left|\sup_{x \in \mathbb{R}^n} \dfrac{|f_j(\cdot - z)|}{1 + |d_j z|^{\frac{n}{r}}}\right|^{\beta}\right)^{1/\beta}\right\|_{M_q^p} \leq C\left\|\left(\sum_{j=0}^{\infty}|f_j|^{\beta}\right)^{1/\beta}\right\|_{M_q^p}$

Proof: We only prove for the case of $MF_{p,q}^{s,\beta}$ space. The result for $MB_{p,q}^{s,\beta}$ can be obtained by interchanging $\left\|\cdot\right|_{l_q}\right\|$ and $\|\cdot\|_{M_q^p}$ in the proof presented below.

First, let $\{f_j\}_{j=0}^{\infty} \in M_{p,q}^{\beta,\Omega}$, $y^j \in \Omega$ satisfying (i), then $\{h_j\}_j$ also do, where Ω is replaced by $\{\Omega_j - y^j\}_{j=0}^{\infty}$, and the converse also holds. Thus we may let $0 \in \Omega_j$, it is sufficient to consider the case $\Omega_j = D_j = \{y : |y| \leq d_j\}$.

Second, we have to prove that (ii) holds when $\Omega_j = d_j = \{y : |y| \leq d_j\}$ and $d_j > 0$. If $\{f_j\}_{j=0}^{\infty} \in M_{p,q}^{\beta,\Omega}$, $y^j \in \Omega$, then $f_j \in L^{q,\Omega_j}$. If $g_j(x) = f_j(d^{-1}x)$, then $(\mathcal{F}g_j)(x) = d_j^n(\mathcal{F}f_j)(d_j x)$ and $\text{supp}\,\mathcal{F}g_j \subset \{y : |y| \leq 1\}$.

For $x, z \in \mathbb{R}^n$, we have

$$\frac{|g_j(x-z)|}{1 + |z|^{\frac{n}{r}}} \leq C[\mathcal{M}(|g_j|^r)(x)]^{1/r} \tag{13}$$

see [41].

From (13), we obtain

$$\frac{|f_j(x-z)|}{1 + |d_j z|^{\frac{n}{r}}} \leq C[\mathcal{M}(|f_j|^r)(x)]^{1/r}, \qquad \text{for all } x, z \in \mathbb{R}^n \tag{14}$$

where the constant C independent of x, z, j.

If $0 < \beta < \infty$, then by (14),

$$\left\| \sup_{z\in\mathbb{R}^n} \frac{|f_j(\cdot -z)|}{1+|d_j z|^{\frac{n}{r}}} \right\|_{M_{p,q}^\beta} \leq C \left\| \left[\mathcal{M}\left(|f_j|^r\right)\right]^{1/r} \right\|_{M_{p,q}^\beta} = C \left\| \mathcal{M}\left(|f_j|^r\right) \right\|_{M_{p/r,q/r}^{\beta/r}}^{1/r}$$

Since $0 < r < \min\{q, \beta\}$, we have $p/r \geq q/r > 1$, $\beta/r > 1$. By Lemma (2.3.4),

$$\left\| \sup_{z\in\mathbb{R}^n} \frac{|f_j(\cdot -z)|}{1+|d_j z|^{\frac{n}{r}}} \right\|_{M_{p,q}^\beta} \leq C \left\| \mathcal{M}\left(|f_j|^r\right) \right\|_{M_{p/r,q/r}^{\beta/r}}^{1/r} \leq C \|f_j\|_{M_{p,q}^\beta}$$

If $\beta = \infty$, by (14), we have

$$\sup_{j\in\mathbb{N}_0} \sup_{z\in\mathbb{R}^n} \frac{|f_j(x - z)|}{1+|d_j z|^{\frac{n}{r}}} \leq C \sup_j \left[\mathcal{M}\left(|f_j|^r\right)(x)\right]^{1/r} \leq C \left[\mathcal{M}\left(\sup_j |f_j|^r\right)(x)\right]^{1/r}.$$

Thus,

$$\left\| \sup_{j\in\mathbb{N}_0} \sup_{z\in\mathbb{R}^n} \frac{|f_j(\cdot -z)|}{1+|d_j z|^{\frac{n}{r}}} \right\|_{M_q^p} \leq C \left\| \sup_j \left[\mathcal{M}\left(|f_j|^r\right)(\cdot)\right]^{1/r} \right\|_{M_q^p} \leq C \left\| \mathcal{M}\left(\sup_j |f_j|^r\right)(\cdot) \right\|_{M_{q/r}^{p/r}}^{1/r}.$$

Using Lemma (2.3.4), then we obtain

$$\left\| \sup_{j\in\mathbb{N}_0} \sup_{z\in\mathbb{R}^n} \frac{|f_j(\cdot -z)|}{1+|d_j z|^{\frac{n}{r}}} \right\|_{M_q^p} \leq C \left\| \sup_j |f_j| \right\|_{M_q^p}$$

Thus (ii) holds when $\beta = \infty$.

Theorem (2.3.6) [55]: Let $0 < q \leq p < \infty$, $0 < \beta \leq \infty$, let $\Omega = \{\Omega_j\}_{j=0}^\infty$ be a sequence of compact sets on $\mathbb{R}^n$ and $f_j \in L^{q,\Omega_j}$, $j \in \mathbb{N}_0$. Let $d_j > 0$ be the radius of Ω_j. If $v > n/2 + n/\min\{q,\beta\}$, then exists a constant C such that

$$\left\| \mathcal{F}^{-1} \mathcal{M}_j \mathcal{F} f_j \right\|_{M_{p,q}^\beta} \leq C \sup_j \left\| \mathcal{M}_j(d_j \cdot) \right\|_{H_2^v} \|f_j\|_{M_{p,q}^\beta}$$

and

$$\left(\sum_{j=0}^\infty \left\| \mathcal{F}^{-1} \mathcal{M}_j \mathcal{F} f_j \right\|_{M_q^p}^\beta \right)^{1/\beta} \leq C \sup_j \left\| \mathcal{M}_j(d_j \cdot) \right\|_{H_2^v} \left(\sum_{j=0}^\infty \|f_j\|_{M_q^p}^\beta \right)^{1/\beta}$$

for any sequence $\{M_j\}_{j=0}^\infty \in H_2^v(\mathbb{R}^n)$.

Theorem (2.3.7) [55]: Let $\phi = \{\phi_j(x)\}_{j=0}^\infty \in \Phi(\mathbb{R}^n)$ and $\varphi = \{\varphi_j(x)\}_{j=0}^\infty \in \Phi(\mathbb{R}^n)$.

(i) If $-\infty < s < \infty$, $0 < q \leq p < \infty$ and $0 < \beta \leq \infty$, then $\|f\|_{MF_{p,q}^{s,\beta}}^\phi$ and $\|f\|_{MF_{p,q}^{s,\beta}}^\varphi$ are equivalent quasi-norms on $MF_{p,q}^{s,\beta}$.

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(ii) If $-\infty < s < \infty$, $0 < q \le p < \infty$ and $0 < \beta \le \infty$, then $\|f\|^{\phi}_{MB^{s,\beta}_{p,q}}$ and $\|f\|^{\varphi}_{MB^{s,\beta}_{p,q}}$ are equivalent quasi-norms on $MB^{s,\beta}_{p,q}$.

Proof: We first prove (i).

If $\phi_0 = 0$, then we have $\phi_j = \phi_j \sum_{r=-1}^{1} \varphi_{j+r}$ for $j \in \mathbb{N}_0$. Therefore,

$$\mathcal{F}^{-1}\phi_j \mathcal{F}f = \sum_{j=-1}^{1} \mathcal{F}^{-1}\phi_j \mathcal{F}\mathcal{F}^{-1}\varphi_{j+r}\mathcal{F}f$$

Now we choose $0 < v < \min\{q, \beta\}$ and $u > n/2 + n/v$. If we replace f_j and M_j in Theorem (2.3.6) by $\mathcal{F}^{-1}\varphi_{j+r}\mathcal{F}f$ and ϕ_j respectively, then we obtain

$$\left\|\mathcal{F}^{-1}\phi_j \mathcal{F}\mathcal{F}^{-1}\varphi_{j+r}\mathcal{F}f\right\|_{MF^{\beta}_{p,q}} \le C \sup_k \left\|\phi_k(2^k \cdot)\right\|_{H_2^u} \left\|\mathcal{F}^{-1}\varphi_{j+r}\mathcal{F}f\right\|_{MF^{\beta}_{p,q}}$$

where C is independent of j.

By Definition (2.3.3), existing a constant C for $r = -1, 0, 1$, we have

$$\left\|\mathcal{F}^{-1}\phi_j\varphi_{j+r}\mathcal{F}f\right\|_{M^{\beta}_{p,q}} \le C\left\|\mathcal{F}^{-1}\varphi_{j+r}\mathcal{F}f\right\|_{M^{\beta}_{p,q}}.$$

Thus,

$$\left\|\mathcal{F}^{-1}\phi_j\mathcal{F}f\right\|_{M^{\beta}_{p,q}} \le C\left\|\mathcal{F}^{-1}\varphi_j\mathcal{F}f\right\|_{M^{\beta}_{p,q}}.$$

Thus, (i) is proved. The proof of (ii) is similar.

Definition (2.3.8) [55]: Let $L \in \mathbb{N}$, and $A_L(\mathbb{R}^n)$ be the collection of all functions with compact and satisfying

$$L(\phi) = \sup_{x \in \mathbb{R}^n} |x|^L \sum_{|\gamma| \le L} |D^\gamma \phi_0(x)| + \sup_{x \in \mathbb{R}^n \setminus \{0\}, j \in \mathbb{N}} (|x|^L + |x|^{-L}) \sum_{|\gamma| \le L} |D^\gamma \phi_j(2^j x)| < \infty.$$

Definition (2.3.9) [55]: Let $L \in \mathbb{N}$, $\phi = \{\phi_j\}_{j=0}^{\infty} \in A_L(\mathbb{R}^n)$, $f \in \mathcal{S}'(\mathbb{R}^n)$ and $a > 0$, then we define the maximal function

$$(\phi_j^*(x)f)(x) = \sup_{y \in \mathbb{R}^n} \frac{|(\mathcal{F}^{-1}\phi_j\mathcal{F}f)(x-y)|}{1 + |2^j y|^a}, \qquad x \in \mathbb{R}^n$$

where $j \in \mathbb{N}_0$.

By Theorem (2.3.4) and [41], we have

Proposition (2.3.10) [55]: Let $s \in \mathbb{R}$, $0 < q \le p < \infty$, $0 < \beta \le \infty$, $a > n/\min\{q, \beta\}$. If the maximal number $L > |s| + 3a + n + 2$, then exists a positive constant C such that

$$\left\|2^{js} \sup_{0<\tau<1} (\phi_j^{\tau*}f)\right\|_{M^{\beta}_{p,q}} \le C \sup_{0<\tau<1} L(\phi^\tau) \|f\|^{\phi}_{MF^{s,\beta}_{p,q}} \tag{15}$$

and

$$\left(\sum_{j=0}^{\infty} 2^{js} \left\| \sup_{0<\tau<1} (\phi_j^{\tau*} f) \right\|_{M_q^p}^{\beta}\right)^{1/\beta} \leq C \sup_{0<\tau<1} L(\phi^{\tau}) \|f\|_{MB_{p,q}^{s,\beta}}^{\phi} \tag{16}$$

for all $\phi = \{\phi_k\}_{k=0}^{\infty} \in \Phi(\mathbb{R}^n)$ and $\phi^{\tau} = \{\phi_k^{\tau}\}_{k=0}^{\infty} \in A_L(\mathbb{R}^n), 0 < \tau < 1$.

We consider Fourier multiplier.

Definition (2.3.11) [55]: Let m be in $C^{\infty}(\mathbb{R}^n)$, then M is called a Fourier multiplier on $MF_{p,q}^{s,\beta}$ if there exists a constant C such that

$$\|\mathcal{F}^{-1} m \mathcal{F} f\|_{MF_{p,q}^{s,\beta}} \leq C\|f\|_{MF_{p,q}^{s,\beta}}$$

for all $f \in MF_{p,q}^{s,\beta}$.

Similarly, we can define the Fourier multiplier on $MB_{p,q}^{s,\beta}$.

Let $N \in \mathbb{N}$, we write

$$\|m\|_N = \sup_{|\gamma|\leq N} \sup_{x\in\mathbb{R}^n} (1 + |x|^2)^{N/2} |D^{\gamma} m(x)|.$$

Proposition (2.3.12) [55]: Let $s \in \mathbb{R}, 0 < q \leq p < \infty, 0 < \beta \leq \infty$. If N is sufficiently large, then there exists a constant C such that for all $m \in C^{\infty}(\mathbb{R}^n)$, $f \in MF_{p,q}^{s,\beta}$ and $MB_{p,q}^{s,\beta}$ we have

$$\|\mathcal{F}^{-1} m \mathcal{F} f\|_{MF_{p,q}^{s,\beta}} \leq C\|m\|_N \|f\|_{MF_{p,q}^{s,\beta}} \tag{17}$$

and

$$\|\mathcal{F}^{-1} m \mathcal{F} f\|_{MB_{p,q}^{s,\beta}} \leq C\|m\|_N \|f\|_{MB_{p,q}^{s,\beta}} \tag{18}$$

Proof: For $MF_{p,q}^{s,\beta}$ space, since $\phi = \{\phi_k\}_{k=0}^{\infty} \in \Phi(\mathbb{R}^n)$, we have

$$\mathcal{F}^{-1} \phi_k \mathcal{F} [\mathcal{F}^{-1} m \mathcal{F} f] = \mathcal{F}^{-1} \phi_k \mathcal{F} f.$$

By (15), $\phi_k^{\tau} = m\phi_k$ and the following fact

$$|(\mathcal{F}^{-1} \phi^{\tau} \mathcal{F} f)(x)| \leq (\phi^{\tau*} f)(x).$$

When $N > |s| + 3n/\min\{q, \beta\} + n + 2$, then (17) holds. The proof of (18) is similar to that of (17). Thus Proposition (2.3.12) is proved.

Now, we consider the lifting properties. If $\sigma \in \mathbb{R}$, then operator I_{σ} is defined by

$$I_{\sigma} f = \mathcal{F}^{-1}(1 + |x|^2)^{\frac{\sigma}{2}} \mathcal{F} f, \qquad f \in \mathcal{S}'(\mathbb{R}^n).$$

It is well-known that I_{σ} is an one to one mapping on $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ respectively. Obviously, $I_{\sigma} I_x = I_{\sigma+x}$.

Theorem (2.3.13) [55]: Let $s, \sigma \in \mathbb{R}, m \in \mathbb{N}, 0 < \beta \leq \infty, 0 < q \leq p < \infty$. Then

(i) I_σ is an isomorphic mapping from $\mathrm{MF}_{p,q}^{s,\beta}$ to $\mathrm{MF}_{p,q}^{s-\sigma,\beta}$. Moreover,

$$\sum_{|\gamma|\leq m} \|D^\gamma f\|_{\mathrm{MF}_{p,q}^{s-m,\beta}} \quad \text{and} \quad \|f\|_{\mathrm{MF}_{p,q}^{s-m,\beta}} + \sum_{j=1}^{n}\left\|\frac{\partial^m f}{\partial x_j^m}\right\|_{\mathrm{MF}_{p,q}^{s-m,\beta}} \tag{19}$$

are equivalent quasi-norms on $\mathrm{MF}_{p,q}^{s,\beta}$.

(ii) I_σ is an isomorphic mapping from $\mathrm{MB}_{p,q}^{s,\beta}$ to $\mathrm{MB}_{p,q}^{s-\sigma,\beta}$. Moreover,

$$\sum_{|\gamma|\leq m} \|D^\gamma f\|_{\mathrm{MB}_{p,q}^{s-m,\beta}} \quad \text{and} \quad \|f\|_{\mathrm{MB}_{p,q}^{s-m,\beta}} + \sum_{j=1}^{n}\left\|\frac{\partial^m f}{\partial x_j^m}\right\|_{\mathrm{MB}_{p,q}^{s-m,\beta}} \tag{20}$$

are equivalent quasi-norms on $\mathrm{MB}_{p,q}^{s,\beta}$.

Proof: We only prove (i), the proof of (ii) is similar. First, if $\phi \in \{\phi_k\}_{k=0}^\infty \in \Phi(\mathbb{R}^n)$, then $\phi = \{\phi_k\}_{k=0}^\infty \in A_L(\mathbb{R}^n)$, where L may be arbitrary large maximal number and $\phi_k(x) = 2^{-k\sigma}(1+|x|^2)^{\frac{n}{2}}\phi_k(x)$.

If $f \in \mathrm{MF}_{p,q}^{s,\beta}$, by Proposition (2.3.10) and the estimate $|(\mathcal{F}^{-1}\phi_k\mathcal{F}f)(x)| \leq \phi_k^*(x)$, we have

$$\|I_\sigma f\|_{\mathrm{MF}_{p,q}^{s-\sigma,\beta}} = \left\|2^{-(s-\sigma)k}\mathcal{F}^{-1}(1+|\cdot|)^{\sigma/2}\phi_k\mathcal{F}f\right\|_{\mathrm{MF}_{p,q}^{\beta}} = \left\|2^{ks}\mathcal{F}^{-1}\phi_k\mathcal{F}f\right\|_{\mathrm{MF}_{p,q}^{\beta}}$$
$$\leq C\|f\|_{\mathrm{MF}_{p,q}^{s,\beta}} \tag{21}$$

where C is independent of f. Thus, I_σ maps continuously $\mathrm{MF}_{p,q}^{s,\beta}$ into $\mathrm{MF}_{p,q}^{s-\sigma,\beta}$.

If $g \in \mathrm{MF}_{p,q}^{s,\beta}$ and

$$\|f\|_{\mathrm{MF}_{p,q}^{s,\beta}} \leq C\|g\|_{\mathrm{MF}_{p,q}^{s-m,\beta}} = C\|I_\sigma f\|_{\mathrm{MF}_{p,q}^{s-\sigma,\beta}} \tag{22}$$

Since I_σ is an one to one mapping on $\mathcal{S}'(\mathbb{R}^n)$. So it is also an one to one mapping from $\mathrm{MF}_{p,q}^{s,\beta}$ into $\mathrm{MF}_{p,q}^{s-\sigma,\beta}$. By (21) and (22), $\|I_\sigma f\|_{\mathrm{MF}_{p,q}^{s-\sigma,\beta}}$ is an equivalent quasi-norm on $\mathrm{MF}_{p,q}^{s,\beta}$.

Next, we prove that the quasi-norm in (19) is an equivalent quasi-norm on $\mathrm{MF}_{p,q}^{s,\beta}$. If $x = (x_1,\cdots,x_n) \in \mathbb{R}^n$, let $x^\sigma = \prod_{j=1}^n x_j^{\gamma_j}$, where $\gamma = (\gamma_1,\cdots\gamma_n)$. By Proposition (2.3.12), we have that $x^\gamma(1+|x|^2)^{-m/2}$ is a Fourier-multiplier on $\mathrm{MF}_{p,q}^{s,\beta}$. If $|\gamma| \leq m$, $m \in \mathrm{MF}_{p,q}^{s,\beta}(\mathbb{R}^n)$, then we have

$$\sum_{|\gamma|\leq m}\|D^\gamma f\|_{\mathrm{MF}_{p,q}^{s,\beta}} = \sum_{|\gamma|\leq m}\|\mathcal{F}^{-1}x^\gamma\mathcal{F}f\|_{\mathrm{MF}_{p,q}^{s-m,\beta}}$$
$$= \sum_{|\gamma|\leq m}\left\|\mathcal{F}^{-1}x^\gamma(1+|x|^2)^{-m/2}\mathcal{F}\mathcal{F}^{-1}(1+|x|^2)^{m/2}\mathcal{F}f\right\|_{\mathrm{MF}_{p,q}^{s-m,\beta}} \leq C\|I_m f\|_{\mathrm{MF}_{p,q}^{s-m,\beta}}$$
$$\leq C\|f\|_{\mathrm{MF}_{p,q}^{s,\beta}}, \tag{23}$$

where the last inequality is obtained by (21).

We assume that $m \in MF_{p,q}^{s-m,\beta}$ and $\frac{\partial^m}{\partial x_j^m} f \in MF_{p,q}^{s-m,\beta}$ for $j = 1, \cdots n$, we hope to prove that $f \in MF_{p,q}^{s,\beta}$. We claim that there exist Fourier multipliers $\rho_1(x), \cdots, \rho_n(x)$ on $MF_{p,q}^{s-m,\beta}$, and a positive constant C such that $1 + \sum_{j=1}^{n} \rho_j(x) x_j^m \geq C(1 + |x|^2)^{m/2}$, for all $x \in \mathbb{R}^n$; see [41]. By Proposition (2.3.12), we know that

$$M(x) = (1 + |x|^2)^{m/2} \left[1 + \sum_{j=1}^{n} \rho_j(x) x_j^m \right]^{-1}$$

is also a Fourier-multiplier on $MF_{p,q}^{s-m,\beta}$. Then

$$\|f\|_{MF_{p,q}^{s-m,\beta}} \leq C \left\| \mathcal{F}^{-1}(1 + |x|)^{m/2} \mathcal{F}f \right\|_{MF_{p,q}^{s-m,\beta}}$$

$$\leq C \left\| \mathcal{F}^{-1} M(x) \mathcal{F} \mathcal{F}^{-1} \left[1 + \sum_{j=1}^{n} \rho_j(x) x_j^m \right]^{m/2} \mathcal{F}f \right\|_{MF_{p,q}^{s-m,\beta}}$$

$$\leq C \|f\|_{MF_{p,q}^{s-m,\beta}} + C \sum_{j=1}^{n} \left\| \mathcal{F}^{-1} \rho_j(x) x_j^m \mathcal{F}f \right\|_{MF_{p,q}^{s-m,\beta}}.$$

However, $x_j^m \mathcal{F}f = \mathcal{F} \frac{\partial^m f}{\partial x_j^m}$. By Fourier multiplier properties of $\rho_j(x)$, we obtain

$$\|f\|_{MF_{p,q}^{s,\beta}} \leq C \|f\|_{MF_{p,q}^{s-m,\beta}} + C \sum_{j=1}^{n} \left\| \frac{\partial^m f}{\partial x_j^m} \right\|_{MF_{p,q}^{s-m,\beta}}. \tag{24}$$

By (23) and (24), we prove that the quasi-norms in (21) are equivalent quasi-norms on $MF_{p,q}^{s,\beta}$. This proved Theorem (2.3.13).

We give the discrete characterization of Morrey type Besov-Triebel spaces and we generalize the discrete characterization of standard Besov-Triebel spaces [76]. Next, we will use the idea of [76].

With $MF_{p,q}^{s,\beta}$ we associate the space $Mf_{p,q}^{s,\beta}$ of complex sequences $\alpha = \langle \alpha_k^j \rangle_{k \in \mathbb{Z}^n}^{j \in \mathbb{N}_0}$ for which the quasi-norm

$$\|\alpha\|_{Mf_{p,q}^{s,\beta}} := \left\| \left\| \langle 2^{js} \sum_{k \in \mathbb{Z}^n} |\alpha_k^j| \chi_k^j \rangle_{j \in \mathbb{N}_0} \right\|_{l_\beta} \right\|_{M_q^p}$$

is finite. Here χ_k^j is the characteristic of the parallelepiped

$$\Delta_k^j := 2^{-j}[k + [-1/2, 1/2]^n], \qquad j \in \mathbb{N}_0, \qquad k \in \mathbb{Z}^n \cdots$$

The set Δ_k^j from a disjoint decomposition of $\mathbb{R}^n$ for fixed j.

To give a discrete characterization of $MB_{p,q}^{s,\beta}$ we need the sequence space $Mb_{p,q}^{s,\beta}$ of the complex sequences $\alpha = \langle \alpha_k^j \rangle_{k\in\mathbb{Z}^n}^{j\in\mathbb{N}_0}$ for which the quasi-norm

$$\|\alpha\|_{Mb_{p,q}^{s,\beta}} := \left\| \left\| \left\| \langle 2^{js} \sum_{k\in\mathbb{Z}^n} |\alpha_k^j| |\chi_k^j| \rangle_{j\in\mathbb{N}_0} \right\|_{M_q^p} \right\| \right\|_{l_\beta}$$

is finite.

For ϕ_0, ϕ_1 defined as in Definition (2.3.2), we write

$$\psi_0(x) := \frac{\phi_0(x)}{\phi_0^2(x) + \phi_1(x)}, \qquad |x| \le 2,$$

$$\psi_1(x) := \frac{\phi_1(2x)}{\phi_1^2(x) + \phi_1^2(2x) + \phi_1^2(4x)}, \qquad 1/2 \le |x| \le 2.$$

which are extended by 0 to $\mathbb{R}^n$. These are C^∞-functions with properties

$$\text{supp } \psi_0 \subset \{\xi \in \mathbb{R}^n : |\xi| \le 2\},$$

$$\text{supp } \psi_1 \subset \{\xi \in \mathbb{R}^n : 1/2 \le |\xi| \le 2\}.$$

Lemma (2.3.14) [55]: Let $-\infty < s < \infty$, $0 < q \le p < \infty$, $0 < \beta \le \infty$. If $d > n/\min\{q,\beta\}$, then there exists a positive constant C such that

$$\left\| \left\| \left\| 2^{js} \sup_{y\in\mathbb{R}^n} \frac{\left(\mathcal{F}^{-1}[\phi_j \mathcal{F}f](\cdot - y)\right)}{[1 + 2^j|y|]^d} \right\|_{l_\beta} \right\|_{M_q^p} \right\| \le C\|f\|_{MF_{p,q}^{s,\beta}} \tag{25}$$

$$\left\| \left\| \left\| 2^{js} \sup_{y\in\mathbb{R}^n} \frac{\left(\mathcal{F}^{-1}[\phi_j \mathcal{F}f](\cdot - y)\right)}{[1 + 2^j|y|]^d} \right\|_{M_q^p} \right\|_{l_\beta} \right\| \le C\|f\|_{MB_{p,q}^{s,\beta}} \tag{26}$$

hold, where ϕ_j defined as in Definition (2.3.2).

Lemma (2.3.15) [55]: For $\alpha = \langle \alpha_k^j \rangle_{k\in\mathbb{Z}^n}^{j\in\mathbb{N}_0}$, let

$$f_{j,\sigma}(x) := \sum_{r=-1}^{1} \sum_{k\in\mathbb{Z}^n} |\sigma_k^{j+r}| \left[1 + \|2^{j+r}x - k\|_{l_\infty}\right]^{-L}, \qquad x \in \mathbb{R}^n$$

when $L > n/\min\{1,q,\beta\}$, then

$$\left\| \left\| 2^{js} f_{j,\sigma} \right\|_{l_\beta} \right\|_{M_q^p} \le C\|\sigma\|_{MF_{p,q}^{s,\beta}}$$

and

$$\left\| \left\| 2^{js} f_{j,\sigma} \right\|_{M_q^p} \right\|_{l_\beta} \leq C \|\sigma\|_{MB_{p,q}^{s,\beta}}$$

for $-\infty < s < \infty$, $0 < q \leq p < \infty$, $0 < \beta < \infty$, where the constant C is independent of σ.

Proof: Let $x \in \Delta_{k_0}^j$ and write

$$K_l := \left\{ k \in \mathbb{Z}^n : 2^l \leq \|k - k_0\|_{l_\infty} < 2^{l+1} \right\}, \qquad l \in \mathbb{N}_0.$$

If $k \in K_l$, then $\left\| 2^j - k \right\|_{l_\infty} \geq 2^{l-1}$. By $l_1 \hookrightarrow l_t$, $0 < t < 1$, we obtain

$$g_{j\sigma}(x) := \sum_{k \in \mathbb{Z}^n} |\sigma_k^j| \left[1 + \left\| 2^j x - k \right\|_{l_\infty} \right]^{-L} \leq |\sigma_k^j| + \sum_{l=0}^{\infty} \sum_{k \in K_l} |\sigma_k^j| 2^{-lL}$$

$$\leq \left(|\sigma_k^j|^t \right)^{1/t} + \sum_{l=0}^{\infty} \left(\sum_{k \in K_l} |\sigma_k^j| t \right)^{1/t} 2^{-lL}$$

$$= \left(2^{jn} \int_{\mathbb{R}^n} \left[|\sigma_k^j| \chi_{k_0}^j(y) \right]^t dy \right)^{1/t} + \sum_{l=0}^{\infty} 2^{-lL} \left(2^{jn} \int_{\mathbb{R}^n} \left[\sum_{k \in K_l} |\sigma_k^j| \chi_k^j(y) \right]^t dy \right)^{1/t}. \quad (27)$$

Since the Δ_k^j's are disjoint to each other and the measure is 2^{-jn}. If $y \in \Delta_k^j$, $k \in K_l$, we have

$$|x - y| \leq |x - 2^{-j} k_0| + |2^{-j} k_0 - 2^{-j} k| + |2^{-j} k - y| \leq C2^{-j} + C2^{-j} 2^l + C2^{-j}$$
$$\leq C2^{-j+l} \quad (28)$$

Therefore,

$$\int_{\mathbb{R}^n} \left[\sum_{k \in K_L} |\sigma_k^j| \chi_k^j \right]^t dy \leq C2^{(-j+l)n} \mathcal{M} \left[\sum_{k \in \mathbb{Z}^n} |\sigma_k^j| \chi_k^j \right]^t (x).$$

Since $L > n/\min\{1, q, \beta\}$, we take t such that $0 < t < \min\{1, q, \beta\}$ and $L > n/t$. Thus,

$$g_{j\sigma}(x) \leq C \left(\mathcal{M} \left[\sum_{k \in K_L} |\sigma_k^j| \chi_k^j(x) \right]^t (x) \right)^{1/t}.$$

By the maximal inequality in Lemma (2.3.4), we obtain

$$\left\| \left\| 2^{js} f_{j,\sigma} \right\|_{l_\beta} \right\|_{M_q^p} \leq C \left\| \left\| 2^{js} g_{j,\sigma} \right\|_{l_\beta} \right\|_{M_q^p} \leq C \left\| \left\| \mathcal{M} \left[\sum_{k \in \mathbb{Z}^n} |\sigma_k^j| \chi_k^j \right]^t \right\|_{l_{\beta/t}} \right\|_{M_{q/t}^{p/t}}^{1/t}$$

$$\leq C \left\| \left\| \left[\sum_{k \in \mathbb{Z}^n} |\sigma_k^j| \chi_k^j \right]^t \right\|_{l_{\beta/t}} \right\|_{M_{q/t}^{p/t}}^{1/t} = C \|\sigma\|_{Mf_{p,q}^{s,\beta}}. \quad (29)$$

For $Mb_{p,q}^{s,\beta}$ the proof is simpler.

Theorem (2.3.16) [55]: Let $s \in \mathbb{R}^n$, $0 < \beta < \infty$, $0 < q \le p < \infty$. The operators

$$se : MF_{p,q}^{s,\beta} \to Mf_{p,q}^{s,\beta}$$

and

$$fu : Mf_{p,q}^{s,\beta} \to MF_{p,q}^{s,\beta}$$

(these are abbreviations for sequence and function, respectively), defined by

$$se\, f =: \langle (2\pi)^{-\frac{n}{2}} \mathcal{F}^{-1}[\phi_j \mathcal{F} f](2^{-j}k) \rangle_{k \in \mathbb{Z}^n}^{j \in \mathbb{N}_0} \tag{30}$$

$$fu\, \sigma =: \sum_{k \in \mathbb{Z}^n} \sigma_k^0 \big(\mathcal{F}^{-1}\psi_0(\cdot - k) \big) + \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}^n} \sigma_k^j \left(\mathcal{F}^{-1}\psi_1(2^j \cdot - k) \right) \tag{31}$$

are bounded. Furthermore, $fu \cdot se = id$ and $\| se \cdot \|_{Mf_{p,q}^{s,\beta}}$ is an equivalent quasi-norm on $Mf_{p,q}^{s,\beta}$. the same result holds on Besov spaces when replacing $MF_{p,q}^{s,\beta}$ by $MB_{p,q}^{s,\beta}$ and $Mf_{p,q}^{s,\beta}$ by $Mb_{p,q}^{s,\beta}$.

Proof: We only consider $MF_{p,q}^{s,\beta}$. For $MB_{p,q}^{s,\beta}$ the proof is similar.

First, we prove that se is bounded. Let $x \in \Delta_k^j$, then $|x - 2^{-j}k_0| \le C2^{-j}$ and we have

$$2^{js} \sum_{k \in \mathbb{Z}^n} (sef)_k^j \chi_k^j(x) \le (2\pi)^{-n/2} \sup_{|\gamma| \le C2^{-j}} 2^{js} |\mathcal{F}^{-1}[\phi_j \mathcal{F} f](x - z)|$$

$$\le C \sup_{z \in \mathbb{R}^n} \frac{2^{js} |\mathcal{F}^{-1}[\phi_j \mathcal{F} f](x - z)|}{[1 + 2^j |z|]^N}. \tag{32}$$

By Lemma (2.3.14),

$$\left\| \left\| 2^{js} \sum_{k \in \mathbb{Z}^n} (sef)_k^j \chi_k^j(\cdot) \right\|_{l_\beta} \right\|_{M_q^p} \le \left\| \left\| \sup_{z \in \mathbb{R}^n} \frac{2^{js} |\mathcal{F}^{-1}[\phi_j \mathcal{F} f](\cdot - z)|}{[1 + 2^j |z|]^N} \right\|_{l_\beta} \right\|_{M_q^p} \le \| f \|_{MF_{p,q}^{s,\beta}} \tag{33}$$

Thus, se is bounded.

Second, let

$$\| fu\sigma \|_{Mf_{p,q}^{s,\beta}} = \left\| \left\| 2^{js} I_j \right\|_{l_\beta} \right\|_{M_q^p}, \qquad I_j(x) := |\mathcal{F}^{-1}[\phi_j \mathcal{F} fu\sigma](x)|$$

We have the fact that $I_j(x) \le C f_{j\sigma}(x)$; see [76]. By the fact above and Lemma (2.3.15), we know that fu is bounded.

Third, the proof of the rest is similar to that in [76]. Theorem (2.3.16) is proved.

Corollary (2.3.17) [55]: Assume $0 < q \le p < \infty$, $0 < \beta < \infty$, and $s_0, s_1 \in \mathbb{R}$. Then a linear operator $T : MF_{p,q}^{s_0,\beta} \to MF_{p,q}^{s_1,\beta}$ is bounded, iff the operator seTfu $: Mf_{p,q}^{s_0,\beta} \to Mf_{p,q}^{s_1,\beta}$ is bounded. The respective quasi-norms of the operators are equivalent.

The operators sefu are given by matrices. Assume that $T : \mathcal{S} \to \mathcal{S}'$ is bounded. Then

$$(seTfu\sigma)_m^l = \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^n} \beta_{k,m}^{j,l} \alpha_k^j$$

where

$$\beta_{k,m}^{j,l} \le \begin{cases} \left(se(T[(\mathcal{F}\Phi_0)(- \cdot k)])\right)_M^l, & j = 0. \\ \left(se(T[(\mathcal{F}\Phi_J)(- \cdot k)])\right)_M^l, & j \ge 1. \end{cases}$$

The matrix associated with the pseudo-deferential operator A corresponding to the symbol a is defined by $\beta(a)$. The boundedness of this matrix between sequence spaces depends on the relative size of its coefficient.

The Hörmander class $S_{1,\delta}^v$ with $v \in \mathbb{R}$ and $0 \le \delta < 1$ consists of all functions $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ which satisfy

$$\left| D_\xi^\alpha D_x^\beta a(x, \xi) \right| \le C_{\alpha,\beta} (1 + |\xi|)^{v - |\alpha| + \delta|\beta|}, \qquad x, \xi \in \mathbb{R}^n$$

for all multi-indices α and β with $a \in S_{1,\beta}^v$.

We associate the pseudo-deferential operator T defined by

$$Tf(x) := \mathcal{F}^{-1}[a(x,\cdot)\mathcal{F}f](x), \qquad f \in \mathcal{S}, \qquad x \in \mathbb{R}^n.$$

The function a is called the symbol of T and the class of operators arising in this way from $S_{1,\delta}^v$ is denoted by $\psi_{1,\delta}^v$. If $\delta = 1$ one calls $\psi_{1,1}^0$ the exotic class.

Lemma (2.3.18) [55]: Assume $0 \le \delta < 1$, $v \in \mathbb{R}$ and $a \in S_{1,\delta}^v$. Then for $N \in \mathbb{N}$ there exists a constant $C > 0$ such that

$$\left| \beta(a)_{k,m}^{j,l} \right| := C \begin{cases} 2^{jv} 2^{(j-l)N} \left(1 + |2^{j-l}m - k|\right)^{-(N+n)}, & j < l. \\ 2^{jv} 2^{(l-j)N} \left(1 + |2^{l-j}k - m|\right)^{-(N+n)}, & j \ge l. \end{cases}$$

holds for all $j, l \in \mathbb{N}_0$ and $k, m \in \mathbb{Z}^n$.

Lemma (2.3.19) [55]: Assume $s, v \in \mathbb{R}$, $0 < \beta < \infty$, $0 < q \le p < \infty$ and $L > n/\min\{1, q, \beta\}$.

(i) If $k > s$, then

$$\left\| \langle \sum_{j=0}^{l-1} \sum_{k \in \mathbb{Z}^n} |\sigma_k^j| 2^{jv} 2^{(j-l)k} \left(1 + |2^{j-l}m - k|\right)^{-L} \rangle_{m \in \mathbb{Z}^n}^{l \in \mathbb{N}_0} \right\|_{Mf_{p,q}^{s,\beta}} \le C\|\sigma\|_{Mf_{p,q}^{s+v,\beta}} \qquad (34)$$

holds, where C is independent of σ.

(ii) If $\lambda > n/\min\{1, q, \beta\} - s$, then

$$\left\|\left\langle \sum_{j=l}^{\infty} \sum_{k\in\mathbb{Z}^n} |\sigma_k^j| 2^{jv} 2^{(l-j)\lambda}\left(1+|2^{l-j}k-m|\right)^{-L}\right\rangle_{m\in\mathbb{Z}^n}^{l\in\mathbb{N}_0}\right\|_{Mf_{p,q}^{s,\beta}} \leq C\|\sigma\|_{Mf_{p,q}^{s+v,\beta}} \tag{35}$$

holds, where C is independent of σ.

Proof: Step 1. We prove (34). Let $x \in \Delta_m^l$, put $K_r := \{k \in \mathbb{Z}^n : 2^r - 1 \leq |2^{j-l}m - k| < 2^{r+1} - 1\}$, $r \in \mathbb{N}_0$ with $0 < t < 1$. Then it follows

$$r_m^l := \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^n} |\sigma_k^j| 2^{j-l} 2^{(j-l)k}\left(1+|2^{j-l}m-k|\right)^{-L} \leq \sum_{j=0}^{l-1} 2^{jv} 2^{(j-k)k} \sum_{r=0}^{\infty} 2^{-Lr} \left(\sum_{k\in K_r} |\sigma_k^j|^t\right)^{1/t}.$$

If $y \in \Delta_k^j$, then $|x - y| \leq C2^{-j+r}$. Since $j < l$, thus

$$\sum_{k\in K_r} |\sigma_k^j|^t = 2^{jn} \int_{\mathbb{R}^n} \left[\sum_{k\in K_r} |\sigma_k^j| \chi_k^j(y)\right]^t dy \leq C2^{rn} \mathcal{M}\left[\sum_{k\in K_r} |\sigma_k^j| \chi_k^j(y)\right]^t(x).$$

If we write $\mathcal{M}_j(x) = \mathcal{M}\left[\sum_{k\in K_r} |\sigma_k^j| \chi_k^j(y)\right]^t(x)$, then

$$2^{js} \sum_{m\in\mathbb{Z}^n} r_m^l \chi_m^l(x) \leq C \sum_{j=0}^{l-1} 2^{(l-1)(k-s)} 2^{j(s+v)} \mathcal{M}_j^{1/t}(x).$$

Since $L > n/\min\{1, q, \beta\}$, so we can take t such that $0 < t < \min\{1, q, \beta\}$ and $L > n/t$. thus

$$\text{Left side of (34)} \leq \left\|\left\|\left\langle 2^{ls} \sum_{k\in\mathbb{Z}^n} r_m^l \chi_m^l(x)\right\rangle_{l\in\mathbb{N}_0}\right\|_{l_\beta}\right\|_{M_q^p}$$

$$\leq C \left\|\left\|\left\langle \left(\sum_{j=0}^{l-1} 2^{(j-l)(k-s)t} 2^{j(s+v)t} \mathcal{M}_j\right)\right\rangle_{l\in\mathbb{N}_0}\right\|_{l_{\beta/t}}\right\|_{M_{q/t}^{p/t}}$$

$$\leq C \left\|\left\|\left\langle \left(2^{j(s+v)t} \mathcal{M}_j\right)\right\rangle_{l\in\mathbb{N}_0}\right\|_{l_{\beta/t}}\right\|_{M_{q/t}^{p/t}} \leq C\|\sigma\|_{Mf_{p,q}^{s+v,\beta}}. \tag{36}$$

In the third inequality in (36), we use the following fact

$$\left\|\left\langle \sum_{j=0}^{l-1} 2^{(j-l)\epsilon} a_j\right\rangle_{l\in\mathbb{N}_0}\right\|_{l_\beta} \leq C\|\langle a_j\rangle_{j\in\mathbb{N}_0}\|_{l_\beta}, \qquad 1 \leq \beta < \infty, \qquad \epsilon > 0$$

where a_j is a complex-sequence. In the last inequality in (36), we use the maximal inequality in Lemma (2.3.4).

Step 2. To prove (35), we decompose $\mathbb{R}^n$ into subsets

$$\bar{K}_r := \{k \in \mathbb{Z}^n : 2^r - 1 \leq |m - 2^{l-j}k| < 2^{r+1} - 2\}, \qquad r \in \mathbb{N}_0.$$

The rest is similar to step 1 and by the following fact

$$\left\|\langle\sum_{j=l}^{\infty}2^{(l-j)\epsilon}a_j\rangle_{l\in\mathbb{N}_0}\right\|_{l_\beta}\leq C\|\langle a_j\rangle_{j\in\mathbb{N}_0}\|_{l_\beta},\qquad 1\leq\beta<\infty,\qquad\epsilon>0,$$

where a_j is a complex-sequence.

Theorem (2.3.20) [55]: Assume $-\infty<s<\infty,\ 0<q\leq p<\infty,\ 0<\beta<\infty$. Let $T\in\psi_{1,\delta}^v$ with $v\in\mathbb{R}$.

(i) $T:=MF_{p,q}^{s+v,\beta}\to MF_{p,q}^{s,\beta}$ is bounded, if $0\leq\delta<1$.
$\quad T:=MB_{p,q}^{s+v,\beta}\to MB_{p,q}^{s,\beta}$ is bounded, if $0\leq\delta<1$.
(ii)$T:=MF_{p,q}^{s+v,\beta}\to MF_{p,q}^{s,\beta}$ is bounded, if $s<n(1/\min\{1,q,\beta\}-1),\ \delta=1$.
$\quad T:=MB_{p,q}^{s+v,\beta}\to MB_{p,q}^{s,\beta}$ is bounded, if $s<n(1/\min\{1,q,\beta\}-1),\ \delta=1$.

Proof : By Lemma (2.3.18), we have

$$\|seTfu\sigma\|_{Mf_{p,q}^{s,\beta}}\leq\left\|\langle\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n}|\beta_{k,m}^{j,l}|\sigma_k^j\rangle_{k\in\mathbb{Z}^n}^{l\in\mathbb{N}_0}\right\|_{Mf_{p,q}^{s,\beta}}$$

$$\leq C\left\|\langle\sum_{j=0}^{l-1}\sum_{k\in\mathbb{Z}^n}|\sigma_k^j|2^{jv}2^{(l-j)N}(1+|2^{j-l}m-k|)^{-(N+n)}\rangle_{k\in\mathbb{Z}^n}^{l\in\mathbb{N}_0}\right\|_{Mf_{p,q}^{s,\beta}}$$

$$+C\left\|\langle\sum_{j=l}^{\infty}\sum_{k\in\mathbb{Z}^n}|\sigma_k^j|2^{jv}2^{(j-l)N}(1+|2^{j-l}k-m|)^{-(N+n)}\rangle_{k\in\mathbb{Z}^n}^{l\in\mathbb{N}_0}\right\|_{Mf_{p,q}^{s,\beta}}.$$

By Lemma (2.3.19), we obtain

$$\|seTfu\sigma\|_{Mf_{p,q}^{s,\beta}}\leq C\|\sigma\|_{Mf_{p,q}^{s+v,\beta}}.$$

If $s>I_{q,\beta}:=n(1/\min\{1,q,\beta\}-1)$ and N is sufficient large, by Corollary (2.3.17), the operator $T:MF_{p,q}^{s+v,\beta}\to MF_{p,q}^{s,\beta}$ is bounded.

When $s\leq I_{q,\beta}$, by lifting properties, we have that

$$I_\tau:=\mathcal{F}^{-1}[(1+|\cdot|^2)^{\tau/2}\mathcal{F}]\in\psi_{1,\delta}^\tau:MF_{p,q}^{s+v,\beta}\to MF_{p,q}^{s,\beta}$$

is bounded for $s,\tau\in\mathbb{R}$. By the fact that $\psi_{1,\delta}^{v_1}\cdot\psi_{1,\delta}^{v_2}\subset\psi_{1,\delta}^{v_1+v_2},0\leq\delta<1$, then the operator

$$T^\tau:=I_\tau TI_{-\tau}\in\psi_{1,\delta}^v:MF_{p,q}^{I_{q,\beta}+1+v,\beta}\to MF_{p,q}^{I_{q,\beta}+1,\beta}$$

is bounded.

Therefore,

$$T=I_{-\tau}T^\tau I_\tau\in\psi_{1,\delta}^v:MF_{p,q}^{I_{q,\beta}+1+v+\tau,\beta}\to MF_{p,q}^{I_{q,\beta}+1+\tau,\beta}$$

is bounded for $\tau \in \mathbb{R}$.

From $\mathrm{MB}_{\mathrm{p,q}}^{\mathrm{s,\beta}}$ is similar, we omit the details here.

Thus, Theorem (2.3.20) is proved.

Chapter 3
Variable Smoothness and Integrability

We give molecular and atomic decomposition results and show that the space is well-defined, i.e., independent of the choice of basic functions. As in the classical case, a unified scale of spaces permits clearer results in cases where smoothness and integrability interact, such as Sobolev embedding and trace theorems. As an application of the decomposition we show optimal trace theorems in the variable indices case. We give Sobolev-type embeddings, and show that the scale contains variable order Hölder-Zygmund spaces as special cases. We provide an alternative characterization of the Besov space using approximations by analytic functions.

Section (3.1): Function Spaces

From a vast array of different function spaces a well ordered superstructure appeared in the 1960's and 70's based on two three-index spaces: the Besov space $B_{p,q}^{\alpha}$ and the Triebel–Lizorkin space $F_{p,q}^{\alpha}$. There has been a growing interest in generalizing classical spaces such as Lebesgue and Sobolev spaces to the case with either variable integrability (e.g., $W^{1,p(\cdot)}$) or variable smoothness (e.g., $W^{m(\cdot),2}$). These generalized spaces are obviously not covered by the superstructures with fixed indices.

It is well-known from the classical case that smoothness and integrability often interact, for instance, in trace and embedding theorems. There has so far been no attempt to treat spaces with variable integrability and smoothness in one scale. We address this issue by introducing Triebel–Lizorkin spaces with variable indices, denoted $F_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$.

Spaces of variable integrability can be traced back to 1931 and W. Orlicz [124], but the modern development started with [113] of Kováčik and Rákosník in 1991. Apart from interesting theoretical considerations, the motivation to study such function spaces comes from applications to fluid dynamics, image processing, PDE and the calculus of variation.

The first concrete application arose from a model of electrorheological fluids in [128] (cf. [85, 86, 130, 131] for mathematical treatments of the model). We mention that an electrorheological fluid is a so-called smart material in which the viscosity depends on the external electric field. This dependence is expressed through the variable exponent p; specifically, the motion of the fluid is described by a Navier–Stokes-type equation where the Laplacian 4u is replaced by the $p(x)$-Laplacian $\mathrm{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)$. By standard arguments, this means that the natural energy space of the problem is $W^{1,p(\cdot)}$, the Sobolev space of variable integrability.

An application to image restoration was proposed by Chen, Levine & Rao [94, 123]. Their model combines isotropic and total variation smoothing. And, their model requires the minimization over u of the energy

$$\int_{\Omega} |\nabla u(x)|^{p(x)} + \lambda |u(x) - I(x)|^2 \, dx$$

where I is given input. Recall that in the constant exponent case, the power $p \equiv 2$ corresponds to isotropic smoothing, whereas $p \equiv 1$ gives total variation smoothing. Hence the exponent varies between these two extremes in the variable

exponent model. This variational problem has an Euler-Lagrange equation, and the solution can be found by solving a corresponding evolutionary PDE.

Partial differential equations have also been studied from a more abstract and general point of view in the variable exponent setting. We can approach boundary value problems through a suitable trace space, which, by definition, is a space consisting of restrictions of functions to the boundary. For the Sobolev space $W^{1,p(\cdot)}$, the trace space was first characterized by an intrinsic norm, see [100]. In analogy with the classical case, this trace space can be formally denoted $F_{P(\cdot),P(\cdot)}^{1-1/p(\cdot)}$, so it is an example of a space with variable smoothness and integrability, albeit on with a very special relationship between the two exponents. Already somewhat earlier Almeida & Samko [88] and Gurka, Harjulehto & Nekvinda [109] had extended variable integrability Sobolev spaces to Bessel potential spaces $W^{1,p(\cdot)}$ for constant but non-integer α^1.

Along a different line of study, Leopold [117, 118, 119, 120] and Leopold & Schrohe [121] studied pseudo-differential operators with symbols of the type $\langle \xi^{m(x)} \rangle$, and defined related function spaces of Besov-type with variable smoothness, formally $B_{p,q}^{m(\cdot)}$. In the case $p = 2$, this corresponds to the Sobolev space $H^{m(\cdot)} = W^{m(x),2}$. Function spaces of variable smoothness have been studied by Besov [89, 90, 91, 91]. He generalized Leopold's work by considering both Triebel-Lizorkin spaces $F_{p,q}^{\alpha(\cdot)}$ and Besov spaces $B_{p,q}^{\alpha(\cdot)}$ in $\mathbb{R}^n$. Schneider and Schwab [135] used $H^{m(\cdot)}(\mathbb{R})$ in the analysis of certain Black-Scholes equations. The variable smoothness corresponds to the volatility of the market, which surely should change with time.

We define and study a generalized scale of Triebel-Lizorkin type spaces with variable smoothness $\alpha(x)$, and variable primary and secondary indices of integrability, $p(x)$ and $q(x)$. By setting some of the indices to appropriate values we recover all previously mentioned spaces as special cases, except the Besov spaces (which, like in the classical case, form a separate scale).

Apart from the value added through unification, our new space allows treating traces and embeddings in a uniform and comprehensive manner, rather than doing them case by case.

When generalizing Triebel-Lizorkin spaces, we have several obstacles to overcome. The main difficulty is the absence of the vector-valued maximal function inequalities. It turns out that the inequalities are not only missing, rather, they do not even hold in the variable indices case. As a consequence of this, the Hörmander-Mikhlin multiplier theorem does not apply in the case of variable indices. The solution is to work in closer connection with the actual structure of the space with what we call η-functions and to derive suitable estimates directly for these functions.

We state the main results: atomic and molecular decomposition of Triebel-Lizorkin spaces, a trace theorem, and a multiplier theorem. We show that the new scale is indeed a unification of previous spaces, in that it includes them all as special cases with appropriate choices of the indices. We formulate and prove an appropriate version of the multiplier theorem. We give the proofs of the main decompositions theorems, and we discuss the trace theorem.

For $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B^n(x, r)$ the open ball in $\mathbb{R}^n$ with center x and radius r. By B^n we denote the unit ball $B^n(0,1)$. We use c as a generic constant, i.e., a constant whose values may change from appearance to appearance. The inequality $f \approx g$ means that $\frac{1}{c}g \leq f \leq c\,g$ for some suitably independent constant c. By χ_A we denote the characteristic function of the set A. If $a \in \mathbb{R}^n$, then we use the notation a_+ for the positive part of a, i.e., $a_+ = \max\{0, a\}$. By $\mathbb{N}$ and $\mathbb{N}_0$ we denote the sets of positive and non-negative integers. For $x \in \mathbb{R}$ we denote by $\lfloor x \rfloor$ the largest integer less than or equal to x.

We denote the mean-value of the integrable function f, defined on a set A of finite, non-zero measure, by

$$\fint_A f(x)\mathrm{d}x = \frac{1}{|A|}\int_A f(x)\mathrm{d}x.$$

The Hardy-Littlewood maximal operator M is defined on $L^1_{\mathrm{loc}}(\mathbb{R}^n)$

$$Mf(x) = \sup_{r>0} \fint_{B^n(x,r)} |f(y)|\mathrm{d}y$$

By $\mathrm{supp}\, f$ we denote the support of the function f, i.e., the closure of its zero set.

By $\Omega \subset \mathbb{R}^n$ we always denote an open set. By a variable exponent we mean a measurable bounded function $p : \Omega \to (0, \infty)$ which is bounded away from zero. For $A \subset \Omega$ we denote $P_A^+ = \mathrm{ess\,sup}_A p(x)$ and $P_A^- = \mathrm{ess\,inf}_A p(x)$; we abbreviate $p^+ = p_\Omega^+$ and $p^- = p_\Omega^-$. We define the modular of a measurable function f to be

$$\varrho_{L^{p(\cdot)}(\Omega)}(f) \int_\Omega |f(x)|^{p(x)}\,\mathrm{d}x$$

The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ consists of all measurable functions $f : \Omega \to \mathbb{R}^n$ for which $\varrho_{L^{p(\cdot)}(\Omega)}(f) < \infty$. We define the Luxemburg norm on this space by

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 : \varrho_{L^{p(\cdot)}(\Omega)}(f/\lambda) \leq 1\right\},$$

which is the Minkowski functional of the absolutely convex set $\{f : \varrho_{L^{p(\cdot)}(\Omega)}(f) \leq 1\}$. In the case when $\Omega = \mathbb{R}^n$ we replace the $L^{p(\cdot)}(\mathbb{R}^n)$ in subscripts simply by $p(\cdot)$. The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is the subspace of $L^{1,p(\cdot)}(\Omega)$ of functions f whose distributional gradient exists and satisfies $|\nabla f| \in L^{p(\cdot)}(\Omega)$. The norm

$$\|f\|_{W^{1,p(\cdot)}(\Omega)} = \|f\|_{L^{p(\cdot)}(\Omega)} + \|\nabla f\|_{L^{p(\cdot)}(\Omega)}$$

makes $W^{1,p(\cdot)}(\Omega)$ a Banach space.

For fixed exponent spaces we have a very simple relationship between the norm and the modular. In the variable exponent case this is not so. However, we have the following useful property: $\varrho_{L^{p(\cdot)}}(f) \leq 1$ if and only if $\|f\|_{p(\cdot)} \leq 1$. This and many other basic results were proven in [113].

Definition (3.1.1) [140]: Le $g \in C(\mathbb{R}^n)$. We say that g is locally log-Hölder continuous, abbreviated $g \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$, if there exists $C_{\log} > 0$ such that

$$|g(x) - g(y)| \leq \frac{C_{\log}}{\log(e + 1/|x - y|)}$$

for all $x, y \in \mathbb{R}^n$.

We say that g is globally log-Hölder continuous, abbreviated $g \in C^{\log}(\mathbb{R}^n)$, if it is locally log-Hölder continuous and there exists $g_\infty \in \mathbb{R}$ such that

$$|g(x) - g_\infty| \leq \frac{C_{\log}}{\log(e + 1/|x|)}$$

Note that g is globally log-Hölder continuous if and only if

$$|g(x) - g(y)| \leq \frac{c}{\left|\log\frac{1}{2}q(x,y)\right|}$$

for all $x, y \in \overline{\mathbb{R}^n}$, where q denotes the spherical-chordal metric (the metric inherited from a projection to the Riemann sphere), hence the name, global log-Hölder continuity.

Building on [96] and [97] it is shown in [99] that

$$M : L^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p(\cdot)}(\mathbb{R}^n)$$

is bounded if $p \in C^{\log}(\mathbb{R}^n)$ and $1 < p^- \leq p^+ \leq \infty$. Global log-Hölder continuity is the best possible modulus of continuity to imply the boundedness of the maximal operator, see [96]. However, if one moves beyond assumptions based on continuity moduli, it is possible to derive results also under weaker assumptions, see [98, 122, 126].

Definition (3.1.2) [140]: We say a pair (φ, Φ) is admissible if $\varphi, \Phi \in \mathcal{S}(\mathbb{R}^n)$ satisfy

(i) $\sup\widehat{\varphi} \subseteq \left\{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2\right\}$ and $|\widehat{\varphi}(\xi)| \geq c > 0$ when $\frac{3}{5} \leq |\xi| \leq \frac{5}{3}$

(ii) $\sup\widehat{\Phi} \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq 2\}$ and $\left|\widehat{\Phi}(\xi)\right| \geq c > 0$ when $|\xi| \leq \frac{5}{3}$.

We set $\varphi_v(x) = 2^{vn}\varphi(2^v x)$ for $v \in \mathbb{N}$ and $\varphi_0(x) = \Phi(X)$. For $Q \in \mathcal{D}_v$ we set

$$\varphi_Q(x) = \begin{cases} |Q|^{1/2}\varphi_v(x - x_Q) & \text{if } v \geq 1 \\ |Q|^{1/2}\Phi(x - x_Q) & \text{if } v = 0 \end{cases}$$

We define ψ_v and ψ_Q analogously.

Following [106], given an admissible pair (φ, Φ) we can select another admissible pair (ψ, Ψ) such that

$$\widehat{\widehat{\Phi}}(\xi) \cdot \widehat{\Psi}(\xi) + \sum_{v \geq 1} \widehat{\widetilde{\varphi}}(2^{-v}\xi) \cdot \widehat{\psi}(2^{-v}\xi) = 1 \quad \text{for all } \xi.$$

Here, $\widehat{\Phi}(x) = \overline{\Phi(-x)}$ and similarly for $\widetilde{\varphi}$.

For each $f \in \dot{\mathcal{S}}(\mathbb{R}^n)$ we define the (inhomogeneous) φ-transform $\mathcal{S}_\varphi$ as the map taking f to the sequence $(\mathcal{S}_\varphi f)_{Q \in \mathcal{D}^+}$ by setting $(\mathcal{S}_\varphi f)_Q = \langle f, \varphi_Q\rangle$. Here, $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $L^2(\mathbb{R}^n; \mathbb{C})$. For later purposes note that $(\mathcal{S}_\varphi f)_Q = |Q|^{1/2}\widetilde{\varphi}_v * f(2^{-v}k)$ for $l(Q) = 2^{-v} < 1$ and $(\mathcal{S}_\varphi f)_Q = |Q|^{1/2}\widetilde{\Phi} * f(2^{-v}k)$ for $l(Q) = 1$.

The inverse (inhomogeneous) φ-transform T_ψ is the map taking a sequence $s = \{s_Q\}_{l(Q) \leq 1}$ to $T_\psi s = \sum_{l(Q)=1} s_Q \Psi_Q + \sum_{l(Q)<1} s_Q \psi_Q$. We have the following identity for $f \in \dot{\mathcal{S}}(\mathbb{R}^n)$:

$$f = \sum_{Q \in \mathcal{D}_0} \langle f, \Phi_Q\rangle \Psi_Q + \sum_{v=1}^{\infty} \sum_{Q \in \mathcal{D}_v} \langle f, \varphi_Q\rangle \psi_Q \tag{1}$$

We consider all distributions in $\dot{\mathcal{S}}(\mathbb{R}^n)$ (rather than $\dot{\mathcal{S}}/\mathcal{P}$ as in the homogeneous case), since $\widehat{\Phi}(0) \neq 0$.

Using the admissible functions (φ, Φ) we can define the norms

$$\|f\|_{F^\alpha_{p,q}} = \left\|\|2^{v\alpha}\varphi_v * f\|_{l^q}\right\|_{L^p} \quad \text{and} \quad \|f\|_{B^\alpha_{p,q}} = \left\|\|2^{v\alpha}\varphi_v * f\|_{L^p}\right\|_{l^q},$$

for constants $p, q \in (0, \infty)$ and $\alpha \in \mathbb{R}$. The Triebel-Lizorkin space $F^{\alpha}_{p,q}$ and the Besov $B^{\alpha}_{p,q}$ consists of distributions $f \in \dot{S}$ for which $\|f\|_{F^{\alpha}_{p,q}} < \infty$ and $\|f\|_{B^{\alpha}_{p,q}} < \infty$, respectively. The classical theory of these spaces is in Triebel [41, 56, 136]. The discrete representation as sequence spaces through the φ-transform is due to Frazier and Jawerth [45, 106]. Recently, anisotropic and weighted versions of these spaces have been studied by many people, see, e.g., Bownik and Ho [93], Frazier and Roudenko [129, 107], Kühn, Leopold, Sickel and Skrzypczak [114]. We now move on to generalizing these definitions to the variable index case.

We assume that p, q are positive functions on $\mathbb{R}^n$ such that $\frac{1}{p}, \frac{1}{q} \in C^{\log}(\mathbb{R}^n)$. This implies, in particular, $0 < p^- \le p^+ < \infty$ and $0 < q^- \le q^+ < \infty$. We also assume that $\alpha \in C^{\log}_{loc}(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)$ with $\alpha \ge 0$ and that α has a limit at infinity.

One of the central classical tools that we are missing in the variable integrability setting is a general multiplier theorem of Mikhlin-Hörmander type. We show that a general theorem does not hold, and instead prove the following result which is still sufficient to work with Triebel-Lizorkin spaces. For a family of functions $f_v : \mathbb{R}^n \to \mathbb{R}$, $v \ge 0$, we define

$$\|f_v(x)\|_{l^{q(x)}_v} = \left(\sum_{v \ge 0} |f_v(x)|^{q(x)} \right)^{1/q(x)}.$$

Note that this is just an ordinary discrete Lebesgue space, since $q(x)$ does not depend on v. The mapping $x \mapsto \|f_v(x)\|_{l^{q(x)}_v}$ is a function of x and can be measured in $L^{p(\cdot)}$. We write $L^{p(\cdot)}_x$ to indicate that the integration variable is x. We define

$$\eta_m(x) = (1 + |x|)^{-m} \quad \text{and} \quad \eta_{v,m}(x) = 2^{nv} \eta_m(2^v x) \tag{2}$$

Definition (3.1.3) [140]: Let $\varphi_v, v \in \mathbb{N}_0$, be as in Definition (3.1.2). The Triebel-Lizorkin space $F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is defined to be the space of all distributions $f \in \dot{S}$ with $\|f\|_{F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} < \infty$, where

$$\|f\|_{F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \left\| \left\| 2^{v\alpha(x)} \varphi_v * f(x) \right\|_{l^{q(x)}_v} \right\|_{L^{p(\cdot)}_x}$$

In the case of $p = q$ we use the notation $F^{\alpha(\cdot)}_{p(\cdot)}(\mathbb{R}^n) := F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

In the classical case it has proved very useful to express the Triebel-Lizorkin norm in terms of two sums, rather than a sum and an integral, thus, giving rise to discrete Triebel-Lizorkin spaces $f^{\alpha}_{p,q}$. This is achieved by viewing the function as a constant on dyadic cubes. The size of the appropriate dyadic cube varies according to the level of smoothness.

We next present a formulation of the Triebel-Lizorkin norm which is similar in spirit. For a sequence of real numbers $\{s_Q\}_Q$ we define

$$\left\| \{s_Q\}_Q \right\|_{f^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \left\| \left\| 2^{v\alpha(x)} \sum_{Q \in \mathcal{D}_v} |s_Q| |Q|^{-\frac{1}{2}} \chi_Q \right\|_{l^{q(x)}_v} \right\|_{L^{p(\cdot)}_x}.$$

The space $f_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$ consists of all those sequences $\{S_Q\}_Q$ for which this norm is finite. We are ready to state our first decomposition result, which says that $S_\varphi : F_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \hookrightarrow f_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$ is a bounded operator.

Definition (3.1.4) [140]: Let $v \in \mathbb{N}_0$, $Q \in \mathcal{D}_v$, and $k \in \mathbb{Z}, l \in \mathbb{N}_0$ k and $M \geq n$. A function m_Q is said to be a (k, l, M)-smooth molecule for Q if it satisfies the following conditions for some $m > M$:

(i) if $v > 0$, then $\int_{\mathbb{R}^n} x^\gamma m_Q(x) dx = 0$ for all $|\gamma| \leq k$; and

(ii) $|\mathcal{D}^\gamma m_Q(x)| \leq 2^{|\gamma||v|} |Q|^{1/2} \eta_{v,m}(x + x_Q)$ for all multi-indices $\gamma \in \mathbb{R}^n$ with $|\gamma| \leq l$

The conditions (i) and (ii) are called the moment and decay conditions, respectively.

Definition (3.1.5) [140]: Let $K, L : \mathbb{R}^n \to \mathbb{R}$ and $M > n$. The family $\{m_Q\}_Q$ is said to be a family of (K, L, M)-smooth molecules if m_Q is $(\lfloor K_Q^- \rfloor, \lfloor L_Q^- \rfloor, M)$-smooth for every $Q \in \mathcal{D}^+$.

Definition (3.1.6) [140]: We say that $\{m_Q\}_Q$ is a family of smooth molecules for $F_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$ if it is a family of $(N + \varepsilon, \alpha + \varepsilon + 1, M)$-smooth molecules, where

$$N(x) := \frac{n}{\min\{1, p(x), q(x)\}} - n - \alpha(x)$$

for some constant $\varepsilon > 0$, and M is a sufficiently large constant.

The number M needs to be chosen sufficiently large, for instance

$$2 \frac{n + c_{\log}(\alpha)}{\min\{1, p^-, q^-\}}$$

will do, where $c_{\log}(\alpha)$ denotes the log-Hölder continuity constant of α. Since M can be fixed depending on the parameters we will usually omit it from our notation of molecules.

Theorems (3.1.22) and (3.1.24) yield an isomorphism between $F_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$ and a subspace of $f_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$ via the S_φ transform:

Corollary (3.1.7) [140]: If the functions p, q and α are as in the Standing Assumptions, then

$$\|f\|_{F_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} \approx \|S_\varphi f\|_{f_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}$$

for every $f \in F_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

We can prove that the space $F_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ is well-defined.

Theorem (3.1.8) [140]: The space $F_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ is well-defined, i.e., the definition does not depend on the choice of the functions φ and Φ satisfying the conditions of Definition (3.1.2), up to the equivalence of norms.

Proof: Let $\widetilde{\varphi}_v$ and φ_v be different basis functions as in Definition (3.1.2). Let $\|\cdot\|_{\widetilde{\varphi}}$ and $\|\cdot\|_\varphi$ denote the corresponding norms of $F_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$. By symmetry, it suffices to prove $\|f\|_{\widetilde{\varphi}} \leq c\|f\|_\varphi$ for all $f \in \dot{S}$. Let $\|f\|_\varphi < \infty$. Then by (1) and Theorem (3.1.22) we have $f = \sum_{Q \in \mathcal{D}^+} (S_\varphi f)_Q \psi_Q$ and $\|S_\varphi f\|_{f_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} \leq c\|f\|_\varphi$.

Since $\{\psi_Q\}_Q$ is a family of smooth molecules, $\|f\|_{\widetilde{\varphi}} \le c\|\mathcal{S}_\varphi f\|_{f^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}$ by Theorem (3.1.24), which completes the proof.

It is often convenient to work with compactly supported basis functions. Thus, we say that the molecule a_Q concentrated on Q is an atom if it satisfies $\operatorname{supp} a_Q \subset 3Q$. The downside of atoms is that we need to choose a new set of them for each function f that we represent. Note that this coincides with the definition of atoms in [106] in the case when p, q and α as constants.

If the maximal operator is bounded and $1 < p^- \le p^+ < \infty$, then it follows easily that $C_0^\infty(\mathbb{R}^n)$ (the space of smooth functions with compact support) is dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$, since it is then possible to use convolution. However, density can be achieved also under more general circumstances, see [105, 112, 139]. The standing assumptions are strong enough to give the density directly:

Corollary (3.1.9) [140]: Let the functions p, q and α be as in the Standing Assumptions. Then $C_0^\infty(\mathbb{R}^n)$ is dense in $F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. Another consequence of our atomic decomposition is the analogue of the standard trace theorem. Note that the assumption $\alpha - \frac{1}{p} - (n-1)\left(\frac{1}{p} - 1\right)_+ > 0$ is optimal also in the constant smoothness and integrability case, cf. [45]

Proof: Choose K so large that $F^K_{p^+,2} \hookrightarrow F^{\alpha^+}_{p^+,1}$. This is possible by classical, fixed exponent, embedding results.

Let $f \in F^{\alpha(\cdot)}_{p(\cdot)q(\cdot)}$ and choose smooth atoms $a_Q \in C^k(\mathbb{R}^n)$ so that $f = \sum_{Q\in\mathcal{D}^+} t_Q a_Q$ in $\acute{S}$. Define

$$f_{\mathrm m} = \sum_{\nu=0}^{m} \sum_{Q\in\mathcal{D}_\nu,|x_Q|<m} t_Q a_Q$$

Then clearly $f_{\mathrm m} \in C_0^K$ and $f_{\mathrm m} \to f$ in $F^{\alpha(\cdot)}_{p(\cdot)q(\cdot)}$.

We can chose a sequence of functions $\varphi_{m,k} \in C_0^\infty$ so that $\|f_{\mathrm m} - \varphi_{m,k}\|_{W^{K,p^+}} \to 0$ as $k \to \infty$ and the support of $\varphi_{m,k}$ is lies in the ball $B(0, r_{\mathrm m})$. By the choice of K we conclude that

$$\left\|f_{\mathrm m} - \varphi_{m,k}\right\|_{F^{\alpha^+}_{p^+,1}} \le c\left\|f_{\mathrm m} - \varphi_{m,k}\right\|_{F^K_{p^+,2}} = c\left\|f_{\mathrm m} - \varphi_{m,k}\right\|_{W^{K,p^+}}.$$

By Proposition (3.1.26) we conclude that

$$\left\|f_{\mathrm m} - \varphi_{m,k}\right\|_{F^{\alpha(\cdot)}_{p(\cdot)q(\cdot)}} \le c\left\|f_{\mathrm m} - \varphi_{m,k}\right\|_{F^{\alpha^+}_{p^+,1}}$$

Now we show how the Triebel-Lizorkin scale $F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ includes as special cases previously studied spaces with variable differentiability or integrability.

We begin with the variable exponent Lebesgue spaces, which were originally introduced by Orlicz in [124]. We show that $F^0_{p(\cdot),2} \cong L^{p(\cdot)}$ under suitable assumptions on p. We use an extrapolation result for $L^{p(\cdot)}$. Recall, that a weight ω is in the Muckenhoupt class A_1 if $M_\omega \le k_\omega$ for some such $K > 0$. The smallest K is the A_1 constant of ω.

Lemma (3.1.10) [95]: Let $p \in C^{\log}(\mathbb{R}^n)$ with $1 < p^- \le p^+ < \infty$ and let $\mathcal{G}$ denote a family of tuples (f, g) of measurable functions on $\mathbb{R}^n$. Suppose that there exists a constant $r_0 \in (0, p^-)$ so that

$$\left(\int_{\mathbb{R}^n}|f(x)|^{r_0}\omega(x)\mathrm{d}x\right)^{1/r_0}\le c_0\left(\int_{\mathbb{R}^n}|g(x)|^{r_0}\omega(x)\mathrm{d}x\right)^{1/r_0}$$

for all $(f,g)\in\mathcal{G}$ and every weight $\omega\in A_1$, where c_0 is independent of f and g and depends on ω only via its A_1-constant. Then

$$\|f\|_{L^{(\cdot)}(\mathbb{R}^n)}\le c_1\|g\|_{L^{(\cdot)}(\mathbb{R}^n)}$$

for all $(f,g)\in\mathcal{G}$ with $\|f\|_{L^{(\cdot)}(\mathbb{R}^n)}<\infty$.

Theorem (3.1.11) [140]: Let $p\in C^{\log}(\mathbb{R}^n)$ with $1<p^-\le p^+<\infty$. Then $L^{p(\cdot)}(\mathbb{R}^n)\cong F^0_{p(\cdot),2}(\mathbb{R}^n)$ particular,

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\approx\left\|\|\varphi_v*f\|_{l_v^2}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

for all $f\in L^{p(\cdot)}(\mathbb{R}^n)$.

Proof: Since $C_0^\infty(\mathbb{R}^n)$ is dense in $L^{p(\cdot)}(\mathbb{R}^n)$ (see [113]) and also in $F^0_{p(\cdot),2}(\mathbb{R}^n)$ by Corollary (3.1.9), it suffices to prove the claim for all $f\in C_0^\infty(\mathbb{R}^n)$. Fix $r\in(1,p^-)$. Then

$$\left\|\|\varphi_v*f\|_{l_v^2}\right\|_{L^{r_0}(\mathbb{R}^n;\omega)}\approx\|f\|_{L^{r_0}(\mathbb{R}^n;\omega)},$$

for all $\omega\in A_1$ by [115], where the constant depends only on the A_1-constant of the weight ω, so the assumptions of Lemma (3.1.10) are satisfied. Applying the lemma with $\mathcal{G}$ equal to either

$$\left\{\left(\|\varphi_v*f\|_{l_v^2},f\right):f\in C_0^\infty(\Omega)\right\}\quad\text{or}\quad\left\{\left(\|\varphi_v*f\|_{l_v^2}\right):f\in C_0^\infty(\Omega)\right\}$$

completes the proof.

Theorem (3.1.11) generalizes the equivalence of $L^p(\mathbb{R}^n)\cong F^0_{p,q}$ for constant $p\in(1,\infty)$ to the setting of variable exponent Lebesgue spaces. If $p\in(0,1]$, then the spaces $L^p(\mathbb{R}^n)$ have to be replaced by the Hardy spaces $h^p(\mathbb{R}^n)$. This suggests the following definition:

Definition (3.1.12) [140]: Let $p\in C^{\log}(\mathbb{R}^n)$ with $1<p^-\le p^+<\infty$. Then we define the variable exponent Hardy space $h^{p(\cdot)}(\mathbb{R}^n)$ by $h^{p(\cdot)}(\mathbb{R}^n):=F^0_{p(\cdot),2}$.

Let $\mathcal{B}^\sigma$ denote the Bessel potential operator $\mathcal{B}^\sigma=\mathcal{F}^{-1}(1+|\xi|^2)^{-\frac{\sigma}{2}}\mathcal{F}$ for $\sigma\in\mathbb{R}$. Then the variable exponent Bessel potential space is defined by

$$\mathcal{L}^{\alpha,p(\cdot)}(\mathbb{R}^n):=\mathcal{B}^\alpha\left(L^{p(\cdot)}(\mathbb{R}^n)\right)=\left\{\mathcal{B}^\alpha g:g\in L^{p(\cdot)}(\mathbb{R}^n)\right\},$$

equipped with the norm $\|g\|_{\mathcal{L}^{\alpha,p(\cdot)}}:=\|\mathcal{B}^{-\alpha}g\|_{p(\cdot)}$. It was shown independently in [88] and [109] that $\mathcal{L}^{k,p(\cdot)}(\mathbb{R}^n)\cong W^{k,p(\cdot)}(\mathbb{R}^n)$ for $k\in\mathbb{N}_0$ when $p\in C^{\log}(\mathbb{R}^n)$ with $1<p^-\le p^+<\infty$.

We will show that $\mathcal{L}^{k,p(\cdot)}(\mathbb{R}^n)\cong W^{k,p(\cdot)}(\mathbb{R}^n)$ under suitable assumptions on p for $\alpha\ge0$ and that $\mathcal{L}^{k,p(\cdot)}(\mathbb{R}^n)\cong W^{k,p(\cdot)}(\mathbb{R}^n)\cong F^k_{p(\cdot),2}(\mathbb{R}^n)$ for $k\in\mathbb{N}_0$. It is clear by the definition of $\mathcal{L}^{\alpha,p(\cdot)}(\mathbb{R}^n)$ that $\mathcal{B}^\sigma$ with $\sigma>0$ is an isomorphism between $\mathcal{L}^{\alpha,p(\cdot)}(\mathbb{R}^n)$ and $\mathcal{L}^{\alpha+\sigma,p(\cdot)}(\mathbb{R}^n)$, i.e., it has a lifting property. Therefore, in view of Theorem (3.1.11) and $\mathcal{L}^{0,p(\cdot)}(\mathbb{R}^n)=L^{p(\cdot)}(\mathbb{R}^n)\cong F^0_{p(\cdot),2}(\mathbb{R}^n)$, we will complete the circle by proving a lifting property for the scale $F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Lemma (3.1.13) [140]: Let p,q and α be as in the Standing Assumptions and $\sigma\ge0$. Then the Bessel potential operator $\mathcal{B}^\sigma$ is an isomorphism between $F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ and $F^{\alpha(\cdot)+\sigma}_{p(\cdot),q(\cdot)}$.

Proof: Let $f\in F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$. We know that $\{\varphi_Q\}$ is a family of smooth molecules, thus, by Theorem (3.1.22)

$$\left\|\{s_Q\}_Q\right\|_{f_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} \approx \|f\|_{F_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}},$$

where $f = \sum_{Q \in \mathcal{D}^+} s_Q \varphi_Q$. Therefore,

$$\mathcal{B}^\sigma f = \sum_{Q \in \mathcal{D}^+} s_Q \mathcal{B}^\sigma \varphi_Q = \sum_{Q \in \mathcal{D}^+} \underbrace{2^{-\nu\sigma} s_Q}_{=: \acute{s}_Q} \underbrace{2^{\nu\sigma} \mathcal{B}^\sigma \varphi_Q}_{=: \acute{\varphi}_Q}.$$

Let us check that $\{K\acute{\varphi}_Q\}_Q$ is a family of smooth molecules of an arbitrary order for a suitable constant K. Let $Q \in \mathcal{D}^+$. Without loss of generality we may assume that $x_Q = 0$. Then

$$\widehat{\acute{\varphi}_Q}(\xi) = \frac{2^{\nu\sigma}\widehat{\varphi_Q}(\xi)}{(1+|\xi|)^\sigma} = \frac{2^{\nu\sigma}|Q|^{1/2}\widehat{\varphi}(2^{-\nu}\xi)}{(1+|\xi|)^\sigma}.$$

Since $\widehat{\varphi}$ has support in the annulus $B^n(0,2)\backslash B^n(0,1/2)$, it is clear that $\widehat{\acute{\varphi}_Q} \equiv 0$ in a neighborhood of the origin when $l(Q) < 1$, so the family satisfies the moment condition in Definition (3.1.4) for an arbitrarily high order.

We consider the decay condition for molecules. Let $\mu \in \mathbb{N}_0^n$ be a multi-index with $|\mu| = m$. We estimate

$$\left|D_\xi^\mu \widehat{\acute{\varphi}_Q}(\xi)\right| \le 2^{\nu\sigma}|Q|^{1/2}\left|D_\xi^\mu\left[\frac{\widehat{\varphi}(2^{-\nu}\xi)}{(1+|\xi|)^\sigma}\right]\right|$$

$$= |Q|^{1/2}2^{-\nu m}\left|D_\zeta^\mu\left[\frac{\widehat{\varphi}(\zeta)}{(2^{-\nu}+|\zeta|)^\sigma}\right]\right| \le c|Q|^{1/2}2^{-\nu m}\left|D_\zeta^\mu[\widehat{\varphi}(\zeta)|\zeta|^{-\sigma}]\right|$$

where $\zeta = 2^{-\nu}\xi$ and we used that the support of $\widehat{\varphi}$ lies in the annulus $B^n(0,2)\backslash B^n(0,1/2)$ for the last estimate. Define

$$K_m = \sup_{|\mu|=m, \zeta \in \mathbb{R}^n} 2^{-\nu m}\left|D_\zeta^\mu[\widehat{\varphi}(\zeta)|\zeta|^{-\sigma}]\right|.$$

Since $\sigma \ge 0$ and $\widehat{\varphi}$ vanishes in a neighborhood of the origin, we conclude that $K_m < \infty$ for every m. From the estimate

$$|x^\mu \psi(x)| = c\left|\int_{\mathbb{R}^n} (-1)^m D_\zeta^\mu \widehat{\psi}(\xi) e^{ix\cdot\xi} d\xi\right| \le c|\text{supp}\widehat{\psi}| \sup_\xi \left|D_\zeta^\mu \widehat{\psi}(\xi)\right|,$$

we conclude that

$$|x|^m |\acute{\varphi}_Q(x)| \le c2^{\nu n}|Q|^{1/2}2^{-\nu m}K_m \text{ and } |\acute{\varphi}_Q(x)| \le c2^{\nu n}|Q|^{1/2}2^{\nu m}K_0.$$

Multiplying the former of the two inequalities by $2^{\nu m}$ and adding it to the latter gives

$$(1 + 2^{\nu m}|x|^m)|\acute{\varphi}_Q(x)| \le c2^{\nu n}|Q|^{1/2}(K_0 + K_m).$$

Finally, this implies that

$$|\acute{\varphi}_Q(x)| \le c\frac{2^{\nu n}}{(1 + 2^\nu|x|^m)}|Q|^{1/2}(K_0 + K_m)\eta_{\nu,m}(x),$$

from which we conclude that the family $\{K\acute{\varphi}_Q\}_Q$ satisfy the decay condition when $K \le (|Q|^{1/2}(K_0 + K_m)^{-1})$. A similar argument yields the decay condition for $D_x^\mu \acute{\varphi}_Q$

Since $\{K\acute{\varphi}_Q\}_Q$ is a family of smooth molecules for $F_{p(\cdot),q(\cdot)}^{\alpha(\cdot)+\sigma}$, we can apply Theorem (3.1.24) to conclude that

$$\|\mathcal{B}^\sigma f\|_{F_{p(\cdot),q(\cdot)}^{\alpha(\cdot)+\sigma}} \le c\left\|\{\acute{s}_Q/K\}_Q\right\|_{f_{p(\cdot),q(\cdot)}^{\alpha(\cdot)+\sigma}} \le c\left\|\{s_Q\}_Q\right\|_{f_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} \approx \|f\|_{F_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}.$$

The reverse inequality is handled similarly.

Theorem (3.1.14) [140]: Let $p \in C^{\log}(\mathbb{R}^n)$ with $1 < p^- \le p^+ < \infty$ and $\alpha \in [0,\infty)$. Then $F^{\alpha}_{p(\cdot),2}(\mathbb{R}^n) \cong \mathcal{L}^{\alpha,p(\cdot)}(\mathbb{R}^n)$. If $k \in \mathbb{N}_0$, then $F^{k}_{p(\cdot),2}(\mathbb{R}^n) \cong W^{k,p(\cdot)}(\mathbb{R}^n)$

Proof: Suppose that $f \in F^{k}_{p(\cdot),2}(\mathbb{R}^n)$. By Lemma (3.1.13), $\mathcal{B}^{-\alpha}f \in F^{0}_{p(\cdot),2}(\mathbb{R}^n)$, so we conclude by Theorem (3.1.11) that $\mathcal{B}^{-\alpha}f \in L^{p(\cdot)}(\mathbb{R}^n) = \mathcal{L}^{0,p(\cdot)}(\mathbb{R}^n)$. Then it follows by the definition of the Bessel space that $f = \mathcal{B}^{\alpha}[\mathcal{B}^{-\alpha}f] \in \mathcal{L}^{\alpha,p(\cdot)}(\mathbb{R}^n)$. The reverse inclusion follows by reversing these steps. The claim regarding the Sobolev spaces follows from this and the equivalence $\mathcal{L}^{k,p(\cdot)}(\mathbb{R}^n) \cong W^{k,p(\cdot)}(\mathbb{R}^n)$ for $k \in \mathbb{N}_0$ (see [88] or [109]).

Now we come to spaces of variable smoothness as introduced by Besov [89], following Leopold [116]. Let $p, q \in (1,\infty)$ and let $\alpha \in C^{\log}_{loc}(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)$ with $\alpha \ge 0$. Then Besov defines the following spaces of variable smoothness

$$F^{\alpha(\cdot),\text{Besov}}_{p,q}(\mathbb{R}^n) := \left\{ f \in L^{p}_{loc}(\mathbb{R}^n) : \|f\|_{F^{\alpha(\cdot),\text{Besov}}_{p,q}} < \infty \right\},$$

$$\|f\|_{F^{\alpha(\cdot),\text{Besov}}_{p,q}} := \left\| \left\| 2^{\nu\alpha(x)} \int_{|h|\le 1} |\Delta^M(2^{-k}h, f)(x)| \, dh \right\|_{l^q_{\nu}} \right\|_{L^p_x} + \|f\|_{L^p_x},$$

where

$$\Delta^M(y, f)(x) := \sum_{k=0}^{M} (-1)^{M-k} \binom{M}{k} f(x + ky).$$

In [91] Besov proved that $F^{\alpha(\cdot),\text{Besov}}_{p,q}(\mathbb{R}^n)$ can be renormed by

$$\left\| \left\| 2^{\nu\alpha(x)} \varphi_{\nu} * f(x) \right\|_{l^q_{\nu}} \right\|_{L^p_x} \approx \|f\|_{F^{\alpha(\cdot),\text{Besov}}_{p,q}},$$

which agrees with our definition of the norm of $F^{\alpha(\cdot)}_{p,q}$, since p and q are constants. This immediately implies the following result:

Theorem (3.1.15) [140]: Let $p, q \in (1,\infty)$, $\alpha \in C^{\log}_{loc} \cap L^{\infty}$ and $\alpha \ge 0$. Then $\|f\|_{F^{\alpha(\cdot),\text{Besov}}_{p,q}}(\mathbb{R}^n) \approx \|f\|_{F^{\alpha(\cdot)}_{p,q}}(\mathbb{R}^n)$.

In his works, Besov also studied Besov spaces of variable differentiability. For $p, q \in (1,\infty)$, and $\alpha \in C^{\log}_{loc} \cap L^{\infty}$ with $\alpha \ge 0$, he defines

$$B^{\alpha(\cdot),\text{Besov}}_{p,q}(\mathbb{R}^n) := \left\{ f \in L^{p}_{loc}(\mathbb{R}^n) : \|f\|_{F^{\alpha(\cdot),\text{Besov}}_{p,q}(\mathbb{R}^n)} < \infty \right\},$$

$$\|f\|_{B^{\alpha(\cdot),\text{Besov}}_{p,q}(\mathbb{R}^n)} := \left\| \left\| \sup_{|h|\le 1} |\Delta^M(2^{-\nu\alpha(x)}h, f)(x)| \right\|_{L^p_x} \right\|_{l^q_{\nu}} + \|f\|_{L^p_x}.$$

In the classical case the scale of Triebel-Lizorkin spaces and the scale of Besov spaces agree if $p = q$. Besov showed in [92] that this is also the case for his new scales of Triebel-Lizorkin and Besov spaces, i.e., $F^{\alpha(\cdot),\text{Besov}}_{p,q}(\mathbb{R}^n) = B^{\alpha(\cdot),\text{Besov}}_{p,q}(\mathbb{R}^n)$ for $p \in (1,\infty)$, $\alpha \in C^{\log}_{loc} \cap L^{\infty}$. and $\alpha \ge 0$. This enables us to point out a connection to another family of spaces. By means of the symbols of pseudodifferential operators, Leopold [116] introduced Besov spaces with variable differentiability $B^{\alpha(\cdot),\text{Leopold}}_{p,q}(\mathbb{R}^n)$. He further showed that if $0 < \alpha^- \le \alpha^+ < \infty$ and $\alpha \in C^{\infty}(\mathbb{R}^n)$, then the spaces $B^{\alpha(\cdot),\text{Leopold}}_{p,q}(\mathbb{R}^n)$ can be characterized by means of finite differences. This characterization agrees with the

one that later Besov [90] used in the definition of the spaces $B_{p,q}^{\alpha(\cdot),\text{Besov}}(\mathbb{R}^n)$. In particular, we have $B_{p,q}^{\alpha(\cdot),\text{Leopold}}(\mathbb{R}^n) = B_{p,q}^{\alpha(\cdot),\text{Besov}}(\mathbb{R}^n) = F_{p,q}^{\alpha(\cdot)}(\mathbb{R}^n)$ for such α.

It should be mentioned that there have recently also been some extensions of variable integrability spaces in other directions, not covered by the Triebel-Lizorkin scale that we introduce here. For instance, Harjulehto & Hästö [110] modified the Lebesgue space scale on the upper end to account for the fact that $W^{1,n}$ does not map to L^∞ under the Sobolev embedding. Similarly, in the image restoration model by Chen, Levine and Rao mentioned above, one has the problem that the exponent p takes values in the closed interval [1,2], including the lower bound, so that one is not working with reflexive spaces. It is well-known that the space BV of functions of bounded variation is often a better alternative than $W^{1,1}$ when studying differential equations. Consequently, it was necessary to modify the scale $W^{1,p(\cdot)}$ so that the lower end corresponded to BV. This was done by Harjulehto, Hästö & Latvala in [111]. Schneider [133, 134] has also investigated spaces of variable smoothness, but these spaces are not included in the scale of Leopold and Besov. Diening, Harjulehto, Hästö, Mizuta & Shimomura [99] have studied Sobolev embeddings when $p \to 1$ using Lebesgue spaces with an L log L-character on the lower end in place of L^1.

Cruz-Uribe, Fiorenza, Martell and Pérez [95] proved a very general extrapolation theorem, which implies among other things the following vector-valued maximal inequality, for variable p but constant q:

Lemma (3.1.16) [140]: Let $p \in C^\infty(\mathbb{R}^n)$, with $1 < p^- \leq p^+ < \infty$ and $1 < q < \infty$. Then

$$\||Mf_i\|_{l^q}\|_{p(\cdot)} \leq c \||f_i\|_{l^q}\|_{p(\cdot)}.$$

It would be very nice to generalize this estimate to the variable q case. In particular, this would allow us to use classical machinery to deal with Triebel-Lizorkin spaces. Unfortunately, it turns out that it is not possible: if q is not constant, then the inequality

$$\left\|\|Mf_i\|_{l_x^{q(x)}}\right\|_{L_x^{p(\cdot)}} \leq c \left\|\|f_i\|_{l_x^{q(x)}}\right\|_{L_x^{p(\cdot)}}.$$

does not hold, even if p is constant or $p(\cdot) = q(\cdot)$. For a concrete counter-example consider q with $q|_{\Omega_j}, j = 0, 1$, and $q_0 \neq q_1$ and a constant p. Set $f_k := a_k \chi_{\Omega_0}$. Then $Mf_k|_{\Omega_1} \geq c a_k \chi_{\Omega_1}$. This shows that $l^{q_0} \hookrightarrow l^{q_1}$. The opposite embedding follows in the same way, hence, we would conclude that $l^{q_0} \cong l^{q_1}$, which is of course false.

In view of a vector-valued maximal inequality, we show estimates which take into account that there is a clear stratification in the Triebel-Lizorkin space, namely, a given magnitude of cube size is used in exactly one term in the sum. Recall that $\eta_m(x) = (1 + |x|)^{-m}$ and $\eta_{v,m}(x) = 2^{nv}\eta_m(2^v x)$. For a measurable set Q and an integrable function g we denote

$$M_Q g := \int_Q |g(x)| dx.$$

Lemma (3.1.17) [140]: For every $m > n$ there exists $c = c(m, n) > 0$ such that

$$\eta_{v,m} * |g|(x) \leq c \sum_{j \geq 0} 2^{-j(m-n)} \sum_{Q \in \mathcal{D}_{v-j}} \chi_{3Q}(x) M_Q g$$

for all $v \geq 0, g \in L_{loc}^1$, and $x \in \mathbb{R}^n$.

Proof: Fix $v \geq 0, g \in L_{loc}^1$, and $x, y \in \mathbb{R}^n$. If $|x - y| \leq 2^{-v}$, then we choose $Q \in \mathcal{D}_v$ which contains x and y. If $\|x - y\| > 2^{-v}$, then we choose $j \in \mathbb{N}_0$ such that $2^{v-j} \leq |x - y| \leq 2^{v-j+1}$ and let $Q \in \mathcal{D}_{v-j}$ be the cube containing y. Note that $x \in 3Q$. In either case, we conclude that

$$2^{vn}(1 + 2^v|x - y|)^{-m} \leq c2^{-j(m-n)}\chi_{3Q}(x)\frac{\chi_Q(y)}{|Q|}.$$

we multiply this inequality by $|g(y)|$ and integrate with respect to y over $\mathbb{R}^n$. This gives $\eta_{v,m} * |g|(x) \leq c2^{j(m-n)}\chi_{3Q}(x)M_Q g$, which clearly implies the claim.

For the proof of the Lemma (3.1.19) we need the following result on the maximal operator. It follows from [99], since $p^+ < \infty$ in our case.

Lemma (3.1.18) [140]: Let $p \in C^{\log}(\mathbb{R}^n)$, with $1 < p^- \leq p^+ < \infty$. Then there exists $h \in \text{weaL}^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that

$$Mf(x)^{p(x)} \leq cM\big(|f(\cdot)|^{p(\cdot)}\big)(x) + \min\{|Q|, 1\}h(x)$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ with $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq 1$.

Lemma (3.1.19) [140]: Let $p, q \in C^{\log}(\mathbb{R}^n)$, with $1 < p^- \leq p^+ < \infty$, $1 < q^- \leq q^+ < \infty$, and $(p/q)^- \cdot q^- > 1$. Then there exists $m > n$ such that

$$\Big\| \|\eta_{v,m} * f_v\|_{l_x^{q(x)}} \Big\|_{L_x^{p(\cdot)}} \leq c \Big\| \|f_v\|_{l_x^{q(x)}} \Big\|_{L_x^{p(\cdot)}}$$

for every sequence $\{f_v\}_{v \in \mathbb{N}_0}$ of L_{loc}^1-functions.

Proof: By homogeneity, it suffices to consider the case

$$\Big\| \|f_v\|_{l_x^{q(x)}} \Big\|_{L_x^{p(\cdot)}} \leq 1.$$

Then, in particular,

$$\int_{\mathbb{R}^n} |f_v(x)|^{p(x)}dx \leq 1 \tag{3}$$

for every $v \geq 1$. Using Lemma (3.1.17) and Jensen's inequality (i.e., the embedding in weighted discrete Lebesgue spaces), we estimate

$$\int_{\mathbb{R}^n} \left(\sum_{v \geq 0} |\eta_{v,m} * f_v(x)|^{q(x)}\right)^{\frac{p(x)}{q(x)}} dx \leq \int_{\mathbb{R}^n} \left(\sum_{v \geq 0} \left(\sum_{j \geq 0} 2^{-j(m-n)} \sum_{Q \in \mathcal{D}_{v-j}} \chi_{3Q}(x)M_Q f_v\right)^{q(x)}\right)^{\frac{p(x)}{q(x)}} dx$$

$$\leq c \int_{\mathbb{R}^n} \left(\sum_{v \geq 0} \sum_{j \geq 0} 2^{-j(m-n)} \left(\sum_{Q \in \mathcal{D}_{v-j}} \chi_{3Q}(x)M_Q f_v\right)^{q(x)}\right)^{\frac{p(x)}{q(x)}} dx$$

$$\leq c \int_{\mathbb{R}^n} \left(\sum_{v \geq 0} \sum_{j \geq 0} 2^{-j(m-n)}c \sum_{Q \in \mathcal{D}_{v-j}} \chi_{3Q}(x) \left(M_Q f_v\right)^{q(x)}\right)^{\frac{p(x)}{q(x)}} dx.$$

For the last inequality we used the fact that the innermost sum contains only a finite, uniformly bounded number of non-zero terms.

It follows from (3) and $p(x) \geq \frac{q(x)}{q^-}$ that $\|f_v\|_{L^{\frac{q(\cdot)}{q^-}}} \leq c$. Thus, by Lemma (3.1.18),

$$\left(M_Q f_v\right)^{\frac{q(x)}{q^-}} \leq cM_Q\left(|f_v|^{\frac{q}{q^-}}\right) + c \min\{|Q|, 1\}h(x)$$

for all $Q \in \mathcal{D}_{v-j}$ and $x \in Q$. Combining this with the estimates above, we get

$$\int_{\mathbb{R}^n}\left(\sum_{v\geq 0}|\eta_v * f_v(x)|^{q(x)}\right)^{\frac{p(x)}{q(x)}}dx \leq c\int_{\mathbb{R}^n}\left(\sum_{v\geq 0}\sum_{j\geq 0}2^{-j(m-n)}\sum_{Q\in\mathcal{D}_{v-j}}\chi_{3Q}(x)\left[M_Q\left(|f_v|^{\frac{q}{q^-}}\right)\right]^{q^-}\right)^{\frac{p(x)}{q(x)}}dx$$

$$+\ c\int_{\mathbb{R}^n}\left(\sum_{v\geq 0}\sum_{j\geq 0}2^{-j(m-n)}\sum_{Q\in\mathcal{D}_{v-j}}\chi_{3Q}(x)\,(\min\{|Q|,1\}h(x))^{q^-}\right)^{\frac{p(x)}{q(x)}}dx =: (\mathrm{I}) + (\mathrm{II})\,.$$

Now we easily estimate that

$$(\mathrm{I}) \leq \int_{\mathbb{R}^n}\left(\sum_{v\geq 0}\left[M\left(|f_v|^{\frac{q}{q^-}}\right)(x)\right]^{q^-}\sum_{j\geq 0}2^{-j(m-n)}\sum_{Q\in\mathcal{D}_{v-j}}\chi_{3Q}(x)\right)^{\frac{p(x)}{q(x)}}dx$$

$$\leq c\int_{\mathbb{R}^n}\left(\sum_{v\geq 0}\left[M\left(|f_v|^{\frac{q}{q^-}}\right)(x)\right]^{q^-}\right)^{\frac{p(x)}{q(x)}}dx \quad = c\int_{\mathbb{R}^n}\left\|M\left(|f_v|^{\frac{q}{q^-}}\right)(x)\right\|_{l_v^{q^-}}^{\frac{p(x)}{q(x)}q^-}dx.$$

The vector valued maximal inequality, Lemma (3.1.16), with $(p/q)^-\cdot q^- > 1$ and $q^- > 1$, implies that the last expression is bounded since

$$\int_{\mathbb{R}^n}\left(\sum_{v\geq 0}\left(|f_v|^{\frac{q(x)}{q^-}}\right)^{q^-}\right)^{\frac{p(x)}{q(x)}}dx = \int_{\mathbb{R}^n}\left(\sum_{v\geq 0}|f_v|^{q(x)}\right)^{\frac{p(x)}{q(x)}}dx \leq 1.$$

For the estimation of (II) we first note the inequality

$$\sum_{v\geq 0}\sum_{j\geq 0}2^{-j(m-n)}\sum_{Q\in\mathcal{D}_{v-j}}\chi_{3Q}(x)\min\{|Q|,1\}^{q^-} \leq \sum_{v\geq 0}\sum_{j\geq 0}2^{-j(m-n)}\min\{2^{n(j-v)q^-},1\}$$

$$\leq \sum_{j\geq 0}2^{-j(m-n)}\left(j+\sum_{v>j}2^{n(j-v)q^-}\right)$$

$$\leq \sum_{j\geq 0}2^{-j(m-n)}(j+1) \leq c.$$

We then estimate (II) as follows:

$$(\mathrm{II}) \leq c\int_{\mathbb{R}^n}\left(h(x)^{q^-}\sum_{v\geq 0}\sum_{j\geq 0}2^{-j(m-n)}\sum_{Q\in\mathcal{D}_{v-j}}\chi_{3Q}(x)\min\{|Q|,1\}^{q^-}\right)^{\frac{p(x)}{q(x)}}dx \leq c\int_{\mathbb{R}^n}h(x)^{\frac{p(x)}{q(x)}q^-}dx.$$

Since $(p/q)^-q^- > 1$ and $h\in$ weak-$L^1\cap L^\infty$, the last expression is bounded.

Theorem (3.1.20) [140]: Let $p,q\in C^{\log}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$ and $1 < q^- \leq q^+ < \infty$ Then the inequality

$$\left\|\left\|\eta_{v,m}(x) * f_v\right\|_{l_v^{q(x)}}\right\|_{l_x^{p(\cdot)}} \leq c\left\|\left\|f_v\right\|_{l_v^{q(x)}}\right\|_{l_x^{p(\cdot)}}$$

holds for every sequence $\{f_v\}_{v\in\mathbb{N}_0}$ of L^1_{loc}-functions and constant $m > n$.

Proof: Because of the uniform continuity of p and q, we can choose a finite cover $\{\Omega_i\}$ of $\mathbb{R}^n$ with the following properties:

(i) each $\Omega_i \subset \mathbb{R}^n, 1 \leq i \leq k$, is open;

(ii) the sets Ω_i cover $\mathbb{R}^n$, i.e., $\bigcup_i\Omega_i = \mathbb{R}^n$;

(iii) non-contiguous sets are separated in the sense that $d(\Omega_i,\Omega_i) > 0$ if $|i - j| > 1$; and

(iv)   we have $(p/q)_{A_i}^- q_{A_i}^- > 1$ for $1 \le i \le k$,   where $A_i := \cup_{j=i-1}^{i+1} \Omega_i$ (with the understanding that $\Omega_0 = \Omega_{k+1} = \emptyset$.

Let us choose an integer l so that $2^i \le \min_{|i-j|>1} 3\, d(\Omega_i, \Omega_i) < 2^{i+1}$. Since there are only finitely many indices, the third condition implies that such an l exists.

Next we split the problem and work with the domains Ω_i. In each of these we argue as in the previous lemma to conclude that

$$\int_{\mathbb{R}^n} \left(\sum_{v \ge 0} |\eta_{v,m} * f_v(x)|^{q(x)} \right)^{\frac{p(x)}{q(x)}}$$

$$\le \sum_{i=1}^k \int_{\Omega_i} \left(\sum_{v \ge 0} |\eta_{v,m} * f_v(x)|^{q(x)} \right)^{\frac{p(x)}{q(x)}} dx$$

$$\le c \sum_{i=1}^k \int_{\Omega_i} \left(\sum_{v \ge 0} \sum_{j \ge 0} 2^{-j(m-n)} \sum_{Q \in \mathcal{D}_{v-j}} \chi_{3Q}(x) \left(M_Q f_v \right)^{q(x)} \right)^{\frac{p(x)}{q(x)}} dx.$$

From this we get

$$\int_{\Omega_i} \left(\sum_{v \ge 0} |\eta_{v,m} * f_v(x)|^{q(x)} \right)^{\frac{p(x)}{q(x)}} dx \le c \int_{\Omega_i} \left(\sum_{v \ge 0} \sum_{j=0}^{v+l} 2^{-j(m-n)} \sum_{Q \in \mathcal{D}_{v-j}} \chi_{3Q}(x) \left(M_Q f_v \right)^{q(x)} \right)^{\frac{p(x)}{q(x)}} dx$$

$$+ c \int_{\Omega_i} \left(\sum_{v \ge 0} \sum_{j \ge v+l} 2^{-j(m-n)} M f_v(x)^{q(x)} \right)^{\frac{p(x)}{q(x)}} dx.$$

The first integral on the right hand side is handled as in the previous proof. This is possible, since the cubes in this integral are always in A_i and $(p/q)_{A_i}^- q_{A_i}^- > 1$.
So it remains only to bound

$$\int_{\Omega_i} \left(\sum_{v \ge 0} \sum_{j \ge v+l} 2^{-j(m-n)} M f_v(x)^{q(x)} \right)^{\frac{p(x)}{q(x)}} dx \le c \int_{\Omega_i} \left(\sum_{v \ge 0} 2^{-(m-n)v} M f_v(x)^{q(x)} \right)^{\frac{p(x)}{q(x)}} dx.$$

For a non-negative sequence x_i we have

$$\left(\sum_{v \ge 0} 2^{-j(m-n)} x_i \right)^r \le \begin{cases} c(r) \sum_{i \ge 0} 2^{-i(m-n)} x_i^r & \text{if } r \ge 1 \\ \sum_{i \ge 0} 2^{-i(m-n)} x_i^r & \text{if } r \le 1. \end{cases}$$

We apply this estimate for $r = \dfrac{p(x)}{q(x)}$ and conclude that

$$\int_{\Omega_i} \left(\sum_{v \ge 0} 2^{-j(m-n)v} M f_v(x)^{q(x)} \right)^{\frac{p(x)}{q(x)}} dx \le c \sum_{v \ge 0} 2^{-(m-n)v \min\left\{1, \left(\frac{p}{q}\right)^-\right\}} \int_{\Omega_i} M f_v(x)^{q(x)} dx.$$

The boundedness of the maximal operator implies that the integral may be estimated by a constant, since $\int |f_v(x)|^{p(x)} dx \le 1$. We are left with a geometric sum, which certainly converges.

Lemma (3.1.21) [140]: Let α be as in the Standing Assumptions. There exists $d \in (n, \infty)$ such that if $m > d$, then
$$2^{v\alpha(x)}\eta_{v,2m}(x-y) \leq c2^{v\alpha(y)}\eta_{v,m}(x-y)$$
for all $x, y \in \mathbb{R}^n$.

Proof: Choose $k \in \mathbb{N}_0$ as small as possible subject to the condition that $|x-y| \leq 2^{-v+k}$. Then $1 + 2^v|x-y| \approx 2^k$. We estimate that
$$\frac{\eta_{v,2m}(x-y)}{\eta_{v,m}(x-y)} \leq c\left(1+2^k\right)^{-m} \leq c2^{-km}.$$
On the other hand, the log-Hölder continuity of α implies that
$$2^{v(\alpha(x)-\alpha(y))} \geq 2^{-vc_{\log}/\log(e+1/|x-y|)} \geq 2^{-kc_{\log}|x-y|^{-c_{\log}/\log(e+1/|x-y|)}} \geq c2^{-km}.$$
The claim follows from these estimates provided we choose $m \geq c_{\log}$.

Theorem (3.1.22) [140]: If p, q and α are as in the Standing Assumptions, then
$$\left\|\mathcal{S}_\varphi f\right\|_{f^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leq c\|f\|_{F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Proof: Let $f \in F^{\alpha(\cdot)}_{p(\cdot)q(\cdot)}$. Then we have the representation
$$f = \sum_{Q \in \mathcal{D}^+} \langle \varphi_Q, f \rangle \psi_Q = \sum_{Q \in \mathcal{D}^+} |Q|^{\frac{1}{2}} \varphi_v * f(x_Q)\psi_Q.$$
Let $r \in (0, \min[p^-, q^-])$ and let m be so large that Lemma (3.1.21) applies, so
$$\|\varphi_v * f\|_{f^{\alpha(\cdot)}_{p(\cdot)q(\cdot)}} = \left\|\left\|2^{v\alpha(x)}\sum_{Q \in \mathcal{D}_x}\varphi_v * f(x_Q)\chi_Q\right\|_{l^{q(x)}_v}\right\|_{L^{p(\cdot)}_x}$$
$$\leq c\left\|\left\|2^{v\alpha(x)r}\left(\eta_{v,2m} * |\varphi_v * f|^r\right)^{1/r}\right\|_{l^{q(x)}_v}\right\|_{L^{p(\cdot)}_x}$$
$$= c\left\|\left\|2^{v\alpha(x)r}\left(\eta_{v,2m} * |\varphi_v * f|^r\right)^{1/r}\right\|_{l^{q(x)/r}_v}\right\|_{L^{p(\cdot)/r}_x}.$$
By Lemma (3.1.21) and Theorem (3.1.20), we further conclude that
$$\|\mathcal{S}_v f\|_{f^{\alpha(\cdot)}_{p(\cdot)q(\cdot)}} \leq c\left\|\left\|\left(\eta_{v,m} * 2^{v\alpha(\cdot)}|\varphi_v * f|\right)^r\right\|_{l^{q(x)/r}_v}\right\|_{L^{p(\cdot)/r}_x}^{1/r} \leq c\left\|\left\|2^{v\alpha(x)r}|\varphi_v * f|^r\right\|_{l^{q(x)/r}_v}\right\|_{L^{p(\cdot)/r}_x}^{1/r}$$
$$= c\left\|\left\|2^{v\alpha(x)}\varphi_v * f\right\|_{l^{q(x)}_v}\right\|_{L^{p(\cdot)}_x}.$$
This proves the theorem.

Lemma (3.1.23) [140]: Let p, q, and α be as in the Standing Assumptions and define functions $J = n/\min\{1, p, q\}$ and $N = J - n - \alpha$ Let Ω be a cube or the complement of a finite collection of cubes and suppose that $\{m_Q\}_Q, Q \subset \Omega$, is a family of $(J^+ - n - \alpha^- + \varepsilon, \alpha^+ + 1 + \varepsilon)$-smooth molecules, for some $\varepsilon > 0$. Then
$$\|f\|_{F^{\alpha(\cdot)}_{p(\cdot)q(\cdot)}(\Omega)} \leq c\left\|\{s_Q\}_Q\right\|_{f^{\alpha(\cdot)}_{p(\cdot)q(\cdot)}(\Omega)}, \qquad \text{where } f = \sum_{\substack{v \geq 0 \\ Q \subset \Omega}}\sum_{\substack{Q \in \mathcal{D}_v}} s_Q m_Q$$
and $c > 0$ is independent of $\{s_Q\}_Q$ and $\{m_Q\}_Q$.

Theorem (3.1.24) [140]: Let the functions p, q and α be as in the Standing Assumptions. Suppose that $\{m_Q\}_Q$ is a family of smooth molecules for $F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$, and that $\{m_Q\}_Q \in f^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$. Then

$$\|f\|_{F_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} \le c \left\|\{m_Q\}_Q\right\|_{f_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}, \text{where } f = \sum_{v\ge 0}\sum_{Q\in\mathcal{D}_v} s_Q\, m_Q.$$

Proof: We will reduce the claim to the previous lemma.

By assumption there exists $\varepsilon > 0$ so that the molecules m_Q are $(N+4\varepsilon, \alpha+3\varepsilon)$-smooth. By the uniform continuity of p, q and α, we may choose $\mu_0 \ge 0$ so that $N_Q^- > J_Q^+ - \alpha_Q^- - n + \varepsilon$ and $\alpha_Q^- > \alpha_Q^+ - \varepsilon$ for every dyadic cube Q of level μ_0. Note that if Q_0 is a dyadic cube of level μ_0 and $Q \subset Q_0$ is another dyadic cube, then

$$N_Q^- \ge N_{Q_0}^- > J_{Q_0}^+ - \alpha_{Q_0}^- - n - \varepsilon \ge J_Q^+ - \alpha_Q^- - n - \varepsilon,$$

similarly for α. Thus we conclude that m_Q is a $(J_Q^+ - \alpha_Q^- - n + 3\varepsilon, \alpha_Q^+ + 1 + 2\varepsilon)$-smooth when Q is of level at most μ_0.

Since p, q and α have a limit at infinity, we conclude that $N_{\mathbb{R}^n\setminus k}^- > J_{\mathbb{R}^n\setminus k}^+ - \alpha_{\mathbb{R}^n\setminus k}^- - n + -\varepsilon$ and $\alpha_{\mathbb{R}^n\setminus k}^- > \alpha_{\mathbb{R}^n\setminus k}^+ - \varepsilon$ for some compact set $K \subset \mathbb{R}^n$. We denote by $\Omega_i, i = 1, \dots, M$, those dyadic cubes of level μ_0 which intersect K, and define $\Omega_i = \mathbb{R}^n\setminus\cup_{i=1}^M \Omega_i$.

For every integer $i \in [0, M]$ choose $r_i \in \left(0, \min\{1, p_{\Omega_i}^-, q_{\Omega_i}^-\}\right)$ so that $\frac{n}{r_i} < J_Q^+ + \varepsilon$, and set $k_i = \frac{n}{r_i} - n - \alpha_{\Omega_i}^- + 2\varepsilon$ and $K_i = \alpha_{\Omega_i}^+ + 2\varepsilon$. Then m_Q is a $(k_i, K_i + 1)$-smooth molecule when Q is of level at most μ_0. Define $k_i(v,\mu) := K_i(v-\mu)_+ + k_i(\mu-v)_+$ and $\widetilde{s_{Q_\mu}} := s_{Q_\mu}|Q_\mu|^{-1/2}$. Finally, let $r \in (0, \min\{p^-, q^-\})$.

Note that the constants k_i and K_i have been chosen so that in each set Ω_i we may argue as in the previous lemma. Thus we get

$$\left|\varphi_v * m_{Q_\mu}(x)\right| \le c\, 2^{-k(v,\mu)} s_{Q_\mu}|Q_\mu|^{-1/2}\left(\eta_{v,2m}*\eta_{\mu,2m}*\chi_{Q_\mu}\right)(x).$$

From this we conclude that

$$\|f\|_{F_{p(\cdot)q(\cdot)}^{\alpha(\cdot)}} \le c\left\|\|2^{v\alpha}\varphi_v * f\|_{l_v^{q(x)}}\right\|_{L_x^{p(\cdot)}}$$

$$\le \left\|\left\|\sum_{\mu=0}^{\mu_0-1}\sum_{Q_\mu\in\mathcal{D}_\mu}|\widetilde{s_Q}|^r 2^{\mu\alpha(x)-k(v,\mu)}\eta_{v,2m}*\eta_{\mu,2m}*\chi_{Q_\mu}\right\|_{l_v^{q(x)}}\right\|_{L_x^{p(\cdot)}}$$

$$+ \sum_{i=0}^M\left\|\left\|\sum_{\mu\ge\mu_0-1}\sum_{Q_\mu\in\mathcal{D}_\mu}|\widetilde{s_Q}|^r 2^{\mu\alpha(x)-k(v,\mu)}\eta_{v,2m}*\eta_{\mu,2m}*\chi_{Q_\mu}\right\|_{l_v^{q(x)}}\right\|_{L_x^{p(\cdot)}(Q_i)}.$$

By the previous lemma, each term in the last sum is dominated by $\left\|\{s_Q\}_Q\right\|_{f_{p(\cdot)q(\cdot)}^{\alpha(\cdot)}}$, so we conclude that

$$\|f\|_{F_{p(\cdot)q(\cdot)}^{\alpha(\cdot)}} \le \left\|\left\|\sum_{\mu=0}^{\mu_0-1}\sum_{Q_\mu\in\mathcal{D}_\mu}|\widetilde{s_Q}|^r 2^{\mu\alpha(x)-k(v,\mu)}\eta_{v,2m}*\eta_{\mu,2m}*\chi_{Q_\mu}\right\|_{l_v^{q(x)}}\right\|_{L_x^{p(\cdot)}}$$

$$+ c(M+1)\left\|\{s_Q\}_Q\right\|_{f_{p(\cdot)q(\cdot)}^{\alpha(\cdot)}}.$$

It remains only to take care of the first term on the right hand side. In the current case we get instead see [311]

$$2^{\mu\alpha(x)r-rk(v,\mu)}\left(\eta_{v,2mr} * \eta_{\mu,2mr} * \chi_{Q_\mu}\right)^r \le c2^{\mu\alpha(x)r-2\varepsilon|v-\mu|+n(1-r)_++(\mu-v)_+}\eta_{v,2mr} * \eta_{\mu,2mr} * \chi_{Q_\mu},$$

since we have no control of k_2. However, since $\mu \le \mu_0$ and $v \ge 0$, the extra term satisfies $2^{n(1-r)_++(\mu-v)_+} \le 2^{n(1-r)_++\mu_0}$, so it is just a constant.

Theorem (3.1.25) [140]: Let the functions p, q and α be as in the Standing Assumptions and let $f \in F_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$. Then there exists a family of smooth atoms $\{a_Q\}_Q$ and a sequence of coefficients $\{t_Q\}_Q$ such that

$$f = \sum_{Q\in\mathcal{D}^+} t_Q\, a_Q \text{ in } \mathcal{S}' \text{ and } \left\|\{t_Q\}_Q\right\|_{f_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} \approx F_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}.$$

Moreover, the atoms can be chosen to satisfy conditions (i) and (ii) in Definition (3.1.4) for arbitrarily high, given order.

Proof: Define constants $K = n/\min\{1, p^-, q^-\} - n + \varepsilon$ and $L = \alpha^+ + 1 + \varepsilon$. We construct (K, L)-smooth atoms $\{a_Q\}_{Q\in\mathcal{D}^+}$ (see [106]). Note that we may use the constant indices construction, since the constants K and L give sufficient smoothness at every point. These atoms are also atoms for the space $F_{p(\cdot)q(\cdot)}^{\alpha(\cdot)}$.

Let $f \in F_{p(\cdot)q(\cdot)}^{\alpha(\cdot)}$. With functions as in Definition (3.1.2), we represent f as $f = \sum_{Q\in\mathcal{D}^+} t_Q\varphi_Q$, where $t_Q = \langle f, \psi_Q\rangle$. Next, we define

$$(t_r^*)_{Q_{vk}} = \left(\sum_{P\in\mathcal{D}_v} \frac{|t_P|^r}{(1 + 2^v|x_P - x_Q|)^m}\right)^{1/r},$$

for $Q = Q_{vk}, v \in \mathbb{N}^0$, and $k \in \mathbb{Z}^n$. For there numbers $(t_r^*)_Q$ we know that $f = \sum_Q (t_r^*)_Q\, a_Q$ where $\{a_Q\}_Q$ are atoms (molecules with support in $3Q$), by the construction of [106]. Technically, the atoms from the construction of [106] satisfy our inequalities for molecules only up to a constant (independent of the cube and scale).

For $v \in \mathbb{N}_0$ define $T_v := \sum_{Q\in\mathcal{D}_v} t_Q\chi_Q$. The definition of t_r^* is a discrete convolution of T_v with $\eta_{v,m}$. Changing to the continuous version, we see that $(t_r^*)_{Q_{vk}} \approx \left(\eta_{v,M} * (|T_v|^r)(x)\right)^{1/r}$ for $x \in Q_{vk}$. By this point-wise estimate we conclude that

$$\|t_r^*\|_{f_{p(\cdot)q(\cdot)}^{\alpha(\cdot)}} = \left\|\left\|\left\{2^{v\alpha(x)}\sum_{Q\in\mathcal{D}_v} |Q|^{-\frac{1}{2}}(t_r^*)_Q\chi_Q\right\}_v\right\|_{l_v^{q(x)}}\right\|_{L_x^{p(\cdot)}}$$

$$\approx \left\|\left\|\left\{2^{v\alpha(x)+v/2}\eta_{v,M} * (|T_v|^r)(x)\right\}_v\right\|_{l_v^{q(x)/r}}\right\|_{L_x^{p(\cdot)/r}}.$$

Next we use Lemma (3.1.21) and Theorem (3.1.20) to conclude that

$$\left\|\left\|\left\{2^{v\alpha(x)+v/2}\eta_{v,M} * (|T_v|^r)\right\}_v\right\|_{l_v^{\frac{q(x)}{r}}}\right\|_{L_x^{\frac{p(\cdot)}{r}}}^{1/r} \le c\left\|\left\|\left\{2^{v\alpha(x)+v/2}T_v\right\}_v\right\|_{l_v^{q(x)}}\right\|_{L_x^{p(\cdot)}}$$

$$= \left\|\left\|\left\{2^{v\alpha(x)}\sum_{Q\in\mathcal{D}_v} |Q|^{-\frac{1}{2}}t_Q\chi_Q\right\}_v\right\|_{l_v^{q(x)}}\right\|_{L_x^{p(\cdot)}}.$$

Since $f = \sum_{Q \in \mathcal{D}^+} t_Q \, \varphi_Q$, Theorem (3.1.22) implies that this is bounded by a constant times $\|f\|_{F^{\alpha(\cdot)}_{p(\cdot)q(\cdot)}}$.

This completes one direction. The other direction,

$$\|f\|_{F^{\alpha(\cdot)}_{p(\cdot)q(\cdot)}} \leq c \left\| \{s_Q\}_Q \right\|_{f^{\alpha(\cdot)}_{p(\cdot)q(\cdot)}},$$

follows from Theorem (3.1.24), since every family of atoms is in particular a family of molecules.

We consider a general embedding lemma. The local classical scale of Triebel-Lizorkin spaces is increasing in the primary index p and decreasing in the secondary index q. This is a direct consequence of the corresponding properties of L^p and l^q. In the variable exponent setting we have the following global result provided we assume that p stays constant at infinity:

Proposition (3.1.26) [140]: Let p_j, q_j, and α_j be as in the Standing Assumptions, $j = 0, 1$.

(i) If $p_0 \geq p_1$ and $(p_0)_\infty = (p_1)_\infty$, , then $L^{p_0(\cdot)} \hookrightarrow L^{p_1(\cdot)}$.

(ii) If $\quad \alpha_0 \geq \alpha_1, \ p_0 \geq p_1, (p_0)_\infty = (p_1)_\infty, \quad$ and $\quad q_0 \leq q_1, \quad$ then $\quad F^{\alpha_0(\cdot)}_{p_0(\cdot)q_0(\cdot)} \hookrightarrow F^{\alpha_1(\cdot)}_{p_1(\cdot)q_1(\cdot)}$.

Proof: In Lemma 2.2 of [97] it is shown that $L^{p_0(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p_1(\cdot)}(\mathbb{R}^n)$. if and only if $p_0 \geq p_1$ almost everywhere and $1 \in L^{r(\cdot)}(\mathbb{R}^n)$, where $\frac{1}{r(x)} := \frac{1}{p_1(x)} - \frac{1}{p_0(x)}$. Note that $r(x) = \infty$ if $p_1(x) = p_0(x)$. The condition $1 \in L^{r(\cdot)}(\mathbb{R}^n)$ means in this context (since r is usually unbounded) that $\lim_{\lambda \searrow 0} \varrho_{r(\cdot)}(\lambda) = 0$, where we use the convention that $\lambda^{r(x)} = 0$ if $r(x) = \infty$ and $\lambda \in (0, 1]$. Due to the assumptions on p_0 and p_1, we have $\frac{1}{r} \in C^{\log}, \frac{1}{r} \geq 0$, and $\frac{1}{r_\infty} = 0$. In particular, $\left| \frac{1}{r(x)} \right| \leq \frac{A}{\log(e+|x|)}$ for some $A > 0$ and all $x \in \mathbb{R}^n$. Thus,

$$\varrho_{r(\cdot)}\big(\exp(-2nA)\big) = \int_{\mathbb{R}^n} \exp\left(\frac{-2nA}{\left|\frac{1}{r(x)}\right|} \right) dx \leq \int_{\mathbb{R}^n} (e + |x|)^{-2n} \, dx < \infty.$$

The convexity of $\varrho_{r(\cdot)}$ implies that $\varrho_{r(\cdot)}\big(\exp(-2nA)\big) \to 0$ as $\lambda \searrow 0$ and (i) follows.

For (ii) we argue as follows. Since $\alpha_0 \geq \alpha_1$, we have $2^{v\alpha_0(x)} \leq 2^{v\alpha_1(x)}$ for all $v \geq 0$ and all $x \in \mathbb{R}^n$. Moreover, $q_0 \leq q_1$ implies $\|\cdot\|_{l^{q_1}} \leq \|\cdot\|_{l^{q_0}}$ and (i) implies $L^{p_0(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p_1(\cdot)}(\mathbb{R}^n)$. Now, the claim follows immediately from the definitions of the norms of $F^{\alpha_0(\cdot)}_{p_0(\cdot)q_0(\cdot)}$ and $F^{\alpha_1(\cdot)}_{p_1(\cdot)q_1(\cdot)}$.

Now we deal with trace theorems for Triebel-Lizorkin spaces. We write $\mathcal{D}^n$ and $\mathcal{D}^n_v$ for the families of dyadic cubes in $\mathcal{D}^+$ when we want to emphasize the dimension of the underlying space. The idea of the proof of the main trace theorem is to use the localization afforded by the atomic decomposition, and express a function as a sum of only those atoms with support intersecting the hyperplane $\mathbb{R}^{n-1} \subset \mathbb{R}^n$. In the classical case, this approach is due to Frazier and Jawerth [45].

There have been other approaches to deal with traces and an extension operator using wavelet decomposition instead of atomic decomposition, which utilizes compactly supported Daubechies wavelets, and thus, conveniently gives

trace theorems. However, for that one would need to define and establish properties of almost diagonal operators and almost diagonal matrices for the $F_{p(\cdot)q(\cdot)}^{\alpha(\cdot)}$, and $f_{p(\cdot)q(\cdot)}^{\alpha(\cdot)}$, spaces. The following lemma shows that it does not matter much for the norm if we shift around the mass a bit in the sequence space.

Lemma (3.1.27) [140]: Let p, q and α be as in the Standing Assumptions, $\varepsilon > 0$, and let $\{E_Q\}_Q$ be a collection of sets with $E_Q \subset 3Q$ and $|E_Q| \geq \varepsilon|Q|$ Then

$$
\left\| \{s_Q\}_Q \right\|_{f_{p(\cdot)q(\cdot)}^{\alpha(\cdot)}} \approx \left\| \left\| 2^{v\alpha(x)} \sum_{Q \in \mathcal{D}_v} |s_Q| |Q|^{-\frac{1}{2}} \chi_Q \right\|_{l_v^{q(x)}} \right\|_{L_x^{p(\cdot)}}
$$

for all $\{s_Q\}_Q \in f_{p(\cdot)q(\cdot)}^{\alpha(\cdot)}$.

Proof: We start by proving the inequality " $\leq$ ". Let $r \in (0, \min\{p^-, q^-\})$. We express the norm as

$$
\left\| \{s_Q\}_Q \right\|_{f_{p(\cdot)q(\cdot)}^{\alpha(\cdot)}} = \left\| \left\| 2^{v\alpha(x)r} \sum_{Q \in \mathcal{D}_v} |s_Q|^r |Q|^{-\frac{r}{2}} \chi_Q \right\|_{l_v^{q(x)}} \right\|_{L_x^{p(\cdot)/r}}^{1/r}
$$

since the sum has only one non-zero term. We use the estimate $\chi_Q \leq c\, \eta_{v,m} * \chi_{E_Q}$ for all $Q \in \mathcal{D}_v$. Now Lemma (3.1.21) implies that

$$
\left\| \{s_Q\}_Q \right\|_{f_{p(\cdot)q(\cdot)}^{\alpha(\cdot)}} = \left\| \left\| 2^{v\alpha(x)r} \sum_{Q \in \mathcal{D}_v} |s_Q|^r |Q|^{-\frac{r}{2}} \chi_Q \right\|_{l_v^{q(x)}} \right\|_{L_x^{p(\cdot)/r}}^{1/r}
$$

$$
\leq c \left\| \left\| \eta_v * \left(2^{v\alpha(\cdot)r} \sum_{Q \in \mathcal{D}_v} |s_Q|^r |Q|^{-\frac{r}{2}} \chi_{E_Q} \right) \right\|_{l_v^{q(x)/r}} \right\|_{L_x^{p(\cdot)/r}}^{1/r}.
$$

Then Theorem (3.1.20) completes the proof of the first direction:

$$
\left\| \{s_Q\}_Q \right\|_{f_{p(\cdot)q(\cdot)}^{\alpha(\cdot)}} \leq c \left\| \left\| 2^{v\alpha(x)r} \sum_{Q \in \mathcal{D}_v} |s_Q|^r |Q|^{-\frac{r}{2}} \chi_{E_Q} \right\|_{l_v^{q(x)/r}} \right\|_{L_x^{p(\cdot)/r}}^{1/r}
$$

$$
= c \left\| \left\| 2^{v\alpha(x)} \sum_{Q \in \mathcal{D}_v} |s_Q| |Q|^{-\frac{1}{2}} \chi_{E_Q} \right\|_{l_v^{q(x)/r}} \right\|_{L_x^{p(\cdot)/r}}^{1/r}
$$

The other direction follows by the same argument, since $\chi_{E_Q} \leq c\, \eta_v * \chi_Q$.

Next we use the embedding proposition to show that the trace space does not really depend on the secondary index of integration.

Lemma (3.1.28) [140]: Let p_1, p_2, q_1, α_1 and α_2 be as in the Standing Assumptions and let $q_2 \in (0, \infty)$. Assume that $\alpha_1 = \alpha_2$ and $p_1 = p_2$ in the upper or lower half space, and that $\alpha_1 \geq \alpha_2$ and $p_1 \leq p_2$. Then

$$
\operatorname{tr} F_{p_1(\cdot)q_1(\cdot)}^{\alpha_1(\cdot)}(\mathbb{R}^n) = \operatorname{tr} F_{p_2(\cdot)q_2}^{\alpha_2(\cdot)}(\mathbb{R}^n).
$$

Proof: We assume without loss of generality that $\alpha_1 = \alpha_2$ and $p_1 = p_2$ in the upper half space. We define $r_0 = \min\{q_2, q_1^-\}$ and $r_1 = \max\{q_2, q_1^+\}$. It follows from Proposition (3.1.26) that

$$\text{tr } F_{p_2(\cdot),r_0}^{\alpha_2(\cdot)} \hookrightarrow \text{tr } F_{p_1(\cdot),q_1(\cdot)}^{\alpha_2(\cdot)} \hookrightarrow \text{tr } F_{p_1(\cdot),r_1}^{\alpha_1(\cdot)}$$

and

$$\text{tr } F_{p_2(\cdot),r_0}^{\alpha_2(\cdot)} \hookrightarrow \text{tr } F_{p_2(\cdot),q_2}^{\alpha_2(\cdot)} \hookrightarrow \text{tr } F_{p_1(\cdot),r_1}^{\alpha_1(\cdot)}.$$

We complete the proof by showing that $\text{tr } F_{p_1(\cdot),r_1}^{\alpha_1(\cdot)} \hookrightarrow \text{tr } F_{p_2(\cdot),r_0}^{\alpha_2(\cdot)}$. Let $f \in F_{p_1(\cdot),r_1}^{\alpha_1(\cdot)}$. According to Theorem (3.1.25) we have the representation

$$f = \sum_{Q \in \mathcal{D}^+} t_Q a_Q \quad \text{with} \quad \left\| \{t_Q\}_Q \right\|_{f_{p_1(\cdot),r_1}^{\alpha_1(\cdot)}} \leq c \|f\|_{F_{p_1(\cdot),r_1}^{\alpha_1(\cdot)}}$$

where the a_Q are smooth atoms for $F_{p_1(\cdot),r_1}^{\alpha_1(\cdot)}$ satisfying (i) and (ii) up to high order. Then they are also smooth atoms for $F_{p_2(\cdot),r_0}^{\alpha_2(\cdot)}$.

Let $A := \{Q \in \mathcal{D}^+ : 3\bar{Q} \cap \{x_n = 0\} \neq \emptyset\}$. If $Q \in A$ is contained in the closed upper half space, then we write $Q \in A^+$, otherwise $Q \in A^-$. We set $\tilde{t}_Q = t_Q$ when $Q \in A$, and $\tilde{t}_Q = 0$ otherwise. Then we define $\tilde{f} = \sum_{Q \in \mathcal{D}^+} \tilde{t}_Q \widetilde{a_Q}$. It is clear that $\text{tr } f = \text{tr } \tilde{f}$, since all the atoms of f whose support intersects $\mathbb{R}^{n-1}$ are included in $\tilde{f}$. For $Q \in A^+$, we define

$$E_Q = \left\{ x \in Q : \frac{3}{4} \ell(Q) \leq x_n \leq \ell(Q) \right\}$$

for $Q \in A^-$ we define

$$E_Q = \left\{ (\acute{x}, x_n) \in \mathbb{R}^n : (\acute{x}, -x_n) \in Q, \quad \frac{1}{2} \ell(Q) \leq x_n \leq \frac{3}{4} \ell(Q) \right\}$$

for all other cubes $E_Q = \emptyset$. If $Q \in A$, then $|Q| = 4|E_Q|$; moreover, $\{E_Q\}_Q$ covers each point at most three times.

By Theorem (3.1.24) and Lemma (3.1.27) we conclude that

$$\|\tilde{f}\|_{F_{p_2(\cdot),r_0}^{\alpha_2(\cdot)}} \leq c \left\| \{\tilde{t}_Q\}_Q \right\|_{f_{p_2(\cdot),r_0}^{\alpha_2(\cdot)}} \leq c \left\| \left\| 2^{\nu \alpha_2(x)} \sum_{Q \in \mathcal{D}_\nu} |t_Q| |Q|^{-\frac{r}{2}} \chi_{E_Q} \right\|_{l_\nu^{r_0}} \right\|_{L_x^{p_2(\cdot)}}.$$

The inner norm consists of at most three non-zero members for each $x \in \mathbb{R}^n$. Therefore, we can replace r_0 by r_1. Moreover, each E_Q is supported in the upper half space, where α_2 and α_1, and p_2 and p_1 agree. Thus,

$$\|\tilde{f}\|_{F_{p_2(\cdot),r_0}^{\alpha_2(\cdot)}} \leq c \left\| \left\| 2^{\nu \alpha_1(x)} \sum_{Q \in \mathcal{D}_\nu} |t_Q| |Q|^{-\frac{r}{2}} \chi_{E_Q} \right\|_{l_\nu^{r_1}} \right\|_{L_x^{p_1(\cdot)}}.$$

The right hand side is bounded by $\|f\|_{F_{p_1(\cdot),r_1}^{\alpha_1(\cdot)}}$ according to Theorem (3.1.24) and Lemma (3.1.27) Therefore, $\text{tr } F_{p_1(\cdot),r_1}^{\alpha_1(\cdot)} \hookrightarrow \text{tr } F_{p_2(\cdot),r_0}^{\alpha_2(\cdot)}$, and the claim follows.

For the next proposition we recall the common notation $F_{p(\cdot)}^{\alpha(\cdot)} = F_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$ for the Triebel-Lizorkin space with identical primary and secondary indices of integrability. The next result shows that the trace space depends only on the values of the indices at the boundary, as should be expected.

Proposition (3.1.29) [140]: Let p_1, p_2, q_1, α_1, and α_2 be as in the Standing Assumptions. Assume that $\alpha_1(x) = \alpha_2(x)$ and $p_1(x) = p_2(x)$ for all $x \in \mathbb{R}^{n-1} \times \{0\}$. Then

$$\operatorname{tr} F^{\alpha_1(\cdot)}_{p_1(\cdot), q_1(\cdot)}(\mathbb{R}^n) = \operatorname{tr} F^{\alpha_2(\cdot)}_{p_2(\cdot)}(\mathbb{R}^n).$$

Proof: By Lemma (3.1.28) we conclude that $\operatorname{tr} F^{\alpha_1(\cdot)}_{p_1(\cdot), q_1(\cdot)} = \operatorname{tr} F^{\alpha_1(\cdot)}_{p_1(\cdot)}$. Therefore, we can assume that $q_1 = p_1$.

We define $\tilde{\alpha}_j$ to equal α_j on the lower half space and $\min\{\alpha_1, \alpha_2\}$ on the upper half space and let $\tilde{\alpha} = \min\{\alpha_1, \alpha_2\}$. Similarly, we define $\tilde{p}_j$ and $\tilde{p}$. Applying Lemma (3.1.28) four times in the following chain

$$\operatorname{tr} F^{\alpha_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n) = \operatorname{tr} F^{\tilde{\alpha}_1(\cdot)}_{\tilde{p}_1(\cdot)}(\mathbb{R}^n) = \operatorname{tr} F^{\tilde{\alpha}(\cdot)}_{\tilde{p}(\cdot)}(\mathbb{R}^n) = \operatorname{tr} F^{\tilde{\alpha}_2(\cdot)}_{\tilde{p}_2(\cdot)}(\mathbb{R}^n) = \operatorname{tr} F^{\alpha_2(\cdot)}_{p_2(\cdot)}(\mathbb{R}^n),$$

gives the result.

Theorem (3.1.30) [140]: Let the functions p, q and α be as in the Standing Assumptions. If

$$\alpha - 1/p - (n-1)(1/p - 1)_+ > 0, \quad \text{then} \quad \operatorname{tr} F^{\alpha(\cdot)}_{p(\cdot), q(\cdot)}(\mathbb{R}^n) = F^{\alpha(\cdot)-1/p(\cdot)}_{p(\cdot)}(\mathbb{R}^{n-1}).$$

Proof: By Proposition (3.1.29) it suffices to consider the case $q = p$ with p and α independent of the n-th coordinate for $|x_n| \leq 2$. Let $f \in F^{\alpha(\cdot)}_{p(\cdot)}$ with $\|f\|_{F^{\alpha(\cdot)}_{p(\cdot)}} \leq 1$ and let $f = \sum s_Q a_Q$ be an atomic decomposition as in Theorem (3.1.25).

We denote by π the orthogonal projection of $\mathbb{R}^n$ onto $\mathbb{R}^{n-1}$, and $(\acute{x}, x_n) \in \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$. For $J \in \mathcal{D}^{n-1}_\mu$, a dyadic cube in $\mathbb{R}^{n-1}$, we define $Q_i(J) \in \mathcal{D}^n_\mu, i = 1, \ldots, 6 \cdot 5^{n-1}$, to be all the dyadic cubes satisfying $J \subset 3Q_i$. We define $t_J = |Q_1(J)|^{-\frac{1}{2n}} \sum_i |s_{Q_i(J)}|$ and $h_J(\acute{x}) = t_J^{-1} \sum_i s_{Q_i Q} a_{Q_i}$. By $Q_+(J)$ we denote the cube $Q_i(J)$ which has J as a face (i.e. $J \subset \partial Q_+(J)$).
Then we have

$$\operatorname{tr} f(\acute{x}) = \sum_\mu \sum_{J \in \mathcal{D}^{n-1}_\mu} t_J h_J(x')$$

with convergence in S'. The condition $\alpha - 1/p - (n-1)(1/p - 1)_+ > 0$ implies that molecules in $F^{\alpha(\cdot)-1/p(\cdot)}_{p(\cdot)}(\mathbb{R}^{n-1})$ are not required to satisfy any moment conditions. Therefore, h_J is a family of smooth molecules for this space. Consequently, by Theorem (3.1.24), we find that

$$\|\operatorname{tr} f\|_{F^{\alpha(\cdot)-1/p(\cdot)}_{p(\cdot)}(\mathbb{R}^{n-1})} \leq c \left\|\{t_J\}_J\right\|_{f^{\alpha(\cdot)-1/p(\cdot)}_{p(\cdot)}(\mathbb{R}^{n-1})}.$$

Thus, we conclude the proof by showing that the right hand side is bounded by a constant. Since the norm is bounded if and only if the modular is bounded, we see that it suffices to show that

$$\int_{\mathbb{R}^{n-1}} \sum_\mu \sum_{J \in \mathcal{D}^{n-1}_\mu} \left(2^{\mu\left(\alpha(x',0) - \frac{1}{p(x',0)}\right)} |t_J| |J|^{-1/2} \chi_J(x', 0) \right) dx'$$

$$= \sum_\mu \sum_{J \in \mathcal{D}^{n-1}_\mu} 2^{-\mu} \int_J \left(2^{\mu\alpha(x',0)} |t_J| |J|^{-1/2} \right)^{p(\acute{x},0)} dx'$$

is bounded. For the integral we calculate

$$2^{-\mu}\int_J\left(2^{\mu\alpha(x',0)}|t_J||J|^{-1/2}\right)^{p(x',0)}dx' = 2^{-\mu}\int_{\mathcal{Q}_+(J)}\left(2^{\mu\alpha(x',0)}|t_J||J|^{-1/2}\right)^{p(x',0)}d(x',x_n)$$

$$\leq c\int_{\mathcal{Q}_+(J)}\left(2^{\mu\alpha(x)}\sum_i|\mathcal{Q}_i||\mathcal{Q}|^{-\frac{1}{2n}-\frac{n-1}{2n}}\right)^{p(x)}dx$$

$$= c\int_{\mathcal{Q}_+(J)}\left(2^{\mu\alpha(x)}\sum_i|s_{\mathcal{Q}_i}||\mathcal{Q}|^{-1/2}\right)^{p(x)}dx$$

Hence, we obtain

$$\sum_\mu\sum_{J\in\mathcal{D}_\mu^{n-1}}2^{-\mu}\int_J\left(2^{\mu\alpha(x',0)}|t_J||J|^{-1/2}\right)^{p(x',0)}dx'$$

$$\leq c\sum_\mu\sum_{\mathcal{Q}\in\mathcal{D}_\mu^n}2^{-\mu}\int_{\mathcal{Q}}\left(2^{\mu\alpha(x)}\sum_i|s_{\mathcal{Q}_i}||\mathcal{Q}|^{-1/2}\right)^{p(x)}dx$$

$$\leq c\int_{\mathbb{R}^n}\sum_v\sum_{\mathcal{Q}\in\mathcal{D}_\mu^n}\left(2^{v\alpha(x)}|s_{\mathcal{Q}}||\mathcal{Q}|^{-1/2}\chi_{\mathcal{Q}}(x)\right)^{p(x)}dx$$

Where were again swapped the integral and the sums. Since $\|f\|_{F_{p(\cdot)}^{\alpha(\cdot)}}\leq 1$, the right hand side quantity is bounded, and we are done.

Section (3.2): Besov Spaces

Spaces of variable integrability, also known as variable exponent function spaces, can be traced back to 1931 and W. Orlicz [124], but the modern development started with [113] of Kováčik and Rákosník in 1991. Corresponding PDE with non-standard growth have been studied since the same time. For an overview we refer to the surveys [101, 128, 132, 148] and the monograph [145]. Apart from interesting theoretical considerations, the motivation to study such function spaces comes from applications to fluid dynamics [86, 130, 141], image processing [94], PDE and the calculus of variation [87, 103, 111, 147, 151, 155, 159].

We complete the picture of the variable exponent Lebesgue and Sobolev spaces, Almeida and Samko [88] and Gurka, Harjulehto and Nekvinda [109] introduced variable exponent Bessel potential spaces $\mathcal{L}^{\alpha p(\cdot)}$ with constant $\alpha\in\mathbb{R}$. As in the classical case, this space coincides with the Lebesgue-Sobolev space for integer α. There was taken a step further by Xu [137, 138, 159], who considered Besov $B_{p(\cdot),q}^\alpha$ and Triebel-Lizorkin $F_{p(\cdot),q}^\alpha$ spaces with variable p, but fixed q and α.

Along a different line of inquiry, Leopold [117, 119, 120] studied pseudo-differential operators with symbols of the type $\langle\xi^{m(x)}\rangle$, and defined related function spaces of Besov-type with variable smoothness, $B_{p,p}^{m(\cdot)}$. Beauzamy [144] had studied similar ψDEs already in the beginning of the 70s. Function spaces of variable smoothness have recently been studied by Besov [90, 91, 92]: he generalized Leopold's work by considering both Triebel-Lizorkin spaces $F_{p(\cdot),q}^\alpha$ and Besov spaces $B_{p(\cdot),q}^\alpha$ in $\mathbb{R}^n$. By way of application, Schneider and Schwab [135] used $B_{2,2}^{m(\cdot)}(\mathbb{R})$ in the analysis of certain Black-Scholes equations. For further considerations of ψDEs, see Hoh [149].

Integrating the above mentioned spaces into a single larger scale promises similar gains and simplifications as were seen in the constant exponent case in the 60s and 70s with the advent of the full Besov and Triebel-Lizorkin scales. Most of the advantages of

unification do not occur with only one index variable: for instance, traces or Sobolev embeddings cannot be covered in this case, since they involve an interaction between integrability and smoothness. To tackle this, Diening, Hästö and Roudenko [140] introduced Triebel-Lizorkin spaces with all three indices variable, $F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ and showed that they behaved nicely with respect to trace. Subsequently, Vybíral [157] proved Sobolev (Jawerth) type embeddings in these spaces; they were also studied by Kempka [150]. These studies were all restricted to bounded exponents p and q.

Vybíral [157] and Kempka [150] also considered Besov spaces $B^{\alpha(\cdot)}_{p(\cdot),q}$ note that only the case of constant q was included. This is quite natural, since the norm in the Besov space is usually defined via the iterated space $\ell^q(L^p)$ so that the space integration in L^p is done first, followed by the sum over frequency scales in ℓ^q. Therefore, it is not obvious how q could depend on x, which has already been integrated out. It is the purpose to propose a method making this dependence possible and thus completing the unification process in the variable integrability-smoothness case by introducing the Besov space $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ with all three indices variable.

The space includes the previously mentioned spaces of Besov-type, as well as the Hölder-Zygmund space $C^{\alpha(\cdot)}$. As in the constant exponent case, it is possible to consider unbounded exponents p and q in the Besov space case, while for the Triebel-Lizorkin space one needs p to be bounded. Another advantage of the Besov space for constant exponent is its simplicity compared to the Triebel-Lizorkin space. This is not true for the generalization with variable q. We will see that working in the Besov space is relatively simple once some basic tools have been established for dealing in the ''iterated'' space $\ell^{q(\cdot)}(L^{p(\cdot)})$.

We then define the Besov space $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ and give several basic properties establishing the soundness of our definition. We show elementary embeddings between Besov and Triebel-Lizorkin spaces, as well as Sobolev embeddings in the Besov scale. We show that our scale includes the variable order Hölder-Zygmund space as a special case: $B^{\alpha(\cdot)}_{\infty,\infty} = C^{\alpha(\cdot)}$ for $0 < \alpha < 1$. We give an alternative characterization of the Besov space by means of approximations by analytic functions.

So far, complex interpolation has been considered in the variable exponent in [101, 145]. Real interpolation, however, is more difficult in this setting. We have, for constant exponents,

$$(L^{p_0}, L^{p_1})_{\theta,q} = L^{p_\theta,q}$$

where $1/p_\theta := \theta/p_0 + (1-\theta)/p_1$ and $L^{p_\theta,q}$ is the Lorenz space. To obtain interpolation of Lebesgue spaces one simply chooses $q = p_\theta$. It seems that there are no major difficulties in letting p_0 and p_1 be variable here, i.e.

$$\left(L^{p_0(\cdot)}, L^{p_1(\cdot)}\right)_{\theta,q} = L^{p_\theta(\cdot),q}$$

where p_θ is defined point-wise by the same formula as before. However, this time we do not obtain an interpolation result in Lebesgue spaces, since we cannot set the constant q equal to the function p_θ. In fact, the role of q in the real interpolation method is quite similar to the role of q in the Besov space $B^\alpha_{p,q}$. Therefore, we hope that the approach introduced for Besov spaces with variable q will also allow us to generalize real interpolation properly to the variable exponent context. Another interesting challenge is to extend extrapolation [95] to the setting of Besov spaces.

The expression $f \approx g$ means that $\frac{1}{c}g \leq f \leq cg$ for some suitably independent constant c. By χ_A we denote the characteristic function of $A \subset \mathbb{R}^n$. By $\operatorname{supp} f$ we denote the support of the function f, i.e. the closure of its zero set. The notation $X \hookrightarrow Y$ denotes continuous embeddings from X to Y.

The spaces studied fit into the framework of so-called semimodular spaces. For an exposition of these concepts see [145, 151]. We recall the following definition:

Definition (3.2.1) [160]: Let X be a vector space over $\mathbb{R}$ or $\mathbb{C}$. A function $\varrho : X \to [0, \infty]$ is called a semimodular on X if the following properties hold:

(i) $\varrho(0) = 0$.

(ii) $\varrho(\lambda f) = \varrho(f)$ for all $f \in X$ and $|\lambda| = 1$.

(iii) $\varrho(\lambda f) = 0$ for all $|\lambda| > 1$ implies $f = 0$.

(iv) $\lambda \mapsto \varrho(\lambda f)$ is left-continuous on $[0, \infty)$ for every $f \in X$.

A semimodular ϱ is called a modular if

(v) $\varrho(f) = 0$ implies $f = 0$.

A semimodular ϱ is called continuous if

(vi) For every $f \in X$ the mapping $\lambda \mapsto \varrho(\lambda f)$ is continuous on $[0, \infty)$.

A semimodular ϱ can be additionally qualified by the term (quasi)convex. This means, as usual, that

$$\varrho(\theta f + (1 - \theta)g) \leq A[\theta\varrho(f) + (1 - \theta)\varrho(g)]$$

for all $f, g \in X$; here $A = 1$ in the convex case, and $A \in [1, \infty)$ in the quasiconvex case.

We obtain a normed space in a standard way:

Definition (3.2.2) [160]: If ϱ is a (semi)modular on X, then

$$X_\varrho := \{x \in X : \exists \lambda > 0, \varrho(\lambda x) < \infty\}$$

is called a (semi)modular space.

Theorem (3.2.3) [160]: Let ϱ be a (quasi)convex semimodular on X. Then X_ϱ is a (quasi)normed space with the Luxemburg (quasi)norm given by

$$\|x\|_\varrho := \inf\left\{\lambda > 0 : \varrho\left(\frac{1}{\lambda}x\right) \leq 1\right\}.$$

For simplicity we will refer to semimodulars as modulars except when special clarity is needed; similarly, we later drop the word "quasi".

For dealing with the somewhat complicated definition of a norm is the following relationship which follows from the definition and left-continuity: $\varrho(f) \leq 1$ if and only if $\|f\|_\varrho \leq 1$.

The variable exponents that we consider are always measurable functions on $\mathbb{R}^n$ with range $(c, \infty]$ for some $c > 0$. We denote the set of such functions by $\mathcal{P}_0$. The subset of variable exponents with range $[1, \infty]$ is denoted by $\mathcal{P}$. For $A \subset \mathbb{R}^n$ and $p \in \mathcal{P}_0$ we denote $p_A^+ = \operatorname{ess\,sup}_A p(x)$ and $p_A^- = \operatorname{ess\,inf}_A p(x)$; we abbreviate $p^+ = p_{\mathbb{R}^n}^+$ and $p^- = p_{\mathbb{R}^n}^-$.

The function φ_p is defined as follows:

$$\varphi_p(t) = \begin{cases} t^p & \text{if } p \in (0, \infty), \\ 0 & \text{if } p = \infty \text{ and } t \leq 1, \\ \infty & \text{if } p = \infty \text{ and } t > 1. \end{cases}$$

The convention $1^\infty = 0$ is adopted in order that φ_p be left-continuous. In what follows we write t^p instead of $\varphi_p(t)$, with this convention implied. The variable exponent modular is defined by

$$\varrho_{p(\cdot)}(f) := \int\limits_{\mathbb{R}^n} \varphi_p(|f(x)|)dx.$$

The variable exponent Lebesgue space $L^{p(\cdot)}$ and its norm $\|f\|_{p(\cdot)}$ are defined by the modular as explained. The variable exponent Sobolev space $W^{k,p(\cdot)}$ is the subspace of $L^{p(\cdot)}$ consisting of functions f whose distributional k-th order derivative exists and satisfies $|D^k f| \in L^{p(\cdot)}$ with norm

$$\|f\|_{W^{k,p(\cdot)}} = \|f\|_{p(\cdot)} + \|D^k f\|_{p(\cdot)}.$$

We say that $g : \mathbb{R}^n \to \mathbb{R}$ is locally log-Hölder continuous, abbreviated $g \in C_{loc}^{log}$, if there exists $c_{log} > 0$ such that

$$|g(x) - g(y)| \le \frac{c_{log}}{\log(e + 1/|x - y|)}$$

for all $x, y \in \mathbb{R}^n$. We say that g is globally log-Hölder continuous, abbreviated $g \in C^{log}$, if it is locally log-Hölder continuous and there exists $g_\infty \in \mathbb{R}^n$ such that

$$|g(x) - g_\infty| \le \frac{c_{log}}{\log(e + |x|)}$$

for all $x \in \mathbb{R}^n$. The notation $\mathcal{P}^{log}$ is used for those variable exponents $p \in \mathcal{P}$ with $\frac{1}{p} \in C^{log}$. The class $\mathcal{P}_0^{log}$ is defined analogously. If $p \in \mathcal{P}^{log}$, then convolution with a radially decreasing L^1-function is bounded on $L^{p(\cdot)}$:

$$\|\varphi * f\|_{p(\cdot)} \le c\|\varphi\|_1 \|f\|_{p(\cdot)}.$$

Now we introduce a generalization of the iterated function space $\ell^q(L^{p(\cdot)})$ for the case of variable q, which allows us to define Besov spaces with variable q. We give a general but quite strange looking definition for the mixed Lebesgue-sequence space modular. This is not strictly an iterated function space indeed, it cannot be, since then there would be no space variable left in the outer function space. To motivate our definition, we show that it has several sensible properties and that it concurs with the iterated space when q is constant (Proposition (3.2.5)). Then we show that our modular in fact is a semimodular in the sense defined and conclude that it defines a normed space.

Definition (3.2.4) [160]: Let $p, q \in \mathcal{P}_0$. The mixed Lebesgue-sequence space $\ell^q(L^{p(\cdot)})$ is defined on sequences of $L^{p(\cdot)}$-functions by the modular

$$\varrho_{\ell^q(L^{p(\cdot)})}((f_v)_v) := \sum_v \inf\left\{\lambda_v > 0 \,\Big|\, \varrho_{\ell^q(L^{p(\cdot)})}\left(f_v / \lambda_v^{1/q(\cdot)}\right) \le 1\right\}.$$

Here we use the convention $\lambda^{1/\infty} = 1$. The norm is defined from this as usual:

$$\|(f_v)_v\|_{\ell^q(L^{p(\cdot)})} := \inf\left\{\mu > 0 \,\Big|\, \varrho_{\ell^q(L^{p(\cdot)})}\left(\frac{1}{\mu}(f_v)_v\right) \le 1\right\}.$$

If $q^+ < \infty$, then

$$\inf\{\lambda > 0 \,|\, \varrho_{p(\cdot)}(f/\lambda^{1/q(\cdot)}) \le 1\} = \||f|^{q(\cdot)}\|_{p(\cdot)/q(\cdot)}.$$

Since the right-hand side expression is much simpler, we use this notation to stand for the left-hand side even when $q^+ = \infty$. We often use the notation

$$\varrho_{\ell^q(L^{p(\cdot)})}((f_v)_v) = \sum_v \||f_v|^{q(\cdot)}\|_{p(\cdot)/q(\cdot)}$$

for the modular.

Proposition (3.2.5) [160]: If $q \in (0, \infty)$ is constant, then

$$\|(f_v)_v\|_{\ell^q(L^{p(\cdot)})} = \|\|f_v\|_{p(\cdot)}\|_{\ell^q}.$$

Proof: Suppose first that $q \in (0, \infty)$. Since q is constant,

$$\left\||f_v|^q\right\|_{p(\cdot)/q} = \|f_v\|_{p(\cdot)}^q$$

and thus

$$\varrho_{\ell^q(L^{p(\cdot)})}((f_v)_v) = \sum_v \|f_v\|_{p(\cdot)}^q = \left\|\|f_v\|_{p(\cdot)}\right\|_{\ell^q}^q$$

from which the claim follows.

In the case $q = \infty$, we find

$$\varrho_{\ell^\infty(L^{p(\cdot)})}((f_v)_v) = \sum_v \inf\{\lambda_v > 0 | \varrho_{p(\cdot)}(f_v/\lambda_v^0) \leq 1\}.$$

Here the infimum is zero, unless at least one of the sets over which it is taken is empty, in which case it is infinite. Therefore, the inequality in the definition of the norm,

$$\|(f_v)_v\|_{\ell^\infty(L^{p(\cdot)})} = \inf\left\{\mu > 0 \,\middle|\, \varrho_{\ell^\infty(L^{p(\cdot)})}\left(\frac{(f_v)_v}{\mu}\right) \leq 1\right\}$$

holds if and only if μ is such that $\varrho_{p(\cdot)}(f_v/\mu) \leq 1$ for every v, which means that

$$\inf \mu = \sup\{\|f_v\|_{p(\cdot)}\} = \left\|\|f_v\|_{p(\cdot)}\right\|_{\ell^\infty}.$$

Proposition (3.2.6) [160]: Let $p, q \in \mathcal{P}_0$. Then $\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ is a semimodular. Additionally,
(i) it is a modular if $p^+ < \infty$; and
(ii) it is continuous if $p^+, q^+ < \infty$.

Proof: We need to check properties (i)-(iv) of Definition (3.2.1) and properties (v)-(vi) under the appropriate additional assumptions. Properties (i) and (ii) are clear. To prove (iii), we suppose that

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}(\lambda(f_v)_v) = 0$$

for all $\lambda > 0$. Clearly, $\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}(0, \cdots, 0, \lambda f_{v_0}, 0 \cdots) \leq \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}(\lambda(f_v)_v) = 0$. Thus $\|f_{v_0}\|_{p(\cdot)} = 0$ see [323], and so $f = 0$. If p is bounded, then the same argument implies (v).

To prove the left-continuity we start by noting that $\mu \mapsto \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}(\mu(f_v)_v)$ in nondecreasing. By relabeling the function if necessary, we see that it suffices to show that

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}(\mu(f_v)_v) \nearrow \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v)$$

$\mu \nearrow 1$. We assume that

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) < \infty$$

the other case is similar. We fix $\varepsilon > 0$ and choose $N > 0$ such that

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) - \varepsilon < \sum_{v=0}^N \inf\left\{\lambda_v > 0 \,\middle|\, \varrho_{p(\cdot)}\left(f_v/\lambda_v^{1/q(\cdot)}\right) \leq 1\right\}.$$

By the left-continuity of $\mu \mapsto \varrho_{p(\cdot)}(\mu f)$, we then choose $\mu^* < 1$ such that

$$\sum_{v=0}^N \inf\left\{\lambda_v > 0 \,\middle|\, \varrho_{p(\cdot)}\left(f_v/\lambda_v^{1/q(\cdot)}\right) \leq 1\right\} - \varepsilon < \sum_{v=0}^N \inf\left\{\lambda_v > 0 \,\middle|\, \varrho_{p(\cdot)}\left(\mu f_v/\lambda_v^{1/q(\cdot)}\right) \leq 1\right\}$$

for all $\mu \in (\mu^*, 1)$. Then $\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) < \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}(\mu(f_v)_v) + 2\varepsilon$ in the same range, which proves (iv). When $q^+ < \infty$, a similar argument reduces (vi) to the continuity of $\varrho_{p(\cdot)}$, which holds when $p^+ < \infty$.

We would have shown that the modular is quasiconvex as part of the previous theorem. Then Theorem (3.2.3) would immediately imply that the modular in $\ell^{q(\cdot)}(L^{p(\cdot)})$ defines a quasinorm. Unfortunately, we do not know whether the modular is quasiconvex when $q^+ = \infty$. Therefore, we prove the quasiconvexity of the norm directly; we do this in two steps, beginning with the true convexity. Notice that our assumption when q is non-constant is not as expected. We also do not know if it is necessary.

Theorem (3.2.7) [160]: Let $p, q \in \mathcal{P}$. If either $1/p + 1/q \leq 1$ point-wise, or q is a constant, then $\|\cdot\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ is a norm.

Proof: Theorem (3.2.3) implies all the other claims, except the convexity. If $p \in \mathcal{P}$ and $q \in [1, \infty]$ is a constant, then by Proposition (3.2.3), the convexity follows directly from the convexity of the modulars in ℓ^q and $L^{p(\cdot)}$.

Thus it remains only to consider $\frac{1}{p} + \frac{1}{q} \leq 1$ and to show that

$$\|(f_v)_v + (g_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leq \|(f_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} + \|(g_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}.$$

Let $\lambda > \|(f_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ and $\mu > \|(g_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$. Then the claim follows from left-continuity if we show that

$$\left\| \frac{(f_v)_v + (g_v)_v}{\lambda + \mu} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leq 1.$$

Moving to the modular, we get the equivalent condition

$$\sum_v \left\| \left| \frac{f_v + g_v}{\lambda + \mu} \right|^{q(\cdot)} \right\|_{p(\cdot)/q(\cdot)} \leq 1$$

with our usual convention regarding the case $p/q = 0$. Since

$$\sum_v \left\| \left| \frac{f_v}{\lambda} \right|^{q(\cdot)} \right\|_{p(\cdot)/q(\cdot)} \leq 1 \quad \text{and} \quad \sum_v \left\| \left| \frac{g_v}{\mu} \right|^{q(\cdot)} \right\|_{p(\cdot)/q(\cdot)} \leq 1$$

the claim follows provided we show that

$$\left\| \left| \frac{f_v + g_v}{\lambda + \mu} \right|^{q(\cdot)} \right\|_{p(\cdot)/q(\cdot)} \leq \frac{\lambda}{\lambda + \mu} \left\| \left| \frac{f_v}{\lambda} \right|^{q(\cdot)} \right\|_{p(\cdot)/q(\cdot)} + \frac{\mu}{\lambda + \mu} \left\| \left| \frac{g_v}{\mu} \right|^{q(\cdot)} \right\|_{p(\cdot)/q(\cdot)}$$

for every v. Fix now one v. Denote the norms on the right-hand side of the previous inequality by σ and τ. Then what we need to show reads

$$\int_{\mathbb{R}^n} \left| \frac{f_v + g_v}{\lambda + \mu} \right|^{p(x)} \left(\frac{\lambda\sigma + \mu\tau}{\lambda + \mu} \right)^{-p(x)/q(x)} dx \leq 1. \tag{5}$$

We use Hölder's inequality (with two-point atomic measure and weights (λ, μ)) as follows:

$$|f_v| + |g_v| = \lambda\sigma^{1/q(x)} \frac{|f_v|/\lambda}{\sigma^{1/q(x)}} + \mu\tau^{1/q(x)} \frac{|g_v|/\mu}{\tau^{1/q(x)}}$$

$$\leq (\lambda + \mu)^{1 - 1/p(x) - 1/q(x)} (\lambda\sigma + \mu\tau)^{1/q(x)} \left(\lambda \left(\frac{|f_v|/\lambda}{\sigma^{1/q(x)}} \right)^{p(x)} + \mu \left(\frac{|g_v|/\mu}{\tau^{1/q(x)}} \right)^{p(x)} \right)^{1/p(x)}.$$

With this, we obtain

$$\left| \frac{f_v + g_v}{\lambda + \mu} \right|^{p(x)} \left(\frac{\lambda\sigma + \mu\tau}{\lambda + \mu} \right)^{-p(x)/q(x)} \leq \frac{\lambda}{\lambda + \mu} \left(\frac{|f_v|/\lambda}{\sigma^{1/q(x)}} \right)^{p(x)} + \frac{\mu}{\lambda + \mu} \left(\frac{|g_v|/\mu}{\tau^{1/q(x)}} \right)^{p(x)}.$$

Integrating the inequality over $\mathbb{R}^n$ and taking into account that σ is the norm of f_v/λ and τ the norm of g_v/μ gives us (5), which completes the proof.

Theorem (3.2.8) [160]: If $p, q \in \mathcal{P}_0$, then $\|\cdot\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ is a quasinorm on $\ell^{q(\cdot)}(L^{p(\cdot)})$.

Proof: By Theorem (3.2.3), we only need to cosinder quasiconvexity. Let $r \in (0, \frac{1}{2}\min\{p^-, q^-, 2\}]$ and define $\tilde{p} = p/r$ and $\tilde{q} = q/r$. Then clearly $1/p + 1/q \leq 1$. Thus we obtain by the previous theorem that

$$\|(f_v)_v + (g_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = \||(f_v)_v + (g_v)_v|^r\|_{\ell^{\tilde{q}(\cdot)}(L^{\tilde{p}(\cdot)})}^{\frac{1}{r}} \leq \|(|f_v|^r)_v + (|g_v|^r)_v\|_{\ell^{\tilde{q}(\cdot)}(L^{\tilde{p}(\cdot)})}^{\frac{1}{r}}$$

$$\leq \left(\|(|f_v|^r)_v\|_{\ell^{\tilde{q}(\cdot)}(L^{\tilde{p}(\cdot)})} + \|(|g_v|^r)_v\|_{\ell^{\tilde{q}(\cdot)}(L^{\tilde{p}(\cdot)})}\right)^{\frac{1}{r}}$$

$$= \left(\|(f_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}^r + \|(g_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}^r\right)^{\frac{1}{r}}$$

$$\leq 2^{\frac{1}{r}-1}\left(\|(f_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} + \|(g_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}\right).$$

This completes the proof.

The condition $p, q \geq 1$ is not sufficient to guarantee that the modular $\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ be convex! Although it is not true that the modular $\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ is never convex when q is non-constant (see, [160]).

Since q is constant when f is non-zero, we conclude by Proposition (3.2.5) that
$$\varrho_{\ell^{q(\cdot)}(L^p)}((f_v)_v) = \varrho_{\ell^{q_1}(L^p(Q_1))}((f_v)_v) = \left\|a^{1/q_1}\chi_{Q_1}\right\|_p^{q_1} = a.$$

Similarly, $\varrho_{\ell^{q(\cdot)}(L^p)}((g_v)_v) = b$. Then we consider the modular of $\frac{1}{2}(f + g)$:
$$\varrho_{\ell^{q(\cdot)}(L^p)}\left(\frac{1}{2}(f_v + g_v)_v\right) = \inf\left\{\lambda > 0 \,\Big|\, \varrho_p\left(\frac{1}{2}(f + g)/\lambda^{\frac{1}{q(\cdot)}}\right) \leq 1\right\}.$$

The condition in the infimum translates to
$$1 \geq \int_{\mathbb{R}^n} \left(\frac{f + g}{2\lambda^{1/q(x)}}\right)^p dx = \frac{1}{2^p} \int_{\mathbb{R}^n} \left(\frac{a}{\lambda}\right)^{\frac{p}{q_1}}\chi_{Q_1} + \left(\frac{b}{\lambda}\right)^{\frac{p}{q_2}}\chi_{Q_2} dx = \frac{1}{2^p}\left(\frac{a}{\lambda}\right)^{\frac{p}{q_1}} + \frac{1}{2^p}\left(\frac{b}{\lambda}\right)^{\frac{p}{q_2}}.$$

Since the right hand side is continuous and decreasing in λ, we see that there exists a unique $\lambda_0 > 0$ for which equality holds. This number is the value of the modular of $\frac{1}{2}(f + g)$. Therefore the convexity inequality for the modular,
$$\varrho_{\ell^{q(\cdot)}(L^p)}\left(\frac{1}{2}(f_v + g_v)_v\right) \leq \frac{1}{2}\left[\varrho_{\ell^{q(\cdot)}(L^p)}((f_v)_v) + \varrho_{\ell^{q(\cdot)}(L^p)}((g_v)_v)\right],$$

can be written as
$$\lambda_0 \leq \frac{a + b}{2} \quad \text{where} \quad \left(\frac{a}{\lambda_0}\right)^{\frac{p}{q_1}} + \left(\frac{b}{\lambda_0}\right)^{\frac{p}{q_2}} = 2^p.$$

Let us denote $x := a/\lambda_0$ and $y := b/\lambda_0$. Then the convexity condition becomes
$$2 \leq x + y \quad \text{when} \quad x^{\frac{p}{q_1}} + y^{\frac{p}{q_2}} = 2^p.$$

By monotonicity, we may reformulate this as follows:
$$x^{\frac{p}{q_1}} + y^{\frac{p}{q_2}} \leq 2^p \quad \text{when} \quad 2 = x + y. \tag{6}$$

Thus we need to look for the maximum of $x^{\frac{p}{q_1}} + (2 - x)^{\frac{p}{q_2}}$ on $[0, 2]$.

Suppose first that $p = 1$. Then (6) holds with equality at $x = y = 1$, but this is not a maximum if $q_1 \neq q_2$. Thus we see that the inequality $x^{1/q_1} + y^{1/q_2} \leq 2$ does not hold in this case, which means that the modular is non-convex for arbitrarily small $|q_1 - q_2|$.

On the other hand, fix $p > 1$ and choose $q_1 = 0$. Then we can choose $x \in (0, 2)$ so large that $2^p - x^{p/q_1} = 1/2$. Since $y = 2 - x > 0$, we can choose q_1 so large that $y^{p/q_2} = 1/2$. Thus we see that there exists q_1 and q_2 for every p such that (6) does not hold.

We use a Fourier approach to the Besov and Triebel-Lizorkin space. For this we need some general definitions, well-known from the constant exponent case.

Definition (3.2.9) [160]: We say a pair (φ, Φ) is admissible if $\varphi, \Phi \in \mathcal{S}$ satisfy

(i) $\text{supp }\hat{\varphi} \subseteq \left\{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2\right\}$ and $|\hat{\varphi}(\xi)| \geq c > 0$ when $\frac{3}{5} \leq |\xi| \leq \frac{5}{3}$,

(ii) supp $\widehat{\Phi} \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq 2\}$ and $\left|\widehat{\Phi}(\xi)\right| \geq c > 0$ when $|\xi| \leq \frac{5}{3}$.

We set $\varphi_v(x) := 2^{vn}\varphi(2^v x)$ for $v \in \mathbb{N}$ and $\varphi_0(x) := \Phi(x)$.

We always denote by φ_v and ψ_v admissible functions in the sense of the previous definition. Usually, the Besov space is defined using the functions φ_v; when this is not the case, it will be explicitly marked, e.g. $\|\cdot\|_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}^{\psi}$.

Using the admissible functions (φ, Φ) we can define the norms

$$\|f\|_{F^\alpha_{p,q}} := \left\|\|2^{v\alpha}\varphi_v * f\|_{\ell^q}\right\|_p \quad \text{and} \quad \|f\|_{B^\alpha_{p,q}} := \left\|\|2^{v\alpha}\varphi_v * f\|_p\right\|_{\ell^q},$$

for constants $\alpha \in \mathbb{R}^n$ and $p, q \in (0, \infty]$ (excluding $p = \infty$ for the F-scale). The Triebel-Lizorkin space $F^\alpha_{p,q}$ and the Besov space $B^\alpha_{p,q}$ consist of all distributions $f \in \mathcal{S}'$ for which $\|f\|_{F^\alpha_{p,q}} < \infty$ and $\|f\|_{B^\alpha_{p,q}} < \infty$, respectively. It is well-known that these spaces do not depend on the choice of the initial system (φ, Φ) (up to equivalence of quasinorms). Further details on the classical theory of these spaces can be found in Triebel [40, 41]; see also [136].

Definition (3.2.10) [160]: Let φ_v be as in Definition (3.2.9). For $\alpha : \mathbb{R}^n \to \mathbb{R}$ and $p, q \in \mathcal{P}_0$, the Besov space $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ consists of all distributions $f \in \mathcal{S}'$ such that

$$\|f\|_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}^{\varphi} := \left\|\left(2^{v\alpha(\cdot)}\varphi_v * f\right)_v\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

In the case of $p = q$ we use the notation $B^{\alpha(\cdot)}_{p(\cdot)} := B^{\alpha(\cdot)}_{p(\cdot),p(\cdot)}$.

To the Besov space we can also associate the following modular:

$$\varrho_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}^{\varphi}(f) := \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\left(2^{v\alpha(\cdot)}\varphi_v * f\right)_v\right),$$

which can be used to define the norm. By Proposition (3.2.5) we directly obtain the following simplification in the case when q is constant:

Corollary (3.2.11) [160]: If q is a constant, then

$$\|f\|_{B^{\alpha(\cdot)}_{p(\cdot),q}}^{\varphi} := \left\|\left\|2^{v\alpha(\cdot)}\varphi_v * f\right\|_{p(\cdot)}\right\|_{\ell^q}.$$

An important special case of the Besov space is when $p = q$. In this case we show that the Besov space agrees with the corresponding Triebel-Lizorkin space studied in [140]. This space is defined via the norm

$$\|f\|_{F^{\alpha(\cdot)}_{p(\cdot),q}}^{\varphi} := \left\|\left\|2^{v\alpha(\cdot)}\varphi_v * f\right\|_{\ell^{q(\cdot)}}\right\|_{p(\cdot)}.$$

Notice that there is no difficulty with q depending on the space variable x here, since the $\ell^{q(\cdot)}$-norm is inside the $L^{p(\cdot)}$-norm.

Proposition (3.2.12) [160]: Let $p \in \mathcal{P}_0$ and $\alpha \in L^\infty$. Then $B^{\alpha(\cdot)}_{p(\cdot)} = F^{\alpha(\cdot)}_{p(\cdot)}$.

Proof: The claim follows from the following calculation:

$$\varrho_{B^{\alpha(\cdot)}_{p(\cdot)}}^{\varphi}(f) = \sum_v \left\|\left\|2^{v\alpha(\cdot)}\varphi_v * f\right\|^{p(\cdot)}\right\|_1 = \sum_v \int_{\mathbb{R}^n} \left|2^{v\alpha(x)}\varphi_v * f(x)\right|^{p(x)} dx$$

$$= \int_{\mathbb{R}^n} \sum_v \left|2^{v\alpha(x)}\varphi_v * f(x)\right|^{p(x)} dx = \int_{\mathbb{R}^n} \left\|2^{v\alpha(x)}\varphi_v * f(x)\right\|_{\ell^{q(x)}}^{p(x)} dx = \varrho_{F^{\alpha(\cdot)}_{p(\cdot)}}^{\varphi}(f).$$

So far we have not considered whether the space given by Definition (3.2.10) depends on the choice of (φ, Φ). Therefore, the previous result has to be understood in the sense that the Besov space defined from a certain (φ, Φ) equals the Triebel-Lizorkin space defined by the same φ. This is not entirely satisfactory. In [140] it was shown that the Triebel-Lizorkin space is independent of the basic functions, essentially assuming that

$p, q, \alpha \in \mathcal{P}_0^{\log} \cap L^\infty$. We prove now a corresponding result for the Besov space, but with more general assumptions; namely we allow $p, q \in \mathcal{P}_0^{\log}$ to be unbounded, and assume of $\alpha \in L^\infty$ only local log-Hölder continuity.

Theorem (3.2.13) [160]: Let $p, q \in \mathcal{P}_0^{\log}$ and $\alpha \in C_{loc}^{\log} \cap L^\infty$. Then the space $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$ does not depend on the admissible basis functions φ_v, i.e. different functions yield equivalent quasinorms.

Proof: Let (φ, Φ) and (ψ, Ψ) be two pairs of admissible functions. By symmetry, it suffices to prove that

$$\|f\|_{B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}^{\varphi} \leq c\|f\|_{B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}^{\psi}.$$

Define $K := \{-1, 0, 1\}$. Following classical lines, and using that $\widehat{\varphi}_v \widehat{\psi}_\mu = 0$ when $|\mu - v| > 1$, we have

$$\varphi_v * f = \sum_{k \in K} \varphi_v * \psi_{v+k} * f.$$

Fix $r \in (0, \min\{1, p^-\})$ and $m > n$ large. Since $|\varphi_v| \leq c\, \eta_{v,2m/r}$, with $c > 0$ independent of v, we obtain

$$|\varphi_v * \psi_{v+k} * f| \leq c\, \eta_{v,2m/r} * |\psi_{v+k} * f| \leq c\, \eta_{v,2m/r} * \left(\eta_{v+k,2m} * |\psi_{v+k} * f|^r \right)^{1/r},$$

where in the second inequality we used the r-trick. By Minkowski's integral inequality (with exponent $1/r > 1$) and Lemma A.3, [140] we further obtain

$$|\varphi_v * \psi_{v+k} * f|^r \leq c\left[\eta_{v,2m/r} * \eta_{v+k,2m}^{1/r} \right]^r * |\psi_{v+k} * f|^r \approx \eta_{v+k,2m} * |\psi_{v+k} * f|^r.$$

This, together with Lemma 6.1 [140] and Lemma 4.7 [160] gives

$$\left\| \left(2^{v\alpha(\cdot)}\varphi_v * f \right)_v \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = \left\| \left(2^{v\alpha(\cdot)r}|\varphi_v * f|^r \right)_v \right\|_{\ell^{\frac{q(\cdot)}{r}}\left(L^{\frac{p(\cdot)}{r}} \right)}^{1/r}$$

$$\leq c \sum_{k \in K} \left\| \left(2^{v\alpha(\cdot)r}\eta_{v+k,2m} * |\psi_{v+k} * f|^r \right)_v \right\|_{\ell^{\frac{q(\cdot)}{r}}\left(L^{\frac{p(\cdot)}{r}} \right)}^{1/r}$$

$$\leq c \sum_{k \in K} \left\| \left(\eta_{v+k,m} * \left(2^{v\alpha(\cdot)r}|\psi_{v+k} * f|^r \right) \right)_v \right\|_{\ell^{\frac{q(\cdot)}{r}}\left(L^{\frac{p(\cdot)}{r}} \right)}^{1/r}$$

$$\leq c \sum_{k \in K} \left\| \left(2^{v\alpha(\cdot)r}|\psi_{v+k} * f|^r \right)_v \right\|_{\ell^{\frac{q(\cdot)}{r}}\left(L^{\frac{p(\cdot)}{r}} \right)}^{1/r} = c \sum_{k \in K} \left\| \left(2^{v\alpha(\cdot)}\psi_{v+k} * f \right)_v \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}.$$

By the shift invariance of the mixed Lebesgue sequence space, the last sum equals $3\|f\|_{B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}^{\psi}$, which completes the proof.

Although one would obviously like to work in the variable index Besov space independent of the choice of basic functions φ_v, the assumptions needed in the previous theorem are quite strong in the sense that many of the later results work under much weaker assumptions. In the interest of clarity, we state those results only with the assumptions actually needed in their proofs. They should then be understood to hold with any particular choice of basic functions. For simplicity, we will not explicitly include the dependence on φ, thus omitting φ in the notation of the norm and modular.

The following theorem gives basic embeddings between Besov spaces and Triebel-Lizorkin spaces.

Theorem (3.2.14) [160]: Let $\alpha, \alpha_0, \alpha_1 \in L^\infty$ and $p, q_0, q_1 \in \mathcal{P}_0$.
(i) If $q_0 \leq q_1$, then

$$B_{p(\cdot),q_0(\cdot)}^{\alpha(\cdot)} \hookrightarrow B_{p(\cdot),q_1(\cdot)}^{\alpha(\cdot)}.$$

(ii) If $(\alpha_0 - \alpha_1)^- > 0$, then
$$B_{p(\cdot),q_0(\cdot)}^{\alpha_0(\cdot)} \hookrightarrow B_{p(\cdot),q_1(\cdot)}^{\alpha_1(\cdot)}.$$

(iii) If $p^+, q^+ < \infty$, then
$$B_{p(\cdot),\min\{p(\cdot),q(\cdot)\}}^{\alpha(\cdot)} \hookrightarrow F_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \hookrightarrow B_{p(\cdot),\max\{p(\cdot),q(\cdot)\}}^{\alpha(\cdot)}.$$

Proof: Assume that $q_0 \le q_1$. We note that $\lambda^{\frac{1}{q_0(x)}} \le \lambda^{\frac{1}{q_1(x)}}$ when $\lambda \le 1$. By the definition it follows that
$$\varrho_{B_{p(\cdot),q_0(\cdot)}^{\alpha(\cdot)}}(f/\mu) \ge \varrho_{B_{p(\cdot),q_1(\cdot)}^{\alpha(\cdot)}}(f/\mu)$$
for every $\mu > 0$, which implies (i).
By (i),
$$B_{p(\cdot),q_0(\cdot)}^{\alpha_0(\cdot)} \hookrightarrow B_{p(\cdot),q_0^+(\cdot)}^{\alpha_0(\cdot)} \quad \text{and} \quad B_{p(\cdot),q_1^-(\cdot)}^{\alpha_1(\cdot)} \hookrightarrow B_{p(\cdot),q_1(\cdot)}^{\alpha_1(\cdot)}.$$
Therefore, it suffices to prove (ii) for constant exponents q_0^+ and q_0^-, which we denote again by $q_0, q_1 \in (0, \infty]$ for simplicity. Then the proof is similar to the constant exponent situation. Indeed,
$$\left\| \left\| 2^{\nu\alpha_1(\cdot)} \varphi_\nu * f \right\|_{p(\cdot)} \right\|_{\ell^{q_1}} \le c_1 \left\| \left\| 2^{\nu\alpha_0(\cdot)} \varphi_\nu * f \right\|_{p(\cdot)} \right\|_{\ell^\infty} \le c_1 \left\| \left\| 2^{\nu\alpha_0(\cdot)} \varphi_\nu * f \right\|_{p(\cdot)} \right\|_{\ell^{q_0}}$$
with $c_1^{q_1} = \sum_{\nu \ge 0} 2^{\nu q_1(\alpha_0 - \alpha_1)^-} < \infty$.

To prove the first embedding in (iii), let $r := \min\{p, q\}$ and $f_\nu(x) := 2^{\nu\alpha(x)}|\varphi_\nu * f(x)|$. We assume that $\varrho_{B_{p(\cdot),r(\cdot)}^{\alpha(\cdot)}}(f) \le 1$. Then it suffices to show that $\varrho_{F_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}(f) \le c$. Since $\ell^{r(x)} \hookrightarrow \ell^{q(x)}$, we obtain

$$\varrho_{p(\cdot)}\left(\|f_\nu\|_{\ell^{q(x)}} \right) \le \varrho_{p(\cdot)}\left(\|f_\nu\|_{\ell^{r(x)}} \right) = \int_{\mathbb{R}^n} \left(\sum_\nu f_\nu^{r(x)} \right)^{\frac{p(x)}{r(x)}} dx = \varrho_{\frac{p(\cdot)}{r(\cdot)}}\left(\sum_\nu f_\nu^{r(\cdot)} \right).$$

Thus it suffices to show that the right hand side is bounded by a constant, which follows if the corresponding norm is bounded. Using the triangle inequality, we obtain just this:
$$\left\| \sum_\nu f_\nu^{r(\cdot)} \right\|_{\frac{p(\cdot)}{r(\cdot)}} \le \sum_\nu \|f_\nu^{r(\cdot)}\|_{\frac{p(\cdot)}{r(\cdot)}} = \varrho_{B_{p(\cdot),r(\cdot)}^{\alpha(\cdot)}}(f) \le 1.$$

For the second embedding in (iii), we use a similar derivation, with $s = \max\{p, q\}$. We assume that $\varrho_{F_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}(f) \le 1$. Then we estimate the modular in the Besov space with a reverse triangle inequality which holds since $p/s \le 1$:
$$\varrho_{B_{p(\cdot),s(\cdot)}^{\alpha(\cdot)}}(f) = \sum_\nu \|f_\nu^{s(\cdot)}\|_{\frac{p(\cdot)}{s(\cdot)}} \le \left\| \sum_\nu f_\nu^{s(\cdot)} \right\|_{\frac{p(\cdot)}{s(\cdot)}} = \left\| \|f_\nu\|_{\ell^{s(\cdot)}}^{s(\cdot)} \right\|_{\frac{p(\cdot)}{s(\cdot)}}.$$

Since p/s is bounded, the right hand side is bounded if and only if the corresponding modular is bounded. In fact,
$$\varrho_{\frac{p(\cdot)}{s(\cdot)}}\left(\|f_\nu\|_{\ell^{s(\cdot)}}^{s(\cdot)} \right) = \int_{\mathbb{R}^n} \|f_\nu\|_{\ell^{s(x)}}^{p(x)} dx = \varrho_{F_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}(f) \le 1,$$
so we are done.

We next consider embeddings of Sobolev-type which trade smoothness for integrability. For constant exponents it is well-known that
$$B_{p_0,q}^{\alpha_0} \hookrightarrow B_{p_1,q}^{\alpha_1} \tag{7}$$

if $\alpha_0 - \frac{n}{p_0} = \alpha_1 - \frac{n}{p_1}$, where $0 < p_0 \leq p_1 \leq \infty$, $0 < q \leq \infty$, $-\infty < \alpha_1 \leq \alpha_0 < \infty$ (see e.g. [41]). This is a consequence of certain Nikolskii inequalities for entire analytic functions of exponential type (cf. [41]), which we now generalize to the variable exponent setting.

Lemma (3.2.15) [160]: Let $p_1, p_0, q \in \mathcal{P}_0$ with $\alpha - n/p_1$ and $1/q$ locally log-Hölder continuous. If $p_1 \geq p_0$, then there exists $c > 0$ such that

$$\left\| \left| c 2^{v\alpha(\cdot)} g \right|^{q(\cdot)} \right\|_{\frac{p_1(\cdot)}{q(\cdot)}} \leq \left\| \left| 2^{v\left(\alpha(\cdot) + \frac{n}{p_0(\cdot)} - \frac{n}{p_1(\cdot)}\right)} g \right|^{q(\cdot)} \right\|_{\frac{p_0(\cdot)}{q(\cdot)}} + 2^{-v}$$

for all $v \in \mathbb{N}_0$ and $g \in L^{p_0(\cdot)} \cap \mathcal{S}'$ with $\operatorname{supp} \hat{g} \subset \{\xi : |\xi| \leq 2^{v+1}\}$ such that the norm on the right hand side is at most one.

Proof: Let us denote $\beta := \alpha - n/p_1$ and

$$\lambda := \left\| \left| 2^{v\left(\beta(\cdot) + \frac{n}{p_0(\cdot)}\right)} g \right|^{q(\cdot)} \right\|_{\frac{p_0(\cdot)}{q(\cdot)}} + 2^{-v}.$$

Note that the assumption on the norm implies that $\lambda \in [2^{-v}, 1 + 2^{-v}]$. Using the r-trick and [140, Lemma 6.1], we get

$$\lambda^{-\frac{r}{q(x)}} 2^{vr\beta(x)} |g(x)|^r \leq c \, \lambda^{-\frac{r}{q(x)}} 2^{vr\beta(x)} \left(\eta_{v,2m} * |g|^r\right)(x) \leq c \, \eta_{v,m} * \left(\lambda^{-\frac{1}{q(\cdot)}} 2^{v\beta(\cdot)} |g|\right)^r (x)$$

for large m. Fix $r \in (0, p_0^-)$ and set $s = p_0/r \in \mathcal{P}_0$. An application of Hölder's inequality with exponent s yields

$$\lambda^{-\frac{1}{q(x)}} 2^{v\beta(x)} |g(x)| \leq c \left\| 2^{-\frac{vn}{s(\cdot)}} \eta_{v,m}(x - \cdot) \right\|_{s'(\cdot)}^{1/r} \left\| \lambda^{-\frac{1}{q(\cdot)}} 2^{v\left(\beta(\cdot) + \frac{n}{p_0(\cdot)}\right)} g \right\|_{p_0(\cdot)}.$$

The second norm on the right hand side is bounded by 1 due to the choice of λ. To show that the first norm is also bounded, we investigate the corresponding modular:

$$\varrho_{s'(\cdot)}\left(2^{-\frac{vn}{s(\cdot)}} \eta_{v,m}(x - \cdot)\right) = \int_{\mathbb{R}^n} 2^{vn} (1 + 2^v |x - y|)^{-ms'(y)} dy \leq \int_{\mathbb{R}^n} (1 + |2^v x - z|)^{-m(s')^-} dz < \infty,$$

since $m(s')^- > n$. Now with the appropriate choice of $c_0 \in (0, 1]$, we find that

$$\left(c_0 \lambda^{-\frac{1}{q(x)}} 2^{v\alpha(x)} |g(x)|\right)^{p_1(x)}$$

$$\leq c_0^{p_0(x)} \left[c_0 \frac{2^{v\beta(x)} |g(x)|}{\lambda^{-1/q(x)}}\right]^{p_1(x) - p_0(x)} \left(\lambda^{-\frac{1}{q(x)}} 2^{v\left(\beta(x) + \frac{n}{p_0(x)}\right)} |g(x)|\right)^{p_0(x)}$$

$$\leq \left(\lambda^{-\frac{1}{q(x)}} 2^{v\left(\beta(x) + \frac{n}{p_0(x)}\right)} |g(x)|\right)^{p_0(x)}.$$

Integrating this inequality over $\mathbb{R}^n$ and taking into account the definition of λ gives us the claim.

Applying the previous lemma, we obtain the following generalization of (7).

Theorem (3.2.16) (Sobolev inequality) [160]: Let $p_0, p_1, q \in \mathcal{P}_0$ and $\alpha_0, \alpha_1 \in L^\infty$ with $\alpha_0 \geq \alpha_1$. If $1/q$ and

$$\alpha_0(x) - \frac{n}{p_0(x)} = \alpha_1(x) - \frac{n}{p_1(x)}$$

are locally log-Hölder continuous, then

$$B_{p_0(\cdot), q(\cdot)}^{\alpha_0(\cdot)} \hookrightarrow B_{p_1(\cdot), q(\cdot)}^{\alpha_1(\cdot)}.$$

Proof: Suppose without loss of generality that the $B_{p_0(\cdot),q(\cdot)}^{\alpha_0(\cdot)}$-modular of a function is less than 1. Then an application of the previous lemma with $\alpha(x) = \alpha_1(x)$ and $g = \varphi_v * f$, shows that the $B_{p_1(\cdot),q(\cdot)}^{\alpha_1(\cdot)}$-modular is bounded by a constant.

Corollary (3.2.17) [160]: Let $p_0, p_1, q_0, q_1 \in \mathcal{P}_0$ and $\alpha_0, \alpha_1 \in L^\infty$ with $\alpha_0 \geq \alpha_1$. If

$$\alpha_0(x) - \frac{n}{p_0(x)} = \alpha_1(x) - \frac{n}{p_1(x)} + \varepsilon(x)$$

is locally log-Hölder continuous and $\varepsilon^- > 0$, then

$$B_{p_0(\cdot),q_0(\cdot)}^{\alpha_0(\cdot)} \hookrightarrow B_{p_1(\cdot),q_1(\cdot)}^{\alpha_1(\cdot)}.$$

Proof: By Theorems (3.2.14) (i) and (3.2.16),

$$B_{p_0(\cdot),q_0(\cdot)}^{\alpha_0(\cdot)} \hookrightarrow B_{p_0(\cdot),\infty}^{\alpha_0(\cdot)} \hookrightarrow B_{p_1(\cdot),\infty}^{\alpha_1(\cdot)+\varepsilon(\cdot)}.$$

We combine this with the embedding $B_{p_1(\cdot),\infty}^{\alpha_1(\cdot)+\varepsilon(\cdot)} \hookrightarrow B_{p_1(\cdot),q_1(\cdot)}^{\alpha_1(\cdot)}$ from Theorem (3.2.14) (ii) to conclude the proof.

Let C_u be the space of all bounded uniformly continuous functions on $\mathbb{R}^n$ equipped with the sup norm. Concerning embeddings into C_u, we have the following result.

Corollary (3.2.18) [160]: Let $\alpha \in C_{loc}^{log}$, $p \in \mathcal{P}^{log}$ and $q \in \mathcal{P}_0$. If

$$\alpha(x) - \frac{n}{p(x)} \geq \delta \max\left\{1 - \frac{1}{q(x)}, 0\right\}$$

for some fixed $\delta > 0$ and every $x \in \mathbb{R}^n$, then

$$B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \hookrightarrow C_u$$

Proof: Let $\gamma(x) := \alpha(x) - \frac{n}{p(x)}$. By Theorem (3.2.14) (i), we may replace q with the larger exponent $\max\{1, \delta/(\delta - \gamma)\} \in \mathcal{P}^{log}$. It then follows from Theorem (3.2.16) that

$$B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \hookrightarrow B_{\infty,q(\cdot)}^{\gamma(\cdot)}.$$

Since $B_{\infty,1}^0 \hookrightarrow C_u$ by classical results, we will complete the proof by showing that

$$B_{\infty,q(\cdot)}^{\gamma(\cdot)} \hookrightarrow B_{\infty,1}^0.$$

Denote $f_v := \varphi_v * f$. The remaining embedding can be written, using homogeneity in the usual manner, as

$$\sum_v \sup_x |f_v| \leq c \quad \text{whenever} \quad \sum_v \sup_x \left|2^{v\gamma(x)} f_v\right|^{q(x)} \leq 1.$$

We choose x_v such that $\sup_x |f_v| \leq 2|f_v(x_v)|$ for each v. Then it follows from Young's inequality that

$$\sum_v \sup_x |f_v| \approx \sum_v |f_v(x_v)| \leq \sum_v \left|2^{v\gamma(x_v)} f_v(x_v)\right|^{q(x_v)} + 2^{-v\gamma(x_v)q'(x_v)} \leq 1 + \sum_v 2^{-v\delta} \leq c,$$

which completes the proof of the remaining embedding.

Let $\mathcal{L}^{\alpha,p(\cdot)}$, $\alpha \in \mathbb{R}$, be the Bessel potential space modeled in $L^{p(\cdot)}$. It was shown in [140] that $F_{p(\cdot),2}^\alpha = \mathcal{L}^{\alpha,p(\cdot)}$ when $\alpha \geq 0$, $1 < p^- \leq p^+ < \infty$ and $p \in \mathcal{P}^{log}$. Under the same assumptions on p, by Theorem (3.2.14) one gets the embedding

$$B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \hookrightarrow \mathcal{L}^{\sigma,p(\cdot)}$$

for $\sigma^- > \sigma^+ \geq 0$. In particular, we have $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \hookrightarrow L^{p(\cdot)}$ if $\alpha^- > 0$ (cf. [88] or [109]). Next we derive a stronger version of this.

Let us define

$$\sigma_p(x) := n\left(\frac{1}{\max\{1, p(x)\}} - 1\right) \quad \text{and} \quad \bar{p}(x) := \max\{1, p(x)\}, \qquad x \in \mathbb{R}^n. \tag{8}$$

If $\alpha - n/p = \alpha - \sigma_p - n/\bar{p}$ is log-Hölder continuous, $p, q \in \mathcal{P}_0$, $\alpha \in L^\infty$ and $\left(\alpha - \sigma_p\right)^- > 0$, then by Corollary (3.2.17) we get

$$B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \hookrightarrow B_{\bar{p}(\cdot),1}^0.$$

We further conclude that

$$\|f\|_{\bar{p}(\cdot)} \leq \sum_{v \geq 0} \|\varphi_v * f\|_{\bar{p}(\cdot)} = \|f\|_{B_{\bar{p}(\cdot),1}^0} \leq c\|f\|_{B_{\bar{p}(\cdot),1}^{\alpha(\cdot)-\sigma_{p(\cdot)}}}.$$

Proposition (3.2.19) [160]: Assume that $p, q \in \mathcal{P}_0$ and $\alpha \in L^\infty$ are such that $\alpha - n/p$ is log-Hölder continuous. Let σ_p and $\bar{p}$ be as in (8). If $\left(\alpha - \sigma_p\right)^- > 0$, then

$$B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \hookrightarrow L^{\bar{p}(\cdot)}.$$

Let $p, q \in \mathcal{P}_0$ and $\alpha \in L^\infty$. Define $\alpha_0 := \left(\alpha - \frac{n}{p}\right)^-$. Then $\alpha \geq \alpha_0 + \frac{n}{p} =: \alpha_1 \in L^\infty$. It is clear that $\alpha_1 + \frac{n}{p} = \alpha_0$ is log-Hölder continuous. Therefore we obtain by Theorem (3.2.16) that

$$B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \hookrightarrow B_{p(\cdot),\infty}^{\alpha_1(\cdot)} \hookrightarrow B_{\infty,\infty}^{\alpha_1(\cdot)-\frac{n}{p(\cdot)}} = B_{\infty,\infty}^{\alpha_0} \hookrightarrow \mathcal{S}'.$$

Now we show that our scale of Besov spaces includes also the Hölder-Zygmund spaces of continuous functions. This application requires in particular that we include the case of unbounded p and q.

We start by generalizing the definition of Hölder-Zygmund spaces to the variable order setting. Such spaces have been considered e.g. in [142, 143, 153].

Recall that C_u denotes the set of all bounded uniformly continuous functions.

Definition (3.2.20) [160]: Let $\alpha : \mathbb{R}^n \to (0, 1]$. The Zygmund space $\mathcal{C}^{\alpha(\cdot)}$ consists of all $f \in C_u$ such that $\|f\|_{\mathcal{C}^{\alpha(\cdot)}} < \infty$, where

$$\|f\|_{\mathcal{C}^{\alpha(\cdot)}} := \|f\|_\infty + \sup_{x \in \mathbb{R}^n, h \in \mathbb{R}^n \setminus \{0\}} \frac{|\Delta_h^2 f(x)|}{|h|^{\alpha(x)}}$$

For $\alpha < 1$, the Hölder space $C^{\alpha(\cdot)}$ is defined analogously but with the norm given by

$$\|f\|_{C^{\alpha(\cdot)}} := \|f\|_\infty + \sup_{x \in \mathbb{R}^n, h \in \mathbb{R}^n \setminus \{0\}} \frac{|\Delta_h^1 f(x)|}{|h|^{\alpha(x)}}$$

Here Δ_h^j is the j-th order difference operator $h \in \mathbb{R}^n, j \in \mathbb{N}$:

$$\Delta_h^1 f(x) = f(x + h) - f(x), \quad \Delta_h^{j+1} f = \Delta_h^1\left(\Delta_h^j f\right).$$

One can easily derive the point-wise inequality

$$\sup_h |h|^{-\alpha(x)} |\Delta_h^1 f(x)| \leq \frac{1}{2 - 2^{\alpha^+}} \sup_h |h|^{-\alpha(x)} |\Delta_h^2 f(x)|, \quad x \in \mathbb{R}^n.$$

Hence we have $\mathcal{C}^{\alpha(\cdot)} \hookrightarrow C^{\alpha(\cdot)}$ for $\alpha^+ < 1$. In fact, these two spaces coincide for such α, as in the classical case. This is one consequence of the following result.

Theorem (3.2.21) [160]: For α locally log-Hölder continuous with $\alpha^- > 0$,

$$B_{\infty,\infty}^{\alpha(\cdot)} = \mathcal{C}^{\alpha(\cdot)} \quad (\alpha \leq 1) \quad \text{and} \quad B_{\infty,\infty}^{\alpha(\cdot)} = C^{\alpha(\cdot)} \quad (\alpha^+ < 1).$$

Proof: The proof is naturally divided into two parts. First we consider the claim that

$$\mathcal{C}^{\alpha(\cdot)} \hookrightarrow B_{\infty,\infty}^{\alpha(\cdot)} \quad (\alpha \leq 1) \quad \text{and} \quad B_{\infty,\infty}^{\alpha_0} \hookrightarrow C^{\alpha(\cdot)} \quad (\alpha^+ < 1).$$

We prove only the first embedding; the second is similar. We estimate the absolute value on the right hand side of

$$\|f\|_{B_{\infty,\infty}^{\alpha(\cdot)}} = \sup_v \sup_x \left|2^{v\alpha(x)} \varphi_v * f(x)\right|$$

by $\|f\|_{\mathcal{C}^{\alpha(\cdot)}}$. The term $v = 0$ is easily estimated in terms of $\|f\|_\infty$, so we consider in what follows $v > 0$.

Since the Besov space is independent of the choice of admissible φ, we may assume without loss of generality that $\varphi(-y) = \varphi(y)$. Then

$$\varphi_v * f(x) = \frac{1}{2} \int_{\mathbb{R}^n} \varphi_v(h)[f(x+h) + f(x-h)]dh = \frac{1}{2} \int_{\mathbb{R}^n} \varphi_v(h)\Delta_h^2 f(x-h)dh,$$

where we used the fact that $\int \varphi_v(y)dy = \widehat{\varphi_v}(0) = 0$ in the second step. By definition, $|\Delta_h^2 f(x-h)| \le \|f\|_{\mathcal{C}^{\alpha(\cdot)}}|h|^{\alpha(x-h)}$. For small h, the log-Hölder continuity implies that $|h|^{\alpha(x-h)} \le c|h|^{\alpha(x)}$. Thus we obtain

$$|\varphi_v * f(x)| \le c \int_{|h|<1} |\varphi_v(h)||h|^{\alpha(x)}dh + c \int_{|h|\ge 1} |\varphi_v(h)||h|^{\alpha^+}dh$$

$$= c \int_{|h|<2^v} |\varphi_v(h)||2^{-v}h|^{\alpha(x)}dh + c \int_{|h|\ge 2^v} |\varphi_v(h)||2^{-v}h|^{\alpha^+}dh$$

$$\le c2^{-v\alpha(x)} \int_{\mathbb{R}^n} |\varphi_v(h)|\big[|h|^{\alpha^+} + |h|^{\alpha^-}\big]dh,$$

where in the second step we used a change of variables. Since φ decays faster than any polynomial (as $\operatorname{supp} \widehat{\varphi}$ is bounded), the integral on the right-hand side is finite, and so we are done.

We then move on to the second part of the proof of the theorem, and consider the claim

$$B_{\infty,\infty}^{\alpha(\cdot)} \hookrightarrow \mathcal{C}^{\alpha(\cdot)} \quad (\alpha \le 1) \quad \text{and} \quad B_{\infty,\infty}^{\alpha(\cdot)} \hookrightarrow C^{\alpha(\cdot)} \quad (\alpha^+ < 1).$$

First we note that

$$\sup_{0<|h|\le 1} \sup_x \frac{|\Delta_h^M f(x)|}{|h|^{\alpha(x)}} \le 2^{\alpha^+} \sup_{k\ge 0} \sup_{|h|\le 2^{-k}} \sup_x |2^{k\alpha(x)}\Delta_h^M f(x)|.$$

(We restrict ourselves to $|h| \le 1$ since large h are easily handled.) For $a > 0$ and $M > 1$ there exists $c > 0$ such that

$$|\Delta_h^M(\varphi_v * f)(x)| \le c \min\{1, 2^{(v-k)M}\} (\varphi_v^* f)_a(x),$$

for every $v, k \in \mathbb{N}_0$ and $|h| \le 2^{-k}$, where $(\varphi_v^* f)_a(x) := \sup_y \frac{|\varphi_v * f(x-y)|}{1+|2^v y|^a}$ is the Peetre maximal function, cf. [40]. Since $f = \sum_v \varphi_v * f$ with convergence in L^1, we can use the previous estimate to obtain

$$\sup_{|h|\le 2^{-k}} |2^{k\alpha(x)}\Delta_h^M f(x)|$$

$$\le c \sum_{v<k} 2^{(v-k)(M-\alpha(x))} 2^{v\alpha(x)} (\varphi_v^* f)_a(x) + c \sum_{v\ge k+1} 2^{(v-k)\alpha(x)} 2^{v\alpha(x)} (\varphi_v^* f)_a(x) \quad (9)$$

Therefore, we need to estimate $2^{v\alpha(x)}(\varphi_v^* f)_a(x)$. Let us denote $K := \sup_x 2^{v\alpha(x)}|\varphi_v * f(x)|$. Then

$$2^{v\alpha(x)}(\varphi_v^* f)_a(x) = \sup_y 2^{v\alpha(x)} \frac{|\varphi_v * f(x-y)|}{1+|2^v y|^a} \le K \sup_y \frac{2^{v(\alpha(x)-\alpha(x-y))}}{1+|2^v y|^a}.$$

When $|y| < 2^{-v/2}$, it follows from the log-Hölder continuity of α that $v(\alpha(x) - \alpha(x-y)) \le c$. When $|y| \ge 2^{-v/2}$, the right-hand side is bounded by $K2^{v(\alpha^+ - \alpha^- - a/2)}$, which remains bounded provided we choose $a > (\alpha^+ - \alpha^-)$. Therefore we have shown that

$$2^{v\alpha(x)}(\varphi_v^* f)_a(x) \le c \sup_x 2^{v\alpha(x)}|\varphi_v * f(x)| \le c\|f\|_{B_{\infty,\infty}^{\alpha(\cdot)}}.$$

Using this in (9), we find that

$$\sup_{|h|\leq 2^{-k}}|2^{k\alpha(x)}\Delta_h^M f(x)|\leq c\left[\sum_{v<k}2^{(v-k)(M-\alpha^+)}+\sum_{v\geq k+1}2^{(k-v)\alpha^-}\right]\|f\|_{B_{\infty,\infty}^{\alpha(\cdot)}}.$$

If $M=1$, then we have assumed that $\alpha^+<1$; for $M=2$, $M-\alpha^+\geq 1$. Thus the terms in the brackets are bounded, so we have estimated the main part of the norm. Since we also have $\|f\|_\infty\leq c\|f\|_{B_{\infty,\infty}^{\alpha(\cdot)}}$ for $\alpha^->0$, the proof is complete.

We characterize the elements from $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$ in terms of Nikolskii representations involving sequences of entire analytic functions. Let
$$\mathcal{U}^{p(\cdot)}:=\{(u_v)_v\subset\mathcal{S}'\cap L^{p(\cdot)}:\text{supp }\hat{u}_v\subset\{\xi:|\xi|\leq 2^{v+1}\},v\in\mathbb{N}_0\}.$$

Theorem (3.2.22) [160]: Let $p,q\in\mathcal{P}_0^{\log}$ and $\alpha\in C_{loc}^{\log}\cap L^\infty$ with $\alpha^->0$. Then $f\in\mathcal{S}'$ belongs to $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$ if and only if there exists $u=(u_v)_v\in\mathcal{U}^{p(\cdot)}$ such that
$$f=\lim_{v\to\infty}u_v\quad\text{in }\mathcal{S}' \tag{10}$$

and
$$\|f\|^u:=\|u_0\|_{p(\cdot)}+\left\|\left(2^{v\alpha(\cdot)}(f-u_v)\right)_v\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}<\infty.$$

Moreover,
$$\|f\|^*:=\inf_u\|f\|^u$$

is an equivalent quasinorm in $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$, where the infimum is taken over all possible representations $(u_v)_v\in\mathcal{U}^{p(\cdot)}$ satisfying (10).

Proof: First we show that $\|f\|^*\leq\|f\|_{B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}$. If $(\varphi_v)_v$ is an admissible system, then
$$u_v:=\sum_{j=0}^v\varphi_j*f\to f\quad\text{in }\mathcal{S}'\quad\text{as }v\to\infty.$$

Thus $(u_v)_v\in\mathcal{U}^{p(\cdot)}$ and
$$\left(2^{v\alpha(\cdot)}(f-u_v)\right)_v=\sum_{j\geq 0}2^{-j\alpha(\cdot)}\left(2^{(j+v)\alpha(\cdot)}\varphi_{j+v}*f\right)_v\quad\text{in }\mathcal{S}'.$$

Observe that $2^{-j\alpha(\cdot)}\leq 2^{-j\alpha^-}$ and that $\alpha^->0$ by assumption. Let $r\in\left(0,\frac{1}{2}\min\{p,q,2\}\right)$. Using the previous expression and the triangle inequality in the mixed Lebesgue-sequence space, we obtain

$$\left\|\left(2^{v\alpha(\cdot)}(f-u_v)\right)_v\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}=\left\|\left|\sum_{j\geq 0}2^{-j\alpha(\cdot)}\left(2^{(j+v)\alpha(\cdot)}\varphi_{j+v}*f\right)_v\right|^r\right\|_{\ell^{\frac{q(\cdot)}{r}}\left(L^{\frac{p(\cdot)}{r}}\right)}^{1/r}$$

$$\leq\left\|\sum_{j\geq 0}2^{-jr\alpha(\cdot)}\left(2^{(j+v)r\alpha(\cdot)}|\varphi_{j+v}*f|^r\right)_v\right\|_{\ell^{\frac{q(\cdot)}{r}}\left(L^{\frac{p(\cdot)}{r}}\right)}^{1/r}$$

$$\leq\left(\sum_{j\geq 0}2^{-jr\alpha^-}\left\|\left(2^{(j+v)r\alpha(\cdot)}|\varphi_{j+v}*f|^r\right)_v\right\|_{\ell^{\frac{q(\cdot)}{r}}\left(L^{\frac{p(\cdot)}{r}}\right)}\right)^{1/r}$$

$$\leq c\left\|\left(2^{v\alpha(\cdot)}\varphi_v*f\right)_v\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})},$$

where the last step follows from the invariance of the norm under shifts in the v direction. Since $\|u_0\|_{p(\cdot)}=\|\varphi_0*f\|_{p(\cdot)}\leq\|f\|_{B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}$, we have shown that

$$\|f\|^{\mathrm{u}} \leq c\|f\|_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Now we prove the opposite inequality. Let $(u_\nu)_\nu \in \mathcal{U}^{p(\cdot)}$ be such that $f = \lim\limits_{k\to\infty} u_k$ and $\|f\|^{\mathrm{u}} < \infty$. Then $\varphi_\nu * f = \sum_{k\geq-1} \varphi_\nu * (u_{\nu+k} - u_{\nu+k-1})$, $\nu \in \mathbb{N}_0$ (with $u_{-1} = 0$). Using the r-trick, with r as before, we find that

$$2^{\nu\alpha(x)}|\varphi_\nu * f| \leq 2^{\nu\alpha(x)} \sum_{k\geq-1} |\varphi_\nu * (u_{\nu+k} - u_{\nu+k-1})| \leq \sum_{k\geq-1} \left[\eta_{\nu,m} * \left(2^{\nu\alpha(\cdot)r}|u_{\nu+k} - u_{\nu+k-1}|^r\right)\right]^{\frac{1}{r}}.$$

Since $2^{\nu\alpha(\cdot)} \leq 2^{(\nu+k)\alpha(\cdot)} 2^{-k\alpha^-}$, we obtain

$$\left\|\left(2^{\nu\alpha(\cdot)}\varphi_\nu * f\right)_\nu\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leq c \sum_{k\geq-1} 2^{-k\alpha^-} \left\|\left(\eta_{\nu,m} * \left(2^{(\nu+k)\alpha(\cdot)r}|u_{\nu+k} - u_{\nu+k-1}|^r\right)\right)_\nu\right\|^{\frac{1}{r}}_{\ell^{\frac{q(\cdot)}{r}}\left(L^{\frac{p(\cdot)}{r}}\right)}.$$

Then we can get rid of the function η (see, [160]). Using
$$|u_{\nu+k} - u_{\nu+k-1}| \leq |f - u_{\nu+k}| + |f - u_{\nu+k-1}|,$$
we find that

$$\left\|\left(2^{\nu\alpha(\cdot)}\varphi_\nu * f\right)_\nu\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leq c \sum_{k\geq-1} 2^{-k\alpha^-} \left\|\left(2^{(\nu+k)\alpha(\cdot)}(f - u_{\nu+k})\right)_\nu\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}.$$

Using again the invariance of the sequence space with respect to shifts, we see that the left hand side can be estimated by a constant times $\|f\|^{\mathrm{u}}$. Taking the infimum over u, we obtain $\|f\|_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leq c\|f\|^*$.

Chapter 4
Besov-Type Spaces and Decomposition of Besov-Hausdorff Spaces

We introduce certain Hardy-Hausdorff spaces $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and show that the dual space of $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is just $\dot{B}_{p',q'}^{-s,\tau}(\mathbb{R}^n)$, where t' denotes the conjugate index of $t \in (1,\infty)$. Moreover, using their atomic and molecular decomposition characterizations, we investigate the trace properties and the boundedness of pseudo-differential operators with homogeneous symbols in $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ $(q > 1)$, which generalize the corresponding classical results on homogeneous Besov and Triebel-Lizorkin spaces when $p \in (1,\infty)$ and $q \in [1,\infty)$ by taking $\tau = 0$.

Section (4.1): Triebel-Lizorkin-Type Spaces Including Q Spaces

The most general scales, known so far, are the scales of Besov spaces and Triebel-Lizorkin spaces. Besov spaces $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ respectively domains in $\mathbb{R}^n$ for the full range of parameters $s \in \mathbb{R}$ and $p, q \in (0,\infty]$ were introduced between 1959 and 1975 (see [41]). They cover many well-known classical concrete function spaces such as Hölder-Zygmund spaces, Sobolev spaces, fractional Sobolev spaces (also often referred to as Bessel-potential spaces), Hardy spaces and BMO($\mathbb{R}^n$), which have their own history. Acomprehensive treatment of these function spaces and their history can be found in Triebel's monographes [56, 136].

Let $\mathcal{S}(\mathbb{R}^n)$ be the space of all Schwartz functions on $\mathbb{R}^n$. Let φ and ψ be functions on $\mathbb{R}^n$ satisfying

$$\varphi, \psi \in \mathcal{S}(\mathbb{R}^n) \tag{1}$$
$$\text{supp } \widehat{\varphi}, \widehat{\psi} \subset \{\xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2\}, \tag{2}$$
$$|\widehat{\varphi}(\xi)|, |\widehat{\psi}(\xi)| \geq C > 0 \;\; if \;\; 3/5 \leq |\xi| \leq 5/3, \tag{3}$$

And

$$\sum_{j\in\mathbb{Z}} \overline{\widehat{\varphi}(2^j\xi)}\widehat{\psi}(2^j\xi) = 1 \;\text{ if } \xi \neq 0, \tag{4}$$

where $\hat{f}(\xi) \equiv \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx$. For all $j \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, we put $\varphi_j(x) \equiv 2^{jn}\varphi(2^j x)$. As in [176], we set

$$\mathcal{S}_\infty(\mathbb{R}^n) \equiv \left\{\varphi \in \mathcal{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \varphi(x)x^\gamma dx = 0 \text{ for all multi} - \text{indices } \gamma \in (\mathbb{N} \cup \{0\})^n\right\}.$$

Following Triebel [41], we consider $\mathcal{S}_\infty(\mathbb{R}^n)$ as a subspace of $\mathcal{S}(\mathbb{R}^n)$, including the topology. Thus, $\mathcal{S}_\infty(\mathbb{R}^n)$ is a complete metric space (see [181]). Equivalently, $\mathcal{S}_\infty(\mathbb{R}^n)$ can be defined as a collection of all $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that semi-norms $\|\varphi\|_M \equiv \sup_{|\gamma|\leq M} \sup_{\xi\in\mathbb{R}^n} |\partial^\gamma\widehat{\varphi}(\xi)|(|\xi|^M + |\xi|^{-M}) < \infty$ for all $M \in \mathbb{N} \cup \{0\}$ (see [93]), where and in what follows, $\gamma = (\gamma_1, \ldots, \gamma_n) \in (\mathbb{N} \cup \{0\})^n$, $|\gamma| = \gamma_1 + \ldots + \gamma_n$ and $\partial^\gamma = \left(\frac{\partial}{\partial\xi_1}\right)\gamma_1 \ldots \left(\frac{\partial}{\partial\xi_1}\right)\gamma_n\partial$. The semi-norms $\{\|\cdot\|_M\}_{M\in\mathbb{N}\cup\{0\}}$ generate a topology of a locally convex space on $\mathcal{S}_\infty(\mathbb{R}^n)$ which coincides with the topology of $\mathcal{S}_\infty(\mathbb{R}^n)$ as a subspace of a locally convex space $\mathcal{S}(\mathbb{R}^n)$. Let $\mathcal{S}_\infty'(\mathbb{R}^n)$ be the topological dual of $\mathcal{S}_\infty(\mathbb{R}^n)$, namely, the set of all continuous linear functionals $\mathcal{S}_\infty(\mathbb{R}^n)$. We endow $\mathcal{S}_\infty'(\mathbb{R}^n)$ with the weak $*$-topology. Then $\mathcal{S}_\infty'(\mathbb{R}^n)$ is complete; see [177].

Let $s \in \mathbb{R}$, $p, q \in (0,\infty]$ and φ satisfy (1) through (3). The Besov space $\dot{B}_{p,q}^s(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}_\infty'(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} \equiv \left\{ \sum_{j\in\mathbb{Z}} 2^{jsq} \|\varphi_j * f\|^q_{L^p(\mathbb{R}^n)} \right\}^{1/q} < \infty;$$

the Triebel-Lizorkin space $\dot{F}^s_{p,q}(\mathbb{R}^n)$ for $p < \infty$ is defined to be the set of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \equiv \left\| \left\{ \sum_{j\in\mathbb{Z}} (2^{js}|\varphi_j * f|)^q \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} < \infty;$$

see [41, 106]. For $s \in \mathbb{R}$ and $q \in (0, \infty]$, the Triebel-Lizorkin space $\dot{F}^s_{\infty,q}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{F}^s_{\infty,q}(\mathbb{R}^n)} \equiv \sup_{P \text{ dyadic}} \left\{ \frac{1}{|P|} \int_P \sum_{j=j_P}^{\infty} (2^{js}|\varphi_j * f(x)|)^q \, dx \right\}^{1/q} < \infty,$$

where $l(P)$ is the side length of dyadic cube P, $j_P \equiv -\log_2 l(P)$ and the supremum is taken over all dyadic cubes P; see [106]. It is well known that the spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$ and $\dot{F}^s_{p,q}(\mathbb{R}^n)$ are independent of the choices of φ (see, for example, [41, 106, 176]).

There has been increasing interest in a new family of function spaces, called $\mathcal{Q}_\alpha$ spaces, where $\alpha \in \mathbb{R}$. These spaces were originally defined by Aulaskari et al. [163] as spaces of holomorphic functions on the unit disk, which are geometric in the sense that they transform naturally under conformal mappings (see [185]). Following the works of Essén and Xiao [174] and Janson [178] on the boundary values of these functions on the unit circle, Essén et al. [173] extended these spaces to the n-dimensional Euclidean space $\mathbb{R}^n$. Very recently, Xiao [186] found some applications of these spaces in Navier-Stokes equations. Recall that for $\alpha \in \mathbb{R}$, the space $\mathcal{Q}_\alpha(\mathbb{R}^n)$ is defined to be the space of all measurable functions $f \in L^2_{loc}(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{Q}_\alpha(\mathbb{R}^n)} \equiv \sup_I \left\{ \frac{1}{|I|^{1-\frac{2\alpha}{n}}} \int_I \int_I \frac{|f(x)-f(y)|^2}{|x-y|^{n+2\alpha}} \, dx \, dy \right\}^{1/2} < \infty, \tag{5}$$

where I ranges over all cubes in $\mathbb{R}^n$. Since every cube I is contained in a cube J with dyadic length (namely, $l(J) \in \{2^j : j \in \mathbb{Z}\}$) such that $l(J) < 2l(I)$, we obtain an equivalent norm if we consider only cubes of dyadic edge lengths in (5).

In Dafni and Xiao [167], asked what are the relations among $\mathcal{Q}_\alpha(\mathbb{R}^n)$, Besov spaces and Triebel-Lizorkin spaces? Let $s \in \mathbb{R}, \tau \in [0, \infty), p \in (1, \infty)$ and $q \in (1, \infty]$. In [189], based on the Carleson measure characterizations of $\mathcal{Q}_\alpha$ spaces and $\dot{F}^s_{\infty,q}(\mathbb{R}^n)$ spaces (see [106]), we introduced a new class of function spaces $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$, which unify and generalize the Triebel-Lizorkin spaces with both $p < \infty$ [41] and $p = \infty$ [41, 106] and $\mathcal{Q}$ spaces [173] on $\mathbb{R}^n$. Our results establish the relationship between Triebel-Lizorkin spaces and $\mathcal{Q}$ spaces. The spaces also include the space $\mathcal{Q}^{\alpha,q}_p(\mathbb{R}^n)$ with $\alpha \in (0, 1)$ and $2 \leq q < p < \infty$ in [166] as a special case. Furthermore, for $s \in \mathbb{R}, p, q \in [1, \infty), \max\{p,q\} > 1$ and $\tau \in \left[0, \frac{q}{(\max\{p,q\})'}\right]$, via the Hausdorff capacity, we introduced in [189] a new class of tent spaces $F\dot{T}^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_\mathbb{Z})$, and determined their dual spaces $F\dot{W}^{-s,\tau/q}_{p',q'}(\mathbb{R}^{n+1}_\mathbb{Z})$; as an application of this, we further introduced certain Hardy-Hausdorff spaces $F\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$ and proved that the dual space of $F\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$ is just $\dot{F}^{-s,\tau/q}_{p',q'}(\mathbb{R}^n)$ when $p, q \in (1, \infty)$, where

herein and in what follows, t' denotes the conjugate index of $t \in [1, \infty]$. Also, herein and in what follows, we set $\mathbb{R}_{\mathbb{Z}}^{n+1} \equiv \mathbb{R}^n \times \{2^k : k \in \mathbb{Z}\}$.

Let $s, \tau \in \mathbb{R}$. We continue study of the Triebel-Lizorkin-type spaces $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ by also considering the cases $p, q \leq 1$; moreover, we also introduce some new class of Besov-type spaces $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ for $p, q \in (0, \infty]$ as follows.

Definition (4.1.1) [190]: Let $s, \tau \in \mathbb{R}$, $q \in (0, \infty]$ and φ be a Schwartz function satisfying (1) through (3).

(i) Let $p \in (0, \infty]$. The Besov-type space $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}_\infty'(\mathbb{R}^n)$ such that $\|f\|_{\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)} < \infty$, where

$$\|f\|_{\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)} \equiv \sup_{P \text{ dyadic}} \frac{1}{|P|^\tau} \left\{ \sum_{j=j_P}^{\infty} \left[\int_P \left(2^{js} |\varphi_j * f(x)| \right)^p dx \right]^{q/p} \right\}^{1/q}$$

with suitable modifications made when $p = \infty$ or $q = \infty$.

(ii) Let $p \in (0, \infty)$. The Triebel-Lizorkin-type space $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}_\infty'(\mathbb{R}^n)$ such that $\|f\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)} < \infty$, where

$$\|f\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)} \equiv \sup_{P \text{ dyadic}} \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{j=j_P}^{\infty} \left(2^{js} |\varphi_j * f(x)| \right)^q \right]^{p/q} dx \right\}^{1/p}$$

with suitable modifications made when $q = \infty$.

Obviously, when $p = q \in (0, \infty)$, $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n) = \dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$. For simplicity, in what follows, we use $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$ to denote either $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ or $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$. If $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$ means $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$, then the case $p = \infty$ is excluded. The spaces $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$ unify and generalize the classical Besov spaces [41, 45, 56, 176], Triebel-Lizorkin spaces [41, 56, 106, 176] and $\mathcal{Q}$ spaces [173] on $\mathbb{R}^n$. Thus, we give a complete answer to the question of Dafni and Xiao in [167].

Different from [189], we need some discrete Calderón reproducing formulae, which further yield the φ-transform characterization of these spaces via some subtle modifications on the methods developed by Frazier and Jawerth [45, 106]. As a special case of our Besov spaces $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$, we also obtain the φ-transform characterization of the space $\dot{B}BMO_p^{s,q}(\mathbb{R}^n)$, which was, via the φ-transform and the space of sequences, introduced by Lin and Wang [180] to establish certain $T(1)$ theorem for Besov spaces $\dot{B}_{p,q}^s(\mathbb{R}^n)$. We notice that the Besov-type spaces $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ when $p, q \in [1, \infty)$ were first introduced by El Baraka [169, 170, 171, 172]. El Baraka obtained some embedding and lifting properties and applied in [171] these results to study properties of solutions of certain elliptic systems.

Using the φ-transform characterizations of $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$, we obtain their embedding and lifting properties, which are also new even when $p, q \geq 1$; moreover, for appropriate τ, we also show that almost diagonal operators are bounded on their corresponding sequence spaces $\dot{a}_{p,q}^{s,\tau}(\mathbb{R}^n)$, which further induces the smooth atomic and molecular decomposition characterizations of $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$. For $s \in \mathbb{R}$, $p, q \in [1, \infty)$, $\max\{p, q\} > 1$ and $\tau \in \left[0, \frac{1}{(\max\{p,q\})'}\right]$, via the Hausdorff capacity, we introduce a new class of tent spaces $B\dot{T}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ and determine their dual spaces $BW_{p',q'}^{-s,\tau/q}(\mathbb{R}_{\mathbb{Z}}^{n+1})$; as an application of this, we

further introduce certain Hardy-Hausdorff spaces $B\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$ and show that the dual space of $B\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$ is just $\dot{B}^{-s,\tau}_{p',q'}(\mathbb{R}^n)$ when $p \in (1,\infty)$ and $q \in [1,\infty)$.

We establish some discrete Calderón reproducing formulae in $\mathcal{S}_\infty(\mathbb{R}^n)$ and its dual space $\mathcal{S}'_\infty(\mathbb{R}^n)$, which is a key tool.

For all $s, \tau \in \mathbb{R}$ and $q \in (0,\infty]$, we introduce the sequence spaces, $\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)$. Via the fact that $\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n) = \dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $Q_\alpha(\mathbb{R}^n) = \dot{F}^{\alpha,1/2-\alpha/n}_{2,2}(\mathbb{R}^n)$ (see [189]), we immediately obtain the connection between Besov spaces and Q spaces. Let φ satisfy (1) through (3). The main result is Theorem (4.1.5) below, which establish the φ-transform characterizations of $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$ in the sense of Frazier and Jawerth for all desired indices s, τ, p and q. This result generalizes the classical results on $\dot{B}^s_{p,q}(\mathbb{R}^n)$ and $\dot{F}^s_{p,q}(\mathbb{R}^n)$ in [45, 106]. From this characterization, we deduce that the spaces $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$ are independent of the choices of φ as in (1) through (3). Also, applying the φ-transform characterizations of $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$, we obtain some embedding properties for different metrics and the lifting properties of these spaces. If $\tau = 0$, all these results go back to the classical results. As a by-product, we also obtain the φ-transform characterization of the spaces $\dot{B}BMO^{s,q}_p(\mathbb{R}^n)$.

In Definition (4.1.16) below, for all $\varepsilon \in (0,\infty)$, we introduce a class of ε-almost diagonal operators on $\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)$, and then prove in Theorem (4.1.17) below that for all $s \in \mathbb{R}$, $p, q \in (0,\infty]$ and $\tau \in [0, 1/p + \varepsilon/(2n))$, ε-almost diagonal operators on $\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)$ are bounded on $\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)$. We establish the smooth atomic and molecular decomposition characterizations of $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$.

Let $s \in \mathbb{R}$, $p, q \in [1,\infty)$, $\max\{p,q\} > 1$ and $\tau \in \left[0, \frac{1}{(\max\{p,q\})'}\right]$. Using the Hausdorff capacity, we introduce tent spaces $B\dot{T}^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{Z}})$ and $B\dot{W}^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{Z}})$, which are related to Besov spaces. Then we establish their dual relations. These results generalize the corresponding results in [167] for the spaces with $p = q = 2$ and s and τ taking special values to the full ranges as above. We point out that our restriction on τ as above is optimal in certain sense.

Via the Tent spaces we introduce a new class of Hardy-Hausdorff spaces $B\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$, where $p \in (1,\infty), q \in [1,\infty)$. Via the duality of Tent spaces we further prove that the dual space of $B\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$, is just $\dot{B}^{-s,\tau}_{p',q'}(\mathbb{R}^n)$.

We make some conventions on notation. We denote by, C denotes unspecified positive constants, possibly different at each occurrence; the symbol $X \lesssim CY$ means that there exists a positive constant C such that $X \le CY$, and $X \sim Y$ means $C^{-1}Y \le X \le CY$. For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we set $\tilde{\varphi}(x) \equiv \overline{\varphi(-x)}$ for all $x \in \mathbb{R}^n$. For $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$ and $j \in \mathbb{Z}$, Q_{jk} denotes the dyadic cube $Q_{jk} \equiv \{(x_1, \dots, x_n): k_i \le 2^j x_i < k_i + 1 \text{ for } i = 1, \dots, n\} \in \mathbb{Z}^n$ and $Q \equiv \{Q_{jk}\}_{j,k}$. We denote by x_Q the lower left-corner $2^{-j}k$ of $Q = Q_{jk}$. When dyadic cube Q appears as an index, such as $\sum_Q$ and $\{\cdot\}_Q$, it is understood that Q runs over all dyadic cubes in $\mathbb{R}^n$. For each cube Q, we denote its side length by $l(Q)$, its center by c_Q, and for $r > 0$, we denote by rQ the cube concentric with Q having the side length $rl(Q)$. Let E be a set of $\mathbb{R}^n$. Denote by χ_E its characteristic function and E° its interior. Also, set $\mathbb{N} \equiv \{1, 2, \dots\}$ and $\mathbb{Z}_+ \equiv \mathbb{N} \cup \{0\}$.

Now we establish some Calderón reproducing formulae in $\mathcal{S}_\infty(\mathbb{R}^n)$ and its dual space $\mathcal{S}'_\infty(\mathbb{R}^n)$.

For a function φ and dyadic cube $Q = Q_{jk}$, set $\varphi_Q(x) \equiv |Q|^{-1/2}\varphi(2^j x - k) = |Q|^{1/2}\varphi_j(x - x_Q)$ for all $x \in \mathbb{R}^n$.

We establish the following discrete Calderón reproducing formula via [189]. It is well-known that this type of Calderón reproducing formula plays an important role in the study of Besov and Triebel-Lizorkin spaces when $p \leq 1$ or $q \leq 1$; see, for example, [106]. The difference between these Calderón reproducing formulae with those in [16,18] exists in that here, we use the distribution space $\mathcal{S}'_\infty(\mathbb{R}^n)$ instead of $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ therein.

Lemma (4.1.2) [190]: Let $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (4) such that $\operatorname{supp}\widehat{\varphi}, \operatorname{supp}\widehat{\psi}$ are compact and bounded away from the origin. Then for any $f \in \mathcal{S}_\infty(\mathbb{R}^n)$,

$$f = \sum_{j\in\mathbb{Z}} 2^{-jn} \sum_{k\in\mathbb{Z}^n} (\widetilde{\varphi}_j * f)(2^{-j}k)\psi_j(\cdot - 2^{-j}k) = \sum_Q \langle f, \varphi_Q\rangle \psi_Q \qquad (6)$$

holds in $\mathcal{S}_\infty(\mathbb{R}^n)$, where $\widetilde{\varphi}(x) \equiv \overline{\varphi(-x)}$ for all $x \in \mathbb{R}^n$. Moreover, for any $f \in \mathcal{S}_\infty(\mathbb{R}^n)$, (6) also holds in $\mathcal{S}'_\infty(\mathbb{R}^n)$.

Proof: Let $f \in \mathcal{S}_\infty(\mathbb{R}^n)$. By [189, Lemma 2.1], we have that $f = \sum_{j\in\mathbb{Z}} \psi_j * \widetilde{\varphi}_j * f$ in $\mathcal{S}_\infty(\mathbb{R}^n)$. Thus, the proof of (6) for $f \in \mathcal{S}_\infty(\mathbb{R}^n)$ is reduced to proving

$$\psi_j * \widetilde{\varphi}_j * f = 2^{-jn} \sum_{k\in\mathbb{Z}^n} (\widetilde{\varphi}_j * f)(2^{-j}k)\psi_j(\cdot - 2^{-j}k) \qquad (7)$$

in $\mathcal{S}_\infty(\mathbb{R}^n)$.

To show this, let $g \equiv \widetilde{\varphi}_j * f$. Obviously, $g \in \mathcal{S}_\infty(\mathbb{R}^n)$. Since $\operatorname{supp}\widehat{g} \subset 2^j[-\pi, \pi]^n$ expanding $\widehat{g}$ in the Fourier orthonormal basis $\left\{\frac{2^{-jn/2}}{(2\pi)^{n/2}} e^{-i\langle 2^{-j}k,\xi\rangle}\right\}_{k\in\mathbb{Z}^n}$ of $L^2(2^j[-\pi, \pi]^n)$, then for any $\xi \in 2^j[-\pi, \pi]^n$, we have

$$\widehat{g} = \sum_{k\in\mathbb{Z}^n} \frac{2^{-jn}}{(2\pi)^n}\left\{\int_{2^j[-\pi,\pi]^n} \widehat{g}(y)e^{i\langle 2^{-j}k,y\rangle}dy\right\}e^{-i\langle 2^{-j}k,\xi\rangle} = \sum_{k\in\mathbb{Z}^n} 2^{-jn}g(2^{-j}k)e^{-i\langle 2^{-j}k,\xi\rangle}. \qquad (8)$$

Notice that $\operatorname{supp}\widehat{\psi}_j \subset 2^j[-\pi, \pi]^n$, we can replace $\widehat{g}$ by its periodic extension without altering the product $\widehat{g}\widehat{\psi}_j$. Using $g * \psi_j = (\widehat{g}\widehat{\psi}_j)^\vee$ and (8), we obtain that for all $x \in \mathbb{R}^n$,

$$(g * \psi_j)(x) = \sum_{k\in\mathbb{Z}^n} 2^{-jn}g(2^{-j}k)\left(e^{-i\langle 2^{-j}k,\cdot\rangle}\widehat{\psi}_j\right)^\vee(x) = 2^{-jn}\sum_{k\in\mathbb{Z}^n}(\widetilde{\varphi}_j * f)(2^{-j}k)\psi_j(x - 2^{-j}k),$$

where $f^\vee(x) = \widehat{f}(-x)$ for all $x \in \mathbb{R}^n$. Thus, (7) holds pointwise.

To prove that (7) holds in $\mathcal{S}_\infty(\mathbb{R}^n)$, by the chain rule, for any $M \in \mathbb{Z}_+$ and $k \in \mathbb{Z}^n$, we have

$$\left\|g(2^{-j}k)\psi_j(\cdot - 2^{-j}k)\right\|_M \lesssim |g(2^{-j}k)| \sup_{\xi\in\mathbb{R}^n}\sup_{|\gamma|\leq M}\left|\partial^\gamma\left(e^{-i\langle 2^{-j}k,\xi\rangle}\right)\right| \sup_{|\gamma|\leq M}\left|\partial^\gamma\widehat{\psi}_j(\xi)\right|(|\xi|^M + |\xi|^{-M})$$

$$\sim |2^{-j}k|^M|g(2^{-j}k)|\|\psi_j\|_M.$$

Since g is a Schwartz function, then $|g(2^{-j}k)| \lesssim (1 + |(2^{-j}k)|)^{-M-n-1}$. Hence

$$\sum_{k\in\mathbb{Z}^n}\left\|2^{-jn}g(2^{-j}k)\psi_j(\cdot - 2^{-j}k)\right\|_M \lesssim \sum_{k\in\mathbb{Z}^n} 2^{-jn}|2^{-j}k|^M|g(2^{-j}k)|\|\psi_j\|_M < \infty,$$

which together with the completion of $\mathcal{S}_\infty(\mathbb{R}^n)$ implies that $\sum_{k\in\mathbb{Z}^n} 2^{-jn}g(2^{-j}k)\psi_j(\cdot - 2^{-j}k) \in \mathcal{S}_\infty(\mathbb{R}^n)$, and then (7) holds in $\mathcal{S}_\infty(\mathbb{R}^n)$. This shows that (6) holds in $\mathcal{S}_\infty(\mathbb{R}^n)$. To verify that (6) also holds for any $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$, by [189, Lemma 2.1] again, we only need to show that (7) holds in $\mathcal{S}'_\infty(\mathbb{R}^n)$.

To this end, let $g \equiv \widetilde{\varphi}_j * f$ again. Then it is well-known that g is a slowly increasing C^∞ function. For any $\delta > 0$, let $g_\delta(x) \equiv \gamma(\delta x)g(x)$, where $\gamma \in \mathcal{S}(\mathbb{R}^n)$ satisfies $\gamma(0) = 1$,

and supp $\hat{\gamma}$ is compact. Then $g_\delta \in \mathcal{S}(\mathbb{R}^n)$. For sufficient small δ, by the proof of (7), we have that for all $x \in \mathbb{R}^n$,

$$\psi_j * g_\delta(x) = 2^{-jn} \sum_{k \in \mathbb{Z}^n} g_\delta(2^{-j}k)\psi_j(x - 2^{-j}k) \tag{9}$$

in $\mathcal{S}_\infty(\mathbb{R}^n)$.

Assume that g is at most polynomially increasing with order $m \in \mathbb{Z}_+$. Since $\psi_j \in \mathcal{S}_\infty(\mathbb{R}^n)$, for any fixed $x \in \mathbb{R}^n$, we have

$$\left|\psi_j * g_\delta(x)\right| \le \int_{\mathbb{R}^n} \left|g_\delta(x - y)\psi_j(y)\right| dy \le C_{\gamma,j}(1 + |x|)^m < \infty,$$

$$\left|g_\delta(2^{-j}k)\right|\left|\psi_j(x - 2^{-j}k)\right| \le C_{\gamma,j}(2^{-j}k)^m(1 + |x - 2^{-j}k|)^{-(m+n+1)}$$

and

$$2^{-jn} \sum_{k \in \mathbb{Z}^n} \left|2^{-j}k\right|^m(1 + |x - 2^{-j}k|)^{-(m+n+1)} \le C_{\gamma,j} \int_{\mathbb{R}^n} |y|^m(1 + |x - y|)^{-(m+n+1)} dy < \infty.$$

Then applying the Lebesgue dominated convergence theorem and taking the limit as $\delta \to 0$ in (9), we obtain that (7) converges pointwise.

Noticing that for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\left|\langle\psi_j(\cdot - 2^{-j}k), \phi\rangle\right| \le C_j(1 + |x - 2^{-j}k|)^{-(m+n+1)}$, by $\left|g_\delta(2^{-j}k)\right| \le C_\gamma(1 + |2^{-j}k|)^m$, we have

$$\sum_{k \in \mathbb{Z}^n} 2^{-jn}\left|g_\delta(2^{-j}k)\right|\left|\langle\psi_j(\cdot - 2^{-j}k), \phi\rangle\right| \le C_{\gamma,j} \sum_{k \in \mathbb{Z}^n} (1 + |2^{-j}k|)^m(1 + |2^{-j}k|)^{-(m+n+1)} < \infty.$$

This observation together with the Lebesgue dominated convergence theorem and (9) implies that for any $\varphi \in \mathcal{S}_\infty(\mathbb{R}^n)$,

$$\langle\psi_j * g, \phi\rangle = \lim_{\delta \to 0}\langle\psi_j * g_\delta, \phi\rangle = \lim_{\delta \to 0} \sum_{k \in \mathbb{Z}^n} 2^{-jn}g_\delta(2^{-j}k)\langle\psi_j(\cdot - 2^{-j}k), \phi\rangle$$

$$= \sum_{k \in \mathbb{Z}^n} 2^{-jn}g(2^{-j}k)\langle\psi_j(\cdot - 2^{-j}k), \phi\rangle.$$

Thus (6) holds in $\mathcal{S}'_\infty(\mathbb{R}^n)$, which completes the proof of Lemma (4.1.2).

For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $M \in \mathbb{Z}_+$, set $\|\varphi\|_{\mathcal{S}_M} \equiv \sup_{|\gamma| \le M} \sup_{x \in \mathbb{R}^n} |\partial^\gamma\varphi(x)|(1 + |x|)^{n+M+|\gamma|}$. The following basic estimate is used; for its proof, see [189].

Lemma (4.1.3) [190]: For any $M \in \mathbb{N}$, there exists a positive constant $C = C(M, n)$ such that for all $\varphi, \psi \in \mathcal{S}_\infty(\mathbb{R}^n)$, $i, j \in \mathbb{Z}$ and $x \in \mathbb{R}^n$,

$$\left|\varphi_j * \psi_i(x)\right| \le C\|\varphi\|_{\mathcal{S}_{M+1}}\|\psi\|_{\mathcal{S}_{M+1}}2^{-|i-j|M}\frac{2^{-(i\wedge j)M}}{(2^{-(i\wedge j)} + |x|)^{n+M}},$$

where $i \wedge j \equiv \min\{i, j\}$.

Now we establish the φ-transform characterizations of the spaces $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$. To this end, we introduce their corresponding sequence spaces as follows.

Definition (4.1.4): Let $\tau, s \in \mathbb{R}$ and $q \in (0, \infty]$.

(i) Let $p \in (0, \infty]$. The sequence space $\dot{b}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is defined to be the set of all sequences $\{t_Q\}_Q \subset \mathbb{C}$ such that $\|t\|_{\dot{b}_{p,q}^{s,\tau}(\mathbb{R}^n)} < \infty$, where

$$\|t\|_{\dot{b}_{p,q}^{s,\tau}(\mathbb{R}^n)} \equiv \sup_{P \text{ dyadic}} \frac{1}{|P|^\tau}\left\{\sum_{j=j_P}^\infty 2^{jsq}\left[\int_P \left(\sum_{l(Q)=2^{-j}} |t_Q||\tilde{\chi}_Q(x)|\right)^p dx\right]^{q/p}\right\}^{1/q}$$

and $\tilde{\chi}_Q \equiv |Q|^{-1/2}\chi_Q$.

(ii) Let $p \in (0, \infty)$. The sequence space $\dot{f}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is defined to be the set of all sequences $\{t_Q\}_Q \subset \mathbb{C}$ such that $\|t\|_{\dot{f}_{p,q}^{s,\tau}(\mathbb{R}^n)} < \infty$, where

$$\|t\|_{\dot{f}_{p,q}^{s,\tau}(\mathbb{R}^n)} \equiv \sup_{P \text{ dyadic}} \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{Q \subset P} \left(|Q|^{-s/n} |t_Q| \tilde{\chi}_Q(x) \right)^q \right]^{p/q} dx \right\}^{1/p}.$$

Obviously, we have

$$\|t\|_{\dot{b}_{p,q}^{s,\tau}(\mathbb{R}^n)} \equiv \sup_{P \text{ dyadic}} \frac{1}{|P|^\tau} \left\{ \sum_{j=j_P}^{\infty} \left[\sum_{\substack{l(Q)=2^{-j} \\ Q \subset P}} \left(|Q|^{-s/n-1/2+1/p} |t_Q| \right)^p \right]^{q/p} \right\}^{1/q} \tag{10}$$

Similarly to the case of $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$, we use $\dot{a}_{p,q}^{s,\tau}(\mathbb{R}^n)$ to denote either $\dot{b}_{p,q}^{s,\tau}(\mathbb{R}^n)$ or $\dot{f}_{p,q}^{s,\tau}(\mathbb{R}^n)$. If $\dot{a}_{p,q}^{s,\tau}(\mathbb{R}^n)$ means $\dot{f}_{p,q}^{s,\tau}(\mathbb{R}^n)$, then the case $p = \infty$ is excluded.

Let φ and ψ satisfy (1) through (4). Recall that the φ-transform S_φ is defined to be the map taking each $f \in \mathcal{S}_\infty(\mathbb{R}^n)$ to the sequence $S_\varphi f = \{(S_\varphi f)_Q\}_Q$, where $(S_\varphi f)_Q \equiv \langle f, \varphi_Q \rangle$ for all dyadic cubes Q; the inverse φ-transform T_ψ is defined to be the map taking a sequence $t = \{t_Q\}_Q$ to $T_\psi t = \sum_Q t_Q \psi_Q$; see, for example, [45, 175]. Then we have the following result.

Theorem (4.1.5) [190]: Let $s \in \mathbb{R}$, $\tau \in [0, \infty)$, $p, q \in (0, \infty]$ and φ and ψ satisfy (1) through (4). Then the operators $S_\varphi : \dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n) \to \dot{a}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $T_\psi : \dot{a}_{p,q}^{s,\tau}(\mathbb{R}^n) \to \dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$ are bounded. Furthermore, $T_\psi \circ S_\varphi$ is the identity on $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$.

Lemma (4.1.6) [190]: Let $\delta \in \mathbb{R}$. Then there exist positive constants L_0 and C such that for all $j \in \mathbb{Z}$,

$$\sum_{Q \in \mathcal{Q}, l(Q)=2^{-j}} \frac{|Q|^\delta}{\left(1 + |x_Q|^n / \max\{1, |Q|\} \right)^{L_0}} \le C 2^{n(2|\delta|+1)|j|}.$$

To show that T_ψ is well defined for all $t \in \dot{a}_{p,q}^{s,\tau}(\mathbb{R}^n)$, we have the following conclusions.

Lemma (4.1.7) [190]: Let $s \in \mathbb{R}$, $\tau \in [0, \infty)$, $p, q \in (0, \infty]$ and $\psi \in \mathcal{S}_\infty(\mathbb{R}^n)$. Then for all $t \in \dot{a}_{p,q}^{s,\tau}(\mathbb{R}^n)$, $T_\psi t = \sum_Q t_Q \psi_Q$ converges in $\mathcal{S}_\infty'(\mathbb{R}^n)$; moreover, $T_\psi : \dot{a}_{p,q}^{s,\tau}(\mathbb{R}^n) \to \mathcal{S}_\infty'(\mathbb{R}^n)$ is continuous.

Proof: By Minkowski's inequality, we see that for all $s \in \mathbb{R}$, $\tau \in [0, \infty)$, $p \in (0, \infty)$ and $q \in (0, \infty]$, $\dot{b}_{p,\min(p,q)}^{s,\tau}(\mathbb{R}^n) \subset \dot{f}_{p,q}^{s,\tau}(\mathbb{R}^n) \subset \dot{b}_{p,\max\{p,q\}}^{s,\tau}(\mathbb{R}^n)$, which implies that to prove Lemma (4.1.7), it suffices to show it for the space $\dot{b}_{p,q}^{s,\tau}(\mathbb{R}^n)$.

Let $t \in \dot{b}_{p,q}^{s,\tau}(\mathbb{R}^n)$. We need to show that there exists an $M \in \mathbb{Z}_+$ such that for all $\phi \in \mathcal{S}_\infty(\mathbb{R}^n)$, $\sum_Q |t_Q| |\langle \psi_Q, \phi \rangle| \lesssim \|\phi\|_{S_M}$. Indeed, observe that for all dyadic cubes Q, $|t_Q| \le \|t\|_{\dot{b}_{p,q}^{s,\tau}(\mathbb{R}^n)} |Q|^{s/n+1/2+\tau-1/p}$. We have

$$\sum_Q |t_Q| |\langle \psi_Q, \phi \rangle| \le \|t\|_{\dot{b}_{p,q}^{s,\tau}(\mathbb{R}^n)} \sum_Q |Q|^{s/n+1/2+\tau-1/p} |\langle \psi_Q, \phi \rangle|.$$

To complete the proof, we need the following estimate that for any $L > 0$, there exists an $M \in \mathbb{Z}_+$ such that for all $Q, P \in \mathcal{Q}$,

$$|\langle \psi_Q, \phi_P \rangle| \lesssim \|\psi\|_{S_M} \|\phi\|_{S_M} \left(1 + \frac{|x_Q - x_P|^n}{\max\{|P|, |Q|\}} \right)^{-L} \left(\min\left\{ \frac{|Q|}{|P|}, \frac{|P|}{|Q|} \right\} \right)^L. \tag{11}$$

Recall that if $P = [0,1)^n$, then $\phi_P \equiv \phi$. Applying (11) for $P = [0,1)^n$, $Q = Q_{jk}$ and $L > \max\{1/p + 1/2 - s/n - \tau, \ 1/p + 3/2 + s/n + \tau, \ L_0\}$, where L_0 is as in Lemma (4.1.6), we obtain

$$|\langle \psi_Q, \phi \rangle| \lesssim \|\phi\|_{S_M} \left(1 + \frac{|x_Q|^n}{\max\{1, |Q|\}}\right)^{-L} (\min\{2^{-jn}, 2^{jn}\})^L.$$

Then applying Lemma (4.1.6) yields that

$$\sum_Q |t_Q| |\langle \psi_Q, \phi \rangle|$$

$$\lesssim \|t\|_{\dot{b}^{s,\tau}_{p,q}(\mathbb{R}^n)} \|\phi\|_{S_M} \sum_{j \in \mathbb{Z}} 2^{-|j|nL} 2^{-jn\left(\frac{s}{n}+\frac{1}{2}+\tau-1/p\right)} |\langle \psi_Q, \phi \rangle|$$

$$\times \sum_{Q \in \mathcal{Q}, l(Q) = 2^{-j}} \left(1 + |x_Q|^n / \max\{1, |Q|\}\right)^{-L} \lesssim \|t\|_{\dot{b}^{s,\tau}_{p,q}(\mathbb{R}^n)} \|\phi\|_{S_M}.$$

Therefore, $T_\psi t = \sum_Q t_Q \psi_Q$ converges in $\mathcal{S}'_\infty(\mathbb{R}^n)$; moreover, for all $t \in \dot{b}^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}_\infty(\mathbb{R}^n)$, $|\langle T_\psi t, \phi \rangle| \lesssim \|t\|_{\dot{b}^{s,\tau}_{p,q}(\mathbb{R}^n)} \|\phi\|_{S_M}$, which completes the proof of Lemma (4.1.7).

For a sequence $t = \{t_Q\}_Q$, $r \in (0, \infty]$ and a fixed $\lambda \in (0, \infty)$, set

$$(t^*_{r,\lambda})_Q \equiv \left(\sum_{\{R: l(R) = l(Q)\}} \frac{|t_R|^r}{(1 + l(R)^{-1}|x_R - x_Q|)^\lambda}\right)^{1/r}$$

and $t^*_{r,\lambda} \equiv \left\{(t^*_{r,\lambda})_Q\right\}_Q$. We have the following estimates.

Lemma (4.1.8) [190]: Let $s \in \mathbb{R}$, $\tau \in [0, \infty)$, $p, q \in (0, \infty]$ and $\lambda \in (n, \infty)$. Then there exists a constant $C \in [1, \infty)$ such that for all $t \in \dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)$, $\|t\|_{\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)} \leq \left\|t^*_{\min\{p,q\},\lambda}\right\|_{\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)} \leq C\|t\|_{\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)}$.

Proof: Notice that $|t_Q| \leq (t^*_{r,\lambda})_Q$ holds for all dyadic cubes Q. This observation immediately implies that $\|t\|_{\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)} \leq \left\|t^*_{p \wedge q, \lambda}\right\|_{\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)}$, where $p \wedge q = \min\{p, q\}$.

To see the converses, fix a dyadic cube P. Let $r_Q \equiv t_Q$ if $Q \subset 3P$ and $r_Q \equiv 0$ otherwise, and let $u_Q \equiv t_Q - r_Q$. Set $r \equiv \{r_Q\}_Q$ and $u \equiv \{u_Q\}_Q$. Then for all dyadic cubes Q, we have

$$\left(t^*_{p \wedge q, \lambda}\right)^{p \wedge q}_Q = \left(r^*_{p \wedge q, \lambda}\right)^{p \wedge q}_Q + \left(u^*_{p \wedge q, \lambda}\right)^{p \wedge q}_Q. \tag{12}$$

Applying the fact that for each sequence $t = \{t_Q\}_Q$, $\left\|t^*_{p \wedge q, \lambda}\right\|_{\dot{b}^s_{p,q}(\mathbb{R}^n)} \sim \|t\|_{\dot{b}^s_{p,q}(\mathbb{R}^n)}$ and $\left\|t^*_{p \wedge q, \lambda}\right\|_{\dot{f}^s_{p,q}(\mathbb{R}^n)} \sim \|t\|_{\dot{f}^s_{p,q}(\mathbb{R}^n)}$ (see [106]), we then have

$$I_P \equiv \frac{1}{|P|^\tau} \left\{\sum_{j=j_P}^\infty \left[\sum_{\substack{l(Q)=2^{-j} \\ Q \subset P}} \left[|Q|^{-s/n-1/2+1/p}\left(r^*_{p \wedge q, \lambda}\right)_Q\right]^p\right]^{q/p}\right\}^{1/q} \leq \frac{1}{|P|^\tau} \left\|r^*_{p \wedge q, \lambda}\right\|_{\dot{b}^s_{p,q}(\mathbb{R}^n)}$$

$$\lesssim \frac{1}{|P|^\tau} \|r\|_{\dot{b}^s_{p,q}(\mathbb{R}^n)} \lesssim \|t\|_{\dot{b}^{s,\tau}_{p,q}(\mathbb{R}^n)}$$

and similarly,

$$\tilde{I}_P \equiv \frac{1}{|P|^\tau} \left\{\int_P \left[\sum_{Q \subset P} \left[|Q|^{-s/n-1/2}\left(r^*_{p \wedge q, \lambda}\right)_Q \chi_Q(x)\right]^q\right]^{p/q} dx\right\}^{1/p} \lesssim \|t\|_{\dot{f}^{s,\tau}_{p,q}(\mathbb{R}^n)}.$$

On the other hand, let $Q \subset P$ be a dyadic cube with $l(Q) = 2^{-j}l(P)$ for some $i \in \mathbb{Z}_+$. Suppose $\tilde{Q}$ is any dyadic cube with $l(\tilde{Q}) = l(Q) = 2^{-j}l(P)$ and $\tilde{Q} \subset P + kl(P) \nsubseteq 3P$ for some $k \in \mathbb{Z}^n$, where $P + kl(Q) \equiv \{x + kl(P) : x \in P\}$. Then $|k| \geq 2$ and $1 + l(\tilde{Q})^{-1}|x_Q - x_{\tilde{Q}}| \sim 2^i|k|$. Thus,

$$J_P \equiv \frac{1}{|P|^\tau} \left\{ \sum_{i=0}^\infty \left[\sum_{\substack{l(Q)=2^{-i}l(P) \\ Q \subset P}} \left[|Q|^{-s/n-1/2+1/p} \left(u_{p\wedge q,\lambda}^* \right)_Q \right]^p \right]^{q/p} \right\}^{1/q}$$

$$\lesssim \frac{1}{|P|^\tau} \left\{ \sum_{i=0}^\infty 2^{inq/p - i\lambda q/(p\wedge q)} \left[\sum_{\substack{k\in\mathbb{Z}^n \\ |k|\geq 2}} |k|^{-\lambda} \sum_{\substack{l(\tilde{Q})=2^{-i}l(P) \\ \tilde{Q}\subset P+kl(P)}} \left(|\tilde{Q}|^{-s/n-1/2+1/p} |t_{\tilde{Q}}| \right)^{p\wedge q} \right]^{\frac{q}{p\wedge q}} \right\}^{1/q}.$$

When $p \leq q$, by $\lambda > n$, we have

$$J_P \lesssim \|t\|_{\dot{b}_{p,q}^{s,\tau}(\mathbb{R}^n)} \left\{ \sum_{i=0}^\infty 2^{inq/p - i\lambda q/p} \left(\sum_{\substack{k\in\mathbb{Z}^n \\ |k|\geq 2}} |k|^{-\lambda} \right)^{q/p} \right\}^{1/q} \lesssim \|t\|_{\dot{b}_{p,q}^{s,\tau}(\mathbb{R}^n)};$$

when $p > q$, by Hölder's inequality and $\lambda > n$, we obtain

$$J_P \lesssim \frac{1}{|P|^\tau} \left\{ \sum_{i=0}^\infty 2^{inq/p - i\lambda} \left[\sum_{\substack{k\in\mathbb{Z}^n \\ |k|\geq 2}} |k|^{-\lambda} \sum_{\substack{l(\tilde{Q})=2^{-i}l(P) \\ \tilde{Q}\subset P+kl(P)}} \left(|\tilde{Q}|^{-s/n-1/2+1/p} |t_{\tilde{Q}}| \right)^q \right] \right\}^{1/q}$$

$$\lesssim \|t\|_{\dot{b}_{p,q}^{s,\tau}(\mathbb{R}^n)} \left\{ \sum_{i=0}^\infty 2^{in - i\lambda} \left(\sum_{\substack{k\in\mathbb{Z}^n \\ |k|\geq 2}} |k|^{-\lambda} \right) \right\}^{1/q} \lesssim \|t\|_{\dot{b}_{p,q}^{s,\tau}(\mathbb{R}^n)}.$$

Therefore, by (12), $\left\| t_{\min\{p,q\},\lambda}^* \right\|_{\dot{b}_{p,q}^{s,\tau}(\mathbb{R}^n)} \lesssim \sup_{P\in Q}(I_P + J_P) \lesssim \|t\|_{\dot{b}_{p,q}^{s,\tau}(\mathbb{R}^n)}$.

To complete the proof, for any $i \in \mathbb{Z}_+$, $k \in \mathbb{Z}_+^n$ and dyadic cube P, set

$$A(i,k,P) \equiv \{\tilde{Q} \in Q : l(\tilde{Q}) = 2^{-i}l(P), \quad \tilde{Q} \subset P + kl(P), \quad \tilde{Q} \cap (3P) = \emptyset\}.$$

Recall that $1 + l(\tilde{Q})^{-1}|x_Q - x_{\tilde{Q}}| \sim 2^i|k|$ for any $Q \subset P$ and $\tilde{Q} \in A(i,k,P)$, and that for all $d \in [0,1]$ and $\{a_j\}_j \subset \mathbb{C}$,

$$\left(\sum_j |a_j| \right)^d \leq \sum_j |a_j|^d. \tag{13}$$

By (13), we obtain that for all $x \in P$ and $a \in (0, p \wedge q]$,

$$\sum_{\tilde{Q}\in A(i,k,P)} \frac{\left(|\tilde{Q}|^{-s/n-1/2} |t_{\tilde{Q}}| \right)^{p\wedge q}}{\left(1 + l(\tilde{Q})^{-1}|x_Q - x_{\tilde{Q}}| \right)^\lambda}$$

$$\lesssim 2^{-i(\lambda - n(p\wedge q)/a)} |k|^{-\lambda} \left[M_{HL} \left(\sum_{\substack{l(\tilde{Q})=2^{-i}l(P) \\ \tilde{Q}\subset P+kl(P)}} \left(|\tilde{Q}|^{-s/n} |t_{\tilde{Q}}| \tilde{\chi}_{\tilde{Q}} \right)^a \right) (x + kl(P)) \right]^{\frac{p\wedge q}{a}},$$

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where herein and in what follows, M_{HL} denotes the Hardy-Littlewood maximal function on $\mathbb{R}^n$. Let $a \equiv \frac{2n(p \wedge q)}{n+\lambda}$. Then $a \in (0, p \wedge q)$. Applying Minkowski's inequality, Fefferman-Stein's vector-valued inequality and Hölder's inequality, we have

$$\breve{J}_P \equiv \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{Q \subset P} \left[|Q|^{-\frac{s}{n}-\frac{1}{2}} \left(u^*_{p \wedge q, \lambda} \right)_Q \chi_Q(x) \right]^q \right]^{p/q} dx \right\}^{1/p}$$

$$\lesssim \frac{1}{|P|^\tau} \left\{ \int_P \left| \sum_{i=0}^{\infty} \left(\sum_{\substack{k \in \mathbb{Z}^n \\ |k| \geq 2}} 2^{-i(\lambda - n(p \wedge q)/a)} |k|^{-\lambda} \right. \right. \right.$$

$$\left. \left. \left. \times \left[M_{HL} \left(\sum_{\substack{l(\tilde{Q})=2^{-i}l(P) \\ \tilde{Q} \subset P + kl(P)}} \left(|\tilde{Q}|^{-s/n} |t_{\tilde{Q}}| \tilde{\chi}_{\tilde{Q}} \right)^a \right) (x + kl(P)) \right]^{\frac{p \wedge q}{a}} \right)^{\frac{q}{p \wedge q}} \right]^{p/q} \right| dx \right\}^{1/p}$$

$$\lesssim \|t\|_{\dot{f}^{s,\tau}_{p,q}(\mathbb{R}^n)}.$$

Therefore, by (12) again, $\|t^*_{\min\{p,q\},\lambda}\|_{\dot{f}^{s,\tau}_{p,q}(\mathbb{R}^n)} \lesssim \sup_{P \in Q}(\breve{I}_P + \breve{J}_P) \lesssim \|t\|_{\dot{f}^{s,\tau}_{p,q}(\mathbb{R}^n)}$, which completes the proof of Lemma (4.1.8).

Let φ satisfy (1) through (3). Since $\widetilde{\varphi}(x) \equiv \overline{\phi(-x)}$ also satisfy (1) through (3), we may take $\widetilde{\varphi}$ in place of φ in the definition of $\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)$. For any $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ and $Q \in Q$ with $l(Q) = 2^{-j}$, define the sequence $\sup(f) \equiv \left\{ \sup_Q(f) \right\}_Q$ by setting $\sup_Q(f) \equiv |Q|^{1/2} \sup_{y \in Q} |\widetilde{\varphi}_j * f(y)|$, and for any $\gamma \in \mathbb{Z}_+$, the sequence $\inf_\gamma(f) \equiv \left\{ \inf_{Q,\gamma}(f) \right\}_Q$ by setting $\inf_{Q,\gamma}(f) \equiv |Q|^{1/2} \max\left\{ \inf_{y \in \tilde{Q}} |\widetilde{\varphi}_j * f(y)| : l(\tilde{Q}) = 2^{-\gamma}l(Q), \tilde{Q} \subset Q \right\}$. We have the following estimates.

Lemma (4.1.9) [190]: Let $s \in \mathbb{R}$, $\tau \in [0, \infty)$, $p, q \in (0, \infty]$ and $\gamma \in \mathbb{Z}_+$ be sufficiently large. Then there exists a constant $C \in [1, \infty)$ such that for all $f \in \dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$,

$$C^{-1} \left\| \inf_\gamma(f) \right\|_{\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)} \leq \|f\|_{\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)} \leq \|\sup(f)\|_{\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)} \leq C \left\| \inf_\gamma(f) \right\|_{\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)}.$$

Proof: From Definitions (4.1.1), (4.1.4) and the definition of $\sup(f)$, it immediately follows that $\|f\|_{\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)} \leq \|\sup(f)\|_{\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)}$.

To prove the converses, define a sequence $t \equiv \{t_J\}_J$ by setting $t_J \equiv |J|^{1/2} \inf_{y \in J} |\widetilde{\varphi}_{i-\gamma} * f(y)|$ for all $J \in Q$ with $l(J) = 2^{-j}$. Then for all $r \in (0, \infty)$, dyadic cubes Q with $l(Q) = 2^{-j}$ and a fixed $\lambda > n$, we have

$$\inf_{Q,\gamma}(f) \tilde{\chi}_Q \lesssim 2^{\gamma \lambda / r} \sum_{\substack{\tilde{Q} \subset Q \\ l(\tilde{Q})=2^{-\gamma}l(Q)}} (t^*_{r,\lambda})_{\tilde{Q}} \tilde{\chi}_{\tilde{Q}}.$$

Picking $r = \min(p, q)$, then by Lemma (4.1.8), we obtain $\left\|\inf_\gamma(f)\right\|_{\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)} \lesssim$ $\|t^*_{r,\lambda}\|_{\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)} \lesssim \|t\|_{\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)}$. Finally, for each $j \in \mathbb{Z}$, applying Lemma A.4 in [106] to the function $\widetilde{\varphi}_j * f(2^{-j}x)$, we obtain that for all dyadic cubes Q with $l(Q) = 2^{-j}$,

$$\left(\sup(f)^*_{r,\lambda}\right)_Q \sim \left(\inf_\gamma(f)^*_{r,\lambda}\right)_Q, \text{ where } r = \min(p, q).$$

Thus, $\left\|\sup(f)^*_{r,\lambda}\right\|_{\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)} \sim \left\|\inf_\gamma(f)^*_{r,\lambda}\right\|_{\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)}$, which together with Lemma (4.1.8) yields

$\|\sup(f)\|_{\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)} \sim \left\|\inf_\gamma(f)\right\|_{\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)}$. This finishes the proof of Lemma (4.1.9).

Corollary (4.1.10) [190]: With all the notations as in Definition (4.1.1), then the spaces $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$ are independent of the choice of φ.

Let φ satisfy (1) through (3). In Lin and Wang [180], introduced spaces $\dot{bbmo}^{s,q}_p(\mathbb{R}^n)$, and $\dot{BBMO}^{s,q}_p(\mathbb{R}^n)$ to establish certain T(1) theorem for Besov spaces. Recall that the sequence space $\dot{bbmo}^{s,q}_p(\mathbb{R}^n)$ is defined to be the set of all sequences $\{t_Q\}_Q \subset \mathbb{C}$ such that

$$\|t\|_{\dot{bbmo}^{s,q}_p(\mathbb{R}^n)} = \sup_{P \text{ dyadic}} \left\{ \sum_{j=j_P}^\infty \left[\frac{1}{|P|} \sum_{\substack{l(Q)=2^{-j} \\ Q \subset P}} \left(|Q|^{-s/n-1/2+1/p}|t_Q|\right)^p \right]^{q/p} \right\}^{1/q} < \infty,$$

and the space $\dot{BBMO}^{s,q}_p(\mathbb{R}^n)$ in [180] is defined to be the set of all distributions f such that $\|f\|_{\dot{BBMO}^{s,q}_p(\mathbb{R}^n)} \equiv \|S_\varphi f\|_{\dot{bbmo}^{s,q}_p(\mathbb{R}^n)} < \infty$. Obviously, $\dot{bbmo}^{s,q}_p(\mathbb{R}^n) \equiv \dot{bbmo}^{s,1/p}_{p,q}(\mathbb{R}^n)$, which together with Theorem (4.1.5) yields that $\dot{BBMO}^{s,q}_p(\mathbb{R}^n) \equiv \dot{BBMO}^{s,1/p}_{p,q}(\mathbb{R}^n)$. Thus, Theorem (4.1.5) also gives the φ-transform characterization of the spaces $\dot{BBMO}^{s,q}_p(\mathbb{R}^n)$.

From Definition (4.1.1), it is easy to deduce the following basic properties of the spaces $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$; see also [41]. In what follows, the symbol $\subset$ stands for continuous embedding.

Proposition (4.1.11) [190]: Let $\tau, s \in \mathbb{R}$ and $p, q \in (0, \infty]$.

(i) If $q_1 \leq q_2$, then $\dot{A}^{s,\tau}_{p,q_1}(\mathbb{R}^n) \subset \dot{A}^{s,\tau}_{p,q_2}(\mathbb{R}^n)$;

(ii) $\dot{A}^{s,\tau+\frac{1}{p_2}-\frac{1}{p_1}}_{p_2,q}(\mathbb{R}^n) \subset \dot{A}^{s,\tau}_{p_1,q}(\mathbb{R}^n)$ for $0 < p_1 \leq p_2 \leq \infty$;

(iii) If $\varepsilon \in (0, \infty)$ and $q_1, q_2 \in (0, \infty]$, then $\dot{A}^{s+\varepsilon,\tau-\frac{\varepsilon}{n}}_{p,q_1}(\mathbb{R}^n) \subset \dot{A}^{s,\tau}_{p,q_2}(\mathbb{R}^n)$;

(iv) If $\tau \in [0, 1/p)$, then $\dot{A}^{s}_{\frac{p}{1-\tau p},q}(\mathbb{R}^n) \subset \dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$;

(v) If $\tau = 0$, then $\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n) = \dot{B}^{s}_{p,q}(\mathbb{R}^n)$ for $p \in (0, \infty]$ and $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n) = \dot{F}^{s}_{p,q}(\mathbb{R}^n)$ for $p \in (0, \infty)$ with equivalent norms;

(vi) If $\tau \in (-\infty, 0)$, then $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n) = \mathcal{P}(\mathbb{R}^n)$, where $\mathcal{P}(\mathbb{R}^n)$ denotes the set of all polynomials on $\mathbb{R}^n$;

(vii) For all $p \in (0, \infty)$, $\dot{B}^{s,\tau}_{p,\min\{p,q\}}(\mathbb{R}^n) \subset \dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n) \subset \dot{B}^{s,\tau}_{p,\max\{p,q\}}(\mathbb{R}^n)$;

(viii) For each $r \in (0, \infty)$, $\dot{F}^{s,\frac{1}{r}}_{r,q}(\mathbb{R}^n) = \dot{F}^{s}_{\infty,q}(\mathbb{R}^n)$ with equivalent norms;

(ix) Let $\tau \in [0, \infty)$. Then $\mathcal{S}_\infty(\mathbb{R}^n) \subset \dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$.

Proof: The properties (i) through (vii) are simple corollaries of both the monotonicity of the l^q-norm on q (see (13)) and Hölder's inequality. We leave the details to the reader. Property (viii) is just [106, Corollary 5.7].

To prove (ix), let $f \in \mathcal{S}_\infty(\mathbb{R}^n)$ and φ be as in Definition (4.1.1). Then by Lemma (4.1.3), we obtain that for all $k \in \mathbb{Z}$,

$$|\varphi_k * f(x)| \lesssim \|f\|_{S_{M+1}} \|\varphi\|_{S_{M+1}} 2^{-|k|M} \frac{2^{-(0\wedge k)M}}{(2^{-(0\wedge k)} + |x|)^{n+M}},$$

where $M > 0$ will be determined later. We now show $f \in \dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$.

Let P be an arbitrary dyadic cube. If $j_P \geq 0$, choosing $M > \max\{0, n(1/p - 1), s + n\tau\}$, we then have

$$J_P \equiv \frac{1}{|P|^\tau} \left\{ \sum_{j=j_P}^\infty 2^{jsq} \left[\int_P \sum_{l(Q)=2^{-j}} |\varphi_j * f(x)|^p \, dx \right]^{q/p} \right\}^{1/q} \lesssim \|f\|_{S_{M+1}} \|\varphi\|_{S_{M+1}}.$$

If $j_P < 0$, then $|P|^{-\tau} \leq 1$ for all $\tau \in [0, \infty)$. Letting $M > \max\{0, s, n(1/p - 1), +n(1/p - 1) - s\}$, we see that

$$J_P \leq \left\{ \sum_{j=0}^\infty 2^{jsq} \left[\int_P |\varphi_j * f(x)|^p dx \right]^{q/p} \right\}^{1/q} + \left\{ \sum_{j=j_P}^{-1} 2^{jsq} \dots \right\}^{1/q} \lesssim \|f\|_{S_{M+1}} \|\varphi\|_{S_{M+1}}.$$

Thus, $\|f\|_{\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)} = \sup_{P \in \mathcal{Q}} J_P \lesssim \|f\|_{S_{M+1}} \|\varphi\|_{S_{M+1}}$, namely, $\mathcal{S}_\infty(\mathbb{R}^n) \subset \dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ for all $s \in \mathbb{R}$, $\tau \in [0, \infty)$ and $p, q \in (0, \infty]$.

Applying Proposition (4.1.11) (vii), we also have $\mathcal{S}_\infty(\mathbb{R}^n) \subset \dot{B}_{p,\min\{p,q\}}^{s,\tau}(\mathbb{R}^n) \subset \dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$, which completes the proof of Proposition (4.1.11).

Proposition (4.1.12) [190]: Let $s \in \mathbb{R}$ and $q \in (0, \infty]$.

(i) If $r \in (0, \infty)$, then $\dot{B}_{\infty,q}^s(\mathbb{R}^n) \subsetneq \dot{B}_{r,q}^{s,1/r}(\mathbb{R}^n)$ and particularly $\dot{B}_{\infty,q}^s(\mathbb{R}^n) \subsetneq \widetilde{\dot{B}_{\infty,q}^s}(\mathbb{R}^n)$.

(ii) If $r \in [q, \infty]$, then $\dot{B}_{r,q}^{s,1/r}(\mathbb{R}^n) \subset \widetilde{\dot{B}_{\infty,q}^s}(\mathbb{R}^n)$; if $r \in (0, q]$, then $\widetilde{\dot{B}_{\infty,q}^s}(\mathbb{R}^n) \subset \dot{B}_{r,q}^{s,1/r}(\mathbb{R}^n)$.

Proof: By Theorem (4.1.5), it suffices to prove the corresponding conclusions on sequence spaces $\dot{b}_{p,q}^{s,\tau}(\mathbb{R}^n)$.

(i) The proof of $\dot{b}_{\infty,q}^s(\mathbb{R}^n) \subset \dot{b}_{r,q}^{s,1/r}(\mathbb{R}^n)$ is trivial. Next we show that there exists certain $t \in \dot{b}_{r,q}^{s,1/r}(\mathbb{R}^n)$ but $\|t\|_{\dot{b}_{\infty,q}^s(\mathbb{R}^n)} = \infty$. Indeed, for all $j \in \mathbb{Z}$ and $Q \in \mathcal{Q}$ with $l(Q) = 2^{-j}$, let $t_Q \equiv 2^{-jn(s/n+1/2)}$ when $Q = [0, 2^{-j})^n$ and $t_Q \equiv 0$ otherwise. Obviously, $\|t\|_{\dot{b}_{\infty,q}^s(\mathbb{R}^n)} = \infty$. Observe that if a dyadic cube P contains $[0, 2^{-j})^n$, then $P = [0, 2^{-j_P})^n$ and $j_P \leq j$. Hence, $\|t\|_{\dot{b}_{r,q}^{s,1/r}(\mathbb{R}^n)} = \sup_{j_P \in \mathbb{Z}} \{ \sum_{j=j_P}^\infty 2^{-(j-j_P)nq/r} \}^{1/q} \lesssim 1$. That is, $t \in \dot{b}_{r,q}^{s,1/r}(\mathbb{R}^n)$.

(ii) From Hölder's inequality, it is easy to deduce that if $r \in [q, \infty]$, then $\|t\|_{\widetilde{\dot{b}_{\infty,q}^s}(\mathbb{R}^n)} \leq \|t\|_{\dot{b}_{r,q}^{s,1/r}(\mathbb{R}^n)}$, and if $r \in (0, q]$, then $\|t\|_{\dot{b}_{r,q}^{s,1/r}(\mathbb{R}^n)} \leq \|t\|_{\widetilde{\dot{b}_{\infty,q}^s}(\mathbb{R}^n)}$, which completes the proof of Proposition (4.1.12).

We establish some embedding results, which generalize the classical results on Besov spaces $\dot{B}_{p,q}^s(\mathbb{R}^n)$ and Triebel-Lizorkin spaces $\dot{F}_{p,q}^s(\mathbb{R}^n)$ (see [41]). In fact, if $\tau = 0$, then Proposition (4.1.13) is just [41, Theorem 2.7.1].

Proposition (4.1.13) [190]: Let $\tau \in [0, \infty), r, q \in (0, \infty]$ and $-\infty < s_1 < s_0 < \infty$.

(i) If $0 < p_0 < p_1 \leq \infty$ such that $s_0 - n/p_0 = s_1 - n/p_1$, then $\dot{B}^{s_0,\tau}_{p_0,q}(\mathbb{R}^n) \subset \dot{B}^{s_1,\tau}_{p_1,q}(\mathbb{R}^n)$.

(ii) If $0 < p_0 < p_1 < \infty$ such that $s_0 - n/p_0 = s_1 - n/p_1$, then $\dot{F}^{s_0,\tau}_{p_0,r}(\mathbb{R}^n) \subset \dot{F}^{s_1,\tau}_{p_1,q}(\mathbb{R}^n)$.

Proof: By Theorem (4.1.5), it suffices to prove the corresponding conclusions on sequence spaces $\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)$.

The embedding $\dot{b}^{s_0,\tau}_{p_0,q}(\mathbb{R}^n) \subset \dot{b}^{s_1,\tau}_{p_1,q}(\mathbb{R}^n)$ is immediately deduced from (10) and (13). To prove $\dot{f}^{s_0,\tau}_{p_0,r}(\mathbb{R}^n) \subset \dot{f}^{s_1,\tau}_{p_1,q}(\mathbb{R}^n)$, by Proposition (4.1.11)(i), we only need to show that $\dot{f}^{s_0,\tau}_{p_0,\infty}(\mathbb{R}^n) \subset \dot{f}^{s_1,\tau}_{p_1,q}(\mathbb{R}^n)$. Let $t \in \dot{f}^{s_0,\tau}_{p_0,\infty}(\mathbb{R}^n)$. By the homogeneity of $\|\cdot\|_{\dot{f}^{s,\tau}_{p,q}(\mathbb{R}^n)}$, without loss of generality, we may assume that $\|t\|_{\dot{f}^{s_0,\tau}_{p_0,\infty}(\mathbb{R}^n)} = 1$.

For any $\lambda \in (0, \infty)$ and $P \in Q$, pick $N \in \mathbb{Z}$ such that

$$\{1 - 2^{-qn/p_1}\}^{1/q} \frac{\lambda}{2^{1+n/p_1}} < |P|^\tau 2^{nN/p_1} \leq \{1 - 2^{-qn/p_1}\}^{1/q} \frac{\lambda}{2}.$$

If $N \geq j_P$, since $|Q|^{-s_0/n-1/2}|t_Q| \leq |Q|^{\tau-1/p_0}\|t\|_{\dot{f}^{s_0,\tau}_{p_0,\infty}(\mathbb{R}^n)} = 2^{-jn(\tau-1/p_0)}$ for all $Q \in Q$ with $l(Q) = 2^{-j}$, this together with $s_0 - n/p_0 = s_1 - n/p_1$ yields that

$$\left\{\sum_{j=j_P}^{N} 2^{-jq(s_0-s_1)} \sup_{\substack{l(Q)=2^{-j} \\ Q \subset P}} \left(|Q|^{-s_0/n}|t_Q|\tilde{\chi}_Q(x)\right)^q\right\}^{1/q} \leq |P|^\tau 2^{nN/p_1}\{1 - 2^{-qn/p_1}\}^{-1/q} \leq \lambda/2,$$

and

$$\left\{\sum_{j=N+1}^{\infty} 2^{-jq(s_0-s_1)} \sup_{\substack{l(Q)=2^{-j} \\ Q \subset P}} \left(|Q|^{-s_0/n}|t_Q|\tilde{\chi}_Q(x)\right)^q\right\}^{1/q}$$
$$\leq 2^{-p_1(s_0-s_1)/n}\{1 - 2^{-qn/p_1}\}^{-p_1(s_0-s_1)/qn}\{1 - 2^{-q(s_0-s_1)}\}^{-1/q}$$
$$\times |P|^{\tau p_1(s_0-s_1)/n}\lambda^{-p_1(s_0-s_1)/n} \sup_{Q \subset P}\left(|Q|^{-s_0/n}|t_Q|\tilde{\chi}_Q(x)\right).$$

Notice that for all dyadic cubes P,

$$\left\{\sum_{Q \subset P}\left(|Q|^{-s_0/n}|t_Q|\tilde{\chi}_Q(x)\right)^q\right\}^{1/q} = \left\{\sum_{j=j_P}^{\infty} 2^{-jq(s_0-s_1)} \sup_{\substack{l(Q)=2^{-j} \\ Q \subset P}} \left(|Q|^{-s_0/n}|t_Q|\tilde{\chi}_Q(x)\right)^q\right\}^{1/q}.$$

Then by $s_0 - n/p_0 = s_1 - n/p_1$, we have

$$\left|\left\{x \in P : \left\{\sum_{Q \subset P}\left(|Q|^{-s_0/n}|t_Q|\tilde{\chi}_Q(x)\right)^q\right\}^{1/q} > \lambda\right\}\right|$$
$$\lesssim \left|\left\{x \in P : \sup_{Q \subset P}\left(|Q|^{-s_1/n}|t_Q|\tilde{\chi}_Q(x)\right)\right.\right.$$
$$> 2^{-1-p_1(s_0-s_1)/n}$$
$$\left.\left.\times \{1 - 2^{-qn/p_1}\}^{-p_1(s_0-s_1)/qn}\{1 - 2^{-q(s_0-s_1)}\}^{1/q}|P|^{-\tau(p_1/p_0-1)}\lambda^{p_1/p_0}\right\}\right|,$$

and hence

$$\|t\|^{p_1}_{\dot{f}^{s_1,\tau}_{p_1,q}(\mathbb{R}^n)} \lesssim \sup_{P \text{ dyadic}} \frac{1}{|P|^{-\tau p_0}} \int_0^\infty \lambda^{p_0-1}\left|\left\{x \in P : \sup_{Q \subset P}\left(|Q|^{-s_1/n}|t_Q|\tilde{\chi}_Q(x)\right) > \lambda\right\}\right| d\lambda \sim \|t\|^{p_0}_{\dot{f}^{s_0,\tau}_{p_1,\infty}(\mathbb{R}^n)} \sim 1.$$

For the case $N < j_P$, notice that

$$\left\{\sum_{j=j_P}^{\infty} \sup_{\substack{l(Q)=2^{-j}\\Q\subset P}} \left(|Q|^{-s_1/n}|t_Q|\tilde{\chi}_Q(x)\right)^q\right\}^{1/q} \leq \left\{\sum_{j=N+1}^{\infty} \sup_{\substack{l(Q)=2^{-j}\\Q\subset P}} \left(|Q|^{-s_1/n}|t_Q|\tilde{\chi}_Q(x)\right)^q\right\}^{1/q}.$$

By the argument same as above, we also obtain $\|t\|_{\dot{f}^{s_1,\tau}_{p_1,q}(\mathbb{R}^n)} \lesssim 1$, which completes the proof of Proposition (4.1.13).

From Proposition (4.1.13), we deduce the following properties of the space $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$. Recall that the symbol $\subset$ stands for continuous embedding

Proposition (4.1.12) [190]: Let $s \in \mathbb{R}, \tau \in [0, \infty)$ and $p, q \in (0, \infty]$. Then $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n) \subset \mathcal{S}'_\infty(\mathbb{R}^n)$.

Proof: We first show that $\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n) \subset \mathcal{S}'_\infty(\mathbb{R}^n)$, namely, there exists an $M \in \mathbb{N}$ such that for all $f \in \dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $\phi \in \mathcal{S}_\infty(\mathbb{R}^n)$, $|\langle f, \phi\rangle| \lesssim \|f\|_{\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)}\|\phi\|_{\mathcal{S}_{M+1}}$. Indeed, let φ be as in Definition (4.1.1). By [176, Lemma (6.9)], there exists a function ψ such that φ and ψ satisfy (1) through (4). Then by [189, Lemma 2.1] and Lemma (4.1.3), we obtain

$$|\langle f, \phi\rangle| \leq \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} |\psi_j * \phi(x)||\varphi_j * f(x)|\,dx$$

$$\lesssim \|\phi\|_{\mathcal{S}_{M+1}} \sum_{j\in\mathbb{Z}} 2^{-|j|M} \sum_{k\in\mathbb{Z}^n} \int_{Q_{j\wedge0,k}} \frac{2^{-(j\wedge0)M}}{(2^{-(j\wedge0)M}+|x|)^{n+M}}|\varphi_j * f(x)|\,dx,$$

where $Q_{j\wedge0,k}$ denotes the dyadic cube $2^{-(j\wedge0)}(k+[0,1)^n)$ and $M \in \mathbb{N}$ will be determined later.

Notice that there exist 2^n disjoint dyadic cubes $\{Q^l_{j\wedge0}\}_{l=1}^{2^n}$ with $l(Q^l_{j\wedge0}) = 2^{-(j\wedge0)}$ such that the ball $B(0, 2^{-(j\wedge0)}) \subset (\cup_{l=1}^{2^n} Q^l_{j\wedge0})$. Obviously, if $Q_{j\wedge0,k} \notin \{Q^l_{j\wedge0}\}_{l=1}^{2^n}$ and $x \in Q_{j\wedge0,k}$, then $|x| \geq 2^{-(j\wedge0)}$. Moreover, if setting

$$\chi_{j,m}(k) \equiv \chi_{\{k\in\mathbb{Z}^n : 2^{m-(j\wedge0)} \leq |c_{Q_{j\wedge0,k}}| < 2^{m+1-(j\wedge0)}\}}(k),$$

where $c_{Q_{j\wedge0,k}}$ denotes the center of $Q_{j\wedge0,k}$ we then have $\sum_{k\in\mathbb{Z}^n} \chi_{j,m}(k) \lesssim 2^{mn}$. If $p \in [1, \infty]$, choose $M > \max\{-s, s + n\tau - n/p\}$. Then applying Hölder's inequality, we obtain

$$|\langle f, \phi\rangle| \lesssim \|\phi\|_{\mathcal{S}_{M+1}} \sum_{j\in\mathbb{Z}} 2^{-|j|M} \left\{\sum_{l=1}^{2^n} \int_{Q^l_{j\wedge0}} \frac{2^{-(j\wedge0)M}}{(2^{-(j\wedge0)M}+|x|)^{n+M}}|\varphi_j * f(x)|\,dx\right.$$

$$\left. + \sum_{m=0}^{\infty} \sum_{k\in\mathbb{Z}^n} \chi_{j,m}(k) \int_{Q_{j\wedge0,k}} \cdots\right\} \lesssim \|\phi\|_{\mathcal{S}_{M+1}}\|f\|_{\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)}.$$

The case $p \in (0, 1)$ is deduced from the embedding $\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n) \subset \dot{B}^{s-n/p+n,\tau}_{1,q}(\mathbb{R}^n)$ in Proposition (4.1.13).

For Triebel-Liorkin-type spaces $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$, applying Proposition (4.1.11)(vii), we obtain $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n) \subset \dot{B}^{s,\tau}_{p,\max\{p,q\}}(\mathbb{R}^n) \subset \mathcal{S}'_\infty(\mathbb{R}^n)$, which completes the proof of Proposition (4.1.14).

Now we have the following lifting property. For $\sigma \in \mathbb{R}$, recall that the lifting operator $\dot{I}_\sigma$ (see, for example, [41]) is defined by $(\dot{I}_\sigma f)(x) \equiv (|\cdot|^\sigma \hat{f})^\vee(x)$ for all $x \in \mathbb{R}^n$

and $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$, where the symbol $^\vee$ denotes the inverse Fourier transform. It is well-known that $\dot{I}_\sigma$ maps $\mathcal{S}'_\infty(\mathbb{R}^n)$ onto itself.

Proposition (4.1.15) [190]: Let $s, \sigma \in \mathbb{R}, \tau \in [0, \infty)$ and $p, q \in (0, \infty]$. Then $\dot{I}_\sigma$ maps $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$ isomorphically onto $\dot{A}^{s-\sigma,\tau}_{p,q}(\mathbb{R}^n)$; moreover, $\|f\|_{\dot{A}^{s-\sigma,\tau}_{p,q}(\mathbb{R}^n)}$ is an equivalent quasi-norm on $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$.

The proof of Proposition (4.1.15) is standard (see [41]).
As an application of Theorem (4.1.5), we study boundedness of operators in $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$ by first considering their boundedness in corresponding $\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)$. We show that almost diagonal operators are bounded on $\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)$ for appropriate indices, which generalize the classical results on $\dot{b}^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $\dot{f}^{s,\tau}_{p,q}(\mathbb{R}^n)$; see [106, 176].

Definition (4.1.16) [190]: Let $s \in \mathbb{R}, p, q \in (0, \infty]$ and $\varepsilon \in (0, \infty)$. Let $J \equiv n/\min\{1, p\}$ when $\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n) \equiv \dot{b}^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $J \equiv n/\min\{1, p, q\}$ when $\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n) \equiv \dot{f}^{s,\tau}_{p,q}(\mathbb{R}^n)$. An operator A associated with a matrix $\{a_{QP}\}_{Q,P}$, namely, for all sequences $t = \{t_Q\}_Q \subset \mathbb{C}$, $At \equiv \{(At)_Q\}_Q \equiv \{\sum_P a_{QP} t_P\}_Q$, is called ε-almost diagonal on $\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)$ if the matrix $\{a_{QP}\}_{Q,P}$ satisfies $\sup_{Q,P} |a_{QP}|/\omega_{QP}(\varepsilon) < \infty$, where

$$\omega_{QP}(\varepsilon) \equiv \left(\frac{l(Q)}{l(P)}\right)^s \left(1 + \frac{|x_Q - x_P|}{\max(l(P), l(Q))}\right)^{-J-\varepsilon} \times \min\left[\left(\frac{l(Q)}{l(P)}\right)^{(n+\varepsilon)/2}, \left(\frac{l(P)}{l(Q)}\right)^{(n+\varepsilon)/2+J-n}\right].$$

Theorem (4.1.17) [190]: Let $\varepsilon \in (0, \infty), s \in \mathbb{R}, p, q \in (0, \infty]$ and $\tau \in [0, 1/p + \varepsilon/(2n))$. Then all ε-almost diagonal operators on $\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)$ are bounded on $\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)$.

Proof: Let $t = \{t_Q\}_Q \in \dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)$ and A be a ε-almost diagonal operator on $\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)$ associated with the matrix $\{a_{QR}\}_{Q,R}$ and $\varepsilon \in (0, \infty)$. Without loss of generality, we may assume $s = 0$. Indeed, if the conclusion holds for $s = 0$, let $\tilde{t}_R \equiv l(R)^{-s} t_R$ and B be the operator associated with the matrix $\{b_{QR}\}_{Q,R}$, where $b_{QR} \equiv (l(R)/l(Q))^s a_{QR}$ for all $Q, R \in \mathcal{Q}$. Then we have $\|At\|_{\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)} = \|B\tilde{t}\|_{\dot{a}^{0,\tau}_{p,q}(\mathbb{R}^n)} \lesssim \|\tilde{t}\|_{\dot{a}^{0,\tau}_{p,q}(\mathbb{R}^n)} \sim \|t\|_{\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)}$, which deduces the desired conclusions.

We now consider the space $\dot{b}^{0,\tau}_{p,q}(\mathbb{R}^n)$ in the case $\min(p, q) > 1$. For all $Q \in \mathcal{Q}$, we write $A \equiv A_0 + A_1$ with $(A_0 t)_Q \equiv \sum_{\{R:l(R) \geq l(Q)\}} a_{QR} t_R$ and $(A_1 t)_Q \equiv \sum_{\{R:l(R) < l(Q)\}} a_{QR} t_R$. By Definition (4.1.16), we see that for all $Q \in \mathcal{Q}$,

$$|(A_0 t)_Q| \lesssim \sum_{\{R:l(R) \geq l(Q)\}} \left(\frac{l(Q)}{l(R)}\right)^{\frac{(n+\varepsilon)}{2}} \frac{|t_R|}{\left(1 + l(R)^{-1}|x_Q - x_R|\right)^{n+\varepsilon}},$$

and therefore

$$\|A_0 t\|_{\dot{b}^{0,\tau}_{p,q}(\mathbb{R}^n)} \lesssim \sup_{P \text{ dyadic}} \frac{1}{|P|^\tau} \left\{ \sum_{j=j_P}^{\infty} \left[\int_P \left(\sum_{l(Q)=2^{-j}} \sum_{\{R:l(Q)\le l(R)\le l(P)\}} \left(\frac{l(Q)}{l(R)}\right)^{(n+\varepsilon)/2} \right. \right. \right.$$

$$\left. \left. \left. \times \frac{|t_R|\tilde{\chi}_Q(x)}{\left(1+l(R)^{-1}|x_Q - x_R|\right)^{n+\varepsilon}} \right)^p dx \right]^{q/p} \right\}^{1/q}$$

$$+ \sup_{P \text{ dyadic}} \frac{1}{|P|^\tau} \left\{ \sum_{j=j_P}^{\infty} \left[\int_P \left(\sum_{l(Q)=2^{-j}} \sum_{\{R:l(R)>l(P)\}} \cdots \right)^p dx \right]^{q/p} \right\}^{1/q} \equiv I_1 + I_2.$$

For all $i \in \mathbb{Z}$ and $m \in \mathbb{N}$, set $U_{0,i} \equiv \{R \in \mathcal{Q} : l(R) = 2^{-i} \text{ and } |x_Q - x_R| < l(R)\}$ and, $U_{m,i} \equiv \{R \in \mathcal{Q} : l(R) = 2^{-i} \text{ and } 2^{m-1}l(R) \le |x_Q - x_R| < 2^m l(R)\}$. Then we have $\#U_{m,i} \lesssim 2^{mn}$, where $\#U_{m,i}$ denotes the cardinality of $U_{m,i}$. Notice that $|t_R| \le |R|^{1/2-1/p+\tau}\|t\|_{\dot{b}^{0,\tau}_{p,q}(\mathbb{R}^n)}$ for all $R \in \mathcal{Q}$. Thus, by $0 \le \tau < \frac{1}{p} + \frac{\varepsilon}{2n}$,

$$I_2 \lesssim \|t\|_{\dot{b}^{0,\tau}_{p,q}(\mathbb{R}^n)} \sup_{P \text{ dyadic}} \frac{1}{|P|^\tau} \left\{ \sum_{j=j_P}^{\infty} \left[\int_P \left(\sum_{l(Q)=2^{-j}} \sum_{i=-\infty}^{j_P-1} \sum_{m=0}^{\infty} \sum_{R\in U_{m,i}} \left(\frac{l(Q)}{l(R)}\right)^{(n+\varepsilon)/2} \right. \right. \right.$$

$$\left. \left. \left. \times \frac{|R|^{1/2-1/p+\tau}\tilde{\chi}_Q(x)}{\left(1+l(R)^{-1}|x_Q - x_R|\right)^{n+\varepsilon}} \right)^p dx \right]^{q/p} \right\}^{1/q} \lesssim \|t\|_{\dot{b}^{0,\tau}_{p,q}(\mathbb{R}^n)}$$

For I_1, let r and u be the same as in the proof of Lemma (4.1.8). We see that

$$I_1 \lesssim \sup_{P \text{ dyadic}} \frac{1}{|P|^\tau} \left\{ \sum_{j=j_P}^{\infty} \left[\int_P \left(\sum_{l(Q)=2^{-j}} \sum_{i=j_P}^{j} 2^{(i-j)(n+\varepsilon)/2} \times \sum_{l(R)=2^{-i}} \frac{|r_R|\tilde{\chi}_Q(x)}{\left(1+l(R)^{-1}|x_Q-x_R|\right)^{n+\varepsilon}} \right)^p dx \right]^{q/p} \right\}^{1/q}$$

$$+ \sup_{P \text{ dyadic}} \frac{1}{|P|^\tau} \left\{ \sum_{j=j_P}^{\infty} \left[\int_P \left(\sum_{l(Q)=2^{-j}} \sum_{i=j_P}^{j} 2^{(i-j)(n+\varepsilon)/2} \right. \right. \right.$$

$$\left. \left. \left. \times \sum_{l(R)=2^{-i}} \frac{|u_R|\tilde{\chi}_Q(x)}{\left(1+l(R)^{-1}|x_Q-x_R|\right)^{n+\varepsilon}} \right)^p dx \right]^{q/p} \right\}^{1/q} \equiv J_1 + J_2$$

Applying [106, Remark A.3] with $a = 1$, for all $x \in Q$, we have

$$\sum_{l(R)=2^{-i}} \frac{|r_R|\tilde{\chi}_Q(x)}{\left(1+l(R)^{-1}|x_Q-x_R|\right)^{n+\varepsilon}} \lesssim M_{HL}\left(\sum_{l(R)=2^{-i}} |r_R|\chi_R\right)(x).$$

Hence Hölder's inequality and the $L^p(\mathbb{R}^n)$-boundedness for $p \in (1, \infty]$ of the Hardy-Littlewood maximal operator yield

$$J_1 \lesssim \sup_{P \text{ dyadic}} \frac{1}{|P|^\tau} \left\{ \sum_{j=j_P}^\infty \left[\int_P \left(\sum_{i=j_P}^j 2^{(i-j)\varepsilon/2} M_{HL}\left(\sum_{l(R)=2^{-i}} |r_R|\tilde{\chi}_R \right)(x) \right)^p dx \right]^{q/p} \right\}^{1/q}$$

$$\lesssim \sup_{P \text{ dyadic}} \frac{1}{|P|^\tau} \left\{ \sum_{i=j_P}^\infty \left[\int_{3P} \left(\sum_{l(R)=2^{-i}} |t_R|\tilde{\chi}_R(x) \right)^p dx \right]^{q/p} \right\}^{1/q} \lesssim \|t\|_{\dot{b}_{p,q}^{0,\tau}(\mathbb{R}^n)},$$

where the last inequality follows from Minkowski's inequality if $q > p$ or (13) if $q \le p$. To estimate J_2, we notice that if $R \cap (3P) = \emptyset$, then $R \subset P + kl(P)$ and $(P + kl(P)) \cap (3P) = \emptyset$ for some $k \in \mathbb{Z}^n$ with $|k| \ge 2$ and $1 + l(R)^{-1}|x_Q - x_R| \sim |k|l(P)/l(R)$. Therefore, by Hölder's inequality,

$$J_2 \lesssim \sup_{P \text{ dyadic}} \frac{1}{|P|^{\tau+1+\frac{\varepsilon}{n}}} \left\{ \sum_{j=j_P}^\infty 2^{-jq\varepsilon/2} \left[\int_P \left(\sum_{l(Q)=2^{-j}} \sum_{i=j_P}^j 2^{-i(n+\varepsilon)/2} \right. \right. \right.$$

$$\left. \left. \left. \times \sum_{\substack{k \in \mathbb{Z}^n \\ |k| \ge 2}} |k|^{-n-\varepsilon} \sum_{\substack{l(R)=2^{-i} \\ R \subset P + kl(P)}} |t_R|\chi_R(x) \right)^p dx \right]^{q/p} \right\}^{1/q}$$

$$\lesssim \|t\|_{\dot{b}_{p,q}^{0,\tau}(\mathbb{R}^n)} \sup_{P \text{ dyadic}} \frac{1}{|P|^{\frac{\varepsilon}{n}}} \left\{ \sum_{j=j_P}^\infty 2^{-jq\varepsilon/2} \left[\sum_{i=j_P}^j 2^{-i\varepsilon/2} \sum_{\substack{k \in \mathbb{Z}^n \\ |k| \ge 2}} |k|^{-n-\varepsilon} \right]^q \right\}^{1/q} \lesssim \|t\|_{\dot{b}_{p,q}^{0,\tau}(\mathbb{R}^n)}.$$

Hence, $\|A_0 t\|_{\dot{b}_{p,q}^{0,\tau}(\mathbb{R}^n)} \lesssim \|t\|_{\dot{b}_{p,q}^{0,\tau}(\mathbb{R}^n)}$. Some similar computations to I_1 also yield that $\|A_1 t\|_{\dot{b}_{p,q}^{0,\tau}(\mathbb{R}^n)} \lesssim \|t\|_{\dot{b}_{p,q}^{0,\tau}(\mathbb{R}^n)}$.

For Triebel-Lizorkin-type spaces $\dot{f}_{p,q}^{s,\tau}(\mathbb{R}^n)$, we also have

$$\|A_0 t\|_{\dot{f}_{p,q}^{s,\tau}(\mathbb{R}^n)} \lesssim \sup_{P \text{ dyadic}} \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{Q \subset P} \left(\sum_{\{R: l(Q) \le l(R) \le l(P)\}} \left(\frac{l(Q)}{l(R)} \right)^{(n+\varepsilon)/2} \right. \right. \right.$$

$$\left. \left. \left. \times \frac{|t_R|\tilde{\chi}_Q(x)}{(1 + l(R)^{-1}|x_Q - x_R|)^{n+\varepsilon}} \right)^q \right]^{p/q} dx \right\}^{1/p}$$

$$+ \sup_{P \text{ dyadic}} \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{Q \subset P} \left(\sum_{\{R: l(R) > l(P)\}} \cdots \right)^q \right]^{p/q} dx \right\}^{1/p} \equiv \tilde{I}_1 + \tilde{I}_2.$$

Similarly to the estimate for I_2, applying the fact that $|t_R| \le |R|^{1/2-1/p+\tau}\|t\|_{\dot{f}_{p,q}^{0,\tau}(\mathbb{R}^n)}$ and $\tau \in [0, \frac{1}{p} + \frac{\varepsilon}{2n})$, we obtain

$$\tilde{I}_2 \lesssim \|t\|_{\dot{f}^{0,\tau}_{p,q}(\mathbb{R}^n)} \sup_{P \text{ dyadic}} \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{j=j_P}^\infty \sum_{l(Q)=2^{-j}} \left(\sum_{i=-\infty}^{j_P-1} \sum_{m=0}^\infty \sum_{R \in U_{m,i}} \left(\frac{l(Q)}{l(R)} \right)^{(n+\varepsilon)/2} \right. \right. \right.$$

$$\left. \left. \left. \times \frac{|R|^{1/2-1/p+\tau} \tilde{\chi}_Q(x)}{\left(1 + l(R)^{-1}|x_Q - x_R|\right)^{n+\varepsilon}} \right)^q \right]^{p/q} dx \right\}^{1/p} \lesssim \|t\|_{\dot{f}^{0,\tau}_{p,q}(\mathbb{R}^n)}.$$

To estimate $\tilde{I}_1$, notice that

$$\tilde{I}_1 \lesssim \sup_{P \text{ dyadic}} \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{j=j_P}^\infty \sum_{l(Q)=2^{-j}} \left(\sum_{i=j_P}^j 2^{(i-j)(n+\varepsilon)/2} \right. \right. \right.$$

$$\left. \left. \left. \times \sum_{l(R)=2^{-i}} \frac{|r_R| \tilde{\chi}_Q(x)}{\left(1 + l(R)^{-1}|x_Q - x_R|\right)^{n+\varepsilon}} \right)^q \right]^{p/q} dx \right\}^{1/p}$$

$$+ \sup_{P \text{ dyadic}} \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{j=j_P}^\infty \sum_{l(Q)=2^{-j}} \left(\sum_{i=j_P}^j 2^{(i-j)(n+\varepsilon)/2} \right. \right. \right.$$

$$\left. \left. \left. \times \sum_{l(R)=2^{-i}} \frac{|u_R| \tilde{\chi}_Q(x)}{\left(1 + l(R)^{-1}|x_Q - x_R|\right)^{n+\varepsilon}} \right)^q \right]^{p/q} dx \right\}^{1/p} \equiv \tilde{J}_1 + \tilde{J}_2.$$

Similarly to the estimate of J_1, by [106, Remark A.3], Hölder's inequality and the Fefferman-Stein vector valued inequality, we obtain

$$\tilde{J}_1 \lesssim \sup_{P \text{ dyadic}} \frac{1}{|P|^\tau} \left\{ \int_P \left(\sum_{j=j_P}^\infty \left[\sum_{i=j_P}^j 2^{(i-j)\varepsilon/2} M_{HL} \left(\sum_{l(R)=2^{-i}} |r_R| \tilde{\chi}_R \right)(x) \right]^q \right)^{p/q} dx \right\}^{1/p}$$

$$\lesssim \sup_{P \text{ dyadic}} \frac{1}{|P|^\tau} \left\{ \int_{3P} \left(\sum_{i=j_P}^\infty \left[\left(\sum_{l(R)=2^{-i}} |t_R| \tilde{\chi}_R \right)(x) \right]^q \right)^{p/q} dx \right\}^{1/p} \lesssim \|t\|_{\dot{f}^{0,\tau}_{p,q}(\mathbb{R}^n)}.$$

For $\tilde{J}_2$, similarly to the estimate for J_2, by Minkowski's inequality and the Fefferman-Stein vector-valued maximal inequality, we obtain

$$\tilde{J}_2 \lesssim \sum_{\substack{k \in \mathbb{Z}^n \\ |k| \geq 2}} |k|^{-(n+\varepsilon)} \sup_{P \text{ dyadic}} \frac{1}{|P|^{\tau+\varepsilon/n}} \left\{ \int_P \left(\sum_{j=j_P}^\infty \left[\sum_{i=j_P}^j 2^{(i-j)\varepsilon/2} 2^{-i\varepsilon} \right. \right. \right.$$

$$\left. \left. \left. \times M_{HL} \left(\sum_{\substack{l(R)=2^{-i} \\ R \subset P+kl(P)}} |R|^{-1/2} |t_R| \chi_R \right)(x + kl(P)) \right]^q \right)^{p/q} dx \right\}^{1/p} \lesssim \|t\|_{\dot{f}^{0,\tau}_{p,q}(\mathbb{R}^n)}.$$

Hence $\|A_0 t\|_{\dot{f}^{0,\tau}_{p,q}(\mathbb{R}^n)} \lesssim \|t\|_{\dot{f}^{0,\tau}_{p,q}(\mathbb{R}^n)}.$

Some similar estimates to $\tilde{I}_1$ also yield that $\|A_1 t\|_{\dot{f}_{p,q}^{0,\tau}(\mathbb{R}^n)} \lesssim \|t\|_{\dot{f}_{p,q}^{0,\tau}(\mathbb{R}^n)}$. Thus, we obtain the desired conclusion for the case $\min\{p, q\} > 1$.

The case $\min\{p, q\} \leq 1$ is a simple consequence of the case $\min\{p, q\} > 1$. Indeed, choosing a $\delta \in (0, p \wedge q)$, then $p/\delta > 1$ and $q/\delta > 1$. Let $\widetilde{A}$ be an operator on $\dot{a}_{p,q}^{0,\tau}(\mathbb{R}^n)$ associated with the matrix $\{\tilde{a}_{QP}\}_{Q,P} \equiv \{|a_{QP}|^{\delta}(l(Q)/l(P))^{n/2-\delta n/2}\}_{Q,P}$. Then $\widetilde{A}$ is a ε-almost diagonal operator on $\dot{a}_{p/\delta,q/\delta}^{0,\tau\delta}(\mathbb{R}^n)$ with $\tilde{\varepsilon} = \delta\varepsilon$.

Define $\tilde{t} \equiv \{l(Q)^{n/2-\delta n/2}|t_Q|^{\delta}\}_Q$. Then $\|\tilde{t}\|_{\dot{a}_{p/\delta,q/\delta}^{0,\tau\delta}(\mathbb{R}^n)}^{1/\delta} = \|t\|_{\dot{a}_{p,q}^{0,\tau}(\mathbb{R}^n)}$. Since $\delta < 1$, by (13), we see that $\|At\|_{\dot{a}_{p,q}^{0,\tau}(\mathbb{R}^n)} \lesssim \|\widetilde{A}\tilde{t}\|_{\dot{a}_{p/\delta,q/\delta}^{0,\tau\delta}(\mathbb{R}^n)}^{1/\delta}$. Applying the conclusions for $\min\{p, q\} > 1$ yields $\|At\|_{\dot{a}_{p,q}^{0,\tau}(\mathbb{R}^n)} \lesssim \|\widetilde{A}\tilde{t}\|_{\dot{a}_{p/\delta,q/\delta}^{0,\tau\delta}(\mathbb{R}^n)}^{1/\delta} \lesssim \|\tilde{t}\|_{\dot{a}_{p/\delta,q/\delta}^{0,\tau\delta}(\mathbb{R}^n)}^{1/\delta} \sim \|t\|_{\dot{a}_{p,q}^{0,\tau}(\mathbb{R}^n)}$, which completes the proof of Theorem (4.1.17).

Definition (4.1.18) [190]: Let $s \in \mathbb{R}$, $\tau \in [0, \infty)$, $p, q \in (0, \infty]$ and $J \equiv n/\min\{1, p\}$ when $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n) \equiv \dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ or $J \equiv n/\min\{1, p, q\}$ when $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n) \equiv \dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$. Let $N \equiv \max(\lfloor J - n - s \rfloor, -1)$ and $s^* = s - \lfloor s \rfloor$ denotes the maximal integer no more than s.

(i) A function m_Q is called a smooth synthesis molecule for $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$ supported near dyadic cube Q if there exist a $\delta \in (\max\{s^*, (s + n\tau)^*\}, 1]$ and an $M \in (J, \infty)$ such that
$$\int_{\mathbb{R}^n} x^{\gamma} m_Q(x) dx = 0 \text{ if } |\gamma| \leq N, \left|m_Q(x)\right| \leq |Q|^{-1/2}\left(1 + l(Q)^{-1}|x - x_Q|\right)^{-\max(M, M-s)},$$
$$\left|\delta^{\gamma} m_Q(x)\right| \leq |Q|^{-1/2-|\gamma|/n}\left(1 + l(Q)^{-1}|x - x_Q|\right)^{-M} \text{ if } |\gamma| \leq \lfloor s + n\tau \rfloor, \quad (14)$$
and
$$\left|\delta^{\gamma} m_Q(x) - \delta^{\gamma} m_Q(y)\right|$$
$$\leq |Q|^{-1/2-|\gamma|/n-\delta/n}|x - y|^{\delta} \times \sup_{|z| \leq |x-y|} \left(1 + l(Q)^{-1}|x - z - x_Q|\right)^{-M} \text{ if } |\gamma|$$
$$= \lfloor s + n\tau \rfloor. \quad (15)$$
A collection $\{m_Q\}_Q$ is called a family of smooth synthesis molecules for $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$, if each m_Q is a smooth synthesis molecule for $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$ supported near Q.

(ii) A function b_Q is called a smooth analysis molecule for $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$ supported near Q if there exist a $\rho \in ((J - s)^*, 1]$ and an $M \in (J, \infty)$ such that $\int_{\mathbb{R}^n} x^{\gamma} b_Q(x) dx = 0$ if $|\gamma| \leq \lfloor s + n\tau \rfloor$, $\left|b_Q(x)\right| \leq |Q|^{-1/2}\left(1 + l(Q)^{-1}|x - x_Q|\right)^{-\max(M, M+n+s+n\tau-J)}$,
$$\left|\delta^{\gamma} b_Q(x)\right| \leq |Q|^{-1/2-|\gamma|/n}\left(1 + l(Q)^{-1}|x - x_Q|\right)^{-M} \text{ if } |\gamma| \leq N, \quad (16)$$
and
$$\left|\delta^{\gamma} b_Q(x) - \delta^{\gamma} b_Q(y)\right|$$
$$\leq |Q|^{-1/2-|\gamma|/n-\rho/n}|x - y|^{\rho} \times \sup_{|z| \leq |x-y|} \left(1 + l(Q)^{-1}|x - z - x_Q|\right)^{-M} \text{ if } |\gamma|$$
$$= N. \quad (17)$$
A collection $\{b_Q\}_Q$ is called a family of smooth synthesis molecules for $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$, if each b_Q is a smooth analysis molecule for $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$ supported near Q.

Lemma (4.1.19) [190]: Let s, p, q, J, M, N and ρ be as in Definition (4.1.18). Assume that $\tau \in [0, \left(\frac{1}{p} + \frac{M-J}{2n}\right) \wedge \left(\frac{1}{p} + \frac{\rho-(J-s)^*}{n}\right))$ if $N \geq 0$, $\tau \in [0, \left(\frac{1}{p} + \frac{M-J}{2n}\right) \wedge \left(\frac{1}{p} + \frac{s+n-J}{n}\right))$ if $N < 0$, and $\delta \in (\max\{(s + n\tau)^*, s^*\}, 1]$. Then there exist a positive real number ε_1 and a positive

constant C such that $\varepsilon_1 > 2n\left(\tau - \frac{1}{p}\right)$ and for all families $\{m_Q\}_Q$ of smooth synthesis molecules for $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and families $\{b_Q\}_Q$ of smooth analysis molecules for $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$,

$$|\langle m_P, b_Q\rangle| \leq C\omega_{Q\,P}(\varepsilon_1).$$

Namely, the operator associated with the matrix $\{a_{Q\,P}\}_{Q,P} \equiv \{\langle m_P, b_Q\rangle\}_{Q,P}$ is ε_1-almost diagonal on $\dot{a}_{p,q}^{s,\tau}(\mathbb{R}^n)$.

As an immediate consequence of Lemma (4.1.19), we have the following corollary

Corollary (4.1.20) [190]: Let s, p, q, τ and ε_1 be as in Lemma (4.1.19) and φ satisfy (1) through (3). Suppose that $\{m_Q\}_Q$ and $\{b_Q\}_Q$ are families of smooth synthesis and analysis molecules for $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$, respectively. Then the operators associated with the matrices $\{a_{Q\,P}\}_{Q,P} = \{\langle m_Q, \varphi_P\rangle\}_{Q,P}$ and $\{b_{Q\,P}\}_{Q,P} = \{\langle \varphi_P, b_Q\rangle\}_{Q,P}$ are, respectively, ε_1-almost diagonal operators on $\dot{a}_{p,q}^{s,\tau}(\mathbb{R}^n)$.

Lemma (4.1.21) [190]: Let $s \in \mathbb{R}, p, q \in (0, \infty], \tau$ and ε_1 be as in Lemma (4.1.19), $f \in \dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and Φ be a smooth analysis molecule for $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$ supported near Q. Then $\langle f, \Phi\rangle$ is well defined. Indeed, for φ, ψ satisfy (1) through (4),

$$\langle f, \Phi\rangle \equiv \sum_j \langle \widetilde{\varphi}_j * \psi_j * f, \Phi\rangle = \sum_P \langle f, \varphi_P\rangle\langle \psi_P, \Phi\rangle \tag{18}$$

converges absolutely and its value is independent of the choices of φ and ψ.

Proof: By similarity, we only consider the space $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$. Let Φ be a smooth analysis molecule for $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ supported near Q and φ, ψ satisfy (1) through (4). We claim that there exists a matrix $\{a_{Q\,P}\}_{Q,P}$ such that $|\langle f, \varphi_P\rangle||\langle \psi_P, \Phi\rangle| \leq a_{Q\,P}$ and $\sum_P a_{Q\,P} < \infty$. In fact, Corollary (4.1.20) yields that there exists a positive constant C such that $|\psi_P, \Phi| \leq C\omega_{Q\,P}(\varepsilon_1)$. Then $a_{Q\,P} \equiv C|\langle f, \varphi_P\rangle|\omega_{Q\,P}(\varepsilon_1)$ does the job. Furthermore, by Theorem (4.1.5), the sequence $\{|\langle f, \varphi_P\rangle|\}_P \in \dot{b}_{p,q}^{s,\tau}(\mathbb{R}^n)$, and hence by Theorem (4.1.17), $\sum_P a_{Q\,P} < \infty$. This shows the absolutely convergence of (18).

Next we claim that for $f \in \dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$, $\sum_{j=0}^\infty \widetilde{\psi}_j * \varphi_j * f$ converges in $\mathcal{S}'(\mathbb{R}^n)$. To see this, we need the following estimate that there exists an $M \in \mathbb{Z}_+$ such that for all $\psi \in \mathcal{S}_\infty(\mathbb{R}^n), \varphi \in \mathcal{S}(\mathbb{R}^n), j \in \mathbb{Z}_+$ and $x \in \mathbb{R}^n$,

$$|\psi_j * \phi(x)| \lesssim \|\phi\|_{\mathcal{S}_{M+1}}\|\psi\|_{\mathcal{S}_{M+1}} 2^{-jM} \frac{1}{(1+|x|)^{n+M}}. \tag{19}$$

The proof of (19) is similar to that of Lemma (4.1.3) (see also, [189]). We omit the details.

Let $M > \max\{0, -s\}$. If $p \in [1, \infty]$, then by (19) and Hölder's inequality, we obtain that for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\sum_{j=0}^\infty |\langle \widetilde{\psi}_j * \varphi_j * f, \phi\rangle| \lesssim \|\phi\|_{\mathcal{S}_{M+1}}\|\psi\|_{\mathcal{S}_{M+1}} \sum_{j=0}^\infty 2^{-jM} \int_{\mathbb{R}^n} \frac{|\varphi_j * f(x)|}{(1+|x|)^{n+M}} dx$$

$$\lesssim \|\phi\|_{\mathcal{S}_{M+1}}\|\psi\|_{\mathcal{S}_{M+1}}\|f\|_{\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)}. \tag{20}$$

If $p \in (0, 1)$, by Corollary (4.1.10), $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n) \subset \dot{B}_{1,q}^{s+\left(1-\frac{1}{p}\right)n,\tau}(\mathbb{R}^n)$, and hence

$$\sum_{j=0}^\infty |\langle \widetilde{\psi}_j * \varphi_j * f, \phi\rangle| \lesssim \|\phi\|_{\mathcal{S}_{M+1}}\|\psi\|_{\mathcal{S}_{M+1}}\|f\|_{\dot{B}_{1,q}^{s+\left(1-\frac{1}{p}\right)n,\tau}(\mathbb{R}^n)}$$

$$\lesssim \|\phi\|_{\mathcal{S}_{M+1}}\|\psi\|_{\mathcal{S}_{M+1}}\|f\|_{\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)}. \tag{21}$$

The estimates (20) and (21) imply that $\sum_{j=0}^\infty \widetilde{\psi}_j * \varphi_j * f$ converges in $\mathcal{S}'(\mathbb{R}^n)$.

Since $\psi \in \mathcal{S}_\infty(\mathbb{R}^n)$, for all $x \in \mathbb{R}^n, j \in \mathbb{Z}$ and multi-indices γ, we have

$$\left|(\partial^\gamma \widetilde{\psi}_j) * \varphi_j * f(x)\right| \lesssim \|\psi\|_{\mathcal{S}_{M+1}} 2^{j(n+\gamma)} \int_{\mathbb{R}^n} \frac{\left|\varphi_j * f(y)\right|}{(1 + |x - y|)^{n+M+|\gamma|}} \, dy.$$

Then, if $p \in [1, \infty]$, applying Hölder's inequality, we obtain $\left|(\partial^\gamma \widetilde{\psi}_j) * \varphi_j * f(x)\right| \lesssim$ $\|\psi\|_{\mathcal{S}_M} 2^{j(|\gamma|+n/p-s-n\tau)} \|f\|_{\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)}$. Similarly to the estimate of (21), applying Proposition (4.1.13) again, we know this estimate still holds when $p \in (0, 1)$. Thus, if $|\gamma| \in (s + n\tau - n/p, \infty)$, then for all $x \in \mathbb{R}^n$, $\sum_{j=-\infty}^{1}\left|(\partial^\gamma \widetilde{\psi}_j) * \varphi_j * f(x)\right| \lesssim \|\psi\|_{\mathcal{S}_M} \|f\|_{\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)}$, which together with (20) and (21) implies that there exist a sequence of polynomials, $\{P_N\}_{N=1}^\infty$, with degree no more than $L \equiv s + n\tau - n/p$ and $g \in \mathcal{S}'(\mathbb{R}^n)$ such that $g = \lim_{N\to\infty}\left(\sum_{j=-N}^\infty \widetilde{\psi}_j * \varphi_j * f + P_N\right)$ in $\mathcal{S}'(\mathbb{R}^n)$ and g is a representative of the equivalence class $f + \mathcal{P}(\mathbb{R}^n)$; see [106]. Using [93, Lemma 5.4] and repeating the argument in [106] then completes the proof of Lemma (4.1.21).

Using Lemmas (4.1.19) and (4.1.21), by the method pioneered by Frazier and Jawerth [93, 106], we obtain the following Theorem (4.1.22).

Theorem (4.1.22) [190]: Let $s \in \mathbb{R}, p, q \in (0, \infty], \tau$ and ε_1 be as in Lemma (4.1.19).

(i) If $\{m_Q\}_Q$ is a family of smooth synthesis molecules for $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$, then there exists a positive constant C such that for all $t = \{t_Q\}_Q \in \dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n), \left\|\sum_Q t_Q m_Q\right\|_{\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)} \leq C\|t\|_{\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)}$;

(ii) If $\{b_Q\}_Q$ is a family of smooth analysis molecules for $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$, then there exists a positive constant C such that for all $f \in \dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n), \left\|\{\langle f, b_Q\rangle\}_Q\right\|_{\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)} \leq C\|f\|_{\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)}$.

We establish smooth atomic decomposition characterizations of $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$. For the classical results on $\dot{B}^s_{p,q}(\mathbb{R}^n)$ and $\dot{F}^s_{p,q}(\mathbb{R}^n)$, see [93, 106, 164].

Definition (4.1.23) [190]: Let s, τ, p, q and J be as in Definition (4.1.18). A function a_Q is called a smooth atom for $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$ supported near a dyadic cube Q if there exist $\widetilde{K}$ and $\widetilde{N}$ with $\widetilde{K} \geq \max\{\lfloor s + n\tau + 1\rfloor, 0\}$ and $\widetilde{N} \geq \max(\lfloor J - n - s\rfloor, -1)$ such that $\operatorname{supp} a_Q \subset 3Q$, $\int_{\mathbb{R}^n} x^\gamma a_Q(x)dx = 0$ if $|\gamma| \leq \widetilde{N}$, and $\left|\partial^\gamma a_Q(x)\right| \leq |Q|^{-1/2-|\gamma|/n}$ if $|\gamma| \leq \widetilde{K}$.

A collection $\{a_Q\}_Q$ is called a family of smooth atoms for $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$, if each a_Q is a smooth atom for $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$ supported near Q.

It is clear that every smooth atom for $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$ is a multiple of a smooth synthesis molecule for $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$ supported near Q. Using Theorem (4.1.22) and repeating the argument as in [106] or [93] yield the following result.

Theorem (4.1.24) [190]: Let $s \in \mathbb{R}, p, q \in (0, \infty], \tau$ be as in Lemma (4.1.19). Then for each $f \in \dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$, there exist smooth atoms $\{a_Q\}_Q$ for $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$, coefficients $t = \{t_Q\}_Q \in \dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)$ such that $f = \sum_Q t_Q a_Q$ in $\mathcal{S}'_\infty(\mathbb{R}^n)$ and $\|t\|_{\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)} \leq C\|f\|_{\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)}$, where C is a positive constant independent of f and t.

Conversely, there exists a positive constant C such that for all families $\{a_Q\}_Q$ of smooth atoms for $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $t = \{t_Q\}_Q \in \dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n), \left\|\sum_Q t_Q a_Q\right\|_{\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)} \leq C\|t\|_{\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)}$.

In [189], some tent spaces $F\dot{T}^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{Z}})$ were introduced, which are used to determine the predual spaces of $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$. We introduce a class of tent spaces

$B\dot{T}^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{Z}})$, which is used to determine the predual space of $\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)$. First we recall the notion of Hausdorff capacities; see, for example, [161, 162]. In what follows, for $x \in \mathbb{R}^n$ and $r > 0$, $B(x,r) \equiv \{y \in \mathbb{R}^n : |x - y| < r\}$.

Definition (4.1.25) [190]: Let $d \in (0, \infty)$ and $E \subset \mathbb{R}^n$. Then the d-dimensional Hausdorff capacity of E is defined by

$$\Lambda^{(\infty)}_d(E) \equiv \inf\left\{\sum_j r_j^d : E \subset \bigcup_j B(x_j, r_j)\right\}, \tag{22}$$

where the infimum is taken over all covers of E by countable families of open balls with radius r_j .

The notion of $\Lambda^{(\infty)}_d$ can be extended to $d = 0$, namely, in (22), let $d = 0$. Then $\Lambda^{(\infty)}_d$ is monotone and countable sub additive; moreover, $\Lambda^{(\infty)}_d$ does not vanish on the empty set, it has the property that for all sets $E \subset \mathbb{R}^n$, $\Lambda^{(\infty)}_0(E) \geq 1$ and $\Lambda^{(\infty)}_0(E) = 1$ if E is bounded.

Adyadic version of the Hausdorff capacity, $\widetilde{\Lambda}^{(\infty)}_d$, which is defined by

$$\widetilde{\Lambda}^{(\infty)}_d(E) \equiv \inf\left\{\sum_j l(I_j)^d : E \subset \left(\bigcup_j I_j\right)^\circ\right\}, \tag{23}$$

where now the infimum ranges only over covers of E by dyadic cubes $\{I_j\}_j$. Recall that A° denotes the interior of the set A.

It was proved in [188] that $\widetilde{\Lambda}^{(\infty)}_d$ is a Choquet capacity and that $\Lambda^{(\infty)}_d$ and $\widetilde{\Lambda}^{(\infty)}_d$ are equivalent, namely, there exist positive constants C_1 and C_2, only depending on the dimension n, such that

$$C_1 \Lambda^{(\infty)}_d(E) \leq \widetilde{\Lambda}^{(\infty)}_d(E) \leq C_2 \Lambda^{(\infty)}_d(E) \quad \text{for all } E \subset \mathbb{R}^n. \tag{24}$$

Next we recall the notions of Choquet integral with respect to the Hausdorff capacities $\Lambda^{(\infty)}_d$ and $\widetilde{\Lambda}^{(\infty)}_d$; see [161, 162]. For any function $f : \mathbb{R}^n \to [0, \infty]$, define

$$\int_{\mathbb{R}^n} f \, d\Lambda^{(\infty)}_d \equiv \int_0^\infty \Lambda^{(\infty)}_d(\{x \in \mathbb{R}^n : f(x) > \lambda\}) d\lambda.$$

This functional is not sub linear, so sometimes we need to use an equivalent integral with respect to $\widetilde{\Lambda}^{(\infty)}_d$, which is sublinear and satisfies Fatou's lemma. See [188, 189] for more properties on the Hausdorff capacities and their Choquet integrals.

Let $\mathbb{R}^{n+1}_+ \equiv \mathbb{R}^n \times (0, \infty)$. For $x \in \mathbb{R}^n$, let $\Gamma(x) \equiv \{(y, t) \in \mathbb{R}^{n+1}_+ : |y - x| < t\}$ be the cone at x. Define the nontangential maximal function $N(f)$ of any measurable function f on $\mathbb{R}^{n+1}_+$ by $Nf(x) \equiv \sup_{(y,t) \in \Gamma(x)} |f(y,t)|$ for all $x \in \mathbb{R}^n$. Since a point $(x,t) \in \mathbb{R}^{n+1}_+$ belongs to $\Gamma(y)$ for every $y \in B(x,t)$, we see that

$$|f(x,t)| \leq \inf_{y \in B(x,t)} N(f)(y). \tag{25}$$

Recall that $\mathbb{R}^{n+1}_{\mathbb{Z}} \equiv \mathbb{R}^n \times \{2^k : k \in \mathbb{Z}\}$. For all functions f on $\mathbb{R}^{n+1}_+$ or $\mathbb{R}^{n+1}_{\mathbb{Z}}$ and $k \in \mathbb{Z}$, we set $f^k(x) \equiv f(x, 2^{-k})$. For any set A, define $T(A) \equiv \{(x,t) \in \mathbb{R}^{n+1}_+ : B(x,t) \subset A\}$.

Definition (4.1.26) [190]: Let $s \in \mathbb{R}$.

(i) Let $p, q \in [1, \infty)$, $(p \vee q) > 1$ and $\tau \in (0, \frac{1}{(p \vee q)'}]$. The space $B\dot{T}^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{Z}})$ is defined to be the set of all functions f on $\mathbb{R}^{n+1}_{\mathbb{Z}}$ such that $\{f^k\}_{k \in \mathbb{Z}}$ are Lebesgue measurable and

$$\|f\|_{B\dot{T}^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{Z}})} \equiv \inf_{\omega}\left\{\sum_{k\in\mathbb{Z}} 2^{ksq}\left\|f^k[\omega^k]^{-1}\right\|^q_{L^p(\mathbb{R}^n)}\right\}^{1/q} < \infty,$$

(ii) where the infimum is taken over all nonnegative Borel measurable functions ω on $\mathbb{R}^{n+1}_+$ with

$$\int_{\mathbb{R}^n} \left(N\omega(x)\right)^{(p\vee q)'} d\Lambda^{(\infty)}_{n\tau(p\vee q)'} \leq 1 \qquad (26)$$

and with the restriction that ω is allowed to vanish only where f vanishes.

(iii) Let $p, q \in (0, \infty]$ and $\tau \in [0, \infty)$. The space $B\dot{W}^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{Z}})$ is defined to be the set of all functions f on $\mathbb{R}^{n+1}_{\mathbb{Z}}$ such that $\{f^k\}_{k\in\mathbb{Z}}$ are Lebesgue measurable and

$$\|f\|_{B\dot{W}^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{Z}})} \equiv \sup_{B} \frac{1}{|B|^\tau}\left\{\sum_{k\in\mathbb{Z}} 2^{ksq}\left[\int_{\mathbb{R}^n}|f^k(x)|^p\chi_{T(B)}(x,2^{-k})dx\right]^{q/p}\right\}^{1/q} < \infty$$

where B runs over all balls in $\mathbb{R}^n$.

Definition (4.1.27) [190]: Let s, τ, p, q be as in Definition (4.1.25) (i). A function a on $\mathbb{R}^{n+1}_{\mathbb{Z}}$ is called a $B\dot{T}^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{Z}})$-atom associated with a ball B, if a is supported in T(B) and satisfies

$$\sum_{k\in\mathbb{Z}} 2^{ksq}\left(\int_{\mathbb{R}^n}|a^k(x)|^p\chi_{T(B)}(x,2^{-k})dx\right)^{q/p} \leq |B|^{-\tau q}.$$

Lemma (4.1.28) [190]: Let s, τ, p, q be as in Definition (4.1.26) (i). Then there exists a positive constant C such that all $B\dot{T}^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{Z}})$-atoms a belong to $B\dot{T}^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{Z}})$ and $\|a\|_{B\dot{T}^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{Z}})} \leq C$.

Lemma (4.1.29) [190]: Let $s \in \mathbb{R}, p, q \in (1, \infty], (p \wedge q) < \infty, \tau \in (0, \frac{1}{p\wedge q}]$ and $a \in (0, \infty)$. Then there exists a positive constant C such that for all $f \in B\dot{W}^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{Z}})$ and nonnegative Borel measurable functions ω on $\mathbb{R}^{n+1}_+$, when $q \leq p$,

$$\sum_{k\in\mathbb{Z}} 2^{ksq}\left\{\int_{\mathbb{R}^n}|f^k(x)|^p[\omega^k(x)]^{ap}dx\right\}^{q/p} \leq C\|f\|^q_{B\dot{W}^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{Z}})}\int_{\mathbb{R}^n}\left(N\omega(x)\right)^{ap}d\Lambda^{(\infty)}_{n\tau q}(x);$$

and when $p < q$,

$$\left\{\sum_{k\in\mathbb{Z}} 2^{ksq}\left[\int_{\mathbb{R}^n}|f^k(x)|^p[\omega^k(x)]^{ap}dx\right]^{q/p}\right\}^{1/q} \leq C\|f\|_{B\dot{W}^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{Z}})}\left\{\int_{\mathbb{R}^n}\left(N\omega(x)\right)^{ap}d\Lambda^{(\infty)}_{n\tau q}(x)\right\}^{1/p}.$$

In Theorem (4.1.30) below, we establish the dual relation between tent spaces $B\dot{T}^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{Z}})$ and $B\dot{W}^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{Z}})$, whose continuous variant when $s = \frac{n-d}{2}, \tau = \frac{d}{2n}$ and $p = q = 2$ was obtained by Dafni and Xiao [167]. The proof of Theorem (4.1.30) is a slight modification of the proof of [189, Theorem 4.1] by replacing Lemmas 4.2 and 4.3 in [189] with Lemmas (4.1.28) and (4.1.29) here.

Theorem (4.1.30) [190]: Let s, τ, p, q be as in Definition (4.1.26) (i).

(i) If $f \in B\dot{T}^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{Z}})$, then there exist $B\dot{T}^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{Z}})$-atoms $\{a_j\}_j$ and an l^1-sequence $\{\lambda_j\}$ such that $f = \sum_j \lambda_j a_j$ pointwise; moreover, $\sum_j|\lambda_j| \leq C\|f\|_{B\dot{T}^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{Z}})}$. In particular, if $p = q \in (1, \infty)$, then $f = \sum_j \lambda_j a_j$ also in $B\dot{T}^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_{\mathbb{Z}})$.

Conversely, if $p = q \in (1, \infty)$, and there exist $B\dot{T}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$-atoms $\{a_j\}_j$ and an l^1-sequence $\{\lambda_j\}$ such that $f = \sum_j \lambda_j a_j$ pointwise, then $f = \sum_j \lambda_j a_j$ also in $B\dot{T}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ and $\|f\|_{B\dot{T}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})} \leq C\sum_j |\lambda_j|$, where C is a positive constant independent of f.

(ii) There exists a positive constant C such that for all $f \in B\dot{T}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ and $g \in B\dot{W}_{p',q'}^{-s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$, $\left|\sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} f^k(x)g^k(x)dx\right| \leq C\|f\|_{B\dot{T}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})}\|g\|_{B\dot{W}_{p',q'}^{-s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})}$.

(iii) The dual space of $B\dot{T}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ is $B\dot{W}_{p',q'}^{-s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ under the following pairing

$$\langle f, g \rangle = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} f^k(x)g^k(x)dx.$$

Section (4.2): Triebel-Lizorkin-Hausdorff Spaces and their Applications

To establish the connections between Besov and Triebel-Lizorkin spaces with Q spaces, which was an open problem (see, [167]), Yang and Yuan [189, 190] introduced new classes of Besov-type spaces $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and Triebel-Lizorkin-type spaces $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$, which unify and generalize the Besov spaces $\dot{B}_{p,q}^s(\mathbb{R}^n)$, Triebel-Lizorkin spaces $\dot{F}_{p,q}^s(\mathbb{R}^n)$, Morrey spaces, Morrey-Triebel-Lizorkin spaces and Q spaces. We pointed out that the Q spaces on $\mathbb{R}^n$ were originally introduced in [193]; see also [167, 184, 185, 193] for the history of Q spaces and their properties.

Let $p \in (1, \infty), q \in [1, \infty)$, $s \in \mathbb{R}$ and $\tau \in [0, 1 - \frac{1}{\max\{p,q\}}]$, where and in what follows, $\grave{t}$ denotes the conjugate index of $t \in [1, \infty)$. The Besov-Hausdorff spaces $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and Triebel-Lizorkin-Hausdorff spaces $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ $(q > 1)$ were also introduced in [189, 190]; moreover, it was proved therein that they are, respectively, the predual spaces of $\dot{B}_{\grave{p},\grave{q}}^{-s,\tau}(\mathbb{R}^n)$ and $\dot{F}_{\grave{p},\grave{q}}^{-s,\tau}(\mathbb{R}^n)$. The spaces $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ were originally called the Hardy-Hausdorff spaces in [189, 190]. However, it seems that it is more reasonable to call them, respectively, the Besov-Hausdorff spaces and the Triebel-Lizorkin-Hausdorff spaces. The spaces $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ unify and generalize the Besov space $\dot{B}_{p,q}^s(\mathbb{R}^n)$, the Triebel-Lizorkin space $\dot{F}_{p,q}^s(\mathbb{R}^n)$ and the Hardy-Hausdorff space $HH^1_{-\alpha}(\mathbb{R}^n)$, where $HH^1_{-\alpha}(\mathbb{R}^n)$ was introduced in [167] and was proved to be the predual space of the space $Q_\alpha(\mathbb{R}^n)$ therein.

It is well known that the wavelet decomposition plays an important role in the study of function spaces and their applications; see, for example, [198, 199]. The φ-transform decomposition of Frazier and Jawerth [45, 106, 175] is very similar in spirit to the wavelet decomposition, which is also proved to be a powerful tool in the study of function spaces and boundedness of operators, and was further developed by Bownik [93, 165]. We establish the φ-transform characterizations of the spaces $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$; via these characterizations, we also obtain their embedding properties (which on $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is also sharp), smooth atomic and molecular decomposition characterizations for suitable τ. Using their atomic and molecular decomposition characterizations, we investigate the trace properties and the boundedness of pseudo-differential operators with homogeneous symbols (see [197]) in $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, which generalizes the corresponding classical results on homogeneous Besov and Triebel-Lizorkin spaces when $p \in (1, \infty)$ and $q \in [1, \infty)$ by taking $\tau = 0$; see, for example [7, 17, 106, 197]. Recall that the study of pseudo-differential operators with non-homogeneous symbols on non-homogeneous Besov and Triebel-Lizorkin spaces using φ-transform arguments was started by Torres [201, 202]; the results in [197] are based on these works. See Pseudo-differential operators

on Triebel-Lizorkin spaces using more classical methods. We will concentrate here on φ-transform arguments.

Let $\mathcal{S}(\mathbb{R}^n)$ be the space of all Schwartz functions on $\mathbb{R}^n$. Following Triebel's [41], set

$$\mathcal{S}_\infty(\mathbb{R}^n) = \left\{\varphi \in \mathcal{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \varphi(x)x^\gamma dx = 0 \text{ for all multi} - \text{indices } \gamma \in (\mathbb{N} \cup \{0\})^n\right\}$$

and use $\mathcal{S}'_\infty(\mathbb{R}^n)$ to denote the topological dual of $\mathcal{S}_\infty(\mathbb{R}^n)$, namely, the set of all continuous linear functionals on $\mathcal{S}_\infty(\mathbb{R}^n)$ endowed with weak $*$-topology. Recall that $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ and $\mathcal{S}'_\infty(\mathbb{R}^n)$ are topologically equivalent, where $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{P}(\mathbb{R}^n)$ denote, respectively, the space of all Schwartz distributions and the set of all polynomials on $\mathbb{R}^n$.

For each cube Q in $\mathbb{R}^n$, we denote its side length by $\ell(Q)$, its center by $c\,Q$, and set $j_Q \equiv -\log_2 \ell(Q)$. For $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$ and $j \in \mathbb{Z}$, let Q_{jk} be the dyadic cube $\{(x_1, \dots, x_n) : k_i \leq 2^j x_i < k_i + 1 \text{ for } i = 1, \dots, n\} \subset \mathbb{R}^n$, x_Q be the lower left-corner $2^j k$ of $Q = Q_{jk}$, $\mathcal{D}(\mathbb{R}^n) \equiv \{Q_{jk}\}_{j,k}$, and $\mathcal{D}_j(\mathbb{R}^n) \equiv \{Q \in \mathcal{D}(\mathbb{R}^n) : \ell(Q) = 2^{-j}\}$. When dyadic cube Q appears as an index, such as $\sum_{Q \in \mathcal{D}(\mathbb{R}^n)}$ and $\{\cdot\}_{Q \in \mathcal{D}(\mathbb{R}^n)}$, it is understood that Q runs over all dyadic cubes in $\mathbb{R}^n$.

For $x \in \mathbb{R}^n$ and $r > 0$, we write $B(x, r) \equiv \{y \in \mathbb{R}^n : |x - y| < r\}$. Next we recall the notion of Hausdorff capacities; see, for example, [161, 188]. Let $E \subset \mathbb{R}^n$ and $d \in (0, n]$. The d-dimensional Hausdorff capacity of E is defined by

$$H^d(E) \equiv \inf\left\{\sum_j r_j^d : E \subset \bigcup_j B(x_j, r_j)\right\},$$

where the infimum is taken over all covers $\{B(x_j, r_j)\}_{j=1}^\infty$ of E by countable families of open balls. It is well known that H^d is monotone, countably subadditive and vanishes on empty set. Moreover, the notion of H^d can be extended to $d = 0$. In this case, H^0 has the property that for all sets $E \subset \mathbb{R}^n$, $H^0(E) \geq 1$, and $H^0(E) = 1$ if and only if E is bounded.

For any function $f : \mathbb{R}^n \longmapsto [0, \infty]$, the Choquet integral of f with respect to H^d is defined by

$$\int_{\mathbb{R}^n} f dH^d \equiv \int_0^\infty H^d(\{x \in \mathbb{R}^n : f(x) > \lambda\})d\lambda.$$

This functional is not sublinear, so sometimes we need to use an equivalent integral with respect to the d-dimensional dyadic Hausdorff capacity $\widetilde{H}^d$, which is sublinear; see [188] for the definition of dyadic Hausdorff capacities and their properties.

Set $\mathbb{R}^{n+1}_+ \equiv \mathbb{R}^n \times (0, \infty)$. For any measurable function ω on $\mathbb{R}^{n+1}_+$ and $x \in \mathbb{R}^n$, we define its nontangential maximal function $N\omega(x)$ by setting $N\omega(x) \equiv \sup_{|y-x|<t}|\omega(y,t)|$.

In what follows, for any $\varphi \in \mathbb{R}^n$, we use $\mathcal{F}\varphi$ to denote its Fourier transform, namely, for all $\xi \in \mathbb{R}^n$, $\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^n} e^{-i\xi x}\varphi(x)dx$. For all $j \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, let $\varphi_j(x) = 2^{jn}\varphi(2^j x)$. For any $p, q \in (0, \infty]$, let $(p \vee q) \equiv \max\{p, q\}$; and for any $t \in [1, \infty]$, we denote by t' the conjugate index, namely, $\frac{1}{t} + 1/t' = 1$.

We now recall the notions of $B\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $F\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$ in [189, 190].

Definition (4.2.1) [204]: Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that supp $\mathcal{F}\varphi \subset \{\xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2\}$ and $\mathcal{F}\varphi$ never vanishes on $\{\xi \in \mathbb{R}^n : 3/5 \leq |\xi| \leq 5/3\}$. Let $p \in (1, \infty)$ and $s \in \mathbb{R}^n$.

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(i) If $q \in [1, \infty)$ and $\tau \in \left[0, \frac{1}{(p \vee q)'}\right]$, the Besov-Hausdorff space $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is then defined to be the set of all $f \in \mathcal{S}_\infty'(\mathbb{R}^n)$ such that

$$\|f\|_{B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \equiv \operatorname{Inf}_\omega \left\{ \sum_{j \in \mathbb{Z}} 2^{jsq} \left\| \varphi_j * f [\omega(\cdot, 2^{-j})]^{-1} \right\|_{L^p(\mathbb{R}^n)}^q \right\}^{1/q} < \infty,$$

where ω runs over all nonnegative Borel measurable functions on $\mathbb{R}_+^{n+1}$ such that

$$\int_{\mathbb{R}^n} [N\omega(x)]^{(p \vee q)'} dH^{n\tau(p \vee q)'}(x) \leq 1 \tag{27}$$

and with the restriction that for any $j \in \mathbb{Z}$, $\omega(\cdot, 2^{-j})$ is allowed to vanish only where $\varphi_j * f$ vanishes.

(ii) If $q \in (1, \infty)$ and $\tau \in \left[0, \frac{1}{(p \vee q)'}\right]$, the Triebel-Lizorkin-Hausdorff space $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is then defined to be the set of all $f \in \mathcal{S}_\infty'(\mathbb{R}^n)$ such that

$$\|f\|_{F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \equiv \operatorname{Inf}_\omega \left\| \left\{ \sum_{j \in \mathbb{Z}} 2^{jsq} \left| \varphi_j * f [\omega(\cdot, 2^{-j})]^{-1} \right|^q \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} < \infty,$$

where ω runs over all nonnegative Borel measurable functions on $\mathbb{R}_+^{n+1}$ such that ω satisfies (27) and with the restriction that for any $j \in \mathbb{Z}$, $\omega(\cdot, 2^{-j})$ is allowed to vanish only where $\varphi_j * f$ vanishes.

To simplify the presentation, in what follows, we use $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ to denote either $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ or $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$. When $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ denotes $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, then it will be understood tacitly that $q \in (1, \infty)$. It was proved in [189, 190] that the space $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is independent of the choices of φ. We also remark that when $\tau = 0$, then $B\dot{H}_{p,q}^{s,0}(\mathbb{R}^n) \equiv \dot{B}_{p,q}^s(\mathbb{R}^n)$ and $F\dot{H}_{p,q}^{s,0}(\mathbb{R}^n) \equiv \dot{F}_{p,q}^s(\mathbb{R}^n)$; when $\alpha \in (0,1)$, $s = -\alpha$, $p = q = 2$ and $\tau = 1/2 - \alpha/n$, then $A\dot{H}_{2,2}^{-\alpha,1/2-\alpha}(\mathbb{R}^n) \equiv HH_{-\alpha}^1(\mathbb{R}^n)$, which is the predual space of $\mathcal{Q}_\alpha(\mathbb{R}^n)$.

We now recall the notions of Besov-type spaces $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and Triebel-Lizorkin-type spaces $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ in [189, 190].

Definition (4.2.2) [204]: Let $s \in \mathbb{R}$, $\tau \in [0, \infty)$, $q \in (0, \infty]$ and φ be as in Definition (4.2.1).

(i) If $p \in (0, \infty]$, the Besov-type space $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}_\infty'(\mathbb{R}^n)$ such that $\|f\|_{\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)} < \infty$, where

$$\|f\|_{\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)} \equiv \sup_{P \in \mathcal{D}(\mathbb{R}^n)} \frac{1}{|P|^\tau} \left\{ \sum_{j=j_P}^{\infty} \left[\int_P (2^{js} |\varphi_j * f(x)|)^P dx \right]^{q/p} \right\}^{1/q}$$

with suitable modifications made when $p = \infty$ or $q = \infty$.

(ii) If $p \in (0, \infty)$, the Triebel-Lizorkin-type space $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}_\infty'(\mathbb{R}^n)$ such that $\|f\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)} < \infty$, where

$$\|f\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)} \equiv \sup_{P \in \mathcal{D}(\mathbb{R}^n)} \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{j=j_P}^{\infty} (2^{js} |\varphi_j * f(x)|)^q \right]^{p/q} dx \right\}^{1/p}$$

with suitable modifications made when $q = \infty$.

Similarly, we use $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$ to denote $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ or $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$. If $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$ means $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$, then the case $p = \infty$ is excluded. It was proved in [190] that the space $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is independent of the choices of φ. Also, [189, 190] show that $\left(\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)\right)^* = \dot{A}_{p',q'}^{-s,\tau}(\mathbb{R}^n)$ for all $s \in \mathbb{R}$, $p \in (1, \infty)$, $q \in [1, \infty)$ and $\tau \in \left[0, \frac{1}{(p \vee q)'}\right]$. This result partially extends the well-known dual results on Besov spaces, Triebel-Lizorkin spaces and the recent result that $\left(HH_{-\alpha}^1(\mathbb{R}^n)\right)^* = \mathcal{Q}_\alpha(\mathbb{R}^n)$ obtained in [167].

We establish the φ-transform characterizations and embedding properties of $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$. In particular, we show that the embedding property of $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is sharp. Using these φ-transform characterizations. We obtain the boundedness of almost diagonal operators and the smooth atomic and molecular decomposition characterizations of $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$. As applications of these decomposition characterizations, we investigate the trace properties and the boundedness of pseudo-differential operators with homogeneous symbols in $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$. We pointed out that the method used in the proof of Theorem (4.2.27) comes from [194, 195].

We denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. The symbol $A \lesssim B$ means that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$. If E is a subset of $\mathbb{R}^n$, we denote by χ_E the characteristic function of E. For all $Q \in \mathcal{D}(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, set $\varphi_Q(x) \equiv |Q|^{-1/2}\varphi\left(2^{jQ}(x - x_Q)\right)$ and $\tilde{\chi}_Q(x) \equiv |Q|^{-1/2}\chi_Q(x)$ for all $x \in \mathbb{R}^n$. We also set $\mathbb{N} \equiv \{1, 2, \dots\}$ and $\mathbb{Z}_+ \equiv (\mathbb{N} \cup \{0\})$.

We establish the φ-transform characterizations of the spaces $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ in the sense of Frazier and Jawerth; see, for example, [45, 106, 175, 176]. We begin with the definition of the corresponding sequence space of $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$.

Definition (4.2.3) [204]: Let $p \in (1, \infty)$ and $s \in \mathbb{R}$.

(i) If $q \in [1, \infty)$ and $\tau \in \left[0, \frac{1}{(p \vee q)'}\right]$, the sequence space $b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is then defined to be the set of all $t = \{t_Q\}_{Q \in \mathcal{D}(\mathbb{R}^n)} \subset \mathbb{C}$ such that

$$\|t\|_{b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \equiv \inf_\omega \left\{\sum_{j \in \mathbb{Z}} 2^{jsq} \left\|\sum_{Q \in \mathcal{D}(\mathbb{R}^n)} |t_Q| * \tilde{\chi}_Q[\omega(\cdot, 2^{-j})]^{-1}\right\|_{L^p(\mathbb{R}^n)}^q\right\}^{1/q} < \infty,$$

where the infimum is taken over all nonnegative Borel measurable functions ω on $\mathbb{R}_+^{n+1}$ such that ω satisfies (27) and with the restriction that for any $j \in \mathbb{Z}$, $\omega(\cdot, 2^{-j})$ is allowed to vanish only where $\sum_{Q \in \mathcal{D}(\mathbb{R}^n)}|t_Q| * \tilde{\chi}_Q$ vanishes.

(iii) If $q \in (1, \infty)$ and $\tau \in \left[0, \frac{1}{(p \vee q)'}\right]$, the sequence space $f\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is then defined to be the set of all $t = \{t_Q\}_{Q \in \mathcal{D}(\mathbb{R}^n)} \subset \mathbb{C}$ such that

$$\|t\|_{f\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \equiv \inf_\omega \left\|\left\{\sum_{j \in \mathbb{Z}} 2^{jsq}\left(\sum_{Q \in \mathcal{D}(\mathbb{R}^n)} |t_Q| * \tilde{\chi}_Q[\omega(\cdot, 2^{-j})]^{-1}\right)^q\right\}^{1/q}\right\|_{L^p(\mathbb{R}^n)} < \infty,$$

where the infimum is taken over all nonnegative Borel measurable functions ω on $\mathbb{R}_+^{n+1}$ with the same restrictions as in (i).

Similarly, in what follows, we use $a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ to denote either $b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ or $f\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$. When $a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ denotes $f\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, then it will be understood tacitly that

$q \in (1, \infty)$. We remark that $\|\cdot\|_{a\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)}$ is a quasi-norm, namely, there exists a nonnegative constant $\rho \in [0, 1]$ such that for all $t_1, t_2 \in a\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$,

$$\|t_1 + t_2\|_{a\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)} \le 2^\rho \left(\|t_1\|_{a\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)} + \|t_2\|_{a\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)} \right). \tag{28}$$

Let φ be as in Definition (4.2.1). For all $x \in \mathbb{R}^n$, set $\widetilde{\varphi}(x) \equiv \overline{\varphi(x)}$. Then by [176], there exists a function $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that $\operatorname{supp} \mathcal{F}\psi \subset \{\xi \in \mathbb{R}^n : 1/2 \le |\xi| \le 2\}$, $\mathcal{F}\psi$ never vanishes on $\{\xi \in \mathbb{R}^n : 3/5 \le |\xi| \le 5/3\}$ and that for all $\xi \in \mathbb{R}^n$, $\sum_{j \in \mathbb{Z}} \mathcal{F}\widetilde{\varphi}(2^{-j}\xi)\mathcal{F}\psi(2^{-j}\xi) = \chi_{\mathbb{R}^n \setminus \{0\}}(\xi)$. Furthermore, we have the following Calderón reproducing formula which asserts that for all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$,

$$f = \sum_{j \in \mathbb{Z}} \psi_j * \widetilde{\varphi}_j * f = \sum_{Q \in \mathcal{D}(\mathbb{R}^n)} \langle f, \varphi_Q \rangle \psi_Q \tag{29}$$

in $\mathcal{S}'_\infty(\mathbb{R}^n)$; see [190].

We recall the notion of the φ-transform; see, for example, [45, 106, 175, 176].

Definition (4.2.4) [204]: Let $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ such that $\operatorname{supp} \mathcal{F}\psi \subset \{\xi \in \mathbb{R}^n : 1/2 \le |\xi| \le 2\}$, $\mathcal{F}\varphi, \mathcal{F}\psi$ never vanish on $\{\xi \in \mathbb{R}^n : 3/5 \le |\xi| \le 5/3\}$ and $\sum_{j \in \mathbb{Z}} \mathcal{F}(\widetilde{\varphi}_j)\mathcal{F}(\psi_j) \equiv \chi_{\mathbb{R}^n \setminus \{0\}}$.

(i) The φ-transform $\mathcal{S}_\varphi$ is defined to be the map taking each $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ to the sequence $\mathcal{S}_\varphi f \equiv \left\{ (\mathcal{S}_\varphi f)_Q \right\}_{Q \in \mathcal{D}(\mathbb{R}^n)}$, where $(\mathcal{S}_\varphi f)_Q \equiv \langle f, \varphi_Q \rangle$ for all $Q \in \mathcal{D}(\mathbb{R}^n)$.

(ii) The inverse φ-transform T_ψ is defined to be the map taking a sequence $t = \{t_Q\}_{Q \in \mathcal{D}(\mathbb{R}^n)} \subset \mathbb{C}$ to $T_\psi t \equiv \sum_{Q \in \mathcal{D}(\mathbb{R}^n)} t_Q \psi_Q$.

To show that T_ψ is well defined for all $t \in a\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$, we need the following conclusion.

Lemma (4.2.5) [204]: Let $p \in (1, \infty), q \in [1, \infty), s \in \mathbb{R}$ and $\tau \in \left[0, \frac{1}{(p \vee q)} \right]$. Then for all $t \in a\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n), T_\psi t \equiv \sum_{Q \in \mathcal{D}(\mathbb{R}^n)} t_Q \psi_Q$ converges in $\mathcal{S}'_\infty(\mathbb{R}^n)$; moreover, $T_\psi : a\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n) \to \mathcal{S}'_\infty(\mathbb{R}^n)$ is continuous.

Proof: By similarity, we only consider the space $b\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$.

Let $t = \{t_Q\}_{Q \in \mathcal{D}(\mathbb{R}^n)} \in b\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$. We need to show that there exists an $M \in \mathbb{Z}_+$ such that for all $\phi \in \mathcal{S}_\infty(\mathbb{R}^n)$, $\sum_{Q \in \mathcal{D}(\mathbb{R}^n)} |t_Q| |\langle \psi_Q, \phi \rangle| \lesssim \|\phi\|_{\mathcal{S}_M}$, where and in what follows, for all $M \in \mathbb{Z}_+$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we set $\|\phi\|_{\mathcal{S}_M} \equiv \sup_{|\gamma| \le M} \sup_{x \in \mathbb{R}^n} |\partial^\gamma \varphi(x)| (1 + |x|)^{n + M + |\gamma|}$.

Choose a Borel function ω that almost attains the infimum in Definition (4.2.3)(i). That is, ω is a function on $\mathbb{R}^{n+1}_+$ satisfying (27) as well as

$$\left\{ \sum_{j \in \mathbb{Z}} 2^{jsq} \left\| \sum_{Q \in \mathcal{D}_j(\mathbb{R}^n)} |t_Q| * \widetilde{\chi}_Q [\omega(\cdot, 2^{-j})]^{-1} \right\|_{L^p(\mathbb{R}^n)}^q \right\}^{1/q} \le 2\|t\|_{b\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)}. \tag{30}$$

A simple consequence obtained from (27) is that for all $(x, s) \in \mathbb{R}^{n+1}_+, \omega(x, s) \lesssim s^{-n\tau}$; see [189]. Then for all $Q \in \mathcal{D}_j(\mathbb{R}^n)$, by Hölder's inequality and (30), we have

$$|t_Q| \le |Q|^{-\tau - \frac{1}{p}} |t_Q| \left(\int_Q [\omega(x, 2^{-j})]^{-p} dx \right)^{1/p} \lesssim |Q|^{\frac{s}{n} + \frac{1}{2} - \tau - \frac{1}{p}} \|t\|_{b\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)}. \tag{31}$$

Recall that as a special case of [165], there exists a positive constant L_0 such that for all $j \in \mathbb{Z}$,

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$$\sum_{\mathcal{Q}\in\mathcal{D}_j(\mathbb{R}^n)}\left(1+\frac{|x_{\mathcal{Q}}|^n}{\max\{1,|\mathcal{Q}|\}}\right)^{-L_0} \lesssim 2^{n|j|}. \tag{32}$$

Furthermore, it was proved in [190] that if $L > \max\{1/p + 1/2 - s/n - \tau, 1/p + 3/2 + s/n + \tau, L_0\}$, then there exists an $M \in \mathbb{Z}_+$ such that for all $\mathcal{Q} \in \mathcal{D}_j(\mathbb{R}^n)$,

$$|\langle\psi_{\mathcal{Q}},\phi\rangle| \lesssim \|\phi\|_{\mathcal{S}_M}\left(1+\frac{|x_{\mathcal{Q}}|^n}{\max\{1,|\mathcal{Q}|\}}\right)^{-L}(\min\{2^{-jn},2^{jn}\})^L \tag{33}$$

see also [165]. Using (31), (32) and (33), we conclude that

$$\sum_{\mathcal{Q}\in\mathcal{D}(\mathbb{R}^n)}|t_{\mathcal{Q}}||\langle\psi_{\mathcal{Q}},\phi\rangle| \lesssim \|t\|_{b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}\|\phi\|_{\mathcal{S}_M} \lesssim \sum_{\mathcal{Q}\in\mathcal{D}(\mathbb{R}^n)}|\mathcal{Q}|^{\frac{s}{n}+\frac{1}{2}-\tau-\frac{1}{p}}\left(1+\frac{|x_{\mathcal{Q}}|^n}{\max\{1,|\mathcal{Q}|\}}\right)^{-L}2^{-L|j_{\mathcal{Q}}|n}$$

$$\lesssim \|t\|_{b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}\|\phi\|_{\mathcal{S}_M},$$

which completes the proof of Lemma (4.2.5).

Lemma (4.2.6) [204]: Let s, p, q, τ be as in Theorem (4.2.8) and $\lambda \in (n, \infty)$ be sufficiently large. Then there exists a positive constant C such that for all $t \in a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, $\|t\|_{a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \leq \|t_{p\wedge q,\lambda}^*\|_{a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \leq C\|t\|_{a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}$.

Proof: The inequality $\|t\|_{a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \leq \|t_{p\wedge q,\lambda}^*\|_{a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}$ being trivial, we only need to concentrate on $\|t_{p\wedge q,\lambda}^*\|_{a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \lesssim \|t\|_{a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}$. Also, by similarity, we only consider the spaces $b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$.

Let $t = \{t_{\mathcal{Q}}\}_{\mathcal{Q}\in\mathcal{D}(\mathbb{R}^n)} \in b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$. We choose a Borel function ω as in the proof of Lemma (4.2.5). For all cubes $\mathcal{Q} \in \mathcal{D}_j(\mathbb{R}^n)$ and $m \in \mathbb{N}$, we set $A_0(\mathcal{Q}) \equiv \{P \in \mathcal{D}_j(\mathbb{R}^n) : 2^j|x_P - x_{\mathcal{Q}}| \leq 1\}$ and $A_m(\mathcal{Q}) \equiv \{P \in \mathcal{D}_j(\mathbb{R}^n) : 2^{m-1} < 2^j|x_P - x_{\mathcal{Q}}| \leq 2^m\}$. The triangle inequality that $|x - y| \leq |x - x_{\mathcal{Q}}| + |x_{\mathcal{Q}} - x_P| + |x_P - y|$ gives us that $|x - y| \leq 3\sqrt{n}2^{m-j}$ provided $x \in \mathcal{Q}$, $y \in P$ and $P \in A_m(\mathcal{Q})$.

For all $m \in \mathbb{Z}_+$ and $(x, s) \in \mathbb{R}_+^{n+1}$, we set

$$\omega_m(x, s) \equiv 2^{-mn(\lfloor(p\vee q)'\rfloor+2)}\sup\{\omega(y, s) : y \in \mathbb{R}^n, |y - x| < \sqrt{n}2^{m+2}s\},$$

where and in what follows, $\lfloor s \rfloor$ denotes the maximal integer no more than s. By the argument in [189], we know that ω_m still satisfies (27) modulo multiplicative constants independent of m. Also it follows from the definition of ω_m that for all $x \in \mathcal{Q}$, $y \in P$ with $P \in A_m(\mathcal{Q})$, $\omega(y, 2^{-j}) \leq 2^{-mn(\lfloor(p\vee q)'\rfloor+2)}\omega_m(x, 2^{-j})$. For all $r \in (0, \infty)$ and $a \in (0, r))$, using this estimate and the monotonicity of $l^{a/r}$, we obtain that for all $x \in \mathcal{Q}$,

$$\sum_{P \in \mathcal{D}_{j_Q}(\mathbb{R}^n)} \frac{|t_P|^r}{\left(1 + 2^j |x_Q - x_P|\right)^{\lambda}} \left[\omega_m\left(x, 2^{-j}\right)\right]^{-r}$$

$$\leq \left\{ \sum_{P \in \mathcal{D}_{j_Q}(\mathbb{R}^n)} \frac{|t_P|^a}{\left(1 + 2^j |x_Q - x_P|\right)^{\lambda a/r}} \left[\omega_m\left(x, 2^{-j}\right)\right]^{-a} \right\}^{r/a}$$

$$\lesssim 2^{-m\lambda + jnr/a} \left\{ \int_{\mathbb{R}^n} \sum_{P \in A_m(Q)} |t_P|^a \chi_P(y) \left[\omega_m\left(x, 2^{-j}\right)\right]^{-a} dy \right\}^{r/a}$$

$$\lesssim 2^{-m\lambda + mnr(1/a + \lfloor (p \vee q)' \rfloor + 2)} \left\{ \int_{\mathbb{R}^n} \sum_{P \in A_m(Q)} |t_P|^a \chi_P(y) \left[\omega_m\left(y, 2^{-j}\right)\right]^{-a} dy \right\}^{r/a}$$

$$\lesssim 2^{-m\lambda + mnr(1/a + \lfloor (p \vee q)' \rfloor + 2)} \left\{ HL\left(\sum_{P \in A_m(Q)} |t_P|^a \chi_P \left[\omega_m\left(\cdot, 2^{-j}\right)\right]^{-a} \right)(x) \right\}^{r/a},$$

where HL denotes the Hardy-Littlewood maximal operator on $\mathbb{R}^n$.

For all $m \in \mathbb{Z}_+$, set $t_{r,\lambda}^{*,m} \equiv \left\{ \left(t_{r,\lambda}^{*,m}\right)_Q \right\}_{Q \in \mathcal{D}(\mathbb{R}^n)}$ with

$$\left(t_{r,\lambda}^{*,m}\right)_Q \equiv \left(\sum_{P \in A_m(Q)} \frac{|t_P|^r}{\left(1 + 2^j [\ell(P)]^{-1} |x_P - x_Q|\right)^{\lambda}} \right)^{1/r}.$$

In what follows, choose $a \in (0, p \wedge q)$ and $\lambda > (p \wedge q)[n(1/a + \lfloor (p \vee q)' \rfloor + 2) + \rho]$, where ρ is a nonnegative constant as in (30). By (30), the previous pointwise estimate and the $L^{\frac{p}{a}}(\mathbb{R}^n)$-boundedness of HL, we obtain

$$\left\| t_{p \wedge q, \lambda}^{*} \right\|_{b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \lesssim \sum_{m=0}^{\infty} 2^{\rho m} \left\| t_{p \wedge q, \lambda}^{*,m} \right\|_{b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}$$

$$\lesssim \sum_{m=0}^{\infty} 2^{\rho m} \left\{ \sum_{j \in \mathbb{Z}} 2^{jsq} \left[\int_{\mathbb{R}^n} \sum_{Q \in \mathcal{D}_j(\mathbb{R}^n)} \left(\sum_{P \in A_m(Q)} \frac{|t_P|^{p \wedge q}}{\left(1 + 2^j [\ell(P)]^{-1} |x_P - x_Q|\right)^{\lambda}} \right)^{\frac{p}{p \wedge q}} \frac{\tilde{\chi}_Q(x)^p}{[\omega_m(x, 2^{-j})]^p} dx \right]^{\frac{q}{p}} \right\}^{\frac{1}{q}}$$

$$\lesssim \sum_{m=0}^{\infty} 2^{-\frac{m}{p \wedge q}\{\lambda - (p \wedge q)[n(1/a + \lfloor (p \vee q)' \rfloor + 2) + \rho]\}} \left[\sum_{j \in \mathbb{Z}} 2^{jsq} \left\{ \int_{\mathbb{R}^n} \left[HL\left(\sum_{P \in \mathcal{D}_j(\mathbb{R}^n)} \frac{(|t_P| \tilde{\chi}_P)^a}{[\omega_m(\cdot, 2^{-j})]^a} \right)(x) \right]^{\frac{p}{a}} dx \right\}^{\frac{q}{p} \cdot \frac{1}{q}} \right]$$

$$\lesssim \|t\|_{b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)},$$

which completes the proof of Lemma (4.2.6).

For any $f \in \mathcal{S}_{\infty}'(\mathbb{R}^n)$, $\gamma \in \mathbb{Z}$ and $Q \in \mathcal{D}_j(\mathbb{R}^n)$, set $\sup_Q(f) \equiv |Q|^{1/2} \sup_{y \in Q} |\tilde{\varphi}_j * f(y)|$ and

$$\inf_{Q,\gamma}(f) \equiv |Q|^{1/2} \max \left\{ \inf_{y \in \tilde{Q}} |\tilde{\varphi}_j * f(y)| : \ell(\tilde{Q}) = 2^{-\gamma} \ell(Q), \tilde{Q} \subset Q \right\}.$$

Let $\sup(f) \equiv \{\sup_Q(f)\}_{Q \in \mathcal{D}(\mathbb{R}^n)}$ and $\inf_{\gamma}(f) \equiv \{\inf_{Q,\gamma}(f)\}_{Q \in \mathcal{D}(\mathbb{R}^n)}$. We have the following conclusion, whose proof is similar to [106].

Lemma (4.2.7) [204]: Let s, p, q, τ be as in Theorem (4.2.8) and $\gamma \in \mathbb{Z}_+$ be sufficiently large. Then there exists a constant $C \in [1, \infty)$ such that for all $f \in A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$,

$$C^{-1}\|\inf_{\gamma}(f)\|_{a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \leq \|f\|_{A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \leq \|\sup(f)\|_{a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \leq C\|\inf_{\gamma}(f)\|_{a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}.$$

Theorem (4.2.8) [204]: Let $p \in (1, \infty), q \in [1, \infty), s \in \mathbb{R}, \tau \in \left[0, \frac{1}{(p\vee q)'}\right], \varphi$ and Ψ be as in Definition (4.2.4). Then $S_\varphi : A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n) \to a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $T_\Psi : a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n) \to A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ are bounded; moreover, $T_\Psi \circ S_\varphi$ is the identity on $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$.

Proof: To prove Theorem (4.2.8), we need some technical lemmas. For a sequence $t = \{t_Q\}_{Q\in\mathcal{D}(\mathbb{R}^n)}$, $Q \in \mathcal{D}(\mathbb{R}^n)$, $r \in (0, \infty]$ and $\lambda \in (0, \infty)$, define

$$(t_{r,\lambda}^*)_Q \equiv \left(\sum_{P\in\mathcal{D}_{j_Q}(\mathbb{R}^n)} \frac{|t_P|^r}{\left(1 + [\ell(P)]^{-1}|x_P - x_Q|\right)^{\lambda}}\right)^{1/r}$$

and $t_{r,\lambda}^* \equiv \left\{(t_{r,\lambda}^*)_Q\right\}_{Q\in\mathcal{D}(\mathbb{R}^n)}$. For any $p, q \in (0, \infty]$, let $p \wedge q \equiv \min\{p, q\}$. The following estimate is crucial in that this corresponds to the maximal operator estimate.

With the Calderón reproducing formula (29), Lemmas (4.2.6) and (4.2.7), the proof of Theorem (4.2.8) follows (see [106, 165]). We omit the details.

Recall that the corresponding sequence spaces $\dot{a}_{p,q}^{s,\tau}(\mathbb{R}^n)$ of $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$ in [190] were defined as follows.

Definition (4.2.9) [204]: Let $s \in \mathbb{R}$, $q \in (0, \infty]$ and $\tau \in (0, \infty)$. The sequence space $\dot{a}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is defined to be the set of all $t = \{(t)_Q\}_{Q\in\mathcal{D}(\mathbb{R}^n)} \subset \mathbb{C}$ such that $\|t\|_{\dot{a}_{p,q}^{s,\tau}(\mathbb{R}^n)} < \infty$, where if $\dot{a}_{p,q}^{s,\tau}(\mathbb{R}^n) \equiv \dot{b}_{p,q}^{s,\tau}(\mathbb{R}^n)$ for $p \in (0, \infty]$, then

$$\|t\|_{\dot{b}_{p,q}^{s,\tau}(\mathbb{R}^n)} \equiv \sup_{Q\in\mathcal{D}(\mathbb{R}^n)} \frac{1}{|P|^{\tau}} \left\{\sum_{j=j_P}^{\infty} 2^{jsq} \left[\int_P \left(\sum_{l(Q)=2^{-j}} |t_Q||\tilde{\chi}_Q(x)|\right)^p dx\right]^{q/p}\right\}^{1/q}$$

and if $\dot{a}_{p,q}^{s,\tau}(\mathbb{R}^n) \equiv \dot{f}_{p,q}^{s,\tau}(\mathbb{R}^n)$ for $p \in (0, \infty)$, then

$$\|t\|_{\dot{f}_{p,q}^{s,\tau}(\mathbb{R}^n)} \equiv \sup_{Q\in\mathcal{D}(\mathbb{R}^n)} \frac{1}{|P|^{\tau}} \left\{\int_P \left[\sum_{Q\subset P} \left(|Q|^{-s/n}|t_Q||\tilde{\chi}_Q(x)|\right)^q\right]^{p/q} dx\right\}^{1/p}.$$

We now establish the duality between $\dot{a}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $\dot{a}_{p',q'}^{-s,\tau}(\mathbb{R}^n)$. In what follows, for any quasi-Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$, the symbol $\mathcal{B}_1 \hookrightarrow \mathcal{B}_1$ means that there exists a positive constant C such that for all $f \in \mathcal{B}_1$, then $f \in \mathcal{B}_2$ and $\|f\|_{\mathcal{B}_2} \leq C\|f\|_{\mathcal{B}_1}$.

Proposition (4.2.10) [204]: Let s, p, q, τ be as in Theorem (4.2.8). Then $\left(a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)\right)^* = \dot{a}_{p',q'}^{-s,\tau}(\mathbb{R}^n)$ in the following sense.

If $t = \{t_Q\}_{Q\in\mathcal{D}(\mathbb{R}^n)} \in \dot{a}_{p',q'}^{-s,\tau}(\mathbb{R}^n)$, then the map

$$\lambda = \{\lambda_Q\}_{Q\in\mathcal{D}(\mathbb{R}^n)} \mapsto \langle\lambda, t\rangle \equiv \sum_{Q\in\mathcal{D}(\mathbb{R}^n)} \lambda_Q \tilde{t}_Q$$

defines a continuous linear functional on $a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ with operator norm no more than a constant multiple of $\|t\|_{\dot{a}_{p',q'}^{-s,\tau}(\mathbb{R}^n)}$. Conversely, every $L \in \left(a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)\right)^*$ is of this form for a certain $t \in \dot{a}_{p',q'}^{-s,\tau}(\mathbb{R}^n)$ and $\|t\|_{\dot{a}_{p',q'}^{-s,\tau}(\mathbb{R}^n)}$ is no more than a constant multiple of the operator norm of L.

Proof: We only consider the spaces $b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ because the assertion for $f\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ can be proved similarly. Below we write $\mathbb{R}_{\mathbb{Z}}^{n+1} \equiv \{(x,a) \in \mathbb{R}_+^{n+1} : \log_2 a \in \mathbb{Z}\}$.

For $t = \{t_Q\}_{Q\in\mathcal{D}(\mathbb{R}^n)} \in \dot{b}_{p',q'}^{-s,\tau}(\mathbb{R}^n)$ and $\lambda = \{\lambda_Q\}_{Q\in\mathcal{D}(\mathbb{R}^n)} \in b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, let F and G be functions on $\mathbb{R}_{\mathbb{Z}}^{n+1}$ defined by setting, for all $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}$, $F(x,2^{-j}) \equiv \sum_{Q\in\mathcal{D}(\mathbb{R}^n)}|\lambda_Q|\tilde{\chi}_Q$ and $G(x,2^{-j}) \equiv \sum_{P\in\mathcal{D}_j(\mathbb{R}^n)}|\lambda_P|\tilde{\chi}_P$. Since

$$\|F\|_{B\dot{T}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})} \sim \|\lambda\|_{b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}$$

and $\|G\|_{B\dot{W}_{p',q'}^{-s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})} \sim \|t\|_{\dot{b}_{p',q'}^{-s,\tau}(\mathbb{R}^n)}$, where $B\dot{T}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ and $B\dot{W}_{p',q'}^{-s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ are tent spaces introduced in [190], by the duality of tent spaces obtained in [190] that $\left(B\dot{T}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})\right)^* = B\dot{W}_{p',q'}^{-s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$, we have

$$\left|\sum_{Q\in\mathcal{D}(\mathbb{R}^n)}\lambda_Q\tilde{t}_Q\right| \leq \sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n}\sum_{Q\in\mathcal{D}(\mathbb{R}^n)}\sum_{P\in\mathcal{D}_j(\mathbb{R}^n)}|\lambda_Q|\tilde{\chi}_Q|t_P|\tilde{\chi}_P(x)\,dx = \sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n}F(x,2^{-j})G(x,2^{-j})\,dx$$

$$\lesssim \|F\|_{B\dot{T}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})}\|G\|_{B\dot{W}_{p',q'}^{-s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})} \sim \|\lambda\|_{b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}\|t\|_{\dot{b}_{p',q'}^{-s,\tau}(\mathbb{R}^n)}$$

which implies that $\dot{b}_{p',q'}^{-s,\tau}(\mathbb{R}^n) \hookrightarrow \left(b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)\right)^*$.

Conversely, since sequences with finite non-vanishing elements are dense in $b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, we know that every $L \in \left(b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)\right)^*$ is of the form $\lambda \mapsto \sum_{Q\in\mathcal{D}(\mathbb{R}^n)}\lambda_Q\tilde{t}_Q$ for a certain $t = \{t_Q\}_{Q\in\mathcal{D}(\mathbb{R}^n)} \subset \mathbb{C}$. It remains to show that $\|t\|_{\dot{b}_{p',q'}^{-s,\tau}(\mathbb{R}^n)} \lesssim \|t\|_{\left(b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)\right)^*}$.

Fix $P \in \mathcal{D}(\mathbb{R}^n)$ and $a \in \mathbb{R}$. For $j \geq j_P$, let X_j be the set of all $Q \in \mathcal{D}_j(\mathbb{R}^n)$ satisfying $Q \subset P$ and let μ be a measure on X_j such that the μ-measure of the "point" Q is $|Q|/|P|^{\tau a}$. Also, let l_P^q denote the set of all $\{a_j\}_{j\geq j_P} \subset \mathbb{C}$ with $\left\|\{a_j\}_{j\geq j_P}\right\|_{l_P^q} \equiv \left(\sum_{j=j_P}^\infty|a_j|^q\right)^{1/q}$ and $l_P^q\left(l^p(X_j,d\mu)\right)$ denote the set of all $\{a_{Q,j}\}_{Q\in\mathcal{D}_j(\mathbb{R}^n),Q\subset P,j\geq j_P} \subset \mathbb{C}$ with

$$\left\|\{a_{Q,j}\}_{Q\in\mathcal{D}_j(\mathbb{R}^n),Q\subset P,j\geq j_P}\right\|_{l_P^q\left(l^p(X_j,d\mu)\right)} \equiv \left(\sum_{j=j_P}^\infty\left[\sum_{Q\in\mathcal{D}_j(\mathbb{R}^n),Q\subset P}|a_{Q,j}|^p\frac{|Q|}{|P|^{\tau a}}\right]^{q/p}\right)^{1/q}.$$

It is easy to see that the dual space of $l_P^q\left(l^p(X_j,d\mu)\right)$ is $l_P^{q'}\left(l^{p'}(X_j,d\mu)\right)$; see [41]. Via this observation and the already proved conclusion of this proposition, we see that

$$\frac{1}{|P|^\tau}\left\{\sum_{j=j_P}^\infty\left[\sum_{Q\in\mathcal{D}_j(\mathbb{R}^n),Q\subset P}\left(|Q|^{-\frac{s}{n}-\frac{1}{2}}|t_Q|\right)^{p'}|Q|\right]^{\frac{p'}{q'}}\right\}^{\frac{1}{q'}} = \left\|\{|Q|^{-\frac{s}{n}-\frac{1}{2}}|t_Q|\}_{Q\in\mathcal{D}_j(\mathbb{R}^n),Q\subset P,j\geq j_P}\right\|_{l_P^{q'}\left(l^{p'}(X_j,d\mu)\right)}$$

$$= \sup_{\left\|\{\lambda_Q\}_{Q\in\mathcal{D}_j(\mathbb{R}^n),Q\subset P,j\geq j_P}\right\|_{l_P^q\left(l^p(X_j,d\mu)\right)}\leq 1}\left|\sum_{j=j_P}^\infty\sum_{Q\in\mathcal{D}_j(\mathbb{R}^n),Q\subset P}\lambda_Q|Q|^{-\frac{s}{n}-\frac{1}{2}}|t_Q||Q|/|P|^{\tau p'}\right|$$

$$\leq \sup_{\left\|\{\lambda_Q\}_{Q\in\mathcal{D}_j(\mathbb{R}^n),Q\subset P,j\geq j_P}\right\|_{l_P^q\left(l^p(X_j,d\mu)\right)}\leq 1}\|t\|_{\left(b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)\right)^*}\left\|\{\lambda_Q|Q|^{-\frac{s}{n}-\frac{1}{2}}|Q|/|P|^{\tau p'}\}_{Q\in\mathcal{D}_j(\mathbb{R}^n),Q\subset P,j\geq j_P}\right\|_{b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}.$$

To finish the proof of this proposition, it suffices to show that

$$\left\|\left\{\lambda_Q|Q|^{-\frac{s}{n}-\frac{1}{2}}|Q|/|P|^{\tau p'}\right\}_{Q\in\mathcal{D}_j(\mathbb{R}^n),Q\subset P,j\geq j_P}\right\|_{b\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)}\lesssim 1$$

for all sequences λ satisfying $\left\|\{\lambda_Q\}_{Q\in\mathcal{D}_j(\mathbb{R}^n),Q\subset P,j\geq j_P}\right\|_{l^q_P\left(l^p(X_j,d\mu)\right)}\leq 1$. In fact, let $B\equiv$ $B\left(C_P,\sqrt{n}\ell(P)\right)$ and ω be as in the proof of [189] associated with B, then ω satisfies (27) and for all $x\in P$ and $j\geq j_P$, $\left[\omega\left(x,2^{-j}\right)\right]^{-1}\sim[\ell(P)]^{n\tau}$. We then obtain that

$$\left\|\left\{\lambda_Q|Q|^{-\frac{s}{n}-\frac{1}{2}}|Q|/|P|^{\tau p'}\right\}_{Q\in\mathcal{D}_j(\mathbb{R}^n),Q\subset P,j\geq j_P}\right\|_{b\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)}$$

$$\lesssim\left\{\sum_{j=j_P}^{\infty}2^{jsq}\left[\sum_{Q\in\mathcal{D}_j(\mathbb{R}^n),Q\subset P}|Q|^{-\frac{p}{2}}\left(|\lambda_Q||Q|^{-\frac{s}{n}-\frac{1}{2}}\frac{|Q|}{|P|^{\tau p'}}\right)^p\int_Q\left[\omega\left(x,2^{-j}\right)\right]^{-p}dx\right]^{\frac{q}{p}}\right\}^{\frac{1}{q}}$$

$$\sim\left\{\sum_{j=j_P}^{\infty}\left[\sum_{Q\in\mathcal{D}_j(\mathbb{R}^n),Q\subset P}|\lambda_Q|^p|Q|/|P|^{\tau p'}\right]^{\frac{q}{p}}\right\}^{\frac{1}{q}}\sim\left\|\{\lambda_Q\}_{Q\in\mathcal{D}_j(\mathbb{R}^n),Q\subset P,j\geq j_P}\right\|_{b\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)}\lesssim 1,$$

which completes the proof of Proposition (4.2.10).

Applying Theorem (4.2.8), we establish the following Sobolev-type embedding properties of $A\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$. For the corresponding results on $\dot{B}^s_{p,q}(\mathbb{R}^n)$ and $\dot{F}^s_{p,q}(\mathbb{R}^n)$, see [41].

Proposition (4.2.11) [204]: Let $1<p_0<p_1<\infty$ and $-\infty<s_1<s_0<\infty$ Assume in addition that $s_0-n/p_0=s_1-n/p_1$.

(i) If $q\in[1,\infty)$ and $\tau\in\left[0,\min\left\{\frac{1}{(p_0\vee q)'},\frac{1}{(p_1\vee q)'}\right\}\right]$ such that $\tau(p_0\vee q)'=\tau(p_1\vee q)'$, then $B\dot{H}^{s_0,\tau}_{p_0,q}(\mathbb{R}^n)\hookrightarrow B\dot{H}^{s_1,\tau}_{p_1,q}(\mathbb{R}^n)$.

(ii) If $q,r\in(1,\infty)$ and $\tau\in\left[0,\min\left\{\frac{1}{(p_0\vee r)'},\frac{1}{(p_1\vee q)'}\right\}\right]$ such that $\tau(p_0\vee r)'=\tau(p_1\vee q)'$, then $F\dot{H}^{s_0,\tau}_{p_0,r}(\mathbb{R}^n)\hookrightarrow F\dot{H}^{s_1,\tau}_{p_1,q}(\mathbb{R}^n)$.

Proof: By Theorem (4.2.8) and similarity, it suffices to prove the corresponding conclusions on sequence spaces $f\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$, namely, to show that $\|t\|_{f\dot{H}^{s_1,\tau}_{p_1,q}(\mathbb{R}^n)}\lesssim\|t\|_{f\dot{H}^{s_0,\tau}_{p_0,r}(\mathbb{R}^n)}$ for all $t\in f\dot{H}^{s_0,\tau}_{p_0,r}(\mathbb{R}^n)$. When $\tau=0$, this is a classic conclusion on Triebel-Lizorkin spaces.

In the case when $\tau>0$, we have $(p_0\vee r)'\leq(p_1\vee q)'$. Let $t\in f\dot{H}^{s_0,\tau}_{p_0,r}(\mathbb{R}^n)$ and ω satisfy

$$\int_{\mathbb{R}^n}[N\omega(x)]^{(p_0\vee r)'}dH^{n\tau(p_0\vee r)'}(x)\leq 1 \tag{34}$$

and

$$\left\{\int_{\mathbb{R}^n}\left[\sum_{j\in\mathbb{Z}}\sum_{Q\in\mathcal{D}_j(\mathbb{R}^n)}|Q|^{-\frac{s_0}{n}-\frac{r}{2}}|t_Q|^r\chi_Q[\omega(x,2^{-j})]^{-r}\right]^{p_0/r}dx\right\}^{1/p_0}\lesssim\|t\|_{f\dot{H}^{s_0,\tau}_{p_0,r}(\mathbb{R}^n)}.$$

For all $(x,t) \in \mathbb{R}^{n+1}_+$, we set $\widetilde{\omega}(x,s) \equiv \sup\{\omega(y,s) : y \in \mathbb{R}^n, |y-x| < \sqrt{n}s\}$. Then by the argument in [189], we know that a constant multiple of $\widetilde{\omega}$ also satisfies (34). Since $(p_0 \vee r)' \leq (p_1 \vee q)'$, we note that $\widetilde{\omega}$ satisfies

$$\int_{\mathbb{R}^n} [N\widetilde{\omega}(x)]^{(p_1 \vee q)'} dH^{n\tau(p_1 \vee q)'}(x) \lesssim 1.$$

For all Q with $\ell(Q) = 2^{-j}$, set $\tilde{t}_Q \equiv |t_Q| \sup_{y \in Q}\{[\widetilde{\omega}(y, 2^{-j})]^{-1}\}$. Observe that for all $x \in Q$ with $\ell(Q) = 2^{-j}$, $[\widetilde{\omega}(y, 2^{-j})]^{-1} \lesssim \inf_{y \in Q}[\omega(y, 2^{-j})]^{-1}$, and hence, $\sup_{x \in Q}[\widetilde{\omega}(x, 2^{-j})]^{-1} \lesssim \inf_{y \in Q}[\omega(y, 2^{-j})]^{-1}$. This observation together with $p_0 < p_1$, $s_0 - n/p_0 = s_1 - n/p_1$ and the corresponding embedding property for Triebel-Lizorkin spaces (see, [41]) yields that

$$\left\{ \int_{\mathbb{R}^n} \left[\sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_j(\mathbb{R}^n)} |Q|^{-\frac{s_1 q}{n} - \frac{q}{2}} |t_Q|^q \chi_Q(x) [\widetilde{\omega}(x, 2^{-j})]^{-q} \right]^{p_1/q} dx \right\}^{1/p_1}$$

$$\leq \left\{ \int_{\mathbb{R}^n} \left[\sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_j(\mathbb{R}^n)} |Q|^{-\frac{s_1 q}{n} - \frac{q}{2}} |t_Q|^q \chi_Q(x) \sup_{y \in Q}\{[\widetilde{\omega}(y, 2^{-j})]^{-q}\} \right]^{p_1/q} dx \right\}^{1/p_1}$$

$$= \left\{ \int_{\mathbb{R}^n} \left[\sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_j(\mathbb{R}^n)} |Q|^{-\frac{s_1 q}{n} - \frac{q}{2}} |\tilde{t}_Q|^q \chi_Q(x) \right]^{p_1/q} dx \right\}^{1/p_1} = \|t\|_{\dot{f}^{s_1}_{p_1,q}(\mathbb{R}^n)} \lesssim \|t\|_{\dot{f}^{s_0}_{p_0,r}(\mathbb{R}^n)}$$

$$\sim \left\{ \int_{\mathbb{R}^n} \left[\sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_j(\mathbb{R}^n)} |Q|^{-\frac{s_0 r}{n} - \frac{r}{2}} |\tilde{t}_Q|^r \chi_Q(x) \right]^{p_0/r} dx \right\}^{1/p_0}$$

$$\sim \left\{ \int_{\mathbb{R}^n} \left[\sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_j(\mathbb{R}^n)} |Q|^{-\frac{s_0 r}{n} - \frac{r}{2}} |t_Q|^r \chi_Q(x) \sup_{y \in Q}\{[\widetilde{\omega}(y, 2^{-j})]^{-r}\} \right]^{p_0/r} dx \right\}^{1/p_0}$$

$$\lesssim \left\{ \int_{\mathbb{R}^n} \left[\sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_j(\mathbb{R}^n)} |Q|^{-\frac{s_0 r}{n} - \frac{r}{2}} |t_Q|^r \chi_Q(x) [\omega(x, 2^{-j})]^{-r} \right]^{p_0/r} dx \right\}^{1/p_0} \lesssim f\dot{H}^{s_0,\tau}_{p_0,r}(\mathbb{R}^n);$$

see [106] for the definition of the sequence spaces $\dot{f}^s_{p,q}(\mathbb{R}^n)$. Therefore, $\|t\|_{\dot{f}^{s_1,\tau}_{p_1,q}(\mathbb{R}^n)} \lesssim \|t\|_{\dot{f}^{s_0,\tau}_{p_0,r}(\mathbb{R}^n)}$, which completes the proof of Proposition (4.2.11).

When $\tau = 0$, Proposition (4.2.11) recovers the corresponding results on $\dot{B}^s_{p,q}(\mathbb{R}^n)$ and $\dot{F}^s_{p,q}(\mathbb{R}^n)$ in [41], which are known to be sharp; see [203]. We further show that the restriction that $\tau(p_0 \vee q)' = \tau(p_1 \vee q)'$ in Proposition (4.2.11)(i) is also sharp. To see this, we need the following geometrical observation on the Hausdorff capacity.

Lemma (4.2.12) [204]: Let $d \in (0,n)$. Suppose that $\{E_j\}_{j=1}^{\infty}$ are given subsets of $\mathbb{R}^n$ such that $E_j \subset B((A_j, 0, \ldots, 0), n)$, where $\{A_j\}_{j=1}^{\infty}$ is an increasing sequence of natural numbers satisfying that $A_1 \geq 10$ and for all $j, l \in \mathbb{N}, A_{j+l} - A_j \geq 4nl^{1/d}$. Then $H^d(\cup_{j=1}^{\infty} E_j)$ and $\sum_{j=1}^{\infty} H^d(E_j)$ are equivalent.

Proof: The inequality $H^d\left(\bigcup_{j=1}^\infty E_j\right) \le \sum_{j=1}^\infty H^d(E_j)$ is trivial. Let us prove the reverse inequality. To this end, let us first notice the following geometric observation that when a ball $B \equiv (x_B, r_B)$ intersects E_j and E_{j+l} for some $j, l \in \mathbb{N}$, then $2B$ engulfs $E_j, E_{j+l}, \ldots, E_{j+l}$. Thus, $4r_B$ is greater than $A_{j+l} - A_j$ and hence, $r_B^d \ge \left((A_{j+l} - A_j)/4\right)^d \ge l n^d$. Therefore, instead of using B we can use $B\left((A_j, 0, \ldots, 0), n\right), \ldots, B\left((A_j, 0, \ldots, 0), n\right)$ to cover E_j and E_{j+l}. Notice that $\left\{B\left((A_j, 0, \ldots, 0), n\right)\right\}_{j=1}^\infty$ are disjoint. Based on these observations, without loss of generality, we may assume, in estimating $H^d\left(\bigcup_{j=1}^\infty E_j\right)$, that each ball in the ball covering meets only one E_j. From this, it is easy to follow that $H^d\left(\bigcup_{j=1}^\infty E_j\right) \gtrsim \sum_{j=1}^\infty H^d(E_j)$, which completes the proof of Lemma (4.2.12).

Lemma (4.2.13) [204]: Let $s \in \mathbb{R}, p \in (1, \infty), q \in [1, \infty), \tau \in (0, \frac{1}{(p \vee q)'}]$ and $\{A_k\}_{k=1}^\infty$ be as in Lemma (4.2.12) such that $\mathcal{Q}_k \equiv (A_k, 0, \ldots, 0) + 2^{-k}[0,1)^n \in \mathcal{D}_k(\mathbb{R}^n)$ for all $k \in \mathbb{N}$ (the existence of $\{A_k\}_{k=1}^\infty$ is obvious). Define $t_j \equiv \left\{(t_j)_\mathcal{Q}\right\}_{\mathcal{Q} \in \mathcal{D}(\mathbb{R}^n)}$ so that $(t_j)_\mathcal{Q} \equiv 2^{-\frac{kn}{2} - k\left(s - \frac{n}{p}\right)}$ if $\mathcal{Q} = \mathcal{Q}_k$ and $k \in \{1, \ldots, j\}$, $(t_j)_\mathcal{Q} \equiv 0$ otherwise. Then for all $j \in \mathbb{N}$, $\|t_j\|_{b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}$ is equivalent to $j^{\frac{1}{q} + \frac{1}{(p \vee q)'}}$ and $\|t_j\|_{f\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}$ is equivalent to $j^{\frac{1}{p} + \frac{1}{(p \vee q)'}}$.

Proof: For the Besov-Hausdorff space, let us minimize

$$\left(\sum_{k=1}^j 2^{ksq} \left\| \left|(t_j)_{\mathcal{Q}_k}\right| \tilde{\chi}_{\mathcal{Q}_k}(x)\left[\omega(\cdot, 2^{-k})\right]^{-1}\right\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q}$$

under the condition (27). By the definition of t_j and the assumption on ω in Definition (4.2.3), we may assume that $\omega \equiv 0$ outside $\bigcup_{k=1}^j \left(\mathcal{Q}_{0,(A_k,0,\ldots,0)} \times \{2^{-k}\}\right)$ and for all $\mathcal{Q} \in \mathcal{D}_k(\mathbb{R}^n)$, $\mathcal{Q} \subset \mathcal{Q}_{0,(A_k,0,\ldots,0)}$ and $k \in \{1, \ldots, j\}$, $\sup_{x \in \mathcal{Q}} \omega(x, 2^{-k}) = \sup_{x \in \mathcal{Q}_k} \omega(x, 2^{-k})$, where $\mathcal{Q}_{0,(A_j,0,\ldots,0)} \equiv (A_j, 0, \ldots, 0) + [0,1)^n \in \mathcal{D}_0(\mathbb{R}^n)$. Also, we can replace ω with the maximal function $\tilde{\omega}$ given by $\tilde{\omega}(x, 2^{-k}) \equiv \sup_{y \in \mathcal{Q}_{k,x}} \omega(y, 2^{-k})$, where $k \in \{1, \ldots, j\}$ and $\mathcal{Q}_{k,x} \in \mathcal{D}_k(\mathbb{R}^n)$ is a unique cube containing x. This construction implies that $\tilde{\omega}$ equals a constant on $\mathcal{Q}_{0,(A_j,0,\ldots,0)}$ for each $k \in \{1, \ldots, j\}$, namely, $\tilde{\omega}(\cdot, 2^{-k}) \equiv \alpha_k \chi_{\mathcal{Q}_{0,(A_j,0,\ldots,0)}}$. Notice that if $N\tilde{\omega}(x) \ne 0$, then $x \in B\left((A_k, 0, \ldots, 0), n\right)$ for some $k \in \{1, \ldots, j\}$. This combined with Lemma (4.2.12) yields that

$$\int_{\mathbb{R}^n} [N\tilde{\omega}(x)]^{(p \vee q)'} dH^{n\tau(p \vee q)'}(x)$$

$$= \int_0^\infty H^{n\tau(p \vee q)'}\left(\left\{x \in \left(\bigcup_{k=1}^j B((A_k, 0, \ldots, 0), n)\right) : [N\tilde{\omega}(x)]^{(p \vee q)'}\right.\right.$$

$$\left.\left. > \lambda\right\}\right) d\lambda \sim \sum_{k=1}^j \int_0^\infty H^{n\tau(p \vee q)'}\left(\{x \in B((A_k, 0, \ldots, 0), n) : [N\tilde{\omega}(x)]^{(p \vee q)'}\right.$$

$$\left. > \lambda\}\right) d\lambda \sim \sum_{k=1}^j \int_{B((A_k,0,\ldots,0),n)} [N\tilde{\omega}(x)]^{(p \vee q)'} dH^{n\tau(p \vee q)'}(x) \sim \sum_{k=1}^j (\alpha_k)^{(p \vee q)'}.$$

On the other hand,

$$\left(\sum_{k=1}^{j} 2^{ksq} \left\| \left|(t_j)_{Q_k}\right| \tilde{\chi}_{Q_k}(x)[\tilde{\omega}(\cdot, 2^{-k})]^{-1} \right\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} = \left[\sum_{k=1}^{j}(\alpha_k)^{-q}\right]^{1/q}$$

In summary (modulo a multiplicative constant), we need to minimize $\left(\sum_{k=1}^{j}(\alpha_k)^{-q}\right)^{1/q}$ under the condition $\sum_{k=1}^{j}(\alpha_k)^{(p\vee q)'} \lesssim 1$. This can be achieved as follows: By using the geometric mean, we have

$$\left(\sum_{k=1}^{j}(\alpha_k)^{-q}\right)^{1/q} \gtrsim \left(\sum_{k=1}^{j}(\alpha_k)^{-q}\right)^{1/q} \left(\sum_{k=1}^{j}(\alpha_k)^{(p\vee q)'}\right)^{1/(p\vee q)'}$$

$$\gtrsim \left(j\sqrt[j]{\prod_{k=1}^{j}(\alpha_k)^{-q}}\right)^{1/q} \left(j\sqrt[j]{\prod_{k=1}^{j}(\alpha_k)^{(p\vee q)'}}\right)^{1/(p\vee q)'} \sim j^{1/q+1/(p\vee q)'}.$$

In particular, $\left(\sum_{k=1}^{j}(\alpha_k)^{-q}\right)^{\frac{1}{q}} \sim j^{\frac{1}{q}+\frac{1}{(p\vee q)'}}$ when $\sum_{k=1}^{j}(\alpha_k)^{(p\vee q)'} \sim 1$ and the α_k 's are identical. Thus, for all $j \in \mathbb{N}$, $\|t_j\|_{b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \sim j^{\frac{1}{q}+\frac{1}{(p\vee q)'}}$.

For the Triebel-Lizorkin-Hausdorff space, similarly to the above arguments, we see that

$$\left(\int_{\mathbb{R}^n} \left[\sum_{k=1}^{j} |Q_k|^{-(s/n+1/2)q} \left|(t_j)_{Q_k}\right|^q \chi_{Q_k}(x)[\tilde{\omega}(x, 2^{-k})]^{-q}\right]^{p/q} dx\right)^{1/p}$$

$$= \left(\int_{\mathbb{R}^n} \sum_{k=1}^{j} |Q_k|^{-(s/n+1/2)q} \left|(t_j)_{Q_k}\right|^p \chi_{Q_k}(x)(\alpha_k)^{-p} dx\right)^{1/p} = \left[\sum_{k=1}^{j}(\alpha_k)^{-q}\right]^{1/q}.$$

Applying the geometric mean again, we have

$$\left(\sum_{k=1}^{j}(\alpha_k)^{-p}\right)^{1/p} \gtrsim \left(\sum_{k=1}^{j}(\alpha_k)^{-p}\right)^{1/p} \left(\sum_{k=1}^{j}(\alpha_k)^{(p\vee q)'}\right)^{1/(p\vee q)'}$$

$$\gtrsim \left(j\sqrt[j]{\prod_{k=1}^{j}(\alpha_k)^{-p}}\right)^{1/p} \left(j\sqrt[j]{\prod_{k=1}^{j}(\alpha_k)^{(p\vee q)'}}\right)^{1/(p\vee q)'} \sim j^{1/p+1/(p\vee q)'}.$$

In particular, $\left(\sum_{k=1}^{j}(\alpha_k)^{-p}\right)^{\frac{1}{p}} \sim j^{\frac{1}{p}+\frac{1}{(p\vee q)'}}$ when $\sum_{k=1}^{j}(\alpha_k)^{(p\vee q)'} \sim 1$ and the α_k 's are identical, which implies that for all $j \in \mathbb{N}$, $\|t_j\|_{b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \sim j^{\frac{1}{p}+\frac{1}{(p\vee q)'}}$. This finishes the proof of Lemma (4.2.13).

Proposition (4.2.14) [204]: Let s, τ, p_0, p_1, q, r be as in Proposition (4.2.11).

(i) If $b\dot{H}_{p_0,q}^{s_0,\tau} \hookrightarrow b\dot{H}_{p_1,q}^{s_1,\tau}$, then $\tau(p_0 \vee q)' = \tau(p_1 \vee q)'$.

(ii) If $f\dot{H}_{p_0,r}^{s_0,\tau} \hookrightarrow f\dot{H}_{p_1,q}^{s_1,\tau}$, then $\tau(p_0 \vee r)' \leq \tau(p_1 \vee q)' + \tau\left(\frac{1}{p_0} - \frac{1}{p_1}\right)(p_0 \vee r)'(p_1 \vee q)'$.

Proof: By similarity, we only consider the Besov-Hausdorff space. Let t_j be as in Lemma (4.2.13) with s, p replaced, respectively, by s_0 and p_0. Since $s_0 - n/p_0 = s_1 - n/p_1$, by

Lemma (4.2.13), we have $\|t_j\|_{b\dot{H}_{p_0,q}^{s_0,\tau}} \sim j^{\frac{1}{p}+\frac{1}{(p\vee q)'}}$ and $\|t_j\|_{b\dot{H}_{p_1,q}^{s_1,\tau}} \sim j^{\frac{1}{p}+\frac{1}{(p\vee q)'}}$ for all $j \in \mathbb{N}$, which together with $b\dot{H}_{p_0,q}^{s_0,\tau} \hookrightarrow b\dot{H}_{p_1,q}^{s_1,\tau}$ implies that $j^{\frac{1}{q}+\frac{1}{(p_1\vee q)'}} \lesssim j^{\frac{1}{p}+\frac{1}{(p_0\vee q)'}}$ for all $j \in \mathbb{N}$. Therefore, $(p_0 \vee q)' \le (p_1 \vee q)'$. Meanwhile it is trivial that $(p_0 \vee q)' \ge (p_1 \vee q)'$ since $p_1 > p_0$. We then have $(p_0 \vee q)' = (p_1 \vee q)'$. This finishes the proof of Proposition (4.2.14).

We begin with considering the boundedness of almost diagonal operators on $a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, which is applied to establish the smooth atomic and molecular decomposition characterizations of $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$.

Definition (4.2.15) [204]: Let $p \in (1,\infty), q \in [1,\infty)$, $s \in \mathbb{R}, \tau \in (0, \frac{1}{(p\vee q)'}]$ and $\varepsilon \in (0,\infty)$. For all $Q, P \in \mathcal{D}(\mathbb{R}^n)$, define

$$\omega_{Q\,P}(\varepsilon) \equiv \left(\frac{\ell(Q)}{\ell(P)}\right)^s \left(1 + \frac{|x_P - x_Q|}{\max(\ell(Q),\ell(P))}\right)^{-n-\varepsilon} \min\left(\left(\frac{\ell(P)}{\ell(Q)}\right)^{\frac{n+\varepsilon}{2}}, \left(\frac{\ell(Q)}{\ell(P)}\right)^{\frac{n+\varepsilon}{2}}\right).$$

An operator A associated with a matrix $\{a_{Q\,P}\}_{Q,P\in\mathcal{D}(\mathbb{R}^n)}$, namely, for all sequences $t = \{t_Q\}_{Q\in\mathcal{D}(\mathbb{R}^n)} \subset \mathbb{C}$, $At \equiv \{(At)_Q\}_{Q\in\mathcal{D}(\mathbb{R}^n)} \equiv \{\sum_{P\in\mathcal{D}(\mathbb{R}^n)} a_{Q\,P} t_P\}_{Q\in\mathcal{D}(\mathbb{R}^n)}$, is called ε-almost diagonal on $a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, if the matrix $\{a_{Q\,P}\}_{Q,P\in\mathcal{D}(\mathbb{R}^n)}$ satisfies

$$\sup_{Q,P\in\mathcal{D}(\mathbb{R}^n)} |a_{Q\,P}|/\omega_{Q\,P}(\varepsilon) < \infty.$$

Lemma (4.2.16) [204]: Let $d \in (0,n]$ and Ω be an open set in $\mathbb{R}^n$ such that $\Omega = \cup_{j=1}^\infty B_j$, where $\{B_j\}_{j=1}^\infty \equiv \{B(X_j, R_j)\}_{j=1}^\infty$ is a countable collection of balls. Define

$$H^d\left(\Omega, \{B_j\}_{j=1}^\infty\right) \equiv \inf\left\{\sum_{k=1}^\infty r_k^d : \Omega \subset \bigcup_{k=1}^\infty B(X_k, R_k), B(X_k, R_k) \supset B_j \text{ if } B_j \cap B(X_k, R_k) \neq \emptyset\right\}$$

Then there exists a positive constant C, independent of Ω, $\{B_j\}_{j=1}^\infty$ and d, such that

$$H^d(\Omega) \le H^d\left(\Omega, \{B_j\}_{j=1}^\infty\right) \le C(46)^d H^d(\Omega).$$

Proof: The first inequality is trivial. We only need to prove the second one. Without loss of generality, we may assume $\sup_{j\in\mathbb{N}} R_j < \infty$. By the well-known $(5r)$-covering lemma, there exists a subset j^* of $\mathbb{N}$ such that $\cup_{j=1}^\infty(3B_j) \subset \cup_{j\in j^*}(15B_j)$ and $\chi_{j\in j^*} * \chi_{(3B_j)} \le 1$. Furthermore, by its construction, if $B_{j'}, j' \in \mathbb{N}$, intersects B_j for some $j \in j^*$, we have that $(3B_{j'}) \subset (15B_j)$.

Let $\{B(x_k, r_k)\}_{k\in\mathbb{N}}$ be a collection of balls such that $\Omega \subset \cup_{k=1}^\infty B(x_k, r_k)$ and $\sum_{k=1}^\infty r_k^d \le 2H^d(\Omega)$. Set

$K_1 \equiv \{k \in \mathbb{N} : \text{when } B(x_k, 45r_k) \cap B_j \neq \emptyset \text{ for any } j \in \mathbb{N}, \quad \text{then } r_k \ge 135R_j\}$

And $J_1 \equiv \{j \in \mathbb{N} : B_j \cap B(x_k, 45r_k) \neq \emptyset \text{ for some } k \in K_1\}$. Also define $J_2 \equiv (\mathbb{N}\backslash J_1)$ and $K_2 \equiv (\mathbb{N}\backslash K_1)$. We remark that if $k \in K_2$, then there exists $j \in J_2$ such that $B_j \cap B(x_k, 45r_k) \neq \emptyset$ and $135R_j > r_k$. Notice that $B_j \subset \Omega \subset (\cup_{k=1}^\infty B(x_k, r_k))$. Hence, for each $j \in J_2$, we have $B_j \subset \Omega \subset \left(\cup_{k\in K_2, B(x_k,r_k)\cap B_j\neq\emptyset} B(x_k, r_k)\right)$, and then, by $d \le n$ and the monotonicity of $l^{\frac{d}{n}}$, we see that

$$\sum_{k\in K_2} r_k^d \sim \sum_{k\in K_2} |B(x_k, r_k)|^{\frac{d}{n}} \gtrsim \sum_{j\in J^*\cap J_2} \sum_{k\in K_2, B_j\cap B(x_k, 45r_k)\neq\emptyset} |B(x_k, r_k)|^{\frac{d}{n}}$$

$$\gtrsim \sum_{j\in J^*\cap J_2} \left(\sum_{k\in K_2, B_j\cap B(x_k, 45r_k)\neq\emptyset} |B(x_k, r_k)| \right)^{\frac{d}{n}} \gtrsim \sum_{j\in J^*\cap J_2} R_j^d,$$

which further yields that

$$\sum_{k\in K_1} r_k^d + \sum_{j\in J^*\cap J_2} R_j^d \lesssim \sum_{k\in K} r_k^d.$$

On the other hand, we have

$$\Omega \subset \bigcup_{j=1}^{\infty} B_j \subset \bigcup_{j\in J^*}(15B_j) = \left\{ \bigcup_{j\in J^*\cap J_1}(15B_j) \right\} \cup \left\{ \bigcup_{j\in J^*\cap J_2}(15B_j) \right\}$$

$$\subset \left\{ \bigcup_{k\in K_1} B(x_k, 46r_k) \right\} \cup \left\{ \bigcup_{j\in J^*\cap J_2}(15B_j) \right\}.$$

Notice that for $k \in K_1$, $B(x_k, 45r_k)$ meets B_j for some $j \in \mathbb{N}$ gives us $r_k \geq 135R_j$, which further implies that $B(x_k, 46r_k) \supset B_j$. Also, for $j \in J^*$ and $j' \in \mathbb{N}$, if $B_j \cap B_{j'} \neq \emptyset$, then $(15B_j) \supset B_{j'}$. As a result, we conclude that $\{B(x_k, 46r_k)\}_{k\in K_1} \cup \{15B_j\}_{j\in J^*\cap J_2}$ is the desired covering of Ω and hence,

$$H^d\left(\Omega, \{B_j\}_{j=1}^{\infty}\right) \leq \sum_{k\in K_1}(46r_k)^d + \sum_{j\in J^*\cap J_2}(15R_j)^d \lesssim (46)^d H^d(\Omega),$$

which completes the proof of Lemma (4.2.16).

Applying Lemma (4.2.16), we have the following conclusion.

Lemma (4.2.17)[204]: Let $\beta \in [1, \infty)$, $\lambda \in (0, \infty)$ and ω be a nonnegative Borel measurable function on $\mathbb{R}_+^{n+1}$. Then there exists a positive constant C, independent of β, ω and λ, such that

$$H^d\left(\{x \in \mathbb{R}^n : N_\beta\omega(x) > \lambda\}\right) \leq C\beta^d H^d\left(\{x \in \mathbb{R}^n : N\omega(x) > \lambda\}\right),$$

where $N_\beta\omega(x) \equiv \sup_{|y-x|<\beta t} \omega(y, t)$.

Proof: Observe that

$$\{x \in \mathbb{R}^n : N\omega(x) > \lambda\} = \bigcup_{\substack{t\in(0,\infty)}} \bigcup_{\substack{y\in\mathbb{R}^n \\ \omega(y,t)>\lambda}} B(y, t)$$

and that

$$\{x \in \mathbb{R}^n : N_\beta\omega(x) > \lambda\} = \bigcup_{\substack{t\in(0,\infty)}} \bigcup_{\substack{y\in\mathbb{R}^n \\ \omega(y,t)>\lambda}} B(y, \beta t).$$

By the Linderöf covering lemma, there exists a countable subset $\{B_l\}_{l=0}^{\infty}$ of $\{B(y, t) : t \in (0, \infty), y \in \mathbb{R}^n$ satisfying $\omega(y, t) > \lambda\}$ such that $\{x \in \mathbb{R}^n : N_\beta\omega(x) > \lambda\}$. By Lemma (4.2.16), it suffices to prove that

$$H^d\left(\{x \in \mathbb{R}^n : N_\beta\omega(x) > \lambda\}, \{\beta B_l\}_{l=0}^{\infty}\right) \lesssim \beta^d H^d\left(\bigcup_{l=0}^{\infty} B_l, \{B_l\}_{l=0}^{\infty}\right).$$

Let $\{B_k^*\}_{k=0}^{\infty}$ be a ball covering of $\bigcup_{l\in\mathbb{N}} B_l$ such that $\sum_{k=0}^{\infty} r_{B_k^*}^d \leq 2H^d(\bigcup_{l=0}^{\infty} B_l, \{B_l\}_{l=0}^{\infty})$, and that B_k^* engulfs B_l whenever they intersect, where $r_{B_k^*}$ denotes the radius of B_k^*.

Therefore, βB_k^* engulfs βB_l whenever they intersect and $\{x \in \mathbb{R}^n : N_\beta \omega(x) > \lambda\} \subset \{\bigcup_{k=0}^\infty (\beta B_k^*)\}$. We then have

$$2\beta^d H^d\left(\bigcup_{l=0}^\infty B_l, \{B_l\}_{l=0}^\infty\right) \geq \sum_{l=0}^\infty (\beta r_{B_k^*})^d \geq H^d(\{x \in \mathbb{R}^n : N_\beta \omega(x) > \lambda\}, \{\beta B_l\}_{l=0}^\infty),$$

which completes the proof of Lemma (4.2.17).

As an immediate consequence of Lemma (4.2.17), we have the following result.

Corollary (4.2.18) [204]: Let $d \in (0, n]$, $\beta \in [1, \infty)$ and ω be a nonnegative measurable function on $\mathbb{R}_+^{n+1}$. Define $\omega_\beta(x, t) \equiv \sup_{y \in B(x, \beta t)} \omega(y, t)$. Then there exists a positive constant C such that

$$\int_{\mathbb{R}^n} N_\beta \omega(x) dH^d(x) \leq C\beta^d \leq \int_{\mathbb{R}^n} N\omega(x) dH^d(x).$$

Theorem (4.2.19) [204]: Let $p \in (1, \infty)$, $q \in [1, \infty)$, $s \in \mathbb{R}$, $\varepsilon \in (0, \infty)$ and $\tau \in (0, \frac{1}{(p \vee q)'}]$. Then all the ε-almost diagonal operators on $a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ are bounded if $\varepsilon > 2n\tau$.

Proof: By similarity, we only consider $f\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$. Similarly to the proof of [190, Theorem 4.1], without loss of generality, we may assume $s = 0$, since this case implies the general case.

By the Aoki theorem (see [191]), there exists a $\kappa \in (0, 1]$ such that $\|\cdot\|_{f\dot{H}_{p,q}^{0,\tau}(\mathbb{R}^n)}^\kappa$ becomes a norm in $f\dot{H}_{p,q}^{0,\tau}(\mathbb{R}^n)$. Let $t \in f\dot{H}_{p,q}^{0,\tau}(\mathbb{R}^n)$. For $Q \in \mathcal{D}(\mathbb{R}^n)$, we write $A \equiv A_0 + A_1$ with $(A_0 t)_Q \equiv \sum_{\{P \in \mathcal{D}(\mathbb{R}^n) : \ell(Q) \leq \ell(P)\}} a_{Q P} t_P$ and $(A_1 t)_Q \equiv \sum_{\{P \in \mathcal{D}(\mathbb{R}^n) : \ell(P) < l(Q)\}} a_{Q P} t_P$. By Definition (4.2.15), we see that for $Q \in \mathcal{D}(\mathbb{R}^n)$,

$$|(A_0 t)_Q| \lesssim \sum_{\{P \in \mathcal{D}(\mathbb{R}^n) : \ell(Q) \leq \ell(P)\}} \left(\frac{\ell(Q)}{\ell(P)}\right)^{\frac{n+\varepsilon}{2}} \frac{|t_P|}{(1 + [\ell(P)]^{-1}|x_Q - x_P|)^{n+\varepsilon}}.$$

Thus, we have

$\|A_0 t\|_{f\dot{H}_{p,q}^{0,\tau}(\mathbb{R}^n)}$

$$\lesssim \inf_\omega \left\|\left\{\sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_j(\mathbb{R}^n)} |Q|^{-\frac{q}{2}} \chi_Q \left[\sum_{i=-\infty}^j \sum_{P \in \mathcal{D}_i(\mathbb{R}^n)} 2^{(i-j)\frac{n+\varepsilon}{2}} \frac{|t_P|[\omega(\cdot, 2^{-j})]^{-1}}{(1 + 2^i|x_Q - x_P|)^{n+\varepsilon}}\right]^q\right\}^{1/q}\right\|_{L^p(\mathbb{R}^n)}.$$

Let ω be a nonnegative Borel measurable function satisfying (27) and

$$\left\|\left\{\sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_j(\mathbb{R}^n)} |t_Q|^q [\tilde{\chi}_Q \omega(\cdot, 2^{-j})]^{-q}\right\}^{1/q}\right\|_{L^p(\mathbb{R}^n)} \lesssim \|t\|_{f\dot{H}_{p,q}^{0,\tau}(\mathbb{R}^n)}.$$

Let $A_{0,i}(Q) \equiv \{P \in \mathcal{D}_i(\mathbb{R}^n) : 2^i|x_Q - x_P| \leq \sqrt{n}/2\}$ and $A_{m,i}(Q) \equiv \{P \in \mathcal{D}_i(\mathbb{R}^n) : 2^{m-1}\sqrt{n}/2 < 2^i|x_P - x_Q| \leq 2^m \sqrt{n}/2\}$ for all $i \in \mathbb{Z}$ and $m \in \mathbb{Z}_+$. Define $\omega_m(x, t) \equiv 2^{-mn\tau} \sup_{y \in B(x, \sqrt{n}2^{m+1}t)} \omega(y, t)$ for all $(x, t) \in \mathbb{R}_n^{n+1}$. Then $N\omega_m \lesssim 2^{-mn\tau}N_{\sqrt{n}2^{m+2}}\omega$ and $[\omega_m(x, 2^{-j})]^{-1}\omega(y, 2^{-j}) \lesssim 2^{mn\tau}$ for $m \in \mathbb{Z}_+$, $x \in Q$ with $Q \in \mathcal{D}_j(\mathbb{R}^n)$, $y \in P$ with $P \in A_{m,i}(Q)$ and $i \leq j$. Moreover, using Corollary (4.2.18), we see that a constant multiple of ω_m also satisfies (27). Similarly to the proof of Lemma (4.2.6), we have that for all $x \in Q$,

$$\sum_{P \in A_{m,i}(Q)} \frac{|t_P|[\omega(x, 2^{-j})]^{-1}}{(1 + 2^i|x_Q - x_P|)^{n+\varepsilon}} \lesssim 2^{-m\varepsilon + mn\tau} HL\left(\sum_{P \in A_{m,i}(Q)} |t_P|[\chi_P \omega(\cdot, 2^{-i})]^{-1}\right)(x).$$

Hence, choosing $\varepsilon > n\tau$, by Fefferman-Stein's vector-valued inequality, we obtain

$\|A_0 t\|_{f\dot{H}_{p,q}^{0,\tau}(\mathbb{R}^n)}^{\kappa}$

$$\lesssim \sum_{m=0}^{\infty} \left\{ \inf_{\omega} \left\| \left\{ \sum_{j\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_j(\mathbb{R}^n)} |Q|^{-\frac{q}{2}}\chi_Q \right. \right. \right.$$

$$\left. \times \left[\sum_{i=-\infty}^{j} \sum_{P\in A_{m,i}(Q)} 2^{(i-j)\frac{n+\varepsilon}{2}} \frac{|t_P|[\omega(\cdot,2^{-j})]^{-1}}{(1+2^i|x_Q-x_P|)^{n+\varepsilon}} \right]^q \right\}^{1/q} \left. \left. \right\|_{L^p(\mathbb{R}^n)} \right\}^{\kappa}$$

$$\lesssim \sum_{m=0}^{\infty} \left\| \left\{ \sum_{j\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_j(\mathbb{R}^n)} |Q|^{-\frac{q}{2}}\chi_Q \left[\sum_{i=-\infty}^{j} \sum_{P\in A_{m,i}(Q)} 2^{(i-j)\frac{n+\varepsilon}{2}} \frac{|t_P|[\omega_m(\cdot,2^{-j})]^{-1}}{(1+2^i|x_Q-x_P|)^{n+\varepsilon}} \right]^q \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)}^{\kappa}$$

$$\lesssim \sum_{m=0}^{\infty} 2^{m(n\tau-\varepsilon)\kappa} \left\| \left\{ \sum_{j\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_j(\mathbb{R}^n)} \chi_Q \left[\sum_{i=-\infty}^{j} 2^{(i-j)\varepsilon/2} \right. \right. \right.$$

$$\left. \left. \times \mathrm{HL}\left(\sum_{P\in A_{m,i}(Q)} |t_P|[\tilde{\chi}_P\omega(\cdot,2^{-i})]^{-1} \right) \right]^q \right\}^{1/q} \left\|_{L^p(\mathbb{R}^n)}^{\kappa} \lesssim \|t\|_{f\dot{H}_{p,q}^{0,\tau}(\mathbb{R}^n)}^{\kappa}. \right.$$

The proof for $A_1 t$ is similar. Indeed, we have

$$\left|(A_0 t)_Q\right| \lesssim \sum_{\{P\in\mathcal{D}(\mathbb{R}^n):\ell(Q)\leq\ell(P)\}} \left(\frac{\ell(Q)}{\ell(P)}\right)^{\frac{n+\varepsilon}{2}} \frac{|t_P|}{(1+[\ell(P)]^{-1}|x_Q-x_P|)^{n+\varepsilon}}.$$

Thus,

$\|A_1 t\|_{f\dot{H}_{p,q}^{0,\tau}(\mathbb{R}^n)}$

$$\lesssim \inf_{\omega} \left\| \left\{ \sum_{j\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_j(\mathbb{R}^n)} |Q|^{-\frac{q}{2}}\chi_Q \left[\sum_{l=0}^{\infty} \sum_{P\in\mathcal{D}_{j+l}(\mathbb{R}^n)} 2^{-l\frac{n+\varepsilon}{2}} \frac{|t_P|[\omega(\cdot,2^{-j})]^{-1}}{(1+2^j|x_Q-x_P|)^{n+\varepsilon}} \right]^q \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)}.$$

Let $\tilde{A}_{0,j,l}(Q) \equiv \{P \in \mathcal{D}_{j+l}(\mathbb{R}^n) : 2^j|x_P - x_Q| \leq \sqrt{n}/2\}$ and $\tilde{A}_{m,j,l}(Q) \equiv \{P \in \mathcal{D}_{j+l}(\mathbb{R}^n) : 2^{m-1}\sqrt{n}/2 < 2^j|x_P - x_Q| \leq 2^m\sqrt{n}/2\}$ for all $j \in \mathbb{Z}$ and $l \in \mathbb{Z}_+$. Set

$$\tilde{\omega}_m(x,s) \equiv 2^{-(m+l)n\tau} \sup\{\omega(y,s) : y \in \mathbb{R}^n, |y-x| < \sqrt{n}2^{m+l+1}s\}$$

for all $m \in \mathbb{Z}_+$ and $(x,s) \in \mathbb{R}_+^{n+1}$. Similarly, we have that a constant multiple of $\tilde{\omega}_m$ satisfies (27) and $[\tilde{\omega}_m(x,2^{-j})]^{-1} \times \omega(y,2^{-j-l}) \lesssim 2^{(m+l)n\tau}$ for $m, l \in \mathbb{Z}_+$, $x \in Q$ with $Q \in \mathcal{D}_j(\mathbb{R}^n)$, $y \in P$ with $P \in \tilde{A}_{m,j,l}(Q)$. Similarly to the proof of Lemma (4.2.6) again, we see that for all $x \in Q$,

$$\sum_{P\in\tilde{A}_{m,j,l}(Q)} \frac{|t_P|[\tilde{\omega}_m(x,2^{-j})]^{-1}}{(1+2^i|x_Q-x_P|)^{n+\varepsilon}} \lesssim 2^{-m\varepsilon+ln+(m+l)n\tau}\mathrm{HL}\left(\sum_{P\in\tilde{A}_{m,j,l}(Q)} \frac{|t_P|\chi_P}{\omega(\cdot,2^{-i})} \right)(x).$$

Hence, choosing $\varepsilon > 2n\tau$, similarly to the estimate of $\|A_0 t\|_{f\dot{H}_{p,q}^{0,\tau}(\mathbb{R}^n)}$, we also have

$$\|A_0 t\|^\kappa_{f\dot{H}^{0,\tau}_{p,q}(\mathbb{R}^n)}$$

$$\lesssim \sum_{m=0}^{\infty} \left\| \left\{ \sum_{j\in\mathbb{Z}} \sum_{\mathcal{Q}\in\mathcal{D}_j(\mathbb{R}^n)} |\mathcal{Q}|^{-\frac{q}{2}} \chi_\mathcal{Q} \left[\sum_{l=0}^{\infty} \sum_{P\in\tilde{A}_{m,j,i}(\mathcal{Q})} 2^{-l\frac{n+\varepsilon}{2}} \frac{|t_P|[\omega_m(\cdot,2^{-j})]^{-1}}{(1+2^j|x_\mathcal{Q}-x_P|)^{n+\varepsilon}} \right]^q \right\}^{1/q} \right\|^\kappa_{L^p(\mathbb{R}^n)}$$

$$\lesssim \sum_{m=0}^{\infty} 2^{m(n\tau-\varepsilon)\kappa} \left\| \left\{ \sum_{j\in\mathbb{Z}} \sum_{\mathcal{Q}\in\mathcal{D}_j(\mathbb{R}^n)} \chi_\mathcal{Q} \left[\sum_{l=0}^{\infty} 2^{-l(\varepsilon/2-n\tau)} \right. \right. \right.$$

$$\left. \left. \left. \times \mathrm{HL}\left(\sum_{P\in\tilde{A}_{m,j,i}(\mathcal{Q})} |t_P|[\tilde{\chi}_P \omega(\cdot,2^{-i})]^{-1} \right) \right]^q \right\}^{1/q} \right\|^\kappa_{L^p(\mathbb{R}^n)} \lesssim \|t\|^\kappa_{f\dot{H}^{0,\tau}_{p,q}(\mathbb{R}^n)},$$

which completes the proof of Theorem (4.2.19).

As applications of Theorem (4.2.19), we establish the smooth atomic and molecular decomposition characterizations of $A\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$.

Definition (4.2.20) [204]: Let $p\in(1,\infty)$, $q\in[1,\infty)$, $s\in\mathbb{R}$, $\tau\in[0,\frac{1}{(p\vee q)'})$ and $\mathcal{Q}\in\mathcal{D}(\mathbb{R}^n)$. Set $N\equiv\max(\lfloor-s+2n\tau\rfloor,-1)$ and $s^*\equiv s-\lfloor s\rfloor$.

(i) A function $m_\mathcal{Q}$ is called a smooth synthesis molecule for $A\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$ supported near $\mathcal{Q}$, if there exist a $\delta\in(\max\{s^*,(s+n\tau)^*\},1]$ and $M>n+2n\tau$ such that $\int_{\mathbb{R}^n} x^\gamma m_\mathcal{Q}(x)dx=0$ if $|\gamma|\leq N$, $|m_\mathcal{Q}(x)|\leq|\mathcal{Q}|^{-\frac{1}{2}}(1+[\ell(\mathcal{Q})]^{-1}|x-x_\mathcal{Q}|)^{-\max(M,M-s)}$,

$$\left|\partial^\gamma m_\mathcal{Q}(x)\right|\leq|\mathcal{Q}|^{-\frac{1}{2}-\frac{|\gamma|}{n}}\left(1+[\ell(\mathcal{Q})]^{-1}|x-x_\mathcal{Q}|\right)^{-M} \quad \text{if } |\gamma|\leq\lfloor s+3n\tau\rfloor \quad (35)$$

and

$$\left|\partial^\gamma m_\mathcal{Q}(x)-\partial^\gamma m_\mathcal{Q}(y)\right|\leq|\mathcal{Q}|^{-\frac{1}{2}-\frac{|\gamma|}{n}-\frac{\delta}{n}}|x-y|^\delta \sup_{|z|\leq|x-y|}\left(1+[\ell(\mathcal{Q})]^{-1}|x-z-x_\mathcal{Q}|\right)^{-M} \quad (36)$$

if $|\gamma|=\lfloor s+3n\tau\rfloor$.

A set $\{m_\mathcal{Q}\}_{\mathcal{Q}\in\mathcal{D}(\mathbb{R}^n)}$ of functions is called a family of smooth synthesis molecules for $A\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$, if each $m_\mathcal{Q}$ is a smooth synthesis molecule for $A\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$ supported near $\mathcal{Q}$.

(ii) A function $b_\mathcal{Q}$ is called a smooth analysis molecule for $A\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$ supported near near $\mathcal{Q}$, if there exist a $\rho\in((n-s)^*,1]$ and $M>n+2n\tau$ such that $\int_{\mathbb{R}^n}x^\gamma b_\mathcal{Q}(x)dx=0$ if $|\gamma|\leq\lfloor s+3n\tau\rfloor$, $|b_\mathcal{Q}(x)|\leq|\mathcal{Q}|^{-\frac{1}{2}}(1+[\ell(\mathcal{Q})]^{-1}|x-x_\mathcal{Q}|)^{-\max(M,M+s+n\tau)}$,

$$\left|\partial^\gamma b_\mathcal{Q}(x)\right|\leq|\mathcal{Q}|^{-\frac{1}{2}-\frac{|\gamma|}{n}}\left(1+[\ell(\mathcal{Q})]^{-1}|x-x_\mathcal{Q}|\right)^{-M} \quad \text{if } |\gamma|\leq N \quad (37)$$

and if $|\gamma|=N$

$$\left|\partial^\gamma b_\mathcal{Q}(x)-\partial^\gamma b_\mathcal{Q}(y)\right|\leq|\mathcal{Q}|^{-\frac{1}{2}-\frac{|\gamma|}{n}-\frac{\delta}{n}}|x-y|^\delta \sup_{|z|\leq|x-y|}\left(1+[\ell(\mathcal{Q})]^{-1}|x-z-x_\mathcal{Q}|\right)^{-M} \quad (38)$$

A set $\{b_\mathcal{Q}\}_{\mathcal{Q}\in\mathcal{D}(\mathbb{R}^n)}$ of functions is called a family of smooth analysis molecules for $A\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$, if each $b_\mathcal{Q}$ is a smooth analysis molecule for $A\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$ supported near $\mathcal{Q}$.

Lemma (4.2.21) [204]: Let $p\in(1,\infty)$, $q\in[1,\infty)$, $s\in\mathbb{R}$, $\tau\in[0,\frac{1}{(p\vee q)'})$. Then there exist $\varepsilon_1>2n\tau$ and a positive constant C such that for all families $\{m_\mathcal{Q}\}_{\mathcal{Q}\in\mathcal{D}(\mathbb{R}^n)}$ of smooth synthesis molecules for $A\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$ and families $\{b_\mathcal{Q}\}_{\mathcal{Q}\in\mathcal{D}(\mathbb{R}^n)}$ of smooth analysis molecules for $A\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$, $|\langle m_P,b_\mathcal{Q}\rangle_{L^2(\mathbb{R}^n)}|\leq C\omega_{\mathcal{Q}\,P}(\varepsilon_1)$.

To formulate the molecular decomposition, the following lemma is indispensable.

Lemma (4.2.22) [204]: Retain the same assumptions as in Lemma (4.2.21). Let $f \in$ $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and Φ be a smooth analysis molecule for $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ supported near a dyadic cube Q. Then $\langle f, \Phi \rangle$ is well defined. Indeed, let $\phi, \psi \in \mathcal{S}(\mathbb{R}^n)$ be as in (29). Then the series

$$\langle f, \Phi \rangle \equiv \sum_{j \in \mathbb{Z}} \langle \widetilde{\varphi}_j * \psi_j * f, \Phi \rangle = \sum_{P \in \mathcal{D}(\mathbb{R}^n)} \langle f, \varphi_P \rangle \langle \psi_P, \Phi \rangle \tag{39}$$

converges absolutely and its value is independent of the choices of ϕ and ψ.

Proof: The same proof as that of [190, Lemma 4.2] works for the absolute convergence of (39). We only need to prove that the value of (39) is independent of the choices of ϕ and ψ. By similarity again, we only consider the spaces $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$.

Let $f \in B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$. We claim that $\sum_{j=0}^{\infty} \widetilde{\varphi}_j * \psi_j * f, \Phi$ converges in $\mathcal{S}'(\mathbb{R}^n)$. In fact, similarly to the proof of [189, Lemma 4.2.6], we have that for all $\psi \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\left| \varphi_j * \phi(x) \right| \lesssim \|\varphi\|_{\mathcal{S}_{M+1}} \|\phi\|_{\mathcal{S}_{M+1}} \frac{2^{-jM}}{(1+|x|)^{n+M}},$$

where $M \in \mathbb{N}$ is determined later. Thus,

$$\sum_{j=0}^{\infty} \left| \langle \widetilde{\varphi}_j * \psi_j * f, \Phi \rangle \right| \lesssim \|\varphi\|_{\mathcal{S}_{M+1}} \|\phi\|_{\mathcal{S}_{M+1}} \sum_{j=0}^{\infty} 2^{-jM} \int_{\mathbb{R}^n} \frac{|\psi_j * f(x)|}{(1+|x|)^{n+M}} \, dx.$$

Recall again that $\omega(x,t) \lesssim t^{-n\tau}$ for all nonnegative Borel measurable functions ω on $\mathbb{R}_+^{n+1}$ satisfying (27). Letting $M > \max(0, n\tau - s)$, by Hölder's inequality, we then obtain

$$\sum_{j=0}^{\infty} \left| \langle \widetilde{\varphi}_j * \psi_j * f, \Phi \rangle \right| \lesssim \|\varphi\|_{\mathcal{S}_{M+1}} \|\phi\|_{\mathcal{S}_{M+1}} \sum_{j=0}^{\infty} 2^{-jM+jn\tau} \int_{\mathbb{R}^n} \frac{|\psi_j * f(x)|[\omega(x, 2^{-j})]^{-1}}{(1+|x|)^{n+M}} \, dx$$

$$\lesssim \|\varphi\|_{\mathcal{S}_{M+1}} \|\phi\|_{\mathcal{S}_{M+1}} \|f\|_{B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)},$$

which implies that $\sum_{j=0}^{\infty} \widetilde{\varphi}_j * \psi_j * f$ at converges in $\mathcal{S}'(\mathbb{R}^n)$. Thus, the claim is true.

We need to handle carefully the remaining summation: $\sum_{j=-\infty}^{-1} \widetilde{\varphi}_j * \psi_j * f$. In general it is not possible to prove that $\sum_{j=-\infty}^{-1} \widetilde{\varphi}_j * \psi_j * f$ is convergent in $\mathcal{S}'(\mathbb{R}^n)$. Therefore, we pass to its partial derivatives. Choose $\gamma \in \mathbb{Z}_+^n$ such that $|\gamma| > s - n\tau - n/p$. Then using Hölder's inequality, similarly to the previous estimate, we obtain that for all $x \in \mathbb{R}^n$,

$$\sum_{j=-\infty}^{-1} \left| \partial^\gamma (\widetilde{\varphi}_j * \psi_j * f)(x) \right| \lesssim \sum_{j=-\infty}^{-1} 2^{j(+|\gamma|)} \|\varphi\|_{\mathcal{S}_{M+1}} \int_{\mathbb{R}^n} \frac{|\psi_j * f(x)|}{(1+2^j|x-y|)^{n+M+|\gamma|}} \, dy$$

$$\lesssim \sum_{j=-\infty}^{-1} 2^{j\left(|\gamma|-s+n\tau+\frac{n}{p}\right)} \|\varphi\|_{\mathcal{S}_{M+1}} \|f\|_{B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \lesssim \|\varphi\|_{\mathcal{S}_{M+1}} \|f\|_{B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}.$$

Therefore, it follows from the well-known result in [93] that there exist a sequence $\{P_N\}_{N \in \mathbb{N}}$ of polynomials on $\mathbb{R}^n$ with degree no more than $\max(-1, \lfloor s - n\tau - n/p \rfloor)$ and $g \in \mathcal{S}'(\mathbb{R}^n)$ such that $g = \lim_{N \to \infty} \left(\sum_{j=-N}^{\infty} |\widetilde{\varphi}_j * \psi_j * f + P_N| \right)$ in $\mathcal{S}'(\mathbb{R}^n)$ and g is a representative of the equivalence class $f + \mathcal{P}(\mathbb{R}^n)$; see [106]. Using [93, Lemma 5.4] and repeating the argument in [106], we obtain that the value of (39) is independent of the choices of ϕ and ψ, which completes the proof of Lemma (4.2.22).

Theorem (4.2.23) [204]: Let s, p, q and τ be as in Lemma (4.2.21).

(i) If $\left\{ m_Q \right\}_{Q \in \mathcal{D}(\mathbb{R}^n)}$ is a family of smooth synthesis molecules for $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, then there exists a positive constant C such that for all $t = \left\{ t_Q \right\}_{Q \in \mathcal{D}(\mathbb{R}^n)} \in a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$,

$$\left\|\sum_{Q\in\mathcal{D}(\mathbb{R}^n)} t_Q m_Q\right\|_{A\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)} \leq C\|t\|_{a\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)}.$$

(ii) If $\{b_Q\}_{Q\in\mathcal{D}(\mathbb{R}^n)}$ is a family of smooth analysis molecules for $A\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$, then there exists a positive constant C such that for all $f \in a\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$,

$$\left\|\{\langle f, b_Q\rangle\}_{Q\in\mathcal{D}(\mathbb{R}^n)}\right\|_{a\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)} \leq C\|f\|_{A\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)}.$$

Theorem (4.2.23) generalizes the well-known results on $\dot{B}^s_{p,q}(\mathbb{R}^n)$ and $\dot{F}^s_{p,q}(\mathbb{R}^n)$ in [45, 106, 164, 165, 175, 176] by taking $\tau = 0$. We establish the smooth atomic decomposition characterizations of $A\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$.

Definition (4.2.24) [204]: Let $s \in \mathbb{R}, p \in (1, \infty), q \in [1, \infty), \tau$ and N be as in Definition (4.2.20). A function a_Q is called a smooth atom for $A\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$ supported near a dyadic cube Q, if there exist $\widetilde{K}$ and $\widetilde{N}$ with $\widetilde{K} \geq \max(\lfloor s + 3n\tau + 1\rfloor, 0)$ and $\widetilde{N} \geq N$ such that a_Q satisfies the following support, regularity and moment conditions: supp $a_Q \subset 3Q, \|\partial^\gamma a_Q\|_{L^\infty(\mathbb{R}^n)} \leq |Q|^{-\frac{1}{2}-\frac{|\gamma|}{n}}$ if $|\gamma| \leq \widetilde{K}$, and $\int_{\mathbb{R}^n} x^\gamma a_Q(x)dx = 0$ if $|\gamma| \leq \widetilde{N}$.

A set $\{a_Q\}_{Q\in\mathcal{D}(\mathbb{R}^n)}$ of functions is called a family of smooth atoms for $A\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$, if each a_Q is a smooth atom for $A\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$ supported near Q.

Theorem (4.2.25) [204]: Let s, p, q, τ be as in Lemma (4.2.21). Then for each $f \in A\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$, there exist a family $\{a_Q\}_{Q\in\mathcal{D}(\mathbb{R}^n)}$ of smooth atoms for $A\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$, a coefficient sequence $t \equiv \{t_Q\}_{Q\in\mathcal{D}(\mathbb{R}^n)} \in a\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$, and a positive constant C such that $f = \sum_{Q\in\mathcal{D}(\mathbb{R}^n)} t_Q a_Q$ in $\mathcal{S}'_\infty(\mathbb{R}^n)$ and $\|t\|_{a\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)} \leq C\|f\|_{A\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)}$.

Conversely, there exists a positive constant C such that for all families $\{a_Q\}_{Q\in\mathcal{D}(\mathbb{R}^n)}$ of smooth atoms for $A\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$ and coefficient sequences $t \equiv \{t_Q\}_{Q\in\mathcal{D}(\mathbb{R}^n)} \in a\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$, $\left\|\sum_{Q\in\mathcal{D}(\mathbb{R}^n)} t_Q a_Q\right\|_{A\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)} \leq C\|t\|_{a\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)}$.

We give some applications of the smooth atomic and molecular decomposition characterizations of $A\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$, including the boundedness of pseudo-differential operators with homogeneous symbols in these spaces and their trace properties. We first recall the notion of homogeneous symbols; see, [197].

Definition (4.2.26) [204]: Let $m \in \mathbb{Z}$. A smooth function a defined on $\mathbb{R}^n_x \times \left(\mathbb{R}^n_\xi\backslash\{0\}\right)$ belongs to the class $\dot{S}^m_{1,1}(\mathbb{R}^n)$, if a satisfies the following differential inequalities that for all $\alpha, \beta \in \mathbb{Z}^n_+$,

$$\sup_{x\in\mathbb{R}^n, \xi\in(\mathbb{R}^n\backslash\{0\})} |\xi|^{-m-|\alpha|+|\beta|}\left|\partial^\alpha_x \partial^\beta_\xi a(x,\xi)\right| < \infty.$$

As an application of the smooth molecular decomposition of $A\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$ (Theorem (4.2.23)) and the Calderón reproducing formula (29), we have the following conclusion.

Theorem (4.2.27) [204]: Let $m \in \mathbb{Z}, s \in \mathbb{R}, p \in (1, \infty), q \in [1, \infty)$ and $\tau \in \left[0, \frac{1}{(p\vee q)'}\right]$. Let a be a symbol in $\dot{S}^m_{1,1}(\mathbb{R}^n)$ and $a(x, D)$ be the pseudodifferential operator such that

$$a(x, D)f(x) \equiv \int_{\mathbb{R}^n} a(x,\xi)(\mathcal{F}f)(\xi)e^{-ix\xi}d\xi$$

for all smooth synthesis molecules for $A\dot{H}^{s+m,\tau}_{p,q}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. Assume that its formal adjoint $a(x, D)^*$ satisfies $a(x, D)^*(x^\beta) = 0$ in $\mathcal{S}'_\infty(\mathbb{R}^n)$ for all $\beta \in \mathbb{Z}^n_+$ with $|\beta| \leq$

max$\{-s + 2n\tau, -1\}$. Then $a(x, D)$ is a bounded linear operator from $A\dot{H}_{p,q}^{s+m,\tau}(\mathbb{R}^n)$ to $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$.

Proof: The proof is similar to that in [194, 195]; see also [200]. We abbreviate $T \equiv a(x, D)$ for simplicity. Let $f \in A\dot{H}_{p,q}^{s+m,\tau}(\mathbb{R}^n)$ and ϕ be as in Definition 4.2.1 such that for all $\xi \in \mathbb{R}^n$, $\sum_{j \in \mathbb{Z}}|\mathcal{F}\varphi(2^{-j}\xi)|^2 = \chi_{\mathbb{R}\setminus\{0\}}(\xi)$. Then by the Calderón reproducing formula (29), we have $f \equiv \sum_{Q \in \mathcal{D}(\mathbb{R}^n)}\langle f, \varphi_Q\rangle\varphi_Q$ in $\mathcal{S}'_\infty(\mathbb{R}^n)$; moreover, by the φ -transform characterization of $A\dot{H}_{p,q}^{s+m,\tau}(\mathbb{R}^n)$,we see that $\left\|\{\langle f, \varphi_Q\rangle\}_{Q \in \mathcal{D}(\mathbb{R}^n)}\right\|_{a\dot{H}_{p,q}^{s+m,\tau}(\mathbb{R}^n)} \lesssim$ $\|f\|_{A\dot{H}_{p,q}^{s+m,\tau}(\mathbb{R}^n)}$, or equivalently, $\left\|\left\{|Q|^{-\frac{m}{n}}\langle f, \varphi_Q\rangle\right\}_{Q \in \mathcal{D}(\mathbb{R}^n)}\right\|_{a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \lesssim \|f\|_{A\dot{H}_{p,q}^{s+m,\tau}(\mathbb{R}^n)}$.

We claim that $T(f) \equiv \sum_{Q \in \mathcal{D}(\mathbb{R}^n)}\langle f, \varphi_Q\rangle T(\varphi_Q)$ in $\mathcal{S}'_\infty(\mathbb{R}^n)$ with $\|T(f)\|_{A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \lesssim$ $\|f\|_{A\dot{H}_{p,q}^{s+m,\tau}(\mathbb{R}^n)}$. To this end it suffices to show that every $|Q|^{\frac{m}{n}}T(\varphi_Q)$ is a constant multiple of a synthesis molecule for $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ supported near Q. This fact was established by Grafakos and Torres [197]. We then conclude that T is bounded from $A\dot{H}_{p,q}^{s+m,\tau}(\mathbb{R}^n)$ to $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, which completes the proof of Theorem (4.2.27).

Lemma (4.2.28) [204]: Let $d \in (0, n]$ and Ω be an open set in $\mathbb{R}^n$. Define

$$H_*^d(\Omega) \equiv \inf\left\{\sum_{j=1}^\infty r_j^d : \Omega \subset \bigcup_{j=1}^\infty B(x_r, r_j), r_j > \frac{\text{dist}(x_j, \partial\Omega)}{10000}\right\}.$$

Then $H^d(\Omega)$ and $H_*^d(\Omega)$ are equivalent for all Ω.

Proof: The inequality $H^d(\Omega) \leq H_*^d(\Omega)$ is trivial from the definitions. To prove the converse, we choose a ball covering $\{B(x_j, r_j)\}_{j=1}^\infty$ of Ω such that $\sum_{j=1}^\infty r_j^d :\leq 2H^d(\Omega)$. Let $\{B(X_j, R_j)\}_{j=1}^\infty$ be a Whitney covering of Ω satisfying $\Omega = \cup_{j=1}^\infty B(X_j, R_j)$, $R_j/1000 \leq$ dist $(X_j, \partial(\Omega)) \leq R_j/100$ and $\sum_{j \in \mathbb{N}}\chi_{R_j} \leq C_n$; see, [196]. Set

$$J_1 \equiv \left\{j \in \mathbb{N}: \left(B(X_j, R_j) \cap B(x_k, r_k)\right) \neq \emptyset \text{ and } R_j \leq 4r_k \text{ for some } k \in \mathbb{N}\right\}$$

and $J_2 \equiv (\mathbb{N}\setminus J_1)$. Notice that if $k \in \mathbb{N}$ satisfies $\left(B(X_j, R_j) \cap B(x_k, r_k)\right) \neq \emptyset$ for some $j \in J_2$, then $B(x_k, r_k) \subset B(X_j, 2R_j)$ since $r_k < R_j/4$. With this in mind, we define

$$K_1 \equiv \left\{k \in \mathbb{N} : \left(B(x_k, r_k) \cap B(X_j, R_j)\right) \neq \emptyset \text{ for some } j \in J_2\right\},$$

and $K_1 \equiv (\mathbb{N}\setminus K_2)$. It is easy to see that

$$\bigcup_{k=1}^\infty B(x_k, r_k) \subset \left(\bigcup_{k \in K_1} B(x_k, r_k) \cup \bigcup_{j \in J_2} B(X_j, 2R_j)\right). \tag{40}$$

Furthermore, for each $k \in \mathbb{N}$, the cardinality of the set $\left\{j \in J_2 : \left(B(x_k, r_k) \cap B(X_j, R_j)\right) \neq \emptyset\right\}$ is bounded by a constant depending only on the dimension. Hence, we have

$$\sum_{k=1}^{\infty} r_k^d = \sum_{k\in K_1} r_k^d + \sum_{k\in K_2} r_k^d \sim \sum_{k\in K_1} r_k^d + \sum_{j\in J_2}\left(\sum_{k\in K_2,\,\left(B(x_k,r_k)\cap B(X_j,R_j)\right)\neq\emptyset} r_k^d\right) \sim \sum_{k\in K_1} r_k^d$$

$$+ \sum_{j\in J_2}\left(\sum_{k\in K_2,\,\left(B(x_k,r_k)\cap B(X_j,R_j)\right)\neq\emptyset} |B(x_k,r_k)|^{\frac{d}{n}}\right).$$

Notice that $B(X_j, R_j) \subset \Omega \subset (\cup_{k=1}^{\infty} B(x_k, r_k))$. Then for each $j \in J_2$, we have

$$B(X_j, R_j) \subset \left\{\bigcup_{k\in K_2,\,\left(B(x_k,r_k)\cap B(X_j,R_j)\right)\neq\emptyset} B(x_k, r_k)\right\}.$$

Since $d \in (0, n]$, by the monotonicity of $l^{\frac{d}{n}}$, we see that

$$\left(\sum_{k\in K_2,\,\left(B(x_k,r_k)\cap B(X_j,R_j)\right)\neq\emptyset} |B(x_k, r_k)|^{\frac{d}{n}}\right) \geq \left(\sum_{k\in K_2,\,\left(B(x_k,r_k)\cap B(X_j,R_j)\right)\neq\emptyset} |B(x_k, r_k)|\right)^{\frac{d}{n}}$$

$$\geq |B(X_j, R_j)|^{\frac{d}{n}}.$$

As a consequence, $\sum_{k=0}^{\infty} r_k^d \gtrsim \sum_{k\in K_1} r_k^d + \sum_{j\in J_2} R_j^d$, which combined with (40) yields that $H_*^d(\Omega) \leq \sum_{k\in K_1} r_k^d + \sum_{j\in J_2}(2R_j)^d \lesssim \sum_{k\in K_1} r_k^d + \sum_{j\in J_2} R_j^d \lesssim \sum_{k=0}^{\infty} r_k^d \lesssim H^d(\Omega)$. This finishes the proof of Lemma (4.2.28).

As an application of smooth atomic decomposition of $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, we are now going to show the trace theorem. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we set $x' \equiv (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$.
Theorem (4.2.29) [204]: Let $n \geq 2$, $p \in (1, \infty), q \in [1, \infty)$, $\tau \in \left[0, \frac{n-1}{n(p\vee q)'}\right]$ and $s \in \left(\frac{1}{p} + 2n\tau, \infty\right)$. Then there exists a surjective and continuous operator

$$\text{Tr} : f \in A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n) \mapsto \text{Tr}(f) \in A\dot{H}_{p,q}^{s-\frac{1}{p}\frac{n}{n-1}\tau}(\mathbb{R}^{n-1})$$

such that $\text{Tr}(f)(x') = f(x', 0)$ for all $x' \in \mathbb{R}^{n-1}$ and smooth atoms f for $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$.
Proof: For similarity, we concentrate on the space $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$. By Theorem (4.2.25), any $f \in B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ admits a smooth atomic decomposition $f = \sum_{Q\in\mathcal{D}(\mathbb{R}^n)} t_Q a_Q$ in $\mathcal{S}_\infty'(\mathbb{R}^n)$, where each a_Q is a smooth atom for $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $t \equiv \{t_Q\}_{Q\in\mathcal{D}(\mathbb{R}^n)} \subset \mathbb{C}$ satisfies $\|t\|_{b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \lesssim \|f\|_{B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}$. Since $s > 1/p + 2n\tau$, there is no need to postulate any momentcondition on a_Q. Define

$$\text{Tr}(f)(*') \equiv \sum_{Q\in\mathcal{D}(\mathbb{R}^n)} t_Q a_Q(*',0) = \sum_{Q\in\mathcal{D}(\mathbb{R}^n)} \frac{t_Q}{[\ell(Q)]^{\frac{1}{2}}} [\ell(Q)]^{\frac{1}{2}} a_Q(*',0).$$

By the support condition of smooth atoms, the above summation can be re-written as

$$\text{Tr}(f)(*') \equiv \sum_{i=0}^{2} \sum_{Q'\in\mathcal{D}(\mathbb{R}^{n-1})} \frac{t_{Q'\times[(i-1)\ell(Q'),\,i\ell(Q'))}}{[\ell(Q')]^{\frac{1}{2}}} [\ell(Q')]^{\frac{1}{2}} a_{Q'\times[(i-1)\ell(Q'),\,i\ell(Q'))}(*',0). \quad (41)$$

We need to show that (41) converges in $\mathcal{S}_\infty'(\mathbb{R}^{n-1})$ and

$$\|\text{Tr}(f)\|_{B\dot{H}_{p,q}^{s-\frac{1}{p}\frac{n}{n-1}\tau}(\mathbb{R}^{n-1})} \lesssim \|f\|_{B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}.$$

To this end, by Theorem (4.2.25), it suffices to prove that each $[\ell(Q')]^{\frac{1}{2}}a_{Q'\times[(i-1)\ell(Q'),\,i\ell(Q')]}(*',0)$ is a smooth atom for $B\dot{H}_{p,q}^{s-\frac{1}{p'}\frac{n}{n-1}\tau}(\mathbb{R}^{n-1})$ supported near Q' and for all $i \in \{0,1,2\}$,

$$\left\|\left\{[\ell(Q')]^{\frac{1}{2}}t_{Q'\times[(i-1)\ell(Q'),\,i\ell(Q')]}\right\}_{Q'\in\mathcal{D}(\mathbb{R}^{n-1})}\right\|_{b\dot{H}_{p,q}^{s-\frac{1}{p'}\frac{n}{n-1}\tau}(\mathbb{R}^{n-1})} < \infty. \tag{42}$$

Indeed, it was already proved in [200] that $[\ell(Q')]^{\frac{1}{2}}a_{Q'\times[(i-1)\ell(Q'),\,i\ell(Q')]}(*',0)$ is a smooth atom for $B\dot{H}_{p,q}^{s-\frac{1}{p'}\frac{n}{n-1}\tau}(\mathbb{R}^{n-1})$. By similarity, we only prove (42) when $=1$. Let ω be a nonnegative function on $\mathbb{R}_+^{n+1}$ satisfying (27) and

$$\left\{\sum_{j\in\mathbb{Z}}\left[\sum_{Q\in\mathcal{D}(\mathbb{R}^n)}|Q|^{-(\frac{s}{n}+\frac{1}{2})p}|t_Q|^p\int_Q[\omega(x,2^{-j})]^{-p}dx\right]^{\frac{q}{p}}\right\}^{\frac{1}{q}} \lesssim \|t\|_{b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}.$$

For all $\lambda \in (0,\infty)$, set $E_\lambda \equiv \{x \in \mathbb{R}^n : [N\omega(x)]^{(p\vee q)'} > \lambda\}$. Then there exists a ball covering $\{B_m\}_m$ of E_λ such that

$$H^{n\tau(p\vee q)'}(E_\lambda) \sim \sum_m r_{B_m}^{n\tau(p\vee q)'}, \tag{43}$$

where r_{B_m} denotes the radius of B_m. Let $\widetilde{H}^{n\tau(p\vee q)'}$ be the $(n-1)\frac{n\tau}{n-1}(p\vee q)'$-Hausdorff capacity in $\mathbb{R}^{n-1}$ and define $\widetilde{\omega}$ on $\mathbb{R}_+^n$ by setting, for all $x' \in \mathbb{R}_+^{n+1}$ and $t \in (0,\infty)$, $\widetilde{\omega}(x',t) \equiv \widetilde{C}\sup_{\{x_n\in\mathbb{R}:|x_n|<t\}}\omega((x',x_n),t)$, where $\widetilde{C}$ is a positive constant chosen so that $N\widetilde{\omega}(x') \le N\omega(x',0)$ for all $x' \in \mathbb{R}^{n-1}$. Therefore, if $[N\widetilde{\omega}(x')]^{(p\vee q)'} > \lambda$, then $[N\omega(x',0)]^{(p\vee q)'} > \lambda$, and hence $(x',0) \in B_m$ for some m, which further implies that $\widetilde{E}_\lambda \equiv \{x' \in \mathbb{R}^{n-1} : [N\widetilde{\omega}(x')]^{(p\vee q)'} > \lambda\} \subset (\cup_m B_m^*)$, where B_m^* is the projection of B_m from $\mathbb{R}^n$ to $\mathbb{R}^{n-1}$. This combined with (43) further yields that

$$\int_{\mathbb{R}^{n-1}}[N\widetilde{\omega}(x')]^{(p\vee q)'}d\widetilde{H}^{n\tau(p\vee q)'}(x') = \int_0^\infty \widetilde{H}^{n\tau(p\vee q)'}(\widetilde{E}_\lambda)d\lambda \lesssim \int_0^\infty H^{n\tau(p\vee q)'}(E_\lambda)d\lambda \lesssim 1.$$

Furthermore,

$$\left\|\left\{[\ell(Q')]^{\frac{1}{2}}t_{Q'\times[0,\,\ell(Q')]}\right\}_{Q'\in\mathcal{D}(\mathbb{R}^{n-1})}\right\|_{b\dot{H}_{p,q}^{s-\frac{1}{p'}\frac{n}{n-1}\tau}(\mathbb{R}^{n-1})}$$

$$\lesssim \left\{\sum_{j\in\mathbb{Z}}\left[\sum_{Q'\in\mathcal{D}_j(\mathbb{R}^{n-1})}|Q|^{-sp-\frac{np}{2}+1}\left|t_{Q'\times[0,\,\ell(Q')]}\right|^p\int_{Q'}[\widetilde{\omega}(x',2^{-j})]^{-p}dx'\right]^{q/p}\right\}^{1/q}$$

$$\lesssim \left\{\sum_{j\in\mathbb{Z}}\left[\sum_{Q'\in\mathcal{D}_j(\mathbb{R}^{n-1})}|Q|^{-sp-\frac{np}{2}}\left|t_{Q'\times[0,\,\ell(Q')]}\right|^p\int_{Q}[\omega(x,2^{-j})]^{-p}dx\right]^{q/p}\right\}^{1/q}$$

$$\lesssim \|t\|_{b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)},$$

which implies that Tr is well defined and bounded from $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ to $B\dot{H}_{p,q}^{s-\frac{1}{p'}\frac{n}{n-1}\tau}(\mathbb{R}^{n-1})$.

Let us show that Tr is surjective. To this end, for any $f \in B\dot{H}_{p,q}^{s-\frac{1}{p'}\frac{n}{n-1}\tau}(\mathbb{R}^{n-1})$, by Theorem (4.2.25), there exist smooth atoms $\{a_{Q'}\}_{Q'\in\mathcal{D}(\mathbb{R}^{n-1})}$ for $B\dot{H}_{p,q}^{s-\frac{1}{p'}\frac{n}{n-1}\tau}(\mathbb{R}^{n-1})$ and coefficients $t \equiv \{t_{Q'}\}_{Q'\in\mathcal{D}(\mathbb{R}^{n-1})}$ such that $f = \sum_{Q'\in\mathcal{D}(\mathbb{R}^{n-1})} t_{Q'} a_{Q'}$ in $\mathcal{S}'_\infty(\mathbb{R}^{n-1})$ and $\|t\|_{b\dot{H}_{p,q}^{s-\frac{1}{p'}\frac{n}{n-1}\tau}(\mathbb{R}^{n-1})} \lesssim \|f\|_{B\dot{H}_{p,q}^{s-\frac{1}{p'}\frac{n}{n-1}\tau}(\mathbb{R}^{n-1})}$. Let $\varphi \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}\varphi \subset \left(-\frac{1}{2},\frac{1}{2}\right)$ and $\varphi(0) = 1$. For all $Q' \in \mathcal{D}(\mathbb{R}^{n-1})$ and $x \in \mathbb{R}$, set $\varphi_{Q'}(x) \equiv \varphi(2^{-\log_2 \ell(Q')}x)$. Under this notation, we define $F \equiv \sum_{Q'\in\mathcal{D}(\mathbb{R}^{n-1})} t_{Q'} a_{Q'} \otimes \varphi_{Q'}$. It is easy to check that for all $Q' \in \mathcal{D}(\mathbb{R}^{n-1})$, $[\ell(Q')]^{-\frac{1}{2}}a_{Q'} \otimes \varphi_{Q'}$ is a smooth atom for $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ supported near $Q' \times [0, \ell(Q'))$. Hence, to show $F \in B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, by Theorem (4.2.25), it suffices to prove that

$$\left\|\left\{[\ell(Q')]^{\frac{1}{2}}t_{Q'}\right\}_{Q'\in\mathcal{D}(\mathbb{R}^{n-1})}\right\|_{b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \lesssim \|f\|_{B\dot{H}_{p,q}^{s-\frac{1}{p'}\frac{n}{n-1}\tau}(\mathbb{R}^{n-1})}.$$

Let $\tilde\omega$ satisfy $\int_{\mathbb{R}^{n-1}}[N\tilde\omega(x')]^{(p\vee q)'}d\tilde{H}^{n\tau(p\vee q)'}(x') \leq 1$ and

$$\left\{\sum_{j\in\mathbb{Z}}\left[\sum_{Q'\in\mathcal{D}_j(\mathbb{R}^{n-1})}|Q'|^{-\left(\frac{s-1/p}{n-1}+\frac{1}{2}\right)}|t_{Q'}|^p\int_{Q'}[\tilde\omega(x',2^{-j})]^{-p}dx'\right]^{q/p}\right\}^{1/q} \lesssim \|t\|_{b\dot{H}_{p,q}^{s-\frac{1}{p'}\frac{n}{n-1}\tau}(\mathbb{R}^{n-1})}.$$

By Lemma (4.2.28), for each $\lambda \in (0,\infty)$ there exists a ball covering $\{B_m^*\}_m \equiv \{B(x_{B_m^*}, r_{B_m^*})\}_m$ of $\tilde{E}_\lambda \equiv \{x' \in \mathbb{R}^{n-1} : [N\tilde\omega(x')]^{(p\vee q)'} > \lambda\}$ such that $\sum_m r_{B_m^*}^{n\tau(p\vee q)'} \sim \tilde{H}_*^{n\tau(p\vee q)'}(\tilde{E}_\lambda) \sim \tilde{H}^{n\tau(p\vee q)'}(\tilde{E}_\lambda)$ and that $r_{B_m^*} > \operatorname{dist}(x_{B_m^*},\partial\tilde{E}_\lambda)/10000$ for all m. For all $x = (x',x_n) \in \mathbb{R}^n$ and $t \in (0,\infty)$, define $\omega(x,t) \equiv \tilde\omega(x',t)\chi_{[0,t)}(x_n)$. Notice that if $N\omega(x',x_n) > \lambda^{\frac{1}{(p\vee q)'}}$, then $\tilde\omega(y',t) = \omega((y',y_n),t) > \lambda^{\frac{1}{(p\vee q)'}}$ for some $|(y',y_n) - (x',x_n)| < t$ and $y_n \in [0,t)$. Then $N\tilde\omega(y') > \lambda^{\frac{1}{(p\vee q)'}}$ and thus, $y' \in B_m^*$ for some m. Since for all $z' \in B(y',t)$, $N\tilde\omega(z') \geq \tilde\omega(y',t) > \lambda^{\frac{1}{(p\vee q)'}}$, we see that $B(y',t) \subset \tilde{E}_\lambda \subset (\cup_m B_m^*)$, and hence, $t \leq 10000 r_{B_m^*}$. Notice that $x_n \in [0,t)$. We have $(x',x_n) \in (20000B_m^*) \times [0,20000r_{B_m^*})$ and $E_\lambda \subset \cup_m(20000B_m^*) \times [0,20000r_{B_m^*})$, which further implies that $H^{n\tau(p\vee q)'}(E_\lambda) \lesssim \sum_m r_{B_m^*}^{n\tau(p\vee q)'} \lesssim \tilde{H}^{n\tau(p\vee q)'}(\tilde{E}_\lambda)$ and

$$\int_{\mathbb{R}^n}[N\omega(x',x_n)]^{(p\vee q)'}dH^{n\tau(p\vee q)'}(x) = \int_0^\infty H^{n\tau(p\vee q)'}(E_\lambda)d\lambda \lesssim \int_0^\infty \tilde{H}^{n\tau(p\vee q)'}(\tilde{E}_\lambda)d\lambda$$

$$\lesssim \int_{\mathbb{R}^{n-1}}[N\tilde\omega(x')]^{(p\vee q)'}d\tilde{H}^{n\tau(p\vee q)'}(x') \lesssim 1.$$

Therefore, we have

$$\left\| \left\{ [\ell(Q')]^{\frac{1}{2}} t_{Q'} \right\}_{Q' \in \mathcal{D}(\mathbb{R}^{n-1})} \right\|_{b\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)}$$

$$\lesssim \left\{ \sum_{j \in \mathbb{Z}} \left[\sum_{Q' \in \mathcal{D}_j(\mathbb{R}^{n-1})} [\ell(Q')]^{-\left(\frac{s}{n}+\frac{1}{2}\right)pn+\frac{p}{2}} |t_{Q'}|^p \int_{Q' \times [0,\ell(Q'))} [\omega(x,2^{-j})]^{-p} dx' \right]^{q/p} \right\}^{1/q}$$

$$\lesssim \left\{ \sum_{j \in \mathbb{Z}} \left[\sum_{Q' \in \mathcal{D}_j(\mathbb{R}^{n-1})} |Q'|^{-\left(\frac{s-1/p}{n-1}+\frac{1}{2}\right)} |t_{Q'}|^p \int_{Q'} [\widetilde{\omega}(x',2^{-j})]^{-p} dx' \right]^{q/p} \right\}^{1/q}$$

$$\lesssim \|t\|_{b\dot{H}^{s-\frac{1}{p},\frac{n}{n-1}\tau}_{p,q}(\mathbb{R}^{n-1})} \lesssim \|f\|_{B\dot{H}^{s-\frac{1}{p},\frac{n}{n-1}\tau}_{p,q}(\mathbb{R}^{n-1})},$$

which implies that $F \in B\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $\|F\|_{B\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{B\dot{H}^{s-\frac{1}{p},\frac{n}{n-1}\tau}_{p,q}(\mathbb{R}^{n-1})}$. Furthermore, the

definition of F implies $\mathrm{Tr}(F) = f$, which completes the proof of Theorem (4.2.29).

We point out that Theorem (4.2.29) generalizes the corresponding classical results on Besov and Triebel-Lizorkin spaces for $p \in (1, \infty)$ and $q \in [1, \infty)$ by taking $\tau = 0$; see, for example, [7, 17, 106].

Chapter 5
Besov-Morrey Spaces and Characterizations of Besov-Type Spaces

We obtain the characterization of local means, the boundedness of pseudo-differential operators and the characterization of the Hardy-Morrey spaces. By using the maximal estimate and the molecular decomposition, we shall integrate and extend the known results on these spaces. We obtain the local mean characterizations of these function spaces via functions satisfying the Tauberian condition and establish a Fourier multiplier theorem on these spaces. All these results generalize the existing classical results on Besov and Triebel-Lizorkin spaces by taking $\tau = 0$ and are also new even for Q spaces and Hardy-Hausdorff spaces.

Section (5.1): Triebel-Lizorkin-Morrey Spaces

The well-known two scales of spaces B_{pq}^s and F_{pq}^s with $s \in \mathbb{R}$ and $p, q \in (0, \infty]$ ($p < \infty$ for the F-scale) on $\mathbb{R}^n$ are based on the L^p spaces, that is, they can be regarded as variants of the L^p spaces which take into account some smoothness condition. A_{pq}^s, which indicates either B_{pq}^s or F_{pq}^s, is applied to various partial differential equations. Recently there arose some interest to replace L^p by Morrey spaces $\mathcal{M}_q^p$ (see [55, 206, 207]). It is Kozono and Yamazaki that initially investigated Besov-Morrey spaces in connection with the Navier-Stokes equations. These function spaces are investigated by changing scales or extending admissible parameters in [50, 52, 55, 206, 207].

Besov-Morrey spaces and Triebel-Lizorkin-Morrey spaces are defined as follows: let $0 < q \le p < \infty$ and $0 < r \le \infty$. Given a sequence of functions $\{f_j\}_{j \in A}$ indexed by a countable set A, we define

$$\left\| \{f_j\}_{j \in A} : l_r(\mathcal{M}_q^p) \right\| := \left\| f_j : l_r(\mathcal{M}_q^p, A) \right\| := \left(\sum_{j \in A} \left\| f_j : \mathcal{M}_q^p \right\|^r \right)^{1/r},$$

$$\left\| \{f_j\}_{j \in A} : \mathcal{M}_q^p(l_r) \right\| := \left\| f_j : \mathcal{M}_q^p(l_r, A) \right\| := \left\| \left(\sum_{j \in A} |f_j|^r \right)^{1/r} : \mathcal{M}_q^p \right\|,$$

where a natural modification is made if $r = \infty$. If $A = \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, then we write

$$\left\| f_j : l_r(\mathcal{M}_q^p) \right\| = \left\| \{f_j\}_{j \in A} : l_r(\mathcal{M}_q^p) \right\|,$$

$$\left\| f_j : \mathcal{M}_q^p(l_r) \right\| = \left\| \{f_j\}_{j \in A} : \mathcal{M}_q^p(l_r) \right\|.$$

Given a function $t \in \mathcal{S}$ and $j \in \mathbb{N}_0$, we define $t^j(x) := 2^{jn} t(2^j x)$. If $t \in \mathcal{S}$ and $f \in \mathcal{F}$, then we denote $t(D)f := \mathcal{F}^{-1}[t \cdot \mathcal{F}f]$, where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and its inverse respectively. Pick $\psi \in \mathcal{G}$ so that $\chi_{B(1)} \le \psi \le \chi_{B(2)}$, where B(r) is the open ball centered at the origin and of radius r. In this paper we use Q(r) to denote the closed cube centered at the origin and of side length r = Q(r) := $\{x \in \mathbb{R}^n : \max(|x_1|, |x_2|, \cdots, |x_n|) \le$ r$\}$. Returning to the definition of the function spaces, we define $\psi(x) = \Psi(x)$ and $\varphi(x) = \Psi(x) - \Psi(2x)$. Define $\varphi_j(x) := \varphi(2^{-j}x)$ for $j \in \mathbb{N}$. Let $0 < q \le p < \infty, 0 < r \le \infty$ and $s \in \mathbb{R}$. Then the Besov-Morrey norm and the Triebel-Lizorkin-Morrey norm are given respectively as follows:

$$\left\| f : \mathcal{N}_{pqr}^s \right\| := \left\| \psi(D)f : \mathcal{M}_q^p \right\| + \left\| \{2^{js}\varphi_j(D)f\}_{j \in \mathbb{N}} : l_r(\mathcal{M}_q^p) \right\|,$$

$$\left\| f : \mathcal{E}_{pqr}^s \right\| := \left\| \psi(D)f : \mathcal{M}_q^p \right\| + \left\| \{2^{js}\varphi_j(D)f\}_{j \in \mathbb{N}} : \mathcal{M}_q^p(l_r) \right\|$$

for $f \in \mathcal{S}'$. To unify the statement in the sequel we use $\mathcal{A}_{pqr}^s$ to denote either $\mathcal{N}_{pqr}^s$ or $\mathcal{E}_{pqr}^s$. From the definition of the Morrey spaces we have

$$\mathcal{A}_{ppr}^s = \mathcal{A}_{pr}^s, \qquad 0 < p < \infty, \qquad 0 < r \le \infty, \qquad s \in \mathbb{R}.$$

Theorem (5.1.1) [219]: Suppose that $a \in S_{1\delta}^0$ with $0 \le \delta < 1$. Then $a(x, D)$, the pseudo-differential operator with symbol a, is $\mathcal{A}_{pqr}^s$-bounded.

We have the following

Proposition (5.1.2) [219]: Let the parameters $p, q, r, r_1, r_2, s, \varepsilon$ satisfy

$$0 < q \le p < \infty, \qquad 0 < r, r_1, r_2 \le \infty, \qquad s \in \mathbb{R}, \qquad \varepsilon > 0.$$

Then we have

1 $\mathcal{N}_{pqr_1}^{s+\varepsilon} \subset \mathcal{E}_{pqr_2}^s$ and $\mathcal{E}_{pqr_1}^{s+\varepsilon} \subset \mathcal{N}_{pqr_2}^s$,
2 $\mathcal{A}_{pqr_1}^{s+\varepsilon} \subset \mathcal{A}_{pqr_2}^s$, if $r_1 \le r_2$,
3 $\mathcal{N}_{pq\,\min(q,r)}^s \subset \mathcal{E}_{pqr}^s \subset \mathcal{N}_{pq\infty}^s$.

Proposition (5.1.3) [219]: The inclusions in Proposition (5.1.2) are strict in the following sense:

1 Suppose that the parameters p, q, r, r_0, s satisfy

$$0 < q < p < \infty, \qquad 0 < r, r_0 \le \infty, \qquad s \in \mathbb{R}.$$

If the continuous embedding $\mathcal{E}_{pqr}^s \subset \mathcal{N}_{pqr_0}^s$ is true, then $r_0 = \infty$.

2 Suppose that the parameters p, q, s satisfy $0 < q \le p < \infty, s \in \mathbb{R}$. Then the inclusion $\mathcal{E}_{pq\infty}^s \subset \mathcal{N}_{pq\infty}^s$ is strict.

Proposition (5.1.4) [219]: Suppose that the parameters satisfy

$$0 < q_i \le p_i < \infty, \qquad 0 < r_i \le \infty, \qquad s_i \in \mathbb{R}.$$

and let $\mathcal{N}_i := \mathcal{N}_{p_iq_ir_i}^{s_i}$ and $\mathcal{E}_i := \mathcal{E}_{p_iq_ir_i}^{s_i}$ for $i = 0, 1$. The following are true:

1 $\mathcal{N}_0 = \mathcal{N}_1$ if and only if $(p_0, q_0, r_0, s_0) = (p_1, q_1, r_1, s_1)$,
2 $\mathcal{E}_0 = \mathcal{E}_1$ if and only if $(p_0, q_0, r_0, s_0) = (p_1, q_1, r_1, s_1)$,
3 $\mathcal{N}_0 = \mathcal{E}_1$ if and only if $p_0 = p_1 = q_0 = q_1 = r_0 = r_1$ and $s_0 = s_1$.

Now we have the following, which will yield a clue to the analysis of Morrey spaces in connection with partial differential equations (see [206]).

Proposition (5.1.5) [219]: $\mathcal{E}_{pq2}^0 = \mathcal{M}_q^p$ with $1 < q \le p < \infty$.

For example, in [208] some properties connected to partial differential equations such as the trace property of Sobolev-Morrey spaces were obtained.

Following [57], let us define molecules. As usual, for $a \in \mathbb{R}^n$ and $0 < q, r \le \infty$, we define

$$\langle a \rangle := \sqrt{|a|^2 + 1}, \qquad \sigma_q := \frac{n}{\min(1, q)} - n, \qquad \sigma_{qr} := \frac{n}{\min(1, q, r)} - n.$$

Definition (5.1.6) [219]: (Molecule) Let $s \in \mathbb{R}, 0 < q \le p < \infty, 0 < r \le \infty$. Fix $K, L \in \mathbb{Z}$ such that

$$K \ge (1 + [s])_+, \quad L \ge \max\left(-1, [\sigma_q - s]\right)$$

for the $\mathcal{N}$-scale and

$$K \ge (1 + [s])_+, \quad L \ge \max\left(-1, [\sigma_{qr} - s]\right)$$

for the $\mathcal{E}$-scale. A C^K-function m is called an (s, p)-molecule for the function space $\mathcal{A}_{pqr}^s$, if the following oscillation and decay conditions hold for some point $x_0 \in \mathbb{R}^n$ and $\nu \in \mathbb{Z}$, where M is a sufficiently large constant:

1 $\int_{\mathbb{R}^n} x^\alpha m(x) dx = 0$ for $|\alpha| \le L$,
2 $|\partial^\alpha m(x)| \le 2^{-\nu(s-n/p)+\nu|\alpha|} \langle 2^\nu(x - x_0) \rangle^{-M-|\alpha|}$ if $|\alpha| \le K$.

Here and below we call m a molecule centered at x_0 and we always assume that

$$M \geq L + \frac{10n}{\min(1, q, r)},$$

as we have assumed in [207]. One defines

$$\text{Mol}_0 = \{\{M_{vm}\}_{v \in \mathbb{N}_0, m \in \mathbb{Z}^n} : \text{each } M_{vm} \text{ is a molecular with } x_0 = 2^{-v}m\}.$$

Proposition (5.1.7) [219]: (Molecular Decomposition) Suppose that the parameters $K, L \in \mathbb{Z}$ and $p, q, r, s \in \mathbb{R}$ satisfy

$$0 < q \leq p < \infty, \qquad 0 < r \leq \infty, \qquad K \geq (1 + [s])_+, \qquad L \geq \max\left(-1, [\sigma_q - s]\right)$$

for the $\mathcal{N}$-scale and

$$0 < q \leq p < \infty, \qquad 0 < r \leq \infty, \qquad K \geq (1 + [s])_+, \qquad L \geq \max\left(-1, [\sigma_{qr} - s]\right)$$

for the $\mathcal{E}$-scale.

1. Assume that $\{M_{vm}\}_{v \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \text{Mol}_0$ and $\lambda = \{\lambda_{vm}\}_{v \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in a_{pqr}$. Then the sum

$$f := \sum_{v \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{vm} M_{vm}$$

converges in $\mathcal{S}'$ and belongs to $\mathcal{A}_{pqr}^s$ with the norm estimate

$$\|f : \mathcal{A}_{pqr}^s\| \leq c\|\lambda : a_{pqr}\|.$$

Here the constant c does not depend on $\{M_{vm}\}_{v \in \mathbb{N}_0, m \in \mathbb{Z}^n}$ nor λ.

2. Conversely, any $f \in \mathcal{A}_{pqr}^s$ admits the following decomposition

$$f := \sum_{v \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{vm} M_{vm}.$$

The sum converges in $\mathcal{S}'$. We can even arrange that $\{M_{vm}\}_{v \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \text{Mol}_0$ and that the coefficient $\lambda = \{\lambda_{vm}\}_{v \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in a_{pqr}$ fulfills the norm estimate

$$\|\lambda : a_{pqr}\| \leq c\|f : \mathcal{A}_{pqr}^s\|.$$

We frequently use the following lemma, which gives us information on the coefficients of molecular decomposition.

Lemma (5.1.8) [219]: Let $\kappa_0, \kappa_1 \in \mathcal{S}$ supported on $B(4)$ and $B(8) \backslash B(1)$ respectively. Set $\kappa_j(x) = \kappa_1(2^{-j+1}x)$ for $j \geq 2$. Then we have

$$\left\| \left\{ 2^{k\left(s - \frac{n}{p}\right)} \sup_{y \in Q_{km}} |\kappa_k(D)f(y)| \right\}_{k \in \mathbb{N}_0, m \in \mathbb{Z}^n} : a_{pqr} \right\| \leq c\|f : \mathcal{A}_{pqr}^s\|.$$

In view of our actual construction, unfortunately the coefficient λ does not depend linearly on f, see [57, 106, 207].

We reconsider the local means on $\mathcal{A}_{pqr}^s$. The local means for $\mathcal{A}_{pq}^s$ are effective equivalent norms for these function spaces dealt with in [56]. However, we had to be prudent when we use it: we need to check that the function belongs a priori to the function spaces in question.

Now we shall describe the local means. To do this, we set $K(x) = \Psi(x)$ and $k(x) = \Delta^L \Psi(x)$ with L sufficiently large. In [207] the following assertion was established.

Theorem (5.1.9) [219]: Let $0 < q \leq p < \infty$, $0 < r \leq \infty$ and $s \in \mathbb{R}$. For $L \gg 1$ the following are true.

1. Let us define

$$\|f : \mathcal{N}_{pqr}^s\|_* := \|K * f : \mathcal{M}_q^p\| + \|\{2^{js}k^j * f\}_{j \in \mathbb{N}} : l_r(\mathcal{M}_q^p)\|$$

for $f \in \mathcal{S}'$. Then

$$\|f : \mathcal{N}_{pqr}^s\|_* \leq c\|f : \mathcal{N}_{pqr}^s\|$$

for all $f \in \mathcal{S}'$. Furthermore, if $f \in \mathcal{N}_{pqr}^s$. Then we have

$$\|f : \mathcal{N}_{pqr}^s\| \leq c\|f : \mathcal{N}_{pqr}^s\|_*$$

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for some constant $c > 0$ independent of f.

2 Let us define
$$\|f : \mathcal{E}_{pqr}^s\|_* := \|K * f : \mathcal{M}_q^p\| + \|\{2^{js}k^j * f\}_{j\in\mathbb{N}} : \mathcal{M}_q^p(l_r)\|$$
for $f \in \mathcal{S}'$. Then
$$\|f : \mathcal{E}_{pqr}^s\|_* \le c\|f : \mathcal{E}_{pqr}^s\|$$
or all $f \in \mathcal{S}'$. Furthermore, if $f \in \mathcal{E}_{pqr}^s$. Then we have
$$\|f : \mathcal{E}_{pqr}^s\| \le c\|f : \mathcal{E}_{pqr}^s\|_*$$
for some constant $c > 0$ independent of f.

Proposition (5.1.10) [219]: Let $K \in \mathbb{N}_0$ and $0 < q \le p < \infty$. Suppose that A is a C^K-function with compact support. Then $a \in \mathcal{N}_{pq\infty}^K$. In particular if $K > s$, $0 < r \le \infty$, then we have $a \in \mathcal{A}_{pqr}^s$.

Proof: By virtue of the equivalent norm
$$\|a : \mathcal{N}_{pq\infty}^K\| \simeq \|a : \mathcal{N}_{pq\infty}^0\| + \sum_{j=1}^n \|\partial_j^K a : \mathcal{N}_{pq\infty}^0\|,$$
(see [207]) we can assume $K = 0$. Since the family $\{k^j * a\}_{j\in\mathbb{N}}$ is supported on a fixed compact set, owing to the fact that
$$\mathcal{M}_{q_0}^{p_0} \cap \{f : \text{supp}(f) \subset B(R)\} \supset \mathcal{M}_{q_1}^{p_1} \cap \{f : \text{supp}(f) \subset B(R)\}$$
for all $0 < q_0 \le p_0 < q_1 \le p_1 < \infty$ and $R > 0$ we see, by using the local means, the matter is reduced to the case when $1 < q \le p < \infty$. In this case
$$\|a : \mathcal{N}_{pq\infty}^K\| = \|a : \mathcal{N}_{pq\infty}^0\| \le c\|a : \mathcal{E}_{pq2}^0\| \le c\|a : \mathcal{M}_q^p\| < \infty$$
and the proof of the first statement is now complete. The second assertion follows from the embedding $\mathcal{A}_{pqr}^s \supset \mathcal{N}_{pq\infty}^K$, which follows from Proposition (5.1.2).

Definition (5.1.11) [219]: Let $j \in \mathbb{N}$ and $A > 0$. Then define
$$\mathcal{M}_{A,j}f(x) := \sup_{\substack{y\in\mathbb{R}^n \\ l \ge j}} 2^{A(j-l)}\langle 2^j y\rangle^{-A}|k^l * f(x - y)|,$$
$$\mathcal{M}_{A,0}f(x) := \mathcal{M}_{A,1}f(x) + \sup_{y\in\mathbb{R}^n} \langle y\rangle^{-A}|K * f(x - y)|$$
for $f \in \mathcal{S}'$. Also define
$$\mathcal{M}_{A,j;\eta}f(x) := \left(\mathcal{M}_{A,j}f(x)\right)^\eta$$
for $A, \eta > 0$ and $j \in \mathbb{N}_0$.

For $s \in \mathbb{R}$, we denote $s_- := -\min(s, 0)$.

Proposition (5.1.12) [219]: Let $0 < q \le p < \infty$, $0 < r \le \infty$ and $s \in \mathbb{R}$.

1 If $A > -s_- + \frac{n}{\min(1,q)}$, then we have
$$\left\|\{2^{js}\mathcal{M}_{A,j}f\}_{j\in\mathbb{N}_0} : l_r(\mathcal{M}_q^p)\right\| \le c\|f : \mathcal{N}_{pqr}^s\|_*$$
for all $f \in \mathcal{S}'$.

2 If $A > -s_- + \frac{n}{\min(1,q,r)}$, then we have
$$\left\|\{2^{js}\mathcal{M}_{A,j}f\}_{j\in\mathbb{N}_0} : \mathcal{M}_q^p(l_r)\right\| \le c\|f : \mathcal{E}_{pqr}^s\|_*$$
for all $f \in \mathcal{S}'$.

Proof: The proof is identical to [213] except in that we use the Hardy norm. Here, for the sake of convenience for readers, we include the proof. Let $j \in \mathbb{N}$. The term for $j = 0$ can be readily incorporated afterward. Therefore, we shall consider $\mathcal{M}_{A,j}f$ with $j \in \mathbb{N}$. Furthermore, we shall prove (ii), the proof of (i) being simpler.

Choose $\zeta, \theta \in \mathcal{D}$ so that

$$\zeta \cdot \mathcal{F}K + \sum_{l=0}^{\infty} \theta(2^{-l} *) \cdot \mathcal{F}[k^l] \equiv 1, \quad 0 \notin \operatorname{supp}(\theta). \tag{1}$$

Note that $\mathcal{F}^{-1}\theta$ has vanishing moment of infinite order. Then we have

$$k^j * f = (2\pi)^{\frac{n}{2}} \left(K^j * (\mathcal{F}^{-1}\zeta)^j * k^j * f + \sum_{l=j+1}^{\infty} k^j * (\mathcal{F}^{-1}\theta)^l * k^l * f \right).$$

Let $L \gg 1$, where L is a number appearing in the definition of $k : k = \Delta^L \Psi$. Note that

$$\mathcal{F}[k^j * (\mathcal{F}^{-1}\theta)^l](x) = (2\pi)^{\frac{n}{2}} = \mathcal{F}k(2^{-j}x)\theta(2^{-l}x)$$

and that k has vanishing moment up to order $[A] + 1$. Hence, for every $\alpha \in \mathbb{N}_0{}^n$ and $M \in \mathbb{N}$, there exists $c_{\alpha,M}$ independent of l and m such that

$$\left| \partial^\alpha \mathcal{F}[k^j * (\mathcal{F}^{-1}\theta)^l](x) \right| \leq c_{\alpha,M} 2^{([A]+1)(j-l)+jn} \langle 2^j x \rangle^{-M}.$$

Hence it follows that

$$\left| k^j * (\mathcal{F}^{-1}\theta)^l(x) \right| \leq c_K 2^{([A]+1)(j-l)+jn} \langle 2^k x \rangle^{-K}.$$

We remark that this technique was due to Rychkov [213]. As a result, we obtain

$$\left| k^j * (\mathcal{F}^{-1}\theta)^l * k^l * f(x) \right| \leq c 2^{A(j-l)+jn} \int_{\mathbb{R}^n} \frac{|k^l * f(x-y)|}{\langle 2^j y \rangle^A} \, dy.$$

for all $A > 0$. A similar calculation works for the first term, namely, $K^j * (\mathcal{F}^{-1}\zeta)^j * k^j * f$, and we obtain

$$\left| k^j * f(x) \right| \leq c \sum_{l=j}^{\infty} 2^{A(j-l)+ln} \int_{\mathbb{R}^n} \frac{|k^l * f(x-y)|}{\langle 2^j y \rangle^M} \, dy.$$

Now we invoke this pointwise estimate. From this estimate we deduce

$$\mathcal{M}_{A,j} f(x) = \sup_{\substack{y \in \mathbb{R}^n \\ l \geq j}} \frac{2^{A(j-l)} |k^l * f(x-y)|}{\langle 2^j y \rangle^A}$$

$$\leq c \sup_{\substack{y \in \mathbb{R}^n \\ l \geq j}} \sum_{m=l}^{\infty} 2^{A(j-l)+A(l-m)+mn} \int_{\mathbb{R}^n} \frac{|k^m * f(x-y-z)|}{\langle 2^j y \rangle^A \langle 2^l z \rangle^A} \, dz$$

$$= c \sup_{y \in \mathbb{R}^n} \sum_{m=j}^{\infty} 2^{A(j-m)+mn} \int_{\mathbb{R}^n} \frac{|k^m * f(x-y-z)|}{\langle 2^j y \rangle^A \langle 2^j z \rangle^A} \, dz.$$

By virtue of the Peetre inequality $\langle y + z \rangle \leq \sqrt{2} \langle y \rangle \langle z \rangle$ and by changing variables, we obtain

$$\left| k^j * f(x) \right| \leq \mathcal{M}_{A,j} f(x) \leq c \sum_{m=j}^{\infty} 2^{A(j-m)+mn} \int_{\mathbb{R}^n} \frac{|k^m * f(x-z)|}{\langle 2^j z \rangle^A} \, dz. \tag{2}$$

Let us choose $\eta > 0$ so that

$$(A+s)\eta > n, \qquad A\eta > n, \qquad 0 < \eta < \min(1, q, r),$$

which is possible by assumption. From the definition of the maximal operator $\mathcal{M}_{A,j} f(x)$ and the above inequality we deduce

$$\mathcal{M}_{A,m} f(x) \leq c \mathcal{M}_{A,m;1-\eta} f(x) \sum_{l=m}^{\infty} 2^{A\eta(m-l)+ln} \int_{\mathbb{R}^n} \frac{|k^l * f(x-z)|^\eta}{\langle 2^m z \rangle^{A\eta}} \, dz. \tag{3}$$

Here the constant c depends on A. Therefore, if we assume that $\mathcal{M}_{A,m} f(x) < \infty$, then we obtain from (3)

$$\mathcal{M}_{A,m;\eta}f(x) \le c \sum_{l=m}^{\infty} 2^{(m-l)(A\eta-n)+mn} \int_{\mathbb{R}^n} \frac{|k^l * f(x-z)|^{\eta}}{\langle 2^m z \rangle^{A\eta}} dz. \tag{4}$$

By using the Torchinsky technique [214], Rychkov observed that (4) is still the case if we do not assume $\mathcal{M}_{A,m}f(x) < \infty$. To establish this, let us assume that the right-hand side of (4) is finite. Since $f \in \mathcal{S}'$, we see that there exists A_f such that
$$\mathcal{M}_{A,m}f(x) < \infty, \qquad x \in \mathbb{R}^n$$
for all $A \ge A_f$ and $m \in \mathbb{N}_0$. Note that c in (4) depends implicitly on A. Therefore, for all $f \in \mathcal{S}'$, there exists $c_f > 0$ depending on f such that

$$\left| k^j * f(x) \right|^{\eta} \le c_f \sum_{l=m}^{\infty} 2^{(m-l)(A\eta-n)+mn} \int_{\mathbb{R}^n} \frac{|k^l * f(x-z)|^{\eta}}{\langle 2^m z \rangle^{A_f \eta}} dz. \tag{5}$$

Since the right-hand side of (4) is decreasing with respect to A, we see that (5) is valid if we replace A_f with any positive number A less than A_f. Let $A \le A_f$. Then we have

$$\mathcal{M}_{A,m;\eta}f(x) = \sup_{y \in \mathbb{R}^n, l \in \mathbb{Z}: l \ge m} 2^{A\eta(m-l)} \frac{|k^l * f(x-y)|^{\eta}}{\langle 2^m y \rangle^{A\eta}}$$

$$\le c_f \sup_{y \in \mathbb{R}^n, l \in \mathbb{Z}: l \ge m} \sum_{i=l}^{\infty} 2^{(m-i)(A\eta-n)+mn} \int_{\mathbb{R}^n} \frac{\left| k^i * f(x-y-z) \right|^{\eta}}{\langle 2^m y \rangle^{A\eta} \langle 2^l z \rangle^{A\eta}} dz$$

$$\le c_f \sup_{y \in \mathbb{R}^n} \sum_{l \in \mathbb{Z}: i=l}^{\infty} 2^{(m-i)(A\eta-n)+mn} \int_{\mathbb{R}^n} \frac{\left| k^i * f(x-y-z) \right|^{\eta}}{\langle 2^m y \rangle^{A\eta} \langle 2^m z \rangle^{A\eta}} dz$$

$$\le c_f \sup_{y \in \mathbb{R}^n} \sum_{l=m}^{\infty} 2^{(m-l)(A\eta-n)+mn} \int_{\mathbb{R}^n} \frac{|k^l * f(x-y-z)|^{\eta}}{\langle 2^m(y+z) \rangle^{A\eta}} dz < \infty,$$

because we are assuming the right-hand side of (4) is finite. Returning to (3), we obtain (4) for all $f \in \mathcal{S}'$. By using the Hardy-Littlewood maximal operator M, we have

$$2^{js\eta} \mathcal{M}_{A,m;\eta}f(x) \le c \sum_{l=m}^{\infty} 2^{(m-l)((A+s)\eta-n)} M[|2^{ls}k^l * f|^{\eta}](x).$$

By the Fefferman-Stein maximal inequality for the Morrey spaces (see [54] or [55]), we obtain the desired result.

Corollary (5.1.13) [219]: Retain the same condition as Proposition (5.1.12).

1 If $A > -s_- + \frac{n}{\min(1,q)}$, then we have

$$\left\| \sup_{z \in \mathbb{R}^n} \langle z \rangle^{-A} |K * f(*-z)| : \mathcal{M}_q^p \right\| + \left\| \sup_{z \in \mathbb{R}^n} 2^{js} \langle 2^j z \rangle^{-A} |k^j * f(*-z)| : l_r(\mathcal{M}_q^p) \right\|$$

   is dominated by $\|f : \mathcal{N}_{pqr}^s\|_*$ for all $f \in \mathcal{S}'$.

2 If $A > -s_- + \frac{n}{\min(1,q,r)}$, then we have

$$\left\| \sup_{z \in \mathbb{R}^n} \langle z \rangle^{-A} |K * f(*-z)| : \mathcal{M}_q^p \right\| + \left\| \sup_{z \in \mathbb{R}^n} 2^{js} \langle 2^j z \rangle^{-A} |k^j * f(*-z)| : \mathcal{M}_q^p(l_r) \right\|$$

   is dominated by $\|f : \mathcal{E}_{pqr}^s\|_*$ for all $f \in \mathcal{S}'$.

Theorem (5.1.14) [219]: Let $0 < q \le p < \infty$, $0 < r \le \infty$ and $s \in \mathbb{R}$. Then there exists a constant $c > 0$ such that
$$c^{-1} \|f : \mathcal{A}_{pqr}^s\| \le \|f : \mathcal{A}_{pqr}^s\|_* \le c \|f : \mathcal{A}_{pqr}^s\|$$
for all $f \in \mathcal{S}'$.

Proof: From Theorem (5.1.9) it suffices to show the right inequality. Pick $\zeta, \eta \in \mathcal{D}$ so that

$$\zeta \cdot \mathcal{F}l + \sum_{m=0}^{\infty} \eta(2^{-m}*) \cdot \mathcal{F}k^m \equiv 1, \quad 0 \notin \text{supp}(\eta).$$

Note that $\mathcal{F}^{-1}\eta$ has vanishing moment up to order L, since $0 \notin \text{supp}(\eta)$. From this formula, we deduce

$$2^{js}[\mathcal{F}^{-1}\varphi_j] * f = (2\pi)^{\frac{n}{2}} 2^{js}[\mathcal{F}^{-1}\varphi_j] * l^j * \zeta^j * f + (2\pi)^{\frac{n}{2}} \sum_{m=j+1}^{\infty} 2^{js}[\mathcal{F}^{-1}\varphi_j] * (\mathcal{F}\eta)^j * k^j * f.$$

Observe that

$$\left|[\mathcal{F}^{-1}\varphi_j] * (\mathcal{F}\eta)^j(x)\right| \le c 2^{jn+L(j-m)}\langle 2^j x\rangle^{-A},$$

where A satisfies the same condition as Corollary (5.1.13). Therefore, it follows that

$$\left|2^{js}[\mathcal{F}^{-1}\varphi_j] * (\mathcal{F}\eta)^m * k^m * f(x)\right| \le c 2^{js} \int_{\mathbb{R}^n} \left|[\mathcal{F}^{-1}\varphi_j] * (\mathcal{F}\eta)^m(y)\right| \cdot |k^m * f(x-y)| dy$$

$$\le \left(\sup_{z \in \mathbb{R}^n} \langle 2^m z\rangle^{-A} |k^m * f(x-z)|\right) \cdot \left(2^{js} \int_{\mathbb{R}^n} \left|[\mathcal{F}^{-1}\varphi_j] * (\mathcal{F}\eta)^m(y)\right| \cdot \langle 2^m y\rangle^A dy\right)$$

$$\le c 2^{-(s+A-L)(m-j)} \sup_{z \in \mathbb{R}^n} \langle 2^m z\rangle^{-A} |2^{sm} k^m * f(x-z)|,$$

if $m \ge j$. Therefore, if we let $L > A + s$, then this inequality is summable. Hence, we obtain the desired result by Corollary (5.1.13).

Now we deal with pseudo-differential operators.

First, we deal with the class $S_{1\delta}^m$ with $0 \le \delta \le 1$.

Let $0 \le \rho, \delta \le 1$. $a \in C^{\infty}(\mathbb{R}^n \times \mathbb{R}^n)$ is said to be $S_{\rho,\delta}^m$, if for all $\alpha, \beta \in \mathbb{N}_0^n$,

$$\sup_{x,\xi \in \mathbb{R}^n} \left|\partial_x^{\beta} \partial_{\xi}^{\alpha} a(x,\xi)\right|\langle \xi\rangle^{-m+\rho|\beta|-\delta|\alpha|} < \infty.$$

Let $a \in S_{\rho,\delta}^m$ with $m \in \mathbb{R}$ and $0 \le \rho, \delta \le 1$. One defines a continuous linear mapping $a(x,D) : \mathcal{S} \to \mathcal{S}$ by

$$a(x,D)f(x) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} a(x,\xi) \mathcal{F}f(\xi) d\xi.$$

By duality this mapping extends to a continuous mapping from $\mathcal{S}'$ to $\mathcal{S}'$ (see [216]).

Theorem (5.1.15) [219]: Let $0 < q \le p < \infty$, $0 < r \le \infty$, $m \in \mathbb{R}$ and $a \in S_{1,1}^m$.

1 If $s > \sigma_q$, then there exists $c > 0$ such that
$$\|a(x,D)f : \mathcal{N}_{pqr}^s\| \le c\|f : \mathcal{N}_{pqr}^{s+m}\|.$$

2 If $s > \sigma_{qr}$, then there exists $c > 0$ such that
$$\|a(x,D)f : \mathcal{E}_{pqr}^s\| \le c\|f : \mathcal{E}_{pqr}^{s+m}\|.$$

Proof: Let us pick auxiliary functions $\psi, \varphi, \kappa \in \mathcal{S}$ with the following conditions.

1 $\chi_{Q(1)} \le \psi \le \chi_{Q(2)}$;

2 $\varphi(x) = \psi(x) - \psi(2x)$;

3 $\chi_{Q(2)} \le \kappa \le \chi_{Q(3)}$.

We consider $a(x,D)(1 - \psi(D))f$ because $a(x,D)\psi(D)f$ can be dealt with in a similar way.

Let us consider $(1 - \psi) \cdot \mathcal{F}f = \sum_{j=1}^{\infty} \varphi_j \cdot \mathcal{F}f$. Expand $\varphi_j \cdot \mathcal{F}f$ into a Fourier series

$$\varphi_j \cdot \mathcal{F}f = \sum_{m \in \mathbb{Z}^n} \frac{\varphi_j(D)f(2^{-j}m)}{(2\pi)^{\frac{n}{2}} \cdot 2^{jn}} \kappa_j \exp(-i2^{-j}m \cdot *),$$

where $\kappa_j(x) = \kappa(2^{-j}x)$ for $j \in \mathbb{N}$. From this we have

$$a(x,D)[(1 - \psi(D))f](x) = \sum_{j=1}^{\infty} \sum_{m \in \mathbb{Z}^n} \frac{\varphi_j(D)f(2^{-j}m)}{(2\pi \cdot 2^j)^n} \int_{\mathbb{R}^n} a(x,\xi)\kappa_j(\xi) \exp(i(x - 2^{-j}m) \cdot \xi) d\xi.$$

The Leibniz rule gives us that

$$\partial_x^\alpha \int_{\mathbb{R}^n} a(x,\xi)\kappa_j(\xi)\exp(i(x-2^{-j}m)\cdot\xi)\,d\xi$$

$$= \sum_{\beta\le\alpha} c_{\alpha\beta}\int_{\mathbb{R}^n}[\partial_x^\beta a(x,\xi)]\kappa_j(\xi)(i\xi)^{\alpha-\beta}\exp(i(x-2^{-j}m)\cdot\xi)\,d\xi$$

and κ does not contain 0 as its support, where $c_{\alpha\beta}$ is the binomial coefficient. If we carry out integration by parts, then we obtain

$$2^{-j(s+m)}\int_{\mathbb{R}^n} a(x,\xi)\kappa_j(\xi)\exp(i(x-2^{-j}m)\cdot\xi)\,d\xi$$

is a molecule in $\mathscr{A}_{pqr}^s$. Together with Lemma (5.1.8) we obtain the desired result.

A passage to the general case can be achieved by the following well-known lemma (see [216]).

Lemma (5.1.16) [219]: Let $0\le\delta<1$, $a\in S_{1\delta}^0$ and $N\in\mathbb{N}$. Then there exists $b\in S_{1\delta}^0$ such that

$$a(x,D) = (1-\Delta)^N\circ b(x,D)\circ(1-\Delta)^{-N}.$$

Corollary (5.1.17) [219]: Let $0<q\le p<\infty$, $0<r\le\infty$, $s,m\in\mathbb{R}$, $0\le\delta<1$, $a\in S_{1\delta}^0$. Then there exists $c>0$ such that

$$\|a(x,D)f:\mathscr{A}_{pqr}^s\| \le c\|f:\mathscr{A}_{pqr}^{s+m}\|.$$

Definition (5.1.18) [215]: Let $0\le\delta\le1$ and $l>0$. One defines

$$C_*^l S_{1\delta}^m := \{p:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{C}:p(x,*)\in C^\infty, x\in\mathbb{R}^n \text{ and } \|p:C_*^l S_{1\delta}^m\|_* <\infty\},$$

where

$$\|p:C_*^l S_{1\delta}^m\|_* := \sup_{\xi\in\mathbb{R}^n}\langle\xi\rangle^{-m+|\alpha|-l\delta}\|\partial_\xi^\alpha p(*,\xi):C_*^l\| + \sup_{\xi\in\mathbb{R}^n}\langle\xi\rangle^{-m+|\alpha|}\|\partial_\xi^\alpha p(*,\xi):L^\infty\|.$$

An elementary symbol is an expression of the form

$$\sigma(x,\xi) = \sum_{j=1}^\infty \sigma_j(x)\psi_j(\xi),$$

where $\psi_j(\xi) = \psi_1(2^{-j+1}\xi)$ for $j\in\mathbb{N}$ and ψ_1 is a compactly supported function such that ψ_1 is not supported on the origin. Furthermore,

$$\sup_{j\in\mathbb{N}} 2^{-jm}\|\sigma_j:L^\infty\| + 2^{-j(m+l\delta)}\|\sigma_j:C_*^l\| <\infty.$$

Proposition (5.1.19) [41]: Let $f\in L^\infty$ such that $\operatorname{supp}(\mathcal{F}f)\subset B(R)$. Then for all $\eta>0$, there exists $c>0$ such that there holds

$$\sup_{y\in\mathbb{R}^n}\frac{|f(x-y)|}{\langle Ry\rangle^{n/\eta}} < cM[|f|^\eta](x)^{1/\eta}$$

for every $x\in\mathbb{R}^n$.

Theorem (5.1.20) [219]: Suppose that the parameters p,q,r,s,l,δ satisfy

$$0<s<l,\qquad 1\le q\le p<\infty,\qquad 1\le r\le\infty,\qquad 0\le\delta\le1$$

and that $a\in C_l^* S_{1\delta}^0$. Then there exists $c>0$ such that

$$\|a(x,D)f:\mathscr{A}_{pqr}^s\| \le c\|f:\mathscr{A}_{pqr}^{s+m}\|.$$

Proof: Let us concentrate on the case when $\mathscr{A}=\mathcal{E}$, the case when $\mathscr{A}=\mathcal{N}$ is simpler. Note that $(1-\Delta)^{\frac{m}{2}}$ is an isomorphism that composes well with pseudo-differential operators. Therefore, it is enough to examine the case $m=0$. Let $f\in\mathcal{E}_{pqr}^s$ and a be an elementary form as in [215]. Any symbol in $C_l^* S_{1\delta}^0$ can be approximated by elementary symbols. Therefore, it is sufficient to investigate the case when a is an elementary form:

$$a(x,\xi) = \sum_{j=1}^{\infty} \sigma_j(x)\varphi_j(\xi).$$

Define $q_{jk} = \varphi_k(D)\sigma_j$. Then we have

$$\|q_{jk}\|_{\infty} \le c2^{(j-k)l}. \tag{6}$$

As a consequence, we obtain $q(x,D)f(x) = \sum_{j\in\mathbb{N},k\in\mathbb{N}_0} q_{jk}(x) \cdot \varphi_j(D)f(x)$. We decompose $q(x,D)f$ into three pieces. Let us set

$$q_1(x,D)f := \sum_{j=4}^{\infty}\left(\sum_{k=0}^{j-4} q_{jk}\right)\varphi_j(D)f,$$

$$q_2(x,D)f := \sum_{j=0}^{\infty}\left(\sum_{k=\max(j-3,0)}^{j+3} q_{jk}\right)\varphi_j(D)f,$$

$$q_3(x,D)f := \sum_{j=0}^{\infty}\left(\sum_{k=j+4}^{\infty} q_{jk}\right)\varphi_j(D)f.$$

The estimate of $q_2(x,D)f$ is simple. Choose an auxiliary function $\kappa \in \mathcal{S}$ so that $\chi_{Q(3)} \le \kappa \le \chi_{Q(\pi)}$. Then we have

$$\left\|\partial^{\alpha}\left(\sum_{k=\max(j-3,0)}^{j+3} q_{jk}\right)\right\|_{\infty} \le c2^{j|\alpha|}\left\|\left(\sum_{k=\max(j-3,0)}^{j+3} q_{jk}\right)\right\|_{\infty} \le c2^{j|\alpha|}.$$

Here we have used Proposition (5.1.19) for the first inequality and for the second inequality we have used (6) and the fact that at most 7 terms are involved. Therefore,

$$q_2(x,D)f := \sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\varphi_j(D)f(2^{-j}m)\left(\sum_{k=\max(j-3,0)}^{j+3} q_{jk}\right)\mathcal{F}^{-1}\kappa(2^j * -m)$$

can be regarded as a molecular decomposition and hence we conclude that $q_2(x,D)$ is bounded from $\mathcal{A}_{pqr}^s$ to itself.

The first piece is treated in a spirit similar to [41]. We shall make use of the fact that $\mathrm{supp}(f * g) \subset \mathrm{supp}(f) + \mathrm{supp}(g)$ for all compactly supported distributions $f, g \in \mathcal{S}'$, where the right-hand side denotes the algebraic sum. Note that $\left(\sum_{k=0}^{j-4} q_{jk}\right)\varphi_j(D)f$ is concentrated on $B(2^{j+3})\backslash B(2^{j-3})$ in frequency. Hence it follows that

$$\|q_1(x,D)f : \mathcal{E}_{pqr}^s\| \le c\left\|2^{js}\left(\sum_{k=0}^{\max(j-4,0)} q_{jk}\right)\varphi_j(D)f : \mathcal{M}_q^p(l_r)\right\|$$

$$\le c\left(\sup_{j\in\mathbb{N}_0}\left\|\sum_{k=0}^{\max(j-4,0)} q_{jk}\right\|_{\infty}\right)\|2^{js}\varphi_j(D)f : \mathcal{M}_q^p(l_r)\|$$

$$\le c\left(\sup_{j\in\mathbb{N}_0}\left(\sum_{k=0}^{\max(j-4,0)} \|q_{jk}\|_{\infty}\right)\right)\|2^{js}\varphi_j(D)f : \mathcal{M}_q^p(l_r)\|$$

$$\le c\left(\sup_{j\in\mathbb{N}_0}\|\sigma_j\|_{\infty}\right)\|2^{js}\varphi_j(D)f : \mathcal{M}_q^p(l_r)\| \le c\|f : \mathcal{E}_{pqr}^s\|.$$

Let us turn to the estimate of $q_3(x,D)$. Let us rewrite

$$q_3(x, D)f(x) := \sum_{k=4}^{\infty} \left(\sum_{j=0}^{k-4} q_{jk}(x) \right) \varphi_j(D)f(x),$$

where the change of order of the summation will be justified below. Note that the frequency support of $\sum_{j=0}^{k-4} q_{jk}(x) \cdot \varphi_j(D)f(x)$ is concentrated on $B(2^{k+3}) \backslash B(2^{k-3})$. As a result, we obtain

$$\left\| q_3(x, D)f : \mathcal{E}_{pqr}^s \right\| \leq c \left\| \left\{ 2^{ks} \sum_{j=0}^{k-4} q_{jk} \cdot \varphi_j(D)f \right\}_{k=4}^{\infty} : \mathcal{M}_q^p(l_r) \right\|$$

$$\leq c \left\| \left\{ \sum_{j=0}^{k-4} 2^{(k-j)(s-l)} \cdot 2^{js} |\varphi_j(D)f| \right\}_{k=4}^{\infty} : \mathcal{M}_q^p(l_r) \right\| \leq c \| f : \mathcal{E}_{pqr}^s \|,$$

where we have used the fact that $s < l$ for the second inequality. This is the desired result.

Now we are going to characterize $\mathcal{E}_{pq2}^0$ and its homogeneous counterpart. Assume that ψ is a non-degenerate function in the sense that

$$\int \psi \neq 0. \tag{7}$$

Definition (5.1.21) [219] (Hardy-Morrey Spaces)**:** Let $0 < q \leq p < \infty$. Then define

$$\| f : H\mathcal{M}_q^p \| := \left\| \sup_{j \in \mathbb{Z}} |\psi^j * f| : \mathcal{M}_q^p \right\|, \qquad \| f : h\mathcal{M}_q^p \| := \left\| \sup_{j \in \mathbb{N}_0} |\psi^j * f| : \mathcal{M}_q^p \right\|$$

for $f \in \mathcal{S}'$. $H\mathcal{M}_q^p$ (resp. $h\mathcal{M}_q^p$) is a set of all tempered distributions $f \in \mathcal{S}'$ for which the quasi-norm $\| f : H\mathcal{M}_q^p \|$ (resp. $\| f : h\mathcal{M}_q^p \|$) is finite. $H\mathcal{M}_q^p$ is the homogeneous Hardy-Morrey space and $h\mathcal{M}_q^p$ is the nonhomogeneous Hardy-Morrey space. If we invoke the fact that the maximal operator M is bounded from $\mathcal{M}_q^p$ to itself whenever $1 < q \leq p < \infty$, then we obtain

$$H\mathcal{M}_q^p \simeq h\mathcal{M}_q^p \simeq \mathcal{M}_q^p$$

with norm equivalence.

We use the homogeneous norms to formulate our results:

$$\| f : \dot{\mathcal{N}}_{pqr}^s \| := \| 2^{js} \varphi_j(D)f : l_r(\mathcal{M}_q^p, \mathbb{Z}) \| = \left(\sum_{j=-\infty}^{\infty} 2^{jrs} \| \varphi_j(D)f : \mathcal{M}_q^p \|^r \right)^{1/r},$$

$$\| f : \dot{\mathcal{E}}_{pqr}^s \| := \| 2^{js} \varphi_j(D)f : \mathcal{M}_q^p(l_r, \mathbb{Z}) \| = \left\| \left(\sum_{j=-\infty}^{\infty} 2^{jrs} |\varphi_j(D)f|^r \right)^{1/r} : \mathcal{M}_q^p \right\|.$$

Unlike the nonhomogeneous version, we need to consider these norms modulo the set of all polynomials $\mathcal{P}$ (see [106]).

Note we assume that $\psi \in \mathcal{S}$ is a non-degenerate function in the sense that $\int \psi \neq 0$. Define $\varphi(x) := 2^n \psi(2x) - \psi(x)$.

Lemma (5.1.22) [219]: Let $\psi \in \mathcal{S}$ and $L \in \mathbb{N}_0$ be given. Define $\varphi(x) := 2^n \psi(2x) - \psi(x)$. Then there exist $\widetilde{\psi}, \widetilde{\varphi} \in \mathcal{S}$ such that

$$\widetilde{\psi} * \psi + \sum_{j=0}^{\infty} \widetilde{\varphi}^j * \varphi^j = \delta$$

and that $\widetilde{\varphi}$ has vanishing moment up to order L.

Proof: The proof can be obtained with a minor modification of the results due to Rychkov (see [218]), where Rychkov used $\mathcal{D}$ instead of $\mathcal{S}$.

Theorem (5.1.23) [219]: Let $0 < q \le p < \infty$. Then the sets $H\mathcal{M}_q^p$ and $h\mathcal{M}_q^p$ are independent of the choices of admissible ψ satisfying (7).

Proof: for $H\mathcal{M}_q^p$ The following theorem asserts more than Theorem (5.1.23). To formulate the stronger result, for $N \in \mathbb{N}$ we set

$$p_N(\zeta) := \sum_{|\alpha| \le N} \sup_{x \in \mathbb{R}^n} \langle x \rangle^N |\partial^\alpha \zeta(x)|, \qquad \zeta \in \mathcal{S}$$

Note that $\{p_N\}_{N \in \mathbb{N}}$ topologizes $\mathcal{S}$.

Theorem (5.1.24) [219]: Let $0 < q \le p < \infty$. Assume that $\psi \in \mathcal{S}$ satisfies the non-degenerate condition $\int \psi \ne 0$. Then there exist $N \in \mathbb{N}$ and $c > 0$ such that

$$\left\| \sup_{\zeta \in B_{p_N}(1)} \left(\sup_{j \in \mathbb{Z}} |\zeta^j * f| \right) : \mathcal{M}_q^p \right\| \le c \left\| \sup_{j \in \mathbb{Z}} |\psi^j * f| : \mathcal{M}_q^p \right\|,$$

where $B_{p_N}(1) := \{\zeta \in \mathcal{S} : p_N(\zeta) < 1\}$.

Proof: Fix $\zeta \in B_{p_N}(1)$ and $j \in \mathbb{Z}$. Then there exist $\widetilde{\psi}, \widetilde{\varphi} \in \mathcal{S}$ such that $\widetilde{\varphi}$ has vanishing moment up to order L with L large enough and that

$$\widetilde{\psi} * \psi + \sum_{l=0}^{\infty} \widetilde{\varphi}^l * \varphi^l = \delta$$

by virtue of Lemma (5.1.22). Using this formula, we obtain

$$|\zeta^j * f(x)| \le |\zeta^j * \widetilde{\psi}^j * \psi^j * f(x)| + \sum_{l=0}^{\infty} |\zeta^j * \widetilde{\varphi}^{j+l} * \varphi^{j+l} * f(x)|.$$

Let us set

$$\varphi^{j+l,*}f(x) := \sup_{y \in \mathbb{R}^n} \langle 2^{j+l}y \rangle^{-\frac{n}{r}} |\varphi^{j+l} * f(x-y)|,$$

$$\psi^{j+l,*}f(x) := \sup_{y \in \mathbb{R}^n} \langle 2^{j+l}y \rangle^{-\frac{n}{r}} |\psi^{j+l} * f(x-y)|$$

for $j, l \in \mathbb{N}_0$ and $0 < r \ll 1$. Then we have

$$|\zeta^j * \widetilde{\varphi}^{j+l} * \varphi^{j+l} * f(x)| \le \int_{\mathbb{R}^n} |\zeta^j * \widetilde{\varphi}^{j+l}(y)| \cdot |\varphi^{j+l} * f(x-y)| dy$$

$$\le \varphi^{j+l,*}f(x) \int_{\mathbb{R}^n} |\zeta^j * \widetilde{\varphi}^{j+l}(y)| \cdot \langle 2^{j+l}y \rangle^{\frac{n}{r}} dy = \varphi^{j+l,*}f(x) \int_{\mathbb{R}^n} |\zeta * \widetilde{\varphi}^l(y)| \cdot \langle 2^l y \rangle^{\frac{n}{r}} dy.$$

Now that $\widetilde{\varphi}$ has vanishing moment up to order L, we have $|\mathcal{F}\widetilde{\varphi}(x)| \le c|x|^L$. Therefore, it follows that

$$|\zeta * \widetilde{\varphi}^l(y)| \le c2^{-Ll} \langle y \rangle^{-n\left(1+\frac{1}{r}\right)-1}.$$

Inserting this estimate, we obtain

$$|\zeta^j * \widetilde{\varphi}^{j+l} * \varphi^{j+l} * f(x)| \le c2^{-\left(L-\frac{n}{r}\right)l} \varphi^{j+l,*}f(x). \tag{8}$$

Since $\varphi = \psi^1 - \psi$, it follows that

$$|\zeta^j * \widetilde{\varphi}^{j+l} * \varphi^{j+l} * f(x)| \le c2^{-\left(L-\frac{n}{r}\right)l} \left(\psi^{j+l+1,*}f(x) + \psi^{j+l,*}f(x) \right).$$

A similar estimate is valid for $|\zeta^j * \widetilde{\psi}^j * \psi^j * f(x)|$. As a result, adding these estimates over $l \in \mathbb{N}_0$, we have

$$|\zeta^j * f(x)| \le c \sup_{k \in \mathbb{Z}} \psi^{k,*}f(x).$$

From this formula, we obtain

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$$\left\| \sup_{\zeta \in B_{p_N}(1)} \left(\sup_{j \in \mathbb{Z}} |\zeta^j * f| \right) : \mathcal{M}_q^p \right\| \leq c \left\| \sup_{j \in \mathbb{Z}} \psi^{j,*} f : \mathcal{M}_q^p \right\|.$$

Since we can deduce

$$\psi^{j,*} f(x)^\eta \leq c \sum_{l=0}^{\infty} 2^{-l\delta} M[|\psi^{j+l,*} f(x)|^\eta](x), \qquad \delta > 0$$

in the same way as before, another application of the Hardy-Littlewood maximal inequality for the Morrey spaces gives us

$$\left\| \sup_{j \in \mathbb{Z}} \psi^{j,*} f : \mathcal{M}_q^p \right\| \leq c \left\| \sup_{j \in \mathbb{Z}} |\psi^j * f| : \mathcal{M}_q^p \right\|.$$

Putting together these observations, we obtain the desired result.

We consider the Fourier multiplier. Recall that Bui proved the following result on the weighted Hardy spaces. As for A_∞-weights, see [216].

Proposition (5.1.25) [217]: Let $\omega \in A_\infty$ and define

$$\|f : H^p(\omega)\| := \left\| \sup_{j \in \mathbb{Z}} |\psi_j * f| : L^p(\omega) \right\|.$$

for $f \in \mathcal{S}'$ and $0 < p < \infty$. Assume in addition that $\tau \in \mathcal{S}$ and that

$$c_\alpha := \sup_{x \in \mathbb{R}^n} |x|^{|\alpha|} \cdot |\partial^\alpha \tau(x)| < \infty.$$

Then there exists c, depending on $c_\alpha, \alpha \in \mathbb{N}$ and on the A_∞-constant, such that

$$\|\tau(D)f : H^p(\omega)\| \leq c\|f : H^p(\omega)\|.$$

With this proposition in mind, we show that the Fourier multiplier operators are bounded on $H\mathcal{M}_q^p$.

Proposition (5.1.26) [219]: Let $0 < q \leq p < \infty$ and $\tau \in \mathcal{D}$. Assume

$$c_\alpha := \sup_{x \in \mathbb{R}^n} |x|^{|\alpha|} \cdot |\partial^\alpha \tau(x)| < \infty.$$

Then there exists a constant c depending only on $c_\alpha, \alpha \in \mathbb{N}_0{}^n$ such that

$$\|\tau(D)f : H\mathcal{M}_q^p\| \leq c\|f : H\mathcal{M}_q^p\|$$

for all $f \in H\mathcal{M}_q^p$.

Proof: We adopt the following norm in view of Theorem (5.1.24): For $f \in \mathcal{S}'$, we define

$$\|f : H\mathcal{M}_q^p\| = \left\| \sup_{j \in \mathbb{Z}} |\psi_j(D)f| : \mathcal{M}_q^p \right\|,$$

where ψ is a bump function satisfying $\chi_{B(1)} \leq \psi \leq \chi_{B(2)}$.

Since $\left(M\chi_Q \right)^{1-\frac{q}{2p}}$ is an A_1-weight, we obtain

$$|Q|^{\frac{q}{p}-1} \int_Q \left(\sup_{j \in \mathbb{Z}} |\psi_j(D)\tau(D)f|^q \right) \leq |Q|^{\frac{q}{p}-1} \left\| \tau(D)f : H^q \left(\left(M\chi_Q \right)^{1-\frac{q}{2p}} \right) \right\|^q$$

$$\leq c|Q|^{\frac{q}{p}-1} \left\| f : H^q \left(\left(M\chi_Q \right)^{1-\frac{q}{2p}} \right) \right\|^q,$$

where for the last inequality we have used Proposition (5.1.25). $2^k Q$ denotes the cube concentric to Q with sidelength $2^k |Q|^{\frac{1}{n}}$. Now we use the two-sided estimate

$$M\chi_Q(x) \simeq \sum_{k=0}^{\infty} \frac{\chi_{2^k Q}(x)}{2^{kn}},$$

which yields

$$|Q|^{\frac{q}{p}-1}\int_Q\left(\sup_{j\in\mathbb{Z}}|\psi_j(D)\tau(D)f|^q\right)\le c|Q|^{\frac{q}{p}-1}\sum_{k=0}^\infty 2^{k\left(\frac{q}{2p}-1\right)}\int_{2^kQ}\left(\sup_{j\in\mathbb{Z}}|\psi_j(D)f|^q\right)$$

$$\le c\sum_{k=0}^\infty 2^{-\frac{q\,n}{2p}}\cdot|2^kQ|^{\frac{q}{p}-1}\int_{2^kQ}\left(\sup_{j\in\mathbb{Z}}|\psi_j(D)f|^q\right)\le c\|f:H\mathcal{M}_q^p\|^q.$$

The cubes being chosen arbitrarily, we obtain the desired estimate.

Now we turn to Local Hardy-Morrey spaces.

Lemma (5.1.27) [219]: Let $0<q\le p<\infty$. The definition of $h\mathcal{M}_q^p$ does not depend on the admissible choices of $\psi\in\mathcal{S}$ with $\int\psi\ne 0$.

Proof: The proof is analogous to homogeneous Hardy spaces (just mimic the proof of Theorem (5.1.24)).

Local Hardy spaces and Hardy spaces are related as follows :

Proposition (5.1.28) [219]: Let $0<q\le p<\infty$. Assume in addition that $\mathcal{F}_\psi\equiv 1$ on a neighborhood of 0. Then we have
$$\|f:h\mathcal{M}_q^p\|\simeq\|\psi(D)f:\mathcal{M}_q^p\|+\|(1-\psi(D))f:H\mathcal{M}_q^p\|.$$

Proof: Following the definition we calculate
$$\|f:h\mathcal{M}_q^p\|\simeq\left\|\sup_{j\in\mathbb{N}_0}|\psi_j(D)f|:\mathcal{M}_q^p\right\|$$
$$\simeq\left\|\sup_{j\in\mathbb{N}_0}|\psi_j(D)\psi(D)f|:\mathcal{M}_q^p\right\|+\left\|\sup_{j\in\mathbb{N}_0}|\psi_j(D)(1-\psi(D))f|:\mathcal{M}_q^p\right\|$$
$$\simeq\|\psi(D)f:\mathcal{M}_q^p\|+\|(1-\psi(D))f:H\mathcal{M}_q^p\|$$
where we have used Proposition (5.1.19) for the second and the third equivalences.

Definition (5.1.29) [219] (Vector-valued Spaces)**:** Let $E\subset\mathbb{Z}$. We define $M(\mathbb{C},E)$ as the set of all $\mathbb{C}$-valued $E\times E$-matrices for a set E. Now we shall consider the vector-valued space. Let us denote by $\widetilde{\mathcal{S}(E)}$ the set of all $M(\mathbb{C},E)$-valued functions Φ of the form
$$\Phi(x)=\left\{\Phi^{ee'}(x)\right\}_{e,e'\in E'}$$
where $\Phi^{ee'}\in\mathcal{S}$ and $\Phi^{ee'}\equiv 0$ with finite exception. In this case we denote $\Phi=\left\{\Phi^{ee'}\right\}_{e,e'\in\mathbb{N}_0}$.

Recall we defined $\tau_j(x)=\tau(2^{-j}x)$ for $j\in\mathbb{Z}$ and $\tau\in\mathcal{S}$. Given $\Phi=\left\{\Phi^{ee'}\right\}_{e,e'\in E}\in\widetilde{\mathcal{S}(E)}$ and $j\in\mathbb{Z}$, we define $\Phi_j:=\left\{\left(\Phi^{ee'}\right)_j\right\}_{e,e'\in E}\in\widetilde{\mathcal{S}(E)}$. If we are given $\Phi=\left\{\Phi^{ee'}\right\}_{e,e'\in E}\in\widetilde{\mathcal{S}(E)}$ and $F=\{f_e\}_{e\in E}$, then we define
$$\Phi*F:=\left\{\sum_{e'\in E}\Phi^{ee'}*f_{e'}\right\}_{e\in E}.$$

Denote by $H\mathcal{M}_q^p(l_2,E)$ the set of all sequences of distributions $F=\{f_e\}_{e\in E}$ such that
$$\|F:H\mathcal{M}_q^p(l_2(E))\|:=\left\|\sup_{j\in\mathbb{Z}}\|\Phi_j*F:l_2(E)\|:\mathcal{M}_q^p\right\|<\infty.$$

For $\Phi\in\widetilde{\mathcal{S}(E)}$, we define
$$p_N(\Phi):=\sup_{x\in\mathbb{R}^n}\left(\sum_{|\alpha|,|\beta|\le N}\left\|\{x^\alpha\partial^\beta\Phi^{ee'}(x)\}_{e,e'\in E}:B(l_2(E))\right\|\right).$$

Denote by $h\mathcal{M}_q^p(l_2(E))$ the set of all sequences of distributions $F=\{f_e\}_{e\in E}$ for which the quasi-norm

$$\|F : h\mathcal{M}_q^p(l_2(E))\| := \left\|\sup_{j\in\mathbb{N}}\left\|\{|\psi_j * f_e|\}_{e\in E} : l_2(E)\right\| : \mathcal{M}_q^p\right\| < \infty.$$

It is the same as the scalar case where the following theorems hold.

Theorem (5.1.30) [219]: Let $E \subset \mathbb{Z}$. Let $0 < q \le p < \infty$ and $\Phi \in \widetilde{\mathcal{S}(E)}_0$. Then there exists c such that

$$\left\|\sup_{\Phi\in B_{p_N}(1),j\in\mathbb{N}}\left\|\Phi_j * F(\cdot) : l_2(E)\right\| : \mathcal{M}_q^p\right\| \le c\left\|\sup_{j\in\mathbb{N}}\left\|\{|\psi_j * f_k|\}_{k\in E} : l_2(E) : \mathcal{M}_q^p\right\|\right\|$$

for all $F = \{f_e\}_{e\in E} \in \mathcal{S}'^E = \{\{f_e\}_{e\in E} : f_e \in \mathcal{S}', e \in E\}$.

Theorem (5.1.31) [219]: Let $\psi \in \widetilde{\mathcal{S}(E)}$. Set

$$c_\alpha := \sup_{x\in\mathbb{R}^n}|x|^{|\alpha|}\|\partial^\alpha\Psi(x) : B(l_2)\|.$$

Then there exists $c > 0$ depending only on $c_\alpha, \alpha \in \mathbb{N}_0{}^n$ such that

$$\|\Psi * F : H\mathcal{M}_q^p(l_2(E))\| \le c\|F : H\mathcal{M}_q^p(l_2(E))\|.$$

Having defined the nonhomogeneous Hardy spaces, we are now in a position of proving an equivalence theorem (Theorem (5.1.32)). Now assume that $\psi \in C_c^\infty$ is a bump function that equals 1 on a neighborhood of 0. Let us set $\varphi(x) = \psi(2^{-1}x) - \psi(x)$ for $x \in \mathbb{R}^n$.

Proposition (5.1.32) [219]: Let $f \in \mathcal{S}'$ satisfy $\|f : H\mathcal{M}_q^p\| < \infty$. Then we have $f = \sum_{j\in\mathbb{Z}}\varphi_j(D)f$ in the topology of $\mathcal{S}'$.

Proof: It is easy to see that $f = c_\psi \lim_{j\to\infty}\psi_j(D)f$ for some $c_\psi > 0$. Therefore, we can assume that f itself is a band-limited distribution. As a result there exists $j_0 \in \mathbb{N}$ such that $f = \psi_{j_0}(D)f$. From this and the assumption that $f \in H\mathcal{M}_q^p$ we deduce that $f \in \mathcal{M}_q^p$. In [207] we have shown that

$$\|\varphi_j(D)f : L^\infty\| \le c2^{\frac{jn}{p}}\cdot\|\psi_j(D)f : \mathcal{M}_q^p\| \le c2^{\frac{jn}{p}}\cdot\|\psi_{j_0}(D)f : \mathcal{M}_q^p\|.$$

Therefore, we conclude that

$$g := \lim_{j\to-\infty}\sum_{k=j}^\infty\varphi_k(D)f$$

converges in L^∞. However, $\mathcal{F}(f - g)$ is supported on the origin and hence $f - g$ is a polynomial. Furthermore, $\{f - \sum_{k=j}^\infty\varphi_k(D)f\}_{j\in\mathbb{Z}}$ is a uniformly bounded set in $\mathcal{M}_q^p$. As a result we have $f = g$, which is the desired result.

Theorem (5.1.33) [219]: Let $0 < q \le p < \infty$. Then we have

$$\|f : H\mathcal{M}_q^p\| \simeq \|f : \dot{\mathcal{E}}_{pq2}^0\|,$$
$$\|f : h\mathcal{M}_q^p\| \simeq \|f : \dot{\mathcal{E}}_{pq2}^0\|$$

for all $f \in \mathcal{S}'$.

Proof: see Proposition (5.1.23) for $H\mathcal{M}_q^p$.

Let $f \in \mathcal{S}'$. Then we have

$$\|f : h\mathcal{M}_q^p\| \simeq \|\psi(D)f : h\mathcal{M}_q^p\| + \|(1 - \psi(D))f : h\mathcal{M}_q^p\|$$
$$\simeq \|\psi(D)f : \mathcal{M}_q^p\| + \|(1 - \psi(D))f : H\mathcal{M}_q^p\|$$
$$\simeq \|\psi(D)f : \mathcal{M}_q^p\| + \|\varphi_j(D)f : \mathcal{M}_q^p(l_2, \mathbb{N})\| = \|f : \mathcal{E}_{pq2}^0\|,$$

Let $\tau \in \mathcal{D}$ be a function such that $\text{supp}(\tau) \ni 0$ and $\tau \equiv 1$ on a neighborhood of the origin. We shall consider the following operators:

$$T_1, T_2 : H\mathcal{M}_q^p(l_2, \mathbb{Z}) \to H\mathcal{M}_q^p(l_2, \mathbb{Z}),$$

which are given by

$$T_1\left(\{f_j\}_{j\in\mathbb{Z}}\right) = \left\{\delta_{j0}\sum_{k\in\mathbb{Z}}\tau_k(D)f_k\right\}_{j\in\mathbb{Z}},$$

$$T_2\left(\{f_j\}_{j\in\mathbb{Z}}\right) = \left\{\tau_j(D)f_0\right\}_{j\in\mathbb{Z}}.$$

By Theorem (5.1.31) and a simple limiting argument we have

$$\left\|\sum_{k\in\mathbb{Z}}\tau_k(D)f_k : \mathcal{M}_q^p(l_2,\mathbb{Z})\right\| \leq c\|\{f_k\}_{k\in\mathbb{Z}} : \mathcal{M}_q^p(l_2)\|,$$

$$\left\|\tau_j(D)f_0 : \mathcal{M}_q^p(l_2,\mathbb{Z})\right\| \leq c\|f_0 : \mathcal{M}_q^p\|$$

for all $\{f_k\}_{k\in\mathbb{Z}} \in \mathcal{M}_q^p(l_2)$ such that $f_k = 0$ for all sufficiently large k. By virtue of Proposition (5.1.32), we have

$$\|f : H\mathcal{M}_q^p\| = \left\|\sum_{j\in\mathbb{Z}}\varphi_j(D)f : H\mathcal{M}_q^p\right\| = \left\|\sum_{j\in\mathbb{Z}}\tau_j(D)\left[\varphi_j(D)f\right] : H\mathcal{M}_q^p\right\|$$

$$\simeq \left\|\varphi_j(D)f : H\mathcal{M}_q^p(l_2,\mathbb{Z})\right\|.$$

It is not so hard by using Proposition (5.1.19) to show that

$$\left\|\varphi_j(D)f : H\mathcal{M}_q^p(l_2,\mathbb{Z})\right\| \simeq \left\|f : \dot{\mathcal{E}}_{pq2}^0\right\|.$$

The proof is therefore complete.

Corollary (5.1.34) [299]: Let $K \in \mathbb{N}_0$ and $0 \leq \epsilon \leq \infty$. Suppose that A is a C^K-function with compact support. Then $a \in \mathcal{N}_{(1+2\epsilon)(1+\epsilon)\infty}^K$. In particular if $K > s$, then we have $a \in \mathcal{A}_{(1+2\epsilon)(1+\epsilon)^2}^s$.

Proof: By virtue of the equivalent norm

$$\left\|a : \mathcal{N}_{(1+2\epsilon)(1+\epsilon)\infty}^K\right\| \simeq \left\|a : \mathcal{N}_{(1+2\epsilon)(1+\epsilon)\infty}^0\right\| + \sum_{j=1}^n\left\|\partial_j^K a : \mathcal{N}_{(1+2\epsilon)(1+\epsilon)\infty}^0\right\|,$$

(see [207]) we can assume $K = 0$. Since the family $\{k^j * a\}_{j\in\mathbb{N}}$ is supported on a fixed compact set, owing to the fact that

$$\mathcal{M}_{1+\epsilon_0}^{1+2\epsilon_0} \cap \{f^2 : \text{supp}(f^2) \subset B(R)\} \supset \mathcal{M}_{1+\epsilon_1}^{1+2\epsilon_1} \cap \{f^2 : \text{supp}(f^2) \subset B(R)\}$$

for all $0 \leq \epsilon_0 \leq \epsilon_1 \leq \infty$ and $R > 0$ we see, by using the local means, the matter is reduced to the case when $0 \leq \epsilon \leq \infty$. In this case

$$\left\|a : \mathcal{N}_{(1+2\epsilon)(1+\epsilon)\infty}^K\right\| = \left\|a : \mathcal{N}_{(1+2\epsilon)(1+\epsilon)\infty}^0\right\| \leq (1+\epsilon)\left\|a : \mathcal{E}_{(1+2\epsilon)(1+\epsilon)\infty}^0\right\|$$

$$\leq (1+\epsilon)\left\|a : \mathcal{M}_{(1+\epsilon)}^{(1+2\epsilon)}\right\| < \infty$$

and the proof of the first statement is now complete. The second assertion follows from the embedding $\mathcal{A}_{(1+2\epsilon)(1+\epsilon)^2}^s \supset \mathcal{N}_{(1+2\epsilon)(1+\epsilon)\infty}^K$, which follows from Proposition (5.1.2).

Corollary (5.1.35) [299]: Let $0 \leq \epsilon \leq \infty$ and $s \in \mathbb{R}$. Then there exists a constant $\epsilon > -1$ such that

$$(1+\epsilon)^{-1}\left\|f^2 : \mathcal{A}_{(1+2\epsilon)(1+\epsilon)^2}^s\right\| \leq \left\|f^2 : \mathcal{A}_{(1+2\epsilon)(1+\epsilon)^2}^s\right\|_* \leq (1+\epsilon)\left\|f^2 : \mathcal{A}_{(1+2\epsilon)(1+\epsilon)^2}^s\right\|$$

for all $f^2 \in \mathcal{S}'$.

Proof: Now we refer back to the proof of Theorem (5.1.14). From Theorem (5.1.9) it suffices to show the right inequality. Pick $\zeta \in \mathcal{D}$ so that

$$\zeta \cdot \mathcal{F}l + \sum_{m=0}^\infty (1+\epsilon)(2^{-m}*)\cdot\mathcal{F}k^m \equiv 1, \quad 0 \notin \text{supp}(1+\epsilon).$$

Note that $\mathcal{F}^{-1}(1+\epsilon)$ has vanishing moment up to order L, since $0 \notin \text{supp}(1+\epsilon)$. From this formula, we deduce

$$2^{js}\left[\mathcal{F}^{-1}\varphi_j\right] * f^2 = (2\pi)^{\frac{n}{2}}2^{js}\left[\mathcal{F}^{-1}\varphi_j\right] * l^j * \zeta^j * f^2 + (2\pi)^{\frac{n}{2}}\sum_{m=j+1}^{\infty}2^{js}\left[\mathcal{F}^{-1}\varphi_j\right] * (\mathcal{F}\eta)^j * k^j * f^2.$$

Observe that

$$\left|\left[\mathcal{F}^{-1}\varphi_j\right] * \left(\mathcal{F}(1+\epsilon)\right)^j(x_n)\right| \leq c2^{jn+L(j-m)}\langle 2^j x_n\rangle^{-(1+\epsilon)},$$

where $(1+\epsilon)$ satisfies the same condition as Corollary (5.1.13). Therefore, it follows that

$$\left|2^{js}\left[\mathcal{F}^{-1}\varphi_j\right] * \left(\mathcal{F}(1+\epsilon)\right)^m * k^m * f^2(x_n)\right|$$

$$\leq (1+\epsilon)2^{js}\int_{\mathbb{R}^n}\left|\left[\mathcal{F}^{-1}\varphi_j\right] * \left(\mathcal{F}(1+\epsilon)\right)^m(x_n)\right|\cdot|k^m * f^2(\epsilon)|dx_n$$

$$\leq \left(\sup_{x_n\in\mathbb{R}^n}\langle 2^m(x_n+2\epsilon)\rangle^{-(1+\epsilon)}|k^m * f^2(2\epsilon)|\right)$$

$$\cdot\left(2^{js}\int_{\mathbb{R}^n}\left|\left[\mathcal{F}^{-1}\varphi_j\right] * \left(\mathcal{F}(1+\epsilon)\right)^m(x_n+\epsilon)\right|\cdot\langle 2^m(x_n+\epsilon)\rangle^{(1+\epsilon)}dx_n\right)$$

$$\leq (1+\epsilon)2^{-(1+\epsilon+s-L)(m-j)}x_n\langle 2^m(x_n+2\epsilon)\rangle^{-(1+\epsilon)}|2^{sm}k^m * f^2(2\epsilon)|,$$

if $m \geq j$. Therefore, if we let $L > 1 + \epsilon + s$, then this inequality is summable. Hence, we obtain the desired result by Corollary (5.1.13).

Corollary (5.1.36) [299]: Let $0 \leq \epsilon \leq \infty$, $m \in \mathbb{R}$ and $a \in S_{1,1}^m$.

(i) If $s > \sigma_{1+\epsilon}$, then there exists $\epsilon > -1$ such that

$$\left\|a(x_n,D)f^2 : \mathcal{N}_{(1+2\epsilon)(1+\epsilon)^2}^s\right\| \leq (1+\epsilon)\left\|f^2 : \mathcal{N}_{(1+2\epsilon)(1+\epsilon)^2}^{s+m}\right\|.$$

(ii) If $s > \sigma_{(1+\epsilon)^2}$, then there exists $\epsilon > -1$ such that

$$\left\|a(x_n,D)f^2 : \mathcal{E}_{(1+2\epsilon)(1+\epsilon)^2}^s\right\| \leq (1+\epsilon)\left\|f^2 : \mathcal{E}_{(1+2\epsilon)(1+\epsilon)^2}^{s+m}\right\|.$$

Proof: Let us pick auxiliary functions $\psi, \varphi, \kappa \in \mathcal{S}$ with the following conditions.

(i) $\chi_{Q(1)} \leq \psi \leq \chi_{Q(2)}$;

(ii) $\varphi(x_n) = \psi(x_n) - \psi(2x_n)$;

(iii) $\chi_{Q(2)} \leq \kappa \leq \chi_{Q(3)}$.

We consider $a(x_n,D)\left(1 - \psi(D)\right)f^2$ because $a(x_n,D)\psi(D)f^2$ can be dealt with in a similar way.

Let us consider $(1-\psi)\cdot\mathcal{F}f^2 = \sum_{j=1}^{\infty}\varphi_j\cdot\mathcal{F}f^2$. Expand $\varphi_j\cdot\mathcal{F}f^2$ into a Fourier series

$$\varphi_j\cdot\mathcal{F}f^2 = \sum_{m\in\mathbb{Z}^n}\frac{\varphi_j(D)f^2(2^{-j}m)}{(2\pi)^{\frac{n}{2}}\cdot 2^{jn}}\kappa_j\exp(-i2^{-j}m\cdot *),$$

where $\kappa_j(x_n) = \kappa(2^{-j}x_n)$ for $j \in \mathbb{N}$. From this we have

$$a(x_n,D)\left[\left(1 - \psi(D)\right)f^2\right](x_n)$$

$$= \sum_{j=1}^{\infty}\sum_{m\in\mathbb{Z}^n}\frac{\varphi_j(D)f^2(2^{-j}m)}{(2\pi\cdot 2^j)^n}\int_{\mathbb{R}^n}a(x_n,\xi)\kappa_j(\xi)\exp(i(x_n - 2^{-j}m)\cdot\xi)\,d\xi.$$

The Leibniz rule gives us that

$$\partial_{x_n}^{\alpha}\int_{\mathbb{R}^n}a(x_n,\xi)\kappa_j(\xi)\exp(i(x_n - 2^{-j}m)\cdot\xi)\,d\xi$$

$$= \sum_{\beta\leq\alpha}(1+\epsilon)_{\alpha\beta}\int_{\mathbb{R}^n}\left[\partial_{x_n}^{\beta}a(x_n,\xi)\right]\kappa_j(\xi)(i\xi)^{\alpha-\beta}\exp(i(x_n - 2^{-j}m)\cdot\xi)\,d\xi$$

and κ does not contain 0 as its support, where $(1+\epsilon)_{\alpha\beta}$ is the binomial coefficient. If we carry out integration by parts, then we obtain

$$2^{-j(s+m)}\int_{\mathbb{R}^n}a(x_n,\xi)\kappa_j(\xi)\exp(i(x_n - 2^{-j}m)\cdot\xi)\,d\xi$$

is a molecule in $\mathcal{A}^s_{(1+2\epsilon)(1+\epsilon)^2}$. Together with Lemma (5.1.8) we obtain the desired result.

Corollary (5.1.37) [299]: Suppose that the parameters ϵ, s, l satisfy

$$0 < s < l, \qquad 0 \le \epsilon \le \infty$$

and that $a \in C^*_l S^0_{1,(1-\epsilon)}$. Then there exists $\epsilon > -1$ such that

$$\left\| a(x_n, D)f^2 : \mathcal{A}^s_{(1+2\epsilon)(1+\epsilon)^2} \right\| \le (1+\epsilon) \left\| f^2 : \mathcal{A}^{s+m}_{(1+2\epsilon)(1+\epsilon)^2} \right\|.$$

Proof: Let us concentrate on the case when $\mathcal{A} = \mathcal{E}$, the case when $\mathcal{A} = \mathcal{N}$ is simpler.

Let $f^2 \in \mathcal{E}^s_{(1+2\epsilon)(1+\epsilon)^2}$ and a be an elementary form as in [215]. Any symbol in $C^*_l S^0_{1,(1-\epsilon)}$ can be approximated by elementary symbols. Therefore, it is sufficient to investigate the case when a is an elementary form:

$$a(x_n, \xi) = \sum_{j=1}^{\infty} \sigma_j(x_n) \varphi_j(\xi).$$

Define $q_{jk} = \varphi_k(D)\sigma_j$. Then we have

$$\left\| q_{jk} \right\|_{\infty} \le (1+\epsilon) 2^{(j-k)l}. \tag{9}$$

As a consequence, we obtain $q(x_n, D)f^2(x_n) = \sum_{j \in \mathbb{N}, k \in \mathbb{N}_0} q_{jk}(x_n) \cdot \varphi_j(D)f^2(x_n)$ .We decompose $q(x_n, D)f^2$ into three pieces. Let us set

$$q_1(x_n, D)f^2 := \sum_{j=4}^{\infty} \left(\sum_{k=0}^{j-4} q_{jk} \right) \varphi_j(D)f^2,$$

$$q_2(x_n, D)f^2 := \sum_{j=0}^{\infty} \left(\sum_{k=\max(j-3,0)}^{j+3} q_{jk} \right) \varphi_j(D)f^2,$$

$$q_3(x_n, D)f^2 := \sum_{j=0}^{\infty} \left(\sum_{k=j+4}^{\infty} q_{jk} \right) \varphi_j(D)f^2.$$

The estimate of $q_2(x_n, D)f^2$ is simple. Choose an auxiliary function $\kappa \in \mathcal{S}$ so that $\chi_{Q(3)} \le \kappa \le \chi_{Q(\pi)}$. Then we have

$$\left\| \partial^\alpha \left(\sum_{k=\max(j-3,0)}^{j+3} q_{jk} \right) \right\|_{\infty} \le (1+\epsilon) 2^{j|\alpha|} \left\| \left(\sum_{k=\max(j-3,0)}^{j+3} q_{jk} \right) \right\|_{\infty} \le (1+\epsilon) 2^{j|\alpha|}.$$

Here we have used Proposition (5.1.19) for the first inequality and for the second inequality we have used (9) and the fact that at most 7 terms are involved. Therefore,

$$q_2(x_n, D)f^2 := \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \varphi_j(D)f^2(2^{-j}m) \left(\sum_{k=\max(j-3,0)}^{j+3} q_{jk} \right) \mathcal{F}^{-1}\kappa(2^j * -m)$$

can be regarded as a molecular decomposition and hence we conclude that $q_2(x_n, D)$ is bounded from $\mathcal{A}^s_{(1+2\epsilon)(1+\epsilon)^2}$ to itself.

The first piece is treated in a spirit similar to [41]. We shall make use of the fact that $\operatorname{supp}(f^2 * g^2) \subset \operatorname{supp}(f^2) + \operatorname{supp}(g^2)$ for all compactly supported distributions $f^2, g^2 \in \mathcal{S}'$, where the right-hand side denotes the algebraic sum. Hence it follows that

$$\left\| q_1(x_n, D)f^2 : \mathcal{E}^s_{(1+2\epsilon)(1+\epsilon)^2} \right\| \leq (1+\epsilon) \left\| 2^{js} \left(\sum_{k=0}^{\max(j-4,0)} q_{jk} \right) \varphi_j(D)f^2 : \mathcal{M}^{(1+2\epsilon)}_{(1+\epsilon)}(l_{1+\epsilon}) \right\|$$

$$\leq (1+\epsilon) \left(\sup_{j\in\mathbb{N}_0} \left\| \sum_{k=0}^{\max(j-4,0)} q_{jk} \right\|_\infty \right) \left\| 2^{js}\varphi_j(D)f^2 : \mathcal{M}^{(1+2\epsilon)}_{(1+\epsilon)}(l_{1+\epsilon}) \right\|$$

$$\leq (1+\epsilon) \left(\sup_{j\in\mathbb{N}_0} \left(\sum_{k=0}^{\max(j-4,0)} \|q_{jk}\|_\infty \right) \right) \left\| 2^{js}\varphi_j(D)f^2 : \mathcal{M}^{(1+2\epsilon)}_{(1+\epsilon)}(l_{1+\epsilon}) \right\|$$

$$\leq (1+\epsilon) \left(\sup_{j\in\mathbb{N}_0} \|\sigma_j\|_\infty \right) \left\| 2^{js}\varphi_j(D)f^2 : \mathcal{M}^{(1+2\epsilon)}_{(1+\epsilon)}(l_{1+\epsilon}) \right\|$$

$$\leq (1+\epsilon) \left\| f^2 : \mathcal{E}^s_{(1+2\epsilon)(1+\epsilon)^2} \right\|.$$

Finally let us turn to the estimate of $q_3(x_n, D)$. Let us rewrite

$$q_3(x_n, D)f^2(x_n) := \sum_{k=4}^{\infty} \left(\sum_{j=0}^{k-4} q_{jk}(x_n) \right) \varphi_j(D)f^2(x_n),$$

where the change of order of the summation will be justified below. As a result, we obtain

$$\left\| q_3(x_n, D)f^2 : \mathcal{E}^s_{(1+2\epsilon)(1+\epsilon)^2} \right\| \leq (1+\epsilon) \left\| \left\{ 2^{ks} \sum_{j=0}^{k-4} q_{jk} \cdot \varphi_j(D)f^2 \right\}_{k=4}^{\infty} : \mathcal{M}^{1+2\epsilon}_{1+\epsilon}(l_{1+\epsilon}) \right\|$$

$$\leq (1+\epsilon) \left\| \left\{ \sum_{j=0}^{k-4} 2^{(k-j)(s-l)} \cdot 2^{js} |\varphi_j(D)f^2| \right\}_{k=4}^{\infty} : \mathcal{M}^{1+2\epsilon}_{1+\epsilon}(l_{1+\epsilon}) \right\|$$

$$\leq (1+\epsilon) \left\| f^2 : \mathcal{E}^s_{(1+2\epsilon)(1+\epsilon)^2} \right\|,$$

where we have used the fact that $s < l$ for the second inequality. This is the desired result.

Corollary (5.1.38) [299]: Let $0 \leq \epsilon \leq \infty$. Then the sets $H\mathcal{M}^{1+2\epsilon}_{1+\epsilon}$ and $h\mathcal{M}^{1+2\epsilon}_{1+\epsilon}$ are independent of the choices of admissible ψ satisfying (7).

Proof: For $H\mathcal{M}^{1+2\epsilon}_{1+\epsilon}$ Corollary (5.1.39) asserts more than Corollary (5.1.38). To formulate the stronger result, for $N \in \mathbb{N}$ we set

$$p_N(\zeta) := \sum_{|\alpha| \leq N} \sup_{x_n \in \mathbb{R}^n} \langle x_n \rangle^N |\partial^\alpha \zeta(x_n)|, \qquad \zeta \in \mathcal{S}$$

Corollary (5.1.39) [299]: Let $0 \leq \epsilon \leq \infty$. Assume that $\psi \in \mathcal{S}$ satisfies the non-degenerate condition $\int \psi \neq 0$. Then there exist $N \in \mathbb{N}$ and $\epsilon > -1$ such that

$$\left\| \sup_{\zeta \in B_{p_N}(1)} \left(\sup_{j\in\mathbb{Z}} |\zeta^j * f^2| \right) : \mathcal{M}^{1+2\epsilon}_{1+\epsilon} \right\| \leq (1+\epsilon) \left\| \sup_{j\in\mathbb{Z}} |\psi^j * f^2| : \mathcal{M}^{1+2\epsilon}_{1+\epsilon} \right\|,$$

where $B_{p_N}(1) := \{\zeta \in \mathcal{S} : p_N(\zeta) < 1\}$.

Proof: Fix $\zeta \in B_{p_N}(1)$ and $j \in \mathbb{Z}$. Then there exist $\widetilde{\psi}, \widetilde{\varphi} \in \mathcal{S}$ such that $\widetilde{\varphi}$ has vanishing moment up to order L with L large enough and that

$$\widetilde{\psi} * \psi + \sum_{l=0}^{\infty} \widetilde{\varphi}^l * \varphi^l = 1 - \epsilon$$

by virtue of Lemma (5.1.22). Using this formula, we obtain

$$|\zeta^j * f^2(x_n)| \leq |\zeta^j * \widetilde{\psi}^j * \psi^j * f^2(x_n)| + \sum_{l=0}^{\infty} |\zeta^j * \widetilde{\varphi}^{j+l} * \varphi^{j+l} * f^2(x_n)|.$$

Let us set

$$\varphi^{j+l,*}f^2(x_n) := \sup_{x_n \in \mathbb{R}^n} \langle 2^{j+l}(x_n + \epsilon)\rangle^{-\frac{n}{1+\epsilon}} \left|\varphi^{j+l} * f^2(\epsilon)\right|,$$

$$\psi^{j+l,*}f^2(x_n) := \sup_{x_n \in \mathbb{R}^n} \langle 2^{j+l}(x_n + \epsilon)\rangle^{-\frac{n}{1+\epsilon}} \left|\psi^{j+l} * f^2(\epsilon)\right|,$$

for $j, l \in \mathbb{N}_0$ and $0 \leq \epsilon \ll 1$. Then we have

$$\left|\zeta^j * \widetilde{\varphi}^{j+l} * \varphi^{j+l} * f^2(x_n)\right| \leq \int_{\mathbb{R}^n} \left|\zeta^j * \widetilde{\varphi}^{j+l}(x_n + \epsilon)\right| \cdot \left|\varphi^{j+l} * f^2(\epsilon)\right| dx_n$$

$$\leq \varphi^{j+l,*}f^2(x_n) \int_{\mathbb{R}^n} \left|\zeta^j * \widetilde{\varphi}^{j+l}(x_n + \epsilon)\right| \cdot \langle 2^{j+l}(x_n + \epsilon)\rangle^{\frac{n}{1+\epsilon}} dx_n$$

$$= \varphi^{j+l,*}f^2(x_n) \int_{\mathbb{R}^n} \left|\zeta * \widetilde{\varphi}^l(x_n + \epsilon)\right| \cdot \langle 2^l(x_n + \epsilon)\rangle^{\frac{n}{1+\epsilon}} dx_n.$$

we have $|\mathcal{F}\widetilde{\varphi}(x_n)| \leq (1 + \epsilon)|x_n|^L$. Therefore, it follows that

$$\left|\zeta * \widetilde{\varphi}^l(x_n + \epsilon)\right| \leq (1 + \epsilon)2^{-Ll}\langle x_n + \epsilon\rangle^{-n\left(\frac{2+\epsilon}{1+\epsilon}\right)-1}.$$

Inserting this estimate, we obtain

$$\left|\zeta^j * \widetilde{\varphi}^{j+l} * \varphi^{j+l} * f^2(x_n)\right| \leq (1 + \epsilon)2^{-\left(L-\frac{n}{1+\epsilon}\right)l}\varphi^{j+l,*}f^2(x_n).$$

Since $\varphi = \psi^1 - \psi$, it follows that

$$\left|\zeta^j * \widetilde{\varphi}^{j+l} * \varphi^{j+l} * f^2(x_n)\right| \leq (1 + \epsilon)2^{-\left(L-\frac{n}{1+\epsilon}\right)l}\left(\psi^{j+l+1,*}f^2(x_n) + \psi^{j+l,*}f^2(x_n)\right).$$

A similar estimate is valid for $\left|\zeta^j * \widetilde{\psi}^j * \psi^j * f^2(x_n)\right|$. As a result, adding these estimates over $l \in \mathbb{N}_0$, we have

$$\left|\zeta^j * f^2(x_n)\right| \leq (1 + \epsilon) \sup_{k \in \mathbb{Z}} \psi^{k,*}f^2(x_n).$$

From this formula, we obtain

$$\left\|\sup_{\zeta \in B_{p_N}(1)}\left(\sup_{j \in \mathbb{Z}}|\zeta^j * f^2|\right) : \mathcal{M}^{1+2\epsilon}_{1+\epsilon}\right\| \leq (1 + \epsilon)\left\|\sup_{j \in \mathbb{Z}} \psi^{j,*}f^2 : \mathcal{M}^{1+2\epsilon}_{1+\epsilon}\right\|.$$

Since we can deduce

$$\psi^{j,*}f^2(x_n)^{1+\epsilon} \leq (1 + \epsilon)\sum_{l=0}^{\infty} 2^{-l(1-\epsilon)}M\left[\left|\psi^{j+l,*}f^2(x_n)\right|^{1+\epsilon}\right](x_n), \qquad \epsilon < 1$$

in the same way as before, another application of the Hardy-Littlewood maximal inequality for the Morrey spaces gives us

$$\left\|\sup_{j \in \mathbb{Z}} \psi^{j,*}f^2 : \mathcal{M}^{1+2\epsilon}_{1+\epsilon}\right\| \leq (1 + \epsilon)\left\|\sup_{j \in \mathbb{Z}}|\psi^j * f^2| : \mathcal{M}^{1+2\epsilon}_{1+\epsilon}\right\|.$$

Putting together these observations, we obtain the desired result.

Corollary (5.1.40) [299]: Let $0 \leq \epsilon \leq \infty$ and $\tau \in \mathcal{D}$. Assume

$$(1 + \epsilon)_\alpha := \sup_{x_n \in \mathbb{R}^n} |x_n|^{|\alpha|} \cdot |\partial^\alpha \tau(x_n)| < \infty.$$

Then there exists a constant $(1 + \epsilon)$ depending only on $(1 + \epsilon)_\alpha, \alpha \in \mathbb{N}_0{}^n$ such that

$$\|\tau(D)f^2 : H\mathcal{M}^{1+2\epsilon}_{1+\epsilon}\| \leq (1 + \epsilon)\|f^2 : H\mathcal{M}^{1+2\epsilon}_{1+\epsilon}\|$$

for all $f^2 \in H\mathcal{M}^{1+2\epsilon}_{1+\epsilon}$.

Proof: We adopt the following norm in view of Theorem (5.1.24): For $f^2 \in \mathcal{S}'$, we define

$$\|f^2 : H\mathcal{M}^{1+2\epsilon}_{1+\epsilon}\| = \left\|\sup_{j \in \mathbb{Z}}|\psi_j(D)f^2| : \mathcal{M}^{1+2\epsilon}_{1+\epsilon}\right\|,$$

where ψ is a bump function satisfying $\chi_{B(1)} \leq \psi \leq \chi_{B(2)}$.

Since $(M\chi_Q)^{\frac{1+3\epsilon}{2(1+2\epsilon)}}$ is an A_1-weight, we obtain

$$|Q|^{\frac{-\epsilon}{1+2\epsilon}} \int_Q \left(\sup_{j \in \mathbb{Z}} \left| \psi_j(D)\tau(D)f^2 \right|^{1+\epsilon} \right) \leq |Q|^{\frac{-\epsilon}{1+2\epsilon}} \left\| \tau(D)f^2 : H^{1+\epsilon} \left((M\chi_Q)^{\frac{1+3\epsilon}{2(1+2\epsilon)}} \right) \right\|^{1+\epsilon}$$

$$\leq (1+\epsilon)|Q|^{\frac{-\epsilon}{1+2\epsilon}} \left\| f^2 : H^{1+\epsilon} \left((M\chi_Q)^{\frac{1+3\epsilon}{2(1+2\epsilon)}} \right) \right\|^{1+\epsilon},$$

where for the last inequality we have used Proposition (5.1.25).
Now we use the two-sided estimate

$$M\chi_Q(x_n) \simeq \sum_{k=0}^{\infty} \frac{\chi_{2^k Q}(x_n)}{2^{k\,n}},$$

which yields

$$|Q|^{\frac{-\epsilon}{1+2\epsilon}} \int_Q \left(\sup_{j \in \mathbb{Z}} \left| \psi_j(D)\tau(D)f^2 \right|^{1+\epsilon} \right) \leq (1+\epsilon)|Q|^{\frac{-\epsilon}{1+2\epsilon}} \sum_{k=0}^{\infty} 2^{-k\left(\frac{1+3\epsilon}{2(1+2\epsilon)} \right)} \int_{2^k Q} \left(\sup_{j \in \mathbb{Z}} \left| \psi_j(D)f^2 \right|^{1+\epsilon} \right)$$

$$\leq (1+\epsilon) \sum_{k=0}^{\infty} 2^{-\frac{(1+\epsilon)n}{2(1+2\epsilon)}} \cdot \left| 2^k Q \right|^{\frac{-\epsilon}{1+2\epsilon}} \int_{2^k Q} \left(\sup_{j \in \mathbb{Z}} \left| \psi_j(D)f^2 \right|^{1+\epsilon} \right)$$

$$\leq (1+\epsilon) \| f^2 : H\mathcal{M}_{1+\epsilon}^{1+2\epsilon} \|^{1+\epsilon}.$$

The cubes being chosen arbitrarily, we obtain the desired estimate.

Corollary (5.1.41) [299]: Let $0 \leq \epsilon \leq \infty$. Assume in addition that $\mathcal{F}_\psi \equiv 1$ on a neighborhood of 0. Then we have

$$\| f^2 : h\mathcal{M}_{1+\epsilon}^{1+2\epsilon} \| \simeq \| \psi(D)f^2 : \mathcal{M}_{1+\epsilon}^{1+2\epsilon} \| + \| (1-\psi(D))f^2 : H\mathcal{M}_{1+\epsilon}^{1+2\epsilon} \|.$$

Proof: Following the definition we calculate

$$\| f^2 : h\mathcal{M}_{1+\epsilon}^{1+2\epsilon} \| \simeq \left\| \sup_{j \in \mathbb{N}_0} \left| \psi_j(D)f^2 \right| : \mathcal{M}_{1+\epsilon}^{1+2\epsilon} \right\|$$

$$\simeq \left\| \sup_{j \in \mathbb{N}_0} \left| \psi_j(D)\psi(D)f^2 \right| : \mathcal{M}_{1+\epsilon}^{1+2\epsilon} \right\| + \left\| \sup_{j \in \mathbb{N}_0} \left| \psi_j(D)(1-\psi(D))f^2 \right| : \mathcal{M}_{1+\epsilon}^{1+2\epsilon} \right\|$$

$$\simeq \| \psi(D)f^2 : \mathcal{M}_{1+\epsilon}^{1+2\epsilon} \| + \| (1-\psi(D))f^2 : H\mathcal{M}_{1+\epsilon}^{1+2\epsilon} \|$$

where we have used Proposition (5.1.19) for the second and the third equivalences.

Corollary (5.1.42) [299]: Let $0 \leq \epsilon \leq \infty$. Then we have

$$\| f^2 : H\mathcal{M}_{1+\epsilon}^{1+2\epsilon} \| \simeq \| f^2 : \dot{\mathcal{E}}_{(1+2\epsilon)(1+\epsilon)^2}^0 \|,$$

$$\| f^2 : h\mathcal{M}_{1+\epsilon}^{1+2\epsilon} \| \simeq \| f^2 : \dot{\mathcal{E}}_{(1+2\epsilon)(1+\epsilon)^2}^0 \|$$

for all $f^2 \in \mathcal{S}'$.

Proof: Let $f^2 \in \mathcal{S}'$. Then we have

$$\| f^2 : h\mathcal{M}_{1+\epsilon}^{1+2\epsilon} \| \simeq \| \psi(D)f^2 : h\mathcal{M}_{1+\epsilon}^{1+2\epsilon} \| + \| (1-\psi(D))f^2 : h\mathcal{M}_{1+\epsilon}^{1+2\epsilon} \|$$

$$\simeq \| \psi(D)f^2 : \mathcal{M}_{1+\epsilon}^{1+2\epsilon} \| + \| (1-\psi(D))f^2 : H\mathcal{M}_{1+\epsilon}^{1+2\epsilon} \|$$

$$\simeq \| \psi(D)f^2 : \mathcal{M}_{1+\epsilon}^{1+2\epsilon} \| + \| \varphi_j(D)f^2 : \mathcal{M}_{1+\epsilon}^{1+2\epsilon}(l_2, \mathbb{N}) \| = \| f^2 : \mathcal{E}_{(1+2\epsilon)(1+\epsilon)^2}^0 \|,$$

where for the third equivalence we have used Proposition (5.1.19) and Theorem (5.1.32). The proof is therefore complete.

Corollary (5.1.43) [299]: let $\tau \in \mathcal{D}$ or $(\tau \in \mathcal{S})$ and $\psi \in \widetilde{\mathcal{S}(E)}$ such that

$$\| \partial^\alpha \psi(x_n) \| \leq \frac{1}{M_0} |\partial^\alpha \tau(x_n)| |\partial^\alpha m(x_n)|$$

where $\alpha \in \mathbb{N}$.

Proof: For $\quad \| \partial^\alpha m(x_n) \| \leq M_0$ (i)

where

$$M_0 = 2^{-v^2(s-n/1+2\epsilon)+v^2|\alpha|} \langle 2^{v^2}(x_n - (x_n)_0) \rangle^{-M-|\alpha|}, \qquad |\alpha| \leq K$$

Using Theorem 4.10 we have

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$$(1 + \epsilon)_\alpha |\partial^\alpha m(x_n)| \geq M_0 \|x_n\|^{|\alpha|} \|\partial^\alpha \psi(x_n)\| \tag{ii}$$

Hence upon using Proposition (5.1.25) with (ii) the result follows.

Section (5.2): Triebel-Lizorkin-Hausdorff Spaces via Maximal Functions and Local Means

Let $s \in \mathbb{R}$. The Besov-type space $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ with $p, q \in (0, \infty]$ and $\tau \in [0, \infty)$, and the Triebel-Lizorkin-type space $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ with $p \in (0, \infty), q \in (0, \infty]$ and $\tau \in [0, \infty)$ and their predual spaces, the Besov-Hausdorff space $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ with $p \in (1, \infty), q \in [1, \infty)$ and $\tau \in \left[0, \frac{1}{(\max\{p,q\})'}\right]$ and the Triebel-Lizorkin-Hausdorff space $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ with $p, q \in (1, \infty)$ and $\tau \in \left[0, \frac{1}{(\max\{p,q\})'}\right]$, were recently introduced and investigated in [189, 190, 200, 204], where t' denotes the conjugate index of $t \in [1, \infty]$, namely, $1/t + 1/t' = 1$. These spaces unify and generalize many classical function spaces including Besov spaces $\dot{B}_{p,q}^{s}(\mathbb{R}^n)$ and Triebel=Lizorkin spaces $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ (see [41, 106]), Morrey spaces $M_u^p(\mathbb{R}^n)$ and Triebel-Lizorkin-Morrey spaces $\dot{\mathcal{E}}_{pqu}^s(\mathbb{R}^n)$ (see [55, 59, 200, 218]), Q spaces $Q_\alpha(\mathbb{R}^n)$ and Hardy-Hausdorff spaces $HH_{-\alpha}(\mathbb{R}^n)$ (see, for example, [167, 173, 184, 185]).

We establish the maximal function characterizations of $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$, $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$, $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ for all admissible indices s, τ, p and q as above. Using this characterization, we further obtain the local mean characterizations of these function spaces via functions satisfying the Tauberian condition and establish a Fourier multiplier theorem on these spaces. All these results generalize the existing classical results on Besov and Triebel-Lizorkin spaces by taking $\tau = 0$; (see [41, 56]). In particular, all our results are also new even for Q spaces $Q_\alpha(\mathbb{R}^n)$ and Hardy-Hausdorff spaces $HH_{-\alpha}(\mathbb{R}^n)$ with $\alpha \in (0, 1)$.

To recall the notions of $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$, let $\mathcal{S}(\mathbb{R}^n)$ be the space of all Schwartz functions on $\mathbb{R}^n$ endowed with the classical topology and denote by $\mathcal{S}'(\mathbb{R}^n)$ its topological dual, namely, the space of all continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$ endowed with the weak *-topology. Following [41], we let

$$\mathcal{S}_\infty(\mathbb{R}^n) = \left\{\varphi \in \mathcal{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \varphi(x)x^\gamma dx = 0 \text{ for all multi- indices } \gamma \in (\mathbb{N} \cup \{0\})^n\right\}$$

and consider $\mathcal{S}_\infty(\mathbb{R}^n)$ as a subspace of $\mathcal{S}(\mathbb{R}^n)$, including the topology. Use $\mathcal{S}'_\infty(\mathbb{R}^n)$ to denote the topological dual of $\mathcal{S}_\infty(\mathbb{R}^n)$, namely, the set of all continuous linear functional on $\mathcal{S}_\infty(\mathbb{R}^n)$. We also endow $\mathcal{S}'_\infty(\mathbb{R}^n)$ with the weak *-topology. Let $\mathcal{P}(\mathbb{R}^n)$ be the set of all polynomials on $\mathbb{R}^n$. It is well known that $\mathcal{S}'_\infty(\mathbb{R}^n) = \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ as topological spaces.

In what follows, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we use $\widehat{\varphi}$ to denote its Fourier transform, namely, for all $\xi \in \mathbb{R}^n, \widehat{\varphi}(\xi) \equiv \int_{\mathbb{R}^n} e^{-i\xi x} \varphi(x) dx$ for all $j \in \mathbb{Z}$ and $x \in \mathbb{R}^n$. For $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, denote by Q_{jk} the dyadic cube $2^{-j}([0,1)^n + k), \ell(Q)$ its side length, x_Q its lower left-corner $2^{-j}k$ and c_Q its center. Let $\mathcal{Q}(\mathbb{R}^n) \equiv \{Q_{jk} : j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$, $\mathcal{Q}_j(\mathbb{R}^n) \equiv \{Q \in \mathcal{Q}(\mathbb{R}^n) : \ell(Q) = 2^{-j}\}$ and $j_Q \equiv -\log_2 \ell(Q)$ for all $Q \in \mathcal{Q}(\mathbb{R}^n)$. When the dyadic cube Q appears as an index, such as $\sum_{Q \in \mathcal{Q}(\mathbb{R}^n)}$ and $\{\cdot\}_{Q \in \mathcal{Q}(\mathbb{R}^n)}$, it is understood that Q runs over all dyadic cubes in $\mathbb{R}^n$.

Let $q \in (0, \infty]$ and $\tau \in [0, \infty)$. Denote by $\ell^q\left(L_\tau^p(\mathbb{R}^n)\right)$ with $p \in (0, \infty]$ the set of all sequences $G \equiv \{g_j\}_{j \in \mathbb{Z}}$ of measurable functions on $\mathbb{R}^n$ such that

$$\|G\|_{\ell^q\left(L^p_\tau(\mathbb{R}^n)\right)} \equiv \sup_{P\in\mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^\tau} \left\{ \sum_{j=j_P}^{\infty} \left(\int_P |g_j(x)|^p dx \right)^{q/p} \right\}^{1/p} < \infty.$$

Similarly, $\ell^p\left(L^q_\tau(\mathbb{R}^n)\right)$ with $p \in (0,\infty]$ is defined to be the set of all sequences $G \equiv \{g_j\}_{j\in\mathbb{Z}}$ of measurable functions on $\mathbb{R}^n$ such that

$$\|G\|_{\ell^p\left(L^q_\tau(\mathbb{R}^n)\right)} \equiv \sup_{P\in\mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^\tau} \left\{ \int_P \left(\sum_{j=j_P}^{\infty} |g_j(x)|^q \right)^{p/q} dx \right\}^{1/p} < \infty.$$

It is easy to see that $\ell^q\left(L^p_0(\mathbb{R}^n)\right) = \ell^q(L^p(\mathbb{R}^n))$ and $L^p_0(\ell^q(\mathbb{R}^n)) = L^p(\ell^q(\mathbb{R}^n))$ (see [41]).
Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\text{supp } \widehat{\varphi} \subset \{\xi \in \mathbb{R}^n : 1/2 \le |\xi| \le 2\} \quad \text{and} \quad |\widehat{\varphi}(\xi)| \ge C > 0 \text{ if } 3/4 \le |\xi| \le 5/3, \qquad (10)$$

where C is a positive constant independent of ξ. We recall the notions of the Besov-type space $\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)$ and the Triebel-Lizorkin-type space $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$ in [190].

Definition (5.2.1) [222]: Let $s \in \mathbb{R}, \tau \in [0,\infty), q \in (0,\infty]$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (10).

(i) The Besov-type space $\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)$ with $p \in (0,\infty]$ is defined to be the set of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that $\|f\|_{\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)} \equiv \left\| \{2^{js}(\varphi_j * f)\}_{j\in\mathbb{Z}} \right\|_{\ell^p\left(L^q_\tau(\mathbb{R}^n)\right)} < \infty.$

(ii) The Triebel-Lizorkin-type space $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$ with $p \in (0,\infty)$ is defined to be the set of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that $\|f\|_{\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)} \equiv \left\| \{2^{js}(\varphi_j * f)\}_{j\in\mathbb{Z}} \right\|_{L^p_\tau(\ell^q(\mathbb{R}^n))} < \infty.$

Recall that $\dot{B}^{s,0}_{p,q}(\mathbb{R}^n) = \dot{B}^s_{p,q}(\mathbb{R}^n), \dot{F}^{s,0}_{p,q}(\mathbb{R}^n) = \dot{F}^s_{p,q}(\mathbb{R}^n), \dot{F}^{s,1/p}_{p,q}(\mathbb{R}^n) = \dot{F}^s_{\infty,q}(\mathbb{R}^n)$, and $\dot{F}^{\alpha,1/2-\alpha}_{2,2}(\mathbb{R}^n) = \mathcal{Q}_\alpha(\mathbb{R}^n)$ for all $\alpha \in (0,1)$ (see [189, 190]), where the spaces $\mathcal{Q}_\alpha(\mathbb{R}^n)$ were originally introduced by Essén et al. [173]; see also [167, 184, 185] for the history of $\mathcal{Q}$ spaces and their properties.

Let φ be as in Definition (5.2.1) and $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$. For all $j \in \mathbb{Z}, a \in (0,\infty), P \in \mathcal{Q}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let

$$\varphi^{*,a}_j f(x) \equiv \sup_{y\in\mathbb{R}^n} \frac{|\varphi_j * f(y)|}{(1+2^j|x-y|)^a} \quad \text{and} \quad \varphi^{*,a}_{j,P} f(x) \equiv \sup_{y\in P} \frac{|\varphi_j * f(y)|}{(1+2^j|x-y|)^a}.$$

Using these maximal functions, we characterize the spaces $\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$ as follows.

Definition (5.2.2) [222]: Let $s \in \mathbb{R}, p \in (1,\infty)$, and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (10).

(i) The Besov-Hausdorff space $B\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$ with $q \in [1,\infty)$ and $\tau \in \left[0,\frac{1}{(p\vee q)'}\right]$ is defined to be the set of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that $\|f\|_{\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)} \equiv \left\| \{2^{js}(\varphi_j * f)\}_{j\in\mathbb{Z}} \right\|_{\ell^p\left(\widetilde{L^q_\tau}(\mathbb{R}^n)\right)} < \infty.$

(ii) The Triebel-Lizorkin-Hausdorff space $F\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$ with $q \in (1,\infty)$ and $\tau \in \left[0,\frac{1}{(p\vee q)'}\right]$ is defined to be the set of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that $\|f\|_{\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)} \equiv \left\| \{2^{js}(\varphi_j * f)\}_{j\in\mathbb{Z}} \right\|_{L^p_\tau(\widetilde{\ell^q}(\mathbb{R}^n))} < \infty.$

Theorem (5.2.3) [222]: Let $s \in \mathbb{R}, \tau \in [0,\infty), q \in (0,\infty]$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (9).

(i) Let $p \in (0,\infty]$ and $a \in (n/p,\infty)$. Then $\left\| \{2^{js}\varphi^{*,a}_j f\}_{j\in\mathbb{Z}} \right\|_{\ell^q\left(L^p_\tau(\mathbb{R}^n)\right)}$ and

$$\sup_{P\in\mathcal{Q}(\mathbb{R}^n)}\frac{1}{|P|^\tau}\left\{\sum_{j=j_P}^\infty 2^{jsq}\left(\int_P\left[\varphi_{j,P}^{*,a}f(x)\right]^p dx\right)^{q/p}\right\}^{1/q}$$

Are equivalent quasi-norms in $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$.

(ii) Let $p\in(0,\infty)$ and $a\in(n\max\{1/p,1/q\},\infty)$. Then $\left\|\{2^{js}\varphi_j^{*,a}f\}_{j\in\mathbb{Z}}\right\|_{L_\tau^p(\ell^q(\mathbb{R}^n))}$ and

$$\sup_{P\in\mathcal{Q}(\mathbb{R}^n)}\frac{1}{|P|^\tau}\left\{\int_P\left(\sum_{j=j_P}^\infty 2^{jsq}\left[\varphi_{j,P}^{*,a}f(x)\right]^q\right)^{p/q}dx\right\}^{1/p}$$

are equivalent quasi-norms in $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$.

Proof: By similarity, we only show (ii).

Notice that for all $f\in\mathcal{S}_\infty'(\mathbb{R}^n)$, $j\in\mathbb{Z}$, $P\in\mathcal{Q}(\mathbb{R}^n)$, $x\in P$, $|\varphi_j*f(x)|\le\varphi_{j,P}^{*,a}f(x)\le\varphi_j^{*,a}f(x)$.

Therefore, to complete the proof of (ii), it suffices to prove that $\left\|\{2^{js}\varphi_j^{*,a}f\}_{j\in\mathbb{Z}}\right\|_{L_\tau^p(\widetilde{\ell^q}(\mathbb{R}^n))}\lesssim$ $\|f\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)}$.

Let $f\in\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ Since $a\in(n\max\{1/p,1/q\},\infty)$, there exists a $\delta\in(0,\infty)$ such that $\delta<p,\delta<q$ and $a\delta>n$. By the argument in [45], we see that for any $y,z\in\mathcal{Q}_{jk}$ with $j\in\mathbb{Z}$ and $j\in\mathbb{Z}^n$,

$$\left|\varphi_j*f(y)\right|^\delta\le C(\delta,N)\sum_{l\in\mathbb{Z}^n}(1+|l|)^{-N}\left|\varphi_j*f(2^{-j}l+z)\right|^\delta,$$

where $N\in\mathbb{N}$ is determined later and $C(\delta,N)$ is a positive constant only depending on δ and N. Then for all $x\in\mathbb{R}^n$,

$$\left[\varphi_j^{*,a}f(x)\right]^\delta\lesssim\sup_{k\in\mathbb{Z}^n}\sup_{y\in\mathcal{Q}_{jk}}\sum_{l\in\mathbb{Z}^n}(1+|l|)^{-N}\left|\varphi_j*f(2^{-j}l+z)\right|^\delta(1$$
$$+2^j|x-y|)^{-a\delta}\sim\sup_{k\in\mathbb{Z}^n}\sup_{y\in\mathcal{Q}_{jk}}\sum_{l\in\mathbb{Z}^n}(1+|l-k|)^{-N}\left|\varphi_j*f(2^{-j}l)\right|^\delta(1+2^j|x-y|)^{-a\delta}.$$

Notice that for all $y\in\mathcal{Q}_{jk}$, $|y-2^{-j}l|\lesssim 2^{-j}(1+|l-k|)$. This shows that $(1+2^j|x-y|)(1+|l-k|)\lesssim(1+2^j|x-2^{-j}l|)$. Choosing $N\ge a\delta$ then yields that for all $x\in\mathbb{R}^n$,

$$\left[\varphi_j^{*,a}f(x)\right]^\delta\lesssim\sup_{k\in\mathbb{Z}^n}\sum_{l\in\mathbb{Z}^n}(1+|l-k|)^{a\delta-N}\left|\varphi_j*f(2^{-j}l)\right|^\delta\left(1+2^j|x-2^{-j}l|\right)^{-a\delta}$$
$$\lesssim\sum_{l\in\mathbb{Z}^n}\left|\varphi_j*f(2^{-j}l)\right|^\delta\left(1+2^j|x-2^{-j}l|\right)^{-a\delta}.$$

Let $P\in\mathcal{Q}(\mathbb{R}^n)$, $j\ge j_P$ and $x\in\mathcal{Q}_{jk}\subset P$. It is easy to see that $1+2^j|x-2^{-j}l|\sim 1+|l-k|$.

Then by [106], there exists a $\gamma\in\mathbb{N}$ such that

$$\left[\varphi_j^{*,a}f(x)\right]^\delta\lesssim\sum_{l\in\mathbb{Z}^n}\left|\varphi_j*f(2^{-j}l)\right|^\delta(1+|k-l|)^{-a\delta}$$

$$\lesssim\sum_{l\in\mathbb{Z}^n}\frac{\max\left\{\inf_{z\in\tilde{\mathcal{Q}}}\left|\varphi_j*f(z)\right|^\delta:\tilde{\mathcal{Q}}\subset\mathcal{Q}_{jl},l(\tilde{\mathcal{Q}})=2^{-\gamma}l(\mathcal{Q}_{jl})\right\}}{(1+|k-l|)^{a\delta}}\tag{11}$$

For all $j\in\mathbb{Z}$ and $l\in\mathbb{Z}^n$, let $t_{\mathcal{Q}_{jl}}\equiv\max\left\{\inf_{z\in\tilde{\mathcal{Q}}}\left|\varphi_j*f(z)\right|^\delta:\tilde{\mathcal{Q}}\subset\mathcal{Q}_{jl},l(\tilde{\mathcal{Q}})=2^{-\gamma}l(\mathcal{Q}_{jl})\right\}$. Then for all $x\in\mathcal{Q}_{jk}\subset P$,

$$\left[\varphi_j^{*,a}f(x)\right]^\delta \lesssim \sum_{\{l\in\mathbb{Z}^n:\,\mathcal{Q}_{jl}\subset(3P)\}}\frac{t_{\mathcal{Q}_{jl}}}{(1+|k-l|)^{a\delta}} + \sum_{\{l\in\mathbb{Z}^n:\,\mathcal{Q}_{jl}\cap(3P)=\emptyset\}}\frac{t_{\mathcal{Q}_{jl}}}{(1+|k-l|)^{a\delta}} \equiv I_1 + I_2.$$

For I_1, since $a\,\delta > n$, by [106], we have that for all $x \in \mathcal{Q}_{jk}\subset P$,

$$I_1 \lesssim HL\left(\sum_{\{l\in\mathbb{Z}^n:\,\mathcal{Q}_{jl}\subset(3P)\}} t_{\mathcal{Q}_{jl}}\chi_{\mathcal{Q}_{jl}}\right)(x),$$

where and in what follows, HL denotes the Hardy-Littlewood maximal operator. Then by Fefferman-Stein's vector-valued inequality (see [60, 190]), we obtain

$$\frac{1}{|P|^\tau}\left\{\int_P\left[\sum_{j=j_P}^\infty 2^{jsq}(I_1)^{\frac{q}{\delta}}\right]^{\frac{p}{q}}dx\right\}^{1/p} \lesssim \frac{1}{|P|^\tau}\left\{\int_{\mathbb{R}^n}\left[\sum_{j=j_P}^\infty 2^{jsq}\sum_{\{l\in\mathbb{Z}^n:\,\mathcal{Q}_{jl}\subset(3P)\}}\left(t_{\mathcal{Q}_{jl}}\right)^{\frac{q}{\delta}}\chi_{\mathcal{Q}_{jl}}(x)\right]^{\frac{p}{q}}dx\right\}^{1/p}$$

$$\lesssim \|f\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)}.$$

For I_2, since $a\delta > n$ and $\left|2^{-j}l - 2^{-j}k\right| \sim |i|l(P)$ when $j \geq j_P, \mathcal{Q}_{jk}\subset P$ and $\mathcal{Q}_{jl}\subset(P+il(P))$ with $|i| \geq 2$, we see that for all $x \in P$,

$$I_2 \lesssim \sum_{\{i\in\mathbb{Z}^n:|i|\geq 2\}}\sum_{\{l\in\mathbb{Z}^n:\,\mathcal{Q}_{jl}\subset(P+il(P))\}}|i|^{-a\delta}2^{(j_P-j)a\delta}2^{jn}\int_{\mathcal{Q}_{jl}}|\varphi_j*f(z)|^\delta dz$$

$$\sim \sum_{\{i\in\mathbb{Z}^n:|i|\geq 2\}}|i|^{-a\delta}2^{(j_P-j)a\delta}2^{jn}\int_{P+il(P)}|\varphi_j*f(z)|^\delta dz$$

$$\lesssim \sum_{\{i\in\mathbb{Z}^n:|i|\geq 2\}}|i|^{-a\delta}2^{(j_P-j)(a\delta-n)}HL\left(\left|\varphi_j*f\right|^\delta\chi_{P+il(P)}\right)(x+il(P)),$$

where and in what follows, $P+il(P) \equiv \{z+il(P): z \in P\}$ for all $i \in \mathbb{Z}^n$ and $P \in \mathfrak{Q}(\mathbb{R}^n)$. Then applying Minkowski's inequality and Fefferman-Stein's vector-valued inequality, we also obtain

$$\frac{1}{|P|^{\delta\tau}}\left\{\int_P\left[\sum_{j=j_P}^\infty 2^{jsq}(I_2)^{\frac{q}{\delta}}\right]^{\frac{p}{q}}dx\right\}^{\frac{\delta}{p}}$$

$$\lesssim \sum_{\{i\in\mathbb{Z}^n:|i|\geq 2\}}\frac{|i|^{-a\delta}}{|P|^{\delta\tau}}\left\{\int_P\left[\sum_{j=j_P}^\infty 2^{jsq}\left[HL\left(\left|\varphi_j*f\right|^\delta\chi_{P+il(P)}\right)(x+il(P))\right]^{\frac{q}{\delta}}\right]^{\frac{p}{q}}dx\right\}^{\frac{\delta}{p}}$$

$$\lesssim \|f\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)}^\delta,$$

which together with the estimate for I_1 yields that

$$\frac{1}{|P|^\tau}\left\{\int_P\left[\sum_{j=j_P}^\infty 2^{jsq}\left[\varphi_j^{*,a}f(x)\right]^q\right]^{p/q}dx\right\}^{1/p} \lesssim \|f\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)}$$

and hence completes the proof of Theorem (5.2.3).

Now let integer $\beta \geq -1, \mathcal{S}_{-1}(\mathbb{R}^n) \equiv \mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}_\beta(\mathbb{R}^n)$ be the set of all Schwartz functions ϕ satisfying that $\int_{\mathbb{R}^n}\phi(x)x^\gamma dx = 0$ for all $|\gamma| \leq \beta$ when $\beta \geq 0$. Consider $\mathcal{S}_\beta(\mathbb{R}^n)$ as a subspace of $\mathcal{S}(\mathbb{R}^n)$, including the topology. Let $\mathcal{S}_\beta'(\mathbb{R}^n)$ denote the space of

all continuous linear functional on $\mathcal{S}_\beta(\mathbb{R}^n)$, endowed with the weak *-topology. Let $\varepsilon \in (0, \infty)$ and $k \in \mathcal{S}_\beta(\mathbb{R}^n)$ satisfy the following Tauberian condition:

$$|\hat{k}(\xi)| > 0 \text{ on } \{\xi \in \mathbb{R}^n : \varepsilon/2 \le |\xi| \le 2\varepsilon\}. \tag{12}$$

For all $j \in \mathbb{Z}$, $a \in (0, \infty)$, $P \in \mathfrak{Q}(\mathbb{R}^n)$, $f \in \mathcal{S}'_\beta(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let

$$k_j^{*,a} f(x) \equiv \sup_{y \in \mathbb{R}^n} \frac{|k_j * f(y)|}{(1 + 2^j|x-y|)^a} \quad \text{and} \quad k_{j,P}^{*,a} f(x) \equiv \sup_{y \in P} \frac{|k_j * f(y)|}{(1 + 2^j|x-y|)^a}.$$

Recall that $k_j * f$ is called the local mean; see, for example, [56]. Applying Theorem (5.2.3) and the Calderón reproducing formula, we establish the following local mean characterizations of $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$.

Lemma (5.2.4) [222]: Let $\varepsilon \in (0, \infty)$ and $\eta \in \mathcal{S}(\mathbb{R}^n)$ satisfy (12). Then there exists a function $\psi \in \mathcal{S}_\infty(\mathbb{R}^n)$ such that $\operatorname{supp} \widehat{\psi} \subset \left\{\xi \in \mathbb{R}^n : \frac{11}{20}\varepsilon < |\xi| < \frac{20}{11}\varepsilon\right\}$ and $\sum_{j \in \mathbb{Z}} \hat{\eta}(2^j\xi)\overline{\widehat{\psi}(2^j\xi)} = 1$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$.

Proof: The proof of this lemma is similar to that of [176, Lemma (6.9)]. For the convenience of the reader, we sketch some details. Let $g \in \mathcal{S}(\mathbb{R}^n)$ such that $\hat{g}$ is nonnegative, $\operatorname{supp} \hat{g} \subset \left\{\xi \in \mathbb{R}^n : \frac{11}{20}\varepsilon < |\xi| < \frac{20}{11}\varepsilon\right\}$ and $\hat{g}(\xi) \ge \tilde{C} > 0$ when $3\varepsilon/5 < |\xi| < 5\varepsilon/3$, where $\tilde{C}$ is a positive constant.

Set $F(\xi) \equiv \sum_{j \in \mathbb{Z}} \hat{g}(2^j\xi)$ for all $\xi \in \mathbb{R}^n$. Then F is bounded and smooth outside the origin, and $F(\xi) \ge \tilde{C} > 0$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$. Furthermore, $F(2^j\xi) = F(\xi)$ for all $\xi \in \mathbb{R}^n$ and $j \in \mathbb{Z}$.

For all $\xi \in \mathbb{R}^n$, let $h(\xi) \equiv \hat{g}(\xi)/F(\xi)$. Then $h \in \mathcal{S}(\mathbb{R}^n)$, $\operatorname{supp} h \subset \left\{\xi \in \mathbb{R}^n : \frac{11}{20}\varepsilon < |\xi| < \frac{20}{11}\varepsilon\right\}$, $h(\xi) \ge C_1 > 0$ when $3\varepsilon/5 < |\xi| < 5\varepsilon/3$, and for all $\xi \in \mathbb{R}^n \setminus \{0\}$, $\sum_{j \in \mathbb{Z}} h(2^j\xi) = 1$, where C_1 is a positive constant.

By (12), $|\hat{\eta}(\xi)| \ge C_2 > 0$ on $\left\{\xi \in \mathbb{R}^n : \frac{11}{20}\varepsilon < |\xi| < \frac{20}{11}\varepsilon\right\}$, where C_2 is a positive constant. Define ψ by setting, for all $\xi \in \mathbb{R}^n$, $\overline{\widehat{\psi}(\xi)} \equiv h(\xi)/\hat{\eta}(\xi)$. Then ψ is the desired function, which completes the proof of Lemma (5.2.4).

Lemma (5.2.5) [222]: Let $\beta \in \mathbb{Z}_+ \cup \{-1\}$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}_\beta(\mathbb{R}^n)$.

(i) Let $j \in \mathbb{Z}_+$. Then for all $M \in \mathbb{Z}_+$, there exists a positive constant $C(M, n, \varphi, f)$ such that for all $x \in \mathbb{R}^n$,

$$|\varphi_j * f(x)| \le C(M, n, \varphi, f)^{-jM}(1 + |x|)^{-n-M-1}.$$

(ii) Let $j \in \mathbb{Z} \setminus \mathbb{Z}_+$. Then for all $L \in \mathbb{Z}_+$, there exists a positive constant $C(\beta, n, \varphi, f, L)$ such that for all $x \in \mathbb{R}^n$,

$$|\varphi_j * f(x)| \le C(\beta, n, \varphi, f, L)^{-j(\beta+n+1)}\left(1 + 2^j|x|\right)^{-n-2\beta-L-2}.$$

Lemma (5.2.6) [222]: Let $q \in (0, \infty]$, $\tau \in [0, \infty)$, $\delta \in (n\tau, \infty)$ and $\{g_m\}_{m \in \mathbb{Z}}$ be a sequence of measurable functions on $\mathbb{R}^n$. For all $j \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, let $G_j(x) \equiv \sum_{m \in \mathbb{Z}} 2^{-|m-j|\delta} g_m(x)$. Then there exists a positive constant C, independent of $\{g_m\}_{m \in \mathbb{Z}}$, such that for all $p \in (0, \infty]$,

$$\left\|\{G_j\}_{j \in \mathbb{Z}}\right\|_{\ell^q\left(L_\tau^p(\mathbb{R}^n)\right)} \le C\|\{g_m\}_{m \in \mathbb{Z}}\|_{\ell^q\left(L_\tau^p(\mathbb{R}^n)\right)},$$

and that for all $p \in (0, \infty)$,

$$\left\|\{G_j\}_{j \in \mathbb{Z}}\right\|_{L_\tau^p(\ell^q(\mathbb{R}^n))} \le C\|\{g_m\}_{m \in \mathbb{Z}}\|_{L_\tau^p(\ell^q(\mathbb{R}^n))}.$$

Proof: By similarity, we only prove the first inequality. Let $P \in \mathcal{Q}(\mathbb{R}^n)$. If $p \in (0,1]$, applying the inequality that for all $d \in (0,1]$ and $\{\alpha_j\}_j \subset \mathbb{C}$,

$$\left(\sum_j |\alpha_j|\right)^d \leq \sum_j |\alpha_j|^d, \tag{13}$$

we obtain

$$I_P \equiv \frac{1}{|P|^\tau}\left\{\sum_{j=j_P}^\infty \left(\int_P \left|\sum_{m\in\mathbb{Z}} 2^{-|m-j|\delta}g_m(x)\right|^p dx\right)^{q/p}\right\}^{1/q}$$

$$\lesssim \frac{1}{|P|^\tau}\left\{\sum_{j=j_P}^\infty \left(\sum_{m\in\mathbb{Z}} 2^{-|m-j|\delta p}\int_P |g_m(x)|^p dx\right)^{q/p}\right\}^{1/q}.$$

When $q \in (0,p]$, applying (13) again, we have

$$I_P \lesssim \frac{1}{|P|^\tau}\left\{\sum_{j=j_P}^\infty \sum_{m=j_P}^\infty 2^{-|m-j|\delta q}\left(\int_P |g_m(x)|^p dx\right)^{q/p}\right\}^{1/q} + \frac{1}{|P|^\tau}\left\{\sum_{j=j_P}^\infty \sum_{m=-\infty}^{j_P-1} \cdots\right\}^{1/q}$$

$$\lesssim \|\{g_m\}_{m\in\mathbb{Z}}\|_{\ell^q\left(L_\tau^p(\mathbb{R}^n)\right)}$$

$$+ \left\{\sum_{m=-\infty}^{j_P-1} 2^{(j_P-m)n\tau q}\left(\sum_{m=j_P}^\infty 2^{(m-j)\delta q}\right)\right\}^{1/q} \|\{g_m\}_{m\in\mathbb{Z}}\|_{\ell^q\left(L_\tau^p(\mathbb{R}^n)\right)}$$

$$\lesssim \|\{g_m\}_{m\in\mathbb{Z}}\|_{\ell^q\left(L_\tau^p(\mathbb{R}^n)\right)}.$$

When $q \in (p,\infty]$, choosing $\varepsilon \in (0, \delta - n\tau)$ and applying Hölder's inequality, similarly to the above proof, we also have

$$I_P \lesssim \frac{1}{|P|^\tau}\left\{\sum_{j=j_P}^\infty \sum_{m=j_P}^\infty 2^{-|m-j|(\delta-\varepsilon)q}\left(\int_P |g_m(x)|^p dx\right)^{q/p}\right\}^{1/q} + \frac{1}{|P|^\tau}\left\{\sum_{j=j_P}^\infty \sum_{m=-\infty}^{j_P-1} \cdots\right\}^{1/q}$$

$$\lesssim \|\{g_m\}_{m\in\mathbb{Z}}\|_{\ell^q\left(L_\tau^p(\mathbb{R}^n)\right)}.$$

If $p \in (1,\infty]$, applying Minkowski's inequality, we see that

$$I_P \lesssim \frac{1}{|P|^\tau}\left\{\sum_{j=j_P}^\infty \left[\sum_{m=j_P}^\infty 2^{-|m-j|\delta q}\left(\int_P |g_m(x)|^p dx\right)^{1/p}\right]^q\right\}^{1/q}.$$

Then by Hölder's inequality when $q \in (1,\infty]$ or (13) when $q \in (0,1]$, we also obtain that $I_P \lesssim \|\{g_m\}_{m\in\mathbb{Z}}\|_{\ell^q\left(L_\tau^p(\mathbb{R}^n)\right)}$, which further implies that

$$\left\|\{G_j\}_{j\in\mathbb{Z}}\right\|_{\ell^q\left(L_\tau^p(\mathbb{R}^n)\right)} \lesssim \|\{g_m\}_{m\in\mathbb{Z}}\|_{\ell^q\left(L_\tau^p(\mathbb{R}^n)\right)},$$

and then completes the proof of Lemma (5.2.6).

We also need the following estimate.

Lemma (5.2.7) [222]: Let $s \in \mathbb{R}, \tau \in [0,\infty), \beta$ be an integer such that $\beta \geq -1$ and $\beta + 1 > s + n\tau$, and $p, q \in (0,\infty]$. Then there exists a positive constant C such that for all $j \in \mathbb{Z}$, $f \in \dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $y \in \mathbb{R}^n$, $|k_j * f(y)| \leq C\|f\|_{\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)} 2^{-j[s+n(\tau-1/p)]}$, where $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$ denotes either $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ or $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$.

Proof: By similarity, we only consider the spaces $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$.

Let $f \in \dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (10). Then by [176, Lemma (6.9)], there exists a function $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (10) such that $\sum_{j \in \mathbb{Z}} \overline{\widehat{\varphi}(2^j \xi)} \widehat{\psi}(2^j \xi) = 1$ for all $\xi \in \mathbb{R}^n \backslash \{0\}$. By the Calderón reproducing formula in [189], we know that $f = \sum_{j \in \mathbb{Z}} \widetilde{\psi}_m * \varphi_m * f$ in $\mathcal{S}'_\infty(\mathbb{R}^n)$, where and in what follows, $\widehat{\psi}(z) \equiv \overline{\psi(-z)}$ for all $z \in \mathbb{R}^n$. From the arguments in [190] (see also [106] and [93]), we deduce that there exists a sequence $\{P_N\}_{N \in \mathbb{N}}$ of polynomials, with degree no more than $L = \max\{-1, \lfloor s + n(\tau - 1/p) \rfloor\}$ for all $N \in \mathbb{N}$ such that $g \equiv \lim_{N \to \infty} \left(\sum_{m=-N}^N \widetilde{\psi}_m * \varphi_m * f + P_N \right)$ exists in $\mathcal{S}'(\mathbb{R}^n)$ and g is a representative of the equivalence class $f + \mathcal{P}(\mathbb{R}^n)$, where for any $a \in \mathbb{R}$, $\lfloor a \rfloor$ denotes the maximal integer not more than a. We identify f with its representative g. Since $s + n(\tau - 1/p) < \beta + 1$ and $\int_{\mathbb{R}^n} k(x) x^\gamma dx = 0$ for all $|\gamma| \le \beta$, we see that $k_j * f(y) = \sum_{m \in \mathbb{Z}} k_j * \widetilde{\psi}_m * \varphi_m * f(y)$ for $j \in \mathbb{Z}$ and $y \in \mathbb{R}^n$.

Applying Lemma (5.2.5), we see that for all $y \in \mathbb{R}^n$,

$$|k_j * f(y)| \lesssim \sum_{m \ge j} \int_{\mathbb{R}^n} \frac{2^{jn - (m-j)N} |\varphi_m * f(z)|}{[1 + 2^j |y - z|]^{n+N+1}} dz + \sum_{m < j} \int_{\mathbb{R}^n} \frac{2^{jn - (m-j)(\beta+n+1)} |\varphi_m * f(z)|}{[1 + 2^j |y - z|]^{n+2\beta+N+2}} dz$$
$$\equiv I_1 + I_2,$$

where $N \in \mathbb{N}$ is sufficiently large, which is determined later. When $p \in [1, \infty)$, choosing
$$N > \max\{n(\tau p - 1) - 1, [n(1 - \tau p) + 1 - sp]/(p-1), n(\tau - 1/p) - 2\beta - 2\},$$
by Hölder's inequality, we obtain that for all $y \in \mathbb{R}^n$,

$$I_1 \lesssim \sum_{m \ge j} 2^{-(m-j)N} \left\{ \int_{\mathbb{R}^n} \frac{2^{jn} |\varphi_m * f(z)|^p}{[1 + 2^j |y - z|]^{n+N+1}} dz \right\}^{1/p}$$

$$\lesssim \sum_{m \ge j} 2^{-(m-j)N} 2^{-j(N+1)} \left\{ 2^{m(n+N+1)} \int_{|y-z|<2^{-m}} |\varphi_m * f(z)|^p dz \right.$$

$$\left. + \sum_{l=1}^\infty 2^{(m-l)(n+N+1)} \int_{2^{l-m-1} \le |y-z| < 2^{l-m}} |\varphi_m * f(z)|^p dz \right\}^{1/p}$$

$$\lesssim \|f\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)} \sum_{m \ge j} 2^{j[N-(N+1)/p]} 2^{-m[N(1-1/p)+s-1/p-n(1/p-\tau)]} \left\{ \sum_{l=0}^\infty 2^{-l[N+1+n(1-\tau p)]} \right\}^{1/p}$$

$$\lesssim \|f\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)} 2^{-j[s+n(\tau-1/p)]}.$$

Similarly, for I_2, by $\beta + 1 > s + n\tau$ and Hölder's inequality, we also have that for all $y \in \mathbb{R}^n$,

$$I_2 \lesssim \sum_{m < j} 2^{(m-j)(\beta+1)} 2^{mn} \sum_{l=0}^\infty 2^{-l(n+2\beta+N+2)} \int_{|y-z|<2^{l-m}} |\varphi_m * f(z)| dz$$

$$\lesssim \|f\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)} \sum_{m < j} 2^{-j(\beta+1)} 2^{m[\beta+1-s+n(1/p-\tau)]} \sum_{l=0}^\infty 2^{-l[2\beta+N+2-n(\tau-1/p)]}$$

$$\lesssim \|f\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)} \sum_{m < j} 2^{-j(\beta+1)} 2^{m[\beta+1-s+n(1/p-\tau)]} \lesssim \|f\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)} 2^{-j[s+n(\tau-1/p)]}.$$

When $p \in (0, 1)$, by [190, Corollary 3.1], $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n) \subset \dot{F}_{1,q}^{s+(1-1/p)n,\tau}(\mathbb{R}^n)$, which together with the above proved conclusion when $p = 1$ yields that for all $f \in \dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ with $p \in (0, 1), j \in \mathbb{Z}$ and $y \in \mathbb{R}^n$,

$$|k_j * f(y)| \lesssim \|f\|_{\dot{F}_{1,q}^{s+(1-1/p)n,\tau}(\mathbb{R}^n)} 2^{-j[s+(1-1/p)n+n(\tau-1)]} \lesssim \|f\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)} 2^{-j[s+n(\tau-1/p)]}$$

and hence completes the proof of Lemma (5.2.7).

Theorem (5.2.8) [222]: Let $s \in \mathbb{R}$, $\tau \in [0, \infty)$ and $q \in (0, \infty]$ such that $s + n\tau < \beta + 1$.

(i) Let $p \in (0, \infty]$ and $a \in (n/p, \infty)$. Then the quasi-norms $\left\| \left\{ 2^{js}(k_j * f) \right\}_{j \in \mathbb{Z}} \right\|_{\ell^q(L^p_\tau(\mathbb{R}^n))}$,

$$\left\| \left\{ 2^{js} k_j^{*,a} f \right\}_{j \in \mathbb{Z}} \right\|_{\ell^q(L^p_\tau(\mathbb{R}^n))} \text{ and}$$

$$\sup_{P \in \mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^\tau} \left\{ \sum_{j=j_P}^{\infty} 2^{jsq} \left(\int_P [k_{j,P}^{*,a} f(x)]^p dx \right)^{q/p} \right\}^{1/q}$$

are equivalent in $\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)$.

(ii) Let $p \in (0, \infty)$ and $a \in (n \max\{1/p, 1/q\}, \infty)$. Then the quasi-norms $\left\| \left\{ 2^{js}(k_j * \right. \right.$

$\left. \left. f) \right\}_{j \in \mathbb{Z}} \right\|_{L^p_\tau(\ell^q(\mathbb{R}^n))}$, $\left\| \left\{ 2^{js} k_j^{*,a} f \right\}_{j \in \mathbb{Z}} \right\|_{L^p_\tau(\ell^q(\mathbb{R}^n))}$ and

$$\sup_{P \in \mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^\tau} \left\{ \int_P \left(\sum_{j=j_P}^{\infty} 2^{jsq} [k_{j,P}^{*,a} f(x)]^q \right)^{p/q} dx \right\}^{1/p}$$

are equivalent in $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$.

Proof: By similarity, we only prove (ii). Notice that for all $f \in \dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$, $j \in \mathbb{Z}$, $P \in \mathcal{Q}(\mathbb{R}^n)$ and $x \in P$, $|k_j * f(x)| \leq k_{j,P}^{*,a} f(x) \leq k_j^{*,a} f(x)$. Therefore, to show (ii), it suffices to prove that $\left\| \left\{ 2^{js} k_j^{*,a} f \right\}_{j \in \mathbb{Z}} \right\|_{L^p_\tau(\ell^q(\mathbb{R}^n))}$ and $\left\| \left\{ 2^{js}(k_j * f) \right\}_{j \in \mathbb{Z}} \right\|_{L^p_\tau(\ell^q(\mathbb{R}^n))}$ are equivalent quasi-norms in $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$. We show this in three steps.

Step 1. Let $f \in \dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$. First we prove

$$\left\| \left\{ 2^{js} k_j^{*,a} f \right\}_{j \in \mathbb{Z}} \right\|_{L^p_\tau(\ell^q(\mathbb{R}^n))} \lesssim \| \{ 2^{ms} \varphi_m^{*,a} f \}_{m \in \mathbb{Z}} \|_{L^p_\tau(\ell^q(\mathbb{R}^n))}. \tag{14}$$

To this end, letting φ and ψ be as in the proof of Lemma (5.2.7), we see that for all $j \in \mathbb{Z}$ and $y \in \mathbb{R}^n$,

$$k_j * f(y) = \sum_{m \in \mathbb{Z}} k_j * \widetilde{\psi}_m * \varphi_m * f(y). \tag{15}$$

Notice that for all $y \in \mathbb{R}^n$,

$$\left| k_j * \widetilde{\psi}_m * \varphi_m * f(y) \right| \leq \varphi_m^{*,a} f \int_{\mathbb{R}^n} \left| k_j * \psi_m(z) \right| (1 + 2^m |z|)^a dz.$$

By Lemma (5.2.5), we see that for all $y \in \mathbb{R}^n$,

$$\left| k_j * \widetilde{\psi}_m * \varphi_m * f(y) \right| \lesssim \min\{ 2^{(m-j)(\beta+1)}, 2^{(j-m)(M-a)} \} \varphi_m^{*,a} f(y), \tag{16}$$

where $M > a$ can be sufficiently large, which is determined later; see also [213] and its proof. On the other hand, for all $x, y \in \mathbb{R}^n$,

$$\varphi_m^{*,a} f(y) \leq \varphi_m^{*,a} f(x)(1 + 2^m |x - y|)^a \leq \max\{1, 2^{(m-j)a}\}(1 + 2^j |x - y|)^a \varphi_m^{*,a} f(x),$$

which together with (16) yields that

$$\left| k_j * \widetilde{\psi}_m * \varphi_m * f(y) \right| \lesssim \min\{ 2^{(m-j)(\beta+1)}, 2^{(j-m)(M-2a)} \} \varphi_m^{*,a} f(x)(1 + 2^j |x - y|)^a.$$

Then, from this and (15), it follows that for all $x \in \mathbb{R}^n$,

$$k_j^{*,a} * f(x) \leq \sum_{m \in \mathbb{Z}} \sup_{y \in \mathbb{R}^n} \frac{\left| k_j * \widetilde{\psi}_m * \varphi_m * f(y) \right|}{(1 + 2^j |x - y|)^a} \lesssim \sum_{m \in \mathbb{Z}} \varphi_m^{*,a} f(x) \min\{ 2^{(m-j)(\beta+1)}, 2^{(j-m)(M-2a)} \}$$

and hence

$$2^{js}k_j^{*,a} * f(x) \lesssim \sum_{m \in \mathbb{Z}} 2^{ms}\varphi_m^{*,a}f(x)\min\{2^{(m-j)(\beta+1-s)}, 2^{(j-m)(M-2a+s)}\}.$$

Choosing $M > 2a - s + n\tau$, by $\beta + 1 > s + n\tau$ and Lemma (5.2.6), we obtain (14).

Step 2. Next we show that for all $f \in \mathcal{S}_\beta'(\mathbb{R}^n)$,

$$\left\|\{2^{js}\varphi_j^{*,a}f\}_{j \in \mathbb{Z}}\right\|_{L_\tau^p(\ell^q(\mathbb{R}^n))} \lesssim \left\|\{2^{ms}k_m^{*,a}f\}_{m \in \mathbb{Z}}\right\|_{L_\tau^p(\ell^q(\mathbb{R}^n))}. \tag{17}$$

Without loss of generality, we may assume that the right-hand side of (17) is finite.
The proof for (17) is similar to that for (14). In fact, by Lemma (5.2.4), there exists a function $\psi \in \mathcal{S}_\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\widehat{\psi} \subset \{\xi \in \mathbb{R}^n : \frac{11}{20}\varepsilon < |\xi| < \frac{20}{11}\varepsilon\}$ and $\sum_{m \in \mathbb{Z}}\widehat{k}(2^m\xi)\overline{\widehat{\psi}(2^m\xi)} = 1$ for all $\xi \in \mathbb{R}^n\backslash\{0\}$. Then by the Calderón reproducing formula in [189], for all $f \in \mathcal{S}_\beta'(\mathbb{R}^n)$, $\sum_{m \in \mathbb{Z}}\widetilde{\psi}_m * k_m * f$ in $\mathcal{S}_\infty'(\mathbb{R}^n)$ and hence $\varphi_j * f(y) = \sum_{m \in \mathbb{Z}}\varphi_j * \widetilde{\psi}_m * k_m * f(y)$ for all $j \in \mathbb{Z}$ and $y \in \mathbb{R}^n$. Since $\varphi, \psi \in \mathcal{S}_\infty(\mathbb{R}^n)$, similarly to the estimate of (16), by [189], we see that for all $y \in \mathbb{R}^n$,

$$\left|\varphi_j * \widetilde{\psi}_m * k_m * f(y)\right| \lesssim \min\{2^{(m-j)M}, 2^{(j-m)(M-a)}\}k_m^{*,a}f(y),$$

where we chose $M > \max\{0, s + n\tau, 2a - s + n\tau\}$. Then repeating the argument in Step 1, we obtain (17).

From Step 1, Step 2 and Theorem (5.2.3), it follows that $\left\|\{2^{js}\varphi_j^{*,a}f\}_{j \in \mathbb{Z}}\right\|_{L_\tau^p(\ell^q(\mathbb{R}^n))}$ is an equivalent quasi-norm in $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$.

Step 3. To complete the proof of Theorem (5.2.8)(ii), we still need to prove that for all $f \in \dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$,

$$\left\|\{2^{js}k_j^{*,a}f\}_{j \in \mathbb{Z}}\right\|_{L_\tau^p(\ell^q(\mathbb{R}^n))} \lesssim \left\|\{2^{ms}(k_m * f)\}_{m \in \mathbb{Z}}\right\|_{L_\tau^p(\ell^q(\mathbb{R}^n))}, \tag{18}$$

since the converse inequality is trivial. Without loss of generality, we may assume that the right-hand side of (18) is finite.

Since $a > (n/\min\{p, q\}, \infty)$, we choose $r < \min\{p, q\}$ such that $ar > n$. By the argument in the proof of Lemma (5.2.7), there exist functions $\varphi, \psi \in \mathcal{S}_\infty(\mathbb{R}^n)$ such that $f = \sum_{m \in \mathbb{Z}}\psi_j * \widetilde{\varphi}_m * f$ in $\mathcal{S}_\infty'(\mathbb{R}^n)$. For all $N \in \mathbb{N}$, set $f_N = \sum_{m=-N}^{N}\psi_m * \widetilde{\varphi}_m * f$. Then $f_N \in \mathcal{S}'(\mathbb{R}^n)$. By [190, (2.2)], we see that $f_N = \sum_{m=-N}^{N}\sum_{l(Q)=2^{-m}}\langle f, \varphi_Q\rangle\psi_Q$ in $\mathcal{S}_\infty'(\mathbb{R}^n)$. Then, similarly to the proof of [190, Lemma 3.2], we obtain that for all $f \in \dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $N \in \mathbb{N}$,

$$\|f_N\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)}. \tag{19}$$

Also, similarly to the proof of Lemma (5.2.7), we know that there exists a sequence $\{P_N\}_{N \in \mathbb{N}}$ of polynomials with degree no more than $L \equiv \max\{-1, \lfloor s + n(\tau - 1/p)\rfloor\}$ for all $N \in \mathbb{N}$ such that $g \equiv \lim_{N \to \infty}(f_N + P_N)$ exists in $\mathcal{S}_\infty'(\mathbb{R}^n)$ and g is a representative of the equivalence class $f + \mathcal{P}(\mathbb{R}^n)$. Then, by $\beta \geq -1$ and $\beta + 1 > s + n\tau$, for any $\phi \in \mathcal{S}_\beta(\mathbb{R}^n)$ and all $y \in \mathbb{R}^n$, $\phi * f(y) = \lim_{N \to \infty}\phi * f_N(y)$. Hence, for all $j \in \mathbb{Z}$ and $y \in \mathbb{R}^n$, $k_j * f(y) = \lim_{N \to \infty}k_j * f_N(y)$. From the discrete version of the Strömberg-Torchinsky type estimate in [71] (see also [73]), we deduce that for any $M \in \mathbb{N}$, there exists a positive constant $\widetilde{C}(k, r, M)$, depending only on k, r and M, such that for all $y \in \mathbb{R}^n$,

$$\left|k_j * f_N(y)\right|^r \leq \widetilde{C}(k, r, M)\sum_{m \geq j}2^{(j-m)Mr}\int_{\mathbb{R}^n}\frac{2^{mn}|k_m * f_N(z)|^r}{(1 + 2^m|x - y|)^{Mr}}\,dz. \tag{20}$$

Applying Lemma (5.2.7) and (19), we see that for all $m \in \mathbb{Z}$ and $z \in \mathbb{R}^n$,

$$|k_m * f_N(z)| \lesssim \|f_N\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)} 2^{-m\lceil s+n(\tau-1/p)\rceil} \lesssim \|f\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)} 2^{-m\lceil s+n(\tau-1/p)\rceil}.$$

If we choose $M > \max\{n/r, n(1/p-\tau)-s, a, a-s\}$, then for all $y \in \mathbb{R}^n$,

$$\sum_{m \geq j} 2^{(j-m)Mr} \int_{\mathbb{R}^n} \frac{2^{mn}|k_m * f_N(z)|^r}{(1 + 2^m|y-z|)^{Mr}} \, dz$$

$$\lesssim \|f\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)}^r \sum_{m \geq j} 2^{jMr-mr\lceil M+s+n(\tau-1/p)\rceil} \int_{\mathbb{R}^n} \frac{2^{mn}}{(1 + 2^m|y-z|)^{Mr}} \, dz$$

$$\lesssim \|f\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)}^r 2^{-j\lceil s+n(\tau-1/p)\rceil}.$$

Therefore, from (20) and Lebesgue's dominated convergence theorem, it follows that for all $y \in \mathbb{R}^n$,

$$\left|k_j * f_N(y)\right|^r \lesssim \sum_{m \geq j} 2^{(j-m)Mr} \int_{\mathbb{R}^n} \frac{2^{mn}|k_m * f_N(z)|^r}{(1 + 2^m|y-z|)^{Mr}} \, dz. \tag{21}$$

Notice that for all $m \geq j$ and $x, y, z \in \mathbb{R}^n$,

$$2^m|x-z| \leq 2^{m-j}(2^j|x-y| + 2^m|y-z|).$$

Then $1 + 2^m|x-z| \leq 2^{m-j}(1 + 2^j|x-y|)(1 + 2^m|y-z|)$, which combined with (21) yields that for all $x, y \in \mathbb{R}^n$,

$$\frac{\left|k_j * f(y)\right|^r}{(1 + 2^j|x-y|)^{ar}} \lesssim \sum_{m \geq j} 2^{(j-m)(M-a)r} \int_{\mathbb{R}^n} \frac{2^{mn}|k_m * f(z)|^r}{(1 + 2^m|x-z|)^{ar}} \, dz.$$

Thus, by taking the supremum on $y \in \mathbb{R}^n$ in the above formula and $a\,r > n$, we obtain that for all $x \in P$ with $P \in \mathfrak{Q}(\mathbb{R}^n)$,

$$\left[k_j^{*,a} f(x)\right]^r \lesssim \sum_{m \geq j} 2^{(j-m)(M-a)r} \int_{\mathbb{R}^n} \frac{2^{mn}|k_m * f(z)|^r}{(1 + 2^m|x-z|)^{ar}} \, dz$$

$$\lesssim \sum_{m \geq j} 2^{(j-m)(M-a)r} \left[\int_{3P} \frac{2^{mn}|k_m * f(z)|^r}{(1 + 2^m|x-z|)^{ar}} \, dz \right.$$

$$+ \sum_{i \in \mathbb{Z}^n, |i| \geq 2} \int_{P+il(P)} \frac{2^{mn}|k_m * f(z)|^r}{(1 + 2^m|x-z|)^{ar}} \, dz \Bigg]$$

$$\lesssim \sum_{m \geq j} 2^{(j-m)(M-a)r} \left[\sum_{i=0}^{\infty} 2^{-lar} 2^{mn} \int_{|x-z| \leq 2^{l-m}} |k_m * f(z)|^r \chi_{3P}(z) \, dz \right.$$

$$+ \sum_{i \in \mathbb{Z}^n, |i| \geq 2} |i|^{-ar} 2^{(j_P-m)(ar-n)} \mathrm{HL}\left(|k_m * f|^r \chi_{P+il(P)}\right)\left(x + il(P)\right) \Bigg]$$

$$\lesssim \sum_{m \geq j} 2^{(j-m)(M-a)r} \left[\mathrm{HL}(|k_m * f|^r \chi_{3P})(x) \right.$$

$$+ \sum_{i \in \mathbb{Z}^n, |i| \geq 2} |i|^{-ar} \mathrm{HL}\left(|k_m * f|^r \chi_{3P+il(P)}\right)\left(x + il(P)\right) \Bigg].$$

Then, choosing $\varepsilon \in (0, \min\{M-a, M-a+s\})$, by Hölder's inequality, Minkowski's inequality, Fefferman-Stein's vector valued inequality and $a\,r > n$, we further obtain

$$\frac{1}{|P|^{\tau r}}\left\{\int_P\left(\sum_{j=j_P}^\infty 2^{jsq}[k_j^{*,a}f(x)]^q\right)^{\frac{p}{q}}dx\right\}^{\frac{r}{p}}$$

$$\lesssim \frac{1}{|P|^{\tau r}}\left\{\int_P\left(\sum_{j=j_P}^\infty 2^{jsq}\sum_{m\geq j}2^{(j-m)(M-a-\varepsilon)q}\left[HL(|k_m*f|^r\chi_{3P})(x)\right.\right.\right.$$

$$\left.\left.\left.+\sum_{i\in\mathbb{Z}^n,|i|\geq 2}|i|^{-ar}HL(|k_m*f|^r\chi_{P+il(P)})(x+il(P))\right]^{\frac{q}{r}}\right)^{\frac{p}{q}}dx\right\}^{\frac{r}{p}}$$

$$\lesssim \frac{1}{|P|^{\tau r}}\left\{\int_P\left(\sum_{m=j_P}^\infty\sum_{j=j_P}^m 2^{(M-a-\varepsilon+s)q}2^{-m(M-a-\varepsilon)q}\left[HL(|k_m*f|^r\chi_{P+il(P)})(x\right.\right.\right.$$

$$\left.\left.\left.+il(P))\right]^{\frac{q}{r}}\right)^{\frac{p}{q}}dx\right\}^{\frac{r}{p}}\lesssim \|\{2^{ms}(k_m*f)\}_{m\in\mathbb{Z}}\|_{L_\tau^p(\ell^q(\mathbb{R}^n))}^r,$$

which implies (18) and hence completes the proof of Theorem (5.2.8).

For the Besov-Hausdorff space $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and the Triebel-Lizorkin-Hausdorff space $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, we also have the maximal function and the local mean characterizations similar to Theorems (5.2.3) and (5.2.8). To state these results, we first recall some notation.

For $x\in\mathbb{R}^n$ and $r>0$, let $B(x,r)\equiv\{y\in\mathbb{R}^n:|x-y|<r\}$. For $E\subset\mathbb{R}^n$ and $d\in(0,n]$, the d-dimensional Hausdorff capacity of E is defined by

$$H^d(E)\equiv\inf\left\{\sum_j r_j^d:E\subset\bigcup_j B(x_j,r_j)\right\}, \tag{22}$$

where the infimum is taken over all countable coverings $\{B(x_j,r_j)\}_{j=1}^\infty$ of open balls of E; see, for example, [161, 188]. It is well known that H^d is monotone, countably subadditive and vanishes on the empty set. Moreover, H^d in (22) when $d=0$ also makes sense, and H^0 has the properties that for all sets $E\subset\mathbb{R}^n$, $H^0(E)\geq 1$, and $H^0(E)=1$ if and only if E is bounded. For any function $f:\mathbb{R}^n\to[0,\infty]$, the Choquet integral of f with respect to H^d is then defined by

$$\int_{\mathbb{R}^n}f\,dH^d\equiv\int_0^\infty H^d(\{x\in\mathbb{R}^n:f(x)>\lambda\})d\lambda.$$

In what follows, for any $p,q\in(0,\infty]$, let $p\vee q\equiv\max\{p,q\}$ and $p\wedge q\equiv\min\{p,q\}$. Set $\mathbb{R}_+^{n+1}\equiv\mathbb{R}^n\times(0,\infty)$. For any measurable function ω on $\mathbb{R}_+^{n+1}$ and $x\in\mathbb{R}^n$, define its nontangential maximal function $N\omega$ by setting $N\omega(x)\equiv\sup_{|y-x|<t}|\omega(y,t)|$.

For $p\in(0,\infty)$ and $\tau\in[0,\infty)$, let $\ell^q\left(\widetilde{L_\tau^p(\mathbb{R}^n)}\right)$ with $q\in[1,\infty)$ be the set of all sequences $G\equiv\{g_j\}_{j\in\mathbb{Z}}$ of measurable functions on $\mathbb{R}^n$ such that

$$\|G\|_{\ell^q\left(\widetilde{L^p_\tau(\mathbb{R}^n)}\right)} \equiv \inf_\omega \left\{ \sum_{j\in\mathbb{Z}} \left(\int_{\mathbb{R}^n} |g_j(x)|^p [\omega(x, 2^{-j})]^{-p} dx \right)^{q/p} \right\}^{1/q} < \infty$$

and $L^p_\tau\left(\widetilde{\ell^q(\mathbb{R}^n)}\right)$ with $q \in (1, \infty)$ the set of all sequences $G \equiv \{g_j\}_{j\in\mathbb{Z}}$ of measurable functions on $\mathbb{R}^n$ such that

$$\|G\|_{L^p_\tau\left(\widetilde{\ell^q(\mathbb{R}^n)}\right)} \equiv \inf_\omega \left\{ \int_{\mathbb{R}^n} \left(\sum_{j\in\mathbb{Z}} |g_j(x)|^q [\omega(x, 2^{-j})]^{-q} \right)^{p/q} dx \right\}^{1/p} < \infty,$$

where the infimum is taken over all nonnegative Borel measurable functions on $\mathbb{R}^{n+1}_+$ satisfying

$$\int_{\mathbb{R}^n} [N\omega(x)]^{(p\vee q)'} dH^{n\tau(p\vee q)'}(x) \le 1 \tag{23}$$

and with the restriction that for any $j \in \mathbb{Z}, \omega(\cdot, 2^{-j})$ is allowed to vanish only where g_j vanishes.

Theorem (5.2.9) [222]: Let $s \in \mathbb{R}, p \in (1, \infty)$, and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (10).

(i) If $q \in [1, \infty), \tau \in [0, 1/(p \vee q)']$ and $a \in (n[1/p + \tau], \infty)$, then $\left\| \left\{ 2^{js} \varphi_j^{*,a} f \right\}_{j\in\mathbb{Z}} \right\|_{\ell^q\left(\widetilde{L^p_\tau(\mathbb{R}^n)}\right)}$ is

   an equivalent quasi-norm in $B\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$.

(ii) If $q \in (1, \infty), \tau \in [0, 1/(p \vee q)']$ and $a \in (n[\max\{1/p, 1/q\} + \tau], \infty)$, then the quasi-norm $\left\| \left\{ 2^{js} \varphi_j^{*,a} f \right\}_{j\in\mathbb{Z}} \right\|_{L^p_\tau\left(\widetilde{\ell^q(\mathbb{R}^n)}\right)}$ is an equivalent quasi-norm in $F\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$.

Proof: By similarity, we only show (ii). Notice that for all $f \in \mathcal{S}'_\infty(\mathbb{R}^n), \in \mathbb{Z}$ and $y \in \mathbb{R}^n$, $|\varphi_j * f(x)| \le \varphi_j^{*,a} f(x)$. Therefore, to complete the proof of (ii), it suffices to prove that $\left\| \left\{ 2^{js} \varphi_j^{*,a} f \right\}_{j\in\mathbb{Z}} \right\|_{L^p_\tau\left(\widetilde{\ell^q(\mathbb{R}^n)}\right)} \lesssim \|f\|_{F\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)}$ for all $f \in F\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)$.

Let ω be a nonnegative function on $\mathbb{R}^{n+1}_+$ satisfying (23) such that

$$\left\{ \int_{\mathbb{R}^n} \left(\sum_{j\in\mathbb{Z}}^\infty 2^{jsq} |\varphi_j * f(x)|^q [\omega(x, 2^{-j})]^{-q} \right)^{p/q} dx \right\}^{1/p} \lesssim \|f\|_{F\dot{H}^{s,\tau}_{p,q}(\mathbb{R}^n)}. \tag{24}$$

Since $a > n(\max\{1/p, 1/q\} + \tau)$, we choose $\delta \in (0, \min\{p, q\})$ such that $a > n(1/\delta + \tau)$. By (11), we know that for all $x \in Q_{jk}$ with $j \in \mathbb{Z}$ and $k, l \in \mathbb{Z}^n$, $1 + 2^j|x - 2^{-j}l| \sim 1 + |k - l|$ and

$$\left[\varphi_j^{*,a} f(x) \right]^\delta \lesssim \sum_{l\in\mathbb{Z}^n} t_{Q_{jk}} (1 + |k - l|)^{-a\delta} \lesssim \sum_{l\in\mathbb{Z}^n} t_{Q_{jk}} \left(1 + 2^j|x - 2^{-j}l| \right)^{-a\delta},$$

where $t_{Q_{jk}}$ is as in the proof of Theorem (5.2.3). Set $A_0 \equiv \{l \in \mathbb{Z}^n : 2^j|x - 2^{-j}l| \le \sqrt{n}\}$ and $A_m \equiv \{l \in \mathbb{Z}^n : 2^{m-1}\sqrt{n} < 2^{-j}|x - 2^{-j}l| \le 2^m\sqrt{n}\}$ for all $m \in \mathbb{N}$. We further have that for all $x \in Q_{jk}$,

$$\varphi_j^{*,a} f(x) \lesssim \left[\sum_{m=0}^\infty 2^{-ma\delta} \sum_{l\in A_m} t_{Q_{jl}} \right]^{1/\delta} \lesssim \sum_{m=0}^\infty 2^{-m(\varepsilon-a)} \left[\sum_{l\in A_m} t_{Q_{jl}} \right]^{1/\delta},$$

where the last inequality follows from (13) if $1/\delta \le 1$ or Hölder's inequality if $1/\delta > 1$, and $\varepsilon \in \left(0, a - n(1/\delta + \tau)\right)$.

Since $\|\cdot\|_{L_\tau^p(\widetilde{\ell^q(\mathbb{R}^n)})}^{v_2}$ is a norm, the last inequality further implies that

$$\left\|\left\{2^{js}\varphi_j^{*,a}f(x)\right\}_{j\in\mathbb{Z}}\right\|_{L_\tau^p(\widetilde{\ell^q(\mathbb{R}^n)})}^{v_2} \lesssim \sum_{m=0}^\infty 2^{mv_2(\varepsilon-a)}\left\|\left\{2^{js}\left[\sum_{l\in A_m}t_{\mathcal{Q}_{jl}}\right]^{1/\delta}\right\}_{j\in\mathbb{Z}}\right\|_{L_\tau^p(\widetilde{\ell^q(\mathbb{R}^n)})}^{v_2}. \quad (25)$$

For each $m\in\mathbb{Z}_+$ and $(y,s)\in\mathbb{R}_+^{n+1}$, define

$$\omega_m(y,s) \equiv 2^{-mn\tau}\sup\{\omega(z,s) : z\in\mathbb{R}^n, |z-y| < \sqrt{n}2^{m+2}s\}.$$

Then by [204, Corollary 3.1], ω_m still satisfies (23) modulo a multiplicative positive constant independent of m. Moreover, for all $y\in\mathcal{Q}_{jl}$ with $l\in A_m$, we have that $\omega(y,2^{-j}) \le 2^{mn\tau}\omega_m(y,2^{-j})$. Therefore, for all $x\in\mathcal{Q}_{jk}$,

$$\sum_{l\in A_m}t_{\mathcal{Q}_{jl}}[\omega_m(x,2^{-j})]^{-\delta} \lesssim 2^{jn}\sum_{l\in A_m}\int_{\mathcal{Q}_{jl}}|\varphi_j * f(y)|^\delta dy\,[\omega_m(x,2^{-j})]^{-\delta}$$

$$\lesssim 2^{jn}2^{mn\tau\delta}\int_{\mathbb{R}^n}\sum_{l\in A_m}|\varphi_j * f(y)|^\delta\chi_{\mathcal{Q}_{jl}}(y)[\omega(y,2^{-j})]^{-\delta}dy$$

$$\lesssim 2^{mn(1+\tau\delta)}\mathrm{HL}\left(\sum_{l\in A_m}|\varphi_j * f|^\delta\chi_{\mathcal{Q}_{jl}}[\omega(\cdot,2^{-j})]^{-\delta}\right)(x),$$

which combined with Fefferman-Stein's vector-valued inequality and (24) yields that

$$\left\|\left\{2^{js}\left[\sum_{l\in A_m}t_{\mathcal{Q}_{jl}}\right]^{1/\delta}\right\}_{j\in\mathbb{Z}}\right\|_{L_\tau^p(\widetilde{\ell^q(\mathbb{R}^n)})} \lesssim \left\{\int_{\mathbb{R}^n}\left[\sum_{j\in\mathbb{Z}}2^{jsq}\left(\sum_{l\in A_m}t_{\mathcal{Q}_{jl}}[\omega_m(x,2^{-j})]^{-\delta}\right)^{q/\delta}\right]^{p/q}dx\right\}^{1/p}$$

$$\lesssim 2^{mn(1/\delta+\tau)}\left\{\int_{\mathbb{R}^n}\left(\sum_{j\in\mathbb{Z}}2^{jsq}\sum_{l\in A_m}|\varphi_j * f(x)|^q\chi_{\mathcal{Q}_{jl}}(x)[\omega(x,2^{-j})]^{-q}\right)^{p/q}dx\right\}^{1/p}$$

$$\lesssim 2^{mn(1/\delta+\tau)}\|f\|_{F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}.$$

By $a > n(1/\delta+\tau)$ and (25), we further have

$$\left\|\left\{2^{js}\varphi_j^{*,a}f\right\}_{j\in\mathbb{Z}}\right\|_{L_\tau^p(\widetilde{\ell^q(\mathbb{R}^n)})}^{v_2} \lesssim \sum_{m=0}^\infty 2^{mv_2(\varepsilon-a+n(1/\delta+\tau))}\|f\|_{L_\tau^p(\widetilde{\ell^q(\mathbb{R}^n)})}^{v_2} \lesssim \|f\|_{L_\tau^p(\widetilde{\ell^q(\mathbb{R}^n)})}^{v_2},$$

which completes the proof of Theorem (5.2.9).

Lemma (5.2.10) [222]: Let $p\in(1,\infty)$, $\delta\in(0,\infty)$ and $\{g_m\}_{m\in\mathbb{Z}}$ be a sequence of measurable functions on $\mathbb{R}^n$. For all $j\in\mathbb{Z}$ and $x\in\mathbb{R}^n$, let $G_j(x) \equiv \sum_{m\in\mathbb{Z}}2^{-|m-j|\delta}g_m(x)$.

(i) If $q\in[1,\infty)$, $\tau\in[0,1/(p\vee q)']$ and $\delta\in(n\tau,\infty)$, then there exists a positive constant C, independent of $\{g_m\}_{m\in\mathbb{Z}}$, such that $\left\|\{G_j\}_{j\in\mathbb{Z}}\right\|_{\ell^q(\widetilde{L_\tau^p(\mathbb{R}^n)})} \lesssim C\|\{g_m\}_{m\in\mathbb{Z}}\|_{\ell^q(\widetilde{L_\tau^p(\mathbb{R}^n)})}$.

(ii) If $q\in(1,\infty)$, $\tau\in[0,1/(p\vee q)']$ and $\delta\in(n\tau,\infty)$, then there exists a positive constant C, independent of $\{g_m\}_{m\in\mathbb{Z}}$, such that $\left\|\{G_j\}_{j\in\mathbb{Z}}\right\|_{L_\tau^p(\widetilde{\ell^q(\mathbb{R}^n)})} \lesssim C\|\{g_m\}_{m\in\mathbb{Z}}\|_{L_\tau^p(\widetilde{\ell^q(\mathbb{R}^n)})}$.

Proof: By similarity, we only prove (ii). Let $\widetilde{\omega}$ be a nonnegative function on $\mathbb{R}_+^{n+1}$ satisfying (23) such that

$$\left\{\int_{\mathbb{R}^n}\left[\sum_{m\in\mathbb{Z}}|g_m(x)|^q[\widetilde{\omega}(x,2^{-m})]^{-q}\right]^{p/q}dx\right\}^{1/p}\lesssim C\|\{g_m\}_{m\in\mathbb{Z}}\|_{L_\tau^p(\widetilde{\ell^q(\mathbb{R}^n)})}. \tag{26}$$

For each $i\in\mathbb{Z}_+$ and all $(x,s)\in\mathbb{R}_+^{n+1}$, let $\omega_i(x,s)\equiv 2^{-in\tau}\sup\{\widetilde{\omega}(x,t):2^{-i}\le s/t\le 2^i\}$. Then by [204, Corollary 3.1], ω_i still satisfies (23) modulo a multiplicative constant independent of i. Moreover, for all $x\in\mathbb{R}^n$, $m,j\in\mathbb{Z}$ with $|j-m|=i$, we have that $[\omega_i(x,2^{-j})]^{-1}\le 2^{in\tau}[\widetilde{\omega}(x,2^{-m})]^{-1}$.

Since $\|\cdot\|_{L_\tau^p(\widetilde{\ell^q(\mathbb{R}^n)})}^{v_2}$ is a norm, by (26), we see that

$$\left\|\{G_j\}_{j\in\mathbb{Z}}\right\|_{L_\tau^p(\widetilde{\ell^q(\mathbb{R}^n)})}^{v_2}=\inf_\omega\left\{\int_{\mathbb{R}^n}\left[\sum_{j\in\mathbb{Z}}\left(\sum_{i=0}^{\infty}\sum_{\substack{m\in\mathbb{Z}\\|m-j|=i}}2^{-i\delta}\left[\frac{|g_m(x)|}{\omega(x,2^{-j})}\right]\right)^q\right]^{\frac{p}{q}}dx\right\}^{\frac{v_2}{p}}$$

$$\lesssim\sum_{i=0}^{\infty}2^{-i\delta v_2}\inf_\omega\left\{\int_{\mathbb{R}^n}\left[\sum_{j\in\mathbb{Z}}\left(\sum_{\substack{m\in\mathbb{Z}\\|m-j|=i}}\left[\frac{|g_m(x)|}{\omega(x,2^{-j})}\right]\right)^q\right]^{\frac{p}{q}}dx\right\}^{\frac{v_2}{p}}$$

$$\lesssim\sum_{i=0}^{\infty}2^{-i\delta v_2}\left[\left\{\int_{\mathbb{R}^n}\left(\sum_{m\in\mathbb{Z}}\left[\frac{|g_m(x)|}{\omega_i(x,2^{-m+i})}\right]^q\right)^{\frac{p}{q}}dx\right\}^{\frac{v_2}{p}}\right.$$

$$\left.+\left\{\int_{\mathbb{R}^n}\left(\sum_{m\in\mathbb{Z}}\left[\frac{|g_m(x)|}{\omega_i(x,2^{-m-i})}\right]^q\right)^{\frac{p}{q}}dx\right\}^{\frac{v_2}{p}}\right]$$

$$\lesssim\sum_{i=0}^{\infty}2^{-iv_2(\delta-n\tau)}\left\{\int_{\mathbb{R}^n}\left(\sum_{m\in\mathbb{Z}}\left[\frac{|g_m(x)|}{\widetilde{\omega}(x,2^{-m})}\right]^q\right)^{\frac{p}{q}}dx\right\}^{\frac{v_2}{p}}\lesssim\|\{g_m\}_{m\in\mathbb{Z}}\|_{L_\tau^p(\widetilde{\ell^q(\mathbb{R}^n)})}^{v_2},$$

which completes the proof of Lemma (5.2.10).

Lemma (5.2.11) [222]: Let $s\in\mathbb{R}$, $p\in(1,\infty)$, $q\in[1,\infty)$, $\tau\in[0,1/(p\vee q)']$ and β be an integer such that $\beta\ge-1$ and $\beta+1\ge s+n\tau$. Then there exists a positive constant C such that for all $j\in\mathbb{Z}$, $f\in A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $y\in\mathbb{R}^n$, $|k_j*f(y)|\le C\|f\|_{A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}2^{-j[s-n(\tau+1/p)]}$, where $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ denotes either $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ or $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$.

Proof: By similarity, we only consider the Triebel-Lizorkin-Hausdorff spaces $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$. Let $f\in F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $\varphi,\psi\in\mathcal{S}(\mathbb{R}^n)$ be as in the proof of Lemma (5.2.7). From the arguments in [204], we deduce that there exists a sequence $\{P_N\}_{N\in\mathbb{N}}$ of polynomials with degree no more than $L\equiv\max\{-1,\lfloor s-n(\tau+1/p)\rfloor\}$ for all $N\in\mathbb{N}$ such that $g\equiv\lim_{N\to0}\left(\sum_{m=-N}^{N}\widetilde{\psi}_m*\varphi_m*f+P_N\right)$ exists in $\mathcal{S}'(\mathbb{R}^n)$ and g is a representative of the equivalence class $f+\mathcal{P}(\mathbb{R}^n)$. Since $\beta+1\ge s+n\tau$ and $\int_{\mathbb{R}^n}k(x)x^\gamma dx=0$ for all $|\gamma|\le\beta$, we obtain that $k_j*f(y)=\sum_{m\in\mathbb{Z}}k_j*\widetilde{\psi}_m*\varphi_m*f(y)$ for all $j\in\mathbb{Z}$ and $y\in\mathbb{R}^n$. Furthermore, we have $|k_j*f(y)|\lesssim I_1+I_2$, where I_1,I_2 are as in the proof of Lemma (5.2.7).

Let ω be as in (24). Recall that if ω satisfies (23), then for all $(x, s) \in \mathbb{R}_+^{n+1}$, $\omega(x, s) \lesssim s^{-n\tau}$ (see [189]). Choosing $N > n\tau - s$ and applying Hölder's inequality and (24) yield that for all $y \in \mathbb{R}^n$,

$$I_1 \lesssim \sum_{m \geq j} 2^{-m(N-n\tau)+jN} \int_{\mathbb{R}^n} \frac{2^{jn}|\varphi_m * f(z)|[\omega(z, 2^{-m})]^{-1}}{[1 + 2^j|y - z|]^{n+N+1}} \, dz$$

$$\lesssim \sum_{m \geq j} 2^{-m(N+s-n\tau)+jN} \left\{ \int_{\mathbb{R}^n} \frac{2^{jn} 2^{msq}|\varphi_m * f(z)|^p[\omega(z, 2^{-m})]^{-p}}{[1 + 2^j|y - z|]^{n+N+1}} \, dz \right\}^{1/p}$$

$$\lesssim \sum_{m \geq j} 2^{-m(N+s-n\tau)+jN} 2^{jn/p} \|f\|_{F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \lesssim \|f\|_{F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} 2^{-j[s-n(\tau+1/p)]}.$$

Similarly, by the assumption that $\beta + 1 \geq s + n\tau$ and Hölder's inequality, we also have that for all $y \in \mathbb{R}^n$, $I_1 \lesssim 2^{-j[s-n(\tau+1/p)]} \|f\|_{F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}$, which completes the proof of Lemma (5.2.11).

Theorem (5.2.12) [222]: Let $a \in (1, \infty)$, $s \in \mathbb{R}$ and $p \in (1, \infty)$.

(i) If $q \in [1, \infty)$, $\tau \in [0, 1/(p \vee q)']$ and $a \in (n[1/p + \tau], \infty)$ such that $s + n\tau < \beta + 1$, then
$$\left\| \left\{ 2^{js} k_j^{*,a} f \right\}_{j \in \mathbb{Z}} \right\|_{\ell^q(\widetilde{L_\tau^p(\mathbb{R}^n)})} \quad \text{and} \quad \left\| \left\{ 2^{js} k_j * f \right\}_{j \in \mathbb{Z}} \right\|_{\ell^q(\widetilde{L_\tau^p(\mathbb{R}^n)})} \quad \text{are equivalent quasi-norms in } B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n).$$

(ii) If $q \in (1, \infty)$, $\tau \in [0, 1/(p \vee q)']$ and $a \in (n[\max\{1/p, 1/q\} + \tau], \infty)$ such that $s + n\tau < \beta + 1$, then $\left\| \left\{ 2^{js} k_j^{*,a} f \right\}_{j \in \mathbb{Z}} \right\|_{L_\tau^p(\widetilde{\ell^q(\mathbb{R}^n)})}$ and $\left\| \left\{ 2^{js} k_j * f \right\}_{j \in \mathbb{Z}} \right\|_{L_\tau^p(\widetilde{\ell^q(\mathbb{R}^n)})}$ are equivalent quasi-norms in $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$.

Proof: By similarity, we only show (ii), namely, we need to prove that $\left\| \left\{ 2^{js} k_j^{*,a} f \right\}_{j \in \mathbb{Z}} \right\|_{L_\tau^p(\widetilde{\ell^q(\mathbb{R}^n)})}$ and $\left\| \left\{ 2^{js}(k_j * f) \right\}_{j \in \mathbb{Z}} \right\|_{L_\tau^p(\widetilde{\ell^q(\mathbb{R}^n)})}$ are equivalent quasi-norms in $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$. We show this in three steps.

Step 1. Let $f \in F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, φ and ψ be as in the proof of Lemma (5.2.7). Similarly to Step 1 of the proof of Theorem (5.2.8), we see that for all $x \in \mathbb{R}^n$,

$$2^{js} k_j^{*,a} f(x) \lesssim \sum_{m \in \mathbb{Z}} 2^{ms} \varphi_m^{*,a} f(x) \min\{2^{(m-j)(\beta+1-s)}, 2^{(j-m)(M-2a+s)}\}.$$

Choosing $M > 2a + s + n\tau$, by $\beta + 1 \geq s + n\tau$ and Lemma (5.2.10), we obtain

$$\left\| \left\{ 2^{js} k_j^{*,a} f \right\}_{j \in \mathbb{Z}} \right\|_{L_\tau^p(\widetilde{\ell^q(\mathbb{R}^n)})} \lesssim \|\{2^{ms} \varphi_m^{*,a} f\}_{m \in \mathbb{Z}}\|_{L_\tau^p(\widetilde{\ell^q(\mathbb{R}^n)})}.$$

Step 2. Next we show that for all $f \in S_\beta'(\mathbb{R}^n)$,

$$\left\| \left\{ 2^{js} \varphi_j^{*,a} f \right\}_{j \in \mathbb{Z}} \right\|_{L_\tau^p(\widetilde{\ell^q(\mathbb{R}^n)})} \lesssim \|\{2^{ms} k_m^{*,a} f\}_{m \in \mathbb{Z}}\|_{L_\tau^p(\widetilde{\ell^q(\mathbb{R}^n)})}. \tag{27}$$

Without loss of generality, we may assume that the right-hand side of (27) is finite. Similarly to Step 2 of the proof of Theorem (5.2.8), there exists a function $\psi \in S_\infty(\mathbb{R}^n)$ such that for all $j \in \mathbb{Z}$, $f \in S_\beta'(\mathbb{R}^n)$, $\varphi \in S_\infty(\mathbb{R}^n)$ and $y \in \mathbb{R}^n$, $\varphi_j * f(y) = \sum_{m \in \mathbb{Z}} \varphi_j * \widetilde{\psi}_m * k_m * f(y)$. By [189, Lemma 2.2], we obtain that for all $y \in \mathbb{R}^n$,

$$|\varphi_j * \widetilde{\psi}_m * k_m * f(y)| \lesssim \min\{2^{(m-j)M}, 2^{(j-m)(M-a)}\} k_j^{*,a} f(y),$$

where we chose $M > \max\{s + n\tau, 2a - s + n\tau\}$. Then, similarly to Step 1 of the proof of Theorem (5.2.8), we obtain that for all $x \in \mathbb{R}^n$,

$$2^{js} \varphi_j^{*,a} f(x) \lesssim \sum_{m \in \mathbb{Z}} 2^{ms} k_m^{*,a} f(x) \min\{2^{(m-j)(M-s)}, 2^{(j-m)(M-2a+s)}\}.$$

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which together with Lemma (5.2.10) yields (27).

Step 3. Combining Step 1, Step 2 and Theorem (5.2.9) yields that $\left\|\{2^{js}k_j^{*,a}f\}_{j\in\mathbb{Z}}\right\|_{L_\tau^p(\widetilde{\ell^q(\mathbb{R}^n)})}$ is an equivalent quasi-norm in $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$. To complete the proof of (ii), it suffices to prove that for all $f\in\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$,

$$\left\|\{2^{js}k_j^{*,a}f\}_{j\in\mathbb{Z}}\right\|_{L_\tau^p(\widetilde{\ell^q(\mathbb{R}^n)})} \lesssim \|\{2^{ms}(k_m*f)\}_{m\in\mathbb{Z}}\|_{L_\tau^p(\widetilde{\ell^q(\mathbb{R}^n)})}. \tag{28}$$

Without loss of generality, we may assume that the right-hand side of (28) is finite.
Since $a > n[\max\{1/p,1/q\}+\tau]$, we choose $1\le r\le\min\{p,q\}$ such that $a > n(1/r+\tau)$. Let $\widetilde{\omega}$ be a nonnegative function on $\mathbb{R}_+^{n+1}$ satisfying (23) such that

$$\left\{\int_{\mathbb{R}^n}\left(\sum_{j\in\mathbb{Z}}2^{jsq}|k_j*f(x)|^q[\widetilde{\omega}(x,2^{-j})]^{-q}\right)^{p/q}dx\right\}^{1/p} \lesssim \|\{2^{ms}(k_m*f)\}_{m\in\mathbb{Z}}\|_{L_\tau^p(\widetilde{\ell^q(\mathbb{R}^n)})}. \tag{29}$$

Let φ and ψ be as in the proof of Lemma (5.2.11). Then $f=\sum_{m\in\mathbb{Z}}\widetilde{\psi}_m*\varphi_j*f$ in $\mathcal{S}_\infty'(\mathbb{R}^n)$. By the argument in [204], we know that there exists a sequence $\{P_N\}_{N\in\mathbb{N}}$ of polynomials with degree no more than $L\equiv\max\{-1,\lfloor s-n(\tau+1/p)\rfloor\}$ for all $N\in\mathbb{N}$ such that $g\equiv\lim_{N\to\infty}(f_N+P_N)$ exists in $\mathcal{S}'(\mathbb{R}^n)$ and g is a representative of the equivalence class $f+\mathcal{P}(\mathbb{R}^n)$, where $f_N\in\mathcal{S}'(\mathbb{R}^n)$ is as in the proof of Theorem (5.2.8). By Lemma (5.2.11) and [204, Lemma 2.1], we know that for all $j\in\mathbb{Z}$, $f\in F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, $y\in\mathbb{R}^n$ and $N\in\mathbb{N}$, $|k_j*f_N(y)|\lesssim\|f_N\|_{F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}2^{-j[s-n(\tau+1/p)]}\lesssim\|f\|_{F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}2^{-j[s-n(\tau+1/p)]}$. On the other hand, since $\beta+1\ge s+n\tau$, we see that for all $j\in\mathbb{Z}$ and $y\in\mathbb{R}^n$, $k_j*f(y)=\lim_{N\to\infty}k_j*f_N(y)$. Similarly to the proof of Theorem (5.2.8), we obtain that for all $j\in\mathbb{Z}$ and $x\in\mathbb{R}^n$,

$$[k_j^{*,a}f(x)]^r \lesssim \sum_{m\ge j}2^{(j-m)(M-a)r}\int_{\mathbb{R}^n}\frac{2^{mn}|k_m*f(z)|^r}{[1+2^m|x-z|]^{ar}}dz,$$

where we chose $M > a+n\tau-s$. Then for all $x\in\mathbb{R}^n$,

$$[2^{js}k_j^{*,a}f(x)]^r \lesssim \sum_{m\ge j}2^{(j-m)(M-a)r}2^{jsr}2^{mn}\int_{\mathbb{R}^n}\frac{|k_m*f(z)|^r}{[1+2^m|x-z|]^{ar}}dz$$

$$\lesssim \sum_{i=0}^\infty 2^{-iar}\left\{\sum_{m\ge j}2^{(j-m)(M-a)r}2^{jsr}2^{mn}\int_{|z-x|\le 2^{i-m}}|k_m*f(z)|^rdz\right\} \equiv \sum_{i=0}^\infty 2^{-iar}J_{j,i},$$

which, together with (13) and the fact that $\|\cdot\|_{L_\tau^p(\widetilde{\ell^q(\mathbb{R}^n)})}^{v_2}$ is a norm, yields that

$$\left\|\{2^{js}k_j^{*,a}f\}_{j\in\mathbb{Z}}\right\|_{L_\tau^p(\widetilde{\ell^q(\mathbb{R}^n)})}^{v_2} \lesssim \sum_{i=0}^\infty 2^{-iav_2}\left\|\{(J_{j,i})^{1/r}\}_{j\in\mathbb{Z}}\right\|_{L_\tau^p(\widetilde{\ell^q(\mathbb{R}^n)})}^{v_2}. \tag{30}$$

Furthermore, by (13) and the fact that $\|\cdot\|_{L_\tau^p(\widetilde{\ell^q(\mathbb{R}^n)})}^{v_2}$ is a norm again, we obtain

$$\left\|\left\{(J_{j,i})^{\frac{1}{r}}\right\}_{j\in\mathbb{Z}}\right\|_{L^p_\tau(\widetilde{\ell^q(\mathbb{R}^n)})}^{v_2} \sim \inf_\omega \left\{\int_{\mathbb{R}^n}\left[\sum_{j\in\mathbb{Z}}\left(\sum_{l=0}^\infty 2^{-l(M-a)r}\,2^{jsr}2^{(j+l)n}\int_{|z-x|\le 2^{i-j-l}}|k_{j+l}\right.\right.\right.$$

$$\left.\left.\left. * f(z)|^r dz\right)^{\frac{q}{r}}[\omega(x,2^{-j})]^{-q}\right]^{\frac{p}{q}}dx\right\}^{\frac{v_2}{p}}$$

$$\lesssim \sum_{l=0}^\infty 2^{-l(M-a)v_2}\inf_\omega\left\{\int_{\mathbb{R}^n}\left[\sum_{j\in\mathbb{Z}}2^{jsq}\left(2^{(j+l)n}\int_{|z-x|\le 2^{i-j-l}}|k_{j+l}\right.\right.\right.$$

$$\left.\left.\left. * f(z)|^r[\omega(x,2^{-j})]^{-r}dz\right)^{\frac{q}{r}}\right]^{\frac{p}{q}}dx\right\}^{\frac{v_2}{p}}. \tag{31}$$

For all $i,l\in\mathbb{Z}_+$ and all $(x,s)\in\mathbb{R}^{n+1}_+$, let $\omega_{i,l}(x,s)\equiv 2^{-(i+l)n\tau}\sup\{\widetilde{\omega}(y,t):$ $|x-y|\le 2^i t, 2^{-l}\le t/s\le 2^l\}$. Then by [204, Corollary 3.1], $\omega_{i,l}$ still satisfies (23) modulo a multiplicative constant independent of i and l. Moreover, for all $m,j\in\mathbb{Z}$ with $|j-m|=l$ and $x,y\in\mathbb{R}^n$ with $|z-x|\le 2^{i-m}$, we have that $[\omega_{i,l}(x,2^{-j})]^{-1}\le 2^{(i+l)n\tau}[\widetilde{\omega}(z,2^{-m})]^{-1}$. Thus, from Fefferman-Stein's vector-valued inequality, (29) and (31), it follows that

$$\left\| \left\{ \left(J_{j,i}\right)^{\frac{1}{r}} \right\}_{j\in\mathbb{Z}} \right\|_{L^p_\tau(\widetilde{\ell^q(\mathbb{R}^n)})}^{v_2}$$

$$\lesssim \sum_{l=0}^{\infty} 2^{-l(M-a+s)v_2} \left\{ \int_{\mathbb{R}^n} \left[\sum_{j\in\mathbb{Z}} 2^{jsq} \left(2^{jn} \int_{|z-x|\leq 2^{i-j}} |k_j \right. \right. \right.$$

$$* \left. \left. \left. f(z)|^r [\omega_{i,l}(x, 2^{-j-l})]^{-r} dz \right)^{\frac{q}{r}} \right]^{\frac{p}{q}} dx \right\}^{\frac{v_2}{p}}$$

$$\lesssim \sum_{l=0}^{\infty} 2^{-l(M-a+s-n\tau)v_2} 2^{in\tau v_2} \left\{ \int_{\mathbb{R}^n} \left[\sum_{j\in\mathbb{Z}} 2^{jsq} \left(2^{jn} \int_{|z-x|\leq 2^{i-j}} |k_j \right. \right. \right.$$

$$* \left. \left. \left. f(z)|^r [\widetilde{\omega}(z, 2^{-j})]^{-r} dz \right)^{\frac{q}{r}} \right]^{\frac{p}{q}} dx \right\}^{\frac{v_2}{p}}$$

$$\lesssim \sum_{l=0}^{\infty} 2^{-l(M-a+s-n\tau)v_2} 2^{i(\tau+1/r)nv_2} \left\{ \int_{\mathbb{R}^n} \left(\sum_{j\in\mathbb{Z}} 2^{jsq} [HL(|k_j \right. \right.$$

$$* \left. f|^r [\widetilde{\omega}(\cdot, 2^{-j})]^{-r})(x)]^{\frac{q}{r}} \right)^{\frac{p}{q}} dx \right\}^{\frac{v_2}{p}} \lesssim 2^{i(\tau+1/r)nv_2} \left\| \left\{ 2^{js}(k_j * f) \right\}_{j\in\mathbb{Z}} \right\|_{L^p_\tau(\widetilde{\ell^q(\mathbb{R}^n)})}^{v_2},$$

which together with (30) yields that

$$\left\| \left\{ 2^{js} k_j^{*,a} f \right\}_{j\in\mathbb{Z}} \right\|_{L^p_\tau(\widetilde{\ell^q(\mathbb{R}^n)})}^{v_2} \lesssim \left\{ \sum_{i=0}^{\infty} 2^{-iv_2[a-n(1/r+\tau)]} \right\} \left\| \left\{ 2^{js}(k_j * f) \right\}_{j\in\mathbb{Z}} \right\|_{L^p_\tau(\widetilde{\ell^q(\mathbb{R}^n)})}^{v_2}$$

$$\lesssim \left\| \left\{ 2^{js}(k_j * f) \right\}_{j\in\mathbb{Z}} \right\|_{L^p_\tau(\widetilde{\ell^q(\mathbb{R}^n)})}^{v_2}$$

and hence completes the proof of Theorem (5.2.12).

We give a simple application of Theorems (5.2.8) and (5.2.9), we establish a Fourier multiplier theorem of the spaces $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, where $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$ denotes either $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ or $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ denotes either $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ or $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$.

Let $s \in \mathbb{R}$, $p, q \in (0, \infty]$, $a \in [0, \infty)$ and φ be as in Definition (5.2.1). The space $\dot{B}_{p,q}^s(a)$ is defined to be the set of all $m \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that

$$\|m\|_{\dot{B}_{p,q}^s(a)} \equiv \left\| \left\{ 2^{js}(1 + 2^j|\cdot|)^a \varphi_j * m \right\}_{j\in\mathbb{Z}} \right\|_{l^q(\widetilde{L^p(\mathbb{R}^n)})} < \infty;$$

see [66]. Then we have the following theorem, whose proof is similar to that of [66, Theorem 5.1].

Theorem (5.2.13) [222]: Let $s \in \mathbb{R}$.

(i) If $p, q \in (0, \infty]$, $\tau \in [0, \infty)$, $a \in (n/p, \infty)$ when $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n) \equiv \dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ or $a \in (n[\max\{1/p, 1/q\}], \infty)$ when $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n) \equiv \dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $m \in \dot{B}_{1,\infty}^0(a)$, then

$$\|m * f\|_{\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)} \leq C\|m\|_{\dot{B}_{1,\infty}^0(a)} \|f\|_{\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)},$$

where C is a positive constant independent of m and f.

(ii) If $\ p \in (1, \infty)$, $\ q \in [1, \infty)$, $\ \tau \in \left[0, \frac{1}{(p \vee q)'}\right]$, $\ a \in (n[1/p + \tau], \infty)$ when $\ A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n) \equiv$
$B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ or $\ a \in (n[\max\{1/p, 1/q\} + \tau], \infty)$ when $\ A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n) \equiv F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, and $\ m \in$
$\dot{B}_{1,\infty}^0(a)$, then
$$\|m * f\|_{A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \le C\|m\|_{\dot{B}_{1,\infty}^0(a)}\|f\|_{A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)},$$
where C is a positive constant independent of m and f.

Proof: By similarity, we only consider the spaces $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (10).
Then $\varphi * \varphi$ also satisfies (10). Recall that $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is independent of the choice of φ.
Thus, $\left\|\left\{2^j(\varphi_j * \varphi_j * f)\right\}_{j \in \mathbb{Z}}\right\|_{L_\tau^p(\ell^q(\mathbb{R}^n))}$ is an equivalent quasi-norm of $f \in \dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$. Notice
that for all $x \in \mathbb{R}^n$,
$$\left|\varphi_j * \varphi_j * m * f(x)\right| \le \left\{\int_{\mathbb{R}^n}\left|\varphi_j * m(y)\right|(1 + 2^j|y|)^a dy\right\}\varphi_j^{*,a}f(x) = \|m\|_{\dot{B}_{1,\infty}^0(a)}\varphi_j^{*,a}f(x).$$
Then applying Theorem (5.2.3) yields that
$$\|m * f\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)} \lesssim \left\|\left\{2^j(\varphi_j * \varphi_j * m * f)\right\}_{j \in \mathbb{Z}}\right\|_{L_\tau^p(\ell^q(\mathbb{R}^n))} \lesssim \|m\|_{\dot{B}_{1,\infty}^0(a)}\|f\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)},$$
which completes the proof of Theorem (5.2.13).

Chapter 6
Atomic Decomposition and Besov-Type Spaces

We characterize the Besov spaces with variable smoothness and integrability by so-called Peetre maximal functions. We use these results to show the atomic decomposition for these spaces. As an application of their atomic characterization, we obtain a trace theorem of these variable Besov-type spaces.

Section (6.1): Besov Spaces with Variable Smoothness and Integrability

Besov spaces of variable smoothness and integrability, $B^{\alpha(\cdot)}_{p(\cdot)q(\cdot)}$, initially appeared in [160]. Several basic properties were established, such as the Fourier analytical characterization. When p, q, α are constants they coincide with the usual function spaces $B^{\alpha}_{p,q}$. Also Sobolev type embeddings and the characterization by approximations of these function spaces were obtained. Some properties of such a type are well known with variable p, but fixed q and α. J. Vybiral [157] proved Sobolev type embeddings in these spaces. H. Kempka [150, 264] has studied so-called micro-local versions of variable index Besov spaces, when local means characterizations, atomic, molecular and wavelet decomposition of these spaces are given. This setting includes also some range of weights as well as slightly more general smoothness. These studies were all restricted to variable p, but fixed q. Also J.-S. Xu [138, 158] has studied Besov spaces with variable p, but fixed q and α.

The interest in these spaces comes not only from theoretical reasons but also from their applications to several classical problems in analysis. For the range of parameters $p = q = 2$, the spaces $B^{m(\cdot)}_{2,2}(\mathbb{R})$ have been considered in the analysis of certain Black-Scholes equations, see Schneider, Reichmann and Schwab [135]. For further considerations of PDEs, see [227].

We present a decomposition by atoms for $B^{\alpha(\cdot)}_{p(\cdot)q(\cdot)}$. All these results generalize the existing classical results on Besov spaces by taking p, q and α as constants.

We define the Besov spaces $B^{\alpha(\cdot)}_{p(\cdot)q(\cdot)}$ and repeat some results from [160]. We show a useful characterization of these spaces based on the so-called local means. The theorem on local means that proved for Besov spaces of variable smoothness and integrability is highly technical and its proved required (based on maximal functions and the classical situation) new techniques and ideas. Using the results, we show the atomic decomposition for $B^{\alpha(\cdot)}_{p(\cdot)q(\cdot)}$.

As usual, $\mathbb{R}^n$ the n-dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The letter $\mathbb{Z}$ stands for the set of all integer numbers. For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$, we write $|\alpha| = \alpha_1 + \cdots + \alpha_n$. The Euclidean scalar product of $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ is given by $x \cdot y = x_1 y_1, + \cdots + x_n y_n$. For $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B(x, r)$ the open ball in $\mathbb{R}^n$ with center x and radius r. By $\operatorname{supp} f$ we denote the support of the function f, i.e., the closure of its non-zero set. By $\mathcal{S}(\mathbb{R}^n)$ we denote the Schwartz space of all complex-valued, infinitely differentiable and rapidly decreasing functions on $\mathbb{R}^n$ and by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$. We define the Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}(f)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) dx.$$

Its inverse is denoted by $\mathcal{F}^{-1}f$. Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended to the dual Schwartz space $\mathcal{S}'(\mathbb{R}^n)$ in the usual way. The Hardy-Littlewood maximal operator $\mathcal{M}$ is defined on L^1_{loc} by

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy,$$

and $\mathcal{M}_t f = \mathcal{M}|f|^t$ for any $0 < t \le 1$. The variable exponents that we consider are always measurable functions on $\mathbb{R}^n$ with range in $[c, \infty]$ for some $c > 0$. We denote the set of such functions by $\mathcal{P}_0$. The subset of variable exponents with range $[1, \infty]$ is denoted by $\mathcal{P}$.

$$p^- = \operatorname*{ess\,inf}_{x \in \mathbb{R}^n} p(x), \qquad p^+ = \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} p(x).$$

We define

$$\rho_p(t) = \begin{cases} t^p & \text{if} \quad p \in (0, \infty), \\ 0 & \text{if } p = \infty \text{ and } t \le 1, \\ \infty & \text{if } p = \infty \text{ and } t > 1. \end{cases}$$

The convention $1^\infty = 0$ is adopted in order that ρ_p be left-continuous. The variable exponent modular is defined by

$$\varrho_{p(\cdot)}(f) = \int_{\mathbb{R}^n} \rho_{p(x)}(|f(x)|)dx.$$

The variable exponent Lebesgue space $L^{p(\cdot)}$ consists of measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ with $\varrho_{p(\cdot)}(\lambda f) < \infty$ for some $\lambda > 0$.

We define the Luxemburg (quasi)-norm on this space by the formula

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \varrho_{p(\cdot)}\left(\frac{f}{\lambda}\right) \le 1\right\}.$$

As is known, the following inequalities hold (see [227])

$$\|f\|_{p(\cdot)} \le 1 \quad \Leftrightarrow \quad \varrho_{p(\cdot)}(f) \le 1 \tag{1}$$

and

$$\|f\|_{p(\cdot)} \le \varrho_{p(\cdot)}(f) + 1. \tag{2}$$

Let $p, q \in \mathcal{P}_0$. The mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$ is defined on sequences of $L^{p(\cdot)}$-functions by the modular

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) = \sum_v \inf\left\{\lambda_v > 0 : \varrho_{p(\cdot)}\left(\frac{f_v}{\lambda_v^{1/q(\cdot)}}\right) \le 1\right\}.$$

The norm is defined from this as usual:

$$\|(f_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = \inf\left\{\mu > 0 : \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\frac{1}{\mu}(f_v)_v\right) \le 1\right\}.$$

We will use the notation

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) = \sum_v \left\|\,|f_v|^{q(\cdot)}\,\right\|_{\frac{p(\cdot)}{q(\cdot)}}$$

for the modular. In (1) and (2) $\|f\|_{p(\cdot)}$ and $\varrho_{p(\cdot)}(f)$ can be replaced by $\|(f_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ and $\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v)$, respectively.

We say that $g : \mathbb{R}^n \to \mathbb{R}$ is locally log-Hölder continuous, abbreviated $g \in C^{\log}_{\text{loc}}$, if there exists $c_1 > 0$ such that

$$|g(x) - g(y)| \le \frac{c_1}{\log(e + 1/|x - y|)}$$

for all $x, y \in \mathbb{R}^n$. We say that g satisfies the log-Hölder decay condition, if there exists $g_\infty > 0$ and a constant $c_2 > 0$ such that

$$|g(x) - g_\infty| \leq \frac{c_2}{\log(e + |x|)}$$

for all $x \in \mathbb{R}^n$. We say that g is globally-log-Hölder continuous, abbreviated $g \in C^{\log}$, if it is locally log-Hölder continuous and satisfies the log-Hölder decay condition. The constants c_1 and c_2 are called the locally log-Hölder constant and the log-Hölder decay constant, respectively. The maximum $\max\{c_1, c_2\}$ is just called the log-Hölder constant of g and it is denoted by $c_{\log}(g)$.

We note that all functions $g \in C_{\text{loc}}^{\log}$ always belong to L^∞.

We define the following class of variable exponents

$$\mathcal{P}^{\log} = \left\{ p \in \mathcal{P} : \frac{1}{p} \text{ is globally-log-Hölder continuous} \right\}.$$

The class $\mathcal{P}_0^{\log}$ is defined analogously. It was shown in [227] that $\mathcal{M} : L^{p(\cdot)} \to L^{p(\cdot)}$ is bounded if $p \in \mathcal{P}^{\log}$ and $p^- > 1$, see [95] and [97], where various results on maximal function in variable Lebesgue spaces were obtained.

Recall that $\eta_{v,m}(x) = 2^{nv}(1 + 2^v|x|)^{-m}$, for any $x \in \mathbb{R}^n$, $v \in \mathbb{N}_0$ and $m > 0$. Note that $\eta_{v,m} \in L^1$ when $m > n$ and that $\|\eta_{v,m}\|_1 = c_m$ is independent of v.

By c we denote generic positive constants, which may have different values at different occurrences. Although the exact values of the constants are usually irrelevant for our purposes, sometimes we emphasize their dependence on certain parameters (e.g. $c(p)$ means that c depends on p, etc.).

Lemma (6.1.1) [258]: If $\alpha \in C_{\text{loc}}^{\log}$, then there exists $d \in (n, \infty)$ such that if $m > d$, then
$$2^{v\alpha(x)}\eta_{v,2m}(x - y) \leq c2^{v\alpha(y)}\eta_{v,m}(x - y)$$
with $c > 0$ independent of $x, y \in \mathbb{R}^n$ and $v \in \mathbb{N}_0$.

The previous lemma allows us to treat the variable smoothness in many cases as if it were not variable at all, namely we can move the term inside the convolution as follows:
$$2^{v\alpha(x)}\eta_{v,2m} * f(x) \leq c\, \eta_{v,m} * \left(2^{v\alpha(\cdot)}f\right)(x).$$

The next lemma often allows us to deal with exponents which are smaller than 1.

Lemma (6.1.2) [258]: Let $r > 0$, $v \in \mathbb{N}_0$ and $m > n$. Then there exists $c = c(r, m, n) > 0$ such that for all $g \in \mathcal{S}'(\mathbb{R}^n)$ with $\text{supp}\, \mathcal{F}g \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 2^{v+1}\}$, we have
$$|g(x)| \leq c\left(\eta_{v,m} * |g|^r(x)\right)^{1/r}, \qquad x \in \mathbb{R}^n .$$

The next lemma is a Hardy-type inequality which is easy to prove.

Lemma (6.1.3) [258]: Let $0 < a < 1$ and $0 < q \leq \infty$. Let $\{\varepsilon_k\}_{k \in \mathbb{N}_0}$ be a sequence of positive real numbers, such that
$$\left\|\{\varepsilon_k\}_{k \in \mathbb{N}_0}\right\|_{\ell^q} = I < \infty.$$
The sequence $\left\{\delta_k : \delta_k = \sum_{j=0}^\infty a^{|k-j|}\varepsilon_j\right\}_{k \in \mathbb{N}_0}$ is in ℓ^q with
$$\left\|\{\delta_k\}_{k \in \mathbb{N}_0}\right\|_{\ell^q} \leq cI.$$
c depends only on a and q.

Lemma (6.1.4) [258]: Let $\omega, \mu \in \mathcal{S}(\mathbb{R}^n)$ and $M \geq -1$, an integer such that $\int_{\mathbb{R}^n} x^\alpha \mu(x)dx = 0$ for all $|\alpha| \leq M$. Then for any $N > 0$, there is a constant $c_N > 0$ so that
$$\sup_{z \in \mathbb{R}^n} |t^{-n}\mu(t^{-1} \cdot) * \omega(z)|(1 + |z|)^N \leq c_N t^{M+1}.$$

Lemma (6.1.5) [258]: Let $0 < r \leq 1$, and let $\{b_j\}_{j \in \mathbb{N}_0}$, $\{d_j\}_{j \in \mathbb{N}_0}$ be two sequences taking values in $(0, +\infty)$. Assume that for some $N_0 > 0$
$$d_j = O(2^{jN_0}), \qquad j \to \infty, \tag{3}$$

and that for any $N > 0$

$$d_j \leq C_N \sum_{k=j}^{\infty} 2^{(j-k)N} b_k d_k^{1-r}, \qquad j \in \mathbb{N}_0.$$

Then for any $N > 0$

$$d_j^r \leq C_N \sum_{k=j}^{\infty} 2^{(j-k)Nr} b_k, \qquad j \in \mathbb{N}_0$$

with the same constants C_N.

We present the Fourier analytical definition of the spaces $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$ and recall their basic properties. We first need the concept of a smooth dyadic resolution of unity.

Definition (6.1.6) [258]: Let ψ be a function in $S(\mathbb{R}^n)$ satisfying $\psi(x) = 1$ for $|x| \leq 1$ and $\psi(x) = 0$ for $|x| \geq 2$. We put $\mathcal{F}\varphi_0(x) = \psi(x)$, $\mathcal{F}\varphi_1(x) = \psi\left(\frac{x}{2}\right) - \psi(x)$ and

$$\mathcal{F}\varphi_v(x) = \mathcal{F}\varphi_1(2^{-v+1}x) \quad \text{for } v = 2, 3, \cdots$$

Then $\{\mathcal{F}\varphi_v\}_{v \in \mathbb{N}_0}$ is a smooth dyadic resolution of unity,

$$\sum_{v=0}^{\infty} \mathcal{F}\varphi_v(x) = 1$$

for all $x \in \mathbb{R}^n$. Thus we obtain the Littlewood-Paley decomposition

$$f = \sum_{v=0}^{\infty} \varphi_v * f$$

of all $f \in S'(\mathbb{R}^n)$ (convergence in $S'(\mathbb{R}^n)$).

Definition (6.1.7) [258]: Let φ_v be as in Definition (6.1.6). For $\alpha : \mathbb{R}^n \to \mathbb{R}$ and $p, q \in \mathcal{P}_0$, the Besov space $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$ consists of all distributions $f \in S'(\mathbb{R}^n)$ such that

$$\|f\|_{B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} = \left\| \left(2^{v\alpha(\cdot)} \varphi_v * f\right)_v \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty. \tag{4}$$

For any $p, q \in \mathcal{P}_0^{\log}$ and $\alpha \in C_{loc}^{\log}$, the space $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$ does not depend on the chosen smooth dyadic resolution of unity $\{\mathcal{F}\varphi_v\}_{v \in \mathbb{N}_0}$ (in the sense of equivalent quasi-norms). They are quasi-Banach spaces, and

$$S(\mathbb{R}^n) \hookrightarrow B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \hookrightarrow S'(\mathbb{R}^n).$$

Moreover, if p, q, α are constants, we re-obtain the usual Besov spaces $B_{p,q}^{\alpha}$ studied in detail by H. Triebel in [41, 56, 57, 136].

The full treatment of the spaces $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$ can be found in [160] and [227], see [138, 150, 157, 264], for further results on the variable Besov spaces $B_{p(\cdot),q}^{\alpha(\cdot)}$ (only the case of constant q was considered, see also [90, 91]).

We characterize the spaces $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$ by so-called local means (see [213]). Therefore, we define for $a > 0$, $\alpha : \mathbb{R}^n \to \mathbb{R}$ and $f \in S'(\mathbb{R}^n)$, the Peetre maximal function

$$\varphi_v^{*,a} 2^{v\alpha(\cdot)} f(x) = \sup_{y \in \mathbb{R}^n} \frac{2^{v\alpha(y)} |\varphi_v * f(y)|}{(1 + 2^v |x - y|)^a}, \quad v \in \mathbb{N}_0.$$

Theorem (6.1.8) [258]: Let $\alpha \in C_{loc}^{\log}$, $p, q \in \mathcal{P}_0^{\log}$ and $a > \frac{n}{p^-}$. Then

$$\left\| \left(\varphi_v^{*,a} 2^{v\alpha(\cdot)} f\right)_v \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \tag{5}$$

is an equivalent quasi-norm in $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$.

Proof: We will do the proof in two steps.

Step 1. It is easy to see that for any $f \in \mathcal{S}'(\mathbb{R}^n)$ and any $x \in \mathbb{R}^n$ we have
$$2^{v\alpha(x)}|\varphi_v * f(x)| \leq \varphi_v^{*,a} 2^{v\alpha(\cdot)} f(x).$$
This shows that the right-hand side in (4) is less than or equal (5).

Step 2. We will prove in this step that there is a constant $c > 0$ such that for every $f \in \mathcal{S}'(\mathbb{R}^n)$
$$\left\| \left(\varphi_v^{*,a} 2^{v\alpha(\cdot)} f \right)_v \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leq c \left\| \left(2^{v\alpha(\cdot)} \varphi_v * f \right)_v \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}.$$
By a scaling argument, we see that it suffices to consider the case
$$\left\| \left(2^{v\alpha(\cdot)} \varphi_v * f \right)_v \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = 1$$
and show that the modular of a constant times the function on the left-hand side is bounded. In particular, we will show that
$$\sum_{v=0}^{\infty} \left\| \left| c\varphi_v^{*,a} 2^{v\alpha(\cdot)} f \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq C \quad \text{whenever} \quad \sum_{v=0}^{\infty} \left\| \left| 2^{v\alpha(\cdot)} \varphi_v * f \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} = 1.$$
This clearly follows from the inequality
$$\left\| \left| c\varphi_v^{*,a} 2^{v\alpha(\cdot)} f \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq \left\| \left| 2^{v\alpha(\cdot)} \varphi_v * f \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} + 2^{-\sigma v} = \delta,$$
for some $\sigma > 0$. This claim can be reformulated as showing that
$$\left\| \delta^{-1} \left| c\varphi_v^{*,a} 2^{v\alpha(\cdot)} f \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq 1,$$
which is equivalent to
$$\left\| c\delta^{-\frac{1}{q(\cdot)}} \varphi_v^{*,a} 2^{v\alpha(\cdot)} f \right\|_{p(\cdot)} \leq 1.$$
We choose $t > 0$ such that $a > \frac{n}{t} > \frac{n}{p^-}$. By Lemmas (6.1.2) and (6.1.1) the estimates
$$2^{v\alpha(y)}|\varphi_v * f(y)| \leq C 2^{v\alpha(y)} \left(\eta_{v,2m} * |\varphi_v * f|^t(y) \right)^{1/t} \leq C \left(\eta_{v,m} * \left(2^{v\alpha(\cdot)}|\varphi_v * f| \right)^t (y) \right)^{1/t} \quad (6)$$
are true for any $y \in \mathbb{R}^n$, $v \in \mathbb{N}_0$ and any $m > d$ (with d as in Lemma (6.1.1)). Divide both sides of (6) by $(1 + 2^v|x - y|)^a$, in the right-hand side we use the inequality
$$(1 + 2^v|x - y|)^{-a} \leq (1 + 2^v|x - z|)^{-a}(1 + 2^v|y - z|)^a, \quad x, y, z \in \mathbb{R}^n,$$
in the left-hand side take the supremum over $y \in \mathbb{R}^n$ and get for all $f \in \mathcal{S}'(\mathbb{R}^n)$, any $x \in \mathbb{R}^n$, $m > \max(d, at)$ and any $v \in \mathbb{N}_0$
$$\left(\varphi_v^{*,a} 2^{v\alpha(\cdot)} f(x) \right)^t \leq C 2^{vn} \int_{\mathbb{R}^n} \frac{2^{v\alpha(z)t}|\varphi_v * f(z)|^t}{(1 + 2^v|x - z|)^{at}} \, dz$$
$$= C \int_{B(x,2^{-v/2})} \cdots dz + C \sum_{i=0}^{\infty} \int_{B(x,2^{-v/2+i+1}) \setminus B(x,2^{-v/2+i})} \cdots dz = J_v^1(x) + \sum_{i \geq 0} J_{v-i}^2(x),$$
where $C > 0$ is independent of x, v and f. We choose $\sigma > 0$ such that
$$0 < \sigma < \frac{a - n/t}{4(1/q^- - 1/q^+)}.$$
Since $1/q$ is log-Hölder continuous and $\delta \in [2^{-\sigma v}, 1 + 2^{-\sigma v}]$, we have
$$\delta^{\left(\frac{1}{q(z)} - \frac{1}{q(x)} \right)} \leq (2^{\sigma v}\delta)^{\left| \frac{1}{q(z)} - \frac{1}{q(x)} \right|} 2^{v\sigma \left| \frac{1}{q(z)} - \frac{1}{q(x)} \right|} \leq c 2^{2c_{\log}(q)\sigma v / \log(e+1/|x-z|)} \leq c \quad (7)$$
for any $z \in B\left(x, 2^{-v/2}\right)$. Hence
$$\delta^{-\frac{t}{q(x)}} J_v^1(x) \leq C 2^{vn} \int_{\mathbb{R}^n} \frac{\delta^{-\frac{t}{q(z)}} 2^{v\alpha(z)t}|\varphi_v * f(z)|^t}{(1 + 2^v|x - z|)^{at}} \, dz.$$

Now the function $z \mapsto \frac{1}{(1+|z|)^{at}}$ is in L^1 (since $a > n/t$), then using the majorant property for the Hardy-Littlewood maximal operator $\mathcal{M}$, see [181],

$$\left(|g|^t * \frac{1}{(1+|\cdot|)^{at}}\right)(x) \le C \left\|\frac{1}{(1+|\cdot|)^{at}}\right\|_1 \mathcal{M}_t(g)(x),$$

it follows that for any $x \in \mathbb{R}^n$

$$\delta^{-\frac{t}{q(x)}}J_v^1(x) \le C\mathcal{M}_t\left(\delta^{-\frac{1}{q(\cdot)}}2^{v\alpha(\cdot)}\varphi_v * f\right)(x),$$

where the constant $C > 0$ is independent of x and v. Since $|x - z| \ge 2^{-v/2+i}$ and the right-hand side of (7) can be estimated by $c\, 2^{2v\sigma(1/q^- - 1/q^+)}$, then for any $x \in \mathbb{R}^n$ and any $v \in \mathbb{N}_0$, $\delta^{-t/q(x)}J_{v-i}^2(x)$ is bounded by

$$C2^{vt(2\sigma(1/q^- - 1/q^+) - a/2 + n/t)}2^{-iat} \int\limits_{B(x,2^{-v/2+i+1})} \delta^{-t/q(z)}2^{v\alpha(z)t}|\varphi_v * f(z)|^t dz$$

$$\le C2^{vt(2\sigma(1/q^- - 1/q^+) - a/2 + n/2t)}2^{i(n-at)}\mathcal{M}_t\left(\delta^{-1/q(\cdot)}2^{v\alpha(\cdot)}\varphi_v * f\right)(x)$$

$$\le C2^{i(n-at)}\mathcal{M}_t\left(\delta^{-1/q(\cdot)}2^{v\alpha(\cdot)}\varphi_v * f\right)(x),$$

due to our choice of σ. Hence,

$$\sum_{i=0}^{\infty} \delta^{-t/q(x)}J_{v-i}^2(x) \le C\sum_{i=0}^{\infty} 2^{i(n-at)}\mathcal{M}_t\left(\delta^{-1/q(\cdot)}2^{v\alpha(\cdot)}\varphi_v * f\right)(x)$$

$$\le C\mathcal{M}_t\left(\delta^{-1/q(\cdot)}2^{v\alpha(\cdot)}\varphi_v * f\right)(x),$$

since again $a > n/t$. Consequently we have proved that

$$\left(\delta^{-1/q(x)}\varphi_v^{*,a}2^{v\alpha(\cdot)}f(x)\right)^t \le C\mathcal{M}_t\left(\delta^{-1/q(\cdot)}2^{v\alpha(\cdot)}\varphi_v * f\right)(x),$$

for all $x \in \mathbb{R}^n$. Taking the $L^{\frac{p(\cdot)}{t}}$-norm and using the fact that $\mathcal{M} : L^{\frac{p(\cdot)}{t}} \to L^{\frac{p(\cdot)}{t}}$ is bounded we obtain that

$$\left\|c\delta^{-1/q(\cdot)}\varphi_v^{*,a}2^{v\alpha(\cdot)}f\right\|_{p(\cdot)}^t = \left\||c\delta^{-1/q(\cdot)}\varphi_v^{*,a}2^{v\alpha(\cdot)}f|^t\right\|_{p(\cdot)/t} \le \left\|\delta^{-t/q(\cdot)}2^{v\alpha(\cdot)t}|\varphi_v * f|^t\right\|_{p(\cdot)/t}$$

$$= \left\|\delta^{-1/q(\cdot)}2^{v\alpha(\cdot)}\varphi_v * f\right\|_{p(\cdot)}^t,$$

with an appropriate choice of $c > 0$. Now the right-hand side is less than or equal to one if and only if

$$\left\||\delta^{-1/q(\cdot)}2^{v\alpha(\cdot)}\varphi_v * f|^{q(\cdot)}\right\|_{p(\cdot)/q(\cdot)} \le 1,$$

which follows immediately from the definition of δ.
The proof is completed.

In order to formulate the main result, let us consider $k_0, k \in \mathcal{S}(\mathbb{R}^n)$ and $S \ge -1$ an integer such that for an $\varepsilon > 0$

$$|\mathcal{F}k_0(\xi)| > 0 \quad \text{for } |\xi| < 2\varepsilon, \tag{8}$$

$$|\mathcal{F}k(\xi)| > 0 \quad \text{for } \frac{\varepsilon}{2} < |\xi| < 2\varepsilon, \tag{9}$$

and

$$\int_{\mathbb{R}^n} x^\alpha k(x)dx = 0 \quad \text{for any } |\alpha| \le S. \tag{10}$$

Here (8) and (9) are Tauberian conditions, while (10) are moment conditions on k. We recall the notation

$$k_t(x) = t^{-n}k(t^{-1}x), \quad k_j(x) = k_{2^{-j}}(x), \quad \text{for } t > 0 \text{ and } j \in \mathbb{N}.$$

For any $a > 0$, $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ we denote

$$k_j^{*,a} 2^{j\alpha(\cdot)} f(x) = \sup_{y \in \mathbb{R}^n} \frac{2^{j\alpha(y)} |k_j * f(y)|}{(1 + 2^j |x - y|)^a}, \qquad j \in \mathbb{N}_0. \tag{11}$$

Usually $k_j * f$ is called local mean.

Theorem (6.1.9) [258]: Let $\alpha \in C_{loc}^{log}$ and $p, q \in \mathcal{P}_0^{log}$ with $q^+ < \infty$. Let $a > \frac{n}{p^-}$ and $\alpha^+ < S + 1$. Then

$$\left\| f | B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \right\|' = \left\| \left(k_j^{*,a} 2^{j\alpha(\cdot)} f \right)_j \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \tag{12}$$

and

$$\left\| f | B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \right\|'' = \left\| \left(2^{j\alpha(\cdot)} k_j * f \right)_j \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}, \tag{13}$$

are equivalent quasi-norms on $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$.

Proof: First H. Kempka [150] proved this result, but only the case of constant q was included. J.-S. Xu [138] has proved this result with variable p, but fixed q and α. H. Kempka and J. Vybiral [265], independently, proved this result with $2^{j\alpha(\cdot)} k_j^{*,a} f$, $\frac{n + c_{log}(1/q)}{p^-} + c_{log}(\alpha)$ in place of $k_j^{*,a} 2^{j\alpha(\cdot)} f, \frac{n}{p^-}$ respectively. The idea of the proof is from V.S. Rychkov [213].

Step 1. Take any pair of functions ϕ_0 and $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that
$$|\mathcal{F}\phi_0(\xi)| > 0 \qquad \text{for } |\xi| < 2\varepsilon,$$
$$|\mathcal{F}\phi(\xi)| > 0 \qquad \text{for } \frac{\varepsilon}{2} < |\xi| < 2\varepsilon.$$

We will prove that there is a constant $c > 0$ such that for any $f \in \mathcal{S}'(\mathbb{R}^n)$
$$\left\| f | B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \right\|' \le c \left\| \left(\phi_j^{*,a} 2^{j\alpha(\cdot)} f \right)_j \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}. \tag{14}$$

By a scaling argument, we see that it suffices to consider the case
$$\left\| \left(\phi_j^{*,a} 2^{j\alpha(\cdot)} f \right)_j \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = 1$$

and show that the modular of a constant times the function on the left-hand side is bounded. In particular, we will show that
$$\sum_{j=0}^{\infty} \left\| |ck_j^{*,a} 2^{j\alpha(\cdot)} f|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \le C \quad \text{whenever} \quad \sum_{j=0}^{\infty} \left\| |\phi_j^{*,a} 2^{j\alpha(\cdot)} f|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} = 1.$$

Let $\Lambda, \lambda \in \mathcal{S}(\mathbb{R}^n)$ so that
$$\text{supp}\, \mathcal{F}\Lambda \subset \{\xi \in \mathbb{R}^n : |\xi| < 2\varepsilon\}, \qquad \text{supp}\, \mathcal{F}\lambda \subset \{\xi \in \mathbb{R}^n : \varepsilon/2 < |\xi| < 2\varepsilon\},$$
$$\mathcal{F}\Lambda(\xi) \mathcal{F}\phi_0(\xi) + \sum_{v=1}^{\infty} \mathcal{F}\lambda(2^{-v}\xi) \mathcal{F}\phi(2^{-v}\xi) = 1, \qquad \xi \in \mathbb{R}^n. \tag{15}$$

In particular, for any $f \in \mathcal{S}'(\mathbb{R}^n)$ the identity is true
$$f = \Lambda * \phi_0 * f + \sum_{v=1}^{\infty} \lambda_v * \phi_v * f. \tag{16}$$

Hence we can write
$$k_j * f = k_j * \Lambda * \phi_0 * f + \sum_{v=1}^{\infty} k_j * \lambda_v * \phi_v * f.$$

We have
$$2^{j\alpha(y)} |k_j * \lambda_v * \phi_v * f(y)| \le 2^{j\alpha(y)} \int_{\mathbb{R}^n} |k_j * \lambda_v(z)| |\phi_v * f(y - z)| dz. \tag{17}$$

First let $v \leq j$. Writing for any $z \in \mathbb{R}^n$
$$k_j * \lambda_v(z) = 2^{vn} k_{2^{v-j}} * \lambda(2^v z),$$
we get by Lemma (6.1.4), that for any integer $S \geq -1$ and any $N > 0$ there is a constant $c > 0$ independent of j and v
$$\left| k_j * \lambda_v(z) \right| \leq c \frac{2^{(v-j)(S+1)+vn}}{(1 + 2^v|z|)^{2N}}, \quad z \in \mathbb{R}^n.$$
So the right-hand side of (17) can be estimated from above by
$$c2^{j\alpha(y)+(v-j)(S+1)+vn} \int_{\mathbb{R}^n} (1 + 2^v|z|)^{-2N} |\phi_v * f(y - z)| dz = c2^{(v-j)(S+1)} 2^{j\alpha(y)} \eta_{v,2N} * |\phi_v * f|(y).$$
By Lemma (6.1.1) the estimates
$$2^{j\alpha(y)} \eta_{v,2N} * |\phi_v * f(y)| \leq 2^{(j-v)\alpha^+} \eta_{v,N} * \left(2^{v\alpha(\cdot)} |\phi_v * f| \right)(y) \leq 2^{(j-v)\alpha^+} \phi_v^{*,a} 2^{v\alpha(\cdot)} f(y) \|\eta_{v,N-a}\|_1$$
$$\leq c2^{(j-v)\alpha^+} \phi_v^{*,a} 2^{v\alpha(\cdot)} f(y),$$
are true for any $N > \max(d, n + a)$ and any $v \leq j$ (with d as in Lemma (6.1.1)).

Let now $v \geq j$. Then, again by Lemma (6.1.4) we have for any $z \in \mathbb{R}^n$ and any $L > 0$
$$\left| k_j * \lambda_v(z) \right| = 2^{jn} \left| k * \lambda_{2^{j-v}}(2^j z) \right| \leq c \frac{2^{(j-v)(M+1)+jn}}{(1 + 2^j|z|)^{2L}},$$
where $M \geq -1$ an integer can be taken arbitrarily large, since $D^\alpha \mathcal{F}\lambda(0) = 0$ for all α. Therefore, for $v \geq j$, the right-hand side of (17) can be estimated from above by
$$c2^{j\alpha(y)+(j-v)(M+1)+jn} \int_{\mathbb{R}^n} (1 + 2^j|z|)^{-2L} |\phi_v * f(y - z)| dz = c2^{j\alpha(y)+(j-v)(M+1)} \eta_{j,2L} * |\phi_v * f|(y).$$
We have for any $v \geq j$
$$(1 + 2^j|z|)^{-2L} \leq 2^{2(v-j)L} (1 + 2^v|z|)^{-2L}.$$
Then, again, the right-hand side of (17) is dominated by
$$c2^{j\alpha(y)+(j-v)(M-2L+1+n)} \eta_{v,2L} * |\phi_v * f|(y) \leq c2^{(j-v)(M-2L+1+\alpha^-+n)} \eta_{v,L} * \left(2^{v\alpha(\cdot)} |\phi_v * f| \right)(y)$$
$$\leq c2^{(j-v)(M-2L+1+\alpha^-+n)} \phi_v^{*,a} 2^{v\alpha(\cdot)} f(y) \|\eta_{v,L-a}\|_1$$
$$\leq c2^{(j-v)(M-2L+1+\alpha^-+n)} \phi_v^{*,a} 2^{v\alpha(\cdot)} f(y),$$
where in the first inequality we have used Lemma (6.1.1) (by taking $L > \max(d, n + a)$). Taking $M > 2L - \alpha^- + a - n$ to estimate the last expression by
$$c2^{(j-v)(a+1)} \phi_v^{*,a} 2^{v\alpha(\cdot)} f(y),$$
where $c > 0$ is independent of j, v and f. Further, note that for all $x, y \in \mathbb{R}^n$ and all $j, v \in \mathbb{N}$
$$\phi_v^{*,a} 2^{v\alpha(\cdot)} f(y) \leq \phi_v^{*,a} 2^{v\alpha(\cdot)} f(x)(1 + 2^v|x - y|)^a$$
$$\leq \phi_v^{*,a} 2^{v\alpha(\cdot)} f(x) \max\left(1, 2^{(v-j)a}\right)(1 + 2^j|x - y|)^a.$$
Hence
$$\sup_{y \in \mathbb{R}^n} \frac{2^{j\alpha(y)} |k_j * \lambda_v * \phi_v * f(y)|}{(1 + 2^j|x - y|)^a} \leq C\phi_v^{*,a} 2^{v\alpha(\cdot)} f(x) \times \begin{cases} 2^{(v-j)(S+1-\alpha^+)} & \text{if } v \leq j, \\ 2^{j-v} & \text{if } v \geq j. \end{cases}$$
Using the fact that for any $z \in \mathbb{R}^n$, any $N > 0$ and any integer $S \geq -1$
$$\left| k_j * \Lambda(z) \right| = \left| k_{2^{-j}} * \Lambda(z) \right| \leq c \frac{2^{-j(S+1)}}{(1 + |z|)^{2N}},$$
we obtain by the similar arguments that for any $j \in \mathbb{N}$
$$\sup_{y \in \mathbb{R}^n} \frac{2^{j\alpha(y)} |k_j * \Lambda * \phi_0 * f(y)|}{(1 + 2^j|x - y|)^a} \leq C2^{-j(S+1-\alpha^+)} \phi_0^{*,a} f(x).$$
Hence with $\delta = \min(1, S + 1 - \alpha^+) > 0$ for all $f \in \mathcal{S}'(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, $j \in \mathbb{N}$

$$k_v^{*,a} 2^{j\alpha(\cdot)} f(y) \leq C 2^{-j\delta} \phi_0^{*,a} f(x) + C \sum_{v=1}^{\infty} 2^{-|j-v|\delta} \phi_v^{*,a} 2^{v\alpha(\cdot)} f(x) = C \sum_{v=0}^{\infty} 2^{-|j-v|\delta} \phi_v^{*,a} 2^{v\alpha(\cdot)} f(x).$$

Also for j = 0, we use the fact that for $v \geq 1$, any $z \in \mathbb{R}^n$, any N > 0 and any integer M ≥ −1

$$|k_0 * \lambda_v(z)| = |k_0 * \lambda_{2^{-v}}(z)| \leq c \frac{2^{-v(M+1)}}{(1+|z|)^{2N}}$$

and

$$|k_0 * \Lambda(z)| \leq c \frac{1}{(1+|z|)^{2N}}$$

to get for any $x \in \mathbb{R}^n$

$$k_0^{*,a} f(x) \leq C \phi_0^{*,a} f(x) + C \sum_{v=1}^{\infty} 2^{-v\delta} \phi_v^{*,a} 2^{v\alpha(\cdot)} f(x) = C \sum_{v=0}^{\infty} 2^{-v\delta} \phi_v^{*,a} 2^{v\alpha(\cdot)} f(x).$$

Let $\tau > \max(q^+, q^+/p^-)$. Then by Lemma (6.1.3)

$$\sum_{j=0}^{\infty} \left\| |c k_j^{*,a} 2^{j\alpha(\cdot)} f|^{q(\cdot)} \right\|_{p(\cdot)/q(\cdot)} = \sum_{j=0}^{\infty} \left\| |c k_j^{*,a} 2^{j\alpha(\cdot)} f|^{q(\cdot)/\tau} \right\|_{\tau p(\cdot)/q(\cdot)}^{\tau}$$

$$\leq \sum_{j=0}^{\infty} \left(\sum_{v=0}^{\infty} 2^{-|j-v|\delta} \left\| |\phi_v^{*,a} 2^{v\alpha(\cdot)} f|^{q(\cdot)/\tau} \right\|_{\tau p(\cdot)/q(\cdot)} \right)^{\tau}$$

$$\leq C \sum_{j=0}^{\infty} \left\| |\phi_j^{*,a} 2^{j\alpha(\cdot)} f|^{q(\cdot)/\tau} \right\|_{\tau p(\cdot)/q(\cdot)}^{\tau} = C \sum_{j=0}^{\infty} \left\| |\phi_j^{*,a} 2^{j\alpha(\cdot)} f|^{q(\cdot)} \right\|_{p(\cdot)/q(\cdot)} \leq C,$$

with an appropriate choice of c > 0.

Step 2. We will prove in this step that there is a constant c > 0 such that for any $f \in \mathcal{S}'(\mathbb{R}^n)$

$$\left\| f | B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \right\|' \leq c \left\| f | B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \right\|''. \tag{18}$$

Analogously to (15), (16) find two functions $\Lambda, \psi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\operatorname{supp} \mathcal{F}\Lambda \subset \{\xi \in \mathbb{R}^n : |\xi| < 2\varepsilon\}, \qquad \operatorname{supp} \mathcal{F}\psi \subset \{\xi \in \mathbb{R}^n : \varepsilon/2 < |\xi| < 2\varepsilon\},$$

and for all $f \in \mathcal{S}'(\mathbb{R}^n)$ and $j \in \mathbb{N}_0$

$$f = \Lambda_j * (k_0)_j * f + \sum_{m=j+1}^{\infty} \psi_m * k_m * f.$$

Hence

$$k_j * f = \Lambda_j * (k_0)_j * k_j * f + \sum_{m=j+1}^{\infty} k_j * \psi_m * k_m * f.$$

By a scaling argument, we see that it suffices to consider the case

$$\left\| f | B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \right\|'' = 1$$

and show that the modular of a constant times the function on the left-hand side is bounded. In particular, we will show that

$$\sum_{j=0}^{\infty} \left\| |c k_j^{*,a} 2^{j\alpha(\cdot)} f|^{q(\cdot)} \right\|_{p(\cdot)/q(\cdot)} \leq C \quad \text{whenever} \quad \sum_{j=0}^{\infty} \left\| |2^{j\alpha(\cdot)} k_j * f|^{q(\cdot)} \right\|_{p(\cdot)/q(\cdot)} = 1.$$

Writing for any $z \in \mathbb{R}^n$

$$k_j * \psi_m(z) = 2^{jn} (k * \psi_{2^{j-m}})(2^j z),$$

we get by Lemma (6.1.4), that for any integer $K \geq -1$ and any $M > 0$ there is a constant $c > 0$ independent of j and m

$$\left|k_j * \psi_m(z)\right| \leq c \frac{2^{(j-m)(K+1)+jn}}{(1 + 2^j|z|)^{2M}}, \quad z \in \mathbb{R}^n.$$

Analogous estimate

$$\left|\Lambda_j * (k_0)_j(z)\right| \leq c \frac{2^{jn}}{(1 + 2^j|z|)^{2M}}, \quad z \in \mathbb{R}^n,$$

is obvious. From this it follows that

$$2^{j\alpha(y)}\left|k_j * f(y)\right| \leq c \sum_{m=j}^{\infty} 2^{(j-m)(K+1+\alpha^-)} 2^{m\alpha(y)} \eta_{j,2M} * |k_m * f|(y)$$

$$= c \sum_{m=j}^{\infty} 2^{(j-m)(K+1+\alpha^-)+jn} \int_{\mathbb{R}^n} \frac{2^{m\alpha(y)}|k_m * f(z)|}{(1 + 2^j|y - z|)^{2M}} \, dz.$$

Since

$$(1 + 2^j|y - z|)^{-2M} \leq 2^{2(m-j)M}(1 + 2^m|y - z|)^{-2M},$$

then by Lemma (6.1.1) we have

$$2^{j\alpha(y)}\left|k_j * f(y)\right| \leq c \sum_{m=j}^{\infty} 2^{(j-m)(K+1+\alpha^--2M+n)} 2^{m\alpha(y)} \eta_{m,2M} * |k_m * f|(y)$$

$$\leq c \sum_{m=j}^{\infty} 2^{(j-m)(K+1+\alpha^--2M+n)} \eta_{m,a} * \left(2^{m\alpha(\cdot)}|k_m * f|\right)(y), \tag{19}$$

by taking $M > \max(d, a)$. Using the elementary estimates

$$(1 + 2^j|x - y|)^{-a} \leq (1 + 2^j|x - z|)^{-a}(1 + 2^j|y - z|)^a$$
$$\leq 2^{(m-j)a}(1 + 2^m|x - z|)^{-a}(1 + 2^m|y - z|)^a, \tag{20}$$

to get

$$k_j^{*,a} 2^{j\alpha(\cdot)} f(x) \leq c \sum_{m=j}^{\infty} 2^{(j-m)(K+1-a-\alpha^--2M+n)+mn} \int_{\mathbb{R}^n} \frac{2^{m\alpha(z)}|k_m * f(z)|}{(1 + 2^m|x - z|)^a} \, dz.$$

Fix any $r \in (0, 1]$. We have

$$2^{m\alpha(z)}|k_m * f(z)| = \left(2^{m\alpha(z)}|k_m * f(z)|\right)^r \left(2^{m\alpha(z)}|k_m * f(z)|\right)^{1-r}$$

$$= \left(2^{m\alpha(z)}|k_m * f(z)|\right)^r \left(\frac{2^{m\alpha(z)}|k_m * f(z)|}{(1 + 2^m|x - z|)^a}\right)^{1-r} (1 + 2^m|x - z|)^{a(1-r)}$$

$$\leq \left(2^{m\alpha(z)}|k_m * f(z)|\right)^r \left(k_m^{*,a} 2^{m\alpha(\cdot)} f(x)\right)^{1-r} (1 + 2^m|x - z|)^{a(1-r)}.$$

Then

$$k_j^{*,a} 2^{j\alpha(\cdot)} f(x) \leq c \sum_{m=j}^{\infty} 2^{(j-m)N'+mn} \int_{\mathbb{R}^n} \frac{2^{mr\alpha(z)}|k_m * f(z)|^r}{(1 + 2^m|x - z|)^{ar}} \, dz \left(k_m^{*,a} 2^{m\alpha(\cdot)} f(x)\right)^{1-r},$$

where $N' = K + 1 - a + n - \alpha^- - 2M$ can be still be taken arbitrarily large. Quite analogously one proves for all $f \in \mathcal{S}'(\mathbb{R}^n)$ the estimate

$$k_0^{*,a} f(x) \leq C \sum_{m=0}^{\infty} 2^{-mN'+mn} \int_{\mathbb{R}^n} \frac{2^{mr\alpha(z)}|k_m * f(z)|^r}{(1 + 2^m|x - z|)^{ar}} \, dz \left(k_m^{*,a} 2^{m\alpha(\cdot)} f(x)\right)^{1-r}.$$

We now fix any $x \in \mathbb{R}^n$ and apply Lemma (6.1.5) with

$$d_j = k_j^{*,a} 2^{j\alpha(\cdot)} f(x), \quad j \in \mathbb{N}_0,$$

$$b_m = \int_{\mathbb{R}^n} \frac{2^{mr\alpha(z)+mn}|k_m * f(z)|^r}{(1 + 2^m|x - z|)^{ar}}dz, \qquad m \in \mathbb{N}_0.$$

The assumption (3) is satisfied with $N_0 = N_1 + n + [\max(0, \alpha^+)] + 1$, where N_1 is the order of the distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ ($[a]$ the integer part of the real number a). We conclude that for any $f \in \mathcal{S}'(\mathbb{R}^n)$, any $N > 0$ and any $j \in \mathbb{N}_0$

$$\left(k_j^{*,a}2^{j\alpha(\cdot)}f(x)\right)^r \le C \sum_{m=j}^{\infty} 2^{(j-m)Nr+mn} \int_{\mathbb{R}^n} \frac{2^{mr\alpha(z)}|k_m * f(z)|^r}{(1 + 2^m|x - z|)^{ar}}dz.$$

This estimate is also true for $r > 1$, with much simpler proof. It suffices to take (19) with $a + n$ instead of a, apply Hölder's inequalities in m and z, and finally the inequality (20). We omit the details.

Since $a > n/p^-$, it possible to take $n/a < r < p^-$. Let $\tau > q^+/r$. We see that

$$\left\||ck_j^{*,a}2^{j\alpha(\cdot)}f|^{q(\cdot)}\right\|_{p(\cdot)/q(\cdot)} = \left\||ck_j^{*,a}2^{j\alpha(\cdot)}f|^{rq(\cdot)/r}\right\|_{p(\cdot)/q(\cdot)} = \left\||ck_j^{*,a}2^{j\alpha(\cdot)}f|^{rq(\cdot)/r\tau}\right\|_{\tau p(\cdot)/q(\cdot)}^{\tau}$$

$$\le C \left(\sum_{m=j}^{\infty} 2^{(j-m)Nq^-/\tau} \left\||c\eta_{m,ar} * \left(2^{m\alpha(\cdot)}|k_m * f|\right)^r|^{q(\cdot)/r\tau}\right\|_{\tau p(\cdot)/q(\cdot)}\right)^{\tau}.$$

By the same method given in the proof of Theorem (6.1.8) (with $m, q(\cdot)/\tau, r$ in place of $v, q(\cdot), t$ respectively) we can prove that

$$\left\||c\eta_{m,ar} * \left(2^{m\alpha(\cdot)}|k_m * f|\right)^r|^{q(\cdot)/r\tau}\right\|_{\tau p(\cdot)/q(\cdot)} \le \left\||2^{m\alpha(\cdot)}k_m * f|^{q(\cdot)/\tau}\right\|_{\tau p(\cdot)/q(\cdot)} + 2^{-m\sigma}$$

$$= \left\||2^{m\alpha(\cdot)}k_m * f|^{q(\cdot)}\right\|_{p(\cdot)/q(\cdot)}^{1/\tau} + 2^{-m\sigma},$$

with an appropriate choice of $c > 0$ and here $0 < \sigma < \frac{a-n/r}{4\tau(1/q^--1/q^+)}$. Then for any $f \in \mathcal{S}'(\mathbb{R}^n)$ and any $j \in \mathbb{N}_0$

$$\left\||ck_j^{*,a}2^{j\alpha(\cdot)}f|^{q(\cdot)}\right\|_{p(\cdot)/q(\cdot)} \le \left(\sum_{m=j}^{\infty} 2^{(j-m)Nq^-/\tau}\left(\left\||2^{m\alpha(\cdot)}k_m * f|^{q(\cdot)}\right\|_{p(\cdot)/q(\cdot)}^{1/\tau} + 2^{-m\sigma}\right)\right)^{\tau}.$$

By Lemma (6.1.3) we get

$$\sum_{j=0}^{\infty} \left\||ck_j^{*,a}2^{j\alpha(\cdot)}f|^{q(\cdot)}\right\|_{p(\cdot)/q(\cdot)} \le c \sum_{j=0}^{\infty} \left(\left\||2^{j\alpha(\cdot)}k_j * f|^{q(\cdot)}\right\|_{p(\cdot)/q(\cdot)}^{1/\tau} + 2^{-j\sigma}\right)^{\tau}$$

$$\le c \sum_{j=0}^{\infty} \left\||2^{j\alpha(\cdot)}k_j * f|^{q(\cdot)}\right\|_{p(\cdot)/q(\cdot)} + c \sum_{j=0}^{\infty} 2^{-j\sigma\tau} \le C.$$

Step 3. We will prove in this step that for all $f \in \mathcal{S}'(\mathbb{R}^n)$ the following estimates are true:

$$\left\|f|B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}\right\|' \le c\left\|f|B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}\right\| \le c\left\|f|B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}\right\|''.$$

Let $\{\varphi_j\}_{j\in\mathbb{N}_0}$ be as in Definition (6.1.6) and let $\phi_j = \varphi_j$. The first inequality is proved by the chain of the estimates

$$\left\|f|B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}\right\|' \le c \left\|\left(\varphi_j^{*,a}2^{j\alpha(\cdot)}f\right)_j\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \le c \left\|\left(2^{j\alpha(\cdot)}\phi_j * f\right)_j\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \le c\left\|f|B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}\right\|,$$

where the first inequality is (14), see Step 1, the second inequality is (18) (with ϕ and ϕ_0 instead of k and k_0), see Step 2, and finally the third inequality is obvious. Now the second inequality can be obtained by the following chain

$$\left\|f|B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}\right\| \leq c \left\|\left(\varphi_j^{*,a} 2^{j\alpha(\cdot)} f\right)_j\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leq c\left\|f|B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}\right\|' \leq c\left\|f|B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}\right\|'',$$

where the first inequality is obvious, the second inequality is (14), see Step 1, with the roles of k_0 and k respectively φ_0 and φ interchanged, and finally the last inequality is (18), see Step 2. Hence the theorem is proved.

Now let $\mathbb{Z}^n$ be the lattice of all points in $\mathbb{R}^n$ with integer-valued components. If $v \in \mathbb{N}_0$ and $m = (m_1, \cdots, m_n) \in \mathbb{Z}^n$ we denote Q_{vm} the dyadic cube in $\mathbb{R}^n$ centred at $2^{-v}m$ which has sides parallel to the axes and side length 2^{-v}. If Q_{vm} is such a cube in $\mathbb{R}^n$ and $c > 0$ then cQ_{vm} is the cube in $\mathbb{R}^n$ concentric with Q_{vm} and with side length $c2^{-v}$. By χ_{vm} we denote the characteristic function of the cube Q_{vm}. The main goal is to show an atomic decomposition result for $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$.

Definition (6.1.10) [258]: Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$ and let $\alpha \in C^{\log}_{loc}$. Then for all complex-valued sequences $\lambda = \{\lambda_{vm} \in \mathbb{C} : v \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$ we define

$$b^{\alpha(\cdot)}_{p(\cdot),q(\cdot)} = \left\{\lambda : \left\|\lambda|b^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}\right\| < \infty\right\}$$

where

$$\left\|\lambda|b^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}\right\| = \left\|\left(\sum_{m\in\mathbb{Z}^n} 2^{v\alpha(\cdot)}\lambda_{vm}\chi_{vm}\right)_v\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}.$$

Definition (6.1.11) [258]: Let $K, L \in \mathbb{N}_0$ and let $\gamma > 1$. A K-times continuous differentiable function $a \in C^k(\mathbb{R}^n)$ is called $[K, L]$-atom centered at Q_{vm}, $v \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$, if

$$\text{supp } a \subseteq \gamma Q_{vm}, \tag{21}$$

$$\left|D^\beta a(x)\right| \leq 2^{v|\beta|}, \quad \text{for } 0 \leq |\beta| \leq K, \quad x \in \mathbb{R}^n \tag{22}$$

and if

$$\int_{\mathbb{R}^n} x^\beta a(x)dx = 0, \quad \text{for } 0 \leq |\beta| < L \text{ and } v \geq 1. \tag{23}$$

If the atom a located at Q_{vm}, that means if it fulfills (21), then we will denote it by a_{vm}. For $v = 0$ or $L = 0$ there are no moment conditions (23) required.

Lemma (6.1.12) [45]: Let $\{\mathcal{F}\varphi_j\}, j \in \mathbb{N}_0$ be a resolution of unity and let ρ_{vm} be an $[K, L]$-atom. Then

$$\left|\varphi_j * \rho_{vm}(x)\right| \leq c2^{(v-j)K}(1 + 2^v|x - 2^{-v}m|)^{-M}$$

if $v \leq j$, and

$$\left|\varphi_j * \rho_{vm}(x)\right| \leq c2^{(j-v)(L+n+1)}(1 + 2^j|x - 2^{-v}m|)^{-M}$$

if $v \geq j$, where M is sufficiently large.

Lemma (6.1.13) [45]: Let $\{\mathcal{F}\varphi_j\}, j \in \mathbb{N}_0$ be a resolution of unity and let $R \in \mathbb{N}$. Then there exist functions $\theta_0, \theta \in \mathcal{S}(\mathbb{R}^n)$ with:

$$\text{supp } \theta_0, \text{supp } \theta \subset \{x \in \mathbb{R}^n : |x| \leq 1\}, \tag{24}$$

$$|\mathcal{F}\theta_0(\xi)| > c_0 \text{ for } |\xi| \leq 2, \quad |\mathcal{F}\theta(\xi)| > c \text{ for } \frac{1}{2} \leq |\xi| \leq 2,$$

$$\int_{\mathbb{R}^n} x^\beta \theta(x)dx = 0, \quad \text{for } 0 \leq |\beta| < R, \tag{25}$$

such that

$$\mathcal{F}\theta_0(\xi)\mathcal{F}\psi_0(\xi) + \sum_{j\geq 1} \mathcal{F}\theta(2^{-j}\xi)\mathcal{F}\psi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^n,$$

where the functions $\psi_0, \psi \in \mathcal{S}(\mathbb{R}^n)$ are defined via

$$\mathcal{F}\psi_0(\xi) = \frac{\mathcal{F}\varphi_0(\xi)}{\mathcal{F}\theta_0(\xi)} \quad \text{and} \quad \mathcal{F}\psi(\xi) = \frac{\mathcal{F}\varphi_1(2\xi)}{\mathcal{F}\theta(\xi)}.$$

Theorem (6.1.14) [258]: Let $\alpha \in C_{loc}^{log}$ and $p, q \in \mathcal{P}_0^{log}$ with $q^+ < \infty$. Further, let $K, L \in \mathbb{N}_0$ such that

$$K > \alpha^+, \quad L > n\left(\frac{1}{\min(1, p^-)} - 1\right) - 1 - \alpha^-. \tag{26}$$

Then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$, if and only if, it can be represented as

$$f = \sum_{v=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{vm}\rho_{vm}, \tag{27}$$

convergence being in $\mathcal{S}'(\mathbb{R}^n)$, where ρ_{vm} are $[K, L]$-atoms and $\lambda = \{\lambda_{vm} \in \mathbb{C} : v \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$. Furthermore, $\inf \left\| \lambda \left| b_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \right| \right\|$, where the infimum is taken over admissible representations (27), is an equivalent quasi-norm in $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$.

Proof: The proof follows the ideas in [45].

Step 1.　　Assume that $f \in B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$ and let θ_0, θ, ψ_0 and ψ be the functions introduced in Lemma (6.1.13). We have

$$f = \theta_0 * \psi_0 * f + \sum_{v=1}^{\infty} \theta_v * \psi_v * f$$

and using the definition of the cubes $\mathcal{Q}_{vm}$ we obtain

$$f(x) = \sum_{m \in \mathbb{Z}^n} \int_{\mathcal{Q}_{0m}} \theta_0(x-y)\psi_0 * f(y)dy + \sum_{v=1}^{\infty} 2^{vn} \sum_{m \in \mathbb{Z}^n} \int_{\mathcal{Q}_{vm}} \theta(2^v(x-y))\psi_v * f(y)dy,$$

with convergence in $\mathcal{S}'(\mathbb{R}^n)$. We define for every $v \in \mathbb{N}$ and all $m \in \mathbb{Z}^n$

$$\lambda_{vm} = C_\theta \sup_{y \in \mathcal{Q}_{vm}} |\psi_v * f(y)| \tag{28}$$

where

$$C_\theta = \max\left\{\sup_{|y| \leq 1} |D^\alpha \theta(y)| : |\alpha| \leq K\right\}.$$

Define also

$$\rho_{vm}(x) = \frac{1}{\lambda_{vm}} 2^{vn} \int_{\mathcal{Q}_{vm}} \theta(2^v(x-y))\psi_v * f(y)dy. \tag{29}$$

Similarly we define for every $m \in \mathbb{Z}^n$ the numbers λ_{0m} and the functions ρ_{0m} taking in (28) and (29) $v = 0$ and replacing ψ_v and θ by ψ_0 and θ_0, respectively. Let us now check that such ρ_{vm} are atoms in the sense of Definition (6.1.11). Note that the support and moment conditions are clear by (24) and (25), respectively. It thus remains to check (22) in Definition (6.1.11). We have

$$\left|D^\beta \rho_{vm}(x)\right| \leq \frac{2^{v(n+|\beta|)}}{C_\theta} \int_{\mathcal{Q}_{vm}} \left|(D^\beta \theta)(2^v(x-y))\right| |\psi_v * f(y)| dy \left(\sup_{y \in \mathcal{Q}_{vm}} |\psi_v * f(y)|\right)^{-1}$$

$$\leq \frac{2^{v(n+|\beta|)}}{C_\theta} \int_{\mathcal{Q}_{vm}} \left|(D^\beta \theta)(2^v(x-y))\right| dy \leq 2^{v(n+|\beta|)}|\mathcal{Q}_{vm}| \leq 2^{v|\beta|}.$$

The modifications for the terms with $v = 0$ are obvious.

Step 2.　　Next we show that there is a constant $c > 0$ such that

$$\left\|\lambda \left| b_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \right.\right\| \le c \left\|f \left| B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \right.\right\|.$$

For that reason we exploit the equivalent quasi-norms given in Theorem (6.1.9) involving Peetre's maximal function. Let $v \in \mathbb{N}$. Taking into account that $|x - y| \le c2^{-v}$ for $x, y \in Q_{vm}$ we obtain

$$2^{v(\alpha(x)-\alpha(y))} \le 2^{\frac{c_{\log}(\alpha)v}{\log(e+1/|x-y|)}} \le 2^{\frac{c_{\log}(\alpha)v}{\log(e+2^v/c)}} \le c$$

if $v \ge [\log_2 c] + 2$. If $0 < v < [\log_2 c] + 2$, then $2^{v(\alpha(x)-\alpha(y))} \le 2^{v(\alpha^+ - \alpha^-)} \le c$. Therefore,

$$2^{v\alpha(x)} |\psi_v * f(y)| \le c2^{v\alpha(y)} |\psi_v * f(y)|$$

for any $x, y \in Q_{vm}$ and any $v \in \mathbb{N}$. Hence,

$$\sum_{m \in \mathbb{Z}^n} \lambda_{vm} 2^{v\alpha(x)} \chi_{vm}(x) = C_\theta \sum_{m \in \mathbb{Z}^n} 2^{v\alpha(x)} \sup_{y \in Q_{vm}} |\psi_v * f(y)| \chi_{vm}(x)$$

$$\le c \sum_{m \in \mathbb{Z}^n} \sup_{|z| \le c2^{-v}} \frac{2^{v\alpha(x-z)} |\psi_v * f(x-z)|}{(1 + 2^v |z|)^a} (1 + 2^v |z|)^a \chi_{vm}(x)$$

$$\le c\psi_v^{*,a} 2^{v\alpha(\cdot)} f(x) \sum_{m \in \mathbb{Z}^n} \chi_{vm}(x) = c\psi_v^{*,a} 2^{v\alpha(\cdot)} f(x),$$

where we have used $\sum_{m \in \mathbb{Z}^n} \chi_{vm}(x) = 1$. This estimate and its counterpart for $v = 0$ (which can be obtained by a similar calculation) give

$$\left\|\lambda \left| b_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \right.\right\| \le c \left\| \left(\psi_v^{*,a} 2^{v\alpha(\cdot)} f\right)_v \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \le c \left\|f \left| B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \right.\right\|,$$

by Theorem (6.1.9) (since $\psi_0 \in \mathcal{S}(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$ are two kernels which fulfill Tauberian conditions (8) and (9) and the moment conditions (10)).

Step 3.   Assume that f can be represented by (27), with K and L satisfying (26). We will show that $f \in B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$ and that for some $c > 0$

$$\left\|f \left| B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \right.\right\| \le c \left\|\lambda \left| b_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \right.\right\|.$$

By a scaling argument, we see that it suffices to consider the case $\left\|\lambda \left| b_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \right.\right\| = 1$ and show that the modular of a constant times the function on the left-hand side is bounded. In particular, we will show that

$$\sum_{j=0}^{\infty} \left\| |c2^{j\alpha(\cdot)} \varphi_j * f|^{q(\cdot)} \right\|_{p(\cdot)/q(\cdot)} \le C \quad \text{whenever} \quad \sum_{j=0}^{\infty} \left\| \left| \sum_{m \in \mathbb{Z}^n} 2^{j\alpha(\cdot)} \lambda_{jm} \chi_{jm} \right|^{q(\cdot)} \right\|_{p(\cdot)/q(\cdot)} = 1, \quad (30)$$

where $\{\mathcal{F}\varphi_j\}_{j \in \mathbb{N}_0}$ is the resolution of unity. We write

$$f = \sum_{v=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{vm} \rho_{vm} = \sum_{v=0}^{j} + \cdots + \sum_{v=j+1}^{\infty} \cdots.$$

Let $0 < r < \max\left(\frac{1}{q^+}, \frac{p^-}{q^+}\right)$. We have

$$\sum_{j=0}^{\infty}\left\||c2^{j\alpha(\cdot)}\varphi_j*f|^{q(\cdot)}\right\|_{p(\cdot)/q(\cdot)} \le \sum_{j=0}^{\infty}\left(\sum_{v=0}^{\infty}\left\|\left|c\sum_{m\in\mathbb{Z}^n}2^{j\alpha(\cdot)}\lambda_{vm}\varphi_j*\rho_{vm}\right|^{rq(\cdot)}\right\|_{p(\cdot)/rq(\cdot)}\right)^{1/r}$$

$$\le c\sum_{j=0}^{\infty}\left(\sum_{v=0}^{j}\left\|\left|c\sum_{m\in\mathbb{Z}^n}2^{j\alpha(\cdot)}\lambda_{vm}\varphi_j*\rho_{vm}\right|^{rq(\cdot)}\right\|_{p(\cdot)/rq(\cdot)}\right)^{1/r}$$

$$+c\sum_{j=0}^{\infty}\left(\sum_{v=j+1}^{\infty}\left\|\left|c\sum_{m\in\mathbb{Z}^n}2^{j\alpha(\cdot)}\lambda_{vm}\varphi_j*\rho_{vm}\right|^{rq(\cdot)}\right\|_{p(\cdot)/rq(\cdot)}\right)^{1/r} = I + II.$$

For each $k \in \mathbb{N}$ we define $\Omega_k = \left\{m \in \mathbb{Z}^n : 2^{k-1} < 2^{\min(v,j)}|x - 2^{-v}m| \le 2^k\right\}$ and $\Omega_0 = \left\{m \in \mathbb{Z}^n : 2^{\min(v,j)}|x - 2^{-v}m| \le 1\right\}$.

    Estimate of I. From Lemma (6.1.12), we have for any M sufficiently large

$$\sum_{m\in\mathbb{Z}^n}2^{j\alpha(x)}|\lambda_{vm}||\varphi_j*\rho_{vm}(x)| \le c2^{(v-j)(k-\alpha^+)}\sum_{m\in\mathbb{Z}^n}2^{v\alpha(x)}|\lambda_{vm}|(1+2^v|x-2^{-v}m|)^{-M}.$$

We claim that there exists $c > 0$ such that

$$\left\|\left|c\sum_{m\in\mathbb{Z}^n}2^{v\alpha(\cdot)}\lambda_{vm}(1+2^v|\cdot-2^{-v}m|)^{-M}\right|^{rq(\cdot)}\right\|_{\frac{p(\cdot)}{rq(\cdot)}} \le \left\|\left|c\sum_{m\in\mathbb{Z}^n}2^{j\alpha(\cdot)}\lambda_{vm}\chi_{vm}\right|^{rq(\cdot)}\right\|_{\frac{p(\cdot)}{rq(\cdot)}} + 2^{-v} = \delta \quad (31)$$

Therefore, by Lemma (6.1.3) (with the help of (26)) we obtain

$$I \le c\sum_{j=0}^{\infty}\left(\left\|\left|\sum_{m\in\mathbb{Z}^n}2^{j\alpha(\cdot)}\lambda_{jm}\chi_{jm}\right|^{rq(\cdot)}\right\|_{\frac{p(\cdot)}{rq(\cdot)}} + 2^{-j}\right)^{1/r} \le c\sum_{j=0}^{\infty}\left\|\left|\sum_{m\in\mathbb{Z}^n}2^{j\alpha(\cdot)}\lambda_{jm}\chi_{jm}\right|^{rq(\cdot)}\right\|_{\frac{p(\cdot)}{rq(\cdot)}}^{1/r} + c\sum_{j\ge0}2^{-j/r}$$

$$= c\sum_{j=0}^{\infty}\left\|\left|\sum_{m\in\mathbb{Z}^n}2^{j\alpha(\cdot)}\lambda_{jm}\chi_{jm}\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} + c \le C.$$

Let us prove (31). This claim can be reformulated as showing that

$$\left\|\left|\delta^{-1}\left|c\sum_{m\in\mathbb{Z}^n}2^{v\alpha(\cdot)}\lambda_{vm}(1+2^v|\cdot-2^{-v}m|)^{-M}\right|^{rq(\cdot)}\right\|\right\|_{p(\cdot)/rq(\cdot)} \le 1,$$

which is equivalent to

$$\left\|c\delta^{-1/rq(\cdot)}\sum_{m\in\mathbb{Z}^n}2^{v\alpha(\cdot)}\lambda_{vm}(1+2^v|\cdot-2^{-v}m|)^{-M}\right\|_{p(\cdot)} \le 1.$$

We have, with $M = R + T$,

$$\sum_{m\in\mathbb{Z}^n}\delta^{-\frac{1}{rq(x)}}2^{v\alpha(x)}|\lambda_{vm}|(1+2^v|x-2^{-v}m|)^{-M} = \sum_{k=0}^{\infty}\sum_{m\in\Omega_k}\delta^{-\frac{1}{rq(x)}}2^{v\alpha(x)}|\lambda_{vm}|(1+2^v|x-2^{-v}m|)^{-M}$$

$$\le c\sum_{k=0}^{\infty}\sum_{m\in\Omega_k}\delta^{-\frac{1}{rq(x)}}2^{v\alpha(x)}2^{-Mk}|\lambda_{vm}| = \sum_{k=0}^{\infty}2^{-(T-n/t)k}\sum_{m\in\Omega_k}\delta^{-\frac{1}{rq(x)}}2^{v\alpha(x)}2^{-(R+n/t)k}|\lambda_{vm}|$$

$$\le \sup_{k\in\mathbb{N}_0}\sum_{m\in\Omega_k}\delta^{-\frac{1}{rq(x)}}2^{v\alpha(x)}|\lambda_{vm}|2^{-(R+n/t)k},$$

for any T sufficiently large such that $T > n/t$. For any $0 < t \leq 1$, the last expression is bounded by

$$\sup_{k \in \mathbb{N}_0} \left(\sum_{m \in \Omega_k} \delta^{-t/rq(x)} 2^{v\alpha(x)t} |\lambda_{vm}|^t 2^{-(Rt+n)k} \right)^{1/t}$$

$$= \left(\sup_{k \in \mathbb{N}_0} 2^{-Rtk+(v-k)n} \int_{\bigcup_{m \in \Omega_k} \mathcal{Q}_{vm}} \left(\sum_{m \in \Omega_k} \delta^{-1/rq(x)} 2^{v\alpha(x)} |\lambda_{vm}| \chi_{vm}(y) \right)^t dy \right)^{1/t}. \tag{32}$$

Let $y \in \bigcup_{m \in \Omega_k} \mathcal{Q}_{vm}$ then $y \in \mathcal{Q}_{vm}$ for some $m \in \Omega_k$ and $2^{k-1} < 2^v |x - 2^{-v}m| \leq 2^k$. From this it follows that

$$|y - x| \leq |y - 2^{-v}m| + |x - 2^{-v}m| \leq \sqrt{n}2^{-v} + 2^{k-v} \leq 2^{k-v+h_n}, \qquad h_n \in \mathbb{N},$$

which implies that y is located in some ball $B(x, 2^{k-v+h_n})$. Therefore, (32) does not exceed

$$c \left(\sup_{k \in \mathbb{N}_0} \frac{2^{-Rtk}}{|B(x, 2^{k-v+h_n})|} \int_{B(x, 2^{k-v+h_n})} \left(\sum_{m \in \Omega_k} \delta^{-1/rq(x)} 2^{v\alpha(x)} |\lambda_{vm}| \chi_{vm}(y) \right)^t dy \right)^{1/t}. \tag{33}$$

Since $1/q$ is log-Hölder continuous and $\delta \in [2^{-v}, 1 + 2^{-v}]$, we have

$$\delta^{1/q(x)-1/q(y)} = (2^v \delta)^{1/q(x)-1/q(y)} 2^{(1/q(x)-1/q(y))v} \leq 2^{|1/q(x)-1/q(y)|(2v+1)} \leq 2^{\frac{c_{\log}(q)(2v+1)}{\log(e+1/|x-y|)}}$$

$$\leq 2^{\frac{c_{\log}(q)(2v+1)}{v-k-h_n}} \leq c2^{2c_{\log}(q)k},$$

for any $k < \max(0, v - h_n)$ and any $y \in B(x, 2^{k-v+h_n})$. If $k \geq \max(0, v - h_n)$ then since again $\delta \in [2^{-v}, 1 + 2^{-v}]$,

$$\delta^{1/q(x)-1/q(y)} \leq c2^{|1/q(x)-1/q(y)|(2v+1)} \leq c2^{2(1/q^--1/q^+)k}.$$

Also since α is log-Hölder continuous we can prove that

$$2^{v(\alpha(x)-\alpha(y))} \leq c \times \begin{cases} 2^{c_{\log}(\alpha)k} & \text{if} \quad k < \max(0, v - h_n), \\ 2^{(\alpha^+-\alpha^-)k} & \text{if} \quad k \geq \max(0, v - h_n), \end{cases}$$

where $c > 0$ not depending on v and k. Hence with R sufficiently large such that

$$R > \max\left(2/r\, c_{\log}(q) + c_{\log}(\alpha), \qquad 2/r\, (1/q^- - 1/q^+) + \alpha^+ - \alpha^-\right),$$

we get that (33) is bounded by

$$c \left(\mathcal{M}_t \left(\sum_{m \in \Omega_k} \delta^{-1/rq(\cdot)} 2^{v\alpha(\cdot)} |\lambda_{vm}| \chi_{vm} \right) (x) \right)^{1/t}, \quad x \in \mathbb{R}^n.$$

Now taking $0 < t < \min(1, p^-)$ and using the fact that $\mathcal{M} : L^{\frac{p(\cdot)}{t}} \to L^{\frac{p(\cdot)}{t}}$ is bounded we obtain

$$\left\| c \sum_{m \in \mathbb{Z}^n} \delta^{-1/rq(\cdot)} 2^{v\alpha(\cdot)} |\lambda_{vm}| (1 + 2^v |\cdot - 2^{-v}m|)^{-L} \right\|_{p(\cdot)}$$

$$\leq c \left\| \mathcal{M}_t \left(\sum_{m \in \Omega_k} \delta^{-1/rq(\cdot)} 2^{v\alpha(\cdot)} |\lambda_{vm}| \chi_{vm} \right) \right\|_{p(\cdot)/t}^{1/t}$$

$$\leq \left\| \sum_{m \in \mathbb{Z}^n} \delta^{-1/rq(\cdot)} 2^{v\alpha(\cdot)} |\lambda_{vm}| \chi_{vm} \right\|_{p(\cdot)},$$

with an appropriate choice of $c > 0$. Now this expression is less than or equal to one if and only if

$$\left\| \left| \sum_{m \in \mathbb{Z}^n} \delta^{-1/rq(\cdot)} 2^{v\alpha(\cdot)} \lambda_{vm} \chi_{vm} \right|^{rq(\cdot)} \right\|_{p(\cdot)/rq(\cdot)} \leq 1,$$

which follows immediately from the definition of δ.

Estimate of II. From Lemma (6.1.6), we have for any M sufficiently large

$$\sum_{m \in \mathbb{Z}^n} 2^{j\alpha(x)} |\lambda_{vm}| \left| \varphi_j * \rho_{vm}(x) \right| \leq c 2^{(j-v)(L+n+1)} \sum_{m \in \mathbb{Z}^n} 2^{j\alpha(x)} |\lambda_{vm}| (1 + 2^j |x - 2^{-v} m|)^{-M}.$$

Let $0 < t < \min(1, p^-)$ be a real number such that $L > n/t - 1 - n - \alpha^-$. Using a combination of the arguments used in the estimate of I, we arrive at the inequality

$$\left\| \left| c 2^{(j-v)(n/t - \alpha^-)} \sum_{m \in \mathbb{Z}^n} 2^{j\alpha(\cdot)} \lambda_{vm} (1 + 2^j |\cdot - 2^{-v} m|)^{-M} \right|^{rq(\cdot)} \right\|_{p(\cdot)/rq(\cdot)}$$

$$\leq \left\| \left| \sum_{m \in \mathbb{Z}^n} 2^{v\alpha(\cdot)} \lambda_{vm} \chi_{vm} \right|^{rq(\cdot)} \right\|_{p(\cdot)/rq(\cdot)} + 2^{-j},$$

with some positive constant c. Hence II can be estimated by

$$\sum_{j=0}^{\infty} \left(\sum_{v=j+1}^{\infty} 2^{(j-v)(L+n+1-n/t+\alpha^-)rq^-} \left(\left\| \left| \sum_{m \in \mathbb{Z}^n} 2^{v\alpha(\cdot)} \lambda_{vm} \chi_{vm} \right|^{rq(\cdot)} \right\|_{p(\cdot)/rq(\cdot)} + 2^{-j} \right) \right)^{1/r}.$$

Observing that $L > n/t - 1 - n - \alpha^-$, an application of Lemma (6.1.3) yields the desired inequality, i.e.

$$\text{II} \leq \sum_{j=0}^{\infty} \left\| \left| \sum_{m \in \mathbb{Z}^n} 2^{j\alpha(\cdot)} \lambda_{jm} \chi_{jm} \right|^{rq(\cdot)} \right\|_{p(\cdot)/rq(\cdot)}^{1/r} + c = c \sum_{j=0}^{\infty} \left\| \left| \sum_{m \in \mathbb{Z}^n} 2^{j\alpha(\cdot)} \lambda_{jm} \chi_{jm} \right|^{q(\cdot)} \right\|_{p(\cdot)/q(\cdot)} + c \leq C.$$

The proof is completed.

Corollary (6.1.15) [299]: Let $j \in \mathbb{N}_0$ and $x^r \in \mathbb{R}^n$. There exists $\epsilon > 0$ such that for all $g : \mathbb{R}^n \to \mathbb{R}$ we have

$$|g(y^r)| \leq (1 + \epsilon) \left(\frac{1}{\log(e + 1/|x^r - y^r|)} - \left(\eta_{j+\epsilon, 1+\epsilon} * |g|^{1-\epsilon}(x^r) \right)^{1/1-\epsilon} \right)$$

Proof. Since $|g(x^r) - g(y^r)| \leq |g(x^r)| + |g(y^r)|$

Then by Lemma (6.1.2), we have

$$|g(y^r)| \leq \frac{1 + \epsilon}{\log(e + 1/|x^r - y^r|)} - |g(x^r)|$$

$$= (1 + \epsilon) \left(\frac{1}{\log(e + 1/|x^r - y^r|)} - \left(\eta_{j+\epsilon, 1+\epsilon} * |g|^{1-\epsilon}(x^r) \right)^{1/1-\epsilon} \right)$$

Corollary (6.1.16) [299]: Let $\alpha \in C_{loc}^{\log}$, $p, q \in \mathcal{P}_0^{\log}$ and $1 + \epsilon > \frac{n}{p^-}$. Then

$$\left\| \left(\varphi_{j+\epsilon}^{*, 1+\epsilon} 2^{(j+\epsilon)\alpha(\cdot)} f^2 \right)_{j+\epsilon} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \tag{34}$$

is an equivalent quasi-norm in $B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$.

Proof. We will do the proof in two steps.

Step 1. It is easy to see that for any $f^2 \in \mathcal{S}'(\mathbb{R}^n)$ and any $x^r \in \mathbb{R}^n$ we have

$$2^{(j+\epsilon)\alpha(x^r)} \left| \varphi_{j+\epsilon} * f^2(x^r) \right| \leq \varphi_{j+\epsilon}^{*, 1+\epsilon} 2^{(j+\epsilon)\alpha(\cdot)} f^2(x^r).$$

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This shows that the right-hand side in (4) is less than or equal (34).

Step 2.   We will prove in this step that there is a constant $\epsilon \geq 0$ such that for every $f^2 \in \mathcal{S}'(\mathbb{R}^n)$

$$\left\|\left(\varphi_{j+\epsilon}^{*,1+\epsilon} 2^{(j+\epsilon)\alpha(\cdot)} f^2\right)_{j+\epsilon}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leq (1+\epsilon)\left\|\left(2^{(j+\epsilon)\alpha(\cdot)}\varphi_{j+\epsilon} * f^2\right)_{j+\epsilon}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}.$$

By a scaling argument, we see that it suffices to consider the case

$$\left\|\left(2^{(j+\epsilon)\alpha(\cdot)}\varphi_{j+\epsilon} * f^2\right)_{j+\epsilon}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = 1$$

and show that the modular of a constant times the function on the left-hand side is bounded. In particular, we will show that

$$\sum_{j+\epsilon=0}^{\infty} \left\|\left|(1+\epsilon)\varphi_{j+\epsilon}^{*,1+\epsilon} 2^{(j+\epsilon)\alpha(\cdot)} f^2\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq 1+\epsilon \text{ whenever } \sum_{j+\epsilon=0}^{\infty} \left\|\left|2^{(j+\epsilon)\alpha(\cdot)}\varphi_{j+\epsilon} * f^2\right|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} = 1$$

This clearly follows from the inequality

$$\left\|\left|(1+\epsilon)\varphi_{j+\epsilon}^{*,1+\epsilon} 2^{(j+\epsilon)\alpha(\cdot)} f^2\right|^{q(\cdot)}\right\|_{p(\cdot)/q(\cdot)} \leq \left\|\left|2^{(j+\epsilon)\alpha(\cdot)}\varphi_{j+\epsilon} * f^2\right|^{q(\cdot)}\right\|_{p(\cdot)/q(\cdot)} + 2^{-(1+\epsilon)(j+\epsilon)} = \delta,$$

for some $\epsilon > 0$. This claim can be reformulated as showing that

$$\left\|\delta^{-1}\left|(1+\epsilon)\varphi_{j+\epsilon}^{*,1+\epsilon} 2^{(j+\epsilon)\alpha(\cdot)} f^2\right|^{q(\cdot)}\right\|_{p(\cdot)/q(\cdot)} \leq 1,$$

which is equivalent to

$$\left\|(1+\epsilon)\delta^{-1/q(\cdot)}\varphi_{j+\epsilon}^{*,1+\epsilon} 2^{(j+\epsilon)\alpha(\cdot)} f^2\right\|_{p(\cdot)} \leq 1.$$

We choose $\epsilon \geq 0$ such that $1+\epsilon > n/1 + \epsilon > n/p^-$. By Lemmas (6.1.1) and (6.1.2) the estimates

$$2^{(j+\epsilon)\alpha(y^r)}\left|\varphi_{j+\epsilon} * f^2(y^r)\right| \leq (1+\epsilon)2^{(j+\epsilon)\alpha(y^r)}\left(\eta_{j+\epsilon,2(n+\epsilon)} * \left|\varphi_{j+\epsilon} * f^2\right|^{1+\epsilon}(y^r)\right)^{1/1+\epsilon}$$

$$\leq (1+\epsilon)\left(\eta_{j+\epsilon,(n+\epsilon)} * \left(2^{(j+\epsilon)\alpha(\cdot)}\left|\varphi_{j+\epsilon} * f^2\right|\right)^{1+\epsilon}(y^r)\right)^{1/1+\epsilon} \tag{35}$$

are true for any $y^r \in \mathbb{R}^n$, $j \in \mathbb{N}_0$. Divide both sides of (35) by $(1 + 2^{j+\epsilon}|x^r - y^r|)^{1+\epsilon}$, in the right-hand side we use the inequality

$$(1 + 2^{j+\epsilon}|x^r - y^r|)^{-(1+\epsilon)} \leq (1 + 2^{j+\epsilon}|x^r - z^r|)^{-(1+\epsilon)}(1 + 2^{j+\epsilon}|y^r - z^r|)^{1+\epsilon}, \quad x^r, y^r, z^r \in \mathbb{R}^n,$$

in the left-hand side take the supremum over $y^r \in \mathbb{R}^n$ and get for all $f^2 \in \mathcal{S}'(\mathbb{R}^n)$, any $x^r \in \mathbb{R}^n$, and any $j \in \mathbb{N}_0$

$$\left(\varphi_{j+\epsilon}^{*,1+\epsilon} 2^{(j+\epsilon)\alpha(\cdot)} f^2(x^r)\right)^{1-\epsilon} \leq (1+\epsilon)2^{(j+\epsilon)n} \int_{\mathbb{R}^n} \frac{2^{(j+\epsilon)(1-\epsilon)\alpha(z^r)}\left|\varphi_{j+\epsilon} * f^2(z^r)\right|^{1-\epsilon}}{(1 + 2^{j+\epsilon}|x^r - z^r|)^{(1-\epsilon^2)}} dz^r$$

$$= (1+\epsilon)\int_{B(x^r,2^{-j+\epsilon/2})} \cdots dz^r + (1+\epsilon)\sum_{i=0}^{\infty} \int_{B(x^r,2^{-(j+\epsilon)/2+i+1})\backslash B(x^r,2^{-(j+\epsilon)/2+i})} \cdots dz^r$$

$$= J_{j+\epsilon}^1(x^r) + \sum_{i\geq 0} J_{j+\epsilon-i}^2(x^r),$$

where $\epsilon > 0$ is independent of x^r, j and f^2, such that

$$0 < \frac{1+\epsilon}{(1+\epsilon)^2 - n} < 1/4(1/q^- - 1/q^+).$$

Since $1/q$ is log-Hölder continuous and $\delta \in \left[2^{-(1+\epsilon)(j+\epsilon)}, 1 + 2^{-(1+\epsilon)(j+\epsilon)}\right]$, we have

$$\delta^{(1/q(z^2)-1/q(x^r))} \leq \left(2^{(1+\epsilon)(j+\epsilon)}\delta\right)^{|1/q(z^r)-1/q(x^r)|} 2^{(j+\epsilon)(1+\epsilon)|1/q(z^r)-1/q(x^r)|}$$

$$\leq (1+\epsilon)2^{\frac{2c_{\log}q(1+\epsilon)(j+\epsilon)}{\log(e+1/|x^r-z^r|)}}$$

$$\leq 1 + \epsilon \tag{36}$$

for any $z^r \in B(x^r, 2^{-(j+\epsilon)/2})$. Hence

$$\delta^{-(1+\epsilon)/q(x^r)}J^1_{j+\epsilon}(x^r) \le (1+\epsilon)2^{(j+\epsilon)n}\int_{\mathbb{R}^n}\frac{\delta^{-(1+\epsilon)/q(z^r)}2^{(j+\epsilon)\alpha(z^r)(1+\epsilon)}\left|\varphi_{(j+\epsilon)}*f^2(z^r)\right|^{1+\epsilon}}{(1+2^{j+\epsilon}|x^r-z^r|)^{(1+\epsilon)2}}dz^r.$$

Now the function $z^r \mapsto \frac{1}{(1+|z^r|)^{(1+\epsilon)2}}$ is in L^1 (since $(1+\epsilon)^2 > n$), then using the majorant property for the Hardy-Littlewood maximal operator $\mathcal{M}$

$$\left(|g|^{1+\epsilon}*1/(1+|\cdot|)^{(1+\epsilon)^2}\right)(x^r) \le (1+\epsilon)\left\|1/(1+|\cdot|)^{(1+\epsilon)^2}\right\|_1\mathcal{M}_{1+\epsilon}(g)(x^r),$$

it follows that for any $x^r \in \mathbb{R}^n$

$$\delta^{-(1+\epsilon)/q(x^r)}J^1_{j+\epsilon}(x^r) \le (1+\epsilon)\mathcal{M}_{1+\epsilon}\left(\delta^{-1/q(\cdot)}2^{(j+\epsilon)\alpha(\cdot)}\varphi_{j+\epsilon}*f^2\right)(x^r),$$

where the constant $\epsilon \ge 0$ is independent of x^r and j. Since $|x^r - z^r| \ge 2^{-(j+\epsilon)/2+i}$ and the right-hand side of (36) can be estimated by $(1+\epsilon)2^{2(j+\epsilon)(1+\epsilon)(1/q^- -1/q^+)}$, then for any $x^r \in \mathbb{R}^n$ and any $j+\epsilon \in \mathbb{N}_0$, $\delta^{-(1+\epsilon)/q(x^r)}J^2_{(j+\epsilon)-i}(x^r)$ is bounded by

$$(1+\epsilon)2^{(j+\epsilon)(1+\epsilon)^2(2(1/q^- -1/q^+)-n-(1+\epsilon)^2/2)}2^{-i(1+\epsilon)^2}\int_{B(x^r,2^{-(j+\epsilon)/2+i+1})}\delta^{-(1+\epsilon)/q(z^r)}2^{(j+\epsilon)\alpha(z^r)(1+\epsilon)}\big|\varphi_{j+\epsilon}$$

$$*f^2(z^r)\big|^{1+\epsilon}dz^r$$

$$\le (1+\epsilon)2^{(j+\epsilon)(1+\epsilon)^2(2(1/q^- -1/q^+)+n-(1+\epsilon)^2/2)}2^{i(n-(1+\epsilon)^2)}\mathcal{M}_{1+\epsilon}\left(\delta^{-1/q(\cdot)}2^{(j+\epsilon)\alpha(\cdot)}\varphi_{j+\epsilon}\right.$$

$$\left.*f^2\right)(x^r) \le (1+\epsilon)2^{i(n-(1+\epsilon)^2)}\mathcal{M}_{1+\epsilon}\left(\delta^{-1/q(\cdot)}2^{(j+\epsilon)\alpha(\cdot)}\varphi_{j+\epsilon}*f^2\right)x^r,$$

Hence,

$$\sum_{i=0}^{\infty}\delta^{-1+\epsilon/q(x^r)}J^2_{j+\epsilon-i}(x^r) \le (1+\epsilon)\sum_{i=0}^{\infty}2^{i(n-(1+\epsilon)^2)}\mathcal{M}_{1+\epsilon}\left(\delta^{-1/q(\cdot)}2^{(j+\epsilon)\alpha(\cdot)}\varphi_{j+\epsilon}*f^2\right)(x^r)$$

$$\le (1+\epsilon)\mathcal{M}_{1+\epsilon}\left(\delta^{-1/q(\cdot)}2^{(j+\epsilon)\alpha(\cdot)}\varphi_{j+\epsilon}*f^2\right)(x^r)$$

since again $(1+\epsilon)^2 > n$. Consequently we have proved that

$$\left(\delta^{-1/q(x^r)}\varphi^{*,1+\epsilon}_{j+\epsilon}2^{(j+\epsilon)\alpha(\cdot)}f^2(x^r)\right)^{1+\epsilon} \le (1+\epsilon)\mathcal{M}_{1+\epsilon}\left(\delta^{-1/q(\cdot)}2^{(j+\epsilon)\alpha(\cdot)}\varphi_{j+\epsilon}*f^2\right)(x^r),$$

for all $x^r \in \mathbb{R}^n$. Taking the $L^{\frac{p(\cdot)}{1+\epsilon}}$-norm and using the fact that $\mathcal{M}: L^{\frac{p(\cdot)}{1+\epsilon}} \to L^{\frac{p(\cdot)}{1+\epsilon}}$ is bounded we obtain that

$$\left\|(1+\epsilon)\delta^{-1/q(\cdot)}\varphi^{*,1+\epsilon}_{j+\epsilon}2^{(j+\epsilon)\alpha(\cdot)}f^2\right\|^{1+\epsilon}_{p(\cdot)} = \left\|\left|(1+\epsilon)\delta^{-1/q(\cdot)}\varphi^{*,1+\epsilon}_{j+\epsilon}2^{(j+\epsilon)\alpha(\cdot)}f^2\right|^{1+\epsilon}\right\|_{p(\cdot)/(1+\epsilon)}$$

$$\le \left\|\delta^{-(1+\epsilon)/q(\cdot)}2^{(j+\epsilon)\alpha(\cdot)1+\epsilon}\left|\varphi_{j+\epsilon}*f^2\right|^{1+\epsilon}\right\|_{p(\cdot)/(1+\epsilon)}$$

$$= \left\|\delta^{-1/q(\cdot)}2^{(j+\epsilon)\alpha(\cdot)}\varphi_{j+\epsilon}*f^2\right\|^{1+\epsilon}_{p(\cdot)}$$

with an appropriate choice of $\epsilon > 0$. Now the right-hand side is less than or equal to one if and only if

$$\left\|\left|\delta^{-1/q(\cdot)}2^{(j+\epsilon)\alpha(\cdot)}\varphi_{j+\epsilon}*f^2\right|^{q(\cdot)}\right\|_{p(\cdot)/q(\cdot)} \le 1,$$

which follows immediately from the definition of δ.
The proof is completed.

Corollary (6.1.17) [299]: Let $\alpha \in C^{\log}_{loc}$ and $p, q \in \mathcal{P}^{\log}_0$ with $q^+ < \infty$. Let $1+\epsilon > n/p^-$ and $\alpha^+ < \epsilon$. Then

$$\left\|f^2|B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}\right\|' = \left\|\left(k^{*,1+\epsilon}_j2^{j\alpha(\cdot)}f^2\right)_j\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \tag{37}$$

and

$$\left\|f^2|B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}\right\|'' = \left\|\left(2^{j\alpha(\cdot)}k_j*f^2\right)_j\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}, \tag{38}$$

are equivalent quasi-norms on $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$.

Proof:

Step 1.   Take any pair of functions ϕ_0 and $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that
$$|\mathcal{F}\phi_0(\xi)| > 0 \qquad \text{for } |\xi| < 2\varepsilon,$$
$$|\mathcal{F}\phi(\xi)| > 0 \qquad \text{for } \frac{\varepsilon}{2} < |\xi| < 2\varepsilon.$$

We will prove that there is a constant $\epsilon > 0$ such that for any $f^2 \in \mathcal{S}'(\mathbb{R}^n)$
$$\left\| f^2 | B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)} \right\|' \leq (1 + \epsilon) \left\| \left(\phi_j^{*,1+\epsilon} 2^{j\alpha(\cdot)} f^2 \right)_j \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}. \tag{39}$$

By a scaling argument, we see that it suffices to consider the case
$$\left\| \left(\phi_j^{*,1+\epsilon} 2^{j\alpha(\cdot)} f^2 \right)_j \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = 1$$

and show that the modular of a constant times the function on the left-hand side is bounded. In particular, we will show that

$$\sum_{j=0}^{\infty} \left\| \left| (1 + \epsilon) k_j^{*,1+\epsilon} 2^{j\alpha(\cdot)} f^2 \right|^{q(\cdot)} \right\|_{p(\cdot)/q(\cdot)} \leq 1 + \epsilon \quad \text{when} \quad \sum_{j=0}^{\infty} \left\| \left| \phi_j^{*,1+\epsilon} 2^{j\alpha(\cdot)} f^2 \right|^{q(\cdot)} \right\|_{p(\cdot)/q(\cdot)} = 1.$$

Let $\Lambda, \lambda \in \mathcal{S}(\mathbb{R}^n)$ so that
$$\operatorname{supp} \mathcal{F}\Lambda \subset \{\xi \in \mathbb{R}^n : |\xi| < 2\varepsilon\}, \qquad \operatorname{supp} \mathcal{F}\lambda \subset \{\xi \in \mathbb{R}^n : \varepsilon/2 < |\xi| < 2\varepsilon\},$$

$$\mathcal{F}\Lambda(\xi)\mathcal{F}\phi_0(\xi) + \sum_{j+\epsilon=1}^{\infty} \mathcal{F}\lambda\left(2^{-(j+\epsilon)}\xi\right)\mathcal{F}\phi\left(2^{-(j+\epsilon)}\xi\right) = 1, \qquad \xi \in \mathbb{R}^n. \tag{40}$$

In particular, for any $f^2 \in \mathcal{S}'(\mathbb{R}^n)$ the identity is true
$$f^2 = \Lambda * \phi_0 * f^2 + \sum_{j+\epsilon=1}^{\infty} \lambda_{j+\epsilon} * \phi_{j+\epsilon} * f^2. \tag{41}$$

Hence we can write
$$k_j * f^2 = k_j * \Lambda * \phi_0 * f^2 + \sum_{j+\epsilon=1}^{\infty} k_j * \lambda_{j+\epsilon} * \phi_{j+\epsilon} * f^2.$$

We have
$$2^{j\alpha(y^r)}\left| k_j * \lambda_{j+\epsilon} * \phi_{j+\epsilon} * f^2(y^r) \right| \leq 2^{j\alpha(y^r)} \int_{\mathbb{R}^n} \left| k_j * \lambda_{j+\epsilon}(z^r) \right| \left| \phi_{j+\epsilon} * f^2(y^r - z^r) \right| dz^r. \tag{42}$$

First let $\epsilon \geq 0$. Writing for any $z^r \in \mathbb{R}^n$
$$k_j * \lambda_{j+\epsilon}(z^r) = 2^{(j+\epsilon)n} k_{2\epsilon} * \lambda(2^{j+\epsilon} z^r),$$

we get by Lemma (6.1.4),
$$\left| k_j * \lambda_{j+\epsilon}(z^r) \right| \leq (1 + \epsilon) \frac{2^{\epsilon^2 + (j+\epsilon)n}}{(1 + 2^{j+\epsilon}|z^r|)^{2(1+\epsilon)}}, \qquad z^r \in \mathbb{R}^n.$$

So the right-hand side of (42) can be estimated from above by
$$(1 + \epsilon) 2^{j\alpha(y^r) + \epsilon^2 + (j+\epsilon)n} \int_{\mathbb{R}^n} (1 + 2^{1+\epsilon}|z^r|)^{-2(1+\epsilon)} |\phi_{1+\epsilon} * f^2(y^r - z^r)| dz^r$$

$$= (1 + \epsilon) 2^{\epsilon^2} 2^{j\alpha(y^r)} \eta_{1+\epsilon, 2(1+\epsilon)} * |\phi_{1+\epsilon} * f^2|(y^r).$$

By Lemma (6.1.1) the estimates
$$2^{j\alpha(y^r)} \eta_{1+\epsilon, 2(1+\epsilon)} * |\phi_{j+\epsilon} * f^2(y^r)| \leq 2^{\epsilon\alpha^+} \eta_{j+\epsilon, 1+\epsilon} * \left(2^{(j+\epsilon)\alpha(\cdot)} |\phi_{j+\epsilon} * f^2| \right)(y^r)$$

$$\leq 2^{\epsilon\alpha^+} \phi_{j+\epsilon}^{*,1+\epsilon} 2^{(j+\epsilon)\alpha(\cdot)} f^2(y^r) \|\eta_{j+\epsilon}\|_1 \leq (1 + \epsilon) 2^{\epsilon\alpha^+} \phi_{j+\epsilon}^{*,1+\epsilon} 2^{(j+\epsilon)\alpha(\cdot)} f^2(y^r),$$

Let now $\epsilon \geq 0$. Then, again by Lemma (6.1.4) we have for any $z^r \in \mathbb{R}^n$

$$\left|k_j * \lambda_{j+\epsilon}(z^r)\right| = 2^{jn}\left|k * \lambda_{2^\epsilon}(2^j z^r)\right| \le (1+\epsilon)\frac{2^{\epsilon^2+jn}}{(1+2^j|z^r|)^{2(1+\epsilon)}},$$

where $\epsilon \ge 0$ an integer can be taken arbitrarily large, since $D^\alpha \mathcal{F}\lambda(0) = 0$ for all α. Therefore, the right-hand side of (42) can be estimated from above by

$$(1+\epsilon)2^{j\alpha(y^r)+\epsilon^2+jn}\int_{\mathbb{R}^n}(1+2^j|z^r|)^{-2(1+\epsilon)}\left|\phi_{j+\epsilon} * f^2(y^r-z^r)\right|dz^r$$

$$= (1+\epsilon)2^{j\alpha(y^r)+\epsilon^2}\eta_{j,2(1+\epsilon)} * \left|\phi_{j+\epsilon} * f^2\right|(y^r).$$

We have

$$(1+2^j|z^r|)^{-2(1+\epsilon)} \le 2^{2\epsilon(1+\epsilon)}(1+2^{j+\epsilon}|z^r|)^{-2(1+\epsilon)}.$$

Then, again, the right-hand side of (42) is dominated by

$$(1+\epsilon)2^{j\alpha(y^r)-\epsilon(n-\epsilon-1)}\eta_{j+\epsilon,2L} * \left|\phi_{j+\epsilon} * f^2\right|(y^r)$$

$$\le (1+\epsilon)2^{\epsilon(\alpha^-+n-\epsilon)}\eta_{j+\epsilon,L} * \left(2^{(j+\epsilon)\alpha(\cdot)}\left|\phi_{j+\epsilon} * f^2\right|\right)(y^r)$$

$$\le (1+\epsilon)2^{\epsilon(\alpha^-+n-\epsilon)}\phi_{j+\epsilon}^{*,1+\epsilon}2^{(j+\epsilon)\alpha(\cdot)}f^2(y^r)\left\|\eta_{j+\epsilon,L-(1+\epsilon)}\right\|_1$$

$$\le (1+\epsilon)2^{\epsilon(\alpha^-+n-\epsilon)}\phi_{j+\epsilon}^{*,1+\epsilon}2^{(j+\epsilon)\alpha(\cdot)}f^2(y^r),$$

where in the first inequality we have used Lemma (6.1.1). Taking $\epsilon < \frac{\alpha^--n+4}{2}$ to estimate the last expression by

$$(1+\epsilon)2^{\epsilon(\epsilon+2)}\phi_{j+\epsilon}^{*,1+\epsilon}2^{(j+\epsilon)\alpha(\cdot)}f^2(y^r),$$

where $\epsilon \ge 0$ is independent of j and f^2. Further, note that for all $x^r, y^r \in \mathbb{R}^n$ and all $j \in \mathbb{N}$

$$\phi_{j+\epsilon}^{*,1+\epsilon}2^{(j+\epsilon)\alpha(\cdot)}f^2(y^r) \le \phi_{j+\epsilon}^{*,1+\epsilon}2^{(j+\epsilon)\alpha(\cdot)}f^2(x^r)(1+2^{j+\epsilon}|x^r-y^r|)^{1+\epsilon}$$

$$\le \phi_{j+\epsilon}^{*,j+\epsilon}2^{(j+\epsilon)\alpha(\cdot)}f^2(x^r)\max\left(1,2^{\epsilon(1+\epsilon)}\right)(1+2^j|x^r-y^r|)^{1+\epsilon}.$$

Hence

$$\sup_{x^r \in \mathbb{R}^n}\frac{2^{j\alpha(y^r)}\left|k_j * \lambda_{j+\epsilon} * \phi_{j+\epsilon} * f^2(y^r)\right|}{(1+2^j|x^r-y^r|)^{1+\epsilon}} \le (1+\epsilon)\phi_{j+\epsilon}^{*,1+\epsilon}2^{(j+\epsilon)\alpha(\cdot)}f^2(x^r) \times \begin{cases}2^{\epsilon(\epsilon-\alpha^+)} & \text{if } \epsilon \le 0, \\ 2^\epsilon & \text{if } \epsilon \ge 0.\end{cases}$$

Using the fact that for any $z^r \in \mathbb{R}^n$, any integer $\epsilon \ge 0$

$$\left|k_j * \Lambda(z^r)\right| = \left|k_{2^{-j}} * \Lambda(z^r)\right| \le (1+\epsilon)\frac{2^{-j\epsilon}}{(1+|z^r|)^{2(1+\epsilon)}},$$

we obtain by the similar arguments that for any $j \in \mathbb{N}$

$$\sup_{y^r \in \mathbb{R}^n}\frac{2^{j\alpha(y^r)}\left|k_j * \Lambda * \phi_0 * f^2(y^r)\right|}{(1+2^j|x^r-y^2|)^{1+\epsilon}} \le (1+\epsilon)2^{-j(\epsilon-\alpha^+)}\phi_0^{*,1+\epsilon}f^2(x^r).$$

Hence with $\delta = \min(1,\epsilon-\alpha^+) > 0$ for all $f^2 \in S'(\mathbb{R}^n)$, $x^r \in \mathbb{R}^n$, $j \in \mathbb{N}$

$$k_{j+\epsilon}^{*,1+\epsilon}2^{j\alpha(\cdot)}f^2(y^r) \le (1+\epsilon)2^{-j\delta}\phi_0^{*,1+\epsilon}f^2(x^r) + (1+\epsilon)\sum_{j+\epsilon=1}^{\infty}2^{-|\epsilon|\delta}\phi_{j+\epsilon}^{*,1+\epsilon}2^{(j+\epsilon)\alpha(\cdot)}f^2(x^r)$$

$$= (1+\epsilon)\sum_{j+\epsilon=0}^{\infty}2^{-|\epsilon|\delta}\phi_{j+\epsilon}^{*,1+\epsilon}2^{(j+\epsilon)\alpha(\cdot)}f^2(x^r).$$

Also for $j = 0$, we use the fact that for $\epsilon \ge 0$, any $z^r \in \mathbb{R}^n$

$$\left|k_0 * \lambda_{j+\epsilon}(z^r)\right| = \left|k_0 * \lambda_{2^{-(j+\epsilon)}}(z^r)\right| \le (1+\epsilon)\frac{2^{-\epsilon(j+\epsilon)}}{(1+|z^r|)^{2(1+\epsilon)}}$$

and

$$\left|k_0 * \Lambda(z^r)\right| \le (1+\epsilon)\frac{1}{(1+|z^r|)^{2(1+\epsilon)}}$$

to get for any $x^r \in \mathbb{R}^n$

$$k_0^{*,1+\epsilon} f^2(x^r) \leq (1 + \epsilon)\left(\phi_0^{*,1+\epsilon} f^2(x^r) + \sum_{j+\epsilon=1}^{\infty} 2^{-(j+\epsilon)\delta} \phi_{j+\epsilon}^{*,1+\epsilon} 2^{(j+\epsilon)\alpha(\cdot)} f^2(x^r)\right)$$

$$= (1 + \epsilon) \sum_{j+\epsilon=0}^{\infty} 2^{-(j+\epsilon)\delta} \phi_{j+\epsilon}^{*,1+\epsilon} 2^{j+\epsilon\alpha(\cdot)} f^2(x^r).$$

Let $\tau > \max(q^+, q^+/p^-)$. Then by Lemma (6.1.3)

$$\sum_{j=0}^{\infty} \left\| |(1+\epsilon) k_j^{*,1+\epsilon} 2^{j\alpha(\cdot)} f^2|^{q(\cdot)} \right\|_{p(\cdot)/q(\cdot)} = \sum_{j=0}^{\infty} \left\| |(1+\epsilon) k_j^{*,1+\epsilon} 2^{j\alpha(\cdot)} f^2|^{q(\cdot)/\tau} \right\|_{\tau p(\cdot)/q(\cdot)}^{\tau}$$

$$\leq \left(\sum_{j=0}^{\infty} 2^{-|\epsilon|\delta} \left\| |\phi_{j+\epsilon}^{*,1+\epsilon} 2^{(j+\epsilon)\alpha(\cdot)} f^2|^{q(\cdot)/\tau} \right\|_{\tau p(\cdot)/q(\cdot)}\right)^{\tau}$$

$$\leq (1 + \epsilon) \sum_{j=0}^{\infty} \left\| |\phi_j^{*,1+\epsilon} 2^{j\alpha(\cdot)} f^2|^{q(\cdot)/\tau} \right\|_{\tau p(\cdot)/q(\cdot)}^{\tau}$$

$$= (1 + \epsilon) \sum_{j=0}^{\infty} \left\| |\phi_j^{*,1+\epsilon} 2^{j\alpha(\cdot)} f^2|^{q(\cdot)} \right\|_{p(\cdot)/q(\cdot)} \leq 1 + \epsilon,$$

with an appropriate choice of $\epsilon > 0$.

Step 2.   We will prove in this step that there is a constant $\epsilon > 0$ such that for any $f^2 \in \mathcal{S}'(\mathbb{R}^n)$

$$\left\| f^2 | B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \right\|' \leq (1 + \epsilon) \left\| f^2 | B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \right\|''. \tag{43}$$

Analogously to (40), (41) find two functions $\Lambda, \psi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\operatorname{supp} \mathcal{F}\Lambda \subset \{\xi \in \mathbb{R}^n : |\xi| < 2\varepsilon\}, \qquad \operatorname{supp} \mathcal{F}\psi \subset \{\xi \in \mathbb{R}^n : \varepsilon/2 < |\xi| < 2\varepsilon\},$$

and for all $f^2 \in \mathcal{S}'(\mathbb{R}^n)$ and $j \in \mathbb{N}_0$

$$f^2 = \Lambda_j * (k_0)_j * f^2 + \sum_{n=j-\epsilon+1}^{\infty} \psi_{n+\epsilon} * k_{n+\epsilon} * f^2.$$

Hence

$$k_j * f^2 = \Lambda_j * (k_0)_j * k_j * f^2 + \sum_{n=j-\epsilon+1}^{\infty} k_j * \psi_{n+\epsilon} * k_{n+\epsilon} * f^2.$$

By a scaling argument, we see that it suffices to consider the case

$$\left\| f^2 | B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \right\|'' = 1$$

and show that the modular of a constant times the function on the left-hand side is bounded. In particular, we will show that

$$\sum_{j=0}^{\infty} \left\| |(1+\epsilon) k_j^{*,1+\epsilon} 2^{j\alpha(\cdot)} f^2|^{q(\cdot)} \right\|_{p(\cdot)/q(\cdot)} \leq 1 + \epsilon \quad \text{when} \quad \sum_{j=0}^{\infty} \left\| |2^{j\alpha(\cdot)} k_j * f^2|^{q(\cdot)} \right\|_{p(\cdot)/q(\cdot)} = 1.$$

Writing for any $z^r \in \mathbb{R}^n$

$$k_j * \psi_{n+\epsilon}(z^r) = 2^{jn}\left(k * \psi_{2^{j-(n+\epsilon)}}\right)(2^j z^r),$$

we get by Lemma (6.1.5), that for any integer $\epsilon \geq 0$ independent of j

$$\left|k_j * \psi_{n+\epsilon}(z^r)\right| \leq (1 + \epsilon) \frac{2^{(j-n-\epsilon)\epsilon + jn}}{(1 + 2^j |z^r|)^{2(\epsilon-1)}}, \quad z^r \in \mathbb{R}^n.$$

Analogous estimate

$$\left|\Lambda_j * (k_0)_j(z^r)\right| \leq (1 + \epsilon) \frac{2^{jn}}{(1 + 2^j |z^r|)^{2(\epsilon-1)}}, \quad z^r \in \mathbb{R}^n,$$

is obvious. From this it follows that

$$2^{j\alpha(y^r)}\big|k_j * f^2(y^r)\big| \le (1+\epsilon) \sum_{j=0}^{\infty} 2^{(j-n-\epsilon)(\epsilon+\alpha^-)} 2^{(n+\epsilon)\alpha(y^r)} \eta_{j,2(\epsilon-1)} * |k_{n+\epsilon} * f^2|(y^r)$$

$$= (1+\epsilon) \sum_{n=j-\epsilon}^{\infty} 2^{(j-n-\epsilon)(\epsilon+\alpha^-)+jn} \int_{\mathbb{R}^n} \frac{2^{(n+\epsilon)\alpha(y^r)}|k_{n+\epsilon} * f^2(z^r)|}{(1+2^j|y^r-z^r|)^{2(\epsilon-1)}} dz^r.$$

Since

$$(1 + 2^j|y^r - z^r|)^{-2(\epsilon-1)} \le 2^{2(n+\epsilon-j)(\epsilon-1)}(1 + 2^{n+\epsilon}|y^r - z^r|)^{-2(\epsilon-1)},$$

then by Lemma (6.1.1) we have

$$2^{j\alpha(y^r)}\big|k_j * f^2(y^r)\big| \le (1+\epsilon) \sum_{n=j-\epsilon}^{\infty} 2^{(j-n-\epsilon)(\alpha^- - 2\epsilon + n - 2)+(n+\epsilon)\alpha(y^r)} \eta_{n+\epsilon,2(\epsilon-1)} * |k_{n+\epsilon} * f^2|(y^r)$$

$$\le (1+\epsilon) \sum_{n=j-\epsilon}^{\infty} 2^{(j-n-\epsilon)(\alpha^- - 2\epsilon + n - 2)} \eta_{n+\epsilon,1+\epsilon} * \big(2^{(n+\epsilon)\alpha(\cdot)}|k_{n+\epsilon} * f^2|\big)(y^r), \quad (44)$$

Using the elementary estimates

$$(1 + 2^j|x^r - y^r|)^{-(1+\epsilon)} \le (1 + 2^j|x^r - z^r|)^{-(1+\epsilon)}(1 + 2^j|y^r - z^r|)^{1+\epsilon}$$
$$\le 2^{(n+\epsilon-j)(1+\epsilon)}(1 + 2^{n+\epsilon}|x^r - z^r|)^{-(1+\epsilon)}(1 + 2^{n+\epsilon}|y^r - z^r|)^{1+\epsilon}, \quad (45)$$

to get

$$k_j^{*,1+\epsilon} 2^{j\alpha(\cdot)} f^2(x^r) \le (1+\epsilon) \sum_{n=j-\epsilon}^{\infty} 2^{(j-n-\epsilon)(-\alpha^- - \epsilon + n + 1)+(n+\epsilon)n} \int_{\mathbb{R}^n} \frac{2^{(n+\epsilon)\alpha(z^r)}|k_{n+\epsilon} * f^2(z^r)|}{(1+2^{n+\epsilon}|x^r-z^r|)^{1+\epsilon}} dz^r.$$

Fix any $\epsilon \ge 0$. We have

$$2^{(n+\epsilon)\alpha(z^r)}|k_{n+\epsilon} * f^2(z^r)| = \big(2^{(n+\epsilon)\alpha(z^r)}|k_{n+\epsilon} * f^2(z^r)|\big)^{1-\epsilon}\big(2^{(1+\epsilon)\alpha(z^r)}|k_{n+\epsilon} * f^2(z^r)|\big)^{\epsilon}$$

$$= \big(2^{(n+\epsilon)\alpha(z^r)}|k_{n+\epsilon} * f^2(z^r)|\big)^{1-\epsilon} \left(\frac{2^{(1+\epsilon)\alpha(z^r)}|k_{n+\epsilon} * f^2(z^r)|}{(1+2^{n+\epsilon}|x^r-z^r|)^{1+\epsilon}}\right)^{\epsilon} (1$$

$$+ 2^{n+\epsilon}|x^r - z^r|)^{(1+\epsilon)\epsilon}$$

$$\le \big(2^{(1+\epsilon)\alpha(z^r)}|k_{n+\epsilon} * f^2(z^r)|\big)^{1-\epsilon} \left(k_{n+\epsilon}^{*,1+\epsilon} 2^{(1+\epsilon)\alpha(\cdot)} f^2(x^r)\right)^{\epsilon} (1$$

$$+ 2^{n+\epsilon}|x^r - z^r|)^{(1+\epsilon)\epsilon}.$$

Then

$$k_j^{*,1+\epsilon} 2^{j\alpha(\cdot)} f^2(x^r)$$

$$\le (1+\epsilon) \sum_{n=j-\epsilon}^{\infty} 2^{(j-n-\epsilon)N'+(n+\epsilon)n} \int_{\mathbb{R}^n} \frac{2^{n+\epsilon(1-\epsilon)\alpha(z^r)}|k_{n+\epsilon} * f^2(z^r)|^{1-\epsilon}}{(1+2^{n+\epsilon}|x^r-z^r|)^{1-\epsilon^2}} dz^r \left(k_{n+\epsilon}^{*,1+\epsilon} 2^{n+\epsilon\alpha(\cdot)} f^2(x^r)\right)^{\epsilon},$$

where $N' = n - \alpha^- - 2\epsilon - 1$ can be still be taken arbitrarily large. Quite analogously one proves for all $f^2 \in \mathcal{S}'(\mathbb{R}^n)$ the estimate

$$k_0^{*,1+\epsilon} f^2(x^r)$$

$$\le (1+\epsilon) \sum_{n=-\epsilon}^{\infty} 2^{-(n+\epsilon)(N'-n)} \int_{\mathbb{R}^n} \frac{2^{(n+\epsilon)(1-\epsilon)\alpha(z^r)}|k_{n+\epsilon} * f^2(z^r)|^{1-\epsilon}}{(1+2^{n+\epsilon}|x^r-z^r|)^{1-\epsilon^2}} dz^r \left(k_{n+\epsilon}^{*,1+\epsilon} 2^{(n+\epsilon)\alpha(\cdot)} f^2(x^r)\right)^{\epsilon}.$$

We now fix any $x^r \in \mathbb{R}^n$ and apply Lemma (6.1.5) with

$$(n+\epsilon)_j = k_j^{*,1+\epsilon} 2^{j\alpha(\cdot)} f^2(x^r), \qquad j \in \mathbb{N}_0,$$

$$(n+2\epsilon)_{n+\epsilon} = \int_{\mathbb{R}^n} \frac{2^{(n+\epsilon)(1-\epsilon)\alpha(z^r)+(n+\epsilon)n}|k_{n+\epsilon} * f^2(z^r)|^{1-\epsilon}}{(1+2^{n+\epsilon}|x^r-z^r|)^{1-\epsilon^2}} dz^r, \qquad n \in \mathbb{N}_0.$$

The assumption (3) is satisfied with $N_0 = N_1 + n + [\max(0, \alpha^+)] + 1$, where N_1 is the order of the distribution $f^2 \in \mathcal{S}'(\mathbb{R}^n)$. We conclude that for any $f^2 \in \mathcal{S}'(\mathbb{R}^n)$, and any $j \in \mathbb{N}_0$

$$\left(k_j^{*,1+\epsilon}2^{j\alpha(\cdot)}f^2(x^r)\right)^{1-\epsilon}$$

$$\leq (1+\epsilon)\sum_{j=n+\epsilon}^{\infty} 2^{(j-n-\epsilon)(1-\epsilon^2)+(n+\epsilon)n}\int_{\mathbb{R}^n}\frac{2^{(n+\epsilon)(1-\epsilon)\alpha(z^r)}|k_{n+\epsilon}*f^2(z^r)|^{1-\epsilon}}{(1+2^{n+\epsilon}|x^r-z^r|)^{1-\epsilon^2}}dz^r.$$

This estimate is also true for $\epsilon < 0$, with much simpler proof. It suffices to take (44) with $1 + \epsilon + n$ instead of $1 + \epsilon$, apply Hölder's inequalities in z^r, and finally the inequality (45). We omit the details.

Since $1 + \epsilon > \frac{n}{p^-}$, it possible to take $\frac{n}{1+\epsilon} < 1 - \epsilon < p^-$. Let $\tau > \frac{q^+}{1-\epsilon}$. We see that

$$\left\|\left|(1+\epsilon)k_j^{*,1+\epsilon}2^{j\alpha(\cdot)}f^2\right|^{q(\cdot)}\right\|_{p(\cdot)/q(\cdot)} = \left\|\left|(1+\epsilon)k_j^{*,1+\epsilon}2^{j\alpha(\cdot)}f^2\right|^{q(\cdot)/\tau}\right\|_{p(\cdot)/q(\cdot)}$$

$$= \left\|\left|(1+\epsilon)k_j^{*,1+\epsilon}2^{j\alpha(\cdot)}f^2\right|^{q(\cdot)/\tau}\right\|_{\tau p(\cdot)/q(\cdot)}^{\tau}$$

$$\leq (1+\epsilon)\left(\sum_{n=j-\epsilon}^{\infty}2^{(j-n-\epsilon)Nq^-/\tau}\left\|\left|(1+\epsilon)\eta_{n+\epsilon,1-\epsilon^2}*\left(2^{(n+\epsilon)\alpha(\cdot)}|k_{n+\epsilon}*f^2|\right)^{1-\epsilon}\right|^{q(\cdot)/(1-\epsilon)\tau}\right\|_{\tau p(\cdot)/q(\cdot)}\right)^{\tau}$$

By the same method given in the proof of Theorem (6.1.8) we can prove that

$$\left\|\left|(1+\epsilon)\eta_{n+\epsilon,1-\epsilon^2}*\left(2^{(n+\epsilon)\alpha(\cdot)}|k_{n+\epsilon}*f^2|\right)^{1-\epsilon}\right|^{q(\cdot)/(1-\epsilon)\tau}\right\|_{\tau p(\cdot)/q(\cdot)}$$

$$\leq \left\|\left|2^{(n+\epsilon)\alpha(\cdot)}k_{n+\epsilon}*f^2\right|^{q(\cdot)/\tau}\right\|_{\tau p(\cdot)/q(\cdot)} + 2^{-(n+\epsilon)\sigma}$$

$$= \left\|\left|2^{(n+\epsilon)\alpha(\cdot)}k_{n+\epsilon}*f^2\right|^{q(\cdot)}\right\|_{p(\cdot)/q(\cdot)}^{1/\tau} + 2^{-(n+\epsilon)\sigma},$$

with an appropriate choice of $\epsilon \geq 0$ and here $0 < \sigma < \frac{1+\epsilon-n/(1-\epsilon)}{4\tau(1/q^--1/q^+)}$. Then for any $f^2 \in \mathcal{S}'(\mathbb{R}^n)$ and any $j \in \mathbb{N}_0$

$$\left\|\left|(1+\epsilon)k_j^{*,1+\epsilon}2^{j\alpha(\cdot)}f^2\right|^{q(\cdot)}\right\|_{p(\cdot)/q(\cdot)}$$

$$\leq \left(\sum_{n=j-\epsilon}^{\infty}2^{(j-n-\epsilon)(1+\epsilon)q^-/\tau}\left(\left\|\left|2^{(n+\epsilon)\alpha(\cdot)}k_{n+\epsilon}*f^2\right|^{q(\cdot)}\right\|_{p(\cdot)/q(\cdot)}^{1/\tau} + 2^{-(n+\epsilon)(1+\epsilon)}\right)\right)^{\tau}.$$

By Lemma (6.1.4) we get

$$\sum_{j=0}^{\infty}\left\|\left|(1+\epsilon)k_j^{*,1+\epsilon}2^{j\alpha(\cdot)}f^2\right|^{q(\cdot)}\right\|_{p(\cdot)/q(\cdot)} \leq (1+\epsilon)\sum_{j=0}^{\infty}\left(\left\|\left|2^{j\alpha(\cdot)}k_j*f^2\right|^{q(\cdot)}\right\|_{p(\cdot)/q(\cdot)}^{1/\tau} + 2^{-j(1+\epsilon)}\right)^{\tau}$$

$$\leq (1+\epsilon)\sum_{j=0}^{\infty}\left\|\left|2^{j\alpha(\cdot)}k_j*f^2\right|^{q(\cdot)}\right\|_{p(\cdot)/q(\cdot)} + (1+\epsilon)\sum_{j=0}^{\infty}2^{-j(1+\epsilon)\tau} \leq 1 + \epsilon.$$

Step 3. We will prove in this step that for all $f^2 \in \mathcal{S}'(\mathbb{R}^n)$ the following estimates are true:

$$\left\|f^2|B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}\right\|' \leq (1+\epsilon)\left\|f^2|B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}\right\| \leq (1+\epsilon)\left\|f^2|B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}\right\|''.$$

Let $\{\varphi_j\}_{j\in\mathbb{N}_0}$ be as in Definition (6.1.1) and let $\phi_j = \varphi_j$. The first inequality is proved by the chain of the estimates

$$\left\|f^2|B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}\right\|' \leq (1+\epsilon)\left\|\left(\varphi_j^{*,1+\epsilon}2^{j\alpha(\cdot)}f^2\right)_j\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leq (1+\epsilon)\left\|\left(2^{j\alpha(\cdot)}\phi_j*f^2\right)_j\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$$

$$\leq (1+\epsilon)\left\|f^2|B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}\right\|,$$

where the first inequality is (39), see Step 1, the second inequality is (43) (with ϕ and ϕ_0 instead of k and k_0), see Step 2, and finally the third inequality is obvious. Now the second inequality can be obtained by the following chain

$$\left\| f^2 | B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \right\| \leq (1+\epsilon) \left\| \left(\varphi_j^{*,1+\epsilon} 2^{j\alpha(\cdot)} f^2 \right)_j \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leq (1+\epsilon) \left\| f^2 | B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \right\|'$$

$$\leq (1+\epsilon) \left\| f^2 | B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \right\|'',$$

where the first inequality is obvious, the second inequality is (39), see Step 1, with the roles of k_0 and k respectively φ_0 and φ interchanged, and finally the last inequality is (43), see Step 2. Hence the Corollary is proved.

Corollary (6.1.18) [299]: Let $\alpha \in C_{loc}^{log}$ and $p, q \in \mathcal{P}_0^{log}$ with $q^+ < \infty$. Further, let $K, L \in \mathbb{N}_0$ such that

$$K > \alpha^+, \quad L > n \left(\frac{1}{\min(1, p^-)} - 1 \right) - 1 - \alpha^-. \tag{46}$$

Then $f^2 \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$, if and only if, it can be represented as

$$f^2 = \sum_{j=0}^{\infty} \sum_{n+\epsilon \in \mathbb{Z}^n} \lambda_A \rho_A, \tag{47}$$

where $A = (j+\epsilon)(n+\epsilon)$
convergence being in $\mathcal{S}'(\mathbb{R}^n)$, where ρ_A are $[K, L]$-atoms and $\lambda = \{\lambda_{(j+\epsilon)(n+\epsilon)} \in \mathbb{C} : j \in \mathbb{N}_0, \epsilon \geq 0, n+\epsilon \in \mathbb{Z}^n \}$. Furthermore, $\inf \left\| \lambda \middle| b_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \right\|$, where the infimum is taken over admissible representations (47), is an equivalent quasi-norm in $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$.

Proof: The proof follows the ideas in [45].

Step 1.   Assume that $f^2 \in B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$ and let θ_0, θ, ψ_0 and ψ be the functions introduced in Lemma (6.1.13). We have

$$f^2 = \theta_0 * \psi_0 * f^2 + \sum_{j=1}^{\infty} \theta_{j+\epsilon} * \psi_{j+\epsilon} * f^2$$

and using the definition of the cubes $\mathcal{Q}_{(j+\epsilon)(1+\epsilon)}$ we obtain

$$f^2(x^r) = \sum_{n+\epsilon \in \mathbb{Z}^n} \int_{\mathcal{Q}_{0(n+\epsilon)}} \theta_0(x^r - y^r) \psi_0 * f^2(y^r) dy^r$$

$$+ \sum_{j=1}^{\infty} 2^{(j+\epsilon)n} \sum_{n+\epsilon \in \mathbb{Z}^n} \int_{\mathcal{Q}_A} \theta\left(2^{j+\epsilon}(x^r - y^r) \right) \psi_{j+\epsilon} * f^2(y^r) dy^r,$$

with convergence in $\mathcal{S}'(\mathbb{R}^n)$. We define for every $j \in \mathbb{N}$

$$\lambda_A = (1+\epsilon)_\theta \sup_{y^r \in \mathcal{Q}_A} \left| \psi_{j+\epsilon} * f^2(y^r) \right| \tag{48}$$

where

$$(1+\epsilon)_\theta = \max \left\{ \sup_{|y^r| \leq 1} |D^\alpha \theta(y^r)| : |\alpha| \leq K \right\}.$$

Define also

$$\rho_A(x^r) = \frac{1}{\lambda_A} 2^{(j+\epsilon)n} \int_{\mathcal{Q}_A} \theta\left(2^{j+\epsilon}(x^r - y^r) \right) \psi_{j+\epsilon} * f^2(y^r) dy^r. \tag{49}$$

Similarly we define for every $n + \epsilon \in \mathbb{Z}^n$ the numbers $\lambda_{\epsilon(n+\epsilon)}$ and the functions $\rho_{\epsilon(n+\epsilon)}$ taking in (48) and (49) $j = -\epsilon$ and replacing $\psi_{j+\epsilon}$ and θ by $\psi_{-\epsilon}$ and $\theta_{-\epsilon}$, respectively. Let us now check that such $\rho_{(j+\epsilon)(n+\epsilon)}$ are atoms in the sense of Definition (6.1.11). We have

$$\left|D^\beta \rho_A(x^r)\right| \leq \frac{2^{(j+\epsilon)(n+|\beta|)}}{(1+\epsilon)_\theta} \int_{Q_A} \left|\left(D^\beta \theta\right)\left(2^{j+\epsilon}(x^r - y^r)\right)\right| \left|\psi_{j+\epsilon}\right.$$

$$* f^2(y^r)\left|dy^r \left(\sup_{y^r \in Q_A} \left|\psi_{j+\epsilon} * f^2(y^r)\right|\right)^{-1}\right.$$

$$\leq \frac{2^{(j+\epsilon)(n+|\beta|)}}{(1+\epsilon)_\theta} \int_{Q_A} \left|\left(D^\beta \theta\right)\left(2^{j+\epsilon}(x^r - y^r)\right)\right| dy^r \leq 2^{(j+\epsilon)(n+|\beta|)}|Q_A| \leq 2^{(j+\epsilon)|\beta|}.$$

The modifications for the terms with $j = -\epsilon$ are obvious.

Step 2.   Next we show that there is a constant $\epsilon \geq 0$ such that

$$\left\|\lambda \left|b_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}\right.\right\| \leq (1 + \epsilon) \left\|f^2 \left|B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}\right.\right\|.$$

Let $j \in \mathbb{N}$. Taking into account that $|x^r - y^r| \leq (1+\epsilon)2^{-(j+\epsilon)}$ for $x^r, y^r \in Q_A$ we obtain

$$2^{(j+\epsilon)(\alpha(x^r)-\alpha(y^r))} \leq 2^{\frac{(1+\epsilon)_{\log}(\alpha)(j+\epsilon)}{\log(e+1/|x^r - y^r|)}} \leq 2^{\frac{(1+\epsilon)_{\log}(\alpha)(j+\epsilon)}{\log(e+2^{j+\epsilon}/(1+\epsilon))}} \leq 1 + \epsilon$$

if $j + \epsilon \geq [\log_2(1+\epsilon)] + 2$. If $0 < j + \epsilon < [\log_2(1+\epsilon)] + 2$, then $2^{(j+\epsilon)(\alpha(x^r)-\alpha(y^r))} \leq 2^{(j+\epsilon)(\alpha^+ - \alpha^-)} \leq 1 + \epsilon$. Therefore,

$$2^{(j+\epsilon)\alpha(x^r)}\left|\psi_{j+\epsilon} * f^2(y^r)\right| \leq (1+\epsilon)2^{(j+\epsilon)\alpha(y^r)}\left|\psi_{j+\epsilon} * f^2(y^r)\right|$$

for any $x^r, y^r \in Q_A$ and any $j \in \mathbb{N}$. Hence,

$$\sum_{n+\epsilon \in \mathbb{Z}^n} \lambda_A 2^{(j+\epsilon)\alpha(x^r)} \chi_A(x^r) = (1+\epsilon)_\theta \sum_{n+\epsilon \in \mathbb{Z}^n} 2^{(j+\epsilon)\alpha(x^r)} \sup_{y^r \in Q_A} \left|\psi_{j+\epsilon} * f^2(y^r)\right| \chi_A(x^r)$$

$$\leq (1+\epsilon) \sum_{n+\epsilon \in \mathbb{Z}^n} \sup_{|z^r| \leq (1+\epsilon)2^{-(j+\epsilon)}} \frac{2^{(j+\epsilon)\alpha(x^r - z^r)}\left|\psi_{j+\epsilon} * f^2(x^r - z^r)\right|}{(1 + 2^{j+\epsilon}|z^r|)^{1+\epsilon}} (1 + 2^{j+\epsilon}|z^r|)^{1+\epsilon} \chi_A(x^r)$$

$$\leq (1+\epsilon)\psi_{j+\epsilon}^{*,1+\epsilon} 2^{(j+\epsilon)\alpha(\cdot)} f^2(x^r) \sum_{n+\epsilon \in \mathbb{Z}^n} \chi_A(x^r) = (1+\epsilon)\psi_{j+\epsilon}^{*,1+\epsilon} 2^{(j+\epsilon)\alpha(\cdot)} f^2(x^r),$$

where we have used $\sum_{n+\epsilon \in \mathbb{Z}^n} \chi_A(x^r) = 1$. This estimate and its counterpart for $j = -\epsilon$ give

$$\left\|\lambda \left|b_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}\right.\right\| \leq (1+\epsilon) \left\|\left(\psi_{j+\epsilon}^{*,1+\epsilon} 2^{(j+\epsilon)\alpha(\cdot)} f^2\right)_{j+\epsilon}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leq (1+\epsilon) \left\|f^2 \left|B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}\right.\right\|,$$

by Theorem (6.1.9) and $\psi \in \mathcal{S}(\mathbb{R}^n)$ are two kernels which fulfill Tauberian conditions (8) and (9) and the moment conditions (10).

Step 3.   Assume that f^2 can be represented by (27), with K and L satisfying (26). We will show that $f^2 \in B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$ and that for some $\epsilon \geq 0$

$$\left\|f^2 \left|B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}\right.\right\| \leq (1+\epsilon) \left\|\lambda \left|b_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}\right.\right\|.$$

By a scaling argument, we see that it suffices to consider the case $\left\|\lambda \left|b_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}\right.\right\| = 1$ and show that the modular of a constant times the function on the left-hand side is bounded. In particular, we will show that

$$\sum_{j=0}^\infty \left\|\left|(1+\epsilon)2^{j\alpha(\cdot)}\varphi_j * f^2\right|^{q(\cdot)}\right\|_{p(\cdot)/q(\cdot)}$$

$$\leq 1 + \epsilon \text{ when } \sum_{j=0}^\infty \left\|\left|\sum_{n+\epsilon \in \mathbb{Z}^n} 2^{j\alpha(\cdot)}\lambda_{j(n+\epsilon)}\chi_{j(n+\epsilon)}\right|^{q(\cdot)}\right\|_{p(\cdot)/q(\cdot)}$$

$$= 1, \tag{50}$$

where $\{\mathcal{F}\varphi_j\}_{j\in\mathbb{N}_0}$ is the resolution of unity. We write

$$f^2 = \sum_{j=0}^{\infty}\sum_{n+\epsilon\in\mathbb{Z}^n}\lambda_A\rho_A = \sum_{j=0}^{j} + \cdots + \sum_{j=\epsilon+1}^{\infty}\cdots$$

Let $0 < 1 - \epsilon < \max(1/q^+, p^-/q^+)$. We have

$$\sum_{j=0}^{\infty}\left\||(1+\epsilon)2^{j\alpha(\cdot)}\varphi_j * f^2|^{q(\cdot)}\right\|_{p(\cdot)/q(\cdot)}$$

$$\leq \sum_{j=0}^{\infty}\left(\sum_{j=0}^{\infty}\left\|\left|(1+\epsilon)\sum_{n+\epsilon\in\mathbb{Z}^n}2^{j\alpha(\cdot)}\lambda_A\varphi_j * \rho_A\right|^{(1-\epsilon)q(\cdot)}\right\|_{p(\cdot)/(1-\epsilon)q(\cdot)}\right)^{1/(1-\epsilon)}$$

$$\leq (1+\epsilon)\sum_{j=0}^{\infty}\left(\sum_{j=0}^{\infty}\left\|\left|(1+\epsilon)\sum_{n+\epsilon\in\mathbb{Z}^n}2^{j\alpha(\cdot)}\lambda_A\varphi_j * \rho_A\right|^{(1-\epsilon)q(\cdot)}\right\|_{p(\cdot)/(1-\epsilon)q(\cdot)}\right)^{1/(1-\epsilon)}$$

$$+ (1+\epsilon)\sum_{j=0}^{\infty}\left(\sum_{j=\epsilon+1}^{\infty}\left\|\left|(1+\epsilon)\sum_{n+\epsilon\in\mathbb{Z}^n}2^{j\alpha(\cdot)}\lambda_A\varphi_j * \rho_A\right|^{(1-\epsilon)q(\cdot)}\right\|_{p(\cdot)/(1-\epsilon)q(\cdot)}\right)^{1/(1-\epsilon)}$$

$$= I + II.$$

For each $k \in \mathbb{N}$ we define $\Omega_k = \{n + \epsilon \in \mathbb{Z}^n : 2^{k-1} < 2^{\min(j+\epsilon,j)}|x^r - 2^{-(j+\epsilon)}(n+\epsilon)| \leq 2^k\}$ and $\Omega_0 = \{n + \epsilon \in \mathbb{Z}^n : 2^{\min(j+\epsilon,j)}|x^r - 2^{-(j+\epsilon)}(n+\epsilon)| \leq 1\}$.

Estimate of I. From Lemma (6.1.12), we have for any ϵ sufficiently large

$$\sum_{n+\epsilon\in\mathbb{Z}^n}2^{j\alpha(x^r)}|\lambda_A||\varphi_j * \rho_A(x^r)|$$

$$\leq (1+\epsilon)2^{\epsilon(k-\alpha^+)}\sum_{n+\epsilon\in\mathbb{Z}^n}2^{(j+\epsilon)\alpha(x^r)}|\lambda_A|\left(1 + 2^{j+\epsilon}|x^r - 2^{-(j+\epsilon)}(n+\epsilon)|\right)^{1-\epsilon}.$$

We claim that there exists $\epsilon \geq 0$ such that

$$\left\|\left|(1+\epsilon)\sum_{n+\epsilon\in\mathbb{Z}^n}2^{(j+\epsilon)\alpha(\cdot)}\lambda_A\left(1 + 2^{j+\epsilon}|\cdot - 2^{-(j+\epsilon)}(n+\epsilon)|\right)^{1-\epsilon}\right|^{(1-\epsilon)q(\cdot)}\right\|_{\frac{p(\cdot)}{(1-\epsilon)q(\cdot)}}$$

$$\leq \left\|\left|(1+\epsilon)\sum_{n+\epsilon\in\mathbb{Z}^n}2^{j\alpha(\cdot)}\lambda_A\chi_A\right|^{(1-\epsilon)q(\cdot)}\right\|_{\frac{p(\cdot)}{(1-\epsilon)q(\cdot)}} + 2^{-(j+\epsilon)} = \delta \qquad (51)$$

Therefore, by Lemma (6.1.3) we obtain

$$I \leq (1+\epsilon)\sum_{j=0}^{\infty}\left(\left\|\left|\sum_{n+\epsilon\in\mathbb{Z}^n}2^{j\alpha(\cdot)}\lambda_{j(n+\epsilon)}\chi_{j(n+\epsilon)}\right|^{(1-\epsilon)q(\cdot)}\right\|_{p(\cdot)/(1-\epsilon)q(\cdot)} + 2^{-j}\right)^{1/(1-\epsilon)}$$

$$\leq (1+\epsilon)\sum_{j=0}^{\infty}\left\|\left|\sum_{n+\epsilon\in\mathbb{Z}^n}2^{j\alpha(\cdot)}\lambda_{j(n+\epsilon)}\chi_{j(n+\epsilon)}\right|^{(1-\epsilon)q(\cdot)}\right\|_{p(\cdot)/(1-\epsilon)q(\cdot)}^{1/(1-\epsilon)}$$

$$+ (1+\epsilon)\sum_{j\geq 0}2^{-j/(1-\epsilon)} = (1+\epsilon)\sum_{j=0}^{\infty}\left\|\left|\sum_{n+\epsilon\in\mathbb{Z}^n}2^{j\alpha(\cdot)}\lambda_{j(n+\epsilon)}\chi_{j(n+\epsilon)}\right|^{q(\cdot)}\right\|_{p(\cdot)/q(\cdot)} \leq 0.$$

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Let us prove (51). This claim can be reformulated as showing that

$$\left\| \delta^{-1} \left| (1+\epsilon) \sum_{n+\epsilon \in \mathbb{Z}^n} 2^{(j+\epsilon)\alpha(\cdot)} \lambda_A \left(1 + 2^{j+\epsilon} \left| \cdot -2^{-(j+\epsilon)}(n+\epsilon)\right|\right)^{1-\epsilon} \right|^{(1-\epsilon)q(\cdot)} \right\|_{p(\cdot)/(1-\epsilon)q(\cdot)} \le 1,$$

which is equivalent to

$$\left\| (1+\epsilon)\delta^{-1/(1-\epsilon)q(\cdot)} \sum_{n+\epsilon \in \mathbb{Z}^n} 2^{(j+\epsilon)\alpha(\cdot)} \lambda_A \left(1 + 2^{j+\epsilon} \left| \cdot -2^{-(j+\epsilon)}(n+\epsilon)\right|\right)^{1-\epsilon} \right\|_{p(\cdot)} \le 1.$$

We have, with $\epsilon - 1 = R + T$,

$$\sum_{n+\epsilon \in \mathbb{Z}^n} \delta^{-1/(1-\epsilon)q(x^r)} 2^{(j+\epsilon)\alpha(x^r)} |\lambda_A| \left(1 + 2^{j+\epsilon}\left|x^r - 2^{-(j+\epsilon)}(n+\epsilon)\right|\right)^{1-\epsilon}$$

$$= \sum_{k=0}^{\infty} \sum_{n+\epsilon \in \Omega_k} \delta^{-1/(1-\epsilon)q(x^r)} 2^{(j+\epsilon)\alpha(x^r)} |\lambda_A| \left(1 + 2^{j+\epsilon}\left|x^r - 2^{-(j+\epsilon)}(n+\epsilon)\right|\right)^{1-\epsilon}$$

$$\le (1+\epsilon) \sum_{k=0}^{\infty} \sum_{n+\epsilon \in \Omega_k} \delta^{-1/(1-\epsilon)q(x^r)} 2^{(j+\epsilon)\alpha(x^r)} 2^{(1-\epsilon)k} |\lambda_A|$$

$$= \sum_{k=0}^{\infty} 2^{-(T-n/(1-\epsilon))k} \sum_{n+\epsilon \in \Omega_k} \delta^{-1/(1-\epsilon)q(x^r)} 2^{(j+\epsilon)\alpha(x^r)} 2^{-(R+n/(1-\epsilon))k} |\lambda_A|$$

$$\le \sup_{k \in \mathbb{N}_0} \sum_{n+\epsilon \in \Omega_k} \delta^{-1/(1-\epsilon)q(x^r)} 2^{(j+\epsilon)\alpha(x^r)} |\lambda_A| 2^{-(R+n/(1-\epsilon))k},$$

for any T sufficiently large such that $T > n/(1-\epsilon)$. For any $0 \le \epsilon < 1$, the last expression is bounded by

$$\sup_{k \in \mathbb{N}_0} \left(\sum_{n+\epsilon \in \Omega_k} \delta^{-\frac{1-\epsilon}{(1-\epsilon)q(x^r)}} 2^{(j+\epsilon)(1-\epsilon)\alpha(x^r)} |\lambda_A|^{1-\epsilon} 2^{-(R(1-\epsilon)+n)k} \right)^{\frac{1}{(1-\epsilon)}} =$$

$$\left(\sup_{k \in \mathbb{N}_0} 2^{-Rk(1-\epsilon)+(j+\epsilon-k)n} \int_{\bigcup \mathcal{Q}_A} \left(\sum_{n+\epsilon \in \Omega_k} \delta^{-\frac{1}{(1-\epsilon)q(x^r)}} 2^{(j+\epsilon)\alpha(x^r)} |\lambda_A| \chi_A(y^r) \right)^{1-\epsilon} dy^r \right)^{\frac{1}{(1-\epsilon)}}. \quad (52)$$

Let $y^r \in \bigcup_{n+\epsilon \in \Omega_k} \mathcal{Q}_A$ then $y^r \in \mathcal{Q}_A$ for some $n + \epsilon \in \Omega_k$ and $2^{k-1} < 2^{j+\epsilon}\left|x^r - 2^{-(j+\epsilon)}(n+\epsilon)\right| \le 2^k$. From this it follows that

$$|y^r - x^r| \le \left|y^r - 2^{-(j+\epsilon)}(1+\epsilon)\right| + \left|x^r - 2^{-(j+\epsilon)}(n+\epsilon)\right| \le \sqrt{n}2^{-(j+\epsilon)} + 2^{k-(j+\epsilon)}$$

$$\le 2^{k-(j+\epsilon)+h_n}, \ h_n \in \mathbb{N},$$

which implies that y^r is located in some ball $B\left(x^r, 2^{k-(j+\epsilon)+h_n}\right)$. Therefore, (52) does not exceed

$$\left(\sup_{k \in \mathbb{N}_0} \frac{2^{-R(1+\epsilon)k}}{|B(x^r, 2^{k-(j+\epsilon)+h_n})|} \int_{B(x^r, 2^{k-(j+\epsilon)+h_n})} \left(\sum_{n+\epsilon \in \Omega_k} \delta^{-\frac{1}{(1-\epsilon)q(x^r)}} 2^{(j+\epsilon)\alpha(x^r)} |\lambda_A| \chi_A(y^r) \right)^{1-\epsilon} dy^r \right)^{\frac{1}{(1-\epsilon)}} \quad (53)$$

Since $1/q$ is log-Hölder continuous and $\delta \in \left[2^{-(j+\epsilon)}, 1 + 2^{-(j+\epsilon)}\right]$, we have

$$\delta^{1/q(x^r)-1/q(y^r)} = (2^{j+\epsilon}\delta)^{1/q(x^r)-1/q(y^r)} 2^{(1/q(x^r)-1/q(y^r))(j+\epsilon)} \le 2^{|1/q(x^r)-1/q(y^r)|(2(j+\epsilon)+1)}$$

$$\le 2^{\frac{c_{\log}(q)(2(j+\epsilon)+1)}{\log(e+1/|x^r-y^r|)}} \le 2^{\frac{c_{\log}(q)(2(j+\epsilon)+1)}{j+\epsilon-k-h_n}} \le (1+\epsilon)2^{2c_{\log}(q)k},$$

for any $k < \max(0, j + \epsilon - h_n)$ and any $y^r \in B\left(x^r, 2^{k-(j+\epsilon)+h_n}\right)$. If $k \ge \max(0, j + \epsilon - h_n)$ then since again $\delta \in \left[2^{-(j+\epsilon)}, 1 + 2^{-(j+\epsilon)}\right]$,

$$\delta^{1/q(x^r)-1/q(y^r)} \leq (1+\epsilon)2^{|1/q(x^r)-1/q(y^r)|(2(j+\epsilon)+1)} \leq (1+\epsilon)2^{2(1/q^--1/q^+)k}.$$

Also since α is log-Hölder continuous we can prove that

$$2^{(j+\epsilon)(\alpha(x^r)-\alpha(y^r))} \leq (1+\epsilon) \times \begin{cases} 2^{c_{\log}(\alpha)k} & \text{if} \quad k < \max(0, j+\epsilon-h_n), \\ 2^{(\alpha^+-\alpha^-)k} & \text{if} \quad k \geq \max(0, j+\epsilon-h_n), \end{cases}$$

where $\epsilon \geq 0$ not depending on j and k. Hence with R sufficiently large such that

$$R > \max\left(2/(1-\epsilon)\,c_{\log}(q) + c_{\log}(\alpha), \quad 2/(1-\epsilon)\left(\frac{1}{q^-} - \frac{1}{q^+}\right) + \alpha^+ - \alpha^-\right),$$

we get that (53) is bounded by

$$(1+\epsilon)\left(\mathcal{M}_{1+\epsilon}\left(\sum_{n+\epsilon\in\Omega_k} \delta^{-1/(1-\epsilon)q(\cdot)}2^{(j+\epsilon)\alpha(\cdot)}|\lambda_A|\chi_A\right)(x^r)\right)^{1/(1-\epsilon)}, \quad x^r \in \mathbb{R}^n.$$

Now taking $0 \leq \epsilon < \min(1, p^-)$ and using the fact that $\mathcal{M} : L^{\frac{p(\cdot)}{1-\epsilon}} \to L^{\frac{p(\cdot)}{1-\epsilon}}$ is bounded we obtain

$$\left\| (1+\epsilon) \sum_{n+\epsilon\in\mathbb{Z}^n} \delta^{-1/(1-\epsilon)q(\cdot)}2^{(j+\epsilon)\alpha(\cdot)}|\lambda_A|\left(1 + 2^{j+\epsilon}|\cdot - 2^{-(j+\epsilon)}(n+\epsilon)|\right)^{-L} \right\|_{p(\cdot)}$$

$$\leq (1+\epsilon)\left\| \mathcal{M}_{1+\epsilon}\left(\sum_{n+\epsilon\in\Omega_k} \delta^{-1/(1-\epsilon)q(\cdot)}2^{(j+\epsilon)\alpha(\cdot)}|\lambda_A|\chi_A\right) \right\|_{p(\cdot)/(1-\epsilon)}^{1/(1-\epsilon)}$$

$$\leq \left\| \sum_{n+\epsilon\in\mathbb{Z}^n} \delta^{-1/(1-\epsilon)q(\cdot)}2^{(j+\epsilon)\alpha(\cdot)}|\lambda_A|\chi_A \right\|_{p(\cdot)},$$

with an appropriate choice of $\epsilon \geq 0$. Now this expression is less than or equal to one if and only if

$$\left\| \left| \sum_{n+\epsilon\in\mathbb{Z}^n} \delta^{-1/(1-\epsilon)q(\cdot)}2^{(j+\epsilon)\alpha(\cdot)}\lambda_A\chi_A \right|^{(1-\epsilon)q(\cdot)} \right\|_{p(\cdot)/(1-\epsilon)q(\cdot)} \leq 1,$$

which follows immediately from the definition of δ.

Section (6.2): Variable Smoothness and Integrability

Spaces of variable integrability, also known as variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$, can be traced back to Orlicz [124, 284], and studied by Musielak [45] and Nakano [278, 279], but the modern development started with [31] of Kováčik and Rákosník as well as [8] of Cruz-Uribe and [13] of Diening. The variable Lebesgue spaces have already widely used in the study of harmonic analysis. Apart from theoretical considerations, such function spaces have interesting applications in fluid dynamics [85, 130], image processing [7], partial differential equations and variational calculus [87, 103, 111, 283, 285].

Function spaces with variable exponents attract many attentions, especially based on classical Besov and Triebel-Lizorkin spaces (see Triebel's monographes [41, 56, 136] for the history of these two spaces). When Leopold [117, 118, 119, 120] and Leopold and Schrohe [37] studied pseudo-differential operators, they introduced related Besov spaces with variable smoothness, $B_{p,q}^{s(\cdot)}(\mathbb{R}^n)$, which were further generalized to the case that $q \neq p$, including $B_{p,q}^{s(\cdot)}(\mathbb{R}^n)$ and $F_{p,q}^{s(\cdot)}(\mathbb{R}^n)$, by Besov [90, 91, 92]. Along a different line of study, Xu [137, 138] studied Besov spaces $B_{p(\cdot),q}^{s}(\mathbb{R}^n)$ and Triebel-Lizorkin spaces

$F^s_{p(\cdot),q}(\mathbb{R}^n)$ with variable exponent $p(\cdot)$ but fixed q and s. As was well known from the trace theorem (see [22]) and Sobolev-type embeddings (see [60]) of classical function spaces, the smoothness and the integrability often interact each other. However, the unification of both trace theorems and Sobolev-type embeddings does not hold true on function spaces with only one variable index; for example, the trace space of Sobolev space $W^{k,p(\cdot)}$ is no longer a space of the same type (see [15]). Thus, function spaces with full ranges of variable smoothness and variable integrability are needed.

The concept of function spaces with variable smoothness and variable integrability was firstly mixed up by Diening, Hästö and Roudenko in [16]; they introduced Triebel-Lizorkin spaces with variable exponents $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and proved a trace theorem as follows:

$$\operatorname{Tr} F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = F^{s(\cdot,0)-1/p(\cdot,0)}_{p(\cdot,0),p(\cdot,0)}(\mathbb{R}^{n-1}),$$

(see [16]), which shows that these spaces behaved nicely with respect to the trace operator. Subsequently, Vybíral [65] established Sobolev-Jawerth embeddings of these spaces. On the other hand, Almeida and Hästö [3] introduced the Besov space with variable smoothness and integrability $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, which makes a further step in completing the unification process of function spaces with variable smoothness and integrability. Later, Drihem [17] established the atomic characterization of $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and Noi et al. [280, 281, 282] also studied the space $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ including the boundedness of trace and extension operators, duality and complex interpolation. Here we point out that vector-valued convolution inequalities developed in [3] and [16] supply well remedy for the absence of the Fefferman-Stein vector-valued inequality for the mixed Lebesgue sequence spaces $\ell^{q(\cdot)}\left(L^{p(\cdot)}(\mathbb{R}^n)\right)$ and $L^{p(\cdot)}\left(\ell^{q(\cdot)}(\mathbb{R}^n)\right)$, respectively, in studying Besov spaces and Triebel-Lizorkin spaces with variable smoothness and integrability.

More generally, 2-microlocal Besov and Triebel-Lizorkin spaces with variable, $B^{\omega(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $F^{\omega(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, were introduced by Kempka [150, 264] and provided a unified approach that cover the classical Besov and Triebel-Lizorkin spaces as well as versions of variable smoothness and integrability. Afterwards, Kempka and Vybíral [29] characterized these spaces by local means and ball means of differences. The trace spaces of 2-microlocal type spaces were studied very recently by Moura et al. [44] and Gonçalves et al. [24].

Besov-type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ and Triebel-Lizorkin spaces $F^{s,\tau}_{p,q}(\mathbb{R}^n)$ and their homogeneous counterparts for all admissible parameters were introduced in [189, 190, 221] in order to clarify the relations among Besov spaces, Triebel-Lizorkin spaces and Q space (see [12, 19]). Various properties and equivalent characterizations of Besov-type and Triebel-Lizorkin-type spaces, including smoothness atomic, molecular or wavelet decompositions, characterizations, respectively, via differences, oscillations, Peetre maximal functions, Lusin area functions or g^*_λ functions, have already been established in [222, 261, 271, 289, 290, 291, 292, 295]. Moreover, these function spaces, including some of their special cases related to Q spaces, have been used to study the existence and the regularity of solutions of some partial differential equations such as (fractional) Navier-Stokes equations. Based on $F^{s,\tau}_{p,q}(\mathbb{R}^n)$, we introduced the Triebel-Lizorkin-type space with variable exponent $F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)p(\cdot)$ in [76] with a measurable function ϕ on $\mathbb{R}^{n+1}_+$ and obtained a related trace theorem (see [294]).

We based on Besov-type spaces $B_{p,q}^{s,\tau}(\mathbb{R}^n)$ and variable Besov spaces $B_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$, we are aimed to introduce another more generalized scale of function spaces with variable smoothness $s(\cdot)$, variable integrability $p(\cdot)$ and $q(\cdot)$, and a measurable function ϕ on $\mathbb{R}_+^{n+1}$, denoted by $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$, which covers both Besov spaces with variable smoothness and integrability and Besov-type spaces. We then establish their φ-transform characterization in the sense of Frazier and Jawerth. We also characterize these spaces by smooth atoms or Peetre maximal functions and we give some basic properties and Sobolev-type embeddings. We show a trace theorem of $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$ and obtain several equivalent norms of these spaces.

We give some conventions and notation such as semimodular spaces, variable and mixed Lebesgue-sequence spaces, and also introduce variable Besov-type spaces $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$. We point out that the function spaces studied fit into the framework of so-called semimodular spaces. We point out that, in general, the scale of Besov-type spaces with variable smoothness and integrability and the scale of Musielak-Orlicz Besov-type spaces in [75] do not cover each other.

We devote to the φ-transform characterization of $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$ in the sense of Frazier and Jawerth [22], which is then applied to show that $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$ is well defined. This is different from [3], in which the space $B_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$ was proved to be well defined via the Calderón reproducing formula. We point out that the method used is originally from Frazier and Jawerth [22], which is smartly modified, via a subtle decomposition of dyadic cubes, so that it is suitable to the present setting. Observe that the r-trick lemma from [16] plays a key role in establishing a convolutional estimate so that we can use the convolutional inequality from [3] to obtain the desired conclusion.

By making full use of the r-trick lemma from [16] again, we mainly give out the Sobolev-type embedding property of $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$. Some other basic embeddings and properties of the spaces $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$ are also presented.

We characterize the space $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$ via Peetre maximal functions. A key step to obtain this is to establish a technical lemma, which indicates that the Peetre maximal function can be controlled, via semimodulars, by the approximation to the identity in a suitable way. We further obtain two equivalent characterizations of $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$. By applying a Hardy-type inequality from [18] and the Sobolev-type embedding theorem obtained, together with some ideas from the proof of Lemma (6.2.23), we establish the smooth atomic characterization of $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$.

The symbols $A \lesssim B$ means $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$. For all $a, b \in \mathbb{R}$, let $a \vee b := \max\{a, b\}$. For all $k := (k_1, \cdots, k_n) \in \mathbb{Z}^n$, let $|k| := |k_1| + \cdots + |k_n|$. Let $\mathbb{Z}_+ := \{0, 1, \cdots\}$, $\mathbb{N} := \{1, 2, \cdots\}$ and $\mathbb{K} := \mathbb{R}$ or $\mathbb{C}$. Let $\mathbb{R}_+^{n+1} := \mathbb{R}^n \times [0, \infty)$. If E is a subset of $\mathbb{R}^n$, we denote by χ_E its characteristic function and $\tilde{\chi}_E := |E|^{-1/2}\chi_E$. For all $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, denote by $Q(x, r)$ the cube centered at x with side length r, whose sides parallel axes of coordinate. For all cube $Q \subset \mathbb{R}^n$, we denote its center by c_Q and its side length by $\ell(Q)$ and, for $a \in (0, \infty)$, we denote by aQ the cube concentric with Q having the side length with $a\ell(Q)$.

For an exposition of these concepts, see [15]. The function spaces studied fit into the framework of so-called semimodular spaces. In what follows, let X be a vector space over $\mathbb{K}$.

Definition (6.2.1) [298]: A function $\varrho : X \to [0, \infty]$ is called a semimodular on X if it satisfies:

(i) $\varrho(0) = 0$ and, for all $f \in X$ and $\lambda \in \mathbb{K}$ with $|\lambda| = 1$, $\varrho(\lambda f) = \rho(f)$;

(ii) If $\varrho(\lambda, f) = 0$ for all $\lambda \in (0, \infty)$, then $f = 0$;

(iii) ρ is quasiconvex, namely, there exists $A \in [1, \infty)$ such that, for all $f, g \in X$,
$$\varrho(\theta f + (1 - \theta)g) \le A[\theta\varrho(f) + (1 - \theta)\varrho(g)];$$

(iv) $\lambda \mapsto \varrho(\lambda f)$ is left continuous on $[0, \infty)$ for every $f \in X$, namely, $\lim_{\lambda < 1, \lambda \to 1} \varrho(\lambda f) = \varrho(f)$.

A semimodular ϱ is called a modular if it satisfies that $\varrho(f) = 0$ implies $f = 0$, and is called continuous if, for every $f \in X$, the mapping $\lambda \mapsto \varrho(\lambda f)$ is continuous on $[0, \infty)$, namely, $\lim_{\lambda \to 1} \varrho(\lambda f) = \varrho(f)$.

Definition (6.2.2) [298]: Let ϱ be a (semi)modular on X. Then
$$X_\varrho := \{f \in X : \exists \lambda \in (0, \infty) \text{ such that } \varrho(\lambda f) < \infty\}$$
is called a (semi)modular space with the norm
$$\|f\|_\varrho := \inf\{\lambda \in (0, \infty) : \varrho(f/\lambda) \le 1\}.$$

Lemma (6.2.3) [298]: Let ϱ be a semimodular on X. Then $\|f\|_\varrho \le 1$ if and only if $\varrho(f) \le 1$; moreover, if ϱ is continuous, then $\|f\|_\varrho < 1$ if and only if $\varrho(f) < 1$, as well as $\|f\|_\varrho = 1$ if and only if $\varrho(f) = 1$.

We recall some definitions and notation for the space with variable integrability. For a measurable function $p(\cdot) : \mathbb{R}^n \to (0, \infty]$, let
$$p_- := \operatorname*{ees\,inf}_{x \in \mathbb{R}^n} p(x) \quad \text{and} \quad p_+ := \operatorname*{ees\,sup}_{x \in \mathbb{R}^n} p(x).$$

The set of variable exponents, denoted by $\mathcal{P}(\mathbb{R}^n)$, is the set of all measurable functions $p(\cdot) : \mathbb{R}^n \to (0, \infty]$ satisfying $p_- \in (0, \infty]$. For $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, define the function $\rho_{p(x)}$ by setting, for all $t \in [0, \infty)$,
$$\rho_{p(x)}(t) := \begin{cases} t^{p(x)}, & \text{if } p(x) \in (0, \infty), \\ 0, & \text{if } p(x) = \infty \text{ and } t \in [0, 1], \\ \infty, & \text{if } p(x) = \infty \text{ and } t \in (1, \infty). \end{cases}$$

The variable exponent modular of a measurable function f on $\mathbb{R}^n$ is defined by
$$\rho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} \rho_{p(x)}(|f(x)|)dx.$$

Definition (6.2.4) [298]: Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and E be a measurable subset of $\mathbb{R}^n$. Then the variable exponent Lebesgue space $L^{p(\cdot)}(E)$ is defined to be the set of all measurable functions f such that
$$\|f\|_{L^{p(\cdot)}(E)} := \inf\{\lambda \in (0, \infty) : \varrho_{p(\cdot)}(f\chi_E/\lambda) \le 1\} < \infty.$$

Definition (6.2.5) [298]: Let $p, q \in \mathcal{P}(\mathbb{R}^n)$ and E be a measurable subset of $\mathbb{R}^n$. Then the mixed Lebesgue-sequence space $\ell^{q(\cdot)}\left(L^{p(\cdot)}(E)\right)$ is defined to be the set of all sequences $\{f_v\}_{v \in \mathbb{N}}$ of functions in $L^{p(\cdot)}(E)$ such that
$$\|\{f_v\}_{v \in \mathbb{N}}\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}(E)\right)} := \inf\left\{\lambda \in (0, \infty) : \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}(\{f_v\chi_E/\lambda\}_{v \in \mathbb{N}}) \le 1\right\} < \infty,$$
where, for all sequences $\{g_v\}_{v \in \mathbb{N}}$ of measurable functions,
$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}(\{g_v\}_{v \in \mathbb{N}}) := \sum_{v \in \mathbb{N}} \inf\left\{\mu_v \in (0, \infty) : \varrho_{p(\cdot)}\left(\{g_v/\mu_v^{1/q(\cdot)}\}_{v \in \mathbb{N}}\right) \le 1\right\} \tag{54}$$

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with the convention $\lambda^{1/\infty} = 1$ for all $\lambda \in (0, \infty)$.

A measurable function $g \in \mathcal{P}(\mathbb{R}^n)$ is said to satisfy the locally log-Hölder continuous condition, denoted by $g \in C_{\mathrm{loc}}^{\log}(\mathbb{R}^n)$, if there exists a positive constant $C_{\log}(g)$ such that, for all $x, y \in \mathbb{R}^n$,

$$|g(x) - g(y)| \leq \frac{C_{\log}(g)}{\log(e + 1/|x - y|)}, \tag{55}$$

and g is said to satisfy the globally log-Hölder continuous condition, denoted by $g \in C^{\log}(\mathbb{R}^n)$, if $g \in C_{\mathrm{loc}}^{\log}(\mathbb{R}^n)$ and there exist positive constants C_∞ and g_∞ such that, for all $x \in \mathbb{R}^n$,

$$|g(x) - g_\infty| \leq \frac{C_\infty}{\log(e + |x|)}. \tag{56}$$

Now let $\mathcal{G}(\mathbb{R}_+^{n+1})$ be the set of all measurable functions $\phi : \mathbb{R}_+^{n+1} \to (0, \infty)$ having the following properties: there exist positive constants c_1 and c_2 such that, for all $x \in \mathbb{R}^n$ and $r \in (0, \infty)$,

$$c_1^{-1}\phi(x, 2r) \leq \phi(x, r) \leq c_1\phi(x, 2r) \tag{57}$$

and, for all $x, y \in \mathbb{R}^n$ and $r \in (0, \infty)$ with $|x - y| \leq r$,

$$c_1^{-1}\phi(y, r) \leq \phi(x, r) \leq c_2\phi(y, 2r) \tag{58}$$

In what follows, for $\phi \in \mathcal{G}(\mathbb{R}_+^{n+1})$ and all cubes $Q := Q(x, r) \subset \mathbb{R}^n$ with center $x \in \mathbb{R}^n$ and radius $r \in (0, \infty)$, define $\phi(Q) := \phi(Q(x, r)) := \phi(x, 2r)$. Let $\mathcal{S}(\mathbb{R}^n)$ be the space of all Schwartz functions on $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ its topological dual space. A pair of functions, (φ, Φ), is said to be admissible if $\varphi, \Phi \in \mathcal{S}(\mathbb{R}^n)$ satisfy

$$\mathrm{supp}\,\hat{\varphi} \subset \{\xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2\} \text{ and } |\hat{\varphi}(\xi)| \geq c > 0 \text{ when } 3/5 \leq |\xi| \leq 5/3 \tag{59}$$

and

$$\mathrm{supp}\,\hat{\Phi} \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 2\} \text{ and } |\hat{\Phi}(\xi)| \geq c > 0 \text{ when } |\xi| \leq 5/3, \tag{60}$$

where $\hat{f}(\xi) := \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx$ for all $\xi \in \mathbb{R}^n$ and c is a positive constant independent of $\xi \in \mathbb{R}^n$. For all $j \in \mathbb{N}$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we put $\varphi_j(x) := 2^{jn}\varphi(2^j x)$ and $\tilde{\varphi}(x) := \overline{\varphi(-x)}$. For $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, denote by Q_{jk} the dyadic cube $2^{-j}([0,1)^n + k)$, $x_{Q_{jk}} := 2^{-j}k$ its lower left corner and $\ell(Q_{jk})$ its side length. Let $\mathcal{Q} := \{Q_{jk} : j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$, $\mathcal{Q}^* := \{Q \in \mathcal{Q} : \ell(Q) \leq 1\}$ and $j_Q := -\log_2 \ell(Q)$ for all $Q \in \mathcal{Q}$.

Definition (6.2.6) [298]: Let (φ, Φ) be a pair of admissible functions on $\mathbb{R}^n$. Let $p, q \in C^{\log}(\mathbb{R}^n)$, $s \in C_{\mathrm{loc}}^{\log}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $\phi \in \mathcal{G}(\mathbb{R}_+^{n+1})$. Then the Besov-type space with variable smoothness and integrability, $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$, is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)} := \sup_{P \in \mathcal{Q}} \frac{1}{\phi(P)} \left\| \{2^{js(\cdot)}|\varphi_j * f|\}_{j \geq (j_P \vee 0)} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} < \infty,$$

where the supremum is taken over all dyadic cubes P in $\mathbb{R}^n$.

By comparing Besov-type spaces with variable smoothness and integrability with Musielak-Orlicz Besov-type spaces in [75] we show that, in general, these two scales of Besov-type spaces do not cover each other.

To recall the definition of Musielak-Orlicz Besov-type spaces, we need some notions on Musielak-Orlicz functions. A function $\varphi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ is called a Musielak-Orlicz function if the function $\varphi(x, \cdot) : [0, \infty) \to [0, \infty)$ is an Orlicz function for all $x \in \mathbb{R}^n$, namely, for any given $x \in \mathbb{R}^n$, $\varphi(x, \cdot)$ is nondecreasing, $\varphi(x, 0) = 0$, $\varphi(x, t) \in (0, \infty)$ for all $t \in (0, \infty)$ and $\lim_{t \to \infty} \varphi(x, t) = \infty$, and $\varphi(\cdot, t)$ is a Lebesgue measurable function for all $t \in [0, \infty)$. A Musielak-Orlicz function φ is said to be of

uniformly upper (resp. lower) type p for some $p \in [0, \infty)$ if there exist a positive constant C such that, for all $x \in \mathbb{R}^n$, $t \in [0, \infty)$ and $s \in [1, \infty)$ (resp. $s \in [0,1]$), $\varphi(x, st) \leq Cs^p \varphi(x, t)$ (see [32]). Let

$$i(\varphi) := \sup\{p \in (0, \infty) : \varphi \text{ is uniformly lower type } p\}$$

and

$$I(\varphi) := \inf\{p \in (0, \infty) : \varphi \text{ is uniformly upper type } p\}.$$

The function $\varphi(\cdot, t)$ is said to satisfy the uniformly Muckenhoupt condition for some $r \in [1, \infty)$, denoted by $\varphi \in A_r(\mathbb{R}^n)$, if, when $r \in (1, \infty)$,

$$\sup_{t \in (0,\infty)} \sup_{\text{balls } B \subset \mathbb{R}^n} \frac{1}{|B|^r} \int_B \varphi(x, t) dx \left\{ \int_B [\varphi(y, t)]^{-r'/r} dy \right\}^{r'/r} < \infty,$$

where $1/r + 1/r' = 1$, or, when $r = 1$,

$$\sup_{t \in (0,\infty)} \sup_{\text{balls } B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B \varphi(x, t) dx \left\{ \operatorname*{ess\,sup}_{y \in B} [\varphi(y, t)]^{-1} \right\} < \infty.$$

Let $A_\infty(\mathbb{R}^n) := \bigcup_{r \in [1,\infty)} A_r(\mathbb{R}^n)$.

The Musielak-Orlicz space $L^\varphi(\mathbb{R}^n)$ is defined as the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^\varphi(\mathbb{R}^n)} := \inf\left\{ \lambda \in (0, \infty) : \int_{\mathbb{R}^n} \varphi(x, |f(x)|/\lambda) dx \leq 1 \right\} < \infty.$$

Let $\mathcal{S}_\infty(\mathbb{R}^n)$ be the space of all Schwartz functions h satisfying that, for all multi-indices $\gamma := (\gamma_1, \cdots, \gamma_n) \in \mathbb{Z}_+^n$, $\int_{\mathbb{R}^n} h(x) x^\gamma dx = 0$ and let $\mathcal{S}'_\infty(\mathbb{R}^n)$ be its topological dual space.

Definition (6.2.7) [298]: Let $s \in \mathbb{R}$, $\tau \in [0, \infty)$, $q \in (0, \infty]$ and ψ be a Schwartz function satisfying $\operatorname{supp} \widehat{\varphi} \subset \{\xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2\}$ and $|\widehat{\varphi}(\xi)| \geq C > 0$ if $3/5 \leq |\xi| \leq 5/3$ for some positive constant C independent of $\xi \in \mathbb{R}^n$. For all $j \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, let $\psi_j(x) := 2^{jn}\psi(2^j x)$. Assume that, for $j \in \{1, 2\}$, φ_j is a Musielak-Orlicz function with $0 < i(\varphi_j) \leq I(\varphi_j) < \infty$ and $\varphi_j \in A_\infty(\mathbb{R}^n)$. Then the Musielak-Orlicz Besov-type space $\dot{B}^{s,\tau}_{\varphi_1,\varphi_2,q}(\mathbb{R}^n)$ is defined to be the space of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{B}^{s,\tau}_{\varphi_1,\varphi_2,q}(\mathbb{R}^n)} := \sup_{P \in \mathcal{Q}} \frac{1}{\|\chi_P\|_{L^{\varphi_1}(\mathbb{R}^n)}} \left\| \left\{ \sum_{j=j_P}^{\infty} (2^{js}|\psi_j * f|)^q \right\}^{1/q} \right\|_{L^{\varphi_2}(\mathbb{R}^n)} < \infty$$

with suitable modification made when $q = \infty$, where the supremum is taken over all dyadic cubes P of $\mathbb{R}^n$.

The purpose is to show that $B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is independent of the choice of admissible function pairs (φ, Φ). To this end, we first introduce the sequence space $b^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ with respect to $B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and then establish its φ-transform characterization in the sense of Frazier and Jawerth [22].

Definition (6.2.8) [98]: Let p, q, s and ϕ be as in Definition (6.2.6). Then the sequence space $b^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is defined to be the set of all sequences $t := \{t_Q\}_{Q \in \mathcal{Q}^*} \subset \mathbb{C}$ such that

$$\|t\|_{b^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} := \sup_{P \in \mathcal{Q}} \frac{1}{\phi(P)} \left\| \left\{ \sum_{\substack{Q \in \mathcal{Q}^*, Q \subset P \\ \ell(Q)=2^{-j}}} |Q|^{-\frac{s(\cdot)}{n}} |t_Q| \widetilde{\chi}_Q \right\}_{j \geq (j_P \vee 0)} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} < \infty,$$

where the supremum is taken over all dyadic cubes P in $\mathbb{R}^n$.

Let (φ, Φ) be a pair of admissible functions. Then $(\widetilde{\varphi}, \widetilde{\Phi})$ is also a pair of admissible functions, where $\widetilde{\varphi}(\cdot) := \overline{\varphi(-\cdot)}$ and $\widetilde{\Phi}(\cdot) := \overline{\Phi(-\cdot)}$. Moreover, by [22] or [23], there exist Schwartz functions ψ and Ψ satisfying (59) and (60), respectively, such that, for all $\xi \in \mathbb{R}^n$,

$$\widehat{\Phi}(\xi)\widehat{\Psi}(\xi) + \sum_{j=1}^{\infty} \widehat{\varphi}(2^{-j}\xi)\widehat{\psi}(2^{-j}\xi) = 1. \tag{61}$$

Recall that the φ-transform S_φ is defined to be the mapping taking each $f \in \mathcal{S}'(\mathbb{R}^n)$ to the sequence $S_\varphi(f) := \{(S_\varphi f)_Q\}_{Q \in \mathfrak{Q}^*}$, where $(S_\varphi f)_Q := |Q|^{1/2}\varphi_{j_Q} * f(x_Q)$ with φ_0 replaced by Φ; the inverse φ-transform T_ψ is defined to be the mapping taking a sequence $t := \{t_Q\}_{Q \in \mathfrak{Q}^*} \subset \mathbb{C}$ to

$$T_\psi t := \sum_{Q \in \mathfrak{Q}^*, \ell(Q)=1} t_Q \Psi_Q + \sum_{Q \in \mathfrak{Q}^*, \ell(Q)<1} t_Q \psi_Q; \tag{62}$$

Corollary (6.2.9) [298]: With all notations as in Definition (6.2.6), the space $B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is independent of the choice of the admissible function pairs (φ, Φ).

Lemma (6.2.10) [298]: Let $\phi \in \mathcal{G}(\mathbb{R}^{n+1}_+)$. Then there exist positive constants C and $\widetilde{C}$ such that, for all $j \in \mathbb{Z}_+$ and $k \in \mathbb{Z}^n$, $\phi(Q_{jk}) \leq C2^{j \log_2 c_1}(|k|+1)^{2\log_2 c_1}$ and, for all $Q \in \mathfrak{Q}$ and $l \in \mathbb{Z}^n$,

$$\frac{\phi(Q + l\ell(Q))}{\phi(Q)} \leq \widetilde{C}(1 + |l|)^{2\log_2 c_1},$$

where c_1 is as in (57).

Lemma (6.2.11) [298]: Let $p \in C^{\log}(\mathbb{R}^n)$. Then there exists a positive constant C such that, for all dyadic cubes Q_{jk} with $j \in \mathbb{Z}_+$ and $k \in \mathbb{Z}^n$,

$$C^{-1}2^{-(n/p_-)j}(1+|k|)^{n(1/p_+ - 1/p_-)} \leq \left\|\chi_{Q_{jk}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C2^{-(n/p_+)j}(1+|k|)^{n(1/p_- - 1/p_+)}.$$

In what follows, for all $h \in \mathcal{S}(\mathbb{R}^n)$ and $M \in \mathbb{Z}_+$, let

$$\|h\|_{\mathcal{S}_M(\mathbb{R}^n)} := \sup_{|\gamma| \leq M} \sup_{x \in \mathbb{R}^n} |\partial^\gamma h(x)|(1+|x|)^{n+M+\gamma}.$$

Lemma (6.2.12) [298]: Let p, q, s and ϕ be as in Definition (6.2.6). Then, for all $t \in b^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, $T_\psi t$ in (56) converges in $\mathcal{S}'(\mathbb{R}^n)$; moreover, $T_\psi : b^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is continuous.

Proof: Observe that, for any $Q \in \mathfrak{Q}^*$,

$$|t_Q| \leq \left\||Q|^{-s(\cdot)/n}|t_Q|\widetilde{\chi}_Q\right\|_{L^{p(\cdot)}(Q)}\left\|\chi_Q\right\|_{L^{p(\cdot)}(Q)}^{-1}|Q|^{s_-/n+1/2}$$

$$\leq \left\|\left\{\sum_{\substack{\widetilde{Q}\subset Q, \widetilde{Q}\in\mathfrak{Q}^* \\ \ell(\widetilde{Q})=2^{-j}}} |\widetilde{Q}|^{-s(\cdot)/n}|t_{\widetilde{Q}}|\widetilde{\chi}_{\widetilde{Q}}\right\}_{j\geq(j_Q \vee 0)}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}(Q)\right)} \frac{|Q|^{s_-/n+1/2}}{\left\|\chi_Q\right\|_{L^{p(\cdot)}(Q)}}$$

$$\leq \|t\|_{b^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \frac{\phi(Q)}{\left\|\chi_Q\right\|_{L^{p(\cdot)}(Q)}}|Q|^{s_-/n+1/2}.$$

Then, by this and an argument similar to that used in [294], we conclude that, for all $h \in \mathcal{S}(\mathbb{R}^n)$, $\left|\langle T_\psi t, h\rangle\right| \lesssim \|t\|_{b^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \|h\|_{\mathcal{S}_M(\mathbb{R}^n)}$ with some large $M \in (0,\infty)$ which completes the proof of Lemma (6.2.12).

Lemma (6.2.13) [298]: Let $r \in (0,\infty)$, $v \in \mathbb{Z}_+$ and $m \in (n,\infty)$. Then there exists a positive constant C, only depending on r, m and n, such that, for all $x \in \mathbb{R}^n$ and $g \in \mathcal{S}'(\mathbb{R}^n)$ with $\operatorname{supp}\hat{g} \subset \{\xi : |\xi| \leq 2^{v+1}\}$, $\sup_{z \in Q}|g(z)| \leq C\left[\eta_{v,m} * (|g|^r)(x)\right]^{1/r}$, where $Q \in \mathfrak{Q}$ contains x and $\ell(Q) = 2^{-v}$.

Lemma (6.2.14) [298]: Let $s \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$ and $d \in [C_{\log}(s),\infty)$, where $C_{\log}(s)$ denotes the constant as in (55) with g replaced by s. Then, for all $x,y \in \mathbb{R}^n$ and $v \in \mathbb{N}$, $2^{vs(x)}\eta_{v,m+d}(x-y) \leq C2^{vs(y)}\eta_{v,m}(x-y)$ with C being a positive constant independent of x, y and v; moreover, for all nonnegative measurable functions f, it holds true that
$$2^{vs(x)}\eta_{v,m+d} * f(x) \leq C\eta_{v,m} * \left(2^{vs(\cdot)}f\right)(x), \qquad x \in \mathbb{R}^n.$$

Lemma (6.2.15) [298]: Let $p, q \in C^{\log}(\mathbb{R}^n)$ satisfy $p_-, q_- \in [1,\infty]$ and $m \in \left(n + C_{\log}(1/q),\infty\right)$, where $C_{\log}(1/q)$ denotes the constant as in (55) with g replaced by $1/q$. Then there exists a positive constant C such that, for all sequences $\{f_v\}_{v \in \mathbb{N}}$ of measurable functions,
$$\left\|\{\eta_{v,m} * f_v\}_{v \in \mathbb{N}}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}(\mathbb{R}^n)\right)} \leq C\|\{f_v\}_{v \in \mathbb{N}}\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}(\mathbb{R}^n)\right)}.$$

Lemma (6.2.16) [298]: Let $p, q \in \mathcal{P}(\mathbb{R}^n)$, $q_- \in (1,\infty)$ and f be a measurable function on $\mathbb{R}^n$.

(i) If $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq 1$, then $\left\||f|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\mathbb{R}^n)} \leq \|f\|^{q_-}_{L^{p(\cdot)}(\mathbb{R}^n)}$.

(ii) If $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} > 1$, then $\left\||f|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\mathbb{R}^n)} \leq \|f\|^{q_+}_{L^{p(\cdot)}(\mathbb{R}^n)}$.

(iii)   If $\left\||f|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\mathbb{R}^n)} \geq 1$, then $\|f\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\mathbb{R}^n)} \leq \left\||f|^{q(\cdot)}\right\|^{1/q_-}_{L^{\frac{p(\cdot)}{q(\cdot)}}(\mathbb{R}^n)}$.

(iv)   If $\left\||f|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\mathbb{R}^n)} \geq 1$, then $\|f\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\mathbb{R}^n)} \leq \left\||f|^{q(\cdot)}\right\|^{1/q_+}_{L^{\frac{p(\cdot)}{q(\cdot)}}(\mathbb{R}^n)}$.

Proof: By similarity, we only prove (i) and (iii). Let $f \in L^{p(\cdot)}(\mathbb{R}^n)$. Then, by [298] and the fact that $\left\|f/\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} = 1$, we see that $\varrho_{p(\cdot)}\left(f/\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right) \leq 1$. Thus, if $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq 1$, then
$$\varrho_{p(\cdot)}\left(f/\left(\|f\|^{q_-}_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/q(\cdot)}\right) \leq \varrho_{p(\cdot)}\left(f/\left(\|f\|^{q_-}_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/q_-}\right) = \varrho_{p(\cdot)}\left(f/\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right) \leq 1,$$
which implies that $\left\||f|^{q(\cdot)}\right\|_{L^{p(\cdot)/q(\cdot)}(\mathbb{R}^n)} \leq \|f\|^{q_-}_{L^{p(\cdot)}(\mathbb{R}^n)}$ and then completes the proof of (i).
For (iii), if $\left\||f|^{q(\cdot)}\right\|_{L^{p(\cdot)/q(\cdot)}(\mathbb{R}^n)} \geq 1$, then for all $\lambda > \left\||f|^{q(\cdot)}\right\|_{L^{p(\cdot)/q(\cdot)}(\mathbb{R}^n)}$,
$$\varrho_{p(\cdot)}\left(f/\lambda^{1/q_-}\right) \leq \varrho_{p(\cdot)}\left(f/\lambda^{1/q(\cdot)}\right) \leq 1,$$
which implies that $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq \lambda^{\frac{1}{q_-}}$. By this and the arbitrariness of $\lambda > \left\||f|^{q(\cdot)}\right\|_{L^{p(\cdot)/q(\cdot)}(\mathbb{R}^n)}$, we conclude that (iii) holds true, which completes the proof of Lemma (6.2.16).

For a sequence $t = \{t_Q\}_{Q \in \mathfrak{Q}^*} \subset \mathbb{C}$, $r \in (0,\infty)$ and $\lambda \in (0,\infty)$, let $t^*_{r,\lambda} := \left\{(t^*_{r,\lambda})_Q\right\}_{Q \in \mathfrak{Q}^*}$, where, for all $Q \in \mathfrak{Q}^*$,
$$(t^*_{r,\lambda})_Q := \left\{\sum_{R \in \mathfrak{Q}^*, \ell(R)=\ell(Q)} \frac{|t_R|^r}{[1 + \{\ell(R)\}^{-1}|x_R - x_Q|]^\lambda}\right\}^{1/r}.$$

Lemma (6.2.17) [298]: Let p, q, s and ϕ be as in Definition (6.2.6), $r \in (0, \min\{p_-, q_-\})$ and $\lambda \in \left(2n + C_{\log}(s) + 2r \log_2 c_1, \infty\right)$, where $C_{\log}(s)$ denotes the constant as in (55) with g replaced by s, and c_1 is as in (57). Then there exists a constant $C \in [1, \infty)$ such that, for all $t \in b^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$,

$$\|t\|_{b^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \leq \|t^*_{r,\lambda}\|_{b^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \leq C\|t\|_{b^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}. \tag{63}$$

Proof: To prove this lemma, it suffices to show the second inequality of (63) since the first one holds true obviously. We first claim that, for all $t \in b^{s(\cdot),1}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, $\|t^*_{r,\lambda}\|_{b^{s(\cdot),1}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \lesssim \|t\|_{b^{s(\cdot),1}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}$. Indeed, observe that, for all $r \in (0, \min\{p_-, q_-\})$, $Q \in \mathfrak{Q}^*$ and $x \in Q$,

$$\left(t^*_{r,\lambda}\right)_Q \sim \left\{\eta_{j_Q,\lambda} * \left(\sum_{R \in \mathfrak{Q}^*, \ell(R)=2^{-j_Q}} |t_R|^r \chi_R\right)(x)\right\}^{1/r}.$$

Thus, by Lemma (6.2.14) and Lemma (6.2.15), we see that

$$\|t^*_{r,\lambda}\|_{b^{s(\cdot),1}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \lesssim \left\|\left\{\eta_{j,\lambda-C_{\log}(s)} * \left(\left[2^{js(\cdot)} \sum_{R \in \mathfrak{Q}^*, \ell(R)=2^{-j_Q}} |t_R|^r \tilde{\chi}_R\right]^r\right)\right\}_{j \in \mathbb{Z}_+}\right\|^{1/r}_{\ell^{q(\cdot)/r}\left(L^{p(\cdot)/r}(\mathbb{R}^n)\right)}$$

$$\lesssim \left\|\left\{2^{js(\cdot)} \sum_{R \in \mathfrak{Q}^*, \ell(R)=2^{-j_Q}} |t_R| \tilde{\chi}_R\right\}_{j \in \mathbb{Z}_+}\right\|_{\ell^{q(\cdot)/r}\left(L^{p(\cdot)/r}(\mathbb{R}^n)\right)} \sim \|t\|_{b^{s(\cdot),1}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)},$$

which proves the above claim.

For all $P \in \mathfrak{Q}$ and $Q \in \mathfrak{Q}^*$, let $v^P_Q := t_Q$ if $Q \subset 4P$ and $v^P_Q := 0$ otherwise, and let $u^P_Q := t_Q - v^P_Q$. Let $v^P := \{u^P_Q\}_{Q \in \mathfrak{Q}^*}$. Then we have

$\|t^*_{r,\lambda}\|_{b^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}$

$$\leq \sup_{P \in \mathfrak{Q}} \left\{\frac{1}{\phi(P)} \left\|\left\{\sum_{\substack{Q \in \mathfrak{Q}^*, Q \subset P \\ \ell(Q)=2^{-j}}} |Q|^{-s(\cdot)/n} \left|\left((v^P)^*_{r,\lambda}\right)_Q\right| \tilde{\chi}_Q\right\}_{j \geq (j_P \vee 0)}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}(P)\right)}\right.$$

$$\left. + \frac{1}{\phi(P)} \left\|\left\{\sum_{\substack{Q \in \mathfrak{Q}^*, Q \subset P \\ \ell(Q)=2^{-j}}} |Q|^{-s(\cdot)/n} \left|\left((u^P)^*_{r,\lambda}\right)_Q\right| \tilde{\chi}_Q\right\}_{j \geq (j_P \vee 0)}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}(P)\right)}\right\}$$

$$=: \sup_{P \in \mathfrak{Q}} \left(I_{P,1}, I_{P,2}\right). \tag{64}$$

By the above claim, (57), we find that

$$I_{P,1} \le \frac{1}{\phi(P)} \left\| \left(v^P\right)^*_{r,\lambda} \right\|_{b^{s(\cdot),1}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \lesssim \frac{1}{\phi(P)} \left\| v^P \right\|_{b^{s(\cdot),1}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}$$

$$\lesssim \frac{1}{\phi(4P)} \left\| \left\{ \sum_{\substack{Q \in \mathfrak{Q}^*, Q \subset 4P \\ \ell(Q)=2^{-j}}} |Q|^{-s(\cdot)/n} |t_Q| \tilde{\chi}_Q \right\}_{j \ge (j_{4P} \vee 0)} \right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}(4P)\right)} \lesssim \|t\|_{b^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \quad (65)$$

To estimate $I_{P,2}$, we only consider the case that $q_+ \in (0,\infty)$, since the proof of the case that $q_+ = \infty$ is similar, the details being omitted. Without loss of generality, we may assume that $\|t\|_{b^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} = 1$ and prove that $I_{P,2} \lesssim 1$. To this end, it suffices to show that

$$\left\| \left\{ \sum_{\substack{Q \in \mathfrak{Q}^*, Q \subset P \\ \ell(Q)=2^{-j}}} \frac{\chi_P}{\phi(P)} |Q|^{-s(\cdot)/n} \left((u^P)^*_{r,\lambda}\right)_Q \tilde{\chi}_Q \right\}_{j \ge (j_P \vee 0)} \right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}(\mathbb{R}^n)\right)} \lesssim 1.$$

By (54), we see that the above inequality is equivalent to that there exists some large positive constant C_0 such that

$$\sum_{j=(j_P \vee 0)}^{\infty} \left\| \left[\sum_{\substack{Q \in \mathfrak{Q}^*, Q \subset P \\ \ell(Q)=2^{-j}}} \frac{\chi_P}{C_0 \phi(P)} |Q|^{-s(\cdot)/n} \left((u^P)^*_{r,\lambda}\right)_Q \tilde{\chi}_Q \right]^{q(\cdot)} \right\|_{L^{p(\cdot)/q(\cdot)}(\mathbb{R}^n)} \le 1,$$

which, by Lemma (6.2.16) (i), is a consequence of

$$J_P := \sum_{j=(j_P \vee 0)}^{\infty} \left\| \left[\sum_{\substack{Q \in \mathfrak{Q}^*, Q \subset P \\ \ell(Q)=2^{-j}}} \frac{\chi_P}{C_0 \phi(P)} |Q|^{-s(\cdot)/n} \left((u^P)^*_{r,\lambda}\right)_Q \tilde{\chi}_Q \right]^{q(\cdot)} \right\|_{L^{p(\cdot)/q(\cdot)}(\mathbb{R}^n)} \le 1. \quad (66)$$

Now we show (66). Since $\|t\|_{b^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} = 1$, it follows that, for all $\tilde{P} \in \mathfrak{Q}$,

$$\left\| \left\{ \sum_{\substack{Q \in \mathfrak{Q}^*, Q \subset P \\ \ell(Q)=2^{-j}}} [\phi(\tilde{P})]^{-1} \chi_{\tilde{P}} |Q|^{-s(\cdot)/n} |t_Q| \tilde{\chi}_Q \right\}_{j \ge (j_{\tilde{P}} \vee 0)} \right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}(\tilde{P})\right)} \le 1,$$

which, together with (54), implies that

$$\sum_{j=(j_{\tilde{P}} \vee 0)}^{\infty} \left\| \left[\sum_{\substack{Q \in \mathfrak{Q}^*, Q \subset \tilde{P} \\ \ell(Q)=2^{-j}}} [\phi(\tilde{P})]^{-1} \chi_{\tilde{P}} |Q|^{-s(\cdot)/n} |t_Q| \tilde{\chi}_Q \right]^{q(\cdot)} \right\|_{L^{p(\cdot)/q(\cdot)}(\mathbb{R}^n)} \le 1.$$

From this, and (iii) and (iv) of Lemma (6.2.16), we deduce that, for all $\tilde{P} \in \mathfrak{Q}$ and $j \ge (j_{\tilde{P}} \vee 0)$,

$$\left\| \sum_{\substack{Q \in \mathfrak{Q}^*, Q \subset \tilde{P} \\ \ell(Q)=2^{-j}}} [\phi(\tilde{P})]^{-1} |Q|^{-s(\cdot)/n} |t_Q| \tilde{\chi}_Q \right\|_{L^{p(\cdot)}(\tilde{P})} \le 1. \quad (67)$$

For the given $P \in \mathfrak{Q}$, $i \in \mathbb{Z}_+$ and $l \in \mathbb{Z}^n$, let
$$A(i, l, P) := \{R \in \mathfrak{Q}^* : \ell(R) = 2^{-i}\ell(P), R \subset P + l\ell(Q)\}.$$
Then we see that

$$\tilde{J}_P := \sum_{j=(j_P \vee 0)}^{\infty} \left\| \sum_{\substack{Q \in \mathfrak{Q}^*, Q \subset P \\ \ell(Q)=2^{-j}}} \chi_P [\phi(P)]^{-1} |Q|^{-s(\cdot)/n} \left(u_{r,\lambda}^*\right)_Q \tilde{\chi}_Q \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{q_-}$$

$$\lesssim \sum_{i=0}^{\infty} \left\| \sum_{\substack{Q \in \mathfrak{Q}^*, Q \subset P \\ \ell(Q)=2^{-i}\ell(P)}} \frac{\chi_P}{\phi(P)} |Q|^{-[s(\cdot)/n+1/2]} \right.$$

$$\left. \times \left[\sum_{l \in \mathbb{Z}^n, |l| \geq 2} \sum_{R \in A(i,l,P)} \frac{|u_R|^r}{[1 + \{\ell(Q)\}^{-1}|x_R - x_Q|]^{\lambda}} \right]^{1/r} \chi_Q \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{q_-}.$$

Notice that, for all $i \in \mathbb{Z}_+$, $l \in \mathbb{Z}^n$ and $x \in Q \in \mathfrak{Q}^*$ with $\ell(Q) = 2^{-i}\ell(P)$,

$$\sum_{R \in A(i,l,P)} \frac{|u_R|^r}{[1 + \{\ell(Q)\}^{-1}|x_R - x_Q|]^m} \sim \eta_{j_Q,m} * \left(\left[\sum_{R \in A(i,l,P)} |u_R| \chi_R \right]^r \right)(x),$$

where $m \in (n + C_{\log}(s), \infty)$ is chosen such that $\lambda > m + n + 2r \log_2 c_1$. Notice that, when $|l| \geq 2$, $1 + \{\ell(Q)\}^{-1}|x_R - x_Q| \sim 2^i |l|$. Thus, by Lemma (6.2.15), we know that

$$\tilde{J}_P \lesssim \sum_{i=0}^{\infty} \left\| \sum_{\substack{l \in \mathbb{Z}^n \\ |l| \geq 2}} \frac{(2^i |l|)^{m-\lambda}}{[\phi(P)]^r} \eta_{i+j_P, m-C_{\log}(s)} * \left(\left[\sum_{R \in A(i,l,P)} |R|^{-rs(\cdot)/n} |u_R| \tilde{\chi}_R \right]^r \right) \right\|_{L^{p(\cdot)/r}(\mathbb{R}^n)}^{q_-/r}$$

$$\lesssim \sum_{i=0}^{\infty} \left\{ \sum_{\substack{l \in \mathbb{Z}^n \\ |l| \geq 2}} (2^i |l|)^{m-\lambda} \left[\frac{\phi(P + l\ell(P))}{\phi(P)} \right]^r \left\| \sum_{R \in A(i,l,P)} \frac{\chi_{P+l\ell(P)}}{\phi(P + l\ell(P))} |R|^{-s(\cdot)/n} |t_R| \tilde{\chi}_R \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^r \right\}^{q_-/r},$$

which, combined with (67) and Lemma (6.2.10), implies that

$$\tilde{J}_P \lesssim \sum_{i=0}^{\infty} \left\{ \sum_{l \in \mathbb{Z}^n, |l| \geq 2} 2^{i(m-\lambda)} |l|^{m+2r \log_2 c_1 - \lambda} \right\}^{q_-/r} \sim 1.$$

Therefore, there exists a positive constant C_0 large enough such that (66) holds true for all $P \in \mathfrak{Q}$ and hence

$$I_{P,2} \lesssim \|t\|_{b_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)}. \tag{68}$$

Combining (64), (65) and (68), we conclude that

$$\|t_{r,\lambda}^*\|_{b_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)} \leq \sup_{P \in \mathfrak{Q}} (I_{P,1} + I_{P,2}) \lesssim \|t\|_{b_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)},$$

which completes the proof of Lemma (6.2.17).

Theorem (6.2.18) [298]: Let p, q, s and ϕ be as in Definition (6.2.6) and φ, ψ, Φ and Ψ as in (61). Then operators $S_\varphi : B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n) \to b_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$ and $T_\psi : b_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n) \to B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$ are bounded. Furthermore, $T_\psi \circ S_\varphi$ is the identity on $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$.

Proof: We first show that S_φ is bounded from $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$ to $b_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$. Let $f \in B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$, $r \in \left(0, \frac{1}{2}\min\{p_-, q_-, 2\}\right)$ and $m \in \left(n + C_{\log}(s) + C_{\log}(r/q) + \log_2 c_1, \infty\right)$. Then, by Lemma (6.2.13), we see that, for all $\mathcal{Q}_{jk} \in \mathfrak{Q}^*$ and $x \in \mathcal{Q}_{jk}$,

$$\left|\varphi_j * f\left(x_{\mathcal{Q}_{jk}}\right)\right|^r \lesssim 2^{jn} \sum_{l \in \mathbb{Z}^n} \int_{\mathcal{Q}_{j(k+l)}} \frac{|\varphi_j * f(y)|^r}{(1 + 2^{jn}|x - y|)^{4m}} \, dy,$$

which, together with the fact that $1 + 2^{jn}|x - y| \sim 1 + |l|$ when $x \in \mathcal{Q}_{jk}$ and $y \in \mathcal{Q}_{j(k+l)}$, implies that

$$\left|\varphi_j * f\left(x_{\mathcal{Q}_{jk}}\right)\right| \lesssim \left[\sum_{l \in \mathbb{Z}^n}(1 + |l|)^{-m}\eta_{j,3m} * \left|(\varphi_j * f)\chi_{\mathcal{Q}_{j(k+l)}}\right|^r(x)\right]^{1/r}.$$

From this and Lemma (6.2.14), we deduce that

$\left\|S_\varphi(f)\right\|_{b_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)}$

$$\lesssim \sup_{P \in \mathfrak{Q}} \frac{1}{\phi(P)} \left\|\left\{\sum_{k \in \mathbb{Z}^n}\left[\sum_{l \in \mathbb{Z}^n}\frac{2^{jrs(\cdot)}}{(1 + |l|)^m}\eta_{j,3m} * \left|(\varphi_j * f)\chi_{3n|l|P}\right|^r\right]^{1/r}\right\}_{j \geq (j_P \vee 0)} \chi_{\mathcal{Q}_{jk}}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}(P)\right)}$$

$$\lesssim \sup_{P \in \mathfrak{Q}} \frac{1}{\phi(P)} \left\|\left\{\sum_{l \in \mathbb{Z}^n}\frac{2^{jrs(\cdot)}}{(1 + |l|)^m}\eta_{j,3m} * \left|(\varphi_j * f)\chi_{3n|l|P}\right|^r\right\}_{j \geq (j_P \vee 0)}\right\|_{\ell^{q(\cdot)/r}\left(L^{p(\cdot)/r}(P)\right)}^{1/r}$$

$$\lesssim \left[\sum_{l \in \mathbb{Z}^n}(1 + |l|)^{-m}\sup_{P \in \mathfrak{Q}}\frac{1}{\{\phi(P)\}^r}\left\|\left\{\eta_{j,2m} * \left|2^{js(\cdot)}(\varphi_j * f)\chi_{3n|l|P}\right|^r\right\}_{j \geq (j_P \vee 0)}\right\|_{\ell^{q(\cdot)/r}\left(L^{p(\cdot)/r}(P)\right)}\right]^{1/r}$$

which, combined with Lemmas (6.2.15) and (6.2.10), implies that

$$\left\|S_\varphi(f)\right\|_{b_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)} \lesssim \left[\sum_{l \in \mathbb{Z}^n}(1 + |l|)^{-m}\sup_{P \in \mathfrak{Q}}\frac{1}{\phi(P)^r}\left\|\left\{2^{js(\cdot)}|\varphi_j * f|\right\}_{j \geq (j_P \vee 0)}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}(3n|l|P)\right)}^r\right]^{1/r}$$

$$\lesssim \|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)}\left\{\sum_{l \in \mathbb{Z}^n}(1 + |l|)^{-m}(1 + |l|)^{r\log_2 c_1}\right\}^{1/r} \lesssim \|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)}.$$

Therefore, S_φ is bounded from $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$ to $b_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$.

The boundedness of T_ψ from $b_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$ to $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$ is deduced from an argument similar to that used in [78]. Finally, by the Calderón reproducing formula [78], we know that $T_\psi \circ S_\varphi$ is the identity on $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$, which completes the proof of Theorem (6.2.18).

We show some basic properties and embeddings between $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$ and $F_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$. Recall that the Triebel-Lizorkin-type space with variable exponents, $F_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$, is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{F_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)} := \sup_{P \in \mathfrak{Q}}\frac{1}{\phi(P)}\left\|\left\{\sum_{j=\max\{j_P,0\}}^{\infty}\left[2^{js(\cdot)}|\varphi_j * f(\cdot)|\right]^{q(\cdot)}\right\}^{1/q(\cdot)}\right\|_{L^{p(\cdot)}(P)} < \infty,$$

where φ_0 is replaced by Φ and the supremum is taken over all dyadic cubes P in $\mathbb{R}^n$, which was introduced in [76].

Proposition (6.2.19) [298]: Let $\phi \in \mathcal{G}(\mathbb{R}^{n+1}_+)$, $s, s_0, s_1 \in C^{\log}_{loc}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $p, q, q_0, q_1 \in C^{\log}(\mathbb{R}^n)$.

(i) If $q_0 \leq q_1$, then $B^{s(\cdot),\phi}_{p(\cdot),q_0(\cdot)}(\mathbb{R}^n) \hookrightarrow B^{s(\cdot),\phi}_{p(\cdot),q_1(\cdot)}(\mathbb{R}^n)$.

(ii) If $(s_0 - s_1)_- > 0$, then $B^{s_0(\cdot),\phi}_{p(\cdot),q_0(\cdot)}(\mathbb{R}^n) \hookrightarrow B^{s_1(\cdot),\phi}_{p(\cdot),q_1(\cdot)}(\mathbb{R}^n)$.

(iii) If $p_+, q_+ \in (0,\infty)$, then
$$B^{s(\cdot),\phi}_{p(\cdot),\min\{p(\cdot),q(\cdot)\}}(\mathbb{R}^n) \hookrightarrow F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \hookrightarrow B^{s(\cdot),\phi}_{p(\cdot),\max\{p(\cdot),q(\cdot)\}}(\mathbb{R}^n).$$

In particular, if $p_+ \in (0,\infty)$, then $B^{s(\cdot),\phi}_{p(\cdot),p(\cdot)}(\mathbb{R}^n) = F^{s(\cdot),\phi}_{p(\cdot),p(\cdot)}(\mathbb{R}^n)$.

Proof: We only give the proof of (iii). Let $f_j(x) := 2^{js(x)}|\varphi_j * f(x)|$ for all $x \in \mathbb{R}^n$, $f \in \mathcal{S}'(\mathbb{R}^n)$ and $j \in \mathbb{Z}_+$. To prove the first embedding of (iii), we let $r(\cdot) := \min\{p(\cdot), q(\cdot)\}$ and $f \in B^{s(\cdot),\phi}_{p(\cdot),r(\cdot)}(\mathbb{R}^n)$. Without loss of generality, we may assume that $\|f\|_{B^{s(\cdot),\phi}_{p(\cdot),r(\cdot)}(\mathbb{R}^n)} = 1$ and prove that $\|f\|_{F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \lesssim 1$. Obviously, for all $P \in \mathfrak{Q}$,

$$\left\|\left\{[\phi(P)]^{-1}\chi_P f_j\right\}_{j \geq (j_P \vee 0)}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}(\mathbb{R}^n)\right)} \leq 1,$$

which, together with (54), implies that

$$\left\|\sum_{j=(j_P \vee 0)}^\infty [[\phi(P)]^{-1}\chi_P f_j]^{r(\cdot)}\right\|_{L^{p(\cdot)/r(\cdot)}(\mathbb{R}^n)} \leq \sum_{j=(j_P \vee 0)}^\infty \left\|\{[\phi(P)]^{-1}\chi_P f_j\}^{r(\cdot)}\right\|_{L^{p(\cdot)/r(\cdot)}(\mathbb{R}^n)} \leq 1.$$

Then, by the fact that, for all $d \in (0,1]$ and $\{a_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$,

$$\left(\sum_{j\in\mathbb{N}}|a_j|\right)^d \leq \sum_{j\in\mathbb{N}}|a_j|^d, \tag{69}$$

we find that, for all $P \in \mathfrak{Q}$,

$$\varrho_{p(\cdot)}\left(\left[\sum_{j=(j_P\vee 0)}^\infty [[\phi(P)]^{-1}\chi_P f_j]^{q(\cdot)}\right]^{1/q(\cdot)}\right) \leq \varrho_{p(\cdot)/r(\cdot)}\left(\sum_{j=(j_P\vee 0)}^\infty \{[\phi(P)]^{-1}\chi_P f_j\}^{r(\cdot)}\right) \leq 1,$$

which implies that

$$\frac{1}{\phi(P)}\left\|\left\{\sum_{j=(j_P\vee 0)}^\infty [2^{js(\cdot)}|\varphi_j * f|]^{q(\cdot)}\right\}^{1/q(\cdot)}\right\|_{L^{p(\cdot)}(P)} \leq 1.$$

Therefore, $\|f\|_{F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \lesssim 1$, which completes the proof of the first embedding of (iii).

For the second embedding of (iii), let $f \in F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $\alpha(\cdot) := \max\{p(\cdot), q(\cdot)\}$. Without loss of generality, we may assume that $\|f\|_{F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} = 1$ and show that $\|f\|_{B^{s(\cdot),\phi}_{p(\cdot),\alpha(\cdot)}(\mathbb{R}^n)} \lesssim 1$. Since $\|f\|_{F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} = 1$, we know that, for all $P \in \mathfrak{Q}$,

$$\left\|\left\{\sum_{j=(j_P\vee 0)}^\infty ([\phi(P)]^{-1}\chi_P f_j)^{q(\cdot)}\right\}^{1/q(\cdot)}\right\|_{L^{p(\cdot)}(P)} \leq 1,$$

which, combined with (69), implies that, for all $P \in \mathfrak{Q}$,

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$$\varrho_{p(\cdot)/\alpha(\cdot)}\left(\sum_{j=(j_P\vee 0)}^{\infty}\{[\phi(P)]^{-1}\chi_P f_j\}^{\alpha(\cdot)}\right) \leq \varrho_{p(\cdot)}\left(\left[\sum_{j=(j_P\vee 0)}^{\infty}[[\phi(P)]^{-1}\chi_P f_j]^{q(\cdot)}\right]^{1/q(\cdot)}\right) \leq 1.$$

From this, we deduce that

$$\varrho_{\ell^{\alpha(\cdot)}\left(L^{p(\cdot)}(\mathbb{R}^n)\right)}\left(\{[\phi(P)]^{-1}\chi_P f_j\}_{j\geq(j_P\vee 0)}\right) = \sum_{j=(j_P\vee 0)}^{\infty}\left\|\left([\phi(P)]^{-1}\chi_P f_j\right)^{\alpha(\cdot)}\right\|_{L^{p(\cdot)/\alpha(\cdot)}(\mathbb{R}^n)}$$

$$\leq \left\|\sum_{j=(j_P\vee 0)}^{\infty}\left([\phi(P)]^{-1}\chi_P f_j\right)^{\alpha(\cdot)}\right\|_{L^{p(\cdot)/\alpha(\cdot)}(\mathbb{R}^n)} \leq 1,$$

which implies that $\|f\|_{B_{p(\cdot),\alpha(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)} \leq 1$ and hence completes the proof of Proposition (6.2.19).

The Sobolev-type embedding of $B_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$ (see [3]) shows that it is reasonable and necessary to consider the Besov spaces with variable smoothness and integrability. For $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$, we also have the following Sobolev-type embeddings.

Proposition (6.2.20) [298]: Let $\phi \in \mathcal{G}(\mathbb{R}_+^{n+1})$, $s_0, s_1 \in C_{loc}^{log}(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)$, $p_0, p_1 \in C^{log}(\mathbb{R}^n)$ satisfy that, for all $x \in \mathbb{R}^n$, $s_1(x) \leq s_0(x)$ and $s_0(x) - n/p_0(x) = s_1(x) - n/p_1(x)$. Then

$$b_{p_0(\cdot),\infty}^{s_0(\cdot),\phi}(\mathbb{R}^n) \hookrightarrow b_{p_1(\cdot),\infty}^{s_1(\cdot),\phi}(\mathbb{R}^n); \tag{70}$$

moreover, $B_{p_0(\cdot),\infty}^{s_0(\cdot),\phi}(\mathbb{R}^n) \hookrightarrow B_{p_1(\cdot),\infty}^{s_1(\cdot),\phi}(\mathbb{R}^n)$.

Proof: To prove this proposition, we only need to show (70), since the embedding $B_{p_0(\cdot),\infty}^{s_0(\cdot),\phi}(\mathbb{R}^n) \hookrightarrow B_{p_1(\cdot),\infty}^{s_1(\cdot),\phi}(\mathbb{R}^n)$ is a consequence of (70) and Theorem (6.2.18). To prove (70), let $t := \{t_Q\}_{Q\in\mathfrak{Q}^*} \in b_{p_0(\cdot),\infty}^{s_0(\cdot),\phi}(\mathbb{R}^n)$ and $P \in \mathfrak{Q}$ be any given dyadic cube. For all $Q \in \mathfrak{Q}^*$, let $u_Q := t_Q$ when $Q \subset P$ and $u_Q = 0$ otherwise. Then, by the Sobolev-type embedding of $b_{p(\cdot),\infty}^{s(\cdot)}(\mathbb{R}^n) = b_{p(\cdot),\infty}^{s(\cdot),1}(\mathbb{R}^n)$ (see [28]), namely, $b_{p_0(\cdot),\infty}^{s_0(\cdot),1}(\mathbb{R}^n) \hookrightarrow b_{p_1(\cdot),\infty}^{s_1(\cdot),1}(\mathbb{R}^n)$, we conclude that

$$\sup_{j\geq(j_P\vee 0)}\left\|\sum_{\substack{Q\in\mathfrak{Q}^*,Q\subset P\\\ell(Q)=2^{-j}}}|Q|^{-s_1(\cdot)/n}|t_Q|\tilde{\chi}_Q\right\|_{L^{p_1(\cdot)}(P)} = \sup_{j\geq 0}\left\|\sum_{Q\in\mathfrak{Q}^*,\ell(Q)=2^{-j}}|Q|^{-s_1(\cdot)/n}|u_Q|\tilde{\chi}_Q\right\|_{L^{p_1(\cdot)}(P)}$$

$$= \|u\|_{b_{p_1(\cdot),\infty}^{s_1(\cdot),1}(\mathbb{R}^n)} \lesssim \|u\|_{b_{p_0(\cdot),\infty}^{s_0(\cdot),1}(\mathbb{R}^n)} \sim \sup_{j\geq 0}\left\|\sum_{Q\in\mathfrak{Q}^*,\ell(Q)=2^{-j}}|Q|^{-s_0(\cdot)/n}|u_Q|\tilde{\chi}_Q\right\|_{L^{p_0(\cdot)}(P)}$$

$$\sim \sup_{j\geq(j_P\vee 0)}\left\|\sum_{\substack{Q\in\mathfrak{Q}^*,Q\subset P\\\ell(Q)=2^{-j}}}|Q|^{-s_0(\cdot)/n}|t_Q|\tilde{\chi}_Q\right\|_{L^{p_0(\cdot)}(P)}.$$

From this, we further deduce that

$$\|t\|_{b^{s_1(\cdot),\phi}_{p_1(\cdot),\infty}(\mathbb{R}^n)} = \sup_{P\in\mathcal{Q}}\frac{1}{\phi(P)}\sup_{j\geq(j_P\vee 0)}\left\|\sum_{\substack{Q\in\mathcal{Q}^*,Q\subset P\\ \ell(Q)=2^{-j}}}|Q|^{-s_1(\cdot)/n}|t_Q|\tilde{\chi}_Q\right\|_{L^{p_1(\cdot)}(P)}$$

$$\lesssim \sup_{P\in\mathcal{Q}}\frac{1}{\phi(P)}\sup_{j\geq(j_P\vee 0)}\left\|\sum_{\substack{Q\in\mathcal{Q}^*,Q\subset P\\ \ell(Q)=2^{-j}}}|Q|^{-s_0(\cdot)/n}|t_Q|\tilde{\chi}_Q\right\|_{L^{p_0(\cdot)}(P)} \sim \|t\|_{b^{s_0(\cdot),\phi}_{p_0(\cdot),\infty}(\mathbb{R}^n)},$$

which implies that (69) holds true and hence completes the proof of Proposition (6.2.20).

Theorem (6.2.21) [298]: Let $\phi \in \mathcal{G}(\mathbb{R}^{n+1}_+)$, $s_0, s_1 \in C^{\log}_{\text{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $p_0, p_1, q \in C^{\log}(\mathbb{R}^n)$. Assume that, for all $x \in \mathbb{R}^n$, $s_1(x) \leq s_0(x)$ and

$$s_0(x) - \frac{n}{p_0(x)} = s_1(x) - \frac{n}{p_1(x)} \tag{71}$$

Then $B^{s_0(\cdot),\phi}_{p_0(\cdot),q(\cdot)}(\mathbb{R}^n) \hookrightarrow B^{s_1(\cdot),\phi}_{p_1(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Proof: We only give the proof of the case that $q_+ \in (0,\infty)$, since the case that $q_+ = \infty$ was proved in Proposition (6.2.20). Let $f \in B^{s_0(\cdot),\phi}_{p_0(\cdot),q(\cdot)}(\mathbb{R}^n)$ and, for all $j \in \mathbb{Z}_+$ and $x \in \mathbb{R}^n$, $g_j(x) := \varphi_j * f(x)$. Without loss of generality, we may assume that $\|f\|_{B^{s_0(\cdot),\phi}_{p_0(\cdot),q(\cdot)}(\mathbb{R}^n)} = 1$. Next, we show that $\|f\|_{B^{s_1(\cdot),\phi}_{p_1(\cdot),q(\cdot)}(\mathbb{R}^n)} \lesssim 1$. Obviously, by (54) we find that, for all $R \in \mathcal{D}_0(\mathbb{R}^n)$,

$$\sum_{j=(j_R\vee 0)}^{\infty}\left\|\left[\frac{\chi_R}{\phi(R)}2^{js_0(\cdot)}|g_j|\right]^{q(\cdot)}\right\|_{L^{\frac{p_0(\cdot)}{q(\cdot)}}(\mathbb{R}^n)} \lesssim 1. \tag{72}$$

Let $P \in \mathcal{Q}$ be a given dyadic cube. We claim that there exists $c \in (0,1)$, independent of P, such that, for all $j \geq [j_P \vee 0, \infty)$,

$$\left\|\left[\frac{c\chi_P}{\phi(P)}2^{js_1(\cdot)}|g_j|\right]^{q(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q(\cdot)}}(\mathbb{R}^n)} \leq \sum_{i=1}^{\infty}2^{-i\xi}\left\|\left[\frac{\chi_{P_i}}{\phi(P_i)}2^{js_0(\cdot)}|g_j|\right]^{q(\cdot)}\right\|_{L^{\frac{p_0(\cdot)}{q(\cdot)}}(\mathbb{R}^n)} + 2^{-j} =: \delta_j,$$

where $P_i := 2^{i+1+n}P$ and $\xi \in (0,\infty)$. From this claim and (72), we deduce that

$$\sum_{j=(j_P\vee 0)}^{\infty}\left\|\left[\frac{c\chi_P}{\phi(P)}2^{js_1(\cdot)}|g_j|\right]^{q(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q(\cdot)}}(\mathbb{R}^n)} \lesssim 1.$$

which, together with (54), implies that

$$\|f\|_{B^{s_1(\cdot),\phi}_{p_1(\cdot),q(\cdot)}(\mathbb{R}^n)} \lesssim 1 \sim \|f\|_{B^{s_0(\cdot),\phi}_{p_0(\cdot),q(\cdot)}(\mathbb{R}^n)}.$$

Therefore, it remains to prove the above claim. Observe that, for all $j \geq [j_P \vee 0, \infty)$, $\delta_j \in [2^{-j}, 2^{-j} + \theta]$ with $\theta \in [0,\infty)$. Then, by Lemma (6.2.13), we conclude that, for all $x \in \mathbb{R}^n$, $r \in (0, p_-)$ and $m \in (0,\infty)$ large enough,

$$2^{jr[s_1(x)-n/p_1(x)]}\big/[\phi(P)]^r\delta_j^{r/q(x)}|g_j(x)|^r \lesssim \eta_{j,2m} * \left(\left\{\frac{2^{j[s_1(\cdot)-n/p_1(\cdot)]}}{\phi(P)\delta_j^{1/q(\cdot)}}|g_j|\right\}^r\right)(x)$$

$$\lesssim \sum_{i=0}^{\infty}\int_{D_{i,P}}\frac{2^{jn}\big(2^{j[s_1(y)-n/p_1(y)]}|g_j(y)|\big)^r}{[\phi(P)]^r\delta_j^{r/q(y)}(1+2^j|x-y|)^{2m}}|g_j(x)|^r dy$$

$$=: \sum_{i=0}^{\infty}A_{j,i}(x), \tag{73}$$

where $D_{1,P} := 4\sqrt{n}P$ and, for all $i \in [2, \infty)$, $D_{i,P} := \left(2^{i+1}\sqrt{n}P\right)\backslash\left(2^i\sqrt{n}P\right)$. For $A_{j,1}$, by the Hölder inequality in (71), (57) and Lemma (6.2.16), we see that

$$A_{j,1} \lesssim \left\|\left\|\frac{\chi_{4\sqrt{n}P}}{\phi(P)\delta_j^{1/q(\cdot)}}2^{js_0(\cdot)}|g_j|\right\|^r\right\|_{L^{p_0(\cdot)/r}(\mathbb{R}^n)} \lesssim \left\|\frac{2^{jn}2^{-jnr/p_0(\cdot)}}{(1+2^j|x-\cdot|)^{2m}}\right\|_{L^{(p_0(\cdot)/r)^*}(\mathbb{R}^n)}$$

$$\lesssim \left[\frac{\phi(P_1)}{\phi(P)}\right]^r\left\|\frac{\chi_{4\sqrt{n}P}}{\phi(P_1)\delta_j^{1/q(\cdot)}}2^{js_0(\cdot)}|g_j|\right\|^r_{L^{p_0(\cdot)}(\mathbb{R}^n)} \lesssim 1,$$

where the last inequality follows from the definition of δ_j. Similarly, observe that, for all $x \in P$ and $y \in D_{i,P}$ with $i \geq 2$, $|x-y| \gtrsim 2^{i-j_P}$, then the fact that $j \geq j_P$ further implies that

$$A_{j,i} \lesssim \frac{2^{(j_P-j)m}}{2^{i(m-\xi/q_-)}}\left[\frac{\phi(P_i)}{\phi(P)}\right]^r\left\|\frac{\chi_{P_i}2^{js_0(\cdot)}|g_j|}{\phi(P_i)\{2^{i\xi}\delta_j\}^{1/q(\cdot)}}\right\|^r_{L^{p_0(\cdot)}(\mathbb{R}^n)} \left\|\frac{2^{jn}2^{-jnr/p_0(\cdot)}}{(1+2^j|x-\cdot|)^m}\right\|_{L^{(p_0(\cdot)/r)^*}(\mathbb{R}^n)}$$

$$\lesssim 2^{(j_P-j)m}2^{-i(m-r\log_2 c_1)} \lesssim 2^{-i(m-\xi/q_--r\log_2 c_1)}.$$

Thus, by (73) we conclude that, for all $x \in \mathbb{R}^n$,

$$\chi_P(x)[\phi(P)]^{-1}\delta_j^{-1/q(x)}2^{jr[s_1(x)-n/p_1(x)]}|g_j(x)| \lesssim 1.$$

From this, (71) and an appropriate choice of $c \in (0, 1)$, we deduce that

$$\left[\frac{c\chi_P(x)2^{js_1(x)}}{\phi(P)\delta_j^{1/q(x)}}|g_j(x)|\right]^{p_1(x)}$$

$$= c^{p_0(x)}\left[\frac{\chi_P(x)2^{js_0(x)}}{\phi(P)\delta_j^{1/q(x)}}|g_j(x)|\right]^{p_0(x)}\left[\frac{c\chi_P(x)2^{j[s_1(x)-n/p_1(x)]}}{\phi(P)\delta_j^{1/q(x)}}|g_j(x)|\right]^{p_1(x)-p_0(x)}$$

$$\leq c^{p_0(x)}\left[\frac{\chi_P(x)2^{js_0(x)}}{\phi(P)\delta_j^{1/q(x)}}|g_j(x)|\right]^{p_0(x)} \leq \left[\frac{\chi_{P_1}(x)2^{js_0(x)}}{\phi(P_1)\{2^{\xi}\delta_j\}^{1/q(x)}}|g_j(x)|\right]^{p_0(x)},$$

which, together with the definition of δ_j, implies that the previous claim holds true and hence completes the proof of Theorem (6.2.21).

Now we characterize $B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ in terms of the Peetre maximal functions and establish their atomic characterization via Sobolev embeddings. Following [17], for all $f \in \mathcal{S}'(\mathbb{R}^n)$, $a \in (0, \infty)$ and $s : \mathbb{R}^n \to \mathbb{R}$, the Peetre maximal function of f is defined by setting, for all $j \in \mathbb{Z}_+$,

$$\varphi_j^{*,a}\left(2^{js(\cdot)}f\right)(x) := \sup_{y \in \mathbb{R}^n}\frac{2^{js(y)}|\varphi_j * f(y)|}{(1+2^j|x-y|)^a}.$$

Lemma (6.2.22) [298]: Let $p \in C^{\log}(\mathbb{R}^n)$ with $p_- \in (1, \infty]$. Then there exists a positive constant C, independent of f, such that, for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$, $\|\mathcal{M}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

Lemma (6.2.23) [298]: Let p, q, s, ϕ be as in Definition (6.2.6) and $a \in (n + \log_2 c_1 + \varepsilon/q_-, \infty)$ with $\varepsilon \in (0, \infty)$. Assume that $p_- \in (1, \infty)$, $q_+ \in (0, \infty)$ and $f \in B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}$ with norm 1. Then there exists a positive constant c such that, for all $P \in \mathfrak{Q}$ and $j \in \mathbb{Z}_+$ with $j \geq (j_P \vee 0)$,

$$\inf\left\{\lambda_j \in (0, \infty) : \varrho_{p(\cdot)}\left(\frac{c\chi_P \varphi_j^{*,a}(2^{js(\cdot)}f)}{\phi(P)\lambda_j^{1/q(\cdot)}}\right) \le 1\right\}$$

$$\le \sum_{k=1}^{\infty} \inf\left\{\eta_j \in (0, \infty) : \varrho_{p(\cdot)}\left(\frac{\chi_{P_k^n} 2^{js(\cdot)}|\varphi_j * f|}{2^{k\varepsilon/q(\cdot)}\phi(P_k^n)\eta_j^{1/q(\cdot)}}\right) \le 1\right\} + 2^{-\sigma[j-(j_P \vee 0)]} \qquad (74)$$

where, for all $k \in \mathbb{N}$, $P_k^n := 2^{k+1+n}P$ and $\sigma \in \left(0, \frac{a-n}{4(1/q_- - 1/q_+)}\right)$.

Proof: Let δ_j^P be the right hand side term of (74), we easily see that

$$\delta_j^P \le \sum_{k=1}^{\infty} 2^{-k\varepsilon} \frac{1}{\phi(P_k^n)}\left\|\left\{2^{js(\cdot)}|\varphi_j * f|\right\}_{j \ge (j_{P_k^n} \vee 0)}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P_k^n))} + 2^{-\sigma[j-(j_P \vee 0)]}$$

$$\le \sum_{k=1}^{\infty} 2^{-k\varepsilon}\|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)} + 2^{-\sigma[j-(j_P \vee 0)]} = 1/(2^\varepsilon - 1) + 2^{-\sigma[j-(j_P \vee 0)]},$$

which implies that

$$\delta_j^P \in \left[2^{-\sigma[j-(j_P \vee 0)]}, 1/(2^\varepsilon - 1) + 2^{-\sigma[j-(j_P \vee 0)]}\right]. \qquad (75)$$

Thus, to prove Lemma (6.2.23), we only need to show that, for some positive constant c,

$$\inf\left\{\lambda_j \in (0, \infty) : \varrho_{p(\cdot)}\left(\frac{c\chi_P (\delta_j^P)^{-1/q(\cdot)} \varphi_j^{*,a}(2^{js(\cdot)}f)}{\phi(P)\lambda_j^{1/q(\cdot)}}\right) \le 1\right\} \le 1,$$

which, via Lemma (6.2.16), is a consequence of

$$H_P := \left\|\frac{\chi_P (\delta_j^P)^{-1/q(\cdot)}}{\phi(P)} \varphi_j^{*,a}(2^{js(\cdot)}f)\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim 1. \qquad (76)$$

Next we prove (76). By Lemma (6.2.13) and the inequality that, for all $x, y, z \in \mathbb{R}^n$,

$$\left(1 + 2^{-j}|x - y|\right)^{-a} \le \left(1 + 2^{-j}|x - z|\right)^{-a}\left(1 + 2^{-j}|z - y|\right)^a,$$

we find that, for all $x \in \mathbb{R}^n$,

$$\varphi_j^{*,a}(2^{js(\cdot)}f)(x) \lesssim \sup_{y \in \mathbb{R}^n}\int_{\mathbb{R}^n} \frac{2^{jn}2^{js(z)}|\varphi_j * f(y)|}{(1 + 2^j|y - z|)^{2a}}dz\frac{1}{(1 + 2^j|x - y|)^a} \lesssim \int_{\mathbb{R}^n} \frac{2^{jn}2^{js(z)}|\varphi_j * f(z)|}{(1 + 2^j|x - z|)^a}dz$$

$$\sim \int_{4\sqrt{n}P} \frac{2^{jn}2^{js(z)}|\varphi_j * f(z)|}{(1 + 2^j|x - z|)^a}dz + \sum_{k=2}^{\infty}\int_{D_{k,P}} \cdots dz =: A_j(x) + \sum_{k=2}^{\infty} A_j^k(x),$$

where, for all $k \in \mathbb{N} \cap [2, \infty)$, $D_{k,P} := \left(2^{k+1}\sqrt{n}P \backslash 2^k\sqrt{n}P\right)$. Thus, we obtain

$$H_P \le \left\|\frac{\chi_P A_j(\cdot)}{[\delta_j^P]^{1/q(\cdot)}\phi(P)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} + \left\|\frac{\chi_P A_j(\cdot)}{[\delta_j^P]^{1/q(\cdot)}\phi(P)}\sum_{k=2}^{\infty} A_j^k(\cdot)\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} =: H_{P,1} + H_{P,2}. \qquad (77)$$

We first estimate $H_{P,1}$. For all $x \in P$, we write

$$A_j(x) \sim \left\{\int_{B_{-1}^j(x)} + \sum_{k=2}^{\infty}\int_{B_i^j(x)}\right\}\frac{2^{jn}2^{js(z)}|\varphi_j * f(z)|\chi_{4\sqrt{n}P}(z)}{(1 + 2^j|x - z|)^a}dz =: A_{j,1}(x) + A_{j,2}(x), \qquad (78)$$

where, for all $x \in \mathbb{R}^n$, $B_{-1}^j(x) := B\left(x, 2^{-[j-(j_P \vee 0)]/2}\right)$ and, for all $i \in \mathbb{Z}_+$,

$$B_i^j(x) := B\left(x, 2^{-[j-(j_P \vee 0)]/2+i+1}\right)\backslash B\left(x, 2^{-[j-(j_P \vee 0)]/2+i}\right).$$

From (75), $q \in C^{\log}(\mathbb{R}^n)$, we deduce that, for all $x \in \mathbb{R}^n$ and $z \in B_{-1}^j(x)$,

$$(\delta_j^P)^{\frac{1}{q(z)} - \frac{1}{q(x)}} \le \{2^{\sigma[j-(j_P \vee 0)]}\delta_j^P\}^{\left|\frac{1}{q(z)} - \frac{1}{q(x)}\right|}\{2^{\sigma[j-(j_P \vee 0)]}\}^{\left|\frac{1}{q(z)} - \frac{1}{q(x)}\right|} \lesssim 2^{2\sigma[j-(j_P \vee 0)]\frac{C_{\log}(1/q)}{\log(e+1/|x-z|)}} \lesssim 1.$$

By this, $a \in (n, \infty)$ and [286, (3.9)], we conclude that, for all $x \in P$,

$$\frac{\left(\delta_j^P\right)^{-1/q(x)}}{\phi(P)} A_{j,1}(x) \lesssim \frac{1}{\phi(P)} \int_{B_{-1}^j(x)} \frac{2^{jn} 2^{js(z)} |\varphi_j * f(z)| \chi_{4\sqrt{n}P}(z)}{\left[\delta_j^P\right]^{1/q(z)} (1 + 2^j|x-z|)^a} \, dz$$

$$\lesssim \mathcal{M}\left(\frac{2^{js(\cdot)} |\varphi_j * f| \chi_{4\sqrt{n}P}}{\left[\delta_j^P\right]^{1/q(\cdot)} \phi(P)}\right)(x). \tag{79}$$

On the other hand, by (75), we see that, for all $x \in P$ and $z \in \mathbb{R}^n$ with $i \in \mathbb{Z}_+$,

$$\left(\delta_j^P\right)^{[1/q(z)-1/q(x)]} \lesssim 2^{2\sigma[j-(j_P\vee 0)](1/q_- - 1/q_+)} \tag{80}$$

and $1 + 2^j|x-z| \geq 1 + 2^j 2^{-\frac{j-(j_P\vee 0)}{2}+i}$. Thus, by $\sigma \in \left(0, \frac{a-n}{4(1/q_- - 1/q_+)}\right)$, we conclude that, for all $x \in P$,

$$\frac{\left(\delta_j^P\right)^{-1/q(x)}}{\phi(P)} A_{j,2}(x) \lesssim \sum_{i=0}^{\infty} \frac{2^{2\sigma[j-(j_P\vee 0)](1/q_- - 1/q_+)}}{\phi(P) 2^{\left[\frac{j+(j_P\vee 0)}{2}+i\right]a}} \int_{B_i^j(x)} \frac{2^{jn} 2^{js(z)}}{\left(\delta_j^P\right)^{1/q(z)}} |\varphi_j * f(z)| \chi_{4\sqrt{n}P}(z) dz$$

$$\lesssim 2^{j[2\sigma(1/q_- - 1/q_+)+n/2-a/2]} 2^{(j_P\vee 0)[-2\sigma(1/q_- - 1/q_+)-a/2+n/2]}$$

$$\times \sum_{i=0}^{\infty} 2^{i(n-a)} \mathcal{M}\left(\frac{\chi_{4\sqrt{n}P}}{\left[\delta_j^P\right]^{1/q(\cdot)} \phi(P)} 2^{js(\cdot)} |\varphi_j * f|\right)(x)$$

$$\lesssim \mathcal{M}\left(\frac{\chi_{4\sqrt{n}P}}{\left[\delta_j^P\right]^{1/q(\cdot)} \phi(P)} 2^{js(\cdot)} |\varphi_j * f|\right)(x),$$

which, together with (78) and (79), implies that, for all $x \in P$,

$$\frac{\left(\delta_j^P\right)^{-1/q(x)}}{\phi(P)} A_j(x) \lesssim \mathcal{M}\left(\frac{\chi_{4\sqrt{n}P}}{\left[\delta_j^P\right]^{1/q(\cdot)} \phi(P)} 2^{js(\cdot)} |\varphi_j * f|\right)(x).$$

By this, Lemma (6.2.22) and (57), we further know that

$$H_{P,1} \lesssim \left\| \frac{\chi_{2^{n+2}P}}{\left[\delta_j^P\right]^{1/q(\cdot)} \phi(2^{n+2}P)} 2^{js(\cdot)} |\varphi_j * f| \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

$$\lesssim 2^{\varepsilon/q_-} \left\| \frac{\chi_{2^{n+2}P} 2^{\varepsilon/q(\cdot)}}{\left[\delta_j^P\right]^{1/q(\cdot)} \phi(2^{n+2}P)} 2^{js(\cdot)} |\varphi_j * f| \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim 1,$$

where the last inequality comes from the definition of δ_j^P.

We now estimate $H_{P,2}$. Notice that, when $x \in P$ and $z \in D_{k,P}$ with $k \in \mathbb{N} \cap [2,\infty)$, $1 + 2^j|x-z| \gtrsim 2^k 2^{j-j_P}$. Then, by (80) and (74), we see that, for all $x \in P$,

$\left(\delta_j^P\right)^{-1/q(x)} A_j^k(x)$

$$\lesssim 2^{2\sigma[j-(j_P\vee 0)](1/q_- - 1/q_+)} 2^{-(k+j-j_P)a} 2^{jn} 2^{k\varepsilon/q_-} \int_{D_{k,P}} \frac{2^{-k\varepsilon/q(z)}}{\left[\delta_j^P\right]^{1/q(z)}} 2^{js(z)} |\varphi_j * f(z)| dz$$

$$\lesssim 2^{-(j-j_P)[a-n-2\sigma(1/q_- - 1/q_+)]} 2^{-k(a-n-\varepsilon/q_-)} \mathcal{M}\left(\frac{\chi_{P_k^n} 2^{-k\varepsilon/q(\cdot)}}{\left[\delta_j^P\right]^{1/q(\cdot)} \phi(P)} 2^{js(\cdot)t} |\varphi_j * f|\right)(x)$$

$$\lesssim 2^{-k(a-n-\varepsilon/q_-)} \mathcal{M}\left(\frac{\chi_{P_k^n} 2^{-k\varepsilon/q(\cdot)}}{\left[\delta_j^P\right]^{1/q(\cdot)} \phi(P)} 2^{js(\cdot)} |\varphi_j * f|\right)(x),$$

which, combined with Lemma (6.2.22), (57), the definition of δ_j^P and $a \in (n + \log_2 c_1 + \varepsilon/q_- , \infty)$, implies that

$$H_{P,2} \lesssim \sum_{k=2}^{\infty} 2^{-k(a-n-\varepsilon/q_- - \log_2 c_1)} \left\| \frac{\chi_{P_k^n} 2^{-k\varepsilon/q(\cdot)}}{\phi(P_k^n)[\delta_j^P]^{1/q(\cdot)}} 2^{js(\cdot)} |\varphi_j * f| \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim 1. \tag{82}$$

Combining (77), (81) and (82), we conclude that (76) holds true and then complete the proof of Lemma (6.2.23).

Theorem (6.2.24) [298]: Let p, q, s, ϕ be as in Definition (6.2.6) and

$$a \in ([n + \log_2 c_1]/p_- , \infty). \tag{83}$$

Then $f \in B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)}^* < \infty$, where

$$\|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)}^* = \sup_{P \in \mathcal{Q}} \frac{1}{\phi(P)} \left\| \left\{ \varphi_j^{*,a}(2^{js(\cdot)} f) \right\}_{j \geq (j_P \vee 0)} \right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}(\mathbb{R}^n)\right)}.$$

Moreover, for all $f \in B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}$, $\|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)} \sim \|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)}^*$ with equivalent positive constants independent of f.

Proof: Let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)}^* < \infty$. Then, by the obvious fact that, for all $j \in \mathbb{Z}_+$ and $x \in \mathbb{R}^n$,

$$2^{js(x)} |\varphi_j * f(x)| \leq \varphi_j^{*,a}(2^{js(\cdot)} f)(x),$$

we find that $\|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)} \leq \|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)}^*$ and hence $f \in B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$. Thus, to complete the proof of this theorem, we only need to show that, for all $f \in B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$,

$$\|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)}^* \lesssim \|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)}. \tag{84}$$

Without loss of generality, to prove (84), we may assume that $\|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)} = 1$ and show that $\|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)}^* \lesssim 1$. By (83), we find that there exist $t \in (0, p_-)$ and $\varepsilon \in (0, \infty)$ such that

$$at \in (n + \log_2 c_1 + \varepsilon/q_- , \infty). \tag{85}$$

Let $P \subset \mathbb{R}^n$ be a given dyadic cube. Next we show that

$$\frac{1}{[\phi(P)]^t} \left\| \left\{ [\varphi_j^{*,a}(2^{js(\cdot)} f)]^t \right\}_{j \geq (j_P \vee 0)} \right\|_{\ell^{q(\cdot)/t}\left(L^{p(\cdot)/t}(P)\right)} \lesssim 1 \tag{86}$$

with implicit positive constant independent of P, by Lemma (6.2.3), is equivalent to prove that $\sum_{j=(j_P \vee 0)}^{\infty} I_{P,j} \lesssim 1$, where

$$I_{P,j} := \inf \left\{ \lambda_j \in (0, \infty) : \varrho_{\frac{p(\cdot)}{t}} \left(\frac{c\chi_P [\varphi_j^{*,a}(2^{js(\cdot)} f)]^t}{[\phi(P)]^t \lambda_j^{t/q(\cdot)}} \right) \leq 1 \right\},$$

with c being a positive constant sufficiently small. Since

$$[\varphi_j^{*,a}(2^{js(\cdot)} f)(x)]^t = \sup_{y \in \mathbb{R}^n} \frac{2^{js(y)t} |\varphi_j * f(y)|^t}{(1 + 2^j |x - y|)^{at}},$$

it follows, from Lemma (6.2.23), that, for all $j \in \mathbb{Z}_+ \cap [(j_P \vee 0), \infty)$,

$$I_{P,j} \leq \sum_{k=1}^{\infty} \inf\left\{\eta_j \in (0,\infty) : \varrho_{p(\cdot)/t}\left(\frac{\chi_{P_k^n} 2^{js(\cdot)t}|\varphi_j * f(y)|^t}{2^{k\varepsilon t/q(\cdot)}[\phi(P_k^n)]^t \eta_j^{t/q(\cdot)}}\right) \leq 1\right\} + 2^{-\tilde{\sigma}[j-(j_P\vee 0)]}$$

$$= \sum_{k=1}^{\infty} 2^{-k\varepsilon} \inf\left\{\eta_j \in (0,\infty) : \varrho_{p(\cdot)/t}\left(\frac{\chi_{P_k^n} 2^{js(\cdot)t}|\varphi_j * f(y)|^t}{[\phi(P_k^n)]^t \eta_j^{t/q(\cdot)}}\right) \leq 1\right\} + 2^{-\tilde{\sigma}[j-(j_P\vee 0)]}$$

$$=: \widetilde{\delta_j^P},$$

where $P_k^n := 2^{k+1+n}P$ and $\tilde{\sigma} \in \left(0, \frac{at-n}{4(1/q_- - 1/q_+)}\right)$. From this, we further deduce that

$$\sum_{j=(j_P\vee 0)}^{\infty} I_{P,j} \lesssim \sum_{k=1}^{\infty} \frac{2^{-k\varepsilon}}{[\phi(P_k^n)]^t}\left\|\{2^{js(\cdot)}|\varphi_j * f|\}_{j\geq(j_P\vee 0)}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}(P_k^n)\right)}^t + 1$$

$$\lesssim \sum_{k=1}^{\infty} 2^{-k\varepsilon}\|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)}^t + 1 \lesssim 1,$$

which implies that (86) holds true. This finishes the proof of Theorem (6.2.24).

As applications of Theorem (6.2.24), we obtain more equivalent quasi-norms of Besov-type spaces with variable smoothness and integrability. To this end, for all $f \in \mathcal{S}'(\mathbb{R}^n)$, let

$$\left\|f|B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)\right\|_1 := \sup_{P\in\mathfrak{Q}} \frac{1}{\phi(P)}\left\|\{2^{js(\cdot)}|\varphi_j * f|\}_{j\geq 0}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}(P_k^n)\right)}$$

and

$$\left\|f|B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)\right\|_2 := \sup_{Q\in\mathfrak{Q}^*} \sup_{x\in Q}[\phi(Q)]^{-1}|Q|^{-s(x)/n}\left\|\chi_Q\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}|\varphi_{j_Q} * f(x)|.$$

Theorem (6.2.25) [298]: Let p, q, s, ϕ be as in Definition (6.2.6).

(i) Assume that $p_+ \in (0,\infty)$ and $c_1 \in \left(0, 2^{n/p_+}\right)$. Then $f \in B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\left\|f|B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)\right\|_1 < \infty$; moreover, there exists a positive constant C, independent of f, such that

$$\|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)} \leq \left\|f|B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)\right\|_1 \leq C\|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)}. \tag{87}$$

(ii) Assume that $p_- \in (0,\infty)$ and $c_1 \in \left(0, 2^{-n/p_-}\right)$. Then $f \in B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\left\|f|B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)\right\|_2 < \infty$; moreover, there exists a positive constant C, independent of f, such that

$$C^{-1}\|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)} \leq \left\|f|B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)\right\|_2 \leq C\|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)}. \tag{88}$$

Proof: Let $P \subset \mathbb{R}^n$ be a given dyadic cube and, for all $j \in \mathbb{Z}_+$ and $x \in \mathbb{R}^n$, $f_j(x) := 2^{js(x)}|\varphi_j * f(x)|$.

We first prove (i). Let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\left\|f|B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)\right\|_1 < \infty$. Then, by definitions, we easily find that $\|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)} \leq \left\|f|B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)\right\|_1$ and hence $f \in B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$. Conversely, let $f \in B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$. Then $f \in \mathcal{S}'(\mathbb{R}^n)$. To complete the proof of (i), it suffices to show the second inequality of (87).

When $p_+ \in (0,\infty)$, we have

$$\frac{1}{\phi(P)}\left\|\{f_j\}_{j\geq 0}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}(P)\right)} \lesssim \frac{1}{\phi(P)}\left\|\{f_j\}_{j=0}^{(j_P\vee 0)-1}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}(P)\right)} + \frac{1}{\phi(P)}\left\|\{f_j\}_{j\geq(j_P\vee 0)}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}(P)\right)}$$

$$=: I_{P,1} + I_{P,2}, \tag{89}$$

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where $I_{P,1} = 0$ if $j_P \leq 0$. Obviously, $I_{P,2} \lesssim \|f\|_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}$. To estimate $I_{P,1}$, without loss of generality, we may assume that $\|f\|_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} = 1$ and show that $I_{P,1} \lesssim 1$ in the case that $j_P > 0$. Observe that, for all $j \in \mathbb{Z}_+$ with $j \leq j_P - 1$, there exists a unique dyadic cube P_j such that $P \subset P_j$ and $\ell(P_j) = 2^{-j}$. It follows that, for all $x \in P$,

$$f_j(x) := 2^{js(x)}|\varphi_j * f(x)| \lesssim \inf_{y \in P_j} \varphi_j^{*,a}(2^{js(\cdot)}f)(y) \tag{90}$$

and, moreover,

$$\left\|[\phi(P)]^{-1}\chi_P f_j\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \left\|[\phi(P)]^{-1}\chi_P \inf_{y \in P_j} \varphi_j^{*,a}(2^{js(\cdot)}f)(y)\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

$$\lesssim \frac{\|\chi_P\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{P_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} [\phi(P)]^{-1}\left\|\varphi_j^{*,a}(2^{js(\cdot)}f)\right\|_{L^{p(\cdot)}(P_j)}$$

$$\lesssim \|f\|_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \frac{\|\chi_P\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{P_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \frac{\phi(P_j)}{\phi(P)}, \tag{91}$$

where we used Theorem (6.2.24) in the last inequality. On the other hand, by [297, Lemma 2.6], we find that

$$\left\|\chi_{P_j}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \gtrsim 2^{-j\frac{n}{p_+}}2^{j_P\frac{n}{p_+}}\|\chi_P\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

and, by (57) and (58), we see that $\phi(P) \gtrsim 2^{j\log_2 c_1}2^{-j_P \log_2 c_1}\phi\left(c_{P_j}, 2^{-j}\right)$. Thus, by (5.18), we further conclude that

$$\left\|[\phi(P)]^{-1}\chi_P f_j\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} 2^{(j-j_P)(n/p_+-\log_2 c_1)}, \tag{92}$$

which, together with (i) and (ii) of Lemma (6.2.16), implies that

$$\inf\left\{\lambda_j : \varrho_{p(\cdot)}\left([\phi(P)]^{-1}\lambda_j^{-1/q(\cdot)}\chi_P f_j\right) \leq 1\right\} \lesssim \left\|[\phi(P)]^{-1}\chi_P f_j\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{q_-} + \left\|[\phi(P)]^{-1}\chi_P f_j\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{q_+}$$

$$\lesssim 2^{(j-j_P)(n/p_+-\log_2 c_1)q_-} + 2^{(j-j_P)(n/p_+-\log_2 c_1)q_+}.$$

From this and $c_1 \in \left(0, 2^{n/p_+}\right)$, we deduce that there exists a positive constant C_0 such that

$$\sum_{j=0}^{j_P-1}\inf\left\{\lambda_j : \varrho_{p(\cdot)}\left(\frac{\chi_P f_j}{C_0\phi(P)\lambda_j^{1/q(\cdot)}}\right) \leq 1\right\} \leq 1,$$

namely, $\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))}\left(\left\{[C_0\phi(P)]^{-1}\chi_P f_j\right\}_{j=0}^{j_P-1}\right) \leq 1$ implies that $I_{P,1} \lesssim 1$. Therefore, by (89), we find that

$$\left\|f|B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\right\|_1 \lesssim \sup_{P \in \mathfrak{Q}}(I_{P,1} + I_{P,2}) \lesssim \|f\|_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)},$$

which completes the proof of the second inequality of (87) in the case $q_+ \in (0, \infty)$.

We now consider the case that $q_+ = \infty$. In this case, $q \equiv \infty$. From (92) and $c_1 \in \left(0, 2^{n/p_+}\right)$, we deduce that, for $j_P \in \mathbb{N}$,

$$\sup_{j \in \mathbb{Z}_+, j \leq j_P} [\phi(P)]^{-1}\left\|f_j\right\|_{L^{p(\cdot)}(P)} \lesssim \|f\|_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \sup_{j \in \mathbb{Z}_+, j \leq j_P} 2^{(j-j_P)(n/p_+-\log_2 c_1)} \lesssim \|f\|_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}.$$

By this, we know that

$$\left\|f|B^{s(\cdot),\phi}_{p(\cdot),\infty}(\mathbb{R}^n)\right\|_1 \lesssim \sup_{P \in \mathfrak{Q}}\left\{\sup_{j \in \mathbb{Z}_+, j \leq (j_P \vee 0)} \frac{\|f_j\|_{L^{p(\cdot)}(P)}}{\phi(P)} + \sup_{j \in \mathbb{Z}_+, j \geq (j_P \vee 0)} \frac{\|f_j\|_{L^{p(\cdot)}(P)}}{\phi(P)}\right\} \lesssim \|f\|_{B^{s(\cdot),\phi}_{p(\cdot),\infty}(\mathbb{R}^n)},$$

which completes the proof of the second inequality of (87) in the case that $q_+ = \infty$ and hence (i) of Theorem (6.2.25).

Next, we show (ii). Let $f \in B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. Then $f \in \mathcal{S}'(\mathbb{R}^n)$. On the other hand, for all $Q \in \mathfrak{Q}^*$ and $x \in Q$, by Theorem (6.2.24) and (90), we easily see that

$$\frac{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\phi(Q)} f_{jQ}(x) \lesssim \frac{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\phi(Q)} \inf_{y \in Q} \varphi_{jQ}^{*,a}(2^{jQs(\cdot)}f)(y) \lesssim [\phi(Q)]^{-1} \left\| \varphi_{jQ}^{*,a}(2^{jQs(\cdot)}f) \right\|_{L^{p(\cdot)}(Q)}$$

$$\lesssim \|f\|_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}.$$

This implies that $\left\| f | B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \right\|_2 \lesssim \|f\|_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} < \infty$.

Conversely, let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\left\| f | B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \right\|_2 < \infty$. We need to show the first inequality of (88). To this end, for all $j \geq (j_P \vee 0)$ and $x \in \mathbb{R}^n$, $\mathfrak{Q}^*_{P,j} := \{Q \in \mathfrak{Q}^* : Q \subset P, \ell(Q) = 2^{-j}\}$ and, for all $Q \in \mathfrak{Q}^*_{P,j}$, let

$$g(Q,P)(x) := [\phi(P)]^{-1}\phi(Q)\|\chi_Q\|^{-1}_{L^{p(\cdot)}(\mathbb{R}^n)}\chi_Q(x).$$

When $q_+ \in (0,\infty)$, by [297, Lemma 2.6], (57) and (58), we find that

$$\left\| \sum_{Q \in \mathfrak{Q}^*_{P,j}} g(Q,P) \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim 2^{(j-j_P)(\log_2 c_1 + n/p_-)},$$

which, combined with (i) and (ii) of Lemma (6.2.16), implies that

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\left\{\sum_{Q \in \mathfrak{Q}^*_{P,j}} g(Q,P)\right\}\right) = \sum_{j=(j_P \vee 0)}^{\infty} \inf\left\{\lambda_j \in (0,\infty) : \varrho_{p(\cdot)}\left(\sum_{Q \in \mathfrak{Q}^*_{P,j}} \frac{g(Q,P)}{\lambda_j^{1/q(\cdot)}}\right) \leq 1\right\}$$

$$\leq \sum_{j=(j_P \vee 0)}^{\infty} \left[\left\| \sum_{Q \in \mathfrak{Q}^*_{P,j}} g(Q,P) \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{q_-} + \left\| \sum_{Q \in \mathfrak{Q}^*_{P,j}} g(Q,P) \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{q_+} \right]$$

$$\lesssim \sum_{j=(j_P \vee 0)}^{\infty} \left[2^{(j-j_P)(\log_2 c_1 + n/p_-)q_-} + 2^{(j-j_P)(\log_2 c_1 + n/p_-)q_+} \right] \lesssim 1.$$

By this, we conclude that

$$\left\| \left\{ \sum_{Q \in \mathfrak{Q}^*_{P,j}} g(Q,P) \right\}_{j \geq (j_P \vee 0)} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \lesssim 1.$$

Therefore,

$$\left\| \{[\phi(P)]^{-1}\chi_P f_j\}_{j \geq (j_P \vee 0)} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \lesssim 1 \left\| \left\{ \frac{1}{\phi(P)} \sum_{Q \in \mathfrak{Q}^*_{P,j}} \chi_Q f_j \right\}_{j \geq (j_P \vee 0)} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))}$$

$$\lesssim \left\| f | B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \right\|_2 \left\| \left\{ \sum_{Q \in \mathfrak{Q}^*_{P,j}} g(Q,P) \right\}_{j \geq (j_P \vee 0)} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \lesssim \left\| f | B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \right\|_2,$$

which implies that the first inequality of (88) holds true in the case that $q_+ \in (0,\infty)$. The proof of the case that $q_+ = \infty$ is similar and more simple, the details being omitted. This finishes the proof of (ii) and hence Theorem (6.2.25).

As another application of Theorem (6.2.24), we obtain the following conclusion.

Proposition (6.2.26) [298]: Let p, q, s and ϕ be as in Definition (6.2.6). Then

$$S(\mathbb{R}^n) \hookrightarrow B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n) \hookrightarrow S'(\mathbb{R}^n). \tag{93}$$

Proof: By Proposition (6.2.19), we see that $B_{p(\cdot),q_-}^{s(\cdot),\phi}(\mathbb{R}^n) \hookrightarrow B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n) \hookrightarrow B_{p(\cdot),\infty}^{s(\cdot),\phi}(\mathbb{R}^n)$. Thus, to prove (93), it suffices to show that

$$S(\mathbb{R}^n) \hookrightarrow B_{p(\cdot),q_-}^{s(\cdot),\phi}(\mathbb{R}^n) \quad \text{and} \quad B_{p(\cdot),\infty}^{s(\cdot),\phi}(\mathbb{R}^n) \hookrightarrow S'(\mathbb{R}^n) \tag{94}$$

The first embedding of (94) can be obtained by an argument similar to that used in [294]. We give the proof of the second one. To this end, we only need to show that there exists an $M \in \mathbb{N}$ such that, for all $f \in B_{p(\cdot),\infty}^{s(\cdot),\phi}(\mathbb{R}^n)$ and $h \in S(\mathbb{R}^n)$, $|\langle f, h \rangle| \lesssim \|h\|_{S_{M+1}(\mathbb{R}^n)} \|f\|_{B_{p(\cdot),\infty}^{s(\cdot),\phi}(\mathbb{R}^n)}$.

Let φ, ψ, Φ and Ψ be as in (61). Then, by the Calderón reproducing formula in [221], together with [221, Lemma 2.4], we find that

$$|\langle f, h \rangle| \leq \int_{\mathbb{R}^n} |\Phi * f(x)||\Psi * h(x)|dx + \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} |\varphi_j * f(x)||\psi_j * h(x)|dx$$

$$\lesssim \|h\|_{S_{M+1}(\mathbb{R}^n)} \sum_{j=0}^{\infty} 2^{-jM} \sum_{k \in \mathbb{Z}^n} \int_{Q_{0k}} |\varphi_j * f(x)|(1 + |x|)^{-(n+M)}dx, \tag{95}$$

where we used φ_0 to replace . Notice that, for any $j \in \mathbb{Z}_+$, $k \in \mathbb{Z}^n$, $a \in (0, \infty)$ and $y \in Q_{jk}$,

$$\int_{Q_{0k}} |\varphi_j * f(x)|dx \lesssim \varphi_j^{*,a}(2^{js(\cdot)}f)(y)2^{ja} \int_{Q_{0k}} 2^{-js(x)}(1 + 2^j|x| + 2^j|y|)^a dx$$

$$\lesssim 2^{-js_-}\varphi_{j_Q}^{*,a}(2^{js(\cdot)}f)(y)2^{ja}(1 + |k|)^a.$$

It follows that

$$\int_{Q_{0k}} |\varphi_j * f(x)|dx \lesssim 2^{j(a-s_-)}(1 + |k|)^a \inf_{y \in Q_{jk}} \varphi_{j_Q}^{*,a}(2^{js(\cdot)}f)(y),$$

which, combined with (95), Theorem (6.2.24), Lemmas (6.2.10) and (6.2.11), implies that

$$|\langle f, h \rangle| \lesssim \|h\|_{S_{M+1}(\mathbb{R}^n)} \sum_{j=0}^{\infty} 2^{-j(M+s_--a)} \sum_{k \in \mathbb{Z}^n} (1 + |k|)^{a-n-M} \frac{\left\|\varphi_{j_Q}^{*,a}(2^{js(\cdot)}f)\right\|_{L^{p(\cdot)}(Q_{jk})}}{\left\|\chi_{Q_{jk}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}}$$

$$\lesssim \|h\|_{S_{M+1}(\mathbb{R}^n)} \|f\|_{B_{p(\cdot),\infty}^{s(\cdot),\phi}(\mathbb{R}^n)} \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^n} \frac{2^{-j(M+s_--a)}}{(1 + |k|)^{M+n-a}} \frac{\phi(Q_{jk})}{\left\|\chi_{Q_{jk}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}}$$

$$\lesssim \|h\|_{S_{M+1}(\mathbb{R}^n)} \|f\|_{B_{p(\cdot),\infty}^{s(\cdot),\phi}(\mathbb{R}^n)},$$

where a is as in Theorem (6.2.24) and M is large enough. This finishes the proof of Proposition (6.2.26).

Definition (6.2.27) [298]: Let $k \in \mathbb{Z}_+$ and $L \in \mathbb{Z}$. A measurable function a_Q on $\mathbb{R}^n$ is called a (K, L)-smooth atom supported near $Q := Q_{jk} \in \mathfrak{Q}$ if it satisfies the following conditions:

(A1) (support condition) supp $a_Q \subset 3Q$;

(A2) (vanishing moment) when $j \in \mathbb{N}$, $\int_{\mathbb{R}^n} x^\gamma a_Q(x)dx = 0$ for all $\gamma \in \mathbb{Z}_+^n$ with $|\gamma| < L$;

(A3) (smoothness condition) for all multi-indices $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq K$, $|D^\alpha a_Q(x)| \leq 2^{(|\alpha|+n/2)j}$.

A collection $\{a_Q\}_{Q \in \mathfrak{Q}^*}$ is called a family of (K, L)-smoothness atoms, if each a_Q is a (K, L)-smooth atom supported near Q.

We point out that, if $L \leq 0$, then the vanishing moment condition (A2) is avoid.

Lemma (6.2.28) [298]: Let $\{\varphi_j\}_{j\in\mathbb{Z}_+}$ be as in Definition (6.2.6) and $a_{Q_{vk}}$ with $v \in \mathbb{Z}_+$ and $k \in \mathbb{Z}^n$ be a (K, L)-smooth atom. Then, for all $M \in (0, \infty)$, there exist positive constants C_1 and C_2 such that, for all $x \in \mathbb{R}^n$, when $j \le v$,

$$\left|\varphi_j * a_{Q_{vk}}(x)\right| \le C_1 2^{vn/2} 2^{-(v-j)(L+n)}\left(1 + 2^j\left|x - x_{Q_{vk}}\right|\right)^{-M}$$

and, when $j > v$,

$$\left|\varphi_j * a_{Q_{vk}}(x)\right| \le C_2 2^{vn/2} 2^{-(j-v)K}\left(1 + 2^v\left|x - x_{Q_{vk}}\right|\right)^{-M}.$$

Lemma (6.2.29) [298]: Let $a \in (0, 1)$, $J \in \mathbb{Z}$, $q \in (0, \infty]$ and $\{\varepsilon_k\}_{k\in\mathbb{Z}_+}$ be a sequence of positive real numbers. For all $k \in [J \vee 0, \infty)$, let $\delta_k := \sum_{j=(J\vee 0)}^{k} a^{j-k}\varepsilon_j$ and $\eta_k := \sum_{j=k}^{\infty} a^{k-j}\varepsilon_j$. Then there exists a positive constant C, depending only on a and q, such that

$$\left(\sum_{k=(J\vee 0)}^{\infty} \delta_k^q\right)^{1/q} + \left(\sum_{k=(J\vee 0)}^{\infty} \eta_k^q\right)^{1/q} \le C\left(\sum_{k=(J\vee 0)}^{\infty} \varepsilon_k^q\right)^{1/q}.$$

Theorem (6.2.30) [298]: Let p, q, s and ϕ be as in Definition (6.2.6).
(i) Let $K \in (s_+ + \log_2 c_1, \infty)$ and

$$L \in (n/\min\{1, p_-\} - n - s_-, \infty). \tag{96}$$

Suppose that $\{a_Q\}_{Q\in\mathcal{Q}^*}$ is a family of (K, L)-smooth atoms and $t := \{t_Q\}_{Q\in\mathcal{Q}^*} \in b_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$. Then $f := \sum_{Q\in\mathcal{Q}^*} t_Q a_Q$ converges in $\mathcal{S}'(\mathbb{R}^n)$ and $\|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)} \le C\|t\|_{b_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)}$ with C being a positive constant independent of t.

(ii) Conversely, if $f \in B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$, then, for any given $K, L \in \mathbb{Z}_+$, there exist sequences $t := \{t_Q\}_{Q\in\mathcal{Q}^*} \subset \mathbb{C}$ and $\{a_Q\}_{Q\in\mathcal{Q}^*}$ of (K, L)-smooth atoms such that $f := \sum_{Q\in\mathcal{Q}^*} t_Q a_Q$ in $\mathcal{S}'(\mathbb{R}^n)$ and $\|t\|_{b_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)} \le C\|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)}$ with C being a positive constant independent of f.

Proof: The proof of (ii) is similar to that of [221, Theorem 3.3] (see also [106]). Indeed, by repeating the argument that used in [221], therein replaced by Lemma (6.2.17), we can prove (ii), the details being omitted.

Next we prove (i) by two steps. First, we show that $f := \sum_{Q\in\mathcal{Q}^*} t_Q a_Q$ converges in $\mathcal{S}'(\mathbb{R}^n)$. To this end, it suffices to prove that

$$\lim_{N\to\infty, \Lambda\to\infty} \sum_{j=0}^{N} \sum_{k\in\mathbb{Z}^n, |k|\le\Lambda} t_{Q_{jk}} a_{Q_{jk}} \tag{97}$$

exists in $\mathcal{S}'(\mathbb{R}^n)$. By (96), we find that there exists $r \in (0, \min\{1, p_-\})$ such that $s_- + n/p_- (r - 1) > -L$. Let, for all $x \in \mathbb{R}^n$, $\tilde{p}(x) := p(x)/r$ and $\tilde{s}$ be a measurable function on $\mathbb{R}^n$ such that $s(x) - n/p(x) = \tilde{s}(x) - n/\tilde{p}(x)$. Then $\tilde{s}_- \ge s_- + n/p_- (r - 1) > -L$. Therefore, by Proposition (6.2.20) and an argument similar to that used in [294], we conclude that there exist $\delta_0 \in (\log_2 c_1, \infty)$, $a \in (n, \infty)$, $c_0 \in \mathbb{N}$ and $R \in (0, \infty)$ being large enough such that, for all $h \in \mathcal{S}(\mathbb{R}^n)$ and $j \in \mathbb{Z}_+$,

$$\left|\int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}^n,|k|\leq\Lambda} t_{Q_{jk}} a_{Q_{jk}}(y) h(y) dy\right|$$

$$\lesssim 2^{-j(L+\tilde{s}_-)} \sum_{v=0}^{\infty} 2^{-v\delta_0} \sum_{i=0}^{\infty} 2^{-i(R-L-a)} \left\|\sum_{k\in\mathbb{Z}^n} 2^{j\tilde{s}(\cdot)} \left|t_{Q_{jk}}\right| \tilde{\chi}_{Q_{jk}}\right\|_{L^{\tilde{p}(\cdot)}\left(Q(0,2^{i+v+c_0})\right)}$$

$$\lesssim 2^{-j(L+\tilde{s}_-)} \sum_{v=0}^{\infty} 2^{-v\delta_0} \sum_{i=0}^{\infty} 2^{-i(R-L-a)} \phi\left(Q(0,2^{i+v+c_0})\right) \|t\|_{b_{\tilde{p}(\cdot),\infty}^{\tilde{s}(\cdot),\phi}(\mathbb{R}^n)}$$

$$\lesssim 2^{-j(L+\tilde{s}_-)} \sum_{v=0}^{\infty} 2^{-v(\delta_0-\log_2 c_1)} \sum_{i=0}^{\infty} 2^{-i(R-L-a-\log_2 c_1)} \|t\|_{b_{p(\cdot),\infty}^{s(\cdot),\phi}(\mathbb{R}^n)}$$

$$\lesssim 2^{-j(L+\tilde{s}_-)} \|t\|_{b_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)}.$$

By this and the fact that $L > -\tilde{s}_-$, we find that the limit of (97) exists in $\mathcal{S}'(\mathbb{R}^n)$.

Second, we prove that

$$\|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)} \lesssim \|t\|_{b_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)}. \tag{98}$$

without loss of generality, we may assume that $\|t\|_{b_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)} = 1$ and show $\|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)} \lesssim 1$

Case I) $q_+ \in (0,\infty)$. We see that, for all $R \in \mathcal{D}_0(\mathbb{R}^n)$,

$$\frac{1}{\phi(R)} \left\|\left\{\sum_{Q\in\mathbb{Q}^*,\ell(Q)=2^{-v}} |Q|^{-\frac{s(\cdot)}{n}} |t_Q| \tilde{\chi}_Q\right\}_{v\geq(j_R\vee0)}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}(R)\right)} \lesssim 1 \tag{99}$$

with implicit positive constant independent of R. Since $f = \sum_{Q\in\mathbb{Q}^*} t_Q a_Q$ in $\mathcal{S}'(\mathbb{R}^n)$, it follows that, for all $P \in \mathcal{Q}$,

$$\varphi_j * f = \left\{\sum_{v=0}^{(j_P\vee0)-1} + \sum_{v=(j_P\vee0)}^{j} + \sum_{v=j+1}^{\infty}\right\} \sum_{\ell(Q)=2^{-v}} t_Q \varphi_j * a_Q =: S_{j,1} + S_{j,2} + S_{j,3},$$

Where $\sum_{v=0}^{(j_P\vee0)-1} \cdots = 0$ if $j_P \leq 0$. Thus, we find that

$$I_P := \frac{1}{\phi(P)} \left\|\{2^{js(\cdot)}\varphi_j * f\}_{j\geq(j_P\vee0)}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}(P)\right)} \lesssim \sum_{i=1}^{3} \frac{1}{\phi(P)} \left\|\{2^{js(\cdot)}S_{j,i}\}_{j\geq(j_P\vee0)}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}(P)\right)}$$

$$=: I_{P,1} + I_{P,2} + I_{P,3}.$$

In what follows, let $r \in (0, \min\{1, p_-\})$ satisfy $L + n - n/r + s_- > 0$.

We show that $I_{P,1} \lesssim 1$. To this end, it suffices to consider the case that $j_P > 0$ and prove that there exists a positive constant C such that

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})} \left(\left\{\frac{\chi_P 2^{js(\cdot)}}{C\phi(P)} \sum_{v=0}^{j_P-1} \sum_{\ell(Q)=2^{-v}} |t_Q||\varphi_j * a_Q|\right\}_{j\geq j_P}\right) \leq 1,$$

which is equivalent to show that

$$J_{P,1} := \sum_{j=j_P}^{\infty} \left\|\left[\frac{\chi_P}{C\phi(P)} 2^{js(\cdot)} \sum_{v=0}^{j_P-1} \sum_{\ell(Q)=2^{-v}} |t_Q||\varphi_j * a_Q|\right]^{q(\cdot)}\right\|_{L^{p(\cdot)/q(\cdot)}(\mathbb{R}^n)} \lesssim 1.$$

By Lemma (6.2.28) and (69), we find that

$$J_{P,1} \lesssim \sum_{j=j_P}^{\infty} \left\| \left[\frac{\chi_P}{\{\phi(P)\}^r} 2^{js(\cdot)r} \sum_{v=0}^{j_P-1} \sum_{k\in\mathbb{Z}^n} |t_{Q_{vk}}|^r |Q_{vk}|^{-r/2} \right. \right.$$

$$\left. \left. \times\, 2^{(v-j)Kr}\big(1 + 2^v|\cdot - x_{Q_{vk}}|\big)^{-Mr} \right]^{q(\cdot)} \right\|_{L^{p(\cdot)/rq(\cdot)}(\mathbb{R}^n)}^{1/r} \lesssim 1, \tag{100}$$

where $M \in (0,\infty)$ is large enough. On the other hand, by the proof of [294, Theorem 3.8(i)], we know that, for all $v, j \in \mathbb{Z}_+$ with $v \leq j$ and $x \in P$,

$$2^{js(x)r} \sum_{k\in\mathbb{Z}^n} |t_{Q_{vk}}|^r |Q_{vk}|^{-\frac{r}{2}} \times 2^{(v-j)Kr}\big(1 + 2^v|x - x_{Q_{vk}}|\big)^{-Mr}$$

$$\lesssim 2^{(v-j)(K-s_+)r} \sum_{i=0}^{\infty} 2^{-i\left(M - a - \frac{C_{\log}(s)}{r}\right)r} \eta_{v,ar}$$

$$* \left(\left[\sum_{k\in\mathbb{Z}^n} |t_{Q_{vk}}| 2^{vs(\cdot)} \tilde{\chi}_{Q_{vk}} \chi_{Q(c_P, 2^{i-v+c_0})} \right]^r \right)(x),$$

where $a \in (n/r, \infty)$, c_P is the center of P and $c_0 \in \mathbb{N}$ independent of x, P, i, v and k. From this, (100), and (i) and (ii) of Lemma (6.2.16), we deduce that $J_{P,1} \lesssim \sum_{j=j_P}^{\infty} \left[\left(J_{P,1}^j\right)^{q_-} + \left(J_{P,1}^j\right)^{q_+} \right]$, where

$$J_{P,1}^j := \left\| \frac{\chi_P}{[\phi(P)]^r} \sum_{v=0}^{j_P-1} 2^{(v-j)(K-s_+)r} \sum_{i=0}^{\infty} 2^{-i(M-a-C_{\log}(s)/r)r} \times \eta_{v,ar} \right.$$

$$\left. * \left(\sum_{k\in\mathbb{Z}^n} |t_{Q_{vk}}|^r 2^{vs(\cdot)r} |Q_{vk}|^{-r/2} \chi_{Q_{vk}} \chi_{Q(c_P, 2^{i-v+c_0})} \right) \right\|_{L^{p(\cdot)/r}(\mathbb{R}^n)}^{1/r}$$

we find that

$$J_{P,1}^j \lesssim \left\{ \frac{1}{[\phi(P)]^r} \sum_{v=0}^{j_P-1} 2^{(v-j)(K-s_+)r} \sum_{i=0}^{\infty} 2^{-i(M-a-C_{\log}(s)/r)r} \right.$$

$$\left. \times \left\| \sum_{k\in\mathbb{Z}^n} |t_{Q_{vk}}|^r 2^{vs(\cdot)r} |Q_{vk}|^{-r/2} \chi_{Q_{vk}} \chi_{Q(c_P, 2^{i-v+c_0})} \right\|_{L^{\frac{p(\cdot)}{r}}(\mathbb{R}^n)} \right\}^{1/r}$$

$$\lesssim \left\{ \sum_{v=0}^{j_P-1} 2^{(v-j)(K-s_+)r} \sum_{i=0}^{\infty} 2^{-i(M-a-C_{\log}(s)/r)r} \frac{\left[\phi\big(Q(c_P, 2^{i-v+c_0})\big)\right]^r}{[\phi(P)]^r} \right\}^{1/r}.$$

By this, (57) and the fact that $K \in (s_+ + \log_2 c_1, \infty)$, we know that

$$\sum_{j=j_P}^{\infty} \big(J_{P,1}^j\big)^{q_-} \lesssim 2^{j_P q - \log_2 c_1} \sum_{j=j_P}^{\infty} \left\{ 2^{j(s_+ - K)} \sum_{v=0}^{j_P-1} 2^{(K-s_+ + \log_2 c_1)vr} \right\}^{q_-/r} \lesssim 1$$

and $\sum_{j=j_P}^{\infty} \big(J_{P,1}^j\big)^{q_+} \lesssim 1$, where M is chosen large enough such that $M > a + C_{\log}(s)/r + \log_2 c_1$, which implies $I_{P,1} \lesssim 1$. This is a desired estimate.

We now estimate $I_{P,2}$. By Lemma (6.2.28), we see that, for all $M \in (0,\infty)$ and $x \in \mathbb{R}^n$,

$$\sum_{k\in\mathbb{Z}^n} 2^{js(x)}|t_{Q_{vk}}||\varphi_j * a_{Q_{vk}}(x)| \lesssim 2^{(v-j)(K-s_+)}\sum_{k\in\mathbb{Z}^n}|Q_{vk}|^{-s(x)/n+1/2}|t_{Q_{vk}}|(1+2^v|x-x_{Q_{vk}}|)^{-M}$$

and hence, for all $r \in (0, \min\{1/q_+, p_-/q_+\})$,

$$\left\|\left[\frac{\chi_P}{\phi(P)}2^{js(\cdot)}S_{j,2}\right]^{q(\cdot)}\right\|_{L^{p(\cdot)/q(\cdot)}(\mathbb{R}^n)}$$

$$\lesssim \left\{\sum_{v=(j_P\vee 0)}^{j_P-1} 2^{(v-j)q_-(K-s_+)}\left\|\left[\frac{\chi_P}{\phi(P)}\sum_{\ell(Q)=2^{-v}}\frac{|Q|^{-s(\cdot)/n+1/2}|t_Q|}{(1+2^v|\cdot-x_Q|)^M}\right]^{rq(\cdot)}\right\|_{L^{p(\cdot)/rq(\cdot)}(\mathbb{R}^n)}\right\}^{1/r}. \qquad (101)$$

We claim that there exists a positive constant c such that

$$\left\|\left[\frac{c\chi_P}{\phi(P)}\sum_{\ell(Q)=2^{-v}}\frac{|Q|^{-s(\cdot)/n+1/2}|t_Q|}{(1+2^v|\cdot-x_Q|)^M}\right]^{rq(\cdot)}\right\|_{L^{p(\cdot)/rq(\cdot)}(\mathbb{R}^n)}$$

$$\leq \sum_{i=0}^{\infty} 2^{-i\tau}\left\|\left[\frac{\chi_{Q_i^0}}{\phi(Q_i^0)}\sum_{\ell(Q)=2^{-v}}|Q|^{-s(\cdot)/n}|t_Q|\tilde{\chi}_Q\right]^{rq(\cdot)}\right\|_{L^{p(\cdot)/rq(\cdot)}(\mathbb{R}^n)} + 2^{-v} =: \delta_v^P,$$

$$(102)$$

where $Q_i^0 := Q(c_P, 2^{i-j_P+c_0})$ with some $c_0 \in \mathbb{N}$ and $\tau \in (0, \infty)$.

From the above claim, (101), Lemma (6.2.29), the Minkowski inequality and (99), we deduce that

$$\sum_{j=(j_P\vee 0)}^{\infty}\left\|\left[\frac{\chi_P}{\phi(P)}2^{js(\cdot)}S_{j,2}\right]^{q(\cdot)}\right\|_{L^{p(\cdot)/q(\cdot)}(\mathbb{R}^n)}$$

$$\lesssim \sum_{j=(j_P\vee 0)}^{\infty}\left\{\sum_{i=0}^{0} 2^{-i\tau}\left\|\left[\frac{\chi_{Q_i^0}}{\phi(Q_i^0)}\sum_{\ell(Q)=2^{-j}}|Q|^{-s(\cdot)/n}|t_Q|\tilde{\chi}_Q\right]^{rq(\cdot)}\right\|_{L^{p(\cdot)/rq(\cdot)}(\mathbb{R}^n)}\right\}^{1/r}$$

$$+ \sum_{j=0}^{\infty} 2^{-j/r}$$

$$\lesssim \left\{\sum_{i=0}^{\infty} 2^{-i\tau}\left(\sum_{j=(j_P\vee 0)}^{\infty}\left\|\left[\frac{\chi_{Q_i^0}}{\phi(Q_i^0)}\sum_{\ell(Q)=2^{-j}}|Q|^{-s(\cdot)/n}|t_Q|\tilde{\chi}_Q\right]^{q(\cdot)}\right\|_{L^{p(\cdot)/q(\cdot)}(\mathbb{R}^n)}\right)^r\right\}^{1/r} + 1$$

$$\lesssim \left\{\sum_{i=0}^{\infty} 2^{-i\tau}\right\}^{1/r} + 1 \lesssim 1,$$

which, together with Lemma (6.2.3) again, implies that $I_{P,2} \lesssim 1$.

Let us prove (102) now. Obviously, it suffices to show that

$$\left\|[\delta_v^P]^{-1}\left[\frac{c\chi_P}{\phi(P)}\sum_{\ell(Q)=2^{-v}}\frac{|Q|^{-s(\cdot)/n+1/2}|t_Q|}{(1+2^v|\cdot-x_Q|)^M}\right]^{rq(\cdot)}\right\|_{L^{p(\cdot)/rq(\cdot)}(\mathbb{R}^n)} \leq 1,$$

which, via Lemma (6.2.16), is a consequence of

$$\mathcal{A} := \left\|[\delta_v^P]^{-1/rq(\cdot)}\left[\frac{\chi_P}{\phi(P)}\sum_{\ell(Q)=2^{-v}}\frac{|Q|^{-s(\cdot)/n+1/2}|t_Q|}{(1+2^v|\cdot-x_Q|)^M}\right]^{rq(\cdot)}\right\|_{L^{p(\cdot)/rq(\cdot)}(\mathbb{R}^n)} \lesssim 1.$$

Taking $t \in (0, \min\{1, p_-\})$ and using some arguments similar to those used in [261], we conclude that, for all $x \in \mathbb{R}^n$,

$$\sum_{\ell(Q)=2^{-v}} [\delta_v^P]^{-1/rq(x)} \frac{\chi_P(x)}{\phi(P)} \sum_{\ell(Q)=2^{-v}} \frac{|Q|^{-s(\cdot)/n+1/2}|t_Q|}{(1+2^v|x-x_Q|)^M}$$

$$\lesssim \sum_{i=0}^{(v-c_0)\vee 0} 2^{i\zeta} \left\{ \mathcal{M}\left(\sum_{j=(j_P\vee 0)}^{\infty} \left[\frac{\chi_{Q_i^0}}{\phi(Q_i^0)} \sum_{\ell(Q)=2^{-j}} \frac{2^{vs(\cdot)}}{[\delta_v^P]^{1/rq(\cdot)}} |t_Q|\tilde{\chi}_Q \right]^t \right) \right\}^{1/t}$$

$$+ \sum_{i=(v-c_0)\vee 0}^{\infty} 2^{i\vartheta} \cdots, \tag{103}$$

where $\zeta := -M + \frac{n}{t} + \frac{2}{r}C_{\log}(q) + C_{\log}(s)$ and $\vartheta := -M + n/t + 2/r(1/q_- - 1/q_+) + s_+ - s_-$. Taking M large enough such that
$M > \max\{n/t + 2/r\, C_{\log}(q) + C_{\log}(s) + \log_2 c_1, 2/r(1/q_- - 1/q_+) + s_+ - s_- + \log_2 c_1\} + r$,
then, by (57), Lemmas (6.2.22) and (6.2.16), we know that

$$\sum_{i=0}^{(v-c_0)\vee 0} 2^{i\varsigma t} \left\| \mathcal{M}\left(\sum_{j=(j_P\vee 0)}^{\infty} \frac{\chi_{Q_i^0}}{\phi(P)} \left[\sum_{\ell(Q)=2^{-j}} \frac{2^{vs(\cdot)}}{[\delta_v^P]^{1/rq(\cdot)}} |t_Q|\tilde{\chi}_Q \right]^t \right) \right\|_{L^{p(\cdot)/t}(\mathbb{R}^n)}$$

$$\lesssim \sum_{i=0}^{(v-c_0)\vee 0} 2^{it(\varsigma+\log_2 c_1)} \left\| \frac{\chi_{Q_i^0}}{\phi(Q_i^0)} \sum_{\ell(Q)=2^{-j}} \frac{2^{vs(\cdot)}}{[\delta_v^P]^{1/rq(\cdot)}} |t_Q|\tilde{\chi}_Q \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^t$$

$$\lesssim \sum_{i=0}^{(v-c_0)\vee 0} 2^{it(\varsigma+\log_2 c_1)} \left\| 1/\delta_v^P \left[\frac{\chi_{Q_i^0}}{\phi(Q_i^0)} \sum_{\ell(Q)=2^{-j}} \frac{2^{vs(\cdot)}}{[\delta_v^P]^{1/rq(\cdot)}} |t_Q|\tilde{\chi}_Q \right]^{rq(\cdot)} \right\|_{L^{p(\cdot)/rq(\cdot)}(\mathbb{R}^n)}^{t/rq_+}$$

$$\lesssim \sum_{i=0}^{\infty} 2^{it(\varsigma+\log_2 c_1+\tau)} \lesssim 1,$$

where we used the definition of δ_v^P in the penultimate inequality and, similarly,

$$\sum_{i=(v-c_0)\vee 0}^{\infty} 2^{i\vartheta t} \left\| \mathcal{M}\left(\left[\frac{\chi_{Q_i^0}}{\phi(P)} \sum_{\ell(Q)=2^{-j}} \frac{2^{vs(\cdot)}}{[\delta_v^P]^{1/rq(\cdot)}} |t_Q|\tilde{\chi}_Q \right]^t \right) \right\|_{L^{p(\cdot)/t}(\mathbb{R}^n)} \lesssim 1.$$

From this and (103), we deduce that $\mathcal{A} \lesssim 1$, which implies that (102) holds true and then completes the proof that $I_{P,2} \lesssim 1$.

We next prove that $I_{P,3} \lesssim 1$. To this end, it suffices to show that

$$\varrho_{\ell^{q(\cdot)/r}(L^{p(\cdot)/r})} \left(\left\{ \frac{\chi_P}{\tilde{C}\phi(P)} 2^{js(\cdot)} \sum_{v=(j_P\vee 0)}^{\infty} \sum_{\ell(Q)=2^{-v}} |t_Q||\varphi_j * a_Q| \right\}_{j\geq(j_P\vee 0)}^r \right) \leq 1$$

for some positive constant $\tilde{C}$ large enough independent of P, which, by Definition (6.2.5), is equivalent to show that $\sum_{j=(j_P\vee 0)}^{\infty} \sum_{\ell(Q)=2^{-v}} Y_j^P \lesssim 1$, where, for all $j \in \mathbb{Z}_+ \cap [j_P \vee 0, \infty)$,

$$Y_j^P := \inf\left\{ \lambda_j \in (0,\infty) : \varrho_{p(\cdot)/r} \left(\frac{\chi_P[2^{js(\cdot)}\sum_{v=j}^{\infty}\sum_{\ell(Q)=2^{-v}}|t_Q||\varphi_j * a_Q|]^r}{\tilde{C}[\phi(P)\lambda_j^{1/q(\cdot)}]^r} \right) \leq 1 \right\}.$$

We claim that, for all $P \in \mathfrak{Q}$ and $j \in \mathbb{Z}_+ \cap [j_P \vee 0, \infty)$,

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$$Y_j^P \leq 2^{-j} + \sum_{v=j}^{\infty} 2^{(j-v)d} \sum_{i=0}^{\infty} 2^{-i\tilde{d}} \times \inf\left\{\xi_v \in (0,\infty) : \varrho_{\frac{p(\cdot)}{r}}\left(\frac{\chi_{P_i}\left[\sum_{\ell(Q)=2^{-v}}|t_Q|2^{-vs(\cdot)}\chi_Q\right]^r}{\left[\phi(P_i)\xi_v^{1/q(\cdot)}\right]^r}\right) \leq 1\right\}$$

$$=: 2^{-j} + \sum_{v=j}^{\infty} 2^{(j-v)d} \sum_{i=0}^{\infty} 2^{-i\tilde{d}} Y_{v,2}^P =: \delta_j^P, \tag{104}$$

where $P_i := \mathcal{Q}(c_P, 2^{i-j_P+c_0})$ with $c_0 \in \mathbb{N}$, d is chosen such that $L + n - n/r + s_- - d/q_+ > 0$ and $\tilde{d} \in (0,\infty)$.

From the above claim, (99) and (54), we deduce that

$$\sum_{j=(j_P \vee 0)}^{\infty} Y_j^P \lesssim 1 + \sum_{v=(j_P \vee 0)}^{\infty} \sum_{j=(j_P \vee 0)}^{v} 2^{(j-v)d} \sum_{i=0}^{\infty} 2^{-i\tilde{d}} Y_{v,2}^P \lesssim 1 + \sum_{i=0}^{\infty} 2^{-i\tilde{d}} \sum_{v=(j_P \vee 0)}^{\infty} Y_{v,2}^P \lesssim 1,$$

which implies that $I_{P,3} \lesssim 1$ and $\delta_j^P \in [2^{-j}, 2^{-j} + \theta]$ for some $\theta \in [0,\infty)$

Therefore, to complete the estimate for $I_{P,3}$, it remains to prove the above claim (104). To this end, it suffices to show that, for all $j \in \mathbb{Z}_+ \cap [j_P \vee 0, \infty)$,

$$\inf\left\{\tilde{\lambda}_j \in (0,\infty) : \varrho_{\frac{p(\cdot)}{r}}\left(\frac{\chi_P\left[2^{js(\cdot)}\sum_{v=j}^{\infty}\sum_{\ell(Q)=2^{-v}}|t_Q|\,|\varphi_j * a_Q|\right]^r}{\left[\phi(P)\left(\delta_j^P\tilde{\lambda}_j\right)^{1/q(\cdot)}\right]^r}\right) \leq 1\right\} \lesssim 1,$$

which follows from the following estimate

$$H_j^P := \left\|\frac{\chi_P 2^{js(\cdot)r}}{\left\{\phi(P)\left[\delta_j^P\right]^{1/q(\cdot)}\right\}^r} \sum_{v=j}^{\infty} \sum_{\ell(Q)=2^{-v}} |t_Q|^r |\varphi_j * a_Q|^r\right\|_{L^{p(\cdot)/r}} \lesssim 1. \tag{105}$$

Next we show (105). By Lemma (6.2.28), we find that

$$H_j^P \lesssim \sum_{v=j}^{\infty} 2^{-(v-j)(L+n)r} \left\|\frac{\chi_P 2^{js(\cdot)r}}{\left\{\phi(P)\left[\delta_j^P\right]^{1/q(\cdot)}\right\}^r} \sum_{k \in \mathbb{Z}^n} \frac{|t_{Q_{vk}}|^r |Q_{vk}|^{-r/2}}{(1 + 2^j|\cdot - 2^{-v}k|)^{Rr}}\right\|_{L^{p(\cdot)/r}(\mathbb{R}^n)}, \tag{106}$$

where R can be large enough. For all $x \in P$ and $v \in \mathbb{Z}_+$ with $v \geq j$, let

$$\Omega_{0,j}^{x,v} := \left\{k \in \mathbb{Z}^n : 2^j|x - 2^{-v}k| \leq 1\right\}$$

and, for all $i \in \mathbb{N}$, $\Omega_{0,j}^{x,v} := \{k \in \mathbb{Z}^n : 2^{i-1} < 2^j|x - 2^{-v}k| \leq 2^i\}$. Then, we see that, for all $x \in P$,

$$J(v,j,x,P) := \frac{2^{js(x)r}}{\left(\delta_j^P\right)^{r/q(x)}} \sum_{k \in \mathbb{Z}^n} \frac{|t_{Q_{vk}}|^r |Q_{vk}|^{-r/2}}{(1 + 2^j|x - 2^{-v}k|)^{Rr}}$$

$$\sim \frac{2^{js(x)r}}{\left(\delta_j^P\right)^{r/q(x)}} \sum_{i=0}^{\infty} \sum_{k \in \Omega_{i,j}^{x,v}} |t_{Q_{vk}}|^r |Q_{vk}|^{-iRr}$$

$$\sim \frac{2^{js(x)r}}{\left(\delta_j^P\right)^{r/q(x)}} \sum_{i=0}^{\infty} 2^{-iRr} 2^{vn} \int_{\cup_{\tilde{k} \in \Omega_{i,j}^{x,v}} Q_{v\tilde{k}}} \left[\sum_{k \in \Omega_{i,j}^{x,v}} |t_{Q_{vk}}| \tilde{\chi}_{Q_{vk}}(y)\right]^r dy. \tag{107}$$

Since, for all $i \in \mathbb{Z}_+$, $v \in \mathbb{Z}_+$ with $v \geq j$, $x \in P$ and $y \in \cup_{\tilde{k} \in \Omega_{i,j}^{x,v}} Q_{v\tilde{k}}$, there exists $\tilde{k}_y \in \Omega_{i,j}^{x,v}$ such that $y \in Q_{v\tilde{k}_y}$, it follows that

$$1 + 2^j|x - y| \leq 1 + 2^j\left|x - x_{Q_{v\tilde{k}_y}}\right| + 2^j\left|y - x_{Q_{v\tilde{k}_y}}\right| \lesssim 2^i + 2^{j-v} \lesssim 2^i \tag{108}$$

and hence

$$|y - c_P| \leq \left|y - x_{Q_{v\widetilde{k}_y}}\right| + \left|x - x_{Q_{v\widetilde{k}_y}}\right| + |x - c_P| \lesssim 2^{-v} + 2^{i-j} + 2^{-j_P} \lesssim 2^{i-j_P}. \quad (109)$$

By (109), we see that, for all $i \in \mathbb{Z}_+$, $v \in \mathbb{Z}_+$ with $v \geq j$, $x \in P$,

$$\bigcup_{\widetilde{k} \in \Omega_{i,j}^{x,v}} Q_{v\widetilde{k}} \subset Q(c_P, 2^{i-j_P+c_0}) =: Q_i^0$$

for some constant $c_0 \in \mathbb{N}$, which, combined with (107) and (108), implies that

$$J(v, j, x, P) := \left(\delta_j^P\right)^{-r/q(x)} 2^{js(x)r} \sum_{i=0}^{\infty} 2^{-iRr} 2^{(v-j)n} 2^{i(a-\varepsilon)r} \times \eta_{j,ar+\varepsilon r}$$

$$* \left(\left[\sum_{\widetilde{k} \in \Omega_{i,j}^{x,v}} \left|t_{Q_{v\widetilde{k}}}\right| \widetilde{\chi}_{Q_{v k}} \chi_{Q_i^0}\right]^r\right)(x)$$

$$\lesssim 2^{(v-j)(n-rs_-)} \sum_{i=0}^{\infty} 2^{-ir(R-a-\varepsilon)} \eta_{j,ar} \times$$

$$* \left(\left[\sum_{k \in \Omega_{i,j}^{x,v}} \left(\delta_j^P\right)^{-r/q(\cdot)} 2^{js(\cdot)} \left|t_{Q_{v k}}\right| \widetilde{\chi}_{Q_{v k}} \chi_{Q_i^0}\right]^r\right)(x), \quad (110)$$

where $\varepsilon \in [C_{\log}(s) + C_{\log}(1/q), \infty)$. From this, (106) and Lemma (6.2.15), we deduce that

$$H_j^P \lesssim \sum_{v=j}^{\infty} 2^{-(v-j)(L+n-n/r+s_-)r} \sum_{i=0}^{\infty} 2^{-i(R-a-\varepsilon)r} \frac{[\phi(Q_i^0)]^r}{[\phi(P)]^r}$$

$$\times \left\|\frac{\left(\delta_j^P\right)^{-r/q(\cdot)}}{\phi(Q_i^0)} \sum_{k \in \mathbb{Z}^n} \left|t_{Q_{v k}}\right| 2^{vs(\cdot)} \widetilde{\chi}_{Q_{v k}}\right\|_{L^{p(\cdot)}(Q_i^0)}^r$$

$$\lesssim \sum_{v=j}^{\infty} 2^{-(v-j)(L+n-n/r+s_- -d/q_+)r} \sum_{i=0}^{\infty} 2^{-i(R-a-\varepsilon-\log c_1 - \widetilde{d}/q_-)r}$$

$$\times \left\|\frac{2^{(v-j)dr/q(\cdot)} 2^{-i\widetilde{d}r/q(\cdot)}}{[\phi(Q_i^0)]^r \left(\delta_j^P\right)^{r/q(\cdot)}} \left[\sum_{k \in \mathbb{Z}^n} \left|t_{Q_{v k}}\right| 2^{vs(\cdot)} \widetilde{\chi}_{Q_{v k}}\right]^r\right\|_{L^{p(\cdot)/r}(Q_i^0)}$$

$$\lesssim \sum_{v=j}^{\infty} 2^{-(v-j)(L+n-n/r+s_- -d/q_+)r} \sum_{i=0}^{\infty} 2^{-i(R-a-\varepsilon-\log c_1 - \widetilde{d}/q_-)r} \lesssim 1,$$

where R is chosen large enough such that $R > a + \varepsilon + \log c_1 + \widetilde{d}/q_-$, which completes the proof of that $I_{P,3} \lesssim 1$ and hence the case I.

Case II $q_+ = \infty$.

We see that $q(x) = \infty$ for all $x \in \mathbb{R}^n$. Thus, we see that

$$\|t\|_{b_{p(\cdot),\infty}^{s(\cdot),\phi}(\mathbb{R}^n)} = \sup_{P \in \mathcal{Q}} \frac{1}{\phi(P)} \sup_{j \in \mathbb{Z}_+, j \geq (j_P \vee 0)} \left\|\sum_{Q \in \mathcal{Q}^*, \ell(Q)=2^{-j}} |Q|^{-\frac{s(\cdot)}{n}} |t_Q| \widetilde{\chi}_Q\right\|_{L^{p(\cdot)}(P)}.$$

Let P be a given dyadic cube. Then, by (97), we find that, for all $j \in \mathbb{Z}_+ \cap j \geq [j_P \vee 0, \infty)$,

$$G_P^j := \frac{1}{\phi(P)} \left\| 2^{js(\cdot)} |\varphi_j * f| \right\|_{L^{p(\cdot)}(P)}$$

$$\lesssim \frac{1}{\phi(P)} \left\| 2^{js(\cdot)} \sum_{v=0}^{j-1} \sum_{Q \in \mathfrak{Q}^*, \ell(Q)=2^{-j}} |t_Q| |\varphi_j * a_Q| \right\|_{L^{p(\cdot)}(P)}$$

$$+ \frac{1}{\phi(P)} \left\| 2^{js(\cdot)} \sum_{v=0}^{\infty} \sum_{Q \in \mathfrak{Q}^*, \ell(Q)=2^{-j}} |t_Q| |\varphi_j * a_Q| \right\|_{L^{p(\cdot)}(P)} =: G_{P,1}^j + G_{P,2}^j. \tag{111}$$

To estimate $G_{P,1}^j$ and $G_{P,2}^j$, we let $\varepsilon \in (C_{\log}(s), \infty)$, $r \in (0, \min\{1, q_-\})$ and $a \in (n/r, \infty)$. For $G_{P,1}^j$, by an argument similar to that used in the estimate for $I_{P,1}$, we conclude that there exists a positive constant c_0 such that

$$G_{P,1}^j \lesssim \left\{ \sum_{v=0}^{j} 2^{(v-j)(K-s_+)r} \sum_{i=0}^{\infty} 2^{-i(M-a-\varepsilon/r)} \right.$$

$$\left. \times \frac{1}{\phi(P)} \left\| \sum_{Q \in \mathfrak{Q}^*, \ell(Q)=2^{-v}} |t_Q|^r 2^{vs(\cdot)r} |Q|^{-r/2} \chi_Q \right\|_{L^{p(\cdot)/r}(Q(c_P, 2^{i-v+c_0}))} \right\}^{1/r},$$

which, together with (57) and the facts that $c_1 \in [1, \infty)$ and $j \geq j_P$, implies that

$$G_{P,1}^j \lesssim \|t\|_{b_{p(\cdot),\infty}^{q(\cdot),\phi}(\mathbb{R}^n)} \left\{ \sum_{v=0}^{j} 2^{(v-j)(K-s_+)r} \sum_{i=0}^{\infty} 2^{-i(M-a-\varepsilon/r)} \frac{\left[\phi\left(Q(c_P, 2^{i-v}) \right) \right]^r}{[\phi(P)]^r} \right\}^{1/r}$$

$$\lesssim \|t\|_{b_{p(\cdot),\infty}^{q(\cdot),\phi}(\mathbb{R}^n)} \left\{ \sum_{v=0}^{j} 2^{v(K-s_+-\log_2 c_1)r} \sum_{i=0}^{\infty} 2^{-i(M-a-\varepsilon/r-\log_2 c_1)} 2^{j_P \log_2 c_1} \right\}^{1/r}$$

$$\lesssim \|t\|_{b_{p(\cdot),\infty}^{q(\cdot),\phi}(\mathbb{R}^n)} 2^{-j \log_2 c_1} 2^{j_P \log_2 c_1} \lesssim \|t\|_{\|t\|_{b_{p(\cdot),\infty}^{q(\cdot),\phi}(\mathbb{R}^n)}}. \tag{112}$$

For $G_{P,2}^j$, by an argument similar to that used in the proof of (110), we find that there exists $c_0 \in \mathbb{N}$ such that

$$G_{P,2}^j \lesssim \frac{1}{\phi(P)} \left\{ \sum_{v=j+1}^{\infty} 2^{-(v-j)(L+n)r} \sum_{i=0}^{\infty} 2^{-i(R-a-\varepsilon)r} \right.$$

$$\left. \times \left\| \eta_{j,ar} * \left(\left[\sum_{Q \in \mathfrak{Q}^* \atop \ell(Q)=2^{-v}} |t_Q| |Q|^{-s(\cdot)/n} \tilde{\chi}_Q \chi_{Q(c_P, 2^{i-j_P+c_0})} \right]^r \right) \right\|_{L^{p(\cdot)/r}(\mathbb{R}^n)} \right\}^{1/r},$$

which, combined with (57) and (62), implies that

$$G_{P,2}^j \lesssim \frac{1}{\phi(P)} \left\{ \sum_{v=j+1}^{\infty} 2^{-(v-j)(L+n)r} \sum_{i=0}^{\infty} 2^{-i(R-a-\varepsilon)r} \right.$$

$$\times \left\| \left(\sum_{Q \in \mathfrak{Q}^*, \ell(Q)=2^{-v}} |t_Q| |Q|^{-s(\cdot)/n} \tilde{\chi}_Q \right) \right\|_{L^{p(\cdot)/r}\left(Q(c_P, 2^{i-j_P+c_0})\right)}^r \left. \right\}^{1/r}$$

$$\lesssim \|t\|_{b_{p(\cdot),\infty}^{q(\cdot),\phi}(\mathbb{R}^n)} \left\{ \sum_{v=j+1}^{\infty} 2^{-(v-j)(L+n)r} \sum_{i=0}^{\infty} 2^{-i(R-a-\varepsilon-\log_2 c_1)r} \right\}^{1/r} \lesssim \|t\|_{b_{p(\cdot),\infty}^{q(\cdot),\phi}(\mathbb{R}^n)}.$$

By this, (111) and (112), we conclude that

$$\|f\|_{B_{p(\cdot),\infty}^{s(\cdot),\phi}(\mathbb{R}^n)} \lesssim \sup_{P \in \mathfrak{Q}} \frac{1}{\phi(P)} \sup_{j \in \mathbb{Z}_+ \cap [(j_P \vee 0), \infty)} \left\| 2^{js(\cdot)} |\varphi_j * f| \right\|_{L^{p(\cdot)}(P)}$$

$$\lesssim \sup_{P \in \mathfrak{Q}} \frac{1}{\phi(P)} \sup_{j \in \mathbb{Z}_+ \cap [(j_P \vee 0), \infty)} \left(G_{P,1}^j + G_{P,2}^j \right) \lesssim \|t\|_{b_{p(\cdot),\infty}^{q(\cdot),\phi}(\mathbb{R}^n)},$$

which completes the proof of the case II.

Combining Cases I and II, we conclude that (98) holds true. This finishes the proof of Theorem (6.2.30).

List of Symbols

Symbol		Page
B_p^{sp}:	Besov space	1
H_p:	Hardy space	1
supp:	support	1
sup:	supremum	2
max:	maximum	6
Tr:	trace	6
$\otimes$:	tensor product	8
inf:	infimum	10
BMO:	Bounded mean oscillation	10
L^p:	Lebesgue space	13
loc:	local	18
L^2:	Hilbert space	21
L^∞:	Essential Lebesgue space	22
min:	minimum	26
ℓ^2:	Hilbert space of sequences	27
ℓ^q:	Dual of Hilbert space of sequences	27
$MB_{p,q}^{s,\beta}, MF_{p,q}^{s,\beta}$:	Besov and Triebel-Lizerkin space	41
$W^{1,p(\cdot)}$:	Sobolev space of variable	57
L^1:	Lebesgue space	59
ess:	essential	59
$f_{p,q}^\alpha$:	Triebel-Lizorkin space	61
BV:	Bounded variation	67
$L^{p_0,q}$:	Lorentz space	79
ℓ^∞:	Essential Lebesgue space sequences	82
$BH^{s,\tau}$	Hardy-Hausdorff space	93
Mol:	Molecular	142
M_u^p:	Morrey space	162
$HH_{-\alpha}$:	Hardy-Hausdorff space	162

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