Sudan University of Science and Technology

College of Graduate Studies

Department of Physics

Unification of Gauge Coupling Constants in 5D Pati-Salam Model

توحيد ثوابت الاقتران القياسية في نموذج باتي–سلام في خمسة ابعاد

A dissertation submitted the college of graduate studies for the Degree of M.Sc. in Physics

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الآيـــــــــة

وَقَالَ ٱلَّذِينَ كَفَرُوا۟ لَا تَأْتِينَا ٱلسَّاعَةُ ۖ قُلْ بَلَىٰ وَرَبِّى لَتَأْتِيَنَّكُمْ
عَٰلِمِ ٱلْغَيْبِ ۖ لَا يَعْزُبُ عَنْهُ مِثْقَالُ ذَرَّةٍ فِى ٱلسَّمَٰوَٰتِ وَلَا فِى ٱلْأَرْضِ
وَلَآ أَصْغَرُ مِن ذَٰلِكَ وَلَآ أَكْبَرُ إِلَّا فِى كِتَٰبٍ مُّبِينٍ ﴿٣﴾

صدق الله العظيم

سورة سبأ

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Abstract

The standard model of particle physics is currently an accepted theory at low energies; however, it doesn't address the unification of gauge coupling constants. In this dissertation, the renormalization group equations at one loop order in the standard model, both in 4D and 5D, in addition to Pati-Salam model (4D and 5D) have been studied. We found, both analytically and numerically, that a natural and successful gauge coupling unification could be achieved in 4D Pati-Salam model with $\mathrm{M_R} = 10^{11}\mathrm{GeV}$, and that the unification occurred there at $\Lambda = 10^{13}\mathrm{GeV}$; further in and 5D Pati-Salam model a unification occurred too with $\mathrm{M_R} = 10^{7}\mathrm{GeV}$ and compactification scale $\frac{1}{R} = 10^{6}\mathrm{GeV}$ at high energy scale, these results show that the gauge coupling constants are almost unified at high energy in Pati-Salam model.

الملخص

النظرية القياسية للجسميات الأولية تعتبر حاليا نظرية معتمدة عند طاقات منخفضه, ولكن في هذه النظرية لا يؤخذ في الاعتبار توحيد ثوابت الإقتران القياسية. في هذه الأُطروحة دُرست نظرية معادلات المجموعات المعايرة للرتبة الاولي في النموذج القياسي ونموذج باتي–سلام في اربعة وخمسة أبعاد. وُجد تحليليا وحسابيا بنجاح وطبيعيا أن ثوابت الإقتران توحدت في نموذج باتي سلام في أربعة أبعاد إذا كان $M_R = 10^{11}GeV$ وتوحدت في هذه الطاقة ثوابت الاقتران عند طاقة $\Lambda = 10^{13}GeV$ وفي خمسة أبعاد وجد انه إذا كان $M_R = 10^{7}GeV$ و $\frac{1}{R} = 10^{6}GeV$ عند مقايس طاقة عالية تتوحد ثوابت الاقتران عند طاقة $\Lambda = 10^{7}GeV$ وهذة النتائج توضح ان ثوابت الاقتران القياسية تتوحد في نموذج باتي– سلام عند طاقة عالية.

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Chapter One

General Introduction

(1.1) Introduction

Unification of the forces of nature is one of the most important ideas in the fundamental physics of the twentieth century. Early attempts at a unified field theory, including gravity and electromagnetic interactions, were in hindsight premature (T.~Kaluza, 1921). However, the Standard Model, describing with remarkable success the strong, weak and electromagnetic interactions of the three known families of quarks and leptons, seems to provide a much more propitious starting point for unification (A.D.Martin, 1984). It is entirely possible that there are no new low energy gauge interactions. Moreover quarks and leptons appear to be equally fundamental and elementary. Finally, the merging of the three low energy gauge couplings at a high energy scale $M_{GUT} = 10^{16}\text{GeV}$ provides significant evidence for a grand unification within four dimensional models (L. J. Hall and Y. Nomura, 2002).

A higher dimensional gauge theory is non-renormalize able and should be defined as a cutoff theory in which the theory is valid (L. J. Hall and Y. Nomura, 2002). This implies a highly sensitive dependence on an unknown short distance physics. This is a problem for gauge coupling unification if it affects how the difference of any two gauge couplings runs above the compactification scale.

(1.2) The importance of the research

The ultimate goal in particle physics is the unification of the well-known four forces (weak, electromagnetic, strong and gravity).

(1.3) The main objective of the research

The main purpose of this is dissertation is to unify the gauge coupling constants in Pati-Salam model in extra dimension.

(1.4) The outline of the research

This research project is constructed as follows: In chapter one we gave a concise introduction; in chapter two we studied the Standard Model of particle physics with some details; meanwhile, in chapter three we dealt with the Renormalization group equations and Beta function, and finally, in chapter four we presented our numerical results, discussion and conclusion.

Chapter Two

The Standard Model of Particle Physics

(2.1) Introduction

In this chapter, we will present the complete studies of the standard model and its mathematical construction, and then we will discuss the Higgs mechanism knows as spontaneous symmetry breaking, which had led to understanding of how elementary particles obtain their masses.

(2.2) Definition of standard model (SM)

The SM is a gauge theory, based on the symmetry group $SU(3)_C \times SU(2)_L \times U(1)_Y$ so far provides the most accurate description of nature at subatomic level. It is based on the quantum theory of fields (J.donoghue, 1994). In quantum field theory there is one field for each type of particle, i.e. matter particles and force particles. SM describes strong, weak and electromagnetic interactions, via the exchange of the corresponding spin-1 gauge fields: eight massless gluons and one massless photon, respectively, they stand for the strong and electromagnetic interactions, and three massive bosons, $W^{\pm}$and Z, stand for the weak interaction; while, fermionic matter content is given by the known leptons and quarks (A.D.Martin, 1984).

(2.3) Classification of elementary particle

The particles that have been identified in high-energy experiments fall into distinct classes: They are, the leptons (Electron, Neutrino, and Muon) all of which have

spin $\frac{1}{2}$ and they may be charged or neutral. The charged leptons encounter interactions of both electromagnetic and weak nature, meanwhile the neutral ones encounter a weak interaction (A.D.Martin, 1984). There are three well-defined lepton pairs, thes electron (e^-) and the electron neutrino(ν_e), the muon(μ) and the muon neutrino (ν_μ)and the tau (τ^-) (the much heavier charged lepton) and it tau neutrino (ν_τ). And all these particles have antiparticles, in accordance with the predictions of relativistic quantum mechanics "lepton-type" conservation laws, which state that: the number of plus, the number of minus and the number of the corresponding antiparticles is conserved in weak reactions; and similarly for the muon and tau-type leptons. These conservation laws should be followed automatically in the standard model if the neutrinos are massless. However, recently evidence for a tiny nonzero neutrino masses subtle violation of these conservations laws has been observed.

(2.4) Lagrange of the standard model

In order to obtain the Lagrangian of SM we started from the free particle Lagrangian and replaced the ordinary derivative by the covariant derivative in order to have local gauge invariance (A.D.Martin, 1984).

$$\mathcal{L}_{SM} = \mathcal{L}_{Gauge} + \mathcal{L}_{Fermion} + \mathcal{L}_{Higgs} + \mathcal{L}_{Yukawa} \qquad (2.1)$$

(2.4.1) The fermions sector

The fermions matter content is given by the known leptons and quarks, which are organized in a three-fold family structure (A.D.Martin, 1984):

1/ Leptons:

- Left lepton:

$$l_L \equiv \begin{pmatrix} \nu_e \\ e \end{pmatrix}_{L1^{st}} \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_{L2^{nd}} \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_{L3^{rd}}$$

- Right lepton:

$$l_R \equiv e_R \mu_R \tau_R$$

2/ Quarks:

- Left quarks:

$$q_L \equiv \begin{pmatrix} u \\ d \end{pmatrix}_{L\ 1^{st}} \begin{pmatrix} c \\ s \end{pmatrix}_{L\ 2^{nd}} \begin{pmatrix} t \\ b \end{pmatrix}_{L\ 3^{rd}}$$

- Right quarks:

$$q_R \equiv u_R d_R c_R s_R t_R b_R$$

Fermions are described by the Dirac equation:

$$\mathcal{L}_{Fermions} = i\ \bar{\psi}\gamma^{\mu} D_{\mu}\psi - m\bar{\psi}\psi \qquad (2.2)$$

(2.4.2) The bosons sector

The gauge fields lagrangian is given by (A.D.Martin, 1984):

$$\mathcal{L}_{Gauage} = \frac{-1}{4} F^{a}_{\mu\upsilon} F^{a\mu\upsilon} \frac{-1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} W_{\mu\nu}\, W^{\mu\nu} - \frac{1}{4} G_{\mu\nu}\, G^{\mu\nu} \qquad (2.3)$$

Where:

- Electromagnetic field strength tensor is:

$$B_{\mu\upsilon} = \partial_{\mu} B_{\upsilon} - \partial_{\upsilon} B_{\mu} + g' \frac{Y}{2}\, B_{\mu}\, B_{\upsilon} \qquad (2.4)$$

- Weak interaction field strength tensor is:

$$W_{\mu\nu} = \partial_{\mu} W_{\upsilon} - \partial_{\upsilon} W_{\mu} + g T^{a} W_{\mu}{}^{a} W_{\upsilon}{}^{a=1..3} \qquad (2.5)$$

- Strong interaction field strength tensor is:

$$G_{\mu\upsilon} = \partial_{\mu} G_{\upsilon} - \partial_{\upsilon} G_{\mu} + g_s T^{a} G_{\mu}{}^{a} G_{\upsilon}{}^{a=1\ldots8} \qquad (2.6)$$

(2.5) Higgs mechanism (spontaneous symmetry breaking)

The Higgs mechanism is an important part of the Standard Model of particle Physics; it provides masses for the gauge bosons of the weak interaction and for fermions. The electroweak theory in which the Higgs mechanism plays a prominent role had a ssconvincing experimental verification. The Higgs mechanism is generally described as a case of spontaneous symmetry breaking. Therefore, the put that the Higgs mechanism can take in the lagrangian of the SM is as follows (Quigg, 2007):

$$\mathcal{L}_{Higgs} = -\frac{1}{2}(D_\mu\phi)^\dagger\left(D_\mu\phi\right) - V(\phi) \qquad (2.7)$$

Where

The Higgs potential is given by

$$V(\phi) = -\frac{\mu^2}{2}\phi^*\phi - \frac{\lambda}{4}\phi^4 \qquad (2.8)$$

Therefore equation (2.7) becomes

$$\mathcal{L}_{Higgs} = -\tfrac{1}{2}(D_\mu\phi)^\dagger\left(D_\mu\phi\right) + \tfrac{\mu^2}{2}\phi^*\phi + \tfrac{\lambda}{4}\phi^4 \qquad (2.9)$$

Take the first derivative of $V(\phi)$:

$$\frac{\partial V}{\partial \phi} = -\mu^2\langle\phi\rangle - \lambda\langle\phi^3\rangle = 0 \qquad (2.10)$$

We get:

$$(-\mu^2 - \lambda\langle\phi^2\rangle)\langle\phi\rangle = 0 \qquad (2.11)$$

This equation has two solutions

$$\langle\phi\rangle = 0 \text{(Trivial solution)} \qquad (2.12)$$

Or:

$$-\mu^2 - \lambda\langle\phi^2\rangle = 0 \qquad (2.13)$$

Therefor:

$$\langle\phi^2\rangle = \frac{-\mu^2}{\lambda} \qquad (2.14)$$

Therefore:

$$\langle\phi\rangle = \pm\sqrt{\frac{-\mu^2}{\lambda}} = v \qquad (2.15)$$

Which contains two new real parameters μ *and* λ we require $\lambda > 0$ for the potential to be bounded; otherwise the potential is unbounded from below and there will be no stable vacuum state. But μ takes the following two cases:

• $\mu^2 > 0$ Then the vacuum corresponds to $\Phi = 0$, the potential has a minimum at the origin as picture figure 2.1.

• $\mu^2 < 0$ Then the potential develops a non-zero Vacuum Expectation Value (VEV) and the minimum is along a circle of radius $\frac{v}{\sqrt{2}} = \frac{246}{\sqrt{2}}$ as depicted in figure 2.2.

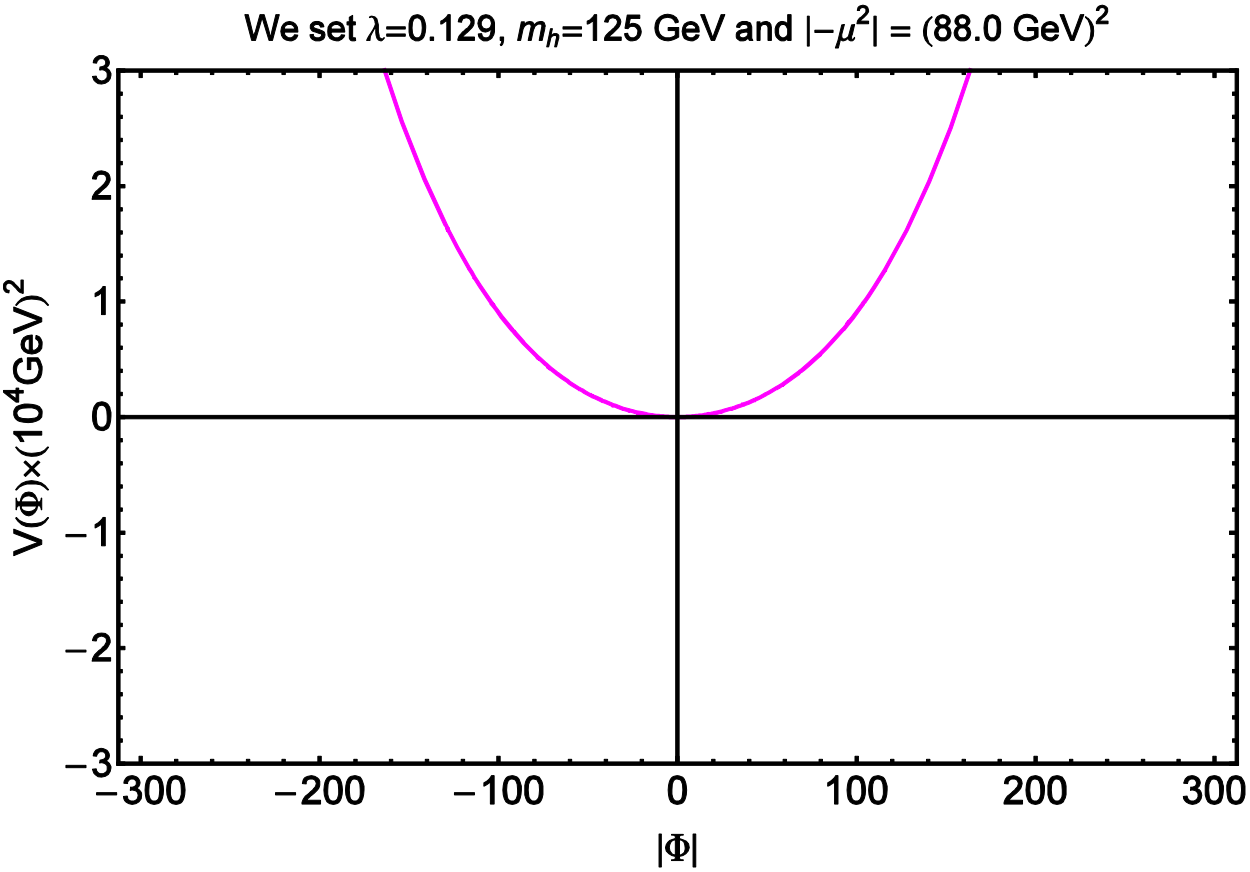

Figure 2.1: The Higgs potential $V(\Phi)$ with, the case $\mu^2 > 0$; as function of $|\Phi| = \sqrt{\Phi^\dagger \Phi}$

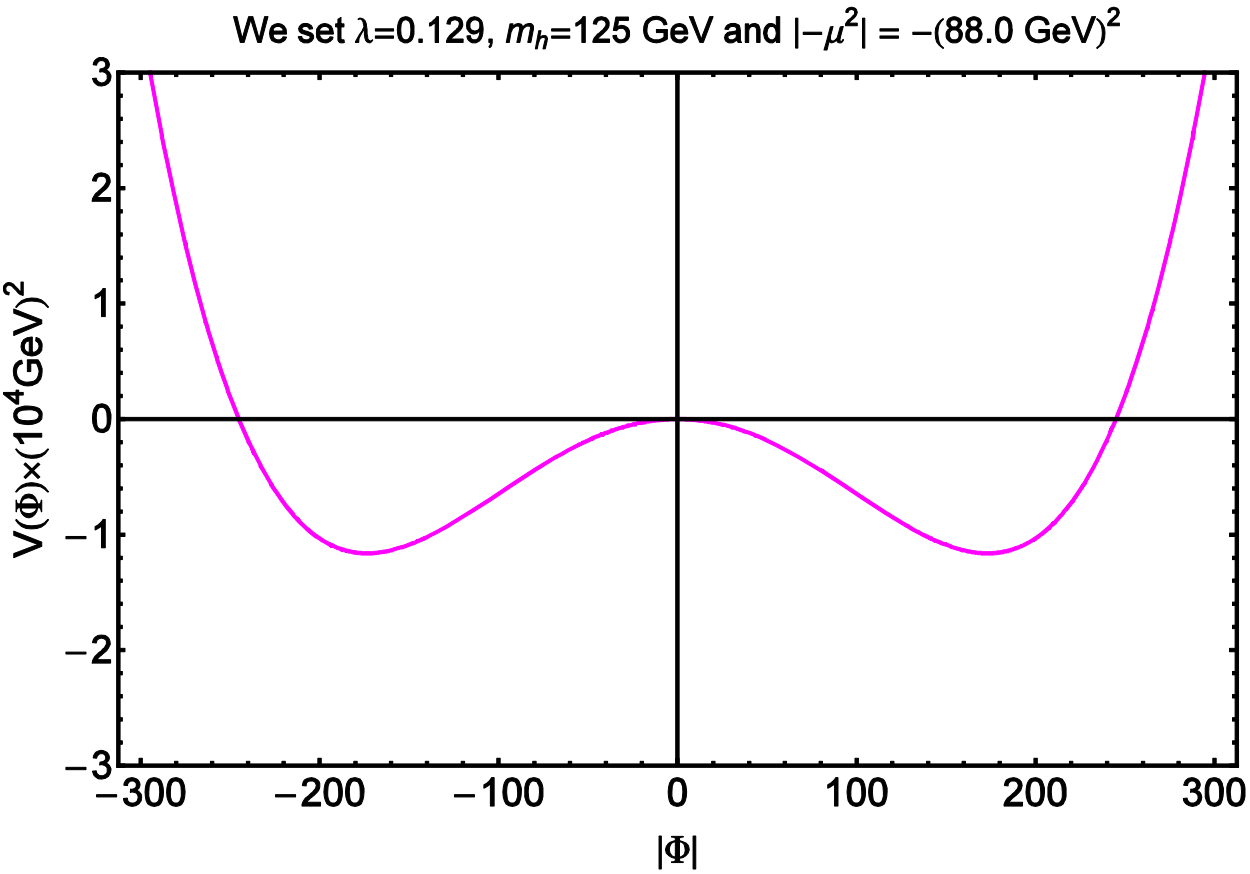

Figure 2.2: The Higgs potential $V(\Phi)$ with, the case $\mu^2 < 0$; as function of$|\Phi| = \sqrt{\Phi^\dagger \Phi}$

Where v is known as the vacuum expectation value (VEV), v=246 GeV.

Hence we can choose the Higgs fields as

$$\phi = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix} \tag{2.16}$$

(2.6) The Higgs boson mass

Fundamental scalar field with self-interactions (H.~-U.~Yee, 2003):

$$V = -\frac{1}{2}\mu^2\phi^\dagger\phi + \frac{1}{4}\lambda\left(\phi^\dagger\phi\right)^2 \quad (2.17)$$

$$\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ h+v \end{pmatrix} \quad (2.18)$$

Since $\phi^\dagger$ is the complex conjugate of ϕ:

$$\phi^\dagger = \frac{1}{\sqrt{2}}(0 \quad h+v) \quad (2.19)$$

Plug equations (2.18) and (2.19) in equation (2.17) we obtain the mass of Higgs boson:

$$\mathrm{m_H} = \sqrt{2\lambda v^2} \quad \text{and} \quad \mathrm{m_H} = \sqrt{2\mu^2} \quad (2.20)$$

Where:

λ Is the Higgs self-coupling parameter in $V(\phi)$.

(2.7) The fermions masses (Yukawa interaction)

A fermonic masses term $\mathcal{L}_m = -m\bar{\psi}\psi = -m(\bar{\psi}_L\psi_R + \psi_R\bar{\psi}_L)$ is not allowed, because it breaks the gauge symmetry however, since we have introduced an additional scalar doublet into the model, and write the following gauge-invariant fermions-scalar coupling (Falcone, 2002).

From Yukawa interaction:

$$\mathcal{L}_{yukawa} = Y\bar{\psi}_L\phi\psi_R + \mathrm{h.c.} = \mathrm{m_\psi}\bar{\psi}_L\psi_R \quad (2.21)$$

Can be written also as:

$$\mathcal{L}_{yukawa} = Y_d \bar{q}_L \phi q_R + Y_u \bar{q}_L \phi^* q_R + Y_e \bar{l}_L \ \phi l_R \tag{2.22}$$

Now using equation (2.16) we obtain

$$\mathcal{L}_{yukawa} = \frac{Y_d}{\sqrt{2}} \begin{pmatrix} \bar{u}_L & \bar{d}_L \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} d_R + \frac{Y_u}{\sqrt{2}} \begin{pmatrix} \bar{u}_L & \bar{d}_L \end{pmatrix} \begin{pmatrix} v \\ 0 \end{pmatrix} u_R + \frac{Y_e}{\sqrt{2}} \begin{pmatrix} \bar{v}_L & \bar{e}_L \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} e_R \tag{2.23}$$

Therefore:

$$\mathcal{L}_{yukawa} = \frac{Y_d}{\sqrt{2}} v \bar{d}_L d_R + \frac{Y_u}{\sqrt{2}} v \bar{u}_L \ u_R + \frac{Y_e}{\sqrt{2}} v \bar{e}_l e_R \tag{2.24}$$

Compare equation (2.24) with equation (2.21) we obtain the masses of the quarks (u, d) and lepton(e):

$$m_d = \frac{Y_d}{\sqrt{2}} v \ \ , m_u = \frac{Y_u}{\sqrt{2}} v \ , \quad m_e = \frac{Y_e}{\sqrt{2}} v \tag{2.25}$$

(2.8) The gauge bosons masses

To obtain the masses for the gauge bosons we need to study the scalar part of; the lagrangian (P.~W.~Higgs, 1964).

$$\mathcal{L} = \frac{1}{2} \left(D_\mu \phi \right)^\dagger (D^\mu \phi) - V(\phi) \tag{2.26}$$

Where:

D_μ is the covariant derivative and given by:

$$D_\mu = \left(\partial_\mu + i g T^a W_\mu^a + i \acute{g} \frac{Y_\phi}{2} B_\mu \right) \tag{2.27}$$

$$T^a W_\mu^a = T^1 W_\mu^1 + T^2 W_\mu^2 + T^3 W_\mu^3 \tag{2.28}$$

$$T^a = \frac{\sigma^2}{2} \tag{2.29}$$

$$T^a W_\mu^a = \frac{1}{2}\left[\begin{pmatrix} 0 & W_\mu^1 \\ W_\mu^1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -iW_\mu^2 \\ iW_\mu^2 & 0 \end{pmatrix} + \begin{pmatrix} W_\mu^3 & 0 \\ 0 & -W_\mu^3 \end{pmatrix}\right]$$

$$= \frac{1}{2}\begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -W_\mu^3 \end{pmatrix} \tag{2.30}$$

$$B_\mu = \begin{pmatrix} B_\mu & 0 \\ 0 & B_\mu \end{pmatrix} \tag{2.31}$$

Putting equations (2.30) and (2.31) in equation (2.27)we get:

$$D_\mu = ig\frac{1}{2}\begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -W_\mu^3 \end{pmatrix} + i\acute{g}\frac{Y_\phi}{2}\begin{pmatrix} B_\mu & 0 \\ 0 & B_\mu \end{pmatrix} \tag{2.32}$$

$$D_\mu = \left(ig\frac{1}{2}\begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -W_\mu^3 \end{pmatrix} + i\acute{g}\frac{1}{2}\begin{pmatrix} B_\mu & 0 \\ 0 & B_\mu \end{pmatrix}\right) \tag{2.33}$$

$$= \left(\frac{i}{2}\begin{pmatrix} gW_\mu^3 & g\left(W_\mu^1 - iW_\mu^2\right) \\ g\left(W_\mu^1 + iW_\mu^2\right) & -gW_\mu^3 \end{pmatrix} + \frac{i}{2}\begin{pmatrix} \acute{g}B_\mu & 0 \\ 0 & \acute{g}B_\mu \end{pmatrix}\right) \tag{2.34}$$

$$D_\mu = \left(\frac{i}{2}\begin{pmatrix} gW_\mu^3 + \acute{g}B_\mu & g\left(W_\mu^1 - iW_\mu^2\right) \\ g\left(W_\mu^1 + iW_\mu^2\right) & -gW_\mu^3 + \acute{g}B_\mu \end{pmatrix}\right) \tag{2.35}$$

And if the covariant derivative in(2.35)operator in a function ϕ and we substitute for ϕ as it defined in equation (2.15) we obtain:

$$D_\mu\phi = \frac{i}{2\sqrt{2}}\begin{pmatrix} gW_\mu^3 + \acute{g}B_\mu & g\left(W_\mu^1 - iW_\mu^2\right) \\ g\left(W_\mu^1 + iW_\mu^2\right) & -gW_\mu^3 + \acute{g}B_\mu \end{pmatrix}\begin{pmatrix} 0 \\ v \end{pmatrix} \tag{2.36}$$

$$D_\mu\phi = \frac{1}{2\sqrt{2}}\begin{pmatrix} igv\left(W_\mu^1 - iW_\mu^2\right) \\ iv\left(-gW_\mu^3 + \acute{g}B_\mu\right) \end{pmatrix} \tag{2.37}$$

Since $\left(D_\mu\phi\right)^\dagger$ is the complex conjugate of $D_\mu\phi$:

$$\left(D_\mu\phi\right)^\dagger = \frac{1}{2\sqrt{2}}\left(-igv\left(W_\mu^1 + iW_\mu^2\right) \quad -iv\left(-gW_\mu^3 - \acute{g}B_\mu\right)\right) \tag{2.38}$$

Therefor:

$$\left(D_\mu\phi\right)^\dagger\left(D_\mu\phi\right) =$$

$$\frac{1}{8}\left(-igv\left(W_\mu^1 + iW_\mu^2\right) \quad -iv\left(gW_\mu^3 - \acute{g}B_\mu\right)\right)\begin{pmatrix} igv\left(W_\mu^1 - iW_\mu^2\right) \\ iv\left(-gW_\mu^3 + \acute{g}B_\mu\right) \end{pmatrix} \tag{2.39}$$

Define $W_\mu^- = \left(W_\mu^1 - iW_\mu^2\right)$ and$W_\mu^+ = \left(W_\mu^1 + iW_\mu^2\right)$, we get:

$$\left(D_\mu\phi\right)^\dagger\left(D_\mu\phi\right) =$$

$$\frac{1}{8}\left(-igvW_\mu^+ \quad -iv\left(gW_\mu^3 - \acute{g}B_\mu\right)\right)\begin{pmatrix} igvW_\mu^- \\ iv\left(-gW_\mu^3 + \acute{g}B_\mu\right) \end{pmatrix} \tag{2.40}$$

$$\left(D_\mu\phi\right)^\dagger\left(D_\mu\phi\right) = \frac{1}{8}\left(g^2v^2W_\mu^+W_\mu^- + v^2\left(gW_\mu^3 - \acute{g}B_\mu\right)\left(-gW_\mu^3 + \acute{g}B_\mu\right)\right) \tag{2.41}$$

Therefore the mass of the Wboson is:

$$m_{W^\pm}^2 = \frac{1}{4}g^2v^2 \rightarrow m_{W^\pm} = \frac{1}{2}gv \tag{2.42}$$

We will not derive the mass of the Z boson here as can be found in any text book,

e.g. (Broken symmetries, massless particles and gauge fields).

Chapter Three

Renormalization Group Equations and Beta function

(3.1) Introduction

In this chapter, we will discuss the Renormalization Group Equations and we will calculate the numerical coefficients values of Beta function for gauge couplings in the standard model, extra dimension model and Pati-Salam model (J.C. Pati and A. Salam, 1974) $SU(4) \times SU(2)_R \times SU(2)_L$ at one loop order (A.J.G.hey, 1993).

(3.2) Renormalization Group Equations (RGEs)

The generic structure of the one loop RGEs for the three gauge couplings constants of the standard model (SM) is given by (A.~Abdalgabar, 2013):

$$16\pi^2 \frac{dg_i}{dt} = b^i g_i^3 \qquad (3.1)$$

Where $t = \ln(\mu/M_Z)$ and contains the energy scale parameter μ. We chose to use the Z mass as a reference scale, so that for $\mu = M_Z$ we have $t = 0$ and we can fix the initial conditions of the running. The coefficients b^i will be calculated for various models.

(3.3) Gauge correction in the standard model at on loop

We will consider first correction the gauge coupling in the standard model in details and the calculation of other models will follow (Weinberg, 1996). The correction of gauge couplings constants receives contributions from the diagrams

in Figure 3.1. The numerical coefficients b^i appear in equation (3.1) can be calculated from the Feynman diagrams presented in figure (3.1). In this work we do not calculate all the diagrams (only their result will be given in next section (Beta function)). We will give a detailed calculation for one diagram, i.e. figure (3.1 b).

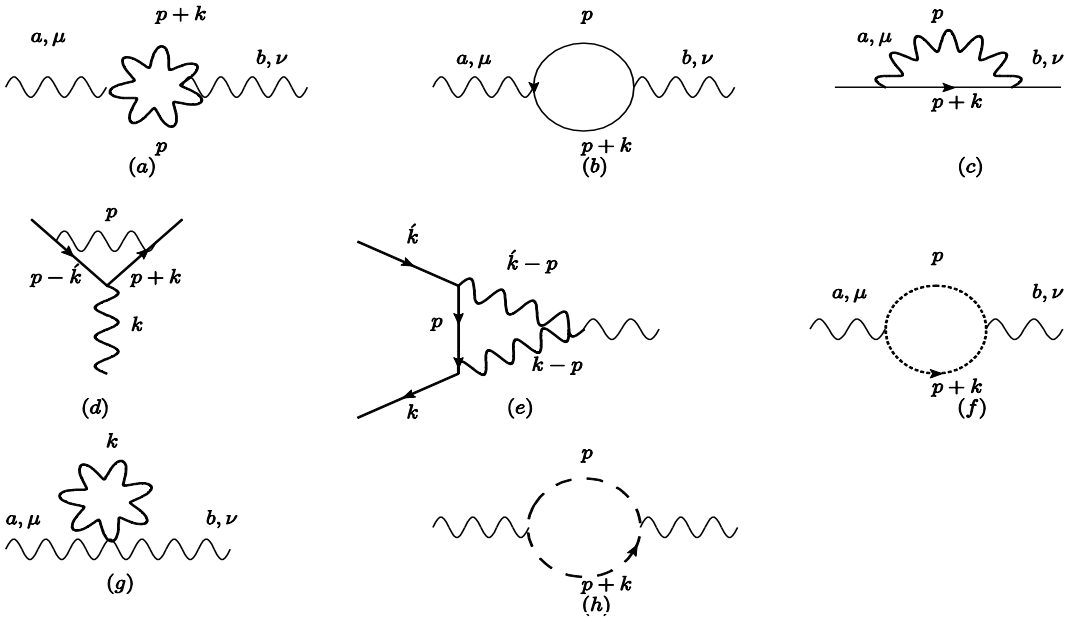

Figure 3.1: Diagrams contributes to the gauge correction in the Standard model

Now convert the diagram in figure (3.1.b) by using Feynman rules as

$$I_{\mu\nu} = Tr \int \frac{d^d p}{(2\pi)^d} \frac{i}{\not{p} - m} ig\gamma^\mu T^a \frac{i}{\not{p} - \not{k} - m} ig\gamma^\nu T^b \tag{3.2}$$

$$= g^2 Tr(T^a T^b) \int \frac{d^d p}{(2\pi)^d} Tr\left(\gamma^\mu \left(\frac{\not{p} + m}{p^2 - m^2}\right)\gamma^\nu \left(\frac{\not{p} + \not{k} + m}{(p-k)^2 - m^2}\right)\right) \tag{3.3}$$

From equation (3.3) take the $Tr\left(\gamma^\mu(\not{p} + m)\gamma^\nu(\not{p} + \not{K} + m)\right)$ to be $N_{\mu\nu}$ and let $m = 0$ we obtained the:

$$N_{\mu\nu} = Tr\left(\gamma^\mu \not{p} \gamma^\nu (\not{p} + \not{K})\right) = Tr(\gamma^\mu \gamma^\rho \gamma^\nu \gamma^\sigma) p_\rho (p+k)_\sigma \tag{3.4}$$

To evaluate the above equation we must use what is so called trace technology (McMahon, 2008)

$$Tr(\gamma^{\mu}\gamma^{\rho}\gamma^{\nu}\gamma^{\sigma}) = 4(g^{\mu\rho}g^{\nu\rho} - g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\rho\nu}) \tag{3.5}$$

Now substituting equation (3.5) into equation (3.4) we obtain:

$$N^{\mu\nu} = 4(\mathrm{g}^{\mu\rho}\mathrm{g}^{\nu\rho} - \mathrm{g}^{\mu\nu}\mathrm{g}^{\rho\sigma} + \mathrm{g}^{\mu\sigma}\mathrm{g}^{\rho\nu})\mathrm{p}_{\rho}(\mathrm{p} + \mathrm{k})_{\sigma} \tag{3.6}$$

After a little algebra the above equation becomes:

$$N^{\mu\nu} = 4\big(p^{\mu}(p^{\nu} + k^{\nu}) - g^{\mu\nu}p(p + k) + p^{\nu}(p^{\mu} + k^{\mu})\big) \tag{3.7}$$

Substitute equation (3.7) in equation (3.3) yields:

$$I_{\mu\nu} = g^2 Tr(T^a T^b) \int \frac{d^d p}{(2\pi)^d} \left(\frac{N^{\mu\nu}}{(p^2 - m^2)((p - k)^2 - m^2)}\right) \tag{3.8}$$

Define:

$$b = p^2 - m^2, \qquad a = (p - k)^2 - m^2, \qquad Tr(T^a T^b) = C_2(R) \tag{3.9}$$

Substituting equation (3.9) in equation (3.8) we get:

$$I_{\mu\nu} = g^2 C_2(R) \int \frac{d^d p}{(2\pi)^d} \left(\frac{N^{\mu\nu}}{\mathrm{ab}}\right) \tag{3.10}$$

(3.3.1) Feynman parameterization Technique

In order to evaluate the integral presented in equation (3.10), we employ the Feynman technique which is given by (Weinberg, 1996)

$$\frac{1}{ab} = \int_0^1 dz \, \frac{1}{(b + (a - b)z)^2} \tag{3.11}$$

Using equation (3.11), equation (3.10) becomes:

$$I_{\mu\nu} = g^2 C_2(R) \int \frac{d^d p}{(2\pi)^d} \int_0^1 dz \frac{N^{\mu\nu}}{(b + (a-b)z)^2} \tag{3.12}$$

Now consider the denominator in equation (3.12)

$$(b + (a-b)z) = p^2 - m^2 - 2pkz + k^2 z \tag{3.13}$$

Let us redefine $p = q + kz$, $dp = dq$ and substitute it in equation (3.13) we obtain

$$(q + kz)^2 - m^2 - 2(q + kz)kz + k^2 z =$$

$$q^2 + 2qkz + k^2 z^2 - m^2 - 2qkz - 2k^2 z^2 + k^2 z = q^2 - k^2 z^2 - m^2 + k^2 z$$

$$= q^2 + k^2 z(1 - z) - m^2 \tag{3.14}$$

Now let us rewrite a as: $a = k^2 z(1 - z) - m^2$ and plug it in equation (3.14) the equation, it becomes $q^2 + a$ and substitute this in equation (3.12), the integration becomes:

$$I_{\mu\nu} = g^2 C_2(R) \int_{-\infty}^{\infty} \frac{d^d q}{(2\pi)^d} \int_0^1 dz \frac{N^{\mu\nu}(q)}{(q^2 + a)^2} \tag{3.15}$$

Similarly the numerator $N^{\mu\nu}$ with this change of variable becomes:

$$
\begin{aligned}
N^{\mu\nu} = 4[&(q^\mu + k^\mu z)(q^\nu + k^\nu z + k^\nu) - g^{\mu\nu}(q + kz)(q + kz + k) \\
&+ (q^\nu + k^\nu z)(q^\mu + k^\mu z + k^\mu)] \\
&= 4[q^\mu q^\nu + q^\mu k^\nu z + q^\mu k^\nu + k^\mu q^\nu z + k^\mu k^\nu z^2 + k^\mu k^\nu z - g^{\mu\nu} q^2 \\
&- g^{\mu\nu} qkz - g^{\mu\nu} qk - g^{\mu\nu} kqz - g^{\mu\nu} k^2 z^2 - g^{\mu\nu} k^2 z + q^\nu q^\mu \\
&+ q^\nu k^\mu z + q^\nu k^\mu + k^\nu q^\mu z + k^\nu k^\mu z^2 \\
&+ k^\mu k^\nu z] \qquad (3.16)
\end{aligned}
$$

Keeping only terms which is quadratic in q as odd power in q gives zero in the integration we arrive at.

$$
N^{\mu\nu} = 4[2q^\mu q^\nu + 2z^2 k^\mu k^\nu + 2zk^\mu k^\nu - g^{\mu\nu} q^2 - g^{\mu\nu} k^2 z(z + 1)] \qquad (3.17)
$$

Substitute equation (3.17) in the integer and in equation (3.15):

$$
\begin{aligned}
I_{\mu\nu} = g^2 C_2(R) \int_{-\infty}^{\infty} \frac{d^d q}{(2\pi)^d} \int_0^1 dz \Bigg(&\frac{8q^\mu q^\nu}{(q^2 + a)^2} \\
&+ \frac{8z^2 k^\mu k^\nu + 8zk^\mu k^\nu - 4g^{\mu\nu} k^2 z(z + 1) - g^{\mu\nu} q^2}{(q^2 + a)^2} \Bigg) \qquad (3.18)
\end{aligned}
$$

To evaluate the above integrals, we will compare them with the standard integrals

(3.3.2) Standard Integrals

$$
\int_{-\infty}^{\infty} d^d q \frac{q^\mu q^\nu}{(q^2 + a)^n} = \frac{i\pi^{d/2}\, \Gamma\left(n - \frac{d}{2} - 1\right) g^{\mu\nu}}{2\Gamma(n) a^{n - \frac{d}{2} - 1}} \qquad (3.19.\,(\mathrm{I}))
$$

$$
\int_{-\infty}^{\infty} d^d q \frac{q^2}{(q^2 + a)^n} = \frac{i\pi^{d/2}\, \Gamma\left(n - \frac{d}{2} - 1\right)}{2\Gamma(n)} \frac{d}{a^{n - \frac{d}{2} - 1}} \qquad (3.19.\,(\mathrm{II}))
$$

$$\int_{-\infty}^{\infty} d^d q \frac{1}{(q^2+a)^n} = \frac{i\pi^{d/2}\,\Gamma\left(n-\frac{d}{2}\right)}{a^{n-\frac{d}{2}}} \qquad \left(3.19(\text{III})\right)$$

Comparing equation (3.18) with the standard integrals in equation (3.19) we get:

$$8\int_{-\infty}^{\infty} d^d q \frac{q^\mu q^\nu}{(q^2+a)^2} = \frac{4i\pi^{d/2}\,\Gamma\left(1-\frac{d}{2}\right) g^{\mu\nu}}{a^{1-\frac{d}{2}}} \qquad (3.20)$$

$$8zk^\mu k^\nu(z+1)\int_{-\infty}^{\infty} dq\ \frac{1}{(q^2+a)^2} = 8zk^\mu k^\nu(z+1)\frac{i\pi^{d/2}\,\Gamma\left(2-\frac{d}{2}\right)}{a^{2-\frac{d}{2}}} \qquad (3.21)$$

$$4g^{\mu\nu}\int_{-\infty}^{\infty} dq\ \frac{q^2}{(q^2+a)^2} = 4g^{\mu\nu}\frac{i\pi^{d/2}\,\Gamma\left(1-\frac{d}{2}\right)}{2}\frac{d}{a^{1-\frac{d}{2}}} \qquad (3.22)$$

$$4g^{\mu\nu}k^2 z(z+1)\int_{-\infty}^{\infty} dq\ \frac{1}{(q^2+a)^2} = \ 4g^{\mu\nu}k^2 z(z+1)\ \frac{i\pi^{d/2}\,\Gamma\left(2-\frac{d}{2}\right)}{a^{2-\frac{d}{2}}} \qquad (3.23)$$

Plugging all the ingredients in our desired integrals we obtain:

$$\int_{-\infty}^{\infty} dq \frac{4[2q^\mu q^\nu + 2z^2 k^\mu k^\nu + 2zk^\mu k^\nu - g^{\mu\nu}q^2 - g^{\mu\nu}k^2 z(z+1)]}{(q^2+a)^2} =$$

$$\frac{4i\pi^{d/2}\,\Gamma\left(1-\frac{d}{2}\right)g^{\mu\nu}}{a^{1-\frac{d}{2}}} + 8zk^{\mu}k^{\nu}(z+1)\frac{i\pi^{d/2}\,\Gamma\left(2-\frac{d}{2}\right)}{a^{2-\frac{d}{2}}} - 4g^{\mu\nu}\frac{i\pi^{d/2}\,\Gamma\left(1-\frac{d}{2}\right)}{2}\frac{d}{a^{1-\frac{d}{2}}} - 4g^{\mu\nu}k^{2}z(z+1)\frac{i\pi^{d/2}\,\Gamma\left(2-\frac{d}{2}\right)}{a^{2-\frac{d}{2}}} \quad (3.24)$$

Substitute equation (3.24) into equation (3.18):

$$\int_0^1 \frac{\mathrm{dz}}{(2\pi)^{\mathrm{d}}}\left(4g^{\mu\nu}\frac{i\pi^{d/2}\,\Gamma\left(1-\frac{d}{2}\right)}{a^{1-\frac{d}{2}}} - 2g^{\mu\nu}\frac{i\pi^{d/2}\,\Gamma\left(1-\frac{d}{2}\right)d}{a^{1-\frac{d}{2}}} + 8zk^{\mu}k^{\nu}(z+1)\frac{i\pi^{d/2}\,\Gamma\left(2-\frac{d}{2}\right)}{a^{2-\frac{d}{2}}} - 4g^{\mu\nu}k^{2}z(z+1)\frac{i\pi^{d/2}\,\Gamma\left(1-\frac{d}{2}\right)}{a^{2-\frac{d}{2}}}\right) \quad (3.25)$$

Now the first term upon integration on z, gives us:

$$\int_0^1 \frac{dz}{(2\pi)^{d}}\left(4g^{\mu\nu}\frac{i\pi^{d/2}\,\Gamma\left(1-\frac{d}{2}\right)}{a^{1-\frac{d}{2}}}\right)\frac{4g^{\mu\nu}\pi^{d/2}\,\Gamma\left(1-\frac{d}{2}\right)}{a^{1-\frac{d}{2}}} = \int_0^1 dz\left(\frac{1}{a^{1-\frac{d}{2}}}\right) \quad (3.26)$$

Rewrite:

$$a^{1-\frac{d}{2}} = a^{2-\frac{d}{2}}.\ a^{-1} \quad (3.27)$$

We get:

$$\frac{4g^{\mu\nu}k^2\pi^{d/2}\,\Gamma\left(1-\frac{d}{2}\right)}{(2\pi)^d\,a^{2-\frac{d}{2}}}\int_0^1 dz\,(z-z^2)=\frac{4g^{\mu\nu}k^2\pi^{d/2}\,\Gamma\left(1-\frac{d}{2}\right)}{(2\pi)^d\,a^{2-\frac{d}{2}}}\left(\frac{1}{2}-\frac{1}{3}\right)$$

$$=\frac{4g^{\mu\nu}k^2\pi^{d/2}\,\Gamma\left(1-\frac{d}{2}\right)}{(2\pi)^d\,a^{2-\frac{d}{2}}}\left(\frac{1}{6}\right)=\frac{2}{3}\frac{\pi^{d/2}\,\Gamma\left(1-\frac{d}{2}\right)}{(2\pi)^d\,a^{2-\frac{d}{2}}}\mathrm{g}^{\mu\nu}\mathrm{k}^2 \qquad (3.28)$$

The second term gives us:

$$\int_0^1\frac{dz}{(2\pi)^d}\left(-2g^{\mu\nu}\,\frac{i\pi^{d/2}\,\Gamma\left(1-\frac{d}{2}\right)d}{a^{1-\frac{d}{2}}}\right)=\frac{-2g^{\mu\nu}k^2\pi^{d/2}\,\Gamma\left(1-\frac{d}{2}\right)d}{(2\pi)^d\,a^{2-\frac{d}{2}}}\int_1^0 dz\,(z-z^2)$$

$$=\frac{-2g^{\mu\nu}k^2\pi^{d/2}\,\Gamma\left(1-\frac{d}{2}\right)d}{(2\pi)^d\,a^{2-\frac{d}{2}}}\left(\frac{1}{6}\right)=-\frac{1}{3}\frac{\pi^{d/2}\,\Gamma\left(1-\frac{d}{2}\right)d}{(2\pi)^d\,a^{2-\frac{d}{2}}}\mathrm{g}^{\mu\nu}\mathrm{k}^2 \qquad (3.29)$$

The third term gives:

$$\int_0^1\frac{dz}{(2\pi)^d}\left(8zk^\mu k^\nu(z+1)\,\frac{i\pi^{d/2}\,\Gamma\left(2-\frac{d}{2}\right)}{a^{2-\frac{d}{2}}}\right)$$

$$=8k^\mu k^\nu\,\frac{i\pi^{d/2}\,\Gamma\left(2-\frac{d}{2}\right)}{a^{2-\frac{d}{2}}}\int_1^0\frac{dz}{(2\pi)^d}\,(z^2+z)=8k^\mu k^\nu\,\frac{i\pi^{d/2}\,\Gamma\left(2-\frac{d}{2}\right)}{(2\pi)^d a^{2-\frac{d}{2}}}\left(\frac{1}{3}+\frac{1}{2}\right)$$

$$=8k^\mu k^\nu\,\frac{i\pi^{d/2}\,\Gamma\left(2-\frac{d}{2}\right)}{(2\pi)^d a^{2-\frac{d}{2}}}\left(\frac{1}{6}\right)$$

$$=\frac{4}{3}\frac{i\pi^{d/2}\,\Gamma\left(2-\frac{d}{2}\right)}{(2\pi)^d a^{2-\frac{d}{2}}}\mathrm{k}^\mu\mathrm{k}^\nu \qquad (3.30)$$

The last term gives us:

$$\int_0^1 \frac{dz}{(2\pi)^d}\left(-4g^{\mu\nu}k^2 z(z+1)\ \frac{i\pi^{d/2}\,\Gamma\left(1-\frac{d}{2}\right)}{a^{2-\frac{d}{2}}}\right)$$

$$= -4g^{\mu\nu}k^2\frac{i\pi^{d/2}\,\Gamma\left(1-\frac{d}{2}\right)}{a^{2-\frac{d}{2}}}\int_1^0 \frac{dz}{(2\pi)^d}(z^2+z)$$

$$= -4g^{\mu\nu}k^2\frac{i\pi^{d/2}\,\Gamma\left(1-\frac{d}{2}\right)}{(2\pi)^d a^{2-\frac{d}{2}}}\left(\frac{1}{3}+\frac{1}{2}\right) = -4g^{\mu\nu}k^2\frac{i\pi^{d/2}\,\Gamma\left(1-\frac{d}{2}\right)}{(2\pi)^d a^{2-\frac{d}{2}}}\left(\frac{1}{6}\right)$$

$$= -\frac{2}{3}\frac{\pi^{d/2}\,\Gamma\left(1-\frac{d}{2}\right)}{(2\pi)^d\ a^{2-\frac{d}{2}}}\mathrm{g}^{\mu\nu}\mathrm{k}^2 \qquad (3.31)$$

We have used $a^{\alpha} = 1 - \alpha\log a$ and neglecting the finite terms, we obtain:

$$\frac{2}{3}\frac{\pi^{d/2}\,\Gamma\left(1-\frac{d}{2}\right)}{(2\pi)^d}g^{\mu\nu}k^2 - \frac{1}{3}\frac{\pi^{d/2}\,\Gamma\left(1-\frac{d}{2}\right)d}{(2\pi)^d}g^{\mu\nu}k^2 + \frac{4}{3}\frac{i\pi^{d/2}\,\Gamma\left(2-\frac{d}{2}\right)}{(2\pi)^d}k^{\mu}k^{\nu}$$

$$-\frac{2}{3}\frac{\pi^{d/2}\,\Gamma\left(1-\frac{d}{2}\right)}{(2\pi)^d}g^{\mu\nu}k^2$$

$$= \frac{4}{3}\frac{i\pi^{d/2}\,\Gamma\left(2-\frac{d}{2}\right)}{(2\pi)^d}k^{\mu}k^{\nu} - \frac{4}{3}\frac{\pi^{d/2}\ \Gamma\left(2-\frac{d}{2}\right)}{(2\pi)^d}g^{\mu\nu}k^2 \qquad (3.32)$$

(3.4)The Beta function

Beta function ($\beta(g)$) gives the rate at which the renormalized coupling constant changes as the renormalization scale (μ) is increased (Collins, 1984).

$$\beta(g) = g\mu\frac{\partial}{\partial\mu}(-\delta_1 + \delta_2 + \delta_3). \qquad (3.33)$$

Where δ contain the contribution of all diagrams in figure 3.1.

$$\delta_1 = -\frac{\mathrm{g}^2}{16\pi^2}\,\frac{\Gamma\left(2-\frac{\mathrm{d}}{2}\right)}{(\mu^2)^{\left(2-\frac{\mathrm{d}}{2}\right)}}\left(C_2(R)+C_2(G)\right) \tag{3.34}$$

$$\delta_2 = \frac{\mathrm{g}^2}{16\pi^2}\,\frac{\Gamma\left(2-\frac{\mathrm{d}}{2}\right)}{(\mu^2)^{\left(2-\frac{\mathrm{d}}{2}\right)}}\,C_2(R) \tag{3.35}$$

$$\delta_3 = \frac{\mathrm{g}^2}{16\pi^2}\,\frac{\Gamma\left(2-\frac{\mathrm{d}}{2}\right)}{(\mu^2)^{\left(2-\frac{\mathrm{d}}{2}\right)}}\left(\frac{5}{3}C_2(G)-\frac{4}{3}n_g C(R)\right) \tag{3.36}$$

Substitute δ in beta function equation we get:

$$\beta(g) = g\mu\frac{\partial}{\partial\mu}[-\frac{\mathrm{g}^2}{16\pi^2}\,\frac{\Gamma\left(2-\frac{\mathrm{d}}{2}\right)}{(\mu^2)^{\left(2-\frac{\mathrm{d}}{2}\right)}}\,\left(C_2(R)+C_2(G)\right)+\frac{\mathrm{g}^2}{16\pi^2}\,\frac{\Gamma\left(2-\frac{\mathrm{d}}{2}\right)}{(\mu^2)^{\left(2-\frac{\mathrm{d}}{2}\right)}}\,C_2(R)$$
$$+\ \frac{\mathrm{g}^2}{16\pi^2}\,\frac{\Gamma\left(2-\frac{\mathrm{d}}{2}\right)}{(\mu^2)^{\left(2-\frac{\mathrm{d}}{2}\right)}}\left(\frac{5}{3}C_2(G)-\frac{4}{3}n_g C(R)\right) \tag{3.37}$$

Now differentiate:

$$\beta(g) = \frac{-\mathrm{g}^3}{16\pi^2}\Gamma\left(2-\frac{\mathrm{d}}{2}\right)\mu\frac{\partial}{\partial\mu}\left(\frac{1}{(\mu^2)^{\left(2-\frac{\mathrm{d}}{2}\right)}}\right)\left\{[C_2(R)+C_2(\mathrm{G})]-\ C_2(R)\right.$$
$$\left.+\ \frac{1}{2}\left(\frac{5}{3}C_2(\mathrm{G})-\frac{4}{3}\mathrm{n_g C(R)}\right)\right\} \tag{3.38}$$

$$\beta(g) = \frac{-\mathrm{g}^3}{16\pi^2}(4-d)\Gamma\left(2-\frac{\mathrm{d}}{2}\right)\left(\frac{1}{\mu^{4-d}}\right)\left(\frac{11}{6}C_2(\mathrm{G})-\frac{4}{6}\mathrm{n_g C(R)}\right) \tag{3.39}$$

Setting $d = 4-\varepsilon$. where ε small value, equation (3.39) becomes

$$\beta(g) =$$

$$\frac{-\mathrm{g}^3}{16\pi^2}(4-(4-\varepsilon))\Gamma\left(2-\left(\frac{4-\varepsilon}{2}\right)\right)\left(\frac{1}{\mu^{4-(4-\varepsilon)}}\right)\left(\frac{11}{6}C_2(\mathrm{G})\right.$$

$$\left.-\frac{4}{6}\mathrm{n_g C(R)}\right) \tag{3.40}$$

$$\beta(g) = \frac{-\mathrm{g}^3}{16\pi^2}(\varepsilon)\Gamma\left(\frac{\varepsilon}{2}\right)\left(\frac{1}{\mu^{\varepsilon}}\right)\left(\frac{11}{6}C_2(G)-\frac{4}{6}n_g C(R)\right) \tag{3.41}$$

So:

$$\beta(g) = \frac{-\mathrm{g}^3}{16\pi^2}(\not{\varepsilon})\left(\frac{2}{\not{\varepsilon}}\right)\left(\frac{1}{\mu^{\varepsilon}}\right)\left(\frac{11}{6}C_2(G)-\frac{4}{6}n_g C(R)\right)$$

$$= \frac{-\mathrm{g}^3}{16\pi^2}(2)\left(\frac{1}{\mu^{\varepsilon}}\right)\left(\frac{11}{6}C_2(G)-\frac{4}{6}n_g C(R)\right) \tag{3.42}$$

Therefore:

$$\beta(g) = \frac{g^3}{16\pi^2}\left(-\frac{11}{3}C_2(G)+\frac{4}{3}n_g C(R)\right) \tag{3.43}$$

Now considering the contribution from the Higgs boson, the equation (3.43) modified to:

$$\beta(g) = \frac{g^3}{16\pi^2}\left(-\frac{11}{3}C_2(G)+\frac{4}{3}n_g C(R)+\frac{1}{3}n_h C_2(R)\right) \tag{3.44}$$

(3.4.1) Calculation of the numerical coefficients b_i^{SM} in the SM

Now we are in the right position to calculate the numerical coefficients in the SM models (Collins, 1984).

$$b_i^{SM} = -\frac{11}{3}C_2(\mathrm{G}) + \frac{4}{3}\mathrm{n_g C(R)} + \frac{1}{3}\mathrm{n_h}C_2(R) \tag{3.45}$$

Consider now the strong Interaction which is based on group SU(3); we have

$C_2(\mathrm{G}) = 3,\ \ \mathrm{n_g} = 3, \mathrm{C(R)} = 1,\ n_h = 1, and$ the Higgs has no color so $C_2(\mathrm{R}) = 0$, therefore:

$$b_3^{SM} = -\frac{11}{3} \times 3 + \frac{4}{3} \times 3 \times 1 + 0 = -7 \tag{3.46}$$

For the Weak Interaction which is based on group $\mathrm{SU}(2)_L$; we have

$$C_2(\mathrm{G}) = 2, \quad \mathrm{n_g} = 3, \quad \mathrm{C(R)} = 1, n_h = 1, \quad C_2(R) = \frac{1}{2}, \text{we get:}$$

$$b_2^{SM} = -\frac{11}{3} \times 2 + \frac{4}{3} \times 3 \times 1 + \frac{1}{3} \times 1 \times \frac{1}{3} = -\frac{19}{6} \tag{3.47}$$

Finally for the Electromagnetic Interaction Hypercharge $\mathrm{U}(1)_Y$; we have to replace in equation (3.43) $\mathrm{C_2(G)}, \mathrm{C(R)}$ and $\mathrm{C_2(R)}$ by $\left(\frac{\mathrm{Y}}{2}\right)^2$ but gauge bosons have no hypercharge, so we set $C_2(\mathrm{G}) = 0$

$$b_1^{SM} = \frac{4}{3} n_g \left(\frac{Y_\psi}{2}\right)^2 + \frac{1}{3} n_h \left(\frac{Y_\Phi}{2}\right)^2 \tag{3.48}$$

Now using the hypercharge of all the standard model particles which can be found in any text book e.g. (Renormalization Group Analysis of the Kobayashi-Maskawa matrix), we get:

$$b_1^{SM} =$$

$$\frac{4}{3}\times 3\times\frac{1}{2}\left(2\times\frac{1}{36}\times 3+\frac{4}{9}\times 3+\frac{1}{9}\times 3+2\times\frac{1}{4}+1\right)+\left(\frac{1}{3}\times 1\times 2\times\frac{1}{4}\right)+\frac{1}{6}=\frac{41}{6} \quad (3.49)$$

The factor $\frac{1}{2}$ appeared in the above equation is representing the helicty.

We use the SU(5) normalization, we end up with:

$$b_1^{SM}=\frac{41}{6}\times\frac{3}{5}=\frac{41}{10} \quad (3.50)$$

(3.4.2) Calculation of the numerical coefficients $b_i^{SM+3scalars}$ with additional scalars

In this model we assume a part of the scalar Higgs boson there is two additional scalar bosons that is $n_h = 3$. Note that the coefficients b_3^{SM} does not change only b_2^{SM} and b_1^{SM} will be modified as follow

$$b_2^{SM+3Higgs}=-\frac{11}{3}(2)+\frac{4}{3}\times 3\times 1+\frac{1}{3}\times 3\times\frac{1}{2}=\frac{-16}{6} \quad (3.51)$$

And

$$b_1^{SM+3Higgs}=\frac{4}{3}\times 3\times\frac{1}{2}\left(2\times\frac{1}{36}\times 3+\frac{4}{9}\times 3+\frac{1}{9}\times 3+2\times\frac{1}{4}+1\right)+\left(\frac{1}{3}\times 3\times 2\times\frac{1}{4}\right)=\frac{43}{6} \quad (3.52)$$

With SU(5) normalization, we get:

$$b_1^{SM+3Higgs} = \frac{43}{6} \times \frac{3}{5} = \frac{43}{10} \tag{3.53}$$

(3.5) The Standard Model in five dimensions (Extra Dimension (EXD))

We will define the SM in a five dimensional flat space time, and the fifth coordinate is compactified in a circle of radius R, where R is the size of the extra dimension, note that, the standard four-dimensional coordinates will be denoted by (x^{μ}) whereas the fifth-dimension coordinate will be represented by (y). As a result of compactification, each of the field expands into a series of modes known as Kaluza–Klein(KK) particles (N.~Maru, 2010).

In the context of universal extra-dimensional (UED), basically we assume that all fields and gauge parameters are periodic functions on this coordinate and expand them in Fourier series with respect to it. In general, for a given field, one has:

$$f(x,y) = \frac{1}{\sqrt{2\pi R}} f^0(x) + \sum_{n=1}^{\infty} f^{n+}(x)\cos(\frac{ny}{R}) + \sum_{n=1}^{\infty} f^{n-}(x)\sin\left(\frac{ny}{R}\right) \tag{3.54}$$

Where the zero modes $f^0(x)$ represent the standard model particles and $f^n(x)$ mode represents the extra dimension KK particles.

We assume all the standard model particles can access the full space time. Therefore all the fields will have KK expansions. The zero-mode will be identified as the SM fields and the rest will be the excited KK modes and will contribute at energy $\geq R^{-1}$.

The gauge coupling constants RGEs evolution in EXD are given by:

$$16\pi^2 \frac{\mathrm{dg_i}}{\mathrm{dt}} = \mathrm{b_i^{SM} g_i^3} + (S(t) - 1)\mathrm{b_i^{5D} g_i^3} \tag{3.55}$$

The beta-function coefficients b_i^{SM} are those of the usual SM given in (3.47-50), which correspond to the zero-mode states, while the new beta-function coefficients b_i^{5D} are given by $b_i^{5D} = \left(\frac{81}{10}, \frac{7}{6}, -\frac{5}{2}\right)$ comes from the excited KK states; and will be calculated in next section and, $S(t) = m_Z R e^t = \mu R$, is the sum of KK modes for $m_Z < \mu < \Lambda$ (where Λ is the cut-off scale in which the theory is valid).

(3.5.1) Calculation of the numerical coefficients b_i^{5D} in Extra Dimensions

In the same way that we used a beta function in the standard model, we used it here to prove that the beta function in extra dimensional models is given by:

$$b_i^{5D} = -\frac{11}{3}C_2(\mathrm{G}) + \frac{8}{3}\mathrm{n_g C(R)} + \frac{1}{3}\mathrm{n_h}C_2(R) + \frac{1}{6}C_2\ (\mathrm{G}) \qquad (3.56)$$

The factor two different in the fermionic contribution in the SM and EXD because both sine and cosine will contribute, and the last term is representing the contribution from the A_5 fields. Following the same procedures as we did for the SM case, we arrive at

$$b_3^{5D} = -\frac{5}{2}\ , \qquad b_2^{5D} = \frac{7}{6}\ and\ \ b_1^{5D} = \frac{81}{10} \qquad (3.57)$$

(3.5.2) Calculation of the numerical coefficients b_i^{5D} in 5D plus additional adjiont scalars

If assume that we have four scalar bosons transforming in the adjoint representation, the last term in equation (3.56) will be changed to $\frac{4}{6}C_2\ (G)$

$$b_i^{5D+4scalars} = -\frac{11}{3}C_2(G) + \frac{8}{3}n_g C(R) + \frac{1}{3}n_h C_2(R) + \frac{4}{6}C_2\ (G) \qquad (3.58)$$

In a similar manner, we obtain:

$$b_3^{5D+4} = -1, \quad b_2^{5D+4} = \frac{13}{6} \quad and\ b_1^{5D+4} = \frac{81}{10} \tag{3.59}$$

(3.6) Pati-Salam (PS) model $SU(4) \times SU(2)_R \times SU(2)_L$ in 4D

In Pati- Salam (J.C. Pati and A. Salam, 1974), the one loop beta coefficients for a $G_1 \times G_2$ gauge group are given by the following equation:

$$\mathrm{b_i}^{PS} = \frac{2}{3}\mathrm{T(R_1)d(R_2)} + \frac{1}{3}\mathrm{T(S_1)d(S_2)} - \frac{11}{3}\mathrm{C_2(G_1)} \tag{3.60}$$

Where (R_1, R_2) is the fermions representation and (S_1, S_2) is the scalars representation. The other symbols that appeared in the above equation(3.60) are; $T(R_i)$ denotes the Dynkin index of the representation R_i, $T(S_i)$ is the Dynkin index of the representation S_i, $C_2(G_i,)$ is the quadratic Casmir operators of the group G_i, $d(R_i)$ and $d(S_i)$ are the dimensions of the representation R_i and S_i respectively.

Considering the theory with gauge group PS: $\mathrm{SU(4) \times SU(2)_R \times SU(2)_L \times Z}$, where Z represents the parity in which $\mathrm{L \leftrightarrow R}$, we have three family in representation of fermions $(4,2,1) \oplus (\overline{4},\ 2, 1)$ plus scalars in $(1, 2, 2) \oplus (10, 3, 1) \oplus (\overline{10},\ 1, 3)$. Here we associate $\mathrm{SU(4)}$ with $\mathrm{G_1}$ and $\mathrm{SU(2)_R}$ with $\mathrm{G_2}$ and $\mathrm{SU(2)_L}$ with $\mathrm{G_3}$. Thus $\mathrm{R_1}$ and $\mathrm{S_1}$ denote the corresponding representation of $\mathrm{SU(4)}$ and the rest will follow. Now plugging all the ingredients in the above equation(3.60) we obtain:

$$b_{SU(4)} = \frac{2}{3}\left(\frac{1}{2}\times 1\times 2+\frac{1}{2}\times 2\times 1\right)(3) + \frac{1}{3}(3\times 1\times 3+0\times 2\times 2+3\times 3\times 1)-\frac{11}{3}\times 4 = -\frac{14}{3} \quad (3.61)$$

Now we associate $\mathrm{SU(2)_R}$ with $\mathrm{G_1}$ and $\mathrm{SU(4)}$ with $\mathrm{G_2}$ and $\mathrm{SU(2)_L}$ with $\mathrm{G_3}$. Thus $\mathrm{R_1}$ and $\mathrm{S_1}$ denote the corresponding representation of $\mathrm{SU(2)_R}$, therefore

$$b_{SU(2)_R} = \frac{2}{3}\left(0\times 4\times 2+\frac{1}{2}\times 4\times 1\right)(3)+\frac{1}{3}\left(2\times 10\times 1+\frac{1}{2}\times 1\times 2+0\times 10\times 3\right) - \frac{11}{3}(2) = \frac{11}{3} \quad (3.62)$$

Finally, we identify $\mathrm{SU(2)_L}$ with $\mathrm{G_1}$ and $\mathrm{SU(4)}$ with $\mathrm{G_2}$ and $\mathrm{SU(2)_R}$ with $\mathrm{G_3}$. Thus $\mathrm{R_1}$ and $\mathrm{S_1}$ denote the corresponding representation of $\mathrm{SU(2)_L}$, thus:

$$b_{SU(2)_L} = \frac{2}{3}\left(\frac{1}{2}\times 4\times 1+0\times 4\times 2\right)(3)+\frac{1}{3}\left(0\times 10\times 3+\frac{1}{2}\times 1\times 2+2\times 10\times 1\right) - \frac{11}{3}(2) = \frac{11}{3} \quad (3.63)$$

(3-7) Pati-Salam model $\mathbf{SU(4) \times SU(2)_R \times SU(2)_L}$ in 5D

Consider now Pati-Salam model in five dimensional, as we saw in 5D SM there will be and additional contribution from A_5 loop and keep in mind that the fermions expansion comes with sine and cosine as result there will be an additional factor two in front of the fermionic contribution, thus we obtain

$$b_i^{5DPS} = \frac{4}{3}T(R_1)d(R_2) + \frac{1}{3}T(S_1)d(S_2) - \frac{11}{3}C_2(G_1) + \frac{1}{6}C_2(G_1) \quad (3.64)$$

Using the same numbers as in 4D PS model we get

$$b_{SU(4)} = \frac{4}{3}\left(\frac{1}{2}\times 1\times 2 + \frac{1}{2}\times 2\times 1\right)(3) + \frac{1}{3}(3\times 1\times 3 + 0\times 2\times 2 + 3\times 3\times 1) - \frac{11}{3}(4) + \frac{1}{6}(4) = 0 \quad (3.65)$$

$$b_{SU(2)_R} = \frac{2}{3}\left(0\times 4\times 2 + \frac{1}{2}\times 4\times 1\right)(3) + \frac{1}{3}\left(2\times 10\times 1 + \frac{1}{2}\times 1\times 2 + 0\times 10\times 3\right) - \frac{11}{3}(2) + \frac{1}{6}(2) = 8 \quad (3.66)$$

$$b_{SU(2)_L} = \frac{4}{3}\left(\frac{1}{2}\times 4\times 1 + 0\times 4\times 2\right)(3) + \frac{1}{3}\left(0\times 10\times 3 + \frac{1}{2}\times 1\times 2 + 2\times 10\times 1\right) - \frac{11}{3}(2) + \frac{1}{6}(2) = 8 \quad (3.67)$$

Chapter Four

Numerical Results, Discussions and Conclusion

(4.1) Introduction

This chapter devoted to our numerical results and discussions for the SM, 5D SM, 4D Pati-Salam model as well as its extension in extra dimension models. We used the method of the RGEs at on-loop level for gauge coupling constants in the SM and Pati-Salam models as well as their extension in extra dimensions; we obtained a set of RGEs and solved them numerically by using the software Mathematica version 9and numerical results are shown here.

(4.2) Numerical results and discussions

We used the initial values adopted at the M_Z scale as follows: for the gauge couplings $g_1(M_Z) = 0.462,\ g_2(M_Z) = 0.651$ and $g_3(M_Z) = 1.22$.

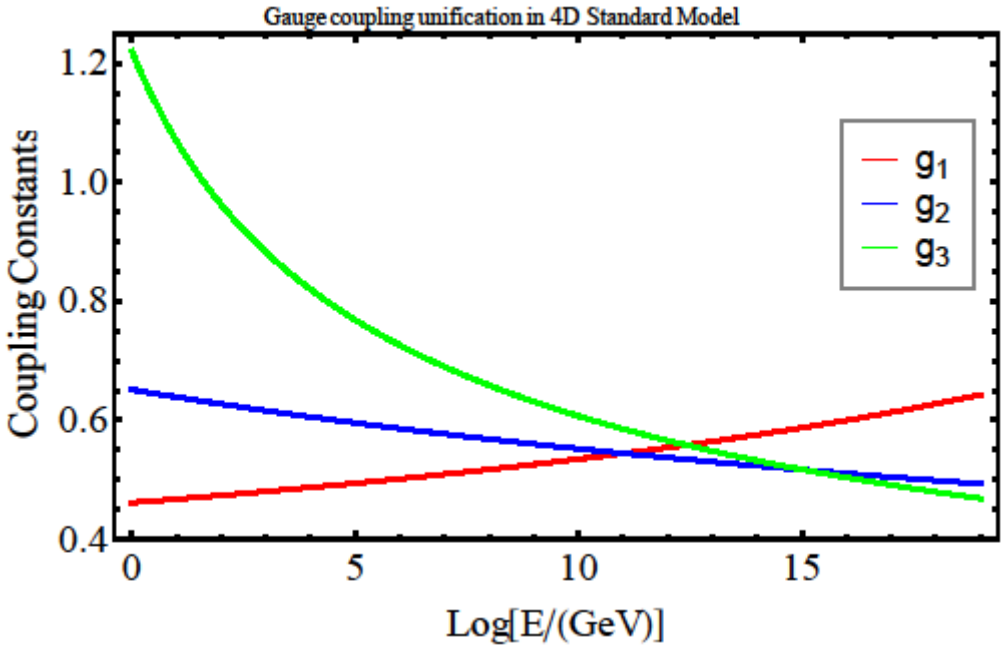

Figure 4.1: Show the running of the gauge coupling constants in the SM

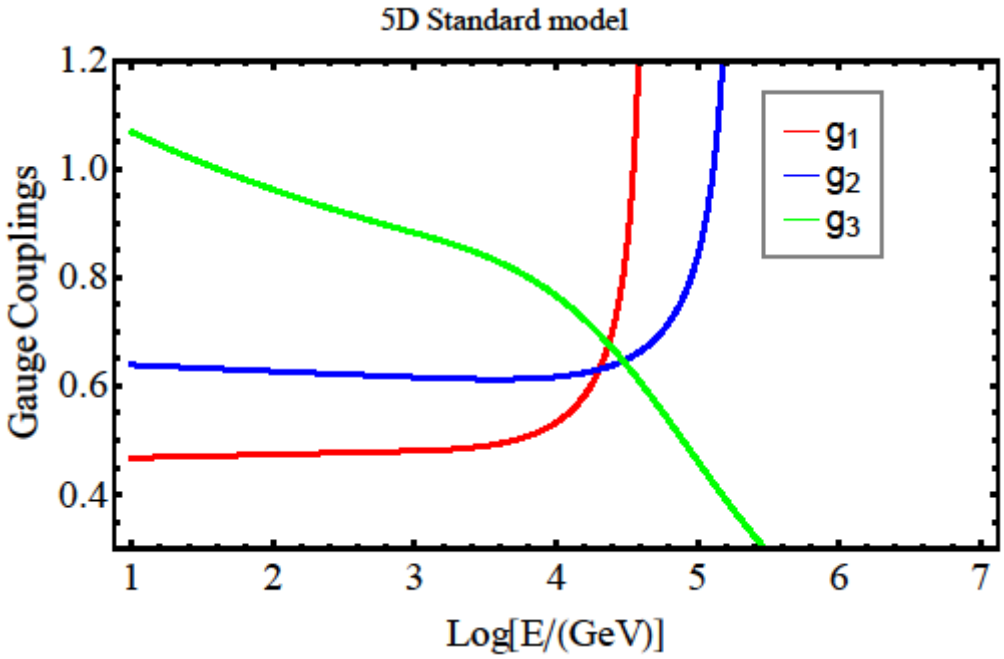

Figure 4.2: Show the running of the gauge coupling constants in the5D SM

In theory of extra dimensions case the SM fields are identified as the zero modes and their result is given in figure 4.1 and figure 4.3 respectively; and Kaluza-Klein (KK) modes will be our new states, so they will contribute to our RGEs and their effect is presented in figure 4.2 and figure 4.4. As can be seen from figure 4.3 for $M_R = 10^{11}\text{GeV}$ there is unification at $\Lambda = 10^{13}\text{GeV}$.

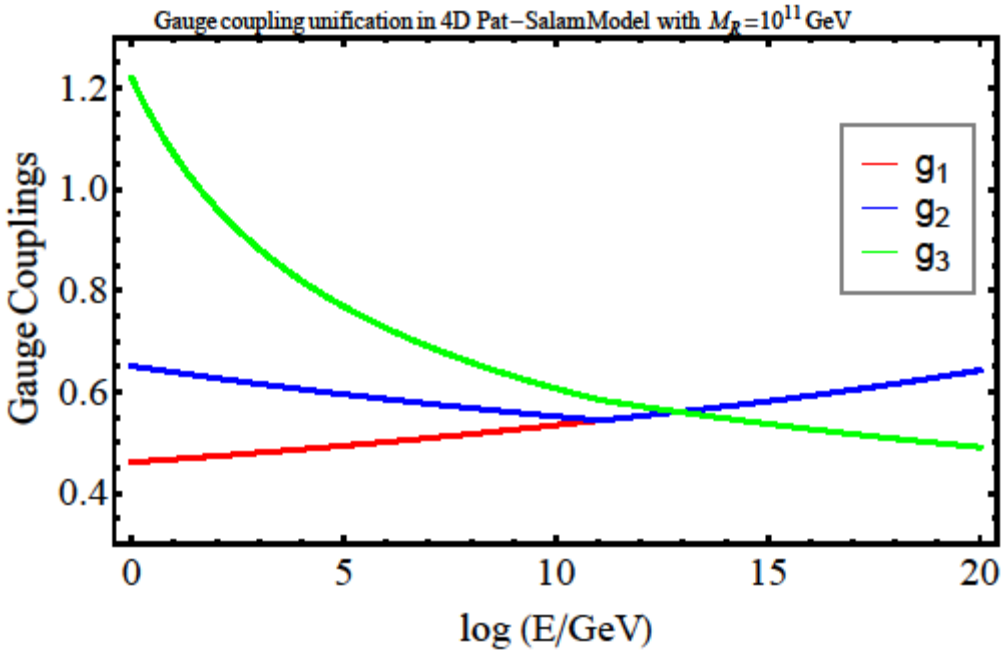

Figure 4.3: Show the running of the gauge coupling constants in 4D Pati-Salam model with $M_R = 10^{11}$GeV.

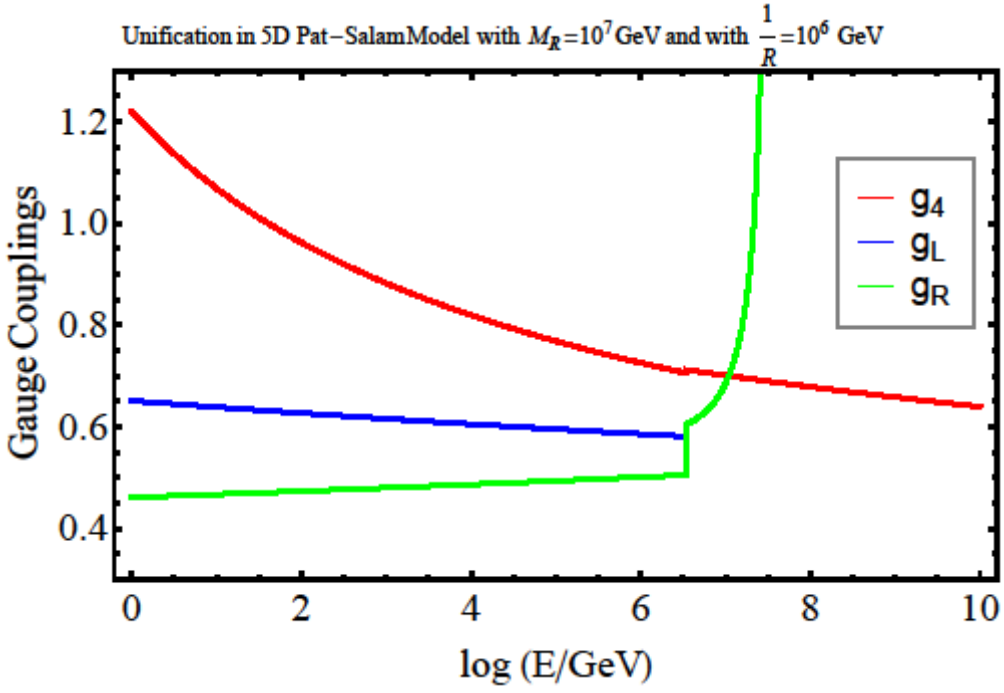

Figure 4.4: Show the running of the gauge coupling constants in 5D Pati-Salam model with $M_R = 10^7$GeV and the compactification scale $\frac{1}{R} = 10^6$GeV

(4.2) Conclusion

We have considered 5D orbifold SM and Pati-Salam models and within them we have addressed the unification issues. Orbifold constructions give an attractive resolution of several outstanding problems of grand unified theories (GUTs), but some extensions are still needed to have full control of difficulties which even appear outside GUTs. Essentially in thesis dissertation we have addressed the question of gauge coupling unification, which in the presence of KK states gets new facets. The effect of these new states as shown in figure 4.3 is power law unification. We have considered the Pati-Salam $SU(4) \times SU(2)_R \times SU(2)_L$ GUTs, and we derived the one loop renormalization group equations for various scenarios i.e SM, 5DSM, PS, 5DPS. Within $SU(4) \times SU(2)_R \times SU(2)_L$, low scale unification can take place is also possible, in some cases the $SU(4) \times SU(2)_R \times SU(2)_L$ predict colored triplet states in the few TeV range.

(4.3) Recommendation

To extend this work, we recommend that one may wish to consider other GUTs models or do the calculation for up to two loop order.

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