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Reachability and Controllability for Time - Invariant Control System with physical Applications

قابلية الوصول والتحكمية لزمن انظمة التحكم اللا متغير مع التطبيقات الفيزيائية

A Thesis submitted in Fulfillment Requirements for the Degree of Ph.D in Mathematics

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Dedication

To My parents.

Who always support

To My Brothers and My Sister

Who always encourage me

To my best friends

Who always stand beside and help me

To Classmates (My batch),

Who don’t find similar to it

Acknowledgments

First, I thank **Allah** for guiding me and taking care of me all the time. My life is so blessed because of hi

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Abstract

The controller is an element which accepts the error in some form and decides the proper corrective action. The block diagrams is shorthand pictorial representation of control system of the cause-and-effect relationship between input and output of physical system or dynamic system is time invariant, is shifting the input of the time axis leads to an equivalent of shifting of the output along the time with no other changes. The aim of this research is study systems that the time invariant system and controllable. The controllability conditions are exactly the same in terms of the matrices for the discrete and continuous case. So we are justified in making multiple inputs, multiple outputs. Control system is a method for using any models of no solution for transfer functions and has extended to multiple inputs, multiple outputs gained selection stability and process. It has been shown that the properties of zeros mean error and single input; single output case can be also achieved. Bounded variance amplification that was seen for the multiple inputs, multiple outputs case. A design procedure is presented to achieved ideal multiple inputs, multiple outputs control case into more sensitive modeling error which are inherent in practical process. Our application is the first solution follows from a general result on the global stabilization of controllable linear system with delay in the input by bounded control laws with a distributed term.

الخلاصة

المتحكم هو عنصر يقبل الخطأ في شكل ما ويقرر الإجراء التصحيحي المناسب. المخططات الصندوقية هي تمثيل مختصر لنظام التحكم لعلاقة السبب والنتيجة بين مدخلات ومخرجات النظام الديناميكي او النظام الفيزيائي، وهو الزمن الثابت، وهو تحويل مدخلات المحور الزمني يؤدي إلى ما يكافئ تحول المخرجات على طول الوقت دون أي تغييرات أخرى. الهدف من هذا البحث هو دراسة الانظمة التي تعمل بنظام الزمن الثابت والسيطرة عليها. شروط التحكم هي نفسها تماما من حيث المصفوفات للحالة المنفصلة والمستمرة. لذلك لابد من التحقق في تقديم انظمة متعددة المدخلات، متعددة المخرجات. نظام التحكم هو طريقة لاستخدام أي نمازج لايمكن حلها بالدوال الناقلة وتمتد الي مدخلات متعددة، مخرجات متعددة اكتسبت استقرار اختياري والمعالجة. وقد تبين أن خصائص الأصفار تعني الخطأ والمدخل الفردي، المخرج الفردي، ويمكن ايضا تحقيق حالة خرج واحدة. تضخيم التباين الممتد الذي شوهد للعديد من المدخلات، حالة المخرجات المتعددة. تم تقديم إجراء التصميم لتحقيق المدخلات المتعددة المثالية، وحالة التحكم في النواتج المتعددة في خطأ النمذجة الأكثر حساسية الكامنة في معاجة العملية. إن تطبيقنا هو الحل الأول من نتيجة عامة على الاستقرار الشامل لنظام خطي خاضع للتحكم مع تأخير في المدخلات من خلال قوانين المراقبة المحددة مع توزيع الفترة.

Introduction

Control theory is an important branch of mathematics that has several applications of distinct area technology, Engineering, Economics, sociology, among others. Control theory has many applications in our life's style Control theories commonly used today are classical control theory (also called conventional .Control theory), modern control theory, and robust control theory. This book, presents comprehensive treatments of the analysis and design of control systems based, on the classical control theory and modern control theory. Automatic control is essential in any field of engineering and science automatic. Control is an important and integral part of space-vehicle systems, robotic systems, modern, manufacturing systems, and any industrial operations involving control of temperature, pressure, humidity, flow, etc. It is desirable that most engineers and scientists are. Familiar with theory and practice of automatic control all necessary background materials are included. Mathematical, background materials related to Laplace transforms and vector-matrix analysis, are presented separately in appendixes. The first significant work in automatic control was James Watt's centrifugal governor, for the speed control of a steam engine in the eighteenth century other, significant works in the early stages of development of control theory were due to Minorsky Hazen, and Nyquist among many others .Minorsky worked on, automatic controllers for steering ships and showed how stability could be determined, from the differential equations describing the system, Nyquist, developed a relatively simple procedure for determining the stability of closed-loop, systems on the basis of open-loop response to steady-state sinusoidal inputs .Hazen, who introduced the term servomechanisms for position control systems, discussed the design of relay servomechanisms capable of closely following a changing, input. Modern control theory is based on time-domain analysis of differential equation, systems. Modern control theory made the design of control systems simpler because. The theory is based on a model of an actual control system. However, the system's, stability is sensitive to the error between the actual system and its model. This means that when the designed controller based on a model is applied to the actual. System, the system may not be stable. To avoid this situation, we design the control. System by first setting up the range of possible errors and then designing the controller.in such a way that, if the error of the system stays within the assumed, range, the designed control system will stay stable. The design method based on this. Principle is called robust control theory. This theory incorporates both the frequency response, approach and the time-domain approach. The theory is mathematically very, complex.

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Chapter one

Introduction

1. Background

The concept of a control system is to sense deviation of the control of the output from the desired value and correct it till the desired output is achieved. The deviation of the actual output from its desired value is called. An error the measurement of error is possible because of feedback. The feedback allows us to compare the actual output with its desired value to generate the error. The error is denoted by as e (t) the desired value of output is also called reference input or a set point the error obtained is required to be analyzed to take the proper corrective action. The controller is an element which accepts the error in some from and decides the proper corrective action. The output of the controller is then applied to the process or final controller element this brings the output back to its desired set point value. The controller is the beart of a control system. The accuracy of the entire system depends on how sensitive is the controller has its own logic to handle the error. Now it is manipulating such an error the controllers such as microprocessors microcontrollers, computers are used such controllers execute certain algorithm to calculate the manipulating sign

Definition of input 1.1

The input is the stimulus excitation or command Applied to a control system typically from an external energy source.

Definition of output 1.2

The output is the actual response obtained from a control system .It may or may not be equal to the specified response implied by the .Inputs and output can have many different forms inputs for examples may be physical variables or more abstract quantities such as reference, set point or desired values for the output of the control system.

Definition of open loop control 1.3

An open loop control system is one in which the control action is independent of the output.

Definition of closed loop control 1.4

A closed loop control system is one in which the control action is somehow dependent on the output closed loop control systems are more commonly called feedback control system.

Definition of feedback control 1.5

Is that property of a closed loop system which permits the output or some other controlled variable to be compared which the input to the system.

Characteristics of feedback 1.6

The presence of feedback typically imparts the following to a system.

A- Increased accuracy.

B- Tendency toward oscillation or in stability

C- Reduced sensitive of the ratio of output to input to variation in system parameters and other Characteristics.

E- Reduced effects of external disturbances or noise

H- Increased band width

The bandwidth of a system is a frequency response measure of how well the system responds to (or filters) variation or frequencies the input signal.

1.1 Control system models or representations

To solve a control systems problem, we must put the specification or description of the system configuration and its components into a form amenable to analysis or design. Three basic representations (models) of components and systems are used extensively in the study of control systems.

1- Mathematical models in the form of differential equations difference equations and or other mathematical relation for example Laplace and

Z-transforms.

2- Block diagrams.

3- Signal flow graphs.

1.1.1 Block diagrams: Fundamentals

A block diagram is a shorthand pictorial representation of the cause and effect relationship between the input and output of a physical system .The simplest form of the block diagram is single block with one input and one output

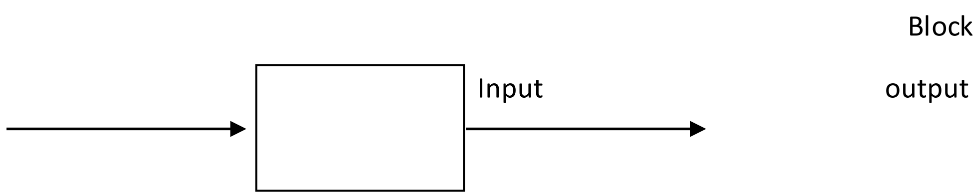


Figure (1.1) block diagram single input and single output

The operation of addition and subtraction have a special representation the block becomes a small circle called a summing point with the appropriate plus or minus sign associated with the arrows entering the circle the output is the algebraic sum of the inputs any number enter a summing point for examples

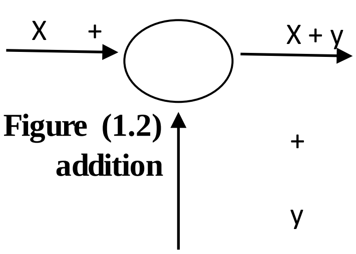


Figure (1.2) addition

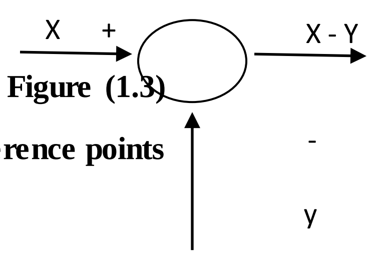


Figure (1.3) Difference points

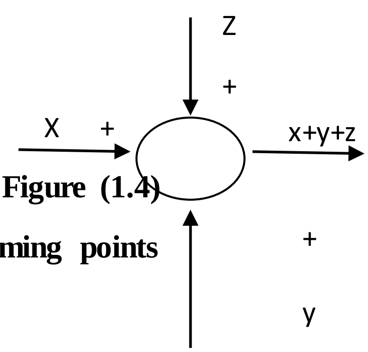


Figure (1.4) Summing points

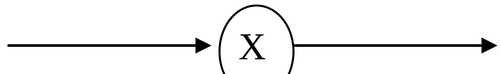


Figure (1.5) some authors put a cross in the circle

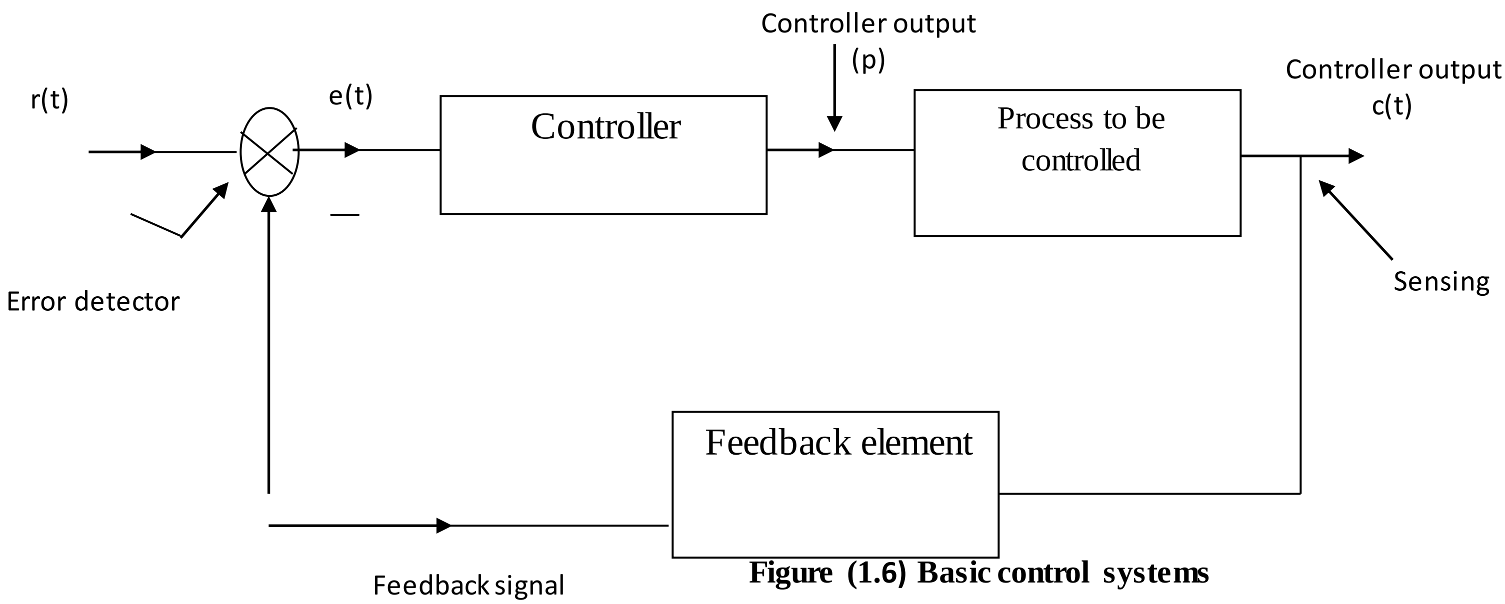


Figure (1.6) Basic control systems

1.1.2 Error

The error detectors compare the feedback signal $b(t)$with the reference input $r(t)$ to generate an error.

$$\therefore e(t) = b(t) - r(t) \qquad (1.1)$$

This gives absolute indication of an error the range of the measured variable $b(t)$

Thus span = $b_{max} - b_{min}$

The Hence error can be expressed as

$$e_P = \frac{r-b}{b_{max}-b_{min}} \times 100 \qquad (1.2)$$

Where $e_P \equiv$ error as % of span

1.Example1

The range of measured variable for ascertains control system is 2 mv to 12 mv and a set point 7 mv. Find the error as percent of span when the measured variable is 6.5 mv.

$$b_{max} = 12\ mv \quad ,b_{min} = 2mv \quad ,b = 6.5\ mv \quad r = 7\ mv$$

$$e_P = \frac{r-b}{b_{max}-b_{min}} \times 100 = \frac{7-6.5}{12-2} \times 100 = 5$$

1.1.3 Variable Range

In practical systems, the controlled variable has a range of values within.

Which the control is required to be maintained. This range specified as the maximum and minimum values allowed for the controlled variable. It can be specified as some nominal values and plus minus tolerance allowed about this value such range is important for the design of controllers

1.1.4 Controller Output Range

Similar to the controller variable a range is associated with a controlled output variable and minimum values. But often the controller output is expressed as a percentage where minimum controller output is 0% and maximum controller out is 100% but 0% controller output does not mean zero output. For example it is necessary of the system that a steam flow corresponding to $\left(\frac{1}{4}\right)^{th}$ opening of the values should be minimum Thus 0% controller output in such case corresponds to the $\left(\frac{1}{4}\right)^{th}$ opening of the value. The controller output as a percent of full scale when the output changes within the specified range is expressed as

$$p = \frac{U - U_{min}}{U_{max} - U_{min}} \times 100 \qquad (1.3)$$

Where

$p \equiv$ Controller output as a percent of full scale

$U \equiv$ Value of the output

$U_{max} \equiv$ Maximum value of controlling variable

$U_{min} \equiv$ Minimum value of controlling variable

1.1.5 Control Lag

The control system can have a large associate with it, the control lag is the time required by the process and controller loop to make the necessary changes to obtain the output at its set point the control lag must be compared with the process lag while designing the controllers for example. In a process value is required to be open or closed for corresponding the output variable physically the of opening. Or closing of the value is very slow and is the part of the process lag. In such a case there is no point in designing a fast controller than the process lag.

1.1.6 Dead Zone

Many a times a dead zone is associated with a process control loop the time corresponding to dead zone is called dead time. The elapsed between the instant when error occurs and instant when first corrective action occurs is called dead time .Nothing happens the error occurs this part is also called dead hand the effect of such dead time must be considered while the design of the controllers.

1.2 classifications of the controllers

The classification of the controllers is based on the response of the controllers and mode of response of the controller

1.2.1 Discontinuous controller Mode

The discontinuous mode controllers are further classified as ON, OFF controllers and multiposition controllers. For example in a simple temperature control of a room the heater is to be controlled it should be switched on or off by the controller when temperature crosses its set point. Such an operation and the mode of operation is called discontinuous mode of controller but in some process control systems simple on/off decision is not sufficient for example controlling the steam slow by opening or closing the value in such case a smooth opening or closing of value is necessary. The controller in such a case is said to be operating in a continuous mode thus the controllers are basically classified discontinuous controllers

1.3.1 Continuous Controller Mode

The continuous mode controllers are further classified as derivative controllers. Some continuous mode controllers can be combined to obtain composite controller mode. For example of such composite controllers are PI, PD and PID controllers. The most of the controllers are placed in the forward path of control system but in some cases input to the controller is controlled though a feedback path. The example of such a controller is rate feedback controller but in the continuous controller output smoothly proportional of the error or proportional to some form of the error. Depending upon which form of the error is used as the input to the controller to product the continuous controller output these controllers are classified as. Proportional control mode, Integral control mode and Derivative control mode

1.3.1.1 Proportional Control Mode

In this control the output of control is simple proportional to the error **e (t)** the relation between the error **e (t)** and controller output P is determined by constant called proportional gain constant denoted by as K_p**,** The output of the controller is a linear function of the error e (t) .Thus each value of the error has error has a unique value of the controller output. The range of the error which covers 0 % to 100% controller output is called proportional hand. Now though there exists linear relation between controller output and the error for a zero error the controller output should not be zero otherwise the process will come to halt. Hence there exists some controller output P_0 for the zero error. Hence mathematically the proportional control mode is expressed as

$$P(t) = k_p e(t) + P_0 \tag{1.4}$$

Where k_p = Proportional gain constant

p_0 = controller output with zero error

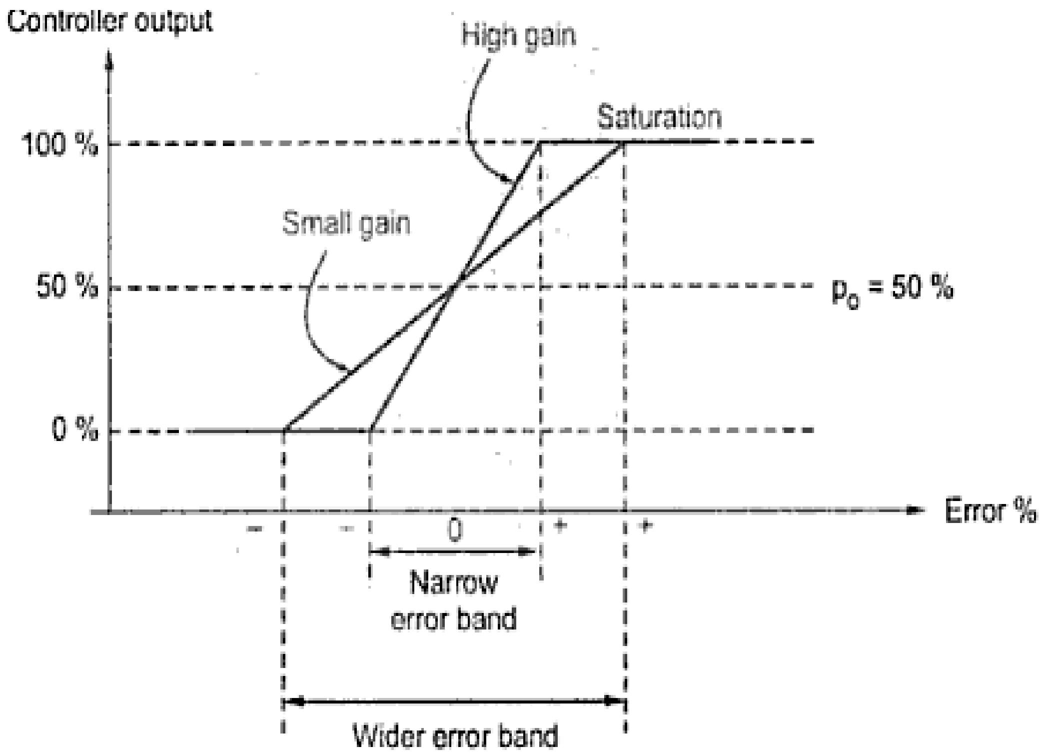


Figure (1.7) the error may be positive or negative the proportional hand

The error may be positive or negative the proportional hand is mathematically defined by

$$P_B = \frac{\mathbf{100}}{\boldsymbol{k_p}} \qquad (\mathbf{1.5})$$

1.3.1.2 Characteristic of proportional control mode

The various characteristics of the proportional mode are

1-When the error of zero, the controller output is constant equal to P_0

2-If the error occurs, then for every 1% of the error the correction of the K_p % is achieved if the error is positive K_p correction gets added to P_0 and if error is negative K_p % correction gets subtracted from P_0

3-The band of error exists for which the output of the controller is the between 0 % to 100 % without saturation

4-The gain P_0 and error band P_B are inversely proportional to each other

1.3.2 Offset

The major disadvantage of the proportional control mode is that it produces an offset error in the output When the load changes the output deviates from the set point such a derivation is called offset error or steady state error such an offset error is the offset error depends upon the relation rate of the controller slow reaction rate produces small offset error while fast reaction rate produces large offset error. The dead time or Transfer lag present in the system further worsens the result it produces not only the large offset at the output but the time required to achieve steady state is also large. The offset error can be minimized by the large proportional gain K_p which reduce the proportional hand. If k_p is made very large the proportional band comes so small that .It acts as an on, off controller producing oscillations about the set point instead of an offset error. The proportional controller can be suitable where

A- Manual reset of the operating point is possible

B- Load changes are small

C- The dead time exists in the system is small

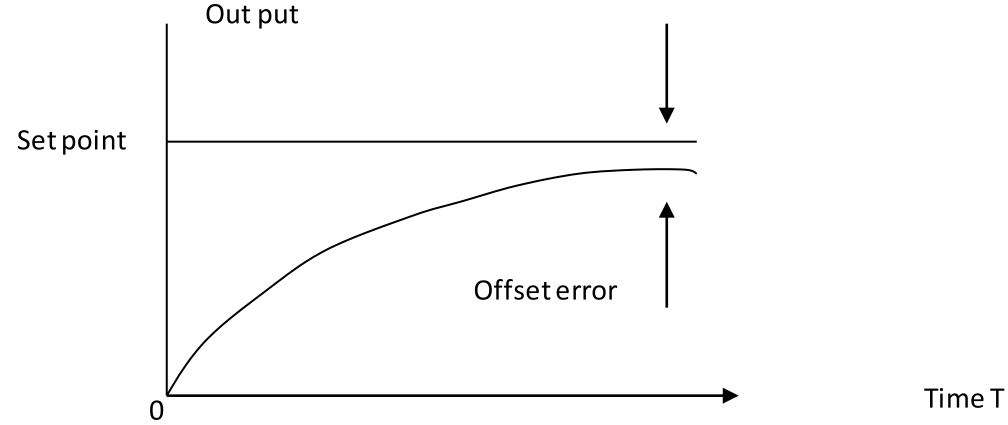


Figure (1.8) offset error in proportional mod**e**

1.3.1.3 Integral Control Mode

In the proportional control mode, error reduces but cannot go to zero. It finally produces an offset an offset error it cannot adapt with the changing load conditions. To avoid this control mode is oftenly used in the control systems which is based on the history of the errors. This mode is called integral mode or reset action controller the value of the controller output p(t) is changed at a rate which is proportional to the actuating error signal e(t) mathematically it is expressed as

$$\frac{dp(t)}{dt} = k_i e(t) \qquad (1.6)$$

Where $k_i \equiv$ constant relating error and rate the constant k_i also integral constant integrating the above equation actual controller output at any time t can be obtained as

$$P(t) = k_i \int e(t)dt + p(0) \qquad (1.7)$$

P (0) $\equiv$ controller output when integral action starts at t=0

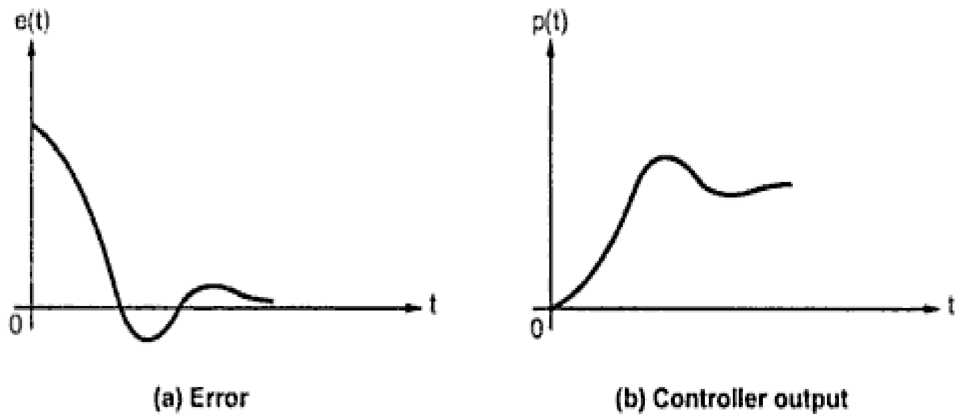


Figure 1.9 Integral mode

The scale factor or constant k_i expresses the scalling between error and the controller output thus a large value of k_i mean that a small error produces a large rate of change of p(t) and viceversa If there is positive error this is shown in the figure 1.8 controller output begins to ramp up. The input error step it can be seen that when error is positive the output p(t) ramp up

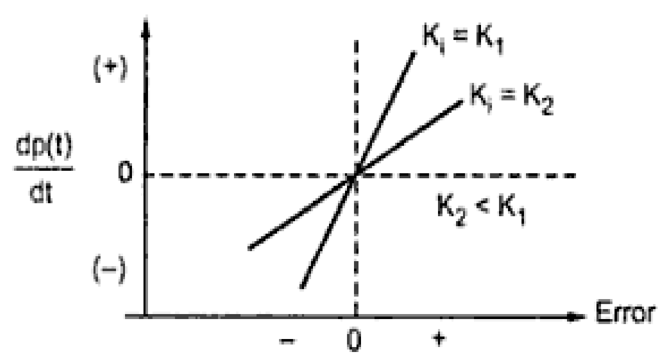


Figure (1.10) the step response of integral control mode

The step response of integral control mode is shown in figure 1.10

The integration time constant is the time taken for the out to the change by an Amount equal to the input error step this is shown in figure 1.11 it can be seen that when error is positive the output ramps up, For zero error there is no change in the output and when error is negative the output p (t) ramps down

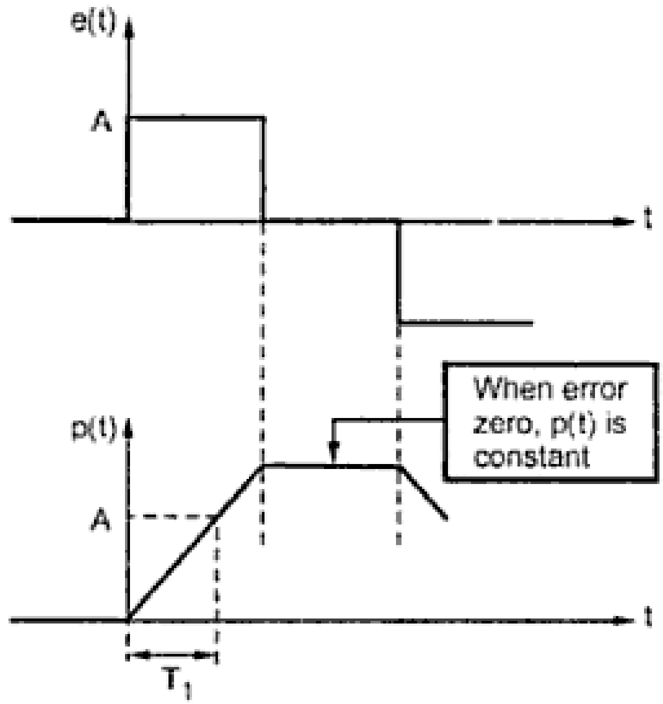


Figure (1.11) step response

1.3.1.4 Characteristics the integral mode

The integrating controller is relatively slow controller its output at a rate which is dependent on the integrating time constant until the error signal is can celled compared to the proportional control. The integral control requires time to build an appreciable output. However it continuous to act till the error signal disappears this corrects the problem of the offset error in the proportional controller. Thus for an integral mode

A- If error is zero, the output remains at a fixed value to what is was when the error became zero.

B- If the error is not zero, then the output begins to increase at a rate k_i % per second for every $\pm 1\%$ for error in some case the inverse of k_i called integral time is specified denoted as T_i

$$T_i = \frac{1}{k_i} = integral\ time \qquad (1.8)$$

It is expressed in minutes instead of second. The comparison of proportional and integral mode behavior at the time of occurrence of an error signal is tabulated below.

Controller	Initial behavior	Steady state behavior
P	Acts Immediately Action according to k_p	Offset error always present larger the k_p smaller the error
1	Acts slowly it is The time integral of the error signal	Error signal always becomes zero

It can be seen that proportional mode is more favorable at the start while the integral is better for steady state response in pure integral mode, error can oscillate about zero and can be cyclic ,Hence is practice integral mode is never used alone but combined with the proportional mode to enjoy the advantages of both the modes.

1.3.1.5 Derivative control Mode

In practice the error is function of time and at a particular instant it can be zero but it may not remain zero for ever after that instant. Hence some action is required corresponding to the rate at which the error is changing. Such

a controller is called derivative controller. In this mode the output of the controller depends on the time rate of change of the actual errors. Hence it is also called rate action mode anticipatory action mode. The mathematical equation for the mode is

$$p(t) = k_d \frac{de(t)}{dt} \tag{1.9}$$

Where $k_d \equiv$ Derivative gain constant indicates by now much % the controller output must change for every % per sec of the change of the error. Generally k_d is expressed in minutes. The important feature of this type of control mode signal there is a unique value of the controller output. The advantage of the derivative control action is that it responds to the rate of change of error and produce of significant correction before the magnitude of the actuating error becomes too large. Derivative control thus anticipates the actuating error initiates an early corrective action and tends to increase stability of the system improving the transient response

1.3.1.6 Characteristics of Derivative Control Mode

The figure 1.12 shows how derivative mode change the controller output for the various rates of the change of the error

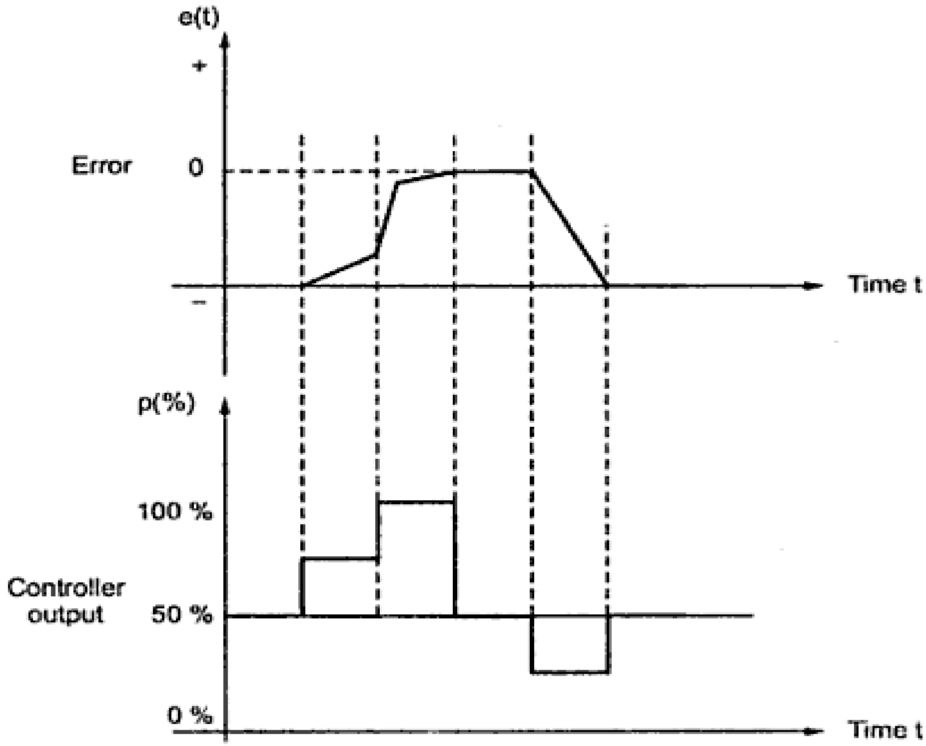


Figure (1.12) the derivative mode changes the controller output for various rates of change of the error.

The derivative mode changes the controller output for various rates of change of the error. The controller output is 50% for the zero error when error starts increasing the controller output suddenly jumps to the higher value it further jumps to higher value for higher rate of increase of error then error becomes constant the output returns to 50% when error is decreasing Having negative slope controller output decrease suddenly to a lower value.

The Various Characteristics of the derivative mode are

A- For a given rate of change of error signal there is a unique value of the controller output.

B- When the error is zero the controller output is zero.

C- When the error is constant - i-e rate of change of error is zero the controller output is zero.

D- When error is changing the controller output changes k_d %

For even 1% per second rate change of error

Note:

Hence it is never used along its gain should be small because faster rate of change of error can cause very large sudden change of controller output this may lead to the instability of the system.

1.3.1.7 Composite Control Modes

As mentionable earlier due to offset error proportional mode is not used alone similarly integral and derivative modes are also not used individually in practice. Thus to take the advantages of various modes together the composite control modes are used the various composite controller modes are.

i. Proportional integral mode (PI).

ii. Proportional Derivative mode (PD).

iii.Proportional + integral + Derivative mode (PID)

Let us see the characteristics of these three modes

Proportional Integral Mode (PI Control Mode)

This is a composite control mode obtained by combining the Proportional mode and the integral mode. The mathematical expression for such a composite control is

$$P(t) = k_p\, e(t) + k_p k_i \int_0^t e(t)dt + P(0) \qquad (1.10)$$

Where $P(0)$= initial value of the output at t=0.The important advantage of this control is that one to one correspondence of proportional mode is available while the offset gets eliminated due to integral mode the integral part of such a composite control provides a reset of the zero error output after load change occurs.

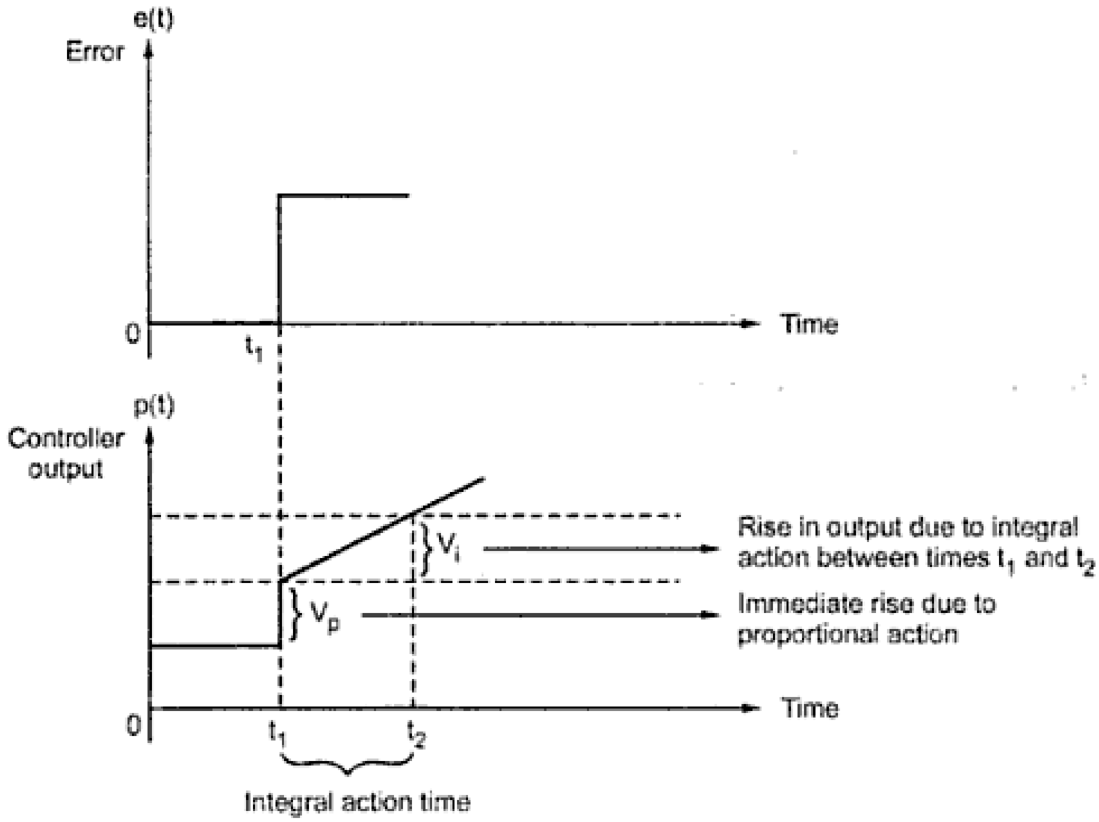


Figure (1.13) Behavior of PI controllers

The composite PI mode completely removes the offset problems of proportional mode such a mode can be used in the system with the frequent or large load changes. But the process must have relatively slow changes in the load to prevent the oscillations.

Proportional + Derivative Mode (PD control mode)

The series combination of proportional and derivative control mode gives proportional plus derivative control mode. The mathematical expression for the PD composite control is

$$P(t) = k_p e(t) + k_p k_d \frac{d\,e(t)}{dt} + P(0) \tag{1.11}$$

The behavior of such a PD control to a ramp type of the input in the figure

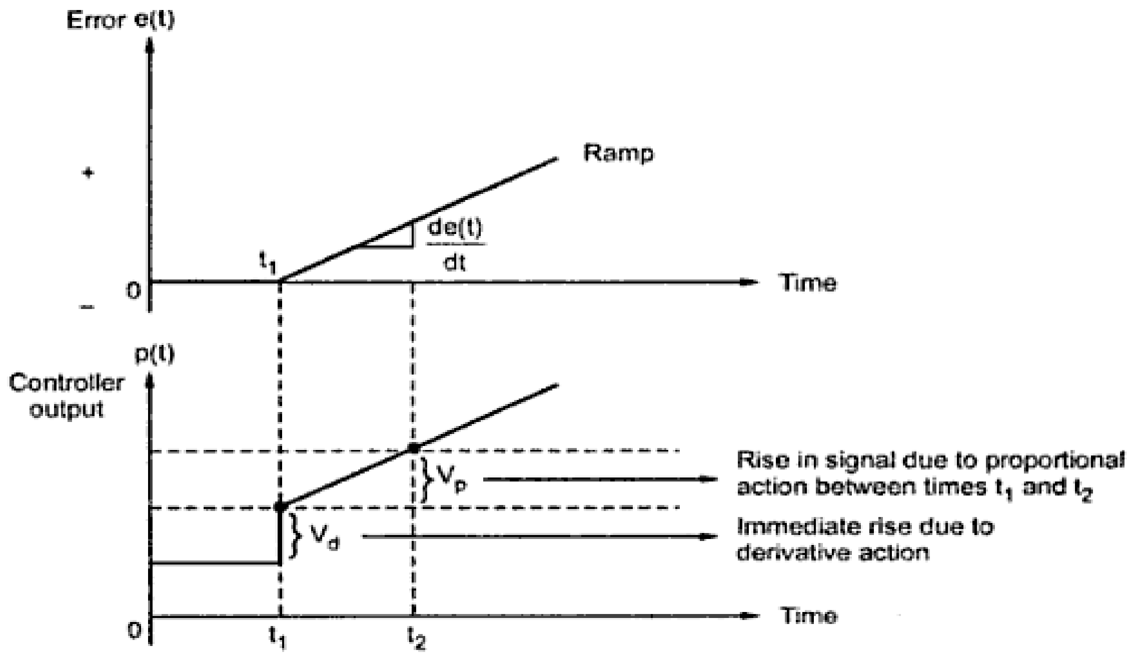


Figure (1.14) Behavior of PD controllers

Rise in signal due to proportional action between time t_1 and t_2

Immediate rise due derivative action

1.3.1.8 Behavior of PD Controller

The ramp function of error occurs at $t = t_1$ the derivative mode cause a step V_d at t_2 and proportional mode cause arise of V_d at t_2.This is for direct action PD control Proportional derivative Type of Controller. A controller in the forward path which changes the controller output corresponding to proportional plus derivative of error signal is PD controller

I.e. output of controller = $ke(t) + T_d \frac{d\,e(t)}{dt}$ (1.12)

Taking Laplace =

$$kE(s) + s\,T_d E(s) = E(s)[k + s\,T_d] \qquad (1.13)$$

the transfer function of such controller is $[k + s\,T_d]$ this can be realized as shown in the igure (15)

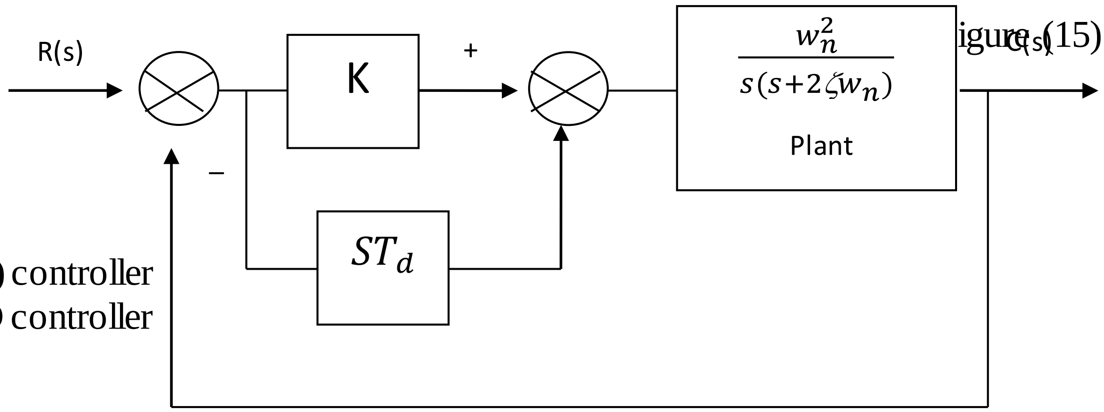


Figure (1.15) controller output of PD controller

Assuming $k = 1$

We can write

$$G(s) = \frac{(1+sT_d)w_n^2}{s(s+2\grave{\zeta}\,w_n)} \qquad (1.14)$$

And

$$\frac{C(s)}{R(s)} = \frac{(1+sT_d)w_n^2}{s^2+s\left[2\zeta w_n+w_n^2\,T_d\right]+w_n^2} \qquad (1.15)$$

Comparing denominator with standard form w_n is same as in the previous p type controller

And

$$2\grave{\zeta}\,w_n = 2\zeta w_n + w_n^2\,T_d \text{ and } \grave{\zeta} = \zeta + \frac{w_n T_d}{2} \qquad (1.16)$$

Because of this controller damping ratio increases by factor $\frac{w_n T_d}{2}$ .I.e. system becomes type in nature. Now as order increase by one system relatively becomes less stable as k_i must be designed in such a way system will remain in stable condition second order system is always stable. Hence transient response gets affected badly if controller is not designed properly.

While

$$k_p = \lim_{s\to 0} G(s)H(s) = \infty \quad ، \; e_{ss} = 0 \qquad (1.17)$$

$$k_v = \lim_{s\to 0} s\, G(s)H(s) = \infty \quad ، \; e_{ss} = 0$$

Hence as type is increased by one error becomes zero for ramp type of inputs steady state of system gets improved and system becomes more accurate in nature. Hence PI Controller has following Effect

i. it increase order of the system

ii. it increase type of the system

iii. Design of k_1 must be proper to maintain stability of system so It makes system relatively state error reduce tremendously for same type of inputs. In general this controller improves steady state part affecting the transient part

1.3.1.9 PID Type of Controller

As PD improves transient and PI improves steady combination of two may be used to improve overall time response of the system this can be realized as shown in the Figure (16)

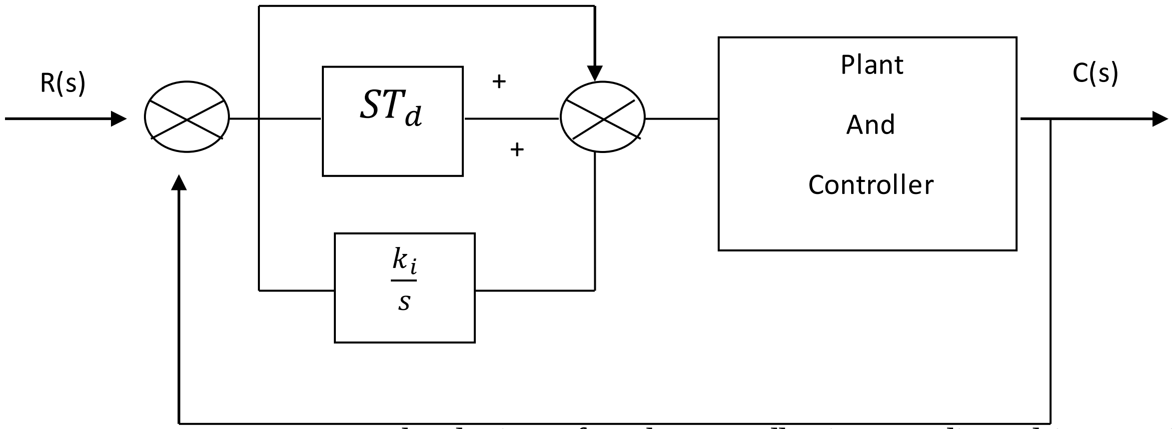


Figure (1.16) the design of such controller is complicated in practice

The design of such controller is complicated in practice, Rate Feedback Controller (output Derivative Controller). This is achieved by feeding back the derivative of output signal internally using a tachogenerator and comparing with the signal proportional to error as shown this is called minor loop feedback compensation output of controller

$$kE(t) - k_t \frac{d\ e(t)}{dt} \quad (1.18)$$

Take Laplace

∴ Output controller $kE(s) - sk_1 c(s)$ (1.19)

This can be realized as shown in the lower Figure (17)

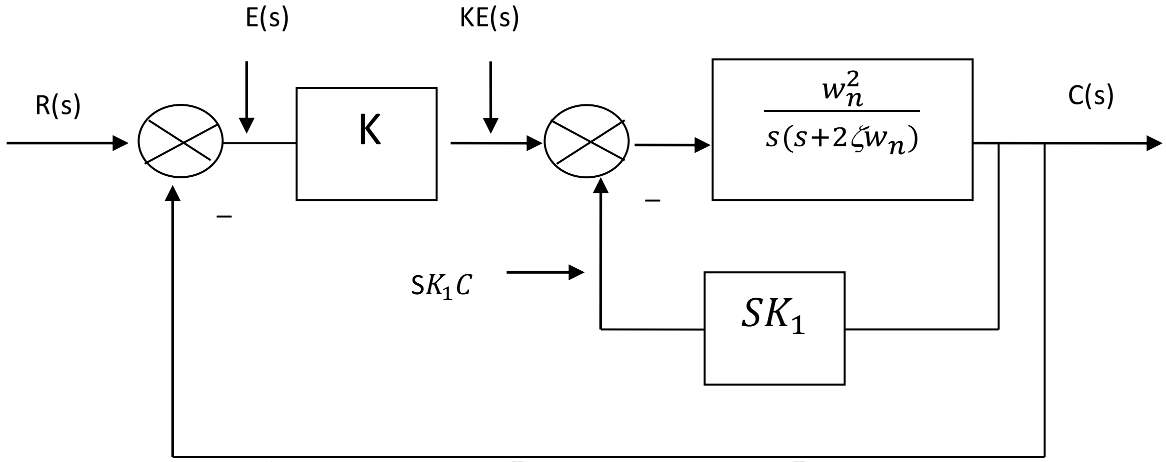


Figure (1.17) controller output PID controller

Assuming $k = 1$

Let us study its effect on same system which is considered earlier

With

$$G(s) = \frac{w_n^2}{s(s+2\zeta w_n)} \quad (1.20)$$

Time constant (T) is the time required by the system output to reach 63.2% of its final value during the first attempt. The equation for the actual response

c (t) is

$$C(t) = 1 - \left\{ \frac{e^{-\zeta w_n t}}{\sqrt{1-\zeta^2}} \sin(w_d t + \theta) \right\} \; and \quad w_d$$

$$= w_n \sqrt{1-\zeta^2} \qquad (1.21)$$

= Damped Frequency of Oscillations and

$$\theta = tan^{-1} \left\{ \frac{\sqrt{1-\zeta^2}}{\zeta} \right\} \qquad (1.22)$$

1.3.6 Steady State Error

Consider a simple closed loop system using negative feedback as shown in the Figure (18)

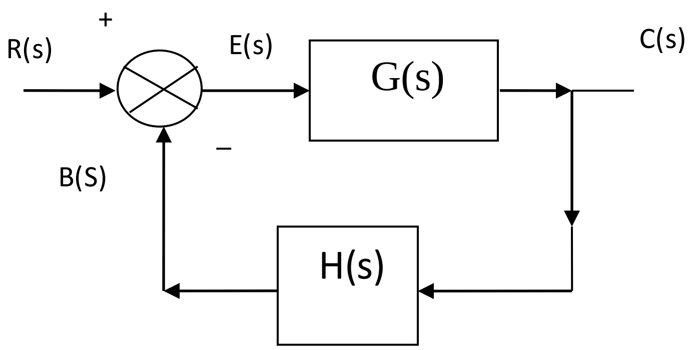


Figure (1.18) Steady State Error

Where

E(s) = Error signal and

B(s) = Feedback signal

$E(s) = R(s) - B(s)\ But\ B(s) = C(s)H(s) and\ \ C(s) = E(s)G(s)$ (1.23)

$$E(s) = R(s) - C(s).H(s) \qquad (1.24)$$

$$E(s) = R(s) - E(s)G(s)H(s) \qquad (1.25)$$

$$R(s) = E(s) + E(s)G(s).H(s) \qquad (1.26)$$

$$\therefore E(s) = \frac{R(s)}{1 + G(s).H(s)} \qquad (1.27)$$

For non-unity Feedback

$$E(s) = \frac{R(s)}{1 + G(s)} \qquad (1.28)$$

For unity Feedback

This E(s) is error in Laplace domain and is expression in‘s’ .We want to calculate the error value in time domain. Corresponding error will be e(t) now steady state of the system is that state which remains as $t \to \infty$

Steady state error

$$e_{ss} = \lim_{t\to\infty} e(t) \tag{1.29}$$

Now we can relate this in Laplace domain by using final value theorem which states that

$$\lim_{t\to\infty} F(t) = \lim_{t\to\infty} sF(s)\mathrm{L}\sum f(t)\} \tag{1.30}$$

Therefore

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) \tag{1.31}$$

Where E(s) is L$\{e(t)\}$

$$e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1 + G(s)H(s)} \tag{1.32}$$

For negative feedback system use positive sign in dominator while use negative sign in denominator if system uses positive feedback From the above expression it can be concluded that stead state error depends on, i. R(s) i.e. Reference input its type and magnitude

ii. G(s).H(s) i.e. open loop transfer function

iii. Dominant non linarites present if any

2.Example 1

The figure (1.19) shown PD controller used for the system .Determine the value of T_d so that system will be critically damped calculate its, settling time

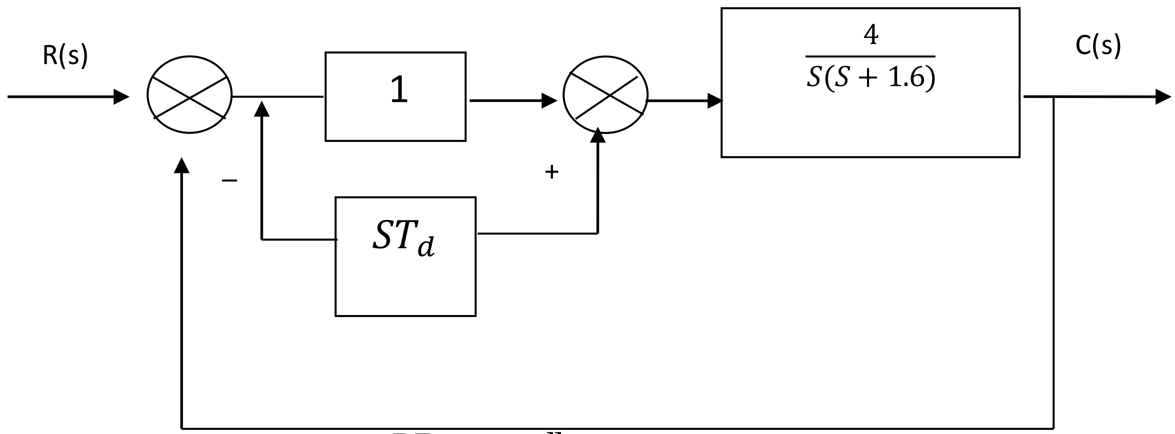


Figure (1.19) PD controller

$G(s) = \frac{(1+s\,T_d)4}{s(s+1.6)}$ H(s) =1

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{(1+sT_d)4}{s(s+1.6)}}{1+\frac{(1+sT_d)4}{s(s+1.6)}} = \frac{(1+s\,T_d)4}{s^2+1.65+4T_d s+4}$$

Comparing denominator with standard form

$w_n^2 = 4,\ \ w_n = 2$ And

$2\zeta w_n = 1.6 + 4T_d$

$$\therefore \zeta = \frac{1.6 + 4T_d}{4}$$

Now system required is critically damped i.e. ζ=1

$$\therefore 1 = \frac{1.6 + 4T_d}{4} \Rightarrow 4 = 1{,}6 + 4T_d$$

$$\therefore 4T_d = 4 - 1.6 = 2.4 \qquad \therefore = T_d = \frac{2.4}{4} = 0.6$$

$\therefore = T_s = \frac{4}{2\times1} = 2\ sec$ And settling time $\frac{4}{\zeta w_n}$

3 .Example 1

APID controller has $k_p = 2.0$, $k_i = 2{,}2\ sec^{-1}$

$k_D = 2\ sec$ And $P_1(0) = 40\%$

Draw the plot of controller output for error

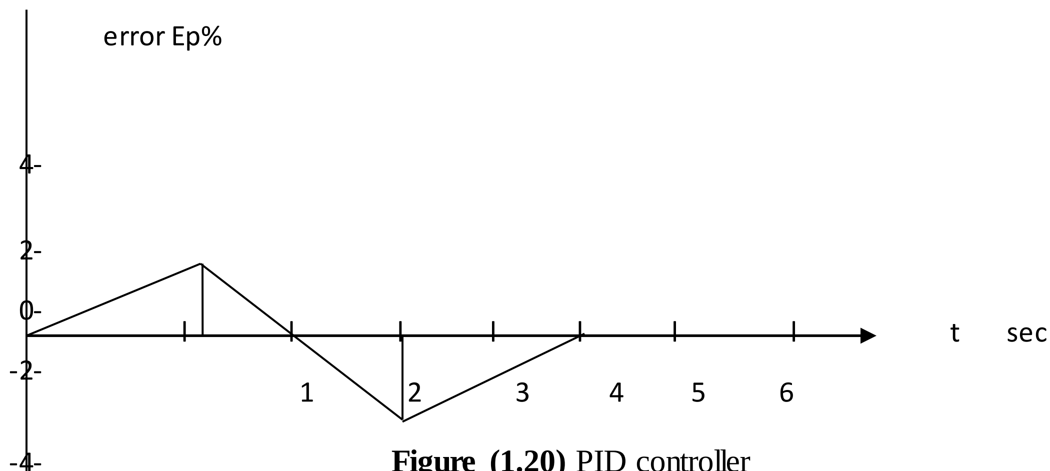


Figure (1.20) PID controller

$k_p = 2,\quad k_1 = 2.2\ sec^{-1} \qquad k_D = 2sec$

$P_1(0) = 40\%$

For 0-2 $E_p = m_1 t$ where $m_1 =$ slope

$E_p = t$ For 0-2 sec $\frac{2-0}{2-0} = 1$ for 2-4 $E_p = m_2 t + c_2$

Two point on the line are (2,2) c_{nd}(4,-3)

$$m_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3-2}{4-2} = -2.5$$

$At(2,2) = -2.5 \times 2 + c_2 = c_2 = 7$

$\therefore E_p = -2.5\ t + 7$ For 2-4 sec

For 4-6 sec

Two points on the line are (4,-3)

$\therefore m_3 = \frac{0-(-3)}{6-4} = \frac{3}{2} = 1.5$

$\therefore E_p = 1.5\ t + c_3$

4+ c_3 ×At (4,-3) =-3 =1.6

$\therefore c_3 = -9$

$\therefore E_p = +1.5\ t - 9$ For 4-6 sec

The mode Equation for PID controller

$$P = k_P E_P + K_P K_i \int_0^t E_P\ dt + K_P K_D\ \frac{dE_p}{dt} + P_1(0)$$

$$\therefore P = 2\ E_P + 4.4 \int_0^t E_P\ dt + 4\ \frac{dE_p}{dt} + 40$$

For 0-2

$$P_1 = 2\ t + 4.4\ \int_0^t t\ dt + 4\ \frac{d}{dt}(t) + 40 = 2\ t + 2.2\ t^2 + 4 + 40$$

This plotted for 0-2 sec

At the end of 2 sec

The integral term has accumulated to

$$P_1(2) = 4.4\ \int_0^t t\ d\ t + 40 = 4.4 \times \left[\frac{t^2}{2}\right] + 40 = 48.8\%$$

For 2-4 sec

$$P_2 = 2(-2.5t + 7) + 4.4\int_2^t(-2.5t + 7)dt + 4\frac{d}{dt}(-2.5t + 7) + 48.8$$

$$-5\,t + 14 + 4.4\,\{-1.25(t^2 - 4) + 7(t - 2)\} - 10 + 48.8$$
$$= -5\,.5\,t^2 + 25.8\,t + 13.2$$

This is plotted for 2-4 sec

At the end of 2 sec the integral term has accumulated to

$$P_1(4\,) = 44(-1.25(t^2 - 4) + 7) + 7(t^2 - 4)\}|_{t=4} + 48.8 = 44.4\%$$

For 4-6 sec

$P_3 = 2(1.5\,t - 9) + 4.4\,\int_4^t(1.5\,t - a)d\,t + 4\,\frac{d}{dt}(1.5\,t - a) + 44.4$ = 3 t - 18 +4.4 [0.75 t^2-a t]$_1^T$ $+ 4 \times 1.5 + 44.4 = 3\,t - 18 + 44\,\{0.75(t^2 - 16) - 9(t - 4)\} + 6 + 44.4 = 3.3\,t^2 - 36.6\,t + 138$

This is plotted for 4-6 sec

After 6 sec error is zero hence the out will simply be the accumulated integral response providing a constant out put

$\therefore\; P_1(6) = 44[075(t^2 - 16) - a(t - 4)|_{t=1}$ + *44.4 =31.2%*

The complete graph of controller output is shown in the figure (21)

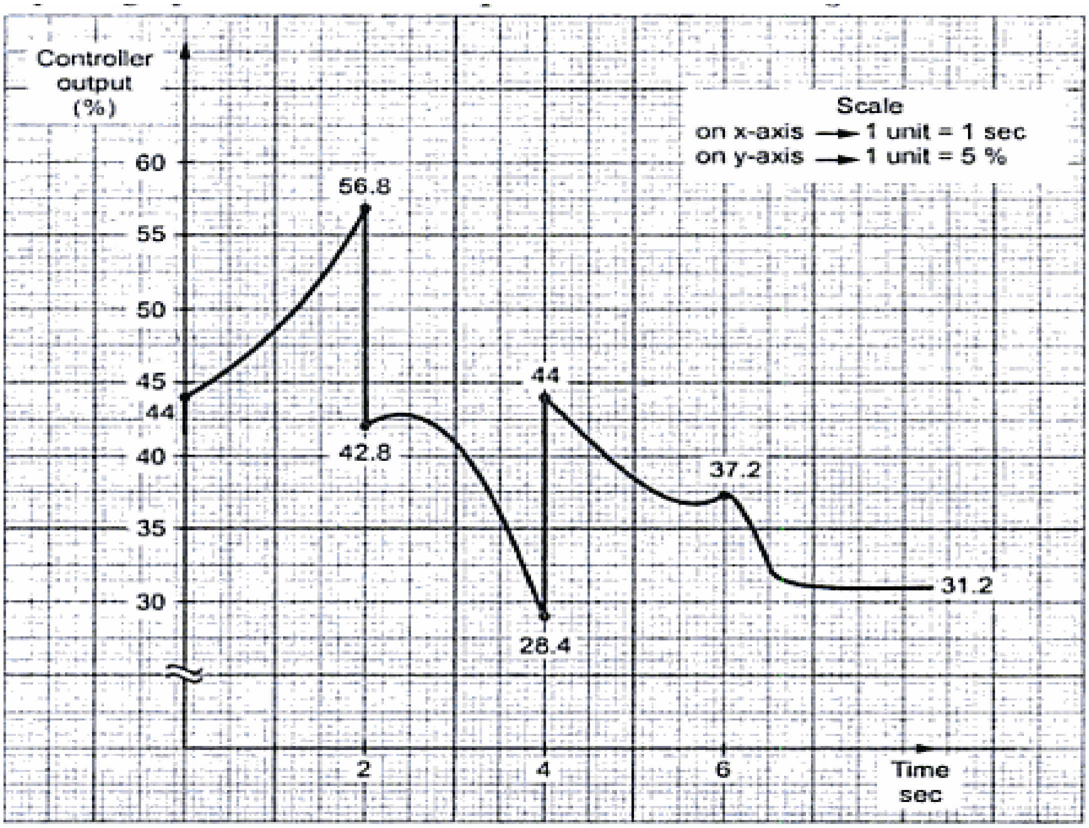


Figure (1.21) graphical solution of PID controller

4.Example 1

Temperature control system has the block diagram given in figure

The input signal is a voltage and represents the desired temperatures θ_r is a unit step the integral term and

(i) D(s) =1 (ii) D(s)= $1+\frac{0.1}{s}$

(iii) D(s) = 1+0.3 s

What is effect of the integral term in the PI controller and derivative term in PD controller on the steady state error

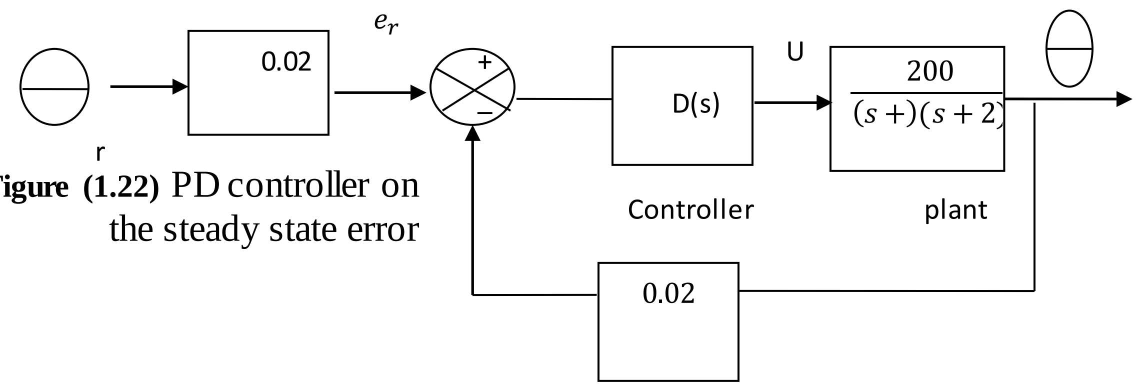


Figure (1.22) PD controller on the steady state error

For given system

$$G(s) = \frac{200D(s)}{(s+1)(s+2)}$$

$H(s) = 0.02$

$R(s) = \frac{0.02}{s}$

$e_{ss} = \lim_{s\to 0} s\,F(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)H(s)}$

D(s) =1

$$\therefore e_{ss} = \lim_{s\to 0} \frac{\frac{s\times 0.02}{s}}{1+\frac{200}{(s+1)(s+2)}\times 0.02} = \frac{0.02}{1+\frac{200\times 0.02}{2}} = 6.66\times 10^{-3}$$

(ii) D(s) = 1+ $\frac{0.1}{s}$

$$\therefore\ e_{ss} = \lim_{s\to 0} \frac{s\times\frac{0.02}{s}}{1+\left[\frac{200(1+0.3s)}{(s+1)(s+2)}\times 0.02\right]} = 6.66\times 10^{-3}$$

Due to PI controller the steady state error reduces drastically while PD controller has no effect on the stead state error

5.Example 1

An integral controller is used for temperature control within arrange 40-60c^o the set point is 48 c^o, the controller output is initially 12% .When error is zero the integral constant $k_1 = -0.2\%$ controller output per second percentage error if the temperature increase 54c^o,Calculate the controller output After 2 sec for a constant error

For integral controller

$$P(t) = k_i \int E_p\, dt + P(0)$$

Controller output initially denoted by $P(0) = 12\%$ the integral constant denoted by $k_1 = -0.29\%$ sec % error

E_P = Error = constant =$\frac{r-b}{b_{max}-b_{min}} \times 100$

Now r= set point =48c^o

b= Actual temperature = 54c^o

$b_{max} = 60c^o$، $b_{m-x} = 40c^o$

$$E_P = \frac{48-54}{60-40}\times 100 = -30$$

$\int E_{Pdt=E_p\ t\ As\ error\ is\ constant}$

P (t) = (-0.2) (-30)(t)+12

At $t = 2 = (0.2)(30 \times 2) + 12 = 24\%$

This controller output after 2 sec

Example 1.5

A proportional controller is employed for the cont. of temperature in the range 50c^o -130c^o with a set point of 73.5 c^o the zero error controller output is 60% what will be the offset error resulting from a change in the controller output to 55% the proportional gain is 2% find the offset in c^o

For proportional control mode

$P = k_P E_P + P(0)$

P (0) = controller output with no error =50%

k_P =Proportional gain = 2% per second

$$E_P = \frac{P - P_0}{k_P} = \frac{55 - 50}{2} = 2.5$$

For the design of a control system it's important to understand how the system of interest behave and how it respond to deferent control design the Laplace transform as discussed in the Laplace transform module is a valuable tool that can be used to solve differential equations and obtain the dynamic

A Transfer function G(s) is defined as the following relation between the output of the system Y(s) and the input to the system U(s)

$$\mathrm{G(s)} = \frac{Y(s)}{U(s)} \qquad (1.33)$$

The roots of polynomial U(s) poles of the system and roots of Y(S) is called the zeros of the system If the input of the system is a unit impulse (U(S) =1)

1.4 Blok Diagrams and Transfer functions

The combination of Blok Diagrams and Transfer functions is a powerful way to represent the control systems. Transfer function relating different signals in the system. Derivation of transfer functions from state model.

Consider a standard state model derived for linear time invariant system as

$$\dot{X}(t) = AX(t) + BU(t) \qquad (1.34)$$

$$Y(t) = CX(t) + DU(t) \qquad (1.35)$$

And taking the Laplace transfer of both sides

$$[SX(s) - X(0)] = AX(s) + BU(s) \; and \; Y(s) = CX(s) + DU(s) \qquad (1.36)$$

Note that as the system is time invariant the coefficient of matrices A, B, C and D are constant while the definition of the transfer function is based on the assumption of zero initial condition

$$X(0 = 0)$$

$$\therefore SX(s) = AX(s) + BU(s) \; and \; SX(s) - AX(s) = BU(s) \qquad (1.37)$$

Now S is an operator while A is matrix of order (n×n), hence to match the orders of two terms on lift hand side multiply "S" by identity matrix I of the same order

$$\therefore SIX(s)AX(s) = BU(S) \; and \; [SI - A]X(s) = BU(s) \qquad (1.38)$$

Multiplying both sides by $[SI - A]^{-1}$

$$[SI - A]^{-1}[SI - A]X(s) = [SI - A]^{-1}BU(s) \quad (1.39)$$

$$\therefore \; X(s) = [SI - A]^{-1}BU(s) \quad (1.40)$$

Substituting in the equation (1.36) we get

$$Y(s) = C[SI - A]^{-1}BU(s) + DU(s) \; and \, Y(s)$$
$$= [C([SI - A]^{-1}B + D)U(s)] \quad (1.41)$$

$$\therefore \; \mathrm{T(s)} = \frac{Y(s)}{U(s)} = C[SI - A]^{-1}B + D \quad But \; [SI - A]^{-1} = \frac{Adj[SI-A]}{|SI-A|} \quad (1.42)$$

The state model of a system is not unique but the transfer function of obtained from any state model is unique. It is independent of the method used to express the system in state model form.

1.4.1 Characteristic equation

It is seen from the excretion of transfer function that the denominator is $[SI - A]$ the equation obtained by equating denominator of transfer function to zero is called characteristic equation the root of this equation are the closed loop poles of the system thus the characteristic equation of the system is $[SI - A] = 0$ The stability of the system depends on the roots, the roots of the equation $[SI - A] = 0$ are called Eigen values of matrix A and this are generally denoted by λ

7 .Example1

Consider a system having state model

$$\begin{bmatrix} x^{\cdot}{}_1 \\ x^{\cdot}{}_2 \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 5 \end{bmatrix} U \text{ And}$$

$$Y = [1 \quad 1]\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

With D=0 obtain

$$[\, s\,I - A\,]^{-1} = \frac{\mathrm{Adj}\,[SI - A]}{|s\ I - A|}$$

$$[SI - A] = \begin{bmatrix} S+2 & 3 \\ -4 & S-2 \end{bmatrix}$$

$$Adj = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T = \begin{bmatrix} S-2 & 4 \\ -3 & S+2 \end{bmatrix}^T = \begin{bmatrix} S-2 & -3 \\ 4 & S+2 \end{bmatrix}$$

$$|s\ I - A| = (S+2)(S-2) + 12 = S^2 - 4S + 12 = S^2 + 8$$

$$[\, s\ I - A\,]^{-1} = \frac{\begin{bmatrix} S-2 & -3 \\ 4 & S+2 \end{bmatrix}}{S^2 + 8}$$

$$\text{T.F} = \frac{[1 \quad 1]\begin{bmatrix} S-2 & -3 \\ 4 & S+2 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix}}{S^2 + 8} = \frac{[8S+1]}{S^2+8}.$$

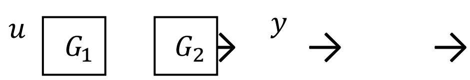


Figure (1.23) product of the transfer functions.

The transfer function of the system is $G = G_1 G_2$ ie the product of the transfer functions.

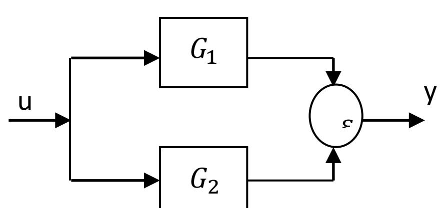


Figure (1.24) a parallel connection of systems

Consider a parallel connection of systems with the transfer functions G_1 and G_2.Let $u = e^{st}$ be the input to the system the pure exponential output of first system $y_1 = G_1 u$ and the output of the second system is $y_2 = G_2 u$

The pure exponential output of the parallel connection is thus

$y = G_1 u + G_2 u = (G_1 + G_2)u$

The transfer function of the parallel connection is thus

$$G = G_1 + G_2$$

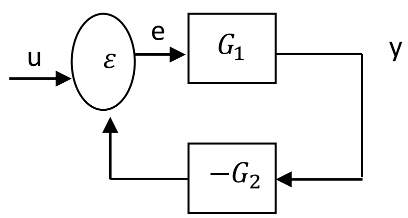


Figure (1.25) feedback connection of systems

Consider a feedback connection of systems with the transfer functions G_1 and G_2.Let $r = e^{st}$ be the input to the system y the pure exponential output and be the pure exponential part of the error. Writing the relations for the different blocks and a summation unit we find

$$y = G_1 e \quad and \ e = r - G_2 y$$

Elimination of e gives $y = G_1(r - G_1 y$.Hence (1+$G_1 G_2$) y= $G_1 r$

Which implies $y = \frac{G_1}{1+G_1 G_2}$ $.r$.The transfer function of the feed Feedback connection is thus

$$G(s) = \frac{G_1(s)}{G_1(s) + G_2(s)}$$

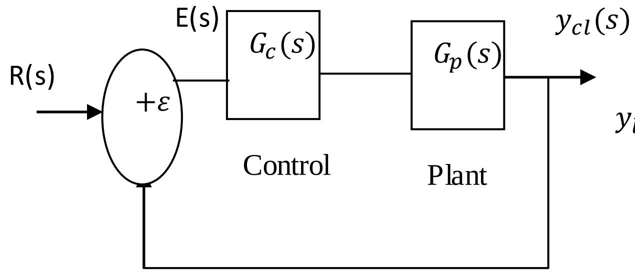


Figure (1.26) Closed loop transfer function feed back

$$y_{cl} = G_p G_c (R - y_{cl}) \quad , \quad T_{(s)} = \frac{y_{cl}(s)}{R(s)}$$

$$R G_p G_c = y_{cl}\left(G_p G_c + 1\right) \qquad , y_{cl} = R \frac{G_p G_c}{1 + G_p G_c}$$

$$\therefore T(s) = \frac{G_p G_c}{1 + G_p G_c} \qquad (1.43)$$

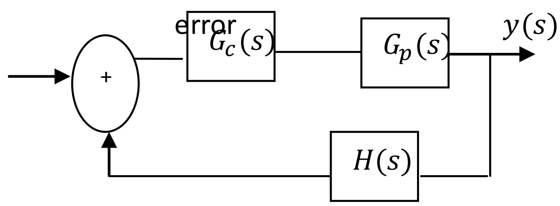


Figure (1.27) Closed loop transfer function with a senor transfer function

$$y = G_p G_c (R - y(s) H(s)) \Rightarrow y = G_p G_c R - G_p G_c H y$$

$$\left(1 + G_p G_c H\right) y = G_p G_c R$$

$$\therefore T(s) = \frac{y(s)}{R(s)} = \frac{G_p G_c}{1 + G_p G_c H} \qquad (1.44)$$

1.5 Signal flow graphs

A signal flow graph is pictorial representation of the simultaneous equations describing a system. It graphically displays the transmission of the signal through the system as does the block diagram. But it is easier draw and therefore easier to manipulate than the block diagram. The properties of signal flow graphs are represented

1.5.1 Fundamental of signal flow

Let us first consider the simplest equation

$$x_i = A_{ij}x_j \qquad (1.45)$$

The variables $x_i\ and\ x_j$ can be functions of time complex frequency or other quantity they many even constants which are variables in mathematical sense.

For signal flow graphs A_{ij} is a mathematical operator mapping $x_i\ into\ x_j$ and is called the transmission function .For example A_{ij} may be a constant in which case x_i is a constant times x_j in equation (1.45) if x_i and x_j are functions of s or z A_{ij} may be transfer function $A_{ij}s\ and\ A_{ij}z$ the signal flow graph for equation (1.45)is given in figure 1this simplest form of a signal flow graph .Note that the variables $x_i and\ x_j$ are represented by a small dot called anode and the transmission function A_{ij} is represented by a line with arrow called a branch

Figure (1.28)

Every variable in a signal flow graph is designated by anode and every transmission function by a branch. Branches are always unidirectional. The arrow denotes the direction of signal flow graph

Example 1.8

Ohm,s law state that $E = RI$ where E is a voltage I a current and R a resistance the signal flow graph for this equation is given by

Figure (1.29)

The addition rule

The value of the variable designated by anode is equal to the sum of the all signals entering the node in the other words the equation represented by

$$x_i = \sum_{j=1}^{n} A_{ij} x_i \tag{1.46}$$

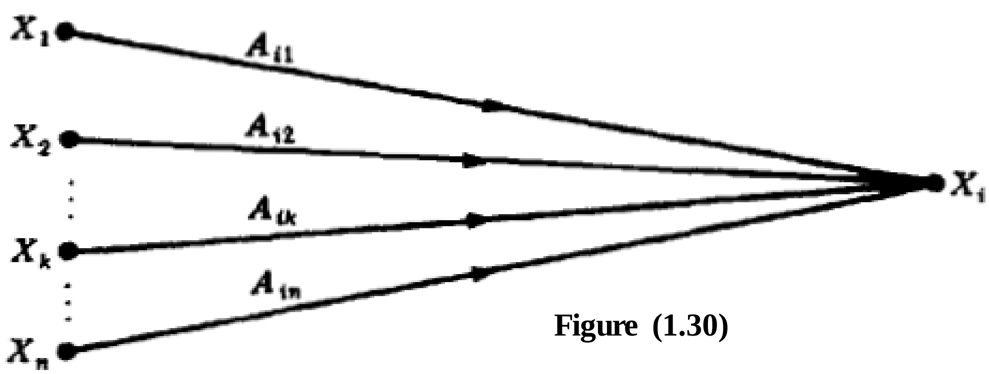


Figure (1.30)

The Transmission Rule

The value of the variable designated by anode is transmitted on every branch leaving that node in other words the equation

$$x_i = A_{ik} x_k \quad i = 1,2,\cdots,n, k \; fixed \tag{1.47}$$

Is represented by

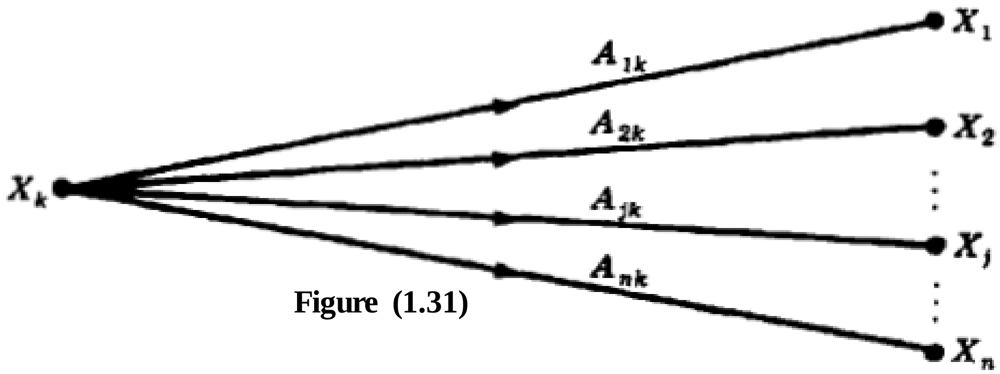


Figure (1.31)

Example 1.9

The signal flow graph of simultaneous equations $Y = 3X$, $Z = -4x$ is given by figure (1.32)

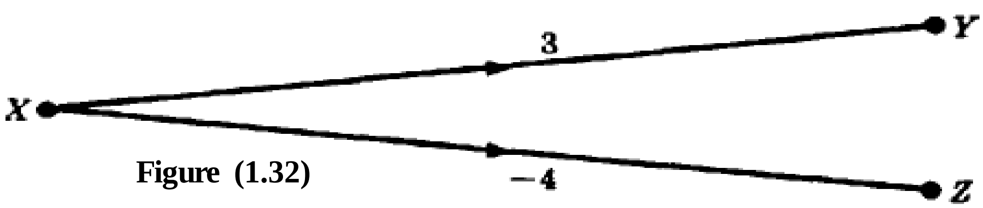


Figure (1.32)

The Multiplication Rule

A cascaded (series) connection of $n - 1$ branches with transmission function equal to product of the old ones that is $x_n = A_{21} \cdot A_{32} \cdot A_{43} \cdot \cdots A_{n(n-1)} \cdot x_1$

The signal flow graph equivalence represented by figure

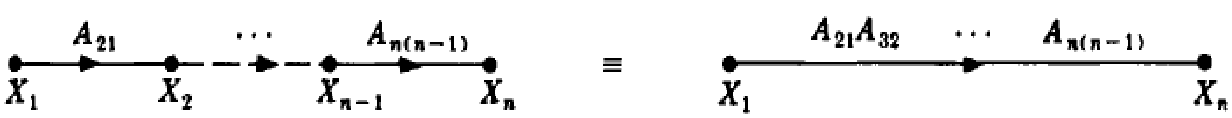


Figure (1.33)

Example 1.10

The signal flow graph of the simultaneous equation $Y = 10x$, $Z = -20x$ is the given by the by figure

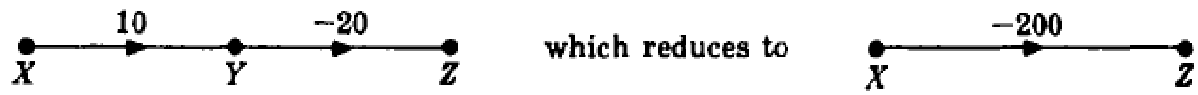


Figure (1.34)

Definitions 1.8

The following terminology is frequently used in signal flow graph theory the example associated with each definition refer to figure

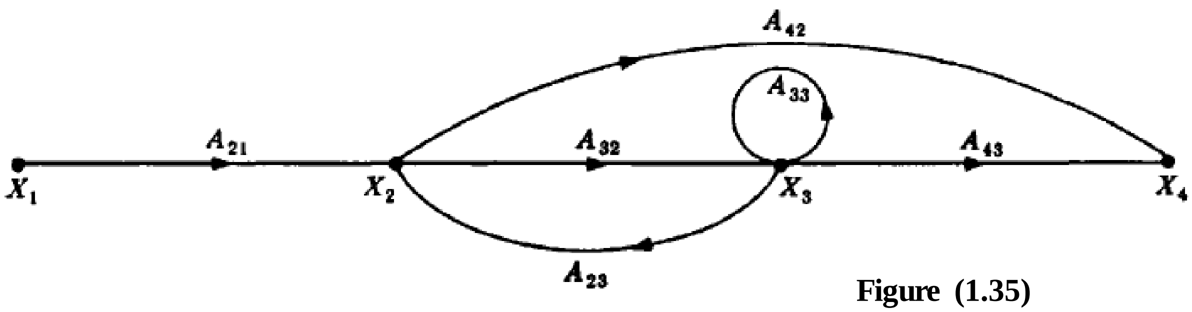


Figure (1.35)

Definition 1.8.1

A path is a continuous, unidirectional succession of branches along which co node is passed more than once

For example $x_1\ to\ x_2\ to\ x_3\ to\ x_4$,$x_2\ to\ x_3$ and back to $x_2\ to\ x_1\ to\ x_2\ to\ x_4$ are paths

Definition 1.8.2

An input node or source is anode with only outgoing branches .For example x_i is an input node

Definition 1.8.3

An output node or sink is anode with only incoming branches For example x_4 is an output node

Definition 1.8.4

A forward path is path from the input node to the output. For example$x_1\ to\ x_2\ to\ x_3\ to\ x_4$ are forward path

Definition1.8.5

A feedback path or feedback loop is a path which originates on the same node

For example $x_2\ to\ x_3\ and\ back\ to\ x_2$ is a feedback path

Definition 1.8.6

A self-loop is a feedback loop consisting of a single branch .For example A_{33} is a self-loop

Definition 1.8.7

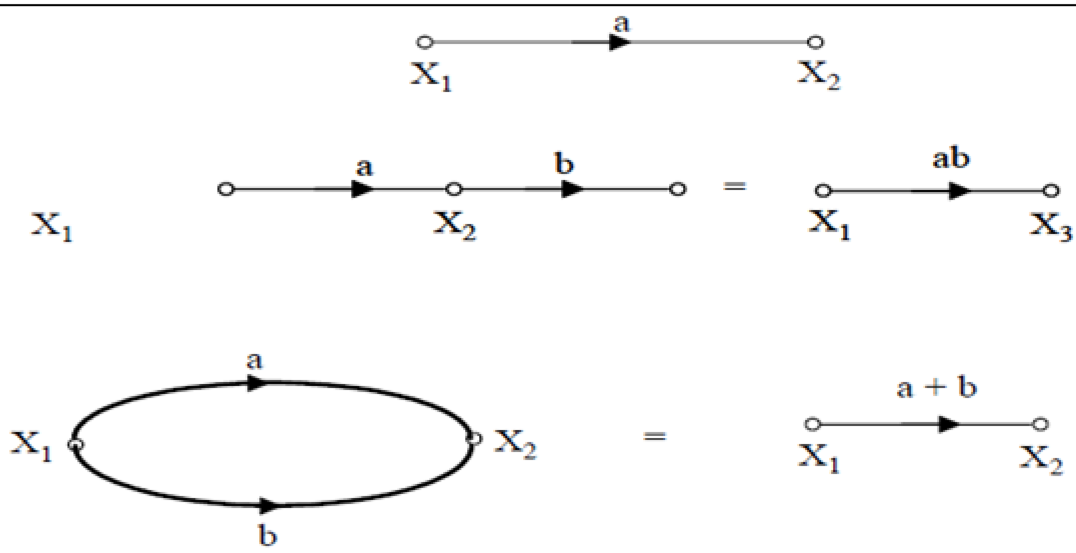
The gain of branch is transmission function of that branch gain encountered in traversing a path .For example the path gain of the forward from $x_1\ to\ x_2\ to\ x_3\ to\ x_4\ is\ A_{21}A_{32}A_{43}$

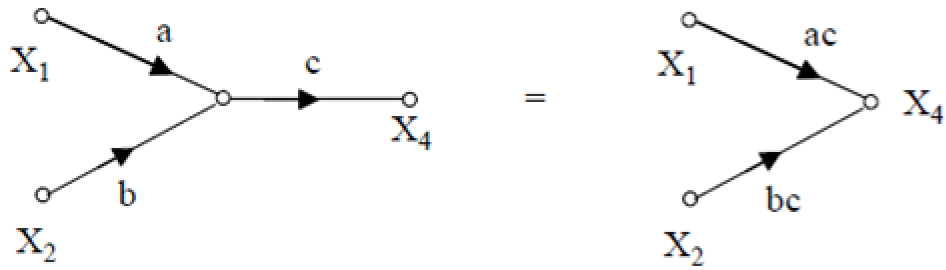
Definition1.8.8

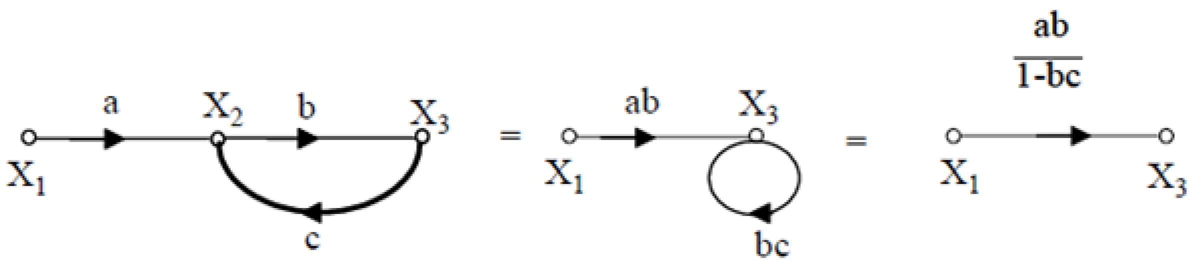
The loop gain is product of the branch gains of the loop .For example the loop gain of the feedback loop form $x_2\ to\ x_3\ and\ back\ x_2\ is\ A_{32}A_{23}$

1.5.2 Construction of signal flow graph

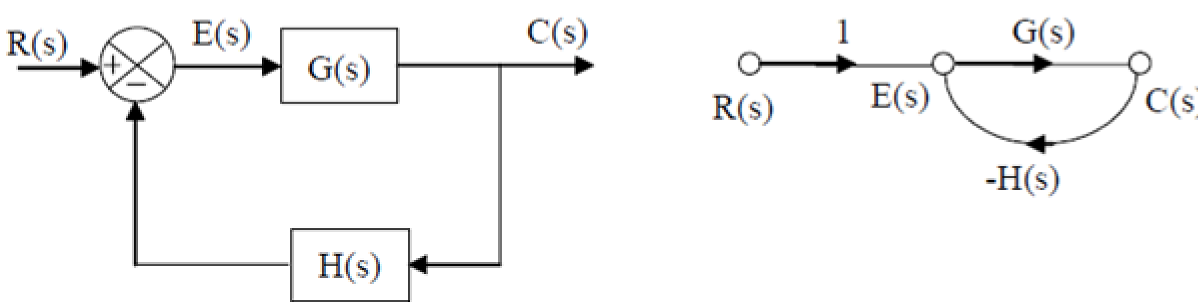
The signal flow graph of linear feedback control system whose components are specified non interacting transfer functions can be constructed by direct reference to block diagram of the system. Each variable of the block diagram becomes anode and each block becomes branch

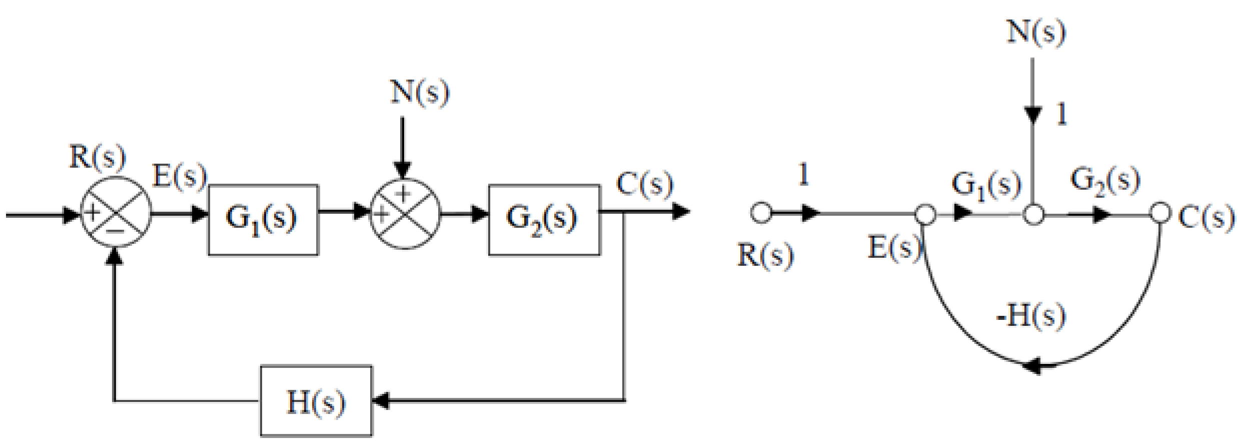



X1
a
c
X4
b
X2
=
X1
ac
X4
bc
X2


a
X2
b
X3
X1
c
=
ab
X3
X1
bc
=
ab
1-bc
X1
X3


R(s)
G(s)
C(s)
G(s)
R(s)
R(s)


R(s)
E(s)
G(s)
C(s)
H(s)
1
G(s)
R(s)
E(s)
C(s)
-H(s)


N(s)
R(s)
E(s)
G1(s)
G2(s)
C(s)
H(s)
N(s)
1
1
G1(s)
G2(s)
R(s)
E(s)
C(s)
-H(s)

1.5.3 Block diagram Reduction using flow graphs and the General input-output gain formula

Often the easiest way to determine the control ratio of complicated block diagram into a signal flow graph

Example 1.11

Let us determine the control ratio $\frac{C}{R}$ and the canonical block diagram of the feedback control system

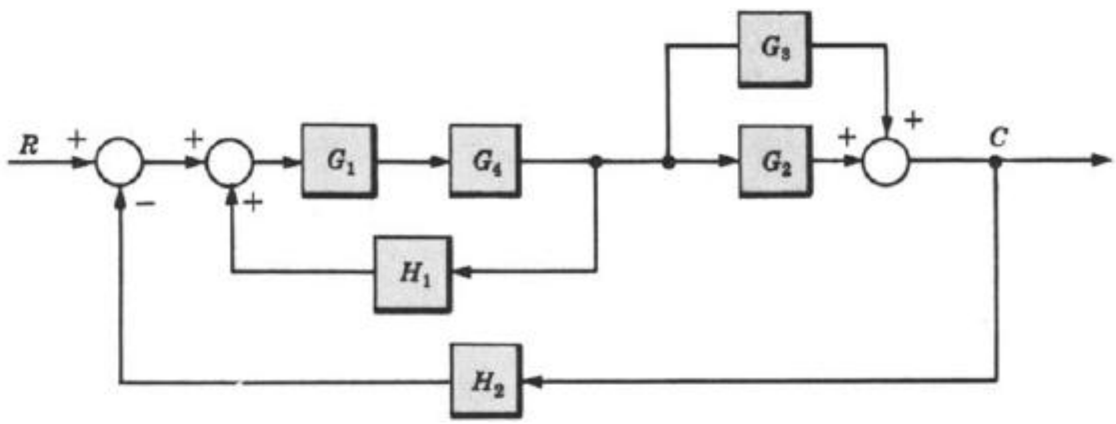


Figure (1.36)

The signal flow graph is gives in figure (1.36) there are forward path

$$P_1 = G_1 G_2 G_4 \quad , P_2 = G_1 G_3 G_4$$

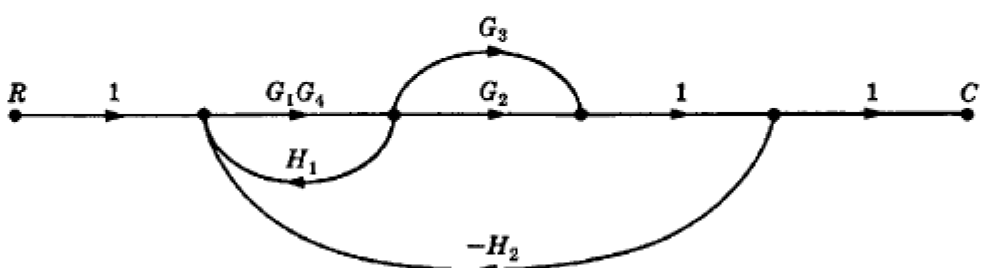


Figure (1.37)

There are three feedback loops

$$p_{11} = G_1 G_2 H_1 \quad , P_{21} = -G_1 G_2 G_4 H_2 \quad , P_{31} = -G_1 G_3 G_4 H_2$$

There are no nontouching loops and all loops touch forward paths than
$\Delta_1 = 1$, $\Delta_2 = 1$

Therefore the control ratio is

$$T = \frac{C}{R} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{G_1G_2G_4 + G_1G_3G_4}{1 - G_1G_4H_1 + G_1G_2G_4H_2 + G_1G_3G_4H_2}$$

$$= \frac{G_1G_4(G_2 + G_3)}{1 - G_1G_4H_1 + G_1G_2G_4H_2 + G_1G_3G_4H_2}$$

Chapter two

Time Invariant system

2. Introduction

A dynamic system is Time invariant if shifting the input on the time axis leads to on equivalent shifting of the output along the time axis with no other changes in other words a time invariant system maps a given input trajectory u(t) no matter when it occurs. Many physical systems can be modeled as linear time invariant (LTI) systems very general signals can be represented as linear combinations of delayed impulses by the principle of super position the response y[n] of a discrete time (TLI) system is the sum of the response to the individual shifted impulses to the individual shifted impulses marking up the input signal x[x]

$$Y(t-\tau) = F[u(t-\tau)] \quad (2.1)$$

The formula above says specifically that if an input signal is delayed by some amount T so will be the output and with no other changes

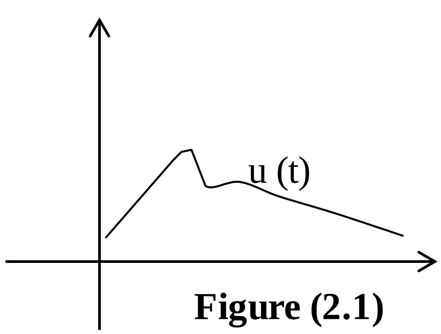


Figure (2.1)

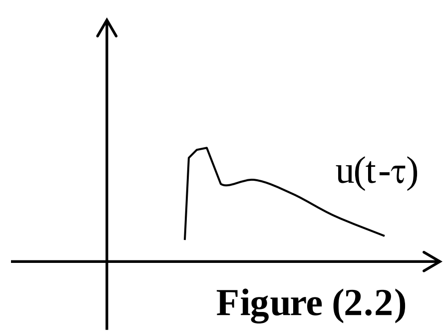


Figure (2.2)

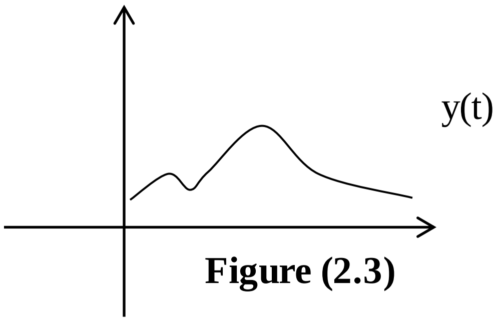


Figure (2.3)

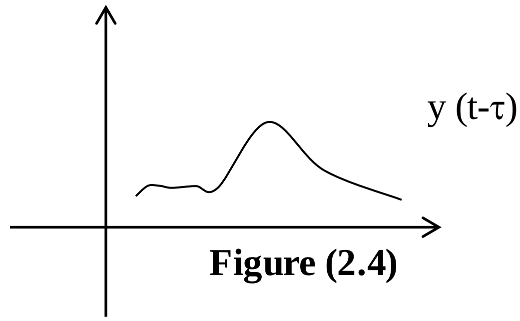


Figure (2.4)

An example of a physical time varying system is the pitch response of a rocket y (t) when the thrusters are being steered by an angle u (t). You can see first that this is an inverted pendulum problem and unstable without .A closed loop controller it is time varying because as the rocket burns fuel its mass is changing and so the pitch responds differently to various inputs throughout its flight. In this case the absolute time coordinate is the time since lift off. To assess whether a system is time varying or not follow these steps replace u (t) with u (t-τ) on one side of equation replace y (t) with y(t-τ) on the other side of the equation and then check if they are equal Hoer several examples.

Y (t) = u (t)$^{3/2}$

This system is clearly time invariant because it is a static map Next

1. Example 2

$$y(t) = \int_0^t \sqrt{u(t)} \;\; dt$$

Replace u (t) with u (t-τ) in the right hand side and it the

$$y(t) = \int_0^t \sqrt{u(t_1 - \tau)} \;\; dt_1 = \int_{-t}^{t-\tau} \sqrt{u(t_2)} \;\; dt_2$$

The left hand side is simply

$$y(t - \tau) = \int_0^{t-\tau} \sqrt{u(t_1)} \;\; dt_1$$

Clearly the right and left hand sides are different hence the system is not time invariant. As another example consider

2.Example 2

$$y(t) = \int_0^t u^2(t_1) \;\; dt_1$$

The right hand side becomes with the time shift

$$\int_{t-5}^{t} u^2(t_1 - \tau) \;\; dt_{1=} \int_{t-5-\tau}^{t-\tau} u^2(t_2) \;\; dt_2$$

Whereas the left hand side is

$$y(t - \tau) = \int_{t-5-\tau}^{t-\tau} u^2 \; (t_1) \; dt_1$$

The two sides of the defining equation are equal under a time shift t and so this system is time invariant. Time invariant also known as shift invariance describe function independence from the location of t = 0 on the time lime By definition a time invariant describe S a function's independence from the location of t = 0 on the time lime By definition a time invariant systems output will shift in tine if its input shifts in time but otherwise will remain exactly the same in other words a time invariant function does not words a time invariant function does not care what time it is we describe time invariance with the following notation

$$Given\, Y(t) = f(t) then\, y(t-s) = f(t-s) \qquad (2.2)$$

System because is a crucial property of real system because it allows us to assume that a system will respond in a predictable manner at any time modeling time dependent systems are often highly influenced by initial condition and system definition

3.Example 2

Is the system $y = t + x(t)$ time invariant?

To prove whether or not the above system is time invariant we must a mathematical technique called proof by contradiction proof by contradiction is often used when two separate conditions can be tested on the same system or mathematical contract. In this case we can compare the results of the time shifted system with the solution assuming that the system is time invariant thus the shifted system is represent the original system as

$$y_1(t) = y(t-s) + x(t-s)$$

Now if we define the input function to be $x(t-s) = x_{shift}(t)$ which we may assume if and only if the system is time invariant then we represent the original system as $y_2(t) = t + x(t-s)$

Since $y_1 \neq y_2$ we may conclude that the system is not time invariant through proof by contradiction

Example 2.4

Is the system $y = \alpha x(t) + \beta$ time invariant? (Note α and β are constant)

We will use of some method a gain on the new system described above

The shifted system is represented as $y_1(t) = y(t-s) = \alpha x(t-s) + \beta$

Again if we $x(t-s) = x_{shift}(t)$ and (other by assume system the system is time in variant).Than we represent the original system as

$$y_2(t) = \alpha x_{shift}(t) + \beta = \alpha x(t-s) + \beta$$

Since $y_1 = y_2$ we may conclude that the system is time invariant

2.1 Solving the time invariant state Equation

In this part, we show obtain the general solution of the linear time invariant system equation .We shall first consider the homogeneous case and then the nonhomogeneous case.

2.1.1 Solution of Homogeneous state Equations

Before we solve vector matrix differential equations .Let u review the solution of the scalar differentia

$$X^{\cdot} = ax \qquad (2.3)$$

In solving this equation, we may assume a solution x (t) of the form

$$x(t) = b_0 + b_1 t + b_2 t^{2+\cdots\cdots+b_k t^k+\cdots} \qquad (2.4)$$

By substituting this assumed solution in to Equation (2.3) we obtain

$$b_1 + 2b_2 t + 3b_3 t^{2+\cdots\cdots+kb_k t^{k-1}+\cdots}$$

$$= \text{a}\left(b_0 + b_1 t + b_2 t^{2+\cdots\cdots+b_k t^k+\cdots}\right)\text{x}(0) \qquad (2.5)$$

If the assumed solution is to be the true solution Equation (2.5) must hold for any t, Hence, equating the coefficients of the equal powers of t , we obtain

$b_1 = ab_0$, $2b_2 = ab_1$, $b_2 = \frac{1}{2}\, ab_1 = \frac{1}{2}\, a^2 b_0$, $b_3 = \frac{1}{3}\, a^2 b_0 = \frac{1}{3\times 2}\, a^3 b_0$, $b_k = \frac{1}{k!}\, a^k\, b_o$

The value of b_0 is determined by substituting t = 0 into equation (2.4), or Hence, the solution x (t) can be written

$$x(t) = \left(1 + at + \frac{1}{2!}\, a^2 t^2 + \cdots + \frac{1}{k!} a^k t^k + \cdots\right) x(0) = e^{at} x(0)$$

We shall now solve the vector matrix differential equation

$$X^{\cdot} = \text{AX} \qquad (2.6)$$

Where X = n – vector

A = n × n constant matrix

By analogy with the scalar case, we assume that the solution is in the form of vector power series in t, or by substituting this assumed solution in to equation (2.5) we obtain

$b_1 + 2b_2t + 3b_3t^{2+\cdots\cdots+kb_kt^{k-1}+\cdots}$ (2.7)

$$= A\left(b_0 + b_1t + b_2t^{2+\cdots\cdots+b_kt^k+\cdots}\right) \quad (2.8)$$

If the assumed solution is to be true solution, equation (2.8) must hold for all t. thus by equating the coefficients of like powers of t on both sides g equation (2.8) we obtain

$b_1 = Ab_0$, $2b_2 = Ab_1$, $b_2 = \frac{1}{2}\, Ab = \frac{1}{2}\, A^2b_0$, $b_3 = \frac{1}{3}\, A^2b_0 = \frac{1}{3\times 2}\, A^3b_0$, $b_k = \frac{1}{k!}\, A^k\, b_o$

By substituting t = 0 into equation (2.7) we obtain

$x(0) = b_0$.Thus, the solution x (t) can be written as

$$x(t) = 1 + At + \frac{1}{2!}A^2t^2 + \cdots + \frac{1}{k!}A^kt^k + \cdots)x(0)$$

The expression in the parentheses on the right hand side of this last equation is an n × n matrix because of the similarity to the infinite power series for a scalar exponential we

$$1 + At + \frac{1}{2!}A^2t^2 + \cdots + \frac{1}{k!}A^kt^k + \cdots = e^{At}$$

In terms of the matrix exponential is very important in state space in state space analysis of linear systems, we shall next examine its properties

Definition 2.1

It can be proved that the matrix exponential of an n × n matrix A

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \quad (2.9)$$

Converges absolutely for all finite t Hence, computes calculations for evaluating the dements e^{At} by using the series expansion can be easily carried out .Because of the convergence of the infinite series

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \quad (2.10)$$

The series can be differential term to give

$$\frac{de^{At}}{dt} = A + A^2 t + \frac{A^3 t^2}{2!} + \cdots + \frac{A^k t^{K-1}}{(k-1)!} \quad (2.11)$$

$$= A\left[1 + At + \frac{A^2 t^2}{2!} + \cdots + \frac{A^{k-1} t^{K-1}}{(k-1)!} + \cdots\right] = Ae^{At}$$

$$= A\left[1 + At + \frac{A^2 t^2}{2!} + \cdots + \frac{A^{k-1} t^{K-1}}{(k-1)!} + \cdots\right] = Ae^{At}$$

The matrix exponential has the property that

$$e^{A(t+S)} = e^{At} e^{As} \quad (2.12)$$

This can be proved as follows

$$e^{At} e^{As} = \left(\sum_{k=0}^{\infty} \frac{A^k t^k}{k!}\right)\left(\sum_{k=0}^{\infty} \frac{A^k s^k}{k!}\right)$$

$$= \sum_{k=0}^{\infty} A^k \left(\sum_{i=0}^{\infty} \frac{t^i s^{k-i}}{!(k-1)!}\right) = \sum_{k=0}^{\infty} A^k \frac{(i+t)^k}{k!} = e^{A(t+S)}$$

In particular, if s = -t, then

$$e^{At}e^{-At} = e^{A(t-t)} = 1$$

Thus, the inverse of e^{At} ise^{-At} since the inverse of e^{At} always exists e^{At}

Is nonsingular, it is very important to remember that

$$e^{t(A+B)} = e^{At}e^{Bt} \quad if \;\; AB = BA \;, e^{t(A+B)} \neq e^{At}e^{Bt} \quad if \; AB \neq BA$$

To prove this note that

$$e^{t(A+B)} = 1 + (A+B)t + \frac{(A+B)^2}{2!}t^2 + \frac{(A+B)^3}{3!}t^3 + \cdots$$

$$e^{At}e^{Bt} = \left(1 + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots\right)\left(1 + Bt + \frac{B^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots\right)$$

$$= 1 + (A+B)t + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \frac{A^2Bt^3}{2!} + \frac{AB^2t^3}{2!} + \frac{B^3t^3}{3!} + \cdots$$

Hence

$$e^{At}e^{Bt} - e^{At}e^{Bt} = \frac{BA - AB}{2!}t^2 + \frac{BA^2 + ABA + B^2A - BAB - 2A^2B - AB^2}{3!}t^3$$

The difference between $e^{At}e^{Bt}$ and $e^{At}e^{Bt}$ vanishes if A and B commute

Laplace transforms Approach to the solution of Homogeneous state equation

Let us first consider the scalar case

$$x^{\cdot} = ax \tag{2.13}$$

Taking the Laplace transform of equation (2.13) we obtain

$$SX(s) - X(0) = aX(s) \quad (2.14)$$

Where $X(s) = \varphi[x]$ for solving (2.14) for X(s) gives

$$X(s) = \frac{x(0)}{s-a} = (s-a)^{-1}x(0)$$

The inverse lap lace transform of this last equation gives the solution

$$x(t) = e^{at}$$

The foregoing to the approach to the solution of the homogeneous scalar differential equation can be extended to the homogeneous state equation

$$X^{\cdot} = AX \quad (2.15)$$

Taking the Laplace transform of both sides of equation (2.15)

$$SX(s) - X(0) = AX(s)$$

Where $X(s) = \varphi[x]$.Hence $(SI - A)X(s)X = X(0)$

Premultiplying both sides of this last equation by $(SI - A)^{-1}$ we obtain

$$X(s) = (SI - A)^{-1}\, x(0)$$

The inverse Laplace transformation of X(s) gives the solution x (t), Thus

$$x(t) = \varphi^{-1}[(SI - A)^{-1}]x(0) \quad (2.16)$$

Note that

$$(SI - A)^{-1} = \frac{1}{S} + \frac{A}{S^2} + \frac{A^2}{S^3} + \cdots$$

Hence the inverse Laplace transform of $(SI-A)^{-1}$ gives

$$\varphi^{-1}[(SI-A)^{-1}] = 1 + At + \frac{1}{2!}A^2t^2 + \cdots + \frac{1}{k!}A^kt^k + \cdots$$
$$= e^{At} \qquad (2.17)$$

The Laplace transform of a matrix is the matrix consisting of the inverse Laplace transform all elements form equation (2.16) and (2.17) the solution of equation (2.15) is obtained as

$X(t) = e^{At}X(0)$

The importance of equation (2.17) lies in the fact that it provides a convenient means for finding the closed solution for matrix exponential

2.2 State Transition Matrix

We can write the solution of the homogeneous state equation

$$X^{\cdot} = AX \qquad (2.18)$$

$$X(t) = \emptyset(t)X(0) \qquad (2.19)$$

Where$\emptyset(t)$ is an n × n matrix and is the unique solution of

$\emptyset^{\cdot}(t) = A\emptyset(t)$, $\emptyset(0) = 1$

To verify this note that

$X(0) = \emptyset(0)X(0) = X(0)$

And

$$X^{\cdot}(t) = \emptyset^{\cdot}(t)X(0) = A\emptyset(t)X(0) = AX(t)$$

We thus confirm that equation (2.19) is the solution of t equation (2.16) from equation (2.17).and (2.18) we obtain

$$\emptyset(t) = e^{At} = \varphi^{-1}[(SI - A)^{-1}]$$

Note that

$$\emptyset^{-1}(t) = e^{-At} = \emptyset(-t)$$

From equation (2.19) we see that the solution of condition, hence the unique matrix $\emptyset(t)$ is called the state information matrix.

The state information contains all the information about the free motions of the system defined by (2.3)

If the Eigen values $\lambda_1, \lambda_2, \cdots, \lambda_n$ of the matrix A are distinct, than $\emptyset(t)$ will contain the n-exponentials

$$e^{\lambda_1 t}, \quad e^{\lambda_2 t}, \cdots, e^{\lambda_n t}$$

In particular, if the matrix A is diagonal, then

$$\varphi(t) = e^{At} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & e^{\lambda_n t} \end{bmatrix} \quad \text{(A:diagonal)}$$

If there is a multiplicity in the Eigen values for example A are $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \ldots, \lambda_n$

Then $\phi(t)$ will contain, in addition to the exponential $e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_n t}$ terms like $te^{\lambda_1 t}$ and $t^2 e^{\lambda_1 t}$

2.2.1 Properties of State Transition Matrix

We shall now some matrix the important properties of state transition matrix $\emptyset(t)$ for the time invariant system

$$X^{\cdot} = AX$$

For which

$$\emptyset(t) = e^{At}$$

We have the following

1. $\emptyset(t) = e^{A0} = 1$

2. $\emptyset(t) = e^{At} = e^{(-At)^{-1}} \quad = [\emptyset(-t)]^{-1} \; or \; \emptyset^{-1}(t) = \emptyset(-t)$

3. $\emptyset(t_1 + t_2) = e^{A(t_1+t_2)} = e^{At_1}e^{At_2} = \emptyset(t_1)\emptyset(t_2) = \emptyset(t_2)\emptyset(t_1)$

4. $[\emptyset(t)]^n = \emptyset(nt)$

5. $\emptyset(t_2 - t_1)\emptyset(t_1 - t_0)\emptyset = \emptyset(t_2 - t_0) = \emptyset(t_1 - t_0)\emptyset(t_2 - t_1)$

5.2 Example

Obtain the state transition matrix $\phi(t)$ of the following system

$$\begin{bmatrix} x^{\cdot}_1 \\ x^{\cdot}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Obtain also the inverse of the state transition matrix $\phi^{-1}(t)$ for system

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

The state transition matrix $\emptyset(t)$is given by

$$\emptyset(t) = e^{At} = \emptyset^{-1}[(SI - A)^{-1}]$$

Since

$$SI\text{-}A = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} S & -1 \\ 2 & S+3 \end{bmatrix}$$

The inverse of (SI – A) is given by

$$(SI\text{-}A)^{-1} = \frac{1}{(S+1)(S+2)} \begin{bmatrix} S+3 & 1 \\ -2 & S \end{bmatrix}$$

$$= \begin{bmatrix} \frac{S+3}{(s+1)(s+2)} & \frac{-1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2} & \frac{s}{(s+1)(s+2} \end{bmatrix}$$

Hence

$$\Phi(t) = e^{At} = \phi^{-1}[SI - A)^{-1}] = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{t} + 2e^{2t} & -e^{t} + 2e^{2t} \end{bmatrix}$$

Noting that $\emptyset^{-1}(t) = \emptyset(-t)$ we obtain the inverse of the state transition matrix as follows

$$\Phi^{-1}(t) = e^{-At} = \begin{bmatrix} 2e^{t} - e^{2t} & e^{t} - e^{2t} \\ -2e^{t} + 2e^{2t} & -e^{t} + 2e^{2t} \end{bmatrix}$$

2.3 Solution of non-homogeneous state equations

We shall begin by considering the scalar case

$$X^{\cdot} = ax + bu \tag{2.20}$$

let us rewrite equation (2.20) as $X^{\cdot} - ax = bu$ multiplying both sides of this equation by e^{-at}we obtain

$$e^{-at}[\,X^{\cdot}(t) - ax(t)] = \frac{d}{dt}[\,e^{-at}x(t)] = e^{-at}bu(t)$$

Integrating this equation between o and t give

$$e^{-at}x(t) - x(0) = \int_0^t e^{-at}bu(\tau)\,d\tau$$

or

$$x(t) = e^{at}x(0) + e^{at}\int_0^t e^{-at}bu(\tau)\,d\tau$$

The first term on the right hand side is the response to the initial condition and the second term is the response to the nonhomogeneous state equation described by

$$X^{\cdot} = AX + BU \tag{2.21}$$

Where X = n – vector

U = r – vector

U = n × n constant matrix

U = n × r constant matrix

By writing equating (2.21) as $X^{\cdot} - AX(t) = BU(t)$ and premultiplying both sides of this equation by e^{-At}we obtain

$$e^{-At}[\,X^{\cdot}(t) - AX(t)] = \frac{d}{dt}[\,e^{-At}X(t)] = e^{-At}BU(t)$$

Integration the preceding equation between o and t gives

$$e^{-At}x(t) - x(0) = \int_0^t e^{-At}Bu(\tau)\,d\tau$$

Or

$$x(t) = e^{At}x(0) + e^{At}\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau \qquad (2.22)$$

Equation above can also between as

$$X(t) - \emptyset(t)X(0) + \int_0^t \emptyset(t-\tau)Bu(\tau)\,d\tau \qquad (2.23)$$

Where $\quad \emptyset(t) = e^{At}$

Equation (2.22) and (2.23) is the solution of equation (2.21) the solution x(t) is clearly the sum of a term consisting of the transition of the initial state and a term arising from the input vector. Laplace transforms Approach to the solution of nonhomogeneous state equation. The solution of the nonhomogeneous state equation

$$X^{\cdot} = AX + BU \qquad (2.24)$$

Can also be obtained by the Laplace transform approach, the Laplace transform of this last equation yields

$SX(s) - X(0) = AX(t) + BU(t)$

Or

$$(SI - A)X(s) = (0) + BU(s) \qquad (2.25)$$

Premultiplying both sides of this last equation by (2.25) $(SI - A)^{-1}$

We obtain

$X(s) = (SI - A)^{-1}\,X(0) + (SI - A)^{-1}\,BU(s)$

Using the relationship given by equation (2.17) gives

$X(s) = \emptyset[e^{At}]X(0) + \emptyset[e^{At}]BU(s)$

The inverse Laplace transform of this last equation can be obtained by use of the convolution integral as follows.

$$x(t) = e^{At}x(0) + e^{At}\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau \qquad (2.24)$$

Solution in term s of $x(t_0)$,thus for we have assumed the initial time to be zero. If however, the initial time is given by to instead of o, them the saluted to above equation must be modified to

$$x(t) = e^{A(t-\tau)}x(0) + e^{At}\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

6.2 Example

Obtain the time response of the following system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Where u (t) is the unit step function occurring at t = 0 or U(t) = I(t)

For this system

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The state transition matrix $\emptyset(t) = e^{At}$ was obtained

The response to the unit step input is then is obtained as

x(t)=

$$e^{At}x(0) + \int_0^t \begin{bmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} [1] d\tau$$

Or

$$\begin{bmatrix} X_1(t) \\ X_1(t) \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} X_1(0) \\ X_1(0) \end{bmatrix} + \begin{bmatrix} \frac{1}{1} - e^{-t} + \frac{1}{1} e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

If the initial state is zero x (0) = 0.Then u (t) can be simplified to

$$\begin{bmatrix} X_1(t) \\ X_1(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{1} - e^{-t} + \frac{1}{1} e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

2.4 State model of linear system

Consider multiple input multiple output nth order

Number of inputs=m

Number of outputs=p

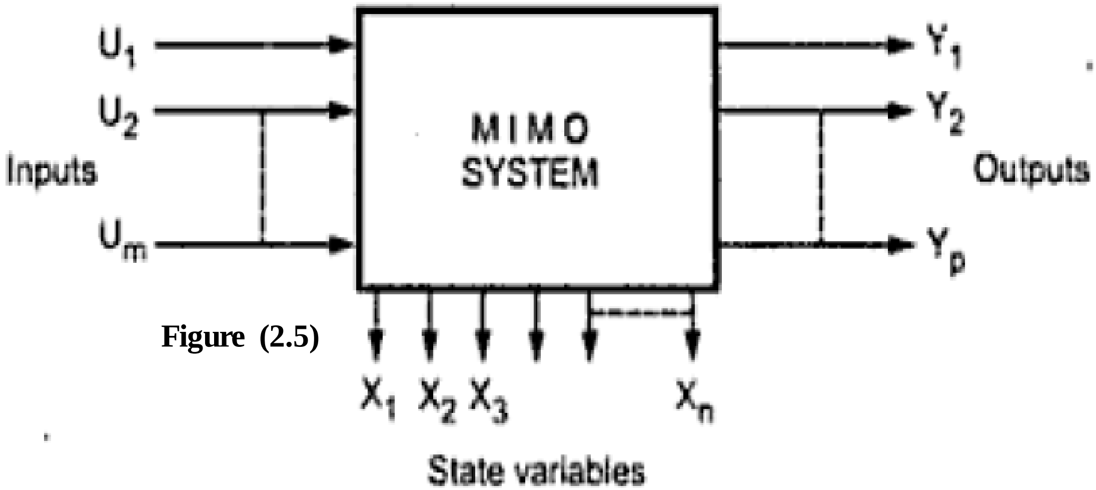


Figure (2.5)

$$U(t)=\begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ \vdots \\ u_m(t) \end{bmatrix} \quad , \quad X(t)=\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ \vdots \\ x_n(t) \end{bmatrix} \quad ,Y(t)=\begin{bmatrix} Y_1(t) \\ Y_2(t) \\ \vdots \\ \vdots \\ Y_P(t) \end{bmatrix}$$

All Colum vectors having order $m \times 1, n \times 1\ and\ p \times 1$ respectively

For such a system the state variable representation can be arranged in the form of n fist order differential equations

$$\frac{dx_1(t)}{dt} = f_{1(x_{1,}x_{2,\ldots\ldots,}x_{n,}U_{1,\ldots\ldots,}U_{m,})}$$

$$\frac{dx_2(t)}{dt} = f_{2(x_{1,}x_{2,\ldots\ldots,}x_{n,}U_{1,\ldots\ldots,}U_{m,})}$$

$$\vdots$$

$$\vdots$$

$$\frac{dx_n(t)}{dt} = f_{n(x_{1,}x_{2,\ldots\ldots,}x_{n,}U_{1,\ldots\ldots,}U_{m,})}$$

Where $f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$ is the functional operator

Integrating the above equation

$$X_i(t) = X_i(t_0) + \int_{t_0}^{t} f_{i(x_{1,}x_{2,\ldots\ldots,}x_{n,}U_{1,\ldots\ldots,}U_{m,})}\, dt \qquad (2.26)$$

, n............Where i=1, 2,

Thus n state variables and hence state vector at any time t can be determined uniquely any n dimensional time invariant system has state equation in functional form as

$$\frac{dx}{dt} = f(X, U) \qquad (2.27)$$

While outputs of such system are dependent on the state and instantaneous Inputs functional output equation can be written as

$$X(t) = g(x, u) \qquad (2.28)$$

Where g is functional output

For the time invariant system the same equation s can be written as

$\frac{dx}{dt} = f(X, U, t) \cdots\cdots$ State equation

$Y(t) = g(X, U, t) \cdots\cdots$ Output equation

Diagrammatically this can be represented in the figure above

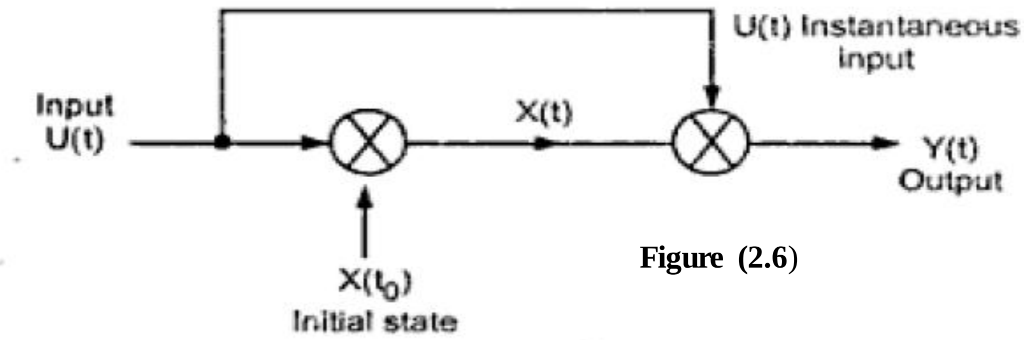


Figure (2.6)

Input-output state description of system

The functional equations can be expressed in terms of linear combination of system states and inputs as

$$\dot{X}_1 = a_{11} X_1 + a_{12}X_2 + \dots + a_{1n}X_n + b_{11}U_1 + b_{12}U_2 + \dots + b_{1m} U_m$$

$$\dot{X}_2 = a_{21} X_1 + a_{22}X_2 \dots + a_{2n}X_n + b_{21}U_1 + b_{22}U_2 + \dots + b_{2m} U_m$$

$$\vdots$$

$$\dot{X}_n = a_{n1} X_1 + a_{n2}X_2 + \dots + a_{nn}X_n + b_{n1}U_1 + b_{n2}U_2 + \dots + b_{nm} U_m$$

For linear time invariant systems the coefficients a_{ij} and b_{ij} thus all equations can be written in the vector matrix form as

$$X^{\cdot}(t) = AX(t) + BU(t) \qquad (2.29)$$

Where

X (t) =state vector matrix of order n$\times$ 1

U(t)=input vector matrix of order m$\times$ 1

A=system matrix or Evaluation matrix of order n$\times$ n

B=input matrix of control matrix of order n$\times$ m

Similarly the output variables at time t can be expressed as the linear combinations of the input variables and state variables at time t as

$$Y_1(t) = c_{11} X_1(t) + \dots + c_{1n} X_n(t) + d_{11} U_1(t) + \dots + d_{1m} U_m(t)$$

$$\vdots$$

$$Y_p(t) = c_{p1} X_1(t) + \dots + c_{p1} X_n(t) + d_{p1} U_1(t) + \dots + d_{pm} U_m(t)$$

For linear time invariant systems, the coefficients C_{ij} and $d_{ij}$are constants thus all the output equations can be written in the vector matrix form as

Y (t) =CX (t) +DU (t)

Where

Y (t) =output vector matrix of order p$\times$ 1

C=output matrix or observation matrix of order p$\times n$

D=Direct transmission matrix of order p$\times m$

The two vector equations together is called state model of linear system

$X^{\cdot}(t) = AX(t) + BU(t)$ State equation

$Y(t) = CX(t) + DU(t)$ Output equation

This state model of a system

For linear time invariant system the matrix A,B,C and D are also time dependent. Thus

$$X^{\cdot}(t) = A(t)X(t) + B(t)U(t) \qquad (2.30)$$

$$Y(t) = C(t)X(t) + D(t)U(t)$$

For linear time invariant system

2.5 State model of single input-single output system

Consider a single input –single output

→But its order n hence n state variable are required to define state of system in such case the state model is

$$X^{\cdot}(t) = AX(t) + BU(t) \qquad (2.31)$$

$$Y(t) = CX(t) + dU(t)$$

Where

$A = n \times n$ Matrix $B = n \times 1$ matrix

$C = 1 \times n$ Matrix d=constant and u(t)=single scalar input variables

In general remember the order of various matrices

A =Evoluation matrix $\rightarrow n \times n$

B =Control matrix→ $n \times m$

c =Observation matrix→ $p \times n$

D =Transimission matrix $\rightarrow p \times m$

Chapter Three

Controllable Pairs of matrice

3. Introduction

A system is said to be controllable at time t_0,if it is possible by means of an unconstrained control vector to transfer the system from. Any initial state $\mathbf{x}(t_0)$ to any other state in a finite interval of time. A system is said to be observable at time t_0 if, with the system in state $\mathbf{x}$ (t_0), it is possible to determine this state from the observation of the output over a finite time interval. The concepts of controllability and observability were introduced by Kalman,s they Play an important role in the design of control systems in state space. In fact, the Conditions of controllability and observability may govern the existence of a complete Solution to the control system design problem. The solution to this problem may not, exist if the system considered is not controllable. Although most physical systems are .Controllable and observable, corresponding mathematical models may not possess the Property of controllability and observability Then it is necessary to know the. Conditions under which a system controllable and observable this deals with controllability and the next discusses observability in what follows, we shall first derive the condition for complete state controllability. Then we derive alternative forms of the condition for complete state controllability followed by discussions of complete output controllability. Finally, we present the concept of stabilizability

3.1 Complete State Controllability of Continuous-Time Systems.

Consider the, Continuous-time system.

1). (3 $\dot{x} = Ax + BU$

Where **x** = state vector (n-vector)

U=control signal (scalar)

n=n×n matrix

B=n× 1matrix

The system described by Equation (3.1) is said to be state controllable at $t = t_0$ if it is possible to construct an unconstrained control signal that will transfer an initial state to any final state in a finite time interval If every state is controllable, then the system is said to be completely state controllable We shall now derive the condition for complete state controllability. Without loss of generality, we can assume that the final state is the origin of the state space and that the Initial time is zero .or $t_0 = 0$.The solution of Equation (3.1) is

X (t) =e^{At}x (0) +$\int_0^t e^{A(t-\tau)} BU(\tau)d\tau$

Applying the definition of complete state controllability just given, we have

X (t_1) =e^{At_1} x (0) +$\int_0^{t_1} e^{A(t_1-\tau)} BU(\tau)d\tau$

Or

X (0)=$-\int_0^{t_1} e^{-A\tau} BU(\tau)d\tau$

2).(3

By definition of e^{-At}can be written

$e^{-At} = \sum_{k=0}^{n-1} \alpha_k(\tau)A^k$

3).(3

Substituting Equation (3.3) into Equation (3.2) gives

$X(0) = -\sum_{k=0}^{n-1} A^k B \int_0^{t_1} \alpha_k(\tau)\, U(\tau)\, d\tau$

4).(3

Let us put $\int_0^{t_1} \alpha_k(\tau)\, u(\tau)\, d\tau \ = B_k$ .Then Equation (3.4) becomes

$X(0) = -\sum_{k=0}^{n-1} A^k B B_k = -\ [B \vdots AB \vdots \cdots A^{n-1}B]$

5).(3

If the system is completely state controllable, then, given any initial state **x** (0) Equation (3.5) must be satisfied. This requires that the rank of the n×n matrix

$$[B \vdots AB \vdots \cdots A^{n-1}B]$$

From this analysis, we can state the condition for complete state controllability as follows. The system given by Equation (3.1) is completely state controllable if and only

If the vectors B, AB,⋯, A^{n-1}B are linearly independent, or the n×n matrix

$$[B \vdots AB \vdots \cdots A^{n-1}B]$$

Is of rank n

The result just obtained can be extended to the case where the control vector **u** is n-dimensional. If the system is described by

$X^{\cdot} = AX + BU$

Where **U** is an r-vector, then it can be proved that the condition for complete state. Controllability is that the n×nr matrix

$$[B \vdots AB \vdots \cdots A^{n-1}B]$$

Be of rank n, or contain n linearly independent column vectors. The matrix

$$[B \vdots AB \vdots \cdots A^{n-1}B]$$

Is commonly called the *controllability matrix*

1 Consider the system given by**.Example 3**

$$\begin{bmatrix} x^{\cdot}{}_1 \\ x^{\cdot}{}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{U}$$

Since $[B \vdots AB] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ =singular matrix

The system is not completely state controllable.

3:1:1 Alternative Form of the Condition for Complete State Controllability. Consider the system defined by

6). $X^{\cdot} = \text{A}X + BU$ (3

Where **x** = state vector (n-vector)

U=control signal (scalar)

n=n×n matrix

B=n× 1 matrix

If the eigenvectors of A are distinct, then it is possible to find a transformation matrix P such that

$$P^{-1}\text{AP=D=}\begin{bmatrix} λ_1 & \cdots & \cdots & 0 \\ \vdots & λ_1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & λ_n \end{bmatrix}$$

Note that if the eigenvalues of **A**are distinct, then the eigenvectors of **A**are distinct; however, the converse is not true. For example, an n×n real symmetric matrix having, multiple eigenvalues has n distinct eigenvectors. Note also that each column of the Matrix is an eigenvector of **A** associated with $λ_i$= (i=1, 2,$\cdots, n$)

Let us define

7). X=PZ (3

Substituting Equation (3.5) into Equation (3.4), we obtain

By defining

$$Z^{\cdot} = P^{-1}APZ + P^{-1}BU \qquad (3.8)$$

By defining

P^{-1}B=F= (f_{ij}).We can rewrite Equation (3.7) as

$$z^{\cdot}_1 = \lambda_1 z_1 + f_{11}u_1 + \quad f_{12}u_2 + \dots + f_{1r}u_r$$

$$z^{\cdot}_2 = \lambda_2 z_2 + f_{21}u_1 + \quad f_{22}u_2 + \dots + f_{2r}u_r$$

$$\vdots$$

$$z^{\cdot}_n = \lambda_n z_n + f_{n1}u_1 + \quad f_{n2}u_2 + \dots + f_{nr}u_r$$

If the elements of any one row of the n×r matrix F are all zero, then the corresponding state variable cannot be controlled by any of the u_i. Hence, the condition of complete state controllability is that if the eigenvectors of A are distinct, then the system is completely State controllable if and only if no row of P^{-1}B has all zero elements. It is important. To note that, to apply this condition for complete state controllability, we must put the matrix $P^{-1}AP$ in Equation (3.8) in diagonal form If the A matrix in Equation (3.6) does not possess distinct eigenvectors, then diagonalization is impossible. In such a case, we may transform A into a Jordan canonical Form. If for example, A has eigenvalues and has n-3 distinct. Eigenvectors, then the Jordan canonical form of A is

$$J = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_4 & 1 & \vdots \\ \vdots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \lambda_n \end{bmatrix}$$

The square sub matrices on the main diagonal are called *Jordan blocks.*

Suppose that we can find a transformation matrix **S** such that

$$S^{-1}AS = J$$

If we define a new state vector **z** by

$X = SZ$ (3.9)

Then substitution of Equation (9) into Equation (6) yields

$$Z^{\cdot} = S^{-1}ASZ + S^{-1} \quad BU$$

$$= JZ + S^{-1}BU \qquad (3.10)$$

The condition for complete state controllability of the system of Equation 6) may. Then be stated as follows: The system is completely state .(3 10) are .controllable if and only if, No two Jordan blocks in **J** of Equation (3 associated with the same eigenvalues, the elements of any row of that 3) .correspond to the last row of each Jordan block. Are not all zero, and (3 the elements of each row of that correspond to distinct, Eigenvalues are not all zero.

2 the following systems are completely state controllable:**.Example3**

$$\begin{bmatrix} x^{\cdot}{}_1 \\ x^{\cdot}{}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} \text{U}$$

Since $[B \vdots AB] = \begin{bmatrix} 2 & -2 \\ 5 & -10 \end{bmatrix} = non$singular matrix the system is completely state controllable

3.1.2 Conditions for Complete State Controllability in the Plane

The condition for complete state controllability can be stated in terms of transfer functions or transfer matrices. It can be proved that a necessary and sufficient condition for complete state controllability is that no cancellation occur in the transfer function or transfer matrix. If cancellation occurs; the system cannot be controlled in the direction of the canceled mode.

3 Consider the following transfer function **.Example 3**

$$\frac{X(S)}{U(S)} = \frac{(s+2,5)}{(s+2,5)(s-1)}$$

Clearly, cancellation of the factor (s+2.5) occurs in the numerator and denominator of this. Transfer function. (Thus one degree of freedom is lost.) Because of this cancellation, this system is not completely state controllable. The same conclusion can be obtained by writing this transfer function in the form of a state, Equation a state-space representation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2,5 & 1,5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} U$$

Since $[B \vdots AB] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

The rank of the matrix is $[B \vdots AB]$ .Therefore, we arrive at the same conclusion: The system is not completely state controllable.

4 Show that the system described by**.3 Example**

$$\dot{X} = AX + BU$$

$$Y = CX$$

Where **x** = state vector (n-vector)

U=control vector(r-vector)

$Y = output\ control$ (m-vector) (m≤ n)

A=n×n matrix

B =n× r matrix

$C = m \times n\ matrix$

Is completely output controllable if and only if the composite m×nr matrix$P$, where

$$P = [CB \vdots CAB \vdots CA^2B \vdots \cdots\cdots \vdots CA^{n-1}B]$$

Is of rank m (Notice) that complete state controllability is neither necessary nor sufficient for complete output controllability

Suppose that the system is output controllable and the output **y** (t) starting from any **y** (0) the initial output can be transferred to the origin of the output space in a finite time interval

0 ≤ t ≤ that is

$$\mathbf{y}\,(T) = Cx(T) = \mathbf{0}$$

Since the solution of Equation P is

$$X(t) = e^{At\left[X(0)+\int_0^t e^{-At}Bu(\tau)d\tau\right]} \qquad (3.11)$$

At t= Twe have

$$X(t) = e^{AT\left[X(0)+\int_0^T e^{-A\tau}Bu(\tau)d\tau\right]} \qquad (3.12)$$

Substituting Equation (3.12) into Equation (3.11), we obtain

Y (T) = Cx(T)

$$= Ce^{AT\left[X(0)+\int_0^T e^{-A\tau}Bu(\tau)\,d\tau\right]} = 0 \qquad (3.13)$$

On the other hand, **y** (0) =Cx (0). Notice that the complete output controllability means that the. Vector Cx(0) spans the m-dimensional output space. Since e^{AT} is nonsingular, if Cx(0) spans the m-dimensional output space, so does Ce^{AT}**x**(0), and vice versa. From Equation (3.3) we obtain

$$\mathbf{C}e^{AT}\mathbf{x}(0) = -\mathbf{C}e^{AT}\int_0^T e^{A\tau}BU(T-\tau)d\tau$$

Note that $e^{AT}\int_0^T e^{A\tau}Bu(T-\tau)d\tau$can be expressed as the sum of A^iB_j; that is

$$\int_0^T e^{A\tau} BU(T-\tau)d\tau = \sum_{i=0}^{p-1} \quad \sum_{j=0}^{r} \gamma_{ij} \; A^i \boldsymbol{B_j}$$

Where

$$\gamma_{ij} = \int_0^T \alpha_{i(\tau)U_j}(\tau)(T-\tau)d\tau = sclar$$

And $\alpha_i(\tau)$ satisfies

$$e^{A\tau = \sum_{i=1}^{p-1} \alpha_i(\tau)} \; A^i$$

(P: degree of the minimal polynomial of (3.12))

And is theB_j; jth column of Therefore, we can write $\mathbf{C}e^{AT}\mathbf{x}$(0 as

$$\mathbf{C}e^{AT}\mathbf{x}(0) = -\sum_{i=0}^{p-1} \quad \sum_{j=0}^{r} \gamma_{ij} \, C \, A^i B_j$$

From this last equation, we see that $\mathbf{C}e^{AT}\mathbf{x}(0)$ is a linear combination of $C \, A^i B_j$ **.** (i=0, 1, 2, p,P-1; j=1, 2, p, r). Note that if the rank of **Q**, where

$$\text{M} = [CB \vdots CAB \vdots CA^2B \vdots \cdots\cdots \vdots CA^{n-1}B] \qquad (p \leq n)$$

Is m, then so is the rank of P, and vice versa. [This is obvious if p=n. If p<n, then the C$A^h B_j$.(Where p ≤ h ≤ n-1) are linearly dependent on CB_j ,CAB_j , C$A^{p-1}B_j$. Hence, the rank of p is equal to that of Q.] If the rank of P is m, then Ce^{AT}x (0) spans the m-dimensional output, space. This means that if the rank of P ism, then Cx(0) also spans the m-dimensional output space, and the system is completely output controllable. Conversely, suppose that the system is completely output controllable, but the rank of P is k, where k<m. Then the set of all initial outputs that can be transferred to the origin is of,

K-dimensional space hence, the dimension of this set is less than m. This contradicts the assumption, that the system is completely output controllable. This completes the proof. Note that it can be immediately proved that, in the system of Equations (3.12) and (3.13), state controllability on $0 \leq t \leq T$ implies complete output controllability on $0 \leq t \leq T$, if and only if m rows of C are linearly in depend

5 obtain a state-space equation and output equation and test **.3 Example** controllable for the system defined by

$$\frac{Y(S)}{U(S)} = \frac{2s^3 + s^2 + s + 2}{s^3 + 4s^2 + 5s + 2}$$

From the given transfer function, the differential equation for the system is

$$\dddot{y} + 4\ddot{y} + 5\dot{y} + 2y = 2\dddot{u} + \ddot{u} + \dot{u} + 2u$$

Comparing this equation with the standard equation given by Equation

$$\dddot{y} + a_1\ddot{y} + a_2\dot{y} + 2y = b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3u$$

We find

$$a_1 = 4, a_2 = 5, a_3 = 2$$

$$b_0 = 2, b_1 = 1, b_2 = 1, b_3 = 2$$

$$and$$

$\beta_0 = b_0 = 2$

$\beta_1 = b_1 - a_1\beta_0 = 1 - 4 \times 2 = -7$

$\beta_2 = b_2 - a_1\beta_1 - a_2\beta_0 = 1 - 4 \times (-7) - 5 \times 2 = 19$

$\beta_3 = b_{3 - a_1\beta_2 - a_2\beta_{1 - a_3\beta_0}} = 2 - 4 \times 19 - 5 \times (-7) - 2 \times 2 = -43$

And

$x_1 = y - \beta_0 u = y - 2u$

$x_{2=}\dot{x}_1 - \beta_1 u = \dot{x}_{1+7u}$

$x_3 = x^{\cdot}{}_2 - \ \beta_2\text{u} = x^{\cdot}{}_2 - 19\text{u}$

And

$$x^{\cdot}{}_1 = x_{2-7u}$$

$$x^{\cdot}{}_2 = x_{2+19u}$$

$x^{\cdot}{}_{3=} - a_3\, x_1 - a_2 x_2 - a_1 x_{3+} \beta_3 u = -2\, x_1 - 5x_2 - 4x_{2\_}43u$

Hence, the state-space representation of the system is

$$\begin{bmatrix} x^{\cdot}{}_1 \\ x^{\cdot}{}_2 \\ x^{\cdot}{}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -7 \\ 19 \\ -43 \end{bmatrix} + 2\text{U}$$

$$\text{Y}=\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Test controllable

Controllability matrix=$[B \vdots AB \vdots A^2 B]$

Since

$$B = \begin{bmatrix} -7 \\ 19 \\ -43 \end{bmatrix}, AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -2 \end{bmatrix} \begin{bmatrix} -7 \\ 19 \\ -43 \end{bmatrix} = \begin{bmatrix} 19 \\ -43 \\ 5 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -5 & -2 \\ 4 & 8 & -1 \end{bmatrix}, A^2 B = \begin{bmatrix} -43 \\ 5 \\ 167 \end{bmatrix}$$

$$\text{M}=[B \vdots AB \vdots A^2 B] == \begin{bmatrix} -7 & 19 & -43 \\ 19 & -43 & 5 \\ -43 & 5 & 167 \end{bmatrix}$$

$|\text{M}|=61492 \neq 0$ is system is complete controllable

Example3.6 the following systems are completely state controllable

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \\ 3 & 1 \end{bmatrix} \qquad A^2 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 9 & 1 \end{bmatrix} \quad A^2B = \begin{bmatrix} 0 & 4 \\ 4 & 0 \\ 9 & 1 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & 1 & 0 & 2 & 0 & 4 \\ 1 & 0 & 2 & 0 & 4 & 0 \\ 0 & 1 & 3 & 1 & 9 & 1 \end{bmatrix}$$

Rank (M)=3 , det(left half)=-1(3)=-3≠0 fully state controllability

3.1.3 Vander Monde Matrix

If the state model is obtained using the phase variables then the matrix A is in bush for or phase variable form as

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$$

And the characteristic equation i.e. denominator of T(s) is

$$F(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0$$

In such a case model matrix takes a form of a special matrix as

$$V = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & & \cdots & \cdots \\ \lambda_1^{n-i} & \lambda_2^{n-2} & \cdots & \lambda_n^{n-1} \end{bmatrix}$$

Such a model matrix for matrix A which in phase variable is called Vander monde matrix

Example 3.7 consider a state model with matrix A as

$$A = \begin{bmatrix} 0 & 2 & 0 \\ 4 & 0 & 1 \\ -48 & -34 & -9 \end{bmatrix}$$

Determine (a) characteristic equation (b) Eigen values (c) Eigen vector and model matrix ,also prove that the transformed $M^{-1}AM$ a diagonal matrix

(a) Characteristic equation is $|\lambda I - A|$

$$A = \lambda \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 2 & 0 \\ 4 & 0 & 1 \\ -48 & -34 & -9 \end{vmatrix} = 0 \quad , \begin{vmatrix} \lambda & -2 & 0 \\ 4 & 0 & 1 \\ 48 & 34 & \lambda + 9 \end{vmatrix} = 0$$

$$\lambda^2(\lambda + 9) + 2 \times 48 + 0 + 0 - 8(\lambda + 9) + 34\lambda = 0$$

$\lambda^3 + 9\lambda^2 + 26\lambda + 24 = 0$

This is required characteristic equation

(b) To find Eigen values test $\lambda = -2$ for root

−2	1	9	26	24
		−2	−14	−24
	1	7	12	0

$$(\lambda + 2)(\lambda^2 + 7\lambda + 12) = 0 \ \ i.e\ (\lambda + 2)(\lambda + 3)(\lambda + 4) = 0$$

$$i.e\ \ \lambda_1 = -2\ ,\lambda_2 = -3\ \ and\ \lambda_3 = -4$$

These are the Eigen values of the matrix A

(c) To find Eigen vectors, obtain matrix $[\lambda_i I - A]$ for each Eigen values by substituting value of λ in above equation

$$For\ \lambda_1 = -2\ ,[\lambda_i I - A] = \begin{vmatrix} -2 & -2 & 0 \\ -4 & -2 & -1 \\ 48 & 34 & 7 \end{vmatrix}$$

$$M_1 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} where\ c_{11,}\ c_{12,}\ c_{13}\ cofactor\ of\ \ row\ 1$$

$$M_1 = \begin{bmatrix} 20 \\ -20 \\ 40 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}\ \ the\ common\ \ factor\ can\ be\ taken\ out$$

$$For\ \lambda_2 = -3, [\lambda_i I - A] = \begin{bmatrix} -3 & -2 & 0 \\ -4 & -3 & -1 \\ 48 & 34 & 6 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \begin{bmatrix} 16 \\ -24 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

$$For\ \lambda_3 = -4\ \ [\lambda_i I - A] = \begin{bmatrix} -4 & -2 & 0 \\ -4 & -4 & -1 \\ 48 & 34 & 5 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \begin{bmatrix} 14 \\ -28 \\ 56 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

$M_1, M_2\ and\ M_3\ are\ the\ eigen\ vectors\ corrsesponding\ to\ the\ eigen\ values$

$$\lambda_1, \lambda_2\ and\ \lambda_3$$

(d) The modal matrix is

$$M = [M_1 : M_2 : M_3] = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -3 & -2 \\ -2 & 1 & 4 \end{bmatrix}$$

Let us prove $M^{-1}AM = is\ adiagonal\ matrix$

$$M^{-1} = \frac{Adj[M]}{|M|} = \frac{[cofactor\ of\ M]^T}{|M|}$$

$$Adj[M] = \begin{bmatrix} -10 & 8 & -7 \\ -7 & 6 & -5 \\ -1 & 1 & -1 \end{bmatrix}^T = \begin{bmatrix} -10 & -7 & -1 \\ 8 & 6 & 1 \\ -7 & -5 & -1 \end{bmatrix}$$

$$|M| = \begin{vmatrix} 1 & 2 & 1 \\ -1 & -3 & -2 \\ -2 & 1 & 4 \end{vmatrix} = -1\ , \qquad \frac{Adj[M]}{|M|} = \begin{bmatrix} 10 & 7 & 1 \\ -8 & -6 & -1 \\ 7 & 5 & 1 \end{bmatrix}$$

$$AM = \begin{bmatrix} 0 & 2 & 0 \\ 4 & 0 & 1 \\ -48 & -34 & -9 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ -1 & -3 & -2 \\ -2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} -2 & -6 & -4 \\ 2 & 9 & 8 \\ 4 & -3 & -16 \end{bmatrix}$$

$$M^{-1}AM = \begin{bmatrix} 10 & 7 & 1 \\ -8 & -6 & -1 \\ 7 & 5 & 1 \end{bmatrix} \begin{bmatrix} -2 & -6 & -4 \\ 2 & 9 & 8 \\ 4 & -3 & -16 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \Delta$$

3.1.4 MIMO System

For multiple inputs multiple output systems a single transfer function does not exist. There exists a mathematical relationship between each output and all the inputs. Hence for such systems there exists a transfer matrix rather than transfer function but method of obtaining transfer matrix remains same as before.

8 Determine the transfer matrix for MIMO system given by**.Example 3**

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

From given state model

A= $\begin{bmatrix} 0 & 3 \\ -2 & -5 \end{bmatrix}$, B= $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, C= $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$, D= [0]

T.M= $C[SI - A]^{-1}B + D$

$[SI - A]$=$\begin{bmatrix} s & -3 \\ 2 & s+3 \end{bmatrix}$

Adj$[SI - A] = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T = \begin{bmatrix} s+5 & -2 \\ 3 & s \end{bmatrix}^T = \begin{bmatrix} s+5 & 3 \\ -2 & s \end{bmatrix}$

$$|SI - A| = s^2 + 5s + 6s = (s+2)(s+3)$$

$\therefore\ [SI - A]^{-1} = \frac{\text{Adj}[SI-A]}{|SI-A|} = \frac{\begin{bmatrix} s+5 & 3 \\ -2 & s \end{bmatrix}}{(s+2)(s+3)}$

$\therefore$ T.M= $C[SI - A]^{-1}B = \frac{\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} s+5 & 3 \\ -2 & s \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}{(s+2)(s+3)}$

$= \frac{\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} s+8 & s+3 \\ s-2 & s-2 \end{bmatrix}}{(s+2)(s+3)} = \frac{\begin{bmatrix} 3s+14 & 3s+14 \\ s+8 & s+8 \end{bmatrix}}{(s+2)(s+3)}$

3.1.5 Gilbert's Test for controllability

For Gilbert's test it is necessary that the matrix A must be in canonical form .Hence the given state model required to be transformed to the canonical form first to apply the Gilbert test

Consider single input linear time invariant system represented by

$$X^{\cdot} = AX(t) + BU(t)$$

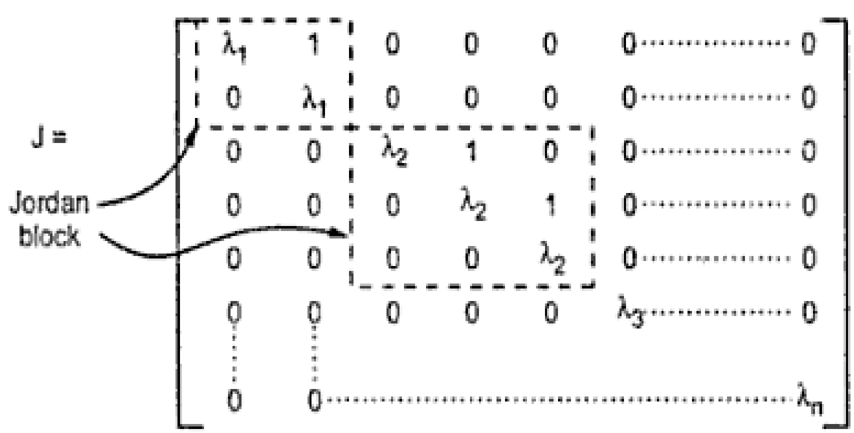
Where A is not in the canonical form then it can be transformed to be canonical form by the transformation $X(t) = MZ(t)$ Where M=model matrix. The transformed state model, as derived earlier take the form

$$Z^{\cdot}(t) = \nabla Z(t) + B^{\sim}U(t)$$

Where $\nabla = diagonal\ matrix$

$$B^{\sim} = M^{-1}B$$

It is assumed that the Eigen values of A are distinct in such a case the necessary and sufficient condition for are complete state controllability is that the vector matrix $B^{\sim}$should not have any zero elements If it has zero elements then the corresponding state variables are not controllable if the Eigen values are repeated then matrix A cannot be transformed to Jordan results Jordan canonical form, as



In such a case the condition for complete state controllability is that the elements of any row of $B^{\sim}$ that corresponds Jordan to the last row of each Jordan block are not all zero

Example 3.9 consider the system with state equation

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} U(t)$$

Estimate the state controllability by (i) kalman,s test and (ii) Gilbert's test

(i) Kaman,s test

$$M = [B : AB : A^2B] \ldots N = 3$$

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -6 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -6 & -11 & -6 \\ 36 & 60 & 25 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & 0 & 1 \\ -6 & -11 & -6 \\ 36 & 60 & 25 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ 25 \end{bmatrix} \quad , M = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 25 \end{bmatrix}$$

$$= 1 \, , thus \, |M| \, is \, nonsingular$$

Hence the rank of M is 3 which is n thus the system is completely state controllable

(ii) Gilbert test. For this it is necessary to express A in the canonical form find the Eigen values of A

$$[\lambda_i I - A] = \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda + 6 \end{bmatrix} = 0 \quad , \lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$$

$$(\lambda + 1)(\lambda + 2)(\lambda + 3) = 0 \ , \lambda_1 = -1, \qquad \lambda_2 = -2, \quad \lambda_3 = -3$$

$$M = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ {\lambda_1}^2 & {\lambda_2}^2 & {\lambda_3}^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

$$M^{-1} = \frac{Adj[M]}{|M|} = \frac{\begin{bmatrix} -6 & 6 & -2 \\ -5 & 8 & -3 \\ -1 & 2 & -1 \end{bmatrix}^T}{-2} = \begin{bmatrix} 3 & 2.5 & O.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix}$$

$$B^{\sim} = M^{-1}B = \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -1 \\ 0.5 \end{bmatrix}$$

As matrix A is in phase variable form the model matrix M is Vander monde matrix.As none of the elements of $B^{\sim}$are zero the system is completely state controllable. As Gilbert’s test requires transforming matrix A into canonical form it is time consuming and hence kalman,s test is popularly used to test controllability

3.1.6 Output controllability

In the practical design of a control system, we may want to control the output rather than the state of the system. Complete state controllability.is neither necessary nor sufficient for controlling the output of the system for this reason, it is desirable to define separately complete output controllability.

Consider the system described by

14). $\dot{x}$ =Ax+Bu (3

$$Y = CX + Du \qquad (3.15)$$

Where **x** = state vector (n-vector)

U=control vector(r-vector)

$Y = output\ control$ (m-vector)

A=n×n matrix

B =n× r matrix

$C = m \times nmatrix$

$D = m \times r \; matrix$

The system described by Equations (3.14) and (3.15) is said to be completely output. Controllable if it is possible to construct an unconstrained control vector **u** (t) that will. Transfer any given initial output **y**At0 B to any final output **y**At1 B in a finite time interval $t_0 \leq$ t $\leq t_1$,It can be proved that the condition for complete output controllability is as follows:

15) is completely output .14) and (3.The system described by Equations (3 controllable. If and only if the m ×(n+1) r matrix

$[C \vdots CAB \vdots CA^2B \vdots \cdots\cdots \vdots CA^{n-1}B \vdots D]$

4.1.7 Uncontrollable System.

An uncontrollable system has a subsystem that is physically disconnected from the, Input

3.2 OBSERVABILITY

We discuss the observability of linear systems. Consider the unforced System described by the following equations

16).$x^{\cdot}$ =Ax (3

$Y = CX$ (3.17)

Where **x** = state vector (n-vector)

$Y = output\ control$ (m-vector)

A=n×n matrix

C =m× n matrix

The system is said to be completely observable if every state **x** (t_0) can be determined from the observation of **y**(t) over a finite time interval, $t_0 \leq$ t $\leq$ t_1

The system is therefore completely observable if every transition of the state eventually affects every element, of the output vector. The concept of observability is useful in solving the problem of reconstructing unmeasurable state variables from measurable variables in the minimum possible length of time we treat only linear, time-invariant systems. Therefore, without loss of generality, we can assume that t_0=0 the concept of observability is very important because, in practice, the difficulty. Encountered with state feedback control is that some of the state variables are not. Accessible for direct measurement, with the result that it becomes necessary to estimate the unmeasurable state variables in order to construct the control signals. It will be. Shown in that such estimate of state variables are possible if and only if the system is completely observable. In discussing observability conditions, we consider the unforced system as given By 17).The reason for this is as follows: If the system is .16) and (3.Equations (3 described

$$\dot{x} = \text{Ax} + BU$$

$$Y = CX + DU$$

Then

X (t) $= e^{At}$x (0) $+\int_0^t e^{A(t-\tau)} Bu(\tau)d\tau$

And y (t) is

Y (t) = e^{At}x (0) $+\int_0^t e^{A(t-\tau)} Bu(\tau)d\tau + DU$

Since the matrices A, B, C, and D are known and u (t) is also known, the last two terms, on the right-hand side of this last equation are known quantities. Therefore they may be subtracted from the observed value of **y** (t). Hence, for investigating a necessary and sufficient condition for complete

Observability, it suffices to consider the system described by Equations 17).16) and (3.(3

3.2.1 Complete Observability of Continuous-Time Systems.

17).The output .16) and (3. Consider the system described by Equations (3 vector Y (t) is

Y (t) =Ce^{At}X (0).

12), we have.10) or (3.Referring to Equation (3

$$e^{At} = \sum_{k=0}^{n-1} \alpha_k(\tau) A^k$$

Where n is the degree of the characteristic polynomial. Note that Equations (3.8) and (3.10) with m replaced by n can be derived using the characteristic polynomial. Hence we obtain.

$$y(t) = \sum_{k=0}^{n-1} \alpha_k(\tau)\, CA^k X(0)$$

$$y(t = \alpha_0(t)CX(0) + \alpha_1(t)CAX(0) + \cdots + \alpha_{n-1}(t)CA^{n-1}X(0 \qquad (3.18)$$

If the system is completely observable, then, given the output **y** (t) over a (3.18) .time interval $t_0 \le$ t $\le t_1$**X** (0) is uniquely determined from Equation It can be shown that this requires the rank of the nm×n matrix

$$\begin{bmatrix} C \\ CA \\ \vdots \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

From this analysis, we can state the condition for complete observability as follows the system described by Equations (3.16)and 3.17) is completely observable if and only if the n×nm matrix

$$[C^T \vdots A^T C^T \vdots \cdots\cdots \vdots (A^T)^{n-1} C^T]$$

Is of rank n or has n linearly independent column vectors. This matrix is called the.*observability matrix*

10.3 Example

consider the system .Is this system controllable and observable. Since the rank of the matrix

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$Y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$[B \vdots AB] = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

Is 2, the system is completely state controllable. For output controllability, let us find the rank of the matrix $[CB \vdots CAB]$ since

$$[CB \vdots CAB] = [1 \quad 0]$$

The rank of this matrix is 1. Hence, the system is completely output controllable. To test the observability condition, examine the rank of

$$[C^T \vdots A^T C^T]$$

Since

$$[C^T \vdots A^T C^T] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

The rank of $[C^T \vdots A^T C^T]$ is 2. Hence, the system is completely observable

3.2.2 Conditions for Complete Observability in the Plane

The conditions for complete observability can also be stated in terms of transfer functions or transfer matrices. The necessary and sufficient conditions for complete observability are that no cancellation, occur in the transfer function or transfer matrix. If cancellation occurs, the canceled, mode cannot be observed in the output

Example3:11

Show that the following system is not completely observable**.**

$$X^{\cdot} = AX + BU$$

$$\text{Y}= C$$

Where

$$\text{X}=\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix},\quad \text{A}=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix},\quad \text{B}=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\quad \text{C}= [4 \quad 5 \quad 1]$$

Note that the control function u does not affect the complete observability of the system to examine complete observability, we may simply set u=0. For this system, we have

$$[C^T \vdots C^T A^T \vdots (A^T)^2 C^T] = \begin{bmatrix} 4 & -6 & 6 \\ 5 & -7 & 5 \\ 1 & -1 & -1 \end{bmatrix}$$

Note that

$$\begin{vmatrix} 4 & -6 & 6 \\ 5 & -7 & 5 \\ 1 & -1 & -1 \end{vmatrix} = 0$$

Hence, the rank of the matrix $[C^T \vdots C^T A^T \vdots \cdots\cdots (A^T)^2 C^T]$ is less than3.Therefore, the system is not, completely observable.

Comments. The transfer function has no cancellation if and only if the system is completely, state controllable and completely observable. This means that the canceled transfer, function does not carry along all the information characterizing the dynamic system.

3.2.3 Alternative Form of the Condition for Complete Observability.

Consider the System described by Equations (13) and (14), rewritten

$$X^{\cdot} = AX \qquad (3.19)$$

$$Y = CX \tag{2.20}$$

Suppose that the transformation matrix **P** transforms **A** into a diagonal matrix, or

$$P^{-1}AP = D$$

Where **D** is a diagonal matrix let us define

$$X = PZ$$

Then Equations (16) and (17) can be written

$$Z^{\cdot} = P^{-1}APZ = DZ$$

$$Y = CPZ$$

Hence

$$Y(t) = CPe^{At}Z(0)$$

The system is completely observable if (1) no two Jordan blocks in **J** are associated with, the same eigenvalues, (2) no columns of **CS** that correspond to the first row of each. Jordan block consist of zero elements, and (3) no columns of **CS** that correspond to, distinct eigenvalues consist of zero elements

3.2.4 Gilbert test observability

It is known that for Gilbert test the state model must be expressed in the canonical form; consider the state model of linear time invariant system, as

$$X^{\cdot}(t) = AX(t) + BU(t) \quad and \;\; Y(t) = CX(t)$$

Use the transformation $X(t) = MZ(t) \;\; where\, M \equiv model\, matrix$

$$\therefore \;\; Y(t) = CMZ(t) = C^{\sim}Z(t) \qquad where \;\; C^{\sim} = CM$$

For a single input single output system

$$Y(t) = C^{\sim}Z(t) = [C^{\sim}{}_{11}\ \ C^{\sim}{}_{12}\ \ \cdots C^{\sim}{}_{1n}]\begin{bmatrix} z_{1(t)} \\ z_2(t) \\ \vdots \\ z_n(t) \end{bmatrix}$$

$$= C^{\sim}{}_{11}z_1(t)\ \ C^{\sim}{}_{12}z_2(t)\ \ \cdots C^{\sim}{}_{1n}z_n(t)$$

Due to the canonical form all the state are decoupled and not like the each other. Hence for the system to be observable each term corresponding to each state must be observed in the output Hence none of the coefficient of the $C^{\sim}$ must be zero thus the system is the complete observable if all the coefficients of $C^{\sim}$ are nonzero coefficient none of the coefficient is zero if any element is zero the corresponding state remains unobservable i.e shielded from observation

Example 3.12 evaluate the observability of the system with using Gilbert's test

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\ and\ C = [3 \quad 4 \quad 1]$$

For Gilbert's test find the Eigen values

$$[\lambda I - A] = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & 2 & \lambda + 3 \end{vmatrix} = 0 \quad , \lambda^3 + 3\lambda^2 + 2\lambda = 0$$

$$\lambda(\lambda^2 + 3\lambda + 2) = 0 \ \ \therefore \lambda(\lambda + 1)(\lambda + 2) = 0$$

$$\lambda_1 = 0\ , \lambda_2 = -1\ , \lambda_3 = -2$$

As the Eigen values are distinct the model matrix M is the Vander Monde matrix

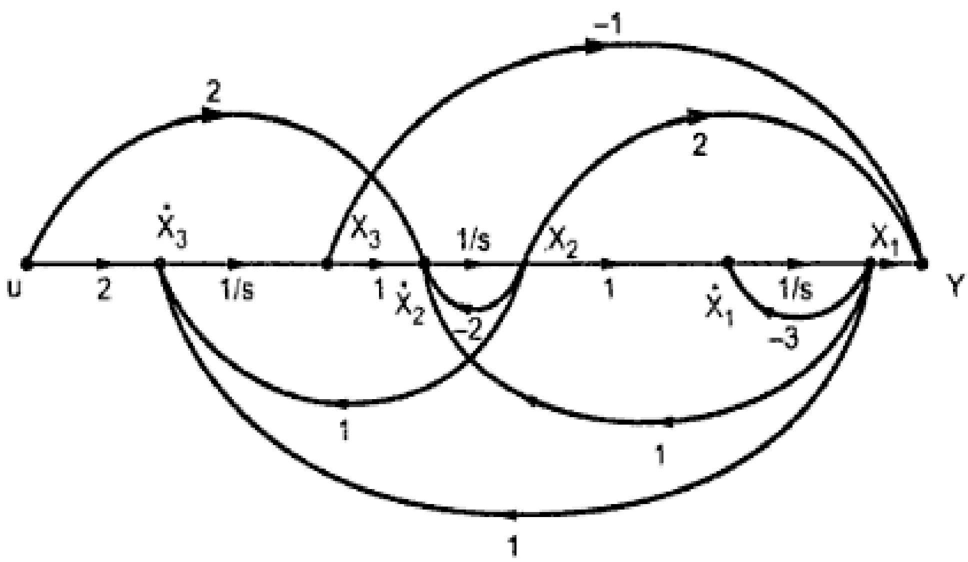
$$M = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_1 \\ {\lambda_1}^2 & {\lambda_2}^2 & {\lambda_3}^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 1 & 4 \end{bmatrix}$$

When the model transformed form

$$C^{\sim} = CM = [3 \quad 4 \quad 1]\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 1 & 4 \end{bmatrix} = [3 \quad 0 \quad -1]$$

As there is one zero elements in $C^{\sim}$ the system is not complete observsble

Example 3.13 Use controllability and observability matrices to determine whether the system represented by the flow graph shown the figure is completely controllability and completely observability



The value of variable at the node is an algebraic sum of all signals entering at the node

$$\therefore \quad \dot{X}_3 = X_1 + X_2 + 2u, \qquad \dot{X}_2 = X_1 - 2X_2 + X_3 + 2u$$

$$\dot{X}_1 = -3X_1 + X_2, \qquad Y = X_1 + 2X_2 - X_3$$

$$\therefore \quad A = \begin{bmatrix} -3 & 1 & 0 \\ 1 & -2 & 1 \\ 1 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, C = [1 \ \ 2 \ -1]$$

For controllability $M = [B: \quad AB: \quad A^2B] \quad for\ n=3$

$$AB = \begin{bmatrix} -3 & 1 & 0 \\ 1 & -2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} -3 & 1 & 0 \\ 1 & -2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -3 & 1 & 0 \\ 1 & -2 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 10 & -5 & 1 \\ -4 & 6 & -2 \\ -2 & -1 & 1 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 10 & -5 & 1 \\ -4 & 6 & -2 \\ -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -8 \\ 8 \\ 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & 2 & -8 \\ 2 & -2 & 8 \\ 2 & 2 & 0 \end{bmatrix} \quad \therefore\ |M| = -32 \neq 0\ hence\ rank\ n = 3 = n$$

The system is fully state controllable

For observablility $M_o\ [C^T: \quad A^TC^T \quad : {A^T}^2C^T] \ ,\ n = 3$

$$A^TC^T = \begin{bmatrix} -3 & 1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix}$$

$${A^T}^2C^T = \begin{bmatrix} -3 & 1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ -4 \end{bmatrix}$$

$$\therefore\ M_o = \begin{bmatrix} 1 & -2 & 4 \\ 2 & -4 & 8 \\ -1 & 2 & -4 \end{bmatrix} \quad , |M_o| = 0\ and\ \begin{vmatrix} 2 & -4 \\ -1 & 2 \end{vmatrix} = 0$$

Thus 2×2 determinant is zero hence the system is not observable

3.3 Principle of Duality

We shall now discuss the relationship between controllability and observability .We shall introduce the principle of duality, due to Kaman, to clarify apparent analogies between controllability and observability.

Consider the system S_1 described by

$$x^{\cdot} = Ax + Bu$$

$$Y = CX$$

Where **x** = state vector (n-vector)

U=control vector(r-vector)

$Y = output\ control$ (m-vector)

A=n×n matrix

B =n× r matrix

$C = m \times nmatrix$

$D = m \times r\ matrix$

And the dual system S_2 defined by

$$Z^{\cdot} = A^{*}Z + C^{*}V$$

$$n = B^{*}Z$$

Where

Z=State vector (n-vector)

V=control vector (m-vector)

n=output vector(r-vector)

$A^{*} = conjugate\ transpose\ of\ A$

$B^{*} = conjugate\ transpose\ of\ B$

$C^{*} = conjugate\ transpose\ of\ C$

The principle of duality states that the system S_1 is completely state controllable.(Observable) if and only if system S_2 is completely observable (state controllable).To verify this principle, let us write down the necessary and sufficient conditions for complete state controllability and complete Observability for systems S_1 and S_2

For system S_1 :

1. A necessary and sufficient condition for complete state controllability is that the. Rank of the n×nr matrix

$$[B \vdots AB \vdots \cdots A^{n-1}B]$$

Be n

2. A necessary and sufficient condition for complete observability is that the rank of the n×nm matrix

$$\left[C^T \vdots A^T\, C^T \vdots \cdots\cdots \vdots A^{T^{n-1}}\, C^T\right]$$

Be n:

For system $\boldsymbol{S_2}$

A. A necessary and sufficient condition for complete state controllability is that the. Rank of the n×nm matrix

$$[C^T \vdots A^T\, C^T \vdots \cdots\cdots \vdots (A^T)^{n-1}\, C^T\,]$$

Be n

B. A necessary and sufficient condition for complete observability is that the rank of the n×nr matrix

$$[B \vdots AB \vdots \cdots A^{n-1}B]$$

Be n.

By comparing these conditions, the truth of this principle is apparent. By use of this, principle, the observability of a given system can be checked by testing the state controllability, of its dual

3.4 Detectability.

For a partially observable system, if the unobservable modes are, stable and the observable modes are unstable, the system is said to be detectable. Note, that the concept of detectability is dual to the concept of stabilizability

Chapter Four

Bounded controls

4. Introduction

Time delay often occurs in engineering system. Since the existence of time delay usually instability of the system the study on the Time delay systems has received considerable attentions and many stability criteria for time delay systems can be found in the literature. Stability criteria for time delay systems. Tend to fall into one of the two categories: delay independent and delay dependent .As the name implies delay independent criteria provide conditions which guarantee stability for any length of the time delay on the other hand delay dependent criteria exploit a priori knowledge of upper bounds on the a mount of time delay, These criteria since more information about the Time delay is assumed to be known

4.1 stability

Definition of stability 4.1

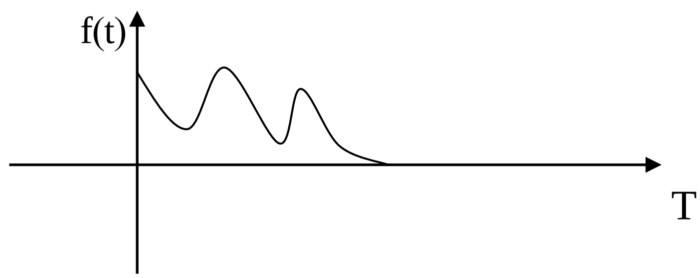
A continuous system (discrete time system) is stable if its impulse response $y_f(t)$ (koneke delta response $y_f(t)$ approaches zero at time approaches infinity.

Alternatively the definition of a stable system can be upon the response of the system to bounded inputs that is inputs whose magnitudes are less than some finite value for all time.

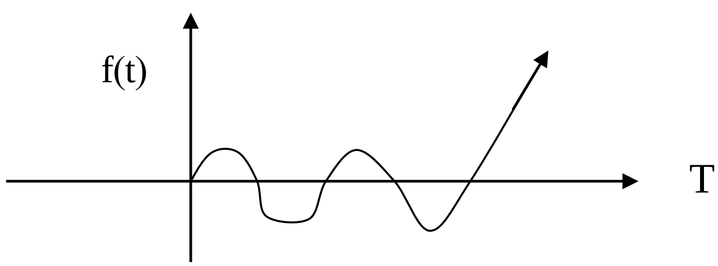
Other definition of stability 4.2

A continuous or discrete time system is stable if every bounded input produces a bounded output **(BIBO)** also definition. The system is stable if

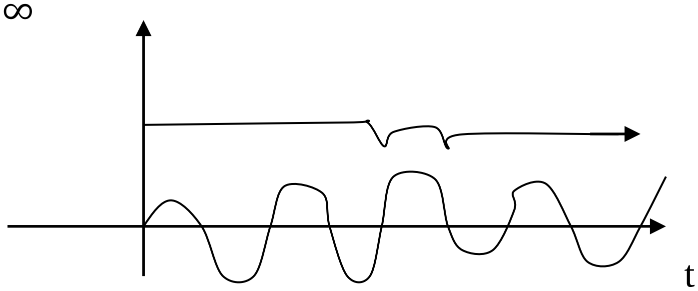
f(t) ⇢ ∞ as t ⇢ ∞



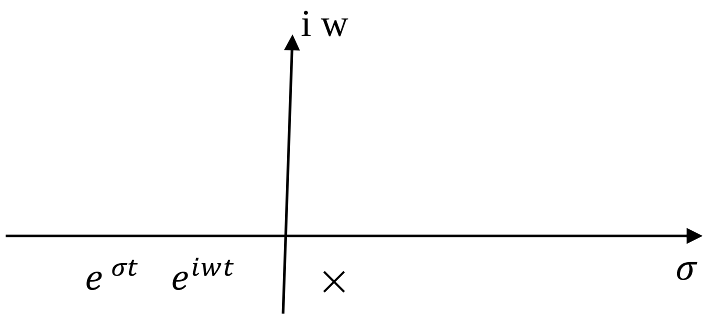
The system is an unstable if f(t) → ∞ as t → ∞



The system is a marginally stable f(t) does not decay to 0 or go to ∞ as t→ ∞



Consideration of degree of stability of a system often provides valuable information about its behavior. That is if it stable how close is it to being unstable and Definition of stability



Poles on the iw axis: marginally stable in above figure we define stability the closed loop transfer function left hand side (**BIBO**).

4.1.1 Characteristic Root locations for continuous systems

A major result of chapter 3 is that the impulse response of a linear time invariant continuous system is a sum of exponential time functions. Whose exponents are the roots of the system characteristic equation A necessary and sufficient condition for the system to be stable is that real parts of the roots of the characteristic equation have negative real parts. This ensures that the impulse response will decay exponentially with time. If the system has some roots with real parts equal to zero but none with positive real parts the system is said to be marginally stable. In this instance the impulse response does not decay to zero although it is bounded but certain other inputs with produce unbounded outputs therefore marginally stable system unstable.

Example 4.1

Differential Equation

$(S^2 + 1)\, y(S) = U(S)$

Has the characteristic equation

$S^2 + 1 = 0$

This equation has the two roots $\pm j$

Since these roots have zero real parts the system is not stable it is however marginally stable since the equation has no roots with positive real parts. In response to most input or output will contain term of the form y= t constant

This is unbounded

4.1.2 Routh stability criterion

The Routh criterion is a method of determining continuous system stability for system with an nth order characteristic equation of the form

$a_n s^n + a_{n-1} s^{n-1} + \ldots\ldots\ldots\ldots + a_1 s + a_0 = 0$

The criterion is applied using a Routh table defined as follows

S^n	a_n	a_{n-2}	a_{n-4}	....
S^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	....
.	b_1	b_2	b_3	....
.	c_1	c_2	c_3	....
				.

Where a_n , a_{n-1} ,…………, a_0 are the coefficient of the characteristic equation and

$$b_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}$$

$$b_2 = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}} \qquad etc$$

$$C_1 = \frac{b_1 a_3 - a_{n-1} b_2}{b_1} \qquad ,C_2 = \frac{b_1 a_{n-5} - a_{n-1} b_3}{b_1} \qquad etc$$

The table is continued horizontally and vertically unit only zero are obtained before the next row is computed without disturbing the properties of the table.

The Routh criterion .all the roots of the characteristic if the elements of the first column of the Routh table have the same sign. Otherwise the number of roots with positive real parts is equal to the number of change of sign

Example 4.2

$S^3 + 6S^2 + 12S + 8 = 0$

s^3	1	12	0
s^2	6	8	0
s^1		$\frac{64}{6}$	0
s^0			8

$$b_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}} = \frac{6\times 12 - 1\times 8}{6} = \frac{64}{6}$$

$C_{1=8}$

Since there are no changes of sign in the first column of the table all the roots of the equation have negative real parts. Often it is desirable to determine arrange of value of a particular system parameter for which the system is stable. This can be accomplished by writing the inequalities the ensure that there is no change of sign in the first column of the Routh table for the system

These inequalities then specify the range of allowable values of the parameter.

For no sign change in the first column it is necessary that the condition

$1 + K > 0 ‹ 8 - K > 0$

Be satisfied .thus the characteristic equation has roots with negative real parts if -1< k <8, the simultaneous solution of these two inequalities. a row of zero the $\grave{S}$ row of the Routh Stable indicates that the polynomial has a pair of roots which satisfy the auxiliary equation formed as flows

AS^2 +B =0

Where A and B are the first and second elements of the S^2 row zero in the

To continue the stable the zero in the $\grave{S}$ row are replaced with the coefficients of the derivative of auxiliary equation is 2AS+0=0

The coefficients 2A and 0 are then entered into the $\grave{S}$row and the table is continued as described above

Example 4.3

$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$

s^4	1	3	5
s^3	2	4	0
s^2	1	5	0
s^1	-6	0	0
s^0			5

The system unstable 2 poles RHP

Special case 1

Example4.4

$s^3 + 2s^2 + s + 2 = 0$

s^3		1	1
s^2		2	2
s^1	0_ϵ	0	
s^0		1	0

E = 0.0001

The system is stable

Special case 2

Example 4.5

$$s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$$

s^5	1	24	-25
s^4	2	48	-50
s^3	0_8	0_{98}	0
s^2	24	-50	o
s^1	112,7	0	
s^0	-50		

The system is unstable

I poles RHP

4.1.3 Hurwitz stability criterion

The Hurwitz criterion is another method for determining whether all the roots of the characteristic equation of a continuous system have negative real parts.

This criterion is applied using determents formed form the coefficients of the characteristic equation.

It is assumed that the first coefficient a_n is positive the determinants A_i I = 1,2,3,………………,n-1

Are formed as the principal minor determinants of the determinant

$$\begin{vmatrix} \ldots\ldots 0\Delta_{n=} & a_{n-1} & a_{n-3}\ldots\ldots & \begin{bmatrix} a_0 \; if\; n\; odd \\ a_1 \; if\; n\; even \end{bmatrix} & 0 \\ \ldots 0\ldots\ldots & a_n & a_{n-2} & \begin{bmatrix} a_1 \; if\; n\; odd \\ a_0 \; if\; n\; even \end{bmatrix} & 0\ldots \\ 0\ldots\ldots & 0 & a_{n-1} & a_{n-3}\ldots\ldots & \\ 0\ldots\ldots & 0 & a_n & a_{n-2}\ldots\ldots & \\ 0\ldots\ldots & & & & a_0 \end{vmatrix}$$

The determinant are thus formed as follows

$\Delta_1 = a_{n-1}$

$$\Delta_2 = \begin{vmatrix} a_{n-1} & a_{n-3} \\ a_n & a_{n-2} \end{vmatrix} = a_{n-1}\,a_{n-2} - a_n\,a_{n-3}$$

$$\Delta_3 = \begin{vmatrix} a_{n-1} & a_{n-3} & a_{n-5} \\ a_n & a_{n-2} & a_{n-4} \\ 0 & a_{n-1} & a_{n-3} \end{vmatrix} = a_{n-1}\,a_{n-2}\,a_{n-3} + a_n\,a_{n-1}\,a_{n-6} - a_n a_{n-3}^2 - a_{n-4}a_{n-1}^2$$

And so on up to Δ_{n-1}

All the roots of the characteristic equation have negative real pats if and only

If n=3

$$\Delta_3 = \begin{vmatrix} a_2 & a_0 & 0 \\ a_3 & a_1 & 0 \\ 0 & a_2 & a_0 \end{vmatrix} = a_2 a_1 a_0 - {a_0}^2 a_3 \qquad , \Delta_2 = \begin{vmatrix} a_2 & a_0 \\ a_3 & a_1 \end{vmatrix} = a_2\,a_1 - a_3\,a_0$$

$\Delta_i > 0$ i= 1,2,……………..,n

$\Delta_1 = a_2$

Thus all the roots of the characteristic equation have negative real parts if

$a_2 a_1 a_0$ - $a_0^2 a_3$>0 ‘$a_2 a_1$ - $a_0 a_1$> 0 ‘a_2> 0

$$\Delta_3 = \begin{vmatrix} a_{n-1} & a_{n-3} & a_{n-5} \\ a_n & a_{n-2} & a_{n-4} \\ & & \vdots \\ 0 & a_{n-1} & a_{n-3} \end{vmatrix} = \begin{vmatrix} 8 & 24 & 0 \\ 1 & 14 & 0 \\ 0 & 8 & 24 \end{vmatrix}$$

Example4.6

Determinant if the characteristic equation below represents a stable or au stable system using the Hurwitz criterion method

$$s^3 + 8s^3 + 24s + 24 = 0$$

$$\Delta_3 = \begin{vmatrix} 8 & 24 & 0 \\ 1 & 14 & 0 \\ 0 & 8 & 24 \end{vmatrix}$$

$$\Delta_3 = 8(14 \times 24 - 0 \times 8) - 24(1 \times 24 - 0) + 0(1 \times 8 - 0) =$$
$$2688 - 576 = 211$$

$$\Delta_2 = \begin{vmatrix} a_2 & a_0 \\ a_3 & a_1 \end{vmatrix} = \begin{vmatrix} 8 & 24 \\ 1 & 14 \end{vmatrix} = 88 \ , \ \Delta_1 = 8$$

In this method the system is a stable iffΔ_3> 0

∴Δ_3= 2112 >0 the system is stable

Δ_1 =a_1 =8

∴Each determinant is positive

∴The system is a stable.

Example4.7

For what range of values of K is system with following characteristic equation stable using Hurwitz method

$S^2 + ks + 2k - 1 = 0$

The Hurwitz determinant for this system

$\Delta_2 = \begin{vmatrix} k & 0 \\ 1 & 2-1k \end{vmatrix} = 2k^2 - k = k(2k-1)$

$\Delta_1 = k$

In order for this determinant stable positive

It is necessary that K >0 an 2K-1> 0

Thus the system is stable if K > ½

Example4.8

Determine the Hurwitz condition for stability of the following general fourth order characteristic equation assuming a_4 is positive

$a_4 S^4 + a_3 S^3 + a_2 S^2 + a_1 S + a_0 = 0$

The Hurwitz determinants are

$$\Delta_4 = \begin{vmatrix} a_3 & a_1 & 0 & 0 \\ a_4 & a_2 & a_0 & 0 \\ 0 & a_3 & a_1 & 0 \\ 0 & a_4 & a_2 & a_0 \end{vmatrix} = a_3\,(a_2\,a_1 a_0 - a_3 a_0^2) - a_1^2 a_0 a_4$$

$$\Delta_3 = \begin{vmatrix} a_3 & a_1 & 0 \\ a_2 & a_0 & 0 \\ 0 & a_2 & a_0 \end{vmatrix} = a_3\,a_2\,a_1 - a_0 a_3^2 - a_4 a_1^2$$

$\Delta_1 = a_3$

$$\Delta_2 = \begin{vmatrix} a_3 & a_1 \\ a_4 & a_2 \end{vmatrix} = a_3\, a_2 \; - a_4\, a_1$$

$a_3\, a_2\, a_1$ - ، $a_3\, a_2$ - $a_4\, a_1 > 0$ ،The conditions for stability are then $a_3 > 0$

$a_0 a_3^2$ -$a_4 a_1^2$>0

$a_3\,(a_2\, a_1 a_0 - a_3 a_0^2)$- $a_1^2 a_0 a_4$>0

Is the system with the following characteristic equation stable?

$S^4+3S^3+6S^2+9S+12=0$

Substituting the appropriate values for the coefficient in the general condition of above Example of Example (3)we have

a_0 =12،a_1=9 ،a_2= 6 ،a_3= 3 ،a_4 =1

$a_3 > 0 \Rightarrow \; 3 > 0$

$a_3 a_2$ - $a_4 a_1 > 0 \;\Rightarrow 3.6 - 1.9 > 0 \Rightarrow 18\text{-}9 = 9 > 0$

$a_3 a_2 a_1$ - $a_0 a_3^2$ -$a_4 a_1^2 > 0 \;\Rightarrow (3.6.a) - (3)^2 - (1)(a)^2 > 0 \Rightarrow 162\text{-}108\text{-}81 > 0 \Rightarrow$

-27< 0

$3((6.9,12) - 3(12)^2) - (9)^2(12)(1) > 0$

$= 3(648) - 432) - 972 > 0 = 648 - 972 = -324 < 0$

Since the last two conditions are not satisfied the system is unstable

4.1.4 Continued Fraction stability criterion

This criterion is applied to the characteristic equation of a continuous system by forming a continued fraction form the odd and even portion of the equation in the following manner

Let $Q(s) = a_n S^n + a_{n-1} S^{n-1} + ... + a_1 S + a_0$

$Q_1(s) = a_n S^n + a_{n-2} S^{n-2} + \dots \dots ..$

$Q_2(s) = a_{n-1} S^{n-1} + a_{n-3} S^{n-3} + \dots \dots ..$

Form the Fraction $\frac{Q_1(S)}{Q_2(S)}$ and then divide the denominator into the numerator and Invert the remainder to form a continued Fraction as follow:

$$\frac{Q_1(S)}{Q_2(S)} = \frac{\frac{a_n S}{a_{n-1}} + (a_{n-2} - \frac{a_n \; a_{n-3}}{a_{n-1}})S^{n-2} + (\frac{a_{n-4} - a_{n-5}}{a_{n-1}})S^{n-4} + \cdots.}{Q_2}$$

$$= h_1 S + \cfrac{1}{\cfrac{1}{h_2 S + \cfrac{1}{\cfrac{1}{h_3 S + \cfrac{1}{\cfrac{1}{\ddots}}}}}}$$

$$\frac{1}{h_n S}$$

h_2 , ………h_n are all positive then all roots of Q(S) =0 have negative ‹If h_1 real parts

Example4.9

Using the continued Fraction stability criterion the polynomial

(S) = S^3 +4S^2+8S+12

Is divided into the two parts

(S) = S^3 +8S

(S) = 4S^2+12

The continued Fraction for is

$$\frac{Q_1(s)}{Q_2(s)} = \frac{s^3+8s}{4\,s^2+12} = ¼\ s + \frac{5s}{4s^2\ +12} = ¼\ s + \frac{1}{\frac{4}{5}s+\frac{1}{\frac{5}{12}s}}$$

∴ All coefficient of s are positive, the polynomial has all roots in the left – plane and the system with the characteristic equation (s)=0

∴The system is stable

Example4.10

Determine bounds the parameter k which a system the following characteristic equation is stable

S^3 +14s^2+56s+K=0

The continued Fraction for is

$$\frac{Q_1(S)}{Q_2(S)} = \frac{s^3+56\ s}{14\ s^2+k} = \frac{1}{14}s + (\frac{56-k}{14})S = \frac{1}{14}s + \left[\frac{14}{56-\frac{k}{14}}\right]S + \frac{1}{\left[\frac{56-k/14}{k}\right]s}$$

For the system to be stable the following condition must be satisfied

$56-\frac{k}{14} > 0$

And k>0 that is $0 < k < 784$

Example 4.11

Derive conditions for all the roots of a general third order polynomial to have negative real parts

For Q (s) = $a_3 s^3 + a_2 s^2 + a_1 s + a_0$

The continued Fraction for is

$$\frac{Q_1(S)}{Q_2(S)} = \frac{a_3 s^3 + a_1 s}{a_2 s^2 + a_0} = \frac{a_3}{a_2} s + \frac{\left[a_1 - \frac{a_3 a_0}{a_2}\right] s}{a_2 s^2 + a_0} = \frac{a_3}{a_2} s + \cfrac{1}{\left[\frac{a_2}{a_1 - \frac{a_3 a_0}{a_2}}\right] s + \cfrac{1}{\left[\frac{a_1 - \frac{a_3 a_0}{a_2}}{a_0}\right] s}}$$

The condition for all the roots of Q(s) to have negative real parts are then

$$\frac{a_1 - \frac{a_3 a_0}{a_2}}{a_0} > 0, \quad \frac{a_2}{a_1 - \frac{a_3 a_0}{a_2}} > 0 \qquad \frac{a_3}{a_2} > 0$$

Thus if a_3 is positive the required conditions are a_2 , a_1 , $a_0 > 0$ and

$a_3 a_0 > 0 — a_1 a_2$

Note that if a_3 is not positive (s) should be multiplied by -1 before checking the above condition

4:1:5 Stability criterions for Discrete Time systems

The stability of discrete systems is determined by the roots of the discrete system characteristic equation

$$a_n z^n + \; a_{n-1} z^{n-1} + \cdots \ldots \ldots \ldots + a_1 z + a_{0=}0$$

However in this case the stability region is defined by the unit circle $|Z|$=1 in the Z- plane .A necessary and sufficient condition for system stability is that all roots of the characteristic equation have a magnitude less than one. That is be within the unit circle this ensures that the kronecker delta response decays with time .A stability criterion for discrete system similar to Routh criterion is called the jury test for this test the coefficients of the characteristic equation are first arranged in the Jury array row.

	zz
1	$a_0 \quad a_1 \quad a_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots a_{n-1} \quad a_n$
2	$a_n \quad a_{n-1} \quad a_{n-2} \quad \ldots\ldots\ldots\ldots a_1 \quad a_0$
3	$b_0 \quad b_1 \quad b_2 \ldots\ldots\ldots\ldots\ldots \quad \ldots.b_{n-1}$
4	$b_{n-1} \quad b_{n-2} \quad b_{n-3} \ldots\ldots\ldots \quad \ldots.b_0$
5	$c_0 \quad c_1 \quad c_2 \ldots\ldots\ldots \quad \ldots.c_{n-1}$
2n-5	$r_0 \quad r_1 \quad r_2 \quad r_3$
2n-4	$r_3 \quad r_2 \quad r_1 \quad r_0$
2n-3	$s_0 \quad s_1 \quad s_2$

Where

$$b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix} \qquad c_k = \begin{vmatrix} b_0 & b_{n-1-k} \\ b_{n-1} & b_k \end{vmatrix}$$

$$S_0 = \begin{vmatrix} r_0 & r_3 \\ r_3 & r_0 \end{vmatrix} \quad ,S_1 = \begin{vmatrix} r_0 & r_2 \\ r_3 & r_1 \end{vmatrix} \quad ,S_2 = \begin{vmatrix} r_0 & r_1 \\ r_3 & r_2 \end{vmatrix}$$

The first two rows are written using the characteristic equation coefficients and the next two rows are computed using the determinant relationships shown above .The process is continued with each succeeding pair of rows having one less column than the previous pair unit row 2n-3 is computed which only has three entries the array is then terminated .Jury test: Necessary and sufficient conditions for the root Q(Z) =0, To have magnitudes less than one are

Q(1)>0

Q(-1)>> o for n even

< 0 for n odd

$|a_0|<a_n$

$|b_0|>|b_{n-1}|$

$|c_0|<|c_{n-2}|$

⋮

$|r_0|>|r_3|$

$|s_0|>|s_2|$

Note that if the Q(1) or Q (-1) condition above are not satisfied the system is unstable and it is not necessary to construct the array

4.2 Stabilization by bounded delayed control

Time delay often occurs in engineering system. Since the existence of time delay usually instability of the system the study on the Time delay systems has received considerable attentions and many stability criteria for time delay systems can be found in the literature. Stability criteria for time delay systems. Tend to fall into one of the two categories: delay independent and delay dependent, As the name implies delay independent criteria provide conditions which guarantee stability for any length of the time delay on the other hand delay dependent criteria exploit a priori knowledge of upper bounds on the a mount of time delay .These criteria since more information about the Time delay is assumed to be known

Consider the following linear time delay stem

$.\dot{X}$ (t) =Bx(t) + $B_d x(t - T(t))$ (4.1)

Where T(t) is a unknown time varying parameter which satisfies

$|\dot{T}(t)| \leq d \quad \forall t \geq 0$ ‹o$\leq$ T(t)$\leq$ h

xϵR^n is the state B and $B_d \epsilon R^{n\times n}$ are constant matrices

In is used to denote n-dimensional identity matrix. Given a matrix M , the transposition an the conjugate transposition are denoted by $\grave{M}$ and M^*

^{m} and $^\times$respectively .A matrix M is called positive definite if m belongs to H^n $\grave{X}$MX > 0 for all xϵC^n x$\neq$ 0,The notation M >0 is used to denote positive definiteness the positive semi definiteness negative definiteness and negative semi definiteness have similar definitions except that the > is replaced by " $\geq$ " , " $<$ " and $\leq$ respective. We use L_2 to denote the space of square assumable Function defined to time interval [0,∞).Given a signal f in L_2 space .We use $\|f\|$ L_2 denote the L_2norm off to denote the L_2induced norm of G $D_e$note the time delay operator and let

ΔBe(D_T-I)° $\frac{1}{s}$ that is

D_T (v): = v(t-T(t)) , and

$\Delta(v) = \int_t^{t-T(t)} v(A)\, d\theta$

Lemma (A)

Operator Δ is bounded L_2space the L_2 induced norm of Δ_T is equal to h and Δ

Satisfies integral quadratic defined by

$\sigma(v,w)= \int_0^{\infty} h^2\, v(t)' X\, v(t) - w(t)' X\, w(t)dt$

Where $X=\grave{X}$ is any positive definite matrix

Proof

$$Let\ X = \grave{X} > o \quad and\ w = \Delta\, we\ have$$

$W(t) = -\int_{t-T(t)}^{t} v(\theta)\, d\theta$ and

$\grave{w}(t)We(t) = -\left(\int_{t-T(t)}^{T} u(\theta)' X v(\eta) d\theta d\eta\right)' X - (\int_{t-T(t)}^{t} v(\eta)d\eta)$

$= \int_{t-T(t)}^{t}\int_{t-T(t)}^{t} v(\theta)' X\, v(\eta)\, d\theta d\eta$

Using the Cauchy Schwartz in equality formula we get.

$\grave{w} \times W \leq \int_{t-h}^{T} \sqrt{(h \times v(\eta)')} \times v(\eta)\sqrt{\int_{t-T(t)}^{t} v}(\theta)' \times v(\eta)\, d\theta d\eta$

$\leq h\left(\sqrt{\int_{t-h}^{t} v}(\theta)' \times v(\eta)\, d\theta d\eta \sqrt{\int_{t-h}^{t} v}(\eta)' X\, v(\eta) d\eta \leq h \int_{t-h}^{t} v(\theta)' \times v(\theta)\, d\theta\right.$

This is turn implies

$\int_0^{\infty} w(t)' \times w(t)\, dt \leq \int_0^{\infty} h\, \grave{v}\, (\theta) \times (\theta) d\theta = h \int_0^{\infty} \left(\int_{-h}^{0} v(t+s)' \times v(t+s)\, ds\right)dt$

$\leq h \int_{-h}^{0} \left(\int_0^{\infty} v(t)' \times v(t)dt\right) d\, s = \int_0^{\infty} h^2\, \grave{v}(t) \times v(t)dt$

$\therefore \Delta$ Bounded by h #

4.3 Stability criteria based on integral quadratic constraints

Lemma (B)

Let $m \in R^{n\times n}$ be a constant matrix then the time delay system

$\dot{X}(t)$= Bx(t)+ B_dx(t-T(t)) can be equivalent formulated as.

$\dot{X}(t)$=(B+mB_d)x t (x)+ (I_n-m)$B_d w_1$(t)+mB_dBw_2(t) + m$B^2 w_3$(t)…(4.2)

Where w_1(t) =x(t-T(t))

$w_3(t)=\int_t^{t-T(t)} w_1(\theta)\, d\theta$ ، $w_2(t)=\int_t^{t-T(t)} x(\theta)\, d\theta$

Using one can put the linear time delay system standard linear fractional transformation setup for robustness analysis as shown in figure (1)

The linear time invariance (LTI) system G has a state space representation

$\dot{X}(t) = \overline{A}_1 x(t) + \overline{B}_1 \breve{w}_1(t) + \overline{B}_2 \breve{w}_2(t)$ (4.3)

$\breve{v}_1(t) = x(t)$

$\breve{v}_2(t) = \begin{bmatrix} x(t) \\ \breve{w}1(t) \end{bmatrix}$

$\breve{w}_2 = D_T(\breve{v}_2)$ and matrices ، Where $\breve{w}_1 = D_T(\breve{v}_1)$

$\overline{B}_2$= [mB_dB mB_d^2] ،$\overline{B}_1$ =(I_n-m)B_d ،$\bar{A}$ =B + mB_d

Since system(4.1) ,(B) and (4.3) are equivalent stability of any one system implies stability of the other Two.

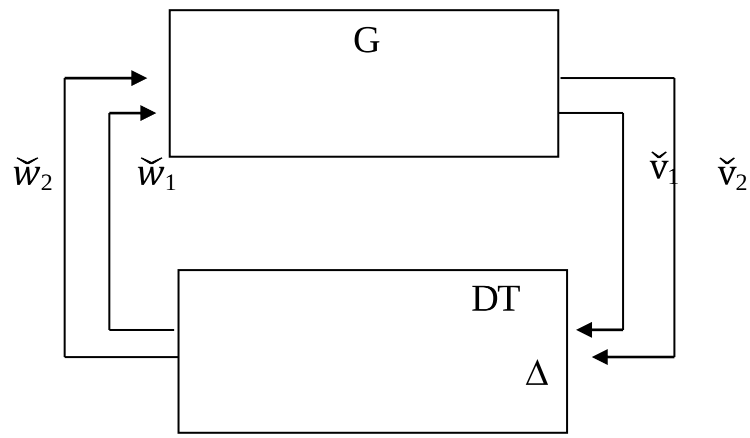


Figure (4.4)

Stability criteria derived in the previous topic are used on simple norm bounded type of integral quadratic constraint for D_T and Δ which might be very conservative .Less conservative criteria can be derived provided (ICS) which better characterize D_T and Δ are a viable in this topic stronger (ICS) for D_T and Δ are derived .

Lemma (C) (swapping lemma for operator D_T)

Let H be a stable linear invariant system with state and let T denote operator of multiplying i- e

$T(v(t))\text{: }=\dot{T}(t)\, v(t)$ then

$D_T \text{ o } H(s)= H(s)\, {}^\circ D_T - H(s)^\circ T^\circ sH(s)$

Proof

Let V be any L_2 Function and defined y and $\underline{Z}$ to be

$y(0)=0 \quad \dot{y}(t) = A_n y(t) + v(t)$

$z(0)=0 \quad \dot{z}= A_n z(t) + v(t-T(t))$

Let $r(t) = y(t-T(t)) - z(t)$ and we have

$\dot{r}(t) = \dot{y}(t- T(t))(1-\dot{T}(t)) - \dot{z}(t)$

$\dot{y}$(t- T(t))0= A_n(y(t-T(t)) - z(t)) - $\dot{T}$(t)

= A_nr(t) - $\dot{T}$(t)oD_T($\dot{y}$(t))

Which implies D_T(HV) = HoD_T(V) - HoD_T($\frac{d}{dt}$(HV))

This conclude the proof

Lemma (D2)(swapping lemma for operator Δ)

Let H and T be the operators as defined in lemma 2 then

ΔoH(s) =H(s)oΔ - H(s)oToD_ToH(s)

Proof

The proof is similar to the of lemma

z and x to be ،Let V be any L_2 Function and defined y

y(0)=0 ،$\dot{y}$(t) = A_ny(t) + v(t)

z(0)=0 ،$\dot{z}$= A_nz(t) + $\int_T^{t-T(t)} v(\theta)\, d\theta$

r(t)=$\int_T^{t-T(t)} y(\theta)\, d\theta - z(t)$

One can easily verify that

$\dot{r}$(t) =y(t- T(t))(1-$\dot{T}$(t)) -y(t)- $\dot{z}$(t)

y(t-T)(t)0=$\int_T^{t-T(t)} \dot{y}(\theta)\, d\theta - \dot{z}(t) - \dot{T}(t)$

=$A_n \int_T^{t-T(t)} y(\theta)\, d\theta - A_n z(t) - \dot{T}(t) 0\, y(t-T)(t)$

=A_nr(t) - $\dot{T}$(t) 0D_T(y(t))

Which implies Δ(HV) = HoΔ(V)- HoToD_r(HV) this concludes the proof

4 .4 Bounded by a neighborhood

$\boldsymbol{Corollary}(\boldsymbol{i})$

Let y$\subseteq R^m$ and pick any T $\geq$ 0 and any integer q $\geq$ 1

Then $R_u^T(0) + e^{TA} R_u^T(0) + \ldots\ldots\ldots + e^{(q-1)TA} R_u^T(0) = R_u^{qT}(0)$

Lemma(E) if C is an open convex sub set of R^n and L is a sub space of R^n contained in C then c +L =C

Proof

so we only need prove the other inclusion pick any ،Cleary c =c +0$\leq$ C +L x $\in$ C and y$\in$ L then for all $\in \neq 0$

$$x+y = \left(\frac{1}{1+\in}\right)[1+\in)x] + \left(\frac{\in}{1+\in}\right)\left[\left(\frac{1+\in}{\in}\right)y\right]$$

Since C is open (1+$\in$)x$\in$C for some sufficiently small $\in > 0$

Since L is a sub space $\left(\frac{1+\in}{\in}\right)y\in$L $\subseteq$ c

That x+ y $\in$C by convexity

Lemma (H)

T$\geq$ 0 then ،Let u $\subseteq R^m$ and pick any Two S

$R_u^T(0) + e^{TA} R_u^s(0) = R_u^{s+T}(0)$

Proof

Pick.

$x_1 = \int_0^T e^{(T-T)A} Bw_1(T)dt = \int_s^{s+T} e^{(s+T-T)A} Bw_1(T-s)dt$ and

x_2 = $\int_0^T$ e$^{(s-T)\,A}$ Bw_2(T)dt

Note that ‹With inputs w_i U- valued

$e^{TA}x_2$ = $\int_0^S$ e$^{(s+T-T)\,A}$ Bw_2(T)dt

Thus $x_1 + e^{TA}x_2$ = $\int_0^{s+T}$ e$^{(s+T-T)\,A}$ B w(T)dt

Where w(s)= $\begin{cases} w_2(\mathrm{T}) & 0 \leq \mathrm{T} \leq \mathrm{S} \\ w_1(\mathrm{T}-\mathrm{s}) & \mathrm{s} \leq \mathrm{T} \leq \mathrm{S}+\mathrm{T} \end{cases}$

w(s)∈U for all T∈[0, s+T]

Thus $R_u^T(0)$ + $e^{TA}R_u^s(0) \leq R_u^{S+T}(0)$

The converse inclusion follows by reversing these steps

There fore $R_u^T(0) + e^{TA}R_u^s(0)$ = $R_u^{S+T}(0)$

Lemma (I)

Assume that (A, B) is controllable and u≤ R^m is an eigh borhood of 0 then

$J_k^R \subseteq R_u$ (0) for all k

Proof

First replacing if necessary U by a convex subset we may assume without loss of generality that u is a convex neighborhood of 0we prove that statement by induction on k the case k=0 being trivial so assume that

$\widetilde{V}$ = any$\widetilde{V}_1$ + i$\widetilde{V}_2\widetilde{V}_1 \in R_u$ (0) ‹ $J_{k-1}^R \subseteq R_u$ (0)and take any$\widetilde{V} \in J_{k},\ \lambda$

First pick any T >o so that $e^{\lambda Tj} = e^{dTj}$ for all j = 0,1,…………………………

If $\beta=0$ one may take any $T>0$

Otherwise, we may use for instance

$T=\frac{2\pi}{|\beta|}$ next choose any $\delta>0$ with the property that $V_1 := \delta V_1^2 \in R_u^T(0)$

There is such a δ because $R_u(0)$ contains 0 in its interior

Since $V \in \mathrm{Ker}(\lambda I-A)^k$ where $V=\delta\widetilde{V}$

$e^{(A-\lambda T)T}V = (1+T(A-\lambda I)+\frac{t^2}{2}(A--\lambda I)^2+\ldots\ldots\ldots\ldots)V = V+W \quad \forall T$

Where $w \in J_{k-1}$

Thus

$e^{\propto T}V = e^{\lambda T}V = e^{TA}V - e^{\lambda T}w = e^{TA}V - e^{\propto T}V \quad \forall T = jT$

j=0,1,…………….

Decomposing into real and imaginary parts

$w = w_1 + i\,w_2$ and taking real parts in above equation

j=0,1,…….. $e^{\propto T}v_1 = e^{TA}v_1 - e^{\propto T}w_1 \forall t = jT$

Now pick any integer $q \geq 1$ then

$\left(\sum_{j=0}^{q-1} e^{\propto jT}\right)v_1 = \sum_{j=0}^{q-1} e^{jTA}\,v_1 + \grave{w}$

Where $\grave{w} = -\sum e^{\propto jT}$ belong to the sub space j_{k-1}^R

Applying First corollary (i) and lemma (A)
We conclude

$P\,V_1 \in R_u^{qT}(0) + J_{k-1}^R \subseteq R_u(0)$

Where $P=\sum_{j=0}^{q} e^{\propto jT} \geq \sum_{j=0}^{q-1} 1 = q \geq \frac{1}{\delta}$

Here is precisely where we used that $\propto \geq 0$

Therefore $\delta P\breve{V}_1 = PV_1 \epsilon R_u(0)$

On the other hand $\delta P \geq 1$ means that

$\breve{V}_1 = \frac{1}{\delta p}\delta p\breve{V}_1 + (1-\frac{1}{\delta p})0$

Is a convex combination

Since $\delta P\breve{V}_1$ and 0 both belong to $R_u(0)$ we conclude by convexity of the latter that indeed $\breve{V}_1 \epsilon R_u(0)$

Corollary (ii)

Assume that (A, B) is controllable and $U \subseteq R^m$ is a convex and bounded neighborhood of 0

There exists a set B such that

$R_u(0) = B + L$ and B is bounded convex and open relative to m

Proof

We claim that $R_u(0) = (R_u(0) \cap m) + L$

On inclusion is clear form

$(R_u(0) \cap m) + L \subseteq R_u(0) + L = R_u(0)$

Applying lemma (B)

Conversely any $V \epsilon R_u(0)$ can by decomposed as $V = x + y \epsilon m + L$

We need to show that $x \epsilon R_u(0)$ But

$x = v - y \epsilon R_u(0) + L = R_u(0)$

Applying the same lemma (B) yet again this establishes the claim

We let $B := R_u(0) \cap m$

This set is convex and open in m because $R_u(0)$ is open and convex

We only need to proof that it is bounded

Let $P: R^{n} \rightarrow R^n$ be the projection on m a long L

$y\epsilon L$ observe that PA =AP because each of L ‹That is P(x+y) = x if $x\epsilon m$
Ay ϵL imply ‹Ax ϵ m ‹and m are A invariant (so v = x + y
PAV =Ax =APV).
Pick any $x\epsilon R_u\ (0) \cap m$
Since x $\epsilon\ R_u\ (0)$ there are some T and some w so that
x= $\int_0^T e^{(T-\tilde{I})A}\, Bw(\tilde{I})d\tilde{I}$
x=px ‹On the other hand since $x\epsilon m$
Thus x= px $=\int_0^T e^{(T-\tilde{I})A}\, Bw(\tilde{I})d\tilde{I}=\int_0^T e^{(T-\tilde{I})A}$ x(Ĩ)dĨ
Where x (Ĩ) =PW(Ĩ)$\epsilon m \cap pB(u)$ for all Ĩ since the restriction of A to m has all its Eigen values negative real part
There are positive constants C, m>0 such that
$\|e^{TA}\| \leq ce^{-mt}\ \|x\|$For all t≥ 0
That if x is also in PB (u), $\|e^{TA}\| \leq c^1 e^{-mt}$ for all t≥ 0
So for as above we conclude

$$\|x\| \leq c^1 \int_0^T e^{-m(T-\tilde{I})} d\tilde{I} \leq \frac{c^1}{m}(1-e^{-mt}) \leq \frac{c^{//}}{m}$$

4.5 Lower Bounded control lyapunov Functions

We will consider systems of the form
$X(\dot{t})$=F(x (t) +u (t))
Where the state x (t) evolves in R^n and the controls u(t) take values in a sub set u R^m containing the origin. For simplicity we assume that v=R^m
The map f:$R^n R^m \rightarrow R^n$ *is locally lipschitzat* (x,v) and f(0,0) =0

To stabilize this system to x =0 we will use energy Function v which can be made to decrease a long system trajectories. A Function v:$R^n \rightarrow$ R≥ 0

Is positive definite if v(0) =0 v(ϑ)> 0 for $\neq 0$

And proper if v (ϑ)$\rightarrow \infty$ as $\|\vartheta\| \rightarrow \infty$ for ϵR^n

We denote by $\|\vartheta\|$theEucliden norm

For a locally Lipchitz continuous Function

V:$R^n \rightarrow$R and P ϵR^n we define the Dini derivative of v in the direction of P At to be

$\overline{D + v}(\vartheta , p)= \lim_{t\rightarrow 0^t} \sup \frac{[v(\vartheta +tp)-v(\vartheta)]}{t}$

In what follows we will be considering closed loop systems of the form

$u_*(\vartheta$))، f (ϑ

Where the feedback controller u_* is measurable and locally bounded but possibly discontinuous hence the classical notion of solution need not apply to deal with this situation we will use the generalized solution due to flipper Set valued Function

ϑ)\ N} ،fu_*(B(£{Ƒ$u_*(\vartheta)$=∩∩$\overline{CO}$

ϑ>0 μ(N) =0

δ) is a ball of radius δ centered at £ ،Whereby (£

$\overline{CO}$Denoted the closure of the convex hull and m is used Lévesque measure on R^n,AFilippov solution on an interval I C R

is absolution continuous on any interval 0)Is Function x:I$\rightarrow R^n$ such that x(

,Al most everywhere on I solution x (t) to (c))$[t_1,t_2]$I, And $\dot{x}$(t) ∈ Ƒ u_*(x (t) are thus state trajectories of the system under. The Feedback controller u = u_*(s) which are differentiable al mostevery where with respect to τ.Since f(£ £)) is measurable and locally bounded .The set valued Function u_* (£) ،u_* ،

is upper semi continuous compact and convex value and locals bounded in particular the differentiable inclusion satisfies ,The basic condition and thus has oFilippov solution for each intrastate .the solution x (t) =0 of a different inclusion $\dot{x}$ (t) = (x)) is called stable if for each $\epsilon > 0$ there exists $\delta > 0$ with the following property ,For each x_0 $such$ that $\|x_0\|< \delta$ each solution x(t) to x(t) to $\dot{x}$ (t) ϵ $(x(t)$ with intial data x(0) =x_0

Exists for 0≤ $t \leq \infty$ and satisfies the inequality

$$\|x\| < \epsilon \ (0 \leq t < \infty)$$

*asymptotically stable if x (t) =0 is stable and in addition, x (t) →0 as t→0

The possibly discontinuous Feedback controller

X→ $u_*(x)$ $stabilize$ systm (A) iff x(t)=0

Is a stable solution of the corresponding differential inclusion

4.5.1 Lower bounded lyapunov pairs

Lipchitz continuous lyapunov pair (v,w) consists of a locally Lipchitz continuous positive definite proper Function v:$R^n \rightarrow R$ and anon negative continuous Function

W:$R^n \rightarrow R$ such that for £≠ 0

V) will،There exists p ϵf(£

p) ≤ -w (£)،$\overline{D+V}$ (£

Definition 4.1

A lower bounded lyapunov pair for the system

$\dot{x}$ (t)= f(x(t), u(t) is alipschitz continuous lyapunov pair (v,w) such that

V) such ،* For £ ≠ 0 there exists p_1, $p_2 \epsilon$ f(£

p_2)…………………………………...(i)،p_1) ≤ -w (£) ≤ $\overline{D+V}$ (£ ،$\overline{D+V}$ (£

**for $\epsilon > 0$ $there\ exists$ δ>0 such that for 0 <‖£‖< (i) holds with

$\|u_i\|<$ ،u_i)،

Definition4.2

A lower bounded lyapunov (v,w) said to be regular if

1. The set valued Function (£) is upper semi continuous, compact and convex valued and locally bounded.
2. There is appositive definite continuous map $\breve{w}$ on R^n such that at points £ where the Function v is not different table (or more generally where

p) is not linear) ‹P → $\overline{D+V}$ (£

p) ≤ -$\breve{w}$(£)∀p∈$\epsilon^{v,w}$(£) ‹$\overline{D+V}$ (£

Definition **4.3**

Alocally lipschtz continuous,positive definite and prper Fuction V is called alower bounded control lyapunov Function (LB- CLF), *if these* Exist apositive definite Function w such that (v,w) is regular lower bounded lyapunov pair the pair (v,w) will be call regular lyapunov pair for

we note that if (v, w)*is aregular pair for V then the differentiable inclusion*

Chapter five

Reachability and controllability under sampling

5. Introduction

When studying the input to state interaction we can take two different points of view in the former we assume that the initial state of the system, the state of the system at t=0 is the fixed and we consider the problem of determining the states of system that can be reached applying a certain input in this case we study the so called Reachability property in the latter .We assume that the final state of the system at some time T is fixed and we aim at determining all initial states that can be steered, by means of a certain input signal to the selected final state. In this case we study the so called Controllability property. In the study of reachability and controllability whenever the input signal that drives a certain initial state to a certain final state is not unique we could impose constraints on such an input signal. E.g. we could consider the input signal with minimum time if minimum amplitude, or the input signal which achieves the transfer are equivalent no constrain is imposed all input signals a achieving the considered transfer in minimum time. For linear systems the properties of reachability and controllability are referred to the state $x = 0$, hence we say that a state is reachable to mean that it is reachable form $x = 0$ and that a state is controllable to mean that it is controllable to $x = 0$.Note more over that, Because these properties are used to describe the input to state interaction they trivially depend only upon properties of the matrices A and B

5.1 Reachability of discrete time systems

Consider a linear, time invariant, discrete time system

Let $x(0) = 0$ and consider an input sequence u(0) , u(1), u(2), . . . , u[k-1] , the state reached at t = k is given

$$x[k] = [\mathrm{B\ AB\ldots\ldots A^{k-1}B}]\begin{bmatrix} u[k-1] \\ u[k-2] \\ \vdots \\ u(0) \end{bmatrix}$$

This implies that the set of states that can be reached at t= k is a linear space i-e it is the subspace $\mathrm{R_k}$ spanned by all linear combinations of the columns of the matrix

$\mathrm{R_k} = [\mathrm{B\ \ AB\ \ldots\ A^{k-1}B}]$

The set $\mathrm{R_k}$ is a vector space, denoted as the reachable subspace ink steps if $\mathrm{R_k} = \mathrm{x}$,I-e rank $\mathrm{R_k} = \mathrm{n}$ then all state of the system are reachable in (at most) k steps and the system is said to be reachable in k steps. Ask varies we have a sequence of subspace namely $\mathrm{R_1, R_2, \ldots, R_k}$. this sequence of subspaces is such that the following properties hold.

Proposition (5.1)

The sequence of subspace (A) is such that $\mathrm{R_1 \subseteq R_2 \subseteq\ \ldots \subseteq R_k \subseteq\ \ldots}$

Moreover if for some $\bar{\mathrm{k}}$, $\mathrm{R_{\bar{k}} = R_{\bar{k}+1}}$ then for all $\mathrm{k \geq \bar{k}}$, $\mathrm{R_k = R_{\bar{k}}}$ finally

$$\mathrm{R_1 \subseteq R_2 \subseteq\ \ldots \subseteq R_n = R_{n+1}}$$

Proof

To prove the first claim note that if a state $\bar{x}$ is reached from zero in k steps, using

The input sequence u(0) , u(1) , …., u[k-1] then the same state is also reached from zero in k+1 steps, using the input sequence 0, u(0),u(1), ...,u(k-1),hence for all $k \geq 1$

$R_k \subseteq R_{k+1}$

To prove the second claim it is enough to show, Or equivalently that if then any, Belongs also to .For let be an element of this mean that there is an input sequence which steers the state of the system from $x = 0$ to $\bar{x}$ in $\bar{k}$+z step consider now the state reached after $\bar{k}$+1 steps using the same input sequence which we denote with $\tilde{x}$ .By assumption hence there is an input sequence which steers the state of the system from $x(0) = 0$ to $\tilde{x}$ in $\tilde{k}$ steps

However by definition of $\tilde{x}$ it is possible to steer $\tilde{x}$ to $\bar{x}$ in one step , hence there is an input sequence which steers $x(0) = 0$ to $\tilde{x}$ in $\bar{k}$+1 steps which prove claim to prove the third claim note that if for some k < n , $R_k = R_{k+1}$ then the claim follows from equation. Suppose now that for all k the dimension of R_{k+1} is strictly larger that the dimension of R_k this implies that the sequences dim R_k.Is strictly increasing at each step however this sequence

is bounded (from above) by n and this proving claim?

Definition5.1

Consider the discrete time system (A) the subspace $\mathfrak{R} = \mathfrak{R}_n$ is reachability subspace of the system, the matrix R= R_n is reachability matrix of the system

The system is said to be reachable if R = x = R^n remark by definition

R = ImR, Hence the discrete time system

$\sigma x = A x + Bu, \quad y = cx + Du$

With $x \epsilon$ x = $\Re$, u(t) $\epsilon\, \Re^m$, y(t) $\epsilon\, \Re^p$ and A,B,C and D matrices of appropriate dimensions and with constant entries. The discrete time system is said to be reachable if and only if .Rank R = n. Equation is known as Kaman reachability rank condition

Remark(5.1)

From the above discussion it is obvious that in an. Dimensional linear discrete time system if a state $\bar{x}$ is reachable, Them it is reachable in at most n steps this does not mean that n steps are necessarily required i-e ,The state $\bar{x}$ could be reached in less than n steps .In a reachable system. The smallest integer k^* such that rank k_{k^*} = n .Is called the reachability index of the system Note that for single input reachable systems necessarily k^*= n

Example 5.1

Consider a discrete time system with $x \epsilon \mathscr{R}$

$$A = \begin{bmatrix} 0 & 0 & \Phi \\ 0 & 0 & \Phi \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Then $R_1 = \text{span B} \qquad R_2 = R_3 = \text{R span} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$

Hence the system is not reachable and its reachable subspace has dimension two

Example5.2

Consider a discrete time system with $x \in \Re^3$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad B= \begin{bmatrix} d \\ B \\ y \end{bmatrix}$$

Then

$$R_3 = \text{span} \begin{bmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & 0 \\ \gamma & 0 & 0 \end{bmatrix}, R_2 = \text{span} \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \\ \gamma & 0 \end{bmatrix} ,R_1 = \text{span } B$$

As a result the system is reachable if and only if $\gamma \neq 0$ moreoverif $\gamma = 0$ and

$\beta \neq 0$ The system is not reachable and the reachable subspace has dimension two. Finally if $\gamma = \beta = 0$ and $\alpha \neq 0$

The system is not reachable and the reachable subspace has dimension one the reachability subspace R has the following important property

The proof of which is a simple consequence of the definition of the subspace

Proposition (5.2)

The reachability sub space contains the subspace span B i-e

Span B$\subseteq$ R

And it is A invariant i.e. AR$\subseteq \Re$ we conclude this case noting that algebraically equivalent system, with state x and $\tilde{x}$ respectively

Let be the coordinates transformation matrix, as a given in equation below consider a continuous time Finite dimensional linear system described by the equations

$\dot{x} = A(t)x + B(t)u$, $y = C(t)x + D(t)u$

y(t) $\in |R^p$ and the change of coordinates ، u(t) $\in \Re^m$ ،With $x \in X = \Re^n$

$x(t) = l(t)\hat{x}(t)$

With l(t) invertible for all t , the state space representation in the new coordinates is given by

$\dot{\hat{x}} = [\, l^{-1}(t)\ (A)x + B(t)(u) + l^{-1}(t)\ x]_x = l(t)\hat{x} = (l(t)^{-1}(t) + \dot{l}^{-1}(t)l(t)\tilde{x} + l^{-1}(t)B(t)u$

And $y(t) = CL(t)\,\tilde{x} + D(t)u$

Similarly for a discrete time finite dimensional linear system described by the state ،the equations change of coordinates, With u(t) invertible for all t space representation in the new coordinates is given by and R_k and $\widetilde{R}_k$ the reachability subspaces and R and $\widetilde{R}$ the reachability matrixes respectively then

Hence .And one of the two systems is reachable if and only if the other is

5.2 Controllability of discrete time system

The result established for the reachability the controllability property

In fact for a linear time invariant, discrete time system .A state x^* is controllability (to zero) in k steps if there exists an input sequence u(0) , u(1) , …………, u(k-1) that drives the state from. This last equation implies that is controllable if the state, is reachable in K steps, Hence if; It is easy to see that the set of all such that equation. Hold is a vector space denotes by and called controllability subspace in k steps, A linear discrete time system is controllable in k steps if Im

Example5.3

Consider the system in Example the system is controllable in two steps in fact

$$A^2 = A \epsilon R_2 =$$

Example5.4

Consider the system in Example the system is controllable in three steps no matrix the values of α , β and γ ,Note in fact that $A^3 = 0$ .The system is controllable in two steps .If $\gamma = 0$ and $\alpha \neq 0$ or $\beta \neq 0$ finally it is controllable in one step if $\gamma = 0$ and $\alpha\ \beta \neq 0$, Similarly to the reachability subspaces ask varies we have a sequence of controllability subspace namely

This sequence of subspace is such that following properties hold.

Proposition (5.3)

The sequence of subspaces is such that, More over if for some the for all

Proof

The proof of this statement is similar to the one of proposition 7-1 we simply remark that if a state is controllable in K steps using the input sequence u(0) ,u(1), …,u(k-1) then the same state is also controllable in k+1 steps using

Input sequence, u (0),u(1), …,u(k-1),

Definition 5.2

Consider the discrete time system

$\sigma x = Ax + Bu \ , y = cx + Du$

With$x \epsilon X = \Re^{n}$,u(t)$\epsilon\Re^{m}$, y(t)$\epsilon \mathscr{R}^{p}$ and A,B,C and D matrices

The subspace C = c_n is the controllability subspace of the system

The system is said to be controllable if C =$x = \Re^{n}$, the discrete time system above is controllable if and only if Im $A^n \subseteq \Re$.In particular if A is nilpotent i.e. $A^q = 0$ For some q$\leq n$ then for any B(even B=0)

The system is controllable

Note:

That reachable system is controllable but the converse statement does not hold in particular

$$\Re \subseteq C \subseteq X = \Re^{n}$$

5.3 Construction of input signals

The study of the properties of reachability and controllability leads to the following equation .Is it possible to explicitly construct an input sequence which steers the state of the system from an initial condition?

$x(0) = x_0$ to a final condition x_f in k – steps i.e $x(k) = x_f$ ‹x_0 .i.e

To answer this equation consider the problem of determining an input sequence u(0) , u(1) ,….,u(k-1) such that

$x_f - A^k x_0 = R_k U_{k-1}$

Where

$$\mathrm{U}_{\mathrm{k}-1} = \begin{bmatrix} u(k-1) \\ u(k-2) \\ \vdots \\ u(1) \\ u(0) \end{bmatrix} \epsilon \Re^{\mathrm{km}}$$

Consider now an input signal defined as $\mathrm{U}_{\mathrm{n}-1} = \mathrm{R}_{\mathrm{n}}^{1}\mathrm{v}$

Where V has to be determined using 1 his definition and setting k = n equation

$$x_f - A^k x_0 = \mathrm{R}_{\mathrm{k}}\mathrm{U}_{\mathrm{k}-1}$$

Becomes $x_f - A^n x_0 = \mathrm{R}_{\mathrm{n}}\mathrm{R}_{\mathrm{n}}^{1}\mathrm{v}$

Where the matrix $\mathrm{R}_{\mathrm{n}}\mathrm{R}_{\mathrm{n}}^{1}$ is square and invertible Hence a control sequence solving the considered problem in n steps is given by

$\mathrm{U}_{n-1} = R_n^1 (R_n R_n^1)^{-1} (x_f - A^k x_0)$

It is possible to show that among all input sequence steering the state of the system from x_0 to x_f in n steps the one constructed has minimal norm (energy).

5.4 Reachability and Controllability of Continuous Time System:

The properties of reachability and controllability for linear, time invariant continuous time systems can be assessed using the same ideas exploited in the case of discrete time systems however the tools are more involved as the input state relation is expressed by means of an integral

$$x(t) = e^{At}x_0 + \int_0^{\tau} e^{(\tau-T)}Bu(T)dT$$

Consider the reachability problem i.e. the initial state of the system is $x(0) = 0$

And we want to characterize all states $\bar{x}$ that can be reached in some interval of time t i.e. all states such that for some input function u(t)

$$\bar{x} = \int_0^{\tau} e^{(\tau-T)}Bu(T)dT$$

Note:

Now that by Clayey –Hamilton theorem

$e^{At} = \delta_0(t)I + \delta_1(t)A + \cdots + \delta_{n-1}(t)A^{n-1}$

For some scalar function $\delta_i(t)$ Hence

This implies that a state $\bar{x}$ is reachable only if

$\bar{x} \epsilon$ Im[B : AB: … : $A^{n-1}B] =$ ImR

We now prove the converse fact i-e that is in the image of R it is reachable to this end define the controllability Gramian

$w_t = \int_0^t e^{A(t-T)}B\grave{B}e^{A(\grave{t}-T)}dT$

With t > 0 and note that ImR =ImWt

Selecting U (T) = $B' e^{\acute{A}(t-T)}\beta$

Where β *is a constant* vector yield $\bar{x} = w_t \beta$

Hence to assess reach ability of the state $\bar{x} \epsilon$ ImR it is sufficient to show that equation $\bar{x} = w_t \, \beta$ has (at least) one solution β

However this fact holds trivially by condition ImR = Imw_t

Remark (5.2)

Unlike the case of discrete time systems where the set of reachable states depends upon the length of the input sequence for continuous time systems if a state is reachable them it is reachable in any (possibly small) interval of time.

Definition of continuous time system 5.3

$\sigma x = Ax + Bu \,, \quad y = Cx + Du$ The subspace R is the reach ability sub space of the system the matrix R is the reach ability matrix of the system the system is said to be reach able if $R = X = \mathfrak{R}^n$ we summarize the above discussion with a formal statement.

Proposition (5.4)

Consider the continuous time system $\sigma x = Ax + Bu \,, \quad y = Cx + Du$ the following statements are equivalent

* The system is reach able

** rank R = n

*** For all t > 0 the controllability Gramian w_t is positive definite

Remark (5.3)

If a system is reachable then it is possible to explicitly determine one input signal which steers the state of the system from x(0) =0 to any $\bar{x}$ is a give time t > 0 in fact determine one such input signal it is sufficient to solve the equation $\bar{x} = w_t \beta$ which if the system is reach able has the unique solution

$\beta = w_t^{-1}\bar{x}$ I.e. the input signal

U(T)= $B' e^{\acute{A}(t-T)} w_t^{-1}\bar{x}$

Steers the state of the system from $x(0) = 0 \ to \ x(t) = \bar{x}$

Similarly to what discussed in construction of input signals it is possible steering the state from 0 to $\bar{x}$ in time t the input signal u(T) = $B' e^{\acute{A}(t-T)} w_t^{-1}\bar{x}$

Similar consideration can be done to determine an input signal steering anon Zero initial state to a given final state to discuss the property of controllability note that a state $\bar{x}$ is controllable (to Zero) in time t > 0 if there exists an input signal such that

0= $e^{At}\bar{x} + \int_0^T e^{A(t-T)} Bu(T)dT$

This however implies that $e^{At}\bar{x} \in \mathfrak{R}$ hence $\bar{x} \in e^{-At}R$

This implies that the set of controllable states in time t > 0 is the set C_t = $e^{-At}R$

Which has the same dimension as R by invariability of e^{-At} for all t and it is contained in R by the fact R is A invariant hence it is trivially e^{-At}.Invariant.

As a consequence for all t > 0 C_t = R which shows that set C of controllable states does not depend upon t > 0 and that continuous time system is controllable if and only if it is reach able C unlike what happens for discrete time system , for which reachability implies but it is not implied by controllability .

Example5.4

Consider the linear electric network in Figure H. Assume $R_1 > 0$,

$R_2 > 0$,

L > 0 and C > 0 the input u is the driving voltage y is current supplied.

Let x_1 be the current through L and x_2 the voltage a cross C

By Kirchhoff, laws we have

$\dot{x}_1 = \frac{-R_1}{L} x_1 - \frac{1}{L} x_2 + \frac{1}{L} u$ ‘ $\dot{x}_2 = \frac{1}{C} x_1$

And y = $x_1 + \frac{1}{R_2} u$

There A = $\begin{bmatrix} \frac{-R_1}{L} & \frac{-1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}$ B = $\begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}$

The reachability matrix is R = $\begin{bmatrix} \frac{1}{L} & \frac{-R_1}{L^2} \\ 0 & \frac{1}{LC} \end{bmatrix}$

Hence the system is reach ability and controllable for any L and C

Example5.5

Consider the linear electric network in Figure G Assume $R_1 > 0$, $R_2 > 0$ and

C > 0 the input u is the deriving voltage and the output y is the current supplied

Let x_1 be the voltage across C and x_2 current through L

By Kirchhoff’s laws we have

$\dot{x}_1 = \frac{1}{R_1 C} x_1 + \frac{1}{R_1 C} u$ ‘ $\dot{x}_2 = \frac{-R_2}{L} x_2 + \frac{1}{L} u$

And y = $\frac{-1}{R_1} x_1 + x_2 + \frac{1}{R_1}$ u

The reach ability net work

$$R = \begin{bmatrix} \frac{1}{R_1 C} & \frac{-1}{R_1{}^2 C^2} \\ \frac{1}{L} & \frac{-R_2}{L^2} \end{bmatrix}$$

And det (R) = $\frac{1}{R_1 CL}\left(\frac{1}{R_1 C} - \frac{-R_2}{L}\right)$

Hence the system is reachable and controllable provided $R_1 R_2 C \neq L$

5:5 Canonical form for reachable systems

we focus on single input systems and we show that the property of reachability allows to write the system in a special from known as reachability canonical from consider the system $6x = Ax + Bu \quad y = Cx + Du$

With $x \epsilon X = R^n$, u(t) ϵ R^m , y(t) ϵ R^P and A,B,C and D matrixes of appropriate dimensions and with constant entries with m=1 and suppose the system is reachable i.e the rank of the reachability matrix is equal to n

By reachability there is (row) vector L such that LB=0, LAB =0, LA^{n-1} B=0

LA^{n-1} B=1

In fact conditions above formula can be rewritten as

IR = [0 0 … .0 1]

Hence L = [0 0 … .0 1]R^{-1}

Is well defined

The vector L has the following important property

Lemma (5.1)

Let L be as in equation LR = [0 0 ... 0 1] R

Then the square matrix

$$T=\begin{bmatrix} L \\ AL \\ . \\ . \\ LA^{n-1} \\ LA^{n-1} \end{bmatrix} \text{ is invertible}$$

Proof

Consider the matrix

$$T_R=\begin{bmatrix} L \\ AL \\ . \\ . \\ LA^{n-1} \\ LA^{n-1} \end{bmatrix} [B: \ AB: \ \ldots\ldots. A^{n-1}B \ : A^{n-2}B]$$

$$=\begin{bmatrix} LB & LAB & \ldots LA^{n-2}B & LA^{n-1}B \\ LAB & LA^2B \cdots & LA^{n-1}B & LA^nB \\ \vdots & \vdots & & \vdots \\ LA^{n-2}B & LA^{n-1}B \ldots\ldots\ldots\ldots\ldots\ldots & & \\ LA^{n-1}B & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & \end{bmatrix}$$

And note that by condition I B =0 LAB =0 $LA^{n-2}B$ $LA^{n-2}B = 1$

$$T_R = \begin{bmatrix} 0 & 0 & \dots\ 0 & 1 \\ 0 & 0 \dots \dots . . 1 & LA^nB & \\ \dots \dots \dots \dots \dots \dots \dots \dots \dots & & & \\ \dots \dots \dots \dots \dots \dots \dots \dots \dots & & & \\ 0 & 1 \dots \dots \dots \dots \dots . . & & \\ 1 & LA^nB \dots \dots \dots \dots & & \end{bmatrix}$$

Which shows that $|\det(T_R)| = 1$

Hence T is invertible

The matrix T can be used to define a new set of coordinate $\hat{x} = T_x$

To derive the state space representation of the system in the $\hat{x}$ coordinates we could use the general discussions however it is easier to proceed in an alternative way.

For consider the auxiliary signal $\tilde{x}_1 = L\,x$

And note that $6\,\tilde{x}_1 = LA\,x = \tilde{x}_2$

for i = 1,….., n-1 and $6\,\tilde{x}_i = L\,A^i x = \tilde{x}_{i+1}$

$$6\,\tilde{x}_n = L\,A^n x + u = LT^{-1}x^2 + u$$

As a result in the new coordinates $\tilde{x}$ one has

$$6\tilde{x} = A_r\tilde{x} + B_r\tilde{x}$$

Where

$$A_r = \begin{bmatrix} 0 & 1 & 0 & \dots\ \dots \dots 0 \\ 0 & 0 & 1 & \dots\ \dots \dots 0 \\ \vdots & & & \\ 0 & 0 & 0 & \dots \dots \dots \\ -\propto_0 & -\propto_1 & -\propto_2 & \dots -\propto_{n-1} \end{bmatrix} \quad B_r = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

And $[-\propto_0\ -\propto_1\ -\propto_2\ \dots -\propto_{n-1}] = L\,A^n T^{-1}$

Note that for any $\propto_i$ the system $6\tilde{x} = A_r\tilde{x} + B_r\tilde{x}$ is reachable hence a system described by equation t $6\tilde{x} = A_r\tilde{x} + B_r\tilde{x}$

Is said to be in reachability canonical form

The matrix A_r is in comanion from it is worth boting that its characteristic polynomial is

P(s) = $s^n + \propto_{n-1} s^{n-1} + \propto_{n-2} s^{n-2} + \cdots + \propto_1 s + \propto_0$

I.e. it depends only upon the elements of the last row.

Example6.5

Consider the system is reachable and controllable

For any L system C

To write the system in reachability canonical from we have to find a(row) vector L such that conditions LB =0 , LAB =0 L$A^{n-2}B = 0$

L$A^{n-1}B = 1$

Hold namely

$L\begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} = 0 \quad L\begin{bmatrix} -\frac{R_1}{L^2} \\ \frac{1}{Lc} \end{bmatrix} = 1$ yielding L = [0 LC]

Finally $T = \begin{bmatrix} L \\ LA \end{bmatrix} \begin{bmatrix} 0 & LC \\ L & 0 \end{bmatrix}$

And the system in the transformed coordinates in described by

$$\dot{\tilde{x}} = \begin{bmatrix} 0 & 1 \\ \frac{-1}{LC} & \frac{R_1}{L} \end{bmatrix} \tilde{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

I.e. it is reachability canonical from

Description of non-reachable system:

We study systems which are not 7-6 reachable i.e. systems described by the equation $6x = \mathrm{A}x + \mathrm{B}u\,, y = \mathrm{C}x + Du$ and such that rank = p < n

Under this assumption consider a set of coordinates $\tilde{x}$ such that $x = \mathrm{L}\tilde{x}$

And the matrix L is constructed as follows .The First P columns of L are P linearly independent columns of the matrix R and the last n-p columns are selected in such a way that the matrix L is invertible the system is the $\tilde{x}\ coordinates$ which is algebraically equivalent to the system in the the x coordinates, i.e. described by the equations

$6\tilde{x} = \tilde{A}\tilde{x}$ + $\tilde{B}$u = $\mathrm{L}^{-1}\mathrm{AL}\tilde{x} + \mathrm{L}^{-1}\mathrm{Au}$

We now show that because of the way in which n L has been constructed the matrices $\tilde{A}$ and $\tilde{B}$ have a special structure to this end note that

$\mathrm{L}\tilde{A} = AL$ $\quad L\tilde{B}$ =B

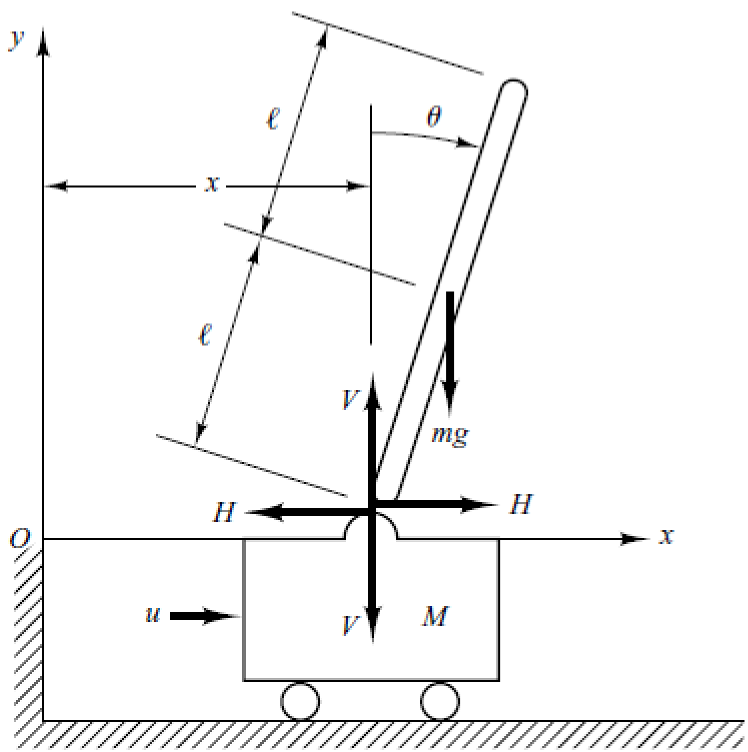
And partition the matrices L,A, $\tilde{A}$,B and $\tilde{B}$ as

P,And have ×P, have dimensions n-P×Where and have dimensions P
an m×dimension p

Chapter six: Results and Conclusions

Model (1) an inverted pendulum mounted

An inverted pendulum mounted on a motor-driven cart is shown in figure 6.1 this is a model, of the attitude control of a space booster on takeoff. The objective of the attitude control problem is to keep the space booster in a vertical position.) The inverted pendulum is unstable in that, it may fall over any time in any direction unless a suitable control force is applied. Here we consider



Figure(6.1)

To derive the equations of motion for the system, consider the free-body diagram shown in Figure 6.1 the rotational motion of the pendulum rod about its center of gravity can be described by

$$I\ddot{\theta} = vl\sin\theta - Hl\cos\theta \qquad (6.1)$$

Where I is the moment of inertia of the rod about its center of gravity the horizontal motion of center of gravity of pendulum rod is given by

$$m\frac{d^2}{dt^2}(x + I\sin\theta) = H \qquad (6.2)$$

The vertical motion of center of gravity of pendulum rod is

$$m\frac{d^2}{dt^2}(I\cos\theta) = v - mg \qquad (6.3)$$

The horizontal motion of cart is described by

$$m\frac{d^2x}{dt^2} = u - H \qquad (6.4)$$

Since we must keep the inverted pendulum vertical, we can assume that $\theta(t) and\ \theta^{\cdot}(t)$ are, small quantities such that

$\sin\theta \doteqdot \theta, \cos\theta = 1,$ and $\theta\dot{\theta}^2 = 0.$

And Then equations (6.1) through (6.3) can be linearized the linearized equations are

$$I\theta^{\cdot\cdot} = vI\theta - Hl \qquad (6.5)$$

$$m(x^{\cdot\cdot} + I\theta^{\cdot\cdot})$$
$$= H \qquad (6.6)$$

$$0$$
$$= v - mg \qquad (6.7)$$

From Equations (6.4) and (6.6), we obtain

$$(M + m)x^{\cdot\cdot} = u \qquad (6.8)$$

From Equations (6.5), (6.6), and (6.7), we have

$$I\ddot{\theta} = mgl\theta - Hl$$
$$= mgl\theta - l(m\ddot{x} + ml\ddot{\theta})$$

Or

$$(I + ml^2)\theta^{\cdot\cdot} + mlx^{\cdot\cdot} = mgl\theta \qquad (6.10)$$

Equations (6.9) and (6.10) describe the motion of the inverted-pendulum-on-the-cart system they constitute a mathematical model of the systems the inverted pendulum is unstable in that, it may fall over any time in any direction unless a suitable control force is applied. Here we consider Model (2) $dashpot\ system\ mounted\ on\ a\ massless\ car$t

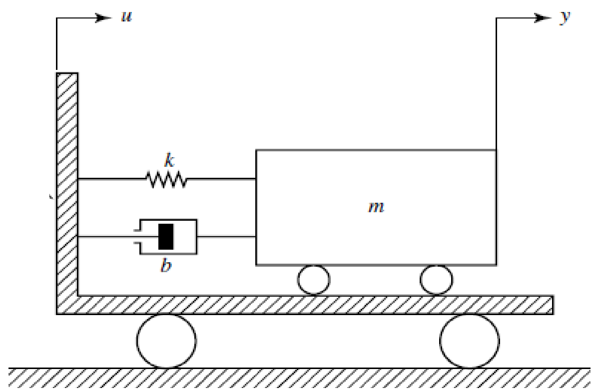


Figure (6.2)

$Consider\ the\ spring - mass -$
$dashpot\ system\ mounted\ on\ a\ massless\ cart\ as\ shown\ in$ figure(6.2) Let us obtain mathematical models of this system by assuming that the cart is standing still for $t < 0$and, the spring-mass-dashpot system on the cart is also standing still for $t < 0$In this system, u(t) is the. Displacement of the cart and is the input to the system. At t=0, the cart is moved at a constant speed, or $\dot{u} =$ constant. The displacement y (t) of the mass is the output the displacement is relative to .The ground.) In this system, m denotes the mass, b denotes the viscous-friction coefficient, and k denotes, the spring constant. We assume that the friction force of the dashpot is proportional to $\dot{y} - \dot{u}$ and that the spring is a linear spring; that is, the spring force is proportional to $y - u$ .For translational systems, Newton's second law states that

$$ma = \sum F$$

where m is amass, a is the acceleration of the mass, and $\sum F$ is the sum of the forces acting on the, mass in the direction of the acceleration a. Applying Newton's second law to the present system, and noting that the cart is massless, we obtain,

$$m\frac{d^2y}{dx^2} = -b\left(\frac{dy}{dt} - \frac{du}{dt}\right) - k(y-u)$$

Or

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = b\frac{du}{dt} + ku$$

This equation represents a mathematical model of the system considered. Taking The Laplace, transform of this last equation, assuming zero initial condition, gives. $(ms^2 + bs + k)Y(s) = (bs + k)U(s)$

Taking the ratio of Y(s) to U(s), we find the transfer function of the system to be transfer function $= G(s) = \frac{Y(s)}{U(s)} = \frac{bs+k}{ms^2+bs+k}$

Such a transfer-function representation of a mathematical model is used very frequently in control engineering. Next we shall obtain a state-space model of this system. We shall first compare the differential, equation for this system

$$\ddot{y} + \frac{b}{m}\dot{y} + \frac{k}{m}y = \frac{b}{m}\dot{u} + \frac{k}{m}u$$

$$\ddot{y} + a_1\dot{y} + a_2 y = b_0\ddot{u} + b_1\dot{u} + b_2 u$$

And identify $a_{1,} a_{2,} b_{0,} b_{1,}$ and $b_{2,}$ as follows

$$a_{1,} = \frac{b}{m} \;, a_{2=\frac{k}{m}}, b_{0,} = 0, b_1 = \frac{b}{m} \;, b_2 = \frac{k}{m}$$

Where

$$\beta_{0\;=b_0\;=0}$$

$$\beta_{1=\,b_1\,-a_1\beta_0} = \frac{b}{m}$$

$$\beta_{2=\,b_2} - a_1\beta_1 - a_2\beta_0 = \frac{k}{m} - \left(\frac{b}{m}\right)^2$$

And

$$x_1 = y - \beta_0 u = y$$

$x_2 = x_1{}^{\cdot} - \beta_1 u = x_1{}^{\cdot} - \frac{b}{m}u$

Also

$$x_1{}^{\cdot} = x_2 + \beta_1 u = x_2 + \frac{b}{m}u$$

$$x_2{}^{\cdot} = -a_{2x_1} - a_{1x_1} + \beta_2 u = -\frac{k}{m}\,x_1 - \frac{b}{m}\,x_2 + \left[\frac{k}{m} - \left(\frac{b}{m}\right)^2\right]u$$

And the output equation becomes

Or

$$\begin{bmatrix} x_1{}^{\cdot} \\ x_2{}^{\cdot} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{b}{m} \\ \frac{k}{m} - \left(\frac{b}{m}\right)^2 \end{bmatrix} \mathrm{u}$$

And $y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Test controllability

$$M = [B \vdots AB]$$

Since

$$B = \begin{bmatrix} \frac{b}{m} \\ \frac{k}{m} - \left(\frac{b}{m}\right)^2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} \frac{b}{m} \\ \frac{k}{m} - \left(\frac{b}{m}\right)^2 \end{bmatrix} = \begin{bmatrix} \frac{k}{m} - \left(\frac{b}{m}\right)^2 \\ \frac{-2kb}{m^2} - \left(\frac{b}{m}\right)^3 \end{bmatrix}$$

$$\therefore M = \begin{bmatrix} \frac{b}{m} & \frac{k}{m} - \left(\frac{b}{m}\right)^2 \\ \frac{k}{m} - \left(\frac{b}{m}\right)^2 & \frac{-2kb}{m^2} - \left(\frac{b}{m}\right)^3 \end{bmatrix}$$

$$|M| = \frac{b}{m}\left(\frac{-2kb}{m^2} - \left(\frac{b}{m}\right)^3\right) - \left(\frac{k}{m} - \left(\frac{b}{m}\right)^2\right)\left(\frac{k}{m} - \left(\frac{b}{m}\right)^2\right) \neq 0$$

The system is fully controllable

Results

It has been shown that properties of zero mean error and bounded variance expansion that were seen for single input-single output (SISO) case a design execution is presented to achieved this ideal result However, the SISO control case inherent in practical makings application the knowledge of these effects presented here as undesirable transmission zero and slower than desired settling times is critical in designing a universal any control system or a dynamic system is Time invariant system is shifting the input on the time axis leads to on equivalent shifting of the output along the time axis with no other changes in other words the condition for complete state controllability can be stated in terms of transfer Functions or transfer matrices if M matrix is square determinant not equal zero and M matrix not square we can find sub square matrix determinant not equal zero and Conditions for Complete Observability M_Q is full rank .A continuous or discrete time system is stable if every bounded input produces a bounded output **(**BIBO**)** and the closed loop transfer function left hand side (BIBO) the Routh criterion of any system is stable all the roots of the characteristic if the elements of the first column of the Routh table have the same sign

Conclusions

A first solution follows from a general result on the global stabilization of null controllable linear system with delay in the input by bounded control laws with a distributed term. Next, it is shown through alaypunov analysis that the stabilization can be achieved as well neglecting the distributed terms. The multiple input-multiple output control system .It more sensitive to modeling errors, which are in bereft in practical process applications. Any system not we cannot representations square matrix controllability test taking sub square matrix detriment not equal zero or full rank, any system is called stable all roots of characteristic equation left hand side

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