Sudan University of Science and Technology
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Young Inequalities in Variable Lebesgue Spaces and Riesz Potential with Maximal Operators in Generalized Orlicz Spaces

متباينات ينق في فضاءات لبيق المتغيرة وجهد ريس مع المؤثرات الأعظمية في فضاءات أورليش المعممة

A Thesis submitted in Fulfillment Requirements for the Degree of Ph.D in Mathematics

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Dedication

To the soul of my father,

To my family,

To my teachers,

And

To my students

Acknowledgments

I am so grateful to my supervisor Prof. Dr. Shawgy Hussein Abdalla, for patient supervising and cooperation to achieve this work. I highly appreciate his sincerity and generosity and above all his humanitarian manner.

I would like to express my wholehearted thanks to my family and for the generous support; they provided me throughout my entire life. They supported me to accomplish this work. Also, would like to thank all my beloved friends for their effort. Thank, extended to my colleagues. Finally, I hope this work may pave the way for others.

Abstract

The Trudinger inequality for Riesz potentials of functions in Musielak-Orlicz spaces, in generalizes Orlicz spaces and Sobolev embedding on generalized Lebesgue and Sobolev spaces are studied. We determine the boundedness of the maximal functions and operators with approximate identities and Sobolev inequalities on Musielak-Orlicz-Morrey spaces. Also the boundedness of the classical operators, local-to-global result and fractional operators in weighted and variable exponent spaces are considered. The Sobolev inequalities and embedding, mean continuity type results, type Young inequalities and regularity for double phase for Orlicz and certain Sobolev spaces are given and characterized.

الخلاصة

تمت دراسة متباينة ترودنجر لأجل جهد ريس للدوال في فضاءات موسلياك-أورليش وفي فضاءات أورليش المعممة وطمر سوبوليف على فضاءات لبيق وسوبوليف المعممة. حددنا المحدودية للمؤثرات والدوال الأعظمية مع متطابقات التقريب ومتباينات سوبوليف على فضاءات موسلياك-أورليش-موري. أيضا اعتبرنا المحدودية للمؤثرات التقليدية ونتيجة المؤثرات الموضوعية-الى-العالمية والمؤثرات الكسرية في فضاءات الأسية المتغيرة والمرجحة. تم إعطاء وتشخيص متباينات وطمر سوبوليف ونتائج نوع الاستمرارية المتوسطة ونوع متباينات ينق والانتظامية لأجل الطور المزدوج لأورليش وفضاءات سوبوليف المؤكدة.

Introduction

We study the Riesz potentials $I_\alpha f$ on the generalized Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^d)$, where $0 < \alpha < d$ and $I_\alpha f := \int_{\mathbb{R}^d} |f(y)||x-y|^{\alpha-d}\, dy$. Under the assumptions that p locally satisfies $|p(x)-p(y)| \le C/\left(-\left|\ln|x-y|\right|\right)$ and is constant outside some large ball, we show that $I_\alpha : L^{p(\cdot)}(\mathbb{R}^d) \to L^{p^{\#}(\cdot)}(\mathbb{R}^d)$, where $\frac{1}{p^{\#}(x)} = \frac{1}{p(x)} - \frac{\alpha}{d}$. If p is given only on abounded domain Ω with Lipschitz boundary we show how to extend p to $\tilde{p}$ on $\mathbb{R}^d$ such that there exists a bounded linear extension operator $\varepsilon : W^{1,p(\cdot)}(\Omega) \hookrightarrow W^{1,\tilde{p}}(\mathbb{R}^d)$, while the bounds and the continuity condition of p are preserved. We show that many classical operators in harmonic analysis such that as maximal operators, singulars, integrals, commutators and fractional integrals are bounded on the variable Lebesgue space $L^{p(\cdot)}$ whenever the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}$. We do so by applying the theory of weighted norm inequalities and extrapolation.

A new method for moving from local to global results in variable exponent function spaces is presented. Several applications of the method are also given. We deal with Sobolev's embeddings for Sobolev Orlicz functions $\nabla u \in \log L^{q(\cdot)}(\Omega)$ for $\Omega \subset \mathbb{R}^n$.

For functions f in Sobolev spaces $W^{1,p(x)}(\Omega)$ which exponent lower semicontinuous, bounded away from 1 and ∞ and with the property of the density of smooth, it is shown that for each open set $\omega \subset\subset \Omega$, for each $h \in \mathbb{R}^N$ such that $\omega + th \subset \Omega \forall t \in [0,1]$, the following inequality holds

$$\|f(x+h) - f(x)\|_{L^{\min p(x,x+h)}(\omega)} \le (1+|\Omega|)\|\nabla f\|_{L^{p(x)}(\Omega)}$$

where $\min p(x, x+h)$ denotes the minimum of p along the segment whose endpoints are $x, x+h$. We study the boundedness of the maximal operator, potential type operators and operators with fixed singularity (of Hardy and Hankel type).

We give conditions for the convergence of approximate identities, both pointwise and in variable L^p spaces. We unify extend results due to Diening [4], Samko [23] and Sharapudinov [145]. We deal with approximate identities in generalized Lebesgue spaces $L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n)$.

We discuss the convergence of approximate identities in Musielek-Orlicz spaces extending the results given by Cruz-Uribe and Fiorenza (2007) and F.-Y. Maeda, Y. Mizuta and T. Ohno (2010). We treat the case where the approximate identity is of compact support. We show a Riesz potential estimate and a Sobolev inequality for general generalized Orlicz spaces. Our assumptions are natural generalizations of the log-Hölder continuity that is commonly used in the variable exponent case.

We give continuity conditions on the exponent function $p(x)$ which are sufficient for the Hardy-Littlewood maximal operator to be bounded on the variable Lebesgue space $L^{p(x)}(\Omega)$, where Ω is any open subset of $\mathbb{R}^n$. Further, our conditions are necessary on $\mathbb{R}$. We present a sufficient condition for the boundedness of the maximal operator on generalized Orlicz spaces.

The Contents

Chapter 1

Riesz Potential and Sobolev Embeding with Boundedness

As an application of Riesz potentials we show the optimal Sobolev embedding $W^{k,p(\cdot)}(\mathbb{R}^d) \hookrightarrow L^{p^*(\cdot)}(\mathbb{R}^d)$ with $\frac{1}{p^*(x)} = \frac{1}{p(x)} - \frac{k}{d}$ and $W^{k,p(\cdot)}(\Omega) \hookrightarrow L^{p^*(\cdot)}(\Omega)$ for $k = 1$. We show compactness of the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$, whenever $q(x) \leq p^*(\cdot) - \varepsilon$ for some $\varepsilon > 0$. As applications we show the Calderón-Zygmund inequality for solutions of $\Delta u = f$ in variable Lebesgue spaces, and show the Calderón extension theorem for variable Sobolev spaces.

Section (1.1): Generalized Lebesque and Sobolev Spaces $L^{p(\cdot)}$ with $W^{k,p(\cdot)}$

The generalized Orlicz-Lebesgue spaces $L^{p(\cdot)}$ (also known as $L^{p(x)}$ and the corresponding generalized Orlicz-Sobolev spaces $W^{k,p(\cdot)}$ have attracted more and more attention. These spaces are special cases of the generalized Orlicz and Orlicz-Sobolev spaces originated by Nakano [19] and developed by Musielak and Orlicz [17], [18]. See Hudzik [12], Kováčik, Rákosník [13], Samko [23], Edmunds, Lang, Nekvinda [7], Růžička [22], Edmunds, Rákosník [8], Fan, Shen, Zhao [10], Diening [3, 5] for properties of the spaces $L^{p(\cdot)}$and $W^{k,p(\cdot)}$ such as reflexivity, denseness of smooth functions, and Sobolev type embeddings. The study of these spaces has been stimulated by problems of elasticity, fluid dynamics, calculus of variations, and differential equations with $p(x)$-growth conditions, where energies of the type $\int |Df(x)|^{p(x)}$ appear (see e.g. Zhikov [26] and Růžička [22]). This energy also appears in the investigations of variational integrals with non-standard growth; see e.g. Zhikov [25], Marcellini [15], Acerbi, Mingione [2].

Since the spaces $L^{p(\cdot)}$ are not invariant to translations, they unfortunately suffer of some undesired properties. So for example the translation operator is not continuous and the convolution with $g \in L^{p(\cdot)}$ is in general not continuous, i.e. in general$\|f * g\|_p \nleq C\|g\|_1 \|g\|_p$. If P satisfies the uniform, local continuity condition $|p(x) - p(y)| \leq \frac{c}{|\ln|x-y||}$, it is still possible to mollify with $\varphi \in C_0^\infty(\mathbb{R}^d)$ function (see Samko [23], Diening [3]). One can reduce this property to the continuity of the Hardy-Littlewood maximal function M (see Diening [3]). If P is constant outside a large ball B_R and satisfies the uniform, local continuity condition above, then M is continuous on $L^{p(\cdot)}(\mathbb{R}^d)$ (see [3]). Especially there holds $f * \varphi_\varepsilon \to f$ in $L^{p(\cdot)}(\mathbb{R}^d)$, where $\varphi_\varepsilon(x) \coloneqq \varepsilon^{-d}\varphi(x/\varepsilon)$, for a large class of mollifiers including $C_0^\infty(\mathbb{R}^d)$ in $W^{k,p(\cdot)}(\mathbb{R}^d)$ and $C^\infty(\overline{\Omega})$ in $W^{k,p(\cdot)}(\Omega)$ for domains (Ω) with Lipschitz boundary. It has been proved by Pick and Růžička [21] that the continuity condition above on p is limiting one. Nekvinda [20] gives a sufficient condition on P, which replaces the assumption on P to be constant outside a large ball B_R by some integral condition, i.e. there exists $p_\infty \coloneqq \lim_{x\to\infty} p(x)$ and constant $c > 0$, $\int c^{1/(p(x)-p_\infty)}\, dx < \infty$.

Růžička and Diening [6] have examined singular operators on $L^{p(\cdot)}(\mathbb{R}^d)$. They showed that if T is a Caldéron-Zygmund operator and p satisfies the local continuity condition above and is constant outside some ball, then there holds $\|Tf\|_p \leq C\|f\|_p$.

Moreover, if T_ε are the truncated operators, then $T_\varepsilon f \to Tf$ almost everywhere and in $L^{p(\cdot)}(\mathbb{R}^d)$.

We examine the Riesz potentials $I_\alpha f$ with $0 < \alpha < d$ and $I_\alpha f \coloneqq \int_{\mathbb{R}^d} |f(y)||x-y|^{\alpha-d}\, dy$ on the spaces $L^{p(\cdot)}(\mathbb{R}^d)$ under the same assumptions on p. For $\sup p < \frac{d}{\alpha}$ we show that $I_\alpha: L^{p(\cdot)}(\mathbb{R}^d) \to L^{p^\#(\cdot)}(\mathbb{R}^d)$, where $\frac{1}{p^\#(x)} = \frac{1}{p(x)} - \frac{\alpha}{d}$. In the case of $0 < \alpha < d$ we will derive a pointwise estimate $I_\alpha$in terms of Mf, see theorem (1.1.13). Note that if $0 < \alpha < d, \Omega$ is a bounded domain, p satisfies the local continuity condition above, and M is continuous on $L^{p(\cdot)}(\Omega)$, then the continuity of $I_\alpha: L^{p(\cdot)}(\mathbb{R}^d) \to L^{p^\#(\cdot)}(\Omega)$ was proved in [24]. Since it has been proved in [4, 3] that the local continuity assumption on p implies the continuity of M on $L^{p(\cdot)}(\Omega)$, these two results imply the continuity $I_\alpha: L^{p(\cdot)}(\mathbb{R}^d) \to L^{p^\#(\cdot)}(\Omega)$ for bounded domainsΩ. In the case of the unbounded domain $\mathbb{R}^d$ the results of [24] cannot be applied and the technique has to be refined

As an application of the results on the Riesz potentials with the same assumptions on p we prove the Sobolev embedding $W^{k,p(\cdot)}(\Omega) \hookrightarrow L^{p^*(\cdot)}(\Omega)$ with $\frac{1}{p^*(x)} = \frac{1}{p(x)} - \frac{k}{d}$. For bounded domain Ω with Lipschitz boundary we prove $W^{1,p(\cdot)}(\Omega) \to L^{p^*(\cdot)}(\Omega)$. The latter result on Sobolev embeddings for bounded domains has also been considered by Edmunds and Rákosník. Under the assumption that $\sup p \le d$ and p is Lipschitz on Ω see [8], or $p \in W^{k,p(\cdot)}(\Omega)$ for some $s > d$, see [9], they prove $W^{1,p(\cdot)}(\Omega) \to L^{p^*(\cdot)}(\Omega)$. Since every $p \in W^{1,s}(\Omega)$ is uniformly Hölder continuous, i.e. $p \in C^{0,\alpha}$ for some $\alpha > 0$, every $p \in W^{1,s}(\Omega)$ satisfies the uniform, local continuity condition $|p(x) - p(y)| \le C/\big(-\big|\ln|x-y|\big|\big)$. Therefore for $\sup\ p > d$ our result is a generalization of the work of Edmunds and Rákosník. Note that Edmunds and Rákosník have also considered the case $\sup p = d$, which implies $\sup p^* = d$. In this case $L^{p^*(\cdot)}$ has to be replaced by a suitable space of Orlicz-Musielak type, which involves a weight depending on $|p - d|$.

If Ω is bounded and has Lipschitz boundary, then we show that the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega)$ is compact whenever $q(x) \le p^*(x) - \varepsilon$. This result is well-known and can also be shown by using only classical Sobolev spaces. See Kováčik. We refer to Kováčik, Rákosník [13] for an alternative proof, who also consider compact embeddings of type $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow C(\Omega)$ in the case $\inf p > d$. The technique used in nevertheless different. For $\varphi \in C_0^\infty(\mathbb{R}^d)$ with $\int \varphi(y) dy = 1$ we show that $\|\varphi_\varepsilon * v - v\|_{p(\cdot)} \le \varepsilon C \|\nabla v\|_{p(\cdot)}$ under the condition that M is continuous on $L^{q(.)}$. Form this we deduce $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega)$. This compact embedding and the embedding $W_0^{1,p(\cdot)}(\mathbb{R}^d) \hookrightarrow\hookrightarrow L^{p^*(\cdot)}(\mathbb{R}^d)$ (under the conditions on p as stated earlier) imply via interpolation the desired compact embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega)$ whenever $q(x) \le p^*(x) - \varepsilon$.

We will now introduce the spaces $L^{q(\cdot)}(\Omega)$ and $W^{k,p(\cdot)}(\Omega)$ and state some fundamental properties of these spaces, which can be found in mentioned above. Let $p : \mathbb{R}^d \to [1,\infty)$ be a measurable function called the exponent or the exponent on $\mathbb{R}^d$. Then for an open set Ω we define $L^{p(\cdot)}(\Omega)$ to consist of measurable functions $f : \Omega \to \mathbb{R}$ such that the modular $\rho_p(f) := \int_\Omega |f(x)|^{p(x)} dx$ is finite. If $p^+ := \sup p < \infty$ (called a bounded exponent) then the expression $\|f\|_{p(\cdot)} := \inf\{\lambda > 0 : \rho_p(f/\lambda) < 1\}$ defines a norm on $L^{p(\cdot)}(\Omega)$. This makes $L^{p(\cdot)}(\Omega)$ a Banach space. Moreover, convergence with respect to the modular is equivalent to convergence with respect to the norm and $\|f\|_{p(\cdot)} \le 1$ iff $\rho_p(f) \le 1$. If $p^- := \inf p > 1$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex and the dual space is isomorphic to $L^{p^*(\cdot)}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Further let $W^{k,p(\cdot)}(\Omega)$ denote the space of measurable functions $f : \Omega \to \mathbb{R}$ such f and its distributional derivatives $\nabla^\alpha f$ up to order k are in $L^{p^*(\cdot)}$. Define the modular $\rho_{k,p}(f) := \sum_{|\alpha|\le k} \rho_p(\nabla^\alpha f)$, then the norm $\|f\|_{k,p(\cdot)} := \inf\{\lambda > 0 : \rho_{k,p}(f/\lambda) < 1\}$ makes $W^{k,p(\cdot)}(\Omega)$ a Banach space. By $W_0^{1,p(\cdot)}(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ in $W^{k,p(\cdot)}(\Omega)$. By B we denote arbitrary ball in $\mathbb{R}^d$. We write $B(x)$ for a ball centered at x and B_r for a ball with radius r. For $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ we denote

$$M_B f := \int_B |f(y)| dy \quad \text{and} \quad M_B^\# := \int_B |f(y) - f_B| dy$$

where $f_B f$ is the mean value integral over B and $f_B := f_B f dx$. By f, resp. $M^\# f$ we denote the Hardy-Lttewood maximal of f, resp. the sharb maximal function f, i.e.

$$Mf(x) := \min_{B(x)} M_{B(x)} f, \quad M^\# f(x) := \min_{B(x)} M^\#_{B(x)} f,$$

Where the supremum is taken over all balls the centered at x. By $\mathcal{P}(\mathbb{R}^d)$ we dente the set of bounded exponent p such that M is bounded on $L^{p(\cdot)}(\mathbb{R}^d)$

Lemma (1.1.1)[1]: Let $p : \mathbb{R}^d \to (0,\infty)$ be continuous with $0 < p^- \le p^+ < \infty$. Then the following conditions are equivalent:

(a) p is uniformly condition on $\mathbb{R}^d$ with
$$|p(x) - p(y)| \le \frac{C_2}{-\ln|x-y|} \text{ for all } |x-y| \le \frac{1}{2}.$$

(b) For all opens ball B there holds
$$|B|^{\inf_B p - \sup_B p} \le C_1.$$

(c) $\frac{1}{p}$ is uniformly continuous on $\mathbb{R}^d$ with
$$\left|\frac{1}{p(x)} - \frac{1}{p(y)}\right| \le \frac{C_2}{-\ln|x-y|} \text{ for all } |x-y| \le \frac{1}{2}.$$

(d) For all open balls B there hold
$$|B|^{\inf_B \frac{1}{p} - \sup_B \frac{1}{p}} \le C_3.$$

Proof. (a) ⇔ (b): This proved in [3].

$(a) \Rightarrow (c)$: Note that for all $|x-y| < \frac{1}{2}$ there holds

$$\left|\frac{1}{p(x)} - \frac{1}{p(y)}\right| = \frac{1}{p(x)p(y)}|p(x)-p(y)| \leq \frac{1}{p(x)p(y)}\frac{C_0}{-\ln|x-y|} \leq \frac{C_0}{-\ln|x-y|}$$

(c) ⇒ (a): Let $q := \frac{1}{\mathrm{p}}$ then q satisfies(a). Thus "$(a) \Rightarrow (c)$" implies

$$|p(x)-p(y)| = \left|\frac{1}{q(x)} - \frac{1}{q(y)}\right| \leq \frac{C}{-\ln|x-y|}$$

$(c) \Leftrightarrow (d)$: Thus follows from (a) ⇔ (b) with replaced p by $\frac{1}{P}$.

Let $L^{1,\infty}(\mathbb{R}^d)$ denote the space of measurable functions f for which there exists a constant $C(f) \geq 0$ such that for all $\lambda > 0$ there holds

$$\left|\{x \in \mathbb{R}^d : |f(x)| > \lambda\}\right| \leq \frac{C(f)}{\lambda}$$

The following propositions have been shown in [3].

Proposition (1.1.2)[1]: Let p be a bounded exponent on $\mathbb{R}^d$ which satisfies the assumptions of Lemma (1.1.1) and is constant outside some ball $B_R(0)$. Then there exist a constant $C(p) > 0$ and $h \in L^{1,\infty}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, such that for all $\|f\|_{p(\cdot)} \leq 1$ there holds

$$(Mf(x))^{\frac{p(x)}{p^-}} \leq C(x)M\left(|f|^{\frac{p}{p^-}}\right)(x) + h(x) \text{ for } a.a.\ x \in \mathbb{R}^d. \qquad (1)$$

Moreover if $p^- > 1$, then (1) implies $p \in p(\mathbb{R}^d)$, i.e. the maximal operator $f \to Mf$ is continuous on $L^{p(\cdot)}(\mathbb{R}^d)$.

Proposition (1.1.3)[1]: Let $p \in \mathcal{P}(\mathbb{R}^d)$. Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be an integrable function and set $\varphi_\varepsilon(x) := \varepsilon^{-d}\varphi(x/\varepsilon)$ for all $\varepsilon > 0$. Assume that least decreasing redial majorant of φ is integrable, i.e. $A := \int_{\mathbb{R}^d} \sup_{|y|\geq|x|}|\varphi(y)|\,dx < \infty$. Then we have

(i) $\sup_{\varepsilon>0}|(f * \varphi_\varepsilon)| \leq 2Mf(x)$ for all $f \in L^{p(\cdot)}(\mathbb{R}^d)$.

(ii) If $\int_{\mathbb{R}^d} \varphi(x)\,dx = 1$, then for $f \in L^{p(\cdot)}(\mathbb{R}^d)$ there holds $f * \varphi_\varepsilon \overset{\varepsilon}{\to} f$ almost everywhere and in $L^{p(\cdot)}(\mathbb{R}^d)$. Furthermore

$$\|f * \varphi_\varepsilon\|_{p(\cdot)} \leq C(A,p)\|\mathrm{M}f\|_{p(\cdot)} \leq C(A,p)\|f\|_{p(\cdot)}$$

Proposition (1.1.4)[1]: Let Ω be a bounded domain with Lipschitz boundary. Further let $p \in p(\mathbb{R}^d)$. Then $\mathrm{C}^\infty(\overline{\Omega})$ is dense in $\mathrm{W}^{1,p(\cdot)}(\overline{\Omega})$.

As a direct consequence of Proposition (1.1.3) there follows

Corollary (1.1.5)[1]: Let $p \in p(\mathbb{R}^d)$ and $\Omega \in \mathbb{R}^d$ be open ball, then $C_0^\infty(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$ and $C_0^\infty(\mathbb{R}^d)$ is dense in $W^{k,p(\cdot)}(\overline{\Omega})$.

Note that the statements of Corollary (1.1.5) are known without the assumption that the maximal operator is bounded in $L^{p(\cdot)}$: see Kováčik and Rákosník [13], for the denseness of $C_0^\infty(\mathbb{R}^d)$ in $L^{p(\cdot)}(\mathbb{R}^d)$ and Samko [23] for the denseness of $C_0^\infty(\mathbb{R}^d)$ in $L^{p(\cdot)}(\mathbb{R}^d)$.

Definition (1.1.6)[1]: Let $0 < \alpha < d$. For $f \in C_0^\infty(\mathbb{R}^d)$ or f measurable with $f \geq 0$. we define $I_\alpha f: \mathbb{R}^d \to [0, \infty)$ by

$$I_\alpha f(x) := \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}} dy.$$

The corresponding kernels $|x|^{\alpha-d}$ are called Riesz kernels.

Definition (1.1.7)[1]: Let p be a bounded exponent. For every ball B we define $\bar{p}_B$ by

$$\frac{1}{\bar{p}_B} := \int_B \frac{1}{p(x)} dy.$$

Lemma (1.1.8)[1]: Let p be a bounded exponent on $\mathbb{R}^d$ and $h \in L^{1,\infty}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, such for all $\|f\|_{p(\cdot)} \leq 1$ there holds

$$(Mf(x))^{\frac{p(x)}{\bar{p}}} \leq C(p) M\left(|f|^{\frac{p}{p^-}}\right)(x) + h(x) \quad \text{for } a.a. \ \ x \in \mathbb{R}^d . \qquad (2)$$

Then for all balls $B(x)$ there hods

$$|B(x)|^{-\frac{p(x)}{\bar{p}_{B(x)}}} \leq \left(C(p)|B|^{-\frac{p}{p^-}} + h(x)\right)^{\frac{p^-}{p(x)}}. \qquad (3)$$

Proof. For $x \in \mathbb{R}^d$ and an open ball $B(x)$ define $f := \mathcal{X}_{B(x)}|B(x)|^{\frac{-1}{\bar{p}}}$. Then $\rho_p(f) = 1$. Sine $t \to s^t$ is convex for all $s > 0$, Jensen's inequality implies

$$|B(x)|^{-\frac{1}{\bar{p}_{B(x)}}} = |B(x)|^{-\int_{B(x)} \frac{1}{p(y)} dy} \leq \int_{B(x)} |B(x)|^{-\frac{1}{\bar{p}_{B(x)}}} dy = M_{B(x)} f \leq Mf(x).$$

Sine $\rho_p(f) \leq 1$ inequality (2) implies

$$|B(x)|^{-\frac{p(x)}{\bar{p}_{B(x)}}} \leq \left(C(p) M\left(|f|^{-\frac{p}{p^-}}\right)(x) + h(x)\right)^{\frac{p^-}{p(x)}}$$

Note $|f|^{\frac{p}{p^-}} \leq |B(x)|^{-\frac{p}{p^-}}$ almost everywhere, so $M\left(|f|^{\frac{p}{p^-}}\right) \leq |B(x)|^{-\frac{p}{p^-}}$ almost everywhere and

$$|B(x)|^{-\frac{p(x)}{\overline{p}_{B(x)}}} \leq \left(C(p)|B|^{-\frac{p}{p^-}} + h(x)\right)^{\frac{p^-}{p(x)}}.$$

This proves the lemma.

Note that in order to prove Lemma (1.1.8) it is sufficient to require that (2) holds for all $f = x_B|B|^{-\frac{1}{p}}$. Note also that if p is bounded exponent on $\mathbb{R}^{\mathbf{d}}$ with $1 < p^- \leq p^+ < \infty$ which satisfies the assumptions of Lemma (1.1.1) and is constant outside some ball $B_R(0)$, then due to Proposition (1.1.2) the requirements of Lemma (1.1.8) are satisfied for all $\|f\|_{p(\cdot)} \leq 1$.

Lemma (1.1.9)[1]: Let $p \in p(\mathbb{R}^d)$, then for all balls B there holds

$$\|\mathcal{X}B\|_{p(\cdot)} \leq C(p)|B|^{\frac{1}{\overline{p}_B}}.$$

Proof. Let $(x) := \mathcal{X}B|B|^{\frac{1}{\overline{p}_B}}$, then $\|f\|_{p(\cdot)} = 1$. Furthermore $x \in B$ there holds

$$CMf(x) \geq M_{B(x)}f = \int_B |B(x)|^{-\frac{1}{p(y)}}\, dy.$$

Since $t \to a^{-t}$ is convex for all $a > 0$, there follows

$$CMf(x) \geq |B|^{-\int_B \frac{1}{p(y)}dy} = |B|^{-\frac{1}{\overline{p}_B}} \quad \text{for all } x \in B. \tag{4}$$

Since M is continuous on $L^{p(\cdot)}(\mathbb{R}^d)$ by assumption, we deduce from (4)

$$\left\|\mathcal{X}B|B|^{-\frac{1}{\overline{p}_B}}\right\|_{p(\cdot)} \leq C\|Mf\|_{p(\cdot)} \leq \|f\|_{p(\cdot)} \leq C..$$

This proves the lemma.

Definition (1.1.10)[1]: Let $0 < \alpha < d$. Then for every bounded exponent p with $p^+ < \frac{d}{\alpha}$, we define $p^\# : \mathbb{R}^d \to [1, \infty)$ by $\frac{1}{p^\#} := \frac{1}{p} - \frac{\alpha}{d}$. Note that due to $p^+ < \frac{\alpha}{d}$ the function $p^\#$ is also a bounded exponent.

Lemma (1.1.11)[1]: Let $0 < \alpha < d$ and let p be a bounded on $\mathbb{R}^d$ with $1 \leq p^- \leq< p^+ < \frac{d}{\alpha}$, and $\frac{d-\alpha}{d}p' \in P(\mathbb{R}^d)$. Moreover, assume that there exists $h \in L^{1.\infty}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ such that for all $\|f\|_{p^\#(\cdot)} \leq 1$ there holds

$$(Mf(x))^{\frac{p^\#(x)}{(p^\#)^-}} \leq C(p^\#)M\left(|f|^{\frac{p^\#}{(p^\#)^-}}\right)(x) + h(x) \quad \text{for } a.a. \ \ x \in \mathbb{R}^d. \tag{5}$$

Then there exists $g \in L^{1,\infty}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, such that for all $\|f\|_{p(\cdot)} \leq 1$ there holds

$$(I_\alpha(|f|))^{\frac{p^\#}{p^-}} \leq C(Mf)^{\frac{p}{p^-}} + g. \tag{6}$$

Proof. First note that $p^+ < \frac{d}{\alpha}$ implies that $\frac{d-\alpha}{d}p'$ and $p^\#$ are bounded exponent. Let $\|f\|_{p(\cdot)} \leq 1$ and $x \in \mathbb{R}^d$. It is well-known (see e.g. Malý, Ziemer [14]) that for all $\delta > 0$ and $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ there holds

$$\int_{B_\delta(x)} \frac{f(y)}{|x-y|^{d-\alpha}} dy \leq C\delta^\alpha Mf(x). \tag{7}$$

Moreover,

$$\int_{B_\delta(x)} \frac{f(y)}{|x-y|^{d-\alpha}} dy \leq C\|f\|_{p(\cdot)} \left\| \mathcal{X}\mathbb{R}^d \backslash B_\delta(x) |x-\cdot|^{\alpha-d} \right\|_{p'(\cdot)}$$

$$= C\|f\|_{p(\cdot)} \left\| \mathcal{X}\mathbb{R}^d \backslash B_\delta(x) |x-\cdot|^{\alpha-d} \right\|_{\frac{d-\alpha}{d}p'(\cdot)}^{\frac{d-\alpha}{d}} \tag{8}$$

Note that for all $y \in \mathbb{R}^d \backslash B_\delta(x)$ there holds

$$M(\mathcal{X}B_\delta(x)|B_\delta(x)|^{-1})(y) \geq \int_{B_{2|x-y|}(y)} \mathcal{X}B_\delta(z)|B_\delta(x)|^{-1} dz$$

$$= \frac{1}{|B_{2|x-y|}(y)|} \quad \text{since } |x-y| \geq \delta$$

$$= C|x-y|^d.$$

Thus for all $y \in \mathbb{R}^d$

$$\mathcal{X}\mathbb{R}^d \backslash B_\delta(x)|x-\cdot|^{\alpha-d} \leq CM(\mathcal{X}B_\delta(x)|B_\delta(x)|^{-1})(y) \tag{9}$$

From (8), (9) and $\|f\|_{p(\cdot)} \leq 1$, there follows

$$\int_{\mathbb{R}^d \backslash B_\delta(x)} \frac{f(y)}{|x-y|^{d-\alpha}} dy \leq C\|M(\mathcal{X}B_\delta(x)|B_\delta(x)|^{-1})\|_{\frac{d-\alpha}{d}p'(\cdot)}^{\frac{d-\alpha}{d}}$$

$$= C|B_\delta(x)|^{\frac{\alpha-d}{d}} \left\| M\big(\mathcal{X}B_\delta(x)\big) \right\|_{\frac{d-\alpha}{d}p'(\cdot)}^{\frac{d-\alpha}{d}}.$$

Since $\frac{d-\alpha}{d}p'(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, there follows

$$\int_{\mathbb{R}^d \backslash B_\delta(x)} \frac{f(y)}{|x-y|^{d-\alpha}} dy \leq C|B_\delta(x)|^{\frac{\alpha-d}{d}} \|\mathcal{X}B_\delta(x)\|_{\frac{d-\alpha}{d}p'(\cdot)}^{\frac{d-\alpha}{d}}.$$

Thus Lemma (1.1.9) applied to $\frac{d-\alpha}{d}p'(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ gives

$$\int_{\mathbb{R}^d \setminus B_\delta(x)} \frac{f(y)}{|x-y|^{d-\alpha}} dy \le C|B_\delta(x)|^{\frac{\alpha-d}{d}} \left(|B_\delta(x)|^{\frac{1}{\frac{\alpha-d}{d} \bar{p}'_{B_\delta(x)}}} \right)^{\frac{d-\alpha}{d}}$$

$$= C|B_\delta(x)|^{\frac{\alpha-d}{d} + \frac{1}{\bar{p}'_{B_\delta(x)}}}$$

$$= C|B_\delta(x)|^{\frac{\alpha}{d} + \frac{1}{\bar{p}_{B_\delta(x)}}}$$

$$= C|B_\delta(x)|^{-\frac{1}{\bar{p}^\#_{B_\delta(x)}}}.$$

Due (5) we apply Lemma (1.1.8) to $p^\#$ and get

$$\int_{\mathbb{R}^d \setminus B_\delta(x)} \frac{f(y)}{|x-y|^{d-\alpha}} dy \le \left(C|B_\delta(x)|^{-\frac{1}{(p^\#)^-}} + h(x) \right)^{\frac{(p^\#)^-}{p^\#(x)}} \tag{10}$$

Thus (7) and (10) implies for all $\delta > 0$

$$I_\alpha(|f|)(x) \le C\delta^\alpha Mf(x) + \left(C|B_\delta(x)|^{-\frac{1}{(p^\#)^-}} + h(x) \right)^{\frac{(p^\#)^-}{p^\#(x)}}$$

$$\le C\delta^\alpha Mf(x) + \left(C\delta^{-\frac{d}{(p^\#)^-}} + h(x) \right)^{\frac{(p^\#)^-}{p^\#(x)}} . \tag{11}$$

Fix $\delta = \delta(x)$ by

$$\delta := \left(Mf(x)\right)^{-\frac{p(x)}{d}},$$

then (11) simplifies to

$$I_\alpha(|f|)(x) \le C\left(Mf(x)\right)^{\frac{p(x)}{p^\#(x)}} + \left(C\left(Mf(x)\right)^{\frac{p(x)}{p^\#(x)}} + h(x) \right)^{\frac{(p^\#)^-}{p^\#(x)}} .$$

Since $1 \le \frac{p^\#}{(p^\#)^-} \le C(p,\alpha) < \infty$, this implies

$$\left(I_\alpha(|f|)(x)\right)^{\frac{p^\#(x)}{(p^\#)^-}} \le C\left(Mf(x)\right)^{\frac{p(x)}{p^\#(x)}} + Ch(x).$$

This proves the lemma.

Theorem (1.1.12)[1]: Let $0 < \alpha < d$ and let p be a bounded exponent on $\mathbb{R}^d$ with $1 < p^- \leq p^+ < \frac{d}{\alpha}$ which satisfies the assumptions of Lemma (1.1.1) and is constant outside some large ball $B_R = B_R(0)$. Then there exists $g \in L^{1,\infty}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, such that for all $\|f\|_{p(\cdot)} \leq 1$ there holds

$$\left(I_\alpha(|f|)\right)^{\frac{p^\#}{p^-}} \leq C(Mf)^{\frac{p}{p^-}} + g. \tag{12}$$

Moreover,

$$\|I_\alpha f\|_{p^\#(.)} \leq C(p,\alpha)\|f\|_{p(.)}. \tag{13}$$

Proof. Since

$$\frac{1}{p^\#} = \frac{1}{p} - \frac{\alpha}{d} \qquad \text{and} \qquad \frac{1}{\frac{\mathrm{d}-\alpha}{\mathrm{d}}p'} = \frac{1}{\mathrm{d}-\alpha}\left(1 - \frac{1}{p}\right)$$

and p fulfills condition (1) of Lemma (1.1.1) , so do the exponent $p^\#$ and $\frac{\mathrm{d}-\alpha}{\mathrm{d}}p'$. Thus form Proposition (1.1.2) applied to $p, p^\#$, and $\frac{\mathrm{d}-\alpha}{\mathrm{d}}p'$ we see that $p \in \mathcal{P}(\mathbb{R}^d)$ and that p fulfills the conditions of Lemma (1.1.11). Therefore there exists $g \in L^{1,\infty}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, such that for all $\|f\|_{p(\cdot)} \leq 1$ there holds (6). Let $\|f\|_{p(\cdot)} \leq 1$, then due (6). There holds $\rho_{p^\#}\left(I_\alpha(|f|)\right) \leq \rho_p(Mf) + \rho_p(g) \leq \rho_p(Mf) + C$. Since $p \in \mathcal{P}$, i.e M is continuous on $L^{p(\cdot)}(\mathbb{R}^d)$, and $\rho_p \leq 1$ there holds $\rho_p(Mf) \leq C$. Hence implies $\rho_{p^\#}(I_\alpha f) \leq C$ and therefore $\|I_\alpha f\|_{p^\#(\cdot)} \leq C$. Overall we have shown that I_α is a bouneded mapping from $L^{p(.)}(\mathbb{R}^d)$ to $L^{p^\#(\cdot)}(\mathbb{R}^d)$. Since I_α is linear, this implies (13). This proves the theorem.

Definition (1.1.13)[1]: Let $\Omega \subset \mathbb{R}^d$ be open and $p : \mathbb{R}^d \to [1,\infty)$ be a bounded exponent. Then we say that Ω is a $\left(1, p(\cdot)\right)$–extension domain if there exists a bounded extension operator ε

$$\varepsilon : \ W^{1,p(\cdot)}(\Omega) \to W^{1,p(\cdot)}(\mathbb{R}^d)$$

Such that $\varepsilon(u)|_\Omega = u$ for all $u \in W^{1,p(\cdot)}$.

Let $q : \Omega \to [1,\infty)$ be measurable bounded. Then we say that (q,Ω) has 1-extension $(q,\mathbb{R}^d)$ if there exist a bounded exponent $q : \mathbb{R}^d \to [1,\infty)$ such that $p|_\Omega = q|_\Omega$ and Ω is a $\left(1, p(\cdot)\right)$–extension domain.

Theorem (1.1.14)[1]: Let $\Omega \subset \mathbb{R}^d$ be an open, bounded set with Lipschitz boundary. Let $p : \mathbb{R}^d \to [1,\infty)$ satisfy the uniform continuity condition

$$|p(x) - p(x)| \leq \rho(|x-y|) \qquad \text{for all} \qquad x, y \in \Omega,$$

where ρ is concave for $t \geq 0$ and $\rho(t) \to 0$ for $t \to 0^+$. Then there exists an 1-extension $(q,\mathbb{R}^d)$ of (q,Ω) and a constant $A > 0$, such that

$$|p(x) - p(x)| \leq \rho(|x - y|) \quad \text{for all} \quad x, y \in \Omega.$$

Moreover, there holds $p^- = p^-$ and $p^- = p^-$. Furthermore the corresponding linear bounded extension operator $\varepsilon : W^{1,p(\cdot)}(\Omega) \to W^{1,p(\cdot)}(\mathbb{R}^d)$ can be chosen in such a way that εf has compact support contained in $\Omega_\beta \coloneqq \{x \in \mathbb{R}^d : \text{dist}(x, \Omega) \leq \beta\}$ for some fixed $\beta > 0$, i.e. $\varepsilon : W^{1,p(\cdot)}(\Omega) \to W^{1,p(\cdot)}(\Omega_\beta)$ continuously.

Proof. In Theorem 4.1 of [8] it is proved via the reflection method of Hestenes [10] that there exists a bounded exponent q with $p|_\Omega = q|_\Omega$ and bounede linear extension operator which satisfies the estimate

$$\|\varepsilon f\|_{W^{1,p(\cdot)}(\mathbb{R}^d)} \leq C\|f\|_{W^{1,q(\cdot)}(\Omega)}$$

This extension operator further satisfies that εf has compact support contained $\{x \in \mathbb{R}^d : \text{dist}(x, \Omega) \leq \beta\}$ for some $\beta \geq 0$. In order to construct the extension Edmunds and Rákosník cover Ω by small open sets $V_j, j = 1, \dots\dots, k$, where they flatten the boundary by bi-Lipschitz maps $T_j : (-\delta, \delta)^{d-1} \times (-\gamma, \gamma) \to \mathbb{R}^d$. To these flattened domains $T_j^{-1}(V_j)$ they apply the reflection operator

$$Ef(x) = \begin{cases} f(x', x_n) & \text{for } x_n \geq 0, \\ f(x', x_n) & \text{for } x_n < 0, \end{cases}$$

both to p and to the f_j, where $f_j(x) = f(x)\varphi_j(x)$ with a sutable partition $\{\varphi_j\}$ of unity. Especially they define $p_j : V_j \cup \Omega$ by

$$p_j(x) = \begin{cases} p(x) & \text{for} \quad x \in \Omega, \\ E_{rj}(T_j^{-1}, (x)) & \text{for } x \in V_j \backslash \Omega, \end{cases}$$

where $rj \coloneqq p_{\,\circ} T_j$. Note that since E, T_j, and T_j^{-1} are Lipschitz there exists $C > 0$ such that

$$|p_j(x) - p_j(y)| \leq \rho(C|x - y|) \quad \text{for all} \quad x, y, \in \Omega. \tag{14}$$

Then Edmunds and Rákosník extend the p_j on Ω to $\widetilde{p_j}$ on $\mathbb{R}^d$ preserving their upper and lower bounds, where they pose no further conditions on extensions of the $\widetilde{p_j}$. Note to Mc shane [16], the p_j can be extended in a way that (15) remains valid for $\widetilde{p_j}$. Here we use that ρ is concave and that$\rho\ (x) \to 0$ for $t \to 0^+$. After that Edmunds and Rákosnık define on $p : \mathbb{R}^d \to [1, \infty)$ by

$$p_j(x) \coloneqq \min_{j=1,\dots,k} \widetilde{p_j}(x) \quad \text{for} \ \ x \in \mathbb{R}^d$$

Thus there holds

$$|p(x) - p(y)| \leq \rho(C|x - y|) \quad \text{for all} \quad x, y \in \Omega. \tag{15}$$

For the rest of the proof we may proceed exactly as Edmunds and Rákosník. This proves the theorem.

Corollary (1.1.15)[1]: Let $\Omega \subset \mathbb{R}^d$ be an open, bounded set with Lipschitz boundary. Let $q : \Omega \to [1, \infty)$ satisfy the local uniform continuity condition

$$|q(x) - q(y)| \leq \frac{C}{-\ln|x-y|} \quad \text{for all} \quad x, y, \in \Omega, \tag{16}$$

Then there exists 1-extension $(p, \mathbb{R}^d)$ of (q, Ω) with $p^- = q^-$and $p^+ = q^+$, which satisfies the same local uniform continuity condition (with a possibly different constant) and is constant outside some ball B_R. Further the corresponding bounded extension operator $\varepsilon :\ W^{1,p(\cdot)}(\Omega) \to W^{1,p(\cdot)}(\mathbb{R}^d)$ can be chosen in such a way that εf has compact support contained in $\Omega_\beta \coloneqq \{x \in \mathbb{R}^d : \mathrm{dist}(x, \Omega) \leq \beta\}$ for some fixed $\beta > 0$, i.e. $\varepsilon :\ W^{1,p(\cdot)}(\Omega) \to W^{1,p(\cdot)}(\Omega_\beta)$ continuously.

If moreover $q^- > 0$, then $p \in \rho(\mathbb{R}^d)$, I.e. M is continuous on $L^{p(\cdot)}(\mathbb{R}^d)$.

Proof. Since the mapping $\psi \coloneqq t \to C/(-\ln|t|)$ is concave on $[0, \exp(-2)\,, \psi(t) \to 0]$ for $t \to 0^+$, and q satisfies uniformly the local continuity condition (16), there exists $\rho : [0, \infty) \to [0, \infty)$ with $\psi|_{[0,t_0]} = C\rho|_{[0,t_0]}$, which is concave on $[0, \infty)$ and $\rho(t) \to 0$ for such that

$$|q(x) - q(y)| \leq \rho(|x - y|) \quad \text{for all} \quad x, y \in \Omega.$$

Due to Theorem (1.1.14) it follows that there exists a 1-extension $(p, \mathbb{R}^d)$ 0f (q, Ω), which possesses all the desired properties save to be constant outside some large ball. For $\beta > 0$ Theorem (1.1.14) choose an open ball B_R with $R\backslash 3 > \beta$ that Ω is a compact subset of $B_{R\backslash 2}$ and let $\eta \in C^\infty(\mathbb{R}^d)$ with $\mathcal{X}B_{R\backslash 2} \leq \eta \leq \mathcal{X}B_R$. Now set $p(x) \coloneqq \big(1 - \eta(x)\big)p^- + \eta(x)s(x)$, then s is constant outside B_R and satisfies the local uniform continuity condition (with possible a possible different constant). Since $\mathrm{supp}\ \varepsilon f \subset \Omega_{R\backslash 2}$ for all $f \in W^{1,p(x)}(\Omega)$ there follwos

$$\|\varepsilon f\|_{W^{1,p(x)}(\mathbb{R}^d)} = \|\varepsilon f\|_{W^{1,p(x)}(\mathbb{R}^d)} \leq \|\varepsilon f\|_{W^{1,p(x)}(\Omega)}. \tag{17}$$

This proves the existence of a suitable 1-extension. If moreover $q^- > 1$, then $p^- > 1$, and Proposition (1.1.2) implies $\eta \in \rho(\mathbb{R}^d)$. This proves the corollary.

We will prove Sobolev embeddings with optimal exponent. In order to do so we need the following result about the maximal sharp function $M^\# f$, which can be found in [6]:

Proposition (1.1.16)[1]: Let $p, p' \in \mathcal{P}(\mathbb{R}^d)$ with $1 < p^- \leq p^+ < \infty$. Then there for all $f \in L^{p(\cdot)}(\mathbb{R}^d)$ there holds

$$\|f\|_{p(\cdot)} \leq C\left\|M^\# f\right\|_{p(\cdot)}$$

We are now prepared to prove the Sobolev embeddings:

Theorem (1.1.17)[1]: Let $k \in \mathbb{N}_0$ with $0 \le k < d$ and let p be a bounded exponent on $\mathbb{R}^d$ with $1 < p^- \le p^+ < \frac{d}{k}$ which satisfies the assumptions of Lemma (1.1.1) and is constant outside some large ball $B_R = B_R(0)$. Define p^* by $\frac{1}{p^*} := \frac{1}{p} - \frac{k}{d}$. Then

$$W^{k,p(\cdot)}(\mathbb{R}^d) \to L^{p^*(\cdot)}(\mathbb{R}^d) \quad \text{continuously.}$$

Proof. Note that $0 \le k < d$ implies $(p^*)^+ < \infty$. We will proceed by induction over k. Case $k = 1$: Let $f \in W^{1,p(\cdot)}(\mathbb{R}^d)$ with $\|f\|_{1,p(\cdot)} \le 1$. We will show $\|f\|_{p(\cdot)} \le C$. Sine $C_0^\infty(\mathbb{R}^d)$ is dense in $W^{1,p(\cdot)}(\mathbb{R}^d)$ and $L^{p(\cdot)}(\mathbb{R}^d)$ we can assume without loss of generality $f \in C_0^\infty(\mathbb{R}^d)$. Due to Theorem (1.1.12) there holds $\|I_1(|\nabla|)f\|_{p*(\cdot)} \le C$. From [14] we deduce that for all $B_r(x)$ there holds

$$M^\#_{B_r(x)} f \le Cr \int_{B_r(x)} |\nabla f(y)| dy \le \int_{B_r(x)} \frac{|\nabla f(y)|}{|x-y|^{d-1}} dy \le C I_1(|\nabla| f)(x).$$

By taking the supremum over all balls $B_r(x)$ we deduce that for all x there holds

$$M^\#_{B_r(x)} \le C I_1(|\nabla| f)(x)$$

This and $\|I_1(|\nabla|)f\|_{p(\cdot)} \le C$ imply $\|M^\# f\|_{p(\cdot)} \le C$. From Proposition (1.1.16) there follows $\|f\|_{p(\cdot)} \le C$, where we have used $f \in L^{p(\cdot)}(\mathbb{R}^d)$ due to $f \in C_0^\infty(\mathbb{R}^d)$. Since $C_0^\infty(\mathbb{R}^d)$ is dense in $W^{1,p(\cdot)}(\mathbb{R}^d)$ and $L^{p(\cdot)}(\mathbb{R}^d)$ (this is a direct consequence of Proposition (1.1.4) of [5]), this prove $\|f\|_{p(\cdot)} \le C$ for all f with $\|f\|_{1,p(\cdot)} \le 1$. This proves the case $k = 1$.

Case $k \to k+1$: Let $f \in W^{k+,1,p(\cdot)}(\mathbb{R}^d)$ with $\|f\|_{k,p(\cdot)} \le 1$, then $\|f\|_{k,p(\cdot)} \le 1$ and $\|\nabla f\|_{k,p(\cdot)} \le 1$. By assumption this implies $\|f\|_{q(\cdot)} \le 1$ and $\|\nabla f\|_{q(\cdot)} \le C$, i.e. $\|f\|_{1,q(\cdot)} \le 1$, with $\frac{1}{q} := \frac{1}{p} - \frac{k-1}{d}$. The case $k = 1$ implies $\|f\|_{p(\cdot)} \le C$. This proves the theorem.

Corollary (1.1.18)[1]: Let $\Omega \subset \mathbb{R}^d$ be an open, bounded set with Lipschitz boundary. Let $p : \Omega \to [1, \infty)$ with $1 < p^- \le p^+ < d$ satisfy the local uniform continuity condition

$$|p(x) - q(y)| \le \frac{C}{-\ln|x-y|} \quad \text{for all } x, y \in \Omega.$$

Then we have the following continuous embeddings

$$W^{1,p(\cdot)}(\mathbb{R}^d) \to L^{p(\cdot)}(\mathbb{R}^d).$$

where $\frac{1}{p^*(x)} = \frac{1}{p(x)} - \frac{1}{d}$.

Proof. Due to corollary (1.1.15) there exists extension of p to $\mathbb{R}^d$, which we still denote by p satisfies the assumptions of Theorem (1.1.17) and there exists a linear bounded extension operator $\varepsilon : W^{1,p(\cdot)}(\Omega) \to W^{1,p(\cdot)}(\mathbb{R}^d)$. Thus

$$W^{1,p(\cdot)}(\mathbb{R}^d) \hookrightarrow L^{p^*(\cdot)}(\mathbb{R}^d)$$

Moreover, there holds

$$W^{1,p(\cdot)}(\Omega) \xrightarrow{\varepsilon} W^{1,p(\cdot)}(\mathbb{R}^d) \hookrightarrow L^{p^*(\cdot)}(\mathbb{R}^d) \to L^{p^*(\cdot)}(\Omega).$$

This proves the corollary.

Lemma (1.1.19)[1]: Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ with $\int \varphi(y)dy = 1$. Then there exists a constant $A = A(\varphi) > 0$ such that for all $v \in W^{1,1}_{\text{loc}}(\mathbb{R}^d)$ and all $\varphi > 0$ there holds

$$|(\varphi_\varepsilon * v)(x) - v(x)| \leq \varepsilon AM(\nabla_v)(x).$$

If $p \in \mathcal{P}(\mathbb{R}^d)$, then there exists $A_2 = A_2(\varphi) > 0$ such that for all $f \in W^{1,p(\cdot)}(\mathbb{R}^d)$.

$$\|\varphi_\varepsilon * v - v\|_{p(\cdot)} \leq \varepsilon A_2 \|\nabla_v\|_{p(\cdot)}.$$

Proof. Without loss of generality we can assume supp $\varphi \subset B_1(0)$. Then

$$\begin{aligned}(\varphi_\varepsilon * v)(x) - v(x) &= \int_{B_\varepsilon(0)} \varphi_\varepsilon(y)\big(v(x-y) - v(x)\big)\, dy \\ &= \int_{B_\varepsilon(0)} \int_0^1 \varphi_\varepsilon(y)\nabla_v(x-ty).\, y\, dt\, dy \\ &= \int_0^1 \int_{B_\varepsilon(0)} \varphi_\varepsilon(y)\nabla_v(x-ty).\, y\, dy\, dt \\ &= \int_0^1 \int_{B_{\varepsilon t}(0)} \varphi_{\varepsilon t}(y)\nabla_v(x-y).\frac{y}{t}\, dy\, dt\, .\end{aligned}$$

Thus

$$\begin{aligned}|(\varphi_\varepsilon * v)(x) - v(x)| &\leq \varepsilon \int_0^1 \int_{B_\varepsilon(0)} |\varphi_{\varepsilon t}(y)||\nabla_v(x-y)|\, dy\, dt \\ &= \varepsilon \int_0^1 (|\varphi_{\varepsilon t}| * |\nabla_v|\,)(x)\, dt\, .\end{aligned}$$

Since $\varphi \in C_0^\infty(\mathbb{R}^d)$ the function $|\varphi|$ satisfies the assumption of proposition (1.12). Thus there exists $A > 0$ with

$$\left|(\varphi_\varepsilon * v)(x) - v(x) \leq \varepsilon A_2 \|\nabla_v\|_{p(\cdot)}\right|$$

Since $p \in \mathcal{P}(\mathbb{R}^d)$, this implies

$$\|\varphi_\varepsilon * v - v\|_{p(\cdot)} \le \varepsilon A \|M(\nabla_v)\|_{p(\cdot)} \le \varepsilon AC \|\nabla_v\|_{p(\cdot)}$$

Thus proves the lemma.

Lemma (1.1.20)[1]: Let $p \in \mathcal{P}(\mathbb{R}^d)$ and let Ω be a bounded domain with Lipschitz boundary, then the mapping

$$W_0^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{1,p(\cdot)}(\Omega)$$

is compact.

Proof. Let $v_k, v \in W_0^{1,p(\cdot)}(\Omega)$ with $v_k \to v$ in $W_0^{1,p(\cdot)}(\Omega)$. We have to show $v_k \to v$ in $L^{1,p(\cdot)}(\Omega)$. Without loss of generality we may assume $v = 0$. Furthermore, we extend the function v_k by zero outside of Ω.

Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ with ≥ 0 , $\operatorname{supp}\varphi \subset B_1(0)$ and $\int \varphi(y)dy = 1$. For $\varepsilon > 0$ define $\varphi_\varepsilon := \varepsilon^{-d}\varphi(x/\varepsilon)$, then

$$v_k(x) = (v_k - \varphi_\varepsilon * v_k)(x) + (\varphi_\varepsilon * v_k)(x)\,. \tag{18}$$

Lemma (1.1.19)

$$\begin{aligned}\|v_k\|_{p(\cdot)} &\le \|v_k - \varphi_\varepsilon * v_k\|_{p(\cdot)} \le \|\varphi_\varepsilon * v_k\|_{p(\cdot)} \\ &\le C\varepsilon \|\nabla v_k\|_{p(\cdot)}\, \|\varphi_\varepsilon * v_k\|_{p(\cdot)}.\end{aligned} \tag{19}$$

Since $v_k \rightharpoonup 0$ there holds $(\varphi_\varepsilon * v_k)(x) = \langle v_k, \varphi_\varepsilon(x - \cdot)\rangle \overset{k}{\to} 0$ almost everywhere. Let $\Omega_\varepsilon := \Omega + B_\varepsilon(0)$, then $(\varphi_\varepsilon * v_k)(x) = 0$ for all $x \in (\mathbb{R}^d)\backslash\Omega_\varepsilon$. Moreover, for all $x \in \Omega_\varepsilon$

$$|(\varphi_\varepsilon * v_k)(x)| = |\langle v_k, \varphi_\varepsilon(x - \cdot)\rangle| \le C\|v_k\|_{p(\cdot)}\, \|\varphi_\varepsilon(x - \cdot)\|_{p'(\cdot)}.$$

Since φ has compact support and is bounded, there holds $\rho_{p'}(\varphi_\varepsilon(x - \cdot) \le C(\varepsilon, p)$ and $\|\varphi_\varepsilon(x - \cdot)\|_{p'(\cdot)} - \cdot) \le C(\varepsilon, p)$ for all $x \in \Omega_\varepsilon$. So $x \in \mathbb{R}^d$

$$|(\varphi_\varepsilon * v_k)(x)| \le C(\varepsilon, p)\chi\Omega_\varepsilon\,(x).$$

Since $\chi\Omega_\varepsilon \in L^{p(\cdot)}(\mathbb{R}^d)$ and $(\varphi_\varepsilon * v_k)(x) \overset{k}{\to} 0$ almost everywhere we can use the theorem convergence which implies $\varphi_\varepsilon * v_k \to 0$ in $L^{p(\cdot)}(\mathbb{R}^d)$. Hence (19) implies

$$\limsup_{k\to\infty}\|v_k\|_{p(\cdot)} \le C\varepsilon \limsup_{k\to\infty}\|v_k\|_{p(\cdot)}.$$

Since $\varepsilon > 0$ was arbitrary this proves $v_k \to 0$ in $L^{p(\cdot)}(\mathbb{R}^d)$. This proves the lemma.

By using a suitable partition of the unity on Ω this implies

$$W_0^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{p(\cdot)}(\Omega)$$

Nevertheless the result for $p \in \mathcal{P}(\mathbb{R}^d)$ is stronger: Indeed, if p is uniformly continuous on $\overline{\Omega}$, then there exists $q : \overline{\Omega} \to [1, \infty)$, with $q \le p \le q^* - \varepsilon$ which satisfies the local, uniform continuity condition $|p(x)-q(y)| \leq \frac{C}{-\ln|x-y|}$. This q can be extended outside Ω such that the local, uniform continuity condition is preserved and q is constant outside some large ball (see [6]). Applying Lemma (1.1.20) to q we get

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow W^{1,q(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{q^*-\varepsilon}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$$

This shows that the result for uniformly continuous p can immediately be deduced from Lemma (1.1.20).

Theorem (1.1.21)[1]: Let Ω, p and p^* as in corollary (1.1.19). Then for all measurable $q : \Omega \to [1,\infty)$ with $q(x) \leq p^*(x) - \varepsilon$ for almost $x \in \mathbb{R}^d$ and some $\varepsilon > 0$ there holds

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega),$$

i.e. the embedding is compact.

Proof. As in Corollary (1.1.19) we extend p to $\mathbb{R}^d$, such that p satisfies the assumptions of Theorem (1.1.18) and there exists a linear bounded extension operator $\varepsilon : W^{1,p(\cdot)}(\Omega) \to W^{1,p(\cdot)}(\mathbb{R}^d)$.

For $0 \leq \theta \leq 1$ define s_θ by $\frac{1}{s_\theta(x)} = \frac{1-\theta}{p(x)} + \frac{\theta}{p^*(x)}$. Since $q(x) \leq p^*(x) - \varepsilon$ and p^* is continuous, there exists $0 < \theta_0 < 1$, such $q \leq s_{\theta_0} \leq p^*$ almost everywhere. Note that s_{θ_0} is abounded exponent. Let $f_n, f \in W^{1,p(\cdot)}(\Omega)$ with $f_n \rightharpoonup f$ in $W^{1,p(\cdot)}(\Omega)$ (weak limit). We have to show that $f_n \to f$ in $L^{q(\cdot)}(\Omega)$ (strong limit). Due to corollary (1.1.18) and Lemma (1.1.20) holds

$$\begin{aligned} f_n &\rightharpoonup f \quad \text{in} \quad L^{p^*(\cdot)}(\Omega), \\ f_n &\to f \quad \text{in} \quad L^{p(\cdot)}(\Omega). \end{aligned}$$

Thus, the generalized Hölder's inequality (see [13]) implies

$$\|f_n - f\|_{s_{\theta_0}} \leq \quad C \underbrace{\|f_n - f\|_{p(\cdot)}^{1-\theta}}_{\to 0} \underbrace{\|f_n - f\|_{p^*(\cdot)}^{\theta}}_{\leq c} \to 0$$

Since $q \leq s_{\theta_0} \leq p^*$ and Ω is bounded, this implies $f_n \to f$ in $L^{q(\cdot)}(\Omega)$. This proves the theorem.

Section (1.2): Classical Operators on Variable $L^{p(\cdot)}$ Spaces

Given an open set $\Omega \subset \mathbb{R}^n$, we consider a measurable function $p : \Omega \to [1,\infty)$, $L^{p(\cdot)}(\Omega)$ denotes the set of measurable functions f on Ω such that for some $\lambda > 0$,

$$\int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx < \infty.$$

This set becomes a Banach function space when equipped with the norm

$$\|f\|_{p(\cdot),\Omega} = \inf\left\{\lambda > 0 : \int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \leq 1\right\}.$$

These spaces are referred to as variable Lebesgue spaces or, more simply, as variable $L^{p(\cdot)}$ spaces since they generalize the standard $L^{p(\cdot)}$ spaces if $p(x) = p_0$ is constant, then $L^{p(\cdot)}(\Omega)$ equals $L^{p_0}(\Omega)$. (Here and below $p(\cdot)$ instead of p to emphasize that exponent is a function and not a constant.) They have many properties in common with the standard $L^{p(\cdot)}$ spaces.

These spaces, and the corresponding variable Sobolev spaces $W^{1,p(\cdot)}(\Omega)$, are of interest in their own right, and also have applications to partial differential equations and the calculus of variations. (See, [4], [6], [8], [13], [25], [27], [59].)

A crucial step has been to show that one of the classical operators of harmonic analysis—e.g., a maximal operator, singular integrals, fractional integrals—is bounded on variable L^p space. We considered the equation of sufficient condition on the exponent function $p(\cdot)$ for given operators to be bounded: see, [1], [6], [24], [48], [49], [51].

We apply techniques from the theory of weighted norm inequalities and extrapolation to show that the boundedness of a wide variety of operators follows from the boundedness of the maximal operator on variable L^p spaces, and from known estimates on weighted Lebesgue spaces. In order to provide the foundation for stating the results, we discuss each of these ideas in turn.

In harmonic analysis, a fundamental operator is the Hardy–Littlewood maximal operator. Given a function f, we define the maximal function, Mf, by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy,$$

where the supremum is taken over all cubes containing x. It is well known that M is bounded on $L^p, 1 < p < \infty$, and it is natural to ask for which exponent functions for which exponent functions $p(\cdot)$ the maximal operator is bounded on $L^{p(\cdot)}(\Omega)$. For conciseness, define $\wp(\Omega)$ to be the set of measurable functions $p : \Omega \to [1, \infty)$ such that

$$p_- = \operatorname{ess\,inf}\{p(x) : x \in \Omega\} > 1, \;\; p_+ = \operatorname{ess\,sup}\{p(x) : x \in \Omega\} < \infty.$$

Let $\mathcal{B}(\Omega)$ be the set of $p(\cdot) \in \wp(\Omega)$ such that M is bounded on $L^{p(\cdot)}(\Omega)$.

Theorem (1.2.1)[66]: Given an open set $\Omega \subset \mathbb{R}^n$, and $p(\cdot) \in \wp(\Omega)$ suppose that $p(\cdot)$ satisfies

$$|p(x) - q(y)| \leq \frac{C}{-\log|x-y|}, \quad x, y \in \Omega, |x - y| \leq 1/2, \tag{20}$$

$$|p(x) - q(y)| \leq \frac{C}{-\log(\mathrm{e}+|x|)}, \quad x, y \in \Omega, |y| \geq |x|, \tag{21}$$

Then $p(\cdot) \in \wp(\Omega)$, that is the Hardy–Littlewood maximal operator is bounded on $L^{p(\cdot)}(\Omega)$.

Theorem (1.2.1) is independently due to Cruz-Uribe, Fiorenza and Neugebauer [36] and to Nekvinda [20]. (In fact, Nekvinda replaced (21) with a slightly more general condition.) Earlier, Diening [4] showed that (20) alone is sufficient if Ω is bounded. Examples show that the continuity conditions (20) and (21) are in some sense close to necessary: see Pick and Růžička [57] and [36]. See also the examples in [54]. The condition $p_- > 1$ is necessary for M to be bounded; see [36].

Diening [38], working in the more general setting of Musielak–Orlicz spaces, has given a necessary and sufficient condition on $p(\cdot)$ for M to be bounded on $L^{p(\cdot)}(\mathrm{R}^n)$. His exact condition is somewhat technical (see [38]).

We proofs rely on duality arguments, we will not need that the maximal operator is bounded on $L^{p(\cdot)}(\Omega)$ but on its associate space $L^{p'(\cdot)}(\Omega)$, where $p'(\cdot)$ is the conjugate exponent function defined by

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \quad x \in \Omega.$$

Since

$$|p'(x) - p'(y)| \le \frac{|p(x) - q(y)|}{(p-1)^2},$$

it follows at once that if $p(\cdot)$ satisfies (20) and (21), then so does $p'(\cdot)$—i.e., if these two conditions hold, then M is bounded on $L^{p(\cdot)}(\Omega)$ and $L^{p'(\cdot)}(\Omega)$. Furthermore, Diening's characterization of variable L^p spaces on which maximal operator is bounded has the following important consequence (see [38,]).

Theorem (1.2.2)[66]: Let $p(\cdot) \in \wp(\mathbb{R}^n)$. Then the following conditions are equivalent:

(i) $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$.

(ii) $p'(\cdot) \in \mathcal{B}(\mathbb{R}^n)$.

(iii) $p(\cdot)/q \in \mathcal{B}(\mathbb{R}^n)$ for some $1 < q < p_-$.

(iv) $(p(\cdot)/q)' \in \mathcal{B}(\mathbb{R}^n)$ for some $1 < q < p_-$.

By a weight we mean a non-negative, locally integrable function ω. There is a weights and weighted norm inequalities; here we will summarize the most important aspects, (see [40], [43]).

Central to the study of weights are the so-called A_p weights, $1 \le p \le \infty$. When $1 < p < \infty$, we say $\omega \in A_p$ if for every cube Q,

$$\left(\frac{1}{|Q|}\int_Q \omega(x)dx\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{1-p'}dx\right)^{p-1} \le C < \infty.$$

We say that $\omega \in A_1$ if $M\omega(x) \le C\omega(x)$ for a.e. x. If $1 < p < q < \infty$, then $A_p \subset A_q$ we let A_∞ denote the union of all the A_p classes, $1 \le p < \infty$. Weighted norm inequalities are generally of two types. The first is

$$\int_{\mathbb{R}^n} |Tf(x)|^{p_0} \omega(x)dx \le C \int_{\mathbb{R}^n} |f(x)|^{p_0} \omega(x)dx, \tag{22}$$

where T is some operator and $\omega \in A_{p_0}$,$1 < p_0 < \infty$. (In other words, T is defined and bounded on $L^{p_0}(\omega)$.) The constant is assumed to depend A_{p_0} constant of ω. The second type is

$$\int_{\mathbb{R}^n} |Tf(x)|^{p_0} \omega(x)dx \le C \int_{\mathbb{R}^n} |Sf(x)|^{p_0} \omega(x)dx, \tag{23}$$

where S and T are operators, $0 < p_0 < \infty, \omega \in A_\infty$, and f is such that the left–hand side in finite. The constant is assumed to depend only on the A_∞ constant of ω. Such inequalities are known for a wide variety of operators and pairs of operators. (See [40], [43].)

Corresponding to these types of inequalities are two extrapolation theorems. Associated (22) is the classical extrapolation theorem of Rubio de Francia [58] (also see [40], [43]). He proved that (22) holds for some operator T, a fixed value $p_0, 1 < p_0 < \infty$, and every weight $\omega \in A_{p_0}$, then (22) holds with p_0 replaced by any $p, 1 < p < \infty$, whenever $\omega \in A_p$. Recently, the analogous extrapolation result for inequalities of the form (23) was proved in [37]: if (23) holds for some $p_0, 1 < p_0 < \infty$, and every $\omega \in A_{p_0}$, then it holds for every $p, 1 < p < \infty$, whenever $\omega \in A_p$.

The proofs of the above extrapolation theorems depend not on the properties of the properties of operators, but rather on duality, the structure of A_p weights, and norm inequalities for the Hardy–Littlewood maximal operator. These ideas can be extended to sitting of the variable L^p spaces to yield our main result, which can be summarized as follows: If an operator T, or a pair of operator (T, S), satisfies weighted norm inequalities on the classical Lebesgue spaces, then it satisfies the corresponding inequality in a variable L^p spaces on which the maximal operators is bounded.

We will adopt the approach taken in [37]. There it was observed that since nothing is assumed about the operators involved (e.g., linearity or sublinearity), better to replace inequalities (22) and (23) with

$$\int_{\mathbb{R}^n} f(x)^{p_0} \omega(x)dx \le C \int_{\mathbb{R}^n} g(x)^{p_0} \omega(x)dx, \tag{24}$$

where the pairs (f, g) are such that the left-hand side of the inequality is finite. One important consequence of adopting this approach is that vector-valued inequalities follow immediately from extrapolation.

F will denote a family of ordered pairs of non-negative, measurable functions (f, g). Whenever we say that an inequality such as (24) holds for any $(f, g) \in F$ and $\omega \in A_q$ (for some $q, 1 < q < \infty$), we mean that it holds for any pair in F such that the left-hand side is finite, and the constant C depends only on p_0 and the A_q constant of ω.

Note that in the classical Lebesgue spaces we can work with L^p where $0 < p < 1$. (Thus, in (23) or (24) we can take $p_0 < 1$.) We would like to consider analogous spaces with variable exponents. Define $F^0(\Omega)$ to be the set of measurable functions $p: \Omega \to (0, \infty)$ such that

$$p_- = \text{ess}\inf\{p(x) : x \in \Omega\} > 0, \quad p_+ = \text{ess}\sup\{p(x) : x \in \Omega\} < \infty.$$

Given $p(\cdot) \in \wp^0(\Omega)$, we can define the Define $L^{p(\cdot)}(\Omega)$ as above. This is equivalent to defining it to the set of all functions Define f such that Define $|f|^{p_0} \in L^{P(\cdot)}(\Omega)$, where $0 < p_0 < p_-$ and $q(x) = p(x)/p_0 \in \wp(\Omega)$. We can define a quiasi-norm on these spaces by

$$\|f\|_{p(\cdot),\Omega} = \||f|^{p_0}\|_{p(\cdot),\Omega}^{1/p_0}.$$

We will not need any other properties of these spaces, so this definition will suffice for our purposes.

Theorem (1.2.3)[66]: Given a family F and an open set $\Omega \subset \mathrm{R}^n$, suppose that for some $p_0, 0 < p_0 < \infty$, and for every weight $\omega \in A_1$,

$$\int_\Omega f(x)^{p_0} \omega(x) dx \leq C_0 \int_\Omega g(x)^{p_0} \omega(x) dx \quad (f, g) \in F, \tag{25}$$

where C_0 depends only on p_0 and the A_1 constant of ω. Let $p(\cdot) \in \wp^0(\Omega)$ be such that $p_0 < p_-$, and $(p(\cdot)/p_0)' \in \mathcal{B}(\Omega)$. Then for all $(f, g) \in F$ such that $f \in L^{P(\cdot)}(\Omega)$,

$$\|f\|_{p(\cdot),\Omega} \leq C\|g\|_{p(\cdot),\Omega}, \tag{26}$$

where the constant C is independent of the pair (f, g).

We want to call attention to two features of Theorem (1.2.3). First, the conclusion (26) is an a priori estimate: that is, it holds for $(f, g) \in F$ such that $f \in L^{P(\cdot)}(\Omega)$. In practice, when applying this theorem in conjunction with inequalities of the form (22) to show that an operator is bounded on variable L^P we will usually need to work with a collection of functions f which satisfy the given weighted Lebesgue space inequality and are dense in $L^{P(\cdot)}(\Omega)$. When working with inequalities of the form (22) the final estimate will hold for a suitable family of "nice" functions.

The family F in the hypothesis of and conclusion of the same, so the goal is to find a large, reasonable family F such that (25) holds with a constant depending only on p_0 and the A_1 constant of ω.

Remark (1.2.4)[66]: In Theorem (1.2.3), (26) holds if $p(\cdot)$ satisfies (20) and (21). By Theorem (1.2.1), setting $q(x) = p(\cdot)/p_0$ we have that $p(\cdot) \in \wp(\Omega)$ and

$$|q'(x) - q'(y)| \leq \frac{|p(x) - p(y)|}{p_0(p - /p_0 - 1)^2}.$$

When $\Omega \in \mathbb{R}^n$, if $1 \leq p_0 < p_-$, then by Theorem (1.2.2) the hypothesis that $(p(\cdot)/p_0)' \in \mathcal{B}(\mathbb{R}^n)$ is equivalent to assuming that $p(\cdot) \in \mathcal{B}(\mathrm{R}^n)$. As we will see below to conclude that a variety of operators are bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ whenever Hardy-Littlewood maximal a operator is.

Using pairs of functions leads to an equivalent formulation of Theorem (1.2.3) in which the exponent p_0 does not play a role. This can be done by defining a new family F_{p_0} consisting of the pairs (f^{p_0}, g^{p_0}) with $(f,g) \in F$. Notice that in this case (26) is satisfied by F_{p_0} with $p_0 = 1$. Thus, the case $p_0 = 1$ will imply that if $1 < p_-$ and $p(\cdot)' \in \mathcal{B}(\mathrm{R}^n)$ the (26) holds. Therefore, if we define $r(x) = p(x)p_0$, we have that $r(\cdot) \in \wp^0(\Omega)$, $p_0 < r_-$, $(p(\cdot)/p_0)' \in \mathcal{B}(\mathbb{R}^n)$ and (26) holds with $r(\cdot)$ in place of $p(\cdot)$. But this is exactly the conclusion of Theorem (1.2.3).

We believe that a more general version of Theorem (1.2.3) is true, one which holds for larger classes of weights and yield inequalities in weighted variable L^p spaces. However, proving such result will require a weighted version of Theorem (1.2.1), and even the statement of such a result has eluded us. For such a weighted extrapolation result the appropriate class of weights is no longer A_1, but A_p (as in [58]) or A_∞ (as in [37]). We emphasize, though, that the class A_1, which is the smallest among the A_p classes, is the natural one to consider when attempting to prove unweighted estimates. Theorem (1.2.3) can be generalized to give "off-diagonal" results. In the classical setting the extrapolation theorem of Rubio de Francia was extended in this manner by Harboure, Macías and Segovia [46].

Theorem (1.2.5)[66]: Given a family F and an open set $\Omega \subset \mathbb{R}^n$, assume that for some p_0 and q_0, $0 < p_0 \le q_0 < \infty$, and every weight $\omega \in A_1$,

$$\left(\int_\Omega f(x)^{q_0}\omega(x)\mathrm{d}x\right)^{1/q_0} \le C_0\left(\int_\Omega g(x)^{q_0}\omega(x)^{p_0/q_{0}0}\mathrm{d}x\right)^{1/p_0} \quad (f,g) \in F. \tag{27}$$

Given $p(\cdot) \in \wp^0(\Omega)$, such that $p_0 < p_- \le p_+ < p_0q_0/(q_0 - p_0)$, define the function $q(\cdot)$ by

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{1}{p_0} - \frac{1}{q_0}, \quad x \in \Omega \tag{28}$$

If $(p(x)/q_0)' \in \mathcal{B}(\Omega)$, then for all $(f,g) \in F$ such that $f \in L^{p(\cdot)}(\Omega)$,

$$\|f\|_{q(\cdot),\Omega} \le C\|g\|_{p(\cdot),\Omega} \tag{29}$$

Corollary (1.2.6)[66]: Given a family F and an open set $\Omega \subset \mathrm{R}^n$, assume that for some p_0, $0 < p_0 < \infty$, and for every $\omega \in A_\infty$,

$$\int_\Omega f(x)^{p_0}\,\omega(x)dx \le C_0\int_\Omega g(x)^{p_0}\,\omega(x)dx, \quad (f,g) \in F. \tag{30}$$

Let $p(\cdot) \in \wp^0(\Omega)$ be such that there exists $0 < p_1 < p_-$ with $(p(\cdot)/q_0)' \in \mathcal{B}(\Omega)$. Then for all $(f,g) \in F$ such that $f \in L^{p(\cdot)}(\Omega)$,

$$\|f\|_{p(\cdot),\Omega} \le C\|g\|_{p(\cdot),\Omega} \tag{31}$$

Furthermore, for every $0 < q < \infty$ and sequence $\{f_j, g_j\}_j \subset F$,

$$\left\|\left(\sum_j (f_j)^q\right)^{1/q}\right\|_{p(\cdot),\Omega} \le C \left\|\left(\sum_j (g_j)^q\right)^{1/q}\right\|_{p(\cdot),\Omega} . \qquad (32)$$

Corollary (1.2.7)[66]: Given a family F and an open set $\Omega \subset \mathrm{R}^n$, assume that (30) holds for some $1 < p_0 < \infty$, for every $\omega \in A_{p_0}$ and for all $(f, g) \in .F$. Let $p(\cdot) \in \wp(\Omega)$ be such that there exists $1 < p_1 < p_-$ with $(p(\cdot)/p_1)' \in \mathcal{B}(\Omega)$. Then (31) holds for all $(f, g) \in F$ such that $f \in L^{p(\cdot)}(\Omega)$. Furthermore, for every $0 < q < \infty$ and sequence $\{f_j, g_j\}_j \subset F$, the vector-valued inequality (32) holds.

We give a number of examples of operators which are bounded on $L^{p(\cdot)}$. These results are immediate consequences of the above results and the theory of weighted norm inequalities. Some of these have been proved by others, but most new. We also prove vector-valued inequalities for these operators, all of which are new results. We present an application to partial differential equations: we extend the Calderón-Zygmund inequality (see [31], [44]) to solution of $\Delta u = f$ with $f \in L^{p(\cdot)}(\Omega)$. We give an application to the theory of Sobolev spaces: we show that the Calderón extension theorem (see [28], [30]) holds in variable Sobolev spaces. We prove Theorems (1.2.3) and (1.2.5). We prove it adeptly from the arguments given in [37]. We prove Corollaries (1.2.6) and (1.2.7).

We will make use of the basic properties of variable $L^{p(\cdot)}$ spaces, and will state some results as needed. For a detailed discussion of these spaces, see Kovácik and Rákosnk [13]. As we noted above, in order to emphasize that are dealing with variable exponent, we will always write $p(\cdot)$ instead of p to denote an exponent function. C will denote a positive constant whose exact value may change at each appearance.

We give a number of applications of Theorems (1.2.3) and (1.2.5), and Corollaries (1.2.6) and (1.2.7), to show that wide variety of classical operators are bounded on the variable L^p spaces. In the following applications, we will impose different conditions on the exponents $p(\cdot)$ to guarantee the corresponding estimates. In most of the cases, it will suffice to assume that $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, or in particular that $p(\cdot)$ satisfies (20) and (21). As we noted in the remarks following Theorem (1.2.3), to prove these applications we will need to use density arguments. In doing so we will use the following facts:

(i) L_c^∞, bounded functions of compact support, and C_c^∞, smooth functions of compact support, are dense $L^{p(\cdot)}(\Omega)$. See Kováčik and Rákosník [13].

(ii) If $p_+ < \infty$ and $f \in L^{p_+}(\Omega) \cap L^{p_-}(\Omega)$ then $f \in L^{p(\cdot)}(\Omega)$ this follows from the fact that $|f(x)|^{p(x)} \le |f(x)|^{p_+}\chi_{\{|f(x)|\ge 1\}} + |f(x)|^{p_-}\chi_{\{|f(x)|<1\}}$.

It is well known that for $0 < p < \infty$ and for $\omega \in A_p$,

$$\int_{\mathbb{R}^n} f(x)^p \omega(x) dx \le C \int_{\mathbb{R}^n} f(x)^p \omega(x) dx.$$

From Corollary (1.2.8) with the pairs $(Mf,|f|)$, we get vector-valued inequalities for M on $L^{p(\cdot)}$, provided there exists $1 < p_1 < p_-$ with $(p(\cdot)/p_1)' \in \mathcal{B}(\mathbb{R}^n)$; by Theorem (1.2.2), this equivalent to $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. To apply Corollary (1.2.8) we need to restrict the pairs to functions $f \in L_c^\infty$, but since these form a dense subset we get the desired estimate for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$.

Corollary (1.2.8)[66]: If $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then for all $0 < p < \infty$,

$$\left\| \left(\sum_j (Mf_j)^q \right)^{1/q} \right\|_{p(\cdot),\mathbb{R}^n} \le C \left\| \left(\sum_j |f_j|^q \right)^{1/q} \right\|_{p(\cdot),\mathbb{R}^n}.$$

From Corollary (1.2.7) we also get one of the implications of Theorem (1.2.2)[66] if $(p(\cdot)/p_1)' \in \mathcal{B}(\mathbb{R}^n)$ then $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. It is very tempting to speculate that all of Theorem (1.2.2) can be proved via extrapolation, but we have been unable to do so.

Given a measurable function f and Q, define

$$f_Q = \frac{1}{|Q|} \int_Q f(y)dy,$$

and the sharp maximal operator by

$$M^\# f(x) = \sup_{x \ni Q} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy.$$

The sharp maximal operator was introduced by Fefferman and Stein [42], who showed that for all $p, 0 < p < \infty$, and $\omega \in A_\infty$,

$$\int_{\mathbb{R}^n} Mf(x)^p\, \omega(x)dx \le C \int_{\mathbb{R}^n} M^\# f(x)^p\, \omega(x)dx$$

(Also see Journé [48].) Therefore, by Corollary (1.2.6) with the pairs $(Mf, M^\# f)$, $f \in L_c^\infty(\mathbb{R}^n)$ and by Theorem (1.2.2) we have the following result.

Corollary (1.2.9)[66]: Let $p(\cdot) \in \wp^0(\mathbb{R}^n)$ be such that there exists $0 < p_1 < p_-$ with $p(\cdot)/p_1 \in \mathcal{B}(\mathbb{R}^n)$. Then

$$\|Mf\|_{p(\cdot),\mathbb{R}^n} \le C \|M^\# f\|_{p(\cdot),\mathbb{R}^n}, \tag{33}$$

and for all $1 < q < \infty$,

$$\left\| \left(\sum_j (Mf_j)^q \right)^{1/q} \right\|_{p(\cdot),\mathbb{R}^n} \le C \left\| \left(\sum_j |f_j|^q \right)^{\frac{1}{q}} \right\|_{p(\cdot),\mathbb{R}^n}. \tag{34}$$

Given a locally integrable function K defined on $\mathbb{R}^n \backslash \{0\}$, suppose that the Fourier transform of K is bounded, and K satisfies

$$|K(x)| \leq \frac{C}{|x|^n}, \quad |\nabla K(x)| \leq \frac{C}{|x|^{n+1}}, \quad x \neq 0 \tag{35}$$

Then the singular integral operator T, defined by $Tf(x) = K * f(x)$, is a bounded operator on weighted L^p. More precisely, given $1 < p < \infty$, and $\omega \in A_p$, then

$$\int_{\mathbb{R}^n} |Tf(x)|^p \, \omega(x) \mathrm{d}x \leq C \int_{\mathbb{R}^n} |f(x)|^p \, \omega(x) \mathrm{d}x. \tag{36}$$

(For details, see [40], [43].)

From Corollary (1.2.7), we get that T is bounded on variable L^p provided there exists $1 < p_1 < p_-$ with $(p(\cdot)/p_1)' \in \mathcal{B}(\mathbb{R}^n)$; by Theorem (1.2.2) this equivalent to $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Again, to apply the corollary we need to restrict ourselves to a suitable dense of functions. We use the fact that C_c^∞ is dense in $L^{p(\cdot)}(\mathbb{R}^n)$, and the fact that if $f \in C_c^\infty$, then $Tf \in \cap_{1<p<\infty} L^p L^{p(\cdot)}(\mathbb{R}^n)$.

Corollary (1.2.10)[66]: If $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then

$$\|Tf\|_{p(\cdot),\mathbb{R}^n} \leq C\|f\|_{p(\cdot),\mathbb{R}^n}, \tag{37}$$

and for all $1 < q < \infty$,

$$\left\|\left(\sum_j |Tf_j|^q\right)^{1/q}\right\|_{p(\cdot),\mathbb{R}^n} \leq C \left\|\left(\sum_j |f_j|^q\right)^{1/q}\right\|_{p(\cdot),\mathbb{R}^n}. \tag{38}$$

We can get estimate on sets Ω in the following way: observe that (36) implies that foe any $\Omega \subset \mathbb{R}^n$ we have

$$\int_\Omega |Tf(x)|^p \, \omega(x)dx \leq \int_{\mathbb{R}^n} |Tf(x)|^p \, \omega(x)dx$$

$$\leq \int_{\mathbb{R}^n} |f(x)|^p \, \omega(x)dx = C \int_\Omega |f(x)|^p \, \omega(x)dx$$

for all f such that $\operatorname{supp}(f) \subset \Omega$ and for all $\omega \in A_p$. That, we can apply Corollary (1.2.9) on Ω in particular, if $p(\cdot) \in \mathcal{P}(\Omega)$ satisfies (20) and (21), then

$$\|Tf\|_{p(\cdot),\Omega} \leq C\|f\|_{p(\cdot),\Omega}.$$

We will use this observation below. Singular integrals satisfy another inequality due to Coifman and Fefferman [33]:

$$\int_{\mathbb{R}^n} |Tf(x)|^p \, \omega(x)dx \leq C \int_{\mathbb{R}^n} |Mf(x)|^p \, \omega(x)dx,$$

for all $1 < q < \infty$ and $\omega \in A_\infty$ and f such that the left-hand side is finite. In particular, if $\omega \in A_1 \subset A_p$, then the left-hand side is finite for all $f \in L_c^\infty(\mathbb{R}^n)$. Thus, by applying Corollary (1.2.6) we can prove the following.

Corollary (1.2.11)[66]: Let $p(\cdot) \in \wp^0(\mathbb{R}^n)$ be such that there exists $0 < p_+ < p_-$ with $p(\cdot)/p_1 \in \mathcal{B}(\mathbb{R}^n)$. Then

$$\|Tf\|_{p(\cdot),\mathbb{R}^n} \le C\|Mf\|_{p(\cdot),\mathbb{R}^n}, \tag{39}$$

and for all $0 < p < \infty$,

$$\left\|\left(\sum_j |Tf_j|^q\right)^{1/q}\right\|_{p(\cdot),\mathbb{R}^n} \le C\left\|\left(\sum_j |Mf_j|^q\right)^{1/q}\right\|_{p(\cdot),\mathbb{R}^n}. \tag{40}$$

Inequality (37) was proved by Diening and Ruzička [6] using (33) and assuming that $p(\cdot), (p(\cdot)/p_1)' \in \mathcal{B}(\mathbb{R}^n)$ for some $0 < s < 1$. More recently, Diening [38] showed that it was enough to assume $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Note that our technique provides an alternative proof which also yields vector-valued inequalities. A weighted version of (33) was proved by Kokilashvili and Samko [50].

The results can be generalized to the Calderón Zygmund operators of Coifman and Meyer. Also, the same estimates hold for T_*, the supremum of the truncated integrals. See [40], [48] for more details

Similar inequalities hold for homogeneous` singular integral operators with "rough" kernels. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, in and suppose

$$K(x) = \frac{\Omega(x/|x|)}{|x|^n} \tag{41}$$

wher $\Omega \in L^r(S^{n-1})$, for some r, $1 < r \le \infty$, and $\int_{S^{n-1}} \Omega(y)dy = 0$. Then, if $r' < p < \infty$ and $\omega \in A_{p/r'}$, inequality (36) holds. (See Duoandikoetxea [39] and Watson [62].) To apply Theorem (1.2.1) we restate these weighed norm estimates as

$$\int_{\mathbb{R}^n} \left(|Tf(x)|^{r'}\right)^s \omega(x)dx \le \int_{\mathbb{R}^n} \left(|f(x)|^{r'}\right)^s \omega(x)dx$$

for every $1 < s < \infty$ and all $\omega \in A_s$. We consider the family of pairs $\left(|Tf|^{r'}, |f|^{r'}\right)$ which satisfy the hypotheses of Corollary (1.2.8). Then $s(\cdot) \in \wp(\mathbb{R}^n)$ such that $(s(\cdot)/s_1)' \in \mathcal{B}(\mathbb{R}^n)$ for some $1 < s_1 < s_-$, we have

$$\left\||Tf|^{r'}\right\|_{p(\cdot),\mathbb{R}^n} \le C\left\||f|^{r'}\right\|_{p(\cdot),\mathbb{R}^n}.$$

By Theorem (1.2.2) the assumptions on $s(\cdot)$ are equivalent to $s(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. If we let $p(x) = s(x)r'$, then we see that T is bounded $L^{p(\cdot)}(\mathbb{R}^n)$ for all $p(\cdot)$ such that $p(\cdot)/r' \in \mathcal{B}(\mathbb{R}^n)$. In the same way can prove l^p-valued inequalities as (38) for all $r' < q < \infty$. Note in particular that all of these estimates hold if $p_- > r'$ and $p(\cdot)$ satisfies (20) and (21).

Similar inequalities also hold for Banach space valued singular integrals, since such operators satisfy weighted norm inequalities with $A_p$weights. For further details, (see [43]). We note one particular application. Let $\varphi \in L^1$ be a non-negative function such that

$$|\varphi(x-y)-\varphi(x)| \leq \frac{C|y|}{|x|^{n+1}}, \quad |x| > 2|y| > 0$$

Let $\varphi_t(x) = t^{-n}\varphi(x/t)$, and define the maximal operator M_φ by

$$M_\varphi f(x) = \sup_{t>0} |\varphi_t * f(x)|$$

If $1 < p < \infty$ and $\omega \in A_p$, then $\|M_\varphi f\|_{L^p(\omega)} \leq C\|f\|_{L^p(\omega)}$. (In the unweightet case, this result is originally due to Zo [65].) Therefore, by Corollary (1.2.7), M_φ is bounded on $L^{p(\cdot)}$ for $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. It is bounded if $p(\cdot)$ satisfies (20) and (21); this gives a positive answer to a conjecture made in [35].

Given a Calderón–Zygmund singular integral operator T, and a function $b \in$ BMO, define the commutator $[b, T]$ to be the operator

$$[b,T]f(x) = b(x)Tf(x) - T(bf)(x)$$

These operators were shown to be bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, $1 < p < \infty$ by Coifman, Rochberg and Weiss [34]. In [56] it was shown that $0 < p < \infty$ and all $\omega \in A_\infty$,

$$\int_{\mathbb{R}^n} |[b,T]f(x)|^p \omega(x)dx \leq C \int_{\mathbb{R}^n} M^2 f(x)^p \omega(x)dx, \tag{42}$$

where $M^2 = M \circ M$. Hence, if and $1 < p < \infty$ and $\omega \in A_\infty$, then $[b,T]$ is bounded on $L^p(\omega)$. Thus, we can apply Corollaries (1.2.6) and (1.2.7) and Theorem (1.2.2) to get the following

Corollary (1.2.12)[66]: Let $p(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$.

(i) If there exists $0 < p_+ < p_-$ with $p(\cdot)/p_1 \in \mathcal{B}(\mathbb{R}^n)$, then

$$\|[T,b]f\|_{p(.),\mathbb{R}^n} \leq C\|M^2 f\|_{p(.),\mathbb{R}^n},$$

and for all $0 < q < \infty$,

$$\left\|\left(\sum_j \left|[T,b]f_j\right|^q\right)^{1/q}\right\|_{p(\cdot),\mathbb{R}^n} \leq C \left\|\left(\sum_j \left|M^2 f_j\right|^q\right)^{\frac{1}{q}}\right\|_{p(\cdot),\mathbb{R}^n}.$$

(ii) If $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then

$$\|[T,b]f\|_{p(\cdot),\mathbb{R}^n} \leq C\|f\|_{p(\cdot),\mathbb{R}^n},$$

and for all $0 < q < \infty$,

$$\left\|\left(\sum_j \left|[T,b]f_j\right|^q\right)^{1/q}\right\|_{p(\cdot),\mathbb{R}^n} \leq C \left\|\left(\sum_j \left|f_j\right|^q\right)^{\frac{1}{q}}\right\|_{p(\cdot),\mathbb{R}^n}.$$

The boundedness of commutators on variable L^p spaces was proved by Karlovich and Lerner [49]. Given a bounded function m, define the operator T_m, (initially on $C_c^\infty(\mathbb{R}^n)$ by $\widehat{T_m f} = m\hat{f}$. The function m is referred to as a multiplier. Here we two important results: the multiplier theorems of Marcinkiewicz and Hörmander.

On the real line, if m has uniformly bounded variation on each dyadic interval in $\mathbb{R}^n$ then for $1 < p < \infty$ and $\omega \in A_p$,

$$\int_{\mathbb{R}} |T_m f(x)|^p \omega(x) dx \leq C \int_{\mathbb{R}} |f(x)|^p \omega(x) dx. \tag{43}$$

(See Kurtz [52].) Therefore, by Corollary (1.2.7), if $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$,

$$\|T_m f\|_{p(\cdot),\mathbb{R}^n} \leq C\|f\|_{p(\cdot),\mathbb{R}^n};$$

we also get the corresponding vector-valued inequalities with $1 < q < \infty$. In dimensions (i.e., $n \geq 2$) let $k = [n/2] + 1$ and suppose that m satisfies $\left|D^\beta m(x)\right| \leq C|x|^{-|\beta|}$ for $x \neq 0$ and every multi-index β with $|\beta| \leq k$. If $n/k < p < \infty$ and $\omega \in A_{pk/n}$ then T_m is bounded on $L^p(\omega)$. (See Kurtz and Wheeden [53].) Proceeding as in the case of the singular integral operators with "rough" kernels we obtain that if $p(\cdot)/(n/k) \in \mathcal{B}(\mathbb{R}^n)$, then

$$\|T_m f\|_{p(\cdot),\mathbb{R}^n} \leq C\|f\|_{p(\cdot),\mathbb{R}^n},$$

with constant C independent of $f \in C_c^\infty(\mathbb{R}^n)$. We also get l^q-valued inequalities with $n/k < p < \infty$ in the same way.

Weighted inequalities also hold for Bochner–Riesz multipliers, so from these we can deduce results on variable L^p paces. See [40].

Let φ be a Schwartz function such that $\int \varphi(x)dx = 0$, and for $t > 0$ let $\varphi_t(x) = t^{-n}\varphi(x/t)$. Given a locally integrable function f, we define two closely related functions: the area integral,

$$S_\varphi f(x) = \left(\int_{|x-y|<t} |\varphi_t * f(y)|^2 \frac{dt\, dy}{t^{n+1}} \mathrm{d}x\right)^{1/2},$$

and for $1 < \lambda < \infty$ the Littlewood–Paley function

$$g_\lambda^* f(x) = \left(\int_0^\infty \int_{\mathbb{R}^n} |\varphi_t * f(y)|^2 \left(\frac{t}{t+|x-y|}\right)^{n\lambda} \frac{dt\, dy}{t^{n+1}} \mathrm{d}x\right)^{1/2},$$

In the classical case, we take φ to be the derivative of the Poisson kernel.

Given $p,\ 1 < p < \infty$, and $\omega \in A_p$, the area integral is bounded on $L^p(\omega)$ in the classical case, this due to Gundy and Wheeden [45]; in the general case it is due to Strömberg and Torchinsky [61]. Therefore, for all $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$,

$$\left\|S_\varphi f\right\|_{p(\cdot),\mathbb{R}^n} \leq C\|f\|_{p(\cdot),\mathbb{R}^n}.$$

The same inequality is true for g_λ^* if $\lambda \geq 2$. If $1 < \lambda < 2$, then for $\lambda/2 < p < \infty$ and $\omega \in A_{\lambda p/2}$, g_λ^* is bounded on $L^p(\omega)$. In the classical case, this due to Muckenhoupt and Wheeden [55]; in general case it due to Strömberg and Torchinsky [61]. Therefore, for all $p(\cdot)/(2/\lambda) \in \mathcal{B}(\mathbb{R}^n)$,

$$\|g_\lambda^* f\|_{p(\cdot),\mathbb{R}^n} \leq C\|f\|_{p(\cdot)\mathbb{R}^n},$$

with constant C independent of $f \in C_c^\infty(\mathbb{R}^n)$. For both kinds of square functions we also get the corresponding vector-valued inequalities.

Given $1 < \alpha < n$ define the fractional integral operator I_α (also known as the Riesz potential) by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

Define the associated fractional maximal operator M_α, by

$$M_\alpha f(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha/n}} \int_{\mathbb{R}^n} |f(y)| dy.$$

Both operators satisfy weighted inequalities. To state them, we need a different class of weights: given p, q such that $1 < p < n/\alpha$ and

$$\frac{1}{p} - \frac{1}{p} = \frac{\alpha}{n},$$

we say that $\omega \in A_{p,q}$ if for all cubes Q,

$$\frac{1}{|Q|}\int_Q \omega(x)dy \left(\frac{1}{|Q|}\int_Q \omega(x)^{-p'/q} dy\right)^{q/p'} \leq C < \infty.$$

Note that this is equivalent to $\omega \in A_r,\ r = 1 + q/p'$, so in particular, if $\omega \in A_1$, then $\omega \in A_{p,q}$. Muckenhoupt and Wheeden [55] showed that if $\omega \in A_{p,q}$ then

$$\left(\int_{\mathbb{R}^n} |I_\alpha f(x)|^q \omega(x) dx\right)^{1/p} \leq C \left(\int_{\mathbb{R}^n} |f(x)|^{p/q} \omega(x) dx\right)^{1/p},$$

$$\left(\int_{\mathbb{R}^n} |M_\alpha f(x)|^q \omega(x) dx\right)^{1/p} \leq C \left(\int_{\mathbb{R}^n} |f(x)|^{p/q} \omega(x) dx\right)^{1/p}.$$

(These results are usually stated with the class $A_{p,q}$ defined slightly differently, with ω replaced by ω^q. Though non-standard, is better for purposes.)

As in these estimates hold with the integrals restricted to any $\Omega \subset \mathbb{R}^n$. Thus Theorems (1.2.7) and (1.2.2) immediately yield the following results in variable L^p spaces.

Corollary (1.2.13)[66]: Let $p(\cdot),\ q(\cdot) \in \wp(\Omega)$ be such that $p_+ < n/\alpha$ and

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}, \quad x \in \Omega.$$

If there exists $q_0,\ n/(n-\alpha) < q_0 < \infty$, such that $p(\cdot)/q_0, \in \mathcal{B}(\mathbb{R}^n)$, then

$$\|I_\alpha f\|_{q(\cdot),\Omega} \leq \|f\|_{p(\cdot),\Omega}. \tag{44}$$

and

$$\|M_\alpha f\|_{q(\cdot),\Omega} \leq \|f\|_{p(\cdot),\Omega}. \tag{45}$$

Corollary (1.2.13) follows automatically from Theorem (1.2.5) applied to the pairs $(|I_\alpha f|, |f|)$ and $(|M_\alpha f|, |f|)$, since the estimates of Muckenhoupt and Wheeden above give (27) for all $1 < p_0 < n/\alpha$ and $n/(n-\alpha) < q_0 < \infty$ with

$$\frac{1}{p_0} - \frac{1}{q_0} = \frac{\alpha}{n}.$$

When $\Omega \subset \mathbb{R}^n$, the condition on $q(\cdot)$ sis equivalent to say that $q(\cdot)(n-\alpha)/n \in \mathcal{B}(\mathbb{R}^n)$, then

$$\frac{q(x)}{n/(n-\alpha)} = \frac{q(x)}{q_0} - \frac{q_0}{n/(n-\alpha)} \in \mathcal{B}(\mathbb{R}^n),$$

since the second ratio is greater than one. (Given $r(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and $\lambda > 1$, then by Jensen's inequality, $r(\cdot)\lambda \in \mathcal{B}(\mathbb{R}^n)$.)

By Theorem (1.2.2), if $q(\cdot)(n-\alpha)/n \in \mathcal{B}(\mathbb{R}^n)$ then there is $\lambda > 1$ such that $q(\cdot)(n-\alpha)/(n\lambda) \in \mathcal{B}(\mathbb{R}^n)$. Taking $q_0 = n\lambda/(n-\alpha)$ we have that $q_0 > n/(n-\alpha)$ and $p(\cdot)/q_0 \in \mathcal{B}(\mathbb{R}^n)$ as desired.

Inequality (44) extends several earlier results. Samko [24] proved (44) assuming that Ω is bounded, $p(\cdot)$ satisfies (20), and the maximal operator is bounded. (Note that given Theorem (1.2.1), his second hypothesis implies his third.) Diening [1] proved it on unbounded domains with (20) replaced by the stronger hypothesis that $p(\cdot)$ is constant outside of a large ball. kokilashvili and Samko [89] proved it on $\mathbb{R}^n$ with $L^{q(\cdot)}$ replaced by a certain weighted variable L^p spaces. (They actually consider a more general operator $I_{\alpha(\cdot)}$ where the constant α in the definition of I_α is replaced by a function $\alpha(\cdot)$.) Implicit in the results the are norm inequalities for M_α in the variable L^p spaces, since $M_\alpha f(x) \le C I_\alpha(|f|)(x)$. This is made explicit by Kokilashvili and Samko [51].

Inequality (45) was proved directly by Capone, Cruz-Uribe and Fiorenza [32]; as in an application they used it to prove (44) and to extend the Sobolev embedding theorem to variable L^pspaces. (Other authors have considered this question; see [32].) We consider the behavior of the solution of Poisson's equation,

$$\Delta u f(x) = f(x), \qquad \text{a.e. } x \in \Omega,$$

when $f \in L^{p(\cdot)}(\Omega)$, $p(\cdot) \in \wp(\Omega)$. We restrict ourselves to the $\Omega \subset \mathbb{R}^n$, $n \ge 3$.

We begin with a few definitions and a lemma. Given $p(\cdot) \in \wp(\Omega)$ and a natural number, define the variable Sobolev space $W^{k,p(\cdot)}(\Omega)$ to be the set of all functions $f \in L^{p(\cdot)}(\Omega)$ such that

$$\sum_{|\alpha| \le k} \|D^\alpha\|_{p(\cdot),\Omega} < +\infty.$$

Where the derivatives are understood in the sense of distributions.

Given a function f which is twice differentiable (in the weak sense), we define for $i = 1, 2,$

$$D^i f = \left(\sum_{|\alpha| \le k} (D^\alpha f)^2 \right)^{1/2}.$$

We need the following auxiliary result whose proof can be found in [13].

Lemma (1.2.14)[66]: If $\Omega \subset \mathbb{R}^n$ is bounded domain, and if $p(\cdot), q(\cdot) \in \wp(\Omega)$ are such that $p(x) \le q(x),\ x \in \Omega$, then $\|f\|_{q(\cdot),\Omega} \le (1 + |\Omega|)\|f\|_{q(\cdot),\Omega}$.

Theorem (1.2.15)[66]: Given an open set $\Omega \subset \mathbb{R}^n$, $n \ge 3$, suppose $p(\cdot) \in \wp(\Omega)$ with $p_+ < n/2$ satisfies (20) and (21). $f \in L^{p(.)}(\Omega)$, then there exists a function $u \in L^{q(.)}(\Omega)$, where

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{2}{n}, \quad x \in \Omega, \tag{46}$$

such that

$$\Delta u f(x) = f(x), \qquad \text{a.e. } x \in \Omega. \tag{47}$$

Furthermore

$$\|D^2 u\|_{q(.),\Omega} \le C\|f\|_{p(.),\Omega}, \tag{48}$$

$$\|D^1 u\|_{q(\cdot),\Omega} \le C\|f\|_{p(\cdot),\Omega}, \tag{49}$$

$$\|u\|_{q(\cdot),\Omega} \le C\|f\|_{p(\cdot),\Omega}. \tag{50}$$

where

$$\frac{1}{p(x)} - \frac{1}{r(x)} = \frac{1}{n}.$$

if Ω is bounded, then $u \in W^{2,p(\cdot)}(\Omega)$.

Proof. We proof roughly follows the proof in the setting of Lebesgue spaces given by Gilbarg and Trudinger [44], but also uses this result in key steps.

Fix $f \in L^{p(\cdot)}(\Omega)$; without loss of generality we may assume that $\|f\|_{p(\cdot),\Omega} = 1$. Decompose f as

$$f = f_1 + f_2 = f\chi_{\{x:|f(x)|>1\}} + f\chi_{\{x:|f(x)|\le 1\}}.$$

Note that $|f_i(x)| \le |f(x)|$ and so$\|f_i\|_{p(.),\Omega} \le 1$. Further, we have $f_i \in L^{p_-}(\Omega)$ and $f_i \in L^{p_-}(\Omega)$ since, by the definition of the norm in $L^{p(\cdot)}(\Omega)$ and since $\|f\|_{p(\cdot),\Omega} = 1$,

$$\int_\Omega f_1(x)^{p_-} dx = \int_{\{x:\in\Omega|f(x)|>1\}} |f(x)|^{p_-} dx \le \int_\Omega |f(x)|^{p(x)} dx \le 1,$$

$$\int_\Omega f_2(x)^{p_-} dx = \int_{\{x:\in\Omega|f(x)|>1\}} |f(x)|^{p_-} dx \le \int_\Omega |f(x)|^{p(x)} dx \le 1,$$

Thus, we can solve Poisson's equation with f_1 and f_2 (see [44]): define

$$u_1(x) = (\Gamma * f_1)(x), \qquad u_2(x) = (\Gamma * f_1)(x)$$

where Γ is the Newtonian potential,

$$\Gamma(x) = \frac{1}{n(2-n)\omega_n}|x|^{2-n}$$

and ω_n is the volume of the unit ball in $\mathbb{R}^n$. Since p_- and p_+ also satisfy (46), by the Caldero´n–Zygmund inequality on classical Lebesgue spaces, $u_1 \in L^{q_-}(\Omega)$. Similarly, since p_- and p_+ satisfy (46), $u_2 \in L^{q_+}(\Omega)$. Let $u = u_1 + u_2$; then $u \in L^{q_-}(\Omega) + L^{q_+}(\Omega)$. Since $u_1$and u_2 are solutions of Poisson's equation,

$$\Delta u f(x) = \Delta u_1 f(x) + \Delta u_1 f(x) = f_1(x) + f_2(x) = f(x), \quad \text{a.e. } x \in \Omega.$$

We show that $u \in L^{q(\cdot)}(\Omega)$ and that (50) holds: by inequality (44),

$$\begin{aligned} \|f\|_{q(\cdot),\Omega} &\leq \|u_1\|_{q(\cdot),\Omega} + \|u_2\|_{q(\cdot),\Omega} \\ &= \frac{1}{n(2-n)\omega_n}\left(\|I_1 u_1\|_{q(\cdot),\Omega} + \|I_2 u_2\|_{q(\cdot),\Omega}\right) \\ &\leq \left(\|f_1\|_{p(\cdot),\Omega} + \|f_2\|_{p(\cdot),\Omega}\right) \\ &\leq C = C\|f\|_{p(.),\Omega}; \end{aligned}$$

the last equality holds since $\|f\|_{p(\cdot),\Omega} = 1$.

Similarly, a direct computation shows that for any multi-index $\alpha, |\alpha| = 1$,

$$|D^\alpha \Gamma(x)| \leq \frac{1}{n\omega_n}|x|^{1-n}$$

Therefore,

$$\begin{aligned} |D^\alpha u(x)| &\leq |D^\alpha(\Gamma * f_1)(x)| + |D^\alpha(\Gamma * f_2)(x)| = \\ &= |(D^\alpha\Gamma * f_1)(x)| + |(D^\alpha\Gamma * f_2)(x)| \\ &\leq \frac{1}{n\,\omega_n}\left(I_1(|f_1|)(x) + I_2(|f_2|)(x)\right). \end{aligned}$$

So again, by inequality (44) we get

$$\|D^\alpha u\|_{r(\cdot),\Omega} \leq C\left(\|f_1\|_{p(\cdot),\Omega} + \|f_2\|_{p(\cdot),\Omega}\right) \leq C,$$

which yields inequality (49).

Given a multi-index $\alpha, |\alpha| = 2$, another computation shows that $D^\alpha\Gamma$ is singular convolution kernels which satisfies (35). Therefore the operator

$$T_\alpha g(x)(D^\alpha\Gamma * g)(x) = D^\alpha(\Gamma * g)(x)$$

is singular integral operator, and as before (48) follows from inequality (37) applied to f_1 and f_2.

If Ω is bounded, since $p(x) \leq q(x)$ and $p(x) \leq r(x),\ x \in \Omega$, by Lemma (1.2.15) we have that $u \in W^{2,p(\cdot)}(\Omega)$.

We could have worked directly with f. We done so, however, we would have had to check that all the integrals appearing were absolutely convergent. The advantage of decomposing $f$as $f_1 + f_2$ is that we did not need to pay attention to this since $f_1 \in L^{q_-}(\Omega),\ f_2 \in L^{q_+}(\Omega)$.

We also want to stress that u_1 and u_2, as solutions of Poisson's equation with $f_1 \in L^{q_-}(\Omega)$ and $f_2 \in L^{q_+}(\Omega)$, satisfy Lebesgue space estimates. For instance, as noted above, $u \in L^{q_-}(\Omega) + L^{q_+}(\Omega)$. We have actually proved more, since $L^{q(\cdot)}(\Omega)$ is smaller space. Similar remarks hold for the first and second derivatives of u.

We state and prove the Calderón extension theorem for variable Sobolev spaces. We proof follows closely the proof of the result in the classical set-ting; see, for example, ℝ. Adams [28] or Calderón [30]. We give two definitions and a lemma.

Definition (1.2.16)[66]: Given a point $x \in \mathbb{R}^n$, a finite cone with vertex at x, C_x, is a set of the form

$$C_x = B_1 \cap \{x + \lambda(y - x) : y \in B_2,\ \lambda > 0\},$$

where B_1 is an open ball centered at x, and B_2 is an open ball which does not contain x.

Definition (1.2.17)[66]: An open set $\Omega \in \mathbb{R}^n$ has uniform cone property if there exists a finite collection of open sets $\{U_j\}$ (not necessarily bounded) and an associated collection $\{C_j\}$ of finite cones such that the following hold:

(i) there exists $\delta > 0$ such that

$$\Omega_\delta = \{x \in \Omega : \text{dist}\,(x, \partial\Omega) < \delta\} \subset \bigcup_j U_j\,;$$

(ii) for every index j and every $x \in \Omega \cap U_j, x + C_j \subset \Omega$.

An example of a set Ω with the uniform cone property is any bounded set whose boundary is locally Lipschitz (See Adams [28].)

In giving extension theorems for variable L^p spaces, we must show worry about extending the exponent function $p(\cdot)$. The following result shows that this is always possible, provided that $p(\cdot)$ satisfies (20) and (21).

Lemma (1.2.18)[66]: Given an open set $\Omega \in \mathbb{R}^n$ and $p(\cdot) \in \wp(\Omega)$ such that (20) and (21) hold, there exists a function $\tilde{p}(\cdot) \in \wp(\mathbb{R}^n)$ such that:

(i) $\tilde{p}$ satisfies (20) and (21);

(ii) $\tilde{p}(x) = p(x), x \in \Omega$;

(iii) $\tilde{p}_- = p_-$ and $\tilde{p}_+ = p_+$.

Remark (1.2.19)[66]: Diening [1] proved an extension theorem for exponent $p(\cdot)$ which satisfy (20), provided that Ω is bounded and has Lipschitz boundary. It would be interesting to determine if every exponent $p(\cdot) \in \mathcal{B}(\Omega)$ can be extended to an exponent function in $\mathcal{B}(\mathbb{R}^n)$.

Proof. Since $p(\cdot)$ is bounded and uniformly continuous, by a well-known result it extends to a continuous function on $\overline{\Omega}$. Straightforward limiting arguments show that this extension satisfies (i), (ii) and (iii).

The extension of $p(\cdot)$ on $\overline{\Omega}$ to $\tilde{p}(x)$ defined on all of $\mathbb{R}^n$ follows from a construction due to Whitney [63] and described in detail in Stein [60]. For ease of reference, we will follow Stein's notation. We first consider the case when $\overline{\Omega}$ is unbounded; the case when Ω is bounded is simpler and will be sketched below.

When $\overline{\Omega}$ is unbounded, (21) is equivalent to the existence of a constant $p_\infty, p_- \le p_\infty \le p_+$, such that for all $x \in \overline{\Omega}$,

$$|p(x) - p_\infty| \le \frac{C}{\log(e + |x|)}.$$

Define a new function $r(\cdot)$ by $r(x) = p(x) - p_\infty$. Then $r(\cdot)$ is still bounded (though no longer necessarily positive), still satisfies

$$|r(x)| \le \frac{C}{\log(e + |x|)}. \qquad ((51)$$

We will extend r to all of $\mathbb{R}^n$. If we define $\omega(t) = 1/\log(e/2t)$, $0 < t \le 1/2$, and $\omega(t) = 1$ for $t \ge 1/2$, then a straightforward calculation shows that $\omega(2t) \le C\omega(t)$. Further, since $\log(e/2t) \approx \log(1/t)$, $0 < t \le 1/2$, and since r is bounded, $|r(x) - r(y)| \le C\omega|x - y|$ for all $x, y \in \bar{\Omega}$. Therefore, in Stein [60, p. 175], there exists a function $\tilde{r}(\cdot)$ Satisfies (20). For $x \in \mathbb{R}^n \backslash \bar{\Omega}, \tilde{r}(x)$ is defined by sum

$$\tilde{r}(x) = \sum_k r(p_k)\varphi_k^*(x),$$

Where $\{Q_k\}$ are the cubes of the Whitney decomposition of $\mathbb{R}^n \backslash \bar{\Omega}, \{\varphi_k^*\}$ is the partition of unity subordinate to this decomposition, and each point $p_k \in \bar{\Omega}$ is such that $(p_k, Q_k) = \operatorname{dist}(\bar{\Omega}, Q_k)$.

It follows immediately from this definition that for all $x \in \mathbb{R}^n$, $r_- \le \tilde{r}(x) \le r_+$. However, $\tilde{r}(x)$ need not satisfies (51) so we must modify it slightly. To do so we need the following observation: if f_1, f_2 are functions and $\max(f_1, f_2)$ satisfies the same inequality. The proof of this observation consists of a number of very similar cases. For instance, suppose $\min(f_1(x), f_2(x)) = f_1(x)$ and $\min(f_1(y), f_2(y)) = f_1(y)$. Then

$$f_1(x) - f_2(y) \le f_2(x) - f_2(y) \le C\omega(|x - y|),$$
$$f_2(x) - f_1(y) \le f_1(x) - f_1(y) \le C\omega(|x - y|).$$

Hence,

$$|\min(f_1(x), f_2(x)) - \min(f_1(y), f_2(y))| = |f_1(x) - f_2(y)| \le C\omega(|x - y|)$$

It follows immediately from this observation that

$$s(x) = \max(\min(\tilde{r}(x), C/\log(e + |x|)), -C/\log(e + |x|))$$

satisfies (20) and (51). Therefore, if we define

$$\tilde{p}(x) = s(x) + p_\infty$$

Then (i), (ii) and (iii) hold.

If Ω is bounded, we define $r(x) = p(x) - p_+$ and repeat the above argument essentially without change.

Theorem (1.2.20)[66]: Given an open set $\Omega \subset \mathbb{R}^n$ which has the uniform cone property, and given $p(\cdot) \in \wp(\Omega)$ such that (20) and (21) hold, then for any natural number k there exists an extension operator

$$E_k : W^{k,p(\cdot)}(\Omega) \to W^{k,p(\cdot)}(\mathbb{R}^n),$$

such that $E_k u(x) = u(x), a.e.\ x \in \Omega$, and

$$\|E_k u\|_{p(\cdot),\mathbb{R}^n} \leq C(p(\cdot), k, \Omega)\|u\|_{p(\cdot),\Omega}.$$

The proof of Theorem (1.2.20) in variable Sobolev spaces is nearly identical to that in the classical setting. (See Adams [28].) The proof, beyond calculations, requires the following facts hypotheses insure are true

(i) By Lemma (1.2.18), $p(\cdot)$ immediately extends to an exponent function on $\mathbb{R}^n$.

(ii) Functions in $C^\infty(\Omega)$ are dense in $W^{k,p(\cdot)}(\Omega)$. By hypotheses, the maximal operator is bounded on $L^{p(\cdot)}(\Omega)$, and the density of $C^\infty(\Omega)$ follows from this by the standard argument (cf. Ziemer [64]). For more details, see Diening [4] or Cruz-Uribe and Fiorenza [35].

(iii) If φ is smooth function on $\mathbb{R}^n\backslash\{0\}$ with compact support, and if there exists $\varepsilon > 0$ such that on $B_\varepsilon(0), \varphi$ is a homogeneous function of degree $k, k > -n$, then $\|\varphi * f\|_{p(.),\Omega} \leq C(p(\cdot), \varphi)\|f\|_{p(.),\Omega}$ this again follows from the fact that the maximal operator is bounded on $L^{p(.)}(\Omega)$ and from the well-known inequality $|\varphi * f(x)| \leq CMf(x)$. For more details, see Cruz-Uribe and Fiorenza [35].

(iv) Singular integral operators with kernels of the form

$$k(x) = \frac{G(x)}{|x|^n}$$

where G is bounded on $\mathbb{R}^n\backslash\{0\}$, has compact support, is homogenous of degree zero on $B_R\backslash\{0\}$ for some $R > 0$, and has $\int_{S_R} G\,dx = 0$, are bounded on $L^{p(\cdot)}(\Omega)$. Such kernels are essentially the same as those given by (41), and as discussed above, our hypotheses imply that they are bounded.

If $p(\cdot)$satisfies (20), then $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{k,p(\cdot)}(\mathbb{R}^n)$. (See [23], [35].) Hence, if the hypotheses of Theorem (1.2.20) hold, then it follows immediately that the set $\{u_{\chi\Omega} : u \in C_c^\infty(\mathbb{R}^n)\}$ is dense in $W^{k,p(\cdot)}(\Omega)$. However this this result is true under much weaker hypotheses; see [8], [25], [28], [41], [47], for details.

Since Theorem (1.2.3) is particular case of Theorem (1.2.5) with $p_0 = q_0$, it suffices to prove the second result.

We need two facts about variable $L^{p(\cdot)}$ spaces. First, if $p(x), q(x) \in \wp^0(\Omega)$ and $p(x)/q(x) = r$, then it follows from the definition of the norm that

$$\|f\|_{p(\cdot),\Omega}^r \leq \||f|^r\|_{q(\cdot),\Omega}. \tag{52}$$

Second, given $p(x) \in \wp(\Omega)$, we have the we have the generalized Hӧlder's inequality

$$\int_\Omega |f(x)g(x)|dx \leq \left(1 + \frac{1}{p_-} - \frac{1}{p_+}\right) \leq \|f\|_{p(.),\Omega}\|g\|_{p'(.),\Omega}, \tag{53}$$

and the "duality" relationship

$$\|f\|_{p(\cdot),\Omega} \leq \sup_{g} \left| \int_{\Omega} f(x)g(x)dx \right| \leq \left(1 + \frac{1}{p_-} - \frac{1}{p_+}\right) \|f\|_{p(\cdot),\Omega}, \quad (54)$$

where the supremum is taken over all $g \in L^{p'(\cdot)}(\Omega)$ such that $\|g\|_{p'(.),\Omega} = 1$. For proofs of these results, see Kovácik and Rákosník [13].

The proof of Theorem (1.2.5) begins with version of a construction due to Rubio de Francia [58] (also see [37], [43]). Fix $p(\cdot) \in \wp^0(\Omega)$ such that $p_- > p_0$, and let $\bar{p}(x) = p(x)/p_0$. Define as in (173), and let $\bar{q}(x) = q(x)/q_0$. By assumption, the maximal operator is bounded on $L^{\bar{p}'(\cdot)}(\Omega)$, so there exists a positive constant B such that

$$\|Mf\|_{\bar{p}'(.),\Omega} \leq B\|f\|_{\bar{p}'(.),\Omega}.$$

Define a new operator $\mathcal{R}$ on $L^{\bar{p}'(\cdot)}(\Omega)$ by

$$\mathcal{R}h(x) = \sum_{k=0}^{\infty} \frac{M^k h(x)}{2^k\, B^k},$$

where, for $k \geq 1, M^k = M \circ M \circ \dots \dots M$ denotes k iterations of the maximal operator, and M° is the identity operator. It follows immediately from this definition that:

(i) if h is non-negative, $h(x) \leq \mathcal{R}h(x)$;

(ii) $\|\mathcal{R}h\|_{\bar{p}'(\cdot),\Omega} \leq 2\|h\|_{\bar{p}'(\cdot),\Omega}$;

(iii) For every $x \in \Omega$, $M(\mathcal{R}h)(x) \leq 2Bh(x)$, so $\mathcal{R}h \in A_1$ with an A_1 constant that does not depend on h.

We can argue as follows: by (52) and (54),

$$\|f\|_{\bar{p}(\cdot),\Omega}^{q_0} \leq 2\|f^{q_0}\|_{\bar{p}(\cdot),\Omega} \leq \sup \int_{\Omega} f(x)^{q_0} h(x)dx,$$

Where the supremum is taken over all non-negative $h \in L^{\bar{p}'(\cdot)}(\Omega)$ with $\|h\|_{\bar{p}'(\cdot),\Omega} = 1$. Fix any function$h$; it will suffice to show that

$$\int_{\Omega} f(x)^{q_0} h(x)dx \leq C\|g\|_{\bar{p}(\cdot),\Omega}^{q_0}$$

with the constant C independent of h. First note that by (i) above we have that

$$\int_{\Omega} f(x)^{q_0} h(x)dx \leq \int_{\Omega} f(x)^{q_0} \mathcal{R}h(x)dx \quad (55)$$

By (53), (ii), and since $f \in L^{q(\cdot)}(\Omega)$,

$$\int_{\Omega} f(x)^{q_0} \mathcal{R}h(x)dx \leq \|f^{q_0}\|_{\bar{q}(\cdot),\Omega}, \|\mathcal{R}h\|_{\bar{q}'(\cdot),\Omega}$$

$$\leq C\|f\|_{q(\cdot),\Omega}^{q_0}, \|h\|_{\bar{q}'(\cdot),\Omega}$$

$$\leq C\|f\|_{q(\cdot),\Omega}^{q_0} < \infty.$$

Therefore we can apply (27) to the right-hand side of (55) and again apply (53), this time with exponent $\overline{p}(\cdot)$:

$$\int_\Omega f(x)^{q_0}\mathcal{R}h(x)dx \leq C\left(\int_\Omega g(x)^{p_0}\mathcal{R}h(x)^{p_0/q_0}dx\right)^{q_0/p_0}$$
$$\leq C\|g^{p_0}\|_{\overline{p}(.),\Omega}^{p_0/q_0}\left\|(\mathcal{R}h)^{p_0/q_0}\right\|_{\overline{p}'(.),\Omega}^{q_0/p_0}$$
$$= C\|g\|_{p(.),\Omega}^{q_0}\left\|(\mathcal{R}h)^{p_0/q_0}\right\|_{\overline{p}'(.),\Omega}^{q_0/p_0}.$$

To complete the proof, we need to show that $\left\|(\mathcal{R}h)^{p_0/q_0}\right\|_{\overline{p}'(\cdot),\Omega}^{q_0/p_0}$ is bounded by a constant independent of h. But it follows from (24) that for all $x \in \Omega$,

$$\overline{p}'(x) = \frac{p(x)}{p(x)-p_0} = \frac{q_0}{p_0}\frac{q(x)}{q(x)-p_0} = \frac{p_0}{p_0}\overline{p}'(\cdot).$$

Therefore,

$$\left\|(\mathcal{R}h)^{p_0/q_0}\right\|_{\overline{p}'(\cdot),\Omega}^{q_0/p_0} = \|\mathcal{R}h\|_{\overline{q}'(.),\Omega} \leq \|h\|_{\overline{q}'(\cdot),\Omega} = C.$$

This completes the proof.

The proof of Corollaries (1.2.6) and (1.2.7) require the more general version the extrapolation theorem discussed in the introduction.

Theorem (1.2.21)[66]: Given a family Ƒ and an open set $\Omega \subset \mathbb{R}^n$, assume that for some p_0, $0 < p_0 < \infty$ and for every $\omega \in A_\infty$,

$$\int_\Omega f(x)^{p_0}\omega(x)dx \leq C_0\int_\Omega g(x)^{p_0}\omega(x)dx, \quad (f,g) \in Ƒ. \qquad ((56))$$

Then for all $0 < p < \infty$ and $\omega \in A_\infty$,

$$\int_\Omega f(x)^{p_0}\omega(x)dx \leq C_0\int_\Omega g(x)^{p_0}\omega(x)dx, \quad (f,g) \in Ƒ. \qquad ((57)$$

Furthermore, for every $0 < p, q < \infty$, $\omega \in A_\infty$, and sequence $\{(f_j, g_j)\}_j \subset Ƒ$,

$$\left\|\left(\sum_j (f_j)^q\right)^{1/q}\right\|_{L^p(\omega,\Omega)} \leq C\left\|\left(\sum_j (g_j)^q\right)^{1/q}\right\|_{L^p(\omega,\Omega)}. \qquad (58)$$

Theorem (1.2.22)[66]: Given family Ƒ and an open set $\Omega \subset \mathbb{R}^n$, assume that for that some p_0, $1 < p < \infty$, and $\omega \in A_p$, (57) holds. Furthermore, for every $1 < p, q < \infty$, $\omega \in A_p$, and sequence $\{(f_j, g_j)\}_j \subset Ƒ$, (58) holds.

Theorem (1.2.21) is proved in [37]. The original statement of Theorem (1.2.22) is only for pairs of the form$(|Tf|, f)$, and does not include the vector-valued estimate (58). (See [40], [43], [58].) However, an examination of the proofs shows that they hold without change when applied to pairs $(f, g) \in \mathcal{F}$. Furthermore, as we noted before, this approach immediately yields the vector-valued inequalities: given a family $\mathcal{F}$ and $1 < q < \infty$, define the new family $\mathcal{F}_q$ to consist of the pairs (F_q, G_q), where

$$F_q(x) = \left(\sum_j (f_j)^q\right)^{1/q}, \quad G_q(x) = \left(\sum_j (f_j)^q\right)^{1/q}, \quad \{(f_j, g_j)\}_j \subset \mathcal{F}.$$

Clearly, inequality (56) holds for $\mathcal{F}_q$ when $p_0 = q$, so by extrapolation we get (58).

Corollary (1.2.7) follows immediately from Theorems (1.2.3) and (1.2.21). Since (30) holds for some p_0, by Theorem (1.2.21) it holds for all $1 < q < \infty$ and for all $\omega \in A_\infty$. Therefore, we can apply Theorem (1.2.3) with p_1 in place of p_0 to obtain (31). To prove the vector-valued inequality (32), note that by (58) we can apply Theorem (1.2.3) to the family $\mathcal{F}_q$ defined above, again with p_1 in place of p_0.

In exactly the same way, Corollary (1.2.7) follows from Theorems (1.2.21) and (1.2.22).

Chapter 2

Local-to-Global Result and Sobolev Inequalities

We show Sobolev and trace embedding; variable Riesz potential estimates; and maximal function inequalities in Morrey spaces are derived for unbounded domains. For p and q are variable exponents satisfying natural continuity conditions. Also the case when p attains the value 1 in some parts of the domain is included in the results.

Section (2.1): Variable Exponent Spaces

Function spaces with variable exponent and related differential equations have attracted a lot of interest, cf. surveys [75, 97]. Apart from interesting theoretical considerations, these investigations were motivated by a proposed application to modeling electrorheological fluids [22, 27], and, an application to image restoration [67, 73]. We focus on the function space aspect of variable exponent problems. For more information on the PDE aspect see e.g. [68, 69, 71, 78, 80, 83, 85, 91, 98].

The study of variable exponent function spaces in higher dimensions was initiated in a 1991 article by O. Kováčik and J. Rákosník [13], where basic properties such as reflexivity and Hölder inequalities were obtained. The rapid expansion of the field started only in the beginning of the current decade with the advent of techniques, which allowed one to control the Hardy-Littlewood maximal operator, and through it may other operators.

One way to describe the impediment to progress in the 90s is a lack of a Hölder inequality for the modular, i.e. the integral form of the Lebesgue norm. In a classical Lebesgue space, the relation between the modular $\varrho(\cdot)$ and norm $\|.\|$ is very simple:

$$\|f\|_{L^p(\Omega)} = (\varrho_{L^p(\Omega)}(f))^{\frac{1}{p}} \quad \text{where} \quad \varrho_{L^p(\Omega)} = \int_\Omega |f(x)|^p \, dx.$$

In the variable exponent context, we retain the form of the modular, but the norm is defined in the spirit of the Luxemburg norm in Orlicz spaces (or Minkowski functional in abstract spaces):

$$\|u\|_{L^{p(\cdot)}(\Omega)} := \inf\left\{\lambda > 0 \colon \varrho_{L^p(\Omega)}\left(\frac{u}{\lambda}\right) \leq 1\right\}\left((f)\right)^{\frac{1}{p}}$$

where

$$\varrho_{L^{p(\cdot)}(\Omega)}(u) := \int_\Omega |u(x)|^{p(x)} \, dx.$$

Obviously, in this case no functional relationship between norm and modular holds, i.e. $\|u\|_{L^{p(\cdot)}(\Omega)} = F(\varrho_{L^p(\Omega)}(u))$ does not hold for any F. We do not get a Hölder inequality for modular from our inequality for the norm.

The major breakthrough came with L. Diening's work [4], which contained the following weak Hölder-type inequality for the modular:

$$\left(\int_{B(x,r)} |f(y)| \, dy\right)^{p(x)} \lesssim \int_{B(x,r)} |f(y)| \, dy$$

Provided $p$that is bounded away from 1 and ∞ and satisfies the local log-Hölder continuity condition

$$|p(x) - p(y)| \leq \frac{c}{\log(e + 1/|x - y|)}$$

for all $x, y \in \Omega$. Using Diening's result, one can easily prove the boundedness of the maximal operator on bounded sets. However, the additive error prevents us from adding up local estimates to obtain a global result on $\mathbb{R}^n$. (Incidentally, the inequality does not hold without the additive term unless p is constant [87].)

The next quest therefor was to prove a global version of the maximal inequality. Diening [4] achieved this only under the additional, unnatural assumption that constant outside some ball. It did not take long for D. Cruz-Uribe, A. Fiorenza and C. Neugebauer [35] to show that the maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ if the previous assumptions are complemented by a natural decay condition at infinity:

$$|p(x) - p_\infty| \leq \frac{c}{\log(e + |x|)}$$

for some $p_\infty > 1, \mathrm{c} > 0$ and all $x \in \mathbb{R}^n$. A. Nekvinda [94] independently obtained an even slightly stronger result; this result is explained.

The pattern described for the maximal operator was repeated a great many times for instance with the Riesz potential operator, the sharp maximal operator, fractional maximal operators, etc.: first, one proves an easy local result, and then additional, sometimes messy, optimization allows one to prove also the global version.

To introduce a simple and convenient method to pass from local to global results. The idea is simply to generalize the following property of the Lebesgue-norm:

$$\|f\|_{L^p(\mathbb{R}^n)}^p = \sum_i \|f\|_{L^p(\Omega_i)}^p \tag{1}$$

for a partition of $\mathbb{R}^n$into measurable sets Ω_i. Once the idea is stated, it is almost trivial to carry out, cf. Theorem (2.1.3). It proves to be a very powerful tool. Thus, we take results by different teams, which have only been proven in bounded domains and extend them to unbounded domains. As a simple "toy example" of how the method is applied, we prove in the second part of Hardy inequality in unbounded domains using a result in bounded domains from [84].

We reprove the above-mentioned boundedness of the maximal operator in $\mathbb{R}^n$ in order to introduce in a simple setting some techniques that are then applied in Morrey spaces. Apart from that, the problems treated are based on articles published in 2007-2008, which had not been solved in the unbounded case, or solved only under additional assumptions. Specifically, the following problems are considered:

Sobolev inequalities:

(i) in the case when$p$ is not bounded away from 1, generalizing P. Harjulehto and P. Hästö [82]; and

(ii) in trace spaces, generalizing X.– L. Fan [79].

Embeddings of Riesz potentials with weights, generalizing N. & S. Samko and B. Vakulov [96]:

The boundedness of maximal function:

(iii) in $L^{\boldsymbol{p}(\cdot)}(\mathbb{R}^n)$**,** reproving Nekvinda's result [84]; and

(iv) in variable exponent Morrey spaces, generalizing A. Almeida, J. Hasanov and S. Samko [70] and Y. Mizota and T. [92].

Let us consider perhaps the biggest advance in the theory of variable exponent spaces since Diening's trick. An extrapolation method was introduced by D. Cruz-Uribe, A. Fiorenza, J. M. Martell and C. Pérez [66], which allows us to pass from weighted, constant exponent spaces to variable, unweighted spaces. Since there is a constant exponent Lebesgue spaces, this allowed them to directly derive results on a variety of topics, including the sharp maximal operators, singular integral operators and multipliers. These results are also directly obtained for the case of unbounded domains. Despite the impressive record of their method, it does not work in every case. The main advantages of the method presented over the extrapolation method from [66] are:

(i) Extrapolation does not allow weights in the variable exponent case (cf. Riesz potentials).

(ii) Extrapolation is not easy to adapt to other than the Lebesgue-norm (cf. Morrey spaces).

(iii) Extrapolation requires that we know weighted results (e.g. this leads to extraneous assumptions when dealing with multipliers).

(iv) Extrapolation requires that $p^+ < \infty$, whereas the new method can be extended to cover this case, as well.

Extrapolation also has definite advantages:

(i) A local, variable exponent result to start with.

(ii) A non-trivial proof; extrapolation follows by a one-line argument, when all the right elements are in place.

(iii) p is long-Hölder continuous, whereas extrapolation works under the slightly weaker assumption that the maximal operator is bounded.

It is fair to say that the methods are complementary: if extrapolation works and gives a sufficiently good result, then it is the method of choice; when this is not the case, the new method is likely to provide an alternative, which is still much simpler than a direct proof.

The notation $f \lesssim$ g means that $f \leq c$g for some constant c, and $f \approx$ g means $f \lesssim$ g $\lesssim f$. By c we denote a generic constant, whose value may change between appearances even within a single line. By cQ we denote a c-fold dilate of the cube Q.

By f_A and $f_A\, f dx$ we denote the average integral of f over A. The notation $A : X \hookrightarrow Y$, means that A is a continuous embedding from X and Y. Omitting the operator, $A : X \hookrightarrow Y$, means that the identity is a continuous embedding.

By $\Omega \subset \mathbb{R}^n$ we denote an open set. A measurable function $p : \Omega \to [1, \infty)$ is called a variable exponent, and we denote for $A \subset \Omega$

$$p_A^+ \coloneqq \operatorname*{ess\ sup}_{x \in A} p(x), p_A^- \coloneqq \operatorname*{ess\ sup}_{x \in A} p(x), p^+ \coloneqq p_\Omega^+ \text{ and } p^- \coloneqq p_\Omega^+.$$

We always assume that $p^+ < \infty$. We will denote by $\mathcal{P}^{log}(\Omega)$ the class of variable exponents, which are log-Hölder continuous, as defined.

The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ consists of all measurable functions $u : \Omega \to \mathbb{R}$ for which $\|f\|_{L^{p(\cdot)}(\Omega)} < \infty$. Equipped with this norm, $L^{p(\cdot)}(\Omega)$ is a Banach space. The variable exponent Lebesque space is a special case of a Musielak Orliez space. For a constant function, it coincides with the standard Lebesgue space.

Theorem (2.1.3), which allows us to prove global results from local ones.

We need the following result by A. Nekvinda on equivalence of discrete Lebesgue spaces. The space $l^{(q_j)}$ is defined by the modular

$$\varrho_{l^{(q_j)}}((x_j)) \coloneqq \sum_j |x_j|^{q_j}$$

and the norm is defined by $\|(x_j)\|_{l^{(q_j)}} \coloneqq \inf \left\{\lambda > 0 : \varrho_{l^{(q_j)}}\left(\left(\frac{x_j}{\lambda}\right)\right) \le 1\right\}$.

Lemma (2.1.1)[99]: Let (q_j) be a sequence in $(1, \infty)$. If there exists q_∞ and $c > 0$ such that $|q_j - q_\infty| \le \frac{c}{\log(e+i)}$, then $l^{(q_j)} \cong l^{q_\infty}$.

Definition (2.1.2)[99]: Let (Q_j) be a partition of $\mathbb{R}^n$into equal sized cubes, ordered so that $I > j$ if dist$(0, Q_j)$. Let p be log-Hölder continuous. We define a partition norm on $L^{p(\cdot)}(\mathbb{R}^n)$ by

$$\|f\|_{p(\cdot),(Q_j)} \coloneqq \left\| \|f\|_{L^{p(\cdot)}(Q_j)} \right\|_{l^{p_\infty}}.$$

Note that $\|f\|_{p(\cdot),(Q_j)} = \|f\|_p$ if p is a constant, by (1). The only essential property of the norm that we need for the next theorem is the following weak relationship between norm and modular:

$$\min\left\{\varrho_{L^{p(\cdot)}(\Omega)}(f)^{\frac{1}{p^-}}, \varrho_{L^{p(\cdot)}(\Omega)}(f)^{\frac{1}{p^+}}\right\} \le \|f\|_{L^{p(\cdot)}(\Omega)}$$
$$\le \max\left\{\varrho_{L^{p(\cdot)}(\Omega)}(f)^{\frac{1}{p^-}}, \varrho_{L^{p(\cdot)}(\Omega)}(f)^{\frac{1}{p^+}}\right\}. \quad (2)$$

The proof of this well-known fact follows directly from the definition of the norm.

Theorem (2.1.3)[99]: If $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$, then $\|f\|_{p(\cdot),(Q_j)} \approx \|f\|_{p(\cdot)}$ Where the cubes (Q_j) are as in Definition (2.1.2).

Proof. Define $q_j: p_{Q_j}^+$ and $q_\infty := p_\infty$. Since p is log-Hölder continuous, we conclude that $|q_j - q_\infty| \le c \log(e + \operatorname{dist}(0, Q_j))^{-1}$. Since the cubes are in increasing order of distance to the origin, there are at most $(2j)^n$ cubes at distance less than j sidelength (Q_0) from the origin. Hence $|q_j - q_\infty| \le \tilde{c} \log(e + j)^{-1}$, so the condition of Lemma (2.1.1) is satisfied. It follows that

$$\|f\|_{p(\cdot),(Q_j)} \approx \left\| \|f\|_{L^{p(\cdot)}(Q_j)} \right\|_{l^{(q_j)}}.$$

The claim that we are trying to prove is homogeneous, and clearly holds when $\|f\|_{p(\cdot)} = 0$. Therefor we may assume that $\|f\|_{p(\cdot)} = 1$. Then $\|f\|_{p(\cdot),(Q_j)} \lesssim \|f\|_{p(\cdot)}$ follows if we prove that $\left\| \|f\|_{p(\cdot)(Q_j)} \right\|_{l^{(q_j)}} \le c$. Since q_j is a bounded sequence, this is equivalent to showing that

$$\varrho_{l^{(q_j)}}\left(\|f\|_{L^{p(\cdot)}(Q_j)}\right) \le c.$$

Since $\|f\|_{l^{p(\cdot)}(Q_j)} \le \|f\|_{p(\cdot)} = 1$, it follows by (2) that

$$\|f\|_{L^{p(\cdot)}(Q_j)}^{p_{Q_j}^+} \le \varrho_{L^{p(\cdot)}(Q_j)}(f).$$

Therefor,

$$\varrho_{l^{(q_j)}}\left(\|f\|_{p(\cdot)(Q_j)}\right) = \sum_{j=0}^{\infty} \|f\|_{L^{p(\cdot)}(Q_j)}^{q_j} \le \sum_{j=0}^{\infty} \varrho_{L^{p(\cdot)}(Q_j)}(f) = \varrho_{L^{p(\cdot)}(\mathbb{R}^n)}(f) = 1.$$

To prove the opposite inequality, we set $q_j := p_{Q_j}^-$ and use the same steps, with the other inequality in (2).

A. Nekvinda has championed the cause of an integral decay condition, which is slightly weaker that the log-Hölder decay condition, see e.g. [95]. His condition on p may be stated as the existence of a constant $c > 0$ such that

$$\int_{\{p \neq p_\infty\}} C^{\frac{1}{|p(x) - p_\infty|}} \, dx < \infty.$$

Lemma (2.1.3) holds also for the discrete analogue of this condition, and thus Theorem (2.1.3) actually automatically gives slightly stronger results, with Nekvinda's decay condition instead of the log-Hölder decay condition. Hence, all the results works directly under this more general condition. Thus, if one prefers, the class $\mathcal{P}^{\log}(\Omega)$ can be interpreted as locally log-Hölder continuous exponents, which satisfy Nekvinda's decay condition.

We present three examples of how the theorem of the previous can be applied to upgrade local results, proved only on bounded domains, to global results, valid in all of $\mathbb{R}^n$. These results involve variable exponent Sobolev spaces, and to state them we need some definitions.

The variable exponent Sobolev space $W^{l,p(\cdot)}(\Omega)$ consists of functions $u \in L^{p(\cdot)}(\Omega)$ whose distributional gradient ∇u belongs to $L^{p(\cdot)}(\Omega)$. The variable exponent Sobolev space $W^{l,p(\cdot)}(\Omega)$ is a Banach space with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} \coloneqq \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

Since the Sobolev norm is just a sum of Lebesgue norms, it is clear that Theorem (2.1.3) holds for this norm as well. We define the Sobolev space with zero boundary values, $W_0^{1,p(\cdot)}(\Omega)$, as the closure of the set of compactly $W^{1,p(\cdot)}(\Omega)$-functions with respect to the norm $\|\cdot\|_{W^{1p(\cdot)}(\Omega)}$ [81].

Hardy inequalities have been studied by several authors in the variable exponent setting, e.g. [72, 84, 86, 90]. Here we consider the following version of Hardy's inequality proved by P. Hästö and M. Koskenoja [84]:

Lemma (2.1.4)[99]: Let Ω be an open and bounded subset of $\mathbb{R}^n$. Let $p \in \mathcal{P}^{\log}(\Omega)$ with $1 < p^- \leq p^+ < \infty$. Assume that there exists a constant $b > 0$ such that $$|B(z,r) \cap \Omega^c| \geq b|B(z,r)|$$
For every $z \in \partial\Omega$ and $r > 0$. Then

$$\left\|\frac{u}{\delta_\Omega}\right\|_{L^{p(\cdot)}(\Omega)} \lesssim \|u\|_{W^{1,(P)}(\Omega)}$$

Holds for all $u \in W_0^{l,p(\cdot)}(\Omega)$. Here $\delta_\Omega(z) := \operatorname{dist}(z, \partial\Omega)$.

Using Theorem (2.1.3) we can easily remove the boundedness restriction.

Theorem (2.1.5)[99]: Lemma (2.1.4) holds without the assumption that Ω is bounded.

Proof. Let $u \in W_0^{l,p(\cdot)}(\Omega)$. We consider u as a function on $\mathbb{R}^n$ by extending it by 0 to $\mathbb{R}^n \backslash \Omega$. Let (Q_j) be a partition of $\mathbb{R}^n$ into unit cubes which satisfies the condition of Definition (2.1.2). Let Φ_j be Lipschitz function with Lipschitz constant 2 which equals 1 in Q_j and is supported in $2Q_j$. Then $\Phi_j u \in W_0^{1,P(\cdot)}(2Q_j)$ and Lemma (2.1.4)[99] implies that

$$\left\|\frac{u}{\delta_\Omega}\right\|_{L^{p(\cdot)}(Q_j)} \leq \left\|\frac{\Phi_j u}{\delta_\Omega}\right\|_{L^{p(\cdot)}(2Q_j)} \leq \left\|\frac{\Phi_j u}{\delta_\Omega \cap 2Q_j}\right\|_{L^{p(\cdot)}(\Omega \cap 2Q_j)}$$
$$\lesssim \|\Phi_j u\|_{W^{1,(P)}(2Q_j)} \lesssim \|u\|_{W^{1,(P)}(2Q_j)}$$

Next, we apply Theorem (2.1.3) and this inequality:

$$\left\|\frac{u}{\delta_\Omega}\right\|_{L^{p(\cdot)}(\Omega)} \approx \left(\sum_j \left\|\frac{u}{\delta_\Omega}\right\|_{W^{1,(P)}(Q_j)}^{p_\infty}\right)^{1/p_\infty} \lesssim \left(\sum_j \|u\|_{W^{1,(P)}(2Q_j)}^{p_\infty}\right)^{1/p_\infty}$$

Then we note that each cube $2Q_j$ can be covered by 3^n of the cubes Q_k. Using this and Theorem (2.1.3) a second time, we conclude that

$$\left\|\frac{u}{\delta_\Omega}\right\|_{L^{p(\cdot)}(\Omega)} \lesssim \left(\sum_j \|u\|_{W^{1,(P)}(2Q_j)}^{p_\infty}\right)^{1/p_\infty} \approx \left(\sum_j \|u\|_{W^{1,(P)}(2Q_j)}^{p_\infty}\right)^{1/p_\infty}$$
$$\approx \|u\|_{W^{1,(P)}(2Q_j)}.$$

Using Riesz' potential and the Hardy Littlewood maximal function, one can easily prove a Sobolev inequality in $W^{1,p(\cdot)}(\mathbb{R}^n)$. This was done by L. Diening [1]. However, this leads to the extraneous assumption $p^- > 1$. P. Harjulehto and the author [82] devised a method based on a weak-type estimate to circumvent this problem. Unfortunately, it was not possible to get global results with this method:

Lemma (2.1.6)[99]: Suppose that $p \in \mathcal{P}^{\log}(\Omega)$ with $1 \le p(x) \le c < n$ in a bounded open set $\Omega \subset \mathbb{R}^n$. Then $\|u\|_{L^{p^*(\cdot)}(\Omega)} \lesssim \|\nabla u\|_{L^{p(\cdot)}(\Omega)}$ for every $u \in W_0^{1,p(\cdot)}(\Omega)$. Here the constant depends only on n, p and $|\Omega|$.

As usual, p^* denotes the point-wise Sobolev conjugate exponent, $p^*(x) := \frac{np(x)}{n-p(x)}$. Note the p^* is log-Hölder continuous if p is log-Hölder continuous and bounded away from n.

Theorem (2.1.7)[99]: Suppose that $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $1 \le p(x) \le c < n$ in $\mathbb{R}^n$. Then $\|u\|_{L^{p^*(\cdot)}(\mathbb{R}^n)} \lesssim \|u\|_{W^{1,p(\cdot)}(\mathbb{R}^n)}$ for every $u \in W_0^{1,p(\cdot)}(\mathbb{R}^n)$. Here the constant depends only on n and p.

Proof. Let $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$. By homogeneity, it suffices to consider the case $\|u\|_{W^{1,p(\cdot)}}(\mathbb{R}^n) = 1$. Let (Q_j) be a partition of $\mathbb{R}^n$ into unit cubes which satisfies the condition of Definition (2.1.2). Let Φ_j be a Lipschitz function with Lipschitz constant 2 which equals 1 in Q_j and is supported in $2Q_j$. Then $\Phi_j u \in W_0^{1,p(\cdot)}(2Q_j)$ and Lemma (2.1.6) implies that

$$\|u\|_{L^{p^*(\cdot)}(Q_j)} \le \|\Phi_j u\|_{L^{p^*(\cdot)}(2Q_j)} \lesssim \|\Phi_j \nabla u\|_{L^{p(\cdot)}(2Q_j)} \le \|\nabla u\|_{W^{1,p(\cdot)}(2Q_j)}$$

Next, we apply Theorem (2.1.3),

$$\|u\|_{L^{p^*(\cdot)}(\mathbb{R}^n)} \approx \left(\sum_j \|u\|_{L^{p^*(\cdot)}(2Q_j)}^{p_\infty^*}\right)^{1/p_\infty^*} \lesssim \left(\sum_j \|u\|_{W^{1,(p)}(2Q_j)}^{p_\infty^*}\right)^{1/p_\infty^*}$$
$$\lesssim \left(\sum_j \|u\|_{W^{1,(p)}(Q_j)}^{p_\infty^*}\right)^{1/p_\infty^*}$$

In contrast to the case of the Hardy inequality, we here end up with the wrong power after the inequality for using Theorem (2.1.3) we would want the norm to be raised to the power of p_∞ instead of p_∞^*. However, since $\|u\|_{W^{1,p(\cdot)}(Q_j)} \le \|u\|_{W^{1,p(\cdot)}(\mathbb{R}^n)} =$

1 and $p_\infty \leq p_\infty^*$, we conclude that $\|u\|^{p_\infty^*}_{W^{1,p(\cdot)}(Q_j)} \leq \|u\|^{p_\infty}_{W^{1,p(\cdot)}(Q_j)}$. Then we can use Theorem (2.1.3) again:

$$\|u\|_{L^{p^*(\cdot)}(\mathbb{R}^n)} \lesssim \left(\sum_j \|u\|^{p_\infty}_{W^{1,p(\cdot)}(Q_j)}\right)^{1/p_\infty^*} \approx \sum_j \|u\|^{p_\infty/p_\infty^*}_{W^{1,(p)}(\mathbb{R}^n)} = 1.$$

The trace of a function essentially means a restriction of the function to a lower dimensional subset of its original domain of definition. Since Sobolev functions are, a priori, only equivalence classes of measurable functions, some care is needed in making this rigorous.

In the variable exponent Sobolev spaces, traces have been studied in [74, 79, 89]. Since $W^{1,p(\cdot)}(\Omega) \hookrightarrow W^{1,1}_{\text{loc}}(\Omega)$, we know by classical theory that $u|\partial\Omega \in L^1_{\text{loc}}(\Omega)$. X.-L. Fan [79] studied Sobolev embeddings for the traces of Sobolev functions. His Theorem (2.1.1) reads:

Lemma (2.1.8)[99]: Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with Lipschitz boundary. Suppose that $\|p\|_{W^{1,\gamma}(\Omega)} < \infty$ and $1 \leq p^- \leq p^+ < n < \gamma$. Then there is a continuous boundary trace embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{\frac{(n-1)p(\cdot)}{n-p(\cdot)}}(\partial\Omega)$.

X.-L. Fan also gave results in the unbounded case; however, these results were based on a stronger assumption on the domain, which was assumed to satisfy a strong Lipschitz boundary condition.

We can easily upgrade to local trace embedding to a global result without the extra assumption on the boundary. We require only that $\|p\|_{W^{1,\gamma}(3Q)}$ is uniformly bounded over unit cubes Q, whereas Fan needs to assume that $p \in L^{p(\cdot)}(\mathbb{R}^n)$ and $p \in L^\infty(\mathbb{R}^n)$. On the other hand, we need the decay condition at infinity. As pointed out in [79], this does not follow the previous assumptions.

Theorem (2.1.9)[99]: Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with Lipschitz boundary. Suppose that $\|p\|_{W^{1,\gamma}(3Q\cap\Omega)} < \infty$ is uniformly bounded over unit cubes Q, that p satisfies the decay condition, and that $1 \leq p^- \leq p^+ < n < \gamma$. Then there is a continuous boundary trace embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{\frac{(n-1)p(\cdot)}{n-p(\cdot)}}(\partial\Omega)$.

Proof. Let us denote $p^\#(\cdot) = \frac{(n-1)p(\cdot)}{n-p(\cdot)}$. Assume as before that $\|u\|_{W^{1,p(\cdot)}(\Omega)} \leq 1$. Let (Q_i) be a partition of $\mathbb{R}^n$ as in Definition (2.1.2). By Theorem (2.1.3) we then obtain

$$\|u\|_{L^{p^\#(\cdot)}(\partial\Omega)} \approx \left(\sum_j \|u\|^{p_\infty^\#}_{L^{p^*(\cdot)}(\partial\Omega\cap Q_i)}\right)^{1/p_\infty^\#} \lesssim \left(\sum_j \|u\|^{p_\infty^*}_{W^{1,(p)}(\Omega\cap Q_j)}\right)^{1/p_\infty^\#}$$

$$\lesssim \left(\sum_j \|u\|^{p_\infty}_{W^{1,(p)}(\Omega\cap Q_i)}\right)^{1/p_\infty^\#} \approx \|u\|^{p_\infty/p_\infty^\#}_{W^{1,(p)}(\Omega\cap Q_i)} = 1.$$

We consider the variable index Riesz potential on weighted Lebesgue spaces with variable exponent. By a weight, we mean a measurable, non-negative function. The weighted Lebesgue space is defined by norm $\|f\|_{L_\omega^{p(\cdot)}(\Omega)} := \left\|f\omega^{1/p(\cdot)}\right\|_{L^{p(\cdot)}(\partial\Omega)}$. The weighted modular is defined by

$$\varrho_{L_\omega^{p(\cdot)}(\Omega)}(f) := \int_\Omega |f(x)|^{p(x)}\,\omega(x)dx,$$

And it is clear that the following analogue of (2) holds:

$$\min\left\{\varrho_{L_\omega^{p(\cdot)}(\Omega)}(f)^{\frac{1}{p^-}}, \varrho_{L_\omega^{p(\cdot)}(\Omega)}(f)^{\frac{1}{p^+}}\right\} \le \|f\|_{L_\omega^{p(\cdot)}(\Omega)} \le \max\left\{\varrho_{L_\omega^{p(\cdot)}(\Omega)}(f)^{\frac{1}{p^-}}, \varrho_{L_\omega^{p(\cdot)}(\Omega)}(f)^{\frac{1}{p^+}}\right\}.$$

We noted previously that Theorem (2.1.3) depends only on this property of the norm, and hence we conclude that it holds also for weighted Lebesgue spaces with variable exponent.

N. & S. Samko and B. Vakulov [96] studied mapping properties of a variable Riesz potential in weighted Lebesgue spaces with variable exponent. The potential operator is defined by

$$I^{\alpha(x)}f(x) := \int_{\mathbb{R}n} \frac{|f(y)|}{|x-y|^{n-\alpha(x)}}\,dy.$$

Assume that p and α are log-Hölder continuous. When Ω is bounded, they proved in [96] that

$$I^{\alpha(x)}: L_\omega^{p(\cdot)}(\Omega) \hookrightarrow L_\omega^{p^\#(\cdot)}(\Omega),$$

if $\sup \alpha(x)p(x) < n$, where $p^\#(x) = \frac{np(x)}{n-\alpha(x)p(x)}$, ω is weight, and $\omega^\# := \omega^{p^\#/p}$. We will here not get into the details of which weights are allowed, and instead refer to [96, Definition (2.1.2)] for further discussion on this. Suffice it to say by way of example that radial weights with appropriate exponents are allowed.

Note that one could equivalently study the operator

$$I^{\alpha(\cdot)}f(x) := \int_{\mathbb{R}n} \frac{|f(y)|}{|x-y|^{n-\alpha(y)}}\,dy.$$

where $\alpha(x)$ is replaced by $\alpha(y)$, since $|x-y|^{\alpha(x)} \approx |x-y|^{\alpha(y)}$ by the log-Hölder continuity of α [96, Lemma (2.1.5)]

For unbounded domains, N. & S. Samko and B. Vakulov needed to assume that α is constant, in which case they proved that

$$I^{\alpha(x)}: L_\omega^{p(\cdot)}(\mathbb{R}n) \hookrightarrow L_\omega^{p^\#(\cdot)}(\mathbb{R}n),$$

if $\sup p(x) < \frac{n}{\alpha}$, where the radial weight is controlled by polynomials at 0 and ∞. It turns out that there is a good reason that they could not prove that $I^{\alpha(x)}: L_{\omega}^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L_{\omega^\#}^{p^\#(\cdot)}(\mathbb{R}^n)$ this embedding does not hold in general, as we now show.

Let $R > 2$ and let $\alpha : \mathbb{R}^n \to (0, n)$ be Lipschitz continuous with $\alpha|_{B(0,1)} \equiv \alpha_0$ and $\alpha|_{\mathbb{R}^n \setminus B(0,1)} \equiv \alpha_\infty$. The exponent p is defined similarly with values p_0 and p_∞. Set $f(x) := |x|^{-\beta} \chi_{\mathbb{R}^n \setminus B(0,1)}(x)$. For $x \in B(0,1)$ we find that

$$I^{\alpha(x)} f(x) := \int_{\mathbb{R}^n \setminus B(0,1)} \frac{|y|^{-\beta}}{|x-y|^{n-\alpha_0}}\, dy \approx \int_{\mathbb{R}}^{\infty} r^{-\beta-n+\alpha_0+n-1}\, dr = \infty$$

provided $\alpha_0 \geq \beta$. No the other hand,

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \left\| |\cdot|^{-\beta} \right\|_{L^{p_\infty}(\mathbb{R}^n \setminus B(0,R))} \approx \int_{R}^{\infty} r^{-\beta p_\infty + n - 1}\, dr < \infty$$

provided $n < \beta p_\infty$. Therefore we see that there exists a function $f \in L^{p(\cdot)}(\mathbb{R}^n)$ with $I^{\alpha(x)} f \equiv \infty$ in $B(0, R)$ provided $\alpha_0 p_\infty > n$.

For function p and α as before and $\chi_{B(01)}$, similar calculations show that $I^{\alpha(\cdot)} f \notin L^{p^\#(\cdot)}(\mathbb{R}^n)$ if $(\alpha_0 - \alpha_\infty) p'_\infty \geq n$.

There is no point in investigating the global behavior of the potentials $I^{\alpha(x)}$ or $I^{\alpha(\cdot)}$, even in the unweighted case. It is, however, possible to obtain a global result, which encompasses all the previous results. For this we introduce the potential operator

$$I^{\alpha,\wedge} f(x) := \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\alpha(x)\wedge\alpha(y)}}\, dy,$$

where $a \wedge b$ denotes $\min\{a, b\}$. As was noted before, $|x-y|^{n-\alpha(x)\wedge\alpha(y)} \approx |x-y|^{n-\alpha(x)} \approx |x-y|^{n-\alpha(y)}$ in a bounded domain. Therefore

$$I^{\alpha,\wedge} f(x) \lesssim \underbrace{\int_{B(x,1)} \frac{|f(y)|}{|x-y|^{n-\alpha(x)}}\, dy}_{=: I_<^{\alpha(x)} f(x)} + \underbrace{\int_{\mathbb{R}^n \setminus B(x,1)} \frac{|f(y)|}{|x-y|^{n-\alpha(x)\wedge\alpha(y)}}\, dy.}_{=: I_>^{\alpha,\wedge} f(x)}$$

Let us now investigate how the local-to-global result can be applied to prove the general, variable α result in the global case. It suffices to study the mapping properties of $I_<^{\alpha(x)}$ and $I_>^{\alpha_\infty}$ separately. For the former we apply Theorem (2.1.3).

Lemma (2.1.10)[99]: Let $p, \alpha \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $\sup \alpha(x) p(x) < n$. Then

$$I_<^{\alpha(x)}:\ L_{\omega}^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L_{\omega^\#}^{p^\#(\cdot)}(\mathbb{R}^n),$$

where the weight ω satisfies the condition of [96].

Proof. Let (Q_j) be a partition of $\mathbb{R}^n$ into unit cubes which satisfies the condition of Definition (2.1.2) For $x \in Q_j, I_<^{\alpha(x)} f(x)$ is not affected by the values of f outside $3Q_j$. If $0 \in 3Q_j$, then the conditions of [96] are satisfied, and conclude that

$$\left\| I_<^{\alpha(\cdot)} f \right\|_{L_{\omega^\#}^{p^\#(\cdot)}(Q_j)} \lesssim \left\| I_<^{\alpha(\cdot)} (f \chi_3 Q_j) \right\|_{L_{\omega^\#}^{p^\#(\cdot)}(Q_j)} \lesssim \left\| I_<^{\alpha(\cdot)} (f \chi_3 Q_j) \right\|_{L_{\omega^\#}^{p^\#(\cdot)}(3Q_j)}$$
$$\lesssim \|f\|_{L_{\omega^\#}^{p(.)}(3Q_j)}. \quad (3)$$

If $0 \notin 3Q_j$, then $d(0, 3Q_j) \geq 1$ since the cubes Q_j are unit cubes integer coordinates. Since ω satisfies the condition of [96] one easily checks that the weight is locally constant in the sense that $\omega_{3Q_j}^+ \leq c\omega_{3Q_j}^-$ with constant c independent of j. Thus (3) follows in this case from the unweighted estimate. Now that we have our local estimate in place, we use Theorem (2.1.3).

$$\left\| I_<^{\alpha(\cdot)} f \right\|_{L_{\omega^\#}^{p^\#(\cdot)}(\mathbb{R}^n)} \approx \left\| \left\| I_<^{\alpha(\cdot)} f \right\|_{L_{\omega^\#}^{p^\#(\cdot)}(Q_j)} \right\|_{l^{p_\infty^\#}} \lesssim \left\| \|f\|_{L_{\omega^\#}^{p^\#(.)}(3Q_j)} \right\|_{l^{p_\infty^\#}}$$
$$\leq \left(\sum \|f\|_{L_{\omega^\#}^{p^\#(\cdot)}(Q_j)}^{p_\infty^\#} \right)^{1/p_\infty^\#}.$$

By homogeneity we assume that $\|f\|_{L_\omega^{p(.)}(\mathbb{R}^n)} = 1$, so that $\|f\|_{L_\omega^{p(\cdot)}(Q_j)}^{p_\infty^\#} \leq \|f\|_{L_\omega^{p(\cdot)}(Q_j)}^{p_\infty}$. Thus

$$\left\| I_<^{\alpha(\cdot)} f \right\|_{L_{\omega^\#}^{p^\#(\cdot)}(\mathbb{R}^n)} \lesssim \left(\sum \|f\|_{L_\omega^{p(\cdot)}(Q_j)}^{p_\infty} \right)^{1/p_\infty^\#} \approx \|f\|_{L_\omega^{p(\cdot)}(\mathbb{R}^n)}^{p_\infty/p_\infty^\#} = 1.$$

We now continue with the other part of the Riesz potential, $I_>^{\alpha,\wedge}$.

Lemma (2.1.11)[99]: Let $p, \alpha \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $\sup \alpha(x)p(x) < n$. Then

$$I_<^{\alpha(x)}: L_\omega^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L_{\omega^\#}^{p^\#(\cdot)}(\mathbb{R}^n),$$

where the weight ω satisfies the condition of [96].

Proof. Let us here present a simple proof using the stronger, log-Hölder decay condition instead of the more general condition. It is possible to adapt the proof for the more general condition using the techniques introduced.

The decay condition on α implies that $|x|^{\alpha(x)} \approx |x|^{\alpha_\infty}$ for $|x| > 1$. Therefore $|x - y|^{\alpha(x)\wedge\alpha(y)} \lesssim |x - y|^{\alpha_\infty}$, since $1 < |x - y| \leq 2\max\{|x|, |y|\}$. Hence $I_>^{\alpha,\wedge} f \lesssim I_>^{\alpha_\infty} f$. Since $I_<^{\alpha_\infty}: L_\omega^{p(.)}(\mathbb{R}^n) \hookrightarrow L_{\omega^\#}^{p^\#(.)}(\mathbb{R}^n)$ by [96], we conclude that the same property holds for $I_>^{\alpha,\wedge}$.

Combining the previous two lemmas yields:

Corollary (2.1.12)[99]: Let $p, \alpha \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $\sup \alpha(x)p(x) < n$. Then

$$I^{\alpha,\wedge}\colon\ L^{p(\cdot)}_{\omega}(\mathbb{R}^n) \hookrightarrow L^{p^{\#}(\cdot)}_{\omega^{\#}}(\mathbb{R}^n),$$

where the weight ω satisfies the condition of [96].

Let us start by giving a new proof of the global boundedness of the maximal operator in variable exponent spaces which is much shoeter than the original one. It is interesting to note that A. Lerner and C. Pérez [88] have also recently reproved the result of Nekvinda [94]. Their proof is completely different from the one presented here and is based on a general theorem which they prove in quasi-Banach spaces.

We start with the following trivial estimate:

$$Mf(x) \le \underbrace{\sup_{r\in(0,1)} \int_{B(x,r)} |f(y)|\, dy}_{=:M_{<}f(x)} + \underbrace{\sup_{r\in[0,1)} \int_{B(x,r)} |f(y)|\, dy}_{=:M_{>}f(x)}.$$

The part $M_{<}$ is easily handled by Theorem (2.1.3), as we see in Lemma (2.1.16). To deal with the part $M_{>}$ we develop a new method.

We need the following lemma, which is due to L. Diening and S. Samko [77]. Their version has $L^{p^*}(\mathbb{R}^n)$ in place of $L^{p_\infty}(\mathbb{R}^n)$, so a short proof is given here for completeness. Recall that the norm in $X \cap Y$ is given by $\|.\|_X + \|.\|_Y$.

Lemma (2.1.13)[99]: (cf. Lemma (2.1.12), [77]). Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$. Then

$$L^{p(.)}(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n) \cong L^{p_\infty}(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n).$$

Proof. Let $\bar{p}(x) := \min\{p(x), p_\infty\}$. Then $\bar{p}$ is also log-Hölder continuous on $\mathbb{R}^n, \bar{p} \le p$, and $\bar{p}_\infty = p_\infty$. By [76], $L^{p(.)}(\mathbb{R}^n) \hookrightarrow L^{\bar{p}(.)}(\mathbb{R}^n)$. on the other hand, $\bar{p} \le p \le \infty$, so $L^{\bar{p}(\cdot)}(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n) \hookrightarrow L^{p(\cdot)}(\mathbb{R}^n)$. Combining these embeddings yields the claim. (Incidentally, [76] is written for log-Hölder continuous exponents, but it also holds for exponents satisfying only Nekvinda's condition.)

To control $M_{>}$ we need some understanding of the convolution operator. Its boundedness on $L^{p(\cdot)}(\mathbb{R}^n)$ is in general proven using the boundedness of the maximal operator. Since we now want to reprove this fact, we must take a different route.

Lemma (2.1.14)[99]: Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and define $\check{p}(x) := p^{-}_{B(x,1)}$ and $Af(x) := |f|_{B(x,1)}$. Then $\check{p} \in \mathcal{P}^{\log}(\Omega)$, $\check{p}_\infty = p_\infty$ and $L^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{\check{p}(\cdot)}(\mathbb{R}^n)$.

Proof: It is easy to see that $\check{p}$ satisfies the local log-Hölder condition and that $\check{p}_\infty = p_\infty$. The log-Hölder decay condition is also easily checked; if one works with Nekvinda's condition, some more effort is needed, see Lemma (2.1.17).

We start the estimate of $Af(x)$ with a constant exponent Hölder inequality, followed by Young's inequality with exponent $\frac{p(y)}{\check{p}(x)}$:

$$Af(x)^{\breve{p}(x)} = \frac{1}{\lambda}\int_{B(x,1)} \left(\lambda^{\frac{1}{\breve{p}(x)}}|f(y)|\right)^{\breve{p}(x)} dy$$
$$\leq \frac{1}{\lambda}\int_{B(x,1)} |f(y)|^{p(x)} dy + \lambda^{\frac{p(y)}{p(y)-\breve{p}(x)}} dy,$$

where $\lambda \in (0,1)$. For sufficiently small $\lambda, \theta \longmapsto \lambda^{1/\theta}$ is convex on $(0, p^+]$. Thus, we have

$$\lambda^{\frac{p(y)}{p(y)-\breve{p}(x)}} \leq \lambda^{\frac{1}{p(y)-\breve{p}(x)}} \leq \lambda^{\frac{1}{|p(y)-p_\infty|}} \leq \lambda^{\frac{1}{|\breve{p}(x)-p_\infty|}}.$$

Integrating the previous estimate now gives

$$\int_{\mathbb{R}^n} Af(x)^{\breve{p}(x)} dx \leq \int_{\mathbb{R}^n} \frac{1}{\lambda}\int_{B(x,1)} |f(y)|^{p(x)} + \lambda^{\frac{1}{|p(y)-p_\infty|}} + \lambda^{\frac{1}{|\breve{p}(x)-p_\infty|}} dydx$$
$$= \frac{1}{\lambda}\int_{B(x,1)} |f(y)|^{p(x)} + \lambda^{\frac{1}{|p(y)-p_\infty|}} + \lambda^{\frac{1}{|\breve{p}(x)-p_\infty|}} dydx$$

which is finite for sufficiently small λ, by Nekvinda's condition for p and $\breve{p}$.

The following corollary is the globalization of Diening's result [69, Theorem (2.1.8)] which says that $M: L^{p(\cdot)}(\Omega) \hookrightarrow L^{\breve{p}(\cdot)}(\Omega)$ for bounded Ω.

Corollary (2.1.15)[99]: Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ satisfy $1 \leq p^- \leq p^+ < \infty$. Then $M: L^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{\breve{p}(\cdot)}(\mathbb{R}^n)$.

Proof. As noted above, we have $Mf \leq M_< f + M_> f$. Therefore we study the operators $M_<$ and $M_>$ separately. Let (Q_i) be a partition of $\mathbb{R}^n$ as in Definition (2.1.2). Then, by Theorem (2.1.3) and the local boundedness of $M_<$, we obtain

$$\|M_< f\|_{L^{p(\cdot)}(\Omega)} \approx \|M_< f\|_{p(\cdot)(Q_j)} \lesssim \|f\|_{p(\cdot)(3Q_j)} \approx \|f\|_{L^{p(\cdot)}(\Omega)}.$$

For the other part we see start by noting $M_> f \approx M_>(Af)$ with A as in Lemma (2.1.14). From Holder's inequality we infer that $Af \in L^\infty(\mathbb{R}^n)$. Thus, by Lemmas (2.1.14) and (2.1.13),

$$A: L^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n) \cap L^{\breve{p}(\cdot)}(\mathbb{R}^n) \cong L^\infty(\mathbb{R}^n) \cap L^{p_\infty}(\mathbb{R}^n).$$

Since $\|M_< f\|_{p_\infty} \leq \|M_< f\|_{p_\infty} \lesssim \|f\|_{p_\infty}$, we conclude that $L^\infty(\mathbb{R}^n) \cap L^{p_\infty}(\mathbb{R}^n) \hookrightarrow L^{p(\cdot)}(\mathbb{R}^n)$. Thus $M_> \approx M_> \circ A: L^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p(\cdot)}(\mathbb{R}^n)$, as required. Combining the estimates for $M_<$ and $M_>$ yields the result.

Let us now get back to the technical part of Lemma (2.1.14), i.e. the case of Nekvinda's decay condition.

Lemma (2.1.16)[99]: Let p satisfy the local log-Hölder condition and Nekvinda's decay Then $\breve{p}(x) := p^-_{B(x,1)}$ satisfies the same conditions.

Proof. It remains only to check Nekvinda's decay condition. Let $\alpha \in (0,1)$ be such that

$$\int_{\mathbb{R}^n} \alpha^{\frac{1}{|p(x)-p_\infty|}}\, dx < \infty.$$

Let $\beta \in (0, \alpha)$, its value will be specified later. Since $\check{p}(x) \leq p(x)$ we conclude that

$$\int_{\mathbb{R}^n} \beta^{\frac{1}{|p(x)-p_\infty|}} \chi_{\{\check{p}_\infty > p_\infty\}}(x) dx \leq \int_{\mathbb{R}^n} \alpha^{\frac{1}{|p(x)-p_\infty|}} \chi_{\{\check{p}_\infty > p_\infty\}}(x)\, dx < \infty.$$

Thus it remains to consider points where $\check{p}_\infty < p_\infty$. Let (Q_i) be a partition of $\mathbb{R}^n$ into unit cubes. Thus

$$\int_{\mathbb{R}^n} \beta^{\frac{1}{|p(x)-p_\infty|}} \chi_{\{\check{p}_\infty > p_\infty\}}(x) dx \leq \sum \beta^{\frac{1}{p_\infty - p_{3Q_i}^-}} |x \in Q_i : \check{p}_\infty < p_\infty| \leq \sum \beta^{\frac{1}{p_\infty - p_{3Q_i}^-}}.$$

For each $x_i \in \overline{3Q_\iota}$ be such that $p(x_i) = p_{3Q_i}^-$. If $p(x_i) < p_\infty$, then we conclude from the local log-Holder condition that $p(y) < p_\infty - \frac{1}{2}(p_\infty < p(x_i))$ in a ball B_i centered at x_i with radius $\exp\left\{1 - \frac{2c_0}{p_\infty - p(x_i)}\right\}$, where c_0 is the log-Hölder constant. Hence

$$\alpha^{\frac{2}{p_\infty - p(x_i)}} e^{-\frac{2nc_0}{p_\infty - p(x_i)}} \approx \alpha^{\frac{2}{p_\infty - p(x_i)}} |B_i| \leq \int_{B_i} \alpha^{\frac{1}{|p(x)-p_\infty|}} dx.$$

Since any given point can occur at most 4^n times as a point x_i we conclude by choosing $\beta = \alpha^2 e^{-2nc_0}$ that

$$\sum \beta^{\frac{1}{p_\infty - p_{3Q_i}^-}} \leq \sum \alpha^{\frac{2}{p_\infty - p(x_i)}} e^{-\frac{2nc_0}{p_\infty - p(x_i)}} \leq 4^2 \int_{B_i} \alpha^{\frac{1}{|p(x)-p_\infty|}} dx < \infty.$$

Thus we have shown that $\check{p}$ satisfies Nekvinda's condition with constant β.

Variable exponent Morrey spaces have been studied in [70, 92]. The Morrey space is defined by the modular

$$\varrho_{L^{p(\cdot),v(\cdot)}(\Omega)}(f) := \sup_{z \in \Omega, r > 0} r^{-v(z)} \int_{B(z,r)} |f(x)|^{p(x)}\, dx.$$

As usual, $\|f\|_{L^{p(\cdot),v(\cdot)}(\Omega)} := \inf\{\lambda > 0 : \varrho_{L^{p(\cdot),v(\cdot)}(\Omega)}(f/\lambda) < 1\}$. In [70], it is shown that

$$\min\left\{\varrho_{L^{p(\cdot),v(\cdot)}(\Omega)}(f)^{\frac{1}{p^-}}, \varrho_{L^{p(\cdot),v(\cdot)}(\Omega)}(f)^{\frac{1}{p^+}}\right\} \leq \|f\|_{L^{p(\cdot),v(\cdot)}(\Omega)}$$
$$\leq \max\left\{\varrho_{L^{p(\cdot)v(\cdot)}(\Omega)}(f)^{\frac{1}{p^-}}, \varrho_{L^{p(\cdot),v(\cdot)}(\Omega)}(f)^{\frac{1}{p^+}}\right\}.$$

In [70], an alternative expression for the norm is provided:

$$\|f\|_{L^{p(\cdot),v(\cdot)}(\Omega)} = \sup_{z \in \Omega, r > 0} \left\| r^{-\frac{v(z)}{p(\cdot)}} \right\|_{L^{p(\cdot)}(B(z,r))} \tag{4}$$

For (Q_i) as in Definition (2.1.2) we define a partition norm on $L^{p(\cdot),v(\cdot)}(\mathbb{R}^n)$ by

$$\|f\|_{L^{p(\cdot),v(\cdot)}(Q_i)} := \left\| \|f\|_{L^{p(\cdot),v(\cdot)}(Q_i)} \right\|_{l^{p_\infty}},$$

We now have all the ingredients that were needed in the proof of Theorem (2.1.3), and may therefore state the following version for Morrey spaces. Being the same, the proof is omitted.

Corollary (2.1.17)[99]: Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$, then $\|f\|_{L^{p(\cdot),v(\cdot)}(Q_i)} \approx \|f\|_{L^{p(\cdot),v(\cdot)}(Q_i)}$.

The following result is proven. Theorem (2.1.6) contains a different version of the result, with slightly more general spaces, but also more restrictive assumptions on p.

Lemma (2.1.18)[99]: Let $\Omega \subset \mathbb{R}^n$ be bounded and open, $p \in \mathcal{P}^{\log}(\Omega)$, and let $0 \leq v^- \leq v^+ < n$. Then $M: L^{p(\cdot),v(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot),v(\cdot)}(\Omega)$.

Theorem (2.1.19)[99]: Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$, and let $0 \leq v^- \leq v^+ < n$. Then $M: L^{p(\cdot),v(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p(\cdot),v(\cdot)}(\mathbb{R}^n)$.

Proof. As before, we have $Mf \leq M_< f + M_> f$. Therefore we study the operators $M_<$ and $M_>$ separately. Let (Q_i) be a partition of $\mathbb{R}^n$ as in Definition (2.1.2). Then, by Corollary (2.1.17).

$$\|M_< f\|_{L^{p(\cdot),v(\cdot)}(\mathbb{R}^n)} \approx \|M_< f\|_{p(\cdot),v(\cdot)(Q_j)} \lesssim \|f\|_{p(\cdot),v(\cdot)(3Q_j)} \approx \|f\|_{L^{p(\cdot),v(\cdot)}(\mathbb{R}^n)}.$$

We consider $M_>$. For $B = B(z,r)$ with $r \geq 1$ we use Young's inequality:

$$\int_B |f(x)|\,dx = \int_B \left[r^{r^{-\frac{v(z)}{p(x)}}}|f(x)|\right] r^{-\frac{v(z)}{p(x)}-n}dx$$

$$\leq \int_B r^{-v(z)}|f(x)|^{p(x)} + r^{-\frac{v(z)-np(x)}{p(x)-1}}dx$$

$$\leq r^{-v(z)}\int_B |f(x)|^{p(x)} \leq \int_B r^{-\frac{v(z)-np(x)}{p(x)-1}}dx \leq \varrho_{L^{p(\cdot),v(\cdot)}(\mathbb{R}^n)} + C.$$

Hence $M_> f \in L^\infty(\mathbb{R}^n)$. Since $v < n$, it follows that

$$\sup_{z\in\Omega, r>0} r^{-v(z)} \int_{B(z,r)} |M_> f(x)|^{p(x)}\,dx \leq (1 + \|M_< f\|_\infty)^{p^+} \leq C.$$

Hence we need only consider the Morrey norm over $r \in [1,\infty)$. As in Lemma (2.1.14) we obtain

$$\int_B Af(x)^{\check{p}(x)}\,dx \leq \frac{1}{\lambda}\int_{2B} |f(y)|^{p(x)} + \lambda^{\frac{1}{|p(y)-p_\infty|}} + \lambda^{\frac{1}{|\check{p}(x)-p_\infty|}}\,dx$$

for $B$with radius at least 1, where $Af(x) := |f|_{B(x,1)}$. Thus, we conclude that

$$r^{-v(z)}\int_B Af(x)^{\check{p}(x)}\,dx \lesssim (2r)^{-v(z)}\int_{2B} |f(y)|^{p(x)}\,dx + C,$$

And so it follows that $M : L^{p(\cdot),v(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{\check{p}(.),v(.)}(\mathbb{R}^n)$. In view of (4), an analogue of Lemma (2.1.13) holds for Morrey spaces. Hence we conclude as in Corollary (2.1.15) that $M_{>}f \approx M_{>}(Af)$ and

$$A : L^{p(\cdot),v(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{\check{p}(.),v(\cdot)}(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n) \cong L^{p_\infty,v(\cdot)}(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n).$$

Since $\|M_{>}f\|_{p_\infty} \leq \|Mf\|_{p_\infty} \lesssim \|f\|_{p_\infty}$, we conclude (4) that $\|M_{>}f\|_{p_\infty,v(.)} \leq \|Mf\|_{p_\infty,v(.)}$. Hence

$$M_{>}L^{p_\infty,v(\cdot)}(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n) \hookrightarrow L^{p(\cdot),v(.)}(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n) \hookrightarrow L^{p(\cdot),v(\cdot)}(\mathbb{R}^n).$$

Thus $M_{>} \approx M_{>} \circ A: L^{p(\cdot),v(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p(\cdot),v(\cdot)}(\mathbb{R}^n)$, as required.

Section (2.2): Orlicz Spaces of two Variables

Variable exponent spaces have been studied; for a survey see [75, 97]. These investigations have dealt both with the spaces themselves, with related differential equations, and with applications. One typical feature is that the exponent has to be strictly bounded away from various critical values. We consider the example of the Sobolev embedding theorem. Such embeddings and embeddings of Riesz potentials have been studied, e.g., in [1, 75, 102, 100, 106, 105, 106, 109, 110, 115] in the variable exponent setting. Most proofs are based on the Riesz potential and maximal functions, and thus lead to the additional, unnatural restriction $\inf p > 1$.

Due to Edmunds and Rákosník [8, 9] avoided this restriction by a use of ad hoc methods of proofs, but these turned out not to extend conveniently to other situations. Harjulehto and Hästö [82] introduced a method based on a weak type estimate, which covers the case $\inf p = 1$ and can be easily adopted also, to other situations. Their result was extended to the case of unbounded domains in [99].

We consider more general variable exponents following Cruz-Uribe and Fiorenza [103]. To define these spaces let $p : \mathbb{R}^n \to [1,\infty)$ and $q : \mathbb{R}^n \to \mathbb{R}$ be continuous functions. We will be considering spaces of type $L^{p(.)} \log L^{q(.)}(\Omega)$. For simplicity we denote the function defining the space by Φ, i.e. $\Phi(x,t) = t^{p(x)}(\log(c_0 + t))^{q(x)}$. By C we denote a generic constant whose value may change between appearances even within a single line.

We assume throughout that the variable exponents p and q are continuous functions on $\mathbb{R}^n$ or $\Omega \subset \mathbb{R}^n$ satisfying:

(p1) $1 \leq p^- := \inf_{x\in\mathbb{R}^n} p(x) \leq \sup_{x\in\mathbb{R}^n} p(x) =: p^+ < \infty$;

(p2) $|p(x) - p(y)| \leq \frac{C}{\log(e+1/|x-y|)}$ whenever $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$;

(p3) $|p(x) - p(y)| \leq \frac{C}{\log(e+|x|)}$ whenever $|y| \geq |x|/2$;

(q1) $-\infty < q^- := \inf_{x\in\mathbb{R}^n} q(x) \leq \sup_{x\in\mathbb{R}^n} q(x) =: q^+ < \infty$;

(q2) $|q(x) - q(y)| \leq \frac{C}{\log\log(e+1/|x-y|)}$ whenever $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$;

Moreover, we assume that

(Φ_1) there exists $c_0 \in [e,\infty)$ such that $\Phi(x,\cdot)$ is convex on $[0,\infty)$ for every $x \in \mathbb{R}^n$.

If there is a positive constant C_0 such that

$$C_0(p(x) - 1) + q(x) \geq 0,$$

then condition (Φ_1) holds; this follows from a computation of the second derivative $\Phi(x,\cdot)$. For example, this inequality holds if $p^- > 0$ or if $q^- \geq 0$. For later use it is convenient to note that (Φ_1) implies the following condition:
(Φ_2) $t \to t^{-1}\Phi(x,t)$ is non-decreasing on $(0,\infty)$ for fixed $x \in \mathbb{R}^n$.

We define the space $L^{\Phi}(\Omega)$ to consist of all measurable functions f on the open set $\Omega \subset \mathbb{R}^n$ with

$$\int_{\Omega} \Phi\left(x, \frac{|f(x)|}{\lambda}\right) dx < \infty$$

for some $\lambda > 0$. We define the norm

$$\|f\|_{\Phi(\cdot,\cdot)(\Omega)} \inf\left\{\lambda > 0 : \int_{\Omega} \Phi\left(x, \frac{|f(x)|}{\lambda}\right) dx \leq \infty\right\}$$

for $f \in L^{\Phi}(\Omega)$. These spaces have been studied in [9, 105]. Note that $L^{\Phi}(\Omega)$ is a Musielak–Orlicz space [111]. In case $q \equiv 0$, $L^{\Phi}(\Omega)$ reduces to the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$.

We prove a weak type inequality of maximal functions in Theorem (2.2.6). Then we prove in Theorem (2.2.11) a weak-type estimate for the Riesz potential. These enable us to prove the main result, a Sobolev embedding for functions in $W^{1,\Phi}$. The Sobolev space $W^{1,\Phi}(\Omega)$ contains of those functions $u \in L^{\Phi}(\Omega)$ with a distributional gradient satisfying $|\nabla_u| \in L^{\Phi}(\Omega)$. Further, we denote by $W_o^{1,\Phi}(\Omega)$ the closure of $C_o^{\infty}(\Omega)$ in the space $W^{1,\Phi}(\Omega)$ (cf. [83] for definitions of zero boundary value functions in the variable exponent context).

Let $p^{\#}(x)$ denote the Sobolev conjugate of $p(x)$, that is,

$$1/p^{\#}(x) = 1/p(x) - \alpha/n .$$

For the Sobolev embedding in $W^{1,\Phi}$ we need the conjugate exponent with $\alpha = 1$, which is denoted by p^*.

Theorem (2.2.1)[114]: Let p and q satisfy the above conditions. If $p^+ < n$, then

$$\|u\|_{\psi(\cdot,\cdot)(\Omega)} \leq c_1 \|\nabla_u\|_{\Phi(\cdot,\cdot)(\Omega)}$$

for every $u \in W_o^{1,\Phi}(\Omega)$, where $\Phi(x,t) := \left(t \log(c_0 + t)^{q(x)/p(x)}\right)^{p(x)}$ and $\psi(x,t) := \left(t \log(c_0 + t)^{q(x)/p(x)}\right)^{p^*(x)}$.

This extends [105] and [99, Theorem (2.2.10)] which dealt with the case $q \equiv 0$.

Proof. We may split $\mathbb{R}^n$ into a finite number of cubes $\Omega_1, \dots\dots\dots, \Omega_k$ and the complement of a cube, Ω_0, in such a way that $p^+_{\Omega_i} < (p^*)^-_{\Omega_i}$ for each i. Then

$$\|u\|_{\psi(\cdot,\cdot)(\mathbb{R}^n)} \leq \sum_{i=0}^{k} \|u\|_{\psi(\cdot,\cdot)(\Omega_i)} \leq C_1 \sum_{i=0}^{k} \|\nabla_u\|_{\Phi(\cdot,\cdot)(\mathbb{R}^n)} = (k+1)c_1 \|\nabla_u\|_{\Phi(\cdot,\cdot)(\Omega)}.$$

by Lemma (2.2.14).

In order to prove the main result, a weak-type for the maximal function, we start by presenting several preparatory results.

Let $B(x,r)$ denote the open ball centered at x with radius r. For a locally integrable function f on $\mathbb{R}^n$, we consider the maximal function Mf defined by

$$Mf(x) := \sup_B f_B = \sup_B \frac{1}{|B|} \int_B |f(y)| dy,$$

where the supremum is taken over all balls $B = B(x,r)$ and $|B|$ denotes the volume of B.

The following lemma is an improvement of [110].

Lemma (2.2.2)[114]: Let f be a non-negative measurable function on $\mathbb{R}^n$ with $\|f\|_{\Phi(\cdot,\cdot)(\Omega)} \le 1$. Set

$$I := \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)dy$$

and

$$J := \frac{1}{|B(x,r)|}\int_{B(x,r)} \Phi\big(y, f(y)\big)dy.$$

Then

$$I \le C\big\{J^{1/p(x)}(\log(c_0 + J))^{-q(x)/p(x)} + 1\big\}.$$

Proof. By condition (Φ_2),we have for $K > 0$

$$1 \le K + \frac{C}{|B(x,r)|}\int_{B(x,r)} f(y)\left(\frac{f(y)}{K}\right)^{p(x)-1}\left(\frac{\log(c_0 + f(y))}{\log(c_0 + K)}\right)^{q(x)} dy,$$

where the first term, K, represents the contribution to the integral of points where $f(y) < K$. If $J \le 1$, then we take $K = 1$ and obtain

$$1 \le 1 + CJ \le C.$$

Now suppose that $J \ge 1$ and set

$$K := J^{1/p(x)}(\log(c_0 + J))^{-q(x)/p(x)}.$$

Note that $J^{C/\log(CJ^{1/n})} \le C$ and $(\log(c_0 + J))^{C/\log(\log(e+CJ^{1/n}))} \le C$. Since we assumed that $\|f\|_{\Phi(\cdot,\cdot)(\mathbb{R}^n)} \le 1$, we conclude that

$$J \le K + \frac{1}{|B(x,r)|}\int_{\mathbb{R}^n} \Phi\big(y, f(y)\big)\ dy\ \le \frac{1}{|B(x,r)|}.$$

Hence, by conditions (p2) and (q2), we obtain, for $y \in B(x,r)$, that

$$\begin{aligned} K^{-p(y)} &\le \big\{CJ^{1/p(x)}(\log(c_0 + J))^{-q(x)/p(x)}\big\}^{-p(x)+C/\log(1/r)} \\ &\le \big\{CJ^{1/p(x)}(\log(c_0 + J))^{-q(x)/p(x)}\big\}^{-p(x)+C/\log(CJ^{1/n})} \\ &\le CJ^{-1}(\log(c_0 + J))^{p(x)}. \end{aligned}$$

and

$$\begin{aligned} (\log(c_0 + K))^{-q(y)} &\le \{C\log(c_0 + J)\}^{-q(x)+C/\log(\log(e+1/r))} \\ &\le \{C\log(c_0 + J)\}^{-q(x)+C/\log(\log(e+CJ^{1/n}))} \\ &\le (\log(c_0 + J))^{-q(x)}. \end{aligned}$$

Consequently it follows that

$$I \le CJ^{1/p(x)}(\log(c_0 + J))^{-q(x)/p(x)}.$$

Combining this with the estimate $I \le C$ from the previous case yields the claim.

Lemma (2.2.2), for each bounded open set G in $\mathbb{R}^n$ we can find a positive constant C such that

$$\{Mf(x)\}^{p(x)} \le C\left\{Mg(x)\big(\log\big(c_0 + Mg(x)\big)\big)^{-q(x)} + (1 + |x|)^{-n}\right\} \qquad (5)$$

for all $x \in G$ and $g(y) := \Phi\big(y, f(y)\big)$, whenever f is nonnegative measurable function on $\mathbb{R}^n$ with $\|f\|_{\Phi(\cdot,\cdot)(\mathbb{R}^n)} \le 1$.

For later use, it is convenient to note that

$$C^{-1}(1+|x|)^{-n/p_\infty} \le (1+|x|)^{-n/p(x)} \le C(1+|x|)^{-n/p_\infty} \tag{6}$$

in view of (p_3).

Lemma (2.2.3)[114]: Let f is nonnegative measurable function on $\mathbb{R}^n$ with $\|f\|_{\Phi(\cdot,\cdot)(\mathbb{R}^n)} \le 1$. If $J \le 1$, then

$$I_1 := \frac{1}{|B(x,r)|}\int_{B(x,r)\setminus B(0,|x|,/2)} f(y)\,dy \le C\{J^{1/p(x)} + (1+|x|)^{-n/p(x)}\}$$

Proof. By condition (Φ_2), we have for $K > 0$

$$1 \le K + \frac{C}{|B(x,r)|}\int_{B(x,r)\setminus B(0,|x|,/2)} f(y)\left(\frac{f(y)}{K}\right)^{p(x)-1}\left(\frac{\log(c_0+f(y))}{\log(c_0+K)}\right)^{q(x)} dy.$$

Then we take $K := \max\{J^{1/p(x)} + (1+|x|)^{-n/p(x)}\} \le 1$ and find

$$I_1 \le K + CK^{-p(x)+1}J \le CK$$

since $p(y) \le p(x) + C/\log(e+|x|)|)$ for $y \in B(x,r)\setminus B(0,|x|,/2)$ by (p3). Thus the proof is complete.

Lemma (2.2.4)[114]: Let f is nonnegative measurable function on $\mathbb{R}^n$ with $\|f\|_{\Phi(\cdot,\cdot)(\mathbb{R}^n)} \le 1$. If $J \le 1$, then

$$I_2 := \frac{1}{|B(x,r)|}\int_{B(x,r)\cap B(0,|x|,/2)} f(y)\,dy \le C(1+|x|)^{-n/p(x)}.$$

Proof. Since $J \le 1$, we see from Lemma (2.2.2) that I_2 is bounded on $B(0,e)$, so that we have only treat the case when $|x| \ge e$.

If $\le |x|/2$, then the integration set is empty and the claim is trivial. We will show that

$$I' := \frac{1}{|B(0,r)|}\int_{B(0,r)} f(y)\,dy \le Cr^{-n/p(x)} \tag{7}$$

for $r > 1$. Since $I_2 \le I'$ when $r > |x|/2$, the claim then follows.

By condition (Φ_2), we have for a measurable function $K = K(y) > 0$

$$I' \le \frac{1}{|B(0,r)|}\int_{B(0,r)} K(y)\,dy + \frac{C}{|B(0,r)|}\int_{B(0,r)} f(y)\left(\frac{f(y)}{K(y)}\right)^{p(x)-1}\left(\frac{\log(c_0+f(y))}{\log(c_0+K(y))}\right)^{q(x)} dy.$$

If $p_\infty > 1$, then we take $K := (1+|x|)^{-n/p(x)}$ and find that

$$I' \le C\left(r^{-n/p_\infty} + r^{n(p_\infty-1)/p_\infty)}J'\right)$$

By use of (p3), where

$$J' := \frac{1}{|B(0,r)|}\int_{B(0,r)} \Phi(y,f(y))\,dy.$$

If $p_\infty = 1$, then we take $K := (1+|x|)^{-\beta}$ for $\beta > n$ and obtain

$$I' \le C(r^{-n} + J').$$

Noting that $I' \le Cr^{-n}$ completes the proof.

Lemma (2.2.5)[114]: Let f is nonnegative measurable function on an open set Ω with $\|f\|_{\Phi(\cdot,\cdot)(\Omega)} \le 1$. If $J \le 1$. Set

$$N(x) := Mg(x)^{1/p(x)} Mg(x)\big(\log\big(c_0 + Mg(x)\big)\big)^{-q(x)/p(x)},$$
where $g(y) := \Phi\big(y, f(y)\big)$. Then

$$\int_{E_t} \Phi(x,t)\,dx \le C,$$

where $E_t := \{x \in \Omega : N(x) > t, Mg(x) > C_1(1+|x|)^{-n}\}$ and $C_1|B(0,1/2)|^{-1}$.

Proof. By the Besicovitch covering theorem, we can find a countable family of bolls $B_i = B(x_i, r_i)$ with a bounded overlap property such that $E_t \subset \cup_i B_i$,

$$t < g_{B_i}^{1/p(x_i)}\big(\log\big(c_0 + g_{B_i}\big)\big)^{-q(x_i)/p(x_i)}$$

and

$$g_{B_i} > C_1(1+|x_i|)^{-n}.$$

If $1 \le g_{B_i} \le |B_i|^{-1}$, then conditions (p2) and (p2) imply that

$$g_{B_i}^{1/p(x_i)}\big(\log\big(c_0 + g_{B_i}\big)\big)^{-q(x_i)/p(x_i)} \le C g_{B_i}^{1/p(x)}\big(\log\big(c_0 + g_{B_i}\big)\big)^{-q(x)/p(x)}$$

for $x \in B_i$; and if $C_1(1+|x_i|)^{-n} < g_{B_i} \le 1$, then $r_i \le (1+|x_i|)/2$, so that the above inequality by use of (p3). A similar argument holds for changing $q(x_i)$ to $q(x)$. Thus we obtain

$$\begin{aligned}
&\Phi\left(x, g_{B_i}^{1/p(x_i)}\big(\log\big(c_0 + g_{B_i}\big)\big)^{-q(x)/p(x)}\right) \\
&\quad \le C\Phi\left(g_{B_i}^{1/p(x)}\big(\log\big(c_0 + g_{B_i}\big)\big)^{-q(x)/p(x)}\right) \\
&\quad = C g_{B_i}\big(\log\big(c_0 + g_{B_i}\big)\big)^{-q(x)}\left(\log(c_0 + g_{B_i}^{1/p(x_i)})\left(\log\big(c_0 + g_{B_i}\big)\right)^{-q(x_i)/p(x_i)})\right)^{q(x)} \\
&\quad \le C g_{B_i}.
\end{aligned}$$

Hence we see that

$$\begin{aligned}
\int_{E_t} \Phi(x,t)\,dx &\le \sum_i \int_{B_i} \Phi(x,t)\,dx \\
&\le C\sum_i \int_{B_i} g_{B_i}\,dx \le C\sum_i \int_{B_i} g(y)\,dy \\
&\le C\int_{\Omega} g(y)\,dy \le C.
\end{aligned}$$

We are now ready for the first main result, a weak-type estimate for the maximal function. This is an extension of [101] and [84, Theorem (2.2.8)].

Theorem (2.2.6)[114]: Let f is nonnegative measurable function on $\mathbb{R}^n$ with $\|f\|_{\Phi(\cdot,\cdot)(\mathbb{R}^n)} \le 1$. If $J \le 1$, then

$$\int_{\{x\in\mathbb{R}^n:\, Mf(x)>t\}} \Phi(x,t)\,dx \le C.$$

Proof. Lemmas (2.2.2)– (2.2.4) and (6) give

$$I \le C\{J^{1/p(x)}(\log(c_0 + J))^{-q(x)/p(x)} + (1+|x|)^{-n/p_\infty}\} \tag{8}$$

for $x \in \mathbb{R}^n$. Hence

$$\{x \in \mathbb{R}^n : Mf(x) > t\} \subset E_t \cup \{x \in \mathbb{R}^n : (1+|x|)^{-n/p_\infty} > t/C\}$$

with E_t as in Lemma (2.2.5). Note that we may $Mg(x) \geq C_1(1+|x|)^{-n}$ in the first set since if $Mg(x) \leq C_1(1+|x|)^{-n}$ and $N(x) > t/C$, then $C_1(1+|x|)^{-n/p_\infty} > t/C$.

If the second set is empty, the claim follows from Lemma (2.2.5). If this is not the case we define $r > 0$ so that $(1+|x|)^{-n/p_\infty} = t/C$. Note that t is bounded in this case. Then

$$\int_{\{x\in\mathbb{R}^n\,:\,Mf(x)>t\}} \Phi(x,t)\,dx \leq \int_{E_t} \Phi(x,t)\,dx + \int_{B(0,r)} \Phi(x,t)\,dx.$$

The first integral on the right hand side is bounded by Lemma (2.2.5). For the second, we note that $\Phi(x,t)) \leq Ct^{p(x)}$ since t and since q are bounded. By the definition of r we have

$$\int_{B(0,r)} t^{p(x)}\,dx \leq C\int_{B(0,r)} (1+r)^{-np(x)/p_\infty}\,dx$$
$$\leq C\int_{B(0,r)} (1+r)^{-n+(C_n/p_\infty)/\log(e+|x|)}\,dx.$$

For $0 < m < n$, noting that $(1+r)^{-m+(C_n/p_\infty)/\log(e+t)}(1+t)^m$ is bounded on (c_1, r) when $m + (C_n/p_\infty)/\log(e+c_1) < 0$, we find

$$\int_{B(0,r)} t^{p(x)}\,dx \leq C\int_{B(0,c_1)} t^{p(x)}\,dx + C(1+r)^{m-n}\,dx \leq \int_{B(0,r)} (1+r)^{-m}\,dx \leq C.$$

Therefore $\int_{B(0,r)} \Phi(x,t)\,dx \leq C$, and so we obtain the theorem.

Take $\omega \in C^\infty(\mathbb{R})$ such that $0 \leq \omega \leq 1, \omega(r) = 0$ when $r \leq 0$ and $\omega(r) = 1$ when $r \geq 1/2$. Let

$$p(x) := \frac{a\log(e+\log(e+|x|))}{\log(e+|x|)}\,\omega\left(\frac{2x_n - |x|}{1+|x|}\right)$$

for $x = (x_1, \dots\dots, x_n)$, where $a > 0$. Consider the function

$$f(x) := \begin{cases} (e+|x|)^{-n}(\log(e+|x|))^\beta & \text{if } 4x_n > 3|x|+1, \\ 0 & \text{elsewhere.} \end{cases}$$

If $-1 < \beta < an - 1$, then $f \in L^{p(\cdot)}(\mathbb{R}^n)$. Note that

$$Mf(x) \geq (e+|x|)^{-n}(\log(e+|x|))^{\beta+1}$$

for all $x \in \mathbb{R}^n$. There exists a constant $C > 0$ such that if

$$|x| \leq Ct^{-1/n}(\log(e+t^{-1}))^{(\beta+1)/n},$$

Then $Mf(x) > t$, so that

$$\int_{\{x\in\mathbb{R}^n:\,Mf(x)>t\}} t^{p(x)}\,dx \geq t|\{x\in\mathbb{R}^n:\ Mf(x) > t, x_n < 0\}|$$
$$\geq C(\log(e+t^{-1}))^{\beta+1},$$

which tends to ∞ as $t \to 0+$. This example shows that the assumption on the exponent in our weak type estimate is quite sharp.

For $0 < \alpha < n$, we define the Riesz potential of order α for a locally integrable function f on $\mathbb{R}^n$ by

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy.$$

Here it is natural to assume that

$$\int_{\mathbb{R}^n} (1 + |y|^{n-\alpha} |f(y)|)\ dy < \infty\,, \tag{9}$$

which is equivalent to the condition that $I_\alpha |f| \not\equiv \infty$ (see [110, Theorem (2.2.1)]).

To establish weak-type estimates for Riesz potentials of functions in $L^\Phi(\mathbb{R}^n)$, when the exponent p satisfies

$$p^+ < n/\alpha.$$

Let $p^\#(x)$ denote the Sobolev conjugate of $p(x)$, as defined.

Lemma (2.2.7)[114]: Suppose that $p^+ < n/\alpha$. If f is nonnegative measurable function on $\mathbb{R}^n$ with $\|f\|_{\Phi(\cdot,\cdot)(\mathbb{R}^n)} \le 1$, then

$$\int_{\mathbb{R}^n \setminus B(x,r)} \frac{f(y)}{|x-y|^{n-\alpha}}\ dy \le C\left\{ r^{\alpha - n/p(x)} + (1+|x|)^{\alpha - n/p_\infty} \right\}$$

for all $x \in \mathbb{R}^n$ and $r \ge 1/e$

proof. If $|x| \le r$ and $r \ge 1/e$, then (7) gives

$$\begin{aligned}\int_{\mathbb{R}^n \setminus B(x,r)} \frac{f(y)}{|x-y|^{n-\alpha}}\ dy &\le C \int_{\mathbb{R}^n} (r + |y|)^{\alpha - n} f(y) dy \\ &\le C \int_0^\infty \left(\int_{B(0,t)} f(y) dy \right) (r+t)^{\alpha - n - 1}\, dt \\ &\le C r^{\alpha - n/p(x)} \le C(1+|x|)^{\alpha - n/p_\infty}.\end{aligned}$$

Next consider the case $|x| > r \ge 1/e$. Then we have

$$\begin{aligned}\int_{B(0,|x|/2) \setminus B(x,r)} \frac{f(y)}{|x-y|^{n-\alpha}}\ dy &\le C|x|^{\alpha - n} \int_{B(0,|x|/2)} f(y) dy \\ &\le C|x|^{\alpha - n/p_\infty}\,.\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^n \setminus B(0,2|x|)} \frac{f(y)}{|x-y|^{n-\alpha}}\ dy &\le C \int_{\mathbb{R}^n \setminus B(0,2|x|)} |y|^{\alpha - n} f(y) dy \\ &\le C \int_{2|x|}^\infty \left(\int_{B(0,t)} f(y) dy \right) t^{\alpha - n - 1}\, dt \\ &\le C(1+|x|)^{\alpha - n/p_\infty}.\end{aligned}$$

It remains to estimate the integral of $|x-y|^{\alpha - n} f(y)$ over the set $E :=$ $B(0, 2|x|) \setminus \{B(0,|x|/2) \cup B(x,r)\}$. By condition (Φ_2), we have $K(y) := |x - y|^{-n/p(x)}$

$$\begin{aligned}\int_E \frac{f(y)}{|x-y|^{n-\alpha}}\ dy &\le \int_E \frac{K(y)}{|x-y|^{n-\alpha}}\ dy \\ &\quad + \int_E \frac{f(y)}{|x-y|^{n-\alpha}} \left(\frac{f(y)}{K(y)} \right)^{p(x)-1} \left(\frac{\log(c_0 + f(y))}{\log(c_0 + K(y))} \right)^{q(x)} dy \\ &\le C r^{\alpha - \frac{n}{p(x)}} + C r^{\alpha - n + \frac{n(p(x)-1)}{p(x)}} \int_E \Phi\big(y, f(y)\big)\, dy \\ &\le C r^{\alpha - n/p(x)}\end{aligned}$$

since $p(y) \le p(x) + C/\log|x|$ for $y \in \mathbb{R}^n \setminus B(0,|x|/2)$ by (p3) and $\alpha p^+ < n$.

Lemma (2.2.8)[114]: Suppose that $p^+ < n/\alpha$. If f is nonnegative measurable function on $\mathbb{R}^n$ with $\|f\|_{\Phi(\cdot,\cdot)(\mathbb{R}^n)} \le 1$. Then

$$\int_{B(x,1/e)\setminus B(x,\delta)} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy \le C\delta^{\alpha-n/p(x)}(\log(c_0+1/\delta))^{-q(x)/p(x)}$$

for all $x \in \mathbb{R}^n$ and $0 < \delta < 1/e$.

Proof. The proof is similar to the last case in the previous proof. Let us set $E := B(x,1/e)\setminus B(x,\delta)$ and

$$K(y) := |x-y|^{-n/p(x)}(\log(c_0+1/|x-y|))^{-q(x)/p(x)}$$

for $x \in E$. By condition (p2), (q2) and (Φ_2), we obtain

$$\begin{aligned}\int_E \frac{f(y)}{|x-y|^{n-\alpha}}\, dy &\le \int_E \frac{K(y)}{|x-y|^{n-\alpha}}\, dy \\ &+C\int_E \frac{f(y)}{|x-y|^{n-\alpha}}\left(\frac{f(y)}{K(y)}\right)^{p(x)-1}\left(\frac{\log(c_0+f(y))}{\log(c_0+K(y))}\right)^{q(x)} dy \\ &\le C\Bigg(\delta^{\alpha-n/p(x)}+(\log(c_0+1/\delta))^{-q(x)/p(x)} \\ &+\int_E |x-y|^{-n/p(x)}(\log(c_0+1/|x-y|))^{-q(x)/p(x)}\Phi(y,f(y))\, dy\Bigg) \\ &\le C\delta^{\alpha-n/p(x)}(\log(c_0+1/\delta))^{-q(x)/p(x)}\left(1+\int_E \Phi(y,f(y))\, dy\right) \\ &\le C\delta^{\alpha-n/p(x)}(\log(c_0+1/\delta))^{-q(x)/p(x)},\end{aligned}$$

as required.

The next lemma generalization of [110].

Lemma (2.2.9)[114]: Suppose that $p^+ < n/\alpha$. Let $f \in L^{\Phi}(\mathbb{R}^n)$ be nonnegative with $\|f\|_{\Phi(\cdot,\cdot)(\mathbb{R}^n)} \le 1$. Then

$$I_\alpha f(x) \le C\{Mf(x)^{p(x)/p^\#(x)}(\log(c_0+Mf(x)))^{-\alpha q(x)/n} + (1+|x|)^{-n/p^\#_\infty}\}.$$

Proof. By Lemmas (2.2.7) and (2.2.8),

$$\begin{aligned}I_\alpha f(x) = \int_{B(x,\delta)} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy &\le \int_{\mathbb{R}^n\setminus B(x,\delta)} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy \\ &\le C\{\delta^\alpha Mf(x)+\delta^{\alpha-n/p(x)}(\log(c_0+1/\delta))^{-q(x)/p(x)} + (1+|x|)^{\alpha-n/p_\infty}\}\end{aligned}$$

for $\delta > 0$. Here, letting

$$\delta = \min\{Mf(x)^{-p(x)/n}(\log(c_0+Mf(x)))^{-q(x)/n} + 1+|x|\},$$

we find

$$I_\alpha f(x) \le C\{Mf(x)^{p(x)/p^\#(x)}(\log(c_0+Mf(x)))^{-\alpha q(x)/n} + (1+|x|)^{-n/p^\#_\infty}\}.$$

Recall that $\psi(x,t) = \left(t\log(c_0+t)^{q(x)/p(x)}\right)^{p^\#(x)}$.

Lemma (2.2.10)[114]: Suppose that $p^+ < n/\alpha$. Let f be a nonnegative measurable function on an open set Ω with $\|f\|_{\Phi(\cdot,\cdot)(\Omega)} \le 1$. Set

$$N(x) := Mg(x)^{1/p^\#(x)}(\log(c_0+Mg(x)))^{-q(x)/p(x)},$$

where $g(y) := \Phi(y,f(y))$ then

$$\int_{\tilde{E}_t} \psi(x,t)\ dy \le C,$$

where $\tilde{E}_t \coloneqq \{x \in \Omega : N(x) > t, Mg(x) \ge C_1(1+|x|)^{-n}\}$ and $C_1 \coloneqq |B(0,1/2)|^{-1}$.
Proof. By the Besicovitch covering theorem, we can find a countable family of balls $B_i = B(x_i, r_i)$ with abounded over property such that $\tilde{E}_t \subset \cup_i B_i$,

$$t < g_{B_i}^{1/p^{\#}(x_i)}\left(\log\left(c_0 + g_{B_i}\right)\right)^{-q(x_i)/p(x_i)}$$

and

$$g_{B_i} > C_1(1+|x|)^{-n}.$$

As in Lemma (2.2.5), we obtain

$$\begin{aligned}\psi&\left(x, g_{B_i}^{1/p^{\#}(x_i)}\left(\log\left(c_0 + g_{B_i}\right)\right)^{-q(x_i)/p(x_i)}\right)\\ &\le C\psi\left(x, g_{B_i}^{1/p^{\#}(x_i)}\left(\log\left(c_0 + g_{B_i}\right)\right)^{-q(x_i)/p(x_i)}\right)\\ &\le C g_{B_i}\end{aligned}$$

for $x \in B_i$. Hence obtain as before that

$$\begin{aligned}\int_{\tilde{E}_t} \psi(x,t)\,dx &\le \sum_i \int_{B_i} \psi(x,t)\,dx\\ &\le \sum_i \int_{B_i} g_{B_i}\,dx \le C\int_{\Omega} g(y)\,dy \le C.\end{aligned}$$

Now we are ready to show the weak- type estimate for Riesz potentials, as extension of [101] and [82, Theorem (2.2.10)].

Lemma (2.2.11)[114]: Suppose that $p^+ < n/\alpha$. Let f be a nonnegative measurable function on $\mathbb{R}^n$ with $\|f\|_{\Phi(\cdot,\cdot)(\mathbb{R}^n)} \le 1$. Then

$$\int_{\{x\in\mathbb{R}^n:\, I_\alpha f(x) > t\}} \psi(x,t)\,dx \le C.$$

Proof. By Lemmas (2.2.9) and (2.2.2)–(2.2.4) give

$$\begin{aligned}I_\alpha f(x) &\le C\left\{Mf(x)^{p(x)/p^{\#}(x)}(\log(c_0 + Mf(x)))^{-\alpha q(x)/n} + (1+|x|)^{-n/p_\infty^{\#}}\right\}\\ &\le C\left\{Mg(x)^{p(x)/p^{\#}(x)}(\log(c_0 + Mg(x)))^{-q(x)/n} + (1+|x|)^{-n/p_\infty^{\#}}\right\}\end{aligned}$$

for $x \in \mathbb{R}^n$. Hence

$$\{x \in \Omega : I_\alpha f(x) > t\} \subset \tilde{E}_t \cup \left\{x \in \Omega : (1+|x|)^{-n/p_\infty^{\#}} > t/C\right\},$$

where $\tilde{E}_t$ is as in Lemma (2.2.10). If the second set is empty, then the claim follows from Lemma (2.2.10). If this is not the case we define $r > 0$ so that $(1+r)^{-n/p_\infty^{\#}} = t/C$. Then

$$\int_{\{x\in\mathbb{R}^n:\, I_\alpha f(x) > t\}} \psi(x,t)\,dx \le \int_{\tilde{E}_t} \psi(x,t)\,dx \le \int_{B(0,r)} \psi(x,t)\,dx.$$

The first integral on the right hand side is bounded by Lemma (2.2.10). For second we note that $\psi(x,t) \le Ct^{\#(x)}$ since t and $q(\cdot)$ are bounded. Thus

$$\int_{B(0,r)} t^{\#(x)}\,dx \le \int_{B(0,r)} C(1+r)^{-n+(C_n/p_\infty^{\#})/\log(e+|x|)}\,dx \le c,$$

where the last step follows exactly as in the proof of Theorem (2.2.6).
Continuing with the notation, we further see that

$$I_\alpha f(x) \ge C(1+r)^{\alpha-n}(\log(e+|x|))^{\beta+1}$$

for all$x \in \mathbb{R}^n$, so that

$$\int_{\{x\in\mathbb{R}^n:\, I_\alpha f(x)>t\}} t^{\#(x)}\, dx \geq t^{\frac{n}{n-\alpha}}\{x\in\Omega : I_\alpha f(x) > t, x_n < 0\}$$
$$\geq C(\log(e+t^{-1}))^{n(\beta+1)/(n-\alpha)},$$
which tends to ∞ as $t \to 0+$.
In view of [109], for each $\beta > 1$ one can find a constant $C > 1$ such that
$$\int_{\mathbb{R}^n} \{I_\alpha f(x)\}^{\#(x)} (\log(e + I_\alpha f(x)))^{-\beta} (\log(e + I_\alpha f(x)^{-1}))^{-\beta} dx \leq C$$
whenever f is a nonnegative measurable function on $\mathbb{R}^n$ with $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq 1$. This gives a supplement of O'Neil [82].
Let us consider the generalized Orlicz-Sobolev space $W^{1,\Phi}(\Omega)$ with the norm
$$\|u\|_{1,\Phi(\cdot,\cdot)(\Omega)} = \|u\|_{\Phi(\cdot,\cdot)(\Omega)} + \|\nabla_u\|_{\Phi(\cdot,\cdot)(\Omega)} < \infty$$
Further we denote by $W_0^{1,\Phi}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in the space $W^{1,\Phi}(\Omega)$ (cf. [81] for definitions of zero boundary value functions in the variable exponent context). We derive a Sobolev inequality for functions in $W_0^{1,\Phi}(\Omega)$ as the application of Sobolev's weak type inequality for Riesz potentials of functions in $L^\Phi(\Omega)$. Frist note the following lemma:
Lemma (2.2.12)[114]: Set $k(y,t) = (\log(e+t))^y$ for y and $t \geq 0$. Then
$$k(y,at) \leq \tau(y,a)k(y,t)$$
whenever $a, t > 0$, where
$$\tau(y,t) = a \max\{(C\log(e+a))^y, (C\log(e+a^{-1}))^{-y}\}.$$
We define local versions of p^+ and p^- as follows:
$$p_\Omega^- = \operatorname{ess\,inf}_{x\in\Omega} p(x) \quad \text{and} \quad p_\Omega^+ = \operatorname{ess\,inf}_{x\in\Omega} p(x).$$
Using the previous lemma we can derive a scaled version of the weak type estimate from which will be needed below.
Lemma (2.2.13)[114]: Let Ω be an open set in $\mathbb{R}^n$. Suppose that $p^+ < n/\alpha$. Let $f \in L^\Phi(\mathbb{R}^n)$ be a nonnegative with $\|f\|_{\Phi(\cdot,\cdot)(\mathbb{R}^n)} \leq 1$. Then for every $\varepsilon > 0$ there exists a constant $C > 0$ such that
$$\int_{\{x\in\mathbb{R}^n \,:\, I_\alpha f(x)>t\}} \psi(x,t)\ dx \leq C\|f\|_{\Phi(\cdot,\cdot)(\mathbb{R}^n)}^{(p^\#)_\Omega^- - \varepsilon},$$
for every $t > 0$.
Proof. For simplicity we denote $\|f\|_{\Phi(\cdot,\cdot)(\mathbb{R}^n)}$ by $a \in [0,1]$. The case $a = 0$ is clear, so we assume that $a > 0$. We apply Theorem (2.2.11) to the function f/a, which has norm equal to 1. Thus
$$\int_{\{x\in\Omega \,:\, I_\alpha f(x)>s\}} \psi(x,t)\ dx = \int_{\left\{x\in\Omega \,:\, I_\alpha \frac{L}{a}(x)>t\right\}} \psi(x,t)\ dx$$
$$\leq \int_{\left\{x\in\mathbb{R}^n \,:\, I_\alpha \frac{L}{a}(x)>t\right\}} \psi(x,t)\ dx \leq C.$$
With k as in the previous lemma and $r = q(x)p^\#(x)/p(x)$, we have $\psi(x,t) = t^{p^\#(x)-1}k(r,t)$. Hence the lemma implies that
$$\psi(x,s/a) = \psi(x,s)a^{1-p^\#(x)}\frac{k(r,s/a)}{k(r,s)} \geq \psi(x,s)a^{1-p^\#(x)}\tau(r,a)^{-1}.$$

Since τ is logarithmic and $a \leq 1$, it follows $a^{1-p^{\#}(x)}\tau(r,a) \leq Ca^{(p^{\#})_{\Omega}^{-}-\varepsilon}$. Now the claim follows by combining the inequalities derived.

Lemma (2.2.14)[114]. Suppose that $p^{+} < n, p_{\Omega}^{+}$. Let and Ω be an open set. If $u \in W_0^{1,\Phi}(\mathbb{R}^n)$, then there exists a constant $C_1 > 0$ such that

$$\|u\|_{\Phi(\cdot,\cdot)(\Omega)} \leq C_1 \|\nabla_u\|_{\Phi(\cdot,\cdot)(\mathbb{R}^n)} .$$

Proof. We may assume that $\|\nabla_u\|_{\Phi(\cdot,\cdot)(\mathbb{R}^n)} \leq 1$ and u is nonnegative. It follows from [108] that

$$|v(x)| \leq C(n) I_1 |\nabla_v|(x)$$

for $v \in W_0^{1,\Phi}(\mathbb{R}^n)$ and almost every $x \in \mathbb{R}^n$. For $u \in W_0^{1,\Phi}(\mathbb{R}^n)$ and each integer j, we write $U_j = \{x \in \Omega :\ 2^j < u(x) \leq 2^{j+1}\}$ and $v_j = \max\{0, \min\{u - 2^j, 2^j\}\}$. Since $v \in W_0^{1,1}(\Omega)$ and $v_j(x) = 2^j$ for almost every $x \in U_{j+1}$, we have

$$I_1 \left|\nabla_{v_j}\right|(x) \geq C2^j$$

for almost every $x \in U_{j+1}$, it follows that

$$\begin{aligned}
\int_{\Omega} \psi(x, u(x))\, dx &\leq \sum_{j\in\mathbb{Z}} \int_{U_{j+1}} \psi\big(x, u(x)\big)\, dx \\
&\leq C \sum_{j\in\mathbb{Z}} \int_{U_{j+1}} \psi\big(x, u(x)\big)\, dx \\
&\leq C \sum_{j\in\mathbb{Z}} \int_{\left\{x\in U_{j+1}:\, I_1\left|\nabla_{v_j}\right|(x) > C2^j\right\}} \psi\big(x, u(x)\big)\, dx .
\end{aligned}$$

Taking $r \in (p^{+}, (p^{*})_{\Omega}^{-})$, we obtain by Lemma (2.2.13) that

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} \int_{\left\{x\in U_{j+1}:\, I_1\left|\nabla_{v_j}\right|(x) > C2^j\right\}} \psi\big(x, u(x)\big)\, dx &\leq C \sum_{j\in\mathbb{Z}} \|f\|_{\Phi(\cdot,\cdot)(\mathbb{R}^n)}^{r} \\
&\leq C \sum_{j\in\mathbb{Z}} \int_{U_j} \Phi(x, |\nabla_u(x)|)\, dx \leq C,
\end{aligned}$$

Which completes the proof.

Chapter 3

Mean Continuity Type Result with Maximal and Fractional Operators

We show that, if $p(x)$ is also continuous, for mollifiers $(\rho_n)_{n\in Z}$ the liminf and the limsup of

$$\left(\left\|\frac{|f(x)-f(y)|}{|x-y|}[\rho_n(x-y)]^{\frac{1}{\min p(x,y)}}\right\|_{L^{\min p(x,y)}(\Omega\times\Omega)}\right)_{n\in\mathbb{N}}$$

are respectively manrorized by majorized by expressions equitant to $\|\nabla f\|_{L^{p}(\Omega)}$. The spaces $L^{p(\cdot)}(\rho,\Omega)$ over a bounded open set in $\mathbb{R}^n$ with a power weight $\rho(x)=|x-x_0|^{\gamma}$, $x_0\in\bar{\Omega}$, and an exponent $p(x)$ satisfying the Dini-Lipschitz condition.

Section (3.1): Certain Sobolev Spaces with Variable Exponent

Throughout Ω will be a non-empty, open cube in $\mathbb{R}^N, N\in\mathbb{N}$ and $\mathcal{P}(\Omega)$ will be the class of all lower semi continuous functions $p=p(x)$ on Ω such that

$$p_* = \inf_{x\in\Omega} p(x) > 1$$
$$p_* = \sup_{x\in\Omega} p(x) < \infty$$

and satisfying the following condition:

(i) For every $x\in\Omega$ there exists numbers $0<r(x)\le 1$, $\zeta(x)>0$ and a vector $\xi(x)\in\mathbb{R}^N\backslash\{0\}$ such that

$$\zeta(x) < |\xi(x)| \le 1,$$
$$B\big(x,r(x)\big)+\bigcup_{0<t\le 1} B(t\xi(x),t\zeta(x))\subset\Omega,$$
$$p(x)\le p(x-y),\forall x\in\Omega,\qquad \forall y\in\bigcup_{0<t\le 1} B\big(t\xi(x),t\zeta(x)\big),$$

where for $x\in\mathbb{R}^N$ and $r>0$, $B(x,r)$ is open ball centered at x and with radius r.

Unless differently specified, in the sequel we will always assume that any function $p=\mathcal{P}(x)$ in the class $\mathcal{P}(\Omega)$. Examples of functions $p\in\mathcal{P}(\Omega)$ are the constant functions, the convex functions, the functions in the $C^1(\Omega)$ class with gradient different from zero in any point. A special explicit example given in Proposition (3.1.9).

We denote also

$$p_{*n} = \inf_{B\left(x,\frac{1}{n}\right)\cap\Omega} p(x) \qquad p_n^* = \sup_{B\left(x,\frac{1}{n}\right)\cap\Omega} p(x)$$

for all $x\in\Omega, n\in\mathbb{N}$.

For $x,y\in\Omega$ such that the segment with endpoints x,y is all contained in Ω, we set

$$\min p(x,y) = \min_{\lambda\in[0,1]} p(\lambda(x+(1-\lambda)y).$$

Notice that the definition is well-posed by virtue of the lower semi continuity of $p\in\mathcal{P}(\Omega)$.

For any (Lebesgue) measurable function f in Ω we define by

$$\|f\|_{L^{p(x)}(\Omega)} = \|f\|_{p(x)} = \inf\left\{\lambda > 0 : \int_{\Omega} \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}, \qquad (1)$$

The norm of f in Banach space $L^{p(x)}(\Omega)$. The $L^{p(x)}(\Omega)$ is called generalized Lebegue space, because if $p(x) \equiv p$ is constant, then the norm coincides with the usual norm of $L^{p(x)}(\Omega)$. In the sequal we will denote by $L^{p(x)}(\mathbb{R}^N)$ the space defined substituting the cube Ω by $\mathbb{R}^N$; its main properties are discussed in [13] and [23].

Problems involving variable exponents have been considered several times (see e.g., [15, 115, 116, 121, 122, 123, 124] and references therein).

We will consider also the following generalization of the norm (1). For $\alpha, \beta \in \mathcal{P}(\Omega)$, $\alpha(x) \le \beta(x)$ $\forall x \in \Omega$, let

$$\Phi_{\alpha,\beta}(t,x) = \max\left(t^{\alpha(x)}, t^{\beta(x)}\right) \qquad \forall t \ge 0, \qquad x \in \Omega,$$

and let us set

$$\|f\|_{\Phi_{\alpha,\beta}(t,x)} = \inf\left\{\lambda > 0 : \int_{\Omega} \Phi_{\alpha,\beta}\left(\frac{|f(x)|}{\lambda}, x\right) dx \le 1\right\}. \qquad (2)$$

We will write simply $\Phi_{\alpha,\beta}(t)$ in the place of $\Phi_{\alpha,\beta}(t,\ x)$ if α, β are constant functions. Norms (1), (2) are particular cases of norms of Musielak−Orlicz ([18]).

Some properties of the classical Lebesgue spaces can be generalized in the context of the $L^{p(x)}(\Omega)$ spaces. We state them as lemmas; proofs can be found in [13].

Lemma (3.1.1)[125]: Let $p = p(x) \in \mathcal{P}(\Omega)$. If $\|f\|_{p(x)} \le 1$, then $\int_{\Omega} |f(x)|^{p(x)} dx \le \|f\|_{p(x)}$.

Lemma (3.1.2)[125]: (Hölder's Inequality). Let $p = p(x) \in \mathcal{P}(\Omega)$, and let $p'(x) \in \mathcal{P}(\Omega)$ be the conjugate function of $p(x)$ defined by

$$p'(x) = \frac{p(x)}{p(x) - 1}.$$

Then the inequality

$$\int_{\Omega} |f(x)g(x)| \le \left(1 + \frac{1}{p_*} - \frac{1}{p^*}\right) \|f\|_{p(x)} \|g\|_{p'(x)}$$

holds for every $f \in L^{p(x)}(\Omega)$, $g \in L^{p'(x)}(\Omega)$.

Lemma (3.1.3)[125]: (Duality and Reflexivity). Let $p = p(x) \in \mathcal{P}(\Omega)$. The dual space to $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, and the space $L^{p(x)}(\Omega)$ is reflexive.

Lemma (3.1.4)[125]: (Embedding). Let $\alpha(x), \beta(x) \in \mathcal{P}(\Omega)$. The space $L^{\beta(x)}(\Omega)$ is continuously in embedded $L^{\alpha(x)}(\Omega)$ if and only if $(x) \le \beta(x)$ $\forall x \in \Omega$. The norm of the embedding operator does not exceed $1 + |\Omega|$.

Lemma (3.1.5)[125]: (Density). Let $p = p(x) \in \mathcal{P}(\Omega)$. Then sets $C(\Omega) \cap L^{p(x)}(\Omega)$, $C_c^{\infty}(\Omega)$ is $L^{p'(x)}(\Omega)$ are dense $L^{p(x)}(\Omega)$.

The theory on spaces $L^{p(x)}(\Omega)$ has been developed in [13] for exponents $p = p(x)$ lying in a much larger class than the ours, which has been called still by $\mathcal{P}(\Omega)$. Namely, the exponents $p(x)$ are simply measurable functions $p : \Omega \to [1, \infty]$. We stress that in the more general context of [13] the lemmas stated above are generally still true, but with some further conditions to be imposed on $p(x)$ and with less restrictive conditions on Ω. The simplified statements are true only because of the strong assumptions made on $\mathcal{P}(\Omega)$.

Differently from [13], we will write in the sequel, for not necessarily constant functions $p = p(x) \in \mathcal{P}(\Omega)$,

$L^{p(x)}(\Omega)$	instad of	$L^{p}(\Omega)$
$\|\cdot\|_p$	instad of	$\|\cdot\|_p$
$p(x) \in \mathcal{P}(\Omega)$	instad of	$p(x) \in \mathcal{P}(\Omega)$

i.e. we will write explicitly the x-varible. Even if after the generalization of the classical theory made in [3] the symbol $L^p(\Omega)$ is not confessional when it denotes a generalized Lebesgue space with variable exponent, we will write the variable x because is mainly on properties, which are in general not hereditated when the exponents are variable.

The generalized Sobolev space immediately $W^{1,p(x)}(\Omega)$ is defined the Banach space of all measurable functions f on Ω such that $|\nabla f| \in L^{p(x)}(\Omega)$, endowed with the norm

$$\|f\|_{1,p(x)} = \|f\|_{p(x)} + \|\nabla f\|_{p(x)}.$$

Also $W^{1,p(x)}(\Omega)$ spaces are particular cases of Musielak–Orlicz spaces. For its main properties, besides the already quoted [13], see [8, 12, 23, 120].

Fundamental for the sequel is the following result, due to Edmunds and Rá kosnı́k (see [120]), proved also for higher order Sobolev spaces:

Theorem (3.1.6)[125]: Let $A \subset \mathbb{R}^N$ be an open, non-empty set let $p : \Omega \to [1,\infty[$ be a measurable function satisfying the condition (i) (the inequality $p(x) \le p(x+y)$ being satisfied almost everywhere). Then the set $C^\infty(A) \cap W^{1,p(x)}(A)$ is dense in $W^{1,p(x)}(A)$.

A more recent result for functions defined in all the space $\mathbb{R}^N$ appears in Samko ([23]), in which (i) substituted by a certain continuity condition (the Dini–Lipschitz condition).

As declared by Ková čik and Rá kosnı́k in ([13]) about the spaces $L^{p(x)}(\Omega)$, $W^{k,p(x)}(\Omega)$, it appears that spaces $L^{p(x)}(\Omega)$ and $L^p(\Omega)$ $(1 < p < \infty)$ have many common properties except a very important one: the p-mean continuity. A function $f \in L^{p(x)}(\Omega)$ is called $p(x)$-main continuous if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\int_\Omega |f(x+h) - f(x)|^{p(x)} dx < \varepsilon \quad \forall h \in \mathbb{R}^N, \quad |h| < \delta.$$

When $p(x) \equiv p$, we will speak of p-main continuity.

It well-known that functions in $L^p(\Omega)$ p-main continuous:

$$f \in L^p(\Omega) \Rightarrow \|f(x-y) - f(x)\|_p = o(h)$$

and that for functions in classical Sobolev spaces it is possible to estimate the order of $o(h)$ (see e.g. [117]):

$$f \in W^{1,p}(\Omega) \Rightarrow \forall \text{ open set } \omega \subset\subset \Omega,\ \forall h \in \mathbb{R}^N,\ |h| < \operatorname{dist}(\omega, {}^c\Omega), \quad (3)$$
$$\|f(x-y) - f(x)\|_{L^p(\omega)} \le \|\nabla f\|_{L^p(\Omega)}.$$

When the exponent p is a not constant function of x, in general, these properties do not hold true: in [13] it is shown that for any continuous and not constant $p(x) \in \mathcal{P}(\Omega)$ there exist functions in $L^p(\Omega)$ which are not $p(x)$-mean continuous, and, in general, $p(x)$-mean continuity fails also for $W^{1,p}(\Omega)$ functions, therefore (3) cannot have its direct counterpart for variable exponents.

In [118] a new expression for the $L^p(\Omega)$ norm of the gradient for functions in classical Sobolev spaces has been discovered, namely, for $f \in L^p(\Omega)$, $1 < p < \infty$, the following equivalence hods:

$$f \in W^{1,p(x)}(\Omega) \Leftrightarrow \lim_{n\to\infty} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_n(x - y)dxdy < \infty$$

and in this case

$$f \in W^{1,p(x)}(\Omega) \Leftrightarrow \lim_{n\to\infty} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_n(x - y)dxdy$$

$$= K_{p,N} \int_\Omega |\nabla f)|^{p(x)} dx \tag{4}$$

where (ρ_n) denotes a sequence of radial mollifiers and $K_{p,N}$ is a constant depending The proof of (4) is based on (3) and therefore the result cannot be directly generalized to the context of variable exponents. In fact we can explicitly state (see Proposition (3.1.17)) that the natural generalization of (4) (i.e. with p replaced by $p(x)$) is in fact false.

In spite of the above considerations, the goal of the prove that some results of the type (3), (4) hold true.

The first main result of that functions in $W^{1,p(x)}(\Omega)$ are mean continuous with respect to a “penalized” version of the function $p(x)$, namely, the function $\min p(x, x + y)$.

Theorem (3.1.7)[125]: Let $p = p(x) \in \mathcal{P}(\Omega)$ and $f \in W^{1,p(x)}(\Omega)$. Then for each open set $\omega \subset\subset \Omega$, the following inequality holds:

$$\|f(x - y) - f(x)\|_{L^{\min p(x,x+y)}(\omega)} \le (1 + |\Omega|)|h|\|\nabla f\|_{L^{p(x)}(\Omega)}.$$

for every $h \in \mathbb{R}^N$ such that

$$\omega + th \subset \Omega \quad \forall t \in [0, 1]. \tag{5}$$

Proof. Without loss of generality, by Theorem (3.1.6), we may assume $f \in C^\infty(\Omega)$, f not constant almost everywhere.

Fix an open set $\omega \subset\subset \Omega$, such that $\omega + th \subset \Omega\ \forall t \in [0, 1]$, and set

$$\tau_h f(x) = f(x + h),$$
$$u(x) = f(x + h), \quad t \in [0, 1].$$

We have

$$\tau_h f(x) - f(x) = u(1) - u(0) = \int_0^1 u'(t)dt = \int_0^1 h.\nabla f(x + th)dt$$

and therefore

$$|\tau_h f(x) - f(x)|^{\min p(x,x+h)} \le \left(\int_0^1 h.\nabla f(x + th)dt\right)^{\min p(x,x+h)}$$

$$\leq \int_0^1 |h|^{\min p(x,x+h)} |\nabla f(x+th)|^{\min p(x,x+h)} dt.$$

Integrating over ω

$$\int_\omega |\tau_h f(x) - f(x)|^{\min p(x,x+h)} dx$$
$$\leq \int_\omega dx \int_0^1 |h|^{\min p(x,x+h)} |\nabla f(x+th)|^{\min p(x,x+h)} dt$$
$$= \int_0^1 dt \int_\omega |h|^{\min p(x,x+h)} |\nabla f(x+th)|^{\min p(x,x+h)} dx.$$

By Lemma (3.1.10),

$$\int_\omega |\tau_h f(x) - f(x)|^{\min p(x,x+h)} dx$$
$$\leq \int_0^1 dt \left(\int_0^1 |h|^{p(x+th)} |\nabla f(x+th)|^{p(x+th)} dx + |\Omega| \right)$$
$$= \int_0^1 dt \int_{\omega+th}^1 |h|^{p(y)} |\nabla f(y)|^{p(y)} dy + |\Omega|$$
$$\leq \int_\Omega^1 |h|^{p(y)} |\nabla f(y)|^{p(y)} dy + |\Omega|.$$

and therefore

$$\int_\omega \left(\frac{|\tau_h f(x) - f(x)|}{(1+|\Omega|)|h| \|\nabla f\|_{p(x)}} \right)^{\min p(x,x+h)} dx$$
$$\leq \frac{1}{1+|\Omega|} \int_\omega \left| \tau_h \left(\frac{f}{|h| \|\nabla f\|_{p(x)}} \right)(x) - \left(\frac{f}{|h| \|\nabla f\|_{p(x)}} \right)(x) \right|^{\min p(x,x+h)} dx$$
$$= \frac{1}{1+|\Omega|} \int_\Omega \left(\frac{|\nabla f(y)|}{\|\nabla f\|_{p(x)}} \right)^{p(x)} dy + \frac{|\Omega|}{1+|\Omega|}$$
$$\leq \frac{|\Omega|}{1+|\Omega|} + \frac{|\Omega|}{1+|\Omega|} = 1$$

from which we get the assertion.

Similarly to the classical case of constant exponents (see [117]), the proof of this result relies upon a result of density of smooth functions. As far as we know, the theory of functions of Sobolev spaces with variable exponents has not actually results of density of smooth functions completely analogous to the classical theory, because of various conditions to be imposed on the exponent $p(x)$. Among the results obtained in this direction (see e.g. Samko, [23]), we will use that one obtained by Edmunds and Rákosník (see [120]), recalled in Theorem (3.1.6), where the monotonicity condition for $p(x)$ with respect to some cone, considered in our definition of $\mathcal{P}(\Omega)$, has been proved to be sufficient for the density of smooth functions.

We had to insert the assumption (i) of Theorem (3.1.6), made on the exponent $p(x)$, as a condition for the class $\mathcal{P}(\Omega)$. A nice property of the condition (i) will be used in the sequel, namely, that it is preserved when making extensions by the (classical) reflection method (see the proof of Lemma (3.1.12). We stress that the condition (i) could be substituted by any other condition, ensuring the density of the smooth functions, satisfying also this nice property. For instance, we claim that the results could be obtained also by using the Dini–Lipschitz condition used in [23]; however, the use of (i) allows also discontinuous exponents in Theorem (3.1.7).

The second main result is that, substituting p with $\min p(x,y)$ in (4), the limit of the converging sequences are equivalent to the $p(x)$-norm of the gradient. We will prove the following theorem, consequence of Corollary (3.1.16) and Theorem (3.1.20).

Theorem (3.1.8)[125]: Let $\Omega \subset \mathbb{R}^N$ be a cube, $f \in W^{1,p(x)}(\Omega)$, $p(x) \in C(\Omega) \cap \mathcal{P}(\Omega)$, and (ρ_n) be a sequence of mollifiers. Then there exist constants $c' = c'(N, |\Omega|)$, $c'' = c''(N, |\Omega|, p^*, p_*)$, such that

$$\liminf_{n\to\infty} \left\| \frac{|f(x)-f(y)|}{|x-y|} [\rho_n(x-y)]^{\frac{1}{\min p(x,y)}} \right\|_{L^{\min p(x,y)}(\Omega\times\Omega)} \geq c' \|\nabla f\|_{p(x)},$$

$$\limsup_{n\to\infty} \left\| \frac{|f(x)-f(y)|}{|x-y|} [\rho_n(x-y)]^{\frac{1}{\min p(x,y)}} \right\|_{L^{\min p(x,y)}(\Omega\times\Omega)} \leq c'' \|\nabla f\|_{p(x)}.$$

In other word: the sequence

$$A_n = \left\| \frac{|f(x)-f(y)|}{|x-y|} [\rho_n(x-y)]^{\frac{1}{\min p(x,y)}} \right\|_{L^{\min p(x,y)}(\Omega\times\Omega)}$$

contains converging subsequences, and each converging subsequence is such that

$$\lim_{n\to\infty} A_{nk} = K \|\nabla f\|_{p(x)}$$

with K such that $c'(N, |\Omega|) \leq K \leq c''(N, |\Omega|, p^*, p_*)$.

The proofs of the Propositions (3.1.9) and (3.1.17) are refinements of [13]; Theorem (3.1.7) follows adapting classical arguments of variable coefficients; the proof of Theorem (3.1.8), splitted in various statements given, has been inspired by the ideas of [118], and, due to the expression (1) of the norm of $L^{p(x)}(\Omega)$ (which is not of integral-type), involves the Musielak–Orlicz norm (2). A difficulty which is met in generalizing the proof in [118] is that it does not exist, to the knowledge, a satisfactory Poincaré–Wirtinger inequality for Sobolev spaces with variable exponents. To overcome such problem we have to refine the proof of [118], in order to avoid the extension to all the space $\mathbb{R}^N$ (see Lemma (3.1.12) and Theorem (3.1.13).

Proposition (3.1.9)[125]: Let $p(x) \in \mathcal{P}(\Omega)$ having the following properties

$$\exists (x_m), \quad x_m \in \Omega \quad \forall x_m \in N, \qquad x_m \to z \in \partial\Omega,$$
$$\exists (y_m), \quad y_m \in \Omega \quad \forall y_m \in N, \qquad y_m \to z \in \partial\Omega,$$
$$p(y_m) > \frac{Np(x_m)}{N - p(y_m)}, \quad \forall m \in N.$$

Then there exist functions in $W^{1,p(x)}(\Omega)$ which are not $p(x)$-mean continuous.

Proof. At first we give an example of function $p(x)$ satisfying the assumed properties.

Fix $s > 2$, $\sigma > \frac{N_s}{N-s}$, $a_m, b_m \in]0,1]$, $a_m \uparrow 1/2$, $b_m \downarrow 1/2$ and set

$$\Omega =]0,1[\times]0,1[$$

$$p(x) = p(x^1, x^2) = s + \frac{1}{2}\left(1 + \sin\frac{1}{x^2}\right)(\sigma - s) + \left(x^2 + \frac{1}{2}\right)$$

$$x_m = \left(\left(\frac{3}{2}\pi + 2m\pi \right)^{-1}, a_m \right)$$
$$y_m = \left(\left(\frac{3}{2}\pi + 2m\pi \right)^{-1}, b_m \right).$$

The function $p(x) \in \mathcal{P}(\Omega)$ because it is the $C^1(\Omega)$ class, and $|\nabla p(x)| \neq 0$ in every point $x \in \Omega$.

Now we begin the construction of a function f in $W^{1,p(x)}(\Omega)$, which is not $p(x)$-mean continuous.

The continuity of p yields the existence of numbers $r_m > 0$, $\lambda_m, \mu_m > 1$ such that
$$\frac{Np(x)}{N-p(x)} < \frac{N\lambda_m}{N-\lambda_m} < \mu_m < p(y) \ \ \forall x \in B(x_m, r_m) \ \ \forall y \in B(y_m, r_m) \quad (6)$$
Set
$$g_m = |x - x_m|^{-\alpha_m} \chi_{B(x_m,r_m)}(x) \quad \forall x \in \Omega, \ \ \forall m \in \Omega,$$
where $\alpha_m \in]N/\mu_m, (N-\lambda_m)/\lambda_m[$. Since $\alpha_m < (N-\lambda_m)/\lambda_m$, we have $g_m \in W^{1,\lambda_m}(\Omega)$ (see e.g. [119]) and therefore, by Lemma (3.1.4), $g_m \in W^{1,p(x)}(\Omega)$; on the other hand, since $\alpha_m > N/\mu_m$, we have $g_m \notin L^{\mu_m}(\Omega)$. Set
$$f_m = \frac{g_m}{\|g_m\|_{1,\lambda_m}}.$$
Evidently $f_m \in W^{1,\lambda_m}(\Omega)$, $f_m \notin L^{\mu_m}(\Omega)$ and $\|f_m\|_{1,\lambda_m} = 1$.

At is this point we can proceed as in [13]. For completeness, we recall here their argument

Define the function f by
$$f(x) = \sum_{m=1}^{\infty} 2^{-m} f_m(x), \ \ x \in \Omega.$$
By (6) and Lemma (3.1.4) we obtain
$$\|f\|_{1,p(x)} \leq \sum_{m=1}^{\infty} 2^{-m} \|f_m\|_{1,p(x)} \leq (1 + |\Omega|) \sum_{m=1}^{\infty} 2^{-m} \|f_m\|_{1,\lambda_m} \leq 1 + |\Omega| < \infty.$$
On the other hand, we obtain
$$h_m = x_m - y_m,$$
and according it (6) and to Lemma (3.1.4), we have
$$\begin{aligned}\|f(x+h_m) - f(x)\|_{p(x)} &\geq \left\| \left(f(x+h_m) = f(x) \right)_{\chi_{B(y_m,r_m)}} \right\|_{p(x)} \\ &\geq \left\| f(x+h_m)_{\chi_{B(y_m,r_m)}} \right\|_{p(x)} - \left\| f(x)_{\chi_{B(y_m,r_m)}} \right\|_{p(x)} \\ &= \left\| f(y)_{\chi_{B(y_m,r_m)}} \right\|_{p(y-h_m)} - \left\| f(x)_{\chi_{B(y_m,r_m)}} \right\|_{p(x)} \\ &\leq (1+|\Omega|)^{-1} 2^{-m} \|f_m\|_{\mu_m} - \|f\|_{p(x)} = \infty.\end{aligned}$$
Thus, the function
$$x \to f(x+h_m) - f(x)$$
does not belong to $L^{p(x)}(\Omega)$ and since $h_m \to 0$, the function f is not $p(x)$-mean continuous.

Lemma (3.1.10)[125]: Let $p = p(x) \in \mathcal{P}(\Omega)$. The following inequality holds:

$$\int_\Omega |f(x))|^{p(x)}dx \int_\Omega |fx)|^{p(x)q(x)}dx + |\Omega|.$$

Proof.

$$\int_\Omega |fx)|^{p(x)}dx = \int_{\Omega\cap\{|f|\le 1\}} |fx)|^{p(x)}dx + \int_{\Omega\cap\{|f|>1\}} |fx)|^{p(x)q(x)}dx$$

$$\le |\{x \in \Omega : |f(x)| \le 1\}| + \int_{\Omega\cap\{|f|>1\}} |fx)|^{p(x)q(x)}dx$$

$$\le |\Omega| + \int_\Omega |f(x)|^{p(x)q(x)}dx.$$

Corollary (3.1.11)[125]: Let $p = p(x) \in \mathcal{P}(\Omega)$ and $f \in W^{1,p(x)}(\Omega)$ not constant $a.e.$ in Ω. Then for each open set $\omega \subset\subset \Omega$, the following inequality holds:

$$\int_\omega \left(\frac{|\tau_h f(x) - f(x)|}{(1 + |\Omega|)|h|\|\nabla f\|_{p(x)}}\right)^{\min p(x,x+h)} dx,$$

for every $h \in \mathbb{R}^N$ such that

$$\omega + th \subset \Omega \quad \forall t \in [0,1].$$

It is clear from the proofs that Theorem (3.1.7) and Corollary (3.1.11) hold also if the cube Ω is substituted by any open set of finite measure.

We will need an argument which uses an extension operator. The role of the shape of the class of domains we consider is fundamental in this point. We need to extend also the exponent $p(x) \in \mathcal{P}(\Omega)$ in a new one which is not only defined in cube in $\widetilde{\Omega}$ containing Ω,but is also in the class $\mathcal{P}(\widetilde{\Omega})$.

Lemma (3.1.12)[125]: Let $p(x) \in \mathcal{P}(\Omega)$. Then exists a cube $\widetilde{\Omega}$ such that $\Omega \subset\subset \widetilde{\Omega}$, a function $p(\widetilde{\Omega}) \in \mathcal{P}(\widetilde{\Omega})$ and a linear extension operator

$$E : W^{1,p(x)}(\Omega) \to W^{1,\tilde{p}(x)}\left(\widetilde{\Omega}\right)$$

such that

(i) $\tilde{p}(x)_{|\Omega} = p(x)$;

(ii) $(Ef)_{|\Omega} = f \quad a.e.$ in Ω;

(iii) $\|Ef\|_{L^{\tilde{p}(x)}(\widetilde{\Omega})} \ge 2^{\frac{2N}{p^*}}\|Ef\|_{L^{p(x)}(\Omega)}$;

(iv) $\|\nabla(Ef)\|_{L^{\tilde{p}(x)}(\widetilde{\Omega})} \ge 2^{\frac{2N}{p^*}}\|\nabla f\|_{L^{p(x)}(\Omega)}$.

Proof. The extension of the function f can be done by following the classical argument (see e.g. [117]), consisting in a sequence of extensions made by the reflection method. After $2N$ reflections, one extends a function $f \in W^{1,p(x)}(\Omega)$ into a new function defined in a cube $\widetilde{\Omega}$ containing Ω. The function $p(x)$ must be extended by using the same technique, but, since it is defined everywhere in the open cube Ω, we first extend on the boundary of Ω setting it equal to p_*, and then we use the similar process, obtaining a function defined in $\widetilde{\Omega}$. It is clear that the extension $\tilde{p}(x)$ is in $\mathcal{P}(\widetilde{\Omega})$: in fact it preserves obviously the infimum and the supremum of $p(x)$, preserves the lower semicontinuity and also condition (i). Notice that the condition (i) is verified also on the boundary of Ω because the exponent has been defined there as the infimum of $p(x)$. To prove estimate (3), it suffices to observe that

$$\int_{\widetilde{\Omega}} \left|\frac{Ef}{2^{\frac{2N}{p^*}}\|f\|_{p(x)}}\right|^{p(x)} dx = 2^{2N}\int_{\Omega} \left|\frac{f}{2^{\frac{2N}{p^*}}\|f\|_{p(x)}}\right|^{p(x)} dx \le \int_{\Omega} \left|\frac{f}{\|f\|_{p(x)}}\right|^{p(x)} dx \le 1$$

The proof of (4) is completely analogous.

With abuse of notation, in the following we will denote still by $p(x)$ the extension $\tilde{p}(x)$ of the function $p = p(x)$ outside Ω, constructed in the proof.

The purpose now is to extend (20) to the case of variable exponents. We begin by proving that if $f \in W^{1,p(x)}(\Omega)$, then the double integral in (4) is finite when p is replaced by $\min p(x,y)$. We will show later (see Proposition (3.1.16)) that replacing p simply by $p(x)$ the double integral can be ∞ for all $n \in N$. From this point of view, next Theorem (3.1.13) represents a generalization of [118].

Theorem (3.1.13)[125]: Assume $f \in W^{1,p(x)}(\Omega)$, $p(x) \in \mathcal{P}(\Omega)$ and let $\rho \in L^1(\mathbb{R}^N)$, $\rho \ge 0$. Then the following inequality holds:

$$\int_{\Omega}\int_{\Omega} \frac{|f(x)-f(y)|^{\min p(x,y)}}{|x-y|^{\min p(x,y)}}\rho(x-y)dxdy$$
$$\le (1+2^{2N}|\Omega|)^{p^*}\|\rho\|_{1^{\Phi_{p_*,p^*}}}\left(2^{\frac{2N}{p^*}}\|f\nabla\|_{p(x)}\right).$$

Proof. If f is constant t, then the assertion is trivial. Let $\widetilde{\Omega}$, $\Omega \subset\subset \widetilde{\Omega}$, be a cube obtained by the reflection method (see Lemma (3.1.20). We may therefore assume

$$f \in W^{1,p(x)}(\widetilde{\Omega}), \quad \|f\nabla\|_{p(x)} \ne 0$$

with $p(x) \in \mathcal{P}(\widetilde{\Omega})$. By Corollary (3.1.11) we have

$$\int_{\omega} \left(\frac{|\tau_h f(x) - f(x)|}{(1+|\widetilde{\Omega}|)|h|\|\nabla f\|_{L^{p(x)}(\widetilde{\Omega})}}\right)^{\min p(x,x+h)} dx \le 1 \qquad (7)$$

for all $f \in W^{1,p(x)}(\Omega)$, for all $\omega \subset\subset \widetilde{\Omega}$, for all $h \in \mathbb{R}^N$ such that

$$\omega + th \subset \widetilde{\Omega} \quad \forall t \in [0,1].$$

By Fubini's theorem we have

$$\int_{\Omega}\int_{\Omega} \frac{|f(x)-f(y)|^{\min p(x,y)}}{|x-y|^{\min p(x,y)}}\rho(x-y)dxdy$$
$$= \int_{\Omega} dx \int_{x-\Omega} \frac{|f(x)-f(x-h)|^{\min p(x,x+h)}}{|h|^{\min p(x,y)}}\rho(h)dh$$
$$= \int_{\Omega-\Omega} dh \int_{(h+\Omega)\cap\Omega} \frac{|f(x)-f(x-h)|^{\min p(x,x+h)}}{|h|^{\min p(x,y)}}\rho(h)dx$$

$$= \int_{\Omega-\Omega} dh \int_{(h+\Omega)\cap\Omega} \left(\frac{|\tau - hf(x) - f(x)|}{(1+|\widetilde{\Omega}|)|h|\|\nabla f\|_{L^{p(x)}(\widetilde{\Omega})}}\right)^{\min p(x,x+h)}$$
$$\times \left(1+|\widetilde{\Omega}|\right)^{\min p(x,x+h)} \|\nabla f\|_{L^{p(x)}(\widetilde{\Omega})}^{\min p(x,x+h)} \rho(h)dx.$$

Observe that

$$h \in \Omega - \Omega \Rightarrow [(h+\Omega)\cap\Omega] - th \subset \Omega - th \subset 3\Omega \subset \widetilde{\Omega} \quad \forall t \in [0,1],$$

where 3Ω denotes the cube with the same center and triple sidelength (see an example in Remark (3.1.14)). We can therefore apply (7) with $\omega = (h + \Omega) \cap \Omega$ and τh replaced by $\tau - h$. We get, by estimate of Lemma (3.1.12), (4),

$$\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^{\min p(x,y)}}{|x - y|^{\min p(x,y)}} \rho(x - y) dx dy$$
$$\leq \int_{\Omega-\Omega} (1 + 2^{2N}|\Omega|)^{p^*} \Phi_{p_*}, p^* \left(\|f\nabla\|_{L^{p(x)}(\widetilde{\Omega})}\right) \rho(h) dh$$
$$\leq (1 + 2^{2N}|\Omega|)^{p^*} \|\rho\|_{1^{\Phi_{p_*,p^*}}} \left(2^{\frac{2N}{p^*}} \|f\nabla\|_{p(x)}\right).$$

The theorem is therefore proved.

Remark (3.1.14)[125]: Notice that in the statements of Theorem (3.1.20) and Corollary (3.1.11) the assumption (5) appears in the place of the more restrictive condition considered in (4). In the notation of the proof of the Theorem (3.1.13), we considered the following hypothesis on ω and h

$$\omega - th \subset \widetilde{\Omega} \quad \forall t \in [0, 1] \tag{8}$$

instead of

$$|h| < \operatorname{dist}(\omega, {}^c\widetilde{\Omega}). \tag{9}$$

Assumption (8) allows "more $\omega \subset\subset \widetilde{\Omega}$. We need (8) instead of (9) because we are in a context whose prototype is the following. Set, in $\mathbb{R}^2$,

$$\Omega =]-2, 2[\times]-2, 2[\,,$$
$$\widetilde{\Omega} =]-10, 6[\times]-10, 6[\,, \quad \text{(for instance)}$$
$$h = (8) \in \mathbb{R}^2,$$
$$\omega = (h + \Omega) \cap \Omega =]1, 2[\times]1, 2[\,.$$

We have

$$|h| = 3\sqrt{2},$$
$$\operatorname{dist}(\omega, {}^c\widetilde{\Omega}) = 4,$$
$$\omega - th =]1 - 3t, 2 - 3t[\times]1 - 3t, 2 - 3t[\,,$$
$$\omega - h =]-2, -1[\times]-2, -1[\,;$$

therefore (8) is verified, while (9) is not.

Of course (3) holds also in the more general assumption (5).

Corollary (3.1.15)[125]: Assume $f \in W^{1,p(x)}(\Omega)$, $p(x) \in \mathcal{P}(\Omega)$ and let $\rho \in L^1(\mathbb{R}^N)$, $\rho \geq 0$, $\|\rho\|_1 = 1$. Then the following inequality holds:

$$\left\| \frac{|f(x) - f(y)|}{|x - y|} [\rho(x - y)]^{\frac{1}{\min p(x,y)}} \right\|_{L^{\min p(x,y)}(\Omega\times\Omega)}$$
$$\leq 2^{\frac{2N}{p^*}} (1 + 2^{2N}|\Omega|)^{p^*/p_*} \left(\|f\nabla\|_{p(x)}\right).$$

Proof. For λ positive parameter, let us apply Theorem (3.1.13) to function f/λ. We have

$$\int_\Omega \int_\Omega \left(\frac{|f(x) - f(y)|}{\lambda|x - y|}\right)^{\min p(x,y)} \rho(x - y) dx dy$$
$$\leq (1 + 2^{2N}|\Omega|)^{p^*} \Phi_{p_*}, p^* \left(\frac{2^{\frac{2N}{p^*}} \|f\nabla\|_{p(x)}}{\lambda}\right)$$

and therefore

$$\left\| \frac{|f(x)-f(y)|}{|x-y|} [\rho(x-y)]^{\frac{1}{\min p(x,y)}} \right\|_{L^{\min p(x,y)}(\Omega\times\Omega)}$$

$$\leq \inf\left\{ \lambda > 0 : (1+2^{2N}|\Omega|)^{p^*} \Phi_{p_*}, p^* \left(\frac{2^{\frac{2N}{p^*}} \|f\nabla\|_{p(x)}}{\lambda} \right) \leq 1 \right\}$$

$$\leq 2^{\frac{2N}{p^*}} (1+2^{2N}|\Omega|)^{p^*/p_*} \left(\|f\nabla\|_{p(x)} \right).$$

The assertion is proved.

Let (ρ_n) be a sequence of radial mollifiers, i.e.

$$\rho_n(x) = \rho_n(|x|), \quad \rho_n \geq 0, \quad \int_{\mathbb{R}^N} \rho_n(x)dx = 1, \qquad \operatorname{supp} \rho_n \subset B\left(0, \frac{1}{n}\right).$$

We save this (standard) notation for all the sequel.

Next theorem states that the first "reasonable" generalization of (4) cannot hold when p is a variable exponent.

Proposition (3.1.16)[125]: There exist $f \in W^{1,p(x)}(\Omega)$, $p(x) \in \mathcal{P}(\Omega)$ and (ρ_n) a sequence of radial mollifiers such that

$$\lim_{n\to\infty} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^{p(x)}}{|x-y|^{p(x)}} \rho_n(x-y)dxdy = \infty.$$

Proof. Set

$$\rho_n(x) = \frac{1}{\left|B\left(0,\frac{1}{n}\right)\right|} \chi_{B\left(0,\frac{1}{n}\right)}(x) = \frac{n^N}{\omega_N} \chi_{B\left(0,\frac{1}{n}\right)}(x)$$

for all $n \in \mathbb{N}$**,** take like in Proposition (3.1.9), and set

$$f(x) \sum_{m=1}^{\infty} 2^{-m} \frac{|x-x_m|^{-\alpha_m} \chi_{B(x_m,r_m)}(x)}{\left\| |x-x_m|^{-\alpha_m} \chi_{B(x_m,r_m)}(x) \right\|_{W^{1,\lambda_m}(\Omega)}}.$$

We can choose in the expression above the number (r_m) so small that the balls $B(x_m, r_m)$, $B(y_m, r_m)$ are disjoint, and therefore,

$$3r_m \leq |x_m - r_m| \quad \forall m \in \mathbb{N}. \tag{10}$$

We recall from Proposition (3.1.9) that, setting $h_m = (x_m - r_m)$, the function

$$\tau_{h_m} f : x \to f(x - h_m)$$

does not belong to $L^{p(x)}(B(y_m, r_m))$, for all $m \in \mathbb{N}$.

We observe that from the expression of f we deduce that also

$$\tau_h f \notin L^{p(x)}\left(B(y_m, r_m)\right) \qquad \forall h \in h_m + B(0, r_m) \tag{11}$$

Now for each $n \in \mathbb{N}$ let us choose $m \in \mathbb{N}$ such that

$$|x_m - z| < \frac{1}{2n}, \qquad |y_m - z| < \frac{1}{2n},$$

therefore m is such that

$$|h_m| = |x_m - y_m| < \frac{1}{2n},$$

and also, by (10)

$$x \in B(y_m, r_m), \qquad y \in B(x_m, 2r_m) \Longrightarrow |x-y| < \frac{1}{n}. \tag{12}$$

By (12) we have

$$\int_{\Omega} dx \int_{\Omega} \frac{|f(x)-f(y)|^{p(x)}}{|x-y|^{p(x)}} \rho_n(x-y)dy$$
$$> \int_{B(y_m,r_m)} dx \int_{B(x_m,2r_m)} \frac{|f(x)-f(y)|^{p(x)}}{|x-y|^{p(x)}} \rho_n(x-y)dy$$
$$= \frac{n^N}{\omega_N} \int_{B(y_m,r_m)} dx \int_{B(x_m,2r_m)} \frac{|f(x)-f(y)|^{p(x)}}{|x-y|^{p(x)}} dy$$
$$> \frac{n^{N-p_*}}{\omega_N} \int_{B(y_m,r_m)} dx \int_{B(x_m,2r_m)} |f(x)-f(y)|^{p(x)} dy.$$

At this point we observe that it is sufficient to prove that

$$\int_{B(y_m,r_m)} dx \int_{B(x_m,2r_m)} |f(x)-f(y)|^{p(x)} dy = \infty \quad \forall m \in \mathbb{N} \qquad (13)$$

Since $t_{h_m} f \notin L^{p(x)}(B(y_m,r_m))$ we have

$$\int_{B(x_m,r_m)} \left|\tau_{h_m} f(x) - f(y)\right|^{p(x)} dx = \infty,$$

and by (11) we have also

$$\int_{B(x_m,r_m)} \left|\tau_{h_m} f(x) - f(y)\right|^{p(x)} dx = \infty \quad \forall h \in x_m - y_m + B(0,r_m)$$

from which

$$\int_{x_m-y_m+B(0,r_m)} dh \int_{B(x_m,r_m)} |f(x+h)-f(x)|^{p(x)} dx = \infty\,.$$

By Fubini's theorem

$$\int_{B(x_m,r_m)} dx \int_{x_m-y_m+B(0,r_m)} |f(x+h)-f(x)|^{p(x)} dh = \infty\,.$$

Sitting $y = x + h$ in the inner integral, we have

$$y \in x + x_m - y_m + B(0,r_m) \subset B(y_m,r_m) + x_m - y_m + B(0,r_m)$$
$$= B(x_m,2r_m), \quad \forall m \in \mathbb{N}$$

and therefore we get (13), from which the assertion follows.

Lemma (3.1.18)[125]: Let $\alpha(x), \beta(x), \gamma(x) \in \mathcal{P}(\Omega)$ be such that

$$\alpha(x) \le \beta(x) \le \gamma(x), \quad \forall x \in \Omega$$

Then

$$\|f\|_{\Phi_{\alpha,\beta}(t,x)} \le (1+|\Omega|)\|f\|_{\gamma(x)} \quad \forall f \in L^{\gamma(x)}(\Omega).$$

Proof: It suffices to prove the lemma in the case $\|f\|_{\gamma(x)} = 1$, namely

$$\|f\|_{\Phi_{\alpha,\beta}(t,x)} \le 1 + |\Omega| \quad \forall f \in L^{\gamma(x)}(\Omega) : \|f\|_{\gamma(x)} = 1. \qquad (14)$$

By Lemma (3.1.1) we have $\int_{\Omega} |f(x)|^{\gamma(x)} dx \le 1$, and therefore

$$\int_{\Omega} \Phi_{\alpha,\beta}\left(\frac{|f(x)|}{1+|\Omega|}, x\right) dx$$
$$\le \frac{1}{1+|\Omega|} \int_{\Omega} \max\left(|f(x)|^{\alpha(x)}, |f(x)|^{\beta(x)}\right) dx$$

$$\le \frac{1}{1+|\Omega|}\left(\int_{\Omega\cap\{|f|\le 1\}} |f(x)|^{\alpha(x)}dx + \int_{\Omega\cap\{|f|>1\}} |f(x)|^{\beta(x)}dx\right)$$
$$\le \frac{1}{1+|\Omega|}\left(|\Omega| + \int_{\Omega} |f(x)|^{\gamma(x)}dx\right) \le 1.$$
Hence (14) holds.

Lemma (3.1.18)[125]: If $f \in C^{\infty}(\overline{\Omega})$, and $p(x) \in C(\Omega) \cap \mathcal{P}(\Omega)$, then
$$\limsup_{n\to\infty} \|f\|_{p_n^*(x)} \le \|f\|_{p(x)}.$$
Proof. Fix $\sigma > 0$. Note
$$g_n(x) = \left|\frac{f(x)}{\|f\|_{p(x)} + \sigma}\right|^{p_n^*(x)} \le \left|1 + \frac{\max_{\overline{\Omega}} f}{\|f\|_{p(x)} + \sigma}\right|^{p^*(x)} < \infty,$$
therefore the sequence (g_n) is constituted by functions uniformly bounded in n. The continuity of $p(x)$ implies that
$$p_n^*(x) \to p(x) \quad \forall x \in \Omega \quad as\, n \to \infty,$$
and therefore
$$\lim_{n\to\infty} \int_{\Omega} g_n(x)dx = \int_{\Omega} \left|\frac{f(x)}{\|f\|_{p(x)} + \sigma}\right|^{p(x)} < 1.$$
We deduce the existence of $v \in \mathbb{N}$ such that
$$\int_{\Omega} g_n(x)dx < 1 \ \forall n > v,$$
i.e.
$$\|f\|_{p_n^*(x)} \le \|f\|_{p(x)} + \sigma \ \ \forall n > v$$
from which
$$\limsup_{n\to\infty} \|f\|_{p_n^*(x)} \le \|f\|_{p(x)} + \sigma \quad \forall \sigma > 0$$
and therefore the lemma follows.

We have now all the background to prove the following

Theorem (3.1.19)[125]: Assume $f \in L^{p(x)}(\Omega)$, $p(x) \in C(\Omega) \cap \mathcal{P}(\Omega)$. Then there exist a constant c, depending on $\mathbb{N}$ and $|\Omega|$ such that
$$\liminf_{n\to\infty} \left\|\frac{|f(x)-f(y)|}{|x-y|}[\rho_n(x-y)]^{\frac{1}{\min p(x,y)}}\right\|_{L^{\min p(x,y)}(\Omega\times\Omega)} \ge c\|\nabla f\|_{p(x)}.$$
If the left hand side is finite, then in fact $f \in W^{1,p(x)}(\Omega)$.

Proof. let $\varphi \in C_0^{\infty}(\Omega)$ (extended by outside Ω) and let e be unit vector in $\mathbb{R}^N$. As in [118] we have
$$J_n = \left|\int_{\Omega} f(x)\, dx \int_{(y-x).e\ge 0} \frac{\varphi(y)-\varphi(x)}{|y-x|}\rho_n(x-y)dy\right|$$
$$\le \int_{\Omega} dx \int_{\Omega} \frac{|f(x)-f(y)|}{|y-x|}|\varphi(y)|\rho_n(x-y)dy$$
$$+ \int_{\mathbb{R}^N\setminus\Omega} dx \int_{\operatorname{supp}\varphi} |f(y)||\varphi(y)|\frac{\rho_n(x-y)}{|y-x|}dy$$
$$= J_{1,n} + J_{2,n}.$$

Since supp $\rho_n \subset B\left(0, \frac{1}{2}\right)$, it is $J_{2,n} = 0 \;\; \forall n > \; 1/\mathrm{dist}(\mathbb{R}^N \backslash \Omega, \mathrm{supp}\; \varphi)$. Therefore we need to estimate $J_{1,n}$.
By Höider's inequality (see Lemma (3.1.2))

$$J_{1,n} \leq \left\| \frac{|f(x)-f(y)|}{|x-y|} [\rho_n(x-y)]^{\frac{1}{\min p(x,y)}} \right\|_{L^{\min p(x,y)}(\Omega\times\Omega)} \cdot \left\| |\varphi(y)| [\rho_n(x-y)]^{\frac{1}{\min p(x,y)'}} \right\|_{L^{[\min p(x,y)]'}(\Omega\times\Omega)} \qquad (15)$$

Let us show that

$$\limsup_{n\to\infty} \left\| |\varphi(y)| [\rho_n(x-y)]^{\frac{1}{\min p(x,y)'}} \right\|_{L^{[\min p(x,y)]'}(\Omega\times\Omega)} \leq (1+|\Omega|)\|\varphi\|_{p(x)'}. \qquad (16)$$

We have

$$p_{*n}(y) \leq \min p(x,y) \leq p_n^*(y) \quad \forall 0 \leq |x-y| < \frac{1}{n}$$

$$[p_{*n}(y)]' \leq [\min p(x,y)]' \leq [p_n^*(y)]' \quad \forall 0 \leq |x-y| < \frac{1}{n}$$

and therefore

$$\begin{aligned}
&\int_\Omega dy \int_\Omega \left(\frac{|\varphi(y)|}{\lambda}\right)^{[\min p(x,y)]'} \rho_n(x-y)dx \\
&\leq \int_\Omega dy \int_\Omega \Phi_{[p_n^*(y)]',[p_{*n}(y)]'}\left(\frac{|\varphi(y)|}{\lambda}, y\right)\rho_n(x-y)dx \\
&\leq \left[\sup_{y\in\Omega} \int_\Omega \rho_n(x-y)dx\right]\left[\int_\Omega \Phi_{[p_n^*(y)]',[p_{*n}(y)]'}\left(\frac{|\varphi(y)|}{\lambda}, y\right)dy\right] \\
&\leq \int_\Omega \Phi_{[p_n^*]',[p_{*n}]'}\left(\frac{|\varphi(y)|}{\lambda}, y\right)dy,
\end{aligned}$$

from which, by Lemma (3.1.17)

$$\begin{aligned}
&\left\| |\varphi(y)| [\rho_n(x-y)]^{\frac{1}{\min p(x,y)'}} \right\|_{L^{[\min p(x,y)]'}(\Omega\times\Omega)} \\
&\leq \|\varphi(y)\|_{\Phi_{[p_n^*]',[p_{*n}]'}(t)} \\
&\leq (1+|\Omega|)\|\varphi(y)\|_{,[p_{*n}]'} = (1+|\Omega|)\|\varphi(y)\|_{[p'(x)]_n^*}.
\end{aligned}$$

By Lemma (3.1.18) we get (16).
Passing to the limit in the inequality

$$J_n \leq J_{1,n} + J_{2,n}$$

By (15) we get (for the limit of J_n, see [118])

$$\begin{aligned}
&K_N \left| \int_\Omega f(x)(\nabla\varphi(x))\, dx \right| \\
&\leq \liminf_{n\to\infty} \left\| \frac{|f(x)-f(y)|}{|x-y|} [\rho_n(x-y)]^{\frac{1}{\min p(x,y)}} \right\|_{L^{\min p(x,y)}(\Omega\times\Omega)}
\end{aligned}$$

$$\cdot \left\| |\varphi(y)| [\rho_n(x-y)]^{\frac{1}{\overline{\min p(x,y)'}}} \right\|_{L^{[\min p(x,y)]'}(\Omega\times\Omega)}, \tag{17}$$

where K_N depends only on N. By (16) the liminf in the right hand side of (17) can be majorzed:

$$(1+|\Omega|)\|\varphi\|_{p(x)'} . \liminf_{n\to\infty} \left\| \frac{|f(x)-f(y)|}{|x-y|} [\rho_n(x-y)]^{\frac{1}{\overline{\min p(x,y)}}} \right\|_{L^{\min p(x,y)}(\Omega\times\Omega)}$$

and this, together with (17), gives

$$\left| \int_\Omega f \frac{\partial \varphi}{\partial x_i}\, dx \right| \le \frac{(1+|\Omega|)}{K_N} \|\varphi\|_{p(x)'}$$

$$. \liminf_{n\to\infty} \left\| \frac{|f(x)-f(y)|}{|x-y|} [\rho_n(x-y)]^{\frac{1}{\overline{\min p(x,y)}}} \right\|_{L^{\min p(x,y)}(\Omega\times\Omega)} \tag{18}$$

for each $i =, \dots, N$. At this point we can consider the linear form

$$L_i : \varphi \in C_c^\infty(\Omega) \to \int_\Omega f \frac{\partial \varphi}{\partial x_i}\, dx.$$

Notice that $C_c^\infty(\Omega)$ is dense in $L^{p(x)'}(\Omega)$ (by Lemma (3.1.5)) and that by (18) the linear form L_i is continuous for the norm in $L^{p(x)'}(\Omega)$. By the Hahn-Banach theorem it can be extended to a linear continuous form on $L^{p(x)'}(\Omega)$, which we can call again L_i. By Lemma (3.1.3) the form L_i can be represented by a function $g_i \in L^{p(x)}(\Omega)$:

$$\langle L_i, \varphi \rangle = \int_\Omega g_i \varphi dx.$$

Therefore we have

$$\int_\Omega f \frac{\partial \varphi}{\partial x_i}\, dx = \int_\Omega g_i \varphi dx \quad \forall \varphi \in C_c^\infty(\Omega)$$

and from this we deduce that $g_i = \frac{\partial \varphi}{\partial x_i}$, and by (18)

$$\left\| \frac{\partial \varphi}{\partial x_i} \right\|_{p(x))} \le \frac{(1+|\Omega|)}{K_N} \liminf_{n\to\infty} \left\| \frac{|f(x)-f(y)|}{|x-y|} [\rho_n(x-y)]^{\frac{1}{\overline{\min p(x,y)}}} \right\|_{L^{\min p(x,y)}(\Omega\times\Omega)}$$

for each $i =, \dots, N$, and the Theorem is proved.

Section (3.2): Weighted $L^{p(x)}$ Spaces

The investigation of the Lebesgue spaces $L^{p(\cdot)}(\Omega)$ with variable exponent was initiated in [133]. During the last two decades these spaces have intensively studied, see [42], [58], [22]−[134]. The interest on these spaces comes from their mathematical curiosity on the hand and their important in some applications (see [22], [25], [134]) on the other hand.

As the space $L^{p(\cdot)}(\mathbb{R}^n)$ is not invariant with respect to translations, convolution operators do not behave well in these spaces. For example, the Young theorem is in general not valid in these spaces; see [24]. Problem also arise for Mellin convolutions ($n = 1$), since $L^{p(\cdot)}(R_+^1)$ is not invariant with respect to dilations. However, the failure of the Young theorem does not prevent some convolution operators from being bounded operators. Roughly speaking, a convolution operator is bounded in $L^{p(\cdot)}$ if its kernel has singularity at the origin only

There are two examples, whose importance is difficult to overestimate. One is the convolution with the singular kernel $k(x) = \frac{1}{x}(n = 1)$, that is, the well-known singular operator, and the other is the related operator, although the latter is not a convolution.

For the second operator over open bounded sets the problem of boundedness was recently solved by L. Diening [3].

We prove weighted estimates for the maximal operator over bounded open sets and for singular type operators with fixed singularity (of Hardy and Hankel type). We give also weighted estimate for potential type operator of variable order $\alpha(x)$ and show, that the Sobolev theorem with the limiting exponent

$$\frac{1}{\mu(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n}$$

Is valid. We also prove that the potential operator is compact in $L^{p(\cdot)}(\Omega)$.

The main result are formulated as in Theorems (3.2.1)−(3.2.5). Provides necessary preliminaries and contain the proofs of Theorems (3.2.1)−(3.2.5).

Ω is a open bounded set in $\mathbb{R}^n$;

$\mu(\Omega) = |\Omega|$ is the Lebesgue meansure of Ω;

$B_r(x) = \{y \in \mathbb{R}^n : \ |y - x| < r\}$;

$|B_r(x)| = \frac{r^n}{n}|S^{n-1}|$ is the volume of $B_r(x)$;

$q(x) = \frac{p(x)}{p(x)-1}$, $1 < p(x) < \infty$, $\frac{1}{p(x)} + \frac{1}{q(x)} \equiv 1$;

$p_0 = \inf_{x\in\Omega} p(x)\,, P = \sup_{x\in\Omega} p(x)\,;$

$p_0 = \inf_{x\in\Omega} q(x) = \dfrac{P}{P-1}, Q = \sup_{x\in\Omega} q(x) = \dfrac{p_0}{p_0 - 1};$

c may denote different positive constants.

Let Ω be an open bounded set in $\mathbb{R}^n$, $n \geq 1$, and $p(x)$ a function on $\bar{\Omega}$ satisfying the conditions

$$1 < p_0 \leq p(x) \leq P < \infty, \ \ x \in \bar{\Omega} \tag{19}$$

and

$$|p(x) - p(y)| \leq \frac{A}{\ln\frac{1}{|x-y|}}, \qquad |x - y)| \leq \frac{1}{2}, \ \ x, y \in \bar{\Omega}. \tag{20}$$

The condition (20) appears naturally in the theory of the spaces $L^{p(\cdot)}(\Omega)$, see [133]−[134]. In [58] it was shown that this condition is fact necessary for boundedness of the maximal operator in $L^{p(\cdot)}(\Omega)$. Condition (20) also appeared in [131] in case of Hölder spaces $H^{\lambda(x)}$ with variable exponent $\lambda(x)$.

$$M^{\beta}f(x) = |x - x_0|^{\beta} \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)\cap\Omega} \frac{|f(y)|}{|x - x_0|^{\beta}} dy \tag{21}$$

where $x_0 \in \overline{\Omega}$. We write $M = M^0$ in the case where $\beta = 0$.

In the case $x_0 \in \partial\Omega$ and when considering the necessity of boundedness conditions, we shall make use of a restriction of the type

$$|\Omega_r(x_0)| \sim r^n, \tag{22}$$

where $\Omega_r(x_0) = \{y \in \Omega : r < |y - x_0| < 2r\}$.

Theorem (3.2.1)[137]: Let $p(x)$ satisfy conditions (19), (20). the operator M^{β} with $x_0 \in \Omega$ is bounded in $L^{p(\cdot)}(\Omega)$ if and only if

$$\frac{n}{p(x_0)} < \beta < \frac{n}{q(x_0)}. \tag{23}$$

If $x_0 \in \partial\Omega$ condition (23) is sufficient for the boundedness of M^{β}.

If $x_0 \in \partial\Omega$ and condition (22) is satisfied, then condition (23) is also necessary for the boundedness of M^{β}.

Let further

$$I^{\alpha(x)}f(x) = \int_{\Omega} \frac{f(y)}{|x - y|^{n-\alpha(x)}} dy, \quad 0 < \alpha(x) < n. \tag{24}$$

Proof. We have to show that

$$\left\|M^{\beta}f\right\|_{p(\cdot)} \le c$$

in some ball $\|f\|_{p(\cdot)} \le R$, which is equivalent to the inequality

$$I_p\left(M^{\beta}f\right) \le c \quad \text{for} \quad \|f\|_{p(\cdot)} \le R.$$

We observe that

$$|x - x_0|^{\beta p(x)} \sim |x - x_0|^{\beta p(x_0)} \tag{25}$$

in case $p(x)$ satisfied the condition (20).

From (25) we obtain

$$I_p\left(M^{\beta}f\right) \le c \int_{\Omega} |x - x_0|^{\beta p(x_0)} \left|M\left(\frac{f(y)}{|y - x_0|^{\beta}}\right)\right|^{p(x)} dx.$$

Following the idea in [42], we represent this as

$$I_p\left(M^{\beta}f\right) \le c \int_{\Omega} \left(|x - x_0|^{\beta p(x_0)} \left|M\left(\frac{f(y)}{|y - x_0|^{\beta}}\right)\right|^{p(x)}\right)^{p_0} dx, \tag{26}$$

where $r(x) = p(x)/p_0$. In the sequel we distinguish between the cases $\beta \le 0$ and $\beta \ge 0$.

Estimate (80) with $\beta = 0$ says that

$$|M\psi(x)|^{r(x)} \leq c\left(1 + M\left[\psi^{r(\cdot)}\right](x)\right) \tag{27}$$

For $\psi \in L^{r(\cdot)}(\Omega)$ with $\|\psi\|_{r(\cdot)} \leq 1$. For $\psi(x) = \frac{f(x)}{|x-x_0|^{\beta}}$ we have

$$\|\psi\|_{r(\cdot)} \leq a_0\|f\|_{r(\cdot)}, \quad a_0 = (\text{diam } \Omega)^{|\beta|},$$

where we took into account that $\beta \leq 0$. From imbedding (63) we obtain

$$\|\psi\|_{r(\cdot)} \leq a_0 . k\|f\|_{p(\cdot)} \leq a_0 kR.$$

Therefore we choose $R = \frac{1}{a_0 k}$. Then $\|\psi\|_{r(\cdot)} \leq 1$, so that (27) is applicable. From (26) we now get

$$I_p\left(M^{\beta} f\right) \leq c \int_{\Omega} \left(|x - x_0|^{\beta p(x_0)} \left|1 + M\left(\frac{f(y)}{|y - x_0|^{\beta}}\right)\right|^{r(y)}\right)^{p_0} dx.$$

By property (25), this yields

$$I_p\left(M^{\beta} f\right) \leq c \int_{\Omega} \left\{\left(|x - x_0|^{\beta p(x_0)} + M\left(\frac{f(y)|^{r(y)}}{|y - x_0|^{\beta r(x_0)}}\right)\right)^{p_0}\right\} dx$$

$$\leq c + c \int_{\Omega} \left(M^{\Upsilon}|f(\cdot)|^{r(\cdot)}(x)\right)^{p_0} dx,$$

where

$$\Upsilon = \beta r(x_0) = \frac{\beta p(x_0)}{p_0}.$$

As is known [128], the weighed maximal operator M^{Υ} is bounded in L^{p_0} with a constant p_0 if $-\frac{n}{p_0} < \Upsilon < \frac{n}{p_0'}$, which is satisfied since $-\frac{n}{p(x_0)} < \beta \leq 0$. Therefore,

$$I_p\left(M^{\beta} f\right) \leq c + c \int_{\Omega} |f(y)|^{r(y).p_0}\, dy = c + c \int_{\Omega} |f(y)|^{p(y)}\, dy < \infty$$

We represent the functional $I_p\left(M^{\beta} f\right)$ in the form

$$I_p\left(M^{\beta} f\right) = \int_{\Omega} \left(\left|M^{\beta} f(x)\right|^{r(y)}\right)^{\lambda} dy \tag{28}$$

Where $r(y) = \frac{p(x)}{\lambda} > 1$, $\lambda > 1$, where λ will be chosen in the interval $1 < \lambda < p_0$. In (28), we wish to use the pointwise weighted estimate (80):

$$\left|M^{\beta} f(x)\right|^{r(x)} \leq c\left[1 + M\left(f^{r(\cdot)}\right)(x)\right]. \tag{29}$$

This estimate is applicable according to Theorem (3.2.10) if $\|f\|_{r(\cdot)} \leq c$ and

$$\beta < \frac{n}{[r(x_0)]'}. \tag{30}$$

The condition $\|f\|_{r(\cdot)} \le c$ is satisfied since $r(x) \le p(x)$. Condition (30) is fulfilled if $\lambda < \frac{n-\beta}{n} p(x_0)$. Therefore, under the choice

$$1 < \lambda < \min\left(p_0, \frac{n-\beta}{n} p(x_0)\right)$$

We may apply (29) to (28). This yields

$$I_p\left(M^\beta f\right) \le c + c \int_\Omega \left|M\left(|f|^{r(.)}\right)(x)\right|^\lambda dy \le c + c \int_\Omega \left(|f(x)|^{p(x)}\right)^\lambda dx$$

by the boundness of the maximal operator M in $L^\lambda(\Omega)$, $\lambda > 0$. Here

$$I_p\left(M^\beta f\right) \le c + c \int_\Omega |f(x)|^{p(x)} \, dx \le c.$$

Suppose that M^β is bounded in $L^{p(x)}(\Omega)$. Then, given a function $f(x)$ such that

$$I_p(wf) \le c_1, \quad w(x) = |x - x_0|^\beta \tag{31}$$

we have

$$I_p(wMf) \le c \tag{32}$$

(for all f satisfying condition (31)) .

i) We choose $f(x) = |x - x_0|^\mu$ with $\mu > -\beta - \frac{n}{p(x_0)}$. Then

$$I_p(wf) \le c \int_{|x-x_0|<r} |x - x_0|^{(\beta+\mu)p(x)} \, dx \le c \int_{|x|<r} |x|^{(\beta+\mu)p(x_0)} \, dx.$$

Where the integral converges, so that we are in the situation (73). However,

$$I_p(wMf) \ge c \int_{\Omega \cap B_r(x_0)} |x - x_0|^{\beta p(x_0)} \, dx,$$

which diverges if $\beta p(x_0) < -n$; here we take into account Lemma (3.2.9) in the case $x_0 \in \partial\Omega$. Therefore, from (32) it follows that $\beta > \frac{n}{q(x_0)}$.

ii) To show the necessity of the right-hand side bound in (23), suppose that, on the contrary, $\beta \ge \frac{n}{q(x_0)}$. Let first $\beta > \frac{n}{q(x_0)}$. We choose

$$f(x) = \frac{1}{|x - x_0|^n},$$

for which $I_p(wf)$ converges but Mf just does not exist. Let now $\beta = \frac{n}{q(x_0)}$. We choose

$$f(x) = \frac{1}{|x - x_0|^n} \left(\ln \frac{1}{|x - x_0|}\right)^\Upsilon, \qquad |x - x_0| \le \frac{1}{2}.$$

Then $I_p(wf)$ exists under the chose $\Upsilon < -\frac{1}{q(x_0)}$, but Mf just does not exist when $\Upsilon > -1$. Thus, taking $\Upsilon \in \left(-1, -\frac{1}{q(x_0)}\right)$, we arrive at a contradiction.

Let

$$uf(x,y) = \frac{y}{c_n}\int_{\mathbb{R}^n} \frac{f(x-t)dt}{(|t|^2+y^2)^{\frac{n+1}{n}}}, \quad y > 0,$$

Be the Poisson integral. Here

$$c_n = \frac{\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}.$$

The theorem below provides a weighted estimate for the non-tangential supremum of the Poisson integral $u_f(x,y)$. We put

$$\Gamma_a(x) = \{(\xi, y) : |\xi - x| < ay\} \quad \text{with fixed} \quad a > 0.$$

Theorem (3.2.2)[137]: Under conditions (19), (20) and the conditions

$$\inf_{x\in\Omega} \alpha(x) > 0 \quad \text{and} \quad \sup_{x\in\Omega} \alpha(x)p(x) < n \tag{33}$$

The potential $I^{\alpha(.)}$ is bounded from $L^{p(\cdot)}(\Omega)$ into $L^{r(\cdot)}(\Omega)$ with

$$\frac{1}{r(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n}.$$

Proof. This theorem is an immediate consequence of Theorem (3.2.4) and Theorem (3.2.7) (the latter for the case $\beta = 0$).

Theorem (3.2.3)[137]: Under conditions (19), (20) and the conditions $\inf_{x\in\Omega} \alpha(x) > 0$, the operator $I^{\alpha(\cdot)}$ is compact in $L^{p(\cdot)}(\Omega)$.

For the weighted potential operator

$$I_\beta^{\alpha(x)} f(x) = |x-x_0|^\beta \int_\Omega \frac{|f(y)|}{|x-x_0|^\beta |x-y|^{n-\alpha(x)}} dy, \quad x_0 \in \partial\bar{\Omega}. \tag{34}$$

Proof. From Theorem (3.2.4) we already know that the operator $I^{\alpha(x)}$ is bounded in $L^{p(\cdot)}(\Omega)$. To show its compactness, we respect it as

$$I^{\alpha(x)} f(x) = \int_{|x-y|<\varepsilon} \frac{|f(y)|dy}{|x-y|^{n-\alpha(x)}} + \int_{|x-y|>\varepsilon} \frac{|f(y)|dy}{|x-y|^{n-\alpha(x)}}$$

$$= K_\varepsilon f(x) = T_\varepsilon f(x) \tag{35}$$

under the usual assumption that $f(x) \equiv 0$ for $y \notin \Omega$. As in the proof Theorem (3.1.4), we have

$$|K_\varepsilon f(x)| \le \sum_{k=0} \int_{2^{-(k+1)}\varepsilon<|x-y|<2^{-k}\varepsilon} \frac{|f(y)|dy}{|x-y|^{n-\alpha(x)}} \le c\varepsilon^{\alpha_0}(Mf)(x) \tag{36}$$

with $\alpha_0 = \inf_{x\in\Omega} \alpha(x) > 0$.

The compactness of the operator T_ε may be shown via direct approximation by finite-dimensional operators. Indeed, denote $t_\varepsilon(x,y) \equiv 1$ if $|x-y| \geq \varepsilon$ and $t_\varepsilon = 0$ otherwise. As is known, functions of the form

$$f_n(x,y) = \sum_{k=1}^{n} a_k\,(x) b_k(y),$$

where $b_k(y) = \chi_{B_k}(y)$, B_k are non-intersecting sets on Ω, and $a_n(x) \in L^Q(\Omega)$, form a dense set in the mixed norm space $L^p[L^Q](\Omega \times \Omega)$ for all constant exponents P and Q, $1 \leq P < \infty$, $1 \leq Q < \infty$. Therefore for the function $t_\varepsilon(x,y)$ with any fixed $\varepsilon > 0$, there exists a sequence of function $k_n(x,y)$ such that

$$\lim_{n\to\infty} \left\| \|t_\varepsilon(x,y) - k_n(x,y)\|_Q \right\|_p = 0. \tag{37}$$

Then the finite dimensional operators

$$A_n(x) = \int_\Omega k_n(x,y) f(y) dy,$$

which are compact in $L^{p(\cdot)}(\Omega)$, approximate the operator T_ε in the operator norm of $L^{p(\cdot)}(\Omega)$ as $n \to \infty$. Indeed, taking into account imbedding (63), we obtain

$$|(T_\varepsilon - A_n)f(x)| \leq \|f\|_{p(\cdot)} \|k_n(x,\cdot) - t_\varepsilon(x,\cdot)\|_{q(\cdot)} \leq c\|f\|_{p(\cdot)} \|k_n(x,.) - t_\varepsilon(x,.)\|_Q$$

and then

$$\|(T_\varepsilon - A_k)f\|_{q(\cdot)} \leq c\|f\|_{p(\cdot)} \left\| \|k_n(x,\cdot) - t_\varepsilon(x,\cdot)\|_Q \right\|_p.$$

Therefore, by the same imbedding (63)

$$\|(T_\varepsilon - A_k)\|_{L^{p(\cdot)} \to L^{p(\cdot)}} \leq \left\| \|k_n - t_\varepsilon\|_Q \right\|_P \to 0$$

in view of (37). Consequently, the operators T_ε are compact in $L^{p(\cdot)}(\Omega)$.

It remains to observe that, by (35) and (36) and by the boundedness of the maximal operator.

$$\left\| I^{\alpha(\cdot)} - T_\varepsilon \right\|_{L^{p(\cdot)} \to L^{p(\cdot)}} = \|K^\varepsilon\|_{L^{p(\cdot)} \to L^{p(\cdot)}} \leq \varepsilon^{\alpha_0} \|M\|_{L^{p(\cdot)} \to L^{p(\cdot)}} \to 0$$

So that $I^{\alpha(x)}$ is a compact operator as well.

Theorem (3.2.4)[137]: Under conditions (19), (20) and the conditions $\inf_{x\in\Omega} \alpha(x) > 0$, the operator $I_\beta^{\alpha(x)}$ is bounded in $L^{p(\cdot)}(\Omega)$ if

$$\frac{n}{p(x_0)} < \beta < \frac{n}{q(x_0)}. \tag{38}$$

Let now $n = 1$, $\Omega = (0, \ell)$ with $0 < \ell < \infty$ and $x_0 = 0$. We consider the weighed Hardy-type operators

$$H^{\beta}f(x) = x^{\beta-1}\int_0^x \frac{f(t)}{t^{\beta}}dt, \quad H_*^{\beta}f(x) = x^{\beta}\int_x^{\ell} \frac{f(t)}{t^{\beta}}dt \tag{39}$$

and the weighed Hardy-type operator

$$\mathcal{H}^{\beta}f(x) = x^{\beta}\int_0^{\ell} \frac{f(t)}{t^{\beta}(t+x)}dt. \tag{40}$$

Proof. We have

$$I_{\beta}^{\alpha(x)}f(x) = |x-x_0|^{\beta}\int_{|x-y|>1, y\in\Omega} \frac{|f(y)|dy}{|x-y|^{n-\alpha(x)}|y-x_0|^{\beta}}$$

$$+|x-x_0|^{\beta}\int_{|x-y|<1, y\in\Omega} \frac{|f(y)|dy}{|x-y|^{n-\alpha(x)}|y-x_0|^{\beta}}$$

$$= A_1f(x) + A_2f(x) \tag{41}$$

For the first term we have

$$|A_1f(x)| \leq c|x-x_0|^{\beta}\int_{\Omega} \frac{|f(y)|dy}{|y-x_0|^{\beta}} \tag{42}$$

and the Hölder inequality (60)

$$\int_{\Omega} \frac{|f(y)|dy}{|y-x_0|^{\beta}} \leq c\|f\|_{p(\cdot)} \leq c\|f\|_{p(\cdot)}\left\||y-x_0|^{-\beta}\right\|_{q(\cdot)}$$

$$\leq c\left\{I_q\left(|y-x_0|^{-\beta}\right)\right\}^{\theta}\|f\|, \tag{43}$$

Where $\theta = \frac{1}{Q}$ if $I_q(\cdot\cdot\cdot) \leq 1$ and $\theta = \frac{1}{q_0}$ otherwise. Obviously,

$$I_q\left(|y-x_0|^{-\beta}\right) \leq c\int_{\Omega} |y-x_0|^{\beta q(x_0)}\,dy = c < \infty \tag{44}$$

by property (25) and the condition $\beta q(x_0) < n$. Thus from (42)−(44) we get

$$|A_1f(x)| \leq c|x-x_0|^{\beta}\|f\|_{p(\cdot)}. \tag{45}$$

For the term $A_2f(x)$ we have

$$|A_2f(x)| \leq |x-x_0|^{\beta}\sum_{k=0}^{\infty}\int_{2^{-(k+1)}<|x-y|<2^{-k}} \frac{|f(y)|dy}{|x-y|^{n-\alpha(x)}|y-x_0|^{\beta}},$$

where it is assumed that $f(x)$ is continued as zero beyond Ω if necessary.

For those x for which $\alpha(x) \leq n$, we obtain

$$|A_2f(x)| \leq 2^n|x-x_0|^{\beta}\sum_{k=0}^{\infty} 2^{k[n-\alpha(x)]}.\,2^{-kn}\frac{1}{2^{kn}}\int_{|x-y|<2^{-k}} \frac{|f(y)|dy}{|y-x_0|^{\beta}}$$

$$\leq 2^n|x-x_0|^\beta \sum_{k=0}^{\infty} 2^{-k\alpha(x)} M\left(\frac{f(y)}{|y-x_0|^\beta}\right).$$

Therefore,

$$|A_2 f(x)| \leq cM^\beta f(x) \tag{46}$$

with $c = 2^n \sum_{k=0}^{\infty} 2^{-k\alpha_0}$, $\alpha_0 = \inf_{x\in\Omega} \alpha(x)$.

In the case $\alpha(x) \geq n$, the pointwise estimate of $A_0(x)$ is the same as that for $A_1(x)$. Consequently, for all $x \in \Omega$ by means of (45) and (46) we obtain

$$\left|I_\beta^{\alpha(x)} f(x)\right| \leq cM^\beta f(x) + c|x-x_0|^\beta \|f\|_{p(\cdot)}. \tag{47}$$

Therefore,

$$\left\|I_\beta^{\alpha(x)} f\right\|_{p(\cdot)} \leq c\left\|M^\beta f\right\|_{p(\cdot)} + c\left\||x-x_0|^\beta\right\|_{p(\cdot)} . \|f\|_{p(\cdot)}.$$

It remain to apply Theorem (3.2.1) to the first term in (47) and to notice that $\left\||x-x_0|^\beta\right\|_{p(\cdot)}$ is finite, the latter being obtained as in (44).

Theorem (3.2.5)[137]: Suppose $1 \leq p(x) \leq P < \infty$ for $x \in [0, \ell]$.

(i) Let conditions (19), (20) be satisfied on a neighborhood $[0, d]$ of the origin, $d > 0$. If

$$\frac{1}{p(0)} < \beta < \frac{1}{q(0)}. \tag{48}$$

then the all operators H^β, H_*^β and $\mathcal{H}^\beta$ are bounded from $L^{p(\cdot)}(\Omega)$ into $L^{s(\cdot)}(\Omega)$ with any $s(x)$ such that $1 \leq s(x) \leq S < \infty$ for $0 \leq x \leq \ell$,

$$s(0) = p(x) \ \text{ and } \ |s(x) - p(x)| \leq \frac{A}{\ln\frac{1}{x}}, \qquad 0 < x < \delta, \qquad \delta > 0. \tag{49}$$

(ii) If $p(x) \leq p(x)$, $0 \leq x \leq d$, for some $d > 0$, then the same statement on boundedness from $L^{p(\cdot)}(\Omega)$ into $L^{s(\cdot)}(\Omega)$ is true if the requirement of the validity of conditions (19), (20) on $[0, d]$ is replaced by the weaker assumption that

$$p(0) > 1 \ \text{ and } \ |s(x) - p(0)| \leq \frac{A}{\ln\frac{1}{x}}, \qquad 0 < x < \min\left(\ell, \frac{1}{2}\right). \tag{50}$$

Proof. Part (i) Suppose as usual that $\|f\|_{p(\cdot)} \leq 1$. Let $d_0 = \min(d, \delta)$. We have

$$\int_0^l \left|H^\beta f(x)\right|^{s(x)} dx \leq \int_0^{d_0} \left|H^\beta f(x)\right|^{s(x)} dx + \frac{1}{d_0^a} \int_{d_0}^l \left|\int_0^x \frac{f(t)}{t^\beta} dt\right|^{s(x)} dx, \tag{51}$$

where $a = (1-\beta)P$.

The second term may be estimate via the Hölder inequality:

$$\left|\int_0^x \frac{f(t)}{t^\beta}dt\right|^{s(x)} \leq k\|f\|_{p(.)}\left\|t^{-\beta}\right\|_{q(\cdot)} \leq k\left\|t^{-\beta}\right\|_{q(\cdot)} = c \tag{52}$$

under the Dini-Lipschitz condition for $p(x)$ on $[0, d]$ and the assumption $\beta q(0) < 1$.

Foe the first term in (51) we observe that the operator H^β is dominated by the weighted maximal operator M^β since

$$\left|\frac{1}{x}\int_0^x f(t)dt\right| \leq \int_0^{2x} |f(t)|dt = \frac{1}{x}\int_{x-x}^{x+x} |f(t)|dt \leq 2Mf(x).$$

First we have to pass to the exponent $p(x)$ in the first term. To this end, we observe that

$$\left|H^\beta f(x)\right|^{s(x)-p(x)} \leq c,\ \ 0 < x \leq d_0\ \ \left(\|f\|_{p(.)} \leq 1\right), \tag{53}$$

where c does not depend on x and f, if $s(x)$ satisfies condition (49).

indeed, by Hölder inequality (60),

$$\left|H^\beta f(x)\right| \leq kx^{\beta-1}\|f\|_{p(.)}\left\|t^{-\beta}\right\|_{q(.)} \leq c.kx^{\beta-1}\left\|t^{-\beta}\right\|_{q(0)} = cx^{\beta-1}. \tag{54}$$

Hence

$$\left|H^\beta f(x)\right|^{s(x)-p(x)} \leq c^{S-1}x^{(\beta-1)[(x)-p(x)]}, \tag{55}$$

which is observe bounded if $x \geq \frac{1}{2}$. For $0 < x \leq \min\left(d_0, \frac{1}{2}\right)$ from (54) we have

$$\left|H^\beta f(x)\right|^{s(x)-p(x)} \leq c^{S-1}e^{(\beta-1)[(x)-p(x)]\ln\frac{1}{x}} \leq c_1 < \infty$$

$$\int_0^{d_0}\left|H^\beta f(x)\right|^{s(x)}dx \leq c\int_0^{d_0}\left|H^\beta f(x)\right|^{s(x)}dx. \tag{56}$$

It remains to apply theorem on $[0, d_0]$. Then

$$\int_0^d \left|H^\beta f(x)\right|^{s(x)}dx \leq c$$

by (51), (52) and (56).

The operator $H_*^\beta = \left(H^{-\beta}\right)^*$ may be regarded as the operator adjoint to $H^{-\beta}$ treated in $L^{q(\cdot)}([0, l])$. However, we admit the possibility for $q(x)$ to bounded beyond a neighborhood of the point $x = 0$, and hence we should first proceed as in (51):

$$I_s\left(H_*^\beta f\right) \leq \int_0^{d_0}\left|H_*^\beta f(x)\right|^{s(x)}dx + c\int_{d_0}^{l}\left(\int_x^1 |f(x)|\,dt\right)^{p(x)}dx$$

$$\leq \int_0^{d_0}\left|H_*^\beta f(x)\right|^{s(x)-p(x)}.\left|H_*^\beta f(x)\right|^{p(x)}dx + c \tag{57}$$

assuming that $\|f\|_{p(\cdot)} \leq 1$. similarly to (55),

$$\left|H_*^\beta f(x)\right|^{s(x)-p(x)} \leq c, \;\; 0 < x \leq d_0$$

Which is shown as in (54) and (55)

$$\left|H_*^\beta f(x)\right| \leq x^{\beta-1} \int_0^l \frac{f(t)}{t^\beta} dt \qquad \text{etc.}$$

Then from (57)

$$I_s\left(H_*^\beta f\right) \leq c \int_0^{d_0} \left(H_*^\beta f(x)\right)^{p(x)} dx + c \tag{58}$$

It remains to use the duality argument for $H_*^\beta = \left(H^{-\beta}\right)^x$.

Part II**.** We need only to estimate anew the first term in (51). In the case $p(0) \leq p(x)$, $0 \leq x \leq d$, we can avoid the passage to the maximal operator by observing that, similarly to (53),

$$\left|H^\beta f(x)\right|^{s(x)-p(x)} \leq c \quad \left(\|f\|_{p(.)} \leq 1\right)$$

under the second condition in (49). Then the first term in (51) is dominated by

$$c \int_0^d \left|H^\beta f(x)\right|^{p(0)} dx \leq \int_0^d |f(x)|^{p(0)} dx$$

by virtue of the boundedness of the weighted Hardy operator H^β in $L^{p(0)}$ with $p(0) > 0$ and $-\frac{1}{p(0)} < \beta < \frac{1}{q(0)}$. Therefore

$$\int_0^d \left|H^\beta f(x)\right|^{p(0)} dx \leq c \int_0^d |f(x)|^{p(0)} dx$$

by imbedding (63)

For the operator H^β we may again proceed as in (57), (58) and use the boundedness of H_*^β in $L^{p(0)}$.

Let for simplicity $f(x)$ be non-negative. We have

$$\mathcal{H}^\beta f(x) \leq x^{\beta-1} \int_0^x \frac{f(t)}{t^\beta} dt + x^\beta \int_x^l \frac{f(t)}{t^{\beta+1}} dt,$$

that is,

$$\mathcal{H}^\beta f(x) \leq H^\beta f(x) \leq + H_*^\beta f(x).$$

Consequently, the boundedness of $\mathcal{H}^\beta$ follows immediately from that of the operators H^β and H_*^β.

Corollary (3.2.6)[137]: Let $1 \leq p(x) \leq P < \infty$ on $[-1, 1]$. The singular operator with fixed singularity,

$$S^\beta f = \frac{|x|^\beta}{\pi} \int_0^1 \frac{f(t)}{t-x} \frac{dt}{t^\beta}, \qquad x \in [-1, 0],$$

is bounded from $L^{p(x)}([0,1])$ into $L^{p(x)}([-1,0])$ if

(i) $p(x) > 1$;

(ii) $p(x)$ satisfies condition (20) on $[-\delta, \delta]$ for some $\delta > 1$;

(iii) $-\frac{1}{p(0)} < \beta < \frac{1}{q(0)}$.

Proof. We have

$$\int_{-1}^{0} \left|S^\beta f(x)\right|^{p(x)} = \int_0^1 \left|\int_0^1 \left(\frac{x}{t}\right)^\beta \frac{f(t)}{t+x}\right|^{p(-x)} dx$$

Thus, it suffices to make use of theorem for the Hankel operator $\mathcal{H}^\beta$, choosing $s(x) = p(-x)$ in that theorem. The condition

$$|s(x) - p(x)| = |p(-x) - p(x)| \leq \frac{A}{\ln \frac{1}{|x|}}, \quad 0 < |x| \leq \delta$$

of Theorem (3.2.1) is obviously satisfied.

The basics on the spaces $L^{p(\cdot)}$ may be found in [42], [13], [132], [24]−[132]. Here we recall only some important facts and definitions.

Let Ω be an open set in $\mathbb{R}^n$ and $p(x)$ a function on $\bar{\Omega}$ such that

$$1 \leq p(x) \leq P < \infty, \quad x \in \bar{\Omega}.$$

By $L^{p(\cdot)}(\Omega)$ we denote the space of measurable function $f(x)$ on Ω such that

$$I_p(f) := \int_\Omega |f(x)|^{p(x)}\, dx < \infty. \tag{59}$$

This is a Banach space with respect to the norm

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : I_p\left(\frac{f}{\lambda}\right) \leq 1\right\}.$$

The Hölder inequality holds in the form

$$\int_\Omega |f(x)g(x)|dx \leq k\|f\|_{p(\cdot)}.\|f\|_{q(\cdot)} \tag{60}$$

with $k = \frac{1}{p_0} < \beta < \frac{n}{q_0}$.

The functional $I_p(f)$ and the norm $\|f\|_{p(\cdot)}$ are simultaneously greater than one and simultaneously less than 1:

$$\|f\|_{p(\cdot)}^{P} \leq I_p(f) \leq \|f\|_{p(\cdot)}^{p_0} \ \text{ if } \ \|f\|_{p(\cdot)} \leq 1 \tag{61}$$

and

$$\|f\|_{p(\cdot)}^{p_0} \leq I_p(f) \leq \|f\|_{p(\cdot)}^{P} \ \text{ if } \ \|f\|_{p(\cdot)} \geq 1 \tag{62}$$

The imbedding

$$L^{p(x)} \subseteq L^{p(\cdot)}, \ \ 1 \leq r(x) \leq p(x) \leq P < \infty$$

is valid if $|\Omega| < \infty$. In that case

$$\|f\|_{r(\cdot)} \leq m\|f\|_{p(\cdot)}, \quad m = a_2 + (1 - a_1)|\Omega|, \tag{63}$$

where $a_1 = \inf\limits_{x\in\Omega} \frac{r(x)}{p(x)}$ and $a_1 = \sup\limits_{x\in\Omega} \frac{r(x)}{p(x)}$.

We deal with $L^{p(x)}$-spaces on open sets in $\mathbb{R}^n$. We shall give some results on boundedness of singular operator with fixed singularity on curves in the complex plane. We only mention that the space $L^{p(x)}(\Gamma)$ on a rectifiable simple curve

$$\Gamma = \{t \in \mathbb{C} : t = t(s), \qquad 0 \leq s \leq \ell\},$$

where s is the are length, may be introduced in a similar way via the functional

$$I_p(f) = \int_\Gamma \ |f(t)|^{p(x)}|dt| = \int_0^\ell |f[t(s)]|^{p[t(s)]}\, ds.$$

Condition (20) may be imposed either on the function $p(t)$:

$$|p(t_1) - p(t_2)| \leq \frac{A}{\ln \frac{1}{|t_1 - t_2|}}, \ \ |t_1 - t_2| \leq \frac{1}{2}, \qquad t_1, t_2 \in \Gamma \tag{64}$$

or on the function $p_* = p[t(s)]$:

$$|p_*(s_1) - p_*(s_2)| \leq \frac{A}{\ln \frac{1}{|s_1 - s_2|}}, \ \ |s_1 - s_2| \leq \frac{1}{2}, \qquad s_1, s_2 \in [0, \ell]. \tag{64}$$

Since $|p(t_1) - p(t_2)| \leq |s_1 - s_2|$, (64) always implies (65). Conversely, (65) implies (64) if there exists a $\lambda > 0$ such that

$$|s_1 - s_2| \leq c|p(t_1) - p(t_2)|^\lambda.$$

Therefore, conditions (64) and (65) are equivalent on curves satisfying the so-called chord condition, for example.

Let

$$K_\varepsilon f = \frac{1}{\varepsilon^n} \int_\Omega \ \mathcal{K}\left(\frac{x - y}{\varepsilon}\right) f(y)dy,$$

where $\mathcal{K}(x)$ has a compact support in $B_R(0)$. In [23], [134] the uniform estimate

$$\|K_\varepsilon f\|_{L^{p(\cdot)}(\Omega_R)} \leq c\|f\|_{L^{p(\cdot)}(\Omega)}, \tag{66}$$

where $\Omega_R = \{x : \text{dist}(x, \Omega) \leq R\}$, was proved under the assumption that $p(x)$ is defined in Ω_R and satisfies conditions (19), (20) on Ω_R.

For the potential type operator $I^{\alpha(x)}$ defined in (24), the following statement was proved in [24] in the case of a bounded open set Ω.

Theorem (3.2.7)[137]: Under assumptions (19), (20) and (25) the operator $I^{\alpha(\cdot)}$ is bounded from $L^{p(.)}(\Omega)$ into $L^{r(\cdot)}(\Omega)$,

$$\frac{1}{r(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n},$$

unconditionally if $p(x)$ is constant, and under the condition that the maximal operator $M = M^0$ is bounded in $L^{p(\cdot)}(\Omega)$ in the general case.

Let

$$M_r f(x) = \frac{1}{B_r(x)|} \int_{B_r(x)} |f(y)| dy \tag{67}$$

denote the mean of the function f over the ball $B_r(x)$. We also need the weighted means

$$M_r^\beta f(x) = \frac{|x - x_0|^\beta}{|B_r(x)|} \int_{B_r(x)} \frac{|f(y)|}{|y - x_0|^\beta} dy \tag{68}$$

related to the weighted maximal operator (21). In (67), (68) we assume that $f(y) = 0$ for $y \notin \Omega$.

Lemma (3.2.8)[137]: If $0 \le \beta < n$, the inequality

$$M_r^\beta (1) = \frac{|x - x_0|^\beta}{|B_r(x)|} \int_{B_r(x)} \frac{dy}{|y - x_0|^\beta} \le c \tag{69}$$

holds with $c > 0$ not depending on x, r and x_0.

Proof. Let

$$J_r(x) = \int_{|y-x|<r} \frac{dy}{|y - x_0|^\alpha} = \int_{|y-(y-x_0)|<r} \frac{dy}{|y|^\alpha} \tag{70}$$

Without loss of generality we may assume that $x_0 = 0$. The change of variables $y = |x|\xi$ gives

$$J_r(x) = |x|^{n-\alpha} \int_{\left|\xi - \frac{x}{|x|}\right| < \frac{x}{|x|}} \frac{d\xi}{|\xi|^\alpha} = |x|^{n-\alpha} \int_{|u-e_1|<\frac{r}{|x|}} \frac{du}{|u|^\alpha} \tag{71}$$

Where $e_1 = (1, 0, \dots ,0)$ and in the last equationwe made the rotation change of variables

$$\xi = \omega_x(u), \quad |\xi| = |u|,$$

where $\omega_x(u)$ is the rotation of $\mathbb{R}^n$ $\omega_x(e_1) = \frac{x}{|x|}$.

From (71)

$$J_r(x) = |x|^{n-\alpha} g\left(\frac{r}{|x|}\right), \qquad g(t) = \int_{|u-e_1|<t} \frac{dy}{|y|^\alpha} \tag{72}$$

To estimate $g(t)$, we distinguish between the three cases,

$$0 < t \le \frac{1}{2}, \quad t \ge 2 \quad \text{and} \quad \frac{1}{2} \le t \le 2.$$

In the case $0 < t \le \frac{1}{2}$ we have

$$|y| = |y - e_1 + e_1| \ge 1 - |y - e_1| \ge 1 - t \ge \frac{1}{2},$$

so that

$$g(t) \le 2^\alpha \int_{|g-e_1|<t} dy = 2^\alpha |B_r(e_1)| = 2^\alpha |B_r(x)|. \tag{73}$$

If $t \ge 2$, we obtain

$$g(t) = \int_{|y-e_1|<2} \frac{dy}{|y|^\alpha} + \int_{2<|y-e_1|<t} \frac{dy}{|y|^\alpha} = c + \int_{2<|y|<t} \frac{dy}{|y+e_1|^\alpha}.$$

Here

$$|y - e_1| \ge |y| - 1 \ge |y| - \frac{|y|}{2} = \frac{|y|}{2}.$$

Therefore

$$g(t) \le c + 2^\alpha \int_{2<|y|<t} \frac{dy}{|y|^\alpha} = c + 2^\alpha |s^{n-1}| \int_{2<|y|<t} \rho^{n-1-\alpha} d\rho$$
$$= c + c_1 t^{n-\alpha} \le c_2 t^{n-\alpha} \tag{74}$$

Finally, if $\frac{1}{2} \le t \le 2$, we have $g(t) \le g(r) = c_3$. Thus, by (73), (74)

$$g(t) \le \begin{cases} t^n, & 0 < t < 1 \\ t^{n-\alpha}, & t \ge 1. \end{cases}$$

Now we obtain from (72) that

$$J_r(t) \le c \begin{cases} r^n |x|^{-\alpha}, & r \le |x| \\ r^{n-\alpha}, & r \ge x \le c r^n |x|^{-\alpha}. \end{cases}$$

Hence (69) follows.

Lemma (3.2.9)[137]: Suppose that $x_0 \in \partial\Omega$ and condition (22) is satisfied. If the function $|y - x_0|^\Upsilon$ is in $L^1(\Omega)$, then necessarily $\Upsilon > -n$.

Proof. Suppose that $x_0 \in \partial\Omega$ and $|y - x_0|^\Upsilon \in L^1(\Omega)$. We have

$$\int_\Omega |x - x_0|^\Upsilon dx \ge \int_{\Omega_r} |y - x_0|^\Upsilon dx = |\xi - x_0|^\Upsilon |\Omega_r|,$$

where $\xi \in \Omega_r$. Since $|\xi - x_0|^\Upsilon \sim r^\Upsilon$ by (22) we obtain

$$\int_{\Omega_r} |y - x_0|^\Upsilon dx \ge c r^{\Upsilon+n}$$

which is only possible if $\Upsilon + n > 0$.

In what follows, Ω is an open bounded set in $\mathbb{R}^n$ and $x_0 \in \overline{\Omega}$.

Theorem (3.2.10)[137]: Let $p(x)$ satisfy conditions (19) and (20). If

$$0 \leq \beta < \frac{n}{q(x_0)}\,, \tag{75}$$

then

$$\left[M_r^{\beta} f(x)\right]^{p(x)} \leq c\left(1 + \frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y)|^{p(y)}\, dy\right) \tag{76}$$

for all $f \in L^{p(\cdot)}(\Omega)$ such that $\|f\|_{p(x)} \leq 1$, where $c = (p, \beta)$ is a constant not depending on x, r and x_0.

Proof. From (77) and the continuity of $p(x)$ we conclude that there exists a $d > 0$ such that

$$\beta q(x) < n \quad \text{for all } |x - x_0| \leq d. \tag{77}$$

Without loss of generality we assume that $d \leq 1$. Let

$$p_r(x) = \min_{|y-x|\leq r} p(y)$$

and

$$\frac{1}{q_r(x)} = 1 - \frac{1}{p_r(x)}$$

From (75) it is easily seen that

$$\beta q_r(x) < n \quad \text{if } |x - x_0| \leq \frac{d}{2} \quad \text{and } 0 < r \leq \frac{d}{4}. \tag{78}$$

Applying the Hölder inequality with the exponents $p_r(x)$ and $q_r(x)$ to the integral on the right-hand side of the equality

$$\left|M_r\left(\frac{f(y)}{|y - x_0|^{\beta}}\right)\right|^{p(x)} = \frac{c}{r^{np(x)}}\int_{B_r(x)}\left(\frac{|f(y)|}{|y - x_0|^{\beta}}\, dy\right)^{p(x)}$$

and taking into account (77) we get

$$\left|M_r\left(\frac{f(y)}{|y - x_0|^{\beta}}\right)\right|^{p(x)}$$

$$\leq \frac{c}{r^{np(x)}}\left(\int_{B_r(x)} |f(y)|^{p_r(x)} dy\right)^{\frac{p(x)}{p_r(x)}} \cdot \left(\int_{B_r(x)} \frac{dy}{|y - x_0|^{\beta q_r(x)}}\right)^{\frac{p(x)}{q_r(x)}}. \tag{79}$$

Making use of the estimate (69), we obtain

$$\left|M_r\left(\frac{f(y)}{|y - x_0|^{\beta}}\right)\right|^{p(x)} \leq c\,\frac{|y - x_0|^{-\beta p(x)}}{r^{\frac{np(x)}{p_r(x)}}}\left(\int_{B_r(x)} |f(y)|^{p_r(x)} dy\right)^{\frac{p(x)}{p_r(x)}}.$$

Here

$$\int_{B_r(x)} |f(y)|^{p_r(x)} dy \le \int_{B_r(x)\cap\{y:\,|f(y)|\ge 1\}} dy + \int_{B_r(x)} |f(y)|^{p(y)} dy,$$

since $q_r(x) \le p(y)$ for $y \in B_r(x)$. Since $p(x)$ is bounded, we see that

$$\left| M_r \left(\frac{f(y)}{|y - x_0|^\beta} \right) \right|^{p(x)} \le c_1 \frac{|y - x_0|^{-\beta p(x)}}{r^{\frac{np(x)}{p_r(x)}}} \left[r^n + \frac{1}{2} \int_{B_r(x)} |f(y)|^{p_r(x)} dy \right]^{\frac{p(x)}{p_r(x)}}.$$

since $r \le d/2 \le 1/2$ and the second term in the brackets is also less then or equal to $1/2$, we arrive at the estimate

$$\left| M_r^\beta f \right|^{p(x)} \le \frac{c}{r^{\frac{np(x)}{p_r(x)}}} \left[r^n + \int_{B_r(x)} |f(y)|^{p(x)} dy \right]$$
$$\le c r^{n\frac{p_r(x) - p(x)}{p_r(x)}} \left[1 + \frac{1}{r^n} \int_{B_r(x)} |f(y)|^{p(x)} dy \right].$$

From here (76) follows, since

$$r^{n\frac{p_r(x) - p(x)}{p_r(x)}} \le c.$$

Indeed,

$$r^{n\frac{p_r(x) - p(x)}{p_r(x)}} = e^{\frac{n}{p_r}[p(x) - p_r(x)] \ln\frac{1}{r}},$$

where

$$\left| \frac{n}{p_r} [p(x) - p_r(x)] \ln \frac{1}{r} \right| \le n |p(x) - p(\xi_r)| \ln \frac{1}{r}$$

with $\xi_r \in B_r(x)$, and then by (20)

$$\left| \frac{n}{p_r} [p(x) - p_r(x)] \ln \frac{1}{r} \right| \le nA \frac{\ln \frac{1}{r}}{\ln \frac{1}{|x - \xi_r|}} \le nA,$$

since $|x - \xi_r| \le r$.

This case is trivial, because

$$|y - x_0| \ge |x - x_0| - |y - x| \ge \frac{d}{2} - \frac{d}{4} = \frac{d}{4}.$$

Thus $|y - x_0|^\beta \ge \left(\frac{d}{4}\right)^\beta$. Since $|y - x_0|^\beta \le (\text{diam}\,\Omega)^\beta$, it follows that

$$M_r^\beta f(x) \le c M_r f(x),$$

and one may proceed as above for the case $\beta = 0$ (the condition $|y - x_0| \ge \frac{d}{2}$ is not needed).

This case is also easy. It suffices to show that the left-hand side is bounded. We have

$$M_r^\beta f(x) \le \frac{c(\operatorname{diam}\Omega)^\beta}{\left(\frac{d}{4}\right)^n}\left[\int_{|y-x_0|\le\frac{d}{8}} \frac{|f(y)|}{|y-x_0|^\beta}dy + \int_{|y-x_0|\ge\frac{d}{8}} \frac{|f(y)|}{|y-x_0|^\beta}dy\right].$$

Here the first integral is estimated via the Hölder inequality with the exponents

$$p_{\frac{d}{8}} = \min_{|y-x_0|\le\frac{d}{8}} p(y) \quad \text{and} \quad q_{\frac{d}{8}} = p'_{\frac{d}{8}}$$

As in (79), which is possible since $\alpha q_{\frac{d}{8}} < n$. The estimate of second integral is trivial since $|y - x_0| \ge d/8$.

Corollary (3.2.11)[137]: Let $0 \le \beta < n/q(x_0)$. If conditions (19), (20) are satisfied, then

$$\left|M^\beta f(x)\right|^{p(x)} \le c\left(1 + M\left[|f(\cdot)|^{p(\cdot)}\right](x)\right) \tag{80}$$

for all $f \in L^{p(\cdot)}(\Omega)$ such that $\|f\|_{p(\cdot)} \le 1$.

Theorem (3.2.12)[137]: Let $f(x)$ have a compact support in a bounded domain Ω. Under assumptions (19) and (20), for the weighted estimate

$$\left\|\,|x-x_0|^\beta \sup_{(\xi,y)\in\Gamma_a(x)}\left|u_f(\xi,y)\right|\right\|_{L^{p(\cdot)}(\Omega)} \le c\left\||x-x_0|^\beta f(x)\right\|_{L^{p(\cdot)}(\Omega)} \tag{81}$$

with an interior $x_0 \in \Omega$ to be valid, it is necessary and sufficient that $-n/p(x_0) < \beta < n/q(x_0)$.

In the case $x_0 \in \Omega$, this condition is sufficient for any x_0 and necessary if x_0 satisfies condition (22)

Proof. It suffices to refer to the fact that

$$\sup_{(\xi,y)\in\Gamma_a(x)}\left|u_f(\xi,y)\right| \sim Mf(x)$$

(see [41]), and to make use of Theorem (3.2.1).

The following operators may be treated as operators with fixed singularity:

(i) the Hardy type operators (48) on $[0, l]$;
(ii) the Hankel operator (48) on $[0, l]$;

(iii) singular operators on a curve Γ_1 with the ”outer” variable on another curve Γ_1. The latter having a unique common point with Γ_1; commentators of the singular operator with the operators of multiplication by piece-wise continuous functions.

For such operators, in contrast to the maximal and potential operators, the ”global” Dini−Lipschitz condition (20) may be replaced by a ”local” condition at the point of the fixed singularity.

Chapter 4

Approximate Identities

We give criteria for smooth functions to be dense in the variable Sobolev spaces, and we give solutions of the Laplace equation and the heat equation with boundary values in the variable L^p spaces. We study Young type inequalities for convolution with respect to norms in such spaces.

Section (4.1): Variable L^p Spaces

Geven an open set $\Omega \subset \mathbb{R}^n$, and a measurable function $p(\cdot) : \Omega \to [1, \infty]$, the variable L^p space, $L^{p(\cdot)}(\Omega)$, is defined to be the Banach function space of measurable functions f on Ω such that for some $\lambda > 0$,

$$\rho(p(\cdot), \Omega, f/\lambda) = \int_{\Omega \setminus \Omega_{p(\cdot),\infty}} |f(x)/\lambda|^{p(\cdot)} dx + \|f/\lambda\|_{\infty, \Omega_{p(.),\infty}} < \infty,$$

Where $\Omega_{p(\cdot),\infty} = \{x \in \Omega : p(x) = \infty\}$, with norm

$$\|f\|_{p(\cdot),\Omega} = \{\Omega > 0 : \rho(p(\cdot), \Omega, f/\lambda) \leq 1\}.$$

similarly, given a positive integer k, we define the variable Sobolev space $W^{k,p(\cdot)}(\Omega)$, to be the Banach space of measurable functions f such that for every multi-index α with $|\alpha| < k$, the derivatives $D^\alpha f$ (defned in the sense of distributions) are in $L^{p(\cdot)}(\Omega)$.

There has been a great deal of interest in the variable L^p spaces, particularly for their applications to PDEs and variational in tegrals. For more information on their properties, see Ková̌cik and Rákosník [13] or Harjulehto and Hä̈stö [142]; for applications, see [4, 27, 36].

We consider the problem of the convergence of approximate identities in variable L^p spaces. We give sufficient conditions for both pointwise and norm convergence. There problems were considered previously by Diening [4], Samko [23] and Sharapudinov [145]; the results have weaker hypotheses since we do not assume $p(\cdot)$ is bounded away from 1. (Added in proof: approximate identities were also considered by Almeida [138].) We show that smooth functions of compact support are dense in the variable Sobolev spaces, and we give solutions to the Laplace equation with boundary values in variable L^p.

We begin by recalling the definition of approximate identities. Let φ be an integrable function defined on $\mathbb{R}^n$such that $\int \varphi \, dx = 1$. For each $t > 0$, define the function $\varphi_t(x) = t^{-n}\varphi(x/t)$. Note that by a change of variables, $\|\varphi_t\|_1 = \|\varphi\|_1$. The sequence $\{\varphi_t\}$ is referred to as an approximate identity. It is well-known (see Stein [146] or Duoandikoetxea [40] that for $1 \leq p < \infty$, the sequence $\{\varphi_t * f\}$ converges to f in $L^p(\Omega)$:

$$\lim_{t \to 0} \|\varphi_t * f - f\|_{p,\Omega} = 0.$$

As a consequence, a subsequence of $\{\varphi_{t_k} * f\}$ converges to pointwise almost everywhere.

If we impose additional conditions on φ then the entire sequence converges almost everywhere to f. Define the radial majorant of φ to be function

$$\tilde{\varphi} = \sup_{|y|\geq|x|} |\varphi(y)|.$$

If $\tilde{\varphi}$ is integrable, we will say that $\{\varphi_t\}$ is a potential-type approximate identity. (This is the case, for example, if φ is a bounded function of compact support.) In this case we have that for all x,

$$\sup_{t>0}|\varphi_t * f(x)| \leq \|\tilde{\varphi}\|_1 Mf(x), \tag{1}$$

where M is the (centered) Hardy–Littlewood maximal operator:

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)\cap\Omega} |f(y)|dy,$$

where Ω is the domain of f. Since M is bounded on L^p, $1 < p < \infty$, and satisfies a weak (1) inequality, inequality implies that if $\{\varphi_t\}$ is a potential-type approximate identity, then $\{\varphi_{t_k} * f\}$ converges f almost everywhere. (Again see [10, 20] for details.)

All of the results mentioned above remain true in the variable L^p spaces, provided that we impose continuity conditions on the exponent function $p(\cdot)$ and size conditions on the φ. Given an open set $\Omega \subset \mathbb{R}^n$, define $\mathcal{P}(\Omega)$ to be the set of all measurable functions $p(\cdot) : \Omega \to [1,\infty]$. For $p(\cdot) \in \mathcal{P}(\Omega)$ and $E \subset \Omega$, let\

$$p_-(E) = \operatorname*{ess\,inf}_{x\in E} p(x), \qquad p_+(E) = \operatorname*{ess\,sup}_{x\in E} p(x)$$

For brevity, we define $p_- = p_-(\Omega)$ and $p_+ = p_+(\Omega)$. We will often, but not always, restrict ourselves to functions $p(\cdot)$ such that $p_+ < \infty$.

We very often need to assume that $p(\cdot)$ satisfies two log-Hölder continuity conditions, one locally and one at infinity:

$$|p(x) - p(y)| \leq \frac{C}{-\log|x-y|}, \quad x, y \in \Omega, \quad |x-y| < 1/2, \tag{2}$$

and

$$|p(x) - p(y)| \leq \frac{C}{\log(e+|x|)}, \quad x, y \in \Omega, \quad |y| \geq |y|. \tag{3}$$

The importance of these conditions is that they are sufficient for the maximal operator to be bounded on $L^{p(\cdot)}(\Omega)$ (which, as (1) shows, implies that potential-type convolution operators are uniformly bounded).

Theorem (4.1.1)[35]: Given an open set Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose that $1 < p_- \leq p_+ < \infty$ and that either Ω is bounded and (2) holds, or Ω is unbounded and both (2) and (3) hold. Then the Hardy–Littlewood maximal operator is bounded on $L^{p(\cdot)}(\Omega)$.

Theorem (4.1.1) was proved independently by Cruz–Uribe, Fiorenza and Neugebauer [36], Nekvinda [20], and by Deining [4] when Ω is bounded or with (3) replaced by the stronger hypothesis that $p(\cdot)$ is constant outside of a large ball.

The first main result shows that certain approximate identities converge pointwise almost everywhere with no assumptions on the exponent function $p(\cdot)$. To state it we need a definition. Given an exponent function $p(\cdot) \in \mathcal{P}(\Omega)$ we define the conjugate exponent function $p'(\cdot)$ by the equation

$$\frac{1}{p(x)} + \frac{1}{p'(\cdot)} = 1, \quad x \in \Omega,$$

where we set $1/\infty = 0$. By p'_+ we mean $\left(p'(\cdot)\right)_+$, i.e., the supremum of the conjugate exponent function. From the definition, we see that p'_+ and p_- are conjugate exponents.

Theorem (4.1.2)[35]: Given a set Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, let φ be such that either:

(i) $\{\varphi_t\}$ is a potential-type approximate identity,

(ii) φ has compact support and $\varphi \in L^{p'_+}(\Omega)$.

Then for all $f \in L^{p(\cdot)}(\Omega)$, $\{\varphi_t * f\}$ converges to f pointwise almost everywhere.

Proof. Fix $f \in L^{p(\cdot)}(\Omega)$, and decompose f as $f_1 + f_2$, where

$$f_1(x) = \begin{cases} f(x), & |f(x)| \geq 1 \\ 0, & |f(x)| < 1. \end{cases}$$

Then $f_1 \in L^{p_-(\cdot)}(\Omega)$ and $f_2 \in L^{p_+(\cdot)}(\Omega)$.

If $\{\varphi_t\}$ is a potential-type approximate identity, then it follows immediately from Lemma (4.1.12) that for $i = 1, 2$, $\varphi_t * f_i \to f_i$ pointwise almost everywhere. The desired limit follows by linearly.

If $\varphi \in L^{p'_+}(\Omega)$ and has compact support, then the proof is almost easy. Since $L^{p'_+} = (p_-)'$, and since $(p_+)' \leq (p_-)'$, we have that $\varphi \in L^{(p_-)'}(\Omega) \subset L^{(p_+)'}(\Omega)$. Therefore, again by Lemma (4.1.12), for $i = 1, 2$, $\varphi_t * f_i \to f_i$ pointwise almost everywhere.

In the case of Lebesgue spaces, Theorem (4.1.2) is well-known: see Stein [148]; for $\varphi \in L^{p'}$ it is due to Zo [65]. Theorem (4.1.2) is a consequence of these results.

A weaker version of Theorem (4.1.2) for potential-type approximate identities was proved by Diening [4]; his proof required that the maximal function be bounded on $L^{p(\cdot)}(\Omega)$. However, the assumption in Theorem (4.1.1) that $p_- > 1$ is necessary, and examples show that (2) and (3) are essentially necessary. (See [36] for details.) Therefore, Theorem (4.1.2) is substantially more general.

The second main result gives conditions for an approximate identity to converge in norm.

Theorem (4.1.3)[35]: Given an open set Ω, let $p(\cdot) \in \mathcal{P}(\Omega)$ be such that (2) and (3) hold. Suppose that either:

(i) $\{\varphi_t\}$ is a potential-type approximate identity,

(ii) φ has compact support and $\varphi \in L^{p'_+}(\Omega)$.

Then for all $t > 0$,

$$\|\varphi_t * f\|_{p(x),\Omega} \leq C\|f\|_{p(\cdot),\Omega}, \tag{4}$$

and $\{\varphi_t * f\}$ converges to f in $L^{p(\cdot)}(\Omega)$ norm:

$$\lim_{t\to 0}\|\varphi_t * f\|_{p(x),\Omega} = 0. \tag{5}$$

Proof. Fix $f \in L^{p(\cdot)}(\Omega)$. Since we can write $f = f_+ - f_-$, where each 0f $f_\pm$ is nannegative and $\|f_\pm\|_{p(x),\Omega} \leq \|f\|_{p(x),\Omega}$, by linearity we may assume without loss of generality that f is nonnegative. Further, by homogeneity assume that $\|f\|_{p(x),\Omega} = 1$.

Define the function $f_1(x) = f(x)\chi_{\{x\in\Omega: f(x)\geq 1\}}$ and $f_2(x) = f(x) - f_1(x)$. Then (depending on φ) f_1 satisfies the hypotheses of either Lemma (4.1.15) or Lemma (4.1.16), and f_2 satisfies the hypotheses of either Lemma (4.1.18). Therefore,

$$\int_\Omega |\varphi_t * f(x)|^{p(x)}dx \leq C\int_\Omega |\varphi_t * f_1(x)|^{p(x)}dx + C\int_\Omega |\varphi_t * f_2(x)|^{p(x)}dx \leq C$$

Since $p_+ < \infty$, by Lemma (4.1.9),

$$\|\varphi_t * f\|_{p(\cdot),\Omega} \leq C = C\|f\|_{p(\cdot),\Omega}.$$

Norm convergence follows from this by an approximation argument. Fix $\varepsilon > 0$. Since bounded functions of compact support are dense in $L^{p(\cdot)}(\Omega)$, fix such a function g with $\|f - g\|_{p(\cdot),\Omega} < \varepsilon$. Then

$$\begin{aligned}\|\varphi_t * f - f\|_{p(\cdot),\Omega} &\leq \|\varphi_t * (f-g)\|_{p(\cdot),\Omega} + \|\varphi_t * g - g\|_{p(\cdot),\Omega} \leq \|f-g\|_{p(\cdot),\Omega} \\ &\leq C\varepsilon + \|\varphi_t * g - g\|_{p(\cdot),\Omega} = 0;\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, if we take the limit as $t \to 0$, then to complete the proof it will suffice to show that

$$\lim_{t\to 0}\|\varphi_t * g - g\|_{p(\cdot),\Omega} = 0.$$

Since g is bounded, define $g_0(x) = \frac{g(x)}{(2\|g\|_\infty)}$. Then

$$|\varphi_t * g_0(x)| \leq \int_\Omega |\varphi_t(x-y)||g_0(y)|dy \leq \|g_0\|_\infty \int_\Omega |\varphi_t(x-y)|dy \leq \|g_0\|_\infty.$$

Therefore, $\|\varphi_t * g_0 - g_0\|_\infty \leq 2\|g_0\|_\infty \leq 1$. Hence

$$\begin{aligned}&\lim_{t\to 0}\int_\Omega |\varphi_t * g(x) - g(x)|^{p(x)}dx \\ &\qquad = \lim_{t\to 0}\int_\Omega (2\|g\|_\infty)^{p(x)}|\varphi_t * g_0(x) - g_0(x)|^{p(x)}dx \\ &\qquad \leq (2\|g\|_\infty + 1)^{p_+}\lim_{t\to 0}\int_\Omega |\varphi_t * g_0(x) - g_0(x)|^{p(x)}dx\end{aligned}$$

$$\leq (2\|g\|_\infty + 1)^{p_+} \lim_{t\to 0} \int_\Omega |\varphi_t * g_0(x) - g_0(x)|^{p_-} dx.$$

Since $g_0 \in L^{p_-}(\Omega)$, the last integral converges to 0 as $t \to 0$. This complete the proof.

Deining [4] proved Theorem (4.1.3) for potential-type approximate identities with the additional assumption that $p_- > 1$; he was required to assume this since his proof requires the maximal operator to be bounded on $L^{p(\cdot)}(\Omega)$. (This approach, however, is very elegant, and we give a version of his proof.) Samko [23] proved Theorem (4.1.3) for $\varphi \in L^{p'_+}(\Omega)$ when Ω is bounded. In [24] he gave a necessary condition for (4) to hold which shows that some additional hypotheses on φ are required. Sharapudinov [145] proved a somewhat more general result on the unit circle for convolution operators with L^1 kernels.

Theorems (4.1.2) and (4.1.3) are false if $\varphi \in L^{p'_+}(\Omega)$ but φ does not have compact support. We give a counter-example below (Example (4.1.19)). We conjecture that they remain true if we further assume that φ satisfies a gradient condition:

$$|\varphi(x - y)| \leq \frac{C|y|}{|x|^{n-1}}, \qquad |y| > 2|y|.$$

This conjecture is motivated by the results of Zo [65] on the pointwise convergence of approximate identities. This gradient condition is well-known from the study of integrals; for the connection between singular integrals and maximal operators, see [40].

This conjecture is true; it was proved as consequence of a much more general result about extrapolation in the scale of variable L^p spaces by Cruz-Uribe, Fiorenza, Martell and Pérez [66]. It would be interesting to give a direct proof.

We give two sets of application of our results. The first consists of three theorems on the density of smooth functions in the variable Sobolev spaces.

Theorem (4.1.4)[35]: Given an open set Ω, let $p(\cdot) \in \mathcal{P}(\Omega)$ be such that $p_+ < \infty$ and (2) holds. Then for $k \geq 1$, the set

$$C^\infty(\Omega) \cap W^{k,p(\cdot)}(\Omega)$$

is dense in $W^{k,p(\cdot)}(\Omega)$.

In the case of classical Sobolev spaces, Theorem (4.1.4) is due to Meyers and Serrin [143]. The proof is almost identical to the proof in the classical case, using Theorem (4.1.3). (see, for example, Ziemer [66].) The proof, which depends on a partition of the identity, uses φ with compact support and is applied on bounded sets K. The case (3) holds automatically with a constant which depends on $\dim(K)$. Also, the proof uses the monotone convergence theorem for Lebesgue spaces, but this remains true in variable L^p spaces since $p_+ < \infty$.

Let $C_c^\infty(\mathbb{R}^n)$ denote the set of infinitely differentiable functions of compact support in $\mathbb{R}^n$.

Theorem (4.1.5)[35]: Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ be such that $p_+ < \infty$ and (2) holds. Then for $k \geq 1$, $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{k,p(\cdot)}(\mathbb{R}^n)$.

The prof of Theorem (4.1.5) is essentially the same as the standard proof for Lebesgue spaces, using Theorem (4.1.3) to get norm convergence. (See, Stein [146].)

Theorem (4.1.5) was first proved by Samko [23] using his weaker version of Theorem (4.1.3). His proof has two steps. First, he shows that functions with compact support are dense in $W^{k,p(\cdot)}(\mathbb{R}^n)$; given this, he can proceed as in the standard proof but only needs norm convergence on compact sets.

Theorem (4.1.5) was also proved by Diening with the additional assumption that $p_- > 1$ and that the maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ (e.g., if (3) holds).

To state the next result, we need a definition. An open set Ω has the segment property if for every $x \in \partial\Omega$ there exists an open set V_x containing x, and a nonzero vector v_x, such that if $z \in \overline{\Omega} \cap V_x$, then $z + tv_x \in \Omega$, $0 < t < 1$. This condition holds, for example, if Ω is boundary is locally a Lipschitz graph. (See Adams [28].)

Theorem (4.1.6)[35]: Given an open set Ω which satisfies the segment property, let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ be such that $p_+ < \infty$ and (2) holds. Then for $k \geq 1$, the set

$$C^\infty(\overline{\Omega}) \cap W^{k,p(\cdot)}(\Omega)$$

is dense $W^{k,p(\cdot)}(\Omega)$.

The proof of Theorem (4.1.6) again follows the proof for classical Sobolev spaces (see Adams [28]). It first reduces to the case of functions of compact support, and then uses a careful partition of unity. As in the proof of Theorem (4.1.4), Theorem (4.1.3) is only used for φ with compact support and for bounded domains, so (3) holds automatically. The proof also uses the fact that translation is continuous in norm. This is not true in general in variable L^p spaces (see [13, 141]) but it is true for bounded functions of compact support, which is all that is required to prove Theorem (4.1.6).

Theorem (4.1.6) was first proved by Diening [27] assuming that Ω is bounded and has Lipschitz boundary, and $p_- > 1$. (i.e., that the maximal operator is bounded on $L^{p(\cdot)}(\Omega)$).

The second set of applications consists of two solutions to classical boundary value problems. On $\mathbb{R}_+^{n+1}$, let $P_t(x)$ denote the Poission kernel, and let $W_t(x)$ denote the Gauss–Weierstrass kernel. (see [40].)

Theorem (4.1.7)[35]: Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ be such that $p_+ < \infty$ and (2) and (3) hold. If $f \in L^{p(\cdot)}(\mathbb{R}^n)$, then $u(x,t) = P_t * f(x)$ is the solution of the boundary value problem

$$\begin{cases} \Delta u(x,t) = 0, & (x,t) \in \mathbb{R}_+^{n+1}, \\ u(x,t) = f(x), & x \in \mathbb{R}^n, \end{cases}$$

where the second equality is understood in the sense that $u(x,t)$ converges to $f(x)$ as $t \to 0$ pointwsie almost everywhere and in $L^{p(\cdot)}(\mathbb{R}^n)$ norm.

Theorem (4.1.8)[35]: Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ be such that $p_+ < \infty$ and (2) and (3) hold. Given $f \in L^{p(\cdot)}(\mathbb{R}^n)$, define $u(x,t) = W_t * f(x)$and $\overline{\omega}(x,t) = \omega\left(x, \sqrt{4\pi t}\right)$. Then $\overline{\omega}$ is the solution of the initial value problem

$$\begin{cases} \dfrac{\partial \overline{\omega}}{\partial t}(x,t) - \Delta \overline{\omega}(x,t) = 0, \quad (x,t) \in \mathbb{R}^{n+1}_+, \\ \overline{\omega}(x,0) = f(x), \quad x \in \mathbb{R}^n, \end{cases}$$

where the second equality is understood in the sense that $\overline{\omega}(x,t)$ converges to $f(x)$ as $t \to 0$ pointwsie almost everywhere and in $L^{p(\cdot)}(\mathbb{R}^n)$ norm.

Since the Poisson kernel and the Gauss–Weierstrass kernel are both potential-type approximate identities, the proofs of Theorems (4.1.7) and (4.1.8) are identical to the proofs of the corresponding results in Lebesgue spaces. (See [40].)

Sharapudinov [145] proved a version of Theorem (4.1.7) on the unit disk.

We state some basic properties of variable L^p spaces, which will be used in the subsequent sections. We prove Theorem (4.1.2) and make some remarks about dense subsets in $L^{p(\cdot)}(\Omega)$ connected to our original proof. We prove Theorem (4.1.3). Finally, we make some observations about Young's theorem in variable L^p spaces.

We will always write $p(\cdot)$ instead p to denote an exponent function. Unless otherwise specified, C and c will denote positive constants which will depend only on the dimension n, the underlying set Ω,the exponent function $p(\cdot)$, and the function φ but whose value may change at each appearance.

We state some basic properties of variable $L^{p(\cdot)}$ spaces. For further information, including proofs of these results, see Kováčik and Rákosník [13].

Given an open set $\Omega, p(\cdot) \in \mathcal{P}(\Omega)$ and a function f, let

$$|f|_{p(\cdot),\Omega} = \int_{\Omega \setminus \Omega_{p(.),\infty}} |f(y)|^{p(y)}\, dy\,.$$

Note that if $p_+ < \infty$ (or even if $\left|\Omega_{p(\cdot),\infty}\right| = 0$), then $\rho(p(\cdot), \Omega, f) = |f|_{p(\cdot),\Omega}$. We will need the following properties relating $\|\cdot\|_{p(\cdot),\Omega}$ and the modular $|\cdot|_{p(.),\Omega}$.

Lemma (4.1.9)[35]: Given an open set $\Omega, p(\cdot) \in \mathcal{P}(\Omega)$, and $f \in L^{p(\cdot)}(\Omega)$, the following are true.

(i) If $\|f\|_{p(.),\Omega} \leq 1$, then $|f|_{p(\cdot),\Omega} \leq \rho(p(\cdot), \Omega, f) \leq \|f\|_{p(\cdot),\Omega}$.

(ii) If $p_+ < \infty$, then $\|f\|_{p(\cdot),\Omega} \leq C_1$ if and only if $|f|_{p(.),\Omega} \leq C_2$.

(iii) If $p_+ < \infty$, then given a sequence $\{f_n\} \subset L^{p(\cdot)}(\Omega)$, $\|f_n - f\|_{p(\cdot),\Omega} \to 0$ if and only if $|f - f_n|_{p(\cdot),\Omega} \to 0$.

As a consequence of (iii), if $p_+ < \infty$, and if f_n is an increasing to $\|f\|_{p(.),\Omega}$. To see this, first note that since $p_+ < \infty$, $|f(\cdot)|^{p(\cdot)} \in L^1(\Omega)$. Therefore,

$$|f(x) - f_n(x)|^{p(\cdot)} \leq |f(x)|^{p(\cdot)}(\Omega) \in L^1(\Omega).$$

So by the dominated convergence theorem, $|f-f_n|_{p(.),\Omega} \to 0$ as $n \to \infty$. Therefore, by (iii), $\|f_n - f\|_{p(.),\Omega} \to 0$, and the desired conclusion follows from the triangle inequality.

Lemma (4.1.10)[35]: Given an open set $\Omega, |\Omega| < \infty$, and $p(\cdot), q(\cdot) \in \mathcal{P}(\Omega)$, $p(x) \le q(x), x \in \Omega$, then

$$\|f\|_{p(\cdot),\Omega} \le (1 + |\Omega|)\|f\|_{p(\cdot),\Omega}.$$

As a consequence of Lemma (4.1.10), if $p_+ < \infty$, and u is abounded function of compact support, then $u(\cdot + t)$ converges to u as $t \to 0$ in $L^{p(\cdot)}(\Omega)$. To see, fix such a u. Then for all $t, |t| < 1$, there exists a compact set U such that $\operatorname{supp}\big(u(\cdot + t)\big) \subset U$. Then, since translation is continuous on $L^{p_+}(\Omega)$,

$$\lim_{t\to 0}\|u(\cdot + t) - u\|_{p(.),\Omega} \le (1 + |U|) \lim_{t\to 0}\|u(\cdot + t) - u\|_{p_+,\Omega} = 0.$$

The final result we need is a version of Hölder's inequality for variable L^p spaces.

Lemma (4.1.11)[35]: Given a set Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, there exists $C > 1$ such that for all function $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{p'(\cdot)}(\Omega)$,

$$\int_{\Omega} |f(x)g(x)|dx \le \|f\|_{p(\cdot),\Omega}\|f\|_{p'(\cdot),\Omega}.$$

The proof follows from known results in the classical Lebesgue spaces. We first quickly state these results; see [65, 146] for proofs.

Lemma (4.1.12)[35]: Given a set Ω and $p, 1 \le p \le \infty$, suppose $f \in L^{p(\cdot)}(\Omega)$. If φ is such that either:

(i) $\{\varphi_t\}$ is potential-type approximate identity,

(ii) φ has compact support and $\varphi \in L^{p'(\cdot)}(\Omega)$,

then $\{\varphi_t * f\}$ converges to f pointwise almost everywhere.

Remark (4.1.13)[35]: The original proof of Theorem (4.1.2) was significantly more complicated. It was modeled after the proof of Lemma (4.1.11) and used the modular weak-type inequality due to Cruz–Uribe, Fiorenza and Neugebauer:

$$|\{x \in \Omega :\ Mf(x) > t\}| \le C \int_{\Omega} \left(\frac{|f(y)|}{t}\right)^{p(y)} dy. \tag{6}$$

This result required the additional hypothesis that $1/p(\cdot) \in RH_\infty$. A nonnegative function u on $\mathbb{R}^n$ is in RH_∞ if there exists a positive constant C such that for every ball almost every $x \in B$,

$$u(x) \le \frac{C}{|B|}\int_B u(y)dy.$$

Note that if there exist positive constants A, B such that $A \le u(x) \le B$, then $u \in RH_\infty$. More generally, if $u(x) = |x|^a$, $a > 0$, then $u \in RH_\infty$. (For more information on the class RH_∞, see Curz–Uribe and Neugebauer [140] or [36].)

It is note that while Theorem (4.1.2) requires no additional hypotheses on $p(\cdot)$, the modular inequality (6) does. To see this on $\Omega = [0, \infty)$, let $p(x) = e^{-1/x}$ and let $f_n(x) = n\chi_{[0,1/n]}(x)$. Then a straightforward calculation shows that for n lage,

$$|\{x \in \Omega : \ Mf(x) > t\}| = 1,$$

$$\int_0^{1/n} f(x)^{p(x)} dx \leq 2/n.$$

This suggests that while inequalities for the maximal operator will yield sufficient hypotheses for extending a variety of results from classical harmonic analysis to variable $L^{p(\cdot)}$ spaces, these may not always be sharp.

However, if $p_+ < \infty$, then continuous functions of compact support need not be dense, and we include two examples that show this. We first consider unbounded Ω. Let $\Omega = [1, \infty)$ and let $p(x) = x$. Then, since for all $\lambda > 1$.

$$\int_1^\infty \left(\frac{1}{\lambda}\right)^x dx \leq \infty,$$

the constant function $f(x) = 1$ is in $L^{p(\cdot)}(\Omega)$. Now let g be any function of compact support. Then $\operatorname{supp}(g) \subset [1, N]$ for some $N > 1$. But then, by the definition,

$$\|f - g\|_{p(\cdot),\Omega} \geq \inf \left\{\lambda > 0 : \int_N^\infty \left(\frac{1}{\lambda}\right)^x dx \leq 1\right\};$$

the integral is finite only when $\lambda > 1$, so we must have that $\|f - g\|_{p(\cdot),\Omega} \geq 1$. Here, no sequence of functions with compact support converges to f in $L^{p(\cdot)}(\Omega)$.

An example for bounded Ω is equally straightforward. Let $\Omega = (0, 1)$ and define $p(\cdot) : (0, 1) \to [1, \infty]$ by $p(x) = 1/|x - 1/2|$.

Let $f(x) = 7\chi_{(0,1/2)}(x)$. Then $f \in L^{p(\cdot)}(\Omega)$ since

$$\int_\Omega \left|\frac{f(x)}{7}\right|^{p(x)} dx = \int_N^{1/2} 1dx = 1/2,$$

and so $\|f - g\|_{p(\cdot),\Omega} \leq 7$.

Now let φ be any constant function defined on Ω. If $\varphi(1/2) \geq 7/2$ such that if $x \in (1/2, \delta)$, then $\varphi(x) > 3$. If $\varphi(x) < 7/2$, then there exists $\delta < 1/2$ such that if $x \in (\delta, 1/2)$, then $\varphi(x) < 7/2$. If either case, there exists an interval I. One of whose endpoints is $1/2$, such that on it, $|f(x) - \varphi(x)| > 3$. It follows that

$$\int_\Omega |f(x) - \varphi(x)|^{p(x)} dx \geq \int_I |f(x) - \varphi(x)|^{p(x)} dx > \int_I 3^{p(x)} dx > \int_I p(x) dx = \infty.$$

Therefore, either $f - \varphi$ is not in $L^{p(\cdot)}(\Omega)$, or it is and $\|f - g\|_{p(.),\Omega} \geq 1$. Hence, no sequence of continuous functions can converge to f in $L^{p(\cdot)}(\Omega)$.

Given an arbitrary function $p(\cdot)$ Such that $p_+ < \infty$, it is an open problem to find nontrivial dense subset of $L^{p(\cdot)}(\Omega)$, that is, sets $A \subset L^1(\Omega) \cap L^\infty(\Omega)$ such that A is dense in $L^{p(\cdot)}(\Omega)$ but not dense in $L^\infty(\Omega)$.

As we noted, Theorem (4.1.3) has an elegant proof in the case of potential-type approximate identities if we also assume $p_+ > 1$. For completeness we give it here. By the hypotheses, the maximal operator is bounded on $L^{p(\cdot)}(\Omega)$. Therefore, by (1), for all $t > 0$,

$$\|\varphi_t * f\|_{p(\cdot),\Omega} \le C\|Mf\|_{p(\cdot),\Omega} \le C\|f\|_{p(\cdot),\Omega}.$$

Further, again by (1), since $p_+ < \infty$,

$$|\varphi_t * f(x) - f(x)|^{p(x)}dx \le CMf(x)^{p(x)}dx \in L^1(\Omega).$$

Theorem (4.1.2) we have that $\{\varphi_t * f\}$ converges to f pointwise almost everywhere. Hence, by the dominated convergence theorem,

$$\lim_{t\to 0}|\varphi_t * f - f|_{p(\cdot)} = \lim_{t\to 0}\int_\Omega |\varphi_t * f(x) - f(x)|^{p(x)}dx = 0.$$

Therefore, by Lemma (4.1.11), $\{\varphi_t * f\}$ converges to f in norm.

The proof of Theorem (4.1.3) in full generality requires five lemmas is due to Diening [4]. (Also see [66, 146].)

Lemma (4.1.14)[35]: Given a set Ω and $p(\cdot) \in \mathcal{P}(\Omega)$ such that (2) holds, suppose $\{\varphi_t\}$ then for every ball B, and $x \in B$,

$$|B|^{\frac{1-p(x)}{p_-(B\cap\Omega)}} \le C.$$

Lemma (4.1.15)[35]: Given a set Ω and $p(\cdot) \in \mathcal{P}(\Omega)$ such that (2) holds, suppose $\{\varphi_t\}$ is a potential-type approximate identity. If $f \in L^{p(\cdot)}(\Omega)$, $f(x) = 0$ or $f(x) \ge 1$, $x \in \Omega$,

$$|\varphi_t * f(x)|^{p(x)} \le C\big(\tilde{\varphi}_t * f(\cdot)^{p(\cdot)}\big)(x).$$

furthermore,

$$\int_\Omega |\varphi_t * f(x)|^{p(x)}dx \le C < \infty.$$

The proof of lemma is adapted from Nekvinda [146].

Proof. Note that since

$$|\varphi_t * f(x)|^{p(x)} \le C(\tilde{\varphi}_t * f)(x)^{p(x)},$$

It will suffice to bound the right-hand term. Suppose first that

$$\tilde{\varphi}_t(x) = \sum_k a_k \chi_{B_k}(x), \tag{7}$$

where each $a_k \ge 0$ and each B_k is a ball centered at the origin. Then

$$(\tilde{\varphi}_t * f)(x)^{p(x)} = \left(\sum_k a_k \chi_{B_k} * f(x)\right)^{p(x)}$$

$$= \left(\sum_k a_k\, |B_k| \cdot \frac{1}{|B_k|} \int_{(x+B_k)} f(y)dy\right)^{p(x)} ;$$

by Hölder's inequality, first for series and then for integrals,

$$\leq \left(\sum_k a_k |B_k|\right)^{p(x)-1} \sum_k a_k |B_k| \left(\sum_k a_k\, |B_k| \cdot \frac{1}{|B_k|} \int_{(x+B_k)\cap\Omega} f(y)^{p(y)} dy\right)^{\frac{p(x)}{p_-((x+B_k)\cap\Omega)}} ;$$

since $f(y) \geq 1$ and $|\tilde{\varphi}_t|_{p(\cdot),\Omega} \leq 1$,

$$\leq \|\tilde{\varphi}_t\|_1^{p(x)-1} \sum_k a_k |B_k| \left(\sum_k a_k\, |B_k| \cdot \frac{1}{|B_k|} \int_{(x+B_k)\cap\Omega} f(y)^{p(y)} dy\right)^{\frac{p(x)}{p_-((x+B_k)\cap\Omega)}}$$

$$\leq C \sum_k a_k\, |B_k|^{\frac{1-p(x)}{p_-((x+B_k)\cap\Omega)}} \int_{(x+B_k)\cap\Omega} f(y)dy.$$

By Lemma (4.1.14),

$$|B_k|^{\frac{1-p(x)}{p_-((x+B_k)\cap\Omega)}} = |x+B_k|^{\frac{1-p(x)}{p_-((x+B_k)\cap\Omega)}} \leq C.$$

Therefore, we have that

$$(\tilde{\varphi}_t * f)(x)^{p(x)} \leq C \sum_k a_k \int_{(x+B_k)\cap\Omega} f(y)dy$$

$$= C \sum_k a_k \left(\chi_{B_k} * f(\cdot)^{p(\cdot)}\right)(x)$$

$$= C\left(\tilde{\varphi}_t * f(\cdot)^{p(\cdot)}\right)(x).$$

This is the desired inequality.

We now argue for general φ. Since for each $t > 0$, $\tilde{\varphi}_t$ is a radial, decreasing function, it can be approximated by an increasing sequence of function of the form in (7). Hence, by the monotone convergence theorem, we get the desired inequality.

Note that by Fubini's theorem,

$$\int_\Omega |\varphi_t * f(x)|^{p(x)} dx \leq C \int_\Omega \left(\tilde{\varphi}_t * f(\cdot)^{p(\cdot)}\right)(x)dx \leq C\|\tilde{\varphi}_t\|_1 |f|_{p(\cdot),\Omega} < C.$$

Lemma (4.1.16)[35]: Given a set Ω and $p(\cdot) \in \mathcal{P}(\Omega)$ such that (2) holds, suppose $\varphi \in L^{p'_+}(\Omega)$ has compact support. Given $f \in L^{p(\cdot)}(\Omega)$ such that $\|f\|_{p(\cdot),\Omega} \leq 1$ and $f(x) = 0$ or $f(x) \geq 1$, $x \in \Omega$,

$$\int_{\Omega} |\varphi_t * f(x)|^{p(x)} dx \le C < \infty.$$

The proof of this lemma is based on ideas in Samko [23].

Proof. $\Omega_f = \operatorname{supp}(f) = \{x \in \Omega : f(x) \ge 1\}$. Then by Lemma (4.1.9),

$$|\Omega_f| \le |f|_{p(\cdot),\Omega} \le \|f\|_{p(\cdot),\Omega} \le 1. \tag{8}$$

We first want to show that

$$|\varphi_t * f(x)| \le C \max(1, t^{-n}). \tag{9}$$

By Hölder's inequality for variable L^p spaces Lemma (4.1.11),

$$|\varphi_t * f(x)| \le \int_{\Omega} |\varphi_t(x-y)| f(y) dy \le C \|\varphi_t(x-\cdot)\|_{p'(\cdot),\Omega_f} \|f\|_{p(\cdot),\Omega}$$
$$\le C \|\varphi_t(x-\cdot)\|_{p'(\cdot),\Omega_f}.$$

To estimate $\|\varphi_t(x-\cdot)\|_{p'(\cdot),\Omega_f}$, there are two cases.

Case 1: $p'_+ < \infty$. In this case, by Lemma (4.1.10) and inequality (8),

$$\|\varphi_t(x-\cdot)\|_{p'(\cdot),\Omega_f} \le \|\varphi_t(x-\cdot)\|_{\infty,\Omega_f} \left\|\chi_{\Omega_f}\right\|_{p'(\cdot),\Omega_f}$$
$$\le t^{-n} \|\varphi\|_{\infty,\Omega} \left(1 + |\Omega_f|\right) \left\|\chi_{\Omega_f}\right\|_{\infty,\Omega_f} \le C t^{-n}.$$

Case 2: $p'_+ < \infty$. Again by Lemma (4.1.10) and (8)

$$\|\varphi_t(x-\cdot)\|_{p'(\cdot),\Omega_f} \le \left(1 + |\Omega_f|\right) \|\varphi_t(x-\cdot)\|_{p'(\cdot),\Omega_f} \le 2t^{-n+\frac{n}{p_+}} \|\varphi\|_{p'_+,\Omega}$$
$$\le C \max(1, t^{-n}).$$

Hence, in either case (9) holds.

Let the support of φ be contained in a ball of radius R centered at origin. Fix $t > 0$; then the support of φ_t is contained in a ball of radius Rt centered at origin. Partition Ω into union of a countable number of disjoint sets $\{\Omega_k\}$ such that each Ω_k is contained in a ball B_k with radius t. Let $\overline{\Omega}_k$ be the intersection of Ω with a ball with the same center as B_k and radius $(R+1)t$. We will prove that

$$\int_{\Omega} |\varphi_t * f(x)|^{p(x)} dx \le C \sum_k \int_{\Omega_k} |\varphi_t * f(x)|^{p_-(\overline{\Omega}_k)} dx \tag{10}$$

by showing that if $x \in \Omega_k$, then

$$|\varphi_t * f(x)|^{p(x) - p_-(\overline{\Omega}_k)} \le C.$$

If $t > \frac{1}{2(R+2)}$, this follows immediately form (9). If $t \le \frac{1}{2(R+2)}$, then the distance between any two points in $\overline{\Omega}_k$ is at most $(R+2)t \le 1/2$. Hence, by (2),

$$0 \le p(x) - p_-(\bar{\Omega}_k) \le \frac{1}{-\log\big((R+2)t\big)},$$

so again by (9),

$$|\varphi_t * f(x)|^{p(x)-p_-(\bar{\Omega}_k)} \le Ct^{-n\left(p(x)-p_-(\bar{\Omega}_k)\right)} \le C.$$

Therefore, inequality (10) holds.

We now complete the proof. If $x \in \Omega_k$, then $\operatorname{supp}\big(\varphi_t(x-\cdot)\big) \subset \bar{\Omega}_k$. Hence, by Hölder's inequality,

$$|\varphi_t * f(x)|^{p_-(\bar{\Omega}_k)} \le \left(\int_\Omega |\varphi_t(x-y)| f(y)dy\right)^{p_-(\bar{\Omega}_k)}$$

$$\le \left(\int_\Omega |\varphi_t(x-y)|dy\right)^{p_-(\bar{\Omega}_k)} \left(\int_{\Omega_k} |\varphi_t(x-y)| f(y)^{p_-(\bar{\Omega}_k)} dy\right);$$

since $f(x) = 0$ or $f(x) \ge 1$,

$$\le \|\varphi\|_1^{p_-(\bar{\Omega}_k)-1} \int_{\Omega_k} |\varphi_t(x-y)| f(y)^{p(y)} dy$$

$$= C\big(|\varphi_t| f(\cdot)^{p(\cdot)}\big)(x).$$

Therefore, from this inequality and from inequality (10), we get Fubini's theorem that

$$\int_\Omega |\varphi_t * f(x)|^{p(x)} dx \le C \sum_k \int_{\Omega_k} \big(|\varphi_t| * f(\cdot)^{p(\cdot)}\big)(x)dx$$

$$= C \int_\Omega \big(|\varphi_t| * f(\cdot)^{p(\cdot)}\big)(x)dx \le C\|\varphi_t\|_{1,\Omega} |f|_{p(\cdot),\Omega} \le C.$$

A version of the next lemma appeared first in [36]. For completeness, we include the proof here.

Lemma (4.1.17)[35]: Given a set G and two nonnegative functions $r(\cdot)$ and $s(\cdot)$, suppose that for each $y \in G$,

$$|s(y) - r(y)| \le \frac{1}{\log(e + |z(y)|)},$$

where $z(\cdot) : G \to \mathbb{R}^n$ is measurable. Then for every positive measure μ and for every function f such that $|f(y)| \le 1$, $y \in G$,

$$\int_G |f(y)|^{r(y)} d\mu(y) \le C \int_G |f(y)|^{s(y)} d\mu(y) + \int_G \beta\big(z(y)\big)^{r*(G)} d\mu(y),$$

where $\beta(y) = (e + |(y)|)^{-(n+1)}$.

Proof. Let $G^\beta = \{y \in G : |f(y)| \ge \beta\big(z(y)\big)\}$. Then

$$\int_G |f(y)|^{r(y)} d\mu(y) = \int_{G^\beta} |f(y)|^{r(y)} d\mu(y) + \int_{G\setminus G^\beta} |f(y)|^{r(y)} d\mu(y),$$

and we estimate each integral separately. Since $\beta\big(z(y)\big)1$,

$$\int_{G\setminus G^\beta} |f(y)|^{r(y)} d\mu(y) = \int_{G\setminus G^\beta} \beta\big(z(y)\big)^{r(y)} d\mu(y) + \int_{G\setminus G^\beta} \beta\big(z(y)\big)^{r*(G)} d\mu(y).$$

On the other hand, if $y \in G^\beta$, since $|f(y)| \le 1$,

$$|f(y)|^{r(y)} = |f(y)|^{s(y)} |f(y)|^{r(y)-s(y)} \le |f(y)|^{s(y)} |f(y)|^{-|s(y)-r(y)|}$$
$$\le |f(y)|^{s(y)} \beta\big(z(y)\big)^{\frac{-C}{-\log(e+|z(y)|)}} \le C|f(y)|^{s(y)}.$$

The desired inequality now follows immediately.

Lemma (4.1.18)[35]: Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$ such that (3) holds, and given any approximate identity $\{\varphi_t\}$, suppose $f \in L^{p(\cdot)}(\Omega)$ is such that $|f|_{p(\cdot),\Omega} \le 1$ and $0 \le f(x) \le 1$, $x \in \Omega$. Then foe all $x \in \Omega$,

$$|\varphi_t * f(x)|^{p(x)} \le C\big(f(\cdot)^{p(\cdot)} * |\varphi_t|\big)(x) + C\big(|\varphi_t| * f(\cdot)^{p(\cdot)}\big)(x) + C\beta(x), \quad (11)$$

where $\beta(y) = (e + |(y)|)^{-(n+1)}$. Furthermore,

$$\int_\Omega |\varphi_t * f(x)|^{p(x)} dx \le C < \infty. \quad (12)$$

Proof. Fix $x \in \Omega$. Define the function $z(\cdot) : G \to \mathbb{R}^n$ by

$$z(y) = \begin{cases} y, & |x| > |y|, \\ x, & |x| \le |y|. \end{cases}$$

Then condition (3) implies that for all $x \in \Omega$,

$$|p(x) - p(y)| \le \frac{C}{(e + |z(y)|)}.$$

Then by Hölder's inequality and Lemma (4.1.17) (applied with the functions $r(y) = p(y), s(y) - p(y)$ and the measure $d\mu(y) = |\varphi_t(x-y)|dy$),

$$|\varphi_t * f(x)|^{p(x)} \le \left(\int_\Omega |\varphi_t(x-y)|dy\right)^{p(x)-1} \left(\int_\Omega |\varphi_t(x-y)| f(y)^{p(x)} dy\right)$$
$$\le C\int_\Omega |\varphi_t(x-y)| f(y)^{p(x)} dy + C\int_\Omega |\varphi_t(x-y)| \beta\big(z(y)\big)^{p(x)} dy$$
$$\le C\big(|\varphi_t| * f(\cdot)^{p(\cdot)}\big)(x) + C\int_\Omega |\varphi_t(x-y)| \beta\big(z(y)\big) dy.$$

The last integral is bounded by $C(|\varphi_t| * \beta)(x) + C\beta(x)$:

$$\int_\Omega |\varphi_t(x-y)|\beta(z(y))dy$$
$$= \int_{\{y\in G:|x|\geq|y|\}} |\varphi_t(x-y)|\beta(y)dy + \int_{\{y\in G:|x|<|y|\}} |\varphi_t(x-y)|\beta(x)dy$$
$$\leq C(|\varphi_t| * \beta)(x) + C\beta(x).$$

This completes the proof of (11). To proof of (12) we integrate (11) on Ω: since $\beta \in L^{p(\cdot)}(\Omega)$ and $|f|_{p(\cdot),\Omega} \leq 1$, we can apply Fubini's theorem to get

$$\int_\Omega |\varphi_t(x)|f(x)^{p(x)}dy$$
$$\leq C\int_\Omega \left(|\varphi_t| * f(\cdot)^{p(\cdot)}\right)(x)dy + C\int_\Omega (|\varphi_t| * \beta)(x)dx + C\int_\Omega \beta(x)dx$$
$$\leq C\|\varphi_t\|_1 |f|_{p(\cdot),\Omega} + \leq C\|\varphi_t\|_{1,\Omega}|f|_{p(\cdot),\Omega} + C\|\beta\|_{1,\Omega} \leq C.$$

Example (4.1.19)[35]: There exists $p(\cdot) \in \mathcal{P}(\mathbb{R})$ be such that (2) and (3) hold $f \in L^{p(\cdot)}(\mathbb{R})$, and $\varphi \in L^q(\mathbb{R})$, $1 \leq q \leq \infty$, such that

$$\limsup_{t\to 0} \|\varphi_t * f(x)\|_{p(\cdot),\Omega} = \infty, \tag{13}$$

and for all x in a set of positive measure,

$$\limsup_{t\to 0} |\varphi_t * (x)| = \infty. \tag{14}$$

Proof. Define the function φ by

$$\varphi(x) = \sum_{n=1}^{\infty} \varphi_n(x) = \sum_{n=1}^{\infty} \chi_{[-n-1/n^2,-n]}(x).$$

Since $\sum \frac{1}{n^2} < \infty$, $\varphi \in L^q(\mathbb{R})$, $1 \leq q \leq \infty$. Define the exponent function $p(\cdot)$ to be a smooth function $\mathcal{P}(\mathbb{R})$ such that $p(x) = p_0$, $x \leq -1$ and $x \geq 2$, $p(x) = p_1$, $0 \leq x \leq 1$. The exact values of p_0 and p_1 will be chosen below. Let

$$f(x) = x^{-\alpha}\chi_{[0,1]}(x).$$

the exact value of α, $0 < \alpha < 1/p_1$, will be chosen below so that $f \in L^{p(\cdot)}(\mathbb{R})$.

For all $t > 0$ and all x,

$$\varphi_{n,t} * f(x) = \frac{1}{t}\int_{[x+nt,x+nt+t/n^2]\cap[0,1]} y^{-\alpha}dy.$$

If $-nt - t/(2n^2) \leq x \leq -nt$, then

$$\varphi_{n,t} * f(x) \geq \frac{1}{t}\int_{[0,t/(2n^2)]} y^{-\alpha}dy = \frac{1}{1-\alpha}\frac{1}{t}\left(\frac{t}{2n^2}\right)^{1-\alpha}$$

Fix $t = 1/n$ and choose x in the given range. Then $x \leq -1$, and so

$$\int_{\mathbb{R}} |\varphi_t * f(x)|^{p(x)} dx \geq \int_{[-nt-t/(2n^2),-nt]} \left|\varphi_{n,t} * f(x)\right|^{p_0}$$
$$\geq \left[\frac{c}{t^\alpha}(2n^2)^{1-\alpha}\right]^{p_0} \frac{t}{2n^2}$$
$$= cn^{\alpha p_0 + 2(\alpha-1)p_0 - 3}.$$

If we let $p_0 = 10$, $p_1 = 11/10$, and $\alpha = 9/10$, then $\alpha p_0 + 2(\alpha - 1)p_0 - 3 = 4$. Therefore,

$$\limsup_{t\to 0} \int_{\mathbb{R}} |\varphi_t * f(x)|^{p(x)} dx = \infty;$$

Since $p_+ < \infty$, (13) follows at once.

Now fix $x \in [-2, -1]$. Given n if t is such that

$$\frac{|x|}{n + 1/(2n^2)} \leq t \leq \frac{|x|}{n},$$

then $-nt - t/(2n^2) \leq x \leq -nt$. For each n, fix t_n in this range. Then $t_n \to 0$ as $n \to \infty$, ant $1/t_n \geq n/|x| \geq n/2$. Therefore, the above calculation show that

$$\varphi_{tn} * f(x) \geq \varphi_{n,tn} * f(x) \geq \frac{c}{t_n^\alpha}(2n^2)^{\alpha-1} \geq cn^\alpha . n^{2(\alpha-1)} = n^{7/10}.$$

It follows that (14) holds for all x in $[-2, -1]$.

An open problem related to the study of approximate identities is the generalization of Young's inequality to variable L^p spaces. It is natural to conjecture that $p(\cdot), q(\cdot)$ and $r(\cdot)$ are exponent functions which satisfy (2) and (3), and are such that

$$\frac{1}{r(x)} + 1 = \frac{1}{p(x)} + \frac{1}{q(x)}, \quad x \in \Omega,$$

then

$$\|\varphi * f\|_{r(\cdot),\Omega} \leq C\|\varphi\|_{p(\cdot),\Omega} \leq \|f\|_{p(\cdot),\Omega}. \tag{15}$$

This is false in general: see Diening [4]. In the special case φ is radial and decreasing, a careful examination of the constant involved in the proof of Theorem (4.1.3) shows that we proved the following.

Corollary (4.1.20)[35]: Suppose Ω and $p(\cdot) \in \mathcal{P}(\Omega)$ is such that $p_+ < \infty$ and (2) and (3) holds. If $f \in L^1(\Omega)$ is a positive, radially decreasing function, then there exists a constant depending only on $p(\cdot)$ such that

$$\|\varphi * f\|_{p(\cdot),\Omega} \leq C\|\varphi\|_{1,\Omega} \leq \|f\|_{p(\cdot),\Omega}.$$

However, (15) does not hold in the full generality stated even for radial functions: we give an example to show that

$$\|\varphi * f\|_{\infty,\Omega} \leq C\|\varphi\|_{p'(\cdot),\Omega} \leq \|f\|_{p(\cdot),\Omega}.$$

Does not hold. Let $\Omega=\mathbb{R}$, and let $p(\cdot):\mathbb{R}\to[1,\infty)$ be smooth function such that $p(x)=10$, $x\le 1$, and $p(x)=2$. Then $p'(x)=2$, $x\ge 2$. Define

$$f(x)=|x-3|^{-1/3}\chi_{[2,4]},$$
$$\varphi(x)=|x|^{-9/11}\chi_{[-1,1]}.$$

Then $f\in L^{p(\cdot)}(\Omega)$ and $\varphi\in L^{p'(\cdot)}(\Omega)$. However, the function $\varphi*f$ is bounded in neighborhood of 3. To see this, let $E_x=[2,4]\cap[x-1,x+1]$. Then

$$\varphi_t*f(x)=\int_{E_x}|x-y|^{-9/11}|y-3|^{-1/3}dy,$$

and since $-9/11+1/3>1$, this integral diverges as $x\to 3$.

Additional counter examples for various $p(\cdot)$ and $q(\cdot)$ were found by Samko [24]. The question of general hypotheses on φ for (15) to be true remains open. Here we note that a weaker inequality holds. If we assume that $\varphi\in L^{p'_-(\cdot)}(\Omega)\cap L^{p'_+(\cdot)}(\Omega)$, then by Hölder's inequality on variable L^p spaces we have that

$$\begin{aligned}|\varphi*f(x)|&\le C\|\varphi(x-\cdot)\|_{p'(\cdot),\Omega}\|f\|_{p(\cdot),\Omega}\\&\le C\left\|\varphi(x-\cdot)\chi_{\{|\varphi(\cdot)|\le 1\}}\right\|_{p'(\cdot),\Omega}\|f\|_{p(\cdot),\Omega}+C\left\|\varphi(x-\cdot)\chi_{\{|\varphi(\cdot)|>1\}}\right\|_{p'(\cdot),\Omega}\|f\|_{p(\cdot),\Omega}\\&\le C\left(\|\varphi\|_{p'_-(\cdot),\Omega}+\|\varphi\|_{p'_+(\cdot),\Omega}\right)\|f\|_{p(\cdot),\Omega}.\end{aligned}$$

If φ has compact support, we get that there is a constant C (depending on $|\mathrm{supp}\,(\varphi)|$) such that

$$\|\varphi*f\|_{\infty,\Omega}\le C\|\varphi\|_{p'_+(\cdot),\Omega}\le\|f\|_{p(\cdot),\Omega}.$$

We can interpolate between this inequality and the one in Corollary (4.1.20) by adapting an argument due to Bennett and Sharpley (see [139]) to prove the following.

Theorem (4.1.21)[35]: Suppose Ω and $p(\cdot)\in\mathcal{P}(\Omega)$ is such that $p_+<\infty$ and (2) and (3) holds. Let $\varphi\in L^1(\Omega)$ is a positive, radially decreasing function with compact support. For each θ, $0<\theta<1$, define q and $r(\cdot)$ by

$$\frac{1}{r(x)}=\frac{1-\theta}{p(x)},\quad \frac{1}{q}=1-\frac{\theta}{p_-}.$$

Then there exists C depending on $|\mathrm{supp}\,(\varphi)|$ such that

$$\|\varphi*f\|_{r(\cdot),\Omega}\le C\|\varphi\|_{q,\Omega}\le\|f\|_{p(\cdot),\Omega}.$$

This inequality is similar to one proved by Samko [24]. Details are left to the reader.

Section (4.2): Young Type Inequalities in Variable Lebesgue-Orlicz Spaces $L^{p(\cdot)}(\log L)^{q(\cdot)}$

Following Cruz-Uribe and Fiorenza [103], we consider two variable exponents $p(\cdot):\mathbb{R}^n\to[1,\infty)$ and $q(\cdot):\mathbb{R}^n\to\mathbb{R}$, which are continuous functions. Letting $\Phi_{p(\cdot),q(\cdot)}(x,t)=t^{p(x)}(\log(c_0+t))^{q(x)}$, we define the space $L^{p(\cdot)}(\log L)^{q(\cdot)}(\Omega)$ of all measurable functions f on an open set Ω such that

$$\int_{\Omega} \Phi_{p(\cdot),q(\cdot)}\left(y, \frac{|f(y)|}{\lambda}\right) dy < \infty$$

for some $\lambda > 0$; here we assume

(Φ) $\Phi_{p(\cdot),q(\cdot)}(x,\cdot)$ is convex on $[0,\infty)$ for every fixed $x \in \mathbb{R}^n$.

Note that (Φ) holds for some $c_0 \geq e$ if and only if there is a positive constants K such that

$$K(p(x)-1) + q(x) \geq 0 \quad \text{for all } x \in \mathbb{R}^n \tag{16}$$

Further, we see from (Φ) that $t^{-1}\Phi_{p(\cdot),q(\cdot)}(x,t)$ is nondecreasing in t. We define the norm

$$\|f\|_{\Phi_{p(\cdot),q(\cdot)},\Omega} = \inf\left\{\lambda > 0: \int_{\Omega} \Phi_{p(\cdot),q(\cdot)}\left(y, \frac{|f(y)|}{\lambda}\right) dy \leq 1\right\}$$

for $f \in L^{p(\cdot)}(\log L)^{q(\cdot)}(\Omega)$. Note that $L^{p(\cdot)}(\log L)^{q(\cdot)}(\Omega)$ is a Musielak–Orilicz space [150]. Such that spaces have been studied in [103, 149, 151]. In case $q(\cdot) = 0$ on $\mathbb{R}^n$, $L^{p(\cdot)}(\log L)^{q(\cdot)}(\Omega)$ is dented by $L^{p(\cdot)}(\Omega)$ ([13]).

We assume that the variable exponents $p(\cdot)$ and $q(\cdot)$ are continuous functions on $\mathbb{R}^n$ satisfying:

(p1) $1 \leq p_- := \inf_{x\in\mathbb{R}^n} p(x) \leq \sup_{x\in\mathbb{R}^n} p(x) =: p_+ < \infty$;

(p2) $|p(x) - p(y)| \leq \frac{C}{\log\left(e+\frac{1}{|x-y|}\right)}$ whenever $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$;

(p3) $|p(x) - (y)| \leq \frac{C}{\log(e+|x|)}$ whenever $|y| \geq |x|/2$;

(q1) $-\infty < q_- := \inf_{x\in\mathbb{R}^n} q(x) \leq \sup_{x\in\mathbb{R}^n} q(x) =: p_+ < \infty$;

(q2) $|q(x) - q(y)| \leq \frac{C}{\log\left(e+\log\left(e+\frac{1}{|x-y|}\right)\right)}$ whenever $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$

for a positive constant C.

We choose $p_0 \geq 1$ as follows: we take $p_0 = p_-$ if $t^{-p_-}\Phi_{p(\cdot),q(\cdot)}(x,t)$ is uniformly almost increasing in t; more precisely, if there exists $C > 0$ such that $s^{-p_-}\Phi_{p(\cdot),q(\cdot)}(x,s) \leq Ct^{-p_-}\Phi_{p(\cdot),q(\cdot)}(x,t)$ whenever $0 < s < t$ and $x \in \mathbb{R}^n$. Otherwise we choose $0 < p_0 < p_-$. Then note that $t^{-p_0}\Phi_{p(\cdot),q(\cdot)}(x,t)$ is uniformly almost increasing in t in any case.

Let φ e integrable function on $\mathbb{R}^n$ for each $t > 0$, define the function $\varphi_t$by $\varphi_t(x) = t^{-n}\varphi(x/t)$. Note that by a change of variables, $\|\varphi_t\|_{L^1,\mathbb{R}^n} = \|\varphi\|_{L^1,\mathbb{R}^n}$. We say that the family $\{\varphi_t\}$ is an approximate identity if $\int_{\mathbb{R}^n} \varphi(x)dx = 1$. Define the radial majorant of φ to be the function

$$\hat{\varphi}(x) = \sup_{|y|\geq|x|} |\varphi(y)|.$$

If $\hat{\varphi}$ is integrable, we say that the family $\{\varphi_t\}$ is potential-type.

Cruz-Uribe and Fiorenza [35] proved the following results:

Theorem (4.2.1)[153]: Let $\{\varphi_t\}$ be an approximate identity. Suppose that either

(i) $\{\varphi_t\}$ is potential-type or

(ii) $\varphi \in L^{(p_-)'}(\mathbb{R}^n)$ and has compact support.

Then

$$\sup_{0<t\le 1} \|\varphi_t * f\|_{L^{p(\cdot)},\mathbb{R}^n} \le \|f\|_{L^{p(\cdot)},\mathbb{R}^n}$$

and

$$\lim_{t\to +0} \|\varphi_t * f - f\|_{L^{p(\cdot)},\mathbb{R}^n} = 0$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$.

We extend their result to the space $L^{p(\cdot)}(\log L)^{q(\cdot)}(\Omega)$ of two variable exponents.

Theorem (4.2.2)[153]: Let $\{\varphi_t\}$ be potential-type approximate identity. If $f \in L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n)$, then $\{\varphi_t * f\}$ converges to f in $L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n)$:

$$\lim_{t\to 0} \|\varphi_t * f - f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} = 0.$$

Proof. Given $\varepsilon > 0$, we find a bounded function g in $L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n)$ with compact support such that $\|f - g\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} < \varepsilon$. By theorem (4.2.7) we have

$$\|\varphi_t * f - f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n}$$
$$\le \|\varphi_t * (f-g)\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} + \|\varphi_t * g - g\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} + \|\varphi_t * f - g\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n}$$
$$\le C\varepsilon + \|\varphi_t * g - g\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n}.$$

Since $|\varphi_t * g| \le \|g\|_{L^\infty,\mathbb{R}^n}$,

$$\|\varphi_t * g - g\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} \le C'\|\varphi_t * f - g\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} \to 0$$

By Lemma (4.2.4) (Here C' depends on $\|g\|_{L^\infty,\mathbb{R}^n}$.) Hence

$$\limsup_{t\to 0} \|\varphi_t * f - f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} \le C\varepsilon,$$

which complete the proof.

Theorem (4.2.3)[153]: Let $\{\varphi_t\}$ be potential-type approximate identity. Suppose that $\varphi \in L^{(p_0)'}(\mathbb{R}^n)$ and has compact support $f \in L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n)$, then $\{\varphi_t * f\}$ converges to f in $L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n)$:

$$\lim_{t\to 0} \|\varphi_t * f - f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} = 0.$$

Proof. Given $\varepsilon > 0$, choose a bounded function g with compact support such that $\|f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} < \varepsilon$. As in the proof of Theorem (4.2.2), using Theorem (4.2.12) this time, we have

$$\|\varphi_t * f - f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} \le C\varepsilon + \|\varphi_t * g - g\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n}.$$

Obviously, $L^{p_0}(\mathbb{R}^n)$. Hence by Lemma (4.2.9), $\varphi_t * g - g$ almost everywhere in $\mathbb{R}^n$. Since there is a compact set S containing all the support of $\varphi_t * g$,

$$\|\varphi_t * g - g\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} \leq C'\|\varphi_t * g - g\|_{L^{p_+ +1},\mathbb{R}^n}$$

with C' depending on $|S|$, and the Lebesgue convergence theorem implies $\|\varphi_t * g - g\|_{L^{p_+ +1},\mathbb{R}^n} \to 0$ as $t \to \infty$. Hence

$$\limsup_{t\to 0}\|\varphi_t * f - f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} \leq C\varepsilon,$$

which complete the proof.

We show by an example that the condition on φ are necessary.

We give some Young type inequalities for convolution with respect to the norms in $L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n)$.

Let C denote various positive constants independent of the variables in question. Let us begin with the following result due to Stein [152]

Lemma (4.2.4)[153]: Let $1 \leq p < \infty$ and $\{\varphi_t\}$ be a potential-type approximate identity. Then for every $f \in L^{p(\cdot)}(\mathbb{R}^n)$, $\{\varphi_t * f\}$ converges to f in $L^{p(\cdot)}(\mathbb{R}^n)$.

We denote by $B(x,r)$ the ball centered at $x \in \mathbb{R}^n$ and with radius $r > 0$. For a measurable set E, we denote by $|E|$ the Lebesgue measure of E.

Lemma (4.2.5)[153]: Let f be a nonnegative measurable function on $\mathbb{R}^n$ with $\|f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} \leq 1$ such that $f(x) \geq 0$ or $f(x) = 0$ for each $x \in \mathbb{R}^n$. Set

$$J = J(x,r,f) = \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\,dy$$

and

$$L = L(x,r,f) = \frac{1}{|B(x,r)|}\int_{B(x,r)} \Phi_{p(\cdot),q(\cdot)}\big(y, f(y)\big)\,dy.$$

Then

$$J \leq CL^{1/p(x)}(\log(c_0 + L))^{q(x)/p(x)},$$

where $C > 0$ does not depend on x, r, f.

We need the following result.

Lemma (4.2.6)[153]: Let f be a nonnegative measurable function on $\mathbb{R}^n$ with $(1 + |y|)^{-n-1} \leq f(y) \leq 1$ or $f(x) = 0$ for each $x \in \mathbb{R}^n$. Set

$$J = J(x,r,f) = \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\,dy$$

and

$$L = L(x,r,f) = \frac{1}{|B(x,r)|}\int_{B(x,r)} \Phi_{p(\cdot),q(\cdot)}\big(y, f(y)\big)\,dy.$$

Then

$$J \leq C\{L^{1/p(x)} + (1+|y|)^{-n-1}\},$$

where $C > 0$ does not depend on x, r, f.

Proof. We have by Jensen's inequality

$$J \leq \left(\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)^{p(y)}dy\right)^{1/p(x)}$$

$$\leq \left(\frac{1}{|B(x,r)|}\int_{B(x,r)\cap B(0,|x|/2)} f(y)^{p(y)}dy\right)^{1/p(x)}$$

$$+\left(\frac{1}{|B(x,r)|}\int_{B(x,r)\setminus B(0,|x|/2)} f(y)^{p(y)}dy\right)^{1/p(x)}$$

$$= J_1 + J_2$$

We see (p3) that

$$J_1 \leq C\left(\frac{1}{|B(x,r)|}\int_{B(x,r)\cap B(0,|x|/2)} f(y)^{p(y)}dy\right)^{1/p(x)}$$

Similarly, setting $E_2 = \{y \in \mathbb{R}^n : f(x) \geq (1+|y|)^{-n-1}\}$, we see from (p3) that

$$J_2 \leq C\left(\frac{1}{|B(x,r)|}\int_{B(x,r)\cap B(0,|x|/2)\cap E_2} f(y)^{p(y)}dy\right)^{1/p(x)}$$

$$+\left(\frac{1}{|B(x,r)|}\int_{B(x,r)\cap B(0,|x|/2)\setminus E_2} (1+|y|)^{-p(x)(n+1)}dy\right)^{1/p(x)}$$

$$\leq C\left\{\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)^{p(y)}dy\right)^{1/p(x)} + (1+|y|)^{-(n+1)}\right\}.$$

Since $f(y) \leq 1$, $f(y)^{p(y)} \leq C\Phi_{p(\cdot),q(\cdot)}(y, f(y))$. Hence, we have the required estimate.

By using Lemmas (4.2.5) and (4.2.6), we show the following theorem.

Theorem (4.2.7)[153]: Let $\{\varphi_t\}$ be potential-type, then

$$\|\varphi_t * f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} \leq C\|\widehat{\varphi}\|_{L^1,\mathbb{R}^n}\|f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n}$$

for all $t > 0$ and $f \in L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n)$.

Proof. Suppose $\|\widehat{\varphi}\|_{L^1,\mathbb{R}^n} = 1$ and take a nonnegative measursble function f on $\mathbb{R}^n$ such that $\|f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} \leq 1$. Write

$$f = f\chi_{\{y\in\mathbb{R}^n: f(y)\geq 1\}} + f\chi_{\{y\in\mathbb{R}^n:\,(1+|y|)^{-n-1}\leq f(y)<1\}} + f\chi_{\{y\in\mathbb{R}^n: f(y)\leq(1+|y|)^{-n-1})\}}$$

$= f_1 + f_2 + f_3$

where χ_E denotes the characteristic function of a measurable set $E \subset \mathbb{R}^n$.

Since $\widehat{\varphi}_t$ is radial function, we write $\widehat{\varphi}_t(r)$ for $\widehat{\varphi}_t(x)$ when $|x| = r$. Note that

$$|\varphi_t * f(x)| \leq \int_{\mathbb{R}^n} \widehat{\varphi}_t(|x-y|) f_1(y)\, dy$$

$$= \int_0^\infty \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} f_1(y) dy \right) |B(x,r)| d\big(-\widehat{\varphi}_t(r)\big),$$

So that Jensen's inequality and Lemma (4.2.2) yield

$$\Phi_{p(\cdot),q(\cdot)}(x, |\varphi_t * f_1(x)|)$$

$$\leq \int_0^\infty \Phi_{p(\cdot),q(\cdot)} \left(x, \frac{1}{|B(x,r)|} \int_{B(x,r)} f_1(y) dy \right) |B(x,r)| d\big(-\widehat{\varphi}_t(r)\big)$$

$$\leq C \int_0^\infty \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} \Phi_{p(\cdot),q(\cdot)}\big(y, f_1(y)\big) dy \right) |B(x,r)| d\big(-\widehat{\varphi}_t(r)\big)$$

$$= C(\widehat{\varphi}_t * g)(x),$$

where $g(y) = \Phi_{p(\cdot),q(\cdot)}\big(y, f(y)\big)$. The usual Young inequality for convolution gives

$$\int_{\mathbb{R}^n} \Phi_{p(\cdot),q(\cdot)}(x, |\varphi_t * f_1(x)|)\, dx \leq C \int_{\mathbb{R}^n} (\widehat{\varphi}_t * g)(x)\, dx$$

$$\leq C \|\widehat{\varphi}_t\|_{L^1,\mathbb{R}^n} \|g\|_{L^1,\mathbb{R}^n} \leq C.$$

Similarly, noting that $\frac{1}{|B(x,r)|} \int_{B(x,r)} f_2(y) dy \leq 1$ and applying Lemma (4.2.6), we derive the same for f_2.

Noting that $|\varphi_t * f_3(x)| \leq C \|\varphi_t\|_{L^1,\mathbb{R}^n} \leq C$, we obtain

$$\int_{\mathbb{R}^n} \Phi_{p(\cdot),q(\cdot)}(x, |\varphi_t * f_3(x)|)\, dx \leq C \int_{\mathbb{R}^n} |\varphi_t * f_3(x)|\, dx$$

$$\leq C \|\varphi_t\|_{L^1,\mathbb{R}^n} \|f_3\|_{L^1,\mathbb{R}^n} \leq C.$$

required.

As another application of Lemmas (4.2.5) and (4.2.6), we can prove the following result, which is an extension of [36] and [149] (see also [148]).

Let Mf be Hardy−Littlewood maximal function of f.

Proposition (4.2.8)[153]: Suppose $p_- > 1$. Then the operator M is bounded from $L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n)$ to $L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n)$.

Proof. Let f be a nonnegative measurable function on $\mathbb{R}^n$ such that $\|f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} \leq 1$ and write $f = f_1 + f_2 + f_3$ as in the proof of Theorem (4.2.7). Take $1 < p_1 < p_-$

and applying Lemmas (4.2.5) and (4.2.6) with $p(\cdot)$ and $q(\cdot)$ replaced by $p(\cdot)/p_1$ and $q(\cdot)/p_1$, respectively. Then

$$\Phi_{p(\cdot),q(\cdot)}\big(x, Mf_1(x)\big) \leq C[Mg_1(x)]^{p_1}$$

and

$$\Phi_{p(\cdot),q(\cdot)}\big(x, Mf_1(x)\big) \leq C\{[Mg_1(x)]^{p_1} + (1+|y|)^{-n-1}\},$$

where $g_1(y) = \Phi_{p(\cdot)/p_1, q(\cdot)/p_1}\big(y, f(y)\big)$. As to f_3, we have

$$\Phi_{p(\cdot),q(\cdot)}\big(x, Mf_3(x)\big) \leq C[Mf_3(x)]^{p_1}.$$

Then the bondedness of the maximal operator in $L^{p_1}(\mathbb{R}^n)$ proves the proposition.

If $p_- > 1$, then the function $\Phi_{p(\cdot),q(\cdot)}$ is proper N-function and the Proposition (4.2.8) implies that this function is of class A in the sense of Diening [149] (see [147, Lemma (4.2.10)). It would be an interesting problem to see whether "class A" is also a sufficient condition or not for the boundedness of M on $L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n)$.

We know the following result due to Zo [65]; see also [35].

Lemma (4.2.9)[153]: Let $1 \leq p < \infty$, $1/p + 1/p' = 1$ and $\{\varphi_t\}$ be an approximate identity. Suppose that $\varphi \in L^{p'}(\mathbb{R}^n)$ has compact support. Then for every $f \in L^p(\mathbb{R}^n)$, $\{\varphi_t * f\}$ converges to f pointwise almost everywhere.

Set

$$\bar{p}(x) = p(x)/p_0 \quad \text{and} \quad \bar{q}(x) = q(x)/p_0;$$

recall that $p_0 \in [1, p_-]$ is chosen such that $t^{-p_0}\Phi_{p(\cdot),q(\cdot)}(x,t)$ is uniformly almost increasing in t.

Lemma (4.2.10)[153]: Let f be a nonnegative measurable function on $\mathbb{R}^n$ such that $\|f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} \leq 1$ such that $f(x) \geq 1$ or $f(x) = 0$ for each $x \in \mathbb{R}^n$ and φ has compact support in $B(0,R)$ with $\|f\|_{L^{(p_0)'},\mathbb{R}^n} \leq 1$. Set

$$F = F(x,t,f) = |\varphi_t * f(x)|$$

and

$$G = G(x,t,f) = \int_{\mathbb{R}^n} |\varphi_t(x-y)| \Phi_{\bar{p}(\cdot),\bar{q}(\cdot)}(y, f(y))\, dy.$$

Then

$$F \leq CG^{1/\bar{p}(x)}(\log(c_0 + G))^{-\bar{p}(x)/\bar{p}(x)}$$

for all $0 < t \leq 1$, where $C > 0$ depends on $\mathbb{R}$.

Proof. Let f a nonnegative measurable function on $\mathbb{R}^n$ with $\|f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} \leq 1$ such that $f(x) \geq 1$ or $f(x) = 0$ for each $x \in \mathbb{R}^n$ and let φ have compact support in $B(0,R)$ with $\|\varphi\|_{L^{(p_0)'},\mathbb{R}^n} \leq 1$. By Hölder's inequality, we have

$$G \le \|\varphi\|_{L^{(p_0)'},\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\Phi_{p(\cdot),q(\cdot)}\big(y,f(y)\big)dy\right)^{1/p_0} \le t^{-n/p_0}.$$

First consider the case when $G \ge 1$. Since $G \le t^{-n/p_0}$, for $x \in B(x, tR)$ we have by (p2)

$$G^{-p(x)} \le G^{-p(x)+C/\log(e+(tR)^{-1})}G \le CG^{-p(x)}$$

and by (q2)

$$(\log(c_0 + G))^{q(y)} \le (\log(c_0 + G))^{q(x)}.$$

Hence it follows the choice of p_0 that

$$F \le G^{1/\bar{p}(x)} \le (\log(c_0 + G))^{-\bar{q}(x)/\bar{p}(x)}\int_{\mathbb{R}^n}|\varphi_t(x-y)|\,dy$$

$$+C\int_{\mathbb{R}^n}|\varphi_t(x-y)|\,d(y)\left\{\frac{f(y)}{G^{1/\bar{p}(x)}(\log(c_0+G))^{-\bar{q}(x)/\bar{p}(x)}}\right\}^{\bar{p}(x)-1}$$

$$\cdot\left\{\frac{(\log(c_0+f(y))}{G^{1/\bar{p}(x)}(\log(c_0+G))^{-\bar{q}(x)/\bar{p}(x)}}\right\}^{\bar{p}(x)-1} dy$$

$$\le CG^{1/\bar{p}(x)} \le (\log(c_0 + G))^{-\bar{q}(x)/\bar{p}(x)}$$

(cf. the proof of [149]).

In case $G < 1$, noting from the choice of p_0 that $f(y) \le \Phi_{\bar{p}(\cdot),\bar{q}(\cdot)}\big(y, f(y)\big)$ for $y \in \mathbb{R}^n$, we find

$$F \le CG \le CG^{1/\bar{p}(x)} \le CG^{1/\bar{p}(x)}(\log(c_0 + G))^{-\bar{q}(x)/\bar{p}(x)}.$$

Now the result follows.

Lemma (4.2.11)[153]: Suppose that $\|\varphi\|_{L^1,\mathbb{R}^n} \le 1$ Let f be a nonnegative measurable function on $\mathbb{R}^n$ with $\|f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} \le 1$. Set

$$I = I(x,t,f) = \int_{\{y\in\mathbb{R}^n:\,|y|>|x|/2\}}|\varphi_t(x-y)|f(y)dy$$

and

$$H = H(x,t,f) = \int_{\mathbb{R}^n}|\varphi_t(x-y)|\Phi_{p(\cdot),q(\cdot)}(y,f(y))\,dy.$$

If $A > 0$ and $H \le H_0$, then

$$F \le C\big(H^{1/p(x)} + |x|^{-A/p(x)}\big)$$

for $|x| > 1$ and $0 < t \le 1$, where $C > 0$ depends on A and H_0.

Proof. Suppose that $\|\varphi\|_{L^1,\mathbb{R}^n} \le 1$ Let f be a nonnegative measurable function on $\mathbb{R}^n$ with $\|f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} \le 1$.

Let $|x| > 1$. In the case $H_0 \ge H \ge |x|^{-A}$ with $A > 0$, we have by (p3)

$$H^{-p(x)} \leq CG^{-p(x)-C/\log(e+|x|)} \leq CH^{-p(x)}$$

for $|y| > |x|/2$. Hence we find by (Φ)

$$1 \leq C\left\{H^{1/p(x)} + \int_{\{y\in\mathbb{R}^n:\,|y|>|x|/2\}} |\varphi_t(x-y)| f(y) dy \right.$$

$$\left. \cdot \left\{\frac{f(y)}{H^{\frac{1}{p(x)}}}\right\}^{p(y)-1} \left\{\frac{(\log(c_0+f(y))}{\log\left(c_0+H^{\frac{1}{p(x)}}\right)}\right\}^{q(y)} dy\right\}$$

$$\leq CH^{-p(x)}$$

Next note from (p3) that

$$|x|^{p(x)} \leq |x|^{p(x)+C/\log(e+|x|)} \leq |x|^{p(x)}$$

for $|y| > |x|/2$. Hence, when $H \leq |x|^{-A}$, we obtain by (Φ)

$$1 \leq C\left\{|x|^{\frac{1}{p(x)}} + \int_{\{y\in\mathbb{R}^n:\,|y|>|x|/2\}} |\varphi_t(x-y)| f(y) dy \right.$$

$$\left. \cdot \left\{\frac{f(y)}{|x|^{1/p(x)}}\right\}^{p(y)-1} \left\{\frac{(\log(c_0+f(y))}{\log(c_0+|x|^{1/p(x)})}\right\}^{q(y)} dy\right\}$$

$$\leq C|x|^{-p(x)},$$

which completes the proof.

Theorem (4.2.12)[153]: Suppose that $\varphi \in L^{(p_0)'}(\mathbb{R}^n)$ has compact in $B(0,R)$. Then

$$\|\varphi_t * f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} \leq C\|\varphi\|_{L^{(p_0)'},\mathbb{R}^n}\|f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n}$$

for all $0 < t \leq 1$ and $L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n)$, where $C > 0$ depends on $\mathbb{R}$.

Proof. Let f a nonnegative measurable function on $\mathbb{R}^n$ such that $\|f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} \leq 1$ and let φ have compact support in $B(0,R)$ with $\|\varphi\|_{L^{(p_0)'},\mathbb{R}^n} \leq 1$. Write

$$f = f\chi_{\{y\in\mathbb{R}^n: f(y)\geq 1\}} + f\chi_{\{y\in\mathbb{R}^n:\, g(y)<f(y)<1\}} = f_1 + f_2$$

We have by Lemma (4.2.10)

$$|\varphi_t * f_1(x)| \leq C\big(|\varphi_t| * g(x)\big)^{p_0/p(x)}\big(\log\big(c_0 + |\varphi_t| * g(x)\big)\big)^{-q(x)/p(x)},$$

where $g(y) = \Phi_{\bar{p}(\cdot),\bar{q}(\cdot)}\big(y, f(y)\big) = \Phi_{p(\cdot),q(\cdot)}\big(y, f(y)\big)^{1/p(x)}$, sothat

$$\Phi_{p(\cdot),q(\cdot)}(x, |\varphi_t * f_1(x)|) \leq C\big(|\varphi_t| * g(x)\big)^{p_0} dx.$$

Hence, since $g \in L^{(p_0)'}(\mathbb{R}^n)$, the usual Young inequality for convolution gives

$$\int_{\mathbb{R}^n} \Phi_{p(\cdot),q(\cdot)}(x, |\varphi_t * f_1(x)|) \leq C\int_{\mathbb{R}^n} \big(|\varphi_t| * g(x)\big)^{p_0} dx$$

$$\leq C\left(\|\varphi_t\|_{L^1,\mathbb{R}^n}\|g\|_{L^{p_0},\mathbb{R}^n}\right)^{p_0} \leq C.$$

Next we are concerned with f_2. Write

$$f_2 = f_2\chi_{B(0,R)} + f_2\chi_{B(0,R)} = f_2' + f_2''$$

Since $|\varphi_t * f_2(x)| \leq C$ on $\mathbb{R}^n$, we have

$$\int_{B(0,2R)} \Phi_{p(\cdot),q(\cdot)}(x, |\varphi_t * f_2(x)|) \leq C.$$

Further, noting that $\varphi_t * f_2' = 0$ outside $B(0,2R)$, we find

$$\int_{\mathbb{R}^n} \Phi_{p(\cdot),q(\cdot)}(x, |\varphi_t * f_2'(x)|) \leq C.$$

Therefore it suffices to prove

$$\int_{\mathbb{R}^n\setminus B(0,2R)} \Phi_{p(\cdot),q(\cdot)}(x, |\varphi_t * f_2''(x)|) \leq C.$$

Thus, in the rest of the proof, we may assume that $0 \leq f < 1$ on $\mathbb{R}^n$ and $f = 0$ on $B(0,R)$. Note that

$$\int_{B(0,|x|/2)} \varphi_t(x-y)f(y)\,dy = 0$$

for $|x| > 2R$. Hence applying Lemma (4.2.11), we have

$$|\varphi_t * f(x)|^{p(x)} \leq (|\varphi_t| * h(x) + |x|^{-A})$$

for $|x| > 2R$, where $h(y) = \Phi_{p(\cdot),q(\cdot)}\big(y, f(y)\big)$. Thus, the integration yields

$$\int_{\mathbb{R}^n\setminus B(0,2R)} |\varphi_t * f(x)|^{p(x)} \leq C,$$

which compete the proof.

In Theorem (4.2.3) (and in Theorem (4.2.1)), the condition $\varphi \in L^{(p_-)'}(\mathbb{R}^n)$ cannot be weakened to $\varphi \in L^q(\mathbb{R}^n)$ for $1 \leq q < (p_-)'$. For given $p_1 > 1$ and $1 \leq q < (p_1)'$, we can find smooth exponent $p(\cdot)$ on $\mathbb{R}^n$ such that $p_- = p_1$, $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $\varphi \in L^q(\mathbb{R}^n)$ having compact support for which

$$\|\varphi * f\|_{L^{p(\cdot)},\mathbb{R}^n} = \infty$$

For this, let $a \in \mathbb{R}^n$ be a fixed point with $|a| > 1$ and let p_2 satisfy

$$\frac{1}{(p_1)'} + \frac{1}{p_2} < \frac{1}{q}.$$

Then choose a smooth exponent $p(\cdot)$ on $\mathbb{R}^n$ such that

$$p(x) = p_1 \text{ for } x \in B\left(0,\frac{1}{2}\right), \qquad p(x) = p_2 \text{ for } x \in B\left(a,\frac{1}{2}\right),$$

$p_- = p_1$ and $p(x) = \text{const.}$ outside $B(0, |a| + 1)$. Take

$$\varphi_j = j^{\frac{n}{q}} \chi_{B(a, j^{-1})} \quad \text{and} \quad \varphi_j = j^{\frac{n}{q}} \chi_{B(0, j^{-1})}, \qquad j = 2, 3, \ldots.$$

Then

$$\|\varphi_j\|_{L^q, \mathbb{R}^n} = C < \infty \quad \text{and} \quad \|f_j\|_{L^{p(\cdot)}, \mathbb{R}^n} = \|\varphi_j\|_{L^{p_1}, B(0,1/2)} = C < \infty.$$

Note that if $x \in B(0, j^{-1})$, then

$$\varphi_j * f_j(x) = j^{\frac{n}{q} + \frac{n}{p_1}} |B(a, j^{-1}) \cap B(x, j^{-1})| \geq j^{\frac{n}{q} + \frac{n}{p_1}} j^{-n},$$

So that

$$\begin{aligned} \int_{\mathbb{R}^n} \{\varphi_j * f_j(x)\}^{p(x)} dx &\geq \int_{B(a, j^{-1})} \{\varphi_j * f_j(x)\}^{p(x)} dx \\ &\geq C j^{p_2\left(\frac{n}{q} + \frac{n}{p_1} - n\right)} j^{-n} \\ &\geq C j^{p_2 n \left(\frac{1}{q} - \frac{1}{(p_1)'} - \frac{1}{p_2}\right)}. \end{aligned}$$

Note consider

$$\varphi = \sum_{j=2}^{\infty} j^{-2} \varphi_{2^j} \quad \text{and} \quad f = \sum_{j=2}^{\infty} j^{-2} \varphi_{2^j}.$$

Then $\varphi \in L^q(\mathbb{R}^n)$ $f \in L^{p(\cdot)}(\mathbb{R}^n)$. On the other hand

$$\begin{aligned} \int_{\mathbb{R}^n} \{\varphi * f(x)\}^{p(x)} dx &\geq j^{-4} \int_{B(a, j^{-1})} \{\varphi_{2^j} * f_{2^j}(x)\}^{p(x)} dx \\ &\geq C j^{-4} j^{p_2 n j \left(\frac{1}{q} - \frac{1}{(p_1)'} - \frac{1}{p_2}\right)} \to \infty \end{aligned}$$

as $j \to \infty$. Hence $\|\varphi * f\|_{L^{p(\cdot)}, \mathbb{R}^n} = \infty$.

By modifying their example, we can also find $p(\cdot)$ and $\varphi \in L^{(p_-)'}(\mathbb{R})$, whose support is not compact, such that

$$\|\varphi * f\|_{L^{p(\cdot)}, \mathbb{R}} \leq C \|f\|_{L^{p(\cdot)}, \mathbb{R}}$$

does not hold, namely there exists $f_N (N = 1, 2, \ldots)$ such that $\|f_N\|_{L^{p(\cdot)}, \mathbb{R}} \leq 1$ and

$$\lim_{N \to \infty} \|\varphi * f_N\|_{L^{p(\cdot)}, \mathbb{R}} = \infty.$$

For this purpose, choose $p_1 > 1$, $p_2 > p_1$ and $a > 1$ such that

$$-p_2/p_1 - a p_1 + 2 > 0,$$

and let $p(\cdot)$ be a smooth variable exponent on $\mathbb{R}$

$$p(x) = p_1 \text{ for } x \leq 0, \quad p(x) = p_2 \text{ for } x \geq 1$$

and $p_1 \leq p(x) \leq p_2$ for $0 < x < 1$. Set $\varphi = \sum_{j=2}^{\infty} \chi_j$, where $\chi_j = \chi_{[-j, -j + j^{-a})}$. Then

$$\int_{\mathbb{R}} \varphi(x)^q dx = \sum_{j=1}^{\infty} \int_{-\mathrm{j}}^{-j+j^{-a}} \chi_j(x)^q dx = \sum_j j^{-a} \le C(a) < \infty$$

for any $q > 0$. Further set $f_N = N^{-1/p_2}\chi_{[1,N+1]}$. Note that for $1 - j + j^{-a} < x < 0$ and $j \le N$

$$\chi_j * f_N(x) \ge \int_{\mathrm{x+j}-j^{-a}}^{x+j} \chi_j(x-y) f_N(y) dy = N^{-1/p_2} j^{-a},$$

$$\begin{aligned}\int_{\mathbb{R}} \{\varphi * f(x)\}^{p(x)} dx &\ge j^{-4} \int_{-\infty}^{0} \left\{\sum_{j=1}^{\infty} \chi_j * f_N(x)\right\}^{p_1} dx \\ &\ge \sum_{j=1}^{\infty} \int_{1-j-j^{-a}}^{0} \left\{\chi_j * f_N(x)\right\}^{p_1} dx \\ &\ge N^{-p_1/p_2} \sum_j j^{-ap_1}(j - j^{-a} - 1) \to \infty \\ &\ge C N^{-p_1/p_2 - ap_1 + 2} \to \infty \quad (N \to \infty).\end{aligned}$$

Cruz-Uribe and Fiorenza [35] conjectured that Theorem (4.2.1) remains true if φ satisfies the additional condition

$$|\varphi(x-y) - \varphi(x)| \le \frac{|y|}{|x|^{x+1}} \quad \text{when } |x| > 2|y|. \tag{17}$$

Noting that this condition implies

$$\sup_{x,z \in B(0,2^{j+1}) \setminus B(0,2^{j+1})} |\varphi(x) - \varphi(z)| \le C 2^{-nj},$$

We see that $\lim_{|x|\to\infty} \varphi(x) = 0$ since $\varphi \in L^1(\mathbb{R}^n)$ and

$$|\varphi(x)| \le C|x|^{-n} \tag{18}$$

if φ satisfies (17). In this connection we show

Theorem (4.2.13)[153]: Let $p_- > 1$. Suppose that $\varphi \in L^1(\mathbb{R}^n) \cap L^{(p_0)'}\big(B(0,R)\big)$ and φ satisfies (18) for $|x| \ge R$. Then

$$\|\varphi * f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} \le C\left(\|\varphi\|_{L^1,\mathbb{R}^n} + \|\varphi\|_{L^{(p_0)'},B(0,R)}\right) \|f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n}$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^n) \cap (\log L)^{q(\cdot)} L^{(p_0)'}(\mathbb{R}^n)$.

Proof. Let f a nonnegative measurable function on $\mathbb{R}^n$ such that $\|f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} \le 1$. Suppose that φ satisfies (18) for $|x| \ge R$ and $\|\varphi\|_{L^1,\mathbb{R}^n} + \|\varphi\|_{L^{(p_0)'},B(0,R)} \le 1$. Decompose $\varphi = \varphi' + \varphi''$, where $\varphi' = \varphi\chi_{B(0,R)}$. We first note by Theorem (4.2.3) that

$$\|\varphi' * f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} \le C.$$

Hence, it suffices to show that

$$\|\varphi'' * f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} \leq C.$$

For this purpose, write

$$f = f\chi_{\{y\in\mathbb{R}^n: f(y)\geq 1\}} + f\chi_{\{y\in\mathbb{R}^n: f(y)<1\}} = f_1 + f_2,$$

As before. Then we have by (18) and (Φ)

$$|\varphi'' * f_1(x)| \leq C\int_{\mathbb{R}^n\backslash B(0,R)} |x-y|^{-n} f_1(y)dy$$

$$\leq CR^{-n}\int_{\mathbb{R}^n} f_1(y)dy$$

$$\leq CR^{-n}\int_{\mathbb{R}^n} \Phi_{p(\cdot),q(\cdot)}\big(y, f(y)\big)dy \leq C.$$

Noting that $|\varphi'' * f_2(x)| \leq 1$, we obtain

$$\int_{B(0,R)} \Phi_{p(\cdot),q(\cdot)}(x, \varphi'' * f_2(x))dx \leq C.$$

Noting, let $\Phi_{p(\cdot),q(\cdot)}\big(y, f(y)\big)$. Then

$$|\varphi''| * h \leq CR^{-n}\int_{\mathbb{R}^n} h(y)dy \leq CR^{-n}.$$

If $x \in \mathbb{R}^n\backslash B(0,R)$, then we have by (18) and Lemma (4.2.11)

$$|\varphi'' * f(x)| \leq \int_{B(0,|x|/2)} |\varphi''(x-y)| f(y)dy + \int_{\mathbb{R}^n\backslash(0,|x|/2)} |\varphi''(x-y)| f(y)dy$$

$$\leq C\left\{|x|^{-n}\int_{B(0,|x|/2)} f(y)dy + \big(|\varphi''| * (h)\big)^{1/p(x)} + |x|^{-A/p(x)}\right\}$$

$$\leq C\left\{Mf(x) + \big(|\varphi''| * (h)\big)^{1/p(x)} + |x|^{-A/p(x)}\right\}$$

with $A > n$. Now it follows from Proposition (4.2.8) that

$$\int_{\mathbb{R}^n\backslash B(0,R)} \Phi_{p(\cdot),q(\cdot)}(x, \varphi'' * f_2(x))dx$$

$$\leq C\left\{\int_{\mathbb{R}^n} \Phi_{p(\cdot),q(\cdot)}\big(x, Mf(x)\big)dy\right.$$

$$\left. + \int_{\mathbb{R}^n} |\varphi| * h(y)dy + \int_{\mathbb{R}^n\backslash(0,R)} |x|^{-A/p(x)}dx\right\} \leq C,$$

as required.

Remark (4.2.14)[153]: Theorem (4.2.13) does not imply an inequality

$$\|\varphi_t * f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} \le C\|f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n}$$

with a constant C independent of $t \in (0, 1]$ even if φ satisfies (18) for all x, because $\left\{\|\varphi_t\|_{L^{(p_0)'},B(0,R)}\right\}_{0<t\le 1}$ is not bounded.

Theorem (4.2.15)[153]: Let $1 - p_-/p_+ \le \theta < 1$, $1 < \tilde{p} < p_-$,

$$\frac{1}{s} = 1 - \frac{\theta}{\tilde{p}} \quad \text{and} \quad \frac{1}{r(x)} = \frac{1-\theta}{p(x)}$$

Take $v = p_-/\tilde{p}$ if $t^{p_-/\tilde{p}}\Phi_{p(\cdot)/\tilde{p},q(\cdot)}(x, t)$ is uniformly almost increasing in t; other wise choose $1 \le v < p_-/\tilde{p}$. Suppose that $\varphi \in L^1(\mathbb{R}^n) \cap L^s(\mathbb{R}^n) \cap L^{sv'}\big(B(0, R)\big)$ and φ satisfies

$$|\varphi(x)| \le C|x|^{-n/s}$$

for $|x| \ge R$. Then

$$\|\varphi * f\|_{\Phi_{r(\cdot),q(\cdot)},\mathbb{R}^n} \le C\left(\|\varphi\|_{L^1,\mathbb{R}^n} + \|\varphi\|_{L^s,\mathbb{R}^n} + \|\varphi\|_{L^{sv'},B(0,R)}\right)\|f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n}$$

for all $f \in (\log L)^{q(\cdot)}L^{(p_0)'}(\mathbb{R}^n)$.

Proof. Suppose that $\|\varphi\|_{L^1,\mathbb{R}^n} + \|\varphi\|_{L^s,\mathbb{R}^n} + \|\varphi\|_{L^{sv'},B(0,R)} \le 1$ and satisfies

$$|\varphi(x)| \le C|x|^{-n/s}$$

for $|x| \ge R$. Let f be a nonnegative measurable function on $\mathbb{R}^n$ such that $\|\varphi\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} \le 1$, and decompose

$$f = f_1 + f_2,$$

where $f_1 = f\chi_{\{y\in\mathbb{R}^n: f(y)\ge 1\}}$. Let

$$\frac{1}{r} = \frac{1-\theta}{p_-} \quad \text{and} \quad \frac{1}{s_1} = 1 + \frac{1}{r} - \frac{1}{p_+}.$$

By our assumption, $s_1 \ge 1$. It follows from Young's inequality for convolution that

$$\|\varphi * f_2\|_{L^s,\mathbb{R}^n} \le \|\varphi\|_{L^{s_1},\mathbb{R}^n} + \|f_2\|_{L^{p_1}\,\mathbb{R}^n}.$$

Hence note that $1 \le s_1 \le s$, so that $\|\varphi\|_{L^{s_1},\mathbb{R}^n} \le \|\varphi\|_{L^1,\mathbb{R}^n} + \|\varphi\|_{L^s,\mathbb{R}^n} \le 1$. Since $0 \le f_1 \le 1$, $\|\varphi\|_{L^{s_1},\mathbb{R}^n} \le C\|\varphi\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} \le C$. Thus, noting that $|\varphi * f_2| \le 1$ and

$$\frac{1}{r(x)} - \frac{1}{r} = \frac{1-\theta}{p(x)} - \frac{1-\theta}{p_+} \le 0,$$

we see that

$$\|\varphi * f_2\|_{\Phi_{r(\cdot),q(\cdot)},\mathbb{R}^n} \le C\|\varphi * f_2\|_{L^r,\mathbb{R}^n} \le C. \tag{19}$$

On the other hand, we have by Höider inequality

$$|\varphi * f_2| \leq \left(\int_{\mathbb{R}^n} |\varphi(x-y)|^s f_1(y)^{\tilde{p}}\, dy\right)^{\frac{(1-\theta)}{\tilde{p}}} \left(\int_{\mathbb{R}^n} |\varphi(x-y)|^s\, dy\right)^{1-\frac{1}{\tilde{p}}}$$

$$\cdot \left(\int_{\mathbb{R}^n} |f_1(y)|^{\tilde{p}}\, dy\right)^{\frac{\theta}{\tilde{p}}} \leq \left(\int_{\mathbb{R}^n} |\varphi|^s * f_1^{\tilde{p}}(x)\, dy\right)^{\frac{(1-\theta)}{\tilde{p}}} \qquad (20)$$

Noting that $|\varphi|^s \in L^1(\mathbb{R}^n) \cap L^{v'}(B(0,R))$, $|\varphi|^s$ satisfies (18) for $|x| \geq R$ and $\left\|f_1^{\tilde{p}}\right\|_{\Phi_{p(\cdot)/\tilde{p},q(\cdot)},\mathbb{R}^n} \leq C$, we find by Theorem (4.2.13)

$$\left\|\varphi^s * f_1^{\tilde{p}}\right\|_{\Phi_{p(\cdot)/\tilde{p},q(\cdot)},\mathbb{R}^n} \leq C.$$

Since (20) implies

$$\Phi_{r(\cdot),q(\cdot)}, \mathbb{R}^n(x, \varphi * f_1(x)) \leq C\Phi_{p(\cdot)/\tilde{p},q(\cdot)}, \mathbb{R}^n\left(x, |\varphi|^s * f_1^{p_1}(x)\right),$$

it follows that

$$\|\varphi * f_1\|_{\Phi_{r(\cdot),q(\cdot)},\mathbb{R}^n} \leq C.$$

Thus, together with (19), we obtain

$$\|\varphi * f\|_{\Phi_{r(\cdot),q(\cdot)},\mathbb{R}^n} \leq C,$$

as required.

If $p_- > 1$, this conjecture was shown to be true by Cruz-Uribe, Fiorenza, Martell and Pérez in [66], using an extrapolation theorem ([66]). Using Proposition (4.2.8), we can prove the following extension of [66]:

Proposition (4.2.16)[153]: Let $\mathcal{F}$ be a family of ordered pairs (f, g) of nonnegative measurable functions on $\mathbb{R}^n$. Since that for some $0 \leq p_0 \leq p^-$,

$$\int_{\mathbb{R}^n} f(x)^{p_0} \omega\, dx \leq C_0 \int_{\mathbb{R}^n} g(x)^{p_0} \omega\, dx$$

for all $(f, g) \in \mathcal{F}$ and for all A_1-weighet ω, where C_0 depends only on p_0 and the A_1-constant of ω. Then

$$\|f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} \leq C\|g\|_{\Phi_{r(\cdot),q(\cdot)},\mathbb{R}^n} \leq C$$

for all $(f, g) \in \mathcal{F}$ such that $f \in L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n)$.

Then, as in [66], we can prove:

Theorem (4.2.17)[153]: Assume that $p_- > 1$. If φ is an integrable function on $\mathbb{R}^n$ satisfying (17), then

$$\|\varphi_t * f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} \leq C\|f\|_{\Phi_{r(\cdot),q(\cdot)},\mathbb{R}^n}$$

for all $t > 0$ and $f \in L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n)$. If, in addition $\int \varphi(x)dx = 1$, then

$$\lim_{t\to 0}\|\varphi_t * f - f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} = 0.$$

For $p \geq 1$, $q \in \mathbb{R}$ and $c \geq e$, we consider the function

$$\Phi(t) = \Phi(p, q, c; t) = t^p (\log(c+t))^q, \quad t \in [0, \infty).$$

We give a proof of the following:

Theorem (4.2.18)[153]: Let X be anon-empty set and let $p(\cdot)$ and $q(\cdot)$ be real valued functions on X such that $1 \leq p(x) \leq p_0 < \infty$ for all $x \in X$. Then the following (i) and (ii) are equivalent to each other:

(i) There exists $c \geq e$ such that $\Phi(p(x), q(x), c_0; \cdot)$ is a convex on $[0, \infty)$ for every $x \in X$;

(ii) There exists $K > 0$ such that $K(p(x) - 1) + q(x) \geq 0$ for all $x \in X$.

Proposition (4.2.19)[153]:

(i) If

$$(1 + \log c)(p - 1) + q \geq 0,$$

Then Φ is convex on $[0, \infty)$.

(ii) Given $p_0 > 1$ and $c \geq e$, there exists $K = K(p_0, c) > 0$ such that Φ is not convex on $[0, \infty)$ where $1 \leq p \leq p_0$ and $q < -K(p - 1)$.

Proof. By elementary calculation we have

$$\Phi''(t) = t^{p-2}(c+t)^{-2}(\log(c+t))^{q-2} G(t).$$

with

$$G(t) = p(p-1)(c+t)^2(\log(c+t))^2 + 2pq(c+t)\log(c+t) - qp^2 \log(c+t) + q(q-1)^2$$

for $t > 0$. $\Phi(t)$ is convex on $[0, \infty)$ if and only if $G(t) \geq 0$ for all $t \in (0, \infty)$.

(i) If $q \geq 0$, then

$$G(t) \geq qt(2(c+t) - t)\log(c+t) - qt^2 \geq q(2pc + 2(p-1)t) \geq 0$$

for all $t \in (0, \infty)$, so that Φ is convex on $[0, \infty)$.

If $-(1 + \log c)(p - 1) \leq q < 0$, then

$$\begin{aligned} G(t) &= p\left\{\sqrt{p-1}(c+t)\log(c+t) + \frac{q}{\sqrt{p-1}} t\right\}^2 \\ &\quad - \frac{-pq^2}{p-1} t^2 - qt^2 \log(c+t) + q(q-1)^2 \\ &\geq (-q)t^2 \left(\frac{pq}{p-1} + \log c - (q-1)\right) \\ &= (-q)t^2 \left(\frac{q}{p-1} t^2 + \log c + 1\right) \geq 0 \end{aligned}$$

for all $t \in (0, \infty)$, so that Φ is convex on $[0, \infty)$.

(ii) If $p = 1$ and $q < 0$, then

$$G(t) \geq qt\big((t+2c)\log(c+t)+(q-1)t\big) \to \infty$$

as $t \to \infty$. Hence Φ is convex on $[0,\infty)$.

Next, let $1 \leq p \leq p_0$ and $q < -K(p-1)$ with $K > 0$. Then

$$\frac{G(t)}{p-1} = p\big((c+t)\log(c+t)-kt\big)^2 + k(2\log(c+t)-k+1)t^2$$

$$\leq \big((c+t)\log(c+t)-kt\big)^2 + k(\log(c+t)-k+1)t^2.$$

Let $\lambda = 1-1/(2p_0)$. Then $0 < \lambda < 1$. If $k > (\log c)/\lambda$, there is (unique) $t_k > 0$ such that $\log(c-t_k) = \lambda k$. Note that $t_k/k \to \infty$. We have

$$\frac{G(t)}{p-1} = p_0\big((c-t_k)\lambda k - kt_k\big)^2 + k(\lambda k - k + 1)t_k^2$$

$$\leq kt_k^2\left\{p_0\big((1-\lambda)-1\big)(1-\lambda)k + 1 - 2p_0c\lambda(1-\lambda)\frac{k}{t_k} + p_0c^2\lambda^2\frac{k}{t_k^2}\right\}.$$

Since $p_0(1-\lambda)-1 = -1/2$, it follows that there is $k = k(c,p_0) > (\log c)/\lambda$ such that $G(t_k) < 0$ whenever $K \geq K$. Hence Φ is not convex if $1 < p \leq p_0$ and $q \leq -K(p-1)$.

Chapter 5

Approximate Identities and Trudinger's Inequalities with Riesz Potentials

We give a Young type inequality for convolution with respect to the norm in Musielak-Orlicz spaces. We are concerned with Trudinger's inequality for Riesz potentials of function in Musielak-Orlicz spaces. We provide a number of useful auxiliary results including a normalization of the Φ-function and behavior under duality.

Section (5.1): Young type Inequalities in Musielak-Orlicz Spaces

Let k be an integrable function on $\mathbb{R}^N$. for each $t > 0$, define the function k_t by $k_t(x) = t^{-N}k(x/t)$. Note that by a change of variables, $\|k_t\|_{L^1(\mathbb{R}^N)} = \|k\|_{L^1(\mathbb{R}^N)}$. We say that the family $\{k_t\}_{t>0}$ is an approximate identity if $\int_{\mathbb{R}^N} k(x) = 1$. Define the radial majorant of k to be function

$$\widehat{k}(x) = \sup_{|y|\geq|x|} |k(x)|.$$

If $\widehat{k}$ is integrable, we say that the family $\{k_t\}_{t>0}$ is of potential-type.

It is well known (see, e.g., [157]) that if $\{k_t\}_{t>0}$ potential-type approximate identity, then $k_t * f \to f$ in $L^p(\mathbb{R}^N)$ as $t \to 0$ for every $f \in L^p(\mathbb{R}^N)$ $(p \geq 0)$.

Variable exponent Lebesgue spaces and Sobolev spaces were introduced to discuss nonlinear partial differential equations with non-standard growth conditions (see [154]). Cruz-Uribe and Fiorenza [35] gave sufficient conditions for the convergence of approximate identities in variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^N)$ when $p(\cdot)$ is a variable exponent satisfying the log-Hölder conditions on $\mathbb{R}^N$, locally and at ∞, as an extension of [147], [157], etc. In fact, they proved the following:

Theorem (5.1.1)[158]: Let $\{k_t\}_{t>0}$ be an approximate identity. Suppose that either

(i) $\{k_t\}_{t>0}$ is of potential-type, or

(ii) $k \in L^{(p^-)'}(\mathbb{R}^N)$ and has compact support, where $p^- \coloneqq \inf_{x\in\mathbb{R}^N} p(x)\ (\geq 1)$ and $1/p^- + 1/(p^-)' = 1$.

Then

$$\sup_{0<t\leq 1} \|k_t * f\|_{L^{p(\cdot)}(\mathbb{R}^N)} \leq \|f\|_{L^{p(\cdot)}(\mathbb{R}^N)}$$

and

$$\lim_{t\to 0} \|k_t * f - f\|_{L^{p(\cdot)}(\mathbb{R}^N)} = 0$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^N)$.

Theorem (5.1.1) was extended to the two variable exponents spaces $L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^N)$ in [153]. These spaces are special cases of the so-called Musielak-Orlicz spaces ([150]).

To extend these result to the Musielak-Orlicz spaces $L^{\Phi}(\mathbb{R}^N)$ (see the definition of Φ). As a related topic, we also give a Yonug type inequality for convolution with respect to the norm in $L^{\Phi}(\mathbb{R}^N)$.

We consider a function

$$\Phi(x,t) = t_{\varphi}(x,t) : \mathbb{R}^N \times [0,\infty) \to [0,\infty)$$

satisfying the following conditions $(\Phi 1) - (\Phi 4)$:

$(\Phi 1)$ $\varphi(\cdot,t)$ is measurable on $\mathbb{R}^N$ for each $t \geq 0$ and $\varphi(x,t)$ is continuous on $[0,\infty)$ for each $x \in \mathbb{R}^N$;

$(\Phi 2)$ there exists a constant on $A_1 \geq 1$ such that

$$A_1^{-1} \leq \varphi(x,1) \leq A_1 \quad \text{for all } x \in \mathbb{R}^N;$$

$(\Phi 3)$ $\varphi(x,\cdot)$ is uniformly almost increasing, namely there exists a constant on $A_2 \geq 0$ such that

$$\varphi(x,t) \leq A_2 \varphi(x,s) \quad \text{for all } x \in \mathbb{R}^N \text{ whenever } 0 \leq t < s;$$

$(\Phi 4)$ there exists a constant on $A_3 \geq 1$ such that

$$\varphi(x,2t) \leq A_3 \varphi(x,t) \quad \text{for all } x \in \mathbb{R}^N \text{ and } t > 0.$$

Note that $(\Phi 2)$, $(\Phi 3)$ and $(\Phi 4)$ imply

$$\inf_{x\in\mathbb{R}^N} \varphi(x,t) \leq \sup_{x\in\mathbb{R}^N} \varphi(x,t) < \infty$$

for each $t > 0$.

If $\Phi(x,\cdot)$ is convex for each $x \in \mathbb{R}^N$, then $(\Phi 3)$ holds with $A_2 = 1$; namely $\varphi(x,\cdot)$ is non-decreasing for each $x \in \mathbb{R}^N$.

If $p_1(\cdot), p_2(\cdot), q_1(\cdot)$, and $q_1(\cdot)$, are measurable function on $\mathbb{R}^N$ such that

(P1) $$1 \leq p_j^- := \inf_{x\in\mathbb{R}^N} p_j(x) \leq \sup_{x\in\mathbb{R}^N} p_j(x) =: p_j^+ < \infty, \quad j = 1,2,$$

and

(Q1) $$-\infty < q_j^- := \inf_{x\in\mathbb{R}^N} q_j(x) \leq \sup_{x\in\mathbb{R}^N} q_j(x) =: q_j^+ < \infty, \quad j = 1,2,$$

Then

$$\Phi(x,t) = (1+t)^{p_1(x)}(1+1/t)^{-p_2(x)}(\log(e+t))^{q_1(x)}(\log(e+1/t))^{-q_2(x)}$$

satisfies (Φ1), (Φ2) and (Φ4). It satisfies (Φ3) if $p_j^- > 1,\ j = 1,2$, or $p_j^- \geq 0, j = 1,2$. As a matter of fact, it satisfies (Φ3) if and only if $p_j(\cdot), q_j(\cdot)$ Satisfy the following conditions:

(i) $q_j(x) \geq 0$ at points x where $p_j(x) = 1, \quad j = 1,2;$

(ii) $\sup_{x:\, p_j(x)>1} \{\min(q_j(x), 0)\log(p_j(x) - 1)\} < \infty, \quad j = 1,2,$

Let $\bar{\varphi}(x,s) = \sup_{0\leq s\leq t} \varphi(x,s)$ and

$$\bar{\Phi}(x,t) = \int_0^t \bar{\varphi}(x,r)dr$$

for $x \in \mathbb{R}^N$ and $t \geq 0$. Then $\bar{\Phi}(x,\cdot)$ Is convex and

$$\frac{1}{2A_3}\Phi(x,t) \leq \bar{\Phi}(x,t) \leq A_2\Phi(x,t) \tag{1}$$

for all $x \in \mathbb{R}^N$ and $t \geq 0$. In fact, the first inequality is seen as follows:

$$\bar{\Phi}(x,t) \geq \int_{t/2}^t \bar{\varphi}(x,r)dr \geq \frac{t}{2}\varphi(x,t/2) \geq \frac{1}{2A_3}\Phi(x,t).$$

Corresponding to(Φ2) and (Φ4), we have by (1)

$$(2A_1A_3)^{-1} \leq \bar{\Phi}(x,1) \leq A_1A_2 \quad \text{and } \bar{\Phi}(x,2t) \leq 2A_3\bar{\Phi}(x,t) \tag{2}$$

for all $x \in \mathbb{R}^N$ and $t > 0$.

Given $\Phi(x,t)$ as above, the associated Musielak-Orlicz space

$$L^\Phi(\mathbb{R}^N) = \left\{f \in L^1_{loc}(\mathbb{R}^N);\ \int_{\mathbb{R}^N} \bar{\Phi}(y,|f(y)|)dy < \infty\right\}$$

is a Banach space with respect to the norm (cf. [150]).

$$\|f\|_{L^\Phi(\mathbb{R}^N)} = \inf\left\{\lambda > 0; \int_{\mathbb{R}^N} \bar{\Phi}(y,|f(y)|/\lambda)dy \leq 1\right\}$$

By (2), we have the following lemma (see [156]).

Lemma (5.1.2)[158]:

$$\|f\|_{L^\Phi(\mathbb{R}^N)} \leq 2\left(\int_{\mathbb{R}^N} \bar{\Phi}(x,|f(x)|)dx\right)^\sigma \tag{3}$$

with $\sigma = \log 2/\log(2A_3)$, $\|f\|_{L^\Phi(\mathbb{R}^N)} \leq 1$.

We shall also consider the following conditions:

(Φ5) or every $\gamma > 0$, there exists a constant $B_\gamma \geq 1$ such that

$$\varphi(x,t) \leq B_\gamma \varphi(y,t)$$

where $|x-y| \leq \gamma^{t-1/N}$ and $t \geq 1$;

(Φ6) there exists a function $g \in L^1(\mathbb{R}^N)$ and a constant $B_\infty \geq 1$ such that $0 \leq g(x) < 1$ and for $x \in \mathbb{R}^N$

$$B_\infty^{-1}\Phi(x,t) \leq \Phi(x',t) \leq B_\infty \Phi(x,t)$$

whenever $|x'| \geq |x|$ and $g(x) \leq t \leq 1$.

If $\Phi(x,t)$ satisfies (Φ5) (resp. (Φ6)), then so does $\bar{\Phi}(x,t)$ with $\bar{B}_\gamma = 2A_2A_3B_\gamma$ in place of B_γ $(resp.\, \bar{B}_\infty = 2A_2A_3B_\gamma$ in place of $B_\infty)$.

Let $\Phi(x,t)$ be a measurable function on $\mathbb{R}^N$. It satisfies (Φ5) if $(P2)$ $p_1(\cdot)$ is log-Hölder continuous, namely

$$|p_1(x)-p_1(y)| \leq \frac{C_p}{\log\left(\frac{1}{|x-y|}\right)} \quad \text{for } |x-y| \leq \frac{1}{2}$$

with a constant $C_p \geq 0$,

and

$(Q2)$ $q_1(\cdot)$ is log-Hölder continuous, namely

$$|q_1(x)-q_1(y)| \leq \frac{C_q}{\log\left(\log\left(\frac{1}{|x-y|}\right)\right)} \quad \text{for } |x-y| \leq e^{-2}$$

with a constant $C_q \geq 0$.

$\Phi(x,t)$ satisfies (Φ6) with $g(x) = 1/(1+|x|)^{N+1}$ if

$(P3)$ $p_2(\cdot)$ is log-Hölder continuous at ∞, namely

$$|p_2(x)-p_2(x')| \leq \frac{C_{p,\infty}}{\log(e+|x|)} \quad \text{whenever } |x'| \geq |x|$$

with a constant $C_{p,\infty} \geq 0$,

and

$(Q3)$ $q_2(\cdot)$ is log-Hölder continuous at ∞ , namely

$$|q_2(x)-q_2(x')| \leq \frac{C_{q,\infty}}{\log(\mathrm{e}+\log(e+|x|))} \quad \text{whenever } |x'| \geq |x|$$

with a constant $C_{q,\infty} \geq 0$.

If $1/(1+|x|)^{N+1} < t \le 1$, then $(1+t)^{|p_1(x)-p_1(x')|} \le 2^{p_1^+-1}$, $(1+1/t)^{|p_2(x)-p_2(x')|} \le e^{(N+1)C_{p,\infty}}$, $(\log(e+t))^{|q_1(x)-q_1(x')|} \le (\log(e+1)^{q_1^+ + q_1^-}$ and $(\log(e+1/t))^{|q_2(x)-q_2(x')|} \le (N, C_{q,\infty})$ for $|x'| \ge |x|$.

Let C denote various positive constant independent of the variables in question.

First, we recall the following classical result (see, e.g., Stein [157]).

Lemma (5.1.3)[158]: Let $1 \le p < \infty$ and $\{k_t\}_{t>0}$ is of potential-type approximate identity. Then, $k_t * f$ converges f in $L^p(\mathbb{R}^N)$ for every $f \in L^p(\mathbb{R}^N)$.

We denote by $B(x,r)$ the open ball centered at $x \in \mathbb{R}^N$ and with $r > 0$. For a measurable set E, we denote $|E|$ the Lebesgue measure of E.

For a nonnegative $f \in L^1_{\text{loc}}(\mathbb{R}^N)$, $x \in \mathbb{R}^N$ and $r > 0$, let

$$I(f;x,r) = \frac{I}{|B(x,r)|}\int_{B(x,r)} f(y)dy$$

and

$$J(f;x,r) = \frac{I}{|B(x,r)|}\int_{B(x,r)} \overline{\Phi} f(y,f(y))dy.$$

The following lemmas are due to [155].

Lemma (5.1.4)[158]: ([155, Lemma (5.1.2)], [155, Lemma (5.1.3)]). Suppose $\Phi(x,t)$ satisfies (Φ5). Then there exists a constant $C > 0$ such that

$$\overline{\Phi} f(x, I(f;x,r)) \le CJ(f;x,r)$$

for all $x \in \mathbb{R}^N$ and $r > 0$ and for all nonnegative $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ such that $f(y) \ge 1$ or $f(y) = 0$ for each $y \in \mathbb{R}^N$ and $\|f\|_{L^\Phi(\mathbb{R}^N)} \le 1$.

Lemma (5.1.5)[158]: ([155, Lemma (5.1.2)], [155, Lemma (5.1.4)]). Suppose $\Phi(x,t)$ satisfies (Φ6). Then there exists a constant $C > 0$ such that

$$\overline{\Phi} f(x, I(f;x,r)) \le C\{J(f;x,r) + g(x)\}$$

for all $x \in \mathbb{R}^N$ and $r > 0$ and for all nonnegative $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ such that $g(y) \le f(y) \le 0$ or $f(y) = 0$ for each $y \in \mathbb{R}^N$, where g is the function appearing in (Φ6).

By using Lemmas (5.1.4) and (5.1.5), we show the following theorem.

Lemma (5.1.6)[158]: Suppose (Φ5) satisfies (Φ6). If $\{k_t\}_{t>0}$ is of potential-type, then

$$\|k_t * f\|_{L^\Phi(\mathbb{R}^N)} \le C\left\|\hat{k}\right\|_{L^1(\mathbb{R}^N)}\|f\|_{L^\Phi(\mathbb{R}^N)}$$

for all $t > 0$ and $f \in L^\Phi(\mathbb{R}^N)$.

Proof. Suppose $\|\hat{k}\|_{L^1(\mathbb{R}^N)} = 1$ and let f be nonnegative umeaurable function on $\mathbb{R}^N$ such that $\|f\|_{L^1(\mathbb{R}^N)} \le 1$. Write

$$f = f\chi_{\{y\in\mathbb{R}^N: f(y)\ge 1\}} + f\chi_{\{y\in\mathbb{R}^N:\, g(y)<f(y)<1\}} + f\chi_{\{y\in\mathbb{R}^N: f(y)\le g(y)\}} = f_1 + f_2 + f_3,$$

where χ_E denotes the characteristic function of a measurable set $E \subset \mathbb{R}^N$ and g is the function appearing in (Φ6).

Since $\hat{k}_t$ is radial function, we write $\hat{k}_t(r)$ for $\hat{k}_t(x)$ when $|x| = r$. First note that

$$|k_t * f_j(x)| \le \int_{\mathbb{R}^N} \hat{k}_t(|x-y|) f_j(y)dy = \int_0^\infty I(f_j; x, r)\, |B(x,r)| d(-\hat{k}_t(r)),$$

$j = 1, 2$, and

$$\int_{\mathbb{R}^N} (|B(x,r)|) d(-\hat{k}_t(r)) = \|\hat{k}\|_{L^1(\mathbb{R}^N)} = 1,$$

so that Jensen's inequality yields

$$\overline{\Phi} f(x, |k_t * f_j(x)|) \le \int_0^\infty \overline{\Phi} f(x, I(f; x, r)) \le |B(x,r)| d(-\hat{k}_t(r)),$$

$j = 1, 2$.

Hence, by Lemma (5.1.4)

$$\overline{\Phi} f(x, |k_t * f_1(x)|) \le C \int_0^\infty J(f_1; x, r) |B(x,r)| d\left(-\hat{k}_t(r)\right) \le C(\hat{k}_t * h),$$

where $h(y) = \overline{\Phi}(y, f(y))$. The usual Young inequality for convolution gives

$$\int_{\mathbb{R}^N} \overline{\Phi} f(x, |k_t * f_1(x)|)dx \le C \int_{\mathbb{R}^N} (\hat{k}_t * h)(x) dx$$

$$\le C \|\hat{k}\|_{L^1(\mathbb{R}^N)} \|h\|_{L^1(\mathbb{R}^N)} \le C.$$

Similarly, noting that $g \in L^1(\mathbb{R}^N)$ and applying Lemma (5.1.5), we derive the same result for f_2.

Noting that $|k_t * f_3(x)| \le C\|k_t\|_{L^1(\mathbb{R}^N)} \le 1$, we obtain

$$\int_{\mathbb{R}^N} \overline{\Phi} f(x, |k_t * f_3(x)|)dx \le C \int_{\mathbb{R}^N} |k_t * f_3(x)| dx$$

$$\le C \|k_t\|_{L^1(\mathbb{R}^N)} \|g\|_{L^1(\mathbb{R}^N)} \le C.$$

Thus

$$\int_{\mathbb{R}^N} \overline{\Phi} f(x, |k_t * f(x)|) dx \leq C,$$

which implies the required assertion.

Theorem (5.1.7)[158]: Suppose $\Phi(x,t)$ satisfies($\Phi5$) and ($\Phi6$). Let $\{k_t\}_{t>0}$ be a potential-type, approximate identity. Then $k_t * f$ converges to f in $L^\Phi(\mathbb{R}^N)$:

$$\lim_{t\to 0} \|k_t * f - f\|_{L^\Phi(\mathbb{R}^N)} = 0$$

For every $f \in L^\Phi(\mathbb{R}^N)$.

Proof. Given $\varepsilon > 0$, we find s bounded function h in $L^\Phi(\mathbb{R}^N)$ with compact support such that$\|f - h\|_{L^\Phi(\mathbb{R}^N)} < \varepsilon$. By Theorem (5.1.6) we have

$$\begin{aligned}
&\|k_t * f - f\|_{L^\Phi(\mathbb{R}^N)} \\
&\qquad \leq \|k_t * (h - f)\|_{L^\Phi(\mathbb{R}^N)} + \|k_t * h - h\|_{L^\Phi(\mathbb{R}^N)} + \|f - h\|_{L^\Phi(\mathbb{R}^N)} \\
&\qquad \leq (C\|\hat{k}\|_{L^1(\mathbb{R}^N)} + 1) + \|k_t * h - h\|_{L^\Phi(\mathbb{R}^N)}.
\end{aligned}$$

Since $|k_t * h| \leq \|h\|_{L^\infty(\mathbb{R}^N)}$, we have

$$\int_{\mathbb{R}^N} \overline{\Phi} f(x, |k_t * f(x)|) dx \leq C' \int_{\mathbb{R}^N} |k_t * h(x) - h(x)| dx \leq\to 0$$

as $t \to 0$ by Lemma (5.1.3) (Here C' depends on $\|h\|_{L^\infty(\mathbb{R}^N)}$.) Hence $\|k_t * h - h\|_{L^\Phi(\mathbb{R}^N)} \to 0$ as $t \to 0$ by Lemma (5.1.2), that

$$\lim_{t\to 0} \sup \|k_t * f - f\|_{L^\Phi(\mathbb{R}^N)} \leq (C\|\hat{k}\|_{L^1(\mathbb{R}^N)} + 1)\varepsilon,$$

which completes the proof.

We know the following result due to Z_0 [65]; see also [35].

Lemma (5.1.8)[158]: Let $1 \leq p < \infty$, $1/p + 1/p' = 1$ and $\{k_t\}_{t>0}$ be an approximate identity. Suppose that $k \in L^{p'}(\mathbb{R}^N)$ and has compact support. Then for every $f \in L^p(\mathbb{R}^N)$, $k_t * f$ converges to f pointwise almost everywhere as $t \to 0$.

We take $p_0 \geq 0$ as follows. Let p be the set of all $p \geq 0$ such that $t \to t^{-p}\Phi(x,t)$ is uniformly almost increasing, and set $\tilde{p} = \sup P$. Note that $1 \in P$ by ($\Phi3$), so that $\tilde{p} > 0$ if $\tilde{p} \notin P$. Let $p_0 = \tilde{p}$ if $\tilde{p} \in P$ and $1 < p_0 < \tilde{p}$ otherwise.

For $\Phi(x,t)$ and $\tilde{p} = \min\{p_1^-, p_2^-\}$, so that $p_0 = 1$ if $p_1^- = 1$ or $p_2^- = 1$; and $1 < p_0 \leq \min\{p_1^-, p_2^-\}$ if $p_j^- > 1$, $j = 1, 2$ (cf. [153]).

Since $t^{-p_0}\Phi(x,t)$ is uniformly almost increasing in t, there exists a constant $A_2' \geq 1$ such that

$$t^{-p_0}\Phi(x,t) \leq A_2' s^{-p_0}\Phi(x,s) \quad \text{for all } x \in \mathbb{R}^N \text{ whenever } 1 \leq t < s.$$

Set

$$\Phi_0(x,t) = \Phi(x,t)^{1/p_0}.$$

Then $\Phi_0(x,t)$ also satisfies all the conditions (Φj), $j = 1,2,\dots\dots,6$. In fact, it trivially satisfies (Φj) for $j = 1,2,4,5,6$ with the same g for $(\Phi 6)$. Since

$$\Phi_0(x,t) = t\varphi_0(x,t) \quad \text{with} \quad \varphi_0(x,t) = [t^{-p_0}\Phi(x,t)]^{1/p_0},$$

$\Phi_0(x,t)$ satisfies $(\Phi 3)$ with A_0 replaced by $A_4 = (A_2')^{1/p_0}$.

Lemma (5.1.9)[158]: Suppose $\Phi(x,t)$ satisfies$(\Phi 5)$. Let k have compact support contained in $B(0,R)$ and let $\|k\|_{L^{(p_0)'}(\mathbb{R}^N)} \le 1$. Then there exists a constant $C > 0$, which depends on R, such that

$$\Phi_0(x,|k_t * f(x)|) \le C\int_{\mathbb{R}^N}|k_t(x-y)|\Phi_0(y,f(y))dy$$

for all $x \in \mathbb{R}^N$, $0 < t \le 1$ and for all nonnegative $f \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ such that $f(y) \ge 1$ or $f(y) = 0$ for each$y \in \mathbb{R}^N$ and $\|f\|_{L^\Phi(\mathbb{R}^N)} \le 1$.

Proof. Given f as in the statement of the lemma, $x \in \mathbb{R}^N$ and $0 < \mathrm{t} \le 1$, set

$$F = |k_t * f(x)| \quad \text{and} \quad G = \int_{\mathbb{R}^N}|k_t(x-y)|\Phi_0(y,f(y))dy.$$

Note that $\|f\|_{L^\Phi(\mathbb{R}^N)} \le 1$ implies

$$G \le \|k_t\|_{L^{(p_0)'}(\mathbb{R}^N)}\left(\int_{\mathbb{R}^N}\Phi(y,f(y))dy\right)^{1/p_0} \le t^{-N/p_0}(2A_3)^{1/p_0} \le (2A_3)^{1/p_0}t^{-N}$$

by Hölder's inequality and (1).

By $(\Phi 2)$, $\Phi_0(y,f(y)) \ge (A_1A_4)^{-1}f(y)$, since $f(y) \ge 1$ or $f(y) = 0$. Hence $F \le A_1A_4G$. Thus, if $G \le 1$, then

$$\Phi_0(x,F) \le (A_1A_4G)A_4(A_1A_4)^{(1-p_0)/p_0}\varphi(x,A_1A_4)^{1-p_0} \le CG.$$

Next, let $G > 1$. Since $\Phi_0(x,t) \to \infty$ as $t \to \infty$, there exists$K \ge 1$ such that

$$\Phi_0(x,K) = \Phi_0(x,1)G.$$

Then $K \le A_4G$, since $\Phi_0(x,K) \ge A_4^{-1}K\Phi_0(x,1)$. With this K, we have

$$F \le K\int_{\mathbb{R}^N}|k_t(x-y)|dy + A_4\int_{\mathbb{R}^N}|k_t(x-y)|f(y)\frac{\varphi_0(y,f(y))}{\varphi_0(y,K)}dy,$$

Since

$$1 \le K \le A_4G \le A_4(2A_3)^{1/p_0}t^{-N} \le C(tR)^{-N},$$

there is $\beta > 0$, independent of f, x, t, such that

$$\varphi_0(x, K) \le \beta \varphi_0(y, K) \quad \text{for all} \quad y \in B(x, tR)$$

by (Φ5). Thus, we have

$$F \le K\|k_t\|_{L^1(\mathbb{R}^N)} + \frac{A_4 \beta}{\varphi_0(x, K)} \int_{\mathbb{R}^N} |k_t(x-y)| f(y) \varphi_0(y, f(y)) dy$$

$$= K\|k\|_{L^1(\mathbb{R}^N)} + A_4 \beta \frac{G}{\varphi_0(x, K)}$$

$$= K\left(\|k\|_{L^1(\mathbb{R}^N)} + \frac{A_4 \beta}{\varphi_0(x, 1)}\right)$$

$$\le K\left(\|k\|_{L^1(\mathbb{R}^N)} + A_1^{1/p_0} A_4 \beta\right) \le CK.$$

Therefore by (Φ3), (Φ4), the choice of K and (Φ2),

$$\Phi_0(x, F) \le C\Phi_0(x, K) \le CG$$

With constants $C > 0$ independent of f, x, t, as required.

Lemma (5.1.10)[158]: Suppose $\Phi(x, t)$ satisfies (Φ6). Let $M \ge 1$ and assume that $\|k\|_{L^1(\mathbb{R}^N)} \le M$. Then there exists a constant $C > 0$, depending on M, such that

$$\overline{\Phi}(x, |k_t * f(x)|) \le C\left\{\int_{\mathbb{R}^N} |k_t(x-y)| \overline{\Phi}\big(y, f(y)\big) dy + g(x)\right\}$$

for all $x \in \mathbb{R}^N$, $t > 0$ and for all nonnegative $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ such that $g(x) \le f(y) \le 1$ or $f(y) = 0$ for each $y \in \mathbb{R}^N$, where g is the function appearing in (Φ6).

Proof. Let f be as in the statement of the lemma, $x \in \mathbb{R}^N$ and $t > 0$. By (Φ4), there is a constant $c_M \ge 1$ such that $\overline{\Phi}(x, Mt) \le c_M \overline{\Phi}(x, t)$ for all $x \in \mathbb{R}^N$ and $t > 0$. By Jensen's inequality, we have

$$\overline{\Phi}(x, |k_t * f(x)|) \le c_M \overline{\Phi}\left(\int_{\mathbb{R}^N} \left(\frac{|k_t(x-y)|}{M}\right) f(y) dy\right)$$

$$\le (c_M/M) \int_{\mathbb{R}^N} |k_t(x-y)| \overline{\Phi}\big(x, f(y)\big) dy.$$

If $|x| \ge |y|$, then $\overline{\Phi}\big(x, f(y)\big) \le \overline{B}_\infty \overline{\Phi}\big(y, f(y)\big)$ by (Φ6).

If $|x| < |y|$ and $g(x) < f(y)$, $\overline{\Phi}\big(x, f(y)\big) \le \overline{B}_\infty \overline{\Phi}\big(x, f(y)\big)$ by (Φ6) again.

If $|x| < |y|$ and $g(x) \ge f(y)$, then

$$\overline{\Phi}\big(x, f(y)\big) \le \overline{\Phi}\big(x, g(x)\big) \le g(x) \overline{\Phi}(x, 1) \le A_1 A_2 g(x)$$

by (2)

Hence,

$$\overline{\Phi}(x, f(y)) \leq C\{\overline{\Phi}(y, f(x)) + g(x)\}$$

in any case. Therefore, we obtain the required inequality.

Theorem (5.1.11)[158]: Suppose $\Phi(x,t)$ satisfies (Φ5) and (Φ6). Suppose that $k \in L^{(p_0)'}(\mathbb{R}^N)$ and compact support in $B(0,R)$ Then

$$\|k_t * f\|_{L^{\Phi}(\mathbb{R}^N)} \leq C\|k\|_{L^{(p_0)'}(\mathbb{R}^N)}\|f\|_{L^{\Phi}(\mathbb{R}^N)}$$

for all $0 < \mathrm{t} \leq 1$ and $f \in L^{\Phi}(\mathbb{R}^N)$, where $C > 0$ depends on R.

Proof. Let f be a nonnegative measurable function on $\mathbb{R}^N$ such that $\|f\|_{L^{\Phi}(\mathbb{R}^N)} \leq 1$ and assume that $\|k\|_{L^{(p_0)'}(\mathbb{R}^N)} = 1$. Note that $\|k\|_{L^1(\mathbb{R}^N)} \leq |B(0,R)|^{1/p_0}$ by Hölder's inequality.

Write

$$f = f\chi_{\{y\in\mathbb{R}^N: f(y)\geq 1\}} + f\chi_{\{y\in\mathbb{R}^N:\, g(y)<f(y)<1\}} + f\chi_{\{y\in\mathbb{R}^N: f(y)\leq g(y)\}} = f_1 + f_2 + f_3,$$

where g is the function appearing in (Φ6). We have by (1) and Lemma (5.1.9),

$$\overline{\Phi}(x, |k_t * f_1(x)|) \leq A_2\Phi_0(x, |k_t * f_1(x)|)^{p_0} \leq C\big(x, |k_t| * h(x)\big)^{p_0},$$

Where $h(y) = \Phi(y, f(y))^{1/p_0}$. Since $\|h\|_{L^{p_0}(\mathbb{R}^N)}^{p_0} \leq 2A_3$, the usual Young's inequality for convolution gives

$$\int_{\mathbb{R}^N} \overline{\Phi}(x, |k_t * f_1(x)|)dx \leq C\int_{\mathbb{R}^N} (\,|k_t| * h(x))^{p_0} dx$$

$$\leq C\left(\|k_t\|_{L^1(\mathbb{R}^N)}\|h\|_{L^{p_0}(\mathbb{R}^N)}\right)^{p_0} \leq C.$$

Similarly, applying Lemma (5.1.11) with $M = |B(0,R)|^{1/p_0}$ and noting that $g \in L^1(\mathbb{R}^N)$, we derive the same result for f_2.

Since $|k_t * f_3(x)| \leq \|k_t\|_{L^1(\mathbb{R}^N)} \leq M$, we obtain

$$\int_{\mathbb{R}^N} \overline{\Phi}(x, |k_t * f_3(x)|)dx \leq C\int_{\mathbb{R}^N} |k_t * f_3(x)|dx$$

$$\leq C\|k_t\|_{L^1(\mathbb{R}^N)}\|g\|_{L^1(\mathbb{R}^N)} \leq C.$$

Thus, we have shown that

$$\int_{\mathbb{R}^N} \overline{\Phi}(x, |k_t * f(x)|)dx \leq C,$$

which implies the required result.

Theorem (5.1.12)[158]: Suppose $\Phi(x,t)$ satisfies ($\Phi5$) and ($\Phi6$). Let $\{k_t\}_{t>0}$ be an approximate identity that $k \in \|k\|_{L^{(p_0)'}(\mathbb{R}^N)}$ and has compact support. Then $k_t * f$ converges to f in $L^{\Phi}(\mathbb{R}^N)$:

$$\lim_{t\to 0}\|k_t * f - f\|_{L^{\Phi}(\mathbb{R}^N)} = 0$$

for every $f \in L^{\Phi}(\mathbb{R}^N)$.

Proof. Let $f \in L^{\Phi}(\mathbb{R}^N)$. Given $\varepsilon > 0$, choose a bounded function h with compact support such that $\|f-h\|_{L^{\Phi}(\mathbb{R}^N)} < \varepsilon$. As in the proof of Theorem (5.1.7), using Theorem (5.1.11) this time, we have

$$\|k_t * f - f\|_{L^{\Phi}(\mathbb{R}^N)} \le \left(C\|k\|_{L^{(p_0)'}(\mathbb{R}^N)} + 1\right)\varepsilon + \|k_t * h - h\|_{L^{\Phi}(\mathbb{R}^N)}.$$

Obviously, $f \in L^{p_0}(\mathbb{R}^N)$. Hence by Lemma (5.1.8), $k_t * h \to h$ almost everywhere in $\mathbb{R}^N$, and hence

$$\overline{\Phi}(x, |k_t * h(x) - h(x)|) \to 0$$

almost everywhere in $\mathbb{R}^N$. Since $\{k_t * h - h\}$ is uniformly and there is a compact set S containing all the supports of $k_t * h$, $\{\overline{\Phi}(x, |k_t * h(x) - h(x)|)\}$ is uniformly bounded and S contains all the supports of $\overline{\Phi}(x, |k_t * h(x) - h(x)|)$. Hence the Lebesgue convergence theorem implies

$$\int_{\mathbb{R}^N} \overline{\Phi}(x, |k_t * h(x) - h(x)|)dx \to 0$$

as $t \to 0$. Then, by Lemma (5.1.2), we see that $\|k_t * h - h\|_{L^{\Phi}(\mathbb{R}^N)} \to 0$ as $t \to 0$, so that

$$\limsup_{t\to 0}\|k_t * f - f\|_{L^{\Phi}(\mathbb{R}^N)} \le \left(C\|k\|_{L^{(p_0)'}(\mathbb{R}^N)} + 1\right)\varepsilon,$$

which completes the proof.

Lemma (5.1.13)[158]: Suppose $\Phi(x,t)$ satisfies ($\Phi5$) and ($\Phi6$). Let $k \in L^1(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)$ with $k \in L^1(\mathbb{R}^N) \le 1$. For $f \in L^1_{\text{loc}}(\mathbb{R}^N)$, set

$$I(f;x) = \int_{\mathbb{R}^N \setminus B(0,|x|/2)} |k(x-y)f(y)|dy$$

and

$$J(f;x) = \int_{\mathbb{R}^N} |k(x-y)|\overline{\Phi}(y, |f(y)|)dy.$$

Then there exists a constant $C > 0$ (depending on $\|k\|_{L^{\infty}(\mathbb{R}^N)}$) such that

$$\overline{\Phi}\big(x, I(f;x)\big) \le C\{J(f;x) + g(x/2)\}$$

for all $x \in \mathbb{R}^N$ and $f \in L^\Phi(\mathbb{R}^N)$ with $\|f\|_{L^\Phi(\mathbb{R}^N)} \le 1$, where g is the function appearing (Φ6).

Proof. Let $k > 0$. Since $t \to \overline{\Phi}(x,t)/t$ is non-decreasing,

$$J(f;x) \le k\int_{\mathbb{R}^N} |k(x-y)|dy + k\int_{\mathbb{R}^N\setminus B(0,|x|/2)} \frac{|k(x-y)|\overline{\Phi}(y,|f(y)|)}{\overline{\Phi}(y,k)}dy.$$

If $g(x/2) \le k \le 1$, then $\overline{\Phi}(x,k) \le C\overline{\Phi}(y,k)$ for $|y| > |x|/2$ by (Φ6). Hence

$$I(f;x) \le k\left(\frac{CJ(f;x)}{\overline{\Phi}(x,k)}\right) \quad \text{whenever } g\left(\frac{x}{2}\right) \le k \le 1. \tag{4}$$

Since $J(f;x) \le \|k\|_{L^\infty(\mathbb{R}^N)}$, there exists $k_x \in [0,1]$ such that

$$\overline{\Phi}(x,k_x) = \frac{J(f;x)}{\|k\|_{L^\infty(\mathbb{R}^N)}}\overline{\Phi}(x,1).$$

If $k_x \ge g(x/2)$, then taking $k = k_x$ in (4), we have

$$I(f;x) \le k_x\left(1 + \frac{C\|k\|_{L^\infty(\mathbb{R}^N)}}{\overline{\Phi}(x,1)}\right) \le k_x,$$

so that

$$\overline{\Phi}\big(x, I(f;x)\big) \le \overline{\Phi}(x,k_x) \le CJ(f;x).$$

If $k_x < g(x/2)$, then

$$J(f;\ k_x) = \|k\|_{L^\infty(\mathbb{R}^N)}\frac{\overline{\Phi}(x,k_x)}{\overline{\Phi}(x,1)} \le \overline{\Phi}\big(x, g(x/2)\big).$$

Hence, taking $k = g(x/2)$ in (4) we have $I(f;x) \le Cg(x/2)$, so that

$$\overline{\Phi}\big(x, I(f;x)\big) \le C\overline{\Phi}(x,k_x) \le Cg(f;x).$$

Hence, we have the assertion of the lemma.

We recall the following result on the boundedness of maximal operator M on $L^\Phi(\mathbb{R}^N)$ (see [155]):

Lemma (5.1.14)[158]: Suppose $\Phi(x,t)$ satisfies (Φ5), (Φ6) and the other condition $\Phi3^*$ $t \longmapsto t^{\varepsilon_0}\varphi(x,t)$ is uniformly almost on $(0,\infty)$ for some $\varepsilon_0 > 0$.

Then the maximal operator M is bounded from $L^\Phi(\mathbb{R}^N)$ into itself, namely

$$\|Mf\|_{L^\Phi(\mathbb{R}^N)} \le \|f\|_{L^\Phi(\mathbb{R}^N)}$$

for all $f \in L^\Phi(\mathbb{R}^N)$.

Theorem (5.1.15)[158]: Suppose $\Phi(x,t)$ satisfies ($\Phi5$), ($\Phi6$) and ($\Phi3^*$). Let $p_0 = 1+\varepsilon_0$ (>0) and $R>0$. Assume that $k \in L^1(\mathbb{R}^N) \cap L^{(p_0)'}(\mathrm{B}(0,\mathrm{R}))$ and $|k(x)| \le C_k|x|^{-N}$ for $|x| \ge R$. Then there is a constant $C>0$ such that

$$\|k * f\|_{L^\Phi(\mathbb{R}^N)} \le C(\|k\|_{L^1(\mathbb{R}^N)} + \|f\|_{L^{(p_0)'}(\mathrm{B}(0,\mathrm{R}))})\|f\|_{L^\Phi(\mathbb{R}^N)}$$

for all $f \in L^\Phi(\mathbb{R}^N)$.

Proof. Let $f \in L^\Phi(\mathbb{R}^N)$ and $f \ge 0$. Assume that $\|f\|_{L^\Phi(\mathbb{R}^N)} \le 1$ and

$$\|k\|_{L^1(\mathbb{R}^N)} + \|f\|_{L^{(p_0)'}(\mathrm{B}(0,\mathrm{R}))} \le 1.$$

Let $k_0 = k\chi_{B(0,R)}$ and $k_\infty = k\chi_{\mathbb{R}^N \setminus B(0,R)}$.

By Theorem (5.1.11),

$$\|k_0 * f\|_{L^1(\mathbb{R}^N)} \le C.$$

Hence, it is enough to show that

$$\int_{\mathbb{R}^N} \overline{\Phi}\big(x, |k_\infty| * f(x)\big)dx \le C. \tag{5}$$

Write

$$|k_\infty| * f(x) = \int_{B(0,|x|/2)} |k_\infty(x-y)|f(y)dy + \int_{\mathbb{R}^N \setminus B(0,|x|/2)} |k_\infty(x-y)|f(y)dy.$$
$$= I_1(x) + I_2(x).$$

Since $|k_\infty(x-y)| \le C_k|x-y|^{-N}$ and $|x-y| \ge |x|/2$ for $|x| \le |x|/2$,

$$I_1(x) \le 2^N C_k|x|^{-N} \int_{B(0,|x|/2)} f(y)dy \le 2^N C_k |x|^{-N} \int_{B(0,3|x|/2)} f(y)dy.$$

Hence,

$$\int_{\mathbb{R}^N} \overline{\Phi}\big(x, I_1(x)\big)dx \le C$$

by Lemma (5.1.14).

on the other hand, Lemma (5.1.13),

$$\overline{\Phi}\big(x, I_2(x)\big) \le C\left\{|k_\infty| * h(x) + g\left(\frac{x}{2}\right)\right\},$$

where $h(y) = \overline{\Phi}\big(y, f(y)\big)$. Since

$$\||k_\infty| * h\|_{L^1(\mathbb{R}^N)} \le \||k_\infty|\|_{L^1(\mathbb{R}^N)}\|h\|_{L^1(\mathbb{R}^N)} \le 1$$

and $g \in L^1(\mathbb{R}^N)$, it follows that

$$\int_{\mathbb{R}^N} \overline{\Phi}\big(x, I_2(x)\big) dx \le C.$$

Hence, we obtain (5), and the proof is complete.

Corollary (5.1.16)[205]: Suppose (Φ5) satisfies (Φ6). If $\{k_{1+\epsilon}\}_{\epsilon>-1}$ is of potential-type, then

$$\left\| \sum_n k_{1+\epsilon} * f^n \right\|_{L^{\Phi_n}(\mathbb{R}^N)} \le C \|\hat{k}\|_{L^1(\mathbb{R}^N)} \sum_n \|f^n\|_{L^{\Phi_n}(\mathbb{R}^N)}$$

for all $\epsilon > -1$ and $f^n \in L^{\Phi_n}(\mathbb{R}^N)$.

Proof. Suppose $\|\hat{k}\|_{L^1(\mathbb{R}^N)} = 1$ and let f^n be nonnegative measurable function on $\mathbb{R}^N$ such that $\sum_n \|f^n\|_{L^1(\mathbb{R}^N)} \le 1$. Write

$$\sum_n f^n = \sum_n f^n \chi_{\{(x+\epsilon)\in\mathbb{R}^N: f^n(x+\epsilon)\ge 1\}} + \sum_n f^n \chi_{\{(x+\epsilon)\in\mathbb{R}^N:\, g^n(x+\epsilon)<f^n(x+\epsilon)<1\}}$$
$$+ \sum_n f^n \chi_{\{(x+\epsilon)\in\mathbb{R}^N: f^n(x+\epsilon)\le g^n(x+\epsilon)\}} = \sum_n f_1^n + \sum_n f_2^n + \sum_n f_3^n$$

where χ_E denotes the characteristic function of a measurable set $E \subset \mathbb{R}^N$ and g^n is the function appearing in (Φ6).

Since $\hat{k}_{1+\epsilon}$ is radial function, we write $\hat{k}_{1+\epsilon}(1+\epsilon)$ for $\hat{k}_{1+\epsilon}(x)$ when $|x| = (1+\epsilon)$. Note that

$$\left| \sum k_{1+\epsilon} * f_j^n(x) \right| \le \int_{\mathbb{R}^N} \sum_n \hat{k}_{1+\epsilon}(|\epsilon|) f_j^n\,(x+\epsilon) \mathrm{d}(x+\epsilon)$$
$$= \int_0^\infty \sum_n I\left(f_j^n; x, (1+\epsilon)\right) \big|B\big(x,(1+\epsilon)\big)\big| d\left(-\hat{k}_{1+\epsilon}(1+\epsilon)\right)$$

$j = 1, 2$, and

$$\int_{\mathbb{R}^N} \big(\big|B\big(x,(1+\epsilon)\big)\big|\big) d\left(-\hat{k}_{1+\epsilon}(1+\epsilon)\right) = \|\hat{k}\|_{L^1(\mathbb{R}^N)} = 1,$$

so that Jensen's inequality yields

$$\sum_n \overline{\Phi}_n\, f^n\big(x, \big|k_{1+\epsilon} * f_j^n(x)\big|\big) \le \int_0^\infty \sum_n \overline{\Phi}_n\, f^n\left(x, I\big(f^n; x, (1+\epsilon)\big)\right)$$
$$\le \big|B\big(x,(1+\epsilon)\big)\big| d\left(-\hat{k}_{1+\epsilon}(1+\epsilon)\right),$$

$j = 1, 2$.

Hence, by Lemma (5.1.4)

$$\sum_n \overline{\Phi}_n f^n(x, |k_{1+\epsilon} * f_1^n(x)|)$$
$$\leq C \int_0^\infty \sum_n J\big(f_1^n; x, (1+\epsilon)\big)\big|B\big(x, (1+\epsilon)\big)\big| d\left(-\hat{k}_{1+\epsilon}(1+\epsilon)\right)$$
$$\leq C\big(\hat{k}_{1+\epsilon} * h\big),$$

where $h(x+\epsilon) = \sum_n \overline{\Phi}_n\left((x+\epsilon), f(x+\epsilon)\right)$. The usual Young inequality for convolution gives

$$\int_{\mathbb{R}^N} \sum_n \overline{\Phi}_n f^n(x, |k_{1+\epsilon} * f_1^n(x)|)dx \leq C \int_{\mathbb{R}^N} (\hat{k}_{1+\epsilon} * h)(x)dx$$
$$\leq C\|\hat{k}\|_{L^1(\mathbb{R}^N)}\|h\|_{L^1(\mathbb{R}^N)} \leq C.$$

Similarly, noting that $g^n \in L^1(\mathbb{R}^N)$ and applying Lemma (5.1.4), we derive the same result for f_2^n.

Finally, noting that $\sum_n |k_{1+\epsilon} * f_3^n| \leq C\|k_{1+\epsilon}\|_{L^1(\mathbb{R}^N)} \leq 1$, we obtain

$$\int_{\mathbb{R}^N} \sum_n \overline{\Phi}_n f^n(x, |k_{1+\epsilon} * f_3^n(x)|)dx \leq C \int_{\mathbb{R}^N} \sum_n |k_{1+\epsilon} * f_3^n|\, dx$$
$$\leq C\|k_{1+\epsilon}\|_{L^1(\mathbb{R}^N)}\|g^n\|_{L^1(\mathbb{R}^N)} \leq C.$$

Thus

$$\int_{\mathbb{R}^N} \sum_n \overline{\Phi}_n f^n(x, |k_{1+\epsilon} * f^n(x)|)\mathrm{d}x \leq C,$$

which implies the required assertion.

Corollary (5.1.17)[205]: Suppose $\Phi_n\big(x, (1+\epsilon)\big)$ satisfies (Φ5) and (Φ6). Let $\{k_{1+\epsilon}\}_{\epsilon > -1}$ be a potential-type, intensive approximate identity. Then $k_{1+\epsilon} * f^n$ converges to f^n in $L^{\Phi_n}(\mathbb{R}^N)$:

$$\lim_{\epsilon \to -1} \sum_n \|k_{1+\epsilon} * f^n - f^n\|_{L^{\Phi_n}(\mathbb{R}^N)} = 0$$

for every $f^n \in L^{\Phi_n}(\mathbb{R}^N)$.

Proof. Given $\varepsilon > 0$, we find a bounded function h in $L^{\Phi_n}(\mathbb{R}^N)$ with compact support such that$\|f^n - h\|_{L^{\Phi_n}(\mathbb{R}^N)} < \varepsilon$. By Corollary (5.1.17) we have

$$\sum_n \|k_{1+\epsilon} * f^n - f^n\|_{L^{\Phi_n(\mathbb{R}^N)}}$$
$$\leq \sum_n \|k_{1+\epsilon} * (f^n - h)\|_{L^{\Phi_n}(\mathbb{R}^N)} + \sum_n \|k_{1+\epsilon} * h - h\|_{L^{\Phi_n}(\mathbb{R}^N)}$$
$$+ \sum_n \|f^n - h\|_{L^{\Phi_n}(\mathbb{R}^N)} \leq \left(C\|\hat{k}\|_{L^1(\mathbb{R}^N)} + 1\right)$$
$$+ \sum_n \|k_{1+\epsilon} * h - h\|_{L^{\Phi_n}(\mathbb{R}^N)}.$$

Since $|k_{1+\epsilon} * h| \leq \|h\|_{L^\infty(\mathbb{R}^N)}$, we have

$$\int_{\mathbb{R}^N} \sum_n \overline{\Phi}_n f^n(x, |k_{1+\epsilon} * f^n(x)|)\, dx \leq C' \int_{\mathbb{R}^N} |k_{1+\epsilon} * h(x) - h(x)| dx \to 0$$

as $\epsilon \to -1$ by Lemma (5.1.3) (Here C' depends on $\|h\|_{L^\infty(\mathbb{R}^N)}$.) Hence $\|k_{1+\epsilon} * h - h\|_{L^{\Phi_n}(\mathbb{R}^N)} \to 0$ as $\epsilon \to -1$ by Lemma (5.1.2), that

$$\limsup_{\epsilon \to -1} \sum_n \|k_{1+\epsilon} * f^n - f^n\|_{L^{\Phi_n}(\mathbb{R}^N)} \leq (C\|\hat{k}\|_{L^1(\mathbb{R}^N)} + 1)\varepsilon,$$

which completes the proof.

Corollary (5.1.18)[205]: Suppose $\Phi_n(x, (1+\epsilon))$ satisfies ($\Phi5$). Let k have compact support contained in $B(0, R)$ and let $\|k\|_{L^{(1+\epsilon)'}(\mathbb{R}^N)} \leq 1$. Then there exists a constant $C > 0$, which depends on R, such that

$$\sum_n (\Phi_n)_0\, (x, |k_{1+\epsilon} * f^n(x)|)$$
$$\leq C \int_{\mathbb{R}^N} \sum_n |k_{1+\epsilon}(\epsilon)| \sum_n (\Phi_n)_0 \left((x+\epsilon), f^n(x+\epsilon)\right) \mathrm{d}(x+\epsilon)$$

for all $x \in \mathbb{R}^N$, $-1 < \epsilon \leq 0$ and for all nonnegative $f^n \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ such that $f^n(x+\epsilon) \geq 1$ or $f^n(x+\epsilon) = 0$ for each $(x+\epsilon) \in \mathbb{R}^N$ and $\sum_n \|f^n\|_{L^{\Phi_n}(\mathbb{R}^N)} \leq 1$.

Proof. Given f^n as in the statement of the lemma, $x \in \mathbb{R}^N$ and $-1 < \epsilon \leq 0$, set

$$F = \sum_n |k_{1+\epsilon} * f^n(x)| \quad \text{and} \quad G = \int_{\mathbb{R}^N} \sum_n |k_{1+\epsilon}(\epsilon)| \sum_n (\Phi_n)_0\, ((x+\epsilon), f^n(x+\epsilon))\, d(x+\epsilon).$$

Note that $\sum_n \|f^n\|_{L^{\Phi_n}(\mathbb{R}^N)} \leq 1$ implies

$$G \leq \|k_{1+\epsilon}\|_{L^{(1+\epsilon)'}(\mathbb{R}^N)} \left(\int_{\mathbb{R}^N} \sum_n \Phi_n((x+\epsilon), f^n(x+\epsilon)) d(x+\epsilon)\right)^{\frac{1}{1+\epsilon}}$$

$$\leq (1+\epsilon)^{-\frac{N}{1+\epsilon}} \big(2(A_1+2\epsilon)\big)^{\frac{1}{1+\epsilon}} \leq \big(2(A_1+2\epsilon)\big)^{\frac{1}{1+\epsilon}} (1+\epsilon)^{-N}$$

by Hölder's inequality and (1).

By (Φ2), $\sum_n (\Phi_n)_0\,((x+\epsilon), f^n(x+\epsilon)) \geq \big(A_1(A_1+3\epsilon)\big)^{-1} \sum_n f^n(x+\epsilon)$, since $\sum_n f^n(x+\epsilon) \geq 1$ or $f^n(x+\epsilon) = 0$. Hence $F \leq A_1(A_1+3\epsilon)G$. Thus, if $G \leq 1$, then

$$\sum_n (\Phi_n)_0\,(x, F) \leq (A_1(A_1+3\epsilon)G)(A_1+3\epsilon)\big(A_1(A_1+3\epsilon)\big)^{\frac{\epsilon}{1+\epsilon}} \sum_n \varphi_n\big(x, A_1(A_1+3\epsilon)\big)^{\epsilon} \leq CG$$

Next, let $G > 1$. Since $\Phi_0\big(x,(1+\epsilon)\big) \to \infty$ as $\epsilon \to \infty$, there exists $K \geq 1$ such that

$$\sum_n (\Phi_n)_0\,(x, K) = \sum_n (\Phi_n)_0\,(x, 1)G.$$

Then $K \leq (A_1+3\epsilon)G$, since $\sum_n (\Phi_n)_0\,(x, K) \geq (A_1^{-1}+3\epsilon)K \sum_n (\Phi_n)_0\,(x, 1)$. With this K, we have

$$F \leq K \int_{\mathbb{R}^N} |k_{1+\epsilon}(\epsilon)| \mathrm{d}(x+\epsilon)$$

$$+(A_1+3\epsilon) \int_{\mathbb{R}^N} \sum_n |k_{1+\epsilon}(\epsilon)| f(x+\epsilon) \sum_n \frac{(\varphi_n)_0\big((x+\epsilon), f^n(x+\epsilon)\big)}{(\varphi_n)_0\big((x+\epsilon), K\big)} \mathrm{d}(x+\epsilon),$$

Since

$$1 \leq K \leq (A_1+3\epsilon)G \leq (A_1+3\epsilon)\big(2(A_1+2\epsilon)\big)^{\frac{1}{1+\epsilon}}(1+\epsilon)^{-N} \leq C((1+\epsilon)R)^{-N},$$

there is $\beta > 0$, independent of $f^n, x, (1+\epsilon)$, such that

$$\sum_n (\varphi_n)_0\,(x, K) \leq \beta \sum_n (\varphi_n)_0\,((x,\epsilon), K) \quad \text{for all } (x+\epsilon) \in B(x, ((1+\epsilon)R)$$

by (Φ5). Thus, we have

$$F \leq K\|k_{1+}\|_{L^1(\mathbb{R}^N)}$$

$$+ \sum_n \frac{(A_1+3\epsilon)\beta}{(\varphi_n)_0(x,K)} \int_{\mathbb{R}^N} \sum_n |k_{1+\epsilon}(\epsilon)| f^n(x+\epsilon)(\varphi_n)_0((x+\epsilon), f^n(x+\epsilon))\, d(x+\epsilon)$$

$$= K\|k\|_{L^1(\mathbb{R}^N)} + (A_1 + 3\epsilon)\beta \sum_n \frac{G}{(\varphi_n)_0(x, K)}$$

$$= K\left(\|k\|_{L^1(\mathbb{R}^N)} + \sum_n \frac{(A_1 + 3\epsilon)\beta}{(\varphi_n)_0(x,\ 1)}\right)$$

$$\leq K\left(\|k\|_{L^1(\mathbb{R}^N)} + A_1^{\frac{1}{1+\epsilon}}(A_1 + 3\epsilon)\beta\right) \leq CK.$$

Therefore by (Φ3), (Φ4), the choice of K and (Φ2),

$$\sum_n (\Phi_n)_0\,(x, F) \leq C \sum_n (\Phi_n)_0\,(x, K) \leq CG$$

With constants $C > 0$ independent of $f^n, x, (1+\epsilon)$, as required.

Corollary (5.1.19)[205]: Suppose $\Phi_n\big(x, (1+\epsilon)\big)$ satisfies (Φ6). Let $M \geq 1$ and assume that $\|k\|_{L^1(\mathbb{R}^N)} \leq M$. Then there exists a constant $C > 0$, depending on M, such that

$$\sum_n \overline{\Phi}_n(x, |k_{1+\epsilon} * f^n(x)|)$$
$$\leq C\left\{\int_{\mathbb{R}^N} \sum_n |k_{1+\epsilon}(\epsilon)| \overline{\Phi}_n\big((x+\epsilon), f^n(x+\epsilon)\big)\, \mathrm{d}(x+\epsilon) + g^n(x)\right\}$$

for all $x \in \mathbb{R}^N$, $\epsilon > -1$ and for all nonnegative $f^n \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ such that $g^n(x) \leq \sum_n f^n(x+\epsilon) \leq 1$ or $\sum_n f^n(x+\epsilon) = 0$ for each $(x+\epsilon) \in \mathbb{R}^N$, where g^n is the function appearing in (Φ6).

Proof. Let f^n be as in the statement of the corollary, $x \in \mathbb{R}^N$ and $\epsilon > -1$. By (Φ4), there is a constant $c_M \geq 1$ such that $\sum_n \overline{\Phi}_n(x, M(1+\epsilon)) \leq c_M \sum_n \overline{\Phi}_n\big(x, (1+\epsilon)\big)$ for all $x \in \mathbb{R}^N$ and $\epsilon > -1$. By Jensen's inequality, we have

$$\sum_n \overline{\Phi}_n(x, |k_{1+\epsilon} * f^n(x)|)$$
$$\leq c_M \sum_n \overline{\Phi}_n\left(\int_{\mathbb{R}^N} \left(\frac{|k_{1+\epsilon}(x+\epsilon)|}{M}\right) f^n(x+\epsilon) d(x+\epsilon)\right)$$
$$\leq \left(\frac{c_M}{M}\right) \int_{\mathbb{R}^N} \sum_n |k_{1+\epsilon}(\epsilon)| \overline{\Phi}_n\big(x, f^n(x+\epsilon)\big)\, d(x+\epsilon).$$

If $|x| \geq |x+\epsilon|$, then $\sum_n \overline{\Phi}_n\big(x, f^n(x+\epsilon)\big) \leq \overline{B}_\infty \sum_n \Phi_n\big((x+\epsilon), f^n(x+\epsilon)\big)$ by (Φ6).

If $|x| < |x+\epsilon|$ and $g^n(x) < \sum_n f^n(x+\epsilon)$, $\sum_n \overline{\Phi}_n\big(x, f^n(x+\epsilon)\big) \leq \overline{B}_\infty \sum_n \overline{\Phi}_n\big(x, f^n(x+\epsilon)\big)$ by (Φ6) again.

If $|x| < |x+\epsilon|$ and $g^n(x) \geq \sum_n f^n(x+\epsilon)$, then

$$\sum_n \overline{\Phi}_n\left(x, f^n(x+\epsilon)\right) \leq \sum_n \overline{\Phi}_n\left(x, g^n(x)\right) \leq g^n(x) \sum_n \overline{\Phi}_n(x,1) \leq A_1(A_1+\epsilon) g^n(x)$$

by (2)

Hence,

$$\sum_n \overline{\Phi}_n\left(x, f^n(x+\epsilon)\right) \leq C \sum_n \left\{\overline{\Phi}_n\left((x+\epsilon), f^n(x)\right) + g^n(x)\right\}$$

therefore, we obtain the required inequality.

Corollary (5.1.20)[205]: Suppose $\Phi_n\big(x,(1+\epsilon)\big)$ satisfies (Φ5) and (Φ6). Suppose that $k \in L^{(1+\epsilon)'}(\mathbb{R}^N)$ and compact support in $B(0,R)$. Then

$$\left\|\sum_n k_{1+\epsilon} * f^n\right\|_{L^{\Phi_n}(\mathbb{R}^N)} \leq C\|k\|_{L^{(1+\epsilon)'}(\mathbb{R}^N)} \sum_n \|f^n\|_{L^{\Phi_n}(\mathbb{R}^N)}$$

for all $-1 < \epsilon \leq 0$ and $\sum_n \|f^n\|_{L^{\Phi_n}(\mathbb{R}^N)}$, where $C > 0$ depends on R.

Proof. Let f^n be a nonnegative measurable function on $\mathbb{R}^N$ such that $\sum_n \|f^n\|_{L^{\Phi_n}(\mathbb{R}^N)} \leq 1$ and assume that $\|k\|_{L^{(1+\epsilon)'}(\mathbb{R}^N)} = 1$. Note that $\|k\|_{L^1(\mathbb{R}^N)} \leq |B(0,R)|^{\frac{1}{1+\epsilon}}$ by Hölder's inequality.

Write

$$\sum_n f^n = \sum_n f^n \chi_{\{(x+\epsilon)\in\mathbb{R}^N: f^n(x+\epsilon)\geq 1\}} + \sum_n f^n \chi_{\{(x+\epsilon)\in\mathbb{R}^N:\, g^n(x+\epsilon)<f^n(x+\epsilon)<1\}}$$

$$+ \sum_n f^n \chi_{\{(x+\epsilon)\in\mathbb{R}^N: f^n(x+\epsilon)\leq g^n(x+\epsilon)\}} = \sum_n f_1^n + \sum_n f_2^n + \sum_n f_3^n,$$

where g^n is the function appearing in (Φ6). We have by (1) and Corollary (5.1.18),

$$\sum_n \overline{\Phi}_n(x, |k_{1+\epsilon} * f_1^n(x)|) \leq (A_1+\epsilon) \sum_n (\Phi_n)_0 \left(x, |k_{1+\epsilon} * f_1^n(x)|\right)^{1+\epsilon}$$

$$\leq C\big(x, |k_{1+\epsilon}| * h(x)\big)^{1+\epsilon}$$

where $h(x+\epsilon) = \sum_n \Phi_n\big((x+\epsilon), f^n(x+\epsilon)\big)^{\frac{1}{1+\epsilon}}$. Since $\|h\|_{L^{1+\epsilon}(\mathbb{R}^N)}^{1+\epsilon} \leq 2(A_1+\epsilon)$, the usual Young's inequality for convolution gives

$$\int_{\mathbb{R}^N} \sum_n \overline{\Phi}_n(x, |k_{1+\epsilon} * f_1^n(x)|)\, dx$$

$$\leq C\int_{\mathbb{R}^N}\left(|k_{1+\epsilon}| * h(x)\right)^{1+\epsilon}dx \ \leq C\left(\|k_{1+\epsilon}\|_{L^1(\mathbb{R}^N)}\|h\|_{L^{1+\epsilon}(\mathbb{R}^N)}\right)^{1+\epsilon} \leq C.$$

Similarly, applying Corollary (5.1.19) with $M = |B(0,R)|^{\frac{1}{1+\epsilon}}$ and noting that $g^n \in L^1(\mathbb{R}^N)$, we derive the same result for f_2^n.

Since $\sum_n \left|k_{(1+\epsilon)} * f_3^n(x)\right| \leq \|k_{\ 1+\epsilon}\|_{L^1(\mathbb{R}^N)} \leq M$, we obtain

$$\int_{\mathbb{R}^N}\sum_n \overline{\Phi}_n(x, |k_{1+\epsilon} * f_3^n(x)|)\,dx \leq C\int_{\mathbb{R}^N}\sum_n |k_{1+\epsilon} * f_3^n(x)|\,dx$$

$$\leq C\|k_{1+\epsilon}\|_{L^1(\mathbb{R}^N)}\|g^n\|_{L^1(\mathbb{R}^N)} \leq C.$$

Thus, we have shown that

$$\int_{\mathbb{R}^N}\sum_n \overline{\Phi}_n(x, |k_{1+\epsilon} * f^n(x)|)\,dx \leq C,$$

which implies the required result.

Corollary (5.1.21)[205]: Suppose $\Phi_n\big(x,(1+\epsilon)\big)$ satisfies (Φ5) and (Φ6). Let $\{k_{1+\epsilon}\}_{\epsilon>-1}$ be an intensive approximate identity that $k \in \|k\|_{L^{(1+\epsilon)'}(\mathbb{R}^N)}$ and has compact support. Then $k_{1+\epsilon} * f^n$ converges to f^n in $L^{\Phi_n}(\mathbb{R}^N)$:

$$\lim_{\epsilon\to -1}\sum_n \|k_{1+\epsilon} * f^n - f^n\|_{L^{\Phi_n}(\mathbb{R}^N)} = 0$$

for every $f^n \in L^{\Phi_n}(\mathbb{R}^N)$.

Proof. Let $f^n \in L^{\Phi_n}(\mathbb{R}^N)$. Given $\varepsilon > 0$, choose a bounded function h with compact support such that $\sum_n \|f^n - h\|_{L^{\Phi_n}(\mathbb{R}^N)} < \varepsilon$. As in the proof of Corollary (5.1.17), using Corollary (5.1.20) this time, we have

$$\sum_n \|k_{1+\epsilon} * f^n - f^n\|_{L^{\Phi_n}(\mathbb{R}^N)}$$

$$\leq \left(C\|k\|_{L^{(1+\epsilon)'}(\mathbb{R}^N)} + 1\right)\varepsilon + \sum_n \|k_{1+\epsilon} * h - h\|_{L^{\Phi_n}(\mathbb{R}^N)}$$

Obviously, $f^n \in L^{1+\epsilon}(\mathbb{R}^N)$. Hence by Lemma (5.1.8), $k_{1+\epsilon} * h \to h$ almost everywhere in $\mathbb{R}^N$, and hence

$$\overline{\Phi}_n(x, |k_{1+\epsilon} * h(x) - h(x)|) \to 0$$

almost everywhere in $\mathbb{R}^N$. Since $\{k_{1+\epsilon} * h - h\}$ is uniformly and there is a compact set S containing all the supports of $k_{1+\epsilon} * h$, $\{\overline{\Phi}_n(x, |k_{1+\epsilon} * h(x) - h(x)|)\}$ is uniformly bounded and S contains all the supports of $\overline{\Phi}_n(x, |k_{1+\epsilon} * h(x) - h(x)|)$. Hence the Lebesgue convergence theorem implies

$$\int_{\mathbb{R}^N} \overline{\Phi}_n(x, |k_{1+\epsilon} * h(x) - h(x)|)dx \to 0$$

as $\epsilon \to -1$. Then, by Lemma (5.1.2), we see that $\|k_{1+\epsilon} * h - h\|_{L^{\Phi_n}(\mathbb{R}^N)} \to 0$ as $\epsilon \to -1$, so that

$$\limsup_{\epsilon \to -1} \sum_n \|k_{1+\epsilon} * f^n - f^n\|_{L^{\Phi_n}(\mathbb{R}^N)} \leq \left(C\|k\|_{L^{(1+\epsilon)'}(\mathbb{R}^N)} + 1\right)\varepsilon,$$

which completes the proof.

Corollary (5.1.22)[205]: Suppose $\Phi_n\big(x, (1+\epsilon)\big)$ satisfies (Φ5) and (Φ6). Let $k \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $k \in L^1(\mathbb{R}^N) \leq 1$. For $f^n \in L^1_{\text{loc}}(\mathbb{R}^N)$, set

$$\sum_n I(f^n; x) = \int_{\frac{\mathbb{R}^N}{B\left(0, \frac{|x|}{2}\right)}} \sum_n |k(\epsilon) f^n(x+\epsilon)| d(x+\epsilon)$$

and

$$\sum_n J(\Phi_n; x) = \int_{\mathbb{R}^N} \sum_n |k(\epsilon)| \overline{\Phi}_n\big((x+\epsilon), |f^n(x+\epsilon)|\big) d(x+\epsilon).$$

Then there exists a constant $C > 0$ (depending on $\|k\|_{L^\infty(\mathbb{R}^N)}$) such that

$$\sum_n \overline{\Phi}_n\big(x, I(f^n; x)\big) \leq C \sum_n \left\{J(f^n; x) + g^n\left(\frac{x}{2}\right)\right\}$$

for all $x \in \mathbb{R}^N$ and $f^n \in L^{\Phi_n}(\mathbb{R}^N)$ with $\sum_n \|f^n\|_{L^{\Phi_n}(\mathbb{R}^N)} \leq 1$, where g^n is the function appearing (Φ6).

Proof. Let $k > 0$. Since $(1+\epsilon) \to \frac{\overline{\Phi}_n(x,(1+\epsilon))}{(1+\epsilon)}$ is non-decreasing,

$$\sum_n J(f^n; x) \leq k \int_{\mathbb{R}^N} |k(\epsilon)| d(x+\epsilon)$$

$$+ k \int_{\frac{\mathbb{R}^N}{B\left(0, \frac{|x|}{2}\right)}} \sum_n \frac{|k(\epsilon)| \overline{\Phi}_n\big((x+\epsilon), |f^n(x+\epsilon)|\big)}{\overline{\Phi}_n\big((x+\epsilon), k\big)} d(x+\epsilon).$$

If $g^n\left(\frac{x}{2}\right) \leq k \leq 1$, then $\sum_n \overline{\Phi}_n(x, k) \leq C \sum_n \overline{\Phi}_n\big((x+\epsilon), k\big)$ for $|x+\epsilon| > \frac{|x|}{2}$ by (Φ6). Hence

$$\sum_n I(f^n; x) \leq k\left(1 + \sum_n \frac{CJ(f^n; x)}{\overline{\Phi}_n(x, k)}\right) \text{ whenever } g^n\left(\frac{x}{2}\right) \leq k \leq 1. \qquad (6)$$

Since $\sum_n J(f^n;x) \le \|k\|_{L^\infty(\mathbb{R}^N)}$, there exists $k_x \in [0,1]$ such that

$$\sum_n \overline{\Phi}_n(x,k_x) = \sum_n \frac{J(f^n;x)}{\|k\|_{L^\infty(\mathbb{R}^N)}} \overline{\Phi}_n(x,1).$$

If $k_x \ge g^n\left(\frac{x}{2}\right)$, then taking $k = k_x$ in (6) ,we have

$$\sum_n I(f^n;x) \le k_x \left(1 + \sum_n \frac{C\|k\|_{L^\infty(\mathbb{R}^N)}}{\overline{\Phi}_n(x,1)}\right) \le k_x,$$

so that

$$\sum_n \overline{\Phi}_n\big(x, I(f^n;x)\big) \le \sum_n \overline{\Phi}_n(x,k_x) \le C \sum_n J(f^n;x).$$

If $k_x < g^n\left(\frac{x}{2}\right)$, then

$$\sum_n J(f^n;\ k_x) = \|k\|_{L^\infty(\mathbb{R}^N)} \sum_n \frac{\overline{\Phi}_n(x,k_x)}{\overline{\Phi}_n(x,1)} \le \sum_n \overline{\Phi}_n\left(x, g^n\left(\frac{x}{2}\right)\right).$$

Hence, taking $k = g^n\left(\frac{x}{2}\right)$ in (6) we have $\sum_n I(f^n;x) \le C g^n\left(\frac{x}{2}\right)$, so that

$$\sum_n \overline{\Phi}_n\big(x, I(f^n;x)\big) \le C \sum_n \overline{\Phi}_n(x,k_x) \le C \sum_n g^n(f^n;x).$$

Hence, we have the assertion of the corollary.

We recall the following result on the boundedness of maximal operator M on $L^{\Phi_n}(\mathbb{R}^N)$ (see [157]) and [11]:

Corollary (5.1.23)[205]: Suppose $\Phi_n\big(x,(1+\epsilon)\big)$ satisfies (Φ5), (Φ6) and ($\Phi 3^*$). Let $(1+\epsilon) = 1+\varepsilon_0$ (>0) and $R>0$. Assume that $k \in L^1(\mathbb{R}^N) \cap L^{(1+\epsilon)'}(\mathrm{B}(0,\mathrm{R}))$ and $|k(x)| \le C_k |x|^{-N}$ for $|x| \ge R$. Then there is a constant $C>0$ such that

$$\sum_n \|k * f^n\|_{L^{\Phi_n}(\mathbb{R}^N)} \le C(\|k\|_{L^1(\mathbb{R}^N)} + \sum_n \|f^n\|_{L^{(1+\epsilon)'}(\mathrm{B}(0,\mathrm{R}))}) \sum_n \|f^n\|_{L^{\Phi_n}(\mathbb{R}^N)}$$

for all $f^n \in L^{\Phi_n}(\mathbb{R}^N)$.

Proof. Let $f^n \in L^{\Phi_n}(\mathbb{R}^N)$ and $(x+\epsilon) \ge 0$. Assume that $\sum_n \|f^n\|_{L^{\Phi_n}(\mathbb{R}^N)} \le 1$ and

$$\|k\|_{L^1(\mathbb{R}^N)} + \sum_n \|f^n\|_{L^{(1+\epsilon)'}(\mathrm{B}(0,\mathrm{R}))} \le 1.$$

Let $k_0 = k\chi_{B(0,R)}$ and $k_\infty = k\chi_{\frac{\mathbb{R}^N}{B(0,R)}}$.

By Corollary (5.1.20),

$$\left\|\sum_n k_0 * f^n\right\|_{L^1(\mathbb{R}^N)} \leq C.$$

Hence it is enough to show that

$$\int_{\mathbb{R}^N} \sum_n \overline{\Phi}_n\big(x, |k_\infty| * f^n(x)\big)\, dx \leq C. \tag{7}$$

Write

$$|k_\infty| * \sum_n f^n(x) = \int_{B\left(0,\frac{|x|}{2}\right)} \sum_n |k_\infty(\epsilon)| f^n(x+\epsilon) d(x+\epsilon)$$

$$+ \int_{\frac{\mathbb{R}^N}{B\left(0,\frac{|x|}{2}\right)}} \sum_n |k_\infty(\epsilon)| f^n(x+\epsilon) d(x+\epsilon).$$

$$= I_1(x) + I_2(x).$$

Since $|k_\infty(\epsilon)| \leq C_k |\epsilon|^{-N}$ and $|\epsilon| \geq \frac{|x|}{2}$ for $|x+\epsilon| \leq \frac{|x|}{2}$,

$$I_1(x) \leq 2^N C_k |x|^{-N} \int_{B\left(0,\frac{|x|}{2}\right)} \sum_n f^n(x+\epsilon) d(x+\epsilon)$$

$$\leq 2^N C_k |x|^{-N} \int_{B\left(0,\frac{3|x|}{2}\right)} \sum_n f^n(x+\epsilon) d(x+\epsilon).$$

Hence,

$$\int_{\mathbb{R}^N} \sum_n \overline{\Phi}_n\big(x, I_1(x)\big) dx \leq C$$

by Lemma (5.1.14).

on the other hand, Corollary (5.1.22),

$$\sum_n \overline{\Phi}_n\big(x, I_2(x)\big) \leq C\left\{|k_\infty| * h(x) + g^n\left(\frac{x}{2}\right)\right\},$$

where $h(x+\epsilon) = \sum_n \overline{\Phi}_n\big((x+\epsilon), f^n(x+\epsilon)\big)$. Since

$$\||k_\infty| * h\|_{L^1(\mathbb{R}^N)} \leq \||k_\infty|\|_{L^1(\mathbb{R}^N)} \|h\|_{L^1(\mathbb{R}^N)} \leq 1$$

and $g^n \in L^1(\mathbb{R}^N)$, it follows that

$$\int_{\mathbb{R}^N} \sum_n \overline{\Phi}_n\big(x, I_2(x)\big)dx \leq C.$$

Hence, we obtain (7), and the proof is complete.

Section (5.2): Riesz Potentials of Functions in Musielak–Orlicz spaces

A famous Trudinger inequality [173] insists that Sobolev functions in $W^{1,N}(G)$ satisfy finite exponential integrability, where G is an open bounded set in $\mathbb{R}^N$ (see also [64, 159, 160, 172]). Great progress on Trudinger type inequalities has been made for Riesz potentials of order α $(0 < \alpha < N)$ in the limiting case $\alpha p = N$ (see e.g. [162−163]). In [160,169] and [171], Trudinger type exponential integrability was studied on Orlicz spaces, as extensions of [162, 163] and [165].

Variable exponent Lebesgue spaces and Sobolev spaces were introduced to discuss nonlinear partial differential equations with non-standard growth conditions (see [154]). Trudinger type exponential integrability on variable exponent Lebesgue spaces $L^{p(\cdot)}$ was investigated in [104, 166] and [167]. For the two variable exponent's space $L^{p(\cdot)}(\log L)^{p(\cdot)}$, see [168]. These spaces are special cases of so-called Musielak–Orlicz spaces [111].

We give a general version of Trudinger type exponential integrability for Riesz potentials of functions in Musielak–Orlicz spaces as an extension of the above results.

Let G be a bounded open set in $\mathbb{R}^N$. Let $d_G = \text{diam } G$.

We consider a function

$$\Phi(x,t) = t\varphi(x,t) : \times [0,\infty) \to [0,\infty)$$

satisfying the following conditions $(\Phi) - (\Phi)$:

$(\Phi 1)$ $\varphi(\cdot,t)$ is measurable on G for each $t \geq 0$ and $\varphi(\cdot,t)$ is continuous on $[0,\infty)$ for each $x \in G$;

$(\Phi 2)$ there exists a constant $A_1 \geq 0$ such that

$$A_1^{-1} \leq \varphi(x,1) \leq A_1 \quad \text{for all } x \in G;$$

$(\Phi 3)$ $\varphi(x,\cdot)$ is uniformly almost increasing, namely there exists a constant on $A_2 \geq 0$ such that

$$\varphi(x,t) \leq A_2\varphi(x,s) \quad \text{for all } x \in G \text{ whenever } 0 \leq t < s;$$

$(\Phi 4)$ there exists a constant on $A_3 \geq 1$ such that

$$\varphi(x,2t) \leq A_3\varphi(x,t) \quad \text{for all } x \in G \text{ and } t > 0.$$

Note that $(\Phi 2)$, $(\Phi 3)$ and $(\Phi 4)$ imply

$$0 < \inf_{x\in G} \phi(x,s) \leq \sup_{x\in G} \phi(x,s) < \infty$$

for each $t > 0$.

If $\Phi(x,\cdot)$ is convex for each $x \in G$, then $(\Phi 3)$ holds with $A_2 = 1$; namely $\varphi(x,.)$ is non-decreasing for each $x \in G$.

Let $\bar{\phi}(x,t) = \sup_{0\le s\le t} \phi(x,s)$ and

for $x \in G$ and $t \geq 0$. Then $\bar{\Phi}(x,\cdot)$ is convex and

$$\frac{1}{2A_3}\Phi(x,t) \leq \bar{\Phi}(x,t) \leq A_2\Phi(x,t)$$

for $x \in G$ and $t \geq 0$. In fact, the first inequality is seen as follows:

$$\bar{\Phi}(x,t) \geq \int_{t/2}^{t} \bar{\varphi}(x,r)dr \geq \frac{1}{2}\varphi(x,t) \geq \frac{1}{2A_3}\Phi(x,t).$$

We shall also consider the following condition:

($\Phi5$) for every $\gamma > 0$, there exists a constant on $B_\gamma \geq 1$ such that

$$\varphi(x,t) \leq B_\gamma \varphi(x,t)$$

whenever $|x-y| \leq \gamma t^{-1/N}$ and $t \geq 1$.

Let $p(\cdot)$ and $q_j(\cdot), j = 1, \dots \dots, k$, be measurable function on G such that

(P1) $$1 \leq p^- \coloneqq \inf_{x\in G} p(x) \leq \sup_{x\in G} p(x) =: p^+ < \infty$$

and

(Q1) $$-\infty < q_j^- \coloneqq \inf_{x\in G} q_j(x) \leq \sup_{x\in G} q_j(x) =: q_j^+ < \infty$$

for all $j = 1, \dots \dots, k$.

set $L_c(t) = \log(c+t)$ for $c \geq e$ and $t \geq 0$, $L_c^{(1)}(t) = L_c(t)$, $L_c^{(1)}(t) = L_c(L_c^{(j+1)}(t))$ and

$$\Phi(x,t) = t^{p(x)} \prod_{j=1}^{k} \left(L_c^{(j)}(t)\right)^{q_j(x)}$$

Then, $\Phi(x,t)$ satisfies $(\Phi1), (\Phi2)$ and $(\Phi4)$. It satisfies $(\Phi3)$ if there is constant $K \geq 0$ such that $K(p(x)-1) + q_j(x) \geq 0$ for all $x \in G$ and $j = 1, \dots \dots, k$; if $p^- > 1$ or $q_j^+ \geq 0$ for all $j = 1, \dots \dots k$.

$\Phi(x,t)$ satisfies $(\Phi5)$ if

(P2) $p(\cdot)$ is log-Hölder continuous, namely

$$|p(x) - p(y)| \leq \frac{C_p}{L_e(1/|x-y|)}$$

with a constant $C_p \geq 0$ and

(Q2) $q_j(\cdot)$ is j-log-Hölder continuous, namely

$$\left|q_j(x) - q_j(y)\right| \le \frac{C_{q_j}}{L_c^{(j)}(1/|x-y|)}$$

with a constant $C_{q_j} \ge 0, j = 1, \dots \dots, k$.

Given $\Phi(x,t)$ as above, the associated Musielak−Orlicz space

$$L^{\Phi}(G) = \left\{ f \in L^1_{loc}(G); \int_G \Phi(y, |f(y)|)dy < \infty \right\}$$

is a Banch space with respect to the norm

$$\|f\|_{L^{\Phi}(G)} = \inf\left\{ \lambda > 0; \int_G \overline{\Phi}(y, |f(y)|/\lambda)dy \le 1 \right\}$$

(cf. [111]).

Let C denote various constants independent of the variables in question and $C(a, b, \dots)$ be a constant that depends on $a, b, \dots \dots$

We denote by $B(x,r)$ the open ball centered at x of radius r. For a measurable set E, we denote by $|E|$ the Lebesgue measure of E.

For a locally integrable function f on G, the Hardy–Littlewood maximal function Mf is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)\cap G} |f(y)|dy.$$

We know the following of maximal operator on $L^{\Phi}(G)$.

Lemma (5.2.1)[174]: (See [155].) Suppose that $\Phi(x,t)$ satisfies ($\Phi5$) and further assume:

($\Phi3^*$) $t \longmapsto t^{-\varepsilon_0}\varphi(x,t)$ is uniformly almost on $(0. \infty)$ for some $\varepsilon_0 > 0$.

Then the maximal operator M is bounded from $L^{\Phi}(G)$ into itself, namely, there is a constant $C > 0$ such that

$$\|Mf\|_{L^{\Phi}(G)} \le C\|f\|_{L^{\Phi}(G)}$$

for all $f \in L^{\Phi}(G)$.

We consider the function

$$\gamma(x,t): G \times (0, d_G) \to [0. \infty)$$

satisfying the following conditions ($\gamma1$) and ($\gamma2$):

($\gamma1$) $\gamma(\cdot, t)$ is measurable on G for each $0 < t < d_G$ and $\gamma(x, \cdot)$ is continuous on $(0, d_G)$ for each $x \in G$;

($\gamma2$) there exists a constant $B_0 \ge 0$ such that

$$B_0^{-1} \le \gamma(x,t) \le B_0 t^{-N} \quad \text{for all } x \in G \text{ whenever } 0 < t < d_G.$$

further we consider the function

$$\Gamma_\alpha(x,t): G \times [0,\infty) \to [0,\infty)$$

satisfying the following conditions $(\Gamma 1)$ and $(\Gamma 2)$:

$(\Gamma 1)$ $\Gamma_\alpha(\cdot,t)$ is measurable on G for each $t \geq 0$ and $\Gamma_\alpha(x,\cdot)$ is continuous on $[0,\infty)$ for each $x \in G$;

$(\Gamma 2)$ $\Gamma_\alpha(x,\cdot)$ is uniformly almost increasing, namely there exists a constant $B_1 \geq 1$ such that

$$\Gamma_\alpha(x,t) \leq B_1 \Gamma_\alpha(x,s) \quad \text{for all } x \in G \text{ whenever } 0 < t < s;,$$

$(\Gamma 3)$ there exists a constant $\alpha_0 > 0, B_2 \geq 1$ and $B_3 \geq 1$ such that

$$t^{\alpha-N}\varphi(x,\gamma(x,t))^{-1} \leq B_2 \Gamma_\alpha(x,1/t)$$

for all $x \in G$ and $\alpha \geq \alpha_0$ whenever $0 < t < d_G$ and

$$\int_t^{d_G} \rho^\alpha \gamma(x,\rho)\frac{d\rho}{\rho} \leq B_3 \Gamma_\alpha(x,1/t)$$

for all $x \in G$, $0 < t < d_G/2$ and $\alpha \geq \alpha_0$.

Lemma (5.2.2)[174]: Suppose that $\Phi(x,t)$ satisfies $(\Phi 5)$ and $\alpha_0 \leq \alpha < N$. Then there exists a constant $C > 0$ such that

$$\int_{G \setminus B(x,\delta)} |x-y|^{\alpha-N} f(y)dy \leq C\Gamma_\alpha\left(x,\frac{1}{\delta}\right)$$

for all $x \in G$, $0 < \delta < d_G/2$ and nonnegative $f \in L^\Phi(G)$ with $\|f\|_{L^\Phi(G)} \leq 1$.

Proof. Let f be a nonnegative measurable function with $\|f\|_{L^\Phi(G)} \leq 1$. Since

$$\varphi(y,\gamma(x,|x-y|))^{-1} \leq B'\varphi\big(y,\gamma(x,|x-y|)\big)^{-1}.$$

with some constant $B' > 0$ by $(\gamma 2)$, $(\Phi 3)$, $(\Phi 4)$, and $(\Phi 5)$, we have by $(\Phi 3)$, $(\Gamma 2)$ and $(\Gamma 3)$

$$\begin{aligned}
&\int_{G \setminus B(x,\delta)} |x-y|^{\alpha-N} f(y)dy \\
&\leq \int_{G \setminus B(x,\delta)} |x-y|^{\alpha-N} f(y)\gamma(x,|x-y|)dy \\
&+A_2 \int_{G \setminus B(x,\delta)} |x-y|^{\alpha-N} f(y)\frac{\varphi(y,f(y))}{\varphi(y,\gamma(x,|x-y|))}\,dy \\
&\leq \int_\delta^{d_G} \rho^\alpha \gamma(x,\rho)\frac{d\rho}{\rho} \\
&+A_2 B' \int_{G \setminus B(x,\delta)} |x-y|^{\alpha-N}\varphi\big(y,\gamma(x,|x-y|)\big)^{-1}\Phi f(y,f(y))dy
\end{aligned}$$

$$\leq CB_3\Gamma_\alpha(x,1/\delta)+A_2B_1B_2B'\Gamma_\alpha(x,1/\delta)\int_{G\setminus B(x,\delta)}\Phi f(y,f(y))dy$$
$$\leq (CB_3+A_2B_1B_2B')\Gamma_\alpha(x,1/\delta).$$

Thus, we obtain the required results.

Lemma (5.2.3)[174]: Let $\alpha\geq\alpha_0$. Then there exists a constant $C'>0$ such that $\Gamma_\alpha(x,1/d_G)\geq C'$ for all $x\in G$.

Proof. By $(\Gamma 3)$ and $(\gamma 2)$,

$$\Gamma_\alpha(x,2/d_G)\geq B_3^{-1}\int_{d_G/2}^{d_G}\rho^\alpha\gamma(x,\rho)\frac{d\rho}{\rho}\geq B_0^{-1}B_3^{-1}\int_{d_G/2}^{d_G}\rho^\alpha\frac{d\rho}{\rho}$$
$$=B_0^{-1}B_3^{-1}\alpha^{-1}d_G^\alpha(1-2^{-\alpha})=C'$$

for all $x\in G$, as required.

Lemma (5.2.4)[174]: (See [170].) Suppose $\Gamma_\alpha(x,t)$ satisfies the uniform log-type condition:

$(\Gamma_{\log})$ there exists a constant $c_\Gamma>0$ such that

$$c_\Gamma^{-1}\Gamma_\alpha(x,s)\leq\Gamma_\alpha(x,s^2)\leq c_\Gamma\Gamma_\alpha(x,s)$$

for all $x\in G$ and $s>0$.

Then for every $c>1$, then there exists $C>0$ such that $\Gamma_\alpha(x,c\,s)\leq C\Gamma_\alpha(x,s)$ for all $x\in G$ and $s>0$.

For $0<\alpha<N$, we define the Riesz potential of order α for a locally integrable function f on G by

$$I_\alpha f(x)=\int_G|x-y|^{\alpha-N}f(y)dy.$$

Theorem (5.2.5)[174]: Assume that $\Phi(x,t)$ satisfies $(\Phi 5)$ and $(\Phi 3^*)$. Suppose that $\Gamma_\alpha(x,t)$ satisfies $(\Gamma_{\log})$. For each $x\in G$, let $\gamma_\alpha(x)=\sup_{s>0}\Gamma_\alpha(x,s)$. Suppose $\psi_\alpha(x,t):G\times[0,\infty)\to[0,\infty]$ satisfies the following conditions:

$(\psi_\alpha 1)$ $\psi_\alpha(\cdot,t)$ is measurable on G for each $f\in[0,\infty)$; $\psi_\alpha(x,.)$ is continuous on $[0,\infty)$ for $t<s$;

$(\psi_\alpha 2)$ there is a constant $B_4\geq 1$ such that $\psi_\alpha(x,t)\leq\psi_\alpha(x,B_4 s)$ for all $x\in G$ whenever $0<t<s$;

$(\psi_\alpha 3)$ there are constants $B_5,B_6\geq 1$ and $t_0>0$ such that $\psi_\alpha(x,\Gamma_\alpha(x,t)/B_5)\leq B_6 t$ for all $x\in G$ and $t\geq t_0$.

Then there exists constants $c_1,c_2>0$ such that $I_\alpha f(x)/c_1<\gamma_\alpha(x)$ for $a.e.\,x\in G$ and

$$\int_G\psi_\alpha\left(x,\frac{I_\alpha f(x)}{c_1}\right)dx\leq c_2$$

for all $\alpha_0\leq\alpha<N$ and $f\geq 0$ satisfing $\|f\|_{L^\Phi(G)}\leq 1$.

Proof. Let $f \geq 0$ and $\|f\|_{L^{\Phi}(G)} \leq 1$. Note from Lemma (5.2.3) that

$$\int_G Mf(x)dx \leq |G| + A_1 A_2 \int_G \Phi\big(x, Mf(y)\big)dx \leq C_M \quad (8)$$

Fix $x \in G$. For $0 < \delta \leq d_G/2$, Lemma (5.2.2) implies

$$I_\alpha f(x) = \int_{B(x,\delta)} |x-y|^{\alpha-N} f(y)dy + \int_{G\setminus B(x,\delta)} |x-y|^{\alpha-N} f(y)dy$$
$$\leq C\left\{Mf(x) + \Gamma_\alpha\left(x, \frac{1}{\delta}\right)\right\}$$

with constants $C > 0$ independent of x.

If $Mf(x) \leq 2/d_G$, then we take $\delta = d_G/2$. Then, by Lemma (5.2.3)

$$I_\alpha f(x) \leq \Gamma_\alpha\left(x, \frac{1}{\delta}\right).$$

By Lemma (5.2.4), there exists $C_1^* > 0$ indepentent x such that

$$I_\alpha f(x) \leq C_1^* \Gamma_\alpha(x, t_0) \quad \text{if} \quad Mf(x) \leq 2/d_G. \quad (9)$$

Next, suppose $2/d_G < Mf(x) < \infty$. Let $m = \sup_{s \geq 2/d_G, x \in G} \Gamma_\alpha(x,s)/s$. By ($\Gamma_{\log}$), $m < \infty$. Define δ by

$$\delta^\alpha = \frac{(2/d_G)^\alpha}{m} \Gamma_\alpha(x, Mf(x))(Mf(x))^{-1}.$$

Since $\Gamma_\alpha(x, Mf(x))(Mf(x))^{-1} \leq m, 0 < \delta \leq d_G/2$. Lemma (5.2.3)

$$\frac{1}{\delta} \leq C\Gamma_\alpha\big(x, Mf(x)\big)^{-\frac{1}{\alpha}}\big(Mf(x)\big)^{-\frac{1}{\alpha}}$$

$$\leq C\Gamma_\alpha\left(x, \frac{d_G}{2}\right)^{-\frac{1}{\alpha}}\big(Mf(x)\big)^{\frac{1}{\alpha}} \leq \big(Mf(x)\big)^{\frac{1}{\alpha}}.$$

Hence, using ($\Gamma_{\log}$) and Lemma (5.2.4), we obtain

$$\Gamma_\alpha\left(x, \frac{1}{\delta}\right) \leq C\Gamma_\alpha\big(x, (Mf(x))^{1/\alpha}\big) \leq C\Gamma_\alpha(Mf(x)).$$

By Lemma (5.2.4) again, see that there exists a constant $C_1^* > 0$ indepentent x such that

$$I_\alpha f(x) \leq C_1^* \Gamma_\alpha\left(x, \frac{t_0 d_G}{2} Mf(x)\right) \ \text{if} \ 2/d_G < Mf(x) < \infty \quad (10)$$

Now, let $c_1 = B_4 B_5 \max(C_1^*, C_2^*)$. Then, by (9) and (10),

$$\frac{I_\alpha f(x)}{c_1} \leq \frac{1}{B_4 B_5} \max\left\{\Gamma_\alpha(x, t_0) + \Gamma_\alpha\left(x, \frac{t_0 d_G}{2} Mf(x)\right)\right\}$$

whenever $Mf(x) < \infty$. Since $Mf(x) < \infty$ for $a.e.\, x \in G$ by Lemma (5.2.1), $I_\alpha f(x)/c_1 < \gamma_\alpha(x)$ $a.e.\, x \in G$, by ($\psi_\alpha 2$) and ($\psi_\alpha 3$), we have

$$\psi_\alpha\left(x,\frac{I_\alpha f(x)}{c_1}\right)$$

$$\leq \max\left\{\psi_\alpha(x,\Gamma_\alpha(x,t_0)/B_5),\psi_\alpha\left(x,\Gamma_\alpha\left(x,\frac{t_0 d_G}{2}Mf(x)\right)/B_5\right)\right\}$$

$$\leq B_6 t_0+\frac{B_6 t_0 d_G}{2}Mf(x)$$

for $a.e.\, x\in G$. Thus, we have by (6)

$$\int_G \psi_\alpha\left(x,\frac{I_\alpha f(x)}{c_1}\right)dx\leq B_6 t_0|G|+\frac{B_6 t_0 d_G}{2}\int_G Mf(x)dx$$

$$\leq B_6 t_0|G|+\frac{B_6 t_0 d_G}{2}=c_2.$$

Appling Theorem (5.2.5) to special Φ given, we obtain the following corollary.

Corollary (5.2.6)[174]: Let Φ be as above:

(i) Suppose there exists an integer $1\leq j_0\leq k$ such that

and

$$\inf_{x\in G}\left(p(x)-q_{j_0}(x)-1\right)>0 \tag{11}$$

$$\sup_{x\in G}\left(p(x)-q_j(x)-1\right)\leq 0 \tag{12}$$

for all $j\leq j_0-1$ in case $j_0\geq 2$. Then there exist constant $c_{1,}c_2>0$ such that

$$\int_G E_+^{(j_0)}\left(\left(\frac{I_\alpha f(x)}{c_1}\right)^{p(x)/\left(p(x)-q_{j_0}(x)-1\right)}\right.$$

$$\left.\times\prod_{j=1}^{k-j_0}\left(L_e^{(j)}\left(\frac{I_\alpha f(x)}{c_1}\right)\right)^{q_{j_0}+j(x)/\left(p(x)-q_{j_0}(x)-1\right)}\right)\leq c_2$$

for all $N/p^-\leq\alpha<N$ and $f\geq 0$ satisfing $\|f\|_{L^\Phi(G)}\leq 1$, where $E^{(1)}(t)=e^t-e$, $E^{(j+1)}(t)=\exp(E^{(j)}(t))-e$ and $\max(E^{(j)}(t),0)$.

(ii) If

$$\sup_{x\in G}\left(p(x)-q_j(x)-1\right)\leq 0$$

for all $j=1,\ldots\ldots,k$, then there exist constant $c_{1,}c_2>0$ such that

$$\int_G E^{(k+1)}\int_G\left(\left(\frac{I_\alpha f(x)}{c_1}\right)^{p(x)/(p(x)-1)}\right)dx\leq c_2$$

for all $N/p^- \le \alpha < N$ and $f \ge 0$ satisfing $\|f\|_{L^{\Phi}(G)} \le 1$.

Proof. We show the case (i). In this case, set

$$\gamma(x,t) = t^{-N/p(x)} \left(\prod_{j=1}^{j_0-1} \left[L_e^{(j)}(1/t \right]^{-1} \right) \left[L_e^{(j_0)}(1/t \right]^{-(q_{j_0}(x)+1)/p(x)} \times \left(\prod_{j=j_0+1}^{k} \left[L_e^{(j)}(1/t) \right]^{-q_j(x)/p(x)} \right)$$

and

$$\Gamma_\alpha(x,t) = \left[L_e^{(j_0)}(1/t) \right]^{(p(x)-q_{j_0}(x)-1)/p(x)} \left(\prod_{j=j_0+1}^{k} \left[L_e^{(j)}(1/t) \right]^{-q_j(x)/p(x)} \right).$$

Here note that $\gamma(x,t)$ satisfies $(\gamma 2)$ and $\Gamma_\alpha(x,t)$ is uniformly almost increasing on t and satisfies and $(\Gamma_{\log})$ by (11). We have $N/p^- \le \alpha$ and (12)

$$\begin{aligned} t^{\alpha-N}\varphi(x,\gamma(x,t))^{-1} &\le Ct^{-N/p(x)} \left(\prod_{j=1}^{j_0-1} \left[L_e^{(j)}(1/t) \right]^{p(x)-q_j(x)-1} \right) \\ &\times \left[L_e^{(j_0)}(1/t) \right]^{(p(x)-q_{j_0}(x)-1)/p(x)} \left(\prod_{j=j_0+1}^{k} \left[L_e^{(j)}(1/t) \right]^{-q_j(x)/p(x)} \right) \\ &\le C \left[L_e^{(j_0)}(1/t) \right]^{(p(x)-q_{j_0}(x)-1)/p(x)} \left(\prod_{j=j_0+1}^{k} \left[L_e^{(j)}(1/t) \right]^{-q_j(x)/p(x)} \right) \\ &= C\Gamma_\alpha(x,1/t) \end{aligned}$$

for all $x \in G$ and $\alpha_0 = N/p^- \le \alpha < N$ whenever $0 < t < d_G$. By (11), we find $\varepsilon_0 > 0$ such that $\inf_{x\in G}\{1-(q_{j_0}+1)/p(x)\} > \varepsilon_0$. We see from $N/p^- \le \alpha$, (11) and (12) that

$$\begin{aligned} \int_t^{d_G} \rho^\alpha \gamma(x,\rho) \frac{d\rho}{\rho} &\le C \int_t^{d_G} \left(\prod_{j=1}^{j_0-1} \left[L_e^{(j)}(1/\rho) \right]^{-1} \right) \left[L_e^{(j_0)}(1/\rho) \right]^{-(q_{j_0}(x)+1)/p(x)} \\ &\times \left(\prod_{j=j_0+1}^{k} \left[L_e^{(j)}(1/\rho) \right]^{-q_j(x)/p(x)} \right) \frac{d\rho}{\rho} \end{aligned}$$

$$\le C\left[L_e^{(j_0)}(1/t)\right]^{1-(q_{j_0}(x)+1)/p(x)-\varepsilon_0}\left(\prod_{j=j_0+1}^{k}\left[L_e^{(j)}(1/t)\right]^{-q_j(x)/p(x)}\right)$$

$$\times\int_t^{d_G}\left(\prod_{j=1}^{j_0-1}\left[L_e^{(j)}(1/\rho)\right]^{-1}\right)\left[L_e^{(j_0)}(1/\rho)\right]^{-1+\varepsilon_0}\frac{d\rho}{\rho}$$

$$\le C\Gamma_\alpha(x,1/t)$$

for all $0 < t \le d_G/2$ and $N/p^- \le \alpha < N$. Hence, $\Gamma_\alpha(x,t)$ satisfies $(\Gamma 3)$.

Now, set

$$\psi(x,t) = t^{p(x)/(p(x)-q_{j_0}(x)-1)}\prod_{i=1}^{k-j_0}\left[L_e^{(i)}(1/\rho)\right]^{q_{j_0}+i(x)/(p(x)-q_{j_0}(x)-1)}$$

for all $x \in G$ and $t > 0$. Then

$$\psi\big(x,\Gamma_\alpha(x,s)\big) \le C_1 L_e^{(j_0)}(s)$$

for $s > 0$.

Since $\inf_{x\in G} p(x)/p(x) - q_{j_0} - 1)/p(x) > 0$, there are constants $0 < \theta \le 1$ and $C_2 \ge 1$ such that

$$\psi(x,ct) \le C_2 c^\theta \psi(x,t) \tag{13}$$

for all $x \in G$, $t > 0$ and $0 < c \le 1$. Hence, choosing $B \ge 1$ such that $C_1C_2B^{-\theta} \le 1$, we have

$$\psi(x,\Gamma_\alpha(x,s)/B) \le C_2B^{-\theta}\psi\big(x,\Gamma_\alpha(x,s)\big) \le C_2B^{-\theta}C_1L_e^{(j_0)}(s) \le L_e^{(j_0)}(s)$$

for $s > 0$. Thus,

$$E^{(j_0)}\big(\psi(x,\Gamma_\alpha(x,s)/B)\big) \le s \quad \text{for } s > 0. \tag{14}$$

Let $u_0 > 0$ be unique solution of equation $e^u - e = u$. Then $E^{(1)}(u) \ge u_0$ if and only if $u \ge u_0$. Choose $t_0 > 0$ such that $\psi(x,t) \ge u_0$ for $t \ge t_0$ and define

$$\psi(x,t) = \begin{cases} E^{(j_0)}\psi(x,t) & \text{for } \ t \ge t_0, \\ \psi(x,t_0)\dfrac{t}{t_0} & \text{for } \ 0 < t < t_0. \end{cases}$$

Noting

$$\psi(x,t) = \psi\left(x,\frac{t}{C_2^{1/\theta}s}C_2^{1/\theta}s\right) \le \psi\left(x,C_2^{1/\theta}s\right)$$

for $0 < t \le s$ by (13), $\psi(x,t)$ satisfies $(\psi_\alpha 1), (\psi_\alpha 2)$ (with $B_4 = C_2^{1/\theta}$, say) and $(\psi_\alpha 3)$, in view of (13) and (14)

Thus Theorem (5.2.5) implies the existence of constants $c_1, C_3 > 0$ such that

$$\int_t^{d_G} \psi\left(x, \frac{I_\alpha f(x)}{c_1}\right) dx \le C_3$$

for all $N/p^- \le \alpha < N$ and $f \ge 0$ satisfing $\|f\|_{L^\Phi(G)} \le 1$, which shows the assertion of (i).

In the case (ii), setting

$$\gamma(x,t) = t^{-N/p(x)} \left(\prod_{j=1}^{k} \left[L_e^{(j)}(1/t) \right]^{-1} \right) \left[L_e^{(k+1)}(1/t) \right]^{-1/p(x)},$$

$$\Gamma_\alpha(x,t) = \left[L_e^{(k+1)}(1/t) \right]^{1-1/p(x)}$$

and

$$\psi(x,t) = t^{p(x)/(p(x)-1)},$$

the above discussion yields the required result.

Section (5.3): Generalized Orlicz spaces

Generalized Orlicz spaces $L^{\varphi(\cdot)}$ have been studied since the 1940's. A major synthesis of functional analysis in these spaces is given in Musielak [150] from 1983 and so the spaces have also been called Musielak–Orlicz spaces. These spaces are similar to Orlicz spaces, but defined by a more general function $\varphi(x,t)$ which may vary with the location in space: the norm is defined by means of the integral

$$\int_{\mathbb{R}^n} \varphi(x, |f(x)|)\, dx,$$

whereas in an Orlicz spaces φ would be independent of $x, \varphi(|f(x)|)$. In the special case $\varphi(t) = t^p$ we obtain the Lebesgue space L^p.

Minimization problems in the calculus of variations have had a similar course of generalization (e.g. 184, 192]): from

$$\min_u \int |\nabla_u|^2 dx \;\text{ to }\; \min_u \int |\nabla_u|^p dx \;\text{ to }\; \min_u \int \varphi(x, |\nabla_u|) dx.$$

Usually, the function φ is assumed to have p-growth conditions, i.e. $\varphi(x,t) \approx t^p$ uniformly. This restriction means that the full complexity of the minimization problem is avoided.

The special case $\varphi(x,t) = t^{p(x)}$, so-called variable exponent spaces $L^{p(\cdot)}$, and corresponding differential equations with non-standard growth have been studied [154, 181, 187]. The spaces were introduced by Orlicz already in 1931 [151], but the field lay dormant for a long time. Some 70 years later, key results in harmonic analysis (e.g., [4, 36, 193]) and regularity theory (e.g., [2, 180]) were established.

To being a natural generalization, which covers results from both variable exponent and Orlicz spaces, the study of generalized Orlicz spaces can be motivated by applications to differential equations, image processing and fluid dynamics.

Chen, Levine and Rao [73] introduced a model for image restoration based on a particular type of generalized Φ-function:

$$\varphi(x,t) = \begin{cases} \dfrac{1}{q(x)} t^{q(x)}, & \text{when } t \leq \beta \\ t - \beta + \dfrac{1}{q(x)} \beta^{q(x)}, & \text{when } t \geq \beta. \end{cases}$$

Since they only consider a bounded domain, the space needed actually turns out to be L^1 (or, more precisely, BV), see Proposition(5.3.13). In [188] we analyzed the $L^{p(\cdot)}$-variant of this model, Alaouia, Nabilab and Altanjia [175] have considered a general structure PDE in the image processing context, but again work in BV.

Wróblewska-Kamińska [195] has studied fluid dynamics models with generalized Orlicz-type structure conditions, and Świerczewska-Gwiazda [196] studied existence of solutions to parabolic equations with generalized Orlicz growth. Giannetti and Passarelli di Napoli [183] and Baroni, Colombo and Mingione [176, 177, 179] studied the regularity of solutions to the minimization problems

$$\min_u \int |\nabla_u|^{p(x)} \log(e + |\nabla_u|)\, dx \quad \text{and} \quad \min_u \int |\nabla_u|^p + a(x)|\nabla_u|^q dx,$$

respectively. The regularity of minimizers depends on the regularity of the exponents p and q, and the weight a.

Giannetti and Passarelli di Napoli studied a very special form of functional. Also in the function space setting the first steps from $L^{p(\cdot)}$ were Φ-functions of type $t^{p(\cdot)} \log(e+t)^{q(\cdot)}$ which were studied, e.g., [114, 149]. Hopefully, the tools presented in this will allow the research community to bypass the stage of special log-type variants in the study of PDE and move directly to the general form, including, among others, those studied by Colombo and Mingione.

A key tool for harmonic analysis is the (Hardy–Littlewood) maximal operator M. Maeda, Mizuta, Ohno and Shimomura [174, 190, 191] were first to study it in $L^{\varphi(\cdot)}$, with somewhat heavy machinery. Their result on the boundedness of M was generalized by Hästö [188] by removing unnecessary assumptions and simplifying the proof.

The Sobolev embedding has been studied in generalized Orlicz spaces by Fan [182]. He uses a reduction to the $W^{1,1}$-case based on direct differentiation of the Φ-function. This leads to extraneous assumptions concerning the derivative φ', we prove the Sobolev embedding by Hedberg's method, establishing the boundedness of the Riesz potential. A similar approach was used in [190]. The proof is more versatile and requires fewer assumptions than the previously known ones, and provide a new perspective even in Orlicz spaces. We hope that our simple and clear results and techniques will allow most of the results that have been derived in $L^{p(\cdot)}$ over the past 15 years to be established in $L^{\varphi(\cdot)}$ as well.

The function $f \lesssim g$ means that there exists a constant $C > 0$ such that $f \leq Cg$. The notation $f \approx g$ means that $f \lesssim g \lesssim f$. The space $A \cap B$ is endowed with norm $\|f\|_{A\cap B} = \max\{\|f\|_A, \|f\|_B\}$. For a real function f we denote

$$f(x^-) := \lim_{\varepsilon\to 0^+} f(x-\varepsilon) \text{ and } f(x^+) := \lim_{\varepsilon\to 0^+} f(x+\varepsilon).$$

By $L^0(\mathbb{R}^n)$ we denote the set of (Lebesgue) measurable functions on $\mathbb{R}^n$. The (Hardy–Littlewood) maximal operator is defined for $f \in L^0(\mathbb{R}^n)$ by

$$Mf(x) := \sup_{r>0} \int_{B(x,r)} |f(x)|dy$$

where $B(x,r)$ is the open ball with center x and radius r, and $⨍$ denote the average integral.

We recall some definitions pertaining to generalized Orlicz spaces. For proofs and further properties see [154] and [150].

Definition (5.3.1)[196]: A convex function $\varphi \in C([0,\infty);[0,\infty])$ with $\varphi(0) = \varphi(0^+) = 0$, and $\lim_{t\to\infty} \varphi(t) = \infty$ is called a Φ-function. This set of Φ-functions is denoted by Φ.

Instead of the usual left-continuity, we have assumed that every Φ-function is continuous in the compactification $[0,\infty]$. This is not restriction as every function satisfying the former condition is equivalent to one satisfying the latter, see [185]. Recall that two functions φ and ψ are equivalent, $\varphi \simeq \psi$, equivalent if there exists $L \geq 1$ such that $\psi\left(\frac{t}{L}\right) \leq \varphi(t) \leq \psi(Lt)$ for relevant all t. Equivalent Φ-functions give rise to the same space with comparable norms.

Note that every Φ-function is increasing on $[0,\infty)$ and strictly increasing on $\{x : \varphi(x) \in [0,\infty)\}$. By φ^{-1} we denote the left-continuous inverse of $\varphi \in \Phi$,

$$\varphi^{-1}(\tau) := \inf\{t \geq 0 : \varphi(t) \geq \tau\}.$$

It follows directly from this definition that $\varphi^{-1}(\varphi(t)) \leq t$ and equality holds if φ is strictly increasing. To be more precise, if $t_0 := \max\{t|\varphi(t) = 0\}$ and $t_\infty := \max\{t|\varphi(t) < \infty\}$, then

$$\varphi^{-1}((t)) = \begin{cases} 0, & t \leq t_0 \\ t, & t_0 < t \leq t_\infty, \\ t_\infty, & t \geq t_\infty. \end{cases} \tag{15}$$

Note $\varphi^{-1}(\varphi(t)) = t$ if $\varphi(t) \in (0,\infty)$. In the opposite order thing work better, since the continuous of φ implies that

$$\varphi(\varphi^{-1}(s)) = s. \tag{16}$$

Note that $\varphi \simeq \psi$ if and only if $\varphi^{-1} \approx \psi^{-1}$.

If $\varphi \simeq \psi$, then by convexity $\varphi \simeq \psi$ we say that φ is doubling if $\varphi(2t) \leq A\varphi(t)$ for every $t > 0$. For a doubling Φ-function $\simeq$ and $\approx$ are equivalent. A Φ-function can be represented as

$$\varphi(t) = \int_0^t \varphi'(s)ds$$

in the set $\{\varphi(t) < \infty\}$, where φ' is the right-continuous right-derivative of the convex function φ.

Definition (5.3.2)[196]: The set $\Phi(\mathbb{R}^n)$ consists of $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ with

(i) $\varphi(y,\cdot) \in \Phi$ for every $y \in \mathbb{R}^n$; and

(ii) $\varphi(\cdot,t) \in L^0(\mathbb{R}^n)$ for every $t \geq 0$.

Also the function in $\Phi(\mathbb{R}^n)$ will be called Φ-functions. In sub- and superscripts the dependence on x will be emphasized by $\varphi(\cdot)$: L^{φ}(Orlicz) vs $L^{\varphi(\cdot)}$ (generalized Orlicz).

Properties and definitions of Φ-functions carry over to generalized Φ-functions point-wise. If

$$\varphi^{-1}(x,\tau) \coloneqq \inf\{t \geq 0 : \varphi(x,t) \geq \tau\}$$

is the left-continuous inverse with respect to the second parameter.

Definition (5.3.3)[196]: Let $\varphi \in \Phi(\mathbb{R}^n)$ and define the modular $\varrho_{\varphi(\cdot)}$ for $f \in L^0(\mathbb{R}^n)$ by

$$\varrho_{\varphi(\cdot)}(f) \coloneqq \int_{\mathbb{R}^n} \varphi(x,|f(x)|)dx.$$

The generalized Orlicz space, also called Musielak−Orlicz space, is defined as the set

$$L^{\varphi(\cdot)}(\mathbb{R}^n) = \left\{f \in L^0(\mathbb{R}^n) : \lim_{\lambda\to 0} \varrho_{\varphi(\cdot)}(\lambda f) = 0\right\}$$

equipped with the (Luxemburg) norm

$$\|f\|_{\varphi(\cdot)} \coloneqq \inf\left\{\lambda > 0 : \varrho_{\varphi(\cdot)}\left(\frac{f}{\lambda}\right) \leq 1\right\}.$$

A problem when modifying Φ-funcyions is that we easily move out of the domain of convex functions. The next lemma often allows us to rectify this.

Lemma (5.3.4)[196]: (Lemma (5.3.6), [188]). Let $\varphi : [0,\infty) \to [0,\infty]$ be a left-continuous function with $\varphi(0) = \varphi(0^+) = 0$, and $\lim_{t\to\infty} \varphi(t) = \infty$. If $s \mapsto \frac{\varphi(s)}{s}$ increasing, then φ is equivalent to a convex function $\psi \in \Phi$.

Define $\varphi_B^-(t) \coloneqq \inf_{x\in B} \varphi(x,t)$ and $\varphi_B^+(t) \coloneqq \sup_{x\in B} \varphi(x,t)$. We state three assumption, which together imply the boundedness of the maximal operator [188].

(A0M) There exists $\beta > 0$ such that $\varphi(x,\beta) \leq 1$ and $\varphi(x,1) \geq 1$ for all $f \in \mathbb{R}^n$.

(A1M) There exists $\beta \in (0,1)$ such that

$$\varphi_B^-(\beta t) \leq \varphi_B^+(t)$$

for every $t \in \left[1, (\varphi_B^-)^{-1}\left(\frac{1}{|B|}\right)\right]$ and every ball B with $1/|B| \geq \varphi_B^-(1)$.

(A2M) There exists $\beta > 0$ and $h \in L^1_{\text{weak}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that, for every $t \in [0,1]$,

$$\varphi(x,\beta t) \lesssim \varphi(y,t) + h(x) + h(y).$$

Theorem (5.3.5)[196]: (Theorem (5.3.16), [188]). Let $\varphi \in \Phi(\mathbb{R}^n)$ satisfying assumptions (A0M)−(A2M), and assume that there exists $\gamma > 1$ such that $s \mapsto s^{-\gamma}\varphi(x,s)$ is increasing for every $x \in \mathbb{R}^n$. Then

$$M : L^{\varphi(\cdot)}(\mathbb{R}^n) \to L^{\varphi(\cdot)}(\mathbb{R}^n)$$

is bounded.

Note that the assumption that $s \mapsto s^{-\gamma}\varphi(x,s)$ is increasing, is a natural generalization of the Lebesgue space condition $p > 1$.

Some examples of generalized Φ-functions:

$$\varphi_1(x,t) = t^{p(x)}\log(1+t), \quad \varphi_2(x,t) = t^p + a(x)^q, \quad \varphi_3(x,t) = e^{p(x)t} - 1.$$

The first and second Φ-functions have been recently studied in [179, 183], while $\varphi_3$is an example of non-doubling Φ-function.

The boundedness of the maximal operator in [188] covers all of them, as do the auxiliary result, including normalization and duality. For the Riesz potential we need to assume that $t^{\frac{\varepsilon-n}{\alpha}}\varphi(t)$ is decreasing. This is a natural generalization of the Lebesgue space condition $p<n$, and it implies that φ is doubling (with constant $2^{\frac{n-\varepsilon}{\alpha}}$).

For the study of generalized Orlicz spaces, we need three main assumptions, which are variants of (AxM).

(A0) $\varphi^{-1}(x,1)\approx 1$.

(A1) there exists $\beta\in(0,1)$ such that $\beta\varphi^{-1}(x,t)\le\varphi^{-1}(y,t)$ for every $t\in\left[1,\frac{1}{|B|}\right]$, every $x,y\in B$ and every ball B with $|B|\le 1$.

(A2) $L^{\varphi(.)}(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)=L^{\varphi_\infty}(\mathbb{R}^n)\to L^\infty(\mathbb{R}^n)$, with $\varphi_\infty:=\limsup_{|x|\to\infty}\varphi(x,t)$.

We elaborate on those and add some technical details. Recalling that $\varphi\simeq\psi$ if and only if $\varphi^{-1}\approx\psi^{-1}$, we establish the following invariance.

Lemma (5.3.6)[196]: These assumptions are invariant under equivalence of Φ-funcyions, i.e. if $\varphi\simeq\psi$, then φ satisfies (Ax) if and only if ψ does.

We convert in three steps the original φ function to an equivalent Φ-funcyion $\bar{\varphi}$ which is more regular. Let us investigate each assumption in turn.

We study relation between (A0M) and (A0).

Lemma (5.3.7)[196]: Assumption (A0M) implies (A0).

Proof. By the definition of φ^{-1}, the inequality $\varphi(x,1)\ge 1$ yield $\varphi^{-1}(x,1)\le 1$. If $\varphi(x,\beta)<1$, then $\varphi^{-1}(x,1)\ge\beta$. If $\varphi(x,\beta)=1$, then by convexity $\varphi(x,\beta/2)<1$ and thus $\varphi^{-1}(x,1)\ge\beta/2$.

The converse is not true. If (A0) holds, so that $c_1\le\varphi^{-1}(x,1)\le c_2$, then $\varphi(x,c_1)\le 1$ and $\varphi(x,c_2)\ge 1$. But it is not necessary $\varphi(x,1)\ge 1$ as the following example shows: if $\varphi(t):=t^2/2$, then $\varphi(x,1)=\frac{1}{2}<1$ but $\varphi^{-1}(x,1)=2$.

We use the assumption (A0) to find an equivalent Φ-funcyion that behaves better than the original one. We set

$$\varphi_1(x,t):=\varphi(x,\varphi^{-1}(x,1)t)$$

Then $\varphi_1$is equivalent to $\varphi_1^{-1}(x,1)\equiv\varphi_1(x,1)\equiv 1$ (by (16)). The set of Φ-functions with $\varphi_1^{-1}(x,1)\equiv 1$ will be denoted $\Phi_1(\mathbb{R}^n)$. Note that every $\Phi_1(\mathbb{R}^n)$-function satisfies assumption (A1) implies (A0M).

Let us reformulating (A1) when $\varphi\in\Phi_1(\mathbb{R}^n)$.

Lemma (5.3.8)[196]: Let $\varphi\in\Phi_1$ condition (A1) holds if and only if there exists $\beta>0$ such that

$$\varphi(x,\beta t)\le\varphi(y,t)$$

for every $t\in\left[1,\varphi^{-1}\left(y,\frac{1}{|B|}\right)\right]$, every $x,y\in B$ and every ball B with $|B|\le 1$.

proof. Let the condition of the lemma hold and assume $t\in\left[1,\frac{1}{|B|}\right]$. Then $\varphi^{-1}(y,t)\in\left[1,\varphi^{-1}\left(y,\frac{1}{|B|}\right)\right]$ and so

$$\varphi\big(x,\beta\varphi^{-1}(y,t)\big)\le\varphi\big(y,\varphi^{-1}(y,t)\big)=t.$$

Let t_0 and t_∞ be as in (15) and abbreviate $s := \beta\varphi^{-1}(y,t)$. If $s(t_0, t_\infty]$, then (A1) follows from the previous inequality, since $\varphi^{-1}(x, \varphi(x,s)) = s$. And if $s > t_\infty$, then $\varphi(x,s) = \infty \leq t$, a contradiction, so this is not possible. If $s \leq t_0$, then $s \leq \varphi^{-1}(x,t)$ since $\varphi^{-1}(x,t) > t_0 (t > 0)$. Thus in each case (A1) holds.

Assume then that (A1) holds and let $t \in \left[1, \frac{1}{|B|}\right]$. By (A1) and (16),

$$\varphi(x, \beta\varphi^{-1}(y,t)) \leq \varphi(x, \varphi^{-1}(x,t)) = t = \varphi(y, \varphi^{-1}(y,t)).$$

Let $s := \varphi^{-1}(y,t)$. Thus $\varphi(x, \beta s) \leq \varphi(y,s)$ in the range of $\varphi^{-1}(y,.)$, including (t_0, t_∞). When $s \to t_0^+$, this gives that $\varphi(y, \beta t_0) \leq \varphi(y, t_0) = 0$, so the inequality holds for $s \leq t_0$, as well. Finally, if $s \geq t_\infty$, then $\varphi(y,s) = \infty$, so the inequality certainly holds.

Corollary (5.3.9)[196]: If $\varphi \in \Phi_1(\mathbb{R}^n)$ satisfies (A1), then it satisfies assumption (A1M).

Proof. Let B be a ball $|B| \leq 1$. We must show that $\varphi_B^{-1}(\beta t) \leq \varphi_B^{-}(t)$ when $t \in \left[1, (\varphi_B^{-1})^{-1}\left(\frac{1}{|B|}\right)\right]$.

Suppose first that t is not the upper end-point of the interval. For such t, there exists $y_i \in B$ such that $t \in \left[1, \varphi^{-1}\left(y_i, \frac{1}{|B|}\right)\right]$ and $\varphi_B^{-}(t) = \lim_j \varphi(y_i, t)$. Then by Lemma (5.3.8)

$$\varphi(x, \beta t) \leq \varphi(y_i, t).$$

We let $j \to \infty$ and take supremum over $x \in B$ to arrive (A1M).

It remains to consider $t = (\varphi_B^{-1})^{-1}\left(\frac{1}{|B|}\right)$. Suppose first $\varphi_B^{+}(\beta t) < \infty$. Let $\varepsilon > 0$ and choose $x \in B$ such that $\varphi_B^{+}(\beta t) \leq (1+\varepsilon)\varphi(x, \beta t)$. Since $\varphi(x, \cdot)$ Is left-continuous, we can choose $t' < t$ such that $\varphi(x, \beta t) \leq (1+\varepsilon)\varphi(x, \beta t')$. Combining this with the previous case, we obtain that

$$\varphi_B^{+}(\beta t) \leq (1+\varepsilon)^2 \varphi(x, \beta t') \leq (1+\varepsilon)^2 \varphi(y, t') \leq (1+\varepsilon)^2 \varphi(y,t).$$

Taking infimum over y and letting $\varepsilon \to 0$, we obtain the desired inequality. The case $\varphi_B^{+}(\beta t) = \infty$ is handled analogously.

When $\varphi(x,t) = t^{p(x)}$, (A1) corresponds to the local log-Hölder continuity condition of $\frac{1}{p}$. Namely let $x, y \in \mathbb{R}^n$ with $|x-y| \leq \frac{1}{2}$. Let B be such a ball that $x, y \in B$ and $\operatorname{diam}(B) = 2|x-y|$. By symmetry, we may assume that $p(x) < p(y)$. Since $\varphi^{-1}(x,t) = t^{1/p(x)}$, assumption (A1) reads $\beta(\omega_n |x-y|^n)^{-1/p(x)} \leq (\omega_n |x-y|^n)^{-1/p(x)}$, where ω_n is the measure of the unit ball. In other words,

$$(\omega_n^{-1}|x-y|^{-n})^{\frac{1}{p(x)} - \frac{1}{p(y)}} \leq \frac{1}{\beta}.$$

Taking the logarithm, we find that

$$\frac{1}{p(x)} - \frac{1}{p(y)} \leq \frac{\log \frac{1}{\beta}}{n \log(|x-y|^{-1}) - \log \omega_n} \lesssim \frac{1}{\log(e + |x-y|^{-1})}.$$

Again, the assumption $\varphi \in \Phi_1(\mathbb{R}^n)$ allows us to reformulate (A2).

Lemma (5.3.10)[196]: Let $\varphi \in \Phi_1$. If φ satisfies (A2), then it satisfies (A2M).

Proof. By theorem of [154], $L^{\psi(\cdot)}(\mathbb{R}^n) \subset L^{\varphi(\cdot)}(\mathbb{R}^n)$ if and only if there exist $\beta > 0$ and $h \in L^1(\mathbb{R}^n)$ such that $\varphi(x, \beta t) \leq \psi(x, t) + h(x)$. Hence (A2) implies that for every t_0 there exists β and $h \in L^1(\mathbb{R}^n)$ such that

$$\varphi(x, \beta t) \leq \varphi_\infty(t) + h(x) \quad \text{and} \quad \varphi_\infty(\beta t) \leq \varphi(x, t) + h(x)$$

for all $t \in [0, 1]$ (the restricted range of t is due to the intersection with $L^\infty(\mathbb{R}^n)$ in the assumption). From these we obtain that

$$\varphi(x, \beta^2 t) \leq \varphi_\infty(t) + h(x) \leq \varphi(y, t) + h(y) + h(x)$$

for $t \in [0, 1]$. Since $\varphi(x, 1) \equiv 1$. So the inequality also holds when we replace h by $\min\{h, 1\} \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, as required by (A2M).

Corollary (5.3.11)[196]: If $\varphi \in \Phi_1(\mathbb{R}^n)$ satisfies (A0)−(A2) and there exists $\gamma > 0$ such that $s \mapsto s^{-\gamma}\varphi(x, s)$ is increasing for every $x \in \mathbb{R}^n$, then $M : L^{\varphi(\cdot)}(\mathbb{R}^n) \to L^{\varphi(\cdot)}(\mathbb{R}^n)$ is bounded.

Proof. As in (A0), we find $\varphi_1 \in \Phi_1$ with $\varphi_1 \simeq \varphi$. then φ_1 satisfies (A0M). by Lemma (5.3.6), φ_1 satisfies (A1) and (A2). A short calculation gives that $s \mapsto s^{-\gamma}\varphi_1(x, s)$ is increasing. By Corollary (5.3.9) and Lemma (5.3.10), (A1M) and (A2M) hold. Therefore by thus also on $L^{\varphi(\cdot)\pm}(\mathbb{R}^n)$.

Φ-function are not totally well-behaved with respect to taking limits. Consider for instance t^p. As $p \to \infty$, the point-wise limit is $\infty\chi_{(1,\infty)} + \chi_{\{1\}}$, which is not left-continuous. For the equivalent Φ-function $\frac{1}{p}t^p$ we have $\lim \infty\chi_{(1,\infty)}$, which is what we want. Therefore, we need to chose the equivalent Φ-function suitably to get a good limit.

We are especially interested in the behavior of φ_∞ when $t \leq 1$. To this end we define

$$\varphi_2(x, t) \coloneqq \max\{\varphi_1(x, t), 2t - 1\}.$$

Clearly $\varphi_1 \leq \varphi_2$. For $t \leq \frac{1}{2}$, $\varphi_2 = \varphi_1$. Since $\varphi \in \Phi_1(\mathbb{R}^n)$ we have $\varphi_1(x, 1) = 1$ and $\varphi_1(x, t) \geq t$ for $t \geq 1$ by convexity. Thus $\varphi_2(x, t) \leq \varphi_2(x, 1) = 1 \leq \varphi_1(x, 2t)$ for $t \in \left[\frac{1}{2}, 1\right]$ and $\varphi_2(x, t) \leq 2\varphi_1(x, t) \leq \varphi_1(x, 2t)$ for $t > 1$. In sum, obtain $\varphi_2 \simeq \varphi_1 \simeq \varphi$ with $\varphi_2(x, 1) \equiv 1 \equiv \varphi_2^{-1}(x, 1)$.

Note that the right-derivative satisfies $\varphi_2'(x, 1^-) \in [1, 2]$: here the lower bound follows from convexity $\varphi_2'(x, 1) \geq \varphi_2(x, t) = 1$ and the upper bound holds since if $\varphi_2'(x, 1^-) > 2$, then $\varphi_2(x, t) < 2t - 1$ for some $t < 1$ contrary to construction of φ_2.

We consider then limt$(\varphi_2)_\infty(t) = \limsup_{|x|\to\infty} \varphi_2(x, t)$. Cleary $(\varphi_2)_\infty(0) = 0$ and $(\varphi_2)_\infty(1) = 1$. For $t \in (0, 1)$, $(\varphi_2)_\infty(1) \geq (\varphi_2)_\infty(t) \geq 2t - 1$ and hence $(\varphi_2)_\infty$ is left-continuous at 1. By convexity of $\varphi_2, \varphi_2(x, t) \leq t\varphi_2(x, 1)$ on $[0, 1]$ and hence $(\varphi_2)_\infty(0^+) = 0$. Since $(\varphi_2)_\infty(t) \geq t$ for $t \geq 1$, we have $\lim_{t\to\infty}(\varphi_2)_\infty(t) = \infty$.

To show that $(\varphi_2)_\infty$ is convex let $0 \leq t_1 < t_2$ and $\theta \in (0, 1)$. Choose $x_i \to \infty$ such that

$$(\varphi_2)_\infty(\theta t_1 + (1 - \theta)t_2) = \lim_i \varphi_2(x_i, \theta t_1 + (1 - \theta)t_2).$$

By convexity of φ_2,

$$\begin{aligned}\lim_i \varphi_2(x_i, \theta t_1 + (1 - \theta)t_2) &\leq \lim_i [\theta\varphi_2(x_i, t_1) + (1 - \theta)(x_i, t_2)] \\ &\leq \theta(\varphi_2)_\infty(t_1) + (1 - \theta)(\varphi_2)_\infty(t_2),\end{aligned}$$

So $(\varphi_2)_\infty$ is convex, as well.

Since $(\varphi_2)_\infty$ is convex and increasing on $[0,1]$, and left-continuous at 1, it is actually continuous on $[0,1]$.

In the variable exponent setting, $(A2)$ is equivalent to Nekvinda's decay condition (see [154]), which is weaker version of the log-Hölder condition.
Note that $(A2)$ implies also the equivalence of norms: indeed, this is a general property of solid Banach spaces, as the following well-known argument shows.
(Recall that a space is solid if $|f| \le |g|$ implies $\|f\| \le \|g\|$.) If $\|f\|_A \not\lesssim \|f\|_B$, then we can choose f_i such that $\|f_i\|_A \ge 3^i$ but $\|f_i\|_B \le 1$. Now for $g := \sum_i 2^{-i}|f_i|$ we have

$$\|g\|_A \ge \left\|2^{-i}f_i\right\|_A \ge (3/2)^i \to \infty$$

and $\|g\|_B \ge \sum_i 2^{-i} = 1$ so that $g \in B\backslash A$. Hence $A \ne B$. The implication $A = B \Rightarrow \|\cdot\|_A \approx \|\cdot\|_B$ follows by contraposition.
Next we make the final normalization of φ satisfying $(A0)-(A2)$ by setting

$$\bar{\varphi}(x,t) = \begin{cases} 2\varphi_2(x,t) - 1, & \text{if } t \ge 1, \\ (\varphi_2)_\infty(t), & \text{if } t < 1. \end{cases}$$

Lemma (5.3.12)[196]: If $\varphi \in \Phi(\mathbb{R}^n)$ satisfies (A0)−(A2), then $\bar{\varphi} \in \Phi_1(\mathbb{R}^n)$.
Proof. For the convexity we have to show that the $\bar{\varphi}'(x,\cdot)$ is increasing for every $x \in \mathbb{R}^n$. We have

$$\bar{\varphi}'(x,t) = \begin{cases} 2\varphi_2'(x,t), & \text{if } t \ge 1, \\ (\varphi_2)_\infty'(t), & \text{if } t < 1. \end{cases}$$

By convexity each of the parts is increasing. At, $2\varphi_2'(x,1) \ge 2$ and $\lim_{t\to 1^-}(\varphi_2)_\infty'(t) \le 2$ (see discussion regarding) (A2), so the right-derivative is increasing also there.

The function $\bar{\varphi}$ is continuous since both φ_2 and $(\varphi_2)_\infty$ are continuous and $\varphi_2(x,1) = (\varphi_2)_\infty(1^-) = 1$. Thus we have that $\bar{\varphi}^{-1}(x,1) \le 1$. In the discussion on (A2), we noted that $\varphi_2(x,t) \le t$ on $[0,1]$. These together give $\bar{\varphi}^{-1}(x,1) \equiv 1$.

The conditions $\bar{\varphi}(x,0) = \bar{\varphi}(x,0^+) = 0$ and $\lim_{t\to\infty}\bar{\varphi}(t) = \infty$ follow from the same continuous for φ_2 and $(\varphi_2)_\infty$.
Proposition (5.3.13)[196]: If $\varphi \in \Phi(\mathbb{R}^n)$ satisfies (A0)−(A2), then $L^{\varphi(\cdot)} = L^{\bar{\varphi}(\cdot)}$ with equivalent norms.
Proof. Since $\varphi \simeq \varphi_2$, it suffices to show that $L^{\varphi_2(\cdot)} = L^{\bar{\varphi}(\cdot)}$.

Let $g \in L^{\varphi_2(\cdot)}(\mathbb{R}^n)$ and set $f := g/\|g\|_{\varphi_2(\cdot)}$. we divide f into two parts $f_1 = \chi_{\{|f|<1\}}$ and $f_2 = \chi_{\{|f|\ge 1\}}$. By (A2), and since $\|f\|_{\varphi_2(\cdot)} = 1$,

$$\begin{aligned} \|f_1\|_{\bar{\varphi}(.)} = \|f_1\|_{(\varphi_2)_\infty} \le \|f_1\|_{L^{(\varphi_2)_\infty} \cap L^\infty} \\ \approx \|f_1\|_{L^{(\varphi_2)_\infty} \cap L^\infty} = \max\{\|f_1\|_{\varphi_2(\cdot)}, \|f_1\|_\infty\} \le 1. \end{aligned}$$

If $|f_2(x)| \ge 1$, then otherwise $|f_2(x)| = 0$, and the inequality holds as well. Thus $\|f_1\|_{\bar{\varphi}(\cdot)} \lesssim \|f_2\|_{\varphi_2(\cdot)} \le 1$ and hence

$$\left\|g/\|g\|_{\varphi_2(\cdot)}\right\|_{\bar{\varphi}(\cdot)} = \|f\|_{\bar{\varphi}(\cdot)} \le \|f_1\|_{\bar{\varphi}(\cdot)} + \|f_1\|_{\bar{\varphi}(\cdot)} \lesssim 1,$$

so that $\|g\|_{\bar{\varphi}(\cdot)} \lesssim \|g\|_{\varphi_2(\cdot)}$. the opposite inequality is proved similarly.

While the spaces in the previous proposition are the same, it is not necessary that $\varphi \simeq \varphi_2$. For instance, if $\varphi(x,t) := \max\left\{t - \frac{1}{2+|x|}, 0\right\}$ then $\varphi\left(0, \frac{1}{2}\right) = 0$ yet $\varphi_\infty(t) =$

$t > 0$ for all $t > 0$. Then for every $\beta \in (0,2)$, $\bar{\varphi}\left(0, \beta\frac{1}{2}\right) = \beta\frac{1}{2} \nleq \varphi\left(0, \frac{1}{2}\right)$, so the Φ-function are not equivalent.

Corollary (5.3.14)[196]: If $\varphi \in \Phi(\mathbb{R}^n)$ satisfies (A0)−(A2) and there exists $\gamma > 1$ such that $s \mapsto s^{-\gamma}\varphi(x,s)$ is increasing for every $x \in \mathbb{R}^n$, $M : L^{\overline{\varphi}(\cdot)}(\mathbb{R}^n) = L^{\overline{\varphi}(\cdot)}(\mathbb{R}^n)$ is bounded.

Note the range of permissible values t in the following proposition, including also $[0,1]$. This is sometimes very useful, e.g. in Proposition (5.3.17).

Proposition (5.3.15)[196]: If $\varphi \in \Phi(\mathbb{R}^n)$ satisfies (A0)−(A2), then there exists $\beta \in (0,2)$ such that

$$\beta\bar{\varphi}^{-1}(x,t) \leq \bar{\varphi}^{-1}(y,t) \tag{17}$$

for every $t \in \left[0, \frac{1}{|\beta|}\right]$, every $x, y \in B$ and every ball B.

Proof. If $t \leq 1$, then $\bar{\varphi}$ is independent of x, so the claim is trivial. Thus it remains only to consider the case $t > 1$. Then by Lemma (5.3.8) the inequality holds if and only if

$$\bar{\varphi}(x, \beta s) \leq \bar{\varphi}(y,s)$$

for every $t \in \left[1, \bar{\varphi}^{-1}\left(y, \frac{1}{|\beta|}\right)\right]$, since $\bar{\varphi} \geq \varphi_2$, such s satisfies $s \in \left[1, \varphi_2^{-1}\left(y, \frac{1}{|\beta|}\right)\right]$. if $\beta s \geq 1$, then using Lemma (5.3.8) for φ_2 we calculate

$$\bar{\varphi}(x, \beta s) = 2\varphi_2(x, \beta s) - 1 \leq 2\varphi_2(y,s) - 1 = \bar{\varphi}(y,s).$$

If $\beta s < 1$, then $\bar{\varphi}(x, \beta s) \leq \bar{\varphi}(x, 1) = 1 \leq \bar{\varphi}(y, 1) \leq \bar{\varphi}(y,s)$, so the inequality holds in both cases.

In view of the previous proposition and the observation, we make the following definition. Note that a normalized Φ-function satisfies assumptions (A0)−(A2).

Definition (5.3.16)[196]: We say that $\varphi \in \Phi(\mathbb{R}^n)$ is normalized Φ-function if $\varphi(x,t) = \varphi_\infty(t)$ for $\in [0,1]$ $\varphi_\infty(1) \in (0,\infty)$, and there exists $\beta > 1$ such that

$$\beta\varphi^{-1}(x,t) \leq \varphi^{-1}(y,t)$$

for every $t \in \left[0, \frac{1}{|\beta|}\right]$, every $x, y \in B$ and every ball B.

Proposition (5.3.15) says that instead of studying $\varphi \in \Phi(\mathbb{R}^n)$ which satisfies (A0)−(A2) we can study the normalized Φ-function $\bar{\varphi}$. This sometimes leads to great simplifications in proofs, as the following result shows.

Proposition (5.3.17)[196]: Suppose that $\varphi \in \Phi(\mathbb{R}^n)$ is normalized. Let $B \ni x$ be a ball. Then

$$\|\chi_B\|_{\varphi(.)} \leq \frac{1}{\beta\varphi^{-1}\left(x, \frac{1}{|\beta|}\right)}.$$

Proof. By assumption

$$\varphi\left(y, \beta\varphi^{-1}\left(x, \frac{1}{|\beta|}\right)\right) \leq \varphi\left(y, \beta\varphi^{-1}\left(y, \frac{1}{|\beta|}\right)\right) \leq \frac{1}{|\beta|}$$

when $x, y \in B$, and hence

$$\varrho_{\varphi(.)}\left(\beta\varphi^{-1}\left(x, \frac{1}{|\beta|}\right)\chi_B\right) = \int_B \varphi\left(y, \beta\varphi^{-1}\left(x, \frac{1}{|\beta|}\right)\right) dy \leq 1.$$

In some regard, it is actually easier to study general normalized Φ-function then the special case of variable exponent spaces: the normalized allows us to omit error term, which commonly appears in the variable exponent case. This is consequence of the fact $\varphi(x,t)=\varphi_\infty(t)$ in the normalized case for small t, whereas only $t^{p(x)} \le t^{p_\infty}+h(x)$ holds in the variable exponent case; the function h leads to the error term.

The conjugate Φ-function of φ is defined by

$$\varphi^*(t) := \sup_{s>0}\big(st-\varphi(s)\big).$$

Note that $\varphi^{**}=\varphi$ [156]. For $\gamma>1$, the Hölder conjugate γ' is defined by $\frac{1}{\gamma}+\frac{1}{\gamma'}=1$. If $\varphi(t)=\frac{1}{p}t^{p(x)}$, and we get the usual Lebesgue duality. The dual is defined for generalized Φ-functions point-wise. Note that conjugating preserves equivalence. i.e. if $\varphi \simeq \psi$, then $\varphi^* \simeq \psi^*$ [156].

Differentiating $st-\varphi(s)$ to find the maximum, we obtain that $\varphi^*(t)=t(\varphi')^{-1}-\varphi\big((\varphi')^{-1}(t)\big)$, where $(\varphi')^{-1}(t)$ is the right-continuous inverse:

$$(\varphi')^{-1}(t) := \sup\{\tau \ge 0 | \varphi'(\tau) \le t\}.$$

For duality arguments we often need function nicer than Φ-functions: N-functions are those (continuous) Φ-functions which satisfy $\varphi(t)\in(0,\infty)$ when $t>1$, $\lim_{t\to 0^+}\frac{\varphi(t)}{t}=0$ and $\lim_{t\to\infty}\frac{\varphi(t)}{t}=\infty$. The set of N-functions is denoted by N. Note that N-functions are strictly increasing. For example, if $\varphi(t)=t^p$, then $\varphi\in N$ if and only if $p\in(1,\infty)$.

We say that $\varphi\in\Phi(\mathbb{R}^n)$ is a (generalized) uniform N-function if there exists $\eta,\xi\in N$ such that

$$\eta(t)\le\varphi(x,t)\le\xi(t)$$

for every $x\in\mathbb{R}^n$ and $t\ge 0$. The set of uniform N-functions is denoted by $N(\mathbb{R}^n)$. We set $N_1(\mathbb{R}^n) := \Phi_1(\mathbb{R}^n)\cap N(\mathbb{R}^n)$.

In the variable exponent case, t^p is uniform only $1<p<\infty$ point-wise. The latter condition has turned out to be nearly useless in $L^{p(\cdot)}$-research, so it is natural to consider here only the uniform case.

Proposition (5.3.18)[196]: If $\varphi\in N(\mathbb{R}^n)$ satisfies (A0)$-$(A2), then $\bar\varphi\in N_1(\mathbb{R}^n)$.

Proof. By Lemma (5.3.12), $\bar\varphi\in\Phi_1(\mathbb{R}^n)$. We need to check that the normalizations do not destroy the functions η and ξ. By (A0), there exists $\beta\in(0,1)$ such that $\beta\le\varphi^{-1}(x,1)\le 1/\beta$. First we set $\eta_1(t):=\eta(\beta t)$ and $\xi_1(t):=\xi(t/\beta)$. Then $\eta_1\le\varphi_1\le\xi_1$. As before $\eta_2(t)=\max\{\eta_1(t),2t-1\}$, similarly for ξ. Then also $\eta_2\le\varphi_2\le\xi_2$, and we easily see that η_2 and ξ_2 are still N-functions. Furthermore, $\bar\eta=\max\{\eta_2,2\eta_2-1\}$ is an N-function minorizing $\bar\varphi$, similarly for $\bar\xi$.

Lemma (5.3.19)[196]: Let $\varphi\in N$ and $\gamma>1$. Then $s\mapsto s^{-\gamma}\varphi(s)$ is increasing if and only if $s\mapsto s^{-\gamma'}\varphi^*(s)$ is decreasing.

Proof. We note that $t\mapsto t^{-\gamma}\varphi(t)$ is increasing if and only if $D\big(t^{-\gamma}\varphi(t)\big)\ge 0$, i.e. $t\varphi'(t)\ge\gamma\varphi(t)$. Since φ is continuous, we conclude from this that

$$t\varphi'(t^-)\ge\gamma\varphi(t^-)=\gamma\varphi(t).$$

On the other hand, as noted after the definition of φ^*, with $t:=(\varphi')^{-1}(s)$,

$$\varphi^*(s)=st-\varphi(t)\ge st-\frac{1}{\gamma}t\varphi'(t^-).$$

By [156], $t=(\varphi^*)'(s)$ and by [156], $\varphi'(\varphi^*)'(s)\le s$ for all $\varepsilon>0$, so that $\varphi'(t^-)\le s$. In the previous inequality, this gives $\varphi^*(s)\ge\frac{1}{\gamma'}st=\frac{1}{\gamma'}s(\varphi^*)'(s)$, which is equivalent to $D\left(s^{-\gamma'}\varphi^*(s)\right)\le 0$, as was to be shown.

Proposition (5.3.20)[196]: If $\varphi \in N_1(\mathbb{R}^n)$ is normalized, then also $\varphi^* \in N(\mathbb{R}^n)$ is normalized.

Proof. First we note that $\eta^*, \xi^* \in N$ by [154]. The inequalities $\eta(t) \le \varphi(x,t) \le \xi(t)$ yield $\xi^*(t) \le \varphi^*(x,t) \le \eta^*(t)$ by [156], and thus $\varphi^* \in N(\mathbb{R}^n)$.

By [154], $t \le \psi^{-1}(t)(\psi^*)^{-1}(t) \le 2t$ for $\psi \in N$. Let $x, y \in B$ and $t \le \frac{1}{|B|}$. Then

$$\frac{\beta}{2}(\varphi^*)^{-1}(x,t) \le \frac{\beta t}{\varphi^{-1}(x,t)} \le \frac{t}{\varphi^{-1}(x,t)} \le (\varphi^*)^{-1}(y,t).$$

Furthermore, $\varphi(x,s) \ge \varphi(x,1)s = s$ when $s \ge 1$ (since φ is convex). When $t \le 1$ and $s \ge 1$, it follows that $st - \varphi(x,s) \le s(t-1) \le 0$. Hence, for $t \le 1$.

$$\begin{aligned}\varphi^*(x,t) &= \sup_{s>0}\big(st - \varphi(x,s)\big) = \sup_{s\in[0,1]}\big(st - \varphi(x,s)\big) = \sup_{s\in[0,1]}\big(st - \varphi_\infty(s)\big)\\ &= \varphi_\infty^*(t)\end{aligned}$$

is independent of x. Since $0 = \lim_{t\to 0^+}\frac{\varphi(x,t)}{t} = \lim_{t\to 0^+}\frac{\varphi_\infty(t)}{t}$ we obtain $\varphi_\infty^*(1) = \sup_{s\in[0,1]}\big(st - \varphi_\infty(s)\big) > 0$. Therefore we have shown that it is a normalized N-function.

Let $0 < \alpha < n$. For measurable f we define $I_\alpha f : \mathbb{R}^n \to [0,\infty]$ by

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy.$$

The operator I_α is called the Riesz potential operator.

Lemma (5.3.21)[196]: For $\varphi \in \Phi(\mathbb{R}^n)$ we write $\hat{\varphi}(x,t) := \varphi^*\left(x, t^{\frac{n-\alpha}{n}}\right)$. Assume that M is bounded from $L^{\hat{\varphi}(\cdot)}(\mathbb{R}^n)$ to itself. Let $x \in \mathbb{R}^n$, $\delta > 0$, and $f \in L^{\varphi(\cdot)}(\mathbb{R}^n)$ with $\|f\|_{\varphi(\cdot)} \le 1$. Then

$$\int_{\mathbb{R}^n\setminus B(x,\delta)} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy \lesssim |B(x,\delta)|^{\frac{n-\alpha}{n}} \left\|\chi_{B(x,\delta)}\right\|_{\varphi^*(\cdot)}.$$

Proof. $B := B(x,\delta)$. We start with Hölder's inequality and take into account that $\|f\|_{\varphi(\cdot)} \le 1$:

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy &\le 2\|f\|_{\varphi(\cdot)}\left\|\chi_{\mathbb{R}^n\setminus B}|x-\cdot|^{n-\alpha}\right\|_{\varphi^*(\cdot)}\\ &\le 2\left\|\chi_{\mathbb{R}^n\setminus B}|x-\cdot|^{-n}\right\|_{\hat{\varphi}(\cdot)}^{\frac{n-\alpha}{n}}.\end{aligned}$$

Next we note that, for $x \in \mathbb{R}^n\setminus B$,

$$M(\chi_B|B|^{-1})(y) \ge \int_{B(y,2|x-y|)} \chi_B(z)|B|^{-1}dz = |B(y,2|x-y|)|^{-1} = c|x-y|^{-1}.$$

Therefore $\chi_{\mathbb{R}^n\setminus B}(y)|x-y|^n \lesssim M(\chi_B|B|^{-1})(y)$ for all $y \in \mathbb{R}^n$. Combing the previous estimates and using the boundedness of M, we find that

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy &\lesssim \|M(\chi_B|B|^{-1})\|_{\hat{\varphi}(\cdot)}^{\frac{n-\alpha}{n}} = |B|^{\frac{n-\alpha}{n}}\|M(\chi_B)\|_{\hat{\varphi}(\cdot)}^{\frac{n-\alpha}{n}}\\ &\lesssim |B|^{\frac{n-\alpha}{n}}\|\chi_B\|_{\hat{\varphi}(\cdot)}^{\frac{n-\alpha}{n}} = |B|^{\frac{n-\alpha}{n}}\|\chi_B\|_{\varphi^*(\cdot)}.\end{aligned}$$

Recall that a function is almost decreasing if $f(x) \le Qf(y)$ when $x > y$, for some fixed $Q \in [1,\infty)$. Almost increasing defined analogously.

Lemma (5.3.22)[196]: Let $\varphi \in N_1(\mathbb{R}^n)$ be normalized and suppose that $s \mapsto s^{\frac{\varepsilon-n}{\alpha}}\varphi(x,s)$ is almost decreasing for every $x \in \mathbb{R}^n$. Then

$$I_\alpha f \lesssim \varphi\big(x, Mf(x)\big)^{\frac{-\alpha}{n}} Mf(x) \quad a.e.$$

for ever y $f \in L^{\varphi(\cdot)}(\mathbb{R}^n)$ with $\|f\|_{\varphi(\cdot)} \le 1$.

Proof. Let us $B := B(x,\delta)$. We divide the Riesz-potential into two parts:

$$I_\alpha f(x) := \int_B \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy + \int_{\mathbb{R}^n \setminus B} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy.$$

In the first part we split the integration domain into annuli and use the definition of M:

$$\begin{aligned}\int_B \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy &\le \sum_{k=1}^{\infty} (\delta 2^{-k})^{\alpha-n} \int_{2^{-k}\delta \le |x-y| < 2^{-k+1}\delta} |f(y)|\,dy \\ &\lesssim \sum_{k=1}^{\infty} (\delta 2^{-k})^{\alpha} \int_{|x-y|<2^{-k+1}\delta} |f(y)|\,dy \\ &\le \delta^\alpha \sum_{k\in\mathbb{N}}^{\infty} 2^{-\alpha k}\, Mf(x) = c|B|^{\frac{\alpha}{n}} Mf(x).\end{aligned}$$

Let $\hat{\varphi}(x,t) = \varphi^*\left(x, t^{\frac{n-\alpha}{n}}\right)$ be as is Lemma (5.3.21). By Proposition (5.3.20), φ^* is normalized. Thus φ^* satisfies $(A0)-(A2)$. Further $\hat{\varphi}^{-1}(x,t) = ((\varphi^*)^{-1}(x,t))^{\frac{n}{n-\alpha}}$, and so $\hat{\varphi}$ inherits $(A0)-(A2)$ from φ^*.

Set $\gamma := \frac{n-\varepsilon}{\alpha}$ and define $\psi(x,t) := t^\gamma \sup_{s\ge t} s^{-\gamma}\,\varphi(x,s)$ for $t \ge 0$. Thus definition directly implies that $t^{-\gamma}\psi(x,t)$ is decreasing and $\psi \ge \varphi$. Since $s^{-\gamma}(x,s)$ is almost decreasing by assumption, $\psi \le Q\varphi$, so $\varphi \simeq \psi$. By Lemma (5.3.19), $t^{-\gamma'}\psi^*(x,t)$ is increasing, and since $\varphi^* \simeq \psi^*$ it follows that $t^{-\gamma'}\varphi^*(x,t)$ is almost increasing. Therefore with $s = t^{\frac{n-\alpha}{n}}$,

$$t^{-\gamma'\frac{n-\alpha}{n}}\hat{\varphi}(x,t) = t^{-\gamma'\frac{n-\alpha}{n}}\varphi^*\left(x, t^{\frac{n-\alpha}{n}}\right) = s^{-M}\varphi^*(x,s)$$

is almost increasing. A calculation yields that $\hat{\gamma} := \gamma'\,\frac{n-\alpha}{n} > 1$. Therefore $\hat{\varphi}$ equivalent to Φ-function ξ with $t^{-\gamma'}\xi(x,t)$ increasing (cf. [190]). Since $\hat{\varphi} \simeq \xi$, also $(A0)-(A2)$ holds. By Corollary (5.3.11), M is bounded on $L^{\xi(\cdot)}$, and hence also on $L^{\hat{\varphi}(\cdot)}$.

Therefore, the assumptions of Lemma (5.3.21) hold, and follows that

$$\int_{\mathbb{R}^n \setminus B} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy \lesssim |B|^{\frac{\alpha-n}{n}} \|\chi_B\|_{\varphi^*(\cdot)} \tag{18}$$

provided $\|f\|_{\varphi(\cdot)} \le 1$.

We combine (18) with Proposition (5.3.17) and (5.3.20), and obtain

$$I_\alpha f(x) \lesssim |B|^{\frac{\alpha}{n}} Mf(x) + |B|^{\frac{\alpha-n}{n}} \|\chi_B\|_{\varphi^*(\cdot)} \le |B|^{\frac{\alpha}{n}} Mf(x) + \frac{|B|^{\frac{\alpha-n}{n}}}{\beta(\varphi^*)^{-1}\left(x, \frac{1}{|B|}\right)}.$$

Now $(\varphi^*)^{-1}(x,t) \approx t/\varphi^{-1}(x,t)$ by [156] and so

$$I_\alpha f(x) \lesssim |B|^{\frac{\alpha}{n}} Mf(x) + |B|^{\frac{\alpha}{n}} \varphi^{-1}\left(x, \frac{1}{|B|}\right).$$

when $Mf(x) < \infty$, we choose the radius δ such that $Mf(x) = \varphi^{-1}\left(x, \frac{1}{|B|}\right)$, i.e. $|B| = 1/\varphi(x, Mf(x))$. Thus

$$I_\alpha f(x) \lesssim \varphi\big(x, Mf(x)\big)^{-\frac{\alpha}{n}} Mf(x) \quad a.e.$$

Lemma (5.3.23)[196]: Let $\alpha > 0$, $\varphi \in N(\mathbb{R}^n)$ with $t \mapsto t^{-\frac{n}{\alpha}} \varphi(x,t)$ strictly decreasing to 0 for every $x \in \mathbb{R}^n$ and let $\lambda(x,t) \coloneqq t\varphi(x,t)^{-\frac{n}{\alpha}}$. Then $\varphi\ o\ (\lambda^{-1})$ is equivalent to convex Φ-function.

By $\varphi\ o\ (\lambda^{-1})$ we main the function $(x,t) \mapsto \varphi\big(x, \lambda^{-1}(x,t)\big)$.

Proof. Since the claim is point-wise in nature, we drop the variable x for the rest of the proof.

Let us denote $\psi \coloneqq \varphi\ o\ (\lambda^{-1})$. Since $t^{-\frac{n}{\alpha}} \varphi(t) \to 0$ we find that $\lambda(t) \to \infty$ as $t \to \infty$. Thus also $\psi(t) \to \infty$ as $t \to \infty$. The function

$$\lambda(s)^{\frac{n}{\alpha}} = \frac{s^{n/\alpha}}{\varphi(s)}$$

is strictly increasing by assumption, so the same holds for λ^{-1}. Furthermore, with $s = \lambda(t)$, the fraction

$$\frac{\lambda^{-1}(s)}{s} = \frac{t}{\lambda(t)} = \varphi(t)^{\frac{\alpha}{n}}$$

is increasing t (since φ is increasing), hence is s as well. Since $t \mapsto \varphi(t)/t$ is increasing (due to convex of φ and $\varphi(0)$ this yields that

$$\frac{\psi(t)}{t} = \frac{\varphi\big(\lambda^{-1}(t)\big)}{\lambda^{-1}(t)} \frac{\lambda^{-1}(t)}{t}$$

is also increasing. Since $\frac{\psi(t)}{t}$ is increasing, we obtain $\lim_{t \to 0^+} \psi(t) = 0$. Thus it follows from Lemma (5.3.4)[196] that ψ is equivalent to a convex function.

The previous lemma shows that the next definition makes sense.

Definition (5.3.24)[196]: Let $\alpha > 0$, $\varphi \in N(\mathbb{R}^n)$ with $t \mapsto t^{-\frac{n}{\alpha}} \varphi(x,t)$ strictly decreasing to 0 for every fixed x. We define $\lambda(x,t) \coloneqq t\varphi(x,t)^{-\frac{n}{\alpha}}$ and let $\varphi_\alpha^\# \in \Phi(\mathbb{R}^n)$ be a Φ-function equivalent $\varphi\ o\ (\lambda^{-1})$ (which exists by Lemma (5.3.23).)

Lemma (5.3.25)[196]: If $\varphi \in \Phi(\mathbb{R}^n)$ satisfies assumptions (A0)$-$(A2) and $t \mapsto t^\gamma \varphi(x,t)$, $\gamma < 0$, is decreasing, $t \mapsto t^\gamma \bar{\varphi}(x,t)$ is almost decreasing.

Proof. We prove first $t \mapsto t^\gamma \varphi_2(x,t)$ is almost decreasing. Since $\varphi \simeq \varphi_2$, for $s < t$,

$$\begin{aligned} s^\gamma \varphi_2(x,s) &\geq s^\gamma \varphi(x, s/L) = L^\gamma (s/L)^\gamma \varphi(x, s/L) \geq L^\gamma (t/L)^\gamma \varphi(x, t/L) \\ &\geq L^\gamma (Lt)^\gamma \varphi(x, Lt) \geq L^\gamma (Lt)^\gamma \varphi_2(x,t) = L^{2\gamma} t^\gamma \varphi_2(x,t). \end{aligned}$$

Using this we obtain the same property for $\bar{\varphi}$: If $0 < s < t \leq 1$, then

$$s^\gamma \bar{\varphi}(x,s) = \limsup_{|z| \to \infty} s^\gamma \varphi_2(z,s) \geq L^{2\gamma} \limsup_{|z| \to \infty} t^\gamma \varphi_2(z,t) = L^{2\gamma} t^\gamma \bar{\varphi}(x,t),$$

and $1 \leq s < t$, then

$$\begin{aligned} s^\gamma \varphi_2(x,s) &= s^\gamma (2\varphi_2(x,s) - 1) \geq s^\gamma \varphi_2(x,s) \geq L^{2\gamma} t^\gamma \varphi_2(x,t) \\ &\geq \frac{1}{2} L^{2\gamma} t^\gamma (2\varphi_2(x,s) - 1) \geq \frac{1}{2} L^{2\gamma} t^\gamma \bar{\varphi}(x,t). \end{aligned}$$

Since the function is almost decreasing on $(0,1]$ and $[1,\infty)$, it is almost decreasing on the union as well.

Lemma (5.3.26)[196]: Let $\varphi \in N(\mathbb{R}^n)$ satisfy assumptions (A0)−(A2) and let $t \mapsto t^{-\frac{n}{\alpha}}\varphi(x,t)$ be strictly decreasing to 0. Then $L^{\varphi_\alpha^\#(\cdot)}(\mathbb{R}^n) = L^{\bar{\varphi}_\alpha^\#(\cdot)}(\mathbb{R}^n)$.

Proof. By Theorem of [154] such that $L^{\psi(\cdot)}(\mathbb{R}^n) \subset L^{\varphi(\cdot)}(\mathbb{R}^n)$ if and only if there exist $\beta > 0$ and $h \in L^1(\mathbb{R}^n)$ such that $\varphi(x,\beta t) \le \psi(x,t) + h(x)$. This can be equivalently written $\psi^{-1}(x,s) \lesssim \varphi^{-1}\big(x, s + h(x)\big)$.

Let us show that $L^{\bar{\varphi}_\alpha^\#(\cdot)}(\mathbb{R}^n) \subset L^{\varphi_\alpha^\#(\cdot)}(\mathbb{R}^n)$; the reverse implication follows analogously. The inclusion is equivalent to the inequality

$$(\bar{\varphi}_\alpha^\#)^{-1}(x,s) \lesssim (\varphi_\alpha^\#)^{-1}\big(x, s + h(x)\big)$$

By Lemma (5.3.25), $t \mapsto t^{-\frac{n}{\alpha}}\bar{\varphi}(x,t)$ is almost decreasing which is equivalent to $t \mapsto t^{-\frac{n}{\alpha}}\bar{\varphi}^{-1}(x,t)$ being almost increasing. By the definition of $\bar{\varphi}_\alpha^\#$ and the almost increasing property, we obtain that

$$(\bar{\varphi}_\alpha^\#)^{-1}(x,s) \approx \bar{\lambda}\big(\bar{\varphi}^{-1}(x,s)\big) = \frac{\bar{\varphi}^{-1}(x,s)}{\bar{\varphi}\big(x,\bar{\varphi}^{-1}(x,s)\big)^{\alpha/n}} = \frac{\bar{\varphi}^{-1}(x,s)}{s^{\alpha/n}}$$
$$\lesssim \frac{\bar{\varphi}^{-1}\big(x, s + h(x)\big)}{\big(s + h(x)\big)^{\alpha/n}},$$

Where $\bar{\lambda}(x,t) = t\bar{\varphi}(x,t)^{-\alpha/n}$. By Proposition (5.3.13), $L^{\varphi(\cdot)} = L^{\bar{\varphi}(\cdot)}$, so that $\bar{\varphi}^{-1}(x,s) \lesssim \varphi^{-1}\big(x, s + h(x)\big)$. Using this in the inequality above, and reversing the steps with φ, we get

$$(\bar{\varphi}_\alpha^\#)^{-1}(x,s) \lesssim \frac{\bar{\varphi}^{-1}\big(x, s + h(x)\big)}{\big(s + h(x)\big)^{\alpha/n}} \lesssim \frac{\varphi^{-1}\big(x, s + 2h(x)\big)}{\big(s + h(x)\big)^{\alpha/n}} \lesssim (\bar{\varphi}_\alpha^\#)^{-1}\big(x, s + 2h(x)\big),$$

as required.

Theorem (5.3.27)[196]: Let $\varphi \in N(\mathbb{R}^n)$ satisfy assumptions (A0)−(A2) and suppose that $\varepsilon > 0$ is such that $t \mapsto t^{-(1+\varepsilon)}\varphi(x,t)$ is increasing and $s \mapsto s^{\frac{\varepsilon-n}{\alpha}}\varphi(x,s)$ is decreasing for every $x \in \mathbb{R}^n$. Then

$$\|I_\alpha f\|_{\varphi_\alpha^\#(\cdot)} \lesssim \|f\|_{\varphi(\cdot)}.$$

Note that φ is doubling with constant $2^{-\frac{n-\varepsilon}{\alpha}}$ since $s \mapsto s^{-\frac{n-\varepsilon}{\alpha}}\varphi(x,s)$ is decreasing.

Proof. Let us first note that since $s \mapsto s^{-\frac{n-\varepsilon}{\alpha}}\varphi(x,s)$ is decreasing $t \mapsto t^{-\frac{n}{\alpha}}\varphi(x,t)$ is strictly decreasing to 0.

By Proposition (5.3.13) and Lemma (5.3.26), $L^{\varphi(\cdot)} = L^{\bar{\varphi}(\cdot)}$ and $L^{\varphi_\alpha^\#(\cdot)} = L^{\bar{\varphi}_\alpha^\#(\cdot)}$ with comparable norms. Thus it suffices to show that $\|I_\alpha f\|_{\bar{\varphi}_\alpha^\#(\cdot)} \lesssim \|f\|_{\bar{\varphi}(\cdot)}$.

By Propositions (5.3.15) and (5.3.18) $\bar{\varphi} \in N_1(\mathbb{R}^n)$ is normalized. By Corollary (5.3.14), $M : L^{\bar{\varphi}(\cdot)}(\mathbb{R}^n) \to L^{\bar{\varphi}(\cdot)}(\mathbb{R}^n)$ is bounded. By Lemma (5.3.25), $t \mapsto t^{-\frac{n-\varepsilon}{\alpha}}\varphi(x,t)$ is almost decreasing. Thus, by Lemma (5.3.22), $\bar{\lambda}^{-1}\big(x, I_\alpha f(x)\big) \lesssim Mf(x)$. Applying $\bar{\varphi}$ to both sides, we find that

$$\bar{\varphi}_\alpha^\#\big(x, I_\alpha f(x)\big) \simeq \bar{\varphi}\left(x, \bar{\lambda}^{-1}\big(x, I_\alpha f(x)\big)\right) \lesssim \bar{\varphi}\big(x, Mf(x)\big).$$

From this we deduce by the normal scaling argument that

$$\|I_\alpha f\|_{\bar{\varphi}_\alpha^\#(\cdot)} \lesssim \|Mf\|_{\bar{\varphi}(\cdot)} \lesssim \|f\|_{\bar{\varphi}(\cdot)}.$$

It is well known that $|u| \lesssim I_1|\nabla u|$ for $u \in C_0^\infty(\mathbb{R}^n)$. With this we directly obtain the following result.

Corollary (5.3.28)[196]: (Sobolev inequality)**.** Let $\varphi \in N(\mathbb{R}^n)$ satisfy assumptions (A0)−(A2) and suppose that $\varepsilon > 0$ is such that $s \mapsto s^{-(1+\varepsilon)}\varphi(x,s)$ is increasing and $s \mapsto s^{\frac{\varepsilon-n}{\alpha}}\varphi(x,s)$ is decreasing for every $x \in \mathbb{R}^n$. Then

$$\|u\|_{\varphi_\alpha^\#(\cdot)} \lesssim \|\nabla u\|_{\varphi(\cdot)}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$.

If $\Omega \subset \mathbb{R}^n$ is John domain, then $|u - u_\Omega| \lesssim I_1|\nabla u|$ by [180], and the same arguments yields that

$$\|u - u_\Omega\|_{\varphi_\alpha^\#(\cdot)} \lesssim \|\nabla u\|_{\varphi(\cdot)}$$

for all $u \in W^{1,1}(\Omega)$. Here u_Ω denotes the average of u over Ω.

Let $\alpha = 1$ and $\varphi(x,s) = s^p$ for some $p \in [1,\infty)$. Then $s \mapsto s^{-(1+\varepsilon)}\varphi(x,s)$ is increasing if $p > 1$ and $s \mapsto s^{-(n-\varepsilon)}\varphi(x,s)$ is increasing $p < n$. Thus Theorem (5.3.27) and Corollary (5.3.28) covers the parameter range $p < p < n$ in which case $\varphi_1^\#(s) = s^{p^*}$. The assumption $1 < p$ can probably be relaxed by weak-type estimate (cf. [156]).

Let $\varphi \in N$. Next we discuss the sharpness of $\varphi_1^\#$. Let (r_k) be positive sequence converging to zero. We set

$$v_k \coloneqq r_k\varphi^{-1}(r_k^{-n}),$$

for $k = 1, 2, \ldots$ Define $v_k \in C_0^\infty\big(B(0,3r_k)\big)$ such that it equals v_k in $B(0,r_k)$ and $|\nabla u_k| \le \frac{v_k}{r_k}$. By a straightforward calculation we obtain that

$$\int_{\mathbb{R}^n} \varphi(|\nabla u_k|)dx \lesssim r_k^n\varphi\left(\frac{v_k}{r_k}\right) = 1$$

for $k = 1, 2, \ldots$ Thus $\|\nabla u_k\|_\varphi \le 1$. On the other hand,

$$\int_{\mathbb{R}^n} \eta(|u_k|)dx \gtrsim r_k^n\eta\big(r_k\varphi^{-1}(r_k^{-n})\big)$$

for $\eta \in \Phi$. Thus we find that the Sobolev inequality $\|u\|_\eta \lesssim \|\nabla u\|_\varphi$ does not hold for all $u \in W_0^{1,\varphi}\big(B(0,1)\big)$ if

$$\limsup_{t\to 0^+} t^n\eta\big(t\varphi^{-1}(t^{-n})\big) = \infty.$$

We consider the function $\eta \coloneqq \varphi\ o\ \psi^{-1}$. With the substitution $t = \varphi(r)^{-1/n}$ we obtain

$$\begin{aligned} t^n\eta\big(t\varphi^{-1}(t^{-n})\big) &= \frac{1}{\varphi(r)}\varphi\left(\psi^{-1}\left[\varphi(r)^{-\frac{1}{n}}\varphi^{-1}\big(\varphi(r)\big)\right]\right) \\ &= \frac{1}{\varphi(r)}\varphi\left(\psi^{-1}\left[r\varphi(r)^{-\frac{1}{n}}\right]\right) = \frac{\varphi(\psi^{-1}[\lambda(\mathrm{r})])}{\varphi(r)}. \end{aligned}$$

If the Sobolev inequality holds, then by the previous argument the limit of this expression (as $t \to 0^+, i.e. r \to \infty$) must be finite. Then $\frac{\varphi(\psi^{-1}[\lambda(\mathrm{r})])}{\varphi(r)} < M \in [1,\infty)$ when $r > r_0$ for some r_0. Hence $\varphi(\psi^{-1}[\lambda(\mathrm{r})]) < M\varphi(r) \le \varphi(Mr)$ for some $r > r_0$.here the last inequality follows by the convexity of φ since $M \ge 1$. Consequently, $\psi^{-1}[\lambda(\mathrm{r})] \le Mr$ when $r > r_0$ so that $\psi^{-1}(t) \le M\lambda^{-1}(t)$ when $t > \lambda(r_0)$. If φ is doubling, this implies that $\eta(t) \le \big(M\lambda^{-1}(t)\big) \lesssim \varphi_1^\#(t)$. Hence we obtain the following proposition.

Proposition (5.3.29)[196]: Let $\varphi \in N$ be doubling and let λ be as in Definition (5.3.24). Let $\psi : [0,\infty) \to [0,\infty)$ be such that

$$\lim_{t\to\infty} \frac{\psi^{-1}(t)}{\lambda^{-1}(1)} = \infty.$$

Then there does not exists a constant $c > 0$ such that

$$\|u\|_{\varphi o \psi^{-1}} \le \|f\|_{\varphi}$$

for all $u \in C_0^\infty\big(B(0,1)\big)$.

Corollary (5.3.30)[205]: Assumption (A0M) implies (A0).

Proof. By the definition of φ^{-1}, the inequality $\varphi(x_r,1) \ge 1$ yield $\varphi^{-1}(x_r,1) \le 1$. If $\varphi\big(x_r,(1+\varepsilon)\big) < 1$, then $\varphi^{-1}(x_r,1) \ge (1+\varepsilon)$. If $\varphi\big(x_r,(1+\varepsilon)\big) = 1$, then by convexity $\varphi\left(x_r,\frac{1+\varepsilon}{2}\right) < 1$ and thus $\varphi^{-1}(x_r,1) \ge \left(\frac{1+\varepsilon}{2}\right)$.

The converse is not true. If (A0) holds, so that $c_1 \le \varphi^{-1}(x_r,1) \le c_2$, then $\varphi(x_r,c_1) \le 1$ and $\varphi(x_r,c_2) \ge 1$. But it is not necessary $\varphi(x_r,1) \ge 1$ as the following example shows: if $\varphi(1+\varepsilon) \coloneqq \frac{(1+\varepsilon)^2}{2}$, then $\varphi(x_r,1) = \frac{1}{2} < 1$ but $\varphi^{-1}(x_r,1) = 2$.

We use the assumption (A0) to find an equivalent Φ-function that behaves better than the original one. We set

$$\varphi_1\big(x_r,(1+\varepsilon)\big) \coloneqq \varphi\big(x_r,\varphi^{-1}(x_r,1)(1+\varepsilon)\big)$$

Then $\varphi_1$is equivalent to $\varphi_1^{-1}(x_r,1) \equiv \varphi_1(x_r,1) \equiv 1$ (by (16)). The set of Φ-functions with $\varphi_1^{-1}(x_r,1) \equiv 1$ will be denoted $\Phi_1(\mathbb{R}^n)$. Note that every $\Phi_1(\mathbb{R}^n)$-function satisfies assumption (A1) implies (A0M).

Corollary (5.3.31)[205]: Let $\varphi \in \Phi_1$ condition (A1) holds if and only if there exists $\varepsilon > 0$ such that

$$\varphi(x_r,(1+\varepsilon)^2) \le \varphi\big((x_r,\varepsilon),(1+\varepsilon)\big)$$

for every $(1+\varepsilon) \le \left[\varphi^{-1}\left((x_r+\varepsilon),\frac{1}{|B|}\right)\right]$, every $x_r,(\,x_r+\varepsilon) \in B$ and every ball B with $|B| \le 1$.

proof. Let the condition of the lemma hold and assume $\varepsilon \ge -1$. Then $\varphi^{-1}\big((x_r+\varepsilon),(1+\varepsilon)\big) \in \left[1,\varphi^{-1}\left((x_r+\varepsilon),\frac{1}{|B|}\right)\right]$ and so

$$\varphi\left(x_r,(1+\varepsilon)\varphi^{-1}\big((x_r+\varepsilon),(1+\varepsilon)\big)\right) \le \varphi\left((x_r+\varepsilon),\varphi^{-1}\big((x_r+\varepsilon),(1+\varepsilon)\big)\right) = (1+\varepsilon).$$

Let t_0 and $(t_0+2\varepsilon)$ be as in (15) and abbreviate $(t_0+\varepsilon) \coloneqq (1+\varepsilon)\varphi^{-1}\big((x_r+\varepsilon),(1+\varepsilon)\big)$. If $1 < \varepsilon \le 2\varepsilon$, then (A1) follows from the previous inequality, since $\varphi^{-1}\left(x_r,\varphi\big(x_r,(t_0+\varepsilon)\big)\right) = (t_0+\varepsilon)$. And if $\varepsilon \ge 0$, then $\varphi\big(x_r,(t_0+\varepsilon)\big) = \infty \le (1+\varepsilon)$, a contradiction, so this is not possible. If $\varepsilon \ge 0$, then $(t_0+\varepsilon) \le \varphi^{-1}\big(x_r,(1+\varepsilon)\big)$ since $\varphi^{-1}\big(x_r,(1+\varepsilon)\big) > t_0(\varepsilon \ge 0)$. Thus in each case (A1) holds.

Assume then that (A1) holds and let $\varepsilon \ge -1$. By (A1) and (16),

$$\varphi\left(x_r,(1+\varepsilon)\varphi^{-1}\big((x_r+\varepsilon),(1+\varepsilon)\big)\right) \le \varphi\left(x_r,\varphi^{-1}\big(x_r,(1+\varepsilon)\big)\right) = (1+\varepsilon) = \varphi\left((x_r+\varepsilon),\varphi^{-1}\big((x_r+\varepsilon),(1+\varepsilon)\big)\right).$$

Let $(t_0+\varepsilon) := \varphi^{-1}\big((x_r+\varepsilon),(1+\varepsilon)\big)$. Thus $\varphi\big(x_r,(1+\varepsilon)(t_0+\varepsilon)\big) \le \varphi\big((x_r+\varepsilon),(t_0+\varepsilon)\big)$ in the range of $\varphi^{-1}\big((x_r+\varepsilon),\cdot\big)$, including $(t_0,(t_0+2\varepsilon))$. When $(t_0+\varepsilon)\to t_0^+$, this gives that $\varphi\big((x_r+\varepsilon),(1+\varepsilon)t_0\big) \le \varphi\big((x_r+\varepsilon),t_0\big)=0$, so the inequality holds for $\varepsilon \ge 0$, as well. Finally, if $\varepsilon \ge 0$, then $\varphi\big((x_r+\varepsilon),(t_0+2\varepsilon)\big)=\infty$, so the inequality certainly holds.

Corollary (5.3.32)[205]: If $\varphi \in \Phi_1(\mathbb{R}^n)$ satisfies (A1), then it satisfies assumption (A1M).

Proof. Let B be a ball $|B| \le 1$. We must show that $\varphi_B^{-1}(1+\varepsilon)^2(\beta \le \varphi_B^{-}(1+\varepsilon)$ when $(1+\varepsilon) \le \left[(\varphi_B^{-1})^{-1}\left(\frac{1}{|B|}\right)\right]$.

Suppose first that $(1+\varepsilon)$ is not the upper end-point of the interval. For such $(1+\varepsilon)$, there exists $(x_r)_i+\varepsilon \in B$ such that $(1+\varepsilon) \le \left[\varphi^{-1}\left(((x_r)_i+\varepsilon),\frac{1}{|B|}\right)\right]$ and $\varphi_B^{-}(1+\varepsilon)=\lim_j \varphi\big(((x_r)_i+\varepsilon),(1+\varepsilon)\big)$. Then by Corollary (5.3.31).

$$\varphi(x_r,(1+\varepsilon)^2) \le \varphi\big(((x_r)_i+\varepsilon),(1+\varepsilon)\big).$$

We let $j\to\infty$ and take supremum over $x_r \in B$ to arrive (A1M).

It remains to consider $(t'+\varepsilon)=(\varphi_B^{-1})^{-1}\left(\frac{1}{|B|}\right)$. Suppose first $\varphi_B^{+}\big((1+\varepsilon)(t'+\varepsilon)\big)<\infty$. Let $\varepsilon>0$ and choose $x_r \in B$ such that $\varphi_B^{+}\big((1+\varepsilon)(t'+\varepsilon)\big) \le (1+\varepsilon)\varphi\big(x_r,(1+\varepsilon)(t'+\varepsilon)\big)$. Since $\varphi(x_r,\cdot)$ Is left-continuous, we can choose $\varepsilon<0$ such that $\varphi\big(x_r,(1+\varepsilon)(t'+\varepsilon)\big) \le (1+\varepsilon)\varphi(x_r,(1+\varepsilon)t')$. Combining this with the previous case, we obtain that

$$\begin{aligned}\varphi_B^{+}\big((1+\varepsilon)(t'+\varepsilon)\big) &\le (1+\varepsilon)^2\varphi(x_r,(1+\varepsilon)t')\\ &\le (1+\varepsilon)^2\varphi\big((x_r+\varepsilon),t'\big)\\ &\le (1+\varepsilon)^2\varphi\big((x_r+\varepsilon),(t'+\varepsilon)\big).\end{aligned}$$

Taking infimum over $(x_r+\varepsilon)$ and letting $\varepsilon\to 0$, we obtain the desired inequality. The case $\varphi_B^{+}\big((1+\varepsilon)(t'+\varepsilon)\big)=\infty$ is handled analogously.

Corollary (5.3.33)[205]: Let $\varphi \in \Phi_1$. If φ satisfies (A2), then it satisfies (A2M).

Proof. By theorem of [154], $L^{\psi(\cdot)}(\mathbb{R}^n) \subset L^{\varphi(\cdot)}(\mathbb{R}^n)$ if and only if there exist $\varepsilon \ge 0$ and $h \in L^1(\mathbb{R}^n)$ such that $\varphi(x_r,(1+\varepsilon)^2) \le \psi\big(x_r,(1+\varepsilon)\big)+h(x_r)$. Hence (A2) implies that for every t_0 there exists $(1+\varepsilon)$ and $h \in L^1(\mathbb{R}^n)$ such that

$$\begin{aligned}\varphi(x_r,(1+\varepsilon)^2) \le \varphi_\infty(1+\varepsilon)+h\,(1+\varepsilon) \quad \text{and} \quad \varphi_\infty(1+\varepsilon)^2\\ \le \varphi\big(x_r,(1+\varepsilon)\big)+h(x_r)\end{aligned}$$

for all $-1 \le \varepsilon \le 0$ (the restricted range of $(1+\varepsilon)$ is due to the intersection with $L^\infty(\mathbb{R}^n)$ in the assumption). From these we obtain that

$$\begin{aligned}\varphi\big(x_r,(t_0+\varepsilon)^2(1+\varepsilon)\big) &\le \varphi_\infty(1+\varepsilon)+h(x_r)\\ &\le \varphi\big((x_r+\varepsilon),(1+\varepsilon)\big)+h(x_r+\varepsilon)+h(x_r)\end{aligned}$$

for $0 \le \varepsilon \le 1$. Since $\varphi(x_r,1)\equiv 1$. So the inequality also holds when we replace h by $\min\{h,1\} \in L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$, as required by (A2M).

Corollary (5.3.34)[205]: If $\varphi \in \Phi_1(\mathbb{R}^n)$ satisfies (A0)−(A2) and there exists $\varepsilon \ge 0$ such that $(t_0+\varepsilon)\mapsto (t_0+\varepsilon)^{-(1+\varepsilon)}\varphi\big(x_r,(t_0+\varepsilon)\big)$ is increasing for every $x_r \in \mathbb{R}^n$, then $M : L^{\varphi(\cdot)}(\mathbb{R}^n) \to L^{\varphi(\cdot)}(\mathbb{R}^n)$ is bounded.

Proof. As in (A0), we find $\varphi_1 \in \Phi_1$ with $\varphi_1 \simeq \varphi$. Then φ_1 satisfies (A0M). By Lemma (5.3.6)[196], φ_1 satisfies (A1) and (A2). A short calculation gives that $(t_0+\varepsilon) \mapsto (t_0+\varepsilon)^{-(1+\varepsilon)}\varphi_1\big(x_r,(t_0+\varepsilon)\big)$ is increasing. By Corollary (5.3.32) and Corollary (5.3.33), (A1M) and (A2M) hold. Therefore by thus also on $L^{\varphi(\cdot)}(\mathbb{R}^n)$.

Φ-function are not totally well-behaved with respect to taking limits. Consider for instance $(1+\varepsilon)^{1+\varepsilon}$. As $\varepsilon \to \infty$, the pointwise limit is $\infty\chi_{(1,\infty)} + \chi_{\{1\}}$, which is not left-continuous. For the equivalent Φ-function $\frac{1}{1+\varepsilon}(1+\varepsilon)^{1+\varepsilon}$ we have $\lim \infty\chi_{(1,\infty)}$, which is what we want. Therefore, we need to chose the equivalent Φ-function suitably to get a good limit.

We are especially interested in the behavior of φ_∞ when $\varepsilon \le 0$. To this end we define [196]

$$\varphi_2\left(x_r,\left(\frac{1}{2}-\varepsilon\right)\right) := \max\left\{\varphi_1\left(x_r,\left(\frac{1}{2}-\varepsilon\right)\right),(1-2\varepsilon)\right\}.$$

Clearly $\varphi_1 \le \varphi_2$. For $\varepsilon \le 0$, $\varphi_2 = \varphi_1$. Since $\varphi \in \Phi_1(\mathbb{R}^n)$ we have $\varphi_1(x_r,1) = 1$ and $\varphi_1\left(x_r,\left(\frac{1}{2}-\varepsilon\right)\right) \ge \left(\frac{1}{2}-\varepsilon\right)$ for $\varepsilon \ge 0$ by convexity. Thus $\varphi_2\big(x_r,(1+\varepsilon)\big) \le \varphi_2(x_r,1) = 1 \le \varphi_1\big(x_r,2(1+\varepsilon)\big)$ for $-\frac{1}{2} < \varepsilon \le 0$ and $\varphi_2\big(x_r,(1+\varepsilon)\big) \le 2\varphi_1\big(x_r,(1+\varepsilon)\big) \le \varphi_1\big(x_r,2(1+\varepsilon)\big)$ for $\varepsilon > 0$. In sum, obtain $\varphi_2 \simeq \varphi_1 \simeq \varphi$ with $\varphi_2(x_r,1) \equiv 1 \equiv \varphi_2^{-1}(x_r,1)$.

Note that the right-derivative satisfies $\varphi_2'(x_r,1^-) \in [1,2]$: here the lower bound follows from convexity $\varphi_2'(x_r,1) \ge \varphi_2\big(x_r,(1+\varepsilon)\big) = 1$ and the upper bound holds since if $\varphi_2'(x_r,1^-) > 2$, then $\varphi_2\big(x_r,(1+\varepsilon)\big) < (1+2\varepsilon)$ for some $\varepsilon < 0$ contrary to construction of φ_2.

We consider then $\text{limit}(\varphi_2)_\infty(1+\varepsilon) = \limsup_{|x_r|\to\infty} \varphi_2\big(x_r,(1+\varepsilon)\big)$. Cleary $(\varphi_2)_\infty(0) = 0$ and $(\varphi_2)_\infty(1) = 1$. For $0 < \varepsilon < 1$, $(\varphi_2)_\infty(1) \ge (\varphi_2)_\infty(1+\varepsilon) \ge (1+2\varepsilon)$ and hence $(\varphi_2)_\infty$ is left-continuous at 1. By convexity of $\varphi_2, \varphi_2\big(x_r,(1+\varepsilon)\big) \le (1+\varepsilon)\varphi_2(x_r,1)$ on $[0,1]$ and hence $(\varphi_2)_\infty(0^+) = 0$. Since $(\varphi_2)_\infty(1+\varepsilon) \ge (1+\varepsilon)$ for $\varepsilon \ge 0$, we have $\lim_{\varepsilon\to\infty}(\varphi_2)_\infty(1+\varepsilon) = \infty$.

Corollary (5.3.35)[205]: If $\varphi \in \Phi(\mathbb{R}^n)$ satisfies (A0)−(A2), then $\bar{\varphi} \in \Phi_1(\mathbb{R}^n)$.

Proof. For the convexity we have to show that the $\bar{\varphi}'(x_r,\cdot)$ is increasing for every $x_r \in \mathbb{R}^n$. We have

$$\bar{\varphi}'\big(x_r,(1+\varepsilon)\big) = \begin{cases} 2\varphi_2'\big(x_r,(1+\varepsilon)\big), & \text{if } \varepsilon \ge 0, \\ ((\varphi_2)_\infty'(1-\varepsilon), & \text{if } \varepsilon < 0. \end{cases}$$

By convexity each of the parts is increasing. At, $2\varphi_2'(x_r,1) \ge 2$ and $\lim_{(1+\varepsilon)\to 1^-}(\varphi_2)_\infty'(1+\varepsilon) \le 2$ (see discussion regarding) (A2), so the right-derivative is increasing also there.

The function $\bar{\varphi}$ is continuous since both φ_2 and $(\varphi_2)_\infty$ are continuous and $\varphi_2(x_r,1) = (\varphi_2)_\infty(1^-) = 1$. Thus we have that $\bar{\varphi}^{-1}(x_r,1) \le 1$. In the discussion on (A2), we noted that $\varphi_2\big(x_r,(1+\varepsilon)\big) \le (1+\varepsilon)$ on $[0,1]$. These together give $\bar{\varphi}^{-1}(x_r,1) \equiv 1$.

The conditions $\bar{\varphi}(x_r, 0) = \bar{\varphi}(x_r, 0^+) = 0$ and $\lim_{\varepsilon\to\infty} \bar{\varphi}(1+\varepsilon) = \infty$ follow from the same continuous for φ_2 and $(\varphi_2)_\infty$.

Corollary (5.3.36)[205]: If $\varphi \in \Phi(\mathbb{R}^n)$ satisfies (A0)−(A2), then $L^{\varphi(\cdot)} = L^{\overline{\varphi}(\cdot)}$ with equivalent norms.

Proof. Since $\varphi \simeq \varphi_2$, it suffices to show that $L^{\varphi_2(\cdot)} = L^{\overline{\varphi}(\cdot)}$.

Let $g \in L^{\varphi_2(\cdot)}(\mathbb{R}^n)$ and set $f := g/\|g\|_{\varphi_2(\cdot)}$. we divide f into two parts $f_1 = \chi_{\{|f|<1\}}$ and $f_2 = \chi_{\{|f|\geq 1\}}$. By (A2), and since $\|f\|_{\varphi_2(\cdot)} = 1$,

$$\begin{aligned}\|f_1\|_{\overline{\varphi}(\cdot)} &= \|f_1\|_{(\varphi_2)_\infty} \leq \|f_1\|_{L^{(\varphi_2)_\infty}\cap L^\infty} \approx \|f_1\|_{L^{(\varphi_2)_\infty}\cap L^\infty} \\ &= \max\{\|f_1\|_{\varphi_2(\cdot)}, \|f_1\|_\infty\} \leq 1.\end{aligned}$$

If $|f_2(x_r)| \geq 1$, then

Otherwise $|f_2(x_r)| = 0$, and the inequality holds as well. Thus $\|f_1\|_{\overline{\varphi}(\cdot)} \lesssim \|f_2\|_{\varphi_2(\cdot)} \leq 1$ and hence

$$\left\|g/\|g\|_{\varphi_2(\cdot)}\right\|_{\overline{\varphi}(\cdot)} = \|f\|_{\overline{\varphi}(\cdot)} \leq \|f_1\|_{\overline{\varphi}(\cdot)} + \|f_1\|_{\overline{\varphi}(\cdot)} \lesssim 1,$$

so that $\|g\|_{\overline{\varphi}(\cdot)} \lesssim \|g\|_{\varphi_2(\cdot)}$. the opposite inequality is proved similarly.

Corollary (5.3.37)[205]: If $\varphi \in \Phi(\mathbb{R}^n)$ satisfies (A0)−(A2), then there exists $0 < \varepsilon < 1$ such that

$$(1-\varepsilon)\bar{\varphi}^{-1}\big(x_r, (1+\varepsilon)\big) \leq \bar{\varphi}^{-1}\big((x_r+\varepsilon), (1+\varepsilon)\big) \tag{19}$$

for every $\varepsilon \geq -1$, every $x_r, (r_0+\varepsilon) \in B$ and every ball B.

Proof. If $\varepsilon \leq 0$, then $\bar{\varphi}$ is independent of x_r, so the claim is trivial. Thus it remains only to consider the case $\varepsilon > 0$. Then by Corollary (5.3.31) the inequality holds if and only if

$$\bar{\varphi}\big(x_r, (1+\varepsilon)(t_0+\varepsilon)\big) \leq \bar{\varphi}\big((x_r+\varepsilon), (t_0+\varepsilon)\big)$$

for every $(1+\varepsilon) \leq \left[\bar{\varphi}^{-1}\left((x_r+\varepsilon), \frac{1}{|\beta|}\right)\right]$, since $\bar{\varphi} \geq \varphi_2$, such s satisfies $(t_0+\varepsilon) \in \left[1, \varphi_2^{-1}\left((x_r+\varepsilon), \frac{1}{|\beta|}\right)\right]$. if $(1+\varepsilon)(t_0+\varepsilon) \geq 1$, then using Corollary (5.3.31) for φ_2 we calculate

$$\begin{aligned}\bar{\varphi}\big(x_r, (1+\varepsilon)(t_0+\varepsilon)\big) &= 2\varphi_2\big(x_r, (1+\varepsilon)(t_0+\varepsilon)\big) - 1 \\ &\leq 2\varphi_2\big((x_r+\varepsilon), (t_0+\varepsilon)\big) - 1 \\ &= \bar{\varphi}\big((x_r+\varepsilon), (t_0+\varepsilon)\big).\end{aligned}$$

If $(1+\varepsilon)(t_0+\varepsilon) < 1$, then $\bar{\varphi}\big(x_r, (1+\varepsilon)(t_0+\varepsilon)\big) \leq \bar{\varphi}(x_r, 1) = 1 \leq \bar{\varphi}((x_r+\varepsilon), 1) \leq \bar{\varphi}\big((x_r+\varepsilon), (t_0+\varepsilon)\big)$, so the inequality holds in both cases.

Corollary (5.3.38)[205]: Suppose that $\varphi \in \Phi(\mathbb{R}^n)$ is normalized. Let $B \ni x_r$ be a ball. Then

$$\|\chi_B\|_{\varphi(\cdot)} \leq \frac{1}{(1+\varepsilon)\varphi^{-1}\left(x_r, \frac{1}{|\beta|}\right)}.$$

Proof. By assumption

$$\varphi\left((x_r+\varepsilon), (1+\varepsilon)\varphi^{-1}\left(x_r, \frac{1}{|\beta|}\right)\right) \leq \varphi\left((x_r+\varepsilon), (1+\varepsilon)\varphi^{-1}\left((x_r+\varepsilon), \frac{1}{|\beta|}\right)\right) \leq \frac{1}{|\beta|}$$

when $x_r, (x_r+\varepsilon) \in B$, and hence

$$\varrho_{\varphi(\cdot)}\left((1+\varepsilon)\varphi^{-1}\left(x_r,\tfrac{1}{|\beta|}\right)\chi_B\right) = \int_B \varphi\left((x_r+\varepsilon),(1+\varepsilon)\varphi^{-1}\left(x_r,\frac{1}{|\beta|}\right)\right)d(x_r+\varepsilon) \le 1.$$

Corollary (5.3.39)[205]: If $\varphi \in N(\mathbb{R}^n)$ satisfies (A0)−(A2), then $\bar{\varphi} \in N_1(\mathbb{R}^n)$.

Proof. By Corollary (5.3.35), $\bar{\varphi} \in \Phi_1(\mathbb{R}^n)$. We need to check that the normalizations do not destroy the functions η and ξ. By (A0), there exists $0<\varepsilon<1$ such that $(1+\varepsilon) \le \varphi^{-1}(x_r,1) \le \frac{1}{1+\varepsilon}$. First we set $\eta_1(1+\varepsilon) \coloneqq \eta(1+\varepsilon)^2$ and $\xi_1(1+\varepsilon) \coloneqq \xi(1)$. Then $\eta_1 \le \varphi_1 \le \xi_1$. As before $\eta_2(1+\varepsilon) = \max\{\eta_1(1+\varepsilon),(1+2\varepsilon)\}$, similarly for ξ. Then also $\eta_2 \le \varphi_2 \le \xi_2$, and we easily see that η_2 and ξ_2 are still N-functions. Furthermore, $\bar{\eta} = \max\{\eta_2, 2\eta_2 - 1\}$ is an N-function minorizing $\bar{\varphi}$, similarly for $\bar{\xi}$.

Corollary (5.3.40)[205]: Let $\varphi \in N$ and $\varepsilon > 0$. Then $(t_0+\varepsilon) \mapsto (t_0+\varepsilon)^{-(1+\varepsilon)}\varphi(t_0+\varepsilon)$ is increasing if and only if $(t_0+\varepsilon) \mapsto (t_0+\varepsilon)^{-\frac{1+\varepsilon}{\varepsilon}}\varphi^*(t_0+\varepsilon)$ is decreasing.

Proof. We note that $(1+\varepsilon) \mapsto (1+\varepsilon)^{-(1+\varepsilon)}\varphi(1+\varepsilon)$ is increasing if and only if $D\left((1+\varepsilon)^{-(1+\varepsilon)}\varphi(1+\varepsilon)\right) \ge 0$, i.e. $(1+\varepsilon)\varphi'(1+\varepsilon) \ge (1+\varepsilon)\varphi(1+\varepsilon)$. Since φ is continuous, we conclude from this that

$$(1+\varepsilon)\varphi'(t^-) \ge (1+\varepsilon)\varphi(t^-) = (1+\varepsilon)\varphi(1+\varepsilon).$$

On the other hand, as noted after the definition of φ^*, with $(1+\varepsilon) \coloneqq (\varphi')^{-1}(1+\varepsilon)$,

$$\varphi^*(t_0+\varepsilon) = (t_0+\varepsilon)(1+\varepsilon) - \varphi(1+\varepsilon) \ge (t_0+\varepsilon)(1+\varepsilon) - \varphi'(1).$$

By [154], $(1+\varepsilon) = (\varphi^*)'(t_0+\varepsilon)$ and by [154], $\varphi'(\varphi^*)'(t_0+\varepsilon) \le (t_0+\varepsilon)$ for all $\varepsilon > 0$, so that $\varphi'(t^-) \le (t_0+\varepsilon)$. This gives $\varphi^*(t_0+\varepsilon) \ge \varepsilon(t_0+\varepsilon) = \frac{\varepsilon}{1+\varepsilon}(t_0+\varepsilon)(\varphi^*)'(t_0+\varepsilon)$, which is equivalent to $D\left((t_0+\varepsilon)^{-\frac{1+\varepsilon}{\varepsilon}}\varphi^*(t_0+\varepsilon)\right) \le 0$, as was be shown.

Corollary (5.3.41)[205]: If $\varphi \in N_1(\mathbb{R}^n)$ is normalized, then also $\varphi^* \in N(\mathbb{R}^n)$ is normalized.

Proof. First we note that $\eta^*, \xi^* \in N$ by [154]. The inequalities $\eta(1+\varepsilon) \le \varphi\left(x_r,(1+\varepsilon)\right) \le \xi(1+\varepsilon)$ yield $\xi^*(1+\varepsilon) \le \varphi^*\left(x_r,(1+\varepsilon)\right) \le \eta^*(1+\varepsilon)$ by [154], and thus $\varphi^* \in N(\mathbb{R}^n)$.

By [154], $(1+\varepsilon) \le \psi^{-1}(1+\varepsilon)(\psi^*)^{-1}(1+\varepsilon) \le 2(1+\varepsilon)$ for $\psi \in N$. Let $x_r, (x_r+\varepsilon) \in B$ and $(1+\varepsilon) \le \frac{1}{|B|}$. Then

$$\frac{1+\varepsilon}{2}(\varphi^*)^{-1}\left(x_r,(1+\varepsilon)\right) \le \frac{(1+\varepsilon)^2}{\varphi^{-1}\left(x_r,(1+\varepsilon)\right)} \le \frac{(1+\varepsilon)}{\varphi^{-1}\left(x_r,(1+\varepsilon)\right)}$$
$$\le (\varphi^*)^{-1}\left((x_r+\varepsilon),(1+\varepsilon)\right).$$

Furthermore, $\varphi\left(x_r,(1+\varepsilon)\right) \ge \varphi(x_r,1)(1+\varepsilon) = (1+\varepsilon)$ when $\varepsilon \ge 0$ (since φ is convex). When $\varepsilon \le 0$ and $\varepsilon \ge 0$, it follows that $(1+\varepsilon)^2 - \varphi\left(x_r,(1+\varepsilon)\right) \le (1+\varepsilon)(\varepsilon) \le 0$. Hence, for $\varepsilon \le 0$.

$$\varphi^*\left(x_r,(1+\varepsilon)\right) = \sup_{\varepsilon>-1}\left((1+\varepsilon)^2 - \varphi\left(x_r,(1+\varepsilon)\right)\right)$$
$$= \sup_{-1\le\varepsilon\le 0}\left((1+\varepsilon)^2 - \varphi\left(x_r,(1+\varepsilon)\right)\right)$$

$$= \sup_{-1\le\varepsilon\le 0} \left((1+\varepsilon)^2 - \varphi_\infty(1+\varepsilon)\right)$$
$$= \varphi_\infty^*(1+\varepsilon)$$
is independent of x_r. Since $0 = \lim_{1+\varepsilon\to 0^+} \frac{\varphi(x_r,(1+\varepsilon))}{1+\varepsilon} = \lim_{(1+\varepsilon)\to 0^+} \frac{\varphi_\infty(1+\varepsilon)}{1+\varepsilon}$ we obtain $\varphi_\infty^*(1) = \sup_{-1\le\varepsilon\le 0} \left((1+\varepsilon)^2 - \varphi_\infty(1+\varepsilon)\right) > 0$. Therefore, we have shown that it is a normalized N-function.

Corollary (5.3.42)[205]: For $\varphi \in \Phi(\mathbb{R}^n)$ we write $\hat{\varphi}(x_r,(1+\varepsilon)) \coloneqq \varphi^*\left(x_r,(1+\varepsilon)^{\frac{\varepsilon}{1+2\varepsilon}}\right)$. Assume that M is bounded from $L^{\hat{\varphi}(\cdot)}(\mathbb{R}^n)$ to itself. Let $x_r \in \mathbb{R}^n$, $\delta > 0$, and $f \in L^{\varphi(\cdot)}(\mathbb{R}^n)$ with $\|f\|_{\varphi(\cdot)} \le 1$. Then
$$\int_{\mathbb{R}^n\setminus B(x_r,\delta)} \frac{|f(x_r+\varepsilon)|}{|x_r|^\varepsilon} dy \lesssim |B(x_r,\delta)|^{\frac{\varepsilon}{1+2\varepsilon}} \left\|\chi_{B(x_r,\delta)}\right\|_{\varphi^*(\cdot)}.$$
Proof. $B \coloneqq B(x_r,\delta)$. We start with Hölder's inequality and take into account that $\|f\|_{\varphi(\cdot)} \le 1$:
$$\int_{\mathbb{R}^n\setminus B} \frac{|f(x_r+\varepsilon)|}{|x_r|^\varepsilon} d(x_r+\varepsilon) \le 2\|f\|_{\varphi(\cdot)} \left\|\chi_{\mathbb{R}^n\setminus B}|x_r - \cdot|^\varepsilon\right\|_{\varphi^*(\cdot)}$$
$$\le 2\left\|\chi_{\mathbb{R}^n\setminus B}|x_r - \cdot|^{-n}\right\|_{\hat{\varphi}(\cdot)}^{\frac{\varepsilon}{1+2\varepsilon}}.$$
Next we note that, for $x_r \in \mathbb{R}^n\setminus B$,
$$M(\chi_B|B|^{-1})(x_r+\varepsilon) \ge \int_{B((x_r+\varepsilon),2|\varepsilon|)} \chi_B(x_r+2\varepsilon)|B|^{-1} d(x_r+2\varepsilon)$$
$$= |B((x_r+\varepsilon),2|\varepsilon|)|^{-1} = c|\varepsilon|^{-1}.$$
Therefore $\chi_{\mathbb{R}^n\setminus B}(x_r+\varepsilon)|\varepsilon|^{1+2\varepsilon} \lesssim M(\chi_B|B|^{-1})(x_r+\varepsilon)$ for all $(x_r+\varepsilon) \in \mathbb{R}^n$. Combining the previous estimates and using the boundedness of M, we find that
$$\int_{\mathbb{R}^n\setminus B} \frac{|f(x_r+\varepsilon)|}{|\varepsilon|^\varepsilon} d(x_r+\varepsilon) \lesssim \|M(\chi_B|B|^{-1})\|_{\hat{\varphi}(\cdot)}^{\frac{\varepsilon}{1+2\varepsilon}} = |B|^{\frac{\varepsilon}{1+2\varepsilon}} \|M(\chi_B)\|_{\hat{\varphi}(\cdot)}^{\frac{\varepsilon}{1+2\varepsilon}}$$
$$\lesssim |B|^{\frac{\varepsilon}{1+2\varepsilon}} \|\chi_B\|_{\hat{\varphi}(\cdot)}^{\frac{\varepsilon}{1+2\varepsilon}} = |B|^{\frac{\varepsilon}{1+2\varepsilon}} \|\chi_B\|_{\varphi^*(\cdot)}.$$
Recall that a function is almost decreasing if $f(x_r) \le Qf(x_r+\varepsilon)$ when $\varepsilon < 0$, for some fixed $Q \in [1,\infty)$. Almost increasing defined analogously.

Corollary (5.3.43)[205]: Let $\varphi \in N_1(\mathbb{R}^n)$ be normalized and suppose that $(1+\varepsilon) \mapsto (1+\varepsilon)^{-1}\varphi(x_r,(1+\varepsilon))$ is almost decreasing for every $x_r \in \mathbb{R}^n$. Then
$$I_{1+\varepsilon} f \lesssim \varphi(x_r, Mf(x_r))^{\frac{-(1+\varepsilon)}{1+2\varepsilon}} Mf(x_r) \quad a.e.$$
for every $f \in L^{\varphi(\cdot)}(\mathbb{R}^n)$ with $\|f\|_{\varphi(\cdot)} \le 1$.

Proof. Let us $B \coloneqq B(x_r,\delta)$. We divide the Riesz-potential into two parts:
$$I_{1+\varepsilon} f(x_r) \coloneqq \int_B \frac{|f(x_r+\varepsilon)|}{|\varepsilon|^\varepsilon} d(x_r+\varepsilon) + \int_{\mathbb{R}^n\setminus B} \frac{|f(x_r+\varepsilon)|}{|\varepsilon|^{1+\varepsilon}} d(x_r+\varepsilon).$$
In the first part we split the integration domain into annuli and use the definition of M:

$$\int_B \frac{|f(x_r+\varepsilon)|}{|\varepsilon|^\varepsilon} d(x_r+\varepsilon) \le \sum_{k=1}^{\infty} (\delta 2^{-k})^{-\varepsilon} \int_{2^{-k}\delta \le |\varepsilon| < 2^{-k+1}\delta} |f(x_r+\varepsilon)|\, d(x_r+\varepsilon)$$
$$\lesssim \sum_{k=1}^{\infty} (\delta 2^{-k})^{1+\varepsilon} \int_{|\varepsilon|<2^{-k+1}\delta} |f(x_r+\varepsilon)|\, d(x_r+\varepsilon)$$
$$\le \delta^{1+\varepsilon} \sum_{k\in\mathbb{N}}^{\infty} 2^{-(1+\varepsilon)k}\, Mf(x_r) = c|B|^{\frac{1+\varepsilon}{1+2\varepsilon}} Mf(x_r).$$

Let $\hat{\varphi}(x_r,(1+\varepsilon)) = \varphi^*\left(x_r,(1+\varepsilon)^{\frac{\varepsilon}{1+2\varepsilon}}\right)$ be as is Corollary (5.3.42). By Corollary (5.3.41), φ^* is normalized. Thus φ^* satisfies $(A0)-(A2)$. Further $\hat{\varphi}^{-1}(x_r,(1+\varepsilon)) = \left((\varphi^*)^{-1}(x_r,(1+\varepsilon))\right)^{\frac{1+2\varepsilon}{\varepsilon}}$, and so $\hat{\varphi}$ inherits $(A0)-(A2)$ from φ^*.

Set $\varepsilon := 0$ and define $\psi(x_r,(1+\varepsilon)) := (1+\varepsilon)^{1+\varepsilon} \sup_{\varepsilon} (1+\varepsilon)^{-(1+\varepsilon)} \varphi(x_r,(1+\varepsilon))$ for $\varepsilon \ge 0$. Thus definition directly implies that $(1+\varepsilon)^{-(1+\varepsilon)}\psi(x_r,(1+\varepsilon))$ is decreasing and $\psi \ge \varphi$. Since $(1+\varepsilon)^{-(1+\varepsilon)}(x_r,(1+\varepsilon))$ is almost decreasing by assumption, $\psi \le Q\varphi$, so $\varphi \simeq \psi$. By Corollary (5.3.41), $(1+\varepsilon)^{-\frac{(1+\varepsilon)}{\varepsilon}}\psi^*(x_r,(1+\varepsilon))$ is increasing, and since $\varphi^* \simeq \psi^*$ it follows that $(1+\varepsilon)^{-\frac{(1+\varepsilon)}{\varepsilon}}\varphi^*(x_r,(1+\varepsilon))$ is almost increasing. Therefore with $\varepsilon = (1+\varepsilon)^{\frac{\varepsilon}{1+2\varepsilon}}$,
$$\hat{\varphi}(x_r,(1+\varepsilon)) = \varphi^*(x_r,1) = (1+\varepsilon)^{-(1+\varepsilon)}\varphi^*(x_r,(1+\varepsilon))$$
is almost increasing. A calculation yields that $(\widehat{1+\varepsilon}) := \left(\frac{1+\varepsilon}{\varepsilon}\right)^{\frac{1+\varepsilon}{1+2\varepsilon}} > 1$. Therefore $\hat{\varphi}$ equivalent to Φ-function ξ with $(1+\varepsilon)^{-\frac{1+\varepsilon}{\varepsilon}}\xi(x_r,(1+\varepsilon))$ increasing (cf. [188]). Since $\hat{\varphi} \simeq \xi$, also $(A0)-(A2)$ holds. By Corollary (5.3.34), M is bounded on $L^{\xi(\cdot)}$, and hence also on $L^{\hat{\varphi}(\cdot)}$.

Therefore, the assumptions of Corollary (5.3.42) hold, and follows that
$$\int_{\mathbb{R}^n\setminus B} \frac{|f(x_r+\varepsilon)|}{|\varepsilon|^\varepsilon} d(x_r+\varepsilon) \lesssim |B|^{\frac{-\varepsilon}{1+2\varepsilon}} \|\chi_B\|_{\varphi^*(\cdot)} \tag{20}$$
provided $\|f\|_{\varphi(\cdot)} \le 1$.

We combine (20) with Corollary (5.3.38) and (5.3.41), and obtain
$$I_{1+\varepsilon}f(x_r) \lesssim |B|^{\frac{1+\varepsilon}{1+2\varepsilon}}(1+\varepsilon)f(x_r) + |B|^{\frac{1+\varepsilon}{1+2\varepsilon}}\|\chi_B\|_{\varphi^*(\cdot)}$$
$$\le |B|^{\frac{-\varepsilon}{1+2\varepsilon}}|B|(1+\varepsilon)f(x_r) + \frac{|B|^{\frac{-\varepsilon}{1+2\varepsilon}}}{(1+\varepsilon)(\varphi^*)^{-1}\left(x_r,\frac{1}{|B|}\right)}.$$

Now $(\varphi^*)^{-1}(x_r,(1+\varepsilon)) \approx (1+\varepsilon)/\varphi^{-1}(x_r,(1+\varepsilon))$ by [12] and so
$$I_{1+\varepsilon}f(x_r) \lesssim |B|^{\frac{1+\varepsilon}{1+2\varepsilon}}(1+\varepsilon)f(x_r) + |B|^{\frac{1+\varepsilon}{1+2\varepsilon}}\varphi^{-1}\left(x_r,\frac{1}{|B|}\right).$$
when $(1+\varepsilon)f(x_r) < \infty$, we choose the radius δ such that $(1+\varepsilon)f(x_r) = \varphi^{-1}\left(x_r,\frac{1}{|B|}\right)$, i.e. $|B| = \frac{1}{\varphi(x_r,(1+\varepsilon)f(x_r))}$. Thus

$$I_{1+\varepsilon}f(x_r) \lesssim \varphi\big(x_r,(1+\varepsilon)f(x_r)\big)^{-\frac{1+\varepsilon}{1+2\varepsilon}}(1+\varepsilon)^2 f(x_r) \quad a.e.$$

Corollary (5.3.44)[205]: Let $\varepsilon > -1$, $\varphi \in N(\mathbb{R}^n)$ with $(1+\varepsilon) \mapsto (1+\varepsilon)^{-\frac{1+\varepsilon}{1+2\varepsilon}}\varphi\big(x_r,(1+\varepsilon)\big)$ strictly decreasing to 0 for every $x_r \in \mathbb{R}^n$ and let $\lambda\big(x_r,(1+\varepsilon)\big) := (1+\varepsilon)\varphi\big(x_r,(1+\varepsilon)\big)^{-\frac{1+\varepsilon}{1+2\varepsilon}}$. Then $\varphi\ o\ (\lambda^{-1})$ is equivalent to convex Φ-function.

By $\varphi\ o\ (\lambda^{-1})$ we main the function $\big(x_r,(1+\varepsilon)\big) \mapsto \varphi\Big(x_r,\lambda^{-1}\big(x_r,(1+\varepsilon)\big)\Big)$.

Proof. Since the claim is pointwise in nature, we drop the variable x_r for the rest of the proof.

Let us denote $\psi := \varphi\ o\ (\lambda^{-1})$. Since $(1+\varepsilon)^{-\frac{1+\varepsilon}{1+2\varepsilon}}\varphi(1+\varepsilon) \to 0$ we find that $\lambda(1+\varepsilon) \to \infty$ as $\varepsilon \to \infty$. Thus also $\psi(1+\varepsilon) \to \infty$ as $\varepsilon \to \infty$. The function

$$\lambda(1+\varepsilon) = \frac{(1+\varepsilon)^{\frac{1+\varepsilon}{1+2\varepsilon}}}{\varphi(1+\varepsilon)}$$

is strictly increasing by assumption, so the same holds for λ^{-1}. Furthermore, with $(1+\varepsilon) = \lambda(1+\varepsilon)$, the fraction

$$\frac{\lambda^{-1}(1+\varepsilon)}{1+\varepsilon} = \frac{1+\varepsilon}{\lambda(1+\varepsilon)} = \varphi(1+\varepsilon)^{\frac{1+\varepsilon}{1+2\varepsilon}}$$

is increasing $(1+\varepsilon)$ (since φ is increasing), hence is s as well. Since $(1+\varepsilon) \mapsto \frac{\varphi(1+\varepsilon)}{1+\varepsilon}$ is increasing (due to convex of φ and $\varphi(0) = 0$)this yields that

$$\frac{\psi(1+\varepsilon)}{1+\varepsilon} = \frac{\varphi\big(\lambda^{-1}(1+\varepsilon)\big)}{\lambda^{-1}(1+\varepsilon)}\frac{\lambda^{-1}(1+\varepsilon)}{1+\varepsilon}$$

is also increasing. Since $\frac{\psi(1+\varepsilon)}{1+\varepsilon}$ is increasing, we obtain $\lim_{(1+\varepsilon)\to 0^+}\psi(1+\varepsilon) = 0$. Thus it follows from Lemma (5.3.4) that ψ is equivalent to a convex function (see [196]).

Corollary (5.3.45)[205]: If $\varphi \in \Phi(\mathbb{R}^n)$ satisfies assumptions (A0)−(A2) and $(1+\varepsilon) \mapsto (1-\varepsilon)^{1-\varepsilon}\varphi\big(x_r,(1-\varepsilon)\big)$, $\varepsilon < 1$, is decreasing, $(1-\varepsilon) \mapsto (1-\varepsilon)^{1-\varepsilon}\bar{\varphi}\big(x_r,(1-\varepsilon)\big)$ is almost decreasing.

Proof. We prove first $(1-\varepsilon) \mapsto (1-\varepsilon)^{1-\varepsilon}\varphi_2\big(x_r,(1-\varepsilon)\big)$ is almost decreasing. Since $\varphi \simeq \varphi_2$, for $\varepsilon > 0$,

$$\begin{aligned}(1-2\varepsilon)^{1+\varepsilon}\varphi_2\big(x_r,(1-2\varepsilon)\big) &\geq (1-2\varepsilon)^{1-\varepsilon}\varphi\left(x_r,\frac{1-2\varepsilon}{1+\varepsilon}\right)\\ &= (1+2\varepsilon)^{1-\varepsilon}\varphi\left(x_r,\frac{1-2\varepsilon}{1+\varepsilon}\right)\\ &\geq (1-\varepsilon)^{1-\varepsilon}\varphi\left(x_r,\frac{1-\varepsilon}{1+\varepsilon}\right)\\ &\geq \big((1+\varepsilon)^2(1-\varepsilon)\big)^{1-\varepsilon}\varphi\big(x_r,(1-\varepsilon^2)\big)\\ &\geq \big((1+\varepsilon)^2(1-\varepsilon)\big)^{1-\varepsilon}\varphi_2\big(x_r,(1-\varepsilon)\big)\\ &= (1+\varepsilon)^{2(1-\varepsilon)}(1-\varepsilon)^{1-\varepsilon}\big(x_r,(1-\varepsilon)\big).\end{aligned}$$

Using this we obtain the same property for $\bar{\varphi}$: If $\varepsilon \leq 0$, then

$$(1-2\varepsilon)^{1-\varepsilon}\bar{\varphi}\big(x_r,(1-2\varepsilon)\big)$$

$$
\begin{aligned}
&= \limsup_{|x_r+2\varepsilon|\to\infty} (1-2\varepsilon)^{1-\varepsilon}\varphi_2\big((x_r+2\varepsilon),(1-2\varepsilon)\big) \\
&\geq (1+\varepsilon)^{2(1-\varepsilon)} \limsup_{|x_r+2\varepsilon|\to\infty} (1-\varepsilon)^{1-\varepsilon}\varphi_2\big((x_r+\varepsilon),(1-\varepsilon)\big) \\
&= \big((1+\varepsilon)^2(1-\varepsilon)\big)^{1-\varepsilon}\bar{\varphi}\big(x_r,(1-\varepsilon)\big),
\end{aligned}
$$

and $\varepsilon \geq 0$, then

$$
\begin{aligned}
(1+\varepsilon)^{1-\varepsilon}\varphi_2\big(x_r,(1+\varepsilon)\big) &= (1+\varepsilon)^{1-\varepsilon}\big(2\varphi_2\big(x_r,(1+\varepsilon)\big)-1\big) \\
&\geq (1+\varepsilon)^{1-\varepsilon}\varphi_2\big(x_r,(1+\varepsilon)\big) \\
&\geq \big((1+\varepsilon)^2(1+2\varepsilon)\big)^{1-\varepsilon}\varphi_2\big(x_r,(1+2\varepsilon)\big) \\
&\geq \frac{1}{2}(1+\varepsilon)^{3(1-\varepsilon)}\bar{\varphi}\big(x_r,(1+2\varepsilon)\big).
\end{aligned}
$$

Since the function is almost decreasing on $(0,1]$ and $[1,\infty)$, it is almost decreasing on the union as well.

Corollary (5.3.46)[205]: Let $\varphi \in N(\mathbb{R}^n)$ satisfy assumptions (A0)−(A2) and let $(1+2\varepsilon) \mapsto (1+2\varepsilon)^{-\frac{1+\varepsilon}{1+2\varepsilon}}\varphi\big(x_r,(1+2\varepsilon)\big)$ be strictly decreasing to 0. Then $L^{\varphi^{\#}_{1+\varepsilon}(\cdot)}(\mathbb{R}^n) = L^{\bar{\varphi}^{\#}_{1+\varepsilon}}(\mathbb{R}^n)$.

Proof. By Theorem of [154] such that $L^{\psi(\cdot)}(\mathbb{R}^n) \subset L^{\varphi(\cdot)}(\mathbb{R}^n)$ if and only if there exist $\varepsilon > -1$ and $h \in L^1(\mathbb{R}^n)$ such that $\varphi\big(x_r,(1+\varepsilon)(1+2\varepsilon)\big) \leq \psi\big(x_r,(1+\varepsilon)\big) + h(x_r)$. This can be equivalently written $\psi^{-1}\big(x_r,(1+\varepsilon)\big) \lesssim \varphi^{-1}\big(x_r,(1+\varepsilon)+h(x_r)\big)$.

Let us show that $L^{\bar{\varphi}^{\#}_{1+\varepsilon}(\cdot)}(\mathbb{R}^n) \subset L^{\varphi^{\#}_{1+\varepsilon}(\cdot)}(\mathbb{R}^n)$; the reverse implication follows analogously. The inclusion is equivalent to the inequality

$$
\left(\bar{\varphi}^{\#}_{1+\varepsilon}(\cdot)\right)^{-1}\big(x_r,(1+\varepsilon)\big) \lesssim \left(\varphi^{\#}_{1+\varepsilon}(\cdot)\right)^{-1}\big(x_r,(1+\varepsilon)+h(x_r)\big)
$$

By Corollary (5.3.45), $(1+2\varepsilon) \mapsto (1+2\varepsilon)^{-1}\bar{\varphi}\big(x_r,(1+\varepsilon)\big)$ is almost decreasing which is equivalent to $(1+2\varepsilon) \mapsto (1+2\varepsilon)^{-1}\bar{\varphi}^{-1}\big(x_r,(1+\varepsilon)\big)$ being almost increasing. By the definition of $\bar{\varphi}^{\#}_{1+\varepsilon}$ and the almost increasing property, we obtain that

$$
\begin{aligned}
\left(\bar{\varphi}^{\#}_{1+\varepsilon}\right)^{-1}\big(x_r,(1+\varepsilon)\big) &\approx \bar{\lambda}\left(\bar{\varphi}^{-1}\big(x_r,(1+\varepsilon)\big)\right) \\
&= \frac{\bar{\varphi}^{-1}\big(x_r,(1+\varepsilon)\big)}{\bar{\varphi}\left(x_r,\bar{\varphi}^{-1}\left(x_r,\big((1+\varepsilon)+h(x_r)\big)\right)\right)^{\frac{1+\varepsilon}{1+2\varepsilon}}} \\
&= \frac{\bar{\varphi}^{-1}\big(x_r,(1+\varepsilon)\big)}{(1+\varepsilon)^{\frac{1+\varepsilon}{1+2\varepsilon}}} \lesssim \frac{\bar{\varphi}^{-1}\big(x_r,(1+\varepsilon)+h(x_r)\big)}{\big((1+\varepsilon)+h(x_r)\big)^{\frac{1+\varepsilon}{1+2\varepsilon}}},
\end{aligned}
$$

Where $\bar{\lambda}\big(x_r,(1+2\varepsilon)\big) = (1+2\varepsilon)\bar{\varphi}\big(x_r,(1+2\varepsilon)\big)^{-\frac{1+\varepsilon}{1+2\varepsilon}}$. By Corollary (5.3.36), $L^{\varphi(\cdot)} = L^{\bar{\varphi}(\cdot)}$, so that $\bar{\varphi}^{-1}\big(x_r,(1+\varepsilon)\big) \lesssim \varphi^{-1}\big(x_r,(1+\varepsilon)+h(x_r)\big)$. Using this in the inequality above, and reversing the steps with φ, we get

$$
\begin{aligned}
\left(\bar{\varphi}^{\#}_{1+\varepsilon}\right)^{-1}\big(x_r,(1+\varepsilon)\big) &\lesssim \frac{\bar{\varphi}^{-1}\big(x_r,(1+\varepsilon)+h(x_r)\big)}{\big((1+\varepsilon)+h(x_r)\big)^{\frac{1+\varepsilon}{1+2\varepsilon}}} \lesssim \frac{\varphi^{-1}\big(x_r,(1+\varepsilon)+2h(x_r)\big)}{\big((1+\varepsilon)+h(x_r)\big)^{\frac{1+\varepsilon}{1+2\varepsilon}}} \\
&\lesssim \left(\bar{\varphi}^{\#}_{1+\varepsilon}\right)^{-1}\big(x_r,(1+\varepsilon)+2h(x_r)\big),
\end{aligned}
$$

as required.

Corollary (5.3.47)[205]: Let $\varphi \in N(\mathbb{R}^n)$ satisfy assumptions (A0)−(A2) and suppose that $\varepsilon > 0$ is such that $(1+2\varepsilon) \mapsto (1+2\varepsilon)^{-(1+\varepsilon)}\varphi\big(x_r,(1+\varepsilon)\big)$ is increasing and $(1+\varepsilon) \mapsto (1+\varepsilon)^{-1}\varphi\big(x_r,(1+\varepsilon)\big)$ is decreasing for every $x_r \in \mathbb{R}^n$. Then

$$\|I_{1+\varepsilon}f\|_{\varphi^{\#}_{1+\varepsilon}(\cdot)} \lesssim \|f\|_{\varphi(\cdot)}.$$

Note that φ is doubling with constant 2^{-1} since $(1+\varepsilon) \mapsto (1+\varepsilon)^{-1}\varphi\big(x_r,(1+\varepsilon)\big)$ is decreasing.

Proof. Let us first note that since $(1+\varepsilon) \mapsto (1+\varepsilon)^{-1}\varphi\big(x_r,(1+\varepsilon)\big)$ is decreasing $(1+2\varepsilon) \mapsto (1+2\varepsilon)^{-1}\varphi\big(x_r,(1+\varepsilon)\big)$ is strictly decreasing to 0.

By Corollary (5.3.36) and (5.3.47), $L^{\varphi(\cdot)} = L^{\bar{\varphi}(\cdot)}$ and $L^{\varphi^{\#}_{1+\varepsilon}(\cdot)} = L^{\bar{\varphi}^{\#}_{1+\varepsilon}(\cdot)}$ with comparable norms. Thus, it suffices to show that $\|I_{1+\varepsilon}f\|_{\bar{\varphi}^{\#}_{1+\varepsilon}(\cdot)} \lesssim \|f\|_{\bar{\varphi}(\cdot)}$.

By Corollary (5.3.37) and (5.3.39) $\bar{\varphi} \in N_1(\mathbb{R}^n)$ is normalized. By Corollary (5.3.14), $(1+\varepsilon) : L^{\bar{\varphi}(\cdot)}(\mathbb{R}^n) \to L^{\bar{\varphi}(\cdot)}(\mathbb{R}^n)$ is bounded. By Corollary (5.3.45), $(1+2\varepsilon) \mapsto (1+2\varepsilon)^{-\frac{1}{1+\varepsilon}}\varphi\big(x_r,(1+2\varepsilon)\big)$ is almost decreasing. Thus, by Corollary (5.3.43), $\bar{\lambda}^{-1}\big(x_r, I_{1+\varepsilon}f(x_r)\big) \lesssim Mf(x_r)$. Applying $\bar{\varphi}$ to both sides, we find that

$$\bar{\varphi}^{\#}_{1+\varepsilon}\big(x_r, I_{1+\varepsilon}f(x_r)\big) \simeq \bar{\varphi}\left(x_r, \bar{\lambda}^{-1}\big(x_r, I_{1+\varepsilon}f(x_r)\big)\right) \lesssim \bar{\varphi}\big(x_r,(1+\varepsilon)f(x_r)\big).$$

From this we deduce by the normal scaling argument that

$$\|I_{1+\varepsilon}f\|_{\bar{\varphi}^{\#}_{1+\varepsilon}(\cdot)} \lesssim \|(1+\varepsilon)f\|_{\bar{\varphi}(\cdot)} \lesssim \|f\|_{\bar{\varphi}(\cdot)}.$$

It is well known that $|u_r| \lesssim I_1|\nabla u_r|$ for $u_r \in C_0^{\infty}(\mathbb{R}^n)$.

Chapter 6

Maximal Function and Operators

The result extends the resent work of Pick and Růžička [22], Diening [1] and Nekvinda [21]. We also show that under much weaker assumptions on $p(x)$, the maximal operator satisfies a weak-type modular inequality. We include as special cases the optimal condition for Orlicz spaces as well as the essentially optimal conditions for variable exponent Lebesgue spaces and double-phase functional.

Section (6.1): Maximality and Variable L^p Spaces

Given an open set $\Omega \subset \mathbb{R}^n$, and a measurable function $p : \Omega \to [1, \infty)$, let $L^{p(x)}(\Omega)$ denote the Banach function space of measurable functions f on Ω such that for some $\lambda > 0$,

$$\int_\Omega \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx < \infty,$$

with norm

$$\|f\|_{p(x),\Omega} = \inf\left\{\lambda > 0 : \int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \le 1\right\}$$

These spaces are a special case of the Musielak–Orlicz spaces (cf. Musielak [18]). When $p(x) = p_0$ is constant, $L^{p(x)}(\Omega)$ becomes the standard Lebesgue space $L^{p_0}(\Omega)$.

Functions in these spaces and the associated Sobolev spaces $W^{k,p(x)}(\Omega)$ have been considered: see, for example, [27], [10]–[41], [15]–[123], [22], [23] and [25]. They appear in the study of variational integrals and partial differential equations with non-standard growth conditions.

Some of the properties of the Lebesgue spaces readily generalize to the spaces $L^{p(x)}(\Omega)$: see, for example, Kovacik and Rakosnik [13]. On the other hand, elementary properties, such as the continuity of translation, often fail to hold (see [13] or [125]).

We consider the Hardy–Littlewood maximal operator,

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_{B\cap\Omega} |f(y)| dy, \qquad (1)$$

where the supremum is taken over all balls B which contain x and for which $|B \cap \Omega| > 0$ It is well known (cf. Duoandikoetxea [40]) that the maximal operator satisfies the following weak and strong-type inequalities:

$$|\{x \in \Omega : Mf(x) > t\}| \le \frac{C}{t^p} \int_\Omega |f(y)|^p dy, \qquad 1 \le p < \infty,$$

$$\int_\Omega Mf(y)^p dy \le C \int_\Omega |f(y)|^p dy, \qquad 1 < p < \infty.$$

We show analogous inequalities for functions in $L^{p(x)}(\Omega)$.

Strong-type inequalities have been studied recently. Pick and Ruzicka [21] constructed examples which showed that the following uniform continuity condition on $p(x)$ is necessary (in some sense) for the maximal operator to be bounded on $L^{p(x)}(\Omega)$:

$$|p(x) - p(y)| \le \frac{C}{-\log|x-y|}, \qquad x, y \in \Omega, |x-y| < \frac{1}{2}. \tag{2}$$

This condition appears to be natural in the study of variable L^p spaces; see [21], [27].

Diening [4] has shown that this condition is sufficient on bounded domains. To state his result, let $p_* = \inf\{p(y) : y \in \Omega\}, p^* = \sup\{p(y) : y \in \Omega\}$.

Theorem (6.1.1)[36]: (Diening). Let $\Omega \subset \mathbb{R}^n$ be an open, bounded domain, and let $p : \Omega \to [1, \infty)$ satisfy (2) and be such that $1 < p_* \le p^* < \infty$. Then the maximal operator is bounded on $L^{p(x)}(\Omega) : \|Mf\|_{p(x),\Omega} \le C(p(x), \Omega)\|f\|_{p(x),\Omega}$.

Very recently, Diening [1] has extended Theorem (6.1.1) to all of $\mathbb{R}^n$ with the additional assumption that $p(x)$ is constant outside of a fixed ball. Further, Nekvinda [20] has shown that this hypothesis can be weakened as follows.

Theorem (6.1.2)[36]: (Nekvinda). Let $p : \mathbb{R}^n \to [1, \infty)$ satisfy (2) and be such that $1 < p_* \le p^* < \infty$. Suppose further that there is a constant $p_\infty > 1$ such that $p(x) = p_\infty + \phi(x)$, where there exists $R > 0$ such that $\varphi(x) \ge 0$ if $|x| > R$, and $\beta > 0$ such that

$$\int_{\{x \in \mathbb{R}^n : \varphi(x) > 0\}} \phi(x) \beta^{\frac{1}{\varphi(x)}} dx < \infty. \tag{3}$$

Then the maximal operator is bounded on $L^{p(x)}(\mathbb{R}^n)$.

(Added in proof). We have learned that Nekvinda has improved this result by removing the requirement that φ be nonnegative.

Note that together, conditions (2) and (3) imply $\varphi(x) \to 0$ as $|x| \to \infty$.

The result is the following theorem; it is similar to Theorem (6.1.3) since it is for exponent functions $p(x)$ of the same form (though φ need not be positive). Further, it gives a pointwise characterization of how quickly $\varphi(x)$ must converge to zero at infinity.

Theorem (6.1.3)[36]: Given an open set $\Omega \subset \mathbb{R}^n$, let $p : \Omega \to [1, \infty)$ be such that $1 < p_* \le p^* < \infty$. Suppose that $p(x)$ satisfies (2) and

$$|p(x) - p(y)| \le \frac{C}{\log(e + |x|)}, \qquad x, y \in \Omega, |y| \ge |x|. \tag{4}$$

Then the Hardy–Littlewood maximal operator is bounded on $L^{p(x)}(\Omega)$.

Proof. Without loss of generality, we may assume that f is non-negative. We first show there exists a constant C such that if $|f|_{p(x),\Omega} \le 1$, then $|Mf|_{p(x),\Omega} \le C$. Fix $f, |f|_{p(x),\Omega} \le 1$. Let $f = f_1 + f_2$, where

$$f_1(x) = f(x)\chi_{\{x: f(x) \ge 1\}}(x).$$

Then for $i = 1, 2, |f_i|_{p(x),\Omega} \le 1$. Since $p^* < \infty$,

$$\int_\Omega Mf(y)^{p(y)}dy \le 2^{p^*}\int_\Omega Mf_1(y)^{p(y)}dy + 2^{p^*}\int_\Omega Mf_2(y)^{p(y)}dy.$$

We will show that each integral on the right-hand side is bounded by a constant. Since $|f_2(x)| \le 1$, by Lemma (6.1.11), f_2 satisfies inequality (9). Therefore, if we integrate over Ω we get that

$$\int_\Omega Mf_2(y)^{p(y)}dy \le C\int_\Omega M\left(\frac{f_2(\cdot)^{p(\cdot)}}{p^*}\right)(y)^{p_*}dy + C\int_\Omega \alpha(y)^{p_*}dy \ + C\int_\Omega Hf_2(y)^{p(y)}dy.$$

Since $p_* > 1$, M is bounded on $L^{p_*}(\Omega)$ and $\alpha(x) \in L^{p_*}(\Omega)$, so

$$\le C\int_\Omega f_2(y)^{p(y)}dy + C + C\int_\Omega Hf_2(y)^{p(y)}dy \ \le C + C\int_\Omega Hf_2(y)^{p(y)}dy.$$

Given a function p, define its increasing, radial minorant i_p to be the function

$$i_p(x) = \inf_{|y|\ge|x|} p(y).$$

Clearly, i_p is a radial, increasing function. Further, (4) implies that for all $x \in \Omega$,

$$0 \le p(x) - i_p(x) \le \frac{C}{\log(e+|x|)}.$$

Therefore, since $f_2(x) \le 1$ and $(i_p)_* = p_*$, by Lemmas (6.1.13) and (6.1.9),

$$\int_\Omega Hf_2(y)^{p(y)}dy \le C\int_\Omega Hf_2(y)^{i_{p(y)}}dy \le C\int_\Omega f_2(y)^{i_p(y)}dy$$
$$\le C\int_\Omega f_2(y)^{p(y)}dy + C\int_\Omega \alpha(y)^{p_*}dy \le C.$$

Hence, $|Mf_2|_{p(x),\Omega} \le C$.

A very similar argument using Lemma (6.1.9) shows that $|Mf_1|_{p(x),\Omega} \le C$.

Therefore, we have shown that if $|f|_{p(x),\Omega} \le 1$, then $|Mf|_{p(x),\Omega} \le C$. Since $C > 1$, it follows that

$$\int_\Omega \left(C^{-1}Mf(x)\right)^{p(x)}dx \le 1,$$

which in turn implies that

$$\|Mf\|_{p(x),\Omega} \le C.$$

To complete the proof we fix a function $g \in L^{p(x)}(\Omega)$, and let $f(x) = g(x)/\|g\|_{p(x),\Omega}$. Then $\|f\|_{p(x),\Omega} \le 1$, so $|f|_{p(x),\Omega} \le 1$. Hence,

$$\|Mg\|_{p(x),\Omega} = \|g\|_{p(x),\Omega}\|Mf\|_{p(x),\Omega} \le C\|g\|_{p(x),\Omega}.$$

Condition (4) is the natural analogue of (2) at infinity. It implies that there is some number p_∞ such that $p(x) \to p_\infty$ as $|x| \to \infty$, and this limit holds uniformly in all directions. It is also necessary (in some sense) on $\mathbb{R}$, as the next example shows.

Theorem (6.1.4)[36]: Fix $p_\infty, 1 < p_\infty < \infty$, and let $\varphi : [0,\infty) \to [0, p_\infty - 1)$ be such that $\varphi(0) = 0$, φ is decreasing on $[1,\infty)$, $\varphi(x) \to 0$ as $x \to \infty$, and

$$\lim_{x\to\infty} \varphi(x)\log(x) = \infty. \tag{5}$$

Define the function $p : \mathbb{R} \to [1,\infty)$ by

$$p(x) = \begin{cases} p_\infty, & x \le 0, \\ p_\infty - \varphi(x), & x > 0; \end{cases}$$

then the maximal operator is not bounded on $L^{p(x)}(\mathbb{R})$.

Proof. The proof is closely modeled on the construction given by Pick and Ruzicka in [20].

By inequality (5), we have that

$$\lim_{x\to\infty}\left(1 - \frac{p_\infty}{p(2x)}\right)\log(x) = -\infty,$$

which in turn implies that

$$\lim_{x\to\infty} x^{1-\frac{p_\infty}{p(2x)}} = 0.$$

Therefore, we can form a sequence $\{c_n\}_{n=1}^{\infty}, c_{n+1} < 2c_n \le -1$, such that

$$|c_n|^{1-\frac{p_\infty}{p(2|c_n|)}} \le 2^{-n}.$$

Let $d_n = 2c_n < c_n$, and define the function f on $\mathbb{R}$ by

$$f(x) = \sum_{n=1}^{\infty} |c_n|^{-\frac{1}{p(|d_n|)}} \chi_{(d_n,c_n)}(x).$$

We claim that $|f|_{p(x),\mathbb{R}} \le 1$ and $|Mf|_{p(x),\mathbb{R}} = \infty$; it follows immediately from this that $\|f\|_{p(x),\mathbb{R}} \le 1$ and $\|Mf\|_{p(x),\mathbb{R}} = \infty$, so the maximal operator is not bounded on $L^{p(x)}(\mathbb{R})$. First, we have that

$$|f|_{p(x),\mathbb{R}} = \sum_{n=1}^{\infty}\int_{d_n}^{c_n} |c_n|^{-\frac{p(x)}{p(|d_n|)}}dx = \sum_{n=1}^{\infty}\int_{d_n}^{c_n} |c_n|^{-\frac{p_\infty}{p(|d_n|)}}dx = \sum_{n=1}^{\infty} |c_n|^{1-\frac{p_\infty}{p(|d_n|)}} \le \sum_{n=1}^{\infty} 2^{-n} = 1.$$

On the other hand, if $x \in (|c_n|, |d_n|)$, then

$$Mf(x) \ge \frac{1}{2|d_n|}\int_{d_n}^{|d_n|} f(y)dy \ge \frac{1}{2|d_n|}\int_{d_n}^{c_n} f(y)dy = \frac{|c_n|^{1-\frac{1}{p(|d_n|)}}}{2|d_n|} = \frac{1}{4}|c_n|^{-\frac{1}{p(|d_n|)}}.$$

Therefore, since $p(x)$ is an increasing function and $|c_n| \ge 1$,

$$|Mf|_{p(x),\mathbb{R}} \ge \frac{1}{4}\sum_{n=1}^{\infty}\int_{|c_n|}^{|d_n|} |c_n|^{-\frac{p(x)}{p(|d_n|)}} \ge \frac{1}{4}\sum_{n=1}^{\infty}\int_{|c_n|}^{|d_n|} |c_n|^{-\frac{p(|d_n|)}{p(|d_n|)}} = \frac{1}{4}\sum_{n=1}^{\infty} 1 = \infty.$$

The assumption in Theorem (6.1.3) that $p^* < \infty$ again holds automatically: it follows from (4). However, the assumption that $p_* > 1$ is necessary, as the following example shows.

Theorem (6.1.5)[36]: Let $\Omega \subset \mathbb{R}^n$ be open, and let $p: \Omega \to [1, \infty)$ be upper semicontinuous. If $p_* = 1$ then the maximal operator is not bounded on $L^{p(x)}$ (Ω).

Proof of. Fix $k \geq 1$. Since $p_* = 1, \Omega$ is open and p is upper semi-continuous, there exists $x_k \in \Omega$ and $\varepsilon_k > 0$ such that $B_k = B_{\varepsilon_k}(x_k) \subset \Omega$, and such that if $\in B_k$, $p(x) < 1 + 1/k$. We define the function $f_k(x) = |x_k - x|^{-\frac{nk}{k+1}} \chi_{B_k}(x)$. Then $f_k \in L^{p(x)}(\Omega)$. On the other hand, for $x \in B_k$, let $r = |x - x_k|$; then

$$Mf_k(x) \geq \frac{c}{|B_r(x_k)|} \int_{B_r(x_k)} f_k(y) dy = c(k+1) f_k(x).$$

Hence, $\|Mf_k\|_{p(x),\Omega} \geq c(k+1)\|f_k\|_{p(x),\Omega}$; since we may take k arbitrarily large, the maximal operator is not bounded on $L^{p(x)}(\Omega)$.

We begin with a lemma which, intuitively, plays the role that Hölder's inequality does in the standard proof that the maximal operator is weak (p, p).

We note that an immediate application of Theorem (6.1.3) has been given by Diening [1]: he has shown that if $\partial\Omega$ is Lipschitz, and the maximal operator is bounded on $L^{p(x)}(\Omega)$, then $C^\infty(\overline{\Omega})$ is dense in $W^{1,p(x)}(\Omega)$.

Unlike the case of the strong-type inequalities, we appear to be prove an analogue of the weak (p, p) inequality for the maximal operator. The weak-type result is somewhat surprising, since it requires no continuity assumptions on $p(x)$, and it is satisfied by unbounded functions. To state it, we need a definition. Given a non-negative, locally integrable function u on $\mathbb{R}^n$, we say that $u \in RH_\infty$ if there exists a constant C such that for every ball B,

$$u(x) \leq \frac{C}{|B|} \int_B u(y) dy \qquad a.e.\ x \in B.$$

Denote the smallest constant C such that this inequality holds by $RH_\infty(u)$. The RH_∞ condition is satisfied by a variety of functions u: for instance, if there exist positive constants A and B such that $A \leq u(x) \leq B$ for all x. More generally, $u \in RH_\infty$ if $u(x) = |x|^a, a > 0$, or if there exists $r > 0$ such that u^{-r} is in the Muckenhoupt class A_1. For further information about RH_∞, see Cruz-Uribe and Neugebauer [197].

Theorem (6.1.6)[36]: Given an open set Ω, suppose the function $p: \Omega \to [1, \infty)$ can be extended to $\mathbb{R}^n$ in such a way that $1/p \in RH_\infty$. Then for all $f \in L^{p(x)}(\Omega)$ and $t > 0$,

$$|\{x \in \Omega : Mf(x) > t\}| \leq C \int_\Omega \left(\frac{|f(y)|}{t}\right)^{p(y)} dy. \tag{6}$$

Proof. For each $N > 0$, define the operator M_N by

$$M_N f(x) = \sup \frac{1}{|B|} \int_{B \cap \Omega} |f(y)| dy,$$

where the supremum is taken over all balls containing x such that $|B| \leq N$. The sequence $\{M_N f(x)\}$ is increasing and converges to $Mf(x)$ for each $x \in \Omega$. Thus, by the monotone convergence theorem, for each $t > 0$,

$$|\{x \in \Omega : Mf(x) > t\}| = \lim_{N\to\infty} |\{x \in \Omega: M_N f(x) > t\}|.$$

Therefore, it will suffice to prove (6) with M replaced by M_N , and with a constant independent of N.

Fix $t > 0$ and let $E_N = \{x \in \Omega : M_N f(x) > t\}$. Then for each $x \in E_N$, there exists a ball B_x containing $x, |Bx| \leq N$, such that

$$\frac{1}{|B_x|}\int_{|B_x\cap\Omega|} |f(y)|dy > t.$$

By a weak variant of the Vitali covering lemma (cf. Stein [60]), there exists a collection of disjoint balls, $\{B_k\}$, contained in $\{B_x : x \in E_N\}$, and a constant C depending only on the dimension n, such that

$$|E_n| \leq C\sum_k |B_k|.$$

Therefore, by Lemma (6.1.13),

$$|E_N| \leq C\sum_k |B_k| \leq \sum_k |B_k| \left(\int_{B_k} \frac{dy}{p(y)}\right)^{-1} \int_{B_k} \frac{dy}{p(y)}$$

$$\leq \sum_k \left(\frac{1}{|B_k|}\int_{B_k} \frac{dy}{p(y)}\right)^{-1} \frac{1}{p_*(B_k)}\int_{B_k\cap\Omega} \left(\frac{|f(y)|}{t}\right)^{p(y)} dy\,;$$

since $p_*(B_k)^{-1} = \left(\frac{1}{p}\right)^* (B_k)$, by the definition of RH_∞ ,

$$\leq RH_\infty\left(\frac{1}{p}\right)\sum_k \int_{B_k\cap\Omega}\left(\frac{|f(y)|}{t}\right)^{p(y)} dy \leq\ C\ \int_\Omega \left(\frac{|f(y)|}{t}\right)^{p(y)} dy.$$

We can give an alternative version of Theorem (6.1.6) which does not require extending $p(x)$ to all of $\mathbb{R}^n$, but to do so we must replace the assumption that $1/p \in RH_\infty$ with the following condition: given any ball B, $|B \cap \Omega| > 0$, and $x \in B \cap \Omega$,

$$\frac{1}{p(x)} \leq \frac{C}{|B|}\int_{B\cap\Omega} \frac{dy}{p(y)}.$$

Note, however, that this condition need not hold if $p(x)$ is constant, and so we do not recapture the classical result.

In the case of the Lebesgue spaces, the strong-type inequality is deduced from the weak-type inequality via the Marcinkiewicz interpolation theorem. It would be interesting to generalize this approach to use Theorem (6.1.6) to prove Theorem (6.1.3).

We prove Theorem (6.1.3), Theorems (6.1.5) and (6.1.6), and Theorem (6.1.6). The notation will be standard or defined as needed. In order to emphasize that we are dealing with variable exponents, we will always write $p(x)$ instead of p to denote an exponent function. Given an open set Ω and function $p(x), 1 \le p(x) \le \infty$, define the conjugate function $q(x)$ to satisfy $1/p(x) + 1/q(x) = 1$, where we take $1/\infty = 0$. Given a set E, let $|E|$ denote its Lebesgue measure, and let $p_*(E) = in\,f\{p(y) : y \in E\}$ and $p^*(E) = \sup\{p(y) : y \in E\}$. For brevity, let $p_* = p_*(\Omega)$ and $p^* = p^*(\Omega)$. Given a function f, let

$$|f|_{p(x),\Omega} = \int_\Omega |f(y)|^{p(y)} dy.$$

Finally, C and c will denote positive constants which will depend only on the dimension n, the underlying set Ω and the exponent function $p(x)$, but whose value may change at each appearance.

The proof of Theorem (6.1.3) requires a series of lemmas. Throughout this, let $\alpha(x) = (e + |x|)^{-n}$.

The first lemma is due to Diening [4]. For completeness, we include its short proof.

Lemma (6.1.7)[36]: Given an open set Ω and a function $p : \Omega \to [1, \infty)$ which satisfies (2), then for any ball B such that $|B \cap \Omega| > 0$,

$$|B|^{p_*(B\cap\Omega) - p^*(B\cap\Omega)} \le C.$$

Proof. Since $p_*(B \cap \Omega) - p^*(B \cap \Omega) \le 0$, we may assume that if r is the radius of B, then $r < \frac{1}{4}$. But in that case, (2) implies that

$$p^*(B \cap \Omega) - p_*(B \cap \Omega) \le \frac{C}{\log\left(\frac{1}{2r}\right)}.$$

Therefore,

$$|B|^{p_*(B\cap\Omega) - p^*(B\cap\Omega)} \le cr^{-n\left(p^*(B\cap\Omega) - p_*(B\cap\Omega)\right)} \le cr^{-\frac{nC}{log\left(\frac{1}{2r}\right)}} \le C.$$

Lemma (6.1.8)[36]: Given a set G and two non-negative functions $r(x)$ and $s(x)$, suppose that for each $x \in G$,

$$0 \le s(x) - r(x) \le \frac{C}{\log(e + |x|)}.$$

Then for every function f,

$$\int_G |f(x)|^{r(x)} dx \le C \int_G |f(x)|^{s(x)} dx + \int_G \alpha(x)^{r_*(G)} dx.$$

Proof. Let $G^\alpha = \{x \in G : |f(x)| \ge \alpha(x)\}$. Then

$$\int_G |f(x)|^{r(x)}\, dx = \int_{G^\alpha} |f(x)|^{r(x)} dx + \int_{G\setminus G^\alpha} |f(x)|^{r(x)} dx,$$

and we estimate each integral separately. First, since $\alpha(x) \le 1$,

$$\int_{G\setminus G^\alpha} |f(x)|^{r(x)}dx \le \int_{G\setminus G^\alpha} \alpha(x)^{r(x)}dx \le \int_G \alpha(x)^{r_*(G)}dx.$$

On the other hand, if $x \in G^\alpha$, then

$$|f(x)|^{r(x)} = |f(x)|^{s(x)}|f(x)|^{r(x)-s(x)} \le |f(x)|^{s(x)}\alpha(x)^{-\frac{C}{log(e+|x|)}} \le C|f(x)|^{s(x)}.$$

The desired inequality now follows immediately.

The next two lemmas generalize the key step in Diening's proof of Theorem (6.1.1) (see [4]).

Lemma (6.1.9)[36]: Given Ω and p as in the statement of Theorem (6.1.3), suppose that $|f|_{p(x),\Omega} \le 1$, and $|f(x)| \ge 1$ or $f(x) = 0, x \in \Omega$. Then for all $x \in \Omega$,

$$Mf(x)^{p(x)} \le CM\left(|f(\cdot)|^{\frac{p(\cdot)}{p_*}}\right)(x)^{p_*} + C\alpha(x)^{p_*}, \tag{7}$$

where $\alpha(x) = (e + |x|)^{-n}$.

Proof. Without loss of generality, we may assume that f is non-negative. Fix $x \in \Omega$, and fix a ball B of radius $r > 0$ containing x such that $|B \cap \Omega| > 0$. Let $B\Omega = B \cap \Omega$. It will suffice to show that (2.1) holds with the left-hand side replaced by

$$\left(\frac{1}{|B|}\int_{B_\Omega} f(y)dy\right)^{p(x)},$$

and with a constant independent of B. We will consider three cases.

Case1: $r < |x|/4$. Define $\overline{p}(x) = p(x)/p_*$. Then $\overline{p}(x) \ge 1$, and (4) holds with p replaced by $\overline{p}$. By assumption on , if $y \in B_\Omega$,

$$0 \le \overline{p}(y) - \overline{p}_*(B_\Omega) \le \frac{C}{\log(e+|y|)}. \tag{8}$$

Therefore, by Hölder's inequality and by Lemma (6.1.9) with $r(x)$ replaced by the constant $\overline{p}_*(B_\Omega)$ and $s(x)$ by $\overline{p}(y)$, we have that

$$\left(\frac{1}{|B|}\int_{B_\Omega} f(y)dy\right)^{p(x)} \le \left(\frac{1}{|B|}\int_{B_\Omega} f(y)^{\overline{p}_*(B_\Omega)}dy\right)^{\frac{p(x)}{\overline{p}_*(B_\Omega)}}$$

$$\le \left(\frac{C}{|B|}\int_{B_\Omega} f(y)^{\overline{p}(y)}dy + \frac{1}{|B|}\int_{B_\Omega} \alpha(y)^{\overline{p}_*(B_\Omega)}dy\right)^{\frac{p(x)}{\overline{p}_*(B_\Omega)}}$$

since $r < |x|/4$ and $p(x)/\overline{p}_*(B_\Omega) \le p^* < \infty$,

$$\le \left(\frac{C}{|B|}\int_{B_\Omega} f(y)^{\overline{p}(y)}dy + C\alpha(x)_*^{\overline{p}(B_\Omega)}\right)^{\frac{p(x)}{\overline{p}_*(B_\Omega)}}$$

$$\leq 2^{p^*}C\left(\frac{1}{|B|}\int_{B_\Omega} f(y)\overline{p}(y)dy\right)^{\frac{p(x)}{\overline{p}_*(B_\Omega)}} + 2^{p^*}C\alpha(x)^{p(x)}.$$

If $|B| \geq 1$, then by Hölder's inequality and since $|f|_{p(x),\Omega} \leq 1$,

$$\frac{1}{|B|}\int_{B_\Omega} f(y)^{\overline{p}(y)}dy \leq \left(\frac{1}{|B|}\int_{B_\Omega} f(y)^{p(y)}dy\right)^{\frac{1}{p_*}} \leq \left(\int_{B_\Omega} f(y)^{p(y)}dy\right)^{\frac{1}{p_*}} \leq 1.$$

Hence, since $p(x)/\overline{p}_*(B_\Omega) \geq p_*$ and $\alpha(x) \leq 1$, we have that

$$\left(\frac{1}{|B|}\int_{B_\Omega} f(y)dy\right)^{p(x)} \leq C\left(\frac{1}{|B|}\int_{B_\Omega} f(y)^{\overline{p}(y)}dy\right)^{p}_{*} + C\alpha(x)^{p_*}$$

$$\leq CM\left(f(\cdot)^{\overline{p}(\cdot)}\right)(x)^{p_*} + C\alpha(x)^{p_*}.$$

If, on the other hand, $|B| \leq 1$, then, again since $|f|_{p(x),\Omega} \leq 1$,

$$\int_{B_\Omega} f(y)^{\overline{p}(y)}dy \leq |B_\Omega|^{\frac{1}{p'_*}}\left(\int_{B_\Omega} f(y)^{p(y)}dy\right)^{\frac{1}{p_*}} \leq 1.$$

Therefore,

$$\left(\frac{1}{|B|}\int_{B_\Omega} f(y)dy\right)^{p(x)} \leq C|B|^{-\frac{p(x)}{\overline{p}_*(B_\Omega)}}\left(\int_{B_\Omega} f(y)^{\overline{p}(y)}dy\right)^{\frac{p(x)}{\overline{p}_*(B_\Omega)}} + C\alpha(x)^{p_*}$$

$$\leq C|B|^{-\frac{p(x)}{\overline{p}_*(B_\Omega)}+p_*}\left(\frac{1}{|B|}\int_{B_\Omega} f(y)^{\overline{p}(y)}dy\right)^{p_*} + C\alpha(x)^{p_*}.$$

Since $|B| \leq 1$, and since

$$-p(x)/\overline{p}_*(B_\Omega) + p_* = \left(\frac{p_*}{p_*(B_\Omega)}\right)\left(p_*(B_\Omega) - p(x)\right) \geq \left(\frac{p_*}{p_*(B_\Omega)}\right)(p_*(B_\Omega) - p^*(B_\Omega)$$

by Lemma (6.1.8),

$$\leq C\left(\frac{1}{|B|}\int_{B_\Omega} f(y)^{\overline{p}(y)}dy\right)^{p_*} + C\alpha(x)^{p_*} \leq CM\left(f(\cdot)^{\overline{p}(\cdot)}\right)(x)^{p_*} + C\alpha(x)^{p_*}.$$

This is precisely what we wanted to prove.

Case 2: $|x| \leq 1$ and $r \geq |x|/4$. The proof is essentially the same as in the previous case: since $|x| \leq 1, \alpha(x) \approx 1$, so inequality (8) and the subsequent argument still hold.

Case 3: $|x| \geq 1$ and $r \geq |x|/4$. Since $f(x) \geq 1, p_* \geq 1$ and $|f|_{p(x),\Omega} \leq 1$,

$$\left(\frac{1}{|B|}\int_{B_\Omega} f(y)dy\right)^{p(x)} \leq |B|^{-p(x)}\left(\int_{B_\Omega} f(y)^{p(y)}dy\right)^{p(x)} \leq Cr^{-np(x)}|f|^{p(x)}_{p(x),\Omega}$$

$$\leq C|x|^{-np_*} \leq C\alpha(x)^{p_*} \leq CM\left(f(\cdot)^{\overline{p}(\cdot)}\right)(x)^{p_*} + C\alpha(x)^{p_*}.$$

This completes the proof.

Definition (6.1.10)[36]: Given a function f on Ω, we define the Hardy operator H by

$$Hf(x) = \left|B_{|x|}(0)\right|^{-1} \int_{B_{|x|}(0)\cap\Omega} |f(y)|dy.$$

Lemma (6.1.11)[36]: Given Ω and p as in the statement of Theorem (6.1.3), suppose that $|f|_{p(x),\Omega} \leq 1$, and $|f(x)| \leq 1, x \in \Omega$. Then for all $x \in \Omega$,

$$Mf(x)^{p(x)} \leq CM(|f(\cdot)|^{\frac{p(\cdot)}{p_*}}(x)^{p_*} + C\alpha(x)^{p_*} + CHf(x)^{p(x)}, \qquad (9)$$

where $\alpha(x) = (e + |x|)^{-n}$.

Proof. We may assume without loss of generality that f is non-negative. We argue almost exactly as we did in the proof of Lemma (6.1.9). In that proof we only used the fact that $f(x) \geq 1$ in Case 3, so it will suffice to fix $x \in \Omega, |x| \geq 1$, and a ball B containing x with radius $r > |x|/4$, and prove that

$$\left(\frac{1}{|B|}\int_{B_\Omega} f(y)dy\right)^{p(x)} \leq CM\left(|f(\cdot)|^{\frac{p(\cdot)}{p_*}}\right)(x)^{p_*} + C\alpha(x)^{p_*} + CHf(x)^{p(x)}.$$

Since $p^* < \infty$, we have that

$$\left(\frac{1}{|B|}\int_{B_\Omega} f(y)dy\right)^{p(x)} \leq 2^{p^*}\left(\frac{1}{|B|}\int_{B_\Omega\cap B_{|x|}(0)} f(y)dy\right)^{p(x)} + 2^{p^*}\left(\frac{1}{|B|}\int_{B_\Omega\backslash B_{|x|}(0)} f(y)dy\right)^{p(x)};$$

since $r > |x|/4$,

$$\leq C\left(\left|B_{|x|}(0)\right|^{-1}\int_{B_{|x|}(0)\cap\Omega} |f(y)|dy\right)^{p(x)} + C\left(\frac{1}{|B|}\int_{B_\Omega\backslash B_{|x|}(0)} f(y)dy\right)^{p(x)}$$

$$= CHf(x)^{p(x)} + C\left(\frac{1}{|B|}\int_{B_\Omega\backslash B_{|x|}(0)} f(y)dy\right)^{p(x)}.$$

To estimate the last term, note that if $y \in B_\Omega\backslash B_{|x|}(0)$ then (8) holds and $\alpha(y) \leq \alpha(x)$, so the argument in Case 1 of the proof of Lemma (6.1.9) goes through. This shows that

$$\left(\frac{1}{|B|}\int_{B_\Omega\backslash B_{|x|}(0)} f(y)dy\right)^{p(x)} \leq CM\left(|f(\cdot)|^{\frac{p(\cdot)}{p_*}}\right)(x)^{p_*} + C\alpha(x)^{p_*}$$

and this completes the proof.

Lemma (6.1.12)[36]: If $i(x)$ is a radial, increasing function, $i_* > 1$, and if $|f(x)| \leq 1$, then

$$\int_\Omega Hf(y)^{i(y)}dy \leq C\big(n, i(x)\big)\int_\Omega |f(y)|^{i(y)}dy.$$

Proof. Without loss of generality, we may assume that f is non-negative. Also, for clarity of notation, we extend f to all of $\mathbb{R}^n$ by setting it equal to zero on $\mathbb{R}^n\backslash\Omega$.

We first assume only that $i_* \geq 1$. Recall that $\left|B_{|x|}(0)\right| = |B_1(0)||x|^n$. Let S denote the unit sphere in $\mathbb{R}^n$. Then by switching to polar coordinates and making a change of variables, we get that

$$Hf(x)^{i(x)} = \left(|B_1(0)|^{-1}|x|^{-n}\int_{B_{|x|}(0)} f(y)dy\right)^{i(x)}$$
$$= \left(|B_1(0)|^{-1}|x|^{-n}\int_S\int_0^{|x|} f(r\theta)r^{n-1}dr\, d\theta\right)^{i(x)}$$
$$= \left(|B_1(0)|^{-1}\int_S\int_0^1 f(|x|r\theta)r^{n-1}dr\, d\theta\right)^{i(x)}$$
$$= \left(|B_1(0)|^{-1}\int_{B_1(0)} f(|x|y)dy\right)^{i(x)}$$
$$\leq |B_1(0)|^{-1}\int_{B_1(0)} f(|x|y)^{i(x)}dy,$$

by Holder's inequality.

Now let $r > 1$; the exact value of r will be chosen below. By Minkowski's integral inequality, and again by switching to polar coordinates,

$$\left\|Hf(\cdot)^{i(\cdot)}\right\|_{r,\mathbb{R}^n} \leq C\left(\int_{\mathbb{R}^n}\left(\int_{B_1(0)} f(|x|y)^{i(x)}dy\right)^r dx\right)^{\frac{1}{r}}$$
$$\leq C\int_{B_1(0)}\left(\int_{\mathbb{R}^n} f(|x|y)^{ri(x)}dx\right)^{\frac{1}{r}} dy$$
$$= C\int_S\int_0^1\left(\int_{\mathbb{R}^n} f(|x|s\theta)^{ri(x)}dx\right)^{\frac{1}{r}} s^{n-1}ds\ d\theta$$
$$= C\int_S\int_0^1 s^{-\frac{n}{r}}\left(\int_{\mathbb{R}^n} f(|x|\theta)^{ri\left(\frac{x}{s}\right)}dx\right)^{\frac{1}{r}} s^{n-1}ds\, d\theta,$$

by a change of variables in the inner integral. Since i is a radial increasing function, $i(x/s) \geq i(x)$; since $f(|x|\theta) \leq 1$,

$$\leq C\int_S\int_0^1 s^{-\frac{n}{r}}\left(\int_{\mathbb{R}^n} f(|x|\theta)^{ri(x)}dx\right)^{\frac{1}{r}} s^{n-1}ds\, d\theta \leq C\int_S\left(\int_{\mathbb{R}^n} f(|x|\theta)^{ri(x)}dx\right)^{\frac{1}{r}} d\theta.$$

Since S has constant, finite measure, by Hölder's inequality,

$$\leq C\left(\int_S\int_{\mathbb{R}^n} f(|x|\theta)^{ri(x)}dx\, d\theta\right)^{\frac{1}{r}}.$$

Since i is a radial function, if we rewrite the inner integral in polar coordinates, we get that

$$= C\left(\int_S \int_S \int_0^\infty f(u\theta)^{ri(u)} u^{n-1} du\, d\phi\, d\theta\right)^{\frac{1}{r}}$$

$$= C\left(\int_S \int_0^\infty f(u\theta)^{ri(u)} u^{n-1} du\, d\theta\right)^{\frac{1}{r}} = C\left(\int_{\mathbb{R}^n} f(y)^{ri(y)} dy\right)^{\frac{1}{r}}.$$

To complete the proof, we repeat the above argument with $i(x)$ replaced by $\bar{i}(x) = i(x)/i_*$ and with $r = i_*$, since $i_* > 1$.

While Theorem (6.1.3) shows that we must have $p^* < \infty$ for the norm inequality to be true in general, we do not need this assumption in restricted cases. If f is a bounded, radial, decreasing function, then $Mf(x) \approx Hf(x)$, and so it follows from Lemma (6.1.12) that if p is a radial increasing function, $\|Mf\|_{p(x),\Omega} \leq C\|f\|_{p(x),\Omega}$.

Lemma (6.1.13)[36]: Given an open set Ω, a function $p : \mathbb{R}^n \to [1, \infty)$ such that $1/p$ is locally integrable, f in $L^{p(x)}(\Omega)$ and $t > 0$, suppose that B is a ball such that

$$\frac{1}{|B|}\int_{B\cap\Omega} |f(y)| dy > t.$$

Then

$$\int_B \frac{dx}{p(x)} \leq \frac{1}{p_*(B)} \int_{B\cap\Omega} \left(\frac{|f(y)|}{t}\right)^{p(y)} dy\,.$$

Proof. Fix a sequence of simple functions $\{s_n(x)\}$ on B, such that $s_n(x) \geq p_*(B)$ and such that the sequence increases monotonically to $p(x)$ on B. For each u we have that

$$s_n(x) = \sum_{j=1}^{k_n} \alpha_{n,j} \chi_{A_{n,j}}(x),$$

where the $A_{n,j}$'s are disjoint sets whose union is B. Let $t_n(x)$ be the conjugate function associated to $s_n(x)$; then $t_n(x)$ decreases to $q(x)$, the conjugate function of $p(x)$.

By Hölder's and Young's inequalities,

$$\int_{B\cap\Omega} \frac{|f(y)|}{t} dy = \sum_{j=1}^{k_n} \int_{A_{n,j}\cap\Omega} \frac{|f(y)|}{t} dy$$

$$\leq \sum_{j=1}^{k_n} \left(\int_{A_{n,j}\cap\Omega} \left(\frac{|f(y)|}{t}\right)^{\alpha_{n,j}} dy\right)^{\frac{1}{\alpha_{n,j}}} |A_{n,j}|^{\frac{1}{\alpha'_{n,j}}}$$

$$\leq \sum_{j=1}^{k_n} \left(\frac{1}{\alpha_{n,j}} \int_{A_{n,j}\cap\Omega} \left(\frac{|f(y)|}{t}\right)^{\alpha_{n,j}} dy + \frac{|A_{n,j}|}{\alpha'_{n,j}}\right)$$

$$\leq \sum_{j=1}^{k_n}\left(\frac{1}{p_*(B)}\int_{A_{n,j}\cap\Omega}\left(\frac{|f(y)|}{t}\right)^{s_n(y)} dy + \int_{A_{n,j}}\frac{dy}{t_n(y)}\right)$$

$$\leq \frac{1}{p_*(B)}\int_{B\cap\Omega}\left(\frac{|f(y)|}{t}\right)^{s_n(y)} dy + \int_B\frac{dy}{t_n(y)}.$$

Since this is true for all n, by the monotone convergence theorem,

$$\int_B \frac{|f(y)|}{t}dy \leq \frac{1}{p_*(B)}\int_{B\cap\Omega}\left(\frac{|f(y)|}{t}\right)^{p(y)} dy + \int_B\frac{dy}{q(y)}.$$

Therefore,

$$\int_B\frac{dy}{p(y)} = |B| - \int_B\frac{dy}{q(y)} < \int_{B\cap\Omega}\frac{|f(y)|}{t}dy - \int_B\frac{dy}{q(y)} \leq \frac{1}{p_*(B)}\int_{B\cap\Omega}\left(\frac{|f(y)|}{t}\right)^{p(y)} dy.$$

Section (6.2): Generalized Orlicz Spaces

Generalized Orlicz spaces $L^{\varphi(\cdot)}$ have been studied since the 1940's. A major synthesis of this research is given in of Musielak [18] 1983, hence the alternative name Musielak−Orlicz spaces. These spaces are similar to the better-known Orlicz spaces, but defined by more general function $\varphi(x,t)$ which may vary with the location in spaces: the norm is defined means of the integral

$$\int_{\mathbb{R}^n}\varphi(x,|f(x)|)dx,$$

whereas in an Orlicz spaces φ would be independent of $x, \varphi(|f(x)|)$.

The special case of variable exponent Lebesgue space $L^{p(\cdot)}$, i.e. $\varphi(x,t) \coloneqq t^{p(x)}$, was introduced by Orlicz [202] already 1931. However, in the beginning of the new millennium, there was an explosion in the number of $L^{p(\cdot)}$. It was Diening [4] who opened the floodgate by proving the boundedness of the maximal operator under natural and essentially optimal conditions on the exponent (see also [20, 21, 36]). This result allowed for the development of harmonic analysis and related differential equations in the $p^{p(\cdot)}$ setting.

Note that we present the analogue of this result for $L^{\varphi(\cdot)}$ with a streamlined proof, which is a simplification even in the Orlicz case. Furthermore, this general result has optimal or near optimal conditions in three important special cases:

(i) Orlicz spaces, where the optimal condition of Gallardo [200] is recovered;

(ii) Variable exponent spaces, where the log-Hölder condition is recovered (cf. [21] regarding the optimalily);

(iii) The double phase functional $\varphi(x,t) = t^p + a(x)t^q$ of Mingine and collaborators [176, 177, 179], where the sharp condition for the regularity of minimizers is recovered, namely $\frac{q}{p} < 1 + \frac{\alpha}{a}$ with $a \in C^\alpha$ (Theorem (6.2.11)).

The result and techniques will allow most of the result that have been derived in $L^{p(\cdot)}$ over the past 15 years to be established in $L^{\varphi(\cdot)}$ as well. With these techniques, the Riesz potential has been considered in [196] and the Dirichlet energy integral in [185].

Maeda, Mizuta, Ohno and Shimomura [155, 158, 174] have also studied the boundedness of the maximal operator in $L^{\varphi(\cdot)}$. Their results are special cases of ours, as they deal only with doubling φ and have other restricting assumption as well. Related differential equations have been studied recently by Baroni, Colombo and Mingine [176, 177, 179] and Giannetti and Passarelli di Napoli [183].

Definition (6.2.1)[204]: A convex, left-continuous function $\varphi : [0,\infty) \to [0,\infty]$ with $\varphi(0) = \lim_{t\to 0^+} \varphi(t) = 0$, and $\lim_{t\to 0^+} \varphi(t) = \infty$ is called a Φ-function. The set of Φ-functions is defined by Φ.

Definition (6.2.2)[204]: The set $\Phi(\mathbb{R}^n)$ consists of those $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty]$ with

$\varphi(y,\cdot) \in \Phi$ for every $y \in \mathbb{R}^n$;

$\varphi(\cdot,t) \in L^0(\mathbb{R}^n)$, the set of measurable functions, for every $t \geq 0$.

Also the function in $\Phi(\mathbb{R}^n)$ will be called Φ-functions. In sub- and superscripts the dependence on x will be emphasized by $\varphi(\cdot) : L^{\varphi}$ (Orlicz) vs $L^{\varphi(\cdot)}$ (Musielak−Orlicz).

Definition (6.2.3)[204]: Let $\varphi \in \Phi(\mathbb{R}^n)$ and define $\varrho_{\varphi(\cdot)}$ for $f \in L^0(\mathbb{R}^n)$ by

$$\varrho_{\varphi(\cdot)} \coloneqq \int_{\mathbb{R}^n} \varphi(x, |f(x)|)dx$$

The generalized Orlicz space, also called Musielak−Orlicz space, is defined as the set

$$L^{\varphi(\cdot)}(\mathbb{R}^n) = \left\{ f \in L^0(\mathbb{R}^n) : \lim_{\lambda\to 0} \varrho_{\varphi(\cdot)}(\lambda t) = 0 \right\}$$

equipped with the (Luxemburg) norm

$$\|f\|_{\varphi(\cdot)} \coloneqq \inf\left\{ \lambda > 0 : \varrho_{\varphi(\cdot)}\left(\frac{\lambda}{t}\right) \leq 1 \right\}.$$

Tow functions φ and ψ are equivalent if there exists $L \geq 1$ such that $\psi\left(x, \frac{t}{L}\right) \leq \varphi(x,t) \leq \psi(x, Lt)$ for all x and t. Equivalent Φ-functions give rise to the same space with comparable norms. For further properties (see [179].)

The notation $f \lesssim g$ means that there exists a constant $C > 0$ such that $C \leq Cg$. The (Hardy-Littlewood) maximal operator is defined for $f \in L^0(\mathbb{R}^n)$ by

$$Mf(x) \coloneqq \sup_{r>0} \int_{B(x,r)} |f(y)| dy\,,$$

where $B(x,r)$ is the with is the ball with center x and radius r, and f denotes the average integral. For convex function φ Jensen's inequality states that

$$\varphi\left(\int_A |f(x)| dx\right) \leq \int_A \varphi(|f(x)|)dx.$$

The definition of Φ-functions presupposes convexity, in contrast to that of [155, 158, 174]. However, theirs is an empty generalization, as we show that any Φ-function satisfying their conditions (Φ1)−(Φ5) is equivalent to a convex Φ-function.

Lemma (6.2.4)[204]: Let $\varphi : [0,\infty) \to [0,\infty]$ left-continuous function with $\varphi(0) = \lim_{t\to 0^+} \varphi(t) = 0$, and $\lim_{t\to\infty} \varphi(t) = \infty$. If $s \mapsto s^{-1}\varphi(s)$ is increasing, then there exists $\psi \in \Phi$ equivalent to φ.

Proof. Let ψ be the greatest convex minorant of φ. Since $0 \le \psi \le \varphi$, it follows that $\psi(0) = \lim_{t\to 0^+} \psi(t) = 0$,

Suppose that $\varphi(s) > 0$. Then $\varphi(t) \ge \frac{t}{s}\varphi(s)$ for $t > s$. Thus the function $\left(\frac{t}{s} - 1\right)\varphi(s)$ is a convex minorant of φ on $[0,\infty)$ and since ψ is the greatest convex minorant we conclude that

$$\psi(t) \ge \left(\frac{t}{s} - 1\right)\varphi(s).$$

It follows that $\lim_{t\to\infty} \psi(t) = \infty$. Furthermore, this inequality implies that $\psi(2s) \ge \varphi(s)$. Since also $\psi \ge \varphi$, we see that $\psi \simeq \varphi$.

Finally, since ψ is convex, it is continuous except at the (possible) left-most point t with $\psi(s) = \infty$ for $s > t$. We force ψ to be left-continuous by (re)defining $\psi(s) = \lim_{t\to\infty} \psi(t)$. The properties above still hold: for $\psi(2s) \ge \varphi(s)$ we need the left-continuity of φ.

Lemma (6.2.5)[204]: Let $\varphi \in \Phi$ nad $\beta > 1$ be such that $s \mapsto s^{-\beta}\varphi(s)$ is increasing. Then there exists $\psi \in \Phi$ equivalent to φ such that $\psi^{1/\beta}$ is convex.

Proof. The function $\psi^{1/\beta}$ satisfies all the assumptions of Lemma (6.2.4). Hence there exists $\xi \in \Phi$ such that $\xi \simeq \psi^{1/\beta}$. Set $\psi := \xi^{\beta}$. Since $\beta > 1$, $\psi \in \Phi$ and further $\psi \simeq \varphi$, as required.

Corollary (6.2.6)[204]: Let $\varphi \in \Phi$ and $\beta > 1$ be such that $s \mapsto s^{-\beta}\varphi(s)$ is increasing. Then

$$M : L^{\varphi}(\mathbb{R}^n) \to L^{\varphi}(\mathbb{R}^n)$$

is bounded.

Proof. Let $\varphi \in \Phi$ be as in Lemma (6.2.5). It suffices to show that $M : L^{\psi}(\mathbb{R}^n) \to L^{\psi}(\mathbb{R}^n)$. Since $\psi^{1/\gamma}$ is convex, it follows from Jensen's inequality that

$$\psi(\epsilon Mf) = \left(\psi^{\frac{1}{\gamma}}(\epsilon Mf)\right)^{\gamma} \le \left(M\left(\psi^{\frac{1}{\gamma}}(\epsilon f)\right)\right)^{\gamma}.$$

Let $f \in L^{\varphi}(\mathbb{R}^n)$ and $\epsilon := \|f\|_{\psi}^{-1}$ so that $\varrho_{\psi}(\epsilon f) \le 1$. Since M is bounded in $L^{\gamma}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n}\left(M\left(\psi^{\frac{1}{\gamma}}(\epsilon f)\right)\right)^{\gamma}dx \lesssim \int_{\mathbb{R}^n}\left(\psi^{\frac{1}{\gamma}}(\epsilon f)\right)^{\gamma}dx = \int_{\mathbb{R}^n}\psi(\epsilon f)dx \leq 1.$$

Hence, $\varrho_{\psi}(\epsilon Mf) \lesssim 1$, which implies that $\|\epsilon Mf\|_{\psi} \lesssim 1$. Dividing by ϵ, we find that $\|Mf\|_{\psi} \lesssim \frac{1}{\epsilon} = \|f\|_{\psi}$, which completes the proof.

For $B \subset \mathbb{R}^n$ define $\varphi_B^-(t) := \inf_{x\in B}\varphi(t)$ and $\varphi_B^+(t) := \sup_{x\in B}\varphi(x,t)$. We will use the following assumptions for some common constant $\sigma > 0$. The second corresponds in the $L^{p(\cdot)}$ case to local log-Hölder continuity.

$(A0)$ there exists $\beta > 0$ such that $\varphi(x,\beta) \leq 1$ and $\varphi(x,\sigma) \geq 1$ for every $x \in \mathbb{R}^n$.

$(A1)$ there exists $\beta \in (0,1)$ such that

$$\varphi_B^+(\beta t) \leq \varphi_B^-(t)$$

for every $f \in \left[\sigma, (\varphi_B^-)^{-1}\left(\frac{1}{|B|}\right)\right]$ and every ball B with $\frac{1}{|B|} \geq \varphi_B^-(\sigma)$.

$(A2)$ there exists $\beta > 0$ and $h \in L^1_{\text{weak}}(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)$ such that for every $t \in [0,\sigma]$,

$$\varphi(x,\beta t) \leq \varphi(y,t) + h(x) + h(y).$$

From $(A0)$ we obtain $\psi(x,\beta/L) \leq \varphi(x,\beta) \leq 1$ and $\varphi(x,L\sigma) \geq \varphi(\sigma) \geq 1$, so ψ satisfies $(A0)$ with constant β/L and $\sigma' := L\sigma$ in place of σ. Suppose that $t \in \left[\sigma', (\varphi_B^-)^{-1}\left(\frac{1}{|B|}\right)\right]$. Then

$$\varphi_B^+\left(\frac{\beta}{L^2}t\right) \leq \varphi_B^+\left(\frac{\beta}{L}t\right) \leq \varphi_B^-(t)$$

since $\frac{t}{L} \in \left[\sigma, (\varphi_B^-)^{-1}\left(\frac{1}{|B|}\right)\right]$ so that $(A1)$ of φ could be used. Thus $(A1)$ holds for ψ, as well. For $(A2)$ we estimate, when $f \in [0,\sigma']$,

$$\psi(x,\beta t/L^2) \leq \varphi\left(x,\frac{\beta t}{L}\right) \leq \varphi\left(y,\frac{t}{L}\right) + h(x) + h(y) \leq \psi(y,t) + h(x) + h(y).$$

Lemma (6.2.7)[204]: Let $\varphi \in \Phi(\mathbb{R}^n)$. Then φ_B^- satisfies the Jensen-type inequality

$$\varphi_B^-\left(\frac{1}{2}\int_B fdx\right) \leq \int_A \varphi_B^-(f)dx.$$

Proof. Let ψ be the greatest convex minorant φ_B^-. Since $t \mapsto \frac{\varphi_B^-(t)}{t}$ is increasing, we conclude as in Lemma (6.2.4) that $\varphi_B^-(s) \leq \psi(2s)$. By Jensen's for ψ,

$$\varphi_B^-\left(\frac{1}{2}\int_B fdx\right) \leq \psi\left(\int_B fdx\right) \leq \int_B \psi(f)dx \leq \int_B \varphi_B^-(f)dx.$$

Lemma (6.2.8)[204]: Let $\varphi \in \Phi(\mathbb{R}^n)$ satisfy assumptions (A0)−(A2). If B is a ball and $f \in L^{\varphi(\cdot)}(\mathbb{R}^n)$ with $\varrho_{\varphi(\cdot)}(f\chi_{\{|f|>\sigma\}}) \leq 1$, then

$$\varphi\left(x,\frac{\beta}{4}\int_B |f|dy\right) \le \left(1+\tfrac{1}{\sigma}\right)\int_B \varphi(y,f)dy + h(x) + \int_B h(y)dy.$$

Proof. Fix a ball B. Assume without loss of generality that $f \ge 0$, and denote $f_1 \coloneqq f\chi_{\{f>\sigma\}}$ and $f_2 \coloneqq f - f_1$. Since φ is convex and increasing,

$$\varphi\left(x,\frac{\beta}{4}\int_B fdy\right) \le \varphi\left(x,\frac{\beta}{2}\int_B f_1dy\right) + \varphi\left(x,\beta\int_B f_2dx\right).$$

Consider first part f_1 when $\frac{1}{|B|} \ge \varphi_B^-(\sigma)$ and define

$$\bar{\varphi}(x,t) \coloneqq \begin{cases}\varphi(x,\sigma)t & \text{when } t \le \sigma,\\ \varphi(x,\sigma) & \text{when } t > \sigma.\end{cases}$$

Since $\varphi \le \bar{\varphi}$ is convex, since $f_1 \notin (0,\sigma)$, $\varphi(y,f_1(y)) = \bar{\varphi}(x,f_1(y))$. Therefore it suffices to prove the second inequality in

$$\varphi\left(x,\frac{\beta}{2}\int_B f_1dy\right) \le \bar{\varphi}\left(x,\frac{\beta}{2}\int_B f_1dy\right) \le \int_B \bar{\varphi}(y,f_1)dy = \int_B \varphi(y,f_1)dy.$$

Note that $\bar{\varphi}$ satisfies (A1) on all of $\left[0,(\varphi_B^-)^{-1}\left(\frac{1}{|B|}\right)\right]$. By Lemma (6.2.7),

$$\varphi_B^-\left(\frac{1}{2}\int_B f_1dy\right) \le \int_B \varphi_B^-(f_1)dy \le \int_B \varphi(y,f_1)dy \le \frac{1}{|B|}.$$

Therefore we can use (A1) and Lemma (6.2.7) to conclude that

$$\bar{\varphi}\left(x,\frac{\beta}{2}\int_B f_1dy\right) \le \bar{\varphi}_B^+\left(\frac{\beta}{2}\int_B f_1dy\right) \le \bar{\varphi}_B^-\left(\frac{1}{2}\int_B f_1dy\right) \le \int_B \varphi(y,f_1)dy.$$

Suppose then that $\frac{1}{|B|} \le \varphi_B^-(\sigma)$. Now

$$\int_B f_1dy \le \int_B \varphi_B^-(\sigma)dy \le \int_B \varphi(y,f_1)dy \le 1.$$

By convexity, (A0) and convexity again, we conclude that

$$\varphi\left(x,\beta\int_B f_1dy\right) \le \varphi(x,\beta)\int_B f_1dy \le \int_B \varphi(y,\sigma)f_1dy \le \frac{1}{\sigma}\int_B \varphi(y,f_1)dy.$$

For f_1 we use the convexity of $\varphi(x,\cdot)$ and (A2):

$$\varphi\left(x,\beta\int_B f_2dy\right) \le \int_B \varphi(x,\beta f_2)dy \le \int_B \varphi(y,f_2)dy + \int_B h(x) + h(y)dy.$$

Adding the estimate for f_1 and f_2, we conclude the proof.

Taking the supremum over balls B in the previous lemma, and noting that $h(x) \le Mh(x)$, we obtain the following corollary:

Corollary (6.2.9)[204]: Let $\varphi \in \Phi(\mathbb{R}^n)$ satisfy assumptions (A0)−(A2) and let $f \in L^{\varphi(\cdot)}(\mathbb{R}^n)$ with $\varrho_{\varphi(\cdot)}\left(f\chi_{\{|f|>\sigma\}}\right) \le 1$, then

$$\varphi\left(x,\frac{\beta}{4}Mf\right) \lesssim M\big(\varphi(\cdot,f)\big) + Mh(x).$$

Theorem (6.2.10)[204]: Let $\varphi \in \Phi(\mathbb{R}^n)$ satisfy assumptions (A0)−(A2). Suppose that $\beta > 1$ is such that $s \mapsto s^{-\beta}\varphi(x,s)$ is in creasing for every $x \in \mathbb{R}^n$. Then the maximal operator is bouneded,

$$M:\ L^{\varphi(\cdot)}(\mathbb{R}^n) \to L^{\varphi(\cdot)}(\mathbb{R}^n).$$

Proof. Let $\psi(x,\cdot) \in \Phi$ be related to $\varphi(x,\cdot)$ as in Lemma (6.2.5), for every $x \in \mathbb{R}^n$. Since the Φ-functions φ and ψ are equivalent, it suffices to show that $M:\ L^{\varphi(\cdot)}(\mathbb{R}^n) \to L^{\varphi(\cdot)}(\mathbb{R}^n)$. Note that ψalso satisfies assumptions (A0)−(A2).

Let $f \in L^{\psi(\cdot)}(\mathbb{R}^n)$ and also choose $\epsilon > 0$ such that $\varrho_{\psi(\cdot)}(\epsilon f) \le 1$. When $t > \sigma$, $\psi(x,t) \ge 1$ by (A0) so that $\psi(x,t)^{1/\gamma} \le \psi(x,t)$. Thus $\varrho_{\psi^{1/\gamma}(\cdot)}\big(\epsilon f\chi_{\{|f|>\sigma\}}\big) \le 1$ and we can apply Corollary (6.2.9) to ϵf with Φ-function $\psi^{1/\gamma}$:

$$\psi\left(x,\frac{\beta}{4}\epsilon Mf(x)\right)^{\frac{1}{\gamma}} \lesssim M\left(\psi^{\frac{1}{\gamma}}(\cdot,\epsilon f)\right) + Mh(x).$$

Raising both side to the power γ and integrating, we find that

$$\int_{\mathbb{R}^n}\psi\left(x,\frac{\beta}{4}\epsilon Mf(x)\right)dx \lesssim \int_{\mathbb{R}^n}M\left(\psi^{\frac{1}{\gamma}}(\cdot,\epsilon f)\right)(x)^{\gamma}dx + \int_{\mathbb{R}^n}Mh(x)^{\gamma}dx.$$

Note that $h \in L^1_{\text{weak}}(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n) \subset L^{\gamma}(\mathbb{R}^n)$ since M is bounded on $L^{\gamma}(\mathbb{R}^n)$, we obtain that

$$\int_{\mathbb{R}^n}\psi\left(x,\frac{\beta}{4}\epsilon Mf(x)\right)dx \lesssim \int_{\mathbb{R}^n}\left(\psi^{\frac{1}{\gamma}}(x,\epsilon f)\right)^{\gamma}dx + \int_{\mathbb{R}^n}h(x)^{\gamma}dx$$
$$= \varrho_{\psi(\cdot)}(\epsilon f) + \|h\|_{\gamma}^{\gamma}.$$

Hence $\varrho_{\psi(\cdot)}\left(\frac{\beta}{2}\epsilon f\right) \lesssim 1$, and the proof is completed by a scaling argument like Corollary (6.2.6).

As an example and application we consider the double-phase Φ-function studied by Baroni, Colombo and Mingine [176, 177, 179]. Note that bound $\frac{q}{p} \le 1 + \frac{\alpha}{n}$ is the same as that obtained by these researchers (for some of their results, the strict inequality is required).

Theorem (6.2.11)[204]: Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $\varphi(x,t) := t^p + a(x)t^q$, $q > p > 1$. If $a \in C^{\alpha}(\overline{\Omega})$ is non-negative, then the maximal operator is bounded on $L^{\varphi(\cdot)}(\Omega)$ when $\frac{q}{p} \le 1 + \frac{\alpha}{n}$.

Proof. We extend a y McShane extension to function in $C^{\alpha}(\mathbb{R}^n)$. This extension can be multiplied by a smooth cut-off function $H \in C_0^{\infty}(\mathbb{R}^n)$ with equals 1 in Ω. Since $q > p > 1$ and $a \ge 0$ in $\mathbb{R}^n$, it follows that $t \mapsto t^{-p}\varphi(x,t)$ is increasing.

We show that (A0)−(A2) hold with $\sigma = 1$. For (A0), we note that $1 \le \varphi(x,1) \le 1 + \|a\|_\infty \chi_k$. If K is the support of H, then $\varphi \equiv t^p$ in $\mathbb{R}^n \setminus K$, so (A2) holds with $h \coloneqq \|a\|_\infty \chi_k$.

Let us show that also condition (A1) holds. Note first that $\varphi(x,t) \coloneqq \max\{t^p, a(x)t^q\}$. Denote $a_B^+ \coloneqq \sup_{z\in B} a(z)$ and $a_B^- \coloneqq \inf_{z\in B} a(z)$. It suffices to show that

$$\max\{t^p, a_B^+ t^q\} \lesssim \max\{t^p, a_B^- t^q\}$$

when $\varphi_B^-(t) < \frac{1}{|B|}$. We prove the inequality in the even greater range $t^p < \frac{1}{|B|}$.

The inequality $t^p \lesssim \max\{t^p, a_B^- t^q\}$ is trivial, so we have to show that $a_B^+ \lesssim \max\{t^{p-q}, a_B^-\}$. Using the upper bound on t, we see that it is sufficient to prove that

$$a_B^+ \lesssim \max\left\{|B|^{\frac{q-p}{p}}, a_B^-\right\} \approx \operatorname{diam}(B)^{\frac{q-p}{p}n} + a_B^-.$$

In view of the definition of α, this follows from the assumption $a \in C^\alpha$.

Note that the reverse implication does not hold, i.e. (A1) does imply that $a \in C^\alpha(\overline{\Omega})$. Indeed, if we choose $a = \chi_\Omega + \chi_E$ foe some measurable $E \subset \Omega$, then a is discontinuous but $\varphi_B^+ \le 2\varphi_B^-$. On the other hand, the assumption is sharp in the sense that if $\frac{q}{p} \le 1 + \frac{\alpha}{n}$, then $a \in C^\alpha$ does not imply (A1), as show the example $a(x) = |x|^\alpha$.

In [155, 158, 174], Maeda, Mizuta, Ohno and Shimomura considered Musielak-Orlicz spaces with six conditions on the Φ-function. The first four conditions are, for some constant $D > 1$:

(Φ1) $\varphi : [0,\infty) \to [0,\infty)$ is continuous, $\varphi(0) = \lim_{t\to 0^+} \varphi(t) = \infty$.

(Φ2) $\frac{1}{D} \le \varphi(x,1) \le D$.

(Φ3) $\frac{\varphi(s)}{s}$ is almost increasing, i.e. $\frac{\varphi(s)}{s} \ge \frac{1}{D}\frac{\varphi(t)}{t}$ for every $s > t$.

(Φ4) φ is doubling, i.e. $\varphi(2t) \le D\varphi(t)$ for every $s > t$.

We note that assumption (Φ1) is ostensibly weaker than the assumption in this note, since convexity is not assumed a priori. However, we show below that any function satisfying these condition is equivalent to a convex function.

Assumption (Φ4) does not correspond to any assumption.

Assumption (Φ3) seems to be less stringent than the one used, since it follows from convexity that $\frac{\varphi(s)}{s}$ is increasing, not merely almost increasing.

Let φ satisfy assumption (Φ3) and define $\psi \coloneqq s\sup_{t\le s}\frac{\varphi(t)}{t}$. Clearly $\frac{\psi(s)}{s}$ is increasing and $\varphi \le \psi$. By condition (Φ3), $\sup_{t\le s}\frac{\varphi(t)}{t} \le D\frac{\varphi(s)}{s}$. Therefore

$$\varphi(s) \le \psi(s) \le \varphi(s) \le D\varphi(s) \le \varphi(D^2 s),$$

so the function are equivalent. Thus, there is no added generally in considering almost increasing functions instead of increasing functions. Furthermore, since $s \mapsto \frac{\psi(s)}{s}$ is increasing there exists a convex $\xi \in \Phi$ is equivalent to ψ by Lemma (6.2.4).

Condition ($\Phi5$) in [155, 158, 174] is essentially the same as (A1). However, their decay condition ($\Phi6$) seems more general, until we combine it with ($\Phi2$), which is a stronger version of (A0). The former condition is as follows:

($\Phi6$) there exists a function $g \in L^1(\mathbb{R}^n)$ and a constant $B_\infty \geq 1$ such that $0 \leq g(x) < 1$ for all $x \in \mathbb{R}^n$ and

$$B_\infty^{-1}\varphi(x,t) \leq \varphi(x',t) \leq B_\infty \varphi(x,t)$$

whenever $|x'| \geq |x|$ $g(x) \leq t \leq 1$.

Let us show how this related to condition (A2). First, we define

$$\varphi_\infty(t) \coloneqq \limsup_{x\to\infty} \varphi(x,t).$$

If $t \in [g(x), 1]$, then $\varphi(x,t) \leq B_\infty \varphi_\infty(t)$. If $t \in [0, g(x)]$ then $\varphi(x,t) \leq D\varphi(x,1)t \leq DA(x)$ by condition ($\Phi2$) and ($\Phi3$). Hence

$$\varphi(x,t) \leq B_\infty \varphi_\infty(t) + DAg(x)$$

for all $x \in [0,1]$. Similarly, we may establish

$$\varphi_\infty(t) \leq B_\infty \varphi(y,t) + DAg(y),$$

and this we conclude that

$$\varphi(x,t) \lesssim \varphi(y,t) + g(x) + g(y)$$

as required assumption (A2).

List of Symbols

Symbol		Page
$W^{k,p}$:	Sobolev space	1
L^{p^*}:	Lebesgue space	1
sup:	supremum	2
min:	minimum	3
$L^{1,\infty}$:	Lebesgue space	4
L^{∞}:	Essential Lebesgue	9
dist:	distance	10
loc:	Local	13
ess inf:	essential infimum	16
ess sup:	essential supremum	16
L^{q}:	Dual of Lebesgue space	30
max:	maximum	40
a.e:	almost every where	70
L^{1}:	Lebesgue in the real line	91
diam:	dimeter	93
dim:	dimension	99
supp:	Support	102
BV:	Bounded variation	161

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