Sudan University of Science and Technology

Gauge Couplings Unification in Anti-deSitter Space (AdS_5) at High Energy Scales

توحيد ثوابت القوى في نموذج الفضاء الخماسي النقيض عند مقاييس طاقة عالية

A dissertation submitted to the College of Graduate Studies, Sudan University of Science and Technology, Sudan-Khartoum, in fulfillment of the requirements for the degree of Master of Science in Physics

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Abstract

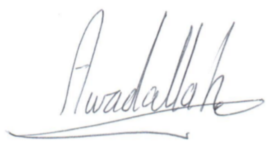
In this dissertation, the gauge couplings unification in AdS_5 was studied. We considered the Standard Model particles to propagate in the bulk of AdS_5. The renormalization group equation for the gauge coupling constants at one loop level were derived. We discussed the evolution of inverse fine structure constants α^{-1} which is related to the gauge coupling constants as well as the evolution of $\sin^2\theta_W$, where θ_W is the weak mixing angle in the Weinberg-Salam theory of the electroweak interaction.

We found logarithmic running for the gauge coupling constants as in the standard model instead of power law running as in case of flat extra dimension models; this result therefore preserving the perturbative unification of couplings. We showed that such theory can be compatible with gauge couplings unification at high energy scales, these scales at which unification occurred were found to be depend strongly on the cutoff Λ and on the number of Kaluza-Klein states. Furthermore, we found that $\sin^2\theta_W$ at this high scales rises up to 0.74 in our model where its value in QFT-subtraction scheme is equals to 0.23, which is a good indicator for the success of our model.

These results may be very useful, at least from a model building perspective of gauge couplings unification.

Declaration

I, the undersigned, hereby declare that the work contained in this thesis is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.



Awadallah MEKKI AWADALLAH, 15 April 2018

الملخص

في هذه الاطروحه، تم دراسة توحيد ثوابت القوى في نموذج الفضاء الخماسي النقيض عند مقاييس عالية. حيث تم السماح لجسيمات النموذج القياسي بالانتشار في فضاء الزمكان الخماسي النقيض $Bulk$. اشتقت معادلات المجموعة المعايرة عند الرتب الأولي لثوابت القوى. ونوقش مقلوب ثوابت البنية التركيبية α^{-1} والتي ترتبط بثوابت القوى القياسية وكذلك نوقش نشأة $\sin^2\theta_W$ بينما θ_W تمثل زاوية الاندماج الضعيفة في نظرية عبد السلام و واينبيرج للتفاعلات الضعيفة.

وجد في هذا النموذج أن نشأة ثوابت القوى تسلك سلوكا لوغريثميا كما في النموذج القياسي علي عكس نماذج الابعاد الزائدة المسطحة والتي نشأتها أسيه. وهذا يدل علي استقرارية النموذج حسب نظرية الاضطرابات. أُثبت أن هذه النظرية تتوافق مع نشأة ثوابت القوى عند مقاييس طاقة عالية، هذه المقاييس عند التوحيد تعتمد بقوة علي المقياس الحرج Λ وعلي عدد جسيمات كالوزا وكلاين. ايضا وجد أن نشأة $\sin^2\theta_W$ ارتفعت الي 0.74 في نموذجنا بينما قيمتها في نظرية $QFT - subtraction$ تساوي 0.23 وهذا مؤشر جيد لنجاح نموذجنا.

هذه النتائج قد تكون مفيدة جدا، علي الاقل لبناء نماذج توحيد ثوابت القوى.

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1. Introduction

It has been known that the ultimate goal of particle physics is to identify what appear to be structureless units of matter and to understand the nature of the forces acting between them. Stated thus, the enterprise has a two-fold aspect: matter on the one hand and forces on the other.

The expectation is that the smallest units of matter should interact in the simplest way, or there is a deep connection between the basic units of matter and the basic forces. Hence, this joint between matter and force nature of the inquiry is properly shown by the discovery of the electron by Thomson and Maxwell theory of the electromagnetic field, which together mark the birth of modern particle physics.

The duality nature of the electron; that is as an elementary source of the electromagnetic field, with its motion constituting an electromagnetic current as well as an important constituent of matter encouraged the particle physicists to investigate deeper in order to know more and more about the ultimate constituents of matter and to understand the interaction between these particles by their force fields. These have consisted in the discovery and study of two new (non-electromagnetic) forces; the weak and the strong forces.
This effort has recently culminated in crucial progress: the identification of matter blocks which are indeed in analogous to the electron and the highly convincing experimental verification of theories of the associated strong and weak force fields. Briefly, the picture as follows; there are two types of matter units: leptons and quarks. Both have spin-$\frac{1}{2}$ and are structureless at the smallest distances, currently these particles are probed by the highest-energy accelerators like CERN and Fermi-Lab. The leptons are directly generalization of the electron, if the particles are charged then they will interact both weakly and electromagnetically. However if they are neutral then they will only interact weakly [Aitchison et al., 2004].

By contrast, the quarks which are the fermions constituents of hadrons, and thence of nuclei-are more like strongly interacting analogues of the electron, since they interact via all three interactions, strong, electromagnetic and weak. This, actually, is the basis for distinguishing between these two types of matter unit, a distinction which will eventually disappear if it proves possible to unify all three types of force (out of four known forces) and to some extent we well be going to discuss this in the last chapter of this dissertation. There is more interaction left, which is the gravitational interaction, however studying this is out of the scope of this research because it's neglected compared to the rest of the three interactions and we will interpret this in more details through this work.

At all actions, the electromagnetic and weak interactions of both quarks and leptons are described partially in unified way by the electroweak theory, which is a generalization of quantum electrodynamics (QED); while the strong interactions of quarks are described by quantum chromodynamics (QCD), which is also in analog to QED.

These theories are collectively in the very famous theory in physics, that is "The Standard Model of Particle Physics", the essence of this theory is that; the elementary particles of matter are quarks and leptons, the mathematical framework for the force dynamics are gauge theories and the vacuum is in a sort of superconducting phase [Nagashima & Nambu, 2011]. We will discuss this theory next in this thesis as one of our theoretical phenomenology for the aim of this work.

But before, as an introduction we will review many concepts in High Energy Particle Physics. In the next section we are going to have an overview of the standard model and will study it in greater details in the next chapter.

1.1 Overview on the Standard Model

As we briefly introduced the Standard Model of particle physics which is the theory that describe three forces out of the four known fundamental forces in nature (the electromagnetic, weak, and strong interactions), as well as classifying all known elementary particles so far, this was driven by theoretical and experimental particle physicists alike, but before going into detail of that, we need to know a bit about the elementary particles, and this what we will discuss briefly next in this section and we are going to study that in more detail in the following chapter in this dissertation.

1.1.1 The Elementary Particles. What are the building blocks of matter? The answer for this question can be found by two ways; either by looking for the elementary particles, the ultimate constituents of matter at their smallest scale or on the other hand is to clarify what interactions are acting among them to construct matter as we see it. As the technology develops, the size of microscopic objects becomes smaller and smaller, so what is considered as an elementary particle in an era is turned out to be a composite particle latter on, this happened many time in the previous century in the field of particle physics.

In the 19^{th} century, when modern atomic theory was established, the exploitable size of the microscopic object was $10^{-10}m$ and the atom was "the elementary particle". Then it was recognized as a structured object when Thomson extracted electrons in 1897 from matter in the form of cathode rays. Its real structure (the Rutherford model) was clarified by investigating the scattering pattern of α-particles striking a golden foil. In 1932, Chadwick discovered that the nucleus, the core of the atom, consisted of protons and neutrons. In the same year, Lawrence constructed the first cyclotron. In 1934 Fermi proposed a theory of weak interactions. In 1935 Yukawa proposed the meson theory to explain the nuclear force acting among them.

The protons and neutrons together with their companion pions, which are collectively called hadrons, were considered as elementary particles until 1960, we know now they made from composite particle called quarks. Where the electrons remained as elementary particles as far as the technology can tell and they are collectively dubbed leptons. Quarks and leptons are the fundamental building blocks of matter so far [Nagashima & Nambu, 2011]. These technologies can investigate up to $10^{-19}m$ and there is the string theory which regards the ultimate constituents of matter not like particle but instead as string in the dimension of Plank's scale ($\sim 10^{-35}m$) and this is out of the scope of this dissertation.

1.1.2 The Four Fundamental Forces. It was well known in physics that, there are four fundamental forces in nature, the Strong, Electromagnetic, Weak and Gravitational force. They can be classified in groups as long-rang forces, which include the Gravity and the Electromagnetic

forces and as short-range forces and these are the Strong and Weak forces.
The electromagnetic force play an essential role in the microscopic level and at the same time it can be easy observed in the macroscopic level as well, however it's strength is inversely proportional with the square of distance. In general, the long-range forces their strength decrease only with the inverse power distance laws (Coulomb's and Newton's laws).

The strong force is thousands time stronger than the rest of the other forces, while the weak force is thousandths time weaker than the other forces and that in the ultra microscopic scale which is less than $10^{-15}m$ where both strong and weak force come in.

The electromagnetic and the weak forces were unified by Glashow, Weinberg and Salam in one force which called GWS theory or simply the electroweak theory [Aitchison et al., 2004]. Now a days many theoretical physics trying to unified all of the four forces in single mathematical framework, however this if it is done-even in the long term-would give a birth to a new era in physics, one of these models of unification is called the Grand Unified Theory (GUT) that is at very high energy the three gauge interactions of the forces, are merged into one single force and this unification is characterized by one large gauge symmetry [Buras et al., 1978].

These classifications address the properties of the forces generally however, all the fundamental forces, despite their variety of appearances, are essentially long-range forces and can be treated within the mathematical framework of gauge theories.

The weak force can be described by a gauge theory which contains two kind of charges, the gravity (theory of gravity) also can be described by it as well as the electroweak theory too, so in order to understand the interactions between particles we need to study the gauge theory first and this what we will do next as an introductory chapter, however in the next section briefly we will have an overview for the particles interactions. Considering that the electromagnetic and weak forces are unified and all four fundamental forces work in the same mathematical framework, it is natural to consider that all the forces are unified but show different aspects in different environments, however in this work as we mentioned we will omit the gravity.

1.2 Particle Interactions

In this part we will review the interactions between particles in general and study how the particle interactions developed from a classical point of view to much better point of view and that is the quantum theory. Previously, in classical physics it was thought that the matter and force are clearly separate for example as in Newton's theory of gravity. Latter on, the theory of quantum physics started to arise and give much more accurate interpretations for physical problems that couldn't have been explained by the classical physics such as the photoelectric effect, Compton effect and the duality nature of light. This lead to a conceptual distinction between matter and forces, or between particle and field, this distinction was no longer so clear until in 1927 Dirac gave a fully quantum mechanical explanation to the spontaneous emission process using the QED.

Now in this section we will give - in a general descriptive level - the use of Yukawa's intelligent idea of the quantum nature of force to study these interactions within the standard model, but before that we will introduce the physics behind this idea. Yukawa proposed a theory to understand the strong interaction between the proton and neutron and can explain the β-decay too. This theory built in analog with the electromagnetism interaction where the photon is the mediator of this force. He postulates a new field of force and succeed to understand the particles interact by exchanging virtual quanta, which mediate this force. Now Yukawa's theory of force as virtual quantum exchange can be stated.

He begins by postulating a potential that describe the strong interaction which is not a coulombic potential, but it's a static potential that decrease rapidly for the interaction between these particles, this in the range of $\leq 2fm$. The potential energy of his postulation is in the form

$$U(r) = \frac{-g_s^2}{4\pi}\frac{e^{-\frac{r}{a}}}{r} \tag{1.2.1}$$

where, g_s is a constant, $r = |r|$ and a is the range parameter. This potential satisfy the equation

$$\left(\bigtriangledown^2 - \frac{1}{a^2}\right)U(r) = g_s^2\delta(r) \tag{1.2.2}$$

Which is in analogous of Poisson's equation in electrostatics. Yukawa had generalized eqn:(1.2.2) to non-static case, that is

$$\left(\bigtriangledown^2 - \frac{\partial^2}{c^2\partial t^2} - \frac{1}{a^2}\right)U(r,t) = 0 \tag{1.2.3}$$

Which is relativistically invariant and still U is a classical field, then treat the field quantum mechanically as De-Broglie wave solution of U in the above equation and that is

$$U \propto e^{\left(ip\cdot\frac{r}{\hbar} - \frac{iEt}{\hbar}\right)} \tag{1.2.4}$$

By substituting eqn:(1.2.4) into eqn:(1.2.3) and solving for the positive root of E, one can find:

$$E = \left[c^2p^2 + \frac{c^2\hbar^2}{a^2}\right]^{\frac{1}{2}} \tag{1.2.5}$$

Comparing eqn:(1.2.5) with the result from the special relativity formula of the energy and momentum, we can easy deduce the following relation for the quantum finite field U which has the mass m_U:

$$a = \frac{\hbar}{m_U c} \tag{1.2.6}$$

For $a \sim 2fm \rightarrow m_U \sim 100MeV$, which is Yukawa's famous prediction for the mass of the nuclear force quantum mechanically.

These had lead to many physical interpretations to many problems in particle physics. Now we move to understand more aspects of Yukawa's force mechanism, in particular in the electromagnetic, weak, strong as well as gravity interactions.

Starting by the electromagnetic interactions, it is special case of Yukawa's picture where g_s^2 is replaced by the electromagnetic charges and $m_U \rightarrow m_\gamma = 0$ which is photon, hence $a \rightarrow \infty$ and the potential in eqn:(1.2.1) return to Coulomb potential. The one-photon exchange scattering process is shown in Fig:[1.1] which drawn as an abstract representation of similar interactions in QED known as Feynman's diagrams, the details of the influence of these diagrams will be studied further in the next chapters.

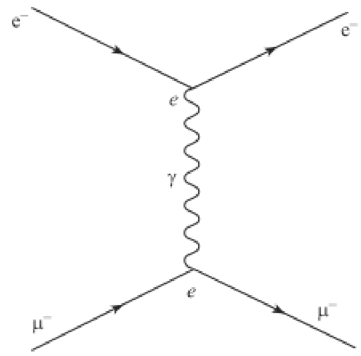


Figure 1.1: One photon exchange mechanism between charged leptons.

The coupling strength in the case of the electromagnetic interaction is e. The expansion parameter of perturbation theory is $\alpha = \dfrac{e^2}{4\pi} \sim \dfrac{1}{137}$. This gives us a general sense about the strength of the electromagnetic interaction.

For the Weak interactions Yukawa extended his theory to describe neutron β-decay too using the hypothesized process shown in the Fig:[1.2] below

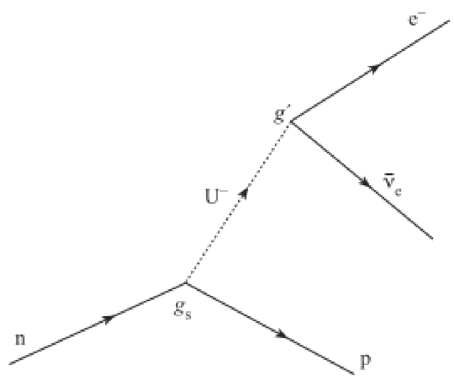


Figure 1.2: Yukawa's U-exchange mechanism for neutron β-decay.

The weak interaction is mediated by $W^{\pm}$ and Z^0 in the same manner as photon does in the electromagnetic interaction, however the photon is massless particle where the $W^{\pm}$ and Z are particles having mass, the $M_W \approx 80GeV$ and $M_Z \approx 91GeV$ and the range of the force in this interaction is $\sim 2.5 \times 10^{-18}m$ which is less than the dimension of the nuclear that is $\sim 10^{-15}m$, however this explain why the weak interaction is so weak because it's only affect tiny part of the hadronic volume.

The weak and strong interactions are mediated by different mediating quantum, but they are both generalization of the gauge theories of the electromagnetism, this arise the possibility of unifying all of the forces.

In the strong interactions Yukawa's theory of interaction viewed to be occurring between quarks rather than nucleons. The interquark force appears to be in analog of the QED, this is clear when probing two nucleons together, regardless to the composites of the nucleons this hard collision scatter quarks to wide angle and breaking up the nucleons in this interaction. The quark has never been found in an isolation (free particle), however what appears to happen is that when two quarks separate from each other, their mutual potential energy increases tremendously so that, at a certain level of the scattering process, the energy stored in the potential converts into a new pair of quarks. This process continues where many pairs of quarks been produced and many other pulled up by unknown mechanism yet, these quarks bidden together into hadrons within an interaction volume of order $1fm^3$, so that no free quarks are finally observed, consistent with 'confinement'. Very strikingly, these hadrons emerge in quite well-collimated 'jets' [Aitchison et al., 2004].

The mediators for the strong interaction are the gluons, unlike the mediators for the weak interaction, the gluon is like photon has no mass, however the QCD is much more complicated than the QED. The phenomenological potential for a pair of quarks is

$$V = -\frac{a}{r} + br \tag{1.2.7}$$

The first term of this potential is dominant for the small r which arise from single gluon exchange and $a \sim g_s^2$ is empirical value. The second term is dominating for the long r and typical value of b is $0.85GeV fm^{-1}$.

In the gravitational interactions, it's natural to expect that Yukawa theory of interaction apply to the gravity, but there is no consistent quantum filed theory of gravity has been constructed and only general framework of string theory offer hope for such combination of gravity and quantum theory and this is beyond the objective of this thesis.

To sum up; for this chapter, as a relatively simple picture for our purpose is to have an overview of the standard model of particle physics, there are quarks and leptons and they don't do gravitational interaction. The quarks form the hadrons and they are the source field of the gluon field. The weak interactions involve pairs of quarks and leptons, these are the source for $W^{\pm}$ and Z^0 fields. Quarks and leptons are also sources for the photon field in the electromagnetic interaction. All the mediating quanta have spin-1 and called bosons. The weak and strong force fields are generalizations of the electromagnetism field and all of them are gauge theories, however they can be interpreted in different ways.

An illustration for the elementary particles according to the standard model can be summarized in below chart:

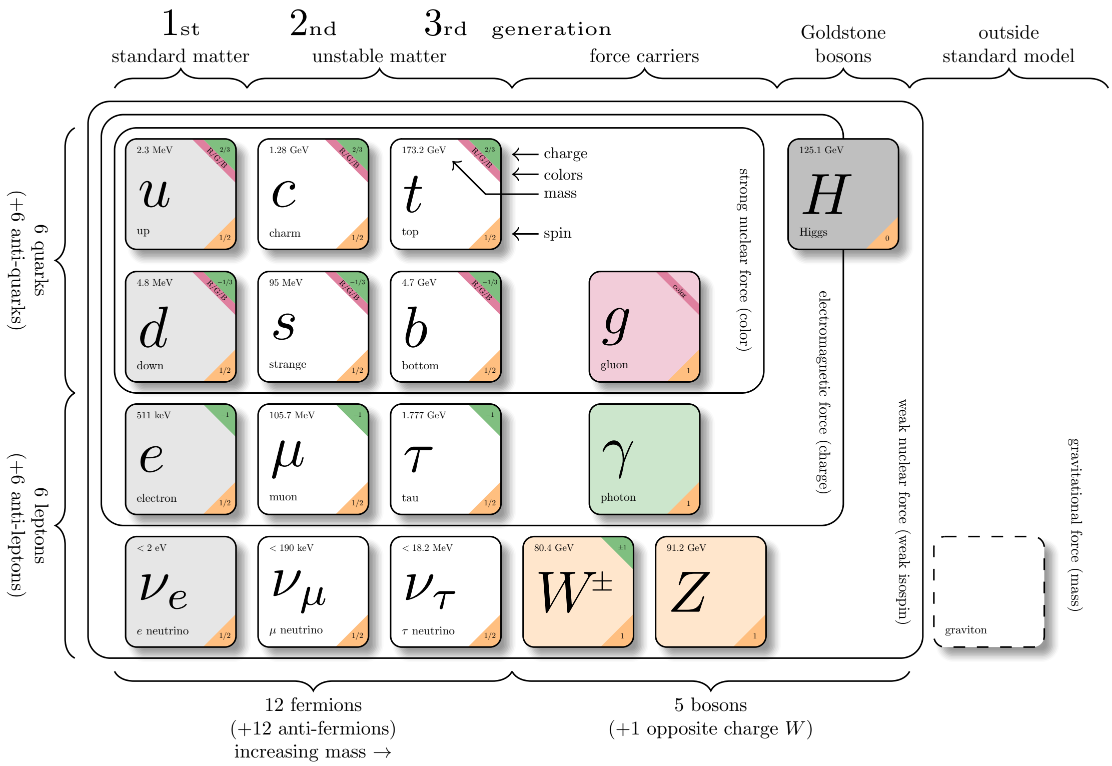


Figure 1.3: This chart summarizes the physical properties of the elementary particles, their interactions and their fields mediators according to the standard model (SM) of particle physics

Where the four fundamental forces nature is shown in this table:

Table 1.1: Shows the four fundamental forces in nature and their physical properties

The force	Charge	Mediator	Mass	Coupling	Range
Strong	Color	8 gulons (g)	0	1	$\simeq 10^{-15}m$
Electromagnetic	Electric charge	Photon (γ)	0	$\frac{1}{137}$	∞
Weak	Weak charge	$W^{\pm}, Z^0$	$\approx 10^2 GeV/c^2$	10^{-2}	$\simeq 10^{-18}m$
Gravitation	Mass	Graviton (g')	0	10^{-4}	∞

To have a deeper sight into the standard model of particle physics we need to understand the fields theories for these interactions and that what we will do in the next chapter of this dissertation which is titled as the fields theories and the standard model.

1.3 Research Problem

Models with extra dimensions might explain some of the unanswered issues of particle physics. One of the candidate proposals of this sort is the Randall-Sundrum (RS) model. where the strong warping of the extra dimensions introduces an exponential hierarchy between the Planck and the weak scales.

1.4 Aim of the Research

We aim to study the Randal Sundrum (RS) model to develop theoretical tools that required to construct model in warped extra dimensions and calculate the gauge couplings constants in this model, and solve the renormalization group equations for it numerically by a dictated numerical program (mathematica version 9).

1.5 Outline of the Research

The dissertation is organized as follow: In chapter two we review the Field Theories and the Standard Model that will be necessary to calculate the effective Lagrangian. In Chapter Three, we develop theoretical tools that required to construct model in warped extra dimensions and match the higher dimensional theory to an effective 4D Lagrangian. Chapter Four is meant for our results, discussions and conclusions.

2. Field Theories and the Standard Model

The aim of this chapter is to study the field theories, gauge theories and the standard model in order to develop the phenomenological basis to see how the generic model can be constructed, introduction to symmetry and the spontaneous symmetry breaking will be discussed too.

2.1 Introduction

In the theoretical phenomenology of particle physics, the particles and their interactions can be studied using the quantum field theory (QFT), this is vital because as we will see the particles are just excitation from the fields and we can study them using it, that why it's an our aim for this chapter and we will study them later in the third section, where it is also can be used to quantize these fields, however we can't do that without quantum mechanics principles [Peskin & Schroeder, 2005, Zee, 2010, Lancaster & Blundell, 2014].

QFT is a tool among others that we will be using to study the gauge coupling unification at a high energy scales which is our aim for this dissertation. What is more, we will study in later section how the continuous symmetries of the Lagrangian is related to the equations of motions that is the famous Noether's theorem, in addition to these in the last section we shall study examples of Lagrangian to construct the Standard Model Lagrangian in the last part of this chapter in order to study briefly the Standard Model of particle physics as an additional topic for this chapter, where at the same time represents our main foundation for the following chapters.

We are reviewing some of the quantum mechanics theory features in the second section of this chapter, then we are going to study the QFT in the following one to see how classical fields can be quantized in order to understand the nature of the particles and their interactions. All these and others are required to know what is meant by the coupling of the gauges at a high scale as our aim of this dissertation. In the first section of this chapter we will present the classical field theories to cope with the purposes of this chapter as we have just stated above.

2.2 Classical Field Theories

As foundation concepts, we will study the concept of the Lagrangian density formalism classically and then see how it can be generalized using QFT.

In the classical paradigm the field is defined as a set of numbers assigned to each space-time point and the dynamics of these fields that is how they evolve with time, can be studied using the Lagrangian and hence the principle of the least action. In the classical mechanics from the

Lagrangian function L, which is defined as the difference between the kinetic and the potential energy

$$L(q, \dot{q}) = T - V \tag{2.2.1}$$

where $\dot{q}$ is the velocity and q is the coordinate [Aitchison et al., 2004], the derivative of the Lagrangian with respect to the velocity is the momentum

$$p = \frac{\partial L}{\partial \dot{q}} \tag{2.2.2}$$

and with respect to the coordinate is the force

$$F = \frac{\partial L}{\partial q} \tag{2.2.3}$$

The quantity

$$H = \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L \tag{2.2.4}$$

Is the Hamiltonian of the system which represents the total energy of that system of many particles.

The least action is defined as

$$S = \int_{t_1}^{t_2} dt\, L(q(t), \dot{q}(t)) \tag{2.2.5}$$

where, $q(t)$ is the generalized coordinates and $\dot{q}(t)$ is the generalized velocity. The equations of motion come from the Hamilton's principle of least action subjected to the constrains t_1 and t_2 which is

$$\delta S = \delta \int_{t_1}^{t_2} dt\, L(q(t), \dot{q}(t)) = 0 \tag{2.2.6}$$

because in many problems in physics, the Lagrangian depends only on the generalized coordinates and their first derivatives, the variation in the action will be

$$\begin{aligned} \delta S &= \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial q(t)} \delta q(t) + \frac{\partial L}{\partial \dot{q}(t)} \delta \dot{q}(t) \right] \\ &= \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial q(t)} \delta q(t) + \frac{\partial L}{\partial \dot{q}(t)} \frac{d}{dt} \delta q(t) \right] \end{aligned} \tag{2.2.7}$$

The second part of eqn:(2.2.7) can be integrated as following:

$$\int_{t_1}^{t_2} \left[dt \frac{\partial L}{\partial \dot{q}(t)} \cdot \frac{d}{dt} \delta q(t) \right] = \frac{\partial L}{\partial \dot{q}(t)} \left[\delta q(t) \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} dt \left[\frac{d}{dt} \frac{\partial L}{\partial \dot{q}(t)} \right] \delta q(t) \tag{2.2.8}$$

Because there are no change at the end points because of the least action concept, hence we have

$$\delta q(t_1) = \delta q(t_2) = 0 \tag{2.2.9}$$

So by substituting eqn:(2.2.9) into eqn:(2.2.8) this yields:

$$\int_{t_1}^{t_2}\left[dt\frac{\partial L}{\partial \dot{q}(t)}\cdot\frac{d}{dt}\delta q(t)\right]=0-\int_{t_1}^{t_2}dt\left[\frac{d}{dt}\frac{\partial L}{\partial \dot{q}(t)}\right]\delta q(t) \tag{2.2.10}$$

The variation in the action in eqn:(2.2.7) will be

$$\begin{aligned}\delta S &= \int_{t_1}^{t_2}dt\left[\frac{\partial L}{\partial q(t)}\delta q(t)-\left[\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right]\delta q(t)\right]\\ &= \int_{t_1}^{t_2}dt\left[\frac{\partial L}{\partial q(t)}\delta q(t)-\left[\frac{d}{dt}\frac{\partial L}{\partial \dot{q}(t)}\right]\delta q(t)\right]\\ &= \int_{t_1}^{t_2}dt\left[\frac{\partial L}{\partial q(t)}-\left[\frac{d}{dt}\frac{\partial L}{\partial \dot{q}(t)}\right]\right]\delta q(t) \end{aligned}\tag{2.2.11}$$

By using eqn:(2.2.9) into eqn:(2.2.11), it deduced to

$$\frac{\partial L}{\partial q(t)}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}(t)}=0 \tag{2.2.12}$$

Which is Euler-Lagrange equations in the generalized coordinates.

These classical generalizations of the Lagrangian is required to study the gauge field theories, i.e, our tools to the objective of this work, because it can give us the physical sense of fields however with different interpretation as we can see later.

From the definition of the classical Lagrangian and with the relativistic invariance transformation of the generalized coordinates for space and time, the classical field theory can be constructed, this begins by defining the Lagrangian density

$$\mathscr{L}(\phi(x),\partial_\mu\phi(x)) \tag{2.2.13}$$

For a functional of the field $\phi(x)$ and its four-components, the Lagrangian depends on the gradient $\partial_\mu\phi(x)=\partial\phi(x)/\partial x^\mu$. This field-often is complex-can be considered as a distinct, generalized coordinate which is the space-time coordinate x, hence the field theory now can be considered as a closed system with an infinite number of degrees of freedom. The classical action S in eqn:(2.2.5) in this case can be written as

$$S=\int_{t_1}^{t_2}dt\int d^3\mathbf{x}\mathscr{L}(\phi,\partial_\mu\phi) \tag{2.2.14}$$

Because the second part is the spacial part, the second integral takes the form of the Lagrangian in classical mechanics, we can compare eqn:(2.2.14) to eqn:(2.2.5), hence,

$$L=\int d^3\mathbf{x}\mathscr{L}(\phi,\partial_\mu\phi) \tag{2.2.15}$$

From the least action concept we are looking for infinitesimal small variation of the field that leaves the action unchanged

$$\begin{aligned} \phi(x) &\rightarrow \phi(x)+\delta\phi(x) && (2.2.16)\\ S &\rightarrow S+\delta S && (2.2.17)\end{aligned}$$

For the same reason that we wrote eqn:(2.2.9) the variation in the action of eqn:(2.2.14) is required to be stationary as stated in [Quigg, 2013]. We can write

$$\delta S = 0 \tag{2.2.18}$$

Or,

$$\delta S = \delta\phi \frac{\partial}{\partial\phi}\int \mathscr{L}\left(\phi,\partial_\mu\phi\right)d^4x + \delta\left(\partial_\mu\phi\right)\frac{\partial}{\partial(\partial_\mu\phi)}\int \mathscr{L}(\phi,\partial_\mu\phi)d^4x \tag{2.2.19}$$

It is clear that

$$\partial_\mu(\delta\phi) = \delta\partial_\mu\phi + \phi\partial_\mu\delta \tag{2.2.20}$$

The second part of the right side of eqn:(2.2.20) is zero, so we will get

$$\delta\partial_\mu\phi = \partial_\mu(\delta\phi) \tag{2.2.21}$$

So by rewriting eqn:(2.2.19)

$$\begin{aligned}\delta S &= \delta\phi\frac{\partial}{\partial\phi}\int \mathscr{L}\left(\phi,\partial_\mu\phi\right)d^4x + \delta\left(\partial_\mu\phi\right)\frac{\partial}{\partial(\partial_\mu\phi)}\int \mathscr{L}(\phi,\partial_\mu\phi)d^4x\\ &= \int\left[\delta\phi\frac{\partial\mathscr{L}}{\partial\phi} + \partial_\mu(\delta\phi)\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\right]d^4x && (2.2.22)\end{aligned}$$

The last term of this equation can be written in the form:

$$\partial_\mu\left[\delta\phi\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\right] = \partial_\mu\left(\delta\phi\right)\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)} + \left(\delta\phi\right)\partial_\mu\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)} \tag{2.2.23}$$

Which can be rearranged to

$$\partial_\mu\left(\delta\phi\right)\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)} = \partial_\mu\left[\delta\phi\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\right] - \left(\delta\phi\right)\partial_\mu\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)} \tag{2.2.24}$$

By substituting eqn:(2.2.24) into eqn:(2.2.22) the variation in the action can be written as

$$\begin{aligned}\delta S &= \int\left[\delta\phi\frac{\partial\mathscr{L}}{\partial\phi} + \partial_\mu(\delta\phi)\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\right]d^4x\\ &= \int\left[\delta\phi\frac{\partial\mathscr{L}}{\partial\phi} + \partial_\mu\left[\delta\phi\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\right] - \left(\delta\phi\right)\partial_\mu\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\right]d^4x\\ &= \int\left[\delta\phi\frac{\partial\mathscr{L}}{\partial\phi} - \left(\delta\phi\right)\partial_\mu\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\right]d^4x && (2.2.25)\end{aligned}$$

The term $\partial_\mu\left(\delta\phi\left(\partial\mathscr{L}/\partial(\partial_\mu\phi)\right)\right)$ vanished because at the boundaries $\delta\phi = 0$ in a similar way to eqn:(2.2.5), where else it is valid for all $\delta\phi$, together with eqn:(2.2.18) eqn:(2.2.25) deduced to

$$\frac{\partial\mathscr{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\phi\right)}\right) = 0 \tag{2.2.26}$$

That is the Euler-Lagrange Equation for the classical fields. It tells us that; for a given Lagrangian density which is defined by a certain classical field the corresponding classical equations of motions for that field can be driven using it.

Next we will examine few examples for some Lagrangian's using eqn:(2.2.26). First for massless and massive scalar fields, then, for a scalar field that is coupled to an external source and lastly for a couple of scalar fields and a complex field, from the indented Lagrangian for each in the classical field theory.

To begin with as first example, consider the relativistically covariant Lagrangian of the massless scalar field $\phi(x)$ which is our first case,

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi = \frac{1}{2}\left(\partial_\mu\phi\right)^2 \tag{2.2.27}$$

This Lagrangian can be expanded to be in the form of eqn:(2.2.1) as following

$$\mathcal{L} = \frac{1}{2}\left(\partial_0\phi\right)^2 - \frac{1}{2}\nabla\phi\cdot\frac{1}{2}\nabla\phi \tag{2.2.28}$$

Where the first part represents the kinetic energy and the second part is the potential energy. From eqn:(2.2.27) we can get

$$\frac{\partial\mathcal{L}}{\partial\phi} = 0, \qquad \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)} = \partial^\mu\phi \tag{2.2.29}$$

Inserting eqn:(2.2.29) terms into eqn:(2.2.26) this yields[1]

$$\partial_\mu\partial^\mu\phi = 0 \tag{2.2.30}$$

In the form of eqn:(2.2.28) with $\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right)$ one can write

$$\partial_\mu\partial^\mu\phi = \frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi = 0 \tag{2.2.31}$$

This is the wave equation for the massless scalar field. So as we have seen, we get the equation of motion for the scalar field from inserting the corresponding Lagrangian into the Euler-Lagrangian equation.

For the massive scalar field consider the Lagrangian given by

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{1}{2}m^2\phi^2 \tag{2.2.32}$$

By simply differentiation this Lagrangian we will get:

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi \tag{2.2.33}$$

$$\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)} = \partial^\mu\phi \quad\Rightarrow\quad \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\right) = \partial_\mu\partial^\mu\phi = \Box\phi \tag{2.2.34}$$

[1] See for example Ref. [Peskin & Schroeder, 2005] for the relativistic covariant notation that we used here.

Substituting the terms from eqn:(2.2.33) and eqn:(2.2.34) into the Euler-Lagrange Equation the later deduced to:

$$\left(\Box + m^2\right)\phi = 0 \tag{2.2.35}$$

This equation is the well known Klein-Gordon equation for the massive scalar field (free particle) driven in this classical regime. It is clear that when $m = 0$ we will get back to eqn:(2.2.30) the massless scalar field case. This classical picture can't explain an aspect of the solution for Klein-Gordon equation, where the quantum paradigm can explain that as we will see later in the next section,

Klein-Gordon equation can be derived in the quantum picture for the field too, to show that both pictures can lead to the same result.

Now, consider the case where the scalar field couples to an external source in one dimension, the simplest case here is that the field is interacting with an external potential, giving a source current $J(x)$ The Lagrangian in this case is

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi(x)\right)^2 - \frac{1}{2}m^2\phi(x)^2 + J(x)\phi(x) \tag{2.2.36}$$

From the above Lagrangian one can get,

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi(x) + J(x) \tag{2.2.37}$$

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi(x)\right)}\right) = \partial_\mu\partial^\mu\phi(x) \tag{2.2.38}$$

Plugging eqns:(2.2.37) and (2.2.38) into eqn:(2.2.26) and rearrange we get:

$$\partial_\mu\partial^\mu\phi(x) + m^2\phi(x) = J(x) \tag{2.2.39}$$

When the $J(x) = 0$ we get back to eqn:(2.2.35) which is Klein-Gordon equation.

Our final illustration is that when we have coupling scalar fields with a complex scalar field. Consider the two defined fields

$$\psi = \frac{1}{\sqrt{2}}\left[\phi_1 + i\phi_2\right] \tag{2.2.40}$$

$$\psi^\dagger = \frac{1}{\sqrt{2}}\left[\phi_1 - i\phi_2\right] \tag{2.2.41}$$

In this case we obtain a new form of Lagrangian defined by

$$\mathcal{L} = \partial^\mu\psi^\dagger\partial_\mu\psi - m^2\psi^\dagger\psi - g\left(\psi^\dagger\psi\right)^2 \tag{2.2.42}$$

We can notice about this field; the complex field ψ created from the two scalar fields ϕ_1 and ϕ_2 as well as $\psi^\dagger$ (having two degree of freedom), the second thing is that the Lagrangian is invariant for the rotations $\psi = \psi e^{i\alpha}$ and $\psi^\dagger = e^{-i\alpha}\psi^\dagger$ as following:

$$\begin{aligned}
\mathcal{L} &= \partial^\mu\psi^\dagger\partial_\mu\psi - m^2\psi^\dagger\psi - g\left(\psi^\dagger\psi\right)^2 \\
&= \partial^\mu e^{-i\alpha}\psi^\dagger\partial_\mu\psi e^{i\alpha} - m^2 e^{-i\alpha}\psi^\dagger\psi e^{i\alpha} - g\left(e^{-i\alpha}\psi^\dagger\psi e^{i\alpha}\right)^2 \\
&= e^{-i\alpha}\partial^\mu\psi^\dagger e^{i\alpha}\partial_\mu\psi - m^2 e^{-i\alpha}\psi^\dagger\psi e^{i\alpha} - g\left(e^{-i\alpha}\psi^\dagger\psi e^{i\alpha}\right)^2 \\
&= e^{-i\alpha+i\alpha}\partial^\mu\psi^\dagger\partial_\mu\psi - m^2\psi^\dagger\psi e^{-i\alpha+i\alpha} - g\left(e^{-i\alpha+i\alpha}\psi^\dagger\psi\right)^2 \\
&= \partial^\mu\psi^\dagger\partial_\mu\psi - m^2\psi^\dagger\psi - g\left(\psi^\dagger\psi\right)^2 \\
&= \mathcal{L}
\end{aligned} \tag{2.2.43}$$

So the Lagrangian has experenceed one dimensional unitary transformation, or in other words in terms of group theory-the formal representation of symmetry-the Lagrangian has encountered $U(1)$ symmetry, which means that the above Lagrangian can give a physical meaning. The relationship between group theory and symmetry is formally stated in Noether's theorem.

We can easily predict for a complicated situation when we have a massive scalar field and a massive complex scalar field and they interact through a potential, the Lagrangian will be,

$$\mathcal{L} = \partial^\mu\psi^\dagger\partial_\mu\psi - \bar{m}^2\psi^\dagger\psi + \frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{1}{2}m^2\phi^2 - g\psi^\dagger\psi\phi \tag{2.2.44}$$

It means that we have just combined the Lagrangian's from eqn:(2.2.42) and eqn:(2.2.32) and we subtracted the interaction term $g\psi^\dagger\psi\phi$ to get the intended Lagrangian.

As an interpretation from the previous discussion, we carefully can claim that: for the intended particles that interacting through a certain fields a Lagrangian is needed to be written in order to construct the physical theory to these interactions, hence, as smart guess we can predict that for many particles, which are interacting through different fields; a generalized form of Lagrangian is important to be written to construct the physical model for these particles and their interactions in one single model, we will experience an example for similar Lagrangian in the last section of this chapter.

So, we have given a clue on how we can apply the same trick to a generalized form of a Lagrangian, which is the Lagrangian of the standard model to study the corresponding theories to the concept of the gauges unification at very high scales which will be studying at the last chapter of this work, because as we mentioned its vital in order to write down a physical model is to construct the appropriate Lagrangian. However, before that we are going to study an alternative picture to this classical picture of the field theory which is more accurate, elegant and sophisticated, but with a shortage than the classical field theory that is the quantum field theory.

The aim of this theory is to construct fields that are similar to the classical fields theories defined over the space and time, but also accommodates the observations of quantum mechanics theory. This is the QFT, where we will have relevant introduction to it later on this chapter.

2.3 Quantum Theories

It is well known that a quantum mechanics state can be described by a state vector in a complex vector space that known as the Hilbert space and one important variable in the quantum theory is time!

There are two pictures of quantum mechanics; Heisenberg picture in which the time dependence is located in the operators and the wave function is constant, while in the other picture that is Schrodinger picture, the wave function only depends on time, we refer to [Griffiths, 2016] & [Zettili, 2009] for further reading.

In Schrodinger picture the way that the wave equation $\psi(x,t)$ develop with time is given by Schrodinger equitation of motion for that wave function

$$i\frac{\partial\psi(x,t)}{\partial t} = \hat{H}\psi(x,t) \tag{2.3.1}$$

Where, $\hat{x} = x$ and $\hat{p} = i\nabla$ are the operators of position and momentum respectively.

In Heisenberg picture the expectation value of an operator $\hat{\mathcal{O}}$ is

$$\left\langle \hat{\mathcal{O}}(t) \right\rangle = \langle\psi(t)|\, \hat{\mathcal{O}}\, |\psi(t)\rangle \tag{2.3.2}$$

and

$$\psi(t) = e^{-i\hat{H}t}\psi(0) \tag{2.3.3}$$

Where $\psi(0)$ is the wave function at $t = 0$, the equation of motion for Heisenberg picture is

$$\frac{d\hat{\mathcal{O}}_H(t)}{dt} = \frac{1}{i\hbar}\left[\hat{\mathcal{O}}_H(t), \hat{H}\right] \tag{2.3.4}$$

Before getting into the quantization of the fields, here we give example on how a classical physical problem can be interpreted using quantum mechanics theory, so we could have a clue about what we will expect in our purpose of study in this part as an alternative paradigm to what we have discussed in the previous section.

The following problem is the famous problem in physics which show how the quantum interpretation for a classical problem could yield a better understanding than the classical one, that is the simple harmonic oscillator. For example consider a mass m attached to spring which has the constant K with x displacement between the mass and the equilibrium point. Classically the momentum and the energy are $P = m\dot{x}$ and $E = Kx^2/2$ respectively.

The total energy of the spring is the sum of kinetic and potential energy of it

$$E = \frac{P^2}{2m} + \frac{1}{2}Kx^2 \tag{2.3.5}$$

Transforming this equation to Schrodinger picture, one can write

$$\hat{H}\psi = E\psi \tag{2.3.6}$$

The $\hat{H}$ is the energy operator (Hamiltonian) and E is the energy eigenvalue corresponding to that operator. Replacing the classical momentum by the operator $\hat{P} = -i\hbar\partial/\partial x$ and the position by the operator $\hat{x}$, gives

$$\therefore \quad \hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}K\hat{x}^2 \tag{2.3.7}$$

So we can rewrite eqn:(2.3.6) as

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}Kx^2\right)\psi = E\psi \tag{2.3.8}$$

Which is the Schrodinger equation for the simple harmonic oscillator. Its full solution can be found in [Griffiths, 2016] because we only interested in the result that is

$$\psi_n(\xi) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} H_n(\xi)e^{-\frac{\xi^2}{2}} \tag{2.3.9}$$

Where, H_n is the Hermite polynomial that gives the wave-like behavior for the wave function and $\xi = \sqrt{m\omega/\hbar x}$.

The eigenvalue is

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega, \qquad n = 0, 1, 2, \cdots \tag{2.3.10}$$

is the particle-like behavior with $\omega = \sqrt{K/m}$. For $n = 0$, it reads $E_0 = \hbar\omega/2$, which known as the zero-point energy.

This duality in the point of view for the same problem is called quantization, which has two aspects; the first quantization: Particles behave like waves. The second quantization: Waves behaves like particles [Lancaster & Blundell, 2014].

Dropping the zero-energy we will have

$$E_{n>0} = n\hbar\omega \tag{2.3.11}$$

We have the oscillator with n quanta on it, where each particle in a certain momentum state. To move to the next energy level in the leader we need to add a quanta $\hbar\omega$ to the zero-point energy and to move from higher energy level to lower one we need to subtract the quanta $\hbar\omega$. This is the quantization of energy.

These can be expressed in terms of energy operators as following:

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right) \tag{2.3.12}$$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right) \tag{2.3.13}$$

$\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators respectively. Which have the commutation rules:

$$[\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0 \quad (2.3.14)$$

$$\left[\hat{a}_i, \hat{a}_j^\dagger\right] = \delta_{ij} \quad (2.3.15)$$

So to move from the zero-point energy to the first energy state we need to operate by $\hat{a}^\dagger$ and to move from higher energy state to a lower state, we need to operate by $\hat{a}$, but when we operate by $\hat{a}$ on the zero-point energy we will get the same energy eigenvalue, which called the ground state energy level.
We can state that the universe is set of simple harmonic oscillators, with the zero-point as the least energy level, so in any other level, if we want to measure the energy for that level, we need to measure the relative energy to the zero-point energy.

There is important question in physics that particle physicists trying to answer; how the particles get their masses!? This seems like a trivial question, but it's not the case indeed! We can find an answer for it from the principal of symmetry later in the next section and when we dive deeper while we use the quantum theory tools to redefine the classical field which known as the quantum field theory, which we will be tackling some of it's aspects next.

In particle physics, particles are classified as hadrons or mesons according to; if they are elementary particles or constituent particles and to their quarks constituents according to the quark model. Also, it's well know that particles in the universe is classified in another way into two types according to their spin; fermions (having half integer spin) and bosons (having integer spin). We can exchange two identical bosons to get the same state but we can't do that for fermions because we will violate Pauli exclusion principle.

Consider a system of particles in the simplest way that is the particles which are confined in a box and then we will recap that because it's more realistic to study particles as continuum system. In natural units[2], the momentum operator can be written as $\hat{p} = -i\partial/\partial x$. The solution for Schrodinger plane wave equation for a particle in a box with side length L is

$$\psi(x) = \frac{1}{\sqrt{L}} e^{ipx} \quad (2.3.16)$$

Operating on that wave function by the momentum operator yields the corresponding momentum state,

$$\hat{p}\psi(x) = p\psi(x) \quad (2.3.17)$$

This wave function must satisfy periodic boundary conditions $\psi(x) = \psi(x + L)$, so eqn:(2.3.16) becomes

$$\psi(x + L) = \frac{1}{\sqrt{L}} e^{ip(x+L)} = \psi(x) \quad (2.3.18)$$

We can't get this unless $e^{ipL} = 1$, from Euler theorem we can deduce that; the momentum for this particles having that wave function is quantized, hence

$$P_m = \frac{2\pi m}{L} \quad (2.3.19)$$

[2]here $\hbar = c = 1$ is used

Where, m is an integer represents the momentum state.

In case of non-interacting particles we can get the total energy for that system by operating by the Hamiltonian on that particles, but how the energy state for each particle can be obtained! This can be easy understood in term of the occupation number representation which can be found when we operate on the vacuum state and from the principal of symmetry (type of the particles). The vacuum state $|0\rangle$ is crucial in quantum field theory in analog of the zero-point energy in the simple harmonic oscillator, hence, to get any particle in particular state we need to operate by the proper operator, as we can see later.

The state label of the particles can be replaced by the creation operator $\hat{a}_i^\dagger$ when we acting on the vacuum state $|0\rangle$ to get the corresponding particle in a particular state. This generally can be written as

$$|n_1 n_2 \cdots\rangle = \prod_m \frac{1}{(n_{pm}!)^{\frac{1}{2}}} (\hat{a}_{pm})^{n_{pm}} |0\rangle \tag{2.3.20}$$

It vary whether the particle in the box are bosons or fermions; in case of bosons we can add as many particles as possible in the same states where they are symmetric for the exchange of any two particles and these particles don't obey to Pauli exclusion principle, where this can't be happen in case of fermions because fermions obey to it, we must conserve the symmetry by not exchanging any two fermions, because they are antisymmetric particles.

In more generalized form, as the box dimension increase the momentum state for the particles will be nearly spaced, taking infinite size in a generalized space variables the momentum will be transformed from being discrete to a continuous state and the δ_{ij} will transformed to Dirac function, see for instance Refs. [Peskin & Schroeder, 2005, Lancaster & Blundell, 2014] and the references therein. Then the summation will transformed to an integral, this is called the continuum limit, which describes particles nature better than the box model which we have just discussed before.

so the commutaion relation in equations (2.3.14) and (2.3.15) will be written in the continuum limit as

$$\left[\hat{a}_p, \hat{a}_q^\dagger\right] = \delta^{(3)}(p-q) \tag{2.3.21}$$

Bosons can be described by commuting operators, while the fermions are described by anticommuting operators where the Hamiltonian in this paradigm is written as

$$\hat{H} = \int d^3p E_p \hat{a}_p^\dagger \hat{a}_p \tag{2.3.22}$$

To sum up, the quantization of the simple harmonic oscillator gave us a clue on how the quantization of the simplest case can be done when the particles is confined in a box using the creation and annihilation operators, this paradigm of quantization states depends on whether the particles are bosons or fermions, however this is generalized to the continuum limit, where it's more realistic in describing particles nature.

We have seen how the particles can be created to a certain momentum state when we operate on the vacuum state by the creation operator. We can dive deeper to study the nature of these field itself in the following sections, this is important because QFT is describe how the particles interact among themselves or with other entities. We study that in order to attempt to answer the questions; are these particles on the first place were confined under one field? or these particles could be confined in one field at a high scale? Are there any models yet? the first question could be answered by cosmologists, but we are more interested to attempt to answer the last question later as our aim for this thesis, but all these depend on the field of these particles that we are studying.

Fields operators create and annihilate particles in different ways than $\hat{a}_p^\dagger$ and $\hat{a}_p$ do, which create/annihilate particle into particular momentum state, but alternatively the field operators create/annihilate particles into particular spatial locations [Lancaster & Blundell, 2014].

These operators are written as $\hat{\Psi}^\dagger(x)$ and $\hat{\Psi}(x)$. They are defined by

$$\hat{\Psi}^\dagger(x) = \frac{1}{\sqrt{\nu}} \sum_p \hat{a}_p^\dagger e^{-ip.x} \tag{2.3.23}$$

$$\hat{\Psi}(x) = \frac{1}{\sqrt{\nu}} \sum_p \hat{a}_p e^{ip.x} \tag{2.3.24}$$

as the creator and annihilator operator respectively. Where, ν is the segment of volume of the space that contain the particle. The filed operators obey the same commutation relations as in eqn:(2.3.21), however instead of two momentum states p and q we have special states x and y and the type of particle is matter here, whether particles are bosons or fermions, this can be written as

$$\left[\hat{\Psi}(x), \hat{\Psi}^\dagger(y)\right] = \delta^3(x-y) \tag{2.3.25}$$

For both types of particles.

We can claim that particles are simply excitation of quantized field as outcomes for the field operators which we have just briefly stated.

As an interpretation for our discussion so far we can claim that the field is quantized and we can get particles in a certain quantum state by making a proper linear combination of operators (field operators) to create/annihilate particle into particular space states instead of the momentum states; in analog of the Hamiltonian in the model of particles in a box, however these time it much more complicated situation than the case of the particles, because we are dealing with fields.

2.4 Quantum Field Theories (QFT)

QFT allows us to describe the interaction between particles together or with other entities, to see that we need to have away to quantize the classical fields because classical fields don't work to describe the elementary particles as it will known in particle physics. These mechanisms which allow us to quantize the classical fields is either the canonical quantization (for non-interacting fields) or the non-canonical quntization (for the interacting fields). For simplicity we only will study the canonical quantization and will leave the other for future work to avoid the complexity, since we will understand the concept from studying the canonical quantization, which allow us to see how particles are added or removed from the system using our fields operators which we have presented previously.

To do the canonical quantization; first we need to write the classical Lagrangian density, because there are many ways to write the Lagrangian, then we need to calculate the momentum density to workout the Hamiltonian in term of the field as in the case of the classical method of writing a model. One, more thing is needed; that to treat the momentum density as quantum operators using the quantum mechanics, so they could have physical meaning and these operators mustn't commute. Finally the field can be expanded in term of creation/annihilation operators (expansion mode). This method of canonical quantization can be seen in many QFT textbooks such as [Lancaster & Blundell, 2014] and [Zee, 2010].

We can illustrate this method with any of our non-interacting previous fields, however only we will consider two fields; the massive scalar field and the massive complex scalar field. To begin with consider the first case, the Lagrangian for the massive field is eqn:(2.2.27) which can be expanded in the $(+ - --)$ metric as

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\left[\partial_\mu\phi(x)\right]^2 - \frac{1}{2}m^2\left[\phi^2(x)\right] \\ &= \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial t}\right)^2 - (\nabla\phi)^2 - m^2\phi^2\right]\end{aligned} \tag{2.4.1}$$

and its equation of motion is Klein-Gordon equation eqn:(2.2.35). The momentum density is

$$\Pi^\mu(x) = \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi(x)\right)} = \partial^\mu\phi(x) \tag{2.4.2}$$

Hence,

$$\Pi^0(x) = \Pi(x) = \partial^0\phi(x) \tag{2.4.3}$$

The Hamiltonian density can be defined in terms of momentum density as

$$\begin{aligned}\mathcal{H} &= \Pi^0(x)\partial_0\phi(x) - \mathcal{L} \\ &= \partial^0\phi(x)\partial_0\phi(x) - \mathcal{L} \\ &= \frac{1}{2}\left[\partial_0\phi(x)\right]^2 + \frac{1}{2}\left[\nabla\phi(x)\right]^2 + \frac{1}{2}m^2\left[\phi(x)\right]^2\end{aligned} \tag{2.4.4}$$

Now we need to turn our field into operators using the quantum mechanics, from which we know that

$$[\hat{x},\hat{p}] = i\hbar \tag{2.4.5}$$

for the single particle. In this regime

$$\phi(x) \rightarrow \hat{\phi}(x) \tag{2.4.6}$$

$$\Pi^0(x) \rightarrow \hat{\Pi}^0(x) \tag{2.4.7}$$

In analog of eqn:(2.3.21) these operators can be written in the form of eqn:(2.4.5) in the equal times, hence,

$$\left[\hat{\phi}(t,x),\hat{\Pi}^0(t,y)\right]=i\delta^{(3)}(x-y) \tag{2.4.8}$$

Where, $\hat{\phi}(x)$ and $\hat{\phi}(y)$ is commute as well as $\hat{\Pi}^0(x)$ and $\hat{\Pi}^0(y)$. From the quantum mechanics see [Griffiths, 2016] the position operator is written as

$$\hat{x}_j=\frac{1}{\sqrt{N}}\sum_k \hat{x}_k e^{ikja} \tag{2.4.9}$$

And the momentum operator is

$$\hat{x}_k=\sqrt{\frac{\hbar}{2m\omega_k}}\left(\hat{a}_k+\hat{a}_{-k}\right) \tag{2.4.10}$$

Where, $\hat{a}_k$ is the annihilate operator into the momentum state k. Substituting eqn:(2.4.10) into eqn:(2.4.9) yields:

$$\hat{x}_j = \frac{1}{\sqrt{N}}\sum_k\sqrt{\frac{\hbar}{2m\omega_k}}\left(\hat{a}_k+\hat{a}_{-k}\right)e^{ikja} \tag{2.4.11}$$

$$= \left(\frac{\hbar}{m}\right)^{\frac{1}{2}}\sum_k\frac{1}{(2\omega_k N)^{\frac{1}{2}}}\left(a_k e^{ikja}+a_k^{\dagger}e^{-ikja}\right) \tag{2.4.12}$$

where, $(\hat{a}_{-k}=\hat{a}_k^{\dagger})$. When $k\to p$ and $\omega\to E_p$ the time-independent field operator can be deduced to

$$\hat{\phi}(x)=\int\frac{d^3p}{(2\pi)^{\frac{3}{2}}}\frac{1}{(2E_p)^{\frac{1}{2}}}\left(\hat{a}_p e^{ip\cdot x}+\hat{a}_p^{\dagger}e^{-ip\cdot x}\right) \tag{2.4.13}$$

which is known as mode expansion where $\left(\frac{1}{2\pi}\right)$ term appears because of the inverse Fourier transformation.

The creation/annihilation operators obey

$$\left[\hat{a}_p,\hat{a}_q^{\dagger}\right]=\delta^{(3)}(p-q) \tag{2.4.14}$$

In the same way as we have got in eqn:(2.3.25).

Our second example of the quantization for the field we will consider studying the complex scalar field. We are going to apply the same method we did for our previous field.

Starting by the Lagrangian of the complex field

$$\mathcal{L}=\partial^{\mu}\psi^{\dagger}(x)\partial_{\mu}\psi(x)-m^2\psi^{\dagger}(x)\psi(x) \tag{2.4.15}$$

This Lagrangian is just addition of two scalar fields in eqn:(2.2.32).

$$\mathcal{L}=\frac{1}{2}\left(\partial_{\mu}\phi_1\right)^2-\frac{1}{2}m^2\phi_1^2+\frac{1}{2}\left(\partial_{\mu}\phi_2\right)^2-\frac{1}{2}m^2\phi_2^2 \tag{2.4.16}$$

Which can be easy transformed to the form as in eqn:(2.4.1). Its clear that it's similar to the Lagrangian of the massive scalar field but without the factor $\frac{1}{2}$. Then the momentum densities are

$$\Pi^0_\psi = \frac{\partial \mathcal{L}}{\partial(\partial_0\psi)} = \partial^0\psi^\dagger \tag{2.4.17}$$

$$\Pi^0_{\psi^\dagger} = \frac{\partial \mathcal{L}}{\partial(\partial_0\psi^\dagger)} = \partial^0\psi \tag{2.4.18}$$

The Hamiltonian can be written as

$$\begin{aligned}\mathcal{H} &= \sum_\sigma \Pi^0_\sigma(x)\partial_0\psi^\sigma(x) - \mathcal{L} \\ &= \partial_0\psi^\dagger(x)\partial_0\psi(x) + \nabla\psi^\dagger(x)\cdot\nabla\psi(x) + m^2\psi^\dagger(x)\psi(x)\end{aligned} \tag{2.4.19}$$

Now putting the fields into quantum operators yields:

$$\left[\hat{\psi}(t,x), \hat{\Pi}^0_\psi(t,y)\right] = \left[\hat{\psi}^\dagger(t,x), \hat{\Pi}^0_{\psi^\dagger}(t,y)\right] = i\delta^{(3)}(x-y) \tag{2.4.20}$$

And the mode expansion of this field operators are

$$\hat{\psi}(x) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_p)^{\frac{1}{2}}} \left(\hat{a}_p e^{-ip\cdot x} + \hat{b}^\dagger_p e^{ip\cdot x}\right) \tag{2.4.21}$$

$$\hat{\psi}^\dagger(x) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_p)^{\frac{1}{2}}} \left(\hat{a}^\dagger_p e^{ip\cdot x} + \hat{b}_p e^{-ip\cdot x}\right) \tag{2.4.22}$$

The operators $\hat{a}_p$ and $\hat{b}_p$ annihilate two different particles and $\hat{a}^\dagger_p$ and $\hat{b}^\dagger_p$ create particle and annhalite antiparticle respectively with the same momentum state as explained by Richard Feynman, see [Feynman et al., 2005]. So the complex scalar fields is quantaized and its excitation are scalar particles and antiparticles [Lancaster & Blundell, 2014].

These two quantizations for the fields (non-interacting fields) were the simplest cases to consider as illustrations for our method of quantization of the non-interacting classical fields, however there are many interacting fields that their quantization are not going to work with this method and we aren't interesting into their quantization because we have simply shown the method, but we are going to study some of the interacting fields for different purposes later in this dissertation.

The field in the relativistic quantum mechanical regime behaves different than in the non-relativistic regime as we will show next.
Consider the field $\phi(x,t)$, in the non-relativistic case, the kinetic energy equation is

$$E = \frac{P^2}{2m} \tag{2.4.23}$$

Transforming the variables E and P to operators yields,

$$\left(E \to \hat{E} = i\hbar\frac{\partial}{\partial t}, p \to \hat{p} = -i\hbar\nabla\right) \tag{2.4.24}$$

Hence,

$$i\hbar\frac{\partial\phi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\phi(x,t) \tag{2.4.25}$$

Which is just Schrodinger equation.

In case of the relativistic field we use the dispersion relation for the energy

$$E = \left(p^2c^2 + m^2c^4\right)^{\frac{1}{2}} \tag{2.4.26}$$

Together with the momentum transformation as we did before in eqn:(2.4.24) we get that

$$\begin{aligned} i\hbar\frac{\partial\phi(x,t)}{\partial t} &= \left(p^2c^2 + m^2c^4\right)^{\frac{1}{2}}\phi(x,t) \\ &= \left(-\hbar^2c^2\nabla^2 + m^2c^4\right)^{\frac{1}{2}}\phi(x,t) \end{aligned} \tag{2.4.27}$$

Unlike eqn:(2.4.25) this equation has two main problems; the problem that comes from the fact that its not covariant - we will know what is meant by that in more details later, as well as the negative root solution in the dispersion relation. A quick solution for these problems could be by just simply squared it to have

$$-\hbar^2\frac{\partial^2\phi(x,t)}{\partial t^2} = \left(-\hbar^2c^2\nabla^2 + m^2c^4\right)\phi(x,t) \tag{2.4.28}$$

In natural units this deduce to Klein-Gordon equation, see eqn:(2.2.35). But Klein-Gordon equation also has another physical problem! When we look at it

$$\left(\partial^2 + m^2\right)\phi(x,t) = 0 \tag{2.4.29}$$

The term ∂^2 isn't defined, as a solution we may think of expanding it, this will contain squared root second order time derivative, which is not an adequate description of nature! Because,

$$\left(\partial^2 + m^2\right) = \left(\sqrt{\partial^2} + im\right)\left(\sqrt{\partial^2} - im\right) \tag{2.4.30}$$

Dirac solved this problem by introducing new four vectors γ^μ whose obey following properties

$$\left(\gamma^0\right)^2 = 0 \tag{2.4.31}$$

$$\left(\gamma^1\right)^2 = \left(\gamma^2\right)^2 = \left(\gamma^3\right)^2 = -1 \tag{2.4.32}$$

These sets of vectors anti-commute and are written as

$$[\gamma^\mu, \gamma^\nu] = 2g^{\mu\nu} \tag{2.4.33}$$

Dirac invented new way to deal with the $\sqrt{\partial^2}$, he introduces new covariant operator $\not{\partial}$ defined by multiplying ∂_μ by γ^μ i.e,

$$\gamma^\mu\partial_\mu \equiv \not{\partial}$$

Squaring it and using equations; (2.4.31) and (2.4.32) yields,

$$\not{\partial}^2 = \gamma^\mu\partial_\mu = \partial^2 \tag{2.4.34}$$

Now eqn:(2.4.30) can be rewritten as

$$\left(\partial^2 + m^2\right) = \left(\not{\partial}^2 + m^2\right) = \left(\not{\partial} - im\right)\left(\not{\partial} + im\right) \tag{2.4.35}$$

Picking up the right bracket above and operating by it on a wave function we get that

$$\left(\not{\partial} + im\right)\psi = 0 \tag{2.4.36}$$

And using $\not{\partial} = \gamma^\mu \partial_\mu$, it gives,

$$\begin{aligned} \left(\gamma^\mu \partial_\mu + im\right)\psi &= 0 \\ or \quad \left(i\gamma^\mu \partial_\mu - m\right)\psi &= 0 \end{aligned} \tag{2.4.37}$$

Which is the famous Dirac equation that describes fermions (such as quarks and leptons). $\gamma^\mu = (\gamma^0, \gamma)$ are four-by-four matrices which can be written in many ways. Writing them is might be a bite tedious as large matrices, but these matrices were reduced to two-by-two matrices where each element is two-by-two matrix itself

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma^\mu} & 0 \end{pmatrix} \tag{2.4.38}$$

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{2.4.39}$$

$$\gamma = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix} \tag{2.4.40}$$

This imply the wave function in Dirac equation has to be two-by-two vectors each with two-by-two element and can be written in the compact form as

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{2.4.41}$$

We can quantize Dirac field using the same method we did for our non-interacting fields before, however we are not going to do that for the same reasons we have mentioned. The excitation for Dirac field is describe fermions and the Lagrangian for this field is

$$\mathcal{L}_{Fermions} = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi \tag{2.4.42}$$

This Lagrangian isn't invariant under some symmetry transformations, but what is the symmetry? and what is meant by the term invariant!? These questions well find their answers in our next section about these topics, which is named gauge symmetries.

2.5 Gauge Symmetries

Before knowing about the gauge symmetries we need to know briefly about symmetry in particle physics to study the gauge symmetries as a foundation for our aim of this dissertation.

In physics symmetry means invariance that is, no change under any type of transformation, such as the coordinates transformation. Symmetry is one of the most effective tools in studying theoretical particle physics, because its allow us to study unrevealed areas of physics and it has become evident that practically all laws of nature originate in symmetries. Because according to Wigner's classification the symmetries of the laws of physics determine the properties of the particles found in nature [Anderson, 1972].

Symmetry formally studied by group theory, because the set of all symmetry transformations represents groups called the symmetry groups. Continuous symmetry is represented by Noether's theorem, which we are stating in this section later.

Symmetry transformation means change in variables so the equation that describing the physical law have the same form before and after this transformation. This equation is said to be covariant with respect to this symmetry transformation. It is classified to geometric symmetries (space-time symmetry) to the internal symmetry. Internal symmetry is classified to global symmetry in which the space and time coordinates are considered to be independent, and to local symmetry where they considered as dependent variables.

Briefly we are going to study the gauge groups which their global symmetry is replaced by local one, because these gauge groups are extremely important in the studying the QFT of elementary particles. The invariance under gauge transformation requires the introduction of gauge vectors fields that interpreted as quanta mediating the interactions of the ultimate constituents of matter [Costa & Fogli, 2012].

In chapter one of this dissertation, we stated that there are four fundamental interactions in nature; weak, electromagnetic, strong and gravitational interaction. In our study we only consider three of them for the reason that when we compare the gravity force to the electromagnetic force; for example we know that

$$F_c = \frac{K_e q^2}{r^2} \tag{2.5.1}$$

$$F_g = \frac{G_N m_e^2}{r^2} \tag{2.5.2}$$

Where, F_c is Coulomb's force and F_g is the gravity force. Substituting the intended quantities above gives us

$$\frac{F_c}{F_g} \simeq 10^{40} \tag{2.5.3}$$

This implies that the electromagnetic interaction is stronger than the gravitational interaction by order of 10^{40}. Gravity compared to the weak force is weaker by factor of $\simeq 10^{24}$ which is known as the hierarchy problem, and of course its much much weaker than the strong interaction. For these reasons and others, that why gravity is negligible in particle physics, however there are some recent attempts to build theories that unified gravity with the other three interactions; such as Theory of everything (ToE) and Supper Symmetry theory (SUSY) which are out of the scope of this dissertation, because we only interested in studying the gauge couplings unification for the three considerable interactions in a space at very high energy scales. We can also recall the properties of these interactions by looking at Table 1.1.

In term of the internal symmetry and its corresponding groups, following we are going to discuss our three considerable interactions of the elementary particles.

To start with, in the QED interaction the gauge field is based on the abelian group $U(1)$. Its gauge vector field is the photon, this gauge field is very successful gauge theory because its theoretical expectations match the experiments perfectly. See [Griffiths, 2017].

For the weak and the strong interactions of the elementary particles, the group $U(1)$ is required to be replaced by larger non-abelian groups, since there are more than one mediators for these interactions and they can't be described using $U(1)$ group. The strong interaction (QCD) is based on the gauge of color $SU(3)$ that describes the interaction between quarks which are mediated by eight vector bosons known as gluons.

There is common thing between the QED and the QCD that is their gauge symmetries are believed to be exact and unbroken up to extreme high energy scales. We will know about these energy scales in our focal point of this thesis, when we will study the gauge coupling at these high energy scales in our third chapter of this dissertaion, but we need to know about the symmetry breaking mechanism first. Simply the spontaneous symmetry breaking mechanism (SSBM) is referring to the situation where the Lagrangian of the field is invariant under specific gauge transformations, however the solution of its equation of motion corresponding to that field will acquire a lower symmetry and some of the fields that including the gauge vector bosons - which are massless in the Lagrangian - will acquire mass [Costa & Fogli, 2012].

This mechanism was applied to attempt to unified the fundamental interactions in the famous theory of the elementary particles of physics that is the Standard Model (SM). According to this theory weak and electromagnetic interactions have been unified in a certain energy scale in gauge field theory that based on the group $SU(2)_L \times U(1)_Y$ [Glashow, 1961]. It has 4-vectors fields, three of them acquire mass through the SSBM we have mentioned above. These gauge vectors are bosons which mediate the weak interaction are $(W^{\pm}, Z)$ where the fourth one is the mediator of the QED (photon).

Some theorists are speculating that the strong and the electroweak interactions may be unified at very high energy scales. We can guess these theories must be based on larger gauge group.

SM and SSBM are our next topics to study, so we could able to go further to study a model in extra dimension to see how these gauges are couple in that dimension, this because in the Standard Model and as its well known between particle physicists, this coupling is not happening. This model is called the Randel Sunderm (RS) model and that extra dimensions space is known as AdS_5. Studding these topics are coming next, but before that we will study the gauge field. So we could had all the necessary tools required to construct our model which is targeting to study the coupling of these gauge bosons mediators for the three interactions at these high scales.

2.5.1 The Gauge field. Field theory may admit different configuration which lead to the same observations, so a selection is needed to be made for particular formulation out of the possible options - choice of the gaug, where the transformation from one configuration to another is known as the gauge transformation and the underline invariance is called the gauge invariance [Lancaster & Blundell, 2014].

To shed the light on this, consider the complex scalar field in eqn:(2.4.15) (without the interaction term) and as we had seen in eqn:(2.2.43) this Lagrangian isn't changed under the global transformation. However, if we transformed the field by different amount at all points in space-time as

$$\psi(x) \rightarrow \psi(x)e^{i\alpha(x)} \tag{2.5.4}$$

Applying this transformation for the Lagrangian

$$\mathcal{L} = (\partial^\mu \psi)^\dagger (\partial_\mu \psi) - m^2 \psi^\dagger \psi \tag{2.5.5}$$

It's clear that the mass term of the Lagrangian won't change under this transformation, and now we are going to experience the other term. So

$$\begin{aligned} \partial_\mu \psi(x) &\rightarrow \partial_\mu \psi(x) e^{i\alpha(x)} \\ &= e^{i\alpha(x)} \partial_\mu \psi(x) + \psi(x) \partial_\mu \left(e^{i\alpha(x)}\right) \\ &= e^{i\alpha(x)} \left[\partial_\mu \psi(x) + i\partial_\mu \alpha(x) \psi(x)\right] \\ &= e^{i\alpha(x)} \left[\partial_\mu + i\partial_\mu \alpha(x)\right] \psi(x) \end{aligned} \tag{2.5.6}$$

In the same way for $\partial^\mu \psi^\dagger(x)$ we get

$$\partial^\mu \psi^\dagger \rightarrow e^{-i\alpha(x)} \left[\partial^\mu - i\partial^\mu \alpha(x)\right] \psi^\dagger(x) \tag{2.5.7}$$

Inserting equations; (2.5.7) and (2.5.6) into the Lagrangian in eqn:(2.5.5) with using of (2.5.4) yields

$$\begin{aligned} \mathcal{L} &= (\partial^\mu \psi)^\dagger (\partial_\mu \psi) - m^2 \psi^\dagger \psi \\ &\neq \left[e^{-i\alpha(x)} \left[\partial^\mu - i\partial^\mu \alpha(x)\right] \psi^\dagger(x)\right] \left[e^{i\alpha(x)} \left[\partial_\mu + i\partial_\mu \alpha(x)\right] \psi(x)\right] \\ &- m^2 \psi(x) e^{-i\alpha(x)} \psi(x) e^{i\alpha(x)} \end{aligned} \tag{2.5.8}$$

So its not invariant under local $U(1)$ transformations. This problem is solved by introducing the field $A^\mu(x)$ via

$$D_\mu = \partial_\mu + iqA_\mu(x) \tag{2.5.9}$$

Which is known as the covariant derivative. The local $U(1)$ symmetry will be fixed if A_μ transformed as

$$A_\mu = A_\mu - \frac{1}{q} \partial_\mu \alpha(x) \tag{2.5.10}$$

Where, q is the coupling strength of that field.

Making these two sets of transformations in equations; (2.5.10) and (2.5.9) to our Lagrangian gives:

$$\mathcal{L} = (\partial^\mu \psi)^\dagger (\partial_\mu \psi) - m^2 \psi^\dagger \psi \tag{2.5.11}$$

In conclusion, making the field transformation and replacing the derivative by the covariant derivative makes the Lagrangian invariant. The field $A^\mu(x)$ (gauge field) which leaves the field theory invariance is known as the gauge theory.

2.6 The Standard Model

The SM is the theory that describes the interactions among elementary particles [Iliopoulos, 1980, Cheng & Li, 1984]. It combines the QCD, based on the group $SU(3)_C$ [Gross & Wilczek, 1973], and the Glashow-Weinberg-Salam theory of the electroweak interaction (that unified the weak and electromagnetic interactions), based on the group $SU(2)_L \times U(1)_Y$. Therefore the SM is an $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge theory. We have the following fermionic assignment for the particle content

$$q_L = \begin{pmatrix} u \\ d \end{pmatrix}_L ; \;\; u_R, d_R, \tag{2.6.1}$$

$$\ell_L = \begin{pmatrix} \nu \\ e \end{pmatrix}_L ; \quad e_R, \tag{2.6.2}$$

where we have omitted the color index for quarks and we present only one generation for simplicity.

The above assignment is the result of maximal parity violation by the weak interaction, since the left handed and right handed fermions transform differently. For instance, the doublet shown in eqn:(2.6.2) is assumed to transform in the fundamental representation of an $SU(2)_L$ group, whereas the right handed partners are taken to be a singlet under this group, and the neutrinos are assumed to be left handed only[3]. The neutrino will not acquire mass as its right handed partner does not exist in this theory [Branco & Senjanovic, 1978], [Senjanovic, 1979], [Mohapatra & Senjanovic, 1981]. As we did previously, in order to construct a model or theory we need to write a Lagrangian, so as to study the standard model of particle physics we need to write the Lagrangian for that theory, which we are going to do next.

The SM Lagrangian can be written as:

$$\mathcal{L}_{SM} = \mathcal{L}_{Gauge} + \mathcal{L}_{Fermions} + \mathcal{L}_{Higgs} + \mathcal{L}_{Yukawa} + \mathcal{L}_{Gauge.fixing} + \mathcal{L}_{Ghost} \,. \tag{2.6.3}$$

We shall now briefly introduce each sector of this Lagrangian that corresponds to the intended field.

2.6.1 Gauge Sector. The gauge sector is composed of 12 gauge fields which mediate the interactions among the fermion fields; the photon (γ, mediates the electromagnetic interactions), the three weak gauge bosons ($W^\pm$ and Z, mediate the weak interactions) and eight gluons (g_α, $\alpha = 1, 2, ..., 8$, mediate the strong interactions). The gauge field dynamics are embedded in the Lagrangian in terms of field strength tensors as

$$\mathcal{L}_{Gauge.Boson} = -\frac{1}{4} G^A_{\mu\nu} G^{A\mu\nu} - \frac{1}{4} W^a_{\mu\nu} W^{a\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \,, \tag{2.6.4}$$

where repeated indices imply a summation over that index, and μ, ν takes 0,1,2,3, where the field strength tensors for non-Abelian theories are given by:

$$G^A_{\mu\nu} = \partial_\mu G^A_\nu - \partial_\nu G^A_\mu - i g_s f^{ABC} G^B_\mu G^C_\nu \,, \tag{2.6.5}$$

[3] Because of maximal parity violation of the electroweak interactions, the neutrinos are massless in the SM. This can be generalised to study mixings in the lepton sector with V_{PMNS}, as we will see in next subsection 2.6.5

being the $SU(3)_C$ field strength, g_s is the coupling strength of the strong interaction, A, B, C run from 1 to 8 and f^{ABC} are the (antisymmetric) structure constants of $SU(3)$, which satisfies the Lie algebra for the group generator t^A

$$[t^A, t^B] = if^{ABC}t^C \ . \tag{2.6.6}$$

$$W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - ig\epsilon^{abc}\ W^b_\mu W^c_\nu \tag{2.6.7}$$

Is the $SU(2)_L$ field strength, a, b, c run from 1 to 3 and ϵ^{abc} is the totally antisymmetric three-index tensor with $\epsilon^{123} = 1$, g is the coupling strength of the weak interaction.

The field strength of the $U(1)_Y$ gauges boson which has the same form as electromagnetism is given by:

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \ . \tag{2.6.8}$$

2.6.2 Fermion Sector. The SM contains three copies of chiral fermions (generations) with different gauge transformations. The fermionic Lagrangian has the usual covariant Dirac form

$$\mathcal{L}_{Fermions} = \sum_f i\bar{f}\gamma_\mu D^\mu f \ , \tag{2.6.9}$$

with the covariant derivatives can be read as

$$D_\mu \begin{pmatrix} u \\ d \end{pmatrix}_L = \left(\partial_\mu - ig_s\frac{\lambda^a}{2}G^a_\mu - ig\frac{\sigma^a}{2}W^a_\mu - ig'\frac{1}{6}B_\mu\right)\begin{pmatrix} u \\ d \end{pmatrix}_L \ , \tag{2.6.10}$$

$$D_\mu u_R = \left(\partial_\mu - ig_s\frac{\lambda^a}{2}G^a_\mu - ig'\frac{2}{3}B_\mu\right)u_R \ , \tag{2.6.11}$$

$$D_\mu d_R = \left(\partial_\mu - ig_s\frac{\lambda^a}{2}G^a_\mu + ig'\frac{1}{3}B_\mu\right)d_R \ , \tag{2.6.12}$$

$$D_\mu \begin{pmatrix} \nu \\ e \end{pmatrix}_L = \left(\partial_\mu - ig\frac{\sigma^a}{2}W^a_\mu + ig'\frac{1}{2}B_\mu\right)\begin{pmatrix} \nu \\ e \end{pmatrix}_L \ , \tag{2.6.13}$$

and

$$D_\mu e_R = \left(\partial_\mu + ig'B_\mu\right)e_R \ . \tag{2.6.14}$$

Here γ_μ are the usual Dirac matrices, see eqn:(2.4.38), g' is the coupling strength of the hypercharge interaction, Y is the hypercharge, σ^a are the generators of $SU(2)_L$ (simply the Pauli matrices), and λ^a are the generators of $SU(3)_C$ (the Gell-Mann matrices).

Note that gauge symmetry forbids a mass term for fermions (quarks and leptons) and gauge bosons. A mass term would break the gauge invariance $SU(2)_L \times U(1)_Y$. However, we observe the mass of gauge bosons $W^\pm$ and Z and the fermions experimentally [Arnison et al., 1983], so a mass is needed to be given for these particles. The masses in the SM are generated through a mechanism known as the the Higgs mechanism, which will be discussed in depth in the next section.

2.6.3 The Higgs Mechanism. As was presented in the previous section, a Dirac mass term will violate the gauge symmetry. As such we need a mechanism that gives mass to the SM particles and keeps the Lagrangian invariant under gauge symmetries. This can be done through the mechanism of spontaneous gauge symmetry breaking also known as the Higgs mechanism. This mechanism adds a new complex scalar field Φ which is a doublet under the $SU(2)_L$ group, a singlet with respect to $SU(3)_C$ and has hypercharge $Y_\Phi = 1$ [Higgs, 1964a, Higgs, 1964b, Englert & Brout, 1964, Nambu, 1960].

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix} , \tag{2.6.15}$$

where ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are real scalars. This new scalar Φ adds extra terms to the SM Lagrangian:

$$\mathcal{L}_{Higgs} = (D_\mu \Phi)^\dagger (D^\mu \Phi) - V(\Phi) , \tag{2.6.16}$$

where the covariant derivative D_μ is defined as

$$D_\mu = \partial_\mu - i\frac{g'}{2} B_\mu - ig\frac{\sigma^a}{2} W_\mu^a . \tag{2.6.17}$$

The general gauge invariant renormalizable potential involving Φ is given by

$$V(\Phi) = -\frac{1}{2}\mu^2 \Phi^\dagger \Phi + \frac{\lambda}{4}(\Phi^\dagger \Phi)^2 . \tag{2.6.18}$$

eqn:(2.6.18) describes the Higgs potential, which involves two new real parameters μ and λ. We demand $\lambda > 0$ for the potential to be bounded; otherwise the potential is unbounded from below and there will be no stable vacuum state. μ takes the following two values:

- $\mu^2 > 0$ then the vacuum corresponds to $\Phi = 0$, the potential has a minimum at the origin (see Fig:[2.1] right panel).
- $\mu^2 < 0$ then the potential develops a non-zero Vacuum Expectation Value (VEV) and the minimum is along a circle of radius $\frac{v}{\sqrt{2}} = \frac{246}{\sqrt{2}}$ (see Fig:[2.1] left panel). Minimizing the potential we get

$$\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 = -\frac{\mu^2}{\lambda} = v^2 . \tag{2.6.19}$$

As such, we need to choose one of these minima as the ground state ($\phi_3 = v$ and $\phi_1 = 0$, $\phi_2 = 0$ and $\phi_4 = 0$). Thus the vacuum does not have the original symmetry of the Lagrangian, and therefore spontaneously breaks the symmetry [Nambu, 1960]. In other words, the Lagrangian is still invariant under the $SU(2)_L \times U(1)_Y$, while the ground state is not. We choose the VEV in the neutral direction as the photon is neutral, so Φ becomes

$$\langle \Phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} . \tag{2.6.20}$$

With this particular choice of the ground state, the electroweak gauge group $SU(2)_L \times U(1)_Y$ is broken to electromagnetism, $U(1)_{em}$,

$$SU(2)_L \times U(1)_Y \overset{\langle \Phi \rangle}{\to} U(1)_{em} . \tag{2.6.21}$$

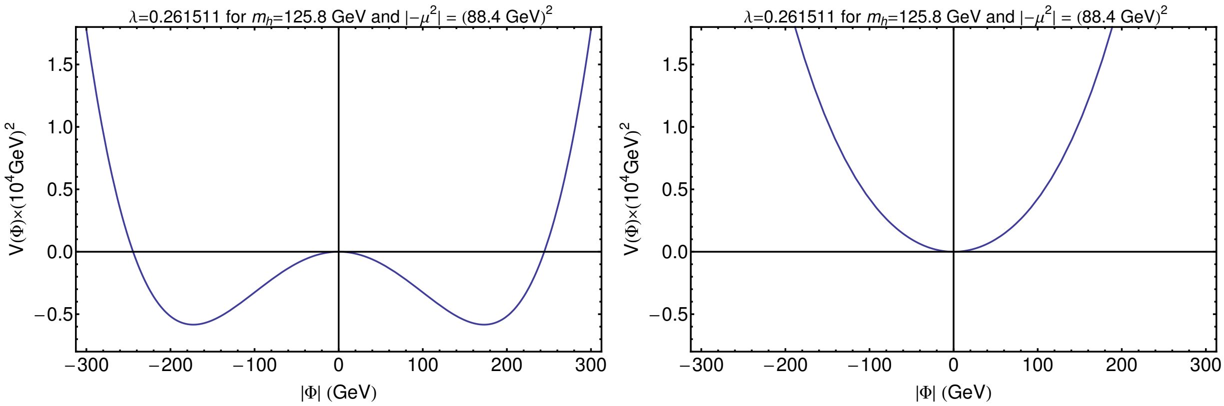


Figure 2.1: *The Higgs potential $V(\Phi)$ with: in the left panel, the case $\mu^2 < 0$; and the right panel for the case $\mu^2 > 0$ as a function of $|\Phi| = \sqrt{\Phi^\dagger \Phi}$.*

2.6.4 Gauge Boson Masses. The gauge boson masses can be obtained from the kinetic term of the Higgs field [Cheng & Li, 1984]. Expanding the Lagrangian about the VEV yields:

$$\mathcal{L}_{Higgs} = \frac{1}{2}\begin{pmatrix}0 & \upsilon\end{pmatrix}(g\frac{\sigma^a}{2}W^a_\mu + \frac{1}{2}g'B_\mu)(g\frac{\sigma^b}{2}W^b_\mu + \frac{1}{2}g'B_\mu)\begin{pmatrix}0\\ \upsilon\end{pmatrix} \ . \tag{2.6.22}$$

From the definition of $W^\pm_\mu = \frac{1}{\sqrt{2}}(W^1_\mu \pm W^2_\mu)$, $Z_\mu = W^3_\mu \cos\theta_W - B_\mu \sin\theta_W$ and $A_\mu = W^3_\mu \sin\theta_W - B_\mu \cos\theta_W$, we get three massive gauge bosons

$$m^2_W = \frac{1}{4}g^2\upsilon^2 \ , \quad m^2_Z = \frac{1}{4}(g'^2 + g^2)\upsilon^2 \ , \tag{2.6.23}$$

and one massless gauge boson (identified as the photon)

$$m^2_A = 0 \ . \tag{2.6.24}$$

The Weinberg angle θ_W or weak mixing angle is defined by

$$\cos\theta_W = \frac{g}{\sqrt{g'^2 + g^2}} \ , \tag{2.6.25}$$

$$\sin\theta_W = \frac{g'}{\sqrt{g^2 + g'^2}} \ . \tag{2.6.26}$$

The Weinberg angle is determined empirically. Where its value varies as a function of the momentum transfer, at which it's measured. This variation, or 'running', is a key prediction of the electroweak theory in the SM and we will compare our model against it in the chapter of the results and discussions later. In QFT-subtraction scheme the Weinberg angle is equals to 0.23120 ± 0.00015.

2.6.5 Fermion Masses. Fermion masses originate from Yukawa interactions, which are the couplings between the fermion doublets and the scalar field Φ [Cheng & Li, 1984]. These Yukawa couplings are uniquely fixed by gauge invariance and the Lagrangian, as given by:

$$\mathcal{L}_{Yukawa} = Y^d_{ij}\bar{q^i_L}\Phi d^j_R + Y^u_{ij}\bar{q^i_L}\tilde{\Phi}u^j_R + Y^e_{ij}\bar{l^i_L}\Phi e^j_R + \text{h.c.} \;, \tag{2.6.27}$$

where the Y's are 3×3 complex matrices, the so called Yukawa coupling constants, h.c. indicates the Hermitian conjugate and $\tilde{\phi}$ is defined by

$$\tilde{\Phi} = \begin{pmatrix} -\phi_2^* \\ \phi_1^* \end{pmatrix} \;. \tag{2.6.28}$$

When the Higgs doublet acquires a non vanishing VEV, eqn:(2.6.27) leads to the mass terms for the fermions as follows:

$$\mathcal{L}_{Yukawa} = m_u\,\bar{u}_L\,u_R + m_d\,\bar{d}_L\,d_R + m_e\,\bar{e}_L e_R \;, \tag{2.6.29}$$

with $m_u = \frac{1}{\sqrt{2}}y_u\upsilon;\; m_d = \frac{1}{\sqrt{2}}y_d\upsilon;\;\; m_e = \frac{1}{\sqrt{2}}y_e\upsilon$.

Note that neutrinos are massless and will never acquire mass, because its chiral partner ν_R does not exist in the theory.

When we consider all the generations of quarks, there are possibilities for their mixing. This mixing is described by the CKM, which has four observable parameters, including three mixing angles and one phase [Cabibbo, 1963]. It appears upon the diagonalisation of Yukawa matrices by using two unitary matrices U and V, where

$$UY_u^\dagger Y_u U^\dagger = \text{diag}(f_u^2, f_c^2, f_t^2); \qquad VY_d^\dagger Y_d V^\dagger = \text{diag}(h_d^2, h_s^2, h_b^2) \;. \tag{2.6.30}$$

The CKM matrix is given by

$$V_{CKM} = UV^\dagger \;. \tag{2.6.31}$$

The form of the CKM matrix that describes the quark sector mixing is parametrised as

$$V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \;,$$

and the standard parametrisation in terms of the three mixing angles and one phase can have the form

$$V_{CKM} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \;, \tag{2.6.32}$$

where $s_{12} = \sin\theta_{12}$, $c_{12} = \cos\theta_{12}$ etc. are the sines and cosines of the three mixing angles θ_{12}, θ_{23} and θ_{13}, and δ is the CP violating phase.

2.6.6 The Higgs Boson. As was discussed in section 2.6.3, the spontaneous symmetry breaking predicted a new particle: the Higgs Boson, which must be a scalar and neutral. The Lagrangian for this new scalar comes from the kinetic term of eqn:(2.6.16) expanded around the VEV

$$\mathcal{L}_{Higgs.Boson} = \frac{1}{2}(\partial_\mu h)(\partial^\mu h) - \frac{1}{2}m_h^2 h^2 + \text{interactions} , \tag{2.6.33}$$

where $m_h^2 = \sqrt{\lambda}v$ is the Higgs boson mass. The interaction terms contain both Higgs self-interactions and interactions with gauge bosons and fermions. Note that as a consequence of the Higgs mechanism all the Higgs couplings are completely determined in terms of the coupling constants and masses. The Higgs boson, which was the last missing piece of the SM, has been confirmed by the ATLAS and CMS experiments, and this is compatible with the SM Higgs expectations with a mass of about 126 GeV [Aad et al., 2012, Chatrchyan et al., 2012].

2.6.7 Gauge Fixing and Ghosts. Gauge fixing is necessary when the gauge fields are quantised. Quantisation means to develop a path integral formalism for the gauge theory. The path integral is diverging as one integrate over an infinite set of gauge-equivalent configuration, here the gauge fixing is used to pick up one arbitrary representative, therefore, giving meaning to the path integral. On other hands, the gauge invariance we look for in gauge theory, a naive path integral approach would spoiled it[Peskin & Schroeder, 2005]. The solution is given by the what is called the Faddev-Popov procedure, where they introduced an identity expression consisting of a functional integral over a gauge fixing condition times a functional determinant over anticommuting fields in the path integral. The latter gives rise to what is known as ghost fields, which keep the gauge freedom within the theory, but are not physical particles (because ghost violate the spin-statistics relation).

As such, we need to add terms in the Lagrangian like

$$\mathcal{L}_{Gauge.fixing} = -\frac{\zeta}{2}(\partial_\mu A^\mu)^2 , \tag{2.6.34}$$

and

$$\mathcal{L}_{Ghost} = \bar{c}_b \partial^\mu D_\mu^{ab} c_a . \tag{2.6.35}$$

Thus, we are now in the right position to write the full SM Lagrangian

$$\mathcal{L}_{SM} = \mathcal{L}_{Gauge} + \mathcal{L}_{Fermions} + \mathcal{L}_{Yukawa} + \mathcal{L}_{Higgs} + \mathcal{L}_{Gauge.fixing} + \mathcal{L}_{Ghost}. \tag{2.6.36}$$

This is the Lagrangian of the SM which used to build the theory and see how the gauges bosons aren't couple according to it. As a solution to that we are going to construct a model and compare these two results against each other. The calculations and results will be studied in the fourth chapter of this dissertation, however next we will be studying the Randall-Sundrum model (5-dimensional warped geometry theory) or more concretely the 5-dimensional anti-de Sitter space. In this model, the elementary particles - without graviton which is a hypothetical particle mediate the gravity interaction - are localized on a $(3+1)$-dimensional brane.

3. Randall Sundrum (RS) Model in Five Dimension

In this chapter, the theoretical tools required to construct models in warped extra dimensions are presented [Randall & Sundrum, 1999, Randall & Schwartz, 2002].

3.1 The Bulk Field Lagrangian

We start by Considering a 5D space time with the AdS_5 metric

$$ds^2 = e^{-2ky}dx^2 - dy^2 = g_{MN}dx^M dx^N, M, N = 0, .., 3, 5 \tag{3.1.1}$$

where k is curvature parameter. The above metric can be written as.

$$ds^2 = \left(\frac{1}{kz}\right)^2 \left(dx^2 - dz^2\right), \quad g_{MN} = \left(\frac{1}{kz}\right)^2 diag(1, -1, -1, -1, -1) \tag{3.1.2}$$

Consider fermion ψ, scalar ϕ and vector A_M bulk fields. We shall take the gauge group to be SU(N), with generator T^a satisfying

$$[T^a, T^b] = if^{abc}T^c \tag{3.1.3}$$

and

$$F^a_{MN} = \partial_M A^a_N - \partial_N A^a_M + gf^{abc}A^b_M A^c_N \tag{3.1.4}$$

In 5D the action is given by

$$S_5 = -\int d^5x\sqrt{g}\left[\frac{1}{4}F^2_{MN} + |D_M\phi|^2 + i\bar{\psi}\Gamma^M\nabla_M\psi + m^2_\phi|\phi|^2 + im_\psi\bar{\psi}\psi\right] \tag{3.1.5}$$

In curved space the gamma matrices are $\Gamma_M = e^A_M\gamma_A$, where e^A_M is funfbein defined by $g_{MN} = e^A_M e^B_N \eta_{AB}$ and $\gamma_A = (\gamma_\alpha, \gamma_5)$ are the usual gamma matrices in flat space. The curved space covariant derivative $\nabla_M = D_M + \omega_M$, where ω_M is the spin connection and D_M is the gauge covariant derivative for fermion and / or scalar fields charged under some gauge symmetry.

3.2 Scalar in the Bulk

Now Consider a free action of complex scalar

$$S_\Phi = \frac{1}{2}\int d^5x\sqrt{-g}\left[g^{MN}\partial_M\Phi^*\partial_N\Phi - m^2\Phi^*\Phi\right] \tag{3.2.1}$$

$$= \frac{1}{2}\int d^4x\int dz\left(\frac{1}{kz}\right)^5\left[(kz)^2\left(\eta^{\mu\nu}\partial_\mu\Phi^*\partial_\nu\Phi - \partial_5\Phi^*\partial_5\Phi\right) - m^2\Phi^*\Phi\right] \tag{3.2.2}$$

Integrate by parts

$$\begin{aligned} S_\Phi &= \frac{1}{2}\int d^4x \int dz \left(\frac{1}{kz}\right)^3 [-\eta^{\mu\nu}\Phi^*\partial_\mu\partial_\nu\Phi] + \Phi^*\partial_5\left(\frac{1}{kz}\right)^3\partial_5\Phi - m^2\left(\frac{1}{kz}\right)^5\Phi^*\Phi \\ &= \frac{1}{2}\int d^4x \int dz \left(\frac{1}{kz}\right)^3 \left[-\eta^{\mu\nu}\Phi^*\partial_\mu\partial_\nu\Phi - \Phi^*\left[\partial_z^2 - \frac{3}{z}\partial_z + \frac{a}{z^2}\right]\Phi\right] \\ &+ \frac{1}{2}\int d^4x \left(\frac{1}{kz}\right)^3 [\eta^{\mu\nu}\partial_\nu\Phi\Phi^* + \partial_5\Phi\Phi^*]_{\frac{1}{k}}^{\acute{R}} \end{aligned} \tag{3.2.3}$$

where $a = (\frac{m}{k})^2$ and the last term is the boundary terms, the first term vanish because ϕ is assumed to vanish at the 4D boundary $x^\mu = \pm\infty$ now decompose our field as

$$\Phi(x^\mu, z) = \sum_{n=0}^{\infty}\phi^{(n)}(x^\mu)f_a^{(n)}(z) \tag{3.2.4}$$

So:

$$\begin{aligned} S_\Phi &= \frac{1}{2}\sum_{n,m=0}^{\infty}\int d^4x \int dz\left(\frac{1}{kz}\right)^3\left[-\eta^{\mu\nu}\phi^{*(n)}(x^\mu)f_a^{(n)}(z)\partial_\mu\partial_\nu\phi^{(m)}(x^\mu)f_a^{(m)}(z)\right] \\ &+ \phi^{*(n)}(x^\mu)f_a^{(n)}(z)\left[\partial_z^2 - \frac{3}{z}\partial_z - \frac{a}{z^2}\right]\phi^{(m)}(x^\mu)f_a^{(m)}(z) \end{aligned} \tag{3.2.5}$$

and constraining those K.K wave functions to behave like

$$\int_{\frac{1}{k}}^{\acute{R}} dz\left(\frac{1}{kz}\right)^3 f_a^{(n)}(z)f_a^{(m)}(z) = \delta_{mn} \tag{3.2.6}$$

and

$$\left[-\partial_z^2 + \frac{3}{z}\partial_z + \frac{a}{z^2}\right]f_a^{(n)}(z) = m_n^2 f_a^{(n)}(z) \tag{3.2.7}$$

Thus, we obtain

$$f_a^{(n)}(z) = z^2\left[c_1J_1(v, zm_n) + c_2Y_1(v, zm_n)\right] \tag{3.2.8}$$

where $v = \sqrt{4+a} = \sqrt{4+\frac{m^2}{k^2}}$. Plug all the ingredients into the action, we obtain the Four-dimensional K.K action

$$S_\Phi = \frac{1}{2}\sum_{n=0}^{\infty}\int d^4x\left[\eta^{\mu\nu}\partial_\mu\phi^{*(n)}(x^\mu)\partial_\nu\phi^{(n)}(x^\mu) - m_n^2\phi^{*(n)}(x^\mu)\phi^{(n)}(x^\mu)\right] \tag{3.2.9}$$

we therefore, define the canonically normalized density wave function as $\tilde{f}_a^{(n)}(z) = \frac{1}{kz}f_a^{(n)}(z)$ so

$$\int_{\frac{1}{k}}^{\acute{R}} dz\frac{1}{kz}\tilde{f}_a^{(n)}(z)\tilde{f}_a^{(m)}(z) = \delta_{mn} \tag{3.2.10}$$

The general solution of the Kaluza-Klein modes corresponding to $(m_0 \neq 0)$ is given by

$$\tilde{f}_a^{(n)}(z) = kz\frac{1}{N_v}\left[J_v(zm_n) + b_v Y_v(zm_n)\right] \quad (3.2.11)$$

N_v and b_v can be obtained from the boundary conditions and normalization condition of our field.

$$b_v = -\frac{J_v\left[\frac{m_n}{k}\right] + J_v'(m_n z)|_{\frac{1}{k}}}{Y_v\left[\frac{m_n}{k}\right] + Y_v'(m_n z)|_{\frac{1}{k}}} \quad (3.2.12)$$

$$N_v^2 = \int dzkz[J_v(m_n z) + b_v Y_v(m_n z)]^2 \quad (3.2.13)$$

where prime means here derivative with respect to z coordinate.

The Kaluza-Klein masses are determined by imposing the boundary conditions and in the limit $\pi kR \gg 1$ lead to the approximate values

$$m_n \approx \left(n + \frac{1}{2}v - \frac{3}{4}\right)\pi k e^{-\pi kR} \quad (3.2.14)$$

3.3 Gauge Fields in the Bulk

$$S_A = \int d^5x\sqrt{-g}g^{MN}g^{AB}F_{AM}F_{BN} + gaugefixing \quad (3.3.1)$$

Integrate by parts.

$$\begin{aligned} S_A &= -\frac{1}{2}\int d^4x\int dz\left(\frac{1}{kz}\right)A^{a\mu}\left[-\partial^2\eta_{\mu\nu} + (1-\lambda)\partial_\nu\partial_\mu - \left(\partial_z^2 - \frac{1}{z}\partial_z\right)\right]A^{a\nu} \\ &+ A_5^a\left[-\partial^\mu\partial_\mu + \frac{1}{\lambda}\left(\partial_z^2 - \frac{1}{z}\partial_z + \frac{1}{z^2}A_\mu\right)\right]A^{a5} \\ &+ gf_{abc}A_\mu^b\left[A_\nu^c(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) + A_5^c(\partial^\mu A^{a5} - \partial^5 A^{a\mu})\right] \\ &+ \frac{1}{2}g^2 f_{abc}f_{ade}\left[A_\mu^b A_\nu^c A^{e\nu}A^{d\mu} + A_\mu^a A_5^c A^{e5}A^{d\mu}\right] \end{aligned} \quad (3.3.2)$$

To fix the gauge, we could add gauge fixing term in a RS bulk metric such way that cancel the mixing term

$$S_{gf} = -\frac{\lambda}{2}\int d^4x\int_{\frac{1}{k}}^{\acute{R}} dz\frac{1}{kz}\left[\partial^\mu A_\mu - \frac{1}{\lambda}(\partial_5 A_5 - \frac{1}{z}A_5)\right]^2 \quad (3.3.3)$$

So:

$$\begin{aligned} S_A &= \frac{1}{2}\int d^4x\int_{\frac{1}{k}}^{\acute{R}} dz\frac{1}{kz}A_\mu\left[\partial^2\eta^{\mu\nu} + (\lambda-1)\partial^\mu\partial^\nu - \left(\partial_z^2 - \frac{1}{z}\partial_z\right)\right] \\ &+ A_5\left[-\partial^\mu\partial_\mu + \frac{1}{\lambda}\left(\partial_z^2 - \frac{1}{z}\partial_z + \frac{1}{z^2}\right)A_5\right] - \int d^4x\frac{1}{kz}A_5\partial^\mu A_\mu|_{\frac{1}{k}}^{\acute{R}} \end{aligned} \quad (3.3.4)$$

where the last term is the boundary term. Decompose our fields as

$$A_\mu(x^\mu, z) = \sum_{n=0}^{\infty} A_5^n(x^\mu) f_{A_\mu}^n(z) \tag{3.3.5}$$

$$A_5(x^\mu, z) = \sum_{n=1}^{\infty} A_5^n(x^\mu) f_{A_5}^n(z) \tag{3.3.6}$$

being constrained by

$$\int_{\frac{1}{k}}^{\acute{R}} dz \frac{1}{kz} f_{A_\mu}^{(n)}(z) f_{A_\mu}^{(m)}(z) = \delta_{nm} \tag{3.3.7}$$

$$\int_{\frac{1}{k}}^{\acute{R}} dz \frac{1}{kz} f_{A_5}^{(n)}(z) f_{A_5}^{(m)}(z) = \delta_{nm} \tag{3.3.8}$$

$$\left[\partial_z^2 - \frac{1}{z}\partial_z\right] f_{A_\mu}^{(n)}(z) = -m_n^2 f_{A_\mu}^{(n)}(z) \tag{3.3.9}$$

$$\left[\partial_z^2 - \frac{1}{z}\partial_z + \frac{1}{z^2}\right] f_{A_5}^{(n)}(z) = -m_n^2 f_{A_5}^{(n)}(z) \tag{3.3.10}$$

K.K mode solution:

$$\left[\partial_z^2 - \frac{1}{z}\partial_z + m_m^2\right] f_{A_\mu}^{(m)}(z) = 0 \tag{3.3.11}$$

$$\left[\partial_z^2 - \frac{1}{z}\partial_z + \frac{m_m^2 z^2 + 1}{z^2}\right] f_{A_5}^{(m)}(z) = 0 \tag{3.3.12}$$

$$f_{A_\mu}^{(n)}(z) = kz \frac{1}{N_{A_\mu}} \left[J_1(zm_n) + b_{A_\mu} Y_1(zm_n)\right] \tag{3.3.13}$$

$$f_{A_5}^{(n)}(z) = kz \frac{1}{N_{A_5}} \left[J_0(zm_n) + b_{A_5} Y_0(zm_n)\right] \tag{3.3.14}$$

Note that those equations of motion can be seen as two linear coupled equations

$$\partial_z f_{A_\mu}^{(n)}(z) = m_n f_{A_5}^{(n)}(z) \tag{3.3.15}$$

$$\left[\partial_z - \frac{1}{z}\right] f_{A_5}^{(n)}(z) = -m_n f_{A_\mu}^{(n)}(z) \tag{3.3.16}$$

a massless zero-mode ($m_0 = 0$) wave function can be easily obtained from the first order equation of motion.

$$\partial_z f_{A_\mu}^{(n)}(z) = 0 \tag{3.3.17}$$

this implies that $f^{(0)}_{A_\mu}(z)$ is constant

$$f^{(0)}_{A_\mu}(z) = \frac{1}{\sqrt{\frac{1}{k}\log\left(k\acute{R}\right)}} = \frac{1}{\sqrt{\pi R}} \tag{3.3.18}$$

and

$$\left[\partial_z - \frac{1}{z}\right] f^{(0)}_{A_5}(z) = 0 \tag{3.3.19}$$

$$f^{(0)}_{A_5}(z) = \sqrt{\frac{2k}{((k\acute{R})^2 - 1)}}kz \tag{3.3.20}$$

$$b_{A_{\mu,5}} = -\frac{J_{1,0}(\frac{m_n}{k}) + J'_{1,0}(m_n z)|_{\frac{1}{k}}}{Y_{1,0}(\frac{m_n}{k}) + Y'_{1,0}(m_n z)|_{\frac{1}{k}}} \tag{3.3.21}$$

$$N^2_{A_{\mu,5}} = \int dzkz[J_{1,0}(m_n z) + b_{A_{\mu,5}} Y_{1,0}(m_n z)]^2 \tag{3.3.22}$$

The Kaluza-Klein masses are determined by imposing the boundary conditions and in the limit $\pi kR \gg 1$ lead to the approximate values

$$m_n \approx \left(n - \frac{1}{4}\right)\pi k e^{-\pi kR} \tag{3.3.23}$$

3.4 Fermions in the Bulk

In five dimensions a fundamental spinors representation has four components, so fermions are described by dirac spinors. Under Z_2 symmetry $z \to -z$ a fermion transforms (up to a phase $\pm$) as

$$\psi(-z) = \pm\gamma_5\psi(z) \tag{3.4.1}$$

consider free action

$$S_\psi = \int d^5x\sqrt{-g}\left[\frac{i}{2}(\overline{\psi}\Gamma^M D_M\psi - D_M\overline{\psi}\Gamma^M\psi) - m\overline{\psi}\psi\right] \tag{3.4.2}$$

$$S_\psi = \int d^4x\int_{\frac{1}{k}}^{\acute{R}} dz\left(\frac{1}{kz}\right)^4\left[\overline{\psi}\left(i\gamma^\mu\partial_\mu - \left(\gamma_5\partial_z + \frac{c-2\gamma_5}{z}\right)\right)\psi\right] \tag{3.4.3}$$

where $m = ck$ and in curved space the covariant derivative is $D_M = \partial_M + \omega_M$, where ω_M is the spin connection. In particular, for our metric, we have

$$\omega_M = \left(\frac{1}{2}\gamma_\mu\gamma_5\frac{1}{z}, 0\right) \tag{3.4.4}$$

it is very easy to show that the spin connection terms cancel each other. Recall that

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{3.4.5}$$

and

$$\overline{\psi} = (\overline{\psi_R} \quad , \overline{\psi_L}) \tag{3.4.6}$$

where $\psi_{L,R}$ are the components of Dirac spinor ψ.

decompose our fields as

$$\psi_L(x^\mu, z) = \sum_n \psi_L^{(n)}(x^\mu) f_{L,c}^{(n)}(z) \tag{3.4.7}$$

$$\psi_R(x^\mu, z) = \sum_n \psi_R^{(n)}(x^\mu) f_{R,c}^{(n)}(z) \tag{3.4.8}$$

where $\psi_{L,R}^{(n)}$ are the Kaluza Klien modes satisfying $\gamma^\mu \partial_\mu \psi_{L,R}^{(n)} = m_n \psi_{L,R}^{(n)}$. We recover our usual four dimension action by constraining our fields as

$$\int_{\frac{1}{k}}^{\acute{R}} dz \left(\frac{1}{kz}\right)^4 f_{L,c}^{(n)}(z) f_{L,c}^{(m)}(z) = \delta_{mn}, \int_{\frac{1}{k}}^{\acute{R}} dz \frac{1}{kz} \tilde{f}_{L,c}^{(n)}(z) \tilde{f}_{L,c}^{(m)}(z) = \delta_{mn} \tag{3.4.9}$$

$$\int_{\frac{1}{k}}^{\acute{R}} dz \left(\frac{1}{kz}\right)^4 f_{R,c}^{(n)}(z) f_{R,c}^{(m)}(z) = \delta_{mn}, \int_{\frac{1}{k}}^{\acute{R}} dz \frac{1}{kz} \tilde{f}_{R,c}^{(n)}(z) \tilde{f}_{R,c}^{(m)}(z) = \delta_{mn} \tag{3.4.10}$$

$$\left(\gamma_5 \partial_z + \frac{c - 2\gamma_5}{z}\right) \begin{pmatrix} f_{L,c}^{(n)}(z) \\ f_{R,c}^{(n)}(z) \end{pmatrix} = m_n \begin{pmatrix} f_{R,c}^{(n)}(z) \\ f_{L,c}^{(n)}(z) \end{pmatrix} \tag{3.4.11}$$

$$\left[\partial_z^2 - \frac{4}{z}\partial_z + \left(m_n^2 - \frac{(c^2 + c - 6)}{z^2}\right)\right] f_{L,c}^{(n)}(z) = 0 \tag{3.4.12}$$

The massless Zero-mode for $m_n = 0$ can be obtained and the general solution is given by

$$\left(\partial_z + \frac{c - 2}{z}\right) f_{L,c}^{(0)}(z) = 0 \tag{3.4.13}$$

$$\tilde{f}_{L,c}^{(0)}(z) = \sqrt{k} \sqrt{\frac{1 - 2c}{(k\acute{R})^{1-2c} - 1}} (kz)^{\frac{1}{2} - c} \tag{3.4.14}$$

and

$$\tilde{f}_{R,c}^{(0)}(z) = \sqrt{k} \sqrt{\frac{1 + 2c}{(k\acute{R})^{1+2c} - 1}} (kz)^{\frac{1}{2} + c} \tag{3.4.15}$$

the Z_2 symmetry implies that one of the components $\psi_{L,R}$ must always be odd. If γ_5 is $diag(1, -1)$, then equation (1.36) implies that $\psi_{R,L}$ is odd and there is no corresponding zero mode for this components of ψ. The Kaluza Klien modes for $m_n \neq 0$ is given by

$$f_{L,c}^{(n)}(z) = (kz)^{\frac{5}{2}} \frac{1}{N_n} \left[J_\alpha(zm_n) + b_n Y_\alpha(zm_n)\right] \tag{3.4.16}$$

where $\alpha = c + \frac{1}{2}$ and in their canonically normalized form

$$f_{L,c}^{(n)}(z) = kz\frac{1}{N_n}\left[J_\alpha(zm_n) + b_n Y_\alpha(zm_n)\right] \tag{3.4.17}$$

we have $f_{R,c}^{(n)}(z) = f_{L,-c}^{(n)}(z)$ therefore

$$\tilde{f}_{R,c}^{(n)}(z) = kz\frac{1}{N_n}\left[J_\beta(zm_n) + b_\beta Y_\beta(zm_n)\right] \tag{3.4.18}$$

where $\beta = c - \frac{1}{2}$

$$b_{c\pm\frac{1}{2}} = -\frac{J_{c\pm\frac{1}{2}}\left(\frac{m_n}{k}\right)}{Y_{c\pm\frac{1}{2}}\left(\frac{m_n}{k}\right)} \tag{3.4.19}$$

$$N_n^2 = \int\left[J_{\pm c+\frac{1}{2}}(m_n z) + b_{c\pm\frac{1}{2}}Y_{\pm c+\frac{1}{2}}(m_n z)\right]^2 \tag{3.4.20}$$

The Kaluza-Klein masses are determined by imposing the boundary conditions and in the limit $\pi kR \gg 1$ lead to the approximate values

$$m_n \approx \left(n + \frac{1}{2}\left(c \pm \frac{1}{2}\right) - \frac{1}{4}\right)\pi k e^{-\pi kR} \tag{3.4.21}$$

3.5 Bulk Interactions

$$S_{Int} = \int d^4x \int_{\frac{1}{k}}^{\acute{R}} \sqrt{-g}\left[g_5\overline{\Psi}\Gamma^M A_M\Psi + Y\overline{\Psi}\Phi\Psi + \lambda\Phi^4\right] \tag{3.5.1}$$

$$S_{Int} = \int d^4x \int_{\frac{1}{k}}^{\acute{R}} \left(\frac{1}{kz}\right)^4\left[g_5\overline{\psi}\gamma^\mu A_\mu\psi + \overline{\psi}\gamma^5 A_5\psi + Y\frac{1}{kz}\overline{\Psi}\Phi\Psi + \lambda\frac{1}{kz}\Phi^4\right] \tag{3.5.2}$$

3.5.1 Gauge Interaction.

$$\begin{aligned} S_{Int} &= \sum_{n,m,l}\int d^4x \int_{\frac{1}{k}}^{\acute{R}}\left(\frac{1}{kz}\right)^4 g_5\Big(\overline{\psi_L}^{(n)}\gamma^\mu A_\mu^{(m)}\psi_L^{(l)} f_{L,c}^{(n)}(z) f_{A_\mu}^{(m)}(z) f_{L,c}^{(l)}(z) \\ &+ \overline{\psi_L}^{(n)}\gamma^\mu A_\mu^{(m)}\psi_R^{(l)} f_{L,c}^{(n)}(z) f_{A_\mu}^{(m)}(z) f_{R,c}^{(l)}(z) + \overline{\psi_R}^{(n)}\gamma^\mu A_\mu^{(m)}\psi_L^{(l)} f_{R,c}^{(n)}(z) f_{A_\mu}^{(m)}(z) f_{L,c}^{(l)}(z) \\ &+ \overline{\psi_R}^{(n)}\gamma^\mu A_\mu^{(m)}\psi_R^{(l)} f_{R,c}^{(n)}(z) f_{A_\mu}^{(m)}(z) f_{R,c}^{(l)}(z)\Big) \end{aligned} \tag{3.5.3}$$

$$
\begin{aligned}
S_{Int} \;=\; & \int d^4x \int \frac{g_{5D}}{\sqrt{\pi R}} \overline{\psi_L}^{(0)} \gamma^\mu A_\mu^{(0)} \psi_L^{(0)} + g_{5D} A^{0n0} \overline{\psi_L}^{(0)} \gamma^\mu \sum_{n=1} A_\mu^{(n)} \psi_L^{(0)} \\
+ \; & g_{5D} A^{0nm} \overline{\psi_L}^{(0)} \gamma^\mu \sum_{n=1} A_\mu^{(n)} \sum_{m=1} \psi_L^{(m)} + \frac{g_{5D}}{\sqrt{\pi R}} \sum_{n=1} \overline{\psi_L}^{(n)} \gamma^\mu A_\mu^{(0)} \psi_L^{(m)} \\
+ \; & g_{5D} A^{nm0} \sum_{n,m=1} \overline{\psi_L}^{(n)} \gamma^\mu A_\mu^{(m)} \psi_L^{(0)} + g_{5D} A^{nml} \sum_{n,m,l=1} \overline{\psi_L}^{(n)} \gamma^\mu \sum_m A_\mu^{(m)} \sum_l \psi_L^{(l)} \\
+ \; & L \longrightarrow R
\end{aligned}
\tag{3.5.4}
$$

Where,

$$
\begin{aligned}
A^{0nm} = A^{nm0} &= \int dz \frac{1}{kz} f_{L,c}^{(0)} f_{A_\mu}^{(n)} f_{L,c}^{(m)} \\
A^{0n0} &= \int dz \frac{1}{kz} f_{L,c}^{(0)} f_{A_\mu}^{(n)} f_{L,c}^{(0)} \\
A^{nml} &= \int dz \frac{1}{kz} f_{L,c}^{(n)} f_{A_\mu}^{(m)} f_{L,c}^{(l)}
\end{aligned}
$$

forA_5 interaction with fermions, we have:

$$
\begin{aligned}
S_{IntA_5} \;=\; & \int d^4x \int g_{5D} B^{0n0} \overline{\psi_L}^{(0)} \gamma^5 \sum_{n=1} A_5^{(n)} \psi_L^{(0)} + g_{5D} B^{0nm} \overline{\psi_L}^{(0)} \gamma^5 \sum_{n=1} A_5^{(n)} \sum_{m=1} \psi_L^{(m)} \\
+ \; & g_{5D} B^{nm0} \sum_{n,m=1} \overline{\psi_L}^{(n)} \gamma^5 A_5^{(m)} \psi_L^{(0)} + g_{5D} B^{nml} \sum_{n,m,l=1} \overline{\psi_L}^{(n)} \gamma^5 \sum_m A_5^{(m)} \sum_l \psi_L^{(l)} \\
+ \; & L \longrightarrow R
\end{aligned}
\tag{3.5.5}
$$

Where,

$$
\begin{aligned}
B^{0nm} = B^{nm0} &= \int dz \frac{1}{kz} f_{L,c}^{(0)} f_{A_5}^{(n)} f_{L,c}^{(m)} \\
B^{0n0} &= \int dz \frac{1}{kz} f_{L,c}^{(0)} f_{A_5}^{(n)} f_{L,c}^{(0)} \\
B^{nml} &= \int dz \frac{1}{kz} f_{L,c}^{(n)} f_{A_5}^{(m)} f_{L,c}^{(l)}
\end{aligned}
$$

These integrals rabidly oscillates and we should find a way to calculate their values, otherwise it would be without physical meaning, so we need to numerically calculate their values according to our model.

3.6 Renormalization group Equations (RGEs)

Studying the non-interacting theories; in that case the parameters of the Lagrangian would give a physical values same as in the experiments, however most of the theories are interacting theories such as the SM and RS models. Adding interaction terms to these Lagrangian would led to divergence, because in the experiment we only measure finite parameters. That means the parameters in the Lagrangian in case of the interaction theories won't correspond to physical quantities and somehow we must find these quantities in our calculations, this tool is the Renormalization Group Equations (RGEs).

The main reasons to use the technique of RGEs is to study the momentum dependent of our desired parameters at high scale. In other words; we use RGEs to study the properties of gauge coupling constants at high scale, because this would allow us to understand better the nature of these interactions. The RGEs of gauge coupling constants evolve with energy as following

$$16\pi^2 \frac{dg_i}{dt} = b_i g_i^3 \tag{3.6.1}$$

The mathematical proof for this equation can be found in [Olli Koskivaara, 2014]. Where, $g_i, \quad i = 1, 2, 3$ are the coupling constant for the desired interactions and b_i are the numerical values that have to be evaluated for a given model for these interactions, in our case for the SM and RS models.

3.6.1 RGEs in SM. To start with, the numerical coefficients b_i is given by

$$b_i^{SM} = \left[\frac{11}{3} C_2(G) - \frac{4}{3} n_g C(R) - \frac{1}{3} n_H C_2(R)\right] \tag{3.6.2}$$

Where the first term represents the gauge fields contribution, the second one represents the fermions contribution and the last term represents the scalar Higgs contribution. $C_2(G)$ is the group structure of adjoin representation, its known that for any $SU(N)$ group is

$$C_2(G) = N \tag{3.6.3}$$

And $C(R)$, $C_2(R)$ are the group structure for the fundamental representation of $SU(N)$; fermion and scalar higgs; given by $C(R) = 1$, and $C_2(R) = \frac{1}{2}$. respectively.

Now we are in the right position to evaluate b_i^{SM}.

Firstly consider b_3^{SM} : $SU(3)$ for strong interaction, here we have

$$C_2(G) = 3 \tag{3.6.4}$$
$$C(R) = 1 \tag{3.6.5}$$
$$C_2(R) = 0 \tag{3.6.6}$$

Note that $C_2(R) = 0$ because the Higgs scalar field does not have color. Substitute these into eqn:(3.6.2), we get:

$$b_3^{SM} = \left[\frac{11}{3}(3) - \frac{4}{3}(3)(1) - 0\right] = 7 \tag{3.6.7}$$

Secondly, b_2^{SM} : $SU(2)$ for Weak interaction, here we have:

$$C_2(G) = 2 \tag{3.6.8}$$
$$C(R) = 1 \tag{3.6.9}$$
$$C_2(R) = \frac{1}{2} \tag{3.6.10}$$

Substitute them into eqn:(3.6.2), we get:

$$b_3^{SM} = \left[\frac{11}{3}(2) - \frac{4}{3}(3)(1) - \frac{1}{3}(1)\left(\frac{1}{2}\right)\right] = \frac{19}{6} \tag{3.6.11}$$

Finally, b_1^{SM} : $U(1)_Y$ Hypercharge, in this case:

$$C_2(G) = 0 \tag{3.6.12}$$

Substitute the hypercharge of the SM particles into eqn:(3.6.2), we get:

$$\begin{aligned} b_1^{SM} &= \Big(-\frac{4}{3}n_g\left(\frac{Y_\psi}{2}\right)^2 - \frac{1}{3}n_H\left(\frac{Y_\phi}{2}\right)^2\Big) \\ &= \Big(-\frac{4}{3}(3)\Big(2\left(\frac{1}{6}\right)^2(3) + \left(\frac{2}{3}\right)^2(3) + \left(\frac{1}{3}\right)^2(3) + \left(\frac{1}{3}\right)^2(3) + 2\left(\frac{-1}{2}\right)^2 + (-1)^2\Big)\frac{1}{2}\Big) \\ &- \Big(\frac{1}{3}(2)\left(\frac{1}{2}\right)^2\Big) \\ &= -\frac{41}{6} \end{aligned} \tag{3.6.13}$$

Normalizing b_1^{SM} with $SU(5)$ normalization we get

$$b_1^{SM} = -\frac{41}{6} \times \frac{3}{5} = -\frac{41}{10} \tag{3.6.14}$$

3.6.2 RGEs in RS. The Numerical Coefficients b_i^{RS} in RS model is given by

$$b_i^{RS} = \left[\frac{11}{3}C_2(G)I^{(1,0)} - \frac{4}{3}n_g C(R)I^{(\frac{1}{2},0)} - \frac{1}{3}n_H C_2(R)I^{(2,0)} - \frac{1}{6}C_2(G)I^{(1,1)}\right] \tag{3.6.15}$$

Where the first term represents the gauge fields contribution, the second one represents the fermions contribution and the third term represents the scalar Higgs contribution and the last term is the A_5^n contribution. Here $I^{(1,0)}$, $I^{(\frac{1}{2},0)}$, $I^{(2,0)}$ and $I^{(1,1)}$ are the sum of KK number contribution for the gauge, fermion, Higgs and adjoint scalar running in the loop. These quantities involving triplet Bessel functions integrals as calculated in section 3.5. We take their values from Ref. [Randall & Schwartz, 2002]; and they are $I^{(1,0)} = 1.024$, $I^{(\frac{1}{2},0)} = 1.009$, $I^{(2,0)} = 1.009$ and $I^{(1,1)} = 0.013$. Now substitute the group structure for each group of the SM, we obtain for $U(1)_Y$

$$b_1^{RS} = \left[-\frac{40}{3}(3)I^{(\frac{1}{2},0)} - \frac{2}{3}\frac{1}{4}I^{(2,0)}\right] \tag{3.6.16}$$

For $SU(2)$ weak interaction, we get

$$b_2^{RS} = \left[\frac{11}{3}(2))I^{(1,0)} - \frac{4}{3}(3)I^{(\frac{1}{2},0)} - \frac{1}{3}\frac{1}{2}I^{(2,0)} - \frac{1}{6}(2)I^{(1,1)}\right] \tag{3.6.17}$$

and for $SU(3)$ strong interaction, we have

$$b_3^{RS} = \left[\frac{11}{3}(3)I^{(1,0)} - \frac{4}{3}(3)I^{(\frac{1}{2},0)} - \frac{1}{6}(3)I^{(1,1)}\right] \tag{3.6.18}$$

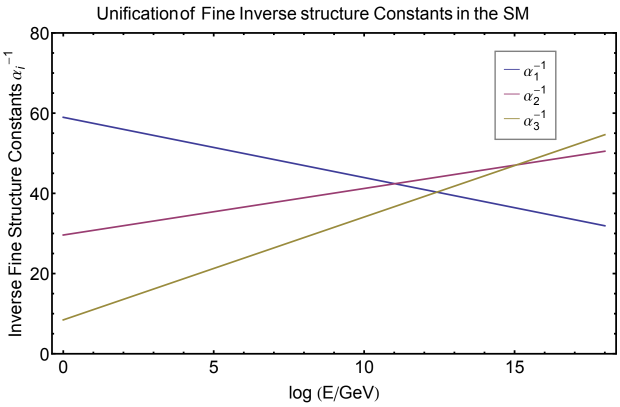
Therefore our RGEs in RS model is given by

$$16\pi^2\frac{dg_i}{dt} = b_i^{RS}g_i^3 \tag{3.6.19}$$

These set of RGEs will be solved and plotted using matheamtica. The results will be discussed in next chapter.

4. Results and Discussions

Chapter four present our numerical results and discussions for the gauge coupling constants in 4D Standard Model and 5D Randall Sundrum model. Only some selected plots will be shown and we will comment on the other similar cases not explicitly presented here. We quantitatively analyzes and explored these quantities in RS model. We employ the technique of renormalization group equations with the initial values adopted at the M_Z scale as follows: for the gauge couplings $g_1(M_Z) = 0.462, g_2(M_Z) = 0.651$ and $g_3(M_Z) = 1.22$. These calculations need to be performed numerically by using dedicated numerical packages. Here in this dissertation we used MATHEMATICA program version 9 to obtain the figures.. We will discuss in both scenarios SM and RS the evolution of the inverse fine structure constants which is related to the gauge couplings by $\alpha^{-1} = \frac{4\pi}{g^2}$.



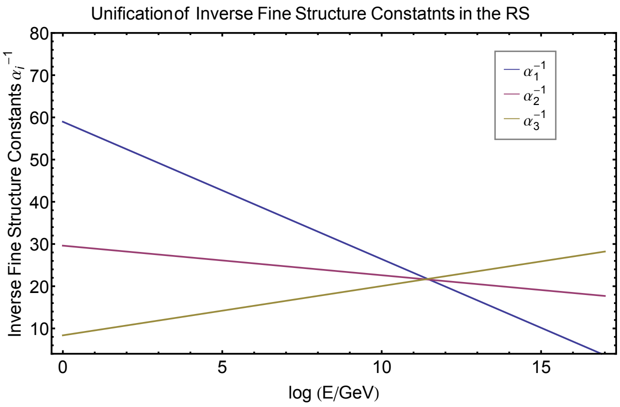


Figure 4.1: Shows the inverse fine structure constants α_1^{-1} (red), α_2^{-1} (blue), α_3^{-1}(gold) with: in the left panel, SM case; and the right panel for RS model as a function of energy scale.

Fig:[4.1] left panel shows that all three standard model inverse fine structure constants are trying to unify at some higher scale for one-loop level and in fact higher order loop correction does not change the result much as its effects is very small that means we need a new physics that have new particles to change the running of gauge coupling. This new physics could be Warped Extra dimension. As depicted in Fig:[4.1] right panel, the unification of inverse fine structure constants in RS model has accelerating and affect the running to lower value, we note that the running is logarithmic as in the SM case. In RS model the unification of inverse fine structure constants is achieved around 10^{10} GeV compare to the SM which is $\approx 10^{14}$ GeV. Furthermore we present the evolution of Weinberg mixing angle as function of energy scale in the SM and RS models. Note that evolution trajectories evolve until the unification scale of gauge couplings unify. Once the KK states begin to contribute the new contributions from the extra dimensions increases until we reach the cutoff scale. This result may be useful, at least from a model building perspective, as many extra-dimensional models such as gauge-Higgs unification models in UED predict for many

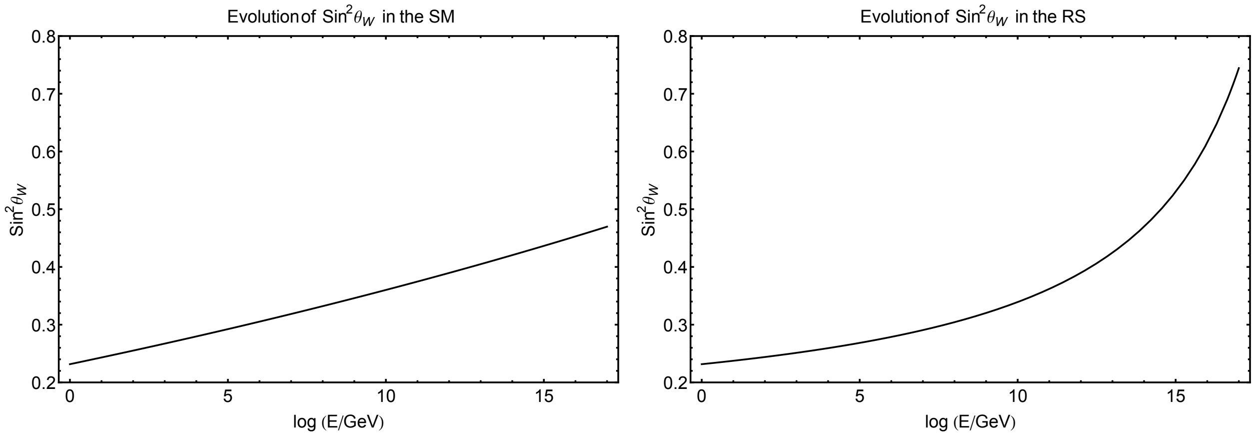


Figure 4.2: *Evolution of Weinberg angle (*$\sin^2\theta_W$ *with: in the left panel, SM case; and the right panel for RS model as a function of energy scale.*

choices of the gauge group large values of $\sin^2\theta_W$ from a group theory point of view. However, this value is the one expected in the energy range of coupling unification, which once evolved back to the electroweak scale and may indeed be close or compatible to the measured value

5. Conclusions

In conclusion, we have studied the possibilities of gauge couplings unification in warped geometry that is RS model in AdS_5.
We derived a set of the renormalization group equations for the gauge coupling constants at one loop level. We also discussed the inverse fine structure constants as well as the evolution of $\sin^2\theta_W$ as indicator of stability of this model. We showed that the running here in this warped geometry is logarithmic running as in the case of the standard model and not a power law like other extra dimension models. What is more, we found that such theory can be compatible with gauge couplings unification at high scale, the scale at which unification occurred depends strongly on the cutoff Λ and the number of Kaluza Klein states. Furthermore, we found that $\sin^2\theta_W$ rises up to 0.74 at high scale.

These results are very useful, at least from a model building perspective.

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